

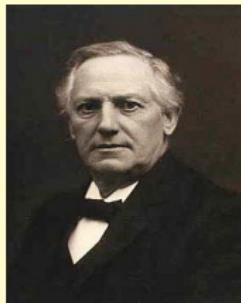
# Lecture 11

- The Gram-Schmidt Process
- Orthogonal Matrices
- Orthogonal Diagonalisation



# The Gram-Schmidt Process

# The Gram-Schmidt Process



Jørgen Pedersen Gram

BORN

27 June 1850

DECEASED

29 April 1916

NATIONALITY

Danish



Erhard Schmidt

BORN

13 January 1876

DECEASED

6 December 1959

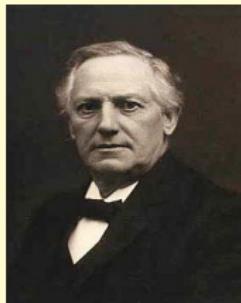
NATIONALITY

Baltic German

## Theorem

*Every nonzero finite-dimensional inner product space has an orthonormal basis.*

# The Gram-Schmidt Process



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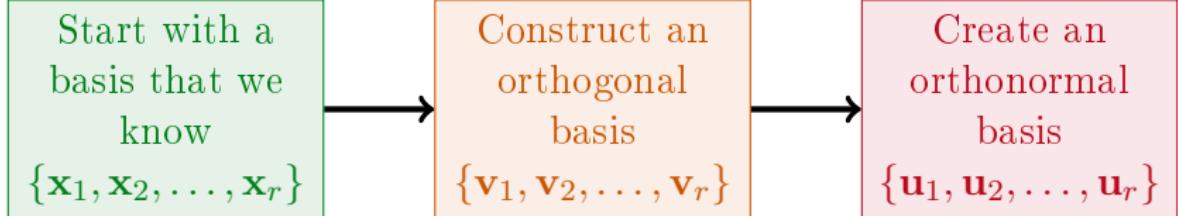
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## Theorem

*Every nonzero finite-dimensional inner product space has an orthonormal basis.*

But how can we find it?

# The Gram-Schmidt Process



# The Gram-Schmidt Process



Just define  $\mathbf{u}_i = \frac{\mathbf{v}_i}{\|\mathbf{v}_i\|}$  for each  $i$ .

Start with a basis that we know  
 $\{\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_r\}$

Construct an orthogonal basis  
 $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_r\}$

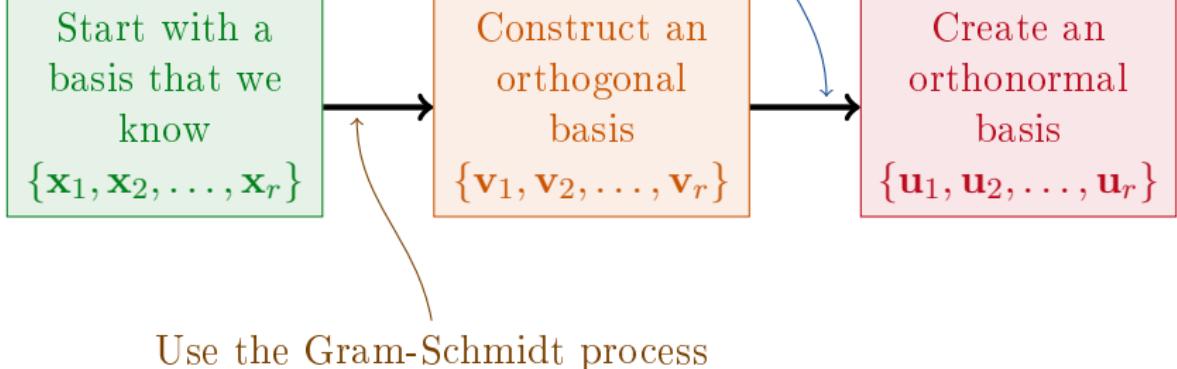
Create an orthonormal basis  
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# The Gram-Schmidt Process



Just define  $\mathbf{u}_i = \frac{\mathbf{v}_i}{\|\mathbf{v}_i\|}$  for each  $i$ .



## The Gram-Schmidt Process

To convert a basis  $\{\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_r\}$  into an orthogonal basis  $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_r\}$ , perform the following steps:

## The Gram-Schmidt Process

To convert a basis  $\{\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_r\}$  into an orthogonal basis  $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_r\}$ , perform the following steps:

1  $\mathbf{v}_1 = \mathbf{x}_1$

2  $\mathbf{v}_2 =$

3  $\mathbf{v}_3 =$

4  $\mathbf{v}_4 =$

## The Gram-Schmidt Process

To convert a basis  $\{\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_r\}$  into an orthogonal basis  $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_r\}$ , perform the following steps:

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2  $\mathbf{v}_2 = \mathbf{x}_2 - \frac{\langle \mathbf{x}_2, \mathbf{v}_1 \rangle}{\|\mathbf{v}_1\|^2} \mathbf{v}_1$

3  $\mathbf{v}_3 =$

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# The Gram-Schmidt Process

## The Gram-Schmidt Process

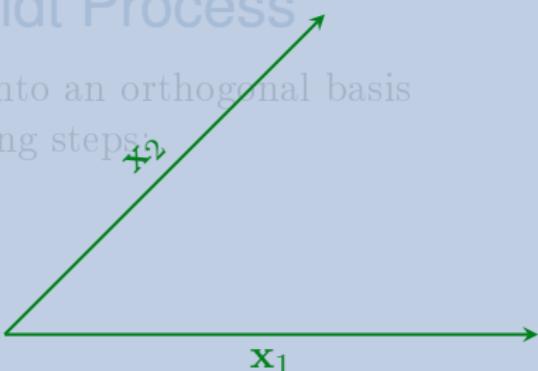
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# The Gram-Schmidt Process

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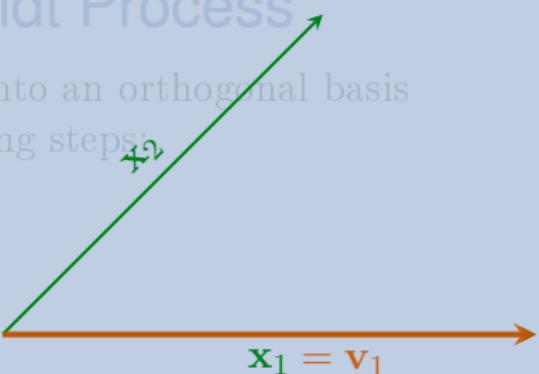
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## The Gram-Schmidt Process

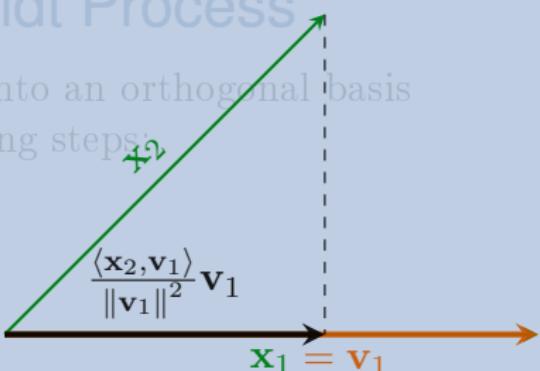
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# The Gram-Schmidt Process



## The Gram-Schmidt Process

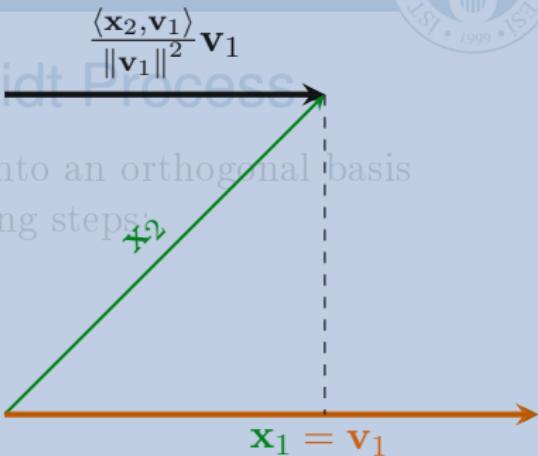
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3  $\mathbf{v}_3 =$

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# The Gram-Schmidt Process



## The Gram-Schmidt Process

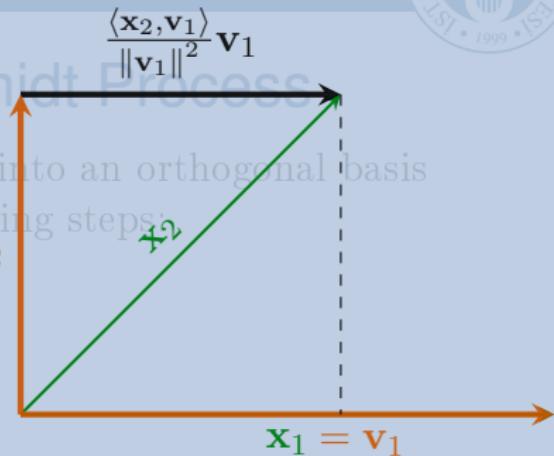
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2  $\mathbf{v}_2 = \mathbf{x}_2 - \frac{\langle \mathbf{x}_2, \mathbf{v}_1 \rangle}{\|\mathbf{v}_1\|^2} \mathbf{v}_1$

3  $\mathbf{v}_3 =$

4  $\mathbf{v}_4 =$



# The Gram-Schmidt Process



Are  $v_1$  and  $v_2$  really orthogonal?

To convert a basis  $\{x_1, x_2, \dots, x_r\}$  into an orthonormal basis  $\{v_1, v_2, \dots, v_r\}$ , follow the following steps:

1  $v_1 = x_1$

2  $v_2 = x_2 - \frac{\langle x_2, v_1 \rangle}{\|v_1\|^2} v_1$

3  $v_3 =$

4  $v_4 =$

$$\frac{\langle x_2, v_1 \rangle}{\|v_1\|^2} v_1$$

$$v_1$$

$$v_2$$

$$v_3$$

$$v_4$$

$$x_1 = v_1$$



# The Gram-Schmidt Process



Are  $\mathbf{v}_1$  and  $\mathbf{v}_2$  really orthogonal?

To convert a basis  $\{\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_r\}$ ,  $\langle \mathbf{v}_1, \mathbf{v}_2 \rangle = \left\langle \mathbf{v}_1, \mathbf{x}_2 - \frac{\langle \mathbf{x}_2, \mathbf{v}_1 \rangle}{\|\mathbf{v}_1\|^2} \mathbf{v}_1 \right\rangle$

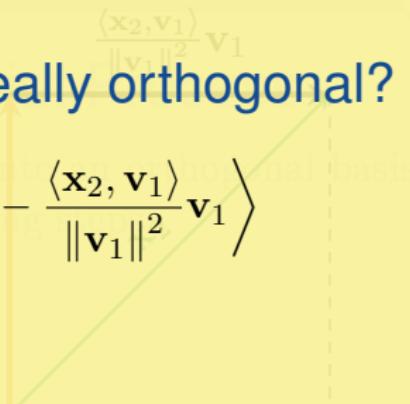
1  $\mathbf{v}_1 = \mathbf{x}_1$

2  $\mathbf{v}_2 = \mathbf{x}_2 - \frac{\langle \mathbf{x}_2, \mathbf{v}_1 \rangle}{\|\mathbf{v}_1\|^2} \mathbf{v}_1 =$

3  $\mathbf{v}_3 =$

4  $\mathbf{v}_4 =$

$$\frac{\langle \mathbf{x}_2, \mathbf{v}_1 \rangle}{\|\mathbf{v}_1\|^2} \mathbf{v}_1$$



# The Gram-Schmidt Process



Are  $\mathbf{v}_1$  and  $\mathbf{v}_2$  really orthogonal?

To convert a basis  $\{\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_r\}$ ,  $\langle \mathbf{v}_1, \mathbf{v}_2 \rangle = \left\langle \mathbf{v}_1, \mathbf{x}_2 - \frac{\langle \mathbf{x}_2, \mathbf{v}_1 \rangle}{\|\mathbf{v}_1\|^2} \mathbf{v}_1 \right\rangle$

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3  $\mathbf{v}_3 =$

=

$\mathbf{x}_1 = \mathbf{v}_1$

4  $\mathbf{v}_4 =$

=

=

# The Gram-Schmidt Process



Are  $\mathbf{v}_1$  and  $\mathbf{v}_2$  really orthogonal?

To convert a basis  $\{\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_r\}$ ,  $\langle \mathbf{v}_1, \mathbf{v}_2 \rangle = \left\langle \mathbf{v}_1, \mathbf{x}_2 - \frac{\langle \mathbf{x}_2, \mathbf{v}_1 \rangle}{\|\mathbf{v}_1\|^2} \mathbf{v}_1 \right\rangle$

1  $\mathbf{v}_1 = \mathbf{x}_1$

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3  $\mathbf{v}_3 =$

4  $\mathbf{v}_4 = \langle \mathbf{v}_1, \mathbf{x}_2 \rangle - \frac{\langle \mathbf{x}_2, \mathbf{v}_1 \rangle \langle \mathbf{v}_1, \mathbf{v}_1 \rangle}{\|\mathbf{v}_1\|^2}$

=

# The Gram-Schmidt Process



Are  $\mathbf{v}_1$  and  $\mathbf{v}_2$  really orthogonal?

To convert a basis  $\{\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_r\}$ ,  $\langle \mathbf{v}_1, \mathbf{v}_2 \rangle = \left\langle \mathbf{v}_1, \mathbf{x}_2 - \frac{\langle \mathbf{x}_2, \mathbf{v}_1 \rangle}{\|\mathbf{v}_1\|^2} \mathbf{v}_1 \right\rangle$

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3  $\mathbf{v}_3 =$

4  $\mathbf{v}_4 = \langle \mathbf{v}_1, \mathbf{x}_2 \rangle - \frac{\langle \mathbf{x}_2, \mathbf{v}_1 \rangle \langle \mathbf{v}_1, \mathbf{v}_1 \rangle}{\|\mathbf{v}_1\|^2}$

$= \langle \mathbf{v}_1, \mathbf{x}_2 \rangle - \langle \mathbf{x}_2, \mathbf{v}_1 \rangle = 0.$

YES!

## The Gram-Schmidt Process

To convert a basis  $\{\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_r\}$  into an orthogonal basis  $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_r\}$ , perform the following steps:

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## The Gram-Schmidt Process

To convert a basis  $\{\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_r\}$  into an orthogonal basis  $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_r\}$ , perform the following steps:

$$1 \quad \mathbf{v}_1 = \mathbf{x}_1$$

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$$4 \quad \mathbf{v}_4 =$$

## The Gram-Schmidt Process

Please check that

To convert a basis  $\{\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_r\}$  into an orthogonal basis  $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_r\}$ , perform the following steps:

and

$$1 \quad \mathbf{v}_1 = \mathbf{x}_1$$

$$2 \quad \mathbf{v}_2 = \mathbf{x}_2 - \frac{\langle \mathbf{x}_2, \mathbf{v}_1 \rangle}{\|\mathbf{v}_1\|^2} \mathbf{v}_1$$

$$\langle \mathbf{v}_2, \mathbf{v}_3 \rangle = 0.$$

$$3 \quad \mathbf{v}_3 = \mathbf{x}_3 - \frac{\langle \mathbf{x}_3, \mathbf{v}_1 \rangle}{\|\mathbf{v}_1\|^2} \mathbf{v}_1 - \frac{\langle \mathbf{x}_3, \mathbf{v}_2 \rangle}{\|\mathbf{v}_2\|^2} \mathbf{v}_2$$

$$4 \quad \mathbf{v}_4 =$$

## The Gram-Schmidt Process

To convert a basis  $\{\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_r\}$  into an orthogonal basis  $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_r\}$ , perform the following steps:

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$$4 \quad \mathbf{v}_4 = \mathbf{x}_4 - \frac{\langle \mathbf{x}_4, \mathbf{v}_1 \rangle}{\|\mathbf{v}_1\|^2} \mathbf{v}_1 - \frac{\langle \mathbf{x}_4, \mathbf{v}_2 \rangle}{\|\mathbf{v}_2\|^2} \mathbf{v}_2 - \frac{\langle \mathbf{x}_4, \mathbf{v}_3 \rangle}{\|\mathbf{v}_3\|^2} \mathbf{v}_3$$

## The Gram-Schmidt Process

To convert a basis  $\{\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_r\}$  into an orthogonal basis  $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_r\}$ , perform the following steps:

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$$4 \quad \mathbf{v}_4 = \mathbf{x}_4 - \frac{\langle \mathbf{x}_4, \mathbf{v}_1 \rangle}{\|\mathbf{v}_1\|^2} \mathbf{v}_1 - \frac{\langle \mathbf{x}_4, \mathbf{v}_2 \rangle}{\|\mathbf{v}_2\|^2} \mathbf{v}_2 - \frac{\langle \mathbf{x}_4, \mathbf{v}_3 \rangle}{\|\mathbf{v}_3\|^2} \mathbf{v}_3$$

$$\vdots$$

(continue until you have  $\mathbf{v}_r$ )

# The Gram-Schmidt Process



## Example (Using the Gram-Schmidt Process)

Assume that the vector space  $\mathbb{R}^3$  has the Euclidean inner product. Apply the Gram–Schmidt process to transform the basis vectors

$$\mathbf{x}_1 = (1, 1, 1), \quad \mathbf{x}_2 = (0, 1, 1), \quad \mathbf{x}_3 = (0, 0, 1)$$

into an orthogonal basis  $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ , and then normalise the orthogonal basis vectors to obtain an orthonormal basis  $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ .

# The Gram-S

$$\mathbf{x}_1 = (1, 1, 1), \quad \mathbf{x}_2 = (0, 1, 1), \quad \mathbf{x}_3 = (0, 0, 1)$$



**Step 1:**  $\mathbf{v}_1 = \mathbf{x}_1 = (1, 1, 1)$

## The Gram-S

$$\mathbf{x}_1 = (1, 1, 1), \quad \mathbf{x}_2 = (0, 1, 1), \quad \mathbf{x}_3 = (0, 0, 1)$$

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## The Gram-Schmidt Process

$$\mathbf{x}_1 = (1, 1, 1), \quad \mathbf{x}_2 = (0, 1, 1), \quad \mathbf{x}_3 = (0, 0, 1)$$

**Step 1:**  $\mathbf{v}_1 = \mathbf{x}_1 = (1, 1, 1)$

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$$\begin{aligned} \langle \mathbf{v}_1, \mathbf{v}_2 \rangle &= (1, 1, 1) \cdot \left(-\frac{2}{3}, \frac{1}{3}, \frac{1}{3}\right) \\ &= (1) \left(-\frac{2}{3}\right) + (1) \left(\frac{1}{3}\right) + (1) \left(\frac{1}{3}\right) = 0 \quad \checkmark \end{aligned}$$

The Gram-Schmidt process

$$\mathbf{x}_1 = (1, 1, 1), \quad \mathbf{x}_2 = (0, 1, 1), \quad \mathbf{x}_3 = (0, 0, 1)$$

**Step 1:**  $\mathbf{v}_1 = \mathbf{x}_1 = (1, 1, 1)$

**Step 2:**  $\mathbf{v}_2 = \mathbf{x}_2 - \frac{\langle \mathbf{x}_2, \mathbf{v}_1 \rangle}{\|\mathbf{v}_1\|^2} \mathbf{v}_1 = (0, 1, 1) - \frac{2}{3}(1, 1, 1) = \left(-\frac{2}{3}, \frac{1}{3}, \frac{1}{3}\right)$

$$\begin{aligned} \langle \mathbf{v}_1, \mathbf{v}_2 \rangle &= (1, 1, 1) \cdot \left(-\frac{2}{3}, \frac{1}{3}, \frac{1}{3}\right) \\ &= (1) \left(-\frac{2}{3}\right) + (1) \left(\frac{1}{3}\right) + (1) \left(\frac{1}{3}\right) = 0 \quad \checkmark \end{aligned}$$

**Step 3:**  $\mathbf{v}_3 = \mathbf{x}_3 - \frac{\langle \mathbf{x}_3, \mathbf{v}_1 \rangle}{\|\mathbf{v}_1\|^2} \mathbf{v}_1 - \frac{\langle \mathbf{x}_3, \mathbf{v}_2 \rangle}{\|\mathbf{v}_2\|^2} \mathbf{v}_2$

$$= (0, 0, 1) - \frac{1}{3}(1, 1, 1) - \frac{\frac{1}{3}}{\frac{2}{3}} \left(-\frac{2}{3}, \frac{1}{3}, \frac{1}{3}\right) = \left(0, -\frac{1}{2}, \frac{1}{2}\right).$$

$$\mathbf{v}_1 = (1, 1, 1), \quad \mathbf{v}_2 = \left(-\frac{2}{3}, \frac{1}{3}, \frac{1}{3}\right), \quad \mathbf{v}_3 = \left(0, -\frac{1}{2}, \frac{1}{2}\right)$$



Let's just finish checking if these are correct:

$$\mathbf{v}_1 = (1, 1, 1), \quad \mathbf{v}_2 = \left(-\frac{2}{3}, \frac{1}{3}, \frac{1}{3}\right), \quad \mathbf{v}_3 = \left(0, -\frac{1}{2}, \frac{1}{2}\right)$$



Let's just finish checking if these are correct:

$$\langle \mathbf{v}_1, \mathbf{v}_3 \rangle = (1, 1, 1) \cdot \left(0, -\frac{1}{2}, \frac{1}{2}\right) = 0 - \frac{1}{2} + \frac{1}{2} = 0 \quad \checkmark$$

$$\langle \mathbf{v}_2, \mathbf{v}_3 \rangle = \left(-\frac{2}{3}, \frac{1}{3}, \frac{1}{3}\right) \cdot \left(0, -\frac{1}{2}, \frac{1}{2}\right) = 0 - \frac{1}{6} + \frac{1}{6} = 0 \quad \checkmark$$

$$\mathbf{v}_1 = (1, 1, 1), \quad \mathbf{v}_2 = \left(-\frac{2}{3}, \frac{1}{3}, \frac{1}{3}\right), \quad \mathbf{v}_3 = \left(0, -\frac{1}{2}, \frac{1}{2}\right)$$



Let's just finish checking if these are correct:

$$\langle \mathbf{v}_1, \mathbf{v}_3 \rangle = (1, 1, 1) \cdot \left(0, -\frac{1}{2}, \frac{1}{2}\right) = 0 - \frac{1}{2} + \frac{1}{2} = 0 \quad \checkmark$$

$$\langle \mathbf{v}_2, \mathbf{v}_3 \rangle = \left(-\frac{2}{3}, \frac{1}{3}, \frac{1}{3}\right) \cdot \left(0, -\frac{1}{2}, \frac{1}{2}\right) = 0 - \frac{1}{6} + \frac{1}{6} = 0 \quad \checkmark$$

Therefore  $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$  is an orthogonal basis for  $\mathbb{R}^3$ .

The Gram-Schmidt Process

$$\mathbf{v}_1 = (1, 1, 1), \quad \mathbf{v}_2 = \left(-\frac{2}{3}, \frac{1}{3}, \frac{1}{3}\right), \quad \mathbf{v}_3 = \left(0, -\frac{1}{2}, \frac{1}{2}\right)$$


The next step is to normalise these three vectors.

$$\mathbf{v}_1 = (1, 1, 1), \quad \mathbf{v}_2 = \left(-\frac{2}{3}, \frac{1}{3}, \frac{1}{3}\right), \quad \mathbf{v}_3 = \left(0, -\frac{1}{2}, \frac{1}{2}\right)$$



The next step is to normalise these three vectors. Since

$$\|\mathbf{v}_1\| = \sqrt{3}, \quad \|\mathbf{v}_2\| = \frac{\sqrt{6}}{3}, \quad \|\mathbf{v}_3\| = \frac{1}{\sqrt{2}},$$

(please check) it follows that

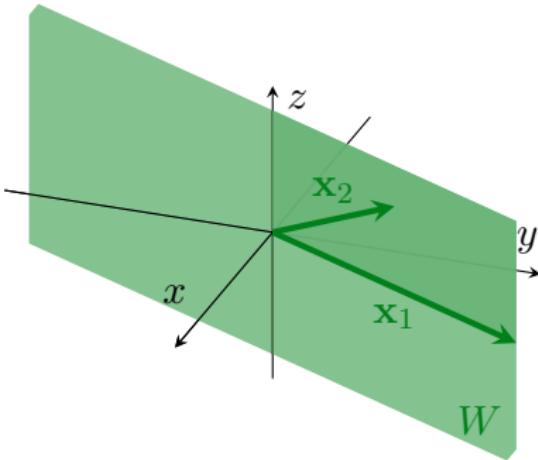
$$\mathbf{u}_1 = \frac{\mathbf{v}_1}{\|\mathbf{v}_1\|} = \left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$$

$$\mathbf{u}_2 = \frac{\mathbf{v}_2}{\|\mathbf{v}_2\|} = \left(-\frac{2}{\sqrt{6}}, \frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}\right)$$

$$\mathbf{u}_3 = \frac{\mathbf{v}_3}{\|\mathbf{v}_3\|} = \left(0, -\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right).$$

Then  $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$  is an orthonormal basis for  $\mathbb{R}^3$ .

# The Gram-Schmidt Process

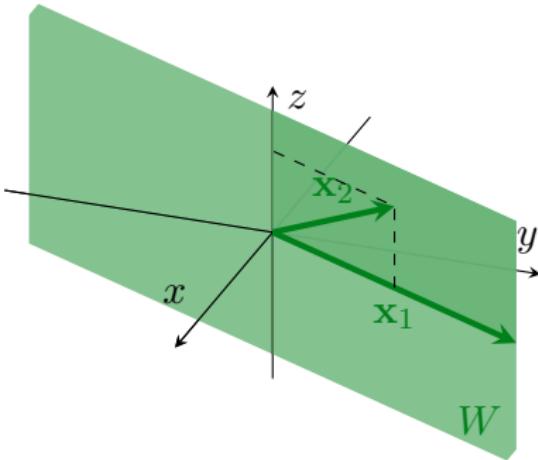


## Example

Let  $V = \mathbb{R}^3$  with the Euclidean inner product and let

$W = \text{span}\{\mathbf{x}_1, \mathbf{x}_2\}$  where  $\mathbf{x}_1 = \begin{bmatrix} 3 \\ 6 \\ 0 \end{bmatrix}$  and  $\mathbf{x}_2 = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$ . Construct an orthogonal basis for  $W$ .

# The Gram-Schmidt Process

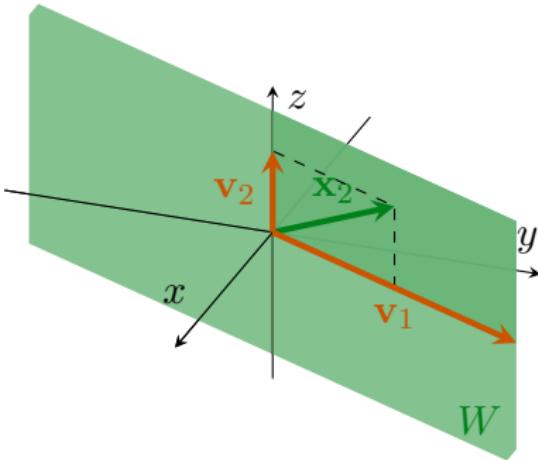


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$$\mathbf{v}_1 = \mathbf{x}_1 = \begin{bmatrix} 3 \\ 6 \\ 0 \end{bmatrix}$$

$$\mathbf{v}_2 =$$

# The Gram-Schmidt Process

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$$\mathbf{v}_1 = \mathbf{x}_1 = \begin{bmatrix} 3 \\ 6 \\ 0 \end{bmatrix}$$

$$\mathbf{v}_2 = \mathbf{x}_2 - \frac{\langle \mathbf{x}_2, \mathbf{v}_1 \rangle}{\|\mathbf{v}_1\|^2} \mathbf{v}_1 = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} - \frac{15}{45} \begin{bmatrix} 3 \\ 6 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix}.$$

# The Gram-Schmidt Process

## Example

Let  $V = \mathbb{R}^3$  with the Euclidean inner product and let

$W = \text{span}\{\mathbf{x}_1, \mathbf{x}_2\}$  where  $\mathbf{x}_1 = \begin{bmatrix} 3 \\ 6 \\ 0 \end{bmatrix}$  and  $\mathbf{x}_2 = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$ . Construct an orthogonal basis for  $W$ .

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$\{\mathbf{v}_1, \mathbf{v}_2\}$  is an orthogonal basis for  $W$ .

# The Gram-Schmidt Process

## Example

Let  $\mathbf{x}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$ ,  $\mathbf{x}_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix}$  and  $\mathbf{x}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}$  be three vectors in  $\mathbb{R}^4$

with the Euclidean inner product. It is easy to see that  $\{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3\}$  is linearly independent (Why?).

# The Gram-Schmidt Process



## Example

Let  $\mathbf{x}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$ ,  $\mathbf{x}_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix}$  and  $\mathbf{x}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}$  be three vectors in  $\mathbb{R}^4$

with the Euclidean inner product. It is easy to see that  $\{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3\}$  is linearly independent (Why?).

Construct an orthogonal basis for  $W = \text{span}\{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3\}$ .

## The Gram-Schmidt Process

$$\mathbf{x}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \quad \mathbf{x}_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \quad \mathbf{x}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}$$



$$\mathbf{v}_1 = \mathbf{x}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \quad \mathbf{v}_2 = \mathbf{x}_2 - \frac{\langle \mathbf{x}_2, \mathbf{v}_1 \rangle}{\|\mathbf{v}_1\|^2} \mathbf{v}_1 = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix} - \frac{3}{4} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -\frac{3}{4} \\ \frac{1}{4} \\ \frac{1}{4} \\ \frac{1}{4} \end{bmatrix}.$$

## The Gram-Schmidt Process

$$\mathbf{x}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \quad \mathbf{x}_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \quad \mathbf{x}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}$$



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At this point we can do an optional step to simplify our calculations. Remember that we are trying to find an orthogonal basis. We don't care how big (long) our vectors are, we only care that they are orthogonal.

## The Gram-Schmidt Process

$$\mathbf{x}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \quad \mathbf{x}_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \quad \mathbf{x}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}$$



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At this point we can do an optional step to simplify our calculations. Remember that we are trying to find an orthogonal basis. We don't care how big (long) our vectors are, we only care that they are orthogonal. So we can multiply  $\mathbf{v}_2$  by 4 to obtain a new  $\mathbf{v}_2$ :

$$\mathbf{v}_2 = \begin{bmatrix} -3 \\ 1 \\ 1 \\ 1 \end{bmatrix}.$$

The

$$\mathbf{x}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \mathbf{x}_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \mathbf{x}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}$$

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} -3 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$



Then

$$\mathbf{v}_3 = \mathbf{x}_3 - \frac{\langle \mathbf{x}_3, \mathbf{v}_1 \rangle}{\|\mathbf{v}_1\|^2} \mathbf{v}_1 - \frac{\langle \mathbf{x}_3, \mathbf{v}_2 \rangle}{\|\mathbf{v}_2\|^2} \mathbf{v}_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix} - \frac{2}{4} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} - \frac{2}{12} \begin{bmatrix} -3 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ -\frac{2}{3} \\ \frac{1}{3} \\ \frac{1}{3} \end{bmatrix}.$$

The

$$\mathbf{x}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \mathbf{x}_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \mathbf{x}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}$$

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} -3 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$



Then

$$\mathbf{v}_3 = \mathbf{x}_3 - \frac{\langle \mathbf{x}_3, \mathbf{v}_1 \rangle}{\|\mathbf{v}_1\|^2} \mathbf{v}_1 - \frac{\langle \mathbf{x}_3, \mathbf{v}_2 \rangle}{\|\mathbf{v}_2\|^2} \mathbf{v}_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix} - \frac{2}{4} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} - \frac{2}{12} \begin{bmatrix} -3 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ -\frac{2}{3} \\ \frac{1}{3} \\ \frac{1}{3} \end{bmatrix}.$$

Again we can multiply  $\mathbf{v}_3$  by a scalar if we want to. Let's multiply it by 3 to get a new vector which we will now call  $\mathbf{v}_3$ :

$$\mathbf{v}_3 = \begin{bmatrix} 0 \\ -2 \\ 1 \\ 1 \end{bmatrix}.$$

# The Gram-Schmidt Process



Hence

$$\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -3 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ -2 \\ 1 \\ 1 \end{bmatrix} \right\}$$

is an orthogonal basis for  $W$ .

# The Gram-Schmidt Process



Adrien-Marie Legendre

BORN

18 September 1752

DECEASED

9 January 1833

NATIONALITY

French

## Example (Legendre Polynomials)

Consider the vector space  $\mathbb{P}^2$  with the inner product

$$\langle \mathbf{p}, \mathbf{q} \rangle = \int_{-1}^1 p(x)q(x) dx.$$

Apply the Gram-Schmidt process to transform the standard basis  $\{1, x, x^2\}$  into an orthogonal basis.

Let  $\mathbf{x}_1 = 1$ ,  $\mathbf{x}_2 = x$  and  $\mathbf{x}_3 = x^2$ .

# The Gram-Schmidt Process

$$\mathbf{x}_1 = 1, \mathbf{x}_2 = x, \mathbf{x}_3 = x^2$$



Step 1:  $\mathbf{v}_1 = \mathbf{x}_1 = 1$

# The Gram-Schmidt Process

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# The Gram-Schmidt Process

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# The Gram-Schmidt Process

$$\mathbf{x}_1 = 1, \mathbf{x}_2 = x, \mathbf{x}_3 = x^2$$



**Step 1:**  $\mathbf{v}_1 = \mathbf{x}_1 = 1$

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**Step 3:**  $\langle \mathbf{x}_3, \mathbf{v}_1 \rangle = \int_{-1}^1 x^2 \, dx = \frac{2}{3}$

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# The Gram-Schmidt Process

$$\mathbf{x}_1 = 1, \mathbf{x}_2 = x, \mathbf{x}_3 = x^2$$



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$$\mathbf{v}_3 = \mathbf{x}_3 - \frac{\langle \mathbf{x}_3, \mathbf{v}_1 \rangle}{\|\mathbf{v}_1\|^2} \mathbf{v}_1 - \frac{\langle \mathbf{x}_3, \mathbf{v}_2 \rangle}{\|\mathbf{v}_2\|^2} \mathbf{v}_2 = x^2 - \frac{1}{3} - 0.$$

# The Gram-Schmidt Process

$$\mathbf{x}_1 = 1, \mathbf{x}_2 = x, \mathbf{x}_3 = x^2$$



Therefore

$$\left\{ 1, x, x^2 - \frac{1}{3} \right\}$$

is an orthogonal basis for  $\mathbb{P}^2$ .



# Orthogonal Matrices

## Definition

An invertible square matrix  $A$  is called *orthogonal* iff

$$A^{-1} = A^T.$$

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This is equivalent to

$$AA^T = A^TA = I.$$

# Orthogonal Matrices



## Example

The matrix

$$A = \begin{bmatrix} \frac{3}{7} & \frac{2}{7} & \frac{6}{7} \\ -\frac{6}{7} & \frac{3}{7} & \frac{2}{7} \\ \frac{2}{7} & \frac{6}{7} & -\frac{3}{7} \end{bmatrix}$$

is orthogonal since

$$A^T A = \begin{bmatrix} \frac{3}{7} & -\frac{6}{7} & \frac{2}{7} \\ \frac{2}{7} & \frac{3}{7} & \frac{6}{7} \\ \frac{6}{7} & \frac{2}{7} & -\frac{3}{7} \end{bmatrix} \begin{bmatrix} \frac{3}{7} & \frac{2}{7} & \frac{6}{7} \\ -\frac{6}{7} & \frac{3}{7} & \frac{2}{7} \\ \frac{2}{7} & \frac{6}{7} & -\frac{3}{7} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

(please check).

# Orthogonal Matrices



Example (Rotation and Reflection Matrices are Orthogonal)

Recall from Lecture 8 that the standard matrices for anticlockwise rotation of  $\mathbb{R}^2$  by an angle of  $\theta$ ; and for reflection about the  $x$ -axis are

$$A = \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

respectively.

# Orthogonal Matrices



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respectively. Note that

$$A^T A = \begin{bmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{bmatrix} \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

# Orthogonal Matrices



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and

$$B^T B = B^2 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I.$$

## Theorem

Let  $A$  be an  $n \times n$  matrix. The following are equivalent:

- 1  $A$  is orthogonal ( $A^{-1} = A^T$ ).
- 2 The row vectors of  $A$  form an orthonormal set in  $\mathbb{R}^n$  with the Euclidean inner product.
- 3 The column vectors of  $A$  form an orthonormal set in  $\mathbb{R}^n$  with the Euclidean inner product.

# Orthogonal Matrices



Proof.

Let

- $\mathbf{r}_i$  be the  $i$ th row vector of  $A$ ;
- $\mathbf{c}_j$  be the  $j$ th column vector of  $A$ ;

# Orthogonal Matrices



Proof.

Let

- $\mathbf{r}_i$  be the  $i$ th row vector of  $A$ ;
- $\mathbf{c}_j$  be the  $j$ th column vector of  $A$ ;
- $\tilde{\mathbf{r}}_i$  be the  $i$ th row vector of  $A^T$ ; and
- $\tilde{\mathbf{c}}_j$  be the  $j$ th column vector of  $A^T$ .

# Orthogonal Matrices



Proof.

Let

- $\mathbf{r}_i$  be the  $i$ th row vector of  $A$ ;
- $\mathbf{c}_j$  be the  $j$ th column vector of  $A$ ;
- $\tilde{\mathbf{r}}_i$  be the  $i$ th row vector of  $A^T$ ; and
- $\tilde{\mathbf{c}}_j$  be the  $j$ th column vector of  $A^T$ .

Because transposing a matrix swaps rows with columns, and columns with rows, we must have

$$\tilde{\mathbf{r}}_i = \mathbf{c}_i \quad \text{and} \quad \tilde{\mathbf{c}}_j = \mathbf{r}_j.$$

Proof Continued.

Note that

$$AA^T = \begin{bmatrix} \mathbf{r}_1 \\ \mathbf{r}_2 \\ \vdots \\ \mathbf{r}_n \end{bmatrix} \begin{bmatrix} \tilde{\mathbf{c}}_1 & \tilde{\mathbf{c}}_2 & \cdots & \tilde{\mathbf{c}}_n \end{bmatrix} =$$

=

Proof Continued.

Note that

$$AA^T = \begin{bmatrix} \mathbf{r}_1 \\ \mathbf{r}_2 \\ \vdots \\ \mathbf{r}_n \end{bmatrix} \begin{bmatrix} \tilde{\mathbf{c}}_1 & \tilde{\mathbf{c}}_2 & \cdots & \tilde{\mathbf{c}}_n \end{bmatrix} = \begin{bmatrix} \mathbf{r}_1 \tilde{\mathbf{c}}_1 & \mathbf{r}_1 \tilde{\mathbf{c}}_2 & \cdots & \mathbf{r}_1 \tilde{\mathbf{c}}_n \\ \mathbf{r}_2 \tilde{\mathbf{c}}_1 & \mathbf{r}_2 \tilde{\mathbf{c}}_2 & \cdots & \mathbf{r}_2 \tilde{\mathbf{c}}_n \\ \vdots & \vdots & & \vdots \\ \mathbf{r}_n \tilde{\mathbf{c}}_1 & \mathbf{r}_n \tilde{\mathbf{c}}_2 & \cdots & \mathbf{r}_n \tilde{\mathbf{c}}_n \end{bmatrix}$$

=

Proof Continued.

Note that

$$\begin{aligned}
 AA^T &= \begin{bmatrix} \mathbf{r}_1 \\ \mathbf{r}_2 \\ \vdots \\ \mathbf{r}_n \end{bmatrix} \begin{bmatrix} \tilde{\mathbf{c}}_1 & \tilde{\mathbf{c}}_2 & \cdots & \tilde{\mathbf{c}}_n \end{bmatrix} = \begin{bmatrix} \mathbf{r}_1 \tilde{\mathbf{c}}_1 & \mathbf{r}_1 \tilde{\mathbf{c}}_2 & \cdots & \mathbf{r}_1 \tilde{\mathbf{c}}_n \\ \mathbf{r}_2 \tilde{\mathbf{c}}_1 & \mathbf{r}_2 \tilde{\mathbf{c}}_2 & \cdots & \mathbf{r}_2 \tilde{\mathbf{c}}_n \\ \vdots & \vdots & & \vdots \\ \mathbf{r}_n \tilde{\mathbf{c}}_1 & \mathbf{r}_n \tilde{\mathbf{c}}_2 & \cdots & \mathbf{r}_n \tilde{\mathbf{c}}_n \end{bmatrix} \\
 &= \begin{bmatrix} \mathbf{r}_1 \cdot \mathbf{r}_1 & \mathbf{r}_1 \cdot \mathbf{r}_2 & \cdots & \mathbf{r}_1 \cdot \mathbf{r}_n \\ \mathbf{r}_2 \cdot \mathbf{r}_1 & \mathbf{r}_2 \cdot \mathbf{r}_2 & \cdots & \mathbf{r}_2 \cdot \mathbf{r}_n \\ \vdots & \vdots & & \vdots \\ \mathbf{r}_n \cdot \mathbf{r}_1 & \mathbf{r}_n \cdot \mathbf{r}_2 & \cdots & \mathbf{r}_n \cdot \mathbf{r}_n \end{bmatrix}.
 \end{aligned}$$

Proof Continued.

Note that

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 AA^T &= \begin{bmatrix} \mathbf{r}_1 \\ \mathbf{r}_2 \\ \vdots \\ \mathbf{r}_n \end{bmatrix} \begin{bmatrix} \tilde{\mathbf{c}}_1 & \tilde{\mathbf{c}}_2 & \cdots & \tilde{\mathbf{c}}_n \end{bmatrix} = \begin{bmatrix} \mathbf{r}_1 \tilde{\mathbf{c}}_1 & \mathbf{r}_1 \tilde{\mathbf{c}}_2 & \cdots & \mathbf{r}_1 \tilde{\mathbf{c}}_n \\ \mathbf{r}_2 \tilde{\mathbf{c}}_1 & \mathbf{r}_2 \tilde{\mathbf{c}}_2 & \cdots & \mathbf{r}_2 \tilde{\mathbf{c}}_n \\ \vdots & \vdots & & \vdots \\ \mathbf{r}_n \tilde{\mathbf{c}}_1 & \mathbf{r}_n \tilde{\mathbf{c}}_2 & \cdots & \mathbf{r}_n \tilde{\mathbf{c}}_n \end{bmatrix} \\
 &= \begin{bmatrix} \mathbf{r}_1 \cdot \mathbf{r}_1 & \mathbf{r}_1 \cdot \mathbf{r}_2 & \cdots & \mathbf{r}_1 \cdot \mathbf{r}_n \\ \mathbf{r}_2 \cdot \mathbf{r}_1 & \mathbf{r}_2 \cdot \mathbf{r}_2 & \cdots & \mathbf{r}_2 \cdot \mathbf{r}_n \\ \vdots & \vdots & & \vdots \\ \mathbf{r}_n \cdot \mathbf{r}_1 & \mathbf{r}_n \cdot \mathbf{r}_2 & \cdots & \mathbf{r}_n \cdot \mathbf{r}_n \end{bmatrix}.
 \end{aligned}$$

We can see from this formula that  $AA^T = I$  if and only if

$$\mathbf{r}_1 \cdot \mathbf{r}_1 = \mathbf{r}_2 \cdot \mathbf{r}_2 = \dots = \mathbf{r}_n \cdot \mathbf{r}_n = 1$$

and

$$\mathbf{r}_i \cdot \mathbf{r}_k = 0 \quad \text{for all } i \neq j.$$

# Orthogonal Matrices



Proof Continued.

This proves that

$$AA^T = I \iff \{\mathbf{r}_1, \mathbf{r}_2, \dots, \mathbf{r}_n\} \text{ is an orthonormal set in } \mathbb{R}^n.$$

# Orthogonal Matrices



Proof Continued.

This proves that

$$AA^T = I \iff \{\mathbf{r}_1, \mathbf{r}_2, \dots, \mathbf{r}_n\} \text{ is an orthonormal set in } \mathbb{R}^n.$$

If we did a similar calculating starting with  $A^T A$  instead of  $AA^T$  then we could prove that

$$A^T A = I \iff \{\mathbf{c}_1, \mathbf{c}_2, \dots, \mathbf{c}_n\} \text{ is an orthonormal set in } \mathbb{R}^n.$$

(I leave this proof for you to write.)



## Theorem

- 1** *The transpose of an orthogonal matrix is orthogonal.*
- 2** *The inverse of an orthogonal matrix is orthogonal.*
- 3** *A product of orthogonal matrices is orthogonal.*
- 4** *If  $A$  is orthogonal, then  $\det(A) = 1$  or  $\det(A) = -1$ .*

I will prove parts **1** and **4**. I leave parts **2** and **3** for you to prove.

# Orthogonal Matrices

Proof.

- 1 Let  $A$  be an orthogonal matrix. Then we know that

$$A^T A = I = AA^T.$$

Let  $B = A^T$ . Then  $B^T = (A^T)^T = A$ . Hence

$$BB^T = I = B^T B$$

which proves that  $B$  is also orthogonal.

# Orthogonal Matrices



Proof.

- 1 Let  $A$  be an orthogonal matrix. Then we know that

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which proves that  $B$  is also orthogonal.

- 4 Suppose that  $A$  satisfies  $AA^T = I$ . Then

$$1 = \det(I) = \det(A) \det(A^T) =$$

# Orthogonal Matrices



Proof.

- 1 Let  $A$  be an orthogonal matrix. Then we know that

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which proves that  $B$  is also orthogonal.

- 4 Suppose that  $A$  satisfies  $AA^T = I$ . Then

$$1 = \det(I) = \det(A) \det(A^T) = \det(A) \det(A) = \det(A)^2.$$

Hence  $\det(A) = \pm 1$ .



# Orthogonal Matrices



## Example

The matrix

$$A = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$$

is orthogonal because its row vectors form an orthonormal set in  $\mathbb{R}^2$  (check!).

# Orthogonal Matrices



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is orthogonal because its row vectors form an orthonormal set in  $\mathbb{R}^2$  (check!).

I leave it to you to check that the determinant of this matrix is

$$\det(A) = 1.$$

# Orthogonal Matrices

## Lemma

If  $\mathbf{u}$  and  $\mathbf{v}$  are vectors in  $\mathbb{R}^n$  with the Euclidean inner product, then

$$\mathbf{u} \cdot \mathbf{v} = \frac{1}{4} \|\mathbf{u} + \mathbf{v}\|^2 - \frac{1}{4} \|\mathbf{u} - \mathbf{v}\|^2.$$

## Proof.

Since

$$\|\mathbf{u} + \mathbf{v}\|^2 = (\mathbf{u} + \mathbf{v}) \cdot (\mathbf{u} + \mathbf{v}) = \|\mathbf{u}\|^2 + 2\mathbf{u} \cdot \mathbf{v} + \|\mathbf{v}\|^2$$

$$\|\mathbf{u} - \mathbf{v}\|^2 = (\mathbf{u} - \mathbf{v}) \cdot (\mathbf{u} - \mathbf{v}) = \|\mathbf{u}\|^2 - 2\mathbf{u} \cdot \mathbf{v} + \|\mathbf{v}\|^2$$

we have

$$\|\mathbf{u} + \mathbf{v}\|^2 - \|\mathbf{u} - \mathbf{v}\|^2 = 4\mathbf{u} \cdot \mathbf{v}.$$



## Theorem

Let  $A$  be an  $n \times n$  matrix. The following are equivalent:

- 1  $A$  is orthogonal.
- 2  $\|Ax\| = \|x\|$  for all  $x$  in  $\mathbb{R}^n$ .
- 3  $Ax \cdot Ay = x \cdot y$  for all  $x$  and  $y$  in  $\mathbb{R}^n$ .

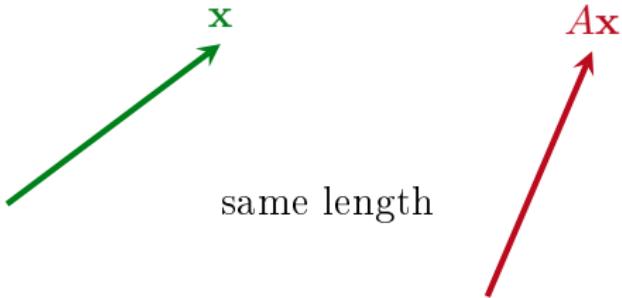
# Orthogonal Matrices



## Theorem

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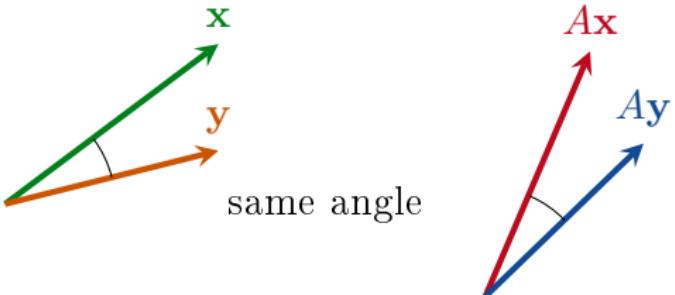
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# Orthogonal Matrices



## Theorem

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## Proof.

I will prove that  $1 \implies 2 \implies 3 \implies 1$ .

- 1  $A$  is orthogonal.
- 2  $\|A\mathbf{x}\| = \|\mathbf{x}\|$  for all  $\mathbf{x}$  in  $\mathbb{R}^n$ .
- 3  $A\mathbf{x} \cdot A\mathbf{y} = \mathbf{x} \cdot \mathbf{y}$  for all  $\mathbf{x}$  and  $\mathbf{y}$  in  $\mathbb{R}^n$ .



Proof Continued.

1  $\implies$  2 : Note first that since  $\mathbf{x} \cdot \mathbf{y} = \mathbf{y}^T \mathbf{x}$  we have

$$A\mathbf{u} \cdot \mathbf{v} = \mathbf{v}^T (A\mathbf{u})$$

- 1  $A$  is orthogonal.
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Proof Continued.

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- 1  $A$  is orthogonal.
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- 3  $A\mathbf{x} \cdot A\mathbf{y} = \mathbf{x} \cdot \mathbf{y}$  for all  $\mathbf{x}$  and  $\mathbf{y}$  in  $\mathbb{R}^n$ .



## Proof Continued.

1  $\implies$  2 : Note first that since  $\mathbf{x} \cdot \mathbf{y} = \mathbf{y}^T \mathbf{x}$  we have

$$A\mathbf{u} \cdot \mathbf{v} = \mathbf{v}^T (A\mathbf{u}) = (\mathbf{v}^T A)\mathbf{u} = (A^T \mathbf{v})^T \mathbf{u}$$

# Orthogonal Matrices

- 1  $A$  is orthogonal.
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# Orthogonal Matrices

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Proof Continued.

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If  $A^T A = I$ , then

$$\|A\mathbf{x}\|^2 = A\mathbf{x} \cdot A\mathbf{x} = \mathbf{x} \cdot A^T A\mathbf{x}$$

# Orthogonal Matrices

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# Orthogonal Matrices

- 1  $A$  is orthogonal.
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Proof Continued.

2  $\implies$  3 : Suppose that  $\|A\mathbf{x}\| = \|\mathbf{x}\|$  for all  $\mathbf{x} \in \mathbb{R}^n$ . By the previous lemma we have

$$\begin{aligned} A\mathbf{x} \cdot A\mathbf{y} &= \\ &= \\ &= \\ &= \mathbf{x} \cdot \mathbf{y}. \end{aligned}$$

- 1  $A$  is orthogonal.
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$$A\mathbf{x} \cdot A\mathbf{y} = \frac{1}{4} \|A\mathbf{x} + A\mathbf{y}\|^2 - \frac{1}{4} \|A\mathbf{x} - A\mathbf{y}\|^2$$

=

=

=

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$$= \frac{1}{4} \|A(\mathbf{x} + \mathbf{y})\|^2 - \frac{1}{4} \|A(\mathbf{x} - \mathbf{y})\|^2$$

=

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# Orthogonal Matrices

- 1  $A$  is orthogonal.
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Proof Continued.

3  $\implies$  1 : Suppose that

$$\mathbf{x} \cdot \mathbf{y} = A\mathbf{x} \cdot A\mathbf{y} = \mathbf{x} \cdot A^T A\mathbf{y}$$

for all  $\mathbf{x}$  and  $\mathbf{y}$ .

# Orthogonal Matrices



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for all  $\mathbf{x}$  and  $\mathbf{y}$ . Then

$$0 = \mathbf{x} \cdot A^T A\mathbf{y} - \mathbf{x} \cdot \mathbf{y}$$

# Orthogonal Matrices



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$$0 = \mathbf{x} \cdot A^T A\mathbf{y} - \mathbf{x} \cdot \mathbf{y} = \mathbf{x} \cdot (A^T A - I)\mathbf{y}.$$

# Orthogonal Matrices



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This equation holds for all  $\mathbf{x} \in \mathbb{R}^n$ . Let  $\mathbf{x} = (A^T A - I)\mathbf{y}$ .

# Orthogonal Matrices



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# Orthogonal Matrices

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Proof Continued.

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$$0 = (A^T A - I)\mathbf{y} \cdot (A^T A - I)\mathbf{y} = \|(A^T A - I)\mathbf{y}\|^2 \implies (A^T A - I)\mathbf{y} = \mathbf{0}.$$

# Orthogonal Matrices



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This equation holds for all  $\mathbf{x} \in \mathbb{R}^n$ . Let  $\mathbf{x} = (A^T A - I)\mathbf{y}$ . Then

$$0 = (A^T A - I)\mathbf{y} \cdot (A^T A - I)\mathbf{y} = \|(A^T A - I)\mathbf{y}\|^2 \implies (A^T A - I)\mathbf{y} = \mathbf{0}.$$

But this equation holds for all  $\mathbf{y} \in \mathbb{R}^n$ , so we must have  $A^T A - I = \mathbf{0}$ .

# Orthogonal Matrices

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This equation holds for all  $\mathbf{x} \in \mathbb{R}^n$ . Let  $\mathbf{x} = (A^T A - I)\mathbf{y}$ . Then

$$0 = (A^T A - I)\mathbf{y} \cdot (A^T A - I)\mathbf{y} = \|(A^T A - I)\mathbf{y}\|^2 \implies (A^T A - I)\mathbf{y} = \mathbf{0}.$$

But this equation holds for all  $\mathbf{y} \in \mathbb{R}^n$ , so we must have  $A^T A - I = \mathbf{0}$ . Therefore  $A^T A = I$  and we are finished.



# Break

We will continue at 3pm

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$$\int \frac{1}{d} \, d\text{house} =$$





# Orthogonal Diagonalisation

# Orthogonal Diagonalisation



Recall that two square matrices  $A$  and  $B$  are called *similar* if there exists an invertible matrix  $P$  such that

$$P^{-1}AP = B.$$

# Orthogonal Diagonalisation



Recall that two square matrices  $A$  and  $B$  are called *similar* if there exists an invertible matrix  $P$  such that

$$P^{-1}AP = B.$$

If  $A$  is similar to a diagonal matrix  $D$ , with

$$P^{-1}AP = D$$

then we say that  $A$  is *diagonalisable* and that  $P$  *diagonalises*  $A$ .

(See Lectures 8 and 9.)

# Orthogonal Diagonalisation



Sometimes it is possible to find an orthogonal matrix  $P$  such that

$$P^{-1}AP = B.$$

# Orthogonal Diagonalisation



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But of course orthogonal means  $P^{-1} = P^T$ .

# Orthogonal Diagonalisation



Sometimes it is possible to find an orthogonal matrix  $P$  such that

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But of course orthogonal means  $P^{-1} = P^T$ .

## Definition

If  $A$  and  $B$  are square matrices, then we say that  $A$  and  $B$  are *orthogonally similar* if there is an orthogonal matrix  $P$  such that

$$P^T AP = B.$$

## Definition

If  $A$  is orthogonally similar to a diagonal matrix  $D$ , with

$$P^T AP = D$$

then we say that  $A$  is *orthogonally diagonalisable* and that  $P$  *orthogonally diagonalises*  $A$ .

# Orthogonal Diagonalisation



## Remark

If  $P^TAP = D$  then we can multiply on the left by  $P$ , and on the right by  $P^T$  to get

$$PP^TAPP^T = PDP^T.$$

# Orthogonal Diagonalisation



## Remark

If  $P^TAP = D$  then we can multiply on the left by  $P$ , and on the right by  $P^T$  to get

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But since  $P^TP = I = PP^T$ , this is just

$$A = PDP^T.$$

# Orthogonal Diagonalisation



## Remark

If  $P^TAP = D$  then we can multiply on the left by  $\textcolor{brown}{P}$ , and on the right by  $P^T$  to get

$$\textcolor{brown}{P}P^TAPP^{\textcolor{brown}{T}} = \textcolor{brown}{P}DP^{\textcolor{brown}{T}}.$$

But since  $P^TP = I = PP^T$ , this is just

$$A = PDP^T.$$

Taking the transpose of both sides gives

$$A^T = (\textcolor{red}{P}DP^{\textcolor{green}{T}})^T$$

# Orthogonal Diagonalisation

## Remark

If  $P^TAP = D$  then we can multiply on the left by  $\textcolor{brown}{P}$ , and on the right by  $P^T$  to get

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But since  $P^TP = I = PP^T$ , this is just

$$A = PDP^T.$$

Taking the transpose of both sides gives

$$A^T = (\textcolor{brown}{P}DP^{\textcolor{green}{T}})^T = (\textcolor{green}{P}^{\textcolor{green}{T}})^T \textcolor{brown}{D}^T \textcolor{brown}{P}^T$$

# Orthogonal Diagonalisation

## Remark

If  $P^TAP = D$  then we can multiply on the left by  $P$ , and on the right by  $P^T$  to get

$$PP^TAPP^T = PDP^T.$$

But since  $P^TP = I = PP^T$ , this is just

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Taking the transpose of both sides gives

$$A^T = (PDP^T)^T = (P^T)^T D^T P^T = PDP^T$$

# Orthogonal Diagonalisation



## Remark

If  $P^TAP = D$  then we can multiply on the left by  $P$ , and on the right by  $P^T$  to get

$$PP^TAPP^T = PDP^T.$$

But since  $P^TP = I = PP^T$ , this is just

$$A = PDP^T.$$

Taking the transpose of both sides gives

$$A^T = (PDP^T)^T = (P^T)^T D^T P^T = PDP^T = A.$$

# Orthogonal Diagonalisation

## Remark

If  $P^TAP = D$  then we can multiply on the left by  $P$ , and on the right by  $P^T$  to get

$$PP^TAPP^T = PDP^T.$$

But since  $P^TP = I = PP^T$ , this is just

$$A = PDP^T.$$

Taking the transpose of both sides gives

$$A^T = (PDP^T)^T = (P^T)^T D^T P^T = PDP^T = A.$$

Hence

$$A \text{ is orthogonally diagonalisable} \quad \implies \quad A \text{ is symmetric.}$$

# Orthogonal Diagonalisation



## Theorem

Let  $A \in \mathbb{R}^{n \times n}$ . The following are equivalent:

- 1  $A$  is orthogonally diagonalisable.
- 2  $A$  has an orthonormal set of  $n$  eigenvectors.
- 3  $A$  is symmetric.

(proof omitted)

# Orthogonal Diagonalisation



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(proof omitted)

## Theorem

Let  $A \in \mathbb{R}^{n \times n}$ . If  $A$  is symmetric then, the eigenvalues of  $A$  are all real numbers.

(proof omitted)

# Orthogonal Diagonalisation



## Theorem

*If  $A \in \mathbb{R}^{n \times n}$  is symmetric, then any two eigenvectors from different eigenspaces are orthogonal.*

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## Proof.

Let

- $\mathbf{x}_1$  be an eigenvector corresponding to the eigenvalue  $\lambda_1$ ; and
- $\mathbf{x}_2$  be an eigenvector corresponding to the eigenvalue  $\lambda_2 \neq \lambda_1$ .

We need to prove that  $\mathbf{x}_1 \cdot \mathbf{x}_2 = 0$ .

# Orthogonal Diagonalisation



Proof Continued.

We calculate that

$$\lambda_1 \mathbf{x}_1 \cdot \mathbf{x}_2 = A\mathbf{x}_1 \cdot \mathbf{x}_2 \quad \text{since } \mathbf{x}_1 \text{ is an eigenvector}$$

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# Orthogonal Diagonalisation



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Proof Continued.

We calculate that

$$\begin{aligned}\lambda_1 \mathbf{x}_1 \cdot \mathbf{x}_2 &= A\mathbf{x}_1 \cdot \mathbf{x}_2 && \text{since } \mathbf{x}_1 \text{ is an eigenvector} \\ &= \mathbf{x}_1 \cdot A^T \mathbf{x}_2 \\ &= \mathbf{x}_1 \cdot A\mathbf{x}_2 && \text{since } A \text{ is symmetric} \\ &= \mathbf{x}_1 \cdot \lambda_2 \mathbf{x}_2 && \text{since } \mathbf{x}_2 \text{ is an eigenvector} \\ &= \end{aligned}$$

# Orthogonal Diagonalisation



Proof Continued.

We calculate that

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# Orthogonal Diagonalisation



Proof Continued.

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It follows that

$$(\lambda_1 - \lambda_2) \mathbf{x}_1 \cdot \mathbf{x}_2 = 0.$$

But since  $\lambda_1 \neq \lambda_2$  we must have  $\mathbf{x}_1 \cdot \mathbf{x}_2 = 0$ . □

## How to Orthogonally Diagonalise an $n \times n$ Symmetric Matrix

Let  $A$  be an  $n \times n$  symmetric matrix with real entries.

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- 3 Write

$$P = \begin{bmatrix} \mathbf{u}_1 & \mathbf{u}_2 & \cdots & \mathbf{u}_n \end{bmatrix}.$$

Then  $P$  will orthogonally diagonalise  $A$ . The eigenvalues on the main diagonal of  $D = P^T A P$  will be in the same order as their corresponding eigenvectors in  $P$ .

## Example

Find an orthogonal matrix  $P$  which diagonalises

$$A = \begin{bmatrix} 4 & 2 & 2 \\ 2 & 4 & 2 \\ 2 & 2 & 4 \end{bmatrix}.$$

# Orthogonal Diagonalisation



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Find an orthogonal matrix  $P$  which diagonalises

$$A = \begin{bmatrix} 4 & 2 & 2 \\ 2 & 4 & 2 \\ 2 & 2 & 4 \end{bmatrix}.$$

Please check that

$$0 = \det(\lambda I - A) = \begin{vmatrix} \lambda - 4 & -2 & -2 \\ -2 & \lambda - 4 & -2 \\ -2 & -2 & \lambda - 4 \end{vmatrix} = (\lambda - 2)^2(\lambda - 8).$$

Thus the eigenvalues of  $A$  are  $\lambda = 2$  and  $\lambda = 8$ .

# Orthogonal Diagonalisation



Please also check that

$$\mathbf{x}_1 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \quad \text{and} \quad \mathbf{x}_2 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

form a basis for the eigenspace corresponding to  $\lambda = 2$ .

# Orthogonal Diagonalisation



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form a basis for the eigenspace corresponding to  $\lambda = 2$ .

We need to use the Gram-Schmidt process on  $\{\mathbf{x}_1, \mathbf{x}_2\}$  to obtain an orthonormal basis.

# Orthogonal Diagonalisation



$$\mathbf{x}_1 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \quad \mathbf{x}_2 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

First we calculate

$$\mathbf{v}_1 = \mathbf{x}_1 =$$

$$\mathbf{v}_2 = \mathbf{x}_2 - \frac{\langle \mathbf{x}_2, \mathbf{v}_1 \rangle}{\|\mathbf{v}_1\|^2} \mathbf{v}_1 =$$

Then we normalise to obtain

$$\mathbf{u}_1 = \frac{\mathbf{v}_1}{\|\mathbf{v}_1\|} = \quad , \quad \mathbf{u}_2 = \frac{\mathbf{v}_2}{\|\mathbf{v}_2\|} =$$

which are orthonormal.

(Please check the calculations on this slide.)

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Then we normalise to obtain

$$\mathbf{u}_1 = \frac{\mathbf{v}_1}{\|\mathbf{v}_1\|} = \begin{bmatrix} -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \end{bmatrix}, \quad \mathbf{u}_2 = \frac{\mathbf{v}_2}{\|\mathbf{v}_2\|} =$$

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which are orthonormal.

(Please check the calculations on this slide.)

# Orthogonal Diagonalisation



Next please check that the eigenspace corresponding to  $\lambda = 8$  has basis

$$\mathbf{x}_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$

# Orthogonal Diagonalisation



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Since we have only one vector, we can skip the Gram-Schmidt process and move on to normalising this vector:

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## Orthogonal

$$\mathbf{u}_1 = \begin{bmatrix} -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \end{bmatrix}, \quad \mathbf{u}_2 = \begin{bmatrix} -\frac{1}{\sqrt{6}} \\ -\frac{1}{\sqrt{6}} \\ \frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{6}} \end{bmatrix}, \quad \mathbf{u}_3 = \begin{bmatrix} \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{bmatrix}$$



Now that we know our  $\mathbf{u}_1$ ,  $\mathbf{u}_2$  and  $\mathbf{u}_3$ , we can write

$$P = \begin{bmatrix} \mathbf{u}_1 & \mathbf{u}_2 & \mathbf{u}_3 \end{bmatrix} =$$

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I leave for you to check that

$$P^T A P = \begin{bmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ -\frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{6}} & \frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \end{bmatrix} \begin{bmatrix} 4 & 2 & 2 \\ 2 & 4 & 2 \\ 2 & 2 & 4 \end{bmatrix} \begin{bmatrix} -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ 0 & \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} \end{bmatrix} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 8 \end{bmatrix}.$$

# Orthogonal Diagonalisation



## Example

Orthogonally diagonalise the matrix  $A = \begin{bmatrix} 3 & -6 & 0 \\ -6 & 0 & 6 \\ 0 & 6 & -3 \end{bmatrix}$ .

# Orthogonal Diagonalisation

$$A = \begin{bmatrix} 3 & -6 & 0 \\ -6 & 0 & 6 \\ 0 & 6 & -3 \end{bmatrix}$$



The characteristic equation of  $A$  is

$$\begin{aligned} 0 &= \det(\lambda I - A) = \begin{vmatrix} \lambda - 3 & 6 & 0 \\ 6 & \lambda & -6 \\ 0 & -6 & \lambda + 3 \end{vmatrix} \\ &= (\lambda - 3) \begin{vmatrix} \lambda & -6 \\ -6 & \lambda + 3 \end{vmatrix} - 6 \begin{vmatrix} 6 & -6 \\ 0 & \lambda + 3 \end{vmatrix} + 0 \\ &= (\lambda - 3)(\lambda(\lambda + 3) - 36) - 6(6\lambda + 18) \\ &= (\lambda - 3)(\lambda^2 + 3\lambda - 36) - 36\lambda - 108 \\ &= \lambda^3 + 3\lambda^2 - 36\lambda - 3\lambda^2 - 9\lambda + 108 - 36\lambda - 108 \\ &= \lambda^3 - 81\lambda = \lambda(\lambda - 9)(\lambda + 9). \end{aligned}$$

The eigenvalues are  $\lambda = 0$ ,  $\lambda = 9$  and  $\lambda = -9$ .

$$\lambda = 0, \lambda = 9, \lambda = -9$$

$$A = \begin{bmatrix} 3 & -6 & 0 \\ -6 & 0 & 6 \\ 0 & 6 & -3 \end{bmatrix}$$



Next we need to find bases for the eigenspaces. Since we have three different eigenvalues, each eigenspace will be one-dimensional.

$$\lambda = 0, \lambda = 9, \lambda = -9$$

$$A = \begin{bmatrix} 3 & -6 & 0 \\ -6 & 0 & 6 \\ 0 & 6 & -3 \end{bmatrix}$$



Since

$$A = \begin{bmatrix} 3 & -6 & 0 \\ -6 & 0 & 6 \\ 0 & 6 & -3 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 & 0 \\ -2 & 0 & 2 \\ 0 & 2 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 & 0 \\ 0 & -4 & 2 \\ 0 & 2 & -1 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & -2 & 0 \\ 0 & 1 & -\frac{1}{2} \\ 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -\frac{1}{2} \\ 0 & 0 & 0 \end{bmatrix},$$

a basis for  $\text{Nul } A$  is given by  $\mathbf{x}_1 = \begin{bmatrix} 1 \\ \frac{1}{2} \\ 1 \end{bmatrix}$ .

(Remember that elementary row operations do not change a null space.)

$$\lambda = 0, \lambda = 9, \lambda = -9$$

$$A = \begin{bmatrix} 3 & -6 & 0 \\ -6 & 0 & 6 \\ 0 & 6 & -3 \end{bmatrix}$$

The RREF of  $A - 9I$  is  $\begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{bmatrix}$ . Hence a basis for

$$\text{Nul}(A - 9I) = \text{Nul}(9I - A) \text{ is given by } \mathbf{x}_2 = \begin{bmatrix} -2 \\ 2 \\ 1 \end{bmatrix}.$$

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$\text{Nul}(A - (-9)I) = \text{Nul}(-9I - A)$  is given by  $\mathbf{x}_3 = \begin{bmatrix} -\frac{1}{2} \\ -1 \\ 1 \end{bmatrix}$ .

(Please check these!)

## Orthogonal Diagonalisation

$$\mathbf{x}_1 = \begin{bmatrix} 1 \\ \frac{1}{2} \\ 1 \end{bmatrix}, \quad \mathbf{x}_2 = \begin{bmatrix} -2 \\ 2 \\ 1 \end{bmatrix}, \quad \mathbf{x}_3 = \begin{bmatrix} -\frac{1}{2} \\ -1 \\ 1 \end{bmatrix}$$



Since we only have one vector in each basis, we can skip the Gram-Schmidt process and move straight to normalisation.

## Orthogonal Diagonalisation

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we have

$$\mathbf{u}_1 = \frac{\mathbf{x}_1}{\|\mathbf{x}_1\|} = \frac{2}{3}\mathbf{x}_1 = \begin{bmatrix} \frac{2}{3} \\ \frac{1}{3} \\ \frac{2}{3} \end{bmatrix}, \quad \mathbf{u}_2 = \frac{\mathbf{x}_2}{\|\mathbf{x}_2\|} = \frac{1}{3}\mathbf{x}_2 = \begin{bmatrix} -\frac{2}{3} \\ \frac{2}{3} \\ \frac{1}{3} \end{bmatrix},$$

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## Orthogonal Diagonalisation

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Hence

$$P = \begin{bmatrix} \mathbf{u}_1 & \mathbf{u}_2 & \mathbf{u}_3 \end{bmatrix} = \begin{bmatrix} \frac{2}{3} & -\frac{2}{3} & -\frac{1}{3} \\ \frac{1}{3} & \frac{2}{3} & -\frac{2}{3} \\ \frac{2}{3} & \frac{1}{3} & \frac{2}{3} \end{bmatrix} = \frac{1}{3} \begin{bmatrix} 2 & -2 & -1 \\ 1 & 2 & -2 \\ 2 & 1 & 2 \end{bmatrix}.$$

## Orthogonal Diagonalisation

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Hence

$$P = \begin{bmatrix} \mathbf{u}_1 & \mathbf{u}_2 & \mathbf{u}_3 \end{bmatrix} = \begin{bmatrix} \frac{2}{3} & -\frac{2}{3} & -\frac{1}{3} \\ \frac{1}{3} & \frac{2}{3} & -\frac{2}{3} \\ \frac{2}{3} & \frac{1}{3} & \frac{2}{3} \end{bmatrix} = \frac{1}{3} \begin{bmatrix} 2 & -2 & -1 \\ 1 & 2 & -2 \\ 2 & 1 & 2 \end{bmatrix}.$$

I leave it to you to check that

$$D = P^T A P = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & -9 \end{bmatrix}$$

or equivalently that

$$A = P D P^T.$$

# Orthogonal Diagonalisation



Example (page 396)

Orthogonally diagonalise the matrix  $A = \begin{bmatrix} 3 & -2 & 4 \\ -2 & 6 & 2 \\ 4 & 2 & 3 \end{bmatrix}$ , whose characteristic equation is

$$0 = \lambda^3 - 12\lambda^2 + 21\lambda + 98 = (\lambda - 7)^2(\lambda + 2).$$

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The usual calculations produce bases for the eigenspaces:

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Although  $\mathbf{x}_1$  and  $\mathbf{x}_2$  are linearly independent, they are not orthogonal. So we must use the Gram-Schmidt process on  $\mathbf{x}_1$  and  $\mathbf{x}_2$ .

$$\text{Or } \lambda = 7 : \mathbf{x}_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \mathbf{x}_2 = \begin{bmatrix} -\frac{1}{2} \\ 1 \\ 0 \end{bmatrix}; \quad \lambda = -2 : \mathbf{x}_3 = \begin{bmatrix} -1 \\ -\frac{1}{2} \\ 1 \end{bmatrix}$$

We calculate that

$$\mathbf{v}_1 = \mathbf{x}_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

$$\mathbf{v}_2 = \quad =$$

The we normalise these vectors to obtain an orthonormal basis for the eigenspace corresponding to  $\lambda = 7$ :

$$\mathbf{u}_1 = \frac{\mathbf{v}_1}{\|\mathbf{v}_1\|} = \quad \mathbf{u}_2 = \frac{\mathbf{v}_2}{\|\mathbf{v}_2\|} =$$

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Then we normalise these vectors to obtain an orthonormal basis for the eigenspace corresponding to  $\lambda = 7$ :

$$\mathbf{u}_1 = \frac{\mathbf{v}_1}{\|\mathbf{v}_1\|} = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ \frac{1}{\sqrt{2}} \end{bmatrix}, \quad \mathbf{u}_2 = \frac{\mathbf{v}_2}{\|\mathbf{v}_2\|} =$$

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An orthonormal basis for the eigenspace corresponding to  $\lambda = -2$  is

$$\mathbf{u}_3 = \frac{\mathbf{x}_3}{\|\mathbf{x}_3\|} = \begin{bmatrix} -\frac{2}{3} \\ -\frac{1}{3} \\ \frac{2}{3} \end{bmatrix}.$$

$$\text{Or } \lambda = 7 : \mathbf{x}_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \mathbf{x}_2 = \begin{bmatrix} -\frac{1}{2} \\ 1 \\ 0 \end{bmatrix}; \quad \lambda = -2 : \mathbf{x}_3 = \begin{bmatrix} -1 \\ -\frac{1}{2} \\ 1 \end{bmatrix}$$

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Let

$$P = \begin{bmatrix} \mathbf{u}_1 & \mathbf{u}_2 & \mathbf{u}_3 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{18}} & -\frac{2}{3} \\ 0 & \frac{4}{\sqrt{18}} & -\frac{1}{3} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{18}} & \frac{2}{3} \end{bmatrix} \quad \text{and} \quad D = \begin{bmatrix} 7 & 0 & 0 \\ 0 & 7 & 0 \\ 0 & 0 & -2 \end{bmatrix}.$$

Then  $P$  orthogonally diagonalises  $A$  and  $A = PDP^T$ .



# Next Time

- More about  $A^T A$
- Singular Value Decomposition