Welcome to

Mathematics III

with Dr Neil Course

MATH113 MATH114 MATH215 MATH216

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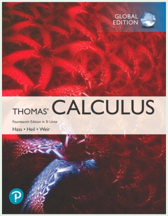
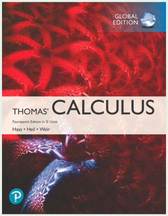


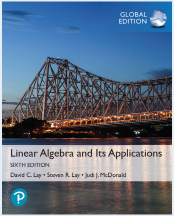
Calculus

MATH113 MATH114



Calculus

MATH215

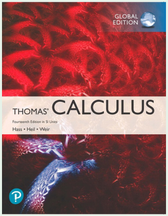


Linear Algebra

MATH216

MATH215 Mathematics III

MATH113 MATH114



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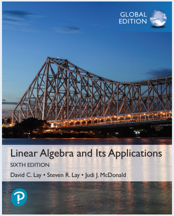

Linear Algebra

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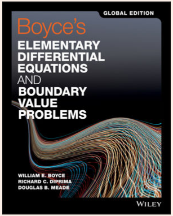
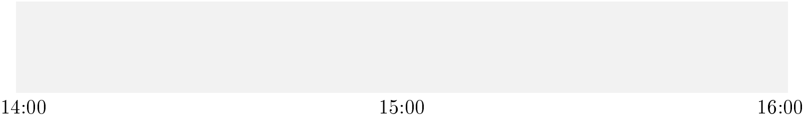


Differential Equations

Information about this course

- $\approx$ 12 classes. Monday afternoons 2pm-4pm.



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- ≈ 12 classes. Monday afternoons 2pm-4pm.
- 2 lectures with a break between.

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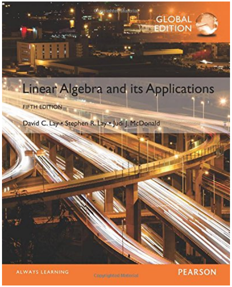
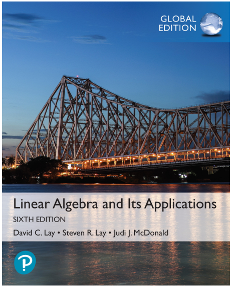
- 2 lectures with a break between.
- Then I will answers your questions.

The Book





David C. Lay, Steven R. Lay and Judi J, McDonald,
Linear Algebra and Its Applications,
5th or 6th Edition, Pearson.

This is a required purchase.

You need to have this book to be able to do the homework.

Syllabus

- linear systems and their solutions
- matrices
- determinants
- inverse matrices
- properties of determinants
- Cramer's Rule
- vector spaces
- subspaces
- linear independence
- basis
- row space
- column space
- null space
- rank and nullity
- linear transformations
- eigenvalues and eigenvectors
- diagonalisation
- inner product spaces
- orthogonality
- the Gram-Schmidt process
- least squares
- orthogonal diagonalisation
- singular value decomposition

Exams and homework

(This information may change based on the University's decisions)

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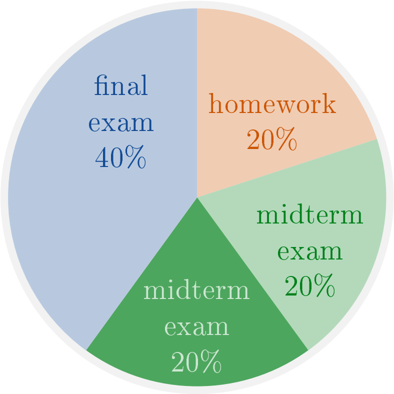

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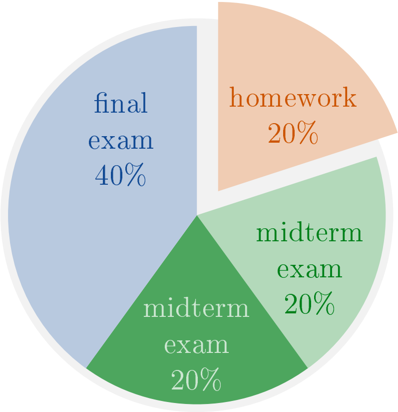

using Pearson MyLab Math

one piece of homework for each lesson

deadline = end of term

more details in O'Learn

Exams and homework

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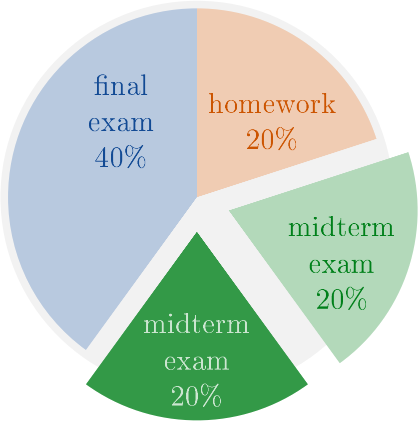

2 midterm exams

face-to-face

Exams and homework

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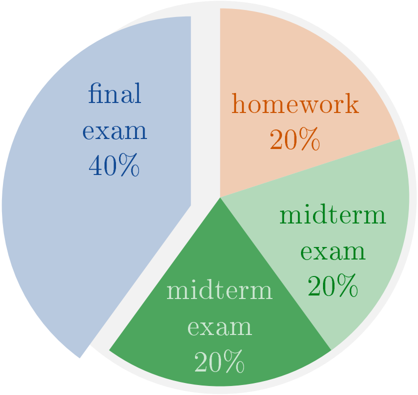
face-to-face



Expectations

If this was a classroom course, the expectation is that for each hour of lectures, you would study an extra 1-2 hours outside of class.

classroom course	lectures (4 hours)	other study (4-8 hours)

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classroom course	lectures (4 hours)	other study (4-8 hours)

For an online course, you are still expected to study a total of 8-12 hours each week.

online course	class (2 hours)	other study (6-10 hours)

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- Read other books?;
- Watch online videos;

$\vdots$

slide number

Lecture 1

- Systems of Linear Equations
- Row Reduction and Echelon Forms

Systems of Linear Equations

Definition

A *linear equation* is an equation of the form

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n = b \tag{1}$$

where

- $x_1, x_2, \ldots, x_n$ are the variables;
- a_j and b are real or complex numbers; and
- n may be any natural number.

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- n may be any natural number.

In this course, n will usually be between 2 and 5. In real-life problems, n might be 50 or 5000, or even larger.

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n = b$$

The equations

$$4x_1 - 5x_2 + 2 = x_1$$

and

$$x_2 = 2\left(\sqrt{6} - x_1\right) + x_3$$

are both linear

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and

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are both linear because they can be rearranged to the standard form

$$3x_1 - 5x_2 = -2$$

and

$$2x_1 + x_2 - x_3 = 2\sqrt{6}.$$

The former is an equation of straight line. The latter is an equation of a plane.

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n = b$$

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are not linear.

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and

$$x_2 = 2\sqrt{x_1} - 6$$

are not linear.

Definition

If an equation is not linear, we say it is *non-linear*.

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n = b$$

Ask the Audience

Are these equations linear or non-linear.

1. $x_1 + x_2 + x_3 + x_4 + x_5 + x_6 = 1$
2. $x_1 + 2x_2 + 3x_3^2 = 4$
3. $x_2 = \cos x_1$
4. $7x_1 - x_3 = 4x_2$
5. $x_1x_2x_3x_4x_5 = 1$

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n = b$$

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Are these equations linear or non-linear.

1. $x_1 + x_2 + x_3 + x_4 + x_5 + x_6 = 1$ Linear
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4. $7x_1 - x_3 = 4x_2$ Linear
5. $x_1x_2x_3x_4x_5 = 1$ Non-linear

Remark

This course is about linear equations.

Definition

A *linear system* is a group of several linear equations:

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots a_{mn}x_n = b_m. \end{cases} \tag{2}$$

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For example

$$\begin{cases} 2x_1 - x_2 + 1.5x_3 = 8 \\ x_1 \qquad - \quad 4x_3 = -7 \end{cases}$$

is a linear system.

Definition

A *solution* of a linear system is a list $(s_1, s_2, \ldots, s_n)$ of numbers which satisfies every equation in the linear system.

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because

$$2x_1 - x_2 + 1.5x_3 = 2(5) - (6.5) + 1.5(3) = 8$$

and

$$x_1 - 4x_3 = (5) - 4(3) = -7.$$

Definition

The set of all possible solutions is called the *solution set* of the linear system

Definition

Two linear systems are called *equivalent* if they have the same solution set.

Example

Find the solution set of

$$\begin{cases} \quad x_1 - 2x_2 = -1 \\ -x_1 + 3x_2 = 3. \end{cases}$$

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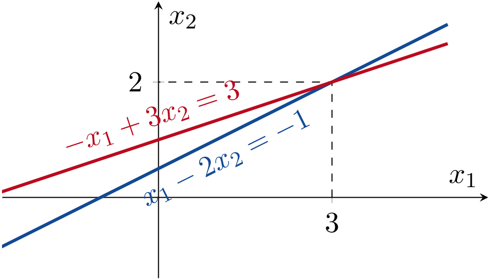
$$\begin{cases} x_1 - 2x_2 = -1 \\ -x_1 + 3x_2 = 3. \end{cases}$$

The graphs of these two equations are lines in $\mathbb{R}^2$.

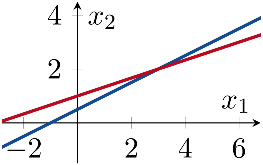


We can see that the only point which is on both lines is $(3, 2)$. So the solution set is the set $\{(3, 2)\}$.

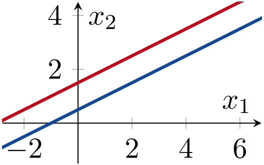
Systems of Linear Equations

There are three possibilities.

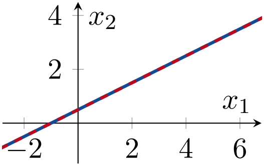
$$\begin{cases} x_1 - 2x_2 = -1 \\ -x_1 + 3x_2 = 3. \end{cases} \qquad \begin{cases} x_1 - 2x_2 = -1 \\ -x_1 + 2x_2 = 3. \end{cases} \qquad \begin{cases} x_1 - 2x_2 = -1 \\ -x_1 + 2x_2 = 1. \end{cases}$$



exactly one solution

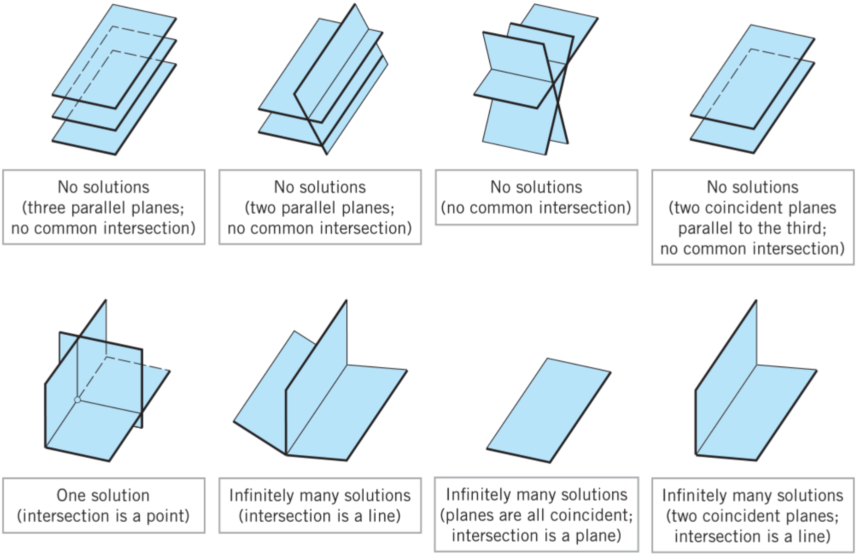


no solutions



infinitely many solutions.

This is also true if we have 3 planes.



Theorem

A linear system has either

1. *zero solutions; or*
2. *exactly one solution; or*
3. *infinitely many solutions.*

There are no other possibilities.

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3. *infinitely many solutions.*

There are no other possibilities.

Definition

A linear system is called *consistent* if it has either one solution or infinitely many solutions.

A linear system is called *inconsistent* if it does not have a solution.

Matrix Notation

Consider the linear system

$$\left\{\begin{array}{rcrcrcr} x_1 & - & 2x_2 & + & x_3 & = & 0 \\ & & 2x_2 & - & 8x_3 & = & 8 \\ -4x_1 & + & 5x_2 & + & 9x_3 & = & -9 \end{array}\right. .$$

What is the important information here?

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What is the important information here?

$$\begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ -4 & 5 & 9 & -9 \end{bmatrix}$$

We don't need the x_j's, the $+$'s or the $=$'s. We only need the numbers.

Definition

$$\begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ -4 & 5 & 9 & -9 \end{bmatrix}$$

is the *augmented matrix* for the linear system

$$\left\{ \begin{array}{rcrcrcr} 1x_1 & - & 2x_2 & + & 1x_3 & = & 0 \\ & & 2x_2 & - & 8x_3 & = & 8 \\ -4x_1 & + & 5x_2 & + & 9x_3 & = & -9 \end{array} \right. .$$

Definition

$$\begin{bmatrix} 1 & -2 & 1 \\ 0 & 2 & -8 \\ -4 & 5 & 9 \end{bmatrix}$$

is the *coefficient matrix* for the linear system

$$\left\{ \begin{array}{rcrcrcr} 1x_1 & - & 2x_2 & + & 1x_3 & = & 0 \\ & & 2x_2 & - & 8x_3 & = & 8 \\ -4x_1 & + & 5x_2 & + & 9x_3 & = & -9 \end{array} \right. .$$

There are three things that we can do to a linear system:

1. Multiply an equation by a number;
2. Swap two equations; or
3. Add a multiple of one equation to another.

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Elementary Row Operations

There are three things that we can do to a matrix:

1. Multiply a row by a number;
2. Swap two rows; or
3. Add a multiple of one row to another.

These are called *elementary row operations*.

Solving a Linear System

Example

Solve

$$\left\{ \begin{array}{rcrcrcr} 1x_1 & - & 2x_2 & + & 1x_3 & = & 0 \\ & & 2x_2 & - & 8x_3 & = & 8 \\ -4x_1 & + & 5x_2 & + & 9x_3 & = & -9 \end{array} \right. .$$

Example

Use elementary row operations to convert the augmented matrix

$$\begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ -4 & 5 & 9 & -9 \end{bmatrix}$$

into

$$\begin{bmatrix} 1 & 0 & 0 & ? \\ 0 & 1 & 0 & ? \\ 0 & 0 & 1 & ? \end{bmatrix}.$$

Remark

There are two things about elementary row operations to learn:

1. **How** to use elementary row operations.
2. **Why** we choose a certain elementary row operation at each step.

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1. **How** to use elementary row operations.
2. **Why** we choose a certain elementary row operation at each step.

I will teach the "how" first. Then in the second hour I will show you the "why".

$$\begin{cases} x_1 - 2x_2 + x_3 = 0 \\ \phantom{-4x_1 + {}} 2x_2 - 8x_3 = 8 \\ -4x_1 + 5x_2 + 9x_3 = -9 \end{cases}$$

1

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$$\begin{array}{rr} & 4 \cdot [\text{equation 1}] \\ + & [\text{equation 3}] \\ \hline & [\text{new equation 3}] \end{array}$$

$$\begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ -4 & 5 & 9 & -9 \end{bmatrix}$$

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1

$$\begin{array}{lr} & 4 \cdot [x_1 - 2x_2 + x_3 = 0] \\ + & [-4x_1 + 5x_2 + 9x_3 = -9] \\ \hline & [\text{new equation } 3] \end{array}$$

$$\begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ -4 & 5 & 9 & -9 \end{bmatrix}$$

1

Systems of Linear Equations

$$\begin{cases} x_1 - 2x_2 + x_3 = 0 \\ \phantom{-4x_1 + {}} 2x_2 - 8x_3 = 8 \\ -4x_1 + 5x_2 + 9x_3 = -9 \end{cases}$$

1

$$\begin{array}{r} 4x_1 - 8x_2 + 4x_3 = 0 \\ \underline{-4x_1 + 5x_2 + 9x_3 = -9} \end{array}$$

$$\begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ -4 & 5 & 9 & -9 \end{bmatrix}$$

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1

$$\begin{array}{r} 4x_1 - 8x_2 + \ \ 4x_3 = 0 \\ -4x_1 + 5x_2 + \ \ 9x_3 = -9 \\ \hline 0x_1 - 3x_2 + 13x_3 = -9 \end{array}$$

1

$$\begin{cases} x_1 - 2x_2 + x_3 = 0 \\ \phantom{x_1 -{}} 2x_2 - 8x_3 = 8 \\ -4x_1 + 5x_2 + 9x_3 = -9 \end{cases}$$

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$$\begin{cases} x_1 - 2x_2 + x_3 = 0 \\ \phantom{x_1 -{}} 2x_2 - 8x_3 = 8 \\ - 3x_2 + 13x_3 = -9 \end{cases}$$

$$\begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ -4 & 5 & 9 & -9 \end{bmatrix}$$

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$$\begin{cases} x_1 - 2x_2 + x_3 = 0 \\ 2x_2 - 8x_3 = 8 \\ -4x_1 + 5x_2 + 9x_3 = -9 \end{cases}$$

$$\begin{array}{cc} \begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ -4 & 5 & 9 & -9 \end{bmatrix} & \begin{array}{l} \leftarrow R_1 \\ \leftarrow R_2 \\ \leftarrow R_3 \end{array} \end{array}$$

1 $4R_1 + R_3 \to \text{new } R_3$

$$\begin{array}{r} 4x_1 - 8x_2 + 4x_3 = 0 \\ \underline{-4x_1 + 5x_2 + 9x_3 = -9} \\ 0x_1 - 3x_2 + 13x_3 = -9 \end{array}$$

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1

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$$\begin{array}{ccccc} & 4 & -8 & 4 & 0 \\ + & -4 & 5 & 9 & -9 \\ \hline & 0 & -3 & 13 & -9 \end{array}$$

$$\begin{cases} x_1 - 2x_2 + x_3 = 0 \\ \phantom{x_1 -{}} 2x_2 - 8x_3 = 8 \\ - 3x_2 + 13x_3 = -9 \end{cases}$$

$$\begin{cases} x_1 - 2x_2 + x_3 = 0 \\ \phantom{-4x_1+{}} 2x_2 - 8x_3 = 8 \\ -4x_1 + 5x_2 + 9x_3 = -9 \end{cases}$$

$$\begin{array}{c} \\ \\ \\ \end{array}\begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ -4 & 5 & 9 & -9 \end{bmatrix}\begin{array}{l} \leftarrow R_1 \\ \leftarrow R_2 \\ \leftarrow R_3 \end{array}$$

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$$\begin{array}{r} 4x_1 - 8x_2 + 4x_3 = 0 \\ -4x_1 + 5x_2 + 9x_3 = -9 \\ \hline 0x_1 - 3x_2 + 13x_3 = -9 \end{array}$$

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$$\begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ 0 & -3 & 13 & -9 \end{bmatrix}\begin{array}{l} \\ \\ \leftarrow \text{new } R_3 \end{array}$$

$$\begin{cases} x_1 - 2x_2 + x_3 = 0 \\ 2x_2 - 8x_3 = 8 \\ - 3x_2 + 13x_3 = -9 \end{cases}$$

$$\begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ 0 & -3 & 13 & -9 \end{bmatrix}$$

2. Multiply [equation 2] by $\frac{1}{2}$.

2. $\frac{1}{2}R_2 \to$ new R_2

$$\begin{cases} x_1 - 2x_2 + x_3 = 0 \\ \phantom{x_1 - {}} 2x_2 - 8x_3 = 8 \\ - 3x_2 + 13x_3 = -9 \end{cases}$$

$$\begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ 0 & -3 & 13 & -9 \end{bmatrix}$$

2 Multiply [equation 2] by $\frac{1}{2}$.

2 $\frac{1}{2}R_2 \to \text{new } R_2$

$$\begin{cases} x_1 - 2x_2 + x_3 = 0 \\ \phantom{x_1 - 2{}} x_2 - 4x_3 = 4 \\ - 3x_2 + 13x_3 = -9 \end{cases}$$

$$\begin{cases} x_1 - 2x_2 + x_3 = 0 \\ 2x_2 - 8x_3 = 8 \\ - 3x_2 + 13x_3 = -9 \end{cases}$$

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$$\begin{cases} x_1 - 2x_2 + x_3 = 0 \\ x_2 - 4x_3 = 4 \\ - 3x_2 + 13x_3 = -9 \end{cases}$$

3

$$\begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 1 & -4 & 4 \\ 0 & -3 & 13 & -9 \end{bmatrix}$$

3 $3R_2 + R_3 \to \text{new } R_3$

$$\begin{cases} x_1 - 2x_2 + x_3 = 0 \\ x_2 - 4x_3 = 4 \\ - 3x_2 + 13x_3 = -9 \end{cases}$$

3

$$\begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 1 & -4 & 4 \\ 0 & -3 & 13 & -9 \end{bmatrix}$$

3 $3R_2 + R_3 \to \text{new } R_3$

$$\begin{array}{rrrrr} & 0 & 3 & -12 & 12 \\ + & 0 & -3 & 13 & -9 \\ \hline \end{array}$$

$$\begin{cases} x_1 - 2x_2 + \quad x_3 = 0 \\ \qquad\quad x_2 - \ 4x_3 = 4 \\ \quad - 3x_2 + 13x_3 = -9 \end{cases}$$

3

$$\begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 1 & -4 & 4 \\ 0 & -3 & 13 & -9 \end{bmatrix}$$

3 $3R_2 + R_3 \rightarrow$ new R_3

$$\begin{array}{ccccc} & 0 & 3 & -12 & 12 \\ + & 0 & -3 & 13 & -9 \\ \hline & 0 & 0 & 1 & 3 \end{array}$$

Systems of Linear Equations

$$\begin{cases} x_1 - 2x_2 + x_3 = 0 \\ x_2 - 4x_3 = 4 \\ - 3x_2 + 13x_3 = -9 \end{cases}$$

$$\begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 1 & -4 & 4 \\ 0 & -3 & 13 & -9 \end{bmatrix}$$

3

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$$\begin{array}{rrrrr} & 0 & 3 & -12 & 12 \\ + & 0 & -3 & 13 & -9 \\ \hline & 0 & 0 & 1 & 3 \end{array}$$

$$\begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 1 & -4 & 4 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

$$\begin{cases} x_1 - 2x_2 + x_3 = 0 \\ x_2 - 4x_3 = 4 \\ - 3x_2 + 13x_3 = -9 \end{cases}$$

3. $3\times$Eq2 + Eq3 $\rightarrow$ new Eq3.

$$\begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 1 & -4 & 4 \\ 0 & -3 & 13 & -9 \end{bmatrix}$$

3. $3R_2 + R_3 \rightarrow$ new R_3

$$\begin{array}{rrrrr} & 0 & 3 & -12 & 12 \\ + & 0 & -3 & 13 & -9 \\ \hline & 0 & 0 & 1 & 3 \end{array}$$

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$$\begin{cases} x_1 - 2x_2 + \quad x_3 = 0 \\ \qquad\quad x_2 - \ \ 4x_3 = 4 \\ \quad - 3x_2 + 13x_3 = -9 \end{cases}$$

3 $3\times$Eq2 + Eq3 $\to$ new Eq3.

$$\begin{array}{r} 3x_2 - 12x_3 = 12 \\ \underline{-3x_2 + 13x_3 = -9} \end{array}$$

$$\begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 1 & -4 & 4 \\ 0 & -3 & 13 & -9 \end{bmatrix}$$

3 $3R_2 + R_3 \to$ new R_3

$$\begin{array}{ccccc} & 0 & 3 & -12 & 12 \\ + & 0 & -3 & 13 & -9 \\ \hline & 0 & 0 & 1 & 3 \end{array}$$

$$\begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 1 & -4 & 4 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

Systems of Linear Equations

$$\begin{cases} x_1 - 2x_2 + \quad x_3 = 0 \\ \qquad\quad x_2 - \ \ 4x_3 = 4 \\ \quad\ - 3x_2 + 13x_3 = -9 \end{cases}$$

3 $3\times$Eq2 + Eq3 $\to$ new Eq3.

$$\begin{array}{r} 3x_2 - 12x_3 = 12 \\ \underline{-3x_2 + 13x_3 = -9} \\ 0x_2 + \quad x_3 = 3 \end{array}$$

$$\begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 1 & -4 & 4 \\ 0 & -3 & 13 & -9 \end{bmatrix}$$

3 $3R_2 + R_3 \to$ new R_3

$$\begin{array}{ccccc} & 0 & 3 & -12 & 12 \\ + & 0 & -3 & 13 & -9 \\ \hline & 0 & 0 & 1 & 3 \end{array}$$

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Systems of Linear Equations

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$$\begin{array}{ccccc} & 0 & 3 & -12 & 12 \\ + & 0 & -3 & 13 & -9 \\ \hline & 0 & 0 & 1 & 3 \end{array}$$

$$\begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 1 & -4 & 4 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

$$\begin{cases} x_1 - 2x_2 + x_3 = 0 \\ x_2 - 4x_3 = 4 \\ x_3 = 3 \end{cases}$$

4 Eq2 $+$ 4$\times$Eq3 $\to$ new Eq2.

$$\begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 1 & -4 & 4 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

4 $R_2 + 4R_3 \to \text{new } R_2$

$$\begin{cases} x_1 - 2x_2 + x_3 = 0 \\ x_2 - 4x_3 = 4 \\ x_3 = 3 \end{cases}$$

4 Eq2 + 4×Eq3 → new Eq2.

$$\begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 1 & -4 & 4 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

4 $R_2 + 4R_3 \rightarrow \text{new } R_2$

$$\begin{array}{rrrrr} & 0 & 1 & -4 & 4 \\ + & 0 & 0 & 4 & 12 \\ \hline & 0 & 1 & 0 & 16 \end{array}$$

$$\begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 1 & 0 & 16 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

$$\begin{cases} x_1 - 2x_2 + x_3 = 0 \\ x_2 - 4x_3 = 4 \\ x_3 = 3 \end{cases}$$

$$\begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 1 & -4 & 4 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

4 Eq2 + 4×Eq3 → new Eq2.

$$\begin{array}{r} x_2 - 4x_3 = 4 \\ 4x_3 = 12 \\ \hline x_2 = 16 \end{array}$$

4 $R_2 + 4R_3 \to \text{new } R_2$

$$\begin{array}{ccccc} & 0 & 1 & -4 & 4 \\ + & 0 & 0 & 4 & 12 \\ \hline & 0 & 1 & 0 & 16 \end{array}$$

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$$\begin{cases} x_1 - 2x_2 + x_3 = 0 \\ x_2 - 4x_3 = 4 \\ x_3 = 3 \end{cases}$$

5 Eq1 $-$ Eq3 $\to$ new Eq1.

$$\begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 1 & 0 & 16 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

5 $R_1 - R_3 \to$ new R_1

$$\begin{cases} x_1 - 2x_2 + x_3 = 0 \\ x_2 - 4x_3 = 4 \\ x_3 = 3 \end{cases}$$

5 Eq1 − Eq3 → new Eq1.

$$\begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 1 & 0 & 16 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

5 $R_1 - R_3 \to$ new R_1

$$\begin{array}{rrrrr} & 1 & -2 & 1 & 0 \\ + & 0 & 0 & -1 & -3 \\ \hline & 1 & -2 & 0 & -3 \end{array}$$

$$\begin{bmatrix} 1 & -2 & 0 & -3 \\ 0 & 1 & 0 & 16 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

$$\begin{cases} x_1 - 2x_2 + \ \ x_3 = 0 \\ \qquad\quad\ x_2 - 4x_3 = 4 \\ \qquad\qquad\qquad x_3 = 3 \end{cases}$$

5 Eq1 − Eq3 → new Eq1.

$$\begin{array}{r} x_1 - 2x_2 + x_3 = 0 \\ - x_3 = -3 \\ \hline x_1 - 2x_2 \qquad = -3 \end{array}$$

$$\begin{cases} x_1 - 2x_2 \qquad = -3 \\ \qquad\quad x_2 \qquad = 16 \\ \qquad\qquad\quad x_3 = 3 \end{cases}$$

$$\begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 1 & 0 & 16 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

5 $R_1 - R_3 \to$ new R_1

$$\begin{array}{rrrrr} & 1 & -2 & 1 & 0 \\ + & 0 & 0 & -1 & -3 \\ \hline & 1 & -2 & 0 & -3 \end{array}$$

$$\begin{bmatrix} 1 & -2 & 0 & -3 \\ 0 & 1 & 0 & 16 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

$$\begin{cases} x_1 - 2x_2 = -3 \\ x_2 - 4x_3 = 4 \\ x_3 = 3 \end{cases}$$

6 Eq1 + 2×Eq2 → new Eq1.

$$\begin{bmatrix} 1 & -2 & 0 & -3 \\ 0 & 1 & 0 & 16 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

6 $R_1 + 2R_2 \to \text{new } R_1$

$$\begin{cases} x_1 - 2x_2 = -3 \\ x_2 - 4x_3 = 4 \\ x_3 = 3 \end{cases}$$

6 Eq1 + 2×Eq2 → new Eq1.

$$\begin{bmatrix} 1 & -2 & 0 & -3 \\ 0 & 1 & 0 & 16 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

6 $R_1 + 2R_2 \to \text{new } R_1$

$$\begin{array}{rrrrr} & 1 & -2 & 0 & -3 \\ + & 0 & 2 & 0 & 32 \\ \hline & 1 & 0 & 0 & 29 \end{array}$$

$$\begin{bmatrix} 1 & 0 & 0 & 29 \\ 0 & 1 & 0 & 16 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

$$\begin{cases} x_1 - 2x_2 \phantom{{}-4x_3} = -3 \\ \phantom{x_1 - {}} x_2 - 4x_3 = 4 \\ \phantom{x_1 - 2x_2 - {}} x_3 = 3 \end{cases}$$

6 Eq1 + 2×Eq2 → new Eq1.

$$\begin{array}{rcr} x_1 - 2x_2 & = & -3 \\ 2x_2 & = & 32 \\ \hline x_1 & = & 29 \end{array}$$

$$\begin{cases} x_1 \phantom{{}+x_2+x_3} = 29 \\ \phantom{x_1 + {}} x_2 \phantom{{}+x_3} = 16 \\ \phantom{x_1 + x_2 + {}} x_3 = 3 \end{cases}$$

$$\begin{bmatrix} 1 & -2 & 0 & -3 \\ 0 & 1 & 0 & 16 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

6 $R_1 + 2R_2 \to \text{new } R_1$

$$\begin{array}{rrrrr} & 1 & -2 & 0 & -3 \\ + & 0 & 2 & 0 & 32 \\ \hline & 1 & 0 & 0 & 29 \end{array}$$

$$\begin{bmatrix} 1 & 0 & 0 & 29 \\ 0 & 1 & 0 & 16 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

Example

Solve

$$\begin{cases} 1x_1 - 2x_2 + 1x_3 = 0 \\ \qquad\quad 2x_2 - 8x_3 = 8 \\ -4x_1 + 5x_2 + 9x_3 = -9 \end{cases}.$$

$$\begin{cases} x_1 = 29 \\ x_2 = 16 \\ x_3 = 3 \end{cases}$$

Example

Use elementary row operations to convert the augmented matrix

$$\begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ -4 & 5 & 9 & -9 \end{bmatrix} \quad \text{into} \quad \begin{bmatrix} 1 & 0 & 0 & ? \\ 0 & 1 & 0 & ? \\ 0 & 0 & 1 & ? \end{bmatrix}.$$

$$\begin{bmatrix} 1 & 0 & 0 & 29 \\ 0 & 1 & 0 & 16 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

Definition

Two matrices A and B are called *row equivalent* if it is possible to change one into the other using elementary row operations.

We write $A \sim B$ if A and B are row equivalent.

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Two matrices A and B are called *row equivalent* if it is possible to change one into the other using elementary row operations.

We write $A \sim B$ if A and B are row equivalent.

Theorem

If the augmented matrices of two linear systems are row equivalent, then the two systems have the same solution set.

Elementary Row Operations (again)

1. Multiply a row by a number (e.g. $cR_3 \to \text{new}R_3$).
2. Swap two rows (e.g. $R_1 \leftrightarrow R_2$).
3. Add a multiple of one row to another
(e.g. $cR_3 + R_4 \to \text{new}R_4$).

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Note that all three of these are invertible.

1. The opposite of multiplying a row by $c \neq 0$ is multiplying it by $\frac{1}{c}$.

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Note that all three of these are invertible.

1. The opposite of multiplying a row by $c \neq 0$ is multiplying it by $\frac{1}{c}$.
2. Swapping two rows is it's own inverse.
3. The opposite of adding something, is subtracting that something.

Existence and Uniqueness Questions

Example

Recall that

$$\begin{cases} x_1 - 2x_2 + x_3 = 0 \\ \phantom{-4x_1 + {}} 2x_2 - 8x_3 = 8 \\ -4x_1 + 5x_2 + 9x_3 = -9 \end{cases} \qquad \text{has solution} \qquad \begin{cases} x_1 = 29 \\ x_2 = 16 \\ x_3 = 3. \end{cases}$$

Therefore this linear system is consistent.

Example

Is

$$\begin{cases} x_2 - 4x_3 = 8 \\ 2x_1 - 3x_2 + 2x_3 = 1 \\ 5x_1 - 8x_2 + 7x_3 = 1 \end{cases}$$

consistent?

Example

Is

$$\begin{cases} \qquad\qquad x_2 - 4x_3 = 8 \\ 2x_1 - 3x_2 + 2x_3 = 1 \\ 5x_1 - 8x_2 + 7x_3 = 1 \end{cases}$$

consistent?

To answer this, we will do elementary row operations on the augmented matrix

$$\begin{bmatrix} 0 & 1 & -4 & 8 \\ 2 & -3 & 2 & 1 \\ 5 & -8 & 7 & 1 \end{bmatrix}$$

and see what we get.

$$\begin{bmatrix} 0 & 1 & -4 & 8 \\ 2 & -3 & 2 & 1 \\ 5 & -8 & 7 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 1 & -4 & 8 \\ 2 & -3 & 2 & 1 \\ 5 & -8 & 7 & 1 \end{bmatrix}$$

1. $R_1 \leftrightarrow R_2$

$$\begin{bmatrix} 2 & -3 & 2 & 1 \\ 0 & 1 & -4 & 8 \\ 5 & -8 & 7 & 1 \end{bmatrix}$$

Systems of Linear Equations

$$\begin{bmatrix} 0 & 1 & -4 & 8 \\ 2 & -3 & 2 & 1 \\ 5 & -8 & 7 & 1 \end{bmatrix}$$

1. $R_1 \leftrightarrow R_2$

$$\begin{bmatrix} 2 & -3 & 2 & 1 \\ 0 & 1 & -4 & 8 \\ 5 & -8 & 7 & 1 \end{bmatrix}$$

2. $R_3 - \frac{5}{2}R_1 \to R_3$

$$\begin{array}{rrrrr} & 5 & -8 & 7 & 1 \\ + & -5 & \frac{15}{2} & -5 & -\frac{5}{2} \\ \hline & 0 & -\frac{1}{2} & 2 & -\frac{3}{2} \end{array} \qquad \begin{bmatrix} 2 & -3 & 2 & 1 \\ 0 & 1 & -4 & 8 \\ 0 & -\frac{1}{2} & 2 & -\frac{3}{2} \end{bmatrix}$$

$$\begin{bmatrix} 2 & -3 & 2 & 1 \\ 0 & 1 & -4 & 8 \\ 0 & -\frac{1}{2} & 2 & -\frac{3}{2} \end{bmatrix}$$

$$\begin{bmatrix} 2 & -3 & 2 & 1 \\ 0 & 1 & -4 & 8 \\ 0 & -\frac{1}{2} & 2 & -\frac{3}{2} \end{bmatrix}$$

3. $\frac{1}{2}R_2 + R_3 \to R_3$

$$\begin{array}{rrrrr} & 0 & \frac{1}{2} & -2 & 4 \\ + & 0 & -\frac{1}{2} & 2 & -\frac{3}{2} \\ \hline & 0 & 0 & 0 & \frac{5}{2} \end{array}$$

$$\begin{bmatrix} 2 & -3 & 2 & 1 \\ 0 & 1 & -4 & 8 \\ 0 & 0 & 0 & \frac{5}{2} \end{bmatrix}$$

$$\begin{bmatrix} 2 & -3 & 2 & 1 \\ 0 & 1 & -4 & 8 \\ 0 & -\frac{1}{2} & 2 & -\frac{3}{2} \end{bmatrix}$$

3. $\frac{1}{2}R_2 + R_3 \to R_3$

$$\begin{array}{ccccc} & 0 & \frac{1}{2} & -2 & 4 \\ + & 0 & -\frac{1}{2} & 2 & -\frac{3}{2} \\ \hline & 0 & 0 & 0 & \frac{5}{2} \end{array} \qquad \begin{bmatrix} 2 & -3 & 2 & 1 \\ 0 & 1 & -4 & 8 \\ 0 & 0 & 0 & \frac{5}{2} \end{bmatrix}$$

4. Convert back to a linear system.

$$\begin{cases} 2x_1 - 3x_2 + 2x_3 = 1 \\ \qquad\quad x_2 - 4x_3 = 8 \\ 0x_1 + 0x_2 + 0x_3 = \frac{5}{2}. \end{cases}$$

$$\begin{cases} \qquad\quad x_2 - 4x_3 = 8 \\ 2x_1 - 3x_2 + 2x_3 = 1 \\ 5x_1 - 8x_2 + 7x_3 = 1 \end{cases} \qquad \to \qquad \begin{cases} 2x_1 - 3x_2 + 2x_3 & = 1 \\ \qquad\quad x_2 - 4x_3 & = 8 \\ & 0 = \frac{5}{2}. \end{cases}$$

But the third equation says

$$0 = \frac{5}{2}$$

which is clearly nonsense.

$$\begin{cases} \phantom{2x_1 - 3x_2 + {}} x_2 - 4x_3 = 8 \\ 2x_1 - 3x_2 + 2x_3 = 1 \\ 5x_1 - 8x_2 + 7x_3 = 1 \end{cases} \quad \rightarrow \quad \begin{cases} 2x_1 - 3x_2 + 2x_3 = 1 \\ \phantom{2x_1 - 3x_2 + {}} x_2 - 4x_3 = 8 \\ 0 = \frac{5}{2}. \end{cases}$$

But the third equation says

$$0 = \frac{5}{2}$$

which is clearly nonsense.

This means that it is not possible to find a solution to this linear system. So the linear system is inconsistent.

Example

For what values of h and k is the following system consistent?

$$\begin{cases} 2x_1 - x_2 = h \\ -6x_1 + 3x_2 = k \end{cases}$$

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To answer this, we do one row operation on this system:

$$\begin{bmatrix} 2 & -1 & h \\ -6 & 3 & k \end{bmatrix} \xrightarrow{3R_1+R_2} \begin{bmatrix} 2 & -1 & h \\ 0 & 0 & 3h+k \end{bmatrix}.$$

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Notice that the latter matrix is equivalent to

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Obviously, we require that $3h + k = 0$ in order for this system to have solutions.

So this linear system is consistent only when $3h + k = 0$.

Remark

It doesn't matter if you use [ ] or ( ) when you write an augmented matrix.

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}$$

Example

Find an equation involving g, h, and k that makes this augmented matrix correspond to a consistent system:

$$\begin{pmatrix} 1 & -4 & 7 & g \\ 0 & 3 & -5 & h \\ -2 & 5 & -9 & k \end{pmatrix}.$$

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Find an equation involving g, h, and k that makes this augmented matrix correspond to a consistent system:

$$\begin{pmatrix} 1 & -4 & 7 & g \\ 0 & 3 & -5 & h \\ -2 & 5 & -9 & k \end{pmatrix}.$$

$$\begin{pmatrix} 1 & -4 & 7 & g \\ 0 & 3 & -5 & h \\ -2 & 5 & -9 & k \end{pmatrix} \xrightarrow{2R_1+R_3} \begin{pmatrix} 1 & -4 & 7 & g \\ 0 & 3 & -5 & h \\ 0 & -3 & 5 & 2g+k \end{pmatrix}$$

$$\xrightarrow{R_2+R_3} \begin{pmatrix} 1 & -4 & 7 & g \\ 0 & 3 & -5 & h \\ 0 & 0 & 0 & 2g+h+k \end{pmatrix}.$$

From the last augmented matrix, we see that we must $2g+h+k=0$ for a consistent system.

Example

Determine if the following system is consistent or inconsistent. If it is consistent, find <u>all</u> the solutions.

$$\begin{cases} 3x_1 + 5x_2 - 4x_3 = 7 \\ -3x_1 - 2x_2 + 4x_3 = -1 \\ 6x_1 + x_2 - 8x_3 = -4. \end{cases}$$

$$\begin{pmatrix} 3 & 5 & -4 & 7 \\ -3 & -2 & 4 & -1 \\ 6 & 1 & -8 & -4 \end{pmatrix}$$

$$\begin{pmatrix} 3 & 5 & -4 & 7 \\ -3 & -2 & 4 & -1 \\ 6 & 1 & -8 & -4 \end{pmatrix} \xrightarrow[-2R_1+R_3]{R_1+R_2} \begin{pmatrix} 3 & 5 & -4 & 7 \\ 0 & 3 & 0 & 6 \\ 0 & -9 & 0 & -18 \end{pmatrix}$$

$$\xrightarrow[\frac{1}{3}R_2]{3R_2+R_3} \begin{pmatrix} 3 & 5 & -4 & 7 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\xrightarrow{-5R_2+R_1} \begin{pmatrix} 3 & 0 & -4 & -3 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

The linear system is equivalent to

$$\begin{cases} 3x_1 \qquad\quad - 4x_3 = -3 \\ \qquad x_2 \qquad\quad = 2 \end{cases}$$

$$\begin{cases} 3x_1 \qquad - 4x_3 = -3 \\ \qquad x_2 \qquad = 2 \end{cases}$$

Thus $x_1 = -1 + \frac{4}{3}x_3$ and every solution has the form of

$$\begin{aligned} (x_1, x_2, x_3) &= (-1 + \tfrac{4}{3}x_3, 2, x_3) \\ &= x_3(\tfrac{4}{3}, 0, 1) + (-1, 2, 0). \end{aligned}$$

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Here, x_3 can take on any real number so there are infinitely many solutions. Geometrically, the solution set is a line passing through the point $(-1, 2, 0)$ parallel to the vector $(\frac{4}{3}, 0, 1)$.

Example

Do the three planes
$$\begin{cases} x_1 + 3x_2 + x_3 = 4 \\ \quad\quad\;\; x_2 - x_3 = 1 \\ x_1 + 4x_2 \quad\quad\; = -2 \end{cases}$$
have a point in common? Explain.

Example

Do the three planes $\begin{cases} x_1 + 3x_2 + x_3 = 4 \\ x_2 - x_3 = 1 \\ x_1 + 4x_2 = -2 \end{cases}$ have a point in common? Explain.

$$\begin{pmatrix} 1 & 3 & 1 & 4 \\ 0 & 1 & -1 & 1 \\ 1 & 4 & 0 & -2 \end{pmatrix} \xrightarrow{-R_1+R_3} \begin{pmatrix} 1 & 3 & 1 & 4 \\ 0 & 1 & -1 & 1 \\ 0 & 1 & -1 & -6 \end{pmatrix}$$

$$\xrightarrow{-R_2+R_3} \begin{pmatrix} 1 & 3 & 1 & 4 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & -7 \end{pmatrix}$$

Example

Do the three planes $\begin{cases} x_1 + 3x_2 + x_3 = 4 \\ \quad\quad\;\; x_2 - x_3 = 1 \\ x_1 + 4x_2 \quad\quad\; = -2 \end{cases}$ have a point in common? Explain.

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The last line of this augmented matrix reads

$$0x_1 + 0x_2 + 0x_3 = -7,$$

which, of course, is nonsense.

Example

Do the three planes $\begin{cases} x_1 + 3x_2 + x_3 = 4 \\ \qquad\quad x_2 - x_3 = 1 \\ x_1 + 4x_2 \qquad\ = -2 \end{cases}$ have a point in common? Explain.

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$$\xrightarrow{-R_2+R_3} \begin{pmatrix} 1 & 3 & 1 & 4 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & -7 \end{pmatrix}$$

The last line of this augmented matrix reads

$$0x_1 + 0x_2 + 0x_3 = -7,$$

which, of course, is nonsense.

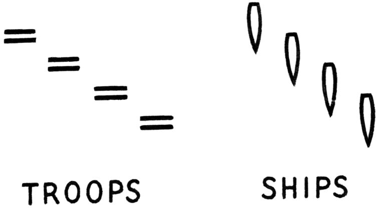
So there is **no** point in common for the three planes.

Break

We will continue at 3pm

Row Reduction and Echelon Forms

Row Reduction and Echelon Forms



An **echelon formation** *is a (usually military) formation in which its units are arranged diagonally.*

Echelon is pronounced “esh-E-lon” because it comes from the French language.

$$\begin{bmatrix} 2 & -3 & 2 & 1 & -7 & 4 \\ 0 & 1 & -4 & 8 & 0 & 0 \\ 0 & 0 & 0 & 5 & 2 & 1 \\ 0 & 0 & 0 & 0 & 0 & 6 \end{bmatrix}$$

Definition

The first non-zero number in each row, is called a *leading entry* or *pivot*.

leading entry/pivot

$$\begin{bmatrix} 2 & -3 & 2 & 1 & -7 & 4 \\ 0 & 1 & -4 & 8 & 0 & 0 \\ 0 & 0 & 0 & 5 & 2 & 1 \\ 0 & 0 & 0 & 0 & 0 & 6 \end{bmatrix}$$

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Definition

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leading entry/pivot

$$\begin{bmatrix} \color{green}{2} & -3 & 2 & 1 & -7 & 4 \\ 0 & \color{green}{1} & -4 & 8 & 0 & 0 \\ 0 & 0 & 0 & \color{green}{5} & 2 & 1 \\ 0 & 0 & 0 & 0 & 0 & 6 \end{bmatrix}$$

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The first non-zero number in each row, is called a *leading entry* or *pivot*.

leading entry/pivot

$$\begin{bmatrix} \color{green}{2} & -3 & 2 & 1 & -7 & 4 \\ 0 & \color{green}{1} & -4 & 8 & 0 & 0 \\ 0 & 0 & 0 & \color{green}{5} & 2 & 1 \\ 0 & 0 & 0 & 0 & 0 & \color{green}{6} \end{bmatrix}$$

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The first non-zero number in each row, is called a *leading entry* or *pivot*.

$$\begin{bmatrix} 2 & 3 & 2 & 1 & 7 & 4 \\ 0 & 1 & 4 & 8 & 0 & 0 \\ 0 & 0 & 0 & 5 & 2 & 1 \\ 0 & 0 & 0 & 0 & 0 & 6 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Definition

A matrix is in *row echelon form* (REF) iff the following 3 rules are satisfied:

Row Reduction and Echelon Forms

$$\begin{bmatrix} 2 & 3 & 2 & 1 & 7 & 4 \\ 0 & 1 & 4 & 8 & 0 & 0 \\ 0 & 0 & 0 & 5 & 2 & 1 \\ 0 & 0 & 0 & 0 & 0 & 6 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{array}{l} \left.\begin{matrix} \\ \\ \\ \\ \end{matrix}\right\} \text{non-zero rows at the top} \\ \left.\begin{matrix} \\ \\ \end{matrix}\right\} \text{zero rows at the bottom} \end{array}$$

Definition

A matrix is in *row echelon form* (REF) iff the following 3 rules are satisfied:

1. All the $00000 \cdots 0$ rows are at the bottom;

Row Reduction and Echelon Forms

$$\begin{bmatrix} 2 & 3 & 2 & 1 & 7 & 4 \\ 0 & 1 & 4 & 8 & 0 & 0 \\ 0 & 0 & 0 & 5 & 2 & 1 \\ 0 & 0 & 0 & 0 & 0 & 6 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

down and right

Definition

A matrix is in *row echelon form* (REF) iff the following 3 rules are satisfied:

1. All the $00000\cdots0$ rows are at the bottom;
2. The leading entries/pivots go $\searrow$; and

$$\begin{bmatrix} 2 & 3 & 2 & 1 & 7 & 4 \\ 0 & 1 & 4 & 8 & 0 & 0 \\ 0 & 0 & 0 & 5 & 2 & 1 \\ 0 & 0 & 0 & 0 & 0 & 6 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

zeros under pivots

Definition

A matrix is in *row echelon form* (REF) iff the following 3 rules are satisfied:

1. All the $00000\cdots0$ rows are at the bottom;
2. The leading entries/pivots go $\searrow$; and
3. All the entries below a leading entry are zero.

Row Reduction and Echelon Forms

Definition

A matrix is in *reduced row echelon form* (RREF) iff the following 5 rules are satisfied:

1. All the $00000\cdots0$ rows are at the bottom;
2. The leading entries/pivots go $\searrow$; and
3. All the entries below a leading entry are zero.

$$\begin{bmatrix} 1 & 0 & 2 & 0 & 7 & 0 \\ 0 & 1 & 4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Definition

A matrix is in *reduced row echelon form* (RREF) iff the following 5 rules are satisfied:

1. All the $00000\cdots0$ rows are at the bottom;
2. The leading entries/pivots go $\searrow$; and
3. All the entries below a leading entry are zero.
4. All the leading entries/pivots are 1

$$\begin{bmatrix} 1 & 0 & 2 & 0 & 7 & 0 \\ 0 & 1 & 4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Row Reduction and Echelon Forms

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1. All the $00000\cdots0$ rows are at the bottom;
2. The leading entries/pivots go $\searrow$; and
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4. All the leading entries/pivots are 1
5. Each leading 1 is the only non-zero entry in its column.

$$\begin{bmatrix} 1 & 0 & 2 & 0 & 7 & 0 \\ 0 & 1 & 4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

REF: rules 1-3 RREF: all 5 rules

Example

Consider

$$\begin{bmatrix} 2 & -3 & 2 & 1 \\ 0 & 1 & -4 & 8 \\ 0 & 0 & 0 & 5/2 \end{bmatrix}.$$

Is this matrix in REF? Is this matrix in RREF?

REF: rules 1-3 RREF: all 5 rules

1. All the $00000\cdots 0$ rows are at the bottom;
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REF: rules 1-3 RREF: all 5 rules

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Is this matrix in REF? Is this matrix in RREF?

REF: rules 1-3 RREF: all 5 rules

1. All the $00000\cdots0$ rows are at the bottom; ✓
2. The leading entries/pivots go ↘; and ✓
3. All the entries below a leading entry are zero. ✓
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Example

Consider

$$\begin{bmatrix} 2 & -3 & 2 & 1 \\ 0 & 1 & -4 & 8 \\ 0 & 0 & 0 & 5/2 \end{bmatrix}.$$

Is this matrix in REF? Is this matrix in RREF?

This matrix is in REF,

REF: rules 1-3 RREF: all 5 rules

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Example

Consider

$$\begin{bmatrix} 2 & -3 & 2 & 1 \\ 0 & 1 & -4 & 8 \\ 0 & 0 & 0 & 5/2 \end{bmatrix}.$$

Is this matrix in REF? Is this matrix in RREF?

This matrix is in REF, but is not in RREF.

REF: rules 1-3 RREF: all 5 rules

Example

Consider

$$\begin{bmatrix} 1 & 0 & 0 & -2 & 3 & 0 & 2 \\ 0 & 0 & 1 & 3 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Is this matrix in REF? Is this matrix in RREF?

REF: rules 1-3 RREF: all 5 rules

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Consider

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Is this matrix in REF? Is this matrix in RREF?

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Example

Consider

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Consider

$$\begin{bmatrix} 1 & 0 & 0 & -2 & 3 & 0 & 2 \\ 0 & 0 & 1 & 3 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Is this matrix in REF? Is this matrix in RREF?

REF: rules 1-3 RREF: all 5 rules

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Example

Consider

$$\begin{bmatrix} 1 & 0 & 0 & -2 & 3 & 0 & 2 \\ 0 & 0 & 1 & 3 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Is this matrix in REF? Is this matrix in RREF?

This matrix is in REF

REF: rules 1-3 RREF: all 5 rules

1. All the $00000\cdots0$ rows are at the bottom; ✓
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Example

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Example

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Is this matrix in REF? Is this matrix in RREF?

This matrix is in REF and in RREF.

REF: rules 1-3 RREF: all 5 rules

Example

Consider

$$\begin{bmatrix} 1 & 0 & 0 & -2 & 3 & 0 & 2 \\ 0 & 0 & 1 & 3 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Is this matrix in REF? Is this matrix in RREF?

This matrix is in REF and in RREF.

In this matrix, the **pivot columns** are the **first**, **third**, and **sixth** columns and the **pivot entries** are the leading nonzero entries in the **first**, **second** and **third** rows.

XAMPLE 1 The following matrices are in echelon form. The leading entries ($\blacksquare$) ay have any nonzero value; the starred entries ($*$) may have any value (including zero).

$$\begin{bmatrix} \blacksquare & * & * & * \\ 0 & \blacksquare & * & * \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & \blacksquare & * & * & * & * & * & * & * & * \\ 0 & 0 & 0 & \blacksquare & * & * & * & * & * & * \\ 0 & 0 & 0 & 0 & \blacksquare & * & * & * & * & * \\ 0 & 0 & 0 & 0 & 0 & \blacksquare & * & * & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \blacksquare & * \end{bmatrix}$$

he following matrices are in reduced echelon form because the leading entries are 1's, nd there are 0's below *and above* each leading 1.

$$\begin{bmatrix} 1 & 0 & * & * \\ 0 & 1 & * & * \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 1 & * & 0 & 0 & 0 & * & * & 0 & * \\ 0 & 0 & 0 & 1 & 0 & 0 & * & * & 0 & * \\ 0 & 0 & 0 & 0 & 1 & 0 & * & * & 0 & * \\ 0 & 0 & 0 & 0 & 0 & 1 & * & * & 0 & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & * \end{bmatrix}$$

Definition

A pivot position in a matrix A is a location in A that corresponds to a leading 1 in the reduced row echelon form of A.

Definition

A pivot column is a column of A that contains a pivot position.

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Definition

A pivot column is a column of A that contains a pivot position.

Theorem (Uniqueness of the row reduced echelon form)

Each matrix is row equivalent to one and only one RREF matrix.

Gaussian Elimination

Example

Row reduce the matrix below to row echelon form (REF) and then to reduced row echelon form (RREF):

$$\begin{bmatrix} 0 & -3 & -6 & 4 & 9 \\ -1 & -2 & -1 & 3 & 1 \\ -2 & -3 & 0 & 3 & -1 \\ 1 & 4 & 5 & -9 & -7 \end{bmatrix}.$$

Gaussian Elimination

Example

Row reduce the matrix below to row echelon form (REF) and then to reduced row echelon form (RREF):

$$\begin{bmatrix} 0 & -3 & -6 & 4 & 9 \\ -1 & -2 & -1 & 3 & 1 \\ -2 & -3 & 0 & 3 & -1 \\ 1 & 4 & 5 & -9 & -7 \end{bmatrix}.$$

So how do we do this? How do we know which elementary row operation to use at each step? We need an algorithm.

- **Step 1:** Find the first non-zero column (from the left).

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- **Step 2:** We don't want a 0 at the top of this column. If necessary swap the first row with another row.

Row Reduction and Echelon Forms

- **Step 1:** Find the first non-zero column (from the left).
- **Step 2:** We don't want a 0 at the top of this column. If necessary swap the first row with another row.
- **Step 3:** Create zeros under the pivot.

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- **Step 4:** Ignore the row and column containing the pivot. Goto step 1.

Row Reduction and Echelon Forms

- **Step 1:** Find the first non-zero column (from the left).
- **Step 2:** We don't want a 0 at the top of this column. If necessary swap the first row with another row.
- **Step 3:** Create zeros under the pivot.
- **Step 4:** Ignore the row and column containing the pivot. Goto step 1.

Keep going until your matrix is in row echelon form (REF). If you want reduced REF, then continue to step 5.

- **Step 1:** Find the first non-zero column (from the left).
- **Step 2:** We don't want a 0 at the top of this column. If necessary swap the first row with another row.
- **Step 3:** Create zeros under the pivot.
- **Step 4:** Ignore the row and column containing the pivot. Goto step 1.

Keep going until your matrix is in row echelon form (REF). If you want reduced REF, then continue to step 5.

- **Step 5:** Make every pivot a 1 by multiplying rows by constants.

Row Reduction and Echelon Forms

- **Step 1:** Find the first non-zero column (from the left).
- **Step 2:** We don't want a 0 at the top of this column. If necessary swap the first row with another row.
- **Step 3:** Create zeros under the pivot.
- **Step 4:** Ignore the row and column containing the pivot. Goto step 1.

Keep going until your matrix is in row echelon form (REF). If you want reduced REF, then continue to step 5.

- **Step 5:** Make every pivot a 1 by multiplying rows by constants.
- **Step 6:** Starting with the bottom-rightmost pivot and working upwards, create zeros above each pivot.

- **Step 1:** Find the first non-zero column (from the left).
- **Step 2:** We don't want a 0 at the top of this column. If necessary swap the first row with another row.
- **Step 3:** Create zeros under the pivot.
- **Step 4:** Ignore the row and column containing the pivot. Goto step 1.

Keep going until your matrix is in row echelon form (REF). If you want reduced REF, then continue to step 5.

- **Step 5:** Make every pivot a 1 by multiplying rows by constants.
- **Step 6:** Starting with the bottom-rightmost pivot and working upwards, create zeros above each pivot.

Your matrix should now by in RREF.

If we want REF we do steps 1-4. This is called *Gaussian Elimination.*

Carl Friedrich Gauss

BORN
30 April 1777

DECEICED
23 February 1855

NATIONALITY
German

If we want REF we do steps 1-4. This is called *Gaussian Elimination*.

If we want RREF, we do steps 1-6. This is called *Gauss-Jordan Elimination*.

Wilhelm Jordan

BORN
1 March 1842

DECEICED
17 April 1899

NATIONALITY
German

Another name is *row reduction*.

Now let's do this:

Example

Row reduce the matrix below to row echelon form (REF) and then to reduced row echelon form (RREF):

$$\begin{bmatrix} 0 & -3 & -6 & 4 & 9 \\ -1 & -2 & -1 & 3 & 1 \\ -2 & -3 & 0 & 3 & -1 \\ 1 & 4 & 5 & -9 & -7 \end{bmatrix}.$$

- **Step 1:** Find the first non-zero column (from the left).

$$\begin{bmatrix} 0 & -3 & -6 & 4 & 9 \\ -1 & -2 & -1 & 3 & 1 \\ -2 & -3 & 0 & 3 & -1 \\ 1 & 4 & 5 & -9 & -7 \end{bmatrix}.$$

This column is not all zeros.

- **Step 2:** We don't want a 0 at the top of this column. If necessary swap the first row with another row.

I don't want 0 here.

$$\begin{bmatrix} 0 & -3 & -6 & 4 & 9 \\ -1 & -2 & -1 & 3 & 1 \\ -2 & -3 & 0 & 3 & -1 \\ 1 & 4 & 5 & -9 & -7 \end{bmatrix}$$

- **Step 2:** We don't want a 0 at the top of this column. If necessary swap the first row with another row.

I don't want 0 here.

$$\begin{bmatrix} 0 & -3 & -6 & 4 & 9 \\ -1 & -2 & -1 & 3 & 1 \\ -2 & -3 & 0 & 3 & -1 \\ 1 & 4 & 5 & -9 & -7 \end{bmatrix}$$

$R_1 \leftrightarrow R_4$

$$\begin{bmatrix} 1 & 4 & 5 & -9 & -7 \\ -1 & -2 & -1 & 3 & 1 \\ -2 & -3 & 0 & 3 & -1 \\ 0 & -3 & -6 & 4 & 9 \end{bmatrix}$$

- **Step 2:** We don't want a 0 at the top of this column. If necessary swap the first row with another row.

I don't want 0 here.

$$\begin{bmatrix} 0 & -3 & -6 & 4 & 9 \\ -1 & -2 & -1 & 3 & 1 \\ -2 & -3 & 0 & 3 & -1 \\ 1 & 4 & 5 & -9 & -7 \end{bmatrix}$$

pivot/leading entry

$R_1 \leftrightarrow R_4$

$$\begin{bmatrix} 1 & 4 & 5 & -9 & -7 \\ -1 & -2 & -1 & 3 & 1 \\ -2 & -3 & 0 & 3 & -1 \\ 0 & -3 & -6 & 4 & 9 \end{bmatrix}$$

- **Step 3:** Create zeros <u>under</u> the pivot.

I want 0 here. (for each of the three entries below the pivot)

$$\begin{bmatrix} 1 & 4 & 5 & -9 & -7 \\ -1 & -2 & -1 & 3 & 1 \\ -2 & -3 & 0 & 3 & -1 \\ 0 & -3 & -6 & 4 & 9 \end{bmatrix}$$

- **Step 3:** Create zeros <u>under</u> the pivot.

$$\begin{array}{c} \\ \\ 0 \\ \text{I want} \quad 0 \quad \text{here.} \\ 0 \end{array} \qquad \begin{bmatrix} 1 & 4 & 5 & -9 & -7 \\ -1 & -2 & -1 & 3 & 1 \\ -2 & -3 & 0 & 3 & -1 \\ 0 & -3 & -6 & 4 & 9 \end{bmatrix}$$

$$R_1 + R_2 \to R_2 \qquad \begin{bmatrix} 1 & 4 & 5 & -9 & -7 \\ 0 & 2 & 4 & -6 & -6 \\ -2 & -3 & 0 & 3 & -1 \\ 0 & -3 & -6 & 4 & 9 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 4 & 5 & -9 & -7 \\ 0 & 2 & 4 & -6 & -6 \\ -2 & -3 & 0 & 3 & -1 \\ 0 & -3 & -6 & 4 & 9 \end{bmatrix}$$

$$2R_1 + R_3 \to R_3 \qquad \begin{bmatrix} 1 & 4 & 5 & -9 & -7 \\ 0 & 2 & 4 & -6 & -6 \\ 0 & 5 & 10 & -15 & -15 \\ 0 & -3 & -6 & 4 & 9 \end{bmatrix}$$

- **Step 4:** Ignore the row and column containing the pivot. Goto step 1.

$$\begin{bmatrix} 1 & 4 & 5 & -9 & -7 \\ 0 & 2 & 4 & -6 & -6 \\ 0 & 5 & 10 & -15 & -15 \\ 0 & -3 & -6 & 4 & 9 \end{bmatrix}$$

- **Step 4:** Ignore the row and column containing the pivot. Goto step 1.

$$\begin{bmatrix} 1 & 4 & 5 & -9 & -7 \\ 0 & 2 & 4 & -6 & -6 \\ 0 & 5 & 10 & -15 & -15 \\ 0 & -3 & -6 & 4 & 9 \end{bmatrix}$$

ignore

Row Reduction and Echelon Forms

- **Step 4:** Ignore the row and column containing the pivot. Goto step 1.

$$\begin{bmatrix} 1 & 4 & 5 & -9 & -7 \\ 0 & 2 & 4 & -6 & -6 \\ 0 & 5 & 10 & -15 & -15 \\ 0 & -3 & -6 & 4 & 9 \end{bmatrix}$$

ignore

ignore

- **Step 4:** Ignore the row and column containing the pivot. Goto step 1.

$$\begin{bmatrix} 1 & 4 & 5 & -9 & -7 \\ 0 & 2 & 4 & -6 & -6 \\ 0 & 5 & 10 & -15 & -15 \\ 0 & -3 & -6 & 4 & 9 \end{bmatrix}$$

ignore

ignore

$$\begin{bmatrix} 1 & 4 & 5 & -9 & -7 \\ 0 & 2 & 4 & -6 & -6 \\ 0 & 5 & 10 & -15 & -15 \\ 0 & -3 & -6 & 4 & 9 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 4 & 5 & -9 & -7 \\ 0 & 2 & 4 & -6 & -6 \\ 0 & 5 & 10 & -15 & -15 \\ 0 & -3 & -6 & 4 & 9 \end{bmatrix}$$

This column is not all zeros.

I don't want 0 here.

$$\begin{bmatrix} 1 & 4 & 5 & -9 & -7 \\ 0 & 2 & 4 & -6 & -6 \\ 0 & 5 & 10 & -15 & -15 \\ 0 & -3 & -6 & 4 & 9 \end{bmatrix}$$

Row Reduction and Echelon Forms

$$\begin{bmatrix} 1 & 4 & 5 & -9 & -7 \\ 0 & 2 & 4 & -6 & -6 \\ 0 & 5 & 10 & -15 & -15 \\ 0 & -3 & -6 & 4 & 9 \end{bmatrix}$$

I want $\begin{matrix} 0 \\ 0 \end{matrix}$ here.

Row Reduction and Echelon Forms

I want $\begin{matrix}0\\0\end{matrix}$ here.

$$\begin{bmatrix} 1 & 4 & 5 & -9 & -7 \\ 0 & 2 & 4 & -6 & -6 \\ 0 & 5 & 10 & -15 & -15 \\ 0 & -3 & -6 & 4 & 9 \end{bmatrix}$$

$$-\tfrac{5}{2}R_2 + R_3 \to R_3 \quad \begin{bmatrix} 1 & 4 & 5 & -9 & -7 \\ 0 & 2 & 4 & -6 & -6 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & -3 & -6 & 4 & 9 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 4 & 5 & -9 & -7 \\ 0 & 2 & 4 & -6 & -6 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & -3 & -6 & 4 & 9 \end{bmatrix}$$

$\frac{3}{2}R_2 + R_4 \to R_4$

$$\begin{bmatrix} 1 & 4 & 5 & -9 & -7 \\ 0 & 2 & 4 & -6 & -6 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -5 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 4 & 5 & -9 & -7 \\ 0 & 2 & 4 & -6 & -6 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -5 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 4 & 5 & -9 & -7 \\ 0 & 2 & 4 & -6 & -6 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -5 & 0 \end{bmatrix}$$

Ignore R_2 and column 2. Find the first non-zero column.

$$\begin{bmatrix} 1 & 4 & 5 & -9 & -7 \\ 0 & 2 & 4 & -6 & -6 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -5 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 4 & 5 & -9 & -7 \\ 0 & 2 & 4 & -6 & -6 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -5 & 0 \end{bmatrix}$$

Ignore R_2 and column 2. Find the first non-zero column.

$$\begin{bmatrix} 1 & 4 & 5 & -9 & -7 \\ 0 & 2 & 4 & -6 & -6 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -5 & 0 \end{bmatrix}$$

This column is all zeros.

Row Reduction and Echelon Forms

$$\begin{bmatrix} 1 & 4 & 5 & -9 & -7 \\ 0 & 2 & 4 & -6 & -6 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -5 & 0 \end{bmatrix}$$

Ignore R_2 and column 2. Find the first non-zero column.

$$\begin{bmatrix} 1 & 4 & 5 & -9 & -7 \\ 0 & 2 & 4 & -6 & -6 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -5 & 0 \end{bmatrix}$$

This column is not all zeros.

$$\begin{bmatrix} 1 & 4 & 5 & -9 & -7 \\ 0 & 2 & 4 & -6 & -6 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -5 & 0 \end{bmatrix}$$

Ignore R_2 and column 2. Find the first non-zero column.

I don't want 0 here.

$$\begin{bmatrix} 1 & 4 & 5 & -9 & -7 \\ 0 & 2 & 4 & -6 & -6 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -5 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 4 & 5 & -9 & -7 \\ 0 & 2 & 4 & -6 & -6 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -5 & 0 \end{bmatrix}$$

$R_3 \leftrightarrow R_4$

$$\begin{bmatrix} 1 & 4 & 5 & -9 & -7 \\ 0 & 2 & 4 & -6 & -6 \\ 0 & 0 & 0 & -5 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

[1] row echelon form

$$\begin{bmatrix} 1 & 4 & 5 & -9 & -7 \\ 0 & 2 & 4 & -6 & -6 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -5 & 0 \end{bmatrix}$$

$R_3 \leftrightarrow R_4$

$$\begin{bmatrix} 1 & 4 & 5 & -9 & -7 \\ 0 & 2 & 4 & -6 & -6 \\ 0 & 0 & 0 & -5 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

The matrix is now in REF[1].

[1] row echelon form

- **Step 5:** Make every pivot a 1 by multiplying rows by constants.

$$\begin{bmatrix} 1 & 4 & 5 & -9 & -7 \\ 0 & 2 & 4 & -6 & -6 \\ 0 & 0 & 0 & -5 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

- **Step 5:** Make every pivot a 1 by multiplying rows by constants.

pivot

$$\begin{bmatrix} 1 & 4 & 5 & -9 & -7 \\ 0 & 2 & 4 & -6 & -6 \\ 0 & 0 & 0 & -5 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

- **Step 5:** Make every pivot a 1 by multiplying rows by constants.

$$\begin{bmatrix} 1 & 4 & 5 & -9 & -7 \\ 0 & 2 & 4 & -6 & -6 \\ 0 & 0 & 0 & -5 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

pivot columns

- **Step 5:** Make every pivot a 1 by multiplying rows by constants.

$$\begin{bmatrix} 1 & 4 & 5 & -9 & -7 \\ 0 & 2 & 4 & -6 & -6 \\ 0 & 0 & 0 & -5 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{matrix} \frac{1}{2}R_2 \to R_2 \\ -\frac{1}{5}R_3 \to R_3 \end{matrix} \qquad \begin{bmatrix} 1 & 4 & 5 & -9 & -7 \\ 0 & 1 & 2 & -3 & -3 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

- **Step 5:** Make every pivot a 1 by multiplying rows by constants.

$$\begin{bmatrix} 1 & 4 & 5 & -9 & -7 \\ 0 & 2 & 4 & -6 & -6 \\ 0 & 0 & 0 & -5 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

WARNING

Be very careful if you do 2 row operations in the same step.

Do not change the same row twice in one step.

$\frac{1}{2}R_2 \to R_2$
$-\frac{1}{5}R_3 \to R_3$

$$\begin{bmatrix} 1 & 4 & 5 & -9 & -7 \\ 0 & 1 & 2 & -3 & -3 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

- **Step 6:** Starting with the bottom-rightmost pivot and working upwards, create zeros <u>above</u> each pivot.

$$\begin{bmatrix} 1 & 4 & 5 & -9 & -7 \\ 0 & 1 & 2 & -3 & -3 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

- **Step 6:** Starting with the bottom-rightmost pivot and working upwards, create zeros <u>above</u> each pivot.

first make this 0

$$\begin{bmatrix} 1 & 4 & 5 & -9 & -7 \\ 0 & 1 & 2 & -3 & -3 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

- **Step 6:** Starting with the bottom-rightmost pivot and working upwards, create zeros above each pivot.

then make this 0

$$\begin{bmatrix} 1 & 4 & 5 & -9 & -7 \\ 0 & 1 & 2 & -3 & -3 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

- **Step 6:** Starting with the bottom-rightmost pivot and working upwards, create zeros <u>above</u> each pivot.

finally make this 0

$$\begin{bmatrix} 1 & 4 & 5 & -9 & -7 \\ 0 & 1 & 2 & -3 & -3 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

- **Step 6:** Starting with the bottom-rightmost pivot and working upwards, create zeros above each pivot.

$$\begin{bmatrix} 1 & 4 & 5 & -9 & -7 \\ 0 & 1 & 2 & -3 & -3 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$3R_3 + R_2 \to R_2$

$$\begin{bmatrix} 1 & 4 & 5 & -9 & -7 \\ 0 & 1 & 2 & 0 & -3 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Row Reduction and Echelon Forms

$$\begin{bmatrix} 1 & 4 & 5 & -9 & -7 \\ 0 & 1 & 2 & 0 & -3 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$9R_3 + R_1 \to R_1$

$$\begin{bmatrix} 1 & 4 & 5 & 0 & -7 \\ 0 & 1 & 2 & 0 & -3 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 4 & 5 & -9 & -7 \\ 0 & 1 & 2 & 0 & -3 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$9R_3 + R_1 \to R_1$

$$\begin{bmatrix} 1 & 4 & 5 & 0 & -7 \\ 0 & 1 & 2 & 0 & -3 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$-4R_2 + R_1 \to R_1$

$$\begin{bmatrix} 1 & 0 & -3 & 0 & 5 \\ 0 & 1 & 2 & 0 & -3 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & -3 & 0 & 5 \\ 0 & 1 & 2 & 0 & -3 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

The matrix is now in RREF[2].

[2] reduced row echelon form

$$\underset{\text{original matrix}}{\begin{bmatrix} 0 & -3 & -6 & 4 & 9 \\ -1 & -2 & -1 & 3 & 1 \\ -2 & -3 & 0 & 3 & -1 \\ 1 & 4 & 5 & -9 & -7 \end{bmatrix}} \xrightarrow[\text{operations}]{\text{row}} \underset{\text{RREF matrix}}{\begin{bmatrix} 1 & 0 & -3 & 0 & 5 \\ 0 & 1 & 2 & 0 & -3 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}}$$

Row Reduction and Echelon Forms

pivots

$$\begin{bmatrix} 0 & -3 & -6 & 4 & 9 \\ -1 & -2 & -1 & 3 & 1 \\ -2 & -3 & 0 & 3 & -1 \\ 1 & 4 & 5 & -9 & -7 \end{bmatrix} \xrightarrow[\text{operations}]{\text{row}} \begin{bmatrix} 1 & 0 & -3 & 0 & 5 \\ 0 & 1 & 2 & 0 & -3 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Row Reduction and Echelon Forms

pivot positions

pivots

$$\begin{bmatrix} 0 & -3 & -6 & 4 & 9 \\ -1 & -2 & -1 & 3 & 1 \\ -2 & -3 & 0 & 3 & -1 \\ 1 & 4 & 5 & -9 & -7 \end{bmatrix} \xrightarrow[\text{operations}]{\text{row}} \begin{bmatrix} 1 & 0 & -3 & 0 & 5 \\ 0 & 1 & 2 & 0 & -3 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Row Reduction and Echelon Forms

pivot positions

pivots

$$\begin{bmatrix} 0 & -3 & -6 & 4 & 9 \\ -1 & -2 & -1 & 3 & 1 \\ -2 & -3 & 0 & 3 & -1 \\ 1 & 4 & 5 & -9 & -7 \end{bmatrix} \xrightarrow[\text{operations}]{\text{row}} \begin{bmatrix} 1 & 0 & -3 & 0 & 5 \\ 0 & 1 & 2 & 0 & -3 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

pivot columns

Next Time

- Solutions of Linear Systems
- Introduction to Matrices
- The Inverse of a Matrix

This is the end of the lecture. Please ask your questions now.
You are welcome to leave whenever you wish.