Lecture 8

- Matrices for Linear Transformations
- Similarity
- Complex Numbers

Matrices for Linear Transformations

Matrices for Linear Transformations

Example

Suppose that $T : \mathbb{R}^2 \to \mathbb{R}^3$ is a linear transformation which satisfies

$$T(\mathbf{e}_1) = \begin{bmatrix} 5 \\ -7 \\ 2 \end{bmatrix} \qquad \text{and} \qquad T(\mathbf{e}_2) = \begin{bmatrix} -3 \\ 8 \\ 0 \end{bmatrix}$$

where $\mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\mathbf{e}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. Find a formula for $T(\mathbf{x})$.

$$T(\mathbf{e}_1) = \begin{bmatrix} 5 \\ -7 \\ 2 \end{bmatrix} \qquad T(\mathbf{e}_2) = \begin{bmatrix} -3 \\ 8 \\ 0 \end{bmatrix}$$

Because T is linear and because

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix} = x_1\mathbf{e}_1 + x_2\mathbf{e}_2$$

$$T(\mathbf{e}_1) = \begin{bmatrix} 5 \\ -7 \\ 2 \end{bmatrix} \qquad T(\mathbf{e}_2) = \begin{bmatrix} -3 \\ 8 \\ 0 \end{bmatrix}$$

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we have that

$$T(\mathbf{x}) = T(x_1\mathbf{e}_1 + x_2\mathbf{e}_2) = x_1 T(\mathbf{e}_1) + x_2 T(\mathbf{e}_2)$$

$$=$$

$$= \qquad =$$

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we have that

$$\begin{aligned} T(\mathbf{x}) &= T(x_1\mathbf{e}_1 + x_2\mathbf{e}_2) = x_1T(\mathbf{e}_1) + x_2T(\mathbf{e}_2) \\ &= x_1 \begin{bmatrix} 5 \\ -7 \\ 2 \end{bmatrix} + x_2 \begin{bmatrix} -3 \\ 8 \\ 0 \end{bmatrix} = \begin{bmatrix} 5x_1 - 3x_2 \\ -7x_1 + 8x_2 \\ 2x_1 \end{bmatrix} \end{aligned}$$

$$= \qquad\qquad =$$

$$T(\mathbf{e}_1) = \begin{bmatrix} 5 \\ -7 \\ 2 \end{bmatrix} \qquad T(\mathbf{e}_2) = \begin{bmatrix} -3 \\ 8 \\ 0 \end{bmatrix}$$

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$$T(\mathbf{e}_1) = \begin{bmatrix} 5 \\ -7 \\ 2 \end{bmatrix} \qquad T(\mathbf{e}_2) = \begin{bmatrix} -3 \\ 8 \\ 0 \end{bmatrix}$$

Because T is linear and because

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix} = x_1\mathbf{e}_1 + x_2\mathbf{e}_2$$

we have that

$$\begin{aligned} T(\mathbf{x}) &= T(x_1\mathbf{e}_1 + x_2\mathbf{e}_2) = x_1T(\mathbf{e}_1) + x_2T(\mathbf{e}_2) \\ &= x_1 \begin{bmatrix} 5 \\ -7 \\ 2 \end{bmatrix} + x_2 \begin{bmatrix} -3 \\ 8 \\ 0 \end{bmatrix} = \begin{bmatrix} 5x_1 - 3x_2 \\ -7x_1 + 8x_2 \\ 2x_1 \end{bmatrix} \\ &= \begin{bmatrix} 5 & -3 \\ -7 & 8 \\ 2 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} T(\mathbf{e}_1) & T(\mathbf{e}_2) \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = A\mathbf{x}. \end{aligned}$$

Theorem

Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation. Then there exists a unique matrix A such that

$$T(\mathbf{x}) = A\mathbf{x}$$

for all $\mathbf{x} \in \mathbb{R}^n$.

Theorem

Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation. Then there exists a unique matrix A such that

$$T(\mathbf{x}) = A\mathbf{x}$$

for all $\mathbf{x} \in \mathbb{R}^n$.

In fact, A is the matrix

$$A = \begin{bmatrix} T(\mathbf{e}_1) & T(\mathbf{e}_2) & \cdots & T(\mathbf{e}_n) \end{bmatrix}$$

where $\mathbf{e}_j$ is the jth column of the identity matrix I_n.

Proof.

First suppose that $\mathbf{x} = x_1\mathbf{e}_1 + x_2\mathbf{e}_2 + \ldots + x_n\mathbf{e}_n$. Then

$$\begin{aligned} T(\mathbf{x}) &= T(x_1\mathbf{e}_1 + x_2\mathbf{e}_2 + \ldots + x_n\mathbf{e}_n) \\ &= x_1T(\mathbf{e}_1) + x_2T(\mathbf{e}_2) + \ldots + x_nT(\mathbf{e}_n) \\ &= \begin{bmatrix} T(\mathbf{e}_1) & T(\mathbf{e}_2) & \cdots & T(\mathbf{e}_n) \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = A\mathbf{x}. \end{aligned}$$

This proves the existance part of the theorem.

Proof continued.

Now we need to prove that such a matrix is unique: Suppose that $T(\mathbf{x}) = A\mathbf{x}$ and $T(\mathbf{x}) = B\mathbf{x}$.

Proof continued.

Now we need to prove that such a matrix is unique: Suppose that $T(\mathbf{x}) = A\mathbf{x}$ and $T(\mathbf{x}) = B\mathbf{x}$. Then for each j we have

$$A\mathbf{e}_j = T(\mathbf{e}_j) = B\mathbf{e}_j =$$

Matrices for Linear Transformations

Proof continued.

Now we need to prove that such a matrix is unique: Suppose that $T(\mathbf{x}) = A\mathbf{x}$ and $T(\mathbf{x}) = B\mathbf{x}$. Then for each j we have

$$\begin{array}{c}\text{the } j\text{th} \\ \text{column of} \\ A\end{array} = A\mathbf{e}_j = T(\mathbf{e}_j) = B\mathbf{e}_j = \begin{array}{c}\text{the } j\text{th} \\ \text{column of} \\ B\end{array}$$

Proof continued.

Now we need to prove that such a matrix is unique: Suppose that $T(\mathbf{x}) = A\mathbf{x}$ and $T(\mathbf{x}) = B\mathbf{x}$. Then for each j we have

$$\begin{matrix}\text{the } j\text{th} \\ \text{column of} \\ A\end{matrix} = A\mathbf{e}_j = T(\mathbf{e}_j) = B\mathbf{e}_j = \begin{matrix}\text{the } j\text{th} \\ \text{column of} \\ B\end{matrix}$$

Hence $A = B$ and we are finished. $\square$

Proof continued.

Now we need to prove that such a matrix is unique: Suppose that $T(\mathbf{x}) = A\mathbf{x}$ and $T(\mathbf{x}) = B\mathbf{x}$. Then for each j we have

$$\begin{array}{c}\text{the } j\text{th} \\ \text{column of} \\ A\end{array} = A\mathbf{e}_j = T(\mathbf{e}_j) = B\mathbf{e}_j = \begin{array}{c}\text{the } j\text{th} \\ \text{column of} \\ B\end{array}$$

Hence $A = B$ and we are finished. □

Definition

The matrix A is called the *standard matrix for* T and is written $[T]$.

EXAMPLE 2 Find the standard matrix A for the dilation transformation $T(\mathbf{x}) = 3\mathbf{x}$, for $\mathbf{x}$ in $\mathbb{R}^2$.

SOLUTION Write

$$T(\mathbf{e}_1) = 3\mathbf{e}_1 = \begin{bmatrix} 3 \\ 0 \end{bmatrix} \quad \text{and} \quad T(\mathbf{e}_2) = 3\mathbf{e}_2 = \begin{bmatrix} 0 \\ 3 \end{bmatrix}$$

$$A = \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}$$

■

EXAMPLE 3 Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the transformation that rotates each point in $\mathbb{R}^2$ about the origin through an angle φ, with counterclockwise rotation for a positive angle. We could show geometrically that such a transformation is linear. (See Fig. 6 in Section 1.8.) Find the standard matrix A of this transformation.

SOLUTION $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ rotates into $\begin{bmatrix} \cos\varphi \\ \sin\varphi \end{bmatrix}$, and $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ rotates into $\begin{bmatrix} -\sin\varphi \\ \cos\varphi \end{bmatrix}$. See Fig. 1. By Theorem 10,

$$A = \begin{bmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{bmatrix}$$

Example 5 in Section 1.8 is a special case of this transformation, with $\varphi = \pi/2$. ■

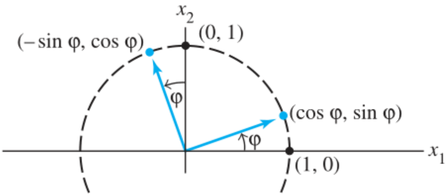
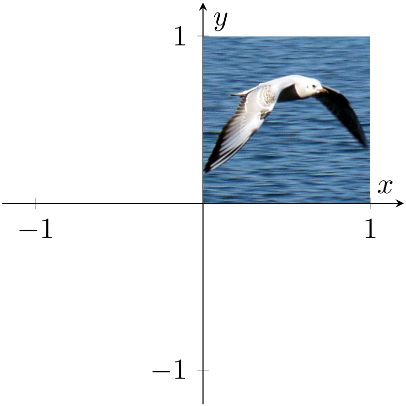
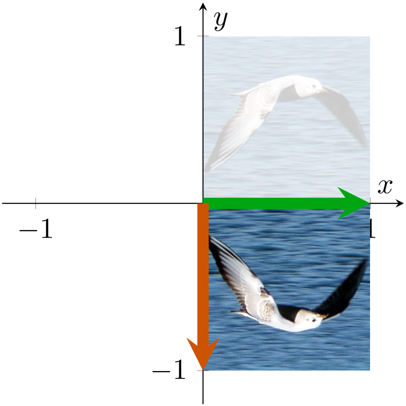


FIGURE 1 A rotation transformation.

Reflections



Reflections



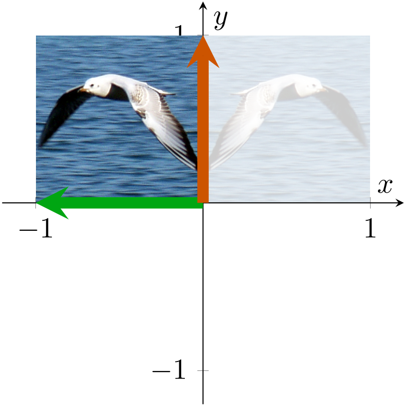
Reflection about the x-axis

$$T(\mathbf{e}_1) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$T(\mathbf{e}_2) = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$$

$$[T] = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

Reflections



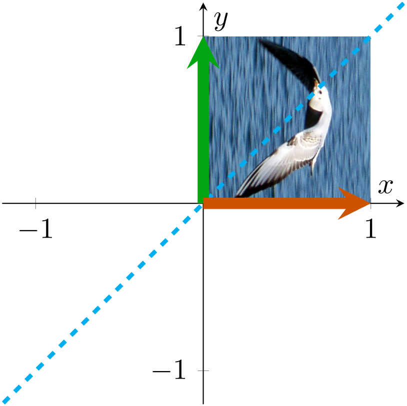
Reflection about the y-axis

$$T(\mathbf{e}_1) = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$$

$$T(\mathbf{e}_2) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$[T] = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$$

Reflections



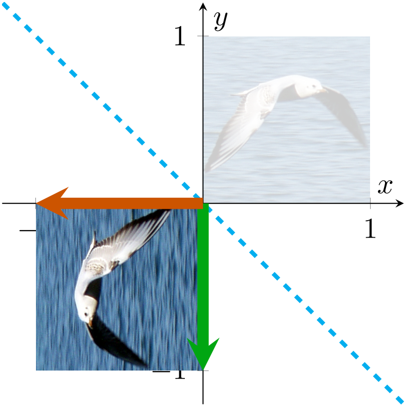
Reflection about the line
$y = x$

$$T(\mathbf{e}_1) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$T(\mathbf{e}_2) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$[T] = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

Reflections



Reflection about the line

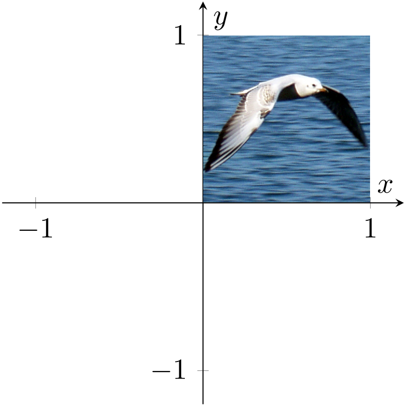
$y = -x$

$$T(\mathbf{e}_1) = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$$

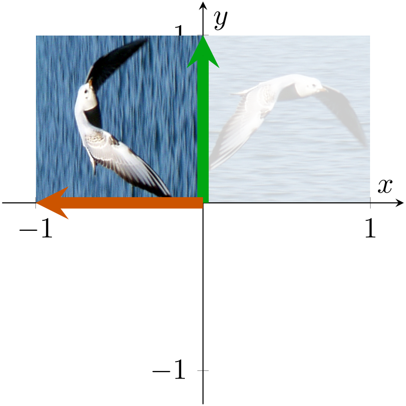
$$T(\mathbf{e}_2) = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$$

$$[T] = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$$

Rotations



Rotations



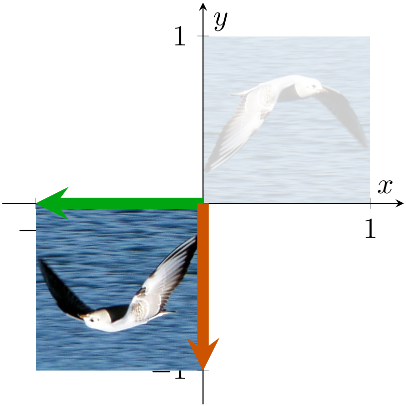
Rotate by $90°$

$$T(\mathbf{e}_1) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$T(\mathbf{e}_2) = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$$

$$[T] = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

Rotations



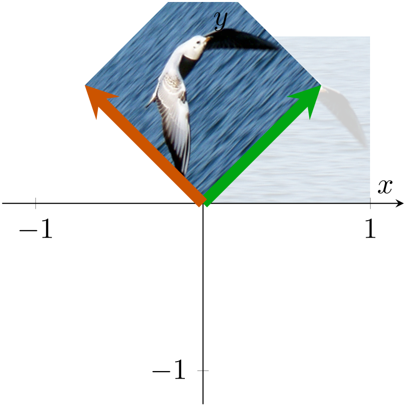
Rotate by $180°$

$$T(\mathbf{e}_1) = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$$

$$T(\mathbf{e}_2) = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$$

$$[T] = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$

Rotations



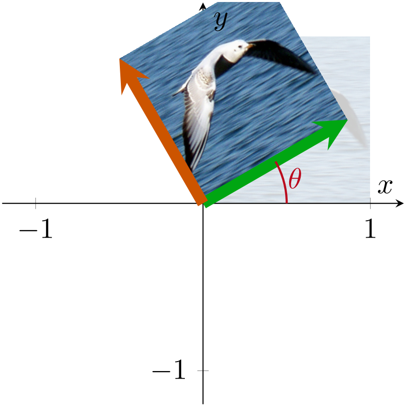
Rotate by $45°$

$$T(\mathbf{e}_1) = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}$$

$$T(\mathbf{e}_2) = \begin{bmatrix} -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}$$

$$[T] = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$$

Rotations



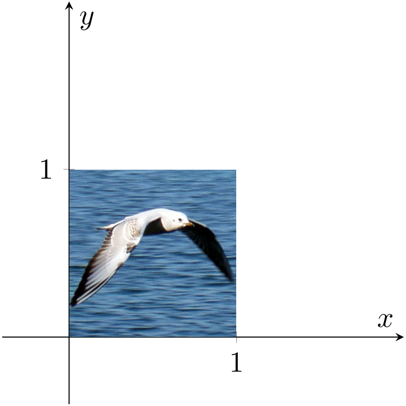
Rotate by θ

$$T(\mathbf{e}_1) = \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}$$

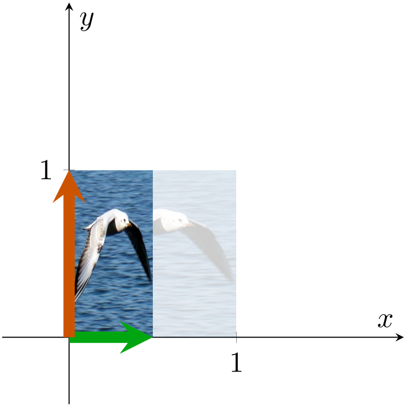
$$T(\mathbf{e}_2) = \begin{bmatrix} -\sin\theta \\ \cos\theta \end{bmatrix}$$

$$[T] = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

Contractions and Expansions



Contractions and Expansions



Horizontal Contraction

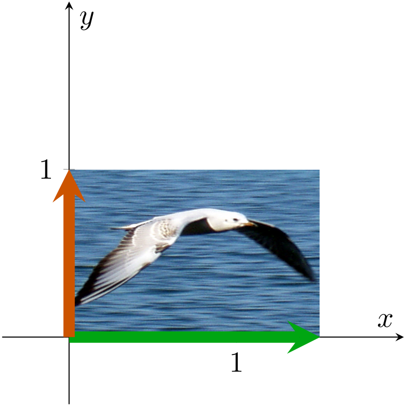
$$T(\mathbf{e}_1) = \begin{bmatrix} k \\ 0 \end{bmatrix}$$

$$T(\mathbf{e}_2) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$[T] = \begin{bmatrix} k & 0 \\ 0 & 1 \end{bmatrix}$$

$$0 < k < 1$$

Contractions and Expansions



Horizontal Expansion

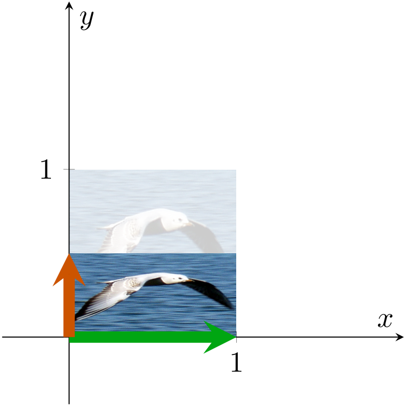
$$T(\mathbf{e}_1) = \begin{bmatrix} k \\ 0 \end{bmatrix}$$

$$T(\mathbf{e}_2) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$[T] = \begin{bmatrix} k & 0 \\ 0 & 1 \end{bmatrix}$$

$$k > 1$$

Contractions and Expansions



Vertical Contraction

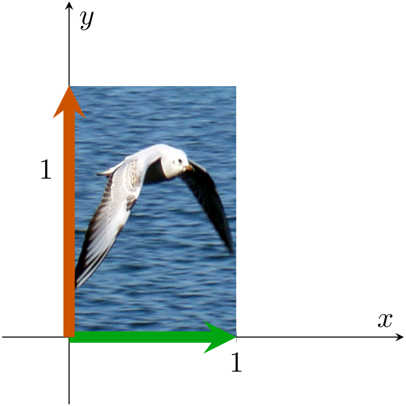
$$T(\mathbf{e}_1) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$T(\mathbf{e}_2) = \begin{bmatrix} 0 \\ k \end{bmatrix}$$

$$[T] = \begin{bmatrix} 1 & 0 \\ 0 & k \end{bmatrix}$$

$$0 < k < 1$$

Contractions and Expansions



Vertical Expansion

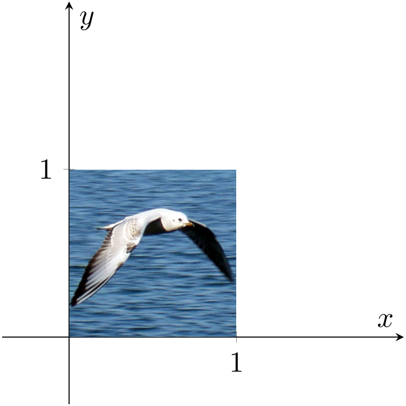
$$T(\mathbf{e}_1) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$T(\mathbf{e}_2) = \begin{bmatrix} 0 \\ k \end{bmatrix}$$

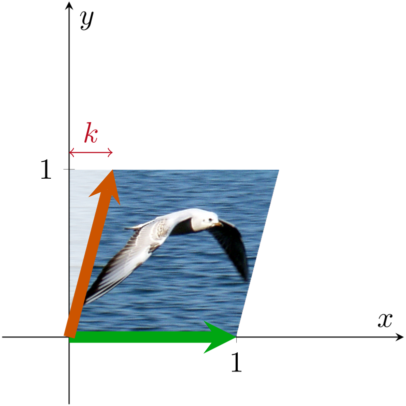
$$[T] = \begin{bmatrix} 1 & 0 \\ 0 & k \end{bmatrix}$$

$$k > 1$$

Shears



Shears



Horizontal Shear

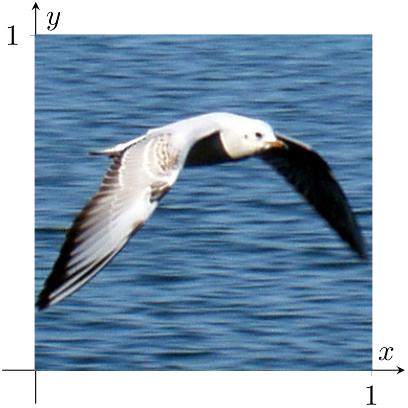
$$T(\mathbf{e}_1) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$T(\mathbf{e}_2) = \begin{bmatrix} k \\ 1 \end{bmatrix}$$

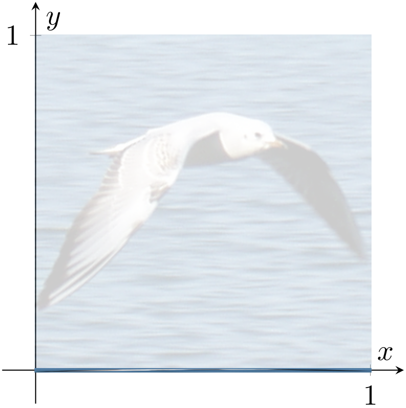
$$[T] = \begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix}$$

$$k > 0$$

Orthogonal Projections



Orthogonal Projections



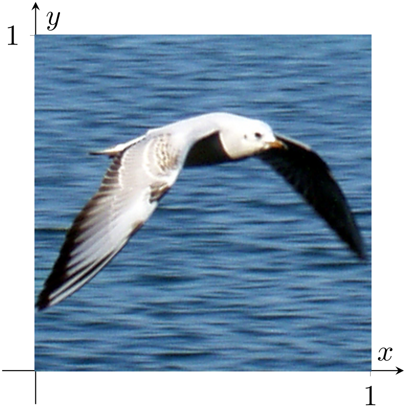
Projection onto the x-axis

$$T(\mathbf{e}_1) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

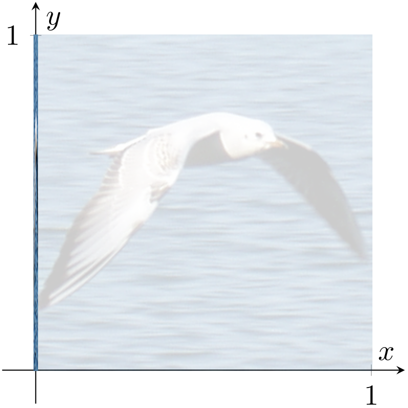
$$T(\mathbf{e}_2) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$[T] = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

Orthogonal Projections



Orthogonal Projections



Projection onto the y-axis

$$T(\mathbf{e}_1) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$T(\mathbf{e}_2) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$[T] = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

A Matrix for T relative to two bases

Let

- V be an n-dimensional vector space;
- $\mathcal{B}$ be a basis for V;
- W be an m-dimensional vector space;
- $\mathcal{C}$ is a basis for W; and
- $T : V \to W$ be a linear transformation.

Matrices for Linear Transformations

a vector in V (n-dimensional)

a vector in W (m-dimensional)

$$\mathbf{x} \xrightarrow{\quad T \quad} T(\mathbf{x})$$

Matrices for Linear Transformations

a vector in V
(n-dimensional)

a vector in W
(m-dimensional)

$$\begin{tikzcd} \mathbf{x} \arrow[rr, "T"] \arrow[d, shift right] & & T(\mathbf{x}) \\ {[\mathbf{x}]_{\mathcal{B}}} \arrow[u, shift right] & & \end{tikzcd}$$

coordinates of $\mathbf{x}$
with respect to $\mathcal{B}$
(a vector in $\mathbb{R}^n$)

Matrices for Linear Transformations

a vector in V
(n-dimensional)

a vector in W
(m-dimensional)

$$
\begin{tikzcd}
\mathbf{x} \arrow[rr, "T"] \arrow[d, shift right] & & T(\mathbf{x}) \arrow[d, shift right] \\
{[\mathbf{x}]_{\mathcal{B}}} \arrow[u, shift right] & & {[T(\mathbf{x})]_{\mathcal{C}}} \arrow[u, shift right]
\end{tikzcd}
$$

coordinates of $\mathbf{x}$
with respect to $\mathcal{B}$
(a vector in $\mathbb{R}^n$)

coordinates of $T(\mathbf{x})$
with respect to $\mathcal{C}$
(a vector in $\mathbb{R}^m$)

Matrices for Linear Transformations

a vector in V
(n-dimensional)

a vector in W
(m-dimensional)

$$\begin{tikzcd}[column sep=huge, row sep=huge]
\mathbf{x} \arrow[r, "T"] \arrow[d, shift right] & T(\mathbf{x}) \arrow[d, shift right] \\
{[\mathbf{x}]_{\mathcal{B}}} \arrow[u, shift right] \arrow[r, "A"] & {[T(\mathbf{x})]_{\mathcal{C}}} \arrow[u, shift right]
\end{tikzcd}$$

coordinates of $\mathbf{x}$
with respect to $\mathcal{B}$
(a vector in $\mathbb{R}^n$)

We want to find a
matrix A such that
$A[\mathbf{x}]_{\mathcal{B}} = [T(\mathbf{x})]_{\mathcal{C}}$

coordinates of $T(\mathbf{x})$
with respect to $\mathcal{C}$
(a vector in $\mathbb{R}^m$)

Matrices for Linear Transformations

a vector in V
(n-dimensional)

a vector in W
(m-dimensional)

$$
\begin{tikzcd}[column sep=huge, row sep=huge]
\mathbf{x} \arrow[r, "T"] \arrow[d, shift right] & T(\mathbf{x}) \arrow[d, shift right] \\
{[\mathbf{x}]_{\mathcal{B}}} \arrow[r, "A"] \arrow[u, shift right] & {[T(\mathbf{x})]_{\mathcal{C}}} \arrow[u, shift right]
\end{tikzcd}
$$

coordinates of $\mathbf{x}$
with respect to $\mathcal{B}$
(a vector in $\mathbb{R}^n$)

coordinates of $T(\mathbf{x})$
with respect to $\mathcal{C}$
(a vector in $\mathbb{R}^m$)

$$T : V \to W$$

We want to find an $m \times n$ matrix A such that

$$A[\mathbf{x}]_{\mathcal{B}} = [T(\mathbf{x})]_{\mathcal{C}}.$$

$$T : V \to W$$

We want to find an $m \times n$ matrix A such that

$$A[\mathbf{x}]_{\mathcal{B}} = [T(\mathbf{x})]_{\mathcal{C}}.$$

Let

- $\mathcal{B} = \{\mathbf{v}_1, \mathbf{v}_2, \dots \mathbf{v}_n\}$ be a basis for the n-dimensional vector space V; and
- $\mathcal{C} = \{\mathbf{w}_1, \mathbf{w}_2, \dots \mathbf{w}_m\}$ be a basis for the m-dimensional vector space W.

(This is on page 307 of your textbook.)

$$A[\mathbf{x}]_{\mathcal{B}} = [T(\mathbf{x})]_{\mathcal{C}}$$

We need A to satisfy

$$A[\mathbf{v}_1]_{\mathcal{B}} = [T(\mathbf{v}_1)]_{\mathcal{C}},\ A[\mathbf{v}_s]_{\mathcal{B}} = [T(\mathbf{v}_s)]_{\mathcal{C}},\ \ldots,\ A[\mathbf{v}_n]_{\mathcal{B}} = [T(\mathbf{v}_n)]_{\mathcal{C}}.$$

Matrices for Linear Transformations

$$A[\mathbf{x}]_{\mathcal{B}} = [T(\mathbf{x})]_{\mathcal{C}}$$

We need A to satisfy

$A[\mathbf{v}_1]_{\mathcal{B}} = [T(\mathbf{v}_1)]_{\mathcal{C}},\ A[\mathbf{v}_s]_{\mathcal{B}} = [T(\mathbf{v}_s)]_{\mathcal{C}},\ \ldots,\ A[\mathbf{v}_n]_{\mathcal{B}} = [T(\mathbf{v}_n)]_{\mathcal{C}}.$

But $\mathbf{v}_1 = 1\mathbf{v}_1 + 0\mathbf{v}_2 + 0\mathbf{v}_3 + \ldots + 0\mathbf{v}_n$ so

$$[\mathbf{v}_1]_{\mathcal{B}} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

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$$[\mathbf{v}_1]_{\mathcal{B}} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

which implies that

$$[T(\mathbf{v}_1)]_{\mathcal{C}} = A[\mathbf{v}_1]_{\mathcal{B}} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} =$$

$$A[\mathbf{x}]_{\mathcal{B}} = [T(\mathbf{x})]_{\mathcal{C}}$$

We need A to satisfy

$$A[\mathbf{v}_1]_{\mathcal{B}} = [T(\mathbf{v}_1)]_{\mathcal{C}},\ A[\mathbf{v}_s]_{\mathcal{B}} = [T(\mathbf{v}_s)]_{\mathcal{C}},\ \ \ldots,\ \ A[\mathbf{v}_n]_{\mathcal{B}} = [T(\mathbf{v}_n)]_{\mathcal{C}}.$$

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$$[\mathbf{v}_1]_{\mathcal{B}} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

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$A[\mathbf{x}]_{\mathcal{B}} = [T(\mathbf{x})]_{\mathcal{C}}$

Similarly

$$[T(\mathbf{v}_2)]_{\mathcal{C}} = A[\mathbf{v}_2]_{\mathcal{B}} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} = \begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{bmatrix}.$$

$$\vdots$$

$$[T(\mathbf{v}_n)]_{\mathcal{C}} = A[\mathbf{v}_n]_{\mathcal{B}} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} = \begin{bmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{bmatrix}.$$

$$A[\mathbf{x}]_{\mathcal{B}} = [T(\mathbf{x})]_{\mathcal{C}}$$

So the $m \times n$ matrix that we want is

$$A = \begin{bmatrix} [T(\mathbf{v}_1)]_{\mathcal{C}} & [T(\mathbf{v}_2)]_{\mathcal{C}} & \cdots & [T(\mathbf{v}_n)]_{\mathcal{C}} \end{bmatrix}.$$

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Definition

This matrix is called the *matrix for T relative to the bases $\mathcal{B}$ and $\mathcal{C}$* and is denoted by $[T]_{\mathcal{C},\mathcal{B}}$.

$$[T]_{\mathcal{C},\mathcal{B}} = \big[[T(\mathbf{v}_1)]_{\mathcal{C}} \quad [T(\mathbf{v}_2)]_{\mathcal{C}} \quad \cdots \quad [T(\mathbf{v}_n)]_{\mathcal{C}}\big]$$

Remark

The linear transformation

$$T: \quad V \quad \to \quad W$$

basis $\mathcal{B}$ basis $\mathcal{C}$

has matrix

$$[T]_{\mathcal{C},\mathcal{B}}.$$

Matrices for Linear Transformations

$$[T]_{\mathcal{C},\mathcal{B}} = \big[[T(\mathbf{v}_1)]_{\mathcal{C}} \quad [T(\mathbf{v}_2)]_{\mathcal{C}} \quad \cdots \quad [T(\mathbf{v}_n)]_{\mathcal{C}}\big]$$

Remark

The linear transformation

$$\begin{array}{cccc} T: & V & \rightarrow & W \\ & \text{basis } \mathcal{B} & & \text{basis } \mathcal{C} \end{array}$$

has matrix

$$[T]_{\mathcal{C},\mathcal{B}}.$$

We will use the formula

$$[T]_{\mathcal{C},\mathcal{B}}[\mathbf{x}]_{\mathcal{B}} = [T(\mathbf{x})]_{\mathcal{C}}.$$

$$[T]_{\mathcal{C},\mathcal{B}}[\mathbf{x}]_{\mathcal{B}} = [T(\mathbf{x})]_{\mathcal{C}}, \quad [T]_{\mathcal{C},\mathcal{B}} = \begin{bmatrix} [T(\mathbf{v}_1)]_{\mathcal{C}} & [T(\mathbf{v}_2)]_{\mathcal{C}} & \cdots & [T(\mathbf{v}_n)]_{\mathcal{C}} \end{bmatrix}$$

Example

I leave it for you to prove that if $T_A : \mathbb{R}^n \to \mathbb{R}^m$ is a matrix transformation ($T_A(\mathbf{x}) = A\mathbf{x}$) and if $\mathcal{B}$ and $\mathcal{C}$ are the standard bases, then

$$[T_A]_{\mathcal{C},\mathcal{B}} = A.$$

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$$[T_A]_{\mathcal{C},\mathcal{B}} = A.$$

Remark

If we are using the standard bases on both V and W, then we can write the matrix for T as just

$$[T].$$

This is called the *standard matrix* for T.

$$[T]_{\mathcal{C},\mathcal{B}}[\mathbf{x}]_{\mathcal{B}} = [T(\mathbf{x})]_{\mathcal{C}}, \quad [T]_{\mathcal{C},\mathcal{B}} = \begin{bmatrix} [T(\mathbf{v}_1)]_{\mathcal{C}} & [T(\mathbf{v}_2)]_{\mathcal{C}} & \cdots & [T(\mathbf{v}_n)]_{\mathcal{C}} \end{bmatrix}$$

Example

Let $T : \mathbb{P}^1 \to \mathbb{P}^2$ be the linear transformation defined by

$$T(\mathbf{p}) = x\mathbf{p}.$$

(For example, if $\mathbf{p} = 1 + x$, then $T(\mathbf{p}) = x + x^2$.)

Find the matrix for T with respect to the bases

$$\mathcal{B} = \{\mathbf{v}_1 = 1, \mathbf{v}_2 = x\} \quad \text{and} \quad \mathcal{C} = \{\mathbf{w}_1 = 1, \mathbf{w}_2 = x, \mathbf{w}_3 = x^2\}.$$

$$[T]_{\mathcal{C},\mathcal{B}}[\mathbf{x}]_{\mathcal{B}} = [T(\mathbf{x})]_{\mathcal{C}}, \quad [T]_{\mathcal{C},\mathcal{B}} = \begin{bmatrix} [T(\mathbf{v}_1)]_{\mathcal{C}} & [T(\mathbf{v}_2)]_{\mathcal{C}} & \cdots & [T(\mathbf{v}_n)]_{\mathcal{C}} \end{bmatrix}$$

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Let $T : \mathbb{P}^1 \to \mathbb{P}^2$ be the linear transformation defined by

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Find the matrix for T with respect to the bases

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First we calculate that

$$T(\mathbf{v}_1) = T(1) = (x)(1) = x = 0\mathbf{w}_1 + 1\mathbf{w}_2 + 0\mathbf{w}_3$$
$$T(\mathbf{v}_2) = T(x) = (x)(x) = x^2 = 0\mathbf{w}_1 + 0\mathbf{w}_2 + 1\mathbf{w}_3.$$

$$[T]_{\mathcal{C},\mathcal{B}}[\mathbf{x}]_{\mathcal{B}} = [T(\mathbf{x})]_{\mathcal{C}}, \quad [T]_{\mathcal{C},\mathcal{B}} = \begin{bmatrix} [T(\mathbf{v}_1)]_{\mathcal{C}} & [T(\mathbf{v}_2)]_{\mathcal{C}} & \cdots & [T(\mathbf{v}_n)]_{\mathcal{C}} \end{bmatrix}$$

$$T(\mathbf{v}_1) = T(1) = (x)(1) = x = 0\mathbf{w}_1 + 1\mathbf{w}_2 + 0\mathbf{w}_3$$
$$T(\mathbf{v}_2) = T(x) = (x)(x) = x^2 = 0\mathbf{w}_1 + 0\mathbf{w}_2 + 1\mathbf{w}_3.$$

Therefore the coordinates of $T(\mathbf{v}_1)$ and $T(\mathbf{v}_2)$ with respect to $\mathcal{C}$ are

$$[T(\mathbf{v}_1)]_{\mathcal{C}} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \qquad \text{and} \qquad [T(\mathbf{v}_2)]_{\mathcal{C}} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

[1] with respect to

$$\begin{aligned} T(\mathbf{v}_1) = T(1) = (x)(1) = x = 0\mathbf{w}_1 + 1\mathbf{w}_2 + 0\mathbf{w}_3 \\ T(\mathbf{v}_2) = T(x) = (x)(x) = x^2 = 0\mathbf{w}_1 + 0\mathbf{w}_2 + 1\mathbf{w}_3. \end{aligned}$$

Therefore the coordinates of $T(\mathbf{v}_1)$ and $T(\mathbf{v}_2)$ with respect to $\mathcal{C}$ are

$$[T(\mathbf{v}_1)]_{\mathcal{C}} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \qquad \text{and} \qquad [T(\mathbf{v}_2)]_{\mathcal{C}} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

Hence the matrix for T wrt[1] $\mathcal{B}$ and $\mathcal{C}$ is

$$[T]_{\mathcal{C},\mathcal{B}} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

[1] with respect to

Example

Let $T : \mathbb{R}^2 \to \mathbb{R}^3$ be the linear transformation defined by

$$T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} x_2 \\ -5x_1 + 13x_2 \\ -7x_1 + 16x_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -5 & 13 \\ -7 & 16 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}.$$

Find the matrix for T wrt the bases $\mathcal{B} = \{\mathbf{v}_1, \mathbf{v}_2\}$ for $\mathbb{R}^2$, and $\mathcal{C} = \{\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3\}$ for $\mathbb{R}^3$, where

$$\mathbf{v}_1 = \begin{bmatrix} 3 \\ 1 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 5 \\ 2 \end{bmatrix}, \quad \mathbf{w}_1 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \quad \mathbf{w}_2 = \begin{bmatrix} -1 \\ 2 \\ 2 \end{bmatrix}, \quad \mathbf{w}_3 = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}.$$

Matrices for Linear Transformations

$$[T]_{\mathcal{C},\mathcal{B}}[\mathbf{x}]_{\mathcal{B}} = [T(\mathbf{x})]_{\mathcal{C}}, \quad [T]_{\mathcal{C},\mathcal{B}} = \begin{bmatrix} [T(\mathbf{v}_1)]_{\mathcal{C}} & [T(\mathbf{v}_2)]_{\mathcal{C}} & \cdots & [T(\mathbf{v}_n)]_{\mathcal{C}} \end{bmatrix}$$

$$T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} x_2 \\ -5x_1 + 13x_2 \\ -7x_1 + 16x_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -5 & 13 \\ -7 & 16 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Note, if we were using the standard bases on $\mathbb{R}^2$ and $\mathbb{R}^3$, then the answer would just be the standard matrix

$$[T] = \begin{bmatrix} 0 & 1 \\ -5 & 13 \\ -7 & 16 \end{bmatrix}.$$

But we are using different bases, so this is not the answer.

$$[T]_{\mathcal{C},\mathcal{B}}[\mathbf{x}]_{\mathcal{B}} = [T(\mathbf{x})]_{\mathcal{C}}, \quad [T]_{\mathcal{C},\mathcal{B}} = \begin{bmatrix} [T(\mathbf{v}_1)]_{\mathcal{C}} & [T(\mathbf{v}_2)]_{\mathcal{C}} & \cdots & [T(\mathbf{v}_n)]_{\mathcal{C}} \end{bmatrix}$$

$$T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} x_2 \\ -5x_1 + 13x_2 \\ -7x_1 + 16x_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -5 & 13 \\ -7 & 16 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$\mathbf{u}_1 = \begin{bmatrix} 3 \\ 1 \end{bmatrix}, \quad \mathbf{u}_2 = \begin{bmatrix} 5 \\ 2 \end{bmatrix}, \quad \mathbf{w}_1 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \quad \mathbf{w}_2 = \begin{bmatrix} -1 \\ 2 \\ 2 \end{bmatrix}, \quad \mathbf{w}_3 = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}$$

We start by calculating

$$T(\mathbf{v}_1) = \begin{bmatrix} 0 & 1 \\ -5 & 13 \\ -7 & 16 \end{bmatrix} \begin{bmatrix} 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \\ -5 \end{bmatrix} = \mathbf{w}_1 - 2\mathbf{w}_3$$

$$T(\mathbf{v}_2) = \begin{bmatrix} 0 & 1 \\ -5 & 13 \\ -7 & 16 \end{bmatrix} \begin{bmatrix} 5 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ -3 \end{bmatrix} = 3\mathbf{w}_1 + \mathbf{w}_2 - \mathbf{w}_3$$

(please check).

$$[T]_{\mathcal{C},\mathcal{B}}[\mathbf{x}]_{\mathcal{B}} = [T(\mathbf{x})]_{\mathcal{C}}, \quad [T]_{\mathcal{C},\mathcal{B}} = \begin{bmatrix} [T(\mathbf{v}_1)]_{\mathcal{C}} & [T(\mathbf{v}_2)]_{\mathcal{C}} & \cdots & [T(\mathbf{v}_n)]_{\mathcal{C}} \end{bmatrix}$$

$$T(\mathbf{v}_1) = \mathbf{w}_1 - 2\mathbf{w}_3 \qquad T(\mathbf{v}_2) = 3\mathbf{w}_1 + \mathbf{w}_2 - \mathbf{w}_3$$

Therefore

$$[T(\mathbf{v}_1)]_{\mathcal{C}} = \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix} \qquad \text{and} \qquad [T(\mathbf{v}_2)]_{\mathcal{C}} = \begin{bmatrix} 3 \\ 1 \\ -1 \end{bmatrix}.$$

Matrices for Linear Transformations

$$[T]_{\mathcal{C},\mathcal{B}}[\mathbf{x}]_{\mathcal{B}} = [T(\mathbf{x})]_{\mathcal{C}}, \quad [T]_{\mathcal{C},\mathcal{B}} = \begin{bmatrix} [T(\mathbf{v}_1)]_{\mathcal{C}} & [T(\mathbf{v}_2)]_{\mathcal{C}} & \cdots & [T(\mathbf{v}_n)]_{\mathcal{C}} \end{bmatrix}$$

$$T(\mathbf{v}_1) = \mathbf{w}_1 - 2\mathbf{w}_3 \qquad\qquad T(\mathbf{v}_2) = 3\mathbf{w}_1 + \mathbf{w}_2 - \mathbf{w}_3$$

Therefore

$$[T(\mathbf{v}_1)]_{\mathcal{C}} = \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix} \qquad \text{and} \qquad [T(\mathbf{v}_2)]_{\mathcal{C}} = \begin{bmatrix} 3 \\ 1 \\ -1 \end{bmatrix}.$$

Hence the matrix for T wrt $\mathcal{B}$ and $\mathcal{C}$ is

$$[T]_{\mathcal{C},\mathcal{B}} = \begin{bmatrix} [T(\mathbf{v}_1)]_{\mathcal{C}} & [T(\mathbf{v}_2)]_{\mathcal{C}} \end{bmatrix} = \begin{bmatrix} 1 & 3 \\ 0 & 1 \\ -2 & -1 \end{bmatrix}.$$

Matrices for Linear Transformations

Remark

$$[T] = \begin{bmatrix} 0 & 1 \\ -5 & 13 \\ -7 & 16 \end{bmatrix} \quad \text{and} \quad [T]_{\mathcal{C},\mathcal{B}} = \begin{bmatrix} 1 & 3 \\ 0 & 1 \\ -2 & -1 \end{bmatrix}$$

are two different matrices for the same linear transformation.

Linear Operators

Recall that if $V = W$, then the linear transformation $T : V \to V$ is called a linear operator.

(If we are using the same basis $\mathcal{B}$ on both the domain and the target of T, then) instead of writing $[T]_{\mathcal{B},\mathcal{B}}$ we just write

$$[T]_{\mathcal{B}}.$$

Linear Operators

Recall that if $V = W$, then the linear transformation $T : V \to V$ is called a linear operator.

(If we are using the same basis $\mathcal{B}$ on both the domain and the target of T, then) instead of writing $[T]_{\mathcal{B},\mathcal{B}}$ we just write

$$[T]_{\mathcal{B}}.$$

The formulae are then

$$[T]_{\mathcal{B}} = \begin{bmatrix} [T(\mathbf{v}_1)]_{\mathcal{B}} & [T(\mathbf{v}_2)]_{\mathcal{B}} & \cdots & [T(\mathbf{v}_n)]_{\mathcal{B}} \end{bmatrix}$$

and

$$[T]_{\mathcal{B}}[\mathbf{x}]_{\mathcal{B}} = [T(\mathbf{x})]_{\mathcal{B}}.$$

Identity Operators

Example

Let V be a finite dimensional vector space and let $\mathcal{B} = \{\mathbf{v}_1, \mathbf{v}_2, \ldots \mathbf{v}_n\}$ be a basis for V. Consider the identity operator $I : V \to V$, $I(\mathbf{x}) = \mathbf{x}$.

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Let V be a finite dimensional vector space and let $\mathcal{B} = \{\mathbf{v}_1, \mathbf{v}_2, \ldots \mathbf{v}_n\}$ be a basis for V. Consider the identity operator $I : V \to V$, $I(\mathbf{x}) = \mathbf{x}$. Note that

$$I(\mathbf{v}_1) = \mathbf{v}_1, \qquad I(\mathbf{v}_2) = \mathbf{v}_2, \quad \ldots, \quad I(\mathbf{v}_n) = \mathbf{v}_n.$$

Identity Operators

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Let V be a finite dimensional vector space and let $\mathcal{B} = \{\mathbf{v}_1, \mathbf{v}_2, \ldots \mathbf{v}_n\}$ be a basis for V. Consider the identity operator $I : V \to V$, $I(\mathbf{x}) = \mathbf{x}$. Note that

$$I(\mathbf{v}_1) = \mathbf{v}_1, \qquad I(\mathbf{v}_2) = \mathbf{v}_2, \quad \ldots, \quad I(\mathbf{v}_n) = \mathbf{v}_n.$$

Hence

$$[I]_{\mathcal{B}} = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix} = I_n.$$

Example

Let $T : \mathbb{P}^2 \to \mathbb{P}^2$ be the linear operator defined by

$$T(\mathbf{p}) = \mathbf{p}(3x - 5).$$

For example, if $\mathbf{p} = 2x + x^2$, then

$$T(\mathbf{p}) = \mathbf{p}(3x - 5) = 2(3x - 5) + (3x - 5)^2.$$

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$$T(\mathbf{p}) = \mathbf{p}(3x - 5) = 2(3x - 5) + (3x - 5)^2.$$

1. Find $[T]_{\mathcal{B}}$ where $\mathcal{B} = \{1, x, x^2\}$.
2. Use this to calculate $T(1 + 2x + 3x^2)$.

1. Since

$$T(1) = 1, \qquad T(x) = (3x-5), \qquad T(x^2) = (3x-5)^2 = 9x^2-30x+25$$

we have

$$[T(1)]_{\mathcal{B}} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \qquad [T(x)]_{\mathcal{B}} = \begin{bmatrix} -5 \\ 3 \\ 0 \end{bmatrix}, \qquad [T(x^2)]_{\mathcal{B}} = \begin{bmatrix} 25 \\ -30 \\ 9 \end{bmatrix}.$$

1. Since

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Hence

$$[T]_{\mathcal{B}} = \begin{bmatrix} 1 & -5 & 25 \\ 0 & 3 & -30 \\ 0 & 0 & 9 \end{bmatrix}.$$

$$[T]_{\mathcal{B}} = \begin{bmatrix} 1 & -5 & 25 \\ 0 & 3 & -30 \\ 0 & 0 & 9 \end{bmatrix}$$

2. The vector $\mathbf{p} = 1 + 2x + 3x^2$ has coordinate vector

$$[\mathbf{p}]_{\mathcal{B}} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}.$$

$$[T]_{\mathcal{B}} = \begin{bmatrix} 1 & -5 & 25 \\ 0 & 3 & -30 \\ 0 & 0 & 9 \end{bmatrix}$$

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Thus

$$[T(\mathbf{p})]_{\mathcal{B}} = [T]_{\mathcal{B}}[\mathbf{p}]_{\mathcal{B}} = \begin{bmatrix} 1 & -5 & 25 \\ 0 & 3 & -30 \\ 0 & 0 & 9 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 66 \\ -84 \\ 27 \end{bmatrix}.$$

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2. The vector $\mathbf{p} = 1 + 2x + 3x^2$ has coordinate vector

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Therefore

$$T(1 + 2x + 3x^2) = 66 - 84x + 27x^2.$$

Compositions and Inverses

Theorem

If $T_1 : U \to V$ and $T_2 : V \to W$ are linear transformations, and if $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$ are bases for U, V, and W, respectively, then

$$[T_2 \circ T_1]_{\mathcal{C},\mathcal{A}} = [T_2]_{\mathcal{C},\mathcal{B}}[T_1]_{\mathcal{B},\mathcal{A}}.$$

Theorem

If $T : V \to V$ is a linear operator, and if $\mathcal{B}$ is a basis for V, then the following are equivalent:

1. *T is one-to-one.*
2. *$[T]_{\mathcal{B}}$ is invertible.*

Theorem

If $T : V \to V$ is a linear operator, and if $\mathcal{B}$ is a basis for V, then the following are equivalent:

1. *T is one-to-one.*
2. *$[T]_{\mathcal{B}}$ is invertible.*

Moreover, when these equivalent conditions hold,

$$[T^{-1}]_{\mathcal{B}} = [T]_{\mathcal{B}}^{-1}$$

Example (Composition)

Let $T_1 : \mathbb{P}^1 \to \mathbb{P}^2$ and $T_2 : \mathbb{P}^2 \to \mathbb{P}^2$ be defined by

$$T_1(\mathbf{p}) = x\mathbf{p} \qquad \text{and} \qquad T_2(\mathbf{p}) = \mathbf{p}(3x-5).$$

Find the matrix of $T_2 \circ T_1$ wrt to the standard bases on $\mathbb{P}^1$ and $\mathbb{P}^2$.

Let

$$\mathcal{B} = \{1, x\} \qquad \text{and} \qquad \mathcal{C} = \{1, x, x^2\}.$$

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Find the matrix of $T_2 \circ T_1$ wrt to the standard bases on $\mathbb{P}^1$ and $\mathbb{P}^2$.

Let

$$\mathcal{B} = \{1, x\} \qquad \text{and} \qquad \mathcal{C} = \{1, x, x^2\}.$$

We have seen today that

$$[T_1]_{\mathcal{C},\mathcal{B}} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad \text{and} \qquad [T_2]_{\mathcal{C}} = \begin{bmatrix} 1 & -5 & 25 \\ 0 & 3 & -30 \\ 0 & 0 & 9 \end{bmatrix}.$$

It follows that

$$[T_2 \circ T_1]_{\mathcal{C},\mathcal{B}} = [T_2]_{\mathcal{C}}[T_1]_{\mathcal{C},\mathcal{B}} = \begin{bmatrix} 1 & -5 & 25 \\ 0 & 3 & -30 \\ 0 & 0 & 9 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} -5 & 25 \\ 3 & -30 \\ 0 & 9 \end{bmatrix}$$

Break

We will continue at 3pm

Similarity

Similarity

Consider the matrix operator $T_A : \mathbb{R}^2 \to \mathbb{R}^2$ where

$$A = \begin{bmatrix} 1 & 1 \\ -2 & 4 \end{bmatrix}.$$

The matrix for T_A relative to the standard basis $\mathcal{B} = \{\mathbf{e}_1, \mathbf{e}_2\}$ for $\mathbb{R}^2$ is

$$[T_A] = [T_A]_{\mathcal{B}} = A = \begin{bmatrix} 1 & 1 \\ -2 & 4 \end{bmatrix}.$$

Similarity

Consider the matrix operator $T_A : \mathbb{R}^2 \to \mathbb{R}^2$ where

$$A = \begin{bmatrix} 1 & 1 \\ -2 & 4 \end{bmatrix}.$$

The matrix for T_A relative to the standard basis $\mathcal{B} = \{\mathbf{e}_1, \mathbf{e}_2\}$ for $\mathbb{R}^2$ is

$$[T_A] = [T_A]_{\mathcal{B}} = A = \begin{bmatrix} 1 & 1 \\ -2 & 4 \end{bmatrix}.$$

I want to calculate the matrix for the same operator T_A relative to a different basis.

$$A = \begin{bmatrix} 1 & 1 \\ -2 & 4 \end{bmatrix}$$

Namely, the basis $\mathcal{C} = \{\mathbf{v}_1, \mathbf{v}_2\}$ for $\mathbb{R}^2$ where

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \qquad \text{and} \qquad \mathbf{v}_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}.$$

$$A = \begin{bmatrix} 1 & 1 \\ -2 & 4 \end{bmatrix}$$

Namely, the basis $\mathcal{C} = \{\mathbf{v}_1, \mathbf{v}_2\}$ for $\mathbb{R}^2$ where

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \qquad \text{and} \qquad \mathbf{v}_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}.$$

Since

$$T_A(\mathbf{v}_1) = A\mathbf{v}_1 = \begin{bmatrix} 1 & 1 \\ -2 & 4 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \end{bmatrix} = 2\mathbf{v}_1$$

and

$$T_A(\mathbf{v}_2) = A\mathbf{v}_2 = \begin{bmatrix} 1 & 1 \\ -2 & 4 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 3 \\ 6 \end{bmatrix} = 3\mathbf{v}_2$$

$$A = \begin{bmatrix} 1 & 1 \\ -2 & 4 \end{bmatrix}$$

Namely, the basis $\mathcal{C} = \{\mathbf{v}_1, \mathbf{v}_2\}$ for $\mathbb{R}^2$ where

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \qquad \text{and} \qquad \mathbf{v}_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}.$$

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and

$$T_A(\mathbf{v}_2) = A\mathbf{v}_2 = \begin{bmatrix} 1 & 1 \\ -2 & 4 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 3 \\ 6 \end{bmatrix} = 3\mathbf{v}_2$$

it follows that

$$[T_A(\mathbf{v}_1)]_{\mathcal{C}} = \begin{bmatrix} 2 \\ 0 \end{bmatrix} \qquad \text{and} \qquad [T_A(\mathbf{v}_2)]_{\mathcal{C}} = \begin{bmatrix} 0 \\ 3 \end{bmatrix}.$$

Therefore the matrix for T_A relative to the basis $\mathcal{C}$ is

$$[T_A]_{\mathcal{C}} = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}.$$

Similarity

Compare these two matrices

$$[T_A]_{\mathcal{B}} = \begin{bmatrix} 1 & 1 \\ -2 & 4 \end{bmatrix} \qquad\qquad [T_A]_{\mathcal{C}} = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}.$$

- The determinant of both matrices is 6.
- Both matrices are invertible.
- The trace of both matrices in 5.
- The latter matrix is diagonal and diagonal matrices are easier to deal with.

Similarity

Definition

If A and B are square matrices, then we say that B is *similar* to A iff there exists an invertible matrix P such that

$$B = P^{-1}AP.$$

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Theorem

$$B \textit{ is similar to } A \quad \implies \quad A \textit{ is similar to } B$$

Proof.

Suppose that $B = P^{-1}AP$. Let $Q = P^{-1}$.

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Theorem

$$B \text{ is similar to } A \quad \Longrightarrow \quad A \text{ is similar to } B$$

Proof.

Suppose that $B = P^{-1}AP$. Let $Q = P^{-1}$. Then $B = QAQ^{-1}$ which rearranges to $A = Q^{-1}BQ$. Hence A is similar to B. $\square$

Similarity

Theorem

If A and B are similar matrices, then

$$\det(A) = \det(B).$$

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Proof.

If $B = P^{-1}AP$, then

$$\det(B) = \det(P^{-1}AP) = \det(P^{-1})\det(A)\det(P)$$

$$=$$

Similarity

Theorem

If A and B are similar matrices, then

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Proof.

If $B = P^{-1}AP$, then

$$\begin{aligned} \det(B) &= \det(P^{-1}AP) = \det(P^{-1})\det(A)\det(P) \\ &= \frac{1}{\det(P)}\det(A)\det(P) = \det(A). \end{aligned}$$

$\square$

Similarity

Lemma

If A and B are square matrices of the same size, then

$$\mathrm{tr}(AB) = \mathrm{tr}(BA)$$

Similarity

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2n} \\ \vdots & \vdots & & \vdots \\ b_{n1} & b_{n2} & \cdots & b_{nn} \end{bmatrix} = \begin{bmatrix} (AB)_{11} & ? & \cdots & ? \\ ? & (AB)_{22} & \cdots & ? \\ \vdots & \vdots & & \vdots \\ ? & ? & \cdots & (AB)_{nn} \end{bmatrix}$$

Proof.

Recall that $\mathrm{tr}(AB)$ is the sum of the entries on the main diagonal of AB. If A and B are $n \times n$ matrices, then

$$\begin{aligned} \mathrm{tr}(AB) &= (AB)_{11} + (AB)_{22} + (AB)_{33} + \ldots + (AB)_{nn} \\ &= \sum_{i=1}^{n} (AB)_{ii} \\ &= \end{aligned}$$

Similarity

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Similarity

$$\mathrm{tr}(AB) = \sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}b_{ji}$$

Proof continued.

Similarly we have that

$$\begin{aligned}\mathrm{tr}(BA) &= \sum_{i=1}^{n}(BA)_{ii} = \sum_{i=1}^{n}\sum_{j=1}^{n} b_{ij}a_{ji} = \sum_{i=1}^{n}\sum_{j=1}^{n} a_{ji}b_{ij} \\ &= \qquad\qquad = \qquad\qquad = \\ &= \end{aligned}$$

$\square$

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□

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Similarity

$$\mathrm{tr}(AB) = \sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}b_{ji}$$

Proof continued.

Similarly we have that

$$\begin{aligned}
\mathrm{tr}(BA) &= \sum_{i=1}^{n}(BA)_{ii} = \sum_{i=1}^{n}\sum_{j=1}^{n} b_{ij}a_{ji} = \sum_{i=1}^{n}\sum_{j=1}^{n} a_{ji}b_{ij} \\
&= \sum_{i=1}^{n}\sum_{k=1}^{n} a_{ki}b_{ik} = \sum_{j=1}^{n}\sum_{k=1}^{n} a_{kj}b_{jk} = \sum_{j=1}^{n}\sum_{i=1}^{n} a_{ij}b_{ji} \\
&= \mathrm{tr}(AB).
\end{aligned}$$

$\square$

$\mathrm{tr}(XY) = \mathrm{tr}(YX)$

Theorem

If A and B are similar matrices, then

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Similarity

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If $B = P^{-1}AP$, then

$$\mathrm{tr}(B) = \mathrm{tr}(P^{-1}AP) = \qquad\qquad = $$

□

Similarity

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□

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□

A New View of Transition Matrices

Suppose that $\mathcal{B} = \{\mathbf{b}_1, \ldots, \mathbf{b}_n\}$ and $\mathcal{C} = \{\mathbf{c}_1, \ldots, \mathbf{c}_n\}$ are two bases for a vector space V. Recall that

- the *change-of-coordinates matrix* or *transition matrix* from $\mathcal{B}$ to $\mathcal{C}$ is the matrix

$$\underset{\mathcal{C}\leftarrow\mathcal{B}}{P} = \begin{bmatrix} [\mathbf{b}_1]_{\mathcal{C}} & [\mathbf{b}_2]_{\mathcal{C}} & \cdots & [\mathbf{b}_n]_{\mathcal{C}} \end{bmatrix}$$

A New View of Transition Matrices

Suppose that $\mathcal{B} = \{\mathbf{b}_1, \ldots, \mathbf{b}_n\}$ and $\mathcal{C} = \{\mathbf{c}_1, \ldots, \mathbf{c}_n\}$ are two bases for a vector space V. Recall that

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- the *change-of-coordinates matrix* or *transition matrix* from $\mathcal{C}$ to $\mathcal{B}$ is the matrix

$$\underset{\mathcal{B}\leftarrow\mathcal{C}}{P} = \begin{bmatrix} [\mathbf{c}_1]_{\mathcal{B}} & [\mathbf{c}_2]_{\mathcal{B}} & \cdots & [\mathbf{c}_n]_{\mathcal{B}} \end{bmatrix}$$

A New View of Transition Matrices

Suppose that $\mathcal{B} = \{\mathbf{b}_1, \ldots, \mathbf{b}_n\}$ and $\mathcal{C} = \{\mathbf{c}_1, \ldots, \mathbf{c}_n\}$ are two bases for a vector space V. Recall that

- the *change-of-coordinates matrix* or *transition matrix* from $\mathcal{B}$ to $\mathcal{C}$ is the matrix

$$\underset{\mathcal{C}\leftarrow\mathcal{B}}{P} = \begin{bmatrix} [\mathbf{b}_1]_{\mathcal{C}} & [\mathbf{b}_2]_{\mathcal{C}} & \cdots & [\mathbf{b}_n]_{\mathcal{C}} \end{bmatrix}$$

- the *change-of-coordinates matrix* or *transition matrix* from $\mathcal{C}$ to $\mathcal{B}$ is the matrix

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- these two matrices are inverses of each other

A New View of Transition Matrices

Suppose that $\mathcal{B} = \{\mathbf{b}_1, \dots, \mathbf{b}_n\}$ and $\mathcal{C} = \{\mathbf{c}_1, \dots, \mathbf{c}_n\}$ are two bases for a vector space V. Recall that

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- the *change-of-coordinates matrix* or *transition matrix* from $\mathcal{C}$ to $\mathcal{B}$ is the matrix

$$\underset{\mathcal{B}\leftarrow\mathcal{C}}{P} = \begin{bmatrix} [\mathbf{c}_1]_{\mathcal{B}} & [\mathbf{c}_2]_{\mathcal{B}} & \cdots & [\mathbf{c}_n]_{\mathcal{B}} \end{bmatrix}$$

- these two matrices are inverses of each other

- $$\underset{\mathcal{C}\leftarrow\mathcal{B}}{P}[\mathbf{v}]_{\mathcal{B}} = [\mathbf{v}]_{\mathcal{C}}$$

A New View of Transition Matrices

Suppose that $\mathcal{B} = \{\mathbf{b}_1, \ldots, \mathbf{b}_n\}$ and $\mathcal{C} = \{\mathbf{c}_1, \ldots, \mathbf{c}_n\}$ are two bases for a vector space V. Recall that

- the *change-of-coordinates matrix* or *transition matrix* from $\mathcal{B}$ to $\mathcal{C}$ is the matrix

$$\underset{\mathcal{C}\leftarrow\mathcal{B}}{P} = \begin{bmatrix} [\mathbf{b}_1]_{\mathcal{C}} & [\mathbf{b}_2]_{\mathcal{C}} & \cdots & [\mathbf{b}_n]_{\mathcal{C}} \end{bmatrix}$$

- the *change-of-coordinates matrix* or *transition matrix* from $\mathcal{C}$ to $\mathcal{B}$ is the matrix

$$\underset{\mathcal{B}\leftarrow\mathcal{C}}{P} = \begin{bmatrix} [\mathbf{c}_1]_{\mathcal{B}} & [\mathbf{c}_2]_{\mathcal{B}} & \cdots & [\mathbf{c}_n]_{\mathcal{B}} \end{bmatrix}$$

- these two matrices are inverses of each other

- $$\underset{\mathcal{C}\leftarrow\mathcal{B}}{P}[\mathbf{v}]_{\mathcal{B}} = [\mathbf{v}]_{\mathcal{C}}$$

- $$\underset{\mathcal{B}\leftarrow\mathcal{C}}{P}[\mathbf{v}]_{\mathcal{C}} = [\mathbf{v}]_{\mathcal{B}}$$

Theorem

If $\mathcal{B} = \{\mathbf{b}_1, \ldots, \mathbf{b}_n\}$ *and* $\mathcal{C} = \{\mathbf{c}_1, \ldots, \mathbf{c}_n\}$ *are bases for a finite-dimensional vector space* V*, and if* $I : V \to V$ *is the identity operator on* V*, then*

$$\underset{\mathcal{C}\leftarrow\mathcal{B}}{P} = [I]_{\mathcal{C},\mathcal{B}} \qquad \text{and} \qquad \underset{\mathcal{B}\leftarrow\mathcal{C}}{P} = [I]_{\mathcal{B},\mathcal{C}}.$$

Proof.

Since $I(\mathbf{v}) = \mathbf{v}$ for all $\mathbf{v} \in V$, we have that

$$[I]_{\mathcal{C},\mathcal{B}} = \begin{bmatrix} [I(\mathbf{b}_1)]_{\mathcal{C}} & [I(\mathbf{b}_2)]_{\mathcal{C}} & \cdots & [I(\mathbf{b}_n)]_{\mathcal{C}} \end{bmatrix}$$

$$= \qquad\qquad =$$

Proof.

Since $I(\mathbf{v}) = \mathbf{v}$ for all $\mathbf{v} \in V$, we have that

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$$= \begin{bmatrix} [\mathbf{b}_1]_{\mathcal{C}} & [\mathbf{b}_2]_{\mathcal{C}} & \cdots & [\mathbf{b}_n]_{\mathcal{C}} \end{bmatrix} =$$

Similarity

Proof.

Since $I(\mathbf{v}) = \mathbf{v}$ for all $\mathbf{v} \in V$, we have that

$$[I]_{\mathcal{C},\mathcal{B}} = \begin{bmatrix} [I(\mathbf{b}_1)]_{\mathcal{C}} & [I(\mathbf{b}_2)]_{\mathcal{C}} & \cdots & [I(\mathbf{b}_n)]_{\mathcal{C}} \end{bmatrix}$$

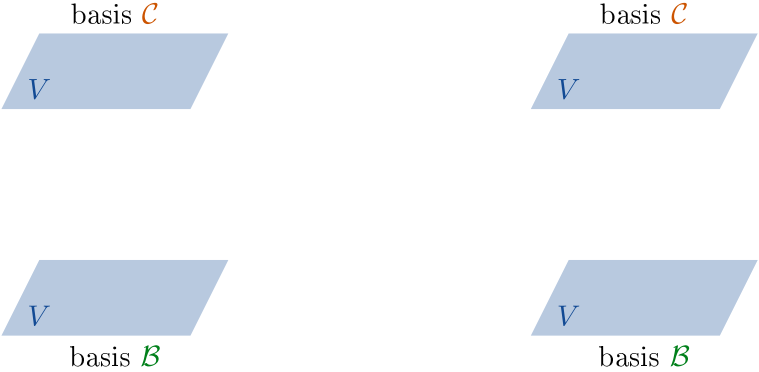
$$= \begin{bmatrix} [\mathbf{b}_1]_{\mathcal{C}} & [\mathbf{b}_2]_{\mathcal{C}} & \cdots & [\mathbf{b}_n]_{\mathcal{C}} \end{bmatrix} = \underset{\mathcal{C} \leftarrow \mathcal{B}}{P}.$$

Similarly $\underset{\mathcal{B} \leftarrow \mathcal{C}}{P} = [I]_{\mathcal{B},\mathcal{C}}$. $\square$

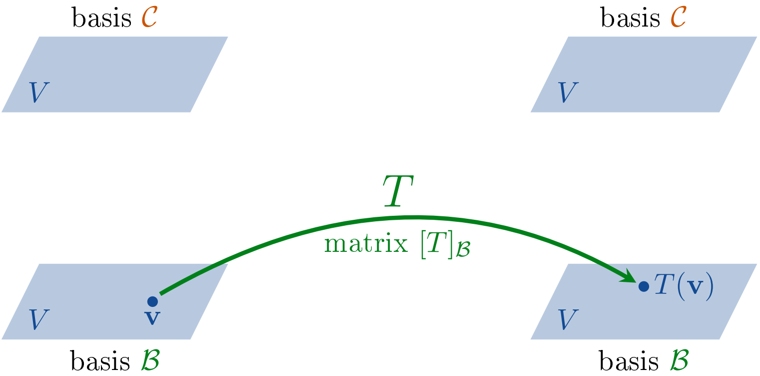
The Effect of Changing Bases on Matrices of Linear Operators

Question: If $\mathcal{B}$ and $\mathcal{C}$ are two bases for a finite-dimensional vector space V, and if $T : V \to V$ is a linear operator, what relationship, if any, exists between the matrices $[T]_{\mathcal{B}}$ and $[T]_{\mathcal{C}}$?

Similarity

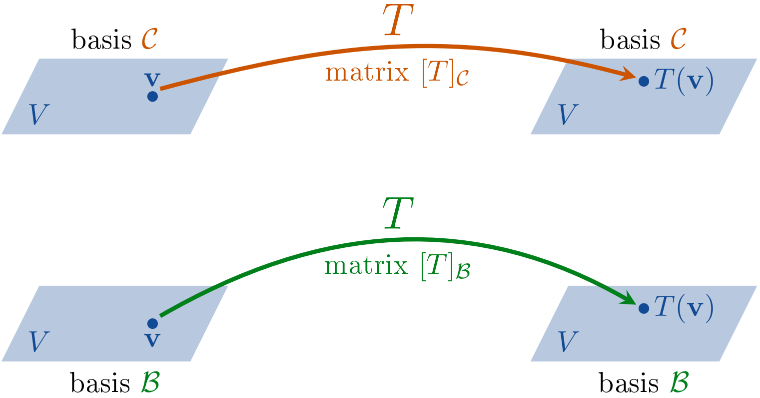

basis $\mathcal{C}$
V
basis $\mathcal{C}$
V
V
basis $\mathcal{B}$
V
basis $\mathcal{B}$

Similarity

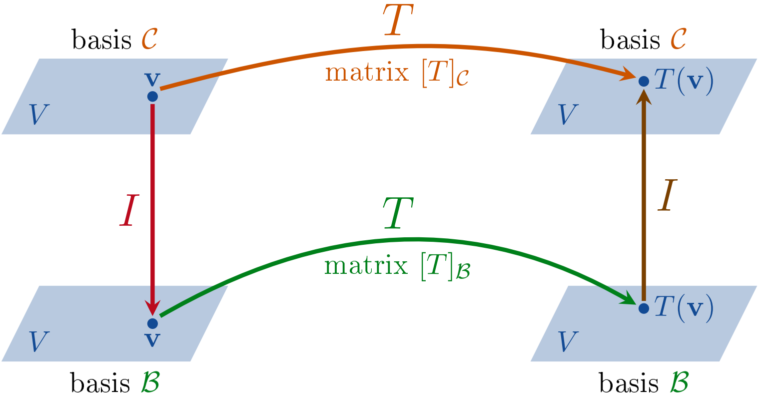


basis $\mathcal{C}$
basis $\mathcal{C}$
V
V
T
matrix $[T]_{\mathcal{B}}$
$T(\mathbf{v})$
$\mathbf{v}$
V
V
basis $\mathcal{B}$
basis $\mathcal{B}$

Similarity

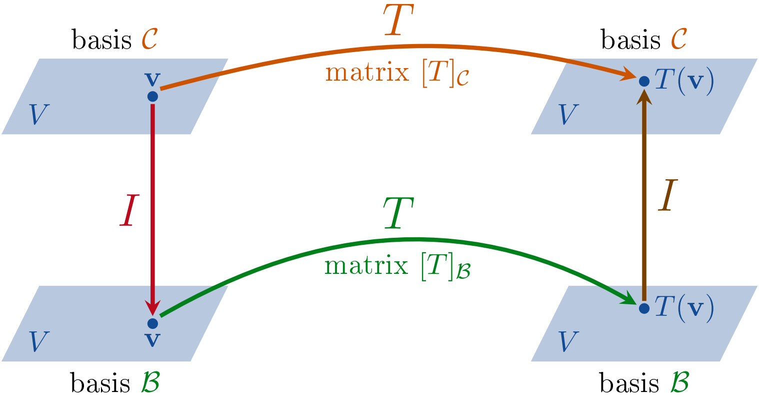


T
basis $\mathcal{C}$
basis $\mathcal{C}$
$\mathbf{v}$
matrix $[T]_{\mathcal{C}}$
$T(\mathbf{v})$
V
V
T
matrix $[T]_{\mathcal{B}}$
$T(\mathbf{v})$
$\mathbf{v}$
V
V
basis $\mathcal{B}$
basis $\mathcal{B}$



Note that $T = I \circ T \circ I$.

Similarity



Note that $T = I \circ T \circ I$. It follows that

$$[T]_\mathcal{C} = [I \circ T \circ I]_\mathcal{C} = [I]_{\mathcal{C},\mathcal{B}}[T]_\mathcal{B}[I]_{\mathcal{B},\mathcal{C}}.$$

Similarity

$$[T]_{\mathcal{C}} = [I \circ T \circ I]_{\mathcal{C}} = [I]_{\mathcal{C},\mathcal{B}}[T]_{\mathcal{B}}[I]_{\mathcal{B},\mathcal{C}}.$$

Since

$$\underset{\mathcal{C}\leftarrow\mathcal{B}}{P} = [I]_{\mathcal{C},\mathcal{B}} \qquad \text{and} \qquad \underset{\mathcal{B}\leftarrow\mathcal{C}}{P} = [I]_{\mathcal{B},\mathcal{C}}$$

we have

Theorem

Let $T : V \to V$ be a linear operator on a finite-dimensional vector space V, and let $\mathcal{B}$ and $\mathcal{C}$ be bases for V. Then

$$[T]_{\mathcal{C}} = P^{-1}[T]_{\mathcal{B}}P$$

where $P = \underset{\mathcal{B}\leftarrow\mathcal{C}}{P}$ and $P^{-1} = \underset{\mathcal{C}\leftarrow\mathcal{B}}{P}$.

Remark

Therefore, two matrices representing the same linear operator, must be similar.

Similarity

Remark

Therefore, two matrices representing the same linear operator, must be similar.

Theorem

Two matrices A and B represent the same linear operator $\iff$ *A and B are similar.*

Complex Numbers

The Natural Numbers

The set

$$\mathbb{N} = \{1, 2, 3, 4, 5, 6, \dots\}$$

is called the set of *natural numbers*. These are the first numbers that children learn. For example

$$\begin{aligned}
2 \in \mathbb{N} &\quad \text{means "2 is a natural number"} \\
7 \in \mathbb{N} &\quad \text{means "7 is a natural number"} \\
\tfrac{1}{2} \notin \mathbb{N} &\quad \text{means "}\tfrac{1}{2}\text{ is } \textbf{not} \text{ a natural number"} \\
0 \notin \mathbb{N} &\quad \text{means "0 is } \textbf{not} \text{ a natural number"} \\
-5 \notin \mathbb{N} &\quad \text{means "}-5\text{ is } \textbf{not} \text{ a natural number"}
\end{aligned}$$

In the natural numbers, we can do "+" and "×"

$$2 + 7 = 9 \in \mathbb{N}, \qquad\qquad 2 \times 7 = 14 \in \mathbb{N}.$$

However we can not do "−" because

$$2 - 7 \notin \mathbb{N}.$$

In the natural numbers, we can do "+" and "×"

$$2 + 7 = 9 \in \mathbb{N}, \qquad\qquad 2 \times 7 = 14 \in \mathbb{N}.$$

However we can not do "−" because

$$2 - 7 \notin \mathbb{N}.$$

So we invent new numbers!

The Integers

The set

$$\mathbb{Z} = \{\dots, -4, -3, -2, -1, 0, 1, 2, 3, 4, \dots\}$$

is called the set of *integers*. We use a $\mathbb{Z}$ for the German word 'zahlen' (numbers).

The Integers

the natural
numbers "exist"
in the world

The set

$$\mathbb{Z} = \{\dots, -4, -3, -2, -1, 0, \underline{1, 2, 3, 4, \dots}\}$$

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The Integers

zero and the negative numbers were invented by humans

the natural numbers “exist” in the world

The set

$$\mathbb{Z} = \{\underline{\ldots, -4, -3, -2, -1, 0}, \underline{1, 2, 3, 4, \ldots}\}$$

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The Integers

The set

$$\mathbb{Z} = \{\ldots, -4, -3, -2, -1, 0, 1, 2, 3, 4, \ldots\}$$

is called the set of *integers*. We use a $\mathbb{Z}$ for the German word 'zahlen' (numbers).

In $\mathbb{Z}$, we can do "$+$", "$-$" and "$\times$" but we can not do "$\div$". For example $3 \in \mathbb{Z}$, $4 \in \mathbb{Z}$, $-5 \in \mathbb{Z}$ and

$$3+4 \in \mathbb{Z}, \qquad 3-4 \in \mathbb{Z}, \qquad 3 \times 4 \in \mathbb{Z}, \qquad 3 \div 4 \notin \mathbb{Z},$$

$$3+(-5) \in \mathbb{Z}, \quad 3-(-5) \in \mathbb{Z}, \quad 3 \times (-5) \in \mathbb{Z}, \quad 3 \div (-5) \notin \mathbb{Z}.$$

The Integers

The set

$$\mathbb{Z} = \{\ldots, -4, -3, -2, -1, 0, 1, 2, 3, 4, \ldots\}$$

is called the set of *integers*. We use a $\mathbb{Z}$ for the German word 'zahlen' (numbers).
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So we invent new numbers!

The Rational Numbers

The set

$$\mathbb{Q} = \{\text{all fractions}\} = \left\{ \frac{a}{b} \,\middle|\, a, b \in \mathbb{Z} \text{ and } b \neq 0 \right\}$$

is called the set of *rational numbers*. We use a $\mathbb{Q}$ for the word 'quotient'.

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$$0 = \frac{0}{1} \in \mathbb{Q} \qquad \frac{100}{13} \in \mathbb{Q}$$

$$1 = \frac{1}{1} \in \mathbb{Q} \qquad \sqrt{2} \notin \mathbb{Q}$$

$$\frac{3}{4} \in \mathbb{Q} \qquad -4 = \frac{8}{-2} \in \mathbb{Q}$$

$$\pi \notin \mathbb{Q} \qquad 0.12345 = \frac{12345}{100000} \in \mathbb{Q}.$$

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In $\mathbb{Q}$ we can do "$+$", "$-$", "$\times$" and "$\div$(by a number $\neq 0$)".

Are we happy now?

Are we happy now?

No!

Are we happy now?

No!

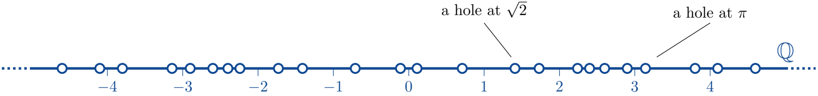
Why?

Are we happy now?

No!

Why?

Because if we draw all the rational numbers in a line, then the line has lots of holes in it. In fact, $\mathbb{Q}$ has ∞ many holes in it.



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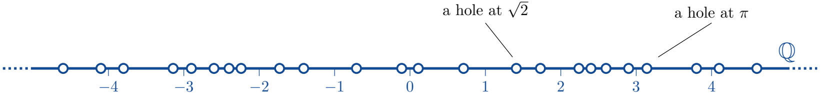

So we invent new numbers!

The Real Numbers

The set

$$\mathbb{R} = \{\text{all numbers which can be written as a decimal}\}$$

is called the set of *real numbers*. For example

$$0 = 0.0 \in \mathbb{R} \qquad \frac{100}{13} = 7.692307\ldots \in \mathbb{R}$$
$$\frac{23}{99} = 0.232323\ldots \in \mathbb{R} \qquad \sqrt{2} = 1.414213\ldots \in \mathbb{R}$$
$$\frac{3}{4} = 0.75 \in \mathbb{R} \qquad \frac{123}{999} = 0.123123\ldots \in \mathbb{R}$$
$$\pi = 3.141592\ldots \in \mathbb{R} \qquad \frac{12345}{100000} = 0.12345 \in \mathbb{R}.$$

The real numbers are complete – this means that if we draw all the real numbers in a line, then there are no holes in the line.

Are we happy now?

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Why?

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Why?

Because not every polynomial has a root in $\mathbb{R}$. For example, there is no solution to

$$x^2 + 1 = 0$$

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Complex Numbers

We invent a new number called i which we say is the solution to

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Complex Numbers

i is a made up number. It was invented by humans.

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i is also useful to us.

Now that we have i, we can start working with numbers like $3+4i$, $2+7i$, $-12i$, etc.

Complex Numbers

Now that we have i, we can start working with numbers like $3 + 4i$, $2 + 7i$, $-12i$, etc.

Definition

A *complex number* is any number that can be written as

$$a + bi$$

where i solves $i^2 + 1 = 0$ and $a, b \in \mathbb{R}$.

Definition

The set of all complex numbers is denoted by $\mathbb{C}$.

$$z = \underbrace{a}_{\text{real part}} + \underbrace{b}_{\text{imaginary part}}i$$

Definition

$\mathrm{Re}(z) = a$ is called the *real* part of z, and $\mathrm{Im}(z) = b$ is called the *imaginary* part of z.

Complex Numbers

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Definition

$\mathrm{Re}(z) = a$ is called the *real* part of z, and $\mathrm{Im}(z) = b$ is called the *imaginary* part of z.

Example

- $-2 + 7i$ has real part -2 and imaginary part 7;
- $7i$ has real part 0 and imaginary part 7; and
- -2 has real part -2 and imaginary part 0.

Definition

An *imaginary number* is a complex number $a + bi$ where $a = 0$.

A *real number* is a complex number $a + bi$ where $b = 0$.

$$4 + 2i \qquad -3 - \sqrt{5}i \qquad 1 - i$$

$$3 \qquad \sqrt{5} \qquad -12.2$$

$$-12.2i \qquad \sqrt{2}i \qquad 7i$$

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complex numbers

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real numbers

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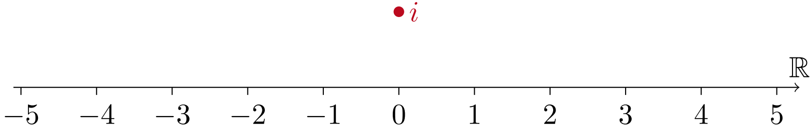
imaginary numbers

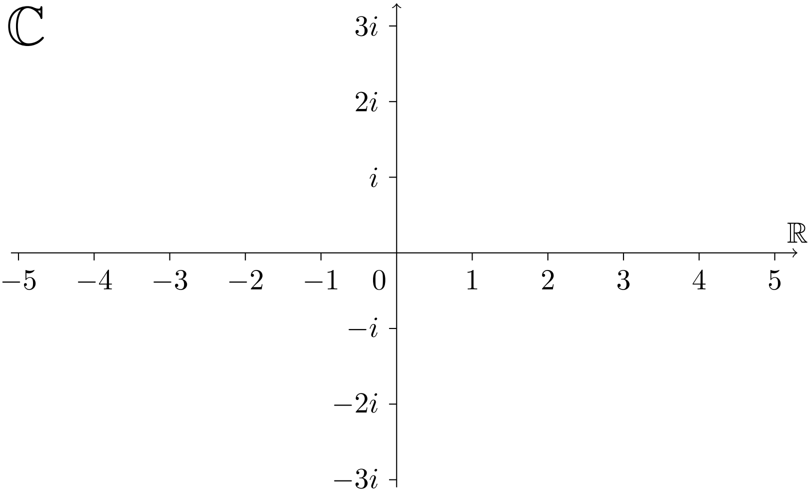
$-12.2i$ $\qquad \sqrt{2}i$ $\qquad 7i$

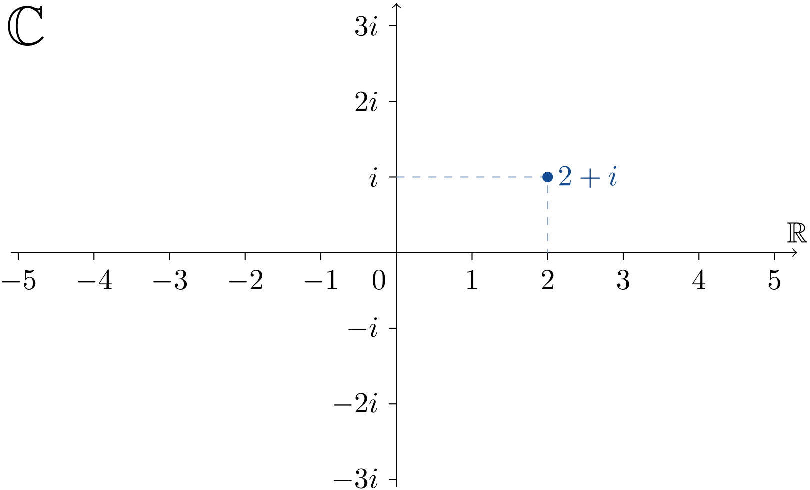
Remark

The names “real number” and “imaginary number” come from the 17th century.

These are really bad names because real numbers are just as imaginary, or made up, as imaginary numbers.


i
ℝ
−5
−4
−3
−2
−1
0
1
2
3
4
5


$\mathbb{C}$
$3i$
$2i$
i
$\mathbb{R}$
-5 -4 -3 -2 -1 0 1 2 3 4 5
$-i$
$-2i$
$-3i$



$\mathbb{C}$
$3i$
$2i$
i
$2+i$
$\mathbb{R}$
-5 -4 -3 -2 -1 0 1 2 3 4 5
$-i$
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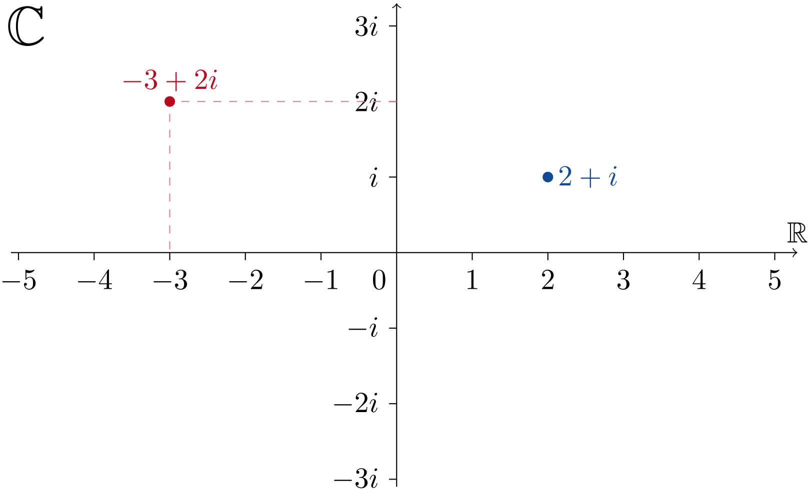

$\mathbb{C}$
$3i$
$-3+2i$
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-5 -4 -3 -2 -1 0 1 2 3 4 5
$-i$
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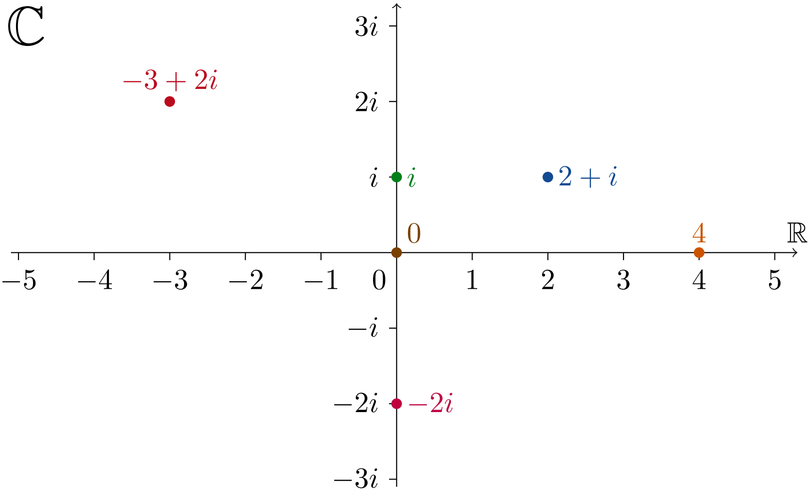
$\mathbb{C}$
$3i$
$-3+2i$
$2i$
i
i
$2+i$
0
4
$\mathbb{R}$
-5 -4 -3 -2 -1 0 1 2 3 4 5
$-i$
$-2i$
$-2i$
$-3i$

Complex Numbers

$$\sqrt{-9} = \sqrt{9(-1)} = \sqrt{9}\sqrt{-1} = 3i$$

Complex Numbers

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$$(2+3i) + (1-6i) = (2+1) + (3i-6i) = 3-3i$$

Complex Numbers

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$$\begin{aligned}(2-i)(3+4i) &= (2)(3) + (2)(4i) + (-i)(3) + (-i)(4i)\\ &= 6 + 8i - 3i - 4i^2\\ &= 6 + 8i - 3i - 4(-1)\\ &= 6 + 5i + 4\\ &= 10 + 5i\end{aligned}$$

Complex Numbers

$$\frac{3-i}{2+i} =$$

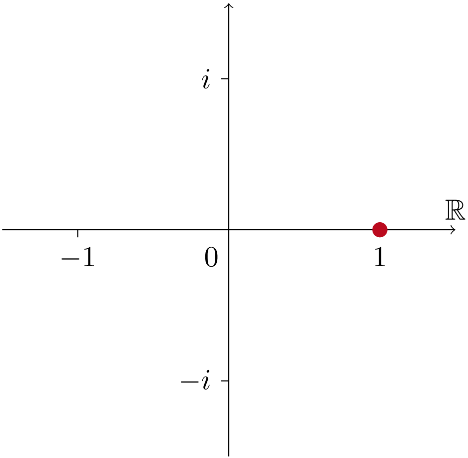
Complex Numbers

$$\frac{3-i}{2+i} = \frac{3-i}{2+i} \cdot \frac{2-i}{2-i}$$

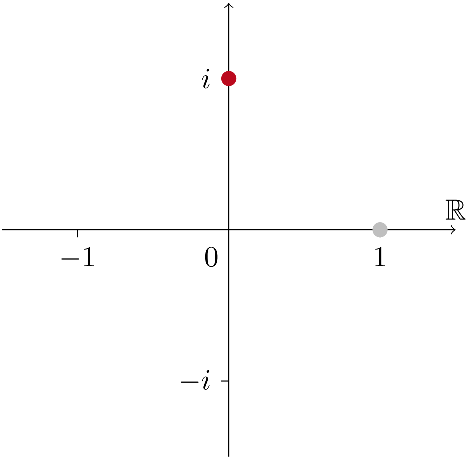
Complex Numbers

$$\frac{3-i}{2+i} = \frac{3-i}{2+i} \cdot \frac{2-i}{2-i} = \frac{6-3i-2i-1}{4-2i+2i+1}$$

$$\frac{3-i}{2+i} = \frac{3-i}{2+i} \cdot \frac{2-i}{2-i} = \frac{6-3i-2i-1}{4-2i+2i+1} = \frac{5-5i}{5} = 1-i.$$



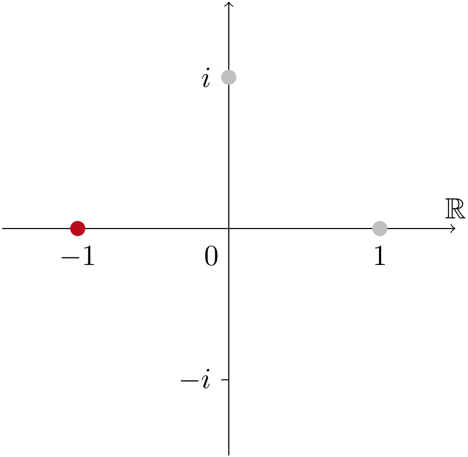
$$i^0 = 1$$



$i^0 = 1$

$i^1 = i$

Complex Numbers

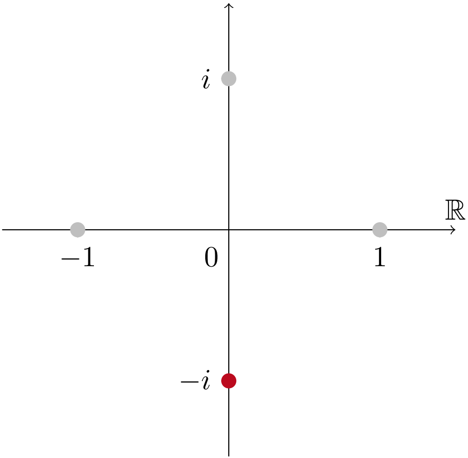


$$i^0 = 1$$

$$i^1 = i$$

$$i^2 = -1$$

Complex Numbers



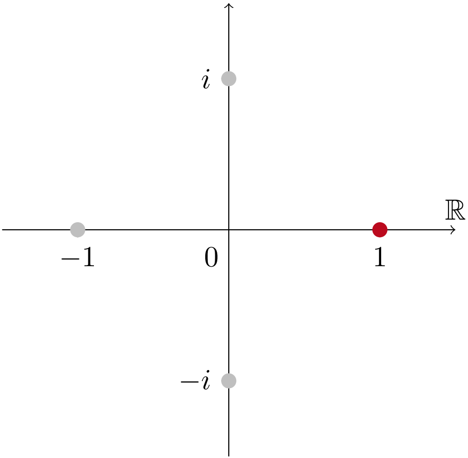
$$i^0 = 1$$

$$i^1 = i$$

$$i^2 = -1$$

$$i^3 = -i$$

Complex Numbers



$i^0 = 1$

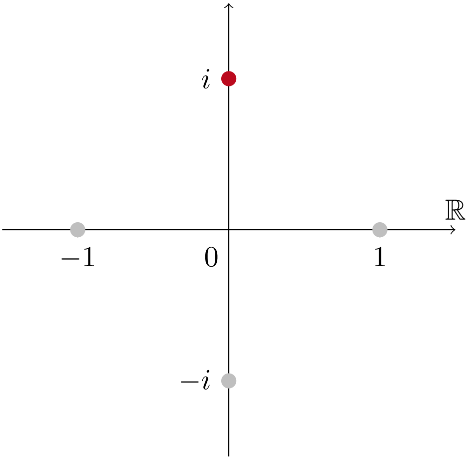
$i^1 = i$

$i^2 = -1$

$i^3 = -i$

$i^4 = 1$

Complex Numbers



$i^0 = 1$

$i^1 = i$

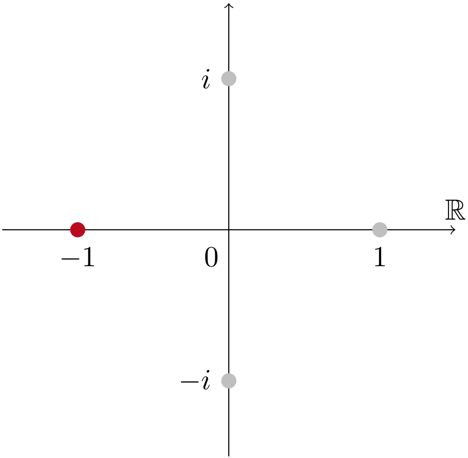
$i^2 = -1$

$i^3 = -i$

$i^4 = 1$

$i^5 = i$

Complex Numbers



$i^0 = 1$

$i^1 = i$

$i^2 = -1$

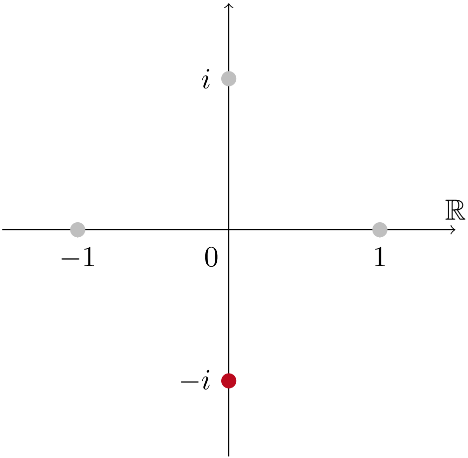
$i^3 = -i$

$i^4 = 1$

$i^5 = i$

$i^6 = -1$

Complex Numbers



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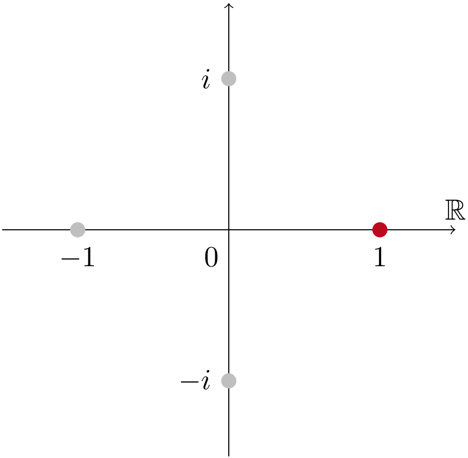
$i^6 = -1$

$i^7 = -i$

Complex Numbers



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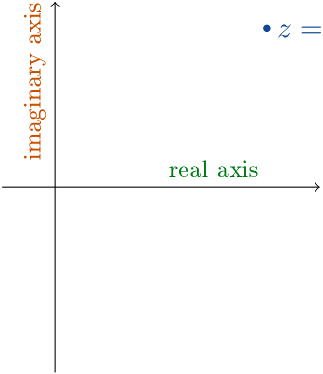
$$i^5 = i$$
$$i^6 = -1$$
$$i^7 = -i$$
$$i^8 = 1$$

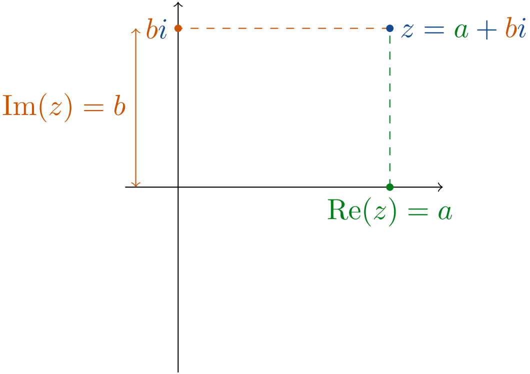
Complex Numbers



- $z = a + bi$

Definition

Complex Numbers



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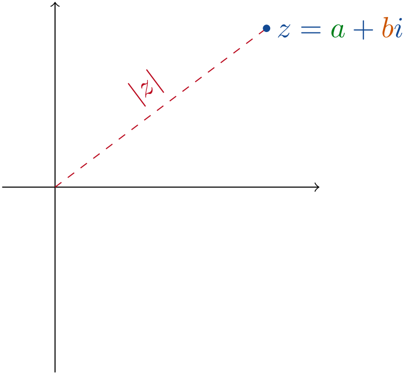
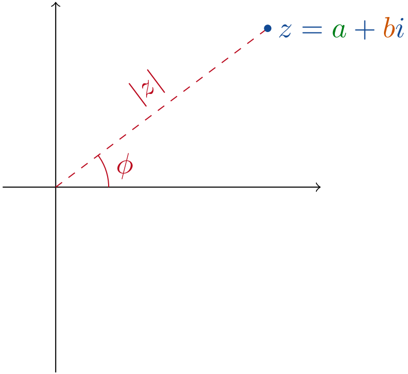


Definition

$|z| = \sqrt{a^2 + b^2}$ is called the *modulus* of z.

Complex Numbers



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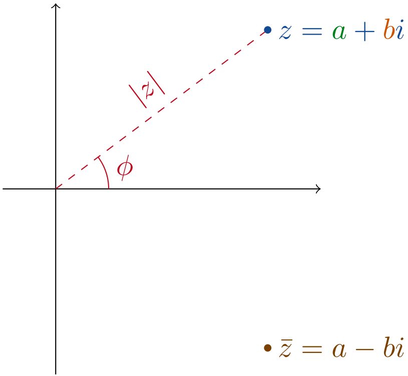
The angle $\arg(z) = \phi$ is called the *argument* of z.

Complex Numbers



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The angle $\arg(z) = \phi$ is called the *argument* of z.

$\bar{z} = a - bi$ is called the *complex conjugate* of z.

Remark

Note that

$$z\bar{z} = (a+bi)(a-bi) = a^2 + b^2 = |z|^2 .$$

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Remark

$$z = \bar{z} \qquad \iff \qquad z \in \mathbb{R}$$

because

$$a + bi = a - bi \qquad \iff \qquad b = 0.$$

Theorem

$$\overline{zw} = \bar{z}\bar{w}.$$

(you prove)

Roots of polynomials

equation	solution	set
$x - 5 = 0$	$x = 5$	$\mathbb{N}$

Roots of polynomials

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Question: Are there any more numbers after $\mathbb{C}$?

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Question: Are there any more numbers after $\mathbb{C}$?

Answer: No.

Theorem (The Fundamental Theorem of Algebra)

Any polynomial with complex coefficients will only have complex roots.

In other words: Using the five operations $+$, $-$, $\times$, $\div$ and a^b, we can not make any more numbers. $\mathbb{C}$ is the end of the line.

Next Time

- Eigenvalues and Eigenvectors
- Diagonalisation
- Complex Vector Spaces