

Lecture 5

- 3.5 Derivatives of Trigonometric Functions
- 3.6 The Chain Rule
- 3.7 Implicit Differentiation



Derivatives of Trigonometric Functions

The Sine and Cosine Functions

First let me remind you of three formulae:

$$\sin(x + h) = \sin x \cos h + \cos x \sin h,$$

$$\lim_{h \rightarrow 0} \frac{\sin h}{h} = 1$$

and

$$\lim_{h \rightarrow 0} \frac{\cos h - 1}{h} = 0.$$

3.5 Derivatives of Trigonometric Functions



The Sine and Cosine Functions

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$$\lim_{h \rightarrow 0} \frac{\sin h}{h} = 1$$

and

$$\lim_{h \rightarrow 0} \frac{\cos h - 1}{h} = 0.$$

Using these, we will calculate the derivative of $f(x) = \sin x$.

3.5 Derivatives of Trigonometric Functions

$$\lim_{h \rightarrow 0} \frac{\sin h}{h} = 1 \quad \lim_{h \rightarrow 0} \frac{\cos h - 1}{h} = 0$$



If $f(x) = \sin x$, then

$$\begin{aligned} f'(x) &= \lim_{h \rightarrow 0} \frac{f(x + h) - f(x)}{h} = \lim_{h \rightarrow 0} \frac{\sin(x + h) - \sin x}{h} \\ &= \lim_{h \rightarrow 0} \frac{\sin x \cos h + \cos x \sin h - \sin x}{h} \end{aligned}$$

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3.5 Derivatives of Trigonometric Functions

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3.5 Derivatives of Trigonometric Functions

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=

3.5 Derivatives of Trigonometric Functions

$$\lim_{h \rightarrow 0} \frac{\sin h}{h} = 1 \quad \lim_{h \rightarrow 0} \frac{\cos h - 1}{h} = 0$$



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3.5 Derivatives of Trigonometric Functions

$$\lim_{h \rightarrow 0} \frac{\sin h}{h} = 1 \quad \lim_{h \rightarrow 0} \frac{\cos h - 1}{h} = 0$$



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$$\begin{aligned} f'(x) &= \lim_{h \rightarrow 0} \frac{f(x + h) - f(x)}{h} = \lim_{h \rightarrow 0} \frac{\sin(x + h) - \sin x}{h} \\ &= \lim_{h \rightarrow 0} \frac{\sin x \cos h + \cos x \sin h - \sin x}{h} \\ &= \lim_{h \rightarrow 0} \frac{\sin x(\cos h - 1) + \cos x \sin h}{h} \\ &= \lim_{h \rightarrow 0} \left(\sin x \cdot \frac{\cos h - 1}{h} \right) + \lim_{h \rightarrow 0} \left(\cos x \cdot \frac{\sin h}{h} \right) \\ &= \sin x \cdot \lim_{h \rightarrow 0} \left(\frac{\cos h - 1}{h} \right) + \cos x \cdot \lim_{h \rightarrow 0} \left(\frac{\sin h}{h} \right) \\ &= \sin x \cdot 0 + \cos x \cdot 1 = \cos x. \end{aligned}$$

$$\begin{aligned}
 \frac{d}{dx}(\cos x) &= \lim_{h \rightarrow 0} \frac{\cos(x + h) - \cos x}{h} \\
 &= \lim_{h \rightarrow 0} \frac{(\cos x \cos h - \sin x \sin h) - \cos x}{h} \\
 &= \lim_{h \rightarrow 0} \frac{\cos x (\cos h - 1) - \sin x \sin h}{h} \\
 &= \lim_{h \rightarrow 0} \left(\cos x \cdot \frac{\cos h - 1}{h} \right) - \lim_{h \rightarrow 0} \left(\sin x \cdot \frac{\sin h}{h} \right) \\
 &= \cos x \cdot \lim_{h \rightarrow 0} \frac{\cos h - 1}{h} - \sin x \cdot \lim_{h \rightarrow 0} \frac{\sin h}{h} \\
 &= \cos x \cdot 0 - \sin x \cdot 1 \\
 &= -\sin x.
 \end{aligned}$$

Derivative definition

Cosine angle sum identity

Example 5a
and Theorem 7,
Section 2.4

3.5 Derivatives of Trigonometric Functions



Theorem

$$\frac{d}{dx} (\sin x) = \cos x$$

and

$$\frac{d}{dx} (\cos x) = -\sin x$$

3.5 Derivatives

$$\frac{d}{dx} (\sin x) = \cos x \quad \frac{d}{dx} (\cos x) = -\sin x$$



Example

Differentiate $y = x^2 - \sin x$.

$$\frac{dy}{dx} = \frac{d}{dx} (x^2) - \frac{d}{dx} (\sin x) = 2x - \cos x.$$

3.5 Derivatives

$$\frac{d}{dx} (\sin x) = \cos x \quad \frac{d}{dx} (\cos x) = -\sin x$$



Example

Differentiate $y = x^2 \sin x$.

We will use the product rule ($(uv)' = u'v + uv'$) with $u = x^2$ and $v = \sin x$.

3.5 Derivatives

$$\frac{d}{dx} (\sin x) = \cos x \quad \frac{d}{dx} (\cos x) = -\sin x$$



Example

Differentiate $y = x^2 \sin x$.

We will use the product rule ($(uv)' = u'v + uv'$) with $u = x^2$ and $v = \sin x$.

$$y' = (\textcolor{orange}{x^2})'(\sin x) + (x^2)(\sin x)' = \textcolor{orange}{2x} \sin x + x^2 \cos x.$$

3.5 Derivatives

$$\frac{d}{dx} (\sin x) = \cos x \quad \frac{d}{dx} (\cos x) = -\sin x$$



Example

Differentiate $y = \frac{\sin x}{x}$.

This time we use the quotient rule $(\left(\frac{u}{v}\right)' = \frac{u'v - uv'}{v^2})$ with $u = \sin x$ and $v = x$.

3.5 Derivatives

$$\frac{d}{dx} (\sin x) = \cos x \quad \frac{d}{dx} (\cos x) = -\sin x$$



Example

Differentiate $y = \frac{\sin x}{x}$.

This time we use the quotient rule $(\left(\frac{u}{v}\right)' = \frac{u'v - uv'}{v^2})$ with $u = \sin x$ and $v = x$.

$$y' = \frac{(\sin x)'x - (\sin x)(x)'}{x^2} = \frac{x \cos x - 1 \sin x}{x^2}.$$

3.5 Derivatives

$$\frac{d}{dx}(\sin x) = \cos x \quad \frac{d}{dx}(\cos x) = -\sin x$$



Example

Differentiate $y = 5x + \cos x$.

$$\frac{dy}{dx} = \frac{d}{dx}(5x) + \frac{d}{dx}(\cos x) = 5 - \sin x.$$

3.5 Derivatives

$$\frac{d}{dx}(\sin x) = \cos x \quad \frac{d}{dx}(\cos x) = -\sin x$$



Example

Differentiate $y = \sin x \cos x$.

By the product rule, we have that

$$\frac{dy}{dx} = \frac{d}{dx}(\sin x) \cos x + \sin x \frac{d}{dx}(\cos x) = \cos^2 x - \sin^2 x.$$

3.5 Derivatives

$$\frac{d}{dx} (\sin x) = \cos x \quad \frac{d}{dx} (\cos x) = -\sin x$$



Example

Differentiate $y = \frac{\cos x}{1-\sin x}$.

By the quotient rule, we have that

$$\frac{dy}{dx} = \frac{\frac{d}{dx}(\cos x)(1 - \sin x) - (\cos x)\frac{d}{dx}(1 - \sin x)}{(1 - \sin x)^2}$$

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3.5 Derivatives

$$\frac{d}{dx} (\sin x) = \cos x \quad \frac{d}{dx} (\cos x) = -\sin x$$



Example

Differentiate $y = \frac{\cos x}{1-\sin x}$.

By the quotient rule, we have that

$$\begin{aligned}\frac{dy}{dx} &= \frac{\frac{d}{dx}(\cos x)(1 - \sin x) - (\cos x)\frac{d}{dx}(1 - \sin x)}{(1 - \sin x)^2} \\ &= \frac{-\sin x(1 - \sin x) - \cos x(0 - \cos x)}{(1 - \sin x)^2}\end{aligned}$$

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3.5 Derivatives

$$\frac{d}{dx}(\sin x) = \cos x \quad \frac{d}{dx}(\cos x) = -\sin x$$



Example

Differentiate $y = \frac{\cos x}{1-\sin x}$.

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$$\begin{aligned}\frac{dy}{dx} &= \frac{\frac{d}{dx}(\cos x)(1 - \sin x) - (\cos x)\frac{d}{dx}(1 - \sin x)}{(1 - \sin x)^2} \\&= \frac{-\sin x(1 - \sin x) - \cos x(0 - \cos x)}{(1 - \sin x)^2} \\&= \frac{-\sin x + \sin^2 x + \cos^2 x}{(1 - \sin x)^2} \\&= \frac{-\sin x + 1}{(1 - \sin x)^2} = \frac{1 - \sin x}{(1 - \sin x)^2} \\&= \frac{1}{1 - \sin x}.\end{aligned}$$

3.5 Derivatives of Trigonometric Functions



Please read Example 3 in your textbook.

The Tangent Function

Theorem

$$\frac{d}{dx} (\tan x) = \sec^2 x.$$

3.5 Derivatives of Trigonometric Functions



The Tangent Function

Theorem

$$\frac{d}{dx} (\tan x) = \sec^2 x.$$

Proof.

Using the quotient rule, we can calculate that

$$\frac{d}{dx} (\tan x) = \frac{d}{dx} \left(\frac{\sin x}{\cos x} \right) = \frac{\frac{d}{dx}(\sin x)(\cos x) - (\sin x)\frac{d}{dx}(\cos x)}{\cos^2 x}$$

=

=



The Tangent Function

Theorem

$$\frac{d}{dx} (\tan x) = \sec^2 x.$$

Proof.

Using the quotient rule, we can calculate that

$$\begin{aligned}\frac{d}{dx} (\tan x) &= \frac{d}{dx} \left(\frac{\sin x}{\cos x} \right) = \frac{\frac{d}{dx}(\sin x)(\cos x) - (\sin x)\frac{d}{dx}(\cos x)}{\cos^2 x} \\ &= \frac{(\cos x)(\cos x) - (\sin x)(-\sin x)}{\cos^2 x} = \frac{\cos^2 x + \sin^2 x}{\cos^2 x} \\ &= \end{aligned}$$



The Tangent Function

Theorem

$$\frac{d}{dx} (\tan x) = \sec^2 x.$$

Proof.

Using the quotient rule, we can calculate that

$$\begin{aligned}\frac{d}{dx} (\tan x) &= \frac{d}{dx} \left(\frac{\sin x}{\cos x} \right) = \frac{\frac{d}{dx}(\sin x)(\cos x) - (\sin x)\frac{d}{dx}(\cos x)}{\cos^2 x} \\&= \frac{(\cos x)(\cos x) - (\sin x)(-\sin x)}{\cos^2 x} = \frac{\cos^2 x + \sin^2 x}{\cos^2 x} \\&= \frac{1}{\cos^2 x} = \sec^2 x.\end{aligned}$$



3.5 Derivatives of Trigonometric Functions



The Other Three

Theorem

$$\frac{d}{dx} (\sec x) = \sec x \tan x$$

$$\frac{d}{dx} (\cot x) = -\operatorname{cosec}^2 x$$

$$\frac{d}{dx} (\operatorname{cosec} x) = -\operatorname{cosec} x \cot x$$

You can use the quotient rule to prove these three rules. We may ask you to prove one of them in an exam.

3.5 Derivatives

$$\frac{d}{dx}(\tan x) = \sec^2 x \quad \frac{d}{dx}(\sec x) = \sec x \tan x$$



Example

Find y'' if $y = \sec x$.

3.5 Derivative

$$\frac{d}{dx}(\tan x) = \sec^2 x \quad \frac{d}{dx}(\sec x) = \sec x \tan x$$



Example

Find y'' if $y = \sec x$.

Since $y' = \sec x \tan x$, we have that

$$y'' = \frac{d}{dx}(y') = \frac{d}{dx}(\sec x \tan x)$$

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3.5 Derivative

$$\frac{d}{dx}(\tan x) = \sec^2 x \quad \frac{d}{dx}(\sec x) = \sec x \tan x$$



Example

Find y'' if $y = \sec x$.

Since $y' = \sec x \tan x$, we have that

$$\begin{aligned}y'' &= \frac{d}{dx}(y') = \frac{d}{dx}(\sec x \tan x) \\&= \frac{d}{dx}(\sec x) \tan x + \sec x \frac{d}{dx}(\tan x) \\&= \\&= \end{aligned}$$

3.5 Derivative

$$\frac{d}{dx}(\tan x) = \sec^2 x \quad \frac{d}{dx}(\sec x) = \sec x \tan x$$



Example

Find y'' if $y = \sec x$.

Since $y' = \sec x \tan x$, we have that

$$\begin{aligned}y'' &= \frac{d}{dx}(y') = \frac{d}{dx}(\sec x \tan x) \\&= \frac{d}{dx}(\sec x) \tan x + \sec x \frac{d}{dx}(\tan x) \\&= (\sec x \tan x)(\tan x) + (\sec x)(\sec^2 x) \\&= \sec x \tan^2 x + \sec^3 x.\end{aligned}$$

The differentiability of the trigonometric functions throughout their domains implies their continuity at every point in their domains (Theorem 1, Section 3.2). So we can calculate limits of algebraic combinations and compositions of trigonometric functions by direct substitution.

EXAMPLE 7 We can use direct substitution in computing limits involving trigonometric functions. We must be careful to avoid division by zero, which is algebraically undefined.

$$\lim_{x \rightarrow 0} \frac{\sqrt{2 + \sec x}}{\cos(\pi - \tan x)} = \frac{\sqrt{2 + \sec 0}}{\cos(\pi - \tan 0)} = \frac{\sqrt{2 + 1}}{\cos(\pi - 0)} = \frac{\sqrt{3}}{-1} = -\sqrt{3}$$





The Chain Rule

3.6 The Chain Rule



How do we differentiate $F(x) = \sin(x^2 - 4)$?

3.6 The Chain Rule



Theorem (The Chain Rule)

Suppose that

- $y = f(u)$ is differentiable at the point $u = g(x)$; and
- $g(x)$ is differentiable at x .

Then $f \circ g$ is differentiable at x and

$$(f \circ g)'(x) = f'(g(x))g'(x).$$

3.6 The Chain Rule



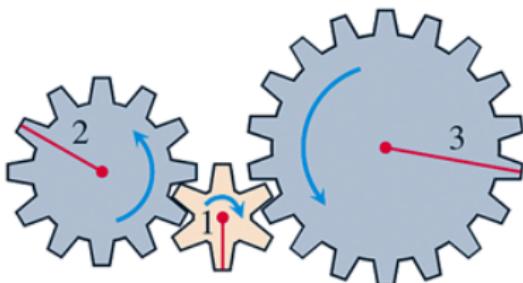
The Chain Rule is easier to remember if we use Leibniz's notation:

$$\frac{dy}{dx} = \frac{dy}{du} \frac{du}{dx}$$

3.6 The Chain Rule

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$$\frac{dy}{dx} = \frac{dy}{du} \frac{du}{dx}$$



C: y turns B: u turns A: x turns

3.6 The Chain Rule

$$\frac{dy}{dx} = \frac{dy}{du} \frac{du}{dx}$$



Example

Differentiate $y = \sin(x^2 - 4)$.

We have $y = \sin u$ with $u = x^2 - 4$.

3.6 The Chain Rule

$$\frac{dy}{dx} = \frac{dy}{du} \frac{du}{dx}$$



Example

Differentiate $y = \sin(x^2 - 4)$.

We have $y = \sin u$ with $u = x^2 - 4$. Now $\frac{dy}{du} = \cos u$ and $\frac{du}{dx} = 2x$.

3.6 The Chain Rule

$$\frac{dy}{dx} = \frac{dy}{du} \frac{du}{dx}$$



Example

Differentiate $y = \sin(x^2 - 4)$.

We have $y = \sin u$ with $u = x^2 - 4$. Now $\frac{dy}{du} = \cos u$ and $\frac{du}{dx} = 2x$. Therefore

$$\begin{aligned}\frac{dy}{dx} &= \frac{dy}{du} \frac{du}{dx} \\&= \\&= \\&= \\&=\end{aligned}$$

by the Chain Rule.

3.6 The Chain Rule

$$\frac{dy}{dx} = \frac{dy}{du} \frac{du}{dx}$$



Example

Differentiate $y = \sin(x^2 - 4)$.

We have $y = \sin u$ with $u = x^2 - 4$. Now $\frac{dy}{du} = \cos u$ and $\frac{du}{dx} = 2x$. Therefore

$$\begin{aligned}\frac{dy}{dx} &= \frac{dy}{du} \frac{du}{dx} \\&= (\cos u)(2x) \\&= 2x \cos u \\&=\end{aligned}$$

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3.6 The Chain Rule

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Example

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$$\begin{aligned}\frac{dy}{dx} &= \frac{dy}{du} \frac{du}{dx} \\&= (\cos u)(2x) \\&= 2x \cos u \\&= 2x \cos(x^2 - 4)\end{aligned}$$

by the Chain Rule.

3.6 The Chain Rule

$$\frac{dy}{dx} = \frac{dy}{du} \frac{du}{dx}$$



Example

Differentiate $\sin(x^2 + x)$.

3.6 The Chain Rule

$$\frac{dy}{dx} = \frac{dy}{du} \frac{du}{dx}$$



Example

Differentiate $\sin(x^2 + x)$.

Let $u = x^2 + x$.

3.6 The Chain Rule

$$\frac{dy}{dx} = \frac{dy}{du} \frac{du}{dx}$$



Example

Differentiate $\sin(x^2 + x)$.

Let $u = x^2 + x$. Then

$$\begin{aligned}\frac{d}{dx} (\sin(x^2 + x)) &= \frac{d}{du} (\sin u) \frac{du}{dx} \\&= (\cos u)(2x + 1) \\&= (2x + 1) \cos(x^2 + x)\end{aligned}$$

by the Chain Rule.

3.6 The Chain Rule

$$\frac{dy}{dx} = \frac{dy}{du} \frac{du}{dx}$$



Example (Using the Chain Rule Two Times)

Differentiate $g(t) = \tan(5 - \sin 2t)$.

Let $u = 5 - \sin 2t$. Then $g(t) = \tan u$. Hence

$$\frac{dg}{dt} = \frac{dg}{du} \frac{du}{dt} = (\sec^2 u) \frac{d}{dt}(5 - \sin 2t).$$

3.6 The Chain Rule

$$\frac{dy}{dx} = \frac{dy}{du} \frac{du}{dx}$$



Example (Using the Chain Rule Two Times)

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We need to use the Chain Rule a second time: Let $w = 2t$.

3.6 The Chain Rule

$$\frac{dy}{dx} = \frac{dy}{du} \frac{du}{dx}$$



Example (Using the Chain Rule Two Times)

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Let $u = 5 - \sin 2t$. Then $g(t) = \tan u$. Hence

$$\frac{dg}{dt} = \frac{dg}{du} \frac{du}{dt} = (\sec^2 u) \frac{d}{dt}(5 - \sin 2t).$$

We need to use the Chain Rule a second time: Let $w = 2t$.
Then

$$\begin{aligned}\frac{dg}{dt} &= (\sec^2 u) \frac{d}{dt}(5 - \sin 2t) \\&= (\sec^2 u) \frac{d}{dw}(5 - \sin w) \frac{dw}{dt} \\&= (\sec^2 u)(-\cos w)(2) \\&= -2 \cos 2t \sec^2(5 - \sin 2t).\end{aligned}$$

3.6 The Chain Rule

$$\frac{dy}{dx} = \frac{dy}{du} \frac{du}{dx}$$



Remark

Your final answer should not have u or w in it.

$$\frac{dy}{dx} = \frac{dy}{du} \frac{du}{dx}$$



The Chain Rule with Powers of a Function

If

- f is a differentiable function of u ;
- u is a differentiable function of x ; and
- $y = f(u)$,

then the Chain Rule $\frac{dy}{dx} = \frac{dy}{du} \frac{du}{dx}$ is the same as

$$\frac{d}{dx} f(u) = f'(u) \frac{du}{dx}.$$

$$\frac{dy}{dx} = \frac{dy}{du} \frac{du}{dx}$$



The Chain Rule with Powers of a Function

If

- f is a differentiable function of u ;
- u is a differentiable function of x ; and
- $y = f(u)$,

then the Chain Rule $\frac{dy}{dx} = \frac{dy}{du} \frac{du}{dx}$ is the same as

$$\frac{d}{dx} f(u) = f'(u) \frac{du}{dx}.$$

Now suppose that $f(u) = u^n$ where $n \in \mathbb{R}$. Then $f'(u) = nu^{n-1}$.

So

$$\frac{d}{dx} (u^n) = nu^{n-1} \frac{du}{dx}.$$

3.6 The Chain Rule

$$\frac{d}{dx}(u^n) = nu^{n-1} \frac{du}{dx}$$



Example

$$\frac{d}{dx} (5x^3 - x^4)^7 =$$

=

3.6 The Chain Rule

$$\frac{d}{dx}(u^n) = nu^{n-1} \frac{du}{dx}$$



Example

$$\begin{aligned}\frac{d}{dx}(5x^3 - x^4)^7 &= 7(5x^3 - x^4)^6 \frac{d}{dx}(5x^3 - x^4) \\ &= \end{aligned}$$

3.6 The Chain Rule

$$\frac{d}{dx}(u^n) = nu^{n-1} \frac{du}{dx}$$



Example

$$\begin{aligned}\frac{d}{dx}(5x^3 - x^4)^7 &= 7(5x^3 - x^4)^6 \frac{d}{dx}(5x^3 - x^4) \\ &= 7(5x^3 - x^4)^6 (15x^2 - 4x^3).\end{aligned}$$

3.6 The Chain Rule

$$\frac{d}{dx}(u^n) = nu^{n-1} \frac{du}{dx}$$



Example

$$\frac{d}{dx}\left(\frac{1}{3x-2}\right) = \frac{d}{dx}(3x-2)^{-1}$$

=

=

3.6 The Chain Rule

$$\frac{d}{dx}(u^n) = nu^{n-1} \frac{du}{dx}$$



Example

$$\begin{aligned}\frac{d}{dx} \left(\frac{1}{3x-2} \right) &= \frac{d}{dx} (3x-2)^{-1} \\&= -1 (3x-2)^{-2} \frac{d}{dx} (3x-2) \\&= -\left(\frac{1}{(3x-2)^2} \right) (2) = \frac{-3}{(3x-2)^2}.\end{aligned}$$

3.6 The Chain Rule

$$\frac{d}{dx}(u^n) = nu^{n-1} \frac{du}{dx}$$



Example

$$\frac{d}{dx}(\sin^5 x) = 5 \sin^4 x \frac{d}{dx}(\sin x) = 5 \sin^4 x \cos x.$$

3.6 The Chain Rule

$$\frac{d}{dx}(u^n) = nu^{n-1} \frac{du}{dx}$$



Example

Differentiate $|x|$.

Since $|x| = \sqrt{x^2}$,

3.6 The Chain Rule

$$\frac{d}{dx}(u^n) = nu^{n-1} \frac{du}{dx}$$



Example

Differentiate $|x|$.

Since $|x| = \sqrt{x^2}$,

$$\sqrt{3^2} = \sqrt{9} = 3 = |3|$$

$$\sqrt{(-3)^2} = \sqrt{9} = 3 = |-3|$$

3.6 The Chain Rule

$$\frac{d}{dx}(u^n) = nu^{n-1} \frac{du}{dx}$$



Example

Differentiate $|x|$.

Since $|x| = \sqrt{x^2}$, we can calculate that if $x \neq 0$ then

$$\begin{aligned}\frac{d}{dx}|x| &= \frac{d}{dx}(\sqrt{x^2}) = \frac{d}{du}(\sqrt{u}) \frac{d}{dx}(x^2) \\ &= \frac{1}{2\sqrt{u}}2x = \frac{x}{\sqrt{x^2}} = \frac{x}{|x|}.\end{aligned}$$

3.6 The Chain Rule

$$\frac{d}{dx}(u^n) = nu^{n-1} \frac{du}{dx}$$



Example

Show that the slope of every line tangent to the curve

$$y = \frac{1}{(1-2x)^3}$$
 is positive.

3.6 The Chain Rule

$$\frac{d}{dx}(u^n) = nu^{n-1} \frac{du}{dx}$$



Example

Show that the slope of every line tangent to the curve $y = \frac{1}{(1-2x)^3}$ is positive.

This is the same as:

Example

Let $y = \frac{1}{(1-2x)^3}$ for $x \neq \frac{1}{2}$. Show that $\frac{dy}{dx} > 0$.

3.6 The Chain Rule

$$\frac{d}{dx}(u^n) = nu^{n-1} \frac{du}{dx}$$



Example

Show that the slope of every line tangent to the curve $y = \frac{1}{(1-2x)^3}$ is positive.

This is the same as:

Example

Let $y = \frac{1}{(1-2x)^3}$ for $x \neq \frac{1}{2}$. Show that $\frac{dy}{dx} > 0$.

First we calculate that

$$\begin{aligned}\frac{dy}{dx} &= \frac{d}{dx}(1-2x)^{-3} = -3(1-2x)^{-4} \frac{d}{dx}(1-2x) \\ &= -3(1-2x)^{-4}(-2) = \frac{6}{(1-2x)^4}\end{aligned}$$

if $x \neq \frac{1}{2}$.

3.6 The Chain Rule

$$\frac{d}{dx}(u^n) = nu^{n-1} \frac{du}{dx}$$



Example

Show that the slope of every line tangent to the curve $y = \frac{1}{(1-2x)^3}$ is positive.

This is the same as:

Example

Let $y = \frac{1}{(1-2x)^3}$ for $x \neq \frac{1}{2}$. Show that $\frac{dy}{dx} > 0$.

First we calculate that

$$\begin{aligned}\frac{dy}{dx} &= \frac{d}{dx}(1-2x)^{-3} = -3(1-2x)^{-4} \frac{d}{dx}(1-2x) \\ &= -3(1-2x)^{-4}(-2) = \frac{6}{(1-2x)^4}\end{aligned}$$

if $x \neq \frac{1}{2}$. Since $(1-2x)^4 > 0$ if $x \neq \frac{1}{2}$ and $6 > 0$, we have that $\frac{dy}{dx} > 0$ if $x \neq \frac{1}{2}$.

3.6 The Chain Rule



Example (Why Do We Use Radians in Calculus?)

Remember that $\frac{d}{dx} \sin x = \cos x$ is true *only if we use radians.*

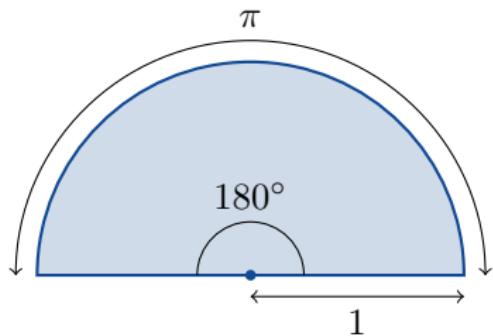
What happens if we use degrees?

3.6 The Chain Rule

Example (Why Do We Use Radians in Calculus?)

Remember that $\frac{d}{dx} \sin x = \cos x$ is true *only if we use radians.*

What happens if we use degrees?



Remember that

$$180 \text{ degrees} = \pi \text{ radians}$$

$$180^\circ = \pi$$

$$1^\circ = \frac{\pi}{180}$$

$$x^\circ = \frac{\pi x}{180}.$$

3.6 The Chain Rule



So

$$\frac{d}{dx} \sin x^\circ = \frac{d}{dx} \sin \left(\frac{\pi x}{180} \right)$$

3.6 The Chain Rule



So

$$\frac{d}{dx} \sin x^\circ = \frac{d}{dx} \sin \left(\frac{\pi x}{180} \right) = \frac{\pi}{180} \cos \left(\frac{\pi x}{180} \right) = \frac{\pi}{180} \cos x^\circ.$$

3.6 The Chain Rule

So

$$\frac{d}{dx} \sin x^\circ = \frac{d}{dx} \sin\left(\frac{\pi x}{180}\right) = \frac{\pi}{180} \cos\left(\frac{\pi x}{180}\right) = \frac{\pi}{180} \cos x^\circ.$$

Therefore we have

$$\frac{d}{dx} \sin x = \cos x$$

a nice formula

and

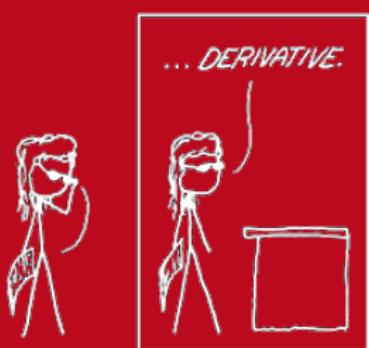
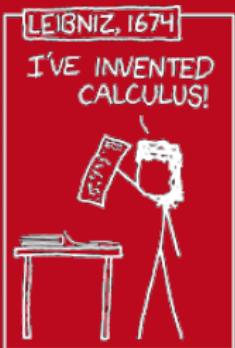
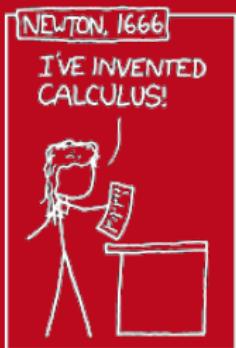
$$\frac{d}{dx} \sin x^\circ = \frac{\pi}{180} \cos x^\circ.$$

not nice

This is why we use radians in Calculus.

Break

We will continue at 3pm





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Implicit Differentiation

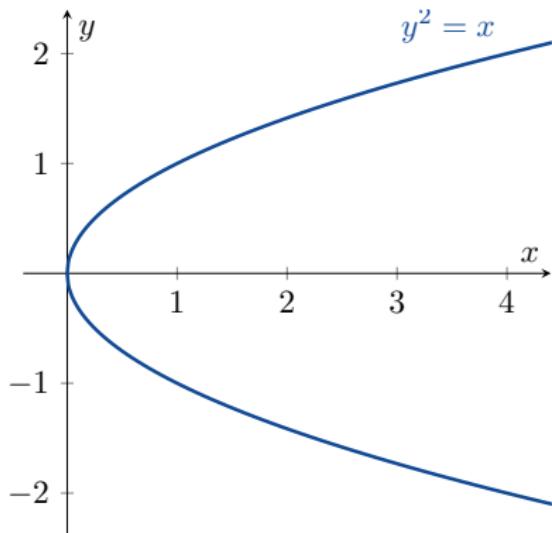
3.7 Implicit Differentiation



Remark

Implicit differentiation seems to scare Calculus students, but it really shouldn't – it is really just the Chain Rule again.

3.7 Implicit Differentiation

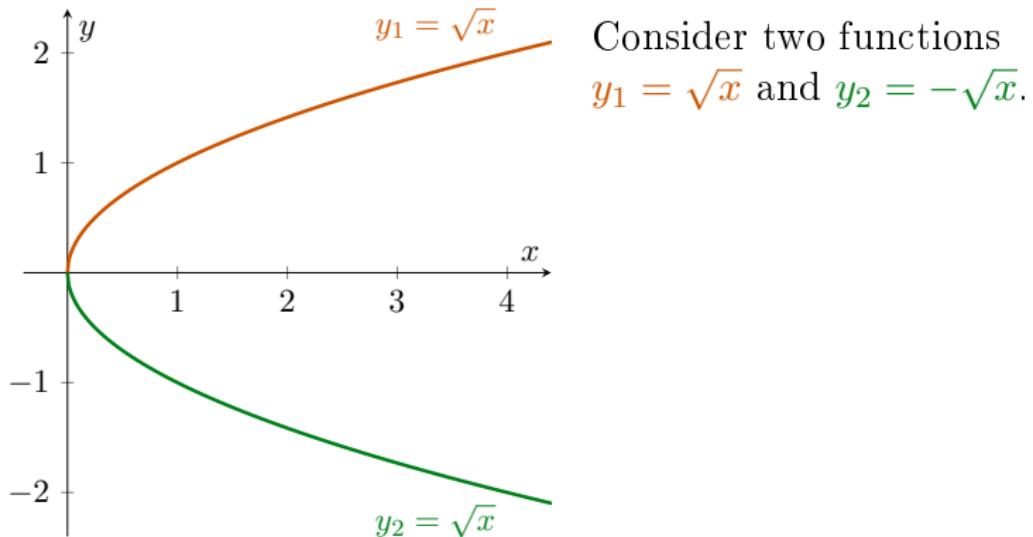


Example

Find $\frac{dy}{dx}$ if $y^2 = x$.

3.7 Implicit Differentiation

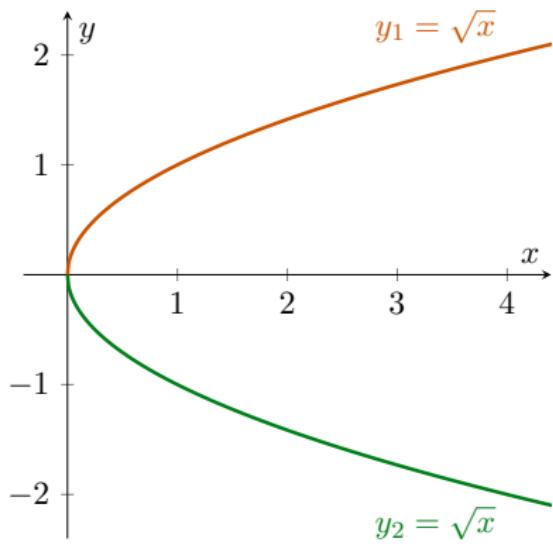
Solution 1: (Don't use Implicit Differentiation)



3.7 Implicit Differentiation



Solution 1: (Don't use Implicit Differentiation)



Consider two functions
 $y_1 = \sqrt{x}$ and $y_2 = -\sqrt{x}$.

Then

$$\frac{dy_1}{dx} = \frac{1}{2\sqrt{x}}$$

and

$$\frac{dy_2}{dx} = -\frac{1}{2\sqrt{x}}.$$

3.7 Implicit Differentiation



Solution 2: (Use Implicit Differentiation)

Since

$$y^2 = x$$

3.7 Implicit Differentiation



Solution 2: (Use Implicit Differentiation)

Since

$$\frac{d}{dx}(y^2) = \frac{d}{dx}(x)$$

3.7 Implicit Differentiation



Solution 2: (Use Implicit Differentiation)

Since

$$\frac{d}{dx}(y^2) = \frac{d}{dx}(x) = 1$$

3.7 Implicit Differentiation



Solution 2: (Use Implicit Differentiation)

Since

$$\frac{d}{dy}(y^2) \frac{dy}{dx} = \frac{d}{dx}(y^2) = \frac{d}{dx}(x) = 1$$

3.7 Implicit Differentiation



Solution 2: (Use Implicit Differentiation)

Since

$$2y \frac{dy}{dx} = \frac{d}{dy}(y^2) \frac{dy}{dx} = \frac{d}{dx}(y^2) = \frac{d}{dx}(x) = 1$$

3.7 Implicit Differentiation



Solution 2: (Use Implicit Differentiation)

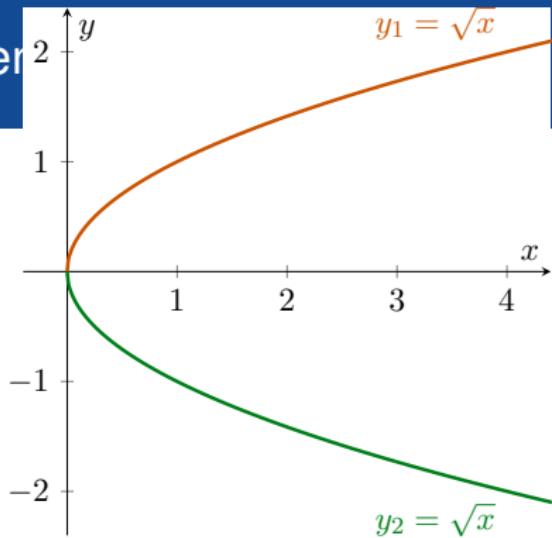
Since

$$2y \frac{dy}{dx} = \frac{d}{dy}(y^2) \frac{dy}{dx} = \frac{d}{dx}(y^2) = \frac{d}{dx}(x) = 1$$

we have

$$\frac{dy}{dx} = \frac{1}{2y}.$$

3.7 Implicit Differentiation



Remark

The formula

$$\frac{dy}{dx} = \frac{1}{2y}$$

gives both

$$\frac{dy_1}{dx} = \frac{1}{2y_1} = \frac{1}{2\sqrt{x}} \quad \text{and} \quad \frac{dy_2}{dx} = \frac{1}{2y_2} = \frac{1}{2(-\sqrt{x})} = -\frac{1}{2\sqrt{x}}.$$

3.7 Implicit Differentiation



Remark

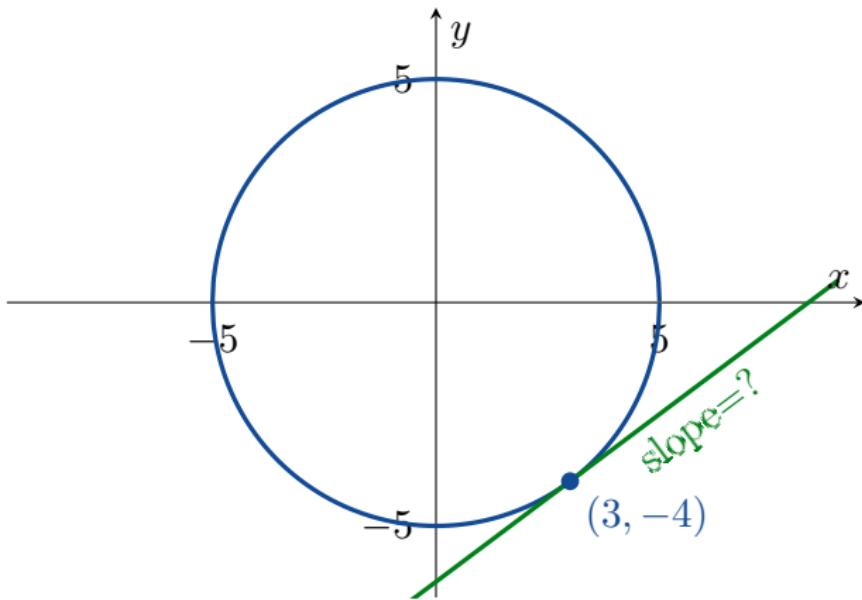
Implicit Differentiation is just

- 1** Differentiate both sides of an equation;
- 2** Use the Chain Rule.

3.7 Implicit Differentiation

Example

Find the slope of the circle $x^2 + y^2 = 25$ at the point $(3, -4)$.



3.7 Implicit Differentiation

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Find the slope of the circle $x^2 + y^2 = 25$ at the point $(3, -4)$.

Using Implicit Differentiation, we calculate that

$$x^2 + y^2 = 25$$

$$\frac{d}{dx}(x^2) + \frac{d}{dx}(y^2) = \frac{d}{dx}(25)$$

$$2x + 2y\frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = -\frac{x}{y}.$$

3.7 Implicit Differentiation

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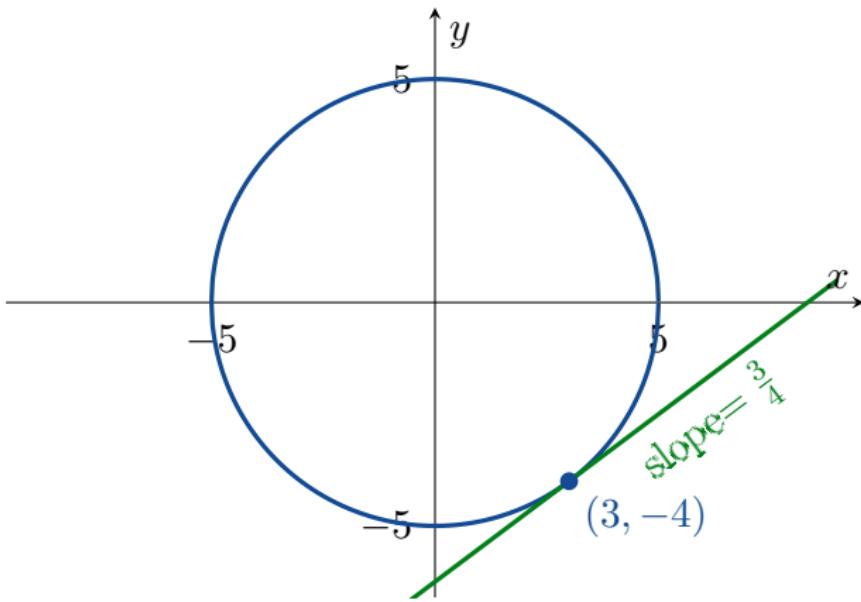
Therefore the slope at $(3, -4)$ is

$$\left. \frac{dy}{dx} \right|_{(3, -4)} = \left. -\frac{x}{y} \right|_{(3, -4)} = -\frac{3}{-4} = \frac{3}{4}.$$

3.7 Implicit Differentiation

Example

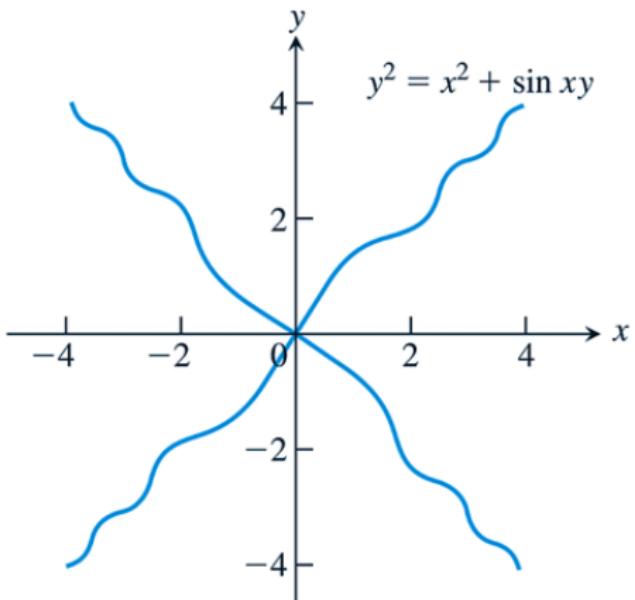
Find the slope of the circle $x^2 + y^2 = 25$ at the point $(3, -4)$.



3.7 Implicit Differentiation

Example

Find $\frac{dy}{dx}$ if $y^2 = x^2 + \sin xy$.



3.7 Implicit Differentiation

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Find $\frac{dy}{dx}$ if $y^2 = x^2 + \sin xy$.

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$$\frac{d}{dx}(y^2) = \frac{d}{dx}(x^2) + \frac{d}{dx}(\sin xy)$$

3.7 Implicit Differentiation

Example

Find $\frac{dy}{dx}$ if $y^2 = x^2 + \sin xy$.

We calculate that

$$\begin{aligned}\frac{d}{dx}(y^2) &= \frac{d}{dx}(x^2) + \frac{d}{dx}(\sin xy) \\ 2y\frac{dy}{dx} &= 2x + (\cos xy)\frac{d}{dx}(xy)\end{aligned}$$

3.7 Implicit Differentiation

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3.7 Implicit Differentiation

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3.7 Implicit Differentiation

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$$(2y - x\cos xy)\frac{dy}{dx} = 2x + y\cos xy$$

$$\frac{dy}{dx} = \frac{2x + y\cos xy}{2y - x\cos xy}.$$

3.7 Implicit Differentiation



Example (Derivatives of Higher Order)

Find $\frac{d^2y}{dx^2}$ if $2x^3 - 3y^2 = 8$.

If $y \neq 0$, then

$$2x^3 - 3y^2 = 8$$

3.7 Implicit Differentiation



Example (Derivatives of Higher Order)

Find $\frac{d^2y}{dx^2}$ if $2x^3 - 3y^2 = 8$.

If $y \neq 0$, then

$$\begin{aligned}\frac{d}{dx} (2x^3 - 3y^2) &= \frac{d}{dx} (8) \\ 6x^2 - 6yy' &= 0\end{aligned}$$

3.7 Implicit Differentiation



Example (Derivatives of Higher Order)

Find $\frac{d^2y}{dx^2}$ if $2x^3 - 3y^2 = 8$.

If $y \neq 0$, then

$$\frac{d}{dx} (2x^3 - 3y^2) = \frac{d}{dx} (8)$$

$$6x^2 - 6yy' = 0$$

$$6yy' = 6x^2$$

$$y' = \frac{x^2}{y}.$$

3.7 Implicit Differentiation



$$y' = \frac{x^2}{y}$$

Now we need to differentiate again.

3.7 Implicit Differentiation



$$y' = \frac{x^2}{y}$$

Now we need to differentiate again. Using the Quotient Rule we calculate that

$$y'' = \frac{d}{dx} \left(\frac{x^2}{y} \right) = \frac{2xy - x^2y'}{y^2} = \frac{2x}{y} - \frac{x^2}{y^2} \cdot y'$$

=

if $y \neq 0$.

3.7 Implicit Differentiation



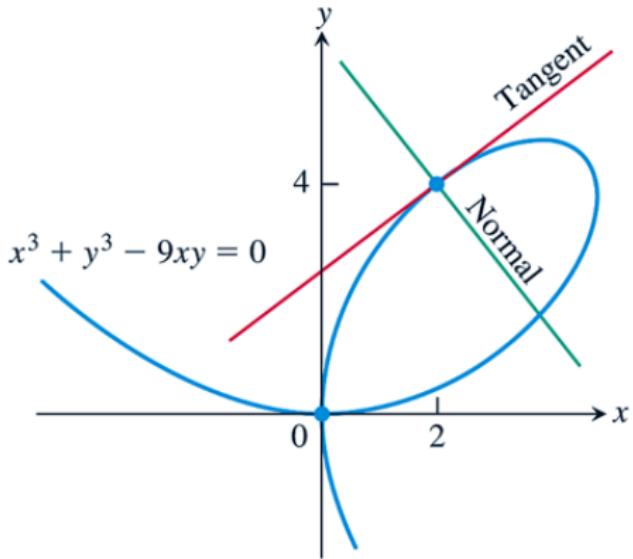
$$y' = \frac{x^2}{y}$$

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$$\begin{aligned}y'' &= \frac{d}{dx} \left(\frac{x^2}{y} \right) = \frac{2xy - x^2y'}{y^2} = \frac{2x}{y} - \frac{x^2}{y^2} \cdot y' \\&= \frac{2x}{y} - \frac{x^2}{y^2} \cdot \frac{x^2}{y} = \frac{2x}{y} - \frac{x^4}{y^3}\end{aligned}$$

if $y \neq 0$.

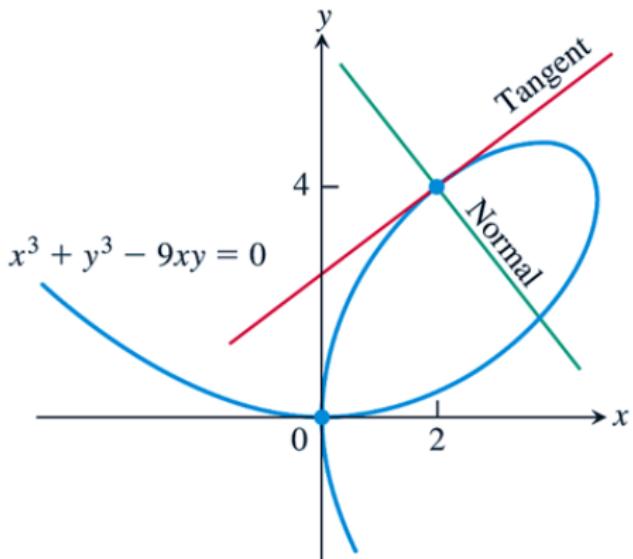
3.7 Implicit Differentiation



Definition

The line passing through a point P which is perpendicular/orthogonal to the tangent line at P is called the *normal line* at P .

3.7 Implicit Differentiation



If the tangent line has slope k , then the normal line has slope $-\frac{1}{k}$.

3.7 Implicit Differentiation



Example

Show that the point $(2, 4)$ lies on the curve $x^3 + y^3 - 9xy = 0$.
Then find the tangent and normal to the curve there.

Since

$$x^3 + y^3 - 9xy = (2)^3 + (4)^3 - 9(2)(4) = 8 + 64 - 72 = 0,$$

the point $(2, 4)$ does lie on this curve.

3.7 Implicit Differentiation



Then we calculate that

$$x^3 + y^3 - 9xy = 0$$

$$\frac{d}{dx}(x^3) + \frac{d}{dx}(y^3) - \frac{d}{dx}(9xy) = \frac{d}{dx}(0)$$

$$3x^2 + 3y^2 \frac{dy}{dx} - 9 \left(x \frac{dy}{dx} + y \frac{dx}{dx} \right) = 0$$

$$(3y^2 - 9x) \frac{dy}{dx} = 9y - 3x^2$$

$$\frac{dy}{dx} = \frac{9y - 3x^2}{3y^2 - 9x} = \frac{3y - x^2}{y^2 - 3x}$$

3.7 Implicit Differentiation



Then we calculate that

$$x^3 + y^3 - 9xy = 0$$

$$\frac{d}{dx}(x^3) + \frac{d}{dx}(y^3) - \frac{d}{dx}(9xy) = \frac{d}{dx}(0)$$

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$$\frac{dy}{dx} = \frac{9y - 3x^2}{3y^2 - 9x} = \frac{3y - x^2}{y^2 - 3x}$$

and

$$\frac{dy}{dx} \Big|_{(2,4)} = \frac{3y - x^2}{y^2 - 3x} \Big|_{(2,4)} = \frac{3 \cdot 4 - 2^2}{4^2 - 3 \cdot 2} = \frac{8}{10} = \frac{4}{5}.$$

3.7 Implicit Differentiation



$$\left. \frac{dy}{dx} \right|_{(2,4)} = \frac{4}{5}$$

So the tangent has slope $\frac{4}{5}$ and the normal has slope $-\frac{5}{4}$.

3.7 Implicit Differentiation



$$\left. \frac{dy}{dx} \right|_{(2,4)} = \frac{4}{5}$$

So the tangent has slope $\frac{4}{5}$ and the normal has slope $-\frac{5}{4}$.

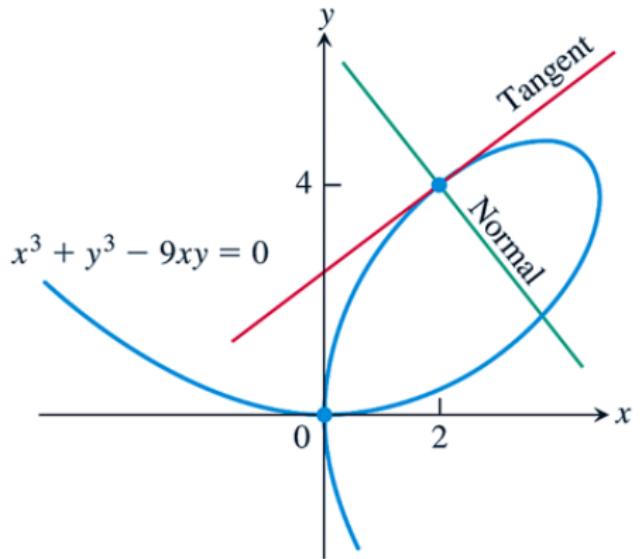
Therefore the tangent line is

$$y = 4 + \frac{4}{5}(x - 2) = \frac{4}{5}x + \frac{12}{5}$$

and the normal line is

$$y = 4 - \frac{5}{4}(x - 2) = -\frac{5}{4}x + \frac{13}{2}.$$

3.7 Implicit Differentiation



The tangent line at $(2, 4)$ is

$$y = 4 + \frac{4}{5}(x - 2) = \frac{4}{5}x + \frac{12}{5}$$

and the normal line at $(2, 4)$ is

$$y = 4 - \frac{5}{4}(x - 2) = -\frac{5}{4}x + \frac{13}{2}.$$



Next Time

- 3.8 Related Rates
- 3.9 Linearisation and Differentials
- 4.1 Extreme Values of Functions on Closed Intervals
- 4.2 The Mean Value Theorem