OKAN ÜNİVERSİTESİ
MÜHENDİSLİK-MİMARLIK FAKÜLTESİ
MÜHENDİSLİK TEMEL BİLİMLERİ BÖLÜMÜ

2016.05.17 **MAT234 Matematik IV – Final Sınavın Çözümleri** N. Course

Soru 1 (Series).

(a) [1p] Please write your student number on every page.

Decide if each of the following series converges or diverges. Justify (prove) your answers.

(b) [8p] $\displaystyle\sum_{n=1}^{\infty} \frac{n^4}{4^n}$.

(c) [8p] $\displaystyle\sum_{n=1}^{\infty} \frac{\cos^2 n}{n^{\frac{5}{2}}}$.

(d) [8p] $\displaystyle\sum_{n=1}^{\infty} n \sin\frac{1}{n}$.

[In this question, you may use any theorem/lemma/test/example/etc. from the course, **but** you must say which one you are using.]

2 pts for correctly stating "converges/diverges" without justification.
2 pts for saying which test is being used (as long as there is some proof given).
Remaining 4 pts for accuracy of proof.

If an answer is incorrect, but the proof is well written and contains only a minor error, then a maximum of 5 points (0+2+3) can be awarded.

(b) Since

$$\frac{a_{n+1}}{a_n} = \frac{(n+1)^4}{4^{n+1}}\frac{4^n}{n^4} = \frac{1}{4}\left(\frac{n+1}{n}\right)^4 = \frac{1}{4}\left(1+\frac{1}{n}\right)^4 \to \frac{1}{4}\times 1^4 = \frac{1}{4} < 1$$

as $n\to\infty$, it follows by the Ratio Test that $\sum_{n=1}^{\infty}\frac{n^4}{4^n}$ converges.

(c) Since $0 \le \cos^2 n \le 1$ for all n, we have that

$$0 \le \frac{\cos^2 n}{n^{\frac{5}{2}}} \le \frac{1}{n^{\frac{5}{2}}} \le \frac{1}{n^2}$$

for all n. Because we know that $\sum_{n=1}^{\infty}\frac{1}{n^2}$ converges, it follows by the Comparison Test that $\sum_{n=1}^{\infty}\frac{\cos^2 n}{n^{\frac{5}{2}}}$ also converges.

(d) Since the Taylor Series for sin, centred at 0, is

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots,$$

we have that

$$\lim_{n\to\infty} n \sin \frac{1}{n} = \lim_{n\to\infty} n \left(\frac{1}{n} - \frac{1}{n^3 3!} + \frac{1}{n^5 5!} - \dots \right) = 1.$$

Therefore $\sum_{n=1}^{\infty} n \sin \frac{1}{n}$ diverges by the Divergence Test.

Soru 2 (Convergent Series).

(a) [3p] Give the definition of the *partial sum* of a series $\displaystyle\sum_{n=1}^{\infty} a_n$.

The partial sum of $\sum_{n=1}^{\infty} a_n$ is defined to be

$$s_n := \sum_{k=1}^{n} a_k.$$

(b) [2p] Give the definition of a *convergent* *series*.

The series $\sum_{n=1}^{\infty} a_n$ is called convergent if and only if the sequence of partial sums, (s_n), is a convergent sequence.

(c) [5p] Use the *Alternating Series Test* to prove that $\displaystyle\sum_{n=2}^{\infty} (-1)^{n+1} \frac{\log n}{n}$ converges.

Clearly $a_n := \frac{\log n}{n} > 0$ and clearly $a_n \geq a_{n+1}$ for all $n \geq 2$. Moreover, it is easy to see that $a_n \to 0$ as $n \to \infty$. Therefore it follows by the Alternating Series Test that $\sum_{n=2}^{\infty} (-1)^{n+1} \frac{\log n}{n}$ converges.

(d) [15p] Give an example of a series $\sum_{n=1}^{\infty} b_n$ which satisfies all of the following conditions:

(a) $b_n \neq 0$ for all $n \in \mathbb{N}$;
(b) $\sum_{n=1}^{\infty} b_n$ is convergent; and
(c) $\sum_{n=1}^{\infty} b_n = 0$.

Show that your series satisfies all three conditions.

Define

$$b_n := \begin{cases} \frac{1}{n} & \text{if } n \text{ is an odd number} \\ -\frac{1}{n-1} & \text{if } n \text{ is an even number.} \end{cases}$$

Then we have the series

$$\sum_{n=1}^{\infty} b_n = 1 - 1 + \frac{1}{3} - \frac{1}{3} + \frac{1}{5} - \frac{1}{5} + \dots.$$

Clearly $b_n \neq 0$ for all n. Moreover, it follows by the Alternating Series Test that $\sum_{n=1}^{\infty} b_n$ converges.

Since $s_{2n} = 0$ for all n, we must have that $s_{2n} \to 0$ as $n \to \infty$. Because every subsequence of a convergent sequence tends to the same limit as the original sequence, we have that $s_n \to 0$ as $n \to \infty$. Hence $\sum_{n=1}^{\infty} b_n = 0$.

Soru 3 (Power Series).

(a) [5p] Let $\sum_{n=0}^{\infty} a_n x^n$ be a power series. Give the definition of the *radius of convergence* of $\sum_{n=0}^{\infty} a_n x^n$.

If $\sum_{n=0}^{\infty} a_n x^n$ converges $\forall |x| < R$ and diverges $\forall |x| > R$, then R is called the radius of convergence of $\sum_{n=0}^{\infty} a_n x^n$.

Define the set

$$S := \left\{ x \in \mathbb{R} : \sum_{n=1}^{\infty} \frac{x^n}{n\,(-5)^n} \text{ converges} \right\} \subseteq \mathbb{R}.$$

(b) [20p]Find S.

For this power series, $a_n = \frac{(-1)^n}{n\ 5^n}$ and

$$\left| \frac{a_n}{a_{n+1}} \right| = \frac{(n+1)\ 5^{n+1}}{n\ 5^n} = \frac{5n+5}{n} \to 5 \qquad \boxed{6}$$

$\boxed{\text{-1 point if candidate omits absolute value signs}}$

as $n \to \infty$. By a theorem from the course $\boxed{2}$, the radius of convergence of this power series is $R = 5$ $\boxed{2}$.

When $x = 5$, the power series becomes $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ which converges by the Alternating Series Test $\boxed{2}$.

When $x = -5$, the power series becomes $\sum_{n=1}^{\infty} \frac{1}{n}$ which diverges $\boxed{2}$.

Therefore $\sum_{n=1}^{\infty} \frac{x^n}{n\ (-5)^n}$ converges $\forall x \in (-5, 5]$ and diverges for all other x $\boxed{2}$. Hence $S = (-5, 5]$ $\boxed{4}$.

$$S := \left\{ x \in \mathbb{R} : \sum_{n=1}^{\infty} \frac{x^n}{n\,(-5)^n} \text{ converges} \right\} \subseteq \mathbb{R}$$

Soru 4 (Taylor Series).

(a) [10p] Calculate the Taylor Series for $f(x) = \sinh x$, centred at $a = 0$.

[You may assume without proof that $\left| \frac{f^n(c)}{n!} x^n \right| \to 0$ as $n \to \infty$ for all $x \in \mathbb{R}$ and for all c between 0 and x.]

Since

$$\frac{d^n}{dx^n} \cosh x = \begin{cases} \sinh x & n = 0, 2, 4, 6, 8, \ldots \\ \cosh x & n = 1, 3, 5, 7, 9, \ldots, \end{cases}$$

we can see that

$$f^n(0) = \begin{cases} 0 & n = 0, 2, 4, 6, 8, \ldots \\ 1 & n = 1, 3, 5, 7, 9, \ldots . \boxed{4} \end{cases}$$

By Taylor's Theorem (and by the hint), we have

$$\begin{aligned} \sinh x &= f(0) + f'(0)x + \frac{f''(0)x^2}{2!} + \frac{f'''(0)x^3}{3!} + \frac{f^{(4)}(0)x^4}{4!} + \ldots \boxed{4} \\ &= x + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} + \frac{x^9}{9!} + \frac{x^{11}}{11!} + \frac{x^{13}}{13!} + \frac{x^{15}}{15!} + \frac{x^{17}}{17!} + \frac{x^{19}}{19!} + \ldots \boxed{5} \end{aligned}$$

(b) [15p] Use your answer to part (a) to calculate $\lim_{t\to 0} \frac{(\sinh t) - t - \frac{t^3}{3!} - \frac{t^5}{5!} - \frac{t^7}{7!}}{t^9}$.

By part (a),

$$\sinh x = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} + \frac{x^9}{9!} + \frac{x^{11}}{11!} + \frac{x^{13}}{13!} + \frac{x^{15}}{15!} + \frac{x^{17}}{17!} + \frac{x^{19}}{19!} + \dots \boxed{5}$$

Therefore

$$\frac{\sinh x - x - \frac{x^3}{3!} - \frac{x^5}{5!} - \frac{x^7}{7!}}{t^9} = \frac{\frac{x^9}{9!} + \frac{x^{11}}{11!} + \frac{x^{13}}{13!} + \frac{x^{15}}{15!} + \frac{x^{17}}{17!} + \frac{x^{19}}{19!} + \dots}{t^9} \quad \boxed{3}$$
$$= \frac{1}{9!} + \frac{t^2}{11!} + \frac{t^4}{13!} + \frac{x^6}{15!} + \frac{x^8}{17!} + \frac{x^{10}}{19!} + \dots. \quad \boxed{5}$$

Hence

$$\lim_{t\to 0} \frac{\sinh t - t - \frac{t^3}{3!} - \frac{t^5}{5!} - \frac{t^7}{7!}}{t^9} = \lim_{t\to 0} \left(\frac{1}{9!} + \frac{t^2}{11!} + \frac{t^4}{13!} + \frac{t^6}{15!} + \frac{t^8}{17!} + \dots \right)$$
$$= \frac{1}{9!} = \frac{1}{362880}. \quad \boxed{7}$$

Soru 5 (Sequences).

(a) [5p] Let (a_n) be a sequence. Give the definition of "$a_n \to l$ as $n \to \infty$".

We say that a_n converges to l if and only if, for all $\varepsilon > 0$ there exists $N = N(\varepsilon) \in \mathbb{N}$ such that

$$n > N \implies |a_n - l| < \varepsilon.$$

Now let $c > 1$. Define a sequence $(h_n)_{n=1}^{\infty}$ by

$$h_n := c^{\frac{1}{n}} - 1.$$

(b) [4p] Show that $h_n > 0$ for all $n \in \mathbb{N}$.

Clearly if $n \in \mathbb{N}$ and if $c > 1$ then $c^{\frac{1}{n}} > 1$. Hence we must have $h_n = c^{\frac{1}{n}} - 1 > 0$ for all $n \in \mathbb{N}$.

(c) [5p] Let $\lambda > 0$. Show that $(1 + \lambda)^n > n\lambda$ for all $n \in \mathbb{N}$.

SOLUTION 1: Since $\lambda > 0$, we have that

$$(1 + \lambda)^n = 1 + n\lambda + \frac{n(n-1)}{2!}\lambda^2 + \frac{n(n-1)(n-2)}{3!}\lambda^3 + \dots + \lambda^n > n\lambda.$$

SOLUTION 2: Use Proof by Induction.

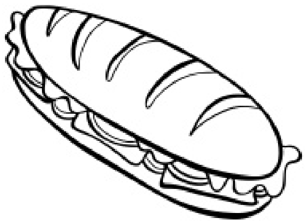
(d) [4p] Use parts (b) and (c) to show that $c > nh_n$ for all $n \in \mathbb{N}$.

Clearly

$$c = \left(c^{\frac{1}{n}}\right)^n = (1 + h_n)^n > nh_n$$

by parts (b) and (c).

(e) [5p] Show that $h_n \to 0$ as $n \to \infty$.

[HINT: ]

Since
$$0 < h_n < \frac{c}{n} \to 0$$
as $n \to \infty$, it follows by the Sandwich Rule that $h_n \to 0$ as $n \to \infty$. (Did you like my hint?)

(f) [2p] Show that $c^{\frac{1}{n}} \to 1$ as $n \to \infty$.

It follows from part (e) that
$$c^{\frac{1}{n}} = 1 + h_n \to 1 + 0 = 1$$
as $n \to \infty$.