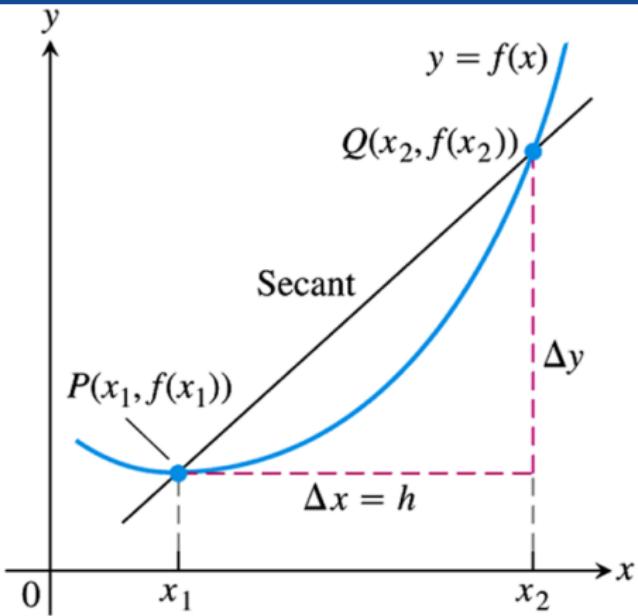


Lecture 2

- 2.1 Rates of Change and Tangents to Curves
- 2.2 Limit of a Function and Limit Laws
- 2.3 The Precise Definition of a Limit



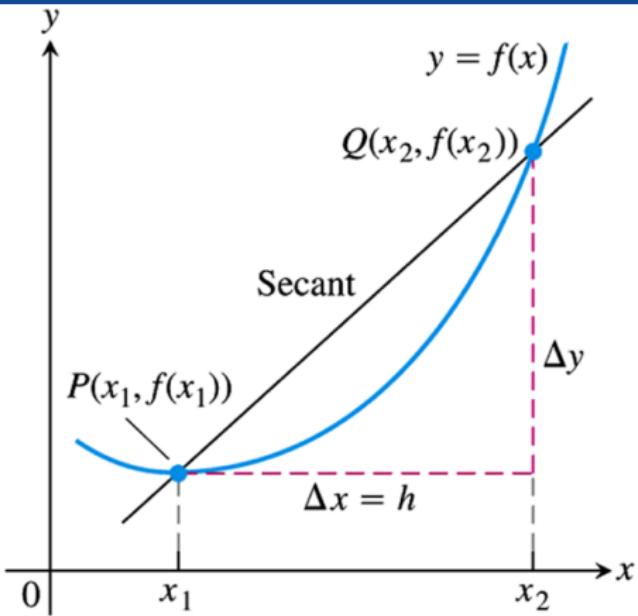
Rates of Change and Tangent Lines to Curves



Definition

The *average rate of change* of $y = f(x)$ with respect to x over the interval $[x_1, x_2]$ is

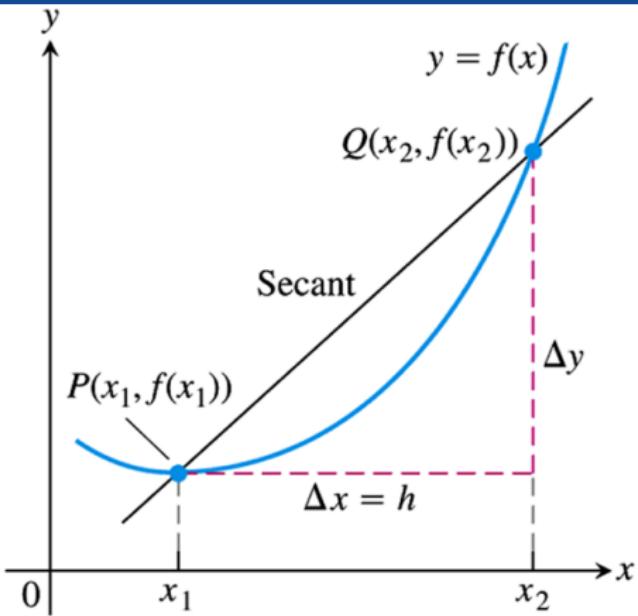
$$\frac{\Delta y}{\Delta x}$$



Definition

The *average rate of change* of $y = f(x)$ with respect to x over the interval $[x_1, x_2]$ is

$$\frac{\Delta y}{\Delta x} = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

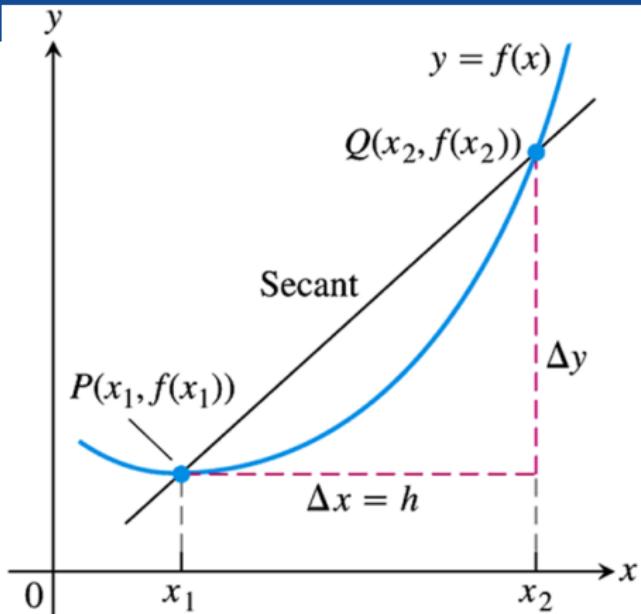


Definition

The *average rate of change* of $y = f(x)$ with respect to x over the interval $[x_1, x_2]$ is

$$\frac{\Delta y}{\Delta x} = \frac{f(x_2) - f(x_1)}{x_2 - x_1} = \frac{f(x_1 + h) - f(x_1)}{h}.$$

2.1 Rates of Change and Tangent Lines to Curves



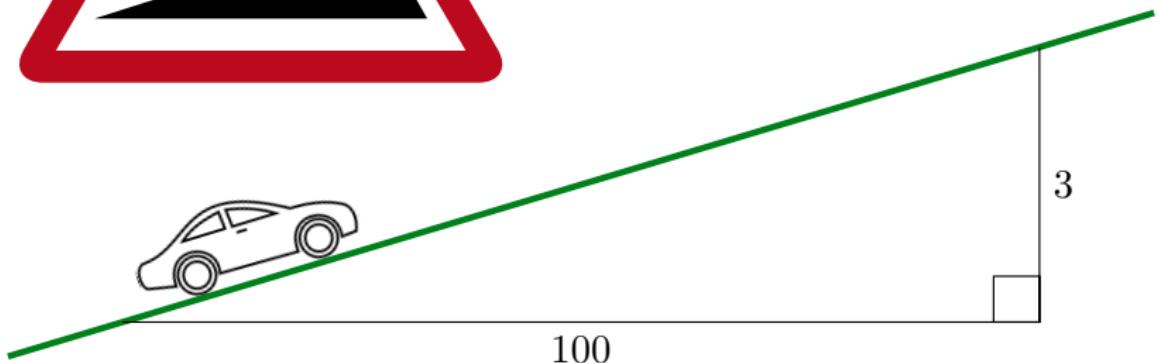
Definition

A line joining 2 points on a curve is called a *secant line*.

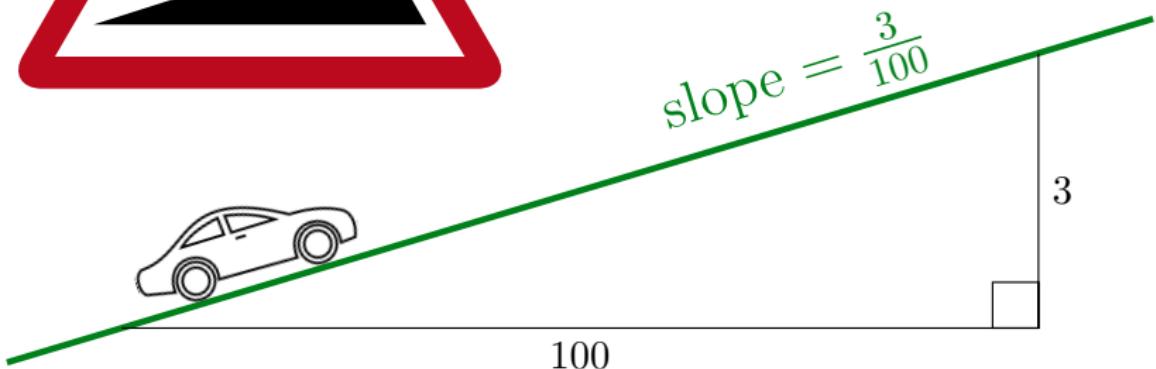
Slopes of Lines



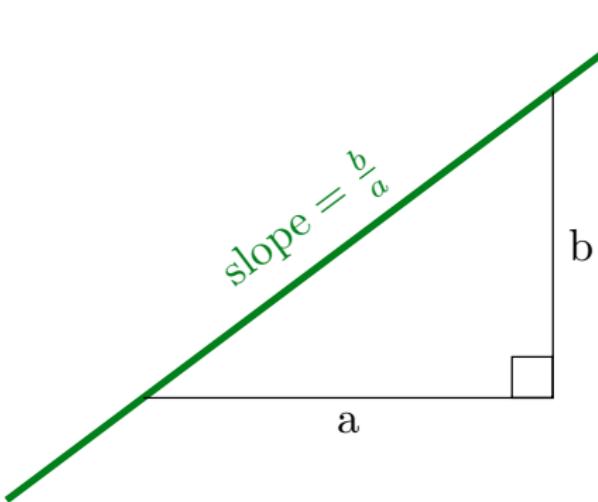
Slopes of Lines



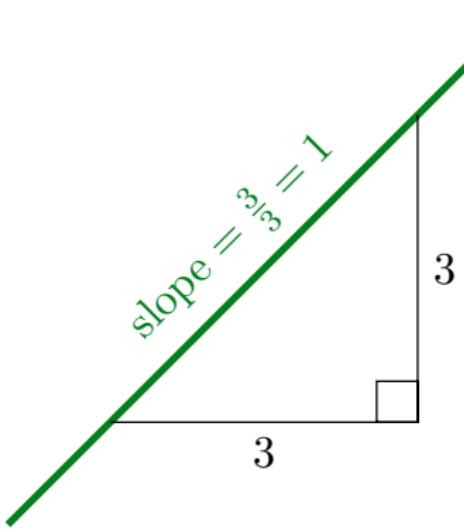
Slopes of Lines



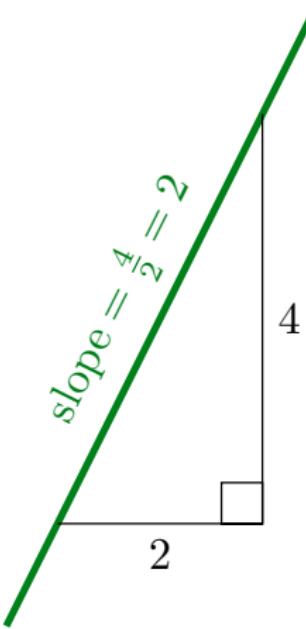
2.1 Rates of Change and Tangent Lines to Curves



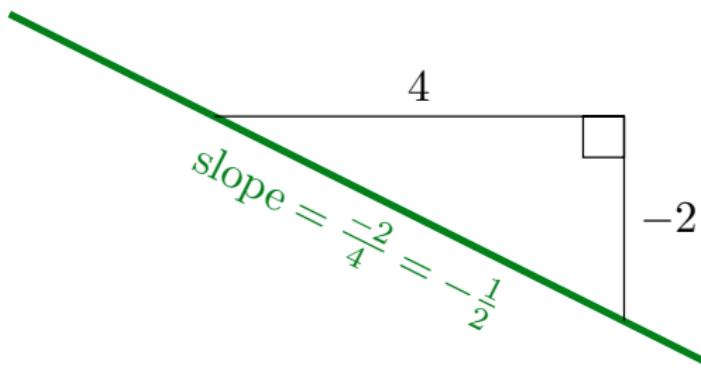
2.1 Rates of Change and Tangent Lines to Curves



2.1 Rates of Change and Tangent Lines to Curves



2.1 Rates of Change and Tangent Lines to Curves



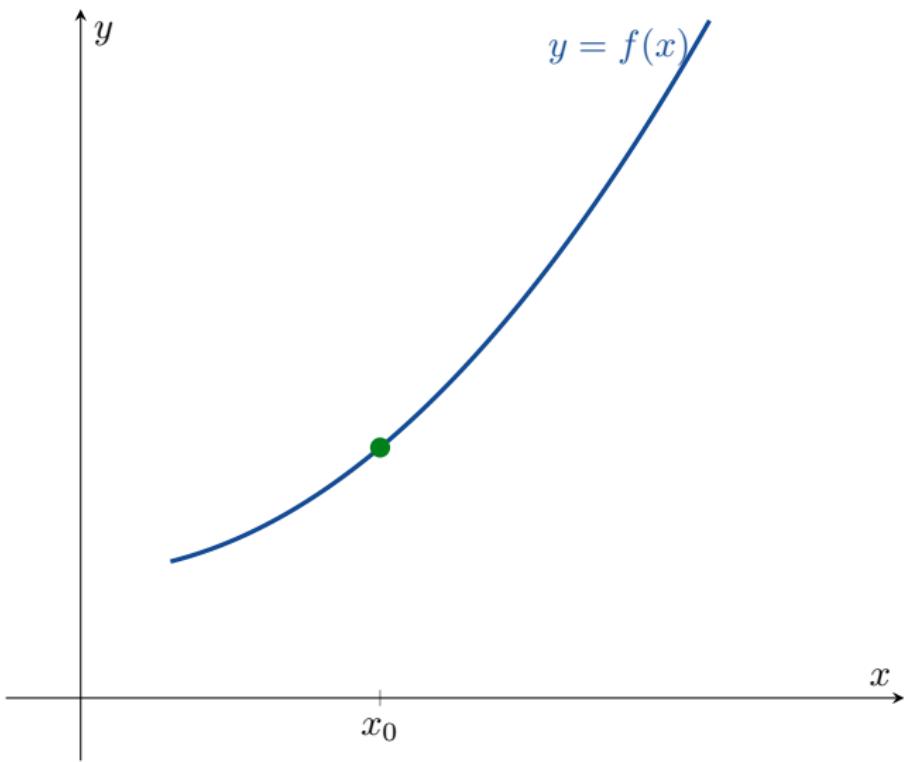
2.1 Rates of Change and Tangent Lines to Curves



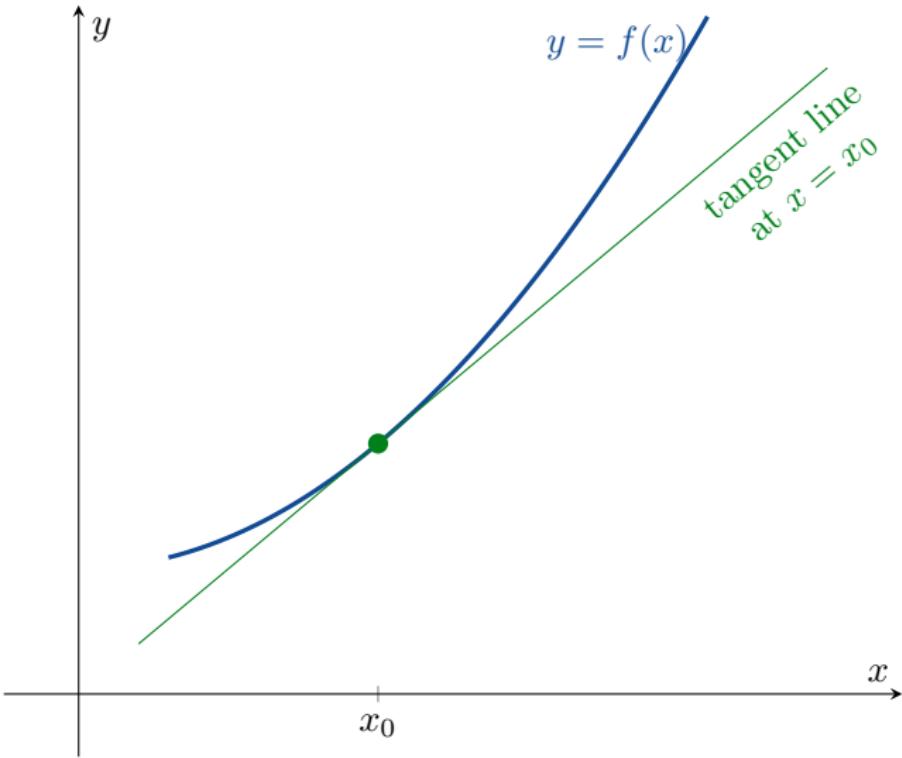
Slopes of Curves

But how can we find the slope of a curve $y = f(x)$ at a point x_0 ?

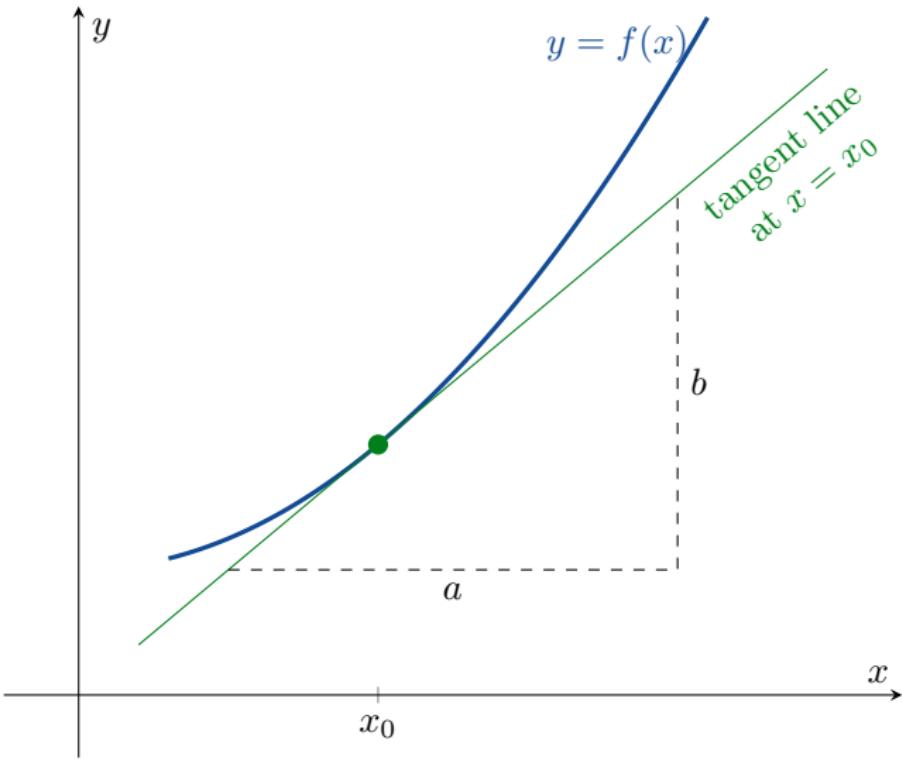
2.1 Rates of Change and Tangent Lines to Curves



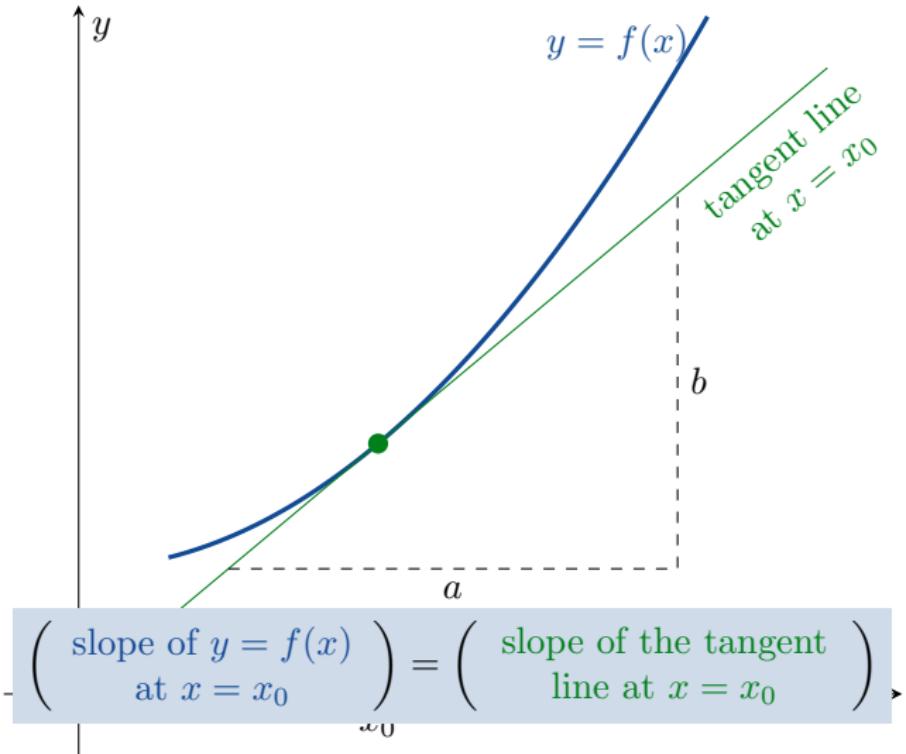
2.1 Rates of Change and Tangent Lines to Curves



2.1 Rates of Change and Tangent Lines to Curves



2.1 Rates of Change and Tangent Lines to Curves



2.1 Rates of Change and Tangent Lines to Curves



We will talk more about the slopes of curves in Lecture 4.



Limit of a Function and Limit Laws

2.2 Limit of a Function and Limit Laws

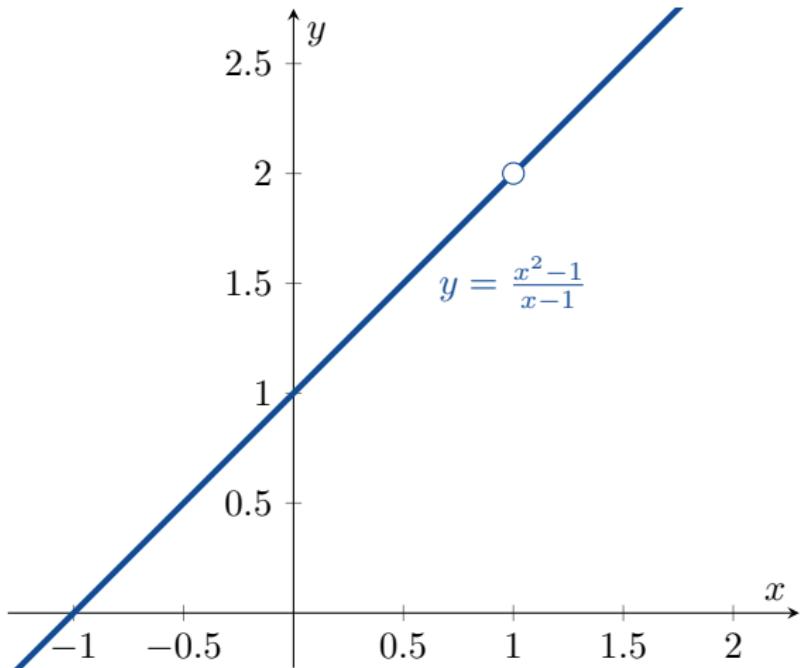


Consider the function $f(x) = \frac{x^2 - 1}{x - 1}$, $f : (-\infty, 1) \cup (1, \infty) \rightarrow \mathbb{R}$.

2.2 Limit of a Function and Limit Laws



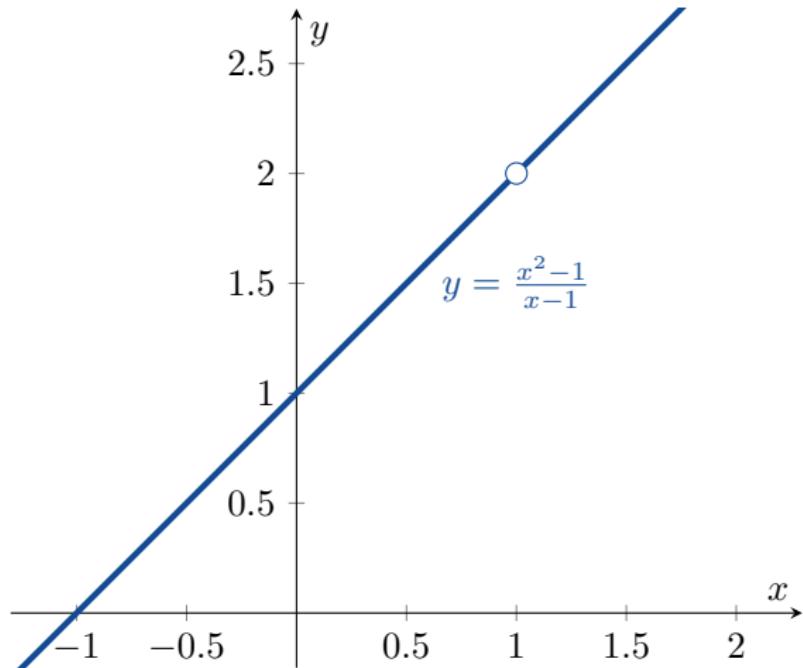
Consider the function $f(x) = \frac{x^2-1}{x-1}$, $f : (-\infty, 1) \cup (1, \infty) \rightarrow \mathbb{R}$.



2.2 Limit of a Function and Limit Laws



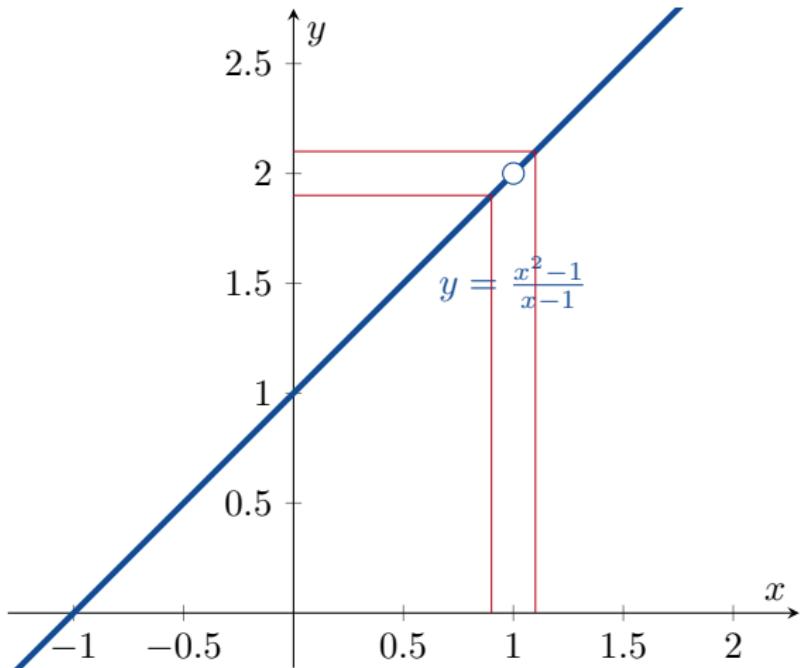
Consider the function $f(x) = \frac{x^2-1}{x-1}$, $f : (-\infty, 1) \cup (1, \infty) \rightarrow \mathbb{R}$.



Question: How does f behave when x is close to 1?

2.2 Limit of a Function and Limit Laws

Consider the function $f(x) = \frac{x^2-1}{x-1}$, $f : (-\infty, 1) \cup (1, \infty) \rightarrow \mathbb{R}$.

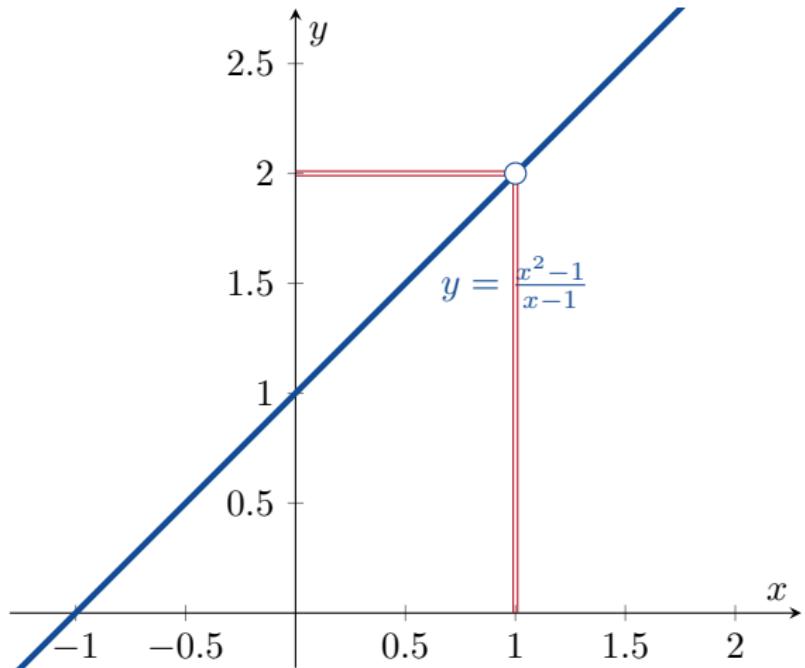


x	$f(x)$
0.9	1.9
1.1	2.1

Question: How does f behave when x is close to 1?

2.2 Limit of a Function and Limit Laws

Consider the function $f(x) = \frac{x^2-1}{x-1}$, $f : (-\infty, 1) \cup (1, \infty) \rightarrow \mathbb{R}$.

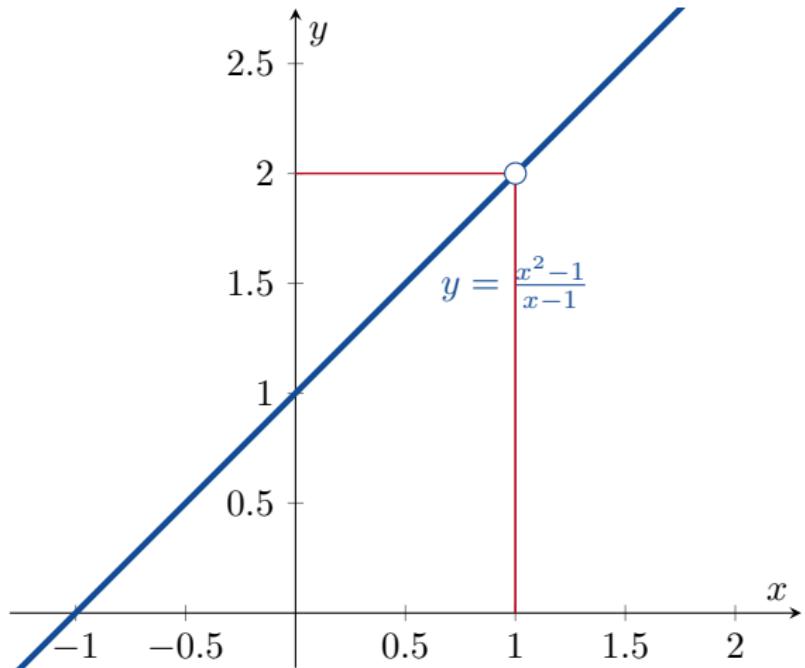


x	$f(x)$
0.9	1.9
1.1	2.1
0.99	1.99
1.01	2.01

Question: How does f behave when x is close to 1?

2.2 Limit of a Function and Limit Laws

Consider the function $f(x) = \frac{x^2-1}{x-1}$, $f : (-\infty, 1) \cup (1, \infty) \rightarrow \mathbb{R}$.

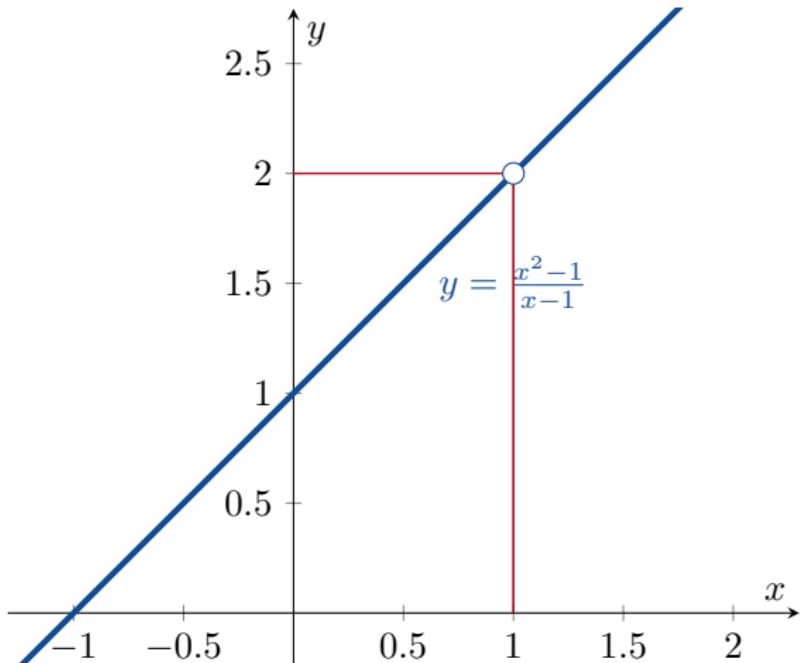


x	$f(x)$
0.9	1.9
1.1	2.1
0.99	1.99
1.01	2.01
0.999	1.999
1.001	2.001

Question: How does f behave when x is close to 1?

2.2 Limit of a Function and Limit Laws

Consider the function $f(x) = \frac{x^2-1}{x-1}$, $f : (-\infty, 1) \cup (1, \infty) \rightarrow \mathbb{R}$.



x	$f(x)$
0.9	1.9
1.1	2.1
0.99	1.99
1.01	2.01
0.999	1.999
1.001	2.001

“If x is close to 1, then $f(x)$ is close to 2.”

2.2 Limit of a Function and Limit Laws



“If x is close to 1, then $f(x)$ is close to 2.”

Mathematically, we write this as

$$\lim_{x \rightarrow 1} f(x) = 2$$

and read it as “the limit, as x tends to 1, of $f(x)$ is equal to 2”.

2.2 Limit of a Function and Limit Laws



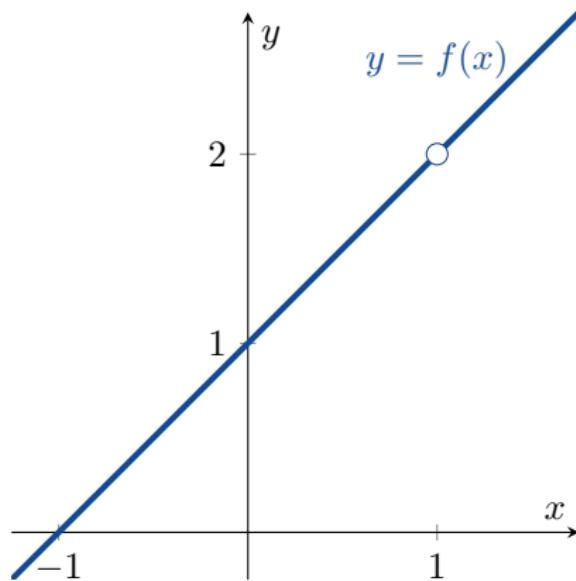
Example

$$f(x) = \frac{x^2 - 1}{x - 1}$$

2.2 Limit of a Function and Limit Laws

Example

$$f(x) = \frac{x^2 - 1}{x - 1}$$



Note that

$$\lim_{x \rightarrow 1} f(x) = 2,$$

but f is not defined at $x = 1$.

2.2 Limit of a Function and Limit Laws

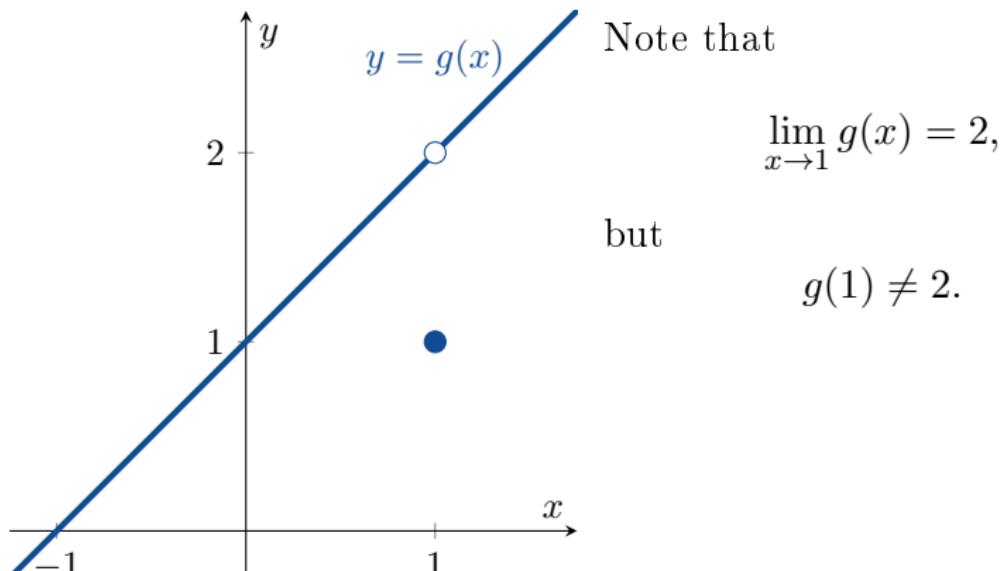
Example

$$g(x) = \begin{cases} \frac{x^2-1}{x-1} & x \neq 1 \\ 1 & x = 1 \end{cases}$$

2.2 Limit of a Function and Limit Laws

Example

$$g(x) = \begin{cases} \frac{x^2-1}{x-1} & x \neq 1 \\ 1 & x = 1 \end{cases}$$



2.2 Limit of a Function and Limit Laws

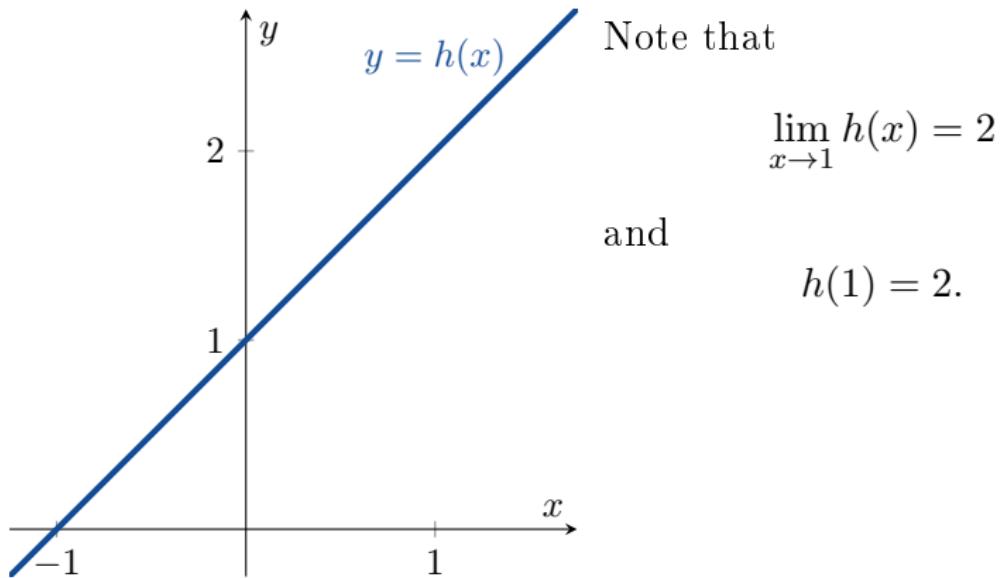
Example

$$h(x) = x + 1$$

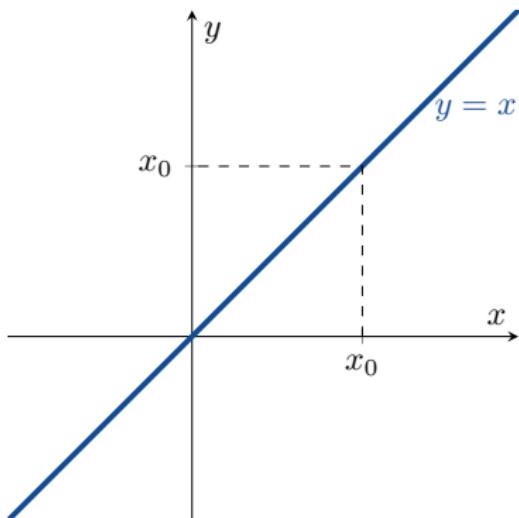
2.2 Limit of a Function and Limit Laws

Example

$$h(x) = x + 1$$



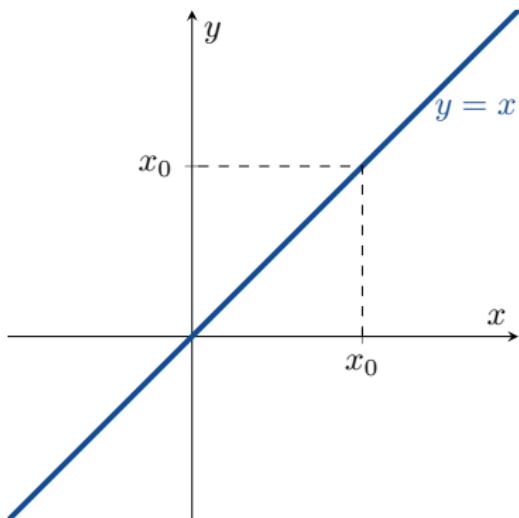
2.2 Limit of a Function and Limit Laws



Example (The Identity Function)

$$f(x) = x$$

2.2 Limit of a Function and Limit Laws

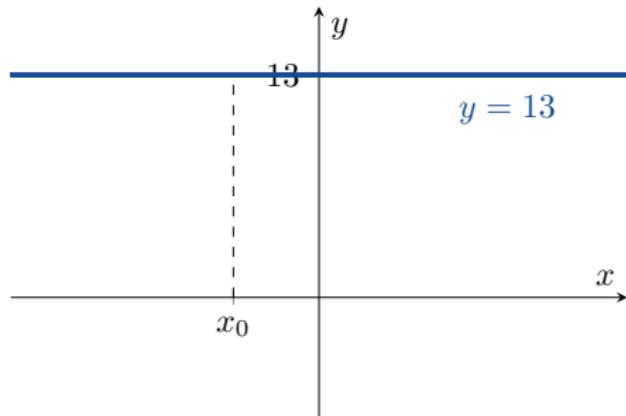


Example (The Identity Function)

$$f(x) = x$$

$$\lim_{x \rightarrow x_0} f(x) = \lim_{x \rightarrow x_0} x = x_0$$

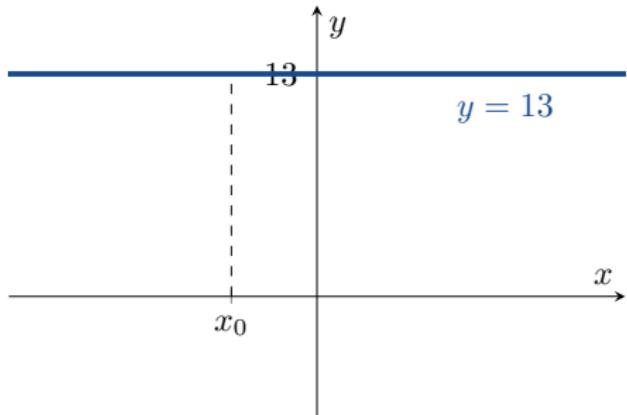
2.2 Limit of a Function and Limit Laws



Example (A Constant Function)

$$f(x) = 13$$

2.2 Limit of a Function and Limit Laws



Example (A Constant Function)

$$f(x) = 13$$

$$\lim_{x \rightarrow x_0} f(x) = \lim_{x \rightarrow x_0} 13 = 13$$

2.2 Limit of a Function and Limit Laws

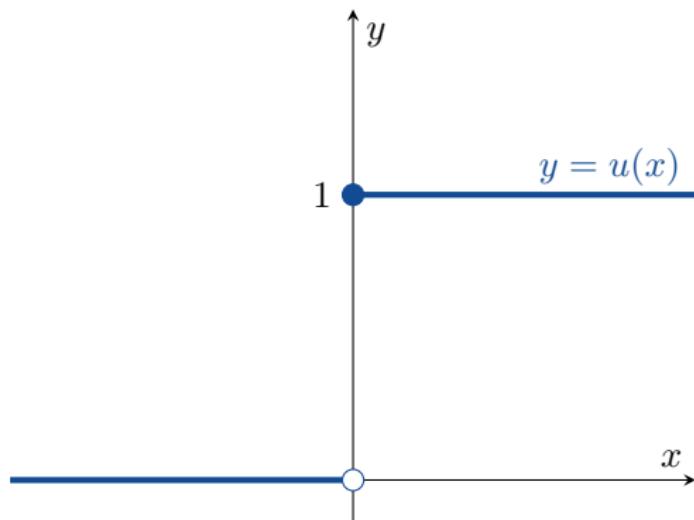


Example (Sometimes Limits Do Not Exist)

Consider the functions

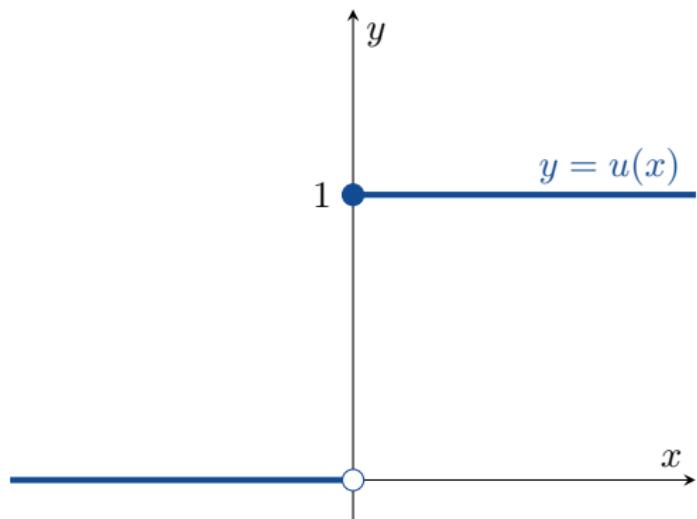
$$u(x) = \begin{cases} 0 & x < 0 \\ 1 & x \geq 0 \end{cases} \quad \text{and} \quad v(x) = \begin{cases} 0 & x \leq 0 \\ \sin \frac{1}{x} & x > 0. \end{cases}$$

2.2 Limit of a Function and Limit Laws



Note that $\lim_{x \rightarrow 0} u(x)$ does not exist.

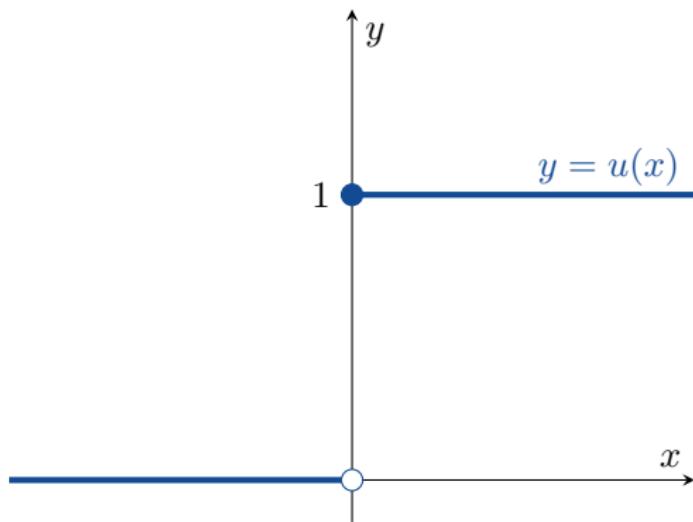
2.2 Limit of a Function and Limit Laws



Note that $\lim_{x \rightarrow 0} u(x)$ does not exist. To understand why, we consider x close to 0:

- If x is close to 0 and $x < 0$, then $u(x) = 0$.
- If x is close to 0 and $x > 0$, then $u(x) = 1$.

2.2 Limit of a Function and Limit Laws

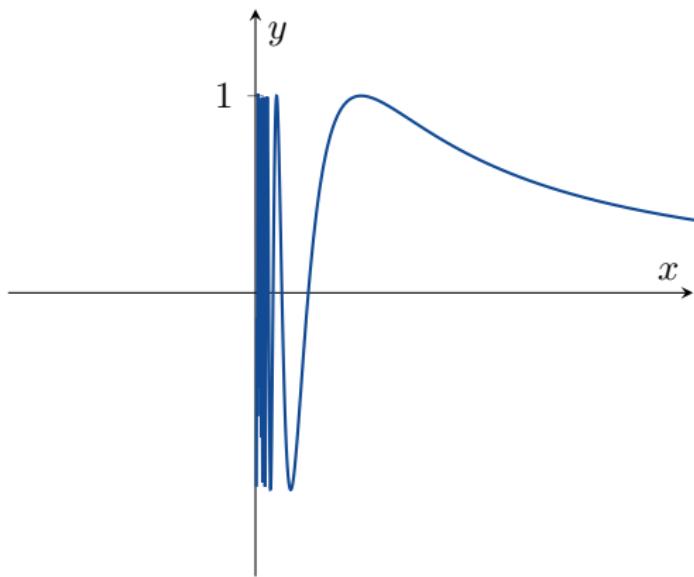


Note that $\lim_{x \rightarrow 0} u(x)$ does not exist. To understand why, we consider x close to 0:

- If x is close to 0 and $x < 0$, then $u(x) = 0$.
- If x is close to 0 and $x > 0$, then $u(x) = 1$.

Because 0 is not close to 1, the limit as $x \rightarrow 0$ can not exist.

2.2 Limit of a Function and Limit Laws



Moreover $\lim_{x \rightarrow 0} v(x)$ does not exist because $v(x)$ oscillates up and down too quickly if $x > 0$ and $x \rightarrow 0$.

2.2 Limit of a Function and Limit Laws



Theorem (The Limit Laws)

Suppose that

- $L, M, c, k \in \mathbb{R}$;
- f and g are functions;
- $\lim_{x \rightarrow c} f(x) = L$; and
- $\lim_{x \rightarrow c} g(x) = M$.

Then

2.2 Limit of a Function and Limit Laws



Theorem (The Limit Laws)

Suppose that

- $L, M, c, k \in \mathbb{R}$;
- f and g are functions;
- $\lim_{x \rightarrow c} f(x) = L$; and
- $\lim_{x \rightarrow c} g(x) = M$.

Then

1 Sum Rule:

$$\lim_{x \rightarrow c} (f(x) + g(x)) = L + M;$$

2.2 Limit of a Function and Limit Laws

Theorem (The Limit Laws)

Suppose that

- $L, M, c, k \in \mathbb{R}$;
- f and g are functions;
- $\lim_{x \rightarrow c} f(x) = L$; and
- $\lim_{x \rightarrow c} g(x) = M$.

Then

2 Difference Rule:

$$\lim_{x \rightarrow c} (f(x) - g(x)) = L - M;$$

2.2 Limit of a Function and Limit Laws



Theorem (The Limit Laws)

Suppose that

- $L, M, c, k \in \mathbb{R}$;
- f and g are functions;
- $\lim_{x \rightarrow c} f(x) = L$; and
- $\lim_{x \rightarrow c} g(x) = M$.

Then

- 3 Constant Multiple Rule:

$$\lim_{x \rightarrow c} (kf(x)) = kL;$$

2.2 Limit of a Function and Limit Laws



Theorem (The Limit Laws)

Suppose that

- $L, M, c, k \in \mathbb{R}$;
- f and g are functions;
- $\lim_{x \rightarrow c} f(x) = L$; and
- $\lim_{x \rightarrow c} g(x) = M$.

Then

- 4 Product Rule:

$$\lim_{x \rightarrow c} (f(x)g(x)) = LM;$$

2.2 Limit of a Function and Limit Laws



Theorem (The Limit Laws)

Suppose that

- $L, M, c, k \in \mathbb{R}$;
- f and g are functions;
- $\lim_{x \rightarrow c} f(x) = L$; and
- $\lim_{x \rightarrow c} g(x) = M$.

Then

- 5 Quotient Rule: if $M \neq 0$, then

$$\lim_{x \rightarrow c} \left(\frac{f(x)}{g(x)} \right) = \frac{L}{M};$$

2.2 Limit of a Function and Limit Laws



Theorem (The Limit Laws)

Suppose that

- $L, M, c, k \in \mathbb{R}$;
- f and g are functions;
- $\lim_{x \rightarrow c} f(x) = L$; and
- $\lim_{x \rightarrow c} g(x) = M$.

Then

- 6 Power Rule: if $n \in \mathbb{N}$, then

$$\lim_{x \rightarrow c} (f(x))^n = L^n;$$

2.2 Limit of a Function and Limit Laws

Theorem (The Limit Laws)

Suppose that

- $L, M, c, k \in \mathbb{R}$;
- f and g are functions;
- $\lim_{x \rightarrow c} f(x) = L$; and
- $\lim_{x \rightarrow c} g(x) = M$.

Then

- 7 Root Rule: if $n \in \mathbb{N}$ and $\sqrt[n]{L}$ exists, then

$$\lim_{x \rightarrow c} \sqrt[n]{f(x)} = \sqrt[n]{L} = L^{\frac{1}{n}}.$$

2.2 Limit of a Function and Limit Laws



Example

$$\text{Find } \lim_{x \rightarrow 2} (x^3 + 4x^2 - 3).$$

2.2 Limit of a Function and Limit Laws



Example

Find $\lim_{x \rightarrow 2} (x^3 + 4x^2 - 3)$.

$$\lim_{x \rightarrow 2} (x^3 + 4x^2 - 3) = (\lim_{x \rightarrow 2} x^3) + (\lim_{x \rightarrow 2} 4x^2) - (\lim_{x \rightarrow 2} 3)$$

(sum and difference rules)

$$= (\lim_{x \rightarrow 2} x)^3 + 4(\lim_{x \rightarrow 2} x)^2 - (\lim_{x \rightarrow 2} 3)$$

(power and constant multiple rules)

$$= 2^3 + 4(2^2) - 3 = 21.$$

2.2 Limit of a Function and Limit Laws



Example

Find $\lim_{x \rightarrow 6} 8(x - 5)(x - 7)$.

$$\lim_{x \rightarrow 6} 8(x - 5)(x - 7) = 8 \lim_{x \rightarrow 6} (x - 5)(x - 7)$$

(constant multiple rule)

$$= 8 \left(\lim_{x \rightarrow 6} (x - 5) \right) \left(\lim_{x \rightarrow 6} (x - 7) \right)$$

(product rule)

$$= 8(1)(-1) = -8.$$

2.2 Limit of a Function and Limit Laws

Example

Find $\lim_{x \rightarrow 5} \frac{x^4 + x^2 - 1}{x^2 + 5}$.

$$\lim_{x \rightarrow 5} \frac{x^4 + x^2 - 1}{x^2 + 5} = \frac{\lim_{x \rightarrow 5}(x^4 + x^2 - 1)}{\lim_{x \rightarrow 5}(x^2 + 5)}$$

(quotient rule)

$$= \frac{\lim_{x \rightarrow 5} x^4 + \lim_{x \rightarrow 5} x^2 - \lim_{x \rightarrow 5} 1}{\lim_{x \rightarrow 5} x^2 + \lim_{x \rightarrow 5} 5}$$

(sum and difference rules)

$$= \frac{5^4 + 5^2 - 1}{5^2 + 5} = \frac{649}{30}.$$

(power rule)

2.2 Limit of a Function and Limit Laws



Example

Find $\lim_{x \rightarrow -5} \frac{x^2 + 3x - 11}{x + 6}$.

$$\lim_{x \rightarrow -5} \frac{x^2 + 3x - 11}{x + 6} = \frac{\lim_{x \rightarrow -5}(x^2 + 3x - 11)}{\lim_{x \rightarrow -5}(x + 6)}$$

(quotient rule)

$$= \frac{\lim_{x \rightarrow -5} x^2 + \lim_{x \rightarrow -5} 3x - \lim_{x \rightarrow -5} 11}{\lim_{x \rightarrow -5} x + \lim_{x \rightarrow -5} 6}$$

(sum and difference rules)

$$= \frac{(-5)^2 - 15 - 11}{-5 + 6} = \frac{-1}{1} = -1.$$

(power rule)

2.2 Limit of a Function and Limit Laws



Is there an easier way?

2.2 Limit of a Function and Limit Laws



Theorem (Limits of Polynomial Functions)

If $P(x) = a_nx^n + a_{n-1}x^{n-1} + \dots + a_1x + a_0$ is a polynomial function, then

$$\lim_{x \rightarrow c} P(x) = P(c).$$

2.2 Limit of a Function and Limit Laws

Theorem (Limits of Polynomial Functions)

If $P(x) = a_nx^n + a_{n-1}x^{n-1} + \dots + a_1x + a_0$ is a polynomial function, then

$$\lim_{x \rightarrow c} P(x) = P(c).$$

Theorem (Limits of Rational Functions)

If $P(x)$ and $Q(x)$ are polynomial functions and if $Q(c) \neq 0$, then

$$\lim_{x \rightarrow c} \frac{P(x)}{Q(x)} = \frac{P(c)}{Q(c)}.$$

2.2 Limit of a Function and Limit Laws



Example

$$\lim_{x \rightarrow -1} \frac{x^3 + 4x^2 - 3}{x^2 + 5} = \frac{(-1)^3 + 4(-1)^2 - 3}{(-1)^2 + 5} = \frac{0}{6} = 0.$$

Example

$$\lim_{x \rightarrow 2} \frac{x^3 + 4x^2 - 3}{x^2 + 5} = \frac{(2)^3 + 4(2)^2 - 3}{(2)^2 + 5} = \frac{8 + 16 - 3}{4 + 5} = \frac{21}{9} = \frac{7}{3}.$$

2.2 Limit of a Function and Limit Laws



Eliminating Zero Denominators Algebraically

$$\lim_{x \rightarrow c} \frac{P(x)}{Q(x)}$$

What can we do if $Q(c) = 0$?

2.2 Limit of a Function and Limit Laws

Example

Find

$$\lim_{x \rightarrow 1} \frac{x^2 + x - 2}{x^2 - x}.$$

2.2 Limit of a Function and Limit Laws



Example

Find

$$\lim_{x \rightarrow 1} \frac{x^2 + x - 2}{x^2 - x}.$$

If we just put in $x = 1$, we would get " $\frac{0}{0}$ " and we never never never want " $\frac{0}{0}$ ". So what can we do?

2.2 Limit of a Function and Limit Laws

Example

Find

$$\lim_{x \rightarrow 1} \frac{x^2 + x - 2}{x^2 - x}.$$

If we just put in $x = 1$, we would get “ $\frac{0}{0}$ ” and we never never never want “ $\frac{0}{0}$ ”. So what can we do?

Instead, we try to factor $x^2 + x - 2$ and $x^2 - x$.

2.2 Limit of a Function and Limit Laws



Example

Find

$$\lim_{x \rightarrow 1} \frac{x^2 + x - 2}{x^2 - x}.$$

If we just put in $x = 1$, we would get “ $\frac{0}{0}$ ” and we never never never want “ $\frac{0}{0}$ ”. So what can we do?

Instead, we try to factor $x^2 + x - 2$ and $x^2 - x$. If $x \neq 1$, we have that

$$\frac{x^2 + x - 2}{x^2 - x} = \frac{(x - 1)(x + 2)}{x(x - 1)} = \frac{x + 2}{x}.$$

2.2 Limit of a Function and Limit Laws



Example

Find

$$\lim_{x \rightarrow 1} \frac{x^2 + x - 2}{x^2 - x}.$$

If we just put in $x = 1$, we would get “ $\frac{0}{0}$ ” and we never never never want “ $\frac{0}{0}$ ”. So what can we do?

Instead, we try to factor $x^2 + x - 2$ and $x^2 - x$. If $x \neq 1$, we have that

$$\frac{x^2 + x - 2}{x^2 - x} = \frac{(x - 1)(x + 2)}{x(x - 1)} = \frac{x + 2}{x}.$$

So

$$\lim_{x \rightarrow 1} \frac{x^2 + x - 2}{x^2 - x} = \lim_{x \rightarrow 1} \frac{x + 2}{x} = \frac{1 + 2}{1} = 3.$$

2.2 Limit of a Function and Limit Laws



Example

Find

$$\lim_{x \rightarrow -5} \frac{x^2 + 3x - 10}{x^2 + 5x}.$$

2.2 Limit of a Function and Limit Laws

Example

Find

$$\lim_{x \rightarrow -5} \frac{x^2 + 3x - 10}{x^2 + 5x}.$$

We must try to factor $x^2 + 3x - 10$ and $x^2 + 5x$. If $x \neq -5$, we have that

$$\frac{x^2 + 3x - 10}{x^2 + 5x} = \frac{(x+5)(x-2)}{x(x+5)} = \frac{x-2}{x}.$$

So

$$\lim_{x \rightarrow -5} \frac{x^2 + 3x - 10}{x^2 + 5x} = \lim_{x \rightarrow -5} \frac{x-2}{x} = \frac{-5-2}{-5} = \frac{7}{5}.$$

2.2 Limit of a Function and Limit Laws



Example

Find

$$\lim_{x \rightarrow 0} \frac{\sqrt{x^2 + 100} - 10}{x^2}.$$

This is another “ $\frac{0}{0}$ ” limit.

2.2 Limit of a Function and Limit Laws



Example

Find

$$\lim_{x \rightarrow 0} \frac{\sqrt{x^2 + 100} - 10}{x^2}.$$

This is another “ $\frac{0}{0}$ ” limit.

There is a trick we can use if we have $A - B$ in a limit: We multiply by $(A + B)$ because $(A - B)(A + B) = A^2 - B^2$.

2.2 Limit of a Function and Limit Laws



Example

Find

$$\lim_{x \rightarrow 0} \frac{\sqrt{x^2 + 100} - 10}{x^2}.$$

This is another “ $\frac{0}{0}$ ” limit.

There is a trick we can use if we have $A - B$ in a limit: We multiply by $(A + B)$ because $(A - B)(A + B) = A^2 - B^2$.

So we multiply top and bottom by $(\sqrt{x^2 + 100} + 10)$.

2.2 Limit of a Function and Limit Laws



$$\lim_{x \rightarrow 0} \frac{\sqrt{x^2 + 100} - 10}{x^2} = \lim_{x \rightarrow 0} \frac{(\sqrt{x^2 + 100} - 10)(\sqrt{x^2 + 100} + 10)}{x^2(\sqrt{x^2 + 100} + 10)}$$

=

=

=

=

2.2 Limit of a Function and Limit Laws



$$\begin{aligned}\lim_{x \rightarrow 0} \frac{\sqrt{x^2 + 100} - 10}{x^2} &= \lim_{x \rightarrow 0} \frac{(\sqrt{x^2 + 100} - 10)(\sqrt{x^2 + 100} + 10)}{x^2(\sqrt{x^2 + 100} + 10)} \\ &= \lim_{x \rightarrow 0} \frac{x^2 + 100 - 100}{x^2(\sqrt{x^2 + 100} + 10)}\end{aligned}$$

=

=

=

2.2 Limit of a Function and Limit Laws



$$\begin{aligned}\lim_{x \rightarrow 0} \frac{\sqrt{x^2 + 100} - 10}{x^2} &= \lim_{x \rightarrow 0} \frac{(\sqrt{x^2 + 100} - 10)(\sqrt{x^2 + 100} + 10)}{x^2(\sqrt{x^2 + 100} + 10)} \\&= \lim_{x \rightarrow 0} \frac{x^2 + 100 - 100}{x^2(\sqrt{x^2 + 100} + 10)} \\&= \lim_{x \rightarrow 0} \frac{x^2}{x^2(\sqrt{x^2 + 100} + 10)} \\&= \\&= \end{aligned}$$

2.2 Limit of a Function and Limit Laws



$$\begin{aligned}\lim_{x \rightarrow 0} \frac{\sqrt{x^2 + 100} - 10}{x^2} &= \lim_{x \rightarrow 0} \frac{(\sqrt{x^2 + 100} - 10)(\sqrt{x^2 + 100} + 10)}{x^2(\sqrt{x^2 + 100} + 10)} \\&= \lim_{x \rightarrow 0} \frac{x^2 + 100 - 100}{x^2(\sqrt{x^2 + 100} + 10)} \\&= \lim_{x \rightarrow 0} \frac{x^2}{x^2(\sqrt{x^2 + 100} + 10)} \\&= \frac{1}{\sqrt{x^2 + 100} + 10} \\&= \frac{1}{\sqrt{0^2 + 100} + 10} \\&= \frac{1}{10 + 10} \\&= \frac{1}{20}\end{aligned}$$

2.2 Limit of a Function and Limit Laws



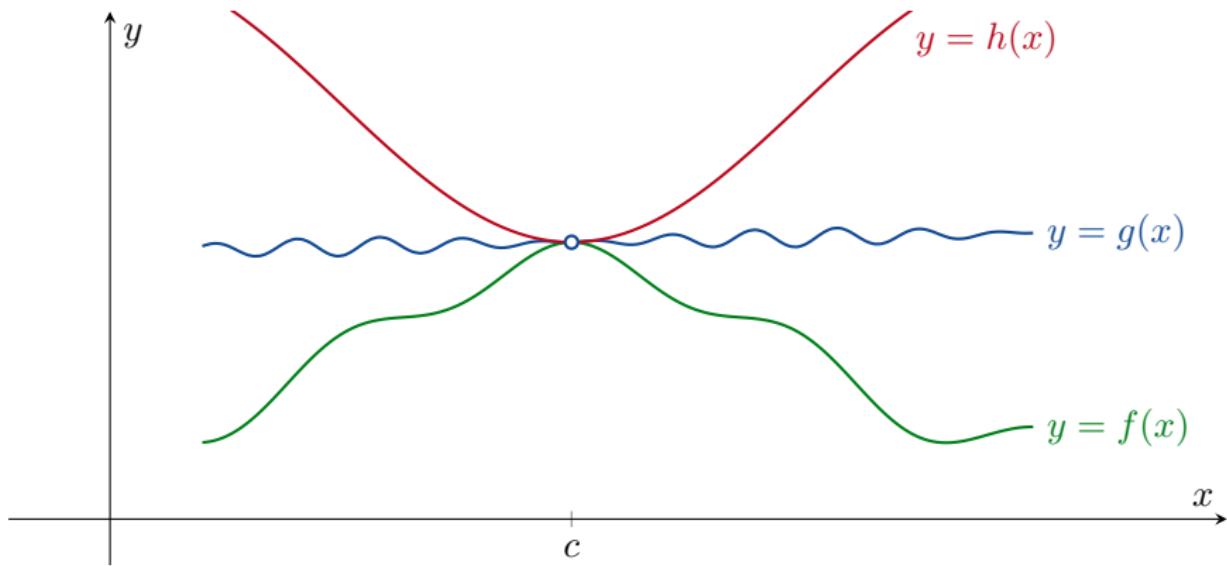
$$\begin{aligned}\lim_{x \rightarrow 0} \frac{\sqrt{x^2 + 100} - 10}{x^2} &= \lim_{x \rightarrow 0} \frac{(\sqrt{x^2 + 100} - 10)(\sqrt{x^2 + 100} + 10)}{x^2(\sqrt{x^2 + 100} + 10)} \\&= \lim_{x \rightarrow 0} \frac{x^2 + 100 - 100}{x^2(\sqrt{x^2 + 100} + 10)} \\&= \lim_{x \rightarrow 0} \frac{x^2}{x^2(\sqrt{x^2 + 100} + 10)} \quad \cancel{x^2} \\&= \lim_{x \rightarrow 0} \frac{1}{\sqrt{x^2 + 100} + 10} \\&= \end{aligned}$$

2.2 Limit of a Function and Limit Laws



$$\begin{aligned}\lim_{x \rightarrow 0} \frac{\sqrt{x^2 + 100} - 10}{x^2} &= \lim_{x \rightarrow 0} \frac{(\sqrt{x^2 + 100} - 10)(\sqrt{x^2 + 100} + 10)}{x^2(\sqrt{x^2 + 100} + 10)} \\&= \lim_{x \rightarrow 0} \frac{x^2 + 100 - 100}{x^2(\sqrt{x^2 + 100} + 10)} \\&= \lim_{x \rightarrow 0} \frac{x^2}{x^2(\sqrt{x^2 + 100} + 10)} \quad \cancel{x^2} \\&= \lim_{x \rightarrow 0} \frac{1}{\sqrt{x^2 + 100} + 10} \\&= \frac{1}{\sqrt{0^2 + 100} + 10} = \frac{1}{20}.\end{aligned}$$

The Sandwich Theorem



2.2 Limit of a Function and Limit Laws



Theorem (The Sandwich Theorem)

Suppose that

- $f(x) \leq g(x) \leq h(x)$ for all x “close” to c ($x \neq c$); and
- $\lim_{x \rightarrow c} f(x) = \lim_{x \rightarrow c} h(x) = L$.

2.2 Limit of a Function and Limit Laws



Theorem (The Sandwich Theorem)

Suppose that

- $f(x) \leq g(x) \leq h(x)$ for all x “close” to c ($x \neq c$); and
- $\lim_{x \rightarrow c} f(x) = \lim_{x \rightarrow c} h(x) = L$.

Then

$$\lim_{x \rightarrow c} g(x) = L$$

also.

2.2 Limit of a Function and Limit Laws

Example

The inequality

$$1 - \frac{x^2}{6} < \frac{x \sin x}{2 - 2 \cos x} < 1$$

holds for all x close to 0 ($x \neq 0$). Calculate $\lim_{x \rightarrow 0} \frac{x \sin x}{2 - 2 \cos x}$.

2.2 Limit of a Function and Limit Laws



Example

The inequality

$$1 - \frac{x^2}{6} < \frac{x \sin x}{2 - 2 \cos x} < 1$$

holds for all x close to 0 ($x \neq 0$). Calculate $\lim_{x \rightarrow 0} \frac{x \sin x}{2 - 2 \cos x}$.

Since $\lim_{x \rightarrow 0} 1 - \frac{x^2}{6} = 1 - \frac{0}{6} = 1$

2.2 Limit of a Function and Limit Laws



Example

The inequality

$$1 - \frac{x^2}{6} < \frac{x \sin x}{2 - 2 \cos x} < 1$$

holds for all x close to 0 ($x \neq 0$). Calculate $\lim_{x \rightarrow 0} \frac{x \sin x}{2 - 2 \cos x}$.

Since $\lim_{x \rightarrow 0} 1 - \frac{x^2}{6} = 1 - \frac{0}{6} = 1$ and $\lim_{x \rightarrow 0} 1 = 1$,

2.2 Limit of a Function and Limit Laws



Example

The inequality

$$1 - \frac{x^2}{6} < \frac{x \sin x}{2 - 2 \cos x} < 1$$

holds for all x close to 0 ($x \neq 0$). Calculate $\lim_{x \rightarrow 0} \frac{x \sin x}{2 - 2 \cos x}$.

Since $\lim_{x \rightarrow 0} 1 - \frac{x^2}{6} = 1 - \frac{0}{6} = 1$ and $\lim_{x \rightarrow 0} 1 = 1$, it follows by the Sandwich Theorem that

$$\lim_{x \rightarrow 0} \frac{x \sin x}{2 - 2 \cos x} = 1$$

also.

2.2 Limit of a Function and Limit Laws



2 important limits

Remember that last lecture we found that

$$-|\theta| \leq \sin \theta \leq |\theta| \quad \text{and} \quad -|\theta| \leq 1 - \cos \theta \leq |\theta| .$$

2.2 Limit of a Function and Limit Laws



2 important limits

Remember that last lecture we found that

$$-|\theta| \leq \sin \theta \leq |\theta| \quad \text{and} \quad -|\theta| \leq 1 - \cos \theta \leq |\theta|.$$

But

$$\lim_{\theta \rightarrow 0} -|\theta| = 0 \quad \text{and} \quad \lim_{\theta \rightarrow 0} |\theta| = 0.$$

2.2 Limit of a Function and Limit Laws



2 important limits

Remember that last lecture we found that

$$-|\theta| \leq \sin \theta \leq |\theta| \quad \text{and} \quad -|\theta| \leq 1 - \cos \theta \leq |\theta|.$$

But

$$\lim_{\theta \rightarrow 0} -|\theta| = 0 \quad \text{and} \quad \lim_{\theta \rightarrow 0} |\theta| = 0.$$

So it follows by the Sandwich Theorem that

$$\lim_{\theta \rightarrow 0} \sin \theta = 0 \quad \text{and} \quad \lim_{\theta \rightarrow 0} (1 - \cos \theta) = 0$$

2.2 Limit of a Function and Limit Laws



2 important limits

Remember that last lecture we found that

$$-|\theta| \leq \sin \theta \leq |\theta| \quad \text{and} \quad -|\theta| \leq 1 - \cos \theta \leq |\theta|.$$

But

$$\lim_{\theta \rightarrow 0} -|\theta| = 0 \quad \text{and} \quad \lim_{\theta \rightarrow 0} |\theta| = 0.$$

So it follows by the Sandwich Theorem that

$$\lim_{\theta \rightarrow 0} \sin \theta = 0 \quad \text{and} \quad \lim_{\theta \rightarrow 0} (1 - \cos \theta) = 0$$

and hence that

$$\lim_{\theta \rightarrow 0} \cos \theta = 1.$$

2.2 Limit of a Function and Limit Laws



Theorem

$$\lim_{\theta \rightarrow 0} \sin \theta = 0 \quad \text{and} \quad \lim_{\theta \rightarrow 0} \cos \theta = 1.$$

We will need these later in the course.

2.2 Limit of a Function and Limit Laws



Theorem

If

- $f(x) \leq g(x)$ for all x close to c ($x \neq c$);
- $\lim_{x \rightarrow c} f(x)$ exists; and
- $\lim_{x \rightarrow c} g(x)$ exists,

then

$$\lim_{x \rightarrow c} f(x) \leq \lim_{x \rightarrow c} g(x).$$

2.2 Limit of a Function and Limit Laws



Remark

Warning:

$$f(x) < g(x) \quad \Rightarrow \quad \lim_{x \rightarrow c} f(x) < \lim_{x \rightarrow c} g(x).$$

2.2 Limit of a Function and Limit Laws



Remark

Warning:

$$f(x) < g(x) \quad \Rightarrow \quad \lim_{x \rightarrow c} f(x) < \lim_{x \rightarrow c} g(x).$$

The actually result is

$$f(x) < g(x) \quad \Rightarrow \quad \lim_{x \rightarrow c} f(x) \leq \lim_{x \rightarrow c} g(x).$$

Break

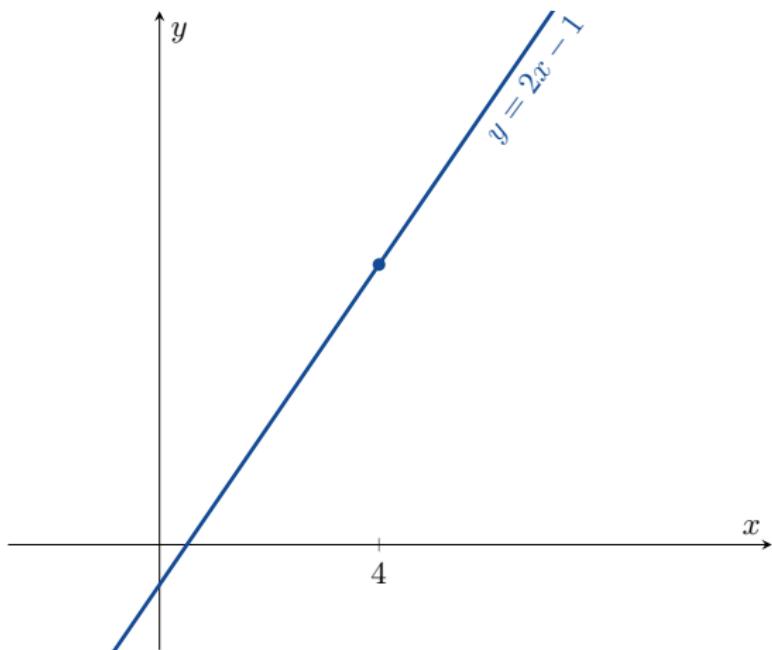
We will continue at 3pm





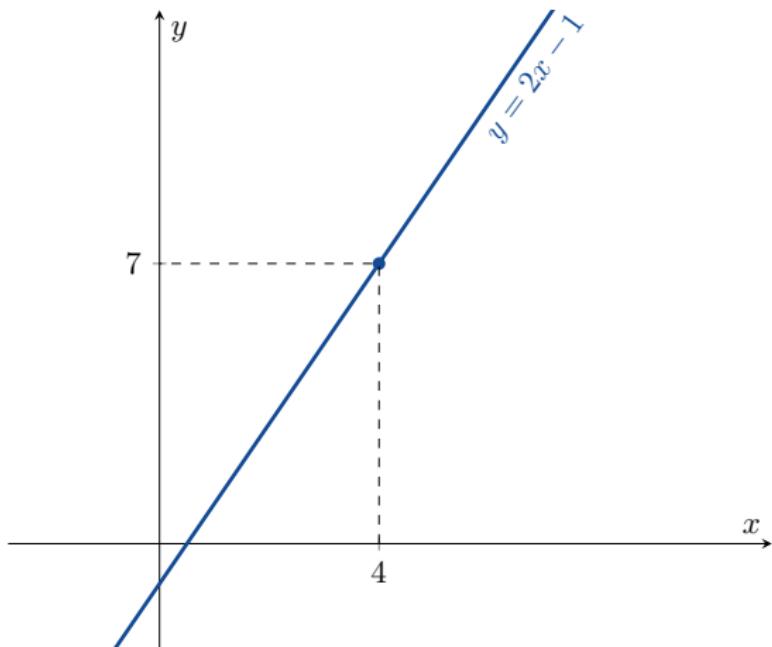
The Precise Definition of a Limit

2.3 The Precise Definition of a Limit



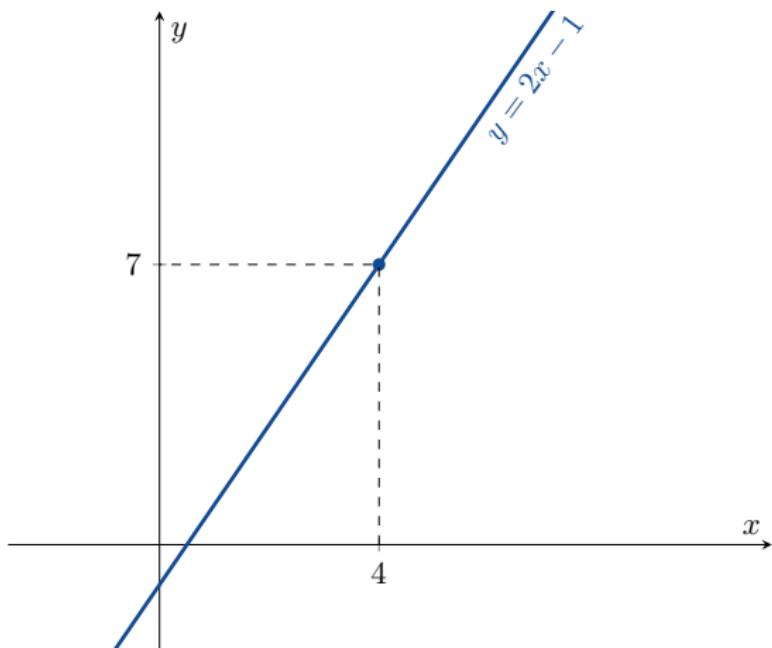
Consider the function $y = 2x - 1$ close to $x = 4$.

2.3 The Precise Definition of a Limit



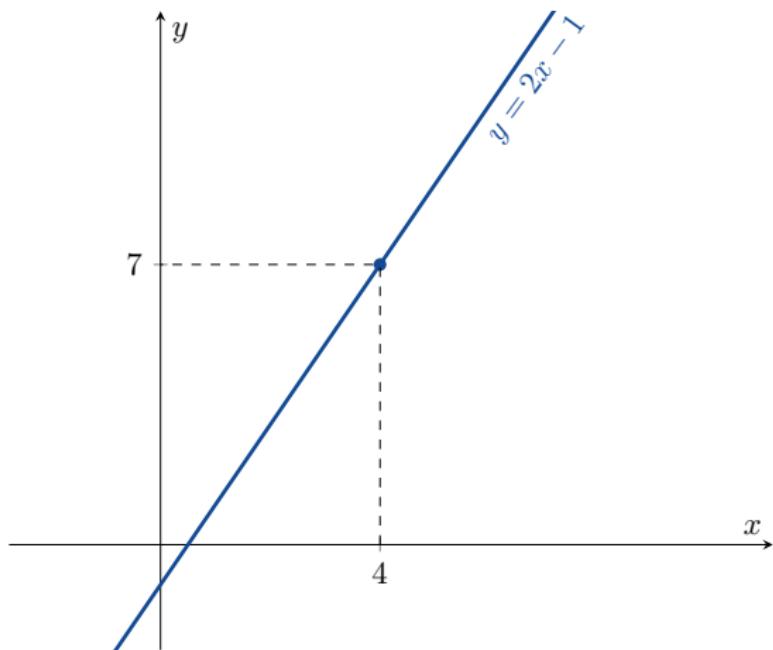
We think that if x is “close to 4” (but $x \neq 4$), then y is “close to 7”.

2.3 The Precise Definition of a Limit



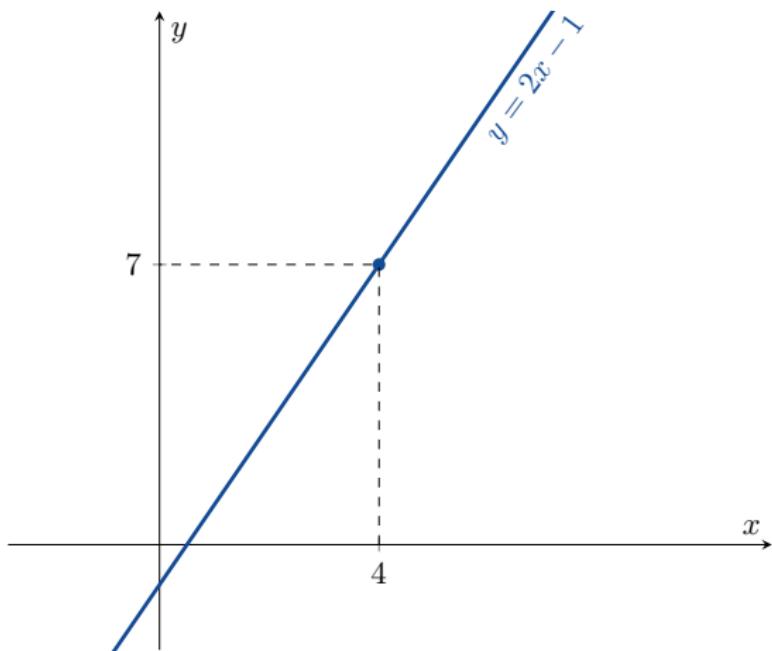
We think that if x is “close to 4” (but $x \neq 4$), then y is “close to 7”. So we think that $\lim_{x \rightarrow 4} (2x - 1) = 7$.

2.3 The Precise Definition of a Limit



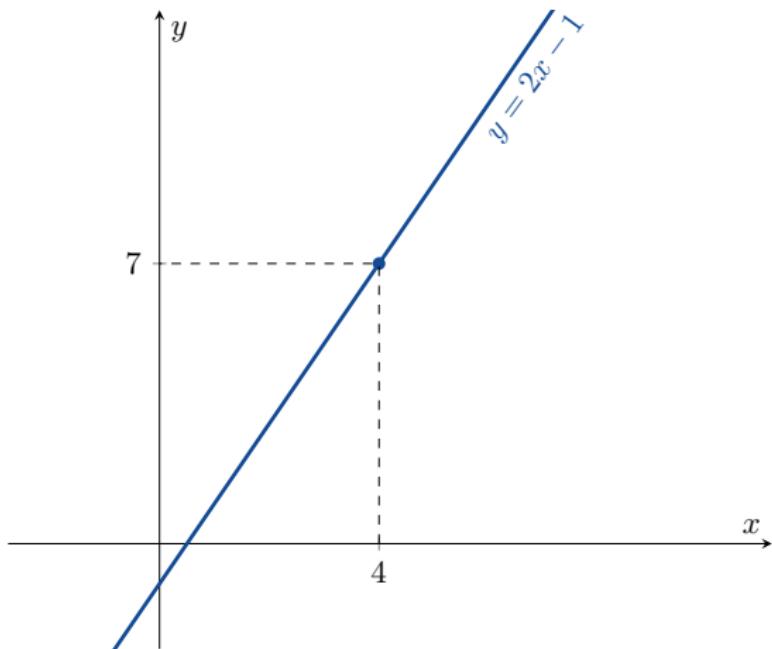
But what does this really mean? How can we make “close to” precise?

2.3 The Precise Definition of a Limit



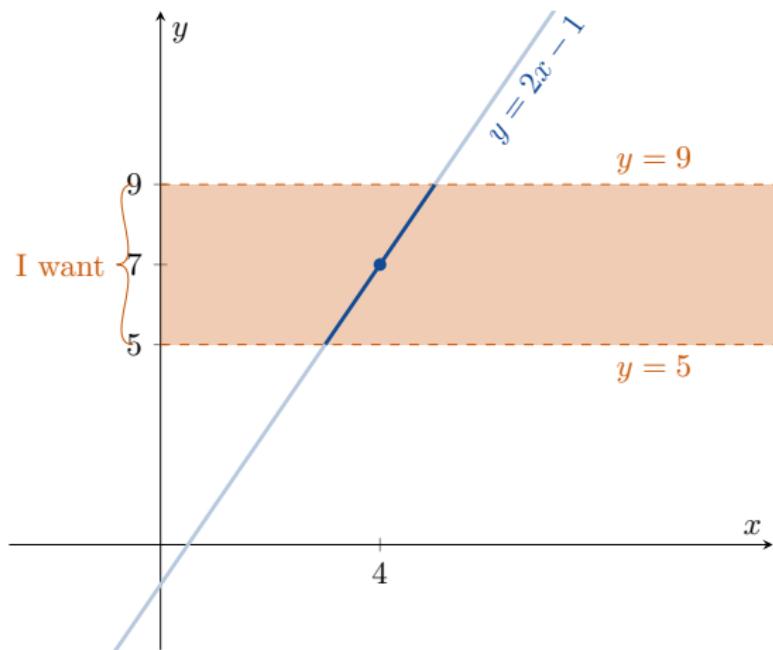
How close to 4 does x need to be to make y close to 7?

2.3 The Precise Definition of a Limit



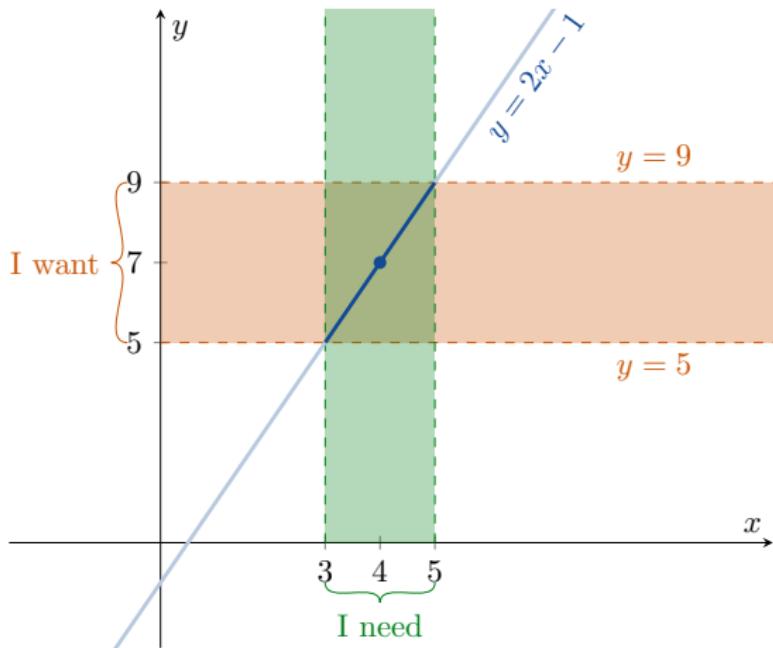
How close to 4 does x need to be to make $|y - 7| < 2$?

2.3 The Precise Definition of a Limit



How close to 4 does x need to be to make $|y - 7| < 2$?

2.3 The Precise Definition of a Limit



If I want $|y - 7| < 2$, then I need to have $3 < x < 5$.

2.3 The Precise Definition

$$y = 2x - 1 \quad \text{close to } x = 4$$



I want $|y - 7| < 2$

2.3 The Precise Definition

$$y = 2x - 1 \quad \text{close to } x = 4$$



I want $|y - 7| < 2$

$$-2 < y - 7 < 2$$

2.3 The Precise Definition

$$y = 2x - 1 \quad \text{close to } x = 4$$



I want $|y - 7| < 2$

$$-2 < y - 7 < 2$$

$$7 - 2 < y < 7 + 2$$

2.3 The Precise Definition

$$y = 2x - 1 \quad \text{close to } x = 4$$



I want $|y - 7| < 2$

$$-2 < y - 7 < 2$$

$$7 - 2 < y < 7 + 2$$

$$5 < y < 9$$

2.3 The Precise Definition

$$y = 2x - 1 \quad \text{close to } x = 4$$



I want $|y - 7| < 2$

$$-2 < y - 7 < 2$$

$$7 - 2 < y < 7 + 2$$

$$5 < y < 9$$

$$5 < 2x - 1 < 9$$

2.3 The Precise Definition

$$y = 2x - 1 \quad \text{close to } x = 4$$



I want $|y - 7| < 2$

$$-2 < y - 7 < 2$$

$$7 - 2 < y < 7 + 2$$

$$5 < y < 9$$

$$5 < 2x - 1 < 9$$

$$6 < 2x < 10$$

2.3 The Precise Definition

$$y = 2x - 1 \quad \text{close to } x = 4$$



I want $|y - 7| < 2$

$$-2 < y - 7 < 2$$

$$7 - 2 < y < 7 + 2$$

$$5 < y < 9$$

$$5 < 2x - 1 < 9$$

$$6 < 2x < 10$$

$$3 < x < 5$$

2.3 The Precise Definition

$$y = 2x - 1 \quad \text{close to } x = 4$$



I want $|y - 7| < 2$

$$-2 < y - 7 < 2$$

$$7 - 2 < y < 7 + 2$$

$$5 < y < 9$$

$$5 < 2x - 1 < 9$$

$$6 < 2x < 10$$

$$3 < x < 5$$

$$3 - 4 < x - 4 < 5 - 4$$

$$-1 < x - 4 < 1$$

2.3 The Precise Definition

$$y = 2x - 1$$

close to $x = 4$



I want $|y - 7| < 2$

$$-2 < y - 7 < 2$$

$$7 - 2 < y < 7 + 2$$

$$5 < y < 9$$

$$5 < 2x - 1 < 9$$

$$6 < 2x < 10$$

$$3 < x < 5$$

$$3 - 4 < x - 4 < 5 - 4$$

$$-1 < x - 4 < 1$$

I need $|x - 4| < 1$

2.3 The Precise Definition

$$y = 2x - 1 \quad \text{close to } x = 4$$



I want $|y - 7| < 2$

$$-2 < y - 7 < 2$$

$$7 - 2 < y < 7 + 2$$

$$5 < y < 9$$

$$5 < 2x - 1 < 9$$

$$6 < 2x < 10$$

$$3 < x < 5$$

$$3 - 4 < x - 4 < 5 - 4$$

$$-1 < x - 4 < 1$$

I need $|x - 4| < 1$

We can write this as

$$|x - 4| < 1 \implies |y - 7| < 2.$$

2.3 The Precise Definition

$$y = 2x - 1$$

close to $x = 4$



I want $|y - 7| < 2$

$$-2 < y - 7 < 2$$

$$7 - 2 < y < 7 + 2$$

$$5 < y < 9$$

$$5 < 2x - 1 < 9$$

$$6 < 2x < 10$$

$$3 < x < 5$$

$$3 - 4 < x - 4 < 5 - 4$$

$$-1 < x - 4 < 1$$

I need $|x - 4| < 1$

rough calculation
(do this on scrap paper)

what we want

We can write this as

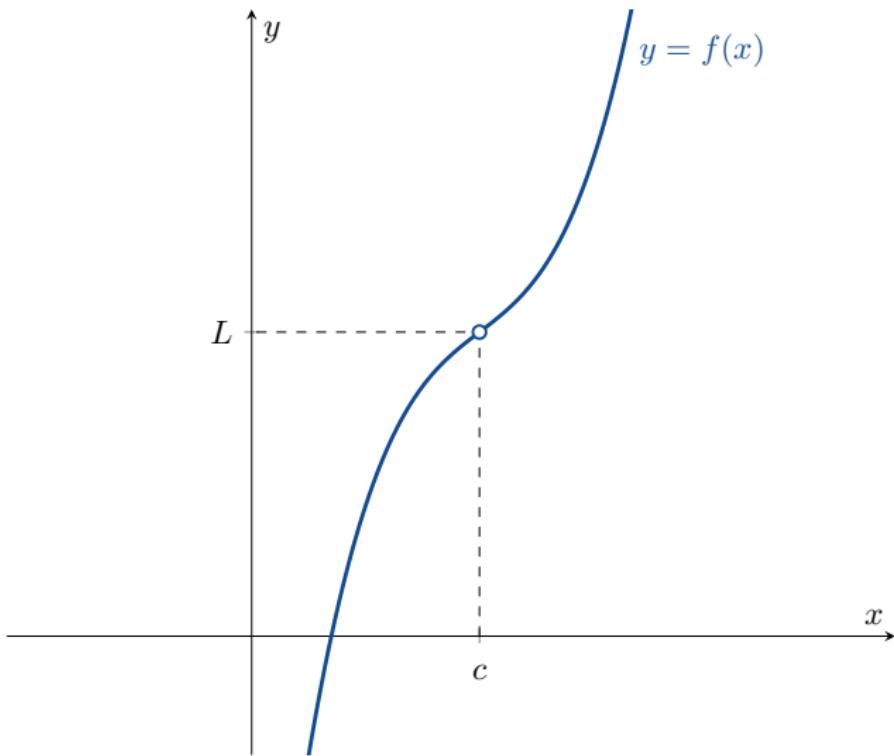
$$|x - 4| < 1 \implies |y - 7| < 2.$$

2.3 The Precise Definition of a Limit

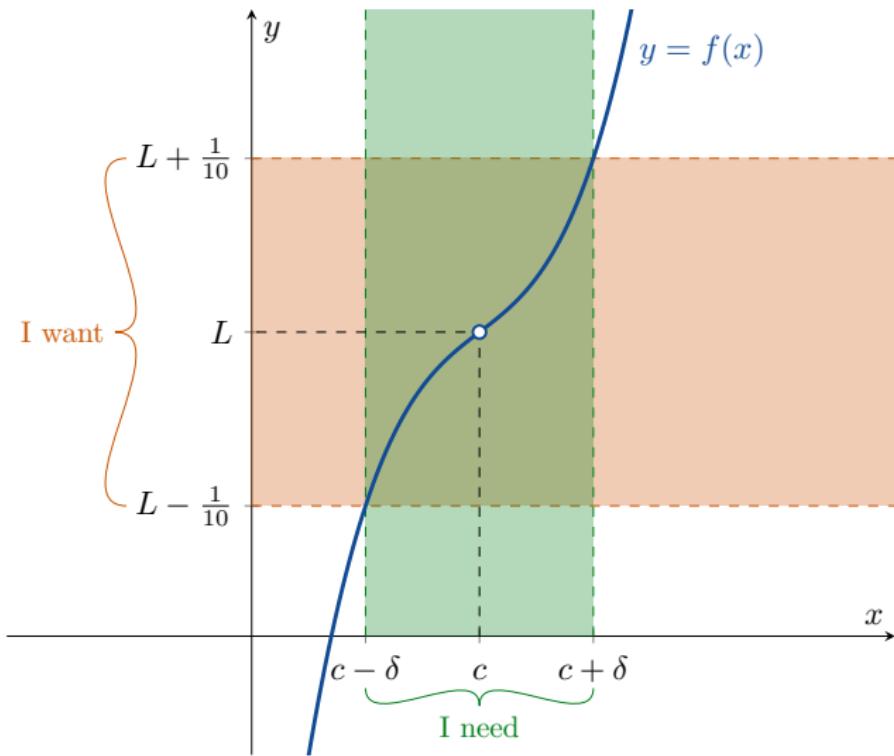


This is the idea that we will use to precisely define what a limit is.

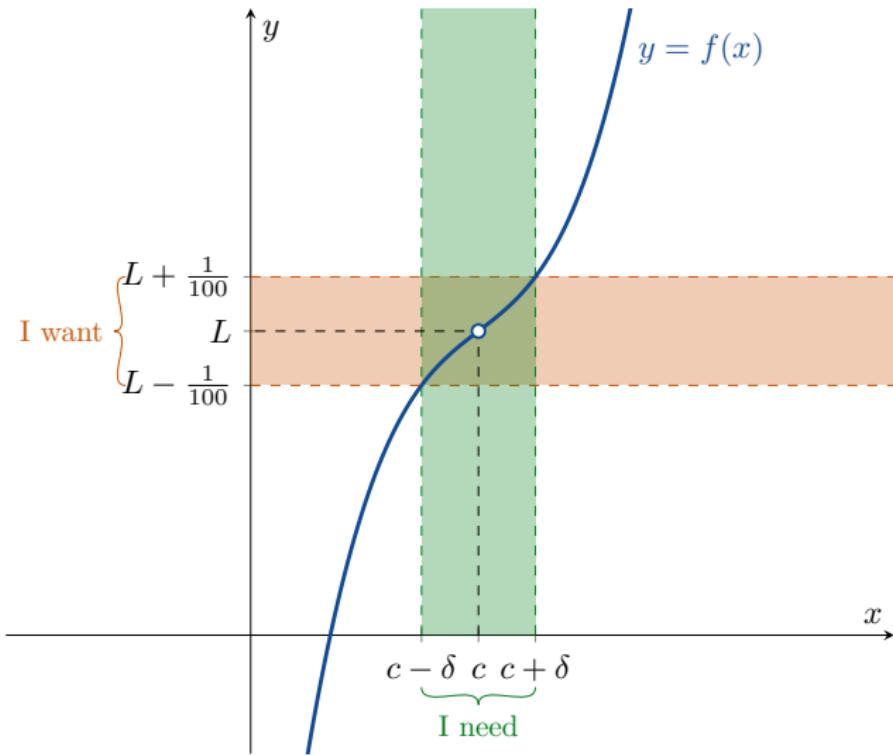
2.3 The Precise Definition of a Limit



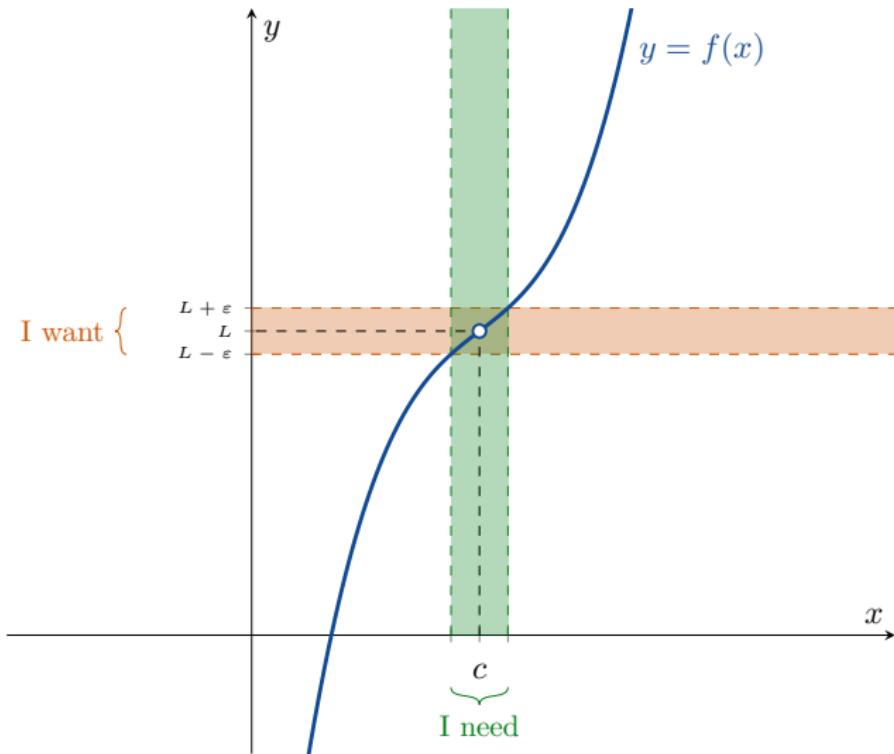
2.3 The Precise Definition of a Limit



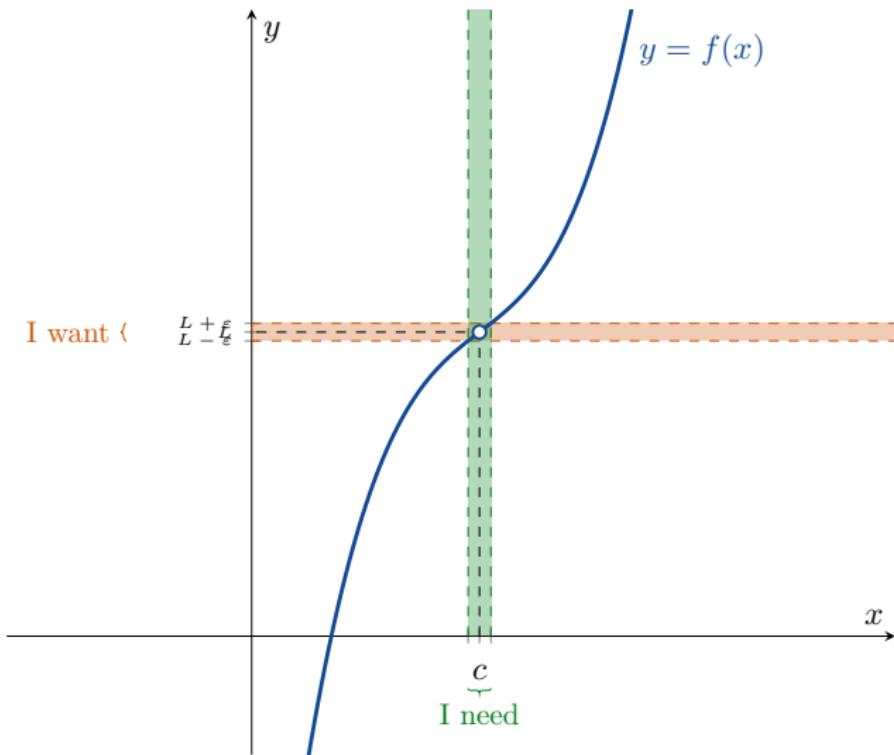
2.3 The Precise Definition of a Limit



2.3 The Precise Definition of a Limit



2.3 The Precise Definition of a Limit



2.3 The Precise Definition of a Limit



Any time you see δ (delta) or ε (epsilon), think “small number”.

2.3 The Precise Definition of a Limit

Any time you see δ (delta) or ε (epsilon), think “small number”.

We want

x is close to c (but $x \neq c$) $\implies f(x)$ is close to L .

“if x is close to c (but $x \neq c$) then $f(x)$ is close to L ”

2.3 The Precise Definition of a Limit



Any time you see δ (delta) or ε (epsilon), think “small number”.

We want

$$x \text{ is close to } c \text{ (but } x \neq c\text{)} \implies f(x) \text{ is close to } L.$$

“if x is close to c (but $x \neq c$) then $f(x)$ is close to L ”

So we want

$$0 < |x - c| < \delta \implies |f(x) - L| < \varepsilon$$

and we want this to **always be true**, no matter how small an $\varepsilon > 0$ we have.

2.3 The Precise Definition of a Limit

Definition

We write $\lim_{x \rightarrow c} f(x) = L$ iff for all $\varepsilon > 0$, there exists $\delta > 0$ such that

$$0 < |x - c| < \delta \implies |f(x) - L| < \varepsilon.$$

2.3 The Precise Definition of a Limit

Definition

We write $\lim_{x \rightarrow c} f(x) = L$ iff for all $\varepsilon > 0$, there exists $\delta > 0$ such that

$$0 < |x - c| < \delta \implies |f(x) - L| < \varepsilon.$$

2.3 The Precise Definition of a Limit

Definition

We write $\lim_{x \rightarrow c} f(x) = L$ iff for all $\varepsilon > 0$, there exists $\delta > 0$ such that

$$0 < |x - c| < \delta \implies |f(x) - L| < \varepsilon.$$

2.3 The Precise Definition of a Limit

Definition

We write $\lim_{x \rightarrow c} f(x) = L$ iff for all $\varepsilon > 0$, there exists $\delta > 0$ such that

$$0 < |x - c| < \delta \quad \Rightarrow \quad |f(x) - L| < \varepsilon.$$

2.3 The Precise Definition of a Limit

Definition

We write $\lim_{x \rightarrow c} f(x) = L$ iff for all $\varepsilon > 0$, there exists $\delta > 0$ such that

$$0 < |x - c| < \delta \implies |f(x) - L| < \varepsilon.$$

Example

Show that $\lim_{x \rightarrow 1} (5x - 3) = 2$.

(Here we have $f(x) = 5x - 3$, $c = 1$ and $L = 2$.)

2.3 The Precise Definition of a Limit

Definition

We write $\lim_{x \rightarrow c} f(x) = L$ iff for all $\varepsilon > 0$, there exists $\delta > 0$ such that

$$0 < |x - c| < \delta \implies |f(x) - L| < \varepsilon.$$

Example

Show that $\lim_{x \rightarrow 1} (5x - 3) = 2$.

(Here we have $f(x) = 5x - 3$, $c = 1$ and $L = 2$.)

Let $\varepsilon > 0$. Choose $\delta = \underline{\hspace{2cm}}$. Then

$$0 < |x - 1| < \delta \implies |(5x - 3) - 2| \underline{\hspace{10cm}} < \varepsilon.$$

Therefore $\lim_{x \rightarrow 1} (5x - 3) = 2$.

2.3 The Precise Definition of a Limit

Definition

We write $\lim_{x \rightarrow c} f(x) = L$ if
that

$$0 < |x - c| < \delta$$

scrap paper

We want

$$|(5x - 3) - 2| < \varepsilon$$

Example

Show that $\lim_{x \rightarrow 1} (5x - 3) =$

(Here we have $f(x) = 5x - 3$)

Let $\varepsilon > 0$. Choose $\delta =$

$$0 < |x - 1| < \delta \implies |(5x - 3) - 2| < \varepsilon.$$

Therefore $\lim_{x \rightarrow 1} (5x - 3) =$

2.3 The Precise Definition of a Limit

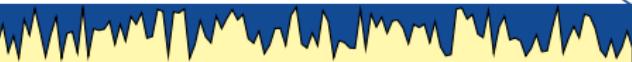
Definition

We write $\lim_{x \rightarrow c} f(x) = L$ if
that

$$0 < |x - c| < \delta$$

Example

Show that $\lim_{x \rightarrow 1} (5x - 3) =$



scrap paper

We want

$$|(5x - 3) - 2| < \varepsilon$$

$$|5x - 5| < \varepsilon$$

$$5|x - 1| < \varepsilon$$

$$|x - 1| < \frac{\varepsilon}{5}$$

(Here we have $f(x) = 5x - 3$)

Let $\varepsilon > 0$. Choose $\delta =$

$$0 < |x - 1| < \delta \implies |(5x - 3) - 2| < \varepsilon.$$

Therefore $\lim_{x \rightarrow 1} (5x - 3) =$



2.3 The Precise Definition of a Limit

Definition

We write $\lim_{x \rightarrow c} f(x) = L$ if
that

$$0 < |x - c| < \delta$$

Example

Show that $\lim_{x \rightarrow 1} (5x - 3) =$

(Here we have $f(x) = 5x - 3$)

Let $\varepsilon > 0$. Choose $\delta =$

$$0 < |x - 1| < \delta \implies |(5x - 3) - 2| <$$

$$\text{Therefore } \lim_{x \rightarrow 1} (5x - 3) =$$

scrap paper

We want

$$|(5x - 3) - 2| < \varepsilon$$

$$|5x - 5| < \varepsilon$$

$$5|x - 1| < \varepsilon$$

$$|x - 1| < \frac{\varepsilon}{5}$$

We can choose $\delta = \frac{\varepsilon}{5}$

$< \varepsilon$.

2.3 The Precise Definition of a Limit

Definition

We write $\lim_{x \rightarrow c} f(x) = L$ iff for all $\varepsilon > 0$, there exists $\delta > 0$ such that

$$0 < |x - c| < \delta \implies |f(x) - L| < \varepsilon.$$

Example

Show that $\lim_{x \rightarrow 1} (5x - 3) = 2$.

(Here we have $f(x) = 5x - 3$, $c = 1$ and $L = 2$.)

Let $\varepsilon > 0$. Choose $\delta = \frac{\varepsilon}{5}$. Then

$$0 < |x - 1| < \delta \implies |(5x - 3) - 2| < \varepsilon.$$

Therefore $\lim_{x \rightarrow 1} (5x - 3) = 2$.

2.3 The Precise Definition of a Limit

Definition

We write $\lim_{x \rightarrow c} f(x) = L$ iff for all $\varepsilon > 0$, there exists $\delta > 0$ such that

$$0 < |x - c| < \delta \implies |f(x) - L| < \varepsilon.$$

Example

Show that $\lim_{x \rightarrow 1} (5x - 3) = 2$.

(Here we have $f(x) = 5x - 3$, $c = 1$ and $L = 2$.)

Let $\varepsilon > 0$. Choose $\delta = \frac{\varepsilon}{5}$. Then

$$0 < |x - 1| < \delta \implies |(5x - 3) - 2| = |5(x - 1)| = 5|x - 1| < \varepsilon.$$

Therefore $\lim_{x \rightarrow 1} (5x - 3) = 2$.

2.3 The Precise Definition of a Limit

Definition

We write $\lim_{x \rightarrow c} f(x) = L$ iff for all $\varepsilon > 0$, there exists $\delta > 0$ such that

$$0 < |x - c| < \delta \implies |f(x) - L| < \varepsilon.$$

Example

Show that $\lim_{x \rightarrow 1} (5x - 3) = 2$.

(Here we have $f(x) = 5x - 3$, $c = 1$ and $L = 2$.)

Let $\varepsilon > 0$. Choose $\delta = \frac{\varepsilon}{5}$. Then

$$0 < |x - 1| < \delta \implies |(5x - 3) - 2| = |5(x - 1)| = 5|x - 1| < \varepsilon.$$

Therefore $\lim_{x \rightarrow 1} (5x - 3) = 2$.

2.3 The Precise Definition of a Limit

Definition

We write $\lim_{x \rightarrow c} f(x) = L$ iff for all $\varepsilon > 0$, there exists $\delta > 0$ such that

$$0 < |x - c| < \delta \implies |f(x) - L| < \varepsilon.$$

Example

Show that $\lim_{x \rightarrow 1} (5x - 3) = 2$.

(Here we have $f(x) = 5x - 3$, $c = 1$ and $L = 2$.)

Let $\varepsilon > 0$. Choose $\delta = \frac{\varepsilon}{5}$. Then

$$0 < |x - 1| < \delta \implies |(5x - 3) - 2| = |5(x - 1)| = 5|x - 1| < \varepsilon.$$

Therefore $\lim_{x \rightarrow 1} (5x - 3) = 2$.

2.3 The Precise Definition of a Limit

Definition

We write $\lim_{x \rightarrow c} f(x) = L$ iff for all $\varepsilon > 0$, there exists $\delta > 0$ such that

$$0 < |x - c| < \delta \implies |f(x) - L| < \varepsilon.$$

Example

Show that $\lim_{x \rightarrow 1} (5x - 3) = 2$.

(Here we have $f(x) = 5x - 3$, $c = 1$ and $L = 2$.)

Let $\varepsilon > 0$. Choose $\delta = \frac{\varepsilon}{5}$. Then

$$0 < |x - 1| < \delta \implies |(5x - 3) - 2| = |5(x - 1)| = 5|x - 1| < \varepsilon.$$

Therefore $\lim_{x \rightarrow 1} (5x - 3) = 2$.

2.3 The Precise Definition of a Limit



Example

For the limit $\lim_{x \rightarrow 5} \sqrt{x - 1} = 2$, find a $\delta > 0$ which works for $\varepsilon = 1$.

2.3 The Precise Definition of a Limit

Example

For the limit $\lim_{x \rightarrow 5} \sqrt{x - 1} = 2$, find a $\delta > 0$ which works for $\varepsilon = 1$.

In other words: Find a $\delta > 0$ such that

$$0 < |x - 5| < \delta \quad \Rightarrow \quad |\sqrt{x - 1} - 2| < 1.$$

2.3 The Precise Definition of a Limit



Example

For the limit $\lim_{x \rightarrow 5} \sqrt{x - 1} = 2$, find a $\delta > 0$ which works for $\varepsilon = 1$.

In other words: Find a $\delta > 0$ such that

$$0 < |x - 5| < \delta \quad \Rightarrow \quad |\sqrt{x - 1} - 2| < 1.$$

First we calculate that

$$|\sqrt{x - 1} - 2| < 1$$

2.3 The Precise Definition of a Limit



Example

For the limit $\lim_{x \rightarrow 5} \sqrt{x - 1} = 2$, find a $\delta > 0$ which works for $\varepsilon = 1$.

In other words: Find a $\delta > 0$ such that

$$0 < |x - 5| < \delta \quad \Rightarrow \quad |\sqrt{x - 1} - 2| < 1.$$

First we calculate that

$$\begin{aligned} |\sqrt{x - 1} - 2| &< 1 \\ -1 &< \sqrt{x - 1} - 2 < 1 \\ 1 &< \sqrt{x - 1} < 3 \end{aligned}$$

2.3 The Precise Definition of a Limit



Example

For the limit $\lim_{x \rightarrow 5} \sqrt{x - 1} = 2$, find a $\delta > 0$ which works for $\varepsilon = 1$.

In other words: Find a $\delta > 0$ such that

$$0 < |x - 5| < \delta \quad \Rightarrow \quad |\sqrt{x - 1} - 2| < 1.$$

First we calculate that

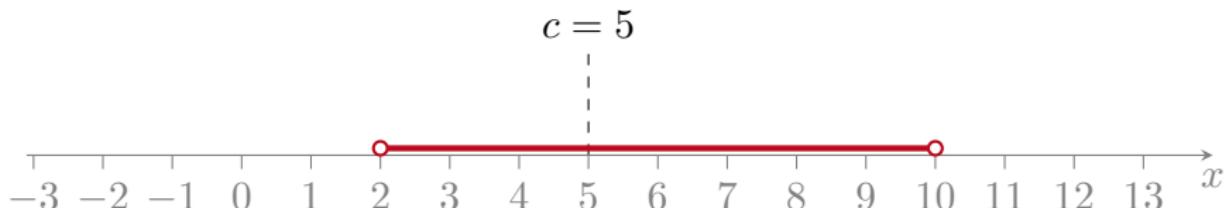
$$\begin{aligned} |\sqrt{x - 1} - 2| &< 1 \\ -1 &< \sqrt{x - 1} - 2 < 1 \\ 1 &< \sqrt{x - 1} < 3 \\ 1 &< x - 1 < 9 \\ 2 &< x < 10. \end{aligned}$$

2.3 The Precise Definition of a Limit



So we need to have

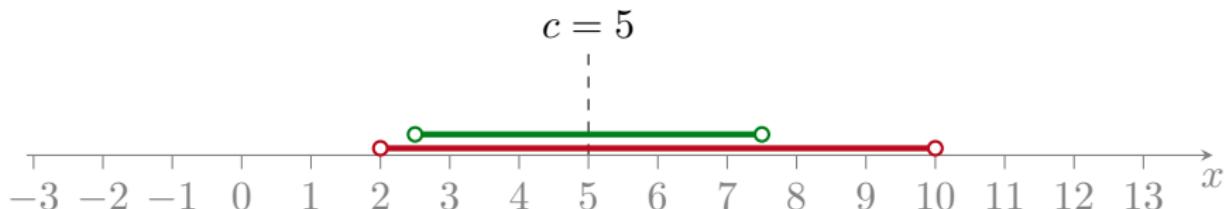
$$2 < x < 10.$$



2.3 The Precise Definition of a Limit

So we need to have

$$2 < x < 10.$$

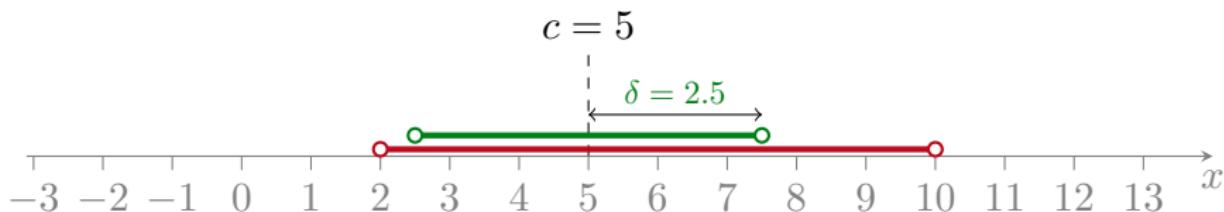


So we need $(5 - \delta, 5 + \delta) \subseteq (2, 10)$.

2.3 The Precise Definition of a Limit

So we need to have

$$2 < x < 10.$$

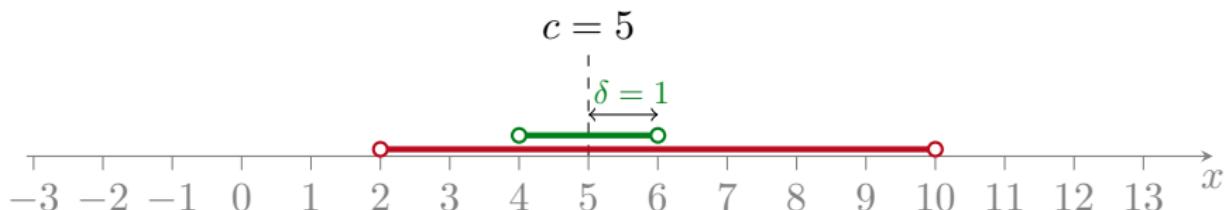


So we need $(5 - \delta, 5 + \delta) \subseteq (2, 10)$.

2.3 The Precise Definition of a Limit

So we need to have

$$2 < x < 10.$$

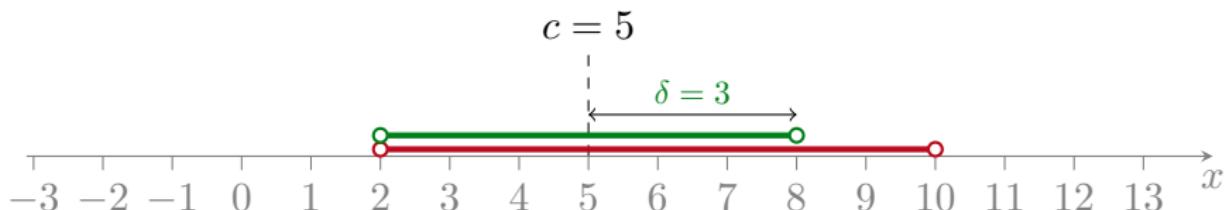


So we need $(5 - \delta, 5 + \delta) \subseteq (2, 10)$.

2.3 The Precise Definition of a Limit

So we need to have

$$2 < x < 10.$$

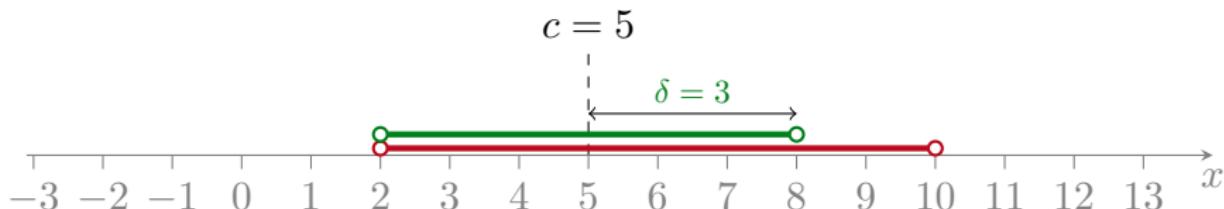


So we need $(5 - \delta, 5 + \delta) \subseteq (2, 10)$.

2.3 The Precise Definition of a Limit

So we need to have

$$2 < x < 10.$$



So we need $(5 - \delta, 5 + \delta) \subseteq (2, 10)$.

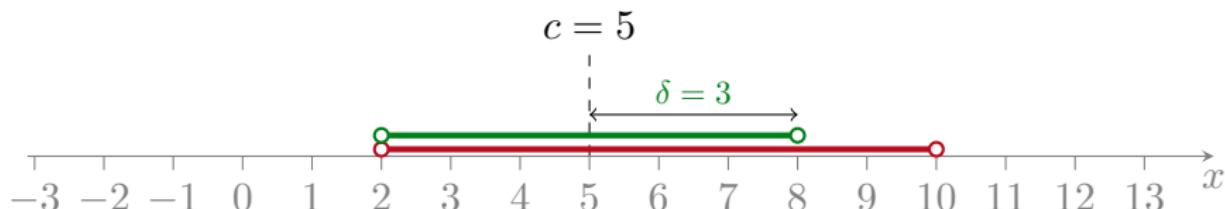
To answer this question, we don't need to find the 'best' δ or 'biggest' δ . We only need to find a $\delta > 0$ which works.

2.3 The Precise Definition of a Limit



So we need to have

$$2 < x < 10.$$



So we need $(5 - \delta, 5 + \delta) \subseteq (2, 10)$.

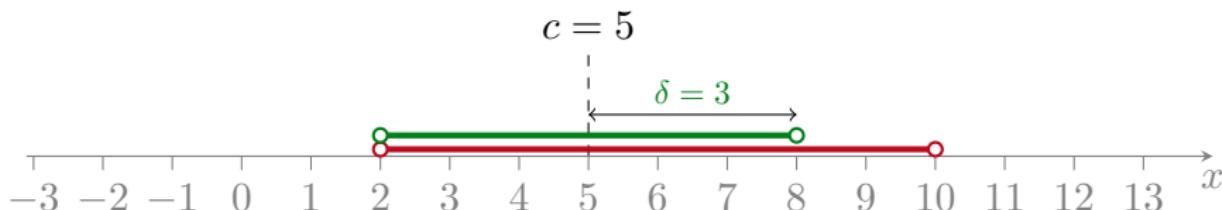
To answer this question, we don't need to find the 'best' δ or 'biggest' δ . We only need to find a $\delta > 0$ which works.

From the picture, we can see that we can choose any δ in $(0, 3]$.

2.3 The Precise Definition of a Limit

So we need to have

$$2 < x < 10.$$



So we need $(5 - \delta, 5 + \delta) \subseteq (2, 10)$.

To answer this question, we don't need to find the 'best' δ or 'biggest' δ . We only need to find a $\delta > 0$ which works.

From the picture, we can see that we can choose any δ in $(0, 3]$. I choose $\delta = 3$.

2.3 The Precise Definition of a Limit

Then

$$\begin{aligned} 0 < |x - 5| < 3 &\implies -3 < x - 5 < 3 \\ &\implies 2 < x < 8 \\ &\implies 2 < x < 10 \\ &\implies 1 < x - 1 < 9 \\ &\implies 1 < \sqrt{x-1} < 3 \\ &\implies -1 < \sqrt{x-1} - 2 < 1 \\ &\implies |\sqrt{x-1} - 2| < 1. \end{aligned}$$

2.3 The Precise Definition of a Limit



Then

$$\begin{aligned} 0 < |x - 5| < 3 &\implies -3 < x - 5 < 3 \\ &\implies 2 < x < 8 \\ &\implies 2 < x < 10 \\ &\implies 1 < x - 1 < 9 \\ &\implies 1 < \sqrt{x-1} < 3 \\ &\implies -1 < \sqrt{x-1} - 2 < 1 \\ &\implies |\sqrt{x-1} - 2| < 1. \end{aligned}$$

Note: $\delta = 2.5$, $\delta = 2$, $\delta = 1$, etc. are also correct answers to this problem. $\delta = 3.0000001$ is not a correct answer.

$\forall \varepsilon > 0$, $\exists \delta > 0$ such that $0 < |x - c| < \delta \implies |f(x) - L| < \varepsilon$.

\forall = “for all”

\exists = “there exists”

Theorem

$$\lim_{x \rightarrow c} x = c$$

$\forall \varepsilon > 0$, $\exists \delta > 0$ such that $0 < |x - c| < \delta \implies |f(x) - L| < \varepsilon$.

\forall = “for all”

\exists = “there exists”

Theorem

$$\lim_{x \rightarrow c} x = c$$

Proof.

Let $\varepsilon > 0$.

$\forall \varepsilon > 0$, $\exists \delta > 0$ such that $0 < |x - c| < \delta \implies |f(x) - L| < \varepsilon$.

\forall = “for all”

\exists = “there exists”

Theorem

$$\lim_{x \rightarrow c} x = c$$

Proof.

Let $\varepsilon > 0$. Choose $\delta = \varepsilon$.

$\forall \varepsilon > 0$, $\exists \delta > 0$ such that $0 < |x - c| < \delta \implies |f(x) - L| < \varepsilon$.

\forall = “for all”

\exists = “there exists”

Theorem

$$\lim_{x \rightarrow c} x = c$$

Proof.

Let $\varepsilon > 0$. Choose $\delta = \varepsilon$. Then

$$0 < |x - c| < \delta \implies$$



$\forall \varepsilon > 0$, $\exists \delta > 0$ such that $0 < |x - c| < \delta \implies |f(x) - L| < \varepsilon$.

\forall = “for all”

\exists = “there exists”

Theorem

$$\lim_{x \rightarrow c} x = c$$

Proof.

Let $\varepsilon > 0$. Choose $\delta = \varepsilon$. Then

$$0 < |x - c| < \delta \implies |x - c| < \delta = \varepsilon.$$



$\forall \varepsilon > 0$, $\exists \delta > 0$ such that $0 < |x - c| < \delta \implies |f(x) - L| < \varepsilon$.

Theorem

$$\lim_{x \rightarrow c} k = k.$$

$\forall \varepsilon > 0$, $\exists \delta > 0$ such that $0 < |x - c| < \delta \implies |f(x) - L| < \varepsilon$.

Theorem

$$\lim_{x \rightarrow c} k = k.$$

Proof.

Let $\varepsilon > 0$.

$\forall \varepsilon > 0$, $\exists \delta > 0$ such that $0 < |x - c| < \delta \implies |f(x) - L| < \varepsilon$.

Theorem

$$\lim_{x \rightarrow c} k = k.$$

Proof.

Let $\varepsilon > 0$. Choose $\delta = 123456789$.

$\forall \varepsilon > 0$, $\exists \delta > 0$ such that $0 < |x - c| < \delta \implies |f(x) - L| < \varepsilon$.

Theorem

$$\lim_{x \rightarrow c} k = k.$$

Proof.

Let $\varepsilon > 0$. Choose $\delta = 123456789$. Then

$$0 < |x - c| < \delta \implies |k - k| = 0 < \varepsilon.$$



$\forall \varepsilon > 0, \exists \delta > 0$ such that $0 < |x - c| < \delta \implies |f(x) - L| < \varepsilon$.

Example

Prove that $\lim_{x \rightarrow 2} f(x) = 4$ if

$$f(x) = \begin{cases} x^2 & x \neq 2 \\ 1 & x = 2. \end{cases}$$

$\forall \varepsilon > 0$, $\exists \delta > 0$ such that $0 < |x - c| < \delta \implies |f(x) - L| < \varepsilon$.

Example

Prove that $\lim_{x \rightarrow 2} f(x) = 4$ if

$$f(x) = \begin{cases} x^2 & x \neq 2 \\ 1 & x = 2. \end{cases}$$

Let $\varepsilon > 0$. Choose $\delta =$. Then

$$0 < |x - 2| < \delta \implies$$

$\forall \varepsilon > 0, \exists \delta > 0$ such that $0 < |x - c| < \delta \implies |f(x) - L| < \varepsilon$.

Example

Prove that $\lim_{x \rightarrow 2} f(x) = 4$

Assume $x \neq 2$.

$f(x)$

$$|f(x) - 4| < \varepsilon$$

$$|x^2 - 4| < \varepsilon$$

Let $\varepsilon > 0$. Choose $\delta =$

$$0 < |x - 2| < \delta \implies$$



$\forall \varepsilon > 0, \exists \delta > 0$ such that $0 < |x - c| < \delta \implies |f(x) - L| < \varepsilon$.

Example

Prove that $\lim_{x \rightarrow 2} f(x) = 4$

Let $\varepsilon > 0$. Choose $\delta =$
 $0 < |x - 2| < \delta \implies$

Assume $x \neq 2$.

$f($

$$|f(x) - 4| < \varepsilon$$

$$|x^2 - 4| < \varepsilon$$

$$-\varepsilon < x^2 - 4 < \varepsilon$$

$$4 - \varepsilon < x^2 < 4 + \varepsilon$$

$$\sqrt{4 - \varepsilon} < x < \sqrt{4 + \varepsilon}$$

$$\sqrt{4 - \varepsilon} - 2 < x - 2 < \sqrt{4 + \varepsilon} - 2$$

$\forall \varepsilon > 0, \exists \delta > 0$ such that $0 < |x - c| < \delta \implies |f(x) - L| < \varepsilon$.

Example

Prove that $\lim_{x \rightarrow 2} f(x) = 4$

As

$f(x)$

$$\sqrt{4 - \varepsilon} < x < \sqrt{4 + \varepsilon}$$

$$\sqrt{4 - \varepsilon} - 2 < x - 2 < \sqrt{4 + \varepsilon} - 2$$

Let $\varepsilon > 0$. Choose $\delta =$

$$0 < |x - 2| < \delta \implies$$



$\forall \varepsilon > 0, \exists \delta > 0$ such that $0 < |x - c| < \delta \implies |f(x) - L| < \varepsilon$.

Example

Prove that $\lim_{x \rightarrow 2} f(x) = 4$

As

$f(x)$

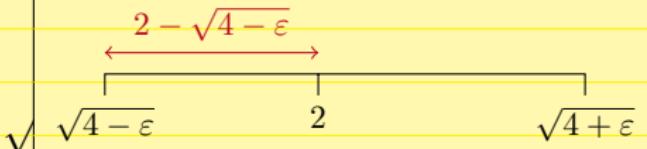
$$\sqrt{4 - \varepsilon} < x < \sqrt{4 + \varepsilon}$$

$$\sqrt{4 - \varepsilon} - 2 < x - 2 < \sqrt{4 + \varepsilon} - 2$$

Let $\varepsilon > 0$. Choose $\delta =$

$$0 < |x - 2| < \delta$$

\implies



$\forall \varepsilon > 0, \exists \delta > 0$ such that $0 < |x - c| < \delta \implies |f(x) - L| < \varepsilon$.

Example

Prove that $\lim_{x \rightarrow 2} f(x) = 4$

As

$f(x)$

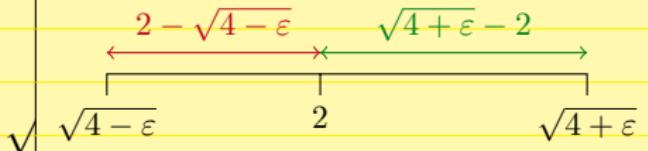
$$\sqrt{4 - \varepsilon} < x < \sqrt{4 + \varepsilon}$$

$$\sqrt{4 - \varepsilon} - 2 < x - 2 < \sqrt{4 + \varepsilon} - 2$$

Let $\varepsilon > 0$. Choose $\delta =$

$$0 < |x - 2| < \delta$$

\implies



$\forall \varepsilon > 0, \exists \delta > 0$ such that $0 < |x - c| < \delta \implies |f(x) - L| < \varepsilon$.

Example

Prove that $\lim_{x \rightarrow 2} f(x) = 4$

As

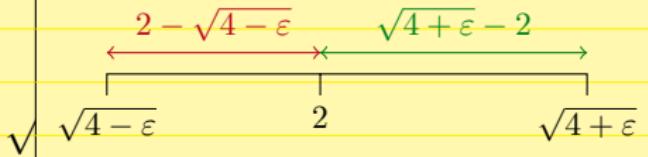
$f(x)$

$$\sqrt{4 - \varepsilon} < x < \sqrt{4 + \varepsilon}$$

$$\sqrt{4 - \varepsilon} - 2 < x - 2 < \sqrt{4 + \varepsilon} - 2$$

Let $\varepsilon > 0$. Choose $\delta =$

$$0 < |x - 2| < \delta \implies$$



Which is smaller?

$$2 - \sqrt{4 - \varepsilon} \text{ or } \sqrt{4 + \varepsilon} - 2?$$

$\forall \varepsilon > 0$, $\exists \delta > 0$ such that $0 < |x - c| < \delta \implies |f(x) - L| < \varepsilon$.

Example

Prove that $\lim_{x \rightarrow 2} f(x) = 4$ if

$$f(x) = \begin{cases} x^2 & x \neq 2 \\ 1 & x = 2. \end{cases}$$

Let $\varepsilon > 0$. Choose $\delta = \min\{2 - \sqrt{4 - \varepsilon}, \sqrt{4 + \varepsilon} - 2\}$. Then

$$0 < |x - 2| < \delta \implies$$

$\forall \varepsilon > 0$, $\exists \delta > 0$ such that $0 < |x - c| < \delta \implies |f(x) - L| < \varepsilon$.

Example

Prove that $\lim_{x \rightarrow 2} f(x) = 4$ if

$$f(x) = \begin{cases} x^2 & x \neq 2 \\ 1 & x = 2. \end{cases}$$

Let $\varepsilon > 0$. Choose $\delta = \min\{2 - \sqrt{4 - \varepsilon}, \sqrt{4 + \varepsilon} - 2\}$. Then

$$\begin{aligned} 0 < |x - 2| < \delta &\implies \sqrt{4 - \varepsilon} - 2 < x - 2 < \sqrt{4 + \varepsilon} - 2 \\ &\implies \sqrt{4 - \varepsilon} < x < \sqrt{4 + \varepsilon} \\ &\implies 4 - \varepsilon < x^2 < 4 + \varepsilon \\ &\implies -\varepsilon < x^2 - 4 < \varepsilon \\ &\implies |x^2 - 4| < \varepsilon. \end{aligned}$$

$\forall \varepsilon > 0$, $\exists \delta > 0$ such that $0 < |x - c| < \delta \implies |f(x) - L| < \varepsilon$.

Example (page 81, exercise 43.)

Prove that $\lim_{x \rightarrow 1} \frac{1}{x} = 1$.

Let $\varepsilon > 0$. Choose $\delta = \dots$. Then

$$0 < |x - 1| < \delta \implies$$

$\forall \varepsilon > 0, \exists \delta > 0$ such that $0 < |x - c| < \delta \implies |f(x) - L| < \varepsilon$.

Example (page 81, exercise 1)

Prove that $\lim_{x \rightarrow 1} \frac{1}{x} = 1$.

Let $\varepsilon > 0$. Choose $\delta =$

$$0 < |x - 1| < \delta$$

$$\left| \frac{1}{x} - 1 \right| < \varepsilon$$

$$-\varepsilon < \frac{1}{x} - 1 < \varepsilon$$

$$1 - \varepsilon < \frac{1}{x} < 1 + \varepsilon$$

$$\frac{1}{1-\varepsilon} > x > \frac{1}{1+\varepsilon}$$

$$\frac{1}{1+\varepsilon} < x < \frac{1}{1-\varepsilon}$$

$$\frac{1}{1+\varepsilon} - 1 < x - 1 < \frac{1}{1-\varepsilon} - 1$$

$$-\frac{\varepsilon}{1+\varepsilon} < x - 1 < \frac{\varepsilon}{1-\varepsilon}$$

Since $\frac{\varepsilon}{1+\varepsilon} < \frac{\varepsilon}{1-\varepsilon}$,
we will choose $\delta = \frac{\varepsilon}{1+\varepsilon}$.

$\forall \varepsilon > 0$, $\exists \delta > 0$ such that $0 < |x - c| < \delta \implies |f(x) - L| < \varepsilon$.

Example (page 81, exercise 43.)

Prove that $\lim_{x \rightarrow 1} \frac{1}{x} = 1$.

Let $\varepsilon > 0$. Choose $\delta = \frac{\varepsilon}{1+\varepsilon}$. Then

$$0 < |x - 1| < \delta \implies$$

$\forall \varepsilon > 0$, $\exists \delta > 0$ such that $0 < |x - c| < \delta \implies |f(x) - L| < \varepsilon$.

Example (page 81, exercise 43.)

Prove that $\lim_{x \rightarrow 1} \frac{1}{x} = 1$.

Let $\varepsilon > 0$. Choose $\delta = \frac{\varepsilon}{1+\varepsilon}$. Then

$$\begin{aligned}
 0 < |x - 1| < \delta &\implies -\frac{\varepsilon}{1+\varepsilon} < x - 1 < \frac{\varepsilon}{1+\varepsilon} < \frac{\varepsilon}{1-\varepsilon} \\
 &\implies 1 - \frac{\varepsilon}{1+\varepsilon} < x < 1 + \frac{\varepsilon}{1-\varepsilon} \\
 &\implies \frac{1}{1+\varepsilon} < x < \frac{1}{1-\varepsilon} \\
 &\implies 1 + \varepsilon > \frac{1}{x} > 1 - \varepsilon \\
 &\implies 1 - \varepsilon < \frac{1}{x} < 1 + \varepsilon \\
 &\implies -\varepsilon < \frac{1}{x} - 1 < \varepsilon \\
 &\implies \left| \frac{1}{x} - 1 \right| < \varepsilon.
 \end{aligned}$$

$\forall \varepsilon > 0$, $\exists \delta > 0$ such that $0 < |x - c| < \delta \implies |f(x) - L| < \varepsilon$.

Example (page 81, exercise 45.)

Prove that $\lim_{x \rightarrow -3} \frac{x^2 - 9}{x + 3} = -6$.

Let $\varepsilon > 0$. Choose $\delta = \dots$. Then

$$0 < |x - (-3)| < \delta \implies$$

$\forall \varepsilon > 0, \exists \delta > 0$ such that $0 < |x - c| < \delta \implies |f(x) - L| < \varepsilon$.

Example (page 81, exercise 1)

Prove that $\lim_{x \rightarrow -3} \frac{x^2 - 9}{x + 3} = -6$

Let $\varepsilon > 0$. Choose $\delta =$

$$0 < |x - (-3)| < \delta$$

$$\left| \frac{x^2 - 9}{x + 3} - (-6) \right| < \varepsilon$$

$$-\varepsilon < \frac{x^2 - 9}{x + 3} + 6 < \varepsilon$$

$$-\varepsilon < \frac{(x+3)(x-3)}{x+3} + 6 < \varepsilon$$

$$-\varepsilon < (x - 3) + 6 < \varepsilon$$

$$-\varepsilon < x + 3 < \varepsilon$$

$$-\varepsilon < x - (-3) < \varepsilon$$

$$|x - (-3)| < \varepsilon$$

Choose $\delta = \varepsilon$.



$\forall \varepsilon > 0$, $\exists \delta > 0$ such that $0 < |x - c| < \delta \implies |f(x) - L| < \varepsilon$.

Example (page 81, exercise 45.)

Prove that $\lim_{x \rightarrow -3} \frac{x^2 - 9}{x + 3} = -6$.

Let $\varepsilon > 0$. Choose $\delta = \varepsilon$. Then

$$0 < |x - (-3)| < \delta \implies$$

$\forall \varepsilon > 0$, $\exists \delta > 0$ such that $0 < |x - c| < \delta \implies |f(x) - L| < \varepsilon$.

Example (page 81, exercise 45.)

Prove that $\lim_{x \rightarrow -3} \frac{x^2 - 9}{x + 3} = -6$.

Let $\varepsilon > 0$. Choose $\delta = \varepsilon$. Then

$$\begin{aligned}
 0 < |x - (-3)| < \delta &\implies -\varepsilon = -\delta < x + 3 < \delta = \varepsilon \\
 &\implies -\varepsilon < (x - 3) + 6 < \varepsilon \\
 &\implies -\varepsilon < \frac{(x-3)(x+3)}{x+3} + 6 < \varepsilon \\
 &\implies -\varepsilon < \frac{x^2-9}{x+3} - (-6) < \varepsilon \\
 &\implies \left| \frac{x^2-9}{x+3} - (-6) \right| < \varepsilon.
 \end{aligned}$$

2.3 The Precise Definition of a Limit



Theorem (Sum Rule for Limits)

Suppose that $\lim_{x \rightarrow c} f(x) = L$ and $\lim_{x \rightarrow c} g(x) = M$. Then

$$\lim_{x \rightarrow c} (f(x) + g(x)) = L + M.$$

2.3 The Precise Definition of a Limit



Proof.

Let $\varepsilon > 0$.

2.3 The Precise Definition of a Limit



Proof.

Let $\varepsilon > 0$.

Since $\lim_{x \rightarrow c} f(x) = L$, we know that there exists a number $\delta_1 > 0$ such that

$$0 < |x - c| < \delta_1 \implies |f(x) - L| < \frac{\varepsilon}{2}.$$

2.3 The Precise Definition of a Limit

Proof.

Let $\varepsilon > 0$.

Since $\lim_{x \rightarrow c} f(x) = L$, we know that there exists a number $\delta_1 > 0$ such that

$$0 < |x - c| < \delta_1 \implies |f(x) - L| < \frac{\varepsilon}{2}.$$

Since $\lim_{x \rightarrow c} g(x) = M$, we know that there exists a number $\delta_2 > 0$ such that

$$0 < |x - c| < \delta_2 \implies |g(x) - M| < \frac{\varepsilon}{2}.$$

2.3 The Precise Definition of a Limit

Proof.

Let $\varepsilon > 0$.

Since $\lim_{x \rightarrow c} f(x) = L$, we know that there exists a number $\delta_1 > 0$ such that

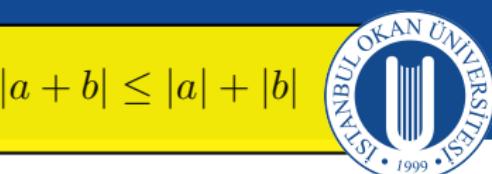
$$0 < |x - c| < \delta_1 \implies |f(x) - L| < \frac{\varepsilon}{2}.$$

Since $\lim_{x \rightarrow c} g(x) = M$, we know that there exists a number $\delta_2 > 0$ such that

$$0 < |x - c| < \delta_2 \implies |g(x) - M| < \frac{\varepsilon}{2}.$$

Choose $\delta = \min\{\delta_1, \delta_2\}$.

2.3 The Precise Definition of a Limit



$$|a + b| \leq |a| + |b|$$

Proof continued.

Then

$$\begin{aligned} 0 < |x - c| < \delta \quad \Rightarrow \quad & |(f(x) + g(x)) - (L + M)| \\ &= |(f(x) - L) + (g(x) - M)| \\ &\leq |f(x) - L| + |g(x) - M| \end{aligned}$$

2.3 The Precise Definition of a Limit

$$|a + b| \leq |a| + |b|$$



Proof continued.

Then

$$\begin{aligned} 0 < |x - c| < \delta \quad \Rightarrow \quad & |(f(x) + g(x)) - (L + M)| \\ &= |(f(x) - L) + (g(x) - M)| \\ &\leq |f(x) - L| + |g(x) - M| \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \end{aligned}$$

2.3 The Precise Definition of a Limit

$$|a + b| \leq |a| + |b|$$



Proof continued.

Then

$$\begin{aligned} 0 < |x - c| < \delta \quad \Rightarrow \quad & |(f(x) + g(x)) - (L + M)| \\ &= |(f(x) - L) + (g(x) - M)| \\ &\leq |f(x) - L| + |g(x) - M| \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \end{aligned}$$

Therefore $\lim_{x \rightarrow c} (f(x) + g(x)) = L + M$. □



Next Time

- 2.4 One-Sided Limits
- 2.5 Continuity
- 2.6 Limits Involving Infinity; Asymptotes
of Graphs