Welcome to

Mathematics III

with Dr Neil Course

MATH113 MATH114 MATH215 MATH216

MATH113 MATH114

Calculus

MATH215

MATH216

MATH113 MATH114

Calculus

MATH215 MATH216

Linear Algebra

MATH113 MATH114

Calculus

MATH215

Linear Algebra

MATH216

Differential Equations

Information about this course

- $\approx$ 12 classes. Monday afternoons 2pm-4pm.

Information about this course

- $\approx$ 12 classes. Monday afternoons 2pm-4pm.
- 2 lectures with a break between.

Information about this course

- $\approx$ 12 classes. Monday afternoons 2pm-4pm.
- 2 lectures with a break between.
- Then I will answers your questions.

The Book

David C. Lay, Steven R. Lay and Judi J, McDonald,
Linear Algebra and Its Applications,
5th or 6th Edition, Pearson.

This is a required purchase.

You need to have this book to be able to do the homework.

Syllabus

- linear systems and their solutions
- matrices
- determinants
- inverse matrices
- properties of determinants
- Cramer's Rule
- vector spaces
- subspaces
- linear independence
- basis
- row space
- column space
- null space
- rank and nullity
- linear transformations
- eigenvalues and eigenvectors
- diagonalisation
- inner product spaces
- orthogonality
- the Gram-Schmidt process
- least squares
- orthogonal diagonalisation
- singular value decomposition

Exams and homework

(This information may change based on the University's decisions)

Exams and homework

(This information may change based on the University's decisions)

Exams and homework

(This information may change based on the University's decisions)

using Pearson MyLab Math

one piece of homework for each lesson

deadline = end of term

more details in O'Learn

Exams and homework

(This information may change based on the University's decisions)

2 midterm exams

face-to-face

Exams and homework

(This information may change based on the University's decisions)

face-to-face

Expectations

If this was a classroom course, the expectation is that for each hour of lectures, you would study an extra 1-2 hours outside of class.

classroom course	lectures (4 hours)	other study (4-8 hours)

Expectations

If this was a classroom course, the expectation is that for each hour of lectures, you would study an extra 1-2 hours outside of class.

classroom course	lectures (4 hours)	other study (4-8 hours)

For an online course, you are still expected to study a total of 8-12 hours each week.

online course	class (2 hours)	other study (6-10 hours)

This may include:

- Do the online homework on MyLab;

This may include:

- Do the online homework on MyLab;
- Rewatch the recorded lectures (O'Learn & YouTube);

This may include:

- Do the online homework on MyLab;
- Rewatch the recorded lectures (O'Learn & YouTube);
- Read the lecture slides (before the lecture? after the lecture?);

This may include:

- Do the online homework on MyLab;
- Rewatch the recorded lectures (O'Learn & YouTube);
- Read the lecture slides (before the lecture? after the lecture?);
- Read the textbook;

This may include:

- Do the online homework on MyLab;
- Rewatch the recorded lectures (O'Learn & YouTube);
- Read the lecture slides (before the lecture? after the lecture?);
- Read the textbook;
- Solve the exercises in the textbook;

This may include:

- Do the online homework on MyLab;
- Rewatch the recorded lectures (O'Learn & YouTube);
- Read the lecture slides (before the lecture? after the lecture?);
- Read the textbook;
- Solve the exercises in the textbook;
- Use the O'Learn Discussion Board;

This may include:

- Do the online homework on MyLab;
- Rewatch the recorded lectures (O'Learn & YouTube);
- Read the lecture slides (before the lecture? after the lecture?);
- Read the textbook;
- Solve the exercises in the textbook;
- Use the O'Learn Discussion Board;
- Read other books?;

This may include:

- Do the online homework on MyLab;
- Rewatch the recorded lectures (O'Learn & YouTube);
- Read the lecture slides (before the lecture? after the lecture?);
- Read the textbook;
- Solve the exercises in the textbook;
- Use the O'Learn Discussion Board;
- Read other books?;
- Watch online videos;

$$\vdots$$

slide number

Lecture 1

- Systems of Linear Equations
- Row Reduction and Echelon Forms

neil.course@okan.edu.tr

Systems of Linear Equations

Definition

A *linear equation* is an equation of the form

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n = b \tag{1}$$

where

- $x_1, x_2, \ldots, x_n$ are the variables;
- a_j and b are real or complex numbers; and
- n may be any natural number.

Systems of Linear Equations

Definition

A *linear equation* is an equation of the form

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n = b \tag{1}$$

where

- $x_1, x_2, \ldots, x_n$ are the variables;
- a_j and b are real or complex numbers; and
- n may be any natural number.

In this course, n will usually be between 2 and 5. In real-life problems, n might be 50 or 5000, or even larger.

Systems of Linear Equations

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n = b$$

The equations

$$4x_1 - 5x_2 + 2 = x_1$$

and

$$x_2 = 2\left(\sqrt{6} - x_1\right) + x_3$$

are both linear

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n = b$$

The equations

$$4x_1 - 5x_2 + 2 = x_1$$

and

$$x_2 = 2\left(\sqrt{6} - x_1\right) + x_3$$

are both linear because they can be rearranged to the standard form

$$3x_1 - 5x_2 = -2$$

and

$$2x_1 + x_2 - x_3 = 2\sqrt{6}.$$

The former is an equation of straight line. The latter is an equation of a plane.

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n = b$$

The equations

$$4x_1 - 5x_2 = x_1x_2$$

and

$$x_2 = 2\sqrt{x_1} - 6$$

are not linear.

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n = b$$

The equations

$$4x_1 - 5x_2 = x_1x_2$$

and

$$x_2 = 2\sqrt{x_1} - 6$$

are not linear.

Definition

If an equation is not linear, we say it is *non-linear*.

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n = b$$

Ask the Audience

Are these equations linear or non-linear.

1. $x_1 + x_2 + x_3 + x_4 + x_5 + x_6 = 1$
2. $x_1 + 2x_2 + 3x_3^2 = 4$
3. $x_2 = \cos x_1$
4. $7x_1 - x_3 = 4x_2$
5. $x_1x_2x_3x_4x_5 = 1$

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n = b$$

Ask the Audience

Are these equations linear or non-linear.

1. $x_1 + x_2 + x_3 + x_4 + x_5 + x_6 = 1$ Linear
2. $x_1 + 2x_2 + 3x_3^2 = 4$
3. $x_2 = \cos x_1$
4. $7x_1 - x_3 = 4x_2$
5. $x_1x_2x_3x_4x_5 = 1$

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n = b$$

Ask the Audience

Are these equations linear or non-linear.

1. $x_1 + x_2 + x_3 + x_4 + x_5 + x_6 = 1$ Linear
2. $x_1 + 2x_2 + 3x_3^2 = 4$ Non-linear
3. $x_2 = \cos x_1$
4. $7x_1 - x_3 = 4x_2$
5. $x_1x_2x_3x_4x_5 = 1$

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n = b$$

Ask the Audience

Are these equations linear or non-linear.

1. $x_1 + x_2 + x_3 + x_4 + x_5 + x_6 = 1$ Linear
2. $x_1 + 2x_2 + 3x_3^2 = 4$ Non-linear
3. $x_2 = \cos x_1$ Non-linear
4. $7x_1 - x_3 = 4x_2$
5. $x_1x_2x_3x_4x_5 = 1$

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n = b$$

Ask the Audience

Are these equations linear or non-linear.

1. $x_1 + x_2 + x_3 + x_4 + x_5 + x_6 = 1$ Linear
2. $x_1 + 2x_2 + 3x_3^2 = 4$ Non-linear
3. $x_2 = \cos x_1$ Non-linear
4. $7x_1 - x_3 = 4x_2$ Linear
5. $x_1x_2x_3x_4x_5 = 1$

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n = b$$

Ask the Audience

Are these equations linear or non-linear.

1. $x_1 + x_2 + x_3 + x_4 + x_5 + x_6 = 1$ Linear
2. $x_1 + 2x_2 + 3x_3^2 = 4$ Non-linear
3. $x_2 = \cos x_1$ Non-linear
4. $7x_1 - x_3 = 4x_2$ Linear
5. $x_1x_2x_3x_4x_5 = 1$ Non-linear

Remark

This course is about linear equations.

Systems of Linear Equations

Definition

A *linear system* is a group of several linear equations:

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots a_{mn}x_n = b_m. \end{cases} \tag{2}$$

Systems of Linear Equations

Definition

A *linear system* is a group of several linear equations:

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots a_{mn}x_n = b_m. \end{cases} \tag{2}$$

For example

$$\begin{cases} 2x_1 - x_2 + 1.5x_3 = 8 \\ x_1 \qquad - \quad 4x_3 = -7 \end{cases}$$

is a linear system.

Definition

A *solution* of a linear system is a list $(s_1, s_2, \ldots, s_n)$ of numbers which satisfies every equation in the linear system.

Systems of Linear Equations

Definition

A *solution* of a linear system is a list $(s_1, s_2, \ldots, s_n)$ of numbers which satisfies every equation in the linear system.

For example, $(5, 6.5, 3)$ is a solution of

$$\begin{cases} 2x_1 - x_2 + 1.5x_3 = 8 \\ x_1 \qquad\quad - \quad 4x_3 = -7 \end{cases}$$

Definition

A *solution* of a linear system is a list $(s_1, s_2, \ldots, s_n)$ of numbers which satisfies every equation in the linear system.

For example, $(5, 6.5, 3)$ is a solution of

$$\begin{cases} 2x_1 - x_2 + 1.5x_3 = 8 \\ x_1 \qquad\quad - \quad 4x_3 = -7 \end{cases}$$

because

$$2x_1 - x_2 + 1.5x_3 = 2(5) - (6.5) + 1.5(3) = 8$$

Definition

A *solution* of a linear system is a list $(s_1, s_2, \ldots, s_n)$ of numbers which satisfies every equation in the linear system.

For example, $(5, 6.5, 3)$ is a solution of

$$\begin{cases} 2x_1 - x_2 + 1.5x_3 = 8 \\ x_1 \qquad\quad - \quad 4x_3 = -7 \end{cases}$$

because

$$2x_1 - x_2 + 1.5x_3 = 2(5) - (6.5) + 1.5(3) = 8$$

and

$$x_1 - 4x_3 = (5) - 4(3) = -7.$$

Definition

The set of all possible solutions is called the *solution set* of the linear system

Definition

Two linear systems are called *equivalent* if they have the same solution set.

Example

Find the solution set of

$$\begin{cases} x_1 - 2x_2 = -1 \\ -x_1 + 3x_2 = 3. \end{cases}$$

Example

Find the solution set of

$$\begin{cases} x_1 - 2x_2 = -1 \\ -x_1 + 3x_2 = 3. \end{cases}$$

The graphs of these two equations are lines in $\mathbb{R}^2$.

We can see that the only point which is on both lines is $(3, 2)$. So the solution set is the set $\{(3, 2)\}$.

Systems of Linear Equations

There are three possibilities.

$$\begin{cases} x_1 - 2x_2 = -1 \\ -x_1 + 3x_2 = 3. \end{cases}$$

exactly one solution

$$\begin{cases} x_1 - 2x_2 = -1 \\ -x_1 + 2x_2 = 3. \end{cases}$$

no solutions

$$\begin{cases} x_1 - 2x_2 = -1 \\ -x_1 + 2x_2 = 1. \end{cases}$$

infinitely many solutions.

Systems of Linear Equations

This is also true if we have 3 planes.

Systems of Linear Equations

Theorem

A linear system has either

1. *zero solutions; or*
2. *exactly one solution; or*
3. *infinitely many solutions.*

There are no other possibilities.

Systems of Linear Equations

Theorem

A linear system has either

1. *zero solutions; or*
2. *exactly one solution; or*
3. *infinitely many solutions.*

There are no other possibilities.

Definition

A linear system is called *consistent* if it has either one solution or infinitely many solutions.

A linear system is called *inconsistent* if it does not have a solution.

Matrix Notation

Consider the linear system

$$\begin{cases} x_1 - 2x_2 + x_3 = 0 \\ 2x_2 - 8x_3 = 8 \\ -4x_1 + 5x_2 + 9x_3 = -9 \end{cases}.$$

What is the important information here?

Matrix Notation

Consider the linear system

$$\begin{cases} 1x_1 - 2x_2 + 1x_3 = 0 \\ 2x_2 - 8x_3 = 8 \\ -4x_1 + 5x_2 + 9x_3 = -9 \end{cases}.$$

What is the important information here?

Matrix Notation

Consider the linear system

$$\begin{cases} 1x_1 - 2x_2 + 1x_3 = 0 \\ 2x_2 - 8x_3 = 8 \\ -4x_1 + 5x_2 + 9x_3 = -9 \end{cases}.$$

What is the important information here?

Matrix Notation

Consider the linear system

$$\begin{cases} 1x_1 - 2x_2 + 1x_3 = 0 \\ 2x_2 - 8x_3 = 8 \\ -4x_1 + 5x_2 + 9x_3 = -9 \end{cases}.$$

What is the important information here?

Matrix Notation

Consider the linear system

$$\begin{cases} 1x_1 - 2x_2 + 1x_3 = 0 \\ 2x_2 - 8x_3 = 8 \\ -4x_1 + 5x_2 + 9x_3 = -9 \end{cases}.$$

What is the important information here?

$$\begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ -4 & 5 & 9 & -9 \end{bmatrix}$$

We don't need the x_j's, the $+$'s or the $=$'s. We only need the numbers.

Definition

$$\begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ -4 & 5 & 9 & -9 \end{bmatrix}$$

is the *augmented matrix* for the linear system

$$\begin{cases} 1x_1 - 2x_2 + 1x_3 = 0 \\ 2x_2 - 8x_3 = 8 \\ -4x_1 + 5x_2 + 9x_3 = -9 \end{cases}.$$

Definition

$$\begin{bmatrix} 1 & -2 & 1 \\ 0 & 2 & -8 \\ -4 & 5 & 9 \end{bmatrix}$$

is the *coefficient matrix* for the linear system

$$\begin{cases} 1x_1 - 2x_2 + 1x_3 = 0 \\ 2x_2 - 8x_3 = 8 \\ -4x_1 + 5x_2 + 9x_3 = -9 \end{cases}.$$

Systems of Linear Equations

There are three things that we can do to a linear system:

1. Multiply an equation by a number;
2. Swap two equations; or
3. Add a multiple of one equation to another.

There are three things that we can do to a linear system:

1. Multiply an equation by a number;
2. Swap two equations; or
3. Add a multiple of one equation to another.

Elementary Row Operations

There are three things that we can do to a matrix:

1. Multiply a row by a number;
2. Swap two rows; or
3. Add a multiple of one row to another.

These are called *elementary row operations*.

Solving a Linear System

Example

Solve

$$\begin{cases} 1x_1 - 2x_2 + 1x_3 = 0 \\ 2x_2 - 8x_3 = 8 \\ -4x_1 + 5x_2 + 9x_3 = -9 \end{cases}.$$

Example

Use elementary row operations to convert the augmented matrix

$$\begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ -4 & 5 & 9 & -9 \end{bmatrix}$$

into

$$\begin{bmatrix} 1 & 0 & 0 & ? \\ 0 & 1 & 0 & ? \\ 0 & 0 & 1 & ? \end{bmatrix}.$$

Remark

There are two things about elementary row operations to learn:

1. **How** to use elementary row operations.
2. **Why** we choose a certain elementary row operation at each step.

Remark

There are two things about elementary row operations to learn:

1. **How** to use elementary row operations.
2. **Why** we choose a certain elementary row operation at each step.

I will teach the "how" first. Then in the second hour I will show you the "why".

Systems of Linear Equations

$$\begin{cases} x_1 - 2x_2 + x_3 = 0 \\ \phantom{x_1 - {}} 2x_2 - 8x_3 = 8 \\ -4x_1 + 5x_2 + 9x_3 = -9 \end{cases}$$

1

$$\begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ -4 & 5 & 9 & -9 \end{bmatrix}$$

1

Systems of Linear Equations

$$\begin{cases} x_1 - 2x_2 + x_3 = 0 \\ \phantom{x_1 - {}} 2x_2 - 8x_3 = 8 \\ -4x_1 + 5x_2 + 9x_3 = -9 \end{cases}$$

1

$$\begin{array}{r} 4 \cdot [\text{equation } 1] \\ + \qquad [\text{equation } 3] \\ \hline [\text{new equation } 3] \end{array}$$

$$\begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ -4 & 5 & 9 & -9 \end{bmatrix}$$

1

Systems of Linear Equations

$$\begin{cases} x_1 - 2x_2 + x_3 = 0 \\ \phantom{x_1 - {}} 2x_2 - 8x_3 = 8 \\ -4x_1 + 5x_2 + 9x_3 = -9 \end{cases}$$

1

$$\begin{array}{rr} & 4 \cdot [x_1 - 2x_2 + x_3 = 0] \\ + & [-4x_1 + 5x_2 + 9x_3 = -9] \\ \hline & [\text{new equation } 3] \end{array}$$

$$\begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ -4 & 5 & 9 & -9 \end{bmatrix}$$

1

Systems of Linear Equations

$$\begin{cases} x_1 - 2x_2 + x_3 = 0 \\ \phantom{x_1 - {}} 2x_2 - 8x_3 = 8 \\ -4x_1 + 5x_2 + 9x_3 = -9 \end{cases}$$

1

$$\begin{array}{r} 4x_1 - 8x_2 + 4x_3 = 0 \\ \underline{-4x_1 + 5x_2 + 9x_3 = -9} \end{array}$$

$$\begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ -4 & 5 & 9 & -9 \end{bmatrix}$$

1

Systems of Linear Equations

$$\begin{cases} x_1 - 2x_2 + x_3 = 0 \\ \phantom{x_1 - {}} 2x_2 - 8x_3 = 8 \\ -4x_1 + 5x_2 + 9x_3 = -9 \end{cases}$$

1

$$\begin{array}{r} 4x_1 - 8x_2 + 4x_3 = 0 \\ -4x_1 + 5x_2 + 9x_3 = -9 \\ \hline 0x_1 - 3x_2 + 13x_3 = -9 \end{array}$$

$$\begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ -4 & 5 & 9 & -9 \end{bmatrix}$$

1

Systems of Linear Equations

$$\begin{cases} x_1 - 2x_2 + x_3 = 0 \\ \phantom{x_1 - {}} 2x_2 - 8x_3 = 8 \\ -4x_1 + 5x_2 + 9x_3 = -9 \end{cases}$$

1

$$\begin{array}{r} 4x_1 - 8x_2 + 4x_3 = 0 \\ -4x_1 + 5x_2 + 9x_3 = -9 \\ \hline 0x_1 - 3x_2 + 13x_3 = -9 \end{array}$$

$$\begin{cases} x_1 - 2x_2 + x_3 = 0 \\ \phantom{x_1 - {}} 2x_2 - 8x_3 = 8 \\ \phantom{x_1 - {}} -3x_2 + 13x_3 = -9 \end{cases}$$

$$\begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ -4 & 5 & 9 & -9 \end{bmatrix}$$

1

Systems of Linear Equations

$$\begin{cases} x_1 - 2x_2 + x_3 = 0 \\ \phantom{x_1 - {}} 2x_2 - 8x_3 = 8 \\ -4x_1 + 5x_2 + 9x_3 = -9 \end{cases}$$

1

$$\begin{array}{r} 4x_1 - 8x_2 + 4x_3 = 0 \\ -4x_1 + 5x_2 + 9x_3 = -9 \\ \hline 0x_1 - 3x_2 + 13x_3 = -9 \end{array}$$

$$\begin{cases} x_1 - 2x_2 + x_3 = 0 \\ \phantom{x_1 - {}} 2x_2 - 8x_3 = 8 \\ - 3x_2 + 13x_3 = -9 \end{cases}$$

$$\begin{array}{c} \begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ -4 & 5 & 9 & -9 \end{bmatrix} \end{array} \begin{array}{l} \leftarrow R_1 \\ \leftarrow R_2 \\ \leftarrow R_3 \end{array}$$

1 $4R_1 + R_3 \to \text{new } R_3$

Systems of Linear Equations

$$\begin{cases} x_1 - 2x_2 + x_3 = 0 \\ \phantom{x_1 - {}} 2x_2 - 8x_3 = 8 \\ -4x_1 + 5x_2 + 9x_3 = -9 \end{cases}$$

1

$$\begin{array}{r} 4x_1 - 8x_2 + 4x_3 = 0 \\ -4x_1 + 5x_2 + 9x_3 = -9 \\ \hline 0x_1 - 3x_2 + 13x_3 = -9 \end{array}$$

$$\begin{cases} x_1 - 2x_2 + x_3 = 0 \\ \phantom{x_1 - {}} 2x_2 - 8x_3 = 8 \\ \phantom{x_1 - {}} -3x_2 + 13x_3 = -9 \end{cases}$$

$$\begin{array}{c} \begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ -4 & 5 & 9 & -9 \end{bmatrix} \end{array} \begin{array}{l} \leftarrow R_1 \\ \leftarrow R_2 \\ \leftarrow R_3 \end{array}$$

1 $4R_1 + R_3 \rightarrow \text{new } R_3$

$$\begin{array}{rrrrr} & 4 & -8 & 4 & 0 \\ + & -4 & 5 & 9 & -9 \\ \hline \end{array}$$

Systems of Linear Equations

$$\begin{cases} x_1 - 2x_2 + x_3 = 0 \\ \phantom{x_1 - {}} 2x_2 - 8x_3 = 8 \\ -4x_1 + 5x_2 + 9x_3 = -9 \end{cases}$$

1

$$\begin{array}{r} 4x_1 - 8x_2 + 4x_3 = 0 \\ -4x_1 + 5x_2 + 9x_3 = -9 \\ \hline 0x_1 - 3x_2 + 13x_3 = -9 \end{array}$$

$$\begin{cases} x_1 - 2x_2 + x_3 = 0 \\ \phantom{x_1 - {}} 2x_2 - 8x_3 = 8 \\ \phantom{x_1 - {}} -3x_2 + 13x_3 = -9 \end{cases}$$

$$\begin{array}{c} \begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ -4 & 5 & 9 & -9 \end{bmatrix} \end{array} \begin{array}{l} \leftarrow R_1 \\ \leftarrow R_2 \\ \leftarrow R_3 \end{array}$$

1 $4R_1 + R_3 \rightarrow \text{new } R_3$

$$\begin{array}{rrrrr} & 4 & -8 & 4 & 0 \\ + & -4 & 5 & 9 & -9 \\ \hline & 0 & -3 & 13 & -9 \end{array}$$

Systems of Linear Equations

$$\begin{cases} x_1 - 2x_2 + x_3 = 0 \\ \phantom{x_1 - {}} 2x_2 - 8x_3 = 8 \\ -4x_1 + 5x_2 + 9x_3 = -9 \end{cases}$$

1

$$\begin{array}{r} 4x_1 - 8x_2 + 4x_3 = 0 \\ -4x_1 + 5x_2 + 9x_3 = -9 \\ \hline 0x_1 - 3x_2 + 13x_3 = -9 \end{array}$$

$$\begin{cases} x_1 - 2x_2 + x_3 = 0 \\ \phantom{x_1 - {}} 2x_2 - 8x_3 = 8 \\ \phantom{x_1 - {}} -3x_2 + 13x_3 = -9 \end{cases}$$

$$\begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ -4 & 5 & 9 & -9 \end{bmatrix} \begin{array}{l} \leftarrow R_1 \\ \leftarrow R_2 \\ \leftarrow R_3 \end{array}$$

1 $4R_1 + R_3 \to \text{new } R_3$

$$\begin{array}{rrrrr} & 4 & -8 & 4 & 0 \\ + & -4 & 5 & 9 & -9 \\ \hline & 0 & -3 & 13 & -9 \end{array}$$

$$\begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ 0 & -3 & 13 & -9 \end{bmatrix} \leftarrow \text{new } R_3$$

Systems of Linear Equations

$$\begin{cases} x_1 - 2x_2 + x_3 = 0 \\ \phantom{x_1 - {}} 2x_2 - 8x_3 = 8 \\ \phantom{x_1 - {}} -3x_2 + 13x_3 = -9 \end{cases}$$

2 Multiply [equation 2] by $\frac{1}{2}$.

$$\begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ 0 & -3 & 13 & -9 \end{bmatrix}$$

2 $\frac{1}{2}R_2 \to \text{new } R_2$

Systems of Linear Equations

$$\begin{cases} x_1 - 2x_2 + x_3 = 0 \\ \phantom{x_1 - {}} 2x_2 - 8x_3 = 8 \\ - 3x_2 + 13x_3 = -9 \end{cases}$$

$$\begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ 0 & -3 & 13 & -9 \end{bmatrix}$$

2 Multiply [equation 2] by $\frac{1}{2}$.

$$\begin{cases} x_1 - 2x_2 + x_3 = 0 \\ \phantom{x_1 - {}} x_2 - 4x_3 = 4 \\ - 3x_2 + 13x_3 = -9 \end{cases}$$

2 $\frac{1}{2}R_2 \to$ new R_2

Systems of Linear Equations

$$\begin{cases} x_1 - 2x_2 + x_3 = 0 \\ \phantom{x_1 - {}} 2x_2 - 8x_3 = 8 \\ - 3x_2 + 13x_3 = -9 \end{cases}$$

$$\begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ 0 & -3 & 13 & -9 \end{bmatrix}$$

2 Multiply [equation 2] by $\frac{1}{2}$.

$$\begin{cases} x_1 - 2x_2 + x_3 = 0 \\ \phantom{x_1 - {}} x_2 - 4x_3 = 4 \\ - 3x_2 + 13x_3 = -9 \end{cases}$$

2 $\frac{1}{2}R_2 \to$ new R_2

$$\begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 1 & -4 & 4 \\ 0 & -3 & 13 & -9 \end{bmatrix}$$

Systems of Linear Equations

$$\begin{cases} x_1 - 2x_2 + \quad x_3 = 0 \\ \qquad x_2 - \ 4x_3 = 4 \\ \quad -3x_2 + 13x_3 = -9 \end{cases}$$

3

$$\begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 1 & -4 & 4 \\ 0 & -3 & 13 & -9 \end{bmatrix}$$

3 $3R_2 + R_3 \to \text{new } R_3$

Systems of Linear Equations

$$\begin{cases} x_1 - 2x_2 + \quad x_3 = 0 \\ \qquad x_2 - \ 4x_3 = 4 \\ \quad -3x_2 + 13x_3 = -9 \end{cases}$$

3

$$\begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 1 & -4 & 4 \\ 0 & -3 & 13 & -9 \end{bmatrix}$$

3 $3R_2 + R_3 \to \text{new } R_3$

$$\begin{array}{rrrrr} & 0 & 3 & -12 & 12 \\ + & 0 & -3 & 13 & -9 \\ \hline \end{array}$$

Systems of Linear Equations

$$\begin{cases} x_1 - 2x_2 + x_3 = 0 \\ x_2 - 4x_3 = 4 \\ -3x_2 + 13x_3 = -9 \end{cases}$$

3

$$\begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 1 & -4 & 4 \\ 0 & -3 & 13 & -9 \end{bmatrix}$$

3 $3R_2 + R_3 \to \text{new } R_3$

$$\begin{array}{rrrrr} & 0 & 3 & -12 & 12 \\ + & 0 & -3 & 13 & -9 \\ \hline & 0 & 0 & 1 & 3 \end{array}$$

Systems of Linear Equations

$$\begin{cases} x_1 - 2x_2 + \quad x_3 = 0 \\ \qquad x_2 - \ 4x_3 = 4 \\ \quad -3x_2 + 13x_3 = -9 \end{cases}$$

3

$$\begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 1 & -4 & 4 \\ 0 & -3 & 13 & -9 \end{bmatrix}$$

3 $3R_2 + R_3 \to \text{new } R_3$

$$\begin{array}{rrrrr} & 0 & 3 & -12 & 12 \\ + & 0 & -3 & 13 & -9 \\ \hline & 0 & 0 & 1 & 3 \end{array}$$

$$\begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 1 & -4 & 4 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

Systems of Linear Equations

$$\begin{cases} x_1 - 2x_2 + \quad x_3 = 0 \\ \qquad x_2 - \ 4x_3 = 4 \\ \quad -3x_2 + 13x_3 = -9 \end{cases}$$

3 $3\times\text{Eq2} + \text{Eq3} \to \text{new Eq3}.$

$$\begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 1 & -4 & 4 \\ 0 & -3 & 13 & -9 \end{bmatrix}$$

3 $3R_2 + R_3 \to \text{new } R_3$

$$\begin{array}{rrrrr} & 0 & 3 & -12 & 12 \\ + & 0 & -3 & 13 & -9 \\ \hline & 0 & 0 & 1 & 3 \end{array}$$

$$\begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 1 & -4 & 4 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

Systems of Linear Equations

$$\begin{cases} x_1 - 2x_2 + \quad x_3 = 0 \\ \qquad x_2 - \ 4x_3 = 4 \\ \quad -3x_2 + 13x_3 = -9 \end{cases}$$

3 $3\times\text{Eq2} + \text{Eq3} \to \text{new Eq3}.$

$$\begin{array}{r} 3x_2 - 12x_3 = 12 \\ \underline{-3x_2 + 13x_3 = -9} \end{array}$$

$$\begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 1 & -4 & 4 \\ 0 & -3 & 13 & -9 \end{bmatrix}$$

3 $3R_2 + R_3 \to \text{new } R_3$

$$\begin{array}{rrrrr} & 0 & 3 & -12 & 12 \\ + & 0 & -3 & 13 & -9 \\ \hline & 0 & 0 & 1 & 3 \end{array}$$

$$\begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 1 & -4 & 4 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

Systems of Linear Equations

$$\begin{cases} x_1 - 2x_2 + \quad x_3 = 0 \\ \qquad x_2 - \ 4x_3 = 4 \\ \quad -3x_2 + 13x_3 = -9 \end{cases}$$

3 $3\times$Eq2 + Eq3 $\rightarrow$ new Eq3.

$$\begin{array}{c} 3x_2 - 12x_3 = 12 \\ -3x_2 + 13x_3 = -9 \\ \hline 0x_2 + \quad x_3 = 3 \end{array}$$

$$\begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 1 & -4 & 4 \\ 0 & -3 & 13 & -9 \end{bmatrix}$$

3 $3R_2 + R_3 \rightarrow$ new R_3

$$\begin{array}{rrrrr} & 0 & 3 & -12 & 12 \\ + & 0 & -3 & 13 & -9 \\ \hline & 0 & 0 & 1 & 3 \end{array}$$

$$\begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 1 & -4 & 4 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

Systems of Linear Equations

$$\begin{cases} x_1 - 2x_2 + x_3 = 0 \\ x_2 - 4x_3 = 4 \\ -3x_2 + 13x_3 = -9 \end{cases}$$

3 $3\times\text{Eq2} + \text{Eq3} \to \text{new Eq3}.$

$$\begin{array}{r} 3x_2 - 12x_3 = 12 \\ -3x_2 + 13x_3 = -9 \\ \hline 0x_2 + x_3 = 3 \end{array}$$

$$\begin{cases} x_1 - 2x_2 + x_3 = 0 \\ x_2 - 4x_3 = 4 \\ x_3 = 3 \end{cases}$$

$$\begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 1 & -4 & 4 \\ 0 & -3 & 13 & -9 \end{bmatrix}$$

3 $3R_2 + R_3 \to \text{new } R_3$

$$\begin{array}{rrrrr} & 0 & 3 & -12 & 12 \\ + & 0 & -3 & 13 & -9 \\ \hline & 0 & 0 & 1 & 3 \end{array}$$

$$\begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 1 & -4 & 4 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

Systems of Linear Equations

$$\begin{cases} x_1 - 2x_2 + x_3 = 0 \\ x_2 - 4x_3 = 4 \\ x_3 = 3 \end{cases}$$

4 Eq2 + 4×Eq3 → new Eq2.

$$\begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 1 & -4 & 4 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

4 $R_2 + 4R_3 \to$ new R_2

Systems of Linear Equations

$$\begin{cases} x_1 - 2x_2 + x_3 = 0 \\ x_2 - 4x_3 = 4 \\ x_3 = 3 \end{cases}$$

4 Eq2 + 4×Eq3 → new Eq2.

$$\begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 1 & -4 & 4 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

4 $R_2 + 4R_3 \to$ new R_2

$$\begin{array}{rrrrr} & 0 & 1 & -4 & 4 \\ + & 0 & 0 & 4 & 12 \\ \hline & 0 & 1 & 0 & 16 \end{array}$$

$$\begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 1 & 0 & 16 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

Systems of Linear Equations

$$\begin{cases} x_1 - 2x_2 + x_3 = 0 \\ x_2 - 4x_3 = 4 \\ x_3 = 3 \end{cases}$$

4 Eq2 + 4×Eq3 → new Eq2.

$$\begin{array}{r} x_2 - 4x_3 = 4 \\ 4x_3 = 12 \\ \hline x_2 = 16 \end{array}$$

$$\begin{cases} x_1 - 2x_2 + x_3 = 0 \\ x_2 = 16 \\ x_3 = 3 \end{cases}$$

$$\begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 1 & -4 & 4 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

4 $R_2 + 4R_3 \to$ new R_2

$$\begin{array}{rrrrr} & 0 & 1 & -4 & 4 \\ + & 0 & 0 & 4 & 12 \\ \hline & 0 & 1 & 0 & 16 \end{array}$$

$$\begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 1 & 0 & 16 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

Systems of Linear Equations

$$\begin{cases} x_1 - 2x_2 + x_3 = 0 \\ x_2 - 4x_3 = 4 \\ x_3 = 3 \end{cases}$$

5 Eq1 $-$ Eq3 $\rightarrow$ new Eq1.

$$\begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 1 & 0 & 16 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

5 $R_1 - R_3 \rightarrow$ new R_1

Systems of Linear Equations

$$\begin{cases} x_1 - 2x_2 + x_3 = 0 \\ x_2 - 4x_3 = 4 \\ x_3 = 3 \end{cases}$$

5 Eq1 − Eq3 → new Eq1.

$$\begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 1 & 0 & 16 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

5 $R_1 - R_3 \to$ new R_1

$$\begin{array}{rrrrr} & 1 & -2 & 1 & 0 \\ + & 0 & 0 & -1 & -3 \\ \hline & 1 & -2 & 0 & -3 \end{array}$$

$$\begin{bmatrix} 1 & -2 & 0 & -3 \\ 0 & 1 & 0 & 16 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

Systems of Linear Equations

$$\begin{cases} x_1 - 2x_2 + x_3 = 0 \\ x_2 - 4x_3 = 4 \\ x_3 = 3 \end{cases}$$

5 Eq1 − Eq3 → new Eq1.

$$\begin{array}{r} x_1 - 2x_2 + x_3 = 0 \\ - x_3 = -3 \\ \hline x_1 - 2x_2 = -3 \end{array}$$

$$\begin{cases} x_1 - 2x_2 = -3 \\ x_2 = 16 \\ = 3 \end{cases}$$

$$\begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 1 & 0 & 16 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

5 $R_1 - R_3 \to$ new R_1

$$\begin{array}{rrrrr} & 1 & -2 & 1 & 0 \\ + & 0 & 0 & -1 & -3 \\ \hline & 1 & -2 & 0 & -3 \end{array}$$

$$\begin{bmatrix} 1 & -2 & 0 & -3 \\ 0 & 1 & 0 & 16 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

Systems of Linear Equations

$$\begin{cases} x_1 - 2x_2 \qquad\quad = -3 \\ \qquad\quad x_2 - 4x_3 = 4 \\ \qquad\qquad\quad x_3 = 3 \end{cases}$$

6 Eq1 + 2×Eq2 → new Eq1.

$$\begin{bmatrix} 1 & -2 & 0 & -3 \\ 0 & 1 & 0 & 16 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

6 $R_1 + 2R_2 \to$ new R_1

Systems of Linear Equations

$$\begin{cases} x_1 - 2x_2 \qquad\quad = -3 \\ \qquad\quad x_2 - 4x_3 = 4 \\ \qquad\qquad\quad x_3 = 3 \end{cases}$$

6 Eq1 + 2×Eq2 → new Eq1.

$$\begin{bmatrix} 1 & -2 & 0 & -3 \\ 0 & 1 & 0 & 16 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

6 $R_1 + 2R_2 \to$ new R_1

$$\begin{array}{rrrrr} & 1 & -2 & 0 & -3 \\ + & 0 & 2 & 0 & 32 \\ \hline & 1 & 0 & 0 & 29 \end{array}$$

$$\begin{bmatrix} 1 & 0 & 0 & 29 \\ 0 & 1 & 0 & 16 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

Systems of Linear Equations

$$\begin{cases} x_1 - 2x_2 \qquad\quad = -3 \\ \qquad\quad x_2 - 4x_3 = 4 \\ \qquad\qquad\quad x_3 = 3 \end{cases}$$

6 Eq1 + 2×Eq2 → new Eq1.

$$\begin{array}{rl} x_1 - 2x_2 & = -3 \\ 2x_2 & = 32 \\ \hline x_1 \qquad\quad & = 29 \end{array}$$

$$\begin{cases} x_1 \qquad\qquad\quad = 29 \\ \qquad x_2 \qquad\quad = 16 \\ \qquad\qquad x_3 = 3 \end{cases}$$

$$\begin{bmatrix} 1 & -2 & 0 & -3 \\ 0 & 1 & 0 & 16 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

6 $R_1 + 2R_2 \to$ new R_1

$$\begin{array}{rrrrr} & 1 & -2 & 0 & -3 \\ + & 0 & 2 & 0 & 32 \\ \hline & 1 & 0 & 0 & 29 \end{array}$$

$$\begin{bmatrix} 1 & 0 & 0 & 29 \\ 0 & 1 & 0 & 16 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

Example

Solve

$$\begin{cases} 1x_1 - 2x_2 + 1x_3 = 0 \\ 2x_2 - 8x_3 = 8 \\ -4x_1 + 5x_2 + 9x_3 = -9 \end{cases}.$$

$$\begin{cases} x_1 = 29 \\ x_2 = 16 \\ x_3 = 3 \end{cases}$$

Example

Use elementary row operations to convert the augmented matrix

$$\begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ -4 & 5 & 9 & -9 \end{bmatrix} \quad \text{into} \quad \begin{bmatrix} 1 & 0 & 0 & ? \\ 0 & 1 & 0 & ? \\ 0 & 0 & 1 & ? \end{bmatrix}.$$

$$\begin{bmatrix} 1 & 0 & 0 & 29 \\ 0 & 1 & 0 & 16 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

Definition

Two matrices A and B are called *row equivalent* if it is possible to change one into the other using elementary row operations.

We write $A \sim B$ if A and B are row equivalent.

Definition

Two matrices A and B are called *row equivalent* if it is possible to change one into the other using elementary row operations.

We write $A \sim B$ if A and B are row equivalent.

Theorem

If the augmented matrices of two linear systems are row equivalent, then the two systems have the same solution set.

Elementary Row Operations (again)

1. Multiply a row by a number (e.g. $cR_3 \to \text{new}R_3$).
2. Swap two rows (e.g. $R_1 \leftrightarrow R_2$).
3. Add a multiple of one row to another (e.g. $cR_3 + R_4 \to \text{new}R_4$).

Elementary Row Operations (again)

1. Multiply a row by a number (e.g. $cR_3 \to \text{new}R_3$).
2. Swap two rows (e.g. $R_1 \leftrightarrow R_2$).
3. Add a multiple of one row to another (e.g. $cR_3 + R_4 \to \text{new}R_4$).

Note that all three of these are invertible.

1. The opposite of multiplying a row by $c \neq 0$ is multiplying it by $\frac{1}{c}$.

Elementary Row Operations (again)

1. Multiply a row by a number (e.g. $cR_3 \to \text{new}R_3$).
2. Swap two rows (e.g. $R_1 \leftrightarrow R_2$).
3. Add a multiple of one row to another (e.g. $cR_3 + R_4 \to \text{new}R_4$).

Note that all three of these are invertible.

1. The opposite of multiplying a row by $c \neq 0$ is multiplying it by $\frac{1}{c}$.
2. Swapping two rows is it's own inverse.

Elementary Row Operations (again)

1. Multiply a row by a number (e.g. $cR_3 \to \text{new}R_3$).
2. Swap two rows (e.g. $R_1 \leftrightarrow R_2$).
3. Add a multiple of one row to another (e.g. $cR_3 + R_4 \to \text{new}R_4$).

Note that all three of these are invertible.

1. The opposite of multiplying a row by $c \neq 0$ is multiplying it by $\frac{1}{c}$.
2. Swapping two rows is it's own inverse.
3. The opposite of adding something, is subtracting that something.

Existence and Uniqueness Questions

Example

Recall that

$$\begin{cases} x_1 - 2x_2 + x_3 = 0 \\ 2x_2 - 8x_3 = 8 \\ -4x_1 + 5x_2 + 9x_3 = -9 \end{cases} \quad \text{has solution} \quad \begin{cases} x_1 = 29 \\ x_2 = 16 \\ x_3 = 3. \end{cases}$$

Therefore this linear system is consistent.

Example

Is

$$\begin{cases} x_2 - 4x_3 = 8 \\ 2x_1 - 3x_2 + 2x_3 = 1 \\ 5x_1 - 8x_2 + 7x_3 = 1 \end{cases}$$

consistent?

Example

Is

$$\begin{cases} x_2 - 4x_3 = 8 \\ 2x_1 - 3x_2 + 2x_3 = 1 \\ 5x_1 - 8x_2 + 7x_3 = 1 \end{cases}$$

consistent?

To answer this, we will do elementary row operations on the augmented matrix

$$\begin{bmatrix} 0 & 1 & -4 & 8 \\ 2 & -3 & 2 & 1 \\ 5 & -8 & 7 & 1 \end{bmatrix}$$

and see what we get.

Systems of Linear Equations

$$\begin{bmatrix} 0 & 1 & -4 & 8 \\ 2 & -3 & 2 & 1 \\ 5 & -8 & 7 & 1 \end{bmatrix}$$

Systems of Linear Equations

$$\begin{bmatrix} 0 & 1 & -4 & 8 \\ 2 & -3 & 2 & 1 \\ 5 & -8 & 7 & 1 \end{bmatrix}$$

1. $R_1 \leftrightarrow R_2$

$$\begin{bmatrix} 2 & -3 & 2 & 1 \\ 0 & 1 & -4 & 8 \\ 5 & -8 & 7 & 1 \end{bmatrix}$$

Systems of Linear Equations

$$\begin{bmatrix} 0 & 1 & -4 & 8 \\ 2 & -3 & 2 & 1 \\ 5 & -8 & 7 & 1 \end{bmatrix}$$

1. $R_1 \leftrightarrow R_2$

$$\begin{bmatrix} 2 & -3 & 2 & 1 \\ 0 & 1 & -4 & 8 \\ 5 & -8 & 7 & 1 \end{bmatrix}$$

2. $R_3 - \frac{5}{2}R_1 \to R_3$

$$\begin{array}{rrrrr} & 5 & -8 & 7 & 1 \\ + & -5 & \frac{15}{2} & -5 & -\frac{5}{2} \\ \hline & 0 & -\frac{1}{2} & 2 & -\frac{3}{2} \end{array} \qquad \begin{bmatrix} 2 & -3 & 2 & 1 \\ 0 & 1 & -4 & 8 \\ 0 & -\frac{1}{2} & 2 & -\frac{3}{2} \end{bmatrix}$$

Systems of Linear Equations

$$\begin{bmatrix} 2 & -3 & 2 & 1 \\ 0 & 1 & -4 & 8 \\ 0 & -\frac{1}{2} & 2 & -\frac{3}{2} \end{bmatrix}$$

$$\begin{bmatrix} 2 & -3 & 2 & 1 \\ 0 & 1 & -4 & 8 \\ 0 & -\frac{1}{2} & 2 & -\frac{3}{2} \end{bmatrix}$$

3 $\frac{1}{2}R_2 + R_3 \to R_3$

$$\begin{array}{rcccc} & 0 & \frac{1}{2} & -2 & 4 \\ + & 0 & -\frac{1}{2} & 2 & -\frac{3}{2} \\ \hline & 0 & 0 & 0 & \frac{5}{2} \end{array} \qquad \begin{bmatrix} 2 & -3 & 2 & 1 \\ 0 & 1 & -4 & 8 \\ 0 & 0 & 0 & \frac{5}{2} \end{bmatrix}$$

$$\begin{bmatrix} 2 & -3 & 2 & 1 \\ 0 & 1 & -4 & 8 \\ 0 & -\frac{1}{2} & 2 & -\frac{3}{2} \end{bmatrix}$$

3 $\frac{1}{2}R_2 + R_3 \to R_3$

$$\begin{array}{rcccc} & 0 & \frac{1}{2} & -2 & 4 \\ + & 0 & -\frac{1}{2} & 2 & -\frac{3}{2} \\ \hline & 0 & 0 & 0 & \frac{5}{2} \end{array} \qquad \begin{bmatrix} 2 & -3 & 2 & 1 \\ 0 & 1 & -4 & 8 \\ 0 & 0 & 0 & \frac{5}{2} \end{bmatrix}$$

4 Convert back to a linear system.

$$\begin{cases} 2x_1 - 3x_2 + 2x_3 = 1 \\ \qquad\qquad x_2 - 4x_3 = 8 \\ 0x_1 + 0x_2 + 0x_3 = \frac{5}{2}. \end{cases}$$

$$\begin{cases} x_2 - 4x_3 = 8 \\ 2x_1 - 3x_2 + 2x_3 = 1 \\ 5x_1 - 8x_2 + 7x_3 = 1 \end{cases} \rightarrow \begin{cases} 2x_1 - 3x_2 + 2x_3 = 1 \\ x_2 - 4x_3 = 8 \\ 0 = \frac{5}{2}. \end{cases}$$

But the third equation says

$$0 = \frac{5}{2}$$

which is clearly nonsense.

$$\begin{cases} x_2 - 4x_3 = 8 \\ 2x_1 - 3x_2 + 2x_3 = 1 \\ 5x_1 - 8x_2 + 7x_3 = 1 \end{cases} \rightarrow \begin{cases} 2x_1 - 3x_2 + 2x_3 = 1 \\ x_2 - 4x_3 = 8 \\ 0 = \frac{5}{2}. \end{cases}$$

But the third equation says

$$0 = \frac{5}{2}$$

which is clearly nonsense.

This means that it is not possible to find a solution to this linear system. So the linear system is inconsistent.

Example

For what values of h and k is the following system consistent?

$$\begin{cases} 2x_1 - x_2 = h \\ -6x_1 + 3x_2 = k \end{cases}$$

Example

For what values of h and k is the following system consistent?

$$\begin{cases} 2x_1 - x_2 = h \\ -6x_1 + 3x_2 = k \end{cases}$$

To answer this, we do one row operation on this system:

$$\begin{bmatrix} 2 & -1 & h \\ -6 & 3 & k \end{bmatrix} \xrightarrow{3R_1+R_2} \begin{bmatrix} 2 & -1 & h \\ 0 & 0 & 3h+k \end{bmatrix}.$$

Example

For what values of h and k is the following system consistent?

$$\begin{cases} 2x_1 - x_2 = h \\ -6x_1 + 3x_2 = k \end{cases}$$

To answer this, we do one row operation on this system:

$$\begin{bmatrix} 2 & -1 & h \\ -6 & 3 & k \end{bmatrix} \xrightarrow{3R_1+R_2} \begin{bmatrix} 2 & -1 & h \\ 0 & 0 & 3h+k \end{bmatrix}.$$

Notice that the latter matrix is equivalent to

$$\begin{cases} 2x_1 - x_2 = h \\ 0x_1 + 0x_2 = 3h + k. \end{cases}$$

Example

For what values of h and k is the following system consistent?

$$\begin{cases} 2x_1 - x_2 = h \\ -6x_1 + 3x_2 = k \end{cases}$$

To answer this, we do one row operation on this system:

$$\begin{bmatrix} 2 & -1 & h \\ -6 & 3 & k \end{bmatrix} \xrightarrow{3R_1+R_2} \begin{bmatrix} 2 & -1 & h \\ 0 & 0 & 3h+k \end{bmatrix}.$$

Notice that the latter matrix is equivalent to

$$\begin{cases} 2x_1 - x_2 = h \\ 0x_1 + 0x_2 = 3h + k. \end{cases}$$

Obviously, we require that $3h + k = 0$ in order for this system to have solutions.

So this linear system is consistent only when $3h + k = 0$.

Remark

It doesn't matter if you use [ ] or ( ) when you write an augmented matrix.

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}$$

Example

Find an equation involving g, h, and k that makes this augmented matrix correspond to a consistent system:

$$\begin{pmatrix} 1 & -4 & 7 & g \\ 0 & 3 & -5 & h \\ -2 & 5 & -9 & k \end{pmatrix}.$$

Example

Find an equation involving g, h, and k that makes this augmented matrix correspond to a consistent system:

$$\begin{pmatrix} 1 & -4 & 7 & g \\ 0 & 3 & -5 & h \\ -2 & 5 & -9 & k \end{pmatrix}.$$

$$\begin{pmatrix} 1 & -4 & 7 & g \\ 0 & 3 & -5 & h \\ -2 & 5 & -9 & k \end{pmatrix} \xrightarrow{2R_1+R_3} \begin{pmatrix} 1 & -4 & 7 & g \\ 0 & 3 & -5 & h \\ 0 & -3 & 5 & 2g+k \end{pmatrix}$$

$$\xrightarrow{R_2+R_3} \begin{pmatrix} 1 & -4 & 7 & g \\ 0 & 3 & -5 & h \\ 0 & 0 & 0 & 2g+h+k \end{pmatrix}.$$

From the last augmented matrix, we see that we must $2g + h + k = 0$ for a consistent system.

Example

Determine if the following system is consistent or inconsistent. If it is consistent, find <u>all</u> the solutions.

$$\begin{cases} 3x_1 + 5x_2 - 4x_3 = 7 \\ -3x_1 - 2x_2 + 4x_3 = -1 \\ 6x_1 + x_2 - 8x_3 = -4. \end{cases}$$

$$\begin{pmatrix} 3 & 5 & -4 & 7 \\ -3 & -2 & 4 & -1 \\ 6 & 1 & -8 & -4 \end{pmatrix}$$

Systems of Linear Equations

$$\begin{pmatrix} 3 & 5 & -4 & 7 \\ -3 & -2 & 4 & -1 \\ 6 & 1 & -8 & -4 \end{pmatrix} \xrightarrow[]{\substack{R_1+R_2 \\ -2R_1+R_3}} \begin{pmatrix} 3 & 5 & -4 & 7 \\ 0 & 3 & 0 & 6 \\ 0 & -9 & 0 & -18 \end{pmatrix}$$

$$\xrightarrow[]{\substack{3R_2+R_3 \\ \frac{1}{3}R_2}} \begin{pmatrix} 3 & 5 & -4 & 7 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\xrightarrow[]{-5R_2+R_1} \begin{pmatrix} 3 & 0 & -4 & -3 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

The linear system is equivalent to

$$\begin{cases} 3x_1 \qquad\quad - 4x_3 = -3 \\ \qquad\quad x_2 \qquad\quad = 2 \end{cases}$$

Systems of Linear Equations

$$\begin{cases} 3x_1 \qquad\quad - 4x_3 = -3 \\ \qquad\quad x_2 \qquad\quad = 2 \end{cases}$$

Thus $x_1 = -1 + \frac{4}{3}x_3$ and every solution has the form of

$$\begin{aligned}(x_1, x_2, x_3) &= (-1 + \tfrac{4}{3}x_3, 2, x_3) \\ &= x_3(\tfrac{4}{3}, 0, 1) + (-1, 2, 0).\end{aligned}$$

$$\begin{cases} 3x_1 \qquad\quad - 4x_3 = -3 \\ \qquad\quad x_2 \qquad\quad = 2 \end{cases}$$

Thus $x_1 = -1 + \frac{4}{3}x_3$ and every solution has the form of

$$\begin{aligned}(x_1, x_2, x_3) &= (-1 + \tfrac{4}{3}x_3, 2, x_3) \\ &= x_3(\tfrac{4}{3}, 0, 1) + (-1, 2, 0).\end{aligned}$$

Here, x_3 can take on any real number so there are infinitely many solutions. Geometrically, the solution set is a line passing through the point $(-1, 2, 0)$ parallel to the vector $(\frac{4}{3}, 0, 1)$.

Example

Do the three planes $\begin{cases} x_1 + 3x_2 + x_3 = 4 \\ \qquad\quad x_2 - x_3 = 1 \\ x_1 + 4x_2 \qquad\quad = -2 \end{cases}$ have a point in common? Explain.

Example

Do the three planes $\begin{cases} x_1 + 3x_2 + x_3 = 4 \\ \qquad\quad x_2 - x_3 = 1 \\ x_1 + 4x_2 \qquad\; = -2 \end{cases}$ have a point in common? Explain.

$$\begin{pmatrix} 1 & 3 & 1 & 4 \\ 0 & 1 & -1 & 1 \\ 1 & 4 & 0 & -2 \end{pmatrix} \xrightarrow{-R_1+R_3} \begin{pmatrix} 1 & 3 & 1 & 4 \\ 0 & 1 & -1 & 1 \\ 0 & 1 & -1 & -6 \end{pmatrix}$$

$$\xrightarrow{-R_2+R_3} \begin{pmatrix} 1 & 3 & 1 & 4 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & -7 \end{pmatrix}$$

Example

Do the three planes $\begin{cases} x_1 + 3x_2 + x_3 = 4 \\ \qquad\quad x_2 - x_3 = 1 \\ x_1 + 4x_2 \qquad\; = -2 \end{cases}$ have a point in common? Explain.

$$\begin{pmatrix} 1 & 3 & 1 & 4 \\ 0 & 1 & -1 & 1 \\ 1 & 4 & 0 & -2 \end{pmatrix} \xrightarrow{-R_1+R_3} \begin{pmatrix} 1 & 3 & 1 & 4 \\ 0 & 1 & -1 & 1 \\ 0 & 1 & -1 & -6 \end{pmatrix}$$

$$\xrightarrow{-R_2+R_3} \begin{pmatrix} 1 & 3 & 1 & 4 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & -7 \end{pmatrix}$$

The last line of this augmented matrix reads

$$0x_1 + 0x_2 + 0x_3 = -7,$$

which, of course, is nonsense.

Example

Do the three planes $\begin{cases} x_1 + 3x_2 + x_3 = 4 \\ \qquad\quad x_2 - x_3 = 1 \\ x_1 + 4x_2 \qquad\; = -2 \end{cases}$ have a point in common? Explain.

$$\begin{pmatrix} 1 & 3 & 1 & 4 \\ 0 & 1 & -1 & 1 \\ 1 & 4 & 0 & -2 \end{pmatrix} \xrightarrow{-R_1+R_3} \begin{pmatrix} 1 & 3 & 1 & 4 \\ 0 & 1 & -1 & 1 \\ 0 & 1 & -1 & -6 \end{pmatrix}$$

$$\xrightarrow{-R_2+R_3} \begin{pmatrix} 1 & 3 & 1 & 4 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & -7 \end{pmatrix}$$

The last line of this augmented matrix reads

$$0x_1 + 0x_2 + 0x_3 = -7,$$

which, of course, is nonsense.

So there is **no** point in common for the three planes.

Break

We will continue at 3pm

Row Reduction and Echelon Forms

Row Reduction and Echelon Forms

An **echelon formation** *is a (usually military) formation in which its units are arranged diagonally.*

Echelon is pronounced "esh-E-lon" because it comes from the French language.

Row Reduction and Echelon Forms

$$\begin{bmatrix} 2 & -3 & 2 & 1 & -7 & 4 \\ 0 & 1 & -4 & 8 & 0 & 0 \\ 0 & 0 & 0 & 5 & 2 & 1 \\ 0 & 0 & 0 & 0 & 0 & 6 \end{bmatrix}$$

Definition

The first non-zero number in each row, is called a *leading entry* or *pivot*.

Row Reduction and Echelon Forms

leading entry/pivot

$$\begin{bmatrix} 2 & -3 & 2 & 1 & -7 & 4 \\ 0 & 1 & -4 & 8 & 0 & 0 \\ 0 & 0 & 0 & 5 & 2 & 1 \\ 0 & 0 & 0 & 0 & 0 & 6 \end{bmatrix}$$

Definition

The first non-zero number in each row, is called a *leading entry* or *pivot*.

Row Reduction and Echelon Forms

leading entry/pivot

$$\begin{bmatrix} 2 & -3 & 2 & 1 & -7 & 4 \\ 0 & 1 & -4 & 8 & 0 & 0 \\ 0 & 0 & 0 & 5 & 2 & 1 \\ 0 & 0 & 0 & 0 & 0 & 6 \end{bmatrix}$$

Definition

The first non-zero number in each row, is called a *leading entry* or *pivot*.

Row Reduction and Echelon Forms

leading entry/pivot

$$\begin{bmatrix} 2 & -3 & 2 & 1 & -7 & 4 \\ 0 & 1 & -4 & 8 & 0 & 0 \\ 0 & 0 & 0 & 5 & 2 & 1 \\ 0 & 0 & 0 & 0 & 0 & 6 \end{bmatrix}$$

Definition

The first non-zero number in each row, is called a *leading entry* or *pivot*.

Row Reduction and Echelon Forms

leading entry/pivot

$$\begin{bmatrix} 2 & -3 & 2 & 1 & -7 & 4 \\ 0 & 1 & -4 & 8 & 0 & 0 \\ 0 & 0 & 0 & 5 & 2 & 1 \\ 0 & 0 & 0 & 0 & 0 & 6 \end{bmatrix}$$

Definition

The first non-zero number in each row, is called a *leading entry* or *pivot*.

Row Reduction and Echelon Forms

$$\begin{bmatrix} 2 & 3 & 2 & 1 & 7 & 4 \\ 0 & 1 & 4 & 8 & 0 & 0 \\ 0 & 0 & 0 & 5 & 2 & 1 \\ 0 & 0 & 0 & 0 & 0 & 6 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Definition

A matrix is in *row echelon form* (REF) iff the following 3 rules are satisfied:

Row Reduction and Echelon Forms

$$\begin{array}{c} \left[\begin{array}{cccccc} 2 & 3 & 2 & 1 & 7 & 4 \\ 0 & 1 & 4 & 8 & 0 & 0 \\ 0 & 0 & 0 & 5 & 2 & 1 \\ 0 & 0 & 0 & 0 & 0 & 6 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right] \end{array} \begin{array}{l} \left.\begin{array}{l} \\ \\ \\ \\ \end{array}\right\} \text{non-zero rows at the top} \\ \left.\begin{array}{l} \\ \\ \end{array}\right\} \text{zero rows at the bottom} \end{array}$$

Definition

A matrix is in *row echelon form* (REF) iff the following 3 rules are satisfied:

1. All the $00000\cdots0$ rows are at the bottom;

Row Reduction and Echelon Forms

$$\begin{bmatrix} 2 & 3 & 2 & 1 & 7 & 4 \\ 0 & 1 & 4 & 8 & 0 & 0 \\ 0 & 0 & 0 & 5 & 2 & 1 \\ 0 & 0 & 0 & 0 & 0 & 6 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

down and right

Definition

A matrix is in *row echelon form* (REF) iff the following 3 rules are satisfied:

1. All the $00000\cdots0$ rows are at the bottom;
2. The leading entries/pivots go $\searrow$; and

Row Reduction and Echelon Forms

zeros under pivots

$$\begin{bmatrix} 2 & 3 & 2 & 1 & 7 & 4 \\ 0 & 1 & 4 & 8 & 0 & 0 \\ 0 & 0 & 0 & 5 & 2 & 1 \\ 0 & 0 & 0 & 0 & 0 & 6 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Definition

A matrix is in *row echelon form* (REF) iff the following 3 rules are satisfied:

1. All the $00000\cdots0$ rows are at the bottom;
2. The leading entries/pivots go $\searrow$; and
3. All the entries below a leading entry are zero.

Row Reduction and Echelon Forms

Definition

A matrix is in *reduced row echelon form* (RREF) iff the following 5 rules are satisfied:

1. All the $00000\cdots0$ rows are at the bottom;
2. The leading entries/pivots go $\searrow$; and
3. All the entries below a leading entry are zero.

$$\begin{bmatrix} 1 & 0 & 2 & 0 & 7 & 0 \\ 0 & 1 & 4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Row Reduction and Echelon Forms

Definition

A matrix is in *reduced row echelon form* (RREF) iff the following 5 rules are satisfied:

1. All the $00000\cdots0$ rows are at the bottom;
2. The leading entries/pivots go $\searrow$; and
3. All the entries below a leading entry are zero.
4. All the leading entries/pivots are 1

$$\begin{bmatrix} 1 & 0 & 2 & 0 & 7 & 0 \\ 0 & 1 & 4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Row Reduction and Echelon Forms

Definition

A matrix is in *reduced row echelon form* (RREF) iff the following 5 rules are satisfied:

1. All the $00000\cdots0$ rows are at the bottom;
2. The leading entries/pivots go $\searrow$; and
3. All the entries below a leading entry are zero.
4. All the leading entries/pivots are 1
5. Each leading 1 is the only non-zero entry in its column.

$$\begin{bmatrix} 1 & 0 & 2 & 0 & 7 & 0 \\ 0 & 1 & 4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

REF: rules 1-3 RREF: all 5 rules

Example

Consider

$$\begin{bmatrix} 2 & -3 & 2 & 1 \\ 0 & 1 & -4 & 8 \\ 0 & 0 & 0 & 5/2 \end{bmatrix}.$$

Is this matrix in REF? Is this matrix in RREF?

REF: rules 1-3

RREF: all 5 rules

1. All the $00000\cdots0$ rows are at the bottom;
2. The leading entries/pivots go $\searrow$; and
3. All the entries below a leading entry are zero.
4. All the leading entries/pivots are 1
5. Each leading 1 is the only non-zero entry in its column.

Example

Consider

$$\begin{bmatrix} 2 & -3 & 2 & 1 \\ 0 & 1 & -4 & 8 \\ 0 & 0 & 0 & 5/2 \end{bmatrix}.$$

Is this matrix in REF? Is this matrix in RREF?

REF: rules 1-3

RREF: all 5 rules

1. All the $00000\cdots0$ rows are at the bottom; ✓
2. The leading entries/pivots go $\searrow$; and
3. All the entries below a leading entry are zero.
4. All the leading entries/pivots are 1
5. Each leading 1 is the only non-zero entry in its column.

Example

Consider

$$\begin{bmatrix} 2 & -3 & 2 & 1 \\ 0 & 1 & -4 & 8 \\ 0 & 0 & 0 & 5/2 \end{bmatrix}.$$

Is this matrix in REF? Is this matrix in RREF?

REF: rules 1-3

RREF: all 5 rules

1. All the $00000\cdots0$ rows are at the bottom; ✓
2. The leading entries/pivots go $\searrow$; and ✓
3. All the entries below a leading entry are zero.
4. All the leading entries/pivots are 1
5. Each leading 1 is the only non-zero entry in its column.

Example

Consider

$$\begin{bmatrix} 2 & -3 & 2 & 1 \\ 0 & 1 & -4 & 8 \\ 0 & 0 & 0 & 5/2 \end{bmatrix}.$$

Is this matrix in REF? Is this matrix in RREF?

REF: rules 1-3

RREF: all 5 rules

1. All the $00000\cdots0$ rows are at the bottom; ✓
2. The leading entries/pivots go $\searrow$; and ✓
3. All the entries below a leading entry are zero. ✓
4. All the leading entries/pivots are 1
5. Each leading 1 is the only non-zero entry in its column.

Example

Consider

$$\begin{bmatrix} 2 & -3 & 2 & 1 \\ 0 & 1 & -4 & 8 \\ 0 & 0 & 0 & 5/2 \end{bmatrix}.$$

Is this matrix in REF? Is this matrix in RREF?

This matrix is in REF,

REF: rules 1-3

RREF: all 5 rules

1. All the $00000\cdots0$ rows are at the bottom; ✓
2. The leading entries/pivots go $\searrow$; and ✓
3. All the entries below a leading entry are zero. ✓
4. All the leading entries/pivots are 1 ✗
5. Each leading 1 is the only non-zero entry in its column.

Example

Consider

$$\begin{bmatrix} 2 & -3 & 2 & 1 \\ 0 & 1 & -4 & 8 \\ 0 & 0 & 0 & 5/2 \end{bmatrix}.$$

Is this matrix in REF? Is this matrix in RREF?

This matrix is in REF, but is not in RREF.

REF: rules 1-3 RREF: all 5 rules

Example

Consider

$$\begin{bmatrix} 1 & 0 & 0 & -2 & 3 & 0 & 2 \\ 0 & 0 & 1 & 3 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Is this matrix in REF? Is this matrix in RREF?

Row Reduction and Echelon Forms

REF: rules 1-3

RREF: all 5 rules

1. All the $00000\cdots0$ rows are at the bottom;
2. The leading entries/pivots go $\searrow$; and
3. All the entries below a leading entry are zero.
4. All the leading entries/pivots are 1
5. Each leading 1 is the only non-zero entry in its column.

Example

Consider

$$\begin{bmatrix} 1 & 0 & 0 & -2 & 3 & 0 & 2 \\ 0 & 0 & 1 & 3 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Is this matrix in REF? Is this matrix in RREF?

Row Reduction and Echelon Forms

REF: rules 1-3 RREF: all 5 rules

1. All the $00000\cdots0$ rows are at the bottom; ✓
2. The leading entries/pivots go $\searrow$; and
3. All the entries below a leading entry are zero.
4. All the leading entries/pivots are 1
5. Each leading 1 is the only non-zero entry in its column.

Example

Consider

$$\begin{bmatrix} 1 & 0 & 0 & -2 & 3 & 0 & 2 \\ 0 & 0 & 1 & 3 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Is this matrix in REF? Is this matrix in RREF?

Row Reduction and Echelon Forms

REF: rules 1-3

RREF: all 5 rules

1. All the $00000\cdots0$ rows are at the bottom; ✓
2. The leading entries/pivots go $\searrow$; and ✓
3. All the entries below a leading entry are zero.
4. All the leading entries/pivots are 1
5. Each leading 1 is the only non-zero entry in its column.

Example

Consider

$$\begin{bmatrix} 1 & 0 & 0 & -2 & 3 & 0 & 2 \\ 0 & 0 & 1 & 3 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Is this matrix in REF? Is this matrix in RREF?

REF: rules 1-3

RREF: all 5 rules

1. All the $00000\cdots0$ rows are at the bottom; ✓
2. The leading entries/pivots go $\searrow$; and ✓
3. All the entries below a leading entry are zero. ✓
4. All the leading entries/pivots are 1
5. Each leading 1 is the only non-zero entry in its column.

Example

Consider

$$\begin{bmatrix} 1 & 0 & 0 & -2 & 3 & 0 & 2 \\ 0 & 0 & 1 & 3 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Is this matrix in REF? Is this matrix in RREF?

This matrix is in REF

REF: rules 1-3

RREF: all 5 rules

1. All the $00000\cdots0$ rows are at the bottom; ✓
2. The leading entries/pivots go $\searrow$; and ✓
3. All the entries below a leading entry are zero. ✓
4. All the leading entries/pivots are 1 ✓
5. Each leading 1 is the only non-zero entry in its column.

Example

Consider

$$\begin{bmatrix} 1 & 0 & 0 & -2 & 3 & 0 & 2 \\ 0 & 0 & 1 & 3 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Is this matrix in REF? Is this matrix in RREF?

This matrix is in REF

REF: rules 1-3

RREF: all 5 rules

1. All the $00000\cdots0$ rows are at the bottom; ✓
2. The leading entries/pivots go $\searrow$; and ✓
3. All the entries below a leading entry are zero. ✓
4. All the leading entries/pivots are 1 ✓
5. Each leading 1 is the only non-zero entry in its column. ✓

Example

Consider

$$\begin{bmatrix} 1 & 0 & 0 & -2 & 3 & 0 & 2 \\ 0 & 0 & 1 & 3 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Is this matrix in REF? Is this matrix in RREF?

This matrix is in REF and in RREF.

REF: rules 1-3 RREF: all 5 rules

Example

Consider

$$\begin{bmatrix} 1 & 0 & 0 & -2 & 3 & 0 & 2 \\ 0 & 0 & 1 & 3 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Is this matrix in REF? Is this matrix in RREF?

This matrix is in REF and in RREF.

In this matrix, the **pivot columns** are the **first**, **third**, and **sixth** columns and the **pivot entries** are the leading nonzero entries in the **first**, **second** and **third** rows.

EXAMPLE 1 The following matrices are in echelon form. The leading entries (■) may have any nonzero value; the starred entries (∗) may have any value (including zero).

$$\begin{bmatrix} \blacksquare & * & * & * \\ 0 & \blacksquare & * & * \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & \blacksquare & * & * & * & * & * & * & * & * \\ 0 & 0 & 0 & \blacksquare & * & * & * & * & * & * \\ 0 & 0 & 0 & 0 & \blacksquare & * & * & * & * & * \\ 0 & 0 & 0 & 0 & 0 & \blacksquare & * & * & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \blacksquare & * \end{bmatrix}$$

The following matrices are in reduced echelon form because the leading entries are 1's, and there are 0's below *and above* each leading 1.

$$\begin{bmatrix} 1 & 0 & * & * \\ 0 & 1 & * & * \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 1 & * & 0 & 0 & 0 & * & * & 0 & * \\ 0 & 0 & 0 & 1 & 0 & 0 & * & * & 0 & * \\ 0 & 0 & 0 & 0 & 1 & 0 & * & * & 0 & * \\ 0 & 0 & 0 & 0 & 0 & 1 & * & * & 0 & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & * \end{bmatrix}$$

■

Definition

A pivot position in a matrix A is a location in A that corresponds to a leading 1 in the reduced row echelon form of A.

Definition

A pivot column is a column of A that contains a pivot position.

Definition

A pivot position in a matrix A is a location in A that corresponds to a leading 1 in the reduced row echelon form of A.

Definition

A pivot column is a column of A that contains a pivot position.

Theorem (Uniqueness of the row reduced echelon form)

Each matrix is row equivalent to one and only one RREF matrix.

Gaussian Elimination

Example

Row reduce the matrix below to row echelon form (REF) and then to reduced row echelon form (RREF):

$$\begin{bmatrix} 0 & -3 & -6 & 4 & 9 \\ -1 & -2 & -1 & 3 & 1 \\ -2 & -3 & 0 & 3 & -1 \\ 1 & 4 & 5 & -9 & -7 \end{bmatrix}.$$

Gaussian Elimination

Example

Row reduce the matrix below to row echelon form (REF) and then to reduced row echelon form (RREF):

$$\begin{bmatrix} 0 & -3 & -6 & 4 & 9 \\ -1 & -2 & -1 & 3 & 1 \\ -2 & -3 & 0 & 3 & -1 \\ 1 & 4 & 5 & -9 & -7 \end{bmatrix}.$$

So how do we do this? How do we know which elementary row operation to use at each step? We need an algorithm.

Row Reduction and Echelon Forms

- **Step 1:** Find the first non-zero column (from the left).

Row Reduction and Echelon Forms

- **Step 1:** Find the first non-zero column (from the left).
- **Step 2:** We don't want a 0 at the top of this column. If necessary swap the first row with another row.

Row Reduction and Echelon Forms

- **Step 1:** Find the first non-zero column (from the left).
- **Step 2:** We don't want a 0 at the top of this column. If necessary swap the first row with another row.
- **Step 3:** Create zeros <u>under</u> the pivot.

Row Reduction and Echelon Forms

- **Step 1:** Find the first non-zero column (from the left).
- **Step 2:** We don't want a 0 at the top of this column. If necessary swap the first row with another row.
- **Step 3:** Create zeros <u>under</u> the pivot.
- **Step 4:** Ignore the row and column containing the pivot. Goto step 1.

Row Reduction and Echelon Forms

- **Step 1:** Find the first non-zero column (from the left).
- **Step 2:** We don't want a 0 at the top of this column. If necessary swap the first row with another row.
- **Step 3:** Create zeros <u>under</u> the pivot.
- **Step 4:** Ignore the row and column containing the pivot. Goto step 1.

Keep going until your matrix is in row echelon form (REF). If you want reduced REF, then continue to step 5.

Row Reduction and Echelon Forms

- **Step 1:** Find the first non-zero column (from the left).
- **Step 2:** We don't want a 0 at the top of this column. If necessary swap the first row with another row.
- **Step 3:** Create zeros <u>under</u> the pivot.
- **Step 4:** Ignore the row and column containing the pivot. Goto step 1.

Keep going until your matrix is in row echelon form (REF). If you want reduced REF, then continue to step 5.

- **Step 5:** Make every pivot a 1 by multiplying rows by constants.

Row Reduction and Echelon Forms

- **Step 1:** Find the first non-zero column (from the left).
- **Step 2:** We don't want a 0 at the top of this column. If necessary swap the first row with another row.
- **Step 3:** Create zeros <u>under</u> the pivot.
- **Step 4:** Ignore the row and column containing the pivot. Goto step 1.

Keep going until your matrix is in row echelon form (REF). If you want reduced REF, then continue to step 5.

- **Step 5:** Make every pivot a 1 by multiplying rows by constants.
- **Step 6:** Starting with the bottom-rightmost pivot and working upwards, create zeros <u>above</u> each pivot.

Row Reduction and Echelon Forms

- **Step 1:** Find the first non-zero column (from the left).
- **Step 2:** We don't want a 0 at the top of this column. If necessary swap the first row with another row.
- **Step 3:** Create zeros <u>under</u> the pivot.
- **Step 4:** Ignore the row and column containing the pivot. Goto step 1.

Keep going until your matrix is in row echelon form (REF). If you want reduced REF, then continue to step 5.

- **Step 5:** Make every pivot a 1 by multiplying rows by constants.
- **Step 6:** Starting with the bottom-rightmost pivot and working upwards, create zeros <u>above</u> each pivot.

Your matrix should now by in RREF.

If we want REF we do steps 1-4. This is called *Gaussian Elimination*.

Carl Friedrich Gauss

BORN
30 April 1777

DECEICED
23 February 1855

NATIONALITY
German

Row Reduction and Echelon Forms

If we want REF we do steps 1-4. This is called *Gaussian Elimination*.

If we want RREF, we do steps 1-6. This is called *Gauss-Jordan Elimination*.

Wilhelm Jordan

BORN
1 March 1842

DECEICED
17 April 1899

NATIONALITY
German

Another name is *row reduction*.

Now let's do this:

Example

Row reduce the matrix below to row echelon form (REF) and then to reduced row echelon form (RREF):

$$\begin{bmatrix} 0 & -3 & -6 & 4 & 9 \\ -1 & -2 & -1 & 3 & 1 \\ -2 & -3 & 0 & 3 & -1 \\ 1 & 4 & 5 & -9 & -7 \end{bmatrix}.$$

- **Step 1:** Find the first non-zero column (from the left).

$$\begin{bmatrix} 0 & -3 & -6 & 4 & 9 \\ -1 & -2 & -1 & 3 & 1 \\ -2 & -3 & 0 & 3 & -1 \\ 1 & 4 & 5 & -9 & -7 \end{bmatrix}.$$

This column is not all zeros.

Row Reduction and Echelon Forms

- **Step 2:** We don't want a 0 at the top of this column. If necessary swap the first row with another row.

I don't want 0 here.

$$\begin{bmatrix} 0 & -3 & -6 & 4 & 9 \\ -1 & -2 & -1 & 3 & 1 \\ -2 & -3 & 0 & 3 & -1 \\ 1 & 4 & 5 & -9 & -7 \end{bmatrix}$$

- **Step 2:** We don't want a 0 at the top of this column. If necessary swap the first row with another row.

I don't want 0 here.

$$\begin{bmatrix} 0 & -3 & -6 & 4 & 9 \\ -1 & -2 & -1 & 3 & 1 \\ -2 & -3 & 0 & 3 & -1 \\ 1 & 4 & 5 & -9 & -7 \end{bmatrix}$$

$R_1 \leftrightarrow R_4$

$$\begin{bmatrix} 1 & 4 & 5 & -9 & -7 \\ -1 & -2 & -1 & 3 & 1 \\ -2 & -3 & 0 & 3 & -1 \\ 0 & -3 & -6 & 4 & 9 \end{bmatrix}$$

- **Step 2:** We don't want a 0 at the top of this column. If necessary swap the first row with another row.

I don't want 0 here.

$$\begin{bmatrix} 0 & -3 & -6 & 4 & 9 \\ -1 & -2 & -1 & 3 & 1 \\ -2 & -3 & 0 & 3 & -1 \\ 1 & 4 & 5 & -9 & -7 \end{bmatrix}$$

pivot/leading entry

$R_1 \leftrightarrow R_4$

$$\begin{bmatrix} 1 & 4 & 5 & -9 & -7 \\ -1 & -2 & -1 & 3 & 1 \\ -2 & -3 & 0 & 3 & -1 \\ 0 & -3 & -6 & 4 & 9 \end{bmatrix}$$

- **Step 3:** Create zeros <u>under</u> the pivot.

$$\text{I want} \quad \begin{matrix} 0 \\ 0 \\ 0 \end{matrix} \quad \text{here.} \qquad \begin{bmatrix} 1 & 4 & 5 & -9 & -7 \\ -1 & -2 & -1 & 3 & 1 \\ -2 & -3 & 0 & 3 & -1 \\ 0 & -3 & -6 & 4 & 9 \end{bmatrix}$$

- **Step 3:** Create zeros <u>under</u> the pivot.

I want $\begin{matrix} 0 \\ 0 \\ 0 \end{matrix}$ here.

$$\begin{bmatrix} 1 & 4 & 5 & -9 & -7 \\ -1 & -2 & -1 & 3 & 1 \\ -2 & -3 & 0 & 3 & -1 \\ 0 & -3 & -6 & 4 & 9 \end{bmatrix}$$

$R_1 + R_2 \to R_2$

$$\begin{bmatrix} 1 & 4 & 5 & -9 & -7 \\ 0 & 2 & 4 & -6 & -6 \\ -2 & -3 & 0 & 3 & -1 \\ 0 & -3 & -6 & 4 & 9 \end{bmatrix}$$

Row Reduction and Echelon Forms

$$\begin{bmatrix} 1 & 4 & 5 & -9 & -7 \\ 0 & 2 & 4 & -6 & -6 \\ -2 & -3 & 0 & 3 & -1 \\ 0 & -3 & -6 & 4 & 9 \end{bmatrix}$$

$$2R_1 + R_3 \to R_3 \qquad \begin{bmatrix} 1 & 4 & 5 & -9 & -7 \\ 0 & 2 & 4 & -6 & -6 \\ 0 & 5 & 10 & -15 & -15 \\ 0 & -3 & -6 & 4 & 9 \end{bmatrix}$$

Row Reduction and Echelon Forms

- **Step 4:** Ignore the row and column containing the pivot. Goto step 1.

$$\begin{bmatrix} 1 & 4 & 5 & -9 & -7 \\ 0 & 2 & 4 & -6 & -6 \\ 0 & 5 & 10 & -15 & -15 \\ 0 & -3 & -6 & 4 & 9 \end{bmatrix}$$

Row Reduction and Echelon Forms

- **Step 4:** Ignore the row and column containing the pivot. Goto step 1.

$$\begin{bmatrix} 1 & 4 & 5 & -9 & -7 \\ 0 & 2 & 4 & -6 & -6 \\ 0 & 5 & 10 & -15 & -15 \\ 0 & -3 & -6 & 4 & 9 \end{bmatrix}$$

ignore

- **Step 4:** Ignore the row and column containing the pivot. Goto step 1.

$$\begin{bmatrix} 1 & 4 & 5 & -9 & -7 \\ 0 & 2 & 4 & -6 & -6 \\ 0 & 5 & 10 & -15 & -15 \\ 0 & -3 & -6 & 4 & 9 \end{bmatrix}$$

ignore

ignore

- **Step 4:** Ignore the row and column containing the pivot. Goto step 1.

$$\begin{bmatrix} 1 & 4 & 5 & -9 & -7 \\ 0 & 2 & 4 & -6 & -6 \\ 0 & 5 & 10 & -15 & -15 \\ 0 & -3 & -6 & 4 & 9 \end{bmatrix}$$

ignore

ignore

$$\begin{bmatrix} 1 & 4 & 5 & -9 & -7 \\ 0 & 2 & 4 & -6 & -6 \\ 0 & 5 & 10 & -15 & -15 \\ 0 & -3 & -6 & 4 & 9 \end{bmatrix}$$

Row Reduction and Echelon Forms

$$\begin{bmatrix} 1 & 4 & 5 & -9 & -7 \\ 0 & 2 & 4 & -6 & -6 \\ 0 & 5 & 10 & -15 & -15 \\ 0 & -3 & -6 & 4 & 9 \end{bmatrix}$$

This column is not all zeros.

Row Reduction and Echelon Forms

I don't want 0 here.

$$\begin{bmatrix} 1 & 4 & 5 & -9 & -7 \\ 0 & 2 & 4 & -6 & -6 \\ 0 & 5 & 10 & -15 & -15 \\ 0 & -3 & -6 & 4 & 9 \end{bmatrix}$$

Row Reduction and Echelon Forms

I want $\begin{matrix} 0 \\ 0 \end{matrix}$ here.

$$\begin{bmatrix} 1 & 4 & 5 & -9 & -7 \\ 0 & 2 & 4 & -6 & -6 \\ 0 & 5 & 10 & -15 & -15 \\ 0 & -3 & -6 & 4 & 9 \end{bmatrix}$$

Row Reduction and Echelon Forms

I want $\begin{matrix} 0 \\ 0 \end{matrix}$ here.

$$\begin{bmatrix} 1 & 4 & 5 & -9 & -7 \\ 0 & 2 & 4 & -6 & -6 \\ 0 & 5 & 10 & -15 & -15 \\ 0 & -3 & -6 & 4 & 9 \end{bmatrix}$$

$$-\tfrac{5}{2}R_2 + R_3 \to R_3 \qquad \begin{bmatrix} 1 & 4 & 5 & -9 & -7 \\ 0 & 2 & 4 & -6 & -6 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & -3 & -6 & 4 & 9 \end{bmatrix}$$

Row Reduction and Echelon Forms

$$\begin{bmatrix} 1 & 4 & 5 & -9 & -7 \\ 0 & 2 & 4 & -6 & -6 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & -3 & -6 & 4 & 9 \end{bmatrix}$$

$$\frac{3}{2}R_2 + R_4 \to R_4 \qquad \begin{bmatrix} 1 & 4 & 5 & -9 & -7 \\ 0 & 2 & 4 & -6 & -6 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -5 & 0 \end{bmatrix}$$

Row Reduction and Echelon Forms

$$\begin{bmatrix} 1 & 4 & 5 & -9 & -7 \\ 0 & 2 & 4 & -6 & -6 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -5 & 0 \end{bmatrix}$$

Row Reduction and Echelon Forms

$$\begin{bmatrix} 1 & 4 & 5 & -9 & -7 \\ 0 & 2 & 4 & -6 & -6 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -5 & 0 \end{bmatrix}$$

Ignore R_2 and column 2. Find the first non-zero column.

$$\begin{bmatrix} 1 & 4 & 5 & -9 & -7 \\ 0 & 2 & 4 & -6 & -6 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -5 & 0 \end{bmatrix}$$

Row Reduction and Echelon Forms

$$\begin{bmatrix} 1 & 4 & 5 & -9 & -7 \\ 0 & 2 & 4 & -6 & -6 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -5 & 0 \end{bmatrix}$$

Ignore R_2 and column 2. Find the first non-zero column.

$$\begin{bmatrix} 1 & 4 & 5 & -9 & -7 \\ 0 & 2 & 4 & -6 & -6 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -5 & 0 \end{bmatrix}$$

This column is all zeros.

Row Reduction and Echelon Forms

$$\begin{bmatrix} 1 & 4 & 5 & -9 & -7 \\ 0 & 2 & 4 & -6 & -6 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -5 & 0 \end{bmatrix}$$

Ignore R_2 and column 2. Find the first non-zero column.

$$\begin{bmatrix} 1 & 4 & 5 & -9 & -7 \\ 0 & 2 & 4 & -6 & -6 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -5 & 0 \end{bmatrix}$$

This column is not all zeros.

Row Reduction and Echelon Forms

$$\begin{bmatrix} 1 & 4 & 5 & -9 & -7 \\ 0 & 2 & 4 & -6 & -6 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -5 & 0 \end{bmatrix}$$

Ignore R_2 and column 2. Find the first non-zero column.

I don't want 0 here.

$$\begin{bmatrix} 1 & 4 & 5 & -9 & -7 \\ 0 & 2 & 4 & -6 & -6 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -5 & 0 \end{bmatrix}$$

Row Reduction and Echelon Forms

$$\begin{bmatrix} 1 & 4 & 5 & -9 & -7 \\ 0 & 2 & 4 & -6 & -6 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -5 & 0 \end{bmatrix}$$

$R_3 \leftrightarrow R_4$

$$\begin{bmatrix} 1 & 4 & 5 & -9 & -7 \\ 0 & 2 & 4 & -6 & -6 \\ 0 & 0 & 0 & -5 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

[1] row echelon form

Row Reduction and Echelon Forms

$$\begin{bmatrix} 1 & 4 & 5 & -9 & -7 \\ 0 & 2 & 4 & -6 & -6 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -5 & 0 \end{bmatrix}$$

$R_3 \leftrightarrow R_4$

$$\begin{bmatrix} 1 & 4 & 5 & -9 & -7 \\ 0 & 2 & 4 & -6 & -6 \\ 0 & 0 & 0 & -5 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

The matrix is now in REF[1].

[1] row echelon form

Row Reduction and Echelon Forms

- **Step 5:** Make every pivot a 1 by multiplying rows by constants.

$$\begin{bmatrix} 1 & 4 & 5 & -9 & -7 \\ 0 & 2 & 4 & -6 & -6 \\ 0 & 0 & 0 & -5 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Row Reduction and Echelon Forms

- **Step 5:** Make every pivot a 1 by multiplying rows by constants.

pivot

$$\begin{bmatrix} 1 & 4 & 5 & -9 & -7 \\ 0 & 2 & 4 & -6 & -6 \\ 0 & 0 & 0 & -5 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Row Reduction and Echelon Forms

- **Step 5:** Make every pivot a 1 by multiplying rows by constants.

$$\begin{bmatrix} 1 & 4 & 5 & -9 & -7 \\ 0 & 2 & 4 & -6 & -6 \\ 0 & 0 & 0 & -5 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

pivot columns

Row Reduction and Echelon Forms

- **Step 5:** Make every pivot a 1 by multiplying rows by constants.

$$\begin{bmatrix} 1 & 4 & 5 & -9 & -7 \\ 0 & 2 & 4 & -6 & -6 \\ 0 & 0 & 0 & -5 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{aligned} \tfrac{1}{2}R_2 &\to R_2 \\ -\tfrac{1}{5}R_3 &\to R_3 \end{aligned} \qquad \begin{bmatrix} 1 & 4 & 5 & -9 & -7 \\ 0 & 1 & 2 & -3 & -3 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

- **Step 5:** Make every pivot a 1 by multiplying rows by constants.

$$\begin{bmatrix} 1 & 4 & 5 & -9 & -7 \\ 0 & 2 & 4 & -6 & -6 \\ 0 & 0 & 0 & -5 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 4 & 5 & -9 & -7 \\ 0 & 1 & 2 & -3 & -3 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$\frac{1}{2}R_2 \to R_2$

$-\frac{1}{5}R_3 \to R_3$

WARNING

Be very careful if you do 2 row operations in the same step.

Do not change the same row twice in one step.

Row Reduction and Echelon Forms

- **Step 6:** Starting with the bottom-rightmost pivot and working upwards, create zeros <u>above</u> each pivot.

$$\begin{bmatrix} 1 & 4 & 5 & -9 & -7 \\ 0 & 1 & 2 & -3 & -3 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Row Reduction and Echelon Forms

- **Step 6:** Starting with the bottom-rightmost pivot and working upwards, create zeros <u>above</u> each pivot.

first make this 0

$$\begin{bmatrix} 1 & 4 & 5 & -9 & -7 \\ 0 & 1 & 2 & -3 & -3 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Row Reduction and Echelon Forms

- **Step 6:** Starting with the bottom-rightmost pivot and working upwards, create zeros <u>above</u> each pivot.

then make this 0

$$\begin{bmatrix} 1 & 4 & 5 & -9 & -7 \\ 0 & 1 & 2 & -3 & -3 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Row Reduction and Echelon Forms

- **Step 6:** Starting with the bottom-rightmost pivot and working upwards, create zeros <u>above</u> each pivot.

finally make this 0

$$\begin{bmatrix} 1 & 4 & 5 & -9 & -7 \\ 0 & 1 & 2 & -3 & -3 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Row Reduction and Echelon Forms

- **Step 6:** Starting with the bottom-rightmost pivot and working upwards, create zeros <u>above</u> each pivot.

$$\begin{bmatrix} 1 & 4 & 5 & -9 & -7 \\ 0 & 1 & 2 & -3 & -3 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$3R_3 + R_2 \to R_2 \qquad \begin{bmatrix} 1 & 4 & 5 & -9 & -7 \\ 0 & 1 & 2 & 0 & -3 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Row Reduction and Echelon Forms

$$\begin{bmatrix} 1 & 4 & 5 & -9 & -7 \\ 0 & 1 & 2 & 0 & -3 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$9R_3 + R_1 \to R_1$

$$\begin{bmatrix} 1 & 4 & 5 & 0 & -7 \\ 0 & 1 & 2 & 0 & -3 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Row Reduction and Echelon Forms

$$\begin{bmatrix} 1 & 4 & 5 & -9 & -7 \\ 0 & 1 & 2 & 0 & -3 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$9R_3 + R_1 \to R_1$

$$\begin{bmatrix} 1 & 4 & 5 & 0 & -7 \\ 0 & 1 & 2 & 0 & -3 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$-4R_2 + R_1 \to R_1$

$$\begin{bmatrix} 1 & 0 & -3 & 0 & 5 \\ 0 & 1 & 2 & 0 & -3 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Row Reduction and Echelon Forms

$$\begin{bmatrix} 1 & 0 & -3 & 0 & 5 \\ 0 & 1 & 2 & 0 & -3 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

The matrix is now in RREF[2].

[2] reduced row echelon form

Row Reduction and Echelon Forms

$$\underset{\text{original matrix}}{\begin{bmatrix} 0 & -3 & -6 & 4 & 9 \\ -1 & -2 & -1 & 3 & 1 \\ -2 & -3 & 0 & 3 & -1 \\ 1 & 4 & 5 & -9 & -7 \end{bmatrix}} \xrightarrow[\text{operations}]{\text{row}} \underset{\text{RREF matrix}}{\begin{bmatrix} 1 & 0 & -3 & 0 & 5 \\ 0 & 1 & 2 & 0 & -3 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}}$$

Row Reduction and Echelon Forms

pivots

$$\begin{bmatrix} 0 & -3 & -6 & 4 & 9 \\ -1 & -2 & -1 & 3 & 1 \\ -2 & -3 & 0 & 3 & -1 \\ 1 & 4 & 5 & -9 & -7 \end{bmatrix} \xrightarrow[\text{operations}]{\text{row}} \begin{bmatrix} 1 & 0 & -3 & 0 & 5 \\ 0 & 1 & 2 & 0 & -3 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Row Reduction and Echelon Forms

pivot positions

pivots

$$\begin{bmatrix} 0 & -3 & -6 & 4 & 9 \\ -1 & -2 & -1 & 3 & 1 \\ -2 & -3 & 0 & 3 & -1 \\ 1 & 4 & 5 & -9 & -7 \end{bmatrix} \xrightarrow[\text{operations}]{\text{row}} \begin{bmatrix} 1 & 0 & -3 & 0 & 5 \\ 0 & 1 & 2 & 0 & -3 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Row Reduction and Echelon Forms

pivot positions

pivots

$$\begin{bmatrix} 0 & -3 & -6 & 4 & 9 \\ -1 & -2 & -1 & 3 & 1 \\ -2 & -3 & 0 & 3 & -1 \\ 1 & 4 & 5 & -9 & -7 \end{bmatrix} \xrightarrow[\text{operations}]{\text{row}} \begin{bmatrix} 1 & 0 & -3 & 0 & 5 \\ 0 & 1 & 2 & 0 & -3 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

pivot columns

Next Time

- Solutions of Linear Systems
- Introduction to Matrices
- The Inverse of a Matrix

This is the end of the lecture. Please ask your questions now.
You are welcome to leave whenever you wish.