Week 2

- 1.5 Classification

First Order Differential Equations

- 2.1 Linear Equations
- 2.2 Separable Equations

1.5 Classification

ODEs

If only ordinary derivatives appear in a differential equation, then it is called an *ordinary differential equation* (ODE) [adi diferansiyel denklem]. For example

$$\frac{dv}{dt} = 9.8 - \frac{v}{5} \qquad \text{(falling object)}$$

and

$$\frac{dp}{dt} = \frac{p}{2} - 450 \qquad \text{(mice and owls)}$$

are ODEs.

PDEs

If the derivatives in a differential equation are partial derivatives, then it is called a *partial differential equation* (PDE) [kısmi türevli diferansiyel denklem]. For example

$$k\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t} \qquad \text{(heat equation)}$$

and

$$c^2\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2} \qquad \text{(wave equation)}$$

are PDEs.

Systems

If there is a single function to be found, then one differential equation is enough. However, if there are two or more unknown functions then we need a *system of differential equations*. For example

$$\begin{cases} \frac{dx}{dt} = ax - \alpha xy \\ \frac{dy}{dt} = -cy + \gamma xy \end{cases} \qquad \text{(Predator-Prey equations)}$$

is a system of differential equations.

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$$\frac{d^2y}{dt^2} + \frac{dy}{dt} + y = 0$$

is a second order ODE.

$$y''' + 2e^t y'' + yy' = t^4$$

is a third order ODE.

Linear and Non-Linear

The ODE

$$F(t, y, y', \ldots, y^{(n)}) = 0$$

is called *linear* iff F is a linear function of y, y', $\ldots$, $y^{(n)}$ (we don't care about t). The *general linear ODE* of order n is

$$a_0(t)y^{(n)} + a_1(t)y^{(n-1)} + \ldots + a_n(t)y = g(t). \qquad (1)$$

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For example(falling object) and (mice and owls) are linear ODEs. An ODE which is not linear is called *non-linear*. For example

$$y''' + 2e^t y'' + yy' = t^4$$

is non-linear due to the yy' term.

Example

For each ODE below, give the order of the equation and state whether it is linear or non-linear:

- $\frac{d^3y}{dx^3} + 2\frac{d^5y}{dt^5} + \frac{dy}{dt} - y - e^x\frac{d^2y}{dx^2} = 0$

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First Order Differential Equations

In this chapter, we will consider equations of the form

$$\frac{dy}{dt} = f(t, y). \tag{2}$$

2.1 Linear Equations

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If the function f in (2) depends linearly on y (we don't care about t), then (2) is a first order *linear* ODE.

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where the coefficients a and b are constants.

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where the coefficients a and b are constants. We will now consider

$$\frac{dy}{dt} + p(t)y = g(t) \tag{4}$$

where the coefficients $p(t)$ and $g(t)$ are functions of t.

We have seen how to solve (3):

$$\begin{aligned}
\frac{dy}{dt} &= -ay + b \\
\int \frac{dy}{y - \frac{b}{a}} &= \int -a\,dt \\
\ln\left|y - \frac{b}{a}\right| &= -at + C \\
&\vdots \\
y &= \frac{b}{a} + ce^{-at}.
\end{aligned}$$

So for example $\frac{dy}{dt} + 2y = 3$ has solution $y = \frac{3}{2} + ce^{-2t}$.

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- Find a special function $\mu(t)$ called an integrating factor;
- Multiply the ODE by $\mu(t)$;
- Integrate.

Example

Use an integrating factor to solve $\dfrac{dy}{dt} + 2y = 3$.

2.1 Linear Equations

$$\frac{dy}{dt} + 2y = 3$$

First we multiply by an unknown function $\mu(t)$:

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$$\frac{d}{dt}\left(\mu(t)y\right) = \mu(t)\frac{dy}{dt} + \frac{d\mu}{dt}(t)y.$$

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$$\frac{d}{dt}\left(\mu(t)y\right) = \mu(t)\frac{dy}{dt} + \frac{d\mu}{dt}(t)y.$$

We want to choose $\mu(t)$ such that

$$\frac{d\mu}{dt} = 2\mu.$$

We know how to solve this equation:

$$\begin{aligned}\int \frac{d\mu}{\mu} &= \int 2\,dt \\ \ln|\mu| &= 2t + C \\ &\vdots \\ \mu(t) &= ce^{2t}.\end{aligned}$$

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Integrating gives

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Therefore

$$\boxed{y = \frac{3}{2} + ce^{-2t}.}$$

Remark

For the ODE $\frac{dy}{dt} + 2y = 3$ we use the integrating factor $\mu(t) = e^{2t}$.

Example

Use an integrating factor to solve $\dfrac{dy}{dt} + {\color{red}a}y = b$.

Example

Use an integrating factor to solve $\frac{dy}{dt} + ay = b$.

If we were to repeat the previous method, we would find that we need the integrating factor $\mu(t) = e^{at}$. (Please check!)

Example

Solve $\dfrac{dy}{dt} + ay = g(t)$.

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The integrating factor depends only on the coefficient of y. So again we use $\mu(t) = e^{at}$.

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By integrating, we obtain

$$e^{at}y = \int^{t} e^{as}g(s)\,ds + c.$$

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Thus

$$\boxed{y = e^{-at}\int^t e^{as}g(s)\,ds + ce^{-at}} \tag{5}$$

Example

Solve

$$\begin{cases} \frac{dy}{dt} + \frac{1}{2}y = 2 + t \\ y(0) = 2. \end{cases}$$

We multiply the ODE by the integrating factor $e^{\frac{t}{2}}$ to obtain

$$e^{\frac{t}{2}}y' + \frac{1}{2}e^{\frac{t}{2}}y = 2e^{\frac{t}{2}} + te^{\frac{t}{2}}$$

and

$$\frac{d}{dt}\left(e^{\frac{t}{2}}y\right) = 2e^{\frac{t}{2}} + te^{\frac{t}{2}}.$$

Integrating gives us

$$e^{\frac{t}{2}}y = 4e^{\frac{t}{2}} + 2te^{\frac{t}{2}} - 4e^{\frac{t}{2}} + c = 2te^{\frac{t}{2}} + c$$

(where we have used $\int u\frac{dv}{dt} = uv - \int \frac{du}{dt}v$ with $u = t$ and $v = 2e^{\frac{t}{2}}$). Therefore

$$y(t) = 2t + ce^{-\frac{t}{2}}.$$

Now

$$2 = y(0) = 0 + c \qquad \Longrightarrow \qquad c = 2.$$

Therefore the solution to the IVP is

$$\boxed{y(t) = 2t + 2e^{-\frac{t}{2}}.}$$

Example

Solve $\dfrac{dy}{dt} - 2y = 4 - t$.

Please check that by using $\mu(t) = e^{-2t}$ we obtain
$y(t) = -\frac{7}{4} + \frac{t}{2} + ce^{2t}$.

Now consider

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WARNING: The integrating factor is NOT $e^{p(t)}$.

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$$\frac{d\mu}{dt} = p(t)\mu.$$

We know how to solve this ODE:

$$\begin{aligned}
\int \frac{d\mu}{\mu} &= \int p(t)\,dt \\
\ln|\mu| &= \int p(t)\,dt + C \\
&\vdots \\
\mu(t) &= c\exp\int p(t)\,dt.
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As before, we can choose $c = 1$ to obtain

$$\mu(t) = \exp \int p(t)\,dt = e^{\int p(t)\,dt}. \tag{6}$$

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and we calculate that

$$\mu y = \int^t \mu(s) g(s)\, ds + c$$

and

$$y(t) = \frac{\int^t \mu(s) g(s)\, ds + c}{\mu(t)}.$$

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First we must write the equation in the standard form:

$$\frac{dy}{dt} + \frac{2}{t}y = 4t.$$

Here $p(t) = \frac{2}{t}$ and $g(t) = 4t$.

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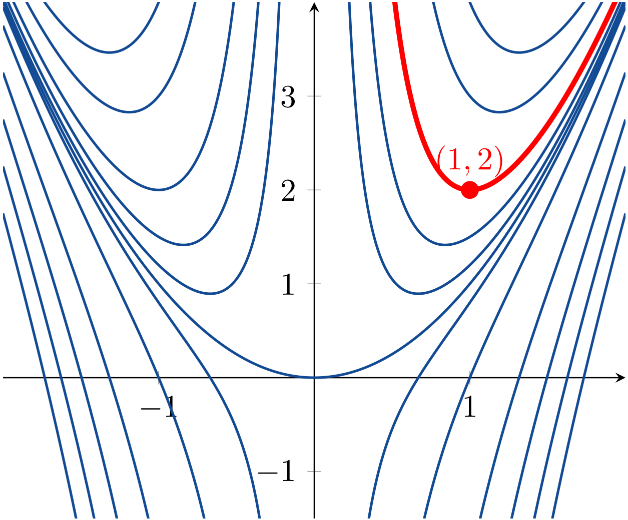
Hence the general solution to the ODE is

$$y(t) = t^2 + \frac{c}{t^2}.$$

To satisfy $y(1) = 2$, we choose $c = 1$. Therefore

$$\boxed{y(t) = t^2 + \frac{1}{t^2} \qquad (t > 0).}$$


3
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(1, 2)

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3. The function $y = t^2 + \frac{1}{t^2}$, $t < 0$ is *not* part of the solution to the IVP. The solution to the IVP only exists for $t \in (0, \infty)$.
4. Solutions for which $c > 0$ (i.e. $y(1) > 1$) are asymptotic to the positive y-axis as $t \searrow 0$. But solutions for which $c < 0$ (i.e. $y(1) < 1$) are asymptotic to the negative y-axis as $t \searrow 0$. So there is an initial value ($y(1) = 0$) where the behaviour changes. This is called a *critical initial value*.

Separable Equations

2.2 Separable Equations

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In the previous section we looked at a special case called "linear equations" – now we will study another special case.

2.2 Separable Equations

$$\frac{dy}{dx} = f(x, y) \tag{7}$$

Equation (7) can *always* be written in the form

$$M(x, y) + N(x, y)\frac{dy}{dx} = 0. \tag{8}$$

One way would be to write $M = -f$ and $N = 1$, but there may be other ways.

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One way would be to write $M = -f$ and $N = 1$, but there may be other ways. *If* we can do this so that $M(x)$ is a function only of x and $N(y)$ is a function only of y, then (8) becomes

$$M(x) + N(y)\frac{dy}{dx} = 0. \tag{9}$$

Definition

A first order ODE is called *separable* if it can be written in the form

$$M(x) + N(y)\frac{dy}{dx} = 0.$$

Remark

Note that we can rearrange $M(x) + N(y)\frac{dy}{dx} = 0$ to

$$\underbrace{M(x)\,dx}_{\text{all } x \text{ terms}} = -\underbrace{N(y)\,dy}_{\text{all } y \text{ terms}}.$$

In other words, it is possible to “separate” the variables.

Example

Consider

$$\frac{dy}{dx} = \frac{x^2}{1-y^2}.$$

1. Show that this ODE is separable.
2. Solve this ODE.

$$\frac{dy}{dx} = \frac{x^2}{1 - y^2}$$

We can rearrange this ODE to

$$-x^2 + (1 - y^2)\frac{dy}{dx} = 0.$$

This is of the form (9). Therefore this ODE is separable.

Note that $\frac{d}{dx}\left(-\frac{1}{3}x^3\right) = -x^2$ and $\frac{d}{dy}\left(y - \frac{1}{3}y^3\right) = 1 - y^2$.

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$$-x^2 + (1 - y^2)\frac{dy}{dx} = 0$$

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Using the Chain Rule, this is

$$\frac{d}{dx}\left(-\frac{1}{3}x^3\right) + \frac{d}{dx}\left(y - \frac{1}{3}y^3\right) = 0$$

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Therefore

$$-\frac{1}{3}x^3 + 1 - \frac{1}{3}y^3 = C$$

or

$$\boxed{x^3 - 3y + y^3 = c.}$$

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$$M(x) + N(y)y' = 0$$

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$$\begin{aligned} M(x) + N(y)\frac{dy}{dx} &= 0 \\ \frac{dH_1}{dx} + \frac{dH_2}{dy}\frac{dy}{dx} &= 0 \\ \frac{dH_1}{dx} + \frac{dH_2}{dx} &= 0 \end{aligned}$$

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by the Chain Rule. Then integrating gives the solution

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So to recap: To solve $M(x) + N(y)y' = 0$ we must integrate M wrt x and integrate N wrt y. But this is basically what we were doing in Chapter 1, where we did the following:

$$\begin{aligned} M(x) + N(y)\frac{dy}{dx} &= 0 \\ M(x) &= -N(y)\frac{dy}{dx} \\ M(x)\,dx &= -N(y)\,dy \\ \int M(x)\,dx &= -\int N(y)\,dy + c. \end{aligned}$$

Example

Solve $\begin{cases} \frac{dy}{dx} = \frac{3x^2+4x+2}{2(y-1)} \\ y(0) = -1. \end{cases}$

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Integrating gives

$$y^2 - 2y = x^3 + 2x^2 + 2x + c.$$

To find c, we use the initial condition $y(0) = 1$ and calculate that

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This is called an *implicit solution*. Sometimes this is the best that we can do. But in this example, it is possible to solve for y. Since

$$y^2 - 2y - (x^3 + 2x^2 + 2x + 3) = 0$$

is a quadratic equation, we find that

$$y = 1 \pm \sqrt{x^3 + 2x^2 + 2x + 4}.$$

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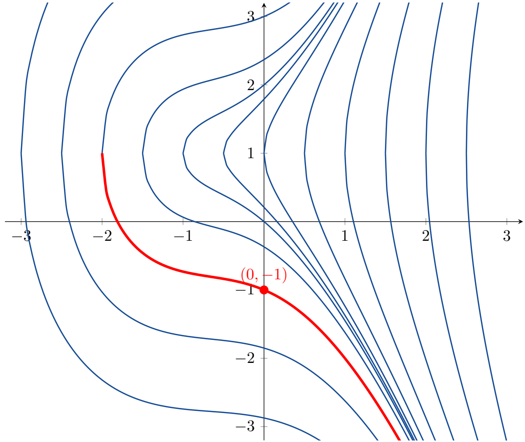
A solution of the form $y = f(x)$ is called an *explicit solution*.



3
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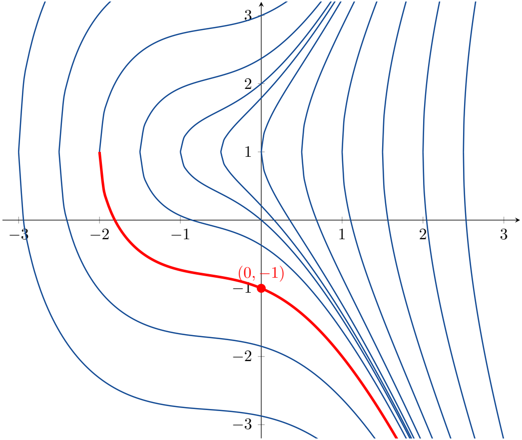

Note that the solution satisfying $y(0) = -1$ is a differentiable function $y : (-2, \infty) \to \mathbb{R}$.

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Solve $\begin{cases} \frac{dy}{dx} = \frac{y\cos x}{1+2y^2} \\ y(0) = 1. \end{cases}$

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$$\int \frac{1+2y^2}{y}\,dy = \int \cos x\,dx$$

$$\ln|y| + y^2 = \sin x + c$$

$$y(0) = 1 \qquad \Longrightarrow \qquad \ln 1 + 1^2 = \sin 0 + c \qquad \Longrightarrow \qquad c = 1.$$

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2. The solution exists on $(-\infty, \infty)$ (left for you to prove).

Next Week

- 2.3 Differences Between Linear and Nonlinear Equations
- 2.4 Autonomous Equations and Population Dynamics