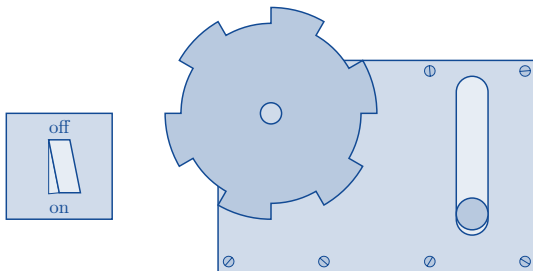


Lecture 9

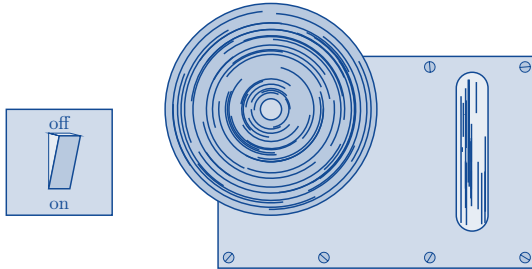
- 4.5 ODEs with Discontinuous Forcing Functions
- 4.6 The Convolution Integral

ODEs with Discontinuous Forcing Functions

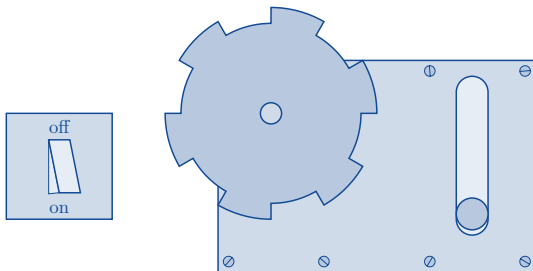
4.5 ODEs with Discontinuous Forcing Functions



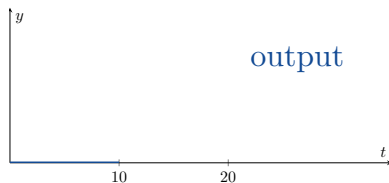
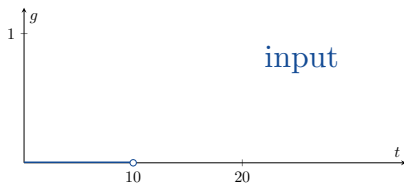
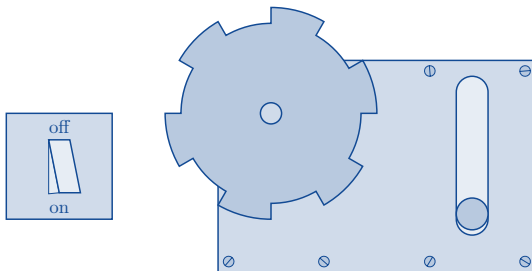
4.5 ODEs with Discontinuous Forcing Functions



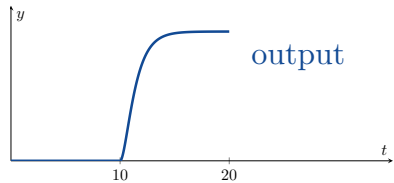
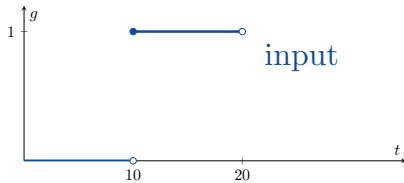
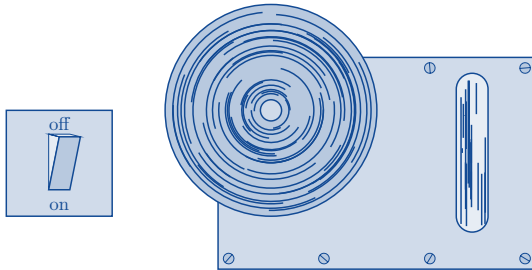
4.5 ODEs with Discontinuous Forcing Functions



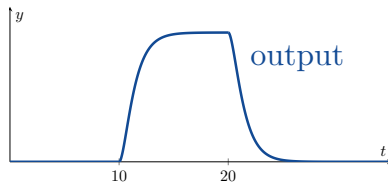
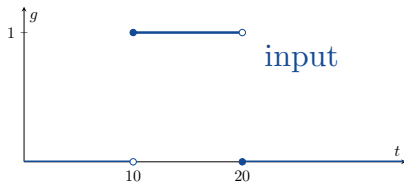
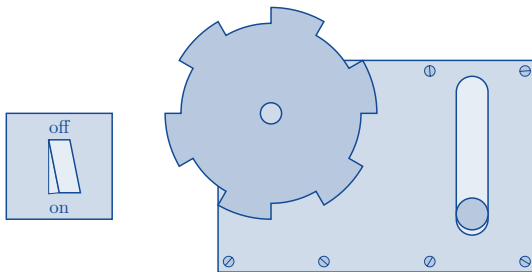
4.5 ODEs with Discontinuous Forcing Functions



4.5 ODEs with Discontinuous Forcing Functions



4.5 ODEs with Discontinuous Forcing Functions



4.5 ODEs with Discontinuous Forcing Functions



Example

Solve

$$\begin{cases} y'' + 4y = f(t) = \begin{cases} 0 & 0 \leq t < 5 \\ \frac{1}{5}(t - 5) & 5 \leq t < 10 \\ 1 & 10 \leq t \end{cases} \\ y(0) = 0 \\ y'(0) = 0. \end{cases}$$

4.5 ODEs with Discontinuous Forcing Functions



Example

Solve

$$\begin{cases} y'' + 4y = f(t) = \begin{cases} 0 & 0 \leq t < 5 \\ \frac{1}{5}(t - 5) & 5 \leq t < 10 \\ 1 & 10 \leq t \end{cases} \\ y(0) = 0 \\ y'(0) = 0. \end{cases}$$

Note that

$$\begin{aligned} f(t) &= 0 + \left(\frac{1}{5}(t - 5) - 0 \right) u_5(t) + \left(1 - \frac{1}{5}(t - 5) \right) u_{10}(t) \\ &= \end{aligned}$$

4.5 ODEs with Discontinuous Forcing Functions



Example

Solve

$$\begin{cases} y'' + 4y = f(t) = \begin{cases} 0 & 0 \leq t < 5 \\ \frac{1}{5}(t - 5) & 5 \leq t < 10 \\ 1 & 10 \leq t \end{cases} \\ y(0) = 0 \\ y'(0) = 0. \end{cases}$$

Note that

$$\begin{aligned} f(t) &= 0 + \left(\frac{1}{5}(t - 5) - 0 \right) u_5(t) + \left(1 - \frac{1}{5}(t - 5) \right) u_{10}(t) \\ &= \frac{1}{5} \left(u_5(t)(t - 5) - u_{10}(t)(t - 10) \right). \end{aligned}$$

4.5 ODEs with Discontinuous Forcing Functions



$$\mathcal{L} [u_c(t)f(t-c)](s) = e^{-cs}F(s) \qquad \mathcal{L} [t] = \frac{1}{s^2}$$

So our IVP is

$$\begin{cases} y'' + 4y = \frac{1}{5} \left(u_5(t)(t-5) - u_{10}(t)(t-10) \right) \\ y(0) = 0 \\ y'(0) = 0. \end{cases}$$

4.5 ODEs with Discontinuous Forcing Functions



$$\mathcal{L} [u_c(t)f(t-c)](s) = e^{-cs}F(s) \qquad \mathcal{L} [t] = \frac{1}{s^2}$$

So our IVP is

$$\begin{cases} y'' + 4y = \frac{1}{5} \left(u_5(t)(t-5) - u_{10}(t)(t-10) \right) \\ y(0) = 0 \\ y'(0) = 0. \end{cases}$$

Taking the Laplace transform of the ODE gives

$$(s^2 + 4)Y = \frac{1}{5} \frac{e^{-5s} - e^{-10s}}{s^2}$$

4.5 ODEs with Discontinuous Forcing Functions



$$\mathcal{L} [u_c(t)f(t-c)](s) = e^{-cs}F(s) \qquad \mathcal{L} [t] = \frac{1}{s^2}$$

So our IVP is

$$\begin{cases} y'' + 4y = \frac{1}{5} \left(u_5(t)(t-5) - u_{10}(t)(t-10) \right) \\ y(0) = 0 \\ y'(0) = 0. \end{cases}$$

Taking the Laplace transform of the ODE gives

$$(s^2 + 4)Y = \frac{1}{5} \frac{e^{-5s} - e^{-10s}}{s^2}$$

and

$$Y = \frac{1}{5} \frac{e^{-5s} - e^{-10s}}{s^2(s^2 + 4)}.$$

4.5 ODEs with Discontinuous Forcing Functions



$$Y = \frac{1}{5} \frac{e^{-5s} - e^{-10s}}{s^2(s^2 + 4)}$$

Let

$$H(s) = \frac{1}{s^2(s^2 + 4)}.$$

Then

$$Y(s) = \frac{1}{5} e^{-5s} H(s) - \frac{1}{5} e^{-10s} H(s).$$

4.5 ODEs with Discontinuous Forcing Functions



$$Y(s) = \frac{1}{5}e^{-5s}H(s) - \frac{1}{5}e^{-10s}H(s)$$

Since

$$\mathcal{L} [u_c(t)h(t - c)] (s) = e^{-cs}H(s)$$

4.5 ODEs with Discontinuous Forcing Functions



$$Y(s) = \frac{1}{5}e^{-5s}H(s) - \frac{1}{5}e^{-10s}H(s)$$

Since

$$\mathcal{L} [u_c(t)h(t-c)](s) = e^{-cs}H(s)$$

we have that

$$u_c(t)h(t-c) = \mathcal{L}^{-1} [e^{-cs}H(s)](t).$$

4.5 ODEs with Discontinuous Forcing Functions



$$Y(s) = \frac{1}{5}e^{-5s}H(s) - \frac{1}{5}e^{-10s}H(s)$$

Since

$$\mathcal{L} [u_c(t)h(t-c)](s) = e^{-cs}H(s)$$

we have that

$$u_c(t)h(t-c) = \mathcal{L}^{-1} [e^{-cs}H(s)](t).$$

If we can find $h(t)$, then we can find $y(t)$.

4.5 ODEs with Discontinuous Forcing Functions



Using partial fractions, we calculate (please check!) that

$$\begin{aligned} H(s) &= \frac{1}{s^2(s^2 + 4)} = \frac{As + B}{s^2} + \frac{Cs + D}{s^2 + 4} \\ &= \frac{As^3 + Bs^2 + 4As + 4B + Cs^3 + Ds^2}{s^2(s^2 + 4)} \\ &= \frac{0s + \frac{1}{4}}{s^2} + \frac{0s - \frac{1}{4}}{s^2 + 4} = \frac{\frac{1}{4}}{s^2} - \frac{\frac{1}{4}}{s^2 + 4}. \end{aligned}$$

4.5 ODEs with Discontinuous Forcing Functions



Using partial fractions, we calculate (please check!) that

$$\begin{aligned} H(s) &= \frac{1}{s^2(s^2 + 4)} = \frac{As + B}{s^2} + \frac{Cs + D}{s^2 + 4} \\ &= \frac{As^3 + Bs^2 + 4As + 4B + Cs^3 + Ds^2}{s^2(s^2 + 4)} \\ &= \frac{0s + \frac{1}{4}}{s^2} + \frac{0s - \frac{1}{4}}{s^2 + 4} = \frac{\frac{1}{4}}{s^2} - \frac{\frac{1}{4}}{s^2 + 4}. \end{aligned}$$

Hence

$$h(t) = \frac{1}{4} \mathcal{L}^{-1} \left[\frac{1}{s^2} \right] - \frac{1}{8} \mathcal{L}^{-1} \left[\frac{2}{s^2 + 4} \right] = \quad .$$

4.5 ODEs with Discontinuous Forcing Functions



Using partial fractions, we calculate (please check!) that

$$\begin{aligned} H(s) &= \frac{1}{s^2(s^2 + 4)} = \frac{As + B}{s^2} + \frac{Cs + D}{s^2 + 4} \\ &= \frac{As^3 + Bs^2 + 4As + 4B + Cs^3 + Ds^2}{s^2(s^2 + 4)} \\ &= \frac{0s + \frac{1}{4}}{s^2} + \frac{0s - \frac{1}{4}}{s^2 + 4} = \frac{\frac{1}{4}}{s^2} - \frac{\frac{1}{4}}{s^2 + 4}. \end{aligned}$$

Hence

$$h(t) = \frac{1}{4} \mathcal{L}^{-1} \left[\frac{1}{s^2} \right] - \frac{1}{8} \mathcal{L}^{-1} \left[\frac{2}{s^2 + 4} \right] = \frac{t}{4} - \frac{1}{8} \sin 2t.$$

4.5 ODEs with Discontinuous Forcing Functions



$$u_c(t)h(t-c)(s) = \mathcal{L}^{-1} [e^{-cs}H(s)]$$

Therefore

$$y(t) = \mathcal{L}^{-1} \left[\frac{1}{5}e^{-5s}H(s) - \frac{1}{5}e^{-10s}H(s) \right]$$

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4.5 ODEs with Discontinuous Forcing Functions



$$u_c(t)h(t-c)(s) = \mathcal{L}^{-1} [e^{-cs}H(s)]$$

Therefore

$$\begin{aligned} y(t) &= \mathcal{L}^{-1} \left[\frac{1}{5}e^{-5s}H(s) - \frac{1}{5}e^{-10s}H(s) \right] \\ &= \frac{1}{5}u_5(t)h(t-5) - \frac{1}{5}u_{10}(t)h(t-10) \\ &= \end{aligned}$$

4.5 ODEs with Discontinuous Forcing Functions



$$u_c(t)h(t-c)(s) = \mathcal{L}^{-1} [e^{-cs}H(s)]$$

Therefore

$$\begin{aligned} y(t) &= \mathcal{L}^{-1} \left[\frac{1}{5}e^{-5s}H(s) - \frac{1}{5}e^{-10s}H(s) \right] \\ &= \frac{1}{5}u_5(t)h(t-5) - \frac{1}{5}u_{10}(t)h(t-10) \\ &= u_5(t) \left(\frac{t-5}{20} - \frac{1}{40}\sin(2t-10) \right) \\ &\quad - u_{10}(t) \left(\frac{t-10}{20} - \frac{1}{40}\sin(2t-20) \right). \end{aligned}$$

4.5 ODEs with Discontinuous Forcing Functions



Example

Solve

$$\begin{cases} y'' + 3y' + 2y = f(t) = \begin{cases} 1 & 0 \leq t < 10 \\ 0 & 10 \leq t \end{cases} \\ y(0) = 1 \\ y'(0) = 0. \end{cases}$$

4.5 ODEs with Discontinuous Forcing Functions



Example

Solve

$$\begin{cases} y'' + 3y' + 2y = f(t) = \begin{cases} 1 & 0 \leq t < 10 \\ 0 & 10 \leq t \end{cases} \\ y(0) = 1 \\ y'(0) = 0. \end{cases}$$

Since $f(t) = 1 - u_{10}(t)$, the Laplace Transform of the ODE is

$$(s^2 + 3s + 2)Y - (s + 3) = \frac{1 - e^{-10s}}{s}.$$

4.5 ODEs with Discontinuous Forcing Functions



Thus

$$\begin{aligned} Y(s) &= \frac{1 - e^{-10s}}{s(s^2 + 3s + 2)} + \frac{s + 3}{s^2 + 3s + 2} \\ &= \frac{(s^2 + 3s + 1) - e^{-10s}}{s(s^2 + 3s + 2)}. \end{aligned}$$

4.5 ODEs with Discontinuous Forcing Functions



Thus

$$\begin{aligned} Y(s) &= \frac{1 - e^{-10s}}{s(s^2 + 3s + 2)} + \frac{s + 3}{s^2 + 3s + 2} \\ &= \frac{(s^2 + 3s + 1) - e^{-10s}}{s(s^2 + 3s + 2)}. \end{aligned}$$

Let

$$G(s) = \frac{s^2 + 3s + 1}{s(s^2 + 3s + 2)} \quad \text{and} \quad H(s) = \frac{1}{s(s^2 + 3s + 2)}.$$

4.5 ODEs with Discontinuous Forcing Functions



Thus

$$\begin{aligned} Y(s) &= \frac{1 - e^{-10s}}{s(s^2 + 3s + 2)} + \frac{s + 3}{s^2 + 3s + 2} \\ &= \frac{(s^2 + 3s + 1) - e^{-10s}}{s(s^2 + 3s + 2)}. \end{aligned}$$

Let

$$G(s) = \frac{s^2 + 3s + 1}{s(s^2 + 3s + 2)} \quad \text{and} \quad H(s) = \frac{1}{s(s^2 + 3s + 2)}.$$

Then $Y = G(s) - e^{-10s}H(s)$.

4.5 ODEs with Discontinuous Forcing Functions



Thus

$$\begin{aligned} Y(s) &= \frac{1 - e^{-10s}}{s(s^2 + 3s + 2)} + \frac{s + 3}{s^2 + 3s + 2} \\ &= \frac{(s^2 + 3s + 1) - e^{-10s}}{s(s^2 + 3s + 2)}. \end{aligned}$$

Let

$$G(s) = \frac{s^2 + 3s + 1}{s(s^2 + 3s + 2)} \quad \text{and} \quad H(s) = \frac{1}{s(s^2 + 3s + 2)}.$$

Then $Y = G(s) - e^{-10s}H(s)$. If we can find $g(t)$ and $h(t)$, then we can find $y(t)$.

4.5 ODEs with Discontinuous Forcing Functions



Using partial fractions we get

$$G(s) = \frac{A}{s} + \frac{B}{s+1} + \frac{C}{s+2} = \frac{\frac{1}{2}}{s} + \frac{1}{s+1} - \frac{\frac{1}{2}}{s+2}$$

and

$$H(s) = \frac{D}{s} + \frac{E}{s+1} + \frac{F}{s+2} = \frac{\frac{1}{2}}{s} - \frac{1}{s+1} + \frac{\frac{1}{2}}{s+2}$$

(please check!).

4.5 ODEs with Discontinuous Forcing Functions



Using partial fractions we get

$$G(s) = \frac{A}{s} + \frac{B}{s+1} + \frac{C}{s+2} = \frac{\frac{1}{2}}{s} + \frac{1}{s+1} - \frac{\frac{1}{2}}{s+2}$$

and

$$H(s) = \frac{D}{s} + \frac{E}{s+1} + \frac{F}{s+2} = \frac{\frac{1}{2}}{s} - \frac{1}{s+1} + \frac{\frac{1}{2}}{s+2}$$

(please check!). It follows that

$$g(t) = \frac{1}{2} (1 + 2e^{-t} - e^{-2t}) \quad \text{and} \quad h(t) = \frac{1}{2} (1 - 2e^{-t} + e^{-2t}).$$

4.5 ODEs with Discontinuous Forcing Functions



$$g(t) = \frac{1}{2} (1 + 2e^{-t} - e^{-2t})$$

$$h(t) = \frac{1}{2} (1 - 2e^{-t} + e^{-2t})$$

Therefore

$$y(t) = \mathcal{L}^{-1} [Y]$$

=

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4.5 ODEs with Discontinuous Forcing Functions



$$g(t) = \frac{1}{2} (1 + 2e^{-t} - e^{-2t})$$

$$h(t) = \frac{1}{2} (1 - 2e^{-t} + e^{-2t})$$

Therefore

$$\begin{aligned} y(t) &= \mathcal{L}^{-1} [Y] \\ &= \mathcal{L}^{-1} [G(s) - e^{-10s}H(s)] \\ &= \\ &= \end{aligned}$$

4.5 ODEs with Discontinuous Forcing Functions



$$g(t) = \frac{1}{2} (1 + 2e^{-t} - e^{-2t})$$

$$h(t) = \frac{1}{2} (1 - 2e^{-t} + e^{-2t})$$

Therefore

$$\begin{aligned} y(t) &= \mathcal{L}^{-1} [Y] \\ &= \mathcal{L}^{-1} [G(s) - e^{-10s} H(s)] \\ &= g(t) - u_{10}(t)h(t - 10) \\ &= \end{aligned}$$

4.5 ODEs with Discontinuous Forcing Functions



$$g(t) = \frac{1}{2} (1 + 2e^{-t} - e^{-2t})$$

$$h(t) = \frac{1}{2} (1 - 2e^{-t} + e^{-2t})$$

Therefore

$$\begin{aligned} y(t) &= \mathcal{L}^{-1} [Y] \\ &= \mathcal{L}^{-1} [G(s) - e^{-10s} H(s)] \\ &= g(t) - u_{10}(t)h(t-10) \\ &= \frac{1}{2} (1 + 2e^{-t} - e^{-2t}) - \frac{1}{2} u_{10}(t) (1 - 2e^{-(t-10)} + e^{-2(t-10)}). \end{aligned}$$

4.5 ODEs with Discontinuous Forcing Functions



Example

Solve

$$\begin{cases} y'' + 4y = u_{\pi}(t) - u_{3\pi}(t) \\ y(0) = 0 \\ y'(0) = 0. \end{cases}$$

4.5 ODEs with Discontinuous Forcing Functions



Example

Solve

$$\begin{cases} y'' + 4y = u_{\pi}(t) - u_{3\pi}(t) \\ y(0) = 0 \\ y'(0) = 0. \end{cases}$$

Taking the Laplace Transform of the ODE gives

$$(s^2 + 4)Y(s) = \frac{e^{-\pi s} - e^{-3\pi s}}{s}.$$

4.5 ODEs with Discontinuous Forcing Functions



Example

Solve

$$\begin{cases} y'' + 4y = u_{\pi}(t) - u_{3\pi}(t) \\ y(0) = 0 \\ y'(0) = 0. \end{cases}$$

Taking the Laplace Transform of the ODE gives

$$(s^2 + 4)Y(s) = \frac{e^{-\pi s} - e^{-3\pi s}}{s}.$$

Thus

$$Y(s) = \frac{e^{-\pi s} - e^{-3\pi s}}{s(s^2 + 4)}.$$

4.5 ODEs with Discontinuous Forcing Functions



Example

Solve

$$\begin{cases} y'' + 4y = u_{\pi}(t) - u_{3\pi}(t) \\ y(0) = 0 \\ y'(0) = 0. \end{cases}$$

Taking the Laplace Transform of the ODE gives

$$(s^2 + 4)Y(s) = \frac{e^{-\pi s} - e^{-3\pi s}}{s}.$$

Thus

$$Y(s) = \frac{e^{-\pi s} - e^{-3\pi s}}{s(s^2 + 4)}.$$

Let

$$H(s) = \frac{1}{s(s^2 + 4)}.$$

4.5 ODEs with Discontinuous Forcing Functions



Using partial fractions, we calculate that

$$\begin{aligned} H(s) &= \frac{1}{s(s^2 + 4)} = \frac{A}{s} + \frac{Bs + C}{s^2 + 4} = \frac{\frac{1}{4}}{s} + \frac{-\frac{1}{4}s + 0}{s^2 + 4} \\ &= \frac{1}{4} \left(\frac{1}{s} \right) - \frac{1}{4} \left(\frac{s}{s^2 + 4} \right) = \frac{1}{4} \mathcal{L} [1] - \frac{1}{4} \mathcal{L} [\cos 2t] . \end{aligned}$$

4.5 ODEs with Discontinuous Forcing Functions



Using partial fractions, we calculate that

$$\begin{aligned} H(s) &= \frac{1}{s(s^2 + 4)} = \frac{A}{s} + \frac{Bs + C}{s^2 + 4} = \frac{\frac{1}{4}}{s} + \frac{-\frac{1}{4}s + 0}{s^2 + 4} \\ &= \frac{1}{4} \left(\frac{1}{s} \right) - \frac{1}{4} \left(\frac{s}{s^2 + 4} \right) = \frac{1}{4} \mathcal{L}[1] - \frac{1}{4} \mathcal{L}[\cos 2t]. \end{aligned}$$

It follows that

$$h(t) = \frac{1}{4} - \frac{1}{4} \cos 2t$$

4.5 ODEs with Discontinuous Forcing Functions



Using partial fractions, we calculate that

$$\begin{aligned} H(s) &= \frac{1}{s(s^2 + 4)} = \frac{A}{s} + \frac{Bs + C}{s^2 + 4} = \frac{\frac{1}{4}}{s} + \frac{-\frac{1}{4}s + 0}{s^2 + 4} \\ &= \frac{1}{4} \left(\frac{1}{s} \right) - \frac{1}{4} \left(\frac{s}{s^2 + 4} \right) = \frac{1}{4} \mathcal{L} [1] - \frac{1}{4} \mathcal{L} [\cos 2t] . \end{aligned}$$

It follows that

$$h(t) = \frac{1}{4} - \frac{1}{4} \cos 2t$$

and the solution to the IVP is

$$\begin{aligned} y(t) &= \mathcal{L}^{-1} [e^{-\pi s} H(s)] - \mathcal{L}^{-1} [e^{-3\pi s} H(s)] \\ &= \\ &= \end{aligned}$$

4.5 ODEs with Discontinuous Forcing Functions



Using partial fractions, we calculate that

$$\begin{aligned} H(s) &= \frac{1}{s(s^2 + 4)} = \frac{A}{s} + \frac{Bs + C}{s^2 + 4} = \frac{\frac{1}{4}}{s} + \frac{-\frac{1}{4}s + 0}{s^2 + 4} \\ &= \frac{1}{4} \left(\frac{1}{s} \right) - \frac{1}{4} \left(\frac{s}{s^2 + 4} \right) = \frac{1}{4} \mathcal{L}[1] - \frac{1}{4} \mathcal{L}[\cos 2t]. \end{aligned}$$

It follows that

$$h(t) = \frac{1}{4} - \frac{1}{4} \cos 2t$$

and the solution to the IVP is

$$\begin{aligned} y(t) &= \mathcal{L}^{-1} [e^{-\pi s} H(s)] - \mathcal{L}^{-1} [e^{-3\pi s} H(s)] \\ &= u_{\pi}(t)h(t - \pi) - u_{3\pi}(t)h(t - 3\pi) \\ &= \end{aligned}$$

4.5 ODEs with Discontinuous Forcing Functions



Using partial fractions, we calculate that

$$\begin{aligned} H(s) &= \frac{1}{s(s^2 + 4)} = \frac{A}{s} + \frac{Bs + C}{s^2 + 4} = \frac{\frac{1}{4}}{s} + \frac{-\frac{1}{4}s + 0}{s^2 + 4} \\ &= \frac{1}{4} \left(\frac{1}{s} \right) - \frac{1}{4} \left(\frac{s}{s^2 + 4} \right) = \frac{1}{4} \mathcal{L}[1] - \frac{1}{4} \mathcal{L}[\cos 2t]. \end{aligned}$$

It follows that

$$h(t) = \frac{1}{4} - \frac{1}{4} \cos 2t$$

and the solution to the IVP is

$$\begin{aligned} y(t) &= \mathcal{L}^{-1} [e^{-\pi s} H(s)] - \mathcal{L}^{-1} [e^{-3\pi s} H(s)] \\ &= u_{\pi}(t)h(t - \pi) - u_{3\pi}(t)h(t - 3\pi) \\ &= \frac{1}{4}u_{\pi}(t)(1 - \cos(2t - 2\pi)) - \frac{1}{4}u_{3\pi}(t)(1 - \cos(2t - 6\pi)). \end{aligned}$$

The Convolution Integral

4.6 The Convolution Integral



Let $f : [0, \infty) \rightarrow \mathbb{R}$ and $g : [0, \infty) \rightarrow \mathbb{R}$ be piecewise continuous functions.

Definition

The *convolution* of f and g is

$$(f * g)(t) = \int_0^t f(\tau)g(t - \tau) d\tau.$$

$$(f * g)(t) = \int_0^t f(\tau)g(t - \tau) d\tau$$



Theorem (Properties)

- $f * g = g * f$
- $f * (g * h) = (f * g) * h$
- $f * (g + h) = (f * g) + (f * h)$
- $f * 0 = 0 = 0 * f$

$$(f * g)(t) = \int_0^t f(\tau)g(t - \tau) d\tau$$



Theorem (Properties)

- $f * g = g * f$
- $f * (g * h) = (f * g) * h$
- $f * (g + h) = (f * g) + (f * h)$
- $f * 0 = 0 = 0 * f$

Example

$$(\cos * 1)(t) = \int_0^t \cos \tau \cdot 1 d\tau = [\sin \tau]_0^t = \sin t - \sin 0 = \sin t$$

$$(1 * \cos)(t) =$$

$$(f * g)(t) = \int_0^t f(\tau)g(t - \tau) d\tau$$



Theorem (Properties)

- $f * g = g * f$
- $f * (g * h) = (f * g) * h$
- $f * (g + h) = (f * g) + (f * h)$
- $f * 0 = 0 = 0 * f$

Example

$$(\cos * 1)(t) = \int_0^t \cos \tau \cdot 1 d\tau = [\sin \tau]_0^t = \sin t - \sin 0 = \sin t$$

$$\begin{aligned}(1 * \cos)(t) &= \int_0^t 1 \cdot \cos(t - \tau) d\tau = [-\sin(t - \tau)]_0^t \\ &= -\sin 0 + \sin t = \sin t\end{aligned}$$

$$(f * g)(t) = \int_0^t f(\tau)g(t - \tau) d\tau$$



Theorem (Properties)

- $f * g = g * f$
- $f * (g * h) = (f * g) * h$
- $f * (g + h) = (f * g) + (f * h)$
- $f * 0 = 0 = 0 * f$

Example

$$(\cos * 1)(t) = \int_0^t \cos \tau \cdot 1 d\tau = [\sin \tau]_0^t = \sin t - \sin 0 = \sin t$$

$$\begin{aligned}(1 * \cos)(t) &= \int_0^t 1 \cdot \cos(t - \tau) d\tau = [-\sin(t - \tau)]_0^t \\ &= -\sin 0 + \sin t = \sin t\end{aligned}$$

Note that $f * 1 \neq f$ in general.

$$(f * g)(t) = \int_0^t f(\tau)g(t - \tau) d\tau$$



Example

$$(\sin * \sin)(t) = \int_0^t \sin \tau \sin(t - \tau) d\tau$$

$$=$$
$$=$$
$$=$$
$$=$$
$$=$$

$$(f * g)(t) = \int_0^t f(\tau)g(t - \tau) d\tau$$



Example

$$\begin{aligned}(\sin * \sin)(t) &= \int_0^t \sin \tau \sin(t - \tau) d\tau \\&= \int_0^t \sin \tau (\sin t \cos \tau - \cos t \sin \tau) d\tau \\&= \\&= \\&= \\&= \end{aligned}$$

$$(f * g)(t) = \int_0^t f(\tau)g(t - \tau) d\tau$$



Example

$$\begin{aligned}(\sin * \sin)(t) &= \int_0^t \sin \tau \sin(t - \tau) d\tau \\&= \int_0^t \sin \tau (\sin t \cos \tau - \cos t \sin \tau) d\tau \\&= \sin t \int_0^t \sin \tau \cos \tau d\tau - \cos t \int_0^t \sin^2 \tau d\tau \\&= \\&= \\&= \end{aligned}$$

$$(f * g)(t) = \int_0^t f(\tau)g(t - \tau) d\tau$$



Example

$$\begin{aligned}(\sin * \sin)(t) &= \int_0^t \sin \tau \sin(t - \tau) d\tau \\&= \int_0^t \sin \tau (\sin t \cos \tau - \cos t \sin \tau) d\tau \\&= \sin t \int_0^t \sin \tau \cos \tau d\tau - \cos t \int_0^t \sin^2 \tau d\tau \\&= \sin t \left[-\frac{1}{2} \cos^2 \tau \right]_0^t - \cos t \left[\frac{1}{2} (\tau - \sin \tau \cos \tau) \right]_0^t \\&= \\&= \end{aligned}$$

$$(f * g)(t) = \int_0^t f(\tau)g(t - \tau) d\tau$$



Example

$$\begin{aligned}(\sin * \sin)(t) &= \int_0^t \sin \tau \sin(t - \tau) d\tau \\&= \int_0^t \sin \tau (\sin t \cos \tau - \cos t \sin \tau) d\tau \\&= \sin t \int_0^t \sin \tau \cos \tau d\tau - \cos t \int_0^t \sin^2 \tau d\tau \\&= \sin t \left[-\frac{1}{2} \cos^2 \tau \right]_0^t - \cos t \left[\frac{1}{2} (\tau - \sin \tau \cos \tau) \right]_0^t \\&= \frac{1}{2} \sin t (1 - \cos^2 t) - \frac{1}{2} \cos t (t - \sin t \cos t) \\&= \end{aligned}$$

$$(f * g)(t) = \int_0^t f(\tau)g(t - \tau) d\tau$$



Example

$$\begin{aligned}(\sin * \sin)(t) &= \int_0^t \sin \tau \sin(t - \tau) d\tau \\&= \int_0^t \sin \tau (\sin t \cos \tau - \cos t \sin \tau) d\tau \\&= \sin t \int_0^t \sin \tau \cos \tau d\tau - \cos t \int_0^t \sin^2 \tau d\tau \\&= \sin t \left[-\frac{1}{2} \cos^2 \tau \right]_0^t - \cos t \left[\frac{1}{2} (\tau - \sin \tau \cos \tau) \right]_0^t \\&= \frac{1}{2} \sin t (1 - \cos^2 t) - \frac{1}{2} \cos t (t - \sin t \cos t) \\&= \frac{1}{2} \sin t - \frac{t}{2} \cos t.\end{aligned}$$

$$(f * g)(t) = \int_0^t f(\tau)g(t - \tau) d\tau$$



Example

$$\begin{aligned}(\sin * \sin)(t) &= \int_0^t \sin \tau \sin(t - \tau) d\tau \\&= \int_0^t \sin \tau (\sin t \cos \tau - \cos t \sin \tau) d\tau \\&= \sin t \int_0^t \sin \tau \cos \tau d\tau - \cos t \int_0^t \sin^2 \tau d\tau \\&= \sin t \left[-\frac{1}{2} \cos^2 \tau \right]_0^t - \cos t \left[\frac{1}{2} (\tau - \sin \tau \cos \tau) \right]_0^t \\&= \frac{1}{2} \sin t (1 - \cos^2 t) - \frac{1}{2} \cos t (t - \sin t \cos t) \\&= \frac{1}{2} \sin t - \frac{t}{2} \cos t.\end{aligned}$$

Note that $f * f \geq 0$ is not true in general.

$$(f * g)(t) = \int_0^t f(\tau)g(t - \tau) d\tau$$



Theorem

$$\mathcal{L}[f * g](s) = F(s)G(s)$$

$$(f * g)(t) = \int_0^t f(\tau)g(t - \tau) d\tau$$



Theorem

$$\mathcal{L} [f * g] (s) = F(s)G(s)$$

This means that $\mathcal{L}^{-1} [FG] = f * g$.

$$(f * g)(t) = \int_0^t f(\tau)g(t - \tau) d\tau$$



Example

Find the inverse Laplace Transform of $H(s) = \frac{a}{s^2(s^2 + a^2)}$.

$$(f * g)(t) = \int_0^t f(\tau)g(t - \tau) d\tau$$



Example

Find the inverse Laplace Transform of $H(s) = \frac{a}{s^2(s^2 + a^2)}$.

Note that $H(s) = \left(\frac{1}{s^2}\right) \left(\frac{a}{s^2 + a^2}\right)$.

$$(f * g)(t) = \int_0^t f(\tau)g(t - \tau) d\tau$$



Example

Find the inverse Laplace Transform of $H(s) = \frac{a}{s^2(s^2 + a^2)}$.

Note that $H(s) = \left(\frac{1}{s^2}\right) \left(\frac{a}{s^2 + a^2}\right)$. We know that $\mathcal{L}[t] = \frac{1}{s^2}$ and $\mathcal{L}[\sin at] = \frac{a}{s^2 + a^2}$.

$$(f * g)(t) = \int_0^t f(\tau)g(t - \tau) d\tau$$



Example

Find the inverse Laplace Transform of $H(s) = \frac{a}{s^2(s^2 + a^2)}$.

Note that $H(s) = \left(\frac{1}{s^2}\right) \left(\frac{a}{s^2 + a^2}\right)$. We know that $\mathcal{L}[t] = \frac{1}{s^2}$ and $\mathcal{L}[\sin at] = \frac{a}{s^2 + a^2}$. So

$$h(t) = \mathcal{L}^{-1} \left[\left(\frac{1}{s^2}\right) \left(\frac{a}{s^2 + a^2}\right) \right] =$$

=

=

$$(f * g)(t) = \int_0^t f(\tau)g(t - \tau) d\tau$$



Example

Find the inverse Laplace Transform of $H(s) = \frac{a}{s^2(s^2 + a^2)}$.

Note that $H(s) = \left(\frac{1}{s^2}\right) \left(\frac{a}{s^2 + a^2}\right)$. We know that $\mathcal{L}[t] = \frac{1}{s^2}$ and $\mathcal{L}[\sin at] = \frac{a}{s^2 + a^2}$. So

$$h(t) = \mathcal{L}^{-1} \left[\left(\frac{1}{s^2}\right) \left(\frac{a}{s^2 + a^2}\right) \right] = \mathcal{L}^{-1} \left[\frac{1}{s^2} \right] * \mathcal{L}^{-1} \left[\frac{a}{s^2 + a^2} \right]$$

=

=

$$(f * g)(t) = \int_0^t f(\tau)g(t - \tau) d\tau$$



Example

Find the inverse Laplace Transform of $H(s) = \frac{a}{s^2(s^2 + a^2)}$.

Note that $H(s) = \left(\frac{1}{s^2}\right) \left(\frac{a}{s^2 + a^2}\right)$. We know that $\mathcal{L}[t] = \frac{1}{s^2}$ and $\mathcal{L}[\sin at] = \frac{a}{s^2 + a^2}$. So

$$\begin{aligned} h(t) &= \mathcal{L}^{-1} \left[\left(\frac{1}{s^2}\right) \left(\frac{a}{s^2 + a^2}\right) \right] = \mathcal{L}^{-1} \left[\frac{1}{s^2} \right] * \mathcal{L}^{-1} \left[\frac{a}{s^2 + a^2} \right] \\ &= t * \sin at = \int_0^t \tau \sin a(t - \tau) d\tau \\ &= \end{aligned}$$

$$(f * g)(t) = \int_0^t f(\tau)g(t - \tau) d\tau$$



Example

Find the inverse Laplace Transform of $H(s) = \frac{a}{s^2(s^2 + a^2)}$.

Note that $H(s) = \left(\frac{1}{s^2}\right) \left(\frac{a}{s^2 + a^2}\right)$. We know that $\mathcal{L}[t] = \frac{1}{s^2}$ and $\mathcal{L}[\sin at] = \frac{a}{s^2 + a^2}$. So

$$\begin{aligned} h(t) &= \mathcal{L}^{-1} \left[\left(\frac{1}{s^2}\right) \left(\frac{a}{s^2 + a^2}\right) \right] = \mathcal{L}^{-1} \left[\frac{1}{s^2} \right] * \mathcal{L}^{-1} \left[\frac{a}{s^2 + a^2} \right] \\ &= t * \sin at = \int_0^t \tau \sin a(t - \tau) d\tau \\ &= \frac{at - \sin at}{a^2}. \end{aligned}$$

$$(f * g)(t) = \int_0^t f(\tau)g(t - \tau) d\tau$$



Example

Solve

$$\begin{cases} y'' + 4y = g(t) \\ y(0) = 3 \\ y'(0) = -1. \end{cases}$$

$$(f * g)(t) = \int_0^t f(\tau)g(t - \tau) d\tau$$



Example

Solve

$$\begin{cases} y'' + 4y = g(t) \\ y(0) = 3 \\ y'(0) = -1. \end{cases}$$

Taking the Laplace Transform of the ODE gives

$$(s^2Y - 3s + 1) + 4Y = G(s)$$

$$(f * g)(t) = \int_0^t f(\tau)g(t - \tau) d\tau$$



Example

Solve

$$\begin{cases} y'' + 4y = g(t) \\ y(0) = 3 \\ y'(0) = -1. \end{cases}$$

Taking the Laplace Transform of the ODE gives

$$(s^2Y - 3s + 1) + 4Y = G(s)$$

which rearranges to

$$\begin{aligned} Y(s) &= \frac{3s - 1}{s^2 + 4} + \frac{G(s)}{s^2 + 4} \\ &= 3 \left(\frac{s}{s^2 + 4} \right) - \frac{1}{2} \left(\frac{2}{s^2 + 4} \right) + \frac{1}{2} \left(\frac{2}{s^2 + 4} \right) G(s). \end{aligned}$$

$$(f * g)(t) = \int_0^t f(\tau)g(t - \tau) d\tau$$



$$Y(s) = 3 \left(\frac{s}{s^2 + 4} \right) - \frac{1}{2} \left(\frac{2}{s^2 + 4} \right) + \frac{1}{2} \left(\frac{2}{s^2 + 4} \right) G(s)$$

Hence the solution to the IVP is

$$\begin{aligned} y(t) &= 3\mathcal{L}^{-1} \left[\frac{s}{s^2 + 4} \right] - \frac{1}{2}\mathcal{L}^{-1} \left[\frac{2}{s^2 + 4} \right] + \frac{1}{2}\mathcal{L}^{-1} \left[\left(\frac{2}{s^2 + 4} \right) G(s) \right] \\ &= \\ &= \end{aligned}$$

$$(f * g)(t) = \int_0^t f(\tau)g(t - \tau) d\tau$$



$$Y(s) = 3 \left(\frac{s}{s^2 + 4} \right) - \frac{1}{2} \left(\frac{2}{s^2 + 4} \right) + \frac{1}{2} \left(\frac{2}{s^2 + 4} \right) G(s)$$

Hence the solution to the IVP is

$$\begin{aligned} y(t) &= 3\mathcal{L}^{-1} \left[\frac{s}{s^2 + 4} \right] - \frac{1}{2}\mathcal{L}^{-1} \left[\frac{2}{s^2 + 4} \right] + \frac{1}{2}\mathcal{L}^{-1} \left[\left(\frac{2}{s^2 + 4} \right) G(s) \right] \\ &= 3 \cos 2t - \frac{1}{2} \sin 2t + \frac{1}{2} \sin 2t * g(t) \\ &= \end{aligned}$$

$$(f * g)(t) = \int_0^t f(\tau)g(t - \tau) d\tau$$



$$Y(s) = 3 \left(\frac{s}{s^2 + 4} \right) - \frac{1}{2} \left(\frac{2}{s^2 + 4} \right) + \frac{1}{2} \left(\frac{2}{s^2 + 4} \right) G(s)$$

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$$\begin{aligned} y(t) &= 3\mathcal{L}^{-1} \left[\frac{s}{s^2 + 4} \right] - \frac{1}{2}\mathcal{L}^{-1} \left[\frac{2}{s^2 + 4} \right] + \frac{1}{2}\mathcal{L}^{-1} \left[\left(\frac{2}{s^2 + 4} \right) G(s) \right] \\ &= 3 \cos 2t - \frac{1}{2} \sin 2t + \frac{1}{2} \sin 2t * g(t) \\ &= 3 \cos 2t - \frac{1}{2} \sin 2t + \frac{1}{2} \int_0^t \sin 2(t - \tau)g(\tau) d\tau. \end{aligned}$$

$$(f * g)(t) = \int_0^t f(\tau)g(t - \tau) d\tau$$



Example

Find the inverse Laplace Transform of $\frac{2}{(s-1)(s^2+4)}$.

$$(f * g)(t) = \int_0^t f(\tau)g(t - \tau) d\tau$$



Example

Find the inverse Laplace Transform of $\frac{2}{(s-1)(s^2+4)}$.

$$\mathcal{L}^{-1} \left[\frac{2}{(s-1)(s^2+4)} \right] = \mathcal{L}^{-1} \left[\left(\frac{2}{s^2+4} \right) \left(\frac{1}{s-1} \right) \right]$$

$$(f * g)(t) = \int_0^t f(\tau)g(t - \tau) d\tau$$



Example

Find the inverse Laplace Transform of $\frac{2}{(s-1)(s^2+4)}$.

$$\mathcal{L}^{-1} \left[\frac{2}{(s-1)(s^2+4)} \right] = \mathcal{L}^{-1} \left[\left(\frac{2}{s^2+4} \right) \left(\frac{1}{s-1} \right) \right] = \sin 2t * e^t$$

$$(f * g)(t) = \int_0^t f(\tau)g(t - \tau) d\tau$$



Example

Find the inverse Laplace Transform of $\frac{2}{(s-1)(s^2+4)}$.

$$\begin{aligned}\mathcal{L}^{-1} \left[\frac{2}{(s-1)(s^2+4)} \right] &= \mathcal{L}^{-1} \left[\left(\frac{2}{s^2+4} \right) \left(\frac{1}{s-1} \right) \right] = \sin 2t * e^t \\ &= \int_0^t e^{t-\tau} \sin 2\tau d\tau\end{aligned}$$

$$(f * g)(t) = \int_0^t f(\tau)g(t - \tau) d\tau$$



Example

Find the inverse Laplace Transform of $\frac{2}{(s-1)(s^2+4)}$.

$$\begin{aligned}\mathcal{L}^{-1}\left[\frac{2}{(s-1)(s^2+4)}\right] &= \mathcal{L}^{-1}\left[\left(\frac{2}{s^2+4}\right)\left(\frac{1}{s-1}\right)\right] = \sin 2t * e^t \\ &= \int_0^t e^{t-\tau} \sin 2\tau d\tau = e^t \int_0^t e^{-\tau} \sin 2\tau d\tau\end{aligned}$$

$$(f * g)(t) = \int_0^t f(\tau)g(t - \tau) d\tau$$



Example

Find the inverse Laplace Transform of $\frac{2}{(s-1)(s^2+4)}$.

$$\begin{aligned}\mathcal{L}^{-1} \left[\frac{2}{(s-1)(s^2+4)} \right] &= \mathcal{L}^{-1} \left[\left(\frac{2}{s^2+4} \right) \left(\frac{1}{s-1} \right) \right] = \sin 2t * e^t \\ &= \int_0^t e^{t-\tau} \sin 2\tau d\tau = e^t \int_0^t e^{-\tau} \sin 2\tau d\tau \\ &= e^t \left[\frac{e^{-\tau}}{5} (-\sin 2\tau - 2 \cos 2\tau) \right]_0^t\end{aligned}$$

$$(f * g)(t) = \int_0^t f(\tau)g(t - \tau) d\tau$$



Example

Find the inverse Laplace Transform of $\frac{2}{(s-1)(s^2+4)}$.

$$\begin{aligned}\mathcal{L}^{-1} \left[\frac{2}{(s-1)(s^2+4)} \right] &= \mathcal{L}^{-1} \left[\left(\frac{2}{s^2+4} \right) \left(\frac{1}{s-1} \right) \right] = \sin 2t * e^t \\&= \int_0^t e^{t-\tau} \sin 2\tau d\tau = e^t \int_0^t e^{-\tau} \sin 2\tau d\tau \\&= e^t \left[\frac{e^{-\tau}}{5} (-\sin 2\tau - 2 \cos 2\tau) \right]_0^t \\&= \frac{2}{5}e^t - \frac{1}{5} \sin 2t - \frac{2}{5} \cos 2t.\end{aligned}$$

$$(f * g)(t) = \int_0^t f(\tau)g(t - \tau) d\tau$$



Example

Solve

$$\begin{cases} 4y'' + y = g(t) \\ y(0) = 3 \\ y'(0) = -7. \end{cases}$$

$$(f * g)(t) = \int_0^t f(\tau)g(t - \tau) d\tau$$



Example

Solve

$$\begin{cases} 4y'' + y = g(t) \\ y(0) = 3 \\ y'(0) = -7. \end{cases}$$

$$4y'' + y = g(t)$$

$$(f * g)(t) = \int_0^t f(\tau)g(t - \tau) d\tau$$



Example

Solve

$$\begin{cases} 4y'' + y = g(t) \\ y(0) = 3 \\ y'(0) = -7. \end{cases}$$

$$\mathcal{L}[4y'' + y] = \mathcal{L}[g(t)]$$

$$(f * g)(t) = \int_0^t f(\tau)g(t - \tau) d\tau$$



Example

Solve

$$\begin{cases} 4y'' + y = g(t) \\ y(0) = 3 \\ y'(0) = -7. \end{cases}$$

$$\mathcal{L}[4y'' + y] = \mathcal{L}[g(t)]$$

$$4(s^2Y - sy(0) - y'(0)) + Y = G(s)$$

$$(f * g)(t) = \int_0^t f(\tau)g(t - \tau) d\tau$$



Example

Solve

$$\begin{cases} 4y'' + y = g(t) \\ y(0) = 3 \\ y'(0) = -7. \end{cases}$$

$$\mathcal{L}[4y'' + y] = \mathcal{L}[g(t)]$$

$$4(s^2Y - 3s + 7) + Y = G(s)$$

$$(f * g)(t) = \int_0^t f(\tau)g(t - \tau) d\tau$$



Example

Solve

$$\begin{cases} 4y'' + y = g(t) \\ y(0) = 3 \\ y'(0) = -7. \end{cases}$$

$$\mathcal{L}[4y'' + y] = \mathcal{L}[g(t)]$$

$$(4s^2 + 1)Y - 12s + 28 = G(s)$$

$$(f * g)(t) = \int_0^t f(\tau)g(t - \tau) d\tau$$



Example

Solve

$$\begin{cases} 4y'' + y = g(t) \\ y(0) = 3 \\ y'(0) = -7. \end{cases}$$

$$\mathcal{L}[4y'' + y] = \mathcal{L}[g(t)]$$

$$(4s^2 + 1)Y = 12s - 28 + G(s)$$

$$(f * g)(t) = \int_0^t f(\tau)g(t - \tau) d\tau$$



Example

Solve

$$\begin{cases} 4y'' + y = g(t) \\ y(0) = 3 \\ y'(0) = -7. \end{cases}$$

$$\mathcal{L}[4y'' + y] = \mathcal{L}[g(t)]$$

$$4 \left(s^2 + \frac{1}{4} \right) Y = 12s - 28 + G(s)$$

$$(f * g)(t) = \int_0^t f(\tau)g(t - \tau) d\tau$$



Example

Solve

$$\begin{cases} 4y'' + y = g(t) \\ y(0) = 3 \\ y'(0) = -7. \end{cases}$$

$$\mathcal{L}[4y'' + y] = \mathcal{L}[g(t)]$$

$$Y = \frac{12s}{4(s^2 + \frac{1}{4})} - \frac{28}{4(s^2 + \frac{1}{4})} + \frac{G(s)}{4(s^2 + \frac{1}{4})}$$

$$(f * g)(t) = \int_0^t f(\tau)g(t - \tau) d\tau$$



Example

Solve

$$\begin{cases} 4y'' + y = g(t) \\ y(0) = 3 \\ y'(0) = -7. \end{cases}$$

$$Y = \frac{3s}{s^2 + \frac{1}{4}} - \frac{7}{s^2 + \frac{1}{4}} + G(s) \frac{\frac{1}{4}}{s^2 + \frac{1}{4}}$$

$$(f * g)(t) = \int_0^t f(\tau)g(t - \tau) d\tau$$



Example

Solve

$$\begin{cases} 4y'' + y = g(t) \\ y(0) = 3 \\ y'(0) = -7. \end{cases}$$

$$Y = 3 \left(\frac{s}{s^2 + \frac{1}{4}} \right) - 14 \left(\frac{\frac{1}{2}}{s^2 + \frac{1}{4}} \right) + \frac{1}{2}G(s) \left(\frac{\frac{1}{2}}{s^2 + \frac{1}{4}} \right)$$

$$(f * g)(t) = \int_0^t f(\tau)g(t - \tau) d\tau$$



Example

Solve

$$\begin{cases} 4y'' + y = g(t) \\ y(0) = 3 \\ y'(0) = -7. \end{cases}$$

$$Y = 3\mathcal{L}\left[\cos\frac{t}{2}\right] - 14\left(\frac{\frac{1}{2}}{s^2 + \frac{1}{4}}\right) + \frac{1}{2}G(s)\left(\frac{\frac{1}{2}}{s^2 + \frac{1}{4}}\right)$$

$$(f * g)(t) = \int_0^t f(\tau)g(t - \tau) d\tau$$



Example

Solve

$$\begin{cases} 4y'' + y = g(t) \\ y(0) = 3 \\ y'(0) = -7. \end{cases}$$

$$Y = 3\mathcal{L}\left[\cos \frac{t}{2}\right] - 14\mathcal{L}\left[\sin \frac{t}{2}\right] + \frac{1}{2}G(s)\mathcal{L}\left[\sin \frac{t}{2}\right]$$

$$(f * g)(t) = \int_0^t f(\tau)g(t - \tau) d\tau$$



Example

Solve

$$\begin{cases} 4y'' + y = g(t) \\ y(0) = 3 \\ y'(0) = -7. \end{cases}$$

$$Y = 3\mathcal{L}\left[\cos \frac{t}{2}\right] - 14\mathcal{L}\left[\sin \frac{t}{2}\right] + \frac{1}{2}G(s)\mathcal{L}\left[\sin \frac{t}{2}\right]$$

$$y(t) =$$

$$(f * g)(t) = \int_0^t f(\tau)g(t - \tau) d\tau$$



Example

Solve

$$\begin{cases} 4y'' + y = g(t) \\ y(0) = 3 \\ y'(0) = -7. \end{cases}$$

$$Y = 3\mathcal{L}\left[\cos \frac{t}{2}\right] - 14\mathcal{L}\left[\sin \frac{t}{2}\right] + \frac{1}{2}G(s)\mathcal{L}\left[\sin \frac{t}{2}\right]$$

$$y(t) = 3 \cos \frac{t}{2}$$

$$(f * g)(t) = \int_0^t f(\tau)g(t - \tau) d\tau$$



Example

Solve

$$\begin{cases} 4y'' + y = g(t) \\ y(0) = 3 \\ y'(0) = -7. \end{cases}$$

$$Y = 3\mathcal{L}\left[\cos \frac{t}{2}\right] - 14\mathcal{L}\left[\sin \frac{t}{2}\right] + \frac{1}{2}G(s)\mathcal{L}\left[\sin \frac{t}{2}\right]$$

$$y(t) = 3 \cos \frac{t}{2} - 14 \sin \frac{t}{2}$$

$$(f * g)(t) = \int_0^t f(\tau)g(t - \tau) d\tau$$



Example

Solve

$$\begin{cases} 4y'' + y = g(t) \\ y(0) = 3 \\ y'(0) = -7. \end{cases}$$

$$Y = 3\mathcal{L}\left[\cos \frac{t}{2}\right] - 14\mathcal{L}\left[\sin \frac{t}{2}\right] + \frac{1}{2}G(s)\mathcal{L}\left[\sin \frac{t}{2}\right]$$

$$y(t) = 3\cos \frac{t}{2} - 14\sin \frac{t}{2} + \frac{1}{2}g(t) * \sin \frac{t}{2}.$$

Next Time

- 5.1 Introduction
- 5.2 Basic Theory of Systems of First Order Linear Equations
- 5.3 Homogeneous Linear Systems with Constant Coefficients
- 5.4 Complex Eigenvalues