

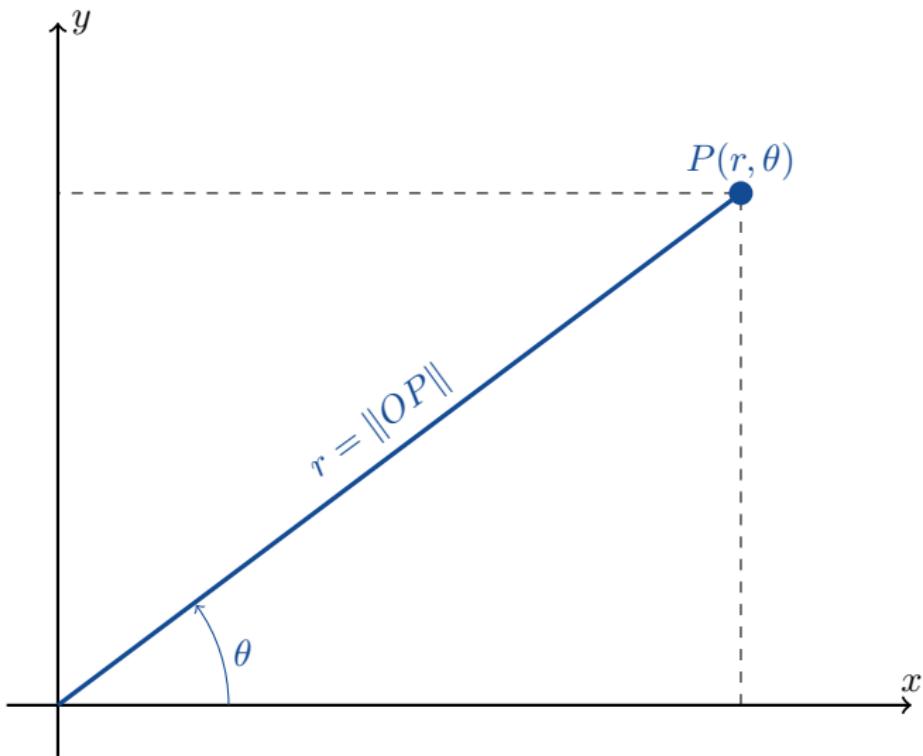
# Lecture 3

- 8. Polar Coordinates
- 9. Conic Sections
- 10. Three Dimensional Cartesian Coordinates



# Polar Coordinates

## 8. Polar Coordinates



## 8. Polar Coordinates

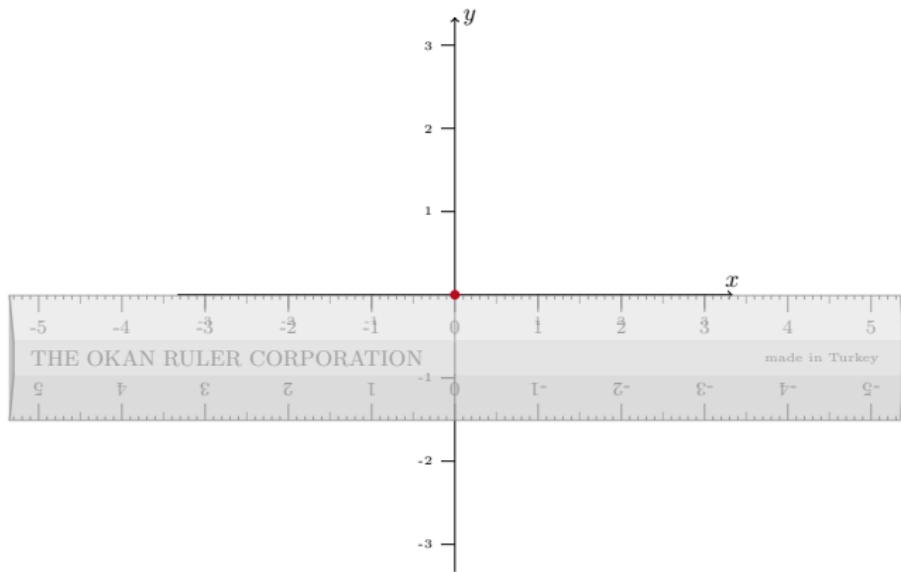


anticlockwise = positive angle  
saat yönünün tersi = pozitif açı

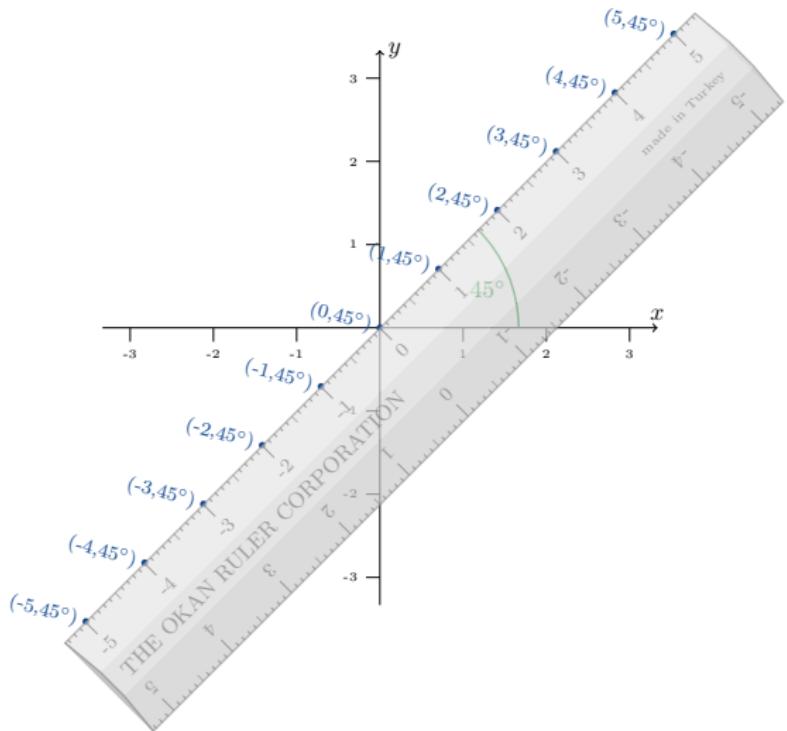


clockwise = negative angle  
saat yönünde = negatif açı

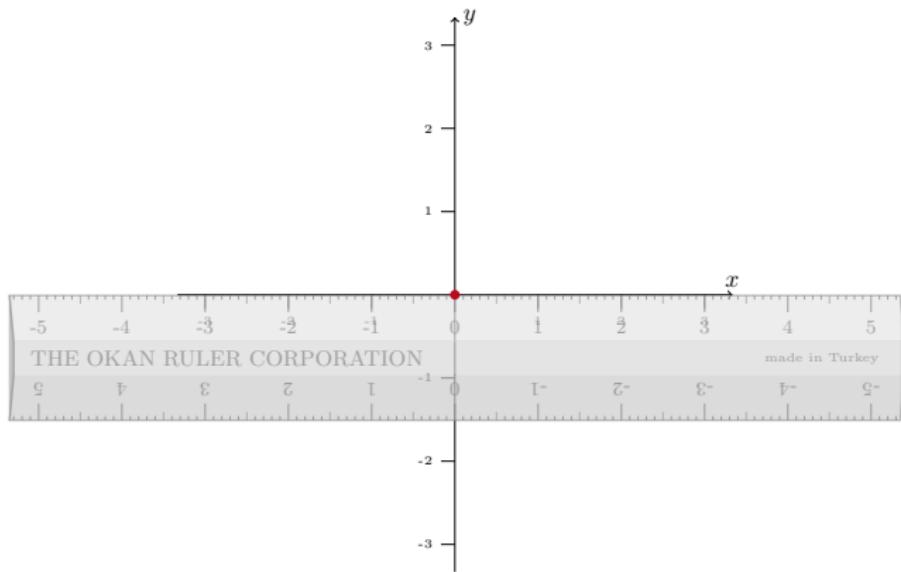
## 8. Polar Coordinates



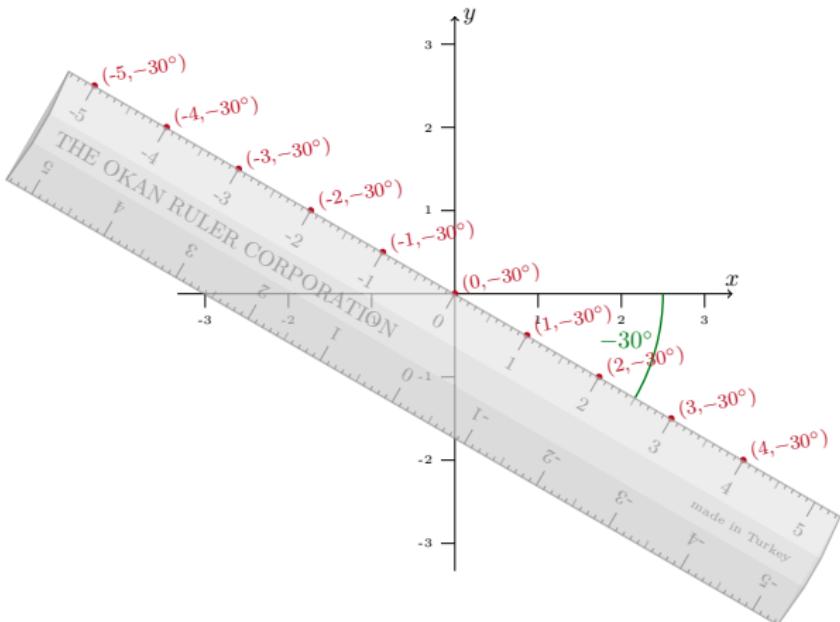
## 8. Polar Coordinates



## 8. Polar Coordinates

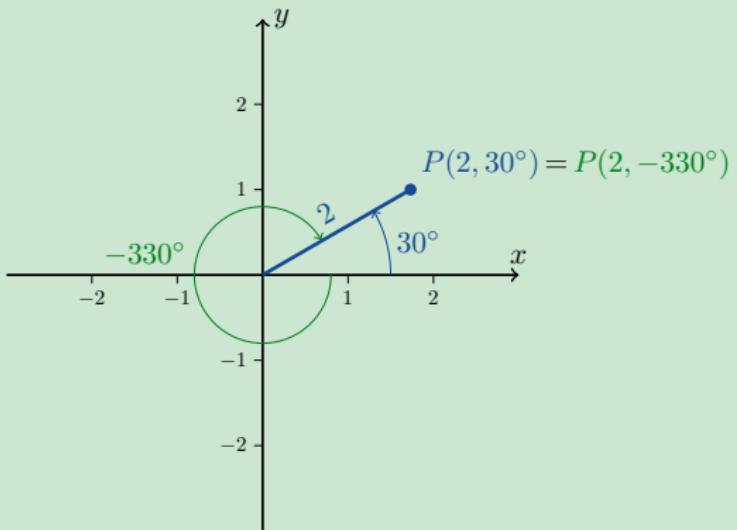


## 8. Polar Coordinates



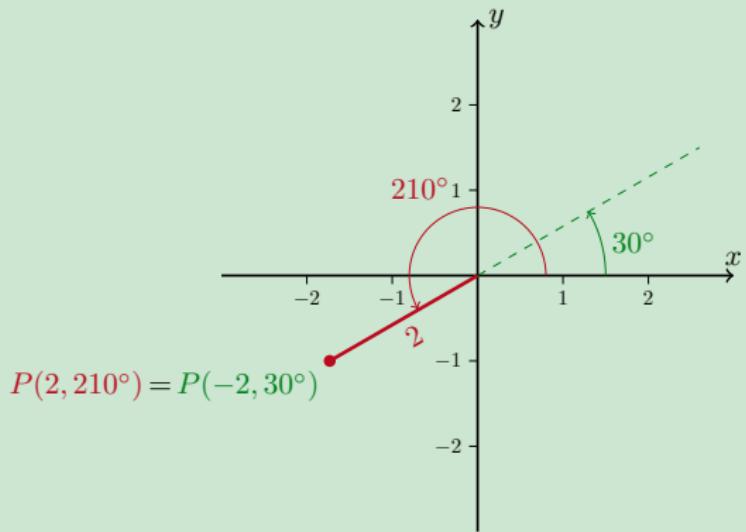
## 8. Polar Coordinates

### Example



## 8. Polar Coordinates

### Example



## 8. Polar Coordinates

### Example

Find all the polar coordinates of  $P(2, 30^\circ)$ .

*solution:* We can have either  $r = 2$  or  $r = -2$ . For  $r = 2$ , we can have

$$\theta = 30^\circ, 30^\circ \pm 360^\circ, 30^\circ \pm 720^\circ, 30^\circ \pm 1080^\circ, \dots$$

For  $r = -2$ , we can have

$$\theta = 210^\circ, 210^\circ \pm 360^\circ, 210^\circ \pm 720^\circ, 210^\circ \pm 1080^\circ, \dots$$

Therefore

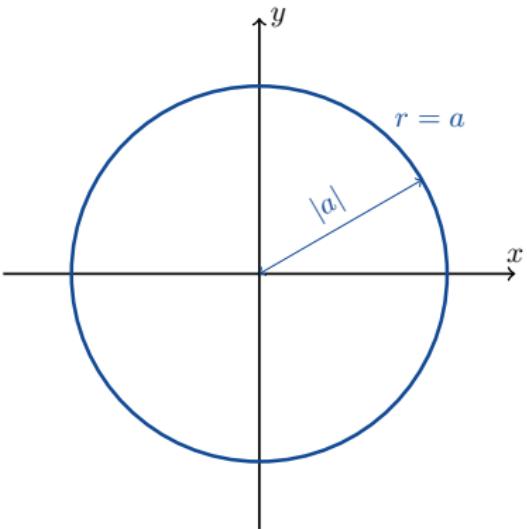
$$P(2, 30^\circ) = P(2, (30 + 360n)^\circ) = P(-2, (210 + 360m)^\circ)$$

for all  $m, n \in \mathbb{Z}$ .

## 8. Polar Coordinates

### Example

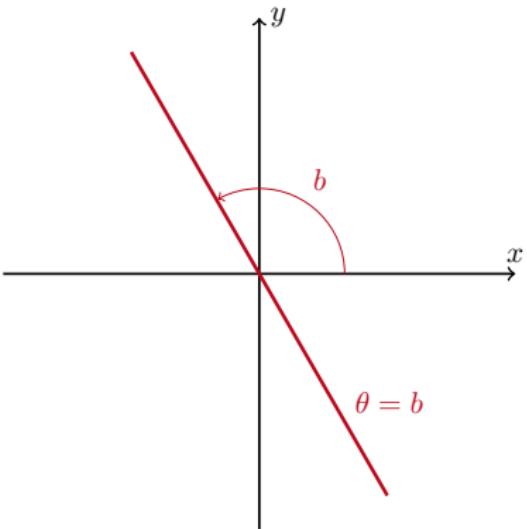
Draw the graph of  $r = a$ .



## 8. Polar Coordinates

### Example

Draw the graph of  $\theta = b$ .



## 8. Polar Coordinates



### Example

- 1  $r = 1$  and  $r = -1$  are both equations for a circle of radius 1 centred at the origin.
- 2  $\theta = 30^\circ$ ,  $\theta = 210^\circ$  and  $\theta = -150^\circ$  are all equations for the same line.

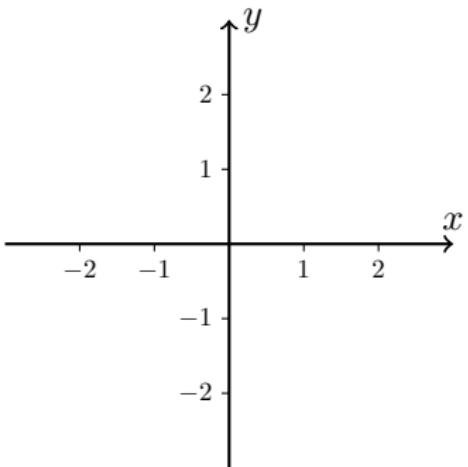
## 8. Polar Coordinates



### Example

Draw the sets of points whose polar coordinates satisfy the following:  $1 \leq r \leq 2$  and  $0 \leq \theta \leq 90^\circ$ .

*solution:*

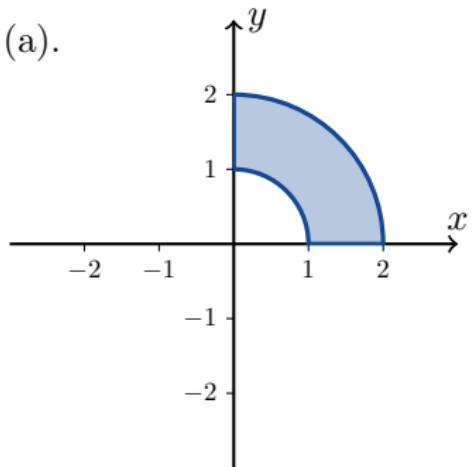


## 8. Polar Coordinates

### Example

Draw the sets of points whose polar coordinates satisfy the following:  $1 \leq r \leq 2$  and  $0 \leq \theta \leq 90^\circ$ .

*solution:*



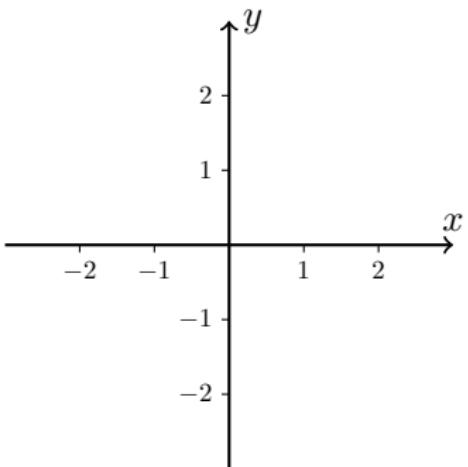
## 8. Polar Coordinates



### Example

Draw the sets of points whose polar coordinates satisfy the following:  $-3 \leq r \leq 2$  and  $\theta = 45^\circ$ .

*solution:*

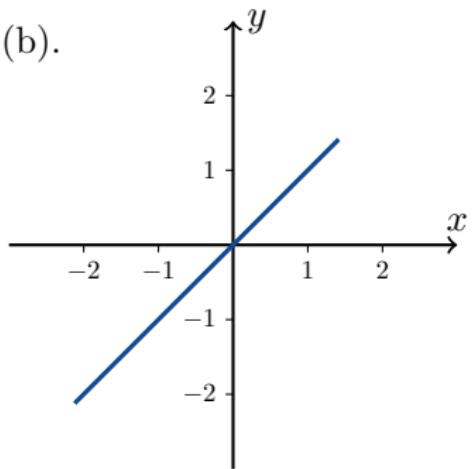


## 8. Polar Coordinates

### Example

Draw the sets of points whose polar coordinates satisfy the following:  $-3 \leq r \leq 2$  and  $\theta = 45^\circ$ .

*solution:*



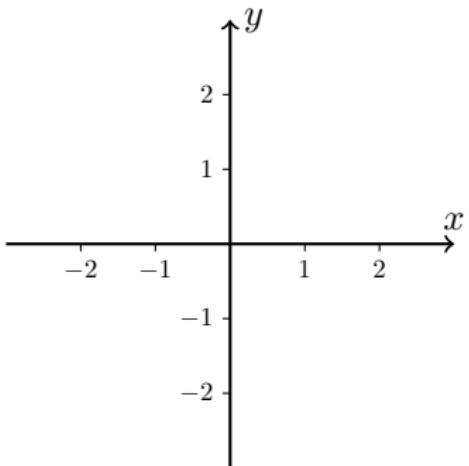
## 8. Polar Coordinates



### Example

Draw the sets of points whose polar coordinates satisfy the following:  $r \leq 0$  and  $\theta = 60^\circ$ .

*solution:*



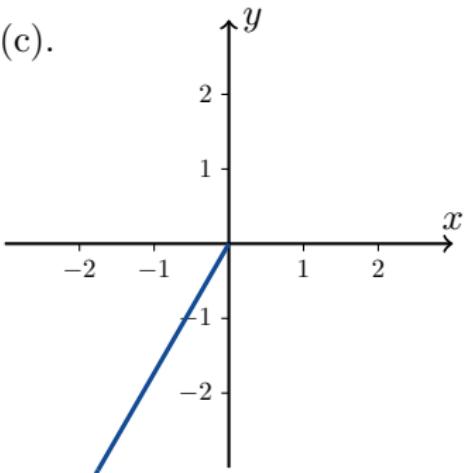
## 8. Polar Coordinates



### Example

Draw the sets of points whose polar coordinates satisfy the following:  $r \leq 0$  and  $\theta = 60^\circ$ .

*solution:*



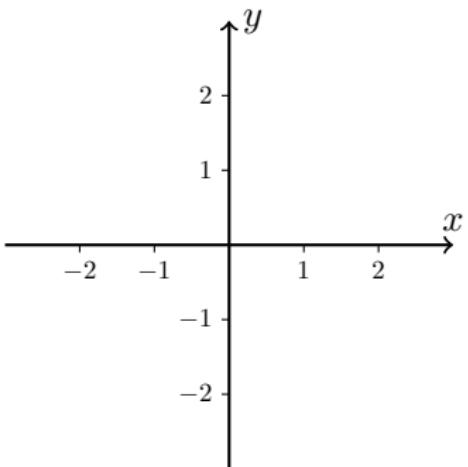
## 8. Polar Coordinates



### Example

Draw the sets of points whose polar coordinates satisfy the following:  $120^\circ \leq \theta \leq 150^\circ$ .

*solution:*

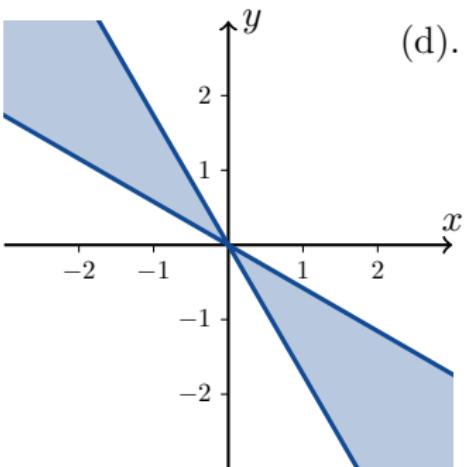


## 8. Polar Coordinates

### Example

Draw the sets of points whose polar coordinates satisfy the following:  $120^\circ \leq \theta \leq 150^\circ$ .

*solution:*



## 8. Polar Coordinates

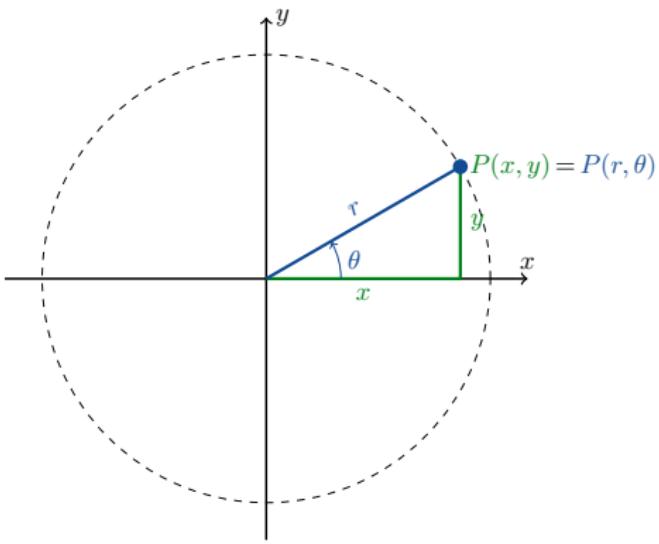
# Relating Polar and Cartesian Coordinates

$$x = r \cos \theta$$

$$x^2 + y^2 = r^2$$

$$y = r \sin \theta$$

$$\tan \theta = \frac{y}{x}$$



## 8. Polar Coordinates



$$x = r \cos \theta \quad y = r \sin \theta \quad x^2 + y^2 = r^2 \quad \tan \theta = \frac{y}{x}$$

### Example

Convert the polar coordinates  $(r, \theta) = (-3, 90^\circ)$  into Cartesian coordinates.

*solution:*

$$(x, y) = (r \cos \theta, r \sin \theta) = (-3 \cos 90^\circ, -3 \sin 90^\circ) = (0, -3).$$

## 8. Polar Coordinates

### Example

Find polar coordinates for the Cartesian coordinates  $(x, y) = (5, -12)$ .

*solution:* Choosing  $r > 0$ , we calculate that

$$r = \sqrt{x^2 + y^2} = \sqrt{25 + 144} = \sqrt{169} = 13.$$

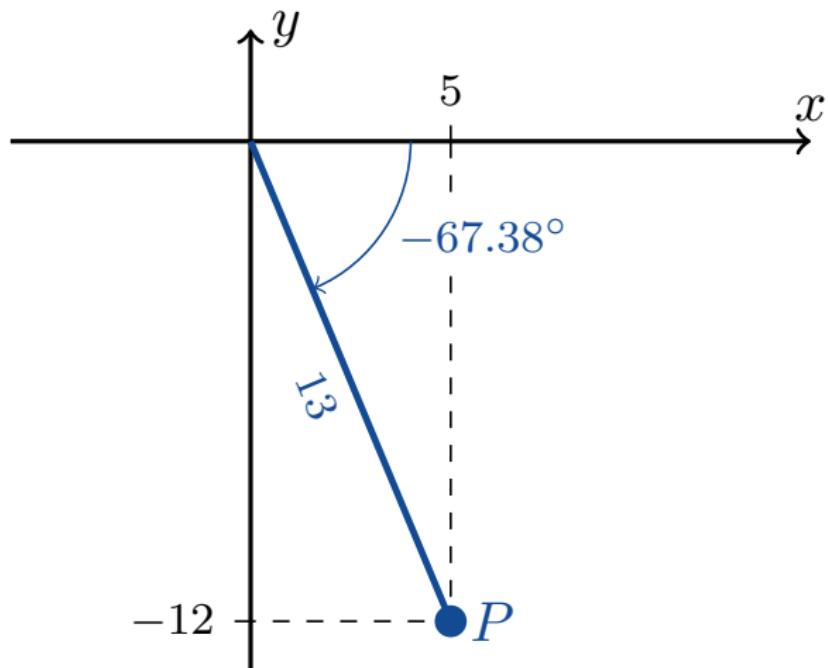
To find  $\theta$  we use the equation  $y = r \sin \theta$  to calculate that

$$\theta = \sin^{-1} \frac{y}{r} = \sin^{-1} \frac{-12}{13} \approx -67.38^\circ.$$

Therefore

$$(r, \theta) = (13, -67.38^\circ).$$

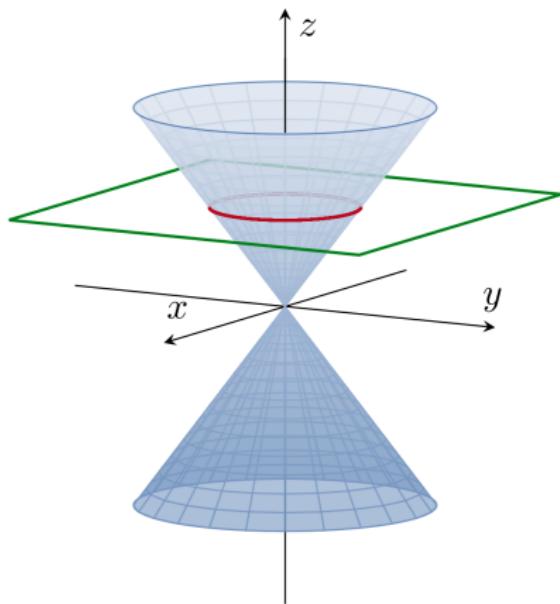
## 8. Polar Coordinates



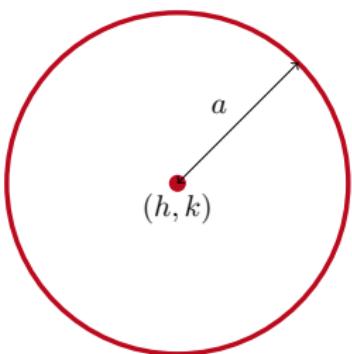
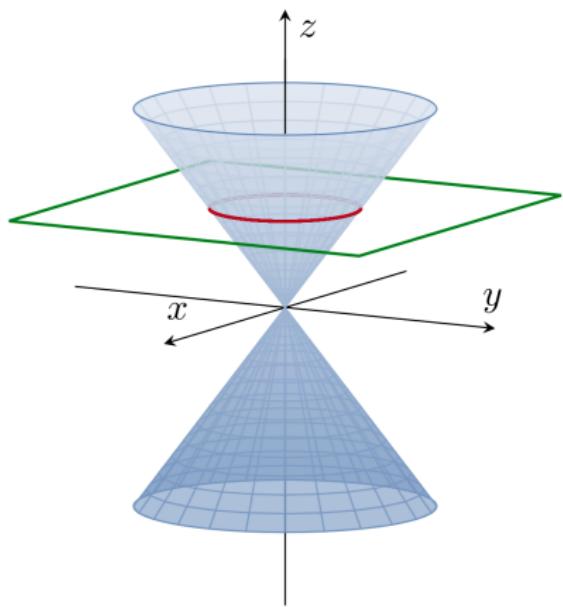


# Conic Sections

## 9. Conic Sections



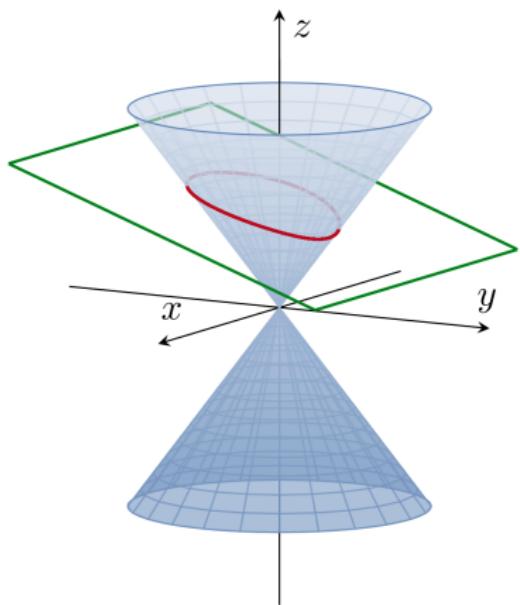
## 9. Conic Sections



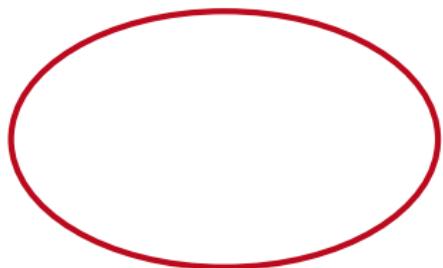
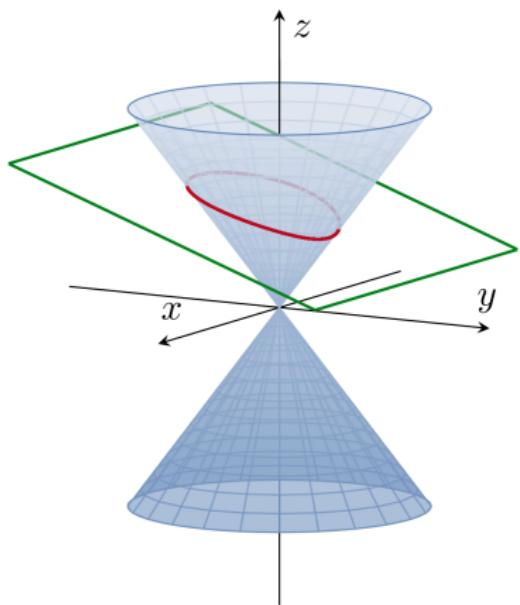
a circle

$$(x - h)^2 + (y - k)^2 = a^2$$

## 9. Conic Sections

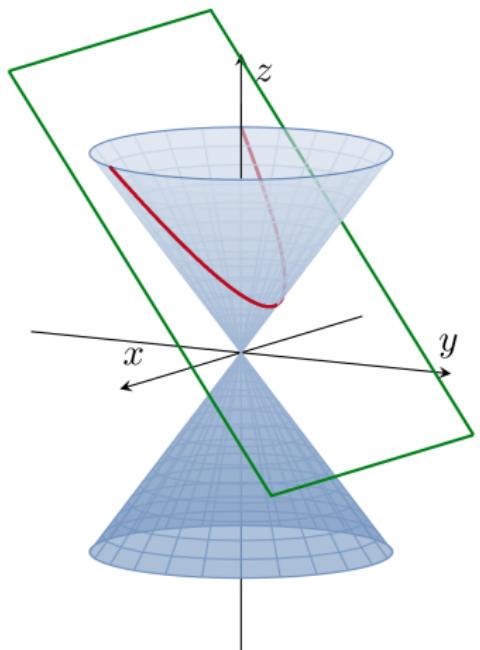


## 9. Conic Sections

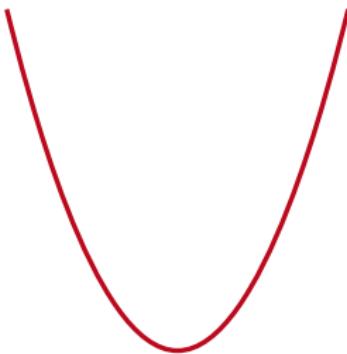
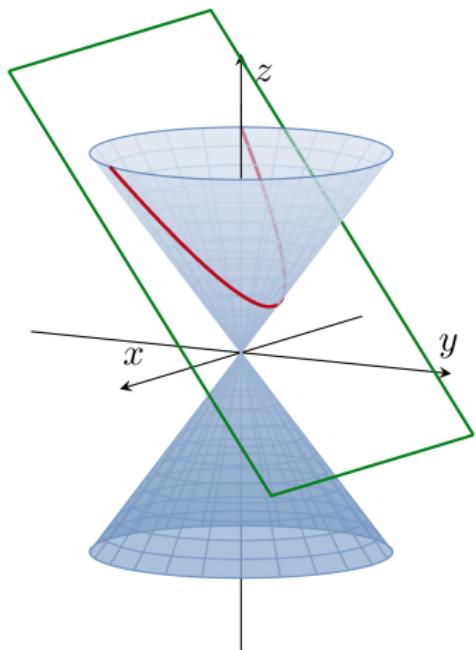


an ellipse

## 9. Conic Sections

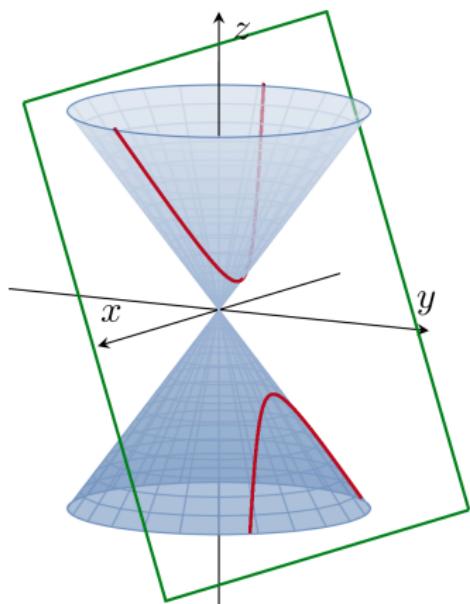


## 9. Conic Sections

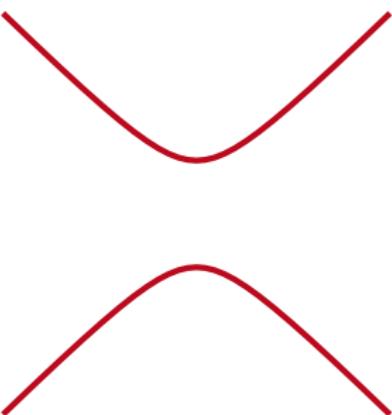
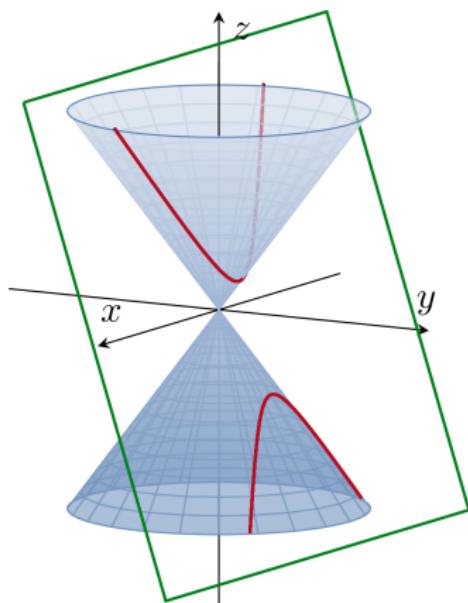


a parabola

## 9. Conic Sections

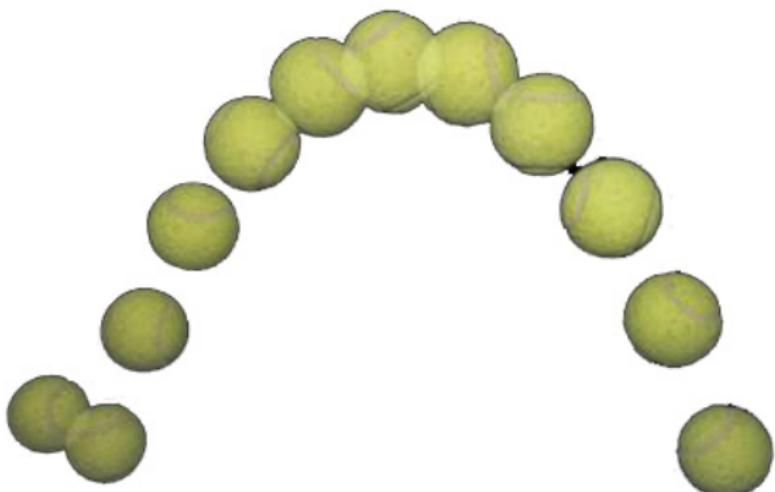


## 9. Conic Sections



a hyperbola

# Parabolas



## 9. Conic Sections



Clifton suspension bridge, Bristol, UK.

The cables of a suspension bridges hang in a shape which is almost (but not exactly) a parabola.

## 9. Conic Sections

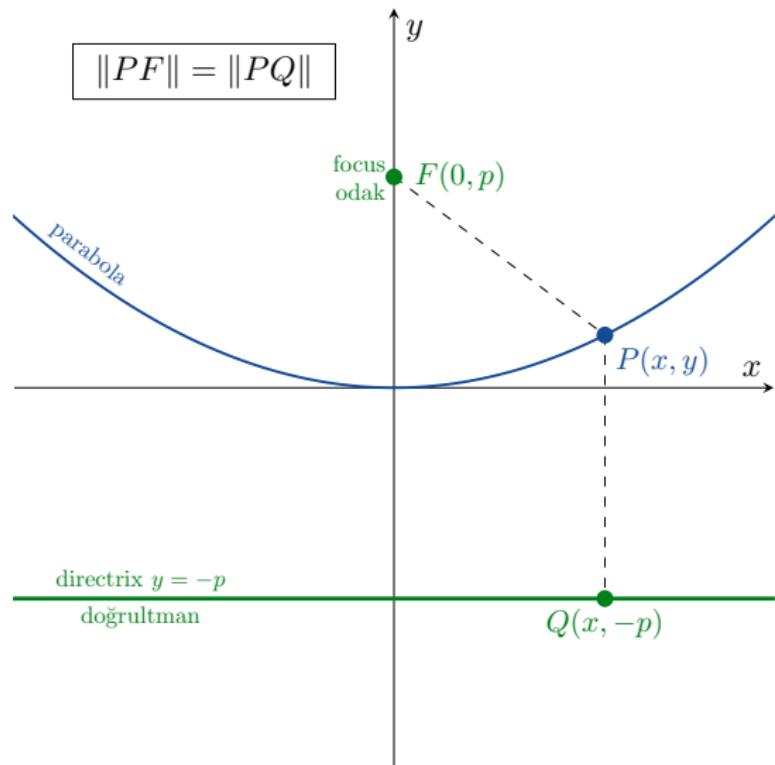


## 9. Conic Sections



To describe a parabola, we need a point called a *focus* and a line called a *directrix*.

## 9. Conic Sections



## 9. Conic Sections



### Definition

A point  $P(x, y)$  lies on the *parabola* if and only if

$$\|PF\| = \|PQ\|.$$

## 9. Conic Sections

Now

$$\begin{aligned}\|PF\| &= \text{distance between } P(x, y) \text{ and } F(0, p) \\ &= \sqrt{(x - 0)^2 + (y - p)^2} = \sqrt{x^2 + (y - p)^2}\end{aligned}$$

and

$$\begin{aligned}\|PQ\| &= \text{distance between } P(x, y) \text{ and } Q(x, -p) \\ &= \sqrt{(x - x)^2 + (y + p)^2} = \sqrt{(y + p)^2} = y + p.\end{aligned}$$

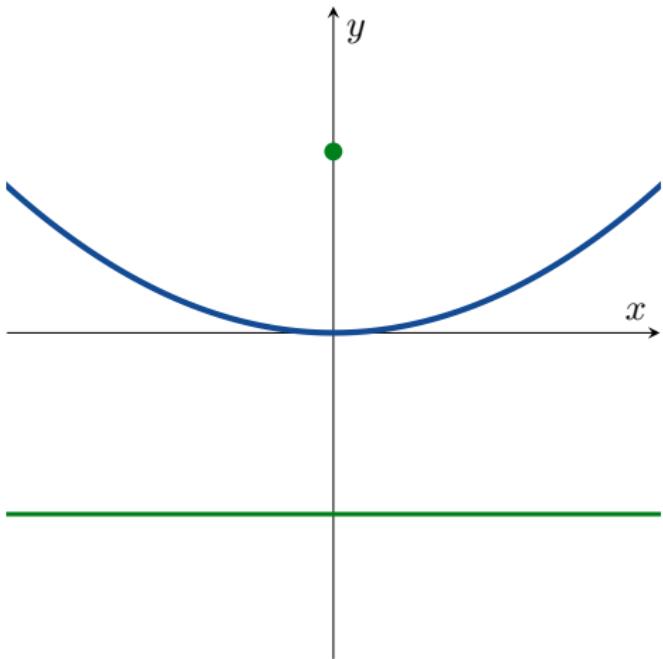
Therefore

$$\begin{aligned}\|PF\| &= \|PQ\| \\ \sqrt{x^2 + (y - p)^2} &= y + p \\ x^2 + (y - p)^2 &= (y + p)^2 \\ x^2 + y^2 - 2py + p^2 &= y^2 + 2py + p^2 \\ x^2 - 2py &= 2py \\ \boxed{x^2 = 4py}\end{aligned}$$

## 9. Conic Sections



equation:  $x^2 = 4py$   
focus:  $F(0, p)$   
directrix:  $y = -p$



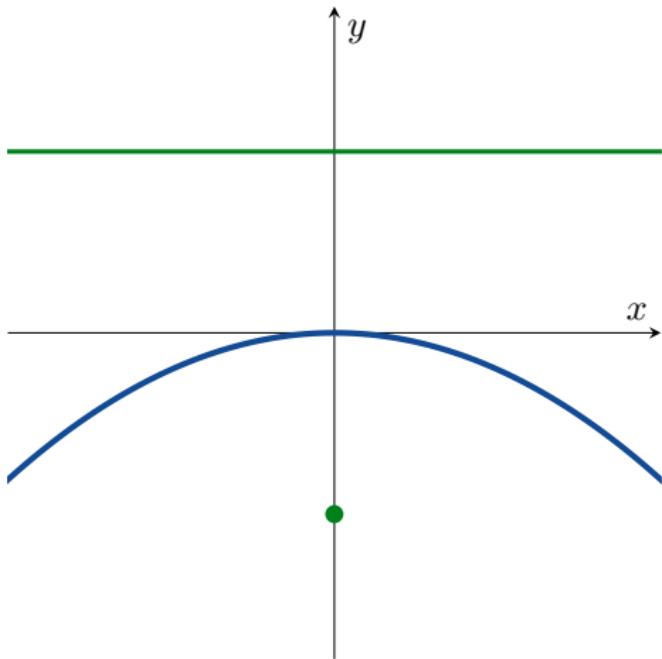
## 9. Conic Sections



equation:  $x^2 = -4py$

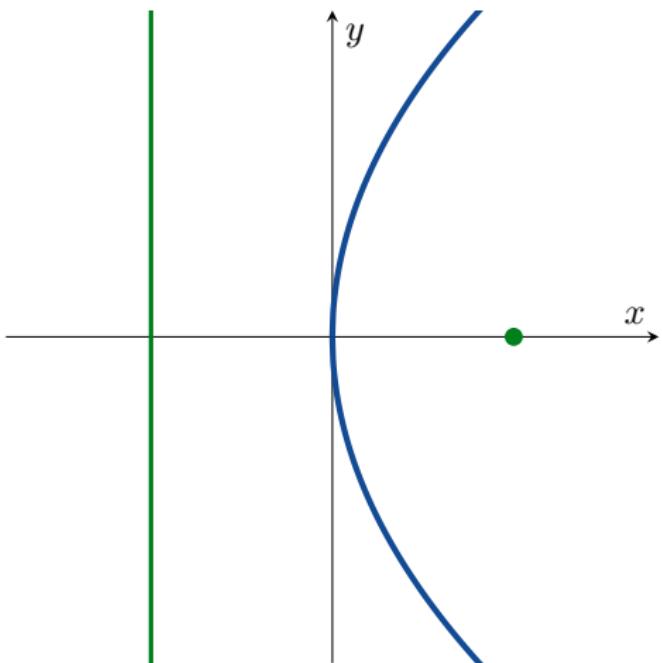
focus:  $F(0, -p)$

directrix:  $y = p$



## 9. Conic Sections

equation:  $y^2 = 4px$   
focus:  $F(p, 0)$   
directrix:  $x = -p$



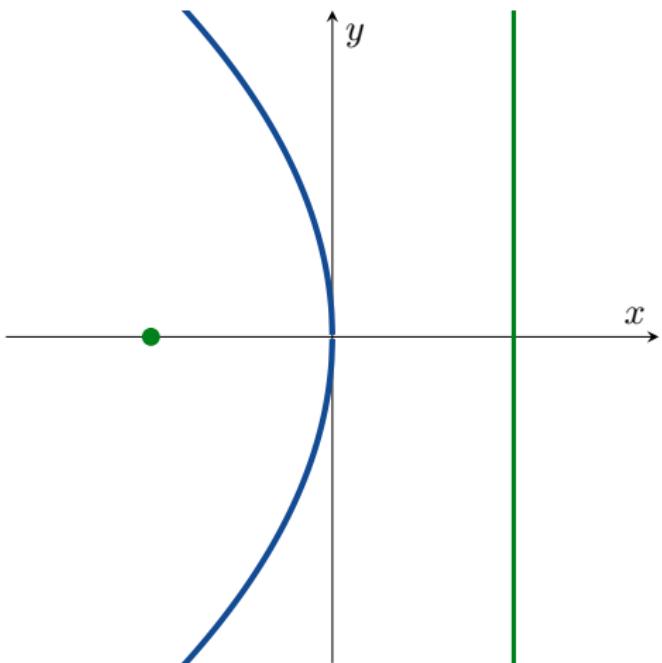
## 9. Conic Sections



equation:  $y^2 = -4px$

focus:  $F(-p, 0)$

directrix:  $x = p$



## 9. Conic Sections



### Example

Find the focus and directrix of the parabola  $y^2 = 10x$ .

*solution:* Our equation  $y^2 = 10x$  looks like  $y^2 = 4px$  with  $p = \frac{10}{4} = 2.5$ . Therefore the focus is at the point  $F(2.5, 0)$  and the directrix is the line  $x = -2.5$ .

## 9. Conic Sections

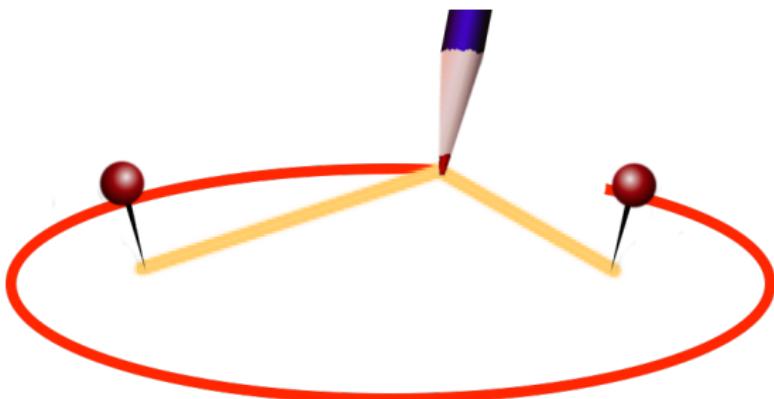


### Example

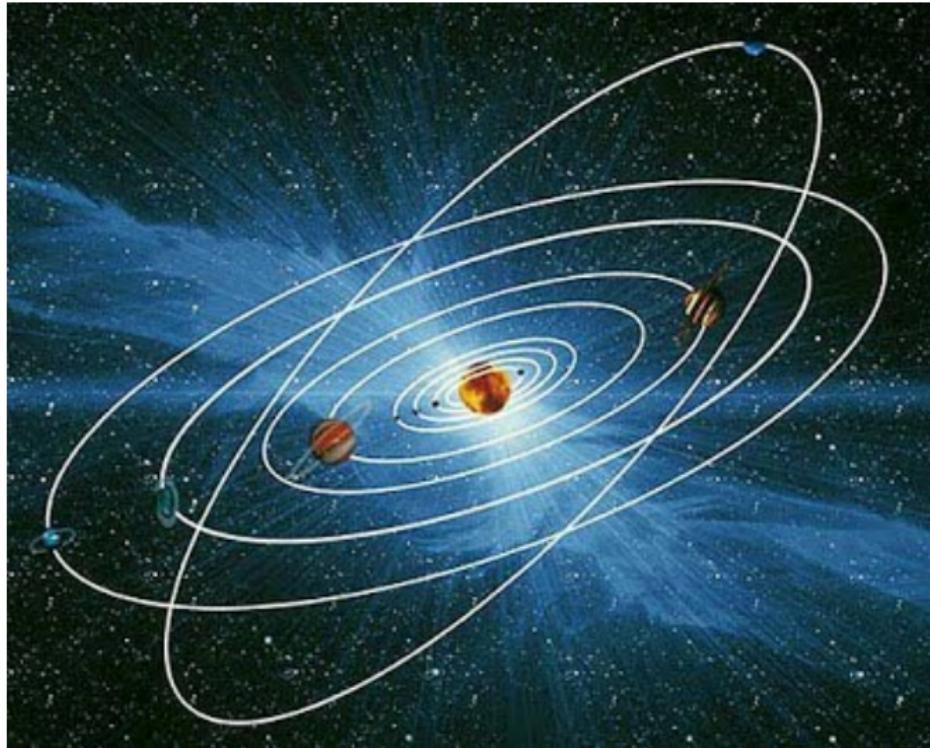
Find the equation for the parabola which has focus  $F(0, -10)$  and directrix  $y = 10$ .

*solution:* Clearly  $p = 10$  and  $x^2 = -4py$ . Therefore the answer is  $x^2 = -40y$ .

# Ellipses



## 9. Conic Sections

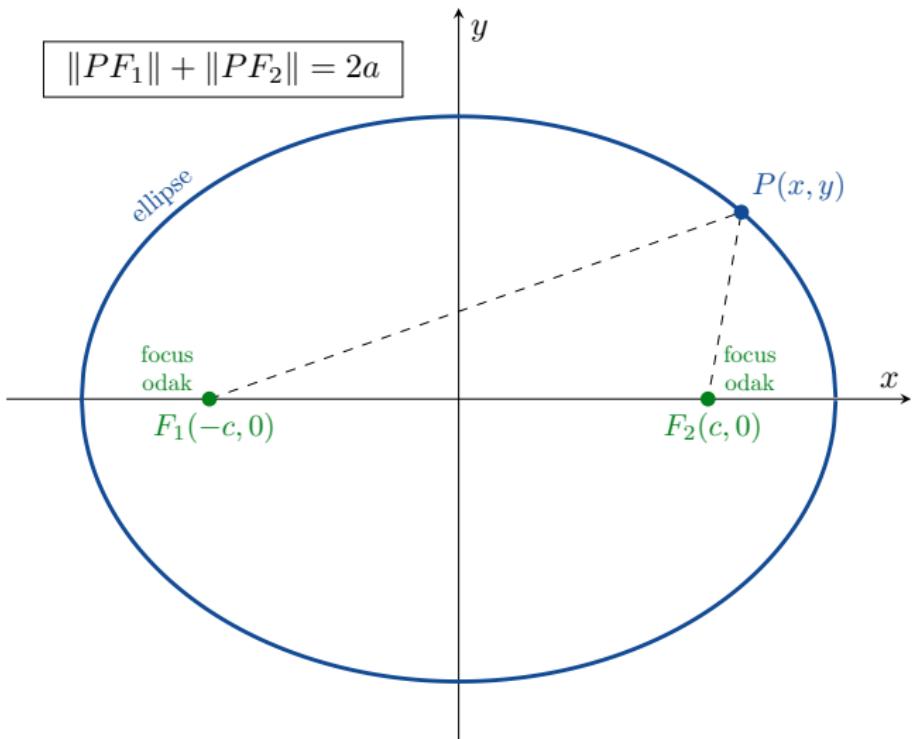


## 9. Conic Sections



Tycho Brahe Planetarium, Copenhagen, Denmark.

## 9. Conic Sections



To describe an ellipse, we need two *foci*.

## 9. Conic Sections

### Definition

A point  $P(x, y)$  is on the *ellipse* if and only if

$$\|PF_1\| + \|PF_2\| = 2a.$$

So

$$\sqrt{(x + c)^2 + y^2} + \sqrt{(x - c)^2 + y^2} = 2a.$$

This rearranges to

$$\frac{x^2}{a^2} + \frac{y^2}{a^2 - c^2} = 1.$$

If we set  $b = \sqrt{a^2 - c^2}$ , then we have

$$\boxed{\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1} \quad (0 < b < a).$$

## 9. Conic Sections

equation:

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \quad (0 < b < a)$$

centre-to-focus distance:

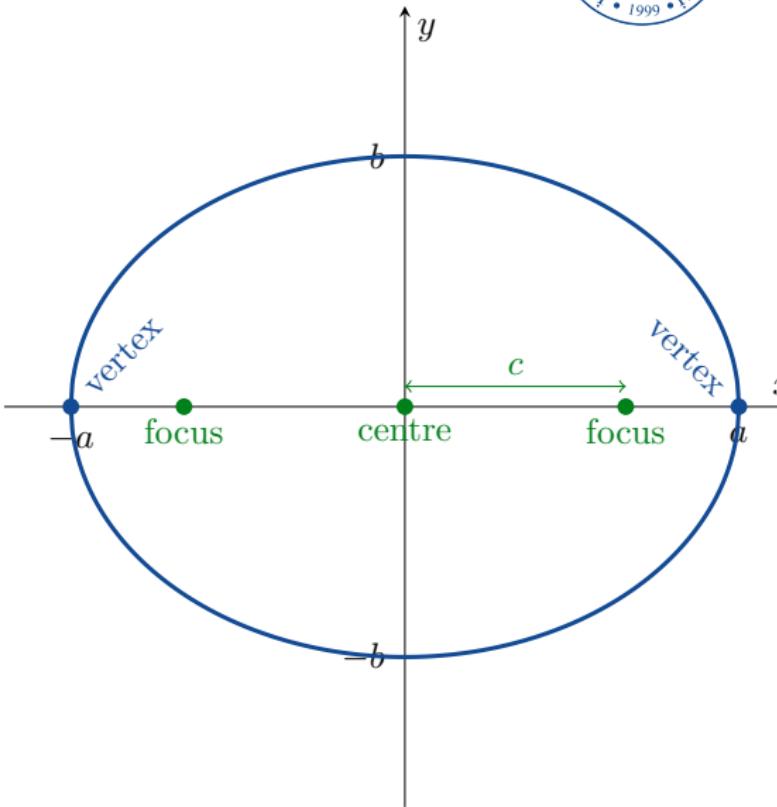
$$c = \sqrt{a^2 - b^2}$$

foci:

$$F_1(-c, 0) \quad \& \quad F_2(c, 0)$$

vertices :

$$(-a, 0) \quad \& \quad (a, 0)$$



## 9. Conic Sections

equation:

$$\frac{x^2}{b^2} + \frac{y^2}{a^2} = 1 \quad (0 < b < a)$$

centre-to-focus distance:

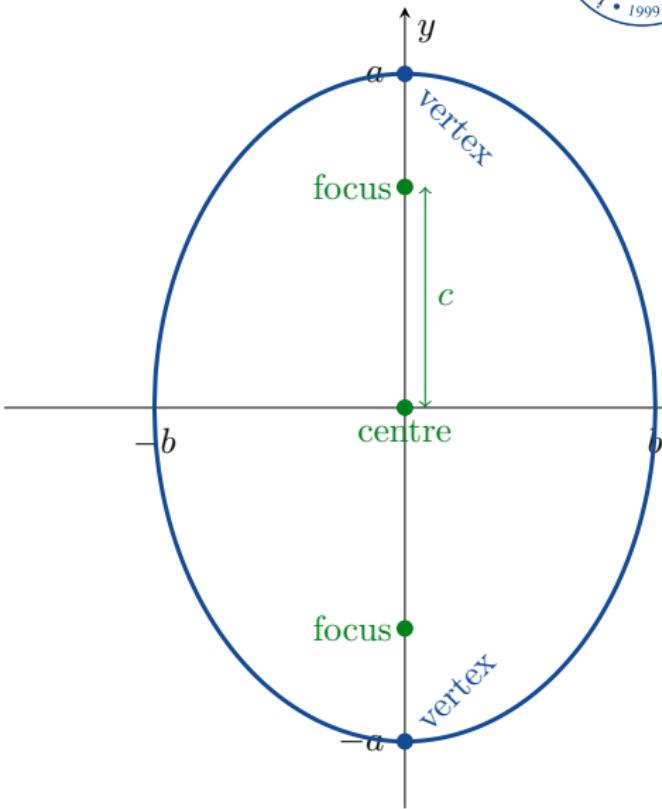
$$c = \sqrt{a^2 - b^2}$$

foci:

$$F_1(0, -c) \quad \& \quad F_2(0, c)$$

vertices :

$$(0, -a) \quad \& \quad (0, a)$$



## 9. Conic Sections



### Example

The ellipse  $\frac{x^2}{16} + \frac{y^2}{9} = 1$  has

- $a = 4$  and  $b = 3$ ;
- centre-to-focus distance  $c = \sqrt{a^2 - b^2} = \sqrt{16 - 9} = \sqrt{7}$ ;
- centre at  $(0, 0)$ ;
- foci at  $(-\sqrt{7}, 0)$  and  $(\sqrt{7}, 0)$ ; and
- vertices at  $(-4, 0)$  and  $(4, 0)$ .

## 9. Conic Sections



### Example

The ellipse  $\frac{x^2}{16} + \frac{y^2}{25} = 1$  has

- $a = 5$  and  $b = 4$ ;
- centre-to-focus distance  
 $c = \sqrt{a^2 - b^2} = \sqrt{25 - 16} = \sqrt{9} = 3$ ;
- centre at  $(0, 0)$ ;
- foci at  $(0, -3)$  and  $(0, 3)$ ; and
- vertices at  $(0, -5)$  and  $(0, 5)$ .

# Hyperbolas

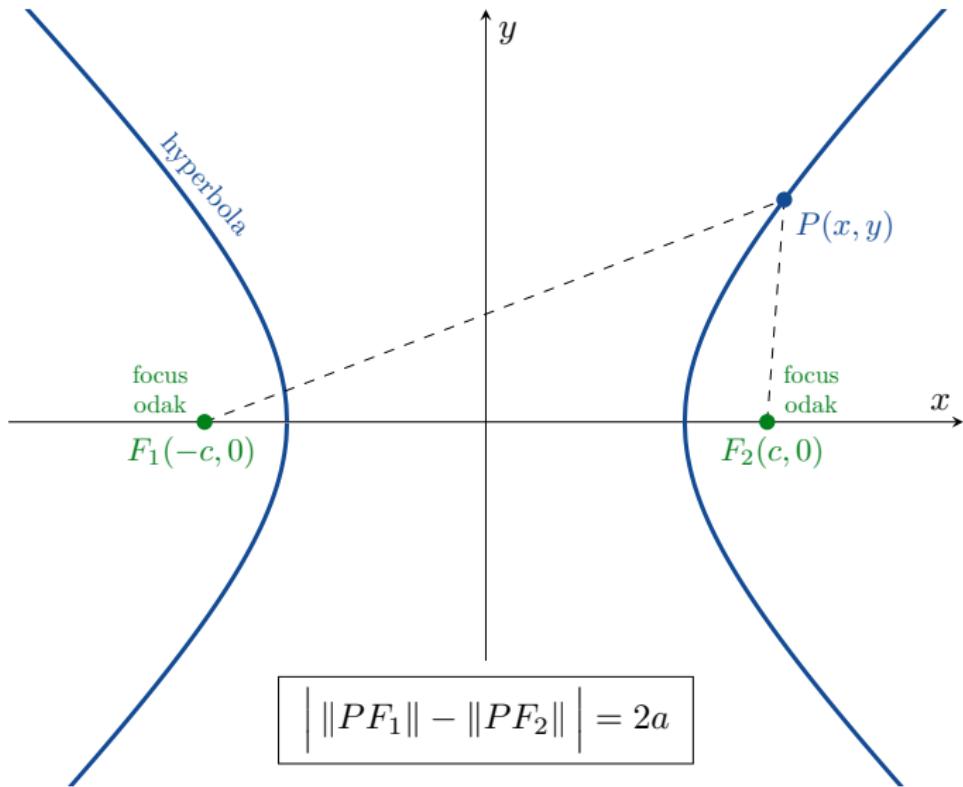


## 9. Conic Sections



Twin Arch 138, Ichinomiya City, Japan.

## 9. Conic Sections



## 9. Conic Sections



To describe a hyperbola, we again need two foci.

### Definition

A point  $P(x, y)$  is on the *hyperbola* if and only if

$$\left| \|PF_1\| - \|PF_2\| \right| = 2a.$$

## 9. Conic Sections



So

$$\sqrt{(x + c)^2 + y^2} - \sqrt{(x - c)^2 + y^2} = \pm 2a.$$

This rearranges to

$$\frac{x^2}{a^2} + \frac{y^2}{a^2 - c^2} = 1$$

where  $c > a > 0$ . If we set  $b = \sqrt{c^2 - a^2}$ , then

$$\boxed{\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1.}$$

## 9. Conic Sections

equation:

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

centre-to-focus distance:

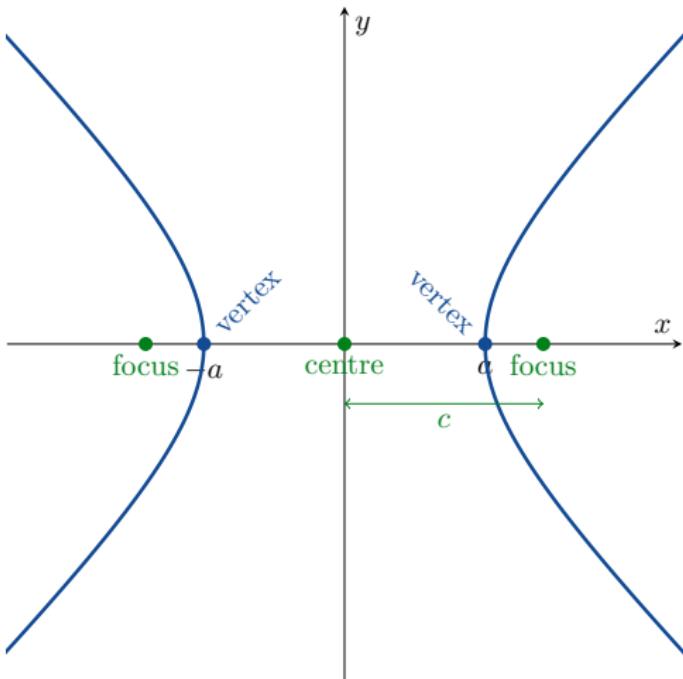
$$c = \sqrt{a^2 + b^2}$$

foci:

$$F_1(-c, 0) \quad \& \quad F_2(c, 0)$$

vertices :

$$(-a, 0) \quad \& \quad (a, 0)$$



## 9. Conic Sections

equation:

$$\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$$

centre-to-focus distance:

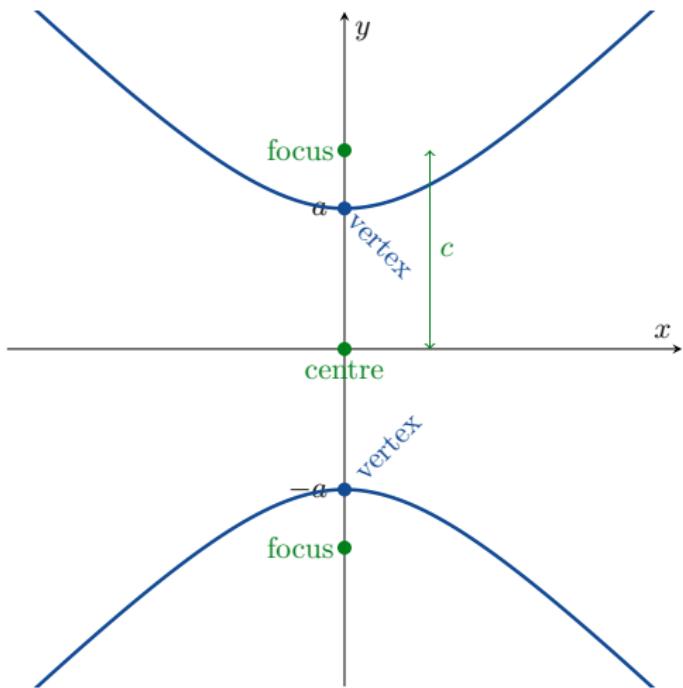
$$c = \sqrt{a^2 + b^2}$$

foci:

$$F_1(0, -c) \quad \& \quad F_2(0, c)$$

vertices :

$$(0, -a) \quad \& \quad (0, a)$$



## 9. Conic Sections



### Example

The hyperbola  $\frac{x^2}{4} - \frac{y^2}{5} = 1$  has

- $a = 2$  and  $b = \sqrt{5}$ ;
- centre at  $(0, 0)$ ;
- centre-to-focus distance  $c = \sqrt{a^2 + b^2} = \sqrt{4 + 5} = 3$ ;
- foci at  $(-3, 0)$  and  $(3, 0)$ ; and
- vertices at  $(-2, 0)$  and  $(2, 0)$ .

## 9. Conic Sections



### Example

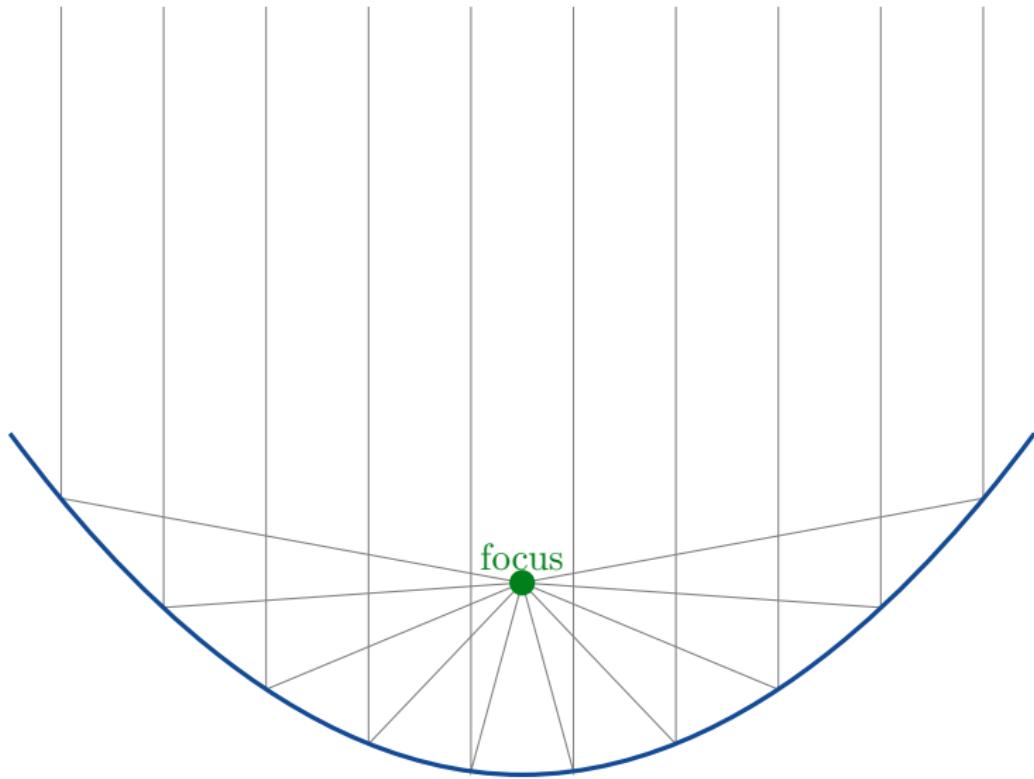
The hyperbola  $\frac{y^2}{9} - \frac{x^2}{16} = 1$  has

- $a = 3$  and  $b = 4$ ;
- centre at  $(0, 0)$ ;
- centre-to-focus distance  $c = \sqrt{a^2 + b^2} = \sqrt{9 + 16} = 5$ ;
- foci at  $(0, -5)$  and  $(0, 5)$ ; and
- vertices at  $(0, -3)$  and  $(0, 3)$ .

# Reflective Properties

Parabolas, ellipses and hyperbolas are useful in architecture and engineering because of their reflective properties.

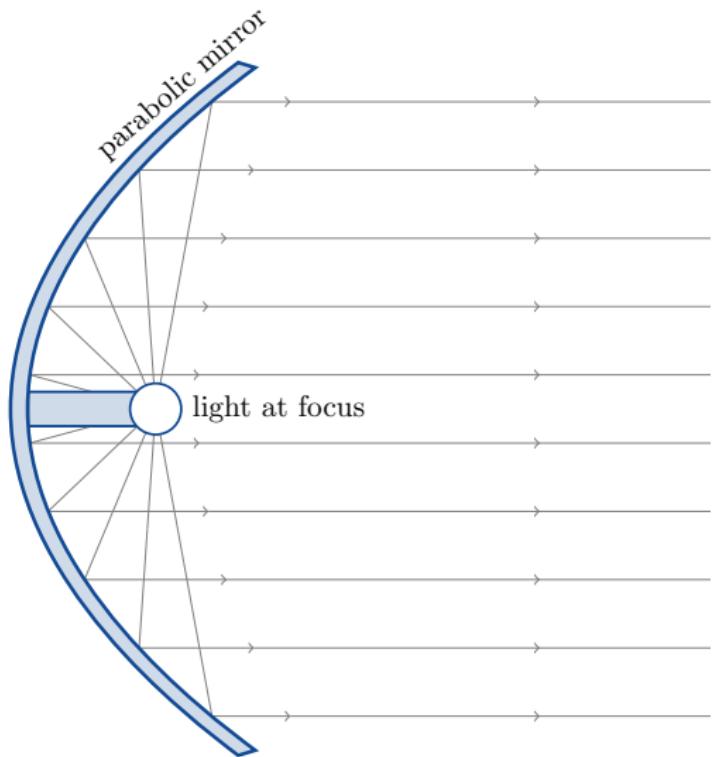
## 9. Conic Sections



## 9. Conic Sections



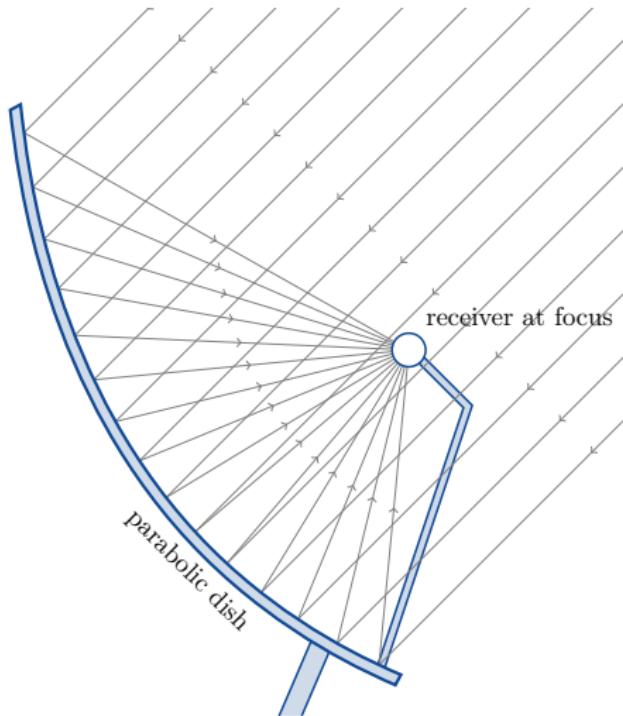
## 9. Conic Sections



## 9. Conic Sections



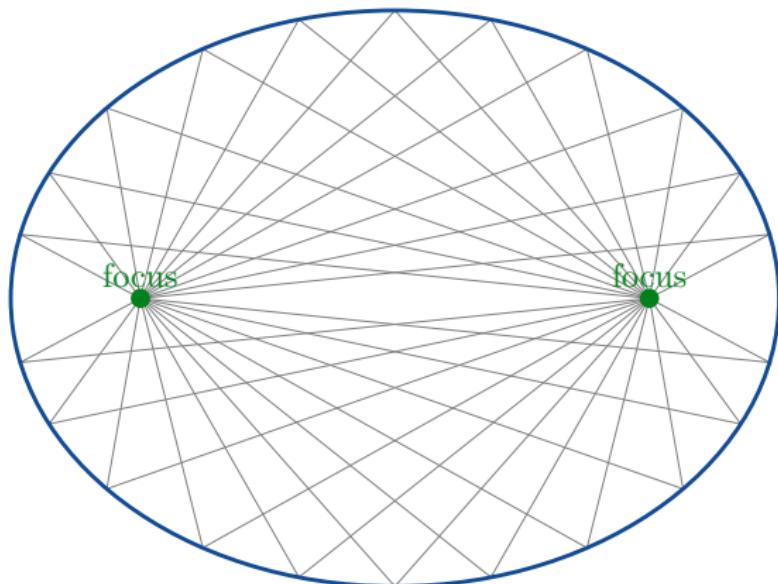
## 9. Conic Sections



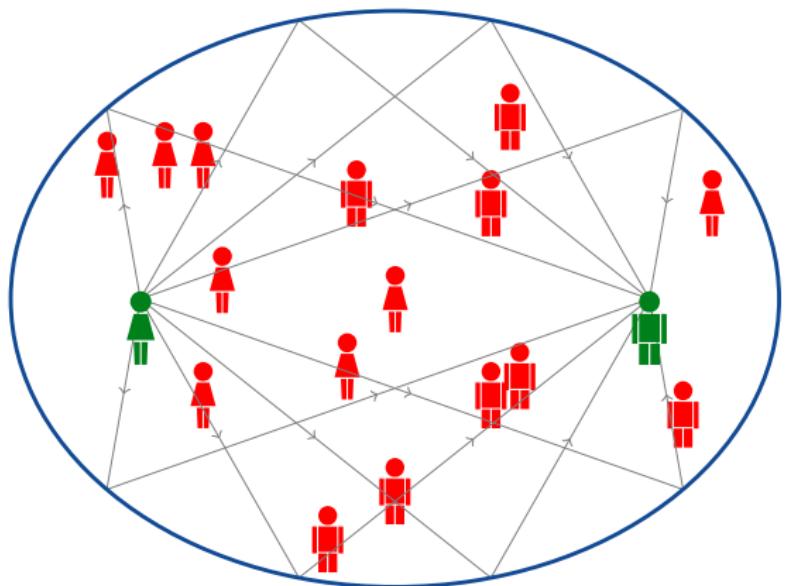
## 9. Conic Sections



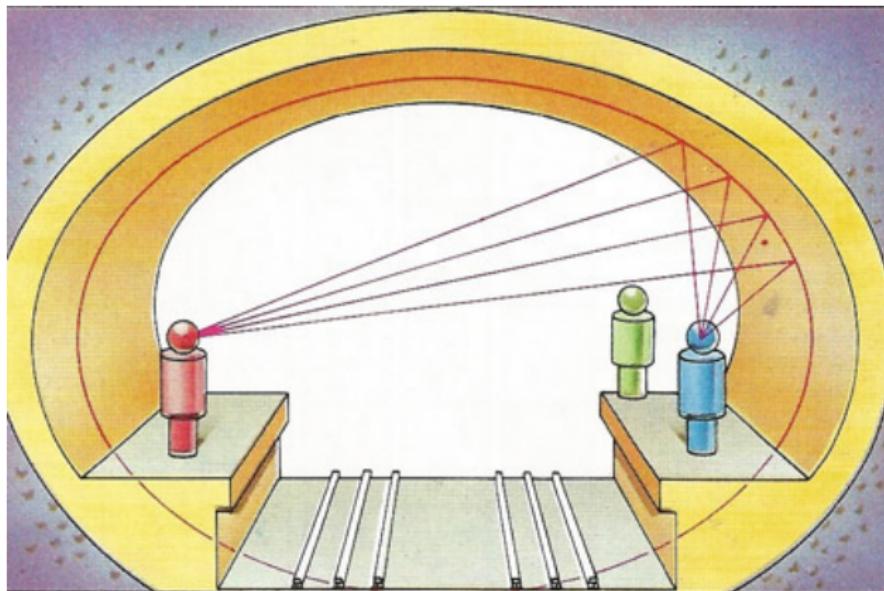
## 9. Conic Sections



## 9. Conic Sections



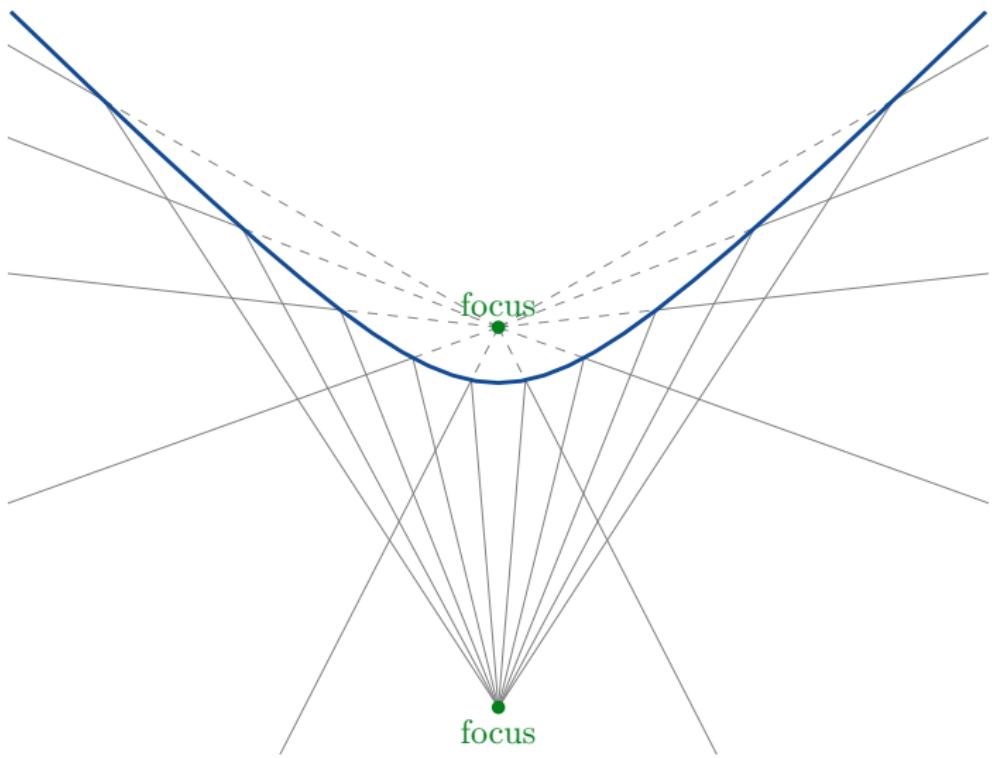
## 9. Conic Sections



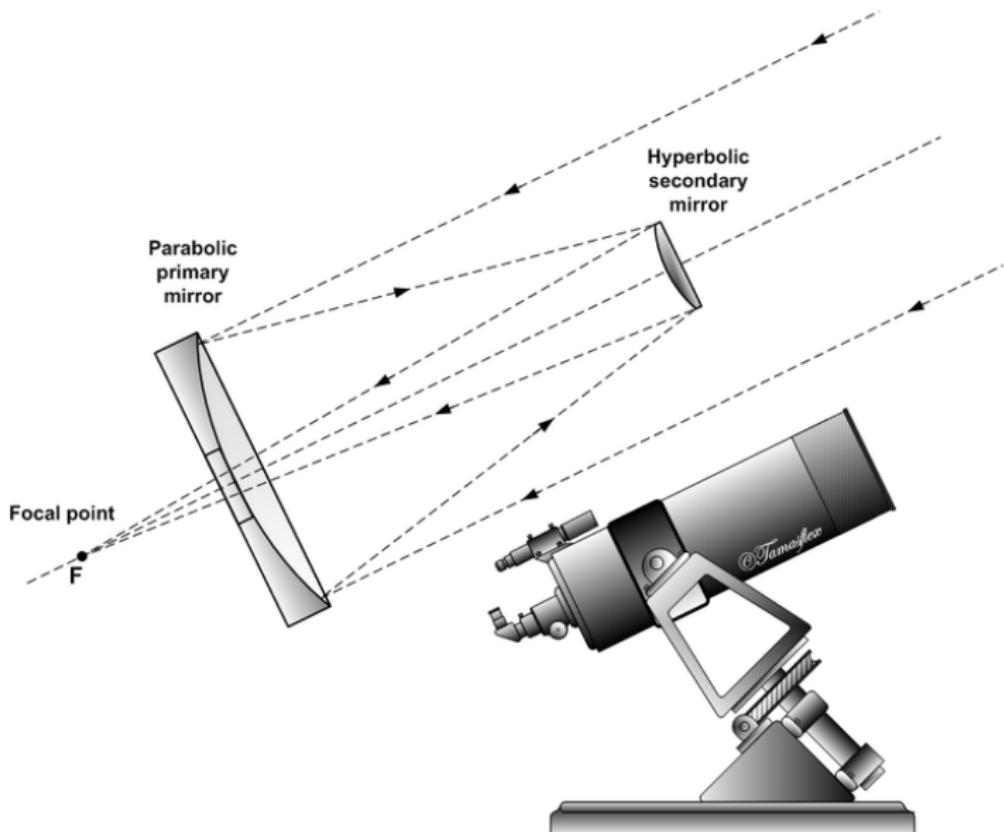
## 9. Conic Sections



## 9. Conic Sections



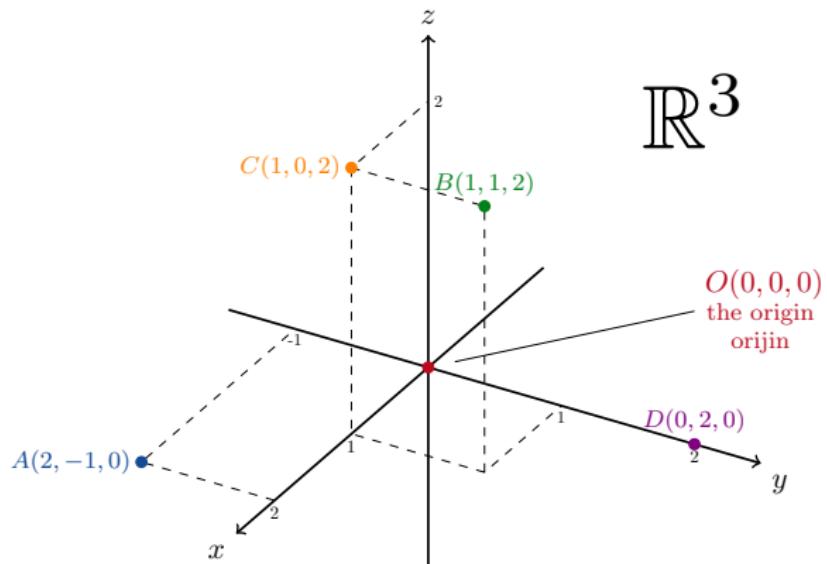
## 9. Conic Sections



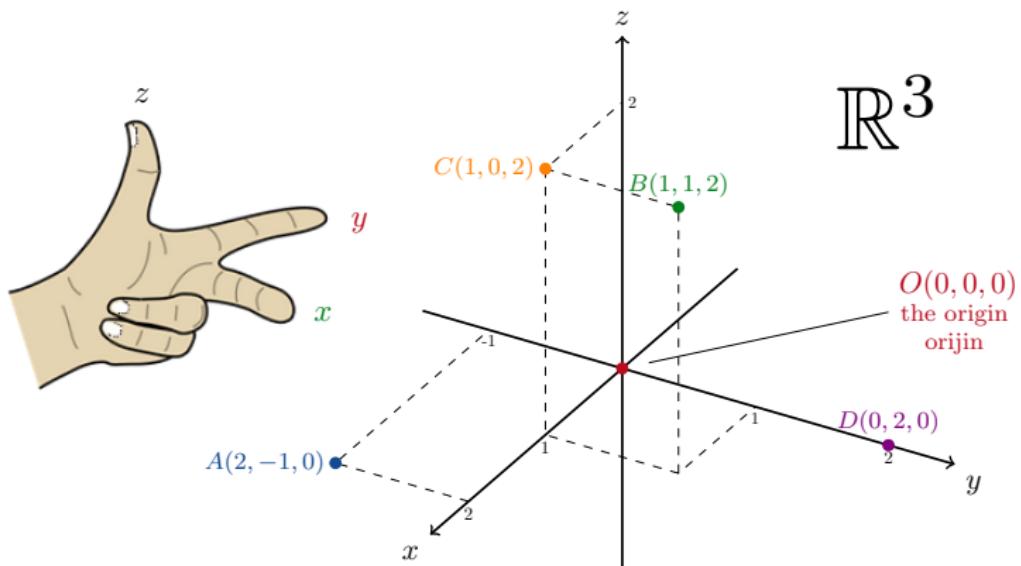


# Three Dimensional Cartesian Coordinates

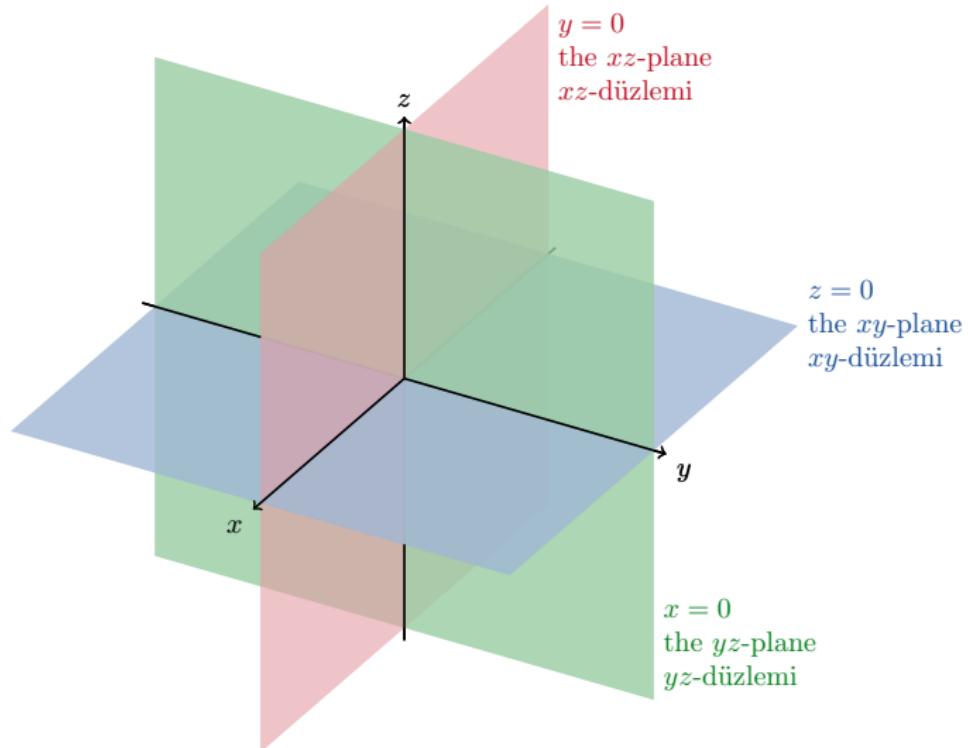
# 10. Three Dimensional Cartesian Coordinates



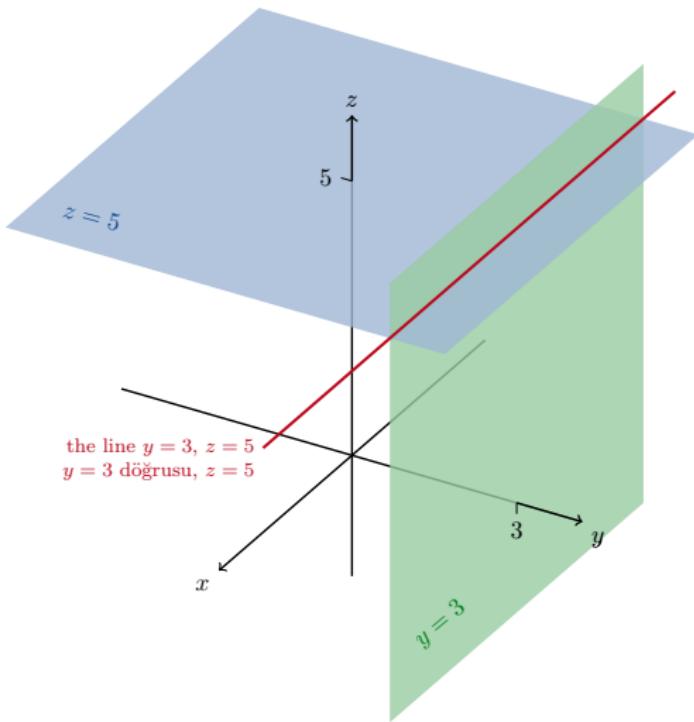
# 10. Three Dimensional Cartesian Coordinates



# 10. Three Dimensional Cartesian Coordinates



# 10. Three Dimensional Cartesian Coordinates



## 10. Three Dimensional Cartesian Coordinates

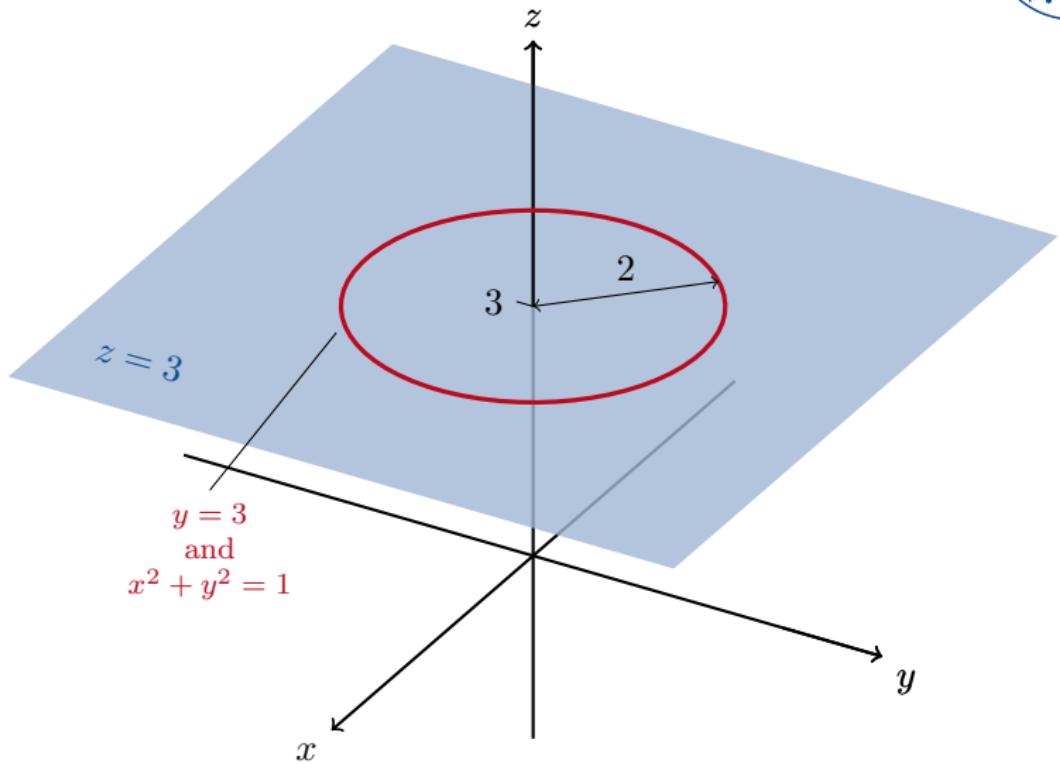


### Example

Which points  $P(x, y, z)$  satisfy  $x^2 + y^2 = 4$  and  $z = 3$ ?

*solution:* We know that  $z = 3$  is a horizontal plane and we recognise that  $x^2 + y^2 = 4$  is the equation of a circle of radius 2. Putting these together, we obtain figure 73.

# 10. Three Dimensional Cartesian Coordinates





## Distance in $\mathbb{R}^3$

### Definition

The set

$$\{(x, y, z) \mid x, y, z \in \mathbb{R}\}$$

is denoted by  $\mathbb{R}^3$ .

# 10. Three Dimensional Cartesian Coordinates



## Definition

The *distance* between  $P_1(x_1, y_1, z_1)$  and  $P_2(x_2, y_2, z_2)$  is

$$\|P_1P_2\| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}.$$

## 10. Three Dimensional Cartesian Coordinates

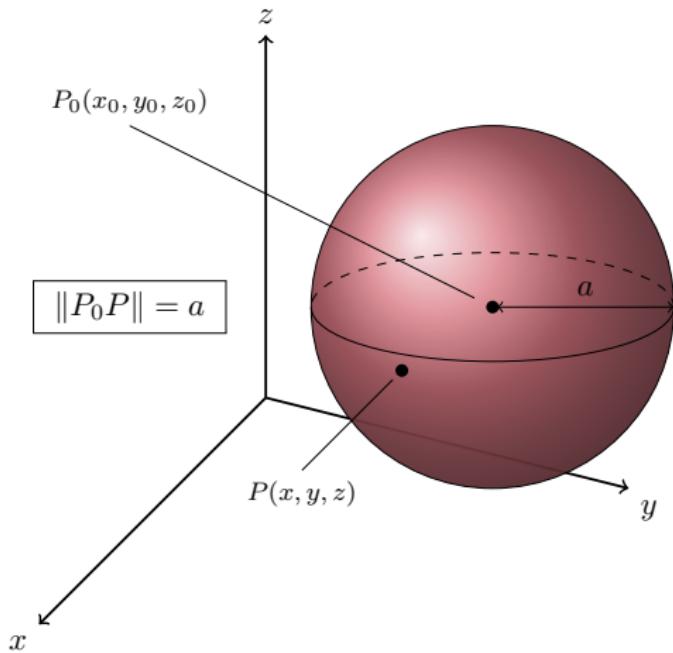


### Example

The distance between  $A(2, 1, 5)$  and  $B(-2, 3, 0)$  is

$$\begin{aligned}\|AB\| &= \sqrt{((-2) - 2)^2 + (3 - 1)^2 + (0 - 5)^2} \\ &= \sqrt{16 + 4 + 25} = \sqrt{45} \\ &= 3\sqrt{5} \approx 6.7.\end{aligned}$$

## Spheres



# 10. Three Dimensional Cartesian Coordinates



## Definition

The *standard equation for a sphere* of radius  $a$  centred at  $P_0(x_0, y_0, z_0)$  is

$$(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2 = a^2.$$

## 10. Three Dimensional Cartesian Coordinates



### Example

Find the centre and radius of the sphere

$$x^2 + y^2 + z^2 + 3x - 4z + 1 = 0.$$

## 10. Three Dimensional Cartesian Coordinates



*solution:* We need to put this equation into the standard form.  
Since  $(x - b)^2 = x^2 - 2bx + b^2$  we have that

$$x^2 + y^2 + z^2 + 3x - 4z + 1 = 0$$

$$(x^2 + 3x) + y^2 + (z^2 - 4z) = -1$$

$$\left(x^2 + 3x + \frac{9}{4}\right) - \frac{9}{4} + y^2 + (z^2 - 4z + 4) - 4 = -1$$

$$\left(x^2 + 3x + \frac{9}{4}\right) + y^2 + (z^2 - 4z + 4) = -1 + \frac{9}{4} + 4$$

$$\left(x + \frac{3}{2}\right)^2 + y^2 + (z - 2)^2 = \frac{21}{4}.$$

The centre is at  $P_0(x_0, y_0, z_0) = P_0(-\frac{3}{2}, 0, 2)$  and the radius is  
 $a = \sqrt{\frac{21}{4}} = \frac{\sqrt{3}\sqrt{7}}{2}$ .



# Next Time

- 11. Vectors
- 12. The Dot Product
- 13. The Cross Product