Lecture 2

- 1.5 Classification
- 2.1 Linear Equations
- 2.2 Separable Equations
- 2.3 Differences Between Linear and Nonlinear Equations

1.5 Classification

ODEs

If only ordinary derivatives appear in a differential equation, then it is called an *ordinary differential equation* (ODE) [adi diferansiyel denklem]. For example

$$\frac{dv}{dt} = 9.8 - \frac{v}{5} \qquad \text{(falling object)}$$

and

$$\frac{dp}{dt} = \frac{p}{2} - 450 \qquad \text{(mice and owls)}$$

are ODEs.

PDEs

If the derivatives in a differential equation are partial derivatives, then it is called a *partial differential equation* (PDE) [kısmi türevli diferansiyel denklem]. For example

$$k\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t} \qquad \text{(heat equation)}$$

and

$$c^2\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2} \qquad \text{(wave equation)}$$

are PDEs.

Systems

If there is a single function to be found, then one differential equation is enough. However, if there are two or more unknown functions then we need a *system of differential equations*. For example

$$\begin{cases} \frac{dx}{dt} = ax - \alpha xy \\ \frac{dy}{dt} = -cy + \gamma xy \end{cases} \qquad \text{(Predator-Prey equations)}$$

is a system of differential equations.

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$$\frac{d^2y}{dt^2} + \frac{dy}{dt} + y = 0$$

is a second order ODE.

$$y''' + 2e^t y'' + yy' = t^4$$

is a third order ODE.

Linear and Non-Linear

The ODE

$$F(t, y, y', \ldots, y^{(n)}) = 0$$

is called *linear* iff F is a linear function of y, y', …, $y^{(n)}$ (we don't care about t). The *general linear ODE* of order n is

$$a_0(t)y^{(n)} + a_1(t)y^{(n-1)} + \ldots + a_n(t)y = g(t). \tag{1}$$

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For example(falling object) and (mice and owls) are linear ODEs. An ODE which is not linear is called *non-linear*. For example

$$y''' + 2e^t y'' + yy' = t^4$$

is non-linear due to the yy' term.

Example

For each ODE below, give the order of the equation and state whether it is linear or non-linear:

- $\dfrac{d^3y}{dx^3} + 2\dfrac{d^5y}{dt^5} + \dfrac{dy}{dt} - y - e^x\dfrac{d^2y}{dx^2} = 0$

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First Order Differential Equations

In this chapter, we will consider equations of the form

$$\frac{dy}{dt} = f(t, y). \tag{2}$$

2.1 Linear Equations

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where the coefficients a and b are constants. We will now consider

$$\frac{dy}{dt} + p(t)y = g(t) \tag{4}$$

where the coefficients $p(t)$ and $g(t)$ are functions of t.

2.1 Linear Equations

We have seen how to solve (3):

$$\begin{aligned}\frac{dy}{dt} &= -ay + b \\ \int \frac{dy}{y - \frac{b}{a}} &= \int -a\,dt \\ \ln\left|y - \frac{b}{a}\right| &= -at + C \\ &\vdots \\ y &= \frac{b}{a} + ce^{-at}.\end{aligned}$$

So for example $\frac{dy}{dt} + 2y = 3$ has solution $y = \frac{3}{2} + ce^{-2t}$.

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- Find a special function $\mu(t)$ called an integrating factor;
- Multiply the ODE by $\mu(t)$;
- Integrate.

Example

Use an integrating factor to solve $\frac{dy}{dt} + 2y = 3$.

$$\frac{dy}{dt} + 2y = 3$$

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We want to choose $\mu(t)$ such that

$$\frac{d\mu}{dt} = 2\mu.$$

We know how to solve this equation:

$$\begin{aligned}
\int \frac{d\mu}{\mu} &= \int 2\,dt \\
\ln|\mu| &= 2t + C \\
&\ \vdots \\
\mu(t) &= ce^{2t}.
\end{aligned}$$

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Integrating gives

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Therefore

$$\boxed{y = \frac{3}{2} + ce^{-2t}.}$$

Remark

For the ODE $\dfrac{dy}{dt} + 2y = 3$ we use the integrating factor $\mu(t) = e^{2t}$.

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If we were to repeat the previous method, we would find that we need the integrating factor $\mu(t) = e^{at}$. (Please check!)

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The integrating factor depends only on the coefficient of y. So again we use $\mu(t) = e^{at}$.

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Thus

$$\boxed{y = e^{-at}\int^t e^{as}g(s)\,ds + ce^{-at}} \tag{5}$$

Example

Solve

$$\begin{cases} \frac{dy}{dt} + \frac{1}{2}y = 2 + t \\ y(0) = 2. \end{cases}$$

We multiply the ODE by the integrating factor $e^{\frac{t}{2}}$ to obtain

$$e^{\frac{t}{2}}y' + \frac{1}{2}e^{\frac{t}{2}}y = 2e^{\frac{t}{2}} + te^{\frac{t}{2}}$$

and

$$\frac{d}{dt}\left(e^{\frac{t}{2}}y\right) = 2e^{\frac{t}{2}} + te^{\frac{t}{2}}.$$

Integrating gives us

$$e^{\frac{t}{2}}y = 4e^{\frac{t}{2}} + 2te^{\frac{t}{2}} - 4e^{\frac{t}{2}} + c = 2te^{\frac{t}{2}} + c$$

(where we have used $\int u\frac{dv}{dt} = uv - \int \frac{du}{dt}v$ with $u = t$ and $v = 2e^{\frac{t}{2}}$). Therefore

$$y(t) = 2t + ce^{-\frac{t}{2}}.$$

Now

$$2 = y(0) = 0 + c \qquad \Longrightarrow \qquad c = 2.$$

Therefore the solution to the IVP is

$$\boxed{y(t) = 2t + 2e^{-\frac{t}{2}}.}$$

Example

Solve $\dfrac{dy}{dt} - 2y = 4 - t$.

Please check that by using $\mu(t) = e^{-2t}$ we obtain $y(t) = -\frac{7}{4} + \frac{t}{2} + ce^{2t}$.

Now consider

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WARNING: The integrating factor is NOT $e^{p(t)}$.

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So we want

$$\frac{d\mu}{dt} = p(t)\mu.$$

We know how to solve this ODE:

$$\int \frac{d\mu}{\mu} = \int p(t)\,dt$$

$$\ln|\mu| = \int p(t)\,dt + C$$

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As before, we can choose $c = 1$ to obtain

$$\mu(t) = \exp\int p(t)\,dt = e^{\int p(t)\,dt}. \tag{6}$$

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and we calculate that

$$\mu y = \int^t \mu(s) g(s)\, ds + c$$

and

$$y(t) = \frac{\int^t \mu(s) g(s)\, ds + c}{\mu(t)}.$$

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First we must write the equation in the standard form:

$$\frac{dy}{dt} + \frac{2}{t}y = 4t.$$

Here $p(t) = \frac{2}{t}$ and $g(t) = 4t$.

Next we must calculate $\mu(t)$:

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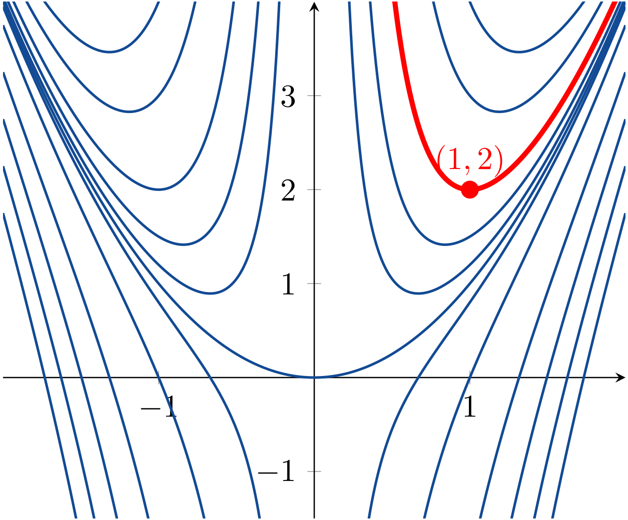
Hence the general solution to the ODE is

$$y(t) = t^2 + \frac{c}{t^2}.$$

To satisfy $y(1) = 2$, we choose $c = 1$. Therefore

$$\boxed{y(t) = t^2 + \frac{1}{t^2} \qquad (t > 0).}$$



3
2
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(1, 2)

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3. The function $y = t^2 + \frac{1}{t^2}$, $t < 0$ is *not* part of the solution to the IVP. The solution to the IVP only exists for $t \in (0, \infty)$.
4. Solutions for which $c > 0$ (i.e. $y(1) > 1$) are asymptotic to the positive y-axis as $t \searrow 0$. But solutions for which $c < 0$ (i.e. $y(1) < 1$) are asymptotic to the negative y-axis as $t \searrow 0$. So there is an initial value ($y(1) = 0$) where the behaviour changes. This is called a *critical initial value*.

2.2 Separable Equations

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In the previous section we looked at a special case called “linear equations” – now we will study another special case.

2.2 Separable Equations

$$\frac{dy}{dx} = f(x, y) \tag{7}$$

Equation (7) can *always* be written in the form

$$M(x, y) + N(x, y)\frac{dy}{dx} = 0. \tag{8}$$

One way would be to write $M = -f$ and $N = 1$, but there may be other ways.

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One way would be to write $M = -f$ and $N = 1$, but there may be other ways. *If* we can do this so that $M(x)$ is a function only of x and $N(y)$ is a function only of y, then (8) becomes

$$M(x) + N(y)\frac{dy}{dx} = 0. \tag{9}$$

Definition

A first order ODE is called *separable* if it can be written in the form

$$M(x) + N(y)\frac{dy}{dx} = 0.$$

Remark

Note that we can rearrange $M(x) + N(y)\frac{dy}{dx} = 0$ to

$$\underbrace{M(x)\,dx}_{\text{all } x \text{ terms}} = -\underbrace{N(y)\,dy}_{\text{all } y \text{ terms}}.$$

In other words, it is possible to “separate” the variables.

Example

Consider

$$\frac{dy}{dx} = \frac{x^2}{1-y^2}.$$

1. Show that this ODE is separable.
2. Solve this ODE.

$$\frac{dy}{dx} = \frac{x^2}{1-y^2}$$

We can rearrange this ODE to

$$-x^2 + (1-y^2)\frac{dy}{dx} = 0.$$

This is of the form (9). Therefore this ODE is separable.

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Using the Chain Rule, this is

$$\frac{d}{dx}\left(-\frac{1}{3}x^3\right) + \frac{d}{dx}\left(y - \frac{1}{3}y^3\right) = 0$$

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Therefore

$$-\frac{1}{3}x^3 + 1 - \frac{1}{3}y^3 = C$$

or

$$\boxed{x^3 - 3y + y^3 = c.}$$

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$$\begin{aligned} M(x) + N(y)\frac{dy}{dx} &= 0 \\ \frac{dH_1}{dx} + \frac{dH_2}{dy}\frac{dy}{dx} &= 0 \\ \frac{dH_1}{dx} + \frac{dH_2}{dx} &= 0 \end{aligned}$$

by the Chain Rule.

2.2 Separable Equations

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by the Chain Rule. Then integrating gives the solution

$$H_1(x) + H_2(y) = c.$$

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So to recap: To solve $M(x) + N(y)y' = 0$ we must integrate M wrt x and integrate N wrt y. But this is basically what we were doing in Chapter 1, where we did the following:

$$\begin{aligned} M(x) + N(y)\frac{dy}{dx} &= 0 \\ M(x) &= -N(y)\frac{dy}{dx} \\ M(x)\,dx &= -N(y)\,dy \\ \int M(x)\,dx &= -\int N(y)\,dy + c. \end{aligned}$$

Example

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Integrating gives

$$y^2 - 2y = x^3 + 2x^2 + 2x + c.$$

To find c, we use the initial condition $y(0) = 1$ and calculate that

$$1 + 2 = 0 + 0 + 0 + c \qquad \Longrightarrow \qquad c = 3.$$

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This is called an *implicit solution*. Sometimes this is the best that we can do. But in this example, it is possible to solve for y. Since

$$y^2 - 2y - (x^3 + 2x^2 + 2x + 3) = 0$$

is a quadratic equation, we find that

$$y = 1 \pm \sqrt{x^3 + 2x^2 + 2x + 4}.$$

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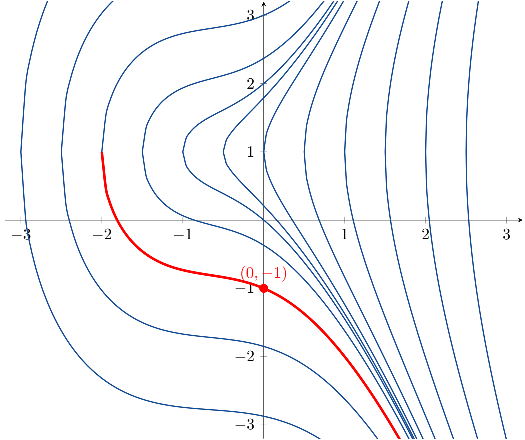
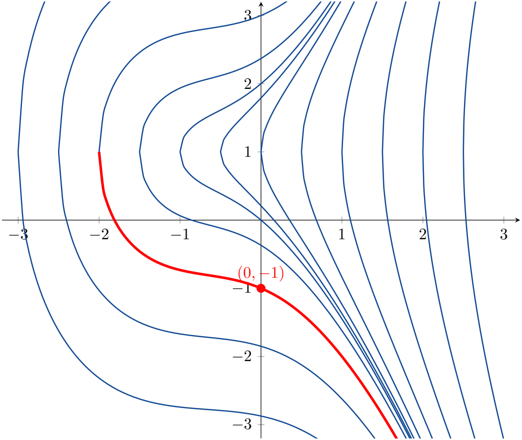
A solution of the form $y = f(x)$ is called an *explicit solution.*


(0, −1)
−3
−2
−1
1
2
3
−3
−2
−1
1
2
3



Note that the solution satisfying $y(0) = -1$ is a differentiable function $y : (-2, \infty) \to \mathbb{R}$.

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Solve $\begin{cases} \frac{dy}{dx} = \frac{y\cos x}{1+2y^2} \\ y(0) = 1. \end{cases}$

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$$\int \frac{1+2y^2}{y}\,dy = \int \cos x\,dx$$

$$\ln|y| + y^2 = \sin x + c$$

$$y(0) = 1 \qquad \Longrightarrow \qquad \ln 1 + 1^2 = \sin 0 + c \qquad \Longrightarrow \qquad c = 1.$$

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1. If $y = 0$, the left-hand side is $-\infty$, but the right-hand side is in $[0, 2]$. This means that $y = 0$ is not possible. Since we know that $y(0) = 1$, we must therefore have $y(x) > 0$ for all x in the domain of the solution.

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2. The solution exists on $(-\infty, \infty)$ (left for you to prove).

Differences Between Linear and Nonlinear Equations

Theorem

Suppose

- p *and* g *are continuous on* (α, β)*;*
- $t_0 \in (\alpha, \beta)$*; and*
- $y_0 \in \mathbb{R}$.

Then there exists a unique solution to

$$\begin{cases} y' + p(t)y = g(t) \\ y(t_0) = y_0 \end{cases}$$

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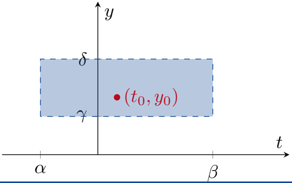
Remark

This theorem says that as long as p and g are continuous, the solution keeps existing. To say this another way: The solution can only stop existing at a discontinuity of either p or g.

Theorem

Suppose that

- f *and* $\frac{\partial f}{\partial y}$ *are continuous for all* $\alpha < t < \beta$ *and* $\gamma < y < \delta$*;*
- $t_0 \in (\alpha, \beta)$*; and*
- $y_0 \in (\gamma, \delta)$.



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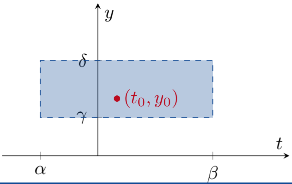

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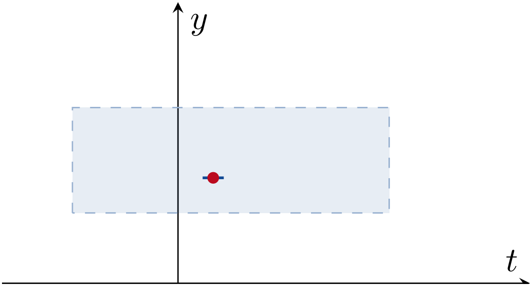
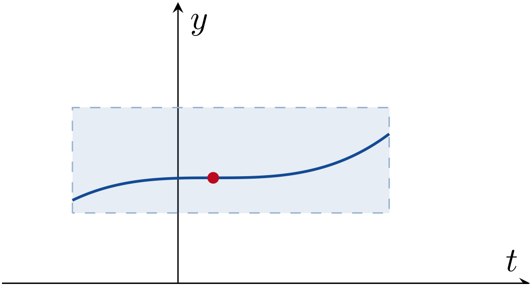
Then in some interval $(t_0 - h, t_0 + h) \subseteq (\alpha, \beta)$, *there exists a unique solution to*

$$\begin{cases} y' = f(t, y) \\ y(t_0) = y_0. \end{cases}$$



Remark

This theorem tells us that "a little bit" of the solution exists. This theorem does not tell us if we only have this little bit of solution or if the solution exists further.



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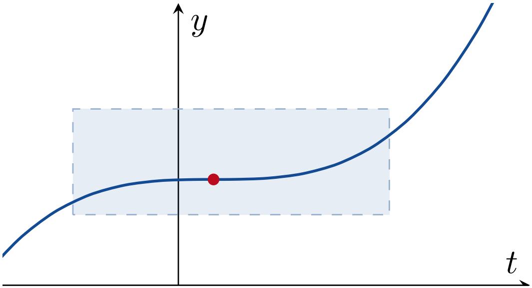

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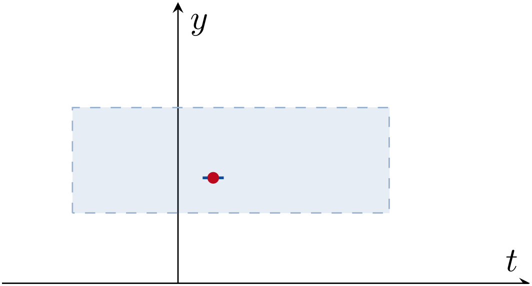

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Solutions to first order ODEs do not intersect !!! (assuming that f and $\frac{\partial f}{\partial y}$ are ...)

Next Time

- 2.4 Autonomous Equations and Population Dynamics
- 2.5 Exact Equations
- 2.6 Substitutions