Lecture 8

- 14.3 Area by Double Integration
- 10.3 Polar Coordinates
- 14.4 Double Integrals in Polar Form

Area by Double Integration

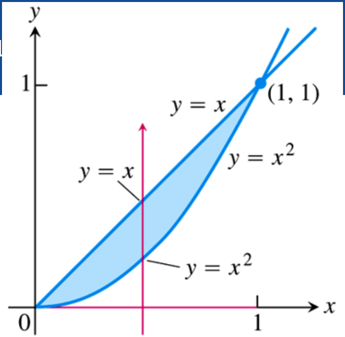
Definition

The *area* of a closed, bounded region R is

$$A = \iint\limits_R dA.$$

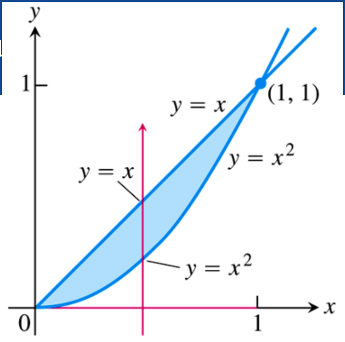
Example

Find the area of the region R bounded by $y = x$ and $y = x^2$ in the first quadrant.



Example

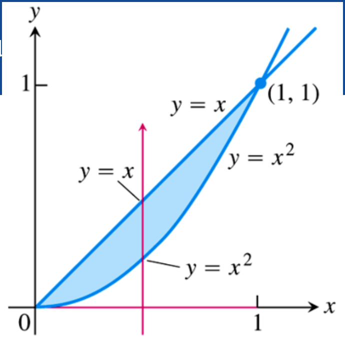
Find the area of the region R bounded by $y = x$ and $y = x^2$ in the first quadrant.



Example

Find the area of the region R bounded by $y = x$ and $y = x^2$ in the first quadrant.

$$A = \int_0^1 \int_{x^2}^{x} dydx$$



Example

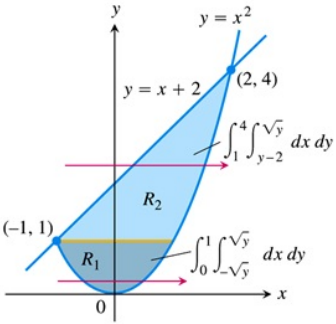
Find the area of the region R bounded by $y = x$ and $y = x^2$ in the first quadrant.

$$A = \int_0^1 \int_{x^2}^{x} dydx = \int_0^1 \Big[y\Big]_{x^2}^{x} dx = \int_0^1 (x - x^2)\, dx$$

$$= \left[\frac{x^2}{2} - \frac{x^3}{3}\right]_0^1 = \frac{1}{6}.$$

Example

Find the area of the region R bounded by $y = x$ and $y = x^2$ in the first quadrant.

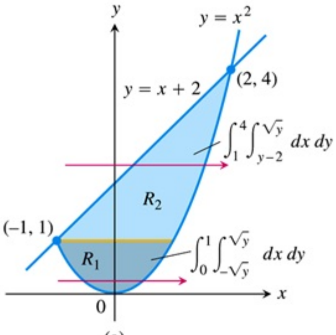


(a)

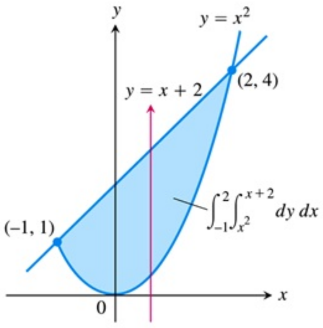
Example

Find the area of the region R bounded by $y = x$ and $y = x^2$ in the first quadrant.

$$A = \iint\limits_{R_1} dA + \iint\limits_{R_2} dA = \int_0^1 \int_{-\sqrt{y}}^{\sqrt{y}} dxdy + \int_1^4 \int_{y-2}^{\sqrt{y}} dxdy = \ldots$$



(a)



(b)

Example

Find the area of the region R bounded by $y = x$ and $y = x^2$ in the first quadrant.

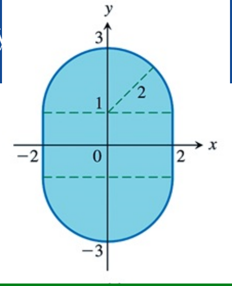
$$A = \iint_{R_1} dA + \iint_{R_2} dA = \int_0^1 \int_{-\sqrt{y}}^{\sqrt{y}} dxdy + \int_1^4 \int_{y-2}^{\sqrt{y}} dxdy = \dots$$

$$A = \int_{-1}^{2} \int_{x^2}^{x+2} dydx = \int_{-1}^{2} \Big[y \Big]_{x^2}^{x+2} = \int_{-1}^{2} (x+2-x^2)\, dx = \dots = \frac{9}{2}.$$

Example

Find the area of the region R described by $-2 \leq x \leq 2$ and $-1-\sqrt{4-x^2} \leq y \leq 1+\sqrt{4-x^2}$.

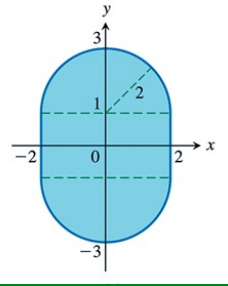
14.3 Area by … ration

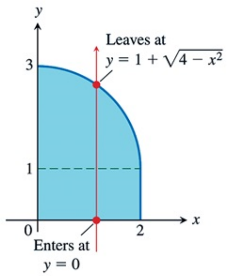


Example

Find the area of the region R described by $-2 \le x \le 2$ and $-1-\sqrt{4-x^2} \le y \le 1+\sqrt{4-x^2}$.

$$A = \iint\limits_R dA$$



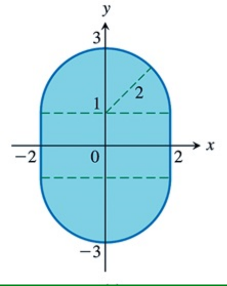


Example

Find the area of the region R described by $-2 \leq x \leq 2$ and $-1 - \sqrt{4 - x^2} \leq y \leq 1 + \sqrt{4 - x^2}$.

$$A = \iint\limits_R dA = 4 \int_0^2 \int_0^{1+\sqrt{4-x^2}} dydx = \ldots = 8 + 4\pi.$$

14.3 Area by ration



Example

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$$A = \iint\limits_R dA = 4\int_0^2 \int_0^{1+\sqrt{4-x^2}} dy\,dx = \ldots = 8+4\pi.$$

or

$$A = \begin{pmatrix}\text{area of a}\\ \text{circle of}\\ \text{radius 2}\end{pmatrix} + \begin{pmatrix}\text{area of a } 4\times 2\\ \text{rectangle}\end{pmatrix} = 4\pi+8.$$

Average Value of a Function

Definition

The *average value* of f over R is

$$\text{av}(f) = \frac{1}{\text{area of } R} \iint\limits_R f \, dA$$

Average Value of a Function

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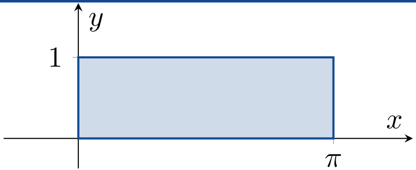
The *average value* of f over R is

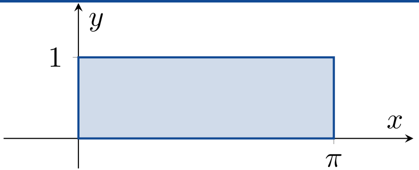
$$\operatorname{av}(f) = \frac{1}{\text{area of } R} \iint\limits_R f \, dA = \frac{\iint_R f \, dA}{\iint_R dA}.$$



Example

Find the average value of $f(x, y) = x \cos xy$ over the rectangle $R = [0, \pi] \times [0, 1]$.

$$\text{av}(f) = \frac{1}{\text{area of } R} \iint_R f \, dA$$



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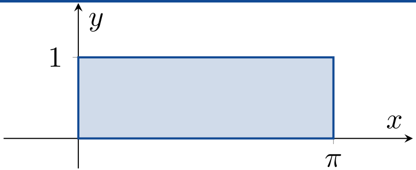
$$\text{av}(f) = \frac{1}{\text{area of } R} \iint_R f \, dA = \frac{1}{\pi} \int_0^{\pi} \int_0^1 x \cos xy \, dy dx$$



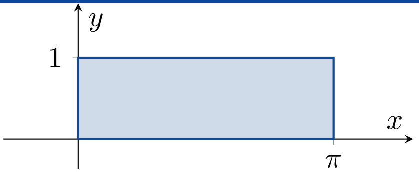
Example

Find the average value of $f(x, y) = x \cos xy$ over the rectangle $R = [0, \pi] \times [0, 1]$.

$$\mathrm{av}(f) = \frac{1}{\text{area of } R} \iint_R f \, dA = \frac{1}{\pi} \int_0^{\pi} \int_0^1 x \cos xy \, dy dx$$

$$= \frac{1}{\pi} \int_0^{\pi} \Big[\sin xy \Big]_{y=0}^{y=1} dx = \frac{1}{\pi} \int_0^{\pi} (\sin x - 0) \, dx$$

14.3 Area by Double Integration

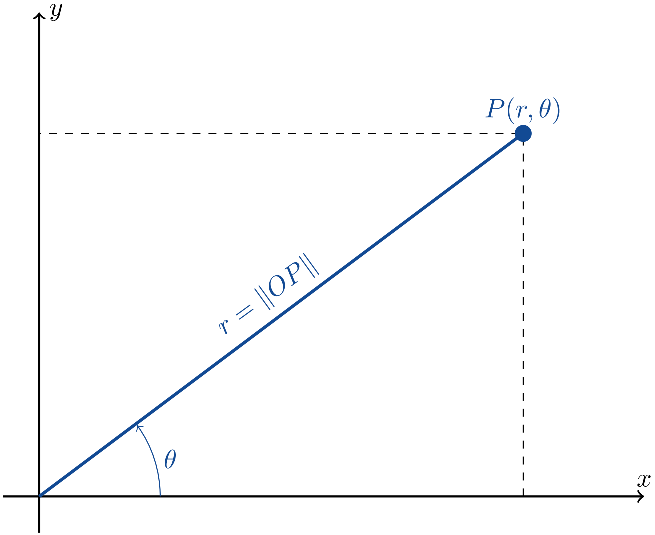


Example

Find the average value of $f(x, y) = x \cos xy$ over the rectangle $R = [0, \pi] \times [0, 1]$.

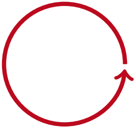
$$\begin{aligned}
\operatorname{av}(f) &= \frac{1}{\text{area of } R} \iint\limits_R f \, dA = \frac{1}{\pi} \int_0^{\pi} \int_0^1 x \cos xy \, dy dx \\
&= \frac{1}{\pi} \int_0^{\pi} \Big[\sin xy \Big]_{y=0}^{y=1} dx = \frac{1}{\pi} \int_0^{\pi} (\sin x - 0) \, dx \\
&= \frac{1}{\pi} \Big[-\cos x \Big]_0^{\pi} = \frac{1}{\pi}(1 + 1) = \frac{2}{\pi}.
\end{aligned}$$

10.3 Polar Coordinates

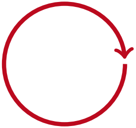


y
$P(r,\theta)$
$r = \|OP\|$
θ
x

10.3 Polar Coordinates

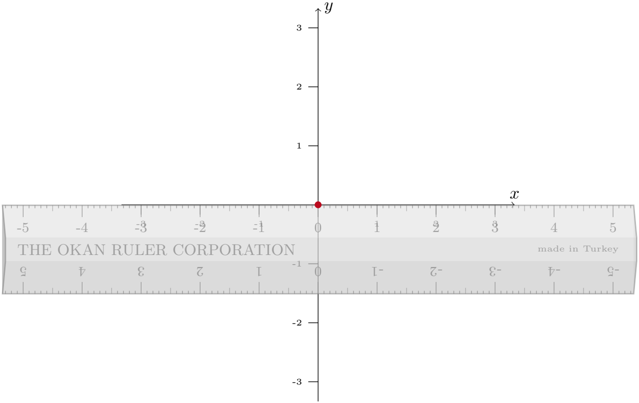


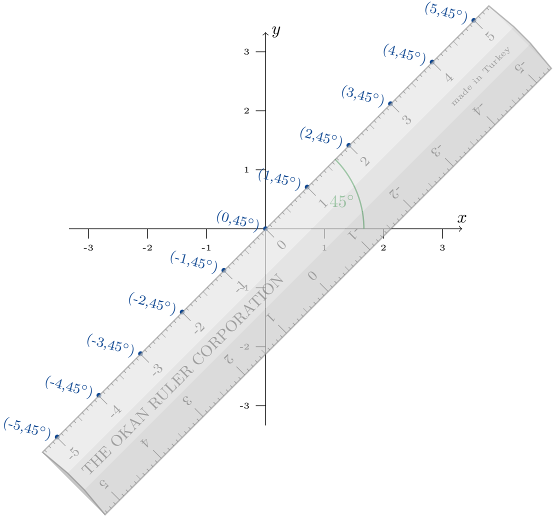
anticlockwise = positive angle
saat yönünün tersi = pozitif açı



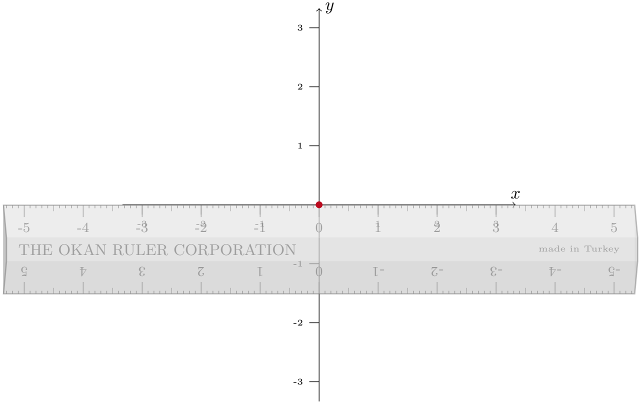
clockwise = negative angle
saat yönünde = negatif açı

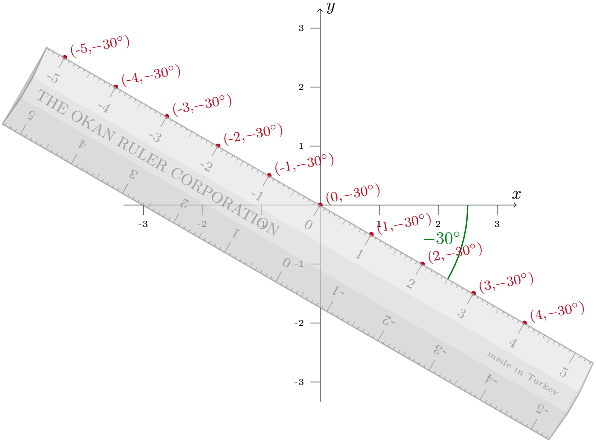
10.3 Polar Coordinates





y
x
3
2
1
-1
-2
-3
-3
-2
-1
1
2
3
(5,45°)
(4,45°)
(3,45°)
(2,45°)
(1,45°)
(0,45°)
(-1,45°)
(-2,45°)
(-3,45°)
(-4,45°)
(-5,45°)
45°
THE OKAN RULER CORPORATION
made in Turkey


y
x
3
2
1
-1
-2
-3
-5 -4 -3 -2 -1 0 1 2 3 4 5
THE OKAN RULER CORPORATION
made in Turkey



y
x
3
2
1
-1
-2
-3
-3
-2
2
3
(-5, −30°)
(-4, −30°)
(-3, −30°)
(-2, −30°)
(-1, −30°)
(0, −30°)
(1, −30°)
(2, −30°)
(3, −30°)
(4, −30°)
−30°
THE OKAN RULER CORPORATION
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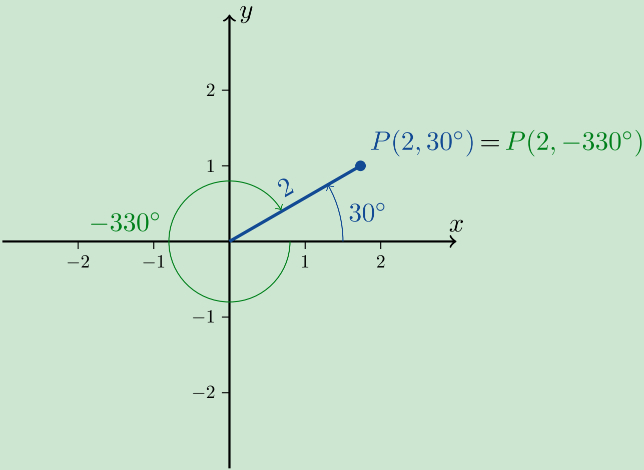
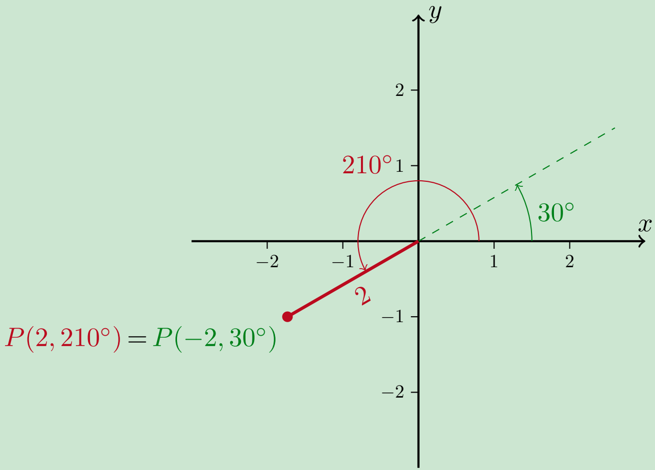
Example



Example



Example

Find all the polar coordinates of $P(2, 30^\circ)$.

We can have either $r = 2$ or $r = -2$.

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We can have either $r = 2$ or $r = -2$. For $r = 2$, we can have

$$\theta = 30^\circ, 30^\circ \pm 360^\circ, 30^\circ \pm 720^\circ, 30^\circ \pm 1080^\circ, \ldots$$

Example

Find all the polar coordinates of $P(2, 30°)$.

We can have either $r = 2$ or $r = -2$. For $r = 2$, we can have

$$\theta = 30°, 30° \pm 360°, 30° \pm 720°, 30° \pm 1080°, \ldots$$

For $r = -2$, we can have

$$\theta = 210°, 210° \pm 360°, 210° \pm 720°, 210° \pm 1080°, \ldots$$

Example

Find all the polar coordinates of $P(2, 30^\circ)$.

We can have either $r = 2$ or $r = -2$. For $r = 2$, we can have

$$\theta = 30^\circ, 30^\circ \pm 360^\circ, 30^\circ \pm 720^\circ, 30^\circ \pm 1080^\circ, \ldots$$

For $r = -2$, we can have

$$\theta = 210^\circ, 210^\circ \pm 360^\circ, 210^\circ \pm 720^\circ, 210^\circ \pm 1080^\circ, \ldots$$

Therefore

$$P(2, 30^\circ) = P\Big(2, (30 + 360n)^\circ\Big) = P\Big(-2, (210 + 360m)^\circ\Big)$$

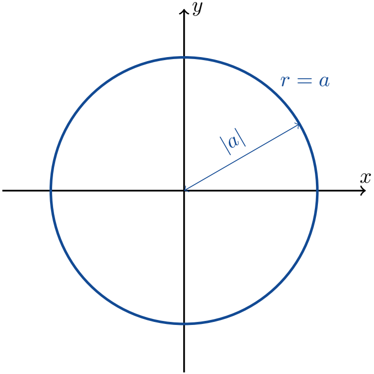
for all $m, n \in \mathbb{Z}$.

Example

Draw the graph of $r = a$.

Example

Draw the graph of $r = a$.

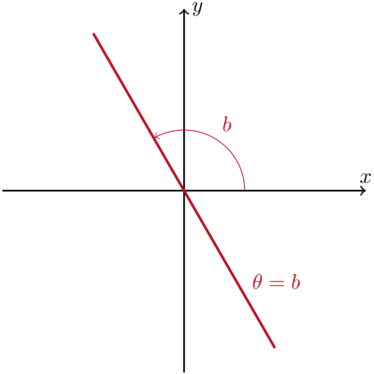


Example

Draw the graph of $\theta = b$.

Example

Draw the graph of $\theta = b$.



Remark

$r = 1$ and $r = -1$ are both equations for a circle of radius 1 centred at the origin.

Remark

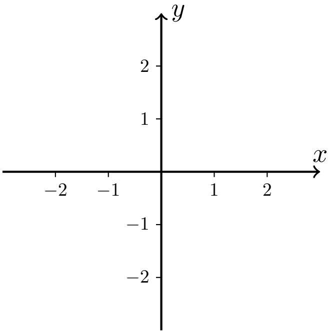
$r = 1$ and $r = -1$ are both equations for a circle of radius 1 centred at the origin.

Remark

$\theta = 30°$, $\theta = 210°$ and $\theta = -150°$ are all equations for the same line.

Example

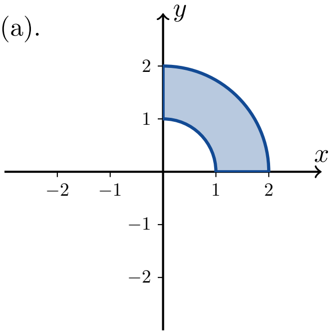
Draw the sets of points whose polar coordinates satisfy the following: $1 \le r \le 2$ and $0 \le \theta \le 90^\circ$.



Example

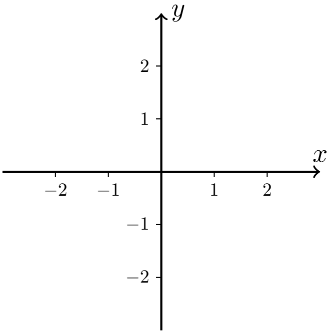
Draw the sets of points whose polar coordinates satisfy the following: $1 \le r \le 2$ and $0 \le \theta \le 90°$.

(a).



Example

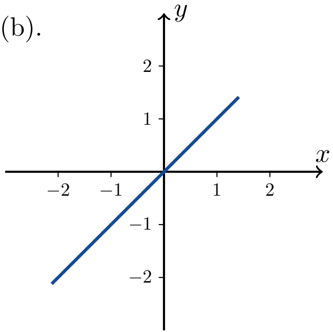
Draw the sets of points whose polar coordinates satisfy the following: $-3 \le r \le 2$ and $\theta = 45°$.



Example

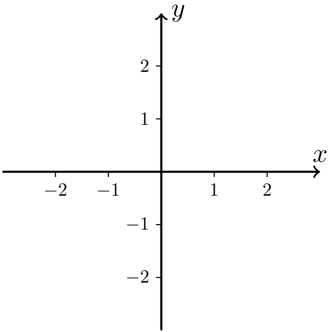
Draw the sets of points whose polar coordinates satisfy the following: $-3 \leq r \leq 2$ and $\theta = 45°$.

(b).



Example

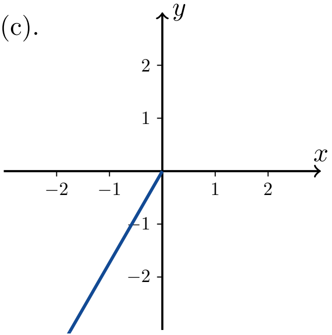
Draw the sets of points whose polar coordinates satisfy the following: $r \leq 0$ and $\theta = 60°$.



Example

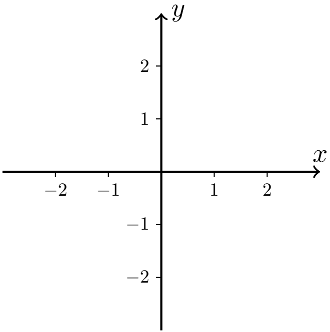
Draw the sets of points whose polar coordinates satisfy the following: $r \leq 0$ and $\theta = 60°$.

(c).



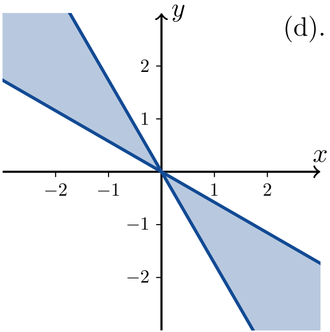
Example

Draw the sets of points whose polar coordinates satisfy the following: $120^\circ \le \theta \le 150^\circ$.



Example

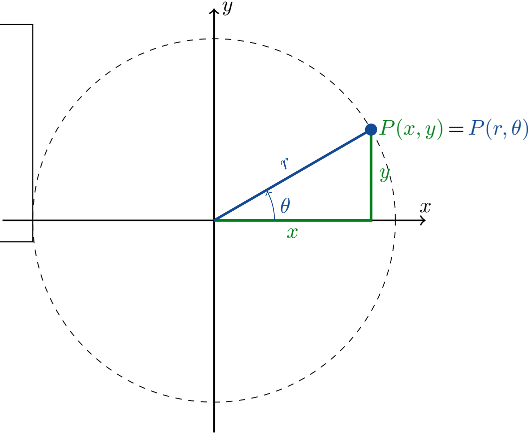
Draw the sets of points whose polar coordinates satisfy the following: $120^\circ \le \theta \le 150^\circ$.



Relating Polar and Cartesian Coordinates

$$x = r\cos\theta \qquad x^2 + y^2 = r^2$$

$$y = r\sin\theta \qquad \tan\theta = \frac{y}{x}$$



$$x = r\cos\theta \quad y = r\sin\theta \quad x^2 + y^2 = r^2 \quad \tan\theta = \frac{y}{x}$$

Example

Convert the polar coordinates $(r, \theta) = (-3, 90°)$ into Cartesian coordinates.

$$x = r\cos\theta \quad y = r\sin\theta \quad x^2 + y^2 = r^2 \quad \tan\theta = \frac{y}{x}$$

Example

Convert the polar coordinates $(r, \theta) = (-3, 90^\circ)$ into Cartesian coordinates.

$$(x, y) = (r\cos\theta, r\sin\theta) = (-3\cos 90^\circ, -3\sin 90^\circ) = (0, -3).$$

$$x = r\cos\theta \quad y = r\sin\theta \quad x^2 + y^2 = r^2 \quad \tan\theta = \frac{y}{x}$$

Example

Find polar coordinates for the Cartesian coordinates $(x, y) = (5, -12)$.

$$x = r\cos\theta \quad y = r\sin\theta \quad x^2 + y^2 = r^2 \quad \tan\theta = \frac{y}{x}$$

Example

Find polar coordinates for the Cartesian coordinates $(x, y) = (5, -12)$.

Choosing $r > 0$, we calculate that

$$r = \sqrt{x^2 + y^2} = \sqrt{25 + 144} = \sqrt{169} = 13.$$

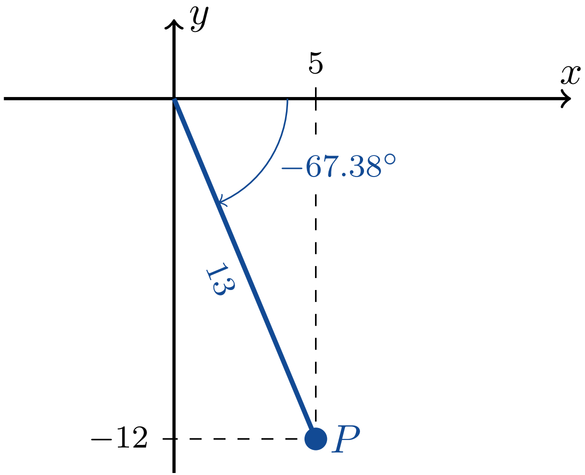
To find θ we use the equation $y = r\sin\theta$ to calculate that

$$\theta = \sin^{-1}\frac{y}{r} = \sin^{-1}\frac{-12}{13} \approx -67.38^\circ.$$

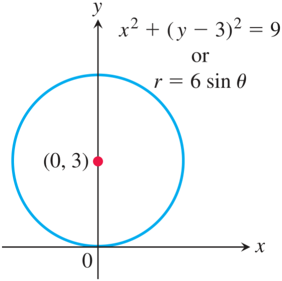
Therefore

$$(r, \theta) = (13, -67.38^\circ).$$

$$x = r\cos\theta \quad y = r\sin\theta \quad x^2 + y^2 = r^2 \quad \tan\theta = \frac{y}{x}$$



$$x = r\cos\theta \quad y = r\sin\theta \quad x^2 + y^2 = r^2 \quad \tan\theta = \frac{y}{x}$$



EXAMPLE 5 Find a polar equation for the circle $x^2 + (y - 3)^2 = 9$

Solution We apply the equations relating polar and Cartesian coordinates:

$$x^2 + (y - 3)^2 = 9$$

$$x^2 + y^2 - 6y + 9 = 9 \qquad \text{Expand } (y-3)^2.$$

$$x^2 + y^2 - 6y = 0 \qquad \text{Cancelation}$$

$$r^2 - 6r\sin\theta = 0 \qquad x^2 + y^2 = r^2,\ y = r\sin\theta$$

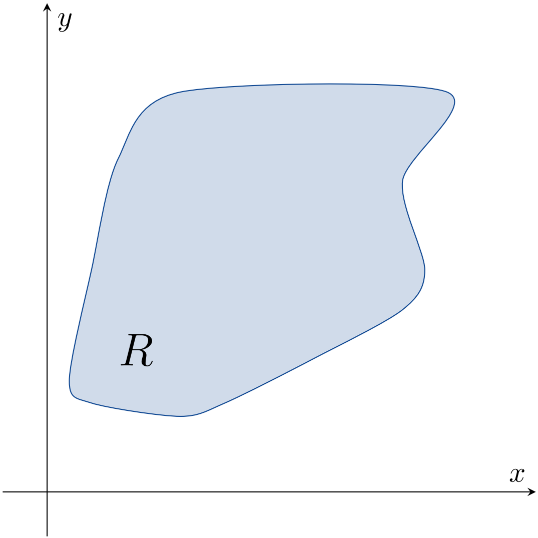
$$r = 0 \quad \text{or} \quad r - 6\sin\theta = 0$$

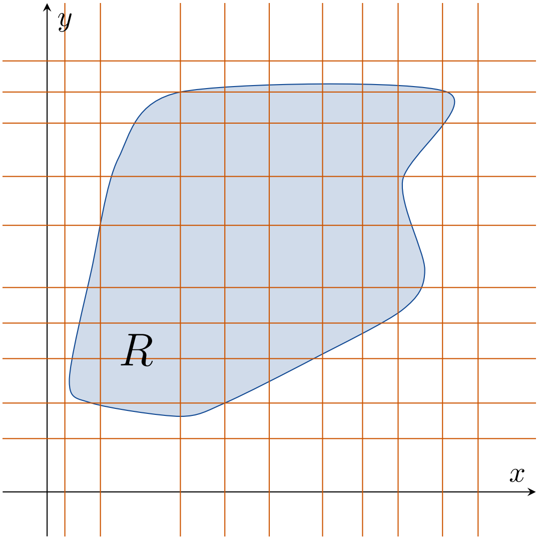
$$r = 6\sin\theta \qquad \text{Includes both possibilities}$$

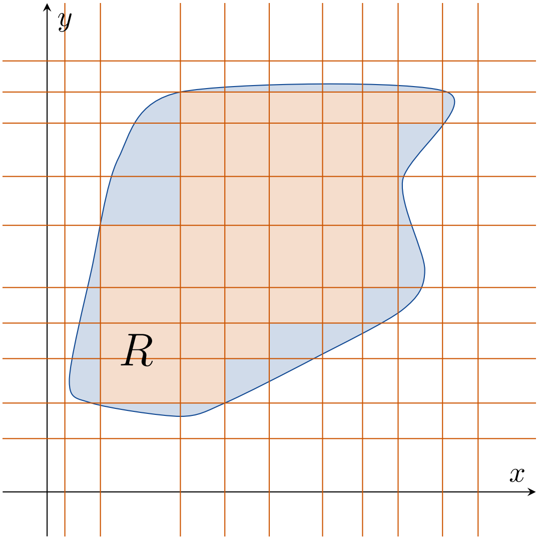
Break

We will continue at 2pm

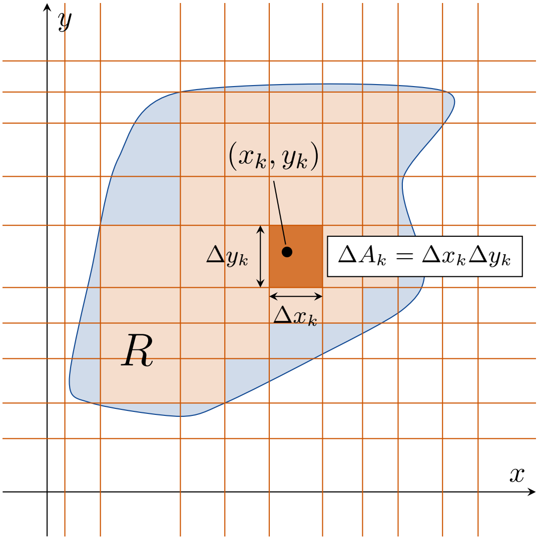
Double Integrals in Polar Form

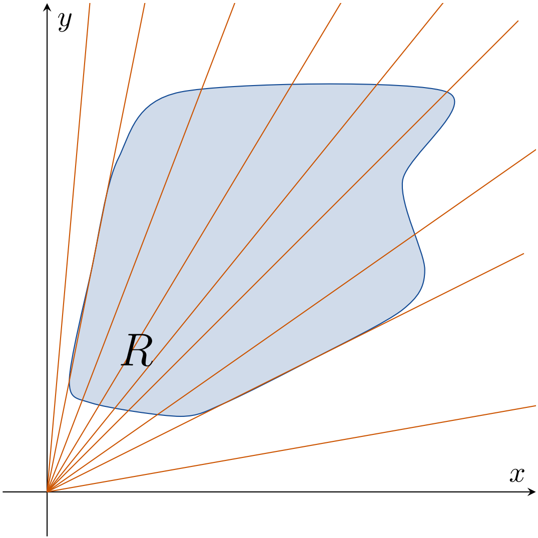

y
x
R


y
x
R

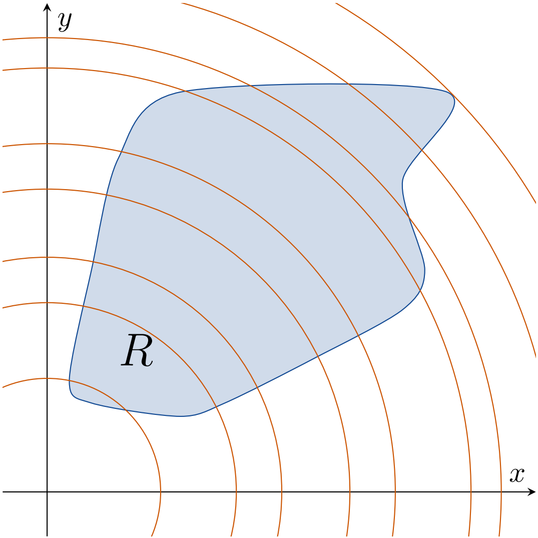


y
x
R


y
(x_k, y_k)
Δy_k
$\Delta A_k = \Delta x_k \Delta y_k$
Δx_k
R
x



y
R
x



y
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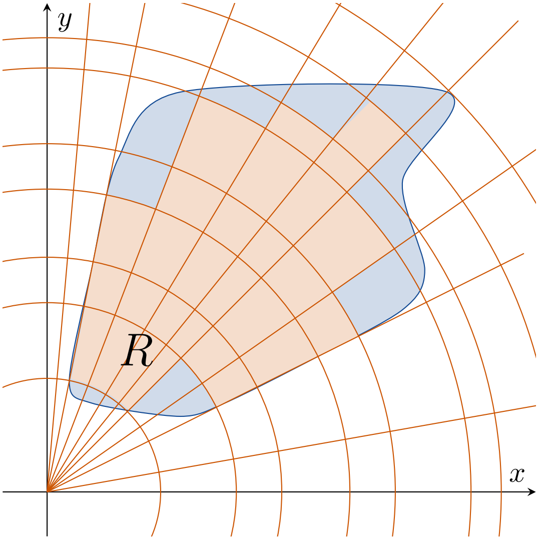
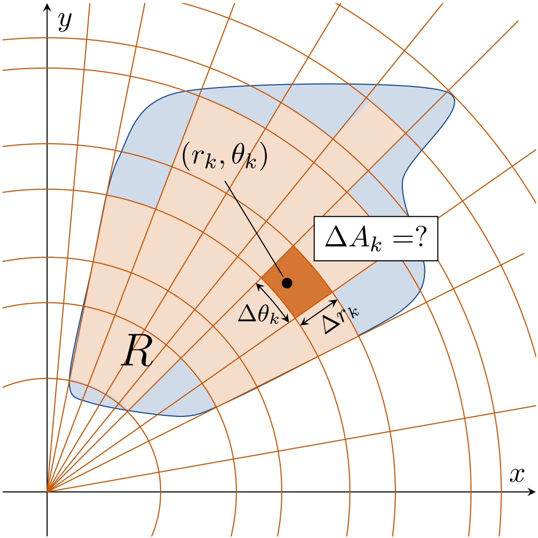
R



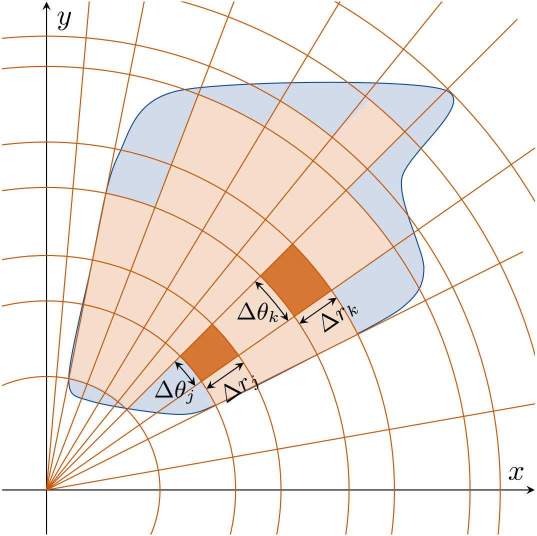
y
x
R



y
(r_k, θ_k)
$\Delta A_k = ?$
$\Delta\theta_k$
Δr_k
R
x

$$\iint_R f(r,\theta)\,dA = \lim_{\|P\|\to 0} \sum_k f(r_k,\theta_k)\Delta A_k$$

But what is ΔA_k?

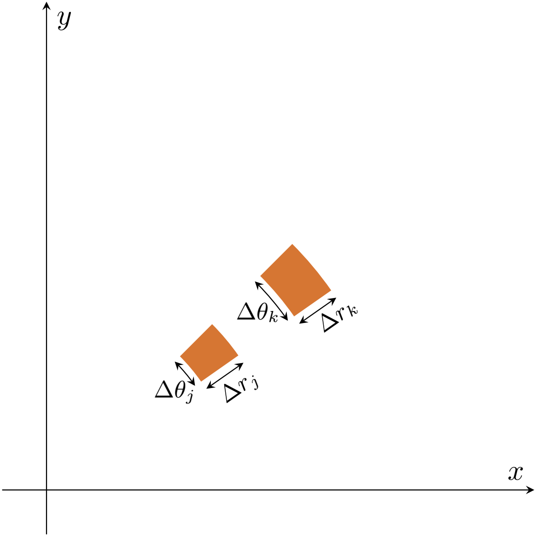


Note that

$$\Delta A_k = \Delta x_k \Delta y_k$$

but

$$\Delta A_k \neq \Delta r_k \Delta \theta_k.$$



Note that

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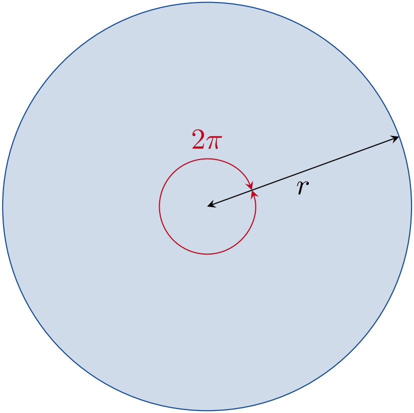
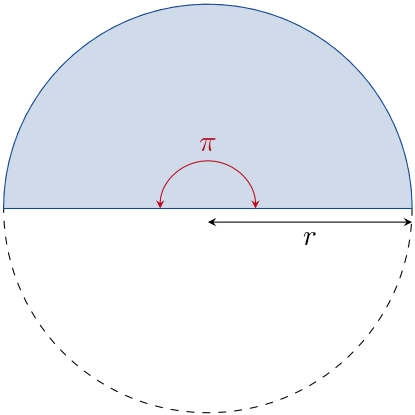
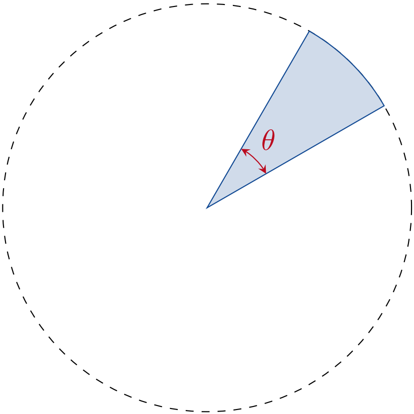


area of a circle $= \pi r^2 = \dfrac{1}{2}(2\pi)r^2$



area of a circle $= \pi r^2 = \frac{1}{2}(2\pi)r^2$

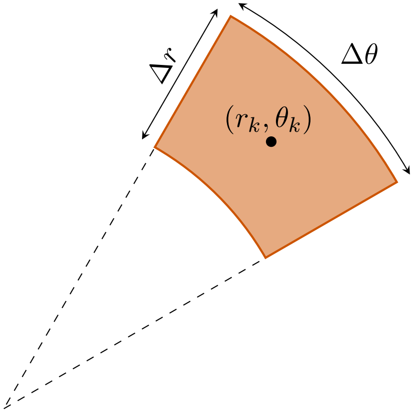
area of a semicircle $= \frac{1}{2}\pi r^2$

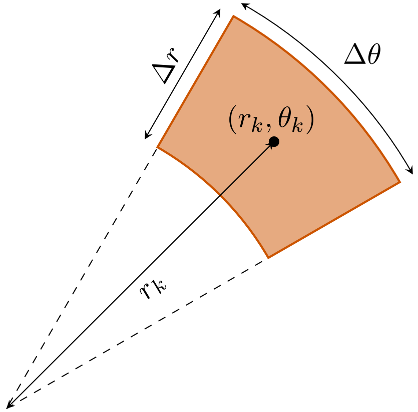


area of a circle $= \pi r^2 = \frac{1}{2}(2\pi)r^2$

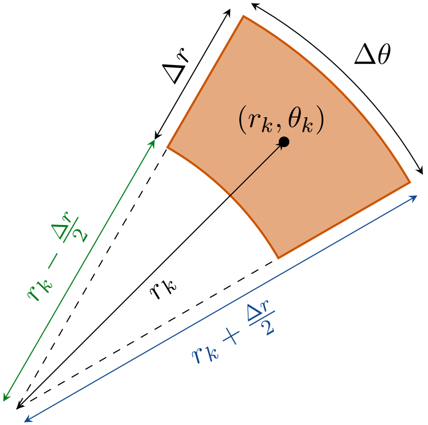
area of a semicircle $= \frac{1}{2}\pi r^2$

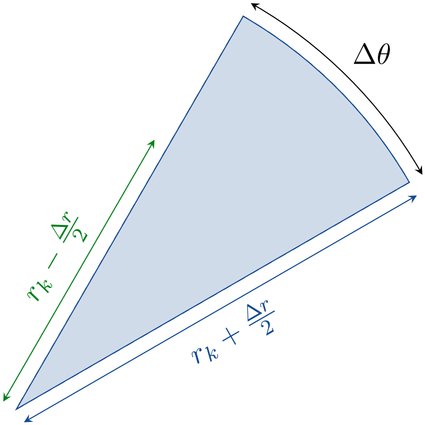
area of a sector $= \frac{1}{2}\theta r^2$

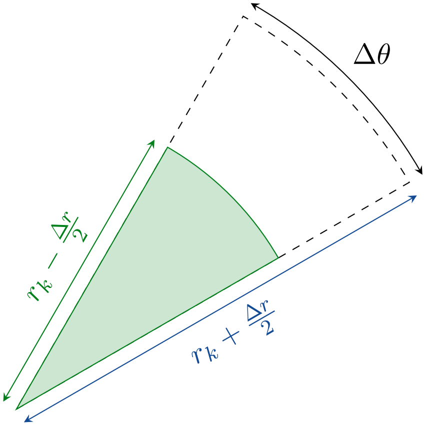

Δr
$\Delta\theta$
(r_k, θ_k)

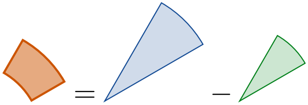


Δr
$\Delta\theta$
(r_k, θ_k)
r_k

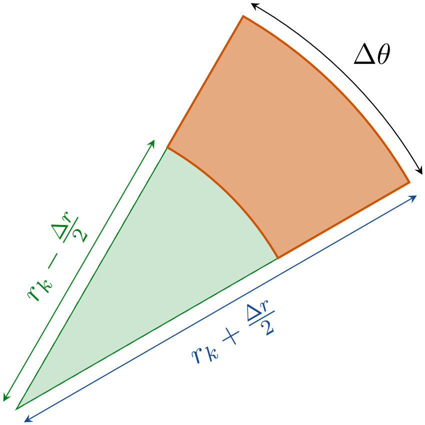

Δr
$\Delta\theta$
(r_k, θ_k)
$r_k - \frac{\Delta r}{2}$
r_k
$r_k + \frac{\Delta r}{2}$

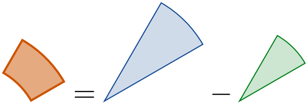

$\Delta\theta$
$r_k - \frac{\Delta r}{2}$
$r_k + \frac{\Delta r}{2}$


$\Delta\theta$
$r_k - \frac{\Delta r}{2}$
$r_k + \frac{\Delta r}{2}$

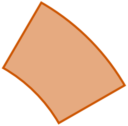


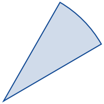
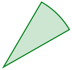
14.4 Double Integrals in Polar Form



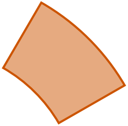


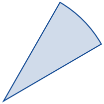
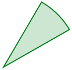
$$= \frac{1}{2}\Delta\theta\left(r_k + \frac{\Delta r}{2}\right)^2 - \frac{1}{2}\Delta\theta\left(r_k - \frac{\Delta r}{2}\right)^2$$



=  − 

$$= \frac{1}{2}\Delta\theta\left(r_k + \frac{\Delta r}{2}\right)^2 - \frac{1}{2}\Delta\theta\left(r_k - \frac{\Delta r}{2}\right)^2$$

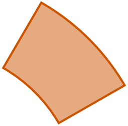


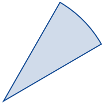
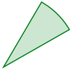
$=$  $-$ 

$$= \frac{1}{2}\Delta\theta\left(r_k + \frac{\Delta r}{2}\right)^2 - \frac{1}{2}\Delta\theta\left(r_k - \frac{\Delta r}{2}\right)^2$$

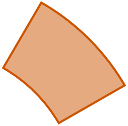
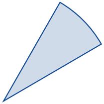
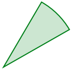
$$= \frac{1}{2}\Delta\theta\left(r_k^2 + 2r_k\frac{\Delta r}{2} + \frac{(\Delta r)^2}{4} - r_k^2 + 2r_k\frac{\Delta r}{2} - \frac{(\Delta r)^2}{4}\right)$$

14.4 Double Integrals in Polar Form



=  − 

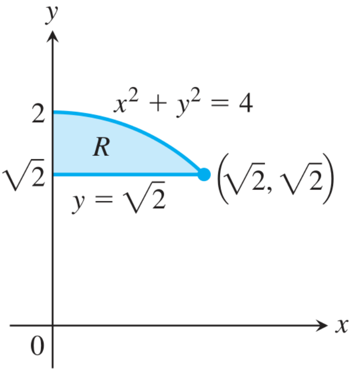
$$
\begin{aligned}
&= \frac{1}{2}\Delta\theta\left(r_k + \frac{\Delta r}{2}\right)^2 - \frac{1}{2}\Delta\theta\left(r_k - \frac{\Delta r}{2}\right)^2 \\
&= \frac{1}{2}\Delta\theta\left(r_k^2 + 2r_k\frac{\Delta r}{2} + \frac{(\Delta r)^2}{4} - r_k^2 + 2r_k\frac{\Delta r}{2} - \frac{(\Delta r)^2}{4}\right) \\
&= \frac{1}{2}\Delta\theta\left(2r_k\Delta r\right)
\end{aligned}
$$

 =  − 

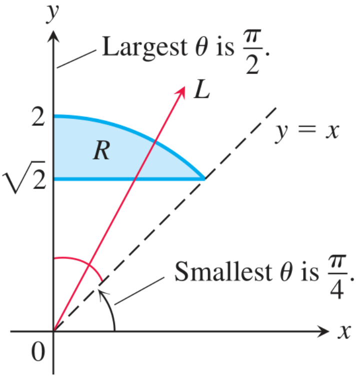
$$
\begin{aligned}
&= \frac{1}{2}\Delta\theta\left(r_k + \frac{\Delta r}{2}\right)^2 - \frac{1}{2}\Delta\theta\left(r_k - \frac{\Delta r}{2}\right)^2 \\
&= \frac{1}{2}\Delta\theta\left(r_k^2 + 2r_k\frac{\Delta r}{2} + \frac{(\Delta r)^2}{4} - r_k^2 + 2r_k\frac{\Delta r}{2} - \frac{(\Delta r)^2}{4}\right) \\
&= \frac{1}{2}\Delta\theta\left(2r_k\Delta r\right) \\
&= r_k\Delta r\Delta\theta.
\end{aligned}
$$

Theorem

$$dA = dxdy = rdrd\theta.$$

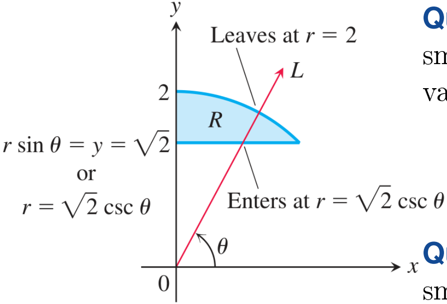


y
x
2
$\sqrt{2}$
0
$x^2 + y^2 = 4$
R
$y = \sqrt{2}$
$(\sqrt{2}, \sqrt{2})$



Question: What are the smallest and biggest possible values of θ in R?

$$\frac{\pi}{4} \leq \theta \leq \frac{\pi}{2}$$

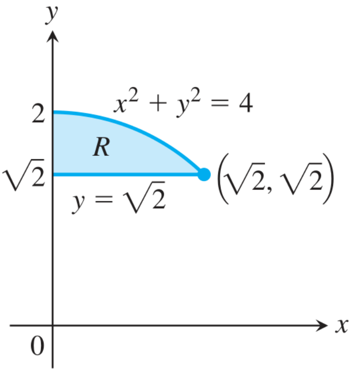


Question: What are the smallest and biggest possible values of θ in R?

$$\frac{\pi}{4} \leq \theta \leq \frac{\pi}{2}$$

Question: What are the smallest and biggest possible values of r in R?

$$\sqrt{2}\,\mathrm{cosec}\,\theta \leq r \leq 2$$



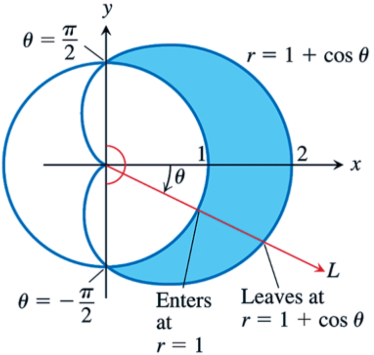
Question: What are the smallest and biggest possible values of θ in R?

$$\frac{\pi}{4} \leq \theta \leq \frac{\pi}{2}$$

Question: What are the smallest and biggest possible values of r in R?

$$\sqrt{2}\,\mathrm{cosec}\,\theta \leq r \leq 2$$

$$\iint_R f\,dA = \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \int_{\sqrt{2}\,\mathrm{cosec}\,\theta}^{2} f(r,\theta)\,r\,dr\,d\theta.$$



EXAMPLE 1 Find the limits of integration for integrating $f(r, \theta)$ over the region R that lies inside the cardioid $r = 1 + \cos\theta$ and outside the circle $r = 1$.

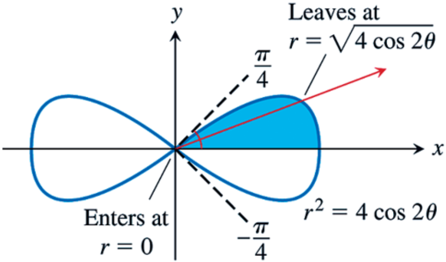
Solution

1. We first sketch the region and label the bounding curves (Figure 15.25).
2. Next we find the *r-limits of integration*. A typical ray from the origin enters R where $r = 1$ and leaves where $r = 1 + \cos\theta$.
3. Finally we find the *θ-limits of integration*. The rays from the origin that intersect R run from $\theta = -\pi/2$ to $\theta = \pi/2$. The integral is

$$\int_{-\pi/2}^{\pi/2}\int_{1}^{1+\cos\theta} f(r, \theta)\, r\, dr\, d\theta.$$

The area of a closed, bounded region R is

$$A = \iint_R dA = \iint_R r dr d\theta.$$



EXAMPLE 2 Find the area enclosed by the lemniscate $r^2 = 4\cos 2\theta$.

Solution We graph the lemniscate to determine the limits of integration (Figure 15.26) and see from the symmetry of the region that the total area is 4 times the first-quadrant portion.

$$A = 4\int_0^{\pi/4}\int_0^{\sqrt{4\cos 2\theta}} r\,dr\,d\theta = 4\int_0^{\pi/4}\left[\frac{r^2}{2}\right]_{r=0}^{r=\sqrt{4\cos 2\theta}} d\theta$$

$$= 4\int_0^{\pi/4} 2\cos 2\theta\,d\theta = 4\sin 2\theta\Big]_0^{\pi/4} = 4.$$

Cartesian Integral $\longrightarrow$ Polar Integral

$$x = r\cos\theta \qquad x^2 + y^2 = r^2$$

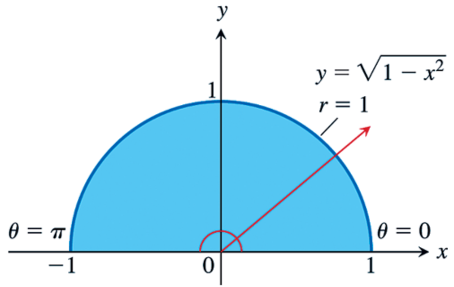
$$y = r\sin\theta \qquad \tan\theta = \frac{y}{x}$$

Cartesian Integral $\longrightarrow$ Polar Integral

$$x = r\cos\theta \qquad x^2 + y^2 = r^2$$

$$dxdy = rdrd\theta$$

$$y = r\sin\theta \qquad \tan\theta = \frac{y}{x}$$

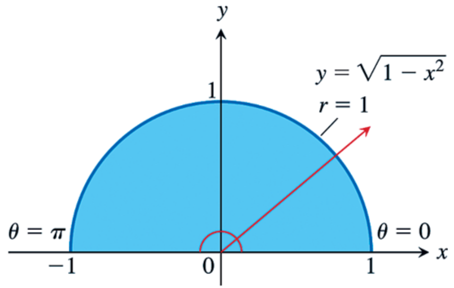


Example

Calculate

$$\iint\limits_R e^{x^2+y^2}\,dy\,dx$$

where R is the region under $y = \sqrt{1-x^2}$.



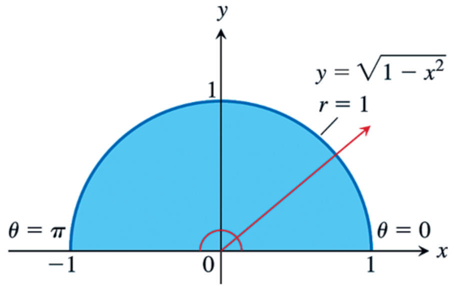
Example

Calculate

$$\iint_R e^{x^2+y^2}\,dydx$$

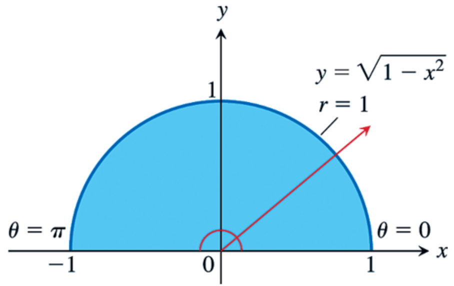
where R is the region under $y = \sqrt{1 - x^2}$.

$$\boxed{\begin{array}{c}\text{difficult}\\ \text{Cartesian}\\ \text{integral}\end{array}} \longrightarrow \boxed{\begin{array}{c}\text{easy polar}\\ \text{integral}\end{array}}$$



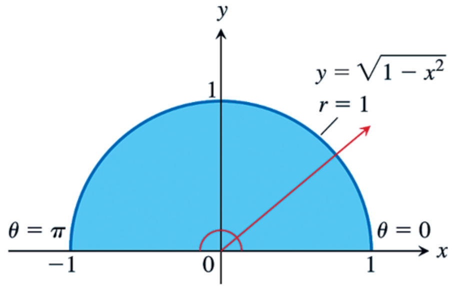
$$\iint\limits_R e^{x^2+y^2}\, dy dx = \int \int \qquad r\, dr d\theta$$

$$=$$



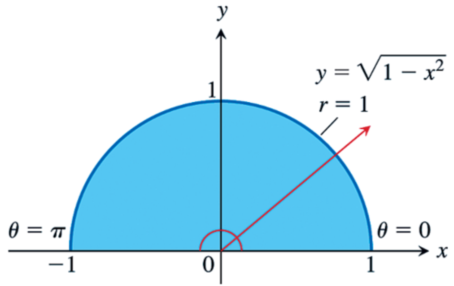
$$\iint\limits_R e^{x^2+y^2}\,dydx = \int \int e^{r^2} r\,drd\theta$$

$$=$$



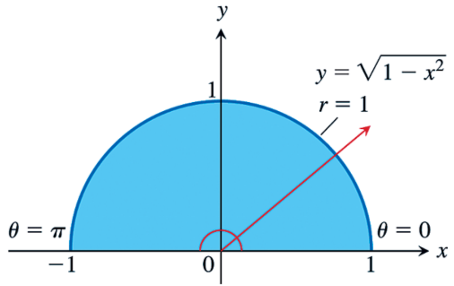
$$\iint\limits_R e^{x^2+y^2}\,dydx = \int_0^{\pi} \int \;\; e^{r^2} r\,drd\theta$$

$$=$$



$$\iint_R e^{x^2+y^2}\,dydx = \int_0^{\pi}\int_0^{1} e^{r^2}\,r\,drd\theta$$

$$=$$



$$\iint\limits_R e^{x^2+y^2}\, dy dx = \int_0^\pi \int_0^1 e^{r^2} r\, dr d\theta$$

$$= \int_0^\pi \left[\frac{1}{2} e^{r^2}\right]_0^1 d\theta = \int_0^\pi \frac{1}{2}(e-1)\, d\theta = \frac{\pi}{2}(e-1).$$

EXAMPLE 4 Evaluate the integral

$$\int_0^1 \int_0^{\sqrt{1-x^2}} (x^2 + y^2)\, dy\, dx.$$

Solution Integration with respect to y gives

$$\int_0^1 \left(x^2 \sqrt{1 - x^2} + \frac{(1 - x^2)^{3/2}}{3} \right) dx,$$

which is difficult to evaluate without tables. Things go better if we change the original integral to polar coordinates. The region of integration in Cartesian coordinates is given by the inequalities $0 \le y \le \sqrt{1 - x^2}$ and $0 \le x \le 1$, which correspond to the interior of the unit quarter circle $x^2 + y^2 = 1$ in the first quadrant. (See Figure 15.27, first quadrant.) Substituting the polar coordinates $x = r\cos\theta$, $y = r\sin\theta$, $0 \le \theta \le \pi/2$, and $0 \le r \le 1$, and replacing $dy\, dx$ by $r\, dr\, d\theta$ in the double integral, we get

$$\begin{aligned} \int_0^1 \int_0^{\sqrt{1-x^2}} (x^2 + y^2)\, dy\, dx &= \int_0^{\pi/2} \int_0^1 (r^2)\, r\, dr\, d\theta \\ &= \int_0^{\pi/2} \left[\frac{r^4}{4} \right]_{r=0}^{r=1} d\theta = \int_0^{\pi/2} \frac{1}{4}\, d\theta = \frac{\pi}{8}. \end{aligned}$$

The polar coordinate transformation is effective here because $x^2 + y^2$ simplifies to r^2 and the limits of integration become constants. ■

EXAMPLE 5 Find the volume of the solid region bounded above by the paraboloid $z = 9 - x^2 - y^2$ and below by the unit circle in the xy-plane.

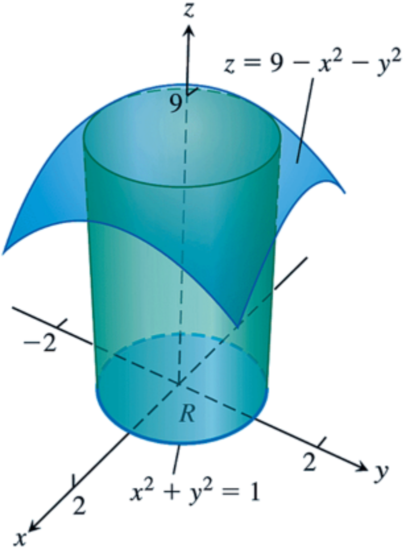
Solution The region of integration R is bo
is described in polar coordinates by $r = 1$,
Figure 15.28. The volume is given by the dou

$$\iint_R (9 - x^2 - y^2)\, dA = \int_0^{2\pi}\int_0^1 (9 -$$

$$= \int_0^{2\pi}\int_0^1 (9r -$$

$$= \int_0^{2\pi}\left[\frac{9}{2}r^2 -\right.$$

$$= \frac{17}{4}\int_0^{2\pi} d\theta =$$



EXAMPLE 5 Find the volume of the solid region bounded above by the paraboloid $z = 9 - x^2 - y^2$ and below by the unit circle in the xy-plane.

Solution The region of integration R is bounded by the unit circle $x^2 + y^2 = 1$, which is described in polar coordinates by $r = 1, 0 \leq \theta \leq 2\pi$. The solid region is shown in Figure 15.28. The volume is given by the double integral

$$\iint_R (9 - x^2 - y^2)\, dA = \int_0^{2\pi}\int_0^1 (9 - r^2)\, r\, dr\, d\theta \qquad r^2 = x^2 + y^2, \quad dA = r\, dr\, d\theta.$$

$$= \int_0^{2\pi}\int_0^1 (9r - r^3)\, dr\, d\theta$$

$$= \int_0^{2\pi} \left[\frac{9}{2} r^2 - \frac{1}{4} r^4\right]_{r=0}^{r=1} d\theta$$

$$= \frac{17}{4}\int_0^{2\pi} d\theta = \frac{17\pi}{2}.$$

Please read Example 6 in the textbook.

Next Time

- 14.5 Triple Integrals in Rectangular Coordinates
- 14.7 Triple Integrals in Cylindrical and Spherical Coordinates
- 14.8 Substitutions in Multiple Integrals