

Lecture 10

- 27. Differentiation Rules
- 28. Derivatives of Trigonometric Functions
- 29. The Chain Rule



Differentiation Rules

27. Differentiation Rules



Constant Function

If $k \in \mathbb{R}$, then

$$\boxed{\frac{d}{dx}(k) = 0.}$$

Power Function

If $n \in \mathbb{R}$, then

$$\frac{d}{dx} (x^n) = nx^{n-1}.$$

Example

$$\frac{d}{dx} (x^3) = 3x^{3-1} = 3x^2$$

27. Differentiation Rules

Example

$$\frac{d}{dx} (\sqrt{x}) = \frac{d}{dx} (x^{\frac{1}{2}}) = \frac{1}{2}x^{\frac{1}{2}-1} = \frac{1}{2}x^{-\frac{1}{2}} = \frac{1}{2\sqrt{x}}$$

Example

$$\frac{d}{dx} \left(\frac{1}{x^4} \right) = \frac{d}{dx} (x^{-4}) = -4x^{-4-1} = -4x^{-5} = -\frac{4}{x^5}$$

The Constant Multiple Rule

If $u(x)$ is differentiable and $k \in \mathbb{R}$, then

$$\frac{d}{dx} (ku) = k \frac{du}{dx}.$$

Proof.

$$\begin{aligned}\frac{d}{dx} (ku) &= \lim_{h \rightarrow 0} \frac{ku(x+h) - ku(x)}{h} \\&= k \lim_{h \rightarrow 0} \frac{u(x+h) - u(x)}{h} = k \frac{du}{dx}\end{aligned}$$



27. Differentiation Rules



Example

$$\frac{d}{dx} (3x^2) = 3 \frac{d}{dx} (x^2) = 3 \times 2x = 6x$$

Example

$$\frac{d}{dx} (-u) = \frac{d}{dx} (-1 \times u) = -1 \times \frac{du}{dx} = -\frac{du}{dx}$$

The Sum Rule

If $u(x)$ and $v(x)$ are differentiable at x_0 , then $u + v$ is also differentiable at x_0 and

$$\frac{d}{dx} (u + v) = \frac{du}{dx} + \frac{dv}{dx}.$$

27. Differentiation Rules



Example

Differentiate $y = x^3 + \frac{4}{3}x^2 - 5x + 1$.

solution:

$$\begin{aligned}\frac{dy}{dx} &= \frac{d}{dx} \left(x^3 + \frac{4}{3}x^2 - 5x + 1 \right) \\&= \frac{d}{dx} (x^3) + \frac{d}{dx} \left(\frac{4}{3}x^2 \right) - \frac{d}{dx} (5x) + \frac{d}{dx} (1) \\&= 3x^2 + \frac{8}{3}x - 5 + 0\end{aligned}$$

27. Differentiation Rules

Example

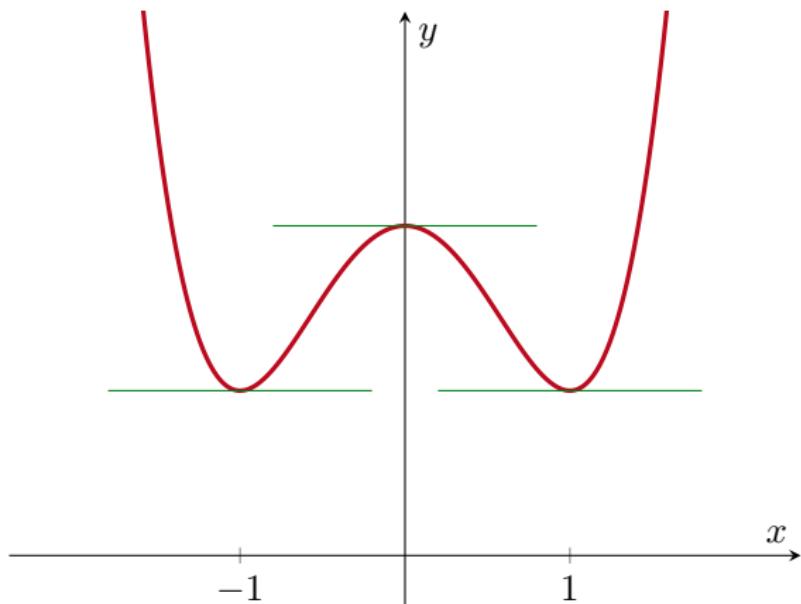
Does the curve $y = x^4 - 2x^2 + 2$ have any points where $\frac{dy}{dx} = 0$?
If so, where?

solution: Since

$$\frac{dy}{dx} = 4x^3 - 4x = 4x(x^2 - 1) = 4x(x - 1)(x + 1),$$

we can see that $\frac{dy}{dx} = 0$ if and only if $x = -1, 0$ or 1 .

27. Differentiation Rules



The Product Rule

If $u(x)$ and $v(x)$ are differentiable at x_0 , then $u(x)v(x)$ is also differentiable at x_0 and

$$\frac{d}{dx} (uv) = u \frac{dv}{dx} + v \frac{du}{dx}.$$

Using prime notation, the product rule is

$$(uv)' = u'v + uv'.$$

27. Differentiation Rules



Example

Differentiate $y = (x^2 + 1)(x^3 + 3)$.

solution 1: We have $y = uv$ with $u = x^2 + 1$ and $v = x^3 + 3$. So

$$\begin{aligned}\frac{dy}{dx} &= (x^2 + 1)'(x^3 + 3) + (x^2 + 1)(x^3 + 3)' \\&= (2x + 0)(x^3 + 3) + (x^2 + 1)(3x^2 + 0) \\&= 2x^4 + 6x + 3x^4 + 3x^2 \\&= 5x^4 + 3x^2 + 6x.\end{aligned}$$

27. Differentiation Rules



solution 2: Since

$$y = (x^2 + 1)(x^3 + 3) = x^5 + x^3 + 3x^2 + 3,$$

we have that

$$\frac{dy}{dx} = 5x^4 + 3x^2 + 6x + 0.$$

The Quotient Rule

If $u(x)$ and $v(x)$ are differentiable at x_0 and if $v(x_0) \neq 0$, then $\frac{u}{v}$ is also differentiable at x_0 and

$$\frac{d}{dx} \left(\frac{u}{v} \right) = \frac{u'v - uv'}{v^2}.$$

27. Differentiation Rules

Example

Differentiate $y = \frac{t^2 - 1}{t^3 + 1}$.

solution: We have $y = \frac{u}{v}$ with $u = t^2 - 1$ and $v = t^3 + 1$.

Therefore

$$\begin{aligned}\frac{dy}{dt} &= \frac{u'v - uv'}{v^2} \\&= \frac{(t^2 - 1)'(t^3 + 1) - (t^2 - 1)(t^3 + 1)'}{(t^3 + 1)^2} \\&= \frac{(2t)(t^3 + 1) - (t^2 - 1)(3t^2)}{(t^3 + 1)^2} \\&= \frac{2t^4 + 2t - 3t^4 + 3t^2}{(t^3 + 1)^2} \\&= \frac{-t^4 + 3t^2 + 2t}{(t^3 + 1)^2}.\end{aligned}$$

27. Differentiation Rules

Example

Differentiate $f(s) = \frac{\sqrt{s}-1}{\sqrt{s}+1}$.

solution: We have $f(s) = \frac{u}{v}$ with $u = \sqrt{s} - 1$ and $v = \sqrt{s} + 1$.

Remember that $\frac{d}{ds}(\sqrt{s}) = \frac{1}{2\sqrt{s}}$. Therefore

$$\begin{aligned}
 \frac{df}{ds} &= \frac{u'v - uv'}{v^2} \\
 &= \frac{(\sqrt{s} - 1)'(\sqrt{s} + 1) - (\sqrt{s} - 1)(\sqrt{s} + 1)'}{(\sqrt{s} + 1)^2} \\
 &= \frac{\left(\frac{1}{2\sqrt{s}}\right)(\sqrt{s} + 1) - (\sqrt{s} - 1)\left(\frac{1}{2\sqrt{s}}\right)}{(\sqrt{s} + 1)^2} \\
 &= \frac{\frac{1}{2} + \frac{1}{2\sqrt{s}} - \frac{1}{2} + \frac{1}{2\sqrt{s}}}{(\sqrt{s} + 1)^2} = \frac{1}{\sqrt{s}(\sqrt{s} + 1)^2}.
 \end{aligned}$$

Second Order Derivatives

If $y = f(x)$ is a differentiable function, then $f'(x)$ is also a function. If $f'(x)$ is also differentiable, then we can differentiate to find a new function called f'' (" f double prime"). f'' is called the *second derivative* of f . We can write

$$f''(x) = \frac{d}{dx} f'(x) = \frac{d}{dx} \left(\frac{dy}{dx} \right) = \frac{d^2y}{dx^2} = \frac{dy'}{dx} = y''$$

↗
"d squared y, dx squared"

27. Differentiation Rules



Example

If $y = x^6$, then $y' = \frac{d}{dx}(x^6) = 6x^5$ and
 $y'' = \frac{d}{dx}(y') = \frac{d}{dx}(6x^5) = 30x^4$. Equivalently, we can write

$$\frac{d^2}{dx^2}(x^6) = \frac{d}{dx}\left(\frac{d}{dx}(x^6)\right) = \frac{d}{dx}(6x^5) = 30x^4.$$

Higher Order Derivatives

If f'' is differentiable, then its derivative $f''' = \frac{d^3 f}{dx^3}$ is the *third derivative* of f .

If f''' is differentiable, then its derivative $f^{(4)} = \frac{d^4 f}{dx^4}$ is the *fourth derivative* of f .

If $f^{(4)}$ is differentiable, then its derivative $f^{(5)} = \frac{d^5 f}{dx^5}$ is the *fifth derivative* of f .

⋮

If $f^{(n-1)}$ is differentiable, then its derivative $f^{(n)} = \frac{d^n f}{dx^n}$ is the *n th derivative* of f .

27. Differentiation Rules



Example

Find the first four derivatives of $y = x^3 - 3x^2 + 2$.

solution:

$$\text{First derivative: } y' = 3x^2 - 6x$$

$$\text{Second derivative: } y'' = 6x - 6$$

$$\text{Third derivative: } y''' = 6$$

$$\text{Fourth derivative: } y^{(4)} = 0.$$

(Note that since $\frac{d}{dx}(0) = 0$, if $n \geq 4$ then $y^{(n)} = 0$ also.)



Derivatives of Trigonometric Functions

Sine and Cosine

$$\frac{d}{dx} (\sin x) = \cos x$$

$$\frac{d}{dx} (\cos x) = -\sin x$$

28. Derivatives of Trigonometric Functions



Example

Differentiate $y = x^2 - \sin x$.

solution:

$$\frac{dy}{dx} = \frac{d}{dx}(x^2) - \frac{d}{dx}(\sin x) = 2x - \cos x.$$

28. Derivatives of Trigonometric Functions



Example

Differentiate $y = x^2 \sin x$.

solution: We will use the product rule $((uv)' = u'v + uv')$ with $u = x^2$ and $v = \sin x$.

$$y' = (x^2)'(\sin x) + (x^2)(\sin x)' = 2x \sin x + x^2 \cos x.$$

28. Derivatives of Trigonometric Functions



Example

Differentiate $y = \frac{\sin x}{x}$.

solution: This time we use the quotient rule ($(\frac{u}{v})' = \frac{u'v - uv'}{v^2}$) with $u = \sin x$ and $v = x$.

$$y' = \frac{(\sin x)'x - (\sin x)(x)'}{x^2} = \frac{x \cos x - \sin x}{x^2}.$$

28. Derivatives of Trigonometric Functions



Example

Differentiate $y = 5x + \cos x$.

solution:

$$\frac{dy}{dx} = \frac{d}{dx}(5x) + \frac{d}{dx}(\cos x) = 5 - \sin x.$$

28. Derivatives of Trigonometric Functions



Example

Differentiate $y = \sin x \cos x$.

solution: By the product rule, we have that

$$\frac{dy}{dx} = \frac{d}{dx}(\sin x) \cos x + \sin x \frac{d}{dx}(\cos x) = \cos^2 x - \sin^2 x.$$

28. Derivatives of Trigonometric Functions



Example

Differentiate $y = \frac{\cos x}{1 - \sin x}$.

solution: By the quotient rule, we have that

$$\begin{aligned}\frac{dy}{dx} &= \frac{\frac{d}{dx}(\cos x)(1 - \sin x) - (\cos x)\frac{d}{dx}(1 - \sin x)}{(1 - \sin x)^2} \\&= \frac{-\sin x(1 - \sin x) - \cos x(0 - \cos x)}{(1 - \sin x)^2} \\&= \frac{-\sin x + \sin^2 x + \cos^2 x}{(1 - \sin x)^2} \\&= \frac{-\sin x + 1}{(1 - \sin x)^2} = \frac{1 - \sin x}{(1 - \sin x)^2} \\&= \frac{1}{1 - \sin x}.\end{aligned}$$

The Tangent Function

$$\boxed{\frac{d}{dx} (\tan x) = \sec^2 x}$$

28. Derivatives of Trigonometric Functions



Proof.

Using the quotient rule, we can calculate that

$$\begin{aligned}\frac{d}{dx}(\tan x) &= \frac{d}{dx}\left(\frac{\sin x}{\cos x}\right) \\&= \frac{\frac{d}{dx}(\sin x)(\cos x) - (\sin x)\frac{d}{dx}(\cos x)}{\cos^2 x} \\&= \frac{(\cos x)(\cos x) - (\sin x)(-\sin x)}{\cos^2 x} \\&= \frac{\cos^2 x + \sin^2 x}{\cos^2 x} \\&= \frac{1}{\cos^2 x} = \sec^2 x.\end{aligned}$$



The Other Three

$$\frac{d}{dx} (\sec x) = \sec x \tan x$$

$$\frac{d}{dx} (\cot x) = -\operatorname{cosec}^2 x$$

$$\frac{d}{dx} (\operatorname{cosec} x) = -\operatorname{cosec} x \cot x$$

You can use the quotient rule to prove these three rules. We may ask you to prove one of them in an exam.

28. Derivatives of Trigonometric Functions



Example

Find y'' if $y = \sec x$.

solution: Since $y' = \sec x \tan x$, we have that

$$\begin{aligned}y'' &= \frac{d}{dx}(y') = \frac{d}{dx}(\sec x \tan x) \\&= \frac{d}{dx}(\sec x) \tan x + \sec x \frac{d}{dx}(\tan x) \\&= (\sec x \tan x)(\tan x) + (\sec x)(\sec^2 x) \\&= \sec x \tan^2 x + \sec^3 x.\end{aligned}$$



The Chain Rule

29. The Chain Rule



How do we differentiate $F(x) = \sin(x^2 - 4)$?

29. The Chain Rule



Theorem (The Chain Rule)

Suppose that

- $y = f(u)$ is differentiable at the point $u = g(x)$; and
- $g(x)$ is differentiable at x .

Then $f \circ g$ is differentiable at x and

$$(f \circ g)'(x) = f'(g(x))g'(x).$$

29. The Chain Rule



The Chain Rule is easier to remember if we use Leibniz's notation:

$$\frac{dy}{dx} = \frac{dy}{du} \frac{du}{dx}$$

29. The Chain Rule

Example

Differentiate $y = \sin(x^2 - 4)$.

solution: We have $y = \sin u$ with $u = x^2 - 4$. Now $\frac{dy}{du} = \cos u$ and $\frac{du}{dx} = 2x$. Therefore

$$\begin{aligned}\frac{dy}{dx} &= \frac{dy}{du} \frac{du}{dx} = (\cos u)(2x) \\ &= 2x \cos u = 2x \cos(x^2 - 4)\end{aligned}$$

by the Chain Rule.

29. The Chain Rule



Example

Differentiate $\sin(x^2 + x)$.

solution: Let $u = x^2 + x$. Then

$$\begin{aligned}\frac{d}{dx} (\sin(x^2 + x)) &= \frac{d}{du} (\sin u) \frac{du}{dx} \\&= (\cos u)(2x + 1) \\&= (2x + 1) \cos(x^2 + x)\end{aligned}$$

by the Chain Rule.

29. The Chain Rule

Example (Using the Chain Rule Two Times)

Differentiate $g(t) = \tan(5 - \sin 2t)$.

solution: Let $u = 5 - \sin 2t$. Then $g(t) = \tan u$. Hence

$$\frac{dg}{dt} = \frac{dg}{du} \frac{du}{dt} = (\sec^2 u) \frac{d}{dt}(5 - \sin 2t).$$

We need to use the Chain Rule a second time: Let $w = 2t$. Then

$$\begin{aligned}\frac{dg}{dt} &= (\sec^2 u) \frac{d}{dt}(5 - \sin 2t) \\&= (\sec^2 u) \frac{d}{dw}(5 - \sin w) \frac{dw}{dt} \\&= (\sec^2 u)(-\cos w)(2) \\&= -2 \cos 2t \sec^2(5 - \sin 2t).\end{aligned}$$

29. The Chain Rule



(Note: Your final answer should not have u or w in it.)

29. The Chain Rule

Powers of a Function

If

- f is a differentiable function of u ;
- u is a differentiable function of x ; and
- $y = f(u)$,

then the Chain Rule $\frac{dy}{dx} = \frac{dy}{du} \frac{du}{dx}$ is the same as

$$\frac{d}{dx} f(u) = f'(u) \frac{du}{dx}.$$

Now suppose that $n \in \mathbb{R}$ and $f(u) = u^n$. Then $f'(u) = nu^{n-1}$.

So

$$\boxed{\frac{d}{dx} (u^n) = nu^{n-1} \frac{du}{dx}.}$$

29. The Chain Rule



Example

$$\begin{aligned}\frac{d}{dx} (5x^3 - x^4)^7 &= 7(5x^3 - x^4)^6 \frac{d}{dx} (5x^3 - x^4) \\&= 7(5x^3 - x^4)^6 (15x^2 - 4x^3).\end{aligned}$$

29. The Chain Rule



Example

$$\begin{aligned}\frac{d}{dx} \left(\frac{1}{3x-2} \right) &= \frac{d}{dx} (3x-2)^{-1} = -1 (3x-2)^{-2} \frac{d}{dx} (3x-2) \\&= - \left(\frac{1}{(3x-2)^2} \right) (2) = \frac{-3}{(3x-2)^2}.\end{aligned}$$

29. The Chain Rule



Example

$$\frac{d}{dx} (\sin^5 x) = 5 \sin^4 x \frac{d}{dx}(\sin x) = 5 \sin^4 x \cos x.$$

29. The Chain Rule

Example

Differentiate $|x|$.

solution: Since $|x| = \sqrt{x^2}$, we can calculate that if $x \neq 0$ then

$$\begin{aligned}\frac{d}{dx} |x| &= \frac{d}{dx} (\sqrt{x^2}) = \frac{d}{du} (\sqrt{u}) \frac{d}{dx} (x^2) \\ &= \frac{1}{2\sqrt{u}} 2x = \frac{x}{\sqrt{x^2}} = \frac{x}{|x|}.\end{aligned}$$

29. The Chain Rule

Example

Let $y = \frac{1}{(1-2x)^3}$ for $x \neq \frac{1}{2}$. Show that $\frac{dy}{dx} > 0$.

solution: First we calculate that

$$\begin{aligned}\frac{dy}{dx} &= \frac{d}{dx}(1-2x)^{-3} = -3(1-2x)^{-4} \frac{d}{dx}(1-2x) \\ &= -3(1-2x)^{-4}(-2) = \frac{6}{(1-2x)^4}\end{aligned}$$

if $x \neq \frac{1}{2}$. Since $(1-2x)^4 > 0$ if $x \neq \frac{1}{2}$ and $6 > 0$, we have that $\frac{dy}{dx} > 0$ if $x \neq \frac{1}{2}$.

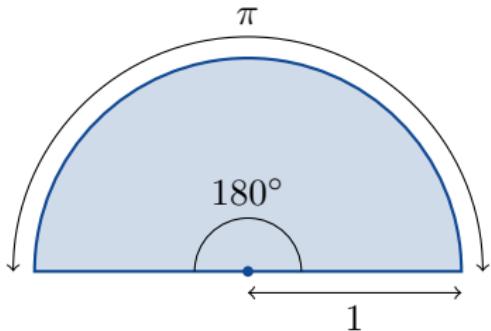
29. The Chain Rule



Example (Why Do We Use Radians in Calculus?)

Remember that $\frac{d}{dx} \sin x = \cos x$ is true *only if we use radians*.
What happens if we use degrees?

29. The Chain Rule



Remember that

$$180 \text{ degrees} = \pi \text{ radians}$$

$$180^\circ = \pi$$

$$1^\circ = \frac{\pi}{180}$$

$$x^\circ = \frac{\pi x}{180}.$$

29. The Chain Rule

So

$$\frac{d}{dx} \sin x^\circ = \frac{d}{dx} \sin\left(\frac{\pi x}{180}\right) = \frac{\pi}{180} \cos\left(\frac{\pi x}{180}\right) = \frac{\pi}{180} \cos x^\circ.$$

Therefore we have

$$\frac{d}{dx} \sin x = \cos x$$

a nice formula

and

$$\frac{d}{dx} \sin x^\circ = \frac{\pi}{180} \cos x^\circ.$$

not nice

This is why we use radians in Calculus.



Next Week: no lesson (Ramadan Feast Eve)

Two Weeks Later: no lesson (Atatürk Commemoration & Youth Day)

Three Weeks Later: ???