Lecture 11

- 7.1 Inverse Functions and Their Derivatives
- 7.2 Natural Logarithms
- 7.3 Exponential Functions

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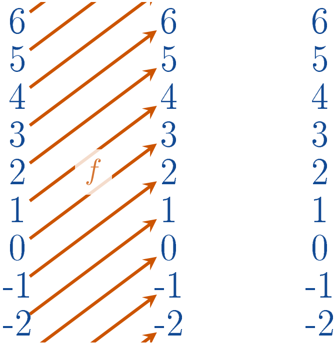
Inverse Functions and Their Derivatives

7.1 Inverse Functions and Their Derivatives

What is the opposite of adding 3 to a number?

What is the inverse of $f : \mathbb{Z} \to \mathbb{Z}$, $f(x) = x + 3$?



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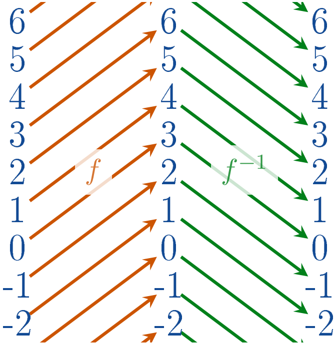

The inverse of f is $f^{-1} : \mathbb{Z} \to \mathbb{Z}$, $f^{-1}(x) = x - 3$.

Note that $f^{-1} \circ f(x) = x$ and $f \circ f^{-1}(x) = x$.

Some functions have an inverse.

For example $g : [0,3] \to [0,9]$, $g(x) = x^2$ has the inverse $g^{-1} : [0,9] \to [0,3]$, $g^{-1}(x) = \sqrt{x}$.

$$\sqrt{4^2} = \sqrt{16} = 4 \qquad \text{and} \qquad \left(\sqrt{9}\right)^2 = 3^2 = 0.$$

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But some functions do not have an inverse.

For example, consider $h : [-3, 3] \to [0, 9]$, $h(x) = x^2$.

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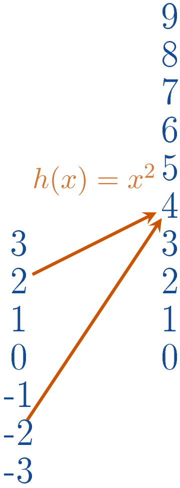
But some functions do not have an inverse.

For example, consider $h : [-3, 3] \to [0, 9]$, $h(x) = x^2$.

$$2^2 = 4 \qquad \text{and} \qquad (-2)^2 = 4.$$

So what do we define $h^{-1}(4)$ to be? 2 or -2?


$h(x) = x^2$
9
8
7
6
5
4
3
2
1
0
3
2
1
0
-1
-2
-3

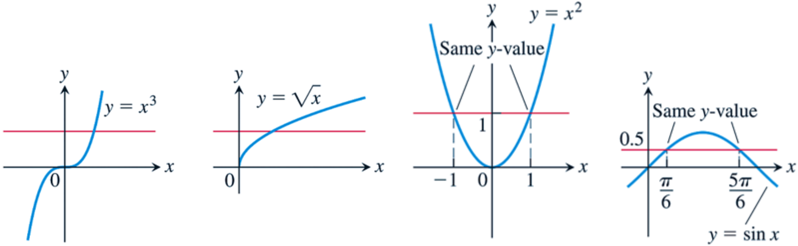
One-to-One Functions

Definition

A function $f(x)$ is called *one-to-one* (or *injective*) (*bire-bir*) on a domain D if

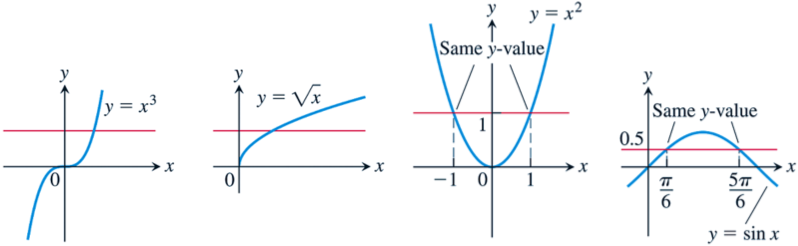
$$x_1 \neq x_2 \quad \Longrightarrow \quad f(x_1) \neq f(x_2)$$

for all $x_1, x_2 \in D$.



Example

- x^3 is one-to-one on any domain.

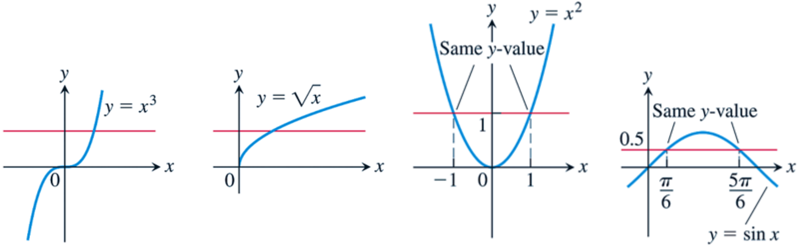


Example

- x^3 is one-to-one on any domain.
- $\sqrt{x}$ is one-to-one on $[0, \infty)$ because

$$x_1 \neq x_2 \quad \Longrightarrow \quad \sqrt{x_1} \neq \sqrt{x_2}.$$

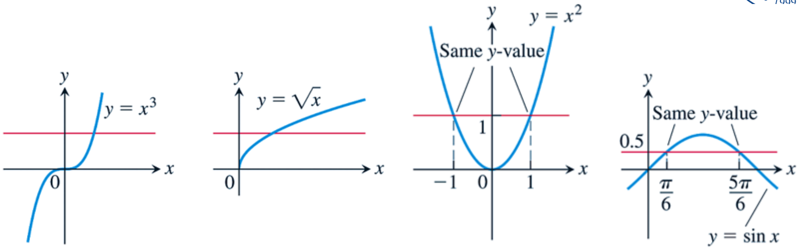
7.1 Inverse Functions and Their Derivatives



Example

- x^2 is not one-to-one on $\mathbb{R}$ because

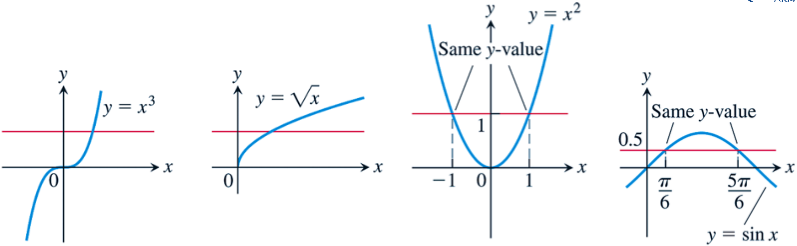
$$(-1)^2 = 1^2.$$



Example

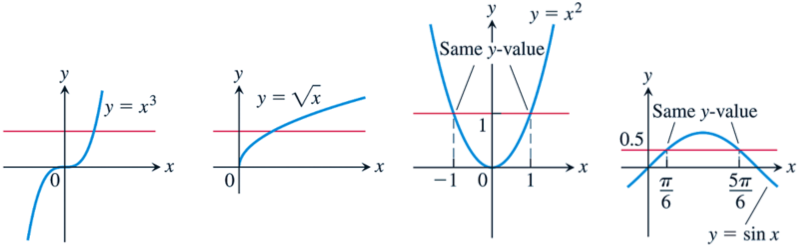
- $\sin x$ is not one-to-one on $[0, \pi]$ because

$$\sin\frac{\pi}{6} = \frac{1}{2} = \sin\frac{5\pi}{6}.$$



Example

- $\sin x$ is not one-to-one on $[0, \pi]$ because
$$\sin\frac{\pi}{6} = \frac{1}{2} = \sin\frac{5\pi}{6}.$$
- However, $\sin x$ is one-to-one on $[0, \frac{\pi}{2}]$.



Theorem (The Horizontal Line Test for One-to-One Functions)

A function $y = f(x)$ is one-to-one if and only if its graph intersects each horizontal line at most once.

Inverse Functions

Definition

Suppose that f is a one-to-one function on a domain D with range R. The *inverse function* f^{-1} is defined by

$$f^{-1}(b) = a \quad \text{if} \quad f(a) = b.$$

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Suppose that f is a one-to-one function on a domain D with range R. The *inverse function* f^{-1} is defined by

$$f^{-1}(b) = a \quad \text{if} \quad f(a) = b.$$

The domain of f^{-1} is R and the range of f^{-1} is D.

We read f^{-1} as "f inverse".

Please note that f^{-1} does not mean $\frac{1}{f(x)}$.

EXAMPLE 2 Suppose a one-to-one function $y = f(x)$ is given by a table of values

x	1	2	3	4	5	6	7	8
$f(x)$	3	4.5	7	10.5	15	20.5	27	34.5

A table for the values of $x = f^{-1}(y)$ can then be obtained by simply interchanging the values in each column of the table for f:

y	3	4.5	7	10.5	15	20.5	27	34.5
$f^{-1}(y)$	1	2	3	4	5	6	7	8

■

Remark

If f^{-1} is the inverse of f, then

$$f^{-1} \circ f(x) = x \qquad \text{(for all } x \text{ in the domain of } f\text{)}$$

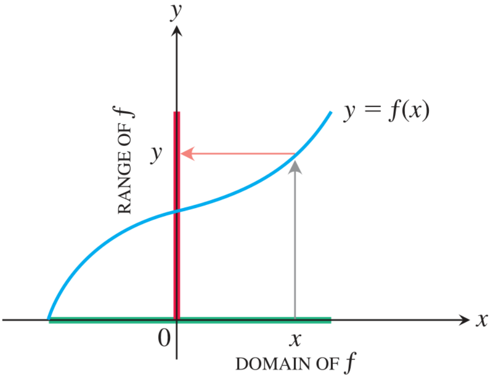
and

$$f \circ f^{-1}(y) = y \qquad \text{(for all } y \text{ in the range of } f\text{).}$$

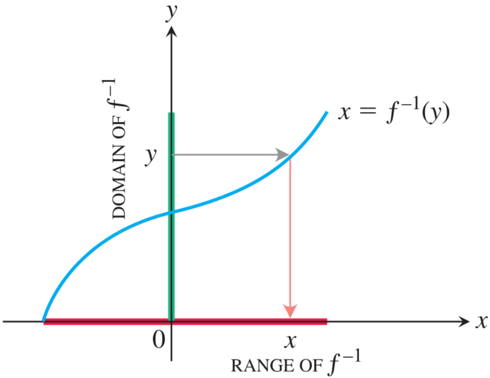
Finding Inverses

Now let's talk about how to find an inverse of a function.

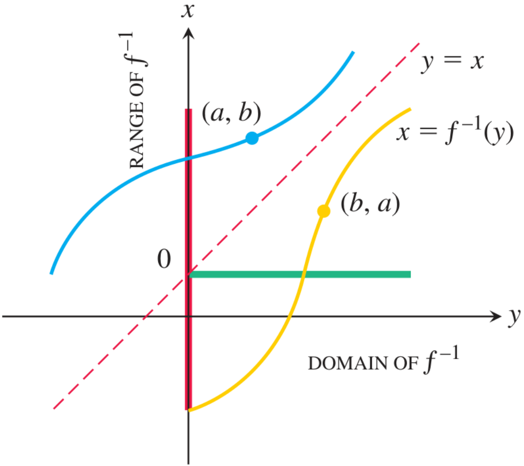
7.1 Inverse Functions and Their Derivatives



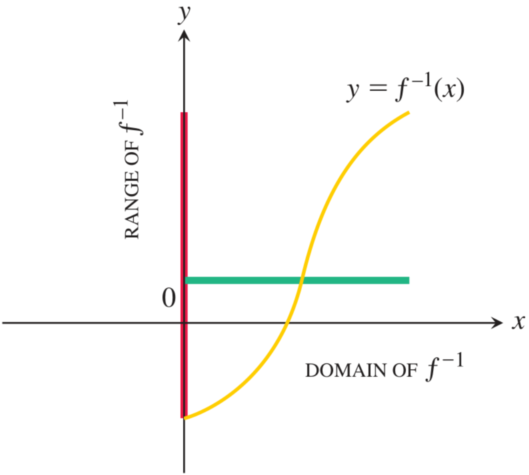
(a) To find the value of f at x, we start at x, go up to the curve, and then over to the y-axis.



(b) The graph of f^{-1} is the graph of f, but with x and y interchanged. To find the x that gave y, we start at y and go over to the curve and down to the x-axis. The domain of f^{-1} is the range of f. The range of f^{-1} is the domain of f.



(c) To draw the graph of f^{-1} in the more usual way, we reflect the system across the line $y = x$.



(d) Then we interchange the letters x and y. We now have a normal-looking graph of f^{-1} as a function of x.

The process of passing from f to f^{-1} can be summarized as a two-step procedure.

1. Solve the equation $y = f(x)$ for x. This gives a formula $x = f^{-1}(y)$ where x is expressed as a function of y.

2. Interchange x and y, obtaining a formula $y = f^{-1}(x)$ where f^{-1} is expressed in the conventional format with x as the independent variable and y as the dependent variable.

Example

Find the inverse of $y = \frac{1}{2}x + 1$, expressed as a function of x.

1. Solve for x in terms of y:

2. Swap x and y:

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The inverse of $f(x) = \frac{1}{2}x + 1$ is $f^{-1}(x) = 2x - 2$.

Example

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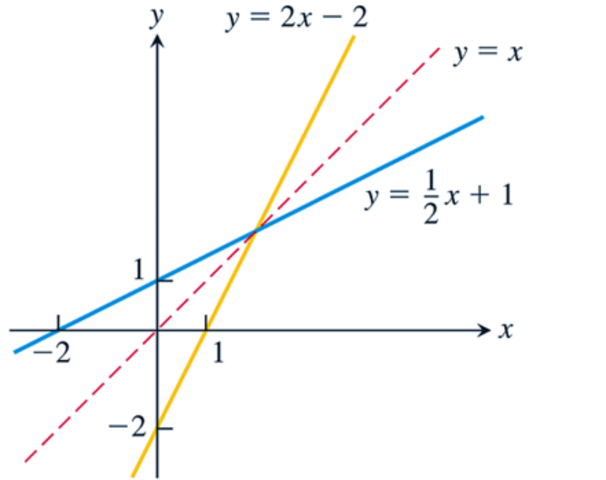

The inverse of $f(x) = \frac{1}{2}x + 1$ is $f^{-1}(x) = 2x - 2$.

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Find the inverse of the function $y = x^2$, $x \geq 0$, expressed as a function of x.

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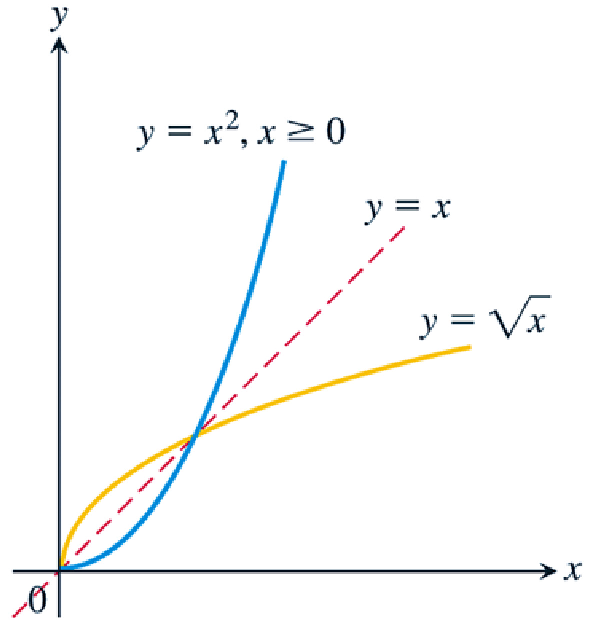

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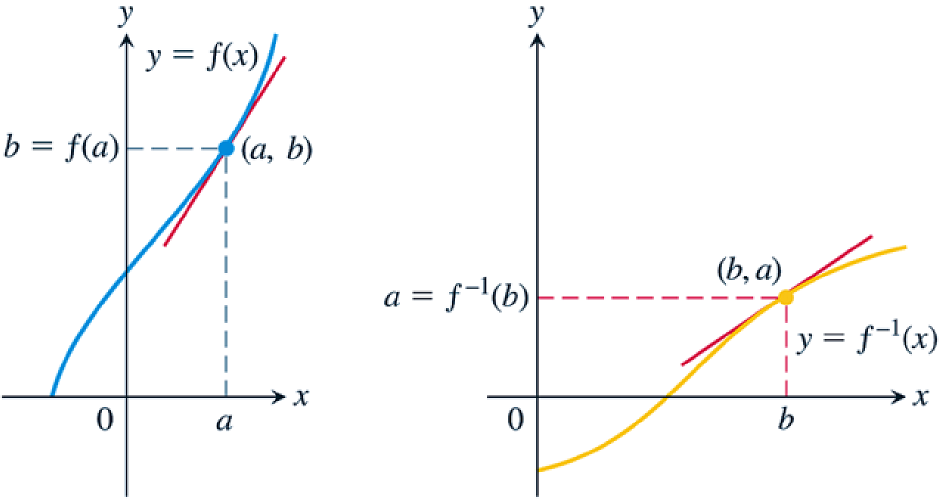

The slopes are reciprocal: $(f^{-1})'(b) = \dfrac{1}{f'(a)}$ or $(f^{-1})'(b) = \dfrac{1}{f'(f^{-1}(b))}$

Note that

$$f\left(f^{-1}(x)\right) = x$$

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Note that

$$\begin{aligned} f\left(f^{-1}(x)\right) &= x \\ \frac{d}{dx} f\left(f^{-1}(x)\right) &= \frac{d}{dx} x = 1 \\ f'\left(f^{-1}(x)\right) \cdot \frac{d}{dx} f^{-1}(x) &= 1 \\ \frac{d}{dx} f^{-1}(x) &= \frac{1}{f'\left(f^{-1}(x)\right)}. \end{aligned}$$

Theorem

Suppose that

- $f : I \to \mathbb{R}$ *for an interval* I*;*
- f' *exists on* I*; and*
- $f'(x) \neq 0$ *on* I*,*

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Moreover

$$\left(f^{-1}\right)'(x) = \frac{1}{f'\left(f^{-1}(x)\right)}.$$

Remark

This theorem says that

$$\left.\frac{df^{-1}}{dx}\right|_{x=b} = \frac{1}{\left.\frac{df}{dx}\right|_{x=f^{-1}(b)}}.$$

$$\left(f^{-1}\right)'(x) = \frac{1}{f'\left(f^{-1}(x)\right)}$$

Example

Let $f(x) = x^2$, $x \geq 0$. Find $\left(f^{-1}\right)'(4)$.

$$\left(f^{-1}\right)'(x) = \frac{1}{f'\left(f^{-1}(x)\right)}$$

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Let $f(x) = x^2$, $x \geq 0$. Find $\left(f^{-1}\right)'(4)$.

Method 1:

The inverse of $f(x) = x^2$, $x \geq 0$ is $f^{-1}(x) = \sqrt{x}$ and the derivative of $\sqrt{x}$ is $\frac{1}{2\sqrt{x}}$.

$$\left(f^{-1}\right)'(x) = \frac{1}{f'\left(f^{-1}(x)\right)}$$

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Method 2: Using the theorem (with $f'(x) = 2x$), we calculate that

$$\left(f^{-1}\right)'(x) = \frac{1}{f'\left(f^{-1}(x)\right)} = \frac{1}{2\left(f^{-1}(x)\right)} = \frac{1}{2\left(\sqrt{x}\right)}.$$

$$\left(f^{-1}\right)'(x) = \frac{1}{f'\left(f^{-1}(x)\right)}$$

Example

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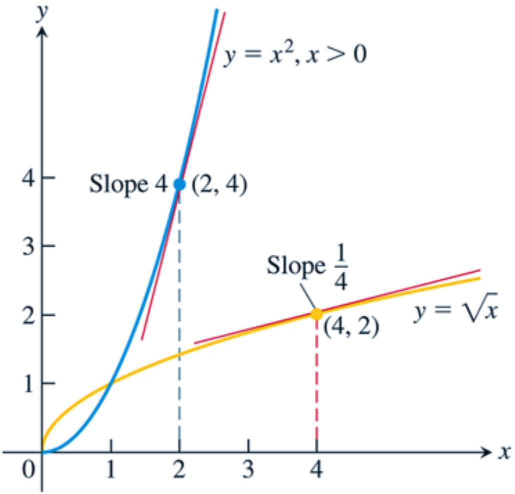
$$\left(f^{-1}\right)'(x) = \frac{1}{f'\left(f^{-1}(x)\right)} = \frac{1}{2\left(f^{-1}(x)\right)} = \frac{1}{2\left(\sqrt{x}\right)}.$$

Either way,

$$\left(f^{-1}\right)'(4) = \frac{1}{2\sqrt{4}} = \frac{1}{4}.$$

$$\left(f^{-1}\right)'(x) = \frac{1}{f'\left(f^{-1}(x)\right)}$$



$$\left.\frac{df^{-1}}{dx}\right|_{x=b} = \frac{1}{\left.\frac{df}{dx}\right|_{x=f^{-1}(b)}}$$

Example

Let $f(x) = x^3 - 2$, $x > 0$. Find the value of $\frac{df^{-1}}{dx}$ at $x = 6 = f(2)$ without finding a formula for $f^{-1}(x)$.

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Look at the formula in the yellow box. We want to calculate $\frac{df^{-1}}{dx}$ at $x = 6$. So we need $\frac{df}{dx}$ at $x = 2$.

$$\left.\frac{df^{-1}}{dx}\right|_{x=b} = \frac{1}{\left.\frac{df}{dx}\right|_{x=f^{-1}(b)}}$$

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Since

$$\left.\frac{df}{dx}\right|_{x=2} = 3x^2\big|_{x=2} = 12,$$

$$\left.\frac{df^{-1}}{dx}\right|_{x=b} = \frac{1}{\left.\frac{df}{dx}\right|_{x=f^{-1}(b)}}$$

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we have that

$$\left.\frac{df^{-1}}{dx}\right|_{x=6} = \frac{1}{\left.\frac{df}{dx}\right|_{x=2}} =$$

$$\left.\frac{df^{-1}}{dx}\right|_{x=b} = \frac{1}{\left.\frac{df}{dx}\right|_{x=f^{-1}(b)}}$$

Example

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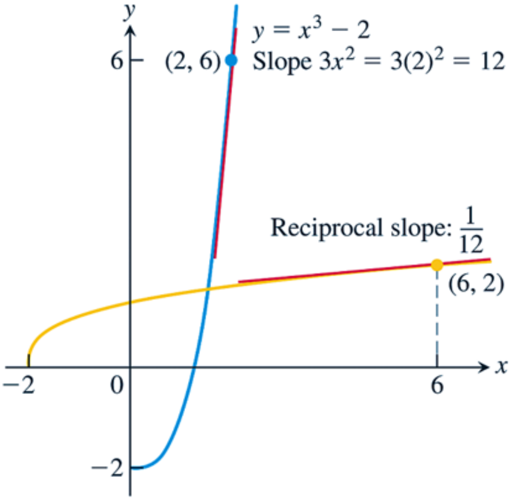
we have that

$$\left.\frac{df^{-1}}{dx}\right|_{x=6} = \frac{1}{\left.\frac{df}{dx}\right|_{x=2}} = \frac{1}{12}.$$


y
x
6
0
−2
6
−2
$y = x^3 - 2$
(2, 6)
Slope $3x^2 = 3(2)^2 = 12$
Reciprocal slope: $\frac{1}{12}$
(6, 2)

7.2 Natural Logarithms

Definition of the Natural Logarithm Function

Definition

The *natural logarithm* function $\ln : (0, \infty) \to \mathbb{R}$ is defined by

$$\ln x = \int_1^x \frac{1}{t}\, dt.$$

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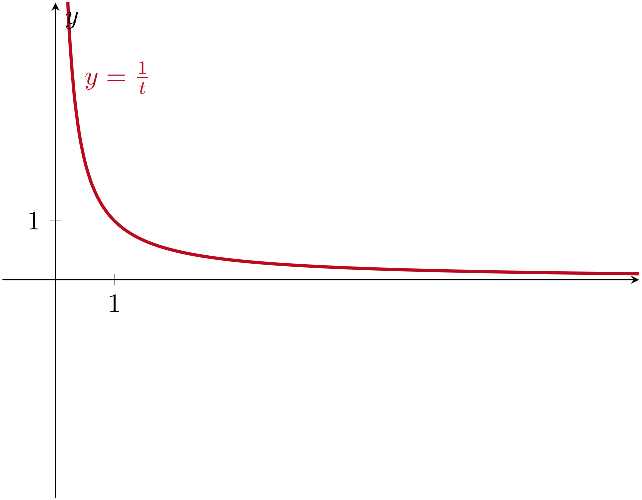
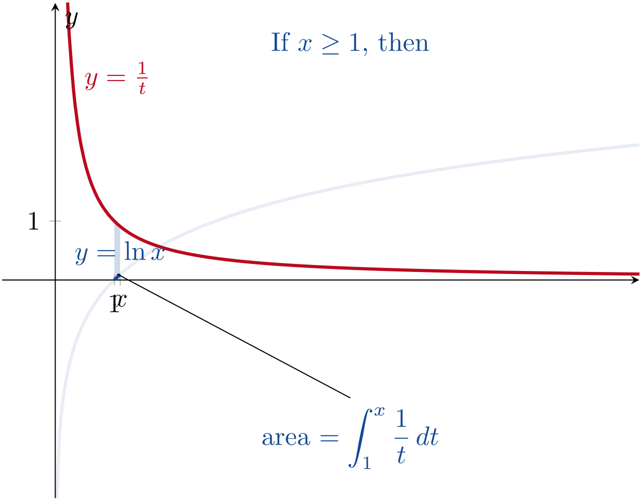


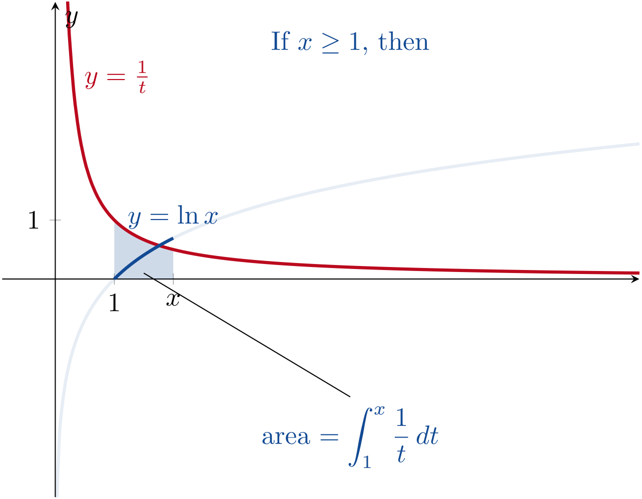
$$\ln x = \int_1^x \frac{1}{t}\,dt$$



If $x \geq 1$, then

$$\text{area} = \int_1^x \frac{1}{t}\,dt$$

$$\ln x = \int_1^x \frac{1}{t}\,dt$$



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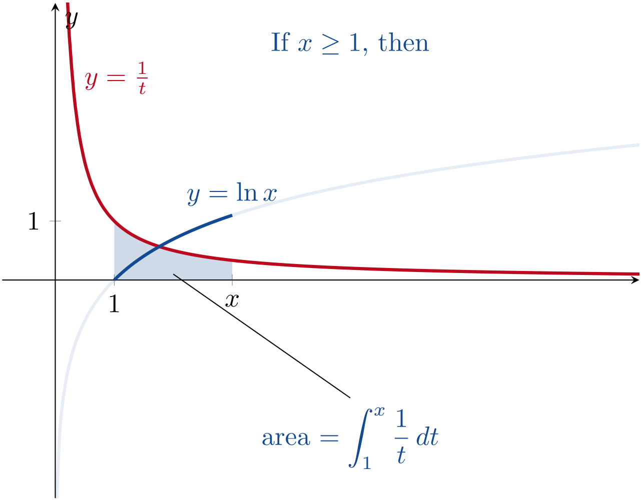

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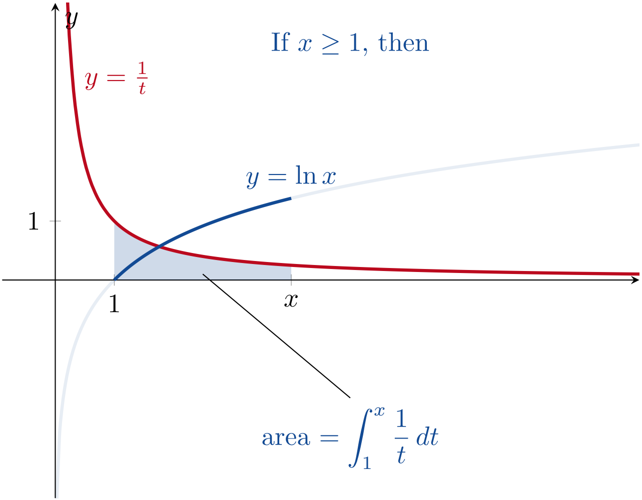

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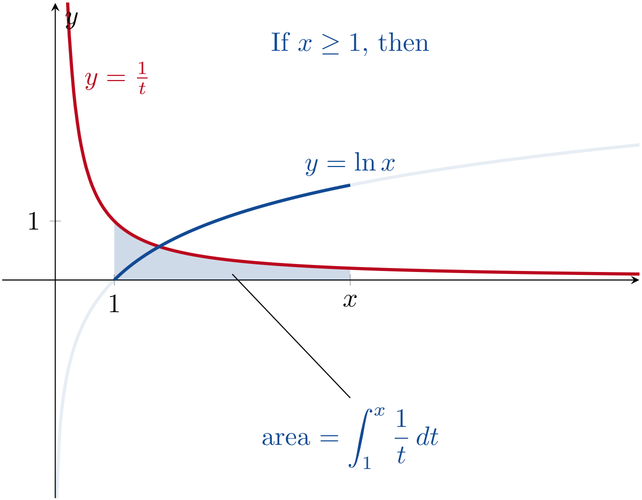

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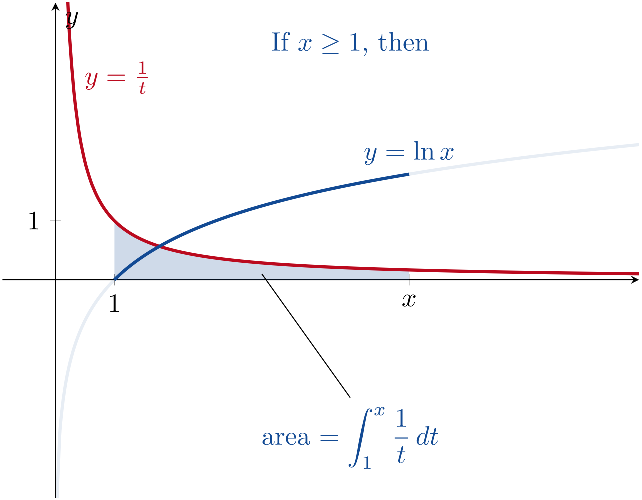

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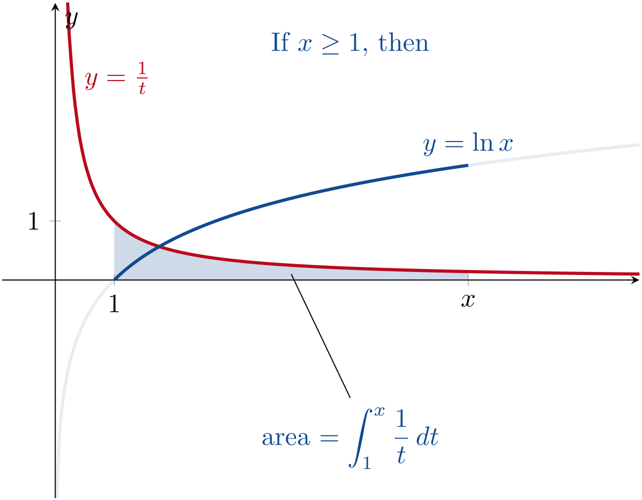

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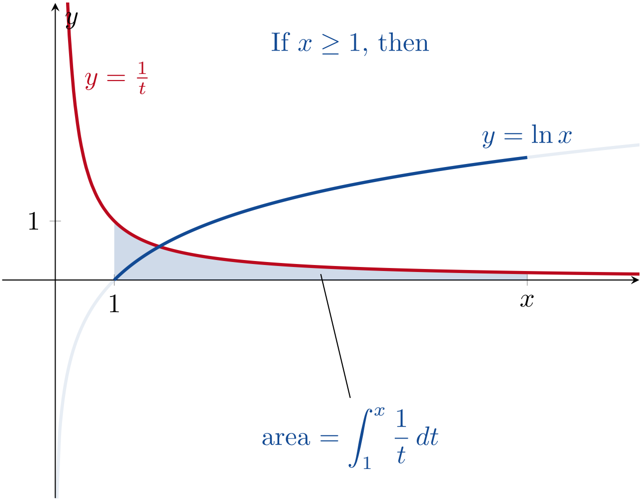

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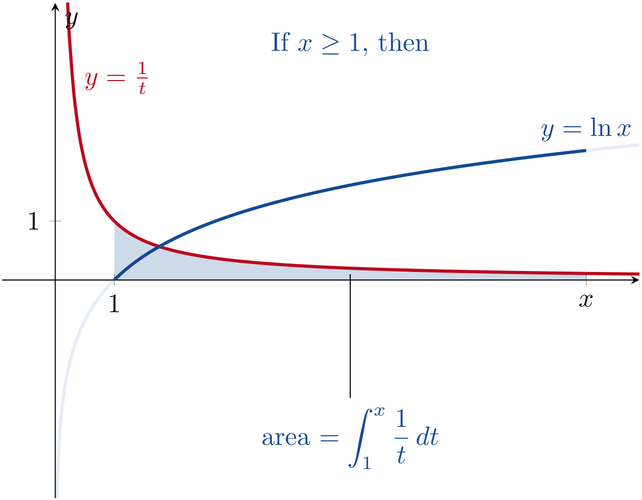

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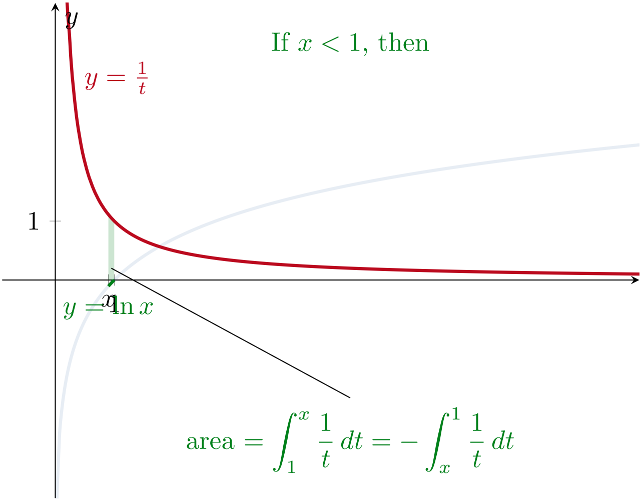
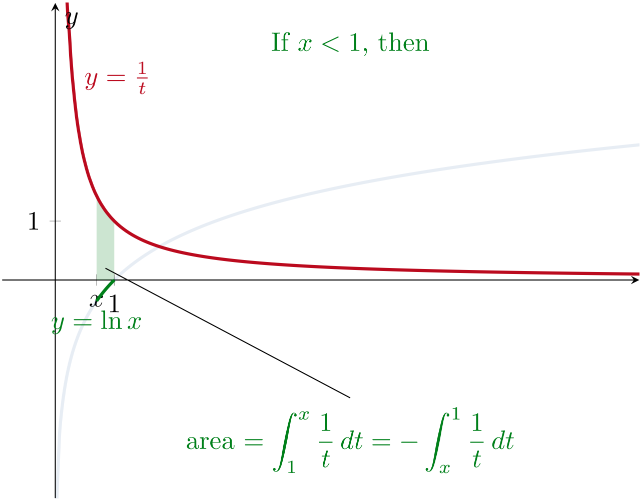


If $x < 1$, then

$$\text{area} = \int_1^x \frac{1}{t}\,dt = -\int_x^1 \frac{1}{t}\,dt$$

$$\ln x = \int_1^x \frac{1}{t}\,dt$$



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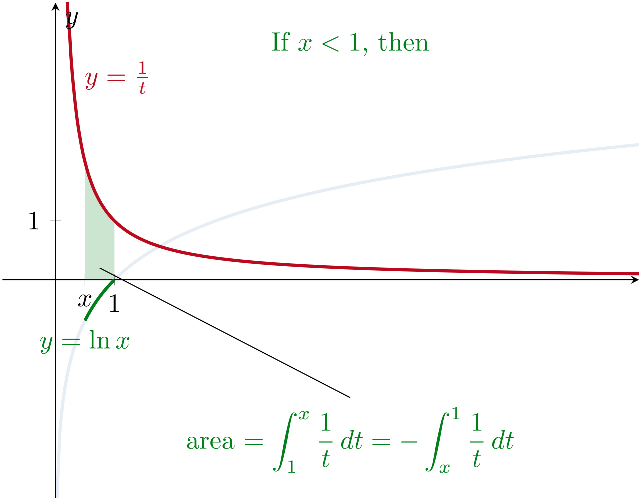

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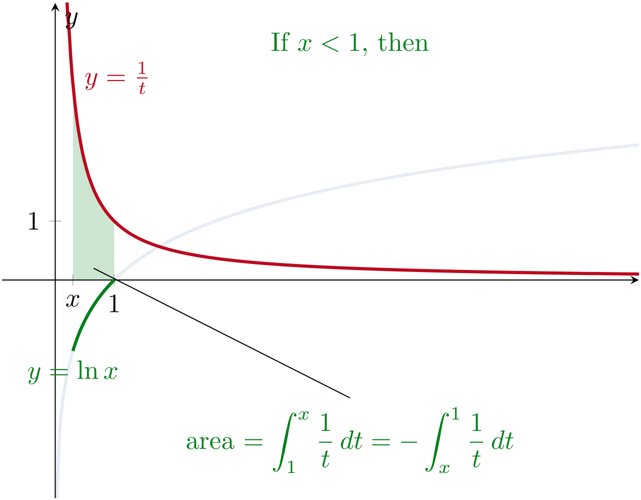

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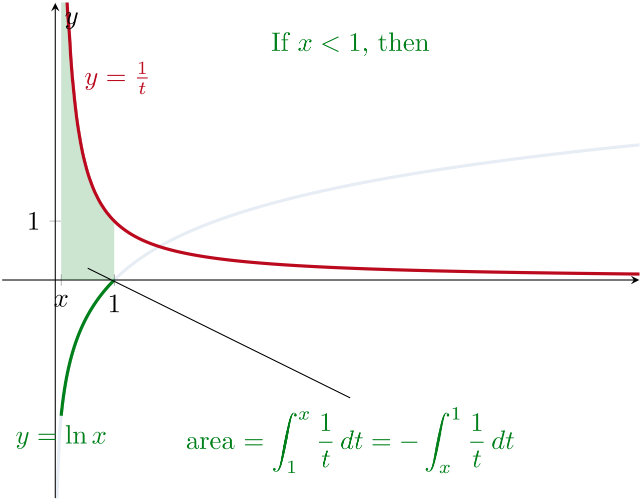

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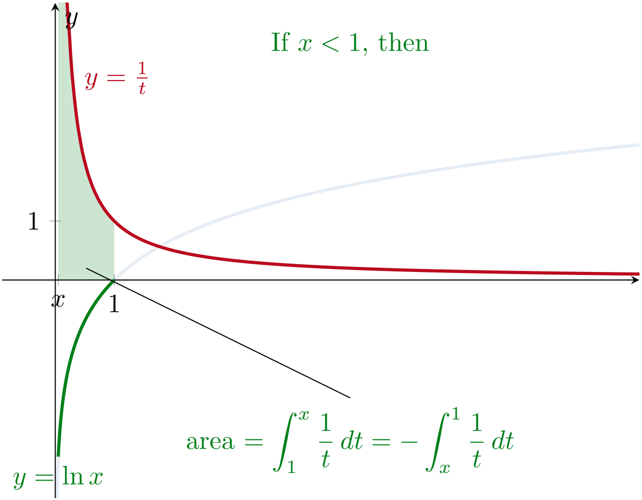

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$$\ln x = \int_1^x \frac{1}{t}\,dt$$

x	$\ln x$
0	undefined
0.05	−3.00
0.5	−0.69
1	0
2	0.69
3	1.10
4	1.39
10	2.30

$$\ln x = \int_1^x \frac{1}{t}\,dt$$

x	$\ln x$
0	undefined
0.05	−3.00
0.5	−0.69
1	0
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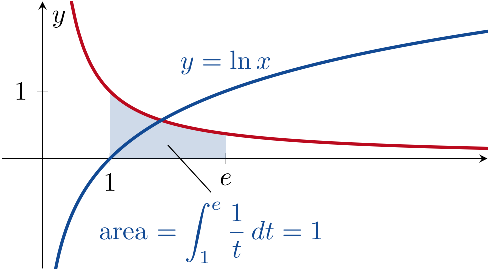
There is an important number between 2 and 3 where the natural logarithm is equal to 1.

$$\ln x = \int_1^x \frac{1}{t}\,dt$$



Definition

The *number* e is the number which satisfies

$$\ln(e) = \int_1^e \frac{1}{t}\,dt = 1.$$

$$\ln x = \int_1^x \frac{1}{t}\,dt$$

The Derivative of $y = \ln x$

Recall the first part of the Fundamental Theorem of Calculus:

$$\frac{d}{dx}\int_a^x f(t)\,dt = f(x).$$

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Recall the first part of the Fundamental Theorem of Calculus:

$$\frac{d}{dx}\int_a^x f(t)\,dt = f(x).$$

It follows that

$$\frac{d}{dx}\ln x = \frac{d}{dx}\int_1^x \frac{1}{t}\,dt = \frac{1}{x}.$$

If

- $u(x)$ is differentiable; and
- $u(x) > 0$,

then it follows by the Chain Rule that

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then it follows by the Chain Rule that

$$\boxed{\frac{d}{dx}\ln u = \frac{1}{u}\frac{du}{dx}.}$$

EXAMPLE 1 We use Equation (2) to find derivatives.

(a) $\dfrac{d}{dx}\ln 2x = \dfrac{1}{2x}\dfrac{d}{dx}(2x) = \dfrac{1}{2x}(2) = \dfrac{1}{x}, \quad x > 0$

(b) Equation (2) with $u = x^2 + 3$ gives

$$\frac{d}{dx}\ln(x^2+3) = \frac{1}{x^2+3}\cdot\frac{d}{dx}(x^2+3) = \frac{1}{x^2+3}\cdot 2x = \frac{2x}{x^2+3}.$$

(c) Equation (2) with $u = |x|$ gives an important derivative:

$$\begin{aligned}
\frac{d}{dx}\ln|x| &= \frac{d}{du}\ln u\cdot\frac{du}{dx} && u = |x|, x \neq 0 \\
&= \frac{1}{u}\cdot\frac{x}{|x|} && \frac{d}{dx}(|x|) = \frac{x}{|x|} \\
&= \frac{1}{|x|}\cdot\frac{x}{|x|} && \text{Substitute for } u. \\
&= \frac{x}{x^2} \\
&= \frac{1}{x}.
\end{aligned}$$

So $1/x$ is the derivative of $\ln x$ on the domain $x > 0$, and the derivative of $\ln(-x)$ on the domain $x < 0$. ■

THEOREM 2—Algebraic Properties of the Natural Logarithm

For any numbers $b > 0$ and $x > 0$, the natural logarithm satisfies the following rules:

1. *Product Rule:* $\ln bx = \ln b + \ln x$
2. *Quotient Rule:* $\ln \frac{b}{x} = \ln b - \ln x$
3. *Reciprocal Rule:* $\ln \frac{1}{x} = -\ln x$ Rule 2 with $b = 1$
4. *Power Rule:* $\ln x^r = r \ln x$ For r rational

EXAMPLE 2 We apply the rules in Theorem 2.

(a) $\ln 4 + \ln \sin x = \ln(4 \sin x)$ Product Rule

(b) $\ln \dfrac{x+1}{2x-3} = \ln(x+1) - \ln(2x-3)$ Quotient Rule

(c) $\ln \dfrac{1}{8} = -\ln 8$ Reciprocal Rule

$= -\ln 2^3 = -3 \ln 2$ Power Rule

Proof that $\ln bx = \ln b + \ln x$.

First note that

$$\frac{d}{dx}\ln(bx) = \frac{b}{bx} = \frac{1}{x}$$

Proof that $\ln bx = \ln b + \ln x$.

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By the second corollary of the Mean Value Theorem, this means that

$$\ln bx = \ln x + C$$

for some constant C.

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for some constant C. Putting in $x = 1$, we have

$$\ln(b \cdot 1) = \ln 1 + C$$

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$$\ln b = \ln(b \cdot 1) = \ln 1 + C = 0 + C = C.$$

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$$\ln b = \ln(b \cdot 1) = \ln 1 + C = 0 + C = C.$$

Therefore

$$\ln bx = \ln x + \ln b.$$

$\square$

Proof that $\ln x^r = r \ln x$ (assuming $r \in \mathbb{Q}$.)

Using the Chain Rule with $u(x) = x^r$, we find that

$$\frac{d}{dx} \ln x^r = \frac{d}{dx} \ln u = \frac{1}{u}\frac{du}{dx} = \frac{1}{x^r} r x^{r-1} = r \cdot \frac{1}{x} = \frac{d}{dx} r \ln x.$$

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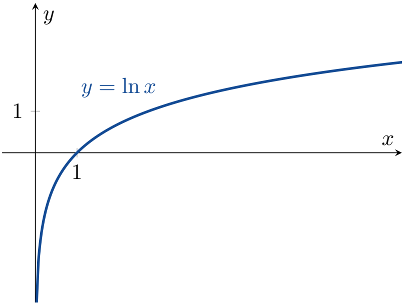
So we must have

$$\ln x^r = r \ln x + C$$

for some constant C. Putting in $x = 1$, we find $C = 0$ and we are finished.

$\square$

The Graph and Range of $\ln x$

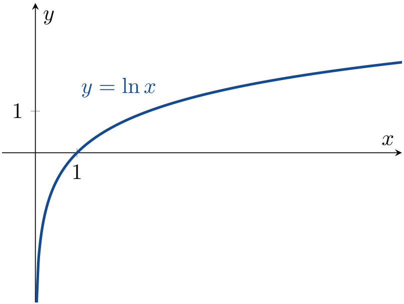


- Note first that

$$\frac{d}{dx}\ln x = \frac{1}{x} > 0$$

for all $x > 0$. So $\ln x$ is an increasing function.

The Graph and Range of $\ln x$



- Note first that

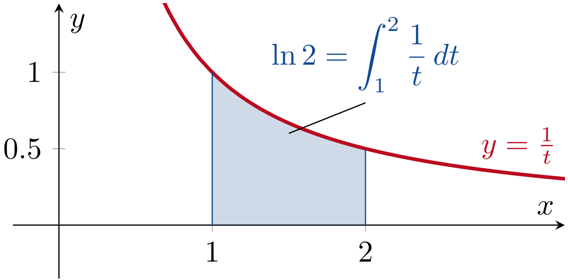
$$\frac{d}{dx}\ln x = \frac{1}{x} > 0$$

for all $x > 0$. So $\ln x$ is an increasing function.

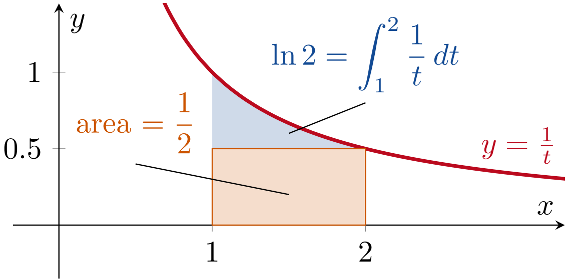
- Moreover

$$\frac{d^2}{dx^2}\ln x = \frac{d}{dx}\frac{1}{x} = -\frac{1}{x^2} < 0.$$

So $y = \ln x$ is concave down.

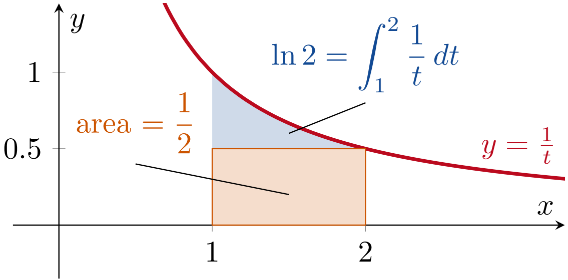


y
1
0.5
$\ln 2 = \int_1^2 \frac{1}{t}\,dt$
$y = \frac{1}{t}$
x
1
2



Note that

$$\ln 2 > \frac{1}{2}.$$

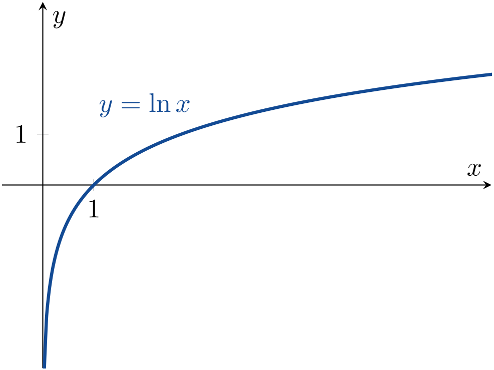


Note that

$$\ln 2 > \frac{1}{2}.$$

It follows that

$$\ln 2^n = n \ln 2 > n\left(\frac{1}{2}\right) = \frac{n}{2}.$$

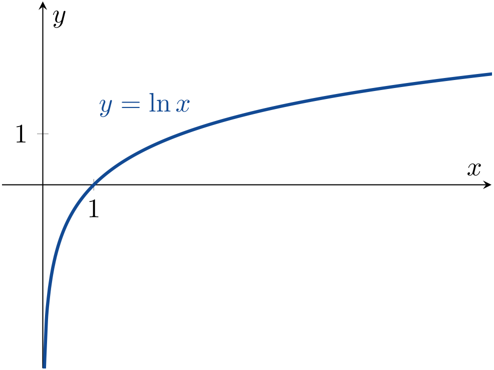


So

$$\lim_{n\to\infty} \ln 2^n \geq \lim_{n\to\infty} \frac{n}{2} = \infty.$$

Since $\ln x$ is an increasing function, we also have that

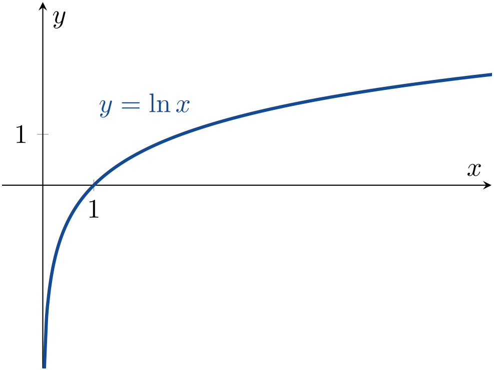
$$\lim_{x\to\infty} \ln x = \infty.$$



Using $t = \frac{1}{x}$, we also have that

$$\lim_{x \to 0^+} \ln x = \lim_{t \to \infty} \ln \frac{1}{t} = \lim_{t \to \infty} \ln t^{-1} = \lim_{t \to \infty} -\ln t = -\infty.$$

7.2 Natural Logarithms



- The domain of $\ln x$ is $(0, \infty)$.
- The range of $\ln x$ is $\mathbb{R}$.

The Integral $\int \frac{1}{u}\,du$

If

- $u(x)$ is differentiable; and
- $u(x) \neq 0$,

then

$$\boxed{\int \frac{1}{u}\,du = \ln|u| + C.}$$

The Integral $\int \frac{1}{u}\,du$

If

- $u(x)$ is differentiable; and
- $u(x) \neq 0$,

then

$$\boxed{\int \frac{1}{u}\,du = \ln|u| + C.}$$

If $u = f(x)$, then $du = \frac{du}{dx}\,dx = f'(x)\,dx$. Thus

$$\boxed{\int \frac{f'(x)}{f(x)}\,dx = \ln|f(x)| + C.}$$

$$\int \frac{du}{u} = \ln|u| + C$$

Example

Calculate $\displaystyle\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{4\cos\theta}{3+2\sin\theta}\,d\theta$.

The idea is to rewrite this integral in the form $\displaystyle\int \frac{du}{u}$.

$$\int \frac{du}{u} = \ln|u| + C$$

Example

Calculate $\displaystyle\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{4\cos\theta}{3+2\sin\theta}\, d\theta$.

The idea is to rewrite this integral in the form $\displaystyle\int \frac{du}{u}$.

Let $u = 3 + 2\cos\theta$. Then $du = 2\sin\theta\, d\theta$.

$$\int \frac{du}{u} = \ln|u| + C$$

Example

Calculate $\displaystyle\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{4\cos\theta}{3+2\sin\theta}\,d\theta.$

The idea is to rewrite this integral in the form $\displaystyle\int \frac{du}{u}$.

Let $u = 3 + 2\cos\theta$. Then $du = 2\sin\theta\,d\theta$. Moreover

$$\begin{aligned} \theta &= \frac{\pi}{2} \\ \theta &= -\frac{\pi}{2} \end{aligned} \quad\Longrightarrow\quad \begin{aligned} u &= 3 + 2\cos\frac{\pi}{2} = 3 + 2 = 5 \\ u &= 3 + 2\cos\left(-\frac{\pi}{2}\right) = 3 - 2 = 1. \end{aligned}$$

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Therefore

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{2\cdot 2\cos\theta}{3+2\sin\theta}\,d\theta = \int_1^5 \frac{2}{u}\,du$$

$$\int \frac{du}{u} = \ln|u| + C$$

Example

Calculate $\displaystyle\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{4\cos\theta}{3+2\sin\theta}\, d\theta.$

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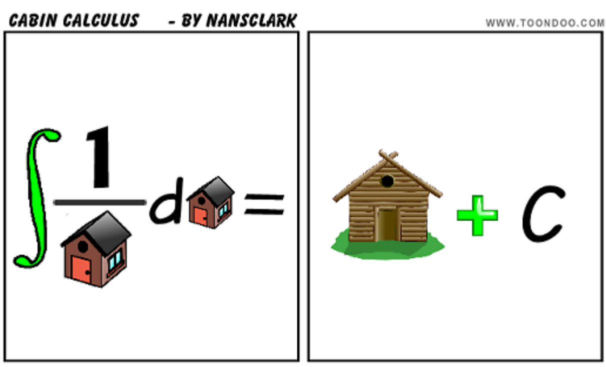
Therefore

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{2 \cdot 2\cos\theta}{3+2\sin\theta}\, d\theta = \int_1^5 \frac{2}{u}\, du = \Big[2\ln|u|\Big]_1^5 = 2\ln|5| - 2\ln|1| = 2\ln 5.$$

Break

We will continue at 3pm



The Integrals of $\tan x$, $\cot x$, $\sec x$ and $\operatorname{cosec} x$

We can also use

$$\int \frac{1}{u}\, du = \ln|u| + C$$

to integrate $\tan x$, $\cot x$, $\sec x$ and $\operatorname{cosec} x$.

$$\int \frac{du}{u} = \ln|u| + C$$

The Integral of $\tan x$

If $u = \cos x$, then $du = -\sin x\,dx$ and

$$\begin{aligned}\int \tan x\,dx = \int \frac{\sin x}{\cos x}\,dx &= \\ &= \\ &= \\ &= \\ &= \qquad\qquad .\end{aligned}$$

$$\int \frac{du}{u} = \ln|u| + C$$

The Integral of $\tan x$

If $u = \cos x$, then $du = -\sin x\, dx$ and

$$\begin{aligned}\int \tan x\, dx &= \int \frac{\sin x}{\cos x}\, dx = \int \frac{-du}{u} \\ &= \\ &= \\ &= \\ &= \qquad\qquad\qquad .\end{aligned}$$

$$\int \frac{du}{u} = \ln|u| + C$$

The Integral of $\tan x$

If $u = \cos x$, then $du = -\sin x\, dx$ and

$$\begin{aligned} \int \tan x\, dx &= \int \frac{\sin x}{\cos x}\, dx = \int \frac{-du}{u} \\ &= -\ln|u| + C = -\ln|\cos x| + C \\ &= \\ &= \\ &= \qquad\qquad\qquad . \end{aligned}$$

$$\int \frac{du}{u} = \ln|u| + C$$

The Integral of $\tan x$

If $u = \cos x$, then $du = -\sin x\, dx$ and

$$\begin{aligned}
\int \tan x\, dx = \int \frac{\sin x}{\cos x}\, dx = \int \frac{-du}{u} \\
&= -\ln|u| + C = -\ln|\cos x| + C \\
&= \ln|\cos x|^{-1} + C \\
&= \ln \frac{1}{|\cos x|} + C \\
&= \ln|\sec x| + C.
\end{aligned}$$

$$\int \frac{du}{u} = \ln|u| + C$$

The Integral of $\cot x$

If $u = \sin x$, then

$$\int \cot x \, dx = \int \frac{\cos x}{\sin x} \, dx = \qquad = \ldots = \qquad .$$

(you fill in the details)

$$\int \frac{du}{u} = \ln|u| + C$$

The Integral of $\cot x$

If $u = \sin x$, then

$$\int \cot x \, dx = \int \frac{\cos x}{\sin x} \, dx = \int \frac{du}{u} = \ldots = \qquad .$$

(you fill in the details)

The Integral of $\cot x$

If $u = \sin x$, then

$$\int \cot x \, dx = \int \frac{\cos x}{\sin x} \, dx = \int \frac{du}{u} = \ldots = -\ln|\operatorname{cosec} x| + C.$$

(you fill in the details)

$$\int \frac{du}{u} = \ln|u| + C \quad (\tan x)' = \sec^2 x \quad (\sec x)' = \sec x \tan x$$

The Integral of $\sec x$

This time, we need to do an extra step. We will multiply and divide by $u = (\sec x + \tan x)$.

$$\int \frac{du}{u} = \ln|u| + C \quad (\tan x)' = \sec^2 x \quad (\sec x)' = \sec x \tan x$$

The Integral of $\sec x$

This time, we need to do an extra step. We will multiply and divide by $u = (\sec x + \tan x)$. We calculate that

$$\int \sec x \, dx = \int \sec x \left(\frac{\sec x + \tan x}{\sec x + \tan x} \right) dx$$

$$=$$

$$=$$

$$= \qquad .$$

$$\int \frac{du}{u} = \ln|u| + C \quad (\tan x)' = \sec^2 x \quad (\sec x)' = \sec x \tan x$$

The Integral of $\sec x$

This time, we need to do an extra step. We will multiply and divide by $u = (\sec x + \tan x)$. We calculate that

$$\begin{aligned}
\int \sec x \, dx &= \int \sec x \left(\frac{\sec x + \tan x}{\sec x + \tan x} \right) dx \\
&= \int \frac{\sec^2 x + \sec x \tan x}{\sec x + \tan x} \, dx \\
&= \\
&= \qquad\qquad .
\end{aligned}$$

$$\int \frac{du}{u} = \ln|u| + C \quad (\tan x)' = \sec^2 x \quad (\sec x)' = \sec x \tan x$$

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&= \int \frac{\sec^2 x + \sec x \tan x}{\sec x + \tan x} \, dx \\
&= \int \frac{du}{u} \\
&= \qquad\qquad\qquad\qquad .
\end{aligned}$$

$$\int \frac{du}{u} = \ln|u| + C \quad (\tan x)' = \sec^2 x \quad (\sec x)' = \sec x \tan x$$

The Integral of $\sec x$

This time, we need to do an extra step. We will multiply and divide by $u = (\sec x + \tan x)$. We calculate that

$$\begin{aligned}
\int \sec x \, dx &= \int \sec x \left(\frac{\sec x + \tan x}{\sec x + \tan x} \right) dx \\
&= \int \frac{\sec^2 x + \sec x \tan x}{\sec x + \tan x} \, dx \\
&= \int \frac{du}{u} \\
&= \ln|u| + C = \ln|\sec x + \tan x| + C.
\end{aligned}$$

$\int \frac{du}{u} = \ln|u| + C \quad (\cot x)' = -\operatorname{cosec}^2 x \quad (\operatorname{cosec} x)' = -\operatorname{cosec} x \cot x$

The Integral of $\operatorname{cosec} x$

This is similar, except that we will multiply and divide by $u = (\operatorname{cosec} x + \cot x)$ instead. We calculate that

$$\int \operatorname{cosec} x \, dx = \int \operatorname{cosec} x \left(\frac{\operatorname{cosec} x + \cot x}{\operatorname{cosec} x + \cot x} \right) dx$$

$$=$$

$$=$$

$$= \qquad .$$

$\int \frac{du}{u} = \ln|u| + C \quad (\cot x)' = -\operatorname{cosec}^2 x \quad (\operatorname{cosec} x)' = -\operatorname{cosec} x \cot x$

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&= \int \frac{\operatorname{cosec}^2 x + \operatorname{cosec} x \cot x}{\sec x + \tan x} \, dx \\
&= \\
&= \qquad .
\end{aligned}$$

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&= \int \frac{-du}{u} \\
&= \qquad .
\end{aligned}$$

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&= \int \frac{\operatorname{cosec}^2 x + \operatorname{cosec} x \cot x}{\sec x + \tan x} \, dx \\
&= \int \frac{-du}{u} \\
&= -\ln|u| + C = -\ln|\operatorname{cosec} x + \cot x| + C.
\end{aligned}$$

Integrals of the tangent, cotangent, secant, and cosecant functions

$$\int \tan u \, du = \ln|\sec u| + C \qquad \int \sec u \, du = \ln|\sec u + \tan u| + C$$

$$\int \cot u \, du = \ln|\sin u| + C \qquad \int \csc u \, du = -\ln|\csc u + \cot u| + C$$

EXAMPLE 4

$$\int_0^{\pi/6} \tan 2x \, dx = \int_0^{\pi/3} \tan u \, \frac{du}{2} = \frac{1}{2}\int_0^{\pi/3} \tan u \, du$$

$$= \frac{1}{2}\ln |\sec u| \Big]_0^{\pi/3} = \frac{1}{2}(\ln 2 - \ln 1) = \frac{1}{2}\ln 2$$

Substitute $u = 2x$,
$dx = du/2$,
$u(0) = 0$,
$u(\pi/6) = \pi/3$

Logarithmic Differentiation

Sometimes when we are trying to find a derivative, it becomes easier if we take ln of both sides of an equation.

Logarithmic Differentiation

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Example

Find $\dfrac{dy}{dx}$ if $y = \dfrac{(x^2+1)(x+3)^{\frac{1}{2}}}{x-1}$ for $x > 1$.

We are going to take our equation and apply ln to both sides.

$$
\begin{aligned}
y &= \frac{(x^2+1)(x+3)^{\frac{1}{2}}}{x-1} \\
&= \\
&= \qquad\qquad\qquad\qquad .
\end{aligned}
$$

We are going to take our equation and apply ln to both sides.

$$\ln y = \ln \frac{(x^2+1)(x+3)^{\frac{1}{2}}}{x-1}$$

$$=$$

$$= \qquad .$$

We are going to take our equation and apply ln to both sides.

$$\begin{aligned}\ln y &= \ln \frac{(x^2+1)(x+3)^{\frac{1}{2}}}{x-1} \\ &= \ln\left((x^2+1)(x+3)^{\frac{1}{2}}\right) - \ln(x-1) \\ &= \qquad\qquad\qquad\qquad\qquad\qquad .\end{aligned}$$

7.2 Natural Logarithms

We are going to take our equation and apply ln to both sides.

$$
\begin{aligned}
\ln y &= \ln \frac{(x^2+1)(x+3)^{\frac{1}{2}}}{x-1} \\
&= \ln\left((x^2+1)(x+3)^{\frac{1}{2}}\right) - \ln(x-1) \\
&= \ln(x^2+1) + \frac{1}{2}\ln(x+3) - \ln(x-1).
\end{aligned}
$$

We are going to take our equation and apply ln to both sides.

$$\begin{aligned}\ln y = \ln \frac{(x^2+1)(x+3)^{\frac{1}{2}}}{x-1} &= \ln\left((x^2+1)(x+3)^{\frac{1}{2}}\right) - \ln(x-1) \\ &= \ln(x^2+1) + \frac{1}{2}\ln(x+3) - \ln(x-1).\end{aligned}$$

Now we can differentiate both sides.

$$\frac{d}{dx}\ln y = \frac{d}{dx}\left(\ln(x^2+1) + \frac{1}{2}\ln(x+3) - \ln(x-1)\right)$$

We are going to take our equation and apply ln to both sides.

$$\begin{aligned}\ln y &= \ln \frac{(x^2+1)(x+3)^{\frac{1}{2}}}{x-1} \\ &= \ln\left((x^2+1)(x+3)^{\frac{1}{2}}\right) - \ln(x-1) \\ &= \ln(x^2+1) + \frac{1}{2}\ln(x+3) - \ln(x-1).\end{aligned}$$

Now we can differentiate both sides.

$$\begin{aligned}\frac{d}{dx}\ln y &= \frac{d}{dx}\left(\ln(x^2+1) + \frac{1}{2}\ln(x+3) - \ln(x-1)\right) \\ \frac{1}{y}\frac{dy}{dx} &= \frac{2x}{x^2+1} + \frac{1}{2}\cdot\frac{1}{x+3} - \frac{1}{x-1}\end{aligned}$$

We are going to take our equation and apply ln to both sides.

$$\begin{aligned}\ln y &= \ln \frac{(x^2+1)(x+3)^{\frac{1}{2}}}{x-1} \\ &= \ln\left((x^2+1)(x+3)^{\frac{1}{2}}\right) - \ln(x-1) \\ &= \ln(x^2+1) + \frac{1}{2}\ln(x+3) - \ln(x-1).\end{aligned}$$

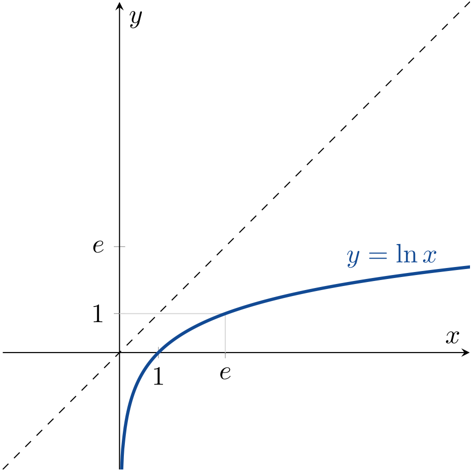
Now we can differentiate both sides.

$$\begin{aligned}\frac{d}{dx}\ln y &= \frac{d}{dx}\left(\ln(x^2+1) + \frac{1}{2}\ln(x+3) - \ln(x-1)\right) \\ \frac{1}{y}\frac{dy}{dx} &= \frac{2x}{x^2+1} + \frac{1}{2}\cdot\frac{1}{x+3} - \frac{1}{x-1} \\ \frac{dy}{dx} &= y\left(\frac{2x}{x^2+1} + \frac{1}{2x+6} - \frac{1}{x-1}\right).\end{aligned}$$

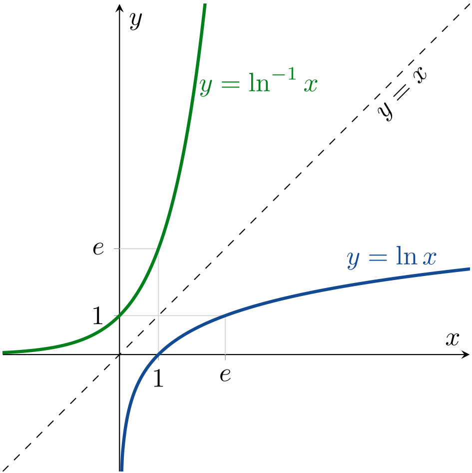
Thus

$$\begin{aligned}\frac{dy}{dx} &= y\left(\frac{2x}{x^2+1}+\frac{1}{2x+6}-\frac{1}{x-1}\right)\\ &= \left(\frac{(x^2+1)(x+3)^{\frac{1}{2}}}{x-1}\right)\left(\frac{2x}{x^2+1}+\frac{1}{2x+6}-\frac{1}{x-1}\right).\end{aligned}$$

7.3 Exponential Functions



ln is a one-to-one function with domain $(0, \infty)$ and range $(-\infty, \infty)$.



So there exists and inverse function $\ln^{-1}$ with domain $(-\infty, \infty)$ and range $(0, \infty)$.

Definition

The inverse of the function $\ln x$ is called the *exponential function* and is written

$$\exp x = e^x.$$

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$$\exp x = e^x.$$

Remark

We have

$$e^{\ln x} = x \qquad (\text{for all } x > 0)$$

and

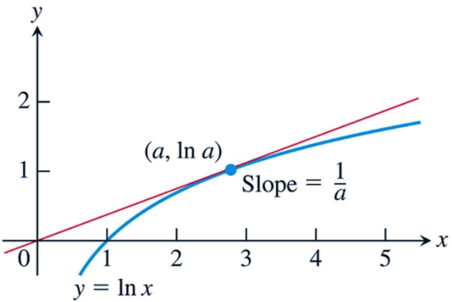
$$\ln\left(e^x\right) = x \qquad (\text{for all } x).$$

EXAMPLE 1 Solve the equation $e^{2x-6} = 4$ for x.

Solution We take the natural logarithm of both sides of the equation and use the second inverse equation:

$$
\begin{aligned}
\ln(e^{2x-6}) &= \ln 4 \\
2x - 6 &= \ln 4 && \text{Inverse relationship} \\
2x &= 6 + \ln 4 \\
x &= 3 + \frac{1}{2}\ln 4 = 3 + \ln 4^{1/2} \\
x &= 3 + \ln 2
\end{aligned}
$$

■



EXAMPLE 2 A line with slope m passes through the origin and is tangent to the graph of $y = \ln x$. What is the value of m?

Solution Suppose the point of tangency occurs at the unknown point $x = a > 0$. Then we know that the point $(a, \ln a)$ lies on the graph and that the tangent line at that point has slope $m = 1/a$ (Figure 7.11). Since the tangent line passes through the origin, its slope is

$$m = \frac{\ln a - 0}{a - 0} = \frac{\ln a}{a}.$$

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$$m = \frac{\ln a - 0}{a - 0} = \frac{\ln a}{a}.$$

Setting these two formulas for m equal to each other, we have

$$\begin{aligned} \frac{\ln a}{a} &= \frac{1}{a} \\ \ln a &= 1 \\ e^{\ln a} &= e^1 \\ a &= e \\ m &= \frac{1}{e}. \end{aligned}$$

The Derivative and Integral of e^x

$$\ln\left(e^x\right) = \quad x$$

The Derivative and Integral of e^x

$$\frac{d}{dx}\ln\left(e^x\right) = \frac{d}{dx}x$$

The Derivative and Integral of e^x

$$\frac{d}{dx}\ln\left(e^x\right) = \frac{d}{dx}x$$

$$\frac{1}{e^x}\frac{d}{dx}e^x = 1$$

The Derivative and Integral of e^x

$$\frac{d}{dx}\ln\left(e^x\right) = \frac{d}{dx}x$$
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by the Chain Rule.

The Derivative and Integral of e^x

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$$\frac{d}{dx}e^x = e^x.$$

by the Chain Rule.

Theorem

$$\boxed{\frac{d}{dx}e^x = e^x.}$$

EXAMPLE 3 We find derivatives of the exponential using Equation (2).

(a) $\dfrac{d}{dx}(5e^x) = 5\dfrac{d}{dx}e^x = 5e^x$

(b) $\dfrac{d}{dx}e^{-x} = e^{-x}\dfrac{d}{dx}(-x) = e^{-x}(-1) = -e^{-x}$ Eq. (2) with $u = -x$

(c) $\dfrac{d}{dx}e^{\sin x} = e^{\sin x}\dfrac{d}{dx}(\sin x) = e^{\sin x} \cdot \cos x$ Eq. (2) with $u = \sin x$

(d) $\dfrac{d}{dx}\left(e^{\sqrt{3x+1}}\right) = e^{\sqrt{3x+1}} \cdot \dfrac{d}{dx}\left(\sqrt{3x+1}\right)$ Eq. (2) with $u = \sqrt{3x+1}$

$$= e^{\sqrt{3x+1}} \cdot \frac{1}{2}(3x+1)^{-1/2} \cdot 3 = \frac{3}{2\sqrt{3x+1}}e^{\sqrt{3x+1}}$$

■

Since $\dfrac{d}{dx}e^x = e^x$, we also have

Theorem

$$\boxed{\int e^x \, dx = e^x + C.}$$

EXAMPLE 4

(a) $$\int_0^{\ln 2} e^{3x}\,dx = \int_0^{\ln 8} e^u \cdot \frac{1}{3}\,du$$

$u = 3x, \quad \frac{1}{3}du = dx, \quad u(0) = 0,$
$u(\ln 2) = 3\ln 2 = \ln 2^3 = \ln 8$

$$= \frac{1}{3}\int_0^{\ln 8} e^u\,du$$

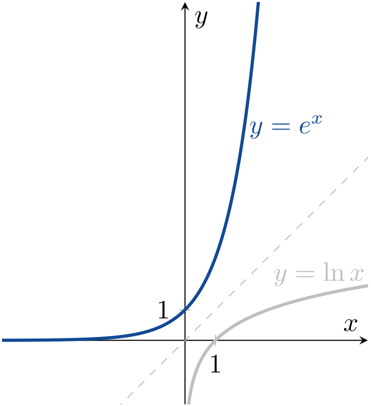
$$= \frac{1}{3}e^u\Big]_0^{\ln 8}$$

$$= \frac{1}{3}(8 - 1) = \frac{7}{3}$$

(b) $$\int_0^{\pi/2} e^{\sin x}\cos x\,dx = e^{\sin x}\Big]_0^{\pi/2}$$

Antiderivative from Example 2c

$$= e^1 - e^0 = e - 1$$



- e^x has domain $(-\infty, \infty)$ and range $(0, \infty)$;
- $\lim_{x\to\infty} e^x = \infty$; and
- $\lim_{x\to-\infty} e^x = 0$

Laws of Exponents

Theorem

1. $e^a e^b = e^{a+b}$

(proof in textbook)

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1. $e^a e^b = e^{a+b}$
2. $e^{-x} = \dfrac{1}{e^x}$
3. $\dfrac{e^a}{e^b} = e^{a-b}$

(proof in textbook)

Laws of Exponents

Theorem

1. $e^a e^b = e^{a+b}$
2. $e^{-x} = \dfrac{1}{e^x}$
3. $\dfrac{e^a}{e^b} = e^{a-b}$
4. $(e^a)^b = e^{ab}$.

(proof in textbook)

The General Exponential Function a^x

Definition

For any numbers $a > 0$ and x, the *exponential function with base* a is

$$a^x = e^{x \ln a}.$$

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Definition

For and $x > 0$ and for any $n \in \mathbb{N}$,

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Now we can prove

Theorem

For any $x > 0$ and any $n \in \mathbb{R}$,

$$\frac{d}{dx} x^n = n x^{n-1}.$$

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Theorem

For any $x > 0$ and any $n \in \mathbb{R}$,

$$\frac{d}{dx}x^n = nx^{n-1}.$$

If $x \leq 0$, then the formula holds whenever $\frac{d}{dx}x^n$, x^n and x^{n-1} all exist.

Proof.

1. If $x > 0$, then we calculate

$$\frac{d}{dx}x^n = \frac{d}{dx}e^{n\ln x} = e^{n\ln x}\cdot\frac{d}{dx}(n\ln x) = x^n\cdot\frac{n}{x} = nx^{n-1}.$$

$\square$

Proof.

2. If $x < 0$, and if y', $y = x^n$ and x^{n-1} all exists, then

$$\ln|y| = \ln|x|^n = n\ln|x|.$$

$\square$

Proof.

2. If $x < 0$, and if y', $y = x^n$ and x^{n-1} all exists, then

$$\ln|y| = \ln|x|^n = n \ln|x|.$$

Using Implicit Differentiation, we have

$$\ln|y| = \quad n \ln|x|$$

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$$\ln|y| = \ln|x|^n = n\ln|x|.$$

Using Implicit Differentiation, we have

$$\frac{d}{dx}\ln|y| = \frac{d}{dx}n\ln|x|$$

□

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$$\ln|y| = \ln|x|^n = n\ln|x|.$$

Using Implicit Differentiation, we have

$$\begin{aligned} \frac{d}{dx}\ln|y| &= \frac{d}{dx}n\ln|x| \\ \frac{y'}{y} &= \frac{n}{x} \end{aligned}$$

$\square$

Proof.

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$$\ln|y| = \ln|x|^n = n\ln|x|.$$

Using Implicit Differentiation, we have

$$\begin{aligned}\frac{d}{dx}\ln|y| &= \frac{d}{dx}n\ln|x| \\ \frac{y'}{y} &= \frac{n}{x} \\ y' &= n\frac{y}{x} = n\frac{x^n}{x} = nx^{n-1}.\end{aligned}$$

$\square$

Proof.

3. For $x = 0$ and $n \geq 1$, we just use the definition of the derivative to show that

$$\left(x^n\right)'(0) = \lim_{h \to 0} \frac{(0+h)^n - 0^n}{h} = \lim_{h \to 0} h^{n-1} = 0.$$

(Note 0^{n-1} does not exist if $n < 1$.)

$\square$

EXAMPLE 5 Differentiate $f(x) = x^x$, $x > 0$.

Solution We cannot apply the power rule here because the exponent is the *variable* x rather than being a constant value n (rational or irrational). However, from the definition of the general exponential function we note that $f(x) = x^x = e^{x \ln x}$, and differentiation gives

$$\begin{aligned}
f'(x) &= \frac{d}{dx}\left(e^{x \ln x}\right) \\
&= e^{x \ln x} \frac{d}{dx}(x \ln x) && \text{Eq. (2) with } u = x \ln x \\
&= e^{x \ln x}\left(\ln x + x \cdot \frac{1}{x}\right) && \text{Product Rule} \\
&= x^x (\ln x + 1). && x > 0
\end{aligned}$$

■

The number e Expressed as a Limit

Theorem

$$e = \lim_{x \to 0} (1 + x)^{\frac{1}{x}}.$$

The number e Expressed as a Limit

Theorem

$$e = \lim_{x \to 0} (1 + x)^{\frac{1}{x}}.$$

Proof.

Let $f(x) = \ln x$. Then $f'(x) = \frac{1}{x}$ and $f'(1) = 1$.

The number e Expressed as a Limit

Theorem

$$e = \lim_{x \to 0} (1+x)^{\frac{1}{x}}.$$

Proof.

Let $f(x) = \ln x$. Then $f'(x) = \frac{1}{x}$ and $f'(1) = 1$. Therefore

$$1 = f'(1) = \lim_{h \to 0} \frac{f(1+h) - f(1)}{h} =$$

$$= \qquad =$$

$$= \qquad = \qquad .$$

The number e Expressed as a Limit

Theorem

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Proof.

Let $f(x) = \ln x$. Then $f'(x) = \frac{1}{x}$ and $f'(1) = 1$. Therefore

$$1 = f'(1) = \lim_{h\to 0}\frac{f(1+h)-f(1)}{h} = \lim_{x\to 0}\frac{f(1+x)-f(1)}{x}$$

$$=$$

$$=$$

$$=$$

$$= \qquad .$$

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Proof.

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$$\begin{aligned} 1 = f'(1) &= \lim_{h \to 0} \frac{f(1+h) - f(1)}{h} = \lim_{x \to 0} \frac{f(1+x) - f(1)}{x} \\ &= \lim_{x \to 0} \frac{\ln(1+x) - 0}{x} = \lim_{x \to 0} \frac{1}{x} \ln(1+x) \\ &= \lim_{x \to 0} \ln(1+x)^{\frac{1}{x}} = \qquad\qquad\qquad . \end{aligned}$$

The number e Expressed as a Limit

Theorem

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Proof.

Let $f(x) = \ln x$. Then $f'(x) = \frac{1}{x}$ and $f'(1) = 1$. Therefore

$$\begin{aligned} 1 = f'(1) = \lim_{h \to 0} \frac{f(1+h) - f(1)}{h} &= \lim_{x \to 0} \frac{f(1+x) - f(1)}{x} \\ = \lim_{x \to 0} \frac{\ln(1+x) - 0}{x} &= \lim_{x \to 0} \frac{1}{x} \ln(1+x) \\ = \lim_{x \to 0} \ln(1+x)^{\frac{1}{x}} &= \ln \left(\lim_{x \to 0} (1+x)^{\frac{1}{x}} \right). \end{aligned}$$

Proof continued.

$$1 = \ln\left(\lim_{x\to 0}(1+x)^{\frac{1}{x}}\right)$$

Taking the exponential of both sides gives

$$e = \lim_{x\to 0}(1+x)^{\frac{1}{x}}$$

as required. $\square$

Proof continued.

$$1 = \ln\left(\lim_{x\to 0}(1+x)^{\frac{1}{x}}\right)$$

Taking the exponential of both sides gives

$$e = \lim_{x\to 0}(1+x)^{\frac{1}{x}}$$

as required. □

Remark

By taking very small x in $(1+x)^{\frac{1}{x}}$, we can calculate

$$e \approx 2.718281828459045$$

to 15 decimal places.

The derivative of a^x

$$\frac{d}{dx}a^x = \frac{d}{dx}e^{x\ln a} = e^{x\ln a} \cdot \frac{d}{dx}(x\ln a) = a^x \ln a.$$

The derivative of a^x

$$\frac{d}{dx}a^x = \frac{d}{dx}e^{x\ln a} = e^{x\ln a}\cdot\frac{d}{dx}(x\ln a) = a^x\ln a.$$

e is a special number because

$$\frac{d}{dx}e^x = e^x\ln e = e^x\cdot 1 = e^x.$$

By the Chain Rule, if u is differentiable then

$$\frac{d}{dx}a^{u} = a^{u}\ln a\frac{du}{dx}$$

for $a > 0$.

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for $a > 0$.

Equivalently

$$\int a^u \, du = \frac{a^u}{\ln a} + C.$$

EXAMPLE 6 We find derivatives and integrals using Equations (3) and (4).

(a) $\frac{d}{dx}3^x = 3^x \ln 3$ Eq. (3) with $a = 3$, $u = x$

(b) $\frac{d}{dx}3^{-x} = 3^{-x}(\ln 3)\frac{d}{dx}(-x) = -3^{-x}\ln 3$ Eq. (3) with $a = 3$, $u = -x$

(c) $\frac{d}{dx}3^{\sin x} = 3^{\sin x}(\ln 3)\frac{d}{dx}(\sin x) = 3^{\sin x}(\ln 3)\cos x$ Eq. (3) with $a = 3$, $u = \sin x$

(d) $\int 2^x\,dx = \frac{2^x}{\ln 2} + C$ Eq. (4) with $a = 2$, $u = x$

(e) $\int 2^{\sin x}\cos x\,dx = \int 2^u\,du = \frac{2^u}{\ln 2} + C$ $u = \sin x$, $du = \cos x\,dx$, and Eq. (4)

$= \frac{2^{\sin x}}{\ln 2} + C$ u replaced by $\sin x$ ■

Logarithms with Base a

If $a > 0$ and $a \neq 1$, then a^x is a one-to-one function. Therefore it has an inverse function.

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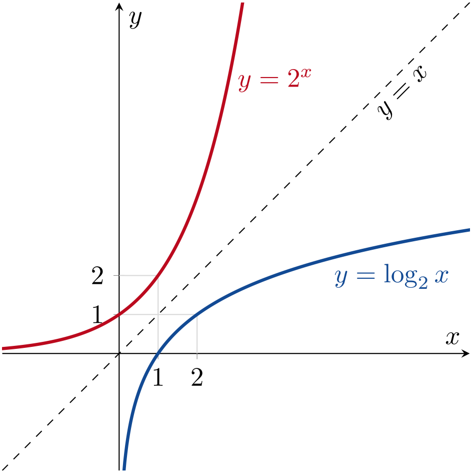
Definition

If $a > 0$ and $a \neq 1$, then the inverse of a^x is called the *logarithm of x with base a* and is denoted by $\log_a x$.



y
$y = 2^x$
$y = x$
$y = \log_2 x$
x
2
1
1
2

Remark

We have

$$a^{\log_a x} = x \qquad \text{(for all } x > 0\text{)}$$

and

$$\log_a(a^x) = x \qquad \text{(for all } x\text{)}.$$

Remark

Note that

$$\begin{aligned} y &= \log_a x \\ a^y &= x \\ \ln a^y &= \ln x \\ y \ln a &= \ln x \\ y &= \frac{\ln x}{\ln a}. \end{aligned}$$

Remark

Note that

$$\begin{aligned} y &= \log_a x \\ a^y &= x \\ \ln a^y &= \ln x \\ y \ln a &= \ln x \\ y &= \frac{\ln x}{\ln a}. \end{aligned}$$

Therefore

$$\boxed{\log_a x = \frac{\ln x}{\ln a}.}$$

Derivatives and Integrals Involving $\log_a x$

Note that

$$\frac{d}{dx}\log_a u = \frac{d}{dx}\left(\frac{\ln u}{\ln a}\right) = \frac{1}{\ln a}\frac{d}{dx}\ln u = \frac{1}{\ln a}\cdot\frac{1}{u}\cdot\frac{du}{dx}.$$

$$\boxed{\frac{d}{dx}\log_a u = \frac{1}{\ln a}\cdot\frac{1}{u}\cdot\frac{du}{dx}}$$

EXAMPLE 7

(a) $$\frac{d}{dx}\log_{10}(3x+1) = \frac{1}{\ln 10}\cdot\frac{1}{3x+1}\frac{d}{dx}(3x+1) = \frac{3}{(\ln 10)(3x+1)}$$

(b) $$\int \frac{\log_2 x}{x}\,dx = \frac{1}{\ln 2}\int \frac{\ln x}{x}\,dx \qquad \log_2 x = \frac{\ln x}{\ln 2}$$

$$= \frac{1}{\ln 2}\int u\,du \qquad u = \ln x,\quad du = \frac{1}{x}dx$$

$$= \frac{1}{\ln 2}\frac{u^2}{2} + C = \frac{1}{\ln 2}\frac{(\ln x)^2}{2} + C = \frac{(\ln x)^2}{2\ln 2} + C$$

Next Time

- 7.5 Indeterminate Forms and L'Hôpital's Rule
- 7.6 Inverse Trigonometric Functions
- 7.7 Hyperbolic Functions