Lecture 10

- 5.3 Homogeneous Linear Systems with Constant Coefficients
- 5.4 Complex Eigenvalues
- 5.5 Fundamental Matrices

Homogeneous Linear Systems with Constant Coefficients

Consider

$$\mathbf{x}' = A\mathbf{x}$$

where $A \in \mathbb{R}^{n\times n}$.

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If $n = 1$, then we just have

$$\frac{dx}{dt} = ax$$

which has general solution $x(t) = ce^{at}$.

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For $n > 1$, we guess that

$$\mathbf{x}(t) = \boldsymbol{\xi} e^{rt}$$

is a solution to $\mathbf{x}' = A\mathbf{x}$, for some number $r \in \mathbb{C}$ and some vector $\boldsymbol{\xi} \in \mathbb{C}^n$.

But if $\mathbf{x}(t) = \boldsymbol{\xi} e^{rt}$, then

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$$\begin{aligned} r\boldsymbol{\xi} e^{rt} &= \mathbf{x}' = A\mathbf{x} = A\boldsymbol{\xi} e^{rt} \\ r\boldsymbol{\xi} &= A\boldsymbol{\xi} \\ (A - rI)\boldsymbol{\xi} &= \mathbf{0} \end{aligned}$$

where I is the identity matrix.

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$$
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\end{aligned}
$$

where I is the identity matrix. Hence r must be an eigenvalue of A and $\boldsymbol{\xi}$ must be a corresponding eigenvector of A.

Remark

So the idea is:

1. Find the eigenvalues;
2. Find the eigenvectors; then
3. Write $\mathbf{x}^{(j)}(t) = \boldsymbol{\xi}^{(j)} e^{r_j t}$.

Example

Solve

$$\mathbf{x}' = \begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix} \mathbf{x}.$$

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First we find the eigenvalues. Since

$$\begin{aligned} 0 = \det(A - rI) &= \begin{vmatrix} 1-r & 1 \\ 4 & 1-r \end{vmatrix} = (1-r)^2 - 4 \\ &= r^2 - 2r - 3 = (r+1)(r-3), \end{aligned}$$

the eigenvalues are $r_1 = 3$ and $r_2 = -1$.

Using the first eigenvalue $r_1 = 3$, we calculate that

$$\mathbf{0} = (A - r_1 I)\boldsymbol{\xi} = \begin{bmatrix} -2 & 1 \\ 4 & -2 \end{bmatrix} \begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix}$$

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Hence we can choose $\boldsymbol{\xi}^{(1)} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$. Then using the second eigenvalue $r_2 = -1$, we calculate that

$$\mathbf{0} = (A - r_2 I)\boldsymbol{\xi} = \begin{bmatrix} 2 & 1 \\ 4 & 2 \end{bmatrix} \begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix}$$

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Hence we can choose $\boldsymbol{\xi}^{(2)} = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$. This gives us two solutions:

$$\mathbf{x}^{(1)}(t) = \begin{bmatrix} 1 \\ 2 \end{bmatrix} e^{3t} \qquad \text{and} \qquad \mathbf{x}^{(2)}(t) = \begin{bmatrix} 1 \\ -2 \end{bmatrix} e^{-t}.$$

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$$W\big(\mathbf{x}^{(1)}, \mathbf{x}^{(2)}\big)(t) = \begin{vmatrix} e^{3t} & e^{-t} \\ 2e^{3t} & -2e^{-t} \end{vmatrix} = -4e^{2t} \neq 0.$$

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Since $W \neq 0$, we have that $\mathbf{x}^{(1)}(t)$ and $\mathbf{x}^{(2)}(t)$ are linearly independent. So $\mathbf{x}^{(1)}(t)$ and $\mathbf{x}^{(2)}(t)$ form a fundamental set of solutions. Therefore the general solution is

$$\boxed{\mathbf{x}(t) = c_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} e^{3t} + c_2 \begin{bmatrix} 1 \\ -2 \end{bmatrix} e^{-t}.}$$

Example

Solve

$$\begin{cases} \mathbf{x}' = \begin{bmatrix} 8 & -1 \\ 6 & 1 \end{bmatrix} \mathbf{x} \\ \\ \mathbf{x}(0) = \begin{bmatrix} 1 \\ -2 \end{bmatrix}. \end{cases}$$

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The eigenvalues are $r_1 = 7$ and $r_2 = 2$.

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The eigenvalues are $r_1 = 7$ and $r_2 = 2$. The corresponding eigenvectors are $\boldsymbol{\xi}^{(1)} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\boldsymbol{\xi}^{(2)} = \begin{bmatrix} 1 \\ 6 \end{bmatrix}$.

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$$\implies \begin{cases} c_1 = \frac{8}{5} \\ c_2 = -\frac{3}{5}. \end{cases}$$

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$$\implies \begin{cases} c_1 = \frac{8}{5} \\ c_2 = -\frac{3}{5}. \end{cases}$$

Therefore the solution to the IVP is

$$\boxed{\mathbf{x}(t) = \frac{8}{5} \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{7t} - \frac{3}{5} \begin{bmatrix} 1 \\ 6 \end{bmatrix} e^{2t}.}$$

Example

Solve

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The eigenvalues are $r_1 = -1$ and $r_2 = -4$.

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The eigenvalues are $r_1 = -1$ and $r_2 = -4$. The corresponding eigenvectors are $\boldsymbol{\xi}^{(1)} = \begin{bmatrix} 1 \\ \sqrt{2} \end{bmatrix}$ and $\boldsymbol{\xi}^{(2)} = \begin{bmatrix} -\sqrt{2} \\ 1 \end{bmatrix}$.

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The eigenvalues are $r_1 = -1$ and $r_2 = -4$. The corresponding eigenvectors are $\boldsymbol{\xi}^{(1)} = \begin{bmatrix} 1 \\ \sqrt{2} \end{bmatrix}$ and $\boldsymbol{\xi}^{(2)} = \begin{bmatrix} -\sqrt{2} \\ 1 \end{bmatrix}$. Hence the general solution is

$$\boxed{\mathbf{x}(t) = c_1 \begin{bmatrix} 1 \\ \sqrt{2} \end{bmatrix} e^{-t} + c_2 \begin{bmatrix} -\sqrt{2} \\ 1 \end{bmatrix} e^{-4t}.}$$

Remark

$$\det(A - rI) = 0$$

There are three possibilities for the eigenvalues of A.

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There are three possibilities for the eigenvalues of A.

1. All the eigenvalues are real and different;
2. Some eigenvalues occur in complex conjugate pairs;
3. Some eigenvalues are repeated.

If all the eigenvalues are real and different, then the eigenvectors are linearly independent:

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So $W\left(\mathbf{x}^{(1)}, \ldots, \mathbf{x}^{(n)}\right)(t) \neq 0$ and $\mathbf{x}^{(1)}(t)$, $\ldots$, $\mathbf{x}^{(n)}(t)$ form a fundamental set of solutions.

If some eigenvalues are repeated, *but there are n linearly independent eigenvectors*, then this is also true: $\mathbf{x}^{(1)}(t), \ldots,$ $\mathbf{x}^{(n)}(t)$ form a fundamental set of solutions.

Example

Solve

$$\mathbf{x}' = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} \mathbf{x}.$$

The eigenvalues and eigenvectors are

$$r_1 = 2 \qquad r_2 = -1 \qquad r_3 = -1$$

$$\boldsymbol{\xi}^{(1)} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \qquad \boldsymbol{\xi}^{(2)} = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \qquad \boldsymbol{\xi}^{(3)} = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$$

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which gives us the following three solutions

$$\mathbf{x}^{(1)}(t) = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} e^{2t} \qquad \mathbf{x}^{(2)}(t) = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} e^{-t} \qquad \mathbf{x}^{(3)}(t) = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} e^{-t}.$$

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You can check that the Wronskian of $\mathbf{x}^{(1)}$, $\mathbf{x}^{(2)}$ and $\mathbf{x}^{(3)}$ is non-zero.

$$\mathbf{x}^{(1)}(t) = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} e^{2t} \qquad \mathbf{x}^{(2)}(t) = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} e^{-t} \qquad \mathbf{x}^{(3)}(t) = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} e^{-t}.$$

You can check that the Wronskian of $\mathbf{x}^{(1)}$, $\mathbf{x}^{(2)}$ and $\mathbf{x}^{(3)}$ is non-zero. Therefore $\mathbf{x}^{(1)}$, $\mathbf{x}^{(2)}$ and $\mathbf{x}^{(3)}$ form a fundamental set of solutions.

$$\mathbf{x}^{(1)}(t) = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} e^{2t} \qquad \mathbf{x}^{(2)}(t) = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} e^{-t} \qquad \mathbf{x}^{(3)}(t) = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} e^{-t}.$$

You can check that the Wronskian of $\mathbf{x}^{(1)}$, $\mathbf{x}^{(2)}$ and $\mathbf{x}^{(3)}$ is non-zero. Therefore $\mathbf{x}^{(1)}$, $\mathbf{x}^{(2)}$ and $\mathbf{x}^{(3)}$ form a fundamental set of solutions. The general solution to the ODE is

$$\boxed{\mathbf{x}(t) = c_1 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} e^{2t} + c_2 \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} e^{-t} + c_3 \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} e^{-t}.}$$

Remark

So if we have repeated eigenvalues with n linearly independent eigenvectors then there is no problem.

In section 5.6 we will study what to do if we do not have enough eigenvectors.

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Remark

Next we will study systems with complex eigenvalues.

5.4 Complex Eigenvalues

Consider

$$\mathbf{x}' = A\mathbf{x}$$

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$$\begin{bmatrix} x_1' \\ x_2' \\ \vdots \\ x_n' \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

Consider

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where $A \in \mathbb{R}^{n\times n}$.

$$\begin{cases} x_1' = a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n \\ x_2' = a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n \\ \qquad \vdots \\ x_n' = a_{n1}x_1 + a_{n2}x_2 + \ldots + a_{nn}x_n \end{cases}$$

Any complex eigenvalues of A must occur in complex conjugate pairs: If $r_1 = \lambda + i\mu$ is an eigenvalue of A, then $r_2 = \overline{r}_1 = \lambda - i\mu$ is also an eigenvalue of A.

Moreover, if $\boldsymbol{\xi}^{(1)}$ is an eigenvector of A corresponding to r_1, then $\boldsymbol{\xi}^{(2)} = \overline{\boldsymbol{\xi}^{(1)}}$ is an eigenvector of A corresponding to $r_2 = \bar{r}_1$.

Two solutions of $\mathbf{x}' = A\mathbf{x}$ are

$$\mathbf{x}^{(1)}(t) = \boldsymbol{\xi}^{(1)} e^{r_1 t} \qquad \text{and} \qquad \mathbf{x}^{(2)}(t) = \overline{\boldsymbol{\xi}^{(1)}} e^{\overline{r}_1 t}.$$

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$$\mathbf{x}^{(1)}(t) = \boldsymbol{\xi}^{(1)} e^{r_1 t} \qquad \text{and} \qquad \mathbf{x}^{(2)}(t) = \overline{\boldsymbol{\xi}^{(1)}} e^{\overline{r}_1 t}.$$

But $\mathbf{x}^{(1)}, \mathbf{x}^{(2)} : \mathbb{R} \to \mathbb{C}^n$ and we want solutions $: \mathbb{R} \to \mathbb{R}^n$.

If $r_1 = \lambda + i\mu$, and $\boldsymbol{\xi}^{(1)} = \mathbf{a} + i\mathbf{b}$ (λ, $\mu \in \mathbb{R}$, $\mathbf{a}$, $\mathbf{b} \in \mathbb{R}^n$), then

$$\mathbf{x}^{(1)}(t) = (\mathbf{a} + i\mathbf{b})e^{(\lambda+i\mu)t}$$

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$$\begin{aligned}\mathbf{x}^{(1)}(t) &= (\mathbf{a} + i\mathbf{b})e^{(\lambda+i\mu)t} \\ &= (\mathbf{a} + i\mathbf{b})e^{\lambda t}\left(\cos\mu t + i\sin\mu t\right)\end{aligned}$$

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$$\begin{aligned}
\mathbf{x}^{(1)}(t) &= (\mathbf{a} + i\mathbf{b})e^{(\lambda + i\mu)t} \\
&= (\mathbf{a} + i\mathbf{b})e^{\lambda t}(\cos\mu t + i\sin\mu t) \\
&= e^{\lambda t}(\mathbf{a}\cos\mu t - \mathbf{b}\sin\mu t) + ie^{\lambda t}(\mathbf{a}\sin\mu t + \mathbf{b}\cos\mu t)
\end{aligned}$$

If $r_1 = \lambda + i\mu$, and $\boldsymbol{\xi}^{(1)} = \mathbf{a} + i\mathbf{b}$ ($\lambda,\ \mu \in \mathbb{R}$, $\mathbf{a},\ \mathbf{b} \in \mathbb{R}^n$), then

$$\begin{aligned}
\mathbf{x}^{(1)}(t) &= (\mathbf{a} + i\mathbf{b})e^{(\lambda + i\mu)t} \\
&= (\mathbf{a} + i\mathbf{b})e^{\lambda t}(\cos\mu t + i\sin\mu t) \\
&= e^{\lambda t}(\mathbf{a}\cos\mu t - \mathbf{b}\sin\mu t) + ie^{\lambda t}(\mathbf{a}\sin\mu t + \mathbf{b}\cos\mu t) \\
&= \mathbf{u}(t) + i\mathbf{v}(t).
\end{aligned}$$

Remark

- The functions $\mathbf{u}(t)$ and $\mathbf{v}(t)$ solve the ODE.

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So we can include $\mathbf{u}(t)$ and $\mathbf{v}(t)$ in our fundamental set of solutions instead of $\mathbf{x}^{(1)}(t)$ and $\mathbf{x}^{(2)}(t)$.

Example

Solve
$$\mathbf{x}' = \begin{bmatrix} -\frac{1}{2} & 1 \\ -1 & -\frac{1}{2} \end{bmatrix} \mathbf{x}.$$

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We calculate that

$$0 = \det(A - rI) = \begin{vmatrix} -\frac{1}{2} - r & 1 \\ -1 & -\frac{1}{2} - r \end{vmatrix} = r^2 + r + \frac{5}{4}$$

and

$$r_{1,2} = \frac{-1 \pm \sqrt{1-5}}{2} = \frac{-1 \pm 2i}{2} = -\frac{1}{2} \pm i.$$

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So we have $r_1 = -\frac{1}{2} + i$ and $r_2 = -\frac{1}{2} - i$.

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So we have $r_1 = -\frac{1}{2} + i$ and $r_2 = -\frac{1}{2} - i$. We will use r_1. We do not need r_2.

Since

$$0 = (A - r_1 I)\boldsymbol{\xi}^{(1)} = \begin{bmatrix} -i & 1 \\ -1 & -i \end{bmatrix} \begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix}$$

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we can choose

$$\boldsymbol{\xi}^{(1)} = \begin{bmatrix} 1 \\ i \end{bmatrix}.$$

Note that we also have

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but we don't need $\boldsymbol{\xi}^{(2)}$.

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$$=$$

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Hence we choose

$$\mathbf{u}(t) = e^{-\frac{t}{2}} \begin{bmatrix} \cos t \\ -\sin t \end{bmatrix} \quad \text{and} \quad \mathbf{v}(t) = e^{-\frac{t}{2}} \begin{bmatrix} \sin t \\ \cos t \end{bmatrix}.$$

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the answer is yes.

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the answer is yes. Therefore $\mathbf{u}(t)$ and $\mathbf{v}(t)$ form a fundamental set of solutions.

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the answer is yes. Therefore $\mathbf{u}(t)$ and $\mathbf{v}(t)$ form a fundamental set of solutions.

Therefore the general solution to $\mathbf{x}' = \begin{bmatrix} -\frac{1}{2} & 1 \\ -1 & -\frac{1}{2} \end{bmatrix} \mathbf{x}$ is

$$\boxed{\mathbf{x}(t) = c_1 \, e^{-\frac{t}{2}} \begin{bmatrix} \cos t \\ -\sin t \end{bmatrix} + c_2 \, e^{-\frac{t}{2}} \begin{bmatrix} \sin t \\ \cos t \end{bmatrix}.}$$

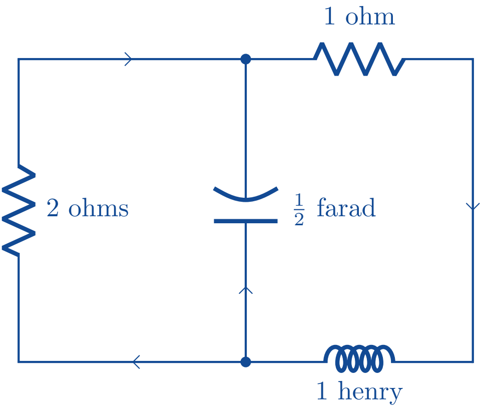
Remark

Our method is

1. Find the eigenvalues;
2. Find the eigenvectors;
3.
   - If r_j is real, just use the solution $\mathbf{x}^{(j)}(t) = \boldsymbol{\xi}^{(j)} e^{r_j t}$;
   - But if r_j is complex, write

$$\mathbf{x}^{(j)}(t) = \boldsymbol{\xi}^{(j)} e^{r_j t} = \left(\begin{array}{c} \text{real valued} \\ \text{function} \end{array} \right) + i \left(\begin{array}{c} \text{real valued} \\ \text{function} \end{array} \right)$$

   and use these two functions.


1 ohm
2 ohms
$\frac{1}{2}$ farad
1 henry

Example

The electric circuit shown above is described by

$$\begin{cases} I' = -I - V \\ V' = 2I - V \end{cases}$$

where

$$I = \text{the current through the inductor}$$
$$V = \text{the voltage drop across the capacitor.}$$

(Ask an Electrical Engineer.)

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Suppose that at time $t = 0$ the current is 2 amperes and the voltage drop is 2 volts. Find $I(t)$ and $V(t)$.

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Suppose that at time $t = 0$ the current is 2 amperes and the voltage drop is 2 volts. Find $I(t)$ and $V(t)$.

We must solve the IVP

$$\begin{cases} \dfrac{d}{dt}\begin{bmatrix} I \\ V \end{bmatrix} = \begin{bmatrix} -1 & -1 \\ 2 & -1 \end{bmatrix}\begin{bmatrix} I \\ V \end{bmatrix} \\[2ex] \begin{bmatrix} I \\ V \end{bmatrix}(0) = \begin{bmatrix} 2 \\ 2 \end{bmatrix}. \end{cases}$$

The eigenvalues of $\begin{bmatrix} -1 & -1 \\ 2 & -1 \end{bmatrix}$ are $r_1 = -1 + i\sqrt{2}$ and $r_2 = -1 - i\sqrt{2}$ (please check). The corresponding eigenvectors are

$$\boldsymbol{\xi}^{(1)} = \begin{bmatrix} 1 \\ -i\sqrt{2} \end{bmatrix} \qquad \text{and} \qquad \boldsymbol{\xi}^{(2)} = \begin{bmatrix} 1 \\ i\sqrt{2} \end{bmatrix}.$$

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Then we calculate that

$$\begin{aligned} \mathbf{x}^{(1)}(t) = \boldsymbol{\xi}^{(1)} e^{r_1 t} &= \begin{bmatrix} 1 \\ -i\sqrt{2} \end{bmatrix} e^{(-1+i\sqrt{2})t} \\ &= \begin{bmatrix} 1 \\ -i\sqrt{2} \end{bmatrix} e^{-t}\big(\cos\sqrt{2}t + i\sin\sqrt{2}t \big) \\ &= e^{-t} \begin{bmatrix} \cos\sqrt{2}t + i\sin\sqrt{2}t \\ -i\sqrt{2}\cos\sqrt{2}t + \sqrt{2}\sin\sqrt{2}t \end{bmatrix} \\ &= e^{-t} \begin{bmatrix} \cos\sqrt{2}t \\ \sqrt{2}\sin\sqrt{2}t \end{bmatrix} + ie^{-t} \begin{bmatrix} \sin\sqrt{2}t \\ -\sqrt{2}\cos\sqrt{2}t \end{bmatrix}. \end{aligned}$$

Hence the general solution to the ODE is

$$\begin{bmatrix} I(t) \\ V(t) \end{bmatrix} = c_1 e^{-t} \begin{bmatrix} \cos\sqrt{2}t \\ \sqrt{2}\sin\sqrt{2}t \end{bmatrix} + c_2 e^{-t} \begin{bmatrix} \sin\sqrt{2}t \\ -\sqrt{2}\cos\sqrt{2}t \end{bmatrix}.$$

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Using the initial condition, we calculate that

$$\begin{bmatrix} 2 \\ 2 \end{bmatrix} = \begin{bmatrix} I(0) \\ V(0) \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ -\sqrt{2} \end{bmatrix} \qquad \Longrightarrow \qquad \begin{cases} c_1 = 2 \\ c_2 = -\sqrt{2}. \end{cases}$$

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Thus

$$\begin{bmatrix} I(t) \\ V(t) \end{bmatrix} = 2\, e^{-t} \begin{bmatrix} \cos\sqrt{2}t \\ \sqrt{2}\sin\sqrt{2}t \end{bmatrix} - \sqrt{2}\, e^{-t} \begin{bmatrix} \sin\sqrt{2}t \\ -\sqrt{2}\cos\sqrt{2}t \end{bmatrix}.$$

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So the answers to this problem are

$$\boxed{I(t) = \qquad\qquad\qquad\qquad\qquad\qquad}$$

and

$$\boxed{V(t) = \qquad\qquad\qquad\qquad\qquad\qquad}$$

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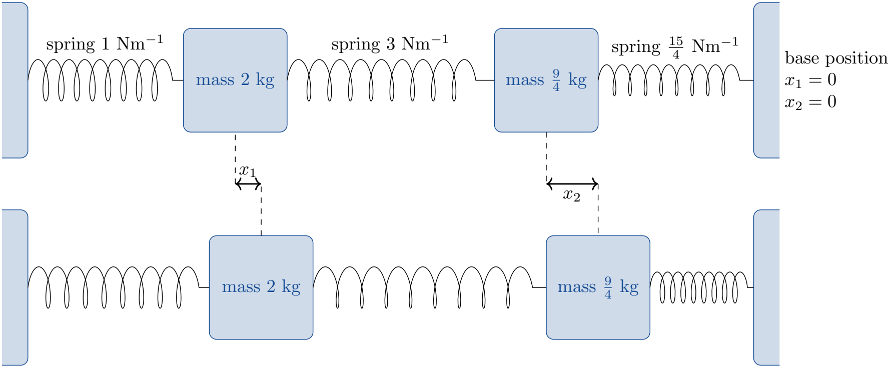
and

$$\boxed{V(t) = 2\sqrt{2}e^{-t}\sin\sqrt{2}t + 2e^{-t}\cos\sqrt{2}t.}$$

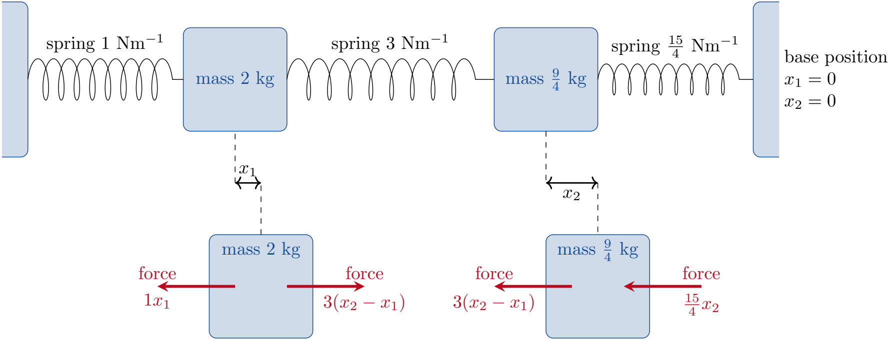


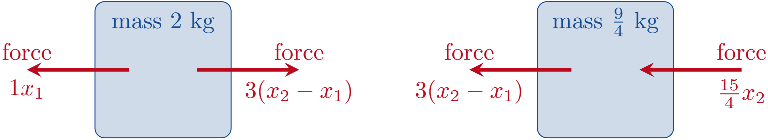
See `https://tinyurl.com/wm2ogdh` for an animated figure.

Example

For the dynamical system shown above, find $x_1(t)$ and $x_2(t)$.

As the springs are stretched and compressed, they apply forces on the blocks as shown below (Hooke's Law).





We calculate that

$$\text{mass} \times \text{acceleration} = \text{force}$$

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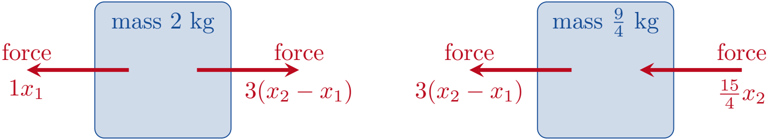
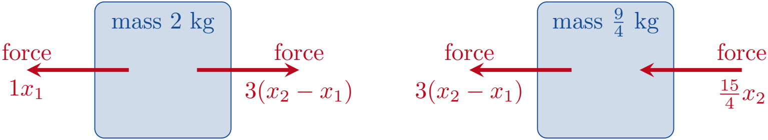


We calculate that

$$2\frac{d^2x_1}{dt^2} = \text{mass} \times \text{acceleration} = \text{force} = -x_1 + 3(x_2 - x_1)$$

$$\text{mass} \times \text{acceleration} = \text{force}$$



We calculate that

$$2\frac{d^2x_1}{dt^2} = \text{mass} \times \text{acceleration} = \text{force} = -x_1 + 3(x_2 - x_1)$$

$$\frac{9}{4}\frac{d^2x_2}{dt^2} = \text{mass} \times \text{acceleration} = \text{force} = -3(x_2 - x_1) - \frac{15}{4}x_2.$$

$$\begin{aligned} 2\frac{d^2x_1}{dt^2} &= -x_1 + 3(x_2 - x_1) \\ \frac{9}{4}\frac{d^2x_2}{dt^2} &= -3(x_2 - x_1) - \frac{15}{4}x_2. \end{aligned}$$

This is a system of 2 second order ODEs.

$$
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$$

This is a system of 2 second order ODEs. We want a system of first order ODEs.

$$2\frac{d^2x_1}{dt^2} = -x_1 + 3(x_2 - x_1)$$
$$\frac{9}{4}\frac{d^2x_2}{dt^2} = -3(x_2 - x_1) - \frac{15}{4}x_2.$$

Now let $y_1 = x_1$, $y_2 = x_2$, $y_3 = x_1'$ and $y_4 = x_2'$.

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y_1' &= x_1' = y_3 \\
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y_4' &=
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$$y_4' =$$

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y_4' &= x_2'' = \frac{4}{9}\left(-3x_2 + 3x_1 - \frac{15}{4}x_2\right) = \frac{4}{3}y_1 - 3y_2.
\end{aligned}$$

So

$$\mathbf{y}' = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -2 & \frac{3}{2} & 0 & 0 \\ \frac{4}{3} & -3 & 0 & 0 \end{bmatrix} \mathbf{y}.$$

The characteristic polynomial of this matrix is

$$0 = r^4 + 5r^2 + 4 = (r^2 + 1)(r^2 + 4).$$

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So $r_1 = i$, $r_2 = -i$, $r_3 = 2i$ and $r_4 = -2i$.

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So $r_1 = i$, $r_2 = -i$, $r_3 = 2i$ and $r_4 = -2i$. We will use r_1 and r_3 (we do not need r_2 and r_4).

The characteristic polynomial of this matrix is

$$0 = r^4 + 5r^2 + 4 = (r^2 + 1)(r^2 + 4).$$

So $r_1 = i$, $r_2 = -i$, $r_3 = 2i$ and $r_4 = -2i$. We will use r_1 and r_3 (we do not need r_2 and r_4).

The corresponding eigenvectors (please check) are

$$\boldsymbol{\xi}^{(1)} = \begin{bmatrix} 3 \\ 2 \\ 3i \\ 2i \end{bmatrix} \qquad \text{and} \qquad \boldsymbol{\xi}^{(3)} = \begin{bmatrix} 3 \\ -4 \\ 6i \\ -8i \end{bmatrix}.$$

It follows that

$$\boldsymbol{\xi}^{(1)} e^{r_1 t} = \begin{bmatrix} 3 \\ 2 \\ 3i \\ 2i \end{bmatrix} (\cos t + i \sin t) = \begin{bmatrix} 3\cos t \\ 2\cos t \\ -3\sin t \\ -2\sin t \end{bmatrix} + i \begin{bmatrix} 3\sin t \\ 2\sin t \\ 3\cos t \\ 2\cos t \end{bmatrix}$$

$$= \mathbf{u}(t) + i\mathbf{v}(t)$$

and

It follows that

$$\boldsymbol{\xi}^{(1)}e^{r_1 t} = \begin{bmatrix} 3 \\ 2 \\ 3i \\ 2i \end{bmatrix} (\cos t + i \sin t) = \begin{bmatrix} 3\cos t \\ 2\cos t \\ -3\sin t \\ -2\sin t \end{bmatrix} + i \begin{bmatrix} 3\sin t \\ 2\sin t \\ 3\cos t \\ 2\cos t \end{bmatrix}$$

$$= \mathbf{u}(t) + i\mathbf{v}(t)$$

and

$$\boldsymbol{\xi}^{(3)}e^{r_3 t} = \begin{bmatrix} 3 \\ -4 \\ 6i \\ -8i \end{bmatrix} (\cos 2t + i \sin 2t) = \begin{bmatrix} 3\cos 2t \\ -4\cos 2t \\ -6\sin 2t \\ +8\sin 2t \end{bmatrix} + i \begin{bmatrix} 3\sin 2t \\ -4\sin 2t \\ 6\cos 2t \\ -8\cos 2t \end{bmatrix}$$

$$= \mathbf{w}(t) + i\mathbf{z}(t)$$

Therefore the general solution is

$$\mathbf{y}(t) = c_1\mathbf{u}(t) + c_2\mathbf{v}(t) + c_3\mathbf{w}(t) + c_4\mathbf{z}(t)$$

Therefore the general solution is

$$\begin{aligned}\mathbf{y}(t) &= c_1\mathbf{u}(t) + c_2\mathbf{v}(t) + c_3\mathbf{w}(t) + c_4\mathbf{z}(t)\\ &= c_1\begin{bmatrix} 3\cos t \\ 2\cos t \\ -3\sin t \\ -2\sin t \end{bmatrix} + c_2\begin{bmatrix} 3\sin t \\ 2\sin t \\ 3\cos t \\ 2\cos t \end{bmatrix} + c_3\begin{bmatrix} 3\cos 2t \\ -4\cos 2t \\ -6\sin 2t \\ 8\sin 2t \end{bmatrix} + c_4\begin{bmatrix} 3\sin 2t \\ -4\sin 2t \\ 6\cos 2t \\ -8\cos 2t \end{bmatrix}.\end{aligned}$$

Example

Suppose that the above system has initial condition

$$\mathbf{y}(0) = \begin{bmatrix} -1 \\ 4 \\ 1 \\ 1 \end{bmatrix}.$$

Sketch graphs of $y_1(t)$ and $y_2(t)$.

The initial value problem

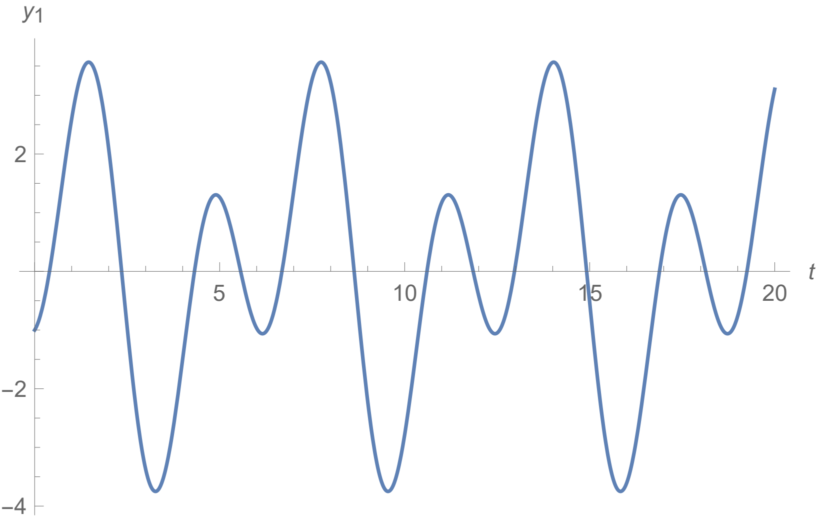
$$\mathbf{y}' = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -2 & \frac{3}{2} & 0 & 0 \\ \frac{4}{3} & -3 & 0 & 0 \end{bmatrix} \mathbf{y}, \qquad \mathbf{y}(0) = \begin{bmatrix} -1 \\ 4 \\ 1 \\ 1 \end{bmatrix}$$

has solution

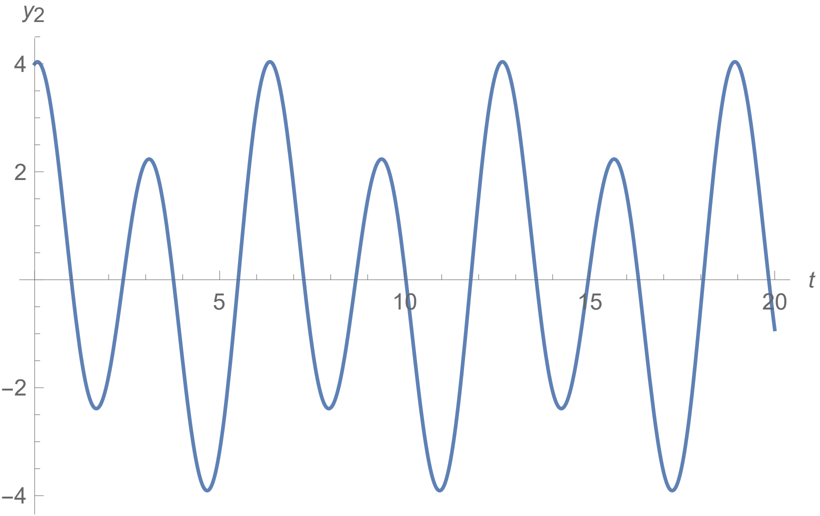
$$\mathbf{y}(t) = \frac{4}{9}\begin{bmatrix} 3\cos t \\ 2\cos t \\ -3\sin t \\ -2\sin t \end{bmatrix} + \frac{7}{18}\begin{bmatrix} 3\sin t \\ 2\sin t \\ 3\cos t \\ 2\cos t \end{bmatrix} - \frac{7}{9}\begin{bmatrix} 3\cos 2t \\ -4\cos 2t \\ -6\sin 2t \\ 8\sin 2t \end{bmatrix} - \frac{1}{36}\begin{bmatrix} 3\sin 2t \\ -4\sin 2t \\ 6\cos 2t \\ -8\cos 2t \end{bmatrix}.$$

Then we can draw the graphs of y_1 and y_2:

5.4 Complex Eigenvalues



5.4 Complex Eigenvalues



5.4 Complex Eigenvalues

Please see `https://tinyurl.com/s7uww7m`

Fundamental Matrices

Now consider

$$\mathbf{x}' = P(t)\mathbf{x}$$

where $P(t)$ is an $n \times n$ matrix function.

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where $P(t)$ is an $n \times n$ matrix function.

Suppose that $\mathbf{x}^{(1)}$, $\mathbf{x}^{(2)}$, $\ldots$, $\mathbf{x}^{(n)}$ are linearly independent solutions to this ODE. In other words, suppose that $\mathbf{x}^{(1)}$, $\mathbf{x}^{(2)}$, $\ldots$, $\mathbf{x}^{(n)}$ form a *fundamental set of solutions* to this ODE.

Definition

The matrix

$$\Psi(t) = \begin{bmatrix} \mathbf{x}^{(1)} & \mathbf{x}^{(2)} & \dots & \mathbf{x}^{(n)} \end{bmatrix} = \begin{bmatrix} x_1^{(1)}(t) & x_1^{(2)}(t) & \dots & x_1^{(n)}(t) \\ x_2^{(1)}(t) & x_2^{(2)}(t) & \dots & x_2^{(n)}(t) \\ \vdots & \vdots & & \vdots \\ x_n^{(1)}(t) & x_n^{(2)}(t) & \dots & x_n^{(n)}(t) \end{bmatrix}$$

is called a *fundamental matrix* of $\mathbf{x}' = P(t)\mathbf{x}$.

Example

Find a fundamental matrix for

$$\mathbf{x}' = \begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix} \mathbf{x}.$$

Example

Find a fundamental matrix for

$$\mathbf{x}' = \begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix} \mathbf{x}.$$

Recall that

$$\mathbf{x}^{(1)}(t) = \begin{bmatrix} 1 \\ 2 \end{bmatrix} e^{3t} \qquad \text{and} \qquad \mathbf{x}^{(2)}(t) = \begin{bmatrix} 1 \\ -2 \end{bmatrix} e^{-t}$$

form a fundamental set of solutions to this ODE.

Example

Find a fundamental matrix for

$$\mathbf{x}' = \begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix} \mathbf{x}.$$

Recall that

$$\mathbf{x}^{(1)}(t) = \begin{bmatrix} 1 \\ 2 \end{bmatrix} e^{3t} \qquad \text{and} \qquad \mathbf{x}^{(2)}(t) = \begin{bmatrix} 1 \\ -2 \end{bmatrix} e^{-t}$$

form a fundamental set of solutions to this ODE. Therefore

$$\Psi(t) = \begin{bmatrix} e^{3t} & e^{-t} \\ 2e^{3t} & -2e^{-t} \end{bmatrix}$$

is a fundamental matrix of this ODE.

Now, the general solution to

$$\mathbf{x}' = A\mathbf{x}$$

is

$$\mathbf{x}(t) = c_1\mathbf{x}^{(1)}(t) + c_2\mathbf{x}^{(2)}(t) + \ldots + c_n\mathbf{x}^{(n)}(t)$$

Now, the general solution to

$$\mathbf{x}' = A\mathbf{x}$$

is

$$\mathbf{x}(t) = c_1\mathbf{x}^{(1)}(t) + c_2\mathbf{x}^{(2)}(t) + \ldots + c_n\mathbf{x}^{(n)}(t) = \Psi(t)\mathbf{c}$$

where

$$\mathbf{c} = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} \in \mathbb{R}^n.$$

Now, the general solution to

$$\mathbf{x}' = A\mathbf{x}$$

is

$$\mathbf{x}(t) = c_1\mathbf{x}^{(1)}(t) + c_2\mathbf{x}^{(2)}(t) + \ldots + c_n\mathbf{x}^{(n)}(t) = \Psi(t)\mathbf{c}$$

where

$$\mathbf{c} = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} \in \mathbb{R}^n.$$

If we have an initial condition $\mathbf{x}(t_0) = \mathbf{x}^0$, then

$$\Psi(t_0)\mathbf{c} = \mathbf{x}^0.$$

$$\mathbf{x}(t) = \Psi(t)\mathbf{c} \qquad\qquad \Psi(t_0)\mathbf{c} = \mathbf{x}^0$$

But

$\mathbf{x}^{(1)}, \ldots, \mathbf{x}^{(n)}$
are linearly
independent

$$\mathbf{x}(t) = \Psi(t)\mathbf{c} \qquad\qquad \Psi(t_0)\mathbf{c} = \mathbf{x}^0$$

But

$$\begin{array}{c} \mathbf{x}^{(1)}, \ldots, \mathbf{x}^{(n)} \\ \text{are linearly} \\ \text{independent} \end{array} \quad \Longrightarrow \quad \Psi(t) \text{ is invertible}$$

$$\mathbf{x}(t) = \Psi(t)\mathbf{c} \qquad\qquad \Psi(t_0)\mathbf{c} = \mathbf{x}^0$$

But

$$\begin{array}{ccc} \mathbf{x}^{(1)}, \ldots, \mathbf{x}^{(n)} & & \\ \text{are linearly} & \Longrightarrow & \Psi(t) \text{ is invertible} \\ \text{independent} & & \\ & \Longrightarrow & \mathbf{c} = \Psi^{-1}(t_0)\mathbf{x}^0. \end{array}$$

$$\mathbf{x}(t) = \Psi(t)\mathbf{c} \qquad\qquad \Psi(t_0)\mathbf{c} = \mathbf{x}^0$$

But

$\mathbf{x}^{(1)}, \ldots, \mathbf{x}^{(n)}$ are linearly independent $\implies$ $\Psi(t)$ is invertible

$\implies$ $\mathbf{c} = \Psi^{-1}(t_0)\mathbf{x}^0.$

Therefore the solution to the IVP

$$\begin{cases} \mathbf{x}' = A\mathbf{x} \\ \mathbf{x}(t_0) = \mathbf{x}^0 \end{cases}$$

is

$$\boxed{\mathbf{x}(t) = \Psi(t)\Psi^{-1}(t_0)\mathbf{x}^0.}$$

Theorem

Suppose that $\Psi(t)$ *is a fundamental matrix for* $\mathbf{x}' = P(t)\mathbf{x}$. *Then* $\Psi(t)$ *solves the differential equation* $\Psi' = P(t)\Psi$.

(You prove)

Remark

It is possible to find a *special fundamental matrix*, $\Phi(t)$, which satisfies

$$\Phi(t_0) = \begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \dots & 1 \end{bmatrix} = I.$$

Remark

It is possible to find a *special fundamental matrix*, $\Phi(t)$, which satisfies

$$\Phi(t_0) = \begin{bmatrix} 1 & 0 & \ldots & 0 \\ 0 & 1 & \ldots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \ldots & 1 \end{bmatrix} = I.$$

We will use Φ for this special fundamental matrix, and Ψ for any fundamental matrix.

Example

Consider

$$\mathbf{x}' = \begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix} \mathbf{x}.$$

Find the special fundamental matrix which satisfies $\Phi(0) = I$.

To find the matrix Φ which satisfies

$$\Phi(0) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

we must solve two IVPs:

$$\begin{cases} \mathbf{x}' = \begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix} \mathbf{x} \\ \\ \mathbf{x}(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \end{cases} \qquad \text{and} \qquad \begin{cases} \mathbf{x}' = \begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix} \mathbf{x} \\ \\ \mathbf{x}(0) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}. \end{cases}$$

The general solution to the ODE is

$$\mathbf{x}(t) = c_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} e^{3t} + c_2 \begin{bmatrix} 1 \\ -2 \end{bmatrix} e^{-t}.$$

We calculate that

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \mathbf{x}(0) = c_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ -2 \end{bmatrix} \implies \begin{aligned} c_1 &= \frac{1}{2} \\ c_2 &= \frac{1}{2} \end{aligned}$$

$$\implies \mathbf{x}(t) = \begin{bmatrix} \frac{1}{2}e^{3t} + \frac{1}{2}e^{-t} \\ e^{3t} - e^{-t} \end{bmatrix}$$

We calculate that

$$\begin{bmatrix}1\\0\end{bmatrix} = \mathbf{x}(0) = c_1\begin{bmatrix}1\\2\end{bmatrix} + c_2\begin{bmatrix}1\\-2\end{bmatrix} \implies \begin{aligned} c_1 &= \frac{1}{2} \\ c_2 &= \frac{1}{2} \end{aligned}$$

$$\implies \mathbf{x}(t) = \begin{bmatrix}\frac{1}{2}e^{3t} + \frac{1}{2}e^{-t} \\ e^{3t} - e^{-t}\end{bmatrix}$$

and

$$\begin{bmatrix}0\\1\end{bmatrix} = \mathbf{x}(0) = c_1\begin{bmatrix}1\\2\end{bmatrix} + c_2\begin{bmatrix}1\\-2\end{bmatrix} \implies \begin{aligned} c_1 &= \frac{1}{4} \\ c_2 &= -\frac{1}{4} \end{aligned}$$

$$\implies \mathbf{x}(t) = \begin{bmatrix}\frac{1}{4}e^{3t} - \frac{1}{4}e^{-t} \\ \frac{1}{2}e^{3t} + \frac{1}{2}e^{-t}\end{bmatrix}.$$

Therefore the special fundamental matrix is

$$\Phi(t) = \begin{bmatrix} \frac{1}{2}e^{3t} + \frac{1}{2}e^{-t} & \frac{1}{4}e^{3t} - \frac{1}{4}e^{-t} \\ e^{3t} - e^{-t} & \frac{1}{2}e^{3t} + \frac{1}{2}e^{-t} \end{bmatrix}.$$

What is e^{At}?

Recall that the solution to

$$\begin{cases} x' = ax \ (a \in \mathbb{R}) \\ x(0) = x_0 \end{cases}$$

is

$$x(t) = x_0 e^{at} = x_0 \exp(at)$$

and recall that

$$\exp(at) = 1 + \sum_{n=1}^{\infty} \frac{a^n t^n}{n!}.$$

Now consider

$$\begin{cases} \mathbf{x}' = A\mathbf{x} \\ \mathbf{x}(0) = \mathbf{x}^0 \end{cases}$$

for $A \in \mathbb{R}^{n \times n}$.

Now consider

$$\begin{cases} \mathbf{x}' = A\mathbf{x} \\ \mathbf{x}(0) = \mathbf{x}^0 \end{cases}$$

for $A \in \mathbb{R}^{n\times n}$.

Definition

$$\exp(At) = I + \sum_{n=1}^{\infty} \frac{A^n t^n}{n!} = I + At + \frac{A^2t^2}{2!} + \frac{A^3t^3}{3!} + \dots.$$

Note that

$$\frac{d}{dt}\exp(At) = \qquad = \qquad$$

$$= \qquad =$$

$$=$$

$$= \qquad = \qquad .$$

Note that

$$\frac{d}{dt}\exp(At) = \frac{d}{dt}\left(I + \sum_{n=1}^{\infty}\frac{A^n t^n}{n!}\right) =$$

$$=\qquad\qquad =$$

$$=$$

$$=\qquad\qquad =\qquad\qquad .$$

5.5 Fundamental Matrices

Note that

$$
\begin{aligned}
\frac{d}{dt}\exp(At) &= \frac{d}{dt}\left(I + \sum_{n=1}^{\infty} \frac{A^n t^n}{n!}\right) = \mathbf{0} + \sum_{n=1}^{\infty} \frac{d}{dt}\left(\frac{A^n t^n}{n!}\right) \\
&= \qquad\qquad = \\
&= \\
&= \qquad\qquad\qquad = \qquad\qquad .
\end{aligned}
$$

Note that

$$\frac{d}{dt}\exp(At) = \frac{d}{dt}\left(I + \sum_{n=1}^{\infty}\frac{A^n t^n}{n!}\right) = \mathbf{0} + \sum_{n=1}^{\infty}\frac{d}{dt}\left(\frac{A^n t^n}{n!}\right)$$

$$= \sum_{n=1}^{\infty}\frac{A^n t^{n-1}}{(n-1)!} =$$

$$=$$

$$= \qquad\qquad = \qquad .$$

Note that

$$\begin{aligned}
\frac{d}{dt}\exp(At) &= \frac{d}{dt}\left(I + \sum_{n=1}^{\infty}\frac{A^n t^n}{n!}\right) = \mathbf{0} + \sum_{n=1}^{\infty}\frac{d}{dt}\left(\frac{A^n t^n}{n!}\right) \\
&= \sum_{n=1}^{\infty}\frac{A^n t^{n-1}}{(n-1)!} = A + \sum_{n=2}^{\infty}\frac{A^n t^{n-1}}{(n-1)!} \\
&= \\
&= \qquad\qquad = \qquad .
\end{aligned}$$

Note that

$$\frac{d}{dt}\exp(At) = \frac{d}{dt}\left(I + \sum_{n=1}^{\infty}\frac{A^n t^n}{n!}\right) = \mathbf{0} + \sum_{n=1}^{\infty}\frac{d}{dt}\left(\frac{A^n t^n}{n!}\right)$$

$$= \sum_{n=1}^{\infty}\frac{A^n t^{n-1}}{(n-1)!} = A + \sum_{n=2}^{\infty}\frac{A^n t^{n-1}}{(n-1)!}$$

$$= A + \sum_{k=1}^{\infty}\frac{A^{k+1} t^k}{k!} \qquad (k = n-1)$$

$$= \qquad\qquad = \qquad .$$

Note that

$$\begin{aligned}
\frac{d}{dt}\exp(At) &= \frac{d}{dt}\left(I + \sum_{n=1}^{\infty}\frac{A^n t^n}{n!}\right) = \mathbf{0} + \sum_{n=1}^{\infty}\frac{d}{dt}\left(\frac{A^n t^n}{n!}\right) \\
&= \sum_{n=1}^{\infty}\frac{A^n t^{n-1}}{(n-1)!} = A + \sum_{n=2}^{\infty}\frac{A^n t^{n-1}}{(n-1)!} \\
&= A + \sum_{k=1}^{\infty}\frac{A^{k+1} t^k}{k!} \qquad\qquad (k = n-1) \\
&= A\left(I + \sum_{k=1}^{\infty}\frac{A^k t^k}{k!}\right) = \qquad\qquad .
\end{aligned}$$

Note that

$$
\begin{aligned}
\frac{d}{dt}\exp(At) &= \frac{d}{dt}\left(I + \sum_{n=1}^{\infty}\frac{A^n t^n}{n!}\right) = \mathbf{0} + \sum_{n=1}^{\infty}\frac{d}{dt}\left(\frac{A^n t^n}{n!}\right) \\
&= \sum_{n=1}^{\infty}\frac{A^n t^{n-1}}{(n-1)!} = A + \sum_{n=2}^{\infty}\frac{A^n t^{n-1}}{(n-1)!} \\
&= A + \sum_{k=1}^{\infty}\frac{A^{k+1} t^k}{k!} \qquad\qquad (k = n-1) \\
&= A\left(I + \sum_{k=1}^{\infty}\frac{A^k t^k}{k!}\right) = A\exp(At).
\end{aligned}
$$

This means that $\exp(At)$ solves

$$\begin{cases} \big(\exp(At)\big)' = A\exp(At) \\ \exp(At)|_{t=0} = I. \end{cases}$$

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But remember that Φ solves

$$\begin{cases} \Phi' = A\Phi \\ \Phi(0) = I. \end{cases}$$

This means that $\exp(At)$ solves

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But remember that Φ solves

$$\begin{cases} \Phi' = A\Phi \\ \Phi(0) = I. \end{cases}$$

Therefore

$$\boxed{\Phi(t) = \exp(At).}$$

Example

Let $A = \begin{bmatrix} 8 & -1 \\ 6 & 1 \end{bmatrix}$. Find $\exp(At)$.

Example

Let $A = \begin{bmatrix} 8 & -1 \\ 6 & 1 \end{bmatrix}$. Find $\exp(At)$.

We have previously found that the general solution to $\mathbf{x}' = A\mathbf{x}$ is

$$\mathbf{x}(t) = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{7t} + c_2 \begin{bmatrix} 1 \\ 6 \end{bmatrix} e^{2t}.$$

To satisfy $\mathbf{x}(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ we require $c_1 = \frac{6}{5}$ and $c_2 = -\frac{1}{5}$. Hence

$$\mathbf{x}(t) = \frac{6}{5} \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{7t} - \frac{1}{5} \begin{bmatrix} 1 \\ 6 \end{bmatrix} e^{2t} = \begin{bmatrix} \frac{6}{5}e^{7t} - \frac{1}{5}e^{2t} \\ \frac{6}{5}e^{7t} - \frac{6}{5}e^{2t} \end{bmatrix}.$$

To satisfy $\mathbf{x}(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ we require $c_1 = \frac{6}{5}$ and $c_2 = -\frac{1}{5}$. Hence

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To satisfy $\mathbf{x}(0) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ we require $c_1 = -\frac{1}{5}$ and $c_2 = \frac{1}{5}$. Hence

$$\mathbf{x}(t) = -\frac{1}{5} \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{7t} + \frac{1}{5} \begin{bmatrix} 1 \\ 6 \end{bmatrix} e^{2t} = \begin{bmatrix} -\frac{1}{5}e^{7t} + \frac{1}{5}e^{2t} \\ -\frac{1}{5}e^{7t} + \frac{6}{5}e^{2t} \end{bmatrix}.$$

Therefore the answer is

$$\boxed{\exp(At) = \Phi(t) = \begin{bmatrix} \frac{6}{5}e^{7t} - \frac{1}{5}e^{2t} & -\frac{1}{5}e^{7t} + \frac{1}{5}e^{2t} \\ \frac{6}{5}e^{7t} - \frac{6}{5}e^{2t} & -\frac{1}{5}e^{7t} + \frac{6}{5}e^{2t} \end{bmatrix}.}$$

Diagonalisable Matrices

If

$$D = \begin{bmatrix} r_1 & 0 & 0 & \dots & 0 \\ 0 & r_2 & 0 & \dots & 0 \\ 0 & 0 & r_3 & \dots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \dots & r_n \end{bmatrix}$$

is a diagonal matrix, then it is easy to calculate $\exp(Dt)$. We simply have

$$\exp(Dt) = \begin{bmatrix} e^{r_1 t} & 0 & 0 & \dots & 0 \\ 0 & e^{r_2 t} & 0 & \dots & 0 \\ 0 & 0 & e^{r_3 t} & \dots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \dots & e^{r_n t} \end{bmatrix}.$$

Now consider

$$\mathbf{x}' = A\mathbf{x}$$

for $A \in \mathbb{R}^{n \times n}$.

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$$\mathbf{x}' = A\mathbf{x}$$

for $A \in \mathbb{R}^{n \times n}$. Recall how we diagonalise a matrix: If $\boldsymbol{\xi}^{(1)}$, $\boldsymbol{\xi}^{(2)}$, $\ldots$, $\boldsymbol{\xi}^{(n)}$ are the eigenvectors of A, we let

$$T = \begin{bmatrix} \boldsymbol{\xi}^{(1)} & \boldsymbol{\xi}^{(2)} & \ldots & \boldsymbol{\xi}^{(n)} \end{bmatrix}.$$

Now consider

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$$T = \begin{bmatrix} \boldsymbol{\xi}^{(1)} & \boldsymbol{\xi}^{(2)} & \dots & \boldsymbol{\xi}^{(n)} \end{bmatrix}.$$

Then

$$\det(T) \neq 0 \implies \begin{array}{c} T^{-1} \\ \text{exists} \end{array}$$

Now consider

$$\mathbf{x}' = A\mathbf{x}$$

for $A \in \mathbb{R}^{n\times n}$. Recall how we diagonalise a matrix: If $\boldsymbol{\xi}^{(1)}$, $\boldsymbol{\xi}^{(2)}$, $\ldots$, $\boldsymbol{\xi}^{(n)}$ are the eigenvectors of A, we let

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Then

$$\det(T) \neq 0 \quad \Longrightarrow \quad \begin{matrix} T^{-1} \\ \text{exists} \end{matrix} \quad \Longrightarrow \quad \begin{matrix} T^{-1}AT \\ \text{is diagonal} \end{matrix}$$

Now consider

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$$T = \begin{bmatrix} \boldsymbol{\xi}^{(1)} & \boldsymbol{\xi}^{(2)} & \dots & \boldsymbol{\xi}^{(n)} \end{bmatrix}.$$

Then

$$\det(T) \neq 0 \quad \Longrightarrow \quad \begin{array}{c} T^{-1} \\ \text{exists} \end{array} \quad \Longrightarrow \quad \begin{array}{c} T^{-1}AT \\ \text{is diagonal} \end{array} \quad \Longrightarrow \quad \begin{array}{c} A \text{ is} \\ \text{diagonalisable.} \end{array}$$

Example

Diagonalise
$$A = \begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix}.$$

Example

Diagonalise

$$A = \begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix}.$$

The eigenvalues are $r_1 = 3$ and $r_2 = -1$. The corresponding eigenvectors are $\boldsymbol{\xi}^{(1)} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $\boldsymbol{\xi}^{(2)} = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$.

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$$T = \begin{bmatrix} 1 & 1 \\ 2 & -2 \end{bmatrix}$$

Example

Diagonalise
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$$T = \begin{bmatrix} 1 & 1 \\ 2 & -2 \end{bmatrix} \qquad \text{and} \qquad T^{-1} = \begin{bmatrix} \frac{1}{2} & \frac{1}{4} \\ \frac{1}{2} & -\frac{1}{4} \end{bmatrix}.$$

Example

Diagonalise

$$A = \begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix}.$$

The eigenvalues are $r_1 = 3$ and $r_2 = -1$. The corresponding eigenvectors are $\boldsymbol{\xi}^{(1)} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $\boldsymbol{\xi}^{(2)} = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$. Thus

$$T = \begin{bmatrix} 1 & 1 \\ 2 & -2 \end{bmatrix} \qquad \text{and} \qquad T^{-1} = \begin{bmatrix} \frac{1}{2} & \frac{1}{4} \\ \frac{1}{2} & -\frac{1}{4} \end{bmatrix}.$$

It follows that

$$D = T^{-1}AT = \begin{bmatrix} 3 & 0 \\ 0 & -1 \end{bmatrix}.$$

Now consider

$$\mathbf{x}' = A\mathbf{x}.$$

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Define a new variable $\mathbf{y}$ by

$$\mathbf{x} = T\mathbf{y} \qquad \text{or} \qquad \mathbf{y} = T^{-1}\mathbf{x}.$$

Now consider

$$\mathbf{x}' = A\mathbf{x}.$$

Define a new variable $\mathbf{y}$ by

$$\mathbf{x} = T\mathbf{y} \qquad \text{or} \qquad \mathbf{y} = T^{-1}\mathbf{x}.$$

Then we calculate that

$$\mathbf{x}' = A\mathbf{x}$$

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Then we calculate that

$$\begin{aligned} \mathbf{x}' &= A\mathbf{x} \\ T\mathbf{y}' &= AT\mathbf{y} \end{aligned}$$

Now consider

$$\mathbf{x}' = A\mathbf{x}.$$

Define a new variable $\mathbf{y}$ by

$$\mathbf{x} = T\mathbf{y} \qquad \text{or} \qquad \mathbf{y} = T^{-1}\mathbf{x}.$$

Then we calculate that

$$\begin{aligned} \mathbf{x}' &= A\mathbf{x} \\ T\mathbf{y}' &= AT\mathbf{y} \\ \mathbf{y}' &= T^{-1}AT\mathbf{y} = D\mathbf{y}. \end{aligned}$$

We know that a fundamental matrix for $\mathbf{y}' = D\mathbf{y}$ is

$$\exp(Dt) = \begin{bmatrix} e^{r_1 t} & 0 & \dots & 0 \\ 0 & e^{r_2 t} & \dots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \dots & e^{r_n t} \end{bmatrix}.$$

We know that a fundamental matrix for $\mathbf{y}' = D\mathbf{y}$ is

$$\exp(Dt) = \begin{bmatrix} e^{r_1 t} & 0 & \dots & 0 \\ 0 & e^{r_2 t} & \dots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \dots & e^{r_n t} \end{bmatrix}.$$

Therefore a fundamental matrix for $\mathbf{x}' = A\mathbf{x}$ is

$$\Psi = T\exp(Dt) = \begin{bmatrix} \boldsymbol{\xi}^{(1)} e^{r_1 t} & \boldsymbol{\xi}^{(2)} e^{r_2 t} & \dots & \boldsymbol{\xi}^{(n)} e^{r_n t} \end{bmatrix}.$$

Example

Find a fundamental matrix for

$$\mathbf{x}' = \begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix} \mathbf{x}.$$

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Find a fundamental matrix for

$$\mathbf{x}' = \begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix} \mathbf{x}.$$

Recall that $T = \begin{bmatrix} 1 & 1 \\ 2 & -2 \end{bmatrix}$.

Example

Find a fundamental matrix for

$$\mathbf{x}' = \begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix} \mathbf{x}.$$

Recall that $T = \begin{bmatrix} 1 & 1 \\ 2 & -2 \end{bmatrix}$. Letting $\mathbf{y} = T^{-1}\mathbf{x}$, we have

$$\mathbf{y}' = \begin{bmatrix} 3 & 0 \\ 0 & -1 \end{bmatrix} \mathbf{y}.$$

A fundamental matrix for $\mathbf{y}' = \begin{bmatrix} 3 & 0 \\ 0 & -1 \end{bmatrix} \mathbf{y}$ is

$$\exp(Dt) = e^{Dt} = \begin{bmatrix} e^{3t} & 0 \\ 0 & e^{-t} \end{bmatrix}.$$

A fundamental matrix for $\mathbf{y}' = \begin{bmatrix} 3 & 0 \\ 0 & -1 \end{bmatrix} \mathbf{y}$ is

$$\exp(Dt) = e^{Dt} = \begin{bmatrix} e^{3t} & 0 \\ 0 & e^{-t} \end{bmatrix}.$$

Hence a fundamental matrix for $\mathbf{x}' = \begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix} \mathbf{x}$ is

$$\Psi(t) = T \exp(Dt)$$

5.5 Fundamental Matrices

A fundamental matrix for $\mathbf{y}' = \begin{bmatrix} 3 & 0 \\ 0 & -1 \end{bmatrix} \mathbf{y}$ is

$$\exp(Dt) = e^{Dt} = \begin{bmatrix} e^{3t} & 0 \\ 0 & e^{-t} \end{bmatrix}.$$

Hence a fundamental matrix for $\mathbf{x}' = \begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix} \mathbf{x}$ is

$$\Psi(t) = T \exp(Dt) = \begin{bmatrix} 1 & 1 \\ 2 & -2 \end{bmatrix} \begin{bmatrix} e^{3t} & 0 \\ 0 & e^{-t} \end{bmatrix}$$

A fundamental matrix for $\mathbf{y}' = \begin{bmatrix} 3 & 0 \\ 0 & -1 \end{bmatrix} \mathbf{y}$ is

$$\exp(Dt) = e^{Dt} = \begin{bmatrix} e^{3t} & 0 \\ 0 & e^{-t} \end{bmatrix}.$$

Hence a fundamental matrix for $\mathbf{x}' = \begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix} \mathbf{x}$ is

$$\Psi(t) = T\exp(Dt) = \begin{bmatrix} 1 & 1 \\ 2 & -2 \end{bmatrix} \begin{bmatrix} e^{3t} & 0 \\ 0 & e^{-t} \end{bmatrix} = \begin{bmatrix} e^{3t} & e^{-t} \\ 2e^{3t} & -2e^{-t} \end{bmatrix}.$$

Next Time

- 5.6 Repeated Eigenvalues
- 5.7 Nonhomogeneous Linear Systems