

Lecture 12

- 7.5 Indeterminate Forms and L'Hôpital's Rule
- 7.6 Inverse Trigonometric Functions
- 7.7 Hyperbolic Functions



Indeterminate Forms and L'Hôpital's Rule

7.5 Indeterminate Forms and L'Hôpital's Rule



Things like " $\frac{0}{0}$ " and " $\frac{\infty}{\infty}$ " are not numbers. We call them *indeterminate forms*.



Guillaume de l'Hôpital

BORN

1661

DECEASED

2 February 1704

NATIONALITY

French

7.5 Indeterminate Forms and L'Hôpital's Rule



Indeterminate Form $\frac{0}{0}$

Theorem (L'Hôpital's Rule)

Suppose that

- $f(a) = g(a) = 0$;
- f and g are differentiable on $(a - \delta, a + \delta)$ for some $\delta > 0$;
- $g'(x) \neq 0$ for all $x \in (a - \delta, a) \cup (a, a + \delta)$.

7.5 Indeterminate Forms and L'Hôpital's Rule



Indeterminate Form $\frac{0}{0}$

Theorem (L'Hôpital's Rule)

Suppose that

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- f and g are differentiable on $(a - \delta, a + \delta)$ for some $\delta > 0$;
- $g'(x) \neq 0$ for all $x \in (a - \delta, a) \cup (a, a + \delta)$.

Then

$$\boxed{\lim_{x \rightarrow a} \frac{f(x)}{g(x)} = \lim_{x \rightarrow a} \frac{f'(x)}{g'(x)}}$$

if the limit $\lim_{x \rightarrow a} \frac{f'(x)}{g'(x)}$ exists.

7.5 Indeterminate Forms and L'Hôpital

$$\lim_{x \rightarrow a} \frac{f(x)}{g(x)} = \lim_{x \rightarrow a} \frac{f'(x)}{g'(x)}$$

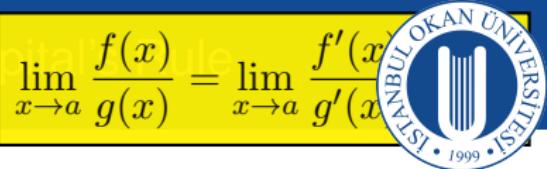

Remark

Note that l'Hôpital's Rule says $\frac{f'}{g'}$. It does not say $\left(\frac{f}{g}\right)'$.

Remark

The ‘H’ in l’Hôpital is silent.

7.5 Indeterminate Forms and L'Hôpital's Rule

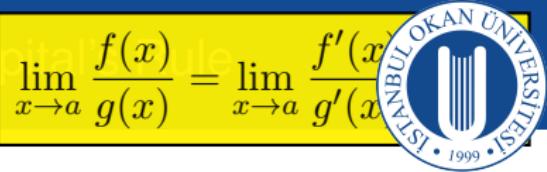


Example

Find $\lim_{x \rightarrow 0} \frac{3x - \sin x}{x}$.

If we just replaced x by 0, we would get the indeterminate form " $\frac{0}{0}$ ". So we use l'Hôpital's Rule.

7.5 Indeterminate Forms and L'Hôpital's Rule



Example

Find $\lim_{x \rightarrow 0} \frac{3x - \sin x}{x}$.

If we just replaced x by 0, we would get the indeterminate form " $\frac{0}{0}$ ". So we use l'Hôpital's Rule.

$$\lim_{x \rightarrow 0} \frac{3x - \sin x}{x} = \lim_{x \rightarrow 0} \frac{(3x - \sin x)'}{(x)'}$$

7.5 Indeterminate Forms and L'Hôpital's Rule

$$\lim_{x \rightarrow a} \frac{f(x)}{g(x)} = \lim_{x \rightarrow a} \frac{f'(x)}{g'(x)}$$

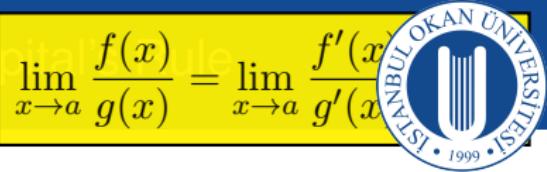

Example

Find $\lim_{x \rightarrow 0} \frac{3x - \sin x}{x}$.

If we just replaced x by 0, we would get the indeterminate form " $\frac{0}{0}$ ". So we use l'Hôpital's Rule.

$$\lim_{x \rightarrow 0} \frac{3x - \sin x}{x} = \lim_{x \rightarrow 0} \frac{(3x - \sin x)'}{(x)'} = \lim_{x \rightarrow 0} \frac{3 - \cos x}{1} = 2.$$

7.5 Indeterminate Forms and L'Hôpital's Rule

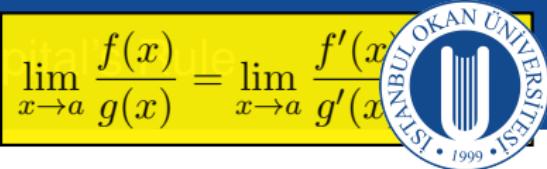


$$\lim_{x \rightarrow a} \frac{f(x)}{g(x)} = \lim_{x \rightarrow a} \frac{f'(x)}{g'(x)}$$

Example

$$\text{Find } \lim_{x \rightarrow 0} \frac{\sqrt{1+x} - 1}{x}.$$

7.5 Indeterminate Forms and L'Hôpital's Rule



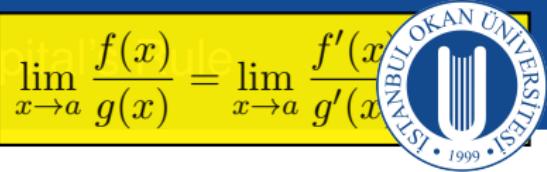
Example

Find $\lim_{x \rightarrow 0} \frac{\sqrt{1+x} - 1}{x}$.

Again we would get the indeterminate form “ $\frac{0}{0}$ ” if we replaced x by 0, so we use l'Hôpital's Rule to calculate

$$\lim_{x \rightarrow 0} \frac{\sqrt{1+x} - 1}{x} = \lim_{x \rightarrow 0} \frac{(\sqrt{1+x} - 1)'}{(x)'}$$

7.5 Indeterminate Forms and L'Hôpital's Rule



Example

Find $\lim_{x \rightarrow 0} \frac{\sqrt{1+x} - 1}{x}$.

Again we would get the indeterminate form “ $\frac{0}{0}$ ” if we replaced x by 0, so we use l'Hôpital's Rule to calculate

$$\lim_{x \rightarrow 0} \frac{\sqrt{1+x} - 1}{x} = \lim_{x \rightarrow 0} \frac{(\sqrt{1+x} - 1)'}{(x)'} = \lim_{x \rightarrow 0} \frac{\frac{1}{2\sqrt{x+1}}}{1} = \frac{1}{2}.$$

7.5 Indeterminate Forms and L'Hôpital's Rule

$$\lim_{x \rightarrow a} \frac{f(x)}{g(x)} = \lim_{x \rightarrow a} \frac{f'(x)}{g'(x)}$$


Example

$$\text{Find } \lim_{x \rightarrow 0} \frac{\sqrt{1+x} - 1 - \frac{x}{2}}{x^2}.$$

Again we would get the indeterminate form “ $\frac{0}{0}$ ” if we replaced x by 0, so we use l'Hôpital's Rule to calculate

7.5 Indeterminate Forms and L'Hôpital's Rule

$$\lim_{x \rightarrow a} \frac{f(x)}{g(x)} = \lim_{x \rightarrow a} \frac{f'(x)}{g'(x)}$$

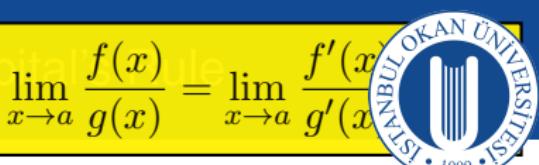

Example

Find $\lim_{x \rightarrow 0} \frac{\sqrt{1+x} - 1 - \frac{x}{2}}{x^2}$.

Again we would get the indeterminate form “ $\frac{0}{0}$ ” if we replaced x by 0, so we use l'Hôpital's Rule to calculate

$$\lim_{x \rightarrow 0} \frac{\sqrt{1+x} - 1 - \frac{x}{2}}{x^2} = \lim_{x \rightarrow 0} \frac{\frac{1}{2}(1+x)^{-\frac{1}{2}} - \frac{1}{2}}{2x}.$$

7.5 Indeterminate Forms and L'Hôpital's Rule



Example

Find $\lim_{x \rightarrow 0} \frac{\sqrt{1+x} - 1 - \frac{x}{2}}{x^2}$.

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But again we would get " $\frac{0}{0}$ " if we replaced x by 0. So we use l'Hôpital's Rule a second time.

7.5 Indeterminate Forms and L'Hôpital's Rule

$$\lim_{x \rightarrow a} \frac{f(x)}{g(x)} = \lim_{x \rightarrow a} \frac{f'(x)}{g'(x)}$$


Example

Find $\lim_{x \rightarrow 0} \frac{\sqrt{1+x} - 1 - \frac{x}{2}}{x^2}$.

Again we would get the indeterminate form " $\frac{0}{0}$ " if we replaced x by 0, so we use l'Hôpital's Rule to calculate

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But again we would get " $\frac{0}{0}$ " if we replaced x by 0. So we use l'Hôpital's Rule a second time.

$$\begin{aligned}\lim_{x \rightarrow 0} \frac{\sqrt{1+x} - 1 - \frac{x}{2}}{x^2} &= \lim_{x \rightarrow 0} \frac{\frac{1}{2}(1+x)^{-\frac{1}{2}} - \frac{1}{2}}{2x} \\ &= \lim_{x \rightarrow 0} \frac{-\frac{1}{4}(1+x)^{-\frac{3}{2}}}{2} = -\frac{1}{8}.\end{aligned}$$

$$\lim_{x \rightarrow 0} \frac{x - \sin x}{x^3}$$

$\frac{0}{0}$; apply l'Hôpital's Rule.

$$= \lim_{x \rightarrow 0} \frac{1 - \cos x}{3x^2}$$

Still $\frac{0}{0}$; apply l'Hôpital's Rule again.

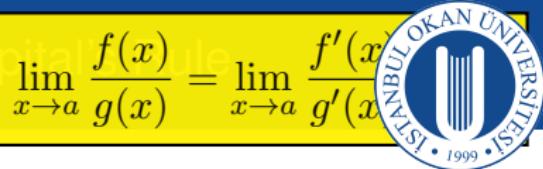
$$= \lim_{x \rightarrow 0} \frac{\sin x}{6x}$$

Still $\frac{0}{0}$; apply l'Hôpital's Rule again.

$$= \lim_{x \rightarrow 0} \frac{\cos x}{6} = \frac{1}{6}$$

Not $\frac{0}{0}$; limit is found.

7.5 Indeterminate Forms and L'Hôpital



$$\lim_{x \rightarrow a} \frac{f(x)}{g(x)} = \lim_{x \rightarrow a} \frac{f'(x)}{g'(x)}$$

Remark

We can only use l'Hôpital's Rule if we have “ $\frac{0}{0}$ ”. If we don't have “ $\frac{0}{0}$ ”, then we can not use this rule.

7.5 Indeterminate Forms and L'Hôpital's Rule

$$\lim_{x \rightarrow a} \frac{f(x)}{g(x)} = \lim_{x \rightarrow a} \frac{f'(x)}{g'(x)}$$
The logo of Istanbul Okan University, featuring a circular emblem with vertical columns and the university's name in English and Turkish.

Example

Find $\lim_{x \rightarrow 0} \frac{1 - \cos x}{x + x^2}$.

7.5 Indeterminate Forms and L'Hôpital's Rule

$$\lim_{x \rightarrow a} \frac{f(x)}{g(x)} = \lim_{x \rightarrow a} \frac{f'(x)}{g'(x)}$$


Example

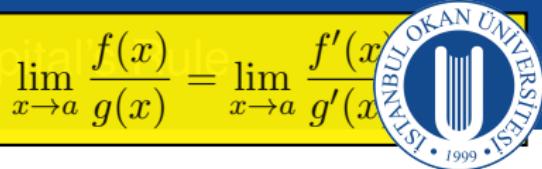
Find $\lim_{x \rightarrow 0} \frac{1 - \cos x}{x + x^2}$.

Wrong Answer:

$$\lim_{x \rightarrow 0} \frac{1 - \cos x}{x + x^2} = \lim_{x \rightarrow 0} \frac{\sin x}{1 + 2x} = \lim_{x \rightarrow 0} \frac{\cos x}{2} = \frac{1}{2}.$$

Why is this wrong?

7.5 Indeterminate Forms and L'Hôpital's Rule



Example

Find $\lim_{x \rightarrow 0} \frac{1 - \cos x}{x + x^2}$.

Wrong Answer:

$$\lim_{x \rightarrow 0} \frac{1 - \cos x}{x + x^2} = \lim_{x \rightarrow 0} \frac{\sin x}{1 + 2x} = \lim_{x \rightarrow 0} \frac{\cos x}{2} = \frac{1}{2}.$$

Why is this wrong?

Because $\frac{\sin x}{1 + 2x}$ does not give " $\frac{0}{0}$ " if we replace x by 0.

7.5 Indeterminate Forms and L'Hôpital's Rule

$$\lim_{x \rightarrow a} \frac{f(x)}{g(x)} = \lim_{x \rightarrow a} \frac{f'(x)}{g'(x)}$$


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Why is this wrong?

Because $\frac{\sin x}{1 + 2x}$ does not give “ $\frac{0}{0}$ ” if we replace x by 0. The correct answer is actually 0. I leave this for you to check.

L'Hôpital's Rule applies to one-sided limits as well.

EXAMPLE 3 In this example the one-sided limits are different.

(a) $\lim_{x \rightarrow 0^+} \frac{\sin x}{x^2}$ $\frac{0}{0}$

$$= \lim_{x \rightarrow 0^+} \frac{\cos x}{2x} = \infty \quad \text{Positive for } x > 0$$

(b) $\lim_{x \rightarrow 0^-} \frac{\sin x}{x^2}$ $\frac{0}{0}$

$$= \lim_{x \rightarrow 0^-} \frac{\cos x}{2x} = -\infty \quad \text{Negative for } x < 0$$

7.5 Indeterminate Forms and L'Hôpital

$$\lim_{x \rightarrow a} \frac{f(x)}{g(x)} = \lim_{x \rightarrow a} \frac{f'(x)}{g'(x)}$$



Indeterminate Form $\frac{\infty}{\infty}$

L'Hôpital's Rule also applies if we would get " $\frac{\infty}{\infty}$ ".

7.5 Indeterminate Forms and L'Hôpital

$$\lim_{x \rightarrow a} \frac{f(x)}{g(x)} = \lim_{x \rightarrow a} \frac{f'(x)}{g'(x)}$$



Indeterminate Form $\frac{\infty}{\infty}$

L'Hôpital's Rule also applies if we would get " $\frac{\infty}{\infty}$ ".

Theorem (L'Hôpital's Rule)

Let $a \in \mathbb{R}$ or $a = \infty$ or $a = -\infty$.

If $\lim_{x \rightarrow a} f(x) = \pm\infty$ and $\lim_{x \rightarrow a} g(x) = \pm\infty$, then

$$\lim_{x \rightarrow a} \frac{f(x)}{g(x)} = \lim_{x \rightarrow a} \frac{f'(x)}{g'(x)}$$

provided the limit $\lim_{x \rightarrow a} \frac{f'(x)}{g'(x)}$ exists.

7.5 Indeterminate Forms and L'Hôpital

$$\lim_{x \rightarrow a} \frac{f(x)}{g(x)} = \lim_{x \rightarrow a} \frac{f'(x)}{g'(x)}$$



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provided the limit $\lim_{x \rightarrow a} \frac{f'(x)}{g'(x)}$ exists.

This theorem is also true for one sided limits $x \rightarrow a^+$ and $x \rightarrow a^-$.

7.5 Indeterminate Forms and L'Hôpital's Rule

$$\lim_{x \rightarrow a} \frac{f(x)}{g(x)} = \lim_{x \rightarrow a} \frac{f'(x)}{g'(x)}$$


Example

Find $\lim_{x \rightarrow \frac{\pi}{2}} \frac{\sec x}{1 + \tan x}$.

7.5 Indeterminate Forms and L'Hôpital's Rule

$$\lim_{x \rightarrow a} \frac{f(x)}{g(x)} = \lim_{x \rightarrow a} \frac{f'(x)}{g'(x)}$$


Example

Find $\lim_{x \rightarrow \frac{\pi}{2}} \frac{\sec x}{1 + \tan x}$.

Note that this is a “ $\frac{\infty}{\infty}$ ” question.

Since $\sec x$ and $\tan x$ are both discontinuous at $\frac{\pi}{2}$, we need to consider one-sided limits.

7.5 Indeterminate Forms and L'Hôpital's Rule

$$\lim_{x \rightarrow a} \frac{f(x)}{g(x)} = \lim_{x \rightarrow a} \frac{f'(x)}{g'(x)}$$


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Since $\sec x$ and $\tan x$ are both discontinuous at $\frac{\pi}{2}$, we need to consider one-sided limits.

$$\lim_{x \rightarrow (\frac{\pi}{2})^-} \frac{\sec x}{1 + \tan x} = \lim_{x \rightarrow (\frac{\pi}{2})^-} \frac{\sec x \tan x}{\sec^2 x}$$

=

7.5 Indeterminate Forms and L'Hôpital's Rule

$$\lim_{x \rightarrow a} \frac{f(x)}{g(x)} = \lim_{x \rightarrow a} \frac{f'(x)}{g'(x)}$$


Example

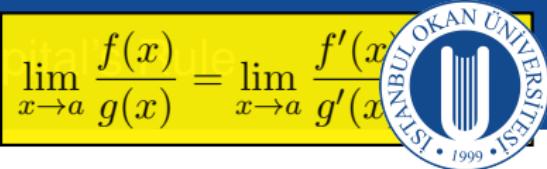
Find $\lim_{x \rightarrow \frac{\pi}{2}} \frac{\sec x}{1 + \tan x}$.

Note that this is a “ $\frac{\infty}{\infty}$ ” question.

Since $\sec x$ and $\tan x$ are both discontinuous at $\frac{\pi}{2}$, we need to consider one-sided limits.

$$\begin{aligned}\lim_{x \rightarrow (\frac{\pi}{2})^-} \frac{\sec x}{1 + \tan x} &= \lim_{x \rightarrow (\frac{\pi}{2})^-} \frac{\sec x \tan x}{\sec^2 x} \\&= \lim_{x \rightarrow (\frac{\pi}{2})^-} \frac{\tan x}{\sec x} = \lim_{x \rightarrow (\frac{\pi}{2})^-} \sin x = 1.\end{aligned}$$

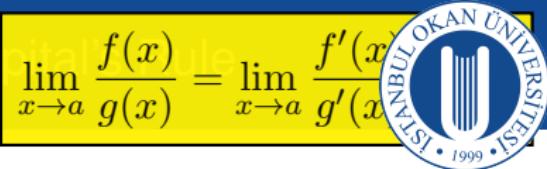
7.5 Indeterminate Forms and L'Hôpital's Rule



$$\lim_{x \rightarrow a} \frac{f(x)}{g(x)} = \lim_{x \rightarrow a} \frac{f'(x)}{g'(x)}$$

I leave it to you to check that $\lim_{x \rightarrow (\frac{\pi}{2})^+} \frac{\sec x}{1 + \tan x} = 1$ also.

7.5 Indeterminate Forms and L'Hôpital's Rule



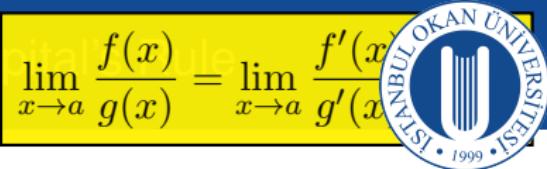
$$\lim_{x \rightarrow a} \frac{f(x)}{g(x)} = \lim_{x \rightarrow a} \frac{f'(x)}{g'(x)}$$

I leave it to you to check that $\lim_{x \rightarrow (\frac{\pi}{2})^+} \frac{\sec x}{1 + \tan x} = 1$ also.

Therefore

$$\lim_{x \rightarrow \frac{\pi}{2}} \frac{\sec x}{1 + \tan x} = 1.$$

7.5 Indeterminate Forms and L'Hôpital's Rule

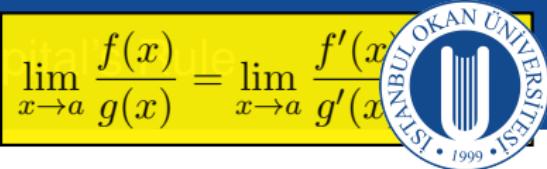


$$\lim_{x \rightarrow a} \frac{f(x)}{g(x)} = \lim_{x \rightarrow a} \frac{f'(x)}{g'(x)}$$

Example

$$\lim_{x \rightarrow \infty} \frac{\ln x}{2\sqrt{x}} = \lim_{x \rightarrow \infty} \frac{\frac{1}{x}}{\frac{1}{\sqrt{x}}}$$

7.5 Indeterminate Forms and L'Hôpital's Rule

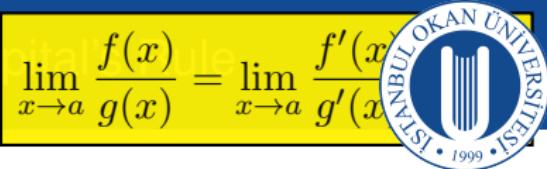


$$\lim_{x \rightarrow a} \frac{f(x)}{g(x)} = \lim_{x \rightarrow a} \frac{f'(x)}{g'(x)}$$

Example

$$\lim_{x \rightarrow \infty} \frac{\ln x}{2\sqrt{x}} = \lim_{x \rightarrow \infty} \frac{\frac{1}{x}}{\frac{1}{\sqrt{x}}} = \lim_{x \rightarrow \infty} \frac{\sqrt{x}}{x}$$

7.5 Indeterminate Forms and L'Hôpital's Rule

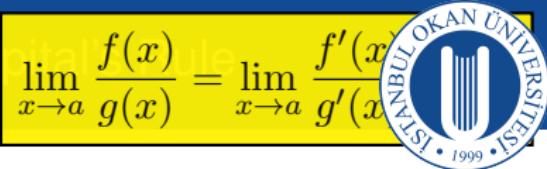


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Example

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7.5 Indeterminate Forms and L'Hôpital's Rule



$$\lim_{x \rightarrow a} \frac{f(x)}{g(x)} = \lim_{x \rightarrow a} \frac{f'(x)}{g'(x)}$$

Example

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7.5 Indeterminate Forms and L'Hôpital's Rule

$$\lim_{x \rightarrow a} \frac{f(x)}{g(x)} = \lim_{x \rightarrow a} \frac{f'(x)}{g'(x)}$$


Example

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Example

$$\lim_{x \rightarrow \infty} \frac{e^x}{x^2} = \lim_{x \rightarrow \infty} \frac{e^x}{2x}$$

7.5 Indeterminate Forms and L'Hôpital's Rule

$$\lim_{x \rightarrow a} \frac{f(x)}{g(x)} = \lim_{x \rightarrow a} \frac{f'(x)}{g'(x)}$$


Example

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Example

$$\lim_{x \rightarrow \infty} \frac{e^x}{x^2} = \lim_{x \rightarrow \infty} \frac{e^x}{2x} = \lim_{x \rightarrow \infty} \frac{e^x}{2}$$

7.5 Indeterminate Forms and L'Hôpital's Rule

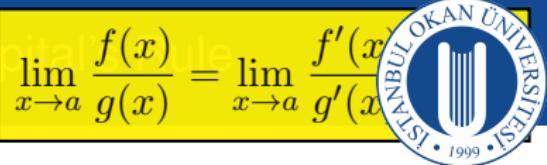
$$\lim_{x \rightarrow a} \frac{f(x)}{g(x)} = \lim_{x \rightarrow a} \frac{f'(x)}{g'(x)}$$


Example

$$\lim_{x \rightarrow \infty} \frac{\ln x}{2\sqrt{x}} = \lim_{x \rightarrow \infty} \frac{\frac{1}{x}}{\frac{1}{\sqrt{x}}} = \lim_{x \rightarrow \infty} \frac{\sqrt{x}}{x} = \lim_{x \rightarrow \infty} \frac{1}{\sqrt{x}} = 0.$$

Example

$$\lim_{x \rightarrow \infty} \frac{e^x}{x^2} = \lim_{x \rightarrow \infty} \frac{e^x}{2x} = \lim_{x \rightarrow \infty} \frac{e^x}{2} = \infty.$$



$$\lim_{x \rightarrow a} \frac{f(x)}{g(x)} = \lim_{x \rightarrow a} \frac{f'(x)}{g'(x)}$$

Indeterminate Forms $\infty \cdot 0$ and $\infty - \infty$

We don't have a l'Hôpital's Rule for " $\infty \cdot 0$ " or " $\infty - \infty$ ", so we will try to rearrange our problem to either a " $\frac{0}{0}$ " problem or a " $\frac{\infty}{\infty}$ " problem.

7.5 Indeterminate Forms and L'Hôpital's Rule

$$\lim_{x \rightarrow a} \frac{f(x)}{g(x)} = \lim_{x \rightarrow a} \frac{f'(x)}{g'(x)}$$


Example

Find $\lim_{x \rightarrow \infty} \left(x \sin \frac{1}{x} \right)$.

This is a “ $\infty \cdot 0$ ” problem.

7.5 Indeterminate Forms and L'Hôpital's Rule

$$\lim_{x \rightarrow a} \frac{f(x)}{g(x)} = \lim_{x \rightarrow a} \frac{f'(x)}{g'(x)}$$


Example

Find $\lim_{x \rightarrow \infty} \left(x \sin \frac{1}{x} \right)$.

This is a “ $\infty \cdot 0$ ” problem. If we let $h = \frac{1}{x}$, then we can change it into a “ $\frac{0}{0}$ ” problem.

7.5 Indeterminate Forms and L'Hôpital's Rule

$$\lim_{x \rightarrow a} \frac{f(x)}{g(x)} = \lim_{x \rightarrow a} \frac{f'(x)}{g'(x)}$$


Example

Find $\lim_{x \rightarrow \infty} \left(x \sin \frac{1}{x} \right)$.

This is a “ $\infty \cdot 0$ ” problem. If we let $h = \frac{1}{x}$, then we can change it into a “ $\frac{0}{0}$ ” problem.

$$\lim_{x \rightarrow \infty} \underbrace{\left(x \sin \frac{1}{x} \right)}_{\infty \cdot 0} = \lim_{h \rightarrow 0^+} \left(\frac{1}{h} \sin h \right) = \lim_{h \rightarrow 0^+} \underbrace{\frac{\sin h}{h}}_{\frac{0}{0}}$$

7.5 Indeterminate Forms and L'Hôpital

$$\lim_{x \rightarrow a} \frac{f(x)}{g(x)} = \lim_{x \rightarrow a} \frac{f'(x)}{g'(x)}$$


Example

Find $\lim_{x \rightarrow \infty} \left(x \sin \frac{1}{x} \right)$.

This is a “ $\infty \cdot 0$ ” problem. If we let $h = \frac{1}{x}$, then we can change it into a “ $\frac{0}{0}$ ” problem.

$$\lim_{x \rightarrow \infty} \underbrace{\left(x \sin \frac{1}{x} \right)}_{\infty \cdot 0} = \lim_{h \rightarrow 0^+} \left(\frac{1}{h} \sin h \right) = \lim_{h \rightarrow 0^+} \underbrace{\frac{\sin h}{h}}_{\frac{0}{0}} = \lim_{h \rightarrow 0^+} \frac{\cos h}{1} = 1.$$

(I didn't need to use l'Hôpital's Rule here because we already know that $\lim_{h \rightarrow 0^+} \frac{\sin h}{h} = 1$.)

7.5 Indeterminate Forms and L'Hôpital's Rule

$$\lim_{x \rightarrow a} \frac{f(x)}{g(x)} = \lim_{x \rightarrow a} \frac{f'(x)}{g'(x)}$$


Example

Find $\lim_{x \rightarrow 0^+} \sqrt{x} \ln x$.

This is another “ $\infty \cdot 0$ ” problem. Actually a “ $0 \cdot -\infty$ ” problem.
We are going to change it into a “ $\frac{\infty}{\infty}$ ” problem.

7.5 Indeterminate Forms and L'Hôpital's Rule

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This is another “ $\infty \cdot 0$ ” problem. Actually a “ $0 \cdot -\infty$ ” problem.
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$$\lim_{x \rightarrow 0^+} \underbrace{\sqrt{x} \ln x}_{0 \cdot -\infty} = \lim_{x \rightarrow 0^+} \frac{\ln x}{\underbrace{x^{-\frac{1}{2}}}_{\frac{-\infty}{\infty}}}$$

7.5 Indeterminate Forms and L'Hôpital's Rule

$$\lim_{x \rightarrow a} \frac{f(x)}{g(x)} = \lim_{x \rightarrow a} \frac{f'(x)}{g'(x)}$$


Example

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$$\lim_{x \rightarrow 0^+} \underbrace{\sqrt{x} \ln x}_{0 \cdot -\infty} = \lim_{x \rightarrow 0^+} \frac{\ln x}{x^{-\frac{1}{2}}} = \lim_{x \rightarrow 0^+} \frac{x^{-1}}{-\frac{1}{2}x^{-\frac{3}{2}}}$$

7.5 Indeterminate Forms and L'Hôpital's Rule

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7.5 Indeterminate Forms and L'Hôpital's Rule

$$\lim_{x \rightarrow a} \frac{f(x)}{g(x)} = \lim_{x \rightarrow a} \frac{f'(x)}{g'(x)}$$


Example

Find $\lim_{x \rightarrow 0} \left(\frac{1}{\sin x} - \frac{1}{x} \right)$.

This is a “ $\infty - \infty$ ” problem.

7.5 Indeterminate Forms and L'Hôpital's Rule

$$\lim_{x \rightarrow a} \frac{f(x)}{g(x)} = \lim_{x \rightarrow a} \frac{f'(x)}{g'(x)}$$


Example

Find $\lim_{x \rightarrow 0} \left(\frac{1}{\sin x} - \frac{1}{x} \right)$.

This is a “ $\infty - \infty$ ” problem. To be more precise:

- If $x \rightarrow 0^+$, then $\frac{1}{\sin x} - \frac{1}{x} \rightarrow \infty - \infty$.
- If $x \rightarrow 0^-$, then $\frac{1}{\sin x} - \frac{1}{x} \rightarrow -\infty + \infty$.

7.5 Indeterminate Forms and L'Hôpital's Rule

$$\lim_{x \rightarrow a} \frac{f(x)}{g(x)} = \lim_{x \rightarrow a} \frac{f'(x)}{g'(x)}$$


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We calculate that

$$\lim_{x \rightarrow 0} \left(\frac{1}{\sin x} - \frac{1}{x} \right) = \lim_{x \rightarrow 0} \frac{x - \sin x}{x \sin x} =$$

$\underbrace{}_{0}$

=

=

7.5 Indeterminate Forms and L'Hôpital's Rule

$$\lim_{x \rightarrow a} \frac{f(x)}{g(x)} = \lim_{x \rightarrow a} \frac{f'(x)}{g'(x)}$$


Example

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We calculate that

$$\begin{aligned} \lim_{x \rightarrow 0} \left(\frac{1}{\sin x} - \frac{1}{x} \right) &= \lim_{x \rightarrow 0} \frac{x - \sin x}{x \sin x} = \lim_{x \rightarrow 0} \frac{1 - \cos x}{\underbrace{\sin x + x \cos x}_{0/0}} \\ &= \end{aligned}$$

=

=

7.5 Indeterminate Forms and L'Hôpital's Rule

$$\lim_{x \rightarrow a} \frac{f(x)}{g(x)} = \lim_{x \rightarrow a} \frac{f'(x)}{g'(x)}$$


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We calculate that

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7.5 Indeterminate Forms and L'Hôpital's Rule

$$\lim_{x \rightarrow a} \frac{f(x)}{g(x)} = \lim_{x \rightarrow a} \frac{f'(x)}{g'(x)}$$


The logo of Istanbul Okan University is located in the top right corner. It features a circular design with vertical columns of text: "İSTANBUL OKAN ÜNİVERSİTESİ" around the top and "1996" at the bottom. In the center is a stylized "U" shape composed of vertical bars.

Example

Find $\lim_{x \rightarrow 0} \left(\frac{1}{\sin x} - \frac{1}{x} \right)$.

This is a “ $\infty - \infty$ ” problem. To be more precise:

- If $x \rightarrow 0^+$, then $\frac{1}{\sin x} - \frac{1}{x} \rightarrow \infty - \infty$.
- If $x \rightarrow 0^-$, then $\frac{1}{\sin x} - \frac{1}{x} \rightarrow -\infty + \infty$.

We calculate that

$$\begin{aligned} \lim_{x \rightarrow 0} \left(\frac{1}{\sin x} - \frac{1}{x} \right) &= \lim_{x \rightarrow 0} \frac{x - \sin x}{x \sin x} = \lim_{x \rightarrow 0} \frac{1 - \cos x}{\underbrace{\sin x + x \cos x}_{\substack{0 \\ 0}}} \\ &= \lim_{x \rightarrow 0} \frac{\sin x}{2 \cos x - x \sin x} = \frac{0}{2} = 0. \end{aligned}$$

7.5 Indeterminate Forms and L'Hôpital's Rule



Ask the audience

One of these calculations is correct. The other 3 are wrong.
Which one is correct?

1 $\lim_{x \rightarrow 0^+} x \ln x = 0 \cdot (-\infty)$
 = 0

3 $\lim_{x \rightarrow 0^+} x \ln x = 0 \cdot (-\infty)$
 = -\infty

2 $\lim_{x \rightarrow 0^+} x \ln x = \lim_{x \rightarrow 0^+} \frac{\ln x}{\frac{1}{x}}$
 = $\frac{-\infty}{\infty} = -1$

4 $\lim_{x \rightarrow 0^+} x \ln x = \lim_{x \rightarrow 0^+} \frac{\ln x}{\frac{1}{x}}$
 = $\lim_{x \rightarrow 0^+} \frac{\frac{1}{x}}{-\frac{1}{x^2}}$
 = $\lim_{x \rightarrow 0^+} (-x) = 0$

7.5 Indeterminate Forms and L'Hôpital's Rule



Ask the audience

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 = 0

3 $\lim_{x \rightarrow 0^+} x \ln x = 0 \cdot (-\infty)$
 = -\infty

2 $\lim_{x \rightarrow 0^+} x \ln x = \lim_{x \rightarrow 0^+} \frac{\ln x}{\frac{1}{x}}$
 = $\frac{-\infty}{\infty} = -1$

4 $\lim_{x \rightarrow 0^+} x \ln x = \lim_{x \rightarrow 0^+} \frac{\ln x}{\frac{1}{x}}$
 = $\lim_{x \rightarrow 0^+} \frac{\frac{1}{x}}{-\frac{1}{x^2}}$
 = $\lim_{x \rightarrow 0^+} (-x) = 0$

Indeterminate Powers 1^∞ , 0^0 and ∞^0

Theorem

Let $a \in \mathbb{R}$ or $a = \infty$ or $a = -\infty$.

If $\lim_{x \rightarrow a} \ln f(x) = L$, then

$$\lim_{x \rightarrow a} f(x) = \exp \left(\lim_{x \rightarrow a} \ln f(x) \right) = e^L.$$

This theorem is also true for one sided limits $x \rightarrow a^+$ and $x \rightarrow a^-$.

7.5 Indeterminate Form

$$\lim_{x \rightarrow a} f(x) = \exp \left(\lim_{x \rightarrow a} \ln f(x) \right) = e$$



Example

Apply l'Hôpital's Rule to show that $\lim_{x \rightarrow 0^+} (1 + x)^{\frac{1}{x}} = e$.

This is a “ 1^∞ ” problem.

7.5 Indeterminate Form

$$\lim_{x \rightarrow a} f(x) = \exp \left(\lim_{x \rightarrow a} \ln f(x) \right) = e$$



Example

Apply l'Hôpital's Rule to show that $\lim_{x \rightarrow 0^+} (1 + x)^{\frac{1}{x}} = e$.

This is a “ 1^∞ ” problem. We will let $f(x) = (1 + x)^{\frac{1}{x}}$ and we will find $\lim_{x \rightarrow 0^+} \ln f(x)$.

7.5 Indeterminate Form

$$\lim_{x \rightarrow a} f(x) = \exp \left(\lim_{x \rightarrow a} \ln f(x) \right) = e$$



Since

$$\ln f(x) = \ln(1 + x)^{\frac{1}{x}} = \frac{1}{x} \ln(1 + x),$$

7.5 Indeterminate Form

$$\lim_{x \rightarrow a} f(x) = \exp \left(\lim_{x \rightarrow a} \ln f(x) \right) = e$$



Since

$$\ln f(x) = \ln(1 + x)^{\frac{1}{x}} = \frac{1}{x} \ln(1 + x),$$

we can use l'Hôpital's Rule to calculate that

$$\lim_{x \rightarrow 0^+} \ln f(x) = \lim_{x \rightarrow 0^+} \underbrace{\frac{\ln(1 + x)}{x}}_{\frac{0}{0}}$$

7.5 Indeterminate Form

$$\lim_{x \rightarrow a} f(x) = \exp \left(\lim_{x \rightarrow a} \ln f(x) \right) = e^{\lim_{x \rightarrow a} \ln f(x)}$$



Since

$$\ln f(x) = \ln(1 + x)^{\frac{1}{x}} = \frac{1}{x} \ln(1 + x),$$

we can use l'Hôpital's Rule to calculate that

$$\lim_{x \rightarrow 0^+} \ln f(x) = \lim_{x \rightarrow 0^+} \frac{\ln(1 + x)}{x} = \lim_{x \rightarrow 0^+} \frac{\frac{1}{1+x}}{1}$$

$\underbrace{\hspace{10em}}$
 $\frac{0}{0}$

7.5 Indeterminate Form

$$\lim_{x \rightarrow a} f(x) = \exp \left(\lim_{x \rightarrow a} \ln f(x) \right) = e$$



Since

$$\ln f(x) = \ln(1 + x)^{\frac{1}{x}} = \frac{1}{x} \ln(1 + x),$$

we can use l'Hôpital's Rule to calculate that

$$\lim_{x \rightarrow 0^+} \ln f(x) = \lim_{x \rightarrow 0^+} \underbrace{\frac{\ln(1 + x)}{x}}_{\frac{0}{0}} = \lim_{x \rightarrow 0^+} \frac{\frac{1}{1+x}}{1} = \frac{1}{1} = 1.$$

7.5 Indeterminate Form

$$\lim_{x \rightarrow a} f(x) = \exp \left(\lim_{x \rightarrow a} \ln f(x) \right) = e$$



Since

$$\ln f(x) = \ln(1 + x)^{\frac{1}{x}} = \frac{1}{x} \ln(1 + x),$$

we can use l'Hôpital's Rule to calculate that

$$\lim_{x \rightarrow 0^+} \ln f(x) = \lim_{x \rightarrow 0^+} \frac{\ln(1 + x)}{x} = \lim_{x \rightarrow 0^+} \frac{\frac{1}{1+x}}{1} = \frac{1}{1} = 1.$$

Therefore

$$\lim_{x \rightarrow 0^+} (1 + x)^{\frac{1}{x}} = \lim_{x \rightarrow 0^+} f(x) = \exp \left(\lim_{x \rightarrow 0^+} \ln f(x) \right)$$

7.5 Indeterminate Form

$$\lim_{x \rightarrow a} f(x) = \exp \left(\lim_{x \rightarrow a} \ln f(x) \right) = e$$



Since

$$\ln f(x) = \ln(1 + x)^{\frac{1}{x}} = \frac{1}{x} \ln(1 + x),$$

we can use l'Hôpital's Rule to calculate that

$$\lim_{x \rightarrow 0^+} \ln f(x) = \lim_{x \rightarrow 0^+} \frac{\ln(1 + x)}{x} = \lim_{x \rightarrow 0^+} \frac{\frac{1}{1+x}}{1} = \frac{1}{1} = 1.$$

Therefore

$$\lim_{x \rightarrow 0^+} (1 + x)^{\frac{1}{x}} = \lim_{x \rightarrow 0^+} f(x) = \exp \left(\lim_{x \rightarrow 0^+} \ln f(x) \right) = e^1 = e.$$

7.5 Indeterminate Forms

$$\lim_{x \rightarrow a} f(x) = \exp \left(\lim_{x \rightarrow a} \ln f(x) \right) = e^{\lim_{x \rightarrow a} \ln f(x)}$$



Example

Find $\lim_{x \rightarrow \infty} x^{\frac{1}{x}}$.

This is an “ ∞^0 ” problem.

7.5 Indeterminate Forms

$$\lim_{x \rightarrow a} f(x) = \exp \left(\lim_{x \rightarrow a} \ln f(x) \right) = e$$



Example

Find $\lim_{x \rightarrow \infty} x^{\frac{1}{x}}$.

This is an “ ∞^0 ” problem. First we use l'Hôpital's Rule to calculate that

$$\lim_{x \rightarrow \infty} \ln x^{\frac{1}{x}}$$

7.5 Indeterminate Forms

$$\lim_{x \rightarrow a} f(x) = \exp \left(\lim_{x \rightarrow a} \ln f(x) \right) = e^{\lim_{x \rightarrow a} \ln f(x)}$$



Example

Find $\lim_{x \rightarrow \infty} x^{\frac{1}{x}}$.

This is an “ ∞^0 ” problem. First we use l'Hôpital's Rule to calculate that

$$\lim_{x \rightarrow \infty} \ln x^{\frac{1}{x}} = \lim_{x \rightarrow \infty} \underbrace{\frac{\ln x}{x}}_{\text{8|8}}$$

7.5 Indeterminate Forms

$$\lim_{x \rightarrow a} f(x) = \exp \left(\lim_{x \rightarrow a} \ln f(x) \right) = e^{\lim_{x \rightarrow a} \ln f(x)}$$



Example

Find $\lim_{x \rightarrow \infty} x^{\frac{1}{x}}$.

This is an “ ∞^0 ” problem. First we use l'Hôpital's Rule to calculate that

$$\lim_{x \rightarrow \infty} \ln x^{\frac{1}{x}} = \lim_{x \rightarrow \infty} \underbrace{\frac{\ln x}{x}}_{\text{8/8}} = \lim_{x \rightarrow \infty} \frac{\frac{1}{x}}{1} = \frac{0}{1} = 0.$$

7.5 Indeterminate Forms

$$\lim_{x \rightarrow a} f(x) = \exp \left(\lim_{x \rightarrow a} \ln f(x) \right) = e$$



Example

Find $\lim_{x \rightarrow \infty} x^{\frac{1}{x}}$.

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It follows that

$$\lim_{x \rightarrow \infty} x^{\frac{1}{x}} = \exp \left(\lim_{x \rightarrow \infty} \ln x^{\frac{1}{x}} \right) = e^0 = 1.$$

7.5 Indeterminate Forms and L'Hôpital's Rule



Theorem (Cauchy's Mean Value Theorem)

Suppose that

- f and g are continuous on $[a, b]$;
- f and g are differentiable on (a, b) ;
- $g'(x) \neq 0$ for all $x \in (a, b)$.

7.5 Indeterminate Forms and L'Hôpital's Rule



Theorem (Cauchy's Mean Value Theorem)

Suppose that

- f and g are continuous on $[a, b]$;
- f and g are differentiable on (a, b) ;
- $g'(x) \neq 0$ for all $x \in (a, b)$.

Then there exists $c \in (a, b)$ such that

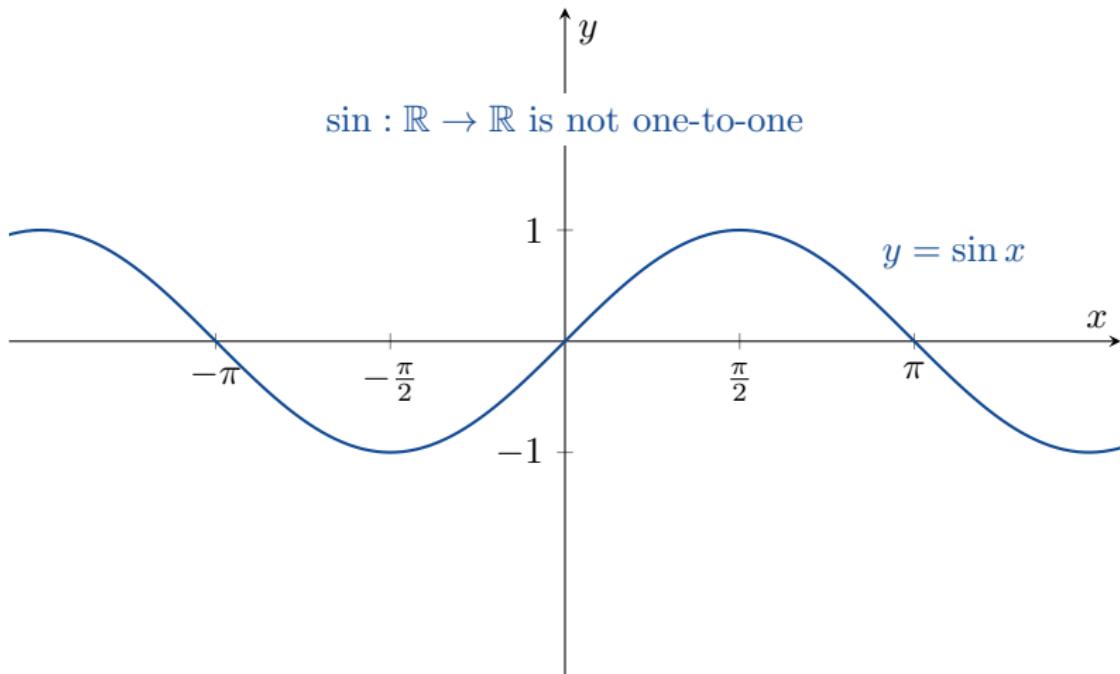
$$\frac{f'(c)}{g'(c)} = \frac{f(b) - f(a)}{g(b) - g(a)}.$$

(proof in textbook)



Inverse Trigonometric Functions

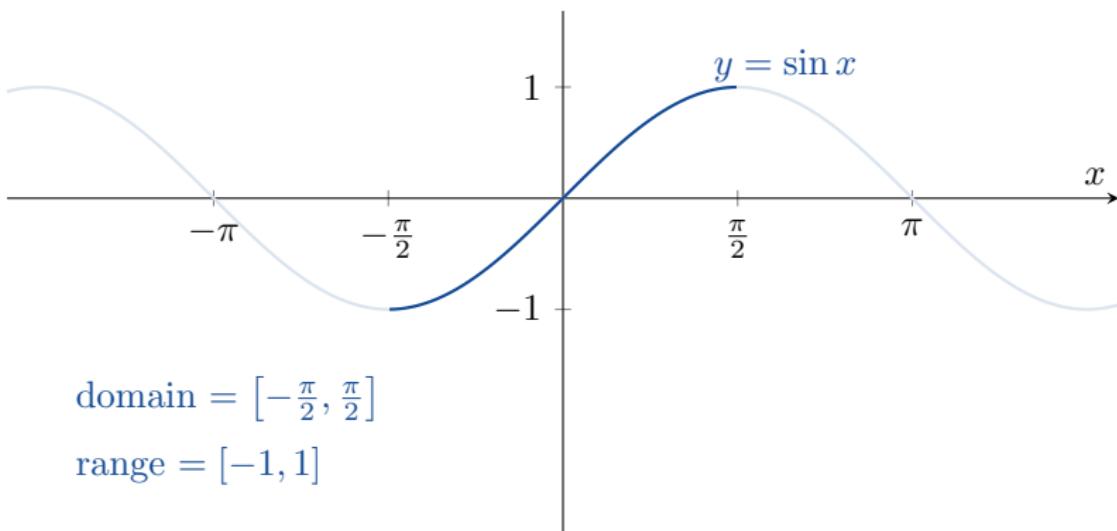
7.6 Inverse Trigonometric Functions



7.6 Inverse Trigonometric Functions



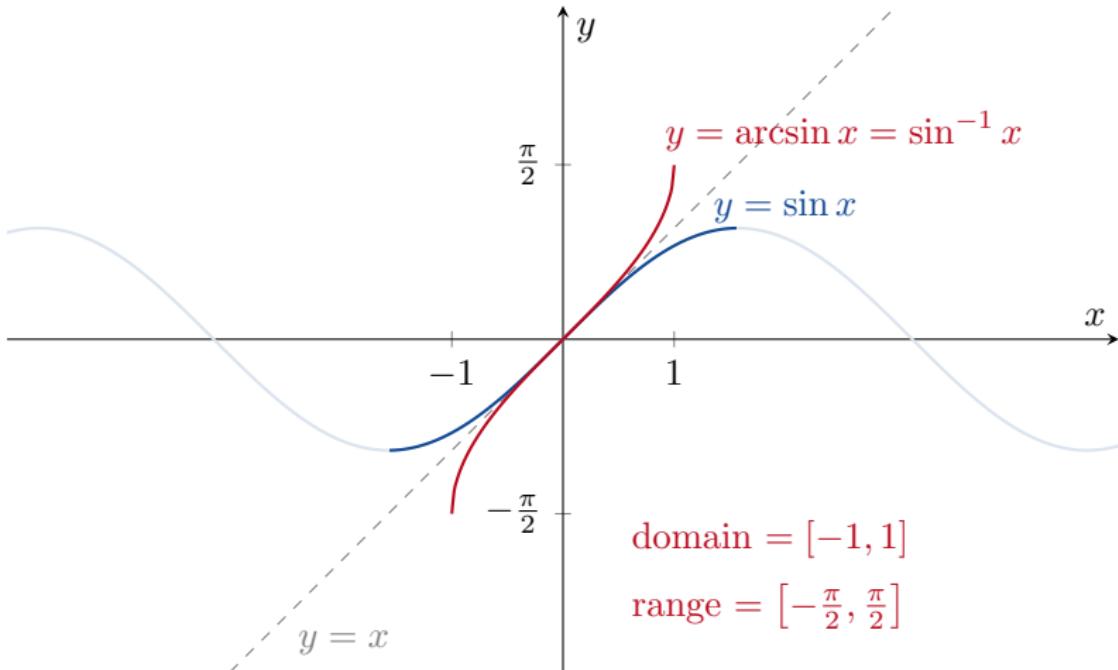
$\sin : \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \rightarrow \mathbb{R}$ is one-to-one



$$\text{domain} = \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$$

$$\text{range} = [-1, 1]$$

7.6 Inverse Trigonometric Functions

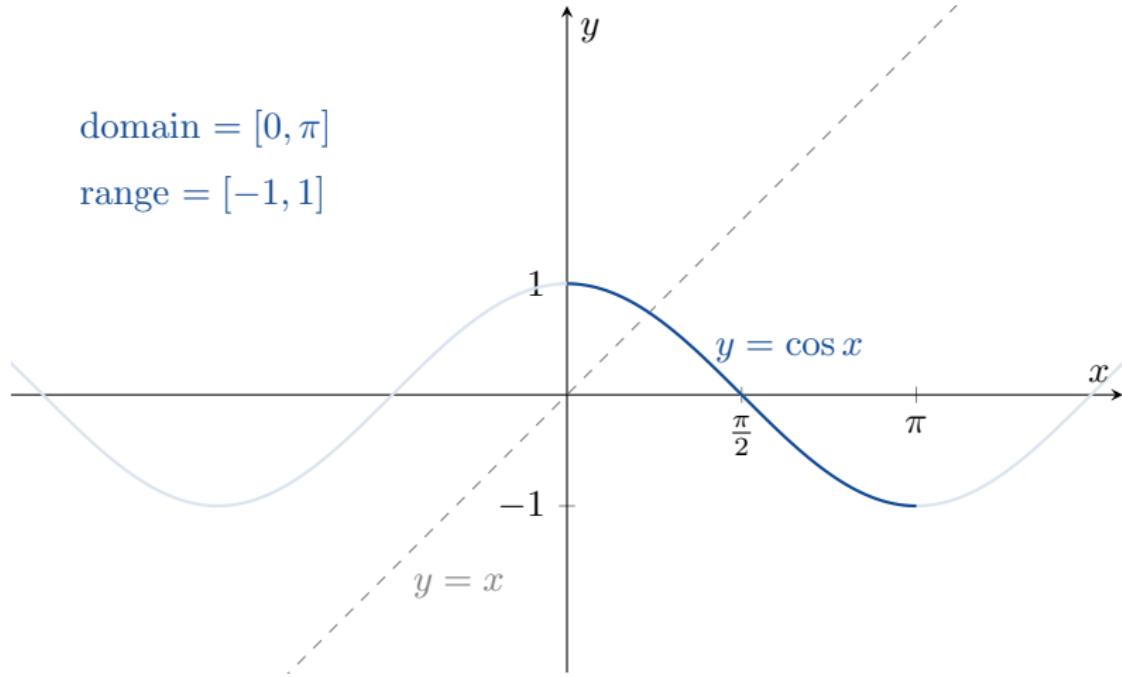


7.6 Inverse Trigonometric Functions



domain = $[0, \pi]$

range = $[-1, 1]$

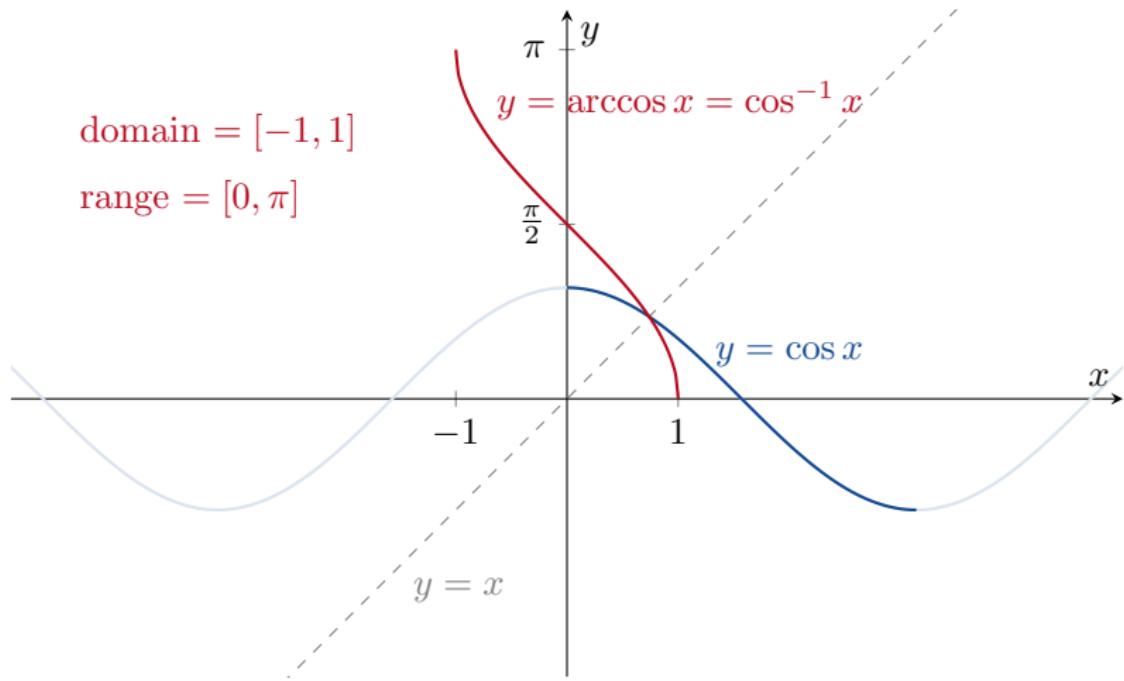


7.6 Inverse Trigonometric Functions



domain = $[-1, 1]$

range = $[0, \pi]$



7.6 Inverse Trigonometric Functions

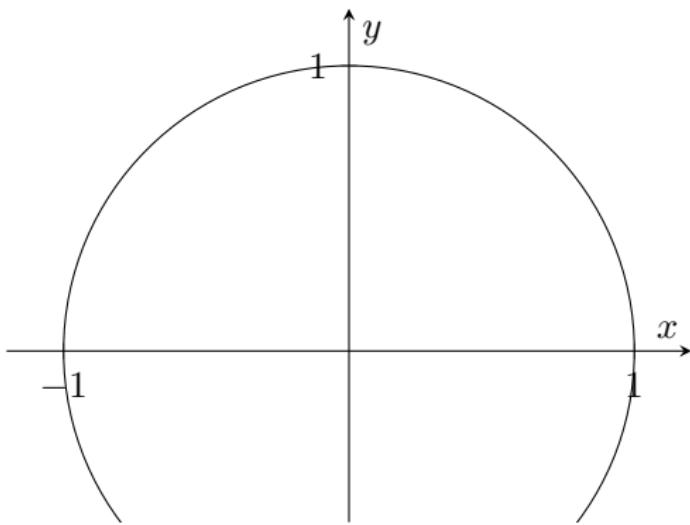


Arcsine and Arccosine

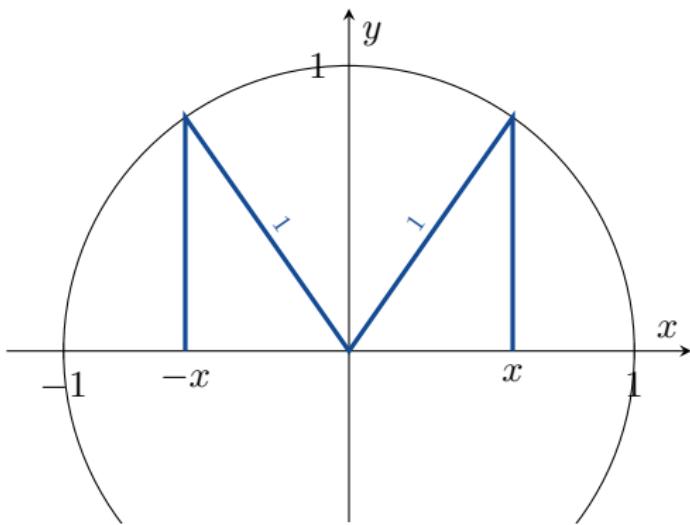
Definition

- $y = \arcsin x$ is the number in $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ for which $\sin y = x$.
- $y = \arccos x$ is the number in $[0, \pi]$ for which $\cos y = x$.

Identities Involving Arcsine and Arccosine



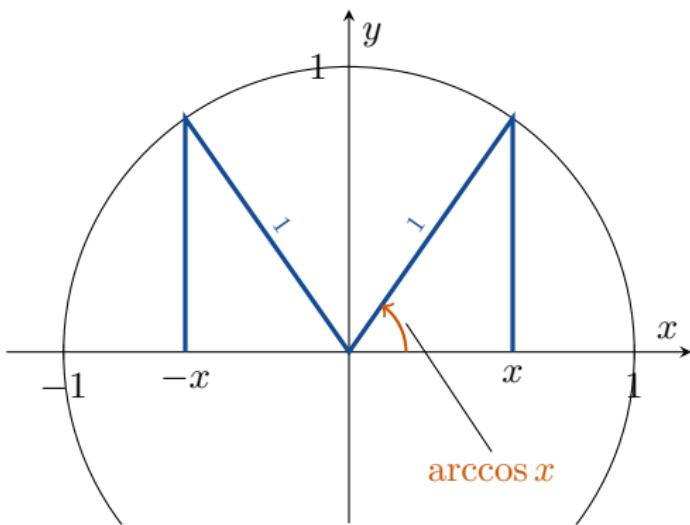
Identities Involving Arcsine and Arccosine



7.6 Inverse Trigonometric Functions



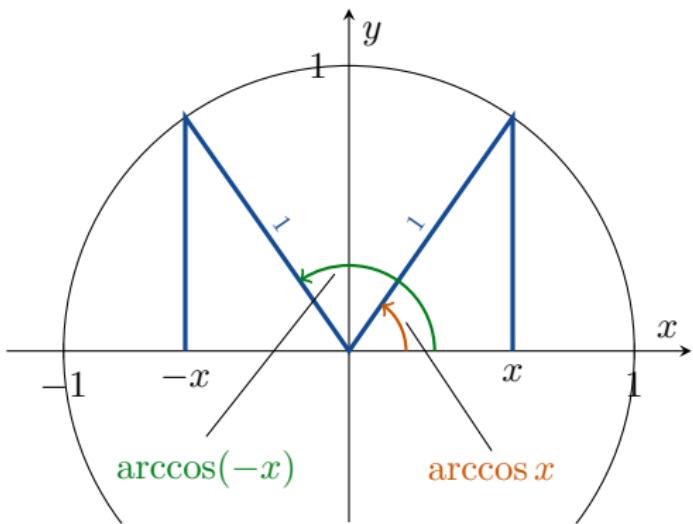
Identities Involving Arcsine and Arccosine



7.6 Inverse Trigonometric Functions



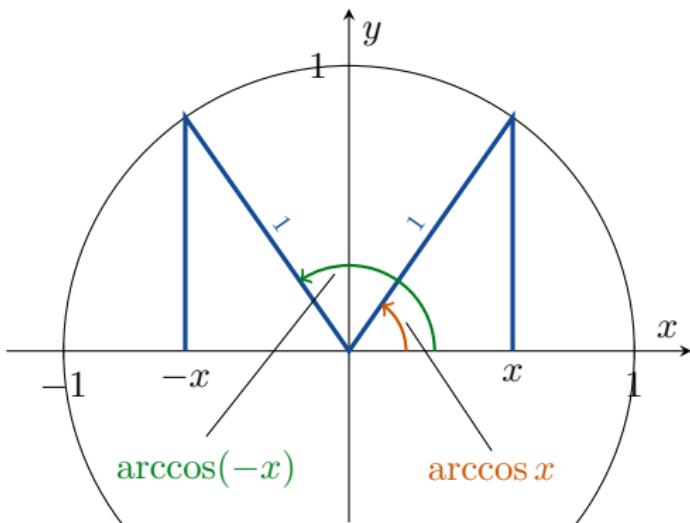
Identities Involving Arcsine and Arccosine



7.6 Inverse Trigonometric Functions



Identities Involving Arcsine and Arccosine

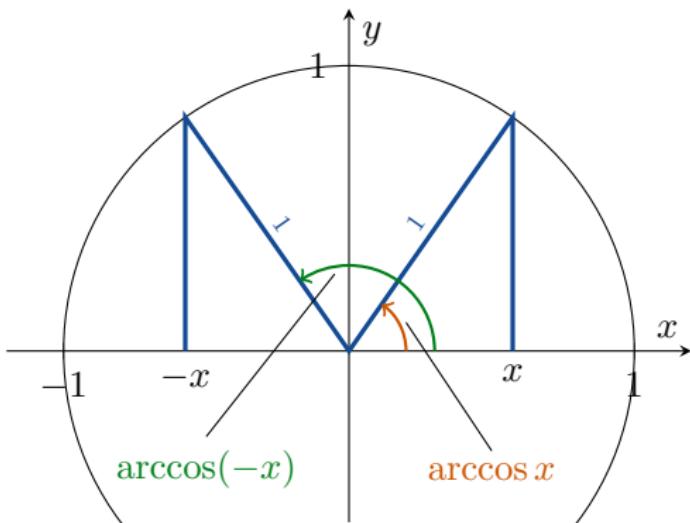


$$\arccos x + \arccos(-x) = \pi$$

7.6 Inverse Trigonometric Functions



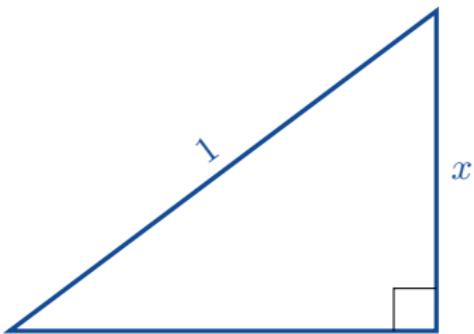
Identities Involving Arcsine and Arccosine



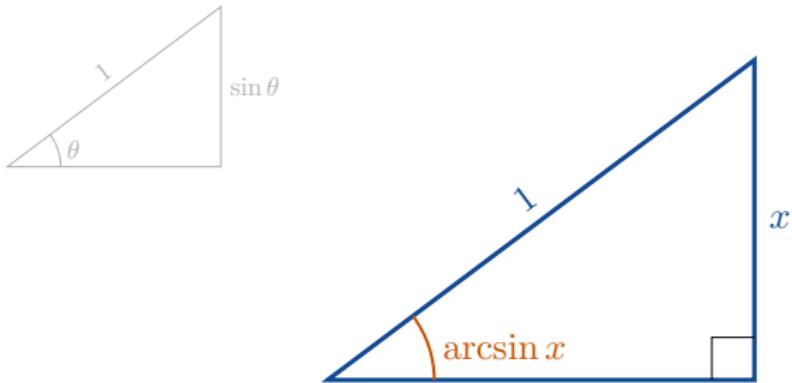
$$\arccos x + \arccos(-x) = \pi$$

$$\boxed{\arccos(-x) = \pi - \arccos x}$$

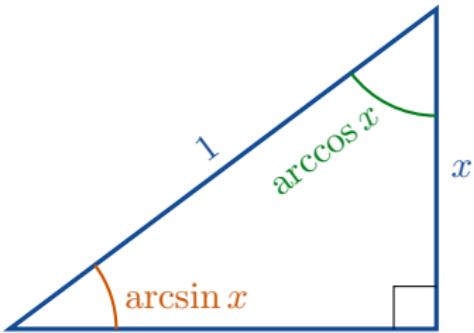
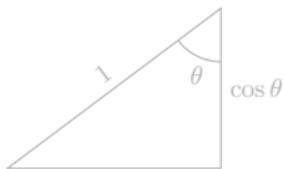
7.6 Inverse Trigonometric Functions



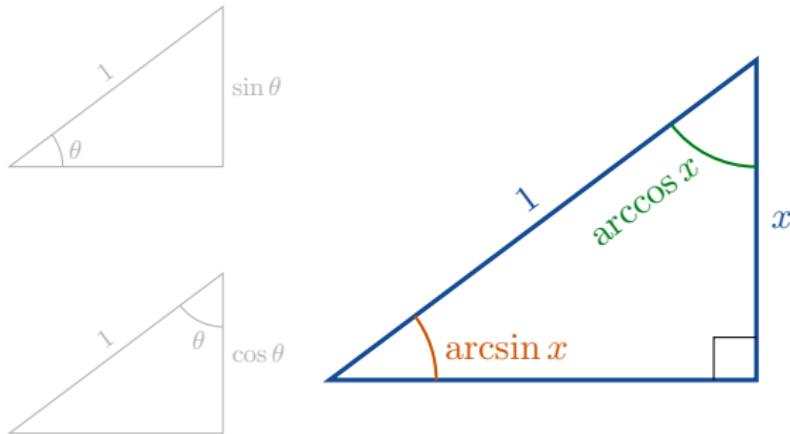
7.6 Inverse Trigonometric Functions



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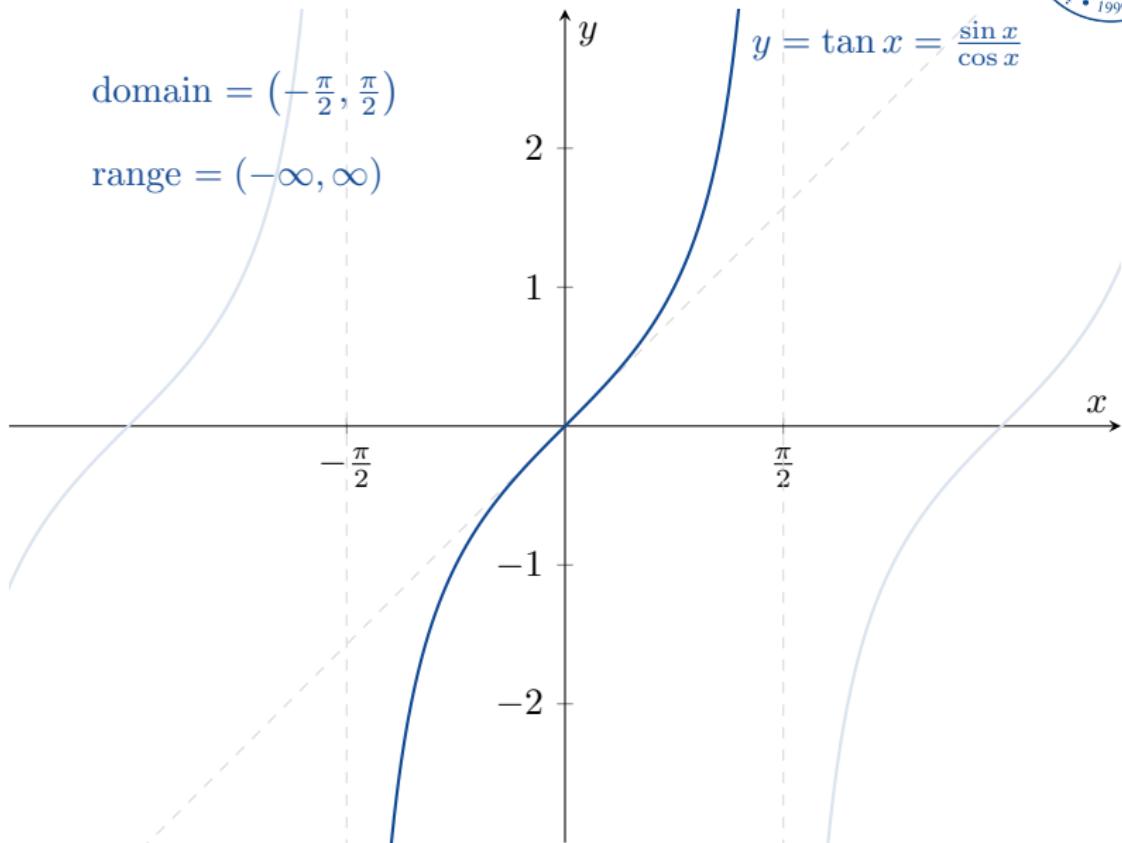


$$\boxed{\arcsin x + \arccos x = \frac{\pi}{2}} \quad x \in [-1, 1]$$

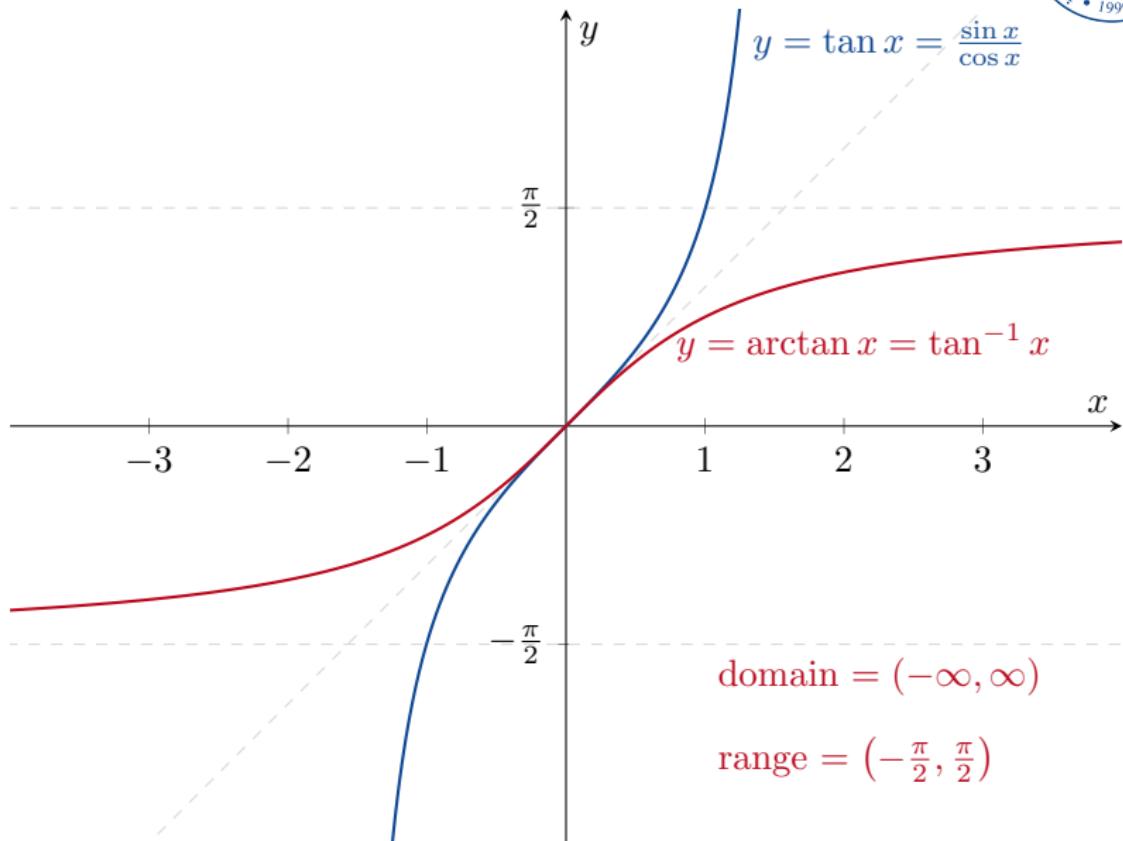
(From this triangle we can see that this is true for $x \in [0, 1]$.

Using the previous identity, we can prove that it is also true for $x \in [-1, 0)$.)

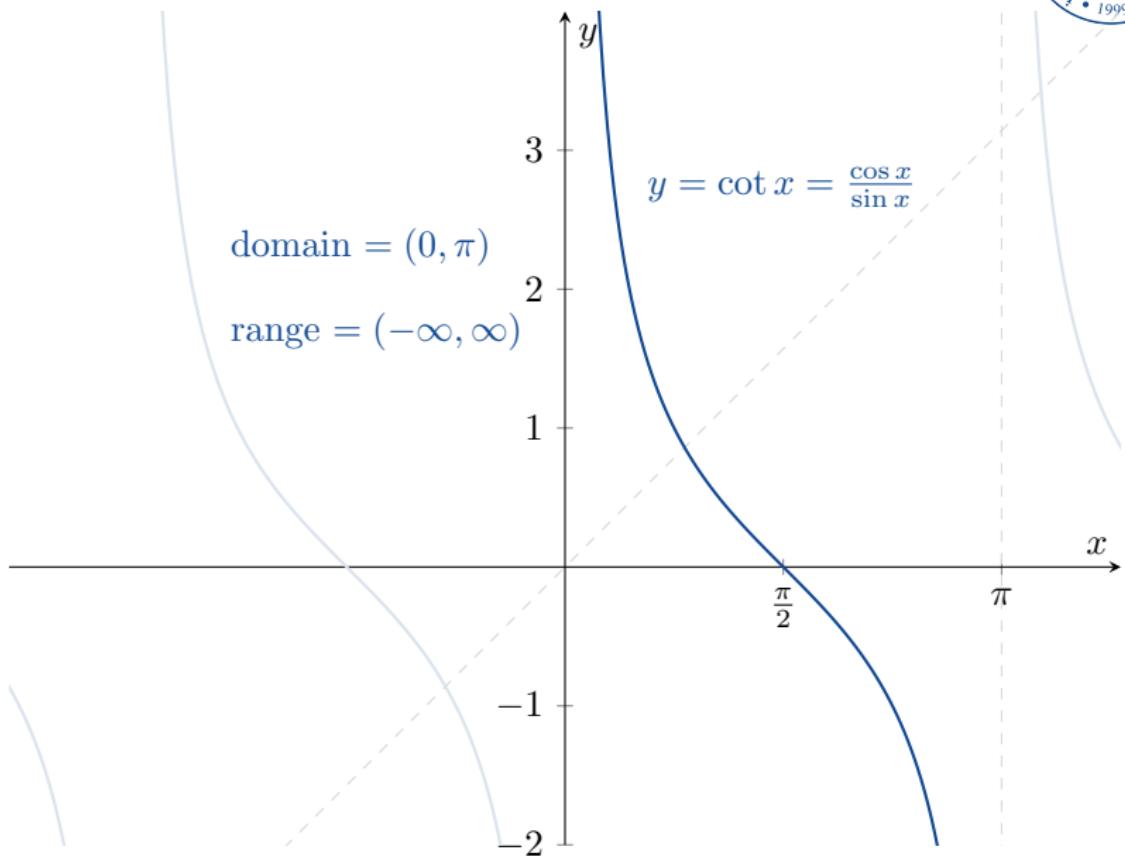
7.6 Inverse Trigonometric Functions



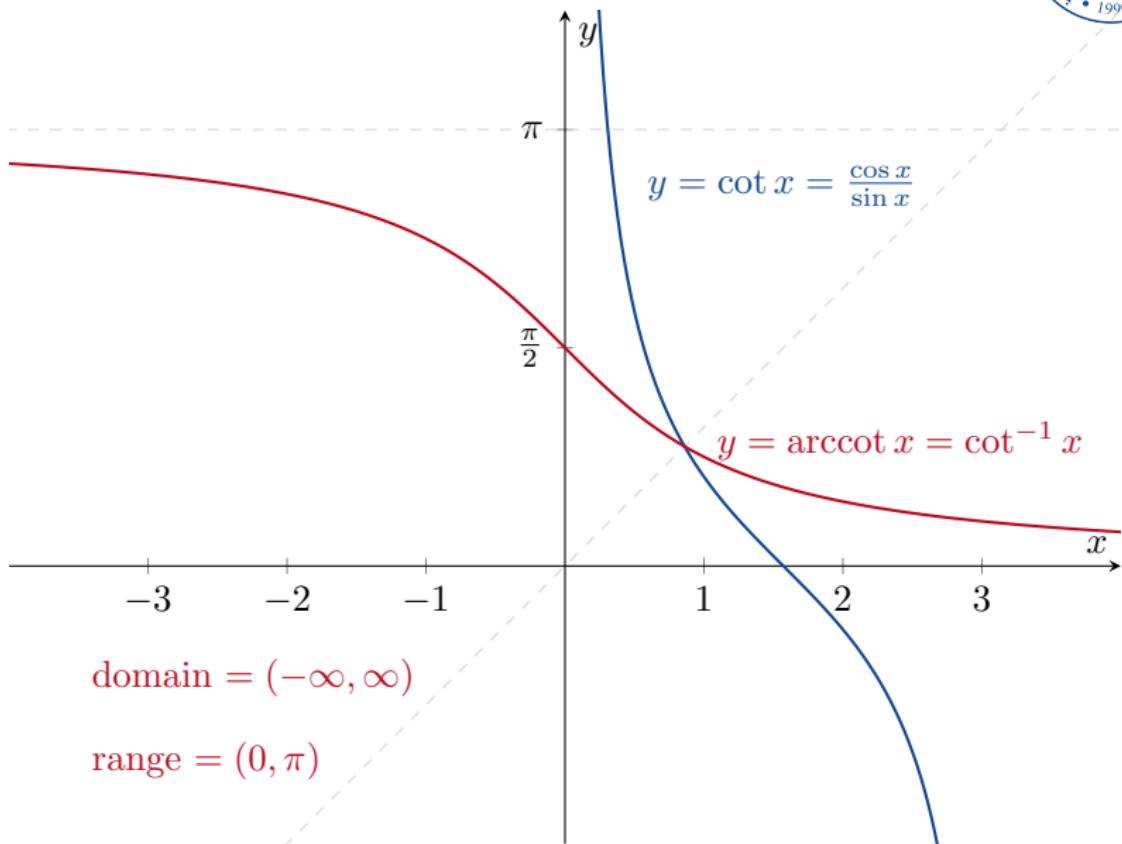
7.6 Inverse Trigonometric Functions



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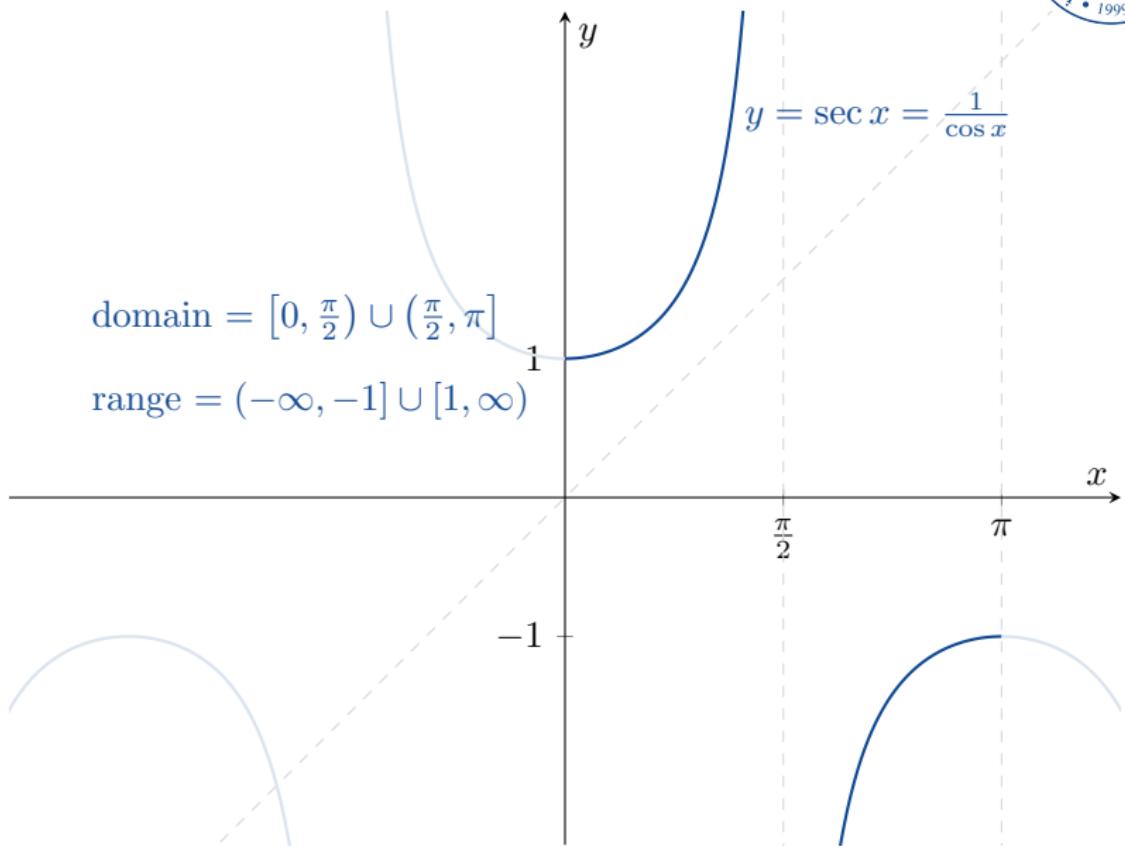
7.6 Inverse Trigonometric Functions



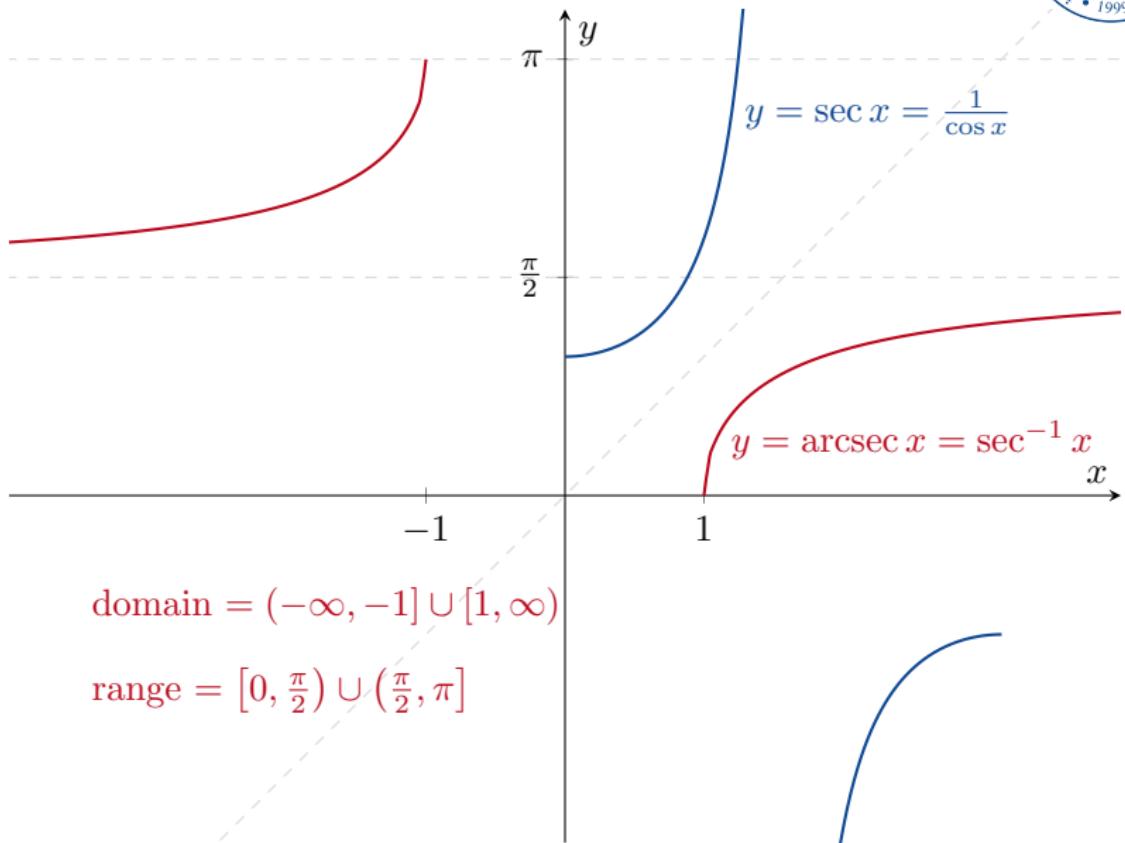
7.6 Inverse Trigonometric Functions

$$\text{domain} = \left[0, \frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \pi\right]$$

$$\text{range} = (-\infty, -1] \cup [1, \infty)$$



7.6 Inverse Trigonometric Functions

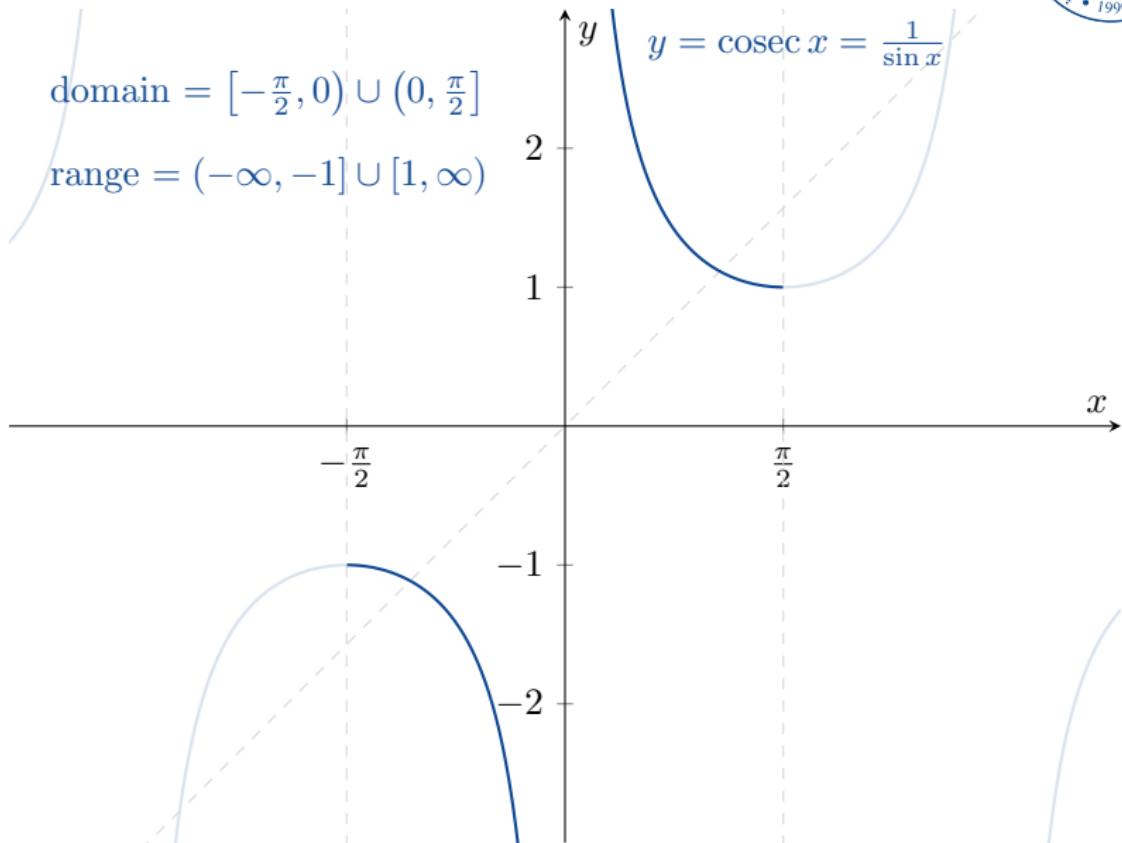


7.6 Inverse Trigonometric Functions

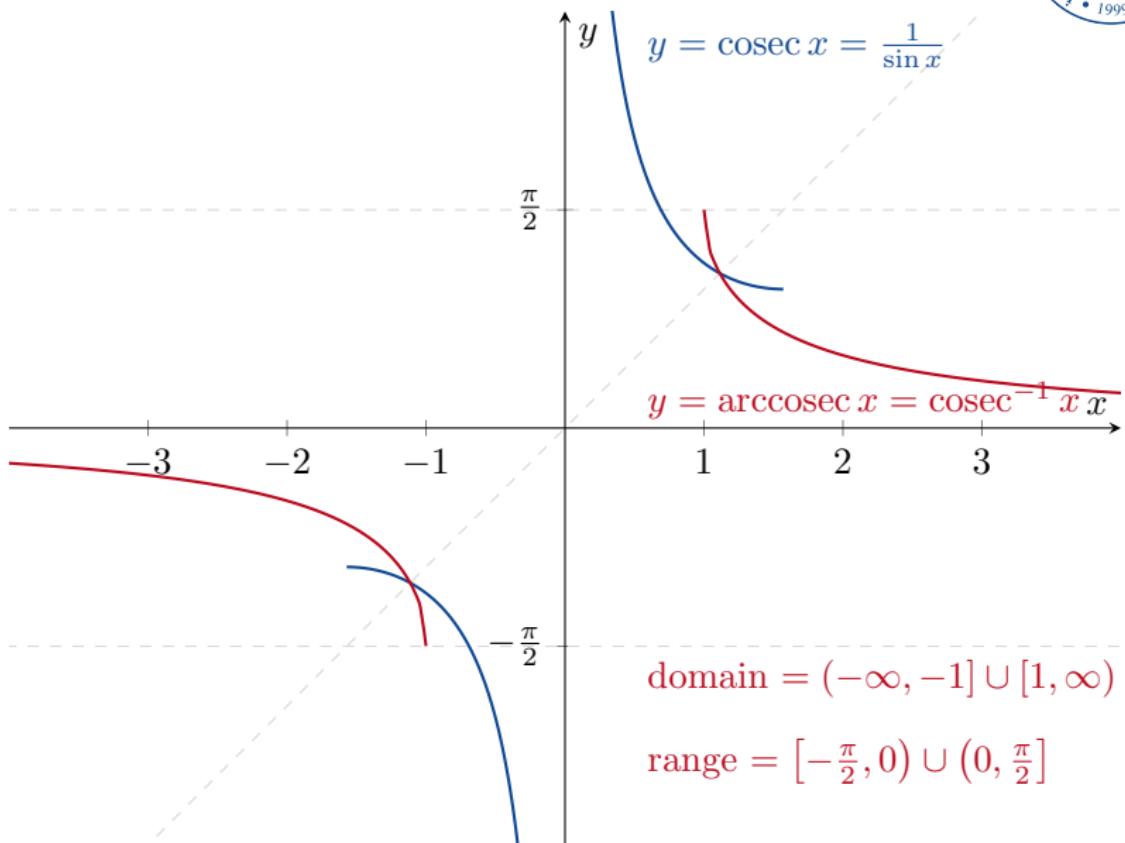


$$\text{domain} = \left[-\frac{\pi}{2}, 0\right) \cup \left(0, \frac{\pi}{2}\right]$$

$$\text{range} = (-\infty, -1] \cup [1, \infty)$$



7.6 Inverse Trigonometric Functions



Arctangent, Arccotangent, Arcsecant and Arccosecant

Definition

- $y = \arctan x$ is the number in $(-\frac{\pi}{2}, \frac{\pi}{2})$ for which $\tan y = x$.
- $y = \operatorname{arccot} x$ is the number in $(0, \pi)$ for which $\cot y = x$.
- $y = \operatorname{arcsec} x$ is the number in $[0, \frac{\pi}{2}) \cup (\frac{\pi}{2}, \pi]$ for which $\sec y = x$.
- $y = \operatorname{arccosec} x$ is the number in $[-\frac{\pi}{2}, 0) \cup (0, \frac{\pi}{2}]$ for which $\operatorname{cosec} y = x$.

$$(f^{-1})'(x) = \frac{1}{f'(f^{-1}(x))}$$



The Derivative of $y = \arcsin x$

Let $f(x) = \sin x$ and $f^{-1}(x) = \arcsin x$.

7.6 Inverse Trigonometric Functions

$$(f^{-1})'(x) = \frac{1}{f'(f^{-1}(x))}$$



The Derivative of $y = \arcsin x$

Let $f(x) = \sin x$ and $f^{-1}(x) = \arcsin x$. Then, if $-1 < x < 1$, we have that

$$(f^{-1})'(x) = \frac{1}{f'(f^{-1}(x))} = \frac{1}{\cos(\arcsin x)}$$

=

=

7.6 Inverse Trigonometric Functions

$$(f^{-1})'(x) = \frac{1}{f'(f^{-1}(x))}$$



The Derivative of $y = \arcsin x$

Let $f(x) = \sin x$ and $f^{-1}(x) = \arcsin x$. Then, if $-1 < x < 1$, we have that

$$\begin{aligned}(f^{-1})'(x) &= \frac{1}{f'(f^{-1}(x))} = \frac{1}{\cos(\arcsin x)} \\&= \frac{1}{\sqrt{1 - \sin^2(\arcsin x)}} \\&\quad (\text{because } \sin^2 \theta + \cos^2 \theta = 1)\end{aligned}$$

=

7.6 Inverse Trigonometric Functions

$$(f^{-1})'(x) = \frac{1}{f'(f^{-1}(x))}$$



The Derivative of $y = \arcsin x$

Let $f(x) = \sin x$ and $f^{-1}(x) = \arcsin x$. Then, if $-1 < x < 1$, we have that

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7.6 Inverse Trigonometric Functions



Theorem

$$\frac{d}{dx} (\arcsin x) = \frac{1}{\sqrt{1-x^2}}, \quad |x| < 1.$$

7.6 Inverse Trigonometric Functions



Theorem

$$\frac{d}{dx} (\arcsin x) = \frac{1}{\sqrt{1-x^2}}, \quad |x| < 1.$$

If $u(x)$ is differentiable and $|u| < 1$, then

$$\frac{d}{dx} (\arcsin u) = \frac{1}{\sqrt{1-u^2}} \frac{du}{dx}.$$

EXAMPLE 4

Using the Chain Rule, we calculate the derivative

$$\frac{d}{dx}(\arcsin x^2) = \frac{1}{\sqrt{1 - (x^2)^2}} \cdot \frac{d}{dx}(x^2) = \frac{2x}{\sqrt{1 - x^4}}.$$

7.6 Inverse Trigonometric Functions

$$(f^{-1})'(x) = \frac{1}{f'(f^{-1}(x))}$$



The Derivative of $y = \arctan x$

This time let $f(x) = \tan x$ and $f^{-1}(x) = \arctan x$. Then

$$(f^{-1})'(x) = \frac{1}{f'(f^{-1}(x))} =$$

=

=

7.6 Inverse Trigonometric Functions

$$(f^{-1})'(x) = \frac{1}{f'(f^{-1}(x))}$$



The Derivative of $y = \arctan x$

This time let $f(x) = \tan x$ and $f^{-1}(x) = \arctan x$. Then

$$(f^{-1})'(x) = \frac{1}{f'(f^{-1}(x))} = \frac{1}{\sec^2(\arctan x)}$$

=

=

7.6 Inverse Trigonometric Functions

$$(f^{-1})'(x) = \frac{1}{f'(f^{-1}(x))}$$



The Derivative of $y = \arctan x$

This time let $f(x) = \tan x$ and $f^{-1}(x) = \arctan x$. Then

$$\begin{aligned}(f^{-1})'(x) &= \frac{1}{f'(f^{-1}(x))} = \frac{1}{\sec^2(\arctan x)} \\&= \frac{1}{1 + \tan^2(\arctan x)} \\&\quad (\text{because } \sec^2 \theta = 1 + \tan^2 \theta)\end{aligned}$$

=

7.6 Inverse Trigonometric Functions

$$(f^{-1})'(x) = \frac{1}{f'(f^{-1}(x))}$$



The Derivative of $y = \arctan x$

This time let $f(x) = \tan x$ and $f^{-1}(x) = \arctan x$. Then

$$\begin{aligned}(f^{-1})'(x) &= \frac{1}{f'(f^{-1}(x))} = \frac{1}{\sec^2(\arctan x)} \\&= \frac{1}{1 + \tan^2(\arctan x)} \\&\quad (\text{because } \sec^2 \theta = 1 + \tan^2 \theta) \\&= \frac{1}{1 + x^2}\end{aligned}$$

7.6 Inverse Trigonometric Functions

$$(f^{-1})'(x) = \frac{1}{f'(f^{-1}(x))}$$



The Derivative of $y = \arctan x$

This time let $f(x) = \tan x$ and $f^{-1}(x) = \arctan x$. Then

$$\begin{aligned}(f^{-1})'(x) &= \frac{1}{f'(f^{-1}(x))} = \frac{1}{\sec^2(\arctan x)} \\&= \frac{1}{1 + \tan^2(\arctan x)} \\&\quad (\text{because } \sec^2 \theta = 1 + \tan^2 \theta) \\&= \frac{1}{1 + x^2}\end{aligned}$$

Theorem

$$\frac{d}{dx} (\arctan x) = \frac{1}{1 + x^2}.$$

7.6 Inverse Trigonometric Functions



The Derivative of $y = \text{arcsec } x$

This time we will use Implicit Differentiation.

7.6 Inverse Trigonometric Functions



The Derivative of $y = \text{arcsec } x$

This time we will use Implicit Differentiation.

$$y = \text{arcsec } x$$

$$\sec y = x$$

7.6 Inverse Trigonometric Functions



The Derivative of $y = \text{arcsec } x$

This time we will use Implicit Differentiation.

$$y = \text{arcsec } x$$

$$\frac{d}{dx} \sec y = \frac{d}{dx} x$$

7.6 Inverse Trigonometric Functions



The Derivative of $y = \text{arcsec } x$

This time we will use Implicit Differentiation.

$$y = \text{arcsec } x$$

$$\frac{d}{dx} \sec y = \frac{d}{dx} x$$

$$\sec y \tan y \frac{dy}{dx} = 1$$

7.6 Inverse Trigonometric Functions



The Derivative of $y = \text{arcsec } x$

This time we will use Implicit Differentiation.

$$y = \text{arcsec } x$$

$$\frac{d}{dx} \sec y = \frac{d}{dx} x$$

$$\sec y \tan y \frac{dy}{dx} = 1$$

$$\frac{dy}{dx} = \frac{1}{\sec y \tan y}.$$

7.6 Inverse Trigonometric Functions



The Derivative of $y = \text{arcsec } x$

This time we will use Implicit Differentiation.

$$y = \text{arcsec } x$$

$$\frac{d}{dx} \sec y = \frac{d}{dx} x$$

$$\sec y \tan y \frac{dy}{dx} = 1$$

$$\frac{dy}{dx} = \frac{1}{\sec y \tan y}.$$

Next we need to use

$$\sec y = x \quad \text{and} \quad \tan y = \pm \sqrt{\sec^2 y - 1} = \pm \sqrt{x^2 - 1}.$$

7.6 Inverse Trigonometric Functions



The Derivative of $y = \text{arcsec } x$

This time we will use Implicit Differentiation.

$$y = \text{arcsec } x$$

$$\frac{d}{dx} \sec y = \frac{d}{dx} x$$

$$\sec y \tan y \frac{dy}{dx} = 1$$

$$\frac{dy}{dx} = \frac{1}{\sec y \tan y}.$$

Next we need to use

$$\sec y = x \quad \text{and} \quad \tan y = \pm \sqrt{\sec^2 y - 1} = \pm \sqrt{x^2 - 1}.$$

So

$$\frac{dy}{dx} = \pm \frac{1}{x \sqrt{x^2 - 1}}.$$

7.6 Inverse Trigonometric Functions



$$\frac{d}{dx} \operatorname{arcsec} x = \pm \frac{1}{x\sqrt{x^2 - 1}}.$$

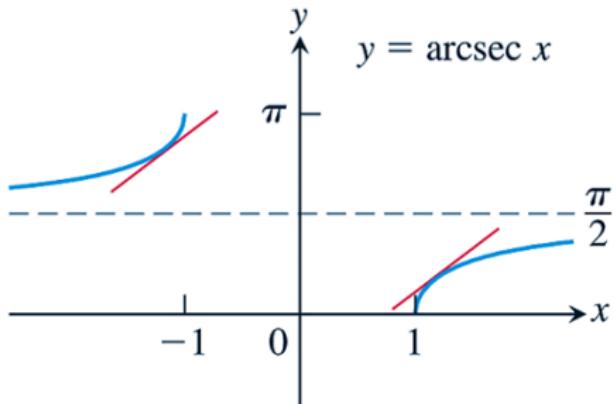
What can we do about the \pm sign?

7.6 Inverse Trigonometric Functions



$$\frac{d}{dx} \operatorname{arcsec} x = \pm \frac{1}{x\sqrt{x^2 - 1}}.$$

What can we do about the \pm sign?



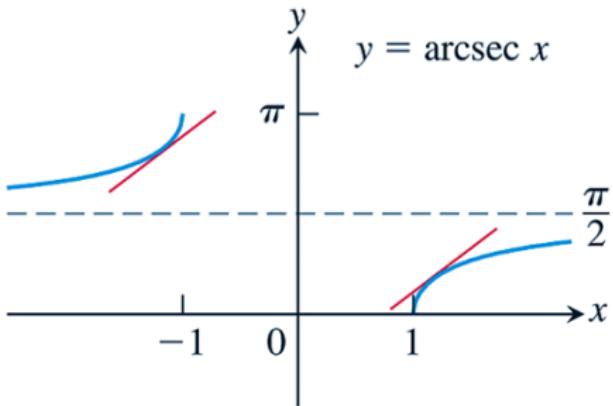
Note that $\frac{d}{dx} \operatorname{arcsec} x$ is always positive.

7.6 Inverse Trigonometric Functions



$$\frac{d}{dx} \operatorname{arcsec} x = \pm \frac{1}{x\sqrt{x^2 - 1}}.$$

What can we do about the \pm sign?



Note that $\frac{d}{dx} \operatorname{arcsec} x$ is always positive. We can replace the $\pm \frac{1}{x}$ by $\frac{1}{|x|}$.

7.6 Inverse Trigonometric Functions

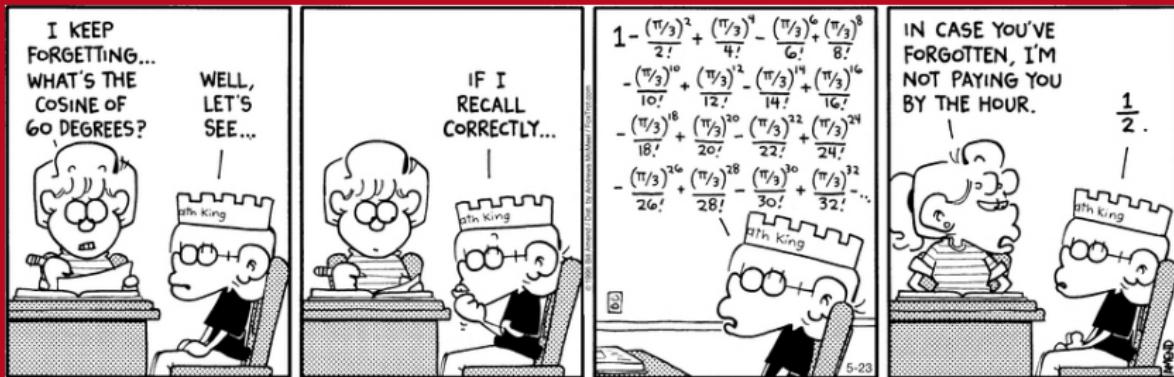


Theorem

$$\frac{d}{dx} (\text{arcsec } x) = \frac{1}{|x| \sqrt{x^2 - 1}}, \quad |x| > 1.$$

Break

We will continue at 2pm



Derivatives of the Other Three Inverse Trigonometric Functions

We can find the derivatives of $\arccos x$, $\text{arccot } x$ and $\text{arcosec } x$ by using the identities

$$\arccos x = \frac{\pi}{2} - \arcsin x$$

$$\text{arccot } x = \frac{\pi}{2} - \arctan x$$

$$\text{arcosec } x = \frac{\pi}{2} - \text{arcsec } x.$$

(I have proved the first one. The others can be derived in similar ways.)

7.6 Inve

$$\arccos x = \frac{\pi}{2} - \arcsin x \quad \frac{d}{dx} (\arcsin x) = \frac{1}{\sqrt{1-x^2}}$$



For example, we can easily calculate that

$$\frac{d}{dx} \arccos x = \frac{d}{dx} \left(\frac{\pi}{2} - \arcsin x \right) = -\frac{1}{\sqrt{1-x^2}}.$$

7.6 Inverse Trigonometric Functions



Derivative Formulae

- $\frac{d}{dx} \arcsin x = \frac{1}{\sqrt{1-x^2}}, |x| < 1$
- $\frac{d}{dx} \arccos x = \frac{-1}{\sqrt{1-x^2}}, |x| < 1$

7.6 Inverse Trigonometric Functions



Derivative Formulae

- $\frac{d}{dx} \arcsin x = \frac{1}{\sqrt{1-x^2}}, |x| < 1$

- $\frac{d}{dx} \arccos x = \frac{-1}{\sqrt{1-x^2}}, |x| < 1$

- $\frac{d}{dx} \arctan x = \frac{1}{1+x^2}$

- $\frac{d}{dx} \operatorname{arccot} x = \frac{-1}{1+x^2}$

7.6 Inverse Trigonometric Functions

Derivative Formulae

- $\frac{d}{dx} \arcsin x = \frac{1}{\sqrt{1-x^2}}, |x| < 1$
- $\frac{d}{dx} \arccos x = \frac{-1}{\sqrt{1-x^2}}, |x| < 1$
- $\frac{d}{dx} \arctan x = \frac{1}{1+x^2}$
- $\frac{d}{dx} \text{arccot } x = \frac{-1}{1+x^2}$
- $\frac{d}{dx} \text{arcsec } x = \frac{1}{|x|\sqrt{x^2-1}}, |x| > 1$
- $\frac{d}{dx} \text{arccosec } x = \frac{-1}{|x|\sqrt{x^2-1}}, |x| > 1$

7.6 Inverse Trigonometric Functions



Derivative Formulae

- $\frac{d}{dx} \arcsin x = \frac{1}{\sqrt{1-x^2}}, |x| < 1$
- $\frac{d}{dx} \arccos x = \frac{-1}{\sqrt{1-x^2}}, |x| < 1$
- $\frac{d}{dx} \arctan x = \frac{1}{1+x^2}$
- $\frac{d}{dx} \text{arccot } x = \frac{-1}{1+x^2}$
- $\frac{d}{dx} \text{arcsec } x = \frac{1}{|x|\sqrt{x^2-1}}, |x| > 1$
- $\frac{d}{dx} \text{arccosec } x = \frac{-1}{|x|\sqrt{x^2-1}}, |x| > 1$

Integral Formulae

- $\int \frac{dx}{\sqrt{a^2 - x^2}} = \arcsin\left(\frac{x}{a}\right) + C$
(valid for $x^2 < a^2$)

7.6 Inverse Trigonometric Functions

Derivative Formulae

- $\frac{d}{dx} \arcsin x = \frac{1}{\sqrt{1-x^2}}, |x| < 1$
- $\frac{d}{dx} \arccos x = \frac{-1}{\sqrt{1-x^2}}, |x| < 1$
- $\frac{d}{dx} \arctan x = \frac{1}{1+x^2}$
- $\frac{d}{dx} \text{arccot } x = \frac{-1}{1+x^2}$
- $\frac{d}{dx} \text{arcsec } x = \frac{1}{|x|\sqrt{x^2-1}}, |x| > 1$
- $\frac{d}{dx} \text{arccosec } x = \frac{-1}{|x|\sqrt{x^2-1}}, |x| > 1$

Integral Formulae

- $\int \frac{dx}{\sqrt{a^2 - x^2}} = \arcsin\left(\frac{x}{a}\right) + C$
(valid for $x^2 < a^2$)
- $\int \frac{dx}{a^2 + x^2} = \frac{1}{a} \arctan\left(\frac{x}{a}\right) + C$
(valid for all x)

7.6 Inverse Trigonometric Functions



Derivative Formulae

- $\frac{d}{dx} \arcsin x = \frac{1}{\sqrt{1-x^2}}, |x| < 1$
- $\frac{d}{dx} \arccos x = \frac{-1}{\sqrt{1-x^2}}, |x| < 1$
- $\frac{d}{dx} \arctan x = \frac{1}{1+x^2}$
- $\frac{d}{dx} \text{arccot } x = \frac{-1}{1+x^2}$
- $\frac{d}{dx} \text{arcsec } x = \frac{1}{|x|\sqrt{x^2-1}}, |x| > 1$
- $\frac{d}{dx} \text{arccosec } x = \frac{-1}{|x|\sqrt{x^2-1}}, |x| > 1$

Integral Formulae

- $\int \frac{dx}{\sqrt{a^2-x^2}} = \arcsin\left(\frac{x}{a}\right) + C$
(valid for $x^2 < a^2$)
- $\int \frac{dx}{a^2+x^2} = \frac{1}{a} \arctan\left(\frac{x}{a}\right) + C$
(valid for all x)
- $\int \frac{dx}{|x|\sqrt{x^2-a^2}} = \frac{1}{a} \text{arcsec}\left|\frac{x}{a}\right| + C$
(valid for $|x| > a > 0$)

7.6 Inverse Trigonometric

$$\int \frac{dx}{\sqrt{a^2 - x^2}} = \arcsin\left(\frac{x}{a}\right) + C$$



Example

Find $\int_{\frac{\sqrt{2}}{2}}^{\frac{\sqrt{3}}{2}} \frac{dx}{\sqrt{1 - x^2}}$.

7.6 Inverse Trigonometric

$$\int \frac{dx}{\sqrt{a^2 - x^2}} = \arcsin\left(\frac{x}{a}\right) + C$$



Example

Find $\int_{\frac{\sqrt{2}}{2}}^{\frac{\sqrt{3}}{2}} \frac{dx}{\sqrt{1 - x^2}}$.

$$\int_{\frac{\sqrt{2}}{2}}^{\frac{\sqrt{3}}{2}} \frac{dx}{\sqrt{1 - x^2}} = \left[\arcsin x \right]_{\frac{\sqrt{2}}{2}}^{\frac{\sqrt{3}}{2}} = \dots = \frac{\pi}{12}.$$

7.6 Inverse Trigonometric

$$\int \frac{dx}{\sqrt{a^2 - x^2}} = \arcsin\left(\frac{x}{a}\right) + C$$



Example

Find $\int \frac{dx}{\sqrt{3 - 4x^2}}$.

7.6 Inverse Trigonometric

$$\int \frac{dx}{\sqrt{a^2 - x^2}} = \arcsin\left(\frac{x}{a}\right) + C$$



Example

Find $\int \frac{dx}{\sqrt{3 - 4x^2}}$.

First we do a substitution: Let $u = 2x$. Then

$$\int \frac{dx}{\sqrt{3 - 4x^2}} = \frac{1}{2} \int \frac{du}{\sqrt{3 - u^2}}.$$

Look at the yellow box at the top: We have $a = \sqrt{3}$.

7.6 Inverse Trigonometric

$$\int \frac{dx}{\sqrt{a^2 - x^2}} = \arcsin\left(\frac{x}{a}\right) + C$$



Example

Find $\int \frac{dx}{\sqrt{3 - 4x^2}}$.

First we do a substitution: Let $u = 2x$. Then

$$\int \frac{dx}{\sqrt{3 - 4x^2}} = \frac{1}{2} \int \frac{du}{\sqrt{3 - u^2}}.$$

Look at the yellow box at the top: We have $a = \sqrt{3}$. So

$$\begin{aligned}\int \frac{dx}{\sqrt{3 - 4x^2}} &= \frac{1}{2} \int \frac{du}{\sqrt{3 - u^2}} = \frac{1}{2} \arcsin\left(\frac{u}{a}\right) + C \\ &= \frac{1}{2} \arcsin\left(\frac{2x}{\sqrt{3}}\right) + C.\end{aligned}$$

7.6 Inverse Trigonometric Functions

$$\int \frac{dx}{|x|\sqrt{x^2 - a^2}} = \frac{1}{a} \operatorname{arcsec} \left| \frac{x}{a} \right| + C$$



Example

Find $\int \frac{dx}{\sqrt{e^{2x} - 6}}$.

Let $u = e^x$. Then $du = e^x dx = u dx$ and $dx = \frac{du}{u}$.

7.6 Inverse Trigonometric Functions

$$\int \frac{dx}{|x|\sqrt{x^2 - a^2}} = \frac{1}{a} \operatorname{arcsec} \left| \frac{x}{a} \right| + C$$



Example

Find $\int \frac{dx}{\sqrt{e^{2x} - 6}}$.

Let $u = e^x$. Then $du = e^x dx = u dx$ and $dx = \frac{du}{u}$. Therefore

$$\int \frac{dx}{\sqrt{e^{2x} - 6}} = \int \frac{\frac{du}{u}}{\sqrt{u^2 - 6}} = \int \frac{du}{u\sqrt{u^2 - 6}} \quad (a = \sqrt{6})$$

=

=

7.6 Inverse Trigonometric Functions

$$\int \frac{dx}{|x|\sqrt{x^2 - a^2}} = \frac{1}{a} \operatorname{arcsec} \left| \frac{x}{a} \right| + C$$



Example

Find $\int \frac{dx}{\sqrt{e^{2x} - 6}}$.

Let $u = e^x$. Then $du = e^x dx = u dx$ and $dx = \frac{du}{u}$. Therefore

$$\begin{aligned}\int \frac{dx}{\sqrt{e^{2x} - 6}} &= \int \frac{\frac{du}{u}}{\sqrt{u^2 - 6}} = \int \frac{du}{u\sqrt{u^2 - 6}} \quad (a = \sqrt{6}) \\ &= \frac{1}{\sqrt{6}} \operatorname{arcsec} \left| \frac{u}{\sqrt{6}} \right| + C\end{aligned}$$

=

7.6 Inverse Trigonometric Functions

$$\int \frac{dx}{|x|\sqrt{x^2 - a^2}} = \frac{1}{a} \operatorname{arcsec} \left| \frac{x}{a} \right| + C$$



Example

Find $\int \frac{dx}{\sqrt{e^{2x} - 6}}$.

Let $u = e^x$. Then $du = e^x dx = u dx$ and $dx = \frac{du}{u}$. Therefore

$$\begin{aligned}\int \frac{dx}{\sqrt{e^{2x} - 6}} &= \int \frac{\frac{du}{u}}{\sqrt{u^2 - 6}} = \int \frac{du}{u\sqrt{u^2 - 6}} \quad (a = \sqrt{6}) \\ &= \frac{1}{\sqrt{6}} \operatorname{arcsec} \left| \frac{u}{\sqrt{6}} \right| + C \\ &= \frac{1}{\sqrt{6}} \operatorname{arcsec} \left| \frac{e^x}{\sqrt{6}} \right| + C.\end{aligned}$$

7.6 Inverse Trigonometric

$$\int \frac{dx}{\sqrt{a^2 - x^2}} = \arcsin\left(\frac{x}{a}\right) + C$$



Example

Find $\int \frac{dx}{\sqrt{4x - x^2}}$.

Since $\sqrt{4x - x^2}$ doesn't match any of these three integration formulae, we must first rewrite this.

7.6 Inverse Trigonometric

$$\int \frac{dx}{\sqrt{a^2 - x^2}} = \arcsin\left(\frac{x}{a}\right) + C$$



Example

Find $\int \frac{dx}{\sqrt{4x - x^2}}$.

Since $\sqrt{4x - x^2}$ doesn't match any of these three integration formulae, we must first rewrite this.

$$x^2 - 4x = x^2 - 4x + 4 - 4 = x^2 - 4x + 4 - 4 = (x - 2)^2 - 4.$$

7.6 Inverse Trigonometric

$$\int \frac{dx}{\sqrt{a^2 - x^2}} = \arcsin\left(\frac{x}{a}\right) + C$$



Example

Find $\int \frac{dx}{\sqrt{4x - x^2}}$.

Since $\sqrt{4x - x^2}$ doesn't match any of these three integration formulae, we must first rewrite this.

$$x^2 - 4x = x^2 - 4x + 4 - 4 = x^2 - 4x + 4 - 4 = (x - 2)^2 - 4.$$

So then we have

$$\int \frac{dx}{\sqrt{4x - x^2}} = \int \frac{dx}{\sqrt{4 - (x - 2)^2}}$$

=

=

7.6 Inverse Trigonometric

$$\int \frac{dx}{\sqrt{a^2 - x^2}} = \arcsin\left(\frac{x}{a}\right) + C$$



Example

Find $\int \frac{dx}{\sqrt{4x - x^2}}$.

Since $\sqrt{4x - x^2}$ doesn't match any of these three integration formulae, we must first rewrite this.

$$x^2 - 4x = x^2 - 4x + 4 - 4 = x^2 - 4x + 4 - 4 = (x - 2)^2 - 4.$$

So then we have

$$\begin{aligned} \int \frac{dx}{\sqrt{4x - x^2}} &= \int \frac{dx}{\sqrt{4 - (x - 2)^2}} \\ &= \int \frac{du}{\sqrt{a^2 - u^2}} \quad (u = x - 2, \ a = 2) \\ &= \dots \end{aligned}$$

7.6 Inverse Trigonometric

$$\int \frac{dx}{a^2 + x^2} = \frac{1}{a} \arctan\left(\frac{x}{a}\right) + C$$



Example

Find $\int \frac{dx}{4x^2 + 4x + 2}$.

Again we need to start by completing the square.

7.6 Inverse Trigonometric

$$\int \frac{dx}{a^2 + x^2} = \frac{1}{a} \arctan\left(\frac{x}{a}\right) +$$



Example

Find $\int \frac{dx}{4x^2 + 4x + 2}$.

Again we need to start by completing the square.

$$\begin{aligned}4x^2 + 4x + 2 &= 4(x^2 + x) + 2 = 4\left(x^2 + x + \frac{1}{4} - \frac{1}{4}\right) + 2 \\&= 4\left(x^2 + x + \frac{1}{4}\right) + 1 = 4\left(x + \frac{1}{2}\right)^2 + 1 \\&= (2x + 1)^2 + 1.\end{aligned}$$

7.6 Inverse Trigonometric

$$\int \frac{dx}{a^2 + x^2} = \frac{1}{a} \arctan\left(\frac{x}{a}\right) +$$



So we need to calculate

$$\int \frac{dx}{(2x+1)^2 + 1}$$

What does that look like?

7.6 Inverse Trigonometric

$$\int \frac{dx}{a^2 + x^2} = \frac{1}{a} \arctan\left(\frac{x}{a}\right) +$$



So we need to calculate

$$\int \frac{dx}{(2x+1)^2 + 1}$$

What does that look like? Let $a = 1$ and $u = (2x+1)$. Then we have

$$\int \frac{dx}{(2x+1)^2 + 1} = \frac{1}{2} \int \frac{du}{u^2 + a^2}$$

=

=

7.6 Inverse Trigonometric

$$\int \frac{dx}{a^2 + x^2} = \frac{1}{a} \arctan\left(\frac{x}{a}\right) + C$$



So we need to calculate

$$\int \frac{dx}{(2x+1)^2 + 1}$$

What does that look like? Let $a = 1$ and $u = (2x+1)$. Then we have

$$\begin{aligned}\int \frac{dx}{(2x+1)^2 + 1} &= \frac{1}{2} \int \frac{du}{u^2 + a^2} \\ &= \frac{1}{2} \frac{1}{a} \arctan\left(\frac{u}{a}\right) + C \\ &= \end{aligned}$$

7.6 Inverse Trigonometric

$$\int \frac{dx}{a^2 + x^2} = \frac{1}{a} \arctan\left(\frac{x}{a}\right) + C$$



So we need to calculate

$$\int \frac{dx}{(2x+1)^2 + 1}$$

What does that look like? Let $a = 1$ and $u = (2x+1)$. Then we have

$$\begin{aligned}\int \frac{dx}{(2x+1)^2 + 1} &= \frac{1}{2} \int \frac{du}{u^2 + a^2} \\&= \frac{1}{2} \frac{1}{a} \arctan\left(\frac{u}{a}\right) + C \\&= \frac{1}{2} \arctan(2x+1) + C.\end{aligned}$$



Hyperbolic Functions

7.7 Hyperbolic Functions

$$\int \frac{dx}{a^2 + x^2} = \frac{1}{a} \arctan\left(\frac{x}{a}\right) + C$$



The hyperbolic sine and hyperbolic cosine functions are defined by

$$\sinh x = \frac{e^x - e^{-x}}{2}$$

and

$$\cosh x = \frac{e^x + e^{-x}}{2}$$

7.7 Hyperbolic Functions

$$\int \frac{dx}{a^2 + x^2} = \frac{1}{a} \arctan\left(\frac{x}{a}\right) + C$$



The hyperbolic sine and hyperbolic cosine functions are defined by

$$\sinh x = \frac{e^x - e^{-x}}{2}$$

and

$$\cosh x = \frac{e^x + e^{-x}}{2}$$

If you forget which is which, try to remember

$$\sinh 0 = 0 = \sin 0$$

and

$$\cosh 0 = 1 = \cos 0.$$

7.7 Hyperbolic Functions

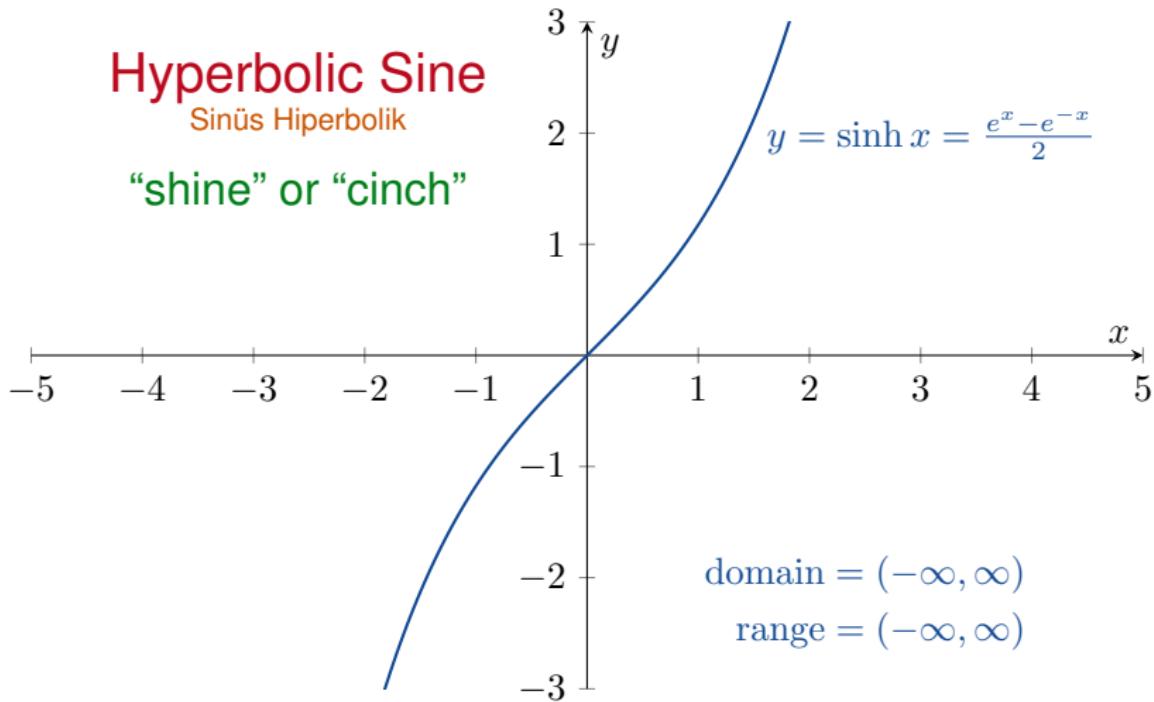
$$\int \frac{dx}{a^2 + x^2} = \frac{1}{a} \arctan\left(\frac{x}{a}\right) + C$$



Hyperbolic Sine

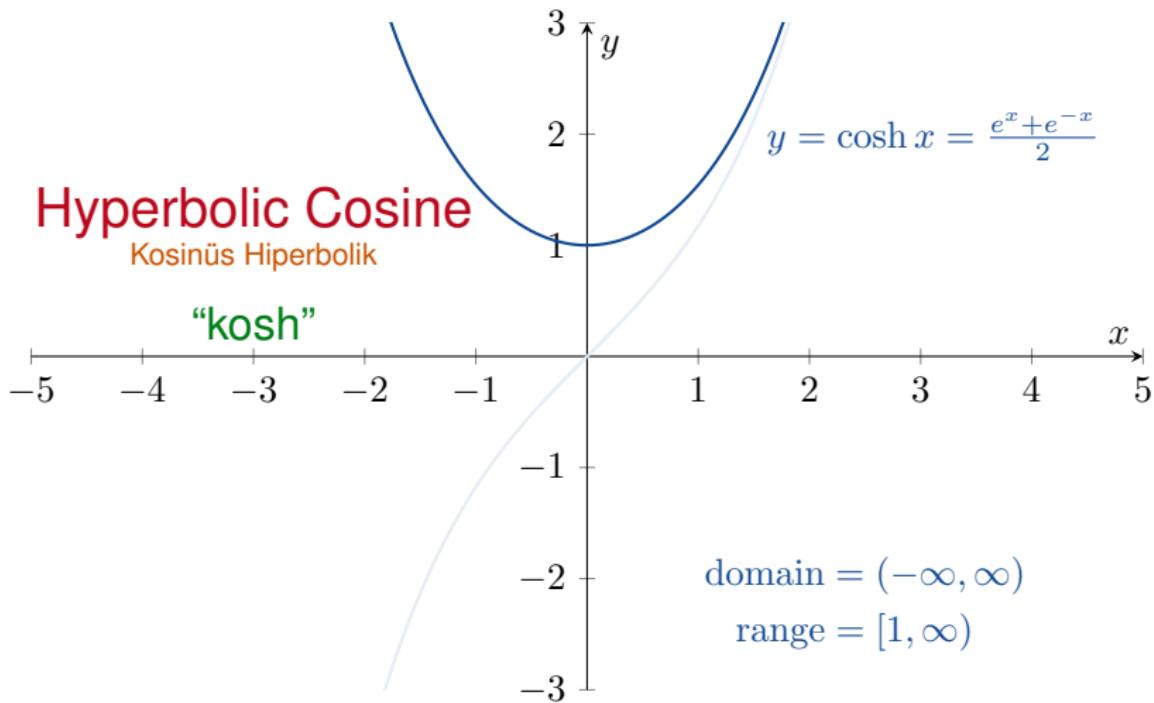
Sinüs Hiperbolik

“shine” or “cinch”



7.7 Hyperbolic Functions

$$\int \frac{dx}{a^2 + x^2} = \frac{1}{a} \arctan\left(\frac{x}{a}\right) + C$$



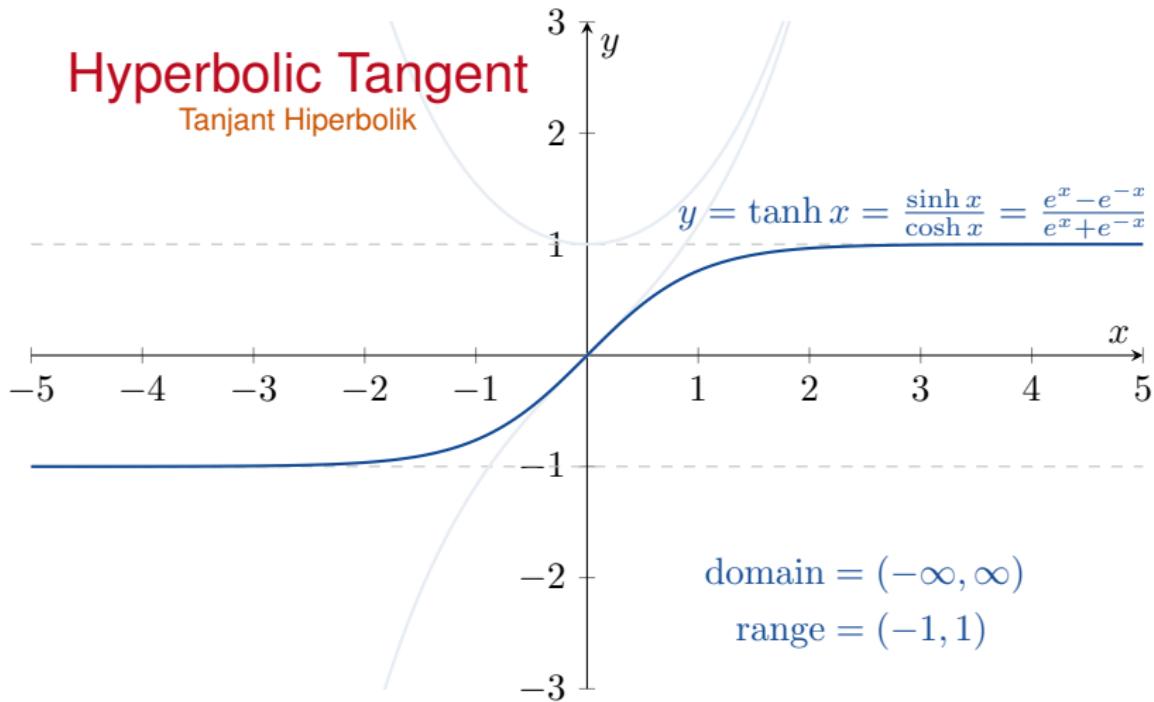
7.7 Hyperbolic Functions

$$\int \frac{dx}{a^2 + x^2} = \frac{1}{a} \arctan\left(\frac{x}{a}\right) + C$$



Hyperbolic Tangent

Tanjant Hiperbolik



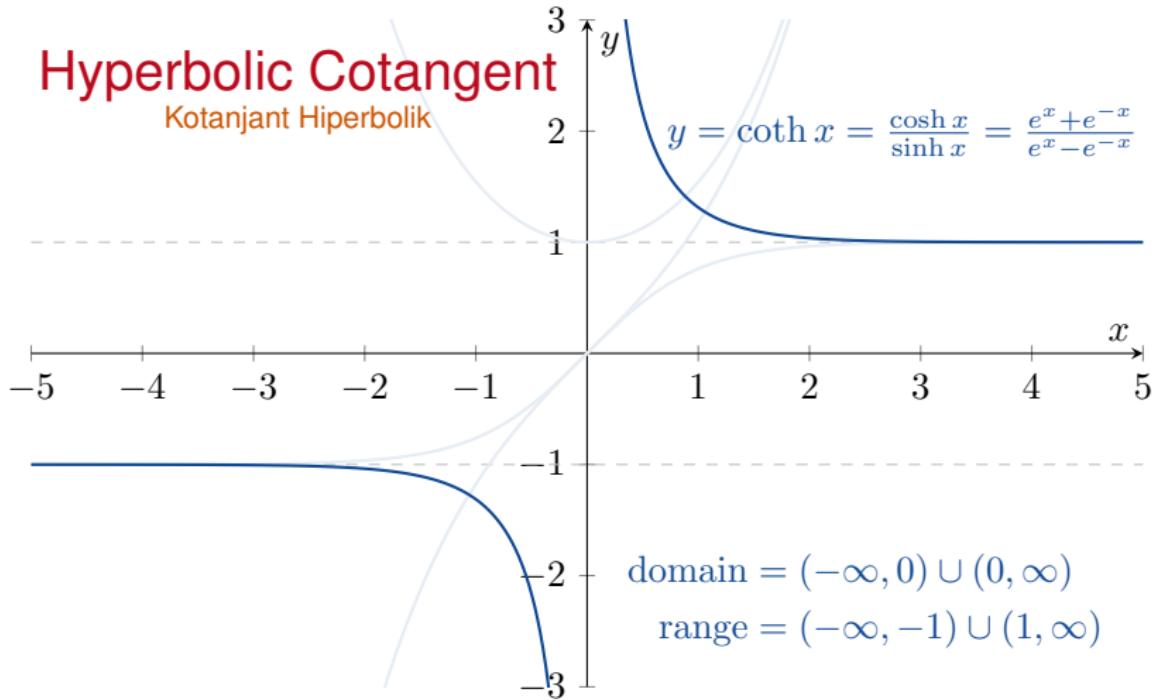
7.7 Hyperbolic Functions

$$\int \frac{dx}{a^2 + x^2} = \frac{1}{a} \arctan\left(\frac{x}{a}\right) + C$$



Hyperbolic Cotangent

Kotanjant Hiperbolik

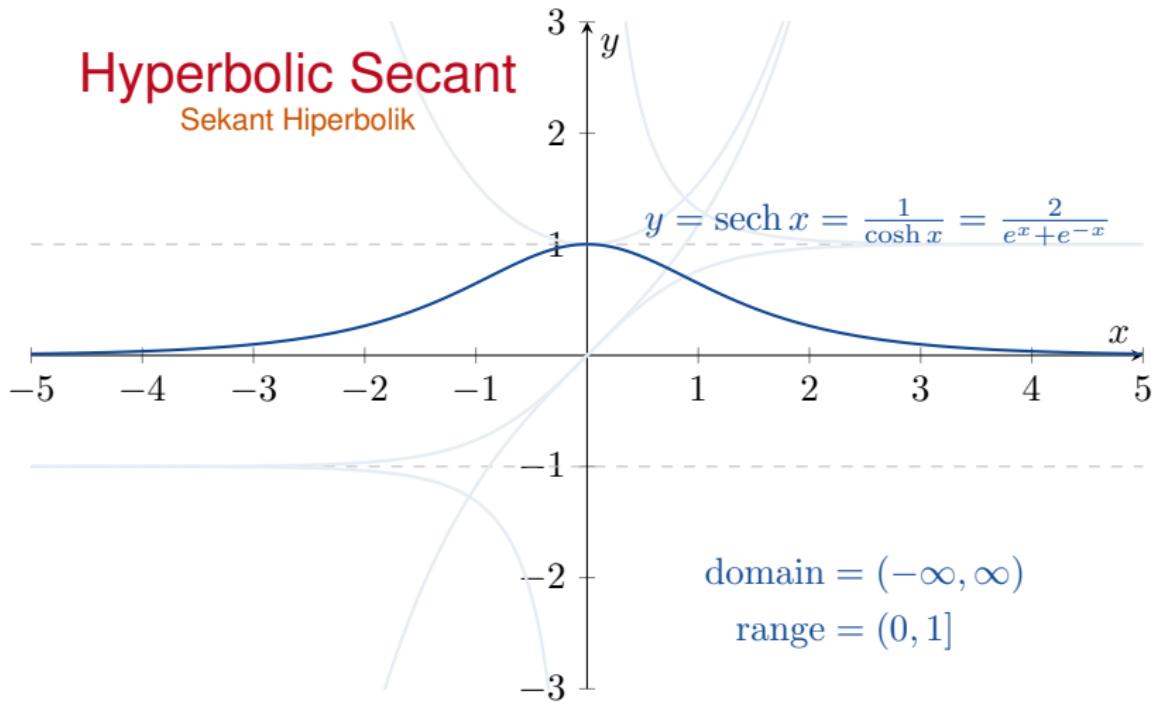


7.7 Hyperbolic Functions

$$\int \frac{dx}{a^2 + x^2} = \frac{1}{a} \arctan\left(\frac{x}{a}\right) + C$$



Hyperbolic Secant Sekant Hiperbolik

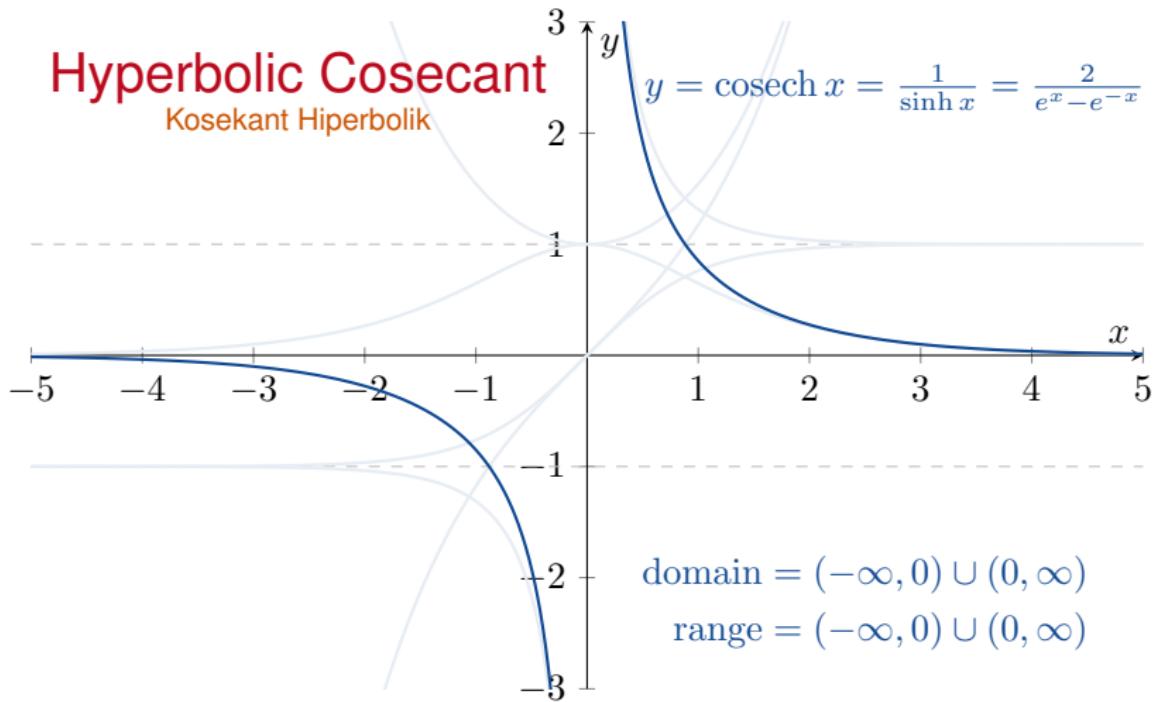


7.7 Hyperbolic Functions

$$\int \frac{dx}{a^2 + x^2} = \frac{1}{a} \arctan\left(\frac{x}{a}\right) + C$$



Hyperbolic Cosecant Kosekant Hiperbolik



7.7 Hyperbolic Functions

$$\int \frac{dx}{a^2 + x^2} = \frac{1}{a} \arctan\left(\frac{x}{a}\right) + C$$



Identities

$$\cosh^2 x - \sinh^2 u = \left(\frac{e^x + e^{-x}}{2} \right)^2 - \left(\frac{e^x - e^{-x}}{2} \right)^2$$
$$=$$

7.7 Hyperbolic Functions

$$\int \frac{dx}{a^2 + x^2} = \frac{1}{a} \arctan\left(\frac{x}{a}\right) + C$$



Identities

$$\begin{aligned}\cosh^2 x - \sinh^2 u &= \left(\frac{e^x + e^{-x}}{2}\right)^2 - \left(\frac{e^x - e^{-x}}{2}\right)^2 \\ &= \frac{1}{4} (e^{2x} + 2 + e^{-2x} - e^{2x} + 2 - e^{-2x}) = 1.\end{aligned}$$

TABLE 7.6 Identities for hyperbolic functions

$$\cosh^2 x - \sinh^2 x = 1$$

$$\sinh 2x = 2 \sinh x \cosh x$$

$$\cosh 2x = \cosh^2 x + \sinh^2 x$$

$$\cosh^2 x = \frac{\cosh 2x + 1}{2}$$

$$\sinh^2 x = \frac{\cosh 2x - 1}{2}$$

$$\tanh^2 x = 1 - \operatorname{sech}^2 x$$

$$\coth^2 x = 1 + \operatorname{csch}^2 x$$

$$\int \frac{dx}{a^2 + x^2} = \frac{1}{a} \arctan\left(\frac{x}{a}\right) + C$$



Derivatives and Integrals of Hyperbolic Functions

We can calculate that

$$\frac{d}{dx} \sinh x =$$

and

$$\frac{d}{dx} \operatorname{cosech} x =$$

$$\int \frac{dx}{a^2 + x^2} = \frac{1}{a} \arctan\left(\frac{x}{a}\right) + C$$



Derivatives and Integrals of Hyperbolic Functions

We can calculate that

$$\frac{d}{dx} \sinh x = \frac{d}{dx} \left(\frac{e^x - e^{-x}}{2} \right)$$

and

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$$\int \frac{dx}{a^2 + x^2} = \frac{1}{a} \arctan\left(\frac{x}{a}\right) + C$$



Derivatives and Integrals of Hyperbolic Functions

We can calculate that

$$\frac{d}{dx} \sinh x = \frac{d}{dx} \left(\frac{e^x - e^{-x}}{2} \right) = \frac{e^x + e^{-x}}{2} = \cosh x$$

and

$$\frac{d}{dx} \operatorname{cosech} x =$$

$$\int \frac{dx}{a^2 + x^2} = \frac{1}{a} \arctan\left(\frac{x}{a}\right) + C$$



Derivatives and Integrals of Hyperbolic Functions

We can calculate that

$$\frac{d}{dx} \sinh x = \frac{d}{dx} \left(\frac{e^x - e^{-x}}{2} \right) = \frac{e^x + e^{-x}}{2} = \cosh x$$

and

$$\frac{d}{dx} \operatorname{cosech} x = \frac{d}{dx} \left(\frac{1}{\sinh x} \right)$$

$$\int \frac{dx}{a^2 + x^2} = \frac{1}{a} \arctan\left(\frac{x}{a}\right) +$$



Derivatives and Integrals of Hyperbolic Functions

We can calculate that

$$\frac{d}{dx} \sinh x = \frac{d}{dx} \left(\frac{e^x - e^{-x}}{2} \right) = \frac{e^x + e^{-x}}{2} = \cosh x$$

and

$$\frac{d}{dx} \operatorname{cosech} x = \frac{d}{dx} \left(\frac{1}{\sinh x} \right) = -\frac{\cosh x}{\sinh^2 x}$$

$$\int \frac{dx}{a^2 + x^2} = \frac{1}{a} \arctan\left(\frac{x}{a}\right) +$$



Derivatives and Integrals of Hyperbolic Functions

We can calculate that

$$\frac{d}{dx} \sinh x = \frac{d}{dx} \left(\frac{e^x - e^{-x}}{2} \right) = \frac{e^x + e^{-x}}{2} = \cosh x$$

and

$$\begin{aligned} \frac{d}{dx} \operatorname{cosech} x &= \frac{d}{dx} \left(\frac{1}{\sinh x} \right) = -\frac{\cosh x}{\sinh^2 x} = -\frac{1}{\sinh x} \frac{\cosh x}{\sinh x} \\ &= \operatorname{cosech} x \coth x. \end{aligned}$$

7.7 Hyperbolic Functions

$$\int \frac{dx}{a^2 + x^2} = \frac{1}{a} \arctan\left(\frac{x}{a}\right) + C$$



Derivative Formulae

- $\frac{d}{dx} \sinh x = \cosh x$
- $\frac{d}{dx} \cosh x = \sinh x$
- $\frac{d}{dx} \tanh x = \operatorname{sech}^2 x$
- $\frac{d}{dx} \coth x = -\operatorname{cosech}^2 x$
- $\frac{d}{dx} \operatorname{sech} x = -\operatorname{sech} x \tanh x$
- $\frac{d}{dx} \operatorname{cosech} x = -\operatorname{cosech} x \coth x$

7.7 Hyperbolic Functions

$$\int \frac{dx}{a^2 + x^2} = \frac{1}{a} \arctan\left(\frac{x}{a}\right) + C$$



Derivative Formulae

- $\frac{d}{dx} \sinh x = \cosh x$
- $\frac{d}{dx} \cosh x = \sinh x$
- $\frac{d}{dx} \tanh x = \operatorname{sech}^2 x$
- $\frac{d}{dx} \coth x = -\operatorname{cosech}^2 x$
- $\frac{d}{dx} \operatorname{sech} x = -\operatorname{sech} x \tanh x$
- $\frac{d}{dx} \operatorname{cosech} x = -\operatorname{cosech} x \coth x$
- $\frac{d}{dx} \sin x = \cos x$
- $\frac{d}{dx} \cos x = -\sin x$
- $\frac{d}{dx} \tan x = \sec^2 x$
- $\frac{d}{dx} \cot x = -\operatorname{cosec}^2 x$
- $\frac{d}{dx} \sec x = +\sec x \tan x$
- $\frac{d}{dx} \operatorname{cosec} x = -\operatorname{cosec} x \cot x$

7.7 Hyperbolic Functions

$$\int \frac{dx}{a^2 + x^2} = \frac{1}{a} \arctan\left(\frac{x}{a}\right) + C$$



Derivative Formulae

- $\frac{d}{dx} \sinh x = \cosh x$
- $\frac{d}{dx} \cosh x = \sinh x$
- $\frac{d}{dx} \tanh x = \operatorname{sech}^2 x$
- $\frac{d}{dx} \coth x = -\operatorname{cosech}^2 x$
- $\frac{d}{dx} \operatorname{sech} x = -\operatorname{sech} x \tanh x$
- $\frac{d}{dx} \operatorname{cosech} x = -\operatorname{cosech} x \coth x$

Integral Formulae

- $\int \sinh x \, dx = \cosh x + C$
- $\int \cosh x \, dx = \sinh x + C$
- $\int \operatorname{sech}^2 x \, dx = \tanh x + C$
- $\int \operatorname{cosech}^2 x \, dx = -\coth x + C$
- $\int \operatorname{sech} x \tanh x \, dx = -\operatorname{sech} x + C$
- $\int \operatorname{cosech} x \coth x \, dx = -\operatorname{cosech} x + C$

7.7 Hyperbolic Functions

$$\frac{d}{dx} \tanh x = \operatorname{sech}^2 x$$



Example

Differentiate $\tanh \sqrt{1 + t^2}$.

7.7 Hyperbolic Functions

$$\frac{d}{dx} \tanh x = \operatorname{sech}^2 x$$



Example

Differentiate $\tanh \sqrt{1 + t^2}$.

$$\begin{aligned}\frac{d}{dt} \tanh \sqrt{1 + t^2} &= \operatorname{sech}^2 \sqrt{1 + t^2} \frac{d}{dt} \sqrt{1 + t^2} \\ &= \frac{t}{\sqrt{1 + t^2}} \operatorname{sech}^2 \sqrt{1 + t^2}.\end{aligned}$$

$$\begin{aligned}\text{(b)} \quad \int \coth 5x \, dx &= \int \frac{\cosh 5x}{\sinh 5x} \, dx = \frac{1}{5} \int \frac{du}{u} && u = \sinh 5x, \\ &= \frac{1}{5} \ln |u| + C = \frac{1}{5} \ln |\sinh 5x| + C && du = 5 \cosh 5x \, dx\end{aligned}$$

$$\begin{aligned} \text{(c)} \quad \int_0^1 \sinh^2 x \, dx &= \int_0^1 \frac{\cosh 2x - 1}{2} \, dx \\ &= \frac{1}{2} \int_0^1 (\cosh 2x - 1) \, dx = \frac{1}{2} \left[\frac{\sinh 2x}{2} - x \right]_0^1 \\ &= \frac{\sinh 2}{4} - \frac{1}{2} \approx 0.40672 \end{aligned}$$

$$\begin{aligned} \text{(d)} \quad \int_0^{\ln 2} 4e^x \sinh x \, dx &= \int_0^{\ln 2} 4e^x \frac{e^x - e^{-x}}{2} \, dx = \int_0^{\ln 2} (2e^{2x} - 2) \, dx \\ &= \left[e^{2x} - 2x \right]_0^{\ln 2} = (e^{2 \ln 2} - 2 \ln 2) - (1 - 0) \\ &= 4 - 2 \ln 2 - 1 \approx 1.6137 \end{aligned}$$

7.7 Hyperbolic Functions

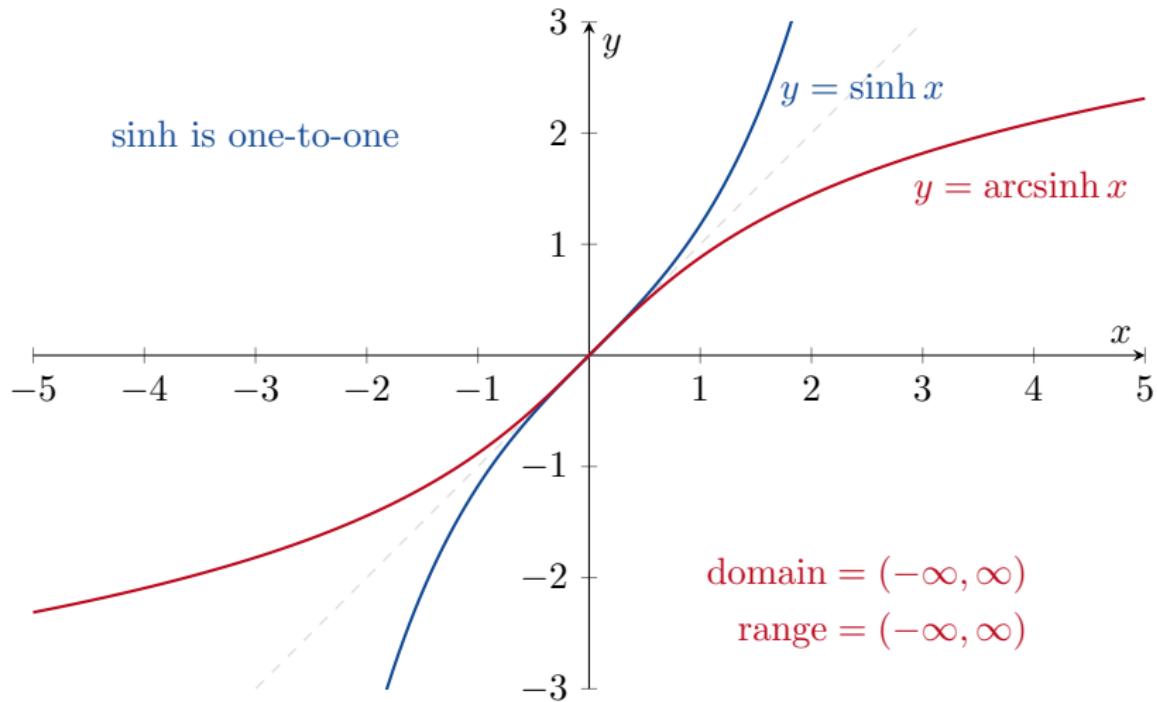


Inverse Hyperbolic Functions

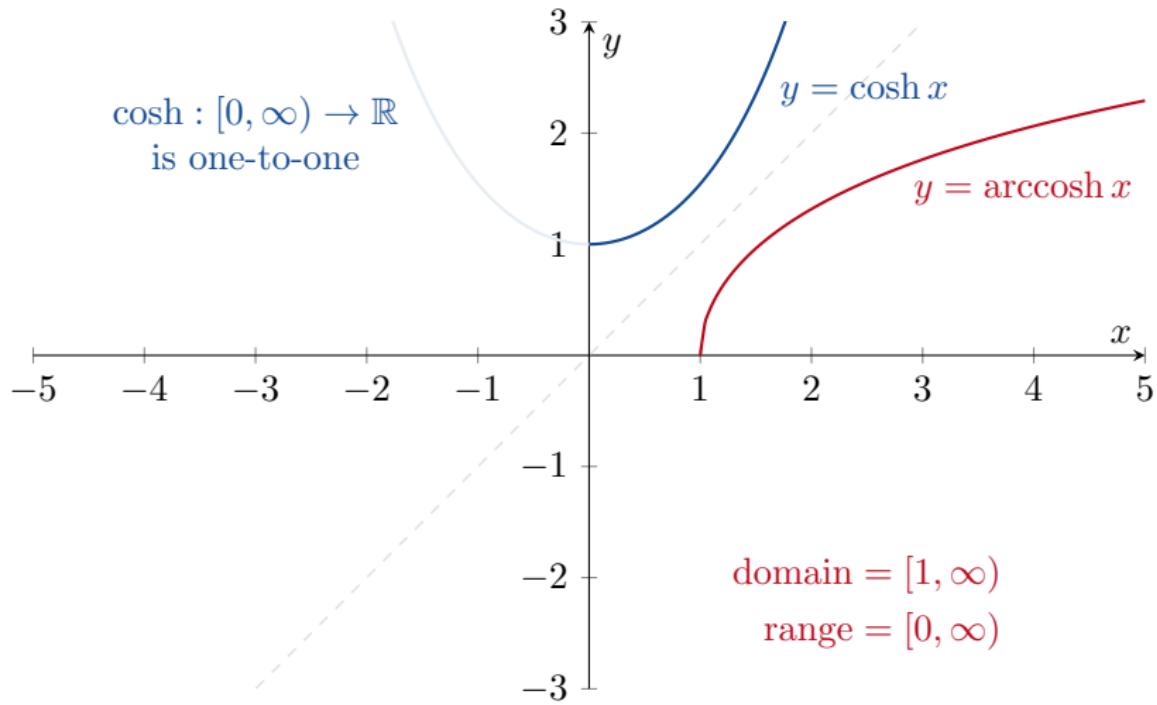
Now it is time to talk about the inverse functions.

\sinh , \tanh , \coth and cosech are all one-to-one functions so have inverses. For \cosh and sech we will need to restrict the domain before we can find the inverse.

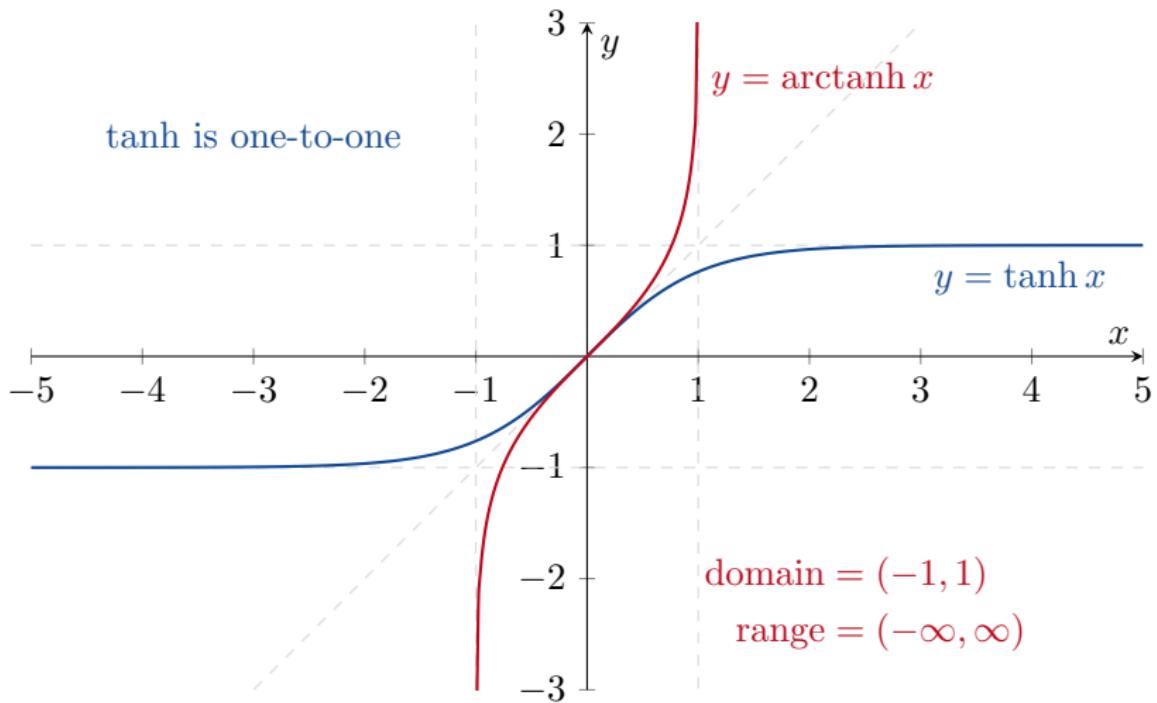
7.7 Hyperbolic Functions



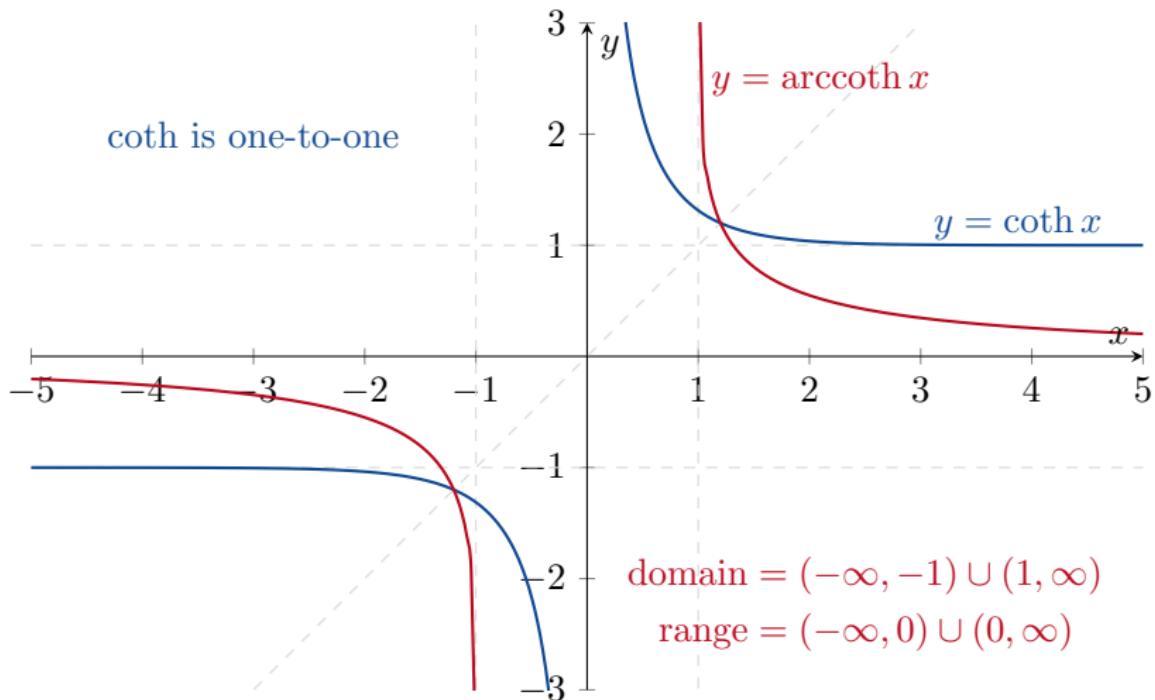
7.7 Hyperbolic Functions



7.7 Hyperbolic Functions

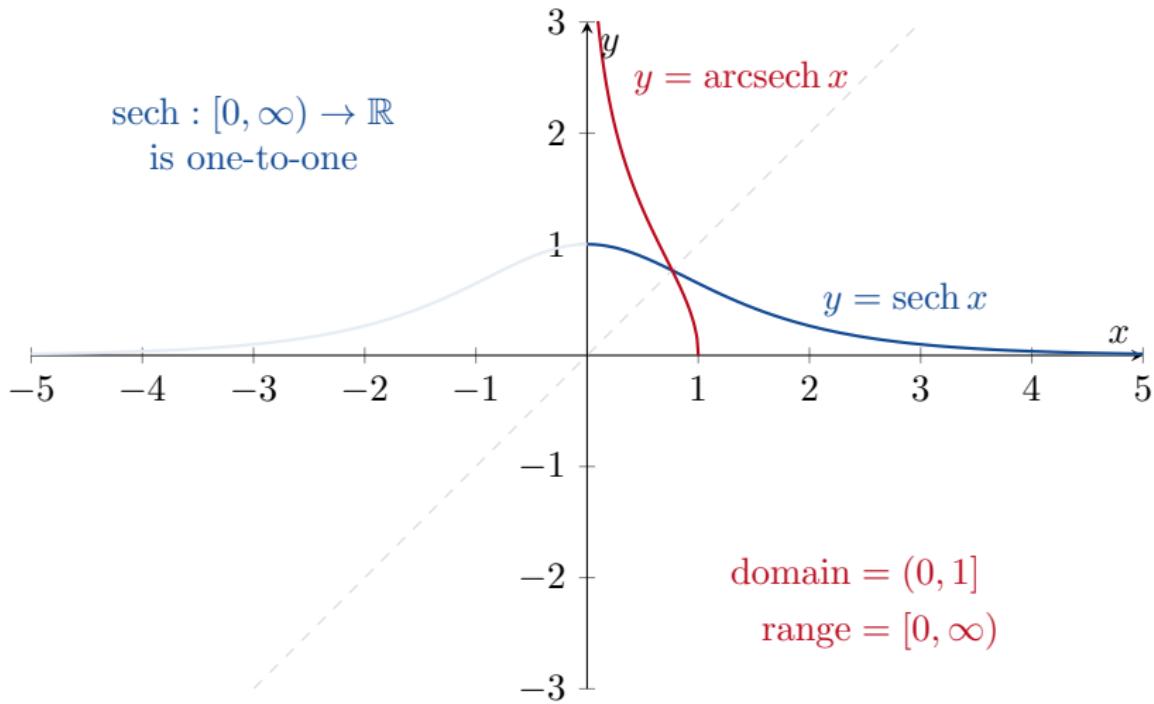


7.7 Hyperbolic Functions



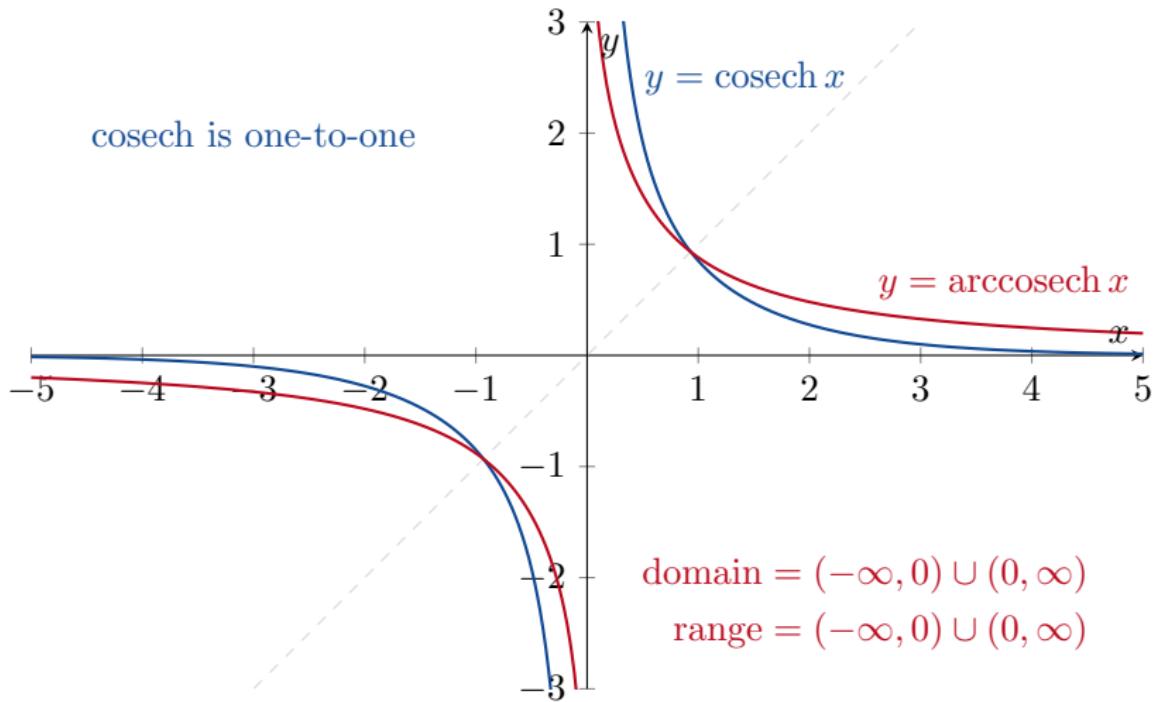
7.7 Hyperbolic Functions

$\operatorname{sech} : [0, \infty) \rightarrow \mathbb{R}$
is one-to-one



7.7 Hyperbolic Functions

cosech is one-to-one



Useful Identities

Note that

$$\operatorname{sech} \left(\cosh^{-1} \left(\frac{1}{x} \right) \right) = \frac{1}{\cosh \left(\cosh^{-1} \left(\frac{1}{x} \right) \right)} = \frac{1}{\left(\frac{1}{x} \right)} = x.$$

Useful Identities

Note that

$$\operatorname{sech} \left(\cosh^{-1} \left(\frac{1}{x} \right) \right) = \frac{1}{\cosh \left(\cosh^{-1} \left(\frac{1}{x} \right) \right)} = \frac{1}{\left(\frac{1}{x} \right)} = x.$$

Taking sech^{-1} of both sides gives

$$\boxed{\cosh^{-1} \left(\frac{1}{x} \right) = \operatorname{sech}^{-1} x.}$$

7.7 Hyperbolic Functions



Useful Identities

Note that

$$\operatorname{sech} \left(\cosh^{-1} \left(\frac{1}{x} \right) \right) = \frac{1}{\cosh \left(\cosh^{-1} \left(\frac{1}{x} \right) \right)} = \frac{1}{\left(\frac{1}{x} \right)} = x.$$

Taking sech^{-1} of both sides gives

$$\boxed{\cosh^{-1} \left(\frac{1}{x} \right) = \operatorname{sech}^{-1} x.}$$

Similarly

$$\boxed{\operatorname{cosech}^{-1} x = \sinh^{-1} \left(\frac{1}{x} \right)}$$

and

$$\boxed{\coth^{-1} x = \tanh^{-1} \left(\frac{1}{x} \right).}$$

7.7 Hyperbolic Functions



(This one is not in the textbook)

Next I want to find a formula for $\text{arcsinh } x = \sinh^{-1} x$.

7.7 Hyperbolic Functions



(This one is not in the textbook)

Next I want to find a formula for $\text{arcsinh } x = \sinh^{-1} x$.

$$\sinh^{-1} x = y$$

7.7 Hyperbolic Functions



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$$\sinh^{-1} x = y$$

$$x = \sinh y$$

7.7 Hyperbolic Functions



(This one is not in the textbook)

Next I want to find a formula for $\text{arcsinh } x = \sinh^{-1} x$.

$$\sinh^{-1} x = y$$

$$x = \sinh y$$

$$x = \frac{e^y - e^{-y}}{2}$$

7.7 Hyperbolic Functions



(This one is not in the textbook)

Next I want to find a formula for $\text{arcsinh } x = \sinh^{-1} x$.

$$\sinh^{-1} x = y$$

$$x = \sinh y$$

$$x = \frac{e^y - e^{-y}}{2}$$

$$2x = e^y - e^{-y}$$

7.7 Hyperbolic Functions



(This one is not in the textbook)

Next I want to find a formula for $\text{arcsinh } x = \sinh^{-1} x$.

$$\sinh^{-1} x = y$$

$$x = \sinh y$$

$$x = \frac{e^y - e^{-y}}{2}$$

$$2x = e^y - e^{-y}$$

$$2xe^y = (e^y)^2 - 1$$

7.7 Hyperbolic Functions



(This one is not in the textbook)

Next I want to find a formula for $\operatorname{arcsinh} x = \sinh^{-1} x$.

$$\sinh^{-1} x = y$$

$$x = \sinh y$$

$$x = \frac{e^y - e^{-y}}{2}$$

$$2x = e^y - e^{-y}$$

$$2xe^y = (e^y)^2 - 1$$

$$0 = (e^y)^2 - 2xe^y - 1.$$

This is a quadratic equation for e^y .

7.7 Hyperbolic Functions



$$\sinh^{-1} x = y \quad (e^y)^2 - 2xe^y - 1 = 0$$

We know that the solution to

$$ar^2 + br + c = 0$$

is given by

$$r = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

7.7 Hyperbolic Functions



$$\sinh^{-1} x = y \quad (e^y)^2 - 2xe^y - 1 = 0$$

We know that the solution to

$$ar^2 + br + c = 0$$

is given by

$$r = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

Therefore

$$e^y = \frac{2x \pm \sqrt{(-2x)^2 + 4}}{2}$$

7.7 Hyperbolic Functions



$$\sinh^{-1} x = y \quad (e^y)^2 - 2xe^y - 1 = 0$$

We know that the solution to

$$ar^2 + br + c = 0$$

is given by

$$r = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

Therefore

$$e^y = \frac{2x \pm \sqrt{(-2x)^2 + 4}}{2} = x \pm \sqrt{x^2 + 1}.$$

But do we want “+” or “−” here?

7.7 Hyperbolic Functions



$$\sinh^{-1} x = y \quad (e^y)^2 - 2xe^y - 1 = 0$$

We know that the solution to

$$ar^2 + br + c = 0$$

is given by

$$r = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

Therefore

$$e^y = \frac{2x \pm \sqrt{(-2x)^2 + 4}}{2} = x \pm \sqrt{x^2 + 1}.$$

But do we want “+” or “−” here? Remember that e^y is always positive. So we must have “+” here.

7.7 Hyperbolic Functions



$$\sinh^{-1} x = y$$

To finish, we take the natural logarithm of

$$e^y = x + \sqrt{x^2 + 1}$$

to obtain

$$\boxed{\sinh^{-1} x = y = \ln \left(x + \sqrt{x^2 + 1} \right)}$$

7.7 Hyperbolic Functions



$$\sinh^{-1} x = y$$

To finish, we take the natural logarithm of

$$e^y = x + \sqrt{x^2 + 1}$$

to obtain

$$\boxed{\sinh^{-1} x = y = \ln \left(x + \sqrt{x^2 + 1} \right)}$$

Similarly

$$\boxed{\cosh^{-1} x = \ln \left(x + \sqrt{x^2 - 1} \right)}$$

but I leave that for you to prove.

$$(f^{-1})'(x) = \frac{1}{f'(f^{-1}(x))}$$



Derivatives of Inverse Hyperbolic Functions

We will use the formula in the yellow box with $f(x) = \cosh x$ and $f^{-1}(x) = \text{arccosh } x = \cosh^{-1} x$.

7.7 Hyperbolic Functions

$$(f^{-1})'(x) = \frac{1}{f'(f^{-1}(x))}$$



Derivatives of Inverse Hyperbolic Functions

We will use the formula in the yellow box with $f(x) = \cosh x$ and $f^{-1}(x) = \text{arccosh } x = \cosh^{-1} x$. Since $\cosh^2 u - \sinh^2 u = 1$, we have that

$$\begin{aligned}(\cosh^{-1})'(x) &= \frac{1}{f'(f^{-1}(x))} = \frac{1}{\sinh(\text{arccosh } x)} \\&= \frac{1}{\sqrt{\cosh^2(\text{arccosh } x) - 1}} = \frac{1}{\sqrt{x^2 - 1}}.\end{aligned}$$

The other five are similar.

7.7 Hyperbolic Functions

Derivative Formulae

- $\frac{d}{dx} \operatorname{arcsinh} x = \frac{1}{\sqrt{1+x^2}}$
- $\frac{d}{dx} \operatorname{arccosh} x = \frac{1}{\sqrt{x^2-1}}, x > 1$
- $\frac{d}{dx} \operatorname{arctanh} x = \frac{1}{1-x^2}, |x| < 1$
- $\frac{d}{dx} \operatorname{arccoth} x = \frac{1}{1-x^2}, |x| > 1$
- $\frac{d}{dx} \operatorname{sech}^{-1} x = -\frac{1}{x\sqrt{1-x^2}}, 0 < x < 1$
- $\frac{d}{dx} \operatorname{cosech}^{-1} x = -\frac{1}{|x|\sqrt{1+x^2}}, x \neq 0$

7.7 Hyperbolic Functions

Derivative Formulae

- $\frac{d}{dx} \operatorname{arcsinh} x = \frac{1}{\sqrt{1+x^2}}$
- $\frac{d}{dx} \operatorname{arccosh} x = \frac{1}{\sqrt{x^2-1}}, \quad x > 1$
- $\frac{d}{dx} \operatorname{arctanh} x = \frac{1}{1-x^2}, \quad |x| < 1$
- $\frac{d}{dx} \operatorname{arccoth} x = \frac{1}{1-x^2}, \quad |x| > 1$
- $\frac{d}{dx} \operatorname{sech}^{-1} x = -\frac{1}{x\sqrt{1-x^2}}, \quad 0 < x < 1$
- $\frac{d}{dx} \operatorname{cosech}^{-1} x = -\frac{1}{|x|\sqrt{1+x^2}}, \quad x \neq 0$

Integral Formulae

- $\int \frac{dx}{\sqrt{a^2+x^2}} = \sinh^{-1}\left(\frac{x}{a}\right) + C, \quad a > 0$
- $\int \frac{dx}{\sqrt{x^2-a^2}} = \cosh^{-1}\left(\frac{x}{a}\right) + C, \quad x > a > 0$
- $\int \frac{dx}{a^2-x^2} = \begin{cases} \frac{1}{a} \tanh^{-1}\left(\frac{x}{a}\right) + C, & x^2 < a^2 \\ \frac{1}{a} \coth^{-1}\left(\frac{x}{a}\right) + C, & x^2 > a^2 \end{cases}$
- $\int \frac{dx}{x\sqrt{a^2-x^2}} = -\frac{1}{a} \operatorname{sech}^{-1}\left(\frac{x}{a}\right) + C, \quad 0 < x < a$
- $\int \frac{dx}{x\sqrt{a^2+x^2}} = -\frac{1}{a} \operatorname{cosech}^{-1}\left|\frac{x}{a}\right| + C, \quad x \neq 0 \text{ and } a > 0$

7.7 Hyperbolic Functions

$$\int \frac{dx}{\sqrt{a^2 + x^2}} = \sinh^{-1} \left(\frac{x}{a} \right) + C$$



Example (Final example of this course)

Calculate $\int_0^1 \frac{2 dx}{\sqrt{3 + 4x^2}}$.

7.7 Hyperbolic Functions

$$\int \frac{dx}{\sqrt{a^2 + x^2}} = \sinh^{-1} \left(\frac{x}{a} \right) + C$$



Example (Final example of this course)

Calculate $\int_0^1 \frac{2 dx}{\sqrt{3 + 4x^2}}$.

We will use the substitution $u = 2x$ and the formula in the yellow box with $a = \sqrt{3}$.

7.7 Hyperbolic Functions

$$\int \frac{dx}{\sqrt{a^2 + x^2}} = \sinh^{-1} \left(\frac{x}{a} \right) + C$$



Example (Final example of this course)

Calculate $\int_0^1 \frac{2 dx}{\sqrt{3 + 4x^2}}$.

We will use the substitution $u = 2x$ and the formula in the yellow box with $a = \sqrt{3}$. The indefinite integral is

$$\begin{aligned}\int \frac{2 dx}{\sqrt{3 + 4x^2}} &= \int \frac{du}{\sqrt{a^2 - u^2}} = \sinh^{-1} \left(\frac{u}{a} \right) + C \\ &= \sinh^{-1} \left(\frac{2x}{\sqrt{3}} \right) + C.\end{aligned}$$

7.7 Hyperbolic Functions

$$\int \frac{dx}{\sqrt{a^2 + x^2}} = \sinh^{-1} \left(\frac{x}{a} \right) + C$$



Example (Final example of this course)

Calculate $\int_0^1 \frac{2 dx}{\sqrt{3 + 4x^2}}$.

We will use the substitution $u = 2x$ and the formula in the yellow box with $a = \sqrt{3}$. The indefinite integral is

$$\begin{aligned}\int \frac{2 dx}{\sqrt{3 + 4x^2}} &= \int \frac{du}{\sqrt{a^2 - u^2}} = \sinh^{-1} \left(\frac{u}{a} \right) + C \\ &= \sinh^{-1} \left(\frac{2x}{\sqrt{3}} \right) + C.\end{aligned}$$

Therefore

$$\begin{aligned}\int_0^1 \frac{2 dx}{\sqrt{3 + 4x^2}} &= \left[\sinh^{-1} \left(\frac{2x}{\sqrt{3}} \right) \right]_0^1 \\ &= \sinh^{-1} \left(\frac{2}{\sqrt{3}} \right) - \sinh^{-1}(0) = \sinh^{-1} \left(\frac{2}{\sqrt{3}} \right) - 0.\end{aligned}$$



The End

