

Lecture 4

- 11. Vectors
- 12. The Dot Product
- 13. The Cross Product

Happy Guy Fawkes Night.

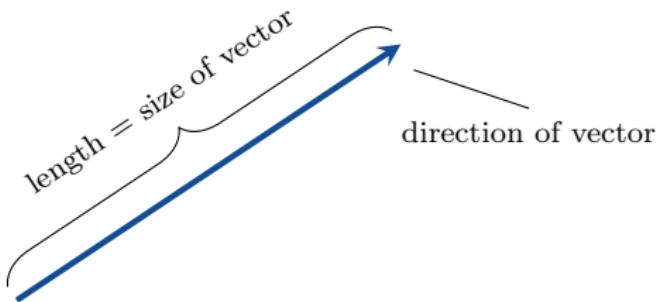


Vectors

11. Vectors

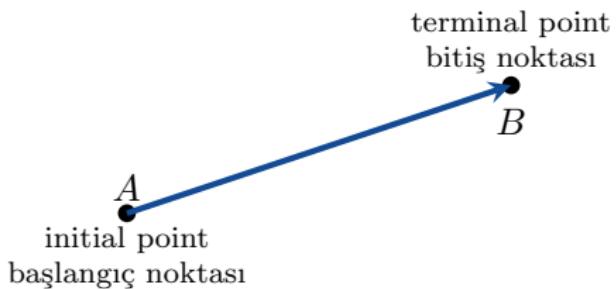


For some quantities (mass, time, distance, ...) we only need a number. For some quantities (velocity, force, ...) we need a number and a direction.



A *vector* is an object which has a size (length) and a direction.

11. Vectors

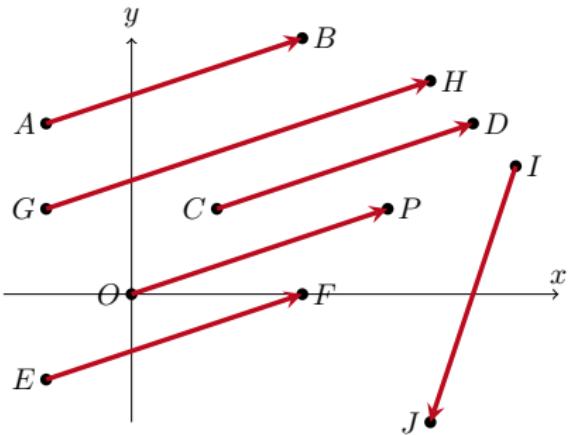


Definition

The vector \overrightarrow{AB} has *initial point* A and *terminal point* B .

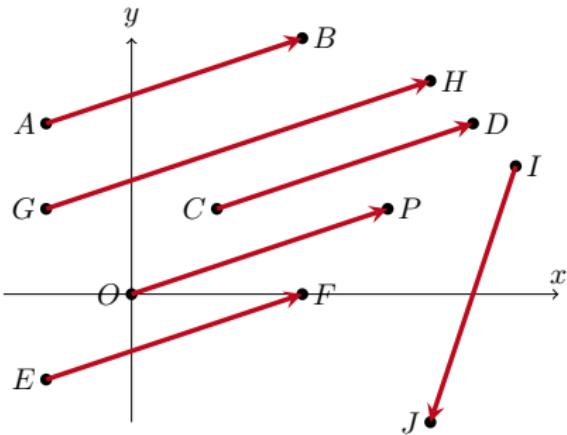
The *length* of \overrightarrow{AB} is written $\|\overrightarrow{AB}\|$ (or $|\overrightarrow{AB}|$).

11. Vectors



Two vectors are equal if they have the same length and the same direction.

11. Vectors

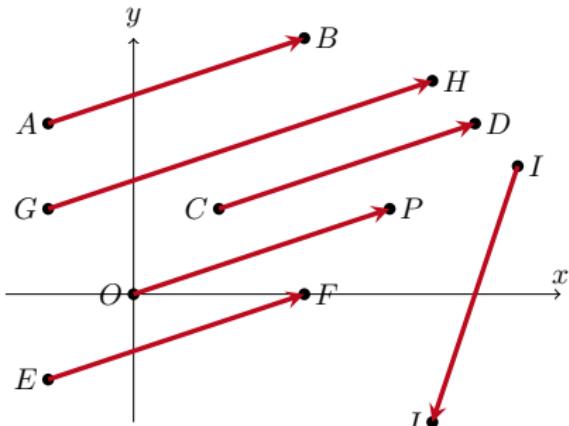


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We can say that

$$\overrightarrow{AB} = \overrightarrow{CD} = \overrightarrow{EF} = \overrightarrow{OP}.$$

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$$\overrightarrow{AB} = \overrightarrow{CD} = \overrightarrow{EF} = \overrightarrow{OP}.$$

Note that $\overrightarrow{AB} \neq \overrightarrow{GH}$ because the lengths are different, and $\overrightarrow{AB} \neq \overrightarrow{IJ}$ because the directions are different.

Notation

When we use a computer, we use bold letters for vectors: \mathbf{u} , \mathbf{v} , \mathbf{w} ,



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When we use a computer, we use bold letters for vectors: \mathbf{u} , \mathbf{v} , \mathbf{w} , . . . When we use a pen, we use underlined letters for vectors: \underline{u} , \underline{v} , \underline{w} , . . .

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If we type $a\mathbf{u} + b\mathbf{v}$ or write $a\underline{u} + b\underline{v}$, then

- a and b are numbers; and
- \mathbf{u} , \mathbf{v} , \underline{u} and \underline{v} are vectors.

11. Vectors



Definition

In \mathbb{R}^2 : If \mathbf{v} has initial point $(0, 0)$ and terminal point (v_1, v_2) , then the *component form* of \mathbf{v} is $\mathbf{v} = (v_1, v_2)$.

11. Vectors

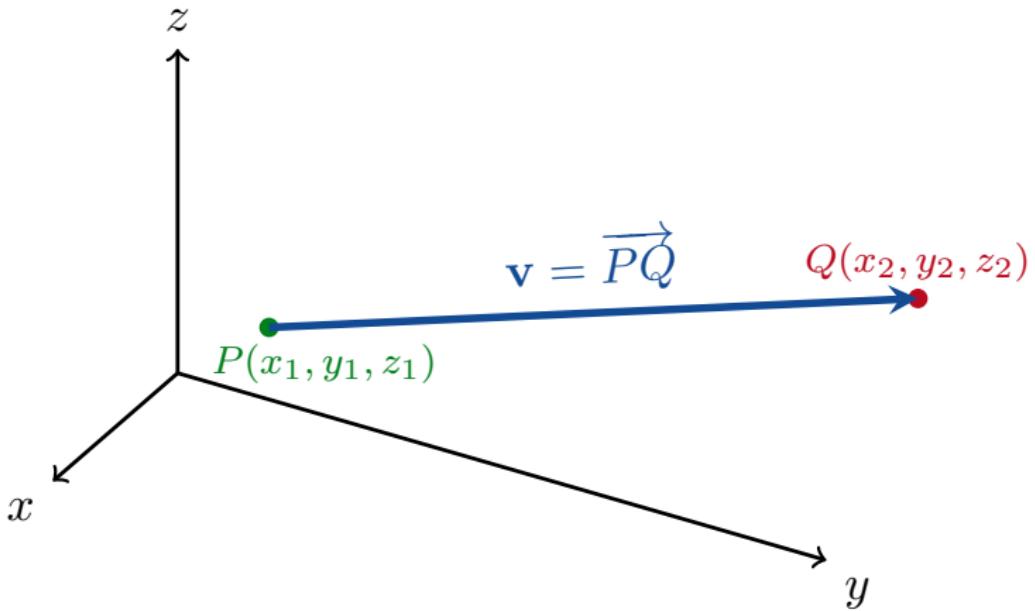


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In \mathbb{R}^2 : If \mathbf{v} has initial point $(0, 0)$ and terminal point (v_1, v_2) , then the *component form* of \mathbf{v} is $\mathbf{v} = (v_1, v_2)$.

In \mathbb{R}^3 : If \mathbf{v} has initial point $(0, 0, 0)$ and terminal point (v_1, v_2, v_3) , then the *component form* of \mathbf{v} is $\mathbf{v} = (v_1, v_2, v_3)$.

11. Vectors



$$(v_1, v_2, v_3) = \mathbf{v} = (x_2 - x_1, y_2 - y_1, z_2 - z_1)$$

11. Vectors



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In \mathbb{R}^2 : The *norm* (or *length*) of $\mathbf{v} = (v_1, v_2)$ is

$$\|\mathbf{v}\| = \sqrt{v_1^2 + v_2^2}$$

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$$\begin{aligned}\|\mathbf{v}\| &= \sqrt{v_1^2 + v_2^2 + v_3^2} \\ &= \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}.\end{aligned}$$

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The vectors $\mathbf{0} = (0, 0)$ and $\mathbf{0} = (0, 0, 0)$ have norm $\|\mathbf{0}\| = 0$. If $\mathbf{v} \neq \mathbf{0}$, then $\|\mathbf{v}\| > 0$.

11. Vectors



Example

Find (1) the component form; and (2) the norm of the vector with initial point $P(-3, 4, 1)$ and terminal point $Q(-5, 2, 2)$.

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 $= (-2, -2, 1).$

2 $\|\mathbf{v}\| = \sqrt{v_1^2 + v_2^2 + v_3^2} = \sqrt{(-2)^2 + (-2)^2 + 1^2} = \sqrt{9} = 3.$

EXAMPLE 2 A small cart is being pulled along a smooth horizontal floor with a 20-lb force \mathbf{F} making a 45° angle to the floor (Figure 12.11). What is the *effective* force moving the cart forward?

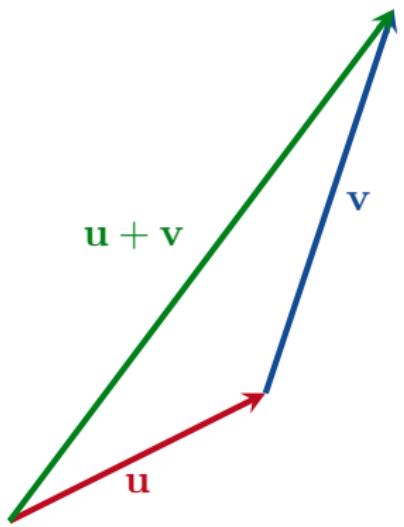
Solution The effective force is the horizontal component of $\mathbf{F} = \langle a, b \rangle$, given by

$$a = |\mathbf{F}| \cos 45^\circ = (20) \left(\frac{\sqrt{2}}{2} \right) \approx 14.14 \text{ lb.}$$

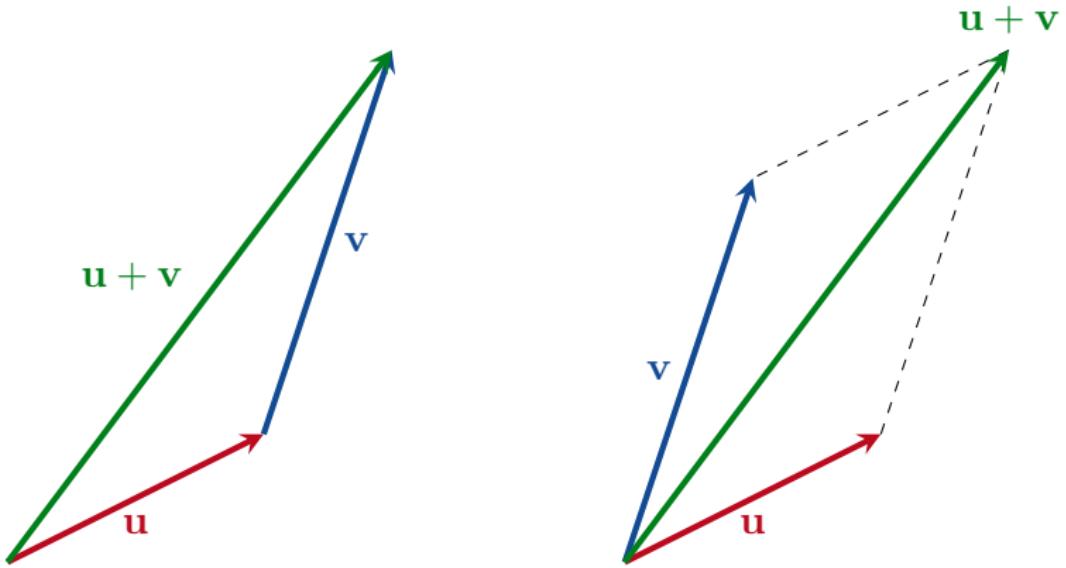
Notice that \mathbf{F} is a two-dimensional vector.



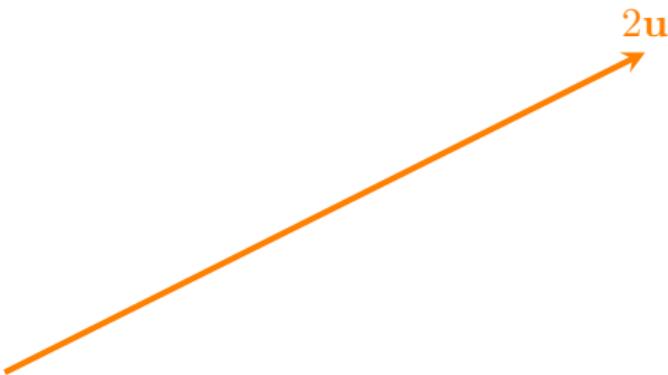
Vector Algebra: Addition



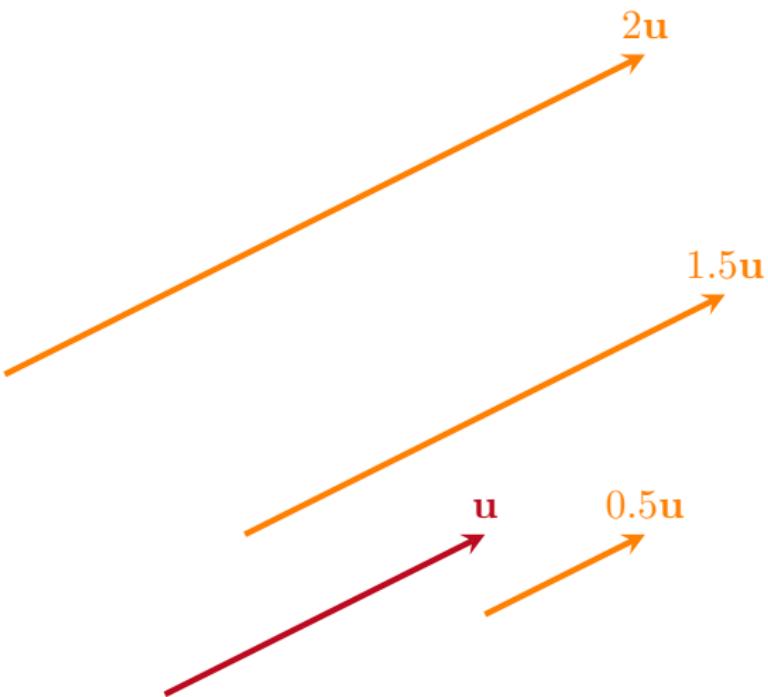
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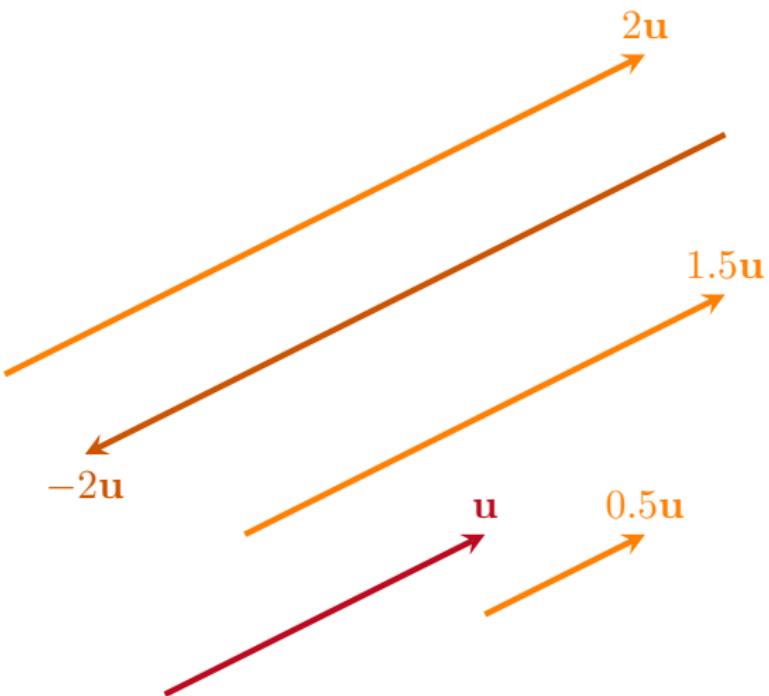
Vector Algebra: Multiplication by a Constant



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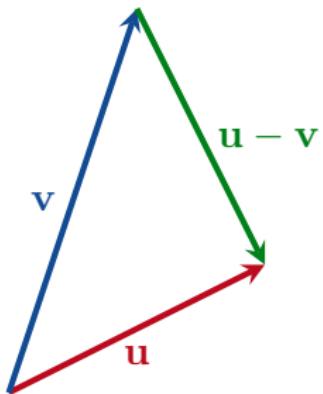


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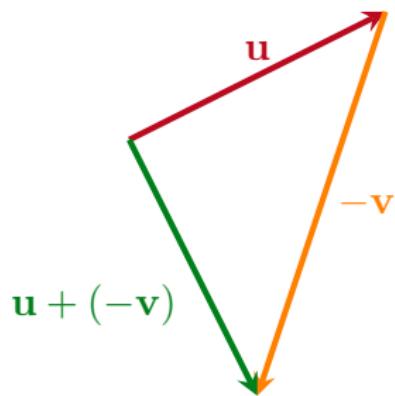
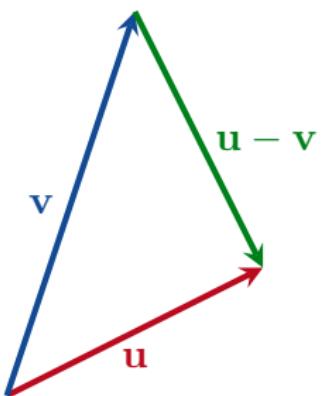
Vector Algebra: Subtraction

$$\mathbf{v} + (\mathbf{u} - \mathbf{v}) = \mathbf{u}$$



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Let $\mathbf{u} = (u_1, u_2, u_3)$ and $\mathbf{v} = (v_1, v_2, v_3)$ be vectors. Let k be a number.

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Let $\mathbf{u} = (u_1, u_2, u_3)$ and $\mathbf{v} = (v_1, v_2, v_3)$ be vectors. Let k be a number. Then

$$\mathbf{u} + \mathbf{v} = (u_1 + v_1, u_2 + v_2, u_3 + v_3)$$

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and

$$k\mathbf{u} = (ku_1, ku_2, ku_3).$$

11. Vectors



Note that

$$\|k\mathbf{u}\| = \|(ku_1, ku_2, ku_3)\|$$

$$=$$

$$=$$

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11. Vectors



Note that

$$\begin{aligned}\|k\mathbf{u}\| &= \|(ku_1, ku_2, ku_3)\| \\ &= \sqrt{(ku_1)^2 + (ku_2)^2 + (ku_3)^2}\end{aligned}$$

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Note that

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$$\begin{aligned}\|k\mathbf{u}\| &= \|(ku_1, ku_2, ku_3)\| \\&= \sqrt{(ku_1)^2 + (ku_2)^2 + (ku_3)^2} \\&= \sqrt{k^2u_1^2 + k^2u_2^2 + k^2u_3^2} \\&= \sqrt{k^2(u_1^2 + u_2^2 + u_3^2)} \\&= \\&= .\end{aligned}$$

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11. Vectors



The vector $-\mathbf{u} = (-1)\mathbf{u}$ has the same length as \mathbf{u} , but points in the opposite direction.

11. Vectors



Example

Let $\mathbf{u} = (-1, 3, 1)$ and $\mathbf{v} = (4, 7, 0)$.

Find $2\mathbf{u} + 3\mathbf{v}$, $\mathbf{u} - \mathbf{v}$, and $\left\| \frac{1}{2}\mathbf{u} \right\|$.

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3 $\left\| \frac{1}{2}\mathbf{u} \right\| = \frac{1}{2} \left\| \mathbf{u} \right\| = \frac{1}{2} \sqrt{(-1)^2 + 3^2 + 1^2} = \frac{1}{2} \sqrt{11}.$

Properties of Vector Operations

Let \mathbf{u} , \mathbf{v} and \mathbf{w} be vectors. Let a and b be numbers. Then

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- 7 $a(b\mathbf{u}) = (ab)\mathbf{u};$
- 8 $a(\mathbf{u} + \mathbf{v}) = a\mathbf{u} + a\mathbf{v};$
- 9 $(a + b)\mathbf{u} = a\mathbf{u} + b\mathbf{u}.$

11. Vectors



Remark

We **can not** multiply vectors. Never never never never write "**uv**".

Unit Vectors

Definition

\mathbf{u} is called a *unit vector* $\iff \|\mathbf{u}\| = 1$.

11. Vectors



Example

$\mathbf{u} = (2^{-\frac{1}{2}}, \frac{1}{2}, -\frac{1}{2})$ is a unit vector because

$$\|\mathbf{u}\| = \sqrt{\left(\frac{1}{\sqrt{2}}\right)^2 + \left(\frac{1}{2}\right)^2 + \left(-\frac{1}{2}\right)^2} = \sqrt{\frac{1}{2} + \frac{1}{4} + \frac{1}{4}} = 1.$$

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Standard Unit Vectors

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In \mathbb{R}^3 : The *standard unit vectors* are $\mathbf{i} = (1, 0, 0)$, $\mathbf{j} = (0, 1, 0)$ and $\mathbf{k} = (0, 0, 1)$. Any vector $\mathbf{v} \in \mathbb{R}^3$ can be written

$$\begin{aligned}\mathbf{v} &= (v_1, v_2, v_3) = (v_1, 0, 0) + (0, v_2, 0) + (0, 0, v_3) \\ &= v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}.\end{aligned}$$

Normalising a Vector

If $\|\mathbf{v}\| \neq 0$, then $\frac{\mathbf{v}}{\|\mathbf{v}\|}$ is a unit vector because

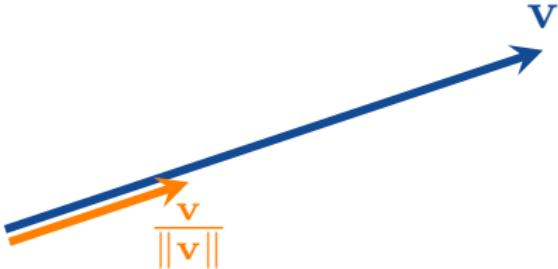
$$\left\| \frac{\mathbf{v}}{\|\mathbf{v}\|} \right\| = \left\| \frac{1}{\|\mathbf{v}\|} \mathbf{v} \right\| = \frac{1}{\|\mathbf{v}\|} \|\mathbf{v}\| = 1.$$

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Clearly $\frac{\mathbf{v}}{\|\mathbf{v}\|}$ and \mathbf{v} point in the same direction.



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We calculate that

$$\overrightarrow{P_1P_2} = P_2 - P_1 = (3, 2, 0) - (1, 0, 1) = (2, 2, -1) = 2\mathbf{i} + 2\mathbf{j} - \mathbf{k}$$

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$$\left\| \overrightarrow{P_1P_2} \right\| = \sqrt{2^2 + 2^2 + (-1)^2} = 3.$$

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and that

$$\left\| \overrightarrow{P_1P_2} \right\| = \sqrt{2^2 + 2^2 + (-1)^2} = 3.$$

The required unit vector is

$$\mathbf{u} = \frac{\overrightarrow{P_1P_2}}{\left\| \overrightarrow{P_1P_2} \right\|} = \frac{2\mathbf{i} + 2\mathbf{j} - \mathbf{k}}{3} = \frac{2}{3}\mathbf{i} + \frac{2}{3}\mathbf{j} - \frac{1}{3}\mathbf{k}.$$

EXAMPLE 5 If $\mathbf{v} = 3\mathbf{i} - 4\mathbf{j}$ is a velocity vector, express \mathbf{v} as a product of its speed times its direction of motion.

Solution Speed is the magnitude (length) of \mathbf{v} :

$$|\mathbf{v}| = \sqrt{(3)^2 + (-4)^2} = \sqrt{9 + 16} = 5.$$

The unit vector $\mathbf{v}/|\mathbf{v}|$ is the direction of \mathbf{v} :

$$\frac{\mathbf{v}}{|\mathbf{v}|} = \frac{3\mathbf{i} - 4\mathbf{j}}{5} = \frac{3}{5}\mathbf{i} - \frac{4}{5}\mathbf{j}.$$

So

$$\mathbf{v} = 3\mathbf{i} - 4\mathbf{j} = 5 \left(\underbrace{\frac{3}{5}\mathbf{i} - \frac{4}{5}\mathbf{j}}_{\substack{\text{Length} \\ (\text{speed})}} \right).$$

■

If $\mathbf{v} \neq \mathbf{0}$, then

1. $\frac{\mathbf{v}}{|\mathbf{v}|}$ is a unit vector called the direction of \mathbf{v} ;
2. the equation $\mathbf{v} = |\mathbf{v}| \frac{\mathbf{v}}{|\mathbf{v}|}$ expresses \mathbf{v} as its length times its direction.

EXAMPLE 6 A force of 6 newtons is applied in the direction of the vector $\mathbf{v} = 2\mathbf{i} + 2\mathbf{j} - \mathbf{k}$. Express the force \mathbf{F} as a product of its magnitude and direction.

Solution The force vector has magnitude 6 and direction $\frac{\mathbf{v}}{|\mathbf{v}|}$, so

$$\begin{aligned}\mathbf{F} &= 6 \frac{\mathbf{v}}{|\mathbf{v}|} = 6 \frac{2\mathbf{i} + 2\mathbf{j} - \mathbf{k}}{\sqrt{2^2 + 2^2 + (-1)^2}} = 6 \frac{2\mathbf{i} + 2\mathbf{j} - \mathbf{k}}{3} \\ &= 6 \left(\frac{2}{3}\mathbf{i} + \frac{2}{3}\mathbf{j} - \frac{1}{3}\mathbf{k} \right).\end{aligned}$$



Midpoint of a Line Segment

Vectors are often useful in geometry. For example, the coordinates of the midpoint of a line segment are found by averaging.

The **midpoint** M of the line segment joining points $P_1(x_1, y_1, z_1)$ and $P_2(x_2, y_2, z_2)$ is the point

$$\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}, \frac{z_1 + z_2}{2} \right).$$

To see why, observe (Figure 12.16) that

$$\begin{aligned}\overrightarrow{OM} &= \overrightarrow{OP}_1 + \frac{1}{2}(\overrightarrow{P_1P_2}) = \overrightarrow{OP}_1 + \frac{1}{2}(\overrightarrow{OP}_2 - \overrightarrow{OP}_1) \\ &= \frac{1}{2}(\overrightarrow{OP}_1 + \overrightarrow{OP}_2) \\ &= \frac{x_1 + x_2}{2}\mathbf{i} + \frac{y_1 + y_2}{2}\mathbf{j} + \frac{z_1 + z_2}{2}\mathbf{k}.\end{aligned}$$

EXAMPLE 7 The midpoint of the segment joining $P_1(3, -2, 0)$ and $P_2(7, 4, 4)$ is

$$\left(\frac{3+7}{2}, \frac{-2+4}{2}, \frac{0+4}{2} \right) = (5, 1, 2). \quad \blacksquare$$

11. Vectors



Please read the final two examples in this section of the textbook.



The Dot Product

12. The Dot Product

Definition

In \mathbb{R}^2 , the *dot product* of $\mathbf{u} = (u_1, u_2) = u_1\mathbf{i} + u_2\mathbf{j}$ and $\mathbf{v} = (v_1, v_2) = v_1\mathbf{i} + v_2\mathbf{j}$ is

$$\mathbf{u} \cdot \mathbf{v} = u_1v_1 + u_2v_2.$$

12. The Dot Product

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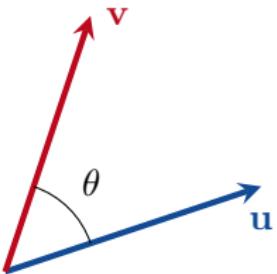
$$\mathbf{u} \cdot \mathbf{v} = u_1v_1 + u_2v_2.$$

Definition

In \mathbb{R}^3 , the *dot product* of $\mathbf{u} = (u_1, u_2, u_3) = u_1\mathbf{i} + u_2\mathbf{j} + u_3\mathbf{k}$ and $\mathbf{v} = (v_1, v_2, v_3) = v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}$ is

$$\mathbf{u} \cdot \mathbf{v} = u_1v_1 + u_2v_2 + u_3v_3.$$

12. The Dot Product

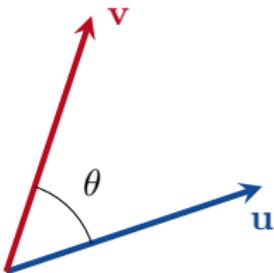


Theorem

The angle between \mathbf{u} and \mathbf{v} is

$$\theta = \cos^{-1} \left(\frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{u}\| \|\mathbf{v}\|} \right).$$

12. The Dot Product



Theorem

The angle between \mathbf{u} and \mathbf{v} is

$$\theta = \cos^{-1} \left(\frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{u}\| \|\mathbf{v}\|} \right).$$

This means that

$$\mathbf{u} \cdot \mathbf{v} = \|\mathbf{u}\| \|\mathbf{v}\| \cos \theta.$$

12. The Dot Product

Example

$$\begin{aligned}(1, -2, -1) \cdot (-6, 2, -3) &= (1 \times -6) + (-2 \times 2) + (-1 \times -3) \\&= -6 - 4 + 3 = -7.\end{aligned}$$

12. The Dot Product

Example

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Example

$$\begin{aligned}\left(\frac{1}{2}\mathbf{i} + 3\mathbf{j} + \mathbf{k}\right) \cdot (4\mathbf{i} - \mathbf{j} + 2\mathbf{k}) &= \left(\frac{1}{2} \times 4\right) + (3 \times -1) + (1 \times 2) \\&= 2 - 3 + 2 = 1.\end{aligned}$$

12.

$$\theta = \cos^{-1} \left(\frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{u}\| \|\mathbf{v}\|} \right).$$



Example

Find the angle between $\mathbf{u} = \mathbf{i} - 2\mathbf{j} - 2\mathbf{k}$ and $\mathbf{v} = 6\mathbf{i} + 3\mathbf{j} + 2\mathbf{k}$.

12.

$$\theta = \cos^{-1} \left(\frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{u}\| \|\mathbf{v}\|} \right).$$



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Since

$$\begin{aligned}\mathbf{u} \cdot \mathbf{v} &= (1, -2, -2) \cdot (6, 3, 2) = (1 \times 6) + (-2 \times 3) + (-2 \times 2) \\ &= 6 - 6 - 4 = -4,\end{aligned}$$

12.

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Find the angle between $\mathbf{u} = \mathbf{i} - 2\mathbf{j} - 2\mathbf{k}$ and $\mathbf{v} = 6\mathbf{i} + 3\mathbf{j} + 2\mathbf{k}$.

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$$\|\mathbf{u}\| = \sqrt{1^2 + (-2)^2 + (-2)^2} = \sqrt{9} = 3$$

and

$$\|\mathbf{v}\| = \sqrt{6^2 + 3^2 + 2^2} = \sqrt{49} = 7,$$

12.

$$\theta = \cos^{-1} \left(\frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{u}\| \|\mathbf{v}\|} \right).$$

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Find the angle between $\mathbf{u} = \mathbf{i} - 2\mathbf{j} - 2\mathbf{k}$ and $\mathbf{v} = 6\mathbf{i} + 3\mathbf{j} + 2\mathbf{k}$.

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we have that

$$\theta = \cos^{-1} \left(\frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{u}\| \|\mathbf{v}\|} \right) =$$

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Example

Find the angle between $\mathbf{u} = \mathbf{i} - 2\mathbf{j} - 2\mathbf{k}$ and $\mathbf{v} = 6\mathbf{i} + 3\mathbf{j} + 2\mathbf{k}$.

Since

$$\begin{aligned}\mathbf{u} \cdot \mathbf{v} &= (1, -2, -2) \cdot (6, 3, 2) = (1 \times 6) + (-2 \times 3) + (-2 \times 2) \\ &= 6 - 6 - 4 = -4,\end{aligned}$$

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and

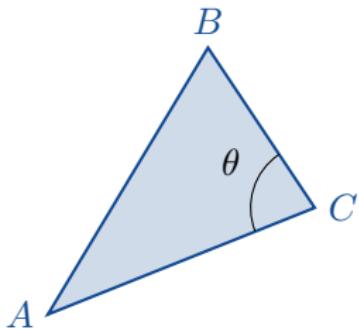
$$\|\mathbf{v}\| = \sqrt{6^2 + 3^2 + 2^2} = \sqrt{49} = 7,$$

we have that

$$\theta = \cos^{-1} \left(\frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{u}\| \|\mathbf{v}\|} \right) = \cos^{-1} \left(-\frac{4}{21} \right) \approx 1.76 \text{ radians} \approx 98.5^\circ.$$

12.

$$\theta = \cos^{-1} \left(\frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{u}\| \|\mathbf{v}\|} \right).$$



Example

If $A(0, 0)$, $B(3, 5)$ and $C(5, 2)$, find $\theta = \angle ACB$.

12. The Dot Product



θ is the angle between \overrightarrow{CA} and \overrightarrow{CB} .

12. The Dot Product

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$$\overrightarrow{CA} = A - C = (0, 0) - (5, 2) = (-5, -2),$$

$$\overrightarrow{CB} = B - C = (3, 5) - (5, 2) = (-2, 3),$$

$$\overrightarrow{CA} \cdot \overrightarrow{CB} = (-5, -2) \cdot (-2, 3) = 4,$$

$$\|\overrightarrow{CA}\| = \sqrt{(-5)^2 + (-2)^2} = \sqrt{29}$$

and

$$\|\overrightarrow{CB}\| = \sqrt{(-2)^2 + 3^2} = \sqrt{13}.$$

12. The Dot Product

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$$\overrightarrow{CA} \cdot \overrightarrow{CB} = (-5, -2) \cdot (-2, 3) = 4,$$

$$\|\overrightarrow{CA}\| = \sqrt{(-5)^2 + (-2)^2} = \sqrt{29}$$

and

$$\|\overrightarrow{CB}\| = \sqrt{(-2)^2 + 3^2} = \sqrt{13}.$$

Therefore

$$\theta = \cos^{-1} \left(\frac{\overrightarrow{CA} \cdot \overrightarrow{CB}}{\|\overrightarrow{CA}\| \|\overrightarrow{CB}\|} \right) = \cos^{-1} \left(\frac{4}{\sqrt{29}\sqrt{13}} \right)$$

$$\approx 78.1^\circ \approx 1.36 \text{ radians.}$$

12. The Dot Product

Definition

\mathbf{u} and \mathbf{v} are *orthogonal* $\iff \mathbf{u} \cdot \mathbf{v} = 0$.

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Remark

Recall that

$$\mathbf{u} \cdot \mathbf{v} = \|\mathbf{u}\| \|\mathbf{v}\| \cos \theta.$$

12. The Dot Product

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\mathbf{u} and \mathbf{v} are *orthogonal* $\iff \mathbf{u} \cdot \mathbf{v} = 0$.

Remark

Recall that

$$\mathbf{u} \cdot \mathbf{v} = \|\mathbf{u}\| \|\mathbf{v}\| \cos \theta.$$

Therefore

$$\mathbf{u} \text{ and } \mathbf{v} \text{ are orthogonal} \iff \begin{cases} \mathbf{u} = \mathbf{0} \\ \text{or} \\ \mathbf{v} = \mathbf{0} \\ \text{or} \\ \theta = 90^\circ. \end{cases}$$

12. The Dot Product

Example

$\mathbf{u} = (3, -2)$ and $\mathbf{v} = (4, 6)$ are orthogonal because

$$\mathbf{u} \cdot \mathbf{v} = (3, -2) \cdot (4, 6) = (3 \times 4) + (-2 \times 6) = 12 - 12 = 0.$$

12. The Dot Product

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Example

$\mathbf{u} = 3\mathbf{i} - 2\mathbf{j} + \mathbf{k}$ and $\mathbf{v} = 2\mathbf{j} + 4\mathbf{k}$ are orthogonal because

$$\mathbf{u} \cdot \mathbf{v} = (3 \times 0) + (-2 \times 2) + (1 \times 4) = 0 - 4 + 4 = 0.$$

12. The Dot Product

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Example

$\mathbf{u} = 3\mathbf{i} - 2\mathbf{j} + \mathbf{k}$ and $\mathbf{v} = 2\mathbf{j} + 4\mathbf{k}$ are orthogonal because

$$\mathbf{u} \cdot \mathbf{v} = (3 \times 0) + (-2 \times 2) + (1 \times 4) = 0 - 4 + 4 = 0.$$

Example

$\mathbf{0}$ is orthogonal to every vector \mathbf{u} because

$$\mathbf{0} \cdot \mathbf{u} = (0, 0, 0) \cdot (u_1, u_2, u_3) = 0u_1 + 0u_2 + 0u_3 = 0.$$

12. The Dot Product



Properties of the Dot Product

Let \mathbf{u} , \mathbf{v} and \mathbf{w} be vectors. Let k be a number. Then

1 $\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u};$

12. The Dot Product



Properties of the Dot Product

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- 1 $\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u}$;
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12. The Dot Product



Properties of the Dot Product

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- 3 $\mathbf{u} \cdot (\mathbf{v} + \mathbf{w}) = (\mathbf{u} \cdot \mathbf{v}) + (\mathbf{u} \cdot \mathbf{w})$;

12. The Dot Product



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- 3 $\mathbf{u} \cdot (\mathbf{v} + \mathbf{w}) = (\mathbf{u} \cdot \mathbf{v}) + (\mathbf{u} \cdot \mathbf{w})$;
- 4 $\mathbf{u} \cdot \mathbf{u} = \|\mathbf{u}\|^2$; and

12. The Dot Product



Properties of the Dot Product

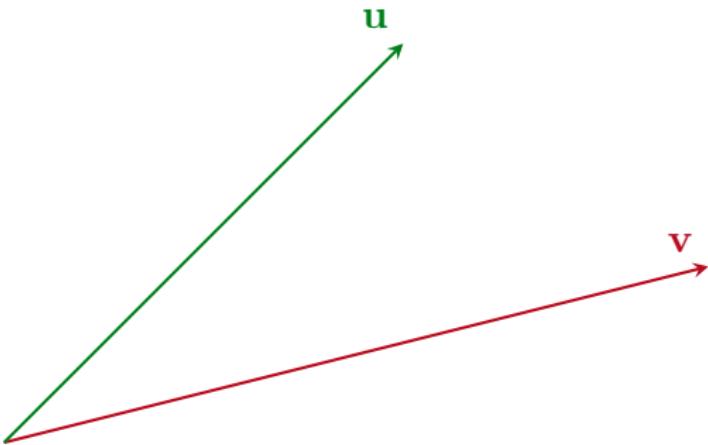
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- 3 $\mathbf{u} \cdot (\mathbf{v} + \mathbf{w}) = (\mathbf{u} \cdot \mathbf{v}) + (\mathbf{u} \cdot \mathbf{w})$;
- 4 $\mathbf{u} \cdot \mathbf{u} = \|\mathbf{u}\|^2$; and
- 5 $\mathbf{0} \cdot \mathbf{u} = 0$.

12. The Dot Product



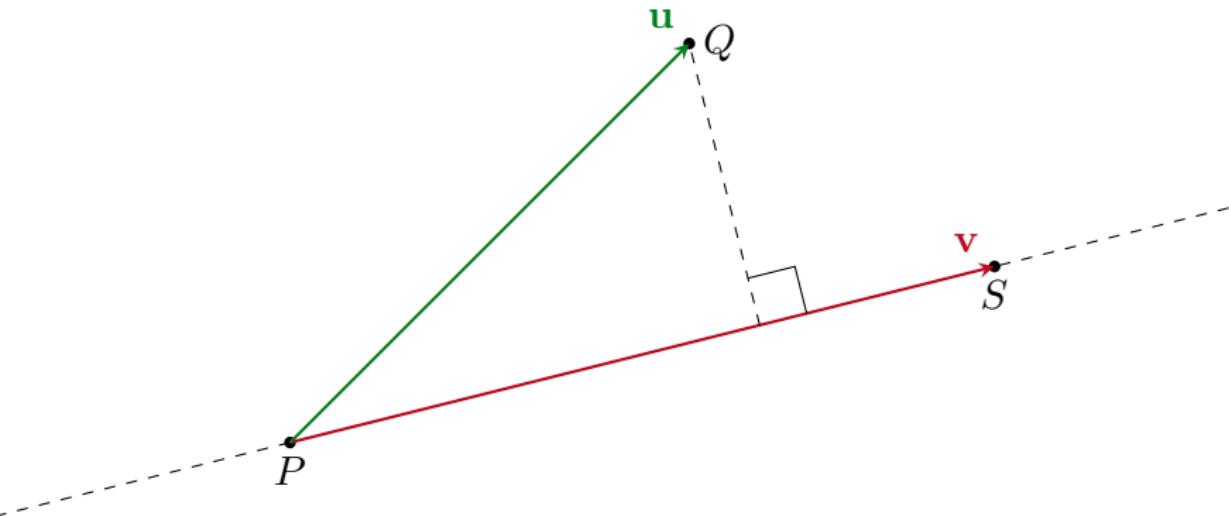
Vector Projections



12. The Dot Product



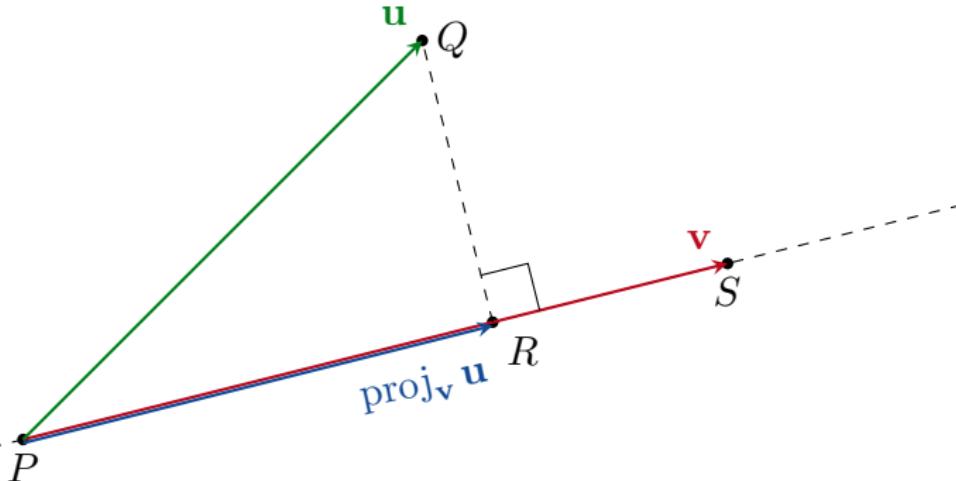
Vector Projections



12. The Dot Product



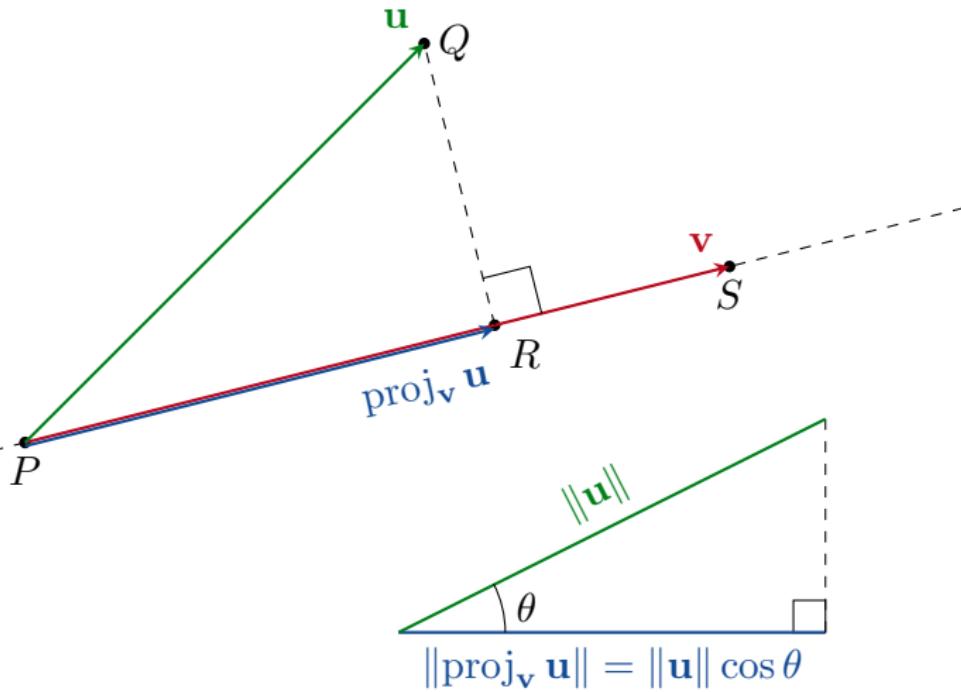
Vector Projections



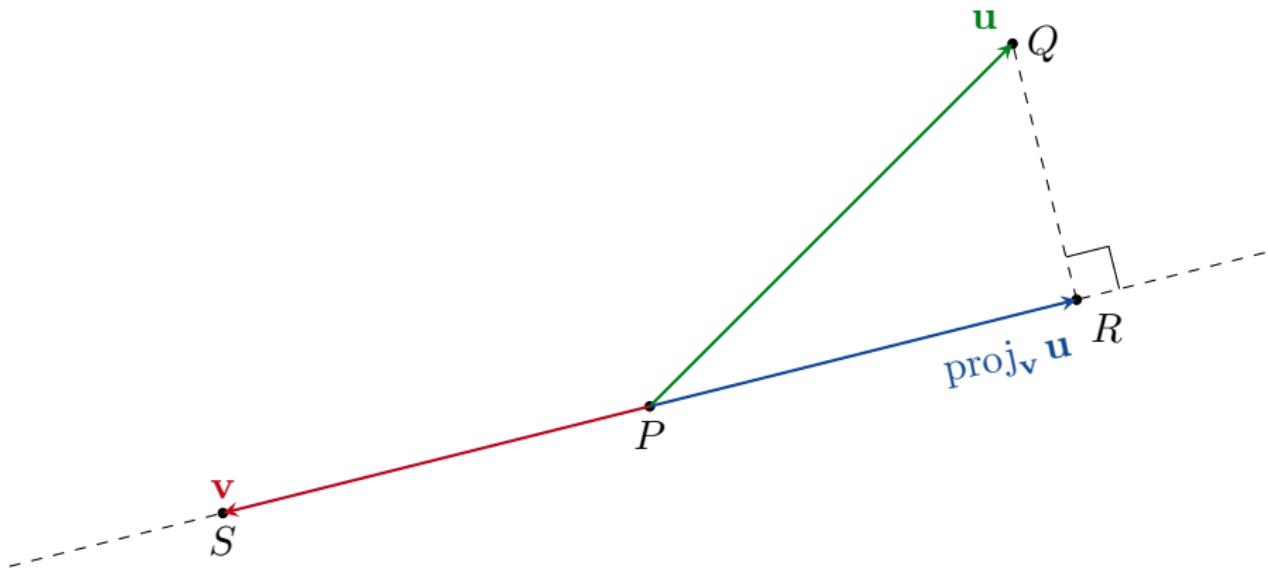
12. The Dot Product



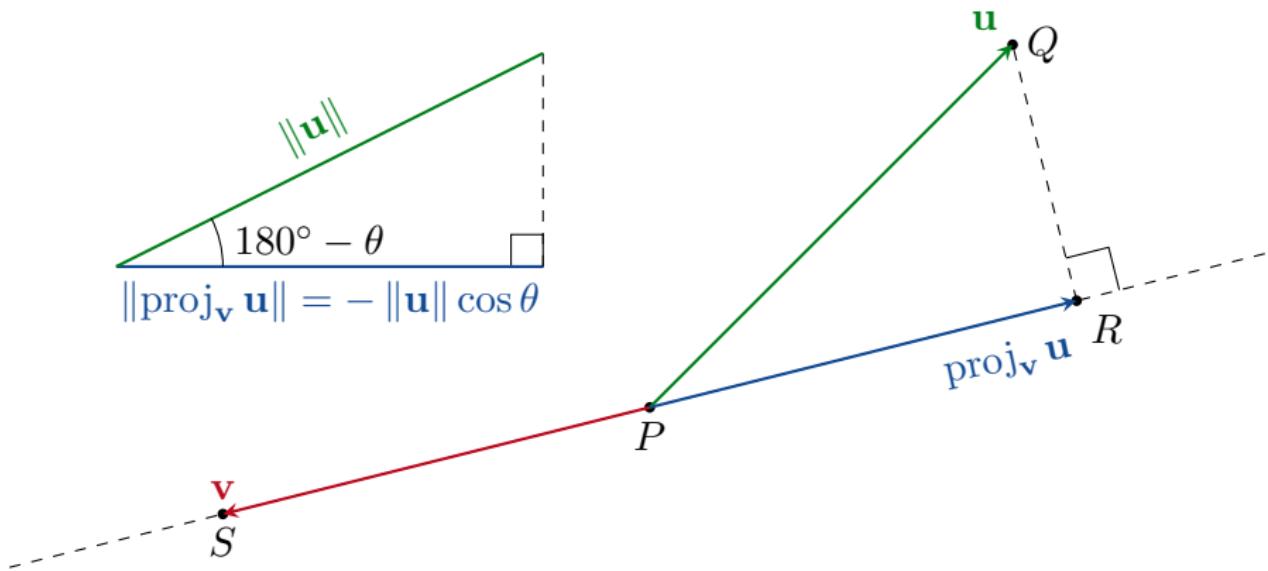
Vector Projections



12. The Dot Product



12. The Dot Product



12. The Dot Product



Definition

The *vector projection* of \mathbf{u} onto \mathbf{v} is the vector

$$\text{proj}_{\mathbf{v}} \mathbf{u} = \overrightarrow{PR}.$$

12. The Dot Product

Now

$$\text{proj}_{\mathbf{v}} \mathbf{u} = (\text{length of } \text{proj}_{\mathbf{v}} \mathbf{u}) \begin{pmatrix} \text{a unit vector in} \\ \text{the same} \\ \text{direction as } \mathbf{v} \end{pmatrix}$$

=

=

=

=

12. The Dot Product

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$$= \|\text{proj}_{\mathbf{v}} \mathbf{u}\| \left(\frac{\mathbf{v}}{\|\mathbf{v}\|} \right)$$

=

=

=



$$\|\text{proj}_{\mathbf{v}} \mathbf{u}\| = \|\mathbf{u}\| \cos \theta$$

12. The Dot Product

Now

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$$= \|\text{proj}_{\mathbf{v}} \mathbf{u}\| \left(\frac{\mathbf{v}}{\|\mathbf{v}\|} \right)$$

$$= \|\mathbf{u}\| (\cos \theta) \left(\frac{\mathbf{v}}{\|\mathbf{v}\|} \right)$$

=

=



$$\|\text{proj}_{\mathbf{v}} \mathbf{u}\| = \|\mathbf{u}\| \cos \theta$$

12. The Dot Product

Now

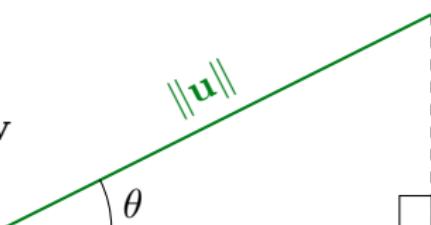
$$\text{proj}_{\mathbf{v}} \mathbf{u} = (\text{length of } \text{proj}_{\mathbf{v}} \mathbf{u}) \begin{pmatrix} \text{a unit vector in} \\ \text{the same} \\ \text{direction as } \mathbf{v} \end{pmatrix}$$

$$= \|\text{proj}_{\mathbf{v}} \mathbf{u}\| \left(\frac{\mathbf{v}}{\|\mathbf{v}\|} \right)$$

$$= \|\mathbf{u}\| (\cos \theta) \left(\frac{\mathbf{v}}{\|\mathbf{v}\|} \right)$$

$$= \left(\frac{\|\mathbf{u}\| \|\mathbf{v}\| \cos \theta}{\|\mathbf{v}\|^2} \right) \mathbf{v}$$

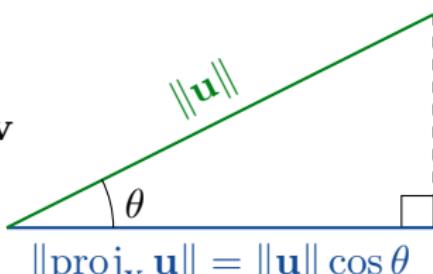
$$= \|\text{proj}_{\mathbf{v}} \mathbf{u}\| = \|\mathbf{u}\| \cos \theta$$



12. The Dot Product

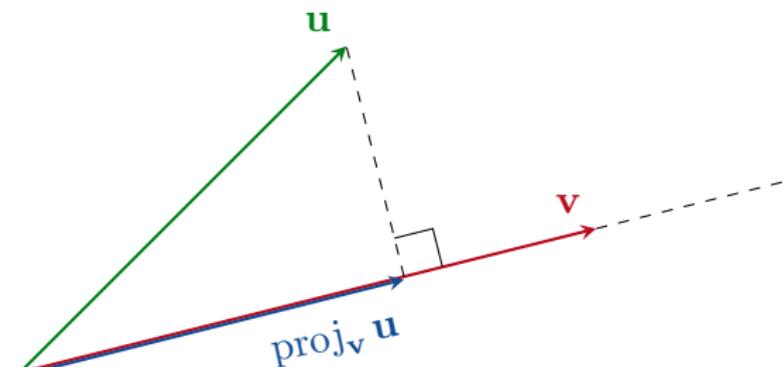
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$$\begin{aligned}\text{proj}_{\mathbf{v}} \mathbf{u} &= (\text{length of } \text{proj}_{\mathbf{v}} \mathbf{u}) \begin{pmatrix} \text{a unit vector in} \\ \text{the same} \\ \text{direction as } \mathbf{v} \end{pmatrix} \\ &= \|\text{proj}_{\mathbf{v}} \mathbf{u}\| \left(\frac{\mathbf{v}}{\|\mathbf{v}\|} \right) \\ &= \|\mathbf{u}\| (\cos \theta) \left(\frac{\mathbf{v}}{\|\mathbf{v}\|} \right) \\ &= \left(\frac{\|\mathbf{u}\| \|\mathbf{v}\| \cos \theta}{\|\mathbf{v}\|^2} \right) \mathbf{v} \\ &= \left(\frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{v}\|^2} \right) \mathbf{v}.\end{aligned}$$



Since this is an important formula, we write it as a theorem.

12. The Dot Product



Theorem

The vector projection of \mathbf{u} onto \mathbf{v} is

$$\text{proj}_{\mathbf{v}} \mathbf{u} = \left(\frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{v}\|^2} \right) \mathbf{v}.$$

12.

$$\text{proj}_{\mathbf{v}} \mathbf{u} = \left(\frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{v}\|^2} \right) \mathbf{v}$$



Example

Find the vector projection of $\mathbf{u} = 6\mathbf{i} + 3\mathbf{j} + 2\mathbf{k}$ onto $\mathbf{v} = \mathbf{i} - 2\mathbf{j} - 2\mathbf{k}$.

12.

$$\text{proj}_{\mathbf{v}} \mathbf{u} = \left(\frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{v}\|^2} \right) \mathbf{v}$$



Example

Find the vector projection of $\mathbf{u} = 6\mathbf{i} + 3\mathbf{j} + 2\mathbf{k}$ onto $\mathbf{v} = \mathbf{i} - 2\mathbf{j} - 2\mathbf{k}$.

$$\begin{aligned}\text{proj}_{\mathbf{v}} \mathbf{u} &= \left(\frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{v}\|^2} \right) \mathbf{v} = \left(\frac{6 - 6 - 4}{1 + 4 + 4} \right) (\mathbf{i} - 2\mathbf{j} - 2\mathbf{k}) \\ &= -\frac{4}{9}\mathbf{i} + \frac{8}{9}\mathbf{j} + \frac{8}{9}\mathbf{k}.\end{aligned}$$

12.

$$\text{proj}_{\mathbf{v}} \mathbf{u} = \left(\frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{v}\|^2} \right) \mathbf{v}$$



Example

Find the vector projection of $\mathbf{F} = 5\mathbf{i} + 2\mathbf{j}$ onto $\mathbf{v} = \mathbf{i} - 3\mathbf{j}$.

$$\begin{aligned}\text{proj}_{\mathbf{v}} \mathbf{F} &= \left(\frac{\mathbf{F} \cdot \mathbf{v}}{\|\mathbf{v}\|^2} \right) \mathbf{v} = \left(\frac{5 - 6}{1 + 9} \right) (\mathbf{i} - 3\mathbf{j}) \\ &= -\frac{1}{10}\mathbf{i} + \frac{3}{10}\mathbf{j}.\end{aligned}$$

12.

$$\text{proj}_{\mathbf{v}} \mathbf{u} = \left(\frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{v}\|^2} \right) \mathbf{v}$$



Example

Verify that the vector $\mathbf{u} - \text{proj}_{\mathbf{v}} \mathbf{u}$ is orthogonal to $\text{proj}_{\mathbf{v}} \mathbf{u}$.

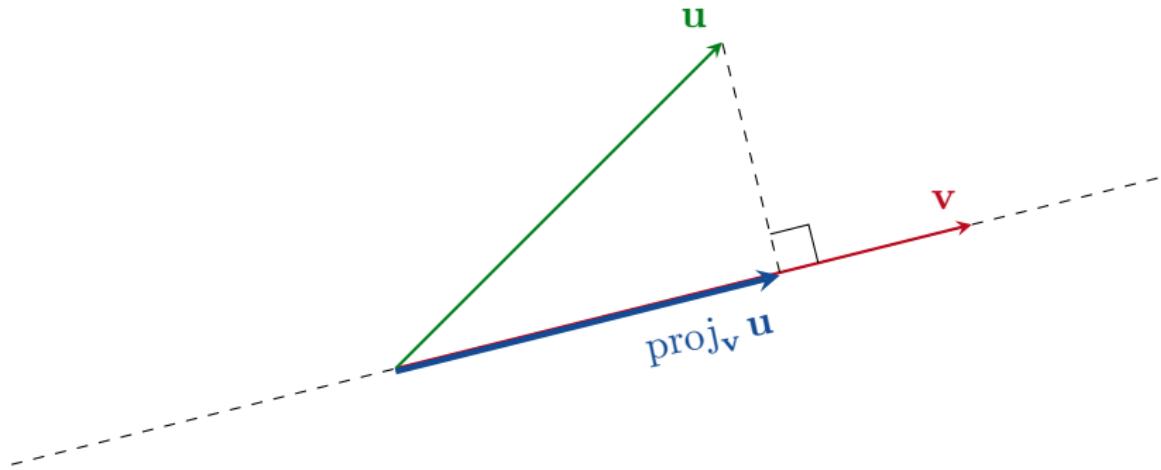
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Example

Verify that the vector $\mathbf{u} - \text{proj}_{\mathbf{v}} \mathbf{u}$ is orthogonal to $\text{proj}_{\mathbf{v}} \mathbf{u}$.



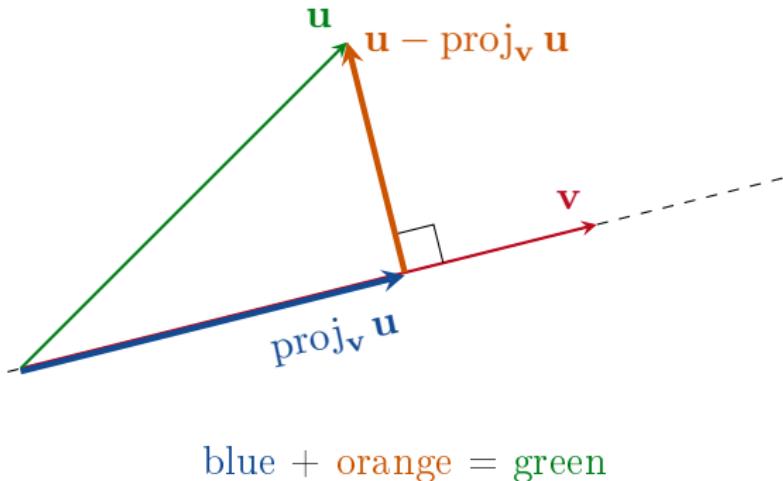
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$$\text{proj}_{\mathbf{v}} \mathbf{u} = \left(\frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{v}\|^2} \right) \mathbf{v}$$



Example

Verify that the vector $\mathbf{u} - \text{proj}_{\mathbf{v}} \mathbf{u}$ is orthogonal to $\text{proj}_{\mathbf{v}} \mathbf{u}$.



12.

$$\text{proj}_{\mathbf{v}} \mathbf{u} = \left(\frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{v}\|^2} \right) \mathbf{v}$$



Clearly

$$\text{proj}_{\mathbf{v}} \mathbf{u} = \left(\frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{v}\|^2} \right) \mathbf{v} = (\text{a number}) \mathbf{v}$$

is parallel to \mathbf{v} .

12.

$$\text{proj}_{\mathbf{v}} \mathbf{u} = \left(\frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{v}\|^2} \right) \mathbf{v}$$



Clearly

$$\text{proj}_{\mathbf{v}} \mathbf{u} = \left(\frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{v}\|^2} \right) \mathbf{v} = (\text{a number}) \mathbf{v}$$

is parallel to \mathbf{v} . So it is enough to show that $\mathbf{u} - \text{proj}_{\mathbf{v}} \mathbf{u}$ is orthogonal to \mathbf{v} .

12.

$$\text{proj}_{\mathbf{v}} \mathbf{u} = \left(\frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{v}\|^2} \right) \mathbf{v}$$



Clearly

$$\text{proj}_{\mathbf{v}} \mathbf{u} = \left(\frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{v}\|^2} \right) \mathbf{v} = (\text{a number}) \mathbf{v}$$

is parallel to \mathbf{v} . So it is enough to show that $\mathbf{u} - \text{proj}_{\mathbf{v}} \mathbf{u}$ is orthogonal to \mathbf{v} .

Since

$$(\mathbf{u} - \text{proj}_{\mathbf{v}} \mathbf{u}) \cdot \mathbf{v} =$$

The diagram illustrates the vector subtraction $\mathbf{u} - \text{proj}_{\mathbf{v}} \mathbf{u}$. A blue vector \mathbf{u} is shown originating from the origin. A red vector \mathbf{v} is also shown. The projection of \mathbf{u} onto \mathbf{v} , represented by an orange vector, is dropped onto the red vector \mathbf{v} . A right-angle symbol at the intersection indicates that the difference vector (the projection removed from \mathbf{u}) is orthogonal to \mathbf{v} . To the right of the diagram, the equation $=$ is followed by three equals signs and the value 0 , indicating the result of the dot product calculation.

$$=$$

$$=$$

$$= 0$$

we have shown that $\mathbf{u} - \text{proj}_{\mathbf{v}} \mathbf{u}$ is orthogonal to \mathbf{v} .

12.

$$\text{proj}_{\mathbf{v}} \mathbf{u} = \left(\frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{v}\|^2} \right) \mathbf{v}$$

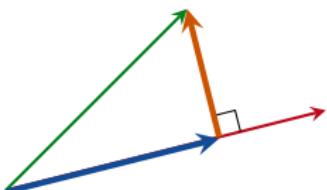
Clearly

$$\text{proj}_{\mathbf{v}} \mathbf{u} = \left(\frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{v}\|^2} \right) \mathbf{v} = (\text{a number}) \mathbf{v}$$

is parallel to \mathbf{v} . So it is enough to show that $\mathbf{u} - \text{proj}_{\mathbf{v}} \mathbf{u}$ is orthogonal to \mathbf{v} .

Since

$$\begin{aligned}
 (\mathbf{u} - \text{proj}_{\mathbf{v}} \mathbf{u}) \cdot \mathbf{v} &= \mathbf{u} \cdot \mathbf{v} - \left(\frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{v}\|^2} \right) \mathbf{v} \cdot \mathbf{v} \\
 &= \mathbf{u} \cdot \mathbf{v} - \frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{v}\|^2} \|\mathbf{v}\|^2 \\
 &= \mathbf{u} \cdot \mathbf{v} - \mathbf{u} \cdot \mathbf{v} \\
 &= 0
 \end{aligned}$$



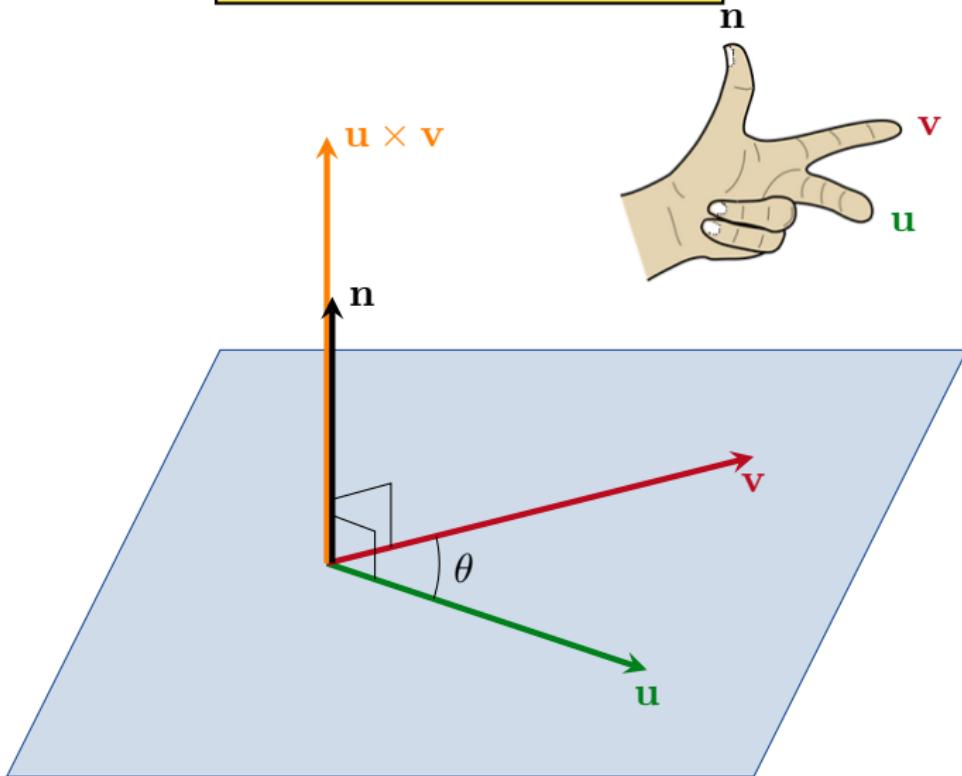
we have shown that $\mathbf{u} - \text{proj}_{\mathbf{v}} \mathbf{u}$ is orthogonal to \mathbf{v} .



The Cross Product

13.

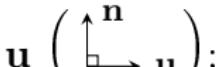
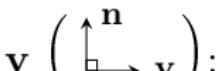
$$\text{proj}_{\mathbf{v}} \mathbf{u} = \left(\frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{v}\|^2} \right) \mathbf{v}$$

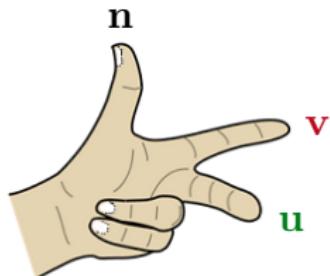


13.

$$\text{proj}_{\mathbf{v}} \mathbf{u} = \left(\frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{v}\|^2} \right) \mathbf{v}$$

Let \mathbf{n} be a unit vector which satisfies

- 1 \mathbf{n} is orthogonal to \mathbf{u} () ;
- 2 \mathbf{n} is orthogonal to \mathbf{v} () ; and
- 3 the direction of \mathbf{n} is chosen using the left-hand rule.



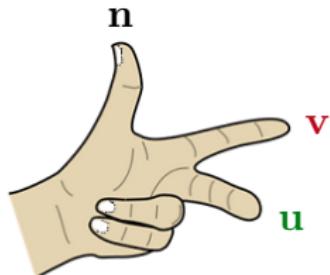
13.

$$\text{proj}_{\mathbf{v}} \mathbf{u} = \left(\frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{v}\|^2} \right) \mathbf{v}$$



Let \mathbf{n} be a unit vector which satisfies

- 1 \mathbf{n} is orthogonal to \mathbf{u} ($\begin{smallmatrix} \mathbf{n} \\ \perp \\ \mathbf{u} \end{smallmatrix}$);
- 2 \mathbf{n} is orthogonal to \mathbf{v} ($\begin{smallmatrix} \mathbf{n} \\ \perp \\ \mathbf{v} \end{smallmatrix}$); and
- 3 the direction of \mathbf{n} is chosen using the left-hand rule.



Definition

The *cross product* of \mathbf{u} and \mathbf{v} is

$$\mathbf{u} \times \mathbf{v} = \|\mathbf{u}\| \|\mathbf{v}\| (\sin \theta) \mathbf{n}.$$

13.

$$\text{proj}_{\mathbf{v}} \mathbf{u} = \left(\frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{v}\|^2} \right) \mathbf{v}$$



Remark

- $\mathbf{u} \cdot \mathbf{v}$ is a number.
- $\mathbf{u} \times \mathbf{v}$ is a vector.

13.

$$\mathbf{u} \times \mathbf{v} = \|\mathbf{u}\| \|\mathbf{v}\| (\sin \theta) \mathbf{n}$$



Remark

$$\begin{pmatrix} \mathbf{u} \text{ and } \mathbf{v} \\ \text{are} \\ \text{parallel} \end{pmatrix} \iff \theta = 0^\circ \text{ or } 180^\circ$$
$$\implies \sin \theta = 0 \implies \mathbf{u} \times \mathbf{v} = \mathbf{0}.$$

13. The Cross Product



Properties of the Cross Product

Let \mathbf{u} , \mathbf{v} and \mathbf{w} be vectors. Let r and s be numbers. Then

1 $(r\mathbf{u}) \times (s\mathbf{v}) = (rs)(\mathbf{u} \times \mathbf{v});$

13. The Cross Product



Properties of the Cross Product

Let \mathbf{u} , \mathbf{v} and \mathbf{w} be vectors. Let r and s be numbers. Then

- 1 $(r\mathbf{u}) \times (s\mathbf{v}) = (rs)(\mathbf{u} \times \mathbf{v});$
- 2 $\mathbf{u} \times (\mathbf{v} + \mathbf{w}) = (\mathbf{u} \times \mathbf{v}) + (\mathbf{u} \times \mathbf{w});$

13. The Cross Product



Properties of the Cross Product

Let \mathbf{u} , \mathbf{v} and \mathbf{w} be vectors. Let r and s be numbers. Then

- 1 $(r\mathbf{u}) \times (s\mathbf{v}) = (rs)(\mathbf{u} \times \mathbf{v});$
- 2 $\mathbf{u} \times (\mathbf{v} + \mathbf{w}) = (\mathbf{u} \times \mathbf{v}) + (\mathbf{u} \times \mathbf{w});$
- 3 $\mathbf{v} \times \mathbf{u} = -\mathbf{u} \times \mathbf{v};$

13. The Cross Product



Properties of the Cross Product

Let \mathbf{u} , \mathbf{v} and \mathbf{w} be vectors. Let r and s be numbers. Then

- 1 $(r\mathbf{u}) \times (s\mathbf{v}) = (rs)(\mathbf{u} \times \mathbf{v});$
- 2 $\mathbf{u} \times (\mathbf{v} + \mathbf{w}) = (\mathbf{u} \times \mathbf{v}) + (\mathbf{u} \times \mathbf{w});$
- 3 $\mathbf{v} \times \mathbf{u} = -\mathbf{u} \times \mathbf{v};$
- 4 $(\mathbf{v} + \mathbf{w}) \times \mathbf{u} = (\mathbf{v} \times \mathbf{u}) + (\mathbf{w} \times \mathbf{u});$

13. The Cross Product



Properties of the Cross Product

Let \mathbf{u} , \mathbf{v} and \mathbf{w} be vectors. Let r and s be numbers. Then

- 1 $(r\mathbf{u}) \times (s\mathbf{v}) = (rs)(\mathbf{u} \times \mathbf{v});$
- 2 $\mathbf{u} \times (\mathbf{v} + \mathbf{w}) = (\mathbf{u} \times \mathbf{v}) + (\mathbf{u} \times \mathbf{w});$
- 3 $\mathbf{v} \times \mathbf{u} = -\mathbf{u} \times \mathbf{v};$
- 4 $(\mathbf{v} + \mathbf{w}) \times \mathbf{u} = (\mathbf{v} \times \mathbf{u}) + (\mathbf{w} \times \mathbf{u});$
- 5 $\mathbf{0} \times \mathbf{u} = \mathbf{0};$ and

13. The Cross Product



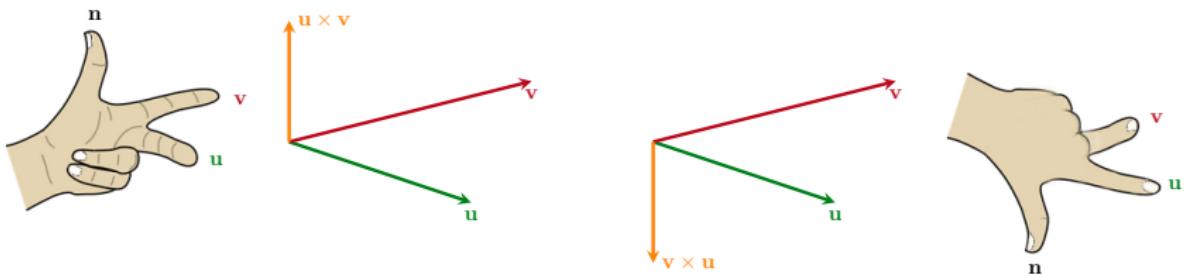
Properties of the Cross Product

Let \mathbf{u} , \mathbf{v} and \mathbf{w} be vectors. Let r and s be numbers. Then

- 1 $(r\mathbf{u}) \times (s\mathbf{v}) = (rs)(\mathbf{u} \times \mathbf{v});$
- 2 $\mathbf{u} \times (\mathbf{v} + \mathbf{w}) = (\mathbf{u} \times \mathbf{v}) + (\mathbf{u} \times \mathbf{w});$
- 3 $\mathbf{v} \times \mathbf{u} = -\mathbf{u} \times \mathbf{v};$
- 4 $(\mathbf{v} + \mathbf{w}) \times \mathbf{u} = (\mathbf{v} \times \mathbf{u}) + (\mathbf{w} \times \mathbf{u});$
- 5 $\mathbf{0} \times \mathbf{u} = \mathbf{0};$ and
- 6 $\mathbf{u} \times (\mathbf{v} \times \mathbf{w}) = (\mathbf{u} \cdot \mathbf{w})\mathbf{v} - (\mathbf{u} \cdot \mathbf{v})\mathbf{w}.$

13. The Cross Product

Property (iii)



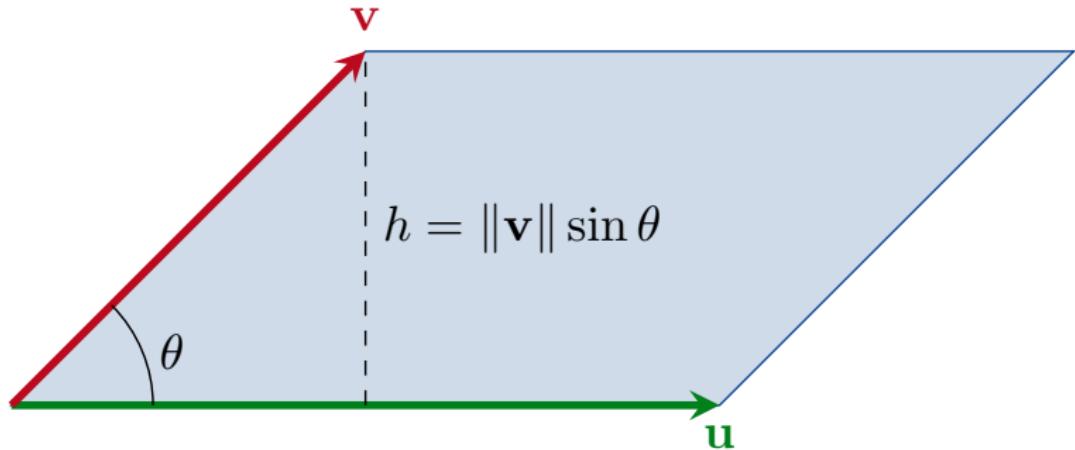
$$\mathbf{v} \times \mathbf{u} = -\mathbf{u} \times \mathbf{v}$$

13.

$$\mathbf{u} \times \mathbf{v} = \|\mathbf{u}\| \|\mathbf{v}\| (\sin \theta) \mathbf{n}$$



Area of a Parallelogram

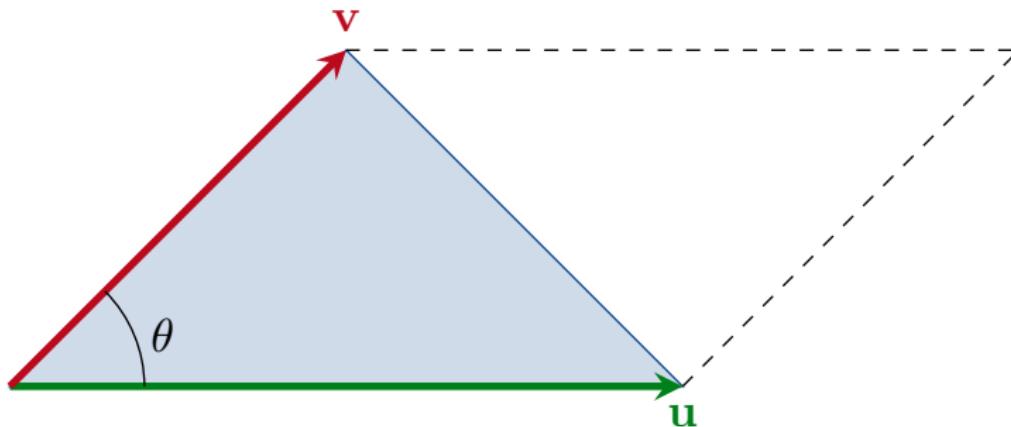


$$\text{area} = (\text{base}) (\text{height}) = \|\mathbf{u}\| \|\mathbf{v}\| \sin \theta = \|\mathbf{u} \times \mathbf{v}\| .$$

13. The Cross Product



Area of a Triangle

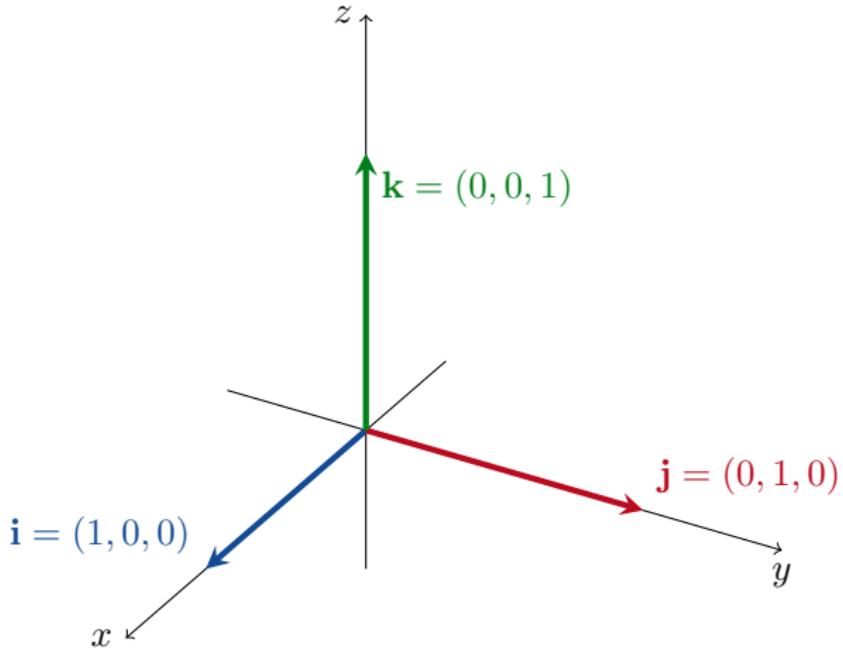


$$\begin{aligned}\text{area of triangle} &= \frac{1}{2} (\text{area of parallelogram}) \\ &= \frac{1}{2} \|\mathbf{u} \times \mathbf{v}\|.\end{aligned}$$

13. The Cross Product



A Formula for $\mathbf{u} \times \mathbf{v}$



13.

$$\mathbf{u} \times \mathbf{v} = \|\mathbf{u}\| \|\mathbf{v}\| (\sin \theta) \mathbf{n}$$



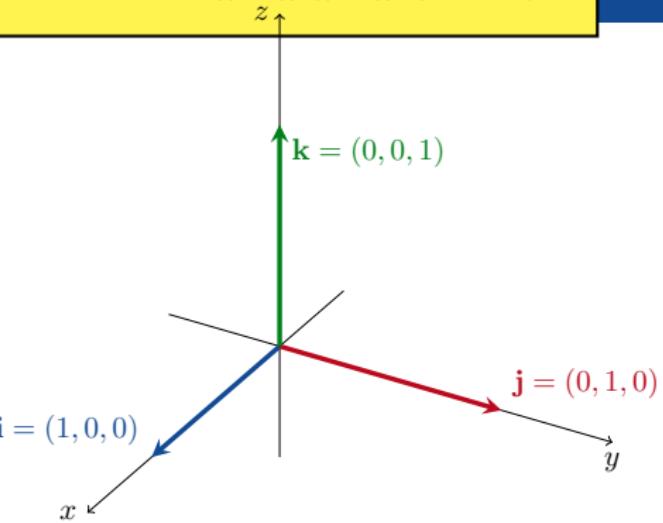
Note first that

$$\mathbf{i} \times \mathbf{i} = \|\mathbf{i}\| \|\mathbf{i}\| \sin 0^\circ \mathbf{n} = \mathbf{0}.$$

Similarly $\mathbf{j} \times \mathbf{j} = \mathbf{0}$ and $\mathbf{k} \times \mathbf{k} = \mathbf{0}$ also.

13.

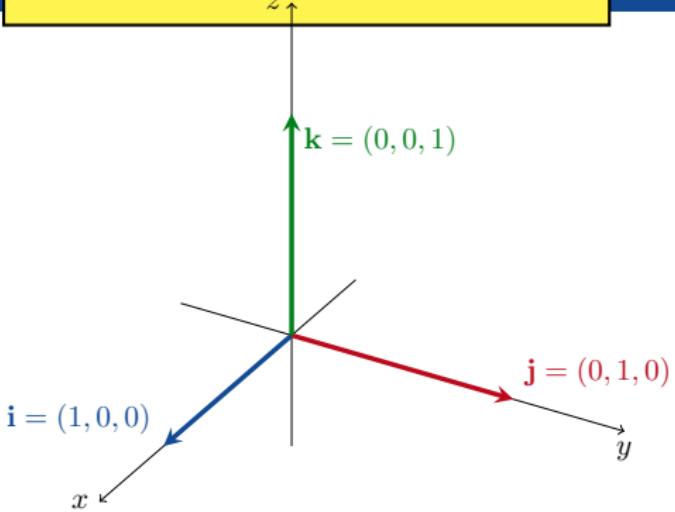
$$\mathbf{u} \times \mathbf{v} = \|\mathbf{u}\| \|\mathbf{v}\| (\sin \theta) \mathbf{n}$$



Next note that $\mathbf{i} \times \mathbf{j}$ must point in the same direction as \mathbf{k} by the left-hand rule.

13.

$$\mathbf{u} \times \mathbf{v} = \|\mathbf{u}\| \|\mathbf{v}\| (\sin \theta) \mathbf{n}$$

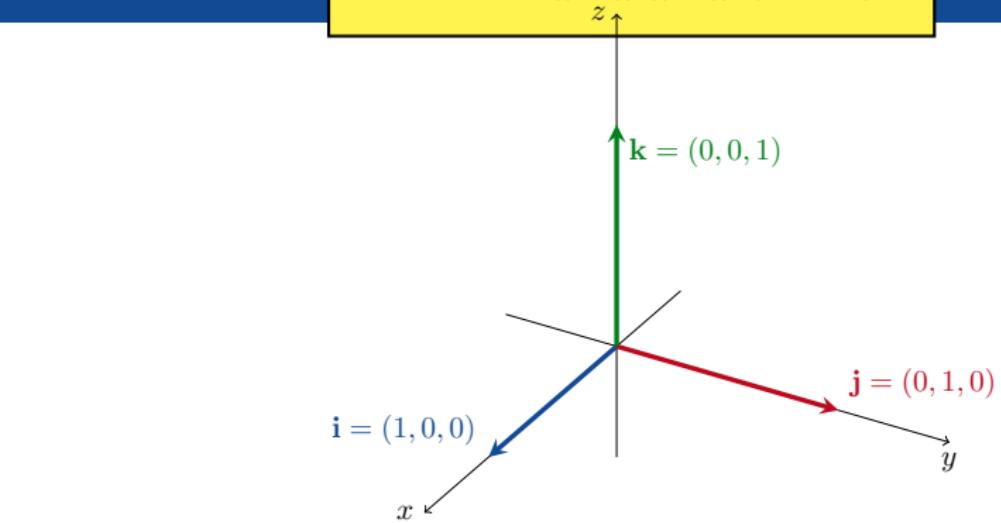


Next note that $\mathbf{i} \times \mathbf{j}$ must point in the same direction as \mathbf{k} by the left-hand rule. Thus

$$\mathbf{i} \times \mathbf{j} = \|\mathbf{i}\| \|\mathbf{j}\| \sin 90^\circ \mathbf{k} = \mathbf{k}.$$

13.

$$\mathbf{u} \times \mathbf{v} = \|\mathbf{u}\| \|\mathbf{v}\| (\sin \theta) \mathbf{n}$$



Next note that $\mathbf{i} \times \mathbf{j}$ must point in the same direction as \mathbf{k} by the left-hand rule. Thus

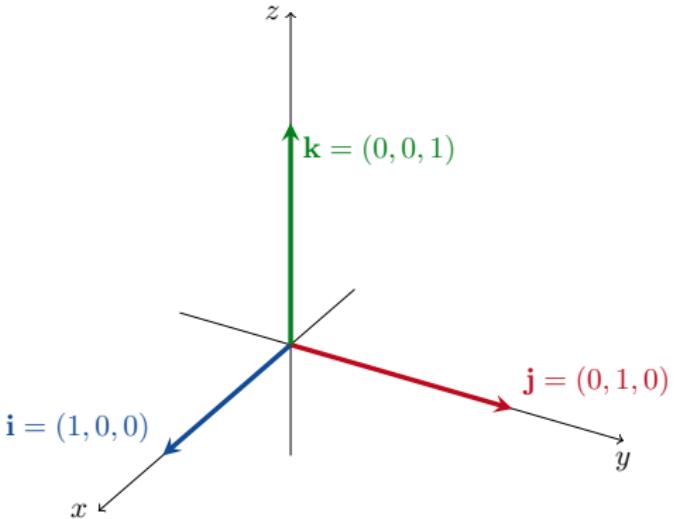
$$\mathbf{i} \times \mathbf{j} = \|\mathbf{i}\| \|\mathbf{j}\| \sin 90^\circ \mathbf{k} = \mathbf{k}.$$

We then immediately also have

$$\mathbf{j} \times \mathbf{i} = -(\mathbf{i} \times \mathbf{j}) = -\mathbf{k}.$$

13.

$$\mathbf{u} \times \mathbf{v} = \|\mathbf{u}\| \|\mathbf{v}\| (\sin \theta) \mathbf{n}$$



I leave it for you to check that

$$\mathbf{j} \times \mathbf{k} = \mathbf{i}, \quad \mathbf{k} \times \mathbf{j} = -\mathbf{i}, \quad \mathbf{i} \times \mathbf{k} = -\mathbf{j} \quad \text{and} \quad \mathbf{k} \times \mathbf{i} = \mathbf{j}.$$

13. The Cross Product



Now suppose that $\mathbf{u} = u_1\mathbf{i} + u_2\mathbf{j} + u_3\mathbf{k}$
and $\mathbf{v} = v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}$.

13. The Cross Product

Now suppose that $\mathbf{u} = u_1\mathbf{i} + u_2\mathbf{j} + u_3\mathbf{k}$
and $\mathbf{v} = v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}$. Then we can calculate
that

$$\mathbf{u} \times \mathbf{v} = (u_1\mathbf{i} + u_2\mathbf{j} + u_3\mathbf{k}) \times (v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k})$$

=

=

=

13. The Cross Product

Now suppose that $\mathbf{u} = u_1\mathbf{i} + u_2\mathbf{j} + u_3\mathbf{k}$
and $\mathbf{v} = v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}$. Then we can calculate
that

$$\begin{aligned}\mathbf{u} \times \mathbf{v} &= (u_1\mathbf{i} + u_2\mathbf{j} + u_3\mathbf{k}) \times (v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}) \\&= u_1v_1\mathbf{i} \times \mathbf{i} + u_1v_2\mathbf{i} \times \mathbf{j} + u_1v_3\mathbf{i} \times \mathbf{k} + u_2v_1\mathbf{j} \times \mathbf{i} + u_2v_2\mathbf{j} \times \mathbf{j} \\&\quad + u_2v_3\mathbf{j} \times \mathbf{k} + u_3v_1\mathbf{k} \times \mathbf{i} + u_3v_2\mathbf{k} \times \mathbf{j} + u_3v_3\mathbf{k} \times \mathbf{k} \\&= \\&= \end{aligned}$$

13. The Cross Product

$$\begin{aligned}\mathbf{i} \times \mathbf{j} &= \mathbf{k} \\ \mathbf{j} \times \mathbf{i} &= -\mathbf{k} \\ \mathbf{j} \times \mathbf{k} &= \mathbf{i} \\ \mathbf{k} \times \mathbf{j} &= -\mathbf{i} \\ \mathbf{i} \times \mathbf{k} &= -\mathbf{j} \\ \mathbf{k} \times \mathbf{i} &= \mathbf{j}\end{aligned}$$

Now suppose that $\mathbf{u} = u_1\mathbf{i} + u_2\mathbf{j} + u_3\mathbf{k}$
and $\mathbf{v} = v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}$. Then we can calculate
that

$$\begin{aligned}\mathbf{u} \times \mathbf{v} &= (u_1\mathbf{i} + u_2\mathbf{j} + u_3\mathbf{k}) \times (v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}) \\ &= u_1v_1\mathbf{i} \times \mathbf{i} + u_1v_2\mathbf{i} \times \mathbf{j} + u_1v_3\mathbf{i} \times \mathbf{k} + u_2v_1\mathbf{j} \times \mathbf{i} + u_2v_2\mathbf{j} \times \mathbf{j} \\ &\quad + u_2v_3\mathbf{j} \times \mathbf{k} + u_3v_1\mathbf{k} \times \mathbf{i} + u_3v_2\mathbf{k} \times \mathbf{j} + u_3v_3\mathbf{k} \times \mathbf{k} \\ &= \\ &= \end{aligned}$$

13. The Cross Product

$$\begin{aligned}\mathbf{i} \times \mathbf{j} &= \mathbf{k} \\ \mathbf{j} \times \mathbf{i} &= -\mathbf{k} \\ \mathbf{j} \times \mathbf{k} &= \mathbf{i} \\ \mathbf{k} \times \mathbf{j} &= -\mathbf{i} \\ \mathbf{i} \times \mathbf{k} &= -\mathbf{j} \\ \mathbf{k} \times \mathbf{i} &= \mathbf{j}\end{aligned}$$

Now suppose that $\mathbf{u} = u_1\mathbf{i} + u_2\mathbf{j} + u_3\mathbf{k}$
and $\mathbf{v} = v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}$. Then we can calculate
that

$$\begin{aligned}\mathbf{u} \times \mathbf{v} &= (u_1\mathbf{i} + u_2\mathbf{j} + u_3\mathbf{k}) \times (v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}) \\ &= u_1v_1\mathbf{i} \times \mathbf{i} + u_1v_2\mathbf{i} \times \mathbf{j} + u_1v_3\mathbf{i} \times \mathbf{k} + u_2v_1\mathbf{j} \times \mathbf{i} + u_2v_2\mathbf{j} \times \mathbf{j} \\ &\quad + u_2v_3\mathbf{j} \times \mathbf{k} + u_3v_1\mathbf{k} \times \mathbf{i} + u_3v_2\mathbf{k} \times \mathbf{j} + u_3v_3\mathbf{k} \times \mathbf{k} \\ &= \mathbf{0} + u_1v_2\mathbf{k} - u_1v_3\mathbf{j} - u_2v_1\mathbf{k} + \mathbf{0} + u_2v_3\mathbf{i} + u_3v_1\mathbf{j} - u_3v_2\mathbf{i} + \mathbf{0} \\ &= \end{aligned}$$

13. The Cross Product

$$\begin{aligned}\mathbf{i} \times \mathbf{j} &= \mathbf{k} \\ \mathbf{j} \times \mathbf{i} &= -\mathbf{k} \\ \mathbf{j} \times \mathbf{k} &= \mathbf{i} \\ \mathbf{k} \times \mathbf{j} &= -\mathbf{i} \\ \mathbf{i} \times \mathbf{k} &= -\mathbf{j} \\ \mathbf{k} \times \mathbf{i} &= \mathbf{j}\end{aligned}$$

Now suppose that $\mathbf{u} = u_1\mathbf{i} + u_2\mathbf{j} + u_3\mathbf{k}$
and $\mathbf{v} = v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}$. Then we can calculate
that

$$\begin{aligned}\mathbf{u} \times \mathbf{v} &= (u_1\mathbf{i} + u_2\mathbf{j} + u_3\mathbf{k}) \times (v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}) \\ &= u_1v_1\mathbf{i} \times \mathbf{i} + u_1v_2\mathbf{i} \times \mathbf{j} + u_1v_3\mathbf{i} \times \mathbf{k} + u_2v_1\mathbf{j} \times \mathbf{i} + u_2v_2\mathbf{j} \times \mathbf{j} \\ &\quad + u_2v_3\mathbf{j} \times \mathbf{k} + u_3v_1\mathbf{k} \times \mathbf{i} + u_3v_2\mathbf{k} \times \mathbf{j} + u_3v_3\mathbf{k} \times \mathbf{k} \\ &= \mathbf{0} + u_1v_2\mathbf{k} - u_1v_3\mathbf{j} - u_2v_1\mathbf{k} + \mathbf{0} + u_2v_3\mathbf{i} + u_3v_1\mathbf{j} - u_3v_2\mathbf{i} + \mathbf{0} \\ &= (u_2v_3 - u_3v_2)\mathbf{i} - (u_1v_3 - u_3v_1)\mathbf{j} + (u_1v_2 - u_2v_1)\mathbf{k}.\end{aligned}$$

13. The Cross Product



Theorem

If $\mathbf{u} = u_1\mathbf{i} + u_2\mathbf{j} + u_3\mathbf{k}$ and $\mathbf{v} = v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}$, then

$$\mathbf{u} \times \mathbf{v} = (u_2v_3 - u_3v_2)\mathbf{i} - (u_1v_3 - u_3v_1)\mathbf{j} + (u_1v_2 - u_2v_1)\mathbf{k}.$$

13. The Cross Product



If you studied matrices and determinants at high school, then you may prefer to use the following symbolic determinant formula instead.

$$\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}.$$

$$\mathbf{u} \times \mathbf{v} = (u_2 v_3 - u_3 v_2) \mathbf{i} - (u_1 v_3 - u_3 v_1) \mathbf{j} + (u_1 v_2 - u_2 v_1) \mathbf{k}$$

Example

Find $\mathbf{u} \times \mathbf{v}$ and $\mathbf{v} \times \mathbf{u}$ if $\mathbf{u} = 2\mathbf{i} + \mathbf{j} + \mathbf{k}$ and $\mathbf{v} = -4\mathbf{i} + 3\mathbf{j} + \mathbf{k}$.

$$\mathbf{u} \times \mathbf{v} = (u_2 v_3 - u_3 v_2) \mathbf{i} - (u_1 v_3 - u_3 v_1) \mathbf{j} + (u_1 v_2 - u_2 v_1) \mathbf{k}$$

Example

Find $\mathbf{u} \times \mathbf{v}$ and $\mathbf{v} \times \mathbf{u}$ if $\mathbf{u} = 2\mathbf{i} + \mathbf{j} + \mathbf{k}$ and $\mathbf{v} = -4\mathbf{i} + 3\mathbf{j} + \mathbf{k}$.

$$\mathbf{u} \times \mathbf{v} = (1 - 3)\mathbf{i} - (2 - -4)\mathbf{j} + (6 - -4)\mathbf{k} = -2\mathbf{i} - 6\mathbf{j} + 10\mathbf{k}$$

$$\mathbf{u} \times \mathbf{v} = (u_2 v_3 - u_3 v_2) \mathbf{i} - (u_1 v_3 - u_3 v_1) \mathbf{j} + (u_1 v_2 - u_2 v_1) \mathbf{k}$$

Example

Find $\mathbf{u} \times \mathbf{v}$ and $\mathbf{v} \times \mathbf{u}$ if $\mathbf{u} = 2\mathbf{i} + \mathbf{j} + \mathbf{k}$ and $\mathbf{v} = -4\mathbf{i} + 3\mathbf{j} + \mathbf{k}$.

$$\mathbf{u} \times \mathbf{v} = (1 - 3)\mathbf{i} - (2 - -4)\mathbf{j} + (6 - -4)\mathbf{k} = -2\mathbf{i} - 6\mathbf{j} + 10\mathbf{k}$$

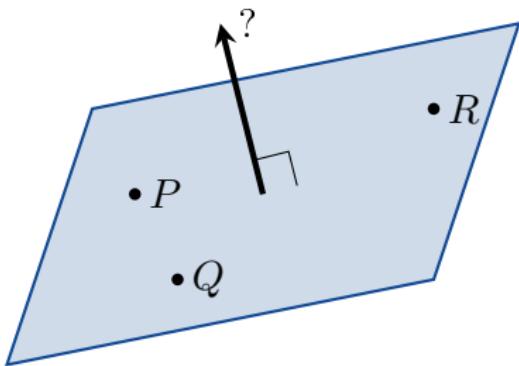
and

$$\mathbf{v} \times \mathbf{u} = -\mathbf{u} \times \mathbf{v} = 2\mathbf{i} + 6\mathbf{j} - 10\mathbf{k}.$$

$$\mathbf{u} \times \mathbf{v} = (u_2 v_3 - u_3 v_2) \mathbf{i} - (u_1 v_3 - u_3 v_1) \mathbf{j} + (u_1 v_2 - u_2 v_1) \mathbf{k}$$

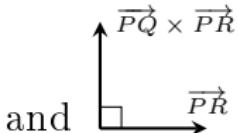
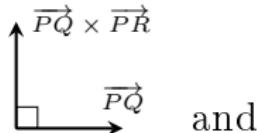
Example

Find a vector perpendicular to the plane containing the three points $P(1, -1, 0)$, $Q(2, 1, -1)$ and $R(-1, 1, 2)$.



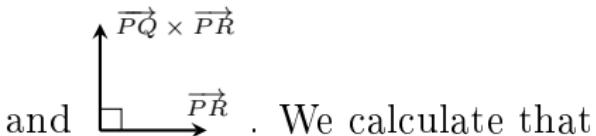
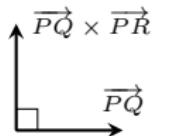
$$\mathbf{u} \times \mathbf{v} = (u_2 v_3 - u_3 v_2) \mathbf{i} - (u_1 v_3 - u_3 v_1) \mathbf{j} + (u_1 v_2 - u_2 v_1) \mathbf{k}$$

The vector $\overrightarrow{PQ} \times \overrightarrow{PR}$ is perpendicular to the plane because



$$\mathbf{u} \times \mathbf{v} = (u_2 v_3 - u_3 v_2) \mathbf{i} - (u_1 v_3 - u_3 v_1) \mathbf{j} + (u_1 v_2 - u_2 v_1) \mathbf{k}$$

The vector $\overrightarrow{PQ} \times \overrightarrow{PR}$ is perpendicular to the plane because



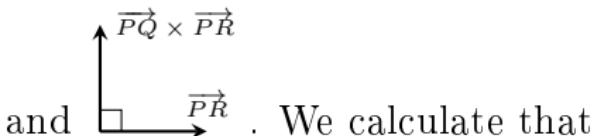
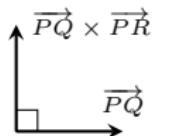
and . We calculate that

$$\begin{aligned}\overrightarrow{PQ} &= Q - P = (2, 1, -1) - (1, -1, 0) \\ &= (2 - 1, 1 + 1, -1 - 0) = \mathbf{i} + 2\mathbf{j} - \mathbf{k}\end{aligned}$$

$$\begin{aligned}\overrightarrow{PR} &= R - P = (-1, 1, 2) - (1, -1, 0) \\ &= (-1 - 1, 1 + 1, 2 - 0) = -2\mathbf{i} + 2\mathbf{j} + 2\mathbf{k}\end{aligned}$$

$$\mathbf{u} \times \mathbf{v} = (u_2 v_3 - u_3 v_2) \mathbf{i} - (u_1 v_3 - u_3 v_1) \mathbf{j} + (u_1 v_2 - u_2 v_1) \mathbf{k}$$

The vector $\overrightarrow{PQ} \times \overrightarrow{PR}$ is perpendicular to the plane because



and . We calculate that

$$\begin{aligned}\overrightarrow{PQ} &= Q - P = (2, 1, -1) - (1, -1, 0) \\ &= (2 - 1, 1 + 1, -1 - 0) = \mathbf{i} + 2\mathbf{j} - \mathbf{k}\end{aligned}$$

$$\begin{aligned}\overrightarrow{PR} &= R - P = (-1, 1, 2) - (1, -1, 0) \\ &= (-1 - 1, 1 + 1, 2 - 0) = -2\mathbf{i} + 2\mathbf{j} + 2\mathbf{k}\end{aligned}$$

and

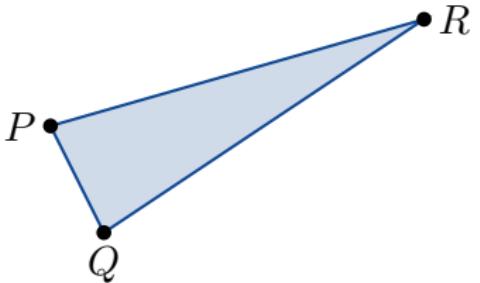
$$\overrightarrow{PQ} \times \overrightarrow{PR} = (4 + 2)\mathbf{i} - (2 - 2)\mathbf{j} + (2 + 4)\mathbf{k} = 6\mathbf{i} + 6\mathbf{k}.$$

13. The Cross Product

Example

Find the area of triangle PQR .

$P(1, -1, 0)$, $Q(2, 1, -1)$ and $R(-1, 1, 2)$

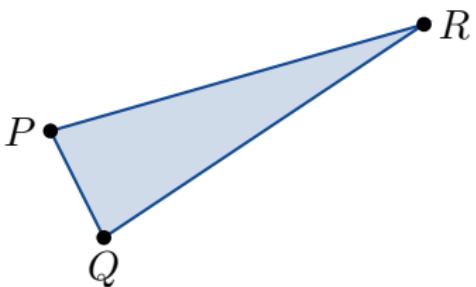


13. The Cross Product

Example

Find the area of triangle PQR .

$P(1, -1, 0)$, $Q(2, 1, -1)$ and $R(-1, 1, 2)$



The area of the triangle is

$$\begin{aligned}\text{area} &= \frac{1}{2} \left\| \overrightarrow{PQ} \times \overrightarrow{PR} \right\| = \frac{1}{2} \|6\mathbf{i} + 6\mathbf{k}\| \\ &= \frac{1}{2} \sqrt{6^2 + 0^2 + 6^2} = 3\sqrt{2}.\end{aligned}$$

13. The Cross Product



Example

Find a unit vector perpendicular to the plane containing P , Q and R .

$P(1, -1, 0)$, $Q(2, 1, -1)$ and $R(-1, 1, 2)$

13. The Cross Product



Example

Find a unit vector perpendicular to the plane containing P , Q and R .

$P(1, -1, 0)$, $Q(2, 1, -1)$ and $R(-1, 1, 2)$

We know that $\overrightarrow{PQ} \times \overrightarrow{PR}$ is perpendicular to the plane. We just need to normalise this vector to find a unit vector.

13. The Cross Product



Example

Find a unit vector perpendicular to the plane containing P , Q and R .

$P(1, -1, 0)$, $Q(2, 1, -1)$ and $R(-1, 1, 2)$

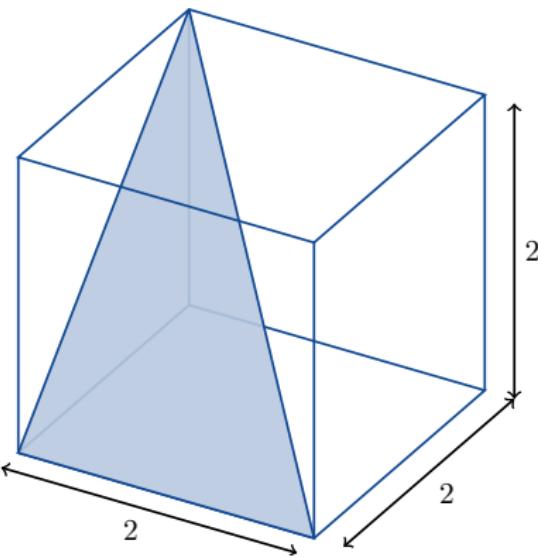
We know that $\overrightarrow{PQ} \times \overrightarrow{PR}$ is perpendicular to the plane. We just need to normalise this vector to find a unit vector.

$$\mathbf{n} = \frac{\overrightarrow{PQ} \times \overrightarrow{PR}}{\|\overrightarrow{PQ} \times \overrightarrow{PR}\|} = \frac{6\mathbf{i} + 6\mathbf{k}}{6\sqrt{2}} = \frac{1}{\sqrt{2}}\mathbf{i} + \frac{1}{\sqrt{2}}\mathbf{k}.$$

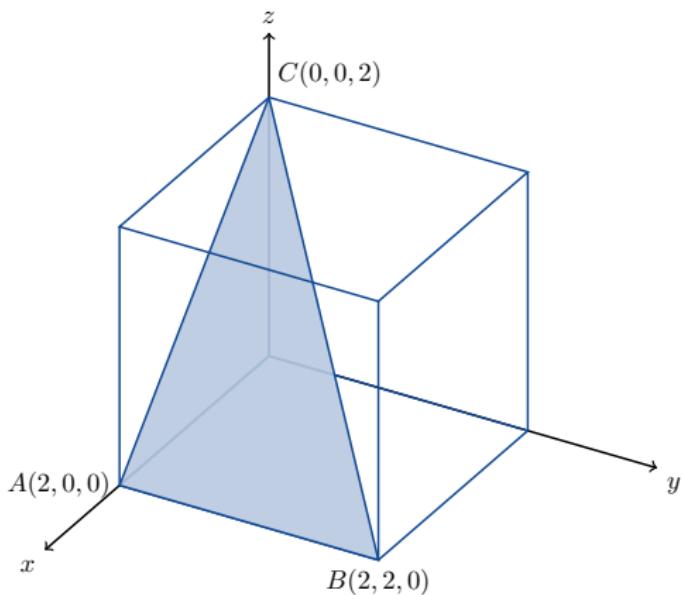
13. The Cross Product

Example

A triangle is inscribed inside a cube of side 2 as shown below.
Use the cross product to find the area of the triangle.



13. The Cross Product



First we draw coordinate axes and assign coordinates to the vertices of the triangle.

13. The Cross Product

Then we can calculate

$$\overrightarrow{AB} = B - A = (2, 2, 0) - (2, 0, 0) = (0, 2, 0) = 2\mathbf{j}$$

and

$$\overrightarrow{AC} = C - A = (0, 0, 2) - (2, 0, 0) = (-2, 0, 2) = -2\mathbf{i} + 2\mathbf{k}.$$

13. The Cross Product

Then we can calculate

$$\overrightarrow{AB} = B - A = (2, 2, 0) - (2, 0, 0) = (0, 2, 0) = 2\mathbf{j}$$

and

$$\overrightarrow{AC} = C - A = (0, 0, 2) - (2, 0, 0) = (-2, 0, 2) = -2\mathbf{i} + 2\mathbf{k}.$$

It follows that

$$\begin{aligned}\overrightarrow{AB} \times \overrightarrow{AC} &= (2\mathbf{j}) \times (-2\mathbf{i} + 2\mathbf{k}) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 2 & 0 \\ -2 & 0 & 2 \end{vmatrix} \\ &= \mathbf{i}(4 - 0) - \mathbf{j}(0 - 0) + \mathbf{k}(0 - -4) = 4\mathbf{i} + 4\mathbf{k}.\end{aligned}$$

13. The Cross Product

Then we can calculate

$$\overrightarrow{AB} = B - A = (2, 2, 0) - (2, 0, 0) = (0, 2, 0) = 2\mathbf{j}$$

and

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Therefore

$$\begin{aligned}\text{area of triangle} &= \frac{1}{2} \left\| \overrightarrow{AB} \times \overrightarrow{AC} \right\| = \frac{1}{2} \sqrt{4^2 + 0^2 + 4^2} \\ &= \frac{1}{2} \sqrt{32} = \frac{1}{2} \sqrt{4} \sqrt{8} = \sqrt{8} = 2\sqrt{2}.\end{aligned}$$

13. The Cross Product



The Triple Scalar Product

Definition

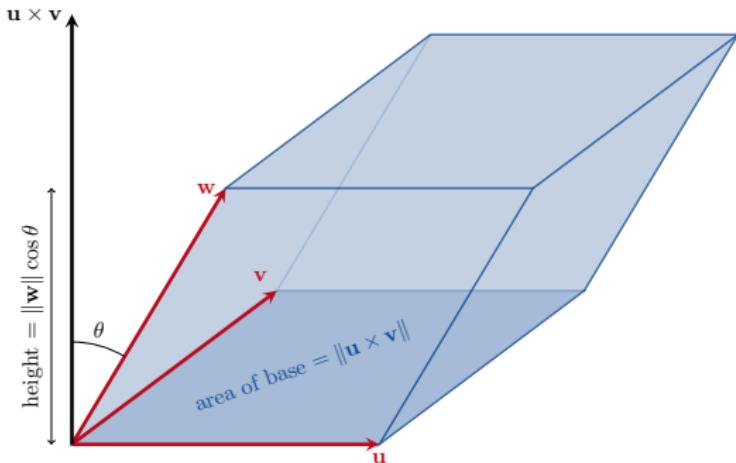
The *triple scalar product* of \mathbf{u} , \mathbf{v} and \mathbf{w} is

$$(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w}.$$

13. The Cross Product



The Volume of a Parallelepiped



$$\text{volume} = (\text{area of base})(\text{height}) = \|\mathbf{u} \times \mathbf{v}\| \|\mathbf{w}\| \cos \theta = |(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w}|$$

13. The Cross Product



One Final Comment

We can do the dot product in both \mathbb{R}^2 and \mathbb{R}^3 . But we can only do the cross product in \mathbb{R}^3 . There is no cross product in \mathbb{R}^2 .



Next Time

- 14. Lines
- 15. Planes
- 16. Projections