

# Lecture 8

- Matrices for Linear Transformations
- Similarity
- Complex Numbers



# Matrices for Linear Transformations

# Matrices for Linear Transformations



## Example

Suppose that  $T : \mathbb{R}^2 \rightarrow \mathbb{R}^3$  is a linear transformation which satisfies

$$T(\mathbf{e}_1) = \begin{bmatrix} 5 \\ -7 \\ 2 \end{bmatrix} \quad \text{and} \quad T(\mathbf{e}_2) = \begin{bmatrix} -3 \\ 8 \\ 0 \end{bmatrix}$$

where  $\mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$  and  $\mathbf{e}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ . Find a formula for  $T(\mathbf{x})$ .

## Matrices for Linear T

$$T(\mathbf{e}_1) = \begin{bmatrix} 5 \\ -7 \\ 2 \end{bmatrix} \quad T(\mathbf{e}_2) = \begin{bmatrix} -3 \\ 8 \\ 0 \end{bmatrix}$$



Because  $T$  is linear and because

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix} = x_1 \mathbf{e}_1 + x_2 \mathbf{e}_2$$

## Matrices for Linear Transformations

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we have that

$$T(\mathbf{x}) = T(x_1 \mathbf{e}_1 + x_2 \mathbf{e}_2) = x_1 T(\mathbf{e}_1) + x_2 T(\mathbf{e}_2)$$

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## Matrices for Linear T

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we have that

$$T(\mathbf{x}) = T(x_1 \mathbf{e}_1 + x_2 \mathbf{e}_2) = x_1 T(\mathbf{e}_1) + x_2 T(\mathbf{e}_2)$$

$$= x_1 \begin{bmatrix} 5 \\ -7 \\ 2 \end{bmatrix} + x_2 \begin{bmatrix} -3 \\ 8 \\ 0 \end{bmatrix} = \begin{bmatrix} 5x_1 - 3x_2 \\ -7x_1 + 8x_2 \\ 2x_1 \end{bmatrix}$$

=

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## Matrices for Linear T

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$$= \begin{bmatrix} 5 & -3 \\ -7 & 8 \\ 2 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} =$$

## Matrices for Linear Transformations

$$T(\mathbf{e}_1) = \begin{bmatrix} 5 \\ -7 \\ 2 \end{bmatrix} \quad T(\mathbf{e}_2) = \begin{bmatrix} -3 \\ 8 \\ 0 \end{bmatrix}$$



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we have that

$$\begin{aligned} T(\mathbf{x}) &= T(x_1 \mathbf{e}_1 + x_2 \mathbf{e}_2) = x_1 T(\mathbf{e}_1) + x_2 T(\mathbf{e}_2) \\ &= x_1 \begin{bmatrix} 5 \\ -7 \\ 2 \end{bmatrix} + x_2 \begin{bmatrix} -3 \\ 8 \\ 0 \end{bmatrix} = \begin{bmatrix} 5x_1 - 3x_2 \\ -7x_1 + 8x_2 \\ 2x_1 \end{bmatrix} \\ &= \begin{bmatrix} 5 & -3 \\ -7 & 8 \\ 2 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} T(\mathbf{e}_1) & T(\mathbf{e}_2) \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = A\mathbf{x}. \end{aligned}$$

# Matrices for Linear Transformations



## Theorem

*Let  $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$  be a linear transformation. Then there exists a unique matrix  $A$  such that*

$$T(\mathbf{x}) = A\mathbf{x}$$

*for all  $\mathbf{x} \in \mathbb{R}^n$ .*

# Matrices for Linear Transformations



## Theorem

Let  $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$  be a linear transformation. Then there exists a unique matrix  $A$  such that

$$T(\mathbf{x}) = A\mathbf{x}$$

for all  $\mathbf{x} \in \mathbb{R}^n$ .

In fact,  $A$  is the matrix

$$A = \begin{bmatrix} T(\mathbf{e}_1) & T(\mathbf{e}_2) & \cdots & T(\mathbf{e}_n) \end{bmatrix}$$

where  $\mathbf{e}_j$  is the  $j$ th column of the identity matrix  $I_n$ .

# Matrices for Linear Transformations



Proof.

First suppose that  $\mathbf{x} = x_1\mathbf{e}_1 + x_2\mathbf{e}_2 + \dots + x_n\mathbf{e}_n$ . Then

$$\begin{aligned} T(\mathbf{x}) &= T(x_1\mathbf{e}_1 + x_2\mathbf{e}_2 + \dots + x_n\mathbf{e}_n) \\ &= x_1T(\mathbf{e}_1) + x_2T(\mathbf{e}_2) + \dots + x_nT(\mathbf{e}_n) \\ &= \begin{bmatrix} T(\mathbf{e}_1) & T(\mathbf{e}_2) & \cdots & T(\mathbf{e}_n) \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = A\mathbf{x}. \end{aligned}$$

This proves the existence part of the theorem.

# Matrices for Linear Transformations



Proof continued.

Now we need to prove that such a matrix is unique: Suppose that  $T(\mathbf{x}) = A\mathbf{x}$  and  $T(\mathbf{x}) = B\mathbf{x}$ .

# Matrices for Linear Transformations



Proof continued.

Now we need to prove that such a matrix is unique: Suppose that  $T(\mathbf{x}) = A\mathbf{x}$  and  $T(\mathbf{x}) = B\mathbf{x}$ . Then for each  $j$  we have

$$A\mathbf{e}_j = T(\mathbf{e}_j) = B\mathbf{e}_j =$$

# Matrices for Linear Transformations



Proof continued.

Now we need to prove that such a matrix is unique: Suppose that  $T(\mathbf{x}) = A\mathbf{x}$  and  $T(\mathbf{x}) = B\mathbf{x}$ . Then for each  $j$  we have

$$\begin{array}{ll} \text{the } j\text{th} & \text{the } j\text{th} \\ \text{column of } & = A\mathbf{e}_j = T(\mathbf{e}_j) = B\mathbf{e}_j = \text{column of} \\ A & B \end{array}$$

# Matrices for Linear Transformations



Proof continued.

Now we need to prove that such a matrix is unique: Suppose that  $T(\mathbf{x}) = A\mathbf{x}$  and  $T(\mathbf{x}) = B\mathbf{x}$ . Then for each  $j$  we have

$$\begin{array}{ccc} \text{the } j\text{th} & & \text{the } j\text{th} \\ \text{column of } & = A\mathbf{e}_j = T(\mathbf{e}_j) = B\mathbf{e}_j = & \text{column of } \\ A & & B \end{array}$$

Hence  $A = B$  and we are finished. □

# Matrices for Linear Transformations



Proof continued.

Now we need to prove that such a matrix is unique: Suppose that  $T(\mathbf{x}) = A\mathbf{x}$  and  $T(\mathbf{x}) = B\mathbf{x}$ . Then for each  $j$  we have

$$\begin{array}{ll} \text{the } j\text{th} & \text{the } j\text{th} \\ \text{column of } & = A\mathbf{e}_j = T(\mathbf{e}_j) = B\mathbf{e}_j = \text{column of} \\ A & B \end{array}$$

Hence  $A = B$  and we are finished. □

## Definition

The matrix  $A$  is called the *standard matrix for  $T$*  and is written  $[T]$ .

**EXAMPLE 2** Find the standard matrix  $A$  for the dilation transformation  $T(\mathbf{x}) = 3\mathbf{x}$ , for  $\mathbf{x}$  in  $\mathbb{R}^2$ .

**SOLUTION** Write

$$T(\mathbf{e}_1) = 3\mathbf{e}_1 = \begin{bmatrix} 3 \\ 0 \end{bmatrix} \quad \text{and} \quad T(\mathbf{e}_2) = 3\mathbf{e}_2 = \begin{bmatrix} 0 \\ 3 \end{bmatrix}$$

The diagram illustrates the mapping of the standard basis vectors  $\mathbf{e}_1$  and  $\mathbf{e}_2$  under the dilation transformation  $T$ . Two blue arrows point from the equations above to the columns of the matrix  $A$  below. The first arrow points to the first column of  $A$ , which is  $\begin{bmatrix} 3 \\ 0 \end{bmatrix}$ . The second arrow points to the second column of  $A$ , which is  $\begin{bmatrix} 0 \\ 3 \end{bmatrix}$ .

$$A = \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}$$

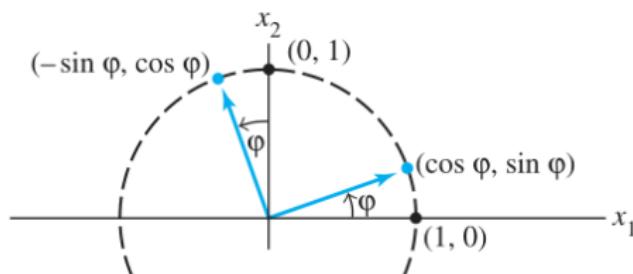
**EXAMPLE 3** Let  $T : \mathbb{R}^2 \rightarrow \mathbb{R}^2$  be the transformation that rotates each point in  $\mathbb{R}^2$  about the origin through an angle  $\varphi$ , with counterclockwise rotation for a positive angle. We could show geometrically that such a transformation is linear. (See Fig. 6 in Section 1.8.) Find the standard matrix  $A$  of this transformation.

**SOLUTION**  $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$  rotates into  $\begin{bmatrix} \cos \varphi \\ \sin \varphi \end{bmatrix}$ , and  $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$  rotates into  $\begin{bmatrix} -\sin \varphi \\ \cos \varphi \end{bmatrix}$ . See Fig. 1.

By Theorem 10,

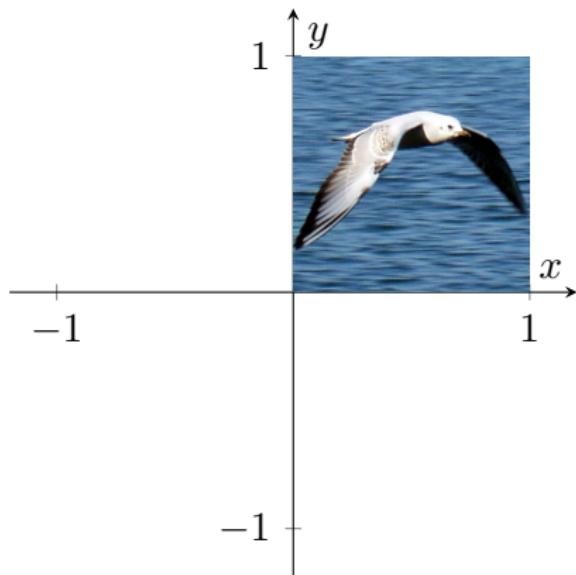
$$A = \begin{bmatrix} \cos \varphi & -\sin \varphi \\ \sin \varphi & \cos \varphi \end{bmatrix}$$

Example 5 in Section 1.8 is a special case of this transformation, with  $\varphi = \pi/2$ . ■

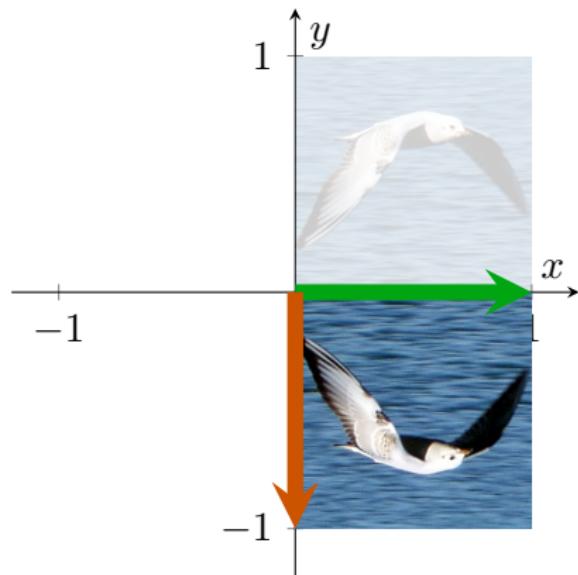


**FIGURE 1** A rotation transformation.

## Reflections



## Reflections



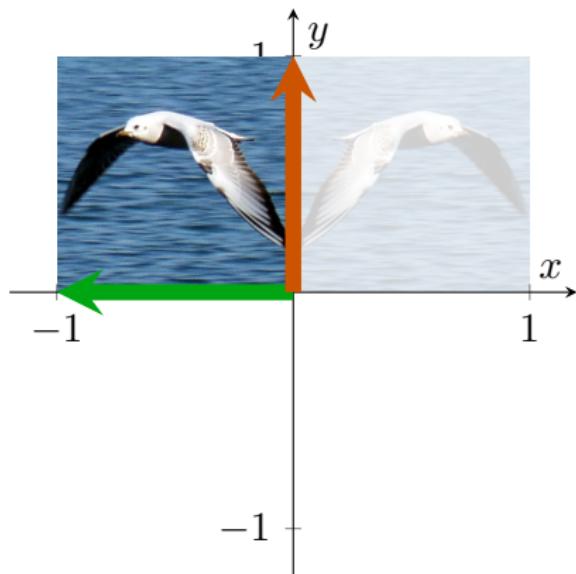
Reflection about the  $x$ -axis

$$T(\mathbf{e}_1) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$T(\mathbf{e}_2) = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$$

$$[T] = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

## Reflections



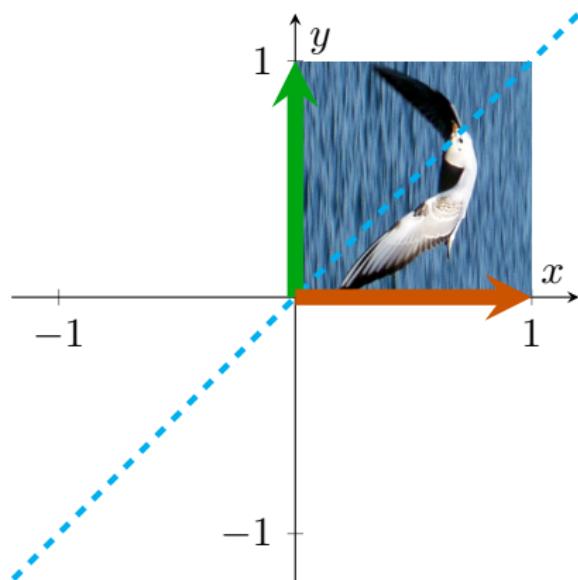
Reflection about the  $y$ -axis

$$T(\mathbf{e}_1) = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$$

$$T(\mathbf{e}_2) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$[T] = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$$

## Reflections



Reflection about the line

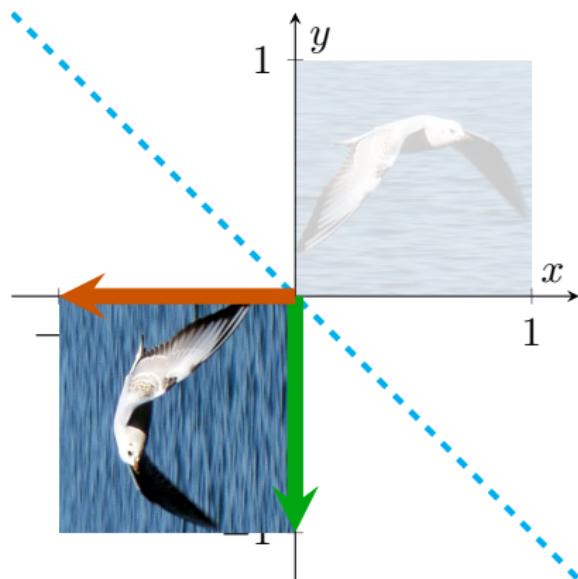
$$y = x$$

$$T(\mathbf{e}_1) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$T(\mathbf{e}_2) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$[T] = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

## Reflections



Reflection about the line

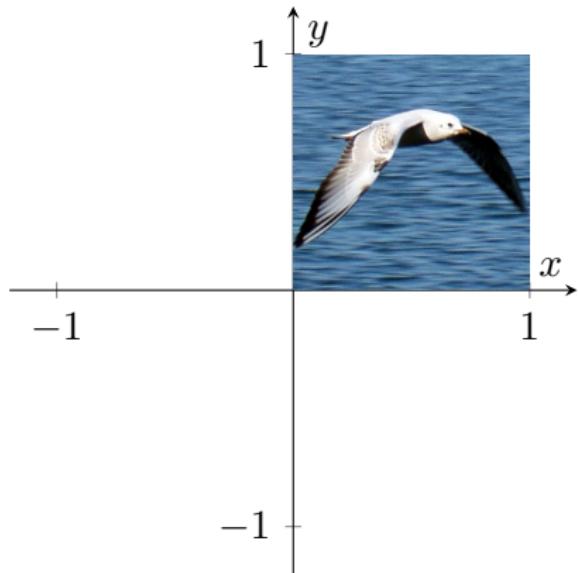
$$y = -x$$

$$T(\mathbf{e}_1) = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$$

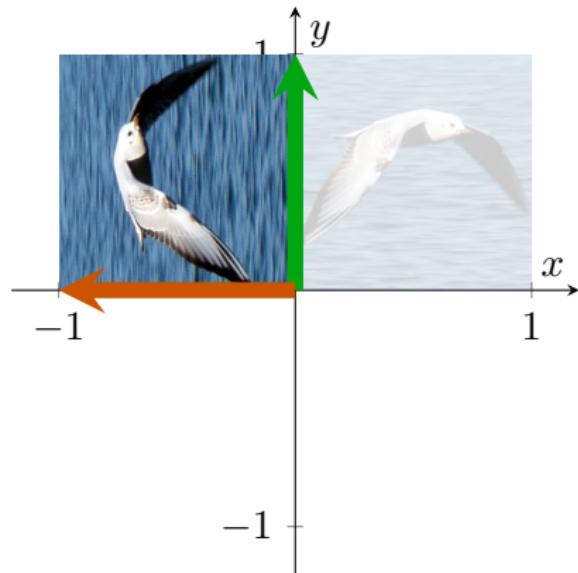
$$T(\mathbf{e}_2) = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$$

$$[T] = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$$

## Rotations



## Rotations



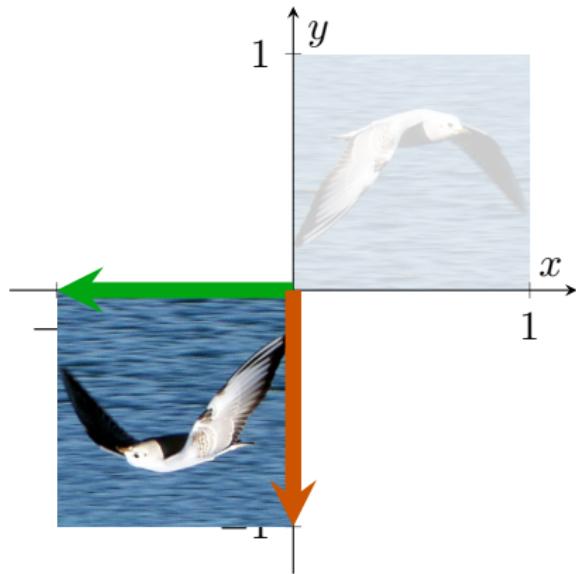
Rotate by  $90^\circ$

$$T(\mathbf{e}_1) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$T(\mathbf{e}_2) = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$$

$$[T] = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

## Rotations



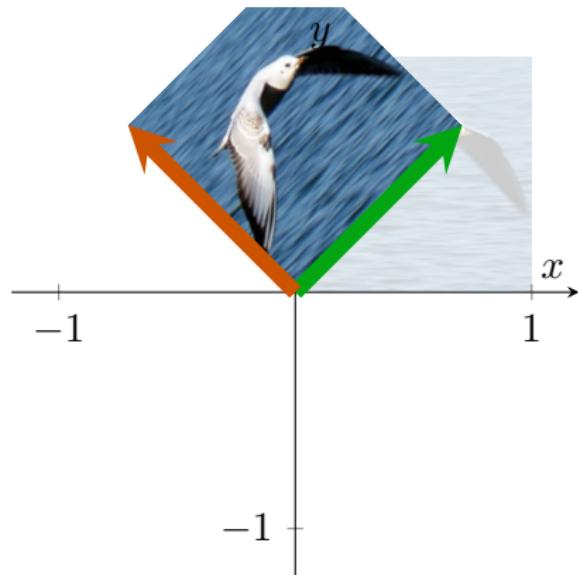
Rotate by 180°

$$T(\mathbf{e}_1) = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$$

$$T(\mathbf{e}_2) = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$$

$$[T] = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$

## Rotations



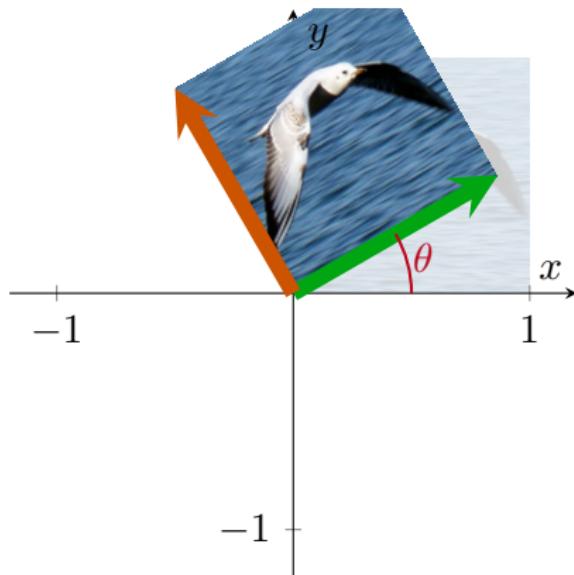
Rotate by  $45^\circ$

$$T(\mathbf{e}_1) = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}$$

$$T(\mathbf{e}_2) = \begin{bmatrix} -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}$$

$$[T] = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$$

## Rotations



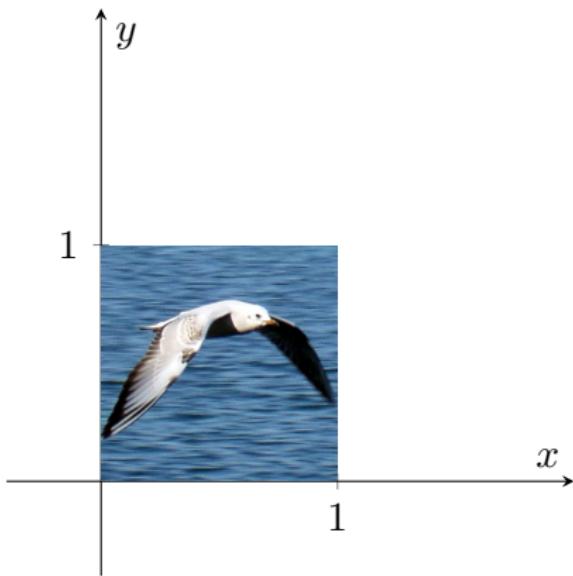
Rotate by  $\theta$

$$T(\mathbf{e}_1) = \begin{bmatrix} \cos \theta \\ \sin \theta \end{bmatrix}$$

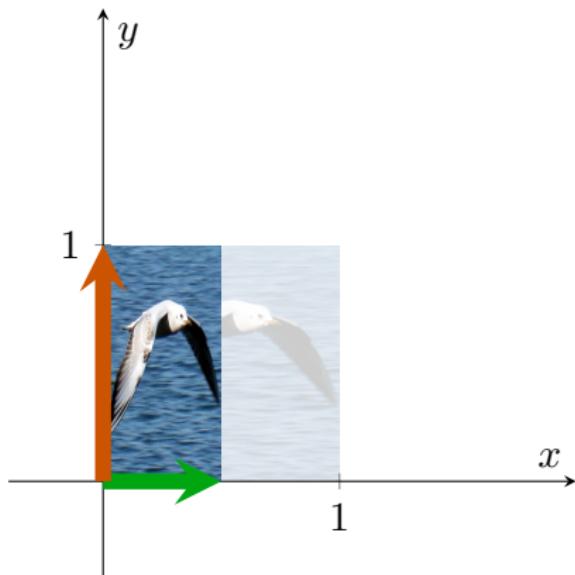
$$T(\mathbf{e}_2) = \begin{bmatrix} -\sin \theta \\ \cos \theta \end{bmatrix}$$

$$[T] = \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix}$$

## Contractions and Expansions



## Contractions and Expansions



Horizontal Contraction

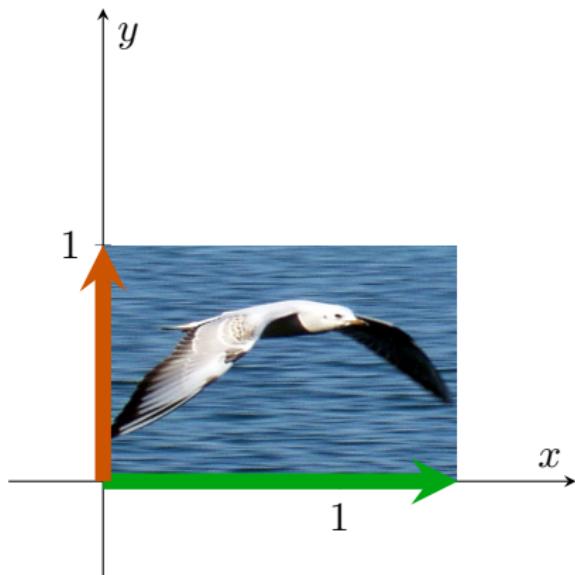
$$T(\mathbf{e}_1) = \begin{bmatrix} k \\ 0 \end{bmatrix}$$

$$T(\mathbf{e}_2) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$[T] = \begin{bmatrix} k & 0 \\ 0 & 1 \end{bmatrix}$$

$$0 < k < 1$$

## Contractions and Expansions



Horizontal Expansion

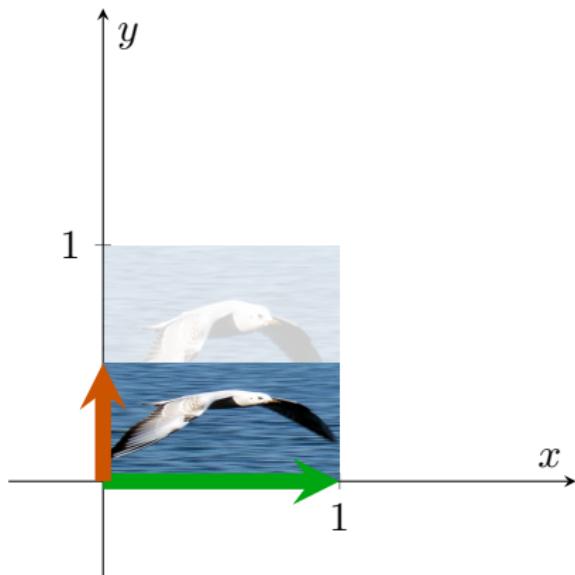
$$T(\mathbf{e}_1) = \begin{bmatrix} k \\ 0 \end{bmatrix}$$

$$T(\mathbf{e}_2) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$[T] = \begin{bmatrix} k & 0 \\ 0 & 1 \end{bmatrix}$$

$$k > 1$$

## Contractions and Expansions



Vertical Contraction

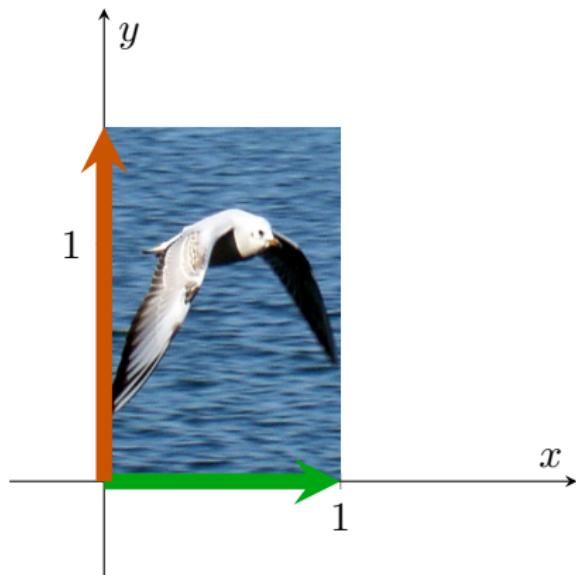
$$T(\mathbf{e}_1) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$T(\mathbf{e}_2) = \begin{bmatrix} 0 \\ k \end{bmatrix}$$

$$[T] = \begin{bmatrix} 1 & 0 \\ 0 & k \end{bmatrix}$$

$$0 < k < 1$$

## Contractions and Expansions



Vertical Expansion

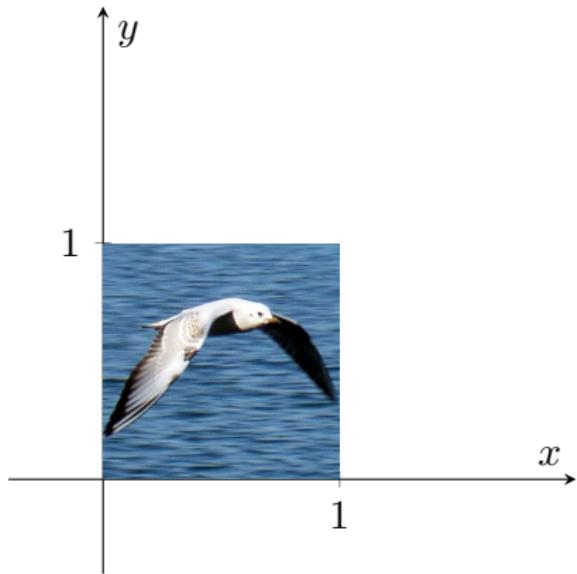
$$T(\mathbf{e}_1) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$T(\mathbf{e}_2) = \begin{bmatrix} 0 \\ k \end{bmatrix}$$

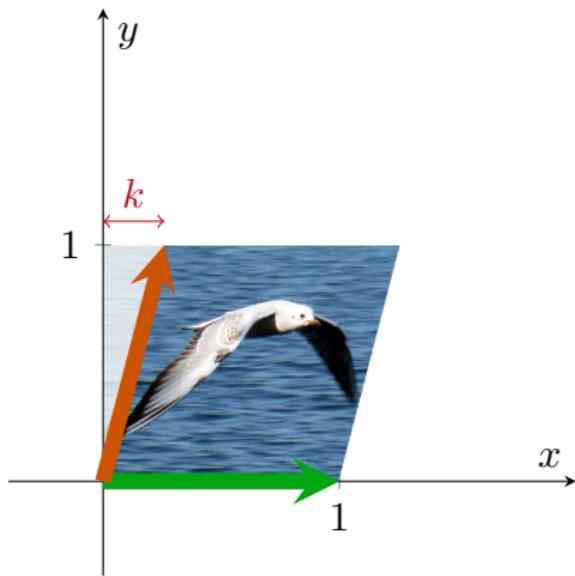
$$[T] = \begin{bmatrix} 1 & 0 \\ 0 & k \end{bmatrix}$$

$$k > 1$$

## Shears



## Shears



Horizontal Shear

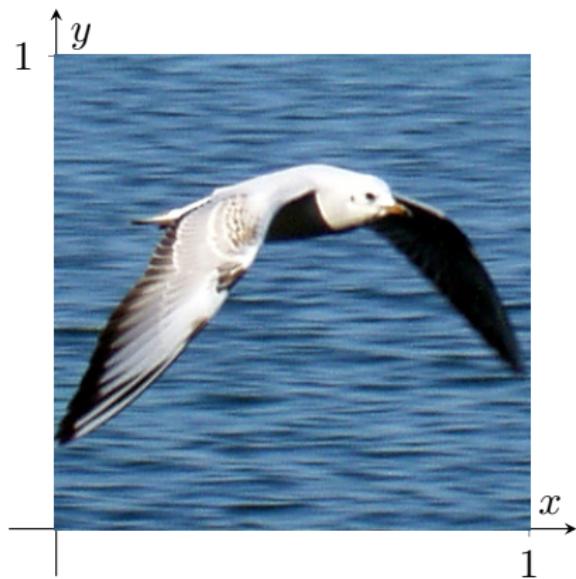
$$T(\mathbf{e}_1) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$T(\mathbf{e}_2) = \begin{bmatrix} k \\ 1 \end{bmatrix}$$

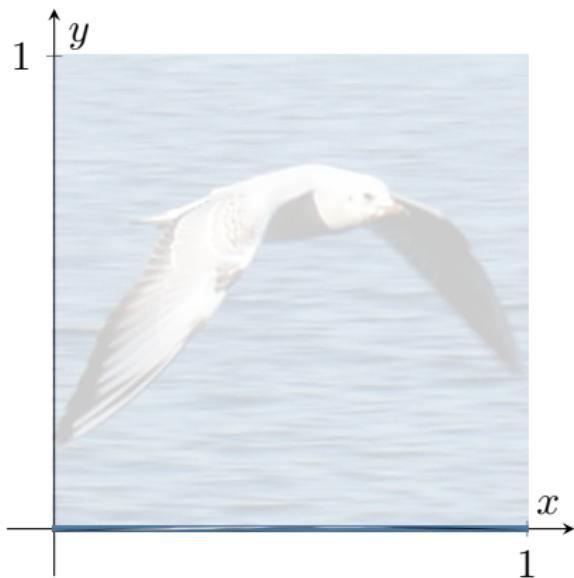
$$[T] = \begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix}$$

$$k > 0$$

## Orthogonal Projections



## Orthogonal Projections



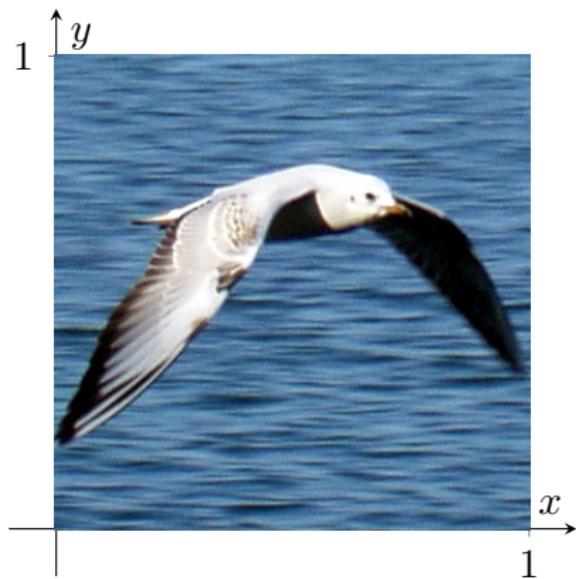
Projection onto the  $x$ -axis

$$T(\mathbf{e}_1) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

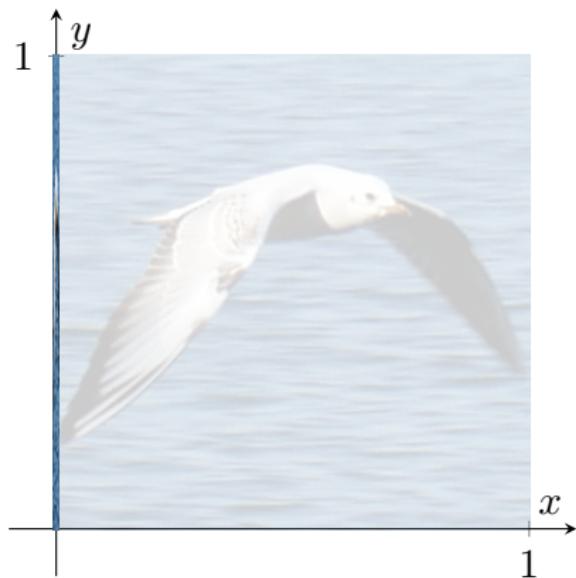
$$T(\mathbf{e}_2) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$[T] = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

## Orthogonal Projections



## Orthogonal Projections



Projection onto the  $y$ -axis

$$T(\mathbf{e}_1) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$T(\mathbf{e}_2) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

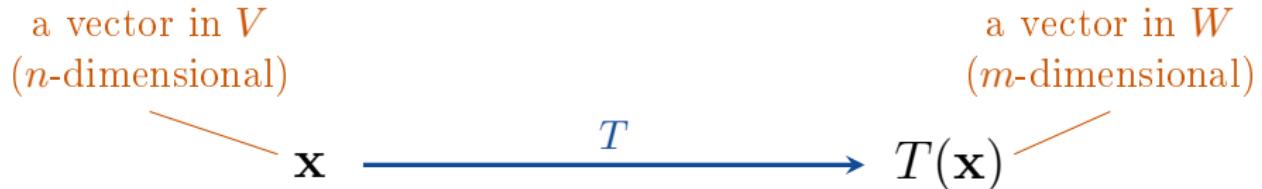
$$[T] = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

## A Matrix for $T$ relative to two bases

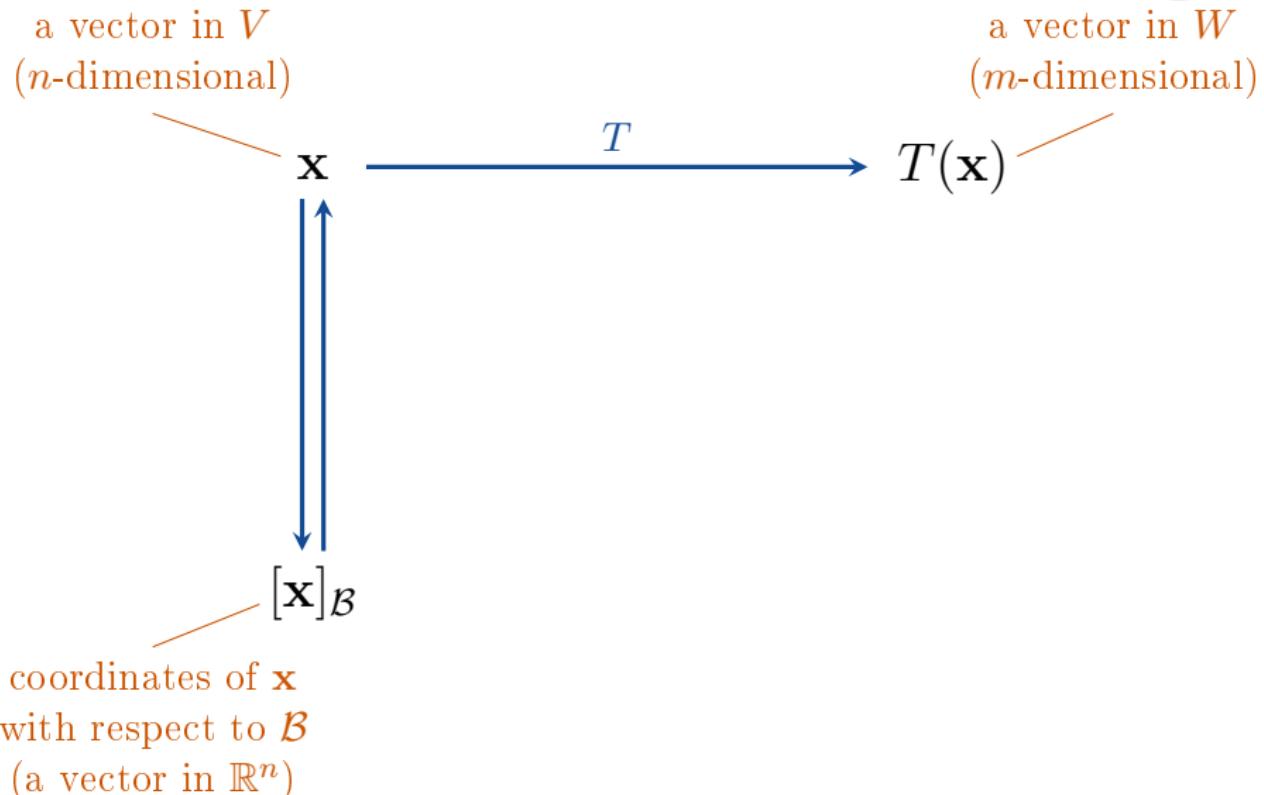
Let

- $V$  be an  $n$ -dimensional vector space;
- $\mathcal{B}$  be a basis for  $V$ ;
- $W$  be an  $m$ -dimensional vector space;
- $\mathcal{C}$  is a basis for  $W$ ; and
- $T : V \rightarrow W$  be a linear transformation.

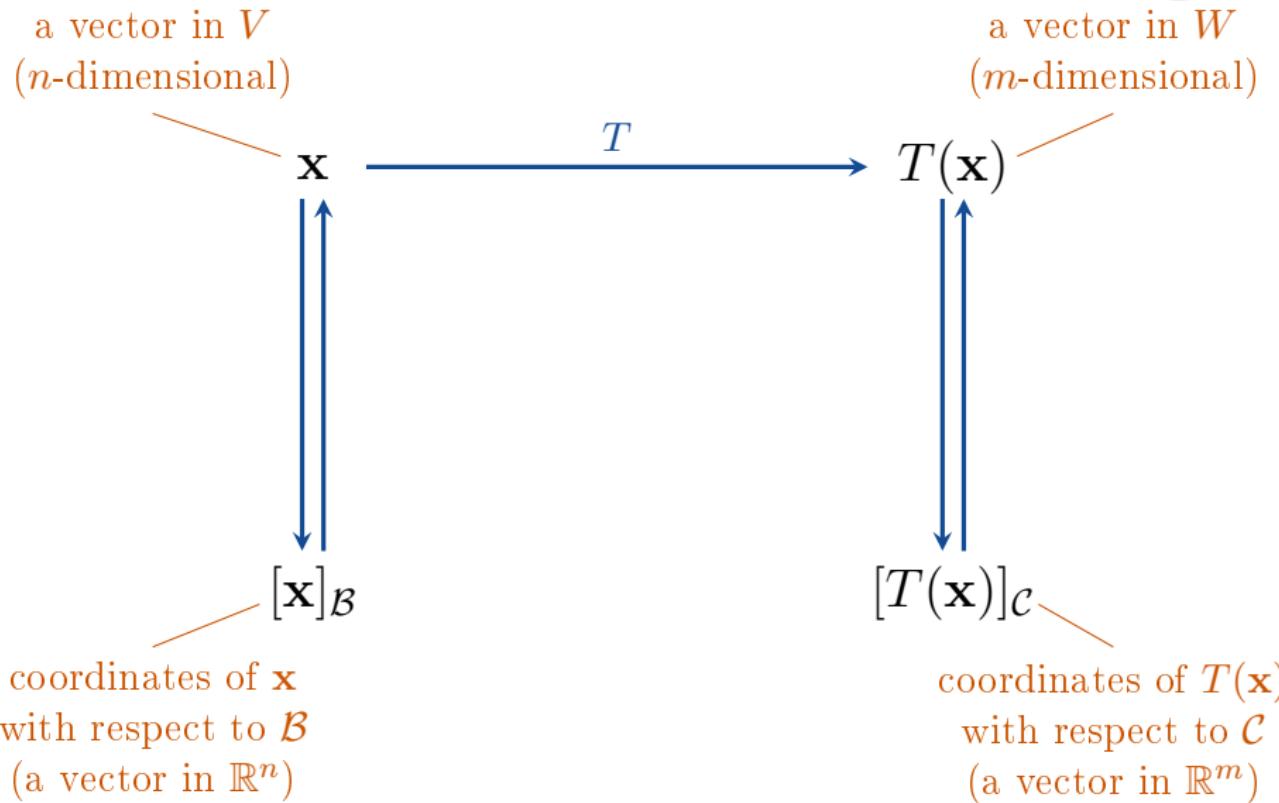
# Matrices for Linear Transformations



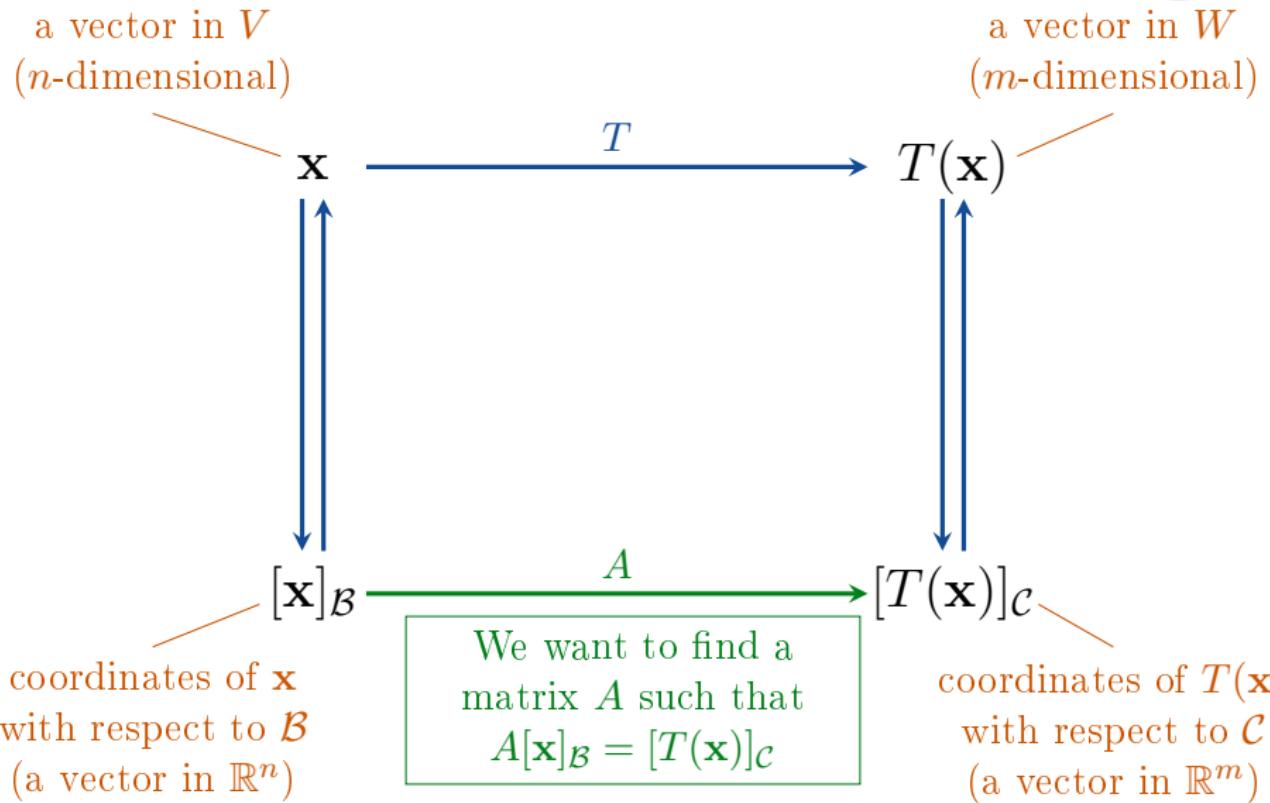
# Matrices for Linear Transformations



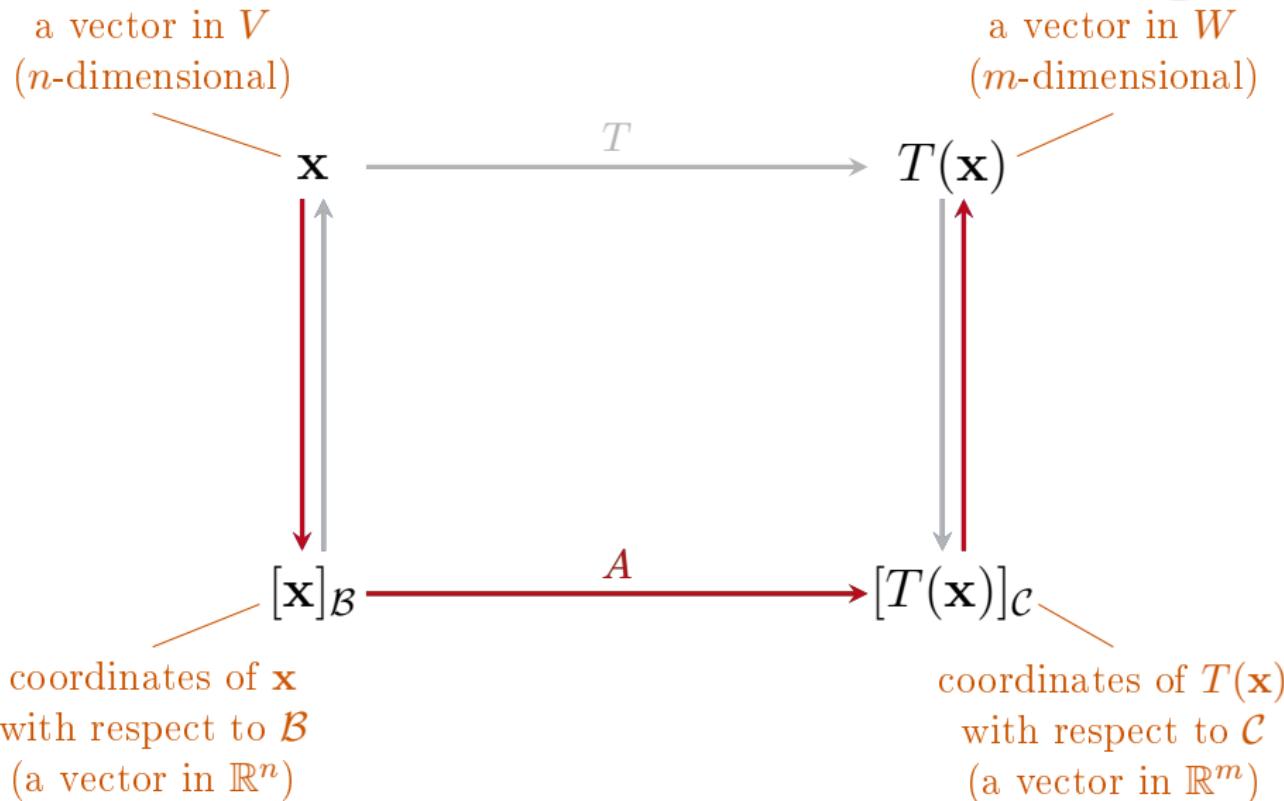
# Matrices for Linear Transformations



# Matrices for Linear Transformations



# Matrices for Linear Transformations



# Matrices for Linear Transformations



$$T : V \rightarrow W$$

We want to find an  $m \times n$  matrix  $A$  such that

$$A[\mathbf{x}]_{\mathcal{B}} = [T(\mathbf{x})]_{\mathcal{C}}.$$

# Matrices for Linear Transformations



$$T : V \rightarrow W$$

We want to find an  $m \times n$  matrix  $A$  such that

$$A[\mathbf{x}]_{\mathcal{B}} = [T(\mathbf{x})]_{\mathcal{C}}.$$

Let

- $\mathcal{B} = \{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\}$  be a basis for the  $n$ -dimensional vector space  $V$ ; and
- $\mathcal{C} = \{\mathbf{w}_1, \mathbf{w}_2, \dots, \mathbf{w}_m\}$  be a basis for the  $m$ -dimensional vector space  $W$ .

(This is on page 307 of your textbook.)

$$A[\mathbf{x}]_{\mathcal{B}} = [T(\mathbf{x})]_{\mathcal{C}}$$



We need  $A$  to satisfy

$$A[\mathbf{v}_1]_{\mathcal{B}} = [T(\mathbf{v}_1)]_{\mathcal{C}}, \quad A[\mathbf{v}_2]_{\mathcal{B}} = [T(\mathbf{v}_2)]_{\mathcal{C}}, \quad \dots, \quad A[\mathbf{v}_n]_{\mathcal{B}} = [T(\mathbf{v}_n)]_{\mathcal{C}}.$$

$$A[\mathbf{x}]_{\mathcal{B}} = [T(\mathbf{x})]_{\mathcal{C}}$$



We need  $A$  to satisfy

$$A[\mathbf{v}_1]_{\mathcal{B}} = [T(\mathbf{v}_1)]_{\mathcal{C}}, \quad A[\mathbf{v}_2]_{\mathcal{B}} = [T(\mathbf{v}_2)]_{\mathcal{C}}, \quad \dots, \quad A[\mathbf{v}_n]_{\mathcal{B}} = [T(\mathbf{v}_n)]_{\mathcal{C}}.$$

But  $\mathbf{v}_1 = 1\mathbf{v}_1 + 0\mathbf{v}_2 + 0\mathbf{v}_3 + \dots + 0\mathbf{v}_n$  so

$$[\mathbf{v}_1]_{\mathcal{B}} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

We need  $A$  to satisfy

$$A[\mathbf{v}_1]_{\mathcal{B}} = [T(\mathbf{v}_1)]_{\mathcal{C}}, \quad A[\mathbf{v}_2]_{\mathcal{B}} = [T(\mathbf{v}_2)]_{\mathcal{C}}, \quad \dots, \quad A[\mathbf{v}_n]_{\mathcal{B}} = [T(\mathbf{v}_n)]_{\mathcal{C}}.$$

But  $\mathbf{v}_1 = 1\mathbf{v}_1 + 0\mathbf{v}_2 + 0\mathbf{v}_3 + \dots + 0\mathbf{v}_n$  so

$$[\mathbf{v}_1]_{\mathcal{B}} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

which implies that

$$[T(\mathbf{v}_1)]_{\mathcal{C}} = A[\mathbf{v}_1]_{\mathcal{B}} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} =$$

We need  $A$  to satisfy

$$A[\mathbf{v}_1]_{\mathcal{B}} = [T(\mathbf{v}_1)]_{\mathcal{C}}, \quad A[\mathbf{v}_2]_{\mathcal{B}} = [T(\mathbf{v}_2)]_{\mathcal{C}}, \quad \dots, \quad A[\mathbf{v}_n]_{\mathcal{B}} = [T(\mathbf{v}_n)]_{\mathcal{C}}.$$

But  $\mathbf{v}_1 = 1\mathbf{v}_1 + 0\mathbf{v}_2 + 0\mathbf{v}_3 + \dots + 0\mathbf{v}_n$  so

$$[\mathbf{v}_1]_{\mathcal{B}} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

which implies that

$$[T(\mathbf{v}_1)]_{\mathcal{C}} = A[\mathbf{v}_1]_{\mathcal{B}} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} = \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix}.$$

$$A[\mathbf{x}]_{\mathcal{B}} = [T(\mathbf{x})]_{\mathcal{C}}$$



Similarly

$$[T(\mathbf{v}_2)]_{\mathcal{C}} = A[\mathbf{v}_2]_{\mathcal{B}} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} = \begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{bmatrix}.$$

⋮

$$[T(\mathbf{v}_n)]_{\mathcal{C}} = A[\mathbf{v}_n]_{\mathcal{B}} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} = \begin{bmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{bmatrix}.$$

$$A[\mathbf{x}]_{\mathcal{B}} = [T(\mathbf{x})]_{\mathcal{C}}$$



So the  $m \times n$  matrix that we want is

$$A = \begin{bmatrix} [T(\mathbf{v}_1)]_{\mathcal{C}} & [T(\mathbf{v}_2)]_{\mathcal{C}} & \cdots & [T(\mathbf{v}_n)]_{\mathcal{C}} \end{bmatrix}.$$

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So the  $m \times n$  matrix that we want is

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## Definition

This matrix is called the *matrix for  $T$  relative to the bases  $\mathcal{B}$  and  $\mathcal{C}$*  and is denoted by  $[T]_{\mathcal{C}, \mathcal{B}}$ .

# Matrices for

$$[T]_{\mathcal{C}, \mathcal{B}} = [[T(\mathbf{v}_1)]_{\mathcal{C}} \quad [T(\mathbf{v}_2)]_{\mathcal{C}} \quad \cdots \quad [T(\mathbf{v}_n)]_{\mathcal{C}}]$$

## Remark

The linear transformation

$$\begin{array}{ccc} T : & V & \rightarrow W \\ & \text{basis } \mathcal{B} & \text{basis } \mathcal{C} \end{array}$$

has matrix

$$[T]_{\mathcal{C}, \mathcal{B}}.$$

# Matrices for

$$[T]_{\mathcal{C}, \mathcal{B}} = [[T(\mathbf{v}_1)]_{\mathcal{C}} \quad [T(\mathbf{v}_2)]_{\mathcal{C}} \quad \cdots \quad [T(\mathbf{v}_n)]_{\mathcal{C}}]$$

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has matrix

$$[T]_{\mathcal{C}, \mathcal{B}}.$$

We will use the formula

$$[T]_{\mathcal{C}, \mathcal{B}}[\mathbf{x}]_{\mathcal{B}} = [T(\mathbf{x})]_{\mathcal{C}}.$$

$$[T]_{\mathcal{C}, \mathcal{B}}[\mathbf{x}]_{\mathcal{B}} = [T(\mathbf{x})]_{\mathcal{C}}, \quad [T]_{\mathcal{C}, \mathcal{B}} = \begin{bmatrix} [T(\mathbf{v}_1)]_{\mathcal{C}} & [T(\mathbf{v}_2)]_{\mathcal{C}} & \cdots & [T(\mathbf{v}_n)]_{\mathcal{C}} \end{bmatrix}$$

## Example

I leave it for you to prove that if  $T_A : \mathbb{R}^n \rightarrow \mathbb{R}^m$  is a matrix transformation ( $T_A(\mathbf{x}) = A\mathbf{x}$ ) and if  $\mathcal{B}$  and  $\mathcal{C}$  are the standard bases, then

$$[T_A]_{\mathcal{C}, \mathcal{B}} = A.$$

$$[T]_{\mathcal{C}, \mathcal{B}}[\mathbf{x}]_{\mathcal{B}} = [T(\mathbf{x})]_{\mathcal{C}}, \quad [T]_{\mathcal{C}, \mathcal{B}} = \begin{bmatrix} [T(\mathbf{v}_1)]_{\mathcal{C}} & [T(\mathbf{v}_2)]_{\mathcal{C}} & \cdots & [T(\mathbf{v}_n)]_{\mathcal{C}} \end{bmatrix}$$

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$$[T_A]_{\mathcal{C}, \mathcal{B}} = A.$$

## Remark

If we are using the standard bases on both  $V$  and  $W$ , then we can write the matrix for  $T$  as just

$$[T].$$

This is called the *standard matrix* for  $T$ .

$$[T]_{\mathcal{C}, \mathcal{B}}[\mathbf{x}]_{\mathcal{B}} = [T(\mathbf{x})]_{\mathcal{C}}, \quad [T]_{\mathcal{C}, \mathcal{B}} = \begin{bmatrix} [T(\mathbf{v}_1)]_{\mathcal{C}} & [T(\mathbf{v}_2)]_{\mathcal{C}} & \cdots & [T(\mathbf{v}_n)]_{\mathcal{C}} \end{bmatrix}$$

## Example

Let  $T : \mathbb{P}^1 \rightarrow \mathbb{P}^2$  be the linear transformation defined by

$$T(\mathbf{p}) = x\mathbf{p}.$$

(For example, if  $\mathbf{p} = 1 + x$ , then  $T(\mathbf{p}) = x + x^2$ .)

Find the matrix for  $T$  with respect to the bases

$$\mathcal{B} = \{\mathbf{v}_1 = 1, \mathbf{v}_2 = x\} \quad \text{and} \quad \mathcal{C} = \{\mathbf{w}_1 = 1, \mathbf{w}_2 = x, \mathbf{w}_3 = x^2\}.$$

$$[T]_{\mathcal{C}, \mathcal{B}}[\mathbf{x}]_{\mathcal{B}} = [T(\mathbf{x})]_{\mathcal{C}}, \quad [T]_{\mathcal{C}, \mathcal{B}} = \begin{bmatrix} [T(\mathbf{v}_1)]_{\mathcal{C}} & [T(\mathbf{v}_2)]_{\mathcal{C}} & \cdots & [T(\mathbf{v}_n)]_{\mathcal{C}} \end{bmatrix}$$

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First we calculate that

$$T(\mathbf{v}_1) = T(1) = (x)(1) = x = 0\mathbf{w}_1 + 1\mathbf{w}_2 + 0\mathbf{w}_3$$

$$T(\mathbf{v}_2) = T(x) = (x)(x) = x^2 = 0\mathbf{w}_1 + 0\mathbf{w}_2 + 1\mathbf{w}_3.$$

$$[T]_{\mathcal{C}, \mathcal{B}}[\mathbf{x}]_{\mathcal{B}} = [T(\mathbf{x})]_{\mathcal{C}}, \quad [T]_{\mathcal{C}, \mathcal{B}} = \begin{bmatrix} [T(\mathbf{v}_1)]_{\mathcal{C}} & [T(\mathbf{v}_2)]_{\mathcal{C}} & \cdots & [T(\mathbf{v}_n)]_{\mathcal{C}} \end{bmatrix}$$

$$\begin{aligned} T(\mathbf{v}_1) &= T(1) = (x)(1) = x = 0\mathbf{w}_1 + 1\mathbf{w}_2 + 0\mathbf{w}_3 \\ T(\mathbf{v}_2) &= T(x) = (x)(x) = x^2 = 0\mathbf{w}_1 + 0\mathbf{w}_2 + 1\mathbf{w}_3. \end{aligned}$$

Therefore the coordinates of  $T(\mathbf{v}_1)$  and  $T(\mathbf{v}_2)$  with respect to  $\mathcal{C}$  are

$$[T(\mathbf{v}_1)]_{\mathcal{C}} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \quad \text{and} \quad [T(\mathbf{v}_2)]_{\mathcal{C}} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

---

<sup>1</sup>with respect to

$$[T]_{\mathcal{C}, \mathcal{B}}[\mathbf{x}]_{\mathcal{B}} = [T(\mathbf{x})]_{\mathcal{C}}, \quad [T]_{\mathcal{C}, \mathcal{B}} = \begin{bmatrix} [T(\mathbf{v}_1)]_{\mathcal{C}} & [T(\mathbf{v}_2)]_{\mathcal{C}} & \cdots & [T(\mathbf{v}_n)]_{\mathcal{C}} \end{bmatrix}$$

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Therefore the coordinates of  $T(\mathbf{v}_1)$  and  $T(\mathbf{v}_2)$  with respect to  $\mathcal{C}$  are

$$[T(\mathbf{v}_1)]_{\mathcal{C}} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \quad \text{and} \quad [T(\mathbf{v}_2)]_{\mathcal{C}} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

Hence the matrix for  $T$  wrt<sup>1</sup>  $\mathcal{B}$  and  $\mathcal{C}$  is

$$[T]_{\mathcal{C}, \mathcal{B}} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

---

<sup>1</sup>with respect to

$$[T]_{\mathcal{C}, \mathcal{B}}[\mathbf{x}]_{\mathcal{B}} = [T(\mathbf{x})]_{\mathcal{C}}, \quad [T]_{\mathcal{C}, \mathcal{B}} = \begin{bmatrix} [T(\mathbf{v}_1)]_{\mathcal{C}} & [T(\mathbf{v}_2)]_{\mathcal{C}} & \cdots & [T(\mathbf{v}_n)]_{\mathcal{C}} \end{bmatrix}$$

## Example

Let  $T : \mathbb{R}^2 \rightarrow \mathbb{R}^3$  be the linear transformation defined by

$$T \left( \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \right) = \begin{bmatrix} x_2 \\ -5x_1 + 13x_2 \\ -7x_1 + 16x_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -5 & 13 \\ -7 & 16 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}.$$

Find the matrix for  $T$  wrt the bases  $\mathcal{B} = \{\mathbf{v}_1, \mathbf{v}_2\}$  for  $\mathbb{R}^2$ , and  $\mathcal{C} = \{\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3\}$  for  $\mathbb{R}^3$ , where

$$\mathbf{v}_1 = \begin{bmatrix} 3 \\ 1 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 5 \\ 2 \end{bmatrix}, \quad \mathbf{w}_1 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \quad \mathbf{w}_2 = \begin{bmatrix} -1 \\ 2 \\ 2 \end{bmatrix}, \quad \mathbf{w}_3 = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}.$$

$$[T]_{\mathcal{C}, \mathcal{B}}[\mathbf{x}]_{\mathcal{B}} = [T(\mathbf{x})]_{\mathcal{C}}, \quad [T]_{\mathcal{C}, \mathcal{B}} = \begin{bmatrix} [T(\mathbf{v}_1)]_{\mathcal{C}} & [T(\mathbf{v}_2)]_{\mathcal{C}} & \cdots & [T(\mathbf{v}_n)]_{\mathcal{C}} \end{bmatrix}$$

$$T \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{bmatrix} x_2 \\ -5x_1 + 13x_2 \\ -7x_1 + 16x_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -5 & 13 \\ -7 & 16 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Note, if we were using the standard bases on  $\mathbb{R}^2$  and  $\mathbb{R}^3$ , then the answer would just be the standard matrix

$$[T] = \begin{bmatrix} 0 & 1 \\ -5 & 13 \\ -7 & 16 \end{bmatrix}.$$

But we are using different bases, so this is not the answer.

$$[T]_{\mathcal{C}, \mathcal{B}}[\mathbf{x}]_{\mathcal{B}} = [T(\mathbf{x})]_{\mathcal{C}}, \quad [T]_{\mathcal{C}, \mathcal{B}} = \begin{bmatrix} [T(\mathbf{v}_1)]_{\mathcal{C}} & [T(\mathbf{v}_2)]_{\mathcal{C}} & \cdots & [T(\mathbf{v}_n)]_{\mathcal{C}} \end{bmatrix}$$

$$T \left( \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \right) = \begin{bmatrix} x_2 \\ -5x_1 + 13x_2 \\ -7x_1 + 16x_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -5 & 13 \\ -7 & 16 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$\mathbf{u}_1 = \begin{bmatrix} 3 \\ 1 \end{bmatrix}, \quad \mathbf{u}_2 = \begin{bmatrix} 5 \\ 2 \end{bmatrix}, \quad \mathbf{w}_1 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \quad \mathbf{w}_2 = \begin{bmatrix} -1 \\ 2 \\ 2 \end{bmatrix}, \quad \mathbf{w}_3 = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}$$

We start by calculating

$$T(\mathbf{v}_1) = \begin{bmatrix} 0 & 1 \\ -5 & 13 \\ -7 & 16 \end{bmatrix} \begin{bmatrix} 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \\ -5 \end{bmatrix} = \mathbf{w}_1 - 2\mathbf{w}_3$$

$$T(\mathbf{v}_2) = \begin{bmatrix} 0 & 1 \\ -5 & 13 \\ -7 & 16 \end{bmatrix} \begin{bmatrix} 5 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ -3 \end{bmatrix} = 3\mathbf{w}_1 + \mathbf{w}_2 - \mathbf{w}_3$$

(please check).

$$[T]_{\mathcal{C}, \mathcal{B}}[\mathbf{x}]_{\mathcal{B}} = [T(\mathbf{x})]_{\mathcal{C}}, \quad [T]_{\mathcal{C}, \mathcal{B}} = \begin{bmatrix} [T(\mathbf{v}_1)]_{\mathcal{C}} & [T(\mathbf{v}_2)]_{\mathcal{C}} & \cdots & [T(\mathbf{v}_n)]_{\mathcal{C}} \end{bmatrix}$$

$$T(\mathbf{v}_1) = \mathbf{w}_1 - 2\mathbf{w}_3 \qquad \qquad T(\mathbf{v}_2) = 3\mathbf{w}_1 + \mathbf{w}_2 - \mathbf{w}_3$$

Therefore

$$[T(\mathbf{v}_1)]_{\mathcal{C}} = \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix} \quad \text{and} \quad [T(\mathbf{v}_2)]_{\mathcal{C}} = \begin{bmatrix} 3 \\ 1 \\ -1 \end{bmatrix}.$$

$$[T]_{\mathcal{C}, \mathcal{B}}[\mathbf{x}]_{\mathcal{B}} = [T(\mathbf{x})]_{\mathcal{C}}, \quad [T]_{\mathcal{C}, \mathcal{B}} = \begin{bmatrix} [T(\mathbf{v}_1)]_{\mathcal{C}} & [T(\mathbf{v}_2)]_{\mathcal{C}} & \cdots & [T(\mathbf{v}_n)]_{\mathcal{C}} \end{bmatrix}$$

$$T(\mathbf{v}_1) = \mathbf{w}_1 - 2\mathbf{w}_3 \qquad \qquad T(\mathbf{v}_2) = 3\mathbf{w}_1 + \mathbf{w}_2 - \mathbf{w}_3$$

Therefore

$$[T(\mathbf{v}_1)]_{\mathcal{C}} = \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix} \quad \text{and} \quad [T(\mathbf{v}_2)]_{\mathcal{C}} = \begin{bmatrix} 3 \\ 1 \\ -1 \end{bmatrix}.$$

Hence the matrix for  $T$  wrt  $\mathcal{B}$  and  $\mathcal{C}$  is

$$[T]_{\mathcal{C}, \mathcal{B}} = \begin{bmatrix} [T(\mathbf{v}_1)]_{\mathcal{C}} & [T(\mathbf{v}_2)]_{\mathcal{C}} \end{bmatrix} = \begin{bmatrix} 1 & 3 \\ 0 & 1 \\ -2 & -1 \end{bmatrix}.$$

# Matrices for Linear Transformations



Remark

$$[T] = \begin{bmatrix} 0 & 1 \\ -5 & 13 \\ -7 & 16 \end{bmatrix} \quad \text{and} \quad [T]_{\mathcal{C}, \mathcal{B}} = \begin{bmatrix} 1 & 3 \\ 0 & 1 \\ -2 & -1 \end{bmatrix}$$

are two different matrices for the same linear transformation.

## Linear Operators

Recall that if  $V = W$ , then the linear transformation  $T : V \rightarrow V$  is called a linear operator.

(If we are using the same basis  $\mathcal{B}$  on both the domain and the target of  $T$ , then) instead of writing  $[T]_{\mathcal{B}, \mathcal{B}}$  we just write

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## Linear Operators

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The formulae are then

$$[T]_{\mathcal{B}} = \begin{bmatrix} [T(\mathbf{v}_1)]_{\mathcal{B}} & [T(\mathbf{v}_2)]_{\mathcal{B}} & \cdots & [T(\mathbf{v}_n)]_{\mathcal{B}} \end{bmatrix}$$

and

$$[T]_{\mathcal{B}}[\mathbf{x}]_{\mathcal{B}} = [T(\mathbf{x})]_{\mathcal{B}}.$$

## Identity Operators

### Example

Let  $V$  be a finite dimensional vector space and let  $\mathcal{B} = \{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\}$  be a basis for  $V$ . Consider the identity operator  $I : V \rightarrow V$ ,  $I(\mathbf{x}) = \mathbf{x}$ .

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$$I(\mathbf{v}_1) = \mathbf{v}_1, \quad I(\mathbf{v}_2) = \mathbf{v}_2, \quad \dots, \quad I(\mathbf{v}_n) = \mathbf{v}_n.$$

## Identity Operators

### Example

Let  $V$  be a finite dimensional vector space and let  $\mathcal{B} = \{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\}$  be a basis for  $V$ . Consider the identity operator  $I : V \rightarrow V$ ,  $I(\mathbf{x}) = \mathbf{x}$ . Note that

$$I(\mathbf{v}_1) = \mathbf{v}_1, \quad I(\mathbf{v}_2) = \mathbf{v}_2, \quad \dots, \quad I(\mathbf{v}_n) = \mathbf{v}_n.$$

Hence

$$[I]_{\mathcal{B}} = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix} = I_n.$$

# Matrices for Linear Transformations



## Example

Let  $T : \mathbb{P}^2 \rightarrow \mathbb{P}^2$  be the linear operator defined by

$$T(\mathbf{p}) = \mathbf{p}(3x - 5).$$

For example, if  $\mathbf{p} = 2x + x^2$ , then

$$T(\mathbf{p}) = \mathbf{p}(3x - 5) = 2(3x - 5) + (3x - 5)^2.$$

# Matrices for Linear Transformations



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$$T(\mathbf{p}) = \mathbf{p}(3x - 5) = 2(3x - 5) + (3x - 5)^2.$$

- 1 Find  $[T]_{\mathcal{B}}$  where  $\mathcal{B} = \{1, x, x^2\}$ .
- 2 Use this to calculate  $T(1 + 2x + 3x^2)$ .

# Matrices for Linear Transformations



1 Since

$$T(1) = 1, \quad T(x) = (3x-5), \quad T(x^2) = (3x-5)^2 = 9x^2 - 30x + 25$$

we have

$$[T(1)]_{\mathcal{B}} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad [T(x)]_{\mathcal{B}} = \begin{bmatrix} -5 \\ 3 \\ 0 \end{bmatrix}, \quad [T(x^2)]_{\mathcal{B}} = \begin{bmatrix} 25 \\ -30 \\ 9 \end{bmatrix}.$$

# Matrices for Linear Transformations



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Hence

$$[T]_{\mathcal{B}} = \begin{bmatrix} 1 & -5 & 25 \\ 0 & 3 & -30 \\ 0 & 0 & 9 \end{bmatrix}.$$

$$[T]_{\mathcal{B}} = \begin{bmatrix} 1 & -5 & 25 \\ 0 & 3 & -30 \\ 0 & 0 & 9 \end{bmatrix}$$



- 2 The vector  $\mathbf{p} = 1 + 2x + 3x^2$  has coordinate vector

$$[\mathbf{p}]_{\mathcal{B}} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}.$$

$$[T]_{\mathcal{B}} = \begin{bmatrix} 1 & -5 & 25 \\ 0 & 3 & -30 \\ 0 & 0 & 9 \end{bmatrix}$$



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Thus

$$[T(\mathbf{p})]_{\mathcal{B}} = [T]_{\mathcal{B}} [\mathbf{p}]_{\mathcal{B}} = \begin{bmatrix} 1 & -5 & 25 \\ 0 & 3 & -30 \\ 0 & 0 & 9 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 66 \\ -84 \\ 27 \end{bmatrix}.$$

## Matrices for Linear Transformations

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Therefore

$$T(1 + 2x + 3x^2) = 66 - 84x + 27x^2.$$

## Compositions and Inverses

### Theorem

If  $T_1 : U \rightarrow V$  and  $T_2 : V \rightarrow W$  are linear transformations, and if  $\mathcal{A}$ ,  $\mathcal{B}$  and  $\mathcal{C}$  are bases for  $U$ ,  $V$ , and  $W$ , respectively, then

$$[T_2 \circ T_1]_{\mathcal{C}, \mathcal{A}} = [T_2]_{\mathcal{C}, \mathcal{B}} [T_1]_{\mathcal{B}, \mathcal{A}}.$$

## Theorem

If  $T : V \rightarrow V$  is a linear operator, and if  $\mathcal{B}$  is a basis for  $V$ , then the following are equivalent:

- 1  $T$  is one-to-one.
- 2  $[T]_{\mathcal{B}}$  is invertible.

## Theorem

If  $T : V \rightarrow V$  is a linear operator, and if  $\mathcal{B}$  is a basis for  $V$ , then the following are equivalent:

- 1  $T$  is one-to-one.
- 2  $[T]_{\mathcal{B}}$  is invertible.

Moreover, when these equivalent conditions hold,

$$[T^{-1}]_{\mathcal{B}} = [T]_{\mathcal{B}}^{-1}$$

# Matrices for Linear Transformations



## Example (Composition)

Let  $T_1 : \mathbb{P}^1 \rightarrow \mathbb{P}^2$  and  $T_2 : \mathbb{P}^2 \rightarrow \mathbb{P}^2$  be defined by

$$T_1(\mathbf{p}) = x\mathbf{p} \quad \text{and} \quad T_2(\mathbf{p}) = \mathbf{p}(3x - 5).$$

Find the matrix of  $T_2 \circ T_1$  wrt to the standard bases on  $\mathbb{P}^1$  and  $\mathbb{P}^2$ .

Let

$$\mathcal{B} = \{1, x\} \quad \text{and} \quad \mathcal{C} = \{1, x, x^2\}.$$

# Matrices for Linear Transformations



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$$\mathcal{B} = \{1, x\} \quad \text{and} \quad \mathcal{C} = \{1, x, x^2\}.$$

We have seen today that

$$[T_1]_{\mathcal{C}, \mathcal{B}} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad [T_2]_{\mathcal{C}} = \begin{bmatrix} 1 & -5 & 25 \\ 0 & 3 & -30 \\ 0 & 0 & 9 \end{bmatrix}.$$

# Matrices for Linear Transformations



It follows that

$$[T_2 \circ T_1]_{\mathcal{C}, \mathcal{B}} = [T_2]_{\mathcal{C}} [T_1]_{\mathcal{C}, \mathcal{B}} = \begin{bmatrix} 1 & -5 & 25 \\ 0 & 3 & -30 \\ 0 & 0 & 9 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} -5 & 25 \\ 3 & -30 \\ 0 & 9 \end{bmatrix}$$



# Break

We will continue at 3pm





# Similarity

# Similarity

Consider the matrix operator  $T_A : \mathbb{R}^2 \rightarrow \mathbb{R}^2$  where

$$A = \begin{bmatrix} 1 & 1 \\ -2 & 4 \end{bmatrix}.$$

The matrix for  $T_A$  relative to the standard basis  $\mathcal{B} = \{\mathbf{e}_1, \mathbf{e}_2\}$  for  $\mathbb{R}^2$  is

$$[T_A] = [T_A]_{\mathcal{B}} = A = \begin{bmatrix} 1 & 1 \\ -2 & 4 \end{bmatrix}.$$

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$$[T_A] = [T_A]_{\mathcal{B}} = A = \begin{bmatrix} 1 & 1 \\ -2 & 4 \end{bmatrix}.$$

I want to calculate the matrix for the same operator  $T_A$  relative to a different basis.

# Similarity

$$A = \begin{bmatrix} 1 & 1 \\ -2 & 4 \end{bmatrix}$$



Namely, the basis  $\mathcal{C} = \{\mathbf{v}_1, \mathbf{v}_2\}$  for  $\mathbb{R}^2$  where

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad \text{and} \quad \mathbf{v}_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}.$$

# Similarity

$$A = \begin{bmatrix} 1 & 1 \\ -2 & 4 \end{bmatrix}$$



Namely, the basis  $\mathcal{C} = \{\mathbf{v}_1, \mathbf{v}_2\}$  for  $\mathbb{R}^2$  where

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad \text{and} \quad \mathbf{v}_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}.$$

Since

$$T_A(\mathbf{v}_1) = A\mathbf{v}_1 = \begin{bmatrix} 1 & 1 \\ -2 & 4 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \end{bmatrix} = 2\mathbf{v}_1$$

and

$$T_A(\mathbf{v}_2) = A\mathbf{v}_2 = \begin{bmatrix} 1 & 1 \\ -2 & 4 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 3 \\ 6 \end{bmatrix} = 3\mathbf{v}_2$$

# Similarity

$$A = \begin{bmatrix} 1 & 1 \\ -2 & 4 \end{bmatrix}$$



Namely, the basis  $\mathcal{C} = \{\mathbf{v}_1, \mathbf{v}_2\}$  for  $\mathbb{R}^2$  where

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad \text{and} \quad \mathbf{v}_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}.$$

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$$T_A(\mathbf{v}_1) = A\mathbf{v}_1 = \begin{bmatrix} 1 & 1 \\ -2 & 4 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \end{bmatrix} = 2\mathbf{v}_1$$

and

$$T_A(\mathbf{v}_2) = A\mathbf{v}_2 = \begin{bmatrix} 1 & 1 \\ -2 & 4 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 3 \\ 6 \end{bmatrix} = 3\mathbf{v}_2$$

it follows that

$$[T_A(\mathbf{v}_1)]_{\mathcal{C}} = \begin{bmatrix} 2 \\ 0 \end{bmatrix} \quad \text{and} \quad [T_A(\mathbf{v}_2)]_{\mathcal{C}} = \begin{bmatrix} 0 \\ 3 \end{bmatrix}.$$

Therefore the matrix for  $T_A$  relative to the basis  $\mathcal{C}$  is

$$[T_A]_{\mathcal{C}} = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}.$$

# Similarity

Compare these two matrices

$$[T_A]_{\mathcal{B}} = \begin{bmatrix} 1 & 1 \\ -2 & 4 \end{bmatrix} \quad [T_A]_{\mathcal{C}} = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}.$$

- The determinant of both matrices is 6.
- Both matrices are invertible.
- The trace of both matrices is 5.
- The latter matrix is diagonal and diagonal matrices are easier to deal with.

## Definition

If  $A$  and  $B$  are square matrices, then we say that  $B$  is *similar* to  $A$  iff there exists an invertible matrix  $P$  such that

$$B = P^{-1}AP.$$

# Similarity

## Definition

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## Theorem

$$B \text{ is similar to } A \implies A \text{ is similar to } B$$

## Proof.

Suppose that  $B = P^{-1}AP$ . Let  $Q = P^{-1}$ .

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## Proof.

Suppose that  $B = P^{-1}AP$ . Let  $Q = P^{-1}$ . Then  $B = QAQ^{-1}$  which rearranges to  $A = Q^{-1}BQ$ . Hence  $A$  is similar to  $B$ .  $\square$

## Theorem

*If  $A$  and  $B$  are similar matrices, then*

$$\det(A) = \det(B).$$

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## Proof.

If  $B = P^{-1}AP$ , then

$$\det(B) = \det(P^{-1}AP) = \det(P^{-1}) \det(A) \det(P)$$

=

# Similarity

## Theorem

If  $A$  and  $B$  are similar matrices, then

$$\det(A) = \det(B).$$

## Proof.

If  $B = P^{-1}AP$ , then

$$\begin{aligned}\det(B) &= \det(P^{-1}AP) = \det(P^{-1})\det(A)\det(P) \\ &= \frac{1}{\det(P)}\det(A)\det(P) = \det(A).\end{aligned}$$



## Lemma

*If  $A$  and  $B$  are square matrices of the same size, then*

$$\text{tr}(AB) = \text{tr}(BA)$$

# Similarity

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2n} \\ \vdots & \vdots & & \vdots \\ b_{n1} & b_{n2} & \cdots & b_{nn} \end{bmatrix} = \begin{bmatrix} (AB)_{11} & ? & \cdots & ? \\ ? & (AB)_{22} & \cdots & ? \\ \vdots & \vdots & & \vdots \\ ? & ? & \cdots & (AB)_{nn} \end{bmatrix}$$

## Proof.

Recall that  $\text{tr}(AB)$  is the sum of the entries on the main diagonal of  $AB$ . If  $A$  and  $B$  are  $n \times n$  matrices, then

$$\begin{aligned} \text{tr}(AB) &= (AB)_{11} + (AB)_{22} + (AB)_{33} + \dots + (AB)_{nn} \\ &= \sum_{i=1}^n (AB)_{ii} \end{aligned}$$

=

# Similarity

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} b_{11} & \color{red}{b_{12}} & \cdots & b_{1n} \\ b_{21} & \color{red}{b_{22}} & \cdots & b_{2n} \\ \vdots & \vdots & & \vdots \\ b_{n1} & \color{red}{b_{n2}} & \cdots & b_{nn} \end{bmatrix} = \begin{bmatrix} (AB)_{11} & ? & \cdots & ? \\ ? & (AB)_{22} & \cdots & ? \\ \vdots & \vdots & & \vdots \\ ? & ? & \cdots & (AB)_{nn} \end{bmatrix}$$

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## Proof.

Recall that  $\text{tr}(AB)$  is the sum of the entries on the main diagonal of  $AB$ . If  $A$  and  $B$  are  $n \times n$  matrices, then

$$\begin{aligned} \text{tr}(AB) &= (AB)_{11} + (AB)_{22} + (AB)_{33} + \dots + (AB)_{nn} \\ &= \sum_{i=1}^n (AB)_{ii} \\ &= \sum_{i=1}^n \sum_{j=1}^n a_{ij}b_{ji}. \end{aligned}$$

# Similarity

$$\text{tr}(AB) = \sum_{i=1}^n \sum_{j=1}^n a_{ij} b_{ji}$$



Proof continued.

Similarly we have that

$$\text{tr}(BA) = \sum_{i=1}^n (BA)_{ii} = \sum_{i=1}^n \sum_{j=1}^n b_{ij} a_{ji} = \sum_{i=1}^n \sum_{j=1}^n a_{ji} b_{ij}$$

=

=

=

=



# Similarity

$$\text{tr}(AB) = \sum_{i=1}^n \sum_{j=1}^n a_{ij}b_{ji}$$



Proof continued.

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$$= \sum_{i=1}^n \sum_{\mathbf{k}=1}^n a_{\mathbf{k}i}b_{i\mathbf{k}} =$$

=



# Similarity

$$\text{tr}(AB) = \sum_{i=1}^n \sum_{j=1}^n a_{ij} b_{ji}$$



Proof continued.

Similarly we have that

$$\begin{aligned}\text{tr}(BA) &= \sum_{i=1}^n (BA)_{ii} = \sum_{i=1}^n \sum_{j=1}^n b_{ij} a_{ji} = \sum_{i=1}^n \sum_{j=1}^n a_{ji} b_{ij} \\ &= \sum_{i=1}^n \sum_{\mathbf{k}=1}^n a_{\mathbf{k}i} b_{i\mathbf{k}} = \sum_{\mathbf{j}=1}^n \sum_{k=1}^n a_{k\mathbf{j}} b_{\mathbf{j}k} = \\ &=\end{aligned}$$



# Similarity

$$\text{tr}(AB) = \sum_{i=1}^n \sum_{j=1}^n a_{ij} b_{ji}$$



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# Similarity

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Proof continued.

Similarly we have that

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$$\text{tr}(\textcolor{red}{X}\textcolor{blue}{Y}) = \text{tr}(\textcolor{blue}{Y}\textcolor{red}{X})$$



## Theorem

*If  $A$  and  $B$  are similar matrices, then*

$$\text{tr}(A) = \text{tr}(B).$$

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If  $A$  and  $B$  are similar matrices, then

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## Proof.

If  $B = P^{-1}AP$ , then

$$\text{tr}(B) = \text{tr}(\textcolor{green}{P}^{-1}\textcolor{red}{A}\textcolor{blue}{P}) = \quad =$$



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$$\text{tr}(\textcolor{red}{X}\textcolor{green}{Y}) = \text{tr}(\textcolor{green}{Y}\textcolor{red}{X})$$



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## A New View of Transition Matrices

Suppose that  $\mathcal{B} = \{\mathbf{b}_1, \dots, \mathbf{b}_n\}$  and  $\mathcal{C} = \{\mathbf{c}_1, \dots, \mathbf{c}_n\}$  are two bases for a vector space  $V$ . Recall that

- the *change-of-coordinates matrix* or *transition matrix* from  $\mathcal{B}$  to  $\mathcal{C}$  is the matrix

$$P_{\mathcal{C} \leftarrow \mathcal{B}} = \begin{bmatrix} [\mathbf{b}_1]_{\mathcal{C}} & [\mathbf{b}_2]_{\mathcal{C}} & \cdots & [\mathbf{b}_n]_{\mathcal{C}} \end{bmatrix}$$

# Similarity

## A New View of Transition Matrices

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- the *change-of-coordinates matrix* or *transition matrix* from  $\mathcal{C}$  to  $\mathcal{B}$  is the matrix

$$P_{\mathcal{B} \leftarrow \mathcal{C}} = \begin{bmatrix} [\mathbf{c}_1]_{\mathcal{B}} & [\mathbf{c}_2]_{\mathcal{B}} & \cdots & [\mathbf{c}_n]_{\mathcal{B}} \end{bmatrix}$$

## A New View of Transition Matrices

Suppose that  $\mathcal{B} = \{\mathbf{b}_1, \dots, \mathbf{b}_n\}$  and  $\mathcal{C} = \{\mathbf{c}_1, \dots, \mathbf{c}_n\}$  are two bases for a vector space  $V$ . Recall that

- the *change-of-coordinates matrix* or *transition matrix* from  $\mathcal{B}$  to  $\mathcal{C}$  is the matrix
- these two matrices are inverses of each other

$$P_{\mathcal{C} \leftarrow \mathcal{B}} = \begin{bmatrix} [\mathbf{b}_1]_{\mathcal{C}} & [\mathbf{b}_2]_{\mathcal{C}} & \cdots & [\mathbf{b}_n]_{\mathcal{C}} \end{bmatrix}$$

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$$P_{\mathcal{C} \leftarrow \mathcal{B}} [\mathbf{v}]_{\mathcal{B}} = [\mathbf{v}]_{\mathcal{C}}$$

- the *change-of-coordinates matrix* or *transition matrix* from  $\mathcal{C}$  to  $\mathcal{B}$  is the matrix

$$P_{\mathcal{B} \leftarrow \mathcal{C}} = \begin{bmatrix} [\mathbf{c}_1]_{\mathcal{B}} & [\mathbf{c}_2]_{\mathcal{B}} & \cdots & [\mathbf{c}_n]_{\mathcal{B}} \end{bmatrix}$$

## A New View of Transition Matrices

Suppose that  $\mathcal{B} = \{\mathbf{b}_1, \dots, \mathbf{b}_n\}$  and  $\mathcal{C} = \{\mathbf{c}_1, \dots, \mathbf{c}_n\}$  are two bases for a vector space  $V$ . Recall that

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$$P_{\mathcal{C} \leftarrow \mathcal{B}} = \begin{bmatrix} [\mathbf{b}_1]_{\mathcal{C}} & [\mathbf{b}_2]_{\mathcal{C}} & \cdots & [\mathbf{b}_n]_{\mathcal{C}} \end{bmatrix}$$

- these two matrices are inverses of each other

$$P_{\mathcal{C} \leftarrow \mathcal{B}} [\mathbf{v}]_{\mathcal{B}} = [\mathbf{v}]_{\mathcal{C}}$$

- the *change-of-coordinates matrix* or *transition matrix* from  $\mathcal{C}$  to  $\mathcal{B}$  is the matrix

$$P_{\mathcal{B} \leftarrow \mathcal{C}} = \begin{bmatrix} [\mathbf{c}_1]_{\mathcal{B}} & [\mathbf{c}_2]_{\mathcal{B}} & \cdots & [\mathbf{c}_n]_{\mathcal{B}} \end{bmatrix}$$

- $P_{\mathcal{B} \leftarrow \mathcal{C}} [\mathbf{v}]_{\mathcal{C}} = [\mathbf{v}]_{\mathcal{B}}$

## Theorem

If  $\mathcal{B} = \{\mathbf{b}_1, \dots, \mathbf{b}_n\}$  and  $\mathcal{C} = \{\mathbf{c}_1, \dots, \mathbf{c}_n\}$  are bases for a finite-dimensional vector space  $V$ , and if  $I : V \rightarrow V$  is the identity operator on  $V$ , then

$$P_{\mathcal{C} \leftarrow \mathcal{B}} = [I]_{\mathcal{C}, \mathcal{B}} \quad \text{and} \quad P_{\mathcal{B} \leftarrow \mathcal{C}} = [I]_{\mathcal{B}, \mathcal{C}}.$$

# Similarity

Proof.

Since  $I(\mathbf{v}) = \mathbf{v}$  for all  $\mathbf{v} \in V$ , we have that

$$[I]_{\mathcal{C}, \mathcal{B}} = \begin{bmatrix} [I(\mathbf{b}_1)]_{\mathcal{C}} & [I(\mathbf{b}_2)]_{\mathcal{C}} & \cdots & [I(\mathbf{b}_n)]_{\mathcal{C}} \end{bmatrix}$$

=

=

# Similarity

Proof.

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Since  $I(\mathbf{v}) = \mathbf{v}$  for all  $\mathbf{v} \in V$ , we have that

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Similarly  ${}_{\mathcal{B} \leftarrow \mathcal{C}}^P = [I]_{\mathcal{B}, \mathcal{C}}$ .

□

## The Effect of Changing Bases on Matrices of Linear Operators

**Question:** If  $\mathcal{B}$  and  $\mathcal{C}$  are two bases for a finite-dimensional vector space  $V$ , and if  $T : V \rightarrow V$  is a linear operator, what relationship, if any, exists between the matrices  $[T]_{\mathcal{B}}$  and  $[T]_{\mathcal{C}}$ ?

# Similarity



basis  $\mathcal{C}$



basis  $\mathcal{C}$



$V$

basis  $\mathcal{B}$

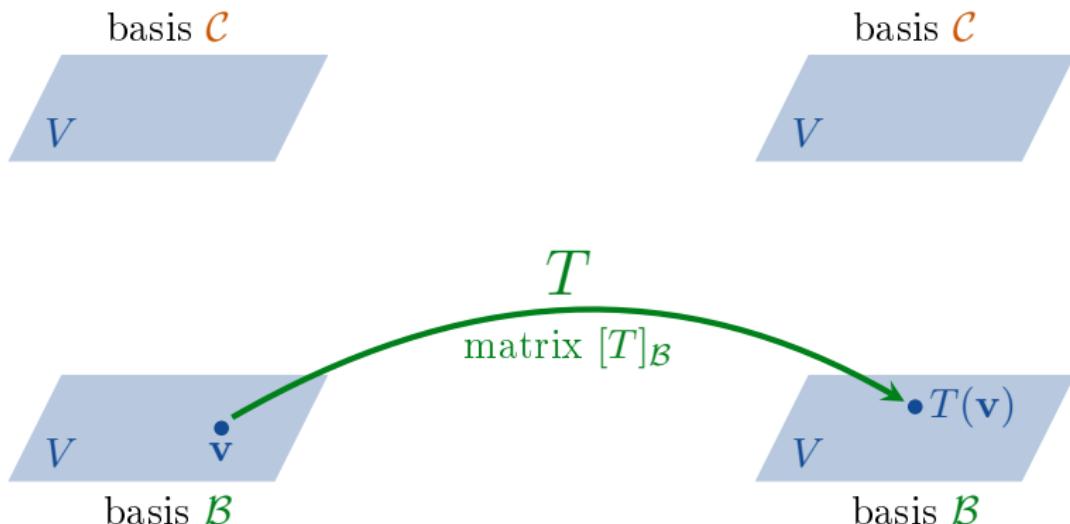


$V$

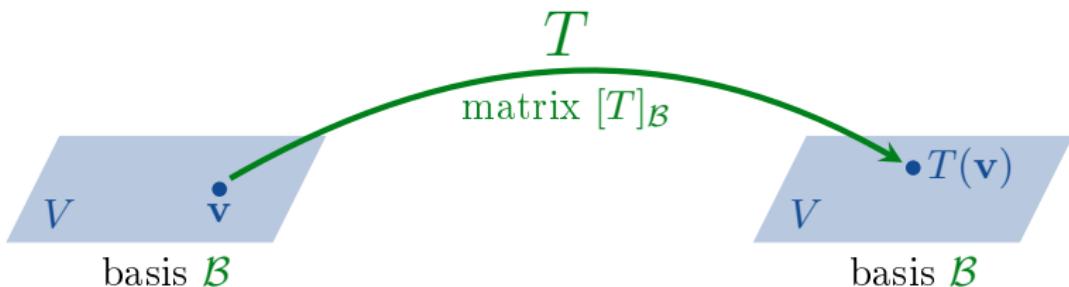
basis  $\mathcal{B}$



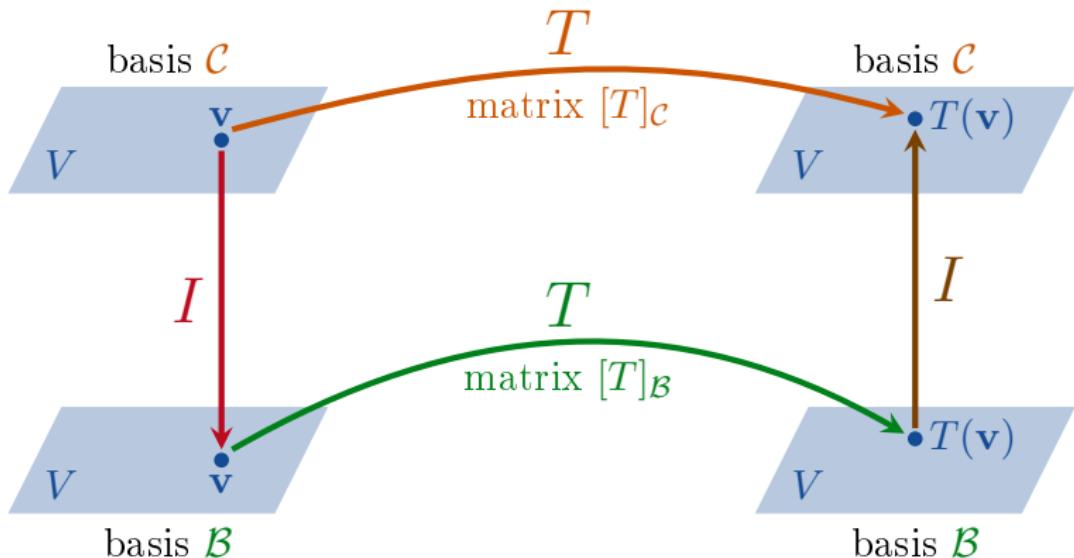
# Similarity



# Similarity

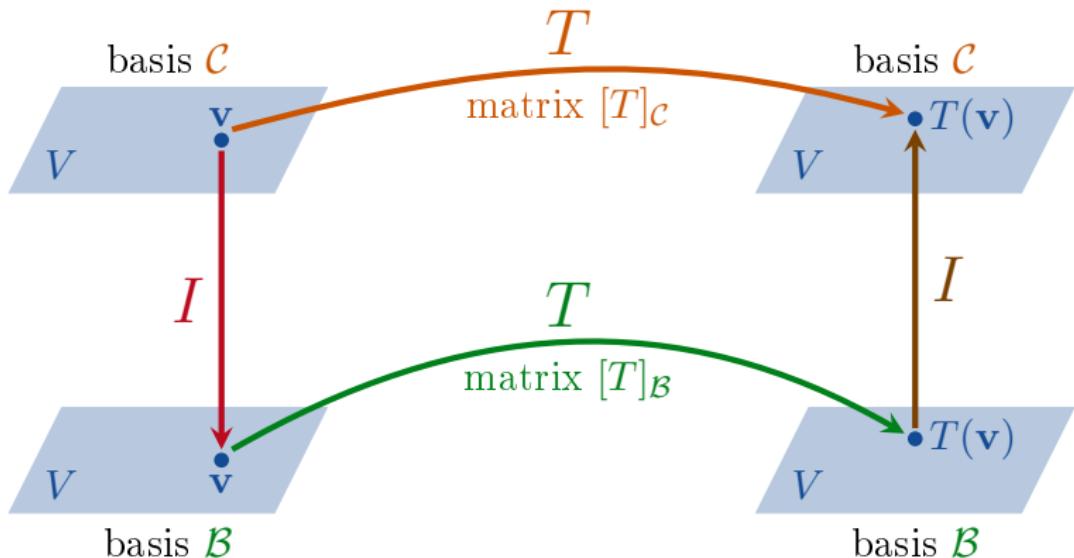


# Similarity



Note that  $\textcolor{brown}{T} = \textcolor{red}{I} \circ \textcolor{green}{T} \circ \textcolor{red}{I}$ .

# Similarity



Note that  $T = I \circ T \circ I$ . It follows that

$$[T]_{\mathcal{C}} = [I \circ T \circ I]_{\mathcal{C}} = [I]_{\mathcal{C}, \mathcal{B}} [T]_{\mathcal{B}} [I]_{\mathcal{B}, \mathcal{C}}.$$

# Similarity

$$[T]_{\mathcal{C}} = [I \circ T \circ I]_{\mathcal{C}} = [I]_{\mathcal{C}, \mathcal{B}} [T]_{\mathcal{B}} [I]_{\mathcal{B}, \mathcal{C}}$$

Since

$$\underset{\mathcal{C} \leftarrow \mathcal{B}}{P} = [I]_{\mathcal{C}, \mathcal{B}} \quad \text{and} \quad \underset{\mathcal{B} \leftarrow \mathcal{C}}{P} = [I]_{\mathcal{B}, \mathcal{C}}$$

we have

## Theorem

*Let  $T : V \rightarrow V$  be a linear operator on a finite-dimensional vector space  $V$ , and let  $\mathcal{B}$  and  $\mathcal{C}$  be bases for  $V$ . Then*

$$[T]_{\mathcal{C}} = P^{-1} [T]_{\mathcal{B}} P$$

*where  $P = \underset{\mathcal{B} \leftarrow \mathcal{C}}{P}$  and  $P^{-1} = \underset{\mathcal{C} \leftarrow \mathcal{B}}{P}$ .*

## Remark

Therefore, two matrices representing the same linear operator, must be similar.

# Similarity

## Remark

Therefore, two matrices representing the same linear operator, must be similar.

## Theorem

*Two matrices  $A$  and  $B$*

*represent the same  
linear operator*       $\iff$        *$A$  and  $B$  are similar.*



# Complex Numbers

## The Natural Numbers

The set

$$\mathbb{N} = \{1, 2, 3, 4, 5, 6, \dots\}$$

is called the set of *natural numbers*. These are the first numbers that children learn. For example

$2 \in \mathbb{N}$  means “2 is a natural number”

$7 \in \mathbb{N}$  means “7 is a natural number”

$\frac{1}{2} \notin \mathbb{N}$  means “ $\frac{1}{2}$  is **not** a natural number”

$0 \notin \mathbb{N}$  means “0 is **not** a natural number”

$-5 \notin \mathbb{N}$  means “−5 is **not** a natural number”

# Complex Numbers



In the natural numbers, we can do “+” and “ $\times$ ”

$$2 + 7 = 9 \in \mathbb{N}, \quad 2 \times 7 = 14 \in \mathbb{N}.$$

However we can not do “−” because

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*So we invent new numbers!*

## The Integers

The set

$$\mathbb{Z} = \{\dots, -4, -3, -2, -1, 0, 1, 2, 3, 4, \dots\}$$

is called the set of *integers*. We use a  $\mathbb{Z}$  for the German word 'zahlen' (numbers).

## The Integers

the natural  
numbers “exist”  
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In  $\mathbb{Z}$ , we can do “+”, “−” and “ $\times$ ” but we can not do “ $\div$ ”. For example  $3 \in \mathbb{Z}$ ,  $4 \in \mathbb{Z}$ ,  $-5 \in \mathbb{Z}$  and

$$3 + 4 \in \mathbb{Z}, \quad 3 - 4 \in \mathbb{Z}, \quad 3 \times 4 \in \mathbb{Z}, \quad 3 \div 4 \notin \mathbb{Z},$$

$$3 + (-5) \in \mathbb{Z}, \quad 3 - (-5) \in \mathbb{Z}, \quad 3 \times (-5) \in \mathbb{Z}, \quad 3 \div (-5) \notin \mathbb{Z}.$$

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## The Rational Numbers

The set

$$\mathbb{Q} = \{\text{all fractions}\} = \left\{ \frac{a}{b} \mid a, b \in \mathbb{Z} \text{ and } b \neq 0 \right\}$$

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$$0 = \frac{0}{1} \in \mathbb{Q}$$

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$$\sqrt{2} \notin \mathbb{Q}$$

$$\frac{3}{4} \in \mathbb{Q}$$

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In  $\mathbb{Q}$  we can do “+”, “-”, “ $\times$ ” and “ $\div$ (by a number  $\neq 0$ )”.

# Complex Numbers



Are we happy now?

# Complex Numbers



Are we happy now?

No!



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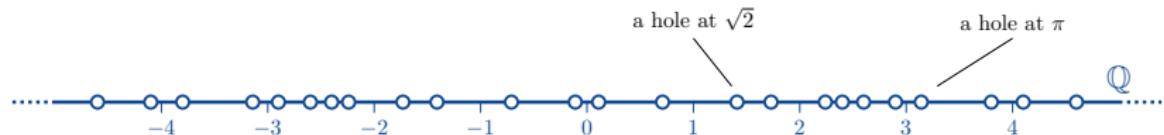
Why?

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## Why?

Because if we draw all the rational numbers in a line, then the line has lots of holes in it. In fact,  $\mathbb{Q}$  has  $\infty$  many holes in it.

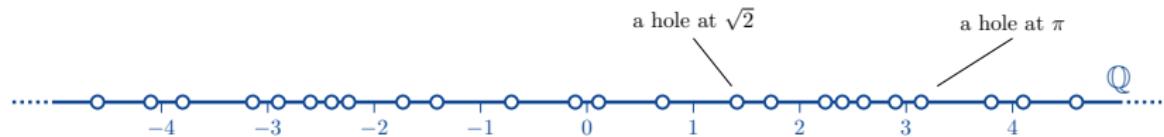


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## The Real Numbers

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$$\mathbb{R} = \{\text{all numbers which can be written as a decimal}\}$$

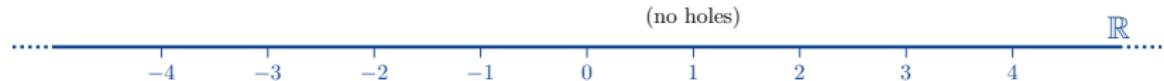
is called the set of *real numbers*. For example

$$\begin{array}{ll} 0 = 0.0 \in \mathbb{R} & \frac{100}{13} = 7.692307\dots \in \mathbb{R} \\ \frac{23}{99} = 0.232323\dots \in \mathbb{R} & \sqrt{2} = 1.414213\dots \in \mathbb{R} \\ \frac{3}{4} = 0.75 \in \mathbb{R} & \frac{123}{999} = 0.123123\dots \in \mathbb{R} \\ \pi = 3.141592\dots \in \mathbb{R} & \frac{12345}{100000} = 0.12345 \in \mathbb{R}. \end{array}$$

# Complex Numbers



The real numbers are complete – this means that if we draw all the real numbers in a line, then there are no holes in the line.



# Complex Numbers



Are we happy now?

# Complex Numbers



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Are we happy now?

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Why?

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Why?

Because not every polynomial has a root in  $\mathbb{R}$ . For example,  
there is no solution to

$$x^2 + 1 = 0$$

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We invent a new number called  $i$  which we say is the solution to

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Or

$$i = \sqrt{-1}.$$

# Complex Numbers



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$i$  is also useful to us.

# Complex Numbers



Now that we have  $i$ , we can start working with numbers like  $3 + 4i$ ,  $2 + 7i$ ,  $-12i$ , etc.

# Complex Numbers



Now that we have  $i$ , we can start working with numbers like  $3 + 4i$ ,  $2 + 7i$ ,  $-12i$ , etc.

## Definition

A *complex number* is any number that can be written as

$$a + bi$$

where  $i$  solves  $i^2 + 1 = 0$  and  $a, b \in \mathbb{R}$ .

## Definition

The set of all complex numbers is denoted by  $\mathbb{C}$ .

# Complex Numbers



$$z = \cancel{a} + \cancel{bi}$$

real part                      imaginary part

## Definition

$\operatorname{Re}(z) = a$  is called the *real* part of  $z$ , and  $\operatorname{Im}(z) = b$  is called the *imaginary* part of  $z$ .

$$z = \cancel{a} + \cancel{bi}$$

real part                      imaginary part

## Definition

$\operatorname{Re}(z) = a$  is called the *real* part of  $z$ , and  $\operatorname{Im}(z) = b$  is called the *imaginary* part of  $z$ .

## Example

- $-2 + 7i$  has real part  $-2$  and imaginary part  $7$ ;
- $7i$  has real part  $0$  and imaginary part  $7$ ; and
- $-2$  has real part  $-2$  and imaginary part  $0$ .

# Complex Numbers

## Definition

An *imaginary number* is a complex number  $a + bi$  where  $a = 0$ .

A *real number* is a complex number  $a + bi$  where  $b = 0$ .

$$4 + 2i \quad -3 - \sqrt{5}i \quad 1 - i$$

$$3 \quad \sqrt{5} \quad -12.2$$

$$-12.2i \quad \sqrt{2}i \quad 7i$$

# Complex Numbers



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complex numbers

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real numbers

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imaginary numbers

$-12.2i$        $\sqrt{2}i$        $7i$

# Complex Numbers

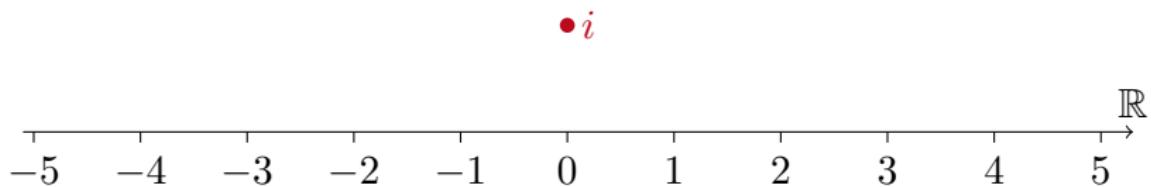


## Remark

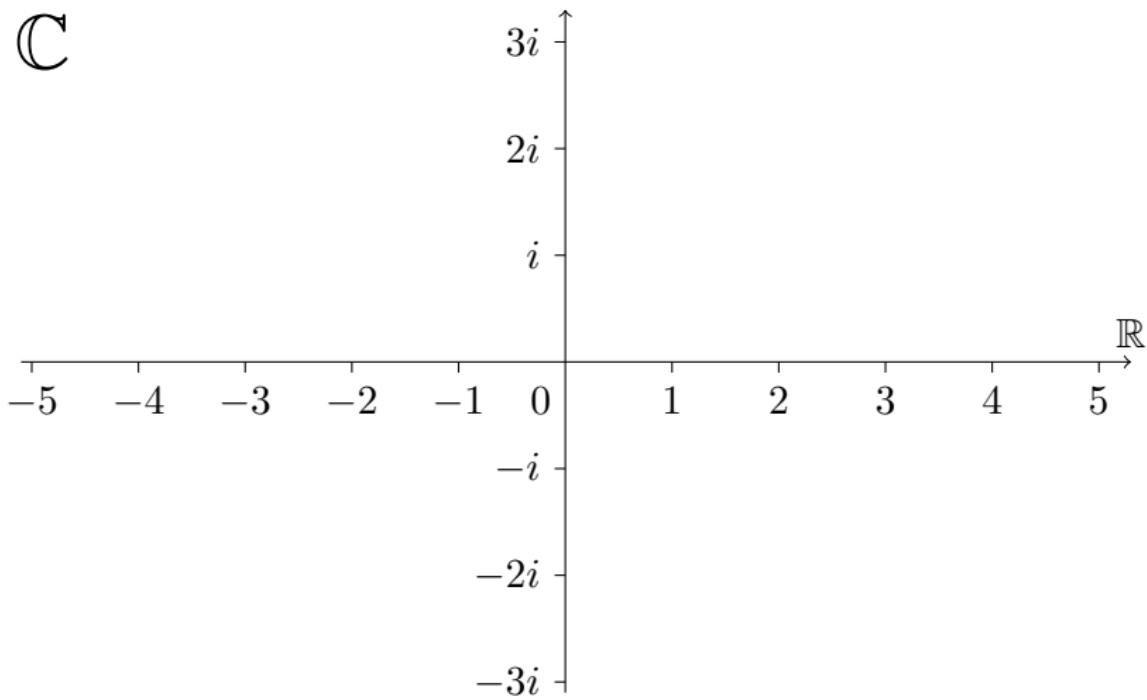
The names “real number” and “imaginary number” come from the 17th century.

These are really bad names because real numbers are just as imaginary, or made up, as imaginary numbers.

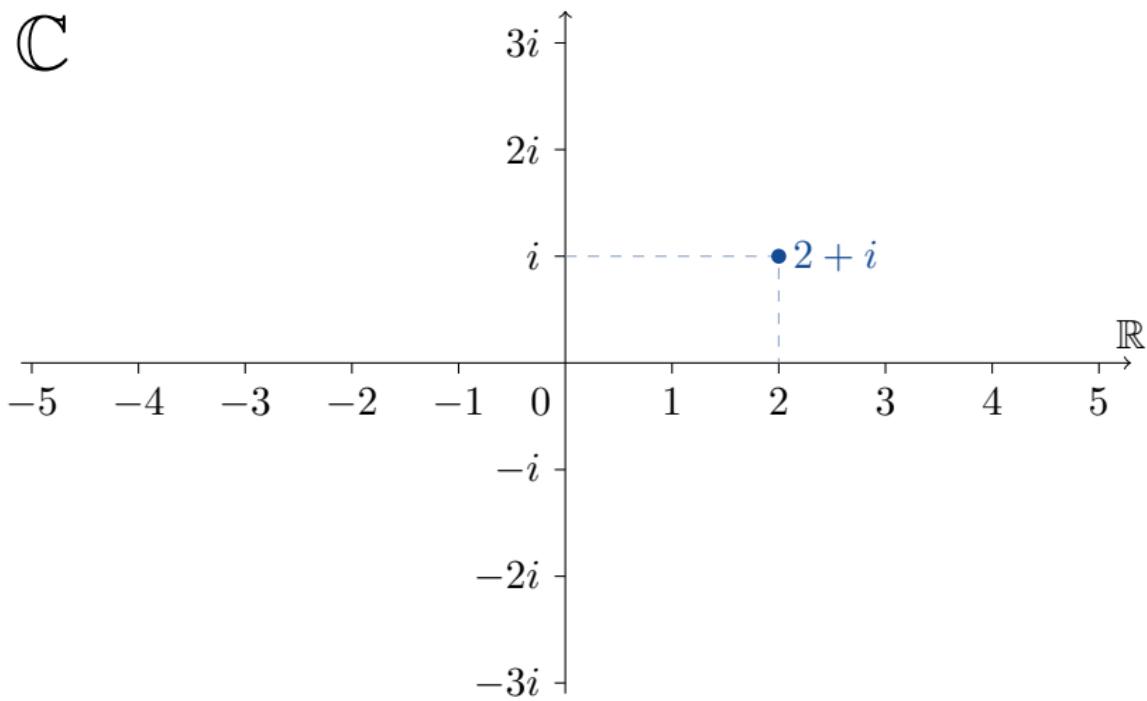
# Complex Numbers



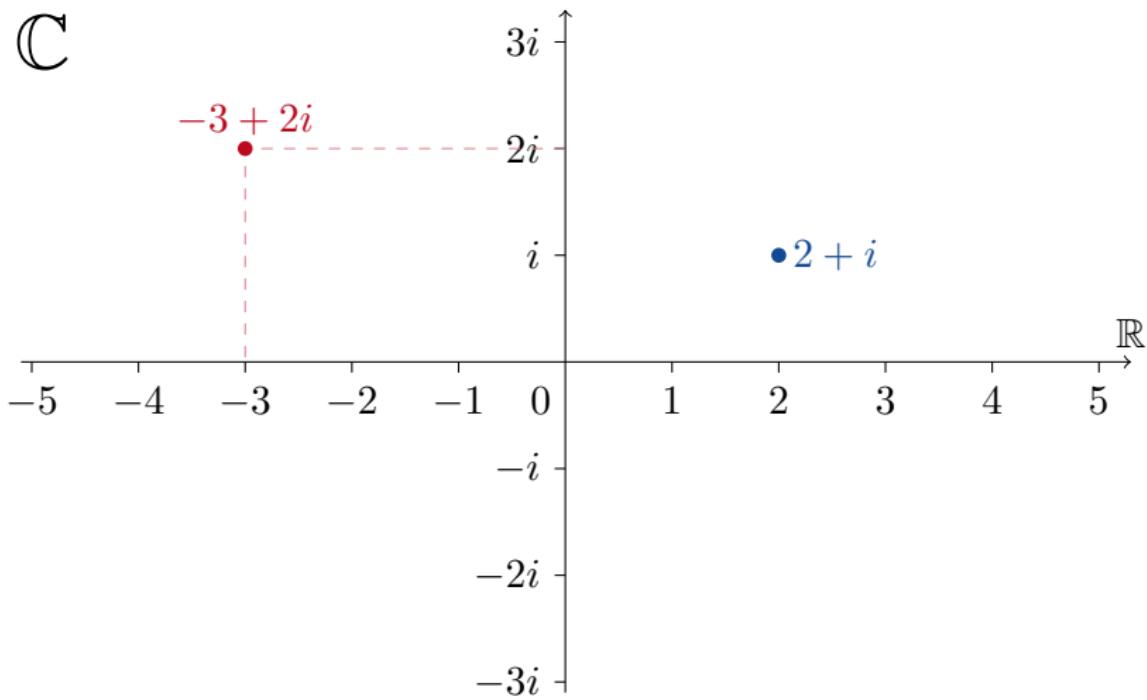
# Complex Numbers



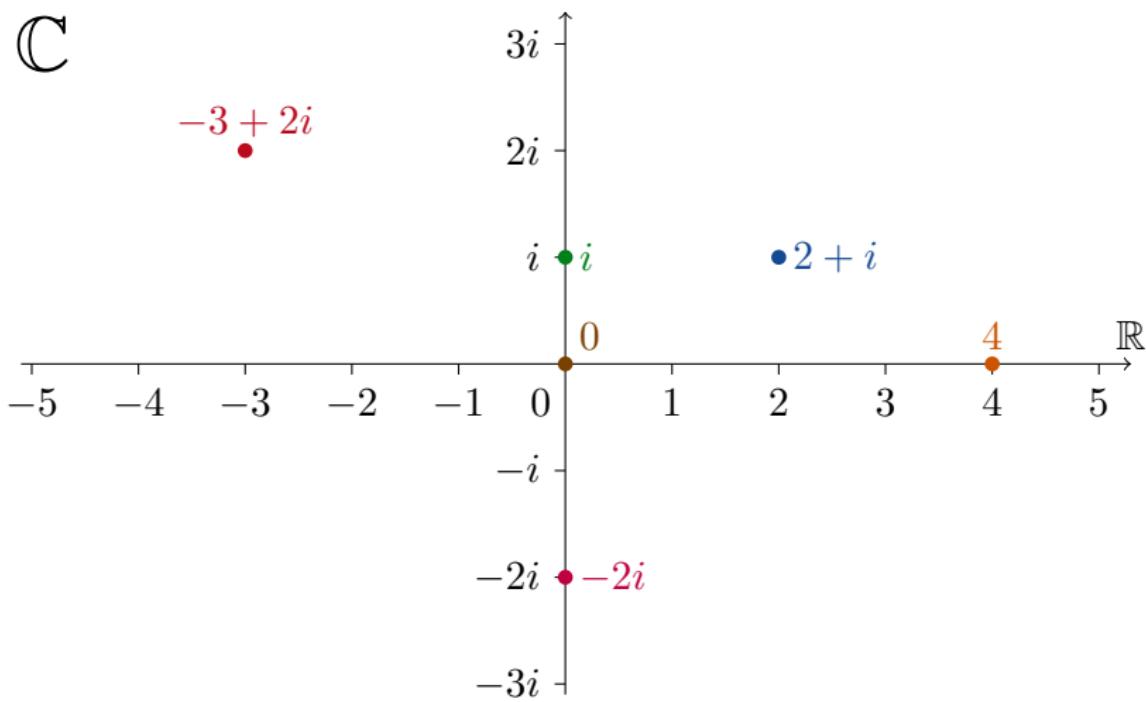
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$$\begin{aligned}(2 - i)(3 + 4i) &= (2)(3) + (2)(4i) + (-i)(3) + (-i)(4i) \\&= 6 + 8i - 3i - 4i^2 \\&= 6 + 8i - 3i - 4(-1) \\&= 6 + 5i + 4 \\&= 10 + 5i\end{aligned}$$

# Complex Numbers



$$\frac{3-i}{2+i} =$$

# Complex Numbers



$$\frac{3-i}{2+i} = \frac{3-i}{2+i} \cdot \frac{\textcolor{brown}{2-i}}{\textcolor{brown}{2-i}}$$

# Complex Numbers



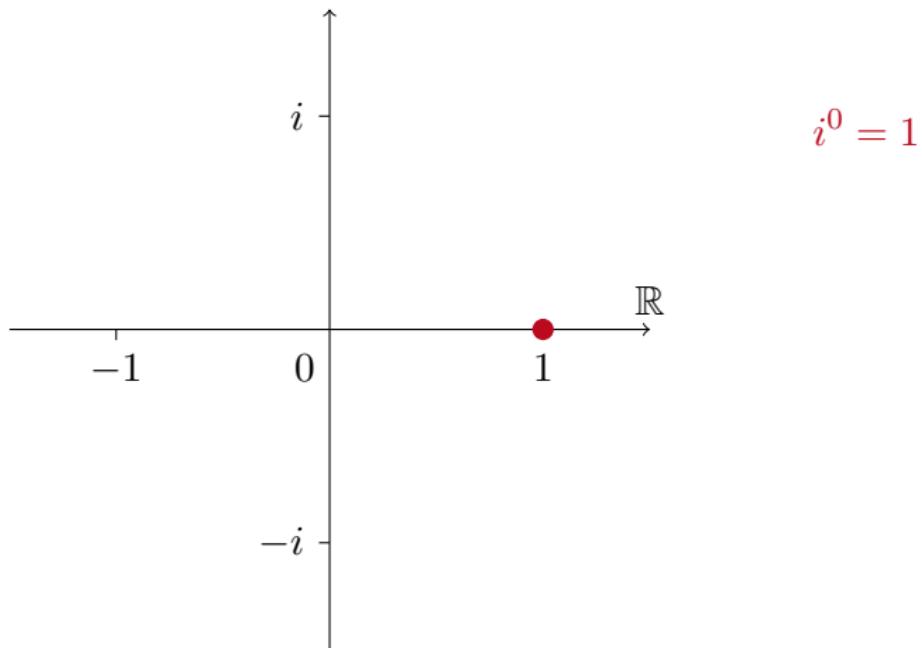
$$\frac{3-i}{2+i} = \frac{3-i}{2+i} \cdot \frac{\textcolor{brown}{2-i}}{\textcolor{brown}{2-i}} = \frac{6-3i-2i-1}{4-2i+2i+1}$$

# Complex Numbers

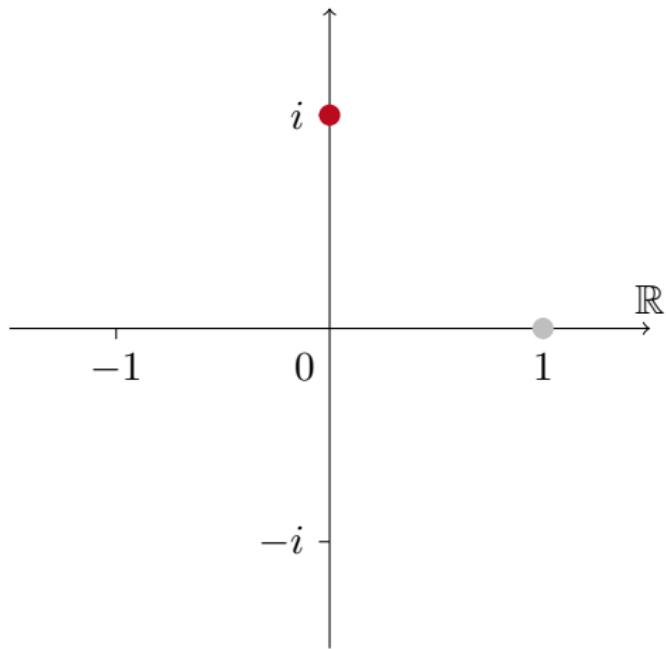


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# Complex Numbers



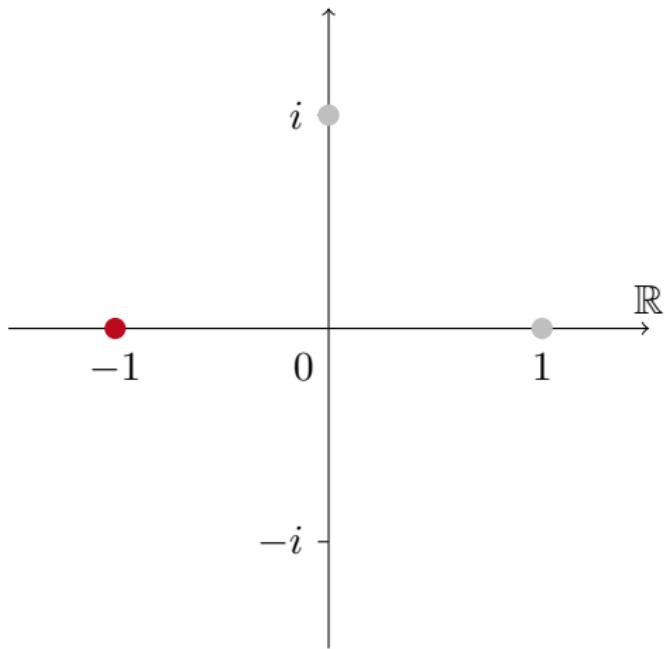
# Complex Numbers



$$i^0 = 1$$

$$i^1 = i$$

# Complex Numbers

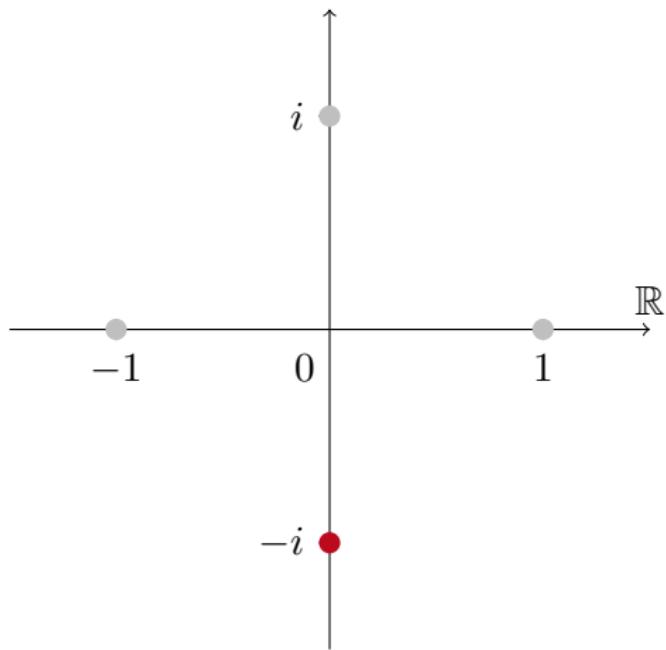


$$i^0 = 1$$

$$i^1 = i$$

$$i^2 = -1$$

# Complex Numbers



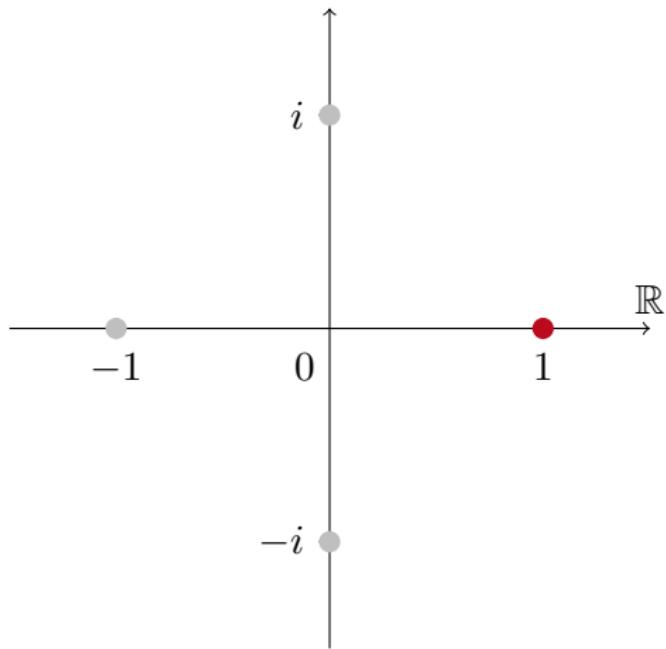
$$i^0 = 1$$

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$$i^3 = -i$$

# Complex Numbers



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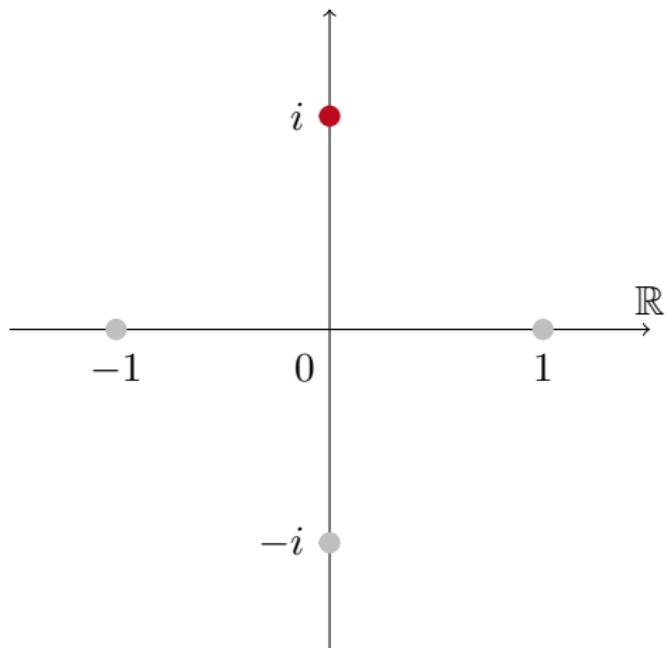
$$i^1 = i$$

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$$i^4 = 1$$

# Complex Numbers



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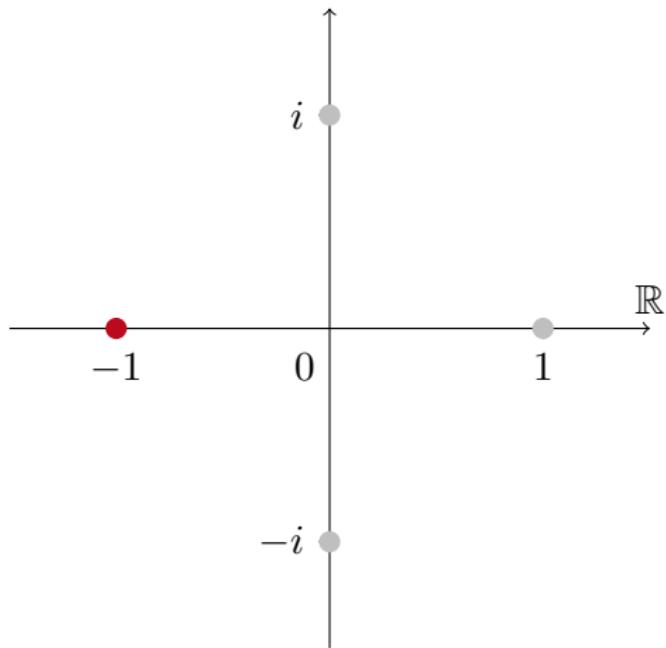
$$i^2 = -1$$

$$i^3 = -i$$

$$i^4 = 1$$

$$i^5 = i$$

# Complex Numbers



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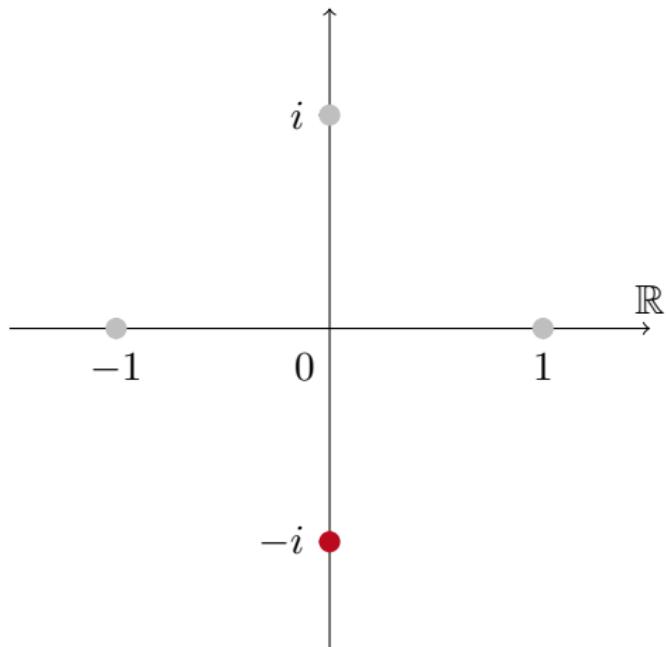
$$i^3 = -i$$

$$i^4 = 1$$

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$$i^6 = -1$$

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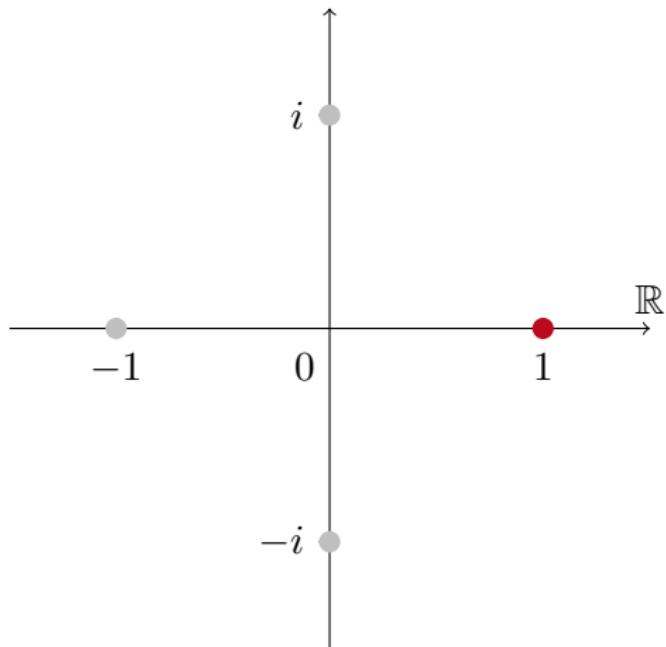
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$$i^7 = -i$$

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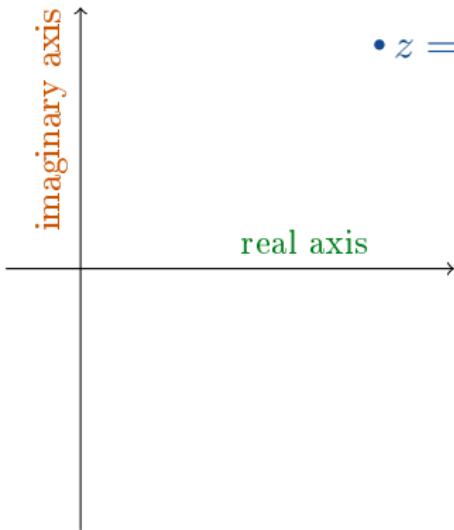
$$i^5 = i$$

$$i^6 = -1$$

$$i^7 = -i$$

$$i^8 = 1$$

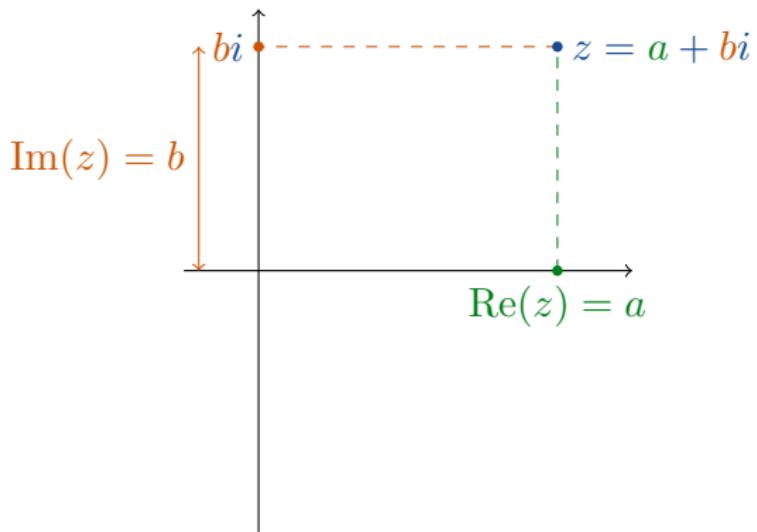
# Complex Numbers



- $z = a + bi$

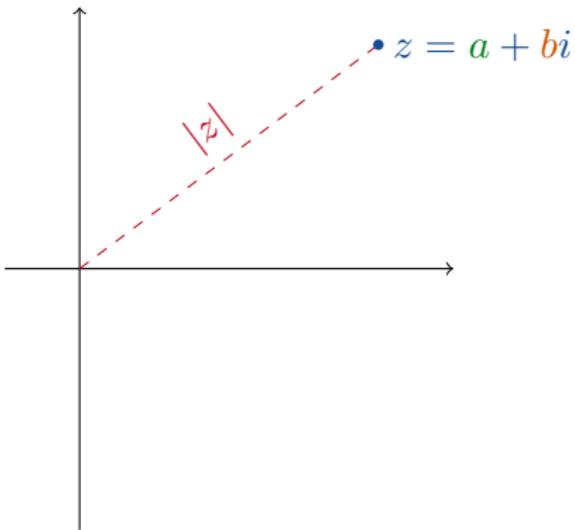
Definition

# Complex Numbers



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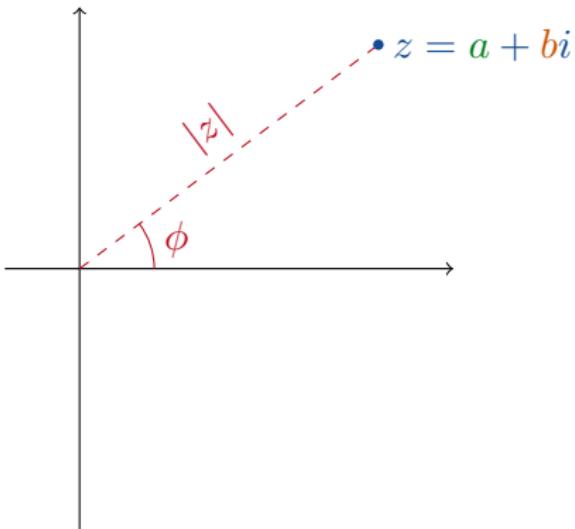
# Complex Numbers



## Definition

$|z| = \sqrt{a^2 + b^2}$  is called the *modulus* of  $z$ .

# Complex Numbers

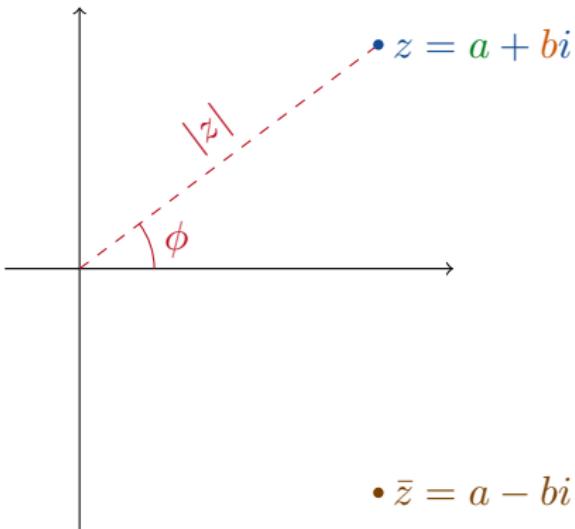


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$\bar{z} = a - bi$  is called the *complex conjugate* of  $z$ .

# Complex Numbers



## Remark

Note that

$$z\bar{z} = (a + bi)(a - bi) = a^2 + b^2 = |z|^2.$$

# Complex Numbers

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## Remark

$$z = \bar{z} \iff z \in \mathbb{R}$$

because

$$a + bi = a - bi \iff b = 0.$$

# Complex Numbers



Theorem

$$\overline{zw} = \bar{z}\bar{w}.$$

(you prove)

## Roots of polynomials

equation	solution	set
$x - 5 = 0$	$x = 5$	$\mathbb{N}$

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**Question:** Are there any more numbers after  $\mathbb{C}$ ?

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**Question:** Are there any more numbers after  $\mathbb{C}$ ?

**Answer:** No.

# Complex Numbers



Theorem (The Fundamental Theorem of Algebra)

*Any polynomial with complex coefficients will only have complex roots.*

In other words: Using the five operations  $+$ ,  $-$ ,  $\times$ ,  $\div$  and  $a^b$ , we can not make any more numbers.  $\mathbb{C}$  is the end of the line.



# Next Time

- Eigenvalues and Eigenvectors
- Diagonalisation
- Complex Vector Spaces