Week 15

- 33. The Fundamental Theorem of Calculus
- 34. The Substitution Method
- 35. Area Between Curves

The Fundamental Theorem of Calculus

We don't want to have to use Riemann sums every time we need to calculate a definite integral – we want a better way.

The following theorem is the most important theorem in Calculus. If you can only memorise one theorem for the exams, it should be this one.

Theorem (The Fundamental Theorem of Calculus)

Suppose that $f : [a, b] \to \mathbb{R}$ *is a continuous function.*

32. The Fundamental Theorem of Calculus

1. Then the function $F : [a, b] \to \mathbb{R}$ defined by

$$F(x) = \int_a^x f(t)\ dt$$

is continuous on $[a, b]$; differentiable on (a, b); and

2. If F is any antiderivative of f on $[a, b]$, then

$$\boxed{\int_a^b f(x)\,dx = F(b) - F(a).}$$

Remark

Part (i) of the theorem tells how to differentiate $\int_a^x f(t)\ dt$.

Example

Find $\frac{dy}{dx}$ if $y = \int_a^x (t^3+1)\ dt$.

solution:

$$\frac{dy}{dx} = \frac{d}{dx}\int_a^x (t^3+1)\ dt = x^3+1.$$

Example

Find $\frac{dy}{dx}$ if $y = \int_1^x \sin t\ dt$.

solution:

$$\frac{dy}{dx} = \frac{d}{dx}\int_1^x \sin t\ dt = \sin x.$$

Example

Find $\frac{dy}{dx}$ if $y = \int_0^x \sin \ln \tan e^{t^2}\ dt$.

solution:

$$\frac{dy}{dx} = \frac{d}{dx} \int_0^x \sin \ln \tan e^{t^2}\ dt = \sin \ln \tan e^{x^2}.$$

Example

Find $\frac{dy}{dx}$ if $y = \int_x^5 3t \sin t \ dt$.

solution:

$$\begin{aligned} \frac{dy}{dx} &= \frac{d}{dx} \int_x^5 3t \sin t \ dt \\ &= \frac{d}{dx} \left(- \int_5^x 3t \sin t \ dt \right) \\ &= -3x \sin x. \end{aligned}$$

Example

Find $\frac{dy}{dx}$ if $y = \int_1^{x^2} \cos t \; dt$.

solution: This time we will need to use the Chain rule. Let $u = x^2$. Then

$$\begin{aligned} \frac{dy}{dx} &= \frac{dy}{du}\frac{du}{dx} \\ &= \left(\frac{d}{du} \int_1^u \cos t \; dt \right) \left(\frac{d}{dx} x^2 \right) \\ &= (\cos u)\,(2x) = 2x \cos x^2. \end{aligned}$$

Remark

Part (ii) of the theorem tells us how to calculate the definite integral of f over $[a, b]$:

1. Find an antiderivative F of f.
2. Calculate $F(b) - F(a)$.

Notation

We will write

$$\Big[F(x)\Big]_a^b = F(b) - F(a).$$

Example

$$
\begin{aligned}
\int_0^\pi \cos x \; dx &= \Big[\sin x \Big]_0^\pi \\
&\qquad \left(\text{because } \tfrac{d}{dx} \sin x = \cos x\right) \\
&= \sin \pi - \sin 0 \\
&= 0 - 0 \\
&= 0
\end{aligned}
$$

32. The Fundamental Theorem of Calculus

Example

$$\int_{-\frac{\pi}{4}}^{0} \sec x \tan x = \Big[\sec x\Big]_{-\frac{\pi}{4}}^{0}$$

$$\text{(because } \tfrac{d}{dx} \sec x = \sec x \tan x\text{)}$$

$$= \sec 0 - \sec -\frac{\pi}{4}$$

$$= 1 - \sqrt{2}.$$

Example

$$\begin{aligned}
\int_1^4 \left(\frac{3}{2}\sqrt{x} - \frac{4}{x^2}\right)\ dx &= \left[x^{\frac{3}{2}} + \frac{4}{x}\right]_1^4 \\
&\qquad \left(\text{because } \tfrac{d}{dx}\left(x^{\frac{3}{2}} + \frac{4}{x}\right) = \frac{3}{2}\sqrt{x} - \frac{4}{x^2}\right) \\
&= \left(4^{\frac{3}{2}} + \frac{4}{4}\right) - \left(1^{\frac{3}{2}} + \frac{4}{1}\right) \\
&= (8+1) - (1+4) \\
&= 4.
\end{aligned}$$

Total Area

Example

Let $f(x) = x^2 - 4$. We have that

$$\int_{-2}^{2} f(x)\ dx = \int_{-2}^{2} (x^2 - 4)\ dx = \left[\frac{x^3}{3} - 4x\right]_{-2}^{2}$$
$$= \left(\frac{8}{3} - 8\right) - \left(\frac{-8}{3} + 8\right) = -\frac{32}{3}.$$

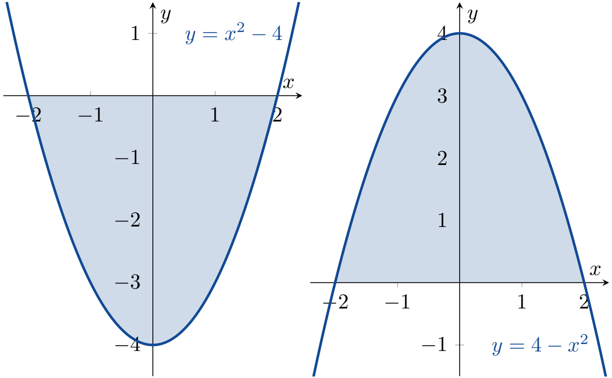
The total area between the graph of $y = f(x)$ and the x-axis, over $[-2, 2]$, is $\left|-\frac{32}{3}\right| = \frac{32}{3}$.

Example

Let $g(x) = 4 - x^2$. We have that

$$\int_{-2}^{2} g(x)\; dx = \int_{-2}^{2} (4 - x^2)\; dx = \left[4x - \frac{x^3}{3} \right]_{-2}^{2}$$
$$= \left(8 - \frac{8}{3} \right) - \left(8 + \frac{-8}{3} \right) = \frac{32}{3}.$$

The total area between the graph of $y = g(x)$ and the x-axis, over $[-2, 2]$, is $\left| \frac{32}{3} \right| = \frac{32}{3}$.



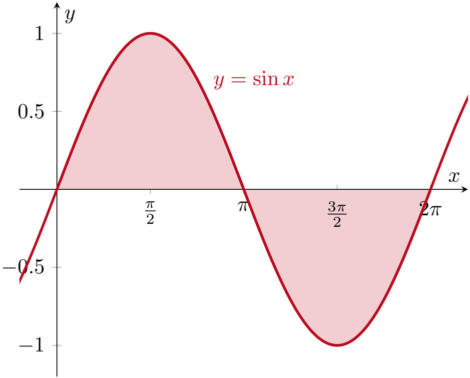
integral $= -\frac{32}{3}$
total area $= \frac{32}{3}$

integral $= \frac{32}{3}$
total area $= \frac{32}{3}$

Example

Let $f(x) = \sin x$. Calculate

1. the definite integral of f over $[0, 2\pi]$; and
2. the total area between the graph of $y = f(x)$ and the x-axis over $[0, 2\pi]$.



y
1
0.5
-0.5
-1
$y = \sin x$
x
$\frac{\pi}{2}$
π
$\frac{3\pi}{2}$
2π

solution:

1

$$\int_0^{2\pi} \sin x \; dx = \Big[-\cos x \Big]_0^{2\pi} = -\cos 2\pi + \cos 0$$
$$= -1 + 1 = 0.$$

2

$$\begin{aligned} \text{total area} &= \int_0^{\pi} \sin x \; dx + \left| \int_{\pi}^{2\pi} \sin x \; dx \right| \\ &= \Big[-\cos x \Big]_0^{\pi} + \left| \Big[-\cos x \Big]_{\pi}^{2\pi} \right| \\ &= -\cos \pi + \cos 0 + \left| -\cos 2\pi + \cos \pi \right| \\ &= -(-1) + 1 + \left| -1 + (-1) \right| = 4. \end{aligned}$$

Summary

To find the *total area* between the graph of $y = f(x)$ and the x-axis over $[a, b]$:

1. Divide $[a, b]$ at the zeroes of f.
2. Integrate f over each subinterval.
3. Add the absolute values of the integrals.

Example

Find the total area between the graph of $y = x^3 - x^2 - 2x$ and the x-axis for $-1 \leq x \leq 2$.

solution:

1. Let $f(x) = x^3 - x^2 - 2x$.

   Since $0 = f(x) = x^3 - x^2 - 2x = x(x+1)(x-2)$ implies that $x = 0$ or $x = -1$ or $x = 2$, we divide $[-1, 2]$ into $[-1, 0]$ and $[0, 2]$.

2. We calculate that

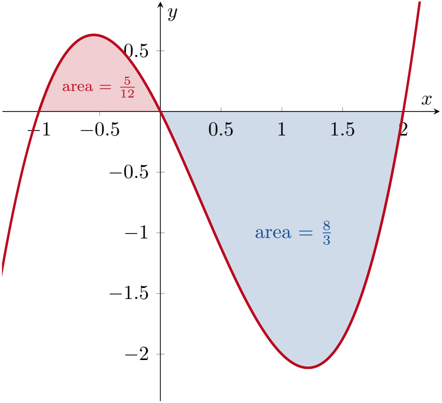
$$\begin{aligned}\int_{-1}^{0}(x^3 - x^2 - 2x)\ dx &= \left[\frac{x^4}{4} - \frac{x^3}{3} - x^2\right]_{-1}^{0} \\ &= (0 - 0 - 0) - \left(\frac{1}{4} + \frac{1}{3} - 1\right) \\ &= \frac{5}{12}\end{aligned}$$

and

$$\begin{aligned}\int_0^2 (x^3 - x^2 - 2x)\ dx &= \left[\frac{x^4}{4} - \frac{x^3}{3} - x^2\right]_0^2 \\ &= \left(\frac{16}{4} - \frac{8}{3} - 4\right) - (0 - 0 - 0) \\ &= -\frac{8}{3}.\end{aligned}$$

3. Therefore

$$\text{total area} = \left|\frac{5}{12}\right| + \left|-\frac{8}{3}\right| = \frac{37}{12}.$$



The Average Value of a Continuous Function

The average of $\{1, 2, 2, 6, 9\}$ is $\frac{1+2+2+6+9}{5} = \frac{20}{5} = 4$. We can also calculate the average value of a continuous function.

Definition

If f is integrable on $[a,b]$, then the *average value of* f *on* $[a,b]$ is

$$\boxed{\operatorname{av}(f) = \frac{1}{b-a}\int_a^b f(x)\,dx.}$$

Example

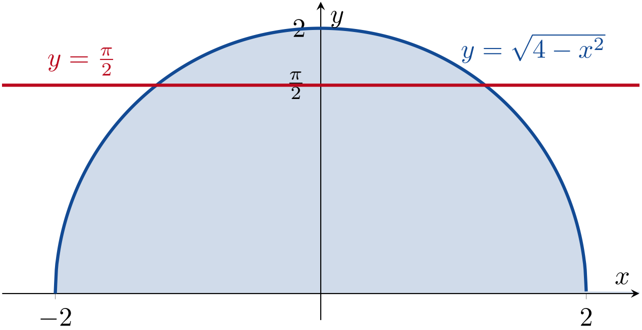
Find the average value of $f(x) = \sqrt{4 - x^2}$ on $[-2, 2]$.

solution: Since

$$\begin{aligned}\int_{-2}^{2} f(x)\ dx &= \frac{1}{2} \times \text{the area of a circle of radius } 2 \\ &= \frac{1}{2}\pi 2^2 = 2\pi,\end{aligned}$$

we have that

$$\mathrm{av}(f) = \frac{1}{2 - (-2)} \int_{-2}^{2} f(x)\ dx = \frac{2\pi}{4} = \frac{\pi}{2}.$$



y
2
$y = \sqrt{4 - x^2}$
$y = \frac{\pi}{2}$
$\frac{\pi}{2}$
x
-2
2

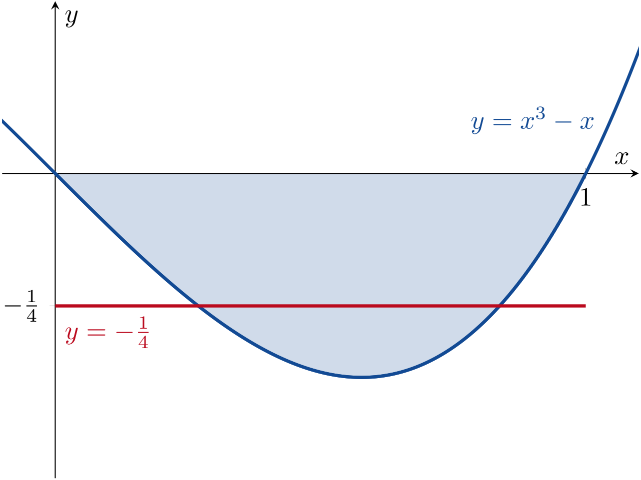
Example

Find the average value of $g(x) = x^3 - x$ on $[0, 1]$.

solution:

$$\begin{aligned}\operatorname{av}(g) &= \frac{1}{1-0}\int_0^1 g(x)\ dx = \int_0^1 (x^3 - x)\ dx \\ &= \left[\frac{x^4}{4} - \frac{x^2}{2}\right]_0^1 = \frac{1}{4} - \frac{1}{2} = -\frac{1}{4}.\end{aligned}$$

32. The Fundamental Theorem of Calculus



Indefinite Integrals & Definite Integrals

Remember that

$$\int f(x)\ dx \text{ is a function.}$$

For example

$$\int x\ dx = \frac{x^2}{2} + C$$

and

$$\int \cos x\ dx = \sin x + C.$$

Remember that

$$\int_a^b f(x)\ dx \text{ is a number.}$$

For example

$$\int_0^1 x\ dx = \frac{1}{2}$$

and

$$\int_0^{\frac{\pi}{2}} \cos x\ dx = 1.$$

The Substitution Method

By the Chain rule,

$$\frac{d}{dx}\left(\frac{u^{n+1}}{n+1}\right) = u^n \frac{du}{dx}.$$

So

$$\int u^n \frac{du}{dx}\ dx = \frac{u^{n+1}}{n+1} + C.$$

But we know that

$$\int u^n\ du = \frac{u^{n+1}}{n+1} + C$$

also. So it looks like

$$\boxed{du = \frac{du}{dx}\ dx.}$$

Example

Find $\int (x^3 + x)^5 (3x^2 + 1)\ dx$.

solution: Let $u = x^3 + x$. Then $du = \frac{du}{dx}\ dx = (3x^2 + 1)\ dx$. By substitution, we have that

$$\begin{aligned}\int (x^3 + x)^5 (3x^2 + 1)\ dx &= \int u^5\ du \\ &= \frac{u^6}{6} + C = \frac{1}{6}(x^3 + x)^6 + C.\end{aligned}$$

Example

Find $\int \sqrt{2x+1}\,dx$.

solution: Let $u = 2x+1$. Then $du = \frac{du}{dx}\,dx = 2dx$. So $dx = \frac{1}{2}\,du$. Therefore

$$\int \sqrt{2x+1}\,dx = \int u^{\frac{1}{2}}\left(\tfrac{1}{2}du\right) = \frac{1}{2}\int u^{\frac{1}{2}}\,du$$
$$= \frac{1}{2}\left(\frac{u^{\frac{3}{2}}}{\frac{3}{2}}\right) + C = \frac{1}{3}(2x+1)^{\frac{3}{2}} + C.$$

Theorem (The Substitution Method)

If

- $u = g(x)$ *is differentiable;*
- $g : \mathbb{R} \to I$*; and*
- $f : I \to \mathbb{R}$ *is continuous,*

then

$$\int f(g(x))g'(x)\ dx = \int f(u)\ du.$$

Example

Find $\displaystyle\int 5\sec^2(5t+1)\ dt$.

solution: Let $u = 5t+1$. Then $du = \frac{du}{dt}\ dt = 5dt$. So

$$\begin{aligned}\int 5\sec^2(5t+1)\ dt &= \int \sec^2 u\ du \\ &= \tan u + C \\ &\qquad \left(\text{because } \frac{d}{du}\tan u = \sec^2 u\right) \\ &= \tan(5t+1) + C.\end{aligned}$$

Example

Find $\displaystyle\int \cos(7\theta + 3)\ d\theta$.

solution: Let $u = 7\theta + 3$. Then $du = \frac{du}{d\theta}\ d\theta = 7d\theta$. So $d\theta = \frac{1}{7}du$ and

$$\begin{aligned}\int \cos(7\theta + 3)\ d\theta &= \frac{1}{7}\int \cos u\ du \\ &= \frac{1}{7}\sin u + C = \frac{1}{7}\sin(7\theta + 3) + C.\end{aligned}$$

Example

Find $\displaystyle\int x^2 \sin(x^3)\ dx$.

solution: Let $u = x^3$. Then $du = \frac{du}{dx}\ dx = 3x^2\ dx$. So $\frac{1}{3} du = x^2\ dx$ and

$$\int x^2 \sin(x^3)\ dx = \int \frac{1}{3} \sin u\ du = -\frac{1}{3} \cos u + C$$
$$= -\frac{1}{3} \cos(x^3) + C.$$

Example

Find $\int x\sqrt{2x+1}\,dx$.

solution: Let $u = 2x+1$. Then $du = \frac{du}{dx}\,dx = 2dx$. So $dx = \frac{1}{2}du$ and

$$\int x\sqrt{2x+1}\,dx = \int x\sqrt{u}\,\frac{1}{2}du.$$

But we still have an x here. We can't integrate until we change all the x terms to u terms. Note that

$$u = 2x+1 \implies u-1 = 2x \implies \frac{1}{2}(u-1) = x.$$

Therefore

$$\begin{aligned}
\int x\sqrt{2x+1}\ dx &= \int \frac{1}{2}(u-1)\sqrt{u}\ \frac{1}{2}du \\
&= \frac{1}{4}\int u^{\frac{3}{2}} - u^{\frac{1}{2}}\ du \\
&= \frac{1}{4}\left(\frac{2}{5}u^{\frac{5}{2}} - \frac{2}{3}u^{\frac{3}{2}}\right) + C \\
&= \frac{1}{10}u^{\frac{5}{2}} - \frac{1}{6}u^{\frac{3}{2}} + C \\
&= \frac{1}{10}(2x+1)^{\frac{5}{2}} - \frac{1}{6}(2x+1)^{\frac{3}{2}} + C.
\end{aligned}$$

Example

Find $\displaystyle\int \frac{2z}{\sqrt[3]{z^2+1}}\ dz$.

solution: Let $u = z^2 + 1$. Then $du = \frac{du}{dx}\ dx = 2z\ dz$ and

$$\begin{aligned}
\int \frac{2z}{\sqrt[3]{z^2+1}}\ dz &= \int \frac{du}{u^{\frac{1}{3}}} \\
&= \int u^{-\frac{1}{3}}\ du \\
&= \frac{u^{\frac{2}{3}}}{\frac{2}{3}} + C \\
&= \frac{3}{2} u^{\frac{2}{3}} + C \\
&= \frac{3}{2}(z^2+1)^{\frac{2}{3}} + C.
\end{aligned}$$

Example

Find $\int \sin^2 x \; dx$.

solution: We use the identity

$$\sin^2 x = \frac{1 - \cos 2x}{2}$$

to calculate that

$$\begin{aligned}
\int \sin^2 x \; dx &= \int \frac{1 - \cos 2x}{2} \; dx \\
&= \frac{1}{2} \int (1 - \cos 2x) \; dx \\
&= \frac{1}{2} \left(x - \frac{1}{2} \sin 2x \right) + C \\
&= \frac{1}{2} x - \frac{1}{4} \sin 2x + C.
\end{aligned}$$

Example

Similarly

$$\int \cos^2 x \; dx = \int \frac{1+\cos 2x}{2} \; dx = \frac{1}{2}x + \frac{1}{4}\sin 2x + C.$$

The Substitution Method for Definite Integrals

Theorem (The Substitution Method)

If

- $u = g(x)$ *is differentiable on* $[a, b]$;
- g' *is continuous on* $[a, b]$; *and*
- f *is continous on the range of* g,

then

$$\int_a^b f(g(x))g'(x)\,dx = \int_{g(a)}^{g(b)} f(u)\,du.$$

Example

Calculate $\displaystyle\int_{-1}^{1} 3x^2\sqrt{x^3+1}\,dx$.

solution 1. We can use the previous theorem to solve this example. Let $u = x^3 + 1$. Then $du = 3x^2\,dx$. Moreover $x = -1 \implies u = 0$ and $x = 1 \implies u = 2$. So

$$\int_{x=-1}^{x=1} 3x^2\sqrt{x^3+1}\,dx = \int_{u=0}^{u=2} \sqrt{u}\,du = \left[\frac{2}{3}u^{\frac{3}{2}}\right]_0^2$$
$$= \frac{2}{3}\left(2^{\frac{3}{2}} - 0^{\frac{3}{2}}\right) = \frac{2}{3}2\sqrt{2} = \frac{4\sqrt{2}}{3}.$$

solution 2. Alternately, we can first find the indefinite integral, then find the required definite integral.
Let $u = x^3 + 1$. Then $du = 3x^2\ dx$. So

$$\int 3x^2\sqrt{x^3+1}\ dx = \int \sqrt{u}\ du = \frac{2}{3}u^{\frac{3}{2}} + C = \frac{2}{3}(x^3+1)^{\frac{3}{2}} + C.$$

Therefore

$$\begin{aligned}\int_{-1}^{1} 3x^2\sqrt{x^3+1}\ dx &= \left[\frac{2}{3}(x^3+1)^{\frac{3}{2}}\right]_{-1}^{1} \\ &= \left(\frac{2}{3}(1+1)^{\frac{3}{2}}\right) - \left(\frac{2}{3}(-1+1)^{\frac{3}{2}}\right) \\ &= \frac{2}{3}\times 2^{\frac{3}{2}} = \frac{4\sqrt{2}}{3}.\end{aligned}$$

Example

Calculate $\displaystyle\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \cot\theta \operatorname{cosec}^2\theta \; d\theta$.

solution: Let $u = \cot\theta$. Then $du = \frac{du}{d\theta} \; d\theta = -\operatorname{cosec}^2\theta \; d\theta$. So $-du = \operatorname{cosec}^2\theta \; d\theta$. Moreover $\theta = \frac{\pi}{4} \implies u = \cot\frac{\pi}{4} = 1$ and $\theta = \frac{\pi}{2} \implies u = \cot\frac{\pi}{2} = 0$. Hence

$$\begin{aligned}\int_{\theta=\frac{\pi}{4}}^{\theta=\frac{\pi}{2}} \cot\theta \operatorname{cosec}^2\theta \; d\theta &= \int_{u=1}^{u=0} u \; (-du) = -\int_1^0 u \; du \\ &= -\left[\frac{u^2}{2}\right]_1^0 = -\left(\frac{0^2}{2} - \frac{1^2}{2}\right) = \frac{1}{2}.\end{aligned}$$

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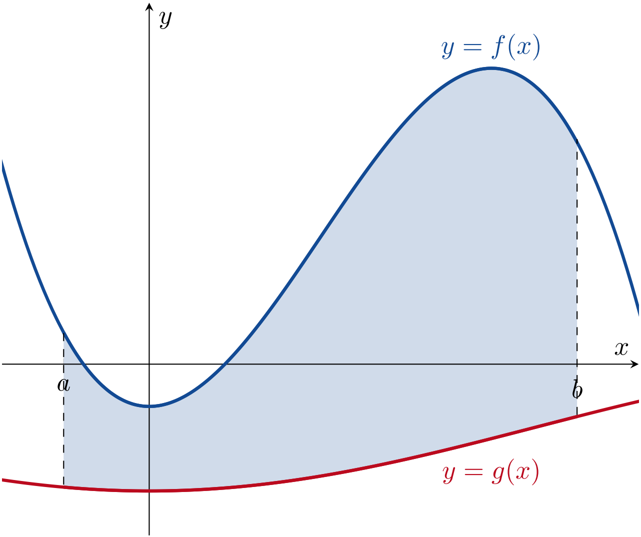
Area Between Curves

34. Area Between Curves



Definition

If

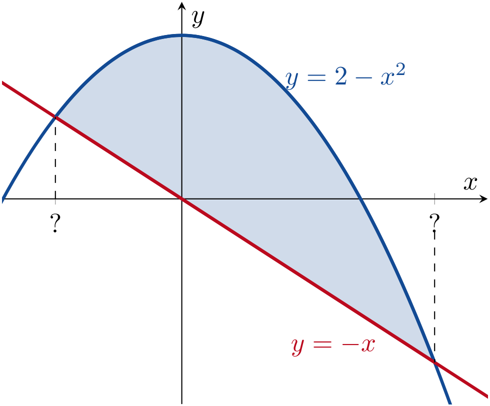
- f is continous;
- g is continous; and
- $f(x) \geq g(x)$ on $[a, b]$,

then the *area of the region between the curves* $y = f(x)$ *and* $y = g(x)$ *for* $a \leq x \leq b$ is

$$\boxed{\text{area} = \int_a^b \Big(f(x) - g(x) \Big) \, dx.}$$

Example

Find the area between $y = 2 - x^2$ and $y = -x$.



solution: First we need to find the limits of integration:

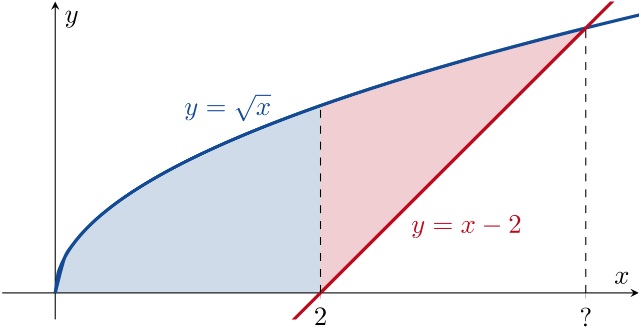
$$\begin{aligned} 2 - x^2 &= -x \\ 0 &= x^2 - x - 2 \\ 0 &= (x+1)(x-2) \quad \Longrightarrow \quad x = -1 \text{ or } 2. \end{aligned}$$

We need to integrate from $x = -1$ to $x = 2$. Therefore

$$\begin{aligned} \text{area} &= \int_{-1}^{2} \Big((2 - x^2) - (-x) \Big) \, dx \\ &= \int_{-1}^{2} (2 + x - x^2) \, dx \\ &= \left[2x + \tfrac{1}{2}x^2 - \tfrac{1}{3}x^3 \right]_{-1}^{2} \\ &= \left(4 + \frac{4}{2} - \frac{8}{3} \right) - \left(-2 + \frac{1}{2} + \frac{1}{3} \right) \\ &= \frac{9}{2}. \end{aligned}$$

Example

Find the area bounded by $y = \sqrt{x}$, $y = x - 2$ and the x-axis, for $x \geq 0$ and $y \geq 0$.



solution: First we calculate that

$$\begin{aligned}
\sqrt{x} &= x - 2 \\
x &= (x-2)^2 = x^2 - 4x + 4 \\
0 &= x^2 - 5x + 4 = (x-1)(x-4) \implies x = 1 \text{ or } 4.
\end{aligned}$$

Since $\sqrt{1} \neq 1 - 2$, we must have $x = 4$.

Therefore

$$\begin{aligned}
\text{area} &= \text{blue area} + \text{red area} \\
&= \int_0^2 \sqrt{x}\ dx + \int_2^4 \left(\sqrt{x} - (x-2)\right)\ dx \\
&= \int_0^2 x^{\frac{1}{2}}\ dx + \int_2^4 (x^{\frac{1}{2}} - x + 2)\ dx \\
&= \left[\frac{2}{3}x^{\frac{3}{2}}\right]_0^2 + \left[\frac{2}{3}x^{\frac{3}{2}} - \frac{1}{2}x^2 + 2x\right]_2^4 \\
&= \left(\frac{2}{3}(2)^{\frac{3}{2}} - 0\right) + \left(\frac{2}{3}(4)^{\frac{3}{2}} - \frac{1}{2}(16) + 2(4)\right) \\
&\qquad - \left(\frac{2}{3}(2)^{\frac{3}{2}} - \frac{1}{2}(4) + 2(2)\right) \\
&= \frac{4\sqrt{2}}{3} + \frac{16}{3} - 8 + 8 - \frac{4\sqrt{2}}{3} + 2 - 4 = \frac{10}{3}.
\end{aligned}$$

The End