



Welcome to

# Mathematics III

with Dr Neil Course



MATH113

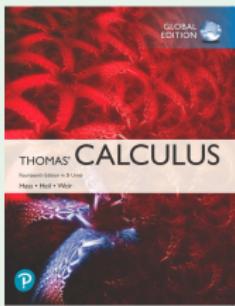
MATH114

MATH215

MATH216

MATH113

MATH114



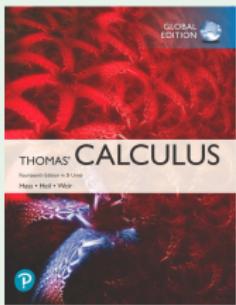
# Calculus

MATH215

MATH216

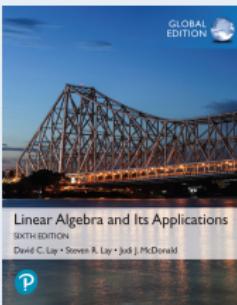
MATH113

MATH114



# Calculus

MATH215

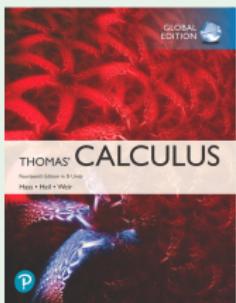


# Linear Algebra

MATH216

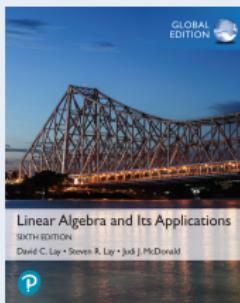
MATH113

MATH114



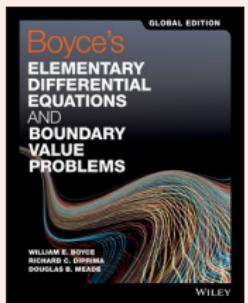
# Calculus

MATH215



# Linear Algebra

MATH216



# Differential Equations

## Information about this course

- $\approx 12$  classes. Monday afternoons 2pm-4pm.

14:00

15:00

16:00

## Information about this course

- $\approx$  12 classes. Monday afternoons 2pm-4pm.
- 2 lectures with a break between.

lecture

lecture

14:00

15:00

16:00

## Information about this course

- $\approx$  12 classes. Monday afternoons 2pm-4pm.
- 2 lectures with a break between.
- Then I will answers your questions.

lecture

14:00

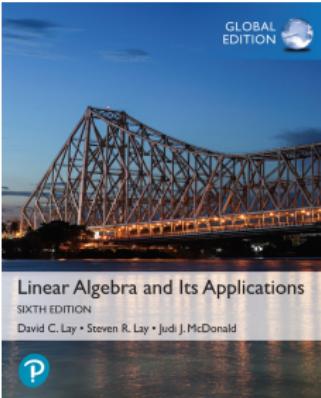
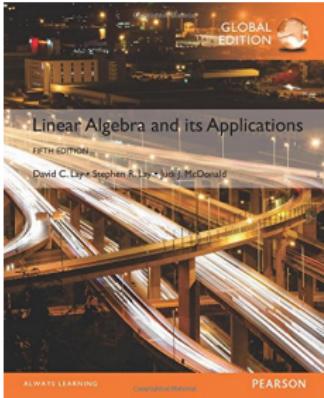
lecture

15:00

questions

16:00

## The Book



David C. Lay, Steven R. Lay and Judi J. McDonald,  
**Linear Algebra and Its Applications,**  
5th or 6th Edition, Pearson.

**This is a required purchase.**  
You need to have this book to be able to do the homework.

## Syllabus

- linear systems and their solutions
- matrices
- determinants
- inverse matrices
- properties of determinants
- Cramer's Rule
- vector spaces
- subspaces
- linear independence
- basis
- row space
- column space
- null space
- rank and nullity
- linear transformations
- eigenvalues and eigenvectors
- diagonalisation
- inner product spaces
- orthogonality
- the Gram-Schmidt process
- least squares
- orthogonal diagonalisation
- singular value decomposition

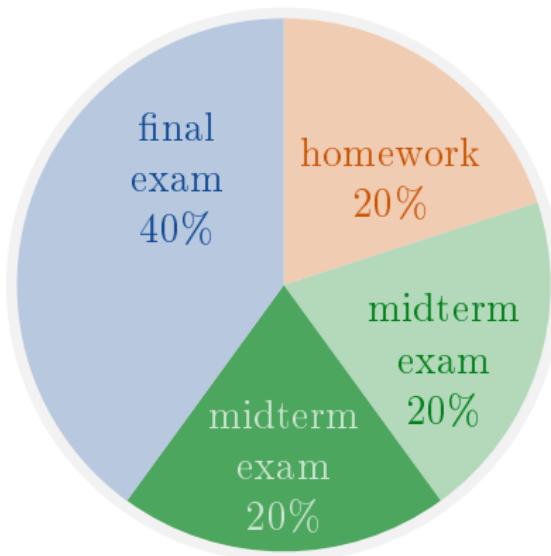
## Exams and homework

(This information may change based on the University's decisions)



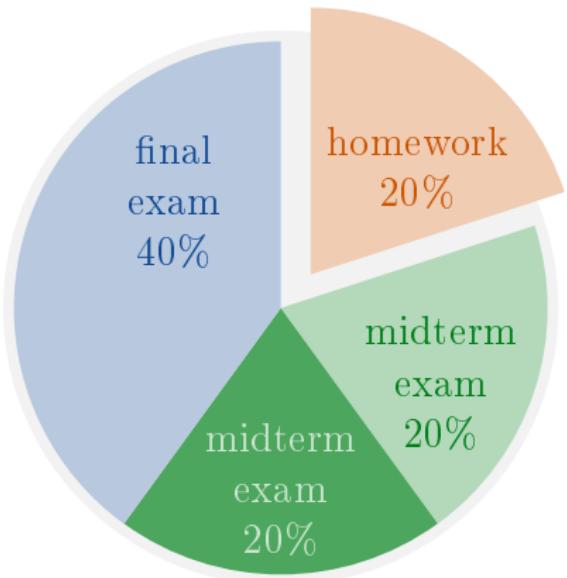
## Exams and homework

(This information may change based on the University's decisions)



## Exams and homework

(This information may change based on the University's decisions)



using Pearson  
MyLab Math

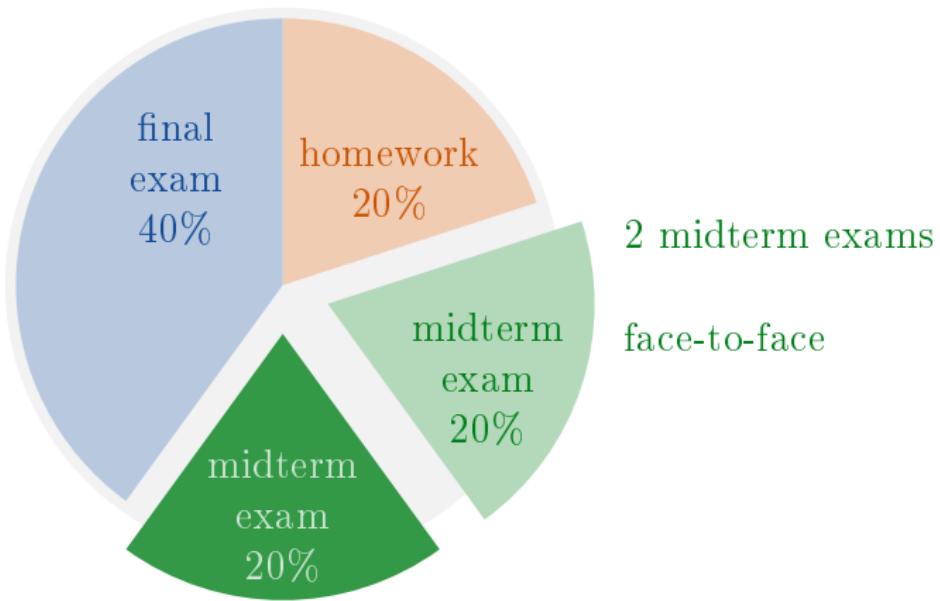
one piece of  
homework for  
each lesson

deadline = end of  
term

more details in  
O'Learn

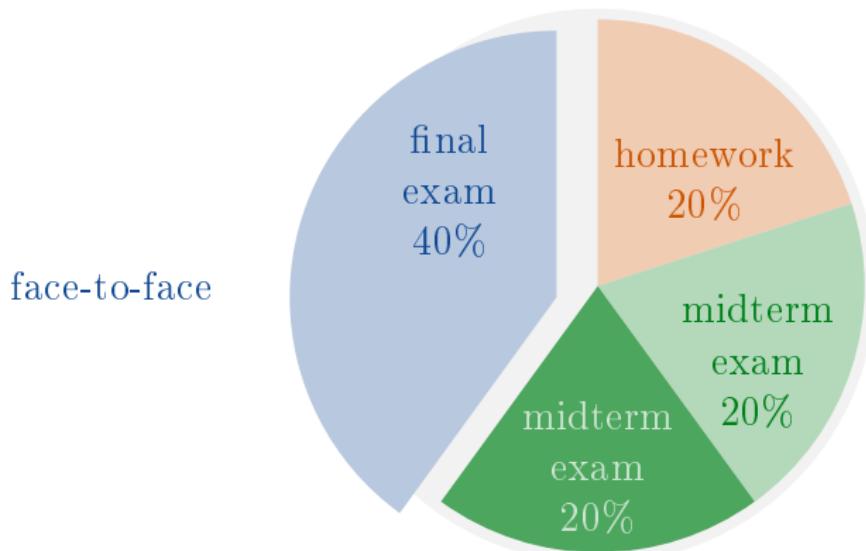
## Exams and homework

(This information may change based on the University's decisions)



## Exams and homework

(This information may change based on the University's decisions)



## Expectations

If this was a classroom course, the expectation is that for each hour of lectures, you would study an extra 1-2 hours outside of class.

classroom  
course

lectures (4 hours)

other study (4-8 hours)

## Expectations

If this was a classroom course, the expectation is that for each hour of lectures, you would study an extra 1-2 hours outside of class.

classroom  
course

lectures (4 hours)

other study (4-8 hours)

For an online course, you are still expected to study a total of 8-12 hours each week.

online  
course

class  
(2 hours)

other study (6-10 hours)

This may include:

- Do the online homework on MyLab;

This may include:

- Do the online homework on MyLab;
- Rewatch the recorded lectures (O'Learn & YouTube);

This may include:

- Do the online homework on MyLab;
- Rewatch the recorded lectures (O'Learn & YouTube);
- Read the lecture slides (before the lecture? after the lecture?);

This may include:

- Do the online homework on MyLab;
- Rewatch the recorded lectures (O'Learn & YouTube);
- Read the lecture slides (before the lecture? after the lecture?);
- Read the textbook;

This may include:

- Do the online homework on MyLab;
- Rewatch the recorded lectures (O’Learn & YouTube);
- Read the lecture slides (before the lecture? after the lecture?);
- Read the textbook;
- Solve the exercises in the textbook;

This may include:

- Do the online homework on MyLab;
- Rewatch the recorded lectures (O'Learn & YouTube);
- Read the lecture slides (before the lecture? after the lecture?);
- Read the textbook;
- Solve the exercises in the textbook;
- Use the O'Learn Discussion Board;

This may include:

- Do the online homework on MyLab;
- Rewatch the recorded lectures (O'Learn & YouTube);
- Read the lecture slides (before the lecture? after the lecture?);
- Read the textbook;
- Solve the exercises in the textbook;
- Use the O'Learn Discussion Board;
- Read other books?;

This may include:

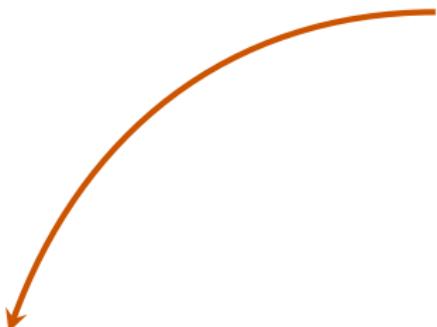
- Do the online homework on MyLab;
- Rewatch the recorded lectures (O'Learn & YouTube);
- Read the lecture slides (before the lecture? after the lecture?);
- Read the textbook;
- Solve the exercises in the textbook;
- Use the O'Learn Discussion Board;
- Read other books?;
- Watch online videos;

⋮

# Section Title



slide number



# Lecture 1

- Systems of Linear Equations
- Row Reduction and Echelon Forms



# Systems of Linear Equations

# Systems of Linear Equations



## Definition

A *linear equation* is an equation of the form

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n = b \quad (1)$$

where

- $x_1, x_2, \dots, x_n$  are the variables;
- $a_j$  and  $b$  are real or complex numbers; and
- $n$  may be any natural number.

# Systems of Linear Equations



## Definition

A *linear equation* is an equation of the form

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n = b \quad (1)$$

where

- $x_1, x_2, \dots, x_n$  are the variables;
- $a_j$  and  $b$  are real or complex numbers; and
- $n$  may be any natural number.

In this course,  $n$  will usually be between 2 and 5. In real-life problems,  $n$  might be 50 or 5000, or even larger.

# Systems of Linear Equations

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n = b$$



The equations

$$4x_1 - 5x_2 + 2 = x_1$$

and

$$x_2 = 2(\sqrt{6} - x_1) + x_3$$

are both linear

# Systems of Linear Equations

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n = b$$



The equations

$$4x_1 - 5x_2 + 2 = x_1$$

and

$$x_2 = 2(\sqrt{6} - x_1) + x_3$$

are both linear because they can be rearranged to the standard form

$$3x_1 - 5x_2 = -2$$

and

$$2x_1 + x_2 - x_3 = 2\sqrt{6}.$$

The former is an equation of straight line. The latter is an equation of a plane.

# Systems of Linear Equations

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n = b$$



The equations

$$4x_1 - 5x_2 = \textcolor{red}{x_1}x_2$$

and

$$x_2 = 2\sqrt{x_1} - 6$$

are not linear.

# Systems of Linear Equations

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n = b$$



The equations

$$4x_1 - 5x_2 = \textcolor{red}{x_1x_2}$$

and

$$x_2 = 2\sqrt{x_1} - 6$$

are not linear.

## Definition

If an equation is not linear, we say it is *non-linear*.

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n = b$$



## Ask the Audience

Are these equations linear or non-linear.

1  $x_1 + x_2 + x_3 + x_4 + x_5 + x_6 = 1$

2  $x_1 + 2x_2 + 3x_3^2 = 4$

3  $x_2 = \cos x_1$

4  $7x_1 - x_3 = 4x_2$

5  $x_1x_2x_3x_4x_5 = 1$

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n = b$$



## Ask the Audience

Are these equations linear or non-linear.

1  $x_1 + x_2 + x_3 + x_4 + x_5 + x_6 = 1$     Linear

2  $x_1 + 2x_2 + 3x_3^2 = 4$

3  $x_2 = \cos x_1$

4  $7x_1 - x_3 = 4x_2$

5  $x_1x_2x_3x_4x_5 = 1$

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n = b$$



## Ask the Audience

Are these equations linear or non-linear.

1  $x_1 + x_2 + x_3 + x_4 + x_5 + x_6 = 1$     Linear

2  $x_1 + 2x_2 + 3x_3^2 = 4$     Non-linear

3  $x_2 = \cos x_1$

4  $7x_1 - x_3 = 4x_2$

5  $x_1x_2x_3x_4x_5 = 1$

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n = b$$



## Ask the Audience

Are these equations linear or non-linear.

1  $x_1 + x_2 + x_3 + x_4 + x_5 + x_6 = 1$     Linear

2  $x_1 + 2x_2 + 3x_3^2 = 4$     Non-linear

3  $x_2 = \cos x_1$     Non-linear

4  $7x_1 - x_3 = 4x_2$

5  $x_1x_2x_3x_4x_5 = 1$

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n = b$$



## Ask the Audience

Are these equations linear or non-linear.

1  $x_1 + x_2 + x_3 + x_4 + x_5 + x_6 = 1$     Linear

2  $x_1 + 2x_2 + 3x_3^2 = 4$     Non-linear

3  $x_2 = \cos x_1$     Non-linear

4  $7x_1 - x_3 = 4x_2$     Linear

5  $x_1x_2x_3x_4x_5 = 1$

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n = b$$



## Ask the Audience

Are these equations linear or non-linear.

1  $x_1 + x_2 + x_3 + x_4 + x_5 + x_6 = 1$  Linear

2  $x_1 + 2x_2 + 3x_3^2 = 4$  Non-linear

3  $x_2 = \cos x_1$  Non-linear

4  $7x_1 - x_3 = 4x_2$  Linear

5  $x_1x_2x_3x_4x_5 = 1$  Non-linear

# Systems of Linear Equations



## Remark

This course is about linear equations.

# Systems of Linear Equations

## Definition

A *linear system* is a group of several linear equations:

$$\left\{ \begin{array}{l} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m. \end{array} \right. \quad (2)$$

# Systems of Linear Equations

## Definition

A *linear system* is a group of several linear equations:

$$\left\{ \begin{array}{l} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m. \end{array} \right. \quad (2)$$

For example

$$\left\{ \begin{array}{l} 2x_1 - x_2 + 1.5x_3 = 8 \\ x_1 - 4x_3 = -7 \end{array} \right.$$

is a linear system.

# Systems of Linear Equations



## Definition

A *solution* of a linear system is a list  $(s_1, s_2, \dots, s_n)$  of numbers which satisfies every equation in the linear system.

# Systems of Linear Equations



## Definition

A *solution* of a linear system is a list  $(s_1, s_2, \dots, s_n)$  of numbers which satisfies every equation in the linear system.

For example,  $(5, 6.5, 3)$  is a solution of

$$\begin{cases} 2x_1 - x_2 + 1.5x_3 = 8 \\ x_1 - 4x_3 = -7 \end{cases}$$

# Systems of Linear Equations



## Definition

A *solution* of a linear system is a list  $(s_1, s_2, \dots, s_n)$  of numbers which satisfies every equation in the linear system.

For example,  $(5, 6.5, 3)$  is a solution of

$$\begin{cases} 2x_1 - x_2 + 1.5x_3 = 8 \\ x_1 - 4x_3 = -7 \end{cases}$$

because

$$2x_1 - x_2 + 1.5x_3 = 2(5) - (6.5) + 1.5(3) = 8$$

# Systems of Linear Equations



## Definition

A *solution* of a linear system is a list  $(s_1, s_2, \dots, s_n)$  of numbers which satisfies every equation in the linear system.

For example,  $(5, 6.5, 3)$  is a solution of

$$\begin{cases} 2x_1 - x_2 + 1.5x_3 = 8 \\ x_1 - 4x_3 = -7 \end{cases}$$

because

$$2x_1 - x_2 + 1.5x_3 = 2(5) - (6.5) + 1.5(3) = 8$$

and

$$x_1 - 4x_3 = (5) - 4(3) = -7.$$

# Systems of Linear Equations



## Definition

The set of all possible solutions is called the *solution set* of the linear system

## Definition

Two linear systems are called *equivalent* if they have the same solution set.

## Systems of Linear Equations

## Example

Find the solution set of

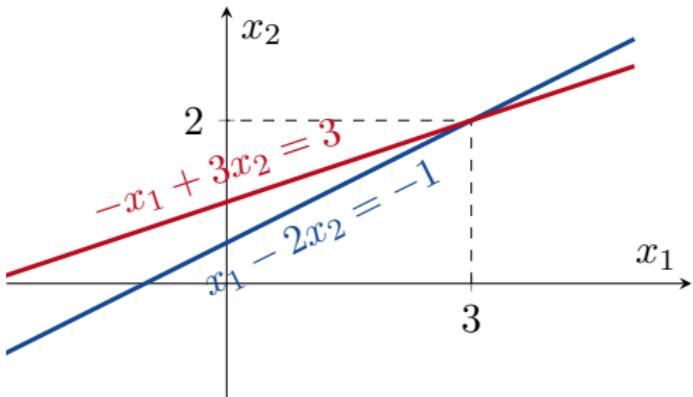
$$\begin{cases} x_1 - 2x_2 = -1 \\ -x_1 + 3x_2 = 3. \end{cases}$$

## Example

Find the solution set of

$$\begin{cases} x_1 - 2x_2 = -1 \\ -x_1 + 3x_2 = 3. \end{cases}$$

The graphs of these two equations are lines in  $\mathbb{R}^2$ .



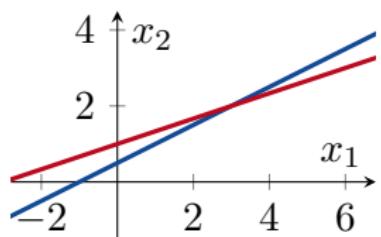
We can see that the only point which is on both lines is  $(3, 2)$ .  
So the solution set is the set  $\{(3, 2)\}$ .

# Systems of Linear Equations



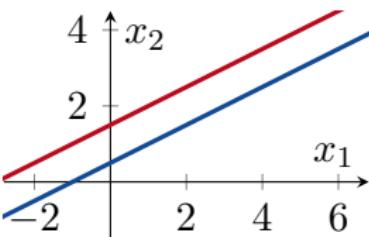
There are three possibilities.

$$\begin{cases} x_1 - 2x_2 = -1 \\ -x_1 + 3x_2 = 3. \end{cases}$$



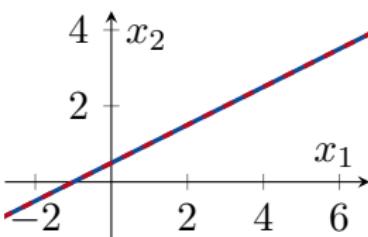
exactly one solution

$$\begin{cases} x_1 - 2x_2 = -1 \\ -x_1 + 2x_2 = 3. \end{cases}$$



no solutions

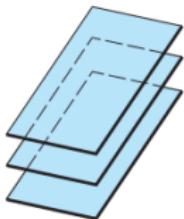
$$\begin{cases} x_1 - 2x_2 = -1 \\ -x_1 + 2x_2 = 1. \end{cases}$$



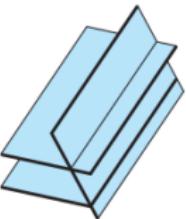
infinitely many solutions.

# Systems of Linear Equations

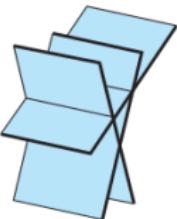
This is also true if we have 3 planes.



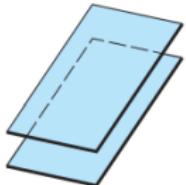
No solutions  
(three parallel planes;  
no common intersection)



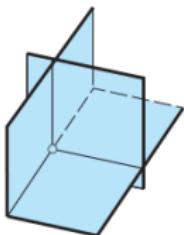
No solutions  
(two parallel planes;  
no common intersection)



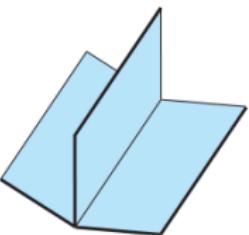
No solutions  
(no common intersection)



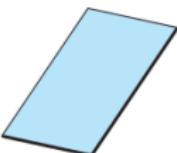
No solutions  
(two coincident planes  
parallel to the third;  
no common intersection)



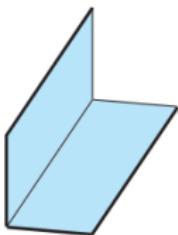
One solution  
(intersection is a point)



Infinitely many solutions  
(intersection is a line)



Infinitely many solutions  
(planes are all coincident;  
intersection is a plane)



Infinitely many solutions  
(two coincident planes;  
intersection is a line)

# Systems of Linear Equations

## Theorem

*A linear system has either*

- 1** *zero solutions; or*
- 2** *exactly one solution; or*
- 3** *infinitely many solutions.*

*There are no other possibilities.*

# Systems of Linear Equations



## Theorem

*A linear system has either*

- 1** *zero solutions; or*
- 2** *exactly one solution; or*
- 3** *infinitely many solutions.*

*There are no other possibilities.*

## Definition

A linear system is called *consistent* if it has either one solution or infinitely many solutions.

A linear system is called *inconsistent* if it does not have a solution.

## Matrix Notation

Consider the linear system

$$\left\{ \begin{array}{rcll} x_1 & - & 2x_2 & + & x_3 = 0 \\ & & 2x_2 & - & 8x_3 = 8 \\ -4x_1 & + & 5x_2 & + & 9x_3 = -9 \end{array} \right.$$

What is the important information here?

## Matrix Notation

Consider the linear system

$$\left\{ \begin{array}{rcll} 1x_1 & - & 2x_2 & + & 1x_3 = 0 \\ & & 2x_2 & - & 8x_3 = 8 \\ -4x_1 & + & 5x_2 & + & 9x_3 = -9 \end{array} \right.$$

What is the important information here?

## Matrix Notation

Consider the linear system

$$\left\{ \begin{array}{rcll} 1x_1 & - & 2x_2 & + & 1x_3 = 0 \\ & & 2x_2 & - & 8x_3 = 8 \\ -4x_1 & + & 5x_2 & + & 9x_3 = -9 \end{array} \right.$$

What is the important information here?

## Matrix Notation

Consider the linear system

$$\left\{ \begin{array}{rcll} 1x_1 & - & 2x_2 & + & 1x_3 = 0 \\ & & 2x_2 & - & 8x_3 = 8 \\ -4x_1 & + & 5x_2 & + & 9x_3 = -9 \end{array} \right.$$

What is the important information here?

## Matrix Notation

Consider the linear system

$$\begin{cases} 1x_1 - 2x_2 + 1x_3 = 0 \\ \quad 2x_2 - 8x_3 = 8 \\ -4x_1 + 5x_2 + 9x_3 = -9 \end{cases}$$

What is the important information here?

$$\begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ -4 & 5 & 9 & -9 \end{bmatrix}$$

We don't need the  $x_j$ 's, the +'s or the ='s. We only need the numbers.

# Systems of Linear Equations



## Definition

$$\begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ -4 & 5 & 9 & -9 \end{bmatrix}$$

is the *augmented matrix* for the linear system

$$\left\{ \begin{array}{rcll} 1x_1 & - & 2x_2 & + & 1x_3 = 0 \\ & & 2x_2 & - & 8x_3 = 8 \\ -4x_1 & + & 5x_2 & + & 9x_3 = -9 \end{array} \right..$$

# Systems of Linear Equations



## Definition

$$\begin{bmatrix} 1 & -2 & 1 \\ 0 & 2 & -8 \\ -4 & 5 & 9 \end{bmatrix}$$

is the *coefficient matrix* for the linear system

$$\left\{ \begin{array}{rcl} 1x_1 - 2x_2 + 1x_3 = 0 \\ 0x_1 + 2x_2 - 8x_3 = 8 \\ -4x_1 + 5x_2 + 9x_3 = -9 \end{array} \right.$$

# Systems of Linear Equations



There are three things that we can do to a linear system:

- 1 Multiply an equation by a number;
- 2 Swap two equations; or
- 3 Add a multiple of one equation to another.

# Systems of Linear Equations



There are three things that we can do to a linear system:

- 1 Multiply an equation by a number;
- 2 Swap two equations; or
- 3 Add a multiple of one equation to another.

## Elementary Row Operations

There are three things that we can do to a matrix:

- 1 Multiply a row by a number;
- 2 Swap two rows; or
- 3 Add a multiple of one row to another.

These are called *elementary row operations*.

## Solving a Linear System

Example

Solve

$$\left\{ \begin{array}{rcl} 1x_1 - 2x_2 + 1x_3 & = & 0 \\ & 2x_2 - 8x_3 & = & 8 \\ -4x_1 + 5x_2 + 9x_3 & = & -9 \end{array} \right.$$

# Systems of Linear Equations



## Example

Use elementary row operations to convert the augmented matrix

$$\begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ -4 & 5 & 9 & -9 \end{bmatrix}$$

into

$$\begin{bmatrix} 1 & 0 & 0 & ? \\ 0 & 1 & 0 & ? \\ 0 & 0 & 1 & ? \end{bmatrix}.$$

# Systems of Linear Equations



## Remark

There are two things about elementary row operations to learn:

- 1 How** to use elementary row operations.
- 2 Why** we choose a certain elementary row operation at each step.

# Systems of Linear Equations



## Remark

There are two things about elementary row operations to learn:

- 1 How** to use elementary row operations.
- 2 Why** we choose a certain elementary row operation at each step.

I will teach the “how” first. Then in the second hour I will show you the “why”.

# Systems of Linear Equations



$$\begin{cases} x_1 - 2x_2 + x_3 = 0 \\ 2x_2 - 8x_3 = 8 \\ -4x_1 + 5x_2 + 9x_3 = -9 \end{cases}$$

$$\left[ \begin{array}{cccc} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ -4 & 5 & 9 & -9 \end{array} \right]$$

1

1

# Systems of Linear Equations

$$\begin{cases} x_1 - 2x_2 + x_3 = 0 \\ 2x_2 - 8x_3 = 8 \\ -4x_1 + 5x_2 + 9x_3 = -9 \end{cases}$$

1

$$\left[ \begin{array}{cccc} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ -4 & 5 & 9 & -9 \end{array} \right]$$

1

$$\begin{array}{r} 4 \cdot [\text{equation 1}] \\ + \quad [\text{equation 3}] \\ \hline [\text{new equation 3}] \end{array}$$

# Systems of Linear Equations

$$\begin{cases} x_1 - 2x_2 + x_3 = 0 \\ 2x_2 - 8x_3 = 8 \\ -4x_1 + 5x_2 + 9x_3 = -9 \end{cases}$$

$$\left[ \begin{array}{cccc} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ -4 & 5 & 9 & -9 \end{array} \right]$$

1

1

$$\begin{array}{r}
 4 \cdot [x_1 - 2x_2 + x_3 = 0] \\
 + [-4x_1 + 5x_2 + 9x_3 = -9] \\
 \hline
 \text{[new equation 3]}
 \end{array}$$

# Systems of Linear Equations

$$\begin{cases} x_1 - 2x_2 + x_3 = 0 \\ 2x_2 - 8x_3 = 8 \\ -4x_1 + 5x_2 + 9x_3 = -9 \end{cases}$$

$$\left[ \begin{array}{cccc} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ -4 & 5 & 9 & -9 \end{array} \right]$$

1

1

$$4x_1 - 8x_2 + 4x_3 = 0$$

$$\underline{-4x_1 + 5x_2 + 9x_3 = -9}$$

# Systems of Linear Equations

$$\begin{cases} x_1 - 2x_2 + x_3 = 0 \\ 2x_2 - 8x_3 = 8 \\ -4x_1 + 5x_2 + 9x_3 = -9 \end{cases}$$

1

$$\left[ \begin{array}{cccc} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ -4 & 5 & 9 & -9 \end{array} \right]$$

1

$$\begin{array}{r} 4x_1 - 8x_2 + 4x_3 = 0 \\ -4x_1 + 5x_2 + 9x_3 = -9 \\ \hline 0x_1 - 3x_2 + 13x_3 = -9 \end{array}$$

# Systems of Linear Equations

$$\begin{cases} x_1 - 2x_2 + x_3 = 0 \\ 2x_2 - 8x_3 = 8 \\ -4x_1 + 5x_2 + 9x_3 = -9 \end{cases}$$

$$\left[ \begin{array}{cccc} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ -4 & 5 & 9 & -9 \end{array} \right]$$

1

1

$$\begin{array}{r} 4x_1 - 8x_2 + 4x_3 = 0 \\ -4x_1 + 5x_2 + 9x_3 = -9 \\ \hline 0x_1 - 3x_2 + 13x_3 = -9 \end{array}$$

$$\begin{cases} x_1 - 2x_2 + x_3 = 0 \\ 2x_2 - 8x_3 = 8 \\ -3x_2 + 13x_3 = -9 \end{cases}$$

# Systems of Linear Equations

$$\begin{cases} x_1 - 2x_2 + x_3 = 0 \\ 2x_2 - 8x_3 = 8 \\ -4x_1 + 5x_2 + 9x_3 = -9 \end{cases}$$

1

$$\begin{array}{r} 4x_1 - 8x_2 + 4x_3 = 0 \\ -4x_1 + 5x_2 + 9x_3 = -9 \\ \hline 0x_1 - 3x_2 + 13x_3 = -9 \end{array}$$

$$\begin{cases} x_1 - 2x_2 + x_3 = 0 \\ 2x_2 - 8x_3 = 8 \\ -3x_2 + 13x_3 = -9 \end{cases}$$

$$\left[ \begin{array}{cccc} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ -4 & 5 & 9 & -9 \end{array} \right] \begin{matrix} \leftarrow R_1 \\ \leftarrow R_2 \\ \leftarrow R_3 \end{matrix}$$

1  $4R_1 + R_3 \rightarrow$  new  $R_3$

# Systems of Linear Equations

$$\begin{cases} x_1 - 2x_2 + x_3 = 0 \\ 2x_2 - 8x_3 = 8 \\ -4x_1 + 5x_2 + 9x_3 = -9 \end{cases}$$

1

$$\begin{array}{r} 4x_1 - 8x_2 + 4x_3 = 0 \\ -4x_1 + 5x_2 + 9x_3 = -9 \\ \hline 0x_1 - 3x_2 + 13x_3 = -9 \end{array}$$

$$\begin{cases} x_1 - 2x_2 + x_3 = 0 \\ 2x_2 - 8x_3 = 8 \\ -3x_2 + 13x_3 = -9 \end{cases}$$

$$\left[ \begin{array}{cccc} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ -4 & 5 & 9 & -9 \end{array} \right] \begin{matrix} \leftarrow R_1 \\ \leftarrow R_2 \\ \leftarrow R_3 \end{matrix}$$

1  $4R_1 + R_3 \rightarrow \text{new } R_3$

$$\begin{array}{r} 4 & -8 & 4 & 0 \\ + & -4 & 5 & 9 & -9 \\ \hline \end{array}$$

# Systems of Linear Equations

$$\begin{cases} x_1 - 2x_2 + x_3 = 0 \\ 2x_2 - 8x_3 = 8 \\ -4x_1 + 5x_2 + 9x_3 = -9 \end{cases}$$

1

$$\begin{array}{r} 4x_1 - 8x_2 + 4x_3 = 0 \\ -4x_1 + 5x_2 + 9x_3 = -9 \\ \hline 0x_1 - 3x_2 + 13x_3 = -9 \end{array}$$

$$\begin{cases} x_1 - 2x_2 + x_3 = 0 \\ 2x_2 - 8x_3 = 8 \\ -3x_2 + 13x_3 = -9 \end{cases}$$

$$\left[ \begin{array}{cccc} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ -4 & 5 & 9 & -9 \end{array} \right] \begin{matrix} \leftarrow R_1 \\ \leftarrow R_2 \\ \leftarrow R_3 \end{matrix}$$

1  $4R_1 + R_3 \rightarrow \text{new } R_3$

$$\begin{array}{r} 4 & -8 & 4 & 0 \\ + -4 & 5 & 9 & -9 \\ \hline 0 & -3 & 13 & -9 \end{array}$$

# Systems of Linear Equations

$$\begin{cases} x_1 - 2x_2 + x_3 = 0 \\ 2x_2 - 8x_3 = 8 \\ -4x_1 + 5x_2 + 9x_3 = -9 \end{cases}$$

1

$$\begin{array}{r} 4x_1 - 8x_2 + 4x_3 = 0 \\ -4x_1 + 5x_2 + 9x_3 = -9 \\ \hline 0x_1 - 3x_2 + 13x_3 = -9 \end{array}$$

$$\begin{cases} x_1 - 2x_2 + x_3 = 0 \\ 2x_2 - 8x_3 = 8 \\ -3x_2 + 13x_3 = -9 \end{cases}$$

$$\left[ \begin{array}{cccc} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ -4 & 5 & 9 & -9 \end{array} \right] \begin{matrix} \leftarrow R_1 \\ \leftarrow R_2 \\ \leftarrow R_3 \end{matrix}$$

1  $4R_1 + R_3 \rightarrow \text{new } R_3$

$$\begin{array}{r} 4 & -8 & 4 & 0 \\ + & -4 & 5 & 9 & -9 \\ \hline 0 & -3 & 13 & -9 \end{array}$$

$$\left[ \begin{array}{cccc} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ 0 & -3 & 13 & -9 \end{array} \right] \leftarrow \text{new } R_3$$

# Systems of Linear Equations



$$\begin{cases} x_1 - 2x_2 + x_3 = 0 \\ 2x_2 - 8x_3 = 8 \\ -3x_2 + 13x_3 = -9 \end{cases}$$

2 Multiply [equation 2] by  $\frac{1}{2}$ .

$$\left[ \begin{array}{cccc} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ 0 & -3 & 13 & -9 \end{array} \right]$$

2  $\frac{1}{2}R_2 \rightarrow \text{new } R_2$

# Systems of Linear Equations

$$\begin{cases} x_1 - 2x_2 + x_3 = 0 \\ 2x_2 - 8x_3 = 8 \\ -3x_2 + 13x_3 = -9 \end{cases}$$

**2** Multiply [equation 2] by  $\frac{1}{2}$ .

$$\left[ \begin{array}{cccc} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ 0 & -3 & 13 & -9 \end{array} \right]$$

**2**  $\frac{1}{2}R_2 \rightarrow \text{new } R_2$

$$\begin{cases} x_1 - 2x_2 + x_3 = 0 \\ x_2 - 4x_3 = 4 \\ -3x_2 + 13x_3 = -9 \end{cases}$$

# Systems of Linear Equations



$$\begin{cases} x_1 - 2x_2 + x_3 = 0 \\ 2x_2 - 8x_3 = 8 \\ -3x_2 + 13x_3 = -9 \end{cases}$$

2 Multiply [equation 2] by  $\frac{1}{2}$ .

$$\left[ \begin{array}{cccc} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ 0 & -3 & 13 & -9 \end{array} \right]$$

2  $\frac{1}{2}R_2 \rightarrow \text{new } R_2$

$$\begin{cases} x_1 - 2x_2 + x_3 = 0 \\ x_2 - 4x_3 = 4 \\ -3x_2 + 13x_3 = -9 \end{cases}$$

$$\left[ \begin{array}{cccc} 1 & -2 & 1 & 0 \\ \color{green}{0} & \color{green}{1} & \color{green}{-4} & \color{green}{4} \\ 0 & -3 & 13 & -9 \end{array} \right]$$

# Systems of Linear Equations

$$\begin{cases} x_1 - 2x_2 + x_3 = 0 \\ x_2 - 4x_3 = 4 \\ -3x_2 + 13x_3 = -9 \end{cases}$$

$$\left[ \begin{array}{cccc} 1 & -2 & 1 & 0 \\ 0 & 1 & -4 & 4 \\ 0 & -3 & 13 & -9 \end{array} \right]$$

3

3  $3R_2 + R_3 \rightarrow \text{new } R_3$

# Systems of Linear Equations

$$\begin{cases} x_1 - 2x_2 + x_3 = 0 \\ x_2 - 4x_3 = 4 \\ -3x_2 + 13x_3 = -9 \end{cases}$$

$$\left[ \begin{array}{cccc} 1 & -2 & 1 & 0 \\ 0 & 1 & -4 & 4 \\ 0 & -3 & 13 & -9 \end{array} \right]$$

3

3  $R_2 + R_3 \rightarrow$  new  $R_3$

$$\begin{array}{r}
 & 0 & 3 & -12 & 12 \\
 + & 0 & -3 & 13 & -9 \\
 \hline
 \end{array}$$

# Systems of Linear Equations

$$\begin{cases} x_1 - 2x_2 + x_3 = 0 \\ x_2 - 4x_3 = 4 \\ -3x_2 + 13x_3 = -9 \end{cases}$$

$$\left[ \begin{array}{cccc} 1 & -2 & 1 & 0 \\ 0 & 1 & -4 & 4 \\ 0 & -3 & 13 & -9 \end{array} \right]$$

3

3  $3R_2 + R_3 \rightarrow \text{new } R_3$

$$\begin{array}{r}
 & 0 & 3 & -12 & 12 \\
 + & 0 & -3 & 13 & -9 \\
 \hline
 & 0 & 0 & 1 & 3
 \end{array}$$

# Systems of Linear Equations

$$\begin{cases} x_1 - 2x_2 + x_3 = 0 \\ x_2 - 4x_3 = 4 \\ -3x_2 + 13x_3 = -9 \end{cases}$$

$$\left[ \begin{array}{cccc} 1 & -2 & 1 & 0 \\ 0 & 1 & -4 & 4 \\ 0 & -3 & 13 & -9 \end{array} \right]$$

3

3  $3R_2 + R_3 \rightarrow$  new  $R_3$

$$\begin{array}{r} 0 \quad 3 \quad -12 \quad 12 \\ + \quad 0 \quad -3 \quad 13 \quad -9 \\ \hline 0 \quad 0 \quad 1 \quad 3 \end{array}$$

$$\left[ \begin{array}{cccc} 1 & -2 & 1 & 0 \\ 0 & 1 & -4 & 4 \\ 0 & 0 & 1 & 3 \end{array} \right]$$

# Systems of Linear Equations

$$\begin{cases} x_1 - 2x_2 + x_3 = 0 \\ x_2 - 4x_3 = 4 \\ -3x_2 + 13x_3 = -9 \end{cases}$$

3  $3 \times \text{Eq2} + \text{Eq3} \rightarrow \text{new Eq3.}$

$$\left[ \begin{array}{cccc} 1 & -2 & 1 & 0 \\ 0 & 1 & -4 & 4 \\ 0 & -3 & 13 & -9 \end{array} \right]$$

3  $3R_2 + R_3 \rightarrow \text{new } R_3$

$$\begin{array}{r} 0 & 3 & -12 & 12 \\ + & 0 & -3 & 13 & -9 \\ \hline 0 & 0 & 1 & 3 \end{array}$$

$$\left[ \begin{array}{cccc} 1 & -2 & 1 & 0 \\ 0 & 1 & -4 & 4 \\ 0 & 0 & 1 & 3 \end{array} \right]$$

# Systems of Linear Equations



$$\begin{cases} x_1 - 2x_2 + x_3 = 0 \\ x_2 - 4x_3 = 4 \\ -3x_2 + 13x_3 = -9 \end{cases}$$

3  $3 \times \text{Eq2} + \text{Eq3} \rightarrow \text{new Eq3.}$

$$\begin{array}{r} 3x_2 - 12x_3 = 12 \\ -3x_2 + 13x_3 = -9 \\ \hline \end{array}$$

$$\left[ \begin{array}{cccc} 1 & -2 & 1 & 0 \\ 0 & 1 & -4 & 4 \\ 0 & -3 & 13 & -9 \end{array} \right]$$

3  $3R_2 + R_3 \rightarrow \text{new } R_3$

$$\begin{array}{rrrr} & 0 & 3 & -12 & 12 \\ + & 0 & -3 & 13 & -9 \\ \hline & \color{green}0 & \color{green}0 & \color{green}1 & \color{green}3 \end{array}$$

$$\left[ \begin{array}{cccc} 1 & -2 & 1 & 0 \\ 0 & 1 & -4 & 4 \\ \color{green}0 & \color{green}0 & \color{green}1 & \color{green}3 \end{array} \right]$$

# Systems of Linear Equations

$$\begin{cases} x_1 - 2x_2 + x_3 = 0 \\ x_2 - 4x_3 = 4 \\ -3x_2 + 13x_3 = -9 \end{cases}$$

3  $3 \times \text{Eq2} + \text{Eq3} \rightarrow \text{new Eq3.}$

$$\begin{array}{r} 3x_2 - 12x_3 = 12 \\ -3x_2 + 13x_3 = -9 \\ \hline 0x_2 + x_3 = 3 \end{array}$$

$$\left[ \begin{array}{cccc} 1 & -2 & 1 & 0 \\ 0 & 1 & -4 & 4 \\ 0 & -3 & 13 & -9 \end{array} \right]$$

3  $3R_2 + R_3 \rightarrow \text{new } R_3$

$$\begin{array}{rrrr} & 0 & 3 & -12 & 12 \\ + & 0 & -3 & 13 & -9 \\ \hline & 0 & 0 & 1 & 3 \end{array}$$

$$\left[ \begin{array}{cccc} 1 & -2 & 1 & 0 \\ 0 & 1 & -4 & 4 \\ 0 & 0 & 1 & 3 \end{array} \right]$$

# Systems of Linear Equations

$$\begin{cases} x_1 - 2x_2 + x_3 = 0 \\ x_2 - 4x_3 = 4 \\ -3x_2 + 13x_3 = -9 \end{cases}$$

3  $\times$  Eq2 + Eq3  $\rightarrow$  new Eq3.

$$\begin{array}{r} 3x_2 - 12x_3 = 12 \\ -3x_2 + 13x_3 = -9 \\ \hline 0x_2 + x_3 = 3 \end{array}$$

$$\begin{cases} x_1 - 2x_2 + x_3 = 0 \\ x_2 - 4x_3 = 4 \\ x_3 = 3 \end{cases}$$

$$\left[ \begin{array}{cccc} 1 & -2 & 1 & 0 \\ 0 & 1 & -4 & 4 \\ 0 & -3 & 13 & -9 \end{array} \right]$$

3  $R_2 + R_3 \rightarrow$  new  $R_3$

$$\begin{array}{rrrr} & 0 & 3 & -12 & 12 \\ + & 0 & -3 & 13 & -9 \\ \hline & 0 & 0 & 1 & 3 \end{array}$$

$$\left[ \begin{array}{cccc} 1 & -2 & 1 & 0 \\ 0 & 1 & -4 & 4 \\ 0 & 0 & 1 & 3 \end{array} \right]$$

# Systems of Linear Equations

$$\begin{cases} x_1 - 2x_2 + x_3 = 0 \\ x_2 - 4x_3 = 4 \\ x_3 = 3 \end{cases}$$

4 Eq2 + 4×Eq3 → new Eq2.

$$\begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 1 & -4 & 4 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

4 R<sub>2</sub> + 4R<sub>3</sub> → new R<sub>2</sub>

# Systems of Linear Equations

$$\begin{cases} x_1 - 2x_2 + x_3 = 0 \\ x_2 - 4x_3 = 4 \\ x_3 = 3 \end{cases}$$

4 Eq2 + 4×Eq3 → new Eq2.

$$\left[ \begin{array}{cccc} 1 & -2 & 1 & 0 \\ 0 & 1 & -4 & 4 \\ 0 & 0 & 1 & 3 \end{array} \right]$$

4 R<sub>2</sub> + 4R<sub>3</sub> → new R<sub>2</sub>

$$\begin{array}{r} 0 & 1 & -4 & 4 \\ + & 0 & 0 & 4 & 12 \\ \hline 0 & 1 & 0 & 16 \end{array}$$

$$\left[ \begin{array}{cccc} 1 & -2 & 1 & 0 \\ 0 & 1 & 0 & 16 \\ 0 & 0 & 1 & 3 \end{array} \right]$$

# Systems of Linear Equations

$$\begin{cases} x_1 - 2x_2 + x_3 = 0 \\ x_2 - 4x_3 = 4 \\ x_3 = 3 \end{cases}$$

4 Eq2 + 4×Eq3 → new Eq2.

$$\begin{array}{rcl} x_2 - 4x_3 &=& 4 \\ 4x_3 &=& 12 \\ \hline x_2 &=& 16 \end{array}$$

$$\begin{cases} x_1 - 2x_2 + x_3 = 0 \\ x_2 = 16 \\ x_3 = 3 \end{cases}$$

$$\left[ \begin{array}{cccc} 1 & -2 & 1 & 0 \\ 0 & 1 & -4 & 4 \\ 0 & 0 & 1 & 3 \end{array} \right]$$

4 R<sub>2</sub> + 4R<sub>3</sub> → new R<sub>2</sub>

$$\begin{array}{rrrr} & 0 & 1 & -4 & 4 \\ + & 0 & 0 & 4 & 12 \\ \hline & 0 & 1 & 0 & 16 \end{array}$$

$$\left[ \begin{array}{cccc} 1 & -2 & 1 & 0 \\ 0 & 1 & 0 & 16 \\ 0 & 0 & 1 & 3 \end{array} \right]$$

# Systems of Linear Equations

$$\begin{cases} x_1 - 2x_2 + x_3 = 0 \\ x_2 - 4x_3 = 4 \\ x_3 = 3 \end{cases}$$

**5** Eq1 – Eq3 → new Eq1.

$$\begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 1 & 0 & 16 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

**5**  $R_1 - R_3 \rightarrow$  new  $R_1$

# Systems of Linear Equations

$$\begin{cases} x_1 - 2x_2 + x_3 = 0 \\ x_2 - 4x_3 = 4 \\ x_3 = 3 \end{cases}$$

5 Eq1 – Eq3 → new Eq1.

$$\left[ \begin{array}{cccc} 1 & -2 & 1 & 0 \\ 0 & 1 & 0 & 16 \\ 0 & 0 & 1 & 3 \end{array} \right]$$

5  $R_1 - R_3 \rightarrow$  new  $R_1$

$$\begin{array}{r} 1 & -2 & 1 & 0 \\ + & 0 & 0 & -1 & -3 \\ \hline 1 & -2 & 0 & -3 \end{array}$$

$$\left[ \begin{array}{cccc} 1 & -2 & 0 & -3 \\ 0 & 1 & 0 & 16 \\ 0 & 0 & 1 & 3 \end{array} \right]$$

# Systems of Linear Equations



$$\begin{cases} x_1 - 2x_2 + x_3 = 0 \\ x_2 - 4x_3 = 4 \\ x_3 = 3 \end{cases}$$

5 Eq1 - Eq3  $\rightarrow$  new Eq1.

$$\begin{array}{rcl} x_1 - 2x_2 + x_3 & = & 0 \\ & - x_3 & = -3 \\ \hline x_1 - 2x_2 & = & -3 \end{array}$$

$$\begin{cases} x_1 - 2x_2 = -3 \\ x_2 = 16 \\ x_3 = 3 \end{cases}$$

$$\left[ \begin{array}{cccc} 1 & -2 & 1 & 0 \\ 0 & 1 & 0 & 16 \\ 0 & 0 & 1 & 3 \end{array} \right]$$

5  $R_1 - R_3 \rightarrow$  new  $R_1$

$$\begin{array}{rrrr} 1 & -2 & 1 & 0 \\ + & 0 & 0 & -1 & -3 \\ \hline 1 & -2 & 0 & -3 \end{array}$$

$$\left[ \begin{array}{cccc} 1 & -2 & 0 & -3 \\ 0 & 1 & 0 & 16 \\ 0 & 0 & 1 & 3 \end{array} \right]$$

# Systems of Linear Equations



$$\begin{cases} x_1 - 2x_2 = -3 \\ x_2 - 4x_3 = 4 \\ x_3 = 3 \end{cases}$$

6 Eq1 + 2×Eq2 → new Eq1.

$$\left[ \begin{array}{cccc} 1 & -2 & 0 & -3 \\ 0 & 1 & 0 & 16 \\ 0 & 0 & 1 & 3 \end{array} \right]$$

6  $R_1 + 2R_2 \rightarrow$  new  $R_1$

# Systems of Linear Equations

$$\begin{cases} x_1 - 2x_2 = -3 \\ x_2 - 4x_3 = 4 \\ x_3 = 3 \end{cases}$$

6 Eq1 + 2×Eq2 → new Eq1.

$$\left[ \begin{array}{cccc} 1 & -2 & 0 & -3 \\ 0 & 1 & 0 & 16 \\ 0 & 0 & 1 & 3 \end{array} \right]$$

6  $R_1 + 2R_2 \rightarrow$  new  $R_1$

$$\begin{array}{r} 1 & -2 & 0 & -3 \\ + & 0 & 2 & 0 & 32 \\ \hline 1 & 0 & 0 & 29 \end{array}$$

$$\left[ \begin{array}{cccc} 1 & 0 & 0 & 29 \\ 0 & 1 & 0 & 16 \\ 0 & 0 & 1 & 3 \end{array} \right]$$

# Systems of Linear Equations

$$\begin{cases} x_1 - 2x_2 = -3 \\ x_2 - 4x_3 = 4 \\ x_3 = 3 \end{cases}$$

6 Eq1 + 2×Eq2 → new Eq1.

$$\begin{array}{rcl} x_1 - 2x_2 & = & -3 \\ 2x_2 & = & 32 \\ \hline x_1 & = & 29 \end{array}$$

$$\begin{cases} x_1 = 29 \\ x_2 = 16 \\ x_3 = 3 \end{cases}$$

$$\left[ \begin{array}{cccc} 1 & -2 & 0 & -3 \\ 0 & 1 & 0 & 16 \\ 0 & 0 & 1 & 3 \end{array} \right]$$

6  $R_1 + 2R_2 \rightarrow$  new  $R_1$

$$\begin{array}{rcl} & & \begin{matrix} 1 & -2 & 0 & -3 \\ + & 0 & 2 & 0 & 32 \\ \hline 1 & 0 & 0 & 29 \end{matrix} \end{array}$$

$$\left[ \begin{array}{cccc} 1 & 0 & 0 & 29 \\ 0 & 1 & 0 & 16 \\ 0 & 0 & 1 & 3 \end{array} \right]$$

# Systems of Linear Equations



## Example

Solve

$$\begin{cases} 1x_1 - 2x_2 + 1x_3 = 0 \\ \quad 2x_2 - 8x_3 = 8 \\ -4x_1 + 5x_2 + 9x_3 = -9 \end{cases}$$

$$\begin{cases} x_1 = 29 \\ x_2 = 16 \\ x_3 = 3 \end{cases}$$

# Systems of Linear Equations



## Example

Use elementary row operations to convert the augmented matrix

$$\left[ \begin{array}{cccc} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ -4 & 5 & 9 & -9 \end{array} \right] \quad \text{into} \quad \left[ \begin{array}{cccc} 1 & 0 & 0 & ? \\ 0 & 1 & 0 & ? \\ 0 & 0 & 1 & ? \end{array} \right].$$

$$\left[ \begin{array}{cccc} 1 & 0 & 0 & 29 \\ 0 & 1 & 0 & 16 \\ 0 & 0 & 1 & 3 \end{array} \right]$$

# Systems of Linear Equations



## Definition

Two matrices  $A$  and  $B$  are called *row equivalent* if it is possible to change one into the other using elementary row operations.

We write  $A \sim B$  if  $A$  and  $B$  are row equivalent.

# Systems of Linear Equations



## Definition

Two matrices  $A$  and  $B$  are called *row equivalent* if it is possible to change one into the other using elementary row operations.

We write  $A \sim B$  if  $A$  and  $B$  are row equivalent.

## Theorem

*If the augmented matrices of two linear systems are row equivalent, then the two systems have the same solution set.*

## Elementary Row Operations (again)

- 1 Multiply a row by a number (e.g.  $cR_3 \rightarrow \text{new}R_3$ ).
- 2 Swap two rows (e.g.  $R_1 \leftrightarrow R_2$ ).
- 3 Add a multiple of one row to another  
(e.g.  $cR_3 + R_4 \rightarrow \text{new}R_4$ ).

## Elementary Row Operations (again)

- 1 Multiply a row by a number (e.g.  $cR_3 \rightarrow \text{new}R_3$ ).
- 2 Swap two rows (e.g.  $R_1 \leftrightarrow R_2$ ).
- 3 Add a multiple of one row to another  
(e.g.  $cR_3 + R_4 \rightarrow \text{new}R_4$ ).

Note that all three of these are invertible.

- 1 The opposite of multiplying a row by  $c \neq 0$  is multiplying it by  $\frac{1}{c}$ .

## Elementary Row Operations (again)

- 1 Multiply a row by a number (e.g.  $cR_3 \rightarrow \text{new}R_3$ ).
- 2 Swap two rows (e.g.  $R_1 \leftrightarrow R_2$ ).
- 3 Add a multiple of one row to another  
(e.g.  $cR_3 + R_4 \rightarrow \text{new}R_4$ ).

Note that all three of these are invertible.

- 1 The opposite of multiplying a row by  $c \neq 0$  is multiplying it by  $\frac{1}{c}$ .
- 2 Swapping two rows is its own inverse.

## Elementary Row Operations (again)

- 1 Multiply a row by a number (e.g.  $cR_3 \rightarrow \text{new}R_3$ ).
- 2 Swap two rows (e.g.  $R_1 \leftrightarrow R_2$ ).
- 3 Add a multiple of one row to another  
(e.g.  $cR_3 + R_4 \rightarrow \text{new}R_4$ ).

Note that all three of these are invertible.

- 1 The opposite of multiplying a row by  $c \neq 0$  is multiplying it by  $\frac{1}{c}$ .
- 2 Swapping two rows is its own inverse.
- 3 The opposite of adding something, is subtracting that something.

## Existence and Uniqueness Questions

### Example

Recall that

$$\begin{cases} x_1 - 2x_2 + x_3 = 0 \\ 2x_2 - 8x_3 = 8 \\ -4x_1 + 5x_2 + 9x_3 = -9 \end{cases} \quad \text{has solution} \quad \begin{cases} x_1 = 29 \\ x_2 = 16 \\ x_3 = 3. \end{cases}$$

Therefore this linear system is consistent.

# Systems of Linear Equations

## Example

Is

$$\begin{cases} x_2 - 4x_3 = 8 \\ 2x_1 - 3x_2 + 2x_3 = 1 \\ 5x_1 - 8x_2 + 7x_3 = 1 \end{cases}$$

consistent?

# Systems of Linear Equations



## Example

Is

$$\begin{cases} x_2 - 4x_3 = 8 \\ 2x_1 - 3x_2 + 2x_3 = 1 \\ 5x_1 - 8x_2 + 7x_3 = 1 \end{cases}$$

consistent?

To answer this, we will do elementary row operations on the augmented matrix

$$\left[ \begin{array}{cccc} 0 & 1 & -4 & 8 \\ 2 & -3 & 2 & 1 \\ 5 & -8 & 7 & 1 \end{array} \right]$$

and see what we get.

# Systems of Linear Equations



$$\begin{bmatrix} 0 & 1 & -4 & 8 \\ 2 & -3 & 2 & 1 \\ 5 & -8 & 7 & 1 \end{bmatrix}$$

# Systems of Linear Equations



$$\begin{bmatrix} 0 & 1 & -4 & 8 \\ 2 & -3 & 2 & 1 \\ 5 & -8 & 7 & 1 \end{bmatrix}$$

1  $R_1 \leftrightarrow R_2$

$$\begin{bmatrix} 2 & -3 & 2 & 1 \\ 0 & 1 & -4 & 8 \\ 5 & -8 & 7 & 1 \end{bmatrix}$$

# Systems of Linear Equations

$$\begin{bmatrix} 0 & 1 & -4 & 8 \\ 2 & -3 & 2 & 1 \\ 5 & -8 & 7 & 1 \end{bmatrix}$$

1  $R_1 \leftrightarrow R_2$

$$\begin{bmatrix} 2 & -3 & 2 & 1 \\ 0 & 1 & -4 & 8 \\ 5 & -8 & 7 & 1 \end{bmatrix}$$

2  $R_3 - \frac{5}{2}R_1 \rightarrow R_3$

$$\begin{array}{r}
 5 & -8 & 7 & 1 \\
 + -5 & \frac{15}{2} & -5 & -\frac{5}{2} \\
 \hline
 0 & -\frac{1}{2} & 2 & -\frac{3}{2}
 \end{array}$$

$$\begin{bmatrix} 2 & -3 & 2 & 1 \\ 0 & 1 & -4 & 8 \\ 0 & -\frac{1}{2} & 2 & -\frac{3}{2} \end{bmatrix}$$

# Systems of Linear Equations



$$\begin{bmatrix} 2 & -3 & 2 & 1 \\ 0 & 1 & -4 & 8 \\ 0 & -\frac{1}{2} & 2 & -\frac{3}{2} \end{bmatrix}$$

# Systems of Linear Equations



$$\begin{bmatrix} 2 & -3 & 2 & 1 \\ 0 & 1 & -4 & 8 \\ 0 & -\frac{1}{2} & 2 & -\frac{3}{2} \end{bmatrix}$$

3  $\frac{1}{2}R_2 + R_3 \rightarrow R_3$

$$\begin{array}{r} & 0 & \frac{1}{2} & -2 & 4 \\ + & 0 & -\frac{1}{2} & 2 & -\frac{3}{2} \\ \hline & 0 & 0 & 0 & \frac{5}{2} \end{array}$$

$$\begin{bmatrix} 2 & -3 & 2 & 1 \\ 0 & 1 & -4 & 8 \\ 0 & 0 & 0 & \frac{5}{2} \end{bmatrix}$$

# Systems of Linear Equations

$$\begin{bmatrix} 2 & -3 & 2 & 1 \\ 0 & 1 & -4 & 8 \\ 0 & -\frac{1}{2} & 2 & -\frac{3}{2} \end{bmatrix}$$

3  $\frac{1}{2}R_2 + R_3 \rightarrow R_3$

$$\begin{array}{rcccc} & 0 & \frac{1}{2} & -2 & 4 \\ + & 0 & -\frac{1}{2} & 2 & -\frac{3}{2} \\ \hline & 0 & 0 & 0 & \frac{5}{2} \end{array}$$

$$\begin{bmatrix} 2 & -3 & 2 & 1 \\ 0 & 1 & -4 & 8 \\ 0 & 0 & 0 & \frac{5}{2} \end{bmatrix}$$

4 Convert back to a linear system.

$$\begin{cases} 2x_1 - 3x_2 + 2x_3 = 1 \\ x_2 - 4x_3 = 8 \\ 0x_1 + 0x_2 + 0x_3 = \frac{5}{2}. \end{cases}$$

# Systems of Linear Equations



$$\begin{cases} x_2 - 4x_3 = 8 \\ 2x_1 - 3x_2 + 2x_3 = 1 \\ 5x_1 - 8x_2 + 7x_3 = 1 \end{cases} \rightarrow \begin{cases} 2x_1 - 3x_2 + 2x_3 &= 1 \\ x_2 - 4x_3 &= 8 \\ 0 &= \frac{5}{2}. \end{cases}$$

But the third equation says

$$0 = \frac{5}{2}$$

which is clearly nonsense.

# Systems of Linear Equations



$$\begin{cases} x_2 - 4x_3 = 8 \\ 2x_1 - 3x_2 + 2x_3 = 1 \\ 5x_1 - 8x_2 + 7x_3 = 1 \end{cases} \rightarrow \begin{cases} 2x_1 - 3x_2 + 2x_3 &= 1 \\ x_2 - 4x_3 &= 8 \\ 0 &= \frac{5}{2}. \end{cases}$$

But the third equation says

$$0 = \frac{5}{2}$$

which is clearly nonsense.

This means that it is not possible to find a solution to this linear system. So the linear system is inconsistent.

## Systems of Linear Equations

## Example

For what values of  $h$  and  $k$  is the following system consistent?

$$\begin{cases} 2x_1 - x_2 = h \\ -6x_1 + 3x_2 = k \end{cases}$$

## Systems of Linear Equations

## Example

For what values of  $h$  and  $k$  is the following system consistent?

$$\begin{cases} 2x_1 - x_2 = h \\ -6x_1 + 3x_2 = k \end{cases}$$

To answer this, we do one row operation on this system:

$$\left[ \begin{array}{ccc} 2 & -1 & h \\ -6 & 3 & k \end{array} \right] \xrightarrow{3R_1+R_2} \left[ \begin{array}{ccc} 2 & -1 & h \\ 0 & 0 & 3h+k \end{array} \right].$$

## Systems of Linear Equations

## Example

For what values of  $h$  and  $k$  is the following system consistent?

$$\begin{cases} 2x_1 - x_2 = h \\ -6x_1 + 3x_2 = k \end{cases}$$

To answer this, we do one row operation on this system:

$$\left[ \begin{array}{ccc} 2 & -1 & h \\ -6 & 3 & k \end{array} \right] \xrightarrow{3R_1+R_2} \left[ \begin{array}{ccc} 2 & -1 & h \\ 0 & 0 & 3h+k \end{array} \right].$$

Notice that the latter matrix is equivalent to

$$\begin{cases} 2x_1 - x_2 = h \\ 0x_1 + 0x_2 = 3h+k. \end{cases}$$

## Systems of Linear Equations

## Example

For what values of  $h$  and  $k$  is the following system consistent?

$$\begin{cases} 2x_1 - x_2 = h \\ -6x_1 + 3x_2 = k \end{cases}$$

To answer this, we do one row operation on this system:

$$\left[ \begin{array}{ccc} 2 & -1 & h \\ -6 & 3 & k \end{array} \right] \xrightarrow{3R_1+R_2} \left[ \begin{array}{ccc} 2 & -1 & h \\ 0 & 0 & 3h+k \end{array} \right].$$

Notice that the latter matrix is equivalent to

$$\begin{cases} 2x_1 - x_2 = h \\ 0x_1 + 0x_2 = 3h+k. \end{cases}$$

Obviously, we require that  $3h+k=0$  in order for this system to have solutions.

So this linear system is consistent only when  $3h+k=0$ .

# Systems of Linear Equations



## Remark

It doesn't matter if you use [ ] or ( ) when you write an augmented matrix.

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}$$

## Example

Find an equation involving  $g$ ,  $h$ , and  $k$  that makes this augmented matrix correspond to a consistent system:

$$\left( \begin{array}{cccc} 1 & -4 & 7 & g \\ 0 & 3 & -5 & h \\ -2 & 5 & -9 & k \end{array} \right).$$

## Example

Find an equation involving  $g$ ,  $h$ , and  $k$  that makes this augmented matrix correspond to a consistent system:

$$\begin{pmatrix} 1 & -4 & 7 & g \\ 0 & 3 & -5 & h \\ -2 & 5 & -9 & k \end{pmatrix}.$$

$$\begin{pmatrix} 1 & -4 & 7 & g \\ 0 & 3 & -5 & h \\ -2 & 5 & -9 & k \end{pmatrix} \xrightarrow{2R_1+R_3} \begin{pmatrix} 1 & -4 & 7 & g \\ 0 & 3 & -5 & h \\ 0 & -3 & 5 & 2g+k \end{pmatrix}$$

$$\xrightarrow{R_2+R_3} \begin{pmatrix} 1 & -4 & 7 & g \\ 0 & 3 & -5 & h \\ 0 & 0 & 0 & 2g+h+k \end{pmatrix}.$$

From the last augmented matrix, we see that we must  $2g + h + k = 0$  for a consistent system.

# Systems of Linear Equations



## Example

Determine if the following system is consistent or inconsistent.  
If it is consistent, find all the solutions.

$$\begin{cases} 3x_1 + 5x_2 - 4x_3 = 7 \\ -3x_1 - 2x_2 + 4x_3 = -1 \\ 6x_1 + x_2 - 8x_3 = -4. \end{cases}$$

$$\left( \begin{array}{ccc|c} 3 & 5 & -4 & 7 \\ -3 & -2 & 4 & -1 \\ 6 & 1 & -8 & -4 \end{array} \right)$$

# Systems of Linear Equations



$$\left( \begin{array}{cccc} 3 & 5 & -4 & 7 \\ -3 & -2 & 4 & -1 \\ 6 & 1 & -8 & -4 \end{array} \right) \xrightarrow{\begin{array}{l} R_1+R_2 \\ -2R_1+R_3 \end{array}} \left( \begin{array}{cccc} 3 & 5 & -4 & 7 \\ 0 & 3 & 0 & 6 \\ 0 & -9 & 0 & -18 \end{array} \right)$$
$$\xrightarrow{\begin{array}{l} 3R_2+R_3 \\ \frac{1}{3}R_2 \end{array}} \left( \begin{array}{cccc} 3 & 5 & -4 & 7 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 0 \end{array} \right)$$
$$\xrightarrow{-5R_2+R_1} \left( \begin{array}{cccc} 3 & 0 & -4 & -3 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 0 \end{array} \right)$$

The linear system is equivalent to

$$\begin{cases} 3x_1 & - 4x_3 = -3 \\ & x_2 = 2 \end{cases}$$

# Systems of Linear Equations



$$\begin{cases} 3x_1 & - 4x_3 = -3 \\ x_2 & = 2 \end{cases}$$

Thus  $x_1 = -1 + \frac{4}{3}x_3$  and every solution has the form of

$$\begin{aligned}(x_1, x_2, x_3) &= (-1 + \frac{4}{3}x_3, 2, x_3) \\ &= x_3(\frac{4}{3}, 0, 1) + (-1, 2, 0).\end{aligned}$$

# Systems of Linear Equations



$$\begin{cases} 3x_1 & - 4x_3 = -3 \\ x_2 & = 2 \end{cases}$$

Thus  $x_1 = -1 + \frac{4}{3}x_3$  and every solution has the form of

$$\begin{aligned}(x_1, x_2, x_3) &= (-1 + \frac{4}{3}x_3, 2, x_3) \\ &= x_3(\frac{4}{3}, 0, 1) + (-1, 2, 0).\end{aligned}$$

Here,  $x_3$  can take on any real number so there are infinitely many solutions. Geometrically, the solution set is a line passing through the point  $(-1, 2, 0)$  parallel to the vector  $(\frac{4}{3}, 0, 1)$ .

## Syst Example

Do the three planes  $\begin{cases} x_1 + 3x_2 + x_3 = 4 \\ x_2 - x_3 = 1 \\ x_1 + 4x_2 = -2 \end{cases}$  have a point in common? Explain.

## Systems Example

Do the three planes  $\begin{cases} x_1 + 3x_2 + x_3 = 4 \\ x_2 - x_3 = 1 \\ x_1 + 4x_2 = -2 \end{cases}$  have a point in common? Explain.

$$\left( \begin{array}{cccc} 1 & 3 & 1 & 4 \\ 0 & 1 & -1 & 1 \\ 1 & 4 & 0 & -2 \end{array} \right) \xrightarrow{-R_1+R_3} \left( \begin{array}{cccc} 1 & 3 & 1 & 4 \\ 0 & 1 & -1 & 1 \\ 0 & 1 & -1 & -6 \end{array} \right)$$
$$\xrightarrow{-R_2+R_3} \left( \begin{array}{cccc} 1 & 3 & 1 & 4 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & -7 \end{array} \right)$$

## Systems Example

Do the three planes  $\begin{cases} x_1 + 3x_2 + x_3 = 4 \\ x_2 - x_3 = 1 \\ x_1 + 4x_2 = -2 \end{cases}$  have a point in common? Explain.

$$\left( \begin{array}{cccc} 1 & 3 & 1 & 4 \\ 0 & 1 & -1 & 1 \\ 1 & 4 & 0 & -2 \end{array} \right) \xrightarrow{-R_1+R_3} \left( \begin{array}{cccc} 1 & 3 & 1 & 4 \\ 0 & 1 & -1 & 1 \\ 0 & 1 & -1 & -6 \end{array} \right)$$

$$\xrightarrow{-R_2+R_3} \left( \begin{array}{cccc} 1 & 3 & 1 & 4 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & -7 \end{array} \right)$$

The last line of this augmented matrix reads

$$0x_1 + 0x_2 + 0x_3 = -7,$$

which, of course, is nonsense.

## Systems Example

Do the three planes  $\begin{cases} x_1 + 3x_2 + x_3 = 4 \\ x_2 - x_3 = 1 \\ x_1 + 4x_2 = -2 \end{cases}$  have a point in common? Explain.

$$\left( \begin{array}{cccc} 1 & 3 & 1 & 4 \\ 0 & 1 & -1 & 1 \\ 1 & 4 & 0 & -2 \end{array} \right) \xrightarrow{-R_1+R_3} \left( \begin{array}{cccc} 1 & 3 & 1 & 4 \\ 0 & 1 & -1 & 1 \\ 0 & 1 & -1 & -6 \end{array} \right)$$
$$\xrightarrow{-R_2+R_3} \left( \begin{array}{cccc} 1 & 3 & 1 & 4 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & -7 \end{array} \right)$$

The last line of this augmented matrix reads

$$0x_1 + 0x_2 + 0x_3 = -7,$$

which, of course, is nonsense.

So there is **no** point in common for the three planes.

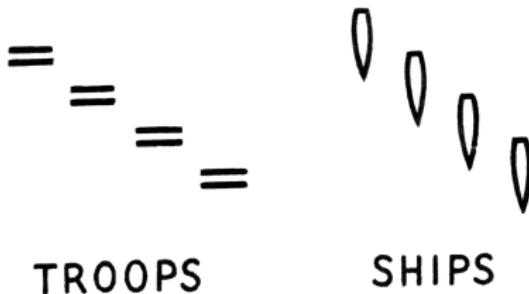
# Break

We will continue at 3pm



# Row Reduction and Echelon Forms

# Row Reduction and Echelon Forms



An **echelon formation** is a (usually military) formation in which its units are arranged diagonally.



Echelon is pronounced “esh-E-lon” because it comes from the French language.

# Row Reduction and Echelon Forms



$$\begin{bmatrix} 2 & -3 & 2 & 1 & -7 & 4 \\ 0 & 1 & -4 & 8 & 0 & 0 \\ 0 & 0 & 0 & 5 & 2 & 1 \\ 0 & 0 & 0 & 0 & 0 & 6 \end{bmatrix}$$

## Definition

The first non-zero number in each row, is called a *leading entry* or *pivot*.

# Row Reduction and Echelon Forms



leading entry/pivot

$$\begin{bmatrix} 2 & -3 & 2 & 1 & -7 & 4 \\ 0 & 1 & -4 & 8 & 0 & 0 \\ 0 & 0 & 0 & 5 & 2 & 1 \\ 0 & 0 & 0 & 0 & 0 & 6 \end{bmatrix}$$

## Definition

The first non-zero number in each row, is called a *leading entry* or *pivot*.

# Row Reduction and Echelon Forms



leading entry/pivot

$$\left[ \begin{array}{cccccc} 2 & -3 & 2 & 1 & -7 & 4 \\ 0 & 1 & -4 & 8 & 0 & 0 \\ 0 & 0 & 0 & 5 & 2 & 1 \\ 0 & 0 & 0 & 0 & 0 & 6 \end{array} \right]$$

## Definition

The first non-zero number in each row, is called a *leading entry* or *pivot*.

# Row Reduction and Echelon Forms



leading entry/pivot

$$\left[ \begin{array}{cccccc} 2 & -3 & 2 & 1 & -7 & 4 \\ 0 & 1 & -4 & 8 & 0 & 0 \\ 0 & 0 & 0 & 5 & 2 & 1 \\ 0 & 0 & 0 & 0 & 0 & 6 \end{array} \right]$$

## Definition

The first non-zero number in each row, is called a *leading entry* or *pivot*.

# Row Reduction and Echelon Forms



leading entry/pivot

$$\left[ \begin{array}{cccccc} 2 & -3 & 2 & 1 & -7 & 4 \\ 0 & 1 & -4 & 8 & 0 & 0 \\ 0 & 0 & 0 & 5 & 2 & 1 \\ 0 & 0 & 0 & 0 & 0 & 6 \end{array} \right]$$

## Definition

The first non-zero number in each row, is called a *leading entry* or *pivot*.

# Row Reduction and Echelon Forms



$$\begin{bmatrix} 2 & 3 & 2 & 1 & 7 & 4 \\ 0 & 1 & 4 & 8 & 0 & 0 \\ 0 & 0 & 0 & 5 & 2 & 1 \\ 0 & 0 & 0 & 0 & 0 & 6 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

## Definition

A matrix is in *row echelon form* (REF) iff the following 3 rules are satisfied:

# Row Reduction and Echelon Forms



$$\left[ \begin{array}{cccccc} 2 & 3 & 2 & 1 & 7 & 4 \\ 0 & 1 & 4 & 8 & 0 & 0 \\ 0 & 0 & 0 & 5 & 2 & 1 \\ 0 & 0 & 0 & 0 & 0 & 6 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array} \right] \quad \left. \begin{array}{l} \text{non-zero rows} \\ \text{at the top} \end{array} \right\} \quad \left. \begin{array}{l} \text{zero rows at} \\ \text{the bottom} \end{array} \right\}$$

## Definition

A matrix is in *row echelon form* (REF) iff the following 3 rules are satisfied:

- 1 All the  $00000 \cdots 0$  rows are at the bottom;

# Row Reduction and Echelon Forms



$$\left[ \begin{array}{cccccc} 2 & 3 & 2 & 1 & 7 & 4 \\ 0 & 1 & 4 & 8 & 0 & 0 \\ 0 & 0 & 0 & 5 & 2 & 1 \\ 0 & 0 & 0 & 0 & 0 & 6 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array} \right]$$

down and right

## Definition

A matrix is in *row echelon form* (REF) iff the following 3 rules are satisfied:

- 1 All the  $00000\cdots 0$  rows are at the bottom;
- 2 The leading entries/pivots go ; and

# Row Reduction and Echelon Forms



zeros under  
pivots

$$\left[ \begin{array}{cccccc} 2 & 3 & 2 & 1 & 7 & 4 \\ 0 & 1 & 4 & 8 & 0 & 0 \\ 0 & 0 & 0 & 5 & 2 & 1 \\ 0 & 0 & 0 & 0 & 0 & 6 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array} \right]$$

## Definition

A matrix is in *row echelon form* (REF) iff the following 3 rules are satisfied:

- 1 All the  $00000\cdots 0$  rows are at the bottom;
- 2 The leading entries/pivots go ; and
- 3 All the entries below a leading entry are **zero**.

# Row Reduction and Echelon Forms



## Definition

A matrix is in *reduced row echelon form* (RREF) iff the following 5 rules are satisfied:

- 1 All the 00000 ··· 0 rows are at the bottom;
- 2 The leading entries/pivots go  $\searrow$ ; and
- 3 All the entries below a leading entry are zero.

$$\begin{bmatrix} 1 & 0 & 2 & 0 & 7 & 0 \\ 0 & 1 & 4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

# Row Reduction and Echelon Forms



## Definition

A matrix is in *reduced row echelon form* (RREF) iff the following 5 rules are satisfied:

- 1 All the 00000 ··· 0 rows are at the bottom;
- 2 The leading entries/pivots go  $\searrow$ ; and
- 3 All the entries below a leading entry are zero.
- 4 All the leading entries/pivots are 1

$$\begin{bmatrix} 1 & 0 & 2 & 0 & 7 & 0 \\ 0 & 1 & 4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

# Row Reduction and Echelon Forms



## Definition

A matrix is in *reduced row echelon form* (RREF) iff the following 5 rules are satisfied:

- 1 All the 00000...0 rows are at the bottom;
- 2 The leading entries/pivots go  $\searrow$ ; and
- 3 All the entries below a leading entry are zero.
- 4 All the leading entries/pivots are 1
- 5 Each leading 1 is the only non-zero entry in its column.

$$\left[ \begin{array}{cccccc} 1 & 0 & 2 & 0 & 7 & 0 \\ 0 & 1 & 4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array} \right]$$



## Example

Consider

$$\begin{bmatrix} 2 & -3 & 2 & 1 \\ 0 & 1 & -4 & 8 \\ 0 & 0 & 0 & 5/2 \end{bmatrix}.$$

Is this matrix in REF? Is this matrix in RREF?

## Row Reduction

REF: rules 1-3

RREF: all 5 rules

## Example

Consider

$$\begin{bmatrix} 2 & -3 & 2 & 1 \\ 0 & 1 & -4 & 8 \\ 0 & 0 & 0 & 5/2 \end{bmatrix}.$$

Is this matrix in REF? Is this

- 1 All the 00000...0 rows are at the bottom;
- 2 The leading entries/pivots go ; and
- 3 All the entries below a leading entry are zero.
- 4 All the leading entries/pivots are **1**
- 5 Each **leading 1** is the only non-zero entry in its column.

# Row Reduction

REF: rules 1-3

RREF: all 5 rules



## Example

Consider

$$\begin{bmatrix} 2 & -3 & 2 & 1 \\ 0 & 1 & -4 & 8 \\ 0 & 0 & 0 & 5/2 \end{bmatrix}.$$

Is this matrix in REF? Is this

matrix in RREF?

- 1 All the 00000...0 rows are at the bottom; ✓
- 2 The leading entries/pivots go ↓; and
- 3 All the entries below a leading entry are zero.
- 4 All the leading entries/pivots are 1
- 5 Each leading 1 is the only non-zero entry in its column.

## Row Reduction

REF: rules 1-3

RREF: all 5 rules

## Example

Consider

$$\begin{bmatrix} 2 & -3 & 2 & 1 \\ 0 & 1 & -4 & 8 \\ 0 & 0 & 0 & 5/2 \end{bmatrix}.$$

Is this matrix in REF? Is this

- 1 All the 00000...0 rows are at the bottom; ✓
- 2 The leading entries/pivots go ↓; and ✓
- 3 All the entries below a leading entry are zero.
- 4 All the leading entries/pivots are 1
- 5 Each leading 1 is the only non-zero entry in its column.

## Row Reduction

REF: rules 1-3

RREF: all 5 rules

## Example

Consider

$$\begin{bmatrix} 2 & -3 & 2 & 1 \\ 0 & 1 & -4 & 8 \\ 0 & 0 & 0 & 5/2 \end{bmatrix}.$$

Is this matrix in REF? Is this

- 1 All the 00000...0 rows are at the bottom; ✓
- 2 The leading entries/pivots go ↓; and ✓
- 3 All the entries below a leading entry are zero. ✓
- 4 All the leading entries/pivots are 1
- 5 Each leading 1 is the only non-zero entry in its column.

This matrix is in REF,

## Row Reduction

REF: rules 1-3

RREF: all 5 rules

## Example

Consider

$$\begin{bmatrix} 2 & -3 & 2 & 1 \\ 0 & 1 & -4 & 8 \\ 0 & 0 & 0 & 5/2 \end{bmatrix}.$$

Is this matrix in REF? Is this

- 1 All the 00000...0 rows are at the bottom; ✓
- 2 The leading entries/pivots go ↓; and ✓
- 3 All the entries below a leading entry are zero. ✓
- 4 All the leading entries/pivots are 1 X
- 5 Each leading 1 is the only non-zero entry in its column.

This matrix is in REF, but is not in RREF.



## Example

Consider

$$\begin{bmatrix} 1 & 0 & 0 & -2 & 3 & 0 & 2 \\ 0 & 0 & 1 & 3 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Is this matrix in REF? Is this matrix in RREF?

## Row Reduction

REF: rules 1-3

RREF: all 5 rules

## Example

Consider

$$\begin{bmatrix} 1 & 0 & 0 & -2 & 3 & 0 & 2 \\ 0 & 0 & 1 & 3 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Is this matrix in REF? Is this

in RREF? Each leading 1 is the only non-zero entry in its column.

- 1 All the 00000...0 rows are at the bottom;
- 2 The leading entries/pivots go ; and
- 3 All the entries below a leading entry are zero.
- 4 All the leading entries/pivots are 1

## Example

Consider

$$\begin{bmatrix} 1 & 0 & 0 & -2 & 3 & 0 & 2 \\ 0 & 0 & 1 & 3 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Is this matrix in REF? Is this

in RREF?

- 1 All the 00000...0 rows are at the bottom; ✓
- 2 The leading entries/pivots go down; and
- 3 All the entries below a leading entry are zero.
- 4 All the leading entries/pivots are 1
- 5 Each leading 1 is the only non-zero entry in its column.

# Row Reduction

REF: rules 1-3

RREF: all 5 rules



## Example

Consider

$$\begin{bmatrix} 1 & 0 & 0 & -2 & 3 & 0 & 2 \\ 0 & 0 & 1 & 3 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Is this matrix in REF? Is this

- 1 All the 00000...0 rows are at the bottom; ✓
  - 2 The leading entries/pivots go down; and ✓
  - 3 All the entries below a leading entry are zero.
  - 4 All the leading entries/pivots are 1
- Is this matrix in RREF? Each leading 1 is the only non-zero entry in its column.

# Row Reduction

REF: rules 1-3

RREF: all 5 rules



## Example

Consider

$$\begin{bmatrix} 1 & 0 & 0 & -2 & 3 & 0 & 2 \\ 0 & 0 & 1 & 3 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Is this matrix in REF? Is this

- 1 All the 00000...0 rows are at the bottom; ✓
- 2 The leading entries/pivots go ↓; and ✓
- 3 All the entries below a leading entry are zero. ✓
- 4 All the leading entries/pivots are 1
- 5 Each leading 1 is the only non-zero entry in its column.

This matrix is in REF

# Row Reduction

REF: rules 1-3

RREF: all 5 rules



## Example

Consider

$$\begin{bmatrix} 1 & 0 & 0 & -2 & 3 & 0 & 2 \\ 0 & 0 & 1 & 3 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Is this matrix in REF? Is this

- 1 All the 00000...0 rows are at the bottom; ✓
- 2 The leading entries/pivots go ↓; and ✓
- 3 All the entries below a leading entry are zero. ✓
- 4 All the leading entries/pivots are 1 ✓
- 5 Each leading 1 is the only non-zero entry in its column.

This matrix is in REF

# Row Reduction

REF: rules 1-3



## Example

Consider

$$\begin{bmatrix} 1 & 0 & 0 & -2 & 3 & 0 & 2 \\ 0 & 0 & 1 & 3 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Is this matrix in REF? Is this

in RREF?

RREF: all 5 rules

- 1 All the 00000...0 rows are at the bottom; ✓
- 2 The leading entries/pivots go down; and ✓
- 3 All the entries below a leading entry are zero. ✓
- 4 All the leading entries/pivots are 1 ✓
- 5 Each leading 1 is the only non-zero entry in its column. ✓

This matrix is in REF and in RREF.



## Example

Consider

$$\begin{bmatrix} 1 & 0 & 0 & -2 & 3 & 0 & 2 \\ 0 & 0 & 1 & 3 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Is this matrix in REF? Is this matrix in RREF?

This matrix is in REF and in RREF.

In this matrix, the **pivot columns** are the **first, third, and sixth** columns and the **pivot entries** are the leading nonzero entries in the **first, second and third** rows.

**EXAMPLE 1** The following matrices are in echelon form. The leading entries ( $\blacksquare$ ) may have any nonzero value; the starred entries (\*) may have any value (including zero).

$$\left[ \begin{array}{cccc} \blacksquare & * & * & * \\ 0 & \blacksquare & * & * \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right], \quad \left[ \begin{array}{cccccccccc} 0 & \blacksquare & * & * & * & * & * & * & * & * \\ 0 & 0 & 0 & \blacksquare & * & * & * & * & * & * \\ 0 & 0 & 0 & 0 & \blacksquare & * & * & * & * & * \\ 0 & 0 & 0 & 0 & 0 & \blacksquare & * & * & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & \blacksquare & * & * & * \end{array} \right]$$

The following matrices are in reduced echelon form because the leading entries are 1's, and there are 0's below *and above* each leading 1.

$$\left[ \begin{array}{cccc} 1 & 0 & * & * \\ 0 & 1 & * & * \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right], \quad \left[ \begin{array}{cccccccccc} 0 & 1 & * & 0 & 0 & 0 & * & * & 0 & * \\ 0 & 0 & 0 & 1 & 0 & 0 & * & * & 0 & * \\ 0 & 0 & 0 & 0 & 1 & 0 & * & * & 0 & * \\ 0 & 0 & 0 & 0 & 0 & 1 & * & * & 0 & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & * \end{array} \right]$$



# Row Reduction and Echelon Forms



## Definition

A pivot position in a matrix  $A$  is a location in  $A$  that corresponds to a leading 1 in the reduced row echelon form of  $A$ .

## Definition

A pivot column is a column of  $A$  that contains a pivot position.

# Row Reduction and Echelon Forms



## Definition

A pivot position in a matrix  $A$  is a location in  $A$  that corresponds to a leading 1 in the reduced row echelon form of  $A$ .

## Definition

A pivot column is a column of  $A$  that contains a pivot position.

## Theorem (Uniqueness of the row reduced echelon form)

*Each matrix is row equivalent to one and only one RREF matrix.*

## Gaussian Elimination

### Example

Row reduce the matrix below to row echelon form (REF) and then to reduced row echelon form (RREF):

$$\begin{bmatrix} 0 & -3 & -6 & 4 & 9 \\ -1 & -2 & -1 & 3 & 1 \\ -2 & -3 & 0 & 3 & -1 \\ 1 & 4 & 5 & -9 & -7 \end{bmatrix}.$$

## Gaussian Elimination

### Example

Row reduce the matrix below to row echelon form (REF) and then to reduced row echelon form (RREF):

$$\begin{bmatrix} 0 & -3 & -6 & 4 & 9 \\ -1 & -2 & -1 & 3 & 1 \\ -2 & -3 & 0 & 3 & -1 \\ 1 & 4 & 5 & -9 & -7 \end{bmatrix}.$$

So how do we do this? How do we know which elementary row operation to use at each step? We need an algorithm.

# Row Reduction and Echelon Forms



- **Step 1:** Find the first non-zero column (from the left).

# Row Reduction and Echelon Forms



- **Step 1:** Find the first non-zero column (from the left).
- **Step 2:** We don't want a 0 at the top of this column. If necessary swap the first row with another row.

# Row Reduction and Echelon Forms



- **Step 1:** Find the first non-zero column (from the left).
- **Step 2:** We don't want a 0 at the top of this column. If necessary swap the first row with another row.
- **Step 3:** Create zeros under the **pivot**.

- **Step 1:** Find the first non-zero column (from the left).
- **Step 2:** We don't want a 0 at the top of this column. If necessary swap the first row with another row.
- **Step 3:** Create zeros under the **pivot**.
- **Step 4:** Ignore the row and column containing the pivot.  
Goto step 1.

- **Step 1:** Find the first non-zero column (from the left).
- **Step 2:** We don't want a 0 at the top of this column. If necessary swap the first row with another row.
- **Step 3:** Create zeros under the **pivot**.
- **Step 4:** Ignore the row and column containing the pivot.  
Goto step 1.

Keep going until your matrix is in row echelon form (REF). If you want reduced REF, then continue to step 5.

- **Step 1:** Find the first non-zero column (from the left).
- **Step 2:** We don't want a 0 at the top of this column. If necessary swap the first row with another row.
- **Step 3:** Create zeros under the **pivot**.
- **Step 4:** Ignore the row and column containing the pivot.  
Goto step 1.

Keep going until your matrix is in row echelon form (REF). If you want reduced REF, then continue to step 5.

- **Step 5:** Make every pivot a **1** by multiplying rows by constants.

# Row Reduction and Echelon Forms



- **Step 1:** Find the first non-zero column (from the left).
- **Step 2:** We don't want a 0 at the top of this column. If necessary swap the first row with another row.
- **Step 3:** Create zeros under the pivot.
- **Step 4:** Ignore the row and column containing the pivot.  
Goto step 1.

Keep going until your matrix is in row echelon form (REF). If you want reduced REF, then continue to step 5.

- **Step 5:** Make every pivot a 1 by multiplying rows by constants.
- **Step 6:** Starting with the bottom-rightmost pivot and working upwards, create zeros above each pivot.

# Row Reduction and Echelon Forms



- **Step 1:** Find the first non-zero column (from the left).
- **Step 2:** We don't want a 0 at the top of this column. If necessary swap the first row with another row.
- **Step 3:** Create zeros under the pivot.
- **Step 4:** Ignore the row and column containing the pivot.  
Goto step 1.

Keep going until your matrix is in row echelon form (REF). If you want reduced REF, then continue to step 5.

- **Step 5:** Make every pivot a 1 by multiplying rows by constants.
- **Step 6:** Starting with the bottom-rightmost pivot and working upwards, create zeros above each pivot.

Your matrix should now be in RREF.

# Row Reduction and Echelon Forms



If we want REF we do steps 1-4. This is called *Gaussian Elimination*.



**Carl Friedrich Gauss**

BORN

30 April 1777

DECEASED

23 February 1855

NATIONALITY

German

# Row Reduction and Echelon Forms



If we want REF we do steps 1-4. This is called *Gaussian Elimination*.

If we want RREF, we do steps 1-6. This is called *Gauss-Jordan Elimination*.



A black and white portrait of Wilhelm Jordan, a man with a full, dark beard and mustache, wearing glasses and a suit.

**Wilhelm Jordan**

**BORN**  
1 March 1842

**DECEASED**  
17 April 1899

**NATIONALITY**  
German

Another name is *row reduction*.

# Row Reduction and Echelon Forms



Now let's do this:

## Example

Row reduce the matrix below to row echelon form (REF) and then to reduced row echelon form (RREF):

$$\begin{bmatrix} 0 & -3 & -6 & 4 & 9 \\ -1 & -2 & -1 & 3 & 1 \\ -2 & -3 & 0 & 3 & -1 \\ 1 & 4 & 5 & -9 & -7 \end{bmatrix}.$$

# Row Reduction and Echelon Forms



- **Step 1:** Find the first non-zero column (from the left).

$$\begin{bmatrix} 0 & -3 & -6 & 4 & 9 \\ -1 & -2 & -1 & 3 & 1 \\ -2 & -3 & 0 & 3 & -1 \\ 1 & 4 & 5 & -9 & -7 \end{bmatrix}.$$



This column is not all zeros.

# Row Reduction and Echelon Forms



■ **Step 2:** We don't want a 0 at the top of this column. If necessary swap the first row with another row.

I don't want 0 here.

$$\left[ \begin{array}{ccccc} 0 & -3 & -6 & 4 & 9 \\ -1 & -2 & -1 & 3 & 1 \\ -2 & -3 & 0 & 3 & -1 \\ 1 & 4 & 5 & -9 & -7 \end{array} \right]$$

# Row Reduction and Echelon Forms



■ **Step 2:** We don't want a 0 at the top of this column. If necessary swap the first row with another row.

I don't want 0 here.

$$\left[ \begin{array}{ccccc} 0 & -3 & -6 & 4 & 9 \\ -1 & -2 & -1 & 3 & 1 \\ -2 & -3 & 0 & 3 & -1 \\ 1 & 4 & 5 & -9 & -7 \end{array} \right]$$

$R_1 \leftrightarrow R_4$

$$\left[ \begin{array}{ccccc} 1 & 4 & 5 & -9 & -7 \\ -1 & -2 & -1 & 3 & 1 \\ -2 & -3 & 0 & 3 & -1 \\ 0 & -3 & -6 & 4 & 9 \end{array} \right]$$

# Row Reduction and Echelon Forms



- **Step 2:** We don't want a 0 at the top of this column. If necessary swap the first row with another row.

I don't want 0 here.

$$\left[ \begin{array}{ccccc} 0 & -3 & -6 & 4 & 9 \\ -1 & -2 & -1 & 3 & 1 \\ -2 & -3 & 0 & 3 & -1 \\ 1 & 4 & 5 & -9 & -7 \end{array} \right]$$

$R_1 \leftrightarrow R_4$

$$\left[ \begin{array}{ccccc} 1 & 4 & 5 & -9 & -7 \\ -1 & -2 & -1 & 3 & 1 \\ -2 & -3 & 0 & 3 & -1 \\ 0 & -3 & -6 & 4 & 9 \end{array} \right]$$

pivot/leading entry

# Row Reduction and Echelon Forms



- **Step 3:** Create zeros under the pivot.

I want 0 here.

$$\left[ \begin{array}{ccccc} 1 & 4 & 5 & -9 & -7 \\ -1 & -2 & -1 & 3 & 1 \\ -2 & -3 & 0 & 3 & -1 \\ 0 & -3 & -6 & 4 & 9 \end{array} \right]$$

The matrix is shown with a green '1' at the top-left. An orange arrow points from the text 'I want 0 here.' to the second row, second column entry '-1'. A red oval encircles the second row.

# Row Reduction and Echelon Forms



■ **Step 3:** Create zeros under the pivot.

I want 0 here.

$$\left[ \begin{array}{ccccc} 1 & 4 & 5 & -9 & -7 \\ -1 & -2 & -1 & 3 & 1 \\ -2 & -3 & 0 & 3 & -1 \\ 0 & -3 & -6 & 4 & 9 \end{array} \right]$$

$$R_1 + R_2 \rightarrow R_2$$
$$\left[ \begin{array}{ccccc} 1 & 4 & 5 & -9 & -7 \\ 0 & 2 & 4 & -6 & -6 \\ -2 & -3 & 0 & 3 & -1 \\ 0 & -3 & -6 & 4 & 9 \end{array} \right]$$

# Row Reduction and Echelon Forms



$$\begin{bmatrix} 1 & 4 & 5 & -9 & -7 \\ 0 & 2 & 4 & -6 & -6 \\ -2 & -3 & 0 & 3 & -1 \\ 0 & -3 & -6 & 4 & 9 \end{bmatrix}$$

$$2R_1 + R_3 \rightarrow R_3$$

$$\begin{bmatrix} 1 & 4 & 5 & -9 & -7 \\ 0 & 2 & 4 & -6 & -6 \\ \textcolor{green}{0} & \textcolor{green}{5} & \textcolor{green}{10} & \textcolor{green}{-15} & \textcolor{green}{-15} \\ 0 & -3 & -6 & 4 & 9 \end{bmatrix}$$

# Row Reduction and Echelon Forms



- **Step 4:** Ignore the row and column containing the pivot.  
Goto step 1.

$$\left[ \begin{array}{ccccc} 1 & 4 & 5 & -9 & -7 \\ 0 & 2 & 4 & -6 & -6 \\ 0 & 5 & 10 & -15 & -15 \\ 0 & -3 & -6 & 4 & 9 \end{array} \right]$$

# Row Reduction and Echelon Forms



- **Step 4:** Ignore the row and column containing the pivot.  
Goto step 1.

$$\left[ \begin{array}{ccccc} 1 & 4 & 5 & -9 & -7 \\ 0 & 2 & 4 & -6 & -6 \\ 0 & 5 & 10 & -15 & -15 \\ 0 & -3 & -6 & 4 & 9 \end{array} \right] \xrightarrow{\text{ignore}} \quad \text{ignore}$$

# Row Reduction and Echelon Forms



- **Step 4:** Ignore the row and column containing the pivot.  
Goto step 1.

$$\left[ \begin{array}{ccccc} 1 & 4 & 5 & -9 & -7 \\ 0 & 2 & 4 & -6 & -6 \\ 0 & 5 & 10 & -15 & -15 \\ 0 & -3 & -6 & 4 & 9 \end{array} \right]$$

ignore

ignore

A 4x5 matrix is shown. The first row has a green '1' at the top-left. A blue arrow points from the text 'ignore' to the second row. Another blue arrow points from the text 'ignore' to the third row.

# Row Reduction and Echelon Forms



- **Step 4:** Ignore the row and column containing the pivot.  
Goto step 1.

$$\left[ \begin{array}{ccccc} 1 & 4 & 5 & -9 & -7 \\ 0 & 2 & 4 & -6 & -6 \\ 0 & 5 & 10 & -15 & -15 \\ 0 & -3 & -6 & 4 & 9 \end{array} \right]$$

ignore

ignore

$$\left[ \begin{array}{ccccc} 1 & 4 & 5 & -9 & -7 \\ 0 & 2 & 4 & -6 & -6 \\ 0 & 5 & 10 & -15 & -15 \\ 0 & -3 & -6 & 4 & 9 \end{array} \right]$$

# Row Reduction and Echelon Forms



$$\begin{bmatrix} 1 & 4 & 5 & -9 & -7 \\ 0 & \textcolor{green}{2} & 4 & -6 & -6 \\ 0 & 5 & 10 & -15 & -15 \\ 0 & -3 & -6 & 4 & 9 \end{bmatrix}$$



This column is not all zeros.

# Row Reduction and Echelon Forms



I don't want 0 here.

$$\left[ \begin{array}{ccccc} 1 & 4 & 5 & -9 & -7 \\ 0 & \textcolor{red}{2} & 4 & -6 & -6 \\ 0 & 5 & 10 & -15 & -15 \\ 0 & -3 & -6 & 4 & 9 \end{array} \right]$$

# Row Reduction and Echelon Forms



I want  $\begin{matrix} 0 \\ 0 \end{matrix}$  here.

$$\left[ \begin{array}{ccccc} 1 & 4 & 5 & -9 & -7 \\ 0 & 2 & 4 & -6 & -6 \\ 0 & 5 & 10 & -15 & -15 \\ 0 & -3 & -6 & 4 & 9 \end{array} \right]$$

# Row Reduction and Echelon Forms



I want  $\begin{matrix} 0 \\ 0 \end{matrix}$  here.

$$\left[ \begin{array}{ccccc} 1 & 4 & 5 & -9 & -7 \\ 0 & 2 & 4 & -6 & -6 \\ 0 & 5 & 10 & -15 & -15 \\ 0 & -3 & -6 & 4 & 9 \end{array} \right]$$

$$-\frac{5}{2}R_2 + R_3 \rightarrow R_3$$
$$\left[ \begin{array}{ccccc} 1 & 4 & 5 & -9 & -7 \\ 0 & 2 & 4 & -6 & -6 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & -3 & -6 & 4 & 9 \end{array} \right]$$

# Row Reduction and Echelon Forms



$$\begin{bmatrix} 1 & 4 & 5 & -9 & -7 \\ 0 & 2 & 4 & -6 & -6 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & -3 & -6 & 4 & 9 \end{bmatrix}$$

$$\frac{3}{2}R_2 + R_4 \rightarrow R_4$$

$$\begin{bmatrix} 1 & 4 & 5 & -9 & -7 \\ 0 & 2 & 4 & -6 & -6 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -5 & 0 \end{bmatrix}$$

# Row Reduction and Echelon Forms



$$\begin{bmatrix} 1 & 4 & 5 & -9 & -7 \\ 0 & 2 & 4 & -6 & -6 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -5 & 0 \end{bmatrix}$$

# Row Reduction and Echelon Forms



$$\begin{bmatrix} 1 & 4 & 5 & -9 & -7 \\ 0 & 2 & 4 & -6 & -6 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -5 & 0 \end{bmatrix}$$

Ignore  $R_2$  and column 2. Find the first non-zero column.

$$\begin{bmatrix} 1 & 4 & 5 & -9 & -7 \\ 0 & 2 & 4 & -6 & -6 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -5 & 0 \end{bmatrix}$$

# Row Reduction and Echelon Forms



$$\begin{bmatrix} 1 & 4 & 5 & -9 & -7 \\ 0 & 2 & 4 & -6 & -6 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -5 & 0 \end{bmatrix}$$

Ignore  $R_2$  and column 2. Find the first non-zero column.

$$\begin{bmatrix} 1 & 4 & 5 & -9 & -7 \\ 0 & 2 & 4 & -6 & -6 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -5 & 0 \end{bmatrix}$$



This column is all zeros.

# Row Reduction and Echelon Forms



$$\begin{bmatrix} 1 & 4 & 5 & -9 & -7 \\ 0 & 2 & 4 & -6 & -6 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -5 & 0 \end{bmatrix}$$

Ignore  $R_2$  and column 2. Find the first non-zero column.

$$\begin{bmatrix} 1 & 4 & 5 & -9 & -7 \\ 0 & 2 & 4 & -6 & -6 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -5 & 0 \end{bmatrix}$$



This column is not all zeros.

# Row Reduction and Echelon Forms



$$\begin{bmatrix} 1 & 4 & 5 & -9 & -7 \\ 0 & 2 & 4 & -6 & -6 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -5 & 0 \end{bmatrix}$$

Ignore  $R_2$  and column 2. Find the first non-zero column.

I don't want 0 here.


$$\begin{bmatrix} 1 & 4 & 5 & -9 & -7 \\ 0 & 2 & 4 & -6 & -6 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -5 & 0 \end{bmatrix}$$

# Row Reduction and Echelon Forms



$$\begin{bmatrix} 1 & 4 & 5 & -9 & -7 \\ 0 & 2 & 4 & -6 & -6 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -5 & 0 \end{bmatrix}$$

$R_3 \leftrightarrow R_4$

$$\begin{bmatrix} 1 & 4 & 5 & -9 & -7 \\ 0 & 2 & 4 & -6 & -6 \\ 0 & 0 & 0 & -5 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

---

<sup>1</sup>row echelon form

# Row Reduction and Echelon Forms



$$\begin{bmatrix} 1 & 4 & 5 & -9 & -7 \\ 0 & 2 & 4 & -6 & -6 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -5 & 0 \end{bmatrix}$$

$R_3 \leftrightarrow R_4$

$$\begin{bmatrix} 1 & 4 & 5 & -9 & -7 \\ 0 & 2 & 4 & -6 & -6 \\ 0 & 0 & 0 & -5 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

The matrix is now in REF<sup>1</sup>.

---

<sup>1</sup>row echelon form

# Row Reduction and Echelon Forms



- **Step 5:** Make every pivot a 1 by multiplying rows by constants.

$$\begin{bmatrix} 1 & 4 & 5 & -9 & -7 \\ 0 & 2 & 4 & -6 & -6 \\ 0 & 0 & 0 & -5 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

# Row Reduction and Echelon Forms



- **Step 5:** Make every pivot a 1 by multiplying rows by constants.

$$\left[ \begin{array}{ccccc} 1 & 4 & 5 & -9 & -7 \\ 0 & 2 & 4 & -6 & -6 \\ 0 & 0 & 0 & -5 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array} \right]$$

A green curved arrow labeled "pivot" points to the number 1 in the first row, first column. Another green curved arrow points to the number 2 in the second row, first column. A third green curved arrow points to the number -5 in the third row, fourth column.

# Row Reduction and Echelon Forms



- **Step 5:** Make every pivot a 1 by multiplying rows by constants.

$$\left[ \begin{array}{ccccc} 1 & 4 & 5 & -9 & -7 \\ 0 & 2 & 4 & -6 & -6 \\ 0 & 0 & 0 & -5 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array} \right]$$

pivot columns

Three orange curved arrows originate from the right side of the matrix and point towards the first three columns. The first arrow points to the first column, the second to the second, and the third to the third. To the right of the matrix, the text "pivot columns" is written in orange, with a small orange arrow pointing towards the arrows above.

# Row Reduction and Echelon Forms



- **Step 5:** Make every pivot a 1 by multiplying rows by constants.

$$\begin{bmatrix} 1 & 4 & 5 & -9 & -7 \\ 0 & 2 & 4 & -6 & -6 \\ 0 & 0 & 0 & -5 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{aligned}\frac{1}{2}R_2 &\rightarrow R_2 \\ -\frac{1}{5}R_3 &\rightarrow R_3\end{aligned}$$

$$\begin{bmatrix} 1 & 4 & 5 & -9 & -7 \\ 0 & 1 & 2 & -3 & -3 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

# Row Reduction and Echelon Forms



- **Step 5:** Make every pivot a 1 by multiplying rows by constants.

$$\left[ \begin{array}{ccccc} 1 & 4 & 5 & -9 & -7 \\ 0 & 2 & 4 & -6 & -6 \\ 0 & 0 & 0 & 5 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array} \right]$$

**WARNING**

Be very careful if you do 2 row operations in the same step.

$$\left[ \begin{array}{ccccc} 1 & 4 & 5 & -9 & -7 \\ 0 & 1 & 2 & -3 & -3 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array} \right]$$

Do not change the same row twice in one step.

$$\begin{aligned} \frac{1}{2}R_2 &\rightarrow R_2 \\ -\frac{1}{5}R_3 &\rightarrow R_3 \end{aligned}$$

# Row Reduction and Echelon Forms



- **Step 6:** Starting with the bottom-rightmost pivot and working upwards, create zeros above each pivot.

$$\begin{bmatrix} 1 & 4 & 5 & -9 & -7 \\ 0 & 1 & 2 & -3 & -3 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

# Row Reduction and Echelon Forms



- **Step 6:** Starting with the bottom-rightmost pivot and working upwards, create zeros above each pivot.

first make this 0

$$\left[ \begin{array}{ccccc} 1 & 4 & 5 & -9 & -7 \\ 0 & 1 & 2 & -3 & -3 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array} \right]$$

# Row Reduction and Echelon Forms



- **Step 6:** Starting with the bottom-rightmost pivot and working upwards, create zeros above each pivot.

then make this 0

$$\left[ \begin{array}{ccccc} 1 & 4 & 5 & -9 & -7 \\ 0 & 1 & 2 & -3 & -3 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array} \right]$$

A red curved arrow points from the text "then make this 0" down to the number -9 in the fourth column of the matrix.

# Row Reduction and Echelon Forms



- **Step 6:** Starting with the bottom-rightmost pivot and working upwards, create zeros above each pivot.

finally make this 0

$$\left[ \begin{array}{ccccc} 1 & 4 & 5 & -9 & -7 \\ 0 & 1 & 2 & -3 & -3 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array} \right]$$


# Row Reduction and Echelon Forms



- **Step 6:** Starting with the bottom-rightmost pivot and working upwards, create zeros above each pivot.

$$\begin{bmatrix} 1 & 4 & 5 & -9 & -7 \\ 0 & 1 & 2 & -3 & -3 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$3R_3 + R_2 \rightarrow R_2$$

$$\begin{bmatrix} 1 & 4 & 5 & -9 & -7 \\ 0 & 1 & 2 & 0 & -3 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

# Row Reduction and Echelon Forms



$$\begin{bmatrix} 1 & 4 & 5 & -9 & -7 \\ 0 & 1 & 2 & 0 & -3 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$9R_3 + R_1 \rightarrow R_1$$

$$\begin{bmatrix} 1 & 4 & 5 & 0 & -7 \\ 0 & 1 & 2 & 0 & -3 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

# Row Reduction and Echelon Forms



$$\begin{bmatrix} 1 & 4 & 5 & -9 & -7 \\ 0 & 1 & 2 & 0 & -3 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$9R_3 + R_1 \rightarrow R_1$$

$$\begin{bmatrix} 1 & 4 & 5 & 0 & -7 \\ 0 & 1 & 2 & 0 & -3 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$-4R_2 + R_1 \rightarrow R_1$$

$$\begin{bmatrix} 1 & 0 & -3 & 0 & 5 \\ 0 & 1 & 2 & 0 & -3 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

# Row Reduction and Echelon Forms



$$\begin{bmatrix} 1 & 0 & -3 & 0 & 5 \\ 0 & 1 & 2 & 0 & -3 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

The matrix is now in RREF<sup>2</sup>.

---

<sup>2</sup>reduced row echelon form

# Row Reduction and Echelon Forms



original matrix

$$\begin{bmatrix} 0 & -3 & -6 & 4 & 9 \\ -1 & -2 & -1 & 3 & 1 \\ -2 & -3 & 0 & 3 & -1 \\ 1 & 4 & 5 & -9 & -7 \end{bmatrix} \xrightarrow{\substack{\text{row} \\ \text{operations}}}$$

RREF matrix

$$\begin{bmatrix} 1 & 0 & -3 & 0 & 5 \\ 0 & 1 & 2 & 0 & -3 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

# Row Reduction and Echelon Forms



$$\begin{bmatrix} 0 & -3 & -6 & 4 & 9 \\ -1 & -2 & -1 & 3 & 1 \\ -2 & -3 & 0 & 3 & -1 \\ 1 & 4 & 5 & -9 & -7 \end{bmatrix} \xrightarrow[\text{operations}]{\text{row}} \begin{bmatrix} 1 & 0 & -3 & 0 & 5 \\ 0 & 1 & 2 & 0 & -3 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

pivots

The diagram illustrates the row reduction process. A green arrow points from the first row of the original matrix to the first row of the reduced matrix, indicating the identity operation. Another green arrow points from the second row of the original matrix to the second row of the reduced matrix, also indicating the identity operation. A third green arrow points from the third row of the original matrix to the third row of the reduced matrix, indicating the third row was multiplied by -1. A fourth green arrow points from the fourth row of the original matrix to the fourth row of the reduced matrix, indicating the fourth row was multiplied by -1. The fifth green arrow points from the first column of the original matrix to the first column of the reduced matrix, indicating the first column was multiplied by -1. The text "row operations" is positioned between the two matrices.

# Row Reduction and Echelon Forms

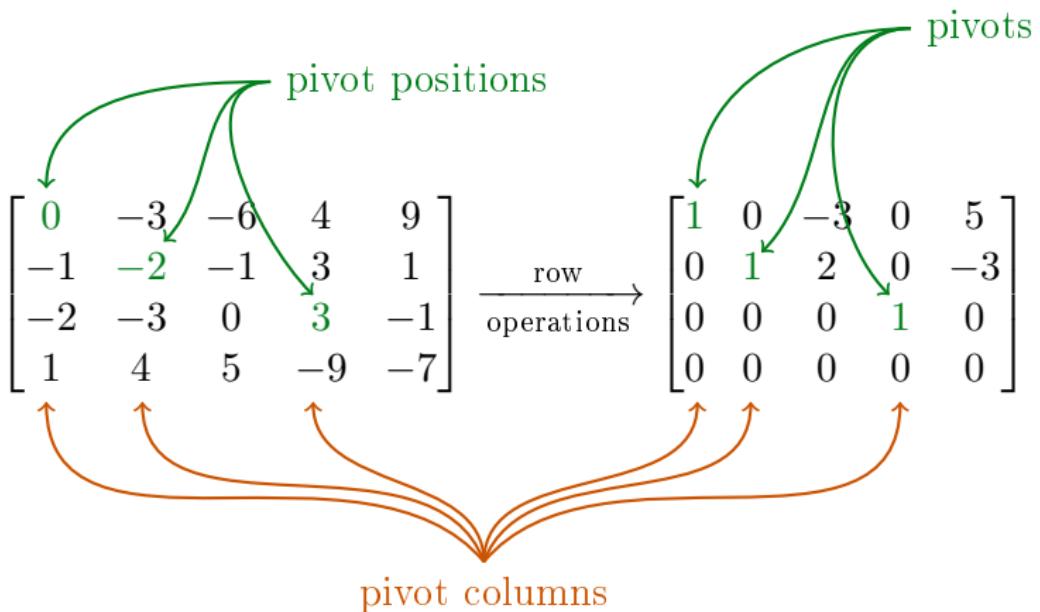


pivot positions

pivots

$$\begin{bmatrix} 0 & -3 & -6 & 4 & 9 \\ -1 & -2 & -1 & 3 & 1 \\ -2 & -3 & 0 & 3 & -1 \\ 1 & 4 & 5 & -9 & -7 \end{bmatrix} \xrightarrow[\text{operations}]{\text{row}} \begin{bmatrix} 1 & 0 & -3 & 0 & 5 \\ 0 & 1 & 2 & 0 & -3 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

# Row Reduction and Echelon Forms





# Next Time

- Solutions of Linear Systems
- Introduction to Matrices
- The Inverse of a Matrix

This is the end of the lecture. Please ask your questions now.  
You are welcome to leave whenever you wish.