

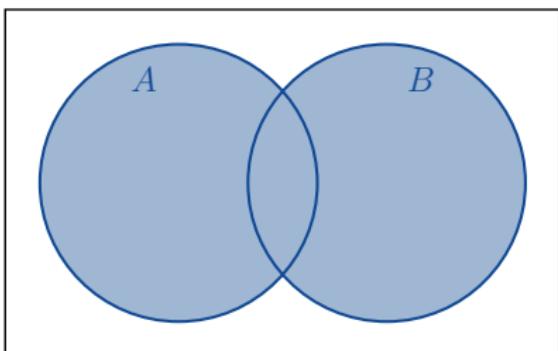
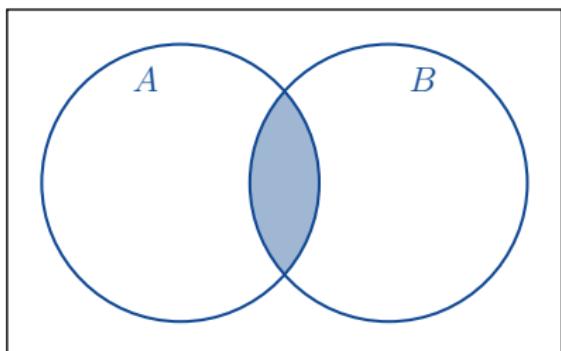
# Lecture 7

- 20. Concepts of Probability
- 21. Conditional Probability
- 22. Probability Trees



# Concepts of Probability

### Union and Intersection


$$A \cup B$$

$$A \cap B$$

## 20. Concepts of Probability



### Example (One die)

The sample space for rolling a single die is

$$S = \{1, 2, 3, 4, 5, 6\}.$$

Assume that each of these simple events are equally likely.

- 1 What is the probability of rolling a number which is even and greater than 3?
- 2 What is the probability of rolling a number which is even or greater than 3?

## 20. Concepts of Probability

$$S = \{1, 2, 3, 4, 5, 6\}$$

*solution:* Let

$$A = \text{even numbers} = \{2, 4, 6\}$$

and

$$B = \text{numbers greater than } 3 = \{4, 5, 6\}.$$

**1 And:** Since  $A \cap B = \{4, 6\}$ , we have that

$$P(A \cap B) = \frac{2}{6} = \frac{1}{3}.$$

**2 Or:** Since  $A \cup B = \{2, 4, 5, 6\}$ , we have that

$$P(A \cup B) = \frac{4}{6} = \frac{2}{3}.$$

## 20. Concepts of Probability

### Remark

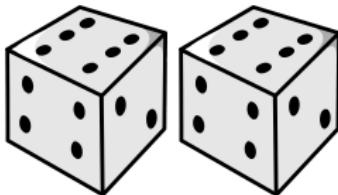
Please recall the Addition Principle which stated that

$$n(A \cup B) = n(A) + n(B) - n(A \cap B).$$

### Theorem

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

## 20. Concepts of Probability



### Example (Two dice)

You roll two dice. What is the probability that:

- 1** the sum is either 5 or 10?
- 2** either the sum is greater than 9, or both dice show the same number?

## 20. Concepts of Probability



*solution:*

**1 5 or 10:** Let

$$\begin{aligned}A &= \text{the sum is 5} \\&= \{(1, 4), (2, 3), (3, 2), (4, 1)\}\end{aligned}$$

and

$$\begin{aligned}B &= \text{the sum is 10} \\&= \{(4, 6), (5, 5), (6, 4)\}.\end{aligned}$$

Since  $A \cap B = \emptyset$ , we have that

$$P(A \cup B) = P(A) + P(B) = \frac{4}{36} + \frac{3}{36} = \frac{7}{36}.$$

## 20. Concepts of Probability



**2 > 9 or same number:** Let

$$\begin{aligned}C &= \text{the sum is greater than 9} \\&= \{(4, 6), (5, 5), (5, 6), (6, 4), (6, 5), (6, 6)\}\end{aligned}$$

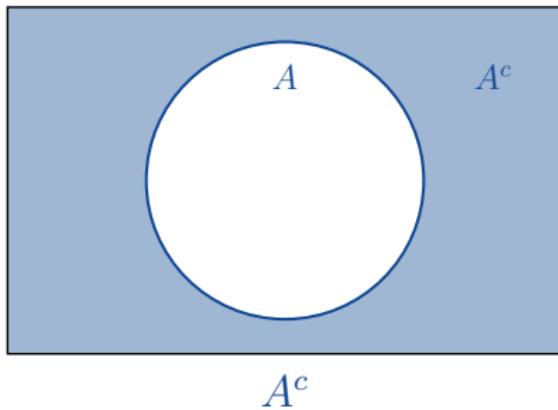
and

$$\begin{aligned}D &= \text{both dice show the same number} \\&= \{(1, 1), (2, 2), (3, 3), (4, 4), (5, 5), (6, 6)\}.\end{aligned}$$

Note that  $C \cap D = \{(5, 5), (6, 6)\}$ . Therefore

$$\begin{aligned}P(C \cup D) &= P(C) + P(D) - P(C \cap D) \\&= \frac{6}{36} + \frac{6}{36} - \frac{2}{36} = \frac{10}{36} = \frac{5}{18}.\end{aligned}$$

# Complements



## 20. Concepts of Probability



### Theorem

$$P(E) = 1 - P(E^c).$$

Sometimes it is easier to calculate  $1 - P(E^c)$ , than to calculate  $P(E)$  directly.

## 20. Concepts of Probability



### Example (Whiteboard Markers)

A box containing 45 whiteboard markers is delivered to Istanbul Okan University. Nine of the markers are red. The remaining markers are black.

Your teacher is given 10 markers at random. He will be happy if one or more of his markers is red.

What is the probability that your teacher will be happy?

## 20. Concepts of Probability

*solution:* Let

$E$  = one or more of the markers is red.

Then

$E^c$  = all 10 markers are black.

Since

$$P(E^c) = \frac{n(E^c)}{n(S)} = \frac{36C_{10}}{45C_{10}},$$

we have that

$$P(E) = 1 - P(E^c) = 1 - \frac{36C_{10}}{45C_{10}} \approx 0.92.$$

## 20. Concepts of Probability

### Example

In a class of 30 students, what is the probability that at least two students have the same birthday? (Same month and day. Ignore 29 February)

**solution:** We assume that there are 365 days in a year and that each day is equally likely. We have

$$n(S) = 365^{30}$$

by the Multiplication Principle. Let

$E = \text{2 or more people have the same birthday.}$

Then

$E^c = \text{all 30 students have different birthdays.}$

## 20. Concepts of Probability



We calculate that

$$n(E^c) = 365 \cdot 364 \cdot 363 \cdot \dots \cdot 336 = \frac{365!}{335!} = {}_{365}P_{30},$$

$$P(E^c) = \frac{n(E^c)}{n(S)} = \frac{365!}{335! \cdot 365^{30}}$$

and

$$P(E) = 1 - P(E^c) = 1 - \frac{365!}{335! \cdot 365^{30}} \approx 0.706$$



# Conditional Probability

## 21. Conditional Probability



Sometimes the probability of an event will depend on another event. For example, suppose that

$A = \text{Ali has cancer}$

and

$B = \text{Ali is a smoker.}$

Clearly the probability that  $A$  occurs depends on  $B$ .

## 21. Conditional Probability



### Definition

The *conditional probability* of  $A$  given  $B$  is

$$P(A|B) = \begin{pmatrix} \text{the probability of A, if we} \\ \text{already know that B occurs} \end{pmatrix}.$$

## 21. Conditional Probability



Theorem

$$P(A|B) = \frac{P(A \cap B)}{P(B)}.$$

## 21. Conditional Probability



### Example (Marbles)

A bag contains red and blue marbles. Two marbles are drawn without replacement.

The probability of selecting a red marble and then a blue marble is 0.28. The probability of selecting a red marble on the first draw is 0.5.

What is the probability of selecting a blue marble on the second draw, given that the first marble drawn was red?

## 21. Conditional Probability



*solution:* Let

$R$  = the first marble is red

and

$B$  = the second marble is blue.

The required probability is

$$P(B|R) = \frac{P(B \cap R)}{P(R)} = \frac{0.28}{0.5} = 0.56$$

## 21. Conditional Probability



### Example (One die)

Your friend says that when she rolled a die, she rolled an odd number. What is the probability that your friend rolled a 3?

*solution:* Let

$$A = \text{your friend rolled a 3}$$

and

$$B = \text{your friend rolled an odd number.}$$

Then  $P(A) = \frac{1}{6}$ ,  $P(A \cap B) = P(A) = \frac{1}{6}$  and  $P(B) = \frac{1}{2}$ . Hence

$$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{\frac{1}{6}}{\frac{1}{2}} = \frac{1}{3}.$$

## 21. Conditional Probability



### Remark

Given that  $P(A|B) = \frac{P(A \cap B)}{P(B)}$ , we have that

$$P(A \cap B) = P(B)P(A|B).$$

Similarly,

$$P(A \cap B) = P(B \cap A) = P(A)P(B|A).$$

Therefore:

## 21. Conditional Probability



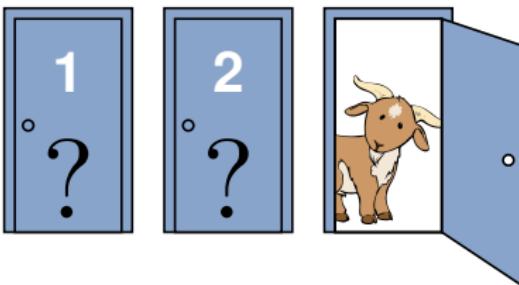
Theorem (Product Rule)

$$P(A \cap B) = P(A)P(B|A) = P(B)P(A|B)$$

and

$$P(A|B) = \frac{P(A)P(B|A)}{P(B)}.$$

### The Monty Hall Problem



*Suppose you're on a TV game show, and you're given the choice of three doors: Behind one door is a car; behind the others, goats. You pick a door, say No. 1, and the host, who knows what's behind the doors, opens another door, say No. 3, which has a goat. He then says to you, "Do you want to pick door No. 2?" Is it to your advantage to switch your choice?*

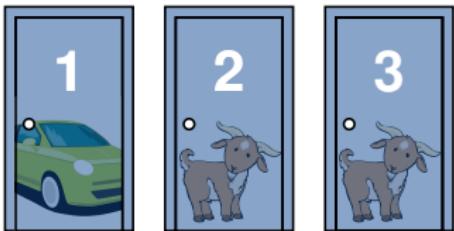
## 21. Conditional Probability



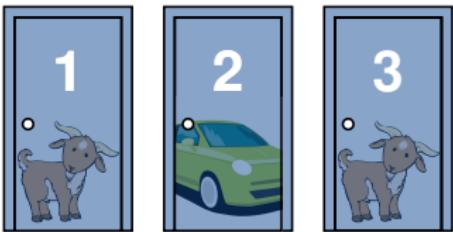
Monty Hall  
USA, 1921-2017.

The *Monty Hall Problem* is one that confuses a lot of people that haven't studied Probability.

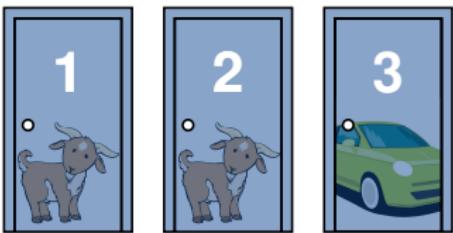
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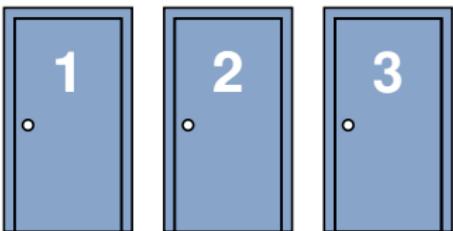
## 21. Conditional Probability



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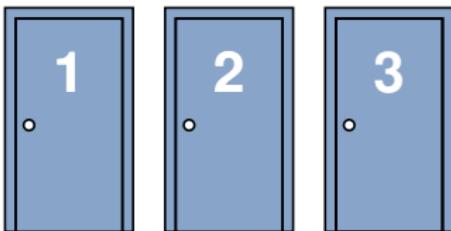


## 21. Conditional Probability



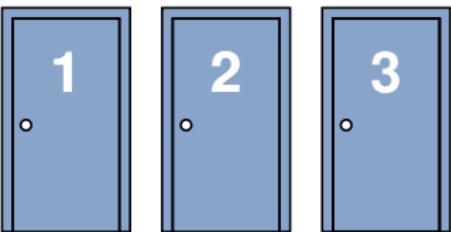
- 1** You choose a door. What is the probability that the car is behind that door?

## 21. Conditional Probability



- 1 You choose a door. What is the probability that the car is behind that door? Easy  $P(\text{car}) = \frac{1}{3}$  right?

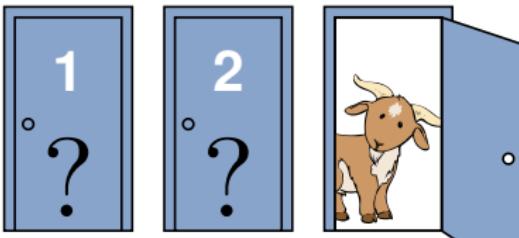
## 21. Conditional Probability



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CORRECT

## 21. Conditional Probability

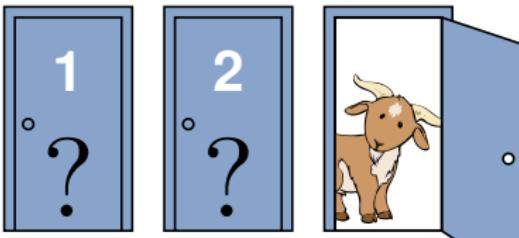


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CORRECT

- 2 Then the host opens another door and shows you a goat. Now there are two closed doors: Behind one is a car and behind the other is a goat.

## 21. Conditional Probability

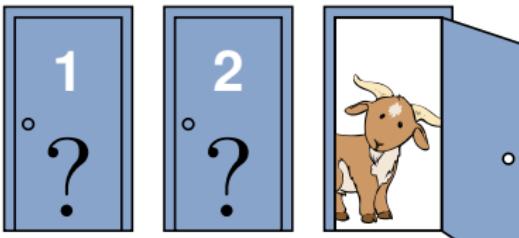


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CORRECT

- 2 Then the host opens another door and shows you a goat. Now there are two closed doors: Behind one is a car and behind the other is a goat.
- 3 What is the probability that the car is behind the door that you chose?

## 21. Conditional Probability

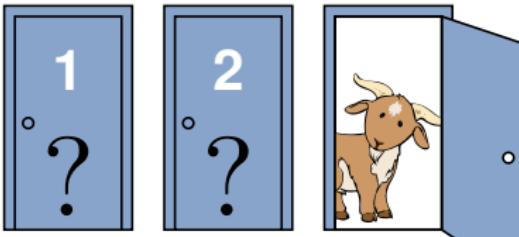


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**CORRECT**

- 2 Then the host opens another door and shows you a goat. Now there are two closed doors: Behind one is a car and behind the other is a goat.
- 3 What is the probability that the car is behind the door that you chose? Two closed doors. One Car. So clearly now  $P(\text{car}) = \frac{1}{2}$  right?

## 21. Conditional Probability



- 1 You choose a door. What is the probability that the car is behind that door? Easy  $P(\text{car}) = \frac{1}{3}$  right?

CORRECT

- 2 Then the host opens another door and shows you a goat. Now there are two closed doors: Behind one is a car and behind the other is a goat.
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WRONG!!!

## 21. Conditional Probability



What? Why?

## 21. Conditional Probability



What? Why?

To explain, let us look at all the possible outcomes if you choose door number 1 first.

## 21. Conditional Probability



What? Why?

To explain, let us look at all the possible outcomes if you choose door number 1 first.

behind door 1	behind door 2	behind door 3	outcome if you don't switch	outcome if you switch
car	goat	goat		

## 21. Conditional Probability



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behind door 1	behind door 2	behind door 3	outcome if you don't switch	outcome if you switch
car	goat	goat	win	lose

## 21. Conditional Probability



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car	goat	goat	win	lose
goat	car	goat		

## 21. Conditional Probability



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car	goat	goat	win	lose
goat	car	goat	lose	win

## 21. Conditional Probability



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To explain, let us look at all the possible outcomes if you choose door number 1 first.

behind door 1	behind door 2	behind door 3	outcome if you don't switch	outcome if you switch
car	goat	goat	win	lose
goat	car	goat	lose	win
goat	goat	car		

## 21. Conditional Probability



What? Why?

To explain, let us look at all the possible outcomes if you choose door number 1 first.

behind door 1	behind door 2	behind door 3	outcome if you don't switch	outcome if you switch
car	goat	goat	win	lose
goat	car	goat	lose	win
goat	goat	car	lose	win

## 21. Conditional Probability

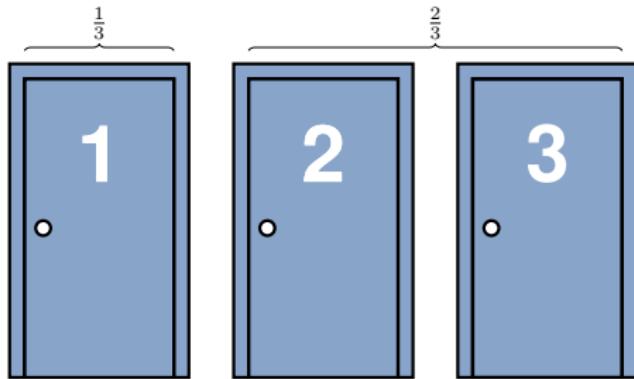


We can see from the table that if you don't switch your choice, then you have a  $\frac{1}{3}$  chance of winning the car, but if you do switch then you have  $\frac{2}{3}$  chance of winning it. These are the probabilities if you choose door number 1, but of course we would get the same results if you choose door 2 or 3 first.

## 21. Conditional Probability

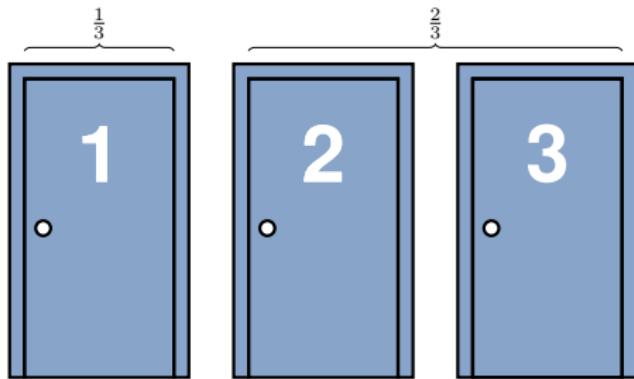


Another way



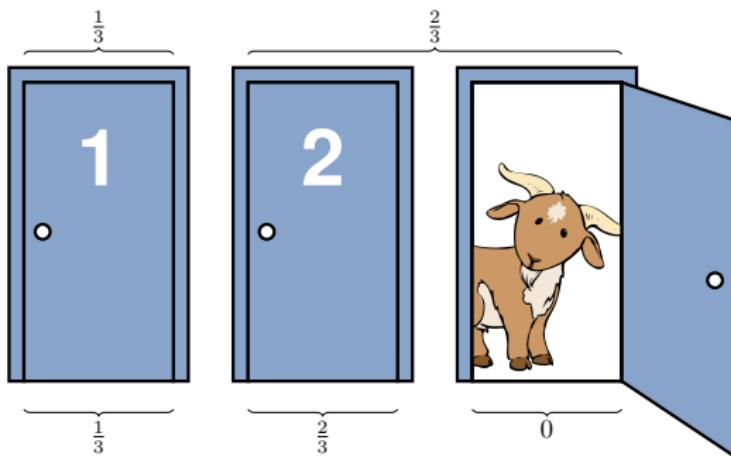
Let's think of this another way. You initially choose door number 1. At the start, the chance of the car being behind door one is  $\frac{1}{3}$  and the chance of the car being behind doors 2 or 3 is  $\frac{2}{3}$ .

## 21. Conditional Probability



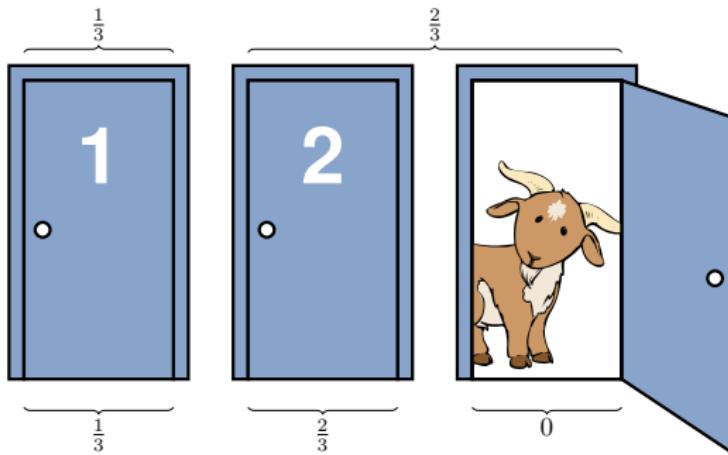
Let's imagine that the host doesn't open a door, he just says you can change your choice for *both* of the other two doors. Would you switch then?

## 21. Conditional Probability



We know that atleast one of doors 2 and 3 hides a goat.

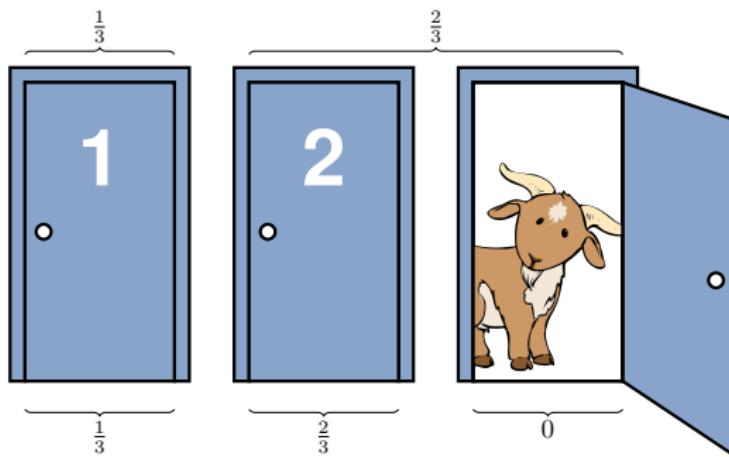
## 21. Conditional Probability



We know that atleast one of doors 2 and 3 hides a goat.

Remember that the host knows where the car is: He doesn't open a door at random, he always opens a door with a goat. So he isn't really giving you any extra information.

## 21. Conditional Probability



We know that atleast one of doors 2 and 3 hides a goat.

Remember that the host knows where the car is: He doesn't open a door at random, he always opens a door with a goat. So he isn't really giving you any extra information.

The probabilities don't change to  $\frac{1}{2}$ ,  $\frac{1}{2}$ , they are still  $\frac{1}{3}$ ,  $\frac{2}{3}$ .

## 21. Conditional Probability



### Using Conditional Probabilities

Let

$C$  = door number 1 has a car behind it

$C^c$  = door number 1 does not have a car behind it  
= door number 1 has a goat behind it

and

$E$  = the host has opened a door with a goat behind it.

## 21. Conditional Probability



### Using Conditional Probabilities

Let

$C$  = door number 1 has a car behind it

$C^c$  = door number 1 does not have a car behind it  
= door number 1 has a goat behind it

and

$E$  = the host has opened a door with a goat behind it.

We have that  $P(C) = \frac{1}{3}$  and  $P(C^c) = 1 - P(C) = \frac{2}{3}$ . Moreover,  $P(E|C) = 1$  and  $P(E|C^c) = 1$  because the host always opens a door with a goat.

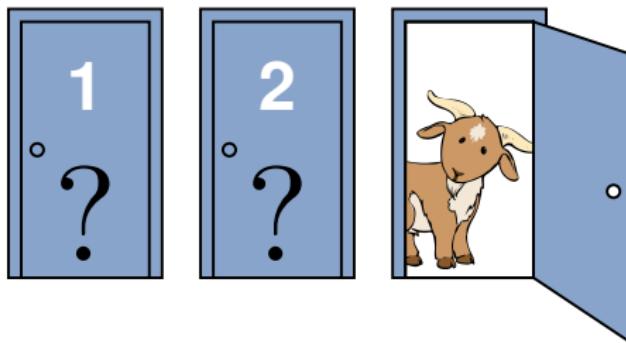
## 21. Conditional Probability

Then we can calculate that

$$\begin{aligned}
 P(C|E) &= \frac{P(C)P(E|C)}{P(E)} \\
 &= \frac{P(C)P(E|C)}{P(E \cap C) + P(E \cap C^c)} \\
 &= \frac{P(C)P(E|C)}{P(C)P(E|C) + P(C^c)P(E|C^c)} \\
 &= \frac{\frac{1}{3} \cdot 1}{\frac{1}{3} \cdot 1 + \frac{2}{3} \cdot 1} = \frac{1}{3}.
 \end{aligned}$$

This means that it doesn't matter if the host opens a door or not, the probability that the car is behind door number 1 is always  $\frac{1}{3}$ .

## 21. Conditional Probability



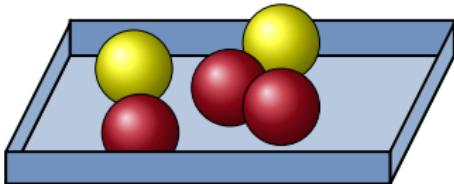
### Conclusion

You should always switch door if you want to win the car.



# Probability Trees

## 22. Probability Trees



### Example (5 balls in a box)

A box contains 3 red and 2 yellow balls. Two balls are randomly drawn without replacement. What is the probability that the second ball is yellow?

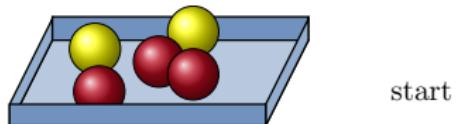
## 22. Probability Trees



*solution:* First we draw a *probability tree*.

first ball

second ball

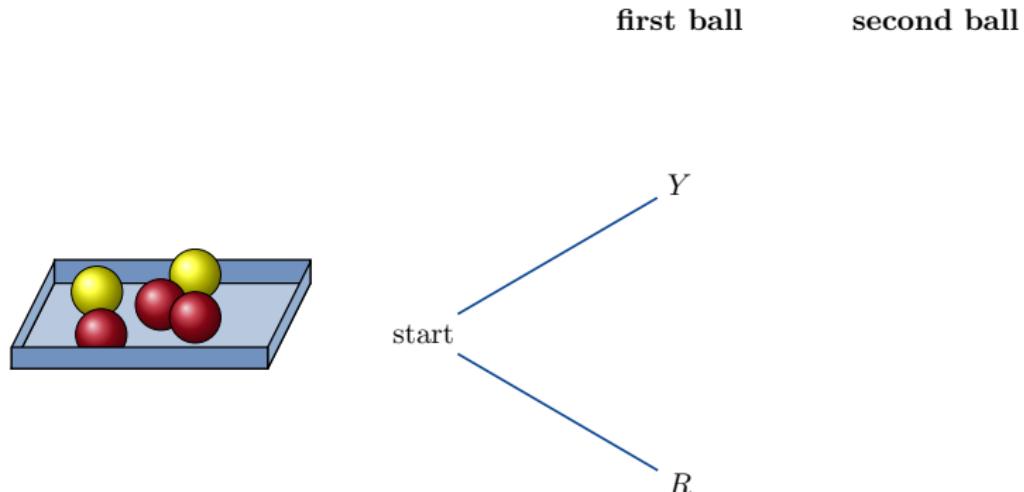


start

## 22. Probability Trees



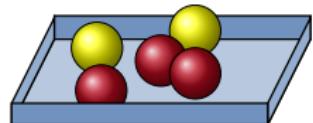
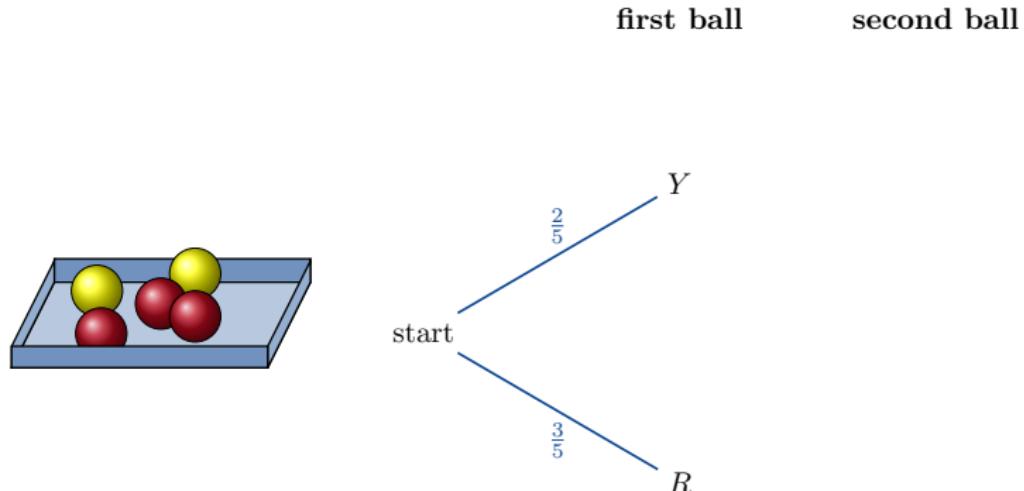
*solution:* First we draw a *probability tree*.



## 22. Probability Trees

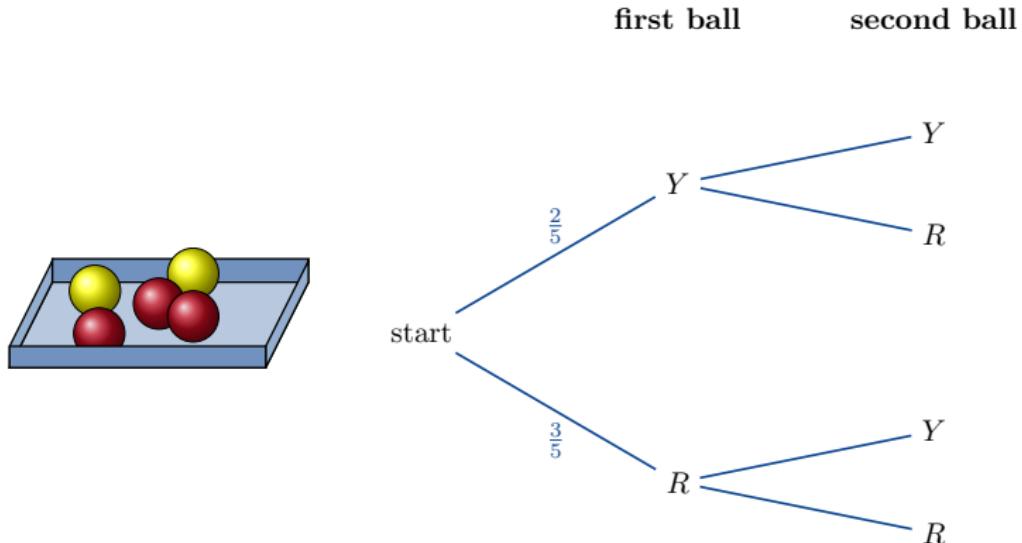


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## 22. Probability Trees

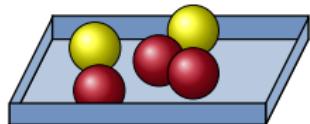
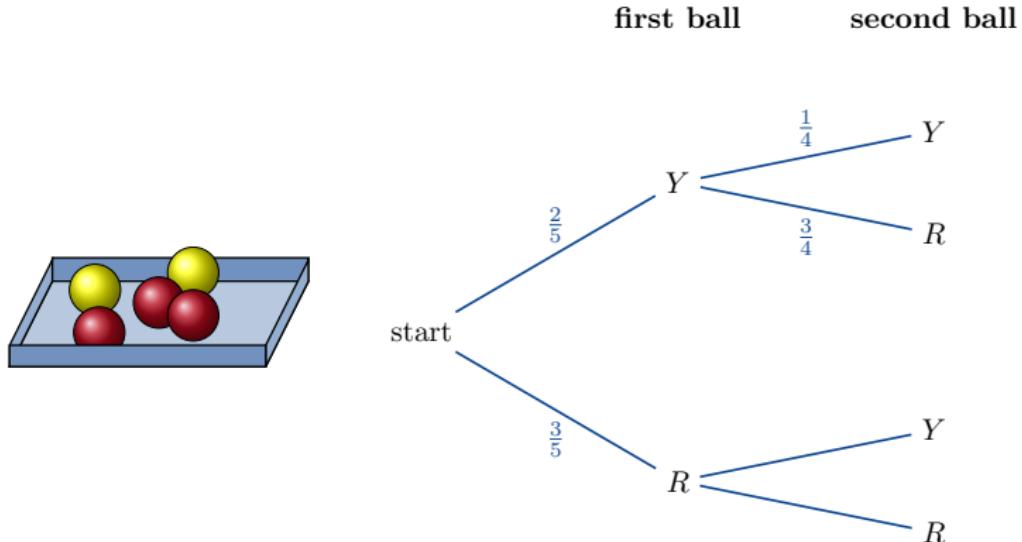
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## 22. Probability Trees



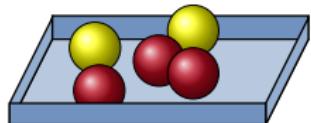
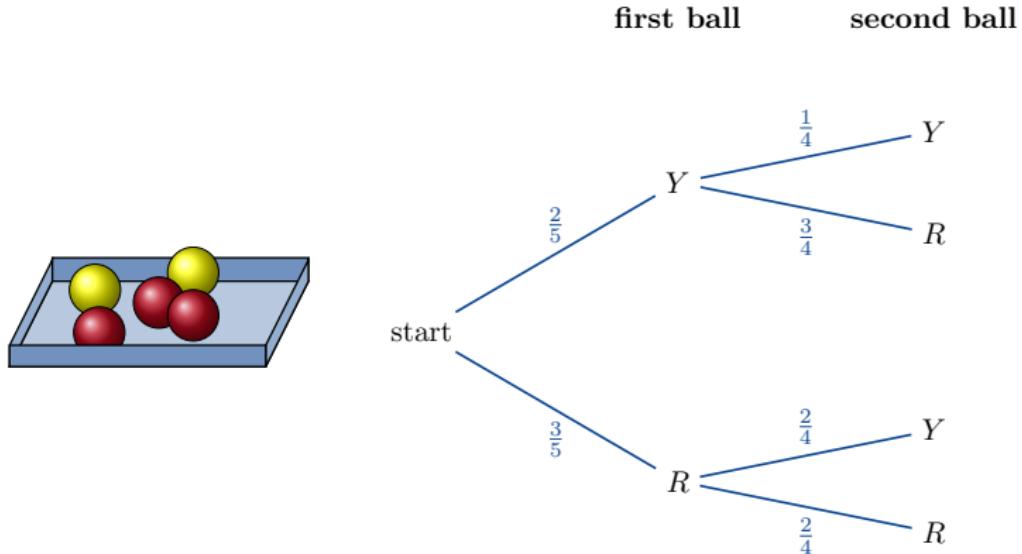
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## 22. Probability Trees



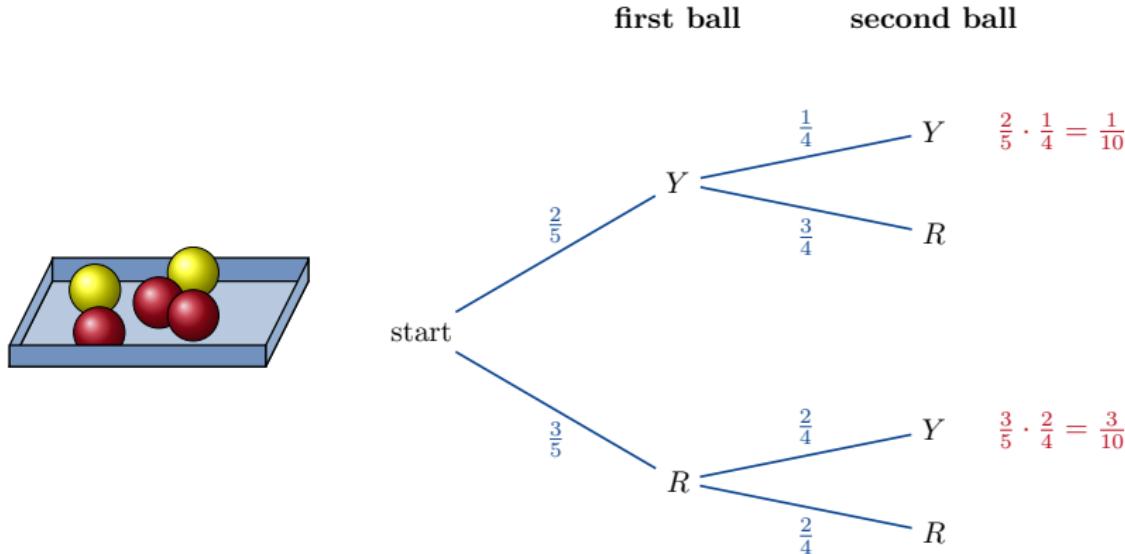
*solution:* First we draw a *probability tree*.



## 22. Probability Trees

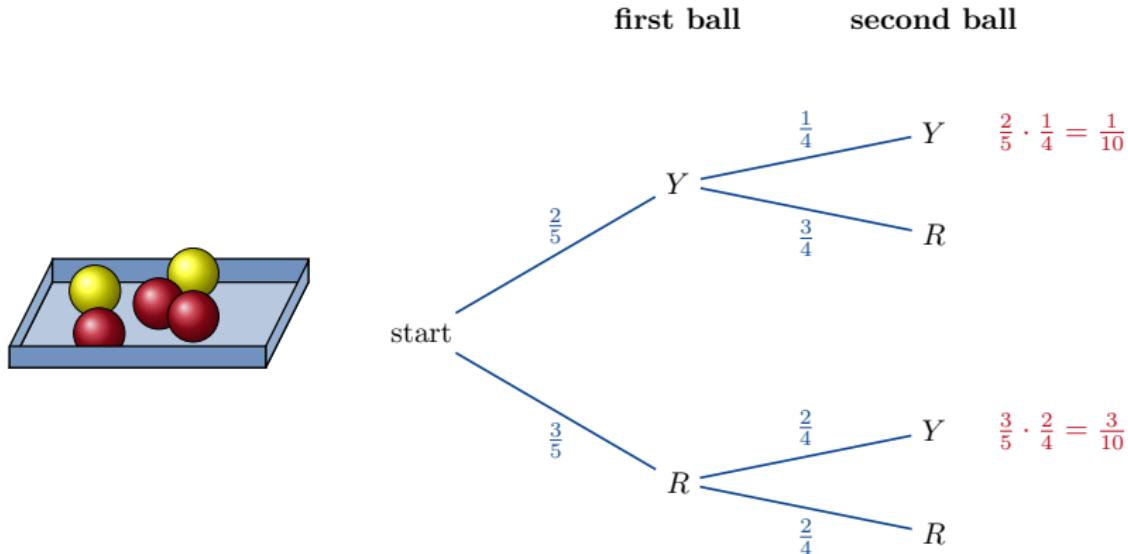


*solution:* First we draw a *probability tree*.



## 22. Probability Trees

*solution:* First we draw a *probability tree*.



From this probability tree, we can see that

$$P(YY) + P(RY) = \frac{1}{10} + \frac{3}{10} = \frac{4}{10} = \frac{2}{5}.$$

The probability that the second ball is yellow is  $\frac{2}{5}$ .

## 22. Probability Trees



### Example (Tennis)

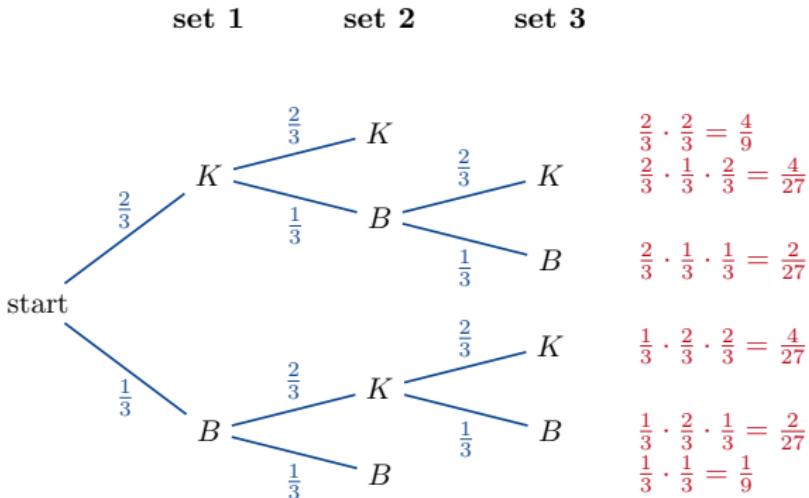
Boris and Keir are playing tennis. The first player to win 2 sets wins the match. In each set, the probability that Keir wins that set is  $\frac{2}{3}$ . Find the probability that:

- 1 Boris wins the match.
- 2 3 sets are played.
- 3 The player who wins the first set, wins the match.

## 22. Probability Trees

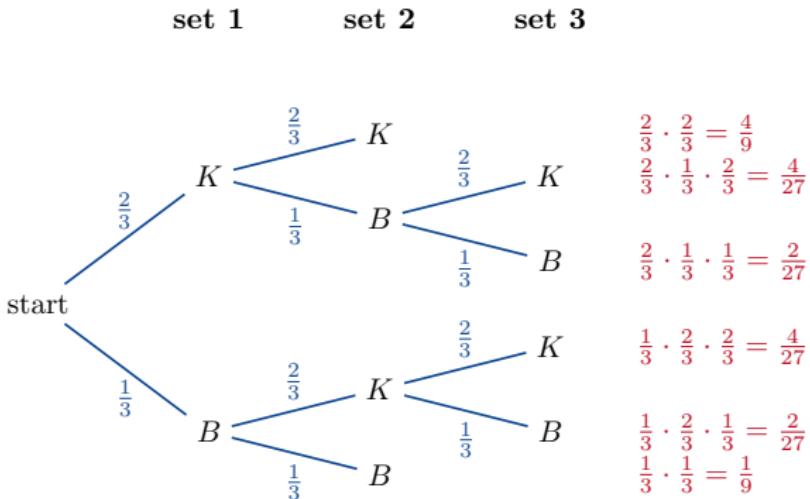


*solution:*



## 22. Probability Trees

*solution:*

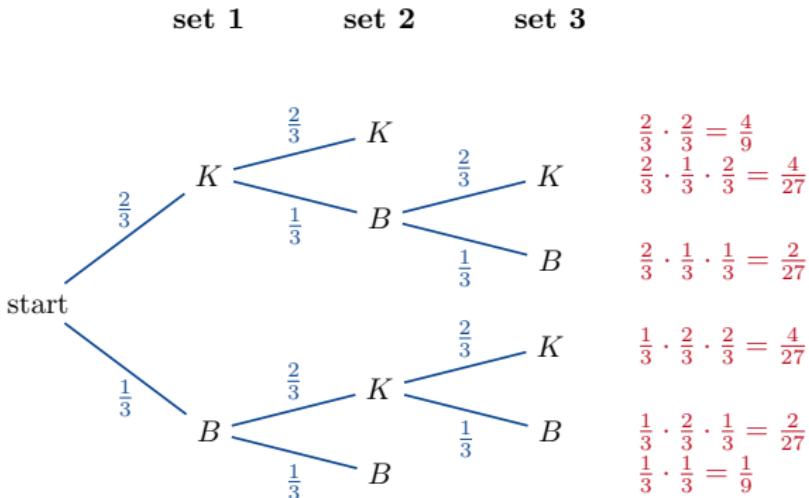


**1 Boris wins:** We calculate that

$$P(KBB) + P(BKB) + P(BB) = \frac{2}{27} + \frac{2}{27} + \frac{1}{9} = \frac{7}{27}.$$

## 22. Probability Trees

*solution:*

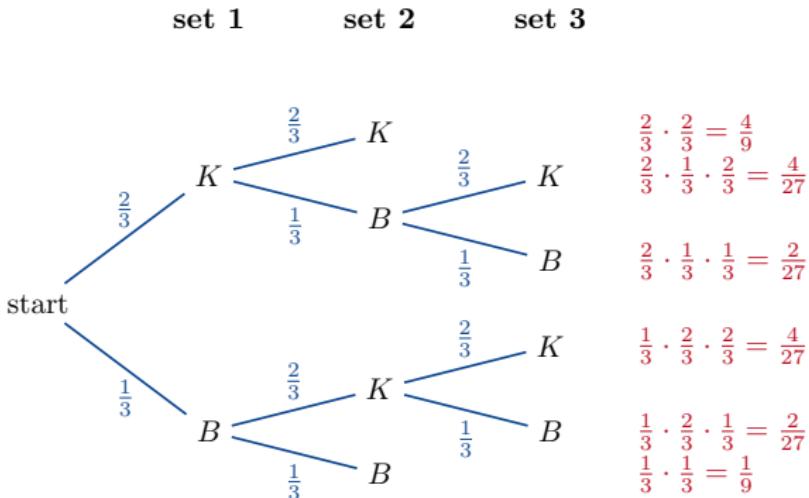


**2** 3 sets are played: Now

$$\begin{aligned}
 & P(KBK) + P(KBB) + P(BKK) + P(BKB) \\
 &= \frac{4}{27} + \frac{2}{27} + \frac{4}{27} + \frac{2}{27} = \frac{4}{9}.
 \end{aligned}$$

## 22. Probability Trees

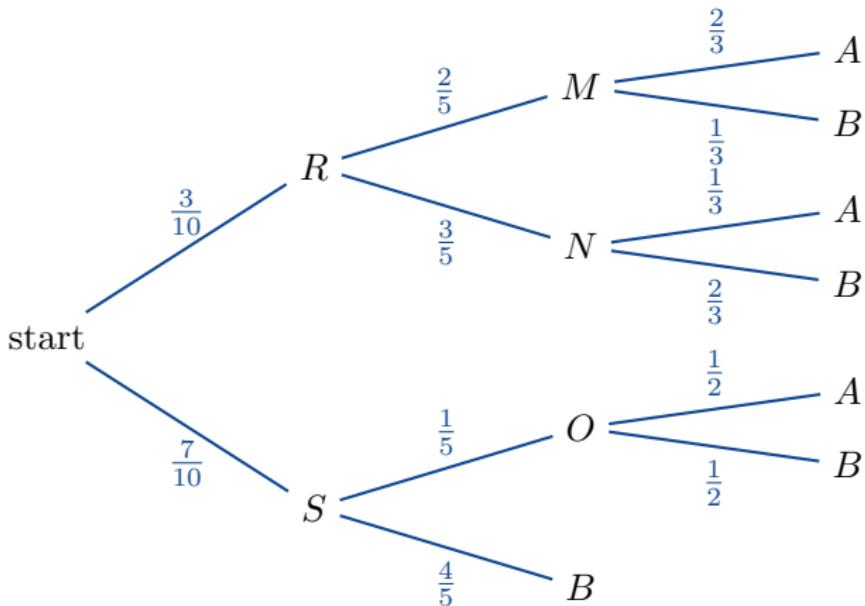
*solution:*



**3 Player who wins first set wins match:** Finally

$$\begin{aligned}
 P(KK) + P(KBK) + P(BKB) + P(BB) \\
 = \frac{4}{9} + \frac{4}{27} + \frac{2}{27} + \frac{1}{9} = \frac{7}{9}.
 \end{aligned}$$

## 22. Probability Trees

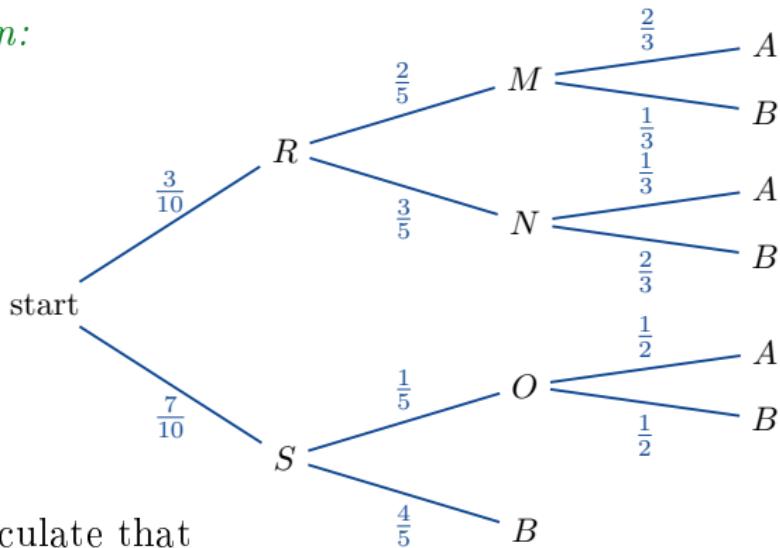


Example

Calculate  $P(B)$ .

## 22. Probability Trees

*solution:*



We calculate that

$$P(B) = P(RMB) + P(RNB) + P(SOB) + P(SB)$$

$$\begin{aligned}
 &= \left( \frac{3}{10} \cdot \frac{2}{5} \cdot \frac{1}{3} \right) + \left( \frac{3}{10} \cdot \frac{3}{5} \cdot \frac{2}{3} \right) + \left( \frac{7}{10} \cdot \frac{1}{5} \cdot \frac{1}{2} \right) + \left( \frac{7}{10} \cdot \frac{4}{5} \right) \\
 &= \frac{6}{150} + \frac{18}{150} + \frac{7}{100} + \frac{28}{50} = \frac{79}{100}.
 \end{aligned}$$

## 22. Probability Trees



### Example

Ron Weasley has a bag with 7 blue sweets and 3 red sweets in it.

He takes a sweet at random from the bag, then puts it back in the bag.

Then he picks a sweet at random from the bag and eats it.

Finally he picks a third sweet at random.

Draw a probability tree to represent this situation.

7 blue sweets and 3 red sweets



*solution:*

**sweet 1**

**sweet 2**

**sweet 3**

start

7 blue sweets and 3 red sweets

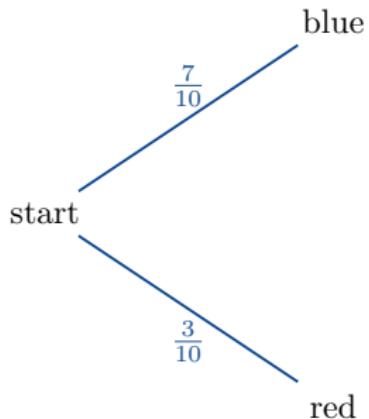


*solution:*

sweet 1

sweet 2

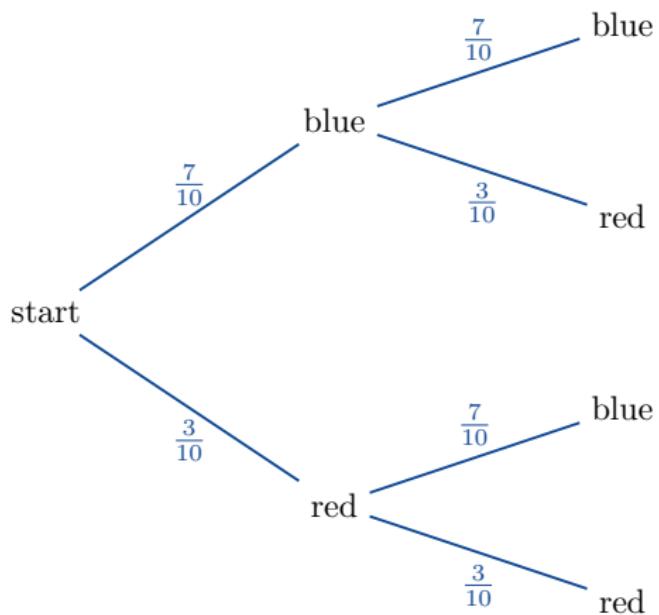
sweet 3



7 blue sweets and 3 red sweets



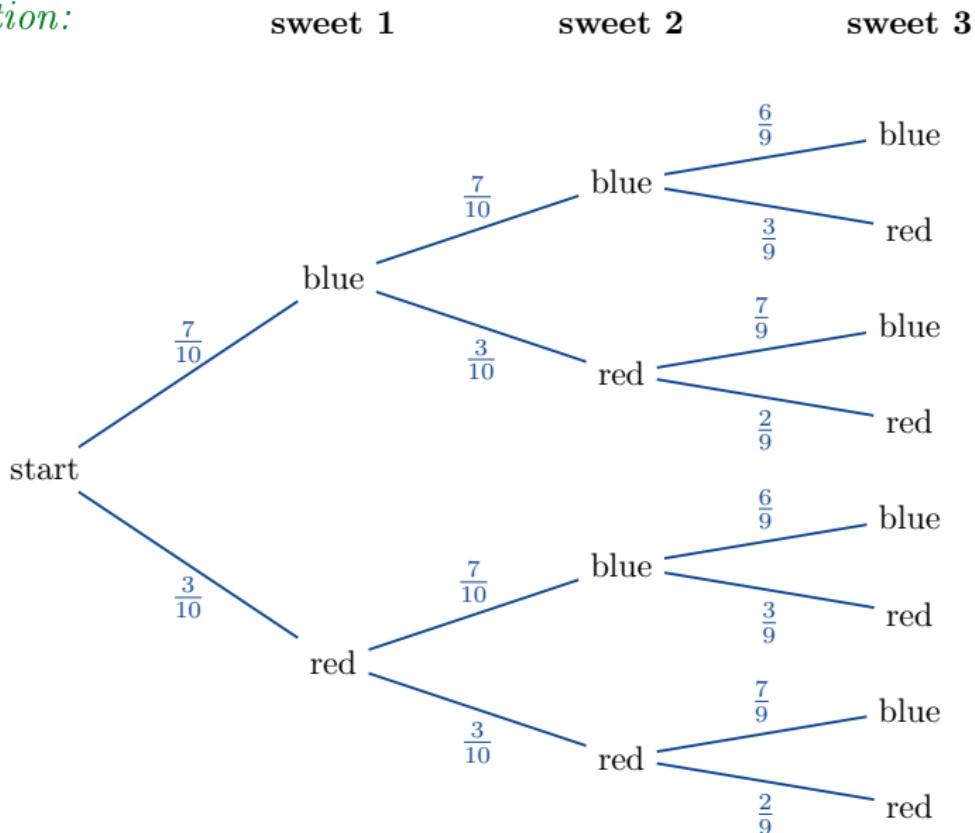
*solution:*



7 blue sweets and 3 red sweets



solution:





# Next Time

- 23. Graph Theory