

Lecture 10

- Inner Product Spaces
- Orthogonality
- Orthogonal Sets and Orthonormal Sets



Inner Product Spaces

Inner Product Spaces



In MATH114 Mathematics II we studied the *dot product* of two vectors in \mathbb{R}^2 or \mathbb{R}^3 .

You will recall that if $\mathbf{u} = (u_1, u_2, u_3), \mathbf{v} = (v_1, v_2, v_3) \in \mathbb{R}^3$ then

$$\mathbf{u} \cdot \mathbf{v} = u_1v_1 + u_2v_2 + u_3v_3$$

and

$$\mathbf{u} \cdot \mathbf{u} = \|\mathbf{u}\|^2.$$

Inner Product Spaces



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and

$$\mathbf{u} \cdot \mathbf{u} = \|\mathbf{u}\|^2.$$

This week will extend these ideas to real vector spaces.

(We will not concern ourselves with complex numbers this week.
Any time you see a scalar k you can assume that it is a real number.)

Inner Product Spaces



Definition

An *inner product* on a (real) vector space V is a function

$$\langle \cdot, \cdot \rangle : V \times V \rightarrow \mathbb{R}$$

that satisfies the following axioms for all vectors \mathbf{u} , \mathbf{v} , and \mathbf{w} in V and all scalars k :

1

2

3

4

Inner Product Spaces



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- 1 $\langle \mathbf{u}, \mathbf{v} \rangle = \langle \mathbf{v}, \mathbf{u} \rangle$ [Symmetry axiom]
- 2
- 3
- 4

Inner Product Spaces



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- 1** $\langle \mathbf{u}, \mathbf{v} \rangle = \langle \mathbf{v}, \mathbf{u} \rangle$ [Symmetry axiom]
- 2** $\langle \mathbf{u} + \mathbf{v}, \mathbf{w} \rangle = \langle \mathbf{u}, \mathbf{w} \rangle + \langle \mathbf{v}, \mathbf{w} \rangle$ [Additivity axiom]
- 3**
- 4**

Inner Product Spaces



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- 1** $\langle \mathbf{u}, \mathbf{v} \rangle = \langle \mathbf{v}, \mathbf{u} \rangle$ [Symmetry axiom]
- 2** $\langle \mathbf{u} + \mathbf{v}, \mathbf{w} \rangle = \langle \mathbf{u}, \mathbf{w} \rangle + \langle \mathbf{v}, \mathbf{w} \rangle$ [Additivity axiom]
- 3** $\langle k\mathbf{u}, \mathbf{v} \rangle = k \langle \mathbf{u}, \mathbf{v} \rangle$ [Homogeneity axiom]
- 4**

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- 1 $\langle \mathbf{u}, \mathbf{v} \rangle = \langle \mathbf{v}, \mathbf{u} \rangle$ [Symmetry axiom]
- 2 $\langle \mathbf{u} + \mathbf{v}, \mathbf{w} \rangle = \langle \mathbf{u}, \mathbf{w} \rangle + \langle \mathbf{v}, \mathbf{w} \rangle$ [Additivity axiom]
- 3 $\langle k\mathbf{u}, \mathbf{v} \rangle = k \langle \mathbf{u}, \mathbf{v} \rangle$ [Homogeneity axiom]
- 4 $\langle \mathbf{v}, \mathbf{v} \rangle \geq 0$ and $\langle \mathbf{v}, \mathbf{v} \rangle = 0$ if and only if $\mathbf{v} = \mathbf{0}$. [Positivity axiom]

Inner Product Spaces



Remark

If we combine

$$1 \quad \langle \mathbf{u}, \mathbf{v} \rangle = \langle \mathbf{v}, \mathbf{u} \rangle \quad [\text{Symmetry axiom}]$$

and

$$2 \quad \langle \mathbf{u} + \mathbf{v}, \mathbf{w} \rangle = \langle \mathbf{u}, \mathbf{w} \rangle + \langle \mathbf{v}, \mathbf{w} \rangle \quad [\text{Additivity axiom}]$$

then we can show that

$$\langle \mathbf{u}, \mathbf{v} + \mathbf{w} \rangle = \langle \mathbf{v} + \mathbf{w}, \mathbf{u} \rangle = \langle \mathbf{v}, \mathbf{u} \rangle + \langle \mathbf{w}, \mathbf{u} \rangle = \langle \mathbf{u}, \mathbf{v} \rangle + \langle \mathbf{u}, \mathbf{w} \rangle .$$

Inner Product Spaces



Remark

Likewise, we can combine

$$1 \quad \langle \mathbf{u}, \mathbf{v} \rangle = \langle \mathbf{v}, \mathbf{u} \rangle \quad [\text{Symmetry axiom}]$$

and

$$3 \quad \langle k\mathbf{u}, \mathbf{v} \rangle = k \langle \mathbf{u}, \mathbf{v} \rangle \quad [\text{Homogeneity axiom}]$$

to prove that

$$\langle \mathbf{u}, k\mathbf{v} \rangle = k \langle \mathbf{u}, \mathbf{v} \rangle$$

Inner Product Spaces



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Likewise, we can combine

1 $\langle \mathbf{u}, \mathbf{v} \rangle = \langle \mathbf{v}, \mathbf{u} \rangle$ [Symmetry axiom]

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to prove that

$$\langle \mathbf{u}, k\mathbf{v} \rangle = k \langle \mathbf{u}, \mathbf{v} \rangle$$

and then we can prove that

$$\langle \mathbf{0}, \mathbf{v} \rangle = 0 = \langle \mathbf{v}, \mathbf{0} \rangle .$$

(proofs left to you.)

Example

Let $V = \mathbb{R}^n$. The function

$$\langle \mathbf{u}, \mathbf{v} \rangle = \mathbf{u} \cdot \mathbf{v} = u_1 v_1 + u_2 v_2 + \dots + u_n v_n$$

is an inner product on \mathbb{R}^n . I leave it to you to prove that all 4 axioms are satisfied. This inner product is called the *Euclidean inner product* on \mathbb{R}^n .

Definition

A (real) vector space with an inner product is called a *(real) inner product space*.

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Example

\mathbb{R}^n with $\langle \mathbf{u}, \mathbf{v} \rangle = \mathbf{u} \cdot \mathbf{v}$ is an inner product space.

Definition

If V is a inner product space, then the *norm* (or *length*) of a vector \mathbf{v} in V is denoted by $\|\mathbf{v}\|$ and is defined by

$$\|\mathbf{v}\| = \sqrt{\langle \mathbf{v}, \mathbf{v} \rangle}.$$

Definition

A vector of norm 1 is called a *unit vector*.

$$\|\mathbf{v}\| = \sqrt{\langle \mathbf{v}, \mathbf{v} \rangle}$$



Theorem

If \mathbf{u} and \mathbf{v} are vectors in a inner product space V , and if k is a scalar, then:

- 1 $\|\mathbf{v}\| \geq 0$
- 2 $\|\mathbf{v}\| = 0$ if and only if $\mathbf{v} = \mathbf{0}$.
- 3 $\|k\mathbf{v}\| = |k| \|\mathbf{v}\|$.

$$\|\mathbf{v}\| = \sqrt{\langle \mathbf{v}, \mathbf{v} \rangle}$$



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Proof.

- 1 \sqrt{x} is always ≥ 0 .

$$\|\mathbf{v}\| = \sqrt{\langle \mathbf{v}, \mathbf{v} \rangle}$$



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- 1 \sqrt{x} is always ≥ 0 .
- 2 $\|\mathbf{v}\| = 0 \iff \langle \mathbf{v}, \mathbf{v} \rangle = 0 \iff \mathbf{v} = \mathbf{0}$ by definition.

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Proof.

- 1 \sqrt{x} is always ≥ 0 .
- 2 $\|\mathbf{v}\| = 0 \iff \langle \mathbf{v}, \mathbf{v} \rangle = 0 \iff \mathbf{v} = \mathbf{0}$ by definition.
- 3 $\|k\mathbf{v}\|^2 = \langle k\mathbf{v}, k\mathbf{v} \rangle = k \langle \mathbf{v}, k\mathbf{v} \rangle = k^2 \langle \mathbf{v}, \mathbf{v} \rangle = k^2 \|\mathbf{v}\|^2$.



$$\|\mathbf{v}\| = \sqrt{\langle \mathbf{v}, \mathbf{v} \rangle}$$



Definition

If V is an inner product space, then the *distance* between two vectors \mathbf{u} and \mathbf{v} in V is

$$\text{dist}(\mathbf{u}, \mathbf{v}) = \|\mathbf{u} - \mathbf{v}\|.$$

Inner Product Spaces



Example (A Weighted Inner Product)

Let $\mathbf{u}, \mathbf{v} \in \mathbb{R}^2$. Show that the function

$$\langle \mathbf{u}, \mathbf{v} \rangle = 3u_1v_1 + 2u_2v_2$$

is an inner product on \mathbb{R}^2 .

We need to show that all four of the axioms are satisfied.

Inner Product Spaces

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- 1 Interchanging \mathbf{u} and \mathbf{v} in the formula does not change the sum on the right side, so $\langle \mathbf{u}, \mathbf{v} \rangle = \langle \mathbf{v}, \mathbf{u} \rangle$.

Inner Product Spaces



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is an inner product on \mathbb{R}^2 .

We need to show that all four of the axioms are satisfied.

2 We have

$$\begin{aligned}\langle \mathbf{u} + \mathbf{v}, \mathbf{w} \rangle &= 3(u_1 + v_1)w_1 + 2(u_2 + v_2)w_2 \\&= 3(u_1w_1 + v_1w_1) + 2(u_2w_2 + v_2w_2) \\&= \dots = \langle \mathbf{u}, \mathbf{w} \rangle + \langle \mathbf{v}, \mathbf{w} \rangle\end{aligned}$$

3 and that

$$\langle k\mathbf{u}, \mathbf{v} \rangle = 3(ku_1)v_1 + 2(ku_2)v_2 = k(3u_1v_1 + 2u_2v_2) = k \langle \mathbf{u}, \mathbf{v} \rangle .$$

Inner Product Spaces



Example (A Weighted Inner Product)

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$$\langle \mathbf{u}, \mathbf{v} \rangle = 3u_1v_1 + 2u_2v_2$$

is an inner product on \mathbb{R}^2 .

We need to show that all four of the axioms are satisfied.

4 Finally note that

$$\langle \mathbf{v}, \mathbf{v} \rangle = 3v_1^2 + 2v_2^2 \geq 0$$

and we can only get “= 0” here if $v_1 = v_2 = 0$.

Therefore $\langle \cdot, \cdot \rangle$ is an inner product. This is called a *weighted Euclidean inner product*. The numbers 3 and 2 are called the *weights*.

Inner Product Spaces



Example (Calculating with a Weighted Euclidean Inner Product)

Let $\mathbf{u} = (1, 0) \in \mathbb{R}^2$. Note that if we use the Euclidean inner product, then we have

$$\|\mathbf{u}\| = \sqrt{\mathbf{u} \cdot \mathbf{u}} = \sqrt{1^2 + 0^2} = 1.$$

Inner Product Spaces



Example (Calculating with a Weighted Euclidean Inner Product)

Let $\mathbf{u} = (1, 0) \in \mathbb{R}^2$. Note that if we use the Euclidean inner product, then we have

$$\|\mathbf{u}\| = \sqrt{\mathbf{u} \cdot \mathbf{u}} = \sqrt{1^2 + 0^2} = 1.$$

However if we use the weighted Euclidean inner product from the last example, then

$$\|\mathbf{u}\| = \sqrt{\langle \mathbf{u}, \mathbf{u} \rangle} = \sqrt{3u_1^2 + 2u_2^2} = \sqrt{3(1)^2 + 2(0)^2} = \sqrt{3}.$$

Unit Circles and Unit Spheres

Definition

If V is an inner product space, then the set of points in V that satisfy

$$\|\mathbf{u}\| = 1$$

is called the *unit circle* or *unit sphere* in V .

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Example (Unusual Unit Circles in \mathbb{R}^2)

- 1 Sketch the unit circle in \mathbb{R}^2 using the Euclidean inner product $\langle \mathbf{u}, \mathbf{v} \rangle = u_1v_1 + u_2v_2$.
- 2 Sketch the unit circle in \mathbb{R}^2 using the weighted Euclidean inner product $\langle \mathbf{u}, \mathbf{v} \rangle = \frac{1}{9}u_1v_1 + \frac{1}{4}u_2v_2$.

Inner Product Spaces



Let $\mathbf{u} = (x, y)$. Then

1

$$1 = \|\mathbf{u}\|^2 = (x, y) \cdot (x, y) = x^2 + y^2$$

and

2

$$1 = \|\mathbf{u}\|^2 = \langle \mathbf{u}, \mathbf{u} \rangle = \frac{1}{9}u_1u_1 + \frac{1}{4}u_2u_2 = \frac{x^2}{9} + \frac{y^2}{4}.$$

Inner Product Spaces



Let $\mathbf{u} = (x, y)$. Then

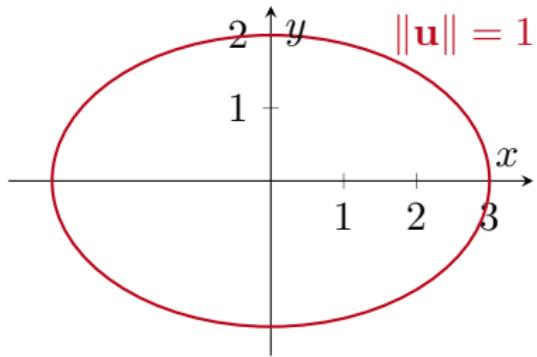
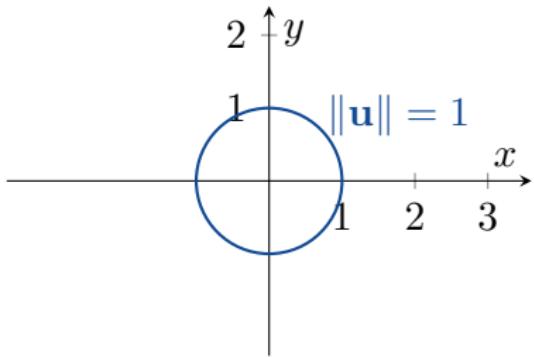
1

$$1 = \|\mathbf{u}\|^2 = (x, y) \cdot (x, y) = x^2 + y^2$$

and

2

$$1 = \|\mathbf{u}\|^2 = \langle \mathbf{u}, \mathbf{u} \rangle = \frac{1}{9}u_1u_1 + \frac{1}{4}u_2u_2 = \frac{x^2}{9} + \frac{y^2}{4}.$$



Inner Products Generated by Matrices

Definition

Let A be an invertible $n \times n$ matrix. Then the formula

$$\langle \mathbf{u}, \mathbf{v} \rangle = A\mathbf{u} \cdot A\mathbf{v}$$

defines an inner product on \mathbb{R}^n called the *inner product generated by A* .

Remark

Note that

$$\langle \mathbf{u}, \mathbf{v} \rangle = A\mathbf{u} \cdot A\mathbf{v} = (A\mathbf{v})^T A\mathbf{u} = \mathbf{v}^T A^T A\mathbf{u}.$$

Inner Product Spaces



Example

Let $A = \begin{bmatrix} \sqrt{3} & 0 \\ 0 & \sqrt{2} \end{bmatrix}$. Then the inner product on \mathbb{R}^2 generated by A is one of the weighted Euclidean inner products that we looked at earlier:

$$\langle \mathbf{u}, \mathbf{v} \rangle = A\mathbf{u} \cdot A\mathbf{v} = \begin{bmatrix} \sqrt{3}u_1 \\ \sqrt{2}u_2 \end{bmatrix} \cdot \begin{bmatrix} \sqrt{3}v_1 \\ \sqrt{2}v_2 \end{bmatrix} = 3u_1v_2 + 2u_1v_2.$$

Note that

$$A^T A = \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}.$$

More Inner Products

Example (The Standard Inner Product on $\mathbb{R}^{n \times n} = M_{nn}$)

If $\mathbf{u} = U$ and $\mathbf{v} = V$ are matrices in the vector space $\mathbb{R}^{n \times n}$, then the formula

$$\langle \mathbf{u}, \mathbf{v} \rangle = \text{tr}(U^T V)$$

defines an inner product on $\mathbb{R}^{n \times n}$ called the *standard inner product* on $\mathbb{R}^{n \times n}$.

More Inner Products

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defines an inner product on $\mathbb{R}^{n \times n}$ called the *standard inner product* on $\mathbb{R}^{n \times n}$.

This can be proved by confirming that the four inner product axioms are satisfied. But there is an easier way:

Inner Product Spaces

If $\mathbf{u} = U = \begin{bmatrix} u_1 & u_2 \\ u_3 & u_4 \end{bmatrix}$ and $\mathbf{v} = V = \begin{bmatrix} v_1 & v_2 \\ v_3 & v_4 \end{bmatrix}$ then

$$\begin{aligned}\langle \mathbf{u}, \mathbf{v} \rangle &= \text{tr}(U^T V) = \text{tr} \begin{bmatrix} u_1 v_1 + u_3 v_3 & u_2 v_2 + u_4 v_4 \\ u_1 v_1 + u_3 v_3 & u_2 v_2 + u_4 v_4 \end{bmatrix} \\ &= u_1 v_1 + u_2 v_2 + u_3 v_3 + u_4 v_4\end{aligned}$$

which is just like the dot product in \mathbb{R}^4 .

Inner Product Spaces

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$$\begin{aligned}\langle \mathbf{u}, \mathbf{v} \rangle &= \text{tr}(U^T V) = \text{tr} \begin{bmatrix} u_1 v_1 + u_3 v_3 & u_2 v_2 + u_4 v_4 \\ u_1 v_1 + u_3 v_3 & u_2 v_2 + u_4 v_4 \end{bmatrix} \\ &= u_1 v_1 + u_2 v_2 + u_3 v_3 + u_4 v_4\end{aligned}$$

which is just like the dot product in \mathbb{R}^4 .

For example, if

$$\mathbf{u} = U = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \quad \text{and} \quad \mathbf{v} = V = \begin{bmatrix} -1 & 0 \\ 3 & 2 \end{bmatrix}$$

then

$$\langle \mathbf{u}, \mathbf{v} \rangle = \text{tr}(U^T V) = 1(-1) + 2(0) + 3(3) + 4(2) = 16$$

and

$$\|\mathbf{u}\| = \sqrt{\langle \mathbf{u}, \mathbf{u} \rangle} = \sqrt{1^2 + 2^2 + 3^2 + 4^2} = \sqrt{30}.$$

Inner Product Spaces

Example (The Standard Inner Product on \mathbb{P}^n)

If

$$\mathbf{p} = a_0 + a_1x + \dots + a_nx^n$$

and

$$\mathbf{q} = b_0 + b_1x + \dots + b_nx^n$$

are polynomials in \mathbb{P}^n , then the following formula defines an inner product on \mathbb{P}^n (please verify) that we will call the *standard inner product* on \mathbb{P}^n :

$$\langle \mathbf{p}, \mathbf{q} \rangle = a_0b_0 + a_1b_1 + \dots + a_nb_n$$

The norm of a polynomial \mathbf{p} relative to this inner product is

$$\|\mathbf{p}\| = \sqrt{\langle \mathbf{p}, \mathbf{p} \rangle} = \sqrt{a_0^2 + a_1^2 + \dots + a_n^2}.$$

Inner Product Spaces



Example (The Evaluation Inner Product on \mathbb{P}^n)

If

$$\mathbf{p} = a_0 + a_1x + \dots + a_nx^n$$

and

$$\mathbf{q} = b_0 + b_1x + \dots + b_nx^n$$

are polynomials in \mathbb{P}^n , and if x_0, x_1, \dots, x_n are distinct real numbers, then the following formula defines an inner product on \mathbb{P}^n that we call the *evaluation inner product* at x_0, x_1, \dots, x_n :

$$\langle \mathbf{p}, \mathbf{q} \rangle = p(x_0)q(x_0) + p(x_1)q(x_1) + \dots + p(x_n)q(x_n)$$

Inner Product Spaces



Example (The Evaluation Inner Product on \mathbb{P}^n)

If

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are polynomials in \mathbb{P}^n , and if x_0, x_1, \dots, x_n are distinct real numbers, then the following formula defines an inner product on \mathbb{P}^n that we call the *evaluation inner product* at x_0, x_1, \dots, x_n :

$$\langle \mathbf{p}, \mathbf{q} \rangle = p(x_0)q(x_0) + p(x_1)q(x_1) + \dots + p(x_n)q(x_n)$$

We can think of this as the dot product of the vector $(p(x_0), p(x_1), \dots, p(x_n))$ with the vector $(q(x_0), q(x_1), \dots, q(x_n))$.

Inner Product Spaces



The first three axioms follow from properties of the dot product.
For the fourth axiom, we have that

$$\langle \mathbf{p}, \mathbf{p} \rangle = [p(x_0)]^2 + [p(x_1)]^2 + \dots + [p(x_n)]^2 \geq 0$$

for all polynomials \mathbf{p} . We can only have “= 0” here if

$$p(x_0) = p(x_1) = \dots = p(x_n) = 0.$$

However the only polynomial, of n th degree or less, with $n + 1$ roots is $\mathbf{p} = \mathbf{0}$.

Example (An Integral Inner Product on $C[a, b]$)

Show that the following function is an inner product on $C[a, b]$:

$$\langle \mathbf{f}, \mathbf{g} \rangle = \int_a^b f(x)g(x) dx.$$

1

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1 $\langle \mathbf{f}, \mathbf{g} \rangle = \int_a^b f(x)g(x) dx = \int_a^b g(x)f(x) dx = \langle \mathbf{g}, \mathbf{f} \rangle .$

2

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2 $\langle \mathbf{f} + \mathbf{g}, \mathbf{h} \rangle = \int_a^b (f(x) + g(x))h(x) dx =$
 $\int_a^b f(x)h(x) dx + \int_a^b g(x)h(x) dx = \langle \mathbf{f}, \mathbf{h} \rangle + \langle \mathbf{g}, \mathbf{h} \rangle .$

3

4

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- 1 $\langle \mathbf{f}, \mathbf{g} \rangle = \int_a^b f(x)g(x) dx = \int_a^b g(x)f(x) dx = \langle \mathbf{g}, \mathbf{f} \rangle .$
- 2 $\langle \mathbf{f} + \mathbf{g}, \mathbf{h} \rangle = \int_a^b (f(x) + g(x))h(x) dx =$
 $\int_a^b f(x)h(x) dx + \int_a^b g(x)h(x) dx = \langle \mathbf{f}, \mathbf{h} \rangle + \langle \mathbf{g}, \mathbf{h} \rangle .$
- 3 $\langle k\mathbf{f}, \mathbf{g} \rangle = \int_a^b kf(x)g(x) dx = k \int_a^b f(x)g(x) dx = k \langle \mathbf{f}, \mathbf{g} \rangle .$
- 4

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$$\langle \mathbf{f}, \mathbf{g} \rangle = \int_a^b f(x)g(x) dx.$$

- 1 $\langle \mathbf{f}, \mathbf{g} \rangle = \int_a^b f(x)g(x) dx = \int_a^b g(x)f(x) dx = \langle \mathbf{g}, \mathbf{f} \rangle .$
- 2 $\langle \mathbf{f} + \mathbf{g}, \mathbf{h} \rangle = \int_a^b (f(x) + g(x))h(x) dx =$
 $\int_a^b f(x)h(x) dx + \int_a^b g(x)h(x) dx = \langle \mathbf{f}, \mathbf{h} \rangle + \langle \mathbf{g}, \mathbf{h} \rangle .$
- 3 $\langle k\mathbf{f}, \mathbf{g} \rangle = \int_a^b kf(x)g(x) dx = k \int_a^b f(x)g(x) dx = k \langle \mathbf{f}, \mathbf{g} \rangle .$
- 4 $\langle \mathbf{f}, \mathbf{f} \rangle = \int_a^b (f(x))^2 dx \geq 0$ since $(f(x))^2 \geq 0$ for all $x \in [a, b]$.
Since f is continuous, we can only have " $= 0$ " here if $\mathbf{f} = \mathbf{0}$.

Calculating with Inner Products

$$\langle \mathbf{u} - 2\mathbf{v}, 3\mathbf{u} + 4\mathbf{v} \rangle = \langle \mathbf{u}, 3\mathbf{u} + 4\mathbf{v} \rangle - \langle 2\mathbf{v}, 3\mathbf{u} + 4\mathbf{v} \rangle$$

=

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Calculating with Inner Products

$$\begin{aligned}\langle \mathbf{u} - 2\mathbf{v}, 3\mathbf{u} + 4\mathbf{v} \rangle &= \langle \mathbf{u}, 3\mathbf{u} + 4\mathbf{v} \rangle - \langle 2\mathbf{v}, 3\mathbf{u} + 4\mathbf{v} \rangle \\&= \langle \mathbf{u}, 3\mathbf{u} \rangle + \langle \mathbf{u}, 4\mathbf{v} \rangle - \langle 2\mathbf{v}, 3\mathbf{u} \rangle - \langle 2\mathbf{v}, 4\mathbf{v} \rangle \\&= \\&= \\&= \end{aligned}$$

Calculating with Inner Products

$$\begin{aligned}\langle \mathbf{u} - 2\mathbf{v}, 3\mathbf{u} + 4\mathbf{v} \rangle &= \langle \mathbf{u}, 3\mathbf{u} + 4\mathbf{v} \rangle - \langle 2\mathbf{v}, 3\mathbf{u} + 4\mathbf{v} \rangle \\&= \langle \mathbf{u}, 3\mathbf{u} \rangle + \langle \mathbf{u}, 4\mathbf{v} \rangle - \langle 2\mathbf{v}, 3\mathbf{u} \rangle - \langle 2\mathbf{v}, 4\mathbf{v} \rangle \\&= 3 \langle \mathbf{u}, \mathbf{u} \rangle + 4 \langle \mathbf{u}, \mathbf{v} \rangle - 6 \langle \mathbf{v}, \mathbf{u} \rangle - 8 \langle \mathbf{v}, \mathbf{v} \rangle \\&= \\&= \end{aligned}$$

Calculating with Inner Products

$$\begin{aligned}\langle \mathbf{u} - 2\mathbf{v}, 3\mathbf{u} + 4\mathbf{v} \rangle &= \langle \mathbf{u}, 3\mathbf{u} + 4\mathbf{v} \rangle - \langle 2\mathbf{v}, 3\mathbf{u} + 4\mathbf{v} \rangle \\&= \langle \mathbf{u}, 3\mathbf{u} \rangle + \langle \mathbf{u}, 4\mathbf{v} \rangle - \langle 2\mathbf{v}, 3\mathbf{u} \rangle - \langle 2\mathbf{v}, 4\mathbf{v} \rangle \\&= 3 \langle \mathbf{u}, \mathbf{u} \rangle + 4 \langle \mathbf{u}, \mathbf{v} \rangle - 6 \langle \mathbf{v}, \mathbf{u} \rangle - 8 \langle \mathbf{v}, \mathbf{v} \rangle \\&= 3 \langle \mathbf{u}, \mathbf{u} \rangle + 4 \langle \mathbf{u}, \mathbf{v} \rangle - 6 \langle \mathbf{u}, \mathbf{v} \rangle - 8 \langle \mathbf{v}, \mathbf{v} \rangle \\&= \end{aligned}$$

Calculating with Inner Products

$$\begin{aligned}\langle \mathbf{u} - 2\mathbf{v}, 3\mathbf{u} + 4\mathbf{v} \rangle &= \langle \mathbf{u}, 3\mathbf{u} + 4\mathbf{v} \rangle - \langle 2\mathbf{v}, 3\mathbf{u} + 4\mathbf{v} \rangle \\&= \langle \mathbf{u}, 3\mathbf{u} \rangle + \langle \mathbf{u}, 4\mathbf{v} \rangle - \langle 2\mathbf{v}, 3\mathbf{u} \rangle - \langle 2\mathbf{v}, 4\mathbf{v} \rangle \\&= 3 \langle \mathbf{u}, \mathbf{u} \rangle + 4 \langle \mathbf{u}, \mathbf{v} \rangle - 6 \langle \mathbf{v}, \mathbf{u} \rangle - 8 \langle \mathbf{v}, \mathbf{v} \rangle \\&= 3 \langle \mathbf{u}, \mathbf{u} \rangle + 4 \langle \mathbf{u}, \mathbf{v} \rangle - 6 \langle \mathbf{u}, \mathbf{v} \rangle - 8 \langle \mathbf{v}, \mathbf{v} \rangle \\&= 3 \|\mathbf{u}\|^2 - 2 \langle \mathbf{u}, \mathbf{v} \rangle - 8 \|\mathbf{v}\|^2.\end{aligned}$$



Orthogonality

Orthogonal Vectors

Definition

Two vectors \mathbf{u} and \mathbf{v} in a (real) inner product space V are *orthogonal* iff

$$\langle \mathbf{u}, \mathbf{v} \rangle = 0.$$

Example (Orthogonality Depends on the Inner Product)

Let $\mathbf{u} = (1, 1)$ and $\mathbf{v} = (1, -1)$. Note that \mathbf{u} and \mathbf{v} are orthogonal with respect to the Euclidean inner product on \mathbb{R}^2 since

$$\mathbf{u} \cdot \mathbf{v} = (1)(1) + (1)(-1) = 0.$$

Orthogonality



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However \mathbf{u} and \mathbf{v} are not orthogonal with respect to the weighted Euclidean inner product $\langle \mathbf{u}, \mathbf{v} \rangle = 3u_1v_1 + 2u_2v_2$ since

$$\langle \mathbf{u}, \mathbf{v} \rangle = 3(1)(1) + 2(1)(-1) = 1 \neq 0.$$

Example

The matrices $U = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$ and $V = \begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix}$ are orthogonal with respect to the standard inner product on $\mathbb{R}^{2 \times 2}$ since

$$\langle \mathbf{u}, \mathbf{v} \rangle = (1)(0) + (0)(2) + (1)(0) + (1)(0) = 0.$$

Orthogonality

Example

Let \mathbb{P}^2 have the inner product

$$\langle \mathbf{p}, \mathbf{q} \rangle = \int_{-1}^1 p(x)q(x) dx$$

and let $\mathbf{p} = x$ and $\mathbf{q} = x^2$.

Orthogonality

Example

Let \mathbb{P}^2 have the inner product

$$\langle \mathbf{p}, \mathbf{q} \rangle = \int_{-1}^1 p(x)q(x) dx$$

and let $\mathbf{p} = x$ and $\mathbf{q} = x^2$. Then

$$\|\mathbf{p}\| = \langle \mathbf{p}, \mathbf{p} \rangle^{\frac{1}{2}} = \left[\int_{-1}^1 x^2 dx \right]^{\frac{1}{2}} = \sqrt{\frac{2}{3}}$$

$$\|\mathbf{q}\| = \langle \mathbf{q}, \mathbf{q} \rangle^{\frac{1}{2}} = \left[\int_{-1}^1 x^4 dx \right]^{\frac{1}{2}} = \sqrt{\frac{2}{5}}$$

$$\langle \mathbf{p}, \mathbf{q} \rangle = \int_{-1}^1 x^3 dx = 0.$$

Orthogonality

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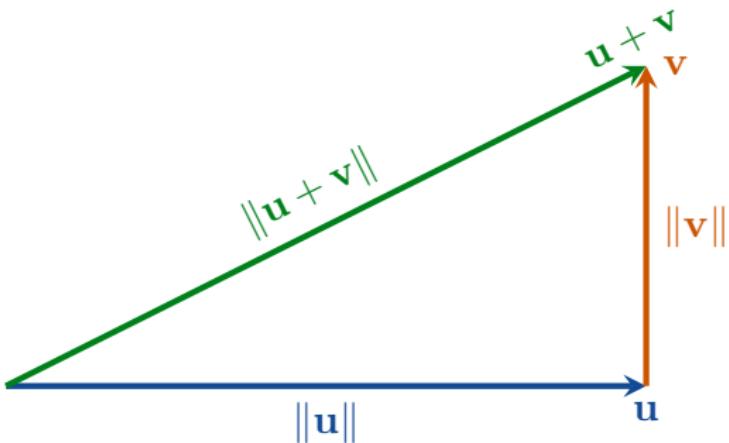
$$\|\mathbf{p}\| = \langle \mathbf{p}, \mathbf{p} \rangle^{\frac{1}{2}} = \left[\int_{-1}^1 x^2 dx \right]^{\frac{1}{2}} = \sqrt{\frac{2}{3}}$$

$$\|\mathbf{q}\| = \langle \mathbf{q}, \mathbf{q} \rangle^{\frac{1}{2}} = \left[\int_{-1}^1 x^4 dx \right]^{\frac{1}{2}} = \sqrt{\frac{2}{5}}$$

$$\langle \mathbf{p}, \mathbf{q} \rangle = \int_{-1}^1 x^3 dx = 0.$$

Because $\langle \mathbf{p}, \mathbf{q} \rangle = 0$, the vectors $\mathbf{p} = x$ and $\mathbf{q} = x^2$ are orthogonal relative to this inner product.

Orthogonality



Theorem (The Pythagorean Theorem)

If \mathbf{u} and \mathbf{v} are orthogonal then

$$\|\mathbf{u} + \mathbf{v}\|^2 = \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2.$$

Orthogonality

Theorem (The Pythagorean Theorem)

If \mathbf{u} and \mathbf{v} are orthogonal then

$$\|\mathbf{u} + \mathbf{v}\|^2 = \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2.$$

Proof.

If $\langle \mathbf{u}, \mathbf{v} \rangle = 0$ then

$$\begin{aligned}\|\mathbf{u} + \mathbf{v}\|^2 &= \langle \mathbf{u} + \mathbf{v}, \mathbf{u} + \mathbf{v} \rangle = \langle \mathbf{u}, \mathbf{u} + \mathbf{v} \rangle + \langle \mathbf{v}, \mathbf{u} + \mathbf{v} \rangle \\ &= \langle \mathbf{u}, \mathbf{u} \rangle + \langle \mathbf{u}, \mathbf{v} \rangle + \langle \mathbf{v}, \mathbf{u} \rangle + \langle \mathbf{v}, \mathbf{v} \rangle \\ &= \|\mathbf{u}\|^2 + 0 + 0 + \|\mathbf{v}\|^2.\end{aligned}$$



Orthogonality



Example

We have seen that $\mathbf{p} = x$ and $\mathbf{q} = x^2$ are orthogonal with respect to the inner product

$$\langle \mathbf{p}, \mathbf{q} \rangle = \int_{-1}^1 p(x)q(x) dx.$$

It follows from the Pythagorean Theorem that

$$\|\mathbf{p} + \mathbf{q}\|^2 = \|\mathbf{p}\|^2 + \|\mathbf{q}\|^2.$$

Let's check this:

Orthogonality



Example

$$\|\mathbf{p}\|^2 + \|\mathbf{q}\|^2 = \left(\sqrt{\frac{2}{3}}\right)^2 + \left(\sqrt{\frac{2}{5}}\right)^2 = \frac{2}{3} + \frac{2}{5} = \frac{16}{15}$$

$$\|\mathbf{p} + \mathbf{q}\|^2 =$$

Orthogonality



Example

$$\|\mathbf{p}\|^2 + \|\mathbf{q}\|^2 = \left(\sqrt{\frac{2}{3}}\right)^2 + \left(\sqrt{\frac{2}{5}}\right)^2 = \frac{2}{3} + \frac{2}{5} = \frac{16}{15}$$

$$\begin{aligned}\|\mathbf{p} + \mathbf{q}\|^2 &= \langle \mathbf{p} + \mathbf{q}, \mathbf{p} + \mathbf{q} \rangle = \int_{-1}^1 (x + x^2)(x + x^2) dx \\&= \int_{-1}^1 x^2 dx + 2 \int_{-1}^1 x^3 dx + \int_{-1}^1 x^4 dx \\&= \frac{2}{3} + 0 + \frac{2}{5} = \frac{16}{15}.\end{aligned}$$

Definition

Let W be a subspace of an inner product space V . The *orthogonal complement* of W is

$$W^\perp = \{\mathbf{v} \in V \mid \langle \mathbf{v}, \mathbf{w} \rangle = 0 \text{ for all } \mathbf{w} \in W\}.$$

Theorem

- 1 W^\perp is also a subspace of V .
- 2 $W \cap W^\perp = \{\mathbf{0}\}$.

Definition

Let W be a subspace of an inner product space V . The *orthogonal complement* of W is

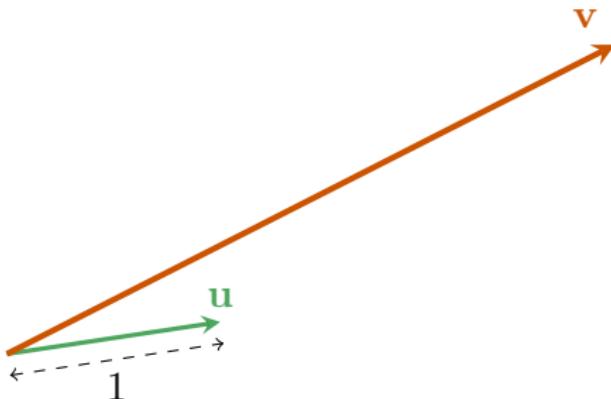
$$W^\perp = \{\mathbf{v} \in V \mid \langle \mathbf{v}, \mathbf{w} \rangle = 0 \text{ for all } \mathbf{w} \in W\}.$$

Theorem

- 1 W^\perp is also a subspace of V .
- 2 $W \cap W^\perp = \{\mathbf{0}\}$.
- 3 If V is finite dimensional then $(W^\perp)^\perp = W$.

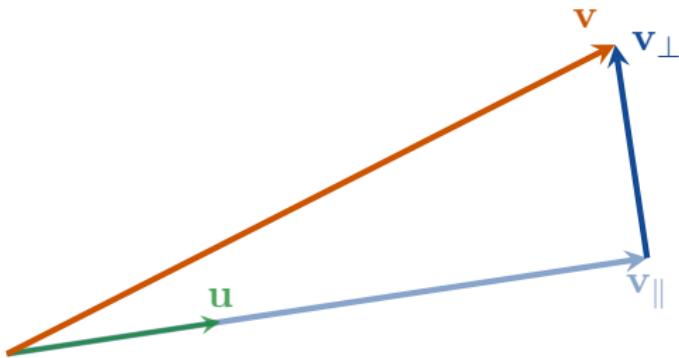
Orthogonal Projection

Let \mathbf{u} be a unit vector and let \mathbf{v} be any nonzero vector in V .



Orthogonal Projection

Let \mathbf{u} be a unit vector and let \mathbf{v} be any nonzero vector in V .



We can write

$$\mathbf{v} = \mathbf{v}_{\parallel} + \mathbf{v}_{\perp}$$

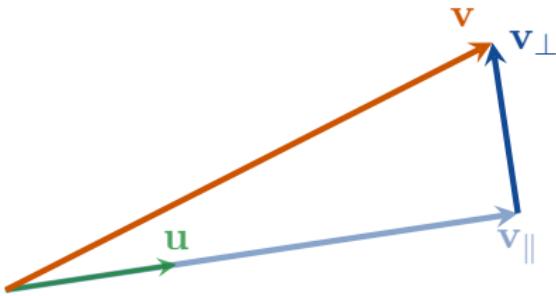
where

$$\mathbf{v}_{\parallel} = \langle \mathbf{u}, \mathbf{v} \rangle \mathbf{u} \quad \text{and} \quad \mathbf{v}_{\perp} = \mathbf{v} - \langle \mathbf{u}, \mathbf{v} \rangle \mathbf{u}.$$

$$\mathbf{v} = \mathbf{v}_{\parallel} + \mathbf{v}_{\perp}$$

$$\mathbf{v}_{\parallel} = \langle \mathbf{u}, \mathbf{v} \rangle \mathbf{u}$$

$$\mathbf{v}_{\perp} = \mathbf{v} - \langle \mathbf{u}, \mathbf{v} \rangle \mathbf{u}$$



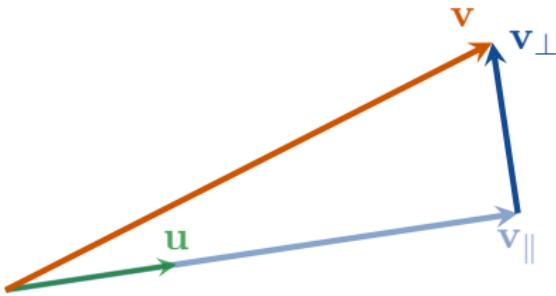
Note that \mathbf{v}_{\perp} is orthogonal to \mathbf{u} because

$$\langle \mathbf{u}, \mathbf{v}_{\perp} \rangle = \langle \mathbf{u}, \mathbf{v} - \langle \mathbf{u}, \mathbf{v} \rangle \mathbf{u} \rangle$$

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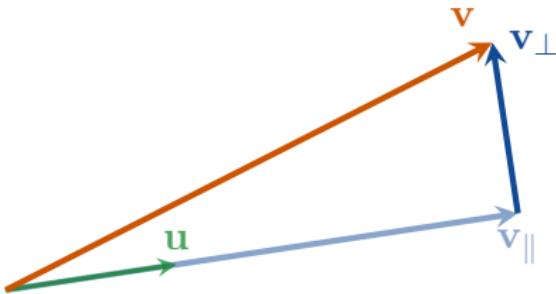
$$\mathbf{v}_{\perp} = \mathbf{v} - \langle \mathbf{u}, \mathbf{v} \rangle \mathbf{u}$$



Note that \mathbf{v}_{\perp} is orthogonal to \mathbf{u} because

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$$\mathbf{v} = \mathbf{v}_{\parallel} + \mathbf{v}_{\perp}$$
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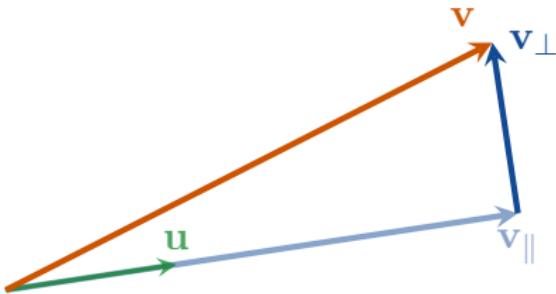
$$\langle \mathbf{u}, \mathbf{v}_{\perp} \rangle = \langle \mathbf{u}, \mathbf{v} - \langle \mathbf{u}, \mathbf{v} \rangle \mathbf{u} \rangle = \langle \mathbf{u}, \mathbf{v} \rangle - \langle \mathbf{u}, \mathbf{v} \rangle \langle \mathbf{u}, \mathbf{u} \rangle = \langle \mathbf{u}, \mathbf{v} \rangle - \langle \mathbf{u}, \mathbf{v} \rangle = 0$$

since \mathbf{u} is a unit vector ($\|\mathbf{u}\| = 1 \implies \langle \mathbf{u}, \mathbf{u} \rangle = 1$).

$$\mathbf{v} = \mathbf{v}_{\parallel} + \mathbf{v}_{\perp}$$

$$\mathbf{v}_{\parallel} = \langle \mathbf{u}, \mathbf{v} \rangle \mathbf{u}$$

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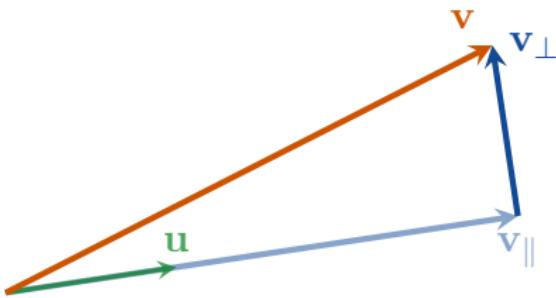
since \mathbf{u} is a unit vector ($\|\mathbf{u}\| = 1 \implies \langle \mathbf{u}, \mathbf{u} \rangle = 1$).

It follows that \mathbf{v}_{\parallel} and \mathbf{v}_{\perp} are orthogonal.

$$\mathbf{v} = \mathbf{v}_{\parallel} + \mathbf{v}_{\perp}$$

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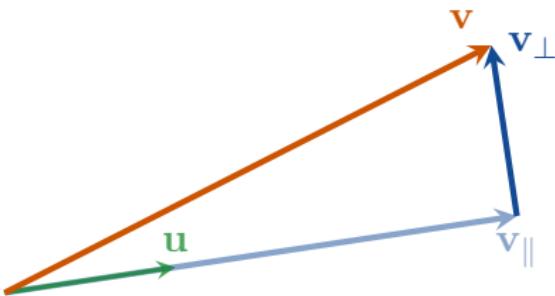
Theorem

- 1 $\|\mathbf{v}_{\parallel}\| \leq \|\mathbf{v}\|$
- 2 $\|\mathbf{v}_{\perp}\| \leq \|\mathbf{v}\|$

$$\mathbf{v} = \mathbf{v}_{\parallel} + \mathbf{v}_{\perp}$$

$$\mathbf{v}_{\parallel} = \langle \mathbf{u}, \mathbf{v} \rangle \mathbf{u}$$

$$\mathbf{v}_{\perp} = \mathbf{v} - \langle \mathbf{u}, \mathbf{v} \rangle \mathbf{u}$$



Theorem

- 1 $\|\mathbf{v}_{\parallel}\| \leq \|\mathbf{v}\|$
- 2 $\|\mathbf{v}_{\perp}\| \leq \|\mathbf{v}\|$
- 3 \mathbf{v}_{\parallel} is the unique vector parallel to \mathbf{u} which is closest to \mathbf{v} .

$$\mathbf{v} = \mathbf{v}_{\parallel} + \mathbf{v}_{\perp} \quad \mathbf{v}_{\parallel} = \langle \mathbf{u}, \mathbf{v} \rangle \mathbf{u} \quad \mathbf{v}_{\perp} = \mathbf{v} - \langle \mathbf{u}, \mathbf{v} \rangle \mathbf{u}$$



Proof.

By the Pythagorean Theorem,

$$\|\mathbf{v}\|^2 = \|\mathbf{v}_{\parallel}\|^2 + \|\mathbf{v}_{\perp}\|^2$$

since \mathbf{v}_{\parallel} and \mathbf{v}_{\perp} are orthogonal, and since $\mathbf{v} = \mathbf{v}_{\parallel} + \mathbf{v}_{\perp}$.

$$\mathbf{v} = \mathbf{v}_{\parallel} + \mathbf{v}_{\perp} \quad \mathbf{v}_{\parallel} = \langle \mathbf{u}, \mathbf{v} \rangle \mathbf{u} \quad \mathbf{v}_{\perp} = \mathbf{v} - \langle \mathbf{u}, \mathbf{v} \rangle \mathbf{u}$$



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By the Pythagorean Theorem,

$$\|\mathbf{v}\|^2 = \|\mathbf{v}_{\parallel}\|^2 + \|\mathbf{v}_{\perp}\|^2$$

since \mathbf{v}_{\parallel} and \mathbf{v}_{\perp} are orthogonal, and since $\mathbf{v} = \mathbf{v}_{\parallel} + \mathbf{v}_{\perp}$. It follows that

$$\|\mathbf{v}_{\parallel}\| \leq \|\mathbf{v}\| \quad \text{and} \quad \|\mathbf{v}_{\perp}\| \leq \|\mathbf{v}\| .$$

$$\mathbf{v} = \mathbf{v}_{\parallel} + \mathbf{v}_{\perp} \quad \mathbf{v}_{\parallel} = \langle \mathbf{u}, \mathbf{v} \rangle \mathbf{u} \quad \mathbf{v}_{\perp} = \mathbf{v} - \langle \mathbf{u}, \mathbf{v} \rangle \mathbf{u}$$

Proof Continued.

For part 3: Let $\alpha\mathbf{u}$ be any other vector which is parallel to \mathbf{u} . Then

$$\|\mathbf{v} - \alpha\mathbf{u}\|^2 = \|\mathbf{v}_{\perp} + (\mathbf{v}_{\parallel} - \alpha\mathbf{u})\|^2$$

$$\mathbf{v} = \mathbf{v}_{\parallel} + \mathbf{v}_{\perp} \quad \mathbf{v}_{\parallel} = \langle \mathbf{u}, \mathbf{v} \rangle \mathbf{u} \quad \mathbf{v}_{\perp} = \mathbf{v} - \langle \mathbf{u}, \mathbf{v} \rangle \mathbf{u}$$

Proof Continued.

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$$\mathbf{v} = \mathbf{v}_{\parallel} + \mathbf{v}_{\perp} \quad \mathbf{v}_{\parallel} = \langle \mathbf{u}, \mathbf{v} \rangle \mathbf{u} \quad \mathbf{v}_{\perp} = \mathbf{v} - \langle \mathbf{u}, \mathbf{v} \rangle \mathbf{u}$$



Proof Continued.

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This distance is smallest when $\alpha = \langle \mathbf{u}, \mathbf{v} \rangle$. Hence $\mathbf{v}_{\parallel} = \langle \mathbf{u}, \mathbf{v} \rangle \mathbf{u}$ is the unique vector parallel to \mathbf{u} which is closest to \mathbf{v} . □



Augustin-Louis Cauchy

BORN

21 August 1789

DECEASED

23 May 1857

NATIONALITY

French



Hermann Schwarz

BORN

25 January 1843

DECEASED

30 November 1921

NATIONALITY

German

Theorem (The Cauchy-Schwarz Inequality)

If \mathbf{u} and \mathbf{v} are vectors in a inner product space V , then

$$|\langle \mathbf{u}, \mathbf{v} \rangle| \leq \|\mathbf{u}\| \|\mathbf{v}\|.$$

This is one of the most important and widely used inequalities in mathematics.

$$|\langle \mathbf{u}, \mathbf{v} \rangle| \leq \|\mathbf{u}\| \|\mathbf{v}\|$$



Proof.

First note that if $\mathbf{u} = \mathbf{0}$ then the Cauchy-Schwarz Inequality is true because $\|\mathbf{u}\| = 0$ and $\langle \mathbf{u}, \mathbf{v} \rangle = 0$.

$$|\langle \mathbf{u}, \mathbf{v} \rangle| \leq \|\mathbf{u}\| \|\mathbf{v}\|$$



Proof.

First note that if $\mathbf{u} = \mathbf{0}$ then the Cauchy-Schwarz Inequality is true because $\|\mathbf{u}\| = 0$ and $\langle \mathbf{u}, \mathbf{v} \rangle = 0$.

Next suppose that \mathbf{u} is a unit vector. Write $\mathbf{v} = \mathbf{v}_{\parallel} + \mathbf{v}_{\perp}$ where $\mathbf{v}_{\parallel} = \langle \mathbf{u}, \mathbf{v} \rangle \mathbf{u}$.

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$$|\langle \mathbf{u}, \mathbf{v} \rangle| \leq \|\mathbf{u}\| \|\mathbf{v}\|$$



Proof.

First note that if $\mathbf{u} = \mathbf{0}$ then the Cauchy-Schwarz Inequality is true because $\|\mathbf{u}\| = 0$ and $\langle \mathbf{u}, \mathbf{v} \rangle = 0$.

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Orthogonality

$$|\langle \mathbf{u}, \mathbf{v} \rangle| \leq \|\mathbf{u}\| \|\mathbf{v}\|$$



Proof.

First note that if $\mathbf{u} = \mathbf{0}$ then the Cauchy-Schwarz Inequality is true because $\|\mathbf{u}\| = 0$ and $\langle \mathbf{u}, \mathbf{v} \rangle = 0$.

Next suppose that \mathbf{u} is a unit vector. Write $\mathbf{v} = \mathbf{v}_{\parallel} + \mathbf{v}_{\perp}$ where $\mathbf{v}_{\parallel} = \langle \mathbf{u}, \mathbf{v} \rangle \mathbf{u}$. Recall that $\|c\mathbf{u}\| = |c| \|\mathbf{u}\|$ for any scalar c . Thus

$$\|\mathbf{v}_{\parallel}\| = \|\langle \mathbf{u}, \mathbf{v} \rangle \mathbf{u}\| = |\langle \mathbf{u}, \mathbf{v} \rangle| \|\mathbf{u}\| = |\langle \mathbf{u}, \mathbf{v} \rangle|.$$

Since $\|\mathbf{v}_{\parallel}\| \leq \|\mathbf{v}\|$ we have

$$|\langle \mathbf{u}, \mathbf{v} \rangle| \leq \|\mathbf{v}\|.$$

$$|\langle \mathbf{u}, \mathbf{v} \rangle| \leq \|\mathbf{u}\| \|\mathbf{v}\|$$



Proof Continued.

Now let \mathbf{u} be any nonzero vector and define $\hat{\mathbf{u}} = \frac{\mathbf{u}}{\|\mathbf{u}\|}$. Then

- $\hat{\mathbf{u}}$ is a unit vector;
- $\mathbf{u} = \|\mathbf{u}\| \hat{\mathbf{u}}$; and
- $|\langle \hat{\mathbf{u}}, \mathbf{v} \rangle| \leq \|\mathbf{v}\|$.

$$|\langle \mathbf{u}, \mathbf{v} \rangle| \leq \|\mathbf{u}\| \|\mathbf{v}\|$$



Proof Continued.

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Therefore

$$|\langle \mathbf{u}, \mathbf{v} \rangle| = |\langle \|\mathbf{u}\| \hat{\mathbf{u}}, \mathbf{v} \rangle|$$

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Proof Continued.

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and we are finished.



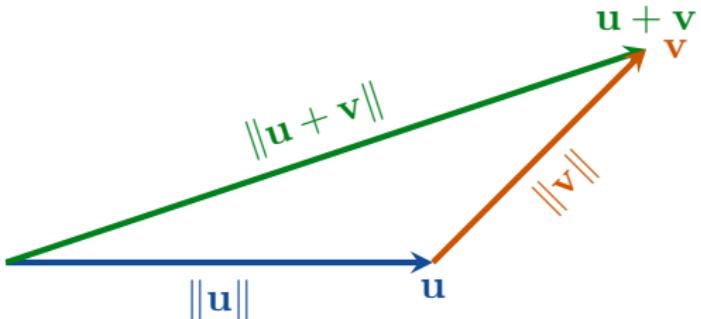
$$|\langle \mathbf{u}, \mathbf{v} \rangle| \leq \|\mathbf{u}\| \|\mathbf{v}\|$$

We can use the Cauchy-Schwarz Inequality to prove the following result:

Theorem (The Triangle Inequality)

For all \mathbf{u}, \mathbf{v} in V ,

$$\|\mathbf{u} + \mathbf{v}\| \leq \|\mathbf{u}\| + \|\mathbf{v}\|.$$



$$|\langle \mathbf{u}, \mathbf{v} \rangle| \leq \|\mathbf{u}\| \|\mathbf{v}\|$$



Theorem (The Triangle Inequality)

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Proof.

Using the Cauchy-Schwarz Inequality we calculate that

$$\|\mathbf{u} + \mathbf{v}\|^2 = \langle \mathbf{u} + \mathbf{v}, \mathbf{u} + \mathbf{v} \rangle =$$

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Orthogonality

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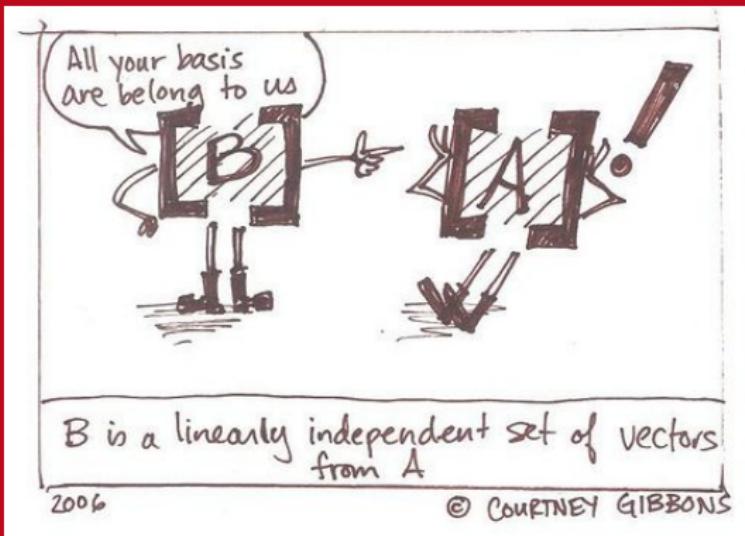
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Break

We will continue at 3pm





Orthogonal Sets and Orthonormal Sets

Orthogonal Sets and Orthonormal Sets



Definition

A set of two or more vectors in a real inner product space is called an *orthogonal set* if all pairs of distinct vectors in the set are orthogonal.

Orthogonal Sets and Orthonormal Sets



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$$\langle \mathbf{u}, \mathbf{v} \rangle = 0$$

for all $\mathbf{u} \neq \mathbf{v}$.

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Definition

An orthogonal set in which each vector is a unit vector is called an *orthonormal set*.

So we must have

- 1 $\langle \mathbf{u}, \mathbf{v} \rangle = 0$ for all $\mathbf{u} \neq \mathbf{v}$; and
- 2 $\langle \mathbf{u}, \mathbf{u} \rangle = 1$ for all $\mathbf{u} \in V$.

Orthogonal Sets and Orthonormal Sets



Example (An Orthogonal Set in \mathbb{R}^3)

Let

$$\mathbf{v}_1 = (0, 1, 0), \quad \mathbf{v}_2 = (1, 0, 1), \quad \mathbf{v}_3 = (1, 0, -1).$$

Assume that \mathbb{R}^3 has the Euclidean inner product (dot product).

Show that $S = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is an orthogonal set.

$$\langle \mathbf{v}_1, \mathbf{v}_2 \rangle = \mathbf{v}_1 \cdot \mathbf{v}_2 =$$

$$\langle \mathbf{v}_1, \mathbf{v}_3 \rangle = \mathbf{v}_1 \cdot \mathbf{v}_3 =$$

$$\langle \mathbf{v}_2, \mathbf{v}_3 \rangle = \mathbf{v}_2 \cdot \mathbf{v}_3 =$$

Orthogonal Sets and Orthonormal Sets



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Assume that \mathbb{R}^3 has the Euclidean inner product (dot product).

Show that $S = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is an orthogonal set.

$$\langle \mathbf{v}_1, \mathbf{v}_2 \rangle = \mathbf{v}_1 \cdot \mathbf{v}_2 = (0)(1) + (1)(0) + (0)(1) = 0$$

$$\langle \mathbf{v}_1, \mathbf{v}_3 \rangle = \mathbf{v}_1 \cdot \mathbf{v}_3 = (0)(1) + (1)(0) + (0)(-1) = 0$$

$$\langle \mathbf{v}_2, \mathbf{v}_3 \rangle = \mathbf{v}_2 \cdot \mathbf{v}_3 = (1)(1) + (0)(0) + (1)(-1) = 0.$$

Therefore S is orthogonal.

Orthogonal Sets and Orthonormal Sets



Recall that if $\mathbf{v} \neq \mathbf{0}$ is any nonzero vector, then $\mathbf{u} = \frac{\mathbf{v}}{\|\mathbf{v}\|}$ is a unit vector because

$$\|\mathbf{u}\| = \left\| \frac{\mathbf{v}}{\|\mathbf{v}\|} \right\| = \left| \frac{1}{\|\mathbf{v}\|} \right| \|\mathbf{v}\| = \frac{\|\mathbf{v}\|}{\|\mathbf{v}\|} = 1.$$

$$\mathbf{v}_1 = (0, 1, 0), \quad \mathbf{v}_2 = (1, 0, 1), \quad \mathbf{v}_3 = (1, 0, -1)$$



Example (Constructing an Orthonormal Set)

We have that $S = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is an orthogonal set in \mathbb{R}^3 with respect to the Euclidean inner product. Note that

$$\|\mathbf{v}_1\| = \sqrt{0^2 + 1^2 + 0^2} = 1 \quad \|\mathbf{v}_2\| = \sqrt{1^2 + 0^2 + 1^2} = \sqrt{2}$$

$$\|\mathbf{v}_3\| = \sqrt{1^2 + 0^2 + (-1)^2} = \sqrt{2}.$$

It follows that if

$$\mathbf{u}_1 = \frac{\mathbf{v}_1}{\|\mathbf{v}_1\|} = (0, 1, 0) \quad \mathbf{u}_2 = \frac{\mathbf{v}_2}{\|\mathbf{v}_2\|} = \left(\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}} \right)$$

$$\mathbf{u}_3 = \frac{\mathbf{v}_3}{\|\mathbf{v}_3\|} = \left(\frac{1}{\sqrt{2}}, 0, -\frac{1}{\sqrt{2}} \right)$$

then $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ is an orthonormal set.

$$\mathbf{v}_1 = (0, 1, 0), \quad \mathbf{v}_2 = (1, 0, 1), \quad \mathbf{v}_3 = (1, 0, -1)$$



Example (Constructing an Orthonormal Set)

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then $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ is an orthonormal set.

I leave it to you to check that

$$\langle \mathbf{u}_1, \mathbf{u}_2 \rangle = \langle \mathbf{u}_1, \mathbf{u}_3 \rangle = \langle \mathbf{u}_2, \mathbf{u}_3 \rangle = 0$$

and

$$\|\mathbf{u}_1\| = \|\mathbf{u}_2\| = \|\mathbf{u}_3\| = 1.$$

Orthogonal Sets and Orthonormal Sets



Theorem

If $S = \{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\}$ is an orthogonal set of nonzero vectors in an inner product space, then S is linearly independent.

Proof.

Suppose that

$$k_1\mathbf{v}_1 + k_2\mathbf{v}_2 + \dots + k_n\mathbf{v}_n = \mathbf{0}.$$

We must prove that $k_1 = k_2 = \dots = k_n = 0$.

Orthogonal Sets and Orthonormal Sets



Proof Continued.

For each $\mathbf{v}_i \in S$ we have

$$0 = \langle \mathbf{0}, \mathbf{v}_i \rangle =$$

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Orthogonal Sets and Orthonormal Sets



Proof Continued.

For each $\mathbf{v}_i \in S$ we have

$$0 = \langle \mathbf{0}, \mathbf{v}_i \rangle = \langle k_1 \mathbf{v}_1 + k_2 \mathbf{v}_2 + \dots + k_n \mathbf{v}_n, \mathbf{v}_i \rangle$$

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Proof Continued.

For each $\mathbf{v}_i \in S$ we have

$$\begin{aligned} 0 &= \langle \mathbf{0}, \mathbf{v}_i \rangle = \langle k_1 \mathbf{v}_1 + k_2 \mathbf{v}_2 + \dots + k_n \mathbf{v}_n, \mathbf{v}_i \rangle \\ &= k_1 \langle \mathbf{v}_1, \mathbf{v}_i \rangle + k_2 \langle \mathbf{v}_2, \mathbf{v}_i \rangle + \dots + k_i \langle \mathbf{v}_i, \mathbf{v}_i \rangle + \dots + k_n \langle \mathbf{v}_n, \mathbf{v}_i \rangle \\ &= \\ &= \end{aligned}$$

Proof Continued.

For each $\mathbf{v}_i \in S$ we have

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Proof Continued.

For each $\mathbf{v}_i \in S$ we have

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Orthogonal Sets and Orthonormal Sets



Proof Continued.

For each $\mathbf{v}_i \in S$ we have

$$\begin{aligned} 0 &= \langle \mathbf{0}, \mathbf{v}_i \rangle = \langle k_1 \mathbf{v}_1 + k_2 \mathbf{v}_2 + \dots + k_n \mathbf{v}_n, \mathbf{v}_i \rangle \\ &= k_1 \langle \mathbf{v}_1, \mathbf{v}_i \rangle + k_2 \langle \mathbf{v}_2, \mathbf{v}_i \rangle + \dots + k_i \langle \mathbf{v}_i, \mathbf{v}_i \rangle + \dots + k_n \langle \mathbf{v}_n, \mathbf{v}_i \rangle \\ &= 0 + 0 + \dots + k_i \langle \mathbf{v}_i, \mathbf{v}_i \rangle + \dots + 0 \\ &= k_i \langle \mathbf{v}_i, \mathbf{v}_i \rangle. \end{aligned}$$

Since $\mathbf{v}_i \neq \mathbf{0}$, we have that $\langle \mathbf{v}_i, \mathbf{v}_i \rangle \neq 0$.

Proof Continued.

For each $\mathbf{v}_i \in S$ we have

$$\begin{aligned} 0 &= \langle \mathbf{0}, \mathbf{v}_i \rangle = \langle k_1 \mathbf{v}_1 + k_2 \mathbf{v}_2 + \dots + k_n \mathbf{v}_n, \mathbf{v}_i \rangle \\ &= k_1 \langle \mathbf{v}_1, \mathbf{v}_i \rangle + k_2 \langle \mathbf{v}_2, \mathbf{v}_i \rangle + \dots + k_i \langle \mathbf{v}_i, \mathbf{v}_i \rangle + \dots + k_n \langle \mathbf{v}_n, \mathbf{v}_i \rangle \\ &= 0 + 0 + \dots + k_i \langle \mathbf{v}_i, \mathbf{v}_i \rangle + \dots + 0 \\ &= k_i \langle \mathbf{v}_i, \mathbf{v}_i \rangle. \end{aligned}$$

Since $\mathbf{v}_i \neq \mathbf{0}$, we have that $\langle \mathbf{v}_i, \mathbf{v}_i \rangle \neq 0$.

Therefore $k_i = 0$ for all i . This proves that S is linearly independent.



Orthogonal and Orthonormal Bases

Definition

In an inner product space, a basis consisting of orthonormal vectors is called an *orthonormal basis*, and a basis consisting of orthogonal vectors is called an *orthogonal basis*.

Example

$$\mathbf{e}_1 = (1, 0, 0, \dots, 0), \mathbf{e}_2 = (0, 1, 0, \dots, 0), \dots, \mathbf{e}_n = (0, 0, 0, \dots, 1)$$

is an orthonormal basis in \mathbb{R}^n with the Euclidean inner product.

(Recall that this basis is called the *standard basis* for \mathbb{R}^n .)

Orthogonal Sets and Orthonormal Sets



Example (An Orthonormal Basis for \mathbb{P}^n)

Consider the vector space of polynomials of degree $\leq n$ with the inner product

$$\langle \mathbf{p}, \mathbf{q} \rangle = a_0 b_0 + a_1 b_1 + \dots + a_n b_n$$

where

$$\mathbf{p} = a_0 + a_1 x + \dots + a_n x^n$$

$$\mathbf{q} = b_0 + b_1 x + \dots + b_n x^n.$$

I leave it to you to prove that the standard basis

$$S = \{1, x, x^2, x^3, \dots, x^n\}$$

is an orthonormal basis with respect to this inner product.

Example

In an earlier example we saw that

$$\mathbf{u}_1 = (0, 1, 0), \quad \mathbf{u}_2 = \left(\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}} \right), \quad \mathbf{u}_3 = \left(\frac{1}{\sqrt{2}}, 0, -\frac{1}{\sqrt{2}} \right)$$

form an orthonormal set with respect to the Euclidean inner product on \mathbb{R}^3 .

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By the previous theorem, these three vectors are linearly independent.

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form an orthonormal set with respect to the Euclidean inner product on \mathbb{R}^3 .

By the previous theorem, these three vectors are linearly independent.

Since \mathbb{R}^3 is three-dimensional, $S = \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ is an orthonormal basis for \mathbb{R}^3 .

Coordinates Relative to Orthonormal Bases

Recall that if $S = \{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\}$ is a basis for a vector space V , and if

$$\mathbf{u} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \dots + c_n\mathbf{v}_n$$

then the coordinates of \mathbf{u} relative to this basis is

$$(\mathbf{u})_S = (c_1, c_2, \dots, c_n).$$

Coordinates Relative to Orthonormal Bases

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then the coordinates of \mathbf{u} relative to this basis is

$$(\mathbf{u})_S = (c_1, c_2, \dots, c_n).$$

If the basis is orthogonal or orthonormal then there is an easy way to find the coefficients c_1, c_2, \dots, c_n .

Theorem

If $S = \{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\}$ is an orthogonal basis for an inner product space V , and if \mathbf{u} is any vector in V , then

$$\mathbf{u} = \frac{\langle \mathbf{u}, \mathbf{v}_1 \rangle}{\|\mathbf{v}_1\|^2} \mathbf{v}_1 + \frac{\langle \mathbf{u}, \mathbf{v}_2 \rangle}{\|\mathbf{v}_2\|^2} \mathbf{v}_2 + \dots + \frac{\langle \mathbf{u}, \mathbf{v}_n \rangle}{\|\mathbf{v}_n\|^2} \mathbf{v}_n.$$

Orthogonal Sets and Orthonormal Sets



Theorem

If $S = \{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\}$ is an orthogonal basis for an inner product space V , and if \mathbf{u} is any vector in V , then

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Proof.

Let

$$\mathbf{u} = c_1 \mathbf{v}_1 + c_2 \mathbf{v}_2 + \dots + c_n \mathbf{v}_n.$$

We need to show that $c_i = \frac{\langle \mathbf{u}, \mathbf{v}_i \rangle}{\|\mathbf{v}_i\|^2}$ for each i .

Orthogonal Sets and Orthonormal Sets



Proof Continued.

Note that

$$\begin{aligned}\langle \mathbf{u}, \mathbf{v}_i \rangle &= \langle c_1 \mathbf{v}_1 + c_2 \mathbf{v}_2 + \dots + c_n \mathbf{v}_n, \mathbf{v}_i \rangle \\&= c_1 \langle \mathbf{v}_1, \mathbf{v}_i \rangle + c_2 \langle \mathbf{v}_2, \mathbf{v}_i \rangle + \dots + c_i \langle \mathbf{v}_i, \mathbf{v}_i \rangle + \dots + c_n \langle \mathbf{v}_n, \mathbf{v}_i \rangle \\&= \\&= \end{aligned}$$

Orthogonal Sets and Orthonormal Sets



Proof Continued.

Note that

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Orthogonal Sets and Orthonormal Sets



Proof Continued.

Note that

$$\begin{aligned}\langle \mathbf{u}, \mathbf{v}_i \rangle &= \langle c_1 \mathbf{v}_1 + c_2 \mathbf{v}_2 + \dots + c_n \mathbf{v}_n, \mathbf{v}_i \rangle \\&= c_1 \langle \mathbf{v}_1, \mathbf{v}_i \rangle + c_2 \langle \mathbf{v}_2, \mathbf{v}_i \rangle + \dots + c_i \langle \mathbf{v}_i, \mathbf{v}_i \rangle + \dots + c_n \langle \mathbf{v}_n, \mathbf{v}_i \rangle \\&= 0 + 0 + \dots + c_i \|\mathbf{v}_i\|^2 + \dots + 0 \\&= c_i \|\mathbf{v}_i\|^2.\end{aligned}$$

Orthogonal Sets and Orthonormal Sets



Proof Continued.

Note that

$$\begin{aligned}\langle \mathbf{u}, \mathbf{v}_i \rangle &= \langle c_1 \mathbf{v}_1 + c_2 \mathbf{v}_2 + \dots + c_n \mathbf{v}_n, \mathbf{v}_i \rangle \\&= c_1 \langle \mathbf{v}_1, \mathbf{v}_i \rangle + c_2 \langle \mathbf{v}_2, \mathbf{v}_i \rangle + \dots + c_i \langle \mathbf{v}_i, \mathbf{v}_i \rangle + \dots + c_n \langle \mathbf{v}_n, \mathbf{v}_i \rangle \\&= 0 + 0 + \dots + c_i \|\mathbf{v}_i\|^2 + \dots + 0 \\&= c_i \|\mathbf{v}_i\|^2.\end{aligned}$$

Hence $c_i = \frac{\langle \mathbf{u}, \mathbf{v}_i \rangle}{\|\mathbf{v}_i\|^2}$ for each i and we are finished. □

Theorem

If $S = \{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\}$ is an orthonormal basis for an inner product space V , and if \mathbf{u} is any vector in V , then

$$\mathbf{u} = \langle \mathbf{u}, \mathbf{v}_1 \rangle \mathbf{v}_1 + \langle \mathbf{u}, \mathbf{v}_2 \rangle \mathbf{v}_2 + \dots + \langle \mathbf{u}, \mathbf{v}_n \rangle \mathbf{v}_n.$$

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Proof.

Just take the previous formula

$$\mathbf{u} = \frac{\langle \mathbf{u}, \mathbf{v}_1 \rangle}{\|\mathbf{v}_1\|^2} \mathbf{v}_1 + \frac{\langle \mathbf{u}, \mathbf{v}_2 \rangle}{\|\mathbf{v}_2\|^2} \mathbf{v}_2 + \dots + \frac{\langle \mathbf{u}, \mathbf{v}_n \rangle}{\|\mathbf{v}_n\|^2} \mathbf{v}_n$$

and replace every $\|\mathbf{v}_i\|$ by 1 since each \mathbf{v}_i is a unit vector. □

Remark

So if S is an orthogonal basis then

$$(\mathbf{u})_S = \left(\frac{\langle \mathbf{u}, \mathbf{v}_1 \rangle}{\|\mathbf{v}_1\|^2}, \frac{\langle \mathbf{u}, \mathbf{v}_2 \rangle}{\|\mathbf{v}_2\|^2}, \dots, \frac{\langle \mathbf{u}, \mathbf{v}_n \rangle}{\|\mathbf{v}_n\|^2} \right)$$

and if S is an orthonormal basis then

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Example

Let

$$\mathbf{v}_1 = (0, 1, 0), \quad \mathbf{v}_2 = \left(-\frac{4}{5}, 0, \frac{3}{5} \right), \quad \mathbf{v}_3 = \left(\frac{3}{5}, 0, \frac{4}{5} \right).$$

I leave it to you to check that $S = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is an orthonormal basis for \mathbb{R}^3 with the Euclidean inner product.

Orthogonal Sets and Orthonormal Sets



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Orthogonal Sets and Orthonormal Sets



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Find $(\mathbf{u})_S$ if $\mathbf{u} = (1, 1, 1)$.

Since

$$\langle \mathbf{u}, \mathbf{v}_1 \rangle = 1, \quad \langle \mathbf{u}, \mathbf{v}_2 \rangle = -\frac{1}{5}, \quad \langle \mathbf{u}, \mathbf{v}_3 \rangle = \frac{7}{5},$$

(please check) we have that

$$(\mathbf{u})_S = (\langle \mathbf{u}, \mathbf{v}_1 \rangle, \langle \mathbf{u}, \mathbf{v}_2 \rangle, \langle \mathbf{u}, \mathbf{v}_3 \rangle) = \left(1, -\frac{1}{5}, \frac{7}{5} \right).$$

Example (An Orthonormal Basis from an Orthogonal Basis)

1 Show that the vectors

$$\mathbf{w}_1 = (0, 2, 0), \quad \mathbf{w}_2 = (3, 0, 3), \quad \mathbf{w}_3 = (-4, 0, 4)$$

form an orthogonal basis for \mathbb{R}^3 with the Euclidean inner product.

2

3

Orthogonal Sets and Orthonormal Sets



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form an orthogonal basis for \mathbb{R}^3 with the Euclidean inner product.

- 2 Normalise each vector above to find an orthonormal basis $S = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$.
- 3 Find $(\mathbf{u})_S$ if $\mathbf{u} = (1, 2, 4)$.

$$\mathbf{w}_1 = (0, 2, 0), \quad \mathbf{w}_2 = (3, 0, 3), \quad \mathbf{w}_3 = (-4, 0, 4)$$



- 1 I leave it to you to check that

$$\langle \mathbf{w}_1, \mathbf{w}_2 \rangle = \langle \mathbf{w}_1, \mathbf{w}_3 \rangle = \langle \mathbf{w}_2, \mathbf{w}_3 \rangle = 0.$$

This shows that $\{\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3\}$ is an orthogonal set.

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Recall that sets of nonzero orthogonal vectors are always linearly independent.

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Since \mathbb{R}^3 is three-dimensional, $\{\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3\}$ must be a basis for \mathbb{R}^3 .

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Recall that sets of nonzero orthogonal vectors are always linearly independent.

Since \mathbb{R}^3 is three-dimensional, $\{\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3\}$ must be a basis for \mathbb{R}^3 .

- 2 We calculate

$$\mathbf{v}_1 = \frac{\mathbf{w}_1}{\|\mathbf{w}_1\|} = (0, 1, 0), \quad \mathbf{v}_2 = \frac{\mathbf{w}_2}{\|\mathbf{w}_2\|} = \left(\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}} \right)$$

$$\mathbf{v}_3 = \frac{\mathbf{w}_3}{\|\mathbf{w}_3\|} = \left(-\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}} \right).$$

$$\mathbf{v}_1 = (0, 1, 0), \quad \mathbf{v}_2 = \left(\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}} \right), \quad \mathbf{v}_3 = \left(-\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}} \right)$$



3 Since

$$\langle \mathbf{u}, \mathbf{v}_1 \rangle = (1, 2, 4) \cdot (0, 1, 0) = 2$$

$$\langle \mathbf{u}, \mathbf{v}_2 \rangle = (1, 2, 4) \cdot \left(\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}} \right) = \frac{5}{\sqrt{2}}$$

$$\langle \mathbf{u}, \mathbf{v}_3 \rangle = (1, 2, 4) \cdot \left(-\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}} \right) = \frac{3}{\sqrt{2}},$$

we have that

$$(\mathbf{u})_S = \left(2, \frac{5}{\sqrt{2}}, \frac{3}{\sqrt{2}} \right).$$

Orthogonal Projections

Theorem

If W is a finite-dimensional subspace of an inner product space V , then every vector \mathbf{u} in V can be expressed in exactly one way as

$$\mathbf{u} = \mathbf{w}_{\parallel} + \mathbf{w}_{\perp}$$

where \mathbf{w}_{\parallel} is in W and \mathbf{w}_{\perp} is in W^{\perp} .

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where \mathbf{w}_{\parallel} is in W and \mathbf{w}_{\perp} is in W^{\perp} .

The vectors \mathbf{w}_{\parallel} and \mathbf{w}_{\perp} are often denoted as

$$\mathbf{w}_{\parallel} = \text{proj}_W \mathbf{u} \quad \text{and} \quad \mathbf{w}_{\perp} = \text{proj}_{W^{\perp}} \mathbf{u}$$

and are called the *orthogonal projection of \mathbf{u} on W* and the *orthogonal projection of \mathbf{u} on W^{\perp}* , respectively.

Orthogonal Sets and Orthonormal Sets



V



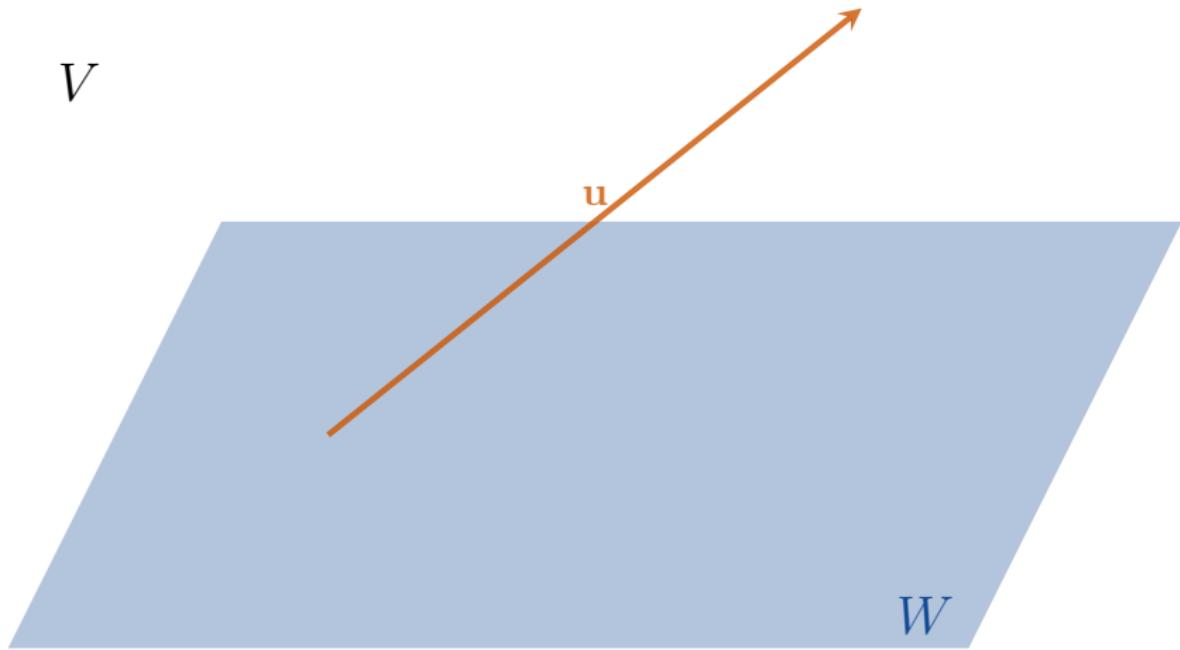
Orthogonal Sets and Orthonormal Sets



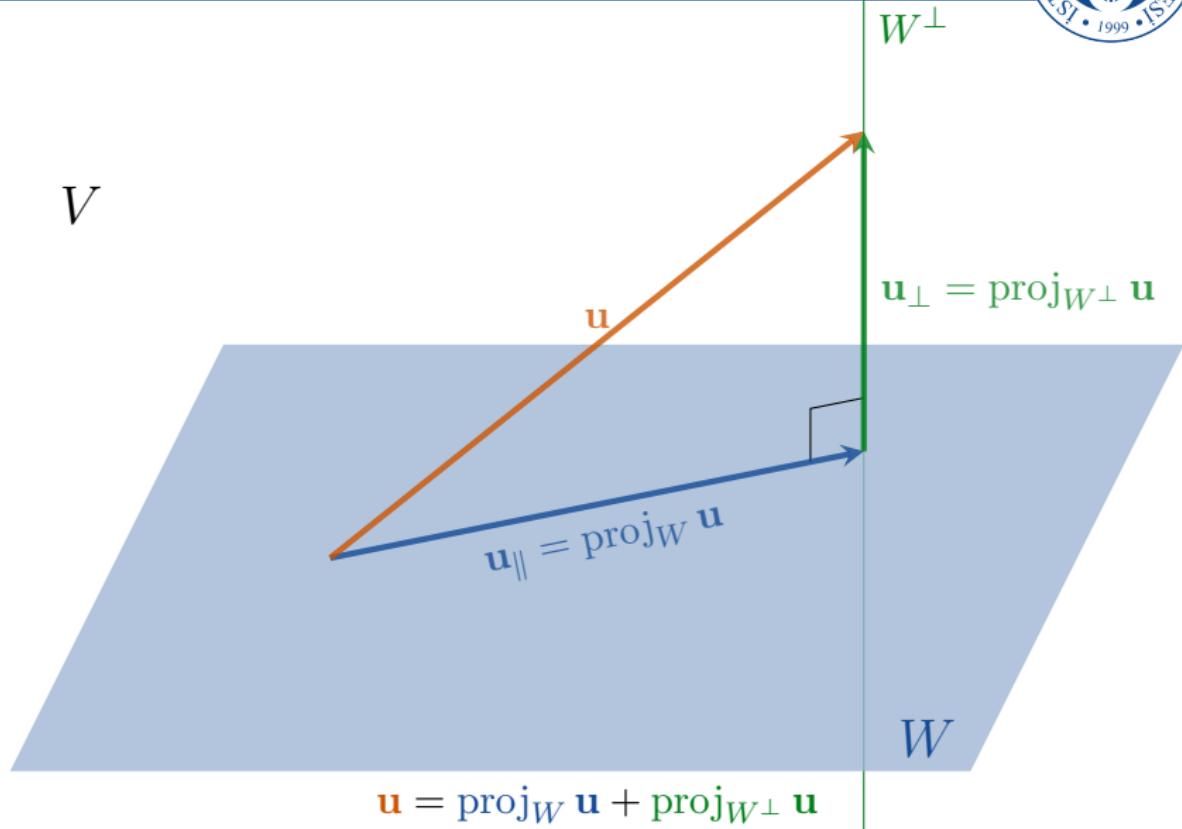
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W



Orthogonal Sets and Orthonormal Sets



Orthogonal Sets and Orthonormal Sets



Theorem

Let W be a finite-dimensional subspace of an inner product space V .

- 1 If $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_r\}$ is an orthogonal basis for W , and \mathbf{u} is any vector in V , then

$$\text{proj}_W \mathbf{u} = \frac{\langle \mathbf{u}, \mathbf{v}_1 \rangle}{\|\mathbf{v}_1\|^2} \mathbf{v}_1 + \frac{\langle \mathbf{u}, \mathbf{v}_2 \rangle}{\|\mathbf{v}_2\|^2} \mathbf{v}_2 + \dots + \frac{\langle \mathbf{u}, \mathbf{v}_r \rangle}{\|\mathbf{v}_r\|^2} \mathbf{v}_r.$$

- 2 If $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_r\}$ is an orthonormal basis for W , and \mathbf{u} is any vector in V , then

$$\text{proj}_W \mathbf{u} = \langle \mathbf{u}, \mathbf{v}_1 \rangle \mathbf{v}_1 + \langle \mathbf{u}, \mathbf{v}_2 \rangle \mathbf{v}_2 + \dots + \langle \mathbf{u}, \mathbf{v}_r \rangle \mathbf{v}_r.$$



Next Time

- The Gram-Schmidt Process
- Orthogonal Matrices
- Orthogonal Diagonalisation