Lecture 5

- 3.5 Derivatives of Trigonometric Functions
- 3.6 The Chain Rule
- 3.7 Implicit Differentiation

Derivatives of Trigonometric Functions

The Sine and Cosine Functions

First let me remind you of three formulae:

$$\sin(x+h) = \sin x \cos h + \cos x \sin h,$$

$$\lim_{h \to 0} \frac{\sin h}{h} = 1$$

and

$$\lim_{h \to 0} \frac{\cos h - 1}{h} = 0.$$

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Using these, we will calculate the derivative of $f(x) = \sin x$.

$$\lim_{h\to 0}\frac{\sin h}{h}=1 \qquad \lim_{h\to 0}\frac{\cos h-1}{h}=0$$

If $f(x)=\sin x$, then

$$\begin{aligned} f'(x) &= \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = \lim_{h\to 0}\frac{\sin(x+h)-\sin x}{h} \\ &= \lim_{h\to 0}\frac{\sin x\cos h+\cos x\sin h-\sin x}{h} \\ &= \\ &= \\ &= \\ &= \end{aligned}$$

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&= \lim_{h\to 0}\frac{\sin x(\cos h-1)+\cos x\sin h}{h}\\
&=\\
&=\\
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f'(x) &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{\sin(x+h) - \sin x}{h} \\
&= \lim_{h\to 0} \frac{\sin x \cos h + \cos x \sin h - \sin x}{h} \\
&= \lim_{h\to 0} \frac{\sin x(\cos h - 1) + \cos x \sin h}{h} \\
&= \lim_{h\to 0} \left(\sin x \cdot \frac{\cos h - 1}{h} \right) + \lim_{h\to 0} \left(\cos x \cdot \frac{\sin h}{h} \right) \\
&= \\
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\end{aligned}$$

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&=
\end{aligned}$$

$$\lim_{h\to 0}\frac{\sin h}{h} = 1 \qquad \lim_{h\to 0}\frac{\cos h - 1}{h} = 0$$

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f'(x) &= \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = \lim_{h\to 0}\frac{\sin(x+h)-\sin x}{h}\\
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&= \lim_{h\to 0}\frac{\sin x(\cos h - 1) + \cos x \sin h}{h}\\
&= \lim_{h\to 0}\left(\sin x \cdot \frac{\cos h - 1}{h}\right) + \lim_{h\to 0}\left(\cos x \cdot \frac{\sin h}{h}\right)\\
&= \sin x \cdot \lim_{h\to 0}\left(\frac{\cos h - 1}{h}\right) + \cos x \cdot \lim_{h\to 0}\left(\frac{\sin h}{h}\right)\\
&= \sin x \cdot 0 + \cos x \cdot 1 = \cos x.
\end{aligned}$$

$$
\begin{aligned}
\frac{d}{dx}(\cos x) &= \lim_{h\to 0} \frac{\cos(x+h) - \cos x}{h} && \text{Derivative definition} \\
&= \lim_{h\to 0} \frac{(\cos x \cos h - \sin x \sin h) - \cos x}{h} && \text{Cosine angle sum identity} \\
&= \lim_{h\to 0} \frac{\cos x(\cos h - 1) - \sin x \sin h}{h} \\
&= \lim_{h\to 0} \left(\cos x \cdot \frac{\cos h - 1}{h}\right) - \lim_{h\to 0} \left(\sin x \cdot \frac{\sin h}{h}\right) \\
&= \cos x \cdot \lim_{h\to 0} \frac{\cos h - 1}{h} - \sin x \cdot \lim_{h\to 0} \frac{\sin h}{h} \\
&= \cos x \cdot 0 - \sin x \cdot 1 && \text{Example 5a and Theorem 7, Section 2.4} \\
&= -\sin x.
\end{aligned}
$$

Theorem

$$\boxed{\frac{d}{dx}(\sin x) = \cos x} \qquad and \qquad \boxed{\frac{d}{dx}(\cos x) = -\sin x}$$

$$\frac{d}{dx}(\sin x) = \cos x \qquad \frac{d}{dx}(\cos x) = -\sin x$$

Example

Differentiate $y = x^2 - \sin x$.

$$\frac{dy}{dx} = \frac{d}{dx}\left(x^2\right) - \frac{d}{dx}(\sin x) = 2x - \cos x.$$

$$\frac{d}{dx}(\sin x) = \cos x \qquad \frac{d}{dx}(\cos x) = -\sin x$$

Example

Differentiate $y = x^2 \sin x$.

We will use the product rule ($(uv)' = u'v + uv'$) with $u = x^2$ and $v = \sin x$.

$$\frac{d}{dx}\left(\sin x\right) = \cos x \qquad \frac{d}{dx}\left(\cos x\right) = -\sin x$$

Example

Differentiate $y = x^2 \sin x$.

We will use the product rule ($(uv)' = u'v + uv'$) with $u = x^2$ and $v = \sin x$.

$$y' = (x^2)'(\sin x) + (x^2)(\sin x)' = 2x \sin x + x^2 \cos x.$$

$$\frac{d}{dx}(\sin x) = \cos x \qquad \frac{d}{dx}(\cos x) = -\sin x$$

Example

Differentiate $y = \frac{\sin x}{x}$.

This time we use the quotient rule ($\left(\frac{u}{v}\right)' = \frac{u'v-uv'}{v^2}$) with $u = \sin x$ and $v = x$.

Example

Differentiate $y = \frac{\sin x}{x}$.

This time we use the quotient rule $\left(\left(\frac{u}{v}\right)' = \frac{u'v-uv'}{v^2}\right)$ with $u = \sin x$ and $v = x$.

$$y' = \frac{(\sin x)'x - (\sin x)(x)'}{x^2} = \frac{x\cos x - 1\sin x}{x^2}.$$

$\frac{d}{dx}(\sin x) = \cos x \qquad \frac{d}{dx}(\cos x) = -\sin x$

Example

Differentiate $y = 5x + \cos x$.

$$\frac{dy}{dx} = \frac{d}{dx}(5x) + \frac{d}{dx}(\cos x) = 5 - \sin x.$$

$$\frac{d}{dx}(\sin x) = \cos x \qquad \frac{d}{dx}(\cos x) = -\sin x$$

Example

Differentiate $y = \sin x \cos x$.

By the product rule, we have that

$$\frac{dy}{dx} = \frac{d}{dx}(\sin x)\cos x + \sin x \frac{d}{dx}(\cos x) = \cos^2 x - \sin^2 x.$$

$$\frac{d}{dx}(\sin x) = \cos x \qquad \frac{d}{dx}(\cos x) = -\sin x$$

Example

Differentiate $y = \frac{\cos x}{1-\sin x}$.

By the quotient rule, we have that

$$\frac{dy}{dx} = \frac{\frac{d}{dx}(\cos x)(1-\sin x) - (\cos x)\frac{d}{dx}(1-\sin x)}{(1-\sin x)^2}$$

$$=$$

$$=$$

$$=$$

$$=$$

$$\frac{d}{dx}(\sin x) = \cos x \qquad \frac{d}{dx}(\cos x) = -\sin x$$

Example

Differentiate $y = \frac{\cos x}{1-\sin x}$.

By the quotient rule, we have that

$$\begin{aligned}\frac{dy}{dx} &= \frac{\frac{d}{dx}(\cos x)(1-\sin x) - (\cos x)\frac{d}{dx}(1-\sin x)}{(1-\sin x)^2} \\ &= \frac{-\sin x(1-\sin x) - \cos x(0-\cos x)}{(1-\sin x)^2} \\ &= \\ &= \\ &= \end{aligned}$$

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&= \frac{-\sin x(1-\sin x) - \cos x(0-\cos x)}{(1-\sin x)^2} \\
&= \frac{-\sin x + \sin^2 x + \cos^2 x}{(1-\sin x)^2} \\
&= \frac{-\sin x + 1}{(1-\sin x)^2} = \frac{1-\sin x}{(1-\sin x)^2} \\
&= \frac{1}{1-\sin x}.
\end{aligned}$$

Please read Example 3 in your textbook.

The Tangent Function

Theorem

$$\boxed{\frac{d}{dx}(\tan x) = \sec^2 x.}$$

The Tangent Function

Theorem

$$\boxed{\frac{d}{dx}(\tan x) = \sec^2 x.}$$

Proof.

Using the quotient rule, we can calculate that

$$\frac{d}{dx}(\tan x) = \frac{d}{dx}\left(\frac{\sin x}{\cos x}\right) = \frac{\frac{d}{dx}(\sin x)(\cos x) - (\sin x)\frac{d}{dx}(\cos x)}{\cos^2 x}$$

$$=$$

$$=$$

$\square$

The Tangent Function

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□

The Tangent Function

Theorem

$$\boxed{\frac{d}{dx}(\tan x) = \sec^2 x.}$$

Proof.

Using the quotient rule, we can calculate that

$$\begin{aligned}
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&= \frac{(\cos x)(\cos x) - (\sin x)(-\sin x)}{\cos^2 x} = \frac{\cos^2 x + \sin^2 x}{\cos^2 x} \\
&= \frac{1}{\cos^2 x} = \sec^2 x.
\end{aligned}$$

□

The Other Three

Theorem

$$\boxed{\frac{d}{dx}(\sec x) = \sec x \tan x}$$

$$\boxed{\frac{d}{dx}(\cot x) = -\operatorname{cosec}^2 x}$$

$$\boxed{\frac{d}{dx}(\operatorname{cosec} x) = -\operatorname{cosec} x \cot x}$$

You can use the quotient rule to prove these three rules. We may ask you to prove one of them in an exam.

$$\frac{d}{dx}(\tan x) = \sec^2 x \qquad \frac{d}{dx}(\sec x) = \sec x \tan x$$

Example

Find y'' if $y = \sec x$.

$$\frac{d}{dx}(\tan x) = \sec^2 x \qquad \frac{d}{dx}(\sec x) = \sec x \tan x$$

Example

Find y'' if $y = \sec x$.

Since $y' = \sec x \tan x$, we have that

$$y'' = \frac{d}{dx}(y') = \frac{d}{dx}(\sec x \tan x)$$

$$=$$

$$=$$

$$=$$

$$\frac{d}{dx}(\tan x) = \sec^2 x \qquad \frac{d}{dx}(\sec x) = \sec x \tan x$$

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Find y'' if $y = \sec x$.

Since $y' = \sec x \tan x$, we have that

$$\begin{aligned} y'' &= \frac{d}{dx}(y') = \frac{d}{dx}(\sec x \tan x) \\ &= \frac{d}{dx}(\sec x) \tan x + \sec x \frac{d}{dx}(\tan x) \\ &= \\ &= \end{aligned}$$

$$\frac{d}{dx}(\tan x) = \sec^2 x \qquad \frac{d}{dx}(\sec x) = \sec x \tan x$$

Example

Find y'' if $y = \sec x$.

Since $y' = \sec x \tan x$, we have that

$$\begin{aligned} y'' &= \frac{d}{dx}(y') = \frac{d}{dx}(\sec x \tan x) \\ &= \frac{d}{dx}(\sec x)\tan x + \sec x \frac{d}{dx}(\tan x) \\ &= (\sec x \tan x)(\tan x) + (\sec x)(\sec^2 x) \\ &= \sec x \tan^2 x + \sec^3 x. \end{aligned}$$

The differentiability of the trigonometric functions throughout their domains implies their continuity at every point in their domains (Theorem 1, Section 3.2). So we can calculate limits of algebraic combinations and compositions of trigonometric functions by direct substitution.

EXAMPLE 7 We can use direct substitution in computing limits involving trigonometric functions. We must be careful to avoid division by zero, which is algebraically undefined.

$$\lim_{x \to 0} \frac{\sqrt{2 + \sec x}}{\cos(\pi - \tan x)} = \frac{\sqrt{2 + \sec 0}}{\cos(\pi - \tan 0)} = \frac{\sqrt{2 + 1}}{\cos(\pi - 0)} = \frac{\sqrt{3}}{-1} = -\sqrt{3}$$ ■

3.6 The Chain Rule

$$\frac{d}{dx}(\tan x) = \sec^2 x \qquad \frac{d}{dx}(\sec x) = \sec x \tan x$$

How do we differentiate $F(x) = \sin(x^2 - 4)$?

$$\frac{d}{dx}(\tan x) = \sec^2 x \qquad \frac{d}{dx}(\sec x) = \sec x \tan x$$

Theorem (The Chain Rule)

Suppose that

- $y = f(u)$ *is differentiable at the point* $u = g(x)$; *and*
- $g(x)$ *is differentiable at* x.

Then $f \circ g$ *is differentiable at* x *and*

$$\boxed{(f \circ g)'(x) = f'\big(g(x)\big)g'(x).}$$

$$\frac{d}{dx}(\tan x) = \sec^2 x \qquad \frac{d}{dx}(\sec x) = \sec x \tan x$$

The Chain Rule is easier to remember if we use Leibniz's notation:

$$\boxed{\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}}$$

$$\frac{d}{dx}(\tan x) = \sec^2 x \qquad \frac{d}{dx}(\sec x) = \sec x \tan x$$

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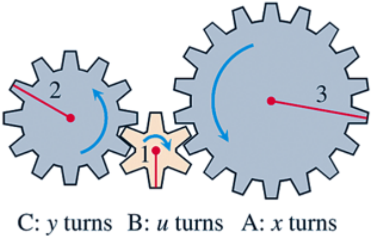

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$$

Example

Differentiate $y = \sin(x^2 - 4)$.

We have $y = \sin u$ with $u = x^2 - 4$.

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$$

Example

Differentiate $y = \sin(x^2 - 4)$.

We have $y = \sin u$ with $u = x^2 - 4$. Now $\frac{dy}{du} = \cos u$ and $\frac{du}{dx} = 2x$.

$$\boxed{\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}}$$

Example

Differentiate $y = \sin(x^2 - 4)$.

We have $y = \sin u$ with $u = x^2 - 4$. Now $\frac{dy}{du} = \cos u$ and $\frac{du}{dx} = 2x$. Therefore

$$\begin{aligned} \frac{dy}{dx} &= \frac{dy}{du}\frac{du}{dx} \\ &= \\ &= \\ &= \end{aligned}$$

by the Chain Rule.

$$\boxed{\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}}$$

Example

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$$\begin{aligned} \frac{dy}{dx} &= \frac{dy}{du}\frac{du}{dx} \\ &= (\cos u)(2x) \\ &= 2x\cos u \\ &= \end{aligned}$$

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$$\begin{aligned} \frac{dy}{dx} &= \frac{dy}{du}\frac{du}{dx} \\ &= (\cos u)(2x) \\ &= 2x\cos u \\ &= 2x\cos(x^2 - 4) \end{aligned}$$

by the Chain Rule.

$$\boxed{\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}}$$

Example

Differentiate $\sin(x^2 + x)$.

$$\boxed{\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}}$$

Example

Differentiate $\sin(x^2 + x)$.

Let $u = x^2 + x$.

$$\boxed{\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}}$$

Example

Differentiate $\sin(x^2 + x)$.

Let $u = x^2 + x$. Then

$$\begin{aligned} \frac{d}{dx}\left(\sin(x^2 + x)\right) &= \frac{d}{du}(\sin u)\,\frac{du}{dx} \\ &= (\cos u)(2x + 1) \\ &= (2x + 1)\cos(x^2 + x) \end{aligned}$$

by the Chain Rule.

$$\boxed{\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}}$$

Example (Using the Chain Rule Two Times)

Differentiate $g(t) = \tan(5 - \sin 2t)$.

Let $u = 5 - \sin 2t$. Then $g(t) = \tan u$. Hence

$$\frac{dg}{dt} = \frac{dg}{du}\frac{du}{dt} = (\sec^2 u)\frac{d}{dt}(5 - \sin 2t).$$

$$\boxed{\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}}$$

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We need to use the Chain Rule a second time: Let $w = 2t$.

$$\boxed{\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}}$$

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We need to use the Chain Rule a second time: Let $w = 2t$. Then

$$\begin{aligned}
\frac{dg}{dt} &= (\sec^2 u)\frac{d}{dt}(5 - \sin 2t) \\
&= (\sec^2 u)\frac{d}{dw}(5 - \sin w)\frac{dw}{dt} \\
&= (\sec^2 u)(-\cos w)(2) \\
&= -2\cos 2t \sec^2(5 - \sin 2t).
\end{aligned}$$

$$\boxed{\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}}$$

Remark

Your final answer should not have u or w in it.

$$\boxed{\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}}$$

The Chain Rule with Powers of a Function

If

- f is a differentiable function of u;
- u is a differentiable function of x; and
- $y = f(u)$,

then the Chain Rule $\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$ is the same as

$$\frac{d}{dx}f(u) = f'(u)\frac{du}{dx}.$$

$$\boxed{\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}}$$

The Chain Rule with Powers of a Function

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then the Chain Rule $\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$ is the same as

$$\frac{d}{dx}f(u) = f'(u)\frac{du}{dx}.$$

Now suppose that $n \in \mathbb{R}$ and $f(u) = u^n$. Then $f'(u) = nu^{n-1}$. So

$$\boxed{\frac{d}{dx}\left(u^n\right) = nu^{n-1}\frac{du}{dx}.}$$

$$\frac{d}{dx}(u^n) = nu^{n-1}\frac{du}{dx}$$

Example

$$\begin{aligned} \frac{d}{dx}\left(5x^3 - x^4\right)^7 &= \\ &= \end{aligned}$$

$$\boxed{\frac{d}{dx}\left(u^n\right) = nu^{n-1}\frac{du}{dx}}$$

Example

$$\frac{d}{dx}\left(5x^3 - x^4\right)^7 = 7\left(5x^3 - x^4\right)^6 \frac{d}{dx}\left(5x^3 - x^4\right)$$
$$=$$

$$\boxed{\frac{d}{dx}\left(u^n\right) = nu^{n-1}\frac{du}{dx}}$$

Example

$$\begin{aligned}\frac{d}{dx}\left(5x^3 - x^4\right)^7 &= 7\left(5x^3 - x^4\right)^6 \frac{d}{dx}\left(5x^3 - x^4\right) \\ &= 7\left(5x^3 - x^4\right)^6 \left(15x^2 - 4x^3\right).\end{aligned}$$

$$\boxed{\frac{d}{dx}\left(u^n\right) = nu^{n-1}\frac{du}{dx}}$$

Example

$$\begin{aligned}\frac{d}{dx}\left(\frac{1}{3x-2}\right) &= \frac{d}{dx}\left(3x-2\right)^{-1} \\ &= \\ &= \end{aligned}$$

$$\boxed{\frac{d}{dx}\left(u^n\right) = nu^{n-1}\frac{du}{dx}}$$

Example

$$\begin{aligned}\frac{d}{dx}\left(\frac{1}{3x-2}\right) &= \frac{d}{dx}\left(3x-2\right)^{-1} \\ &= -1\left(3x-2\right)^{-2}\frac{d}{dx}\left(3x-2\right) \\ &= -\left(\frac{1}{(3x-2)^2}\right)(2) = \frac{-3}{(3x-2)^2}.\end{aligned}$$

$$\frac{d}{dx}\left(u^n\right) = nu^{n-1}\frac{du}{dx}$$

Example

$$\frac{d}{dx}\left(\sin^5 x\right) = 5\sin^4 x \frac{d}{dx}(\sin x) = 5\sin^4 x \cos x.$$

$$\boxed{\frac{d}{dx}\left(u^n\right) = nu^{n-1}\frac{du}{dx}}$$

Example

Differentiate $|x|$.

Since $|x| = \sqrt{x^2}$,

$$\frac{d}{dx}\left(u^n\right) = nu^{n-1}\frac{du}{dx}$$

Example

Differentiate $|x|$.

Since $|x| = \sqrt{x^2}$, we can calculate that if $x \neq 0$ then

$$\begin{aligned} \frac{d}{dx}|x| &= \frac{d}{dx}\left(\sqrt{x^2}\right) = \frac{d}{du}\left(\sqrt{u}\right)\frac{d}{dx}\left(x^2\right) \\ &= \frac{1}{2\sqrt{u}}2x = \frac{x}{\sqrt{x^2}} = \frac{x}{|x|}. \end{aligned}$$

$$\boxed{\frac{d}{dx}\left(u^n\right) = nu^{n-1}\frac{du}{dx}}$$

Example

Show that the slope of every line tangent to the curve $y = \frac{1}{(1-2x)^3}$ is positive.

$$\boxed{\frac{d}{dx}\left(u^n\right) = nu^{n-1}\frac{du}{dx}}$$

Example

Show that the slope of every line tangent to the curve $y = \frac{1}{(1-2x)^3}$ is positive.

This is the same as:

Example

Let $y = \frac{1}{(1-2x)^3}$ for $x \neq \frac{1}{2}$. Show that $\frac{dy}{dx} > 0$.

$$\frac{d}{dx}(u^n) = nu^{n-1}\frac{du}{dx}$$

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Show that the slope of every line tangent to the curve $y = \frac{1}{(1-2x)^3}$ is positive.

This is the same as:

Example

Let $y = \frac{1}{(1-2x)^3}$ for $x \neq \frac{1}{2}$. Show that $\frac{dy}{dx} > 0$.

First we calculate that

$$\begin{aligned}\frac{dy}{dx} &= \frac{d}{dx}(1-2x)^{-3} = -3(1-2x)^{-4}\frac{d}{dx}(1-2x) \\ &= -3(1-2x)^{-4}(-2) = \frac{6}{(1-2x)^4}\end{aligned}$$

if $x \neq \frac{1}{2}$.

$$\frac{d}{dx}(u^n) = nu^{n-1}\frac{du}{dx}$$

Example

Show that the slope of every line tangent to the curve
$y = \frac{1}{(1-2x)^3}$ is positive.

This is the same as:

Example

Let $y = \frac{1}{(1-2x)^3}$ for $x \neq \frac{1}{2}$. Show that $\frac{dy}{dx} > 0$.

First we calculate that

$$\begin{aligned}\frac{dy}{dx} &= \frac{d}{dx}(1-2x)^{-3} = -3(1-2x)^{-4}\frac{d}{dx}(1-2x)\\ &= -3(1-2x)^{-4}(-2) = \frac{6}{(1-2x)^4}\end{aligned}$$

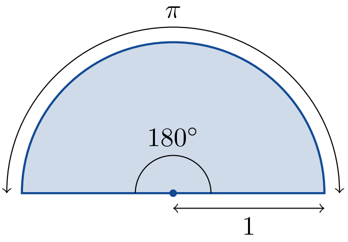
if $x \neq \frac{1}{2}$. Since $(1-2x)^4 > 0$ if $x \neq \frac{1}{2}$ and $6 > 0$, we have that $\frac{dy}{dx} > 0$ if $x \neq \frac{1}{2}$.

Example (Why Do We Use Radians in Calculus?)

Remember that $\frac{d}{dx}\sin x = \cos x$ is true *only if we use radians.* What happens if we use degrees?

Example (Why Do We Use Radians in Calculus?)

Remember that $\dfrac{d}{dx}\sin x = \cos x$ is true *only if we use radians.* What happens if we use degrees?



Remember that

$$\begin{aligned} 180 \text{ degrees} &= \pi \text{ radians} \\ 180^\circ &= \pi \\ 1^\circ &= \frac{\pi}{180} \\ x^\circ &= \frac{\pi x}{180}. \end{aligned}$$

So

$$\frac{d}{dx}\sin x^{\circ} = \frac{d}{dx}\sin\left(\frac{\pi x}{180}\right)$$

So

$$\frac{d}{dx}\sin x^\circ = \frac{d}{dx}\sin\left(\frac{\pi x}{180}\right) = \frac{\pi}{180}\cos\left(\frac{\pi x}{180}\right) = \frac{\pi}{180}\cos x^\circ.$$

So

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Therefore we have

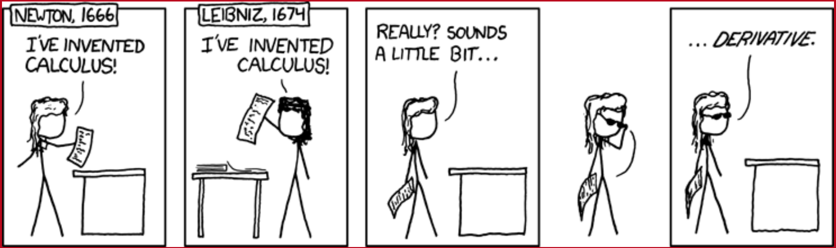
$$\boxed{\frac{d}{dx}\sin x = \cos x} \qquad \text{and} \qquad \boxed{\frac{d}{dx}\sin x^\circ = \frac{\pi}{180}\cos x^\circ.}$$

a nice formula   not nice

This is why we use radians in Calculus.

Break

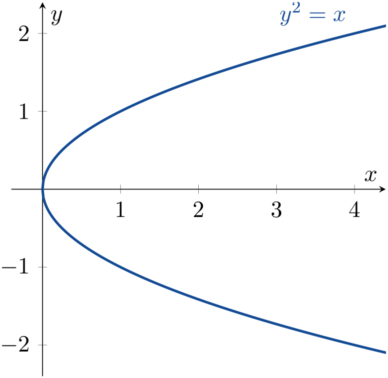
We will continue at 2pm



3.8 Implicit Differentiation

Remark

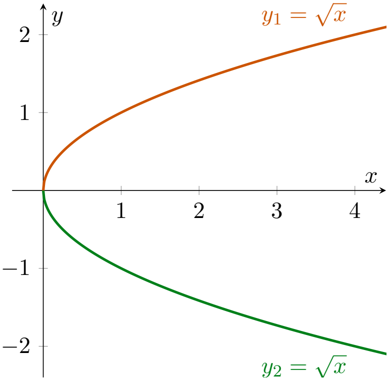
Implicit differentiate seems to scare Calculus students, but it really shouldn't – it is really just the Chain Rule again.



Example

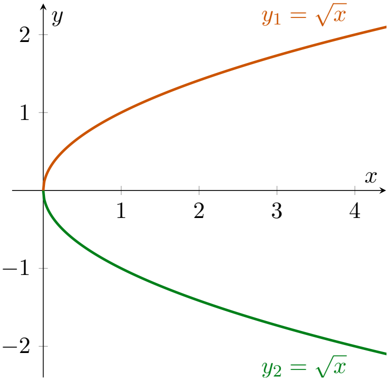
Find $\dfrac{dy}{dx}$ if $y^2 = x$.

Solution 1: (Don't use Implicit Differentiation)



Consider two functions $y_1 = \sqrt{x}$ and $y_2 = -\sqrt{x}$.

Solution 1: (Don't use Implicit Differentiation)



Consider two functions $y_1 = \sqrt{x}$ and $y_2 = -\sqrt{x}$.

Then

$$\frac{dy_1}{dx} = \frac{1}{2\sqrt{x}}$$

and

$$\frac{dy_2}{dx} = -\frac{1}{2\sqrt{x}}.$$

Solution 2: (Use Implicit Differentiation)

Since

$$y^2 = x$$

Solution 2: (Use Implicit Differentiation)

Since

$$\frac{d}{dx}\left(y^2\right) = \frac{d}{dx}\left(x\right)$$

Solution 2: (Use Implicit Differentiation)

Since

$$\frac{d}{dx}\left(y^2\right) = \frac{d}{dx}\left(x\right) = 1$$

Solution 2: (Use Implicit Differentiation)

Since

$$\frac{d}{dy}\left(y^2\right)\frac{dy}{dx} = \frac{d}{dx}\left(y^2\right) = \frac{d}{dx}\left(x\right) = 1$$

Solution 2: (Use Implicit Differentiation)

Since

$$2y\frac{dy}{dx} = \frac{d}{dy}\left(y^2\right)\frac{dy}{dx} = \frac{d}{dx}\left(y^2\right) = \frac{d}{dx}\left(x\right) = 1$$

Solution 2: (Use Implicit Differentiation)

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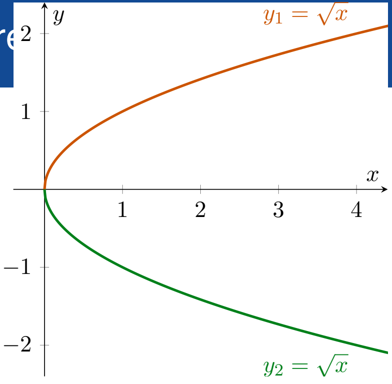
we have

$$\frac{dy}{dx} = \frac{1}{2y}.$$

3.8 Implicit Differe



Remark

The formula

$$\frac{dy}{dx} = \frac{1}{2y}$$

gives both

$$\frac{dy_1}{dx} = \frac{1}{2y_1} = \frac{1}{2\sqrt{x}} \qquad \text{and} \qquad \frac{dy_2}{dx} = \frac{1}{2y_2} = \frac{1}{2(-\sqrt{x})} = -\frac{1}{2\sqrt{x}}.$$

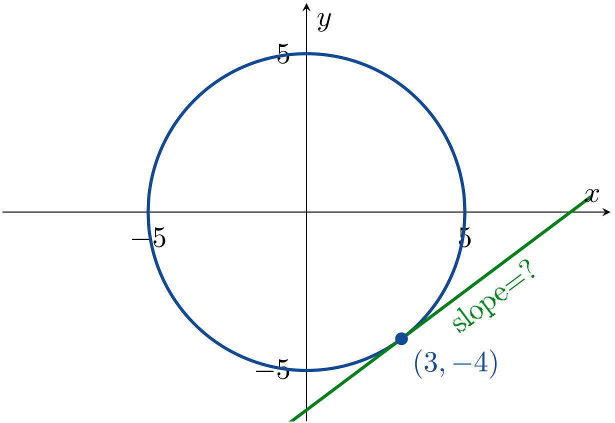
Remark

Implicit Differentiation is just

1. Differentiate both sides of an equation;
2. Use the Chain Rule.

Example

Find the slope of the circle $x^2 + y^2 = 25$ at the point $(3, -4)$.



Example

Find the slope of the circle $x^2 + y^2 = 25$ at the point $(3, -4)$.

Using Implicit Differentiation, we calculate that

$$\begin{aligned} x^2 + y^2 &= 25 \\ \frac{d}{dx}\left(x^2\right) + \frac{d}{dx}\left(y^2\right) &= \frac{d}{dx}\left(25\right) \\ 2x + 2y\frac{dy}{dx} &= 0 \\ \frac{dy}{dx} &= -\frac{x}{y}. \end{aligned}$$

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Therefore the slope at $(3, -4)$ is

$$\left.\frac{dy}{dx}\right|_{(3,-4)} = -\left.\frac{x}{y}\right|_{(3,-4)} = -\frac{3}{-4} = \frac{3}{4}.$$

Example

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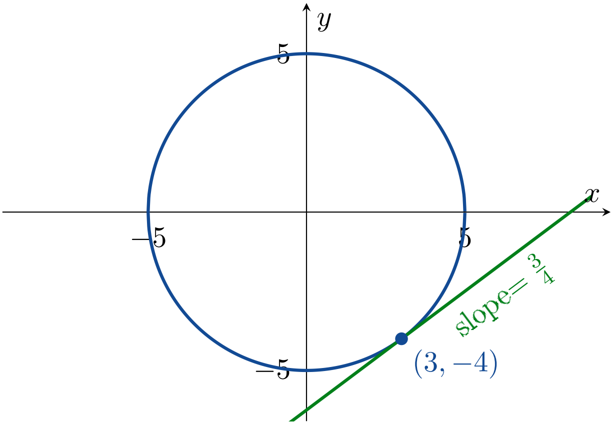
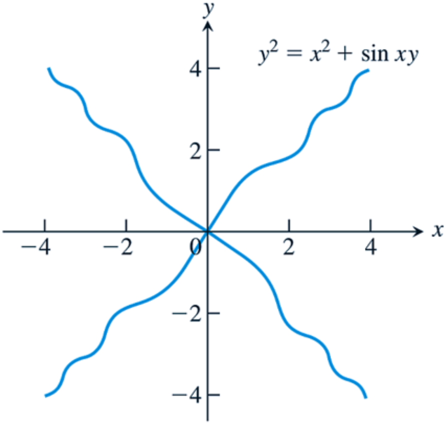


Example

Find $\dfrac{dy}{dx}$ if $y^2 = x^2 + \sin xy$.



Example

Find $\dfrac{dy}{dx}$ if $y^2 = x^2 + \sin xy$.

We calculate that

$$\frac{d}{dx}\left(y^2\right) = \frac{d}{dx}\left(x^2\right) + \frac{d}{dx}\left(\sin xy\right)$$

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Find $\dfrac{dy}{dx}$ if $y^2 = x^2 + \sin xy$.

We calculate that

$$\begin{aligned} \frac{d}{dx}\left(y^2\right) &= \frac{d}{dx}\left(x^2\right) + \frac{d}{dx}\left(\sin xy\right) \\ 2y\frac{dy}{dx} &= 2x + (\cos xy)\frac{d}{dx}(xy) \end{aligned}$$

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\frac{d}{dx}\left(y^2\right) &= \frac{d}{dx}\left(x^2\right) + \frac{d}{dx}\left(\sin xy\right) \\
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2y\frac{dy}{dx} &= 2x + (\cos xy)\left(y + x\frac{dy}{dx}\right)
\end{aligned}$$

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(2y - x\cos xy)\frac{dy}{dx} &= 2x + y\cos xy
\end{aligned}$$

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We calculate that

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(2y - x\cos xy)\frac{dy}{dx} &= 2x + y\cos xy \\
\frac{dy}{dx} &= \frac{2x + y\cos xy}{2y - x\cos xy}.
\end{aligned}$$

Example (Derivatives of Higher Order)

Find $\dfrac{d^2y}{dx^2}$ if $2x^3 - 3y^2 = 8$.

If $y \neq 0$, then

$$2x^3 - 3y^2 = 8$$

Example (Derivatives of Higher Order)

Find $\dfrac{d^2y}{dx^2}$ if $2x^3 - 3y^2 = 8$.

If $y \neq 0$, then

$$\frac{d}{dx}\left(2x^3 - 3y^2\right) = \frac{d}{dx}\left(8\right)$$
$$6x^2 - 6yy' = 0$$

Example (Derivatives of Higher Order)

Find $\dfrac{d^2y}{dx^2}$ if $2x^3 - 3y^2 = 8$.

If $y \neq 0$, then

$$\begin{aligned} \frac{d}{dx}\Big(2x^3 - 3y^2\Big) &= \frac{d}{dx}\Big(8\Big) \\ 6x^2 - 6yy' &= 0 \\ 6yy' &= 6x^2 \\ y' &= \frac{x^2}{y}. \end{aligned}$$

$$y' = \frac{x^2}{y}$$

Now we need to differentiate again.

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$$y'' = \frac{d}{dx}\left(\frac{x^2}{y}\right) = \frac{2xy - x^2y'}{y^2} = \frac{2x}{y} - \frac{x^2}{y^2} \cdot y'$$

$$=$$

if $y \neq 0$.

$$y' = \frac{x^2}{y}$$

Now we need to differentiate again. Using the Quotient Rule we calculate that

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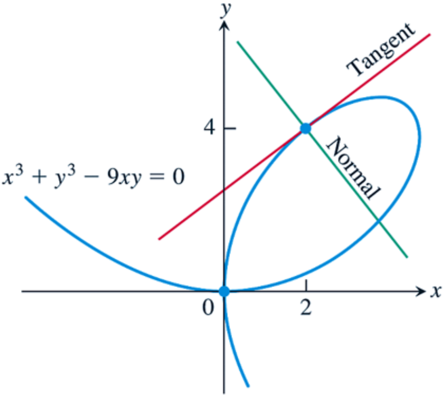
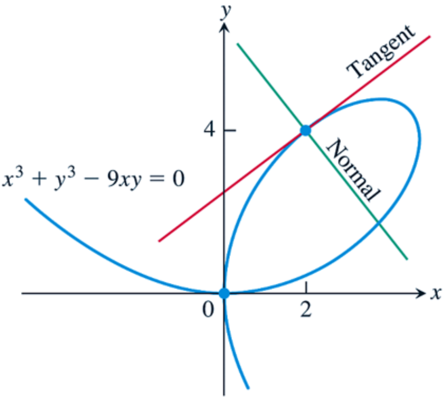
if $y \neq 0$.



Definition

The line passing through a point P which is perpendicular/orthogonal to the tangent line at P is called the *normal line* at P.



If the tangent line has slope k, then the normal line has slope $-\frac{1}{k}$.

Example

Show that the point $(2, 4)$ lies on the curve $x^3 + y^3 - 9xy = 0$. Then find the tangent and normal to the curve there.

Since

$$x^3 + y^3 - 9xy = (2)^3 + (4)^3 - 9(2)(4) = 8 + 64 - 72 = 0,$$

the point $(2, 4)$ does lie on this curve.

Then we calculate that

$$
\begin{aligned}
x^3 + y^3 - 9xy &= 0 \\
\frac{d}{dx}\left(x^3\right) + \frac{d}{dx}\left(y^3\right) - \frac{d}{dx}(9xy) &= \frac{d}{dx}(0) \\
3x^2 + 3y^2\frac{dy}{dx} - 9\left(x\frac{dy}{dx} + y\frac{dx}{dx}\right) &= 0 \\
\frac{dy}{dx} &= 9y - 3x^2 \\
\frac{dy}{dx} &= \frac{9y - 3x^2}{3y^2 - 9x} = \frac{3y - x^2}{y^2 - 3x}
\end{aligned}
$$

Then we calculate that

$$
\begin{aligned}
x^3 + y^3 - 9xy &= 0 \\
\frac{d}{dx}\left(x^3\right) + \frac{d}{dx}\left(y^3\right) - \frac{d}{dx}(9xy) &= \frac{d}{dx}(0) \\
3x^2 + 3y^2\frac{dy}{dx} - 9\left(x\frac{dy}{dx} + y\frac{dx}{dx}\right) &= 0 \\
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\frac{dy}{dx} &= \frac{9y - 3x^2}{3y^2 - 9x} = \frac{3y - x^2}{y^2 - 3x}
\end{aligned}
$$

and

$$\left.\frac{dy}{dx}\right|_{(2,4)} = \left.\frac{3y - x^2}{y^2 - 3x}\right|_{(2,4)} = \frac{3 \cdot 4 - 2^2}{4^2 - 3 \cdot 2} = \frac{8}{10} = \frac{4}{5}.$$

$$\left.\frac{dy}{dx}\right|_{(2,4)} = \frac{4}{5}$$

So the tangent has slope $\frac{4}{5}$ and the normal has slope $-\frac{5}{4}$.

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So the tangent has slope $\frac{4}{5}$ and the normal has slope $-\frac{5}{4}$.

Therefore the tangent line is

$$y = 4 + \frac{4}{5}(x-2) = \frac{4}{5}x + \frac{12}{5}$$

and the normal line is

$$y = 4 - \frac{5}{4}(x-2) = -\frac{5}{4}x + \frac{13}{2}.$$

Next Time

- 3.8 Related Rates
- 3.9 Linearisation and Differentials
- 4.1 Extreme Values of Functions
- 4.2 The Mean Value Theorem