2019–20 **MATH216 Mathematics IV – Solutions to Exercise Sheet 2** N. Course

Exercise 8 (Separable Equations). Solve the following initial value problems:

(a) $\begin{cases} \frac{dy}{dx} = (1-2x)y^2 \\ y(0) = -\frac{1}{6} \end{cases}$ (b) $\begin{cases} x + ye^{-x}\frac{dy}{dx} = 0 \\ y(0) = 1 \end{cases}$ (c) $\begin{cases} \frac{dy}{dx} = \frac{2x}{y+x^2y} \\ y(0) = -2 \end{cases}$

Solution 8.

(a) Rearrange to $\frac{dy}{y^2} = (1-2x)dx$. Then integrate to get $-\frac{1}{y} = x - x^2 + C$, and rearrange to $y = \frac{1}{x^2-x-C}$. To satisfy $y(0) = -\frac{1}{6}$ we must have $y = \frac{1}{x^2-x-6}$. This solution exists for $-2 < x < 3$.

(b) First rearrange to $ydy = -xe^x dx$. Integrating gives $\frac{1}{2}y^2 = (1-x)e^x + C$ which rearranges to $y = \pm\sqrt{2(1-x)e^x + 2C}$. Then we use the initial condition to calculate that $1 = y(0) = +\sqrt{2+2C} \implies 2C = -1$. Therefore $y = \sqrt{2(1-x)e^x - 1}$.

(c) We can rearrange the ODE to $ydy = \frac{2x}{1+x^2}dx$ and then integrate to obtain $\frac{1}{2}y^2 = \ln(1+x^2) + C$. To satisfy $y(0) = -2$ we must have $C = 2$. Therefore $y = -\sqrt{2\ln(1+x^2)+4}$.

Exercise 9 (Stable, Unstable and Semi-Stable Equilibrium Solutions). Each of the following problems involve equations of the form $y' = f(y)$. In each problem, (i) sketch the graph of $f(y)$ versus y; (ii) find the equilibrium solutions (critical points) of the ODE; and (iii) classify each equilibrium solution as asymptotically stable, semi-stable, or unstable.

(a) $\frac{dy}{dt} = ay + by^2$, $a, b > 0$, $y_0 \geq 0$.

(b) $\frac{dy}{dt} = ay + by^2$, $a, b > 0$, $-\infty < y_0 < \infty$.

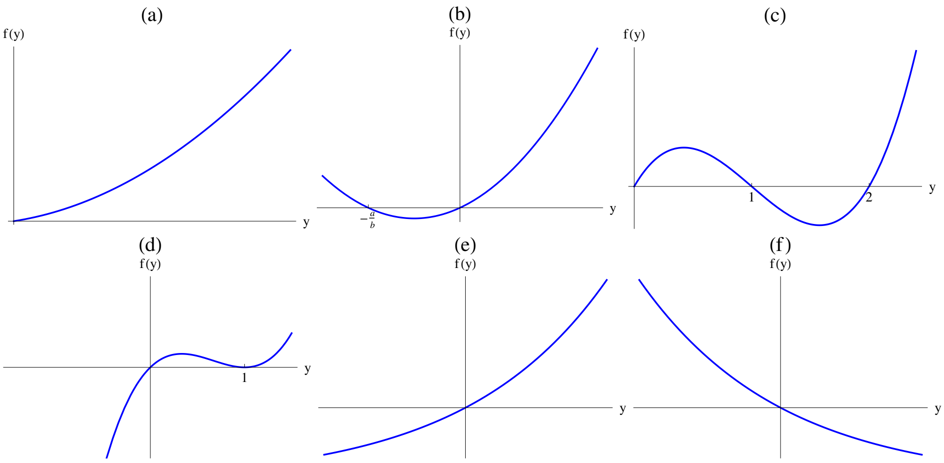
(c) $\frac{dy}{dt} = y(y-1)(y-2)$, $y_0 \geq 0$.

(d) $\frac{dy}{dt} = y(1-y)^2$, $-\infty < y_0 < \infty$.

(e) $\frac{dy}{dt} = e^y - 1$, $-\infty < y_0 < \infty$.

(f) $\frac{dy}{dt} = e^{-y} - 1$, $-\infty < y_0 < \infty$.

Solution 9.



(a) $y = 0$ is unstable,

(b) $y = -a/b$ is asymptotically stable, $y = 0$ is unstable,

(c) $y = 1$ is asymptotically stable, $y = 0$ and $y = 2$ are unstable,

(d) $y = 0$ is unstable, $y = 1$ is semi-stable,

(e) $y = 0$ is unstable,

(f) $y = 0$ is asymptotically stable.

Exercise 10 (Sick Students).

Suppose that the students of İstanbul Okan Üniversitesi can be divided into two groups; those who have the flu virus and can infect others, and those who do not have it but are susceptible. Let x be the proportion of susceptible individuals and y the proportion of infectious individuals; then $x+y=1$.

Assume that the disease spreads by contact between sick students and well students, and that the rate of spread $\frac{dy}{dt}$ is proportional to the number of such contacts. So $\frac{dy}{dt} = k_1 \times$ (number of contacts). Further, assume that members of both groups move about freely among each other, so the number of contacts is proportional to the product of x and y. So (number of contacts) $= k_2xy$. Since $x = 1-y$, we obtain the initial value problem

$$\begin{cases} \frac{dy}{dt} = \alpha y(1-y), \\ y(0) = y_0, \end{cases} \tag{1}$$

where $\alpha > 0$ is a constant, and $0 \leq y_0 \leq 1$ is the initial proportion of infectious individuals.

İstanbul Okan Üniversitesi öğrencilerinin iki gruba ayrıldıklarını varsayın; grip virüsü taşıyan, diğer öğrencilere bulaştırabilecek olanlar ve virüsü taşımayan ancak hastalığa yakalanabilecek olanlar. Hastalığa yakalanabilecek bireylerin oranı x; hastalığı taşıyan ve bulaştırabilecek olanların oranı y'dir. Bu durumda $x+y=1$.

Hastalığın, hasta öğrencilerle sağlıklı öğrenciler arasında etkileşimle yayıldığını, ve $\frac{dy}{dt}$ olan yayılma hızının etkileşim sayısıyla orantılı olduğunu varsayın. Yani $\frac{dy}{dt} = k_1 \times$ (etkileşim sayısı). Ayrıca, her iki grubun üyelerinin birbirlerinin arasında serbestçe dolaştıklarını varsayın; böylece etkileşim sayısı x ve yenin çarpımları ile orantılıdır. Yani, (etkileş im sayısı) $= k_2xy$. $x = 1-y$ olduğundan, (1)'i elde ederiz. $\alpha > 0$ sabit sayıdır, $0 \leq y_0 \leq 1$ hastalık bulaştırabilecek öğrencilerin en baştaki oranıdır.

(a) Find the equilibrium points for the differential equation and determine whether each is asymptotically stable, semi-stable, or unstable.

(b) Draw the graphs of some solutions.

(c) Solve (1).

(d) Suppose that $y_0 > 0$. Show that $\lim_{t\to\infty} y(t) = 1$, which means that ultimately all students catch the disease.

Solution 10.

(a) $y=0$ is unstable, $y=1$ is asymptotically stable.

(b) omitted.

(c) $y(t) = \dfrac{y_0}{y_0 + (1-y_0)e^{-\alpha t}}$.

(d) $\lim_{t\to\infty} y(t) = \lim_{t\to\infty} \dfrac{y_0}{y_0 + (1-y_0)e^{-\alpha t}} = \dfrac{y_0}{y_0+0} = 1.$

Exercise 11 (Exact Equations). Determine if each of the following ODEs is an exact equation. If it is exact, find the solution.

(a) $(2x+4y) + (2x-2y)y' = 0$

(b) $(2x+3) + (2y-2)y' = 0$

(c) $(3x^2-2xy+2) + (6y^2-x^2+3)\dfrac{dy}{dx} = 0$

(d) $(2xy^2+2y) + (2x^2y+2x)y' = 0$

(e) $(e^x \sin y - 2y\sin x) + (e^x\cos y + 2\cos x)\dfrac{dy}{dx} = 0$

(f) $(e^x\sin y + 2y)dx + (3x - e^x \sin y)dy = 0$

Solution 11.

(a) The ODE is not exact.

(b) The ODE is exact and has solution $x^2+3x+y^2-2y = c$.

(c) The ODE is exact and has solution $x^3 - x^2y + 2x + 2y^3 + 3y = c$.

(d) The ODE is exact and has solution $x^2y^2 + 2xy = c$.

(e) The ODE is exact and has solution $e^x \sin y + 2y\cos x = c$.

(f) The ODE is not exact.

Exercise 12 (Exact Equations). The following equations are not exact. For each one, (i) find an integrating factor ($\mu(x)$ or $\mu(y)$) which changes the equation into an exact equation; and (ii) solve the equation.

(a) $(3x^2y + 2xy + y^3) + (x^2+y^2)y' = 0$

(b) $y' = e^{2x} + y - 1$

(c) $dx + \left(\dfrac{x}{y} - \sin y\right) dy = 0$

(d) $y + (2xy - e^{-2y})y' = 0$

Solution 12.

(a) $\mu(x) = e^{3x}$; $\quad (3x^2y + y^3)e^{3x} = c$

(b) $\mu(x) = e^{-x}$; $\quad y = ce^x + 1 + e^{2x}$

(c) $\mu(y) = y$; $\quad xy + y\cos y - \sin y = c$

(d) $\mu(y) = \frac{e^{2y}}{y}$; $\quad xe^{2y} - \ln|y| = c$

Exercise 13 (Homogeneous Equations)**.** Use the substitution $v(x) = \frac{y}{x}$ (or equivalently $y = v(x)x$ and then $y' = v'(x)x + v(x)$) to solve the following ODEs:

(a) $(x^2 + 3xy + y^2)dx - x^2dy = 0$

(b) $\dfrac{dy}{dx} = \dfrac{x^2 - 3y^2}{2xy}$

(c) $x\dfrac{dy}{dx} = y + \sqrt{x^2 - y^2}$

Solution 13.

(a) $\dfrac{x}{x+y} + \ln|x| = c$

(b) $|x|^3\,|x^2 - 5y^2| = c$

(c) $y = x\sin(\ln x + C)$

Exercise 14 (Bernoulli Equations)**.** We can use the substitution $v(x) = y^{1-n}$ to solve $y' + p(t)y = q(t)y^n$. Use this technique to solve the following ODEs:

(a) $t^2y' + 2ty - y^3 = 0$

(b) $y' = ry - ky^2$ (where $r > 0$ and $k > 0$ are constants). This is an autonomous equation called the Logistic Equation.

(c) $y' = \varepsilon y - \sigma y^3$ (where $\varepsilon > 0$ and $\sigma > 0$ are constants). This equation occurs in the study of the stability of fluid flow.

Solution 14.

(a) $y = \pm\sqrt{\dfrac{5t}{2 + 5ct^5}}$

(b) $y = \dfrac{r}{k + cre^{-rt}}$

(c) $y = \pm\sqrt{\dfrac{\varepsilon}{\sigma + c\varepsilon e^{-2\varepsilon t}}}$