

Week 15

- 33. The Fundamental Theorem of Calculus
- 34. The Substitution Method
- 35. Area Between Curves

The Fundamental Theorem of Calculus

32. The Fundamental Theorem of Calculus



We don't want to have to use Riemann sums every time we need to calculate a definite integral – we want a better way.

The following theorem is the most important theorem in Calculus. If you can only memorise one theorem for the exams, it should be this one.

32. The Fundamental Theorem of Calculus



Theorem (The Fundamental Theorem of Calculus)

Suppose that $f : [a, b] \rightarrow \mathbb{R}$ is a continuous function.

32. The Fundamental Theorem of Calculus



1 Then the function $F : [a, b] \rightarrow \mathbb{R}$ defined by

$$F(x) = \int_a^x f(t) \, dt$$

is continuous on $[a, b]$; differentiable on (a, b) ; and its derivative is

$$F'(x) = \frac{d}{dx} \int_a^x f(t) \, dt = f(x).$$

32. The Fundamental Theorem of Calculus



2 If F is any antiderivative of f on $[a, b]$, then

$$\int_a^b f(x) \, dx = F(b) - F(a).$$

32. The Fundamental Theorem of Calculus



Remark

Part (i) of the theorem tells how to differentiate $\int_a^x f(t) dt$.

Example

Find $\frac{dy}{dx}$ if $y = \int_a^x (t^3 + 1) dt$.

solution:

$$\frac{dy}{dx} = \frac{d}{dx} \int_a^x (t^3 + 1) dt = x^3 + 1.$$

Example

Find $\frac{dy}{dx}$ if $y = \int_1^x \sin t dt$.

solution:

$$\frac{dy}{dx} = \frac{d}{dx} \int_1^x \sin t dt = \sin x.$$

32. The Fundamental Theorem of Calculus



Example

Find $\frac{dy}{dx}$ if $y = \int_0^x \sin \ln \tan e^{t^2} dt$.

solution:

$$\frac{dy}{dx} = \frac{d}{dx} \int_0^x \sin \ln \tan e^{t^2} dt = \sin \ln \tan e^{x^2}.$$

32. The Fundamental Theorem of Calculus



Example

Find $\frac{dy}{dx}$ if $y = \int_x^5 3t \sin t \, dt$.

solution:

$$\begin{aligned}\frac{dy}{dx} &= \frac{d}{dx} \int_x^5 3t \sin t \, dt \\ &= \frac{d}{dx} \left(- \int_5^x 3t \sin t \, dt \right) \\ &= -3x \sin x.\end{aligned}$$

32. The Fundamental Theorem of Calculus



Example

Find $\frac{dy}{dx}$ if $y = \int_1^{x^2} \cos t \, dt$.

solution: This time we will need to use the Chain rule. Let $u = x^2$. Then

$$\begin{aligned}\frac{dy}{dx} &= \frac{dy}{du} \frac{du}{dx} \\ &= \left(\frac{d}{du} \int_1^u \cos t \, dt \right) \left(\frac{d}{dx} x^2 \right) \\ &= (\cos u) (2x) = 2x \cos x^2.\end{aligned}$$

32. The Fundamental Theorem of Calculus



Remark

Part (ii) of the theorem tells us how to calculate the definite integral of f over $[a, b]$:

- 1 Find an antiderivative F of f .
- 2 Calculate $F(b) - F(a)$.

Notation

We will write

$$\left[F(x) \right]_a^b = F(b) - F(a).$$

32. The Fundamental Theorem of Calculus



Example

$$\begin{aligned}\int_0^{\pi} \cos x \, dx &= \left[\sin x \right]_0^{\pi} \\ &\quad \left(\text{because } \frac{d}{dx} \sin x = \cos x \right) \\ &= \sin \pi - \sin 0 \\ &= 0 - 0 \\ &= 0\end{aligned}$$

32. The Fundamental Theorem of Calculus



Example

$$\begin{aligned}\int_{-\frac{\pi}{4}}^0 \sec x \tan x \, dx &= \left[\sec x \right]_{-\frac{\pi}{4}}^0 \\ &\quad \text{(because } \frac{d}{dx} \sec x = \sec x \tan x \text{)} \\ &= \sec 0 - \sec -\frac{\pi}{4} \\ &= 1 - \sqrt{2}.\end{aligned}$$

32. The Fundamental Theorem of Calculus



Example

$$\begin{aligned}\int_1^4 \left(\frac{3}{2}\sqrt{x} - \frac{4}{x^2} \right) dx &= \left[x^{\frac{3}{2}} + \frac{4}{x} \right]_1^4 \\ &\quad \left(\text{because } \frac{d}{dx} \left(x^{\frac{3}{2}} + \frac{4}{x} \right) = \frac{3}{2}\sqrt{x} - \frac{4}{x^2} \right) \\ &= \left(4^{\frac{3}{2}} + \frac{4}{4} \right) - \left(1^{\frac{3}{2}} + \frac{4}{1} \right) \\ &= (8 + 1) - (1 + 4) \\ &= 4.\end{aligned}$$

Total Area

Example

Let $f(x) = x^2 - 4$. We have that

$$\begin{aligned}\int_{-2}^2 f(x) \, dx &= \int_{-2}^2 (x^2 - 4) \, dx = \left[\frac{x^3}{3} - 4x \right]_{-2}^2 \\ &= \left(\frac{8}{3} - 8 \right) - \left(\frac{-8}{3} + 8 \right) = -\frac{32}{3}.\end{aligned}$$

The total area between the graph of $y = f(x)$ and the x -axis, over $[-2, 2]$, is $\left| -\frac{32}{3} \right| = \frac{32}{3}$.

32. The Fundamental Theorem of Calculus



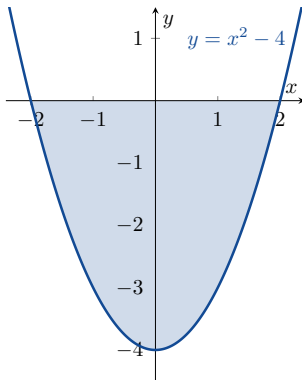
Example

Let $g(x) = 4 - x^2$. We have that

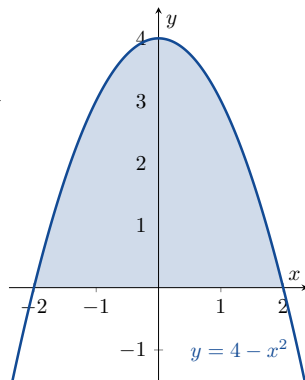
$$\begin{aligned}\int_{-2}^2 g(x) \, dx &= \int_{-2}^2 (4 - x^2) \, dx = \left[4x - \frac{x^3}{3} \right]_{-2}^2 \\ &= \left(8 - \frac{8}{3} \right) - \left(8 + \frac{-8}{3} \right) = \frac{32}{3}.\end{aligned}$$

The total area between the graph of $y = g(x)$ and the x -axis, over $[-2, 2]$, is $\left| \frac{32}{3} \right| = \frac{32}{3}$.

32. The Fundamental Theorem of Calculus



$$\begin{aligned}\text{integral} &= -\frac{32}{3} \\ \text{total area} &= \frac{32}{3}\end{aligned}$$



$$\begin{aligned}\text{integral} &= \frac{32}{3} \\ \text{total area} &= \frac{32}{3}\end{aligned}$$

32. The Fundamental Theorem of Calculus

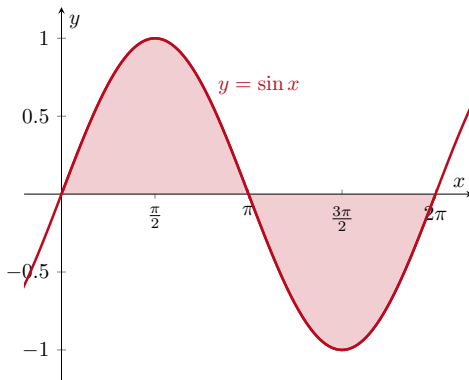


Example

Let $f(x) = \sin x$. Calculate

- 1 the definite integral of f over $[0, 2\pi]$; and
- 2 the total area between the graph of $y = f(x)$ and the x -axis over $[0, 2\pi]$.

32. The Fundamental Theorem of Calculus



32. The Fundamental Theorem of Calculus



solution:

1

$$\begin{aligned}\int_0^{2\pi} \sin x \, dx &= \left[-\cos x \right]_0^{2\pi} = -\cos 2\pi + \cos 0 \\ &= -1 + 1 = 0.\end{aligned}$$

2

$$\begin{aligned}\text{total area} &= \int_0^{\pi} \sin x \, dx + \left| \int_{\pi}^{2\pi} \sin x \, dx \right| \\ &= \left[-\cos x \right]_0^{\pi} + \left| \left[-\cos x \right]_{\pi}^{2\pi} \right| \\ &= -\cos \pi + \cos 0 + |-\cos 2\pi + \cos \pi| \\ &= -(-1) + 1 + |-1 + (-1)| = 4.\end{aligned}$$



Summary

To find the *total area* between the graph of $y = f(x)$ and the x -axis over $[a, b]$:

- 1 Divide $[a, b]$ at the zeroes of f .
- 2 Integrate f over each subinterval.
- 3 Add the absolute values of the integrals.

32. The Fundamental Theorem of Calculus



Example

Find the total area between the graph of $y = x^3 - x^2 - 2x$ and the x -axis for $-1 \leq x \leq 2$.

solution:

- 1 Let $f(x) = x^3 - x^2 - 2x$.

Since $0 = f(x) = x^3 - x^2 - 2x = x(x+1)(x-2)$ implies that $x = 0$ or $x = -1$ or $x = 2$, we divide $[-1, 2]$ into $[-1, 0]$ and $[0, 2]$.

32. The Fundamental Theorem of Calculus



2 We calculate that

$$\begin{aligned}\int_{-1}^0 (x^3 - x^2 - 2x) \, dx &= \left[\frac{x^4}{4} - \frac{x^3}{3} - x^2 \right]_{-1}^0 \\ &= (0 - 0 - 0) - \left(\frac{1}{4} + \frac{1}{3} - 1 \right) \\ &= \frac{5}{12}\end{aligned}$$

32. The Fundamental Theorem of Calculus



and

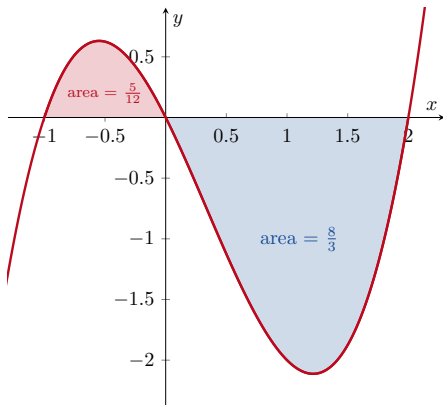
$$\begin{aligned}\int_0^2 (x^3 - x^2 - 2x) \, dx &= \left[\frac{x^4}{4} - \frac{x^3}{3} - x^2 \right]_0^2 \\ &= \left(\frac{16}{4} - \frac{8}{3} - 4 \right) - (0 - 0 - 0) \\ &= -\frac{8}{3}.\end{aligned}$$

32. The Fundamental Theorem of Calculus



3 Therefore

$$\text{total area} = \left| \frac{5}{12} \right| + \left| -\frac{8}{3} \right| = \frac{37}{12}.$$





The Average Value of a Continuous Function

The average of $\{1, 2, 2, 6, 9\}$ is $\frac{1+2+2+6+9}{5} = \frac{20}{5} = 4$. We can also calculate the average value of a continuous function.

32. The Fundamental Theorem of Calculus



Definition

If f is integrable on $[a, b]$, then the *average value of f on $[a, b]$* is

$$\text{av}(f) = \frac{1}{b-a} \int_a^b f(x) \, dx.$$

32. The Fundamental Theorem of Calculus



Example

Find the average value of $f(x) = \sqrt{4 - x^2}$ on $[-2, 2]$.

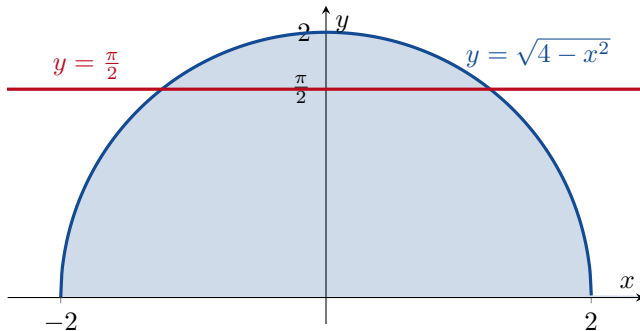
solution: Since

$$\begin{aligned}\int_{-2}^2 f(x) \, dx &= \frac{1}{2} \times \text{the area of a circle of radius 2} \\ &= \frac{1}{2} \pi 2^2 = 2\pi,\end{aligned}$$

we have that

$$\text{av}(f) = \frac{1}{2 - (-2)} \int_{-2}^2 f(x) \, dx = \frac{2\pi}{4} = \frac{\pi}{2}.$$

32. The Fundamental Theorem of Calculus



32. The Fundamental Theorem of Calculus



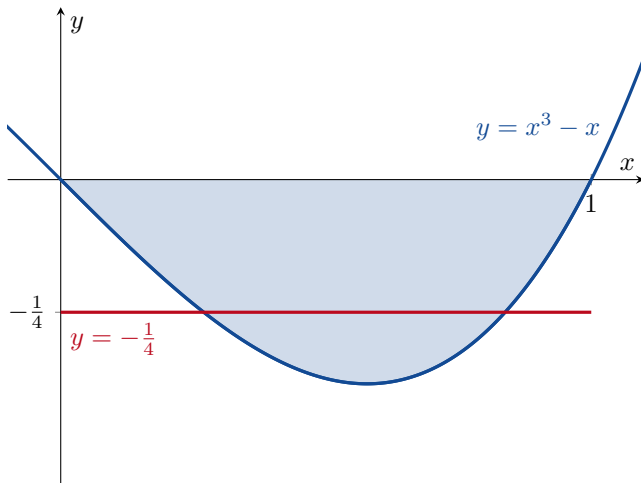
Example

Find the average value of $g(x) = x^3 - x$ on $[0, 1]$.

solution:

$$\begin{aligned}\text{av}(g) &= \frac{1}{1-0} \int_0^1 g(x) \, dx = \int_0^1 (x^3 - x) \, dx \\ &= \left[\frac{x^4}{4} - \frac{x^2}{2} \right]_0^1 = \frac{1}{4} - \frac{1}{2} = -\frac{1}{4}.\end{aligned}$$

32. The Fundamental Theorem of Calculus





Indefinite Integrals & Definite Integrals

Remember that

$\int f(x) dx$ is a function.

For example

$$\int x dx = \frac{x^2}{2} + C$$

and

$$\int \cos x dx = \sin x + C.$$

Remember that

$\int_a^b f(x) dx$ is a number.

For example

$$\int_0^1 x dx = \frac{1}{2}$$

and

$$\int_0^{\frac{\pi}{2}} \cos x dx = 1.$$

The Substitution Method

33. The Substitution Method



By the Chain rule,

$$\frac{d}{dx} \left(\frac{u^{n+1}}{n+1} \right) = u^n \frac{du}{dx}.$$

So

$$\int u^n \frac{du}{dx} dx = \frac{u^{n+1}}{n+1} + C.$$

But we know that

$$\int u^n du = \frac{u^{n+1}}{n+1} + C$$

also. So it looks like

$$\boxed{du = \frac{du}{dx} dx.}$$

33. The Substitution Method



Example

Find $\int (x^3 + x)^5 (3x^2 + 1) dx$.

solution: Let $u = x^3 + x$. Then $du = \frac{du}{dx} dx = (3x^2 + 1) dx$. By substitution, we have that

$$\begin{aligned}\int (x^3 + x)^5 (3x^2 + 1) dx &= \int u^5 du \\ &= \frac{u^6}{6} + C = \frac{1}{6}(x^3 + x)^6 + C.\end{aligned}$$

33. The Substitution Method



Example

Find $\int \sqrt{2x+1} \, dx$.

solution: Let $u = 2x + 1$. Then $du = \frac{du}{dx} \, dx = 2dx$. So $dx = \frac{1}{2} \, du$. Therefore

$$\begin{aligned} \int \sqrt{2x+1} \, dx &= \int u^{\frac{1}{2}} \left(\frac{1}{2} du\right) = \frac{1}{2} \int u^{\frac{1}{2}} \, du \\ &= \frac{1}{2} \left(\frac{u^{\frac{3}{2}}}{\frac{3}{2}} \right) + C = \frac{1}{3} (2x+1)^{\frac{3}{2}} + C. \end{aligned}$$

33. The Substitution Method



Theorem (The Substitution Method)

If

- $u = g(x)$ is differentiable;
- $g : \mathbb{R} \rightarrow I$; and
- $f : I \rightarrow \mathbb{R}$ is continuous,

then

$$\int f(g(x))g'(x) \, dx = \int f(u) \, du.$$

33. The Substitution Method



Example

Find $\int 5 \sec^2(5t + 1) dt$.

solution: Let $u = 5t + 1$. Then $du = \frac{du}{dt} dt = 5dt$. So

$$\begin{aligned}\int 5 \sec^2(5t + 1) dt &= \int \sec^2 u du \\ &= \tan u + C \\ &\quad \left(\text{because } \frac{d}{du} \tan u = \sec^2 u \right) \\ &= \tan(5t + 1) + C.\end{aligned}$$

33. The Substitution Method



Example

Find $\int \cos(7\theta + 3) d\theta$.

solution: Let $u = 7\theta + 3$. Then $du = \frac{du}{d\theta} d\theta = 7d\theta$. So $d\theta = \frac{1}{7}du$ and

$$\begin{aligned}\int \cos(7\theta + 3) d\theta &= \frac{1}{7} \int \cos u du \\ &= \frac{1}{7} \sin u + C = \frac{1}{7} \sin(7\theta + 3) + C.\end{aligned}$$

33. The Substitution Method



Example

Find $\int x^2 \sin(x^3) dx$.

solution: Let $u = x^3$. Then $du = \frac{du}{dx} dx = 3x^2 dx$. So $\frac{1}{3}du = x^2 dx$ and

$$\begin{aligned}\int x^2 \sin(x^3) dx &= \int \frac{1}{3} \sin u du = -\frac{1}{3} \cos u + C \\ &= -\frac{1}{3} \cos(x^3) + C.\end{aligned}$$

33. The Substitution Method



Example

Find $\int x\sqrt{2x+1} \, dx$.

solution: Let $u = 2x + 1$. Then $du = \frac{du}{dx} \, dx = 2dx$. So $dx = \frac{1}{2}du$ and

$$\int x\sqrt{2x+1} \, dx = \int x\sqrt{u} \, \frac{1}{2}du.$$

But we still have an x here. We can't integrate until we change all the x terms to u terms. Note that

$$u = 2x + 1 \implies u - 1 = 2x \implies \frac{1}{2}(u - 1) = x.$$

33. The Substitution Method



Therefore

$$\begin{aligned}\int x\sqrt{2x+1} \, dx &= \int \frac{1}{2}(u-1)\sqrt{u} \, \frac{1}{2}du \\&= \frac{1}{4} \int u^{\frac{3}{2}} - u^{\frac{1}{2}} \, du \\&= \frac{1}{4} \left(\frac{2}{5}u^{\frac{5}{2}} - \frac{2}{3}u^{\frac{3}{2}} \right) + C \\&= \frac{1}{10}u^{\frac{5}{2}} - \frac{1}{6}u^{\frac{3}{2}} + C \\&= \frac{1}{10}(2x+1)^{\frac{5}{2}} - \frac{1}{6}(2x+1)^{\frac{3}{2}} + C.\end{aligned}$$

33. The Substitution Method



Example

Find $\int \frac{2z}{\sqrt[3]{z^2+1}} dz$.

solution: Let $u = z^2 + 1$. Then $du = \frac{du}{dz} dz = 2z dz$ and

$$\begin{aligned}\int \frac{2z}{\sqrt[3]{z^2+1}} dz &= \int \frac{du}{u^{\frac{1}{3}}} \\&= \int u^{-\frac{1}{3}} du \\&= \frac{u^{\frac{2}{3}}}{\frac{2}{3}} + C \\&= \frac{3}{2} u^{\frac{2}{3}} + C \\&= \frac{3}{2} (z^2 + 1)^{\frac{2}{3}} + C.\end{aligned}$$

33. The Substitution Method



Example

Find $\int \sin^2 x \, dx$.

solution: We use the identity

$$\sin^2 x = \frac{1 - \cos 2x}{2}$$

to calculate that

$$\begin{aligned} \int \sin^2 x \, dx &= \int \frac{1 - \cos 2x}{2} \, dx \\ &= \frac{1}{2} \int (1 - \cos 2x) \, dx \\ &= \frac{1}{2} \left(x - \frac{1}{2} \sin 2x \right) + C \\ &= \frac{1}{2}x - \frac{1}{4} \sin 2x + C. \end{aligned}$$

33. The Substitution Method



Example

Similarly

$$\int \cos^2 x \, dx = \int \frac{1 + \cos 2x}{2} \, dx = \frac{1}{2}x + \frac{1}{4}\sin 2x + C.$$

The Substitution Method for Definite Integrals

Theorem (The Substitution Method)

If

- $u = g(x)$ is differentiable on $[a, b]$;
- g' is continuous on $[a, b]$; and
- f is continuous on the range of g ,

then

$$\int_a^b f(g(x))g'(x) dx = \int_{g(a)}^{g(b)} f(u) du.$$

33. The Substitution Method



Example

Calculate $\int_{-1}^1 3x^2 \sqrt{x^3 + 1} \, dx$.

solution 1. We can use the previous theorem to solve this example. Let $u = x^3 + 1$. Then $du = 3x^2 \, dx$. Moreover $x = -1 \implies u = 0$ and $x = 1 \implies u = 2$. So

$$\begin{aligned} \int_{x=-1}^{x=1} 3x^2 \sqrt{x^3 + 1} \, dx &= \int_{u=0}^{u=2} \sqrt{u} \, du = \left[\frac{2}{3} u^{\frac{3}{2}} \right]_0^2 \\ &= \frac{2}{3} \left(2^{\frac{3}{2}} - 0^{\frac{3}{2}} \right) = \frac{2}{3} 2\sqrt{2} = \frac{4\sqrt{2}}{3}. \end{aligned}$$

33. The Substitution Method



solution 2. Alternately, we can first find the indefinite integral, then find the required definite integral.

Let $u = x^3 + 1$. Then $du = 3x^2 dx$. So

$$\int 3x^2 \sqrt{x^3 + 1} dx = \int \sqrt{u} du = \frac{2}{3} u^{\frac{3}{2}} + C = \frac{2}{3} (x^3 + 1)^{\frac{3}{2}} + C.$$

Therefore

$$\begin{aligned} \int_{-1}^1 3x^2 \sqrt{x^3 + 1} dx &= \left[\frac{2}{3} (x^3 + 1)^{\frac{3}{2}} \right]_{-1}^1 \\ &= \left(\frac{2}{3} (1 + 1)^{\frac{3}{2}} \right) - \left(\frac{2}{3} (-1 + 1)^{\frac{3}{2}} \right) \\ &= \frac{2}{3} \times 2^{\frac{3}{2}} = \frac{4\sqrt{2}}{3}. \end{aligned}$$

33. The Substitution Method



Example

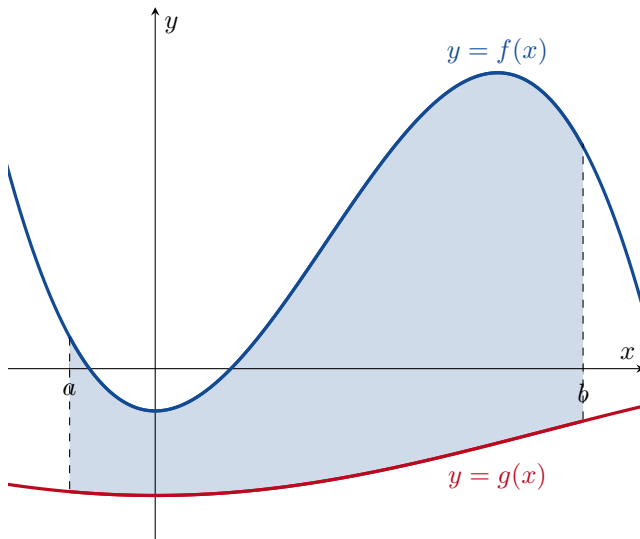
Calculate $\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \cot \theta \operatorname{cosec}^2 \theta \, d\theta$.

solution: Let $u = \cot \theta$. Then $du = \frac{du}{d\theta} d\theta = -\operatorname{cosec}^2 \theta \, d\theta$. So $-du = \operatorname{cosec}^2 \theta \, d\theta$. Moreover $\theta = \frac{\pi}{4} \implies u = \cot \frac{\pi}{4} = 1$ and $\theta = \frac{\pi}{2} \implies u = \cot \frac{\pi}{2} = 0$. Hence

$$\begin{aligned} \int_{\theta=\frac{\pi}{4}}^{\theta=\frac{\pi}{2}} \cot \theta \operatorname{cosec}^2 \theta \, d\theta &= \int_{u=1}^{u=0} u (-du) = - \int_1^0 u \, du \\ &= - \left[\frac{u^2}{2} \right]_1^0 = - \left(\frac{0^2}{2} - \frac{1^2}{2} \right) = \frac{1}{2}. \end{aligned}$$

Area Between Curves

34. Area Between Curves



34. Area Between Curves



Definition

If

- f is continuous;
- g is continuous; and
- $f(x) \geq g(x)$ on $[a, b]$,

then the *area of the region between the curves $y = f(x)$ and $y = g(x)$ for $a \leq x \leq b$* is

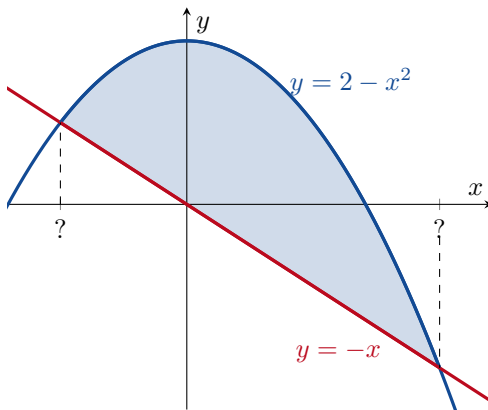
$$\text{area} = \int_a^b (f(x) - g(x)) \, dx.$$

34. Area Between Curves



Example

Find the area between $y = 2 - x^2$ and $y = -x$.



34. Area Between Curves



solution: First we need to find the limits of integration:

$$2 - x^2 = -x$$

$$0 = x^2 - x - 2$$

$$0 = (x + 1)(x - 2) \implies x = -1 \text{ or } 2.$$

We need to integrate from $x = -1$ to $x = 2$. Therefore

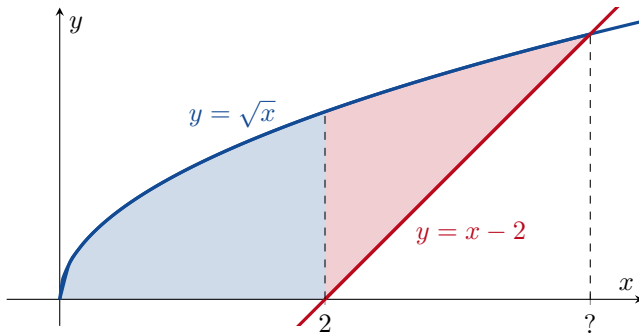
$$\begin{aligned} \text{area} &= \int_{-1}^2 \left((2 - x^2) - (-x) \right) dx \\ &= \int_{-1}^2 (2 + x - x^2) dx \\ &= \left[2x + \frac{1}{2}x^2 - \frac{1}{3}x^3 \right]_{-1}^2 \\ &= \left(4 + \frac{4}{2} - \frac{8}{3} \right) - \left(-2 + \frac{1}{2} + \frac{1}{3} \right) \\ &= \frac{9}{2}. \end{aligned}$$

34. Area Between Curves



Example

Find the area bounded by $y = \sqrt{x}$, $y = x - 2$ and the x -axis, for $x \geq 0$ and $y \geq 0$.



34. Area Between Curves



solution: First we calculate that

$$\sqrt{x} = x - 2$$

$$x = (x - 2)^2 = x^2 - 4x + 4$$

$$0 = x^2 - 5x + 4 = (x - 1)(x - 4) \implies x = 1 \text{ or } 4.$$

Since $\sqrt{1} \neq 1 - 2$, we must have $x = 4$.

34. Area Between Curves



Therefore

$$\text{area} = \text{blue area} + \text{red area}$$

$$\begin{aligned} &= \int_0^2 \sqrt{x} \, dx + \int_2^4 \left(\sqrt{x} - (x - 2) \right) dx \\ &= \int_0^2 x^{\frac{1}{2}} \, dx + \int_2^4 \left(x^{\frac{1}{2}} - x + 2 \right) dx \\ &= \left[\frac{2}{3} x^{\frac{3}{2}} \right]_0^2 + \left[\frac{2}{3} x^{\frac{3}{2}} - \frac{1}{2} x^2 + 2x \right]_2^4 \\ &= \left(\frac{2}{3} (2)^{\frac{3}{2}} - 0 \right) + \left(\frac{2}{3} (4)^{\frac{3}{2}} - \frac{1}{2} (16) + 2(4) \right) \\ &\quad - \left(\frac{2}{3} (2)^{\frac{3}{2}} - \frac{1}{2} (4) + 2(2) \right) \\ &= \frac{4\sqrt{2}}{3} + \frac{16}{3} - 8 + 8 - \frac{4\sqrt{2}}{3} + 2 - 4 = \frac{10}{3}. \end{aligned}$$

The End

A decorative flourish consisting of a series of elegant, flowing loops and curves, rendered in white.