OKAN ÜNİVERSİTESİ
FEN EDEBİYAT FAKÜLTESİ
MATEMATİK BÖLÜMÜ

2013.04.04 **MAT 234 – Matematik IV – Ara Sınavın Çözümleri** N. Course

Question 1 (Subsequences).

(a) [10p] Let (a_n) be a sequence of real numbers. Give the definition of $a_n \to \infty$ as $n \to \infty$.

We say that (a_n) *tends to infinity* ($a_n \to \infty$ as $n \to \infty$) iff $\forall\ A > 0$ $\boxed{2}$, $\exists\ N \in \mathbb{N}$ $\boxed{2}$ such that

$$n > N \ \boxed{2} \implies \boxed{2}\ a_n > A \ \boxed{2}.$$

(b) [15p] Define $b_n = 2n + 2(-1)^n$ for all $n \in \mathbb{N}$. **Use the definition** that you wrote in part (a) to prove that $b_n \to \infty$ as $n \to \infty$.

Let $A > 0$ $\boxed{4}$. Choose $N \geq 1 + \frac{1}{2}A$ $\boxed{4}$. Then

$$n > N \implies b_n = 2n + 2(-1)^n \geq 2n - 2 > 2N - 2 \geq A \ \boxed{6}.$$

Therefore $b_n \to \infty$ as $n \to \infty$ $\boxed{1}$.

(c) [25p] Suppose that

- $(x_n)_{n=1}^{\infty}$ is a sequence of real numbers;
- $x_n \to \infty$ as $n \to \infty$; and
- $(x_{n_k})_{k=1}^{\infty}$ is a subsequence of $(x_n)_{n=1}^{\infty}$.

Show that $x_{n_k} \to \infty$ as $k \to \infty$.

Let $A > 0$ $\boxed{5}$. Since $x_n \to \infty$ as $n \to \infty$, we know that $\exists\ N \in \mathbb{N}$ $\boxed{5}$ such that

$$n > N \implies x_n > A \ \boxed{4}.$$

But since $n_k \geq k$ for all k $\boxed{4}$, it follows that

$$k > N \implies n_k > N \implies x_{n_k} > A \ \boxed{5}.$$

Therefore $x_{n_k} \to \infty$ as $k \to \infty$ $\boxed{2}$.

Question 2 (Limits of sequences).

(a) [11p] Let (a_n) be a sequence of real numbers and let $l \in \mathbb{R}$. Give the definition of $a_n \to l$ as $n \to \infty$.

We say that (a_n) *tends to* l ($a_n \to l$ as $n \to \infty$) iff $\forall\ \varepsilon > 0$ $\boxed{2}$, $\exists\ N \in \mathbb{N}$ $\boxed{2}$ such that

$$n > N \ \boxed{2} \implies \boxed{2}\ |a_n - l| < \varepsilon \ \boxed{3}.$$

Decide if each of the sequences below has a limit, or does not have a limit, as $n \to \infty$. If the limit exists, find it. Give reasons for your answers. (You may use any theorem or lemma from the course.)

(b) [13p] $b_n = \dfrac{6^n + n!}{n + 7^n}$ (c) [13p] $c_n = \dfrac{6^n + n!}{n + (-7)^n}$ (d) [13p] $d_n = \dfrac{6^n + n!}{n! + (-7)^n}$

3pts for correct limit (or correctly saying that the limit doesn't exist)

10pts for reasonable justification.

incorrect limit with incorrect proof can get up to 5pt depending on mistakes in proof

(b) Since

$$b_n = \frac{6^n + n!}{n + 7^n} = \frac{\left(\frac{6}{7}\right)^n + \frac{n!}{7^n}}{\frac{n}{7^n} + 1} \geq \frac{\frac{n!}{7^n}}{1 + 1} = \frac{1}{2}\frac{n!}{7^n} \to \infty$$

as $n \to \infty$, it follows that $b_n \to \infty$ as $n \to \infty$.

(c) Note first that

$$c_n = \frac{6^n + n!}{n + (-7)^n} = \frac{\left(\frac{6}{-7}\right)^n + (-1)^n \frac{n!}{7^n}}{\frac{n}{(-7)^n} + 1}.$$

Since $\left(\frac{6}{-7}\right)^n \to 0$ and $\frac{n}{(-7)^n} \to 0$ as $n \to \infty$, the dominant term is $(-1)^n \frac{n!}{7^n}$. But $\frac{n!}{7^n} \to \infty$ as $n \to \infty$, hence $c_{2n} \to \infty$ and $c_{2n-1} \to -\infty$ as $n \to \infty$. Therefore (c_n) does not have a limit as $n \to \infty$.

(d)

$$d_n = \frac{6^n + n!}{n! + (-7)^n} = \frac{\frac{6^n}{n!} + 1}{1 + \frac{(-7)^n}{n!}} \to \frac{0 + 1}{1 + 0} = 1$$

as $n \to \infty$.

Question 3 (Sequences)**.** Define a sequence of real numbers (a_n) by

$$a_1 = 50 \qquad \text{and} \qquad 110a_{n+1} = a_n^2 + 1000.$$

(a) [13p] Show that $10 \leq a_n \leq 100$ for all $n \in \mathbb{N}$.
[HINT: Use proof by induction.].

Since $10 \leq a_1 = 50 \leq 100$, the statement is true for $n = 1$ $\boxed{3}$. Suppose that it is true for $n = k$. Then $10 \leq a_k \leq 100$ $\boxed{2}$. So $110a_{k+1} = a_k^2 + 1000 \leq 100^2 + 1000 = 11000 \implies a_{k+1} \leq 100$ $\boxed{3}$ and $110a_{k+1} = a_k^2 + 1000 \geq 10^2 + 1000 = 1100 \implies a_{k+1} \geq 10$ $\boxed{3}$. By the principle of mathematical induction $\boxed{2}$, it follows that $10 \leq a_n \leq 100$ $\forall n \in \mathbb{N}$.

(b) [13p] Show that $a_n \geq a_{n+1}$ for all $n \in \mathbb{N}$.

First note that $a_{n+1} - a_n = \frac{1}{110}(a_n^2 + 1000) - a_n = \frac{1}{110}(a_n^2 - 110a_n + 1000) = \frac{1}{110}(a_n - 10)(a_n - 100)$ $\boxed{5}$. Since $10 \leq a_n \leq 100$, $(a_n - 10) \geq 0$ and $(a_n - 100) \leq 0$ $\boxed{4}$. Therefore $a_{n+1} - a_n = \frac{1}{110}(a_n - 10)(a_n - 100) \leq 0$. So $a_{n+1} \leq a_n$ $\forall n \in \mathbb{N}$ $\boxed{4}$.

(c) [12p] Show that (a_n) is a convergent sequence.

By a theorem from the course, "every decreasing sequence which is bounded below is convergent". In part (a), I proved that (a_n) is bounded below. In part (b), I proved that (a_n) is decreasing. Therefore (a_n) is convergent.

(d) [12p] Calculate $\lim_{n\to\infty} a_n$.

Let $a = \lim_{n\to\infty} a_n$. Then $110a \leftarrow 110a_{n+1} = a_n^2 + 1000 \to a^2 + 1000$ as $n \to \infty$ $\boxed{4}$. Because limits are unique, it follows that $0 = a^2 - 110a + 1000 = (a - 10)(a - 100)$. So $a = 10$ or $a = 100$ $\boxed{4}$. Finally, since (a_n) is a decreasing sequence and since $a_1 = 50$, we must have that $a = 10$ $\boxed{4}$.