2016.03.29 **MAT234 Matematik IV – Ara Sınavın Çözümleri** N. Course

Soru 1 (Sequences).

(a) [1p] Please write your student number at the top right of this page.

(b) [9p] Let (a_n) be a sequence of real numbers. Give the definition of "$a_n \to \infty$ as $n \to \infty$".

> We say that (a_n) tends to infinity ($a_n \to \infty$ as $n \to \infty$) iff, for all $A > 0$ there exists $N \in \mathbb{N}$ such that
> $$n > N \implies a_n > A.$$

(c) [15p] Let

$$b_n = \begin{cases} 1 & n \leq 10 \\ \frac{4n^2+1}{2n-2} & n > 10 \end{cases}$$

for all $n \in \mathbb{N}$. **Use the definition** that you wrote in part (b) to prove that $b_n \to \infty$ as $n \to \infty$.

> Let $A > 0$. Choose $N \in \mathbb{N}$ such that $N \geq \max\{11, \frac{1}{2}A\}$. Then
> $$n > N \implies b_n = \frac{4n^2+1}{2n-2} \geq \frac{4n^2}{2n-2} \geq \frac{4n^2}{2n} = 2n > 2N \geq A.$$
> Therefore $b_n \to \infty$ as $n \to \infty$.

(d) [25p] Suppose that

- $(c_n)_{n=1}^\infty$ and $(d_n)_{n=1}^\infty$ are sequences;
- $c_n \to \infty$ as $n \to \infty$; and
- $d_n \to 0.290316$ as $n \to \infty$.

Show that $c_n - d_n \to \infty$ as $n \to \infty$.

> Let $A > 0$. Since $c_n \to \infty$ as $n \to \infty$, $\exists N_1 \in \mathbb{N}$ such that
> $$n > N_1 \implies c_n > A + 1.$$
> Since $d_n \to 0.290316$ as $n \to \infty$, $\exists N_2 \in \mathbb{N}$ such that
> $$n > N_2 \implies |d_n - 0.290316| < 0.1 \implies 0.190316 < d_n < 0.390316 \implies -d_n > -0.390316.$$
> Define $N := \max\{N_1, N_2\}$. Then
> $$n > N \implies c_n - d_n > (A+1) - 0.390316 = A + 0.609684 > A.$$
> Therefore $c_n - d_n \to \infty$ as $n \to \infty$.

Soru 2 (Sequences). Define a sequence of real numbers (a_n) by

$$a_1 = 15 \qquad \text{and} \qquad 30a_{n+1} = a_n^2 + 200.$$

(a) [1p] Please write your student number at the top right of this page.

(b) [13p] Show that $10 \leq a_n \leq 20$ for all $n \in \mathbb{N}$.
[HINT: Use proof by induction.].

Since $10 \leq a_1 = 15 \leq 20$, the statement is true for $n = 1$ $\boxed{3}$. Suppose that it is true for $n = k$. Then $10 \leq a_k \leq 20$ $\boxed{2}$. So

$$30a_{k+1} = a_k^2 + 200 \leq 20^2 + 200 = 600 \implies a_{k+1} \leq 20 \boxed{3}$$

and

$$30a_{k+1} = a_k^2 + 200 \geq 10^2 + 200 = 300 \implies a_{k+1} \geq 10 \boxed{3}.$$

By the principle of mathematical induction $\boxed{2}$, it follows that $10 \leq a_n \leq 20$ $\forall n \in \mathbb{N}$.

(c) [12p] Show that $a_n \geq a_{n+1}$ for all $n \in \mathbb{N}$.

First note that

$$a_{n+1} - a_n = \frac{1}{30}(a_n^2 + 200) - a_n = \frac{1}{30}(a_n^2 - 30a_n + 200) = \frac{1}{30}(a_n - 10)(a_n - 20) \boxed{5}.$$

Since $10 \leq a_n \leq 20$, $(a_n - 10) \geq 0$ and $(a_n - 20) \leq 0$ $\boxed{4}$. Therefore $a_{n+1} - a_n = \frac{1}{30}(a_n - 10)(a_n - 20) \leq 0$. So $a_{n+1} \leq a_n$ $\forall n \in \mathbb{N}$ $\boxed{4}$.

(d) [12p] Show that (a_n) is a convergent sequence.

By a theorem from the course, "every decreasing sequence which is bounded below is convergent". In part (b), I proved that (a_n) is bounded below. In part (c), I proved that (a_n) is decreasing. Therefore (a_n) is convergent.

(e) [12p] Calculate $\lim_{n\to\infty} a_n$.

Let $a = \lim_{n\to\infty} a_n$. Then $30a \leftarrow 30a_{n+1} = a_n^2 + 200 \to a^2 + 200$ as $n \to \infty$ $\boxed{4}$. Because limits are unique, it follows that $0 = a^2 - 30a + 200 = (a - 10)(a - 20)$. So $a = 10$ or $a = 20$ $\boxed{4}$. Finally, since $a_1 = 15$ and (a_n) is decreasing, we must have that $a = 10$ $\boxed{4}$.

Soru 3 (Odds and Sods).

(a) [1p] Please write your student number at the top right of this page.

(b) [10p] Show that $\neg(P \wedge Q) = (\neg P \vee \neg Q)$.

P	Q	$P \wedge Q$	$\neg(P \wedge Q)$	$\neg P$	$\neg Q$	$\neg P \vee \neg Q$
T	T	T	F	F	F	F
T	F	F	T	F	T	T
F	T	F	T	T	F	T
F	F	F	T	T	T	T

Definition A sequence of real numbers (a_n) is *bounded* if and only if there exists $M \in \mathbb{R}$ such that for all $n \in \mathbb{N}$ we have that $|a_n| \leq M$.

(c) [10p] Give the definition of "(a_n) is **not bounded**".
[HINT: Negate the definition above.]

A sequence (a_n) is **not bounded** if and only if for all $M \in \mathbb{R}$, there exists $n \in \mathbb{N}$ such that $|a_n| > M$.

(d) [10p] Let $S \subseteq \mathbb{R}$. Give the definition of the *infinum* of S.

The infinum of S, which we denote by $\inf S$, is the greatest lower bound for S. If S is empty, we define $\inf S = \infty$. If S is not bounded below, we define $\inf S = -\infty$.

Now define $s_n := -\frac{n}{n+1}$ and $S := \{s_1, s_2, s_3, s_4, s_5, \ldots\} = \{-\frac{1}{2}, -\frac{2}{3}, -\frac{3}{4}, -\frac{4}{5}, -\frac{5}{6}, \ldots\} \subseteq \mathbb{R}$.

(e) [19p] Show that $\inf S = -1$.

We must prove (i) that -1 is a lower bound for S, and (ii) that for all $\varepsilon > 0$, $(-1+\varepsilon)$ is not a lower bound for S.

The first part is easy: Clearly $\frac{n+1}{n} = 1 + \frac{1}{n} \geq 1$ which implies that $\frac{n}{n+1} \leq 1$ and we have that $s_n = -\frac{n}{n+1} \geq -1$ for all $n \in \mathbb{N}$. Therefore -1 is a lower bound for S.

Now let $\varepsilon > 0$. Choose $n > \frac{1}{\varepsilon} - 1$. Rearranging this inequality, we have that

$$\varepsilon > \frac{1}{n+1} = \frac{n+1-n}{n+1} = 1 - \frac{n}{n+1}$$

which gives us $-\frac{n}{n+1} < -1 + \varepsilon$. Therefore $(-1+\varepsilon)$ is not a lower bound for S.

Therefore -1 is the greatest lower bound for S.