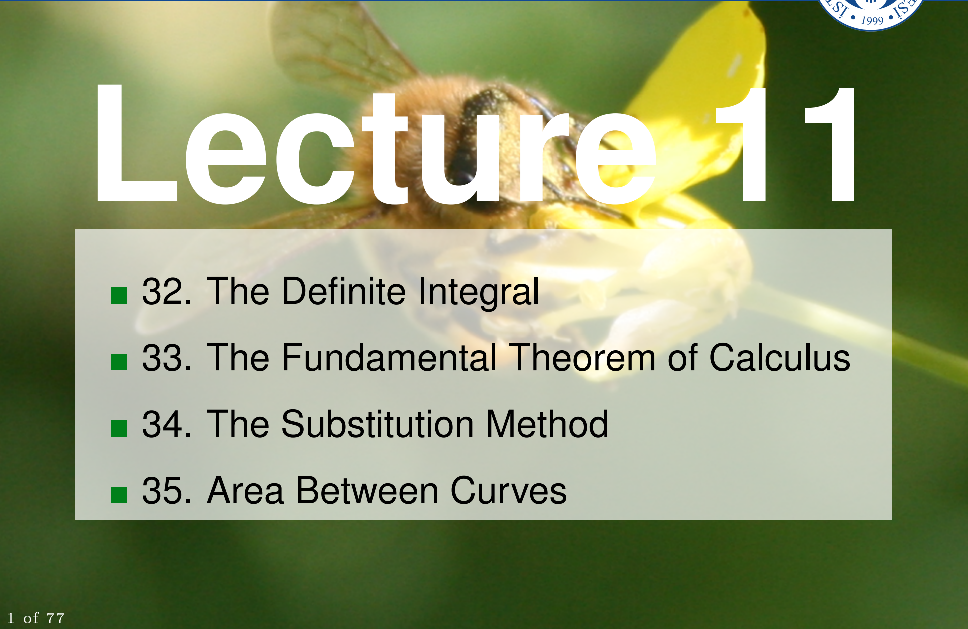


Lecture 11

- 32. The Definite Integral
- 33. The Fundamental Theorem of Calculus
- 34. The Substitution Method
- 35. Area Between Curves

The Definite Integral

Definition

<u>If</u> the limit

$$\lim_{n\to\infty}\sum_{k=1}^{n} f(c_k)\Delta x_k$$

exists, then it is called the *definite integral of f over* $[a,b]$. We write

$$\int_a^b f(x)\,dx = \lim_{n\to\infty}\sum_{k=1}^{n} f(c_k)\Delta x_k$$

if the limit exists.

32. The Definite Integral

upper limit of integration
integralin üst sınırı

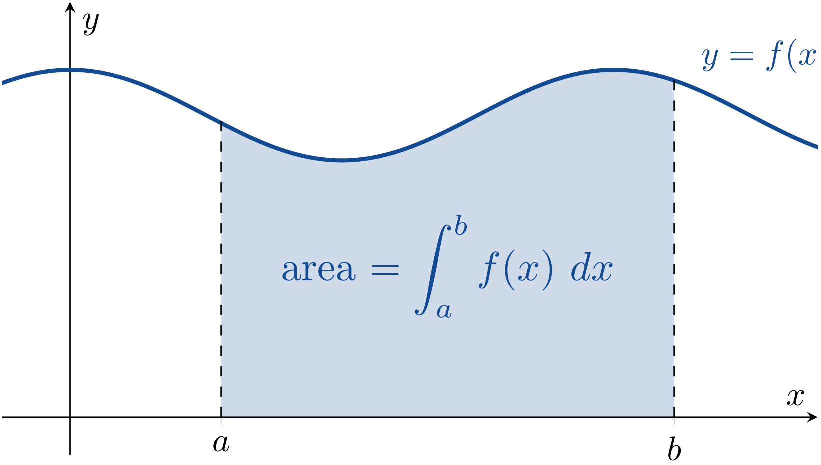
x is the variable of integration
x, integral değişkenidir

integral sign
integral işareti

$$\int_a^b \underbrace{f(x)}_{\substack{\text{the integrand} \\ \text{integralin integrandı}}} \, dx$$

lower limit of integration
integralin alt sınırı

"the integral of f from a to b"



y
$y = f(x$
area $= \int_a^b f(x)\, dx$
x
a
b

Definition

If $\int_a^b f(x)\, dx$ exists, then we say that f *is integrable on* $[a,b]$.

Example

$f(x) = 1 - x^2$ is integrable on $[0, 1]$ and $\int_0^1 (1 - x^2)\ dx = \frac{2}{3}$.

Remark

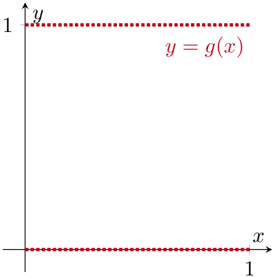
$$\int_a^b f(x)\,dx = \int_a^b f(u)\,du = \int_a^b f(t)\,dt$$

It doesn't matter which letter we use for the *dummy variable*.

Theorem

If f is continuous on $[a, b]$, then f is integrable on $[a, b]$.

If f has finitely many jump discontinuities but is otherwise continuous on $[a, b]$, then f is integrable on $[a, b]$.



Example

Define a function $g : [0, 1] \to \mathbb{R}$ by

$$g(x) = \begin{cases} 1 & x \in \mathbb{Q} \\ 0 & x \notin \mathbb{Q}. \end{cases}$$

This function is not integrable on $[0, 1]$.

Theorem

Suppose that f and g are integrable. Let k be a number.

Theorem

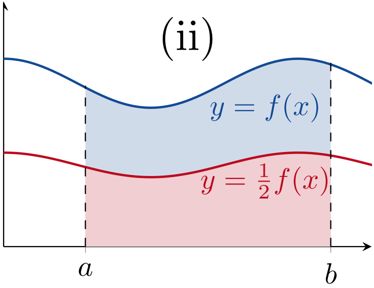
Suppose that f and g are integrable. Let k be a number. Then

1. $$\int_b^a f(x)\ dx = -\int_a^b f(x)\ dx;$$

Theorem

Suppose that f and g are integrable. Let k be a number. Then

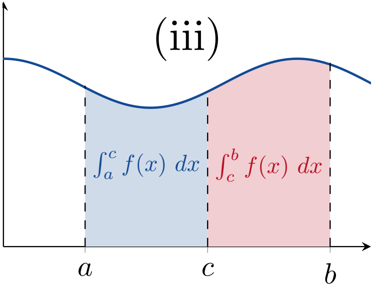
2. $$\int_a^b kf(x)\ dx = k\int_a^b f(x)\ dx;$$



Theorem

Suppose that f and g are integrable. Let k be a number. Then

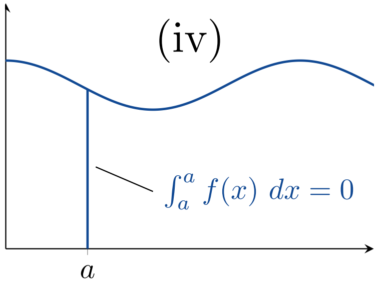
3. $$\int_a^c f(x)\ dx + \int_c^b f(x)\ dx = \int_a^b f(x)\ dx$$



Theorem

Suppose that f and g are integrable. Let k be a number. Then

4. $\displaystyle\int_a^a f(x)\,dx = 0;$



Theorem

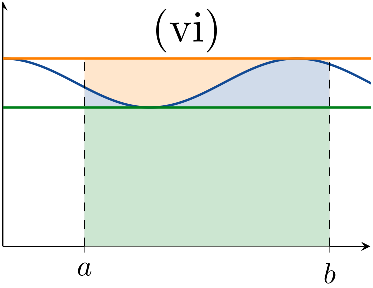
Suppose that f and g are integrable. Let k be a number. Then

5. $$\int_a^b \big(f(x) + g(x)\big)\ dx = \int_a^b f(x)\ dx + \int_a^b g(x)\ dx;$$

Theorem

Suppose that f and g are integrable. Let k be a number. Then

6. $(b-a)\min f \leq \int_a^b f(x)\;dx \leq (b-a)\max f$;

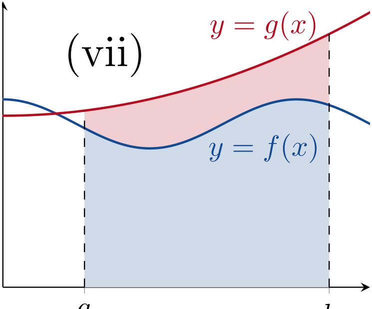


Theorem

Suppose that f and g are integrable. Let k be a number. Then

7. *if $f(x) \leq g(x)$ on $[a, b]$, then*

$$\int_a^b f(x)\ dx \leq \int_a^b g(x)\ dx;$$



Theorem

Suppose that f and g are integrable. Let k be a number. Then

8. *if $g(x) \geq 0$ on $[a, b]$, then*

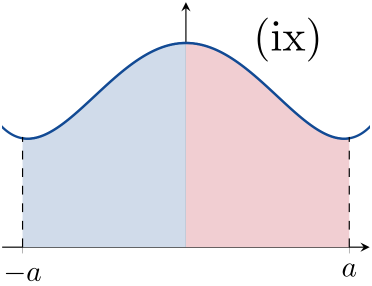
$$\int_a^b g(x)\ dx \geq 0;$$

Theorem

Suppose that f and g are integrable. Let k be a number. Then

9. *if f is an even function, then*

$$\int_{-a}^{a} f(x)\ dx = 2\int_{0}^{a} f(x)\ dx;$$

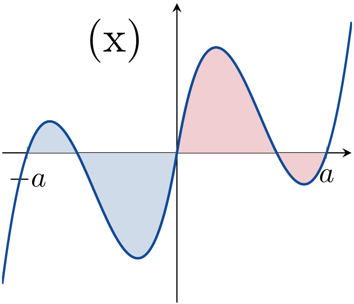


Theorem

Suppose that f and g are integrable. Let k be a number. Then

10 *if f is an odd function, then*

$$\int_{-a}^{a} f(x)\, dx = 0.$$



Example

Suppose that $\int_{-1}^{1} f(x)\ dx = 5$, $\int_{1}^{4} f(x)\ dx = -2$ and $\int_{-1}^{1} h(x)\ dx = 7$. Then

$$\int_{4}^{1} f(x)\ dx = -\int_{1}^{4} f(x)\ dx = 2,$$

$$\begin{aligned}\int_{-1}^{1} \big(2f(x) + 3h(x)\big)\ dx &= 2\int_{-1}^{1} f(x)\ dx + 3\int_{-1}^{1} h(x)\ dx \\ &= 2(5) + 3(7) = 31\end{aligned}$$

and

$$\begin{aligned}\int_{-1}^{4} f(x)\ dx &= \int_{-1}^{1} f(x)\ dx + \int_{1}^{4} f(x)\ dx \\ &= 5 + (-2) = 3.\end{aligned}$$

Example

Show that $\int_0^1 \sqrt{1+\cos x}\, dx \leq \sqrt{2}$.

solution: The maximum value of $\sqrt{1+\cos x}$ on $[0,1]$ is $\sqrt{1+1} = \sqrt{2}$. Therefore

$$\int_0^1 \sqrt{1+\cos x}\, dx \leq (1-0) \max \sqrt{1+\cos x} = 1 \times \sqrt{2}.$$

Example

Calculate $\displaystyle\int_{-2}^{2} (x^3 + x)\ dx$.

solution: Because $(x^3 + x)$ is an odd function, we have that

$$\int_{-2}^{2} (x^3 + x)\ dx = 0.$$

Example

Calculate $\int_{-1}^{1} (1 - x^2)\ dx$.

solution: Because $(1 - x^2)$ is an even function, we have that

$$\int_{-1}^{1} (1 - x^2)\ dx = 2\int_{0}^{1} (1 - x^2)\ dx = 2 \times \frac{2}{3} = \frac{4}{3}.$$

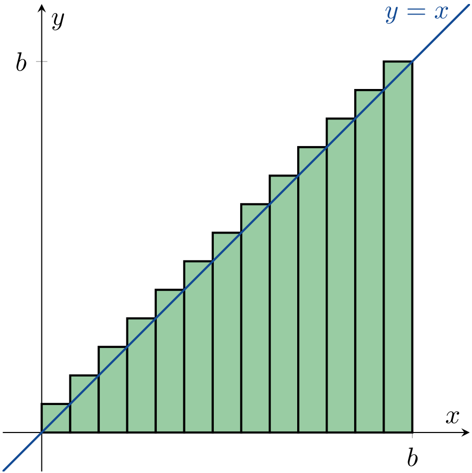
Example

Calculate $\int_0^b x\ dx$ for $b > 0$.

solution 1: We will use a Riemann Sum. First we cut $[0, b]$ in to n pieces using

$$0 < \frac{b}{n} < \frac{2b}{n} < \frac{3b}{n} < \ldots < \frac{(n-1)b}{n} < b$$

and $c_k = \frac{kb}{n}$. Note that $\Delta x_k = \frac{b}{n}$ for all k.



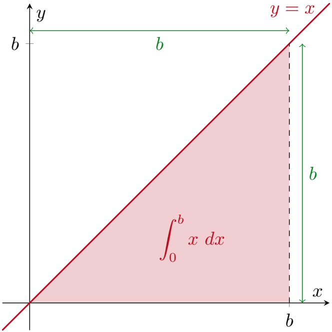
y
$y = x$
b
x
b

Then

$$\begin{aligned}\sum_{k=1}^{n} f(c_k)\Delta x_k = \sum_{k=1}^{n} \frac{kb}{n}\frac{b}{n} = \frac{b^2}{n^2}\sum_{k=1}^{n} k \\ = \frac{b^2}{n^2}\left(\frac{n(n+1)}{2}\right) = \frac{b^2}{2}\left(1+\frac{1}{n}\right).\end{aligned}$$

Then

$$\begin{aligned}\int_0^b x\ dx &= \lim_{n\to\infty}\sum_{k=1}^{n} f(c_k)\Delta x_k \\ &= \lim_{n\to\infty}\frac{b^2}{2}\left(1+\frac{1}{n}\right) = \frac{b^2}{2}.\end{aligned}$$



solution 2: Alternately, we can look at the triangle above and say that

$$\int_0^b x\ dx = \text{area of a triangle} = \frac{1}{2} \times b \times b = \frac{b^2}{2}.$$

Example

$$\begin{aligned}\int_a^b x\ dx &= \int_a^0 x\ dx + \int_0^b x\ dx \\ &= -\int_0^a x\ dx + \int_0^b x\ dx \\ &= -\frac{a^2}{2} + \frac{b^2}{2} \\ &= \frac{b^2}{2} - \frac{a^2}{2}.\end{aligned}$$

The Fundamental Theorem of Calculus

33. The Fundamental Theorem of Calculus

We don't want to have to use Riemann sums every time we need to calculate a definite integral – we want a better way.

The following theorem is the most important theorem in Calculus. If you can only memorise one theorem for the exams, it should be this one.

33. The Fundamental Theorem of Calculus

Theorem (The Fundamental Theorem of Calculus)

Suppose that $f : [a, b] \to \mathbb{R}$ *is a continuous function.*

1. Then the function $F : [a, b] \to \mathbb{R}$ defined by

$$F(x) = \int_a^x f(t)\ dt$$

is continuous on $[a, b]$; differentiable on (a, b); and its derivative is

$$\boxed{F'(x) = \frac{d}{dx}\int_a^x f(t)\ dt = f(x).}$$

2. If F is any antiderivative of f on $[a, b]$, then

$$\boxed{\int_a^b f(x)\, dx = F(b) - F(a).}$$

Remark

Part (i) of the theorem tells how to differentiate $\int_a^x f(t)\ dt$.

Example

Find $\frac{dy}{dx}$ if $y = \int_a^x (t^3 + 1)\ dt$.

solution:

$$\frac{dy}{dx} = \frac{d}{dx} \int_a^x (t^3 + 1)\ dt = x^3 + 1.$$

Example

Find $\frac{dy}{dx}$ if $y = \int_1^x \sin t\ dt$.

solution:

$$\frac{dy}{dx} = \frac{d}{dx} \int_1^x \sin t\ dt = \sin x.$$

Example

Find $\frac{dy}{dx}$ if $y = \int_0^x \sin \ln \tan e^{t^2} \; dt$.

solution:

$$\frac{dy}{dx} = \frac{d}{dx} \int_0^x \sin \ln \tan e^{t^2} \; dt = \sin \ln \tan e^{x^2}.$$

Example

Find $\frac{dy}{dx}$ if $y = \int_x^5 3t \sin t \ dt$.

solution:

$$\begin{aligned} \frac{dy}{dx} &= \frac{d}{dx} \int_x^5 3t \sin t \ dt \\ &= \frac{d}{dx} \left(- \int_5^x 3t \sin t \ dt \right) \\ &= -3x \sin x. \end{aligned}$$

Example

Find $\frac{dy}{dx}$ if $y = \int_1^{x^2} \cos t \; dt$.

solution: This time we will need to use the Chain rule. Let $u = x^2$. Then

$$\begin{aligned} \frac{dy}{dx} &= \frac{dy}{du}\frac{du}{dx} \\ &= \left(\frac{d}{du} \int_1^u \cos t \; dt \right) \left(\frac{d}{dx} x^2 \right) \\ &= (\cos u)(2x) = 2x \cos x^2. \end{aligned}$$

Remark

Part (ii) of the theorem tells us how to calculate the definite integral of f over $[a, b]$:

1. Find an antiderivative F of f.
2. Calculate $F(b) - F(a)$.

Notation

We will write

$$\Big[F(x)\Big]_a^b = F(b) - F(a).$$

Example

$$\int_0^\pi \cos x \; dx = \Big[\sin x \Big]_0^\pi$$

$$\left(\text{because } \tfrac{d}{dx} \sin x = \cos x\right)$$

$$= \sin \pi - \sin 0$$

$$= 0 - 0$$

$$= 0$$

Example

$$\begin{aligned}\int_{-\frac{\pi}{4}}^{0} \sec x \tan x \, dx &= \Big[\sec x \Big]_{-\frac{\pi}{4}}^{0} \\ &\qquad \left(\text{because } \tfrac{d}{dx} \sec x = \sec x \tan x\right) \\ &= \sec 0 - \sec -\frac{\pi}{4} \\ &= 1 - \sqrt{2}.\end{aligned}$$

Example

$$\begin{aligned}\int_1^4 \left(\frac{3}{2}\sqrt{x} - \frac{4}{x^2}\right)\, dx &= \left[x^{\frac{3}{2}} + \frac{4}{x}\right]_1^4 \\ &\qquad \left(\text{because } \tfrac{d}{dx}\left(x^{\frac{3}{2}} + \frac{4}{x}\right) = \frac{3}{2}\sqrt{x} - \frac{4}{x^2}\right) \\ &= \left(4^{\frac{3}{2}} + \frac{4}{4}\right) - \left(1^{\frac{3}{2}} + \frac{4}{1}\right) \\ &= (8+1) - (1+4) \\ &= 4.\end{aligned}$$

Total Area

Example

Let $f(x) = x^2 - 4$. We have that

$$\int_{-2}^{2} f(x)\, dx = \int_{-2}^{2} (x^2 - 4)\, dx = \left[\frac{x^3}{3} - 4x\right]_{-2}^{2}$$
$$= \left(\frac{8}{3} - 8\right) - \left(\frac{-8}{3} + 8\right) = -\frac{32}{3}.$$

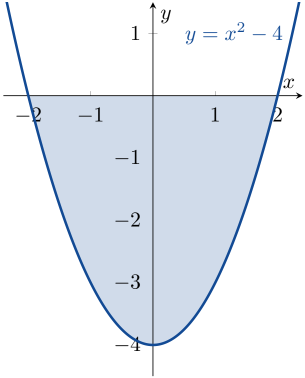
The total area between the graph of $y = f(x)$ and the x-axis, over $[-2, 2]$, is $\left|-\frac{32}{3}\right| = \frac{32}{3}$.

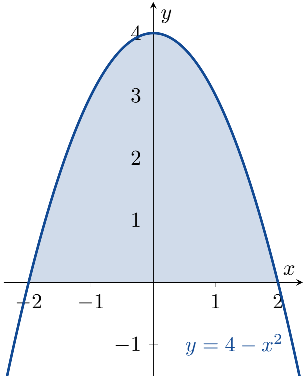
Example

Let $g(x) = 4 - x^2$. We have that

$$\int_{-2}^{2} g(x)\,dx = \int_{-2}^{2} (4 - x^2)\,dx = \left[4x - \frac{x^3}{3}\right]_{-2}^{2}$$
$$= \left(8 - \frac{8}{3}\right) - \left(8 + \frac{-8}{3}\right) = \frac{32}{3}.$$

The total area between the graph of $y = g(x)$ and the x-axis, over $[-2, 2]$, is $\left|\frac{32}{3}\right| = \frac{32}{3}$.





$$\text{integral} = -\frac{32}{3}$$
$$\text{total area} = \frac{32}{3}$$

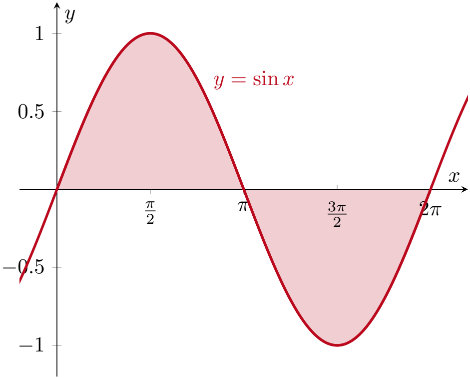
$$\text{integral} = \frac{32}{3}$$
$$\text{total area} = \frac{32}{3}$$

Example

Let $f(x) = \sin x$. Calculate

1. the definite integral of f over $[0, 2\pi]$; and
2. the total area between the graph of $y = f(x)$ and the x-axis over $[0, 2\pi]$.

33. The Fundamental Theorem of Calculus



solution:

1

$$\int_0^{2\pi} \sin x \; dx = \Big[-\cos x \Big]_0^{2\pi} = -\cos 2\pi + \cos 0$$
$$= -1 + 1 = 0.$$

2

$$\text{total area} = \int_0^{\pi} \sin x \; dx + \left| \int_{\pi}^{2\pi} \sin x \; dx \right|$$
$$= \Big[-\cos x \Big]_0^{\pi} + \left| \Big[-\cos x \Big]_{\pi}^{2\pi} \right|$$
$$= -\cos \pi + \cos 0 + |-\cos 2\pi + \cos \pi|$$
$$= -(-1) + 1 + |-1 + (-1)| = 4.$$

Summary

To find the *total area* between the graph of $y = f(x)$ and the x-axis over $[a, b]$:

1. Divide $[a, b]$ at the zeroes of f.
2. Integrate f over each subinterval.
3. Add the absolute values of the integrals.

Example

Find the total area between the graph of $y = x^3 - x^2 - 2x$ and the x-axis for $-1 \leq x \leq 2$.

solution:

1. Let $f(x) = x^3 - x^2 - 2x$.

   Since $0 = f(x) = x^3 - x^2 - 2x = x(x+1)(x-2)$ implies that $x = 0$ or $x = -1$ or $x = 2$, we divide $[-1, 2]$ into $[-1, 0]$ and $[0, 2]$.

2 We calculate that

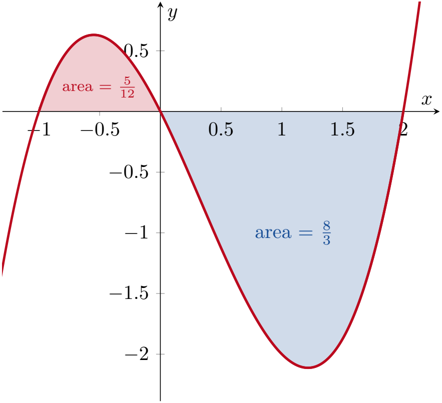
$$\begin{aligned}\int_{-1}^{0} (x^3 - x^2 - 2x)\ dx &= \left[\frac{x^4}{4} - \frac{x^3}{3} - x^2\right]_{-1}^{0} \\ &= (0 - 0 - 0) - \left(\frac{1}{4} + \frac{1}{3} - 1\right) \\ &= \frac{5}{12}\end{aligned}$$

and

$$\begin{aligned}\int_0^2 (x^3 - x^2 - 2x)\ dx &= \left[\frac{x^4}{4} - \frac{x^3}{3} - x^2\right]_0^2 \\ &= \left(\frac{16}{4} - \frac{8}{3} - 4\right) - (0 - 0 - 0) \\ &= -\frac{8}{3}.\end{aligned}$$

3. Therefore

$$\text{total area} = \left|\frac{5}{12}\right| + \left|-\frac{8}{3}\right| = \frac{37}{12}.$$



The Average Value of a Continuous Function

The average of $\{1, 2, 2, 6, 9\}$ is $\frac{1+2+2+6+9}{5} = \frac{20}{5} = 4$. We can also calculate the average value of a continuous function.

Definition

If f is integrable on $[a,b]$, then the *average value of f on* $[a,b]$ is

$$\boxed{\mathrm{av}(f) = \frac{1}{b-a}\int_a^b f(x)\ dx.}$$

Example

Find the average value of $f(x) = \sqrt{4 - x^2}$ on $[-2, 2]$.

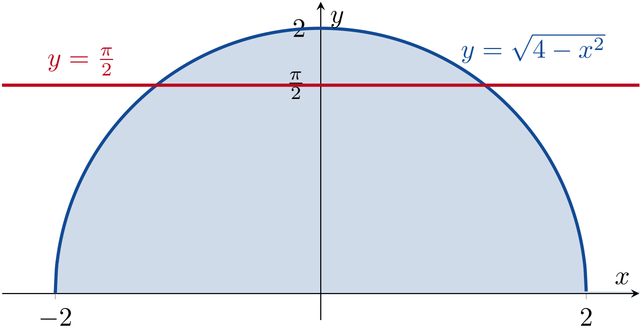
solution: Since

$$\begin{aligned} \int_{-2}^{2} f(x)\ dx &= \frac{1}{2} \times \text{the area of a circle of radius 2} \\ &= \frac{1}{2}\pi 2^2 = 2\pi, \end{aligned}$$

we have that

$$\text{av}(f) = \frac{1}{2 - (-2)} \int_{-2}^{2} f(x)\ dx = \frac{2\pi}{4} = \frac{\pi}{2}.$$

33. The Fundamental Theorem of Calculus

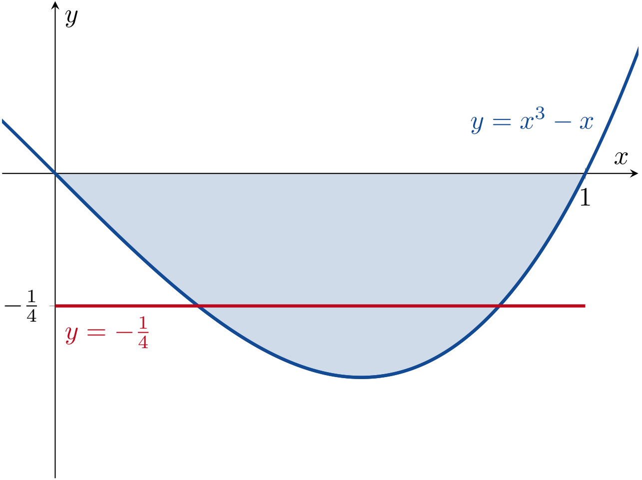


Example

Find the average value of $g(x) = x^3 - x$ on $[0, 1]$.

solution:

$$\begin{aligned}\operatorname{av}(g) &= \frac{1}{1-0}\int_0^1 g(x)\ dx = \int_0^1 (x^3 - x)\ dx \\ &= \left[\frac{x^4}{4} - \frac{x^2}{2}\right]_0^1 = \frac{1}{4} - \frac{1}{2} = -\frac{1}{4}.\end{aligned}$$



y
$y = x^3 - x$
x
1
$-\frac{1}{4}$
$y = -\frac{1}{4}$

Indefinite Integrals & Definite Integrals

Remember that

$$\int f(x)\ dx \text{ is a function.}$$

For example

$$\int x\ dx = \frac{x^2}{2} + C$$

and

$$\int \cos x\ dx = \sin x + C.$$

Remember that

$$\int_a^b f(x)\ dx \text{ is a number.}$$

For example

$$\int_0^1 x\ dx = \frac{1}{2}$$

and

$$\int_0^{\frac{\pi}{2}} \cos x\ dx = 1.$$

The Substitution Method

By the Chain rule,

$$\frac{d}{dx}\left(\frac{u^{n+1}}{n+1}\right) = u^n \frac{du}{dx}.$$

So

$$\int u^n \frac{du}{dx}\ dx = \frac{u^{n+1}}{n+1} + C.$$

But we know that

$$\int u^n\ du = \frac{u^{n+1}}{n+1} + C$$

also. So it looks like

$$\boxed{du = \frac{du}{dx}\ dx.}$$

Example

Find $\int (x^3 + x)^5 (3x^2 + 1)\, dx$.

solution: Let $u = x^3 + x$. Then $du = \frac{du}{dx}\, dx = (3x^2 + 1)\, dx$. By substitution, we have that

$$\begin{aligned}\int (x^3 + x)^5 (3x^2 + 1)\, dx &= \int u^5\, du \\ &= \frac{u^6}{6} + C = \frac{1}{6}(x^3 + x)^6 + C.\end{aligned}$$

Example

Find $\displaystyle\int \sqrt{2x+1}\,dx$.

solution: Let $u = 2x+1$. Then $du = \frac{du}{dx}\,dx = 2dx$. So $dx = \frac{1}{2}\,du$. Therefore

$$\int \sqrt{2x+1}\,dx = \int u^{\frac{1}{2}}\left(\tfrac{1}{2}du\right) = \frac{1}{2}\int u^{\frac{1}{2}}\,du$$

$$= \frac{1}{2}\left(\frac{u^{\frac{3}{2}}}{\frac{3}{2}}\right) + C = \frac{1}{3}(2x+1)^{\frac{3}{2}} + C.$$

Theorem (The Substitution Method)

If

- $u = g(x)$ *is differentiable;*
- $g : \mathbb{R} \to I$*; and*
- $f : I \to \mathbb{R}$ *is continuous,*

then

$$\int f(g(x))g'(x)\ dx = \int f(u)\ du.$$

Example

Find $\int 5\sec^2(5t+1)\ dt$.

solution: Let $u = 5t+1$. Then $du = \frac{du}{dt}\ dt = 5dt$. So

$$\begin{aligned}\int 5\sec^2(5t+1)\ dt &= \int \sec^2 u\ du \\ &= \tan u + C \\ &\quad \left(\text{because } \frac{d}{du}\tan u = \sec^2 u\right) \\ &= \tan(5t+1) + C.\end{aligned}$$

Example

Find $\displaystyle\int \cos(7\theta + 3)\ d\theta$.

solution: Let $u = 7\theta + 3$. Then $du = \frac{du}{d\theta}\ d\theta = 7d\theta$. So $d\theta = \frac{1}{7}du$ and

$$\begin{aligned}\int \cos(7\theta+3)\ d\theta &= \frac{1}{7}\int \cos u\ du \\ &= \frac{1}{7}\sin u + C = \frac{1}{7}\sin(7\theta+3) + C.\end{aligned}$$

Example

Find $\int x^2 \sin(x^3)\ dx$.

solution: Let $u = x^3$. Then $du = \frac{du}{dx}\ dx = 3x^2\ dx$. So $\frac{1}{3}du = x^2\ dx$ and

$$\begin{aligned}\int x^2 \sin(x^3)\ dx &= \int \frac{1}{3}\sin u\ du = -\frac{1}{3}\cos u + C \\ &= -\frac{1}{3}\cos(x^3) + C.\end{aligned}$$

Example

Find $\int x\sqrt{2x+1}\,dx$.

solution: Let $u = 2x+1$. Then $du = \frac{du}{dx}\,dx = 2dx$. So $dx = \frac{1}{2}du$ and

$$\int x\sqrt{2x+1}\,dx = \int x\sqrt{u}\,\frac{1}{2}du.$$

But we still have an x here. We can't integrate until we change all the x terms to u terms. Note that

$$u = 2x+1 \implies u-1 = 2x \implies \frac{1}{2}(u-1) = x.$$

Therefore

$$\begin{aligned}\int x\sqrt{2x+1}\,dx &= \int \frac{1}{2}(u-1)\sqrt{u}\ \frac{1}{2}du\\ &= \frac{1}{4}\int u^{\frac{3}{2}} - u^{\frac{1}{2}}\ du\\ &= \frac{1}{4}\left(\frac{2}{5}u^{\frac{5}{2}} - \frac{2}{3}u^{\frac{3}{2}}\right) + C\\ &= \frac{1}{10}u^{\frac{5}{2}} - \frac{1}{6}u^{\frac{3}{2}} + C\\ &= \frac{1}{10}(2x+1)^{\frac{5}{2}} - \frac{1}{6}(2x+1)^{\frac{3}{2}} + C.\end{aligned}$$

Example

Find $\displaystyle\int \frac{2z}{\sqrt[3]{z^2+1}}\ dz$.

solution: Let $u = z^2 + 1$. Then $du = \frac{du}{dx}\ dx = 2z\ dz$ and

$$\begin{aligned}
\int \frac{2z}{\sqrt[3]{z^2+1}}\ dz &= \int \frac{du}{u^{\frac{1}{3}}} \\
&= \int u^{-\frac{1}{3}}\ du \\
&= \frac{u^{\frac{2}{3}}}{\frac{2}{3}} + C \\
&= \frac{3}{2} u^{\frac{2}{3}} + C \\
&= \frac{3}{2}(z^2+1)^{\frac{2}{3}} + C.
\end{aligned}$$

Example

Find $\int \sin^2 x \, dx$.

solution: We use the identity

$$\sin^2 x = \frac{1 - \cos 2x}{2}$$

to calculate that

$$\begin{aligned}\int \sin^2 x \, dx &= \int \frac{1 - \cos 2x}{2} \, dx \\ &= \frac{1}{2} \int (1 - \cos 2x) \, dx \\ &= \frac{1}{2} \left(x - \frac{1}{2} \sin 2x \right) + C \\ &= \frac{1}{2} x - \frac{1}{4} \sin 2x + C.\end{aligned}$$

Example

Similarly

$$\int \cos^2 x \; dx = \int \frac{1 + \cos 2x}{2} \; dx = \frac{1}{2}x + \frac{1}{4}\sin 2x + C.$$

The Substitution Method for Definite Integrals

Theorem (The Substitution Method)

If

- $u = g(x)$ *is differentiable on* $[a, b]$*;*
- g' *is continuous on* $[a, b]$*; and*
- f *is continous on the range of* g*,*

then

$$\int_a^b f(g(x))g'(x)\;dx = \int_{g(a)}^{g(b)} f(u)\;du.$$

Example

Calculate $\int_{-1}^{1} 3x^2\sqrt{x^3+1}\,dx$.

solution 1. We can use the previous theorem to solve this example. Let $u = x^3 + 1$. Then $du = 3x^2\,dx$. Moreover $x = -1 \implies u = 0$ and $x = 1 \implies u = 2$. So

$$\int_{x=-1}^{x=1} 3x^2\sqrt{x^3+1}\,dx = \int_{u=0}^{u=2} \sqrt{u}\,du = \left[\frac{2}{3}u^{\frac{3}{2}}\right]_0^2$$

$$= \frac{2}{3}\left(2^{\frac{3}{2}} - 0^{\frac{3}{2}}\right) = \frac{2}{3}2\sqrt{2} = \frac{4\sqrt{2}}{3}.$$

solution 2. Alternately, we can first find the indefinite integral, then find the required definite integral.
Let $u = x^3 + 1$. Then $du = 3x^2\ dx$. So

$$\int 3x^2\sqrt{x^3+1}\ dx = \int \sqrt{u}\ du = \frac{2}{3}u^{\frac{3}{2}} + C = \frac{2}{3}(x^3+1)^{\frac{3}{2}} + C.$$

Therefore

$$\begin{aligned}\int_{-1}^{1} 3x^2\sqrt{x^3+1}\ dx &= \left[\frac{2}{3}(x^3+1)^{\frac{3}{2}}\right]_{-1}^{1} \\ &= \left(\frac{2}{3}(1+1)^{\frac{3}{2}}\right) - \left(\frac{2}{3}(-1+1)^{\frac{3}{2}}\right) \\ &= \frac{2}{3} \times 2^{\frac{3}{2}} = \frac{4\sqrt{2}}{3}.\end{aligned}$$

Example

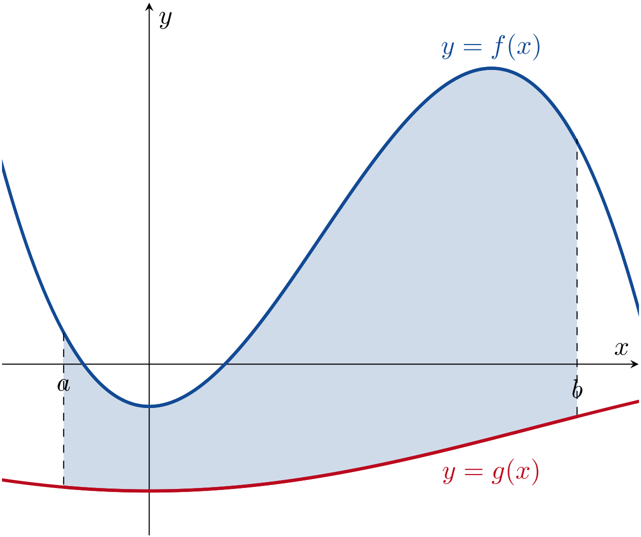
Calculate $\displaystyle\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \cot\theta \operatorname{cosec}^2\theta \; d\theta$.

solution: Let $u = \cot\theta$. Then $du = \frac{du}{d\theta}\; d\theta = -\operatorname{cosec}^2\theta \; d\theta$. So $-du = \operatorname{cosec}^2\theta \; d\theta$. Moreover $\theta = \frac{\pi}{4} \implies u = \cot\frac{\pi}{4} = 1$ and $\theta = \frac{\pi}{2} \implies u = \cot\frac{\pi}{2} = 0$. Hence

$$\begin{aligned}\int_{\theta=\frac{\pi}{4}}^{\theta=\frac{\pi}{2}} \cot\theta \operatorname{cosec}^2\theta \; d\theta &= \int_{u=1}^{u=0} u\; (-du) = -\int_1^0 u\; du \\ &= -\left[\frac{u^2}{2}\right]_1^0 = -\left(\frac{0^2}{2} - \frac{1^2}{2}\right) = \frac{1}{2}.\end{aligned}$$

35 Area Between Curves

35. Area Between Curves



35. Area Between Curves

Definition

If

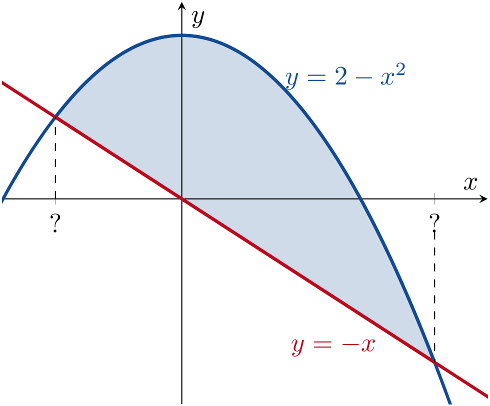
- f is continous;
- g is continous; and
- $f(x) \geq g(x)$ on $[a, b]$,

then the *area of the region between the curves* $y = f(x)$ *and* $y = g(x)$ *for* $a \leq x \leq b$ is

$$\boxed{\text{area} = \int_a^b \Big(f(x) - g(x) \Big) \, dx.}$$

Example

Find the area between $y = 2 - x^2$ and $y = -x$.



solution: First we need to find the limits of integration:

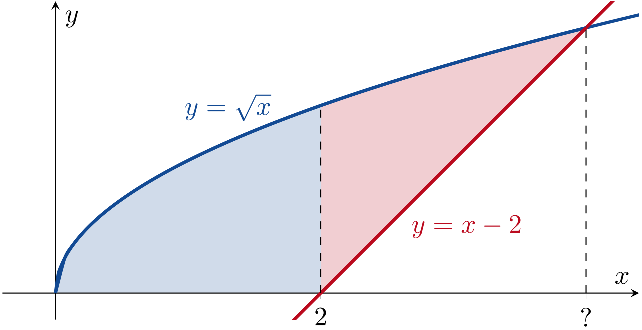
$$
\begin{aligned}
2 - x^2 &= -x \\
0 &= x^2 - x - 2 \\
0 &= (x+1)(x-2) \quad \Longrightarrow \quad x = -1 \text{ or } 2.
\end{aligned}
$$

We need to integrate from $x = -1$ to $x = 2$. Therefore

$$
\begin{aligned}
\text{area} &= \int_{-1}^{2} \Big((2 - x^2) - (-x) \Big) \, dx \\
&= \int_{-1}^{2} (2 + x - x^2) \, dx \\
&= \left[2x + \tfrac{1}{2}x^2 - \tfrac{1}{3}x^3 \right]_{-1}^{2} \\
&= \left(4 + \frac{4}{2} - \frac{8}{3} \right) - \left(-2 + \frac{1}{2} + \frac{1}{3} \right) \\
&= \frac{9}{2}.
\end{aligned}
$$

Example

Find the area bounded by $y = \sqrt{x}$, $y = x - 2$ and the x-axis, for $x \geq 0$ and $y \geq 0$.



solution: First we calculate that

$$\begin{aligned}\sqrt{x} &= x - 2\\ x &= (x-2)^2 = x^2 - 4x + 4\\ 0 &= x^2 - 5x + 4 = (x-1)(x-4) \implies x = 1 \text{ or } 4.\end{aligned}$$

Since $\sqrt{1} \neq 1 - 2$, we must have $x = 4$.

Therefore

$$\begin{aligned}
\text{area} &= \text{blue area} + \text{red area} \\
&= \int_0^2 \sqrt{x}\ dx + \int_2^4 \left(\sqrt{x} - (x-2)\right)\ dx \\
&= \int_0^2 x^{\frac{1}{2}}\ dx + \int_2^4 (x^{\frac{1}{2}} - x + 2)\ dx \\
&= \left[\frac{2}{3}x^{\frac{3}{2}}\right]_0^2 + \left[\frac{2}{3}x^{\frac{3}{2}} - \frac{1}{2}x^2 + 2x\right]_2^4 \\
&= \left(\frac{2}{3}(2)^{\frac{3}{2}} - 0\right) + \left(\frac{2}{3}(4)^{\frac{3}{2}} - \frac{1}{2}(16) + 2(4)\right) \\
&\quad - \left(\frac{2}{3}(2)^{\frac{3}{2}} - \frac{1}{2}(4) + 2(2)\right) \\
&= \frac{4\sqrt{2}}{3} + \frac{16}{3} - 8 + 8 - \frac{4\sqrt{2}}{3} + 2 - 4 = \frac{10}{3}.
\end{aligned}$$

The End