# Lecture 5

- 13.1 Functions of Several Variables
- 13.2 Limits and Continuity in Higher Dimensions
- 13.3 Partial Derivatives
- 13.4 The Chain Rule

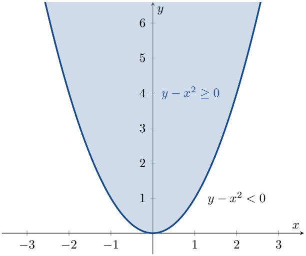
# Functions of Several Variables

We will be considering functions $f : D \subseteq \mathbb{R}^2 \to \mathbb{R}$ and functions $g : D \subseteq \mathbb{R}^3 \to \mathbb{R}$.

## Example

$f : D \to \mathbb{R}$, $f(x, y) = \sqrt{y - x^2}$

domain: $y \geq x^2$




range: $[0, \infty)$

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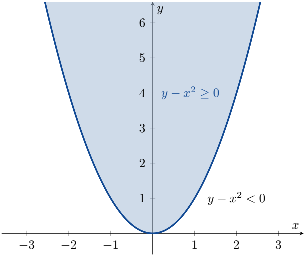
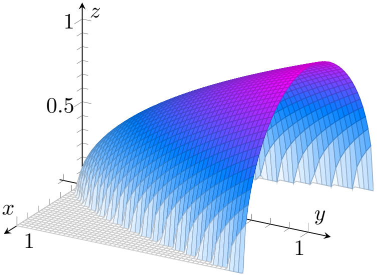
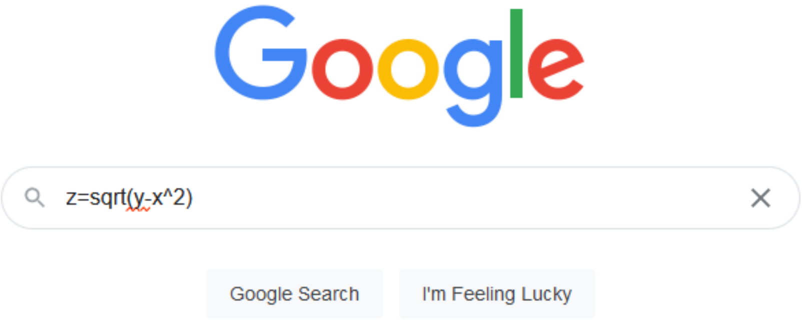



range: $[0, \infty)$

graph:

$z = \sqrt{y - x^2}$




Google
z=sqrt(y-x^2)
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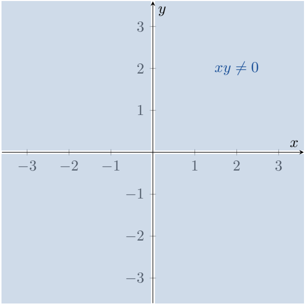
I'm Feeling Lucky

**Example**

$f : D \to \mathbb{R},\ f(x,y) = \dfrac{1}{xy}$

domain: $xy \neq 0$




range: $(-\infty, 0) \cup (0, \infty)$

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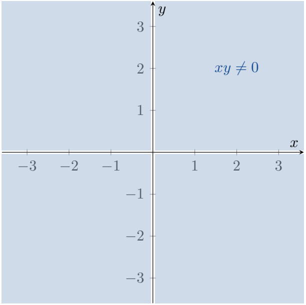


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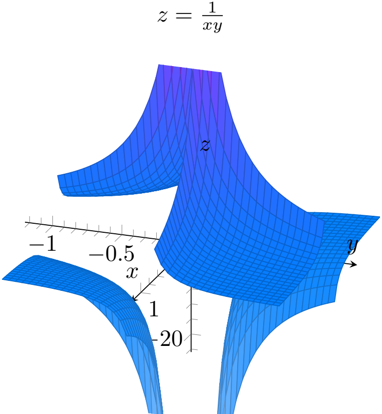
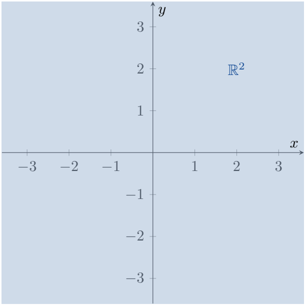


**Example**

$f : D \to \mathbb{R}$, $f(x, y) = \sin xy$

domain: $\mathbb{R}^2$




range: $[-1, 1]$

**Example**

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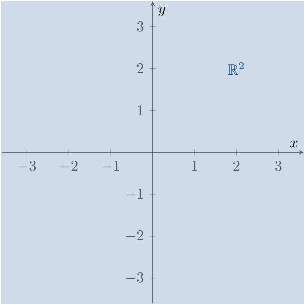
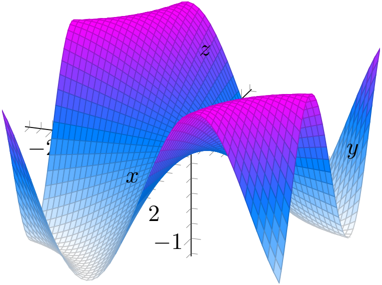



range: $[-1, 1]$

graph:

$z = \sin xy$



**Example**

$g : D \to \mathbb{R}$, $g(x, y, z) = \sqrt{x^2 + y^2 + z^2}$

domain: $\mathbb{R}^3$

range: $[-1, 1]$

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$$g : D \to \mathbb{R},\ g(x, y, z) = \frac{1}{x^2 + y^2 + z^2}$$

domain: $(x, y, z) \neq (0, 0, 0)$ range: $(0, \infty)$

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**Example**

$g : D \to \mathbb{R},\ g(x, y, z) = xy \ln z$

domain: $z > 0$ range: $\mathbb{R}$

# Topology

In $\mathbb{R}$ we have open intervals $(a, b)$, closed intervals $[a, b]$ and half-open intervals $[a, b)$, $(a, b]$.

What do we have in $\mathbb{R}^2$ and $\mathbb{R}^3$?

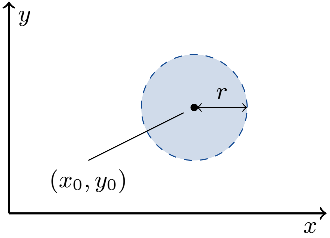
$\mathbb{R}^2$

$\mathbb{R}^3$

$\mathbb{R}^2$ The set

$$\{(x, y) \mid (x - x_0)^2 + (y - y_0)^2 < r^2\}$$

is called an *open disc* of radius $r$ centred at $(x_0, y_0)$.

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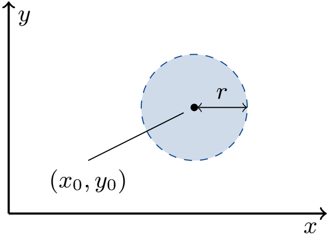


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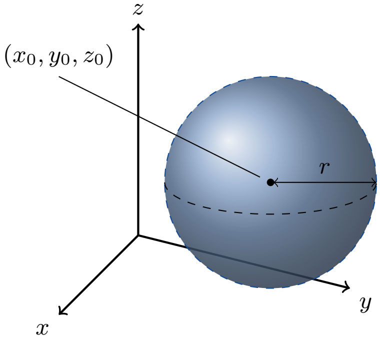
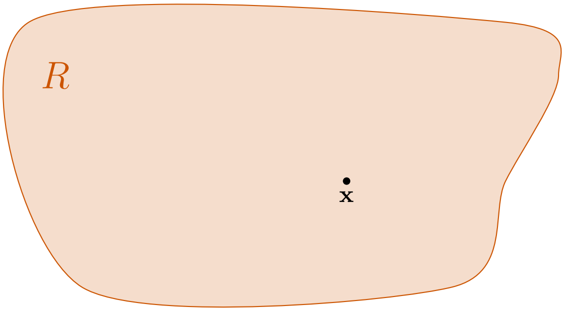
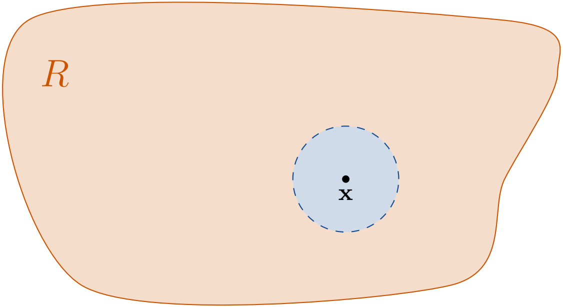
$\mathbb{R}^3$ The set

$$\{(x,y, z) \mid (x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2 < r^2\}$$

is called an *open ball* of radius $r$ centred at $(x_0, y_0, z_0)$.

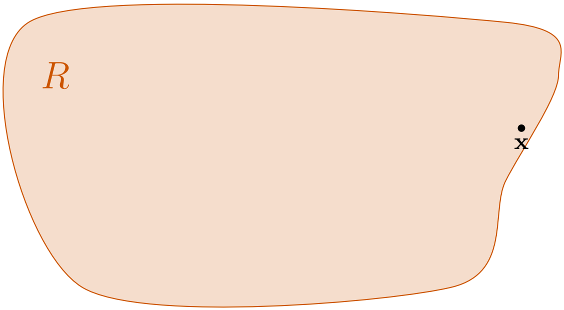



$R$
$\mathbf{x}$




**Definition**

A point $\mathbf{x}$ in $R$ is called an *interior point* of $R$ iff there exists an open disk/ball centred at $\mathbf{x}$ which lies entirely inside $R$.

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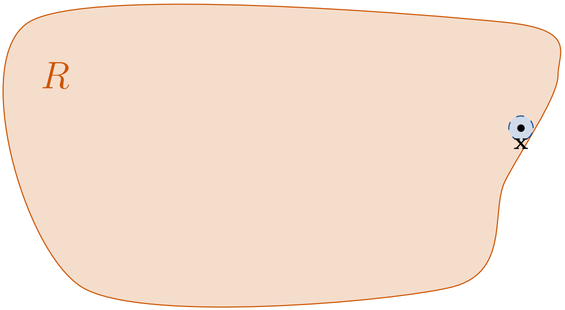


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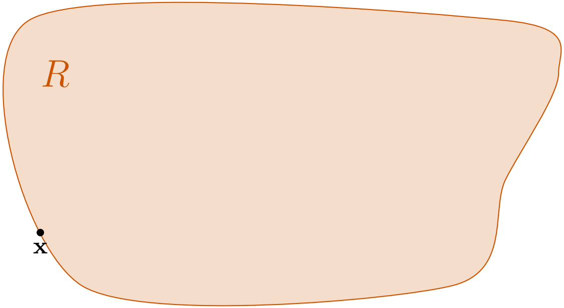
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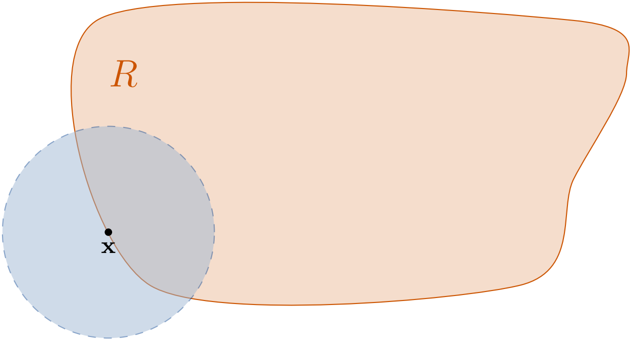
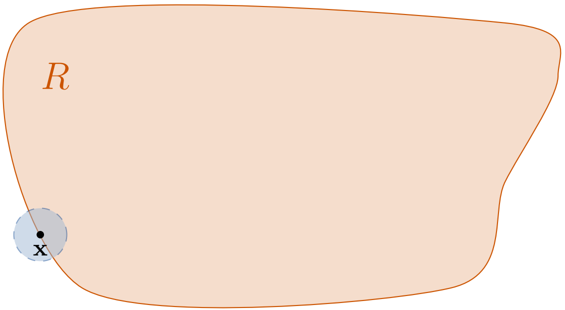



### Definition

A point $\mathbf{x}$ is called a *boundary point* of $R$ iff every open disk/ball centred at $\mathbf{x}$ which partly inside $R$ and partly outside $R$..

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A set $R$ is called *open* if every point in $R$ is an interior point.

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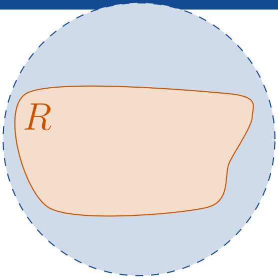
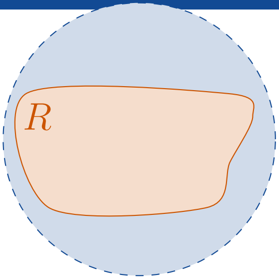
**Definition**

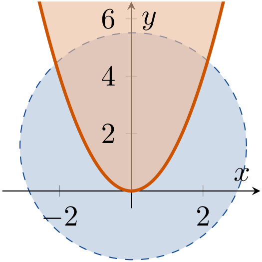
A set $R$ is called *closed* if it contains all of its boundary points.




**Definition**

A set $R$ is called *bounded* iff it lies inside a disk/ball.




**Definition**

A set $R$ is called *bounded* iff it lies inside a disk/ball.

**Definition**

A set $R$ is called *unbounded* iff it is not bounded.

# Limits and Continuity in Higher Dimensions

## Limits

$$\lim_{x \to x_0} f(x) = L$$

"If $x$ is close to $x_0$ (but $x \neq x_0$), then $f(x)$ is close to $L$."

**Definition**

$\mathbb{R}$

We write $\lim_{x \to x_0} f(x) = L$ iff for all $\varepsilon > 0$, there exists $\delta = \delta(\varepsilon) > 0$ such that for all $x \in D$

$$0 < |x - x_0| < \delta \qquad \Longrightarrow \qquad |f(x) - L| < \varepsilon.$$

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We write $\lim_{(x,y) \to (x_0,y_0)} f(x) = L$ iff for all $\varepsilon > 0$, there exists $\delta = \delta(\varepsilon) > 0$ such that for all $(x, y) \in D$

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$$0 < \sqrt{(x - x_0)^2 + (y - y_0)^2} < \delta$$

**Definition**

$\mathbb{R}^2$ We write $\lim_{(x,y)\to(x_0,y_0)} f(x) = L$ iff for all $\varepsilon > 0$, there exists $\delta = \delta(\varepsilon) > 0$ such that for all $(x, y) \in D$

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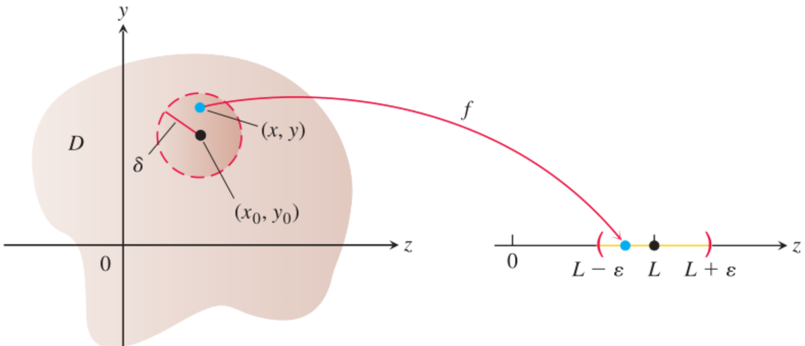

**Definition**

$\mathbb{R}^2$ We write $\displaystyle\lim_{(x,y)\to(x_0,y_0)} f(x) = L$ iff for all $\varepsilon > 0$, there exists $\delta = \delta(\varepsilon) > 0$ such that for all $(x, y) \in D$

$$0 < \|(x,y)-(x_0,y_0)\| < \delta \qquad \Longrightarrow \qquad |f(x) - L| < \varepsilon.$$

**Definition**

$\mathbb{R}^3$ We write $\displaystyle\lim_{(x,y,z)\to(x_0,y_0,z_0)} f(x) = L$ iff for all $\varepsilon > 0$, there exists $\delta = \delta(\varepsilon) > 0$ such that for all $(x, y, z) \in D$

$$0 < \|(x,y,z)-(x_0,y_0,z_0)\| < \delta \qquad \Longrightarrow \qquad |f(x) - L| < \varepsilon.$$

As for functions of a single variable, it can be shown that

$$\lim_{(x, y)\to(x_0, y_0)} x = x_0$$

$$\lim_{(x, y)\to(x_0, y_0)} y = y_0$$

$$\lim_{(x, y)\to(x_0, y_0)} k = k \qquad \text{(any number } k\text{).}$$

For example, in the first limit statement above, $f(x, y) = x$ and $L = x_0$. Using the definition of limit, suppose that $\varepsilon > 0$ is chosen. If we let $\delta$ equal this $\varepsilon$, we see that if

$$0 < \sqrt{(x - x_0)^2 + (y - y_0)^2} < \delta = \varepsilon,$$

then

$$\begin{aligned} \sqrt{(x - x_0)^2} &< \varepsilon & \quad & (x - x_0)^2 \le (x - x_0)^2 + (y - y_0)^2 \\ |x - x_0| &< \varepsilon & & \sqrt{a^2} = |a| \\ |f(x, y) - x_0| &< \varepsilon. & & x = f(x, y) \end{aligned}$$

That is,

$$|f(x, y) - x_0| < \varepsilon \qquad \text{whenever} \qquad 0 < \sqrt{(x - x_0)^2 + (y - y_0)^2} < \delta.$$

So a $\delta$ has been found satisfying the requirement of the definition, and therefore we have proved that

$$\lim_{(x, y)\to(x_0, y_0)} f(x, y) = \lim_{(x, y)\to(x_0, y_0)} x = x_0.$$

**THEOREM 1—Properties of Limits of Functions of Two Variables**

The following rules hold if $L$, $M$, and $k$ are real numbers and

$$\lim_{(x,y)\to(x_0,y_0)} f(x,y) = L \qquad \text{and} \qquad \lim_{(x,y)\to(x_0,y_0)} g(x,y) = M.$$

1. *Sum Rule:* $\displaystyle\lim_{(x,y)\to(x_0,y_0)} (f(x,y) + g(x,y)) = L + M$
2. *Difference Rule:* $\displaystyle\lim_{(x,y)\to(x_0,y_0)} (f(x,y) - g(x,y)) = L - M$
3. *Constant Multiple Rule:* $\displaystyle\lim_{(x,y)\to(x_0,y_0)} kf(x,y) = kL \qquad \text{(any number } k)$
4. *Product Rule:* $\displaystyle\lim_{(x,y)\to(x_0,y_0)} (f(x,y) \cdot g(x,y)) = L \cdot M$
5. *Quotient Rule:* $\displaystyle\lim_{(x,y)\to(x_0,y_0)} \frac{f(x,y)}{g(x,y)} = \frac{L}{M}, \qquad M \neq 0$
6. *Power Rule:* $\displaystyle\lim_{(x,y)\to(x_0,y_0)} [f(x,y)]^n = L^n$, $n$ a positive integer
7. *Root Rule:* $\displaystyle\lim_{(x,y)\to(x_0,y_0)} \sqrt[n]{f(x,y)} = \sqrt[n]{L} = L^{1/n}$,

   $n$ a positive integer, and if $n$ is even, we assume that $L > 0$.

**EXAMPLE 1** In this example, we can combine the three simple results following the limit definition with the results in Theorem 1 to calculate the limits. We simply substitute the $x$- and $y$-values of the point being approached into the functional expression to find the limiting value.

**(a)** $$\lim_{(x,y)\to(0,1)} \frac{x - xy + 3}{x^2y + 5xy - y^3} = \frac{0 - (0)(1) + 3}{(0)^2(1) + 5(0)(1) - (1)^3} = -3$$

**(b)** $$\lim_{(x,y)\to(3,-4)} \sqrt{x^2 + y^2} = \sqrt{(3)^2 + (-4)^2} = \sqrt{25} = 5$$ ■

**Example**

Find $\lim\limits_{(x,y)\to(0,0)} \dfrac{x^2 - xy}{\sqrt{x} - \sqrt{y}}$.

**Example**

Find $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^2 - xy}{\sqrt{x} - \sqrt{y}}$.

We can't use the Quotient Rule here because $\sqrt{x} - \sqrt{y} \to 0$ as $(x, y) \to (0, 0)$.

**Example**

Find $\lim_{(x,y)\to(0,0)} \dfrac{x^2 - xy}{\sqrt{x} - \sqrt{y}}$.

We can't use the Quotient Rule here because $\sqrt{x} - \sqrt{y} \to 0$ as $(x, y) \to (0, 0)$. Instead we calculate that

$$\lim_{(x,y)\to(0,0)} \frac{x^2 - xy}{\sqrt{x} - \sqrt{y}} = \lim_{(x,y)\to(0,0)} \frac{(x^2 - xy)(\sqrt{x} + \sqrt{y})}{(\sqrt{x} - \sqrt{y})(\sqrt{x} + \sqrt{y})}$$

$$=$$

$$=$$

$$=$$

**Example**

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We can't use the Quotient Rule here because $\sqrt{x} - \sqrt{y} \to 0$ as $(x, y) \to (0, 0)$. Instead we calculate that

$$\begin{aligned}
\lim_{(x,y)\to(0,0)} \frac{x^2 - xy}{\sqrt{x} - \sqrt{y}} &= \lim_{(x,y)\to(0,0)} \frac{(x^2 - xy)(\sqrt{x} + \sqrt{y})}{(\sqrt{x} - \sqrt{y})(\sqrt{x} + \sqrt{y})} \\
&= \lim_{(x,y)\to(0,0)} \frac{x(x - y)(\sqrt{x} + \sqrt{y})}{x - y} \\
&= \\
&=
\end{aligned}$$

**Example**

Find $\lim_{(x,y)\to(0,0)} \frac{x^2 - xy}{\sqrt{x} - \sqrt{y}}$.

We can't use the Quotient Rule here because $\sqrt{x} - \sqrt{y} \to 0$ as $(x, y) \to (0, 0)$. Instead we calculate that

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&= \lim_{(x,y)\to(0,0)} \frac{x(x - y)(\sqrt{x} + \sqrt{y})}{x - y} \\
&= \lim_{(x,y)\to(0,0)} x(\sqrt{x} + \sqrt{y}) \\
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\end{aligned}$$

**Example**

Find $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^2 - xy}{\sqrt{x} - \sqrt{y}}$.

We can't use the Quotient Rule here because $\sqrt{x} - \sqrt{y} \to 0$ as $(x, y) \to (0, 0)$. Instead we calculate that

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&= \lim_{(x,y)\to(0,0)} \frac{x(x - y)(\sqrt{x} + \sqrt{y})}{x - y} \\
&= \lim_{(x,y)\to(0,0)} x(\sqrt{x} + \sqrt{y}) \\
&= 0(0 + 0) = 0.
\end{aligned}$$

**EXAMPLE 3** Find $\lim\limits_{(x,y)\to(0,0)} \dfrac{4xy^2}{x^2+y^2}$ if it exists.

**Solution** We first observe that along the line $x = 0$, the function always has value 0 when $y \neq 0$. Likewise, along the line $y = 0$, the function has value 0 provided $x \neq 0$. So if the limit does exist as $(x, y)$ approaches $(0, 0)$, the value of the limit must be 0 (see Figure 14.13). To see if this is true, we apply the definition of limit.

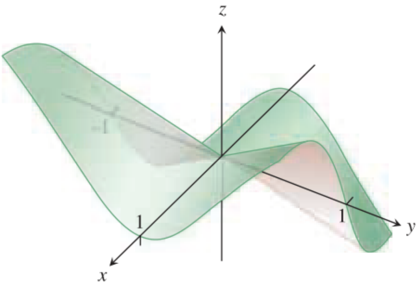



**FIGURE 14.13** The surface graph shows the limit of the function in Example 3 must be 0, if it exists.

Let $\varepsilon > 0$ be given, but arbitrary. We want to find a $\delta > 0$ such that

$$\left|\frac{4xy^2}{x^2+y^2} - 0\right| < \varepsilon \qquad \text{whenever} \qquad 0 < \sqrt{x^2+y^2} < \delta$$

or

$$\frac{4|x|y^2}{x^2+y^2} < \varepsilon \qquad \text{whenever} \qquad 0 < \sqrt{x^2+y^2} < \delta.$$

Since $y^2 \leq x^2 + y^2$ we have that

$$\frac{4|x|y^2}{x^2+y^2} \leq 4|x| = 4\sqrt{x^2} \leq 4\sqrt{x^2+y^2}. \qquad \frac{y^2}{x^2+y^2} \leq 1$$

So if we choose $\delta = \varepsilon/4$ and let $0 < \sqrt{x^2+y^2} < \delta$, we get

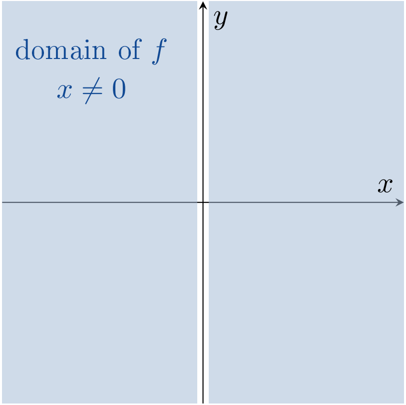
$$\left|\frac{4xy^2}{x^2+y^2} - 0\right| \leq 4\sqrt{x^2+y^2} < 4\delta = 4\left(\frac{\varepsilon}{4}\right) = \varepsilon.$$

It follows from the definition that

$$\lim_{(x,y)\to(0,0)} \frac{4xy^2}{x^2+y^2} = 0.$$

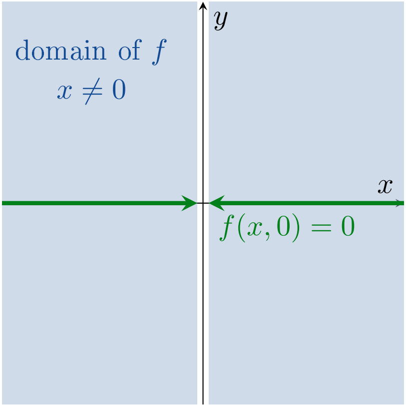
**Example**

If $f(x,y) = \frac{y}{x}$, does $\lim_{(x,y)\to(0,0)} f(x,y)$ exist?



**Example**

If $f(x,y) = \frac{y}{x}$, does $\lim\limits_{(x,y)\to(0,0)} f(x,y)$ exist?




$$\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{(x,y)\to(0,0)} 0 = 0$$

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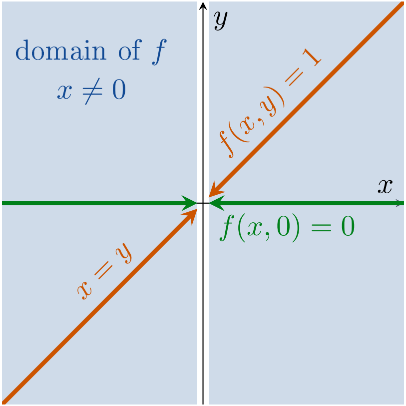


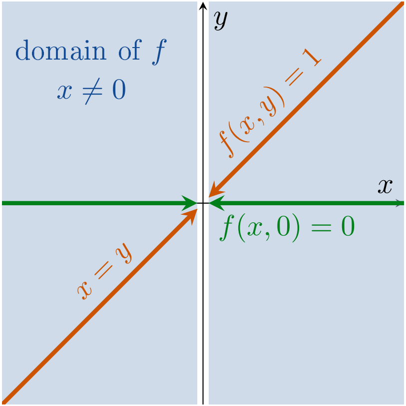

$$\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{(x,y)\to(0,0)} 0 = 0$$

$$\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{(x,y)\to(0,0)} 1 = 1$$

# 13.2 Limits and Continuity in Higher Dimensions

**Example**

If $f(x,y) = \frac{y}{x}$, does $\displaystyle\lim_{(x,y)\to(0,0)} f(x,y)$ exist?




$$\cancel{\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{(x,y)\to(0,0)} 0 = 0}$$

$$\cancel{\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{(x,y)\to(0,0)} 1 = 1}$$

$$\lim_{(x,y)\to(0,0)} f(x,y) \text{ does not exist.}$$

## Continuity

**DEFINITION** A function $f(x, y)$ is **continuous at the point $(x_0, y_0)$** if

1. $f$ is defined at $(x_0, y_0)$,
2. $\lim_{(x, y) \to (x_0, y_0)} f(x, y)$ exists,
3. $\lim_{(x, y) \to (x_0, y_0)} f(x, y) = f(x_0, y_0)$.

A function is **continuous** if it is continuous at every point of its domain.

**EXAMPLE 5** Show that

$$f(x, y) = \begin{cases} \dfrac{2xy}{x^2 + y^2}, & (x, y) \neq (0, 0) \\ 0, & (x, y) = (0, 0) \end{cases}$$

is continuous at every point except the origin (Figure 14.14).

**Solution** The function $f$ is continuous at every point $(x, y)$ except $(0, 0)$ because its values at points other than $(0, 0)$ are given by a rational function of $x$ and $y$, and therefore at those points the limiting value is simply obtained by substituting the values of $x$ and $y$ into that rational expression.

At (0, 0), the value of $f$ is defined, but $f$ has no limit as $(x, y) \to (0, 0)$. The reason is that different paths of approach to the origin can lead to different results, as we now see.

For every value of $m$, the function $f$ has a constant value on the "punctured" line $y = mx$, $x \neq 0$, because

$$f(x, y)\Big|_{y=mx} = \frac{2xy}{x^2 + y^2}\Big|_{y=mx} = \frac{2x(mx)}{x^2 + (mx)^2} = \frac{2mx^2}{x^2 + m^2x^2} = \frac{2m}{1 + m^2}.$$

Therefore, $f$ has this number as its limit as $(x, y)$ approaches (0, 0) along the line:

$$\lim_{\substack{(x, y)\to(0,0) \\ \text{along } y=mx}} f(x, y) = \lim_{(x, y)\to(0,0)} \left[ f(x, y)\Big|_{y=mx} \right] = \frac{2m}{1 + m^2}.$$

This limit changes with each value of the slope $m$. There is therefore no single number we may call the limit of $f$ as $(x, y)$ approaches the origin. The limit fails to exist, and the function is not continuous at the origin. ■

If a function $f(x, y)$ has different limits along two different paths in the domain of $f$ as $(x, y)$ approaches $(x_0, y_0)$, then

$$\lim_{(x,y)\to(x_0,y_0)} f(x, y)$$

does not exist.

**EXAMPLE 6** Show that the function

$$f(x, y) = \frac{2x^2y}{x^4 + y^2}$$

(Figure 14.15) has no limit as $(x, y)$ approaches $(0, 0)$.

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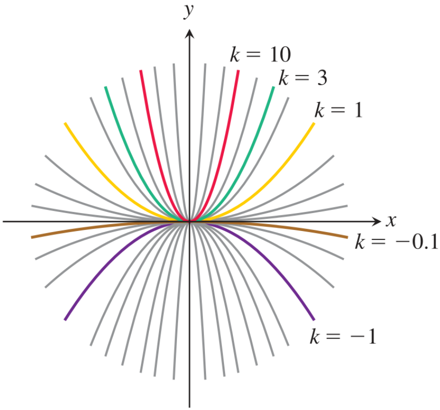


$$y = kx^2$$

**Solution** The limit cannot be found by direct substitution, which gives the indeterminate form $0/0$. We examine the values of $f$ along parabolic curves that end at $(0, 0)$. Along the curve $y = kx^2$, $x \neq 0$, the function has the constant value

$$f(x, y)\Big|_{y=kx^2} = \frac{2x^2y}{x^4 + y^2}\Big|_{y=kx^2} = \frac{2x^2(kx^2)}{x^4 + (kx^2)^2} = \frac{2kx^4}{x^4 + k^2x^4} = \frac{2k}{1 + k^2}.$$

Therefore,

$$\lim_{\substack{(x, y)\to(0,0) \\ \text{along } y=kx^2}} f(x, y) = \lim_{(x, y)\to(0,0)} \left[ f(x, y)\Big|_{y=kx^2} \right] = \frac{2k}{1 + k^2}.$$

This limit varies with the path of approach. If $(x, y)$ approaches $(0, 0)$ along the parabola $y = x^2$, for instance, $k = 1$ and the limit is 1. If $(x, y)$ approaches $(0, 0)$ along the $x$-axis, $k = 0$ and the limit is 0. By the two-path test, $f$ has no limit as $(x, y)$ approaches $(0, 0)$. ■

It can be shown that the function in Example 6 has limit 0 along every straight line path $y = mx$ (Exercise 57). This implies the following observation:

> Having the same limit along all straight lines approaching $(x_0, y_0)$ does not imply that a limit exists at $(x_0, y_0)$.

Almost everything that we have done here has been for functions of two variables. But the same ideas can be extended to functions of three (or more) variables.

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### Example

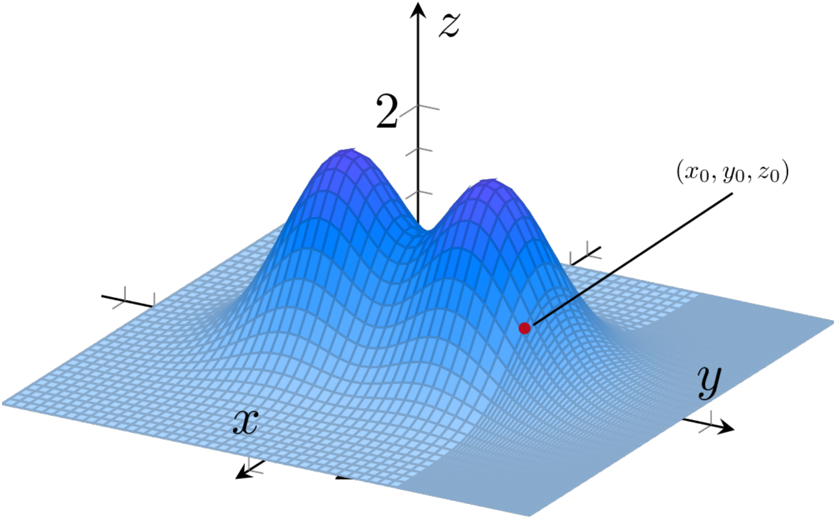
$$\lim_{(x,y,z)\to(1,0,-1)} \frac{e^{x+y+z}}{z^2 + \cos\sqrt{xy}} =$$

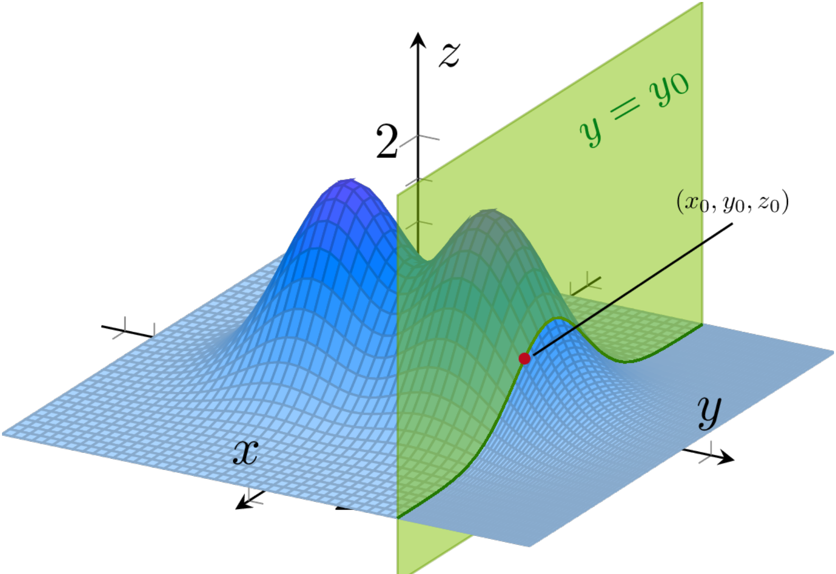
Almost everything that we have done here has been for functions of two variables. But the same ideas can be extended to functions of three (or more) variables.

**Example**

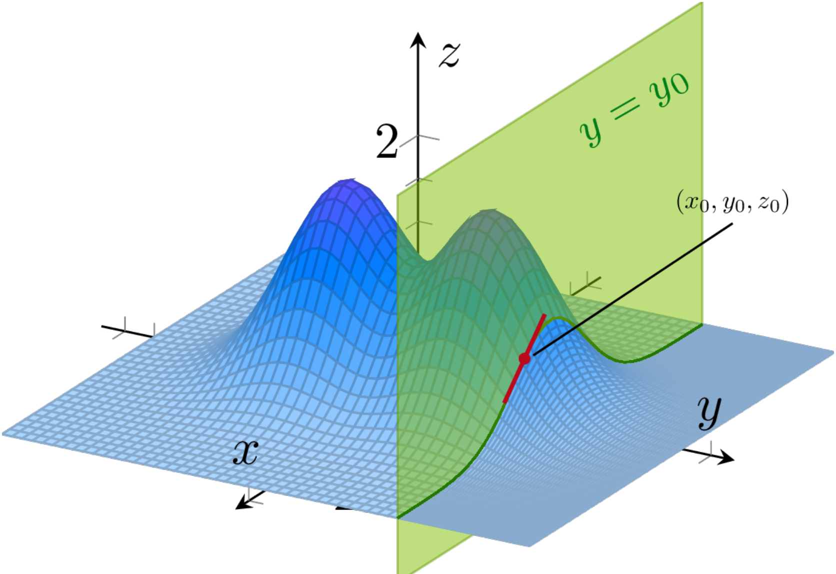
$$\lim_{(x,y,z)\to(1,0,-1)} \frac{e^{x+y+z}}{z^2+\cos\sqrt{xy}} = \frac{e^{1+0-1}}{(-1)^2+\cos 0} = \frac{1}{2}.$$

# 13.3 Partial Derivatives

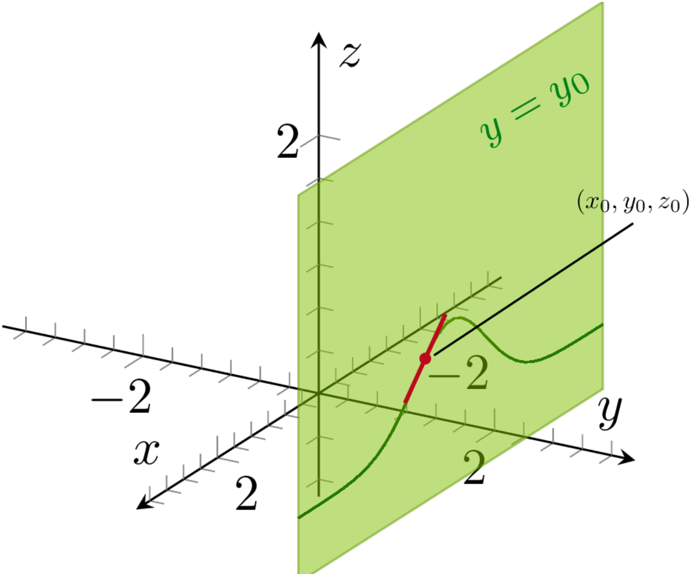

z
2
(x_0, y_0, z_0)
x
y



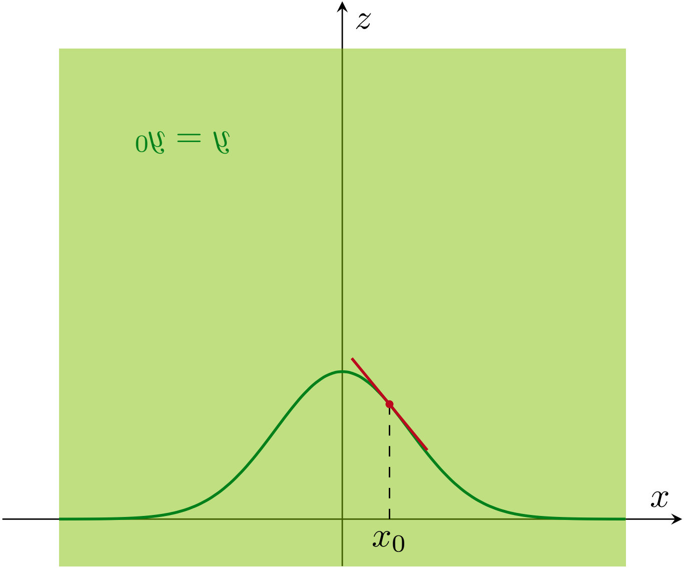
$z$
2
$y = y_0$
$(x_0, y_0, z_0)$
$x$
$y$



$z$
2
$y = y_0$
$(x_0, y_0, z_0)$
$x$
$y$



$z$
$y = y_0$
$(x_0, y_0, z_0)$
$2$
$-2$
$-2$
$x$
$2$
$2$
$y$


$z$
$y = y_0$
$x$
$x_0$

### Definition

The *partial derivative* of $f(x, y)$ with respect to $x$ at the point $(x_0, y_0)$ is

$$\left.\frac{\partial f}{\partial x}\right|_{(x_0,y_0)} = \lim_{h\to 0} \frac{f(x_0+h, y_0) - f(x_0, y_0)}{h}$$

if the limit exists.

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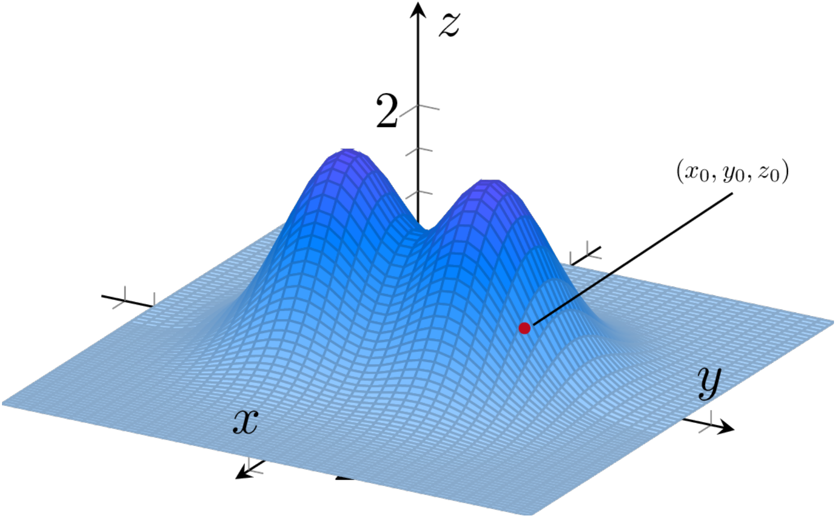
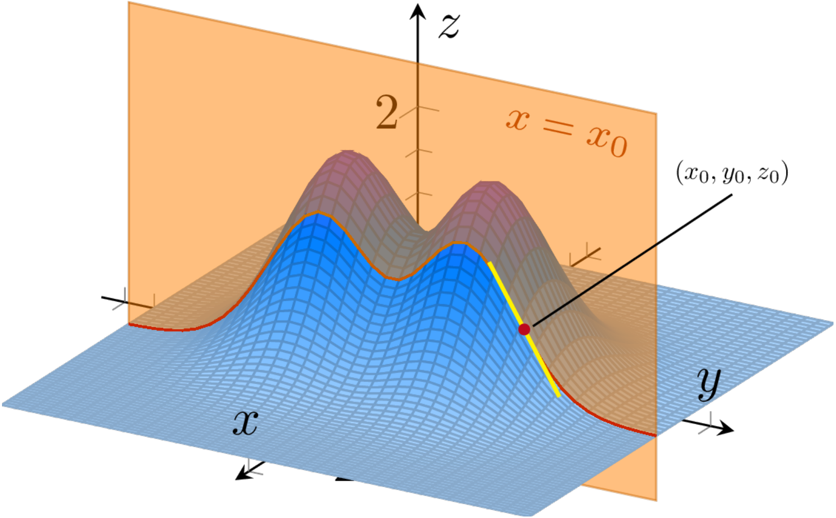
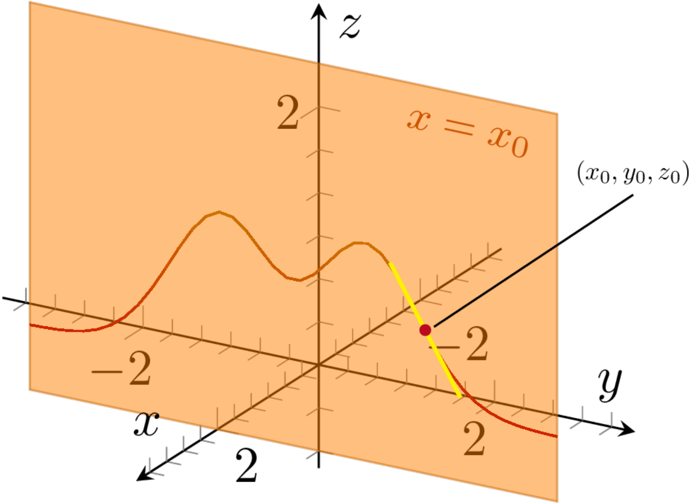
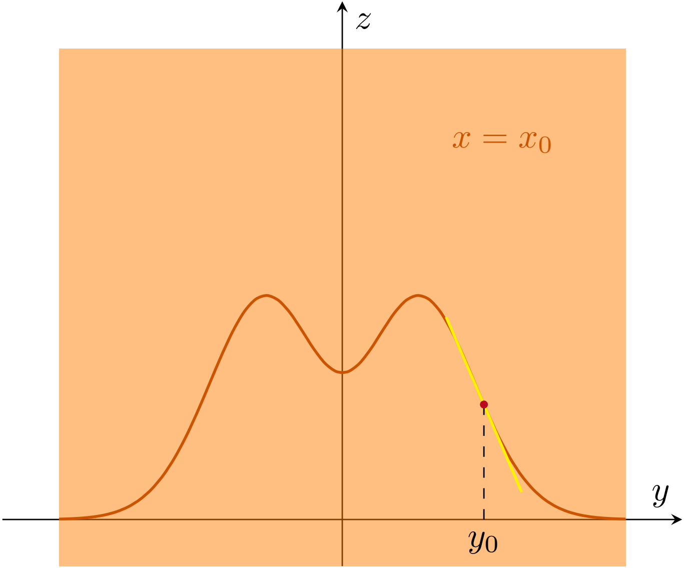


$z$
2
$(x_0, y_0, z_0)$
$x$
$y$



$z$
2
$x = x_0$
$(x_0, y_0, z_0)$
$x$
$y$


$z$
2
$x = x_0$
$(x_0, y_0, z_0)$
$-2$
$-2$
$y$
$x$
2
2



$z$
$x = x_0$
$y$
$y_0$

## Notation

$$\frac{df}{dx} = f' \quad \text{ordinary derivative}$$

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$$\frac{\partial f}{\partial x} = f_x \quad \text{partial derivative wrt } x$$

$$\frac{\partial f}{\partial y} = f_y \quad \text{partial derivative wrt } y$$

**Example**

Let $f(x) = x^2 + 3xy + y - 1$.

Find $\dfrac{\partial f}{\partial x}$ and $\dfrac{\partial f}{\partial y}$ at the point $(4, -5)$.

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$$\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}\left(x^2 + 3xy + y - 1\right) = 2x + 3y + 0 - 0.$$

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$$\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}\left(x^2 + 3xy + y - 1\right) = 2x + 3y + 0 - 0.$$

Therefore

$$\left.\frac{\partial f}{\partial x}\right|_{(4,-5)} = \left. 2x + 3y\right|_{(4,-5)} = -7.$$

$$f(x) = x^2 + 3xy + y - 1$$

And to find $\dfrac{\partial f}{\partial y}$ we pretend that $x$ is a constant.

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$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}\left(x^2 + 3xy + y - 1\right) = 0 + 3x + 1 - 0$$

we have that

$$\left.\frac{\partial f}{\partial y}\right|_{(4,-5)} = 3x + 1\big|_{(4,-5)} = 13.$$

# Break

We will continue at 2pm

### Remark

As long as you "pretend" that one of the variables is a constant and only look at the other variable, then you use the tools that you already know.

**Remark**

As long as you “pretend” that one of the variables is a constant and only look at the other variable, then you use the tools that you already know.

- the product rule;
- the quotient rule;
- implicit differentiation;
- etc.

**EXAMPLE 2** Find $\partial f / \partial y$ as a function if $f(x, y) = y \sin xy$.

**Solution** We treat $x$ as a constant and $f$ as a product of $y$ and $\sin xy$:

$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}(y \sin xy) = y \frac{\partial}{\partial y} \sin xy + (\sin xy) \frac{\partial}{\partial y}(y)$$

$$= (y \cos xy) \frac{\partial}{\partial y}(xy) + \sin xy = xy \cos xy + \sin xy.$$

**EXAMPLE 3** Find $f_x$ and $f_y$ as functions if

$$f(x, y) = \frac{2y}{y + \cos x}.$$

**Solution** We treat $f$ as a quotient. With $y$ held constant, we use the quotient rule to get

$$f_x = \frac{\partial}{\partial x}\left(\frac{2y}{y + \cos x}\right) = \frac{(y + \cos x)\frac{\partial}{\partial x}(2y) - 2y\frac{\partial}{\partial x}(y + \cos x)}{(y + \cos x)^2}$$

$$= \frac{(y + \cos x)(0) - 2y(-\sin x)}{(y + \cos x)^2} = \frac{2y \sin x}{(y + \cos x)^2}.$$

With $x$ held constant and again applying the quotient rule, we get

$$f_y = \frac{\partial}{\partial y}\left(\frac{2y}{y + \cos x}\right) = \frac{(y + \cos x)\frac{\partial}{\partial y}(2y) - 2y\frac{\partial}{dy}(y + \cos x)}{(y + \cos x)^2}$$

$$= \frac{(y + \cos x)(2) - 2y(1)}{(y + \cos x)^2} = \frac{2 \cos x}{(y + \cos x)^2}.$$

■

**EXAMPLE 4** Find $\partial z/\partial x$ assuming that the equation

$$yz - \ln z = x + y$$

defines $z$ as a function of the two independent variables $x$ and $y$ and the partial derivative exists.

**Solution** We differentiate both sides of the equation with respect to $x$, holding $y$ constant and treating $z$ as a differentiable function of $x$:

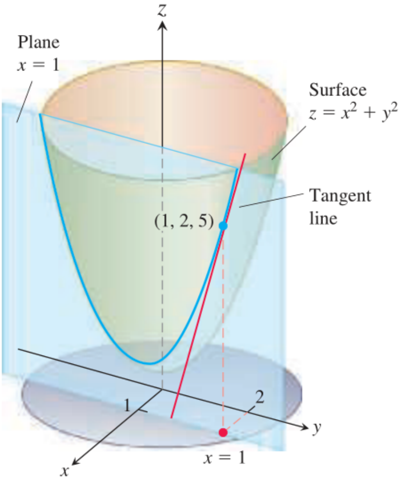
$$\frac{\partial}{\partial x}(yz) - \frac{\partial}{\partial x}\ln z = \frac{\partial x}{\partial x} + \frac{\partial y}{\partial x}$$

$$y\frac{\partial z}{\partial x} - \frac{1}{z}\frac{\partial z}{\partial x} = 1 + 0 \qquad \text{With } y \text{ constant, } \frac{\partial}{\partial x}(yz) = y\frac{\partial z}{\partial x}.$$

$$\left(y - \frac{1}{z}\right)\frac{\partial z}{\partial x} = 1$$

$$\frac{\partial z}{\partial x} = \frac{z}{yz - 1}.$$

**EXAMPLE 5** The plane $x = 1$ intersects the paraboloid $z = x^2 + y^2$ in a parabola. Find the slope of the tangent to the parabola at (1, 2, 5) (Figure 14.19).



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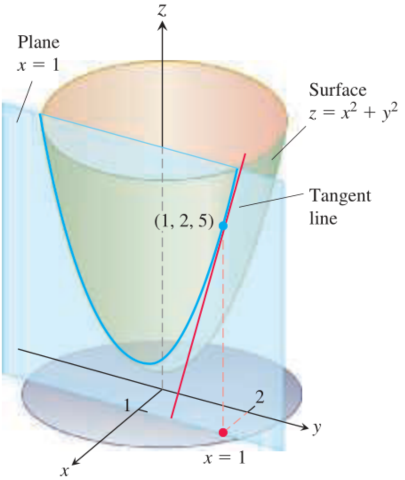


$$\left.\frac{\partial z}{\partial y}\right|_{(1,2)} = \ ?$$

**Solution** The parabola lies in a plane parallel to the $yz$-plane, and the slope is the value of the partial derivative $\partial z/\partial y$ at (1, 2):

$$\left.\frac{\partial z}{\partial y}\right|_{(1,2)} = \left.\frac{\partial}{\partial y}(x^2 + y^2)\right|_{(1,2)} = \left.2y\right|_{(1,2)} = 2(2) = 4.$$

As a check, we can treat the parabola as the graph of the single-variable function $z = (1)^2 + y^2 = 1 + y^2$ in the plane $x = 1$ and ask for the slope at $y = 2$. The slope, calculated now as an ordinary derivative, is

$$\left.\frac{dz}{dy}\right|_{y=2} = \left.\frac{d}{dy}(1 + y^2)\right|_{y=2} = \left.2y\right|_{y=2} = 4.$$

■

**Remark**

We can use the same ideas for functions of three (or more) variables.

**EXAMPLE 6** If $x$, $y$, and $z$ are independent variables and

$$f(x, y, z) = x \sin(y + 3z),$$

then

$$\begin{aligned} \frac{\partial f}{\partial z} &= \frac{\partial}{\partial z}[x \sin(y + 3z)] = x \frac{\partial}{\partial z} \sin(y + 3z) && x \text{ held constant} \\ &= x \cos(y + 3z) \frac{\partial}{\partial z}(y + 3z) && \text{Chain rule} \\ &= 3x \cos(y + 3z). && y \text{ held constant} \end{aligned}$$

Please read Example 7 in the textbook.

**EXAMPLE 8** Let

$$f(x, y) = \begin{cases} 0, & xy \neq 0 \\ 1, & xy = 0 \end{cases}$$

(Figure 14.21).

**(a)** Find the limit of $f$ as $(x, y)$ approaches $(0, 0)$ along the line $y = x$.

**(b)** Prove that $f$ is not continuous at the origin.

**(c)** Show that both partial derivatives $\partial f / \partial x$ and $\partial f / \partial y$ exist at the origin.

## Solution

**(a)** Since $f(x, y)$ is constantly zero along the line $y = x$ (except at the origin), we have

$$\lim_{(x, y) \to (0,0)} f(x, y) \Big|_{y=x} = \lim_{(x, y) \to (0,0)} 0 = 0.$$

**(b)** Since $f(0, 0) = 1$, the limit in part (a) is not equal to $f(0, 0)$, which proves that $f$ is not continuous at $(0, 0)$.

**(c)** To find $\partial f/\partial x$ at $(0, 0)$, we hold $y$ fixed at $y = 0$. Then $f(x, y) = 1$ for all $x$, and the graph of $f$ is the line $L_1$ in Figure 14.21. The slope of this line at any $x$ is $\partial f/\partial x = 0$. In particular, $\partial f/\partial x = 0$ at $(0, 0)$. Similarly, $\partial f/\partial y$ is the slope of line $L_2$ at any $y$, so $\partial f/\partial y = 0$ at $(0, 0)$. ■

## Second-Order Partial Derivatives

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = (f_x)_x = f_{xx}$$

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# Second-Order Partial Derivatives

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = (f_x)_x = f_{xx}$$

$$\frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = (f_y)_y = f_{yy}$$

first diff. wrt $y$
then diff. wrt $x$

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = (f_y)_x = f_{yx}$$

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first diff. wrt $x$
then diff. wrt $y$

**Remark**

Sometimes $f_{xy} = f_{yx}$ and sometimes $f_{xy} \neq f_{yx}$.

**Example**

If $f(x,y) = x\cos y + ye^x$, find the four second order partial derivatives of $f$.

Since

$$f_x = \qquad\qquad \text{and} \qquad f_y = \qquad\qquad ,$$

we have that

$$f_{xx} =$$

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$$f_{yx} =$$

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If $f(x,y) = x\cos y + ye^x$, find the four second order partial derivatives of $f$.

Since

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we have that

$$f_{xx} =$$

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If $f(x,y) = x\cos y + ye^x$, find the four second order partial derivatives of $f$.

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$$f_x = \cos y + ye^x \qquad \text{and} \qquad f_y = -x \sin y + e^x,$$

we have that

$$f_{xx} = \frac{\partial}{\partial x}(f_x) = ye^x$$
$$f_{yy} = \frac{\partial}{\partial y}(f_y) = -x \cos y$$
$$f_{yx} = \frac{\partial}{\partial x}(f_y) = -\sin y + e^x$$
$$f_{xy} = \frac{\partial}{\partial y}(f_x) = -\sin y + e^x = f_{yx}.$$

## Alexis Clairaut

Born
13 May 1713

Deceiced
17 May 1765

Nationality
French

**Theorem (Clairaut's Theorem)**

*If*

- $R$ *is an open region;*
- $(a, b) \in R$*;*
- $f$, $f_x$, $f_y$, $f_{xy}$ *and* $f_{yx}$ *are all defined on* $R$*; and*
- $f$, $f_x$, $f_y$, $f_{xy}$ *and* $f_{yx}$ *are all continuous at* $(a, b)$*;*

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- $f$, $f_x$, $f_y$, $f_{xy}$ *and* $f_{yx}$ *are all continuous at* $(a, b)$*;*

*then*

$$f_{xy}(a, b) = f_{yx}(a, b).$$

**EXAMPLE 10** Find $\dfrac{\partial^2 w}{\partial x\,\partial y}$ if

$$w = xy + \frac{e^y}{y^2 + 1}.$$

**Solution** The symbol $\partial^2 w/\partial x\,\partial y$ tells us to differentiate first with respect to $y$ and then with respect to $x$. However, if we interchange the order of differentiation and differentiate first with respect to $x$ we get the answer more quickly. In two steps,

$$\frac{\partial w}{\partial x} = y \qquad \text{and} \qquad \frac{\partial^2 w}{\partial y\,\partial x} = 1.$$

If we differentiate first with respect to $y$, we obtain $\partial^2 w/\partial x\,\partial y = 1$ as well. We can differentiate in either order because the conditions of Theorem 2 hold for $w$ at all points $(x_0, y_0)$. ■

# Higher Order Partial Derivatives

$$\frac{\partial^3 f}{\partial x \partial y^2} = f_{yyx}$$

$$\frac{\partial^4 f}{\partial x^2 \partial y^2} = f_{yyxx}$$

**EXAMPLE 11** Find $f_{yxyz}$ if $f(x, y, z) = 1 - 2xy^2z + x^2y$.

**Solution** We first differentiate with respect to the variable $y$, then $x$, then $y$ again, and finally with respect to $z$:

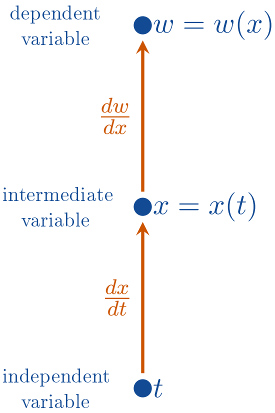
$$\begin{aligned} f_y &= -4xyz + x^2 \\ f_{yx} &= -4yz + 2x \\ f_{yxy} &= -4z \\ f_{yxyz} &= -4. \end{aligned}$$

# 13.4 The Chain Rule

# Functions of One Variable

Recall that if $w(x)$ and $x(t)$ then

$$\frac{dw}{dt} = \frac{dw}{dx}\frac{dx}{dt}.$$



# Functions of Two Variables

But what is $\dfrac{dw}{dt}$ if $w(x, y)$?

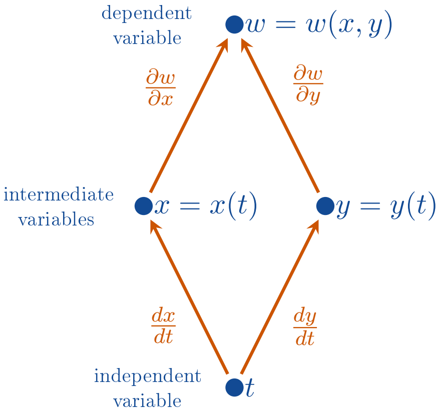
dependent variable $\bullet w = w(x, y)$

intermediate variables $\bullet x = x(t)$ $\bullet y = y(t)$

independent variable $\bullet t$

# Functions of Two Variables

But what is $\dfrac{dw}{dt}$ if $w(x, y)$?



## Functions of Two Variables

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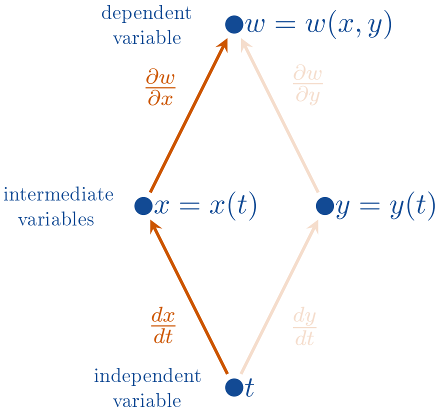
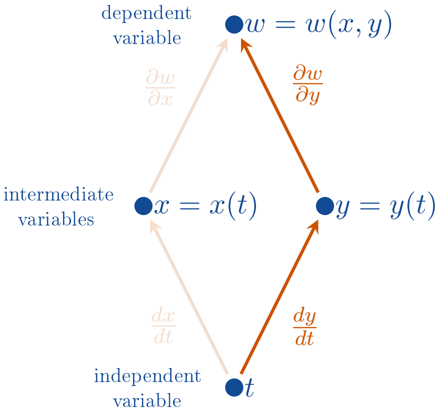


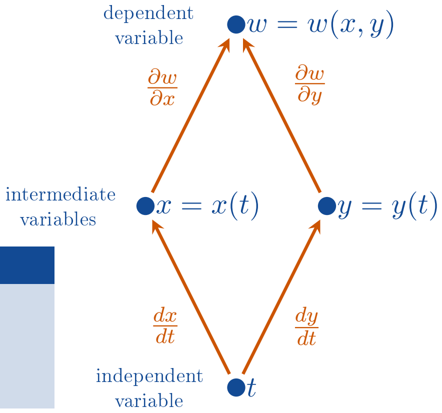
# Functions of Two Variables

But what is $\dfrac{dw}{dt}$ if $w(x, y)$?



# Functions of Two Variables

But what is $\dfrac{dw}{dt}$ if $w(x,y)$?




**Theorem**

$$\frac{dw}{dt} = \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt}.$$

$$\boxed{\frac{dw}{dt} = \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt}}$$

**Example**

Use the Chain Rule to find the derivative of $w = xy$ with respect to $t$ along the path $x = \cos t$, $y = \sin t$. What is the derivative's value at $t = \frac{\pi}{2}$?

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By the Chain Rule

$$\begin{aligned}\frac{dw}{dt} &= \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt} \\ &= \\ &= \\ &= \end{aligned}$$

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### Example

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By the Chain Rule

$$\begin{aligned}\frac{dw}{dt} &= \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt} \\ &= (y)(-\sin t) + (x)(\cos t) \\ &= \\ &= \end{aligned}$$

$$\boxed{\frac{dw}{dt} = \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt}}$$

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&= (\sin t)(-\sin t) + (\cos t)(\cos t) \\
&= -\sin^2 t + \cos^2 t = \cos 2t.
\end{aligned}$$

$$\frac{dw}{dt} = \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt}$$

**Example**

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Therefore

$$\left.\frac{dw}{dt}\right|_{t=\frac{\pi}{2}} = \cos\left(2\frac{\pi}{2}\right) = \cos\pi = -1.$$

$$\boxed{\frac{dw}{dt} = \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt}}$$

**Remark**

Since the example said "Use the Chain Rule...", we needed to do it this way.

If it just said "Find the derivative of...", then we could have just done

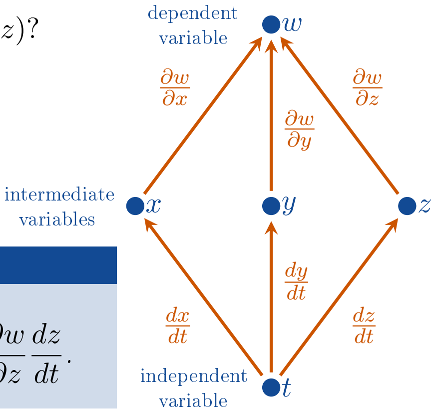
$$w = xy = (\cos t)(\sin t) = \frac{1}{2}\sin 2t$$

and

$$\frac{dw}{dt} = \frac{d}{dt}\left(\frac{1}{2}\sin 2t\right) = \cos 2t.$$

# Functions of Three Variables

And what is $\dfrac{dw}{dt}$ if $w(x,y,z)$?




**Theorem**

$$\frac{dw}{dt} = \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt} + \frac{\partial w}{\partial z}\frac{dz}{dt}.$$

**EXAMPLE 2** Find $dw/dt$ if

$$w = xy + z, \qquad x = \cos t, \qquad y = \sin t, \qquad z = t.$$

In this example the values of $w(t)$ are changing along the path of a helix (Section 13.1) as $t$ changes. What is the derivative's value at $t = 0$?

**Solution** Using the Chain Rule for three intermediate variables, we have

$$\begin{aligned}
\frac{dw}{dt} &= \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt} + \frac{\partial w}{\partial z}\frac{dz}{dt} \\
&= (y)(-\sin t) + (x)(\cos t) + (1)(1) \\
&= (\sin t)(-\sin t) + (\cos t)(\cos t) + 1 \\
&= -\sin^2 t + \cos^2 t + 1 = 1 + \cos 2t,
\end{aligned}$$

Substitute for intermediate variables.

so

$$\left.\frac{dw}{dt}\right|_{t=0} = 1 + \cos(0) = 2.$$

**Theorem**

$$\frac{\partial w}{\partial r} = \qquad ,$$

$$\frac{\partial w}{\partial s} = \qquad .$$

$w$

one dependent variable

$x$ $y$ $z$

three intermidiate variables

$r$ $s$

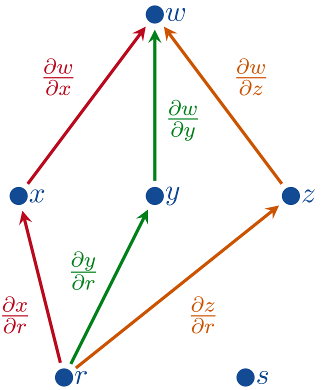
two independent variables

# 13.4 The Chain Rule

**Theorem**

$$\frac{\partial w}{\partial r} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial r} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial r},$$

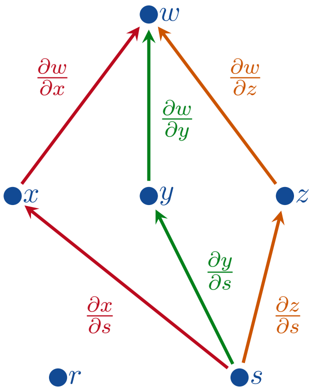
$$\frac{\partial w}{\partial s} = \qquad\qquad\qquad\qquad .$$



**Theorem**

$$\frac{\partial w}{\partial r} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial r} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial r},$$

$$\frac{\partial w}{\partial s} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial s} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial s}.$$




Once you understand this, you can do the Chain Rule for any number of intermediate and independent variables.

**EXAMPLE 3** Express $\partial w/\partial r$ and $\partial w/\partial s$ in terms of $r$ and $s$ if

$$w = x + 2y + z^2, \qquad x = \frac{r}{s}, \qquad y = r^2 + \ln s, \qquad z = 2r.$$

**Solution** Using the formulas in Theorem 7, we find

$$\begin{aligned}
\frac{\partial w}{\partial r} &= \frac{\partial w}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial r} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial r} \\
&= (1)\left(\frac{1}{s}\right) + (2)(2r) + (2z)(2) \\
&= \frac{1}{s} + 4r + (4r)(2) = \frac{1}{s} + 12r \qquad \text{Substitute for intermediate variable } z. \\
\frac{\partial w}{\partial s} &= \frac{\partial w}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial s} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial s} \\
&= (1)\left(-\frac{r}{s^2}\right) + (2)\left(\frac{1}{s}\right) + (2z)(0) = \frac{2}{s} - \frac{r}{s^2}.
\end{aligned}$$

**EXAMPLE 4** Express $\partial w/\partial r$ and $\partial w/\partial s$ in terms of $r$ and $s$ if

$$w = x^2 + y^2, \qquad x = r - s, \qquad y = r + s.$$

**Solution** The preceding discussion gives the following.

$$\begin{aligned}
\frac{\partial w}{\partial r} &= \frac{\partial w}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial r} & \qquad \frac{\partial w}{\partial s} &= \frac{\partial w}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial s} \\
&= (2x)(1) + (2y)(1) & &= (2x)(-1) + (2y)(1) \\
&= 2(r - s) + 2(r + s) & &= -2(r - s) + 2(r + s) \\
&= 4r & &= 4s
\end{aligned}$$

Substitute for the intermediate variables.

If $f$ is a function of $x$ alone, our equations are even simpler.

## Implicit Differentiation Revisited

Suppose that $F(x, y) = 0$ and $y = y(x)$.

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Suppose that $F(x,y)=0$ and $y=y(x)$. Then

$$0 = \quad F(x,y)$$

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$$\frac{d}{dx}0 = \frac{d}{dx}F(x, y)$$

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# Implicit Differentiation Revisited

Suppose that $F(x, y) = 0$ and $y = y(x)$. Then

$$0 = \frac{d}{dx} 0 = \frac{d}{dx} F(x, y) = \frac{dF}{dx}\frac{dx}{dx} + \frac{dF}{dy}\frac{dy}{dx}$$

## Implicit Differentiation Revisited

Suppose that $F(x, y) = 0$ and $y = y(x)$. Then

$$0 = \frac{d}{dx}0 = \frac{d}{dx}F(x, y) = \frac{dF}{dx}\frac{dx}{dx} + \frac{dF}{dy}\frac{dy}{dx} = F_x \cdot 1 + F_y\frac{dy}{dx}.$$

# Implicit Differentiation Revisited

Suppose that $F(x, y) = 0$ and $y = y(x)$. Then

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**Theorem**

*If $F_y \neq 0$, then*

$$\frac{dy}{dx} = -\frac{F_x}{F_y}.$$

$$\boxed{\frac{dy}{dx} = -\frac{F_x}{F_y}}$$

**Example**

Find $\dfrac{dy}{dx}$ if $y^2 - x^2 - \sin xy = 0$.

$$\boxed{\frac{dy}{dx} = -\frac{F_x}{F_y}}$$

**Example**

Find $\dfrac{dy}{dx}$ if $y^2 - x^2 - \sin xy = 0$.

Let $F(x, y) = y^2 - x^2 - \sin xy$.

$$\boxed{\frac{dy}{dx} = -\frac{F_x}{F_y}}$$

**Example**

Find $\dfrac{dy}{dx}$ if $y^2 - x^2 - \sin xy = 0$.

Let $F(x, y) = y^2 - x^2 - \sin xy$. Then

$$\frac{dy}{dx} = -\frac{F_x}{F_y} =$$

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**Example**

Find $\dfrac{dy}{dx}$ if $y^2 - x^2 - \sin xy = 0$.

Let $F(x, y) = y^2 - x^2 - \sin xy$. Then

$$\frac{dy}{dx} = -\frac{F_x}{F_y} = -\frac{-2x - y\cos xy}{2y - x\cos xy} = \frac{2xy + \cos xy}{2y - x\cos xy}.$$

If we start with $F(x, y, z) = 0$ and $z = z(x, y)$, then we obtain

$$\begin{aligned} 0 &= \frac{\partial F}{\partial x}\frac{\partial x}{\partial x} + \frac{\partial F}{\partial y}\frac{\partial y}{\partial x} + \frac{\partial F}{\partial z}\frac{\partial z}{\partial x} \\ &= F_x \cdot 1 + F_y \cdot 0 + F_z \frac{\partial z}{\partial x} \qquad \Longrightarrow \qquad \frac{\partial z}{\partial x} = -\frac{F_x}{F_z} \end{aligned}$$

If we start with $F(x,y,z)=0$ and $z=z(x,y)$, then we obtain

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and

$$\begin{aligned} 0 &= \frac{\partial F}{\partial x}\frac{\partial x}{\partial y} + \frac{\partial F}{\partial y}\frac{\partial y}{\partial y} + \frac{\partial F}{\partial z}\frac{\partial z}{\partial y} \\ &= F_x \cdot 0 + F_y \cdot 1 + F_z \frac{\partial z}{\partial y} \qquad \Longrightarrow \qquad \frac{\partial z}{\partial y} = -\frac{F_y}{F_z}. \end{aligned}$$

$$\frac{\partial z}{\partial x} = -\frac{F_x}{F_z} \qquad \frac{\partial z}{\partial y} = -\frac{F_y}{F_z}$$

Example

Find $\dfrac{\partial z}{\partial x}$ and $\dfrac{\partial z}{\partial y}$ at $(0,0,0)$, if $x^3 + z^2 + ye^{xz} + z\cos y = 0$.

$$\frac{\partial z}{\partial x} = -\frac{F_x}{F_z} \qquad \frac{\partial z}{\partial y} = -\frac{F_y}{F_z}$$

## Example

Find $\dfrac{\partial z}{\partial x}$ and $\dfrac{\partial z}{\partial y}$ at $(0,0,0)$, if $x^3 + z^2 + ye^{xz} + z\cos y = 0$.

Let $F(x,y,z) = x^3 + z^2 + ye^{xz} + z\cos y$. Then

$$\frac{\partial z}{\partial x} = -\frac{F_x}{F_z} = -\frac{3x^2 + zye^{xz}}{2z + xye^{xz} + \cos y}$$

and

$$\frac{\partial z}{\partial y} = -\frac{F_y}{F_z} = -\frac{e^{xz} - z\sin y}{2z + xye^{xz} + \cos y}.$$

$$\frac{\partial z}{\partial x} = -\frac{F_x}{F_z} \qquad \frac{\partial z}{\partial y} = -\frac{F_y}{F_z}$$

**Example**

Find $\dfrac{\partial z}{\partial x}$ and $\dfrac{\partial z}{\partial y}$ at $(0,0,0)$, if $x^3 + z^2 + ye^{xz} + z\cos y = 0$.

Let $F(x,y,z) = x^3 + z^2 + ye^{xz} + z\cos y$. Then

$$\frac{\partial z}{\partial x} = -\frac{F_x}{F_z} = -\frac{3x^2 + zye^{xz}}{2z + xye^{xz} + \cos y}$$

and

$$\frac{\partial z}{\partial y} = -\frac{F_y}{F_z} = -\frac{e^{xz} - z\sin y}{2z + xye^{xz} + \cos y}.$$

At $(0,0,0)$ we have

$$\frac{\partial z}{\partial x} = -\frac{0}{1} = 0 \qquad \text{and} \qquad \frac{\partial z}{\partial y} = -\frac{1}{1} = -1.$$

# Next Time

- 13.5 Directional Derivatives and Gradient Vectors
- 13.6 Tangent Planes and Differentials
- 13.7 Extreme Values and Saddle Points
- 13.8 Lagrange Multipliers