Lecture 5

- Vector Spaces
- Subspaces
- Linear Independence

Vector Spaces

You studied vectors, the dot product, etc. in MATH114. Now it is time to generalise these concepts.

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10. $1\mathbf{u}=\mathbf{u}$.

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- If all the scalars k are real numbers, then V is called a *real vector space*.
- If we allow complex numbers k, then V is called a *complex vector space*.
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- $k\mathbf{u} = \mathbf{0} \implies k = 0$ or $\mathbf{u} = \mathbf{0}$.

To Show That a Set with Two Operations Is a Vector Space

Step 1. Identify the set V of objects that will become vectors.

Step 2. Identify the addition and scalar multiplication operations on V.

Step 3. Verify Axioms 1 and 6; that is, adding two vectors in V produces a vector in V, and multiplying a vector in V by a scalar also produces a vector in V. Axiom 1 is called ***closure under addition***, and Axiom 6 is called ***closure under scalar multiplication***.

Step 4. Confirm that Axioms 2, 3, 4, 5, 7, 8, 9, and 10 hold.

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I leave it to you to check that all 10 axioms are satisfied. We call this the *zero vector space*.

Example (The Vector Space of Sequences)

Let V consist of objects of the form

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We define two infinite sequences to be equal if their corresponding components are equal, and we define

$$\begin{aligned} \mathbf{u} + \mathbf{v} &= (u_1, u_2, \ldots, u_n, \ldots) + (v_1, v_2, \ldots, v_n, \ldots) \\ &= (u_1 + v_1, u_2 + v_2, \ldots, u_n + v_n, \ldots) \\ k\mathbf{u} &= (ku_1, ku_2, \ldots, ku_n, \ldots) \end{aligned}$$

Show that V is a vector space.

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   Clearly $1\mathbf{u} = (1u_1, 1u_2, 1u_3, \dots) = (u_1, u_2, u_3, \dots) = \mathbf{u}$ ✓

Therefore V is a vector space.

Example

Let

$$\begin{aligned}\mathbb{P}^n &= \{\text{all polynomials of degree} \leq n\} \\ &= \{\mathbf{p}(t) = a_0 + a_1 t + a_2 t^2 + \ldots + a_n t^n \mid a_j \in \mathbb{R}\}\end{aligned}$$

for $n \in \mathbb{N} \cup \{0\}$. (See page 210, example 4 in your textbook.) Let addition and scalar multiplication be defined in the obvious way.

Show that $\mathbb{P}^n$ is a vector space.

(Sometimes this set is denoted by $\mathbb{P}_n$ or P_n.)

First let's do axioms 1 and 6. If

$$\mathbf{p}(t) = a_0 + a_1 t + a_2 t^2 + \ldots + a_n t^n$$

and

$$\mathbf{q}(t) = b_0 + b_1 t + b_2 t^2 + \ldots + b_n t^n,$$

then

$$\begin{aligned}(\mathbf{p}+\mathbf{q})(t) &= \mathbf{p}(t) + \mathbf{q}(t) \\ &= (a_0 + b_0) + (a_1 + b_1)t + (a_2 + b_2)t^2 + \ldots + (a_n + b_n)t^n\end{aligned}$$

is a polynomial of degree $\leq n$, and

$$k\mathbf{p}(t) = ka_0 + ka_1 t + ka_2 t^2 + \ldots + ka_n t^n$$

is also a polynomial of degree $\leq n$. So 1. and 6. are satisfied.

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5. For each $\mathbf{u}$ in V, there is an object $-\mathbf{u}$ in V such that $\mathbf{u} + (-\mathbf{u}) = \mathbf{0}$.
   For $\mathbf{p}(t)$ as before, just let

$$-\mathbf{p}(t) = -a_0 - a_1 t - a_2 t^2 - \ldots - a_n t^2.$$

Therefore $\mathbb{P}^n$ is a vector space.

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$$(\mathbf{p} + \mathbf{q})(t) = (1 + x + x^2) + (2 - x^2) = 3 + x$$

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V is not closed under addition, so is not a vector space.

EXAMPLE 5 Let V be the set of all real-valued functions defined on a set $\mathbb{D}$. (Typically, $\mathbb{D}$ is the set of real numbers or some interval on the real line.) Functions are added in the usual way: $\mathbf{f} + \mathbf{g}$ is the function whose value at t in the domain $\mathbb{D}$ is $\mathbf{f}(t) + \mathbf{g}(t)$. Likewise, for a scalar c and an $\mathbf{f}$ in V, the scalar multiple $c\mathbf{f}$ is the function whose value at t is $c\mathbf{f}(t)$. For instance, if $\mathbb{D} = \mathbb{R}$, $\mathbf{f}(t) = 1 + \sin 2t$, and $\mathbf{g}(t) = 2 + .5t$, then

$$(\mathbf{f} + \mathbf{g})(t) = 3 + \sin 2t + .5t \quad \text{and} \quad (2\mathbf{g})(t) = 4 + t$$

Two functions in V are equal if and only if their values are equal for every t in $\mathbb{D}$. Hence the zero vector in V is the function that is identically zero, $\mathbf{f}(t) = 0$ for all t, and the negative of $\mathbf{f}$ is $(-1)\mathbf{f}$. Axioms 1 and 6 are obviously true, and the other axioms follow from properties of the real numbers, so V is a vector space. ■

Example

Let

$$\mathbb{R}^{2\times 2} = \{\text{all } 2\times 2 \text{ matrices with real number entries}\}$$

and let addition and scalar multiplication be the usual operations on matrices: i.e.

$$\mathbf{u}+\mathbf{v} = \begin{bmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{bmatrix} + \begin{bmatrix} v_{11} & v_{12} \\ v_{21} & v_{22} \end{bmatrix} = \begin{bmatrix} u_{11}+v_{11} & u_{12}+v_{12} \\ u_{21}+v_{21} & u_{22}+v_{22} \end{bmatrix}$$

$$k\mathbf{u} = k\begin{bmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{bmatrix} = \begin{bmatrix} ku_{11} & ku_{12} \\ ku_{21} & ku_{22} \end{bmatrix}.$$

Show that $\mathbb{R}^{2\times 2}$ is a vector space.

(Some books use M_{22} instead of $\mathbb{R}^{2\times 2}$.)

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I leave axioms 3,7,8,9 for you to prove. I will just prove 4,5 and 10.

4. Let $\mathbf{0} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$. Then

$$\mathbf{0}+\mathbf{u} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{bmatrix} = \begin{bmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{bmatrix} = \mathbf{u}$$

and similarly $\mathbf{u}+\mathbf{0} = \mathbf{u}$. ✓

5. Let $-\mathbf{u} = \begin{bmatrix} -u_{11} & -u_{12} \\ -u_{21} & -u_{22} \end{bmatrix}$. Then

$$\mathbf{u} + (-\mathbf{u}) = \begin{bmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{bmatrix} + \begin{bmatrix} -u_{11} & -u_{12} \\ -u_{21} & -u_{22} \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}. \checkmark$$

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10 Finally

$$1\mathbf{u} = 1 \begin{bmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{bmatrix} = \begin{bmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{bmatrix} = \mathbf{u}. \checkmark$$

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This proves that $\mathbb{R}^{2\times 2} = M_{22}$ is a vector space.

So far I have been using $\mathbf{u} + \mathbf{v}$ and $k\mathbf{u}$ for addition and scalar multiplication, but this might be confusing for more complicated operations.

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Notation

Let

- $\mathbf{u} \oplus \mathbf{v}$ denote vector addition; and
- $k \odot \mathbf{u}$ denote scalar multiplication.

Example

Let $V = \mathbb{R}^2$ and define $\oplus$ and $\odot$ as follows: If $\mathbf{u} = (u_1, u_2)$ and $\mathbf{v} = (v_1, v_2)$, then we define

$$\mathbf{u} \oplus \mathbf{v} = (u_1 + v_1, u_2 + v_2)$$

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$$\mathbf{u} \oplus \mathbf{v} = (u_1 + v_1, u_2 + v_2) \longleftarrow \text{usual addition}$$

and

$$k \odot \mathbf{u} = (ku_1, 0). \longleftarrow \text{weird scalar multiplication}$$

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$$k \odot \mathbf{u} = (ku_1, 0).$$

For example, if $\mathbf{u} = (2, 4)$, $\mathbf{v} = (-3, 5)$ and $k = 7$, then

$$\mathbf{u} \oplus \mathbf{v} = (2 + (-3), 4 + 5) = (-1, 9)$$
$$k \odot \mathbf{u} = (7 \cdot 2, 0) = (14, 0).$$

Is V a vector space?

$$k \odot \mathbf{u} = (ku_1, 0)$$

Axioms 1-9 are all true. I leave this for you to check.

$$k \odot \mathbf{u} = (ku_1, 0)$$

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10 However axiom 10 fails. For example, if $\mathbf{u} = (0, 1)$ then

$$1 \odot \mathbf{u} = 1 \odot (0, 1) = (1 \cdot 0, 0) = (0, 0) \neq \mathbf{u}.$$

Therefore V is not a vector space with these operations.

Example (An Unusual Vector Space)

Let V be the set of strictly positive real numbers. Let $\mathbf{u} = u$ and $\mathbf{v} = v$ be any vectors in V (i.e. any real numbers > 0) and let k be a scalar. Define $\oplus$ and $\odot$ by

$$\mathbf{u} \oplus \mathbf{v} = uv$$
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Show that V is a vector space with these operations.

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4. The zero vector in this space is the number 1 (i.e. $\mathbf{0} = 1$) since

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5. The negative of a vector $\mathbf{u} = u$ is $-\mathbf{u} = \frac{1}{u}$ since
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7. We have
$$k \odot (\mathbf{u} \oplus \mathbf{v}) = (uv)^k = u^k v^k = (k \odot \mathbf{u}) \oplus (k \odot \mathbf{v}).$$

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Subspaces

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Definition

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Remark

The definition of a vector space has ten conditions.

However, if a particular two of these ten conditions are satisfied, then the other eight are always passed on from V to W. So there are only two things that we need to check to decide if W is a subspace of V.

Theorem

Suppose that

- V *is a vector space; and*
- W *is a subset of* V.

Then

W *is a subspace of* V $\iff$

1. *If* $\mathbf{u}$ *and* $\mathbf{v}$ *are vectors in* W, *then* $\mathbf{u}+\mathbf{v}$ *is in* W.
6. *If* k *is any scalar and* $\mathbf{u}$ *is a vector in* W, *then* $k\mathbf{u}$ *is in* W.

Remark

So we just need to check that our subset W is closed under vector addition and scalar multiplication, for it to be a subspace.

Example (The Zero Subspace)

If V is any vector space, and if $W = \{\mathbf{0}\}$ is the subset of V that contains only the zero vector, then W is closed under addition and scalar multiplication since

$$\mathbf{0} + \mathbf{0} = \mathbf{0} \qquad \text{and} \qquad k\mathbf{0} = \mathbf{0}$$

for any scalar k. We call W the *zero subspace* of V.

Example (Lines Through the Origin)

If W is a line through the origin of $\mathbb{R}^3$ (or $\mathbb{R}^2$), then adding two vectors on the line or multiplying a vector on the line by a scalar produces another vector on the line,

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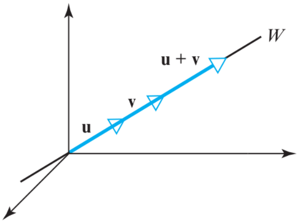
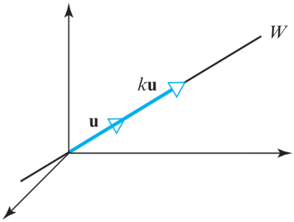
If W is a line through the origin of $\mathbb{R}^3$ (or $\mathbb{R}^2$), then adding two vectors on the line or multiplying a vector on the line by a scalar produces another vector on the line, so W is closed under addition and scalar multiplication. Hence W is a subspace of $\mathbb{R}^3$ (or $\mathbb{R}^2$).



(a) W is closed under addition.



(b) W is closed under scalar multiplication.

Example

Let

$$V = \{ \text{ all functions } f : \mathbb{R} \to \mathbb{R} \ \}$$

and

$$\mathbb{P} = \{ \text{ all polynomials } \}.$$

Then V is a vector space (left for you to prove).

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Because polynomial+polynomial is a polynomial, and scalar×polynomial is a polynomial, $\mathbb{P}$ is closed under both addition and scalar multiplication. Hence $\mathbb{P}$ is a subspace of V.

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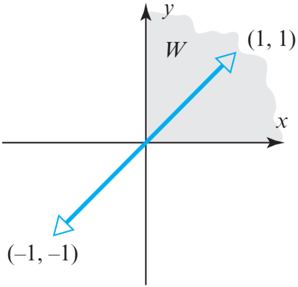
Because polynomial+polynomial is a polynomial, and scalar×polynomial is a polynomial, $\mathbb{P}$ is closed under both addition and scalar multiplication. Hence $\mathbb{P}$ is a subspace of V.

Example

$\mathbb{P}^n$ is a subspace of $\mathbb{P}$.

Example (A subset of $\mathbb{R}^2$ that is NOT a subspace)

Let W be the set of all points (x, y) in $\mathbb{R}^2$ for which $x \geq 0$ and $y \geq 0$ (the shaded region below). This set is not a subspace of $\mathbb{R}^2$ because it is not closed under scalar multiplication. For example, $\mathbf{v} = (1, 1)$ is a vector in W, but $(-1)\mathbf{v} = (-1, -1)$ is not.



Example (Subspaces of $\mathbb{R}^{n\times n}$)

Recall that

$$\begin{pmatrix}\text{symmetric}\\ \text{matrix}\end{pmatrix} + \begin{pmatrix}\text{symmetric}\\ \text{matrix}\end{pmatrix} = \begin{pmatrix}\text{symmetric}\\ \text{matrix}\end{pmatrix}$$

and

$$k\begin{pmatrix}\text{symmetric}\\ \text{matrix}\end{pmatrix} = \begin{pmatrix}\text{symmetric}\\ \text{matrix}\end{pmatrix}.$$

Hence, the set of all symmetric $n \times n$ matrices is a subspace of $\mathbb{R}^{n\times n} = M_{nn}$.

(Similarly, the sets of upper triangular matrices, lower triangular matrices, and diagonal matrices are subspaces of $\mathbb{R}^{n\times n}$.)

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The set W of all invertible matrices is **not** a subspace of $\mathbb{R}^{n\times n}$.

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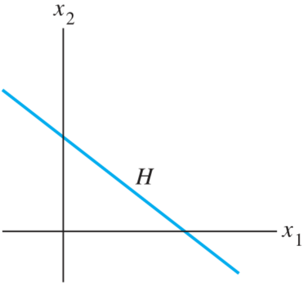
For example, $U = \begin{bmatrix} 1 & 2 \\ 2 & 5 \end{bmatrix}$ and $V = \begin{bmatrix} -1 & 2 \\ -2 & 5 \end{bmatrix}$ are both invertible matrices (please check), but

$$U + V = \begin{bmatrix} 0 & 4 \\ 0 & 10 \end{bmatrix}$$

is not invertible (why?). Hence W is not closed under addition.

Example

A line in $\mathbb{R}^2$ which does not pass through the origin is not a subspace of $\mathbb{R}^2$ because it does not contain the zero vector.



Theorem

If W_1 and W_2 are both subspaces of a vector space V, then $W_1 \cap W_2$ is also a subspace of V.

Proof.

Note first that $\mathbf{0} \in W_1 \cap W_2$ because $\mathbf{0}$ is in both W_1 and W_2. So $W_1 \cap W_2$ is not empty.

Subspaces

Theorem

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We need to prove that $W_1 \cap W_2$ is closed under vector addition and scalar multiplation.

Proof continued.

1. Vector Addition: Since W_1 and W_2 are closed under vector addition, we have

$$\mathbf{u}, \mathbf{v} \in W_1 \cap W_2 \implies \mathbf{u}, \mathbf{v} \in W_1 \implies \mathbf{u} + \mathbf{v} \in W_1$$

6. Scalar Multiplication:

Proof continued.

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6. Scalar Multiplication: Similarly, since W_1 and W_2 are closed under scalar multiplication,

$$\mathbf{u} \in W_1 \cap W_2 \implies \mathbf{u} \in W_1 \implies k\mathbf{u} \in W_1$$

$\square$

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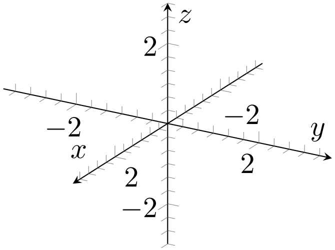
$\square$

Similarly

Theorem

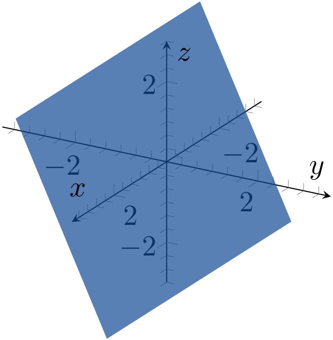
If $W_1, W_2, \ldots, W_r$ are subspaces of a vector space V, then $W_1 \cap W_2 \cap \ldots \cap W_r$ is also a subspace of V.



Example

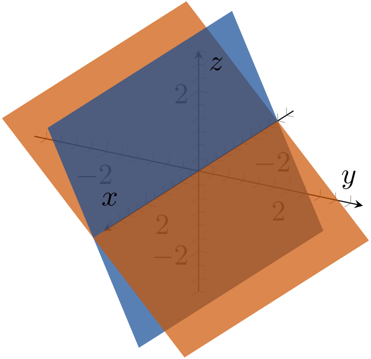
Consider the vector space $V = \mathbb{R}^3$.

Subspaces



Example

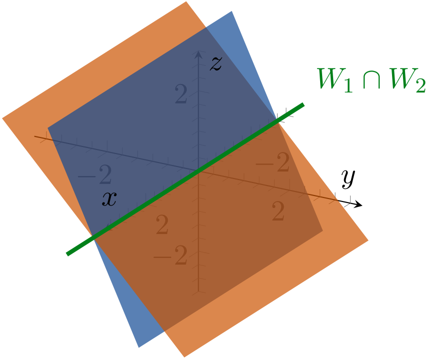
Consider the vector space $V = \mathbb{R}^3$. If W_1 and W_2 are planes passing through the origin,



Example

Consider the vector space $V = \mathbb{R}^3$. If W_1 and W_2 are planes passing through the origin, then W_1 and W_2 are subspaces of $\mathbb{R}^3$.

Subspaces



Example

Consider the vector space $V = \mathbb{R}^3$. If W_1 and W_2 are planes passing through the origin, then W_1 and W_2 are subspaces of $\mathbb{R}^3$. Hence $W_1 \cap W_2$ is also a subspace of $\mathbb{R}^3$.

Linear Combinations

Definition

If $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_r$ are vectors in a vector space V, then

$$k_1\mathbf{v}_1 + k_2\mathbf{v}_2 + \ldots + k_r\mathbf{v}_r$$

is called a *linear combination* of $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_r$. The scalars $k_1, k_2, \ldots, k_r$ are called the *coefficients* of the linear combination.

Theorem

Let $S = \{\mathbf{w}_1, \mathbf{w}_2, \dots, \mathbf{w}_r\}$ *be a nonempty set of vectors in a vector space* V. *Then*

1

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Let $S = \{\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_r\}$ *be a nonempty set of vectors in a vector space* V*. Then*

1. *The set*

$$\begin{aligned} W &= \{ \text{ all possible linear combinations of the vectors in } S \text{ } \} \\ &= \{c_1\mathbf{w}_1 + c_2\mathbf{w}_2 + \ldots + c_r\mathbf{w}_r \mid c_j \in \mathbb{R}\} \end{aligned}$$

   is a subspace of V*.*

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   is a subspace of V.

2. W *is the "smallest"*[1] *subspace of* V *that contains all the vectors in* S.

[1] i.e. any other subspace that contains S also contains W.

Subspaces

Proof.

Let

$$\begin{aligned} W &= \{ \text{ all possible linear combinations of the vectors in } S \ \} \\ &= \{c_1\mathbf{w}_1 + c_2\mathbf{w}_2 + \ldots + c_r\mathbf{w}_r \mid c_j \in \mathbb{R}\}. \end{aligned}$$

1. We need to prove closure under vector addition and scalar multiplication.

Proof continued.

Let

$$\mathbf{u} = a_1\mathbf{w}_1 + a_2\mathbf{w}_2 + \ldots + a_r\mathbf{w}_r$$

and

$$\mathbf{v} = b_1\mathbf{w}_1 + b_2\mathbf{w}_2 + \ldots + b_r\mathbf{w}_r$$

be two vectors in W, and let k be a scalar (number).

Proof continued.

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$$\mathbf{u} = a_1\mathbf{w}_1 + a_2\mathbf{w}_2 + \ldots + a_r\mathbf{w}_r$$

and

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be two vectors in W, and let k be a scalar (number). Then

$$\mathbf{u} + \mathbf{v} = (a_1 + b_1)\mathbf{w}_1 + (a_2 + b_2)\mathbf{w}_2 + \ldots + (a_r + b_r)\mathbf{w}_r$$

and

$$k\mathbf{u} = (ka_1)\mathbf{w}_1 + (ka_2)\mathbf{w}_2 + \ldots + (ka_r)\mathbf{w}_r$$

are both linear combinations of the vectors in S, so are also in W. This proves that W is a subspace of V.

Proof continued.

2. And we need to prove that W is the smallest such subspace.

Let $\tilde{W}$ be any subspace of V that contains all the vectors in S.

Proof continued.

2. And we need to prove that W is the smallest such subspace.

Let $\tilde{W}$ be any subspace of V that contains all the vectors in S.

Since $\tilde{W}$ is closed under vector addition and scalar multiplication, it contains all the linear combinations of $\mathbf{w}_1, \mathbf{w}_2, .\,, \mathbf{w}_r$.

Proof continued.

2. And we need to prove that W is the smallest such subspace.

Let $\tilde{W}$ be any subspace of V that contains all the vectors in S.

Since $\tilde{W}$ is closed under vector addition and scalar multiplication, it contains all the linear combinations of $\mathbf{w}_1, \mathbf{w}_2, .., \mathbf{w}_r$. Hence $W \subseteq \tilde{W}$ and we are finished. $\square$

Definition

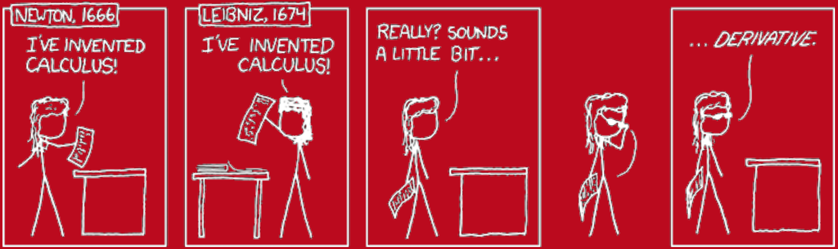
If $S = \{\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_r\}$ is a nonempty set of vectors in a vector space V, then the subspace

$$\begin{aligned}\operatorname{span} S &= \{ \text{ all possible linear combinations of the vectors in } S \text{ } \} \\ &= \{c_1\mathbf{w}_1 + c_2\mathbf{w}_2 + \ldots + c_r\mathbf{w}_r \mid c_j \in \mathbb{R}\}.\end{aligned}$$

is called the *subspace spanned by* S.

Break

We will continue at 3pm



Example (The standard unit vectors span $\mathbb{R}^n$)

The standard unit vectors in $\mathbb{R}^n$ are

$$\mathbf{e}_1 = (1, 0, 0, \ldots, 0),\ \mathbf{e}_2 = (0, 1, 0, \ldots, 0),\ \ldots,\ \mathbf{e}_n = (0, 0, 0, \ldots, 1).$$

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These vectors span $\mathbb{R}^n$ because every vector

$$\mathbf{v} = (v_1, v_2, \ldots, v_n) \in \mathbb{R}^n$$

can be written as

$$\mathbf{v} = v_1\mathbf{e}_1 + v_2\mathbf{e}_2 + \ldots + v_n\mathbf{e}_n$$

which is a linear combinations of $\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n$.

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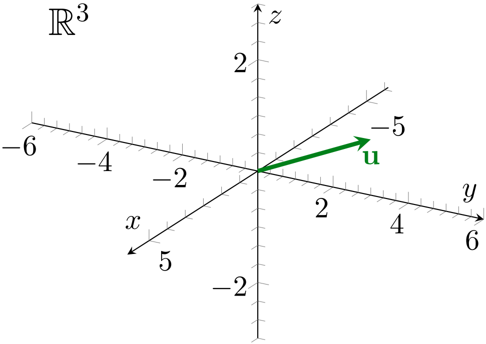
which is a linear combinations of $\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n$.

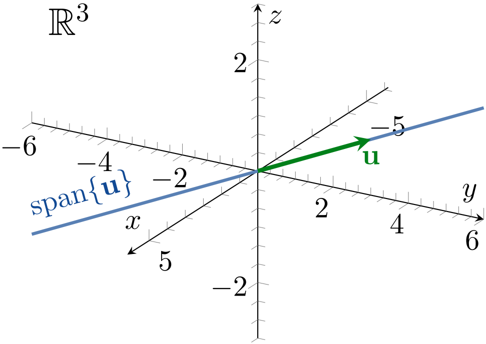
In other words,

$$\mathbb{R}^n = \text{span}\{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n\}.$$



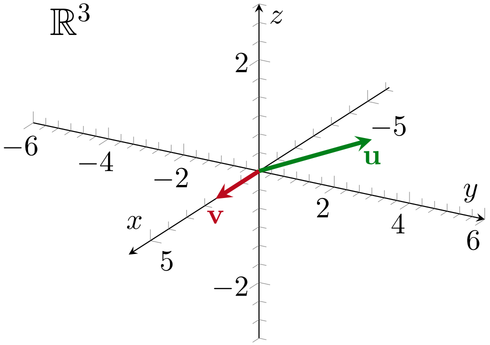
Example (A geometric view)

Let $V = \mathbb{R}^3$. If $\mathbf{u}$ is a nonzero vector in $\mathbb{R}^3$,



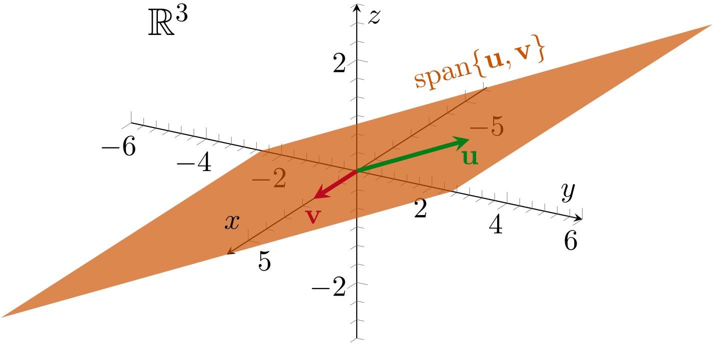
Example (A geometric view)

Let $V = \mathbb{R}^3$. If $\mathbf{u}$ is a nonzero vector in $\mathbb{R}^3$, then span$\{\mathbf{u}\}$ (i.e. the set of all scalar multiples of $\mathbf{u}$) is a line through the origin.



Example (A geometric view)

If $\mathbf{u}$ and $\mathbf{v}$ are two nonzero vectors in $\mathbb{R}^3$,



Example (A geometric view)

If $\mathbf{u}$ and $\mathbf{v}$ are two nonzero vectors in $\mathbb{R}^3$, then span$\{\mathbf{u}, \mathbf{v}\}$ is a plane (or a line if $\mathbf{u}$ and $\mathbf{v}$ are parallel) through the origin.

Example

Let H be the set of all vectors of the form $(a-3b, b-a, a, b)$. That is, let

$$H = \{(a-3b, b-a, a, b) \mid a, b \in \mathbb{R}\} \subseteq \mathbb{R}^4.$$

Show that H is a subspace of $\mathbb{R}^4$.

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$$H = \{(a-3b, b-a, a, b) \mid a, b \in \mathbb{R}\} \subseteq \mathbb{R}^4.$$

Show that H is a subspace of $\mathbb{R}^4$.

If we write $(a-3b, b-a, a, b)$ as a column vector, we can see that

$$\begin{bmatrix} a-3b \\ b-a \\ a \\ b \end{bmatrix} = a \begin{bmatrix} 1 \\ -1 \\ 1 \\ 0 \end{bmatrix} + b \begin{bmatrix} -3 \\ 1 \\ 0 \\ 1 \end{bmatrix}.$$

Hence H is the span of two nonzero vectors. Therefore H is a subspace of $\mathbb{R}^4$.

Example (A spanning set for $\mathbb{P}^n$)

The polynomials $1, x, x^2, \ldots, x^n$ span the vector space $\mathbb{P}^n$ since every polynomial $\mathbf{p}$ in $\mathbb{P}^n$ can be written as

$$\mathbf{p} = a_0 + a_1 x + a_2 x^2 + \ldots + a_n x^n$$

which is a linear combination of $1, x, x^2, \ldots, x^n$.

We write this as

$$\mathbb{P}^n = \text{span}\{1, x, x^2, \ldots, x^n\}.$$

Example

For what value of h will $\mathbf{y}$ be in span$\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ if

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ -1 \\ -2 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 5 \\ -4 \\ -7 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} -3 \\ 1 \\ 0 \end{bmatrix}, \quad \text{and} \quad \mathbf{y} = \begin{bmatrix} -4 \\ 3 \\ h \end{bmatrix}.$$

We want there to exist scalars c_1, c_2, c_3 such that

$$c_1\mathbf{v}_1 \quad + \quad c_2\mathbf{v}_2 \quad + \quad c_3\mathbf{v}_3 \quad = \quad \mathbf{y}$$

Subspaces

We want there to exist scalars c_1, c_2, c_3 such that

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So we want

$$\begin{cases} c_1 + 5c_2 - 3c_3 = -4 \\ -c_1 - 4c_2 + c_3 = 3 \\ -2c_1 - 7c_2 = h \end{cases}$$

to be consistent.

Since (see page 51 of your textbook)

$$\begin{bmatrix} 1 & 5 & -3 & -4 \\ -1 & -4 & 1 & 3 \\ -2 & -7 & 0 & h \end{bmatrix} \sim \begin{bmatrix} 1 & 5 & -3 & -4 \\ 0 & 1 & -2 & -1 \\ 0 & 0 & 0 & h-5 \end{bmatrix}$$

we can see that the linear system is consistent if and only if $h = 5$.

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we can see that the linear system is consistent if and only if $h = 5$.

Therefore $\mathbf{y}$ is in $\operatorname{span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ iff $h = 5$.

Remark

In general, if you want to learn if $\mathbf{y}$ is in $\text{span}\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_r\}$ then start by row reducing the augmented matrix

$$\begin{bmatrix} \\ \mathbf{v}_1 & \mathbf{v}_2 & \cdots & \mathbf{v}_r & \mathbf{y} \\ \\ \end{bmatrix}.$$

Example

Consider the vectors $\mathbf{u} = (1, 2, -1)$ and $\mathbf{v} = (6, 4, 2)$ in $\mathbb{R}^3$. Show that $\mathbf{y} = (9, 2, 7)$ is a linear combination of $\mathbf{u}$ and $\mathbf{v}$ and that $\mathbf{x} = (4, -1, 8)$ is not a linear combination of $\mathbf{u}$ and $\mathbf{v}$.

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We start with

$$\begin{bmatrix} \mathbf{u} & \mathbf{v} & \mathbf{y} & \mathbf{x} \end{bmatrix} = \begin{bmatrix} 1 & 6 & 9 & 4 \\ 2 & 4 & 2 & -1 \\ -1 & 2 & 7 & 8 \end{bmatrix}.$$

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We start with

$$\begin{bmatrix} \mathbf{u} & \mathbf{v} & \mathbf{y} & \mathbf{x} \end{bmatrix} = \begin{bmatrix} 1 & 6 & 9 & 4 \\ 2 & 4 & 2 & -1 \\ -1 & 2 & 7 & 8 \end{bmatrix}.$$

This is row equivalent (please check) to

$$\begin{bmatrix} 1 & 0 & -3 & -\frac{11}{4} \\ 0 & 1 & 2 & \frac{9}{8} \\ 0 & 0 & 0 & 3 \end{bmatrix}.$$

Therefore $\mathbf{y} = -3\mathbf{u} + 2\mathbf{v}$, but $\mathbf{x} = k_1\mathbf{u} + k_2\mathbf{v}$ is inconsistent. Hence $\mathbf{y} \in \operatorname{span}\{\mathbf{u}, \mathbf{v}\}$ but $\mathbf{x} \notin \operatorname{span}\{\mathbf{u}, \mathbf{v}\}$.

Subspaces

Example

Do the vectors $\mathbf{v}_1 = (1, 1, 2)$, $\mathbf{v}_2 = (1, 0, 1)$ and $\mathbf{v}_3 = (2, 1, 3)$ span $\mathbb{R}^3$?

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We need to answer the question: If $\mathbf{y} = (y_1, y_2, y_3)$ is any vector in $\mathbb{R}^3$, can we find scalars k_1, k_2, k_3 such that

$$\mathbf{y} = k_1\mathbf{v}_1 + k_2\mathbf{v}_2 + k_3\mathbf{v}_3 \ ?$$

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$$\mathbf{y} = k_1\mathbf{v}_1 + k_2\mathbf{v}_2 + k_3\mathbf{v}_3 \text{ ?}$$

This is the same as the question: Is

$$\begin{bmatrix} 1 & 1 & 2 \\ 1 & 0 & 1 \\ 2 & 1 & 3 \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$$

consistent for all $\mathbf{y}$?

Or: Is the following matrix invertible?

$$A = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 0 & 1 \\ 2 & 1 & 3 \end{bmatrix}$$

Or: Is the following matrix invertible?

$$A = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 0 & 1 \\ 2 & 1 & 3 \end{bmatrix}$$

The answer is no because $\det(A) = 0$ (please check). Therefore $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ do not span $\mathbb{R}^3$.

Solution Spaces of Homogeneous Linear Systems

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \dots + a_{1n}x_n = 0 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \dots + a_{2n}x_n = 0 \\ \vdots \qquad\qquad \vdots \qquad\qquad \vdots \qquad\qquad\qquad \vdots \qquad \vdots \\ a_{m1}x_1 + a_{m2}x_2 + a_{m3}x_3 + \dots + a_{mn}x_n = 0. \end{cases}$$

Theorem

The solution set of a homogeneous linear system $A\mathbf{x} = \mathbf{0}$ of m equations in n unknowns is a subspace of $\mathbb{R}^n$.

Spanning Sets are not Unique

Note that

$$\mathrm{span}\{(1,0),(0,1)\} = \mathbb{R}^2 = \mathrm{span}\{(1,-1),(1,1)\}.$$

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$$\operatorname{span}\{(1,0),(0,1)\} = \mathbb{R}^2 = \operatorname{span}\{(1,-1),(1,1)\}.$$

Here we have

$$\begin{aligned}(1,0)-(0,1) &= (1,-1)\\ (1,0)+(0,1) &= (1,1)\end{aligned}$$

and

$$\begin{aligned}(1,0) &= \frac{1}{2}(1,-1)+\frac{1}{2}(1,1)\\ (0,1) &= -\frac{1}{2}(1,-1)+\frac{1}{2}(1,1).\end{aligned}$$

Theorem

Suppose that

- V *is a vector space*
- $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_r\}$ *and* $T = \{\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_k\}$ *are nonempty sets of vectors in* V.

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Suppose that

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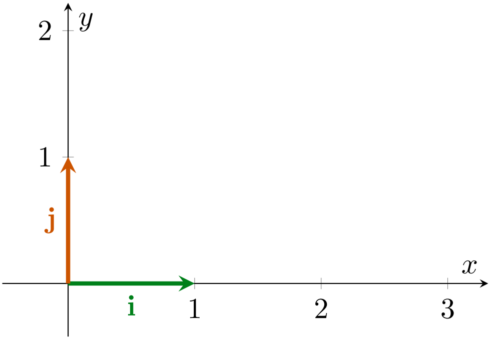
Then

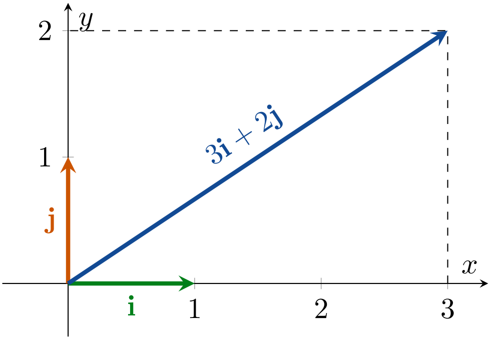
$$\operatorname{span} S = \operatorname{span} T$$

if and only if each vector in S *is a linear combination of those in* T, **and** *each vector in* T *is a linear combination of those in* S.

Linear Independence



In $\mathbb{R}^2$, every vector can be written as a linear combination of $\mathbf{i} = (1, 0)$ and $\mathbf{j} = (0, 1)$ in exactly one way.

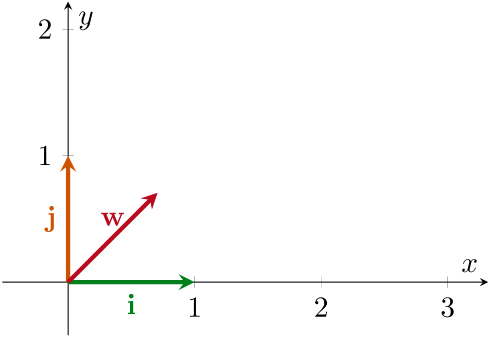


In $\mathbb{R}^2$, every vector can be written as a linear combination of $\mathbf{i} = (1, 0)$ and $\mathbf{j} = (0, 1)$ in exactly one way. For example,

$$(3, 2) = 3(1, 0) + 2(0, 1) = 3\mathbf{i} + 2\mathbf{j}.$$

But what if instead of two vectors, we have 3 vectors?

$$\mathbf{i} = (1, 0), \qquad \mathbf{j} = (0, 1) \quad \text{and} \quad \mathbf{w} = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$$



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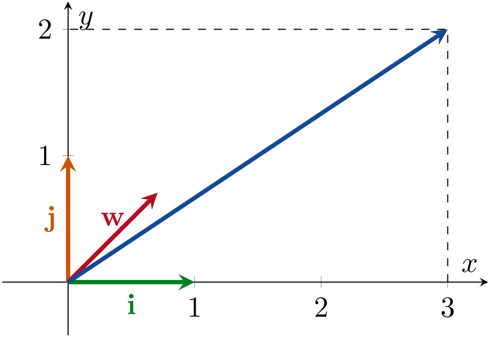
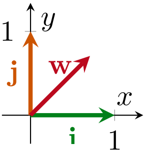


How can we express $(3, 2)$ in terms of $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{w}$?



$$\mathbf{i} = (1, 0)$$
$$\mathbf{j} = (0, 1)$$
$$\mathbf{w} = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$$

There are infinitely many ways. For example

$$(3, 2) = 3\mathbf{i} + 2\mathbf{j} + 0\mathbf{w}$$

or

$$(3, 2) = 3\mathbf{i} + \mathbf{j} + \sqrt{2}\mathbf{w}$$

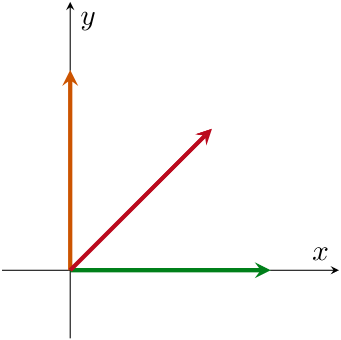
or

$$(3, 2) = 4\mathbf{i} + 3\mathbf{j} - \sqrt{2}\mathbf{w}$$

or

$$\vdots$$

This is happening because each of these three vectors can be written as a linear combination of the other two.



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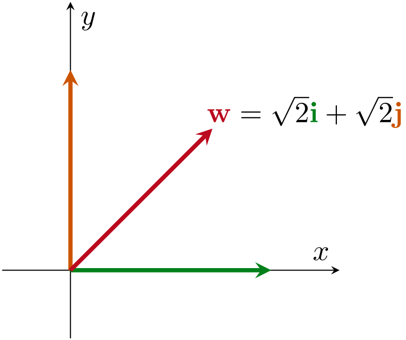

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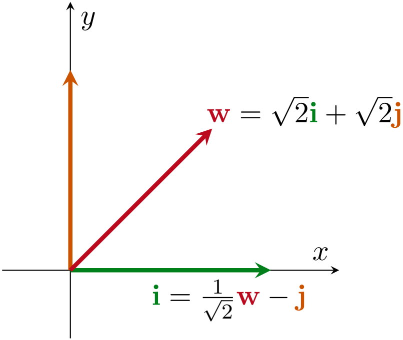

Linear Independence

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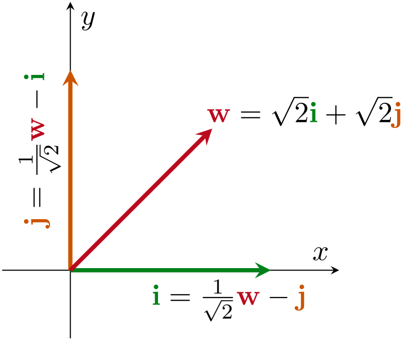

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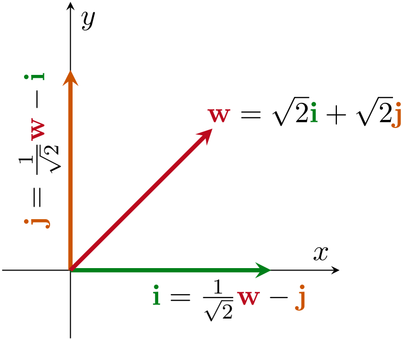

Two is the correct number of vectors for $\mathbb{R}^2$.

Three vectors is too many.

Linear Independence

Definition

If $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_r\}$ is a set of two or more vectors in a vector space V , then S is said to be a *linearly independent set* if no vector in S can be expressed as a linear combination of the others.

A set that is not linearly independent is said to be *linearly dependent*.

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A set that is not linearly independent is said to be *linearly dependent*.

Theorem

A nonempty set $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_r\}$ *in a vector space* V *is linearly independent if and only if the only coefficients satisfying the vector equation*

$$k_1\mathbf{v}_1 + k_2\mathbf{v}_2 + \ldots + k_r\mathbf{v}_r = \mathbf{0}$$

are $k_1 = 0,\ k_2 = 0,\ \ldots,\ k_r = 0.$

$\mathbf{i} = (1, 0, 0) \qquad \mathbf{j} = (0, 1, 0) \qquad \mathbf{k} = (0, 0, 1)$

Example

Let $V = \mathbb{R}^3$. Let $S = \{\mathbf{i}, \mathbf{j}, \mathbf{k}\}$ where $\mathbf{i}, \mathbf{j}, \mathbf{k}$ are the standard unit vectors in $\mathbb{R}^3$. Show that S is a linearly independent set.

To prove linear independence we must show that the only coefficients satisfying the vector equation

$$\mathbf{0} = k_1\mathbf{i} + k_2\mathbf{j} + k_3\mathbf{k}$$

are $k_1 = 0$, $k_2 = 0$ and $k_3 = 0$.

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$$(0,0,0) = k_1(1,0,0) + k_2(0,1,0) + k_3(0,0,1) = (k_1, k_2, k_3).$$

Therefore S is a linearly independent set.

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$$(0, 0, 0) = k_1(1, 0, 0) + k_2(0, 1, 0) + k_3(0, 0, 1) = (k_1, k_2, k_3).$$

Therefore S is a linearly independent set. Equivalently, $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$ are linearly independent vectors.

Example

Are the vectors

$$\mathbf{v}_1 = (1, -2, 3), \qquad \mathbf{v}_2 = (5, 6, -1), \qquad \mathbf{v}_3 = (3, 2, 1)$$

linearly independent or linearly dependent in $\mathbb{R}^3$?

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Are there nontrivial solutions to

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Are there nontrivial solutions to

$$\begin{cases} k_1 + 5k_2 + 3k_3 = 0 \\ -2k_1 + 6k_2 + 2k_3 = 0 \\ 3k_1 - \ k_2 + \ k_3 = 0 \end{cases} ?$$

$$\begin{cases} k_1 + 5k_2 + 3k_3 = 0 \\ -2k_1 + 6k_2 + 2k_3 = 0 \\ 3k_1 - k_2 + k_3 = 0 \end{cases} ?$$

Solution 1: If we solve this linear system, we find that the general solution (please check) is

$$\begin{cases} k_1 = -\frac{1}{2}k_3 \\ k_2 = -\frac{1}{2}k_3 \\ k_3 \text{ is free.} \end{cases}$$

$$\begin{cases} k_1 + 5k_2 + 3k_3 = 0 \\ -2k_1 + 6k_2 + 2k_3 = 0 \\ 3k_1 - k_2 + k_3 = 0 \end{cases} ?$$

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Therefore there are nontrivial solutions, so $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ are linearly dependent vectors.

$$\mathbf{v}_1 = (1, -2, 3), \qquad \mathbf{v}_2 = (5, 6, -1), \qquad \mathbf{v}_3 = (3, 2, 1)$$

Solution 2: Another way to answer this is to calculate the determinant of the coefficient matrix

$$A = \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3 \end{bmatrix} = \begin{bmatrix} 1 & 5 & 3 \\ -2 & 6 & 2 \\ 3 & -1 & 1 \end{bmatrix}.$$

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$$\begin{aligned} \det(A) = 0 \quad &\implies \quad A \text{ is not invertible} \\ &\implies A\mathbf{k} = \mathbf{0} \text{ has infinitely many solutions} \\ &\implies \mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3 \text{ are linearly dependent vectors.} \end{aligned}$$

EXAMPLE 3 Linear Independence in R^4

Determine whether the vectors

$$\mathbf{v}_1 = (1, 2, 2, -1), \quad \mathbf{v}_2 = (4, 9, 9, -4), \quad \mathbf{v}_3 = (5, 8, 9, -5)$$

in R^4 are linearly dependent or linearly independent.

Solution The linear independence or linear dependence of these vectors is determined by whether there exist nontrivial solutions of the vector equation

$$k_1\mathbf{v}_1 + k_2\mathbf{v}_2 + k_3\mathbf{v}_3 = \mathbf{0}$$

or, equivalently, of

$$k_1(1, 2, 2, -1) + k_2(4, 9, 9, -4) + k_3(5, 8, 9, -5) = (0, 0, 0, 0)$$

Equating corresponding components on the two sides yields the homogeneous linear system

$$\begin{aligned} k_1 + 4k_2 + 5k_3 &= 0 \\ 2k_1 + 9k_2 + 8k_3 &= 0 \\ 2k_1 + 9k_2 + 9k_3 &= 0 \\ -k_1 - 4k_2 - 5k_3 &= 0 \end{aligned}$$

We leave it for you to show that this system has only the trivial solution

$$k_1 = 0, \quad k_2 = 0, \quad k_3 = 0$$

from which you can conclude that $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$ are linearly independent.

Example

Show that the polynomials

$$1,\ x,\ x^2,\ \ldots,\ x^n$$

form a linearly independent set in $\mathbb{P}^n$.

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Let

$$\mathbf{p}_0 = 1, \quad \mathbf{p}_1 = x, \quad \mathbf{p}_2 = x^2,\ \ldots,\ \mathbf{p}_n = x^n.$$

We need to show that the only solution to

$$a_0\mathbf{p}_0 + a_1\mathbf{p}_1 + a_2\mathbf{p}_2 + \ldots + a_n\mathbf{p}_n = \mathbf{0}$$

is

$$a_0 = a_1 = a_2 = \ldots = a_n = 0.$$

But

$$a_0\mathbf{p}_0 + a_1\mathbf{p}_1 + a_2\mathbf{p}_2 + \ldots + a_n\mathbf{p}_n = \mathbf{0}$$

is equivalent to the statement that

$$a_0 + a_1x + a_2x^2 + \ldots + a_nx^n = 0$$

for all $x \in \mathbb{R}$.

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Recall that a nonzero polynomial of degree n has at most n roots.

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for all $x \in \mathbb{R}$.

Recall that a nonzero polynomial of degree n has at most n roots. If the a_j are not all zero, then we would have a nonzero polynomial with infinitely many roots (not possible).

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for all $x \in \mathbb{R}$.

Recall that a nonzero polynomial of degree n has at most n roots. If the a_j are not all zero, then we would have a nonzero polynomial with infinitely many roots (not possible).

Hence $a_j = 0$ for all j. Therefore $1,\ x,\ x^2,\ \ldots,\ x^n$ are linearly independent.

EXAMPLE 5 Linear Independence of Polynomials

Determine whether the polynomials

$$\mathbf{p}_1 = 1 - x, \quad \mathbf{p}_2 = 5 + 3x - 2x^2, \quad \mathbf{p}_3 = 1 + 3x - x^2$$

are linearly dependent or linearly independent in P_2.

Solution The linear independence or dependence of these vectors is determined by whether the vector equation

$$k_1\mathbf{p}_1 + k_2\mathbf{p}_2 + k_3\mathbf{p}_3 = \mathbf{0} \tag{7}$$

can be satisfied with coefficients that are not all zero. To see whether this is so, let us rewrite (7) in its polynomial form

$$k_1(1 - x) + k_2(5 + 3x - 2x^2) + k_3(1 + 3x - x^2) = 0 \tag{8}$$

or, equivalently, as

$$(k_1 + 5k_2 + k_3) + (-k_1 + 3k_2 + 3k_3)x + (-2k_2 - k_3)x^2 = 0$$

Since this equation must be satisfied by all x in $(-\infty, \infty)$, each coefficient must be zero (as explained in the previous example). Thus, the linear dependence or independence of the given polynomials hinges on whether the following linear system has a nontrivial solution:

$$\begin{aligned} k_1 + 5k_2 + k_3 &= 0 \\ -k_1 + 3k_2 + 3k_3 &= 0 \\ -2k_2 - k_3 &= 0 \end{aligned} \tag{9}$$

We leave it for you to show that this linear system has nontrivial solutions either by solving it directly or by showing that the coefficient matrix has determinant zero. Thus, the set $\{\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3\}$ is linearly dependent. ◀

Sets with One or Two Vectors

Theorem

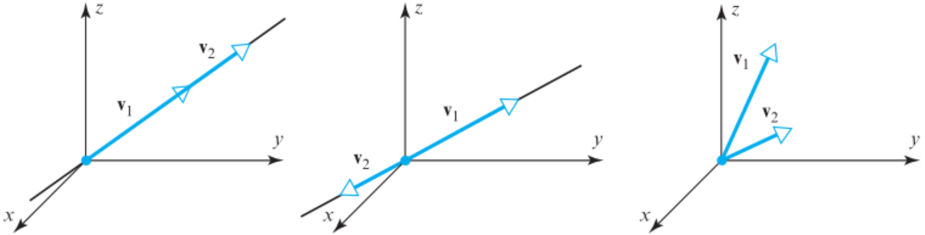
1. *A finite set that contains* $\mathbf{0}$ *is linearly dependent.*
2. *A set with exactly one vector is linearly independent if and only if that vector is not* $\mathbf{0}$.
3. *A set with exactly two vectors is linearly independent if and only if neither vector is a scalar multiple of the other.*

EXAMPLE 6 Linear Independence of Two Functions

The functions $\mathbf{f}_1 = x$ and $\mathbf{f}_2 = \sin x$ are linearly independent vectors in $F(-\infty, \infty)$ since neither function is a scalar multiple of the other. On the other hand, the two functions $\mathbf{g}_1 = \sin 2x$ and $\mathbf{g}_2 = \sin x \cos x$ are linearly dependent because the trigonometric identity $\sin 2x = 2 \sin x \cos x$ reveals that $\mathbf{g}_1$ and $\mathbf{g}_2$ are scalar multiples of each other.

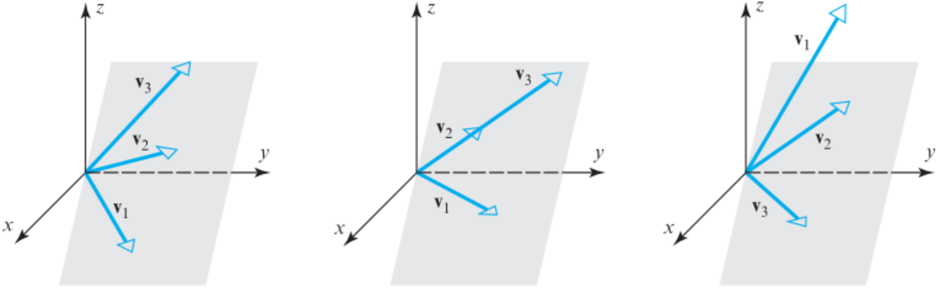
Linear independence has the following useful geometric interpretations in R^2 and R^3:

- Two vectors in R^2 or R^3 are linearly independent if and only if they do not lie on the same line when they have their initial points at the origin. Otherwise one would be a scalar multiple of the other (Figure 4.3.3).



(*a*) Linearly dependent (*b*) Linearly dependent (*c*) Linearly independent

- Three vectors in R^3 are linearly independent if and only if they do not lie in the same plane when they have their initial points at the origin. Otherwise at least one would be a linear combination of the other two (Figure 4.3.4).



(*a*) Linearly dependent (*b*) Linearly dependent (*c*) Linearly independent

Theorem

Let $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_r\}$ be a set of vectors in $\mathbb{R}^n$. If $r > n$, then S is linearly dependent.

Linear Independence of Functions

Are the functions $\mathbf{f}_1 = \sin^2 x$, $\mathbf{f}_2 = \cos^2 x$ and $\mathbf{f}_3 = 5$ linearly independent or linearly dependent?

Linear Independence of Functions

Are the functions $\mathbf{f}_1 = \sin^2 x$, $\mathbf{f}_2 = \cos^2 x$ and $\mathbf{f}_3 = 5$ linearly independent or linearly dependent?

These three functions are linearly dependent because

$$5\mathbf{f}_1 + 5\mathbf{f}_2 - \mathbf{f}_3 = 5\sin^2 x + 5\cos^2 x - 5 = 5 - 5 = \mathbf{0}.$$

This one was easy. For more difficult ones, we need something called the Wronskian.

Definition

If $\mathbf{f}_1 = f_1(x)$, $\mathbf{f}_2 = f_2(x)$, … , $\mathbf{f}_n = f_n(x)$, are functions that are $n-1$ times differentiable on the interval $(-\infty, \infty)$, then the determinant

$$W(x) = \begin{vmatrix} f_1(x) & f_2(x) & \cdots & f_n(x) \\ f_1'(x) & f_2'(x) & \cdots & f_n'(x) \\ f_1''(x) & f_2''(x) & \cdots & f_n''(x) \\ \vdots & \vdots & & \vdots \\ f_1^{(n-1)}(x) & f_2^{(n-1)}(x) & \cdots & f_n^{(n-1)}(x) \end{vmatrix}$$

is called the *Wronskian* of $f_1, f_2, \ldots, f_n$.

Theorem

If the functions $\mathbf{f}_1, \mathbf{f}_2, \ldots, \mathbf{f}_n$ have $n-1$ continuous derivatives on the interval $(-\infty, \infty)$, and if the Wronskian of these functions is not identically zero on $(-\infty, \infty)$, then these functions form a linearly independent set of vectors in $C^{(n-1)}(-\infty, \infty)$.

($C^{(n-1)}(-\infty, \infty)$ denotes the vector space of $n-1$ times continuously differentiable functions.)

▶ EXAMPLE 7 Linear Independence Using the Wronskian

Use the Wronskian to show that $\mathbf{f}_1 = x$ and $\mathbf{f}_2 = \sin x$ are linearly independent vectors in $C^\infty(-\infty, \infty)$.

Solution The Wronskian is

$$W(x) = \begin{vmatrix} x & \sin x \\ 1 & \cos x \end{vmatrix} = x\cos x - \sin x$$

This function is not identically zero on the interval $(-\infty, \infty)$ since, for example,

$$W\left(\frac{\pi}{2}\right) = \frac{\pi}{2}\cos\left(\frac{\pi}{2}\right) - \sin\left(\frac{\pi}{2}\right) = \frac{\pi}{2}$$

Thus, the functions are linearly independent.

EXAMPLE 8 Linear Independence Using the Wronskian

Use the Wronskian to show that $\mathbf{f}_1 = 1$, $\mathbf{f}_2 = e^x$, and $\mathbf{f}_3 = e^{2x}$ are linearly independent vectors in $C^\infty(-\infty, \infty)$.

Solution The Wronskian is

$$W(x) = \begin{vmatrix} 1 & e^x & e^{2x} \\ 0 & e^x & 2e^{2x} \\ 0 & e^x & 4e^{2x} \end{vmatrix} = 2e^{3x}$$

This function is obviously not identically zero on $(-\infty, \infty)$, so $\mathbf{f}_1$, $\mathbf{f}_2$, and $\mathbf{f}_3$ form a linearly independent set. ◀

Remark

We will need to use the Wronskian when we study differential equations in MATH216 Mathematics IV.

Józef Maria Hoëné-Wroński

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DECEICED
9 August 1853

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Next Time

- Coordinates and Basis
- Dimension
- Change of Basis
- Row Space, Column Space, and Null Space