Lecture 10

- Inner Product Spaces
- Orthogonality
- Orthogonal Sets and Orthonormal Sets
- The Gram-Schmidt Process

Inner Product Spaces

In MATH114 Mathematics II we studied the *dot product* of two vectors in $\mathbb{R}^2$ or $\mathbb{R}^3$.

You will recall that if $\mathbf{u} = (u_1, u_2, u_3), \mathbf{v} = (v_1, v_2, v_3) \in \mathbb{R}^3$ then

$$\mathbf{u} \bullet \mathbf{v} = u_1 v_1 + u_2 v_2 + u_3 v_3$$

and

$$\mathbf{u} \bullet \mathbf{u} = \|\mathbf{u}\|^2 .$$

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You will recall that if $\mathbf{u} = (u_1, u_2, u_3), \mathbf{v} = (v_1, v_2, v_3) \in \mathbb{R}^3$ then

$$\mathbf{u} \cdot \mathbf{v} = u_1 v_1 + u_2 v_2 + u_3 v_3$$

and

$$\mathbf{u} \cdot \mathbf{u} = \|\mathbf{u}\|^2 .$$

This week will extend these ideas to real vector spaces.

(We will not concern ourselves with complex numbers this week. Any time you see a scalar k you can assume that it is a real number.)

Definition

An *inner product* on a (real) vector space V is a function

$$\langle \cdot, \cdot \rangle : V \times V \to \mathbb{R}$$

that satisfies the following axioms for all vectors $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ in V and all scalars k:

1.
2.
3.
4.

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1. $\langle \mathbf{u}, \mathbf{v} \rangle = \langle \mathbf{v}, \mathbf{u} \rangle$ [Symmetry axiom]
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2. $\langle \mathbf{u} + \mathbf{v}, \mathbf{w} \rangle = \langle \mathbf{u}, \mathbf{w} \rangle + \langle \mathbf{v}, \mathbf{w} \rangle$ [Additivity axiom]
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3. $\langle k\mathbf{u}, \mathbf{v} \rangle = k \langle \mathbf{u}, \mathbf{v} \rangle$ [Homogeneity axiom]
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Inner Product Spaces

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3. $\langle k\mathbf{u}, \mathbf{v} \rangle = k \langle \mathbf{u}, \mathbf{v} \rangle$ [Homogeneity axiom]
4. $\langle \mathbf{v}, \mathbf{v} \rangle \geq 0$ and $\langle \mathbf{v}, \mathbf{v} \rangle = 0$ if and only if $\mathbf{v} = \mathbf{0}$. [Positivity axiom]

Remark

If we combine

1. $\langle \mathbf{u}, \mathbf{v} \rangle = \langle \mathbf{v}, \mathbf{u} \rangle$ [Symmetry axiom]

and

2. $\langle \mathbf{u} + \mathbf{v}, \mathbf{w} \rangle = \langle \mathbf{u}, \mathbf{w} \rangle + \langle \mathbf{v}, \mathbf{w} \rangle$ [Additivity axiom]

then we can show that

$$\langle \mathbf{u}, \mathbf{v} + \mathbf{w} \rangle = \langle \mathbf{v} + \mathbf{w}, \mathbf{u} \rangle = \langle \mathbf{v}, \mathbf{u} \rangle + \langle \mathbf{w}, \mathbf{u} \rangle = \langle \mathbf{u}, \mathbf{v} \rangle + \langle \mathbf{u}, \mathbf{w} \rangle .$$

Remark

Likewise, we can combine

1. $\langle \mathbf{u}, \mathbf{v} \rangle = \langle \mathbf{v}, \mathbf{u} \rangle$ [Symmetry axiom]

and

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to prove that

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to prove that

$$\langle \mathbf{u}, k\mathbf{v} \rangle = k \langle \mathbf{u}, \mathbf{v} \rangle$$

and then we can prove that

$$\langle \mathbf{0}, \mathbf{v} \rangle = \mathbf{0} = \langle \mathbf{v}, \mathbf{0} \rangle .$$

(proofs left to you.)

Example

Let $V = \mathbb{R}^n$. The function

$$\langle \mathbf{u}, \mathbf{v} \rangle = \mathbf{u} \cdot \mathbf{v} = u_1 v_1 + u_2 + v_2 + \ldots + u_n v_n$$

is an inner product on $\mathbb{R}^n$. I leave it to you to prove that all 4 axioms are satisfied. This inner product is called the *Euclidean inner product* on $\mathbb{R}^n$.

Definition

A (real) vector space with an inner product is called a *(real) inner product space*.

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Example

$\mathbb{R}^n$ with $\langle \mathbf{u}, \mathbf{v} \rangle = \mathbf{u} \cdot \mathbf{v}$ is an inner product space.

Definition

If V is a inner product space, then the *norm* (or *length*) of a vector $\mathbf{v}$ in V is denoted by $\|\mathbf{v}\|$ and is defined by

$$\|\mathbf{v}\| = \sqrt{\langle \mathbf{v}, \mathbf{v} \rangle}.$$

Definition

A vector of norm 1 is called a *unit vector*.

$$\|\mathbf{v}\| = \sqrt{\langle \mathbf{v}, \mathbf{v} \rangle}$$

Theorem

If $\mathbf{u}$ *and* $\mathbf{v}$ *are vectors in a inner product space* V*, and if* k *is a scalar, then:*

1. $\|\mathbf{v}\| \geq 0$
2. $\|\mathbf{v}\| = 0$ *if and only if* $\mathbf{v} = \mathbf{0}$.
3. $\|k\mathbf{v}\| = |k|\,\|\mathbf{v}\|$.

$$\|\mathbf{v}\| = \sqrt{\langle \mathbf{v}, \mathbf{v} \rangle}$$

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Proof.

1. $\sqrt{x}$ is always ≥ 0.

Inner Product Spaces

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1. $\sqrt{x}$ is always ≥ 0.
2. $\|\mathbf{v}\| = 0 \iff \langle \mathbf{v}, \mathbf{v} \rangle = 0 \iff \mathbf{v} = \mathbf{0}$ by definition.

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Proof.

1. $\sqrt{x}$ is always ≥ 0.
2. $\|\mathbf{v}\| = 0 \iff \langle \mathbf{v}, \mathbf{v} \rangle = 0 \iff \mathbf{v} = \mathbf{0}$ by definition.
3. $\|k\mathbf{v}\|^2 = \langle k\mathbf{v}, k\mathbf{v} \rangle = k \langle \mathbf{v}, k\mathbf{v} \rangle = k^2 \langle \mathbf{v}, \mathbf{v} \rangle = k^2 \|\mathbf{v}\|^2$.

$\square$

$$\|\mathbf{v}\| = \sqrt{\langle \mathbf{v}, \mathbf{v} \rangle}$$

Definition

If V is an inner product space, then the *distance* between two vectors $\mathbf{u}$ and $\mathbf{v}$ in V is

$$\text{dist}(\mathbf{u}, \mathbf{v}) = \|\mathbf{u} - \mathbf{v}\|.$$

Example (A Weighted Inner Product)

Let $\mathbf{u}, \mathbf{v} \in \mathbb{R}^2$. Show that the function

$$\langle \mathbf{u}, \mathbf{v} \rangle = 3u_1v_1 = 2u_2v_2$$

is an inner product on $\mathbb{R}^2$.

We need to show that all four of the axioms are satisfied.

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1. Interchanging $\mathbf{u}$ and $\mathbf{v}$ in the formula does not change the sum on the right side, so $\langle \mathbf{u}, \mathbf{v} \rangle = \langle \mathbf{v}, \mathbf{u} \rangle$.

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2. We have

$$\begin{aligned} \langle \mathbf{u}+\mathbf{v}, \mathbf{w} \rangle &= 3(u_1+v_1)w_1 + 2(u_2+v_2)w_2 \\ &= 3(u_1w_1 + v_1w_1) + 2(u_2w_2 + v_2w_2) \\ &= \ldots = \langle \mathbf{u}, \mathbf{w} \rangle + \langle \mathbf{v}, \mathbf{w} \rangle \end{aligned}$$

3. and that

$$\langle k\mathbf{u}, \mathbf{v} \rangle = 3(ku_1)v_1 + 2(ku_2)v_2 = k(3u_1v_1 + 2u_2v_2) = k \langle \mathbf{u}, \mathbf{v} \rangle .$$

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is an inner product on $\mathbb{R}^2$.

We need to show that all four of the axioms are satisfied.

4. Finally note that

$$\langle \mathbf{v}, \mathbf{v} \rangle = 3v_1^2 + 2v_2^2 \geq 0$$

and we can only get "$= 0$" here if $v_1 = v_2 = 0$.

Therefore $\langle \cdot, \cdot \rangle$ is an inner product. This is called a *weighted Euclidean inner product*. The numbers 3 and 2 are called the *weights*.

Example (Calculating with a Weighted Euclidean Inner Product)

Let $\mathbf{u} = (1, 0) \in \mathbb{R}^2$. Note that if we use the Euclidean inner product, then we have

$$\|\mathbf{u}\| = \sqrt{\mathbf{u} \cdot \mathbf{u}} = \sqrt{1^2 + 0^2} = 1.$$

Example (Calculating with a Weighted Euclidean Inner Product)

Let $\mathbf{u} = (1, 0) \in \mathbb{R}^2$. Note that if we use the Euclidean inner product, then we have

$$\|\mathbf{u}\| = \sqrt{\mathbf{u} \cdot \mathbf{u}} = \sqrt{1^2 + 0^2} = 1.$$

However if we use the weighted Euclidean inner product from the last example, then

$$\|\mathbf{u}\| = \sqrt{\langle \mathbf{u}, \mathbf{u} \rangle} = \sqrt{3u_1^2 + 2u_2^2} = \sqrt{3(1)^2 + 2(0)^2} = \sqrt{3}.$$

Unit Circles and Unit Spheres

Definition

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Example (Unusual Unit Circles in $\mathbb{R}^2$)

1. Sketch the unit circle in $\mathbb{R}^2$ using the Euclidean inner product $\langle \mathbf{u}, \mathbf{v} \rangle = u_1 v_1 + u_2 v_2$.
2. Sketch the unit circle in $\mathbb{R}^2$ using the weighted Euclidean inner product $\langle \mathbf{u}, \mathbf{v} \rangle = \frac{1}{9} u_1 v_1 + \frac{1}{4} u_2 v_2$.

Let $\mathbf{u} = (x, y)$. Then

1.
$$1 = \|\mathbf{u}\|^2 = (x, y) \cdot (x, y) = x^2 + y^2$$

and

2.
$$1 = \|\mathbf{u}\|^2 = \langle \mathbf{u}, \mathbf{u} \rangle = \frac{1}{9} u_1 u_1 + \frac{1}{4} u_2 u_2 = \frac{x^2}{9} + \frac{y^2}{4}.$$

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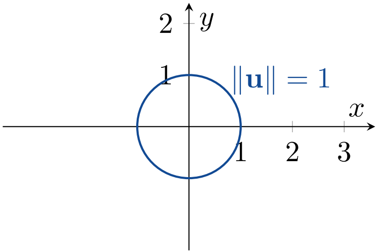

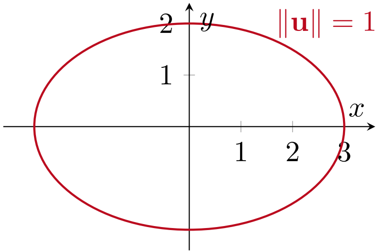

Inner Products Generated by Matrices

Definition

Let A be an invertible $n \times n$ matrix. Then the formula

$$\langle \mathbf{u}, \mathbf{v} \rangle = A\mathbf{u} \cdot A\mathbf{v}$$

defines an inner product on $\mathbb{R}^n$ called the *inner product generated by A*.

Remark

Note that

$$\langle \mathbf{u}, \mathbf{v} \rangle = A\mathbf{u} \cdot A\mathbf{v} = (A\mathbf{v})^T A\mathbf{u} = \mathbf{v}^T A^T A\mathbf{u}.$$

Example

Let $A = \begin{bmatrix} \sqrt{3} & 0 \\ 0 & \sqrt{2} \end{bmatrix}$. Then the inner product on $\mathbb{R}^2$ generated by A is one of the weighted Euclidean inner products that we looked at earlier:

$$\langle \mathbf{u}, \mathbf{v} \rangle = A\mathbf{u} \cdot A\mathbf{v} = \begin{bmatrix} \sqrt{3}u_1 \\ \sqrt{2}u_2 \end{bmatrix} \bullet \begin{bmatrix} \sqrt{3}v_1 \\ \sqrt{2}v_2 \end{bmatrix} = 3u_1v_2 + 2u_1v_2.$$

Note that

$$A^T A = \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}.$$

More Inner Products

Example (The Standard Inner Product on $\mathbb{R}^{n\times n} = M_{nn}$)

If $\mathbf{u} = U$ and $\mathbf{v} = V$ are matrices in the vector space $\mathbb{R}^{n\times n}$, then the formula

$$\langle \mathbf{u}, \mathbf{v} \rangle = \operatorname{tr}(U^T V)$$

defines an inner product on $\mathbb{R}^{n\times n}$ called the *standard inner product* on $\mathbb{R}^{n\times n}$.

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This can be proved by confirming that the four inner product axioms are satisfied. But there is an easier way:

Inner Product Spaces

If $\mathbf{u} = U = \begin{bmatrix} u_1 & u_2 \\ u_3 & u_4 \end{bmatrix}$ and $\mathbf{v} = V = \begin{bmatrix} v_1 & v_2 \\ v_3 & v_4 \end{bmatrix}$ then

$$\begin{aligned} \langle \mathbf{u}, \mathbf{v} \rangle &= \operatorname{tr}(U^T V) = \operatorname{tr} \begin{bmatrix} u_1v_1 + u_3v_3 & u_2v_2 + u_4v_4 \\ u_1v_1 + u_3v_3 & u_2v_2 + u_4 + v_4 \end{bmatrix} \\ &= u_1v_1 + u_2v_2 + u_3v_3 + u_4v_4 \end{aligned}$$

which is just like the dot product in $\mathbb{R}^4$.

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$$= u_1 v_1 + u_2 v_2 + u_3 v_3 + u_4 v_4$$

which is just like the dot product in $\mathbb{R}^4$.

For example, if

$$\mathbf{u} = U = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \qquad \text{and} \qquad \mathbf{v} = V = \begin{bmatrix} -1 & 0 \\ 3 & 2 \end{bmatrix}$$

then

$$\langle \mathbf{u}, \mathbf{v} \rangle = \operatorname{tr}(U^T V) = 1(-1) + 2(0) + 3(3) + 4(2) = 16$$

and

$$\|\mathbf{u}\| = \sqrt{\langle \mathbf{u}, \mathbf{u} \rangle} = \sqrt{1^2 + 2^2 + 3^2 + 4^2} = \sqrt{30}.$$

Example (The Standard Inner Product on $\mathbb{P}^n$)

If

$$\mathbf{p} = a_0 + a_1 x + \ldots + a_n x^n$$

and

$$q = b_0 + b_1 x + \ldots + b_n x^n$$

are polynomials in $\mathbb{P}^n$, then the following formula defines an inner product on $\mathbb{P}^n$ (please verify) that we will call the *standard inner product* on $\mathbb{P}^n$:

$$\langle \mathbf{p}, \mathbf{q} \rangle = a_0 b_0 + a_1 b + 1 + \ldots + a_n b_n$$

The norm of a polynomial $\mathbf{p}$ relative to this inner product is

$$\|\mathbf{p}\| = \langle \mathbf{p}, \mathbf{p} \rangle = \sqrt{a_0^2 + a_1^2 + \ldots + a_n^2}.$$

Example (The Evaluation Inner Product on $\mathbb{P}^n$)

If

$$\mathbf{p} = a_0 + a_1 x + \ldots + a_n x^n$$

and

$$q = b_0 + b_1 x + \ldots + b_n x^n$$

are polynomials in $\mathbb{P}^n$, and if $x_0, x_1, \ldots, x_n$ are distinct real numbers, then the following formula defines an inner product on $\mathbb{P}^n$ (please verify) that we will call the *evaluation inner product* at $x_0, x_1, \ldots, x_n$:

$$\langle \mathbf{p}, \mathbf{q} \rangle = p(x_0)q(x_0) + p(x_1)q(x_1) + \ldots + p(x_n)q(x_n)$$

Example (The Evaluation Inner Product on $\mathbb{P}^n$)

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$$\langle \mathbf{p}, \mathbf{q} \rangle = p(x_0)q(x_0) + p(x_1)q(x_1) + \ldots + p(x_n)q(x_n)$$

We can think of this as the dot product of the vector $\big(p(x_0), p(x_1), \ldots, p(x_n)\big)$ with the vector $\big(q(x_0), q(x_1), \ldots, q(x_n)\big)$.

The first three axioms follow from properties of the dot product. For the fourth axiom, we have that

$$\langle \mathbf{p}, \mathbf{p} \rangle = \big[p(x_0)\big]^2 + \big[p(x_1)\big]^2 + \ldots + \big[p(x_n)\big]^2 \geq 0$$

for all polynomials $\mathbf{p}$. We can only have "$= 0$" here if

$$p(x_0) = p(x_1) = \ldots = p(x_n) = 0.$$

However the only nth degree polynomial with $n+1$ roots is $\mathbf{p} = \mathbf{0}$.

$C[a,b] = \{\text{all continuous functions } f : [a,b] \to \mathbb{R}\}$

Example (An Integral Inner Product on $C[a,b]$)

Show that the following function is an inner product on $C[a,b]$:

$$\langle \mathbf{f}, \mathbf{g} \rangle = \int_a^b f(x)g(x)\,dx.$$

1

2

3

4

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2. $\langle \mathbf{f} + \mathbf{g}, \mathbf{h} \rangle = \int_a^b \big(f(x) + g(x)\big)h(x)\, dx =$
   $\int_a^b f(x)h(x)\, dx + \int_a^b g(x)h(x)\, dx = \langle \mathbf{f}, \mathbf{h} \rangle + \langle \mathbf{g}, \mathbf{h} \rangle$.
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2. $\langle \mathbf{f}+\mathbf{g}, \mathbf{h} \rangle = \int_a^b \big(f(x)+g(x)\big)h(x)\,dx =$ $\int_a^b f(x)h(x)\,dx + \int_a^b g(x)h(x)\,dx = \langle \mathbf{f}, \mathbf{h} \rangle + \langle \mathbf{g}, \mathbf{h} \rangle$.
3. $\langle k\mathbf{f}, \mathbf{g} \rangle = \int_a^b kf(x)g(x)\,dx = k\int_a^b f(x)g(x)\,dx = k\langle \mathbf{f}, \mathbf{g} \rangle$.
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2. $\langle \mathbf{f} + \mathbf{g}, \mathbf{h} \rangle = \int_a^b \big(f(x) + g(x)\big)h(x)\,dx =$
   $\int_a^b f(x)h(x)\,dx + \int_a^b g(x)h(x)\,dx = \langle \mathbf{f}, \mathbf{h} \rangle + \langle \mathbf{g}, \mathbf{h} \rangle$.
3. $\langle k\mathbf{f}, \mathbf{g} \rangle = \int_a^b kf(x)g(x)\,dx = k\int_a^b f(x)g(x)\,dx = k\,\langle \mathbf{f}, \mathbf{g} \rangle$.
4. $\langle \mathbf{f}, \mathbf{f} \rangle = \int_a^b \big(f(x)\big)^2\,dx \geq 0$ since $\big(f(x)\big)^2 \geq 0$ for all $x \in [a,b]$.
   Since f is continuous, we can only have "$= 0$" here if $\mathbf{f} = \mathbf{0}$.

Calculating with Inner Products

$$
\begin{aligned}
\langle \mathbf{u} - 2\mathbf{v}, 3\mathbf{u} + 4\mathbf{v} \rangle &= \langle \mathbf{u}, 3\mathbf{u} + 4\mathbf{v} \rangle - \langle 2\mathbf{v}, 3\mathbf{u} + 4\mathbf{v} \rangle \\
&= \\
&= \\
&= \\
&=
\end{aligned}
$$

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&= \\
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&= 3\langle \mathbf{u}, \mathbf{u} \rangle + 4\langle \mathbf{u}, \mathbf{v} \rangle - 6\langle \mathbf{v}, \mathbf{u} \rangle - 8\langle \mathbf{v}, \mathbf{v} \rangle \\
&= \\
&=
\end{aligned}
$$

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$$
\begin{aligned}
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&=
\end{aligned}
$$

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&= 3 \|\mathbf{u}\|^2 - 2 \langle \mathbf{u}, \mathbf{v} \rangle - 8 \|\mathbf{v}\|^2 .
\end{aligned}$$

Orthogonality

Orthogonal Vectors

Definition

Two vectors $\mathbf{u}$ and $\mathbf{v}$ in a (real) inner product space V are *orthogonal* iff

$$\langle \mathbf{u}, \mathbf{v} \rangle = 0.$$

Orthogonality

Example (Orthogonality Depends on the Inner Product)

Let $\mathbf{u} = (1, 1)$ and $\mathbf{v} = (1, -1)$. Note that $\mathbf{u}$ and $\mathbf{v}$ are orthogonal with respect to the Euclidean inner product on $\mathbb{R}^2$ since

$$\mathbf{u} \cdot \mathbf{v} = (1)(1) + (1)(-1) = 0.$$

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However $\mathbf{u}$ and $\mathbf{v}$ are not orthogonal with respect to the weighted Euclidean inner product $\langle \mathbf{u}, \mathbf{v} \rangle = 3u_1v_1 + 2u_2v_2$ since

$$\langle \mathbf{u}, \mathbf{v} \rangle = 3(1)(1) + 2(1)(-1) = 1 \neq 0.$$

Orthogonality

Example

The matrices $U = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$ and $V = \begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix}$ are orthogonal with respect to the standard inner product on $\mathbb{R}^{2\times 2}$ since

$$\langle \mathbf{u}, \mathbf{v} \rangle = (1)(0) + (0)(2) + (1)(0) + (1)(0) = 0.$$

Example

Let $\mathbb{P}^2$ have the inner product

$$\langle \mathbf{p}, \mathbf{q} \rangle = \int_{-1}^{1} p(x)q(x)\, dx$$

and let $\mathbf{p} = x$ and $\mathbf{q} = x^2$.

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and let $\mathbf{p} = x$ and $\mathbf{q} = x^2$. Then

$$\|\mathbf{p}\| = \langle \mathbf{p}, \mathbf{p} \rangle^{\frac{1}{2}} = \left[\int_{-1}^{1} x^2\,dx\right]^{\frac{1}{2}} = \sqrt{\frac{2}{3}}$$

$$\|\mathbf{q}\| = \langle \mathbf{q}, \mathbf{q} \rangle^{\frac{1}{2}} = \left[\int_{-1}^{1} x^4\,dx\right]^{\frac{1}{2}} = \sqrt{\frac{2}{5}}$$

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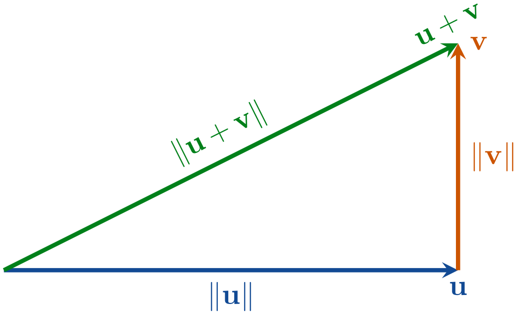
$$\langle \mathbf{p}, \mathbf{q} \rangle = \int_{-1}^{1} x^3\,dx = 0.$$

Because $\langle \mathbf{p}, \mathbf{q} \rangle = 0$, the vectors $\mathbf{p} = x$ and $\mathbf{q} = x^2$ are orthogonal relative to this inner product.

Orthogonality



Theorem (The Pythagorean Theorem)

If $\mathbf{u}$ and $\mathbf{v}$ are orthogonal then

$$\|\mathbf{u}+\mathbf{v}\|^2 = \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2 .$$

Theorem (The Pythagorean Theorem)

If $\mathbf{u}$ *and* $\mathbf{v}$ *are orthogonal then*

$$\|\mathbf{u}+\mathbf{v}\|^2 = \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2 .$$

Proof.

If $\langle \mathbf{u}, \mathbf{v} \rangle = 0$ then

$$\begin{aligned}
\|\mathbf{u}+\mathbf{v}\|^2 &= \langle \mathbf{u}+\mathbf{v}, \mathbf{u}+\mathbf{v} \rangle = \langle \mathbf{u}, \mathbf{u}+\mathbf{v} \rangle + \langle \mathbf{v}, \mathbf{u}+\mathbf{v} \rangle \\
&= \langle \mathbf{u}, \mathbf{u} \rangle + \langle \mathbf{u}, \mathbf{v} \rangle + \langle \mathbf{v}, \mathbf{u} \rangle + \langle \mathbf{v}, \mathbf{v} \rangle \\
&= \|\mathbf{u}\|^2 + 0 + 0 + \|\mathbf{v}\|^2 .
\end{aligned}$$

$\square$

Example

We have seen that $\mathbf{p} = x$ and $\mathbf{q} = x^2$ are orthogonal with respect to the inner product

$$\langle \mathbf{p}, \mathbf{q} \rangle = \int_{-1}^{1} p(x)q(x)\, dx.$$

It follows from the Pythagorean Theorem that

$$\|\mathbf{p} + \mathbf{q}\|^2 = \|\mathbf{p}\|^2 + \|\mathbf{q}\|^2 .$$

Let's check this:

Orthogonality

Example

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Orthogonality

Example

$$\|\mathbf{p}\|^2 + \|\mathbf{q}\|^2 = \left(\sqrt{\frac{2}{3}}\right)^2 + \left(\sqrt{\frac{2}{5}}\right)^2 = \frac{2}{3} + \frac{2}{5} = \frac{16}{15}$$

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Example

$$\|\mathbf{p}\|^2 + \|\mathbf{q}\|^2 = \left(\sqrt{\frac{2}{3}}\right)^2 + \left(\sqrt{\frac{2}{5}}\right)^2 = \frac{2}{3} + \frac{2}{5} = \frac{16}{15}$$

$$\begin{aligned}\|\mathbf{p}+\mathbf{q}\|^2 &= \langle \mathbf{p}+\mathbf{q}, \mathbf{p}+\mathbf{q} \rangle = \int_{-1}^{1} (x+x^2)(x+x^2)\,dx \\ &= \int_{-1}^{1} x^2\,dx + 2\int_{-1}^{1} x^3\,dx + \int_{-1}^{1} x^4\,dx \\ &= \frac{2}{3} + 0 + \frac{2}{5} = \frac{16}{15}.\end{aligned}$$

Orthogonality

Definition

Let W be a subspace of an inner product space V. The *orthogonal complement* of W is

$$W^\perp = \{\mathbf{v} \in V \mid \langle \mathbf{v}, \mathbf{w} \rangle = 0 \text{ for all } \mathbf{w} \in W\}.$$

Theorem

1. *$W^\perp$ is also a subspace of V.*
2. $W \cap W^\perp = \{\mathbf{0}\}$.

Orthogonality

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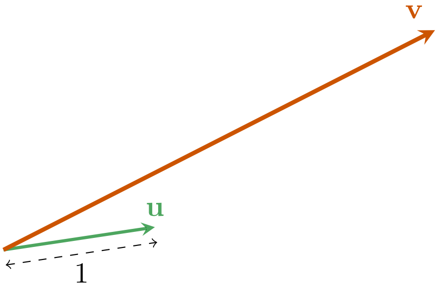
Theorem

1. *$W^{\perp}$ is also a subspace of V.*
2. $W \cap W^{\perp} = \{\mathbf{0}\}$.
3. *If V is finite dimensional then $\left(W^{\perp}\right)^{\perp} = W$.*

Orthogonal Projection

Let $\mathbf{u}$ be a unit vector and let $\mathbf{v}$ be any nonzero vector in V.



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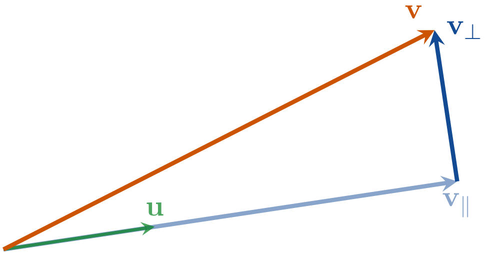


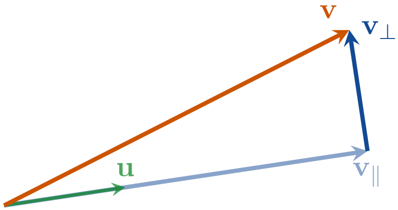
We can write

$$\mathbf{v} = \mathbf{v}_\parallel + \mathbf{v}_\perp$$

where

$$\mathbf{v}_\parallel = \langle \mathbf{u}, \mathbf{v} \rangle \, \mathbf{u} \qquad \text{and} \qquad \mathbf{v}_\perp = \mathbf{v} - \langle \mathbf{u}, \mathbf{v} \rangle \, \mathbf{u}.$$

$$\mathbf{v} = \mathbf{v}_{\parallel} + \mathbf{v}_{\perp} \qquad \mathbf{v}_{\parallel} = \langle \mathbf{u}, \mathbf{v} \rangle \, \mathbf{u} \qquad \mathbf{v}_{\perp} = \mathbf{v} - \langle \mathbf{u}, \mathbf{v} \rangle \, \mathbf{u}$$

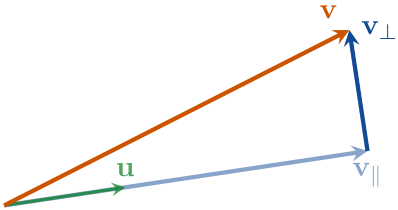


Note that $\mathbf{v}_{\perp}$ is orthogonal to $\mathbf{u}$ because

$$\langle \mathbf{u}, \mathbf{v}_{\perp} \rangle = \langle \mathbf{u}, \mathbf{v} - \langle \mathbf{u}, \mathbf{v} \rangle \, \mathbf{u} \rangle$$

Orthogonality

$$\mathbf{v} = \mathbf{v}_{\parallel} + \mathbf{v}_{\perp} \qquad \mathbf{v}_{\parallel} = \langle \mathbf{u}, \mathbf{v} \rangle \, \mathbf{u} \qquad \mathbf{v}_{\perp} = \mathbf{v} - \langle \mathbf{u}, \mathbf{v} \rangle \, \mathbf{u}$$



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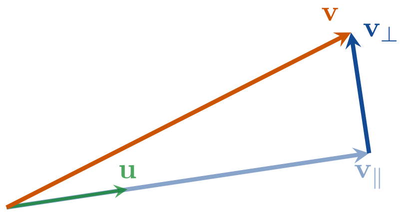
$$\mathbf{v} = \mathbf{v}_\parallel + \mathbf{v}_\perp \qquad \mathbf{v}_\parallel = \langle \mathbf{u}, \mathbf{v} \rangle \, \mathbf{u} \qquad \mathbf{v}_\perp = \mathbf{v} - \langle \mathbf{u}, \mathbf{v} \rangle \, \mathbf{u}$$



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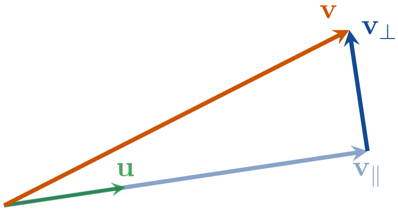
since $\mathbf{u}$ is a unit vector ($\|\mathbf{u}\| = 1 \implies \langle \mathbf{u}, \mathbf{u} \rangle = 1$).

$$\mathbf{v} = \mathbf{v}_{\parallel} + \mathbf{v}_{\perp} \qquad \mathbf{v}_{\parallel} = \langle \mathbf{u}, \mathbf{v} \rangle \, \mathbf{u} \qquad \mathbf{v}_{\perp} = \mathbf{v} - \langle \mathbf{u}, \mathbf{v} \rangle \, \mathbf{u}$$



Note that $\mathbf{v}_{\perp}$ is orthogonal to $\mathbf{u}$ because

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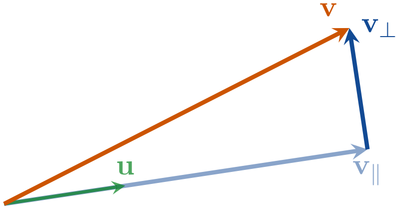
since $\mathbf{u}$ is a unit vector ($\|\mathbf{u}\| = 1 \implies \langle \mathbf{u}, \mathbf{u} \rangle = 1$).

It follows that $\mathbf{v}_{\parallel}$ and $\mathbf{v}_{\perp}$ are orthogonal.

$$\mathbf{v} = \mathbf{v}_{\|} + \mathbf{v}_{\perp} \qquad \mathbf{v}_{\|} = \langle \mathbf{u}, \mathbf{v} \rangle \, \mathbf{u} \qquad \mathbf{v}_{\perp} = \mathbf{v} - \langle \mathbf{u}, \mathbf{v} \rangle \, \mathbf{u}$$



Theorem

1. $\left\|\mathbf{v}_{\|}\right\| \leq \|\mathbf{v}\|$
2. $\|\mathbf{v}_{\perp}\| \leq \|\mathbf{v}\|$
3. $\mathbf{v}_{\|}$ *is the unique vector parallel to* $\mathbf{u}$ *which is closed to* $\mathbf{v}$.

$$\mathbf{v} = \mathbf{v}_{\|} + \mathbf{v}_{\perp} \qquad \mathbf{v}_{\|} = \langle \mathbf{u}, \mathbf{v} \rangle \, \mathbf{u} \qquad \mathbf{v}_{\perp} = \mathbf{v} - \langle \mathbf{u}, \mathbf{v} \rangle \, \mathbf{u}$$

Proof.

By the Pythogorean Theorem,

$$\|\mathbf{v}\|^2 = \left\|\mathbf{v}_{\|}\right\|^2 + \|\mathbf{v}_{\perp}\|^2$$

since $\mathbf{v}_{\|}$ and $\mathbf{v}_{\perp}$ are orthogonal, and since $\mathbf{v} = \mathbf{v}_{\|} + \mathbf{v}_{\perp}$.

$$\mathbf{v} = \mathbf{v}_{\|} + \mathbf{v}_{\perp} \qquad \mathbf{v}_{\|} = \langle \mathbf{u}, \mathbf{v} \rangle \, \mathbf{u} \qquad \mathbf{v}_{\perp} = \mathbf{v} - \langle \mathbf{u}, \mathbf{v} \rangle \, \mathbf{u}$$

Proof.

By the Pythogorean Theorem,

$$\|\mathbf{v}\|^2 = \left\|\mathbf{v}_{\|}\right\|^2 + \|\mathbf{v}_{\perp}\|^2$$

since $\mathbf{v}_{\|}$ and $\mathbf{v}_{\perp}$ are orthogonal, and since $\mathbf{v} = \mathbf{v}_{\|} + \mathbf{v}_{\perp}$. It follows that

$$\left\|\mathbf{v}_{\|}\right\| \leq \|\mathbf{v}\| \qquad \text{and} \qquad \|\mathbf{v}_{\perp}\| \leq \|\mathbf{v}\| .$$

$$\mathbf{v} = \mathbf{v}_{\parallel} + \mathbf{v}_{\perp} \qquad \mathbf{v}_{\parallel} = \langle \mathbf{u}, \mathbf{v} \rangle \, \mathbf{u} \qquad \mathbf{v}_{\perp} = \mathbf{v} - \langle \mathbf{u}, \mathbf{v} \rangle \, \mathbf{u}$$

Proof Continued.

For part 3: Let $\alpha \mathbf{u}$ be any other vector which is parallel to $\mathbf{u}$. Then

$$\|\mathbf{v} - \alpha \mathbf{u}\|^2 = \left\|\mathbf{v}_{\perp} + (\mathbf{v}_{\parallel} - \alpha \mathbf{u})\right\|^2$$

$$\mathbf{v} = \mathbf{v}_{\parallel} + \mathbf{v}_{\perp} \qquad \mathbf{v}_{\parallel} = \langle \mathbf{u}, \mathbf{v} \rangle \, \mathbf{u} \qquad \mathbf{v}_{\perp} = \mathbf{v} - \langle \mathbf{u}, \mathbf{v} \rangle \, \mathbf{u}$$

Proof Continued.

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$$\|\mathbf{v} - \alpha\mathbf{u}\|^2 = \left\|\mathbf{v}_{\perp} + (\mathbf{v}_{\parallel} - \alpha\mathbf{u})\right\|^2 = \|\mathbf{v}_{\perp}\|^2 + |\langle \mathbf{u}, \mathbf{v} \rangle - \alpha|^2 .$$

$$\mathbf{v} = \mathbf{v}_{\parallel} + \mathbf{v}_{\perp} \qquad \mathbf{v}_{\parallel} = \langle \mathbf{u}, \mathbf{v} \rangle \, \mathbf{u} \qquad \mathbf{v}_{\perp} = \mathbf{v} - \langle \mathbf{u}, \mathbf{v} \rangle \, \mathbf{u}$$

Proof Continued.

For part 3: Let $\alpha\mathbf{u}$ be any other vector which is parallel to $\mathbf{u}$. Then

$$\|\mathbf{v} - \alpha\mathbf{u}\|^2 = \left\|\mathbf{v}_{\perp} + (\mathbf{v}_{\parallel} - \alpha\mathbf{u})\right\|^2 = \|\mathbf{v}_{\perp}\|^2 + |\langle \mathbf{u}, \mathbf{v} \rangle - \alpha|^2 .$$

This distance is smallest when $\alpha = \langle \mathbf{u}, \mathbf{v} \rangle$. Hence $\mathbf{v}_{\parallel} = \langle \mathbf{u}, \mathbf{v} \rangle \, \mathbf{u}$ is the unique vector parallel to $\mathbf{u}$ which is closest to $\mathbf{v}$. □

Augustin-Louis Cauchy

Born
21 August 1789

Deceiced
23 May 1857

Nationality
French

Hermann Schwarz

Born
25 January 1843

Deceiced
30 November 1921

Nationality
German

Theorem (The Cauchy-Schwarz Inequality)

If $\mathbf{u}$ *and* $\mathbf{v}$ *are vectors in a inner product space* V, *then*

$$|\langle \mathbf{u}, \mathbf{v} \rangle| \leq \|\mathbf{u}\| \, \|\mathbf{v}\| \, .$$

This is one of the most important and widely used inequalities in mathematics.

$|\langle \mathbf{u}, \mathbf{v} \rangle| \leq \|\mathbf{u}\| \, \|\mathbf{v}\|$

Proof.

First note that if $\mathbf{u} = \mathbf{0}$ then the Cauchy-Schwarz Inequality is true because $\|\mathbf{u}\| = 0$ and $\langle \mathbf{u}, \mathbf{v} \rangle = 0$.

$$|\langle \mathbf{u}, \mathbf{v} \rangle| \leq \|\mathbf{u}\| \, \|\mathbf{v}\|$$

Proof.

First note that if $\mathbf{u} = \mathbf{0}$ then the Cauchy-Schwarz Inequality is true because $\|\mathbf{u}\| = 0$ and $\langle \mathbf{u}, \mathbf{v} \rangle = 0$.

Next suppose that $\mathbf{u}$ is a unit vector. Write $\mathbf{v} = \mathbf{v}_{\|} + \mathbf{v}_{\perp}$ where $\mathbf{v}_{\|} = \langle \mathbf{u}, \mathbf{v} \rangle \, \mathbf{u}$.

$|\langle \mathbf{u}, \mathbf{v} \rangle| \leq \|\mathbf{u}\| \, \|\mathbf{v}\|$

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$$|\langle \mathbf{u}, \mathbf{v} \rangle| \leq \|\mathbf{u}\| \, \|\mathbf{v}\|$$

Proof.

First note that if $\mathbf{u} = \mathbf{0}$ then the Cauchy-Schwarz Inequality is true because $\|\mathbf{u}\| = 0$ and $\langle \mathbf{u}, \mathbf{v} \rangle = 0$.

Next suppose that $\mathbf{u}$ is a unit vector. Write $\mathbf{v} = \mathbf{v}_{\parallel} + \mathbf{v}_{\perp}$ where $\mathbf{v}_{\parallel} = \langle \mathbf{u}, \mathbf{v} \rangle \mathbf{u}$. Recall that $\|c\mathbf{u}\| = |c| \, \|\mathbf{u}\|$ for any scalar c. Thus

$$\left\| \mathbf{v}_{\parallel} \right\| = \| \langle \mathbf{u}, \mathbf{v} \rangle \mathbf{u} \|$$

$|\langle \mathbf{u}, \mathbf{v} \rangle| \leq \|\mathbf{u}\| \, \|\mathbf{v}\|$

Proof.

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$$\left\| \mathbf{v}_{\|} \right\| = \| \langle \mathbf{u}, \mathbf{v} \rangle \, \mathbf{u} \| = |\langle \mathbf{u}, \mathbf{v} \rangle| \, \|\mathbf{u}\|$$

$$|\langle \mathbf{u}, \mathbf{v} \rangle| \leq \|\mathbf{u}\| \, \|\mathbf{v}\|$$

Proof.

First note that if $\mathbf{u} = \mathbf{0}$ then the Cauchy-Schwarz Inequality is true because $\|\mathbf{u}\| = 0$ and $\langle \mathbf{u}, \mathbf{v} \rangle = 0$.

Next suppose that $\mathbf{u}$ is a unit vector. Write $\mathbf{v} = \mathbf{v}_{\|} + \mathbf{v}_{\perp}$ where $\mathbf{v}_{\|} = \langle \mathbf{u}, \mathbf{v} \rangle \mathbf{u}$. Recall that $\|c\mathbf{u}\| = |c| \, \|\mathbf{u}\|$ for any scalar c. Thus

$$\left\| \mathbf{v}_{\|} \right\| = \| \langle \mathbf{u}, \mathbf{v} \rangle \mathbf{u} \| = |\langle \mathbf{u}, \mathbf{v} \rangle| \, \|\mathbf{u}\| = |\langle \mathbf{u}, \mathbf{v} \rangle| \, .$$

Proof.

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Since $\left\|\mathbf{v}_{\|}\right\| \leq \|\mathbf{v}\|$ we have

$$|\langle \mathbf{u}, \mathbf{v} \rangle| \leq \|\mathbf{v}\| \, .$$

$|\langle \mathbf{u}, \mathbf{v} \rangle| \leq \|\mathbf{u}\| \, \|\mathbf{v}\|$

Proof Continued.

Now let $\mathbf{u}$ be any nonzero vector and define $\hat{\mathbf{u}} = \frac{\mathbf{u}}{\|\mathbf{u}\|}$. Then

- $\hat{\mathbf{u}}$ is a unit vector;
- $\mathbf{u} = \|\mathbf{u}\| \, \hat{\mathbf{u}}$; and
- $|\langle \hat{\mathbf{u}}, \mathbf{v} \rangle| \leq \|\mathbf{v}\|$.

Proof Continued.

Now let $\mathbf{u}$ be any nonzero vector and define $\hat{\mathbf{u}} = \frac{\mathbf{u}}{\|\mathbf{u}\|}$. Then

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- $|\langle \hat{\mathbf{u}}, \mathbf{v} \rangle| \leq \|\mathbf{v}\|$.

Therefore

$$|\langle \mathbf{u}, \mathbf{v} \rangle| = |\langle \|\mathbf{u}\| \, \hat{\mathbf{u}}, \mathbf{v} \rangle|$$

$|\langle \mathbf{u}, \mathbf{v} \rangle| \leq \|\mathbf{u}\| \, \|\mathbf{v}\|$

Proof Continued.

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Orthogonality

$$|\langle \mathbf{u}, \mathbf{v} \rangle| \leq \|\mathbf{u}\| \, \|\mathbf{v}\|$$

Proof Continued.

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and we are finished.

$\square$

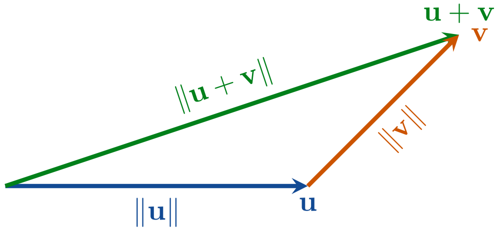
$$|\langle \mathbf{u}, \mathbf{v} \rangle| \le \|\mathbf{u}\| \, \|\mathbf{v}\|$$

We can use the Cauchy-Schwarz Inequality to prove the following result:

Theorem (The Triangle Inequality)

For all $\mathbf{u}$, $\mathbf{v}$ *in* V,

$$\|\mathbf{u} + \mathbf{v}\| \le \|\mathbf{u}\| + \|\mathbf{v}\| .$$



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For all $\mathbf{u}$, $\mathbf{v}$ *in* V,

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Proof.

Using the Cauchy-Schwarz Inequality we calculate that

$$\begin{aligned} \|\mathbf{u} + \mathbf{v}\|^2 = \langle \mathbf{u} + \mathbf{v}, \mathbf{u} + \mathbf{v} \rangle &= \\ &= \\ &= \\ &\leq \\ &\leq \\ &= \end{aligned}$$

$\square$

Orthogonality

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Proof.

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$$\begin{aligned} \|\mathbf{u}+\mathbf{v}\|^2 &= \langle \mathbf{u}+\mathbf{v}, \mathbf{u}+\mathbf{v} \rangle = \langle \mathbf{u}, \mathbf{u}+\mathbf{v} \rangle + \langle \mathbf{v}, \mathbf{u}+\mathbf{v} \rangle \\ &= \langle \mathbf{u}, \mathbf{u} \rangle + \langle \mathbf{u}, \mathbf{v} \rangle + \langle \mathbf{v}, \mathbf{u} \rangle + \langle \mathbf{v}, \mathbf{v} \rangle \\ &= \|\mathbf{u}\|^2 + 2\langle \mathbf{u}, \mathbf{v} \rangle + \|\mathbf{v}\|^2 \\ &\leq \\ &\leq \\ &= \end{aligned}$$

□

$$|\langle \mathbf{u}, \mathbf{v} \rangle| \le \|\mathbf{u}\| \, \|\mathbf{v}\|$$

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For all $\mathbf{u}$, $\mathbf{v}$ *in* V,

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&= \langle \mathbf{u}, \mathbf{u} \rangle + \langle \mathbf{u}, \mathbf{v} \rangle + \langle \mathbf{v}, \mathbf{u} \rangle + \langle \mathbf{v}, \mathbf{v} \rangle \\
&= \|\mathbf{u}\|^2 + 2 \langle \mathbf{u}, \mathbf{v} \rangle + \|\mathbf{v}\|^2 \\
&\le \|\mathbf{u}\|^2 + 2 |\langle \mathbf{u}, \mathbf{v} \rangle| + \|\mathbf{v}\|^2 \\
&\le \\
&=
\end{aligned}$$

□

$$|\langle \mathbf{u}, \mathbf{v}\rangle| \le \|\mathbf{u}\|\,\|\mathbf{v}\|$$

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&= \|\mathbf{u}\|^2 + 2\langle \mathbf{u}, \mathbf{v}\rangle + \|\mathbf{v}\|^2 \\
&\le \|\mathbf{u}\|^2 + 2|\langle \mathbf{u}, \mathbf{v}\rangle| + \|\mathbf{v}\|^2 \\
&\le \|\mathbf{u}\|^2 + 2\|\mathbf{u}\|\,\|\mathbf{v}\| + \|\mathbf{v}\|^2 \\
&=
\end{aligned}$$

□

$$|\langle \mathbf{u}, \mathbf{v} \rangle| \leq \|\mathbf{u}\| \, \|\mathbf{v}\|$$

Theorem (The Triangle Inequality)

For all $\mathbf{u}$, $\mathbf{v}$ *in* V,

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Proof.

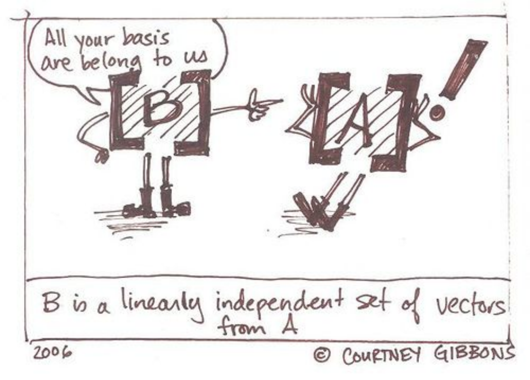
Using the Cauchy-Schwarz Inequality we calculate that

$$\begin{aligned}
\|\mathbf{u} + \mathbf{v}\|^2 &= \langle \mathbf{u} + \mathbf{v}, \mathbf{u} + \mathbf{v} \rangle = \langle \mathbf{u}, \mathbf{u} + \mathbf{v} \rangle + \langle \mathbf{v}, \mathbf{u} + \mathbf{v} \rangle \\
&= \langle \mathbf{u}, \mathbf{u} \rangle + \langle \mathbf{u}, \mathbf{v} \rangle + \langle \mathbf{v}, \mathbf{u} \rangle + \langle \mathbf{v}, \mathbf{v} \rangle \\
&= \|\mathbf{u}\|^2 + 2 \langle \mathbf{u}, \mathbf{v} \rangle + \|\mathbf{v}\|^2 \\
&\leq \|\mathbf{u}\|^2 + 2 |\langle \mathbf{u}, \mathbf{v} \rangle| + \|\mathbf{v}\|^2 \\
&\leq \|\mathbf{u}\|^2 + 2 \|\mathbf{u}\| \, \|\mathbf{v}\| + \|\mathbf{v}\|^2 \\
&= (\|\mathbf{u}\| + \|\mathbf{v}\|)^2 .
\end{aligned}$$

□

Break

We will continue at 3pm



Orthogonal Sets and Orthonormal Sets

Orthogonal Sets and Orthonormal Sets

Definition

A set of two or more vectors in a real inner product space is called an *orthogonal set* if all pairs of distinct vectors in the set are orthogonal.

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for all $\mathbf{u} \neq \mathbf{v}$.

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Definition

An orthogonal set in which each vector is a unit vector is called an *orthonormal set*.

So we must have

1. $\langle \mathbf{u}, \mathbf{v} \rangle = 0$ for all $\mathbf{u} \neq \mathbf{v}$; and
2. $\langle \mathbf{u}, \mathbf{u} \rangle = 1$ for all $\mathbf{u} \in V$.

Example (An Orthogonal Set in $\mathbb{R}^3$)

Let

$$\mathbf{v}_1 = (0, 1, 0), \qquad \mathbf{v}_2 = (1, 0, 1), \qquad \mathbf{v}_3 = (1, 0, -1).$$

Assume that $\mathbb{R}^3$ has the Euclidean inner product (dot product).

Show that $S = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is an orthogonal set.

$$\begin{aligned}
\langle \mathbf{v}_1, \mathbf{v}_2 \rangle &= \mathbf{v}_1 \bullet \mathbf{v}_2 = \\
\langle \mathbf{v}_1, \mathbf{v}_3 \rangle &= \mathbf{v}_1 \bullet \mathbf{v}_3 = \\
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\end{aligned}$$

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Assume that $\mathbb{R}^3$ has the Euclidean inner product (dot product).

Show that $S = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is an orthogonal set.

$$\begin{aligned}
\langle \mathbf{v}_1, \mathbf{v}_2 \rangle &= \mathbf{v}_1 \cdot \mathbf{v}_2 = (0)(1) + (1)(0) + (0)(1) = 0 \\
\langle \mathbf{v}_1, \mathbf{v}_3 \rangle &= \mathbf{v}_1 \cdot \mathbf{v}_3 = (0)(1) + (1)(0) + (0)(-1) = 0 \\
\langle \mathbf{v}_2, \mathbf{v}_3 \rangle &= \mathbf{v}_2 \cdot \mathbf{v}_3 = (1)(1) + (0)(0) + (1)(-1) = 0.
\end{aligned}$$

Therefore S is orthogonal.

Orthogonal Sets and Orthonormal Sets

Recall that if $\mathbf{v} \neq \mathbf{0}$ is any nonzero vector, then $\mathbf{u} = \frac{\mathbf{v}}{\|\mathbf{v}\|}$ is a unit vector because

$$\|\mathbf{u}\| = \left\| \frac{\mathbf{v}}{\|\mathbf{v}\|} \right\| = \left| \frac{1}{\|\mathbf{v}\|} \right| \|\mathbf{v}\| = \frac{\|\mathbf{v}\|}{\|\mathbf{v}\|} = 1.$$

Orthogonal Sets and Orthonormal Sets

$$\mathbf{v}_1 = (0,1,0), \qquad \mathbf{v}_2 = (1,0,1), \qquad \mathbf{v}_3 = (1,0,-1)$$

Example (Constructing an Orthonormal Set)

We have that $S = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is an orthogonal set in $\mathbb{R}^3$ with respect to the Euclidean inner product. Note that

$$\|\mathbf{v}_1\| = \sqrt{0^2+1^2+0^2} = 1 \qquad \|\mathbf{v}_2\| = \sqrt{1^2+0^2+1^2} = \sqrt{2}$$

$$\|\mathbf{v}_3\| = \sqrt{1^2+0^2+(-1)^2} = \sqrt{2}.$$

It follows that if

$$\mathbf{u}_1 = \frac{\mathbf{v}_1}{\|\mathbf{v}_1\|} = (1,0,1) \qquad \mathbf{u}_2 = \frac{\mathbf{v}_2}{\|\mathbf{v}_2\|} = \left(\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right)$$

$$\mathbf{u}_3 = \frac{\mathbf{v}_3}{\|\mathbf{v}_3\|} = \left(\frac{1}{\sqrt{2}}, 0, -\frac{1}{\sqrt{2}}\right)$$

then $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ is an orthonormal set.

$$\mathbf{v}_1 = (0,1,0), \qquad \mathbf{v}_2 = (1,0,1), \qquad \mathbf{v}_3 = (1,0,-1)$$

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$$\mathbf{u}_3 = \frac{\mathbf{v}_3}{\|\mathbf{v}_3\|} = \left(\frac{1}{\sqrt{2}}, 0, -\frac{1}{\sqrt{2}}\right)$$

then $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ is an orthonormal set.

I leave it to you to check that

$$\langle \mathbf{u}_1, \mathbf{u}_2 \rangle = \langle \mathbf{u}_1, \mathbf{u}_3 \rangle = \langle \mathbf{u}_2, \mathbf{u}_3 \rangle = 0$$

and

$$\|\mathbf{u}_1\| = \|\mathbf{u}_2\| = \|\mathbf{u}_3\| = 1.$$

Theorem

If $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is an orthogonal set of nonzero vectors in an inner product space, then S is linearly independent.

Proof.

Suppose that

$$k_1\mathbf{v}_1 + k_2\mathbf{v}_2 + \ldots + k_n\mathbf{v}_n = \mathbf{0}.$$

We must prove that $k_1 = k_2 = \ldots = k_n = 0$.

Orthogonal Sets and Orthonormal Sets

Proof Continued.

For each $\mathbf{v}_i \in S$ we have

$$\begin{aligned} \mathbf{0} = \langle \mathbf{0}, \mathbf{v}_i \rangle &= \\ &= \\ &= \\ &= \end{aligned}$$

Orthogonal Sets and Orthonormal Sets

Proof Continued.

For each $\mathbf{v}_i \in S$ we have

$$\begin{aligned} \mathbf{0} &= \langle \mathbf{0}, \mathbf{v}_i \rangle = \langle k_1\mathbf{v}_1 + k_2\mathbf{v}_2 + \ldots + k_n\mathbf{v}_n, \mathbf{v}_i \rangle \\ &= \\ &= \\ &= \end{aligned}$$

Orthogonal Sets and Orthonormal Sets

Proof Continued.

For each $\mathbf{v}_i \in S$ we have

$$
\begin{aligned}
\mathbf{0} = \langle \mathbf{0}, \mathbf{v}_i \rangle &= \langle k_1\mathbf{v}_1 + k_2\mathbf{v}_2 + \ldots + k_n\mathbf{v}_n, \mathbf{v}_i \rangle \\
&= k_1 \langle \mathbf{v}_1, \mathbf{v}_i \rangle + k_2 \langle \mathbf{v}_2, \mathbf{v}_i \rangle + \ldots + k_i \langle \mathbf{v}_i, \mathbf{v}_i \rangle + \ldots + k_1 \langle \mathbf{v}_1, \mathbf{v}_i \rangle \\
&= \\
&=
\end{aligned}
$$

Orthogonal Sets and Orthonormal Sets

Proof Continued.

For each $\mathbf{v}_i \in S$ we have

$$\begin{aligned}
\mathbf{0} = \langle \mathbf{0}, \mathbf{v}_i \rangle &= \langle k_1\mathbf{v}_1 + k_2\mathbf{v}_2 + \ldots + k_n\mathbf{v}_n, \mathbf{v}_i \rangle \\
&= k_1 \langle \mathbf{v}_1, \mathbf{v}_i \rangle + k_2 \langle \mathbf{v}_2, \mathbf{v}_i \rangle + \ldots + k_i \langle \mathbf{v}_i, \mathbf{v}_i \rangle + \ldots + k_1 \langle \mathbf{v}_1, \mathbf{v}_i \rangle \\
&= 0 + 0 + \ldots + k_i \langle \mathbf{v}_i, \mathbf{v}_i \rangle + \ldots + 0 \\
&=
\end{aligned}$$

Proof Continued.

For each $\mathbf{v}_i \in S$ we have

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\mathbf{0} = \langle \mathbf{0}, \mathbf{v}_i \rangle &= \langle k_1\mathbf{v}_1 + k_2\mathbf{v}_2 + \ldots + k_n\mathbf{v}_n, \mathbf{v}_i \rangle \\
&= k_1 \langle \mathbf{v}_1, \mathbf{v}_i \rangle + k_2 \langle \mathbf{v}_2, \mathbf{v}_i \rangle + \ldots + k_i \langle \mathbf{v}_i, \mathbf{v}_i \rangle + \ldots + k_1 \langle \mathbf{v}_1, \mathbf{v}_i \rangle \\
&= 0 + 0 + \ldots + k_i \langle \mathbf{v}_i, \mathbf{v}_i \rangle + \ldots + 0 \\
&= k_i \langle \mathbf{v}_i, \mathbf{v}_i \rangle .
\end{aligned}$$

Proof Continued.

For each $\mathbf{v}_i \in S$ we have

$$\begin{aligned}\mathbf{0} = \langle \mathbf{0}, \mathbf{v}_i \rangle &= \langle k_1 \mathbf{v}_1 + k_2 \mathbf{v}_2 + \ldots + k_n \mathbf{v}_n, \mathbf{v}_i \rangle \\ &= k_1 \langle \mathbf{v}_1, \mathbf{v}_i \rangle + k_2 \langle \mathbf{v}_2, \mathbf{v}_i \rangle + \ldots + k_i \langle \mathbf{v}_i, \mathbf{v}_i \rangle + \ldots + k_1 \langle \mathbf{v}_1, \mathbf{v}_i \rangle \\ &= 0 + 0 + \ldots + k_i \langle \mathbf{v}_i, \mathbf{v}_i \rangle + \ldots + 0 \\ &= k_i \langle \mathbf{v}_i, \mathbf{v}_i \rangle .\end{aligned}$$

Since $\mathbf{v}_i \neq \mathbf{0}$, we have that $\langle \mathbf{v}_i, \mathbf{v}_i \rangle \neq 0$.

Proof Continued.

For each $\mathbf{v}_i \in S$ we have

$$\begin{aligned}
\mathbf{0} = \langle \mathbf{0}, \mathbf{v}_i \rangle = & \langle k_1\mathbf{v}_1 + k_2\mathbf{v}_2 + \ldots + k_n\mathbf{v}_n, \mathbf{v}_i \rangle \\
&= k_1 \langle \mathbf{v}_1, \mathbf{v}_i \rangle + k_2 \langle \mathbf{v}_2, \mathbf{v}_i \rangle + \ldots + k_i \langle \mathbf{v}_i, \mathbf{v}_i \rangle + \ldots + k_1 \langle \mathbf{v}_1, \mathbf{v}_i \rangle \\
&= 0 + 0 + \ldots + k_i \langle \mathbf{v}_i, \mathbf{v}_i \rangle + \ldots + 0 \\
&= k_i \langle \mathbf{v}_i, \mathbf{v}_i \rangle .
\end{aligned}$$

Since $\mathbf{v}_i \neq \mathbf{0}$, we have that $\langle \mathbf{v}_i, \mathbf{v}_i \rangle \neq 0$.

Therefore $k_i = 0$ for all i. This proves that S is linearly independent.

□

Orthogonal and Orthonormal Bases

Definition

In an inner product space, a basis consisting of orthonormal vectors is called an *orthonormal basis*, and a basis consisting of orthogonal vectors is called an *orthogonal basis*.

Example

$$\mathbf{e}_1 = (1, 0, 0, \ldots, 0),\ \mathbf{e}_2 = (0, 1, 0, \ldots, 0),\ \ldots,\ \mathbf{e}_n = (0, 0, 0, \ldots, 1)$$

is an orthonormal basis in $\mathbb{R}^n$ with the Euclidean inner product.

(Recall that this basis is called the *standard basis* for $\mathbb{R}^n$.)

Example (An Orthonormal Basis for $\mathbb{P}^n$)

Consider the vector space of polynomials of degree $\leq n$ with the inner product

$$\langle \mathbf{p}, \mathbf{q} \rangle = a_0 b_0 + a_1 b_1 + \ldots + a_n b_n$$

where

$$\begin{aligned} \mathbf{p} &= a_0 + a_1 x + \ldots + a_n x^n \\ \mathbf{q} &= b_0 + b_1 x + \ldots + b_n x^n. \end{aligned}$$

I leave it to you to prove that the standard basis

$$S = \{1, x, x^2, x^3, \ldots, x^n\}$$

is an orthonormal basis with respect to this inner product.

Example

In an earlier example we saw that

$$\mathbf{u}_1 = (0, 1, 0), \qquad \mathbf{u}_2 = \left(\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right), \qquad \mathbf{u}_3 = \left(\frac{1}{\sqrt{2}}, 0, -\frac{1}{\sqrt{2}}\right)$$

form an orthonormal set with respect to the Euclidean inner product on $\mathbb{R}^3$.

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By the previous theorem, these three vectors are linearly independent.

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form an orthonormal set with respect to the Euclidean inner product on $\mathbb{R}^3$.

By the previous theorem, these three vectors are linearly independent.

Since $\mathbb{R}^3$ is three-dimensional, $S = \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ is an orthonormal basis for $\mathbb{R}^3$.

Coordinates Relative to Orthonormal Bases

Recall that if $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is a basis for a vector space V, and if

$$\mathbf{u} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \ldots + c_n\mathbf{v}_n$$

then the coordinates of $\mathbf{u}$ relative to this basis is

$$(\mathbf{u})_S = (c_1, c_2, \ldots, c_n).$$

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Recall that if $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is a basis for a vector space V, and if

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then the coordinates of $\mathbf{u}$ relative to this basis is

$$(\mathbf{u})_S = (c_1, c_2, \ldots, c_n).$$

If the basis is orthogonal or orthonormal then there is an easy way to find the coefficients $c_1, c_2, \ldots, c_n$.

Orthogonal Sets and Orthonormal Sets

Theorem

If $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is an orthogonal basis for an inner product space V, and if $\mathbf{u}$ is any vector in V, then

$$\mathbf{u} = \frac{\langle \mathbf{u}, \mathbf{v}_1 \rangle}{\|\mathbf{v}_1\|^2}\mathbf{v}_1 + \frac{\langle \mathbf{u}, \mathbf{v}_2 \rangle}{\|\mathbf{v}_2\|^2}\mathbf{v}_2 + \ldots + \frac{\langle \mathbf{u}, \mathbf{v}_n \rangle}{\|\mathbf{v}_n\|^2}\mathbf{v}_n.$$

Theorem

If $S = \{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\}$ is an orthogonal basis for an inner product space V, and if $\mathbf{u}$ is any vector in V, then

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Proof.

Let

$$\mathbf{u} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \dots + c_n\mathbf{v}_n.$$

We need to show that $c_i = \dfrac{\langle \mathbf{u}, \mathbf{v}_i \rangle}{\|\mathbf{v}_i\|^2}$ for each i.

Orthogonal Sets and Orthonormal Sets

Proof Continued.

Note that

$$\begin{aligned}\langle \mathbf{u}, \mathbf{v}_i \rangle &= \langle c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \ldots + c_n\mathbf{v}_n, \mathbf{v}_i \rangle \\ &= c_1\langle \mathbf{v}_1, \mathbf{v}_i \rangle + c_2\langle \mathbf{v}_2, \mathbf{v}_i \rangle + \ldots + c_i\langle \mathbf{v}_i, \mathbf{v}_i \rangle + \ldots + c_n\langle \mathbf{v}_n, \mathbf{v}_i \rangle \\ &= \\ &= \end{aligned}$$

Proof Continued.

Note that

$$\begin{aligned}
\langle \mathbf{u}, \mathbf{v}_i \rangle &= \langle c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \ldots + c_n\mathbf{v}_n, \mathbf{v}_i \rangle \\
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&= 0 + 0 + \ldots + c_i\|\mathbf{v}_i\|^2 + \ldots + 0 \\
&=
\end{aligned}$$

Proof Continued.

Note that

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Proof Continued.

Note that

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Hence $c_i = \dfrac{\langle \mathbf{u}, \mathbf{v}_i \rangle}{\|\mathbf{v}_i\|^2}$ for each i and we are finished. □

Orthogonal Sets and Orthonormal Sets

Theorem

If $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is an orthonormal basis for an inner product space V, and if $\mathbf{u}$ is any vector in V, then

$$\mathbf{u} = \langle \mathbf{u}, \mathbf{v}_1 \rangle \mathbf{v}_1 + \langle \mathbf{u}, \mathbf{v}_2 \rangle \mathbf{v}_2 + \ldots + \langle \mathbf{u}, \mathbf{v}_n \rangle \mathbf{v}_n.$$

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Proof.

Just take the previous formula

$$\mathbf{u} = \frac{\langle \mathbf{u}, \mathbf{v}_1 \rangle}{\|\mathbf{v}_1\|^2}\mathbf{v}_1 + \frac{\langle \mathbf{u}, \mathbf{v}_2 \rangle}{\|\mathbf{v}_2\|^2}\mathbf{v}_2 + \ldots + \frac{\langle \mathbf{u}, \mathbf{v}_n \rangle}{\|\mathbf{v}_n\|^2}\mathbf{v}_n$$

and replace every $\|\mathbf{v}_i\|$ by 1 since each $\mathbf{v}_i$ is a unit vector. $\square$

Orthogonal Sets and Orthonormal Sets

Remark

So if S is an orthogonal basis then

$$(\mathbf{u})_S = \left(\frac{\langle \mathbf{u}, \mathbf{v}_1 \rangle}{\|\mathbf{v}_1\|^2}, \frac{\langle \mathbf{u}, \mathbf{v}_2 \rangle}{\|\mathbf{v}_2\|^2}, \ldots, \frac{\langle \mathbf{u}, \mathbf{v}_n \rangle}{\|\mathbf{v}_n\|^2} \right)$$

and if S is an orthonormal basis then

$$(\mathbf{u})_S = \big(\langle \mathbf{u}, \mathbf{v}_1 \rangle, \langle \mathbf{u}, \mathbf{v}_2 \rangle, \ldots, \langle \mathbf{u}, \mathbf{v}_n \rangle \big).$$

Example

Let

$$\mathbf{v}_1 = (0, 1, 0), \qquad \mathbf{v}_2 = \left(-\frac{4}{5}, 0, \frac{3}{5}\right), \qquad \mathbf{v}_3 = \left(\frac{3}{5}, 0, \frac{4}{5}\right).$$

I leave it to you to check that $S = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is an orthonormal basis for $\mathbb{R}^3$ with the Euclidean inner product.

Orthogonal Sets and Orthonormal Sets

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Find $(\mathbf{u})_S$ if $\mathbf{u} = (1,1,1)$.

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Find $(\mathbf{u})_S$ if $\mathbf{u} = (1,1,1)$.

Since

$$\langle \mathbf{u}, \mathbf{v}_1 \rangle = 1, \qquad \langle \mathbf{u}, \mathbf{v}_2 \rangle = -\frac{1}{5}, \qquad \langle \mathbf{u}, \mathbf{v}_3 \rangle = \frac{7}{5},$$

(please check) we have that

$$(\mathbf{u})_S = \big(\langle \mathbf{u}, \mathbf{v}_1 \rangle, \langle \mathbf{u}, \mathbf{v}_2 \rangle, \langle \mathbf{u}, \mathbf{v}_3 \rangle \big) = \left(1, -\frac{1}{5}, \frac{7}{5}\right).$$

Example (An Orthonormal Basis from an Orthogonal Basis)

1. Show that the vectors

$$\mathbf{w}_1 = (0, 2, 0), \qquad \mathbf{w}_2 = (3, 0, 3), \qquad \mathbf{w}_3 = (-4, 0, 4)$$

   form an orthogonal basis for $\mathbb{R}^3$ with the Euclidean inner product.

2.

3.

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2. Normalise each vector above to find an orthonormal basis $S = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$.

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Example (An Orthonormal Basis from an Orthogonal Basis)

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   form an orthogonal basis for $\mathbb{R}^3$ with the Euclidean inner product.
2. Normalise each vector above to find an orthonormal basis $S = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$.
3. Find $(\mathbf{u})_S$ if $\mathbf{u} = (1, 2, 4)$.

$$\mathbf{w}_1 = (0, 2, 0), \qquad \mathbf{w}_2 = (3, 0, 3), \qquad \mathbf{w}_3 = (-4, 0, 4)$$

1. I leave it to you to check that

$$\langle \mathbf{w}_1, \mathbf{w}_2 \rangle = \langle \mathbf{w}_1, \mathbf{w}_3 \rangle = \langle \mathbf{w}_2, \mathbf{w}_3 \rangle = 0.$$

This shows that $\{\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3\}$ is an orthogonal set.

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Recall that sets of nonzero orthogonal vectors are always linearly independent.

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Recall that sets of nonzero orthogonal vectors are always linearly independent.

Since $\mathbb{R}^3$ is three-dimensional, $\{\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3\}$ must be a basis for $\mathbb{R}^3$.

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   Recall that sets of nonzero orthogonal vectors are always linearly independent.

   Since $\mathbb{R}^3$ is three-dimensional, $\{\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3\}$ must be a basis for $\mathbb{R}^3$.

2. We calculate
$$\mathbf{v}_1 = \frac{\mathbf{w}_1}{\|\mathbf{w}_1\|} = (0,1,0), \qquad \mathbf{v}_2 = \frac{\mathbf{w}_2}{\|\mathbf{w}_2\|} = \left(\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right)$$
$$\mathbf{v}_3 = \frac{\mathbf{w}_3}{\|\mathbf{w}_3\|} = \left(-\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right).$$

$$\mathbf{v}_1 = (0,1,0), \quad \mathbf{v}_2 = \left(\tfrac{1}{\sqrt{2}}, 0, \tfrac{1}{\sqrt{2}}\right), \quad \mathbf{v}_3 = \left(-\tfrac{1}{\sqrt{2}}, 0, \tfrac{1}{\sqrt{2}}\right)$$

3. Since

$$\begin{aligned}
\langle \mathbf{u}, \mathbf{v}_1 \rangle &= (1,2,4) \cdot (0,1,0) = 2 \\
\langle \mathbf{u}, \mathbf{v}_2 \rangle &= (1,2,4) \cdot \left(\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right) = \frac{5}{\sqrt{2}} \\
\langle \mathbf{u}, \mathbf{v}_3 \rangle &= (1,2,4) \cdot \left(-\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right) = \frac{3}{\sqrt{2}},
\end{aligned}$$

we have that

$$(\mathbf{u})_S = \left(2, \frac{5}{\sqrt{2}}, \frac{3}{\sqrt{2}}\right).$$

Orthogonal Projections

Theorem

If W is a finite-dimensional subspace of an inner product space V, then every vector $\mathbf{u}$ in V can be expressed in exactly one way as

$$\mathbf{u} = \mathbf{w}_{\|} + \mathbf{w}_{\perp}$$

where $\mathbf{w}_{\|}$ is in W and $\mathbf{w}_{\perp}$ is in $W^{\perp}$.

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$$\mathbf{u} = \mathbf{w}_{\|} + \mathbf{w}_{\perp}$$

where $\mathbf{w}_{\|}$ is in W and $\mathbf{w}_{\perp}$ is in $W^{\perp}$.

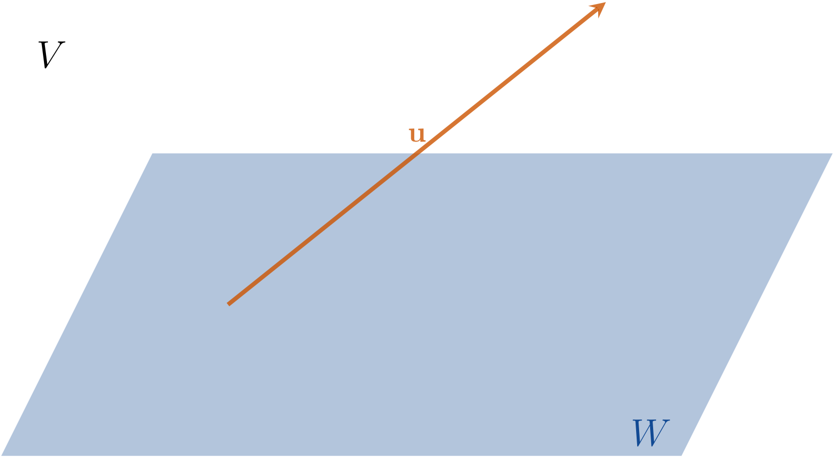
The vectors $\mathbf{w}_{\|}$ and $\mathbf{w}_{\perp}$ are often denoted as

$$\mathbf{w}_{\|} = \operatorname{proj}_W \mathbf{u} \qquad \text{and} \qquad \mathbf{w}_{\perp} = \operatorname{proj}_{W^{\perp}} \mathbf{u}$$

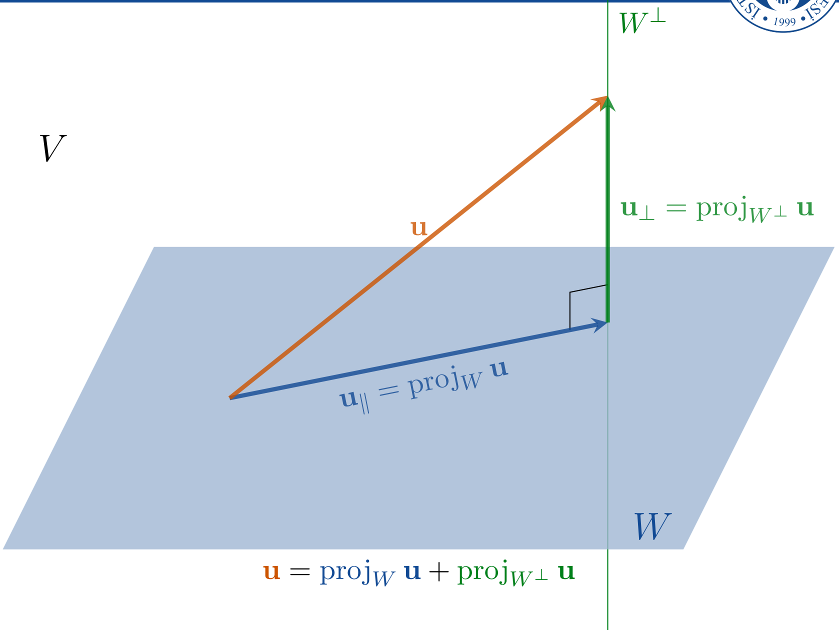
and are called the *orthogonal projection of $\mathbf{u}$ on W* and the *orthogonal projection of $\mathbf{u}$ on $W^{\perp}$*, respectively.

Orthogonal Sets and Orthonormal Sets

Orthogonal Sets and Orthonormal Sets



Orthogonal Sets and Orthonormal Sets



$$\mathbf{u} = \mathrm{proj}_{W}\,\mathbf{u} + \mathrm{proj}_{W^{\perp}}\,\mathbf{u}$$

Orthogonal Sets and Orthonormal Sets

Theorem

Let W be a finite-dimensional subspace of an inner product space V.

1. *If $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_r\}$ is an orthogonal basis for W, and $\mathbf{u}$ is any vector in V, then*

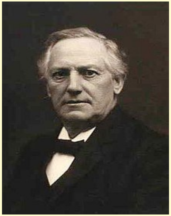
$$\operatorname{proj}_W \mathbf{u} = \frac{\langle \mathbf{u}, \mathbf{v}_1 \rangle}{\|\mathbf{v}_1\|^2}\mathbf{v}_1 + \frac{\langle \mathbf{u}, \mathbf{v}_2 \rangle}{\|\mathbf{v}_2\|^2}\mathbf{v}_2 + \ldots + \frac{\langle \mathbf{u}, \mathbf{v}_n \rangle}{\|\mathbf{v}_r\|^2}\mathbf{v}_r.$$

2. *If $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_r\}$ is an orthonormal basis for W, and $\mathbf{u}$ is any vector in V, then*

$$\operatorname{proj}_W \mathbf{u} = \langle \mathbf{u}, \mathbf{v}_1 \rangle \mathbf{v}_1 + \langle \mathbf{u}, \mathbf{v}_2 \rangle \mathbf{v}_2 + \ldots + \langle \mathbf{u}, \mathbf{v}_r \rangle \mathbf{v}_r.$$

The Gram-Schmidt Process

The Gram-Schmidt Process



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27 June 1850

DECEICED
29 April 1916

NATIONALITY
Danish

Erhard Schmidt

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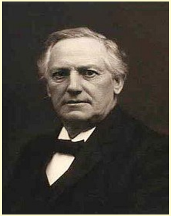
Theorem

Every nonzero finite-dimensional inner product space has an orthonormal basis.

The Gram-Schmidt Process



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Theorem

Every nonzero finite-dimensional inner product space has an orthonormal basis.

But how can we find it?

The Gram-Schmidt Process

Start with a basis that we know $\{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_r\}$ → Construct an orthogonal basis $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_r\}$ → Create an orthonormal basis $\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_r\}$

The Gram-Schmidt Process

Just define $\mathbf{u}_i = \dfrac{\mathbf{v}_i}{\|\mathbf{v}_i\|}$ for each i.

Start with a basis that we know $\{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_r\}$ → Construct an orthogonal basis $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_r\}$ → Create an orthonormal basis $\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_r\}$

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Use the Gram-Schmidt process

The Gram-Schmidt Process

To convert a basis $\{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_r\}$ into an orthogonal basis $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_r\}$, perform the following steps:

The Gram-Schmidt Process

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1. $\mathbf{v}_1 = \mathbf{x}_1$
2. $\mathbf{v}_2 =$
3. $\mathbf{v}_3 =$
4. $\mathbf{v}_4 =$

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To convert a basis $\{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_r\}$ into an orthogonal basis $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_r\}$, perform the following steps:

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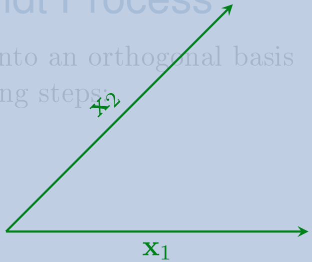

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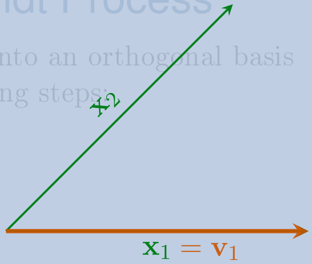

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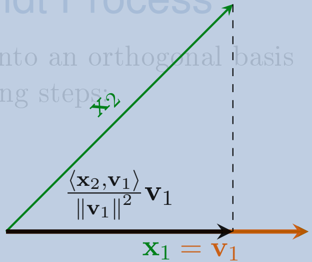

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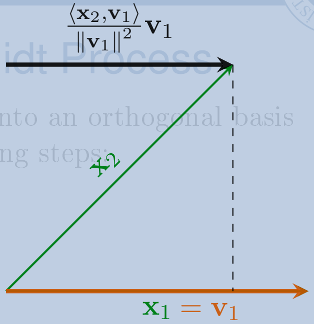

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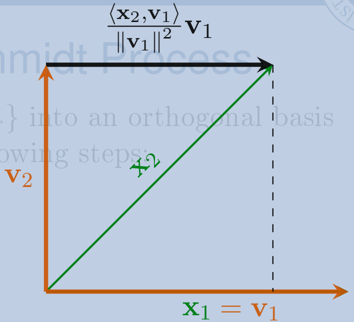

The Gram-Schmidt Process

Are $\mathbf{v}_1$ and $\mathbf{v}_2$ really orthogonal?

To convert a basis $\{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_r\}$ into an orthogonal basis $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_r\}$, perform the following steps:

1. $\mathbf{v}_1 = \mathbf{x}_1$
2. $\mathbf{v}_2 = \mathbf{x}_2 - \frac{\langle \mathbf{x}_2, \mathbf{v}_1 \rangle}{\|\mathbf{v}_1\|^2}\mathbf{v}_1$
3. $\mathbf{v}_3 =$
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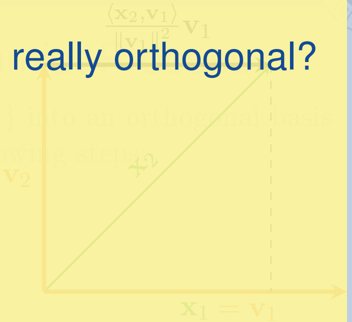
$$\langle \mathbf{v}_1, \mathbf{v}_2 \rangle =$$

$$=$$

$$=$$

$$=$$

$$=$$



The Gram-Schmidt Process

To convert a basis $\{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_r\}$ into an orthogonal basis $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_r\}$, perform the following computations:

1. $\mathbf{v}_1 = \mathbf{x}_1$
2. $\mathbf{v}_2 = \mathbf{x}_2 - \frac{\langle \mathbf{x}_2, \mathbf{v}_1 \rangle}{\|\mathbf{v}_1\|^2}\mathbf{v}_1$
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Are $\mathbf{v}_1$ and $\mathbf{v}_2$ really orthogonal?

$$\langle \mathbf{v}_1, \mathbf{v}_2 \rangle = \left\langle \mathbf{v}_1, \mathbf{x}_2 - \frac{\langle \mathbf{x}_2, \mathbf{v}_1 \rangle}{\|\mathbf{v}_1\|^2}\mathbf{v}_1 \right\rangle$$

$$=$$

$$=$$

$$=$$

$$=$$

To convert a basis $\{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_r\}$ into an orthogonal basis $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_r\}$, perform the following computations:

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3. $\mathbf{v}_3 =$
4. $\mathbf{v}_4 =$

Are $\mathbf{v}_1$ and $\mathbf{v}_2$ really orthogonal?

$$\begin{aligned}
\langle \mathbf{v}_1, \mathbf{v}_2 \rangle &= \left\langle \mathbf{v}_1, \mathbf{x}_2 - \frac{\langle \mathbf{x}_2, \mathbf{v}_1 \rangle}{\|\mathbf{v}_1\|^2}\mathbf{v}_1 \right\rangle \\
&= \langle \mathbf{v}_1, \mathbf{x}_2 \rangle - \left\langle \mathbf{v}_1, \frac{\langle \mathbf{x}_2, \mathbf{v}_1 \rangle}{\|\mathbf{v}_1\|^2}\mathbf{v}_1 \right\rangle \\
&= \\
&= \\
&=
\end{aligned}$$

To convert a basis $\{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_r\}$ into an orthogonal basis $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_r\}$, perform the following computations:

1. $\mathbf{v}_1 = \mathbf{x}_1$
2. $\mathbf{v}_2 = \mathbf{x}_2 - \frac{\langle \mathbf{x}_2, \mathbf{v}_1 \rangle}{\|\mathbf{v}_1\|^2} \mathbf{v}_1$
3. $\mathbf{v}_3 =$
4. $\mathbf{v}_4 =$

Are $\mathbf{v}_1$ and $\mathbf{v}_2$ really orthogonal?

$$
\begin{aligned}
\langle \mathbf{v}_1, \mathbf{v}_2 \rangle &= \left\langle \mathbf{v}_1, \mathbf{x}_2 - \frac{\langle \mathbf{x}_2, \mathbf{v}_1 \rangle}{\|\mathbf{v}_1\|^2} \mathbf{v}_1 \right\rangle \\
&= \langle \mathbf{v}_1, \mathbf{x}_2 \rangle - \left\langle \mathbf{v}_1, \frac{\langle \mathbf{x}_2, \mathbf{v}_1 \rangle}{\|\mathbf{v}_1\|^2} \mathbf{v}_1 \right\rangle \\
&= \langle \mathbf{v}_1, \mathbf{x}_2 \rangle - \frac{\langle \mathbf{x}_2, \mathbf{v}_1 \rangle}{\|\mathbf{v}_1\|^2} \langle \mathbf{v}_1, \mathbf{v}_1 \rangle \\
&= \langle \mathbf{v}_1, \mathbf{x}_2 \rangle - \frac{\langle \mathbf{x}_2, \mathbf{v}_1 \rangle \langle \mathbf{v}_1, \mathbf{v}_1 \rangle}{\|\mathbf{v}_1\|^2} \\
&=
\end{aligned}
$$

To convert a basis $\{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_r\}$ into an orthogonal basis $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_r\}$, perform the following computations:

1. $\mathbf{v}_1 = \mathbf{x}_1$
2. $\mathbf{v}_2 = \mathbf{x}_2 - \frac{\langle \mathbf{x}_2, \mathbf{v}_1 \rangle}{\|\mathbf{v}_1\|^2}\mathbf{v}_1$
3. $\mathbf{v}_3 =$
4. $\mathbf{v}_4 =$

Are $\mathbf{v}_1$ and $\mathbf{v}_2$ really orthogonal?

$$
\begin{aligned}
\langle \mathbf{v}_1, \mathbf{v}_2 \rangle &= \left\langle \mathbf{v}_1, \mathbf{x}_2 - \frac{\langle \mathbf{x}_2, \mathbf{v}_1 \rangle}{\|\mathbf{v}_1\|^2}\mathbf{v}_1 \right\rangle \\
&= \langle \mathbf{v}_1, \mathbf{x}_2 \rangle - \left\langle \mathbf{v}_1, \frac{\langle \mathbf{x}_2, \mathbf{v}_1 \rangle}{\|\mathbf{v}_1\|^2}\mathbf{v}_1 \right\rangle \\
&= \langle \mathbf{v}_1, \mathbf{x}_2 \rangle - \frac{\langle \mathbf{x}_2, \mathbf{v}_1 \rangle}{\|\mathbf{v}_1\|^2}\langle \mathbf{v}_1, \mathbf{v}_1 \rangle \\
&= \langle \mathbf{v}_1, \mathbf{x}_2 \rangle - \frac{\langle \mathbf{x}_2, \mathbf{v}_1 \rangle \langle \mathbf{v}_1, \mathbf{v}_1 \rangle}{\|\mathbf{v}_1\|^2} \\
&= \langle \mathbf{v}_1, \mathbf{x}_2 \rangle - \langle \mathbf{x}_2, \mathbf{v}_1 \rangle = 0.
\end{aligned}
$$

YES!

The Gram-Schmidt Process

To convert a basis $\{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_r\}$ into an orthogonal basis $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_r\}$, perform the following steps:

1. $\mathbf{v}_1 = \mathbf{x}_1$
2. $\mathbf{v}_2 = \mathbf{x}_2 - \dfrac{\langle \mathbf{x}_2, \mathbf{v}_1 \rangle}{\|\mathbf{v}_1\|^2}\mathbf{v}_1$
3. $\mathbf{v}_3 =$
4. $\mathbf{v}_4 =$

The Gram-Schmidt Process

To convert a basis $\{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_r\}$ into an orthogonal basis $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_r\}$, perform the following steps:

1. $\mathbf{v}_1 = \mathbf{x}_1$
2. $\mathbf{v}_2 = \mathbf{x}_2 - \dfrac{\langle \mathbf{x}_2, \mathbf{v}_1 \rangle}{\|\mathbf{v}_1\|^2}\mathbf{v}_1$
3. $\mathbf{v}_3 = \mathbf{x}_3 - \dfrac{\langle \mathbf{x}_3, \mathbf{v}_1 \rangle}{\|\mathbf{v}_1\|^2}\mathbf{v}_1 - \dfrac{\langle \mathbf{x}_3, \mathbf{v}_2 \rangle}{\|\mathbf{v}_2\|^2}\mathbf{v}_2$
4. $\mathbf{v}_4 =$

The Gram-Schmidt Process

To convert a basis $\{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_r\}$ into an orthogonal basis $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_r\}$, perform the following steps:

1. $\mathbf{v}_1 = \mathbf{x}_1$
2. $\mathbf{v}_2 = \mathbf{x}_2 - \dfrac{\langle \mathbf{x}_2, \mathbf{v}_1 \rangle}{\|\mathbf{v}_1\|^2}\mathbf{v}_1$
3. $\mathbf{v}_3 = \mathbf{x}_3 - \dfrac{\langle \mathbf{x}_3, \mathbf{v}_1 \rangle}{\|\mathbf{v}_1\|^2}\mathbf{v}_1 - \dfrac{\langle \mathbf{x}_3, \mathbf{v}_2 \rangle}{\|\mathbf{v}_2\|^2}\mathbf{v}_2$
4. $\mathbf{v}_4 =$

Please check that

$$\langle \mathbf{v}_1, \mathbf{v}_3 \rangle = 0$$

and

$$\langle \mathbf{v}_2, \mathbf{v}_3 \rangle = 0.$$

The Gram-Schmidt Process

To convert a basis $\{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_r\}$ into an orthogonal basis $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_r\}$, perform the following steps:

1. $\mathbf{v}_1 = \mathbf{x}_1$
2. $\mathbf{v}_2 = \mathbf{x}_2 - \dfrac{\langle \mathbf{x}_2, \mathbf{v}_1 \rangle}{\|\mathbf{v}_1\|^2}\mathbf{v}_1$
3. $\mathbf{v}_3 = \mathbf{x}_3 - \dfrac{\langle \mathbf{x}_3, \mathbf{v}_1 \rangle}{\|\mathbf{v}_1\|^2}\mathbf{v}_1 - \dfrac{\langle \mathbf{x}_3, \mathbf{v}_2 \rangle}{\|\mathbf{v}_2\|^2}\mathbf{v}_2$
4. $\mathbf{v}_4 = \mathbf{x}_4 - \dfrac{\langle \mathbf{x}_4, \mathbf{v}_1 \rangle}{\|\mathbf{v}_1\|^2}\mathbf{v}_1 - \dfrac{\langle \mathbf{x}_4, \mathbf{v}_2 \rangle}{\|\mathbf{v}_2\|^2}\mathbf{v}_2 - \dfrac{\langle \mathbf{x}_4, \mathbf{v}_3 \rangle}{\|\mathbf{v}_3\|^2}\mathbf{v}_3$

The Gram-Schmidt Process

To convert a basis $\{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_r\}$ into an orthogonal basis $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_r\}$, perform the following steps:

1. $\mathbf{v}_1 = \mathbf{x}_1$
2. $\mathbf{v}_2 = \mathbf{x}_2 - \dfrac{\langle \mathbf{x}_2, \mathbf{v}_1 \rangle}{\|\mathbf{v}_1\|^2}\mathbf{v}_1$
3. $\mathbf{v}_3 = \mathbf{x}_3 - \dfrac{\langle \mathbf{x}_3, \mathbf{v}_1 \rangle}{\|\mathbf{v}_1\|^2}\mathbf{v}_1 - \dfrac{\langle \mathbf{x}_3, \mathbf{v}_2 \rangle}{\|\mathbf{v}_2\|^2}\mathbf{v}_2$
4. $\mathbf{v}_4 = \mathbf{x}_4 - \dfrac{\langle \mathbf{x}_4, \mathbf{v}_1 \rangle}{\|\mathbf{v}_1\|^2}\mathbf{v}_1 - \dfrac{\langle \mathbf{x}_4, \mathbf{v}_2 \rangle}{\|\mathbf{v}_2\|^2}\mathbf{v}_2 - \dfrac{\langle \mathbf{x}_4, \mathbf{v}_3 \rangle}{\|\mathbf{v}_3\|^2}\mathbf{v}_3$

$$\vdots$$

(continue until you have $\mathbf{v}_r$)

The Gram-Schmidt Process

Example (Using the Gram-Schmidt Process)

Assume that the vector space $\mathbb{R}^3$ has the Euclidean inner product. Apply the Gram–Schmidt process to transform the basis vectors

$$\mathbf{x}_1 = (1,1,1), \qquad \mathbf{x}_2 = (0,1,1), \qquad \mathbf{x}_3 = (0,0,1)$$

into an orthogonal basis $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$, and then normalise the orthogonal basis vectors to obtain an orthonormal basis $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$.

$$\mathbf{x}_1 = (1,1,1), \quad \mathbf{x}_2 = (0,1,1), \quad \mathbf{x}_3 = (0,0,1)$$

Step 1: $\mathbf{v}_1 = \mathbf{x}_1 = (1,1,1)$

$$\mathbf{x}_1 = (1,1,1), \quad \mathbf{x}_2 = (0,1,1), \quad \mathbf{x}_3 = (0,0,1)$$

Step 1: $\mathbf{v}_1 = \mathbf{x}_1 = (1,1,1)$

Step 2: $\mathbf{v}_2 = \mathbf{x}_2 - \dfrac{\langle \mathbf{x}_2, \mathbf{v}_1 \rangle}{\|\mathbf{v}_1\|^2}\mathbf{v}_1 = (0,1,1) - \dfrac{2}{3}(1,1,1) = \left(-\dfrac{2}{3}, \dfrac{1}{3}, \dfrac{1}{3}\right)$

$$\mathbf{x}_1 = (1,1,1), \quad \mathbf{x}_2 = (0,1,1), \quad \mathbf{x}_3 = (0,0,1)$$

Step 1: $\mathbf{v}_1 = \mathbf{x}_1 = (1,1,1)$

Step 2: $\mathbf{v}_2 = \mathbf{x}_2 - \dfrac{\langle \mathbf{x}_2, \mathbf{v}_1 \rangle}{\|\mathbf{v}_1\|^2}\mathbf{v}_1 = (0,1,1) - \dfrac{2}{3}(1,1,1) = \left(-\dfrac{2}{3}, \dfrac{1}{3}, \dfrac{1}{3}\right)$

$$\begin{aligned} \langle \mathbf{v}_1, \mathbf{v}_2 \rangle &= (1,1,1) \bullet \left(-\frac{2}{3}, \frac{1}{3}, \frac{1}{3}\right) \\ &= (1)\left(-\frac{2}{3}\right) + (1)\left(\frac{1}{3}\right) + (1)\left(\frac{1}{3}\right) = 0 \qquad \checkmark \end{aligned}$$

$$\mathbf{x}_1 = (1,1,1), \quad \mathbf{x}_2 = (0,1,1), \quad \mathbf{x}_3 = (0,0,1)$$

Step 1: $\mathbf{v}_1 = \mathbf{x}_1 = (1,1,1)$

Step 2: $\mathbf{v}_2 = \mathbf{x}_2 - \dfrac{\langle \mathbf{x}_2, \mathbf{v}_1 \rangle}{\|\mathbf{v}_1\|^2}\mathbf{v}_1 = (0,1,1) - \dfrac{2}{3}(1,1,1) = \left(-\dfrac{2}{3}, \dfrac{1}{3}, \dfrac{1}{3}\right)$

$$\begin{aligned}\langle \mathbf{v}_1, \mathbf{v}_2 \rangle &= (1,1,1) \cdot \left(-\frac{2}{3}, \frac{1}{3}, \frac{1}{3}\right) \\ &= (1)\left(-\frac{2}{3}\right) + (1)\left(\frac{1}{3}\right) + (1)\left(\frac{1}{3}\right) = 0 \qquad \checkmark\end{aligned}$$

Step 3: $$\begin{aligned}\mathbf{v}_3 &= \mathbf{x}_3 - \frac{\langle \mathbf{x}_3, \mathbf{v}_1 \rangle}{\|\mathbf{v}_1\|^2}\mathbf{v}_1 - \frac{\langle \mathbf{x}_3, \mathbf{v}_2 \rangle}{\|\mathbf{v}_2\|^2}\mathbf{v}_2 \\ &= (0,0,1) - \frac{1}{3}(1,1,1) - \frac{\frac{1}{3}}{\frac{2}{3}}\left(-\frac{2}{3}, \frac{1}{3}, \frac{1}{3}\right) = \left(0, -\frac{1}{2}, \frac{1}{2}\right).\end{aligned}$$

$$\mathbf{v}_1 = (1, 1, 1), \quad \mathbf{v}_2 = \left(-\tfrac{2}{3}, \tfrac{1}{3}, \tfrac{1}{3}\right), \quad \mathbf{v}_3 = \left(0, -\tfrac{1}{2}, \tfrac{1}{2}\right)$$

Let's just finish checking if these are correct:

$$\mathbf{v}_1 = (1,1,1), \quad \mathbf{v}_2 = \left(-\tfrac{2}{3}, \tfrac{1}{3}, \tfrac{1}{3}\right), \quad \mathbf{v}_3 = \left(0, -\tfrac{1}{2}, \tfrac{1}{2}\right)$$

Let's just finish checking if these are correct:

$$\langle \mathbf{v}_1, \mathbf{v}_3 \rangle = (1,1,1) \cdot \left(0, -\frac{1}{2}, \frac{1}{2}\right) = 0 - \frac{1}{2} + \frac{1}{2} = 0 \qquad \checkmark$$

$$\langle \mathbf{v}_2, \mathbf{v}_3 \rangle = \left(-\frac{2}{3}, \frac{1}{3}, \frac{1}{3}\right) \cdot \left(0, -\frac{1}{2}, \frac{1}{2}\right) = 0 - \frac{1}{6} + \frac{1}{6} = 0 \qquad \checkmark$$

$$\mathbf{v}_1 = (1,1,1), \quad \mathbf{v}_2 = \left(-\tfrac{2}{3}, \tfrac{1}{3}, \tfrac{1}{3}\right), \quad \mathbf{v}_3 = \left(0, -\tfrac{1}{2}, \tfrac{1}{2}\right)$$

Let's just finish checking if these are correct:

$$\langle \mathbf{v}_1, \mathbf{v}_3 \rangle = (1,1,1) \cdot \left(0, -\frac{1}{2}, \frac{1}{2}\right) = 0 - \frac{1}{2} + \frac{1}{2} = 0 \qquad \checkmark$$

$$\langle \mathbf{v}_2, \mathbf{v}_3 \rangle = \left(-\frac{2}{3}, \frac{1}{3}, \frac{1}{3}\right) \cdot \left(0, -\frac{1}{2}, \frac{1}{2}\right) = 0 - \frac{1}{6} + \frac{1}{6} = 0 \qquad \checkmark$$

Therefore $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is an orthogonal basis for $\mathbb{R}^3$.

$$\mathbf{v}_1 = (1,1,1), \quad \mathbf{v}_2 = \left(-\tfrac{2}{3}, \tfrac{1}{3}, \tfrac{1}{3}\right), \quad \mathbf{v}_3 = \left(0, -\tfrac{1}{2}, \tfrac{1}{2}\right)$$

The next step is to normalise these three vectors.

$\mathbf{v}_1 = (1, 1, 1), \quad \mathbf{v}_2 = \left(-\frac{2}{3}, \frac{1}{3}, \frac{1}{3}\right), \quad \mathbf{v}_3 = \left(0, -\frac{1}{2}, \frac{1}{2}\right)$

The next step is to normalise these three vectors. Since

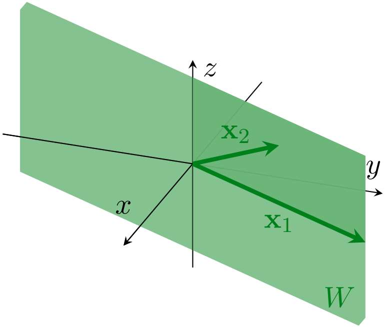
$$\|\mathbf{v}_1\| = \sqrt{3}, \qquad \|\mathbf{v}_2\| = \frac{\sqrt{6}}{3}, \qquad \|\mathbf{v}_3\| = \frac{1}{\sqrt{2}},$$

(please check) it follows that

$$\begin{aligned}
\mathbf{u}_1 &= \frac{\mathbf{v}_1}{\|\mathbf{v}_1\|} = \left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right) \\
\mathbf{u}_2 &= \frac{\mathbf{v}_2}{\|\mathbf{v}_2\|} = \left(-\frac{2}{\sqrt{6}}, \frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}\right) \\
\mathbf{u}_3 &= \frac{\mathbf{v}_3}{\|\mathbf{v}_3\|} = \left(0, -\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right).
\end{aligned}$$

Then $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ is an orthonormal basis for $\mathbb{R}^3$.

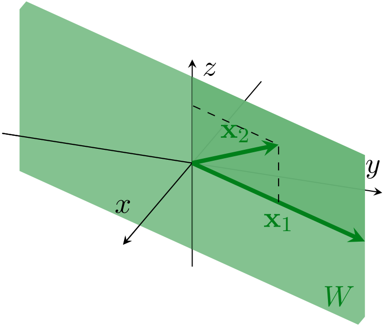
The Gram-Schmidt Process



Example

Let $V = \mathbb{R}^3$ with the Euclidean inner product and let $W = \text{span}\{\mathbf{x}_1, \mathbf{x}_2\}$ where $\mathbf{x}_1 = \begin{bmatrix} 3 \\ 6 \\ 0 \end{bmatrix}$ and $\mathbf{x}_2 = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$. Construct an orthogonal basis for W.

The Gram-Schmidt Process



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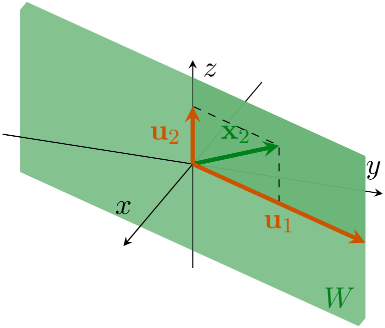
The Gram-Schmidt Process



Example

Let $V = \mathbb{R}^3$ with the Euclidean inner product and let $W = \text{span}\{\mathbf{x}_1, \mathbf{x}_2\}$ where $\mathbf{x}_1 = \begin{bmatrix} 3 \\ 6 \\ 0 \end{bmatrix}$ and $\mathbf{x}_2 = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$. Construct an orthogonal basis for W.

The Gram-Schmidt Process

Example

Let $V = \mathbb{R}^3$ with the Euclidean inner product and let $W = \operatorname{span}\{\mathbf{x}_1, \mathbf{x}_2\}$ where $\mathbf{x}_1 = \begin{bmatrix} 3 \\ 6 \\ 0 \end{bmatrix}$ and $\mathbf{x}_2 = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$. Construct an orthogonal basis for W.

$$\mathbf{v}_1 = \mathbf{x}_1 = \begin{bmatrix} 3 \\ 6 \\ 0 \end{bmatrix}$$

$$\mathbf{v}_2 =$$

Example

Let $V = \mathbb{R}^3$ with the Euclidean inner product and let $W = \operatorname{span}\{\mathbf{x}_1, \mathbf{x}_2\}$ where $\mathbf{x}_1 = \begin{bmatrix} 3 \\ 6 \\ 0 \end{bmatrix}$ and $\mathbf{x}_2 = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$. Construct an orthogonal basis for W.

$$\mathbf{v}_1 = \mathbf{x}_1 = \begin{bmatrix} 3 \\ 6 \\ 0 \end{bmatrix}$$

$$\mathbf{v}_2 = \mathbf{x}_2 - \frac{\langle \mathbf{x}_2, \mathbf{v}_1 \rangle}{\|\mathbf{v}_1\|^2}\mathbf{v}_1 = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} - \frac{15}{45}\begin{bmatrix} 3 \\ 6 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix}.$$

The Gram-Schmidt Process

Example

Let $V = \mathbb{R}^3$ with the Euclidean inner product and let $W = \operatorname{span}\{\mathbf{x}_1, \mathbf{x}_2\}$ where $\mathbf{x}_1 = \begin{bmatrix} 3 \\ 6 \\ 0 \end{bmatrix}$ and $\mathbf{x}_2 = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$. Construct an orthogonal basis for W.

$$\mathbf{v}_1 = \mathbf{x}_1 = \begin{bmatrix} 3 \\ 6 \\ 0 \end{bmatrix}$$

$$\mathbf{v}_2 = \mathbf{x}_2 - \frac{\langle \mathbf{x}_2, \mathbf{v}_1 \rangle}{\|\mathbf{v}_1\|^2}\mathbf{v}_1 = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} - \frac{15}{45}\begin{bmatrix} 3 \\ 6 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix}.$$

$\{\mathbf{v}_1, \mathbf{v}_2\}$ is an orthogonal basis for W.

Example

Let $\mathbf{x}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$, $\mathbf{x}_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix}$ and $\mathbf{x}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}$ be three vectors in $\mathbb{R}^4$ with the Euclidean inner product. It is easy to see that $\{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3\}$ is linearly independent (Why?).

Example

Let $\mathbf{x}_1 = \begin{bmatrix}1\\1\\1\\1\end{bmatrix}$, $\mathbf{x}_2 = \begin{bmatrix}0\\1\\1\\1\end{bmatrix}$ and $\mathbf{x}_3 = \begin{bmatrix}0\\0\\1\\1\end{bmatrix}$ be three vectors in $\mathbb{R}^4$ with the Euclidean inner product. It is easy to see that $\{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3\}$ is linearly independent (Why?).

Construct an orthogonal basis for $W = \text{span}\{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3\}$.

The Gram-Schmidt Process

$$\mathbf{x}_1 = \begin{bmatrix}1\\1\\1\\1\end{bmatrix}, \mathbf{x}_2 = \begin{bmatrix}0\\1\\1\\1\end{bmatrix}, \mathbf{x}_3 = \begin{bmatrix}0\\0\\1\\1\end{bmatrix}$$

$$\mathbf{v}_1 = \mathbf{x}_1 = \begin{bmatrix}1\\1\\1\\1\end{bmatrix}, \quad \mathbf{v}_2 = \mathbf{x}_2 - \frac{\langle \mathbf{x}_2, \mathbf{v}_1 \rangle}{\|\mathbf{v}_1\|^2}\mathbf{v}_1 = \begin{bmatrix}0\\1\\1\\1\end{bmatrix} - \frac{3}{4}\begin{bmatrix}1\\1\\1\\1\end{bmatrix} = \begin{bmatrix}-\frac{3}{4}\\ \frac{1}{4}\\ \frac{1}{4}\\ \frac{1}{4}\end{bmatrix}.$$

$$\mathbf{x}_1 = \begin{bmatrix}1\\1\\1\\1\end{bmatrix}, \mathbf{x}_2 = \begin{bmatrix}0\\1\\1\\1\end{bmatrix}, \mathbf{x}_3 = \begin{bmatrix}0\\0\\1\\1\end{bmatrix}$$

$$\mathbf{v}_1 = \mathbf{x}_1 = \begin{bmatrix}1\\1\\1\\1\end{bmatrix}, \quad \mathbf{v}_2 = \mathbf{x}_2 - \frac{\langle \mathbf{x}_2, \mathbf{v}_1 \rangle}{\|\mathbf{v}_1\|^2}\mathbf{v}_1 = \begin{bmatrix}0\\1\\1\\1\end{bmatrix} - \frac{3}{4}\begin{bmatrix}1\\1\\1\\1\end{bmatrix} = \begin{bmatrix}-\frac{3}{4}\\\frac{1}{4}\\\frac{1}{4}\\\frac{1}{4}\end{bmatrix}.$$

At this point we can do an optional step to simplify our calculations. Remember that we are trying to find an orthogonal basis. We don't care how big (long) our vectors are, we only care that they are orthogonal.

$$\mathbf{x}_1 = \begin{bmatrix}1\\1\\1\\1\end{bmatrix}, \mathbf{x}_2 = \begin{bmatrix}0\\1\\1\\1\end{bmatrix}, \mathbf{x}_3 = \begin{bmatrix}0\\0\\1\\1\end{bmatrix}$$

$$\mathbf{v}_1 = \mathbf{x}_1 = \begin{bmatrix}1\\1\\1\\1\end{bmatrix}, \quad \mathbf{v}_2 = \mathbf{x}_2 - \frac{\langle \mathbf{x}_2, \mathbf{v}_1 \rangle}{\|\mathbf{v}_1\|^2}\mathbf{v}_1 = \begin{bmatrix}0\\1\\1\\1\end{bmatrix} - \frac{3}{4}\begin{bmatrix}1\\1\\1\\1\end{bmatrix} = \begin{bmatrix}-\frac{3}{4}\\\frac{1}{4}\\\frac{1}{4}\\\frac{1}{4}\end{bmatrix}.$$

At this point we can do an optional step to simplify our calculations. Remember that we are trying to find an orthogonal basis. We don't care how big (long) our vectors are, we only care that they are orthogonal. So we can multiply $\mathbf{v}_2$ by 4 to obtain a new $\mathbf{v}_2$:

$$\mathbf{v}_2 = \begin{bmatrix}-3\\1\\1\\1\end{bmatrix}.$$

$$\mathbf{x}_1 = \begin{bmatrix}1\\1\\1\\1\end{bmatrix}, \mathbf{x}_2 = \begin{bmatrix}0\\1\\1\\1\end{bmatrix}, \mathbf{x}_3 = \begin{bmatrix}0\\0\\1\\1\end{bmatrix} \qquad \mathbf{v}_1 = \begin{bmatrix}1\\1\\1\\1\end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix}-3\\1\\1\\1\end{bmatrix}$$

Then

$$\mathbf{v}_3 = \mathbf{x}_3 - \frac{\langle \mathbf{x}_3, \mathbf{v}_1 \rangle}{\|\mathbf{v}_1\|^2}\mathbf{v}_1 - \frac{\langle \mathbf{x}_3, \mathbf{v}_2 \rangle}{\|\mathbf{v}_2\|^2}\mathbf{v}_2 = \begin{bmatrix}0\\0\\1\\1\end{bmatrix} - \frac{2}{4}\begin{bmatrix}1\\1\\1\\1\end{bmatrix} - \frac{2}{12}\begin{bmatrix}-3\\1\\1\\1\end{bmatrix} = \begin{bmatrix}0\\-\frac{2}{3}\\\frac{1}{3}\\\frac{1}{3}\end{bmatrix}.$$

The Gram-Schmidt Process

$$\mathbf{x}_1 = \begin{bmatrix}1\\1\\1\\1\end{bmatrix}, \mathbf{x}_2 = \begin{bmatrix}0\\1\\1\\1\end{bmatrix}, \mathbf{x}_3 = \begin{bmatrix}0\\0\\1\\1\end{bmatrix} \qquad \mathbf{v}_1 = \begin{bmatrix}1\\1\\1\\1\end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix}-3\\1\\1\\1\end{bmatrix}$$

Then

$$\mathbf{v}_3 = \mathbf{x}_3 - \frac{\langle \mathbf{x}_3, \mathbf{v}_1 \rangle}{\|\mathbf{v}_1\|^2}\mathbf{v}_1 - \frac{\langle \mathbf{x}_3, \mathbf{v}_2 \rangle}{\|\mathbf{v}_2\|^2}\mathbf{v}_2 = \begin{bmatrix}0\\0\\1\\1\end{bmatrix} - \frac{2}{4}\begin{bmatrix}1\\1\\1\\1\end{bmatrix} - \frac{2}{12}\begin{bmatrix}-3\\1\\1\\1\end{bmatrix} = \begin{bmatrix}0\\-\frac{2}{3}\\\frac{1}{3}\\\frac{1}{3}\end{bmatrix}.$$

Again we can multiply $\mathbf{v}_3$ by a scalar if we want to. Let's multiply it by 3 to get a new vector which we will now call $\mathbf{v}_3$:

$$\mathbf{v}_3 = \begin{bmatrix}0\\-2\\1\\1\end{bmatrix}.$$

Hence

$$\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -3 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ -2 \\ 1 \\ 1 \end{bmatrix} \right\}$$

is an orthogonal basis for W.

Adrien-Marie Legendre

BORN
18 September 1752

DECEICED
9 January 1833

NATIONALITY
French

Example (Legendre Polynomials)

Consider the vector space $\mathbb{P}^2$ with the inner product

$$\langle \mathbf{p}, \mathbf{q} \rangle = \int_{-1}^{1} p(x)q(x)\, dx.$$

Apply the Gram-Schmidt process to transform the standard basis $\{1, x, x^2\}$ into an orthogonal basis.

Let $\mathbf{x}_1 = 1$, $\mathbf{x}_2 = x$ and $\mathbf{x}_3 = x^2$.

The Gram-Schmidt Process

Step 1: $$\mathbf{v}_1 = \mathbf{x}_1 = 1$$

Step 1: $$\mathbf{v}_1 = \mathbf{x}_1 = 1$$

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The Gram-Schmidt Process

Step 1: $$\mathbf{v}_1 = \mathbf{x}_1 = 1$$

Step 2: $$\begin{aligned} \langle \mathbf{x}_2, \mathbf{v}_1 \rangle &= \int_{-1}^{1} x\,dx = 0 \\ \mathbf{v}_2 &= \mathbf{x}_2 - \frac{\langle \mathbf{x}_2, \mathbf{v}_1 \rangle}{\|\mathbf{v}_1\|^2}\mathbf{v}_1 = x - 0 = x \end{aligned}$$

The Gram-Schmidt Process

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Step 2: $$\langle \mathbf{x}_2, \mathbf{v}_1 \rangle = \int_{-1}^{1} x\,dx = 0$$

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Step 3: $$\langle \mathbf{x}_3, \mathbf{v}_1 \rangle = \int_{-1}^{1} x^2\,dx = \frac{2}{3}$$

$$\langle \mathbf{x}_3, \mathbf{v}_2 \rangle = \int_{-1}^{1} x^3\,dx = 0$$

$$\|\mathbf{v}_1\|^2 = \langle \mathbf{v}_1, \mathbf{v}_1 \rangle = \int_{-1}^{1} 1\,dx = 2$$

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$$\mathbf{v}_3 = \mathbf{x}_3 - \frac{\langle \mathbf{x}_3, \mathbf{v}_1 \rangle}{\|\mathbf{v}_1\|^2}\mathbf{v}_1 - \frac{\langle \mathbf{x}_3, \mathbf{v}_2 \rangle}{\|\mathbf{v}_2\|^2}\mathbf{v}_2 = x^2 - \frac{1}{3} - 0.$$

Therefore

$$\left\{1, x, x^2 - \frac{1}{3}\right\}$$

is an orthogonal basis for $\mathbb{P}^2$.

Next Time

- Orthogonal Matrices
- Orthogonal Diagonalisation