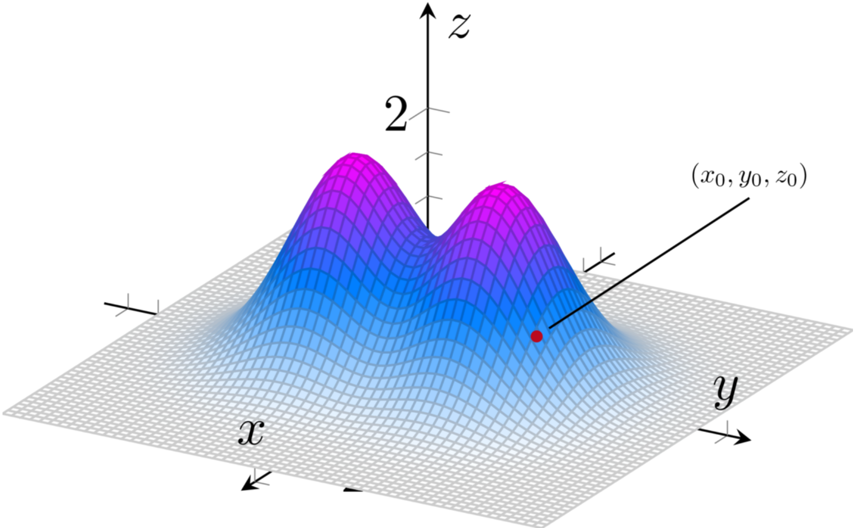
What is a Gradient Vector?



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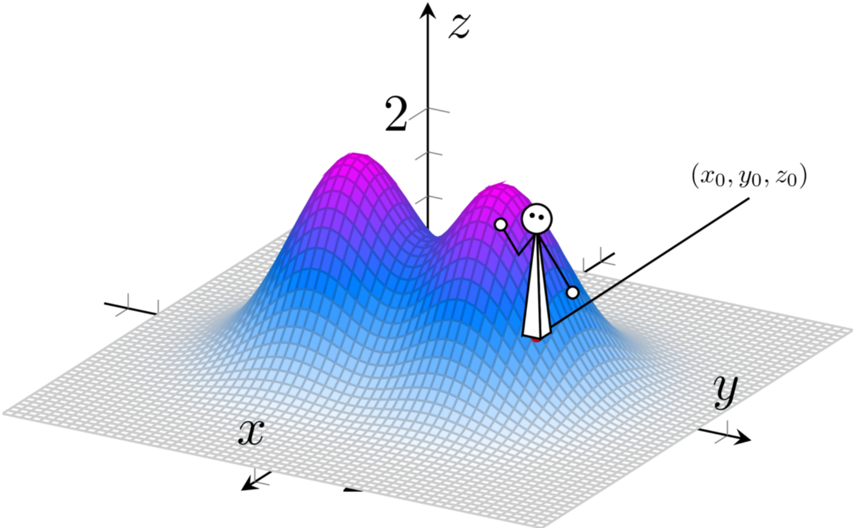

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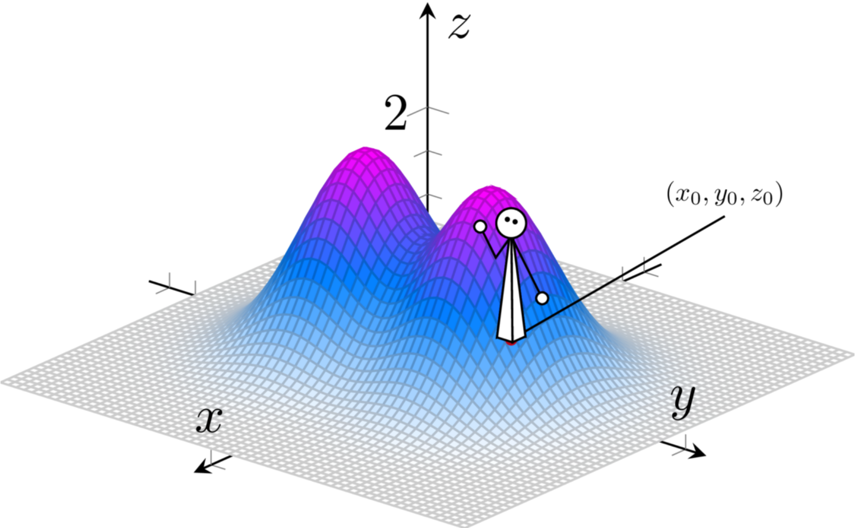

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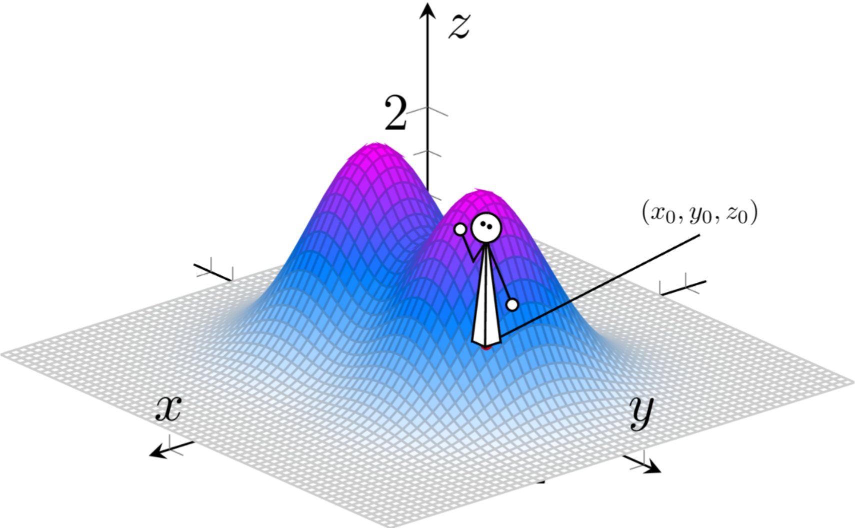

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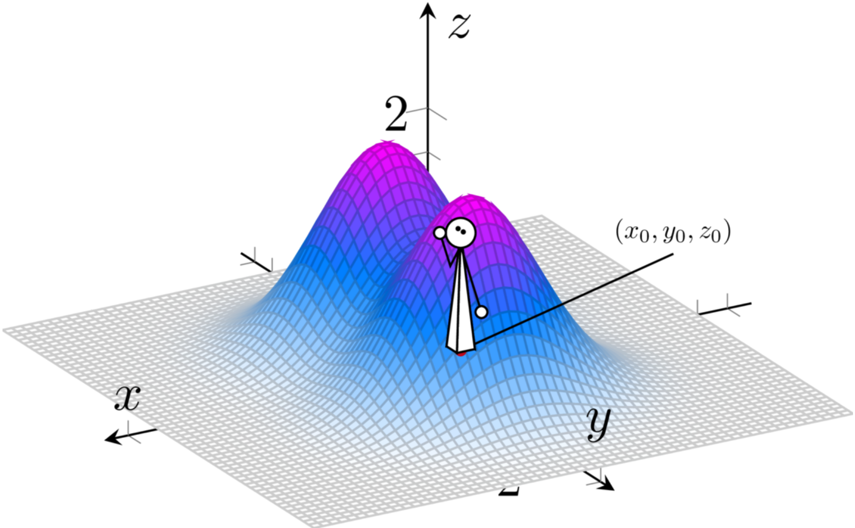

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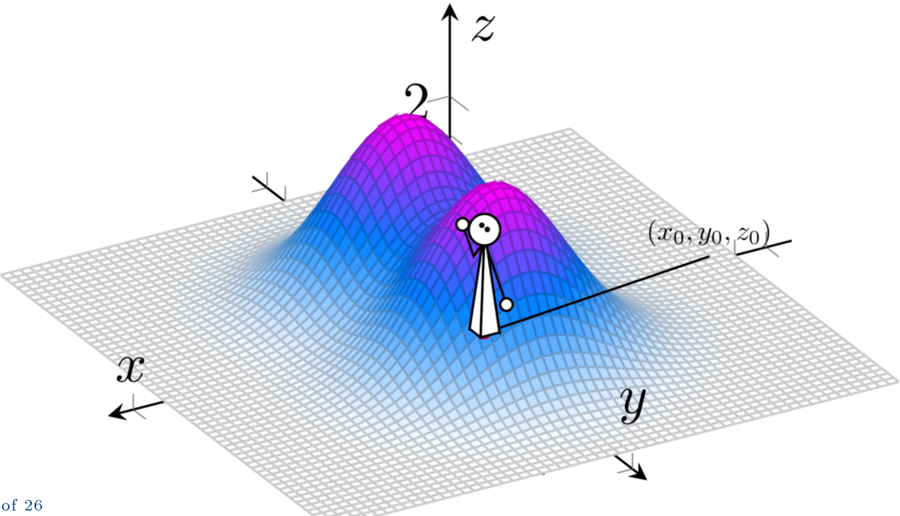

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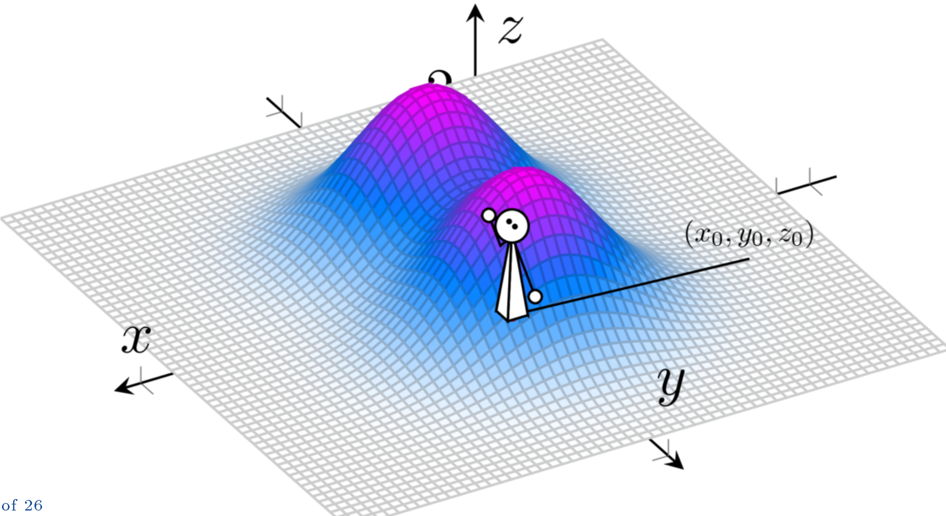

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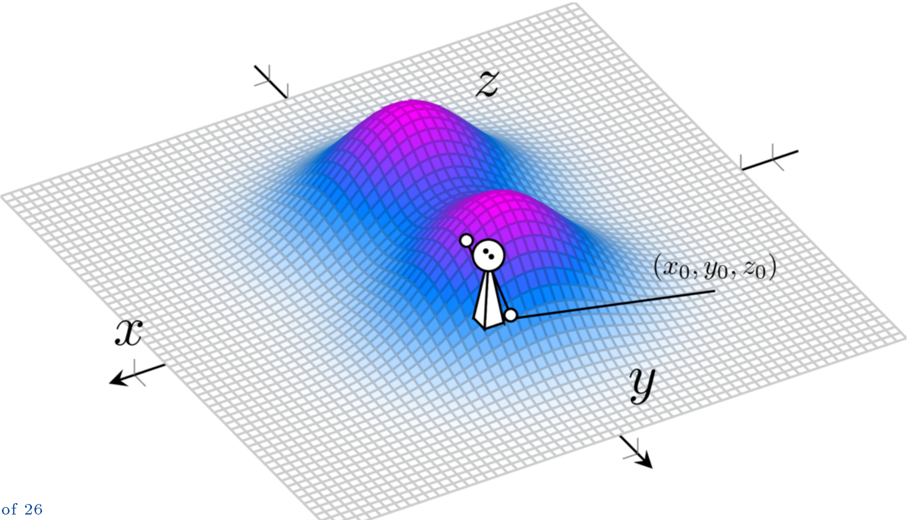

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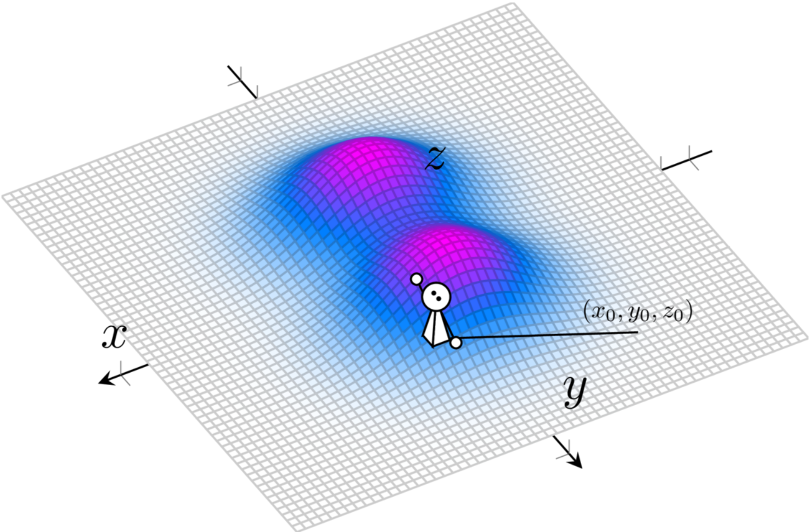

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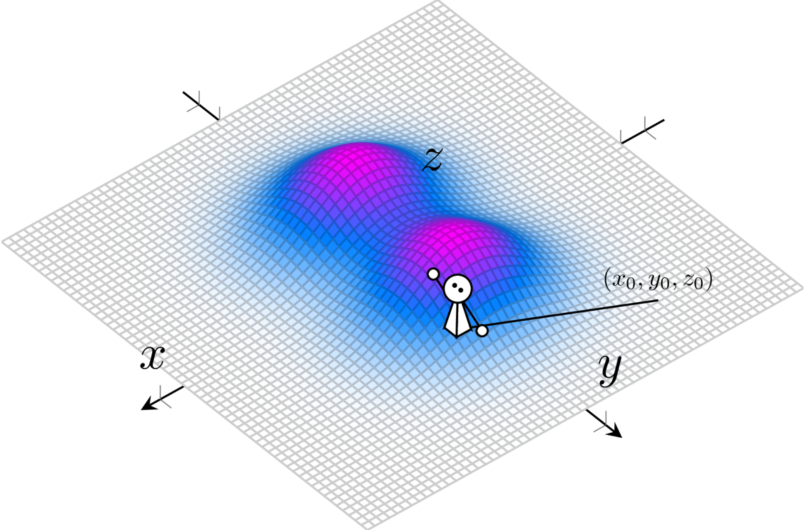

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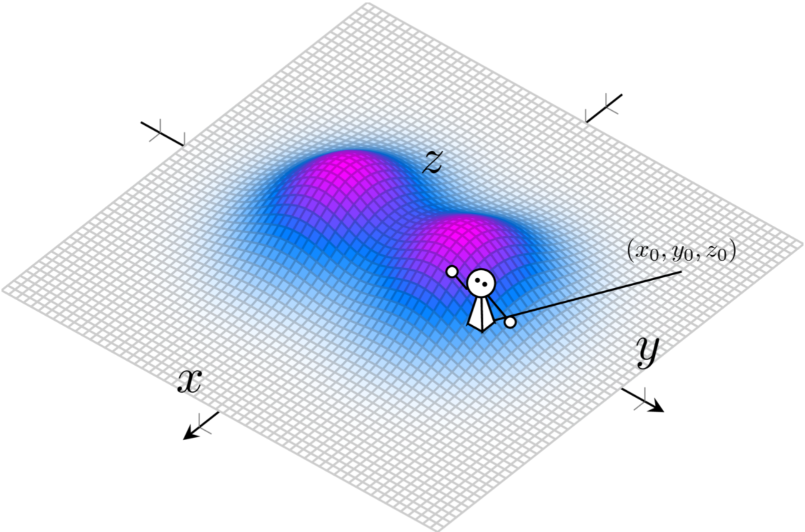

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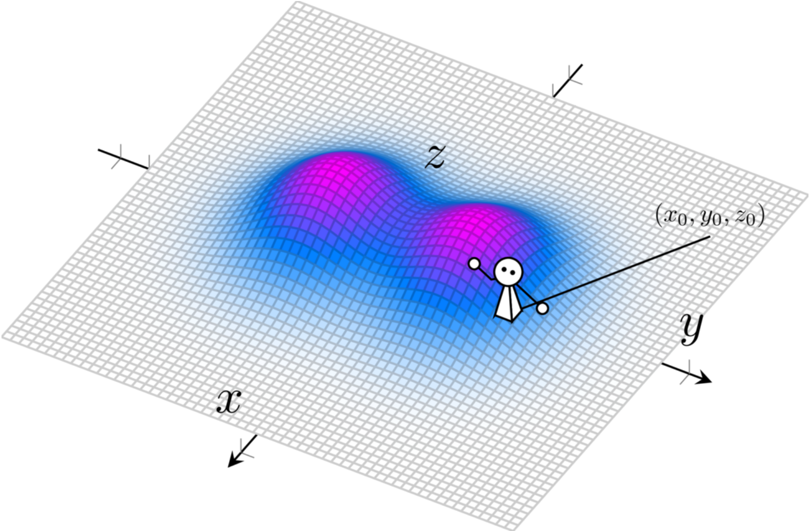

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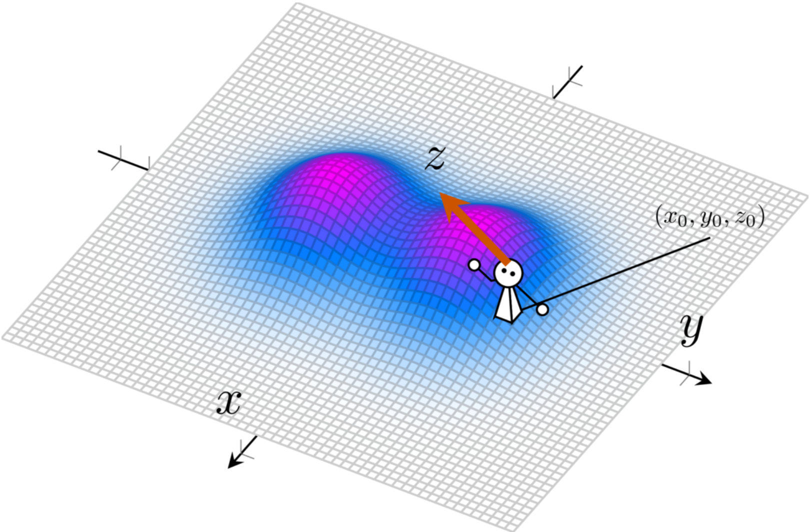

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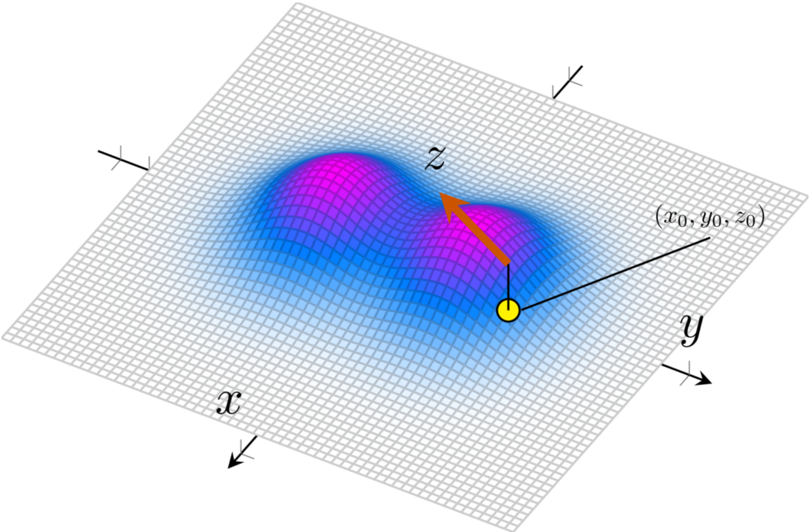

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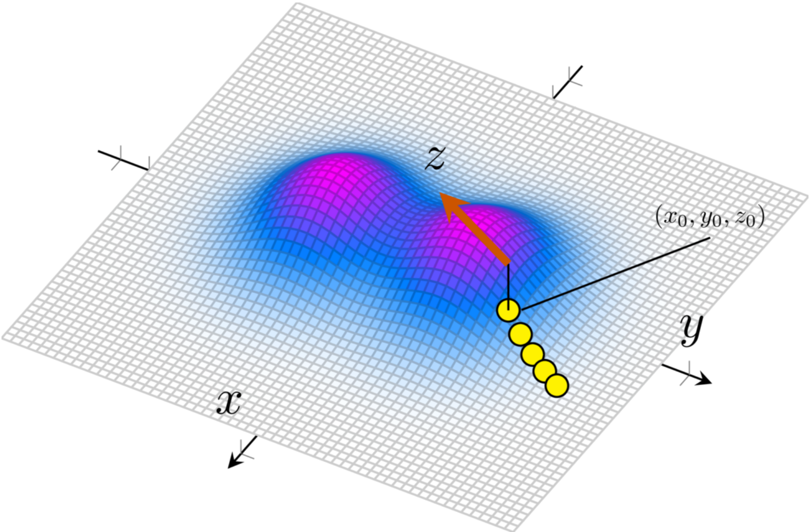

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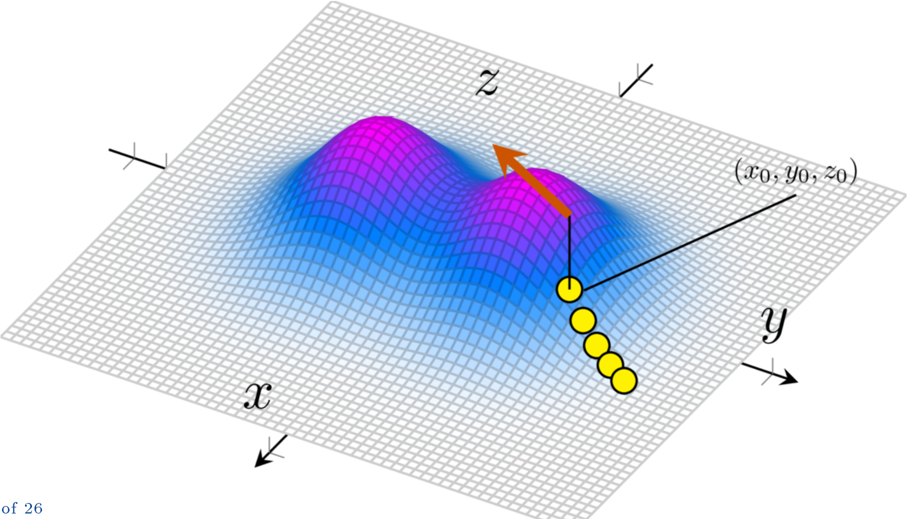

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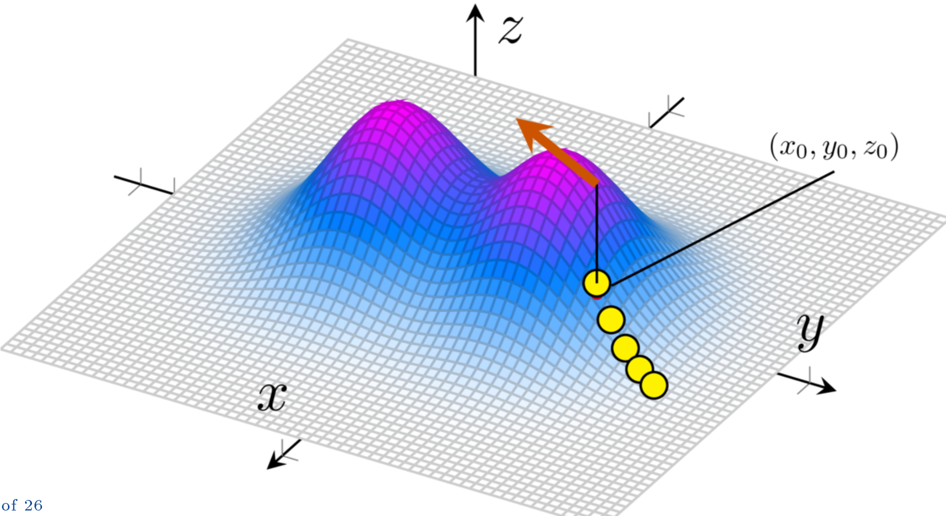

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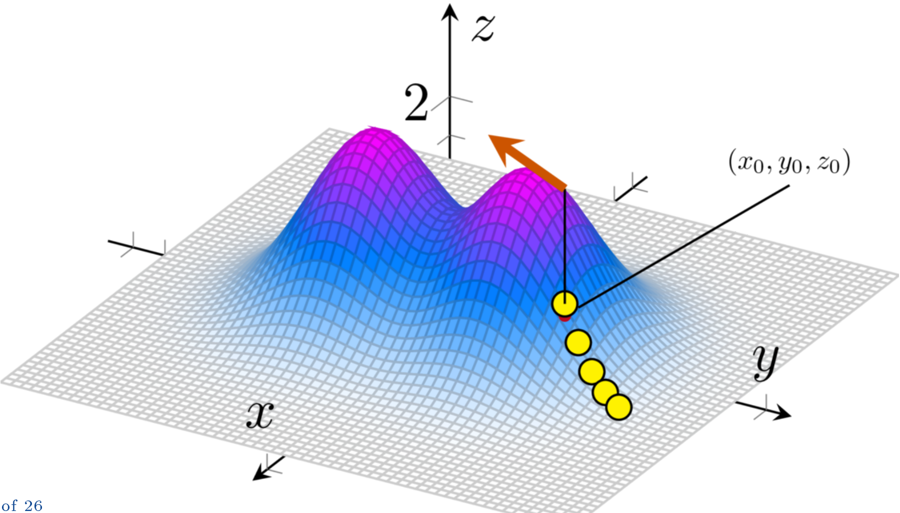

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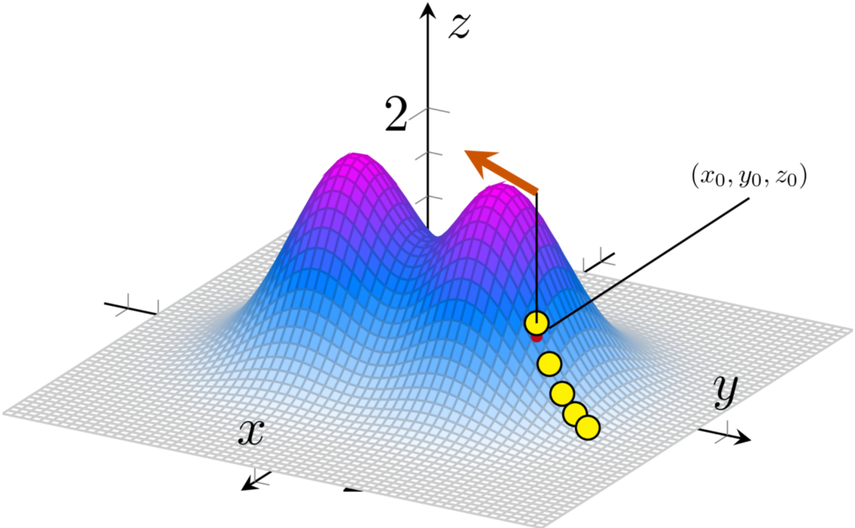

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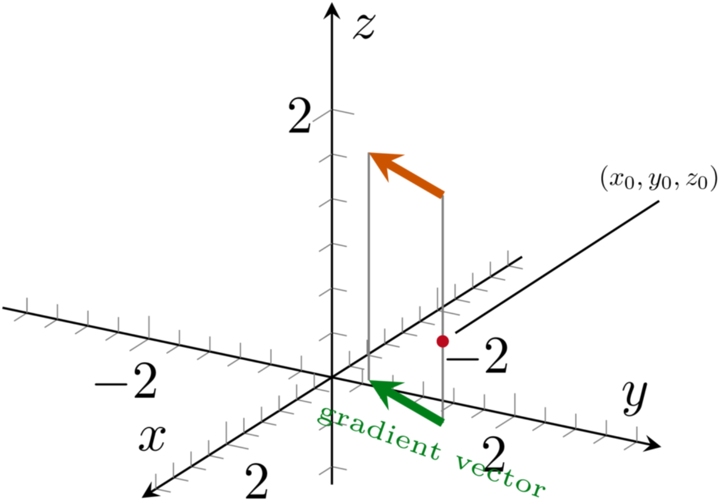

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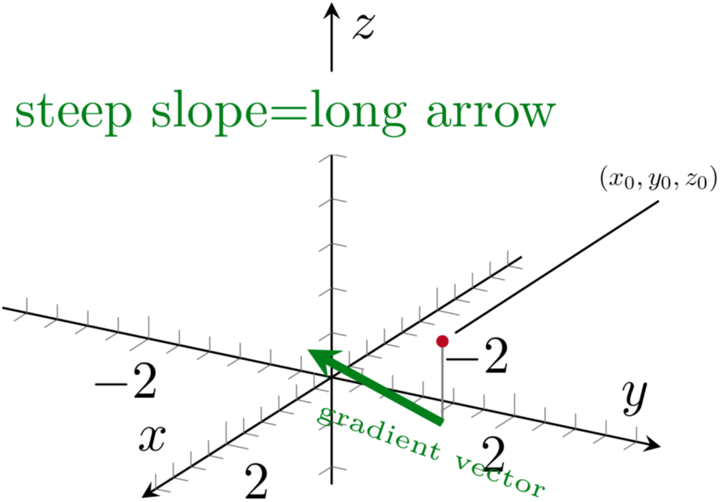

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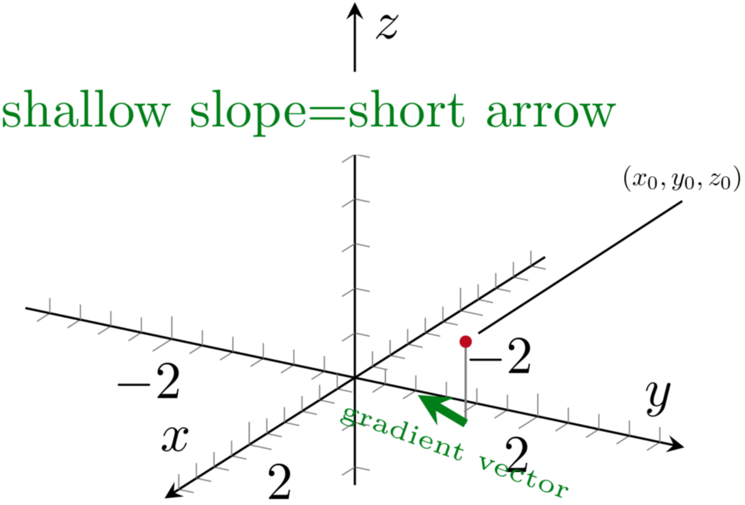

Definition

The *gradient vector* of $f(x,y)$ is the vector

$$\nabla f = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j}.$$

∇ is pronounced “nabla” or “del”.

Harps, p. 984.

$$\nabla f = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j}$$

Example

Find the gradient vector of $f(x, y) = xe^y + \cos(xy)$ at the point $(2, 0)$.

$$\nabla f = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j}$$

Example

Find the gradient vector of $f(x, y) = xe^y + \cos(xy)$ at the point $(2, 0)$.

We calculate that

$$f_x(2, 0) =$$

$$f_y(2, 0) =$$

and

$$\nabla f\Big|_{(2,0)} = f_x(2, 0)\mathbf{i} + f_y(2, 0)\mathbf{j} = \qquad .$$

$$\nabla f = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j}$$

Example

Find the gradient vector of $f(x, y) = xe^y + \cos(xy)$ at the point $(2, 0)$.

We calculate that

$$f_x(2,0) = e^y - y\sin(xy)\Big|_{(2,0)} = e^0 - 0 = 1,$$

$$f_y(2,0) =$$

and

$$\nabla f\Big|_{(2,0)} = f_x(2,0)\mathbf{i} + f_y(2,0)\mathbf{j} = \qquad .$$

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$$f_y(2,0) = xe^y - x\sin(xy)\Big|_{(2,0)} = 2e^0 - 2\cdot 0 = 2$$

and

$$\nabla f\Big|_{(2,0)} = f_x(2,0)\mathbf{i} + f_y(2,0)\mathbf{j} = \qquad .$$

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and

$$\nabla f\Big|_{(2,0)} = f_x(2,0)\mathbf{i} + f_y(2,0)\mathbf{j} = \mathbf{i} + 2\mathbf{j}.$$

So how can we use gradient vectors to find directional derivatives?

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Theorem

$$D_{\mathbf{u}}f = \nabla f \cdot \mathbf{u}.$$

$$D_{\mathbf{u}}f = \nabla f \cdot \mathbf{u} \qquad \nabla f = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j}$$

Example

Find the derivative of $f(x, y) = xe^y + \cos(xy)$ at the point $(2, 0)$ in the direction of $\mathbf{v} = 3\mathbf{i} - 4\mathbf{j}$.

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Example

Find the derivative of $f(x, y) = xe^y + \cos(xy)$ at the point $(2, 0)$ in the direction of $\mathbf{v} = 3\mathbf{i} - 4\mathbf{j}$.

Recall that $\nabla f\big|_{(2,0)} = \mathbf{i} + 2\mathbf{j}$.

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Recall that $\nabla f|_{(2,0)} = \mathbf{i} + 2\mathbf{j}$. We need to find a unit vector $\mathbf{u}$ which points in the same direction as $\mathbf{v}$,

$$D_{\mathbf{u}} f = \nabla f \cdot \mathbf{u} \qquad \nabla f = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j}$$

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Find the derivative of $f(x,y) = xe^y + \cos(xy)$ at the point $(2,0)$ in the direction of $\mathbf{v} = 3\mathbf{i} - 4\mathbf{j}$.

Recall that $\nabla f|_{(2,0)} = \mathbf{i} + 2\mathbf{j}$. We need to find a unit vector $\mathbf{u}$ which points in the same direction as $\mathbf{v}$, so we calculate that

$$\mathbf{u} = \frac{\mathbf{v}}{\|\mathbf{v}\|} = \frac{3\mathbf{i} - 4\mathbf{j}}{\sqrt{3^2 + (-4)^2}} = \frac{3}{5}\mathbf{i} - \frac{4}{5}\mathbf{j}.$$

$$D_{\mathbf{u}}f = \nabla f \bullet \mathbf{u} \qquad \nabla f = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j}$$

Example

Find the derivative of $f(x, y) = xe^y + \cos(xy)$ at the point $(2, 0)$ in the direction of $\mathbf{v} = 3\mathbf{i} - 4\mathbf{j}$.

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Therefore

$$D_{\mathbf{u}}f(2, 0) = \nabla f|_{(2,0)} \bullet \mathbf{u} =$$

$$D_{\mathbf{u}}f = \nabla f \cdot \mathbf{u} \qquad \nabla f = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j}$$

Example

Find the derivative of $f(x,y) = xe^y + \cos(xy)$ at the point $(2,0)$ in the direction of $\mathbf{v} = 3\mathbf{i} - 4\mathbf{j}$.

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Therefore

$$D_{\mathbf{u}}f(2,0) = \nabla f|_{(2,0)} \cdot \mathbf{u} = (\mathbf{i} + 2\mathbf{j}) \cdot \left(\frac{3}{5}\mathbf{i} - \frac{4}{5}\mathbf{j}\right) = \frac{3}{5} - \frac{8}{5} = -1.$$

$$\boxed{\mathbf{a} \bullet \mathbf{b} = \|\mathbf{a}\|\,\|\mathbf{b}\| \cos\theta}$$

Note that

$$D_{\mathbf{u}} f = \nabla f \bullet \mathbf{u} = \|\nabla f\|\,\|\mathbf{u}\| \cos\theta = \|\nabla f\| \cos\theta$$

since $\|\mathbf{u}\| = 1$.

$$\mathbf{a} \cdot \mathbf{b} = \|\mathbf{a}\| \, \|\mathbf{b}\| \cos\theta$$

Note that

$$D_{\mathbf{u}} f = \nabla f \cdot \mathbf{u} = \|\nabla f\| \, \|\mathbf{u}\| \cos\theta = \|\nabla f\| \cos\theta$$

since $\|\mathbf{u}\| = 1$.

So we must always have

$$-\|\nabla f\| \leq D_{\mathbf{u}} f \leq \|\nabla f\| .$$

$$\boxed{D_{\mathbf{u}} f = \|\nabla f\| \cos\theta}$$

Remark

f increases mostly rapidly $\implies \cos\theta = 1 \implies \theta = 0 \implies$ $\mathbf{u}$ points in the same direction as ∇f

∇f points ‘uphill’

$$\boxed{D_{\mathbf{u}} f = \|\nabla f\| \cos\theta}$$

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Remark

f decreases mostly rapidly $\implies \cos\theta = -1 \implies \theta = 180^\circ \implies$ $\mathbf{u}$ points in the opposite direction from ∇f

a ball on a hill rolls in the direction $-\nabla f$

$$\boxed{D_{\mathbf{u}}f = \|\nabla f\| \cos\theta}$$

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Remark

f decreases mostly rapidly $\implies \cos\theta = -1 \implies \theta = 180^\circ \implies$ $\mathbf{u}$ points in the opposite direction from ∇f

a ball on a hill rolls in the direction $-\nabla f$

Remark

$$\theta = 90^\circ \implies D_{\mathbf{u}}f = 0.$$

EXAMPLE 3 Find the directions in which $f(x, y) = (x^2/2) + (y^2/2)$

(a) increases most rapidly at the point (1, 1).

(b) decreases most rapidly at (1, 1).

(c) What are the directions of zero change in f at (1, 1)?

Solution

(a) The function increases most rapidly in the direction of ∇f at (1, 1). The gradient there is

$$(\nabla f)_{(1,1)} = (x\mathbf{i} + y\mathbf{j})_{(1,1)} = \mathbf{i} + \mathbf{j}.$$

Its direction is

$$\mathbf{u} = \frac{\mathbf{i} + \mathbf{j}}{|\mathbf{i} + \mathbf{j}|} = \frac{\mathbf{i} + \mathbf{j}}{\sqrt{(1)^2 + (1)^2}} = \frac{1}{\sqrt{2}}\mathbf{i} + \frac{1}{\sqrt{2}}\mathbf{j}.$$

(b) The function decreases most rapidly in the direction of $-\nabla f$ at (1, 1), which is

$$-\mathbf{u} = -\frac{1}{\sqrt{2}}\mathbf{i} - \frac{1}{\sqrt{2}}\mathbf{j}.$$

(c) The directions of zero change at (1, 1) are the directions orthogonal to ∇f:

$$\mathbf{n} = -\frac{1}{\sqrt{2}}\mathbf{i} + \frac{1}{\sqrt{2}}\mathbf{j} \qquad \text{and} \qquad -\mathbf{n} = \frac{1}{\sqrt{2}}\mathbf{i} - \frac{1}{\sqrt{2}}\mathbf{j}.$$

$$D_{\mathbf{u}}f = \|\nabla f\| \cos\theta$$

Algebra Rules for ∇

Theorem

1. *Sum Rule:* $\nabla(f+g) = \nabla f + \nabla g$
2. *Difference Rule:* $\nabla(f+g) = \nabla f + \nabla g$
3. *Constant Multiple Rule:* $\nabla(kf) = k\nabla f \qquad$ *(for $k \in \mathbb{R}$)*
4. *Product Rule:* $\nabla(fg) = g\nabla f + f\nabla g$
5. *Quotient Rule:* $\nabla\left(\frac{f}{g}\right) = \frac{g\nabla f - f\nabla g}{g^2}$.

EXAMPLE 5 We illustrate two of the rules with

$$f(x, y) = x - y \qquad g(x, y) = 3y$$
$$\nabla f = \mathbf{i} - \mathbf{j} \qquad \nabla g = 3\mathbf{j}.$$

We have

1. $\nabla(f - g) = \nabla(x - 4y) = \mathbf{i} - 4\mathbf{j} = \nabla f - \nabla g$ Rule 2
2. $\nabla(fg) = \nabla(3xy - 3y^2) = 3y\mathbf{i} + (3x - 6y)\mathbf{j}$

   and

$$f\nabla g + g\nabla f = (x - y)3\mathbf{j} + 3y(\mathbf{i} - \mathbf{j}) \qquad \text{Substitute.}$$
$$= 3y\mathbf{i} + (3x - 6y)\mathbf{j}. \qquad \text{Simplify.}$$

We have therefore verified for this example that $\nabla(fg) = f\nabla g + g\nabla f$.

$$\boxed{D_{\mathbf{u}}f = \|\nabla f\| \cos\theta}$$

Functions of Three Variables

$$\nabla f = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} + \frac{\partial f}{\partial z}\mathbf{k}$$

$$D_{\mathbf{u}}f = \nabla f \bullet \mathbf{u}.$$

EXAMPLE 6

(a) Find the derivative of $f(x, y, z) = x^3 - xy^2 - z$ at $P_0(1, 1, 0)$ in the direction of $\mathbf{v} = 2\mathbf{i} - 3\mathbf{j} + 6\mathbf{k}$.

(b) In what directions does f change most rapidly at P_0, and what are the rates of change in these directions?

Solution

(a) The direction of $\mathbf{v}$ is obtained by dividing $\mathbf{v}$ by its length:

$$|\mathbf{v}| = \sqrt{(2)^2 + (-3)^2 + (6)^2} = \sqrt{49} = 7$$

$$\mathbf{u} = \frac{\mathbf{v}}{|\mathbf{v}|} = \frac{2}{7}\mathbf{i} - \frac{3}{7}\mathbf{j} + \frac{6}{7}\mathbf{k}.$$

The partial derivatives of f at P_0 are

$$f_x = (3x^2 - y^2)\Big|_{(1,1,0)} = 2, \qquad f_y = -2xy\Big|_{(1,1,0)} = -2, \qquad f_z = -1\Big|_{(1,1,0)} = -1.$$

The gradient of f at P_0 is

$$\nabla f\big|_{(1,1,0)} = 2\mathbf{i} - 2\mathbf{j} - \mathbf{k}.$$

The derivative of f at P_0 in the direction of $\mathbf{v}$ is therefore

$$D_{\mathbf{u}}f\big|_{(1,1,0)} = \nabla f\big|_{(1,1,0)} \cdot \mathbf{u} = (2\mathbf{i} - 2\mathbf{j} - \mathbf{k}) \cdot \left(\frac{2}{7}\mathbf{i} - \frac{3}{7}\mathbf{j} + \frac{6}{7}\mathbf{k}\right)$$

$$= \frac{4}{7} + \frac{6}{7} - \frac{6}{7} = \frac{4}{7}.$$

(b) The function increases most rapidly in the direction of $\nabla f = 2\mathbf{i} - 2\mathbf{j} - \mathbf{k}$ and decreases most rapidly in the direction of $-\nabla f$. The rates of change in the directions are, respectively,

$$|\nabla f| = \sqrt{(2)^2 + (-2)^2 + (-1)^2} = \sqrt{9} = 3 \qquad \text{and} \qquad -|\nabla f| = -3.$$