Lecture 2

- 8.4 Trigonometric Substitutions
- 8.5 Integration of Rational Functions by Partial Fractions
- 8.8 Improper Integrals

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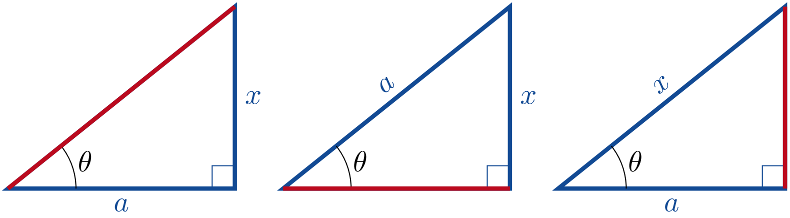
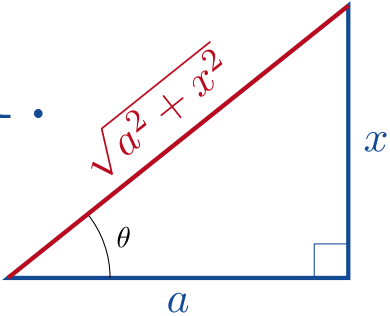
8.4 Trigonometric Substitutions



x
θ
a
a
x
θ
x
θ
a

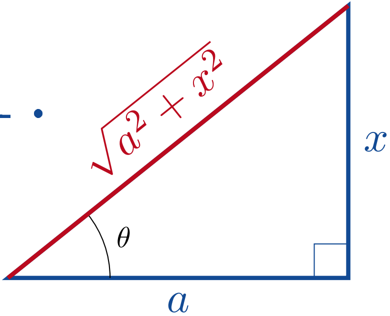
1.



$$x = a\tan\theta$$

$$a^2 + x^2 = \qquad = \qquad = \qquad .$$

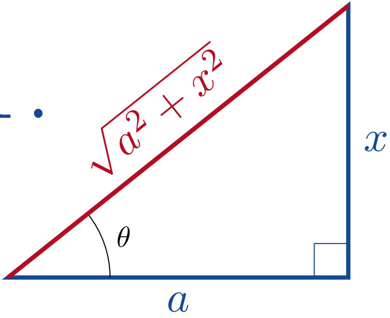
1.



$$x = a\tan\theta$$

$$a^2 + x^2 = a^2 + a^2\tan^2\theta = \qquad = \qquad .$$

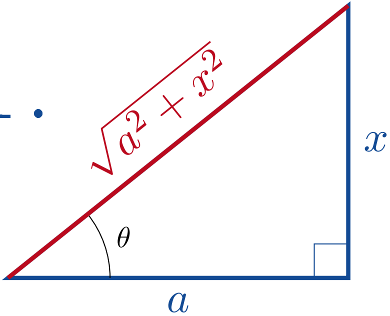
1.



$$x = a \tan\theta$$

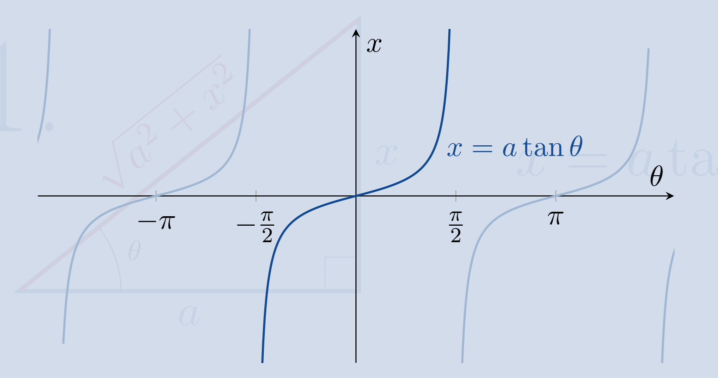
$$a^2 + x^2 = a^2 + a^2 \tan^2\theta = a^2(1 + \tan^2\theta) = \qquad .$$

1.



$$x = a\tan\theta$$

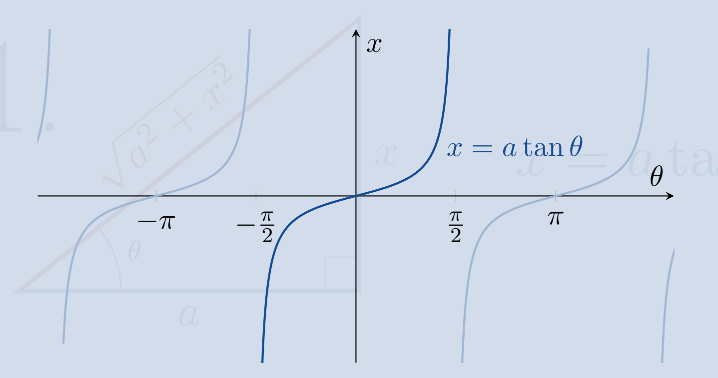
$$a^2 + x^2 = a^2 + a^2\tan^2\theta = a^2(1+\tan^2\theta) = a^2\sec^2\theta.$$



$$a^2 + x^2 = a^2 + a^2\tan^2\theta = a^2(1 + \tan^2\theta) = a^2\sec^2\theta.$$

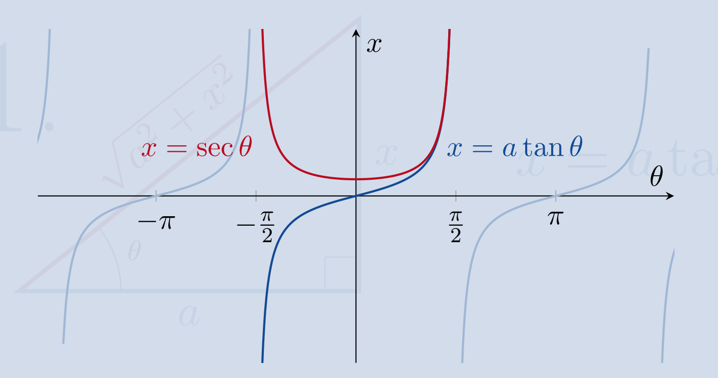
$$\boxed{.}$$

1. $x = a\tan\theta$



$$a^2 + x^2 = a^2 + a^2\tan^2\theta = a^2(1+\tan^2\theta) = a^2\sec^2\theta.$$

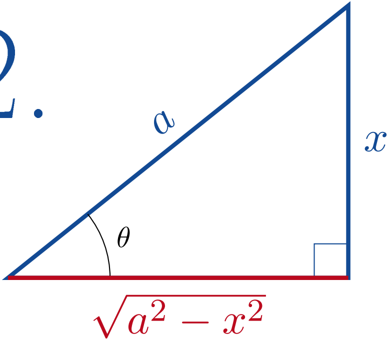
$$\boxed{-\frac{\pi}{2} < \theta < \frac{\pi}{2}.}$$



$$a^2 + x^2 = a^2 + a^2\tan^2\theta = a^2(1+\tan^2\theta) = a^2\sec^2\theta.$$

$$\boxed{\sqrt{a^2+x^2} = a\sec\theta \qquad -\frac{\pi}{2} < \theta < \frac{\pi}{2}.}$$

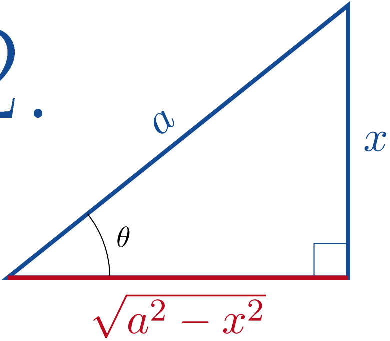
2.



$$x = a\sin\theta$$

$$a^2 - x^2 = \qquad\qquad = \qquad\qquad = \qquad\qquad .$$

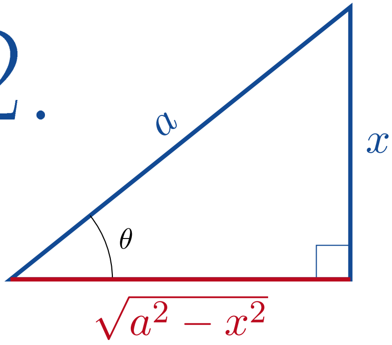
2.



$$x = a\sin\theta$$

$$a^2 - x^2 = a^2 - a^2\sin^2\theta = \qquad\qquad = \qquad\qquad .$$

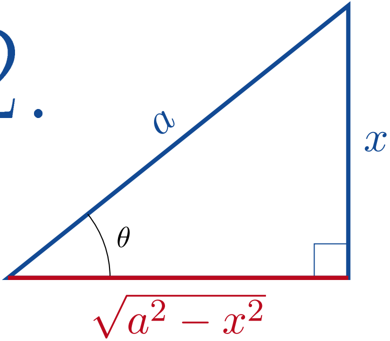
2.



$$x = a\sin\theta$$

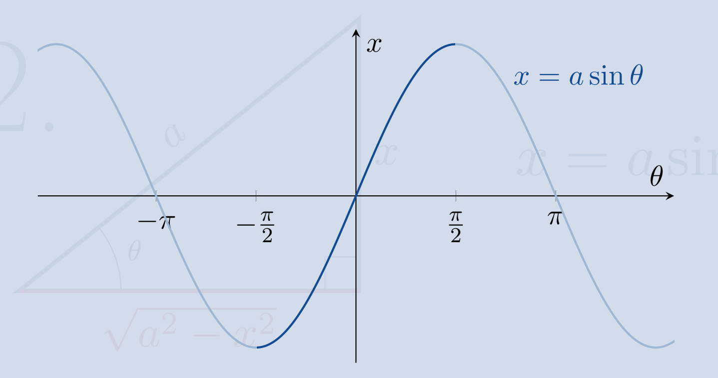
$$a^2 - x^2 = a^2 - a^2\sin^2\theta = a^2(1-\sin^2\theta) = \qquad .$$

2.



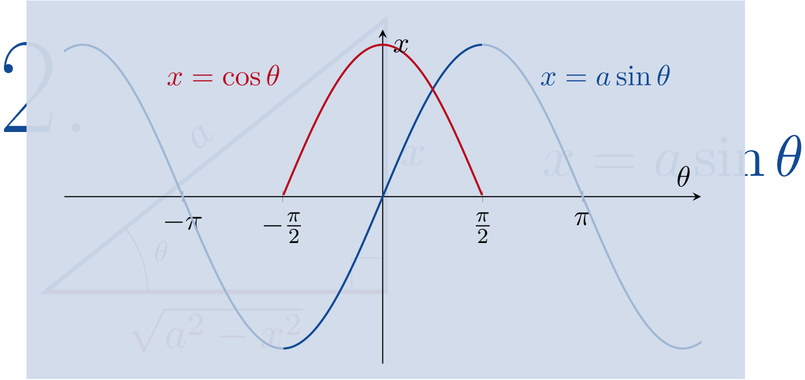
$$x = a \sin\theta$$

$$a^2 - x^2 = a^2 - a^2 \sin^2\theta = a^2(1 - \sin^2\theta) = a^2 \cos^2\theta.$$



$$a^2 - x^2 = a^2 - a^2\sin^2\theta = a^2(1-\sin^2\theta) = a^2\cos^2\theta.$$

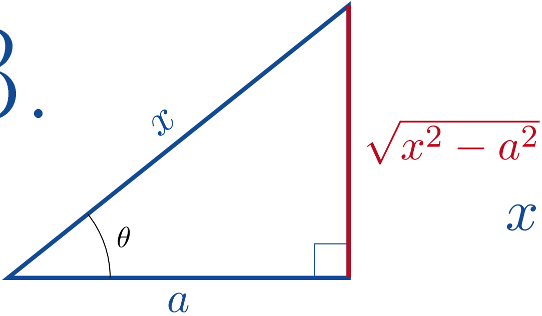
$$\boxed{-\frac{\pi}{2} < \theta < \frac{\pi}{2}.}$$



$$a^2 - x^2 = a^2 - a^2\sin^2\theta = a^2(1-\sin^2\theta) = a^2\cos^2\theta.$$

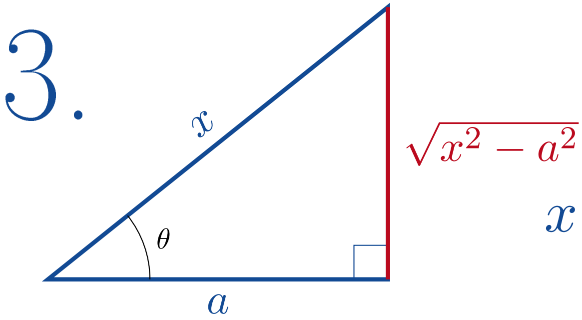
$$\boxed{\sqrt{a^2-x^2} = a\cos x \qquad -\frac{\pi}{2} < \theta < \frac{\pi}{2}.}$$

3.



$$x = a\sec\theta$$

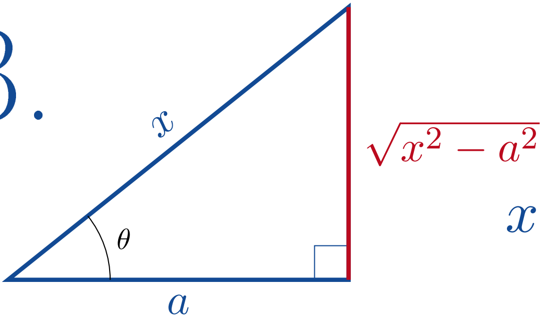
$$x^2 - a^2 = \qquad = \qquad = \qquad .$$



$$x = a\sec\theta$$

$$x^2 - a^2 = a^2\sec^2\theta - a^2 = \qquad\qquad = \qquad .$$

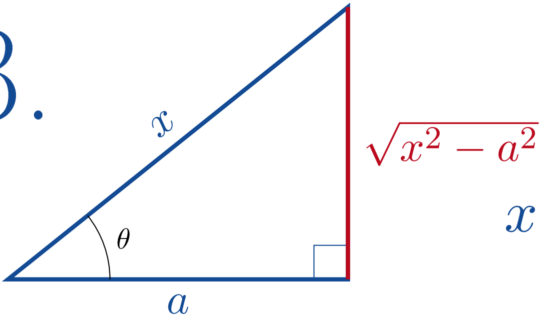
3.



$$x = a \sec \theta$$

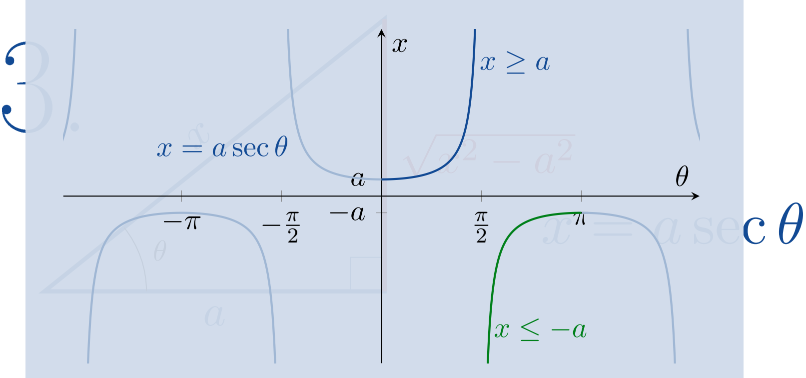
$$x^2 - a^2 = a^2 \sec^2 \theta - a^2 = a^2(\sec^2 \theta - 1) = \qquad .$$

3.



$$x = a\sec\theta$$

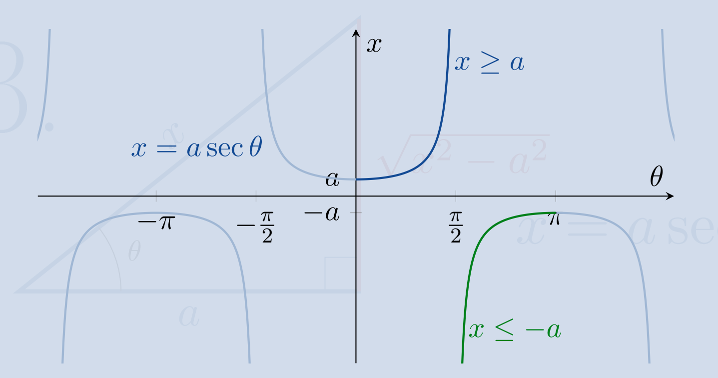
$$x^2 - a^2 = a^2\sec^2\theta - a^2 = a^2(\sec^2\theta - 1) = a^2\tan^2\theta.$$



$$x^2 - a^2 = a^2 \sec^2\theta - a^2 = a^2(\sec^2\theta - 1) = a^2 \tan^2\theta.$$

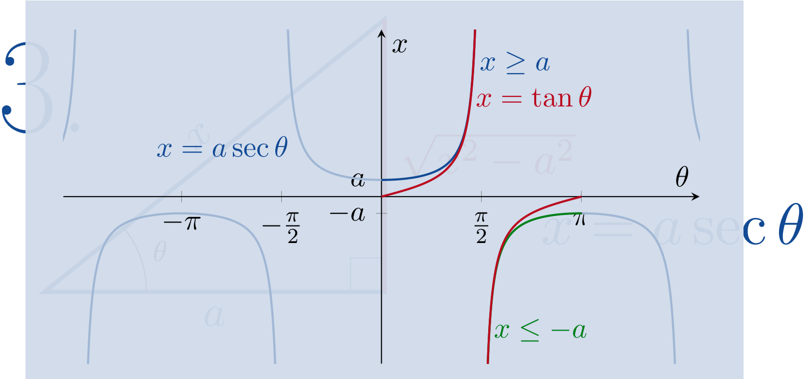
$$\boxed{\qquad .}$$

3. $x = a\sec\theta$



$$x^2 - a^2 = a^2\sec^2\theta - a^2 = a^2(\sec^2\theta - 1) = a^2\tan^2\theta.$$

$$\begin{cases} 0 \leq \theta < \frac{\pi}{2} & x \geq a \\ \frac{\pi}{2} < \theta \leq \pi & x \leq -a \end{cases}.$$



$$x^2 - a^2 = a^2 \sec^2\theta - a^2 = a^2(\sec^2\theta - 1) = a^2\tan^2\theta.$$

$$\boxed{\sqrt{x^2 - a^2} = a\,|\tan x| \qquad \begin{cases} 0 \leq \theta < \frac{\pi}{2} & x \geq a \\ \frac{\pi}{2} < \theta \leq \pi & x \leq -a \end{cases}.}$$

$x = a\tan\theta$	$x = a\sin\theta$	$x = a\sec\theta$
$\sqrt{a^2+x^2} = a\sec\theta$	$\sqrt{a^2-x^2} = a\cos\theta$	$\sqrt{x^2-a^2} = a\left\vert\tan\theta\right\vert$
$-\frac{\pi}{2} < \theta < \frac{\pi}{2}$	$-\frac{\pi}{2} < \theta < \frac{\pi}{2}$	$\begin{cases} 0 \le \theta < \frac{\pi}{2} & x \ge a \\ \frac{\pi}{2} < \theta \le \pi & x \le -a \end{cases}$

$x = a\tan\theta$	$x = a\sin\theta$	$x = a\sec\theta$
$\sqrt{a^2+x^2} = a\sec\theta$	$\sqrt{a^2-x^2} = a\cos\theta$	$\sqrt{x^2-a^2} = a\lvert\tan\theta\rvert$
$-\frac{\pi}{2} < \theta < \frac{\pi}{2}$	$-\frac{\pi}{2} < \theta < \frac{\pi}{2}$	$\begin{cases} 0 \le \theta < \frac{\pi}{2} & x \ge a \\ \frac{\pi}{2} < \theta \le \pi & x \le -a \end{cases}$

Example

Find $\displaystyle\int \frac{dx}{\sqrt{4+x^2}}$.

Let $x = \qquad$.

$x = a\tan\theta$	$x = a\sin\theta$	$x = a\sec\theta$
$\sqrt{a^2+x^2} = a\sec\theta$	$\sqrt{a^2-x^2} = a\cos\theta$	$\sqrt{x^2-a^2} = a\lvert\tan\theta\rvert$
$-\frac{\pi}{2} < \theta < \frac{\pi}{2}$	$-\frac{\pi}{2} < \theta < \frac{\pi}{2}$	$\begin{cases} 0 \le \theta < \frac{\pi}{2} & x \ge a \\ \frac{\pi}{2} < \theta \le \pi & x \le -a \end{cases}$

Example

Find $\displaystyle\int \frac{dx}{\sqrt{4+x^2}}$.

Let $x = 2\tan\theta$.

$x = a\tan\theta$	$x = a\sin\theta$	$x = a\sec\theta$
$\sqrt{a^2+x^2} = a\sec\theta$	$\sqrt{a^2-x^2} = a\cos\theta$	$\sqrt{x^2-a^2} = a\,\lvert\tan\theta\rvert$
$-\frac{\pi}{2} < \theta < \frac{\pi}{2}$	$-\frac{\pi}{2} < \theta < \frac{\pi}{2}$	$\begin{cases} 0 \le \theta < \frac{\pi}{2} & x \ge a \\ \frac{\pi}{2} < \theta \le \pi & x \le -a \end{cases}$

Example

Find $\displaystyle\int \frac{dx}{\sqrt{4+x^2}}$.

Let $x = 2\tan\theta$. Then $dx = 2\sec^2\theta\, d\theta$ and $\sqrt{4+x^2} = 2\sec\theta$.

$x = a\tan\theta$	$x = a\sin\theta$	$x = a\sec\theta$
$\sqrt{a^2+x^2} = a\sec\theta$	$\sqrt{a^2-x^2} = a\cos\theta$	$\sqrt{x^2-a^2} = a\,\lvert\tan\theta\rvert$
$-\frac{\pi}{2} < \theta < \frac{\pi}{2}$	$-\frac{\pi}{2} < \theta < \frac{\pi}{2}$	$\begin{cases} 0 \le \theta < \frac{\pi}{2} & x \ge a \\ \frac{\pi}{2} < \theta \le \pi & x \le -a \end{cases}$

Example

Find $\displaystyle\int \frac{dx}{\sqrt{4+x^2}}$.

Let $x = 2\tan\theta$. Then $dx = 2\sec^2\theta\, d\theta$ and $\sqrt{4+x^2} = 2\sec\theta$. Therefore

$$\int \frac{dx}{\sqrt{4+x^2}} = \int \frac{2\sec^2\theta\, d\theta}{2\sec\theta}$$

$$=$$

$$=$$

$$= \qquad .$$

$x = a\tan\theta$	$x = a\sin\theta$	$x = a\sec\theta$
$\sqrt{a^2+x^2} = a\sec\theta$	$\sqrt{a^2-x^2} = a\cos\theta$	$\sqrt{x^2-a^2} = a\lvert\tan\theta\rvert$
$-\frac{\pi}{2} < \theta < \frac{\pi}{2}$	$-\frac{\pi}{2} < \theta < \frac{\pi}{2}$	$\begin{cases} 0 \le \theta < \frac{\pi}{2} & x \ge a \\ \frac{\pi}{2} < \theta \le \pi & x \le -a \end{cases}$

Example

Find $\displaystyle\int \frac{dx}{\sqrt{4+x^2}}$.

Let $x = 2\tan\theta$. Then $dx = 2\sec^2\theta\, d\theta$ and $\sqrt{4+x^2} = 2\sec\theta$. Therefore

$$\begin{aligned} \int \frac{dx}{\sqrt{4+x^2}} &= \int \frac{2\sec^2\theta\, d\theta}{2\sec\theta} \\ &= \int \sec\theta\, d\theta \\ &= \\ &= \qquad\qquad . \end{aligned}$$

$x = a\tan\theta$	$x = a\sin\theta$	$x = a\sec\theta$
$\sqrt{a^2+x^2} = a\sec\theta$	$\sqrt{a^2-x^2} = a\cos\theta$	$\sqrt{x^2-a^2} = a\lvert\tan\theta\rvert$
$-\frac{\pi}{2} < \theta < \frac{\pi}{2}$	$-\frac{\pi}{2} < \theta < \frac{\pi}{2}$	$\begin{cases} 0 \le \theta < \frac{\pi}{2} & x \ge a \\ \frac{\pi}{2} < \theta \le \pi & x \le -a \end{cases}$

Example

Find $\displaystyle\int \frac{dx}{\sqrt{4+x^2}}$.

Let $x = 2\tan\theta$. Then $dx = 2\sec^2\theta\, d\theta$ and $\sqrt{4+x^2} = 2\sec\theta$. Therefore

$$\begin{aligned} \int \frac{dx}{\sqrt{4+x^2}} &= \int \frac{2\sec^2\theta\, d\theta}{2\sec\theta} \\ &= \int \sec\theta\, d\theta \\ &= \ln\lvert\sec\theta + \tan\theta\rvert + C \\ &= \qquad\qquad\qquad . \end{aligned}$$

$x = a\tan\theta$	$x = a\sin\theta$	$x = a\sec\theta$
$\sqrt{a^2+x^2} = a\sec\theta$	$\sqrt{a^2-x^2} = a\cos\theta$	$\sqrt{x^2-a^2} = a\lvert\tan\theta\rvert$
$-\frac{\pi}{2} < \theta < \frac{\pi}{2}$	$-\frac{\pi}{2} < \theta < \frac{\pi}{2}$	$\begin{cases} 0 \le \theta < \frac{\pi}{2} & x \ge a \\ \frac{\pi}{2} < \theta \le \pi & x \le -a \end{cases}$

Example

Find $\displaystyle\int \frac{dx}{\sqrt{4+x^2}}$.

Let $x = 2\tan\theta$. Then $dx = 2\sec^2\theta\,d\theta$ and $\sqrt{4+x^2} = 2\sec\theta$. Therefore

$$\begin{aligned}
\int \frac{dx}{\sqrt{4+x^2}} &= \int \frac{2\sec^2\theta\,d\theta}{2\sec\theta} \\
&= \int \sec\theta\,d\theta \\
&= \ln\lvert\sec\theta + \tan\theta\rvert + C \\
&= \ln\left|\frac{\sqrt{4+x^2}}{2} + \frac{x}{2}\right| + C.
\end{aligned}$$

$x = a\tan\theta$	$x = a\sin\theta$	$x = a\sec\theta$
$\sqrt{a^2+x^2} = a\sec\theta$	$\sqrt{a^2-x^2} = a\cos\theta$	$\sqrt{x^2-a^2} = a\lvert\tan\theta\rvert$
$-\frac{\pi}{2} < \theta < \frac{\pi}{2}$	$-\frac{\pi}{2} < \theta < \frac{\pi}{2}$	$\begin{cases} 0 \le \theta < \frac{\pi}{2} & x \ge a \\ \frac{\pi}{2} < \theta \le \pi & x \le -a \end{cases}$

Example

Calculate $\displaystyle\int_0^{\frac{3}{2}} \frac{dx}{\sqrt{9-x^2}}$.

Let $x = \qquad\qquad$.

$x = a\tan\theta$	$x = a\sin\theta$	$x = a\sec\theta$
$\sqrt{a^2+x^2} = a\sec\theta$	$\sqrt{a^2-x^2} = a\cos\theta$	$\sqrt{x^2-a^2} = a\lvert\tan\theta\rvert$
$-\frac{\pi}{2} < \theta < \frac{\pi}{2}$	$-\frac{\pi}{2} < \theta < \frac{\pi}{2}$	$\begin{cases} 0 \le \theta < \frac{\pi}{2} & x \ge a \\ \frac{\pi}{2} < \theta \le \pi & x \le -a \end{cases}$

Example

Calculate $\displaystyle\int_0^{\frac{3}{2}} \frac{dx}{\sqrt{9-x^2}}$.

Let $x = 3\sin\theta$.

$x = a\tan\theta$	$x = a\sin\theta$	$x = a\sec\theta$
$\sqrt{a^2+x^2} = a\sec\theta$	$\sqrt{a^2-x^2} = a\cos\theta$	$\sqrt{x^2-a^2} = a\,\lvert\tan\theta\rvert$
$-\frac{\pi}{2} < \theta < \frac{\pi}{2}$	$-\frac{\pi}{2} < \theta < \frac{\pi}{2}$	$\begin{cases} 0 \le \theta < \frac{\pi}{2} & x \ge a \\ \frac{\pi}{2} < \theta \le \pi & x \le -a \end{cases}$

Example

Calculate $\displaystyle\int_0^{\frac{3}{2}} \frac{dx}{\sqrt{9-x^2}}$.

Let $x = 3\sin\theta$. Then $dx = 3\cos\theta\,d\theta$ and $\sqrt{9-x^2} = 3\cos\theta$.

$x = a\tan\theta$	$x = a\sin\theta$	$x = a\sec\theta$
$\sqrt{a^2+x^2} = a\sec\theta$	$\sqrt{a^2-x^2} = a\cos\theta$	$\sqrt{x^2-a^2} = a\,\lvert\tan\theta\rvert$
$-\frac{\pi}{2} < \theta < \frac{\pi}{2}$	$-\frac{\pi}{2} < \theta < \frac{\pi}{2}$	$\begin{cases} 0 \le \theta < \frac{\pi}{2} & x \ge a \\ \frac{\pi}{2} < \theta \le \pi & x \le -a \end{cases}$

Example

Calculate $\displaystyle\int_0^{\frac{3}{2}} \frac{dx}{\sqrt{9-x^2}}$.

Let $x = 3\sin\theta$. Then $dx = 3\cos\theta\, d\theta$ and $\sqrt{9-x^2} = 3\cos\theta$. Moreover

$$x = \frac{3}{2} \quad\Longrightarrow\quad \theta = \sin^{-1}\frac{x}{3} = \sin^{-1}\frac{1}{2} = \frac{\pi}{6}$$
$$x = 0 \quad\Longrightarrow\quad \theta = \sin^{-1}\frac{x}{3} = \sin^{-1} 0 = 0$$

$$\begin{array}{ccc} x = a\tan\theta & x = a\sin\theta & x = a\sec\theta \\ \sqrt{a^2+x^2} = a\sec\theta & \sqrt{a^2-x^2} = a\cos\theta & \sqrt{x^2-a^2} = a\,|\tan\theta| \\ -\frac{\pi}{2} < \theta < \frac{\pi}{2} & -\frac{\pi}{2} < \theta < \frac{\pi}{2} & \begin{cases} 0 \le \theta < \frac{\pi}{2} & x \ge a \\ \frac{\pi}{2} < \theta \le \pi & x \le -a \end{cases} \end{array}$$

Example

Calculate $\displaystyle\int_0^{\frac{3}{2}} \frac{dx}{\sqrt{9-x^2}}$.

Let $x = 3\sin\theta$. Then $dx = 3\cos\theta\, d\theta$ and $\sqrt{9-x^2} = 3\cos\theta$. Moreover

$$x = \frac{3}{2} \implies \theta = \sin^{-1}\frac{x}{3} = \sin^{-1}\frac{1}{2} = \frac{\pi}{6}$$

$$x = 0 \implies \theta = \sin^{-1}\frac{x}{3} = \sin^{-1} 0 = 0$$

Therefore

$$\int_0^{\frac{3}{2}} \frac{dx}{\sqrt{9-x^2}} = \int_0^{\frac{\pi}{6}} \frac{3\cos\theta\, d\theta}{3\cos\theta} = \int_0^{\frac{\pi}{6}} d\theta = \frac{\pi}{6}.$$

$x = a\tan\theta$	$x = a\sin\theta$	$x = a\sec\theta$
$\sqrt{a^2+x^2} = a\sec\theta$	$\sqrt{a^2-x^2} = a\cos\theta$	$\sqrt{x^2-a^2} = a\,\lvert\tan\theta\rvert$
$-\frac{\pi}{2} < \theta < \frac{\pi}{2}$	$-\frac{\pi}{2} < \theta < \frac{\pi}{2}$	$\begin{cases} 0 \le \theta < \frac{\pi}{2} & x \ge a \\ \frac{\pi}{2} < \theta \le \pi & x \le -a \end{cases}$

Example

Calculate $\displaystyle\int_0^{\frac{3}{2}} \frac{dx}{\sqrt{9-x^2}}$.

Let $x = 3\sin\theta$. Then $dx = 3\cos\theta\, d\theta$ and $\sqrt{9-x^2} = 3\cos\theta$.
Moreover

$$x = \frac{3}{2} \implies \theta = \sin^{-1}\frac{x}{3} = \sin^{-1}\frac{1}{2} = \frac{\pi}{6}$$

$$x = 0 \implies \theta = \sin^{-1}\frac{x}{3} = \sin^{-1} 0 = 0$$

Therefore

$$\int_0^{\frac{3}{2}} \frac{dx}{\sqrt{9-x^2}} = \int_0^{\frac{\pi}{6}} \frac{3\cos\theta\, d\theta}{3\cos\theta} = \int_0^{\frac{\pi}{6}} d\theta = \frac{\pi}{6}.$$

Or

$$\int_0^{\frac{3}{2}} \frac{dx}{\sqrt{9-x^2}} = \left[\sin^{-1}\frac{x}{3}\right]_0^{\frac{3}{2}} = \frac{\pi}{6} - 0 = \frac{\pi}{6}.$$

$x = a\tan\theta$	$x = a\sin\theta$	$x = a\sec\theta$
$\sqrt{a^2+x^2} = a\sec\theta$	$\sqrt{a^2-x^2} = a\cos\theta$	$\sqrt{x^2-a^2} = a\lvert\tan\theta\rvert$
$-\frac{\pi}{2} < \theta < \frac{\pi}{2}$	$-\frac{\pi}{2} < \theta < \frac{\pi}{2}$	$\begin{cases} 0 \le \theta < \frac{\pi}{2} & x \ge a \\ \frac{\pi}{2} < \theta \le \pi & x \le -a \end{cases}$

Example

Calculate $\displaystyle\int \frac{x^2\,dx}{\sqrt{9-x^2}}$.

Let $x = \qquad\qquad$.

$$\begin{array}{ccc}
x = a\tan\theta & x = a\sin\theta & x = a\sec\theta \\
\sqrt{a^2+x^2} = a\sec\theta & \sqrt{a^2-x^2} = a\cos\theta & \sqrt{x^2-a^2} = a\left|\tan\theta\right| \\
-\frac{\pi}{2} < \theta < \frac{\pi}{2} & -\frac{\pi}{2} < \theta < \frac{\pi}{2} & \begin{cases} 0 \le \theta < \frac{\pi}{2} & x \ge a \\ \frac{\pi}{2} < \theta \le \pi & x \le -a \end{cases}
\end{array}$$

Example

Calculate $\displaystyle\int \frac{x^2\,dx}{\sqrt{9-x^2}}$.

Let $x = 3\sin\theta$.

$x = a\tan\theta$	$x = a\sin\theta$	$x = a\sec\theta$
$\sqrt{a^2+x^2} = a\sec\theta$	$\sqrt{a^2-x^2} = a\cos\theta$	$\sqrt{x^2-a^2} = a\,\lvert\tan\theta\rvert$
$-\frac{\pi}{2} < \theta < \frac{\pi}{2}$	$-\frac{\pi}{2} < \theta < \frac{\pi}{2}$	$\begin{cases} 0 \le \theta < \frac{\pi}{2} & x \ge a \\ \frac{\pi}{2} < \theta \le \pi & x \le -a \end{cases}$

Example

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Let $x = 3\sin\theta$. Then $dx = 3\cos\theta\,d\theta$ and $\sqrt{9-x^2} = 3\cos\theta$.

$x = a\tan\theta$	$x = a\sin\theta$	$x = a\sec\theta$
$\sqrt{a^2+x^2} = a\sec\theta$	$\sqrt{a^2-x^2} = a\cos\theta$	$\sqrt{x^2-a^2} = a\lvert\tan\theta\rvert$
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Example

Calculate $\displaystyle\int \frac{x^2\,dx}{\sqrt{9-x^2}}$.

Let $x = 3\sin\theta$. Then $dx = 3\cos\theta\,d\theta$ and $\sqrt{9-x^2} = 3\cos\theta$. Hence

$$\int \frac{x^2\,dx}{\sqrt{9-x^2}} = \int \frac{9\sin^2\theta\cdot 3\cos\theta\,d\theta}{3\cos\theta}$$

$$= 9\int \sin^2\theta\,d\theta = \ldots$$

$x = a\tan\theta$	$x = a\sin\theta$	$x = a\sec\theta$
$\sqrt{a^2+x^2} = a\sec\theta$	$\sqrt{a^2-x^2} = a\cos\theta$	$\sqrt{x^2-a^2} = a\,\lvert\tan\theta\rvert$
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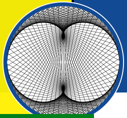

Example

Use $\displaystyle\int \frac{dx}{\sqrt{a^2+x^2}}$ to find an expression for $\sinh^{-1}\frac{x}{a}$ in terms of the natural logarithm.

$x = a\tan\theta$	$x = a\sin\theta$	$x = a\sec\theta$
$\sqrt{a^2+x^2} = a\sec\theta$	$\sqrt{a^2-x^2} = a\cos\theta$	$\sqrt{x^2-a^2} = a\lvert\tan\theta\rvert$
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Example

Use $\displaystyle\int \frac{dx}{\sqrt{a^2+x^2}}$ to find an expression for $\sinh^{-1}\frac{x}{a}$ in terms of the natural logarithm.

This time we use the substitution $x = a\tan\theta$ with $dx = a\sec^2\theta\, d\theta$ and $\sqrt{a^2+x^2} = a\sec\theta$ to calculate

$$\begin{aligned}\int \frac{dx}{\sqrt{a^2+x^2}} = \int \frac{a\sec^2\theta\, d\theta}{a\sec\theta} &= \int \sec\theta\, d\theta \\ &= \ln\lvert\sec\theta + \tan\theta\rvert + C \\ &= \ln\left|\frac{\sqrt{a^2+x^2}}{a} + \frac{x}{a}\right| + C.\end{aligned}$$

$x = a\tan\theta$	$x = a\sin\theta$	$x = a\sec\theta$
$\sqrt{a^2+x^2} = a\sec\theta$	$\sqrt{a^2-x^2} = a\cos\theta$	$\sqrt{x^2-a^2} = a\left\lvert\tan\theta\right\rvert$
$-\frac{\pi}{2} < \theta < \frac{\pi}{2}$	$-\frac{\pi}{2} < \theta < \frac{\pi}{2}$	$\begin{cases} 0 \le \theta < \frac{\pi}{2} & x \ge a \\ \frac{\pi}{2} < \theta \le \pi & x \le -a \end{cases}$

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$$\begin{aligned}\int \frac{dx}{\sqrt{a^2+x^2}} = \int \frac{a\sec^2\theta\,d\theta}{a\sec\theta} &= \int \sec\theta\,d\theta \\ &= \ln\left\lvert\sec\theta + \tan\theta\right\rvert + C \\ &= \ln\left\lvert\frac{\sqrt{a^2+x^2}}{a} + \frac{x}{a}\right\rvert + C.\end{aligned}$$

But we know from MATH113 that $\sinh^{-1}\frac{x}{a}$ is also an antiderivative of $\frac{1}{\sqrt{a^2+x^2}}$.

Recall that if $F(x)$ and $G(x)$ are both antiderivatives of $f(x)$ then we must have

$$F(x) = G(x) + C.$$

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Hence

$$\sinh^{-1}\frac{x}{a} = \ln\left|\frac{\sqrt{a^2+x^2}}{a} + \frac{x}{a}\right| + C.$$

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Hence

$$\sinh^{-1} \frac{x}{a} = \ln \left| \frac{\sqrt{a^2 + x^2}}{a} + \frac{x}{a} \right| + C.$$

What is C?

Recall that if $F(x)$ and $G(x)$ are both antiderivatives of $f(x)$ then we must have

$$F(x) = G(x) + C.$$

Hence

$$\sinh^{-1}\frac{x}{a} = \ln\left|\frac{\sqrt{a^2+x^2}}{a} + \frac{x}{a}\right| + C.$$

What is C? If we put $x = 0$ then we get

$$0 = \sinh^{-1} 0 = \ln\left|\frac{\sqrt{a^2+0^2}}{a} + 0\right| + C = \ln|1| + C = 0 + C.$$

Hence $C = 0$.

$$\sinh^{-1}\frac{x}{a} = \ln\left|\frac{\sqrt{a^2+x^2}}{a} + \frac{x}{a}\right| + C$$

Since $\sqrt{a^2+x^2} > |x|$, we conclude that

$$\boxed{\sinh^{-1}\frac{x}{a} = \ln\left(\frac{\sqrt{a^2+x^2}}{a} + \frac{x}{a}\right).}$$

$x = a\tan\theta$	$x = a\sin\theta$	$x = a\sec\theta$
$\sqrt{a^2+x^2} = a\sec\theta$	$\sqrt{a^2-x^2} = a\cos\theta$	$\sqrt{x^2-a^2} = a\lvert\tan\theta\rvert$
$-\frac{\pi}{2} < \theta < \frac{\pi}{2}$	$-\frac{\pi}{2} < \theta < \frac{\pi}{2}$	$\begin{cases} 0 \le \theta < \frac{\pi}{2} & x \ge a \\ \frac{\pi}{2} < \theta \le \pi & x \le -a \end{cases}$

Example

Calculate $\displaystyle\int \frac{dx}{\sqrt{25x^2-4}}$ for $x > \frac{2}{5}$.

$x = a\tan\theta$	$x = a\sin\theta$	$x = a\sec\theta$
$\sqrt{a^2+x^2} = a\sec\theta$	$\sqrt{a^2-x^2} = a\cos\theta$	$\sqrt{x^2-a^2} = a\,\lvert\tan\theta\rvert$
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First note that

$$\sqrt{25x^2-4} = \sqrt{25\left(x^2-\frac{4}{25}\right)} = 5\sqrt{x^2-\left(\frac{2}{5}\right)^2}.$$

$x = a\tan\theta$	$x = a\sin\theta$	$x = a\sec\theta$
$\sqrt{a^2+x^2} = a\sec\theta$	$\sqrt{a^2-x^2} = a\cos\theta$	$\sqrt{x^2-a^2} = a\left\lvert\tan\theta\right\rvert$
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So we want to integrate $\displaystyle\frac{1}{5}\int \frac{dx}{\sqrt{x^2-a^2}}$ with $a = \frac{2}{5}$.

$x = a\tan\theta$	$x = a\sin\theta$	$x = a\sec\theta$
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So we want to integrate $\displaystyle\frac{1}{5}\int \frac{dx}{\sqrt{x^2-a^2}}$ with $a = \frac{2}{5}$.

Let $x = \frac{2}{5}\sec\theta$.

$x = a\tan\theta$	$x = a\sin\theta$	$x = a\sec\theta$
$\sqrt{a^2+x^2} = a\sec\theta$	$\sqrt{a^2-x^2} = a\cos\theta$	$\sqrt{x^2-a^2} = a\left\lvert\tan\theta\right\rvert$
$-\frac{\pi}{2} < \theta < \frac{\pi}{2}$	$-\frac{\pi}{2} < \theta < \frac{\pi}{2}$	$\begin{cases} 0 \le \theta < \frac{\pi}{2} & x \ge a \\ \frac{\pi}{2} < \theta \le \pi & x \le -a \end{cases}$

$$\int \frac{dx}{\sqrt{25x^2-4}} = \frac{1}{5}\int \frac{dx}{\sqrt{x^2-\left(\frac{2}{5}\right)^2}}$$

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$x = a\tan\theta$	$x = a\sin\theta$	$x = a\sec\theta$
$\sqrt{a^2+x^2} = a\sec\theta$	$\sqrt{a^2-x^2} = a\cos\theta$	$\sqrt{x^2-a^2} = a\lvert\tan\theta\rvert$
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$$\frac{1}{5}\int \frac{dx}{\sqrt{x^2-\left(\frac{2}{5}\right)^2}} = \frac{1}{5}\int \frac{\frac{2}{5}\sec\theta\tan\theta\,d\theta}{\frac{2}{5}\left|\tan\theta\right|}.$$

$x = a\tan\theta$	$x = a\sin\theta$	$x = a\sec\theta$
$\sqrt{a^2+x^2} = a\sec\theta$	$\sqrt{a^2-x^2} = a\cos\theta$	$\sqrt{x^2-a^2} = a\,\lvert\tan\theta\rvert$
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But

$$x > \frac{2}{5} \quad\Longrightarrow\quad 0 \le \theta < \frac{\pi}{2}$$

$x = a\tan\theta$	$x = a\sin\theta$	$x = a\sec\theta$
$\sqrt{a^2+x^2} = a\sec\theta$	$\sqrt{a^2-x^2} = a\cos\theta$	$\sqrt{x^2-a^2} = a\,\lvert\tan\theta\rvert$
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But

$$x > \frac{2}{5} \quad\Longrightarrow\quad 0 \le \theta < \frac{\pi}{2} \quad\Longrightarrow\quad \tan\theta > 0.$$

$x = a\tan\theta$	$x = a\sin\theta$	$x = a\sec\theta$
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$$\frac{1}{5}\int \frac{dx}{\sqrt{x^2-\left(\frac{2}{5}\right)^2}} = \frac{1}{5}\int \frac{\frac{2}{5}\sec\theta\tan\theta\,d\theta}{\frac{2}{5}\lvert\tan\theta\rvert}.$$

But

$$x > \frac{2}{5} \quad\Longrightarrow\quad 0 \le \theta < \frac{\pi}{2} \quad\Longrightarrow\quad \tan\theta > 0.$$

Hence $\lvert\tan\theta\rvert = \tan\theta$.

$x = a\tan\theta$	$x = a\sin\theta$	$x = a\sec\theta$
$\sqrt{a^2+x^2} = a\sec\theta$	$\sqrt{a^2-x^2} = a\cos\theta$	$\sqrt{x^2-a^2} = a\lvert\tan\theta\rvert$
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Thus

$$\frac{1}{5}\int \frac{\frac{2}{5}\sec\theta\tan\theta\,d\theta}{\frac{2}{5}\lvert\tan\theta\rvert} = \frac{1}{5}\int \frac{\frac{2}{5}\sec\theta\tan\theta\,d\theta}{\frac{2}{5}\tan\theta}$$

$$=$$

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Thus

$$\begin{aligned}
\frac{1}{5}\int \frac{\frac{2}{5}\sec\theta\tan\theta\,d\theta}{\frac{2}{5}\left|\tan\theta\right|} &= \frac{1}{5}\int \frac{\frac{2}{5}\sec\theta\tan\theta\,d\theta}{\frac{2}{5}\tan\theta} \\
&= \frac{1}{5}\int \sec\theta\,d\theta \\
&= \\
&=
\end{aligned}$$

$x = a\tan\theta$	$x = a\sin\theta$	$x = a\sec\theta$
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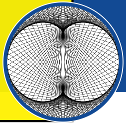

Thus

$$\frac{1}{5}\int \frac{\frac{2}{5}\sec\theta\tan\theta\,d\theta}{\frac{2}{5}\left|\tan\theta\right|} = \frac{1}{5}\int \frac{\frac{2}{5}\sec\theta\tan\theta\,d\theta}{\frac{2}{5}\tan\theta}$$

$$= \frac{1}{5}\int \sec\theta\,d\theta$$

$$= \frac{1}{5}\ln\left|\sec\theta+\tan\theta\right| + C$$

$$=$$

Thus

$$\begin{aligned}
\frac{1}{5}\int \frac{\frac{2}{5}\sec\theta\tan\theta\,d\theta}{\frac{2}{5}\left|\tan\theta\right|} &= \frac{1}{5}\int \frac{\frac{2}{5}\sec\theta\tan\theta\,d\theta}{\frac{2}{5}\tan\theta} \\
&= \frac{1}{5}\int \sec\theta\,d\theta \\
&= \frac{1}{5}\ln\left|\sec\theta+\tan\theta\right| + C \\
&= \frac{1}{5}\ln\left|\frac{5x}{2}+\frac{\sqrt{25x^2-4}}{2}\right| + C
\end{aligned}$$

since we used $x = \frac{2}{5}\sec\theta$ and $\sqrt{25x^2-4} = 2\tan\theta$.

Integration of Rational Functions by Partial Fractions

$$\frac{2}{x+1} + \frac{3}{x-3} = \frac{\text{something?}}{\text{something?}}$$

$$\frac{2}{x+1} + \frac{3}{x-3} = \frac{2}{(x+1)\qquad\qquad} + \frac{3}{(x-3)\qquad\qquad}$$

$$\frac{2}{x+1} + \frac{3}{x-3} = \frac{2(x-3)}{(x+1)(x-3)} + \frac{3}{(x-3)}$$

$$\frac{2}{x+1}+\frac{3}{x-3}=\frac{2(x-3)}{(x+1)(x-3)}+\frac{3(x+1)}{(x-3)(x+1)}$$

$$\begin{aligned}\frac{2}{x+1}+\frac{3}{x-3} &= \frac{2(x-3)}{(x+1)(x-3)}+\frac{3(x+1)}{(x-3)(x+1)}\\ &= \frac{2x-6}{x^2-2x-3}+\frac{3x+3}{x^2-2x-3}\end{aligned}$$

$$\begin{aligned}\frac{2}{x+1}+\frac{3}{x-3} &= \frac{2(x-3)}{(x+1)(x-3)}+\frac{3(x+1)}{(x-3)(x+1)}\\ &= \frac{2x-6}{x^2-2x-3}+\frac{3x+3}{x^2-2x-3}\\ &= \frac{5x-3}{x^2-2x-3}.\end{aligned}$$

$$\begin{aligned}
\frac{2}{x+1} + \frac{3}{x-3} &= \frac{2(x-3)}{(x+1)(x-3)} + \frac{3(x+1)}{(x-3)(x+1)} \\
&= \frac{2x-6}{x^2-2x-3} + \frac{3x+3}{x^2-2x-3} \\
&= \frac{5x-3}{x^2-2x-3}.
\end{aligned}$$

But how do we do the opposite?

$$\frac{13x+1}{x^2-9} = \frac{\text{something?}}{x-3} + \frac{\text{something?}}{x+3}.$$

$$\frac{5x-3}{x^2-2x-3} = \frac{A}{x+1} + \frac{B}{x-3}.$$

Pretend for a moment that we don't know that $A = 2$ and $B = 3$. How can we find A and B?

$$\frac{5x-3}{x^2-2x-3} = \frac{A}{x+1} + \frac{B}{x-3}.$$

Pretend for a moment that we don't know that $A = 2$ and $B = 3$. How can we find A and B?

$$\begin{aligned}\frac{5x-3}{x^2-2x-3} &= \frac{A}{(x+1)} + \frac{B}{(x-3)} \\ &= \\ &= \qquad\qquad\qquad .\end{aligned}$$

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$$\begin{aligned}\frac{5x-3}{x^2-2x-3} &= \frac{A(x-3)}{(x+1)(x-3)} + \frac{B(x+1)}{(x-3)(x+1)} \\ &= \\ &= \qquad\qquad .\end{aligned}$$

$$\frac{5x-3}{x^2-2x-3} = \frac{A}{x+1} + \frac{B}{x-3}.$$

Pretend for a moment that we don't know that $A = 2$ and $B = 3$. How can we find A and B?

$$\begin{aligned}\frac{5x-3}{x^2-2x-3} &= \frac{A(x-3)}{(x+1)(x-3)} + \frac{B(x+1)}{(x-3)(x+1)} \\ &= \frac{Ax-3A}{x^2-2x-3} + \frac{Bx+B}{x^2-2x-3} \\ &= \qquad\qquad\qquad\qquad .\end{aligned}$$

$$\frac{5x-3}{x^2-2x-3} = \frac{A}{x+1} + \frac{B}{x-3}.$$

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Pretend for a moment that we don't know that $A = 2$ and $B = 3$. How can we find A and B?

$$\begin{aligned}\frac{5x-3}{x^2-2x-3} &= \frac{A(x-3)}{(x+1)(x-3)} + \frac{B(x+1)}{(x-3)(x+1)}\\ &= \frac{Ax-3A}{x^2-2x-3} + \frac{Bx+B}{x^2-2x-3}\\ &= \frac{(A+B)x+(B-3A)}{x^2-2x-3}.\end{aligned}$$

$$\frac{5x-3}{x^2-2x-3} = \frac{A}{x+1} + \frac{B}{x-3}.$$

Pretend for a moment that we don't know that $A = 2$ and $B = 3$. How can we find A and B?

$$\begin{aligned}\frac{5x-3}{x^2-2x-3} &= \frac{A(x-3)}{(x+1)(x-3)} + \frac{B(x+1)}{(x-3)(x+1)} \\ &= \frac{Ax-3A}{x^2-2x-3} + \frac{Bx+B}{x^2-2x-3} \\ &= \frac{(A+B)x+(B-3A)}{x^2-2x-3}.\end{aligned}$$

Hence

$$\begin{cases} A+B=5 \\ B-3A=-3 \end{cases}$$

$$\frac{5x-3}{x^2-2x-3} = \frac{A}{x+1} + \frac{B}{x-3}.$$

Pretend for a moment that we don't know that $A = 2$ and $B = 3$. How can we find A and B?

$$\begin{aligned} \frac{5x-3}{x^2-2x-3} &= \frac{A(x-3)}{(x+1)(x-3)} + \frac{B(x+1)}{(x-3)(x+1)} \\ &= \frac{Ax-3A}{x^2-2x-3} + \frac{Bx+B}{x^2-2x-3} \\ &= \frac{(A+B)x+(B-3A)}{x^2-2x-3}. \end{aligned}$$

Hence

$$\begin{cases} A+B=5 \\ B-3A=-3 \end{cases} \implies \begin{cases} A=2 \\ B=3. \end{cases}$$

We can use *partial fractions* on $\dfrac{f(x)}{g(x)}$

We can use *partial fractions* on $\dfrac{f(x)}{g(x)}$ if

- $\left(\begin{array}{c}\text{the degree}\\ \text{of } f(x)\end{array}\right) < \left(\begin{array}{c}\text{the degree}\\ \text{of } g(x)\end{array}\right)$;

We can use *partial fractions* on $\dfrac{f(x)}{g(x)}$ if

- $\begin{pmatrix} \text{the degree} \\ \text{of } f(x) \end{pmatrix} < \begin{pmatrix} \text{the degree} \\ \text{of } g(x) \end{pmatrix}$; and
- we can factorise $g(x)$.

Example

Write $\dfrac{x^2+x+2}{(x+1)(x^2+1)}$ in partial fractions.

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$$\frac{x^2+x+2}{(x+1)(x^2+1)} = \frac{A}{x+1} + \frac{Bx+C}{x^2+1}$$

$$=$$

$$=$$

$$=$$

$$= \qquad .$$

Example

Write $\dfrac{x^2+x+2}{(x+1)(x^2+1)}$ in partial fractions.

$$
\begin{aligned}
\frac{x^2+x+2}{(x+1)(x^2+1)} &= \frac{A}{x+1} + \frac{Bx+C}{x^2+1} \\
&= \frac{A(x^2+1)}{(x+1)(x^2+1)} + \frac{(Bx+C)(x+1)}{(x^2+1)(x+1)} \\
&= \\
&= \\
&= \qquad\qquad\qquad .
\end{aligned}
$$

Example

Write $\dfrac{x^2+x+2}{(x+1)(x^2+1)}$ in partial fractions.

$$
\begin{aligned}
\frac{x^2+x+2}{(x+1)(x^2+1)} &= \frac{A}{x+1} + \frac{Bx+C}{x^2+1} \\
&= \frac{A(x^2+1)}{(x+1)(x^2+1)} + \frac{(Bx+C)(x+1)}{(x^2+1)(x+1)} \\
&= \frac{Ax^2+A+Bx^2+Bx+Cx+C}{(x+1)(x^2+1)} \\
&= \\
&= \qquad .
\end{aligned}
$$

Example

Write $\dfrac{x^2+x+2}{(x+1)(x^2+1)}$ in partial fractions.

$$\begin{aligned}
\frac{x^2+x+2}{(x+1)(x^2+1)} &= \frac{A}{x+1} + \frac{Bx+C}{x^2+1} \\
&= \frac{A(x^2+1)}{(x+1)(x^2+1)} + \frac{(Bx+C)(x+1)}{(x^2+1)(x+1)} \\
&= \frac{Ax^2+A+Bx^2+Bx+Cx+C}{(x+1)(x^2+1)} \\
&= \frac{(A+B)x^2+(B+C)x+(A+C)}{(x+1)(x^2+1)} \\
&= \qquad\qquad\qquad .
\end{aligned}$$

8.5 Integration of Rational Functions by Partial Fractions

Example

Write $\dfrac{x^2+x+2}{(x+1)(x^2+1)}$ in partial fractions.

$$A + B = 1$$
$$B + C = 1$$
$$A + C = 2$$

$$\frac{x^2+x+2}{(x+1)(x^2+1)} = \frac{A}{x+ \ \cdots}$$

$$= \frac{A \ \cdots \ Bx + C)(x+1)}{(x+ \ \cdots \ x^2+1)(x+1)}$$

$$= \frac{Ax^2 + A + Bx^2 + Bx + Cx + C}{(x+1)(x^2+1)}$$

$$= \frac{(A+B)x^2 + (B+C)x + (A+C)}{(x+1)(x^2+1)}$$

$$= \qquad .$$

Example

Write $\dfrac{x^2+x+2}{(x+1)(x^2+1)}$ in partial fractions.

$$\frac{x^2+x+2}{(x+1)(x^2+1)} = \frac{A}{x+\ \cdots}$$

$$= \frac{A\ \cdots\ 1)}{(x+\ \cdots\ 1)}$$

$$= \frac{Ax^2+A+Bx^2+\ \cdots}{(x+1)(x^2+1)}$$

$$= \frac{(A+B)x^2+(B+C)x+(A+C)}{(x+1)(x^2+1)}$$

$$= \qquad .$$

$$A+B=1$$
$$B+C=1$$
$$A+C=2$$

$$A=1$$
$$B=0$$
$$C=1$$

Example

Write $\dfrac{x^2+x+2}{(x+1)(x^2+1)}$ in partial fractions.

$$
\begin{aligned}
\frac{x^2+x+2}{(x+1)(x^2+1)} &= \frac{A}{x+1} + \frac{Bx+C}{x^2+1} \\
&= \frac{A(x^2+1)}{(x+1)(x^2+1)} + \frac{(\cdots 1)}{\cdots 1)} \\
&= \frac{Ax^2+A+Bx^2+\cdots}{(x+1)(x^2+1)} \\
&= \frac{(A+B)x^2+(B+C)x+(A+C)}{(x+1)(x^2+1)} \\
&= \frac{1}{x+1} + \frac{1}{x^2+1}.
\end{aligned}
$$

$$A = 1$$
$$B = 0$$
$$C = 1$$

Example

Write $\dfrac{3x^3 + 2x^2 + x}{(x+3)^4}$ in partial fractions.

Example

Write $\dfrac{3x^3+2x^2+x}{(x+3)^4}$ in partial fractions.

$$\frac{3x^3+2x^2+x}{(x+3)^4} = \frac{A}{x+3} + \frac{B}{(x+3)^2} + \frac{C}{(x+3)^3} + \frac{D}{(x+3)^4} = \dots$$

Example

Write $\dfrac{3x^3+2x^2+x}{(x+3)^4}$ in partial fractions.

$$\frac{3x^3+2x^2+x}{(x+3)^4} = \frac{A}{x+3} + \frac{B}{(x+3)^2} + \frac{C}{(x+3)^3} + \frac{D}{(x+3)^4} = \dots$$

Example

Write $\dfrac{71}{(x+3)(x^2+2x+3)^2}$ in partial fractions.

Example

Write $\dfrac{3x^3+2x^2+x}{(x+3)^4}$ in partial fractions.

$$\frac{3x^3+2x^2+x}{(x+3)^4} = \frac{A}{x+3} + \frac{B}{(x+3)^2} + \frac{C}{(x+3)^3} + \frac{D}{(x+3)^4} = \dots$$

Example

Write $\dfrac{71}{(x+3)(x^2+2x+3)^2}$ in partial fractions.

$$\frac{71}{(x+3)(x^2+2x+3)^2} = \frac{A}{x+3} + \frac{Bx+C}{(x^2+2x+3)} + \frac{Dx+E}{(x^2+2x+3)^2}$$
$$= \dots$$

Method of Partial Fractions When $f(x)/g(x)$ Is Proper

1. Let $x - r$ be a linear factor of $g(x)$. Suppose that $(x - r)^m$ is the highest power of $x - r$ that divides $g(x)$. Then, to this factor, assign the sum of the m partial fractions:

$$\frac{A_1}{(x - r)} + \frac{A_2}{(x - r)^2} + \cdots + \frac{A_m}{(x - r)^m}.$$

   Do this for each distinct linear factor of $g(x)$.

2. Let $x^2 + px + q$ be an irreducible quadratic factor of $g(x)$ so that $x^2 + px + q$ has no real roots. Suppose that $(x^2 + px + q)^n$ is the highest power of this factor that divides $g(x)$. Then, to this factor, assign the sum of the n partial fractions:

$$\frac{B_1x + C_1}{(x^2 + px + q)} + \frac{B_2x + C_2}{(x^2 + px + q)^2} + \cdots + \frac{B_nx + C_n}{(x^2 + px + q)^n}.$$

   Do this for each distinct quadratic factor of $g(x)$.

3. Set the original fraction $f(x)/g(x)$ equal to the sum of all these partial fractions. Clear the resulting equation of fractions and arrange the terms in decreasing powers of x.

4. Equate the coefficients of corresponding powers of x and solve the resulting equations for the undetermined coefficients.

Example

Use partial fractions to find $\displaystyle\int \frac{x^2 + 4x + 1}{(x-1)(x+1)(x+3)}\, dx.$

Since

$$\frac{x^2+4x+1}{(x-1)(x+1)(x+3)} = \frac{A}{x-1} + \frac{B}{x+1} + \frac{C}{x+3}$$

$$=$$

$$=$$

$$=$$

$$=$$

Since

$$\frac{x^2+4x+1}{(x-1)(x+1)(x+3)}$$

$$= \frac{A}{x-1} + \frac{B}{x+1} + \frac{C}{x+3}$$

$$= \frac{A(x+1)(x+3)+B(x-1)(x+3)+C(x-1)(x+1)}{(x-1)(x+1)(x+3)}$$

$$=$$

$$=$$

$$=$$

Since

$$\frac{x^2+4x+1}{(x-1)(x+1)(x+3)}$$

$$= \frac{A}{x-1} + \frac{B}{x+1} + \frac{C}{x+3}$$

$$= \frac{A(x+1)(x+3) + B(x-1)(x+3) + C(x-1)(x+1)}{(x-1)(x+1)(x+3)}$$

$$= \frac{A(x^2+4x+3) + B(x^2+2x-3) + C(x^2-1)}{(x-1)(x+1)(x+3)}$$

$$=$$

$$=$$

Since

$$
\begin{aligned}
\frac{x^2+4x+1}{(x-1)(x+1)(x+3)}
&= \frac{A}{x-1} + \frac{B}{x+1} + \frac{C}{x+3} \\
&= \frac{A(x+1)(x+3) + B(x-1)(x+3) + C(x-1)(x+1)}{(x-1)(x+1)(x+3)} \\
&= \frac{A(x^2+4x+3) + B(x^2+2x-3) + C(x^2-1)}{(x-1)(x+1)(x+3)} \\
&= \frac{(A+B+C)x^2 + (4A+2B)x + (3A-3B-C)}{(x-1)(x+1)(x+3)} \\
&=
\end{aligned}
$$

8.5 Integration of Rational Functions by Partial Fractions

Since

$$\frac{x^2+4x+1}{(x-1)(x+1)(x+3)}$$

$$= \frac{A}{x-1} + \frac{B}{x+1} +$$

$$= \frac{A(x+1)(x+3) \quad (x-1)(x+1)}{}$$

$$= \frac{A(x^2+4x+3) + B(x^2+2x-3) + C(x^2-1)}{(x-1)(x+1)(x+3)}$$

$$= \frac{(A+B+C)x^2 + (4A+2B)x + (3A-3B-C)}{(x-1)(x+1)(x+3)}$$

$$=$$

$$\begin{aligned} A+B+C &= 1 \\ 4A+2B &= 4 \\ 3A-3B-C &= 1 \end{aligned}$$

8.5 Integration of Rational Functions by Partial Fractions

Since

$$\frac{x^2+4x+1}{(x-1)(x+1)(x+3)}$$

$$= \frac{A}{x-1} + \frac{B}{x+1} + \cdots$$

$$= \frac{A(x+1)(x+3) \cdots 1)}{\cdots}$$

$$= \frac{A(x^2+4x+3) + \cdots}{(x-1)(x+1)(x+3)}$$

$$= \frac{(A+B+C)x^2 + (4A+2B)x + (3A-3B-C)}{(x-1)(x+1)(x+3)}$$

$$=$$

$$\begin{aligned} A+B+C &= 1 \\ 4A+2B &= 4 \\ 3A-3B-C &= 1 \end{aligned}$$

$$A = \frac{3}{4} \qquad B = \frac{1}{2} \qquad C = -\frac{1}{4}$$

Since

$$
\begin{aligned}
\frac{x^2+4x+1}{(x-1)(x+1)(x+3)}
&= \frac{A}{x-1} + \frac{B}{x+1} + \frac{C}{x+3} \\
&= \frac{A(x+1)(x+3) + B(x-1)(x+3) + \cdots 1)}{(x-1)(x+1)(x+3)} \\
&= \frac{A(x^2+4x+3) + B(x^2+2x-3) + C \cdots}{(x-1)(x+1)(x+3)} \\
&= \frac{(A+B+C)x^2 + (4A+2B)x + (3A-3B-C)}{(x-1)(x+1)(x+3)} \\
&= \frac{\frac{3}{4}}{x-1} + \frac{\frac{1}{2}}{x+1} + \frac{-\frac{1}{4}}{x+3}
\end{aligned}
$$

$$A = \frac{3}{4}$$

$$B = \frac{1}{2}$$

$$C = -\frac{1}{4}$$

$$\frac{x^2+4x+1}{(x-1)(x+1)(x+3)} = \frac{\frac{3}{4}}{x-1} + \frac{\frac{1}{2}}{x+1} + \frac{-\frac{1}{4}}{x+3}$$

We have that

$$\int \frac{x^2+4x+1}{(x-1)(x+1)(x+3)}\,dx$$

$$= \int \frac{\frac{3}{4}}{x-1} + \frac{\frac{1}{2}}{x+1} + \frac{-\frac{1}{4}}{x+3}\,dx$$

$$=$$

$$= \qquad .$$

$$\frac{x^2+4x+1}{(x-1)(x+1)(x+3)} = \frac{\frac{3}{4}}{x-1} + \frac{\frac{1}{2}}{x+1} + \frac{-\frac{1}{4}}{x+3}$$

We have that

$$\begin{aligned}
\int \frac{x^2+4x+1}{(x-1)(x+1)(x+3)}\,dx \\
&= \int \frac{\frac{3}{4}}{x-1} + \frac{\frac{1}{2}}{x+1} + \frac{-\frac{1}{4}}{x+3}\,dx \\
&= \frac{3}{4}\int \frac{dx}{x-1} + \frac{1}{2}\int \frac{dx}{x+1} - \frac{1}{4}\int \frac{dx}{x+3} \\
&= \qquad\qquad\qquad\qquad .
\end{aligned}$$

$$\frac{x^2+4x+1}{(x-1)(x+1)(x+3)} = \frac{\frac{3}{4}}{x-1} + \frac{\frac{1}{2}}{x+1} + \frac{-\frac{1}{4}}{x+3}$$

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&= \frac{3}{4}\int \frac{dx}{x-1} + \frac{1}{2}\int \frac{dx}{x+1} - \frac{1}{4}\int \frac{dx}{x+3} \\
&= \frac{3}{4}\ln|x-1| + \frac{1}{2}\ln|x+1| - \frac{1}{4}\ln|x+3| + K.
\end{aligned}$$

(I already used C)

Example

Use partial fractions to find $\displaystyle\int \frac{6x+7}{(x+2)^2}\,dx$.

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Because we have $(x+2)^2$ in the denominator, our partial sum will look like

$$\frac{6x+7}{(x+2)^2} = \frac{\text{something}}{x+2} + \frac{\text{something}}{(x+2)^2}$$

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$$6x + 7 = A(x+2) + B = Ax + (2A + B).$$

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So $A = 6$ and $B = 7 - 2A = 7 - 12 = -5$.

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$$6x + 7 = A(x+2) + B = Ax + (2A + B).$$

So $A = 6$ and $B = 7 - 2A = 7 - 12 = -5$. Thus

$$\frac{6x+7}{(x+2)^2} = \frac{6}{x+2} + \frac{-5}{(x+2)^2}.$$

Now we can integrate.

$$\begin{aligned}
\int \frac{6x+7}{(x+2)^2}\,dx &= \int \frac{6}{x+2} + \frac{-5}{(x+2)^2}\,dx \\
&= 6\int \frac{dx}{x+2} - 5\int (x+2)^{-2}\,dx \\
&= \\
&=
\end{aligned}$$

$$
\begin{aligned}
\int \frac{6x+7}{(x+2)^2}\,dx &= \int \frac{6}{x+2} + \frac{-5}{(x+2)^2}\,dx \\
&= 6\int \frac{dx}{x+2} - 5\int (x+2)^{-2}\,dx \\
&= 6\ln|x+2| + 5(x+2)^{-1} + C \\
&= 6\ln|x+2| + \frac{5}{x+2} + C.
\end{aligned}
$$

Example

Use partial fractions to find $\int \frac{2x^3 - 4x^2 - x - 3}{x^2 - 2x - 3}\,dx$.

Example

Use partial fractions to find $\int \frac{2x^3 - 4x^2 - x - 3}{x^2 - 2x - 3}\, dx$.

Here the numerator is a 3rd order polynomial but the denominator is only a 2nd order polynomial.

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Use partial fractions to find $\displaystyle\int \frac{2x^3 - 4x^2 - x - 3}{x^2 - 2x - 3}\,dx.$

Here the numerator is a 3rd order polynomial but the denominator is only a 2nd order polynomial. Remember that we can only use partial fractions on $\frac{f(x)}{g(x)}$ if

$$\begin{pmatrix} \text{the degree} \\ \text{of } f(x) \end{pmatrix} < \begin{pmatrix} \text{the degree} \\ \text{of } g(x) \end{pmatrix}.$$

Example

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So what do we do?

Example

Use partial fractions to find $\int \frac{2x^3 - 4x^2 - x - 3}{x^2 - 2x - 3}\,dx$.

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So what do we do? First we need to write our integrand as

$$\frac{2x^3 - 4x^2 - x - 3}{x^2 - 2x - 3} = (\text{something}) + \frac{\text{something}}{x^2 - 2x - 3}.$$

Example

Use partial fractions to find $\int \frac{2x^3 - 4x^2 - x - 3}{x^2 - 2x - 3}\,dx$.

Here the numerator is a 3rd order polynomial but the denominator is only a 2nd order polynomial. Remember that we can only use partial fractions on $\frac{f(x)}{g(x)}$ if

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$$\frac{2x^3 - 4x^2 - x - 3}{x^2 - 2x - 3} = (\text{linear function}) + \frac{\text{something}}{x^2 - 2x - 3}.$$

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Your textbook really likes "long division" and does this:

$$\begin{array}{r|l} & \quad 2x \\ \cline{2-2} x^2 - 2x - 3 & 2x^3 - 4x^2 - x - 3 \\ & \underline{2x^3 - 4x^2 - 6x} - 3 \\ & \qquad\qquad\quad 5x - 3 \end{array}$$

If you understand this, then great do this.

$$\frac{2x^3 - 4x^2 - x - 3}{x^2 - 2x - 3} = (\text{linear function}) + \frac{\text{something}}{x^2 - 2x - 3}.$$

Your textbook really likes "long division" and does this:

$$\begin{array}{r@{}l} & \;\; 2x \\ x^2 - 2x - 3 \,\big) & \overline{\,2x^3 - 4x^2 - x - 3} \\ & \underline{\,2x^3 - 4x^2 - 6x} - 3 \\ & 5x - 3 \end{array}$$

If you understand this, then great do this. As you know, I prefer to do this a different way.

$$\frac{2x^3 - 4x^2 - x - 3}{x^2 - 2x - 3} = \text{(linear function)} + \frac{\text{something}}{x^2 - 2x - 3}$$

$$\frac{2x^3 - 4x^2 - x - 3}{x^2 - 2x - 3} = (Ax + B) + \frac{Cx + D}{x^2 - 2x - 3}$$

$$\frac{2x^3 - 4x^2 - x - 3}{x^2 - 2x - 3} = (Ax + B) + \frac{Cx + D}{x^2 - 2x - 3}$$

Multiplying both sides by $(x^2 - 2x - 3)$ gives

$$\begin{aligned} 2x^3 - 4x^2 - x - 3 &= (Ax + B)(x^2 - 2x - 3) + (Cx + D) \\ &= Ax^3 - 2Ax^2 - 3Ax + Bx^2 - 2Bx - 3B + Cx + D \\ &= Ax^3 + (-2A + B)x^2 + (-3A - 2B + C)x + D \end{aligned}$$

which implies $A = 2$, $B = 0$, $C = 5$ and $D = -3$ (please check!).

$$\frac{2x^3-4x^2-x-3}{x^2-2x-3} = (Ax+B) + \frac{Cx+D}{x^2-2x-3}$$

Multiplying both sides by (x^2-2x-3) gives

$$\begin{aligned} 2x^3-4x^2-x-3 &= (Ax+B)(x^2-2x-3)+(Cx+D) \\ &= Ax^3-2Ax^2-3Ax+Bx^2-2Bx-3B+Cx+D \\ &= Ax^3+(-2A+B)x^2+(-3A-2B+C)x+D \end{aligned}$$

which implies $A=2$, $B=0$, $C=5$ and $D=-3$ (please check!).

Thus our integrand is

$$\frac{2x^3-4x^2-x-3}{x^2-2x-3} = 2x + \frac{5x-3}{x^2-2x-3}.$$

$$\frac{2x^3 - 4x^2 - x - 3}{x^2 - 2x - 3} = 2x + \frac{5x - 3}{x^2 - 2x - 3}$$

At the start of this section, we found that

$$\frac{5x - 3}{x^2 - 2x - 3} = \frac{2}{x + 2} + \frac{3}{x - 1}.$$

$$\frac{2x^3 - 4x^2 - x - 3}{x^2 - 2x - 3} = 2x + \frac{5x - 3}{x^2 - 2x - 3}$$

At the start of this section, we found that

$$\frac{5x - 3}{x^2 - 2x - 3} = \frac{2}{x + 2} + \frac{3}{x - 1}.$$

Therefore

$$\begin{aligned}\frac{2x^3 - 4x^2 - x - 3}{x^2 - 2x - 3} &= \int 2x\,dx + \int \frac{2}{x + 2}\,dx + \int \frac{3}{x - 1}\,dx \\ &= \end{aligned}$$

$$\frac{2x^3 - 4x^2 - x - 3}{x^2 - 2x - 3} = 2x + \frac{5x - 3}{x^2 - 2x - 3}$$

At the start of this section, we found that

$$\frac{5x - 3}{x^2 - 2x - 3} = \frac{2}{x + 2} + \frac{3}{x - 1}.$$

Therefore

$$\begin{aligned} \frac{2x^3 - 4x^2 - x - 3}{x^2 - 2x - 3} &= \int 2x \, dx + \int \frac{2}{x + 2} \, dx + \int \frac{3}{x - 1} \, dx \\ &= x^2 + 2 \ln |x + 1| + 3 \ln |x - 3| + C. \end{aligned}$$

Example

Use partial fractions to find $\int \frac{-2x+4}{(x^2+1)(x-1)^2}\,dx$.

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We want

$$\frac{-2x+4}{(x^2+1)(x-1)^2} = \frac{\text{something}}{x^2+1} + \frac{\text{something}}{x-1} + \frac{\text{something}}{(x-1)^2}$$

Example

Use partial fractions to find $\int \frac{-2x+4}{(x^2+1)(x-1)^2}\,dx$.

We want

$$\frac{-2x+4}{(x^2+1)(x-1)^2} = \frac{Ax+B}{x^2+1} + \frac{\text{something}}{x-1} + \frac{\text{something}}{(x-1)^2}$$

Example

Use partial fractions to find $\displaystyle\int \frac{-2x+4}{(x^2+1)(x-1)^2}\,dx$.

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Example

Use partial fractions to find $\displaystyle\int \frac{-2x+4}{(x^2+1)(x-1)^2}\,dx$.

We want

$$\frac{-2x+4}{(x^2+1)(x-1)^2} = \frac{Ax+B}{x^2+1} + \frac{C}{x-1} + \frac{D}{(x-1)^2}$$

Please check that multiplying by $(x^2+1)(x-1)^2$ gives

$$\begin{aligned}-2x+4 &= (Ax+B)(x-1)^2 + C(x^2+1)(x-1) + D(x^2+1)\\ &= (A+C)x^3 + (-2A+B-C+D)x^2 + (A-2B+C)x\\ &\qquad + (B-C+D).\end{aligned}$$

$$0x^3 + 0x^2 - 2x + 4 = (A + C)x^3 + (-2A + B - C + D)x^2 + (A - 2B + C)x + (B - C + D).$$

Equating coefficients gives

$$\begin{aligned} \text{coefficient of } x^3: &\quad 0 = A + C \\ \text{coefficient of } x^2: &\quad 0 = -2A + B - C + D \\ \text{coefficient of } x^1: &\quad -2 = A - 2B + C \\ \text{coefficient of } x^0: &\quad 4 = B - C + D. \end{aligned}$$

$$0x^3 + 0x^2 - 2x + 4 = (A + C)x^3 + (-2A + B - C + D)x^2 \\ + (A - 2B + C)x + (B - C + D).$$

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$$
\begin{aligned}
0x^3 + 0x^2 - 2x + 4 = (A + C)x^3 &+ (-2A + B - C + D)x^2 \\
&+ (A - 2B + C)x + (B - C + D).
\end{aligned}
$$

Equating coefficients gives

$$
\begin{aligned}
&\text{coefficient of } x^3: & 0 &= A + C \\
&\text{coefficient of } x^2: & 0 &= -2A + B - C + D \\
&\text{coefficient of } x^1: & -2 &= A - 2B + C \\
&\text{coefficient of } x^0: & 4 &= B - C + D.
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$$

$$0x^3 + 0x^2 - 2x + 4 = (A + C)x^3 + (-2A + B - C + D)x^2 + (A - 2B + C)x + (B - C + D).$$

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\end{aligned}$$

$$\begin{cases} A \phantom{{}+B} + C \phantom{{}+D} = 0 \\ -2A + B - C + D = 0 \\ A - 2B + C \phantom{{}+D} = -2 \\ B - C + D = 4 \end{cases}$$

is a system of four linear equations. You will study how to solve systems like this (and much bigger) in your MATH215 course.

$$\begin{cases} A \quad\quad + C \quad\quad = 0 \\ -2A + \ B - C + D = 0 \\ A - 2B + C \quad\quad = -2 \\ \quad\quad B - C + D = 4 \end{cases}$$

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$$\begin{array}{r} \text{[equation 2]} \\ \underline{-\ \text{[equation 4]}} \end{array}$$

$$\begin{cases} A \qquad\quad + C \qquad\; = 0 \\ -2A + B - C + D = 0 \\ A - 2B + C \qquad\; = -2 \\ \qquad\quad B - C + D = 4 \end{cases}$$

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$$\begin{array}{rr} & [-2A + B - C + D = \quad 0] \\ - & [\qquad\quad B - C + D = \quad 4] \\ \hline \end{array}$$

$$\begin{cases} A \quad\quad + C \quad\quad = 0 \\ -2A + B - C + D = 0 \\ A - 2B + C \quad\quad = -2 \\ \quad\quad B - C + D = 4 \end{cases}$$

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is a system of four linear equations. You will study how to solve systems like this (and much bigger) in your MATH215 course. We can solve it as follows:

$$\begin{array}{rl} & [-2A + B - C + D = \quad 0] \\ - & [\qquad\qquad B - C + D = \quad 4] \\ \hline & -2A \qquad\qquad\qquad\quad = \ -4 \end{array}$$

So $A = 2$.

$$\begin{cases} A \phantom{{}+B} + C \phantom{{}+D} = 0 \\ -2A + B - C + D = 0 \\ A - 2B + C = -2 \\ B - C + D = 4 \end{cases}$$

has solution

$$\begin{cases} A = 2 \\ B = \\ C = \\ D = \end{cases}$$

$$\begin{cases} A \qquad\quad + C \qquad\quad = 0 \\ -2A + \ B - C + D = 0 \\ A - 2B + C \qquad\quad = -2 \\ \qquad\qquad B - C + D = 4 \end{cases}$$

has solution

$$\begin{cases} A = 2 \\ B = \\ C = \ -2 \\ D = \end{cases}$$

$$\begin{cases} A + C = 0 \\ -2A + B - C + D = 0 \\ A - 2B + C = -2 \\ B - C + D = 4 \end{cases}$$

has solution

$$\begin{cases} A = 2 \\ B = 1 \\ C = -2 \\ D = \end{cases}$$

$$\begin{cases} A \phantom{{}-2B} + C \phantom{{}+D} = 0 \\ -2A + B - C + D = 0 \\ A - 2B + C = -2 \\ B - C + D = 4 \end{cases}$$

has solution

$$\begin{cases} A = 2 \\ B = 1 \\ C = -2 \\ D = 1. \end{cases}$$

$$\begin{cases} A \phantom{{}+B} + C \phantom{{}+D} = 0 \\ -2A + B - C + D = 0 \\ A - 2B + C = -2 \\ B - C + D = 4 \end{cases}$$

has solution

$$\begin{cases} A = 2 \\ B = 1 \\ C = -2 \\ D = 1. \end{cases}$$

Hence our integrand is

$$\frac{-2x+4}{(x^2+1)(x-1)^2} = \frac{2x+1}{x^2+1} + \frac{-2}{x-1} + \frac{1}{(x-1)^2}.$$

Now we can integrate

$$\int \frac{-2x+4}{(x^2+1)(x-1)^2}\,dx$$

$$= \int \frac{2x+1}{x^2+1}\,dx + \int \frac{-2}{x-1}\,dx + \int \frac{1}{(x-1)^2}\,dx$$

$$=$$

$$=$$

Now we can integrate

$$\int \frac{-2x+4}{(x^2+1)(x-1)^2}\,dx$$

$$= \int \frac{2x+1}{x^2+1}\,dx + \int \frac{-2}{x-1}\,dx + \int \frac{1}{(x-1)^2}\,dx$$

$$= \int \frac{2x}{x^2+1}\,dx + \int \frac{1}{x^2+1}\,dx - \int \frac{2}{x-1}\,dx + \int \frac{1}{(x-1)^2}\,dx$$

$$=$$

Now we can integrate

$$\begin{aligned}
&\int \frac{-2x+4}{(x^2+1)(x-1)^2}\,dx \\
&\quad = \int \frac{2x+1}{x^2+1}\,dx + \int \frac{-2}{x-1}\,dx + \int \frac{1}{(x-1)^2}\,dx \\
&\quad = \int \frac{2x}{x^2+1}\,dx + \int \frac{1}{x^2+1}\,dx - \int \frac{2}{x-1}\,dx + \int \frac{1}{(x-1)^2}\,dx \\
&\quad = \ln(x^2+1) + \tan^{-1} x - 2\ln|x-1| - \frac{1}{x-1} + C.
\end{aligned}$$

EXAMPLE 5 Use partial fractions to evaluate

$$\int \frac{dx}{x(x^2+1)^2}.$$

Solution The form of the partial fraction decomposition is

$$\frac{1}{x(x^2+1)^2} = \frac{A}{x} + \frac{Bx+C}{x^2+1} + \frac{Dx+E}{(x^2+1)^2}.$$

Multiplying by $x(x^2+1)^2$, we have

$$\begin{aligned} 1 &= A(x^2+1)^2 + (Bx+C)x(x^2+1) + (Dx+E)x \\ &= A(x^4+2x^2+1) + B(x^4+x^2) + C(x^3+x) + Dx^2 + Ex \\ &= (A+B)x^4 + Cx^3 + (2A+B+D)x^2 + (C+E)x + A. \end{aligned}$$

If we equate coefficients, we get the system

$$A+B=0, \qquad C=0, \qquad 2A+B+D=0, \qquad C+E=0, \qquad A=1.$$

Solving this system gives $A=1$, $B=-1$, $C=0$, $D=-1$, and $E=0$. Thus, Solving this system gives $A = 1, B = -1, C = 0, D = -1$, and $E = 0$. Thus,

$$\begin{aligned}
\int \frac{dx}{x(x^2+1)^2} &= \int \left[\frac{1}{x} + \frac{-x}{x^2+1} + \frac{-x}{(x^2+1)^2} \right] dx \\
&= \int \frac{dx}{x} - \int \frac{x\,dx}{x^2+1} - \int \frac{x\,dx}{(x^2+1)^2} \\
&= \int \frac{dx}{x} - \frac{1}{2}\int \frac{du}{u} - \frac{1}{2}\int \frac{du}{u^2} \qquad \begin{aligned} u &= x^2+1, \\ du &= 2x\,dx \end{aligned} \\
&= \ln|x| - \frac{1}{2}\ln|u| + \frac{1}{2u} + K \\
&= \ln|x| - \frac{1}{2}\ln(x^2+1) + \frac{1}{2(x^2+1)} + K \\
&= \ln \frac{|x|}{\sqrt{x^2+1}} + \frac{1}{2(x^2+1)} + K.
\end{aligned}$$

■

Remark

When we have

$$\frac{f(x)}{(x-r_1)(x-r_2)\cdots(x-r_n)},$$

where $r_1, r_2, \ldots, r_n$ are all different, there is a quicker way to find partial fractions.

Example

Find $\displaystyle\int \frac{x+4}{x^3+3x^2-10x}\,dx$.

Example

Find $\displaystyle\int \frac{x+4}{x^3+3x^2-10x}\,dx.$

First we have

$$\frac{x+4}{x^3+3x^2-10x} = \frac{x+4}{x(x-2)(x+5)}$$

$$=$$

Example

Find $\displaystyle\int \frac{x+4}{x^3+3x^2-10x}\,dx.$

First we have

$$\begin{aligned}\frac{x+4}{x^3+3x^2-10x} &= \frac{x+4}{x(x-2)(x+5)}\\ &= \frac{A}{x}+\frac{B}{x-2}+\frac{C}{x+5}.\end{aligned}$$

Example

Find $\displaystyle\int \frac{x+4}{x^3+3x^2-10x}\,dx.$

First we have

$$\begin{aligned}\frac{x+4}{x^3+3x^2-10x} &= \frac{x+4}{x(x-2)(x+5)} \\ &= \frac{A}{x} + \frac{B}{x-2} + \frac{C}{x+5}.\end{aligned}$$

1. multiply by x, then set $x = 0$;
2. multiply by $(x-2)$, then set $x = 2$;
3. multiply by $(x+5)$, then set $x = -5$.

$$\frac{x+4}{x(x-2)(x+5)} = \frac{A}{x} + \frac{B}{x-2} + \frac{C}{x+5}$$

1. multiply by x, then set $x = 0$;

$$\begin{aligned} \frac{x+4}{(x-2)(x+5)} &= A + \frac{Bx}{x-2} + \frac{Cx}{x+5} \\ \frac{4}{(-2)(5)} &= A + 0 + 0 \\ -\frac{2}{5} &= A \end{aligned}$$

$$\frac{x+4}{x(x-2)(x+5)} = \frac{A}{x} + \frac{B}{x-2} + \frac{C}{x+5}$$

2. multiply by $(x-2)$, then set $x=2$;

$$\frac{x+4}{x(x+5)} = \frac{A(x-2)}{x} + B + \frac{C(x-2)}{x+5}$$
$$\frac{2+4}{(2)(7)} = 0 + B + 0$$
$$\frac{3}{7} = B$$

$$\frac{x+4}{x(x-2)(x+5)} = \frac{A}{x} + \frac{B}{x-2} + \frac{C}{x+5}$$

3. multiply by $(x+5)$, then set $x=-5$.

$$\begin{aligned}\frac{x+4}{x(x-2)} &= \frac{A(x+5)}{x} + \frac{B(x+5)}{x-2} + C \\ \frac{-5+4}{(-5)(-7)} &= 0 + 0 + C \\ -\frac{1}{35} &= C\end{aligned}$$

Therefore

$$\frac{x+4}{x(x-2)(x+5)} = \frac{-\frac{2}{5}}{x} + \frac{\frac{3}{7}}{x-2} + \frac{-\frac{1}{35}}{x+5}$$

Therefore

$$\frac{x+4}{x(x-2)(x+5)} = \frac{-\frac{2}{5}}{x} + \frac{\frac{3}{7}}{x-2} + \frac{-\frac{1}{35}}{x+5}$$

and thus

$$\int \frac{x+4}{x(x-2)(x+5)}\,dx = -\frac{2}{5}\ln|x| + \frac{3}{7}\ln|x-2| - \frac{1}{35}\ln|x+5| + C.$$

EXAMPLE 6 Find A, B, and C in the partial fraction expansion

$$\frac{x^2+1}{(x-1)(x-2)(x-3)} = \frac{A}{x-1} + \frac{B}{x-2} + \frac{C}{x-3}. \tag{3}$$

Solution If we multiply both sides of Equation (3) by $(x-1)$ to get

$$\frac{x^2+1}{(x-2)(x-3)} = A + \frac{B(x-1)}{x-2} + \frac{C(x-1)}{x-3}$$

and set $x = 1$, the resulting equation gives the value of A:

$$\frac{(1)^2+1}{(1-2)(1-3)} = A + 0 + 0,$$
$$A = 1.$$

In exactly the same way, we can multiply both sides by $(x-2)$ and then substitute in $x = 2$. This gives

$$\frac{(2)^2+1}{(2-1)(2-3)} = B.$$

So $B = -5$. Finally, we multiply both sides by $(x-3)$ and then substitute in $x = 3$, which yields

$$\frac{(3)^2+1}{(3-1)(3-2)} = C,$$

and $C = 5$. ■

Break

We will continue at 3pm

Remark

We can also use differentiation to find partial fractions.

Example

Find A, B and C in the equation

$$\frac{x-1}{(x+1)^3} = \frac{A}{x+1} + \frac{B}{(x+1)^2} + \frac{C}{(x+1)^3}$$

by clearing fractions, differentiating and setting $x = -1$.

Example

Find A, B and C in the equation

$$\frac{x-1}{(x+1)^3} = \frac{A}{x+1} + \frac{B}{(x+1)^2} + \frac{C}{(x+1)^3}$$

by clearing fractions, differentiating and setting $x = -1$.

First we clear the fractions: Multiplying by $(x+1)^3$ gives

$$x - 1 = A(x+1)^2 + B(x+1) + C.$$

Example

Find A, B and C in the equation

$$\frac{x-1}{(x+1)^3} = \frac{A}{x+1} + \frac{B}{(x+1)^2} + \frac{C}{(x+1)^3}$$

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First we clear the fractions: Multiplying by $(x+1)^3$ gives

$$x - 1 = A(x+1)^2 + B(x+1) + C.$$

Putting in $x = -1$ gives

$$-2 = 0 + 0 + C.$$

Example

Find A, B and C in the equation

$$\frac{x-1}{(x+1)^3} = \frac{A}{x+1} + \frac{B}{(x+1)^2} + \frac{C}{(x+1)^3}$$

by clearing fractions, differentiating and setting $x = -1$.

First we clear the fractions: Multiplying by $(x+1)^3$ gives

$$x - 1 = A(x+1)^2 + B(x+1) + C.$$

Putting in $x = -1$ gives

$$-2 = 0 + 0 + C.$$

Now differentiate the green equation with respect to x to obtain

$$1 = 2A(x+1) + B + 0.$$

$$1 = 2A(x+1) + B$$

Putting in $x = -1$ gives

$$1 = 0 + B.$$

$$1 = 2A(x+1) + B$$

Putting in $x = -1$ gives

$$1 = 0 + B.$$

Finally differentiate the orange equation to obtain

$$0 = 2A + 0.$$

$$1 = 2A(x+1) + B$$

Putting in $x = -1$ gives

$$1 = 0 + B.$$

Finally differentiate the orange equation to obtain

$$0 = 2A + 0.$$

Therefore

$$\begin{cases} A = & 0 \\ B = & 1 \\ C = & -2 \end{cases}$$

is the answer.

Remark

Sometimes we can just try putting in small numbers $x = 0$, $x = \pm 1$, $x = \pm 2$, etc. to find the coefficients A, B, C, ...

Example

Find A, B and C in the equation

$$\frac{x^2+1}{(x-1)(x-2)(x-3)}$$

by assigning numerical values to x.

Example

Find A, B and C in the equation

$$\frac{x^2+1}{(x-1)(x-2)(x-3)}$$

by assigning numerical values to x.

First we clear fractions to get

$$x^2+1 = A(x-2)(x-3) + B(x-1)(x-3) + C(x-1)(x-2)$$

Example

Find A, B and C in the equation

$$\frac{x^2+1}{(x-1)(x-2)(x-3)}$$

by assigning numerical values to x.

First we clear fractions to get

$$x^2+1 = A(x-2)(x-3) + B(x-1)(x-3) + C(x-1)(x-2)$$

Clearly the important numbers here are 1, 2 and 3.

Example

Find A, B and C in the equation

$$\frac{x^2+1}{(x-1)(x-2)(x-3)}$$

by assigning numerical values to x.

First we clear fractions to get

$$x^2+1 = A(x-2)(x-3) + B(x-1)(x-3) + C(x-1)(x-2)$$

Clearly the important numbers here are 1, 2 and 3. So we are going to try $x = 1$, $x = 2$ and $x = 3$.

$$x^2 + 1 = A(x-2)(x-3) + B(x-1)(x-3) + C(x-1)(x-2)$$

$$x = 1 \qquad \begin{aligned} (1)^2 + 1 &= A(-1)(-2) + B(0) + C(0) \\ 2 &= 2A \\ A &= 1 \end{aligned}$$

$$x^2 + 1 = A(x-2)(x-3) + B(x-1)(x-3) + C(x-1)(x-2)$$

$$x = 1 \qquad \begin{aligned} (1)^2 + 1 &= A(-1)(-2) + B(0) + C(0) \\ 2 &= 2A \\ A &= 1 \end{aligned}$$

$$x = 2 \qquad \begin{aligned} (2)^2 + 1 &= A(0) + B(1)(-1) + C(0) \\ 5 &= -B \\ B &= -5 \end{aligned}$$

$$x^2 + 1 = A(x-2)(x-3) + B(x-1)(x-3) + C(x-1)(x-2)$$

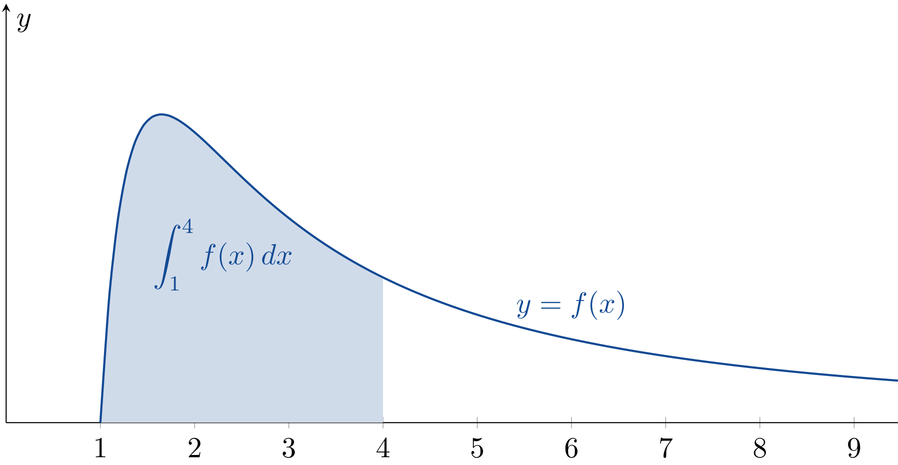
$$x = 1 \qquad \begin{aligned} (1)^2 + 1 &= A(-1)(-2) + B(0) + C(0) \\ 2 &= 2A \\ A &= 1 \end{aligned}$$

$$x = 2 \qquad \begin{aligned} (2)^2 + 1 &= A(0) + B(1)(-1) + C(0) \\ 5 &= -B \\ B &= -5 \end{aligned}$$

$$x = 3 \qquad \begin{aligned} (3)^2 + 1 &= A(0) + B(0) + C(2)(1) \\ 10 &= 2C \\ C &= 5. \end{aligned}$$

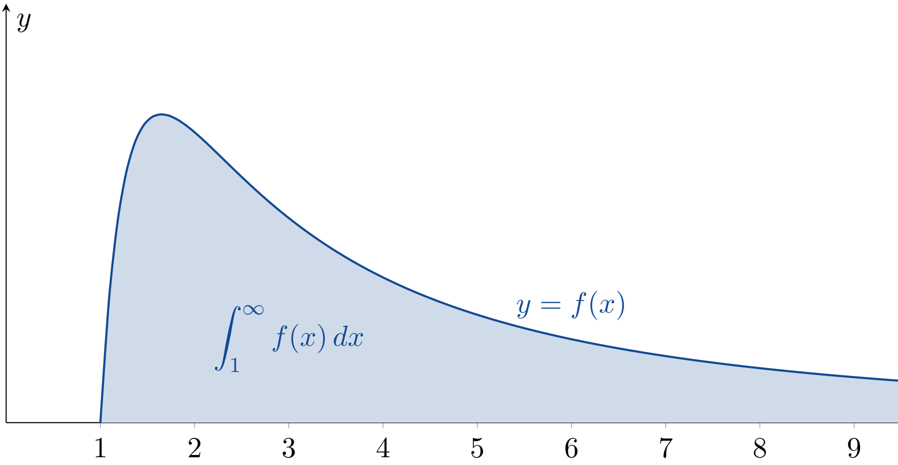
8.8

Improper Integrals



y
$\int_1^4 f(x)\,dx$
$y = f(x)$
1
2
3
4
5
6
7
8
9

8.8 Improper Integrals



We need to use limits.

Example

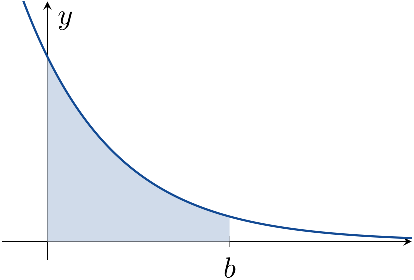
Calculate $\displaystyle\int_0^{\infty} e^{-\frac{x}{2}}\, dx.$

Example

Calculate $\displaystyle\int_0^{\infty} e^{-\frac{x}{2}}\,dx.$

Step 1:

$$\int_0^{b} e^{-\frac{x}{2}}\,dx = ?$$

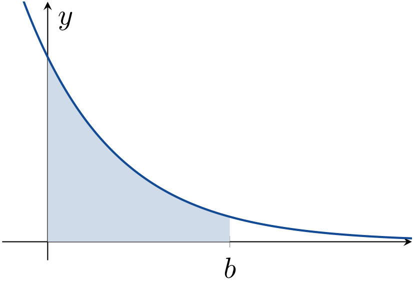


Example

Calculate $\displaystyle\int_0^\infty e^{-\frac{x}{2}}\,dx.$

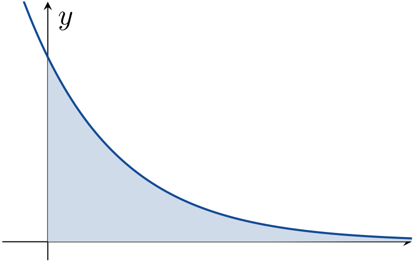
Step 1:

$$\int_0^b e^{-\frac{x}{2}}\,dx = ?$$



Step 2:

$$\int_0^{\infty} e^{-\frac{x}{2}}\,dx = \lim_{b\to\infty}\int_0^b e^{-\frac{x}{2}}\,dx$$



Since

$$\int_0^b e^{-\frac{x}{2}}\,dx = \left[-2e^{-\frac{x}{2}}\right]_0^b = -2e^{-\frac{b}{2}} + 2,$$

Since

$$\int_0^b e^{-\frac{x}{2}}\,dx = \left[-2e^{-\frac{x}{2}}\right]_0^b = -2e^{-\frac{b}{2}} + 2,$$

we have that

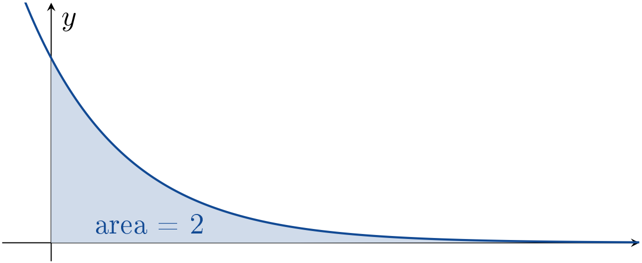
$$\int_0^{\infty} e^{-\frac{x}{2}}\,dx = \lim_{b\to\infty} \int_0^b e^{-\frac{x}{2}}\,dx = \lim_{b\to\infty} \left(-2e^{-\frac{b}{2}} + 2\right) = 2.$$

Since

$$\int_0^b e^{-\frac{x}{2}}\,dx = \left[-2e^{-\frac{x}{2}}\right]_0^b = -2e^{-\frac{b}{2}} + 2,$$

we have that

$$\int_0^\infty e^{-\frac{x}{2}}\,dx = \lim_{b\to\infty} \int_0^b e^{-\frac{x}{2}}\,dx = \lim_{b\to\infty} \left(-2e^{-\frac{b}{2}} + 2\right) = 2.$$



DEFINITION Integrals with infinite limits of integration are **improper integrals of Type I**.

1. If $f(x)$ is continuous on $[a, \infty)$, then

$$\int_a^{\infty} f(x)\,dx = \lim_{b\to\infty} \int_a^b f(x)\,dx.$$

2. If $f(x)$ is continuous on $(-\infty, b]$, then

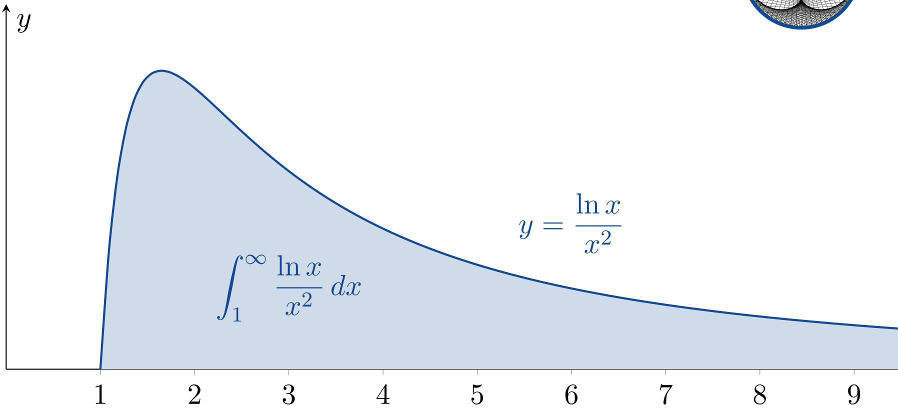
$$\int_{-\infty}^{b} f(x)\,dx = \lim_{a\to-\infty} \int_a^b f(x)\,dx.$$

3. If $f(x)$ is continuous on $(-\infty, \infty)$, then

$$\int_{-\infty}^{\infty} f(x)\,dx = \int_{-\infty}^{c} f(x)\,dx + \int_c^{\infty} f(x)\,dx,$$

where c is any real number.

In each case, if the limit exists and is finite, we say that the improper integral **converges** and that the limit is the **value** of the improper integral. If the limit fails to exist, the improper integral **diverges**.



Example

Is the area under the curve $y = \dfrac{\ln x}{x^2}$, from $x = 1$ to $x = \infty$, finite? Is so, what is its value?.

Since

$$\int_1^b \frac{\ln x}{x^2}\,dx = \left[(\ln x)\left(-\frac{1}{x}\right)\right]_1^b - \int_1^b \left(-\frac{1}{x}\right)\left(\frac{1}{x}\right)\,dx$$

$$= -\frac{\ln b}{b} - \left[\frac{1}{x}\right]_1^b = -\frac{\ln b}{b} - \frac{1}{b} + 1,$$

Since

$$\int_1^b \frac{\ln x}{x^2}\,dx = \left[(\ln x)\left(-\frac{1}{x}\right)\right]_1^b - \int_1^b \left(-\frac{1}{x}\right)\left(\frac{1}{x}\right)dx$$

$$= -\frac{\ln b}{b} - \left[\frac{1}{x}\right]_1^b = -\frac{\ln b}{b} - \frac{1}{b} + 1,$$

we have that

$$\int_1^\infty \frac{\ln x}{x^2}\,dx = \lim_{b\to\infty} \int_1^b \frac{\ln x}{x^2}\,dx = \lim_{b\to\infty}\left(-\frac{\ln b}{b} - \frac{1}{b} + 1\right)$$

$$=$$

$$=$$

.

Since

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$$= \lim_{b\to\infty}\left(-\frac{\ln b}{b}\right) - 0 + 1$$

$$=$$

.

Since

$$\int_1^b \frac{\ln x}{x^2}\,dx = \left[(\ln x)\left(-\frac{1}{x}\right)\right]_1^b - \int_1^b \left(-\frac{1}{x}\right)\left(\frac{1}{x}\right)dx$$

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$$= \lim_{b\to\infty}\left(-\frac{\frac{1}{b}}{1}\right) + 1 \qquad \text{(l'Hôpital's Rule)}$$

.

Since

$$\int_1^b \frac{\ln x}{x^2}\,dx = \left[(\ln x)\left(-\frac{1}{x}\right)\right]_1^b - \int_1^b \left(-\frac{1}{x}\right)\left(\frac{1}{x}\right)\,dx$$

$$= -\frac{\ln b}{b} - \left[\frac{1}{x}\right]_1^b = -\frac{\ln b}{b} - \frac{1}{b} + 1,$$

we have that

$$\int_1^\infty \frac{\ln x}{x^2}\,dx = \lim_{b\to\infty} \int_1^b \frac{\ln x}{x^2}\,dx = \lim_{b\to\infty} \left(-\frac{\ln b}{b} - \frac{1}{b} + 1\right)$$

$$= \lim_{b\to\infty} \left(-\frac{\ln b}{b}\right) - 0 + 1$$

$$= \lim_{b\to\infty} \left(-\frac{\frac{1}{b}}{1}\right) + 1 \qquad \text{(l'Hôpital's Rule)}$$

$$= 0 + 1 = 1.$$

Therefore the integral converges and the area has finite value 1.

EXAMPLE 2 Evaluate

$$\int_{-\infty}^{\infty} \frac{dx}{1 + x^2}.$$

Solution According to the definition (Part 3), we can choose $c = 0$ and write

$$\int_{-\infty}^{\infty} \frac{dx}{1 + x^2} = \int_{-\infty}^{0} \frac{dx}{1 + x^2} + \int_{0}^{\infty} \frac{dx}{1 + x^2}.$$

Next we evaluate each improper integral on the right side of the equation above.

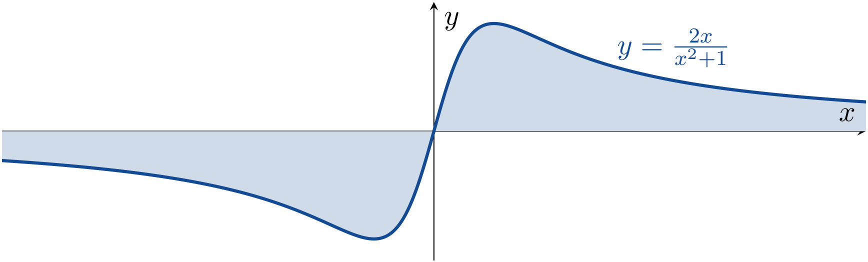
$$\begin{aligned}
\int_{-\infty}^{0} \frac{dx}{1 + x^2} &= \lim_{a \to -\infty} \int_{a}^{0} \frac{dx}{1 + x^2} \\
&= \lim_{a \to -\infty} \tan^{-1} x \Big]_{a}^{0} \\
&= \lim_{a \to -\infty} (\tan^{-1} 0 - \tan^{-1} a) = 0 - \left(-\frac{\pi}{2}\right) = \frac{\pi}{2}
\end{aligned}$$

$$
\begin{aligned}
\int_0^\infty \frac{dx}{1 + x^2} &= \lim_{b\to\infty} \int_0^b \frac{dx}{1 + x^2} \\
&= \lim_{b\to\infty} \tan^{-1} x \Big]_0^b \\
&= \lim_{b\to\infty} (\tan^{-1} b - \tan^{-1} 0) = \frac{\pi}{2} - 0 = \frac{\pi}{2}
\end{aligned}
$$

Thus,

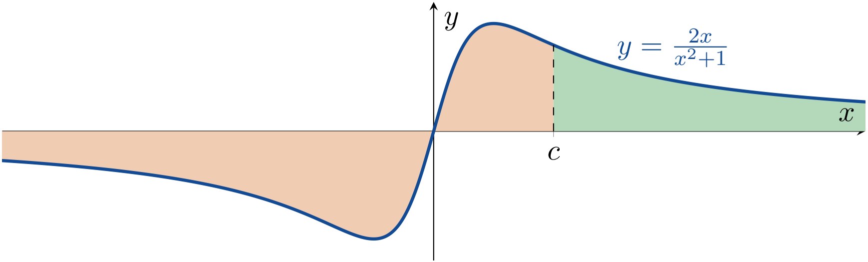
$$
\int_{-\infty}^\infty \frac{dx}{1 + x^2} = \frac{\pi}{2} + \frac{\pi}{2} = \pi.
$$

Since $1/(1 + x^2) > 0$, the improper integral can be interpreted as the (finite) area beneath the curve and above the x-axis (Figure 8.15). ■



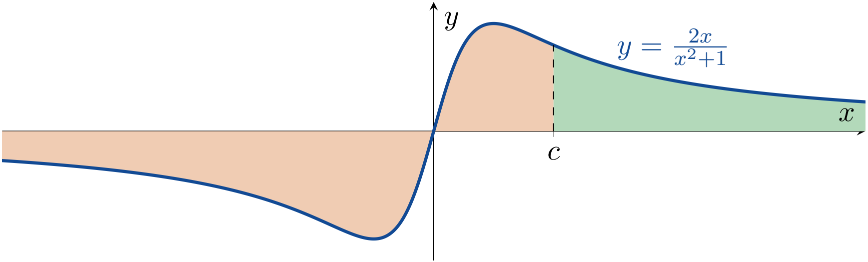
Remark

$$\int_{-\infty}^{\infty} f(x)\, dx$$



Remark

$$\int_{-\infty}^{\infty} f(x)\,dx = \int_{-\infty}^{c} f(x)\,dx + \int_{c}^{\infty} f(x)\,dx.$$

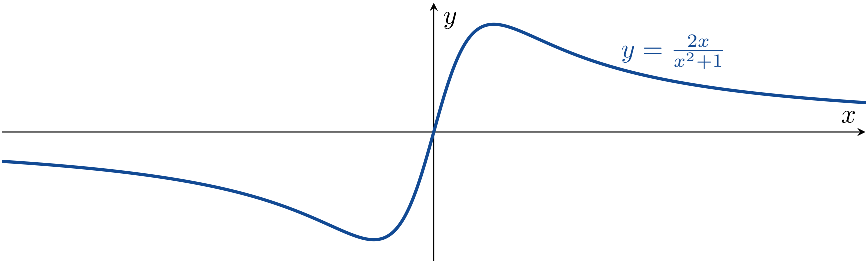


Remark

$$\int_{-\infty}^{\infty} f(x)\,dx = \int_{-\infty}^{c} f(x)\,dx + \int_{c}^{\infty} f(x)\,dx.$$

!!! This is not the same as $\lim\limits_{b\to\infty} \int_{-b}^{b} f(x)\,dx$!!!

Consider $f(x) = \frac{2x}{x^2+1}$.



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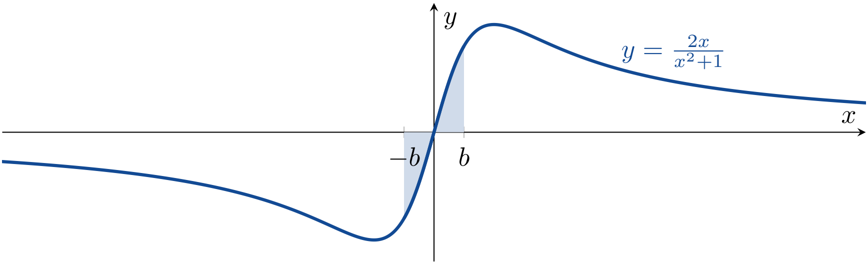

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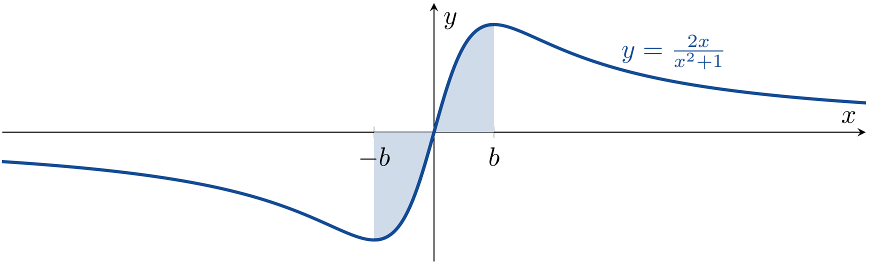

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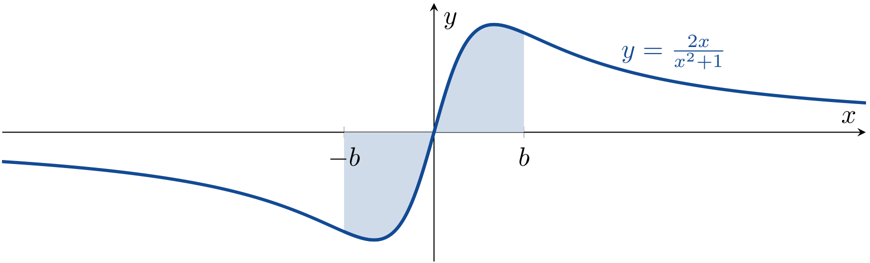

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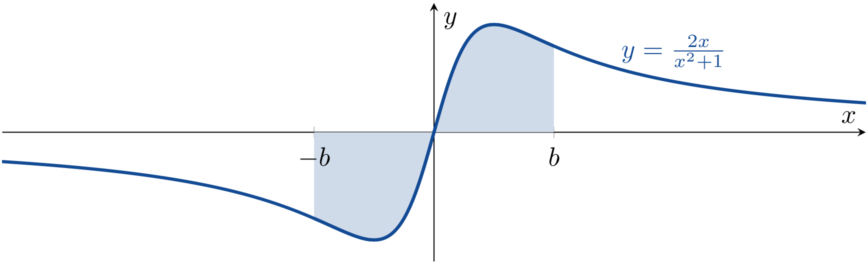

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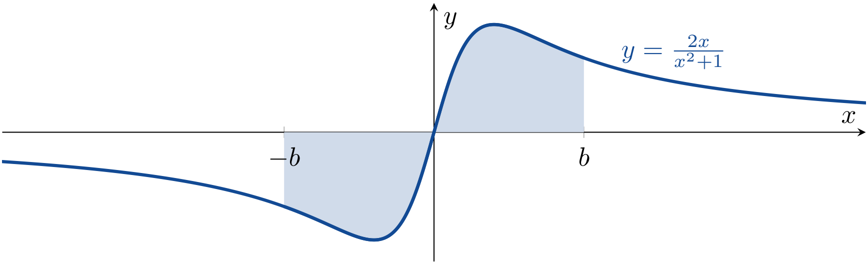

Since f is an odd function,

$$\lim_{b\to\infty} \int_{-b}^{b} \frac{2x}{x^2+1}\, dx = \lim_{b\to\infty} 0 = 0.$$

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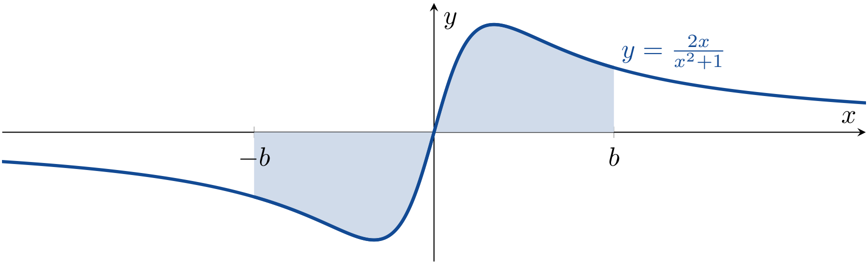

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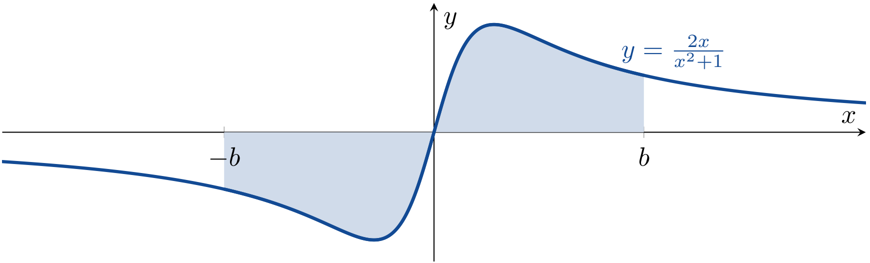

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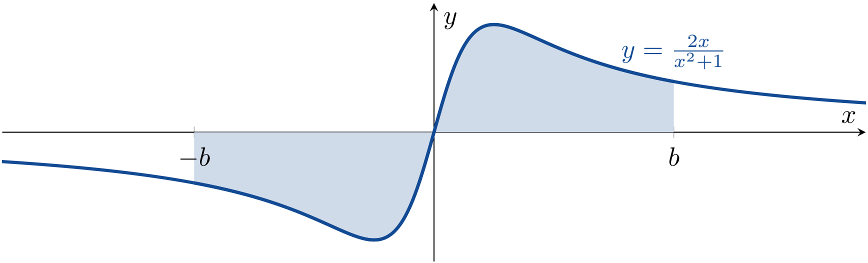

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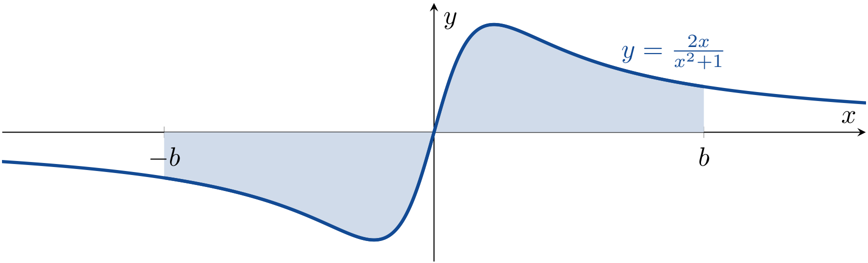

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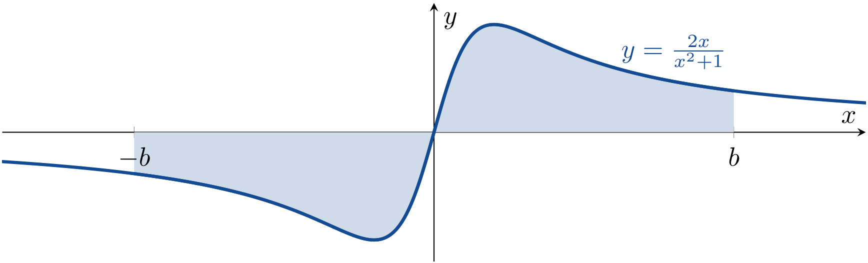

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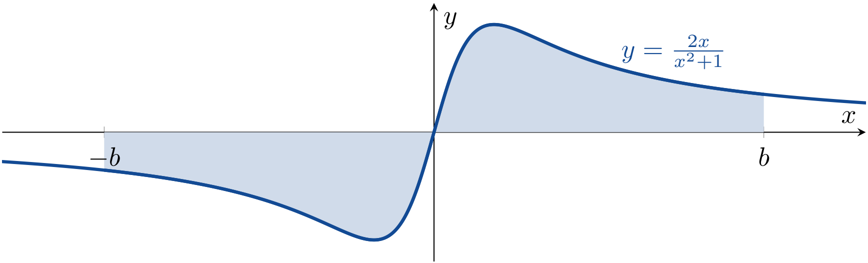

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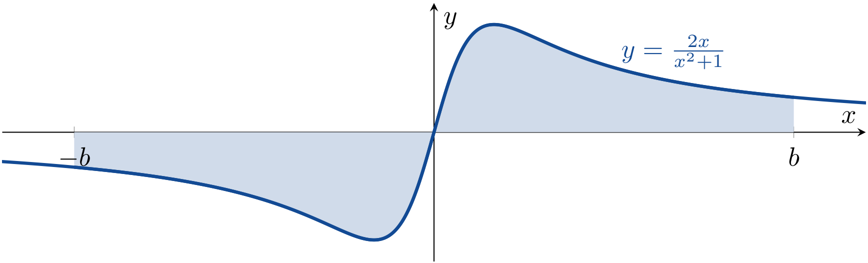

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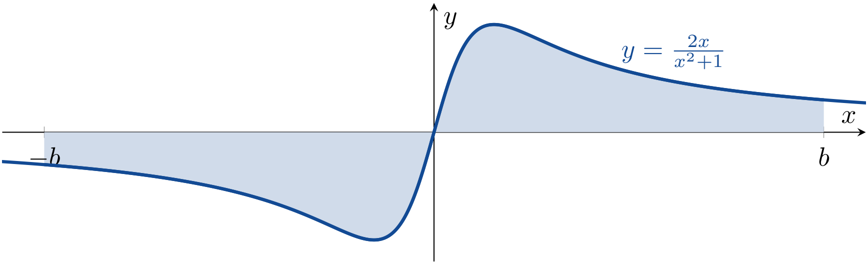

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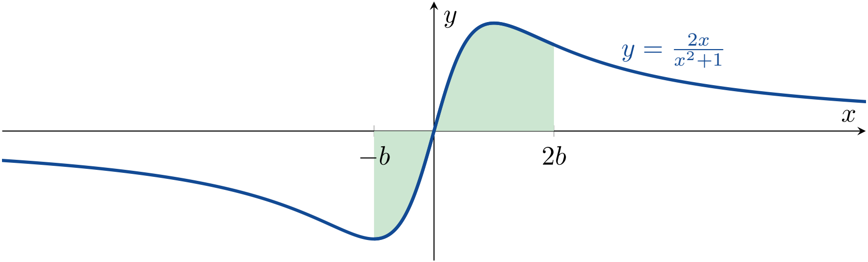
Let's try again. What do we get if we integrate over $[-b, 2b]$ then take the limit?



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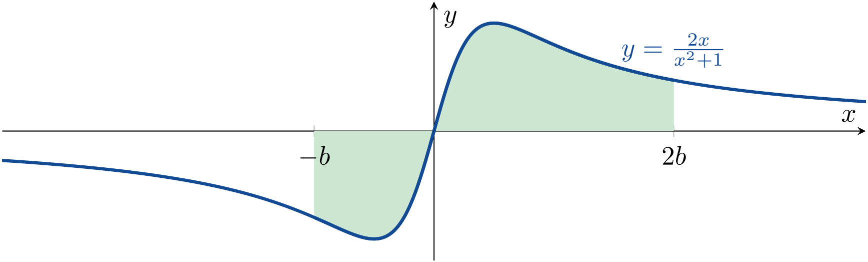

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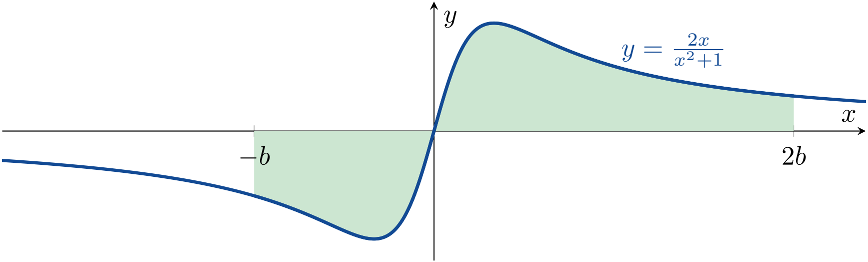

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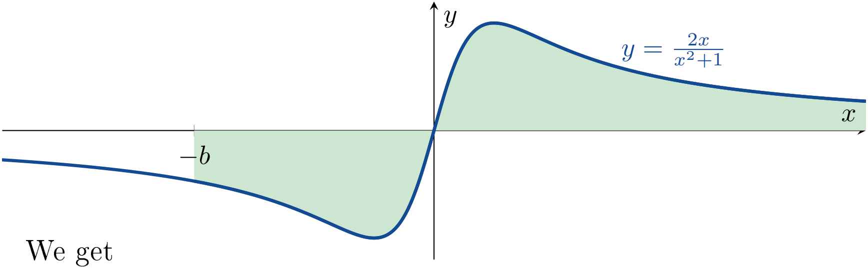

We get

$$\lim_{b\to\infty} \int_{-b}^{2b} \frac{2x}{x^2+1}\,dx = \lim_{b\to\infty} \Big[\ln(x^2+1) \Big]_{-b}^{2b}$$

$$= \lim_{b\to\infty} \ln\left(\frac{4b^2+1}{b^2+1}\right) = \ln 4.$$

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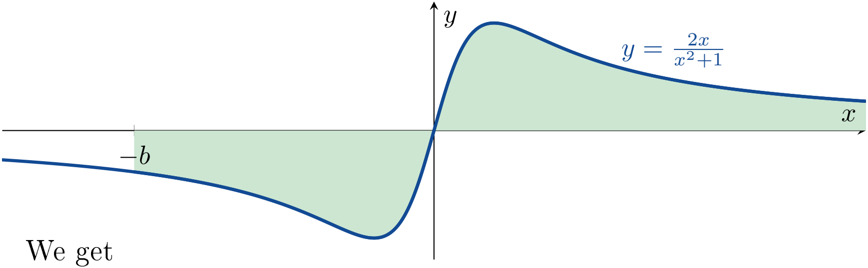

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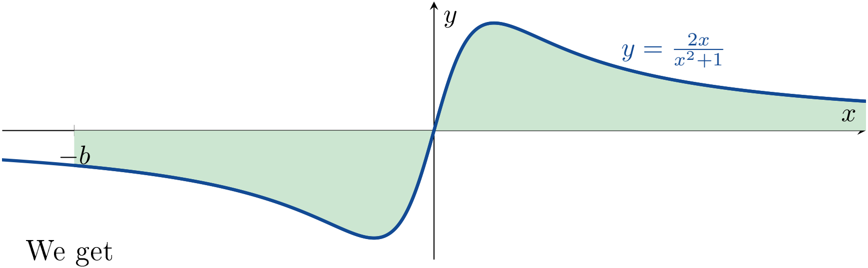

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So we must have $\int_{-\infty}^{\infty} \frac{2x}{x^2+1}\,dx = \ln 4$ right? NO!!!

The correct answer is that since

$$\int_0^\infty \frac{2x\,dx}{x^2+1} = \infty,$$

and

$$\int_{-\infty}^0 \frac{2x\,dx}{x^2+1} = -\infty$$

(you prove)

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and

$$\int_{-\infty}^0 \frac{2x\,dx}{x^2+1} = -\infty$$

(you prove) it follows that

$$\int_{-\infty}^\infty \frac{2x\,dx}{x^2+1} = \int_{-\infty}^0 \frac{2x\,dx}{x^2+1} + \int_0^\infty \frac{2x\,dx}{x^2+1} = \text{“}\underbrace{\infty - \infty}_{\substack{\text{indeterminate}\\ \text{form}}}\text{”}$$

diverges.

8.8 Improper Integrals

Example

For what values of p does the integral $\int_1^\infty \frac{dx}{x^p}$ converge? When the integral does converge, what is its value?

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For what values of p does the integral $\int_1^\infty \frac{dx}{x^p}$ converge? When the integral does converge, what is its value?

If $p \neq 1$, then

$$\int_1^b x^{-p}\,dx = \left[\frac{x^{-p+1}}{1-p}\right]_1^b = \frac{1}{1-p}\left(\frac{1}{b^{p-1}} - 1\right).$$

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$$= \begin{cases} \frac{1}{p-1} & p > 1 \\ \infty & p < 1. \end{cases}$$

So the integral converges to $\frac{1}{p-1}$ if $p > 1$, but diverges if $p < 1$.

What happens if $p = 1$?

What happens if $p = 1$?

$$\int_1^{\infty} \frac{dx}{x} = \lim_{b\to\infty} \int_1^{b} \frac{dx}{x} = \lim_{b\to\infty} \Big[\ln x\Big]_1^b = \lim_{b\to\infty} (\ln b - \ln 1) = \infty.$$

Therefore the integral also diverges for $p = 1$.

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$$\int_1^\infty \frac{dx}{x} = \lim_{b\to\infty} \int_1^b \frac{dx}{x} = \lim_{b\to\infty} \Big[\ln x\Big]_1^b = \lim_{b\to\infty} (\ln b - \ln 1) = \infty.$$

Therefore the integral also diverges for $p = 1$.

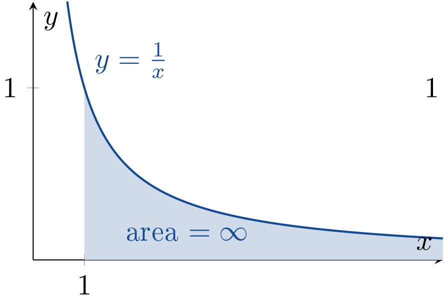
Theorem

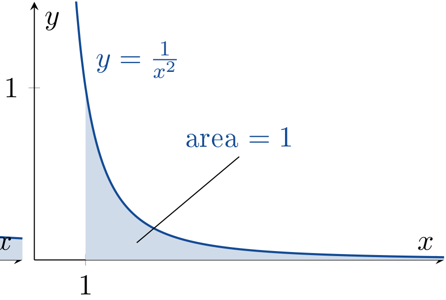
$$\int_1^\infty \frac{dx}{x^p} \begin{cases} \textit{converges if } p > 1, \\ \textit{diverges if } p \leq 1. \end{cases}$$

Remark

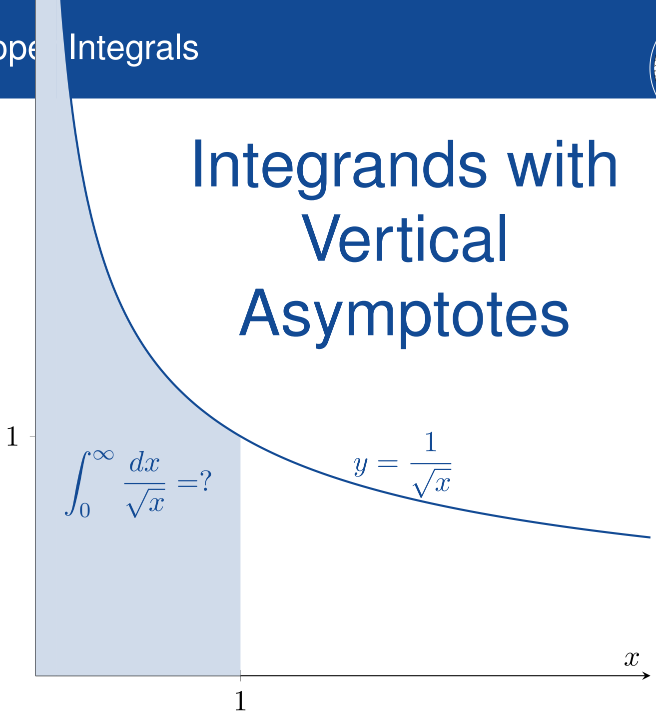
In particular, please remember that

$$\int_1^\infty \frac{dx}{x} \quad \text{diverges} \qquad \text{and} \qquad \int_1^\infty \frac{dx}{x^2} \quad \text{converges.}$$



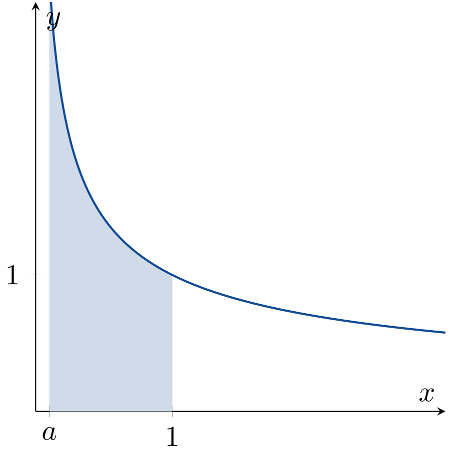


Integrands with Vertical Asymptotes



Step 1:

$$\int_a^1 \frac{dx}{\sqrt{x}} = ?$$



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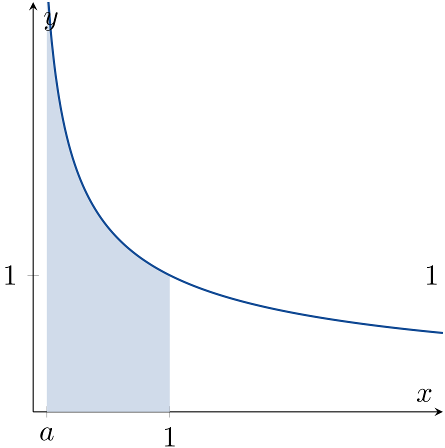
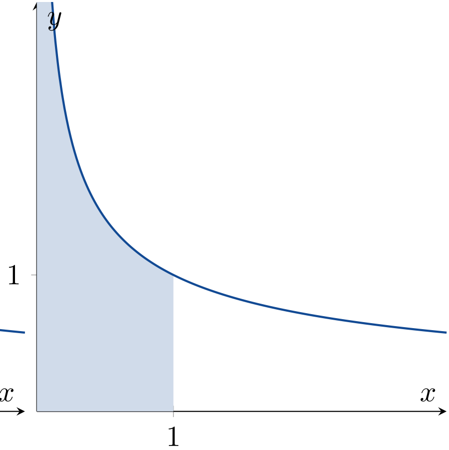
$\int_a^1 \frac{dx}{\sqrt{x}} = ?$



Step 2:

$\int_0^1 \frac{dx}{\sqrt{x}} = \lim_{a \to 0^+} \int_a^1 \frac{dx}{\sqrt{x}}$



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we have that

$$\int_0^1 \frac{dx}{\sqrt{x}} = \lim_{a\to 0^+} \int_a^1 \frac{dx}{\sqrt{x}} = \lim_{a\to 0^+} \left(2 - 2\sqrt{a}\right) = 2.$$

DEFINITION Integrals of functions that become infinite at a point within the interval of integration are **improper integrals of Type II**.

1. If $f(x)$ is continuous on $(a, b]$ and discontinuous at a, then

$$\int_a^b f(x)\,dx = \lim_{c \to a^+} \int_c^b f(x)\,dx.$$

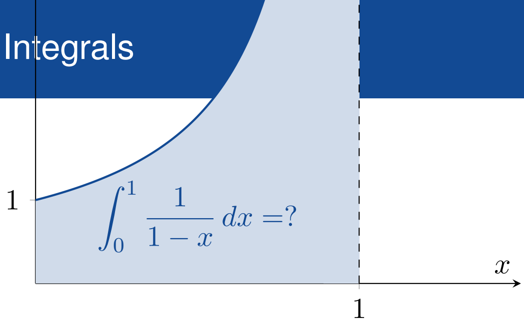
2. If $f(x)$ is continuous on $[a, b)$ and discontinuous at b, then

$$\int_a^b f(x)\,dx = \lim_{c \to b^-} \int_a^c f(x)\,dx.$$

3. If $f(x)$ is discontinuous at c, where $a < c < b$, and continuous on $[a, c) \cup (c, b]$, then

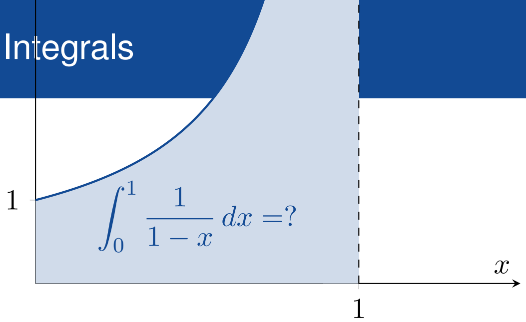
$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx.$$

In each case, if the limit exists and is finite, we say the improper integral **converges** and that the limit is the **value** of the improper integral. If the limit does not exist, the integral **diverges**.



Example

Investigate the convergence of $\int_0^1 \frac{1}{1-x}\,dx$.



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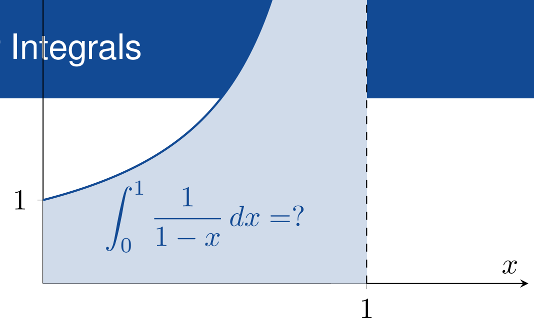
$$\int_0^1 \frac{1}{1-x}\,dx = \lim_{b\to 1^-} \int_0^b \frac{1}{1-x}\,dx =$$
$$=$$

8.8 Improper Integrals

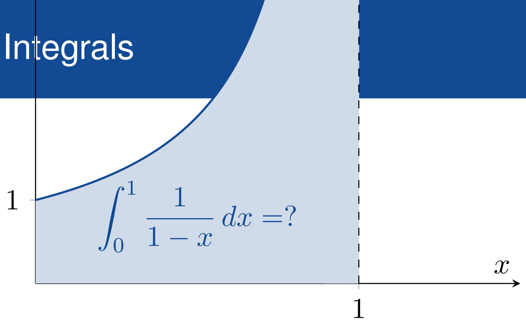


Example

Investigate the convergence of $\displaystyle\int_0^1 \frac{1}{1-x}\, dx$.

$$\int_0^1 \frac{1}{1-x}\, dx = \lim_{b \to 1^-} \int_0^b \frac{1}{1-x}\, dx = \lim_{b \to 1^-} \Big[-\ln|1-x| \Big]_0^b$$

$$=$$



Example

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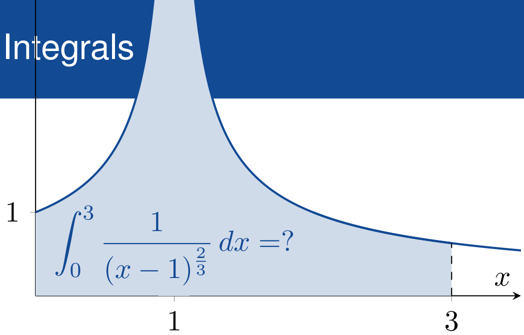
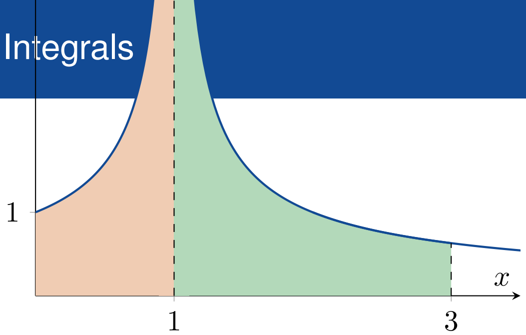
$$= \lim_{b\to 1^-} \big(-\ln(1-b) + 0 \big) = \infty.$$

This integral diverges.



Example

Calculate $\displaystyle\int_0^3 \frac{dx}{(x-1)^{\frac{2}{3}}}$.

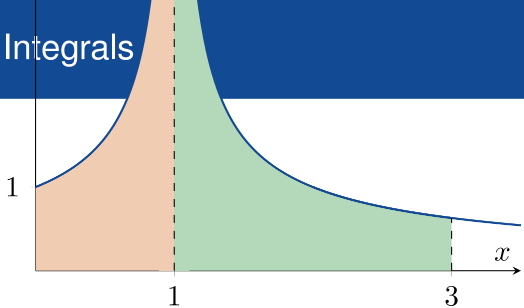


Example

Calculate $\displaystyle\int_0^3 \frac{dx}{(x-1)^{\frac{2}{3}}}$.

Because we have a discontinuity at $x = 1$, we need to consider

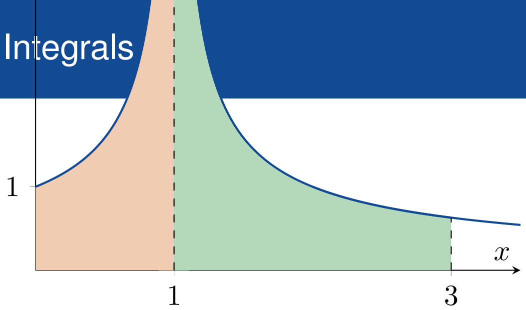
$$\int_0^3 \frac{dx}{(x-1)^{\frac{2}{3}}} = \int_0^1 \frac{dx}{(x-1)^{\frac{2}{3}}} + \int_1^3 \frac{dx}{(x-1)^{\frac{2}{3}}}$$



Since

$$\int_0^1 \frac{dx}{(x-1)^{\frac{2}{3}}} = \lim_{b\to 1^-} \int_0^b \frac{dx}{(x-1)^{\frac{2}{3}}} = \lim_{b\to 1^-} \left[3(x-1)^{\frac{1}{3}}\right]_0^b$$

$$= \lim_{b\to 1^-} \left(3(b-1)^{\frac{1}{3}} + 3\right) = 3$$



Since

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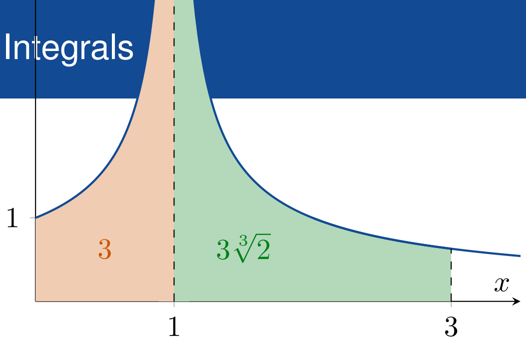
$$= \lim_{b\to 1^-} \left(3(b-1)^{\frac{1}{3}} + 3\right) = 3$$

and

$$\int_1^3 \frac{dx}{(x-1)^{\frac{2}{3}}} = \lim_{b\to 1^+} \int_b^3 \frac{dx}{(x-1)^{\frac{2}{3}}} = \lim_{b\to 1^+} \left[3(x-1)^{\frac{1}{3}}\right]_b^3$$

$$= \lim_{b\to 1^+} \left(3(3-1)^{\frac{1}{3}} - 3(b-1)^{\frac{1}{3}}\right) = 3\sqrt[3]{2}$$



we have that

$$\int_0^3 \frac{dx}{(x-1)^{\frac{2}{3}}} = \int_0^1 \frac{dx}{(x-1)^{\frac{2}{3}}} + \int_1^3 \frac{dx}{(x-1)^{\frac{2}{3}}} = 3 + 3\sqrt[3]{2}.$$

Remark

Sometimes we cannot evaluate an improper integral, but we can still determine whether it converges or diverges.

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Example

Does $\displaystyle\int_1^{\infty} e^{-x^2}\,dx$ converge or diverge?

We can not calculate $\displaystyle\int_1^{b} e^{-x^2}\,dx$ because it is nonelementary.

But we can answer this example another way.

Since $e^{-x^2} > 0$, we know that $I(b) = \int_1^b e^{-x^2}\,dx$ is an increasing function of b.

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So either

- $\lim_{b\to\infty} \int_1^b e^{-x^2}\,dx = \infty$; or
- $\lim_{b\to\infty} \int_1^b e^{-x^2}\,dx$ is a finite number.

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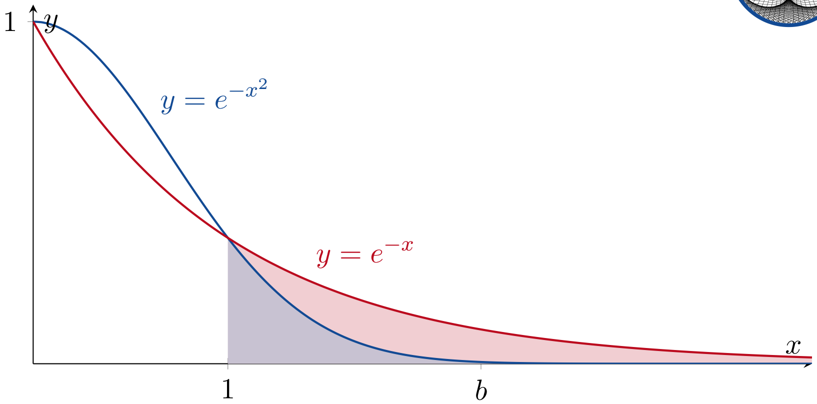
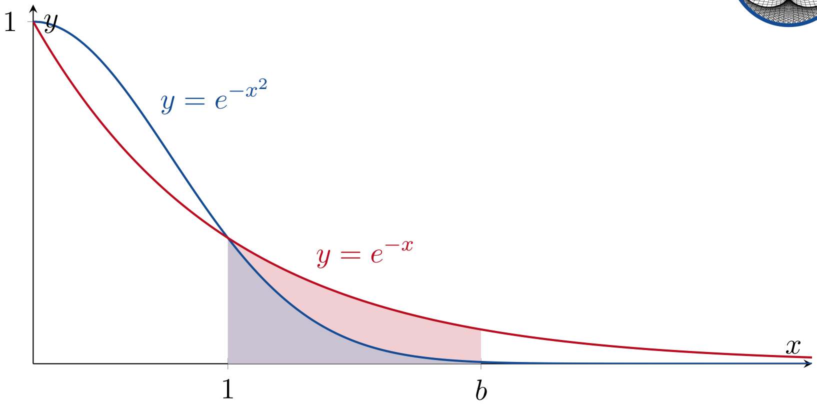
I am going to prove to you that $\lim_{b\to\infty} \int_1^b e^{-x^2}\,dx$ is finite.



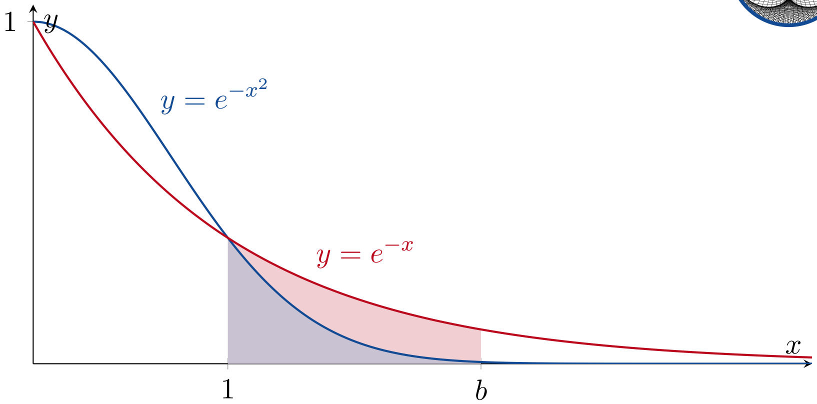
Note that $e^{-x^2} \leq e^{-x}$ for all $x \geq 1$.



Note that $e^{-x^2} \leq e^{-x}$ for all $x \geq 1$. So

$$\int_1^b e^{-x^2}\,dx \leq \int_1^b e^{-x}\,dx$$

for any $b > 1$.



Note that $e^{-x^2} \leq e^{-x}$ for all $x \geq 1$. So

$$\int_1^b e^{-x^2}\,dx \leq \int_1^b e^{-x}\,dx = -e^{-b} + e^{-1} < e^{-1} \approx 0.36788$$

for any $b > 1$.

Therefore

$$\int_1^\infty e^{-x^2}\,dx = \lim_{b\to\infty}\int_1^b e^{-x^2}\,dx$$

converges to a finite value.

Theorem (Direct Comparison Test)

Let $f : [a, \infty) \to \mathbb{R}$ *and* $g : [a, \infty) \to \mathbb{R}$ *be continuous functions. Suppose that*

$$0 \leq f(x) \leq g(x)$$

for all $x \in [a, \infty)$.

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for all $x \in [a, \infty)$. Then

1. $\displaystyle\int_a^\infty g(x)\,dx$ *converges* $\quad\Longrightarrow\quad$ $\displaystyle\int_a^\infty f(x)\,dx$ *converges;*

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for all $x \in [a, \infty)$. Then

1. $\displaystyle\int_a^\infty g(x)\,dx$ *converges* $\implies$ $\displaystyle\int_a^\infty f(x)\,dx$ *converges;*
2. $\displaystyle\int_a^\infty f(x)\,dx$ *diverges* $\implies$ $\displaystyle\int_a^\infty g(x)\,dx$ *diverges.*

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1. $\displaystyle\int_a^\infty g(x)\,dx$ *converges* $\implies$ $\displaystyle\int_a^\infty f(x)\,dx$ *converges;*
2. $\displaystyle\int_a^\infty f(x)\,dx$ *diverges* $\implies$ $\displaystyle\int_a^\infty g(x)\,dx$ *diverges.*

(you can read the proof in the book)

EXAMPLE 7 These examples illustrate how we use Theorem 2.

(a) $\displaystyle\int_1^\infty \frac{\sin^2 x}{x^2}\,dx$ converges because

$$0 \le \frac{\sin^2 x}{x^2} \le \frac{1}{x^2} \quad \text{on} \quad [1, \infty) \quad \text{and} \quad \int_1^\infty \frac{1}{x^2}\,dx \quad \text{converges.}$$ Example 3

(b) $\displaystyle\int_1^\infty \frac{1}{\sqrt{x^2 - 0.1}}\,dx$ diverges because

$$\frac{1}{\sqrt{x^2 - 0.1}} \ge \frac{1}{x} \quad \text{on} \quad [1, \infty) \quad \text{and} \quad \int_1^\infty \frac{1}{x}\,dx \quad \text{diverges.}$$ Example 3

(c) $\displaystyle\int_0^{\pi/2} \frac{\cos x}{\sqrt{x}}\,dx$ converges because

$$0 \le \frac{\cos x}{\sqrt{x}} \le \frac{1}{\sqrt{x}} \quad \text{on} \quad \left[0, \frac{\pi}{2}\right],$$ $0 \le \cos x \le 1$ on $\left[0, \frac{\pi}{2}\right]$

and

$$\begin{aligned}
\int_0^{\pi/2} \frac{dx}{\sqrt{x}} &= \lim_{a\to 0^+} \int_a^{\pi/2} \frac{dx}{\sqrt{x}} \\
&= \lim_{a\to 0^+} \sqrt{4x}\,\Big]_a^{\pi/2} \qquad 2\sqrt{x} = \sqrt{4x} \\
&= \lim_{a\to 0^+} \left(\sqrt{2\pi} - \sqrt{4a}\right) = \sqrt{2\pi} \quad \text{converges.}
\end{aligned}$$ ■

Theorem (Limit Comparison Test)

Suppose that

- $f : [a, \infty) \to \mathbb{R}$ *and* $g : [a, \infty) \to \mathbb{R}$ *are continuous;*
- $f > 0$ *and* $g > 0$*;*
- $\lim_{x \to \infty} \frac{f(x)}{g(x)}$ *exists and* $0 < \lim_{x \to \infty} \frac{f(x)}{g(x)} < \infty.$

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- $\lim_{x\to\infty} \frac{f(x)}{g(x)}$ *exists and* $0 < \lim_{x\to\infty} \frac{f(x)}{g(x)} < \infty$.

Then

$$\int_a^\infty f(x)\,dx \qquad \text{and} \qquad \int_a^\infty g(x)\,dx$$

both converge or both diverge.

Example

Show that the integral $\displaystyle\int_1^\infty \frac{dx}{1+x^2}$ converges, by comparing it with the integral $\displaystyle\int_1^\infty \frac{1}{x^2}\,dx$.

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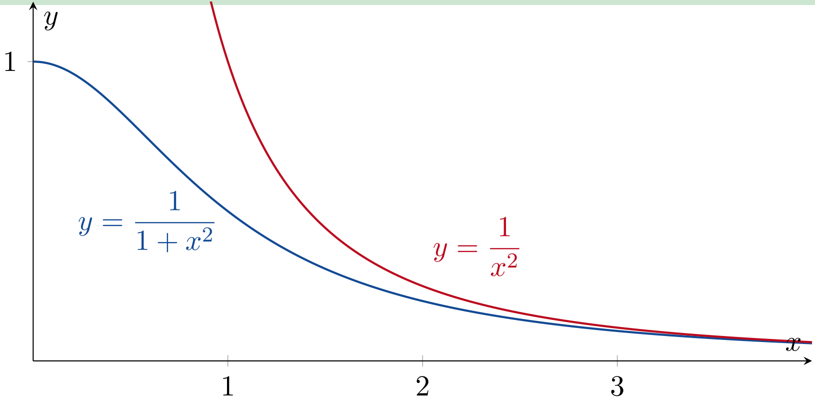

Note first that the functions $f(x) = \dfrac{1}{x^2}$ and $g(x) = \dfrac{1}{1+x^2}$ are both

- continuous on $[1, \infty)$; and
- > 0 on $[1, \infty)$.

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Moreover

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = \lim_{x\to\infty} \frac{\frac{1}{x^2}}{\frac{1}{1+x^2}} = \lim_{x\to\infty} \frac{1+x^2}{x^2} = \lim_{x\to\infty} \left(\frac{1}{x^2} + 1\right) = 0 + 1 = 1 > 0.$$

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This means that the integrals $\displaystyle\int_1^\infty \frac{dx}{1+x^2}$ and $\displaystyle\int_1^\infty \frac{dx}{x^2}$ both converge, or both diverge.

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This means that the integrals $\displaystyle\int_1^\infty \frac{dx}{1+x^2}$ and $\displaystyle\int_1^\infty \frac{dx}{x^2}$ both converge, or both diverge.

Since we already know that $\displaystyle\int_1^\infty \frac{dx}{x^2}$ converges, we must have that $\displaystyle\int_1^\infty \frac{dx}{1+x^2}$ converges as well.

Remark

!!! This does not mean that these two integrals converge to the same value. !!!

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Actually,

$$\int_1^\infty \frac{dx}{x^2} = 1$$

and

$$\int_1^\infty \frac{dx}{1+x^2} = \frac{\pi}{4}.$$

EXAMPLE 9 Investigate the convergence of $\displaystyle\int_1^{\infty} \frac{1 - e^{-x}}{x}\, dx$.

Solution The integrand suggests a comparison of $f(x) = (1 - e^{-x})/x$ with $g(x) = 1/x$. However, we cannot use the Direct Comparison Test because $f(x) \le g(x)$ and the integral of $g(x)$ *diverges*. On the other hand, using the Limit Comparison Test we find that

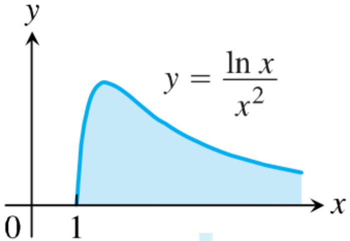
$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = \lim_{x\to\infty} \left(\frac{1 - e^{-x}}{x}\right)\left(\frac{x}{1}\right) = \lim_{x\to\infty} (1 - e^{-x}) = 1,$$

which is a positive finite limit. Therefore, $\displaystyle\int_1^{\infty} \frac{1 - e^{-x}}{x}\, dx$ diverges because $\displaystyle\int_1^{\infty} \frac{dx}{x}$ diverges. Approximations to the improper integral are given in Table 8.5. Note that the values do not appear to approach any fixed limiting value as $b \to \infty$. ■

Let's finish with a quick recap.

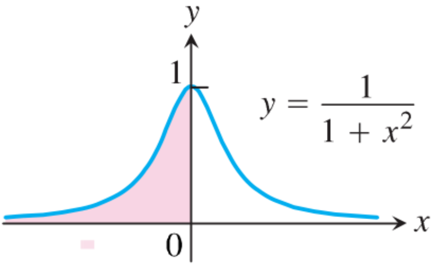
Upper limit

$$\int_1^{\infty} \frac{\ln x}{x^2}\, dx = \lim_{b \to \infty} \int_1^{b} \frac{\ln x}{x^2}\, dx$$



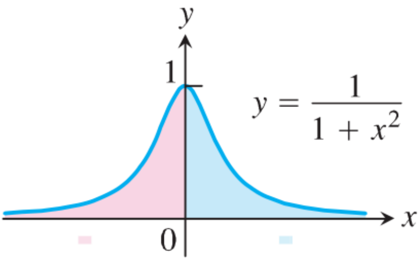
Lower limit

$$\int_{-\infty}^{0} \frac{dx}{1+x^2} = \lim_{a\to-\infty} \int_{a}^{0} \frac{dx}{1+x^2}$$



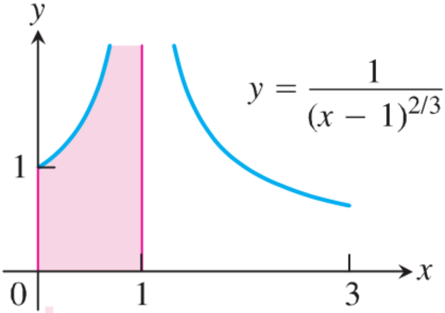
Both limits

$$\int_{-\infty}^{\infty} \frac{dx}{1+x^2} = \lim_{b \to -\infty} \int_b^0 \frac{dx}{1+x^2} + \lim_{c \to \infty} \int_0^c \frac{dx}{1+x^2}$$



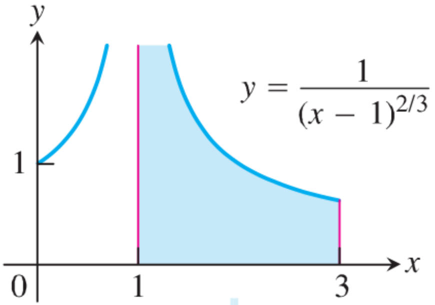
Upper endpoint

$$\int_0^1 \frac{dx}{(x-1)^{2/3}} = \lim_{b \to 1^-} \int_0^b \frac{dx}{(x-1)^{2/3}}$$



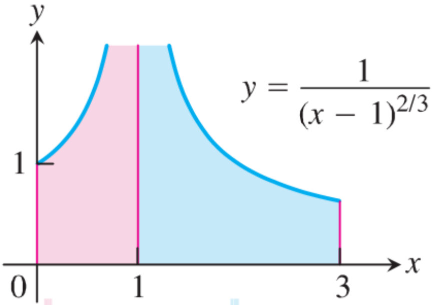
Lower endpoint

$$\int_1^3 \frac{dx}{(x-1)^{2/3}} = \lim_{d \to 1^+} \int_d^3 \frac{dx}{(x-1)^{2/3}}$$



Interior point

$$\int_0^3 \frac{dx}{(x-1)^{2/3}} = \int_0^1 \frac{dx}{(x-1)^{2/3}} + \int_1^3 \frac{dx}{(x-1)^{2/3}}$$



Next Time

- 11.1 Three-Dimensional Coordinate Systems
- 11.2 Vectors
- 11.3 The Dot Product