$$\mathbf{x}' = Ax$$

For repeated eigenvalues (with only one linearly independent eigenvector), the key equations to remember are

$$\boxed{\mathbf{x}^{(2)}(t) = \boldsymbol{\xi} t e^{rt} + \boldsymbol{\eta} e^{rt}} \qquad \text{and} \qquad \boxed{(A - rI)\boldsymbol{\eta} = \boldsymbol{\xi}}.$$

Example. Solve

$$\mathbf{x}' = \begin{pmatrix} -\frac{5}{2} & \frac{3}{2} \\ -\frac{3}{2} & \frac{1}{2} \end{pmatrix} \mathbf{x}, \qquad \mathbf{x}(0) = \begin{pmatrix} 3 \\ -1 \end{pmatrix}.$$

solution: The only eigenvalue of the matrix is $r = -1$. The corresponding eigenvector is $\boldsymbol{\xi} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$. Therefore one solution of the linear system is

$$\mathbf{x}^{(1)}(t) = \boldsymbol{\xi} e^{rt} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-t}.$$

We need to find a second, linearly independent solution. We will consider the ansatz

$$\mathbf{x}^{(2)}(t) = \boldsymbol{\xi} t e^{-t} + \boldsymbol{\eta} e^{-t}$$

where $\boldsymbol{\xi} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ as above and $\boldsymbol{\eta}$ solves $(A - rI)\boldsymbol{\eta} = \boldsymbol{\xi}$. Solving the latter equation,

$$\begin{aligned} (A - rI)\boldsymbol{\eta} &= \boldsymbol{\xi} \\ \begin{pmatrix} -\frac{3}{2} & \frac{3}{2} \\ -\frac{3}{2} & \frac{3}{2} \end{pmatrix} \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} &= \begin{pmatrix} 1 \\ 1 \end{pmatrix} \\ -\frac{3}{2}\eta_1 + \frac{3}{2}\eta_2 &= 1 \\ -\eta_1 + \eta_2 &= \frac{2}{3} \\ \eta_2 &= \eta_1 + \frac{2}{3} \end{aligned}$$

we can see that $\boldsymbol{\eta} = \begin{pmatrix} k \\ k + \frac{2}{3} \end{pmatrix} = k\boldsymbol{\xi} + \begin{pmatrix} 0 \\ \frac{2}{3} \end{pmatrix}$. Because we already have $\mathbf{x}^{(1)}(t) = \boldsymbol{\xi} e^{-t}$, we can choose $k = 0$. Therefore $\boldsymbol{\eta} = \begin{pmatrix} 0 \\ \frac{2}{3} \end{pmatrix}$ and

$$\mathbf{x}^{(2)}(t) = \begin{pmatrix} 1 \\ 1 \end{pmatrix} t e^{-t} + \begin{pmatrix} 0 \\ \frac{2}{3} \end{pmatrix} e^{-t}.$$

Hence the general solution to the linear system is

$$\mathbf{x}(t) = c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-t} + c_2 \left[\begin{pmatrix} 1 \\ 1 \end{pmatrix} t e^{-t} + \begin{pmatrix} 0 \\ \frac{2}{3} \end{pmatrix} e^{-t} \right].$$

The initial condition gives

$$\begin{pmatrix} 3 \\ -1 \end{pmatrix} = \mathbf{x}(0) = c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} 0 \\ \frac{2}{3} \end{pmatrix}$$

which implies that $c_1 = 3$ and $c_2 = -6$. Therefore the solution to the IVP is

$$\mathbf{x}(t) = 3 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-t} - 6 \left[\begin{pmatrix} 1 \\ 1 \end{pmatrix} t e^{-t} + \begin{pmatrix} 0 \\ \frac{2}{3} \end{pmatrix} e^{-t} \right] = \begin{pmatrix} 3 \\ -1 \end{pmatrix} e^{-t} - 6 \begin{pmatrix} 1 \\ 1 \end{pmatrix} t e^{-t}.$$

Example. Solve

$$\mathbf{x}' = \begin{pmatrix} 1 & -4 \\ 4 & -7 \end{pmatrix} \mathbf{x}, \qquad \mathbf{x}(0) = \begin{pmatrix} 3 \\ 2 \end{pmatrix}.$$

solution: The only eigenvalue of the matrix is $r = -3$. The corresponding eigenvector is $\boldsymbol{\xi} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$. Therefore one solution of the linear system is

$$\mathbf{x}^{(1)}(t) = \boldsymbol{\xi} e^{rt} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-3t}.$$

We need to find a second, linearly independent solution. We will consider the ansatz

$$\mathbf{x}^{(2)}(t) = \boldsymbol{\xi} t e^{-3t} + \boldsymbol{\eta} e^{-3t}$$

where $\boldsymbol{\xi} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ as above and $\boldsymbol{\eta}$ solves $(A - rI)\boldsymbol{\eta} = \boldsymbol{\xi}$. Solving the latter equation,

$$\begin{aligned} (A - rI)\boldsymbol{\eta} &= \boldsymbol{\xi} \\ \begin{pmatrix} 4 & -4 \\ 4 & -4 \end{pmatrix} \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} &= \begin{pmatrix} 1 \\ 1 \end{pmatrix} \\ 4\eta_1 - 4\eta_2 &= 1 \\ -\eta_1 + \eta_2 &= -\frac{1}{4} \\ \eta_2 &= \eta_1 - \frac{1}{4} \end{aligned}$$

we can see that $\boldsymbol{\eta} = \begin{pmatrix} k \\ k - \frac{1}{4} \end{pmatrix} = k\boldsymbol{\xi} - \begin{pmatrix} 0 \\ \frac{1}{4} \end{pmatrix}$. Because we already have $\mathbf{x}^{(1)}(t) = \boldsymbol{\xi} e^{-3t}$, we can choose $k = 0$. Therefore $\boldsymbol{\eta} = \begin{pmatrix} 0 \\ -\frac{1}{4} \end{pmatrix}$ and

$$\mathbf{x}^{(2)}(t) = \begin{pmatrix} 1 \\ 1 \end{pmatrix} t e^{-t} + \begin{pmatrix} 0 \\ -\frac{1}{4} \end{pmatrix} e^{-t}.$$

Hence

$$\mathbf{x}(t) = c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-3t} + c_2 \left[\begin{pmatrix} 1 \\ 1 \end{pmatrix} t e^{-3t} + \begin{pmatrix} 0 \\ -\frac{1}{4} \end{pmatrix} e^{-3t} \right].$$

The initial condition gives

$$\begin{pmatrix} 3 \\ 2 \end{pmatrix} = \mathbf{x}(0) = c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} 0 \\ -\frac{1}{4} \end{pmatrix}$$

which implies that $c_1 = 3$ and $c_2 = 4$. Therefore the solution is

$$\mathbf{x}(t) = 3 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-3t} + 4 \left[\begin{pmatrix} 1 \\ 1 \end{pmatrix} t e^{-3t} + \begin{pmatrix} 0 \\ -\frac{1}{4} \end{pmatrix} e^{-3t} \right] = \begin{pmatrix} 3 \\ 2 \end{pmatrix} e^{-3t} + 4 \begin{pmatrix} 1 \\ 1 \end{pmatrix} t e^{-3t} = \begin{pmatrix} 3 + 4t \\ 2 + 4t \end{pmatrix} e^{-3t}.$$