2014.05.29 **MAT234 Matematik IV – Final Sınavın Çözümleri** N. Course

Soru 1 (Sequences). Define a sequence of real numbers (a_n) by

$$a_1 = 11 \qquad \text{and} \qquad 20a_{n+1} = a_n^2 + 91.$$

(a) [7p] Show that $7 \leq a_n \leq 12$ for all $n \in \mathbb{N}$.
[HINT: Use proof by induction.].

> Since $7 \leq a_1 = 11 \leq 12$, the statement is true for $n = 1$ $\boxed{2}$.
>
> Suppose that it is true for $n = k$. Then $7 \leq a_k \leq 12$ $\boxed{1}$. So $20a_{k+1} = a_k^2 + 91 \leq 12^2 + 91 = 235 < 240 \implies a_{k+1} \leq 12$ $\boxed{1}$ and $20a_{k+1} = a_k^2 + 91 \geq 7^2 + 91 = 140 \implies a_{k+1} \geq 7$ $\boxed{1}$.
>
> By the principle of mathematical induction $\boxed{2}$, it follows that $7 \leq a_n \leq 12$ $\forall n \in \mathbb{N}$.

(b) [6p] Is (a_n) an increasing sequence? Is (a_n) a decreasing sequence? Prove your answer.

> First note that
>
> $$a_{n+1} - a_n = \frac{1}{20}(a_n^2 + 91) - a_n = \frac{1}{20}(a_n^2 - 20a_n + 91) = \frac{1}{20}(a_n - 7)(a_n - 13). \qquad \boxed{2}$$
>
> Since $7 \leq a_n \leq 12$, $(a_n - 7) \geq 0$ and $(a_n - 13) < 0$ $\boxed{2}$. Therefore $a_{n+1} - a_n = \frac{1}{20}(a_n - 7)(a_n - 13) < 0$. So $a_{n+1} < a_n$ $\forall n \in \mathbb{N}$. Therefore (a_n) is a decreasing sequence $\boxed{2}$. ((a_n) is not an increasing sequence.)

(c) [6p] Show that (a_n) is a convergent sequence.

> By a theorem from the course, "every decreasing sequence which is bounded below is convergent". In part (a), I proved that (a_n) is bounded below. In part (b), I proved that (a_n) is decreasing. Therefore (a_n) is convergent.

(d) [6p] Calculate $\lim_{n\to\infty} a_n$.

> Let $a = \lim_{n\to\infty} a_n$. Then
>
> $$20a \leftarrow 20a_{n+1} = a_n^2 + 91 \rightarrow a^2 + 91$$
>
> as $n \to \infty$ $\boxed{2}$. Because limits are unique, it follows that $0 = a^2 - 20a + 91 = (a - 7)(a - 13)$. So $a = 7$ or $a = 13$ $\boxed{2}$. Finally, since (a_n) is a decreasing sequence and since $a_1 = 11$, we must have that $a = 7$ $\boxed{2}$.

Soru 2 (Symbolic Logic and Negating a Definition)**.**

(a) [6p] Prove that $\neg(P \implies Q) = P \wedge \neg Q$.

P	Q	$P \implies Q$	$\neg(P \implies Q)$	P	$\neg Q$	$P \wedge \neg Q$
T	T	T	F	T	F	F
T	F	F	T	T	T	T
F	T	T	F	F	F	F
F	F	T	F	F	T	F

-1 point for first mistake, -$\frac{1}{2}$ point for each subsequent mistake.

Definition. A sequence (a_n) is a *Cauchy sequence* if and only if for all $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that for all $n, m \in \mathbb{N}$;

$$n, m > N \implies |a_n - a_m| < \varepsilon.$$

(b) [7p] Give the definition of "(a_n) is **not** a Cauchy sequence".
[HINT: Negate the definition above.]

A sequence (a_n) is **not** a Cauchy sequence if and only if there exists $\varepsilon > 0$ such that for all $N \in \mathbb{N}$, there exist $n, m \in \mathbb{N}$ such that

$$n, m > N \qquad \text{and} \qquad |a_n - a_m| \geq \varepsilon.$$

Let

$$b_n := \frac{(-1)^n(1+n)}{n}$$

for all $n \in \mathbb{N}$.

(c) [12p] Show that (b_n) is **not** a Cauchy sequence.

Choose $\varepsilon = 1$. Let $N \in \mathbb{N}$. Notice that if n is an even number $(n \in \{2, 4, 6, 8, \dots\})$ then $b_n \geq 1$ and if n is an odd number $(n \in \{1, 3, 5, 7, \dots\})$ then $b_n \leq -1$.
Choose $n = N + 1$ and $m = N + 2$. Then $n, m > N$ and

$$|a_n - a_m| = |a_{N+1} - a_{N+2}| \geq 2 > \varepsilon.$$

Therefore (b_n) is not a Cauchy sequence.

Soru 3 (Series)**.** Decide if each of the following series converges or diverges. Justify (prove) your answers.

(a) [8p] $\displaystyle\sum_{n=1}^{\infty} \frac{3^n \; n!}{n \; 2^n \; (n+1)!}$.

(b) [8p] $\displaystyle\sum_{n=1}^{\infty} (-1)^{n+1} \left(\frac{n}{1000000}\right)^n$.

(c) [9p] $\displaystyle\sum_{n=1}^{\infty} \operatorname{sech}^2 n$.

[You may use any theorem/lemma/test/example/etc. from the course, but you must say which one you are using.]

[HINT: $\operatorname{sech} x = \frac{1}{\cosh x}$.] [HINT: $\frac{d}{dx}\left(\frac{\sinh x}{\cosh x}\right) = ?$]

2 pts for "converges/diverges" correct without justification.
2 pts for saying which test is being used (as long as there is some proof given).
Remaining 4/5 pts for accuracy of proof.

If an answer is incorrect, but the proof is well written and contains only a minor error, then a maximum of 5 points (6 points for part (c)) can be awarded.

(a) Since

$$\frac{a_{n+1}}{a_n} = \frac{3^{n+1}\ (n+1)!}{(n+1)\ 2^{n+1}\ (n+2)!} \frac{n\ 2^n\ (n+1)!}{3^n\ n!} = \frac{3n}{2(n+2)} \to \frac{3}{2}$$

as $n \to \infty$, it follows that $\sum_{n=1}^{\infty} \frac{3^n\ n!}{n\ 2^n\ (n+1)!}$ diverges by the Ratio Test.

(b) Since

$$\frac{n^n}{1000000^n} \to \infty$$

as $n \to \infty$, it follows that $\sum_{n=1}^{\infty}(-1)^{n+1}\left(\frac{n}{1000000}\right)^n$ diverges by the Divergence Test.

(c) First note that $\frac{d}{dx}\left(\frac{\sinh x}{\cosh x}\right) = \frac{\cosh x \cosh x - \sinh x \sinh x}{\cosh^2 x} = \frac{1}{\cosh^2 x} = \operatorname{sech}^2 x$.

Since

$$\begin{aligned}\int_1^R \operatorname{sech}^2 x\ dx &= \left[\frac{\sinh x}{\cosh x}\right]_1^R = \frac{\sinh R}{\cosh R} - \frac{\sinh 1}{\cosh 1} \\ &= \frac{e^x - e^{-x}}{e^x + e^{-x}} - \tanh 1 = \frac{1 - e^{-2x}}{1 + e^{-2x}} - \tanh 1 \\ &\to 1 - \tanh 1 < \infty\end{aligned}$$

as $R \to \infty$, it follows that $\sum_{n=1}^{\infty} \operatorname{sech}^2 n$ converges by the Integral Test.

Soru 4 (Power Series)**.**

(a) [5p] Let $\sum_{n=0}^{\infty} a_n x^n$ be a power series. Give the definition of the *radius of convergence* of $\sum_{n=0}^{\infty} a_n x^n$.

If $\sum_{n=0}^{\infty} a_n x^n$ converges $\forall\, |x| < R$ and diverges $\forall\, |x| > R$, then R is called the radius of convergence of $\sum_{n=0}^{\infty} a_n x^n$.

Define the set

$$S := \left\{ x \in \mathbb{R} : \sum_{n=1}^{\infty} \frac{(x-1)^n}{\sqrt{n}} \text{ converges} \right\} \subseteq \mathbb{R}.$$

(b) [20p] Find S.

First consider the power series $\sum_{n=1}^{\infty} \frac{x^n}{\sqrt{n}}$. For this power series, $a_n = \frac{1}{\sqrt{n}}$ and

$$\left|\frac{a_n}{a_{n+1}}\right| = \frac{\sqrt{n+1}}{\sqrt{n}} = \sqrt{1+\frac{1}{n}} \to 1$$

as $n \to \infty$ $\boxed{6}$ $\boxed{\text{-1 point if candidate omits absolute value signs}}$. By a theorem from the course $\boxed{2}$, the radius of convergence of this power series is $R = 1$ $\boxed{2}$.

When $x = 1$, the power series becomes $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$, which diverges $\boxed{2}$. When $x = -1$, the power series becomes $\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}$, which converges $\boxed{2}$.

Therefore $\sum_{n=1}^{\infty} \frac{x^n}{\sqrt{n}}$ converges $\forall x \in [-1,1)$ and diverges for all other x. $\boxed{2}$ Hence $\sum_{n=1}^{\infty} \frac{(x-1)^n}{\sqrt{n}}$ converges $\forall x \in [0,2)$ and diverges for all other x. So $S = [0,2)$ $\boxed{4}$.

$$S := \left\{ x \in \mathbb{R} : \sum_{n=1}^{\infty} \frac{(x-1)^n}{\sqrt{n}} \text{ converges} \right\} \subseteq \mathbb{R}.$$

Soru 5 (Taylor Series).

(a) [10p] Calculate the Taylor Series for $f(x) = \cos x$, centred at $a = 0$.
[You may assume without proof that $\left|\frac{f^n(c)}{n!}x^n\right| \to 0$ as $n \to \infty$ for all $c, x \in \mathbb{R}$.]

Since

$$\frac{d^n}{dx^n}\cos x = \begin{cases} \cos x & n = 0,4,8,\ldots \\ -\sin x & n = 1,5,9,\ldots \\ -\cos x & n = 2,6,10,\ldots \\ \sin x & n = 3,7,11,\ldots, \end{cases}$$

we can see that

$$f^n(0) = \begin{cases} 1 & n = 0,4,8,\ldots \\ 0 & n = 1,3,5,7,9,\ldots \\ -1 & n = 2,6,10,\ldots \boxed{4} \end{cases}$$

By Taylor's Theorem (and by the hint), we have

$$\begin{aligned} \cos x &= f(0) + f'(0)x + \frac{f''(0)x^2}{2!} + \frac{f'''(0)x^3}{3!} + \frac{f^{(4)}(0)x^4}{4!} + \ldots \boxed{3} \\ &= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \frac{x^{10}}{10!} + \frac{x^{12}}{12!} - \frac{x^{14}}{14!} + \frac{x^{16}}{16!} - \frac{x^{18}}{18!} + \ldots \boxed{3} \end{aligned}$$

(b) [15p] Use your answer to part (a) to calculate $\lim_{t\to 0} \frac{1-\cos t - \frac{t^2}{2}}{t^4}$.

By part (a),

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \frac{x^{10}}{10!} + \frac{x^{12}}{12!} - \frac{x^{14}}{14!} + \frac{x^{16}}{16!} - \frac{x^{18}}{18!} + \ldots.$$

Therefore

$$\begin{aligned} \frac{1 - \frac{t^2}{2} - \cos t}{t^4} &= \frac{-\frac{t^4}{4!} + \frac{t^6}{6!} - \frac{t^8}{8!} + \ldots}{t^4} \qquad \boxed{3} \\ &= -\frac{1}{4!} + \frac{t^2}{6!} - \frac{t^4}{8!} + \ldots. \qquad \boxed{5} \end{aligned}$$

Hence

$$\lim_{t\to 0} \frac{1-\cos t - \frac{t^2}{2}}{t^4} = \lim_{t\to 0}\left(-\frac{1}{4!} + \frac{t^2}{6!} - \frac{t^4}{8!} + \ldots\right) = -\frac{1}{4!} = -\frac{1}{24}. \qquad \boxed{7}$$