

Lecture 4

- 3.1 Homogeneous Equations with Constant Coefficients
- 3.2 Fundamental Sets of Solutions
- 3.3 Complex Roots of the Characteristic Equation

Second and Higher Order Linear ODEs

In this chapter we will consider equations of the form

$$y'' + p(t)y' + q(t)y = g(t)$$

or

$$P(t)y'' + Q(t)y' + R(t)y = G(t).$$

Such equations are *linear* second order ODEs.

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Such equations are *linear* second order ODEs.

If $g(t)$ (or $G(t)$) is always zero, then the ODE is called *homogeneous*. Otherwise it is *nonhomogeneous*.

Homogeneous Equations with Constant Coefficients

3.1 Homogeneous Equations with Constant Coefficients



First we will consider the equation

$$ay'' + by' + cy = 0 \quad (1)$$

where $a, b, c \in \mathbb{R}$ are constants.

3.1 Homogeneous Equations with Constant Coefficients



Example

Solve $y'' - y = 0$.

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- And what about $c_1e^t + c_2e^{-t}$?

3.1 Homogeneous Equations with Constant Coefficients



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- What about e^t ? Yes!
- What about e^{-t} ? Yes!
- And what about $c_1e^t + c_2e^{-t}$? Yes! In fact, this is the general solution to $y'' - y = 0$.

3.1 Homogeneous Equations with Constant Coefficients



Example

Solve

$$\begin{cases} y'' - y = 0 \\ y(0) = 2 \\ y'(0) = -1. \end{cases}$$

First note that this IVP has one 2nd order ODE and two initial conditions.

3.1 Homogeneous Equations with Constant Coefficients



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We know that $y(t) = c_1 e^t + c_2 e^{-t}$. We are looking for the solution which passes through the point $(0, 2)$ and has slope -1 at this point.

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First note that this IVP has one 2nd order ODE and two initial conditions.

We know that $y(t) = c_1 e^t + c_2 e^{-t}$. We are looking for the solution which passes through the point $(0, 2)$ and has slope -1 at this point. Using the first initial condition we get that

$$2 = y(0) = c_1 + c_2 \quad \implies \quad c_1 + c_2 = 2.$$

3.1 Homogeneous Equations with Constant Coefficients



Next we need to differentiate $y(t)$:

$$y'(t) = \frac{d}{dt} (c_1 e^t + c_2 e^{-t}) = c_1 e^t - c_2 e^{-t}.$$

3.1 Homogeneous Equations with Constant Coefficients



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$$-1 = y'(0) = c_1 - c_2 \quad \implies \quad c_1 - c_2 = -1.$$

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To satisfy these two conditions we must have $c_1 = \frac{1}{2}$ and $c_2 = \frac{3}{2}$.
Therefore the solution to the IVP is

$$y(t) = \frac{1}{2}e^t + \frac{3}{2}e^{-t}.$$

3.1 Homogeneous Equations with Constant Coefficients



Now let's go back to

$$ay'' + by' + cy = 0. \quad (1)$$

In the previous example, we used exponential functions in our solution. Maybe we always want exponential solutions?

3.1 Homogeneous Equations with Constant Coefficients



We guess that $y(t) = e^{rt}$ might be the solution to (1) for some number r that we don't know yet.

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$$y'' = r^2 e^{rt}$$

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$$0 = ay'' + by' + cy = (ar^2 + br + c)e^{rt}.$$

3.1 Homogeneous Equations with Constant Coefficients



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$$\begin{aligned}y &= e^{rt} \\y' &= re^{rt} \\y'' &= r^2 e^{rt}\end{aligned}$$

and

$$0 = ay'' + by' + cy = (ar^2 + br + c)e^{rt}.$$

Since $e^{rt} \neq 0$ for all t , we must have that

$$ar^2 + br + c = 0. \tag{2}$$

3.1 Homogeneous Equations with Constant Coefficients



$$ay'' + by' + cy = 0 \quad (1)$$

$$ar^2 + br + c = 0 \quad (2)$$

Definition

(2) is called the *characteristic equation* of (1).

3.1 Homogeneous Equations with Constant Coefficients



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Definition

(2) is called the *characteristic equation* of (1).

Theorem

$$e^{rt} \text{ solves (1)} \quad \Longleftrightarrow \quad r \text{ solves (2).}$$

3.1 Homogeneous Equations with Constant Coefficients



$ar^2 + br + c = 0$ has two roots, r_1 and r_2 :

$$r_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

These roots might be

- 1 real numbers and different ($r_1, r_2 \in \mathbb{R}$ and $r_1 \neq r_2$);
- 2 complex conjugates ($r_1, r_2 \in \mathbb{C} \setminus \mathbb{R}$, $\bar{r}_1 = r_2$); or
- 3 real numbers but repeated ($r_1, r_2 \in \mathbb{R}$ and $r_1 = r_2$).

We will study these three cases separately. First we study case 1.

3.1 Homogeneous Equations with Constant Coefficients



Suppose that $r_1, r_2 \in \mathbb{R}$ and $r_1 \neq r_2$. In other words, suppose that $b^2 - 4ac > 0$.

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Suppose that $r_1, r_2 \in \mathbb{R}$ and $r_1 \neq r_2$. In other words, suppose that $b^2 - 4ac > 0$. Then $y_1(t) = e^{r_1 t}$ and $y_2(t) = e^{r_2 t}$ are both solutions to (1). So

$$y(t) = c_1 y_1(t) + c_2 y_2(t) = c_1 e^{r_1 t} + c_2 e^{r_2 t}$$

will also be a solution for any constants $c_1, c_2 \in \mathbb{R}$. This is called the *general solution* to (1).

3.1 Homogeneous Equations with Constant Coefficients



Example

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$$0 = r^2 + 5r + 6$$

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The first thing that we must do is to write down the characteristic equation. The characteristic equation for this ODE is

$$0 = r^2 + 5r + 6 = (r + 2)(r + 3).$$

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The two roots are $r_1 = -2$ and $r_2 = -3$. Therefore the general solution is

$$y(t) = c_1 e^{-2t} + c_2 e^{-3t}.$$

3.1 Homogeneous Equations with Constant Coefficients



Example

Solve

$$\begin{cases} y'' + 5y' + 6y = 0 \\ y(0) = 2 \\ y'(0) = 3. \end{cases}$$

3.1 Homogeneous Equations with Constant Coefficients



We already found that $y(t) = c_1 e^{-2t} + c_2 e^{-3t}$ is the general solution to the ODE. We just need to find c_1 and c_2 .

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We already found that $y(t) = c_1 e^{-2t} + c_2 e^{-3t}$ is the general solution to the ODE. We just need to find c_1 and c_2 . Since $y'(t) = -2c_1 e^{-2t} - 3c_2 e^{-3t}$ we have that

$$2 = y(0) = c_1 + c_2 \quad \implies \quad c_1 = 2 - c_2$$

and

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and

$$\begin{aligned} 3 &= y'(0) = -2c_1 - 3c_2 = -2(2 - c_2) - 3c_2 = -4 - c_2 \\ \implies \quad c_2 &= -7 \\ \implies \quad c_1 &= 9. \end{aligned}$$

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Therefore the solution to the IVP is

$$\boxed{y(t) = 9e^{-2t} - 7e^{-3t}.}$$

3.1 Homogeneous Equations with Constant Coefficients



Example

Solve

$$\begin{cases} 4y'' - 8y' + 3y = 0 \\ y(0) = 2 \\ y'(0) = \frac{1}{2}. \end{cases}$$

3.1 Homogeneous Equations with Constant Coefficients



$$4y'' - 8y' + 3y = 0$$

Since the characteristic equation

$$4r^2 - 8r + 3 = 0$$

has roots,

$$r_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{8 \pm \sqrt{64 - 48}}{8} = 1 \pm \frac{1}{2} = \frac{3}{2} \text{ or } \frac{1}{2},$$

it follows that the general solution to the ODE is

$$y(t) = c_1 e^{\frac{3t}{2}} + c_2 e^{\frac{t}{2}}.$$

3.1 Homogeneous Equations with Constant Coefficients



$$y(t) = c_1 e^{\frac{3t}{2}} + c_2 e^{\frac{t}{2}}$$

Using the initial conditions, we calculate that

$$\begin{aligned} 2 = y(0) &= c_1 + c_2 \\ \frac{1}{2} = y'(0) &= \frac{3}{2}c_1 + \frac{1}{2}c_2 \end{aligned} \quad \Longrightarrow \quad c_1 = -\frac{1}{2} \text{ and } c_2 = \frac{5}{2}.$$

3.1 Homogeneous Equations with Constant Coefficients



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Therefore the solution to the IVP is

$$y = -\frac{1}{2}e^{\frac{3t}{2}} + \frac{5}{2}e^{\frac{t}{2}}.$$

3.1 Homogeneous Equations with Constant Coefficients



Summary

To solve

$$ay'' + by' + cy = 0$$

we need to find two linearly independent solutions.

- 1 If $r_1, r_2 \in \mathbb{R}$ and $r_1 \neq r_2$, then

$$y_1(t) = e^{r_1 t} \quad \text{and} \quad y_2(t) = e^{r_2 t};$$

- 2 If the roots are complex numbers, then ??????????????
- 3 If the roots are repeated, then ??????????????

Fundamental Sets of Solutions

3.2 Fundamental Sets of Solutions



$$y'' + p(t)y' + q(t)y = 0$$

Definition

Let $L = \frac{d^2}{dt^2} + p(t)\frac{d}{dt} + q(t)$.

So

$$L[y] = \frac{d^2 y}{dt^2} + p(t)\frac{dy}{dt} + q(t)y = y'' + p(t)y' + q(t)y$$

and we can write the ODE above as just $L[y] = 0$.

3.2 Fundamental Sets of Solutions



Theorem

If y_1 and y_2 are both solutions of $L[y] = 0$, then $c_1y_1 + c_2y_2$ is also a solution to $L[y] = 0$ for all constants c_1, c_2 .

3.2 Fundamental Sets of Solutions



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Proof.

Since $L[y_1] = 0$ and $L[y_2] = 0$, we have that

$$\begin{aligned} L[y] &= L[c_1y_1 + c_2y_2] \\ &= \frac{d^2}{dt^2} (c_1y_1 + c_2y_2) + p(t) \frac{d}{dt} (c_1y_1 + c_2y_2) + q(t) (c_1y_1 + c_2y_2) \\ &= c_1 (y_1'' + p(t)y_1' + q(t)y_1) + c_2 (y_2'' + p(t)y_2' + q(t)y_2) \\ &= c_1L[y_1] + c_2L[y_2] \\ &= 0 + 0 = 0. \end{aligned}$$



3.2 Fundamental Sets of Solutions



Józef Maria Hoëné-Wronski
POL, 1776-1853

Definition

The *Wronskian* of $y_1(t)$ and $y_2(t)$ is

$$W = W(y_1, y_2)(t) = \begin{vmatrix} y_1(t) & y_2(t) \\ y_1'(t) & y_2'(t) \end{vmatrix}$$

3.2 Fundamental Sets of Solutions



Theorem

Suppose that

- y_1 and y_2 both solve $L[y] = 0$; and
- $\exists t$ s.t. $W(t) \neq 0$.

Then $\{c_1 y_1 + c_2 y_2 : c_1, c_2 \in \mathbb{R}\}$ contains every solution of $L[y] = 0$.

3.2 Fundamental Sets of Solutions



Definition

Since $y(t) = c_1 y_1(t) + c_2 y_2(t)$ contains every solution to $L[y] = 0$, $y(t)$ is called the *general solution* to $L[y] = 0$.

3.2 Fundamental Sets of Solutions



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Since $y(t) = c_1y_1(t) + c_2y_2(t)$ contains every solution to $L[y] = 0$, $y(t)$ is called the *general solution* to $L[y] = 0$.

Definition

In this case, we say that y_1 and y_2 form a *fundamental set of solutions* to $L[y] = 0$.

3.2 Fundamental Sets of Solutions



Example

Show that $y_1(t) = t^{\frac{1}{2}}$ and $y_2(t) = t^{-1}$ form a fundamental set of solutions to

$$2t^2y'' + 3ty' - y = 0$$

for $t > 0$

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$$2t^2y'' + 3ty' - y = 0$$

for $t > 0$

We must show three things:

- 1 that $y_1 = t^{\frac{1}{2}}$ is a solution to the ODE;
- 2 that $y_2 = t^{-1}$ is also a solution to the ODE; and
- 3 that y_1 and y_2 are linearly independent ($W \neq 0$ somewhere).

3.2 Fundamental Sets of Solutions



Since

$$\begin{aligned}2t^2 y_1'' + 3t y_1' - y_1 &= 2t^2 \left(t^{\frac{1}{2}}\right)'' + 3t \left(t^{\frac{1}{2}}\right)' - t^{\frac{1}{2}} \\&= 2t^2 \left(-\frac{1}{4}t^{-\frac{3}{2}}\right) + 3t \left(\frac{1}{2}t^{-\frac{1}{2}}\right) - t^{\frac{1}{2}} \\&= -\frac{1}{2}t^{\frac{1}{2}} + \frac{3}{2}t^{\frac{1}{2}} - t^{\frac{1}{2}} = 0\end{aligned}$$

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and

$$\begin{aligned}2t^2 y_2'' + 3t y_2' - y_2 &= 2t^2 (t^{-1})'' + 3t (t^{-1})' - t^{-1} \\&= 2t^2 (2t^{-3}) + 3t (-t^{-2}) - t^{-1} \\&= 4t^{-1} - 3t^{-1} - t^{-1} = 0,\end{aligned}$$

y_1 and y_2 both solve the ODE.

3.2 Fundamental Sets of Solutions



Moreover since

$$W(y_1, y_2)(t) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} t^{\frac{1}{2}} & t^{-1} \\ \frac{1}{2}t^{-\frac{1}{2}} & -t^{-2} \end{vmatrix} = -t^{-\frac{3}{2}} - \frac{1}{2}t^{-\frac{3}{2}} = -\frac{3}{2}t^{-\frac{3}{2}} \neq 0$$

for all $t > 0$, we have that y_1 and y_2 are linearly independent.

3.2 Fundamental Sets of Solutions



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for all $t > 0$, we have that y_1 and y_2 are linearly independent.

Therefore $y_1 = t^{\frac{1}{2}}$ and $y_2 = t^{-1}$ form a fundamental set of solutions to this ODE.

Complex Roots of the Characteristic Equation

3.3 Complex Roots of the Characteristic Equation



Now consider

$$ay'' + by' + cy = 0 \quad (1)$$

where $b^2 - 4ac < 0$.

3.3 Complex Roots of the Characteristic Equation



Now consider

$$ay'' + by' + cy = 0 \quad (1)$$

where $b^2 - 4ac < 0$. The two roots of the characteristic equation are complex conjugates. We denote them by

$$r_1 = \lambda + i\mu \quad \text{and} \quad r_2 = \lambda - i\mu$$

where $\lambda, \mu \in \mathbb{R}$.

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$$r_1 = \lambda + i\mu \quad \text{and} \quad r_2 = \lambda - i\mu$$

where $\lambda, \mu \in \mathbb{R}$. The corresponding solutions are

$$y_1(t) = e^{r_1 t} = e^{(\lambda + i\mu)t} \quad \text{and} \quad y_2(t) = e^{r_2 t} = e^{(\lambda - i\mu)t}.$$

But what does e to the power of a complex number mean?

3.3 Complex Roots of the Characteristic Equation



Definition

$$e^{(\lambda+i\mu)t} = e^{\lambda t} \cos \mu t + ie^{\lambda t} \sin \mu t.$$

3.3 Complex Roots of the Characteristic Equation



Remark

Please note that

$$\begin{aligned}\frac{d}{dt} (e^{r_1 t}) &= \frac{d}{dt} \left(e^{\lambda t} \cos \mu t + i e^{\lambda t} \sin \mu t \right) \\ &= \\ &= \\ &= \\ &= \\ &= \end{aligned}$$

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3.3 Complex Roots of the Characteristic Equation



Remark

Please note that

$$\begin{aligned}\frac{d}{dt} (e^{r_1 t}) &= \frac{d}{dt} \left(e^{\lambda t} \cos \mu t + i e^{\lambda t} \sin \mu t \right) \\ &= \lambda e^{\lambda t} \cos \mu t - \mu e^{\lambda t} \sin \mu t + i \lambda e^{\lambda t} \sin \mu t + i \mu e^{\lambda t} \cos \mu t \\ &= (\lambda + i \mu) e^{\lambda t} \cos \mu t + (i \lambda - \mu) e^{\lambda t} \sin \mu t \\ &= (\lambda + i \mu) e^{\lambda t} \cos \mu t + i (\lambda + i \mu) e^{\lambda t} \sin \mu t \\ &= (\lambda + i \mu) \left(e^{\lambda t} \cos \mu t + i e^{\lambda t} \sin \mu t \right) \\ &= \end{aligned}$$

3.3 Complex Roots of the Characteristic Equation



Remark

Please note that

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3.3 Complex Roots of the Characteristic Equation



Real Valued Solutions

The solutions $y_1(t) = e^{(\lambda+i\mu)t}$ and $y_2(t) = e^{(\lambda-i\mu)t}$ are functions $y_1, y_2 : \mathbb{R} \rightarrow \mathbb{C}$. But we want solutions $\mathbb{R} \rightarrow \mathbb{R}$.

3.3 Complex Roots of the Characteristic Equation



Consider

$$u(t) = \frac{1}{2} (y_1(t) + y_2(t))$$

and

$$v(t) = \frac{1}{2i} (y_1(t) - y_2(t))$$

3.3 Complex Roots of the Characteristic Equation



Consider

$$\begin{aligned}u(t) &= \frac{1}{2} (y_1(t) + y_2(t)) \\&= \frac{1}{2} e^{\lambda t} (\cos \mu t + i \sin \mu t) + \frac{1}{2} e^{\lambda t} (\cos \mu t - i \sin \mu t) \\&= e^{\lambda t} \cos \mu t\end{aligned}$$

and

$$\begin{aligned}v(t) &= \frac{1}{2i} (y_1(t) - y_2(t)) \\&= \frac{1}{2i} e^{\lambda t} (\cos \mu t + i \sin \mu t) - \frac{1}{2i} e^{\lambda t} (\cos \mu t - i \sin \mu t) \\&= \frac{1}{2i} 2i e^{\lambda t} \sin \mu t = e^{\lambda t} \sin \mu t.\end{aligned}$$

3.3 Complex Roots of the Characteristic Equation



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Note that $u, v : \mathbb{R} \rightarrow \mathbb{R}$ both solve (1). But are they linearly independent?

3.3 Complex Roots of the Characteristic Equation



Since

$$\begin{aligned} W(u, v)(t) &= \begin{vmatrix} u & v \\ u' & v' \end{vmatrix} \\ &= \begin{vmatrix} e^{\lambda t} \cos \mu t & e^{\lambda t} \sin \mu t \\ \lambda e^{\lambda t} \cos \mu t - \mu e^{\lambda t} \sin \mu t & \lambda e^{\lambda t} \sin \mu t + \mu e^{\lambda t} \cos \mu t \end{vmatrix} \\ &= e^{2\lambda t} (\lambda \cos \mu t \sin \mu t + \mu \cos^2 \mu t - \lambda \cos \mu t \sin \mu t + \mu \sin^2 \mu t) \\ &= \mu e^{2\lambda t} \neq 0 \end{aligned}$$

(because $\mu \neq 0$), the answer is YES.

3.3 Complex Roots of the Characteristic Equation



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(because $\mu \neq 0$), the answer is YES. Therefore $u(t)$ and $v(t)$ form a fundamental set of solutions to (1).

3.3 Complex Roots of the Characteristic Equation



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(because $\mu \neq 0$), the answer is YES. Therefore $u(t)$ and $v(t)$ form a fundamental set of solutions to (1). The general solution to (1) is therefore

$$y(t) = c_1 u(t) + c_2 v(t) = c_1 e^{\lambda t} \cos \mu t + c_2 e^{\lambda t} \sin \mu t.$$

3.3 Complex Roots of the Characteristic Equation



Example

Solve $y'' + y' + y = 0$.

3.3 Complex Roots of the Characteristic Equation



Example

Solve $y'' + y' + y = 0$.

The characteristic equation

$$r^2 + r + 1 = 0$$

has roots

$$r = \frac{-1 \pm \sqrt{1-4}}{2} = \frac{-1 \pm \sqrt{(-1)(3)}}{2} = -\frac{1}{2} \pm i\frac{\sqrt{3}}{2}.$$

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3.3 Complex Roots of the Characteristic Equation



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So $\lambda = -\frac{1}{2}$ and $\mu = \frac{\sqrt{3}}{2}$.

Therefore the general solution is

$$y(t) = c_1 e^{-\frac{t}{2}} \cos \frac{\sqrt{3}}{2} t + c_2 e^{-\frac{t}{2}} \sin \frac{\sqrt{3}}{2} t.$$

3.3 Complex Roots of the Characteristic Equation



Example

Solve $y'' + 9y = 0$.

3.3 Complex Roots of the Characteristic Equation



Example

Solve $y'' + 9y = 0$.

Since $0 = r^2 + 9 = (r - 3i)(r + 3i)$ we have $r = \pm 3i$ (i.e. $\lambda = 0$ and $\mu = 3$). Therefore the general solution is

$$y(t) = c_1 \cos 3t + c_2 \sin 3t.$$

3.3 Complex Roots of the Characteristic Equation



Example

Solve

$$\begin{cases} 16y'' - 8y' + 145y = 0 \\ y(0) = -2 \\ y'(0) = 1. \end{cases}$$

3.3 Complex Roots of the Characteristic Equation



Example

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$$\begin{cases} 16y'' - 8y' + 145y = 0 \\ y(0) = -2 \\ y'(0) = 1. \end{cases}$$

The characteristic equation $16r^2 - 8r + 145 = 0$ has roots

$$\begin{aligned} r &= \frac{8 \pm \sqrt{64 - 4(16)(145)}}{32} = \frac{8 \pm \sqrt{(64)(1 - 145)}}{32} \\ &= \frac{8 \pm \sqrt{(-1)(64)(144)}}{32} = \frac{1}{4} \pm 3i. \end{aligned}$$

3.3 Complex Roots of the Characteristic Equation



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Therefore the general solution to the ODE is

$$y(t) = c_1 e^{\frac{t}{4}} \cos 3t + c_2 e^{\frac{t}{4}} \sin 3t.$$

3.3 Complex Roots of the Characteristic Equation



$$y(t) = c_1 e^{\frac{t}{4}} \cos 3t + c_2 e^{\frac{t}{4}} \sin 3t$$

Finally we calculate that

$$y'(t) = \frac{1}{4} c_1 e^{\frac{t}{4}} \cos 3t - 3c_1 e^{\frac{t}{4}} \sin 3t + \frac{1}{4} c_2 e^{\frac{t}{4}} \sin 3t + 3c_2 e^{\frac{t}{4}} \cos 3t$$

3.3 Complex Roots of the Characteristic Equation



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and

$$-2 = y(0) = c_1 + 0 \quad \implies \quad c_1 = -2$$

$$1 = y'(0) = \frac{1}{4}c_1 + 3c_2 = -\frac{1}{2} + 3c_2 \quad \implies \quad c_2 = \frac{1}{2}.$$

3.3 Complex Roots of the Characteristic Equation



$$y(t) = c_1 e^{\frac{t}{4}} \cos 3t + c_2 e^{\frac{t}{4}} \sin 3t$$

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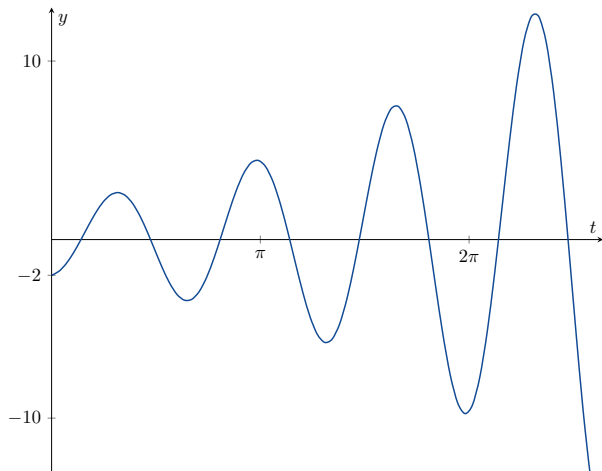
Therefore the solution to the IVP is

$$y = -2e^{\frac{t}{4}} \cos 3t + \frac{1}{2}e^{\frac{t}{4}} \sin 3t.$$

3.3 Complex Roots of the Characteristic Equation



$$\begin{cases} 16y'' - 8y' + 145y = 0 \\ y(0) = -2 \\ y'(0) = 1. \end{cases}$$



3.3 Complex Roots of the Characteristic Equation



Summary

To solve

$$ay'' + by' + cy = 0$$

we need to find two linearly independent solutions.

- 1** If $r_1, r_2 \in \mathbb{R}$ and $r_1 \neq r_2$, then

$$y_1(t) = e^{r_1 t} \quad \text{and} \quad y_2(t) = e^{r_2 t};$$

- 2** If $r_{1,2} = \lambda \pm i\mu$ ($\lambda, \mu \in \mathbb{R}$), then

$$y_1(t) = e^{\lambda t} \cos \mu t \quad \text{and} \quad y_2(t) = e^{\lambda t} \sin \mu t;$$

- 3** If the roots are repeated, then ?????????????

Next Time

- 3.4 Repeated Roots of the Characteristic Equation
- 3.5 Reduction of Order
- 3.6 Nonhomogeneous Equations
- 3.7 The Method of Undetermined Coefficients