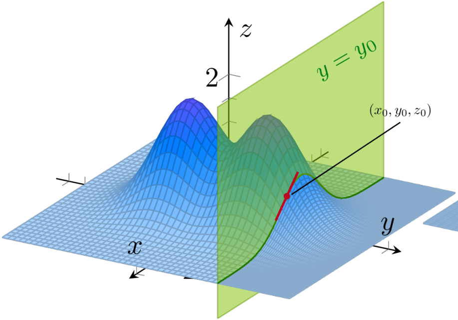
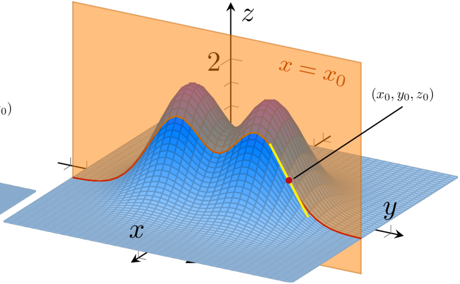
Lecture 6

- 13.5 Directional Derivatives and Gradient Vectors
- 13.6 Tangent Planes and Differentials
- 13.7 Extreme Values and Saddle Points
- 13.8 Lagrange Multipliers

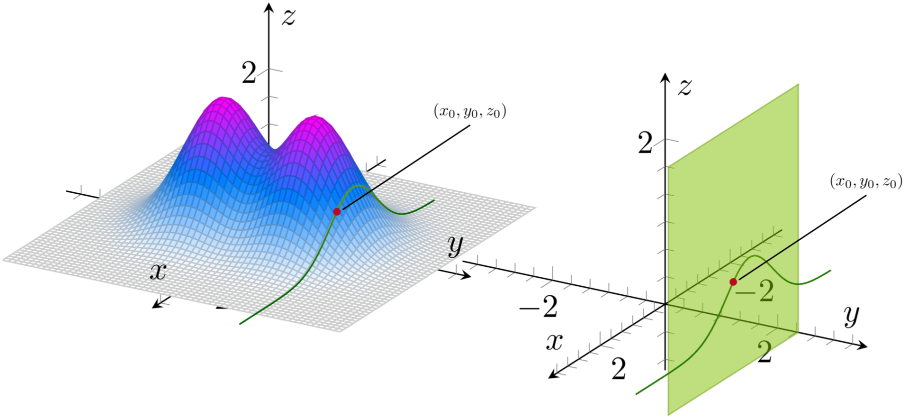
Directional Derivatives and Gradient Vectors

Partial Derivatives (revision)





Directional Derivatives



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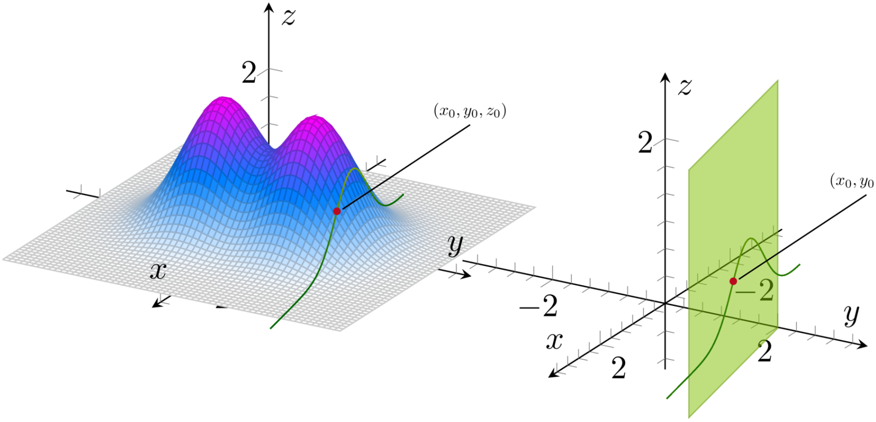

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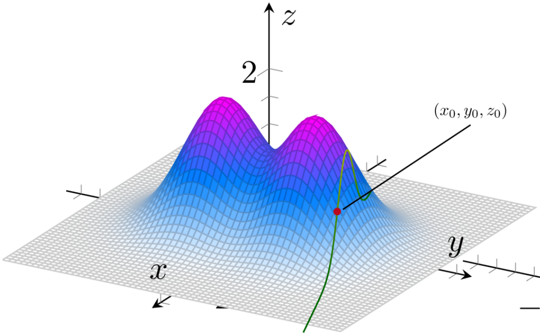

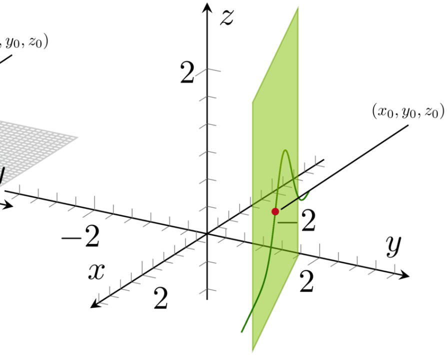

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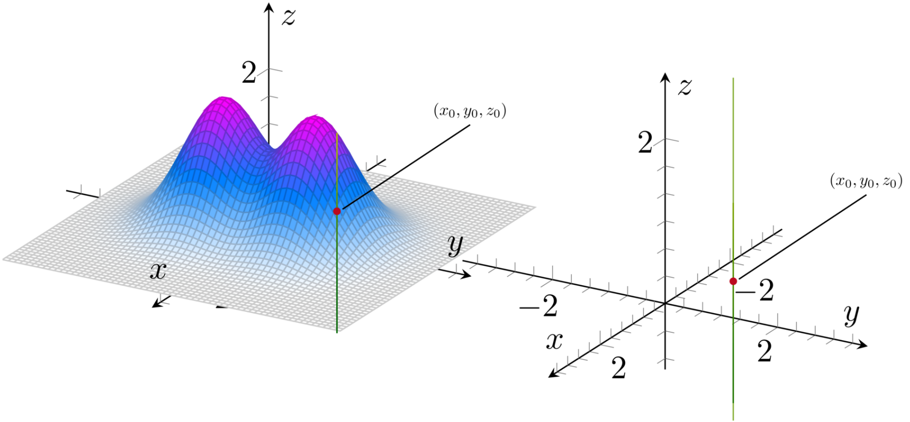

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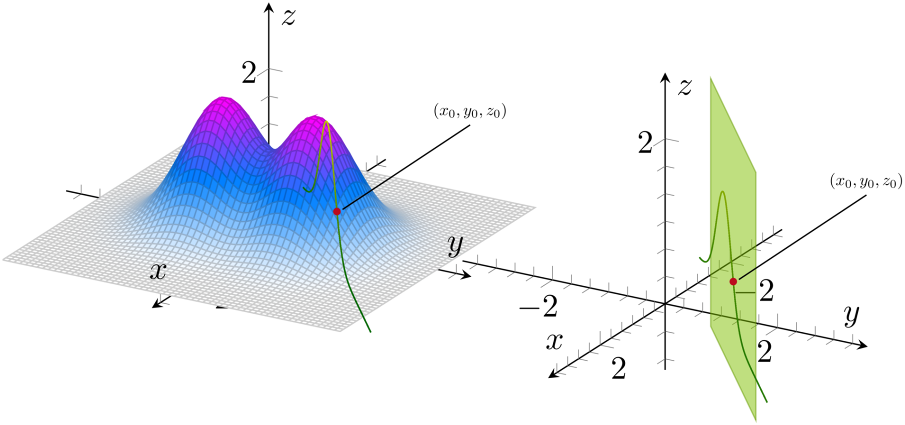

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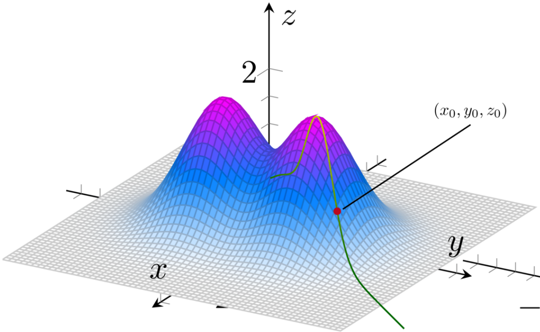

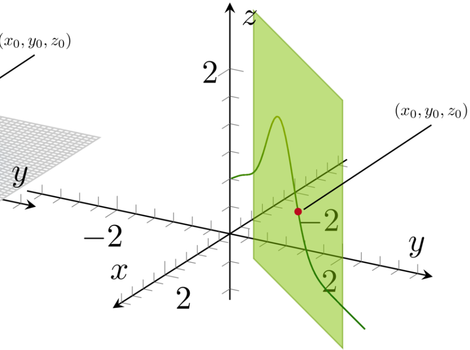

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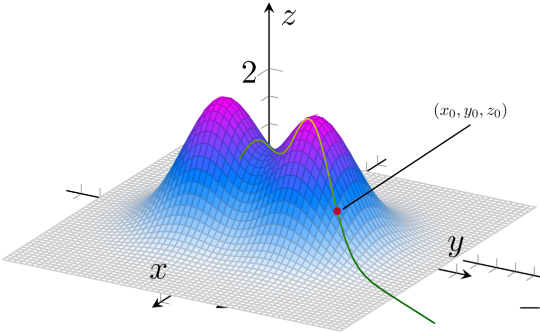

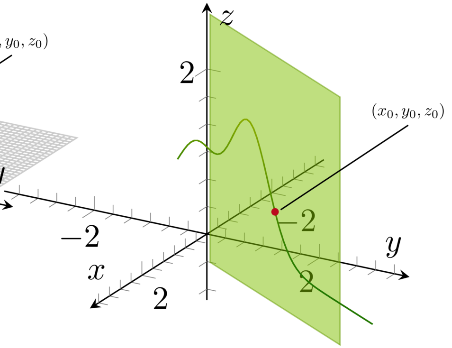

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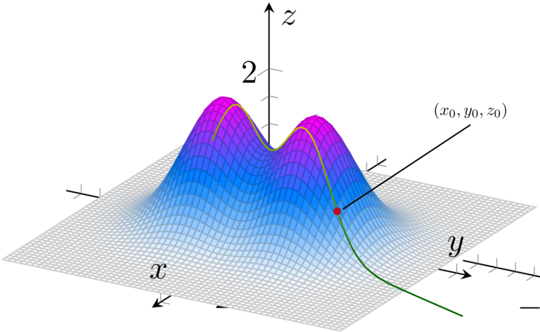

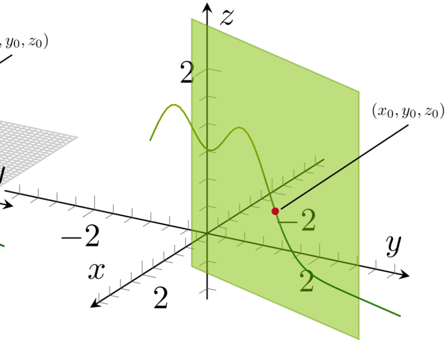

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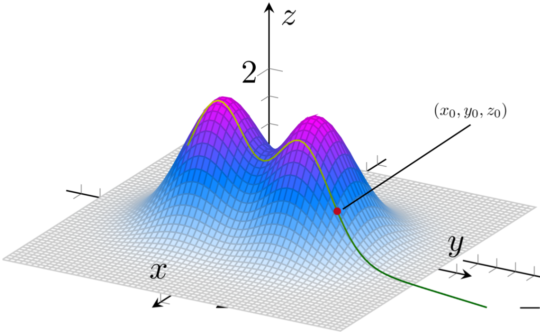

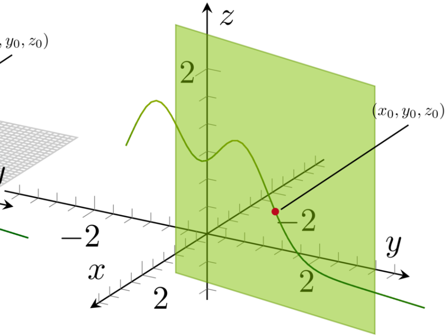

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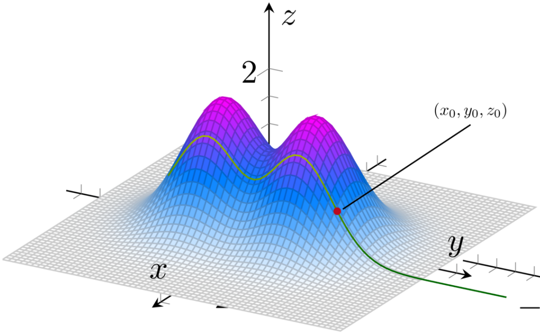

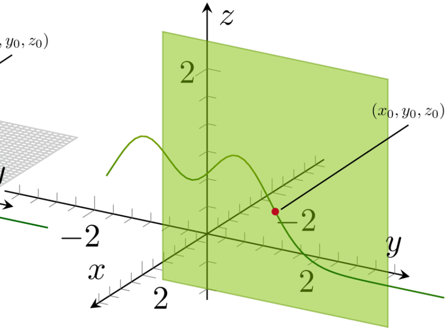

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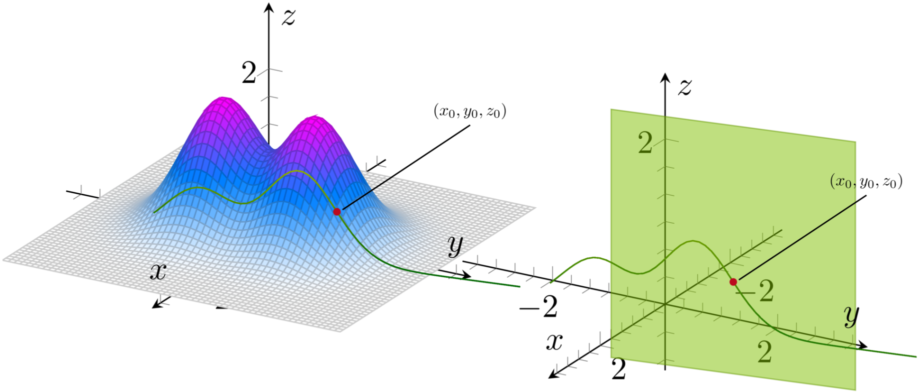

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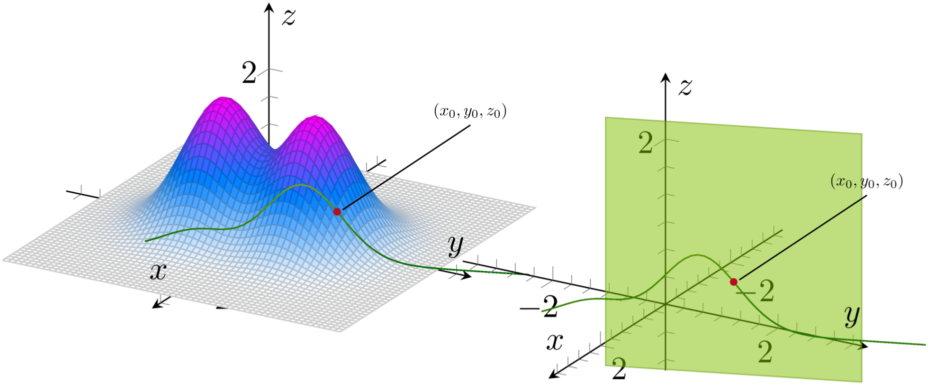

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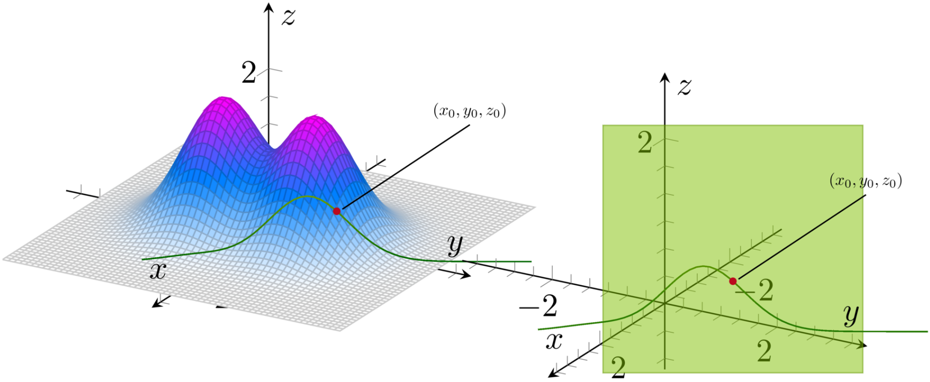

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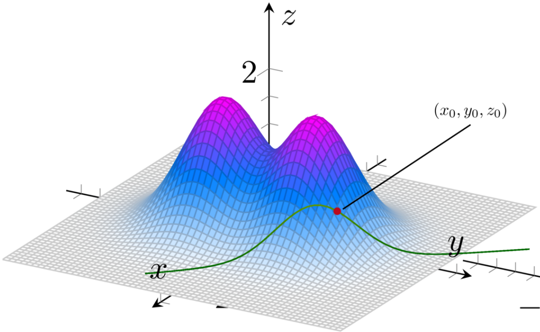

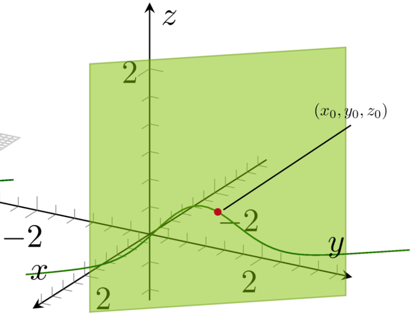

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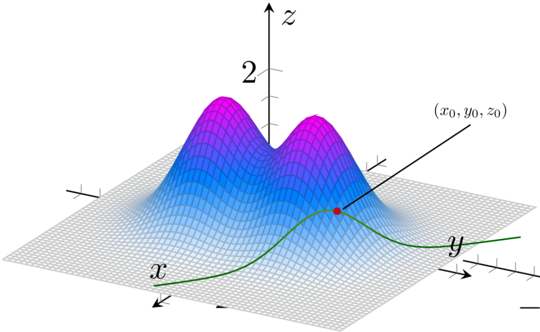

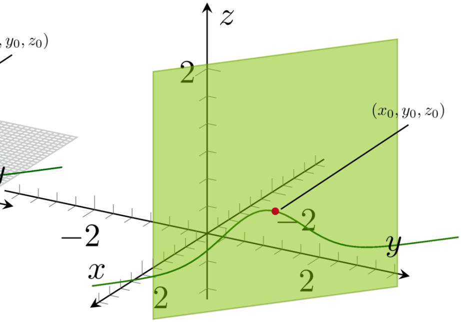

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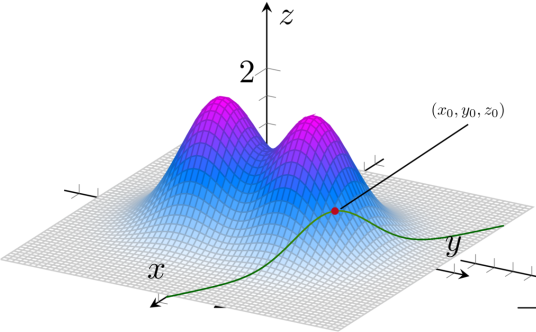

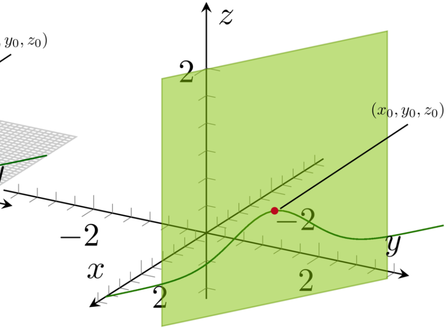

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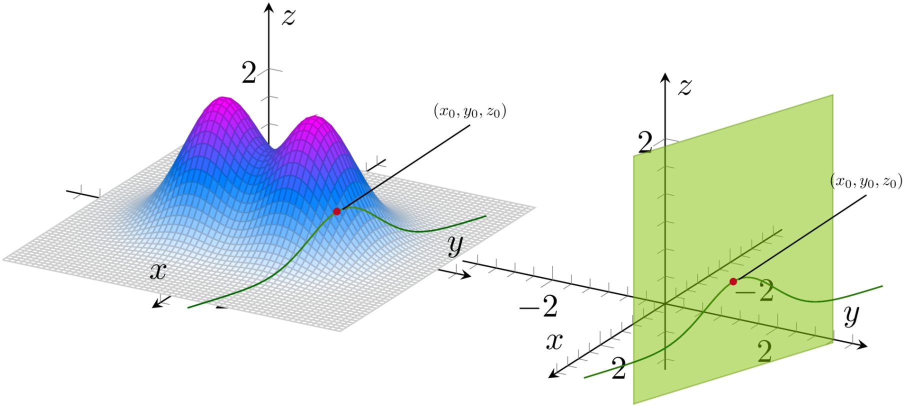
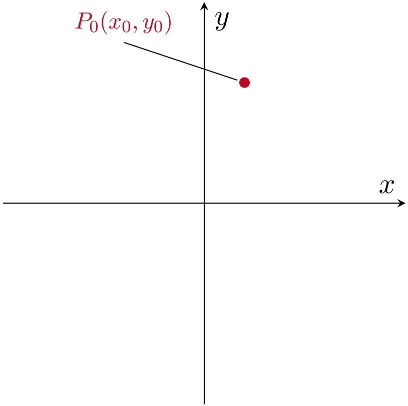


top view



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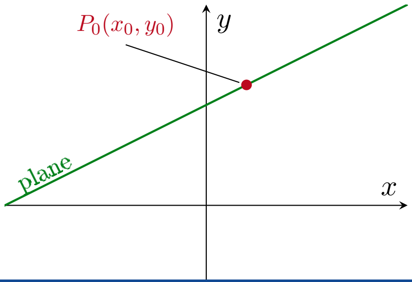
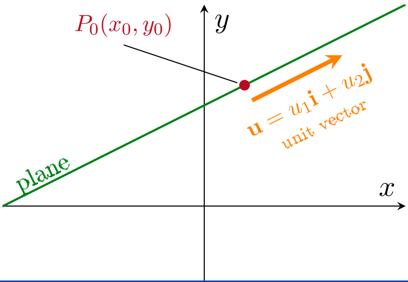


Definition

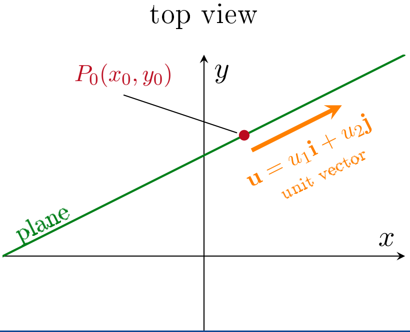
The *derivative of f at* $P_0(x_0, y_0)$

top view



Definition

The *derivative of f at* $P_0(x_0, y_0)$ *in the direction of the unit vector* $\mathbf{u} = u_1\mathbf{i} + u_2\mathbf{j}$



Definition

The *derivative of f at $P_0(x_0, y_0)$ in the direction of the unit vector* $\mathbf{u} = u_1\mathbf{i} + u_2\mathbf{j}$ is

$$D_{\mathbf{u}} f(P_0) = \lim_{s \to 0} \frac{f(x_0 + su_1, y_0 + su_2) - f(x_0, y_0)}{s}$$

provided the limit exists.

$$D_{\mathbf{u}}f(P_0) = \left(\frac{df}{ds}\right)_{\mathbf{u},\ P_0}$$

EXAMPLE 1 Using the definition, find the derivative of

$$f(x, y) = x^2 + xy$$

at $P_0(1, 2)$ in the direction of the unit vector $\mathbf{u} = \left(1/\sqrt{2}\right)\mathbf{i} + \left(1/\sqrt{2}\right)\mathbf{j}$.

Solution Applying the definition in Equation (1), we obtain

$$\left(\frac{df}{ds}\right)_{\mathbf{u},P_0} = \lim_{s\to 0} \frac{f(x_0 + su_1, y_0 + su_2) - f(x_0, y_0)}{s} \qquad \text{Eq. (1)}$$

$$= \lim_{s\to 0} \frac{f\left(1 + s\cdot\frac{1}{\sqrt{2}}, 2 + s\cdot\frac{1}{\sqrt{2}}\right) - f(1, 2)}{s}$$

$$= \lim_{s\to 0} \frac{\left(1 + \frac{s}{\sqrt{2}}\right)^2 + \left(1 + \frac{s}{\sqrt{2}}\right)\left(2 + \frac{s}{\sqrt{2}}\right) - (1^2 + 1\cdot 2)}{s}$$

$$= \lim_{s\to 0} \frac{\left(1 + \frac{2s}{\sqrt{2}} + \frac{s^2}{2}\right) + \left(2 + \frac{3s}{\sqrt{2}} + \frac{s^2}{2}\right) - 3}{s}$$

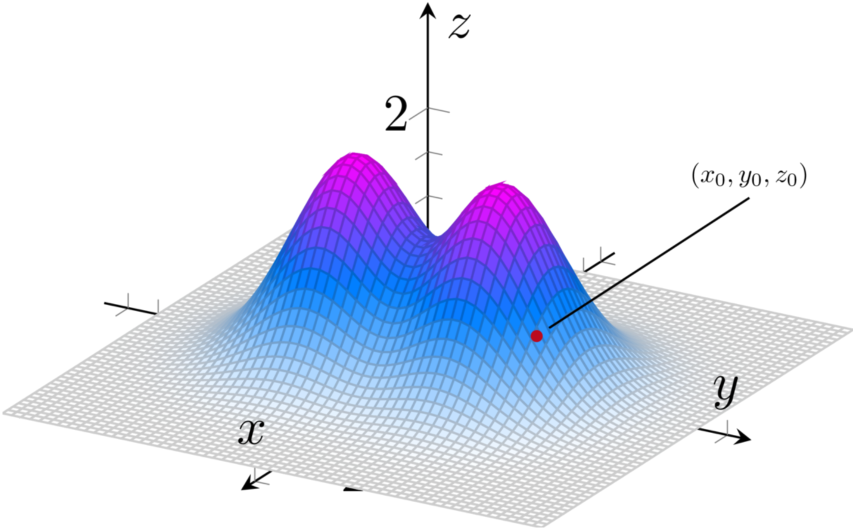
$$= \lim_{s\to 0} \frac{\frac{5s}{\sqrt{2}} + s^2}{s} = \lim_{s\to 0}\left(\frac{5}{\sqrt{2}} + s\right) = \frac{5}{\sqrt{2}}.$$

The rate of change of $f(x, y) = x^2 + xy$ at $P_0(1, 2)$ in the direction $\mathbf{u}$ is $5/\sqrt{2}$.

Remark

But it is easier to calculate directional derivatives if we use gradient vectors.

What is a Gradient Vector?



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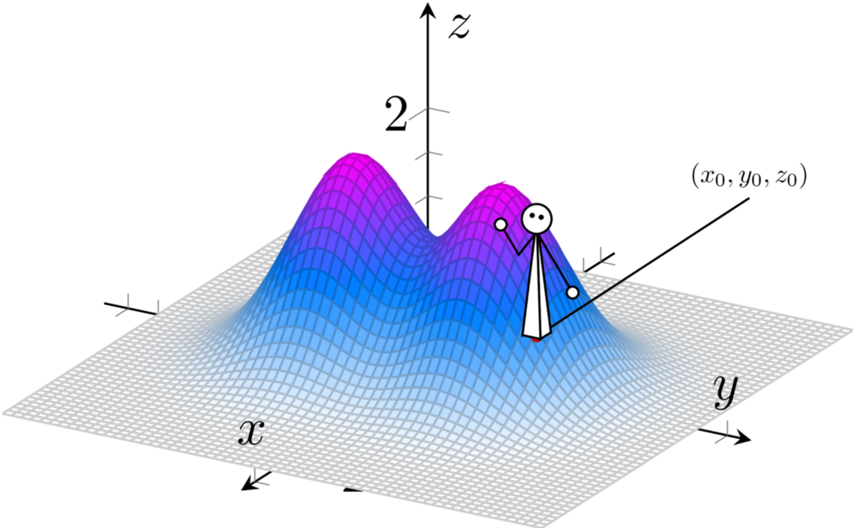

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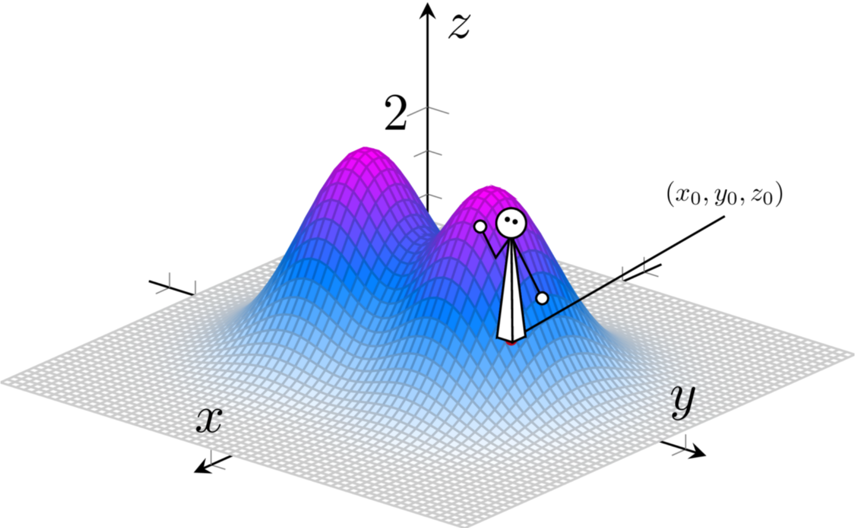

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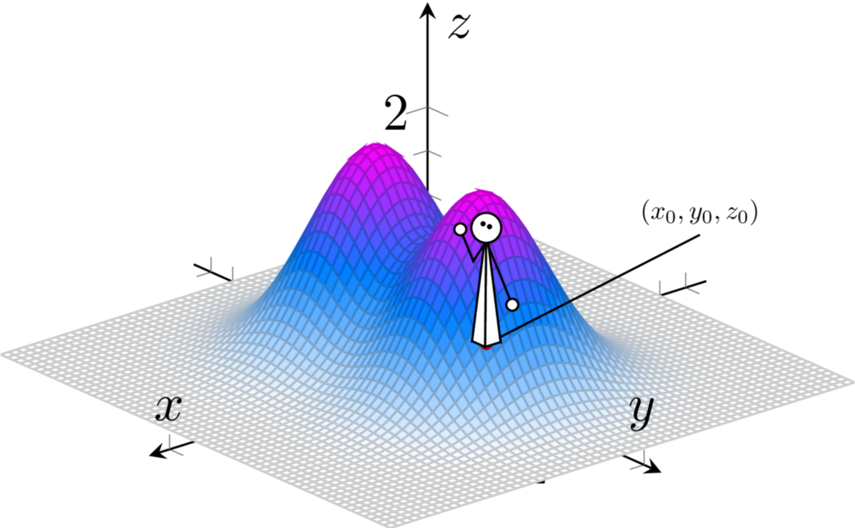

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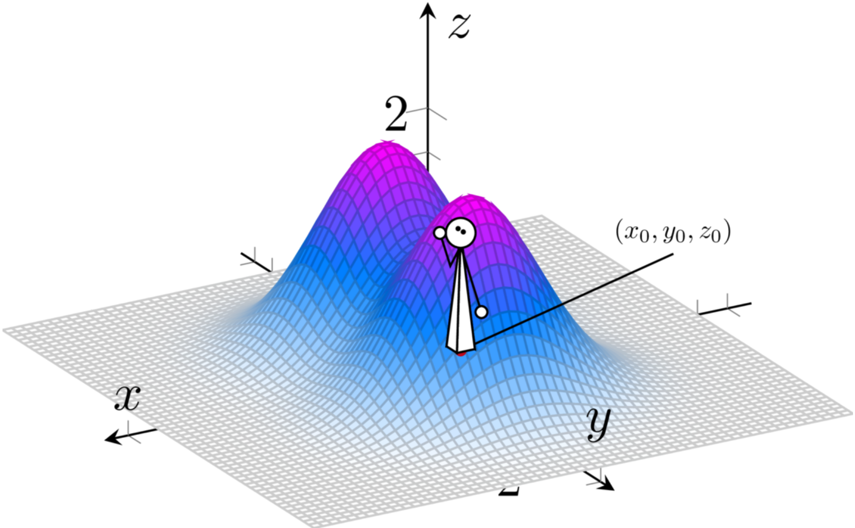

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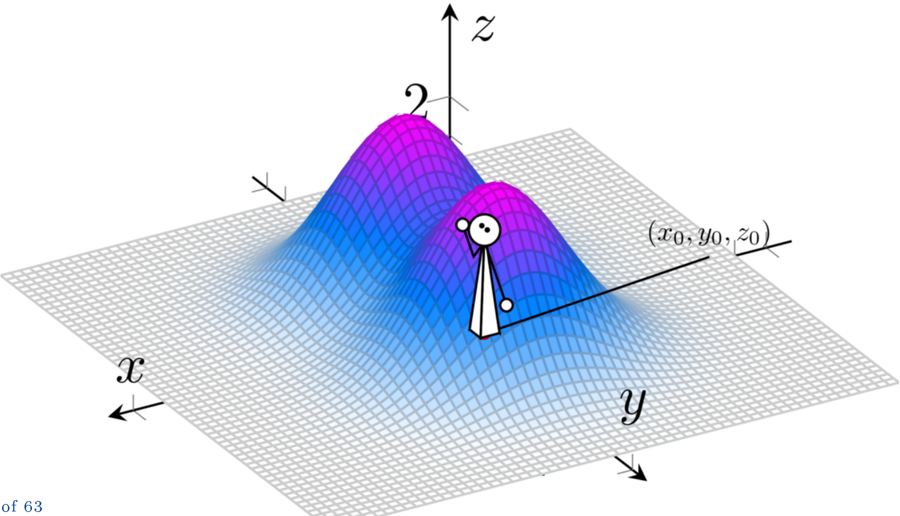

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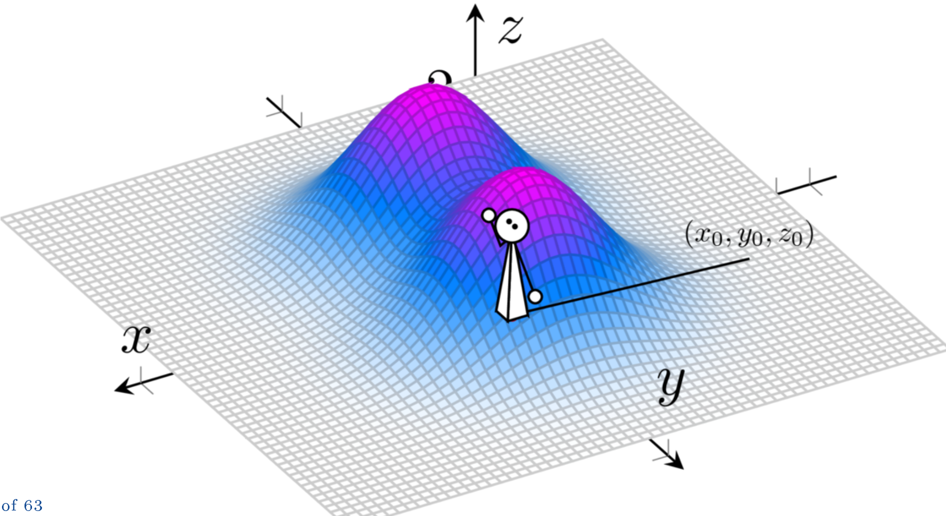

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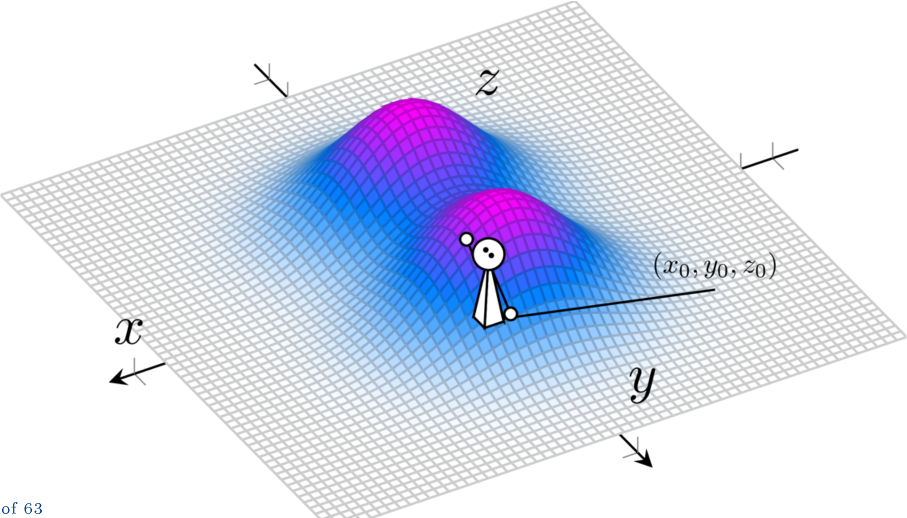

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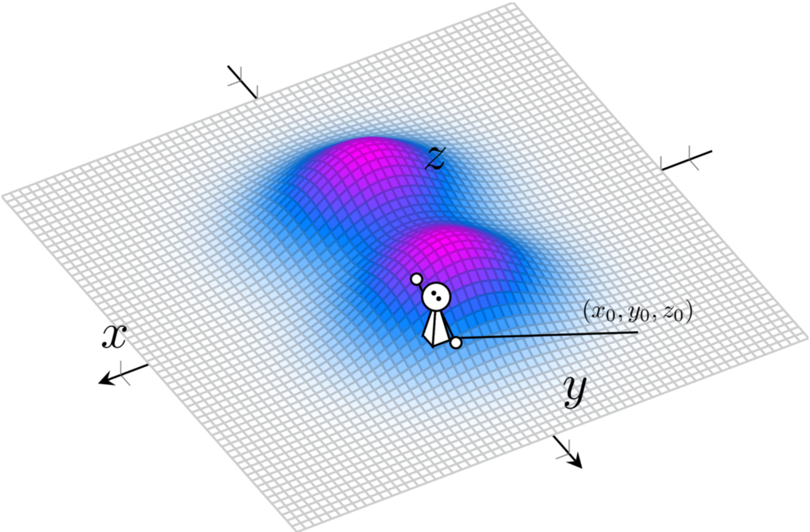

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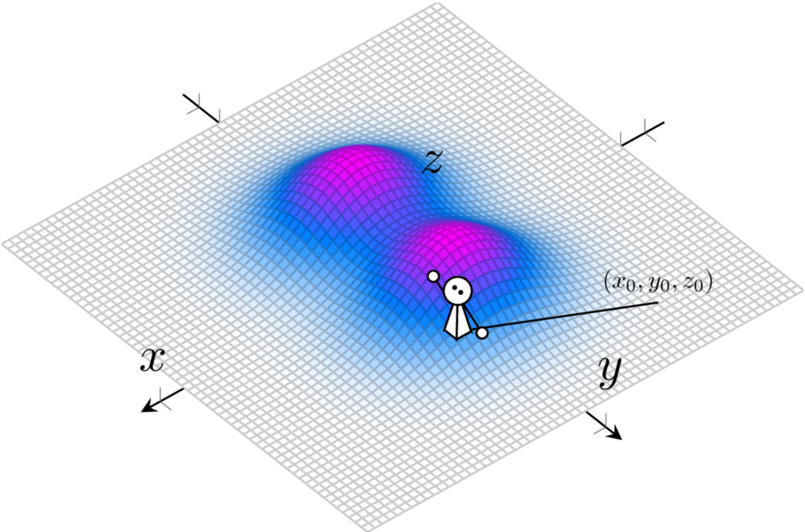

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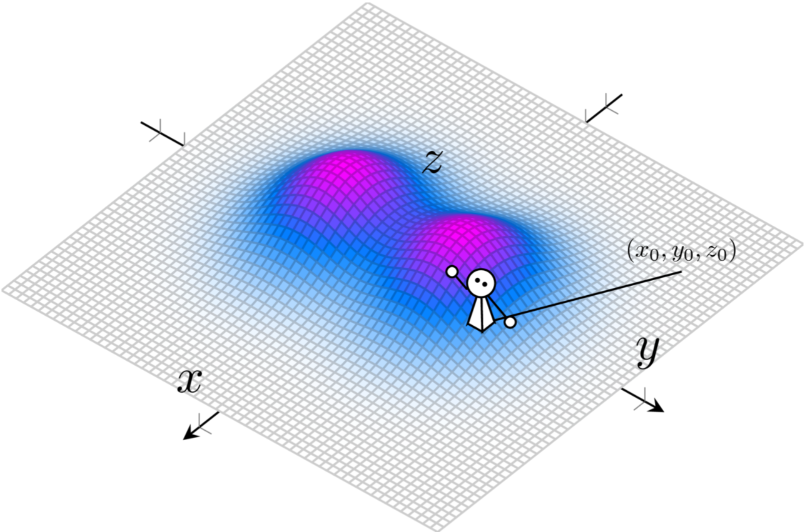

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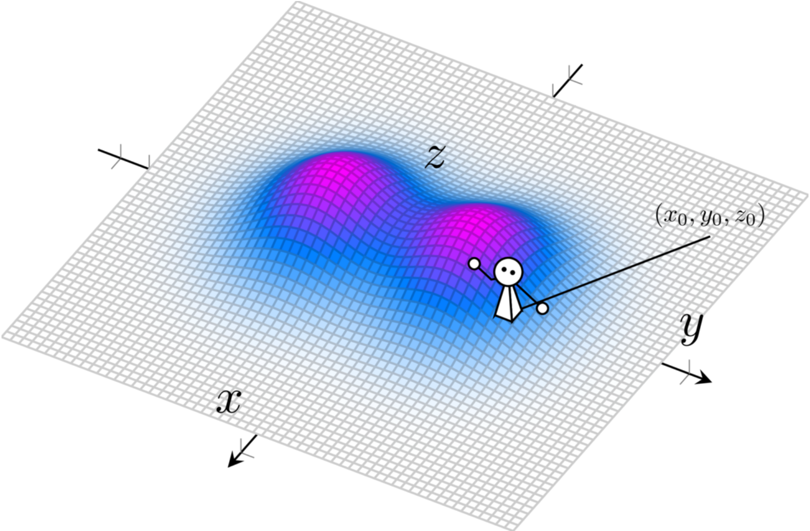

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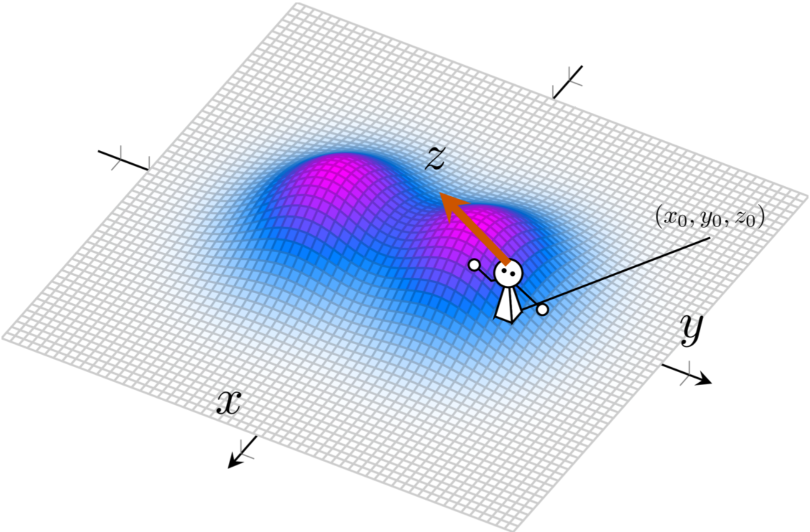

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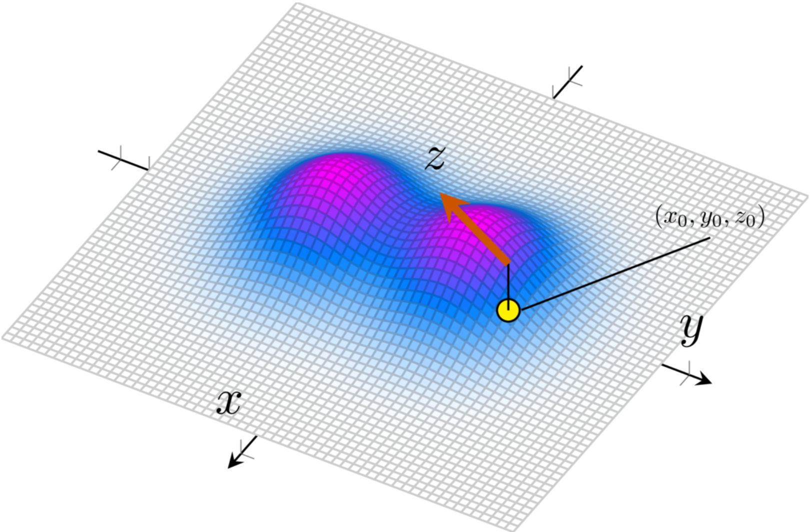

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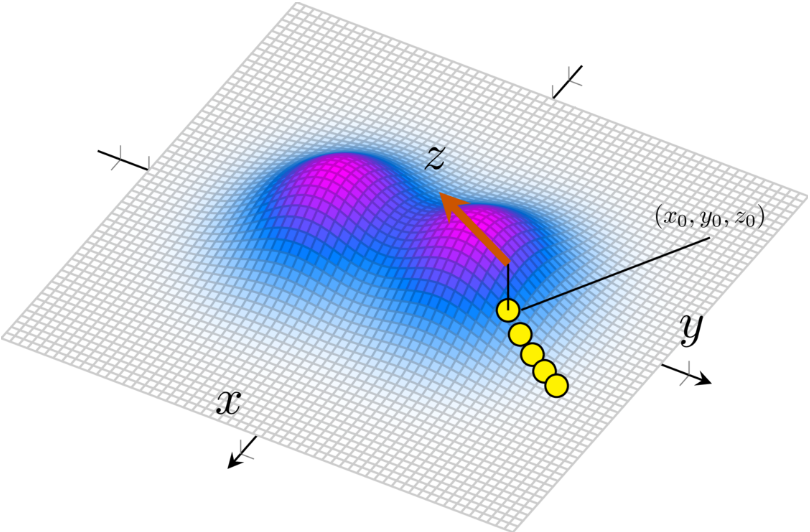

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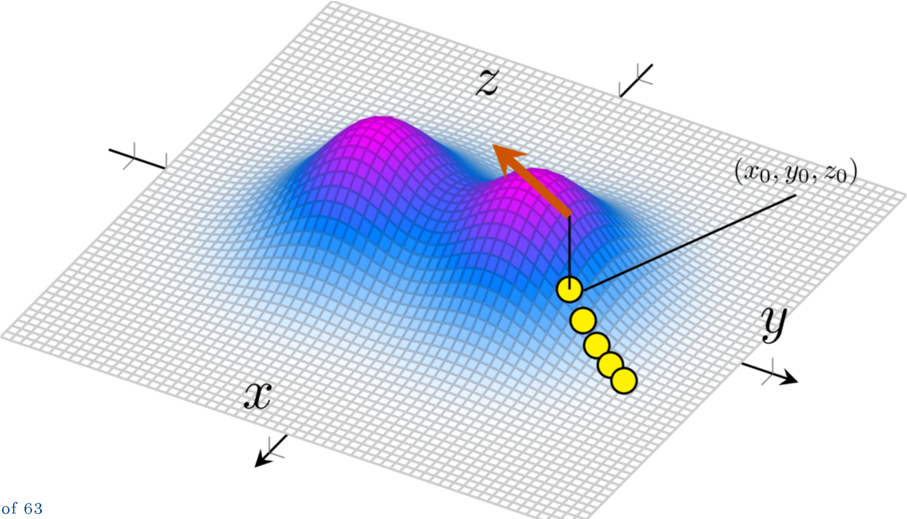

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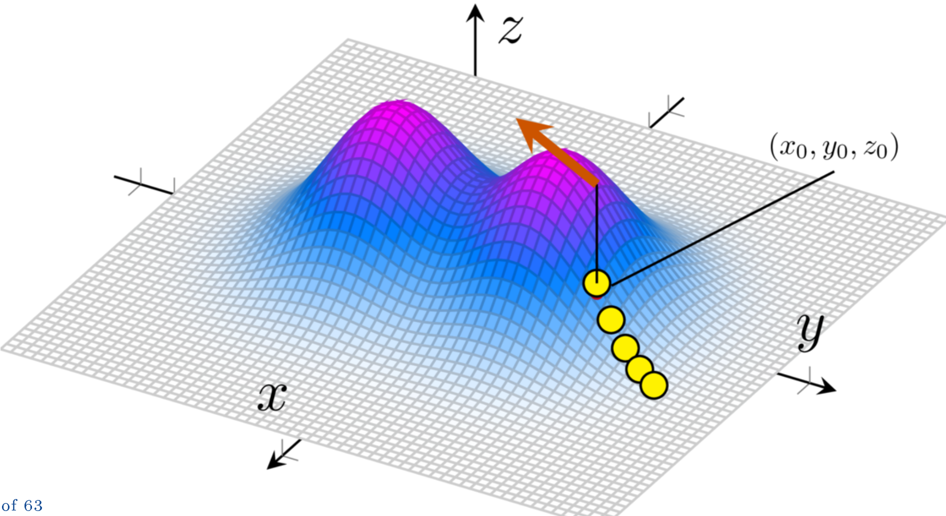

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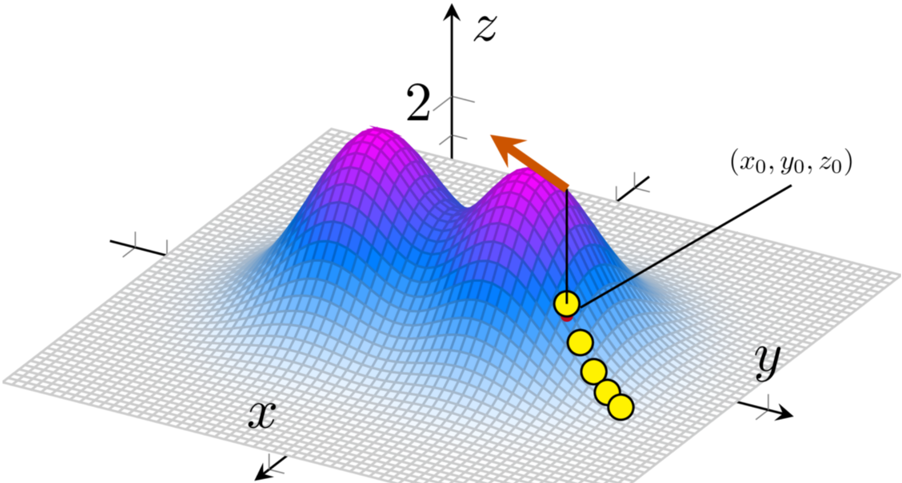

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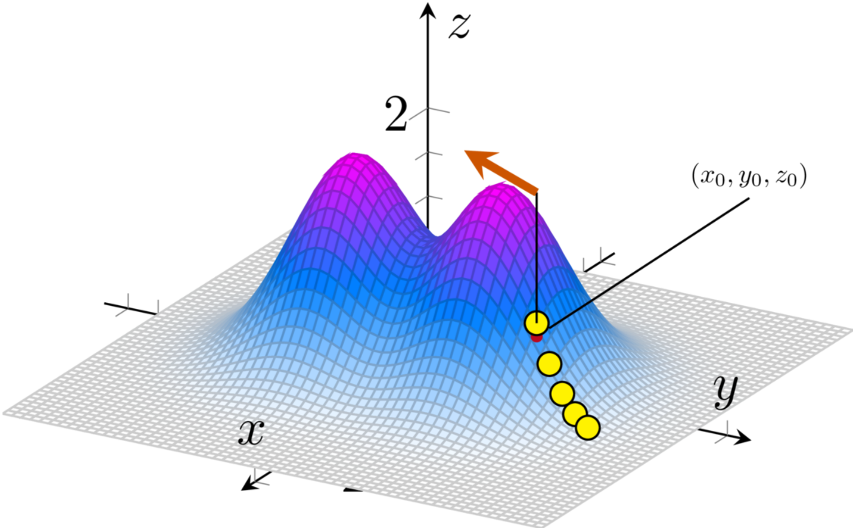

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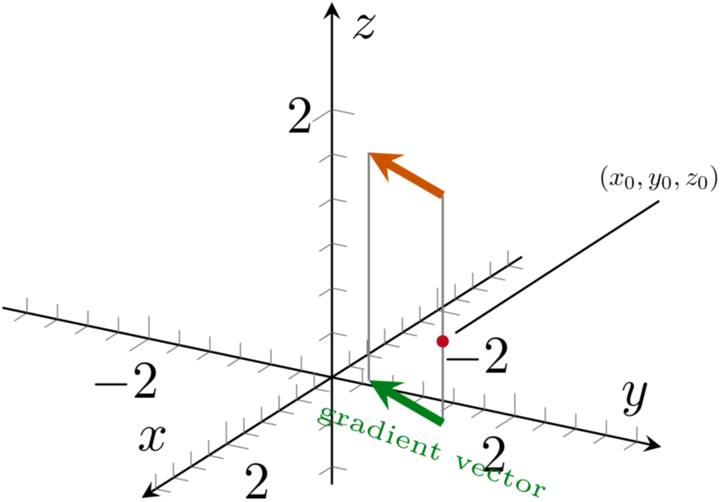

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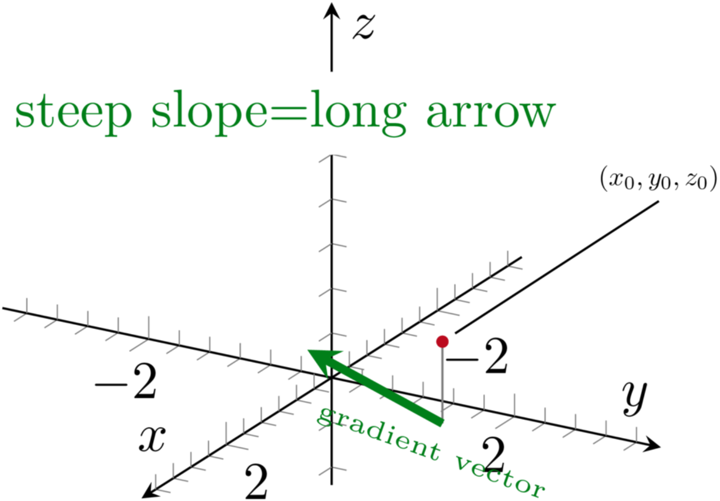

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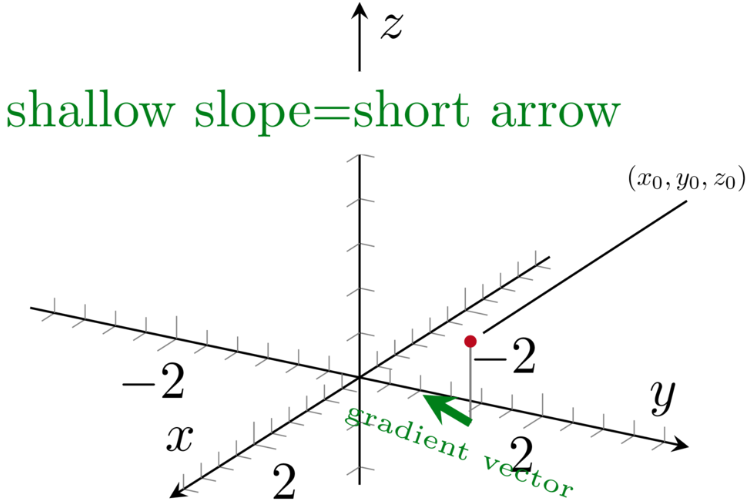

Definition

The *gradient vector* of $f(x, y)$ is the vector

$$\nabla f = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j}.$$

∇ is pronounced “nabla” or “del”.

Harps, p. 984.

$$\nabla f = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j}$$

Example

Find the gradient vector of $f(x, y) = xe^y + \cos(xy)$ at the point $(2, 0)$.

$$\nabla f = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j}$$

Example

Find the gradient vector of $f(x, y) = xe^y + \cos(xy)$ at the point $(2, 0)$.

We calculate that

$$f_x(2, 0) =$$

$$f_y(2, 0) =$$

and

$$\nabla f\Big|_{(2,0)} = f_x(2, 0)\mathbf{i} + f_y(2, 0)\mathbf{j} = \qquad .$$

$$\nabla f = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j}$$

Example

Find the gradient vector of $f(x,y) = xe^y + \cos(xy)$ at the point $(2,0)$.

We calculate that

$$f_x(2,0) = e^y - y\sin(xy)\Big|_{(2,0)} = e^0 - 0 = 1,$$

$$f_y(2,0) =$$

and

$$\nabla f\Big|_{(2,0)} = f_x(2,0)\mathbf{i} + f_y(2,0)\mathbf{j} = \qquad .$$

$$\nabla f = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j}$$

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$$f_y(2,0) = xe^y - x\sin(xy)\Big|_{(2,0)} = 2e^0 - 2\cdot 0 = 2$$

and

$$\nabla f\Big|_{(2,0)} = f_x(2,0)\mathbf{i} + f_y(2,0)\mathbf{j} = \qquad .$$

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and

$$\nabla f\Big|_{(2,0)} = f_x(2,0)\mathbf{i} + f_y(2,0)\mathbf{j} = \mathbf{i} + 2\mathbf{j}.$$

So how can we use gradient vectors to find directional derivatives?

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Theorem

$$D_{\mathbf{u}} f = \nabla f \bullet \mathbf{u}.$$

$$D_{\mathbf{u}}f = \nabla f \cdot \mathbf{u} \qquad \nabla f = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j}$$

Example

Find the derivative of $f(x, y) = xe^y + \cos(xy)$ at the point $(2, 0)$ in the direction of $\mathbf{v} = 3\mathbf{i} - 4\mathbf{j}$.

$$D_{\mathbf{u}}f = \nabla f \cdot \mathbf{u} \qquad \nabla f = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j}$$

Example

Find the derivative of $f(x, y) = xe^y + \cos(xy)$ at the point $(2, 0)$ in the direction of $\mathbf{v} = 3\mathbf{i} - 4\mathbf{j}$.

Recall that $\nabla f|_{(2,0)} = \mathbf{i} + 2\mathbf{j}$.

$$D_{\mathbf{u}}f = \nabla f \cdot \mathbf{u} \qquad \nabla f = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j}$$

Example

Find the derivative of $f(x, y) = xe^y + \cos(xy)$ at the point $(2, 0)$ in the direction of $\mathbf{v} = 3\mathbf{i} - 4\mathbf{j}$.

Recall that $\nabla f|_{(2,0)} = \mathbf{i} + 2\mathbf{j}$. We need to find a unit vector $\mathbf{u}$ which points in the same direction as $\mathbf{v}$,

$$D_{\mathbf{u}}f = \nabla f \bullet \mathbf{u} \qquad \nabla f = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j}$$

Example

Find the derivative of $f(x,y) = xe^y + \cos(xy)$ at the point $(2,0)$ in the direction of $\mathbf{v} = 3\mathbf{i} - 4\mathbf{j}$.

Recall that $\nabla f|_{(2,0)} = \mathbf{i} + 2\mathbf{j}$. We need to find a unit vector $\mathbf{u}$ which points in the same direction as $\mathbf{v}$, so we calculate that

$$\mathbf{u} = \frac{\mathbf{v}}{\|\mathbf{v}\|} = \frac{3\mathbf{i} - 4\mathbf{j}}{\sqrt{3^2 + (-4)^2}} = \frac{3}{5}\mathbf{i} - \frac{4}{5}\mathbf{j}.$$

$$D_{\mathbf{u}}f = \nabla f \bullet \mathbf{u} \qquad \nabla f = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j}$$

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Therefore

$$D_{\mathbf{u}}f(2,0) = \nabla f|_{(2,0)} \bullet \mathbf{u} =$$

$$D_{\mathbf{u}}f = \nabla f \cdot \mathbf{u} \qquad \nabla f = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j}$$

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Therefore

$$D_{\mathbf{u}}f(2,0) = \nabla f|_{(2,0)} \cdot \mathbf{u} = (\mathbf{i} + 2\mathbf{j}) \cdot \left(\frac{3}{5}\mathbf{i} - \frac{4}{5}\mathbf{j}\right) = \frac{3}{5} - \frac{8}{5} = -1.$$

$$\mathbf{a} \bullet \mathbf{b} = \|\mathbf{a}\| \, \|\mathbf{b}\| \cos\theta$$

Note that

$$D_{\mathbf{u}} f = \nabla f \bullet \mathbf{u} = \|\nabla f\| \, \|\mathbf{u}\| \cos\theta = \|\nabla f\| \cos\theta$$

since $\|\mathbf{u}\| = 1$.

$$\mathbf{a} \bullet \mathbf{b} = \|\mathbf{a}\| \, \|\mathbf{b}\| \cos\theta$$

Note that

$$D_{\mathbf{u}} f = \nabla f \bullet \mathbf{u} = \|\nabla f\| \, \|\mathbf{u}\| \cos\theta = \|\nabla f\| \cos\theta$$

since $\|\mathbf{u}\| = 1$.

So we must always have

$$-\|\nabla f\| \leq D_{\mathbf{u}} f \leq \|\nabla f\| .$$

$$\boxed{D_{\mathbf{u}}f = \|\nabla f\| \cos\theta}$$

Remark

f increases mostly rapidly $\implies \cos\theta = 1 \implies \theta = 0 \implies$ $\mathbf{u}$ points in the same direction as ∇f

∇f points 'uphill'

$$D_{\mathbf{u}} f = \|\nabla f\| \cos\theta$$

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Remark

f decreases mostly rapidly $\implies \cos\theta = -1 \implies \theta = 180^\circ \implies$ $\mathbf{u}$ points in the opposite direction from ∇f

a ball on a hill rolls in the direction $-\nabla f$

$$D_{\mathbf{u}}f = \|\nabla f\| \cos\theta$$

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∇f points 'uphill'

Remark

f decreases mostly rapidly $\implies \cos\theta = -1 \implies \theta = 180^\circ \implies$ $\mathbf{u}$ points in the opposite direction from ∇f

a ball on a hill rolls in the direction $-\nabla f$

Remark

$$\theta = 90^\circ \implies D_{\mathbf{u}}f = 0.$$

EXAMPLE 3 Find the directions in which $f(x, y) = (x^2/2) + (y^2/2)$

(a) increases most rapidly at the point (1, 1).

(b) decreases most rapidly at (1, 1).

(c) What are the directions of zero change in f at (1, 1)?

Solution

(a) The function increases most rapidly in the direction of ∇f at (1, 1). The gradient there is

$$(\nabla f)_{(1,1)} = (x\mathbf{i} + y\mathbf{j})_{(1,1)} = \mathbf{i} + \mathbf{j}.$$

Its direction is

$$\mathbf{u} = \frac{\mathbf{i} + \mathbf{j}}{|\mathbf{i} + \mathbf{j}|} = \frac{\mathbf{i} + \mathbf{j}}{\sqrt{(1)^2 + (1)^2}} = \frac{1}{\sqrt{2}}\mathbf{i} + \frac{1}{\sqrt{2}}\mathbf{j}.$$

(b) The function decreases most rapidly in the direction of $-\nabla f$ at (1, 1), which is

$$-\mathbf{u} = -\frac{1}{\sqrt{2}}\mathbf{i} - \frac{1}{\sqrt{2}}\mathbf{j}.$$

(c) The directions of zero change at (1, 1) are the directions orthogonal to ∇f:

$$\mathbf{n} = -\frac{1}{\sqrt{2}}\mathbf{i} + \frac{1}{\sqrt{2}}\mathbf{j} \qquad \text{and} \qquad -\mathbf{n} = \frac{1}{\sqrt{2}}\mathbf{i} - \frac{1}{\sqrt{2}}\mathbf{j}.$$

$$\boxed{D_{\mathbf{u}} f = \|\nabla f\| \cos\theta}$$

Algebra Rules for ∇

Theorem

1. *Sum Rule:* $\nabla(f+g) = \nabla f + \nabla g$
2. *Difference Rule:* $\nabla(f+g) = \nabla f + \nabla g$
3. *Constant Multiple Rule:* $\nabla(kf) = k\nabla f \qquad$ *(for $k \in \mathbb{R}$)*
4. *Product Rule:* $\nabla(fg) = g\nabla f + f\nabla g$
5. *Quotient Rule:* $\nabla\left(\dfrac{f}{g}\right) = \dfrac{g\nabla f - f\nabla g}{g^2}$.

EXAMPLE 5 We illustrate two of the rules with

$$f(x, y) = x - y \qquad g(x, y) = 3y$$
$$\nabla f = \mathbf{i} - \mathbf{j} \qquad \nabla g = 3\mathbf{j}.$$

We have

1. $\nabla(f - g) = \nabla(x - 4y) = \mathbf{i} - 4\mathbf{j} = \nabla f - \nabla g$ Rule 2
2. $\nabla(fg) = \nabla(3xy - 3y^2) = 3y\mathbf{i} + (3x - 6y)\mathbf{j}$

   and

   $$\begin{aligned} f\nabla g + g\nabla f &= (x - y)3\mathbf{j} + 3y(\mathbf{i} - \mathbf{j}) && \text{Substitute.} \\ &= 3y\mathbf{i} + (3x - 6y)\mathbf{j}. && \text{Simplify.} \end{aligned}$$

   We have therefore verified for this example that $\nabla(fg) = f\nabla g + g\nabla f$.

$$D_{\mathbf{u}}f = \|\nabla f\| \cos\theta$$

Functions of Three Variables

$$\nabla f = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} + \frac{\partial f}{\partial z}\mathbf{k}$$

$$D_{\mathbf{u}}f = \nabla f \cdot \mathbf{u}.$$

EXAMPLE 6

(a) Find the derivative of $f(x, y, z) = x^3 - xy^2 - z$ at $P_0(1, 1, 0)$ in the direction of $\mathbf{v} = 2\mathbf{i} - 3\mathbf{j} + 6\mathbf{k}$.

(b) In what directions does f change most rapidly at P_0, and what are the rates of change in these directions?

Solution

(a) The direction of $\mathbf{v}$ is obtained by dividing $\mathbf{v}$ by its length:

$$|\mathbf{v}| = \sqrt{(2)^2 + (-3)^2 + (6)^2} = \sqrt{49} = 7$$

$$\mathbf{u} = \frac{\mathbf{v}}{|\mathbf{v}|} = \frac{2}{7}\mathbf{i} - \frac{3}{7}\mathbf{j} + \frac{6}{7}\mathbf{k}.$$

The partial derivatives of f at P_0 are

$$f_x = (3x^2 - y^2)\Big|_{(1,1,0)} = 2, \qquad f_y = -2xy\Big|_{(1,1,0)} = -2, \qquad f_z = -1\Big|_{(1,1,0)} = -1.$$

The gradient of f at P_0 is

$$\nabla f\big|_{(1,1,0)} = 2\mathbf{i} - 2\mathbf{j} - \mathbf{k}.$$

The derivative of f at P_0 in the direction of $\mathbf{v}$ is therefore

$$D_{\mathbf{u}}f\big|_{(1,1,0)} = \nabla f\big|_{(1,1,0)} \cdot \mathbf{u} = (2\mathbf{i} - 2\mathbf{j} - \mathbf{k}) \cdot \left(\frac{2}{7}\mathbf{i} - \frac{3}{7}\mathbf{j} + \frac{6}{7}\mathbf{k}\right)$$

$$= \frac{4}{7} + \frac{6}{7} - \frac{6}{7} = \frac{4}{7}.$$

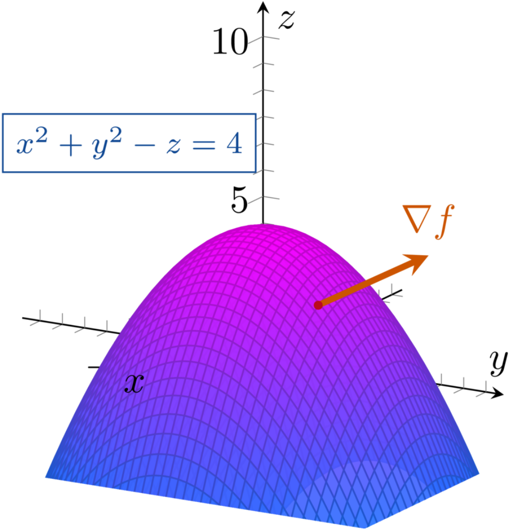
(b) The function increases most rapidly in the direction of $\nabla f = 2\mathbf{i} - 2\mathbf{j} - \mathbf{k}$ and decreases most rapidly in the direction of $-\nabla f$. The rates of change in the directions are, respectively,

$$|\nabla f| = \sqrt{(2)^2 + (-2)^2 + (-1)^2} = \sqrt{9} = 3 \qquad \text{and} \qquad -|\nabla f| = -3.$$

13.6 Tangent Planes and Differentials

$$D_{\mathbf{u}}f = \|\nabla f\| \cos\theta$$

Tangent Planes and Normal Lines



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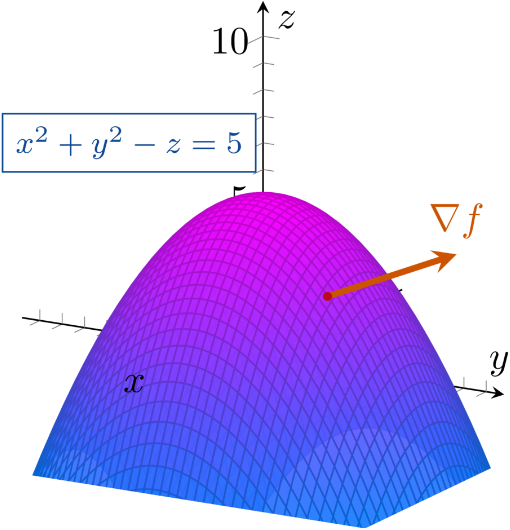
Tangent Planes and Normal Lines



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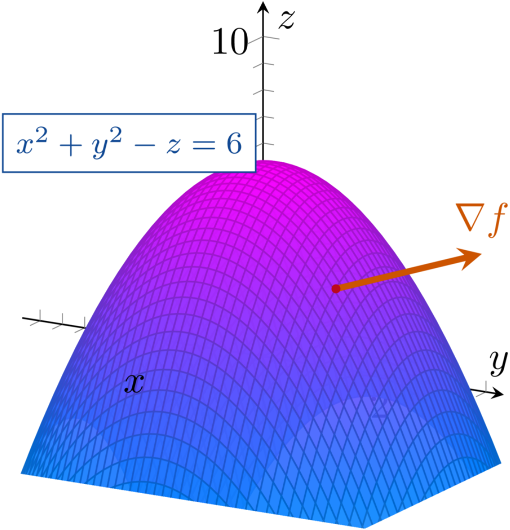
Tangent Planes and Normal Lines



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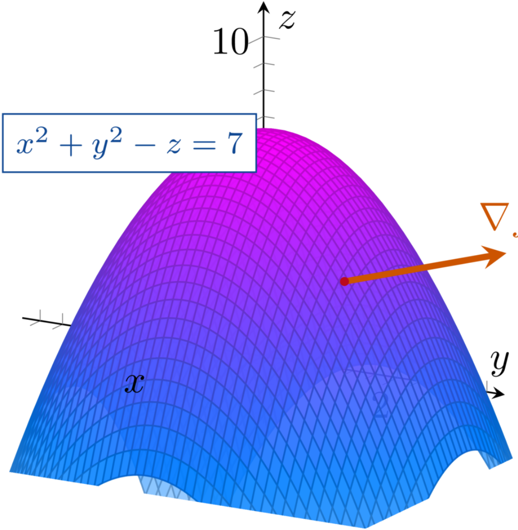

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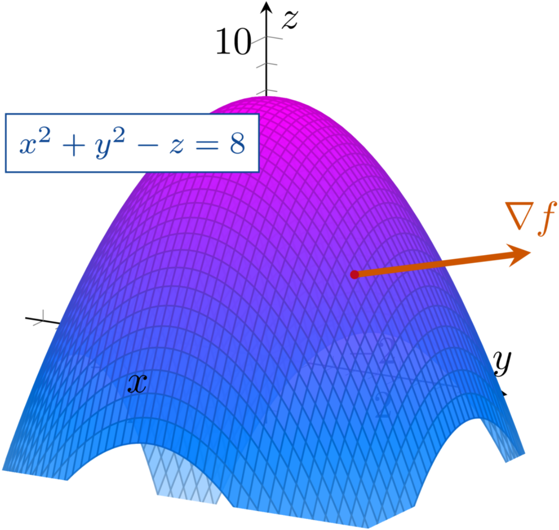
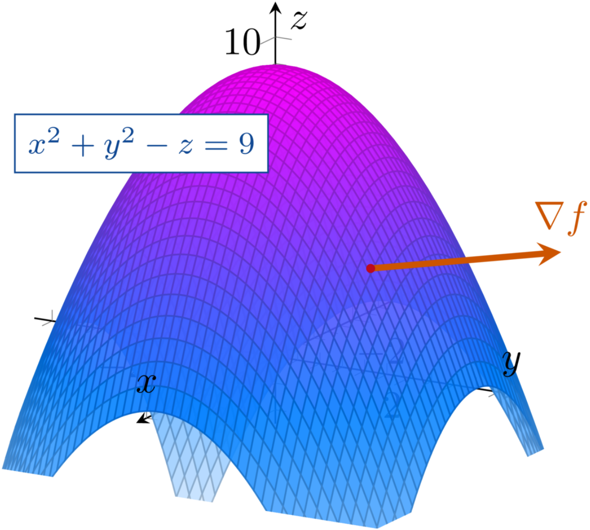


$$\boxed{D_{\mathbf{u}}f = \|\nabla f\| \cos\theta}$$

Tangent Planes and Normal Lines



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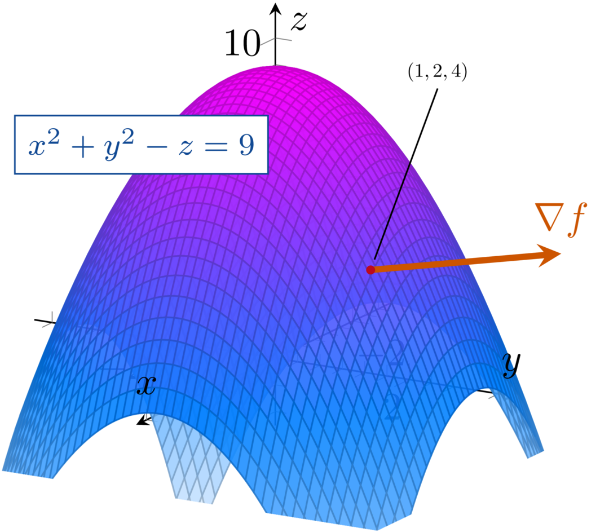

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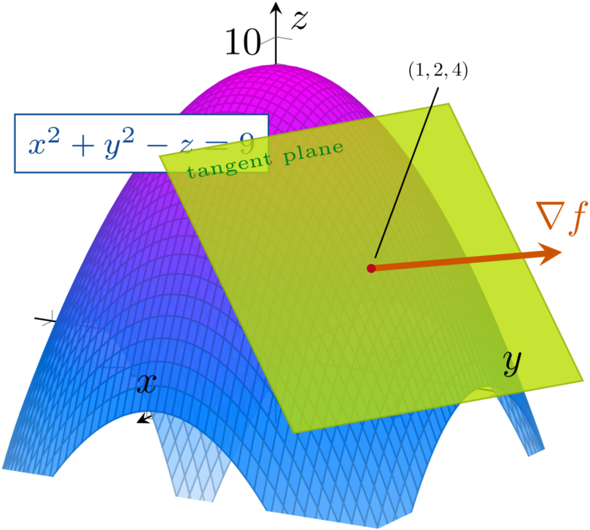

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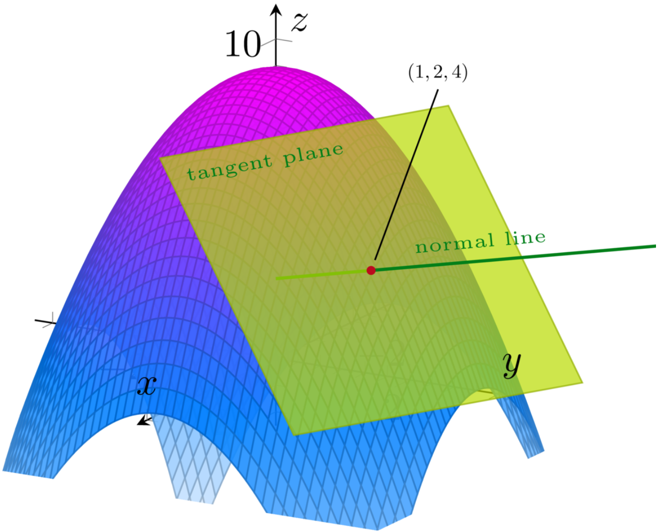

$$D_{\mathbf{u}}f = \|\nabla f\| \cos\theta$$

Definition

The *tangent plane* to the surface $f(x, y, z) = c$ at the point $P(x_0, y_0, z_0)$ (where the gradient is not zero) is the plane through P with normal vector $\nabla f|_P$.

$$D_{\mathbf{u}} f = \|\nabla f\| \cos\theta$$

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$$f_x(P)(x - x_0) + f_y(P)(y - y_0) + f_z(P)(z - z_0) = 0.$$

$$D_{\mathbf{u}}f = \|\nabla f\| \cos\theta$$

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Definition

The *normal line* to the surface $f(x, y, z) = c$ at the point P is the line through P parallel to $\nabla f|_P$.

$$D_{\mathbf{u}}f = \|\nabla f\| \cos\theta$$

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Definition

The *normal line* to the surface $f(x, y, z) = c$ at the point P is the line through P parallel to $\nabla f|_P$.

$$x = x_0 + f_x(P)t \qquad y = y_0 + f_y(P)t \qquad z = z_0 + f_z(P)t.$$

EXAMPLE 1 Find the tangent plane and normal line of the level surface

$$f(x, y, z) = x^2 + y^2 + z - 9 = 0 \qquad \text{A circular paraboloid}$$

at the point $P_0(1, 2, 4)$.

Solution The surface is shown in Figure 14.34.

The tangent plane is the plane through P_0 perpendicular to the gradient of f at P_0. The gradient is

$$\nabla f\big|_{P_0} = (2x\mathbf{i} + 2y\mathbf{j} + \mathbf{k})\Big|_{(1,2,4)} = 2\mathbf{i} + 4\mathbf{j} + \mathbf{k}.$$

The tangent plane is therefore the plane

$$2(x - 1) + 4(y - 2) + (z - 4) = 0, \qquad \text{or} \qquad 2x + 4y + z = 14.$$

The line normal to the surface at P_0 is

$$x = 1 + 2t, \qquad y = 2 + 4t, \qquad z = 4 + t.$$

$$D_{\mathbf{u}}f = \|\nabla f\| \cos\theta$$

Now consider

$$z = f(x, y).$$

Now consider

$$z = f(x, y).$$

This is equivalent to

$$F(x, y, z) = f(x, y) - z = 0.$$

Now consider

$$z = f(x, y).$$

This is equivalent to

$$F(x, y, z) = f(x, y) - z = 0.$$

Definition

The *tangent plane* to the surface $z = f(x, y)$ at the point $P(x_0, y_0, z_0) = (x_0, y_0, f(x_0, y_0))$ is

$$f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0) - (z - z_0) = 0.$$

EXAMPLE 2 Find the plane tangent to the surface $z = x \cos y - ye^x$ at $(0, 0, 0)$.

Solution We calculate the partial derivatives of $f(x, y) = x \cos y - ye^x$ and use Equation (3):

$$f_x(0, 0) = (\cos y - ye^x)\Big|_{(0,0)} = 1 - 0 \cdot 1 = 1$$

$$f_y(0, 0) = (-x \sin y - e^x)\Big|_{(0,0)} = 0 - 1 = -1.$$

The tangent plane is therefore

$$1 \cdot (x - 0) - 1 \cdot (y - 0) - (z - 0) = 0, \qquad \text{Eq. (3)}$$

or

$$x - y - z = 0. \qquad \blacksquare$$

$$D_{\mathbf{u}} f = \|\nabla f\| \cos\theta$$

EXAMPLE 3 The surfaces

$$f(x, y, z) = x^2 + y^2 - 2 = 0 \qquad \text{A cylinder}$$

and

$$g(x, y, z) = x + z - 4 = 0 \qquad \text{A plane}$$

meet in an ellipse E (Figure 14.35). Find parametric equations for the line tangent to E at the point $P_0(1, 1, 3)$.

$$D_{\mathbf{u}} f = \|\nabla f\| \cos\theta$$

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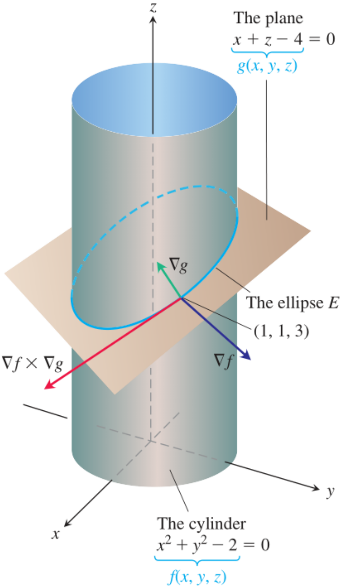
the point $P_0(1, 1, 3)$.



$$D_{\mathbf{u}} f = \|\nabla f\| \cos\theta$$

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meet in an ellipse E (Figure 14.35). Find parametric equations for the line tangent to E at the point $P_0(1, 1, 3)$.

Solution The tangent line is orthogonal to both ∇f and ∇g at P_0, and therefore parallel to $\mathbf{v} = \nabla f \times \nabla g$. The components of $\mathbf{v}$ and the coordinates of P_0 give us equations for the line. We have

$$\nabla f\big|_{(1,1,3)} = (2x\mathbf{i} + 2y\mathbf{j})\Big|_{(1,1,3)} = 2\mathbf{i} + 2\mathbf{j}$$

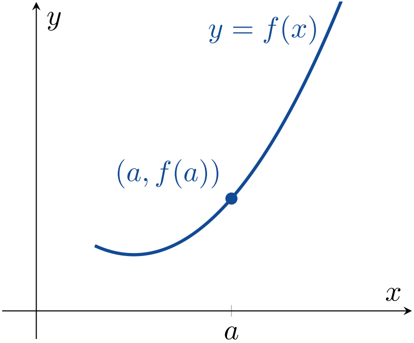
$$\nabla g\big|_{(1,1,3)} = (\mathbf{i} + \mathbf{k})\Big|_{(1,1,3)} = \mathbf{i} + \mathbf{k}$$

$$\mathbf{v} = (2\mathbf{i} + 2\mathbf{j}) \times (\mathbf{i} + \mathbf{k}) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 2 & 0 \\ 1 & 0 & 1 \end{vmatrix} = 2\mathbf{i} - 2\mathbf{j} - 2\mathbf{k}.$$

The tangent line to the ellipse of intersection is

$$x = 1 + 2t, \qquad y = 1 - 2t, \qquad z = 3 - 2t. \qquad \blacksquare$$

Linearisation of a Function of One Variable

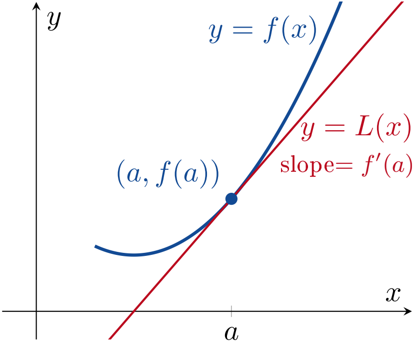


The *linearisation* of f at a is

$L(x) =$

$$D_{\mathbf{u}}f = \|\nabla f\| \cos\theta$$

Linearisation of a Function of One Variable



The *linearisation* of f at a is

$$L(x) = f(a) + f'(a)(x - a).$$

$$L(x) = f(a) + f'(a)(x - a)$$

Linearisation of a Function of Two Variable

Definition

The *linearisation* of a function $f(x, y)$ at a point (x_0, y_0) is

$$L(x, y) = f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0).$$

EXAMPLE 5 Find the linearization of

$$f(x, y) = x^2 - xy + \frac{1}{2}y^2 + 3$$

at the point (3, 2).

Solution We first evaluate f, f_x, and f_y at the point $(x_0, y_0) = (3, 2)$:

$$f(3, 2) = \left(x^2 - xy + \frac{1}{2}y^2 + 3\right)\bigg|_{(3,2)} = 8$$

$$f_x(3, 2) = \frac{\partial}{\partial x}\left(x^2 - xy + \frac{1}{2}y^2 + 3\right)\bigg|_{(3,2)} = (2x - y)\big|_{(3,2)} = 4$$

$$f_y(3, 2) = \frac{\partial}{\partial y}\left(x^2 - xy + \frac{1}{2}y^2 + 3\right)\bigg|_{(3,2)} = (-x + y)\big|_{(3,2)} = -1,$$

giving

$$\begin{aligned} L(x, y) &= f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0) \\ &= 8 + (4)(x - 3) + (-1)(y - 2) = 4x - y - 2. \end{aligned}$$

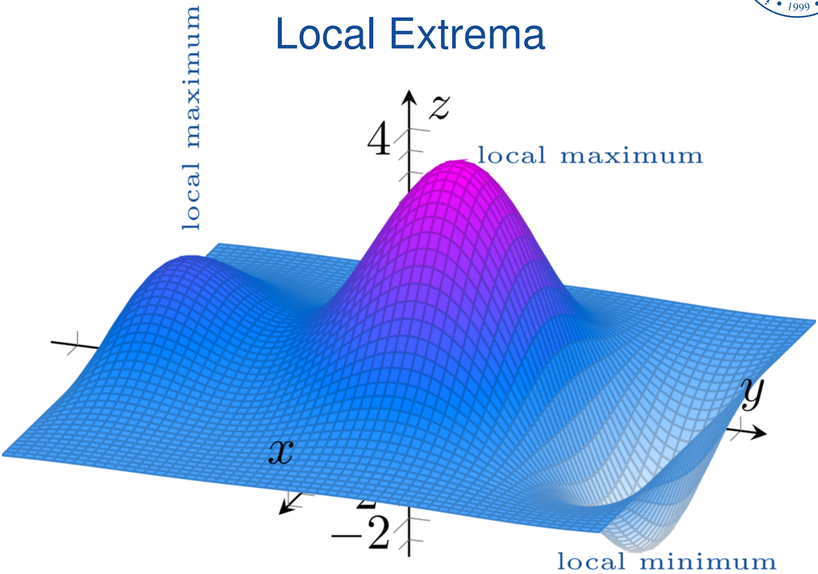
The linearization of f at (3, 2) is $L(x, y) = 4x - y - 2$

Break

We will continue at 2pm

13.7 Extreme Values and Saddle Points

Local Extrema



Definition

1. $f(a,b)$ is a local maximum value of $f(x,y)$ iff
$$f(a,b) \geq f(x,y)$$
for all (x,y) close to (a,b).

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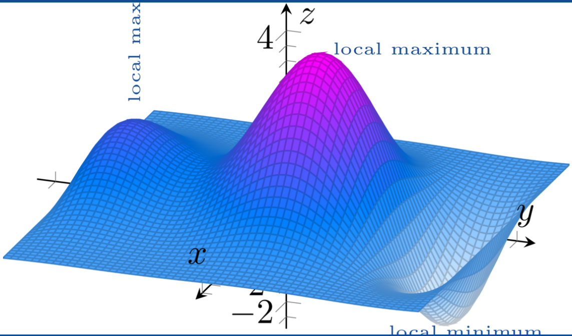


Theorem (First Derivative Test)

$$\begin{pmatrix} f(x,y) \textit{ has a local} \\ \textit{extrema at an interior} \\ \textit{point (a,b) of its} \\ \textit{domain} \end{pmatrix} \implies \begin{matrix} f_x(a,b) = 0 \\ \textit{and} \\ f_y(a,b) = 0 \end{matrix}$$

if $f_x(a,b)$ and $f_y(a,b)$ both exist.

Definition

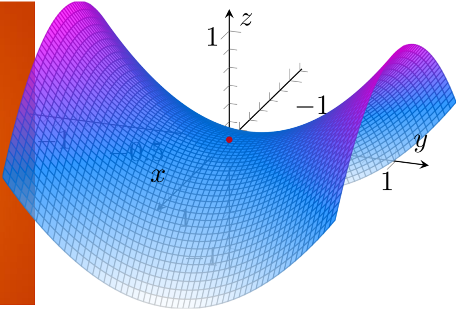
An interior point of the domain of $f(x, y)$ where either

1. $f_x = f_y = 0$;
2. f_x does not exist; or
3. f_y does not exist

is called a *critical point* of f.

Saddle Points

Saddle Points



The point $(0, 0)$ is a *saddle point* of $z = y^2 - x^2$.

Example

Find the local extrema of $f(x,y) = y^2 - 4y + x^2 + 9$.

domain:

partial derivatives:

$$\begin{aligned} 0 &= f_x = \\ 0 &= f_y = \end{aligned} \qquad \Longrightarrow \qquad (x,y) =$$

Example

Find the local extrema of $f(x, y) = y^2 - 4y + x^2 + 9$.

domain: $\mathbb{R}^2$

partial derivatives:

$$\begin{aligned} 0 &= f_x = 2x \\ 0 &= f_y = 2y - 4 \end{aligned} \qquad \Longrightarrow \qquad (x, y) = (0, 2)$$

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Find the local extrema of $f(x, y) = y^2 - 4y + x^2 + 9$.

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partial derivatives:

$$\begin{aligned} 0 &= f_x = 2x \\ 0 &= f_y = 2y - 4 \end{aligned} \qquad \Longrightarrow \qquad (x, y) = (0, 2)$$

Therefore the only possible place for an extrema is $(0, 2)$, where $f(0, 2) = 5$. Is this a local minimum or a local maximum?

Example

Find the local extrema of $f(x, y) = y^2 - 4y + x^2 + 9$.

domain: $\mathbb{R}^2$

partial derivatives:

$$\begin{aligned} 0 &= f_x = 2x \\ 0 &= f_y = 2y - 4 \end{aligned} \qquad \Longrightarrow \qquad (x, y) = (0, 2)$$

Therefore the only possible place for an extrema is $(0, 2)$, where $f(0, 2) = 5$. Is this a local minimum or a local maximum?

Since

$$(y - 2)^2 + x^2 + 5 \geq 5$$

for all (x, y), this must be a local minimum.

Example

Find the local extrema of $f(x,y) = y^2 - 4y - x^2 + 9$.

domain:

partial derivatives:

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Example

Find the local extrema of $f(x, y) = y^2 - 4y - x^2 + 9$.

domain: $\mathbb{R}^2$

partial derivatives:

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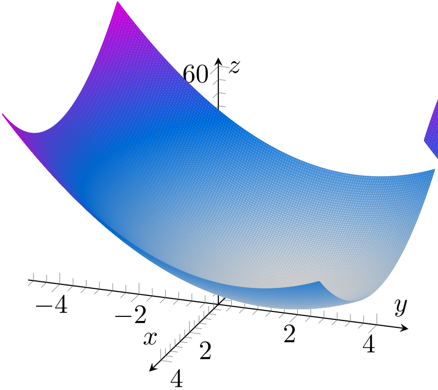
Therefore the only possible place for an extrema is $(0, 2)$, where $f(0, 2) = 5$. Is this a local minimum or a local maximum?

No. Fixing $x = 0$ we have $f(0, y) = (y - 2)^2 + 5$ which curves upwards. But fixing $y = 2$ we have $f(x, 2) = 5 - x^2$ which curves downwards.

So $(0, 2)$ must be a saddle point.

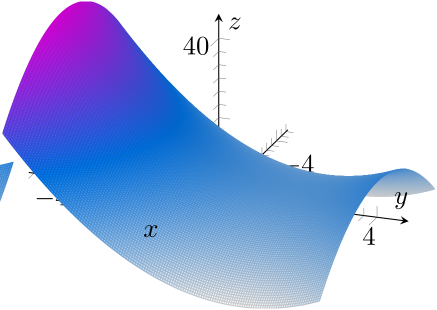
$$z = y^2 - 4y + x^2 + 9$$

has a local minimum at $(0, 2)$.



$$z = y^2 - 4y - x^2 + 9$$

has a saddle point at $(0, 2)$.



13.7 Extreme Values and Saddle Points

Theorem (Second Derivative Test)

Suppose that

- $f(x,y)$, f_x, f_y, f_{xx}, f_{yy} *and* f_{xy} *are all continuous on an open disk centred at* (a,b); *and*
- $f_x(a,b) = 0 = f_y(a,b)$.

If at (a,b) *we have*		*then*

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If at (a, b) *we have*		*then*
$f_{xx} < 0$	$f_{xx}f_{yy} - f_{xy}^2 > 0$	f *has a local maximum at* (a, b)

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If at (a,b) *we have*		*then*
$f_{xx} < 0$	$f_{xx}f_{yy} - f_{xy}^2 > 0$	f *has a local maximum at* (a,b)
$f_{xx} > 0$	$f_{xx}f_{yy} - f_{xy}^2 > 0$	f *has a local minimum at* (a,b)
	$f_{xx}f_{yy} - f_{xy}^2 < 0$	f *has a saddle point at* (a,b)

Theorem (Second Derivative Test)

Suppose that

- $f(x,y)$, f_x, f_y, f_{xx}, f_{yy} *and* f_{xy} *are all continuous on an open disk centred at* (a,b)*; and*
- $f_x(a,b) = 0 = f_y(a,b)$.

If at (a,b) *we have*		*then*
$f_{xx} < 0$	$f_{xx}f_{yy} - f_{xy}^2 > 0$	f *has a local maximum at* (a,b)
$f_{xx} > 0$	$f_{xx}f_{yy} - f_{xy}^2 > 0$	f *has a local minimum at* (a,b)
	$f_{xx}f_{yy} - f_{xy}^2 < 0$	f *has a saddle point at* (a,b)
	$f_{xx}f_{yy} - f_{xy}^2 = 0$	*we don't know*

Otto Hesse

BORN
22 April 1811

DECEICED
4 August 1874

NATIONALITY
German

Definition

$f_{xx}f_{yy} - f_{xy}^2$ is called the *Hessian* (or *discriminant*) of f.

EXAMPLE 3 Find the local extreme values of the function

$$f(x, y) = xy - x^2 - y^2 - 2x - 2y + 4.$$

Solution The function is defined and differentiable for all x and y, and its domain has no boundary points. The function therefore has extreme values only at the points where f_x and f_y are simultaneously zero. This leads to

$$f_x = y - 2x - 2 = 0, \qquad f_y = x - 2y - 2 = 0,$$

or

$$x = y = -2.$$

Therefore, the point $(-2, -2)$ is the only point where f may take on an extreme value. To see if it does so, we calculate

$$f_{xx} = -2, \qquad f_{yy} = -2, \qquad f_{xy} = 1.$$

The discriminant of f at $(a, b) = (-2, -2)$ is

$$f_{xx}f_{yy} - f_{xy}{}^2 = (-2)(-2) - (1)^2 = 4 - 1 = 3.$$

The combination

$$f_{xx} < 0 \qquad \text{and} \qquad f_{xx}f_{yy} - f_{xy}{}^2 > 0$$

tells us that f has a local maximum at $(-2, -2)$. The value of f at this point is $f(-2, -2) = 8$. ■

EXAMPLE 4 Find the local extreme values of $f(x, y) = 3y^2 - 2y^3 - 3x^2 + 6xy$.

Solution Since f is differentiable everywhere, it can assume extreme values only where

$$f_x = 6y - 6x = 0 \qquad \text{and} \qquad f_y = 6y - 6y^2 + 6x = 0.$$

From the first of these equations we find $x = y$, and substitution for y into the second equation then gives

$$6x - 6x^2 + 6x = 0 \qquad \text{or} \qquad 6x(2 - x) = 0.$$

The two critical points are therefore $(0, 0)$ and $(2, 2)$.

To classify the critical points, we calculate the second derivatives:

$$f_{xx} = -6, \qquad f_{yy} = 6 - 12y, \qquad f_{xy} = 6.$$

The discriminant is given by

$$f_{xx}f_{yy} - f_{xy}{}^2 = (-36 + 72y) - 36 = 72(y - 1).$$

At the critical point $(0, 0)$ we see that the value of the discriminant is the negative number -72, so the function has a saddle point at the origin. At the critical point $(2, 2)$ we see that the discriminant has the positive value 72. Combining this result with the negative value of the second partial $f_{xx} = -6$, Theorem 11 says that the critical point $(2, 2)$ gives a local maximum value of $f(2, 2) = 12 - 16 - 12 + 24 = 8$. A graph of the surface is shown in Figure 14.48. ■

EXAMPLE 5 Find the critical points of the function $f(x, y) = 10xye^{-(x^2+y^2)}$ and use the Second Derivative Test to classify each point as one where a saddle, local minimum, or local maximum occurs.

Solution First we find the partial derivatives f_x and f_y and set them simultaneously to zero in seeking the critical points:

$$f_x = 10ye^{-(x^2+y^2)} - 20x^2ye^{-(x^2+y^2)} = 10y(1 - 2x^2)e^{-(x^2+y^2)} = 0 \Rightarrow y = 0 \text{ or } 1 - 2x^2 = 0,$$
$$f_y = 10xe^{-(x^2+y^2)} - 20xy^2e^{-(x^2+y^2)} = 10x(1 - 2y^2)e^{-(x^2+y^2)} = 0 \Rightarrow x = 0 \text{ or } 1 - 2y^2 = 0.$$

Since both partial derivatives are continuous everywhere, the only critical points are

$$(0, 0), \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right), \left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right), \left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right), \text{ and } \left(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right).$$

Next we calculate the second partial derivatives in order to evaluate the discriminant at each critical point:

$$f_{xx} = -20xy(1 - 2x^2)e^{-(x^2+y^2)} - 40xye^{-(x^2+y^2)} = -20xy(3 - 2x^2)e^{-(x^2+y^2)},$$
$$f_{xy} = f_{yx} = 10(1 - 2x^2)e^{-(x^2+y^2)} - 20y^2(1 - 2x^2)e^{-(x^2+y^2)} = 10(1 - 2x^2)(1 - 2y^2)e^{-(x^2+y^2)},$$
$$f_{yy} = -20xy(1 - 2y^2)e^{-(x^2+y^2)} - 40xye^{-(x^2+y^2)} = -20xy(3 - 2y^2)e^{-(x^2+y^2)}.$$

The following table summarizes the values needed by the Second Derivative Test.

Critical Point	f_{xx}	f_{xy}	f_{yy}	Discriminant D
$(0, 0)$	0	10	0	-100
$\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$	$-\frac{20}{e}$	0	$-\frac{20}{e}$	$\frac{400}{e^2}$
$\left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$	$\frac{20}{e}$	0	$\frac{20}{e}$	$\frac{400}{e^2}$
$\left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)$	$\frac{20}{e}$	0	$\frac{20}{e}$	$\frac{400}{e^2}$
$\left(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)$	$-\frac{20}{e}$	0	$-\frac{20}{e}$	$\frac{400}{e^2}$

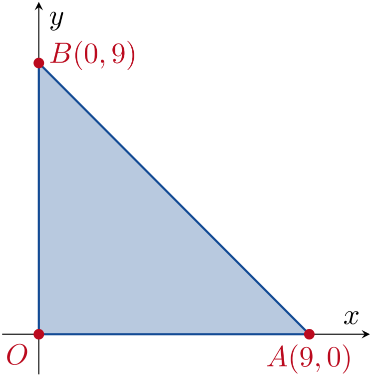
From the table we find that $D < 0$ at the critical point $(0, 0)$, giving a saddle; $D > 0$ and $f_{xx} < 0$ at the critical points $\left(1/\sqrt{2}, 1/\sqrt{2}\right)$ and $\left(-1/\sqrt{2}, -1/\sqrt{2}\right)$, giving local maximum values there; and $D > 0$ and $f_{xx} > 0$ at the critical points $\left(-1/\sqrt{2}, 1/\sqrt{2}\right)$ and $\left(1/\sqrt{2}, -1/\sqrt{2}\right)$, each giving local minimum values. A graph of the surface is shown in Figure 14.49. ■

Example

Find the absolute maximum and minimum values of

$$f(x, y) = 2 + 2x + 4y - x^2 - y^2$$

on the triangular region in the first quadrant bounded by the lines $x = 0$, $y = 0$ and $y = 9 - x$.



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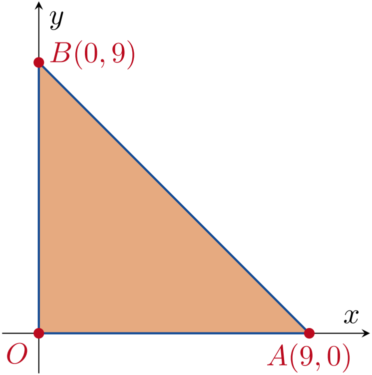

We will look

1. at the interior of the region;

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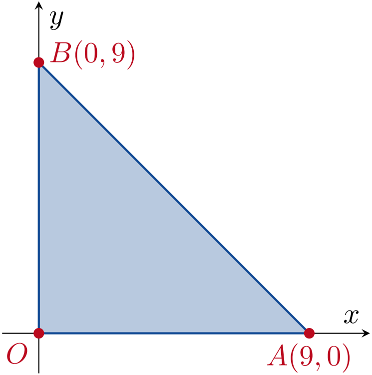

We will look

1. at the interior of the region;
2. at the corners O, A and B;

Example

Find the absolute maximum and minimum values of

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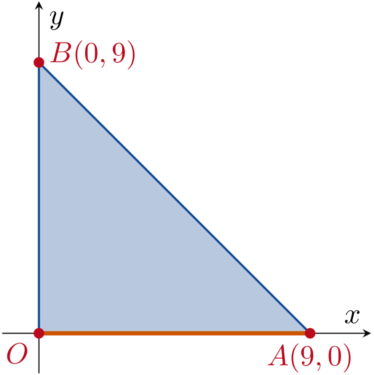


We will look

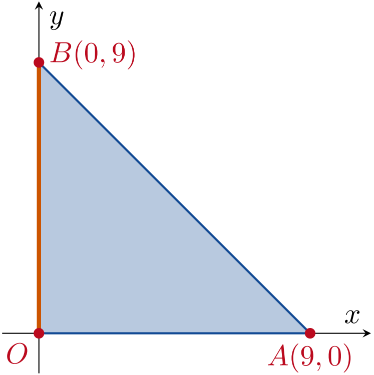
1. at the interior of the region;
2. at the corners O, A and B;
3. at the line OA;

Example

Find the absolute maximum and minimum values of

$$f(x, y) = 2 + 2x + 4y - x^2 - y^2$$

on the triangular region in the first quadrant bounded by the lines $x = 0$, $y = 0$ and $y = 9 - x$.



We will look

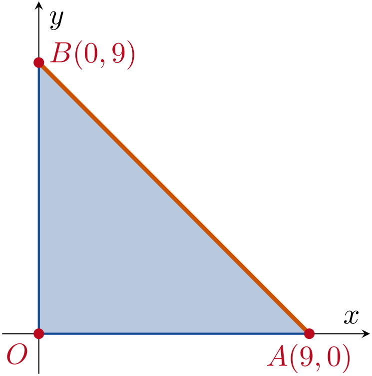
1. at the interior of the region;
2. at the corners O, A and B;
3. at the line OA;
4. at the line OB; and

Example

Find the absolute maximum and minimum values of

$$f(x, y) = 2 + 2x + 4y - x^2 - y^2$$

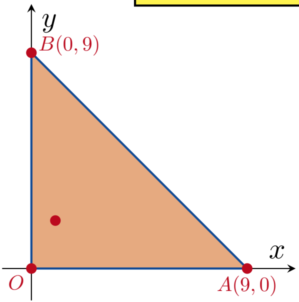
on the triangular region in the first quadrant bounded by the lines $x = 0$, $y = 0$ and $y = 9 - x$.



We will look

1. at the interior of the region;
2. at the corners O, A and B;
3. at the line OA;
4. at the line OB; and
5. at the line AB.

$$f(x, y) = 2 + 2x + 4y - x^2 - y^2$$



1 Consider the interior of the region.

We need to look for the critical points

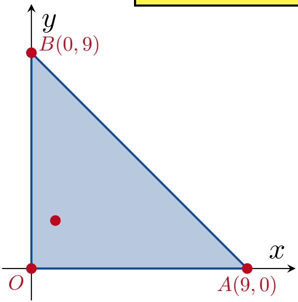
$$\begin{aligned} 0 &= f_x = 2 - 2x \\ 0 &= f_y = 4 - 2y \end{aligned} \quad \Longrightarrow \quad (x, y) = (1, 2).$$

Then we calculate that

$$f(1, 2) = 2 + 2 + 8 - 1 - 4 = 7.$$

$$f(x,y) = 2 + 2x + 4y - x^2 - y^2$$

$f(1,2) = 7$



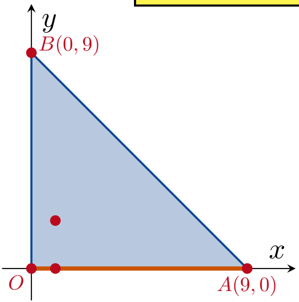
2 Consider the corners of the region.

We calculate that

$$f(0,0) = 2 + 0 + 0 - 0 - 0 = 2$$
$$f(9,0) = 2 + 18 + 0 - 81 - 0 = -61$$
$$f(0,9) = 2 + 0 + 36 - 0 - 81 = -43.$$

$$f(x, y) = 2 + 2x + 4y - x^2 - y^2$$

$f(1, 2) = 7$
$f(0, 0) = 2$
$f(9, 0) = -61$
$f(0, 9) = -43$



3 Consider the line OA.

If we set $y = 0$, then we have a new function

$$g(x) = f(x, 0) = 2 + 2x - x^2.$$

Since

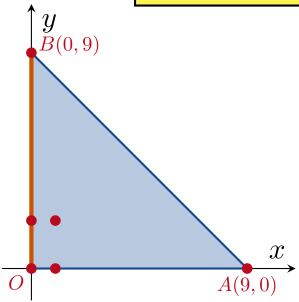
$$0 = g'(x) = 2 - 2x \qquad \Longrightarrow \qquad x = 1$$

we calculate

$$g(1) = f(1, 0) = 2 + 2 - 1 = 3.$$

$$f(x,y) = 2 + 2x + 4y - x^2 - y^2$$

$f(1,2) = 7$
$f(0,0) = 2$
$f(9,0) = -61$
$f(0,9) = -43$
$f(1,0) = 3$



4 Consider the line OB.

If we set $x = 0$, then we have a new function

$$h(y) = f(0,y) = 2 + 4y - y^2.$$

Since

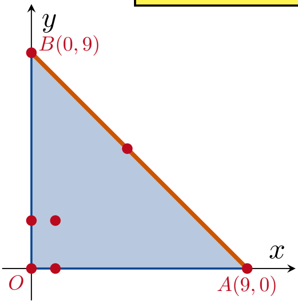
$$0 = h'(x) = 4 - 2y \qquad \Longrightarrow \qquad y = 2$$

we calculate

$$h(2) = f(0,2) = 2 + 8 - 4 = 6.$$

$$f(x,y) = 2 + 2x + 4y - x^2 - y^2$$

$f(1,2) = 7$
$f(0,0) = 2$
$f(9,0) = -61$
$f(0,9) = -43$
$f(1,0) = 3$
$f(0,2) = 6$



5 Finally consider the line AB.

If we set $y = 9 - x$, then we have a new function

$$k(x) = f(x, 9-x) = 2+2x+4(9-x)-x^2-(9-x)^2 = -43+16x-2x^2.$$

Since

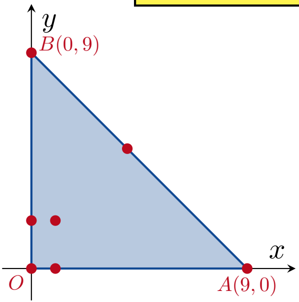
$$0 = k'(x) = 16 - 4x \qquad \Longrightarrow \qquad x = 4$$

we calculate

$$k(4) = f(4,5) = -43 + 64 - 32 = -11.$$

$$f(x,y) = 2 + 2x + 4y - x^2 - y^2$$

$f(1,2) = 7$
$f(0,0) = 2$
$f(9,0) = -61$
$f(0,9) = -43$
$f(1,0) = 3$
$f(0,2) = 6$
$f(4,5) = -11$



We have found the values

$$7,\ 2,\ -61,\ -43,\ 3,\ 6,\ -11.$$

The biggest of these numbers is 7 and the least is -61.

Therefore the absolute maximum value of f on this region is 7 and the absolute minimum value of f on this region is -61.

Please read Example 7 in the textbook.

13.8 Lagrange Multipliers

Example

Find the point $P(x, y, z)$ on the plane $2x + y - z - 5 = 0$ that is closest to the origin.

Example

Find the point $P(x, y, z)$ on the plane $2x + y - z - 5 = 0$ that is closest to the origin.

We need to find the minimum of

$$\left\| \overrightarrow{OP} \right\| = \sqrt{x^2 + y^2 + z^2}$$

subject to the constraint that

$$2x + y - z - 5 = 0.$$

Example

Find the point $P(x, y, z)$ on the plane $2x + y - z - 5 = 0$ that is closest to the origin.

We need to find the minimum of

$$\left\|\overrightarrow{OP}\right\| = \sqrt{x^2 + y^2 + z^2}$$

subject to the constraint that

$$2x + y - z - 5 = 0.$$

Let $f(x, y) = x^2 + y^2 + z^2$. We will study f instead of $\left\|\overrightarrow{OP}\right\|$.

We will let x and y be the independent variables and write

$$z = 2x + y - 5.$$

We will let x and y be the independent variables and write

$$z = 2x + y - 5.$$

So we want to find the minimum of

$$h(x, y) = f(x, y, 2x + y - 5) = x^2 + y^2 + (2x + y - 5)^2.$$

We will let x and y be the independent variables and write

$$z = 2x + y - 5.$$

So we want to find the minimum of

$$h(x, y) = f(x, y, 2x + y - 5) = x^2 + y^2 + (2x + y - 5)^2.$$

We will use the First Derivative Test:

$0 = h_x$

$0 = h_y$

We will let x and y be the independent variables and write

$$z = 2x + y - 5.$$

So we want to find the minimum of

$$h(x, y) = f(x, y, 2x + y - 5) = x^2 + y^2 + (2x + y - 5)^2.$$

We will use the First Derivative Test:

$0 = h_x = 2x + 2(2x + y - 5)(2)$

$0 = h_y = 2y + 2(2x + y - 5)$

We will let x and y be the independent variables and write

$$z = 2x + y - 5.$$

So we want to find the minimum of

$$h(x,y) = f(x,y,2x+y-5) = x^2 + y^2 + (2x+y-5)^2.$$

We will use the First Derivative Test:

$$\begin{aligned} 0 = h_x &= 2x + 2(2x+y-5)(2) \\ 0 = h_y &= 2y + 2(2x+y-5) \end{aligned} \implies \begin{aligned} 10x + 4y &= 20 \\ 4x + 4y &= 10 \end{aligned}$$

We will let x and y be the independent variables and write

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We will use the First Derivative Test:

$$\begin{aligned} 0 = h_x &= 2x + 2(2x + y - 5)(2) \\ 0 = h_y &= 2y + 2(2x + y - 5) \end{aligned} \implies \begin{aligned} 10x + 4y &= 20 \\ 4x + 4y &= 10 \end{aligned} \implies \begin{aligned} x &= \tfrac{5}{3} \\ y &= \tfrac{5}{6}. \end{aligned}$$

Then we have

$$z = 2x + y - 5 = \frac{10}{3} + \frac{5}{6} - 5 = -\frac{5}{6}.$$

We will let x and y be the independent variables and write

$$z = 2x + y - 5.$$

So we want to find the minimum of

$$h(x, y) = f(x, y, 2x + y - 5) = x^2 + y^2 + (2x + y - 5)^2.$$

We will use the First Derivative Test:

$$\begin{array}{l} 0 = h_x = 2x + 2(2x + y - 5)(2) \\ 0 = h_y = 2y + 2(2x + y - 5) \end{array} \implies \begin{array}{r} 10x + 4y = 20 \\ 4x + 4y = 10 \end{array} \implies \begin{array}{l} x = \frac{5}{3} \\ y = \frac{5}{6}. \end{array}$$

Then we have

$$z = 2x + y - 5 = \frac{10}{3} + \frac{5}{6} - 5 = -\frac{5}{6}.$$

The point on this plane which is closest to the origin is

$$P\left(\frac{5}{3}, \frac{5}{6}, -\frac{5}{6}\right).$$

The Method of Lagrange Multipliers

Suppose that we want to find the maximum/minimum of

$$f(x, y, z)$$

subject to the constraint that

$$g(x, y, z) = 0.$$

The Method of Lagrange Multipliers

Suppose that we want to find the maximum/minimum of

$$f(x,y,z)$$

subject to the constraint that

$$g(x,y,z)=0.$$

Theorem (The Method of Lagrange Multipliers)

We only need to find x, y, z and λ which satisfy

$$\nabla f=\lambda\nabla g \qquad \textit{and} \qquad g(x,y,z)=0.$$

The Method of Lagrange Multipliers

Suppose that we want to find the maximum/minimum of

$$f(x, y, \cancel{z})$$

subject to the constraint that

$$g(x, y, \cancel{z}) = 0.$$

Theorem (The Method of Lagrange Multipliers)

We only need to find x, y, $\cancel{z}$ and λ which satisfy

$$\nabla f = \lambda \nabla g \qquad \textit{and} \qquad g(x, y, \cancel{z}) = 0.$$

$$\nabla f = \lambda \nabla g \qquad g(x, y, z) = 0.$$

Example (repeat)

Find the point $P(x, y, z)$ on the plane $2x + y - z - 5 = 0$ that is closest to the origin.

$$\nabla f = \lambda \nabla g \qquad g(x, y, z) = 0.$$

Example (repeat)

Find the point $P(x, y, z)$ on the plane $2x + y - z - 5 = 0$ that is closest to the origin.

Let $f(x, y, z) = x^2 + y^2 + z^2$ and $g(x, y, z) = 2x + y - z - 5$.

$$\nabla f = \lambda \nabla g \qquad g(x,y,z) = 0.$$

Example (repeat)

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$$2x\mathbf{i} + 2y\mathbf{j} + 2z\mathbf{k} = \nabla f = \lambda \nabla g = 2\lambda\mathbf{i} + \lambda\mathbf{j} - \lambda\mathbf{k}$$

$$\nabla f = \lambda \nabla g \qquad g(x, y, z) = 0.$$

Example (repeat)

Find the point $P(x, y, z)$ on the plane $2x + y - z - 5 = 0$ that is closest to the origin.

Let $f(x, y, z) = x^2 + y^2 + z^2$ and $g(x, y, z) = 2x + y - z - 5$. Then

$$2x\mathbf{i} + 2y\mathbf{j} + 2z\mathbf{k} = \nabla f = \lambda \nabla g = 2\lambda\mathbf{i} + \lambda\mathbf{j} - \lambda\mathbf{k} \implies \begin{cases} x = \lambda \\ y = \frac{\lambda}{2} \\ z = -\frac{\lambda}{2}. \end{cases}$$

$$\nabla f = \lambda \nabla g \qquad g(x,y,z) = 0.$$

Example (repeat)

Find the point $P(x,y,z)$ on the plane $2x+y-z-5=0$ that is closest to the origin.

Let $f(x,y,z) = x^2+y^2+z^2$ and $g(x,y,z) = 2x+y-z-5$. Then

$$2x\mathbf{i} + 2y\mathbf{j} + 2z\mathbf{k} = \nabla f = \lambda \nabla g = 2\lambda\mathbf{i} + \lambda\mathbf{j} - \lambda\mathbf{k} \implies \begin{cases} x = \lambda \\ y = \frac{\lambda}{2} \\ z = -\frac{\lambda}{2}. \end{cases}$$

Hence

$$0 = 2x + y - z - 5 =$$

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Hence

$$0 = 2x + y - z - 5 = 2\lambda + \frac{\lambda}{2} - \left(-\frac{\lambda}{2}\right) - 5 = 3\lambda - 5$$

$$\nabla f = \lambda \nabla g \qquad g(x,y,z) = 0.$$

Example (repeat)

Find the point $P(x,y,z)$ on the plane $2x + y - z - 5 = 0$ that is closest to the origin.

Let $f(x,y,z) = x^2 + y^2 + z^2$ and $g(x,y,z) = 2x + y - z - 5$. Then

$$2x\mathbf{i} + 2y\mathbf{j} + 2z\mathbf{k} = \nabla f = \lambda \nabla g = 2\lambda\mathbf{i} + \lambda\mathbf{j} - \lambda\mathbf{k} \implies \begin{cases} x = \lambda \\ y = \frac{\lambda}{2} \\ z = -\frac{\lambda}{2}. \end{cases}$$

Hence

$$0 = 2x + y - z - 5 = 2\lambda + \frac{\lambda}{2} - \left(-\frac{\lambda}{2}\right) - 5 = 3\lambda - 5 \implies \lambda = \frac{5}{3}.$$

$$\nabla f = \lambda \nabla g \qquad g(x,y,z) = 0.$$

Example (repeat)

Find the point $P(x,y,z)$ on the plane $2x+y-z-5=0$ that is closest to the origin.

Let $f(x,y,z) = x^2+y^2+z^2$ and $g(x,y,z) = 2x+y-z-5$. Then

$$2x\mathbf{i} + 2y\mathbf{j} + 2z\mathbf{k} = \nabla f = \lambda \nabla g = 2\lambda\mathbf{i} + \lambda\mathbf{j} - \lambda\mathbf{k} \implies \begin{cases} x = \lambda \\ y = \frac{\lambda}{2} \\ z = -\frac{\lambda}{2}. \end{cases}$$

Hence

$$0 = 2x + y - z - 5 = 2\lambda + \frac{\lambda}{2} - \left(-\frac{\lambda}{2}\right) - 5 = 3\lambda - 5 \implies \lambda = \frac{5}{3}.$$

Therefore

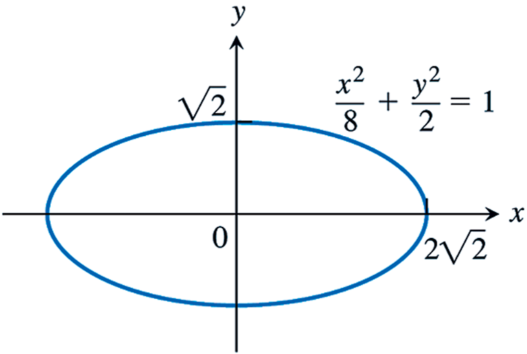
$$P(x,y,z) = \left(\lambda, \frac{\lambda}{2}, -\frac{\lambda}{2}\right) = \left(\frac{5}{3}, \frac{5}{6}, -\frac{5}{6}\right).$$

EXAMPLE 3 Find the greatest and smallest values that the function

$$f(x, y) = xy$$

takes on the ellipse (Figure 14.55)

$$\frac{x^2}{8} + \frac{y^2}{2} = 1.$$



Solution We want to find the extreme values of $f(x, y) = xy$ subject to the constraint

$$g(x, y) = \frac{x^2}{8} + \frac{y^2}{2} - 1 = 0.$$

To do so, we first find the values of x, y, and λ for which

$$\nabla f = \lambda \nabla g \qquad \text{and} \qquad g(x, y) = 0.$$

The gradient equation in Equations (1) gives

$$y\mathbf{i} + x\mathbf{j} = \frac{\lambda}{4} x\mathbf{i} + \lambda y\mathbf{j},$$

from which we find

$$y = \frac{\lambda}{4} x, \qquad x = \lambda y, \qquad \text{and} \qquad y = \frac{\lambda}{4}(\lambda y) = \frac{\lambda^2}{4} y,$$

so that $y = 0$ or $\lambda = \pm 2$. We now consider these two cases.

Case 1: If $y = 0$, then $x = y = 0$. But $(0, 0)$ is not on the ellipse. Hence, $y \neq 0$.

Case 2: If $y \neq 0$, then $\lambda = \pm 2$ and $x = \pm 2y$. Substituting this in the equation $g(x, y) = 0$ gives

$$\frac{(\pm 2y)^2}{8} + \frac{y^2}{2} = 1, \qquad 4y^2 + 4y^2 = 8 \qquad \text{and} \qquad y = \pm 1.$$

The function $f(x, y) = xy$ therefore takes on its extreme values on the ellipse at the four points $(\pm 2, 1)$, $(\pm 2, -1)$. The extreme values are $xy = 2$ and $xy = -2$.

EXAMPLE 4 Find the maximum and minimum values of the function $f(x, y) = 3x + 4y$ on the circle $x^2 + y^2 = 1$.

Solution We model this as a Lagrange multiplier problem with

$$f(x, y) = 3x + 4y, \qquad g(x, y) = x^2 + y^2 - 1$$

and look for the values of x, y, and λ that satisfy the equations

$$\nabla f = \lambda \nabla g: \quad 3\mathbf{i} + 4\mathbf{j} = 2x\lambda\mathbf{i} + 2y\lambda\mathbf{j}$$
$$g(x, y) = 0: \quad x^2 + y^2 - 1 = 0.$$

The gradient equation in Equations (1) implies that $\lambda \neq 0$ and gives

$$x = \frac{3}{2\lambda}, \qquad y = \frac{2}{\lambda}.$$

These equations tell us, among other things, that x and y have the same sign. With these values for x and y, the equation $g(x, y) = 0$ gives

$$\left(\frac{3}{2\lambda}\right)^2 + \left(\frac{2}{\lambda}\right)^2 - 1 = 0,$$

so

$$\frac{9}{4\lambda^2} + \frac{4}{\lambda^2} = 1, \qquad 9 + 16 = 4\lambda^2, \qquad 4\lambda^2 = 25, \qquad \text{and} \qquad \lambda = \pm\frac{5}{2}.$$

Thus,

$$x = \frac{3}{2\lambda} = \pm\frac{3}{5}, \qquad y = \frac{2}{\lambda} = \pm\frac{4}{5},$$

and $f(x, y) = 3x + 4y$ has extreme values at $(x, y) = \pm(3/5, 4/5)$.

By calculating the value of $3x + 4y$ at the points $\pm(3/5, 4/5)$, we see that its maximum and minimum values on the circle $x^2 + y^2 = 1$ are

$$3\left(\frac{3}{5}\right) + 4\left(\frac{4}{5}\right) = \frac{25}{5} = 5 \qquad \text{and} \qquad 3\left(-\frac{3}{5}\right) + 4\left(-\frac{4}{5}\right) = -\frac{25}{5} = -5.$$

Next Time

- 14.1 Double and Iterated Integrals over Rectangles
- 14.2 Double Integrals over General Regions