2019–20 **MATH216 Mathematics IV – Exercise Sheet 2** N. Course

Solutions to Exercise Sheet 1 are now available at `www.neilcourse.co.uk/math216.html`

Exercise 8 (Separable Equations)**.** Solve the following initial value problems:

(a) $\begin{cases} \frac{dy}{dx} = (1-2x)y^2 \\ y(0) = -\frac{1}{6} \end{cases}$ (b) $\begin{cases} x + ye^{-x}\frac{dy}{dx} = 0 \\ y(0) = 1 \end{cases}$ (c) $\begin{cases} \frac{dy}{dx} = \frac{2x}{y+x^2y} \\ y(0) = -2 \end{cases}$

Exercise 9 (Stable, Unstable and Semi-Stable Equilibrium Solutions)**.** Each of the following problems involve equations of the form $y' = f(y)$. In each problem, (i) sketch the graph of $f(y)$ versus y; (ii) find the equilibrium solutions (critical points) of the ODE; and (iii) classify each equilibrium solution as asymptotically stable, semi-stable, or unstable.

(a) $\frac{dy}{dt} = ay + by^2$, $a, b > 0$, $y_0 \geq 0$.

(b) $\frac{dy}{dt} = ay + by^2$, $a, b > 0$, $-\infty < y_0 < \infty$.

(c) $\frac{dy}{dt} = y(y-1)(y-2)$, $y_0 \geq 0$.

(d) $\frac{dy}{dt} = y(1-y)^2$, $-\infty < y_0 < \infty$.

(e) $\frac{dy}{dt} = e^y - 1$, $-\infty < y_0 < \infty$.

(f) $\frac{dy}{dt} = e^{-y} - 1$, $-\infty < y_0 < \infty$.

Exercise 10 (Sick Students)**.**

Suppose that the students of İstanbul Okan Üniversitesi can be divided into two groups; those who have the flu virus and can infect others, and those who do not have it but are susceptible. Let x be the proportion of susceptible individuals and y the proportion of infectious individuals; then $x+y=1$.

Assume that the disease spreads by contact between sick students and well students, and that the rate of spread $\frac{dy}{dt}$ is proportional to the number of such contacts. So $\frac{dy}{dt} = k_1 \times$ (number of contacts). Further, assume that members of both groups move about freely among each other, so the number of contacts is proportional to the product of x and y. So (number of contacts) $= k_2xy$. Since $x = 1-y$, we obtain the initial value problem

$$\begin{cases} \frac{dy}{dt} = \alpha y(1-y), \\ y(0) = y_0, \end{cases} \tag{1}$$

where $\alpha > 0$ is a constant, and $0 \leq y_0 \leq 1$ is the initial proportion of infectious individuals.

İstanbul Okan Üniversitesi öğrencilerinin iki gruba ayrıldıklarını varsayın; grip virüsü taşıyan, diğer öğrencilere bulaştırabilecek olanlar ve virüsü taşımayan ancak hastalığa yakalanabilecek olanlar. Hastalığa yakalanabilecek bireylerin oranı x; hastalığı taşıyan ve bulaştırabilecek olanların oranı y'dir. Bu durumda $x+y=1$.

Hastalığın, hasta öğrencilerle sağlıklı öğrenciler arasında etkileşimle yayıldığını, ve $\frac{dy}{dt}$ olan yayılma hızının etkileşim sayısıyla orantılı olduğunu varsayın. Yani $\frac{dy}{dt} = k_1 \times$ (etkileşim sayısı). Ayrıca, her iki grubun üyelerinin birbirlerinin arasında serbestçe dolaştıklarını varsayın; böylece etkileşim sayısı x ve yenin çarpımları ile orantılıdır. Yani, (etkileş im sayısı) $= k_2xy$. $x = 1-y$ olduğundan, (1)'i elde ederiz. $\alpha > 0$ sabit sayıdır, $0 \leq y_0 \leq 1$ hastalık bulaştırabilecek öğrencilerin en baştaki oranıdır.

(a) Find the equilibrium points for the differential equation and determine whether each is asymptotically stable, semi-stable, or unstable.

(b) Draw the graphs of some solutions.

(c) Solve (1).

(d) Suppose that $y_0 > 0$. Show that $\lim_{t\to\infty} y(t) = 1$, which means that ultimately all students catch the disease.

Exercise 11 (Exact Equations). Determine if each of the following ODEs is an exact equation. If it is exact, find the solution.

(a) $(2x+4y)+(2x-2y)y'=0$

(b) $(2x+3)+(2y-2)y'=0$

(c) $(3x^2-2xy+2)+(6y^2-x^2+3)\dfrac{dy}{dx}=0$

(d) $(2xy^2+2y)+(2x^2y+2x)y'=0$

(e) $(e^x\sin y-2y\sin x)+(e^x\cos y+2\cos x)\dfrac{dy}{dx}=0$

(f) $(e^x\sin y+2y)dx+(3x-e^x\sin y)dy=0$

Exercise 12 (Exact Equations). The following equations are not exact. For each one, (i) find an integrating factor ($\mu(x)$ or $\mu(y)$) which changes the equation into an exact equation; and (ii) solve the equation.

(a) $(3x^2y+2xy+y^3)+(x^2+y^2)y'=0$

(b) $y'=e^{2x}+y-1$

(c) $dx+\left(\dfrac{x}{y}-\sin y\right)dy=0$

(d) $y+(2xy-e^{-2y})y'=0$

Exercise 13 (Homogeneous Equations). Use the substitution $v(x)=\frac{y}{x}$ (or equivalently $y=v(x)x$ and then $y'=v'(x)x+v(x)$) to solve the following ODEs:

(a) $(x^2+3xy+y^2)dx-x^2dy=0$

(b) $\dfrac{dy}{dx}=\dfrac{x^2-3y^2}{2xy}$

(c) $x\dfrac{dy}{dx}=y+\sqrt{x^2-y^2}$

Exercise 14 (Bernoulli Equations). We can use the substitution $v(x)=y^{1-n}$ to solve $y'+p(t)y=q(t)y^n$. Use this technique to solve the following ODEs:

(a) $t^2y'+2ty-y^3=0$

(b) $y'=ry-ky^2$ (where $r>0$ and $k>0$ are constants). This is an autonomous equation called the Logistic Equation.

(c) $y'=\varepsilon y-\sigma y^3$ (where $\varepsilon>0$ and $\sigma>0$ are constants). This equation occurs in the study of the stability of fluid flow.