Week 13

- 27. Differentiation Rules
- 28. Derivatives of Trigonometric Functions
- 29. The Chain Rule

Differentiation Rules

Constant Function

If $k \in \mathbb{R}$, then

$$\boxed{\frac{d}{dx}(k) = 0.}$$

Power Function

If $n \in \mathbb{R}$, then

$$\boxed{\frac{d}{dx}\left(x^n\right) = nx^{n-1}.}$$

Example

$$\frac{d}{dx}\left(x^3\right) = 3x^{3-1} = 3x^2$$

Example

$$\frac{d}{dx}\left(\sqrt{x}\right) = \frac{d}{dx}\left(x^{\frac{1}{2}}\right) = \frac{1}{2}x^{\frac{1}{2}-1} = \frac{1}{2}x^{-\frac{1}{2}} = \frac{1}{2\sqrt{x}}$$

Example

$$\frac{d}{dx}\left(\frac{1}{x^4}\right) = \frac{d}{dx}\left(x^{-4}\right) = -4x^{-4-1} = -4x^{-5} = -\frac{4}{x^5}$$

The Constant Multiple Rule

If $u(x)$ is differentiable and $k \in \mathbb{R}$, then

$$\boxed{\frac{d}{dx}(ku) = k\frac{du}{dx}.}$$

Proof.

$$\begin{aligned}\frac{d}{dx}(ku) &= \lim_{h\to 0}\frac{ku(x+h)-ku(x)}{h} \\ &= k\lim_{h\to 0}\frac{u(x+h)-u(x)}{h} = k\frac{du}{dx}\end{aligned}$$

□

Example

$$\frac{d}{dx}\left(3x^2\right) = 3\frac{d}{dx}\left(x^2\right) = 3 \times 2x = 6x$$

Example

$$\frac{d}{dx}\left(-u\right) = \frac{d}{dx}\left(-1 \times u\right) = -1 \times \frac{du}{dx} = -\frac{du}{dx}$$

The Sum Rule

If $u(x)$ and $v(x)$ are differentiable at x_0, then $u+v$ is also differentiable at x_0 and

$$\boxed{\frac{d}{dx}(u+v)=\frac{du}{dx}+\frac{dv}{dx}.}$$

Example

Differentiate $y = x^3 + \frac{4}{3}x^2 - 5x + 1$.

solution:

$$\begin{aligned}\frac{dy}{dx} &= \frac{d}{dx}\left(x^3 + \frac{4}{3}x^2 - 5x + 1\right)\\ &= \frac{d}{dx}\left(x^3\right) + \frac{d}{dx}\left(\frac{4}{3}x^2\right) - \frac{d}{dx}(5x) + \frac{d}{dx}(1)\\ &= 3x^2 + \frac{8}{3}x - 5 + 0\end{aligned}$$

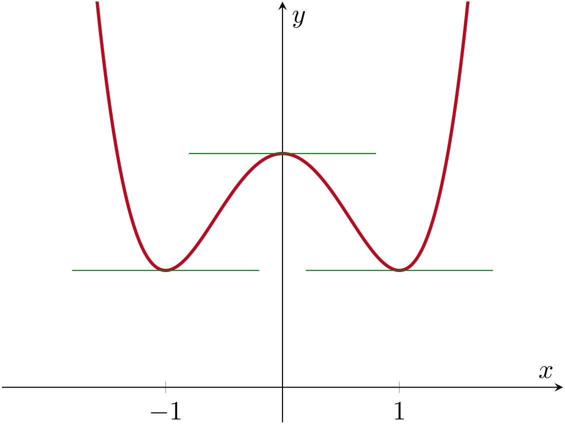
Example

Does the curve $y = x^4 - 2x^2 + 2$ have any points where $\frac{dy}{dx} = 0$? If so, where?

solution: Since

$$\frac{dy}{dx} = 4x^3 - 4x = 4x(x^2 - 1) = 4x(x-1)(x+1),$$

we can see that $\frac{dy}{dx} = 0$ if and only if $x = -1$, 0 or 1.



y
x
−1
1

The Product Rule

If $u(x)$ and $v(x)$ are differentiable at x_0, then $u(x)v(x)$ is also differentiable at x_0 and

$$\boxed{\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}.}$$

Using prime notation, the product rule is

$$\boxed{(uv)' = u'v + uv'.}$$

Example

Differentiate $y = (x^2+1)(x^3+3)$.

solution 1: We have $y = uv$ with $u = x^2+1$ and $v = x^3+3$. So

$$\begin{aligned}
\frac{dy}{dx} &= (x^2+1)'(x^3+3) + (x^2+1)(x^3+3)' \\
&= (2x+0)(x^3+3) + (x^2+1)(3x^2+0) \\
&= 2x^4 + 6x + 3x^4 + 3x^2 \\
&= 5x^4 + 3x^2 + 6x.
\end{aligned}$$

solution 2: Since

$$y = (x^2+1)(x^3+3) = x^5 + x^3 + 3x^2 + 3,$$

we have that

$$\frac{dy}{dx} = 5x^4 + 3x^2 + 6x + 0.$$

The Quotient Rule

If $u(x)$ and $v(x)$ are differentiable at x_0 and if $v(x_0) \neq 0$, then $\frac{u}{v}$ is also differentiable at x_0 and

$$\boxed{\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{u'v - uv'}{v^2}.}$$

Example

Differentiate $y = \frac{t^2-1}{t^3+1}$.

solution: We have $y = \frac{u}{v}$ with $u = t^2 - 1$ and $v = t^3 + 1$. Therefore

$$\begin{aligned}
\frac{dy}{dt} &= \frac{u'v - uv'}{v^2} \\
&= \frac{(t^2-1)'(t^3+1) - (t^2-1)(t^3+1)'}{(t^3+1)^2} \\
&= \frac{(2t)(t^3+1) - (t^2-1)(3t^2)}{(t^3+1)^2} \\
&= \frac{2t^4 + 2t - 3t^4 + 3t^2}{(t^3+1)^2} \\
&= \frac{-t^4 + 3t^2 + 2t}{(t^3+1)^2}.
\end{aligned}$$

Example

Differentiate $f(s) = \frac{\sqrt{s}-1}{\sqrt{s}+1}$.

solution: We have $f(s) = \frac{u}{v}$ with $u = \sqrt{s} - 1$ and $v = \sqrt{s} + 1$. Remember that $\frac{d}{ds}\left(\sqrt{s}\right) = \frac{1}{2\sqrt{s}}$. Therefore

$$
\begin{aligned}
\frac{df}{ds} &= \frac{u'v - uv'}{v^2} \\
&= \frac{(\sqrt{s}-1)'(\sqrt{s}+1) - (\sqrt{s}-1)(\sqrt{s}+1)'}{\left(\sqrt{s}+1\right)^2} \\
&= \frac{\left(\frac{1}{2\sqrt{s}}\right)(\sqrt{s}+1) - (\sqrt{s}-1)\left(\frac{1}{2\sqrt{s}}\right)}{\left(\sqrt{s}+1\right)^2} \\
&= \frac{\frac{1}{2} + \frac{1}{2\sqrt{s}} - \frac{1}{2} + \frac{1}{2\sqrt{s}}}{\left(\sqrt{s}+1\right)^2} = \frac{1}{\sqrt{s}\left(\sqrt{s}+1\right)^2}.
\end{aligned}
$$

Second Order Derivatives

If $y = f(x)$ is a differentiable function, then $f'(x)$ is also a function. If $f'(x)$ is also differentiable, then we can differentiate to find a new function called f'' ("f double prime"). f'' is called the *second derivative* of f. We can write

$$f''(x) = \frac{d}{dx}f'(x) = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d^2y}{dx^2} = \frac{dy'}{dx} = y''$$

"d squared y, dx squared"

Example

If $y = x^6$, then $y' = \frac{d}{dx}\left(x^6\right) = 6x^5$ and $y'' = \frac{d}{dx}\left(y'\right) = \frac{d}{dx}\left(6x^5\right) = 30x^4$. Equivalently, we can write

$$\frac{d^2}{dx^2}\left(x^6\right) = \frac{d}{dx}\left(\frac{d}{dx}\left(x^6\right)\right) = \frac{d}{dx}\left(6x^5\right) = 30x^4.$$

Higher Order Derivatives

If f'' is differentiable, then its derivative $f''' = \frac{d^3 f}{dx^3}$ is the *third derivative* of f.

If f''' is differentiable, then its derivative $f^{(4)} = \frac{d^4 f}{dx^4}$ is the *fourth derivative* of f.

If $f^{(4)}$ is differentiable, then its derivative $f^{(5)} = \frac{d^5 f}{dx^5}$ is the *fifth derivative* of f.

$$\vdots$$

If $f^{(n-1)}$ is differentiable, then its derivative $f^{(n)} = \frac{d^n f}{dx^n}$ is the *nth derivative* of f.

Example

Find the first four derivatives of $y = x^3 - 3x^2 + 2$.

solution:

First derivative: $y' = 3x^2 - 6x$

Second derivative: $y'' = 6x - 6$

Third derivative: $y''' = 6$

Fourth derivative: $y^{(4)} = 0$.

(Note that since $\frac{d}{dx}(0) = 0$, if $n \geq 4$ then $y^{(n)} = 0$ also.)

Derivatives of Trigonometric Functions

Sine and Cosine

$$\boxed{\frac{d}{dx}(\sin x) = \cos x}$$

$$\boxed{\frac{d}{dx}(\cos x) = -\sin x}$$

Example

Differentiate $y = x^2 - \sin x$.

solution:

$$\frac{dy}{dx} = \frac{d}{dx}\left(x^2\right) - \frac{d}{dx}(\sin x) = 2x - \cos x.$$

Example

Differentiate $y = x^2 \sin x$.

solution: We will use the product rule $((uv)' = u'v + uv')$ with $u = x^2$ and $v = \sin x$.

$$y' = (x^2)'(\sin x) + (x^2)(\sin x)' = 2x \sin x + x^2 \cos x.$$

Example

Differentiate $y = \frac{\sin x}{x}$.

solution: This time we use the quotient rule $\left(\left(\frac{u}{v}\right)' = \frac{u'v - uv'}{v^2}\right)$ with $u = \sin x$ and $v = x$.

$$y' = \frac{(\sin x)'x - (\sin x)(x)'}{x^2} = \frac{x\cos x - \sin x}{x^2}.$$

Example

Differentiate $y = 5x + \cos x$.

solution:

$$\frac{dy}{dx} = \frac{d}{dx}(5x) + \frac{d}{dx}(\cos x) = 5 - \sin x.$$

Example

Differentiate $y = \sin x \cos x$.

solution: By the product rule, we have that

$$\frac{dy}{dx} = \frac{d}{dx}(\sin x)\cos x + \sin x \frac{d}{dx}(\cos x) = \cos^2 x - \sin^2 x.$$

Example

Differentiate $y = \frac{\cos x}{1-\sin x}$.

solution: By the quotient rule, we have that

$$\begin{aligned}
\frac{dy}{dx} &= \frac{\frac{d}{dx}(\cos x)(1-\sin x) - (\cos x)\frac{d}{dx}(1-\sin x)}{(1-\sin x)^2} \\
&= \frac{-\sin x(1-\sin x) - \cos x(0-\cos x)}{(1-\sin x)^2} \\
&= \frac{-\sin x + \sin^2 x + \cos^2 x}{(1-\sin x)^2} \\
&= \frac{-\sin x + 1}{(1-\sin x)^2} = \frac{1-\sin x}{(1-\sin x)^2} \\
&= \frac{1}{1-\sin x}.
\end{aligned}$$

The Tangent Function

$$\boxed{\frac{d}{dx}(\tan x) = \sec^2 x}$$

Proof.

Using the quotient rule, we can calculate that

$$\begin{aligned}
\frac{d}{dx}(\tan x) &= \frac{d}{dx}\left(\frac{\sin x}{\cos x}\right) \\
&= \frac{\frac{d}{dx}(\sin x)(\cos x) - (\sin x)\frac{d}{dx}(\cos x)}{\cos^2 x} \\
&= \frac{(\cos x)(\cos x) - (\sin x)(-\sin x)}{\cos^2 x} \\
&= \frac{\cos^2 x + \sin^2 x}{\cos^2 x} \\
&= \frac{1}{\cos^2 x} = \sec^2 x.
\end{aligned}$$

$\square$

The Other Three

$$\boxed{\frac{d}{dx}(\sec x) = \sec x \tan x}$$

$$\boxed{\frac{d}{dx}(\cot x) = -\operatorname{cosec}^2 x}$$

$$\boxed{\frac{d}{dx}(\operatorname{cosec} x) = -\operatorname{cosec} x \cot x}$$

You can use the quotient rule to prove these three rules. We may ask you to prove one of them in an exam.

Example

Find y'' if $y = \sec x$.

solution: Since $y' = \sec x \tan x$, we have that

$$\begin{aligned} y'' &= \frac{d}{dx}\left(y'\right) = \frac{d}{dx}\left(\sec x \tan x\right) \\ &= \frac{d}{dx}\left(\sec x\right)\tan x + \sec x \frac{d}{dx}\left(\tan x\right) \\ &= (\sec x \tan x)(\tan x) + (\sec x)(\sec^2 x) \\ &= \sec x \tan^2 x + \sec^3 x. \end{aligned}$$

The Chain Rule

How do we differentiate $F(x) = \sin(x^2 - 4)$?

Theorem (The Chain Rule)

Suppose that

- $y = f(u)$ *is differentiable at the point* $u = g(x)$; *and*
- $g(x)$ *is differentiable at* x.

Then $f \circ g$ *is differentiable at* x *and*

$$\boxed{(f \circ g)'(x) = f'\big(g(x)\big)g'(x).}$$

29. The Chain Rule

The Chain Rule is easier to remember if we use Leibniz's notation:

$$\boxed{\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}}$$

Example

Differentiate $y = \sin(x^2 - 4)$.

solution: We have $y = \sin u$ with $u = x^2 - 4$. Now $\frac{dy}{du} = \cos u$ and $\frac{du}{dx} = 2x$. Therefore

$$\begin{aligned} \frac{dy}{dx} &= \frac{dy}{du}\frac{du}{dx} = (\cos u)(2x) \\ &= 2x\cos u = 2x\cos(x^2 - 4) \end{aligned}$$

by the Chain Rule.

Example

Differentiate $\sin(x^2 + x)$.

solution: Let $u = x^2 + x$. Then

$$\begin{aligned}\frac{d}{dx}\left(\sin(x^2 + x)\right) &= \frac{d}{du}\left(\sin u\right)\frac{du}{dx} \\ &= (\cos u)(2x + 1) \\ &= (2x + 1)\cos(x^2 + x)\end{aligned}$$

by the Chain Rule.

Example (Using the Chain Rule Two Times)

Differentiate $g(t) = \tan(5 - \sin 2t)$.

solution: Let $u = 5 - \sin 2t$. Then $g(t) = \tan u$. Hence

$$\frac{dg}{dt} = \frac{dg}{du}\frac{du}{dt} = (\sec^2 u)\frac{d}{dt}(5 - \sin 2t).$$

We need to use the Chain Rule a second time: Let $w = 2t$. Then

$$\begin{aligned}\frac{dg}{dt} &= (\sec^2 u)\frac{d}{dt}(5 - \sin 2t)\\ &= (\sec^2 u)\frac{d}{dw}(5 - \sin w)\frac{dw}{dt}\\ &= (\sec^2 u)(-\cos w)(2)\\ &= -2\cos 2t \sec^2(5 - \sin 2t).\end{aligned}$$

(Note: Your final answer should not have u or w in it.)

Powers of a Function

If

- f is a differentiable function of u;
- u is a differentiable function of x; and
- $y = f(u)$,

then the Chain Rule $\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$ is the same as

$$\frac{d}{dx}f(u) = f'(u)\frac{du}{dx}.$$

Now suppose that $n \in \mathbb{R}$ and $f(u) = u^n$. Then $f'(u) = nu^{n-1}$. So

$$\boxed{\frac{d}{dx}\left(u^n\right) = nu^{n-1}\frac{du}{dx}.}$$

Example

$$\begin{aligned}\frac{d}{dx}\left(5x^3 - x^4\right)^7 &= 7\left(5x^3 - x^4\right)^6 \frac{d}{dx}\left(5x^3 - x^4\right) \\ &= 7\left(5x^3 - x^4\right)^6 \left(15x^2 - 4x^3\right).\end{aligned}$$

Example

$$\frac{d}{dx}\left(\frac{1}{3x-2}\right) = \frac{d}{dx}(3x-2)^{-1} = -1\,(3x-2)^{-2}\frac{d}{dx}(3x-2)$$
$$= -\left(\frac{1}{(3x-2)^2}\right)(2) = \frac{-3}{(3x-2)^2}.$$

29. The Chain Rule

Example

$$\frac{d}{dx}\left(\sin^5 x\right) = 5\sin^4 x \frac{d}{dx}(\sin x) = 5\sin^4 x \cos x.$$

Example

Differentiate $|x|$.

solution: Since $|x| = \sqrt{x^2}$, we can calculate that if $x \neq 0$ then

$$\begin{aligned}\frac{d}{dx}|x| &= \frac{d}{dx}\left(\sqrt{x^2}\right) = \frac{d}{du}\left(\sqrt{u}\right)\frac{d}{dx}\left(x^2\right)\\ &= \frac{1}{2\sqrt{u}}2x = \frac{x}{\sqrt{x^2}} = \frac{x}{|x|}.\end{aligned}$$

Example

Let $y = \frac{1}{(1-2x)^3}$ for $x \neq \frac{1}{2}$. Show that $\frac{dy}{dx} > 0$.

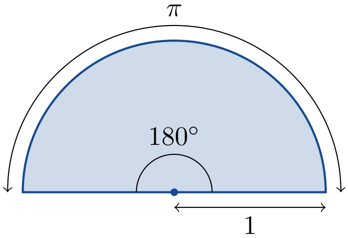
solution: First we calculate that

$$\begin{aligned}\frac{dy}{dx} &= \frac{d}{dx}(1-2x)^{-3} = -3(1-2x)^{-4}\frac{d}{dx}(1-2x) \\ &= -3(1-2x)^{-4}(-2) = \frac{6}{(1-2x)^4}\end{aligned}$$

if $x \neq \frac{1}{2}$. Since $(1-2x)^4 > 0$ if $x \neq \frac{1}{2}$ and $6 > 0$, we have that $\frac{dy}{dx} > 0$ if $x \neq \frac{1}{2}$.

Example (Why Do We Use Radians in Calculus?)

Remember that $\frac{d}{dx}\sin x = \cos x$ is true *only if we use radians*. What happens if we use degrees?



Remember that

$$180 \text{ degrees} = \pi \text{ radians}$$
$$180^\circ = \pi$$
$$1^\circ = \frac{\pi}{180}$$
$$x^\circ = \frac{\pi x}{180}.$$

So

$$\frac{d}{dx}\sin x^\circ = \frac{d}{dx}\sin\left(\frac{\pi x}{180}\right) = \frac{\pi}{180}\cos\left(\frac{\pi x}{180}\right) = \frac{\pi}{180}\cos x^\circ.$$

Therefore we have

$$\boxed{\frac{d}{dx}\sin x = \cos x} \qquad \text{and} \qquad \boxed{\frac{d}{dx}\sin x^\circ = \frac{\pi}{180}\cos x^\circ.}$$

a nice formula — not nice

This is why we use radians in Calculus.

Next Week

- 30. Antiderivatives
- 31. Integration
- 32. The Definite Integral