Welcome to

Mathematics II

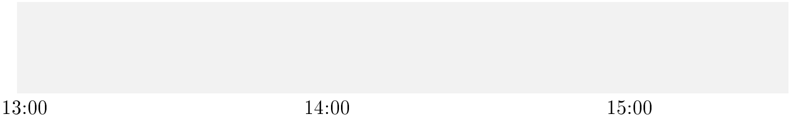
with Dr Neil Course

Lecture 1

- Information about this course
- 8.1 Using Basic Integration Formulae
- 8.2 Integration by Parts
- 8.3 Trigonometric Integrals

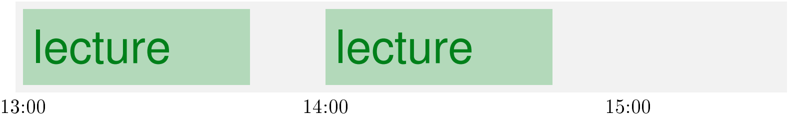
Information about this course

- ≈ 12 classes. Friday afternoons 1pm-3:30pm.



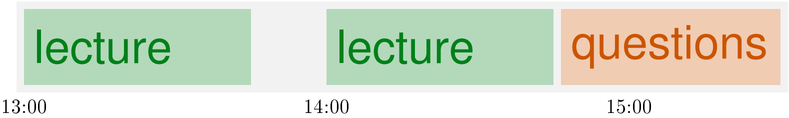
Information about this course

- $\approx$ 12 classes. Friday afternoons 1pm-3:30pm.
- 2 lectures with a break between.

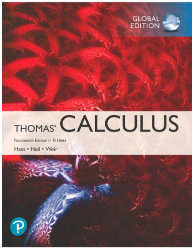


Information about this course

- $\approx$ 12 classes. Friday afternoons 1pm-3:30pm.
- 2 lectures with a break between.
- Then I will answers your questions.



The Book



Joel R. Hass, Christopher E. Heil and Maurice D. Weir,
Thomas' Calculus in SI Units,
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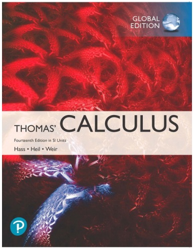

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This is a required purchase.
You need to have this book to be able to do the homework.

8
12
14
15
10

8. Techniques of Integration

12

14

15

10

8. Techniques of Integration

12. Vectors and the Geometry of Space

14

15

10

8. Techniques of Integration

12. Vectors and the Geometry of Space

14. Partial Derivatives

15

10

8. Techniques of Integration

12. Vectors and the Geometry of Space

14. Partial Derivatives

15. Multiple Integrals

10. Infinite Sequences and Series

8. Techniques of Integration	2 weeks
12. Vectors and the Geometry of Space	2 weeks
14. Partial Derivatives	2 weeks
15. Multiple Integrals	3 weeks
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Exams and homework

(This information may change based on the University's decisions)

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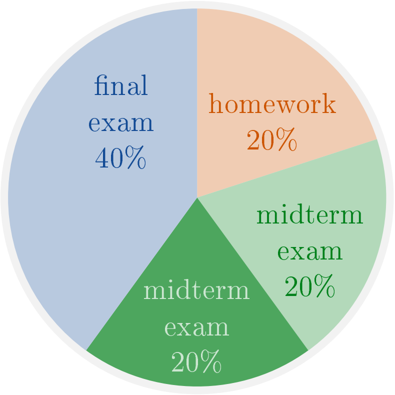

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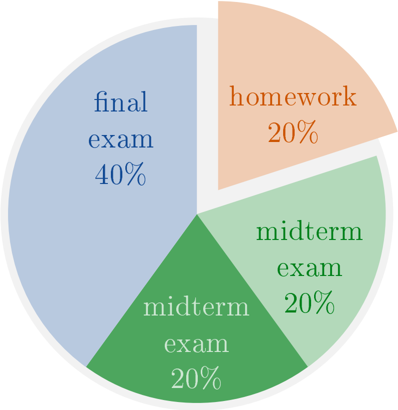

using Pearson MyLab Math

one piece of homework for each lesson

deadline = end of term

more details in O'Learn

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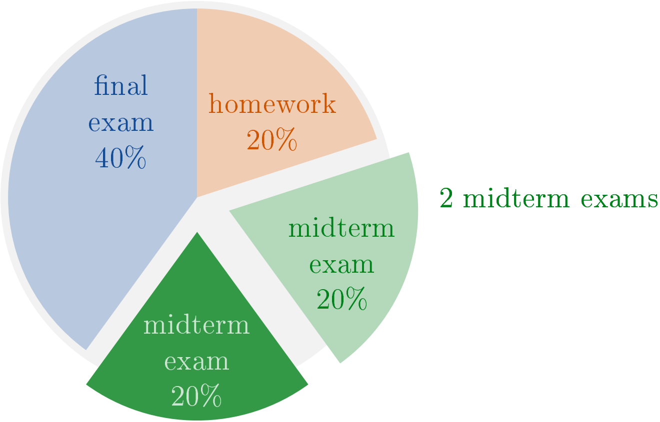

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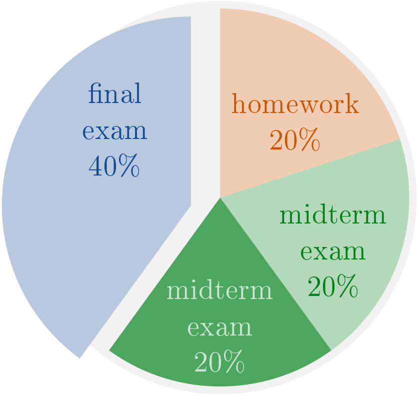

Expectations

If this was a classroom course, the expectation is that for each hour of lectures, you would study an extra 1-2 hours outside of class.

classroom course	lectures (5 hours)	other study (5-10 hours)

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For an online course, you are still expected to study a total of 10-15 hours each week.

online course	class (2.5 hours)	other study (7.5-12.5 hours)

This may include:

- Do the online homework on MyLab;

$\vdots$

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- Use the O'Learn Discussion Board;
- Read other books?;
- Watch online videos;

$\vdots$

slide number

current section

Using Basic Integration Formulae

Table 8.1

Basic integration formulas

1. $\int k\,dx = kx + C$ (any number k)	12. $\int \tan x\,dx = \ln\lvert\sec x\rvert + C$
2. $\int x^n\,dx = \frac{x^{n+1}}{n+1} + C$ $(n \neq -1)$	13. $\int \cot x\,dx = \ln\lvert\sin x\rvert + C$
3. $\int \frac{dx}{x} = \ln\lvert x\rvert + C$	14. $\int \sec x\,dx = \ln\lvert\sec x + \tan x\rvert + C$
4. $\int e^x\,dx = e^x + C$	15. $\int \csc x\,dx = -\ln\lvert\csc x + \cot x\rvert + C$
5. $\int a^x\,dx = \frac{a^x}{\ln a} + C$ $(a > 0, a \neq 1)$	16. $\int \sinh x\,dx = \cosh x + C$
6. $\int \sin x\,dx = -\cos x + C$	17. $\int \cosh x\,dx = \sinh x + C$
7. $\int \cos x\,dx = \sin x + C$	18. $\int \frac{dx}{\sqrt{a^2 - x^2}} = \sin^{-1}\left(\frac{x}{a}\right) + C$
8. $\int \sec^2 x\,dx = \tan x + C$	19. $\int \frac{dx}{a^2 + x^2} = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + C$
9. $\int \csc^2 x\,dx = -\cot x + C$	20. $\int \frac{dx}{x\sqrt{x^2 - a^2}} = \frac{1}{a}\sec^{-1}\left\lvert\frac{x}{a}\right\rvert + C$
10. $\int \sec x \tan x\,dx = \sec x + C$	21. $\int \frac{dx}{\sqrt{a^2 + x^2}} = \sinh^{-1}\left(\frac{x}{a}\right) + C$ $(a > 0)$
11. $\int \csc x \cot x\,dx = -\csc x + C$	22. $\int \frac{dx}{\sqrt{x^2 - a^2}} = \cosh^{-1}\left(\frac{x}{a}\right) + C$ $(x > a > 0)$

Example

Calculate $\displaystyle\int_3^5 \frac{2x-3}{\sqrt{x^2-3x+1}}\,dx.$

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We use the substitution $u = x^2 - 3x + 1$. Then $du = (2x-3)\,dx$ and

$$\begin{aligned} x = 3 &\implies u = 9 - 9 + 1 = 1 \\ x = 5 &\implies u = 25 - 15 + 1 = 11. \end{aligned}$$

Example

Calculate $\displaystyle\int_3^5 \frac{2x-3}{\sqrt{x^2-3x+1}}\,dx$.

We use the substitution $u = x^2 - 3x + 1$. Then $du = (2x-3)\,dx$ and

$$x = 3 \implies u = 9 - 9 + 1 = 1$$
$$x = 5 \implies u = 25 - 15 + 1 = 11.$$

Hence

$$\int_3^5 \frac{2x-3}{\sqrt{x^2-3x+1}}\,dx = \int_1^{11} u^{-\frac{1}{2}}\,du = \left[2\sqrt{u}\right]_1^{11} = 2\left(\sqrt{11}-1\right).$$

$$\boxed{\int \frac{dx}{\sqrt{a^2 - x^2}} = \sin^{-1}\left(\frac{u}{a}\right) + C}$$

Example

Find $\displaystyle\int \frac{dx}{\sqrt{8x - x^2}}$.

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This time we will complete the square of $x^2 - 8x$ and use that to simplify the integral:

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This time we will complete the square of $x^2 - 8x$ and use that to simplify the integral:

$x^2 - 8x = x^2 - 8x + 16 - 16$

$$\int \frac{dx}{\sqrt{a^2 - x^2}} = \sin^{-1}\left(\frac{u}{a}\right) + C$$

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This time we will complete the square of $x^2 - 8x$ and use that to simplify the integral:

$$x^2 - 8x = x^2 - 8x + 16 - 16 = (x^2 - 8x + 16) - 16 = (x - 4)^2 - 16.$$

$$\int \frac{dx}{\sqrt{a^2 - x^2}} = \sin^{-1}\left(\frac{u}{a}\right) + C$$

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So

$$\int \frac{dx}{\sqrt{8x - x^2}} = \int \frac{dx}{\sqrt{16 - (x - 4)^2}}$$

$$=$$

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$$\begin{aligned}\int \frac{dx}{\sqrt{8x - x^2}} &= \int \frac{dx}{\sqrt{16 - (x - 4)^2}} \\ &= \int \frac{du}{\sqrt{16 - u^2}} \\ &= \\ &= \end{aligned}$$

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$$\begin{aligned}
\int \frac{dx}{\sqrt{8x - x^2}} &= \int \frac{dx}{\sqrt{16 - (x-4)^2}} \\
&= \int \frac{du}{\sqrt{16 - u^2}} \\
&= \sin^{-1}\left(\frac{u}{4}\right) + C \\
&= \sin^{-1}\left(\frac{x-4}{4}\right) + C.
\end{aligned}$$

Example

Find $\displaystyle\int \cos x \sin 2x + \sin x \cos 2x \, dx$.

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$$\begin{aligned}
\int \cos x \sin 2x + \sin x \cos 2x \, dx &= \int \sin(x + 2x) \, dx \\
&= \int \sin 3x \, dx \\
&= \ldots
\end{aligned}$$

Example

Find $\displaystyle\int_0^{\frac{\pi}{4}} \frac{dx}{1-\sin x}$.

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Here is a trick for dealing with $\frac{1}{A-B}$: Multiply by $\frac{A+B}{A+B}$. Then we get

$$\frac{1}{A-B} = \left(\frac{1}{A-B}\right)\left(\frac{A+B}{A+B}\right) = \frac{A+B}{A^2-B^2}$$

which is sometimes easier to deal with.

$$
\begin{aligned}
\int_0^{\pi/4} \frac{dx}{1 - \sin x} &= \int_0^{\pi/4} \frac{1}{1 - \sin x} \cdot \frac{1 + \sin x}{1 + \sin x}\, dx && \text{Multiply and divide by conjugate.} \\
&= \int_0^{\pi/4} \frac{1 + \sin x}{1 - \sin^2 x}\, dx && \text{Simplify.} \\
&= \int_0^{\pi/4} \frac{1 + \sin x}{\cos^2 x}\, dx && 1 - \sin^2 x = \cos^2 x \\
&= \int_0^{\pi/4} (\sec^2 x + \sec x \tan x)\, dx && \text{Use Table 8.1, Formulas 8 and 10} \\
&= \Big[\tan x + \sec x\Big]_0^{\pi/4} = \left(1 + \sqrt{2} - (0 + 1)\right) = \sqrt{2}. && \blacksquare
\end{aligned}
$$

Example

Find $\displaystyle\int \frac{3x^2 - 7x}{3x + 2}\, dx$.

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Solution The integrand is an improper fraction since the degree of the numerator is greater than the degree of the denominator. To integrate it, we perform long division to obtain a quotient plus a remainder that is a proper fraction:

$$\frac{3x^2 - 7x}{3x + 2} = x - 3 + \frac{6}{3x + 2}.$$

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Therefore,

$$\int \frac{3x^2 - 7x}{3x + 2}\,dx = \int \left(x - 3 + \frac{6}{3x + 2} \right) dx = \frac{x^2}{2} - 3x + 2\ln|3x + 2| + C. \quad ■$$

$$\boxed{\int \frac{dx}{\sqrt{a^2 - x^2}} = \sin^{-1}\left(\frac{u}{a}\right) + C}$$

Example

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First note that

$$\int \frac{3x+2}{\sqrt{1-x^2}} = 3\int \frac{x\,dx}{\sqrt{1-x^2}} + 2\int \frac{dx}{\sqrt{1-x^2}}$$

$$\boxed{\int \frac{dx}{\sqrt{a^2 - x^2}} = \sin^{-1}\left(\frac{u}{a}\right) + C}$$

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$$\int \frac{dx}{\sqrt{a^2 - x^2}} = \sin^{-1}\left(\frac{u}{a}\right) + C$$

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Find $\displaystyle\int \frac{3x+2}{\sqrt{1-x^2}}$.

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$$\begin{aligned} \int \frac{3x+2}{\sqrt{1-x^2}} &= 3\int \frac{x\,dx}{\sqrt{1-x^2}} + 2\int \frac{dx}{\sqrt{1-x^2}} \\ &= 3\int \frac{x\,dx}{\sqrt{1-x^2}} + 2\sin^{-1} x. \end{aligned}$$

So we just need to calculate $\displaystyle\int \frac{x\,dx}{\sqrt{1-x^2}}$.

$$\int \frac{x\,dx}{\sqrt{1-x^2}}$$

Let $u = 1 - x^2$.

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Let $u = 1 - x^2$. Then $du = -2x\,dx$ and $-\frac{1}{2}\,du = x\,dx$.

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Let $u = 1 - x^2$. Then $du = -2x\,dx$ and $-\frac{1}{2}\,du = x\,dx$. It follows that

$$\int \frac{x\,dx}{\sqrt{1-x^2}} = \int \frac{-\frac{1}{2}\,du}{\sqrt{u}} = -\frac{1}{2}\int u^{-\frac{1}{2}}\,du = \ldots = -\sqrt{1-x^2} + C.$$

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Therefore

$$\begin{aligned}\int \frac{3x+2}{\sqrt{1-x^2}} &= 3\int \frac{x\,dx}{\sqrt{1-x^2}} + 2\int \frac{dx}{\sqrt{1-x^2}} \\ &= -3\sqrt{1-x^2} + 2\sin^{-1} x + C.\end{aligned}$$

Example

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First guess: $u = \sqrt{x}$. But then $du = \frac{1}{2\sqrt{x}}\,dx$ and we would have to deal with this extra $\sqrt{x} = u$ term.

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I want to make a substitution to make this integral easier, but what u should I choose?

First guess: $u = \sqrt{x}$. But then $du = \frac{1}{2\sqrt{x}}\,dx$ and we would have to deal with this extra $\sqrt{x} = u$ term.

Second guess: Instead let us try $u = 1 + \sqrt{x}$.

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First guess: $u = \sqrt{x}$. But then $du = \frac{1}{2\sqrt{x}}\,dx$ and we would have to deal with this extra $\sqrt{x} = u$ term.

Second guess: Instead let us try $u = 1 + \sqrt{x}$. Then again we have $du = \frac{1}{2\sqrt{x}}\,dx$ and $dx = 2\sqrt{x}\,du = 2(u-1)\,du$. Hence

$$\int \frac{dx}{(1+\sqrt{x})^3} = \int \frac{2(u-1)\,du}{u^3} = \int \frac{2}{u^2} - \frac{2}{u^3}\,du = \ldots$$

Example

Calculate $\displaystyle\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} x^3 \cos x \, dx.$

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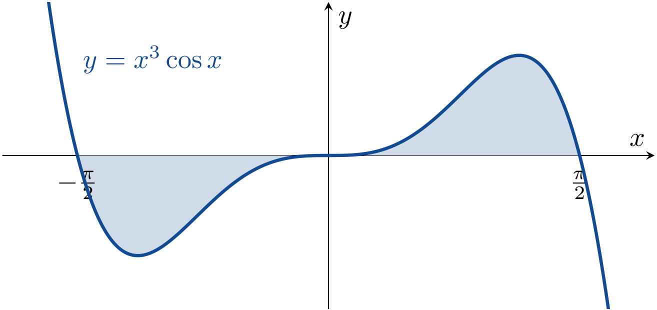

8.2 Integration by Parts

How can we calculate

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Theorem (Integration by Parts)

$$\int u(x) v'(x) \, dx =$$

How can we calculate

$$\int x \cos x \, dx$$

or

$$\int x^2 e^x \, dx \ ?$$

$$\int \text{function} \times \text{function} \, dx$$

Theorem (Integration by Parts)

$$\int u(x)v'(x) \, dx = u(x)v(x) - \int u'(x)v(x) \, dx$$

$$\int uv'\,dx = uv - \int u'v\,dx$$

Example

Find $\displaystyle\int x\cos x\,dx$.

$$\int uv'\,dx = uv - \int u'v\,dx$$

Example

Find $\int x \cos x \, dx$.

We need to choose a $u(x)$ and a $v'(x)$.

$$\boxed{\int uv'\,dx = uv - \int u'v\,dx}$$

Example

Find $\int x\cos x\,dx$.

We need to choose a $u(x)$ and a $v'(x)$.
Let

$$u = x \qquad\qquad v' = \cos x$$

Then

$$\int x\cos x\,dx = \qquad - \int \qquad dx = \qquad .$$

$$\int uv'\,dx = uv - \int u'v\,dx$$

Example

Find $\displaystyle\int x\cos x\,dx$.

We need to choose a $u(x)$ and a $v'(x)$.
Let

$$u = x \qquad\qquad v' = \cos x$$
$$u' = 1$$

Then

$$\int x\cos x\,dx = \qquad - \int \qquad dx = \qquad .$$

$$\boxed{\int uv'\,dx = uv - \int u'v\,dx}$$

Example

Find $\displaystyle\int x\cos x\,dx$.

We need to choose a $u(x)$ and a $v'(x)$.
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$$\begin{aligned} u &= x & v' &= \cos x \\ u' &= 1 & v &= \sin x. \end{aligned}$$

Then

$$\int x\cos x\,dx = \qquad - \int \qquad dx = \qquad .$$

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$$\int x \cos x \, dx = x \sin x - \int \qquad dx = \qquad .$$

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Let

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Then

$$\int x \cos x\,dx = x \sin x - \int 1 \sin x\,dx = \qquad .$$

$$\int uv'\,dx = uv - \int u'v\,dx$$

Example

Find $\displaystyle\int x\cos x\,dx$.

We need to choose a $u(x)$ and a $v'(x)$.
Let

$$\begin{aligned} u &= x & v' &= \cos x \\ u' &= 1 & v &= \sin x. \end{aligned}$$

Then

$$\int x\cos x\,dx = x\sin x - \int 1\sin x\,dx = x\sin x + \cos x + C.$$

$$\int uv' \, dx = uv - \int u'v \, dx$$

Example

Find $\displaystyle\int \ln x \, dx$.

$$\int uv'\,dx = uv - \int u'v\,dx$$

Example

Find $\displaystyle\int \ln x\,dx$.

We will consider $\displaystyle\int \ln x \cdot 1\,dx$.

$$\int uv'\,dx = uv - \int u'v\,dx$$

Example

Find $\displaystyle\int \ln x\,dx$.

We will consider $\displaystyle\int \ln x \cdot 1\,dx$.

Let

$$u = \ln x \qquad\qquad v' = 1$$

$$\int uv'\,dx = uv - \int u'v\,dx$$

Example

Find $\int \ln x\,dx$.

We will consider $\int \ln x \cdot 1\,dx$.

Let

$$u = \ln x \qquad\qquad v' = 1$$
$$u' = \frac{1}{x}$$

$$\int uv'\,dx = uv - \int u'v\,dx$$

Example

Find $\displaystyle\int \ln x\,dx$.

We will consider $\displaystyle\int \ln x \cdot 1\,dx$.

Let

$$\begin{aligned} u &= \ln x & v' &= 1 \\ u' &= \frac{1}{x} & v &= x. \end{aligned}$$

$$\int uv'\,dx = uv - \int u'v\,dx$$

Example

Find $\displaystyle\int \ln x\,dx$.

We will consider $\displaystyle\int \ln x \cdot 1\,dx$.

Let

$$\begin{aligned} u &= \ln x & v' &= 1 \\ u' &= \frac{1}{x} & v &= x. \end{aligned}$$

Then

$$\begin{aligned} \int \ln x \cdot 1\,dx &= \ln x \cdot x - \int \frac{1}{x} \cdot x\,dx \\ &= \\ &= \qquad\qquad . \end{aligned}$$

$$\int uv'\,dx = uv - \int u'v\,dx$$

Example

Find $\displaystyle\int \ln x\,dx$.

We will consider $\displaystyle\int \ln x \cdot 1\,dx$.

Let

$$u = \ln x \qquad\qquad v' = 1$$

$$u' = \frac{1}{x} \qquad\qquad v = x.$$

Then

$$\begin{aligned}\int \ln x \cdot 1\,dx &= \ln x \cdot x - \int \frac{1}{x} \cdot x\,dx \\ &= x \ln x - \int 1\,dx \\ &= \qquad\qquad\qquad .\end{aligned}$$

$$\int uv'\,dx = uv - \int u'v\,dx$$

Example

Find $\displaystyle\int \ln x\,dx$.

We will consider $\displaystyle\int \ln x \cdot 1\,dx$.

Let

$$u = \ln x \qquad\qquad v' = 1$$
$$u' = \frac{1}{x} \qquad\qquad v = x.$$

Then

$$\begin{aligned}\int \ln x \cdot 1\,dx &= \ln x \cdot x - \int \frac{1}{x} \cdot x\,dx \\ &= x\ln x - \int 1\,dx \\ &= x\ln x - x + C.\end{aligned}$$

$$\int uv'\,dx = uv - \int u'v\,dx$$

Sometimes we have to use integration by parts more than once.

$$\boxed{\int uv'\,dx = uv - \int u'v\,dx}$$

Example

Find $\displaystyle\int x^2 e^x\,dx$.

$$\int uv'\,dx = uv - \int u'v\,dx$$

Example

Find $\displaystyle\int x^2 e^x\,dx$.

We calculate that

$$\int x^2 e^x\,dx = x^2 e^x - 2\int x e^x\,dx.$$

$$\int uv'\,dx = uv - \int u'v\,dx$$

Example

Find $\displaystyle\int x^2 e^x\,dx$.

We calculate that

$$\int x^2 e^x\,dx = x^2 e^x - 2\int x e^x\,dx.$$

But what do we do with $\int x e^x\,dx$?

$$\int uv'\,dx = uv - \int u'v\,dx$$

Example

Find $\displaystyle\int x^2 e^x\,dx$.

We calculate that

$$\int x^2 e^x\,dx = x^2 e^x - 2\int x e^x\,dx.$$

But what do we do with $\int x e^x\,dx$?

$$\int x e^x\,dx = x e^x - \int 1 e^x\,dx = x e^x - e^x + C.$$

$$\int uv'\,dx = uv - \int u'v\,dx$$

Example

Find $\int x^2 e^x\,dx$.

We calculate that

$$\int x^2 e^x\,dx = x^2 e^x - 2\int x e^x\,dx.$$

But what do we do with $\int x e^x\,dx$?

$$\int x e^x\,dx = x e^x - \int 1 e^x\,dx = x e^x - e^x + C.$$

Putting it all together, we have

$$\int x^2 e^x\,dx = x^2 e^x - 2\int x e^x\,dx = x^2 e^x - 2x e^x + 2e^x + C.$$

$$\int uv' \, dx = uv - \int u'v \, dx$$

Remark

We can use the same technique to calculate $\int x^n e^x \, dx$.

We would have to do integration by parts n times.

$$\boxed{\int uv'\,dx = uv - \int u'v\,dx}$$

Theorem

$$\int u\,dv = uv - \int v\,du$$

EXAMPLE 4 Evaluate

$$\int e^x \cos x \, dx.$$

Solution Let $u = e^x$ and $dv = \cos x \, dx$. Then $du = e^x \, dx$, $v = \sin x$, and

$$\int e^x \cos x \, dx = e^x \sin x - \int e^x \sin x \, dx.$$

The second integral is like the first except that it has $\sin x$ in place of $\cos x$. To evaluate it, we use integration by parts with

$$u = e^x, \qquad dv = \sin x \, dx, \qquad v = -\cos x, \qquad du = e^x \, dx.$$

Then

$$\begin{aligned} \int e^x \cos x \, dx &= e^x \sin x - \left(-e^x \cos x - \int (-\cos x)(e^x \, dx) \right) \\ &= e^x \sin x + e^x \cos x - \int e^x \cos x \, dx. \end{aligned}$$

The unknown integral now appears on both sides of the equation. Adding the integral to both sides and adding the constant of integration give

$$2\int e^x \cos x \, dx = e^x \sin x + e^x \cos x + C_1.$$

Dividing by 2 and renaming the constant of integration give

$$\int e^x \cos x \, dx = \frac{e^x \sin x + e^x \cos x}{2} + C. \qquad \blacksquare$$

EXAMPLE 5 Obtain a formula that expresses the integral

$$\int \cos^n x \, dx$$

in terms of an integral of a lower power of $\cos x$.

Solution We may think of $\cos^n x$ as $\cos^{n-1} x \cdot \cos x$. Then we let

$$u = \cos^{n-1} x \qquad \text{and} \qquad dv = \cos x \, dx,$$

so that

$$du = (n-1)\cos^{n-2} x\,(-\sin x \, dx) \qquad \text{and} \qquad v = \sin x.$$

Integration by parts then gives

$$\begin{aligned}
\int \cos^n x \, dx &= \cos^{n-1} x \sin x + (n-1)\int \sin^2 x \cos^{n-2} x \, dx \\
&= \cos^{n-1} x \sin x + (n-1)\int (1-\cos^2 x)\cos^{n-2} x \, dx \\
&= \cos^{n-1} x \sin x + (n-1)\int \cos^{n-2} x \, dx - (n-1)\int \cos^n x \, dx.
\end{aligned}$$

If we add

$$(n-1)\int \cos^n x\,dx$$

to both sides of this equation, we obtain

$$n\int \cos^n x\,dx = \cos^{n-1} x \sin x + (n-1)\int \cos^{n-2} x\,dx.$$

We then divide through by n, and the final result is

$$\int \cos^n x\,dx = \frac{\cos^{n-1} x \sin x}{n} + \frac{n-1}{n}\int \cos^{n-2} x\,dx. \qquad \blacksquare$$

The formula found in Example 5 is called a **reduction formula** because it replaces an integral containing some power of a function with an integral of the same form having the power reduced. When n is a positive integer, we may apply the formula repeatedly until the remaining integral is easy to evaluate. For example, the result in Example 5 tells us that

$$\begin{aligned}\int \cos^3 x\,dx &= \frac{\cos^2 x \sin x}{3} + \frac{2}{3}\int \cos x\,dx \\ &= \frac{1}{3}\cos^2 x \sin x + \frac{2}{3}\sin x + C.\end{aligned}$$

$$\int uv' \, dx = uv - \int u'v \, dx$$

Theorem

$$\int_a^b uv' \, dx =$$

$$\boxed{\int uv'\,dx = uv - \int u'v\,dx}$$

Theorem

$$\int_a^b uv'\,dx = \Big[uv\Big]_a^b - \int_a^b u'v\,dx$$

$$\int_a^b uv'\,dx = \Big[uv\Big]_a^b - \int_a^b u'v\,dx$$

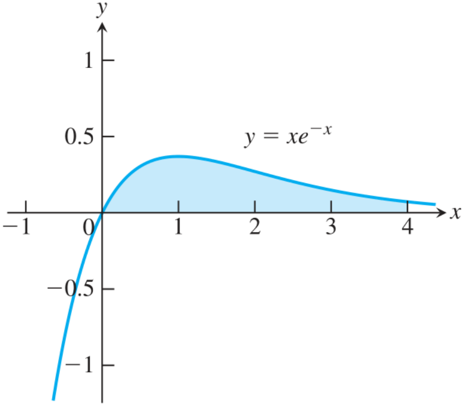
Example

Calculate the area of the region bounded by the curve $y = xe^{-x}$ and the x-axis from $x = 0$ to $x = 4$.

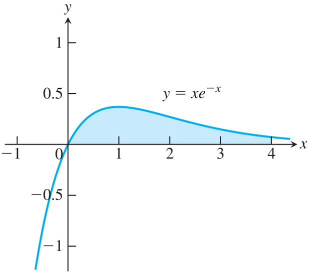
$$\int_a^b uv'\,dx = \Big[uv\Big]_a^b - \int_a^b u'v\,dx$$

Example

Calculate the area of the region bounded by the curve $y = xe^{-x}$ and the x-axis from $x = 0$ to $x = 4$.



$$\int_a^b uv'\,dx = \Big[uv\Big]_a^b - \int_a^b u'v\,dx$$



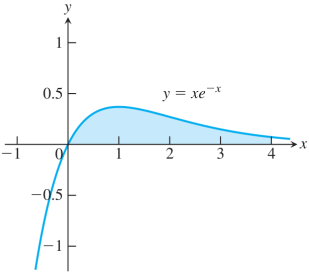
We calculate that

$$\int_0^4 xe^{-x}\,dx =$$

$$=$$

$$= \qquad .$$

$$\int_a^b uv'\,dx = \Big[uv\Big]_a^b - \int_a^b u'v\,dx$$



$$u = x$$
$$u' = 1$$
$$v' = e^{-x}$$
$$v = -e^{-x}$$

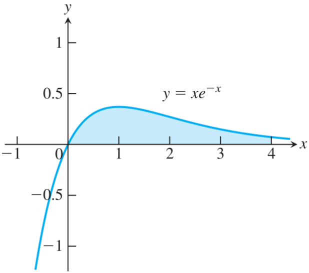
We calculate that

$$\int_0^4 xe^{-x}\,dx =$$

$$=$$

$$= \qquad .$$

$$\int_a^b uv'\,dx = \Big[uv\Big]_a^b - \int_a^b u'v\,dx$$



$u = x$

$u' = 1$

$v' = e^{-x}$

$v = -e^{-x}$

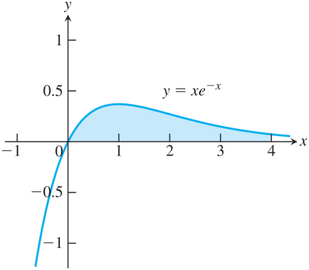
We calculate that

$$\int_0^4 xe^{-x}\,dx = \Big[-xe^{-x}\Big]_0^4 - \int_0^4 1(-e^{-x})\,dx$$

$$=$$

$$= \qquad .$$

$$\int_a^b uv'\,dx = \Big[uv\Big]_a^b - \int_a^b u'v\,dx$$



$$u = x$$
$$u' = 1$$
$$v' = e^{-x}$$
$$v = -e^{-x}$$

We calculate that

$$\begin{aligned}\int_0^4 xe^{-x}\,dx &= \Big[-xe^{-x}\Big]_0^4 - \int_0^4 1(-e^{-x})\,dx \\ &= \left(-4e^{-4}+0\right) + \left[-e^{-x}\right]_0^4 \\ &= -4e^{-4} + \left(-e^{-4}+1\right) = 1 - 5e^{-4}.\end{aligned}$$

$$\int_a^b uv'\,dx = \Big[uv\Big]_a^b - \int_a^b u'v\,dx$$

Example

Find $\displaystyle\int_0^1 \sin^{-1} x\,dx$.

$$\int_a^b uv'\,dx = \Big[uv\Big]_a^b - \int_a^b u'v\,dx$$

Example

Find $\displaystyle\int_0^1 \sin^{-1} x\,dx$.

Recall that $\frac{d}{dx}\sin^{-1} x = \frac{1}{\sqrt{1-x^2}}$.

$$\int_a^b uv'\,dx = \Big[uv\Big]_a^b - \int_a^b u'v\,dx$$

Example

Find $\displaystyle\int_0^1 \sin^{-1} x\,dx$.

Recall that $\frac{d}{dx}\sin^{-1} x = \frac{1}{\sqrt{1-x^2}}$.

Let $u = \sin^{-1} x$ and $v' = 1$.

$$\int_a^b uv'\,dx = \Big[uv\Big]_a^b - \int_a^b u'v\,dx$$

Example

Find $\displaystyle\int_0^1 \sin^{-1} x\,dx$.

Recall that $\frac{d}{dx}\sin^{-1} x = \frac{1}{\sqrt{1-x^2}}$.

Let $u = \sin^{-1} x$ and $v' = 1$. Then $u' = \frac{1}{\sqrt{1-x^2}}$ and $v = x$.

$$\int_a^b uv'\,dx = \Big[uv\Big]_a^b - \int_a^b u'v\,dx$$

Example

Find $\displaystyle\int_0^1 \sin^{-1} x\,dx$.

Recall that $\frac{d}{dx}\sin^{-1} x = \frac{1}{\sqrt{1-x^2}}$.

Let $u = \sin^{-1} x$ and $v' = 1$. Then $u' = \frac{1}{\sqrt{1-x^2}}$ and $v = x$. It follows that

$$\begin{aligned} \int_0^1 \sin^{-1} x \cdot 1\,dx &= \big[x\sin^{-1} x\big]_0^1 - \int_0^1 \frac{x}{\sqrt{1-x^2}}dx \\ &= \\ &= \\ &= \quad . \end{aligned}$$

$$\int_a^b uv'\,dx = \Big[uv\Big]_a^b - \int_a^b u'v\,dx$$

Example

Find $\displaystyle\int_0^1 \sin^{-1} x\,dx$.

Recall that $\frac{d}{dx}\sin^{-1} x = \frac{1}{\sqrt{1-x^2}}$.

Let $u = \sin^{-1} x$ and $v' = 1$. Then $u' = \frac{1}{\sqrt{1-x^2}}$ and $v = x$. It follows that

$$\begin{aligned}
\int_0^1 \sin^{-1} x \cdot 1\,dx &= \big[x\sin^{-1} x\big]_0^1 - \int_0^1 \frac{x}{\sqrt{1-x^2}}dx \\
&= \big[x\sin^{-1} x\big]_0^1 - \Big[-\sqrt{1-x^2}\Big]_0^1 \\
&= \left(\frac{\pi}{2} - 0\right) - (-0+1) \\
&= \frac{\pi}{2} - 1.
\end{aligned}$$

Break

We will continue at 2pm

8.3 Trigonometric Integrals

$$
\begin{aligned}
\cos\theta &= \sin\left(\frac{\pi}{2} - \theta\right) \\
\cos^2\theta + \sin^2\theta &= 1 \\
1 + \tan^2\theta &= \sec^2\theta \\
1 + \cot^2\theta &= \operatorname{cosec}^2\theta \\
\cos(A+B) &= \cos A\cos B - \sin A\sin B \\
\sin(A+B) &= \sin A\cos B + \cos A\sin B \\
\cos 2\theta &= \cos^2\theta - \sin^2\theta \\
\sin 2\theta &= 2\sin\theta\cos\theta \\
\cos^2\theta &= \frac{1}{2}(1+\cos 2\theta) \\
\sin^2\theta &= \frac{1}{2}(1-\cos 2\theta)
\end{aligned}
$$

$$\begin{aligned}
\cos\theta &= \sin\left(\frac{\pi}{2}-\theta\right) \\
\cos^2\theta + \sin^2\theta &= 1 \\
1+\tan^2\theta &= \sec^2\theta \\
1+\cot^2\theta &= \operatorname{cosec}^2\theta \\
\cos(A+B) &= \cos A\cos B - \sin A\sin B \\
\sin(A+B) &= \sin A\cos B + \cos A\sin B \\
\cos 2\theta &= \cos^2\theta - \sin^2\theta \\
\sin 2\theta &= 2\sin\theta\cos\theta \\
\cos^2\theta &= \frac{1}{2}(1+\cos 2\theta) \\
\sin^2\theta &= \frac{1}{2}(1-\cos 2\theta)
\end{aligned}$$

$$\cos\theta = \sin\left(\frac{\pi}{2} - \theta\right)$$

$$\cos^2\theta + \sin^2\theta = 1$$

$$1 + \tan^2\theta = \sec^2\theta$$

$$1 + \cot^2\theta = \operatorname{cosec}^2\theta$$

$$\cos(A+B) = \cos A\cos B - \sin A\sin B$$

$$\sin(A+B) = \sin A\cos B + \cos A\sin B$$

$$\cos 2\theta = \cos^2\theta - \sin^2\theta$$

$$\sin 2\theta = 2\sin\theta\cos\theta$$

$$\cos^2\theta = \frac{1}{2}(1+\cos 2\theta)$$

$$\sin^2\theta = \frac{1}{2}(1-\cos 2\theta)$$

$$\cos\theta = \sin\left(\frac{\pi}{2} - \theta\right)$$

$$\cos^2\theta + \sin^2\theta = 1$$

$$1 + \tan^2\theta = \sec^2\theta$$

$$1 + \cot^2\theta = \operatorname{cosec}^2\theta$$

$$\cos(A+B) = \cos A\cos B - \sin A\sin B$$

$$\sin(A+B) = \sin A\cos B + \cos A\sin B$$

$$\cos 2\theta = \cos^2\theta - \sin^2\theta$$

$$\sin 2\theta = 2\sin\theta\cos\theta$$

$$\cos^2\theta = \frac{1}{2}(1 + \cos 2\theta)$$

$$\sin^2\theta = \frac{1}{2}(1 - \cos 2\theta)$$

$$
\begin{aligned}
\cos\theta &= \sin\left(\frac{\pi}{2}-\theta\right)\\
\cos^2\theta+\sin^2\theta &= 1\\
1+\tan^2\theta &= \sec^2\theta\\
1+\cot^2\theta &= \operatorname{cosec}^2\theta\\
\cos(A+B) &= \cos A\cos B-\sin A\sin B\\
\sin(A+B) &= \sin A\cos B+\cos A\sin B\\
\cos 2\theta &= \cos^2\theta-\sin^2\theta\\
\sin 2\theta &= 2\sin\theta\cos\theta\\
\cos^2\theta &= \frac{1}{2}(1+\cos 2\theta)\\
\sin^2\theta &= \frac{1}{2}(1-\cos 2\theta)
\end{aligned}
$$

$$\cos\theta = \sin\left(\frac{\pi}{2} - \theta\right)$$

$$\cos^2\theta + \sin^2\theta = 1$$

$$1 + \tan^2\theta = \sec^2\theta$$

$$1 + \cot^2\theta = \operatorname{cosec}^2\theta$$

$$\cos(A+B) = \cos A\cos B - \sin A\sin B$$

$$\sin(A+B) = \sin A\cos B + \cos A\sin B$$

$$\cos 2\theta = \cos^2\theta - \sin^2\theta$$

$$\sin 2\theta = 2\sin\theta\cos\theta$$

$$\cos^2\theta = \frac{1}{2}(1 + \cos 2\theta)$$

$$\sin^2\theta = \frac{1}{2}(1 - \cos 2\theta)$$

How can we find

$$\int \sin^m x \, \cos^n x \, dx$$

if $m, n \in \{0, 1, 2, 3, 4, 5, \dots\}$?

$$\int \sin^m x \cos^n x \, dx$$

We need to look at 3 different cases:

$$\int \sin^m x \cos^n x \, dx$$

We need to look at 3 different cases:

1. m **is odd:**

$$\int \sin^m x \cos^n x \, dx$$

We need to look at 3 different cases:

1. **m is odd:**

2. **m is even and n is odd:**

$$\int \sin^m x \cos^n x \, dx$$

We need to look at 3 different cases:

1. **m is odd:**

2. **m is even and n is odd:**

3. **both m and n are even:**

$$\int \sin^m x \cos^n x \, dx \qquad\qquad \cos^2 x + \sin^2 x = 1$$

We need to look at 3 different cases:

1. m **is odd:** Write $m = 2k+1$ and use

   $$\sin^m x = \sin^{2k+1} x = \left(\sin^2 x\right)^k \sin x = \left(1 - \cos^2 x\right)^k \sin x$$

   and the substitution $u = \cos x$.

2. m **is even and** n **is odd:**

3. **both** m **and** n **are even:**

$$\int \sin^m x \cos^n x \, dx \qquad\qquad \cos^2 x + \sin^2 x = 1$$

We need to look at 3 different cases:

1. **m is odd:** Write $m = 2k + 1$ and use
$$\sin^m x = \sin^{2k+1} x = \left(\sin^2 x\right)^k \sin x = \left(1 - \cos^2 x\right)^k \sin x$$
and the substitution $u = \cos x$.
2. **m is even and n is odd:** Write $n = 2k + 1$ use
$$\cos^n x = \cos^{2k+1} x = \left(\cos^2 x\right)^k \cos x = \left(1 - \sin^2 x\right)^k \cos x$$
and the substitution $u = \sin x$.
3. **both m and n are even:**

$$\int \sin^m x \cos^n x \, dx \qquad\qquad \cos^2 x + \sin^2 x = 1$$

We need to look at 3 different cases:

1. **m is odd:** Write $m = 2k+1$ and use

$$\sin^m x = \sin^{2k+1} x = \left(\sin^2 x\right)^k \sin x = \left(1 - \cos^2 x\right)^k \sin x$$

and the substitution $u = \cos x$.

2. **m is even and n is odd:** Write $n = 2k+1$ use

$$\cos^n x = \cos^{2k+1} x = \left(\cos^2 x\right)^k \cos x = \left(1 - \sin^2 x\right)^k \cos x$$

and the substitution $u = \sin x$.

3. **both m and n are even:** Use

$$\sin^2 x = \frac{1 - \cos 2x}{2} \qquad \text{and} \qquad \cos^2 x = \frac{1 + \cos 2x}{2}.$$

Example

Find $\displaystyle\int \sin^3 x \cos^2 x \, dx$.

Example

Find $\int \sin^3 x \cos^2 x \, dx$.

Solution This is an example of Case 1.

$$\begin{aligned}
\int \sin^3 x \cos^2 x \, dx &= \int \sin^2 x \cos^2 x \sin x \, dx && m \text{ is odd.} \\
&= \int (1 - \cos^2 x)(\cos^2 x)(-d(\cos x)) && \sin x \, dx = -d(\cos x) \\
&= \int (1 - u^2)(u^2)(-du) && u = \cos x \\
&= \int (u^4 - u^2) \, du && \text{Multiply terms.} \\
&= \frac{u^5}{5} - \frac{u^3}{3} + C = \frac{\cos^5 x}{5} - \frac{\cos^3 x}{3} + C
\end{aligned}$$

Example

Find $\displaystyle\int \cos^5 x\, dx$.

Example

Find $\int \cos^5 x\,dx$.

Solution This is an example of Case 2, where $m = 0$ is even and $n = 5$ is odd.

$$\int \cos^5 x\,dx = \int \cos^4 x \cos x\,dx = \int (1 - \sin^2 x)^2\,d(\sin x) \qquad \cos x\,dx = d\,(\sin x)$$

$$= \int (1 - u^2)^2\,du \qquad u = \sin x$$

$$= \int (1 - 2u^2 + u^4)\,du \qquad \text{Square } 1 - u^2.$$

$$= u - \frac{2}{3}u^3 + \frac{1}{5}u^5 + C = \sin x - \frac{2}{3}\sin^3 x + \frac{1}{5}\sin^5 x + C$$

Example

Find $\displaystyle\int \sin^2 x \cos^4 x \, dx$.

Example

Find $\displaystyle\int \sin^2 x \cos^4 x \, dx$.

Solution This is an example of Case 3.

$$\int \sin^2 x \cos^4 x \, dx = \int \left(\frac{1 - \cos 2x}{2}\right)\left(\frac{1 + \cos 2x}{2}\right)^2 dx \qquad m \text{ and } n \text{ both even}$$

$$= \frac{1}{8}\int (1 - \cos 2x)(1 + 2\cos 2x + \cos^2 2x)\, dx$$

$$= \frac{1}{8}\int (1 + \cos 2x - \cos^2 2x - \cos^3 2x)\, dx$$

$$= \frac{1}{8}\left[x + \frac{1}{2}\sin 2x - \int (\cos^2 2x + \cos^3 2x)\, dx\right]$$

For the term involving $\cos^2 2x$, we use

$$\int \cos^2 2x \, dx = \frac{1}{2}\int (1 + \cos 4x)\, dx$$

$$= \frac{1}{2}\left(x + \frac{1}{4}\sin 4x\right).$$

Omit constant of integration until final result.

For the $\cos^3 2x$ term, we have

$$\int \cos^3 2x\,dx = \int (1 - \sin^2 2x)\cos 2x\,dx \qquad u = \sin 2x,\ du = 2\cos 2x\,dx$$

$$= \frac{1}{2}\int (1 - u^2)\,du = \frac{1}{2}\left(\sin 2x - \frac{1}{3}\sin^3 2x\right). \qquad \text{Again omit } C.$$

Combining everything and simplifying, we get

$$\int \sin^2 x \cos^4 x\,dx = \frac{1}{16}\left(x - \frac{1}{4}\sin 4x + \frac{1}{3}\sin^3 2x\right) + C.$$

Example

Find $\int_0^{\frac{\pi}{4}} \sqrt{1+\cos 4x}\,dx$.

Example

Find $\displaystyle\int_0^{\frac{\pi}{4}} \sqrt{1+\cos 4x}\,dx.$

Solution To eliminate the square root, we use the identity

$$\cos^2\theta = \frac{1+\cos 2\theta}{2} \qquad \text{or} \qquad 1+\cos 2\theta = 2\cos^2\theta.$$

With $\theta = 2x$, this becomes

$$1+\cos 4x = 2\cos^2 2x.$$

Therefore,

$$\int_0^{\pi/4} \sqrt{1+\cos 4x}\,dx = \int_0^{\pi/4} \sqrt{2\cos^2 2x}\,dx = \int_0^{\pi/4} \sqrt{2}\sqrt{\cos^2 2x}\,dx$$

$$= \sqrt{2}\int_0^{\pi/4} |\cos 2x|\,dx = \sqrt{2}\int_0^{\pi/4} \cos 2x\,dx \qquad \cos 2x \geq 0 \text{ on } [0, \pi/4]$$

$$= \sqrt{2}\left[\frac{\sin 2x}{2}\right]_0^{\pi/4} = \frac{\sqrt{2}}{2}[1-0] = \frac{\sqrt{2}}{2}.$$

$$\sec^2 x = 1 + \tan^2 x \qquad \frac{d}{dx}\tan x = \sec^2 x$$

Example

Find $\displaystyle\int \tan^4 x\,dx$.

$$\sec^2 x = 1 + \tan^2 x \qquad \frac{d}{dx}\tan x = \sec^2 x$$

Example

Find $\displaystyle\int \tan^4 x\,dx$.

Solution

$$\begin{aligned}
\int \tan^4 x\,dx &= \int \tan^2 x \cdot \tan^2 x\,dx = \int \tan^2 x \cdot (\sec^2 x - 1)\,dx \\
&= \int \tan^2 x \sec^2 x\,dx - \int \tan^2 x\,dx \\
&= \int \tan^2 x \sec^2 x\,dx - \int (\sec^2 x - 1)\,dx \\
&= \int \tan^2 x \sec^2 x\,dx - \int \sec^2 x\,dx + \int dx
\end{aligned}$$

In the first integral, we let

$$u = \tan x, \qquad du = \sec^2 x\, dx$$

and have

$$\int u^2\, du = \frac{1}{3} u^3 + C_1.$$

The remaining integrals are standard forms, so

$$\int \tan^4 x\, dx = \frac{1}{3} \tan^3 x - \tan x + x + C. \quad \blacksquare$$

Example

Find $\int \sec^3 x \, dx$.

Solution We integrate by parts using

$$u = \sec x, \qquad dv = \sec^2 x\, dx, \qquad v = \tan x, \qquad du = \sec x \tan x\, dx.$$

Then

$$\begin{aligned} \int \sec^3 x\, dx &= \sec x \tan x - \int (\tan x)(\sec x \tan x\, dx) \\ &= \sec x \tan x - \int (\sec^2 x - 1) \sec x\, dx \qquad \tan^2 x = \sec^2 x - 1 \\ &= \sec x \tan x + \int \sec x\, dx - \int \sec^3 x\, dx. \end{aligned}$$

Combining the two secant-cubed integrals gives

$$2\int \sec^3 x\, dx = \sec x \tan x + \int \sec x\, dx$$

and

$$\int \sec^3 x\, dx = \frac{1}{2} \sec x \tan x + \frac{1}{2} \ln |\sec x + \tan x| + C.$$

■

$$\sec^2 x = 1 + \tan^2 x \qquad \frac{d}{dx}\tan x = \sec^2 x$$

Example

Find $\displaystyle\int \tan^4 x \sec^4 x\, dx$.

$$\sec^2 x = 1 + \tan^2 x \qquad \frac{d}{dx}\tan x = \sec^2 x$$

Example

Find $\displaystyle\int \tan^4 x \sec^4 x \, dx$.

Solution

$$\begin{aligned}
\int (\tan^4 x)(\sec^4 x)\, dx &= \int (\tan^4 x)(1 + \tan^2 x)(\sec^2 x)\, dx && \sec^2 x = 1 + \tan^2 x \\
&= \int (\tan^4 x + \tan^6 x)(\sec^2 x)\, dx \\
&= \int (\tan^4 x)(\sec^2 x)\, dx + \int (\tan^6 x)(\sec^2 x)\, dx \\
&= \int u^4\, du + \int u^6\, du = \frac{u^5}{5} + \frac{u^7}{7} + C && u = \tan x,\ du = \sec^2 x\, dx \\
&= \frac{\tan^5 x}{5} + \frac{\tan^7 x}{7} + C && \blacksquare
\end{aligned}$$

How do we calculate

$$\int \sin mx \sin nx \, dx$$

or

$$\int \sin mx \cos nx \, dx$$

or

$$\int \cos mx \cos nx \, dx$$

?

How do we calculate

$$\int \sin mx \sin nx \, dx$$

or

$$\int \sin mx \cos nx \, dx$$

or

$$\int \cos mx \cos nx \, dx$$

?

It is possible to use integration by parts (twice), but there is an easier way.

$$\cos(A+B) = \cos A \cos B - \sin A \sin B$$

$$\cos(mx - nx) - \cos(mx + nx) =$$

$$\cos(A+B) = \cos A \cos B - \sin A \sin B$$

$$\begin{aligned}\cos(mx-nx) - \cos(mx+nx) = {} & \cos mx \cos nx + \sin mx \sin nx \\ & - \cos mx \cos nx + \sin mx \sin nx\end{aligned}$$

$$\cos(A+B) = \cos A\cos B - \sin A\sin B$$

$$\begin{aligned}\cos(mx-nx) - \cos(mx+nx) &= \cos mx\cos nx + \sin mx\sin nx\\ &\quad - \cos mx\cos nx + \sin mx\sin nx\\ &= 2\sin mx\sin nx\end{aligned}$$

$$\cos(A+B) = \cos A \cos B - \sin A \sin B$$

$$\begin{aligned}\cos(mx-nx) - \cos(mx+nx) &= \cos mx \cos nx + \sin mx \sin nx \\ &\quad - \cos mx \cos nx + \sin mx \sin nx \\ &= 2 \sin mx \sin nx\end{aligned}$$

Therefore

$$\sin mx \sin nx = \frac{1}{2}\big(\cos(m-n)x - \cos(m+n)x\big).$$

$$\cos(A+B) = \cos A \cos B - \sin A \sin B$$

$$\begin{aligned} \cos(mx - nx) - \cos(mx + nx) &= \cos mx \cos nx + \sin mx \sin nx \\ &\quad - \cos mx \cos nx + \sin mx \sin nx \\ &= 2 \sin mx \sin nx \end{aligned}$$

Therefore

$$\sin mx \sin nx = \frac{1}{2}\big(\cos(m-n)x - \cos(m+n)x\big).$$

Similarly

$$\sin mx \cos nx = \frac{1}{2}\big(\sin(m-n)x + \sin(m+n)x\big)$$

and

$$\cos mx \cos nx = \frac{1}{2}\big(\cos(m-n)x + \cos(m+n)x\big).$$

$$\sin mx \cos nx = \frac{1}{2}\big(\sin(m-n)x + \sin(m+n)x\big)$$

Example

Find $\displaystyle\int \sin 3x \cos 5x \, dx$.

$$\sin mx \cos nx = \tfrac{1}{2}\big(\sin(m-n)x + \sin(m+n)x\big)$$

Example

Find $\displaystyle\int \sin 3x \cos 5x \, dx$.

Solution From Equation (4) with $m = 3$ and $n = 5$, we get

$$\begin{aligned}\int \sin 3x \cos 5x \, dx &= \frac{1}{2}\int [\sin(-2x) + \sin 8x]\, dx \\ &= \frac{1}{2}\int (\sin 8x - \sin 2x)\, dx \\ &= -\frac{\cos 8x}{16} + \frac{\cos 2x}{4} + C.\end{aligned}$$

$$\cos mx \cos nx = \frac{1}{2}\big(\cos(m-n)x + \cos(m+n)x\big)$$

Example

Find $\displaystyle\int \cos 3x \cos 2x \, dx$.

$$\cos mx \cos nx = \frac{1}{2}\big(\cos(m-n)x + \cos(m+n)x\big)$$

Example

Find $\displaystyle\int \cos 3x \cos 2x \, dx$.

We have $m = 3$ and $n = 2$. It follows that

$$\begin{aligned}\int \cos 3x \cos 2x \, dx &= \frac{1}{2}\int \cos(3-2)x \, dx + \frac{1}{2}\int \cos(3+2)x \, dx \\ &= \dots\end{aligned}$$

Next Time

- 8.4 Trigonometric Substitutions
- 8.5 Integration of Rational Functions by Partial Fractions
- 8.8 Improper Integrals