Welcome to

Mathematics II

with Dr Neil Course

Lecture 1

- Information about this course
- 8.1 Using Basic Integration Formulae
- 8.2 Integration by Parts
- 8.3 Trigonometric Integrals

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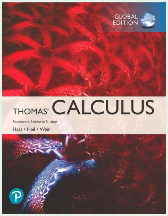
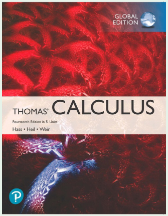
MATH113 MATH114 MATH215 MATH216

MATH113 MATH114 MATH215 MATH216

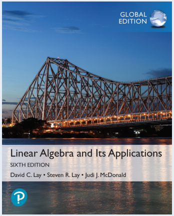


Calculus

MATH113 MATH114



MATH215



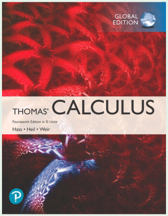
MATH216

Calculus

Linear Algebra

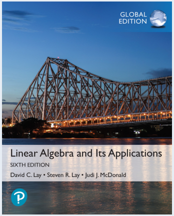
MATH114 Mathematics II

MATH113 MATH114



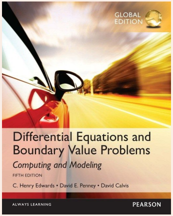
Calculus

MATH215



Linear Algebra

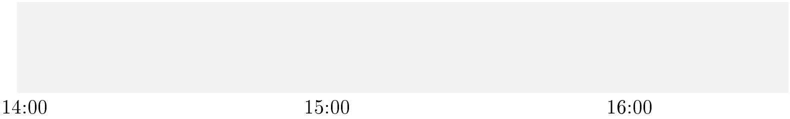
MATH216



Differential Equations

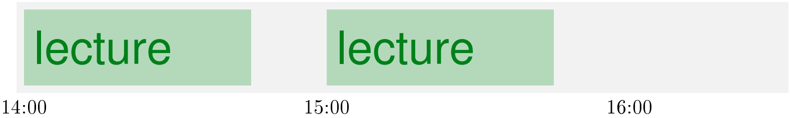
Information about this course

- $\approx$ 12 classes. Friday afternoons 2pm-4:30pm.



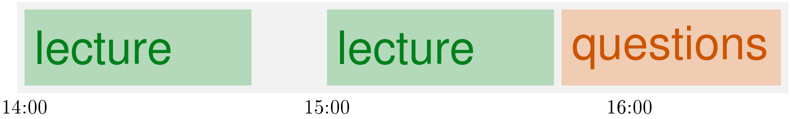
Information about this course

- ≈ 12 classes. Friday afternoons 2pm-4:30pm.
- 2 lectures with a break between.

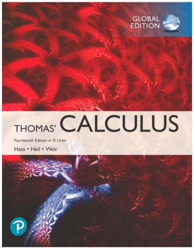


Information about this course

- ≈ 12 classes. Friday afternoons 2pm-4:30pm.
- 2 lectures with a break between.
- Then I will answers your questions.

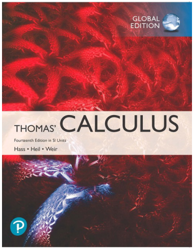


The Book



Joel R. Hass, Christopher E. Heil and Maurice D. Weir,
Thomas' Calculus,
14th Edition in SI Units, Pearson.

The Book



Joel R. Hass, Christopher E. Heil and Maurice D. Weir,
Thomas' Calculus,
14th Edition in SI Units, Pearson.

This is a required purchase.
You need to have this book to be able to do the homework.

8
11
13
14
9

8. Techniques of Integration

11

13

14

9

8. Techniques of Integration

11. Vectors and the Geometry of Space

13

14

9

8. Techniques of Integration

11. Vectors and the Geometry of Space

13. Partial Derivatives

14

9

8. Techniques of Integration

11. Vectors and the Geometry of Space

13. Partial Derivatives

14. Multiple Integrals

9. Infinite Sequences and Series

8. Techniques of Integration	2 weeks
11. Vectors and the Geometry of Space	2 weeks
13. Partial Derivatives	2 weeks
14. Multiple Integrals	3 weeks
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Exams and homework

(This information may change based on the University's decisions)

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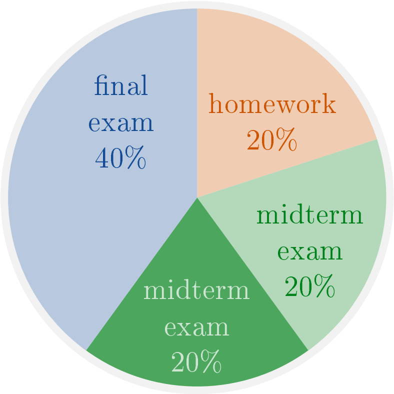

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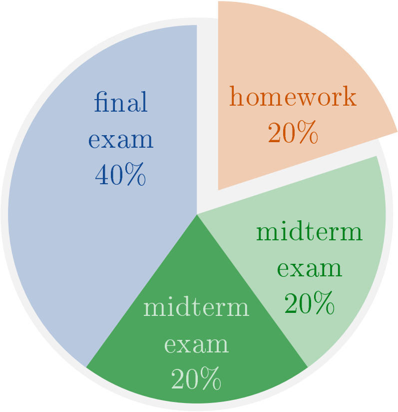

using Pearson's MyLab Math

12 pieces of homework

deadline = one day before the final exam

more details in O'Learn

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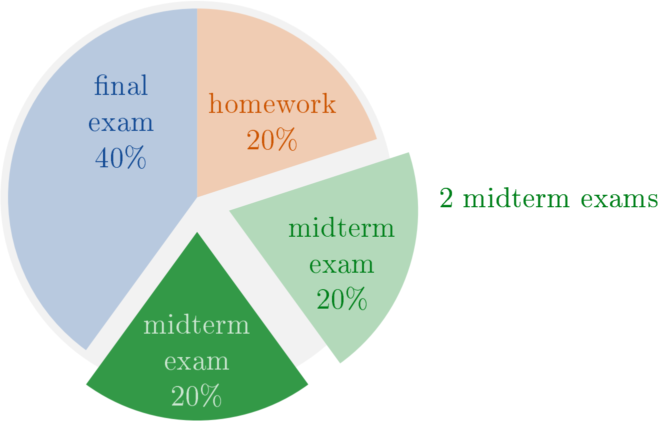

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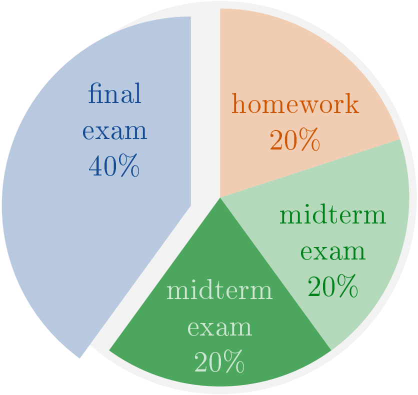

Expectations

If this was a classroom course, the expectation is that for each hour of lectures, you would study an extra 1-2 hours outside of class.

classroom course	lectures (5 hours)	other study (5-10 hours)

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classroom course	lectures (5 hours)	other study (5-10 hours)

For an online course, you are still expected to study a total of 10-15 hours each week.

online course	class (2.5 hours)	other study (7.5-12.5 hours)

This may include:

- Do the online homework on MyLab;

$\vdots$

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$\vdots$

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- Solve the exercises in the textbook;
- Use the O'Learn Discussion Board;
- Read other books?;
- Watch online videos;

⋮

slide number

10 minute break

current section

$$\boxed{1 + 2 = 3}$$

something important

Using Basic Integration Formulae

22 Basic Integration Formulae

Let's start with a little bit of revision

22 Basic Integration Formulae

1. $\displaystyle\int k\,dx = kx + C$ $\qquad$ (k is a number)

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2. $\displaystyle\int x^n\,dx = \frac{x^{n+1}}{n+1} + C$

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3. $\displaystyle\int \frac{1}{x}\,dx = \ln|x| + C$
4. $\displaystyle\int e^x\,dx = e^x + C$

22 Basic Integration Formulae

1. $\displaystyle\int k\,dx = kx + C$ $\quad$ (k is a number)
2. $\displaystyle\int x^n\,dx = \frac{x^{n+1}}{n+1} + C$
3. $\displaystyle\int \frac{1}{x}\,dx = \ln|x| + C$
4. $\displaystyle\int e^x\,dx = e^x + C$
5. $\displaystyle\int a^x\,dx = \frac{a^x}{\ln a} + C$ $\quad$ $(a > 0,\ a \neq 1)$

22 Basic Integration Formulae

6. $$\int \sin x \, dx = -\cos x + C$$
7. $$\int \cos x \, dx = \sin x + C$$

22 Basic Integration Formulae

6 $$\int \sin x \, dx = -\cos x + C$$

7 $$\int \cos x \, dx = \sin x + C$$

8 $$\int \sec^2 x \, dx = \tan x + C$$

9 $$\int \operatorname{cosec}^2 x \, dx = -\cot x + C$$

10 $$\int \sec x \tan x \, dx = \sec x + C$$

11 $$\int \operatorname{cosec} x \cot x \, dx = -\operatorname{cosec} x + C$$

22 Basic Integration Formulae

12 $$\int \tan x \, dx = \ln|\sec x| + C$$

13 $$\int \cot x \, dx = \ln|\sin x| + C$$

14 $$\int \sec x \, dx = \ln|\sec x + \tan x| + C$$

15 $$\int \operatorname{cosec} x \, dx = -\ln|\operatorname{cosec} x + \cot x| + C$$

22 Basic Integration Formulae

16 $$\int \sinh x \, dx = \cosh x + C$$

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22 Basic Integration Formulae

18 $$\int \frac{dx}{\sqrt{a^2 - x^2}} = \sin^{-1}\left(\frac{x}{a}\right) + C$$

22 Basic Integration Formulae

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22 Basic Integration Formulae

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20 $$\int \frac{dx}{x\sqrt{x^2 - a^2}} = \frac{1}{a}\sec^{-1}\left|\frac{x}{a}\right| + C$$

22 Basic Integration Formulae

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22 $$\int \frac{dx}{\sqrt{x^2 - a^2}} = \cosh^{-1}\left(\frac{x}{a}\right) + C \qquad (x > a > 0)$$

Example

Calculate $\displaystyle\int_3^5 \frac{2x-3}{\sqrt{x^2-3x+1}}\,dx.$

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We use the substitution $u = x^2 - 3x + 1$.

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We use the substitution $u = x^2 - 3x + 1$. Then $du = (2x-3)\,dx$

Example

Calculate $\displaystyle\int_3^5 \frac{2x-3}{\sqrt{x^2-3x+1}}\,dx$.

We use the substitution $u = x^2 - 3x + 1$. Then $du = (2x-3)\,dx$ and

$$x = 5 \implies u = x^2 - 3x + 1 = 25 - 15 + 1 = 11$$
$$x = 3 \implies u = x^2 - 3x + 1 = 9 - 9 + 1 = 1.$$

Example

Calculate $\displaystyle\int_3^5 \frac{2x-3}{\sqrt{x^2-3x+1}}\,dx$.

We use the substitution $u = x^2 - 3x + 1$. Then $du = (2x-3)\,dx$ and

$$\begin{aligned} x = 5 &\implies u = x^2 - 3x + 1 = 25 - 15 + 1 = 11 \\ x = 3 &\implies u = x^2 - 3x + 1 = 9 - 9 + 1 = 1. \end{aligned}$$

Hence

$$\int_3^5 \frac{2x-3}{\sqrt{x^2-3x+1}}\,dx = \int_1^{11} u^{-\frac{1}{2}}\,du = \left[2\sqrt{u}\right]_1^{11} = 2\left(\sqrt{11}-1\right).$$

$$\int \frac{dx}{\sqrt{a^2 - x^2}} = \sin^{-1}\left(\frac{x}{a}\right) + C$$

Example

Find $\displaystyle\int \frac{dx}{\sqrt{8x - x^2}}$.

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Find $\displaystyle\int \frac{dx}{\sqrt{8x - x^2}}$.

This time we will complete the square of $x^2 - 8x$ and use that to simplify the integral:

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$x^2 - 8x = x^2 - 8x + 16 - 16$

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$$x^2 - 8x = x^2 - 8x + 16 - 16 = (x^2 - 8x + 16) - 16 = (x - 4)^2 - 16.$$

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So

$$\int \frac{dx}{\sqrt{8x - x^2}} = \int \frac{dx}{\sqrt{16 - (x-4)^2}}$$

$$=$$

$$=$$

$$=$$

$$\int \frac{dx}{\sqrt{a^2 - x^2}} = \sin^{-1}\left(\frac{x}{a}\right) + C$$

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So

$$\begin{aligned}\int \frac{dx}{\sqrt{8x - x^2}} &= \int \frac{dx}{\sqrt{16 - (x - 4)^2}} \\ &= \int \frac{du}{\sqrt{16 - u^2}} \\ &= \\ &= \end{aligned}$$

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So

$$\begin{aligned}\int \frac{dx}{\sqrt{8x - x^2}} &= \int \frac{dx}{\sqrt{16 - (x-4)^2}} \\ &= \int \frac{du}{\sqrt{16 - u^2}} \\ &= \sin^{-1}\left(\frac{u}{4}\right) + C \\ &= \sin^{-1}\left(\frac{x-4}{4}\right) + C.\end{aligned}$$

Example

Find $\displaystyle\int \cos x \sin 2x + \sin x \cos 2x \, dx.$

Example

Find $\displaystyle\int \cos x \sin 2x + \sin x \cos 2x \, dx$.

$$\begin{aligned}\int \cos x \sin 2x + \sin x \cos 2x \, dx &= \int \sin(x + 2x) \, dx \\ &= \int \sin 3x \, dx \\ &= \ldots\end{aligned}$$

Example

Find $\displaystyle\int_0^{\frac{\pi}{4}} \frac{dx}{1-\sin x}$.

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Here is a trick for dealing with $\frac{1}{A-B}$:

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Example

Find $\displaystyle\int_0^{\frac{\pi}{4}} \frac{dx}{1-\sin x}$.

Here is a trick for dealing with $\frac{1}{A-B}$: Multiply by $\frac{A+B}{A+B}$. Then we get

$$\frac{1}{A-B} = \left(\frac{1}{A-B}\right)\left(\frac{A+B}{A+B}\right) = \frac{A+B}{A^2-B^2}$$

which is sometimes easier to deal with.

$$\int_0^{\frac{\pi}{4}} \frac{dx}{1-\sin x} = \int_0^{\frac{\pi}{4}} \frac{1}{1-\sin x} \cdot \qquad dx =$$

$$= \qquad =$$

$$=$$

$$= \qquad = \qquad = \quad .$$

$$\int_0^{\frac{\pi}{4}} \frac{dx}{1-\sin x} = \int_0^{\frac{\pi}{4}} \frac{1}{1-\sin x} \cdot \frac{1+\sin x}{1+\sin x}\, dx =$$

$$=\qquad\qquad =$$

$$=$$

$$=\qquad\qquad =\qquad\qquad = \quad .$$

0.1 Using Basic Integration Formulae

$$\int_0^{\frac{\pi}{4}} \frac{dx}{1-\sin x} = \int_0^{\frac{\pi}{4}} \frac{1}{1-\sin x} \cdot \frac{1+\sin x}{1+\sin x}\, dx = \int_0^{\frac{\pi}{4}} \frac{1+\sin x}{1-\sin^2 x}\, dx$$

$$= \qquad =$$

$$=$$

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$$\int_0^{\frac{\pi}{4}} \frac{dx}{1-\sin x} = \int_0^{\frac{\pi}{4}} \frac{1}{1-\sin x} \cdot \frac{1+\sin x}{1+\sin x}\, dx = \int_0^{\frac{\pi}{4}} \frac{1+\sin x}{1-\sin^2 x}\, dx$$

$$= \int_0^{\frac{\pi}{4}} \frac{1+\sin x}{\cos^2 x}\, dx =$$

$$=$$

$$= \qquad = \qquad = \qquad .$$

0.1 Using Basic Integration Formulae

$$
\begin{aligned}
\int_0^{\frac{\pi}{4}} \frac{dx}{1-\sin x} &= \int_0^{\frac{\pi}{4}} \frac{1}{1-\sin x} \cdot \frac{1+\sin x}{1+\sin x}\, dx = \int_0^{\frac{\pi}{4}} \frac{1+\sin x}{1-\sin^2 x}\, dx \\
&= \int_0^{\frac{\pi}{4}} \frac{1+\sin x}{\cos^2 x}\, dx = \int_0^{\frac{\pi}{4}} \frac{1}{\cos^2 x} + \frac{\sin x}{\cos^2 x}\, dx \\
&= \int_0^{\frac{\pi}{4}} \sec^2 x + \sec x \tan x \, dx \\
&= \qquad\qquad = \qquad\qquad = \qquad .
\end{aligned}
$$

$$
\begin{aligned}
\int_0^{\frac{\pi}{4}} \frac{dx}{1-\sin x} &= \int_0^{\frac{\pi}{4}} \frac{1}{1-\sin x} \cdot \frac{1+\sin x}{1+\sin x}\, dx = \int_0^{\frac{\pi}{4}} \frac{1+\sin x}{1-\sin^2 x}\, dx \\
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&= \int_0^{\frac{\pi}{4}} \sec^2 x + \sec x \tan x \, dx \\
&= \Big[\tan x + \sec x\Big]_0^{\frac{\pi}{4}} = \qquad\qquad = \quad .
\end{aligned}
$$

$$\int_0^{\frac{\pi}{4}} \frac{dx}{1-\sin x} = \int_0^{\frac{\pi}{4}} \frac{1}{1-\sin x} \cdot \frac{1+\sin x}{1+\sin x}\, dx = \int_0^{\frac{\pi}{4}} \frac{1+\sin x}{1-\sin^2 x}\, dx$$
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$$= \int_0^{\frac{\pi}{4}} \sec^2 x + \sec x \tan x\, dx$$
$$= \Big[\tan x + \sec x\Big]_0^{\frac{\pi}{4}} = \left(1+\sqrt{2}\right) - (0+1) = \sqrt{2}.$$

Example

Find $\displaystyle\int \frac{3x^2 - 7x}{3x + 2}\, dx.$

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The integrand is an improper fraction because the degree of $3x^2 - 7x$ (2nd degree) is greater than the degree of $3x + 2$ (1st degree).

Example

Find $\displaystyle\int \frac{3x^2 - 7x}{3x + 2}\, dx$.

The integrand is an improper fraction because the degree of $3x^2 - 7x$ (2nd degree) is greater than the degree of $3x + 2$ (1st degree). We want to write our integrand as

$$\frac{3x^2 - 7x}{3x + 2} = \text{(linear function)} + \frac{\text{something}}{3x + 2}$$

[Do you remember how we studied *oblique asymptotes* in MATH113?]

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Find $\displaystyle\int \frac{3x^2 - 7x}{3x + 2}\, dx.$

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$$\frac{3x^2 - 7x}{3x + 2} = (ax + b) + \frac{\text{something}}{3x + 2}$$

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$$\frac{3x^2 - 7x}{3x + 2} = (ax + b) + \frac{c}{3x + 2}$$

[Do you remember how we studied *oblique asymptotes* in MATH113?]

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Find $\int \frac{3x^2 - 7x}{3x + 2}\, dx$.

The integrand is an improper fraction because the degree of $3x^2 - 7x$ (2nd degree) is greater than the degree of $3x + 2$ (1st degree). We want to write our integrand as

$$\frac{3x^2 - 7x}{3x + 2} \cdot (3x + 2) = (ax + b)(3x + 2) + \frac{c}{3x + 2} \cdot (3x + 2)$$

[Do you remember how we studied *oblique asymptotes* in MATH113?]

Example

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$$3x^2-7x = 3ax^2+(2a+3b)x+(2b+c)$$

$a=1$, $b=-3$, $c=6$.

Example

Find $\displaystyle\int \frac{3x^2-7x}{3x+2}\,dx.$

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Example

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$$\int \frac{3x^2 - 7x}{3x+2}\,dx = \int 2x - 3 + \frac{6}{3x+2}\,dx = \frac{x^2}{2} - 3x + 2\ln|3x+2| + C.$$

$$\int \frac{dx}{\sqrt{a^2 - x^2}} = \sin^{-1}\left(\frac{x}{a}\right) + C$$

Example

Find $\displaystyle\int \frac{3x+2}{\sqrt{1-x^2}}$.

$$\int \frac{dx}{\sqrt{a^2 - x^2}} = \sin^{-1}\left(\frac{x}{a}\right) + C$$

Example

Find $\int \frac{3x + 2}{\sqrt{1 - x^2}}$.

First note that

$$\int \frac{3x + 2}{\sqrt{1 - x^2}} = 3 \int \frac{x\,dx}{\sqrt{1 - x^2}} + 2 \int \frac{dx}{\sqrt{1 - x^2}}$$

$$\int \frac{dx}{\sqrt{a^2 - x^2}} = \sin^{-1}\left(\frac{x}{a}\right) + C$$

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$$\begin{aligned}\int \frac{3x+2}{\sqrt{1-x^2}} &= 3\int \frac{x\,dx}{\sqrt{1-x^2}} + 2\int \frac{dx}{\sqrt{1-x^2}} \\ &= 3\int \frac{x\,dx}{\sqrt{1-x^2}} + 2\sin^{-1} x.\end{aligned}$$

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So we just need to calculate $\displaystyle\int \frac{x\,dx}{\sqrt{1-x^2}}$.

$$\int \frac{x\,dx}{\sqrt{1-x^2}}$$

Let $u = 1 - x^2$.

$$\int \frac{x\,dx}{\sqrt{1-x^2}}$$

Let $u = 1 - x^2$. Then $du = -2x\,dx$ and $-\frac{1}{2}\,du = x\,dx$.

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Let $u = 1 - x^2$. Then $du = -2x\,dx$ and $-\frac{1}{2}\,du = x\,dx$. It follows that

$$\int \frac{x\,dx}{\sqrt{1-x^2}} = \int \frac{-\frac{1}{2}\,du}{\sqrt{u}} = -\frac{1}{2}\int u^{-\frac{1}{2}}\,du = \ldots = -\sqrt{1-x^2} + C.$$

$$\int \frac{x\,dx}{\sqrt{1-x^2}}$$

Let $u = 1 - x^2$. Then $du = -2x\,dx$ and $-\frac{1}{2}\,du = x\,dx$. It follows that

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Therefore

$$\begin{aligned}\int \frac{3x+2}{\sqrt{1-x^2}} &= 3\int \frac{x\,dx}{\sqrt{1-x^2}} + 2\int \frac{dx}{\sqrt{1-x^2}} \\ &= -3\sqrt{1-x^2} + 2\sin^{-1} x + C.\end{aligned}$$

Example

Find $\displaystyle\int \frac{dx}{(1+\sqrt{x})^3}$.

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I want to make a substitution to make this integral easier, but what u should I choose?

First guess: $u = \sqrt{x}$. But then $du = \frac{1}{2\sqrt{x}}\, dx$ and we would have to deal with this extra $\sqrt{x} = u$ term.

Example

Find $\displaystyle\int \frac{dx}{(1+\sqrt{x})^3}$.

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First guess: $u = \sqrt{x}$. But then $du = \frac{1}{2\sqrt{x}}\,dx$ and we would have to deal with this extra $\sqrt{x} = u$ term.

Second guess: Instead let us try $u = 1 + \sqrt{x}$.

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I want to make a substitution to make this integral easier, but what u should I choose?

First guess: $u = \sqrt{x}$. But then $du = \frac{1}{2\sqrt{x}}\,dx$ and we would have to deal with this extra $\sqrt{x} = u$ term.

Second guess: Instead let us try $u = 1 + \sqrt{x}$. Then again we have $du = \frac{1}{2\sqrt{x}}\,dx$ and $dx = 2\sqrt{x}\,du = 2(u-1)\,du$. Hence

$$\int \frac{dx}{(1+\sqrt{x})^3} = \int \frac{2(u-1)\,du}{u^3} = \int \frac{2}{u^2} - \frac{2}{u^3}\,du = \dots$$

Example

Calculate $\displaystyle\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} x^3 \cos x \, dx.$

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This is actually easy: The integrand is an odd function

0.1 Using Basic Integration Formulae

Example

Calculate $\displaystyle\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} x^3 \cos x \, dx$.

This is actually easy: The integrand is an odd function so

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} x^3 \cos x \, dx = 0.$$

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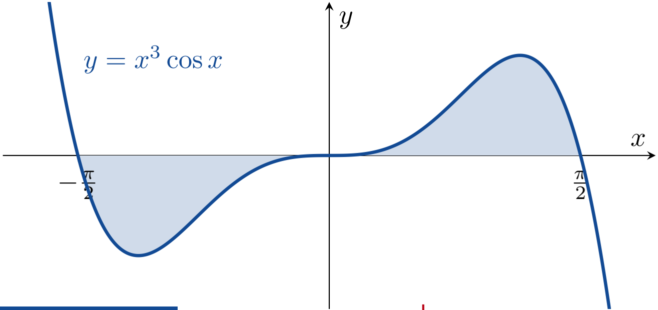

Integration by Parts

How can we calculate

$$\int x \cos x \, dx$$

or

$$\int x^2 e^x \, dx \ ?$$

How can we calculate

$$\int x \cos x \, dx$$

or

$$\int x^2 e^x \, dx \ ?$$

How do we integrate function $\times$ function?

We know how to differentiate function $\times$ function. By the product rule we have

$$(uv)' = u'v + uv'.$$

We know how to differentiate function $\times$ function. By the product rule we have

$$\int (uv)'\,dx = \int u'v\,dx + \int uv'\,dx.$$

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We know how to differentiate function × function. By the product rule we have

$$uv = \int u'v\,dx + \int uv'\,dx.$$

We know how to differentiate function × function. By the product rule we have

$$uv = \int u'v\,dx + \int uv'\,dx.$$

This rearranges to

Theorem (Integration by Parts)

$$\int u(x)v'(x)\,dx = u(x)v(x) - \int u'(x)v(x)\,dx$$

$$\int uv'\,dx = uv - \int u'v\,dx$$

Example

Find $\displaystyle\int x\cos x\,dx.$

$$\int uv'\,dx = uv - \int u'v\,dx$$

Example

Find $\displaystyle\int x\cos x\,dx$.

We need to choose a $u(x)$ and a $v'(x)$.

$$\int uv'\,dx = uv - \int u'v\,dx$$

Example

Find $\displaystyle\int x\cos x\,dx$.

We need to choose a $u(x)$ and a $v'(x)$.
Let

$$u = x \qquad\qquad v' = \cos x$$

Then

$$\int x\cos x\,dx = \qquad - \int \qquad dx = \qquad .$$

$$\int uv'\,dx = uv - \int u'v\,dx$$

Example

Find $\int x\cos x\,dx$.

We need to choose a $u(x)$ and a $v'(x)$.
Let

$$u = x \qquad v' = \cos x$$
$$u' = 1$$

Then

$$\int x\cos x\,dx = \qquad - \int \qquad dx = \qquad .$$

Example

Find $\displaystyle\int x\cos x\,dx$.

We need to choose a $u(x)$ and a $v'(x)$.
Let

$$\begin{aligned} u &= x & v' &= \cos x \\ u' &= 1 & v &= \sin x. \end{aligned}$$

Then

$$\int x\cos x\,dx = \qquad - \int \qquad dx = \qquad .$$

$$\int uv'\,dx = uv - \int u'v\,dx$$

Example

Find $\displaystyle\int x\cos x\,dx$.

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Then

$$\int x\cos x\,dx = x\sin x - \int \qquad dx = \qquad\qquad .$$

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$$\int x\cos x\,dx = x\sin x - \int 1\sin x\,dx = \qquad\qquad .$$

$$\int uv'\,dx = uv - \int u'v\,dx$$

Example

Find $\displaystyle\int x\cos x\,dx$.

We need to choose a $u(x)$ and a $v'(x)$.
Let

$$\begin{aligned} u &= x & \qquad v' &= \cos x \\ u' &= 1 & \qquad v &= \sin x. \end{aligned}$$

Then

$$\int x\cos x\,dx = x\sin x - \int 1\sin x\,dx = x\sin x + \cos x + C.$$

$$\int uv'\,dx = uv - \int u'v\,dx$$

Example

Find $\displaystyle\int \ln x\,dx$.

$$\int uv'\,dx = uv - \int u'v\,dx$$

Example

Find $\displaystyle\int \ln x\,dx$.

We will consider $\displaystyle\int \ln x \cdot 1\,dx$.

$$\int uv'\,dx = uv - \int u'v\,dx$$

Example

Find $\displaystyle\int \ln x\,dx$.

We will consider $\displaystyle\int \ln x \cdot 1\,dx$.

Let

$$u = \ln x \qquad\qquad v' = 1$$

$$\int uv'\,dx = uv - \int u'v\,dx$$

Example

Find $\displaystyle\int \ln x\,dx$.

We will consider $\displaystyle\int \ln x \cdot 1\,dx$.

Let

$$u = \ln x \qquad\qquad v' = 1$$
$$u' = \frac{1}{x}$$

$$\int uv'\,dx = uv - \int u'v\,dx$$

Example

Find $\displaystyle\int \ln x\,dx$.

We will consider $\displaystyle\int \ln x \cdot 1\,dx$.

Let

$$u = \ln x \qquad v' = 1$$
$$u' = \frac{1}{x} \qquad v = x.$$

$$\int uv'\,dx = uv - \int u'v\,dx$$

Example

Find $\displaystyle\int \ln x\,dx$.

We will consider $\displaystyle\int \ln x \cdot 1\,dx$.

Let

$$u = \ln x \qquad\qquad v' = 1$$
$$u' = \frac{1}{x} \qquad\qquad v = x.$$

Then

$$\int \ln x \cdot 1\,dx = \ln x \cdot x - \int \frac{1}{x} \cdot x\,dx$$
$$=$$
$$= \qquad\qquad .$$

$$\int uv'\,dx = uv - \int u'v\,dx$$

Example

Find $\displaystyle\int \ln x\,dx$.

We will consider $\displaystyle\int \ln x \cdot 1\,dx$.

Let

$$u = \ln x \qquad v' = 1$$
$$u' = \frac{1}{x} \qquad v = x.$$

Then

$$\int \ln x \cdot 1\,dx = \ln x \cdot x - \int \frac{1}{x} \cdot x\,dx$$
$$= x \ln x - \int 1\,dx$$
$$= \qquad .$$

$$\int uv'\,dx = uv - \int u'v\,dx$$

Example

Find $\displaystyle\int \ln x\,dx$.

We will consider $\displaystyle\int \ln x \cdot 1\,dx$.

Let

$$u = \ln x \qquad v' = 1$$
$$u' = \frac{1}{x} \qquad v = x.$$

Then

$$\begin{aligned}\int \ln x \cdot 1\,dx &= \ln x \cdot x - \int \frac{1}{x} \cdot x\,dx \\ &= x\ln x - \int 1\,dx \\ &= x\ln x - x + C.\end{aligned}$$

$$\int uv'\,dx = uv - \int u'v\,dx$$

Sometimes we have to use integration by parts more than once.

$$\int uv' \, dx = uv - \int u'v \, dx$$

Example

Find $\displaystyle\int x^2 e^x \, dx$.

$$\int uv'\,dx = uv - \int u'v\,dx$$

Example

Find $\int x^2 e^x\,dx$.

We calculate that

$$\int x^2 e^x\,dx = x^2 e^x - 2\int x e^x\,dx.$$

$$\int uv' \, dx = uv - \int u'v \, dx$$

Example

Find $\displaystyle\int x^2 e^x \, dx$.

We calculate that

$$\int x^2 e^x \, dx = x^2 e^x - 2 \int x e^x \, dx.$$

But what do we do with $\displaystyle\int x e^x \, dx$?

$$\int uv' \, dx = uv - \int u'v \, dx$$

Example

Find $\displaystyle\int x^2 e^x \, dx$.

We calculate that

$$\int x^2 e^x \, dx = x^2 e^x - 2 \int x e^x \, dx.$$

But what do we do with $\displaystyle\int x e^x \, dx$?

$$\int x e^x \, dx = x e^x - \int 1 e^x \, dx = x e^x - e^x + C_1.$$

$$\int uv'\,dx = uv - \int u'v\,dx$$

Example

Find $\int x^2 e^x\,dx$.

We calculate that

$$\int x^2 e^x\,dx = x^2 e^x - 2\int x e^x\,dx.$$

But what do we do with $\int x e^x\,dx$?

$$\int x e^x\,dx = x e^x - \int 1 e^x\,dx = x e^x - e^x + C_1.$$

Putting it all together, we have

$$\int x^2 e^x\,dx = x^2 e^x - 2\int x e^x\,dx = x^2 e^x - 2x e^x + 2e^x + C.$$

Remark

We can use the same technique to calculate $\displaystyle\int x^n e^x\,dx$.

We would have to do integration by parts n times.

$$\int uv'\,dx = uv - \int u'v\,dx$$

Theorem

$$\int u\,dv = uv - \int v\,du$$

$$\int u\,dv = uv - \int v\,du$$

Example (Using Integration by Parts Twice)

Calculate $\displaystyle\int e^x \cos x\,dx$.

$$\int u\,dv = uv - \int v\,du$$

Example (Using Integration by Parts Twice)

Calculate $\displaystyle\int e^x \cos x\,dx$.

Let $u = e^x$ and $dv = \cos x\,dx$.

$$\int u\,dv = uv - \int v\,du$$

Example (Using Integration by Parts Twice)

Calculate $\int e^x \cos x\,dx$.

Let $u = e^x$ and $dv = \cos x\,dx$. Then $du = e^x\,dx$ and $v = \sin x$.

$$\int u\,dv = uv - \int v\,du$$

Example (Using Integration by Parts Twice)

Calculate $\int e^x \cos x\,dx$.

Let $u = e^x$ and $dv = \cos x\,dx$. Then $du = e^x\,dx$ and $v = \sin x$.
Hence

$$\int e^x \cos x\,dx = e^x \sin x - \int \sin x \cdot e^x\,dx.$$

$$\int u\,dv = uv - \int v\,du$$

Example (Using Integration by Parts Twice)

Calculate $\int e^x \cos x\,dx$.

Let $u = e^x$ and $dv = \cos x\,dx$. Then $du = e^x\,dx$ and $v = \sin x$. Hence

$$\int e^x \cos x\,dx = e^x \sin x - \int \sin x \cdot e^x\,dx.$$

But we still have $\int (\text{function}) \times (\text{function})\,dx$. So we need to use Integration by Parts a second time.

$$\int u\,dv = uv - \int v\,du$$

So far we have

$$\int e^x \cos x\,dx = e^x \sin x - \int e^x \sin x\,dx.$$

$$\int u\,dv = uv - \int v\,du$$

So far we have

$$\int e^x \cos x\,dx = e^x \sin x - \int e^x \sin x\,dx.$$

Let $u = e^x$ and $dv = \sin x\,dx$.

$$\int u\,dv = uv - \int v\,du$$

So far we have

$$\int e^x \cos x\,dx = e^x \sin x - \int e^x \sin x\,dx.$$

Let $u = e^x$ and $dv = \sin x\,dx$. Then $du = e^x\,dx$ and $v = -\cos x$.

$$\int u\,dv = uv - \int v\,du$$

So far we have

$$\int e^x \cos x\,dx = e^x \sin x - \int e^x \sin x\,dx.$$

Let $u = e^x$ and $dv = \sin x\,dx$. Then $du = e^x\,dx$ and $v = -\cos x$. Hence

$$\begin{aligned}\int e^x \cos x\,dx &= e^x \sin x - \int e^x \sin x\,dx \\ &= e^x \sin x - \left(-e^x \cos x - \int (-\cos x)\cdot e^x\,dx\right) \\ &= \end{aligned}$$

$$\int u\,dv = uv - \int v\,du$$

So far we have

$$\int e^x \cos x\,dx = e^x \sin x - \int e^x \sin x\,dx.$$

Let $u = e^x$ and $dv = \sin x\,dx$. Then $du = e^x\,dx$ and $v = -\cos x$. Hence

$$\begin{aligned}\int e^x \cos x\,dx &= e^x \sin x - \int e^x \sin x\,dx \\ &= e^x \sin x - \left(-e^x \cos x - \int (-\cos x) \cdot e^x\,dx\right) \\ &= e^x \sin x + e^x \cos x - \int e^x \cos x\,dx.\end{aligned}$$

$$\int u\,dv = uv - \int v\,du$$

So far we have

$$\int e^x \cos x\,dx = e^x \sin x - \int e^x \sin x\,dx.$$

Let $u = e^x$ and $dv = \sin x\,dx$. Then $du = e^x\,dx$ and $v = -\cos x$. Hence

$$\begin{aligned}\int e^x \cos x\,dx &= e^x \sin x - \int e^x \sin x\,dx \\ &= e^x \sin x - \left(-e^x \cos x - \int (-\cos x)\cdot e^x\,dx\right) \\ &= e^x \sin x + e^x \cos x - \int e^x \cos x\,dx.\end{aligned}$$

$$\int u\,dv = uv - \int v\,du$$

$$\int e^x \cos x\,dx = e^x \sin x + e^x \cos x - \int e^x \cos x\,dx.$$

$$\int u\,dv = uv - \int v\,du$$

$$\int e^x \cos x\,dx = e^x \sin x + e^x \cos x - \int e^x \cos x\,dx.$$

It follows that

$$2\int e^x \cos x\,dx = e^x \sin x + e^x \cos x + C_1$$

$$\int u\,dv = uv - \int v\,du$$

$$\int e^x \cos x\,dx = e^x \sin x + e^x \cos x - \int e^x \cos x\,dx.$$

It follows that

$$2\int e^x \cos x\,dx = e^x \sin x + e^x \cos x + C_1$$

and hence that

$$\int e^x \cos x\,dx = \frac{e^x \sin x + e^x \cos x}{2} + C.$$

$$\int u\,dv = uv - \int v\,du$$

Example

Obtain a formula that expresses the integral $\int \cos^n x\,dx$ in terms of an integral of a lower power of $\cos x$.

$$\int u\,dv = uv - \int v\,du$$

Example

Obtain a formula that expresses the integral $\int \cos^n x\,dx$ in terms of an integral of a lower power of $\cos x$.

To use integration by parts, we need (function) $\times$ (function). So think of the integral like this:

$$\int \cos^n x\,dx = \int \cos^{n-1} x \cdot \cos x\,dx.$$

$$\int u\,dv = uv - \int v\,du$$

$$\int \cos^n x\,dx = \int \cos^{n-1} x \cdot \cos x\,dx.$$

If $u = \cos^{n-1} x$ and $dv = \cos x\,dx$,

$$\int u\,dv = uv - \int v\,du$$

$$\int \cos^n x\,dx = \int \cos^{n-1} x \cdot \cos x\,dx.$$

If $u = \cos^{n-1} x$ and $dv = \cos x\,dx$, then we have that $du = -(n-1)\cos^{n-2} x \sin x\,dx$ and $v = \sin x$.

$$\int u\,dv = uv - \int v\,du$$

$$\int \cos^n x\,dx = \int \cos^{n-1} x \cdot \cos x\,dx.$$

If $u = \cos^{n-1} x$ and $dv = \cos x\,dx$, then we have that $du = -(n-1)\cos^{n-2} x \sin x\,dx$ and $v = \sin x$. Hence

$$\begin{aligned}\int \cos^n x\,dx &= \int u\,dv = uv - \int v\,du \\ &= \\ &= \\ &= \\ &= \end{aligned}$$

$$\int u\,dv = uv - \int v\,du$$

$$\int \cos^n x\,dx = \int \cos^{n-1} x \cdot \cos x\,dx.$$

If $u = \cos^{n-1} x$ and $dv = \cos x\,dx$, then we have that $du = -(n-1)\cos^{n-2} x \sin x\,dx$ and $v = \sin x$. Hence

$$\begin{aligned}\int \cos^n x\,dx &= \int u\,dv = uv - \int v\,du \\ &= \cos^{n-1} x \sin x + (n-1)\int \sin x \cos^{n-2} x \sin x\,dx \\ &= \\ &= \\ &= \end{aligned}$$

$$\int u\,dv = uv - \int v\,du$$

$$\int \cos^n x\,dx = \int \cos^{n-1} x \cdot \cos x\,dx.$$

If $u = \cos^{n-1} x$ and $dv = \cos x\,dx$, then we have that $du = -(n-1)\cos^{n-2} x \sin x\,dx$ and $v = \sin x$. Hence

$$\begin{aligned}
\int \cos^n x\,dx &= \int u\,dv = uv - \int v\,du \\
&= \cos^{n-1} x \sin x + (n-1)\int \sin x \cos^{n-2} x \sin x\,dx \\
&= \cos^{n-1} x \sin x + (n-1)\int \sin^2 x \cos^{n-2} x\,dx \\
&= \\
&=
\end{aligned}$$

$$\int u\,dv = uv - \int v\,du$$

$$\int \cos^n x\,dx = \int \cos^{n-1} x \cdot \cos x\,dx.$$

If $u = \cos^{n-1} x$ and $dv = \cos x\,dx$, then we have that $du = -(n-1)\cos^{n-2} x \sin x\,dx$ and $v = \sin x$. Hence

$$\begin{aligned}
\int \cos^n x\,dx &= \int u\,dv = uv - \int v\,du \\
&= \cos^{n-1} x \sin x + (n-1) \int \sin x \cos^{n-2} x \sin x\,dx \\
&= \cos^{n-1} x \sin x + (n-1) \int \sin^2 x \cos^{n-2} x\,dx \\
&= \cos^{n-1} x \sin x + (n-1) \int (1-\cos^2 x) \cos^{n-2} x\,dx \\
&=
\end{aligned}$$

$$\int u\,dv = uv - \int v\,du$$

$$\int \cos^n x\,dx = \int \cos^{n-1} x \cdot \cos x\,dx.$$

If $u = \cos^{n-1} x$ and $dv = \cos x\,dx$, then we have that $du = -(n-1)\cos^{n-2} x \sin x\,dx$ and $v = \sin x$. Hence

$$\begin{aligned}
\int \cos^n x\,dx &= \int u\,dv = uv - \int v\,du \\
&= \cos^{n-1} x \sin x + (n-1) \int \sin x \cos^{n-2} x \sin x\,dx \\
&= \cos^{n-1} x \sin x + (n-1) \int \sin^2 x \cos^{n-2} x\,dx \\
&= \cos^{n-1} x \sin x + (n-1) \int (1-\cos^2 x) \cos^{n-2} x\,dx \\
&= \cos^{n-1} x \sin x + (n-1) \int \cos^{n-2} x\,dx - (n-1) \int \cos^n x\,dx.
\end{aligned}$$

$$\int u\,dv = uv - \int v\,du$$

$$\int \cos^n x\,dx = \cos^{n-1} x \sin x + (n-1)\int \cos^{n-2} x\,dx - (n-1)\int \cos^n x\,dx.$$

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If we add $(n-1)\int \cos^n x\,dx$ to both sides then we obtain

$$n\int \cos^n x\,dx = \cos^{n-1} x \sin x + (n-1)\int \cos^{n-2} x\,dx.$$

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$$n \int \cos^n x\,dx = \cos^{n-1} x \sin x + (n-1) \int \cos^{n-2} x\,dx.$$

Finally we divide by n to obtain our answer

$$\int \cos^n x\,dx = \frac{\cos^{n-1} x \sin x}{n} + \frac{n-1}{n} \int \cos^{n-2} x\,dx.$$

Remark

$$\int \cos^n x\,dx = \frac{\cos^{n-1} x \sin x}{n} + \frac{n-1}{n} \int \cos^{n-2} x\,dx.$$

The formula above is called a *reduction formula* because it replaces it replaces an integral containing $\cos^n x$ with an integral containing a smaller power of $\cos x$.

$$\int u\,dv = uv - \int v\,du$$

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$$\int \cos^n x\,dx = \frac{\cos^{n-1} x \sin x}{n} + \frac{n-1}{n} \int \cos^{n-2} x\,dx.$$

The formula above is called a *reduction formula* because it replaces it replaces an integral containing $\cos^n x$ with an integral containing a smaller power of $\cos x$.

Example ($n = 3$)

$$\begin{aligned}\int \cos^3 x\,dx &= \frac{1}{3}\cos^2 x \sin x + \frac{2}{3}\int \cos x\,dx \\ &= \frac{1}{3}\cos^2 x \sin x + \frac{2}{3}\sin x + C.\end{aligned}$$

Break

We will continue at 3pm

Integration by Parts for Definite Integrals

Theorem

$$\int_a^b uv'\,dx =$$

Integration by Parts for Definite Integrals

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$$\int_a^b uv'\,dx = \Big[uv\Big]_a^b - \int_a^b u'v\,dx$$

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Example

Calculate the area of the region bounded by the curve $y = xe^{-x}$ and the x-axis from $x = 0$ to $x = 4$.

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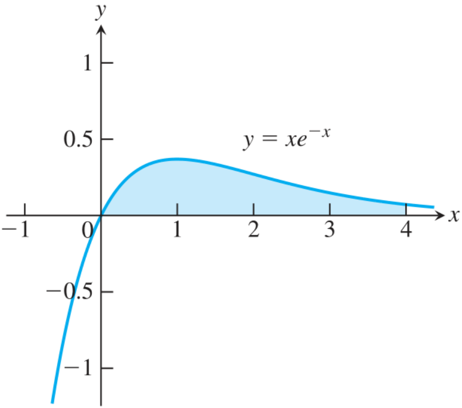
Calculate the area of the region bounded by the curve $y = xe^{-x}$ and the x-axis from $x = 0$ to $x = 4$.



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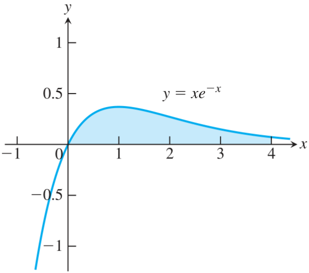


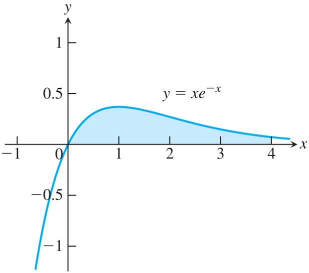
We calculate that

$$\int_0^4 xe^{-x}\,dx =$$

$$=$$

$$= \qquad .$$

$$\int_a^b uv' \, dx = \Big[uv\Big]_a^b - \int_a^b u'v \, dx$$



$$u = x$$
$$u' = 1$$
$$v' = e^{-x}$$
$$v = -e^{-x}$$

We calculate that

$$\int_0^4 xe^{-x} \, dx =$$
$$=$$
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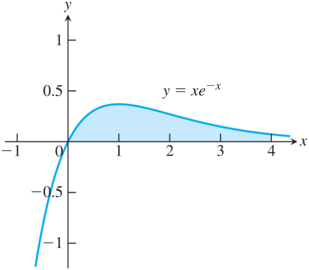
$$\int_a^b uv'\,dx = \Big[uv\Big]_a^b - \int_a^b u'v\,dx$$



$$u = x$$
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We calculate that

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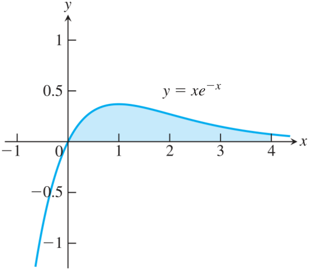

$$u = x$$
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We calculate that

$$\begin{aligned}\int_0^4 xe^{-x}\,dx &= \Big[-xe^{-x}\Big]_0^4 - \int_0^4 1(-e^{-x})\,dx \\ &= \left(-4e^{-4} + 0\right) + \left[-e^{-x}\right]_0^4 \\ &= -4e^{-4} + \left(-e^{-4} + 1\right) = 1 - 5e^{-4}.\end{aligned}$$

$$\int_a^b uv'\,dx = \Big[uv\Big]_a^b - \int_a^b u'v\,dx$$

Example

Find $\displaystyle\int_0^1 \sin^{-1} x\,dx$.

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Find $\displaystyle\int_0^1 \sin^{-1} x\,dx$.

Recall that $\frac{d}{dx}\sin^{-1} x = \frac{1}{\sqrt{1-x^2}}$.

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Find $\displaystyle\int_0^1 \sin^{-1} x\,dx$.

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Let $u = \sin^{-1} x$ and $v' = 1$.

$$\int_a^b uv' \, dx = \Big[uv\Big]_a^b - \int_a^b u'v \, dx$$

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Find $\displaystyle\int_0^1 \sin^{-1} x \, dx$.

Recall that $\frac{d}{dx} \sin^{-1} x = \frac{1}{\sqrt{1-x^2}}$.

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Find $\displaystyle\int_0^1 \sin^{-1} x\,dx$.

Recall that $\frac{d}{dx}\sin^{-1} x = \frac{1}{\sqrt{1-x^2}}$.

Let $u = \sin^{-1} x$ and $v' = 1$. Then $u' = \frac{1}{\sqrt{1-x^2}}$ and $v = x$. It follows that

$$\int_0^1 \sin^{-1} x \cdot 1\,dx = \left[x \sin^{-1} x\right]_0^1 - \int_0^1 \frac{x}{\sqrt{1-x^2}}dx$$

$$=$$

$$=$$

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Find $\displaystyle\int_0^1 \sin^{-1} x\,dx$.

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$$\begin{aligned}
\int_0^1 \sin^{-1} x \cdot 1\,dx &= \big[x\sin^{-1} x\big]_0^1 - \int_0^1 \frac{x}{\sqrt{1-x^2}}dx \\
&= \big[x\sin^{-1} x\big]_0^1 - \Big[-\sqrt{1-x^2}\Big]_0^1 \\
&= \left(\frac{\pi}{2} - 0\right) - (-0+1) \\
&= \frac{\pi}{2} - 1.
\end{aligned}$$

0.3 Trigonometric Integrals

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$$
\begin{aligned}
\cos\theta &= \sin\left(\frac{\pi}{2} - \theta\right) \\
\cos^2\theta + \sin^2\theta &= 1 \\
1 + \tan^2\theta &= \sec^2\theta \\
1 + \cot^2\theta &= \operatorname{cosec}^2\theta \\
\cos(A+B) &= \cos A\cos B - \sin A\sin B \\
\sin(A+B) &= \sin A\cos B + \cos A\sin B \\
\cos 2\theta &= \cos^2\theta - \sin^2\theta \\
\sin 2\theta &= 2\sin\theta\cos\theta \\
\cos^2\theta &= \frac{1}{2}(1 + \cos 2\theta) \\
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How can we find

$$\int \sin^m x \, \cos^n x \, dx$$

if $m, n \in \{0, 1, 2, 3, 4, 5, \ldots\}$?

$$\int \sin^{m} x \, \cos^{n} x \, dx$$

We need to look at 3 different cases:

1. m is an odd number;
2. n is an odd number;
3. both m and n are even.

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1. m is an odd number;
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[If both m and n are odd, then you can choose to use case 1 or case 2.]

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The idea is:

a. Use $\sin^2 x = 1 - \cos^2 x$ to change all but one of the sin terms into cos terms;
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$$\int \sin^m x \cos^n x \, dx = \int \sin^{2k+1} x \cos^n x \, dx$$

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$$\int \sin^m x \cos^n x \, dx = \int (1 - u^2)^k u^n \, (-du)$$

$$\int \sin^m x \, \cos^{n} x \, dx$$

2. n is odd;

Same idea, but with cos and sin swapped:

$$\int \sin^m x \, \cos^{n} x \, dx$$

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a. Use $\cos^2 x = 1 - \sin^2 x$ to change all but one of the cos terms into sin terms;
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$$= \qquad\qquad =$$

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$$\begin{aligned}\int \sin^m x \, \cos^n x \, dx &= \int \sin^m x \cos x (\cos^2 x)^k \, dx \\ &= \int \sin^m x \cos x (1 - \sin^2 x)^k \, dx =\end{aligned}$$

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Same idea, but with cos and sin swapped:

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Since n is odd, we can write $n = 2k + 1$ for $k \in \mathbb{N}$.

$$\begin{aligned}\int \sin^m x \, \cos^n x \, dx &= \int \sin^m x \cos x (\cos^2 x)^k \, dx \\ &= \int \sin^m x \cos x (1 - \sin^2 x)^k \, dx = \int u^m (1 - u^2)^k \, du.\end{aligned}$$

$$\int \sin^{m} x \, \cos^{n} x \, dx$$

3. both m and n are even.

$$\int \sin^{m} x \, \cos^{n} x \, dx$$

3 both m and n are even.

This time the idea is to use

$$\sin^2 x = \frac{1 - \cos 2x}{2} \qquad \text{and} \qquad \cos^2 x = \frac{1 + \cos 2x}{2}$$

to reduce the powers of sin and cos that we need to deal with.

Example

Find $\displaystyle\int \sin^3 x \cos^2 x \, dx.$

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This is an example of case $\boxed{1}$ since 3 is an odd number.

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Find $\int \sin^3 x \cos^2 x \, dx$.

This is an example of case $\boxed{1}$ since 3 is an odd number. We are going to be using the substitution $u = \cos x$ and $-du = \sin x \, dx$.

Example

Find $\int \sin^3 x \cos^2 x \, dx$.

This is an example of case $\boxed{1}$ since 3 is an odd number. We are going to be using the substitution $u = \cos x$ and $-du = \sin x \, dx$.

$$\int \sin^3 x \cos^2 x \, dx = \int \sin^2 x \cos^2 x \sin x \, dx$$

$$=$$

$$=$$

$$=$$

$$=$$

Example

Find $\int \sin^3 x \cos^2 x \, dx$.

This is an example of case $\boxed{1}$ since 3 is an odd number. We are going to be using the substitution $u = \cos x$ and $-du = \sin x \, dx$.

$$\begin{aligned}\int \sin^3 x \cos^2 x \, dx &= \int \sin^2 x \cos^2 x \sin x \, dx \\ &= \int (1 - \cos^2 x) \cos^2 x \sin x \, dx \\ &= \\ &= \\ &= \end{aligned}$$

Example

Find $\int \sin^3 x \cos^2 x \, dx$.

This is an example of case $\boxed{1}$ since 3 is an odd number. We are going to be using the substitution $u = \cos x$ and $-du = \sin x \, dx$.

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Example

Find $\displaystyle\int \sin^3 x \cos^2 x \, dx$.

This is an example of case 1 since 3 is an odd number. We are going to be using the substitution $u = \cos x$ and $-du = \sin x \, dx$.

$$\begin{aligned}
\int \sin^3 x \cos^2 x \, dx &= \int \sin^2 x \cos^2 x \sin x \, dx \\
&= \int (1 - \cos^2 x) \cos^2 x \sin x \, dx \\
&= \int (1 - u^2) \, u^2 \, (-du) \\
&= \int u^4 - u^2 \, du \\
&= \frac{u^5}{5} - \frac{u^3}{3} + C = \frac{\cos^5 x}{5} - \frac{\cos^3 x}{3} + C.
\end{aligned}$$

Example

Find $\int \cos^5 x \, dx$.

Example

Find $\int \cos^5 x\,dx$.

This is an example of case 2 since 5 is an odd number. We are going to be using the substitution $u = \sin x$ and $du = \cos x\,dx$.

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This is an example of case 2 since 5 is an odd number. We are going to be using the substitution $u = \sin x$ and $du = \cos x\,dx$.

$$\int \cos^5 x\,dx = \int \cos^4 x \cos x\,dx$$

$$=$$

$$=$$

$$=$$

Example

Find $\displaystyle\int \cos^5 x\,dx$.

This is an example of case $\boxed{2}$ since 5 is an odd number. We are going to be using the substitution $u = \sin x$ and $du = \cos x\,dx$.

$$\begin{aligned}\int \cos^5 x\,dx &= \int \cos^4 x \cos x\,dx \\ &= \int (\cos^2 x)^2 \cos x\,dx \\ &= \\ &= \end{aligned}$$

Example

Find $\int \cos^5 x\, dx$.

This is an example of case 2 since 5 is an odd number. We are going to be using the substitution $u = \sin x$ and $du = \cos x\, dx$.

$$\begin{aligned}\int \cos^5 x\, dx &= \int \cos^4 x \cos x\, dx \\ &= \int (\cos^2 x)^2 \cos x\, dx \\ &= \int (1 - \sin^2 x)^2 \cos x\, dx \\ &= \end{aligned}$$

Example

Find $\int \cos^5 x\, dx$.

This is an example of case $\boxed{2}$ since 5 is an odd number. We are going to be using the substitution $u = \sin x$ and $du = \cos x\, dx$.

$$\begin{aligned}\int \cos^5 x\, dx &= \int \cos^4 x \cos x\, dx \\ &= \int (\cos^2 x)^2 \cos x\, dx \\ &= \int (1 - \sin^2 x)^2 \cos x\, dx \\ &= \int (1 - u^2)^2\, du = \dots\end{aligned}$$

$$\sin^2 x = \frac{1 - \cos 2x}{2} \qquad \cos^2 x = \frac{1 + \cos 2x}{2}$$

Example

Find $\int \sin^2 x \cos^4 x \, dx$.

Both 2 and 4 are even so this is an example of case 3.

$$\sin^2 x = \frac{1-\cos 2x}{2} \qquad \cos^2 x = \frac{1+\cos 2x}{2}$$

Example

Find $\displaystyle\int \sin^2 x \cos^4 x \, dx$.

Both 2 and 4 are even so this is an example of case 3.

$$\begin{aligned} \int \sin^2 x \cos^4 x \, dx &= \int \left(\frac{1-\cos 2x}{2}\right)\left(\frac{1+\cos 2x}{2}\right)^2 dx \\ &= \\ &= \\ &= \end{aligned}$$

$$\sin^2 x = \frac{1 - \cos 2x}{2} \qquad \cos^2 x = \frac{1 + \cos 2x}{2}$$

Example

Find $\int \sin^2 x \cos^4 x \, dx$.

Both 2 and 4 are even so this is an example of case 3.

$$\begin{aligned}
\int \sin^2 x \cos^4 x \, dx &= \int \left(\frac{1 - \cos 2x}{2} \right) \left(\frac{1 + \cos 2x}{2} \right)^2 dx \\
&= \frac{1}{8} \int (1 - \cos 2x)(1 + 2\cos 2x + \cos^2 2x) \, dx \\
&= \frac{1}{8} \int 1 + \cos 2x - \cos^2 2x - \cos^3 2x \;\, dx \\
&=
\end{aligned}$$

$$\sin^2 x = \frac{1 - \cos 2x}{2} \qquad \cos^2 x = \frac{1 + \cos 2x}{2}$$

Example

Find $\int \sin^2 x \cos^4 x \, dx$.

Both 2 and 4 are even so this is an example of case $\boxed{3}$.

$$\begin{aligned}
\int \sin^2 x \cos^4 x \, dx &= \int \left(\frac{1-\cos 2x}{2}\right)\left(\frac{1+\cos 2x}{2}\right)^2 \, dx \\
&= \frac{1}{8}\int (1-\cos 2x)(1 + 2\cos 2x + \cos^2 2x) \, dx \\
&= \frac{1}{8}\int \underbrace{1 + \cos 2x}_{\text{easy}} - \cos^2 2x - \cos^3 2x \;\, dx \\
&=
\end{aligned}$$

$$\sin^2 x = \frac{1-\cos 2x}{2} \qquad \cos^2 x = \frac{1+\cos 2x}{2}$$

Example

Find $\int \sin^2 x \cos^4 x \, dx$.

Both 2 and 4 are even so this is an example of case $\boxed{3}$.

$$\begin{aligned}
\int \sin^2 x \cos^4 x \, dx &= \int \left(\frac{1-\cos 2x}{2}\right)\left(\frac{1+\cos 2x}{2}\right)^2 dx \\
&= \frac{1}{8}\int (1-\cos 2x)(1+2\cos 2x+\cos^2 2x)\, dx \\
&= \frac{1}{8}\int 1+\cos 2x - \underbrace{\cos^2 2x}_{\text{case } \boxed{3}} - \cos^3 2x \; dx \\
&=
\end{aligned}$$

$$\sin^2 x = \frac{1 - \cos 2x}{2} \qquad \cos^2 x = \frac{1 + \cos 2x}{2}$$

Example

Find $\displaystyle\int \sin^{2} x \cos^{4} x \, dx$.

Both 2 and 4 are even so this is an example of case $\boxed{3}$.

$$\begin{aligned}
\int \sin^{2} x \cos^{4} x \, dx &= \int \left(\frac{1 - \cos 2x}{2}\right) \left(\frac{1 + \cos 2x}{2}\right)^2 dx \\
&= \frac{1}{8} \int (1 - \cos 2x)(1 + 2\cos 2x + \cos^2 2x) \, dx \\
&= \frac{1}{8} \int 1 + \cos 2x - \cos^2 2x - \underbrace{\cos^3 2x}_{\text{case } \boxed{2}} \, dx \\
&=
\end{aligned}$$

$$\sin^2 x = \frac{1 - \cos 2x}{2} \qquad \cos^2 x = \frac{1 + \cos 2x}{2}$$

Example

Find $\int \sin^2 x \cos^4 x \, dx$.

Both 2 and 4 are even so this is an example of case 3.

$$\begin{aligned}
\int \sin^2 x \cos^4 x \, dx &= \int \left(\frac{1 - \cos 2x}{2}\right)\left(\frac{1 + \cos 2x}{2}\right)^2 \, dx \\
&= \frac{1}{8} \int (1 - \cos 2x)(1 + 2\cos 2x + \cos^2 2x) \, dx \\
&= \frac{1}{8} \int 1 + \cos 2x - \cos^2 2x - \cos^3 2x \;\, dx \\
&= \frac{1}{8} \left(x + \frac{1}{2} \sin 2x - \int \cos^2 2x \, dx - \int \cos^3 2x \, dx \right)
\end{aligned}$$

$$\sin^2 x = \frac{1 - \cos 2x}{2} \qquad \cos^2 x = \frac{1 + \cos 2x}{2}$$

For the orange integral we calculate that

$$\int \cos^2 2x \, dx =$$

$$=$$

$$=$$

For the green integral we calculate that

$$\int \cos^3 2x \, dx =$$

$$=$$

$$=$$

$$\sin^2 x = \frac{1 - \cos 2x}{2} \qquad \cos^2 x = \frac{1 + \cos 2x}{2}$$

For the orange integral we calculate that

$$\int \cos^2 2x \, dx = \frac{1}{2} \int 1 + \cos 4x \, dx$$

$$=$$

$$=$$

For the green integral we calculate that

$$\int \cos^3 2x \, dx =$$

$$=$$

$$=$$

$$\sin^2 x = \frac{1-\cos 2x}{2} \qquad \cos^2 x = \frac{1+\cos 2x}{2}$$

For the orange integral we calculate that

$$\begin{aligned}\int \cos^2 2x\,dx &= \frac{1}{2}\int 1+\cos 4x\,dx \\ &= \frac{1}{2}\left(x+\frac{1}{4}\sin 4x\right)+C_1 \\ &= \frac{x}{2}+\frac{1}{8}\sin 4x + C_1.\end{aligned}$$

For the green integral we calculate that

$$\begin{aligned}\int \cos^3 2x\,dx &= \\ &= \\ &= \end{aligned}$$

$$\sin^2 x = \frac{1 - \cos 2x}{2} \qquad \cos^2 x = \frac{1 + \cos 2x}{2}$$

For the orange integral we calculate that

$$\begin{aligned} \int \cos^2 2x \, dx &= \frac{1}{2} \int 1 + \cos 4x \, dx \\ &= \frac{1}{2}\left(x + \frac{1}{4} \sin 4x \right) + C_1 \\ &= \frac{x}{2} + \frac{1}{8} \sin 4x + C_1. \end{aligned}$$

For the green integral we calculate that

$$\begin{aligned} \int \cos^3 2x \, dx &= \int (1 - \sin^2 2x) \cos 2x \, dx \\ &= \\ &= \end{aligned}$$

$$\sin^2 x = \frac{1-\cos 2x}{2} \qquad \cos^2 x = \frac{1+\cos 2x}{2}$$

For the orange integral we calculate that

$$\begin{aligned}\int \cos^2 2x\,dx &= \frac{1}{2}\int 1+\cos 4x\,dx \\ &= \frac{1}{2}\left(x+\frac{1}{4}\sin 4x\right)+C_1 \\ &= \frac{x}{2}+\frac{1}{8}\sin 4x + C_1.\end{aligned}$$

For the green integral we calculate that

$$\begin{aligned}\int \cos^3 2x\,dx &= \int (1-\sin^2 2x)\cos 2x\,dx \\ &= \frac{1}{2}\int (1-u^2)\,du \\ &= \end{aligned}$$

$$\sin^2 x = \frac{1-\cos 2x}{2} \qquad \cos^2 x = \frac{1+\cos 2x}{2}$$

For the orange integral we calculate that

$$\begin{aligned}\int \cos^2 2x\,dx &= \frac{1}{2}\int 1+\cos 4x\,dx \\ &= \frac{1}{2}\left(x+\frac{1}{4}\sin 4x\right)+C_1 \\ &= \frac{x}{2}+\frac{1}{8}\sin 4x + C_1.\end{aligned}$$

For the green integral we calculate that

$$\begin{aligned}\int \cos^3 2x\,dx &= \int (1-\sin^2 2x)\cos 2x\,dx \\ &= \frac{1}{2}\int (1-u^2)\,du \\ &= \frac{1}{2}\left(\sin 2x - \frac{1}{3}\sin^3 2x\right)+C_2\end{aligned}$$

$$\sin^2 x = \frac{1 - \cos 2x}{2} \qquad \cos^2 x = \frac{1 + \cos 2x}{2}$$

Now let's put it all together.

$$\int \sin^2 x \cos^4 x \, dx$$

$$= \frac{1}{8}\left(x + \frac{1}{2}\sin 2x - \int \cos^2 2x \, dx - \int \cos^3 2x \, dx\right)$$

$$=$$

$$=$$

$$\sin^2 x = \frac{1-\cos 2x}{2} \qquad \cos^2 x = \frac{1+\cos 2x}{2}$$

Now let's put it all together.

$$\int \sin^2 x \cos^4 x \, dx$$

$$= \frac{1}{8}\left(x + \frac{1}{2}\sin 2x - \int \cos^2 2x \, dx - \int \cos^3 2x \, dx\right)$$

$$= \frac{1}{8}\left(x + \frac{1}{2}\sin 2x - \left(\frac{x}{2} + \frac{1}{8}\sin 4x\right) - \frac{1}{2}\left(\sin 2x - \frac{1}{3}\sin^3 2x\right)\right) + C$$

$$=$$

$$\sin^2 x = \frac{1-\cos 2x}{2} \qquad \cos^2 x = \frac{1+\cos 2x}{2}$$

Now let's put it all together.

$$\begin{aligned}
&\int \sin^2 x \cos^4 x\, dx \\
&\qquad = \frac{1}{8}\left(x + \frac{1}{2}\sin 2x - \int \cos^2 2x\, dx - \int \cos^3 2x\, dx\right) \\
&\qquad = \frac{1}{8}\left(x + \frac{1}{2}\sin 2x - \left(\frac{x}{2} + \frac{1}{8}\sin 4x\right) - \frac{1}{2}\left(\sin 2x - \frac{1}{3}\sin^3 2x\right)\right) + C \\
&\qquad = \frac{x}{16} - \frac{1}{64}\sin 4x + \frac{1}{48}\sin^3 2x + C.
\end{aligned}$$

$$\sin^2 x = \frac{1 - \cos 2x}{2} \qquad \cos^2 x = \frac{1 + \cos 2x}{2}$$

Example

Find $\displaystyle\int_0^{\frac{\pi}{4}} \sqrt{1 + \cos 4x}\, dx$.

$$\sin^2 x = \frac{1-\cos 2x}{2} \qquad \cos^2 x = \frac{1+\cos 2x}{2}$$

Example

Find $\displaystyle\int_0^{\frac{\pi}{4}} \sqrt{1+\cos 4x}\,dx$.

To get rid of the $\sqrt{}$, we are going to use this .

$$\sin^2 x = \frac{1-\cos 2x}{2} \qquad \cos^2 x = \frac{1+\cos 2x}{2}$$

Example

Find $\int_0^{\frac{\pi}{4}} \sqrt{1+\cos 4x}\, dx$.

To get rid of the $\sqrt{\ }$, we are going to use this . Replacing x by $2x$ and rearranging, this is

$$1 + \cos 4x = 2\cos^2 2x.$$

$$\sin^2 x = \frac{1-\cos 2x}{2} \qquad \cos^2 x = \frac{1+\cos 2x}{2}$$

Example

Find $\int_0^{\frac{\pi}{4}} \sqrt{1+\cos 4x}\, dx$.

To get rid of the $\sqrt{\ }$, we are going to use this . Replacing x by $2x$ and rearranging, this is

$$1+\cos 4x = 2\cos^2 2x.$$

Hence

$$\int_0^{\frac{\pi}{4}} \sqrt{1+\cos 4x}\, dx = \int_0^{\frac{\pi}{4}} \sqrt{2\cos^2 2x}\, dx =$$

$$=$$

$$=$$

$$\sin^2 x = \frac{1-\cos 2x}{2} \qquad \cos^2 x = \frac{1+\cos 2x}{2}$$

Example

Find $\int_0^{\frac{\pi}{4}} \sqrt{1+\cos 4x}\, dx$.

To get rid of the $\sqrt{\ }$, we are going to use this . Replacing x by $2x$ and rearranging, this is

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Hence

$$\int_0^{\frac{\pi}{4}} \sqrt{1+\cos 4x}\, dx = \int_0^{\frac{\pi}{4}} \sqrt{2\cos^2 2x}\, dx = \int_0^{\frac{\pi}{4}} \sqrt{2}\sqrt{\cos^2 2x}\, dx$$

$$=$$

$$=$$

$$\sin^2 x = \frac{1 - \cos 2x}{2} \qquad \cos^2 x = \frac{1 + \cos 2x}{2}$$

Example

Find $\int_0^{\frac{\pi}{4}} \sqrt{1 + \cos 4x}\, dx$.

To get rid of the $\sqrt{\ }$, we are going to use this . Replacing x by $2x$ and rearranging, this is

$$1 + \cos 4x = 2\cos^2 2x.$$

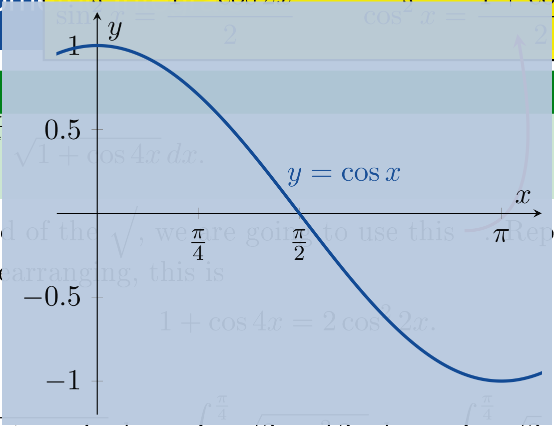
Hence

$$\int_0^{\frac{\pi}{4}} \sqrt{1 + \cos 4x}\, dx = \int_0^{\frac{\pi}{4}} \sqrt{2\cos^2 2x}\, dx = \int_0^{\frac{\pi}{4}} \sqrt{2}\sqrt{\cos^2 2x}\, dx$$

$$= \sqrt{2} \int_0^{\frac{\pi}{4}} |\cos 2x|\ dx$$

$$=$$

$$\sin^2 x = \frac{1 - \cos 2x}{2} \qquad \cos^2 x = \frac{1 + \cos 2x}{2}$$



Example

Find $\displaystyle\int_0^{\frac{\pi}{4}} \sqrt{1 + \cos 4x}\,dx$.

To get rid of the $\sqrt{\ }$, we are going to use this. Replacing x by $2x$ and rearranging, this is

$$1 + \cos 4x = 2\cos^2 2x.$$

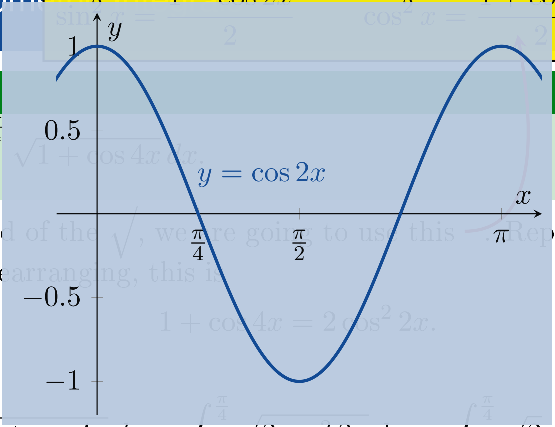
Hence

$$\int_0^{\frac{\pi}{4}} \sqrt{1 + \cos 4x}\,dx = \int_0^{\frac{\pi}{4}} \sqrt{2\cos^2 2x}\,dx = \int_0^{\frac{\pi}{4}} \sqrt{2}\sqrt{\cos^2 2x}\,dx$$

$$= \sqrt{2}\int_0^{\frac{\pi}{4}} |\cos 2x|\,dx$$

$$=$$

$$\sin^2 x = \frac{1-\cos 2x}{2} \qquad \cos^2 x = \frac{1+\cos 2x}{2}$$



Example

Find $\displaystyle\int_0^{\frac{\pi}{4}} \sqrt{1+\cos 4x}\, dx$.

To get rid of the $\sqrt{}$, we are going to use this. Replacing x by $2x$ and rearranging, this is

$$1+\cos 4x = 2\cos^2 2x.$$

Hence

$$\int_0^{\frac{\pi}{4}} \sqrt{1+\cos 4x}\, dx = \int_0^{\frac{\pi}{4}} \sqrt{2\cos^2 2x}\, dx = \int_0^{\frac{\pi}{4}} \sqrt{2}\sqrt{\cos^2 2x}\, dx$$

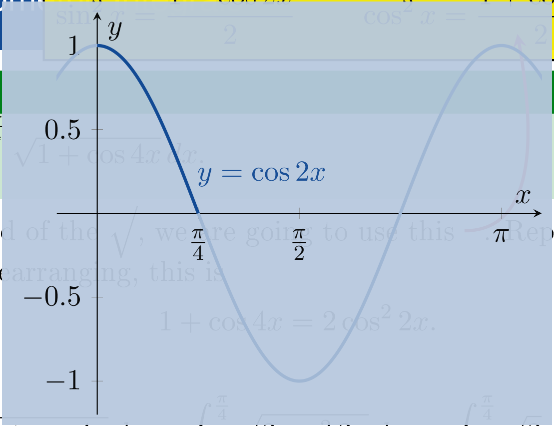
$$= \sqrt{2}\int_0^{\frac{\pi}{4}} \left|\cos 2x\right|\, dx$$

$$=$$

$$\sin^2 x = \frac{1-\cos 2x}{2} \qquad \cos^2 x = \frac{1+\cos 2x}{2}$$

Example

Find $\displaystyle\int_0^{\frac{\pi}{4}} \sqrt{1+\cos 4x}\, dx$.



To get rid of the $\sqrt{\ }$, we are going to use this. Replacing x by $2x$ and rearranging, this is

$$1 + \cos 4x = 2\cos^2 2x.$$

Hence

$$\int_0^{\frac{\pi}{4}} \sqrt{1+\cos 4x}\, dx = \int_0^{\frac{\pi}{4}} \sqrt{2\cos^2 2x}\, dx = \int_0^{\frac{\pi}{4}} \sqrt{2}\sqrt{\cos^2 2x}\, dx$$

$$= \sqrt{2}\int_0^{\frac{\pi}{4}} |\cos 2x|\ dx$$

$$=$$

$$\sin^2 x = \frac{1 - \cos 2x}{2} \qquad \cos^2 x = \frac{1 + \cos 2x}{2}$$

Example

Find $\int_0^{\frac{\pi}{4}} \sqrt{1 + \cos 4x}\, dx$.

To get rid of the $\sqrt{\ }$, we are going to use this . Replacing x by $2x$ and rearranging, this is

$$1 + \cos 4x = 2\cos^2 2x.$$

Hence

$$\int_0^{\frac{\pi}{4}} \sqrt{1 + \cos 4x}\, dx = \int_0^{\frac{\pi}{4}} \sqrt{2\cos^2 2x}\, dx = \int_0^{\frac{\pi}{4}} \sqrt{2}\sqrt{\cos^2 2x}\, dx$$

$$= \sqrt{2} \int_0^{\frac{\pi}{4}} \cos 2x \quad dx$$

$$=$$

$$\sin^2 x = \frac{1-\cos 2x}{2} \qquad \cos^2 x = \frac{1+\cos 2x}{2}$$

Example

Find $\displaystyle\int_0^{\frac{\pi}{4}} \sqrt{1+\cos 4x}\,dx.$

To get rid of the $\sqrt{\ }$, we are going to use this . Replacing x by $2x$ and rearranging, this is

$$1+\cos 4x = 2\cos^2 2x.$$

Hence

$$\int_0^{\frac{\pi}{4}} \sqrt{1+\cos 4x}\,dx = \int_0^{\frac{\pi}{4}} \sqrt{2\cos^2 2x}\,dx = \int_0^{\frac{\pi}{4}} \sqrt{2}\sqrt{\cos^2 2x}\,dx$$

$$= \sqrt{2}\int_0^{\frac{\pi}{4}} \cos 2x \quad dx$$

$$= \sqrt{2}\left[\frac{\sin 2x}{2}\right]_0^{\frac{\pi}{4}} = \sqrt{2}\left(\frac{1}{2}-0\right) = \frac{\sqrt{2}}{2}.$$

$$\sec^2 x = 1 + \tan^2 x \qquad \frac{d}{dx}\tan x = \sec^2 x$$

Example

Find $\displaystyle\int \tan^4 x\,dx$.

$$\sec^2 x = 1 + \tan^2 x \qquad \frac{d}{dx}\tan x = \sec^2 x$$

Example

Find $\displaystyle\int \tan^4 x\, dx$.

Solution

$$\int \tan^4 x\, dx = \int \tan^2 x \cdot \tan^2 x\, dx = \int \tan^2 x \cdot (\sec^2 x - 1)\, dx$$

$$= \int \tan^2 x \sec^2 x\, dx - \int \tan^2 x\, dx$$

$$= \int \tan^2 x \sec^2 x\, dx - \int (\sec^2 x - 1)\, dx$$

$$= \int \tan^2 x \sec^2 x\, dx - \int \sec^2 x\, dx + \int dx$$

In the first integral, we let

$$u = \tan x, \qquad du = \sec^2 x\, dx$$

and have

$$\int u^2\, du = \frac{1}{3} u^3 + C_1.$$

The remaining integrals are standard forms, so

$$\int \tan^4 x\, dx = \frac{1}{3} \tan^3 x - \tan x + x + C. \qquad \blacksquare$$

$$\int u\,dv = uv - \int v\,du$$

Example

Find $\displaystyle\int \sec^3 x\,dx$.

$$\int u\,dv = uv - \int v\,du$$

Example

Find $\int \sec^3 x\,dx$.

Think of this as $\int \sec x \sec^2 x\,dx$. We are going to use Integration by Parts.

$$\int u\,dv = uv - \int v\,du$$

Example

Find $\displaystyle\int \sec^3 x\,dx$.

Think of this as $\displaystyle\int \sec x \sec^2 x\,dx$. We are going to use Integration by Parts. Let $u = \sec x$ and $dv = \sec^2 x\,dx$. Then $du = \sec x \tan x\,dx$ and $v = \tan x$.

$$\int u\,dv = uv - \int v\,du$$

Example

Find $\displaystyle\int \sec^3 x\,dx$.

Think of this as $\displaystyle\int \sec x \sec^2 x\,dx$. We are going to use Integration by Parts. Let $u = \sec x$ and $dv = \sec^2 x\,dx$. Then $du = \sec x \tan x\,dx$ and $v = \tan x$. Hence

$$\begin{aligned}\int \sec x \sec^2 x\,dx &= \sec x \tan x - \int \tan x \sec x \tan x\,dx \\ &= \\ &= \\ &= \end{aligned}$$

$$\int u\,dv = uv - \int v\,du$$

Example

Find $\displaystyle\int \sec^3 x\,dx$.

Think of this as $\displaystyle\int \sec x \sec^2 x\,dx$. We are going to use Integration by Parts. Let $u = \sec x$ and $dv = \sec^2 x\,dx$. Then $du = \sec x \tan x\,dx$ and $v = \tan x$. Hence

$$\begin{aligned}\int \sec x \sec^2 x\,dx &= \sec x \tan x - \int \tan x \sec x \tan x\,dx \\ &= \sec x \tan x - \int \tan^2 x\ \sec x\,dx \\ &= \\ &= \end{aligned}$$

$$\int u\,dv = uv - \int v\,du$$

Example

Find $\displaystyle\int \sec^3 x\,dx$.

Think of this as $\displaystyle\int \sec x \sec^2 x\,dx$. We are going to use Integration by Parts. Let $u = \sec x$ and $dv = \sec^2 x\,dx$. Then $du = \sec x \tan x\,dx$ and $v = \tan x$. Hence

$$\begin{aligned}\int \sec x \sec^2 x\,dx &= \sec x \tan x - \int \tan x \sec x \tan x\,dx \\ &= \sec x \tan x - \int \tan^2 x \sec x\,dx \\ &= \sec x \tan x - \int (\sec^2 x - 1) \sec x\,dx \\ &= \end{aligned}$$

$$\int u\,dv = uv - \int v\,du$$

Example

Find $\displaystyle\int \sec^3 x\,dx$.

Think of this as $\displaystyle\int \sec x \sec^2 x\,dx$. We are going to use Integration by Parts. Let $u = \sec x$ and $dv = \sec^2 x\,dx$. Then $du = \sec x \tan x\,dx$ and $v = \tan x$. Hence

$$\begin{aligned}
\int \sec x \sec^2 x\,dx &= \sec x \tan x - \int \tan x \sec x \tan x\,dx \\
&= \sec x \tan x - \int \tan^2 x \sec x\,dx \\
&= \sec x \tan x - \int (\sec^2 x - 1) \sec x\,dx \\
&= \sec x \tan x + \int \sec x\,dx - \int \sec^3 x\,dx.
\end{aligned}$$

So we have

$$\int \sec^3 x\,dx = \sec x \tan x + \int \sec x\,dx - \int \sec^3 x\,dx.$$

So we have

$$\int \sec^3 x\, dx = \sec x \tan x + \int \sec x\, dx - \int \sec^3 x\, dx.$$

Thus

$$2\int \sec^3 x\, dx = \sec x \tan x + \int \sec x\, dx$$

So we have

$$\int \sec^3 x\,dx = \sec x \tan x + \int \sec x\,dx - \int \sec^3 x\,dx.$$

Thus

$$2\int \sec^3 x\,dx = \sec x \tan x + \int \sec x\,dx$$

and

$$\begin{aligned}\int \sec^3 x\,dx &= \frac{1}{2}\sec x \tan x + \frac{1}{2}\int \sec x\,dx\\ &= \frac{1}{2}\sec x \tan x + \frac{1}{2}\ln|\sec x + \tan x| + C.\end{aligned}$$

$$\sec^2 x = 1 + \tan^2 x \qquad \frac{d}{dx}\tan x = \sec^2 x$$

Example

Find $\displaystyle\int \tan^4 x \sec^4 x \, dx$.

$$\sec^2 x = 1 + \tan^2 x \qquad \frac{d}{dx}\tan x = \sec^2 x$$

Example

Find $\displaystyle\int \tan^4 x \sec^4 x \, dx$.

Solution

$$\int (\tan^4 x)(\sec^4 x)\, dx = \int (\tan^4 x)(1 + \tan^2 x)(\sec^2 x)\, dx \qquad \sec^2 x = 1 + \tan^2 x$$

$$= \int (\tan^4 x + \tan^6 x)(\sec^2 x)\, dx$$

$$= \int (\tan^4 x)(\sec^2 x)\, dx + \int (\tan^6 x)(\sec^2 x)\, dx$$

$$= \int u^4\, du + \int u^6\, du = \frac{u^5}{5} + \frac{u^7}{7} + C \qquad u = \tan x,\ du = \sec^2 x\, dx$$

$$= \frac{\tan^5 x}{5} + \frac{\tan^7 x}{7} + C \qquad \blacksquare$$

How do we calculate

$$\int \sin mx \sin nx \, dx$$

or

$$\int \sin mx \cos nx \, dx$$

or

$$\int \cos mx \cos nx \, dx$$

?

How do we calculate

$$\int \sin mx \sin nx \, dx$$

or

$$\int \sin mx \cos nx \, dx$$

or

$$\int \cos mx \cos nx \, dx$$

?

It is possible to use integration by parts (twice), but there is an easier way.

$$\cos(A+B) = \cos A \cos B - \sin A \sin B$$

$$\cos(mx - nx) - \cos(mx + nx) =$$

$$\cos(A+B) = \cos A \cos B - \sin A \sin B$$

$$\begin{aligned}\cos(mx-nx) - \cos(mx+nx) = {} & \cos mx \cos nx + \sin mx \sin nx \\ & - \cos mx \cos nx + \sin mx \sin nx\end{aligned}$$

$$\cos(A+B) = \cos A \cos B - \sin A \sin B$$

$$\begin{aligned}\cos(mx-nx) - \cos(mx+nx) &= \cos mx \cos nx + \sin mx \sin nx \\ &\quad - \cos mx \cos nx + \sin mx \sin nx \\ &= 2 \sin mx \sin nx\end{aligned}$$

$$\cos(A+B) = \cos A \cos B - \sin A \sin B$$

$$\begin{aligned}\cos(mx-nx) - \cos(mx+nx) &= \cos mx \cos nx + \sin mx \sin nx \\ &\quad - \cos mx \cos nx + \sin mx \sin nx \\ &= 2 \sin mx \sin nx\end{aligned}$$

Therefore

$$\sin mx \sin nx = \frac{1}{2}\big(\cos(m-n)x - \cos(m+n)x\big).$$

$$\cos(A+B) = \cos A \cos B - \sin A \sin B$$

$$\begin{aligned}\cos(mx-nx) - \cos(mx+nx) &= \cos mx \cos nx + \sin mx \sin nx \\ &\quad - \cos mx \cos nx + \sin mx \sin nx \\ &= 2 \sin mx \sin nx\end{aligned}$$

Therefore

$$\sin mx \sin nx = \frac{1}{2}\big(\cos(m-n)x - \cos(m+n)x \big).$$

Similarly

$$\sin mx \cos nx = \frac{1}{2}\big(\sin(m-n)x + \sin(m+n)x \big)$$

and

$$\cos mx \cos nx = \frac{1}{2}\big(\cos(m-n)x + \cos(m+n)x \big).$$

$$\sin mx \cos nx = \frac{1}{2}\big(\sin(m-n)x + \sin(m+n)x\big)$$

Example

Find $\displaystyle\int \sin 3x \cos 5x \, dx$.

$$\sin mx \cos nx = \frac{1}{2}\left(\sin(m-n)x + \sin(m+n)x\right)$$

Example

Find $\displaystyle\int \sin 3x \cos 5x \, dx$.

Solution From Equation (4) with $m = 3$ and $n = 5$, we get

$$\begin{aligned}
\int \sin 3x \cos 5x \, dx &= \frac{1}{2}\int [\sin(-2x) + \sin 8x]\, dx \\
&= \frac{1}{2}\int (\sin 8x - \sin 2x)\, dx \\
&= -\frac{\cos 8x}{16} + \frac{\cos 2x}{4} + C.
\end{aligned}$$

$$\cos mx \cos nx = \frac{1}{2}\big(\cos(m-n)x + \cos(m+n)x\big)$$

Example

Find $\displaystyle\int \cos 3x \cos 2x \, dx$.

$\cos mx \cos nx = \frac{1}{2}\big(\cos(m-n)x + \cos(m+n)x \big)$

Example

Find $\displaystyle\int \cos 3x \cos 2x \, dx$.

We have $m = 3$ and $n = 2$. It follows that

$$\begin{aligned}\int \cos 3x \cos 2x \, dx &= \frac{1}{2}\int \cos(3-2)x \, dx + \frac{1}{2}\int \cos(3+2)x \, dx \\ &= \dots\end{aligned}$$

Next Time

- 8.4 Trigonometric Substitutions
- 8.5 Integration of Rational Functions by Partial Fractions
- 8.8 Improper Integrals