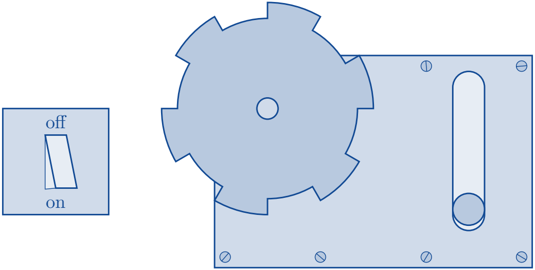
Week 12

- 4.5 ODEs with Discontinuous Forcing Functions
- 4.6 The Convolution Integral

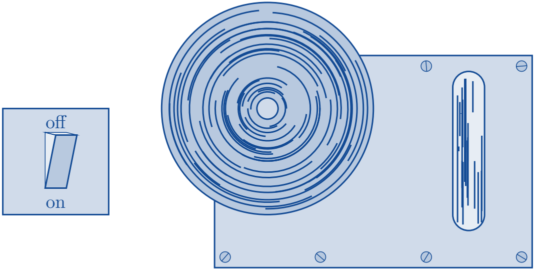
ODEs with Discontinuous Forcing Functions

4.5

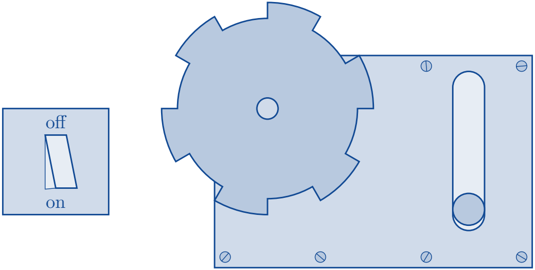
4.5 ODEs with Discontinuous Forcing Functions

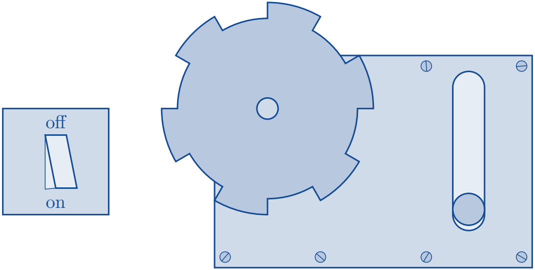


4.5 ODEs with Discontinuous Forcing Functions



4.5 ODEs with Discontinuous Forcing Functions

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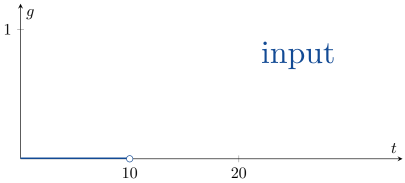
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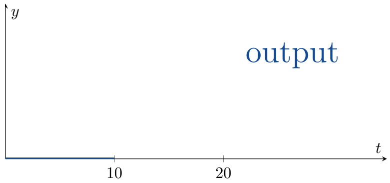
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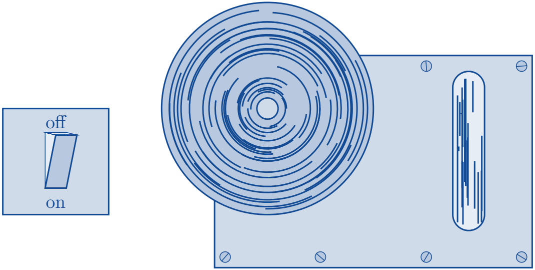
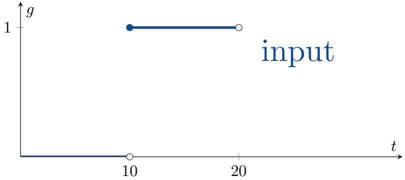
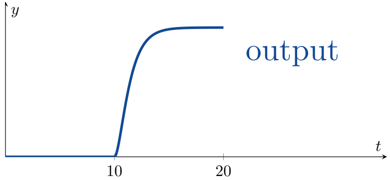
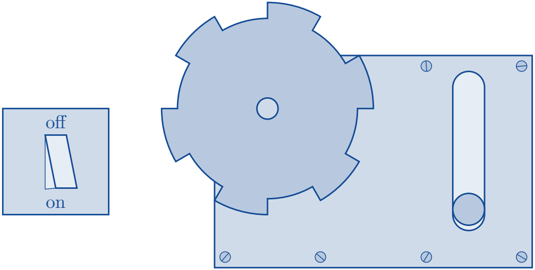
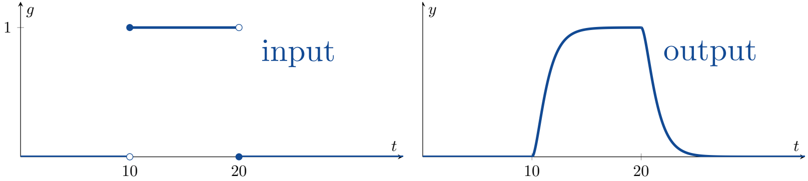






4.5 ODEs with Discontinuous Forcing Functions





Example

Solve

$$\begin{cases} y'' + 4y = f(t) = \begin{cases} 0 & 0 \leq t < 5 \\ \frac{1}{5}(t-5) & 5 \leq t < 10 \\ 1 & 10 \leq t \end{cases} \\ y(0) = 0 \\ y'(0) = 0. \end{cases}$$

Example

Solve

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Note that

$$f(t) = 0 + \left(\frac{1}{5}(t-5) - 0 \right) u_5(t) + \left(1 - \frac{1}{5}(t-5) \right) u_{10}(t)$$

$$=$$

Example

Solve

$$\begin{cases} y'' + 4y = f(t) = \begin{cases} 0 & 0 \leq t < 5 \\ \frac{1}{5}(t-5) & 5 \leq t < 10 \\ 1 & 10 \leq t \end{cases} \\ y(0) = 0 \\ y'(0) = 0. \end{cases}$$

Note that

$$\begin{aligned} f(t) &= 0 + \left(\frac{1}{5}(t-5) - 0\right) u_5(t) + \left(1 - \frac{1}{5}(t-5)\right) u_{10}(t) \\ &= \frac{1}{5}\Big(u_5(t)(t-5) - u_{10}(t)(t-10)\Big). \end{aligned}$$

$$\mathcal{L}\left[u_c(t)f(t-c)\right](s) = e^{-cs}F(s) \qquad \mathcal{L}\left[t\right] = \frac{1}{s^2}$$

So our IVP is

$$\begin{cases} y'' + 4y = \frac{1}{5}\Big(u_5(t)(t-5) - u_{10}(t)(t-10)\Big) \\ y(0) = 0 \\ y'(0) = 0. \end{cases}$$

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Taking the Laplace transform of the ODE gives

$$(s^2+4)Y = \frac{1}{5}\frac{e^{-5s} - e^{-10s}}{s^2}$$

$$\mathcal{L}\left[u_c(t)f(t-c)\right](s) = e^{-cs}F(s) \qquad \mathcal{L}\left[t\right] = \frac{1}{s^2}$$

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Taking the Laplace transform of the ODE gives

$$(s^2+4)Y = \frac{1}{5}\frac{e^{-5s} - e^{-10s}}{s^2}$$

and

$$Y = \frac{1}{5}\frac{e^{-5s} - e^{-10s}}{s^2(s^2+4)}.$$

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Let

$$H(s) = \frac{1}{s^2(s^2+4)}.$$

Then

$$Y(s) = \frac{1}{5}e^{-5s}H(s) - \frac{1}{5}e^{-10s}H(s).$$

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Since

$$\mathcal{L}\left[u_c(t)h(t-c)\right](s) = e^{-cs}H(s)$$

$$Y(s) = \frac{1}{5}e^{-5s}H(s) - \frac{1}{5}e^{-10s}H(s)$$

Since

$$\mathcal{L}\left[u_c(t)h(t-c)\right](s) = e^{-cs}H(s)$$

we have that

$$u_c(t)h(t-c) = \mathcal{L}^{-1}\left[e^{-cs}H(s)\right](t).$$

$$Y(s) = \frac{1}{5}e^{-5s}H(s) - \frac{1}{5}e^{-10s}H(s)$$

Since

$$\mathcal{L}\left[u_c(t)h(t-c)\right](s) = e^{-cs}H(s)$$

we have that

$$u_c(t)h(t-c) = \mathcal{L}^{-1}\left[e^{-cs}H(s)\right](t).$$

If we can find $h(t)$, then we can find $y(t)$.

Using partial fractions, we calculate (please check!) that

$$
\begin{aligned}
H(s) = \frac{1}{s^2(s^2+4)} &= \frac{As+B}{s^2} + \frac{Cs+D}{s^2+4} \\
&= \frac{As^3+Bs^2+4As+4B+Cs^3+Ds^2}{s^2(s^2+4)} \\
&= \frac{0s+\frac{1}{4}}{s^2} + \frac{0s-\frac{1}{4}}{s^2+4} = \frac{\frac{1}{4}}{s^2} - \frac{\frac{1}{4}}{s^2+4}.
\end{aligned}
$$

Using partial fractions, we calculate (please check!) that

$$\begin{aligned} H(s) &= \frac{1}{s^2(s^2+4)} = \frac{As+B}{s^2} + \frac{Cs+D}{s^2+4} \\ &= \frac{As^3+Bs^2+4As+4B+Cs^3+Ds^2}{s^2(s^2+4)} \\ &= \frac{0s+\frac{1}{4}}{s^2} + \frac{0s-\frac{1}{4}}{s^2+4} = \frac{\frac{1}{4}}{s^2} - \frac{\frac{1}{4}}{s^2+4}. \end{aligned}$$

Hence

$$h(t) = \frac{1}{4}\mathcal{L}^{-1}\left[\frac{1}{s^2}\right] - \frac{1}{8}\mathcal{L}^{-1}\left[\frac{2}{s^2+4}\right] = \qquad\qquad .$$

Using partial fractions, we calculate (please check!) that

$$\begin{aligned} H(s) = \frac{1}{s^2(s^2+4)} &= \frac{As+B}{s^2} + \frac{Cs+D}{s^2+4} \\ &= \frac{As^3+Bs^2+4As+4B+Cs^3+Ds^2}{s^2(s^2+4)} \\ &= \frac{0s+\frac{1}{4}}{s^2} + \frac{0s-\frac{1}{4}}{s^2+4} = \frac{\frac{1}{4}}{s^2} - \frac{\frac{1}{4}}{s^2+4}. \end{aligned}$$

Hence

$$h(t) = \frac{1}{4}\mathcal{L}^{-1}\left[\frac{1}{s^2}\right] - \frac{1}{8}\mathcal{L}^{-1}\left[\frac{2}{s^2+4}\right] = \frac{t}{4} - \frac{1}{8}\sin 2t.$$

$$u_c(t)h(t-c)(s) = \mathcal{L}^{-1}\left[e^{-cs}H(s)\right]$$

Therefore

$$\begin{aligned} y(t) &= \mathcal{L}^{-1}\left[\frac{1}{5}e^{-5s}H(s) - \frac{1}{5}e^{-10s}H(s)\right] \\ &= \\ &= \end{aligned}$$

.

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Therefore

$$\begin{aligned} y(t) &= \mathcal{L}^{-1}\left[\frac{1}{5}e^{-5s}H(s) - \frac{1}{5}e^{-10s}H(s)\right] \\ &= \frac{1}{5}u_5(t)h(t-5) - \frac{1}{5}u_{10}(t)h(t-10) \\ &= \end{aligned}$$

.

$$u_c(t)h(t-c)(s) = \mathcal{L}^{-1}\left[e^{-cs}H(s)\right]$$

Therefore

$$\begin{aligned} y(t) &= \mathcal{L}^{-1}\left[\frac{1}{5}e^{-5s}H(s) - \frac{1}{5}e^{-10s}H(s)\right] \\ &= \frac{1}{5}u_5(t)h(t-5) - \frac{1}{5}u_{10}(t)h(t-10) \\ &= u_5(t)\left(\frac{t-5}{20} - \frac{1}{40}\sin(2t-10)\right) \\ &\qquad - u_{10}(t)\left(\frac{t-10}{20} - \frac{1}{40}\sin(2t-20)\right). \end{aligned}$$

Example

Solve

$$\begin{cases} y'' + 3y' + 2y = f(t) = \begin{cases} 1 & 0 \leq t < 10 \\ 0 & 10 \leq t \end{cases} \\ y(0) = 1 \\ y'(0) = 0. \end{cases}$$

Example

Solve

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Since $f(t) = 1 - u_{10}(t)$, the Laplace Transform of the ODE is

$$(s^2 + 3s + 2)Y - (s + 3) = \frac{1 - e^{-10s}}{s}.$$

Thus

$$Y(s) = \frac{1 - e^{-10s}}{s(s^2 + 3s + 2)} + \frac{s + 3}{s^2 + 3s + 2}$$
$$= \frac{(s^2 + 3s + 1) - e^{-10s}}{s(s^2 + 3s + 2)}.$$

Thus

$$\begin{aligned} Y(s) &= \frac{1-e^{-10s}}{s(s^2+3s+2)} + \frac{s+3}{s^2+3s+2} \\ &= \frac{(s^2+3s+1)-e^{-10s}}{s(s^2+3s+2)}. \end{aligned}$$

Let

$$G(s) = \frac{s^2+3s+1}{s(s^2+3s+2)} \qquad \text{and} \qquad H(s) = \frac{1}{s(s^2+3s+2)}.$$

Thus

$$
\begin{aligned}
Y(s) &= \frac{1-e^{-10s}}{s(s^2+3s+2)} + \frac{s+3}{s^2+3s+2} \\
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Then $Y = G(s) - e^{-10s}H(s)$.

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$$G(s) = \frac{s^2 + 3s + 1}{s(s^2 + 3s + 2)} \qquad \text{and} \qquad H(s) = \frac{1}{s(s^2 + 3s + 2)}.$$

Then $Y = G(s) - e^{-10s} H(s)$. If we can find $g(t)$ and $h(t)$, then we can find $y(t)$.

Using partial fractions we get

$$G(s) = \frac{A}{s} + \frac{B}{s+1} + \frac{C}{s+2} = \frac{\frac{1}{2}}{s} + \frac{1}{s+1} - \frac{\frac{1}{2}}{s+2}$$

and

$$H(s) = \frac{D}{s} + \frac{E}{s+1} + \frac{F}{s+2} = \frac{\frac{1}{2}}{s} - \frac{1}{s+1} + \frac{\frac{1}{2}}{s+2}$$

(please check!).

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(please check!). It follows that

$$g(t) = \frac{1}{2}\left(1 + 2e^{-t} - e^{-2t}\right) \qquad \text{and} \qquad h(t) = \frac{1}{2}\left(1 - 2e^{-t} + e^{-2t}\right).$$

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Therefore

$$\begin{aligned} y(t) &= \mathcal{L}^{-1}\left[Y\right] \\ &= \\ &= \\ &= \qquad . \end{aligned}$$

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$$\begin{aligned} y(t) &= \mathcal{L}^{-1}\left[Y\right] \\ &= \mathcal{L}^{-1}\left[G(s) - e^{-10s}H(s)\right] \\ &= \\ &= \qquad . \end{aligned}$$

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Therefore

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Example

Solve

$$\begin{cases} y'' + 4y = u_\pi(t) - u_{3\pi}(t) \\ y(0) = 0 \\ y'(0) = 0. \end{cases}$$

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Taking the Laplace Transform of the ODE gives

$$(s^2+4)Y(s) = \frac{e^{-\pi s} - e^{-3\pi s}}{s}.$$

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$$Y(s) = \frac{e^{-\pi s} - e^{-3\pi s}}{s(s^2 + 4)}.$$

Let

$$H(s) = \frac{1}{s(s^2 + 4)}.$$

Using partial fractions, we calculate that

$$H(s) = \frac{1}{s(s^2+4)} = \frac{A}{s} + \frac{Bs+C}{s^2+4} = \frac{\frac{1}{4}}{s} + \frac{-\frac{1}{4}s+0}{s^2+4}$$
$$= \frac{1}{4}\left(\frac{1}{s}\right) - \frac{1}{4}\left(\frac{s}{s^2+4}\right) = \frac{1}{4}\mathcal{L}\left[1\right] - \frac{1}{4}\mathcal{L}\left[\,\cos 2t\right].$$

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It follows that

$$h(t) = \frac{1}{4} - \frac{1}{4}\cos 2t$$

and the solution to the IVP is

$$y(t) = \mathcal{L}^{-1}\left[e^{-\pi s}H(s)\right] - \mathcal{L}^{-1}\left[e^{-3\pi s}H(s)\right]$$
$$=$$
$$= \qquad .$$

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$$= u_\pi(t)h(t-\pi) - u_{3\pi}(t)h(t-3\pi)$$
$$= \qquad\qquad .$$

Using partial fractions, we calculate that

$$H(s) = \frac{1}{s(s^2+4)} = \frac{A}{s} + \frac{Bs+C}{s^2+4} = \frac{\frac{1}{4}}{s} + \frac{-\frac{1}{4}s+0}{s^2+4}$$
$$= \frac{1}{4}\left(\frac{1}{s}\right) - \frac{1}{4}\left(\frac{s}{s^2+4}\right) = \frac{1}{4}\mathcal{L}\left[1\right] - \frac{1}{4}\mathcal{L}\left[\cos 2t\right].$$

It follows that

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$$\begin{aligned} y(t) &= \mathcal{L}^{-1}\left[e^{-\pi s}H(s)\right] - \mathcal{L}^{-1}\left[e^{-3\pi s}H(s)\right] \\ &= u_\pi(t)h(t-\pi) - u_{3\pi}(t)h(t-3\pi) \\ &= \frac{1}{4}u_\pi(t)\big(1-\cos(2t-2\pi)\big) - \frac{1}{4}u_{3\pi}(t)\big(1-\cos(2t-6\pi)\big). \end{aligned}$$

The Convolution Integral

Let $f : [0, \infty) \to \mathbb{R}$ and $g : [0, \infty) \to \mathbb{R}$ be piecewise continuous functions.

Definition

The *convolution* of f and g is

$$\big(f * g\big)(t) = \int_0^t f(\tau) g(t - \tau)\, d\tau.$$

Theorem (Properties)

- $f * g = g * f$
- $f * (g + h) = (f * g) + (f * h)$
- $f * (g * h) = (f * g) * h$
- $f * 0 = 0 = 0 * f$

Theorem (Properties)

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- $f * (g * h) = (f * g) * h$
- $f*(g+h) = (f*g)+(f*h)$
- $f * 0 = 0 = 0 * f$

Example

$$\big(\cos *1\big)(t) = \int_0^t \cos\tau \cdot 1\,d\tau = [\sin\tau]_0^t = \sin t - \sin 0 = \sin t$$

$\big(1 * \cos\big)(t) =$

Theorem (Properties)

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Example

$$\big(\cos * 1\big)(t) = \int_0^t \cos\tau \cdot 1\,d\tau = [\sin\tau]_0^t = \sin t - \sin 0 = \sin t$$

$$\begin{aligned}\big(1 * \cos\big)(t) &= \int_0^t 1 \cdot \cos(t-\tau)\,d\tau = [-\sin(t-\tau)]_0^t \\ &= -\sin 0 + \sin t = \sin t\end{aligned}$$

Theorem (Properties)

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- $f * (g * h) = (f * g) * h$
- $f * (g + h) = (f * g) + (f * h)$
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Example

$$(\cos * 1)(t) = \int_0^t \cos\tau \cdot 1\,d\tau = [\sin\tau]_0^t = \sin t - \sin 0 = \sin t$$

$$\begin{aligned}(1 * \cos)(t) &= \int_0^t 1 \cdot \cos(t-\tau)\,d\tau = [-\sin(t-\tau)]_0^t \\ &= -\sin 0 + \sin t = \sin t\end{aligned}$$

Note that $f * 1 \neq f$ in general.

Example

$$
\begin{aligned}
\big(\sin * \sin\big)(t) &= \int_0^t \sin\tau \sin(t-\tau)\,d\tau \\
&= \\
&= \\
&= \\
&= \\
&=
\end{aligned}
$$

Example

$$\begin{aligned}
\big(\sin * \sin\big)(t) &= \int_0^t \sin\tau\,\sin(t-\tau)\,d\tau \\
&= \int_0^t \sin\tau\,(\sin t\cos\tau - \cos t\sin\tau)\;d\tau \\
&= \\
&= \\
&= \\
&=
\end{aligned}$$

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Note that $f * f \geq 0$ is *not* true in general.

Theorem

$$\mathcal{L}\left[f * g\right](s) = F(s)G(s)$$

4.6 $\big(f * g\big)(t) = \int_0^t f(\tau)g(t-\tau)\,d\tau$

Theorem

$$\mathcal{L}\left[f * g\right](s) = F(s)G(s)$$

This means that $\mathcal{L}^{-1}\left[FG\right] = f * g$.

Example

Find the inverse Laplace Transform of $H(s) = \dfrac{a}{s^2(s^2+a^2)}$.

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$$h(t) = \mathcal{L}^{-1}\left[\left(\frac{1}{s^2}\right)\left(\frac{a}{s^2+a^2}\right)\right] =$$

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$$h(t) = \mathcal{L}^{-1}\left[\left(\frac{1}{s^2}\right)\left(\frac{a}{s^2+a^2}\right)\right] = \mathcal{L}^{-1}\left[\frac{1}{s^2}\right] * \mathcal{L}^{-1}\left[\frac{a}{s^2+a^2}\right]$$

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Example

Solve

$$\begin{cases} y'' + 4y = g(t) \\ y(0) = 3 \\ y'(0) = -1. \end{cases}$$

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Taking the Laplace Transform of the ODE gives

$$(s^2Y - 3s + 1) + 4Y = G(s)$$

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Taking the Laplace Transform of the ODE gives

$$(s^2 Y - 3s + 1) + 4Y = G(s)$$

which rearranges to

$$\begin{aligned} Y(s) &= \frac{3s-1}{s^2+4} + \frac{G(s)}{s^2+4} \\ &= 3\left(\frac{s}{s^2+4}\right) - \frac{1}{2}\left(\frac{2}{s^2+4}\right) + \frac{1}{2}\left(\frac{2}{s^2+4}\right) G(s). \end{aligned}$$

$$Y(s) = 3\left(\frac{s}{s^2+4}\right) - \frac{1}{2}\left(\frac{2}{s^2+4}\right) + \frac{1}{2}\left(\frac{2}{s^2+4}\right)G(s)$$

Hence the solution to the IVP is

$$y(t) = 3\mathcal{L}^{-1}\left[\frac{s}{s^2+4}\right] - \frac{1}{2}\mathcal{L}^{-1}\left[\frac{2}{s^2+4}\right] + \frac{1}{2}\mathcal{L}^{-1}\left[\left(\frac{2}{s^2+4}\right)G(s)\right]$$

$$=$$

$$=$$

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$$Y(s) = 3\left(\frac{s}{s^2+4}\right) - \frac{1}{2}\left(\frac{2}{s^2+4}\right) + \frac{1}{2}\left(\frac{2}{s^2+4}\right)G(s)$$

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Example

Find the inverse Laplace Transform of $\dfrac{2}{(s-1)(s^2+4)}$.

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$$\mathcal{L}^{-1}\left[\frac{2}{(s-1)(s^2+4)}\right] = \mathcal{L}^{-1}\left[\left(\frac{2}{s^2+4}\right)\left(\frac{1}{s-1}\right)\right]$$

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&= \int_0^t e^{t-\tau}\sin 2\tau\,d\tau = e^t\int_0^t e^{-\tau}\sin 2\tau\,d\tau \\
&= e^t\left[\frac{e^{-\tau}}{5}\left(-\sin 2\tau - 2\cos 2\tau\right)\right]_0^t
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&= e^t\left[\frac{e^{-\tau}}{5}\left(-\sin 2\tau - 2\cos 2\tau\right)\right]_0^t \\
&= \frac{2}{5}e^t - \frac{1}{5}\sin 2t - \frac{2}{5}\cos 2t.
\end{aligned}$$

Example

Solve

$$\begin{cases} 4y'' + y = g(t) \\ y(0) = 3 \\ y'(0) = -7. \end{cases}$$

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$$4y'' + y \;=\; g(t)$$

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$$\mathcal{L}\left[4y'' + y\right] = \mathcal{L}\left[g(t)\right]$$

$$4(s^2Y - sy(0) - y'(0)) + Y = G(s)$$

Example

Solve

$$\begin{cases} 4y'' + y = g(t) \\ y(0) = 3 \\ y'(0) = -7. \end{cases}$$

$$\mathcal{L}\left[4y'' + y\right] = \mathcal{L}\left[g(t)\right]$$

$$4(s^2Y - 3s + 7) + Y = G(s)$$

Example

Solve

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$$\mathcal{L}\left[4y'' + y\right] = \mathcal{L}\left[g(t)\right]$$

$$(4s^2+1)Y - 12s + 28 = G(s)$$

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$$4\left(s^2 + \frac{1}{4}\right)Y = 12s - 28 + G(s)$$

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Solve

$$\begin{cases} 4y'' + y = g(t) \\ y(0) = 3 \\ y'(0) = -7. \end{cases}$$

$$\mathcal{L}\left[4y'' + y\right] = \mathcal{L}\left[g(t)\right]$$

$$Y = \frac{12s}{4(s^2 + \frac{1}{4})} - \frac{28}{4(s^2 + \frac{1}{4})} + \frac{G(s)}{4(s^2 + \frac{1}{4})}$$

Example

Solve

$$\begin{cases} 4y'' + y = g(t) \\ y(0) = 3 \\ y'(0) = -7. \end{cases}$$

$$Y = \frac{3s}{s^2 + \frac{1}{4}} - \frac{7}{s^2 + \frac{1}{4}} + G(s)\frac{\frac{1}{4}}{s^2 + \frac{1}{4}}$$

Example

Solve

$$\begin{cases} 4y'' + y = g(t) \\ y(0) = 3 \\ y'(0) = -7. \end{cases}$$

$$Y = 3\left(\frac{s}{s^2+\frac{1}{4}}\right) - 14\left(\frac{\frac{1}{2}}{s^2+\frac{1}{4}}\right) + \frac{1}{2}G(s)\left(\frac{\frac{1}{2}}{s^2+\frac{1}{4}}\right)$$

Example

Solve

$$\begin{cases} 4y'' + y = g(t) \\ y(0) = 3 \\ y'(0) = -7. \end{cases}$$

$$Y = 3\mathcal{L}\left[\cos\frac{t}{2}\right] - 14\left(\frac{\frac{1}{2}}{s^2+\frac{1}{4}}\right) + \frac{1}{2}G(s)\left(\frac{\frac{1}{2}}{s^2+\frac{1}{4}}\right)$$

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$$\boxed{y(t) = 3\cos\frac{t}{2}}$$

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$$\boxed{y(t) = 3\cos\frac{t}{2} - 14\sin\frac{t}{2} + \frac{1}{2}g(t) * \sin\frac{t}{2}.}$$

Next Week

- 5.1 Introduction
- 5.2 Basic Theory of Systems of First Order Linear Equations
- 5.3 Homogeneous Linear Systems with Constant Coefficients
- 5.4 Complex Eigenvalues