Lecture 9

- 14.5 Triple Integrals in Rectangular Coordinates
- 14.7 Triple Integrals in Cylindrical and Spherical Coordinates
- 14.8 Substitutions in Multiple Integrals

Triple Integrals in Rectangular Coordinates

In the last two lectures we have been studying

$$\iint_R f(x,y)\,dA.$$

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$$\iint_R f(x,y)\,dA.$$

Today we will consider

$$\iiint_D f(x,y,z)\,dV.$$

Definition

The *volume* of a closed, bounded region D is space is

$$V = \iiint\limits_{R} dV.$$

Finding Limits of Integration

$$\int \quad \int \qquad \int \qquad F(x, y, z)\, dz dy dx.$$

Finding Limits of Integration

$$\int_{x=a}^{x=b} \int \int F(x, y, z)\, dz\, dy\, dx.$$

only numbers

Finding Limits of Integration

$$\int_{x=a}^{x=b} \int_{y=g_1(x)}^{y=g_2(x)} \int F(x,y,z)\,dz\,dy\,dx.$$

only numbers

functions of x

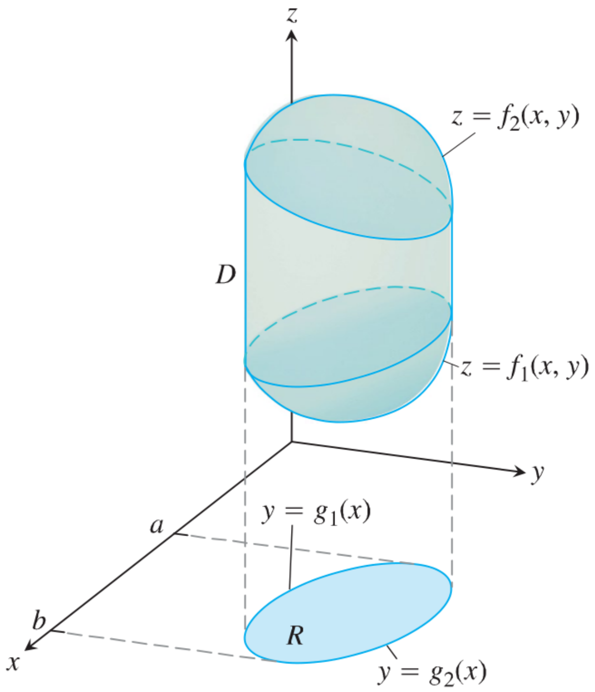
Finding Limits of Integration

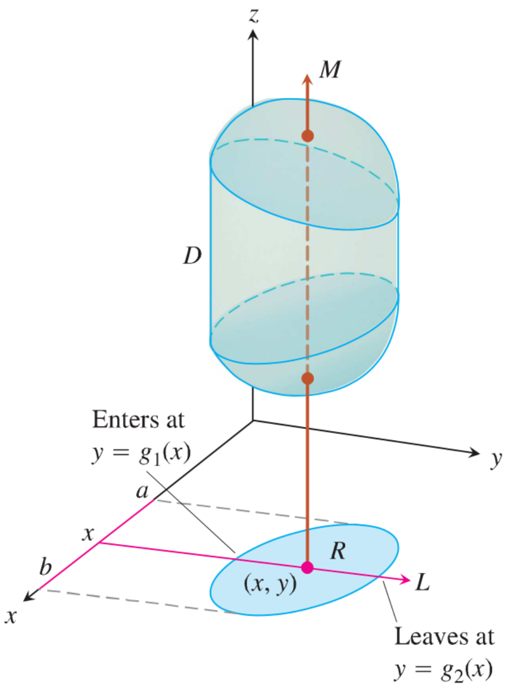
$$\int_{x=a}^{x=b} \int_{y=g_1(x)}^{y=g_2(x)} \int_{z=h_1(x,y)}^{z=h_2(x,y)} F(x,y,z)\, dz\, dy\, dx.$$

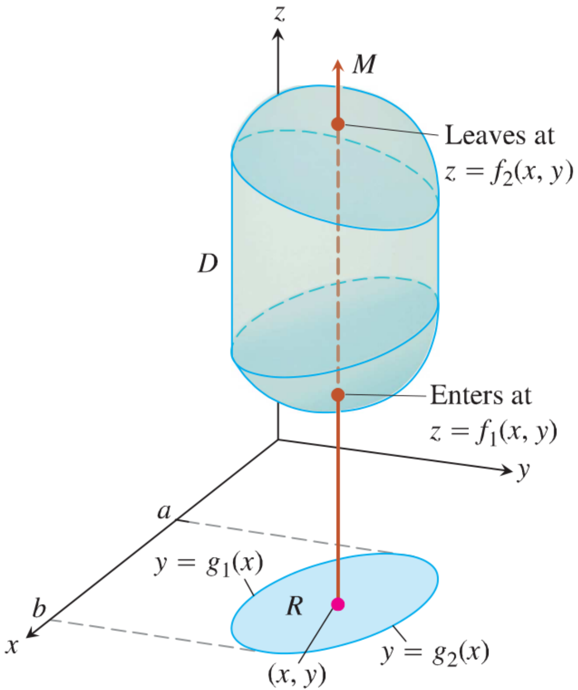
functions of x and y

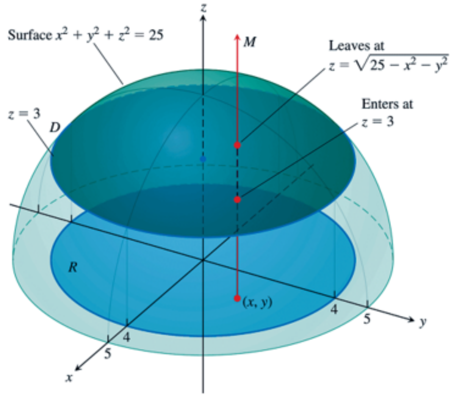
functions of x

only numbers


z
$z = f_2(x, y)$
D
$z = f_1(x, y)$
y
$y = g_1(x)$
a
b
R
x
$y = g_2(x)$

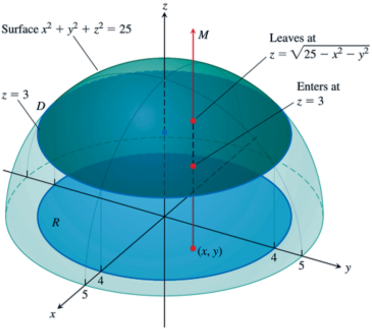

z
M
D
Enters at
y = g1(x)
a
x
b
x
R
(x, y)
L
y
Leaves at
y = g2(x)


z
M
Leaves at
z = f2(x, y)
D
Enters at
z = f1(x, y)
y
a
y = g1(x)
R
b
x
(x, y)
y = g2(x)

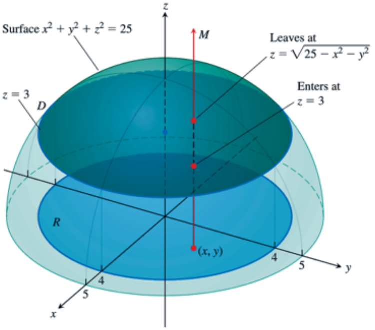


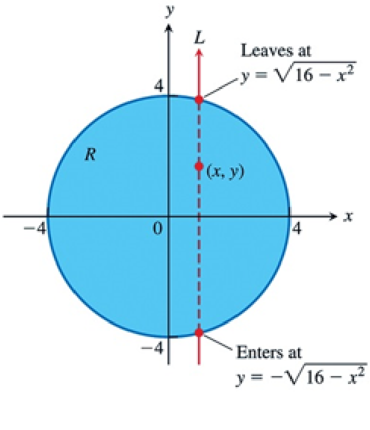
Example

Let S be the sphere of radius 5 centred at the origin. Let D be the region under the sphere and above the plane $z = 3$. Set up the limits of integration over D.



$$z = 3 \quad \Longrightarrow \quad x^2 + y^2 + 9 = 25 \quad \Longrightarrow \quad x^2 + y^2 = 16$$

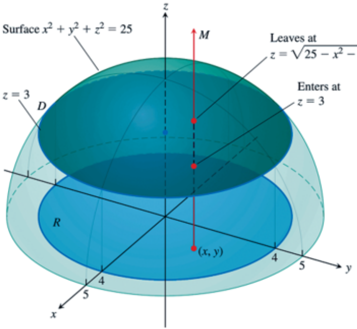


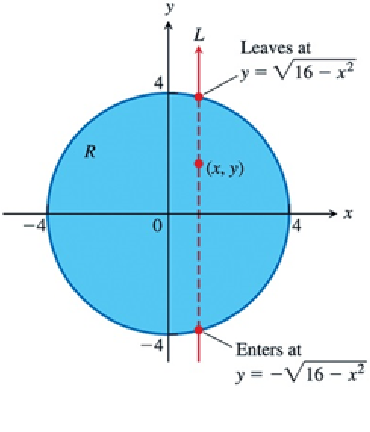


(b)

$$z = 3 \quad \Longrightarrow \quad x^2 + y^2 + 9 = 25 \quad \Longrightarrow \quad x^2 + y^2 = 16$$

$$-4 \leq x \leq 4$$

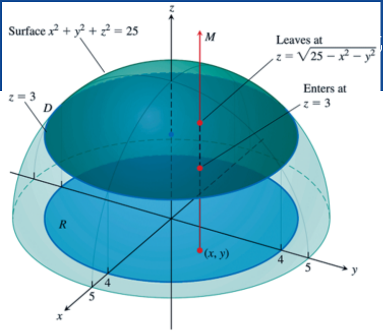


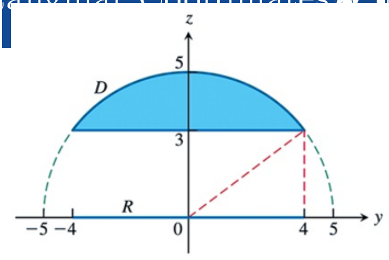


(b)

$$z = 3 \quad \Longrightarrow \quad x^2 + y^2 + 9 = 25 \quad \Longrightarrow \quad x^2 + y^2 = 16$$

$$-4 \leq x \leq 4 \qquad -\sqrt{16 - x^2} \leq y \leq \sqrt{16 - x^2}$$





(a)

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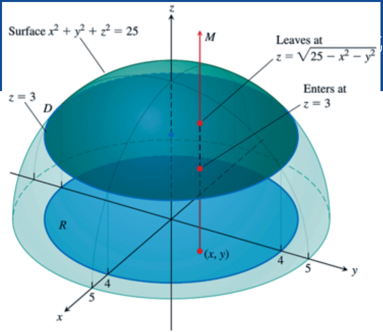
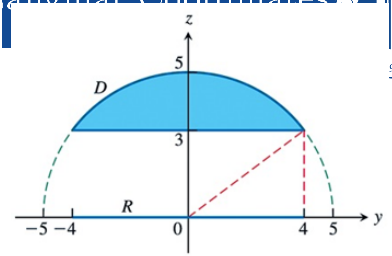
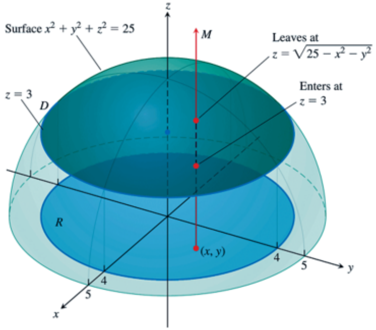


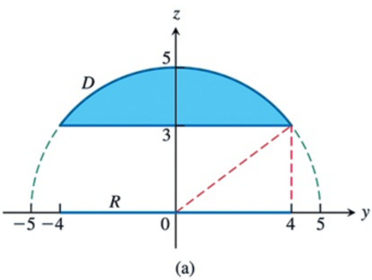


(a)

$$-4 \leq x \leq 4 \qquad -\sqrt{16 - x^2} \leq y \leq \sqrt{16 - x^2}$$

$$3 \leq z \leq \sqrt{25 - x^2 - y^2}$$

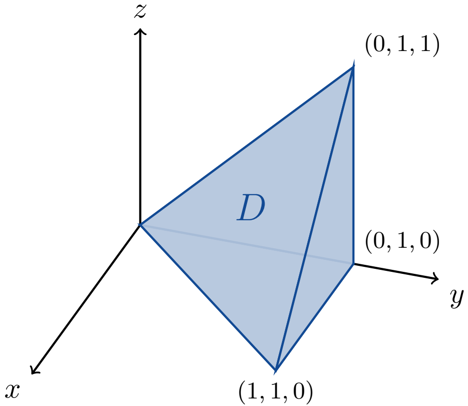




$$-4 \leq x \leq 4 \qquad -\sqrt{16-x^2} \leq y \leq \sqrt{16-x^2}$$

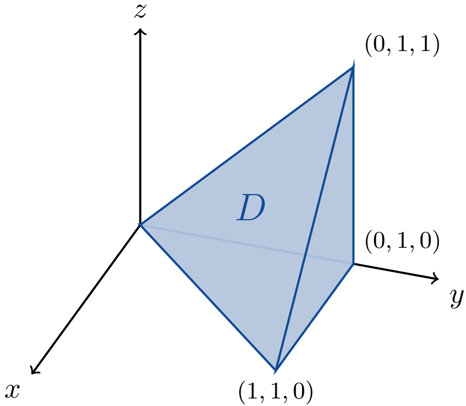
$$3 \leq z \leq \sqrt{25-x^2-y^2}$$

$$\int_{-4}^{4}\int_{-\sqrt{16-x^2}}^{\sqrt{16-x^2}}\int_{3}^{\sqrt{25-x^2-y^2}} F(x,y,z)\,dz\,dy\,dx.$$



Example

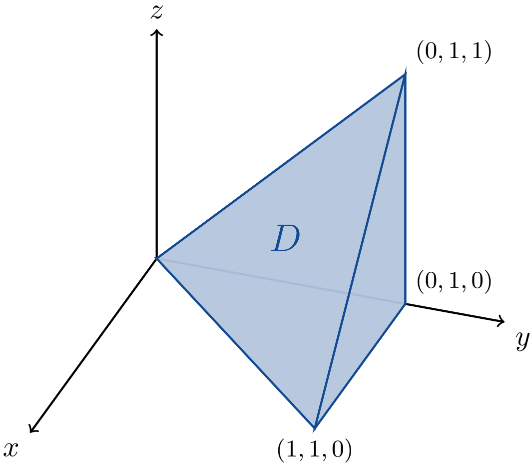
Let D be the tetrahedron whose vertices are $(0,0,0)$, $(1,1,0)$, $(0,1,0)$ and $(0,1,1)$.



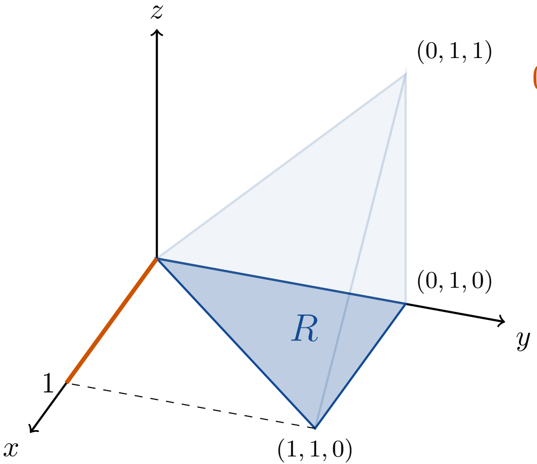
Example

Let D be the tetrahedron whose vertices are $(0, 0, 0)$, $(1, 1, 0)$, $(0, 1, 0)$ and $(0, 1, 1)$. Set up the limits of integration over D using the order $dzdydx$.

$$\iiint_D F(x, y, z)\, dz dy dx =$$

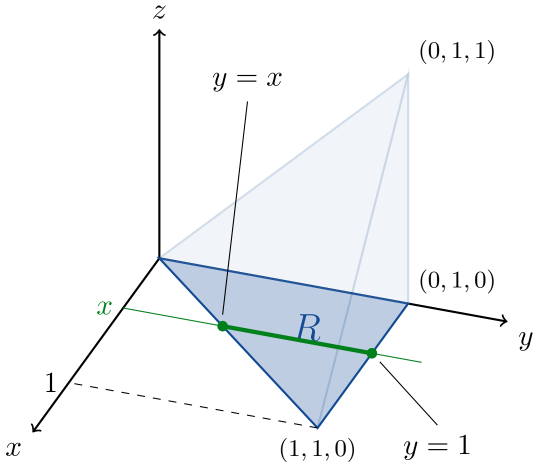


$$\iiint\limits_D F(x,y,z)\,dzdydx =$$



$0 \leq x \leq 1$

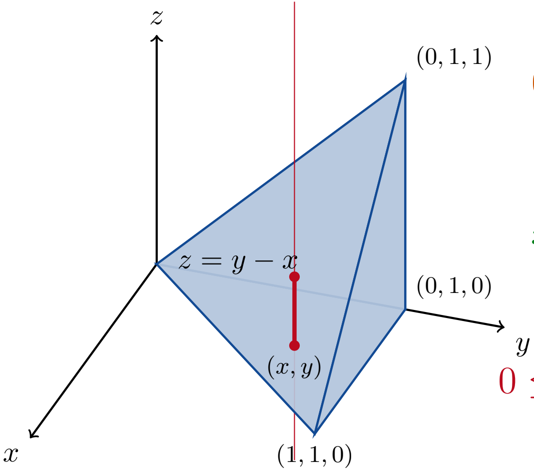
$$\iiint_D F(x,y,z)\,dz\,dy\,dx =$$



$0 \le x \le 1$

$x \le y \le 1$

$$\iiint\limits_{D} F(x,y,z)\, dz dy dx =$$

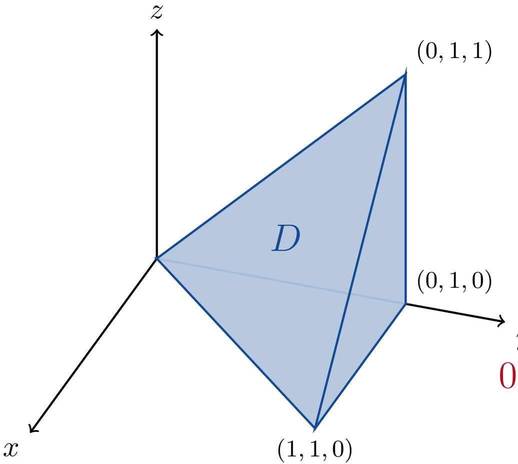


$0 \leq x \leq 1$

$x \leq y \leq 1$

$0 \leq z \leq y - x$

$$\iiint\limits_D F(x,y,z)\,dzdydx = \int_0^1 \int_x^1 \int_0^{y-x} F(x,y,z)\,dzdydx.$$



$0 \leq x \leq 1$

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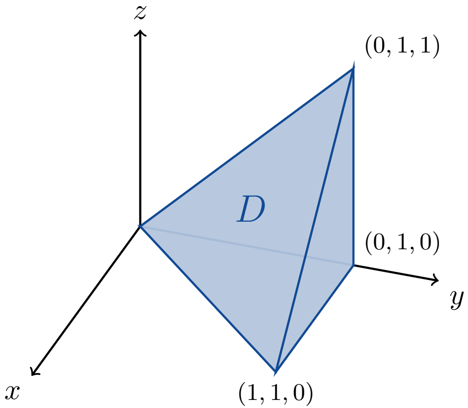
Example

Find the volume of this tetrahedron by integrating the function $F(x, y, z) = 1$ over D using the order $dzdydx$.



Example

Find the volume of this tetrahedron by integrating the function $F(x, y, z) = 1$ over D using the order $dzdydx$.

$$V = \iiint_D dzdydx = \int_0^1 \int_x^1 \int_0^{y-x} dzdydx$$

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Find the volume of this tetrahedron by integrating the function $F(x, y, z) = 1$ over D using the order $dzdydx$.

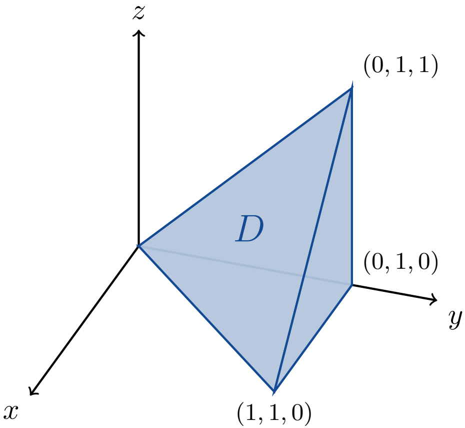
$$V = \iiint\limits_D dzdydx = \int_0^1 \int_x^1 \int_0^{y-x} dzdydx$$

$$= \int_0^1 \int_x^1 (y - x)\, dydx = \int_0^1 \left[\frac{1}{2}y^2 - xy\right]_{y=x}^{y=1} dx$$

$$= \int_0^1 \left(\frac{1}{2} - x + \frac{1}{2}x^2\right) dx = \left[\frac{1}{2}x - \frac{1}{2}x^2 + \frac{1}{6}x^3\right]_0^1 = \frac{1}{6}.$$

Example

Find the volume of this tetrahedron by integrating the function $F(x, y, z) = 1$ over D using the order $dydzdx$.



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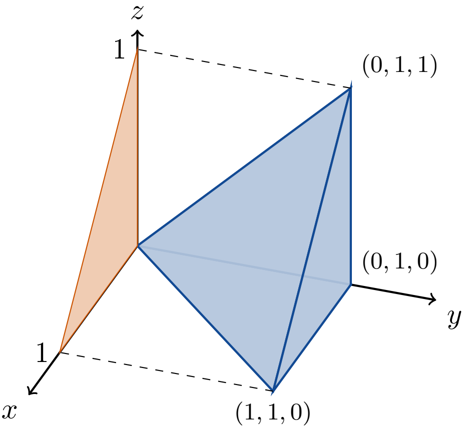

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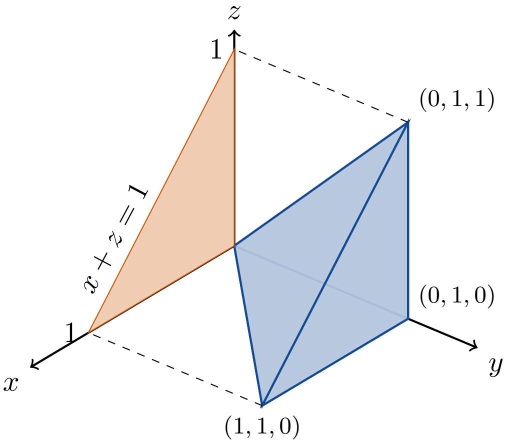

$$0 \leq x \leq 1$$

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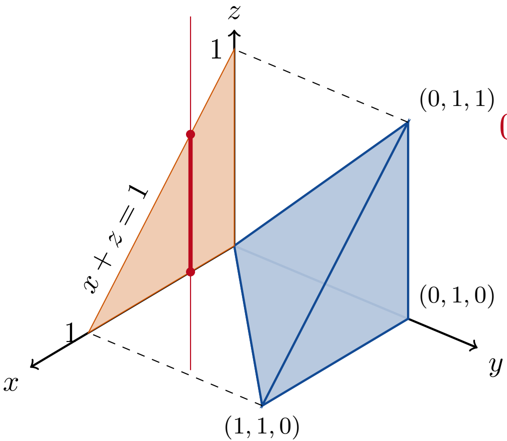

$$0 \leq x \leq 1$$

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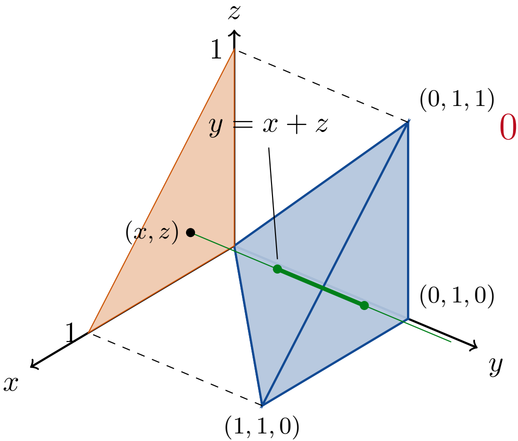

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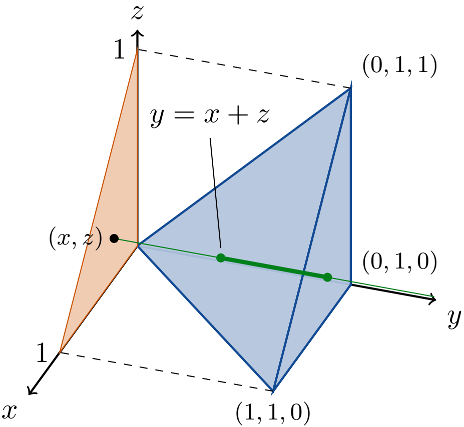


$$0 \leq x \leq 1$$

$$0 \leq z \leq 1 - x$$

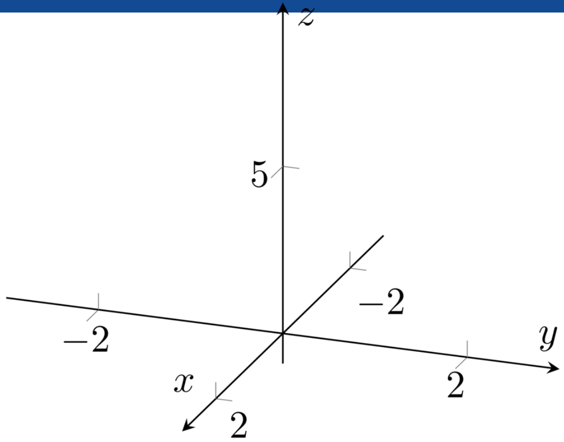
$$x + z \leq y \leq 1$$

$$0 \leq x \leq 1 \qquad 0 \leq z \leq 1 - x \qquad x + z \leq y \leq 1$$

$$V = \iiint\limits_D dz dy dx = \int_0^1 \int_0^{1-x} \int_{x+z}^1 dy dz dx$$

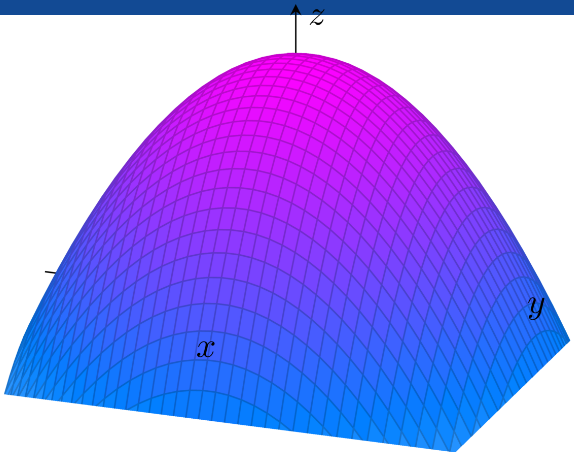
$$0 \leq x \leq 1 \qquad 0 \leq z \leq 1 - x \qquad x + z \leq y \leq 1$$

$$\begin{aligned}
V &= \iiint\limits_D dzdydx = \int_0^1 \int_0^{1-x} \int_{x+z}^1 dydzdx \\
&= \int_0^1 \int_0^{1-x} (1 - x - z)\, dzdx = \int_0^1 \left[z - xz - \frac{1}{2} z^2 \right]_{z=0}^{z=1-x} dx \\
&= \int_0^1 \left(1 - x - x - x^2 - \frac{1}{2}(1-x)^2 \right) dx \\
&= \frac{1}{2} \int_0^1 (1-x)^2 \, dx = \frac{1}{2} \left[-\frac{1}{3}(1-x)^3 \right]_0^1 = \frac{1}{6}.
\end{aligned}$$



Example

Find the volume of the region D enclosed by the surfaces $z = x^2 + 3y^2$ and $z = 8 - x^2 - y^2$.



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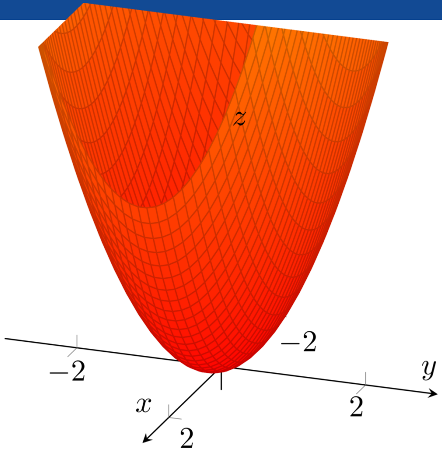

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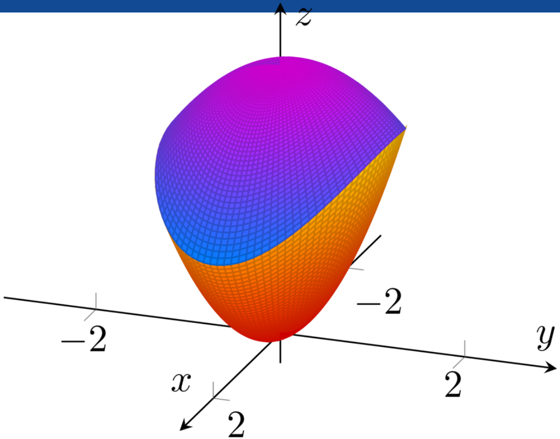

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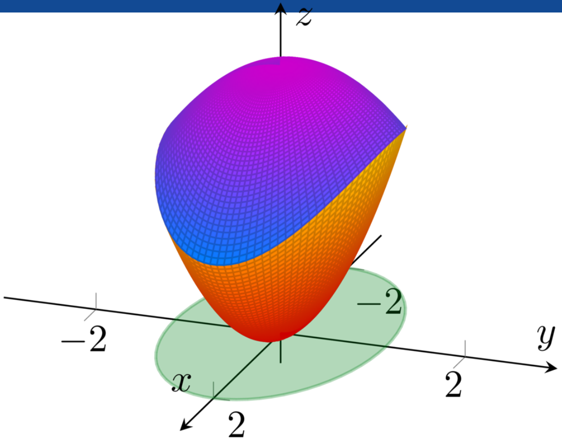

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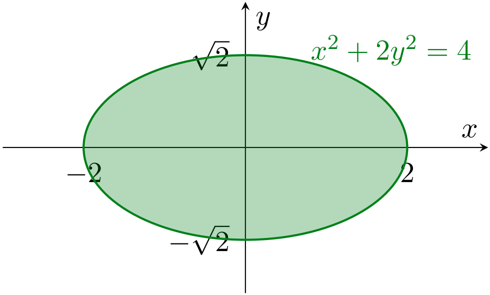
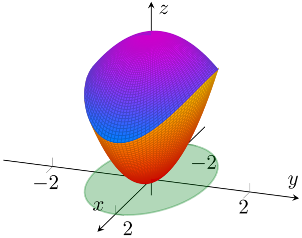
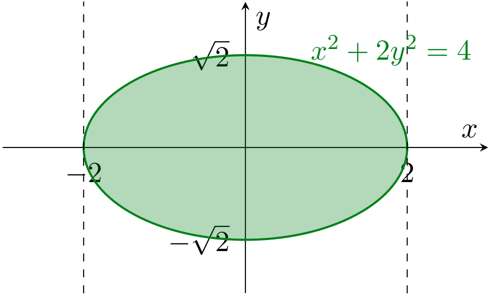
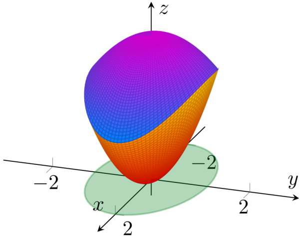




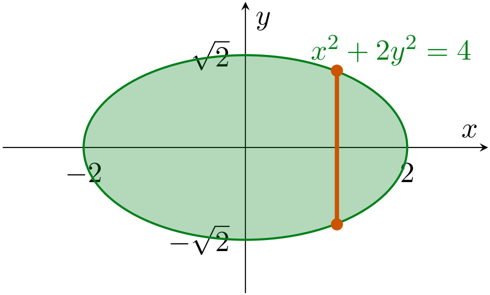
$$
\begin{aligned}
x^2 + 3y^2 &= z = 8 - x^2 - y^2 \\
2x^2 + 4y^2 &= 8 \\
x^2 + 2y^2 &= 4.
\end{aligned}
$$

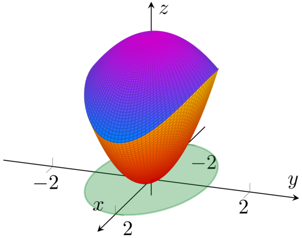




$$-2 \le x \le 2$$

$$\begin{aligned} x^2 + 3y^2 &= z = 8 - x^2 - y^2 \\ 2x^2 + 4y^2 &= 8 \\ x^2 + 2y^2 &= 4. \end{aligned}$$

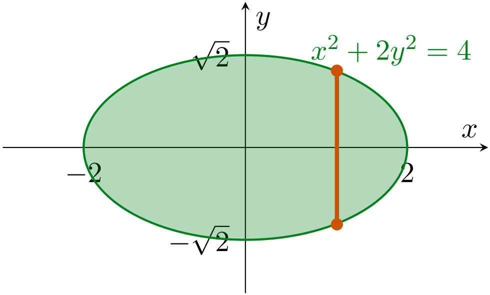


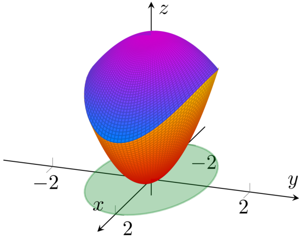


$$-2 \leq x \leq 2$$

$$x^2 + 3y^2 = z = 8 - x^2 - y^2$$
$$2x^2 + 4y^2 = 8$$
$$x^2 + 2y^2 = 4.$$

$$-\sqrt{\frac{4 - x^2}{2}} \leq y \leq \sqrt{\frac{4 - x^2}{2}}$$





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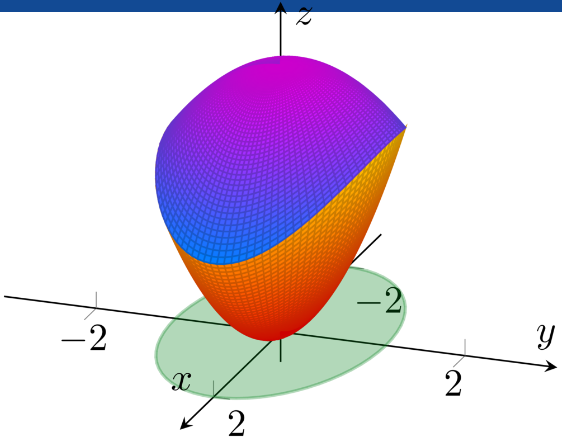
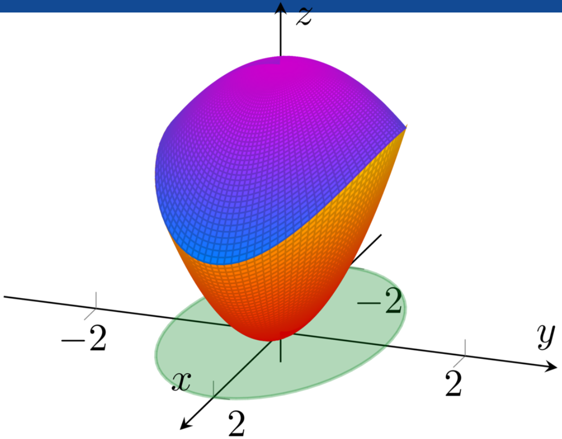
$$-\sqrt{\frac{4 - x^2}{2}} \leq y \leq \sqrt{\frac{4 - x^2}{2}}$$

$$x^2 + 3y^2 \leq z \leq 8 - x^2 - y^2$$



The volume of D is

$$V = \int_{-2}^{2} \int_{-\sqrt{\frac{4-x^2}{2}}}^{\sqrt{\frac{4-x^2}{2}}} \int_{x^2+3y^2}^{8-x^2-y^2} \, dz\,dy\,dx =$$



The volume of D is

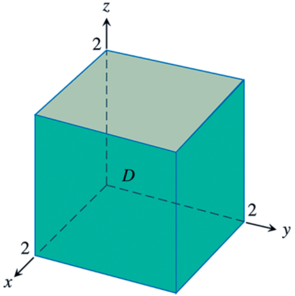
$$V = \int_{-2}^{2} \int_{-\sqrt{\frac{4-x^2}{2}}}^{\sqrt{\frac{4-x^2}{2}}} \int_{x^2+3y^2}^{8-x^2-y^2} dzdydx = \ldots = 8\pi\sqrt{2}.$$

Average Value of a Function

Definition

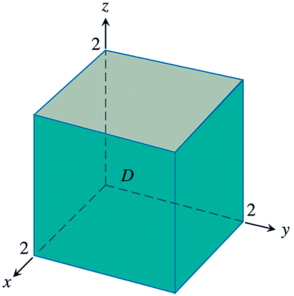
The *average value* of a function F over a region D is

$$\text{av}(F) = \frac{1}{\text{volume of } D} \iiint\limits_D F \, dV.$$



Example

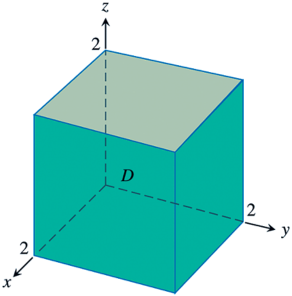
Find the average value of $F(x, y, z) = xyz$ over the cube $[0, 2] \times [0, 2] \times [0, 2]$.



Example

Find the average value of $F(x,y,z) = xyz$ over the cube $[0,2] \times [0,2] \times [0,2]$.

$$\mathrm{av}(F) = \frac{1}{\text{volume of the cube}} \iiint\limits_{\text{cube}} xyz \, dx dy dz$$

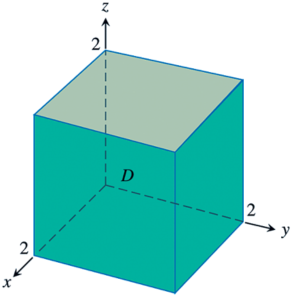


Example

Find the average value of $F(x,y,z) = xyz$ over the cube $[0,2]\times[0,2]\times[0,2]$.

$$\operatorname{av}(F) = \frac{1}{\text{volume of the cube}} \iiint\limits_{\text{cube}} xyz\,dxdydz$$

$$= \frac{1}{8}\int_0^2\int_0^2\int_0^2 xyz\,dxdydz$$



Example

Find the average value of $F(x,y,z) = xyz$ over the cube $[0,2] \times [0,2] \times [0,2]$.

$$\text{av}(F) = \frac{1}{\text{volume of the cube}} \iiint\limits_{\text{cube}} xyz \, dxdydz$$

$$= \frac{1}{8} \int_0^2 \int_0^2 \int_0^2 xyz \, dxdydz = \ldots = 1.$$

Properties of Triple Integrals

Properties of Triple Integrals

The same as for double integrals.

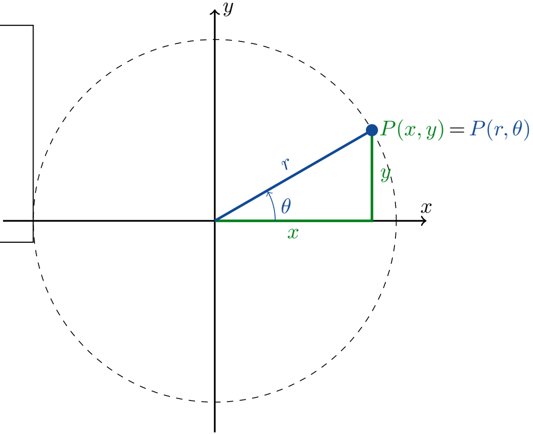
Triple Integrals in Cylindrical and Spherical Coordinates

14.7

Polar Coordinates in $\mathbb{R}^2$

$$x = r\cos\theta \qquad x^2 + y^2 = r^2$$

$$y = r\sin\theta \qquad \tan\theta = \frac{y}{x}$$

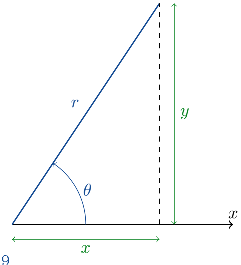


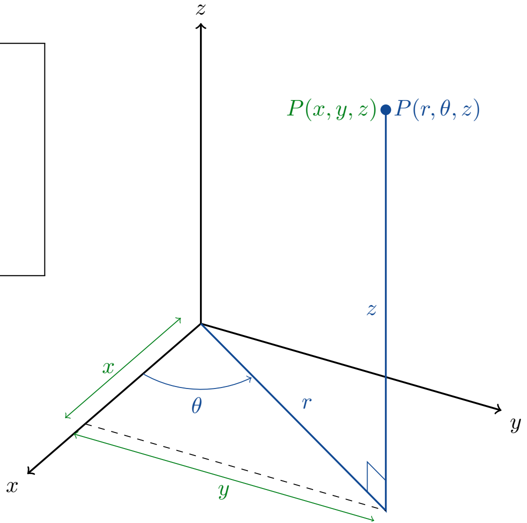
Cylindrical Coordinates in $\mathbb{R}^3$

$x = r\cos\theta \qquad x^2 + y^2 = r^2$

$y = r\sin\theta \qquad \tan\theta = \dfrac{y}{x}$

$z = z$





$$x = r\cos\theta \quad y = r\sin\theta \quad z = z$$

Example

Find cylindrical coordinates for the Cartesian coordiantes $(x, y, z) = (1, 1, 1)$.

$$x = r\cos\theta \quad y = r\sin\theta \quad z = z$$

Example

Find cylindrical coordinates for the Cartesian coordiantes $(x, y, z) = (1, 1, 1)$.

$$\begin{aligned}(r, \theta, z) &= \left(\sqrt{x^2 + y^2}, \tan^{-1}\frac{y}{x}, z\right) \\ &= \left(\sqrt{1^2 + 1^2}, \tan^{-1} 1, 1\right) = \left(\sqrt{2}, \frac{\pi}{4}, 1\right).\end{aligned}$$

$$x = r\cos\theta \qquad y = r\sin\theta \qquad z = z$$

Example

Find cylindrical coordinates for the Cartesian coordiantes $(x, y, z) = (1, 1, 1)$.

$$\begin{aligned}(r, \theta, z) &= \left(\sqrt{x^2 + y^2}, \tan^{-1}\frac{y}{x}, z\right) \\ &= \left(\sqrt{1^2 + 1^2}, \tan^{-1} 1, 1\right) = \left(\sqrt{2}, \frac{\pi}{4}, 1\right).\end{aligned}$$

Example

Convert the cylindrical coordinates $(r, \theta, z) = \left(2, \frac{\pi}{2}, 2\right)$ to Cartesian coordinates.

$$x = r\cos\theta \qquad y = r\sin\theta \qquad z = z$$

Example

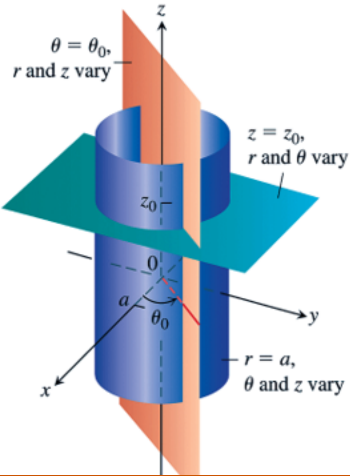
Find cylindrical coordinates for the Cartesian coordiantes $(x, y, z) = (1, 1, 1)$.

$$\begin{aligned}(r, \theta, z) &= \left(\sqrt{x^2 + y^2}, \tan^{-1}\frac{y}{x}, z\right) \\ &= \left(\sqrt{1^2 + 1^2}, \tan^{-1} 1, 1\right) = \left(\sqrt{2}, \frac{\pi}{4}, 1\right).\end{aligned}$$

Example

Convert the cylindrical coordinates $(r, \theta, z) = \left(2, \frac{\pi}{2}, 2\right)$ to Cartesian coordinates.

$$\begin{aligned}(x, y, z) &= (x\cos\theta, y\sin\theta, z) \\ &= \left(2\cos\frac{\pi}{2}, 2\sin\frac{\pi}{2}, 2\right) = (0, 2, 2).\end{aligned}$$



Remark

Cylindrical coordinates are good for describing:

- vertical cylinders with axis on the z-axis ($r = r_0$);
- horizontal planes ($z = z_0$); and
- planes containing the z-axis ($\theta = \theta_0$).

Recall that

$$dA = dxdy = rdrd\theta.$$

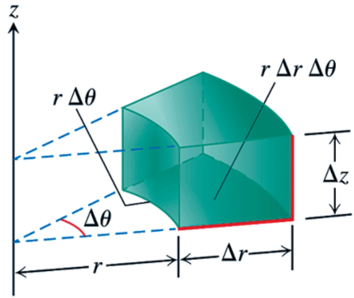
Recall that

$$dA = dxdy = rdrd\theta.$$

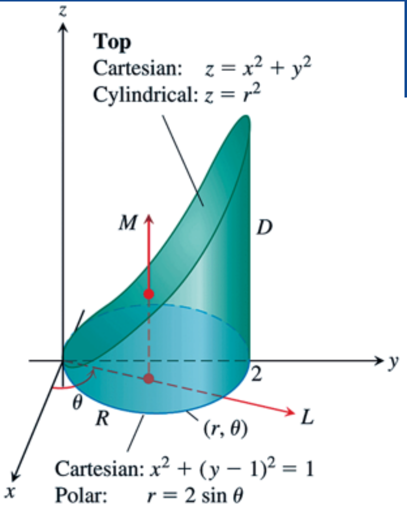
Now we have

Theorem

$$dV = dxdydz = rdrd\theta dz.$$



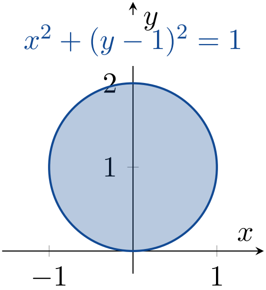
14.7 Triple Inte pherical Coordinates



Example

Let D be the region bounded by $z = 0$, $x^2 + (y-1)^2 = 1$ and $z = x^2 + y^2$. Find the limits of integration in cylindrical coordinates.

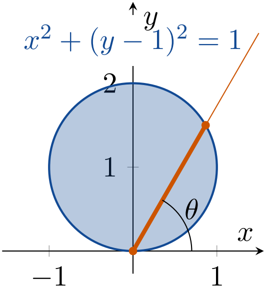
$$x = r\cos\theta \qquad y = r\sin\theta \qquad z = z$$



First note that

$$\begin{aligned} x^2 + (y-1)^2 &= 1 \\ x^2 + y^2 - 2y + 1 &= 1 \\ r^2 - 2r\sin\theta &= 0 \\ r &= 2\sin\theta. \end{aligned}$$

$$x = r\cos\theta \qquad y = r\sin\theta \qquad z = z$$



First note that

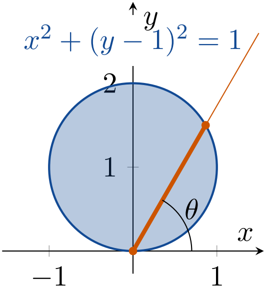
$$\begin{aligned} x^2 + (y-1)^2 &= 1 \\ x^2 + y^2 - 2y + 1 &= 1 \\ r^2 - 2r\sin\theta &= 0 \\ r &= 2\sin\theta. \end{aligned}$$

So

$$0 \le \theta \le \pi$$

$$0 \le r \le 2\sin\theta$$

$$x = r\cos\theta \qquad y = r\sin\theta \qquad z = z$$



First note that

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So

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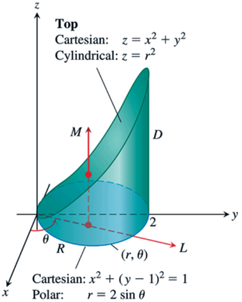
$$0 \le r \le 2\sin\theta$$



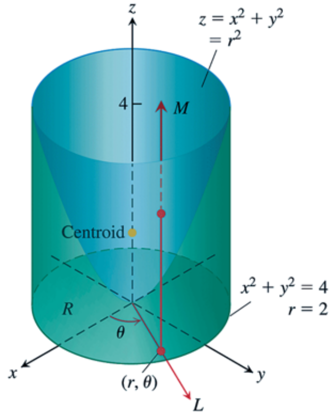
$$0 \le z \le x^2 + y^2 = r^2$$

$$dV = dxdydz = rdrd\theta dz$$

$$0 \leq \theta \leq \pi \qquad 0 \leq r \leq 2\sin\theta \qquad 0 \leq z \leq r^2$$

Therefore

$$\iiint\limits_D F(r,\theta,z)\,dV = \int_0^{\pi}\int_0^{2\sin\theta}\int_0^{r^2} F(r,\theta,z)\,r\,dzdrd\theta.$$

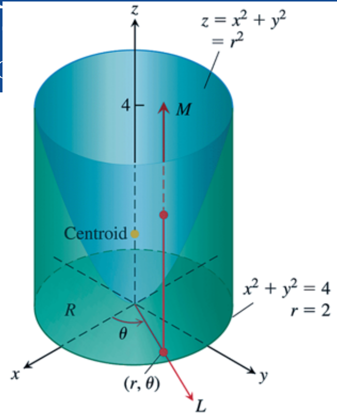


Example

Calculate

$$\iiint_D z\,dV$$

where D is the region enclosed by the cylinder $x^2+y^2=4$, the xy-plane and the paraboloid $z=x^2+y^2$.



$$0 \leq \theta \leq 2\pi$$
$$0 \leq r \leq 2$$
$$0 \leq z \leq r^2.$$

Example

Calculate

$$\iiint_D z \, dV$$

where D is the region enclosed by the cylinder $x^2 + y^2 = 4$, the xy-plane and the paraboloid $z = x^2 + y^2$.

$$dV = dxdydz = rdrd\theta dz$$

$$0 \leq \theta \leq 2\pi \qquad 0 \leq r \leq 2 \qquad 0 \leq z \leq r^2$$

Therefore

$$\iiint\limits_D z\,dV = \int_0^{2\pi}\int_0^2\int_0^{r^2} zr\,dzdrd\theta$$

$$=$$

$$=$$

$$=$$

$$= \qquad = \qquad .$$

$$dV = dxdydz = rdrd\theta dz$$

$$0 \le \theta \le 2\pi \qquad 0 \le r \le 2 \qquad 0 \le z \le r^2$$

Therefore

$$\iiint\limits_D z\,dV = \int_0^{2\pi}\int_0^2\int_0^{r^2} zr\,dzdrd\theta$$

$$= \int_0^{2\pi}\int_0^2 \left[\frac{1}{2}z^2 r\right]_{z=0}^{z=r^2} drd\theta$$

$$=$$

$$=$$

$$= \qquad = \qquad .$$

$$dV = dxdydz = rdrd\theta dz$$

$$0 \le \theta \le 2\pi \qquad 0 \le r \le 2 \qquad 0 \le z \le r^2$$

Therefore

$$\begin{aligned}
\iiint\limits_D z\,dV &= \int_0^{2\pi}\int_0^2\int_0^{r^2} zr\,dzdrd\theta \\
&= \int_0^{2\pi}\int_0^2 \left[\frac{1}{2}z^2 r\right]_{z=0}^{z=r^2} drd\theta \\
&= \int_0^{2\pi}\int_0^2 \frac{1}{2}r^5\,drd\theta \\
&= \\
&= \qquad = \qquad .
\end{aligned}$$

$$dV = dxdydz = rdrd\theta dz$$

$$0 \le \theta \le 2\pi \qquad 0 \le r \le 2 \qquad 0 \le z \le r^2$$

Therefore

$$\begin{aligned}
\iiint\limits_D z\,dV &= \int_0^{2\pi}\int_0^2\int_0^{r^2} zr\,dzdrd\theta \\
&= \int_0^{2\pi}\int_0^2 \left[\frac{1}{2}z^2r\right]_{z=0}^{z=r^2} drd\theta \\
&= \int_0^{2\pi}\int_0^2 \frac{1}{2}r^5\,drd\theta \\
&= \int_0^{2\pi} \left[\frac{1}{12}r^6\right]_0^2 d\theta \\
&= \qquad\qquad = \qquad .
\end{aligned}$$

$$dV = dxdydz = rdrd\theta dz$$

$$0 \leq \theta \leq 2\pi \qquad 0 \leq r \leq 2 \qquad 0 \leq z \leq r^2$$

Therefore

$$\begin{aligned}
\iiint_D z\,dV &= \int_0^{2\pi}\int_0^2\int_0^{r^2} zr\,dzdrd\theta \\
&= \int_0^{2\pi}\int_0^2 \left[\frac{1}{2}z^2 r\right]_{z=0}^{z=r^2} drd\theta \\
&= \int_0^{2\pi}\int_0^2 \frac{1}{2}r^5\,drd\theta \\
&= \int_0^{2\pi}\left[\frac{1}{12}r^6\right]_0^2 d\theta \\
&= \int_0^{2\pi}\frac{16}{3}\,d\theta = \qquad .
\end{aligned}$$

$$dV = dxdydz = rdrd\theta dz$$

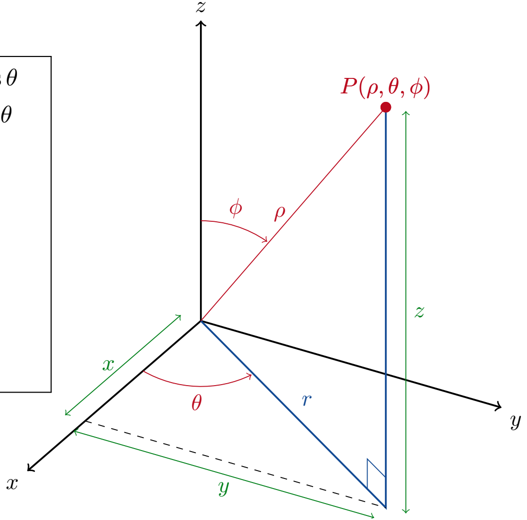
$$0 \le \theta \le 2\pi \qquad 0 \le r \le 2 \qquad 0 \le z \le r^2$$

Therefore

$$\begin{aligned}\iiint_D z\,dV &= \int_0^{2\pi}\int_0^2\int_0^{r^2} zr\,dzdrd\theta \\ &= \int_0^{2\pi}\int_0^2 \left[\frac{1}{2}z^2 r\right]_{z=0}^{z=r^2} drd\theta \\ &= \int_0^{2\pi}\int_0^2 \frac{1}{2}r^5\,drd\theta \\ &= \int_0^{2\pi}\left[\frac{1}{12}r^6\right]_0^2 d\theta \\ &= \int_0^{2\pi}\frac{16}{3}\,d\theta = \frac{32\pi}{3}.\end{aligned}$$

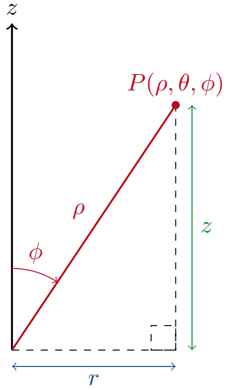
Spherical Coordinates in $\mathbb{R}^3$

$$
\begin{aligned}
x &= r\cos\theta = \rho\sin\phi\cos\theta \\
y &= r\sin\theta = \rho\sin\phi\sin\theta \\
z &= \rho\cos\phi \\
\\
r &= \rho\sin\phi \\
\tan\theta &= \frac{y}{x} \\
\rho &= \sqrt{x^2+y^2+z^2} \\
&= \sqrt{r^2+z^2}
\end{aligned}
$$



Spherical Coordinates in $\mathbb{R}^3$

$$
\begin{aligned}
x &= r\cos\theta = \rho\sin\phi\cos\theta \\
y &= r\sin\theta = \rho\sin\phi\sin\theta \\
z &= \rho\cos\phi \\
\\
r &= \rho\sin\phi \\
\tan\theta &= \frac{y}{x} \\
\rho &= \sqrt{x^2+y^2+z^2} \\
&= \sqrt{r^2+z^2}
\end{aligned}
$$



Typically, we require that $\rho \geq 0$ and $0 \leq \phi \leq \pi$. As before, θ can be any number.

$$\rho = \sqrt{r^2 + z^2} \qquad \theta = \theta \qquad z = \rho \cos \phi$$

Example

Convert the point $P(\sqrt{6}, \frac{\pi}{4}, \sqrt{2})$ from cylindrical to spherical coordinates.

$$\rho = \sqrt{r^2 + z^2} \qquad \theta = \theta \qquad z = \rho\cos\phi$$

Example

Convert the point $P(\sqrt{6}, \frac{\pi}{4}, \sqrt{2})$ from cylindrical to spherical coordinates.

We have that $r = \sqrt{6}$, $\theta = \frac{\pi}{4}$ and $z = \sqrt{2}$. Therefore

$$\begin{aligned}
(\rho, \theta, \phi) &= \left(\sqrt{r^2 + z^2}, \theta, \cos^{-1}\frac{z}{\rho} \right) \\
&= \left(\sqrt{6+2}, \frac{\pi}{4}, \cos^{-1}\frac{\sqrt{2}}{\rho} \right) \\
&= \left(2\sqrt{2}, \frac{\pi}{4}, \cos^{-1}\frac{\sqrt{2}}{2\sqrt{2}} \right) \\
&= \left(2\sqrt{2}, \frac{\pi}{4}, \cos^{-1}\frac{1}{2} \right) = \left(2\sqrt{2}, \frac{\pi}{4}, \frac{\pi}{3} \right).
\end{aligned}$$

$$\rho = \sqrt{x^2 + y^2 + z^2} \qquad y = \rho \sin\phi \sin\theta \qquad z = \rho \cos\phi$$

Example

Convert the point $P(-1, 1, -\sqrt{2})$ from Cartesian to spherical polar coordinates.

$$\rho = \sqrt{x^2+y^2+z^2} \qquad y = \rho\sin\phi\sin\theta \qquad z = \rho\cos\phi$$

Example

Convert the point $P(-1,1,-\sqrt{2})$ from Cartesian to spherical polar coordinates.

First we calculate that

$$\rho = \sqrt{x^2+y^2+z^2} = \sqrt{(-1)^2+1^2+(-\sqrt{2})^2} = \sqrt{4} = 2.$$

Next we calculate that

$$\phi = \cos^{-1}\frac{z}{\rho} = \cos^{-1}\frac{-\sqrt{2}}{2} = \frac{3\pi}{4}$$

because we want $\phi \in [0,\pi]$.

$$\rho = \sqrt{x^2 + y^2 + z^2} \qquad y = \rho \sin\phi \sin\theta \qquad z = \rho\cos\phi$$

Finally we need a θ.

$$\sin\theta = \frac{y}{\rho\sin\phi} = \frac{1}{2\left(\frac{\sqrt{2}}{2}\right)} = \frac{1}{\sqrt{2}}.$$

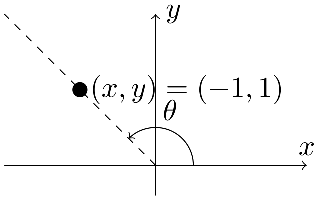
There are infinitely many θ that satisfy this equation. Two possible θ are $\theta = \frac{\pi}{4}$ and $\theta = \frac{3\pi}{4}$.

$$\rho = \sqrt{x^2 + y^2 + z^2} \qquad y = \rho \sin\phi \sin\theta \qquad z = \rho \cos\phi$$

Finally we need a θ.

$$\sin\theta = \frac{y}{\rho\sin\phi} = \frac{1}{2\left(\frac{\sqrt{2}}{2}\right)} = \frac{1}{\sqrt{2}}.$$

There are infinitely many θ that satisfy this equation. Two possible θ are $\theta = \frac{\pi}{4}$ and $\theta = \frac{3\pi}{4}$. Only one of these can be correct.

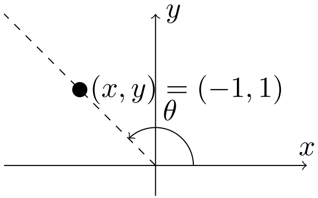


$$\rho = \sqrt{x^2 + y^2 + z^2} \qquad y = \rho \sin\phi \sin\theta \qquad z = \rho\cos\phi$$

Finally we need a θ.

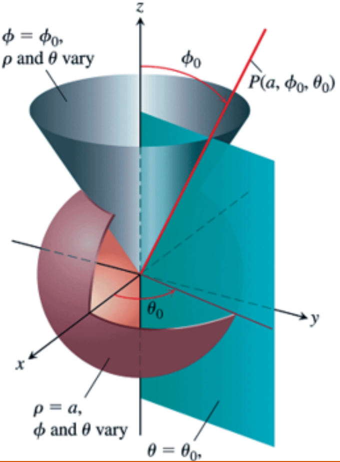
$$\sin\theta = \frac{y}{\rho\sin\phi} = \frac{1}{2\left(\frac{\sqrt{2}}{2}\right)} = \frac{1}{\sqrt{2}}.$$

There are infinitely many θ that satisfy this equation. Two possible θ are $\theta = \frac{\pi}{4}$ and $\theta = \frac{3\pi}{4}$. Only one of these can be correct.



Therefore the answer is

$$(\rho, \theta, \phi) = \left(2, \frac{3\pi}{4}, \frac{3\pi}{4}\right).$$



Remark

Cylindrical coordinates are good for describing:

- spheres centred at the origin ($\rho = \rho_0$);
- cones (with vertex at the origin and axis on the z-axis) ($\phi = \phi_0$); and
- half planes containing the z-axis ($\theta = \theta_0$).

$$x = \rho \sin\phi \cos\theta \qquad y = \rho \sin\phi \sin\theta \qquad z = \rho \cos\phi$$

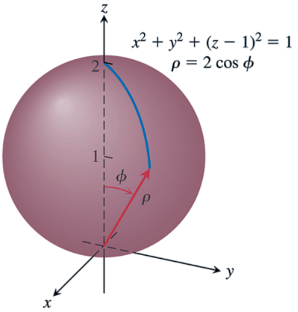
EXAMPLE 3 Find a spherical coordinate equation for the sphere $x^2 + y^2 + (z - 1)^2 = 1$.

Solution We use Equations (1) to substitute for x, y, and z:

$$x^2 + y^2 + (z-1)^2 = 1$$

$$\rho^2 \sin^2\phi \cos^2\theta + \rho^2 \sin^2\phi \sin^2\theta + (\rho\cos\phi - 1)^2 = 1 \qquad \text{Eqs. (1)}$$

$$\rho^2 \sin^2\phi \underbrace{(\cos^2\theta + \sin^2\theta)}_{1} + \rho^2\cos^2\phi - 2\rho\cos\phi + 1 = 1$$



$$\rho^2 \underbrace{(\sin^2\phi + \cos^2\phi)}_{1} = 2\rho\cos\phi$$

$$\rho^2 = 2\rho\cos\phi$$

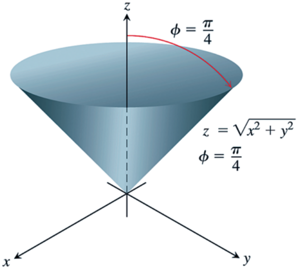
$$\rho = 2\cos\phi. \qquad \rho > 0$$

$$x = \rho \sin\phi \cos\theta \qquad y = \rho \sin\phi \sin\theta \qquad z = \rho \cos\phi$$

EXAMPLE 4 Find a spherical coordinate equation for the cone $z = \sqrt{x^2 + y^2}$.

Solution 1 *Use geometry*. The cone is symmetric with respect to the z-axis and cuts the first quadrant of the yz-plane along the line $z = y$. The angle between the cone and the positive z-axis is therefore $\pi/4$ radians. The cone consists of the points whose spherical coordinates have ϕ equal to $\pi/4$, so its equation is $\phi = \pi/4$. (See Figure 15.54.)

Solution 2 *Use algebra*. If we use Equations (1) to substitute for x, y, and z we obtain the same result:



$$\begin{aligned}
z &= \sqrt{x^2 + y^2} \\
\rho \cos\phi &= \sqrt{\rho^2 \sin^2\phi} && \text{Example 3} \\
\rho \cos\phi &= \rho \sin\phi && \rho > 0,\ \sin\phi \geq 0 \\
\cos\phi &= \sin\phi \\
\phi &= \frac{\pi}{4}. && 0 \leq \phi \leq \pi
\end{aligned}$$

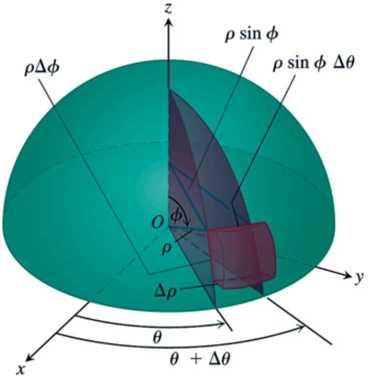
■

Theorem

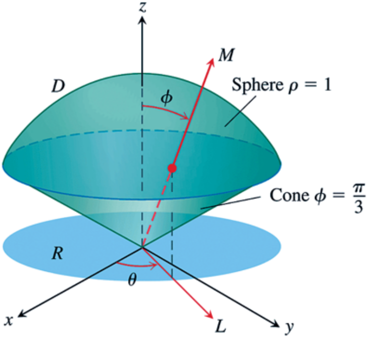
$$dV = dxdydz = r\,drd\theta dz = \qquad\qquad .$$

Theorem

$$dV = dxdydz = r\,drd\theta dz = \rho^2 \sin\phi \, d\rho d\phi d\theta.$$



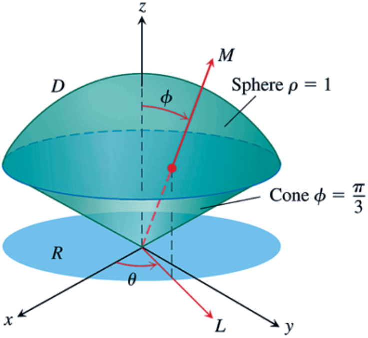
$$dV = \rho^2 \sin\phi \, d\rho d\phi d\theta$$



Example

Calculate the volume of the region enclosed by the sphere $\phi = 1$ and the cone $\phi = \frac{\pi}{3}$.

$$dV = \rho^2 \sin\phi \, d\rho d\phi d\theta$$

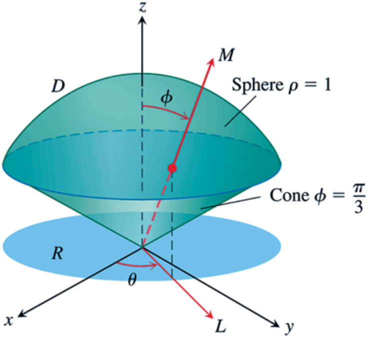


$$0 \leq \theta \leq 2\pi$$
$$0 \leq \phi \leq \frac{\pi}{3}$$
$$0 \leq \rho \leq 1.$$

Example

Calculate the volume of the region enclosed by the sphere $\phi = 1$ and the cone $\phi = \frac{\pi}{3}$.

$$dV = \rho^2 \sin\phi \, d\rho d\phi d\theta$$



$$0 \le \theta \le 2\pi$$
$$0 \le \phi \le \frac{\pi}{3}$$
$$0 \le \rho \le 1.$$

Example

Calculate the volume of the region enclosed by the sphere $\phi = 1$ and the cone $\phi = \frac{\pi}{3}$.

$$\iiint_D dV = \int_0^{2\pi} \int_0^{\frac{\pi}{3}} \int_0^1 \rho^2 \sin\phi \, d\rho d\phi d\theta = \ldots = \frac{\pi}{3}.$$

Example

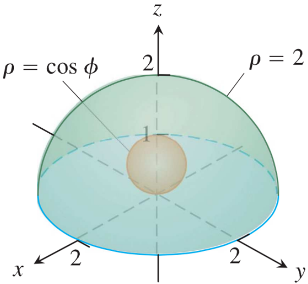
Find the limits of integration for the region shown below: The region is bounded by $\rho = 2$ and the xy-plane, but the sphere $\rho = \cos\phi$ has been removed from this large hemisphere.

A

$$0 \le \theta \le 2\pi$$
$$0 \le \phi \le \pi$$
$$0 \le \rho \le 2$$

B

$$0 \le \theta \le 2\pi$$
$$0 \le \phi \le \frac{\pi}{2}$$
$$0 \le \rho \le 2$$



C

$$0 \le \theta \le 2\pi$$
$$0 \le \phi \le \frac{\pi}{2}$$
$$\cos\phi \le \rho \le 2$$

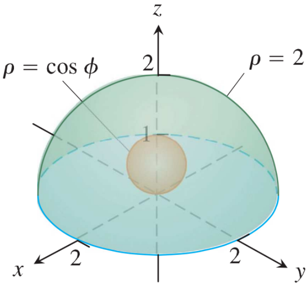
D

$$0 \le \theta \le \pi$$
$$0 \le \phi \le 2$$
$$2 \le \rho \le \cos\phi$$

Example

Find the limits of integration for the region shown below: The region is bounded by $\rho = 2$ and the xy-plane, but the sphere $\rho = \cos\phi$ has been removed from this large hemisphere.

A
$$0 \le \theta \le 2\pi$$
$$0 \le \phi \le \pi$$
$$0 \le \rho \le 2$$



C
$$0 \le \theta \le 2\pi$$
$$0 \le \phi \le \frac{\pi}{2}$$
$$\cos\phi \le \rho \le 2$$

B
$$0 \le \theta \le 2\pi$$
$$0 \le \phi \le \frac{\pi}{2}$$
$$0 \le \rho \le 2$$

D
$$0 \le \theta \le \pi$$
$$0 \le \phi \le 2$$
$$2 \le \rho \le \cos\phi$$

Coordinate Conversion Formulas

Cylindrical to Rectangular	Spherical to Rectangular	Spherical to Cylindrical
$x = r\cos\theta$	$x = \rho\sin\phi\cos\theta$	$r = \rho\sin\phi$
$y = r\sin\theta$	$y = \rho\sin\phi\sin\theta$	$z = \rho\cos\phi$
$z = z$	$z = \rho\cos\phi$	$\theta = \theta$

Corresponding formulas for dV in triple integrals:

$$\begin{aligned} dV &= dx\,dy\,dz \\ &= dz\,r\,dr\,d\theta \\ &= \rho^2\sin\phi\,d\rho\,d\phi\,d\theta \end{aligned}$$

Break

We will continue at 2pm

Substitutions in Multiple Integrals

Substitutions in Single Integrals

You know that if we write $u = 2x + 3$ then $du = 2\,dx$ and

$$\int_0^1 2\sqrt{2x+3}\,dx = \int_3^5 \sqrt{u}\,du.$$

Substitutions in Single Integrals

You know that if we write $u = 2x + 3$ then $du = 2\,dx$ and

$$\int_0^1 2\sqrt{2x+3}\,dx = \int_3^5 \sqrt{u}\,du.$$

Substitutions in Double Integrals

We are going to the same thing for substitutions in double integrals.

Carl Gustav Jacob Jacobi

BORN
10 December 1804

DECEICED
18 February 1851

NATIONALITY
German

Definition

The *Jacobian* of the coordinate transformation $x = g(u, v)$, $y = h(u, v)$ is

$$J(u,v) = \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u}.$$

$$\frac{\partial(x,y)}{\partial(u,v)} = \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u}$$

Example

Find the Jacobian of the polar coordinate transformation $x = r\cos\theta$ and $y = r\sin\theta$.

$$\boxed{\frac{\partial(x,y)}{\partial(u,v)} = \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u}}$$

Example

Find the Jacobian of the polar coordinate transformation $x = r\cos\theta$ and $y = r\sin\theta$.

$$\begin{aligned}\frac{\partial(x,y)}{\partial(r,\theta)} &= \frac{\partial x}{\partial r}\frac{\partial y}{\partial \theta} - \frac{\partial x}{\partial \theta}\frac{\partial y}{\partial r} \\ &= \\ &= \end{aligned}$$

$$\frac{\partial(x,y)}{\partial(u,v)} = \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u}$$

Example

Find the Jacobian of the polar coordinate transformation $x = r\cos\theta$ and $y = r\sin\theta$.

$$\begin{aligned}
\frac{\partial(x,y)}{\partial(r,\theta)} &= \frac{\partial x}{\partial r}\frac{\partial y}{\partial \theta} - \frac{\partial x}{\partial \theta}\frac{\partial y}{\partial r} \\
&= (\cos\theta)\,(r\cos\theta) - (-r\sin\theta)(\sin\theta) \\
&= r.
\end{aligned}$$

$$\frac{\partial(x,y)}{\partial(u,v)} = \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u}$$

Example

Find the Jacobian of the polar coordinate transformation $x = r\cos\theta$ and $y = r\sin\theta$.

$$\begin{aligned}\frac{\partial(x,y)}{\partial(r,\theta)} &= \frac{\partial x}{\partial r}\frac{\partial y}{\partial \theta} - \frac{\partial x}{\partial \theta}\frac{\partial y}{\partial r} \\ &= (\cos\theta)(r\cos\theta) - (-r\sin\theta)(\sin\theta) \\ &= r.\end{aligned}$$

Remark

Remember that

$$dxdy = r\,drd\theta.$$

Theorem

Suppose that $f(x, y)$ is continuous over the region R. Let G be the preimage of R under the transformation $x = g(u, v)$, $y = h(u, v)$, which is assumed to be one-to-one on the interior of G. If the functions g and h have continuous first partial derivatives within the interior of G, then

$$\iint_R f(x, y)\,dxdy = \iint_G f\big(g(u, v), h(u, v)\big) \left|\frac{\partial(x, y)}{\partial(u, v)}\right| dudv.$$

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Suppose that $f(x, y)$ is continuous over the region R. Let G be the preimage of R under the transformation $x = g(u, v)$, $y = h(u, v)$, *which is assumed to be one-to-one on the interior of G. If the functions g and h have continuous first partial derivatives within the interior of G, then*

$$\iint_R f(x, y)\,dxdy = \iint_G f\big(g(u, v), h(u, v)\big)\left|\frac{\partial(x, y)}{\partial(u, v)}\right| dudv.$$

Example

Calculate

$$\int_0^4 \int_{\frac{y}{2}}^{\frac{y}{2}+1} \frac{2x-y}{2}\,dxdy$$

by applying the transformation

$$u = \frac{2x-y}{2}, \qquad v = \frac{y}{2}$$

and integrating over an appropriate region in the uv-plane.

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$$0 \le y \le 4 \quad \text{and} \quad \frac{y}{2} \le x \le \frac{y}{2} + 1$$

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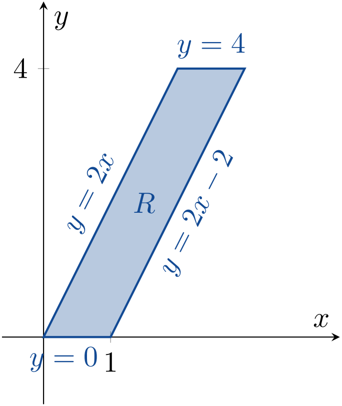
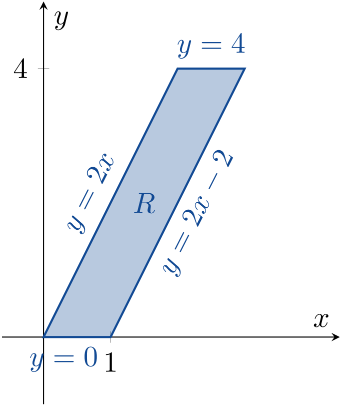
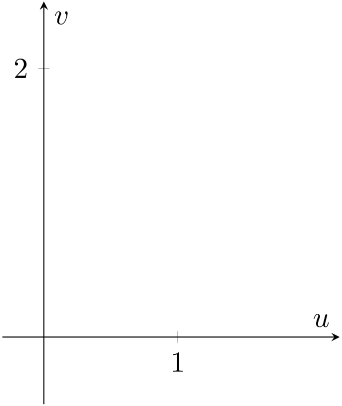


$$0 \leq y \leq 4 \quad \text{and} \quad \frac{y}{2} \leq x \leq \frac{y}{2} + 1$$

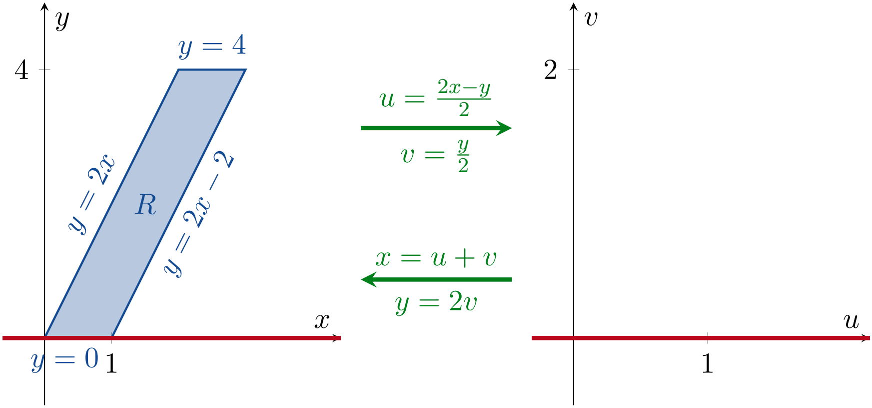


$$u = \frac{2x-y}{2} \qquad v = \frac{y}{2}$$

$$x = u + v \qquad y = 2v$$

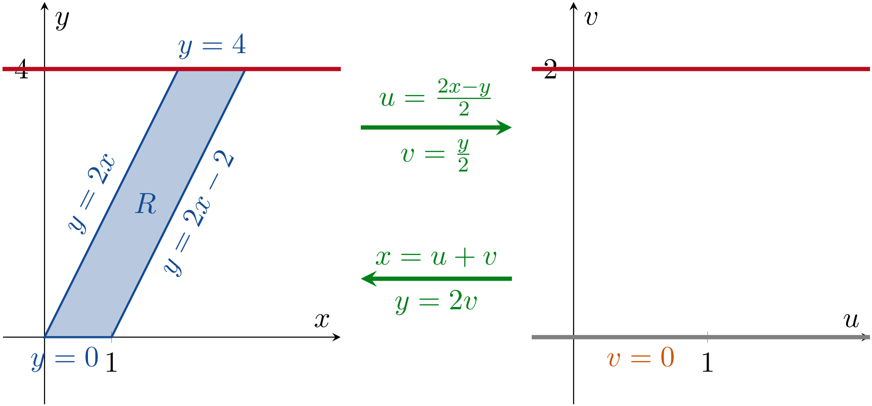


$$0 \le y \le 4 \quad \text{and} \quad \tfrac{y}{2} \le x \le \tfrac{y}{2} + 1$$



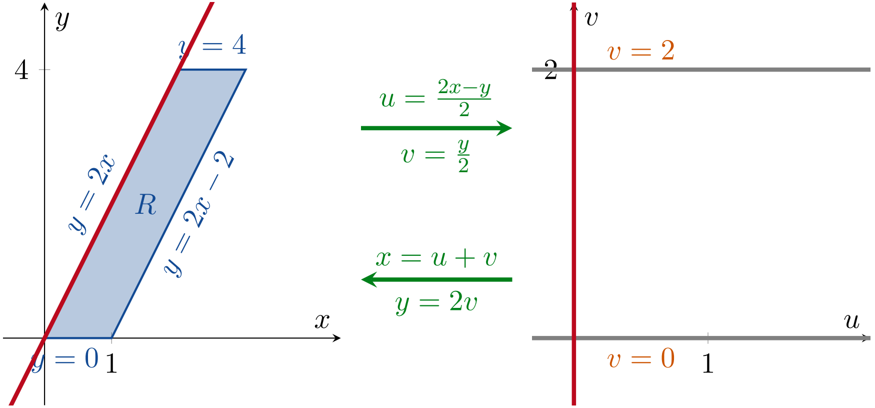
$$y = 0 \quad \Longrightarrow \quad v = \frac{y}{2} = 0$$

$$0 \le y \le 4 \quad \text{and} \quad \frac{y}{2} \le x \le \frac{y}{2} + 1$$



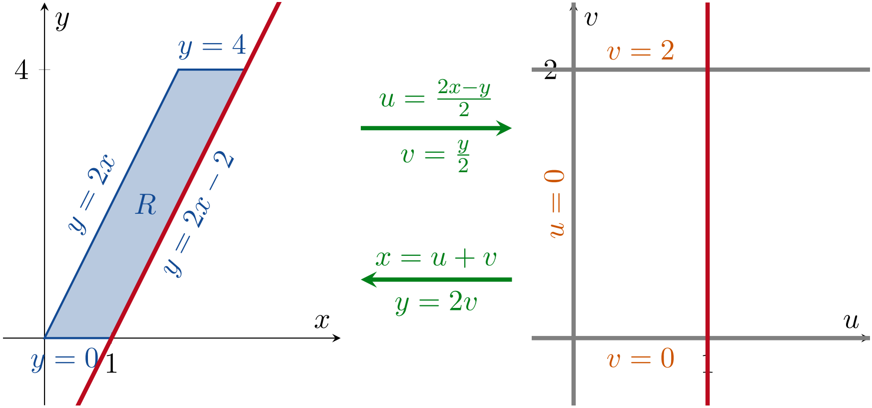
$$y = 4 \quad \Longrightarrow \quad v = \frac{y}{2} = 2$$

$$0 \le y \le 4 \quad \text{and} \quad \frac{y}{2} \le x \le \frac{y}{2} + 1$$



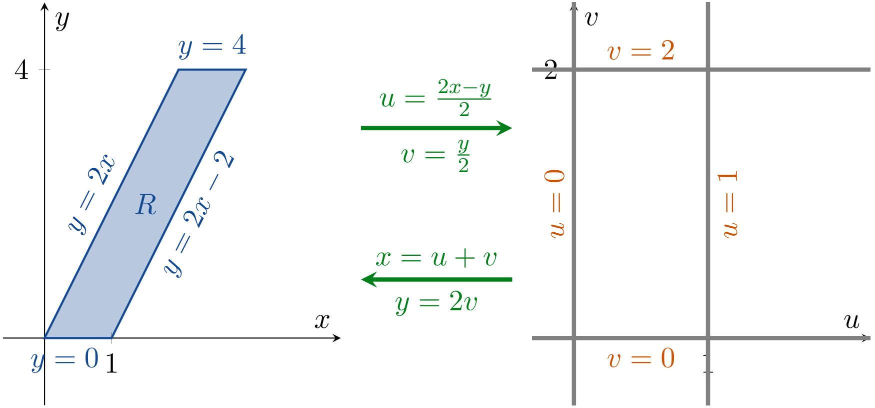
$$y = 2x \quad \Longrightarrow \quad u = \frac{2x - y}{2} = \frac{2x - 2x}{2} = 0$$

$$0 \le y \le 4 \quad \text{and} \quad \frac{y}{2} \le x \le \frac{y}{2} + 1$$



$$y = 2x - 2 \quad \implies \quad u = \frac{2x - y}{2} = \frac{2x - 2x + 2}{2} = 1$$

$$0 \le y \le 4 \quad \text{and} \quad \frac{y}{2} \le x \le \frac{y}{2} + 1$$



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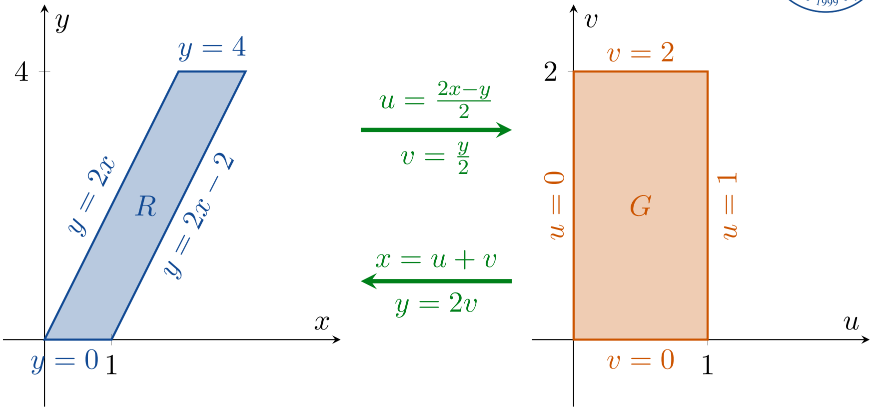

$$0 \le u \le 1 \quad \text{and} \quad 0 \le v \le 2$$

$$\boxed{\frac{\partial(x,y)}{\partial(u,v)} = \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u}}$$

$$x = u + v \quad \text{and} \quad y = 2v$$

Next we need the Jacobian of this coordinate transformation:

$$\boxed{\frac{\partial(x,y)}{\partial(u,v)} = \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u}}$$

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Next we need the Jacobian of this coordinate transformation:

$$\begin{aligned} \frac{\partial(x,y)}{\partial(u,v)} &= \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u} \\ &= \qquad\qquad\qquad\qquad . \end{aligned}$$

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Next we need the Jacobian of this coordinate transformation:

$$\begin{aligned}\frac{\partial(x,y)}{\partial(u,v)} &= \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u} \\ &= (1)(2) - (1)(0) = 2.\end{aligned}$$

$$u = \frac{2x - y}{2}, \qquad v = \frac{y}{2}$$

$$0 \le u \le 1 \qquad 0 \le v \le 2 \qquad \frac{\partial(x, y)}{\partial(u, v)} = 2$$

Therefore

$$\int_0^4 \int_{\frac{y}{2}}^{\frac{y}{2}+1} \frac{2x - y}{2} \, dxdy =$$

$$u = \frac{2x - y}{2}, \qquad v = \frac{y}{2}$$

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Therefore

$$\int_0^4 \int_{\frac{y}{2}}^{\frac{y}{2}+1} \frac{2x - y}{2}\, dxdy = \int_0^2 \int_0^1 u \left| \frac{\partial(x, y)}{\partial(u, v)} \right| dudv$$

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$$= \int_0^2 \int_0^1 2u\, dudv$$

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Therefore

$$\int_0^4 \int_{\frac{y}{2}}^{\frac{y}{2}+1} \frac{2x-y}{2}\,dxdy = \int_0^2 \int_0^1 u \left|\frac{\partial(x,y)}{\partial(u,v)}\right| dudv$$

$$= \int_0^2 \int_0^1 2u\,dudv = \ldots = 2.$$

Remark

To do a substitution, we need to do two things:

1. Calculate the Jacobian and write $dxdy = \left| \frac{\partial(x,y)}{\partial(u,v)} \right| dudv$;

and

2. change the limits of integration.

$$\frac{\partial(x,y)}{\partial(u,v)} = \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u}$$

Example

Calculate $\displaystyle\int_0^1 \int_0^{1-x} \sqrt{x+y}(y-2x)^2 \, dydx.$

$$\boxed{\frac{\partial(x,y)}{\partial(u,v)} = \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u}}$$

Example

Calculate $\displaystyle\int_0^1 \int_0^{1-x} \sqrt{x+y}\,(y-2x)^2\,dydx.$

First we need to choose u and v.

$$\frac{\partial(x,y)}{\partial(u,v)} = \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u}$$

Example

Calculate $\int_0^1 \int_0^{1-x} \sqrt{x+y}(y-2x)^2\,dydx$.

First we need to choose u and v. I choose

$$u = x + y \qquad \text{and} \qquad v = y - 2x.$$

$$\frac{\partial(x,y)}{\partial(u,v)} = \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u}$$

Example

Calculate $\int_0^1 \int_0^{1-x} \sqrt{x+y}(y-2x)^2 \, dydx$.

First we need to choose u and v. I choose

$$u = x + y \qquad \text{and} \qquad v = y - 2x.$$

We can rearrange these to

$$x = \frac{u}{3} - \frac{v}{3} \qquad \text{and} \qquad y = \frac{2u}{3} + \frac{v}{3}.$$

$$\frac{\partial(x,y)}{\partial(u,v)} = \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u}$$

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Calculate $\displaystyle\int_0^1 \int_0^{1-x} \sqrt{x+y}(y-2x)^2\,dydx$.

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We can rearrange these to

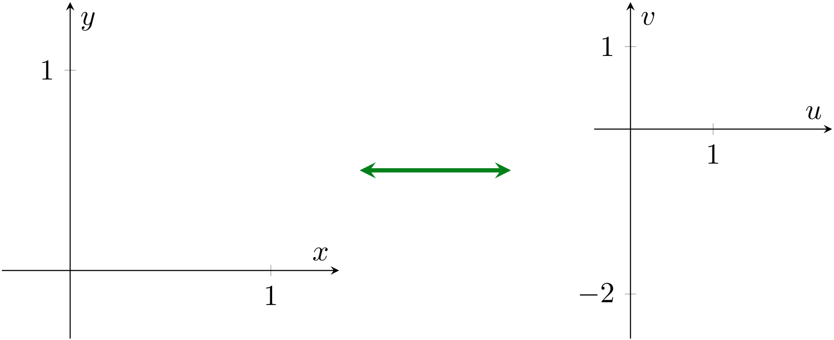
$$x = \frac{u}{3} - \frac{v}{3} \qquad \text{and} \qquad y = \frac{2u}{3} + \frac{v}{3}.$$

Then the Jacobian is

$$\frac{\partial(x,y)}{\partial(u,v)} = \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u} = \left(\frac{1}{3}\right)\left(\frac{1}{3}\right) - \left(-\frac{1}{3}\right)\left(\frac{2}{3}\right) = \frac{1}{3}.$$

$$u = x + y \qquad \int_0^1 \int_0^{1-x} \sqrt{x+y}\,(y-2x)^2 \, dy dx \qquad x = \frac{u}{3} - \frac{v}{3}$$

$$v = y - 2x \qquad\qquad\qquad\qquad\qquad\qquad\qquad y = \frac{2u}{3} + \frac{v}{3}$$

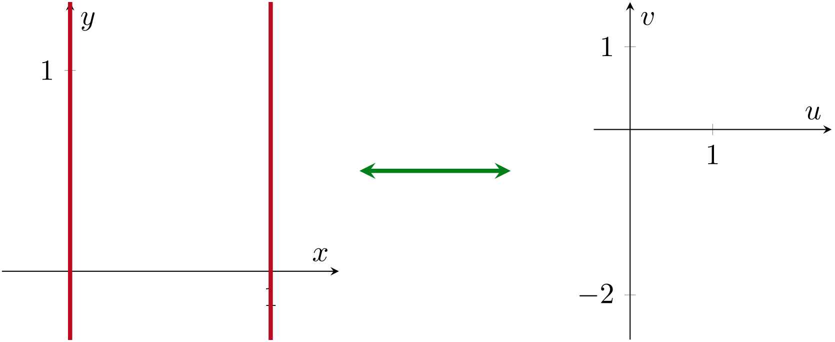


$$u = x + y \qquad \int_0^1 \int_0^{1-x} \sqrt{x+y}(y-2x)^2 \, dy dx \qquad x = \frac{u}{3} - \frac{v}{3}$$

$$v = y - 2x \qquad\qquad\qquad\qquad\qquad\qquad y = \frac{2u}{3} + \frac{v}{3}$$

$$x = 0$$

$$x = 1$$



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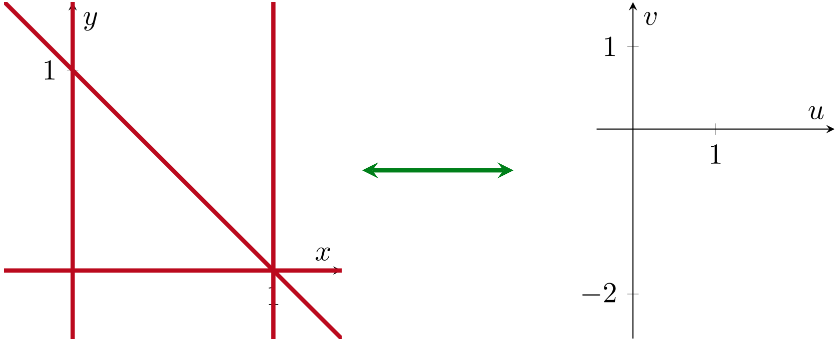
$$v = y - 2x \qquad\qquad\qquad\qquad\qquad\qquad y = \frac{2u}{3} + \frac{v}{3}$$

$$x = 0$$

$$x = 1$$

$$y = 0$$

$$y = 1 - x$$



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$$\cancel{x = 1}$$

$$y = 0$$

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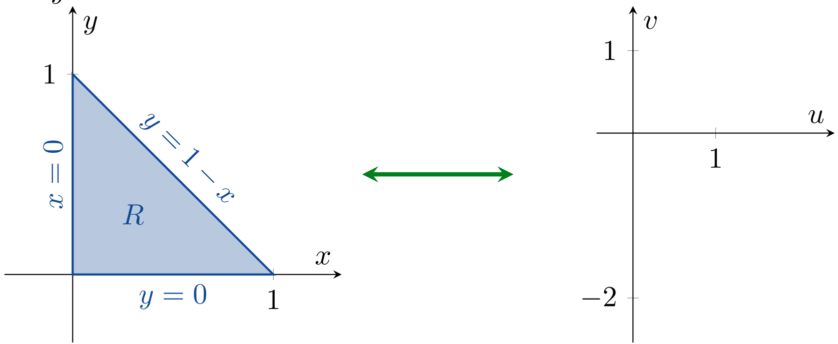

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$$x = 0 \quad \Longrightarrow \quad 0 = \frac{u}{3} - \frac{v}{3} \quad \Longrightarrow \quad u = v$$

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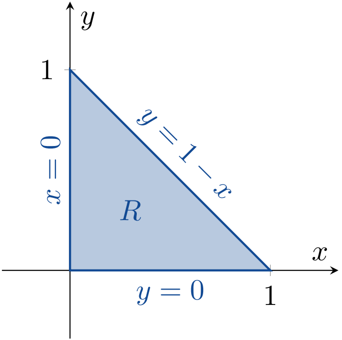
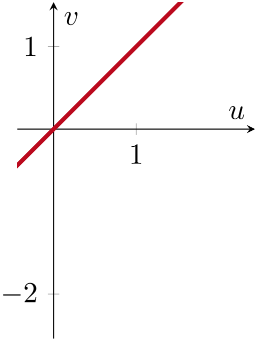


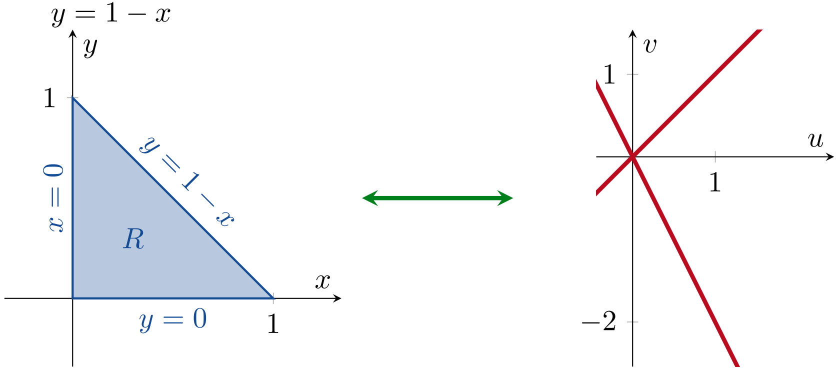


$$u = x + y \qquad v = y - 2x \qquad \int_0^1 \int_0^{1-x} \sqrt{x+y}\,(y-2x)^2\, dy\,dx \qquad x = \frac{u}{3} - \frac{v}{3} \qquad y = \frac{2u}{3} + \frac{v}{3}$$

$$x = 0 \quad \Longrightarrow \quad 0 = \frac{u}{3} - \frac{v}{3} \quad \Longrightarrow \quad u = v$$

$$\cancel{x = 1}$$

$$y = 0 \quad \Longrightarrow \quad 0 = \frac{2u}{3} + \frac{v}{3} \quad \Longrightarrow \quad v = -2u$$



$$u = x + y \qquad \int_0^1 \int_0^{1-x} \sqrt{x+y}\,(y-2x)^2\,dy\,dx \qquad x = \frac{u}{3} - \frac{v}{3}$$
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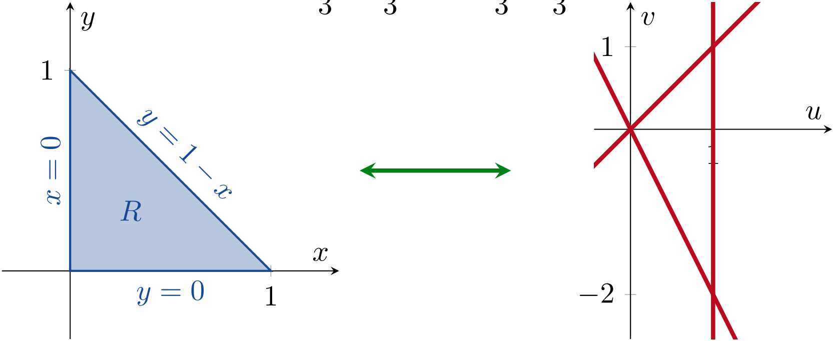
$$x = 0 \implies 0 = \frac{u}{3} - \frac{v}{3} \implies u = v$$

$$\cancel{x = 1}$$

$$y = 0 \implies 0 = \frac{2u}{3} + \frac{v}{3} \implies v = -2u$$

$$y = 1 - x \implies \frac{2u}{3} + \frac{v}{3} = 1 - \frac{u}{3} + \frac{v}{3} \implies u = 1$$



$$u = x + y \qquad \int_0^1 \int_0^{1-x} \sqrt{x+y}\,(y-2x)^2 \, dy dx \qquad x = \frac{u}{3} - \frac{v}{3}$$
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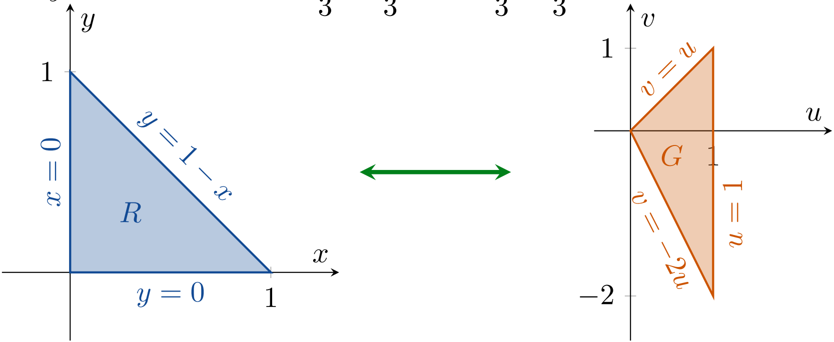
$$x = 0 \implies 0 = \frac{u}{3} - \frac{v}{3} \implies u = v$$

$$\cancel{x = 1}$$

$$y = 0 \implies 0 = \frac{2u}{3} + \frac{v}{3} \implies v = -2u$$

$$y = 1 - x \implies \frac{2u}{3} + \frac{v}{3} = 1 - \frac{u}{3} + \frac{v}{3} \implies u = 1$$



$$u = x + y \qquad \int_0^1 \int_0^{1-x} \sqrt{x+y}\,(y-2x)^2\,dy\,dx \qquad x = \frac{u}{3} - \frac{v}{3}$$
$$v = y - 2x \qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad y = \frac{2u}{3} + \frac{v}{3}$$

$$x = 0 \implies 0 = \frac{u}{3} - \frac{v}{3} \implies u = v$$

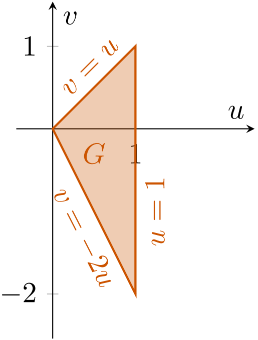
$$\cancel{x = 1}$$

$$y = 0 \implies 0 = \frac{2u}{3} + \frac{v}{3} \implies v = -2u$$

$$y = 1 - x \implies \frac{2u}{3} + \frac{v}{3} = 1 - \frac{u}{3} + \frac{v}{3} \implies u = 1$$

$$0 \le u \le 1$$

$$-2u \le v \le u$$



$$u = x + y \qquad \frac{\partial(x,y)}{\partial(u,v)} = \frac{1}{3} \qquad 0 \leq u \leq 1 \qquad -2u \leq v \leq u$$
$$v = y - 2x$$

Therefore

$$\begin{aligned}\int_0^1 \int_0^{1-x} \sqrt{x+y}(y-2x)^2 \, dydx &= \\ &= \\ &= \\ &= \\ &= \\ &= \end{aligned}$$

$$u = x + y \qquad \frac{\partial(x,y)}{\partial(u,v)} = \frac{1}{3} \qquad 0 \le u \le 1 \qquad -2u \le v \le u$$
$$v = y - 2x$$

Therefore

$$\begin{aligned}
\int_0^1 \int_0^{1-x} \sqrt{x+y}(y-2x)^2 \, dydx &= \int_0^1 \int_{-2u}^{u} \sqrt{u}v^2 \, \frac{1}{3} \, dvdu \\
&= \\
&= \\
&= \\
&= \\
&=
\end{aligned}$$

$$u = x + y \qquad \frac{\partial(x,y)}{\partial(u,v)} = \frac{1}{3} \qquad 0 \le u \le 1 \qquad -2u \le v \le u$$
$$v = y - 2x$$

Therefore

$$\begin{aligned}
\int_0^1 \int_0^{1-x} \sqrt{x+y}(y-2x)^2 \, dydx &= \int_0^1 \int_{-2u}^{u} \sqrt{u}v^2 \, \frac{1}{3} \, dvdu \\
&= \dots \\
&= \dots \\
&= \dots \\
&= \dots \\
&= \frac{2}{9}.
\end{aligned}$$

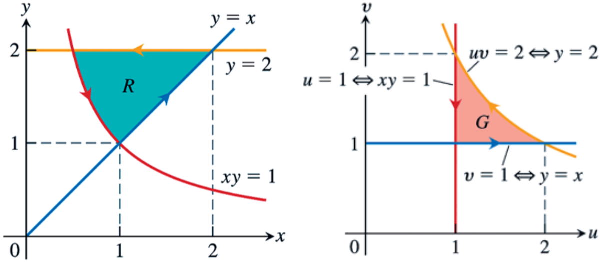
EXAMPLE 4 Evaluate the integral

$$\int_1^2 \int_{1/y}^{y} \sqrt{\frac{y}{x}}\, e^{\sqrt{xy}}\, dx\, dy.$$

Solution The square root terms in the integrand suggest that we might simplify the integration by substituting $u = \sqrt{xy}$ and $v = \sqrt{y/x}$. Squaring these equations gives $u^2 = xy$ and $v^2 = y/x$, which imply that $u^2 v^2 = y^2$ and $u^2/v^2 = x^2$. So we obtain the transformation (in the same ordering of the variables as discussed before)

$$x = \frac{u}{v} \qquad \text{and} \qquad y = uv,$$

with $u > 0$ and $v > 0$.



Let's first see what happens to the integrand itself under this transformation. The Jacobian of the transformation is not constant:

$$J(u, v) = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\[2ex] \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} \dfrac{1}{v} & \dfrac{-u}{v^2} \\[2ex] v & u \end{vmatrix} = \frac{2u}{v}.$$

If G is the region of integration in the uv-plane, then by Equation (2) the transformed double integral under the substitution is

$$\iint_R \sqrt{\frac{y}{x}}\, e^{\sqrt{xy}}\, dx\, dy = \iint_G v e^u \frac{2u}{v}\, du\, dv = \iint_G 2ue^u\, du\, dv.$$

The transformed integrand function is easier to integrate than the original one, so we proceed to determine the limits of integration for the transformed integral.

The region of integration R of the original integral in the xy-plane is shown in Figure 15.61. From the substitution equations $u = \sqrt{xy}$ and $v = \sqrt{y/x}$, we see that the image of the left-hand boundary $xy = 1$ for R is the vertical line segment $u = 1, 2 \geq v \geq 1$, in G (see Figure 15.62). Likewise, the right-hand boundary $y = x$ of R maps to the horizontal line segment $v = 1, 1 \leq u \leq 2$, in G. Finally, the horizontal top boundary $y = 2$ of R

maps to $uv = 2$, $1 \leq v \leq 2$, in G. As we move counterclockwise around the boundary of the region R, we also move counterclockwise around the boundary of G, as shown in Figure 15.62. Knowing the region of integration G in the uv-plane, we can now write equivalent iterated integrals:

$$\int_1^2 \int_{1/y}^{y} \sqrt{\frac{y}{x}}\, e^{\sqrt{xy}}\, dx\, dy = \int_1^2 \int_1^{2/u} 2ue^u\, dv\, du.$$

Note the order of integration.

We now evaluate the transformed integral on the right-hand side,

$$\begin{aligned}
\int_1^2 \int_1^{2/u} 2ue^u\, dv\, du &= 2\int_1^2 \left[vue^u \right]_{v=1}^{v=2/u} du \\
&= 2\int_1^2 (2e^u - ue^u)\, du \\
&= 2\int_1^2 (2 - u)e^u\, du \\
&= 2\Big[(2 - u)e^u + e^u \Big]_{u=1}^{u=2} \\
&= 2(e^2 - (e + e)) = 2e(e - 2).
\end{aligned}$$

Integrate by parts.

■

Substitutions in Triple Integrals

We use

$$x = g(u, v, w) \qquad y = h(u, v, w) \qquad z = k(u, v, w)$$

for

$$\iiint_D F \, dx dy dz = \iiint_R F \left| \frac{\partial(x, y, z)}{\partial(u, v, w)} \right| du dv dw$$

Substitutions in Triple Integrals

We use

$$x = g(u,v,w) \qquad y = h(u,v,w) \qquad z = k(u,v,w)$$

for

$$\iiint\limits_{D} F\, dxdydz = \iiint\limits_{R} F \left| \frac{\partial(x,y,z)}{\partial(u,v,w)} \right| dudvdw$$

where the *Jacobian* is

$$J(u,v,w) = \frac{\partial(x,y,z)}{\partial(u,v,w)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix}.$$

Example

Cartesian coordinates $\longrightarrow$ Cylindrical coordinates.
$x = r\cos\theta,\ y = r\sin\theta,\ z = z.$

$$\frac{\partial(x,y,z)}{\partial(r,\theta,z)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial z} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial z} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial z} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta & 0 \\ \sin\theta & r\cos\theta & 0 \\ 0 & 0 & 1 \end{vmatrix}$$

$$= r\cos^2\theta + r\sin^2\theta = r.$$

Example

Cartesian coordinates $\longrightarrow$ Spherical coordinates.
$x = \rho \sin\phi \cos\theta,\ y = \rho \sin\phi \sin\theta,\ z = \rho\cos\phi.$

$$\frac{\partial(x,y,z)}{\partial(\rho,\phi,\theta)} = \begin{vmatrix} \frac{\partial x}{\partial \rho} & \frac{\partial x}{\partial \phi} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial \rho} & \frac{\partial y}{\partial \phi} & \frac{\partial y}{\partial \theta} \\ \frac{\partial z}{\partial \rho} & \frac{\partial z}{\partial \phi} & \frac{\partial z}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \sin\phi\cos\theta & \rho\cos\phi\cos\theta & -\rho\sin\phi\sin\theta \\ \sin\phi\sin\theta & \rho\cos\phi\sin\theta & \rho\sin\phi\cos\theta \\ \cos\phi & -\rho\sin\phi & 0 \end{vmatrix}$$

$$= \rho^2 \sin\phi.$$

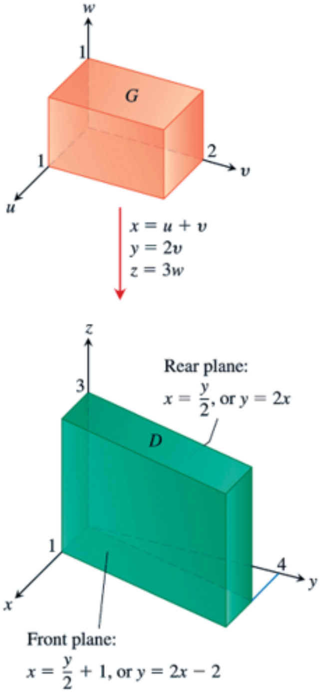
EXAMPLE 5 Evaluate

$$\int_0^3 \int_0^4 \int_{x=y/2}^{x=(y/2)+1} \left(\frac{2x-y}{2} + \frac{z}{3} \right) dx\, dy\, dz$$

by applying the transformation

$$u = (2x - y)/2, \qquad v = y/2, \qquad w = z/3 \tag{8}$$

and integrating over an appropriate region in uvw-space.


w
1
G
1
2
u
v
$x = u + v$
$y = 2v$
$z = 3w$
z
3
Rear plane:
$x = \frac{y}{2}$, or $y = 2x$
D
1
4
x
y
Front plane:
$x = \frac{y}{2} + 1$, or $y = 2x - 2$

Solution We sketch the region D of integration in xyz-space and identify its boundaries (Figure 15.66). In this case, the bounding surfaces are planes.

To apply Equation (7), we need to find the corresponding uvw-region G and the Jacobian of the transformation. To find them, we first solve Equations (8) for x, y, and z in terms of u, v, and w. Routine algebra gives

$$x = u + v, \qquad y = 2v, \qquad z = 3w. \tag{9}$$

We then find the boundaries of G by substituting these expressions into the equations for the boundaries of D:

xyz-equations for the boundary of D	Corresponding uvw-equations for the boundary of G	Simplified uvw-equations
$x = y/2$	$u + v = 2v/2 = v$	$u = 0$
$x = (y/2) + 1$	$u + v = (2v/2) + 1 = v + 1$	$u = 1$
$y = 0$	$2v = 0$	$v = 0$
$y = 4$	$2v = 4$	$v = 2$
$z = 0$	$3w = 0$	$w = 0$
$z = 3$	$3w = 3$	$w = 1$

The Jacobian of the transformation, again from Equations (9), is

$$J(u, v, w) = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix} = \begin{vmatrix} 1 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{vmatrix} = 6.$$

We now have everything we need to apply Equation (7):

$$\begin{aligned}
&\int_0^3 \int_0^4 \int_{x=y/2}^{x=(y/2)+1} \left(\frac{2x - y}{2} + \frac{z}{3} \right) dx\, dy\, dz \\
&= \int_0^1 \int_0^2 \int_0^1 (u + w) \left| J(u, v, w) \right| du\, dv\, dw \\
&= \int_0^1 \int_0^2 \int_0^1 (u + w)(6)\, du\, dv\, dw = 6 \int_0^1 \int_0^2 \left[\frac{u^2}{2} + uw \right]_0^1 dv\, dw \\
&= 6 \int_0^1 \int_0^2 \left(\frac{1}{2} + w \right) dv\, dw = 6 \int_0^1 \left[\frac{v}{2} + vw \right]_0^2 dw = 6 \int_0^1 (1 + 2w)\, dw \\
&= 6 \Big[w + w^2 \Big]_0^1 = 6(2) = 12.
\end{aligned}$$

Next Time

- 9.1 Sequences