Welcome to

Mathematics III

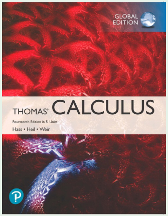
with Dr Neil Course

MATH113 MATH114 MATH215 MATH216

MATH113 MATH114

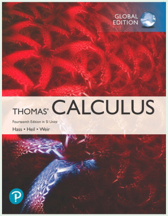
MATH215

MATH216



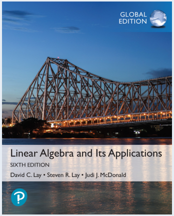
Calculus

MATH113 MATH114



Calculus

MATH215

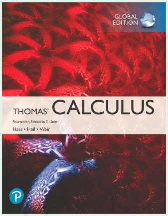


Linear Algebra

MATH216

MATH215 Mathematics III

MATH113 MATH114



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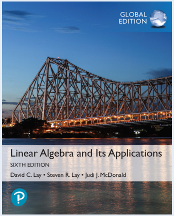

Linear Algebra

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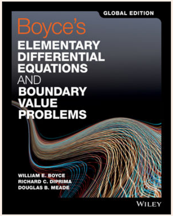
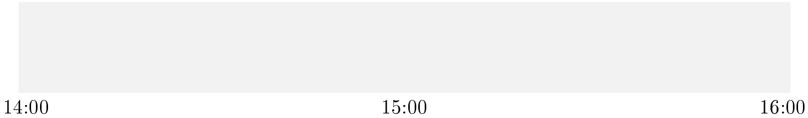


Differential Equations

Information about this course

- ≈ 12 classes. Monday afternoons 2pm-4pm.



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- $\approx$ 12 classes. Monday afternoons 2pm-4pm.
- 2 lectures with a break between.

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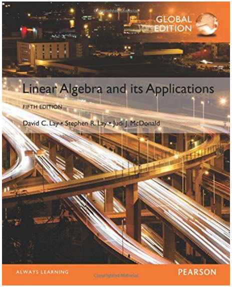
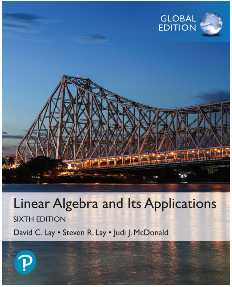
- 2 lectures with a break between.
- Then I will answers your questions.

The Book





David C. Lay, Steven R. Lay and Judi J, McDonald,
Linear Algebra and Its Applications,
5th or 6th Edition, Pearson.

This is a required purchase.

You need to have this book to be able to do the homework.

Syllabus

- linear systems and their solutions
- matrices
- determinants
- inverse matrices
- properties of determinants
- Cramer's Rule
- vector spaces
- subspaces
- linear independence
- basis
- row space
- column space
- null space
- rank and nullity
- linear transformations
- eigenvalues and eigenvectors
- diagonalisation
- inner product spaces
- orthogonality
- the Gram-Schmidt process
- least squares
- orthogonal diagonalisation
- singular value decomposition

Exams and homework

(This information may change based on the University's decisions)

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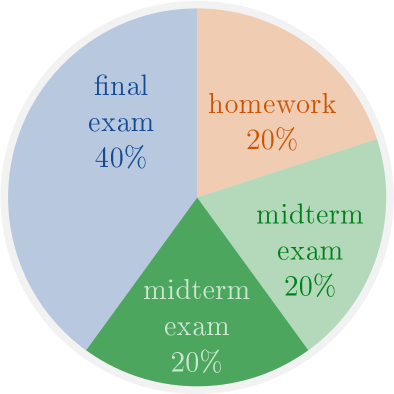

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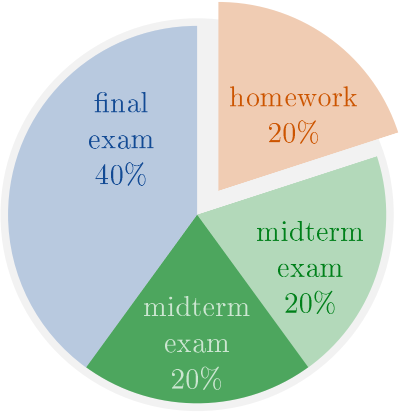

using Pearson MyLab Math

one piece of homework for each lesson

deadline = end of term

more details in O'Learn

Exams and homework

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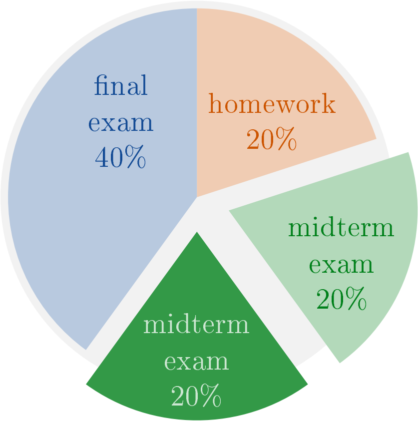

2 midterm exams

face-to-face

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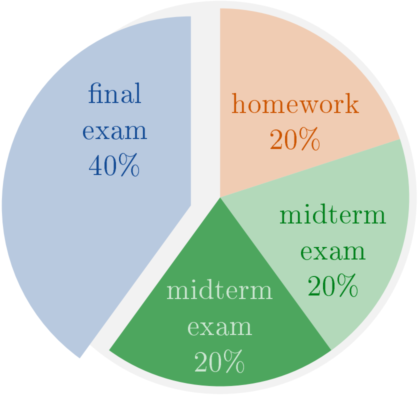
face-to-face



Expectations

If this was a classroom course, the expectation is that for each hour of lectures, you would study an extra 1-2 hours outside of class.

classroom course	lectures (4 hours)	other study (4-8 hours)

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For an online course, you are still expected to study a total of 8-12 hours each week.

online course	class (2 hours)	other study (6-10 hours)

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- Do the online homework on MyLab;

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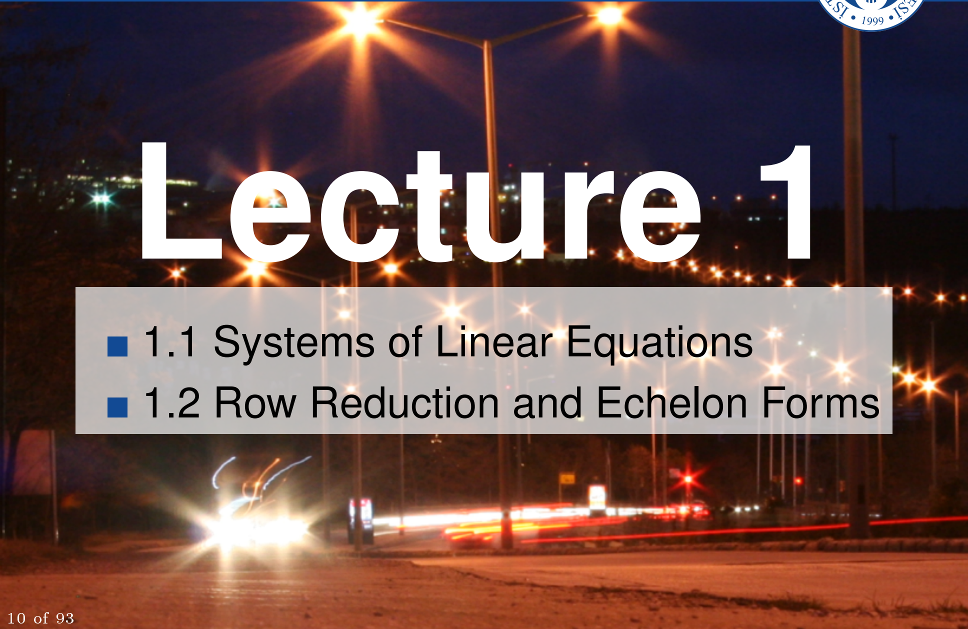
- Read other books?;
- Watch online videos;

$\vdots$

slide number

current section



Lecture 1

- 1.1 Systems of Linear Equations
- 1.2 Row Reduction and Echelon Forms

Systems of Linear Equations

Definition

A *linear equation* is an equation of the form

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n = b \tag{1}$$

where

- $x_1, x_2, \ldots, x_n$ are the variables;
- a_j and b are real or complex numbers; and
- n may be any natural number.

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In this course, n will usually be between 2 and 5. In real-life problems, n might be 50 or 5000, or even larger.

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n =$$

The equations

$$4x_1 - 5x_2 + 2 = x_1$$

and

$$x_2 = 2\left(\sqrt{6} - x_1\right) + x_3$$

are both linear

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and

$$x_2 = 2\left(\sqrt{6} - x_1\right) + x_3$$

are both linear because they can be rearranged to the standard form

$$3x_1 - 5x_2 = -2$$

and

$$2x_1 + x_2 - x_3 = 2\sqrt{6}.$$

These are equations of straight lines.

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n =$$

The equations

$$4x_1 - 5x_2 = x_1x_2$$

and

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are not linear.

$a_1x_1 + a_2x_2 + \cdots + a_nx_n =$

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$$4x_1 - 5x_2 = x_1x_2$$

and

$$x_2 = 2\sqrt{x_1} - 6$$

are not linear.

Definition

If an equation is not-linear, we say it is *non-linear*.

$a_1x_1 + a_2x_2 + \cdots + a_nx_n =$

Ask the Audience

Are these equations linear or non-linear.

1. $x_1 + x_2 + x_3 + x_4 + x_5 + x_6 = 1$
2. $x_1 + 2x_2 + 3x_3^2 = 4$
3. $x_2 = \cos x_1$
4. $7x_1 - x_3 = 4x_2$
5. $x_1x_2x_3x_4x_5 = 1$

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n =$$

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Are these equations linear or non-linear.

1. $x_1 + x_2 + x_3 + x_4 + x_5 + x_6 = 1$ Linear
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4. $7x_1 - x_3 = 4x_2$ Linear
5. $x_1x_2x_3x_4x_5 = 1$ Non-linear

Remark

This course is about linear equations.

Definition

A *linear system* is a group of several linear equations:

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots a_{mn}x_n = b_m. \end{cases} \tag{2}$$

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For example

$$\begin{cases} 2x_1 - x_2 + 1.5x_3 = 8 \\ x_1 \qquad - \quad 4x_3 = -7 \end{cases}$$

is a linear system.

Definition

A *solution* of a linear system is a list $(s_1, s_2, \ldots, s_n)$ of numbers which satisfies every equation in the linear system.

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because

$$2x_1 - x_2 + 1.5x_3 = 2(5) - (6.5) + 1.5(3) = 8$$

and

$$x_1 - 4x_3 = (5) - 4(3) = 7.$$

Definition

The set of all possible solutions is called the *solution set* of the linear system

Definition

Two linear systems are called *equivalent* if they have the same solution set.

Example

Find the solution set of

$$\begin{cases} x_1 - 2x_2 = -1 \\ -x_1 + 3x_2 = 3. \end{cases}$$

Example

Find the solution set of

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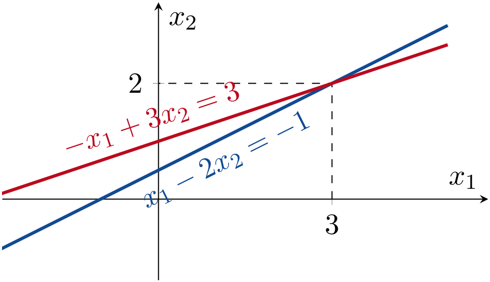
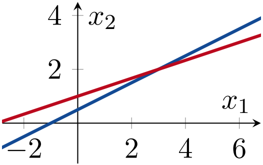
The graphs of these two equations are lines in $\mathbb{R}^2$.



We can see that the only point which is on both lines is $(3, 2)$. So the solution set is the set $\{(3, 2)\}$.

There are three possibilities.

$$\begin{cases} x_1 - 2x_2 = -1 \\ -x_1 + 3x_2 = 3. \end{cases}$$



exactly one solution

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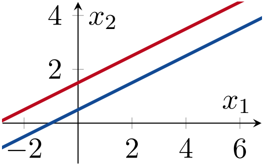
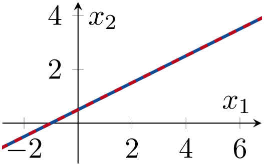


no solutions

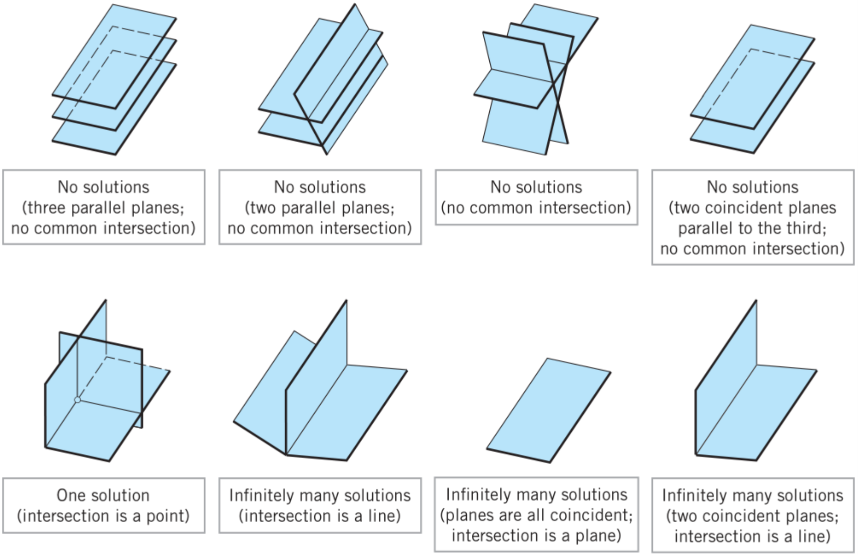
$$\begin{cases} x_1 - 2x_2 = -1 \\ -x_1 + 2x_2 = 1. \end{cases}$$



infinitely many solutions.

1.1 Systems of Linear Equations

This is also true if we have 3 planes.



Theorem

A linear system has either

1. *zero solutions; or*
2. *exactly one solution; or*
3. *infinitely many solutions.*

There are no other possibilities.

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3. *infinitely many solutions.*

There are no other possibilities.

Definition

A linear system is called *consistent* if it has either one solution or infinitely many solutions.

A linear system is called *inconsistent* if it does not have a solution.

Matrix Notation

Consider the linear system

$$\left\{ \begin{array}{rcrcrcr} x_1 & - & 2x_2 & + & x_3 & = & 0 \\ & & 2x_2 & - & 8x_3 & = & 8 \\ -4x_1 & + & 5x_2 & + & 9x_3 & = & -9 \end{array} \right. .$$

What is the important information here?

Matrix Notation

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What is the important information here?

$$\begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ -4 & 5 & 9 & -9 \end{bmatrix}$$

We don't need the x_j's, the $+$'s or the $=$'s. We only need the numbers.

Definition

$$\begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ -4 & 5 & 9 & -9 \end{bmatrix}$$

is the *augmented matrix* for the linear system

$$\begin{cases} 1x_1 & - & 2x_2 & + & 1x_3 & = & 0 \\ & & 2x_2 & - & 8x_3 & = & 8 \\ -4x_1 & + & 5x_2 & + & 9x_3 & = & -9 \end{cases}.$$

Definition

$$\begin{bmatrix} 1 & -2 & 1 \\ 0 & 2 & -8 \\ -4 & 5 & 9 \end{bmatrix}$$

is the *coefficient matrix* for the linear system

$$\left\{ \begin{array}{rcrcrcr} 1x_1 & - & 2x_2 & + & 1x_3 & = & 0 \\ & & 2x_2 & - & 8x_3 & = & 8 \\ -4x_1 & + & 5x_2 & + & 9x_3 & = & -9 \end{array} \right. .$$

1.1 Systems of Linear Equations

There are three things that we can do to a linear system:

1. Multiply an equation by a number;
2. Swap two equations; or
3. Add a multiple of one equation to another.

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Elementary Row Operations

There are three things that we can do to a matrix:

1. Multiply a row by a number;
2. Swap two rows; or
3. Add a multiple of one row to another.

These are called *elementary row operations*.

Solving a Linear System

Example

Solve

$$\begin{cases} 1x_1 - 2x_2 + 1x_3 = 0 \\ \qquad\quad\; 2x_2 - 8x_3 = 8 \\ -4x_1 + 5x_2 + 9x_3 = -9 \end{cases}.$$

Example

Use elementary row operations to convert the augmented matrix

$$\begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ -4 & 5 & 9 & -9 \end{bmatrix}$$

into

$$\begin{bmatrix} 1 & 0 & 0 & ? \\ 0 & 1 & 0 & ? \\ 0 & 0 & 1 & ? \end{bmatrix}.$$

Remark

There are two things about elementary row operations to learn:

1. **How** to use elementary row operations.
2. **Why** we choose a certain elementary row operation at each step.

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1. **How** to use elementary row operations.
2. **Why** we choose a certain elementary row operation at each step.

I will teach the "how" first. Then in the second hour I will show you the "why".

$$\begin{cases} x_1 - 2x_2 + x_3 = 0 \\ 2x_2 - 8x_3 = 8 \\ -4x_1 + 5x_2 + 9x_3 = -9 \end{cases}$$

1

$$\begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ -4 & 5 & 9 & -9 \end{bmatrix}$$

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1

$$\begin{array}{rr} & 4 \cdot [\text{equation } 1] \\ + & [\text{equation } 3] \\ \hline & [\text{new equation } 3] \end{array}$$

$$\begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ -4 & 5 & 9 & -9 \end{bmatrix}$$

1

$$\begin{cases} x_1 - 2x_2 + x_3 = 0 \\ 2x_2 - 8x_3 = 8 \\ -4x_1 + 5x_2 + 9x_3 = -9 \end{cases}$$

1

$$\begin{array}{rr} & 4 \cdot [x_1 - 2x_2 + x_3 = 0] \\ + & [-4x_1 + 5x_2 + 9x_3 = -9] \\ \hline & [\text{new equation 3}] \end{array}$$

$$\begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ -4 & 5 & 9 & -9 \end{bmatrix}$$

1

$$\begin{cases} x_1 - 2x_2 + x_3 = 0 \\ 2x_2 - 8x_3 = 8 \\ -4x_1 + 5x_2 + 9x_3 = -9 \end{cases}$$

1

$$\begin{array}{r} 4x_1 - 8x_2 + 4x_3 = 0 \\ -4x_1 + 5x_2 + 9x_3 = -9 \\ \hline \end{array}$$

$$\begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ -4 & 5 & 9 & -9 \end{bmatrix}$$

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$$\begin{cases} x_1 - 2x_2 + x_3 = 0 \\ \phantom{-4x_1 + {}} 2x_2 - 8x_3 = 8 \\ -4x_1 + 5x_2 + 9x_3 = -9 \end{cases}$$

1

$$\begin{array}{r} 4x_1 - 8x_2 + 4x_3 = 0 \\ -4x_1 + 5x_2 + 9x_3 = -9 \\ \hline 0x_1 - 3x_2 + 13x_3 = -9 \end{array}$$

$$\begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ -4 & 5 & 9 & -9 \end{bmatrix}$$

1

$$\begin{cases} x_1 - 2x_2 + x_3 = 0 \\ \phantom{x_1 -{}} 2x_2 - 8x_3 = 8 \\ -4x_1 + 5x_2 + 9x_3 = -9 \end{cases}$$

1

$$\begin{array}{r} 4x_1 - 8x_2 + 4x_3 = 0 \\ -4x_1 + 5x_2 + 9x_3 = -9 \\ \hline 0x_1 - 3x_2 + 13x_3 = -9 \end{array}$$

$$\begin{cases} x_1 - 2x_2 + x_3 = 0 \\ \phantom{x_1 -{}} 2x_2 - 8x_3 = 8 \\ - 3x_2 + 13x_3 = -9 \end{cases}$$

$$\begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ -4 & 5 & 9 & -9 \end{bmatrix}$$

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$$\begin{cases} x_1 - 2x_2 + x_3 = 0 \\ 2x_2 - 8x_3 = 8 \\ -4x_1 + 5x_2 + 9x_3 = -9 \end{cases}$$

$$\begin{array}{l} \left[\begin{array}{cccc} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ -4 & 5 & 9 & -9 \end{array}\right] \begin{array}{l} \leftarrow R_1 \\ \leftarrow R_2 \\ \leftarrow R_3 \end{array} \end{array}$$

1 $4R_1 + R_3 \rightarrow$ new R_3

$$\begin{array}{r} 4x_1 - 8x_2 + 4x_3 = 0 \\ \underline{-4x_1 + 5x_2 + 9x_3 = -9} \\ 0x_1 - 3x_2 + 13x_3 = -9 \end{array}$$

$$\begin{cases} x_1 - 2x_2 + x_3 = 0 \\ 2x_2 - 8x_3 = 8 \\ -3x_2 + 13x_3 = -9 \end{cases}$$

$$\begin{cases} x_1 - 2x_2 + x_3 = 0 \\ \phantom{x_1 +{}} 2x_2 - 8x_3 = 8 \\ -4x_1 + 5x_2 + 9x_3 = -9 \end{cases}$$

$$\begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ -4 & 5 & 9 & -9 \end{bmatrix} \begin{array}{l} \leftarrow R_1 \\ \leftarrow R_2 \\ \leftarrow R_3 \end{array}$$

1

1 $4R_1 + R_3 \to \text{new } R_3$

$$\begin{array}{r} 4x_1 - 8x_2 + 4x_3 = 0 \\ -4x_1 + 5x_2 + 9x_3 = -9 \\ \hline 0x_1 - 3x_2 + 13x_3 = -9 \end{array}$$

$$\begin{array}{rrrrr} & 4 & -8 & 4 & 0 \\ + & -4 & 5 & 9 & -9 \\ \hline \end{array}$$

$$\begin{cases} x_1 - 2x_2 + x_3 = 0 \\ \phantom{x_1 -{}} 2x_2 - 8x_3 = 8 \\ - 3x_2 + 13x_3 = -9 \end{cases}$$

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$$\begin{array}{rrrrr} & 4 & -8 & 4 & 0 \\ + & -4 & 5 & 9 & -9 \\ \hline & 0 & -3 & 13 & -9 \end{array}$$

$$\begin{cases} x_1 - 2x_2 + x_3 = 0 \\ 2x_2 - 8x_3 = 8 \\ - 3x_2 + 13x_3 = -9 \end{cases}$$

$$\begin{cases} x_1 - 2x_2 + x_3 = 0 \\ \phantom{x_1 - {}} 2x_2 - 8x_3 = 8 \\ -4x_1 + 5x_2 + 9x_3 = -9 \end{cases}$$

$$\begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ -4 & 5 & 9 & -9 \end{bmatrix} \begin{array}{l} \leftarrow R_1 \\ \leftarrow R_2 \\ \leftarrow R_3 \end{array}$$

1 $4R_1 + R_3 \to \text{new } R_3$

$$\begin{array}{r} 4x_1 - 8x_2 + 4x_3 = 0 \\ -4x_1 + 5x_2 + 9x_3 = -9 \\ \hline 0x_1 - 3x_2 + 13x_3 = -9 \end{array}$$

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$$\begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ 0 & -3 & 13 & -9 \end{bmatrix} \begin{array}{l} \\ \\ \leftarrow \text{new } R_3 \end{array}$$

$$\begin{cases} x_1 - 2x_2 + x_3 = 0 \\ \phantom{x_1 - {}} 2x_2 - 8x_3 = 8 \\ - 3x_2 + 13x_3 = -9 \end{cases}$$

2 Multiply [equation 2] by $\frac{1}{2}$.

$$\begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ 0 & -3 & 13 & -9 \end{bmatrix}$$

2 $\frac{1}{2}R_2 \to \text{new } R_2$

$$\begin{cases} x_1 - 2x_2 + x_3 = 0 \\ \phantom{x_1 -{}} 2x_2 - 8x_3 = 8 \\ - 3x_2 + 13x_3 = -9 \end{cases}$$

$$\begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ 0 & -3 & 13 & -9 \end{bmatrix}$$

2 Multiply [equation 2] by $\frac{1}{2}$.

2 $\frac{1}{2}R_2 \rightarrow$ new R_2

$$\begin{cases} x_1 - 2x_2 + x_3 = 0 \\ \phantom{x_1 - 2{}} x_2 - 4x_3 = 4 \\ - 3x_2 + 13x_3 = -9 \end{cases}$$

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2 $\frac{1}{2}R_2 \to$ new R_2

$$\begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 1 & -4 & 4 \\ 0 & -3 & 13 & -9 \end{bmatrix}$$

$$\begin{cases} x_1 - 2x_2 + x_3 = 0 \\ x_2 - 4x_3 = 4 \\ - 3x_2 + 13x_3 = -9 \end{cases}$$

3

$$\begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 1 & -4 & 4 \\ 0 & -3 & 13 & -9 \end{bmatrix}$$

3 $3R_2 + R_3 \to \text{new } R_3$

$$\begin{cases} x_1 - 2x_2 + x_3 = 0 \\ x_2 - 4x_3 = 4 \\ - 3x_2 + 13x_3 = -9 \end{cases}$$

3

$$\begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 1 & -4 & 4 \\ 0 & -3 & 13 & -9 \end{bmatrix}$$

3 $3R_2 + R_3 \rightarrow \text{new } R_3$

$$\begin{array}{ccccc} & 0 & 3 & -12 & 12 \\ + & 0 & -3 & 13 & -9 \\ \hline \end{array}$$

$$\begin{cases} x_1 - 2x_2 + x_3 = 0 \\ x_2 - 4x_3 = 4 \\ - 3x_2 + 13x_3 = -9 \end{cases}$$

3

$$\begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 1 & -4 & 4 \\ 0 & -3 & 13 & -9 \end{bmatrix}$$

3 $3R_2 + R_3 \to \text{new } R_3$

$$\begin{array}{ccccc} & 0 & 3 & -12 & 12 \\ + & 0 & -3 & 13 & -9 \\ \hline & 0 & 0 & 1 & 3 \end{array}$$

$$\begin{cases} x_1 - 2x_2 + \quad x_3 = 0 \\ \qquad\quad x_2 - \ 4x_3 = 4 \\ \quad -3x_2 + 13x_3 = -9 \end{cases}$$

3

$$\begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 1 & -4 & 4 \\ 0 & -3 & 13 & -9 \end{bmatrix}$$

3 $3R_2 + R_3 \to \text{new } R_3$

$$\begin{array}{rrrrr} & 0 & 3 & -12 & 12 \\ + & 0 & -3 & 13 & -9 \\ \hline & 0 & 0 & 1 & 3 \end{array}$$

$$\begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 1 & -4 & 4 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

$$\begin{cases} x_1 - 2x_2 + x_3 = 0 \\ x_2 - 4x_3 = 4 \\ - 3x_2 + 13x_3 = -9 \end{cases}$$

3 $3\times\mathrm{Eq}2 + \mathrm{Eq}3 \rightarrow$ new Eq3.

$$\begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 1 & -4 & 4 \\ 0 & -3 & 13 & -9 \end{bmatrix}$$

3 $3R_2 + R_3 \rightarrow \text{new } R_3$

$$\begin{array}{rrrrr} & 0 & 3 & -12 & 12 \\ + & 0 & -3 & 13 & -9 \\ \hline & 0 & 0 & 1 & 3 \end{array}$$

$$\begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 1 & -4 & 4 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

$$\begin{cases} x_1 - 2x_2 + x_3 = 0 \\ x_2 - 4x_3 = 4 \\ - 3x_2 + 13x_3 = -9 \end{cases}$$

$$\begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 1 & -4 & 4 \\ 0 & -3 & 13 & -9 \end{bmatrix}$$

3. $3\times\text{Eq2} + \text{Eq3} \to \text{new Eq3}$.

$$\begin{array}{r} 3x_2 - 12x_3 = 12 \\ \underline{-3x_2 + 13x_3 = -9} \end{array}$$

3. $3R_2 + R_3 \to \text{new } R_3$

$$\begin{array}{ccccc} & 0 & 3 & -12 & 12 \\ + & 0 & -3 & 13 & -9 \\ \hline & 0 & 0 & 1 & 3 \end{array}$$

$$\begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 1 & -4 & 4 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

$$\begin{cases} x_1 - 2x_2 + \quad x_3 = 0 \\ \qquad\quad x_2 - \ 4x_3 = 4 \\ \quad - 3x_2 + 13x_3 = -9 \end{cases}$$

3. $3\times$Eq2 + Eq3 $\to$ new Eq3.

$$\begin{array}{r} 3x_2 - 12x_3 = 12 \\ \underline{-3x_2 + 13x_3 = -9} \\ 0x_2 + \quad x_3 = 3 \end{array}$$

$$\begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 1 & -4 & 4 \\ 0 & -3 & 13 & -9 \end{bmatrix}$$

3. $3R_2 + R_3 \to$ new R_3

$$\begin{array}{rrrrr} & 0 & 3 & -12 & 12 \\ + & 0 & -3 & 13 & -9 \\ \hline & 0 & 0 & 1 & 3 \end{array}$$

$$\begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 1 & -4 & 4 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

$$\begin{cases} x_1 - 2x_2 + \quad x_3 = 0 \\ \qquad\quad x_2 - \ 4x_3 = 4 \\ \quad - 3x_2 + 13x_3 = -9 \end{cases}$$

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$$\begin{array}{r} 3x_2 - 12x_3 = 12 \\ \underline{-3x_2 + 13x_3 = -9} \\ 0x_2 + \quad x_3 = 3 \end{array}$$

$$\begin{cases} x_1 - 2x_2 + \ x_3 = 0 \\ \qquad\quad x_2 - 4x_3 = 4 \\ \qquad\qquad\quad x_3 = 3 \end{cases}$$

3 $3R_2 + R_3 \rightarrow$ new R_3

$$\begin{array}{ccccc} & 0 & 3 & -12 & 12 \\ + & 0 & -3 & 13 & -9 \\ \hline & 0 & 0 & 1 & 3 \end{array}$$

$$\begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 1 & -4 & 4 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

$$\begin{cases} x_1 - 2x_2 + \ \ x_3 = 0 \\ \qquad\quad\ \ x_2 - 4x_3 = 4 \\ \qquad\qquad\qquad\ \ x_3 = 3 \end{cases}$$

4 Eq2 + 4×Eq3 → new Eq2.

$$\begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 1 & -4 & 4 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

4 $R_2 + 4R_3 \to$ new R_2

$$\begin{cases} x_1 - 2x_2 + x_3 = 0 \\ x_2 - 4x_3 = 4 \\ x_3 = 3 \end{cases}$$

4 Eq2 + 4×Eq3 → new Eq2.

$$\begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 1 & -4 & 4 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

4 $R_2 + 4R_3 \rightarrow$ new R_2

$$\begin{array}{rrrrr} & 0 & 1 & -4 & 4 \\ + & 0 & 0 & 4 & 12 \\ \hline & 0 & 1 & 0 & 16 \end{array}$$

$$\begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 1 & 0 & 16 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

$$\begin{cases} x_1 - 2x_2 + x_3 = 0 \\ x_2 - 4x_3 = 4 \\ x_3 = 3 \end{cases}$$

$$\begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 1 & -4 & 4 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

4 Eq2 + 4×Eq3 → new Eq2.

4 $R_2 + 4R_3 \to \text{new } R_2$

$$\begin{array}{rr} x_2 - 4x_3 &= 4 \\ 4x_3 &= 12 \\ \hline x_2 &= 16 \end{array}$$

$$\begin{array}{ccccc} & 0 & 1 & -4 & 4 \\ + & 0 & 0 & 4 & 12 \\ \hline & 0 & 1 & 0 & 16 \end{array}$$

$$\begin{cases} x_1 - 2x_2 + x_3 = 0 \\ x_2 = 16 \\ x_3 = 3 \end{cases}$$

$$\begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 1 & 0 & 16 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

$$\begin{cases} x_1 - 2x_2 + x_3 = 0 \\ x_2 - 4x_3 = 4 \\ x_3 = 3 \end{cases}$$

5. Eq1 − Eq3 → new Eq1.

$$\begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 1 & 0 & 16 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

5. $R_1 - R_3 \to \text{new } R_1$

$$\begin{cases} x_1 - 2x_2 + \ \ x_3 = 0 \\ \qquad\quad x_2 - 4x_3 = 4 \\ \qquad\qquad\quad\ \ x_3 = 3 \end{cases}$$

5 Eq1 $-$ Eq3 $\rightarrow$ new Eq1.

$$\begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 1 & 0 & 16 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

5 $R_1 - R_3 \rightarrow$ new R_1

$$\begin{array}{rrrrr} & 1 & -2 & 1 & 0 \\ + & 0 & 0 & -1 & -3 \\ \hline & 1 & -2 & 0 & -3 \end{array}$$

$$\begin{bmatrix} 1 & -2 & 0 & -3 \\ 0 & 1 & 0 & 16 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

$$\begin{cases} x_1 - 2x_2 + x_3 = 0 \\ x_2 - 4x_3 = 4 \\ x_3 = 3 \end{cases}$$

$$\begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 1 & 0 & 16 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

5 Eq1 $-$ Eq3 $\rightarrow$ new Eq1.

$$\begin{array}{r} x_1 - 2x_2 + x_3 = 0 \\ -\, x_3 = -3 \\ \hline x_1 - 2x_2 = -3 \end{array}$$

$$\begin{cases} x_1 - 2x_2 = -3 \\ x_2 = 16 \\ x_3 = 3 \end{cases}$$

5 $R_1 - R_3 \rightarrow$ new R_1

$$\begin{array}{ccccc} & 1 & -2 & 1 & 0 \\ + & 0 & 0 & -1 & -3 \\ \hline & 1 & -2 & 0 & -3 \end{array}$$

$$\begin{bmatrix} 1 & -2 & 0 & -3 \\ 0 & 1 & 0 & 16 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

$$\begin{cases} x_1 - 2x_2 = -3 \\ x_2 - 4x_3 = 4 \\ x_3 = 3 \end{cases}$$

6 Eq1 + 2×Eq2 → new Eq1.

$$\begin{bmatrix} 1 & -2 & 0 & -3 \\ 0 & 1 & 0 & 16 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

6 $R_1 + 2R_2 \rightarrow \text{new } R_1$

$$\begin{cases} x_1 - 2x_2 \qquad\qquad = -3 \\ \qquad\quad x_2 - 4x_3 = 4 \\ \qquad\qquad\qquad x_3 = 3 \end{cases}$$

6 Eq1 + 2×Eq2 → new Eq1.

$$\begin{bmatrix} 1 & -2 & 0 & -3 \\ 0 & 1 & 0 & 16 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

6 $R_1 + 2R_2 \to \text{new } R_1$

$$\begin{array}{ccccc} & 1 & -2 & 0 & -3 \\ + & 0 & 2 & 0 & 32 \\ \hline & 1 & 0 & 0 & 29 \end{array}$$

$$\begin{bmatrix} 1 & 0 & 0 & 29 \\ 0 & 1 & 0 & 16 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

$$\begin{cases} x_1 - 2x_2 = -3 \\ x_2 - 4x_3 = 4 \\ x_3 = 3 \end{cases}$$

6 Eq1 + 2×Eq2 → new Eq1.

$$\begin{array}{r} x_1 - 2x_2 = -3 \\ 2x_2 = 32 \\ \hline \color{green}{x_1 \quad\quad = 29} \end{array}$$

$$\begin{cases} x_1 = 29 \\ x_2 = 16 \\ x_3 = 3 \end{cases}$$

$$\begin{bmatrix} 1 & -2 & 0 & -3 \\ 0 & 1 & 0 & 16 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

6 $R_1 + 2R_2 \to \text{new } R_1$

$$\begin{array}{ccccc} & 1 & -2 & 0 & -3 \\ + & 0 & 2 & 0 & 32 \\ \hline & \color{green}{1} & \color{green}{0} & \color{green}{0} & \color{green}{29} \end{array}$$

$$\begin{bmatrix} \color{green}{1} & \color{green}{0} & \color{green}{0} & \color{green}{29} \\ 0 & 1 & 0 & 16 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

Example

Solve

$$\begin{cases} 1x_1 & - & 2x_2 & + & 1x_3 & = & 0 \\ & & 2x_2 & - & 8x_3 & = & 8 \\ -4x_1 & + & 5x_2 & + & 9x_3 & = & -9 \end{cases}.$$

$$\begin{cases} x_1 = 29 \\ x_2 = 16 \\ x_3 = 3 \end{cases}$$

Example

Use elementary row operations to convert the augmented matrix

$$\begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ -4 & 5 & 9 & -9 \end{bmatrix} \quad \text{into} \quad \begin{bmatrix} 1 & 0 & 0 & ? \\ 0 & 1 & 0 & ? \\ 0 & 0 & 1 & ? \end{bmatrix}.$$

$$\begin{bmatrix} 1 & 0 & 0 & 29 \\ 0 & 1 & 0 & 16 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

Definition

Two matrices A and B are called *row equivalent* if it is possible to change one into the other using elementary row operations.

We write $A \sim B$ if A and B are row equivalent.

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Two matrices A and B are called *row equivalent* if it is possible to change one into the other using elementary row operations.

We write $A \sim B$ if A and B are row equivalent.

Theorem

If the augmented matrices of two linear systems are row equivalent, then the two systems have the same solution set.

Elementary Row Operations (again)

1. Multiply a row by a number (e.g. $cR_3 \to \text{new}R_3$).
2. Swap two rows (e.g. $R_1 \leftrightarrow R_2$).
3. Add a multiple of one row to another (e.g. $cR_3 + R_4 \to \text{new}R_4$).

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Note that all three of these are invertible.

1. The opposite of multiplying a row by $c \neq 0$ is multiplying it by $\frac{1}{c}$.

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Note that all three of these are invertible.

1. The opposite of multiplying a row by $c \neq 0$ is multiplying it by $\frac{1}{c}$.
2. Swapping two rows is it's own inverse.
3. The opposite of adding something, is subtracting that something.

Existence and Uniqueness Questions

Example

Recall that

$$\begin{cases} x_1 - 2x_2 + x_3 = 0 \\ 2x_2 - 8x_3 = 8 \\ -4x_1 + 5x_2 + 9x_3 = -9 \end{cases} \qquad \text{has solution} \qquad \begin{cases} x_1 = 29 \\ x_2 = 16 \\ x_3 = 3. \end{cases}$$

Therefore this linear system is consistent.

Example

Is

$$\begin{cases} x_2 - 4x_3 = 8 \\ 2x_1 - 3x_2 + 2x_3 = 1 \\ 5x_1 - 8x_2 + 7x_3 = 1 \end{cases}$$

consistent?

Example

Is

$$\begin{cases} x_2 - 4x_3 = 8 \\ 2x_1 - 3x_2 + 2x_3 = 1 \\ 5x_1 - 8x_2 + 7x_3 = 1 \end{cases}$$

consistent?

To answer this, we will do elementary row operations on the augmented matrix

$$\begin{bmatrix} 0 & 1 & -4 & 8 \\ 2 & -3 & 2 & 1 \\ 5 & -8 & 7 & 1 \end{bmatrix}$$

and see what we get.

$$\begin{bmatrix} 0 & 1 & -4 & 8 \\ 2 & -3 & 2 & 1 \\ 5 & -8 & 7 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 1 & -4 & 8 \\ 2 & -3 & 2 & 1 \\ 5 & -8 & 7 & 1 \end{bmatrix}$$

1. $R_1 \leftrightarrow R_2$

$$\begin{bmatrix} 2 & -3 & 2 & 1 \\ 0 & 1 & -4 & 8 \\ 5 & -8 & 7 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 1 & -4 & 8 \\ 2 & -3 & 2 & 1 \\ 5 & -8 & 7 & 1 \end{bmatrix}$$

1. $R_1 \leftrightarrow R_2$

$$\begin{bmatrix} 2 & -3 & 2 & 1 \\ 0 & 1 & -4 & 8 \\ 5 & -8 & 7 & 1 \end{bmatrix}$$

2. $R_3 - \frac{5}{2}R_1 \rightarrow R_3$

$$\begin{array}{rrrrr} & 5 & -8 & 7 & 1 \\ + & -5 & \frac{15}{2} & -5 & -\frac{5}{2} \\ \hline & 0 & -\frac{1}{2} & 2 & -\frac{3}{2} \end{array}$$

$$\begin{bmatrix} 2 & -3 & 2 & 1 \\ 0 & 1 & -4 & 8 \\ 0 & -\frac{1}{2} & 2 & -\frac{3}{2} \end{bmatrix}$$

$$\begin{bmatrix} 2 & -3 & 2 & 1 \\ 0 & 1 & -4 & 8 \\ 0 & -\frac{1}{2} & 2 & -\frac{3}{2} \end{bmatrix}$$

$$\begin{bmatrix} 2 & -3 & 2 & 1 \\ 0 & 1 & -4 & 8 \\ 0 & -\frac{1}{2} & 2 & -\frac{3}{2} \end{bmatrix}$$

3. $\frac{1}{2}R_2 + R_3 \to R_3$

$$\begin{array}{ccccc} & 0 & \frac{1}{2} & -2 & 4 \\ + & 0 & -\frac{1}{2} & 2 & -\frac{3}{2} \\ \hline & 0 & 0 & 0 & \frac{5}{2} \end{array}$$

$$\begin{bmatrix} 2 & -3 & 2 & 1 \\ 0 & 1 & -4 & 8 \\ 0 & 0 & 0 & \frac{5}{2} \end{bmatrix}$$

$$\begin{bmatrix} 2 & -3 & 2 & 1 \\ 0 & 1 & -4 & 8 \\ 0 & \color{red}{-\frac{1}{2}} & 2 & -\frac{3}{2} \end{bmatrix}$$

3. $\frac{1}{2}R_2 + R_3 \to R_3$

$$\begin{array}{ccccc} & 0 & \frac{1}{2} & -2 & 4 \\ + & 0 & -\frac{1}{2} & 2 & -\frac{3}{2} \\ \hline & 0 & 0 & 0 & \frac{5}{2} \end{array} \qquad \begin{bmatrix} 2 & -3 & 2 & 1 \\ 0 & 1 & -4 & 8 \\ 0 & 0 & 0 & \frac{5}{2} \end{bmatrix}$$

4. Convert back to a linear system.

$$\begin{cases} 2x_1 - 3x_2 + 2x_3 = 1 \\ \qquad\quad x_2 - 4x_3 = 8 \\ 0x_1 + 0x_2 + 0x_3 = \frac{5}{2}. \end{cases}$$

$$\begin{cases} \quad\quad\quad\; x_2 - 4x_3 = 8 \\ 2x_1 - 3x_2 + 2x_3 = 1 \\ 5x_1 - 8x_2 + 7x_3 = 1 \end{cases} \quad \rightarrow \quad \begin{cases} 2x_1 - 3x_2 + 2x_3 & = 1 \\ \quad\quad\quad\; x_2 - 4x_3 & = 8 \\ & 0 = \frac{5}{2}. \end{cases}$$

But the third equation says

$$0 = \frac{5}{2}$$

which is clearly nonsense.

$$\begin{cases} \quad\quad\quad\quad x_2 - 4x_3 = 8 \\ 2x_1 - 3x_2 + 2x_3 = 1 \\ 5x_1 - 8x_2 + 7x_3 = 1 \end{cases} \quad \rightarrow \quad \begin{cases} 2x_1 - 3x_2 + 2x_3 & = 1 \\ \quad\quad\quad x_2 - 4x_3 & = 8 \\ & 0 = \frac{5}{2}. \end{cases}$$

But the third equation says

$$0 = \frac{5}{2}$$

which is clearly nonsense.

This means that it is not possible to find a solution to this linear system. So the linear system is inconsistent.

Example

For what values of h and k is the following system consistent?

$$\begin{cases} 2x_1 - x_2 = h \\ -6x_1 + 3x_2 = k \end{cases}$$

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To answer this, we do one row operation on this system:

$$\begin{bmatrix} 2 & -1 & h \\ -6 & 3 & k \end{bmatrix} \xrightarrow{3R_1+R_2} \begin{bmatrix} 2 & -1 & h \\ 0 & 0 & 3h+k \end{bmatrix}.$$

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Obviously, we require that $3h + k = 0$ in order for this system to have solutions.

So this linear system is consistent only when $3h + k = 0$.

Remark

It doesn't matter if you use [ ] or ( ) when you write an augmented matrix.

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}$$

Example

Find an equation involving g, h, and k that makes this augmented matrix correspond to a consistent system:

$$\begin{pmatrix} 1 & -4 & 7 & g \\ 0 & 3 & -5 & h \\ -2 & 5 & -9 & k \end{pmatrix}.$$

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$$\begin{pmatrix} 1 & -4 & 7 & g \\ 0 & 3 & -5 & h \\ -2 & 5 & -9 & k \end{pmatrix} \xrightarrow{2R_1+R_3} \begin{pmatrix} 1 & -4 & 7 & g \\ 0 & 3 & -5 & h \\ 0 & -3 & 5 & 2g+k \end{pmatrix}$$

$$\xrightarrow{R_2+R_3} \begin{pmatrix} 1 & -4 & 7 & g \\ 0 & 3 & -5 & h \\ 0 & 0 & 0 & 2g+h+k \end{pmatrix}.$$

From the last augmented matrix, we see that we must $2g+h+k=0$ for a consistent system.

Example

Determine if the following system is consistent or inconsistent. If it is consistent, find all the solutions.

$$\begin{cases} 3x_1 + 5x_2 - 4x_3 = 7 \\ -3x_1 - 2x_2 + 4x_3 = -1 \\ 6x_1 + x_2 - 8x_3 = -4. \end{cases}$$

$$\begin{pmatrix} 3 & 5 & -4 & 7 \\ -3 & -2 & 4 & -1 \\ 6 & 1 & -8 & -4 \end{pmatrix}$$

$$\begin{pmatrix} 3 & 5 & -4 & 7 \\ -3 & -2 & 4 & -1 \\ 6 & 1 & -8 & -4 \end{pmatrix} \xrightarrow[-2R_1+R_3]{R_1+R_2} \begin{pmatrix} 3 & 5 & -4 & 7 \\ 0 & 3 & 0 & 6 \\ 0 & -9 & 0 & -18 \end{pmatrix}$$

$$\xrightarrow[\frac{1}{3}R_2]{3R_2+R_3} \begin{pmatrix} 3 & 5 & -4 & 7 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\xrightarrow{-5R_2+R_1} \begin{pmatrix} 3 & 0 & -4 & -3 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

The linear system is equivalent to

$$\begin{cases} 3x_1 \qquad\qquad - 4x_3 = -3 \\ \qquad\quad x_2 \qquad\qquad = 2 \end{cases}$$

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Thus $x_1 = -1 + \frac{4}{3}x_3$ and every solution has the form of

$$\begin{aligned} (x_1, x_2, x_3) &= (-1 + \tfrac{4}{3}x_3, 2, x_3) \\ &= x_3(\tfrac{4}{3}, 0, 1) + (-1, 2, 0). \end{aligned}$$

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Here, x_3 can take on any real number so there are infinitely many solutions. Geometrically, the solution set is a line passing through the point $(-1, 2, 0)$ parallel to the vector $(\frac{4}{3}, 0, 1)$.

Example

Do the three planes $\begin{cases} x_1 + 3x_2 + x_3 = 4 \\ \qquad\quad x_2 - x_3 = 1 \\ x_1 + 4x_2 \qquad\ \ = -2 \end{cases}$ have a point in common? Explain.

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$$\begin{pmatrix} 1 & 3 & 1 & 4 \\ 0 & 1 & -1 & 1 \\ 1 & 4 & 0 & -2 \end{pmatrix} \xrightarrow{-R_1+R_3} \begin{pmatrix} 1 & 3 & 1 & 4 \\ 0 & 1 & -1 & 1 \\ 0 & 1 & -1 & -6 \end{pmatrix}$$

$$\xrightarrow{-R_2+R_3} \begin{pmatrix} 1 & 3 & 1 & 4 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & -7 \end{pmatrix}$$

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The last line of this augmented matrix reads

$$0x_1 + 0x_2 + 0x_3 = -7,$$

which, of course, is nonsense.

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The last line of this augmented matrix reads

$$0x_1 + 0x_2 + 0x_3 = -7,$$

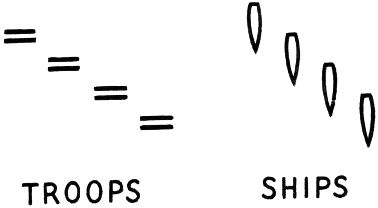
which, of course, is nonsense.

So there is **no** point in common for the three planes.

Break

We will continue at 3pm

Row Reduction and Echelon Forms



An **echelon formation** *is a (usually military) formation in which its units are arranged diagonally.*

Echelon is pronounced "esh-E-lon" because it comes from the French language.

$$\begin{bmatrix} 2 & -3 & 2 & 1 & -7 & 4 \\ 0 & 1 & -4 & 8 & 0 & 0 \\ 0 & 0 & 0 & 5 & 2 & 1 \\ 0 & 0 & 0 & 0 & 0 & 6 \end{bmatrix}$$

Definition

The first non-zero number in each row, is called a *leading entry* or *pivot*.

$$\begin{bmatrix} \overset{\text{leading entry/pivot}}{2} & -3 & 2 & 1 & -7 & 4 \\ 0 & 1 & -4 & 8 & 0 & 0 \\ 0 & 0 & 0 & 5 & 2 & 1 \\ 0 & 0 & 0 & 0 & 0 & 6 \end{bmatrix}$$

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leading entry/pivot

$$\begin{bmatrix} \color{green}{2} & -3 & 2 & 1 & -7 & 4 \\ 0 & \color{green}{1} & -4 & 8 & 0 & 0 \\ 0 & 0 & 0 & 5 & 2 & 1 \\ 0 & 0 & 0 & 0 & 0 & 6 \end{bmatrix}$$

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Definition

A matrix is in *row echelon form* (REF) iff the following 3 rules are satisfied:

$$\begin{bmatrix} 2 & 3 & 2 & 1 & 7 & 4 \\ 0 & 1 & 4 & 8 & 0 & 0 \\ 0 & 0 & 0 & 5 & 2 & 1 \\ 0 & 0 & 0 & 0 & 0 & 6 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{array}{l} \left.\begin{array}{l} \\ \\ \\ \\ \end{array}\right\} \text{non-zero rows at the top} \\ \left.\begin{array}{l} \\ \\ \end{array}\right\} \text{zero rows at the bottom} \end{array}$$

Definition

A matrix is in *row echelon form* (REF) iff the following 3 rules are satisfied:

1. All the $00000\cdots0$ rows are at the bottom;

1.2 Row Reduction and Echelon Forms

$$\begin{bmatrix} 2 & 3 & 2 & 1 & 7 & 4 \\ 0 & 1 & 4 & 8 & 0 & 0 \\ 0 & 0 & 0 & 5 & 2 & 1 \\ 0 & 0 & 0 & 0 & 0 & 6 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

down and right

Definition

A matrix is in *row echelon form* (REF) iff the following 3 rules are satisfied:

1. All the $00000\cdots0$ rows are at the bottom;
2. The leading entries/pivots go $\searrow$; and

zeros under pivots

$$\begin{bmatrix} 2 & 3 & 2 & 1 & 7 & 4 \\ 0 & 1 & 4 & 8 & 0 & 0 \\ 0 & 0 & 0 & 5 & 2 & 1 \\ 0 & 0 & 0 & 0 & 0 & 6 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

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Definition

A matrix is in *reduced row echelon form* (RREF) iff the following 5 rules are satisfied:

1. All the $00000\cdots0$ rows are at the bottom;
2. The leading entries/pivots go $\searrow$; and
3. All the entries below a leading entry are zero.

$$\begin{bmatrix} 1 & 0 & 2 & 0 & 7 & 0 \\ 0 & 1 & 4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

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A matrix is in *reduced row echelon form* (RREF) iff the following 5 rules are satisfied:

1. All the $00000 \cdots 0$ rows are at the bottom;
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3. All the entries below a leading entry are zero.
4. All the leading entries/pivots are 1

$$\begin{bmatrix} 1 & 0 & 2 & 0 & 7 & 0 \\ 0 & 1 & 4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

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1. All the $00000 \cdots 0$ rows are at the bottom;
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4. All the leading entries/pivots are 1
5. Each leading 1 is the only non-zero entry in its column.

$$\begin{bmatrix} 1 & 0 & 2 & 0 & 7 & 0 \\ 0 & 1 & 4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

REF: rules 1-3 RREF: all 5 rules

Example

Consider

$$\begin{bmatrix} 2 & -3 & 2 & 1 \\ 0 & 1 & -4 & 8 \\ 0 & 0 & 0 & 5/2 \end{bmatrix}.$$

Is this matrix in REF? Is this matrix in RREF?

REF: rules 1-3 RREF: all 5 rules

1. All the $00000\cdots0$ rows are at the bottom;
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1. All the $00000\cdots0$ rows are at the bottom; ✓
2. The leading entries/pivots go $\searrow$; and ✓
3. All the entries below a leading entry are zero. ✓
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REF: rules 1-3 RREF: all 5 rules

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Example

Consider

$$\begin{bmatrix} 2 & -3 & 2 & 1 \\ 0 & 1 & -4 & 8 \\ 0 & 0 & 0 & 5/2 \end{bmatrix}.$$

Is this matrix in REF? Is this matrix in RREF?

This matrix is in REF, but is not in RREF.

REF: rules 1-3 RREF: all 5 rules

Example

Consider

$$\begin{bmatrix} 1 & 0 & 0 & -2 & 3 & 0 & 2 \\ 0 & 0 & 1 & 3 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Is this matrix in REF? Is this matrix in RREF?

REF: rules 1-3 RREF: all 5 rules

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Example

Consider

$$\begin{bmatrix} 1 & 0 & 0 & -2 & 3 & 0 & 2 \\ 0 & 0 & 1 & 3 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Is this matrix in REF? Is this matrix in RREF?

This matrix is in REF

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Example

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Is this matrix in REF? Is this matrix in RREF?

This matrix is in REF and in RREF.

REF: rules 1-3 RREF: all 5 rules

Example

Consider

$$\begin{bmatrix} 1 & 0 & 0 & -2 & 3 & 0 & 2 \\ 0 & 0 & 1 & 3 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Is this matrix in REF? Is this matrix in RREF?

This matrix is in REF and in RREF.

In this matrix, the **pivot columns** are the **first**, **third**, and **sixth** columns and the **pivot entries** are the leading nonzero entries in the **first**, **second** and **third** rows.

EXAMPLE 1 The following matrices are in echelon form. The leading entries (■) may have any nonzero value; the starred entries ($*$) may have any value (including zero).

$$\begin{bmatrix} \blacksquare & * & * & * \\ 0 & \blacksquare & * & * \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & \blacksquare & * & * & * & * & * & * & * & * \\ 0 & 0 & 0 & \blacksquare & * & * & * & * & * & * \\ 0 & 0 & 0 & 0 & \blacksquare & * & * & * & * & * \\ 0 & 0 & 0 & 0 & 0 & \blacksquare & * & * & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \blacksquare & * \end{bmatrix}$$

The following matrices are in reduced echelon form because the leading entries are 1's, and there are 0's below *and above* each leading 1.

$$\begin{bmatrix} 1 & 0 & * & * \\ 0 & 1 & * & * \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 1 & * & 0 & 0 & 0 & * & * & 0 & * \\ 0 & 0 & 0 & 1 & 0 & 0 & * & * & 0 & * \\ 0 & 0 & 0 & 0 & 1 & 0 & * & * & 0 & * \\ 0 & 0 & 0 & 0 & 0 & 1 & * & * & 0 & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & * \end{bmatrix}$$

■

Definition

A pivot position in a matrix A is a location in A that corresponds to a leading 1 in the reduced echelon form of A.

Definition

A pivot column is a column of A that contains a pivot position.

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Definition

A pivot column is a column of A that contains a pivot position.

Theorem (Uniqueness of the row reduced echelon form)

Each matrix is row equivalent to one and only one RREF matrix.

Gaussian Elimination

Example

Row reduce the matrix below to row echelon form (REF) and then to reduced row echelon form (RREF):

$$\begin{bmatrix} 0 & -3 & -6 & 4 & 9 \\ -1 & -2 & -1 & 3 & 1 \\ -2 & -3 & 0 & 3 & -1 \\ 1 & 4 & 5 & -9 & -7 \end{bmatrix}.$$

Gaussian Elimination

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Row reduce the matrix below to row echelon form (REF) and then to reduced row echelon form (RREF):

$$\begin{bmatrix} 0 & -3 & -6 & 4 & 9 \\ -1 & -2 & -1 & 3 & 1 \\ -2 & -3 & 0 & 3 & -1 \\ 1 & 4 & 5 & -9 & -7 \end{bmatrix}.$$

So how do we do this? How do we know which elementary row operation to use at each step? We need an algorithm.

- **Step 1:** Find the first non-zero column (from the left).

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- **Step 2:** We don't want a 0 at the top of this column. If necessary swap the first row with another row.

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- **Step 4:** Ignore the row and column containing the pivot. Goto step 1.

- **Step 1:** Find the first non-zero column (from the left).
- **Step 2:** We don't want a 0 at the top of this column. If necessary swap the first row with another row.
- **Step 3:** Create zeros under the pivot.
- **Step 4:** Ignore the row and column containing the pivot. Goto step 1.

Keep going until your matrix is in row echelon form (REF). If you want reduced REF, then continue to step 5.

- **Step 1:** Find the first non-zero column (from the left).
- **Step 2:** We don't want a 0 at the top of this column. If necessary swap the first row with another row.
- **Step 3:** Create zeros under the pivot.
- **Step 4:** Ignore the row and column containing the pivot. Goto step 1.

Keep going until your matrix is in row echelon form (REF). If you want reduced REF, then continue to step 5.

- **Step 5:** Make every pivot a 1 by multiplying rows by constants.

- **Step 1:** Find the first non-zero column (from the left).
- **Step 2:** We don't want a 0 at the top of this column. If necessary swap the first row with another row.
- **Step 3:** Create zeros under the pivot.
- **Step 4:** Ignore the row and column containing the pivot. Goto step 1.

Keep going until your matrix is in row echelon form (REF). If you want reduced REF, then continue to step 5.

- **Step 5:** Make every pivot a 1 by multiplying rows by constants.
- **Step 6:** Starting with the bottom-rightmost pivot and working upwards, create zeros above each pivot.

- **Step 1:** Find the first non-zero column (from the left).
- **Step 2:** We don't want a 0 at the top of this column. If necessary swap the first row with another row.
- **Step 3:** Create zeros under the pivot.
- **Step 4:** Ignore the row and column containing the pivot. Goto step 1.

Keep going until your matrix is in row echelon form (REF). If you want reduced REF, then continue to step 5.

- **Step 5:** Make every pivot a 1 by multiplying rows by constants.
- **Step 6:** Starting with the bottom-rightmost pivot and working upwards, create zeros above each pivot.

Your matrix should now by in RREF.

If we want REF we do steps 1-4. This is called *Gaussian Elimination*.

Carl Friedrich Gauss

BORN
30 April 1777

DECEICED
23 February 1855

NATIONALITY
German

1.2 Row Reduction and Echelon Forms

If we want REF we do steps 1-4. This is called *Gaussian Elimination*.

If we want RREF, we do steps 1-6. This is called *Gauss-Jordan Elimination*.

Wilhelm Jordan

Born
1 March 1842

Deceiced
17 April 1899

Nationality
German

Another name is *row reduction*.

Now let's do this:

Example

Row reduce the matrix below to row echelon form (REF) and then to reduced row echelon form (RREF):

$$\begin{bmatrix} 0 & -3 & -6 & 4 & 9 \\ -1 & -2 & -1 & 3 & 1 \\ -2 & -3 & 0 & 3 & -1 \\ 1 & 4 & 5 & -9 & -7 \end{bmatrix}.$$

- **Step 1:** Find the first non-zero column (from the left).

$$\begin{bmatrix} 0 & -3 & -6 & 4 & 9 \\ -1 & -2 & -1 & 3 & 1 \\ -2 & -3 & 0 & 3 & -1 \\ 1 & 4 & 5 & -9 & -7 \end{bmatrix}.$$

This column is not all zeros.

- **Step 2:** We don't want a 0 at the top of this column. If necessary swap the first row with another row.

I don't want 0 here.

$$\begin{bmatrix} 0 & -3 & -6 & 4 & 9 \\ -1 & -2 & -1 & 3 & 1 \\ -2 & -3 & 0 & 3 & -1 \\ 1 & 4 & 5 & -9 & -7 \end{bmatrix}$$

- **Step 2:** We don't want a 0 at the top of this column. If necessary swap the first row with another row.

I don't want 0 here.

$$\begin{bmatrix} 0 & -3 & -6 & 4 & 9 \\ -1 & -2 & -1 & 3 & 1 \\ -2 & -3 & 0 & 3 & -1 \\ 1 & 4 & 5 & -9 & -7 \end{bmatrix}$$

$R_1 \leftrightarrow R_4$

$$\begin{bmatrix} 1 & 4 & 5 & -9 & -7 \\ -1 & -2 & -1 & 3 & 1 \\ -2 & -3 & 0 & 3 & -1 \\ 0 & -3 & -6 & 4 & 9 \end{bmatrix}$$

- **Step 2:** We don't want a 0 at the top of this column. If necessary swap the first row with another row.

I don't want 0 here.

$$\begin{bmatrix} 0 & -3 & -6 & 4 & 9 \\ -1 & -2 & -1 & 3 & 1 \\ -2 & -3 & 0 & 3 & -1 \\ 1 & 4 & 5 & -9 & -7 \end{bmatrix}$$

pivot/leading entry

$R_1 \leftrightarrow R_4$

$$\begin{bmatrix} 1 & 4 & 5 & -9 & -7 \\ -1 & -2 & -1 & 3 & 1 \\ -2 & -3 & 0 & 3 & -1 \\ 0 & -3 & -6 & 4 & 9 \end{bmatrix}$$

- **Step 3:** Create zeros <u>under</u> the pivot.

$$\begin{bmatrix} 1 & 4 & 5 & -9 & -7 \\ -1 & -2 & -1 & 3 & 1 \\ -2 & -3 & 0 & 3 & -1 \\ 0 & -3 & -6 & 4 & 9 \end{bmatrix}$$

I want $\begin{matrix} 0 \\ 0 \\ 0 \end{matrix}$ here.

- **Step 3:** Create zeros under the pivot.

$$\begin{bmatrix} 1 & 4 & 5 & -9 & -7 \\ -1 & -2 & -1 & 3 & 1 \\ -2 & -3 & 0 & 3 & -1 \\ 0 & -3 & -6 & 4 & 9 \end{bmatrix}$$

I want $\begin{matrix} 0 \\ 0 \\ 0 \end{matrix}$ here.

$$R_1 + R_2 \to R_2 \qquad \begin{bmatrix} 1 & 4 & 5 & -9 & -7 \\ 0 & 2 & 4 & -6 & -6 \\ -2 & -3 & 0 & 3 & -1 \\ 0 & -3 & -6 & 4 & 9 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 4 & 5 & -9 & -7 \\ 0 & 2 & 4 & -6 & -6 \\ -2 & -3 & 0 & 3 & -1 \\ 0 & -3 & -6 & 4 & 9 \end{bmatrix}$$

$$2R_1 + R_3 \to R_3 \qquad \begin{bmatrix} 1 & 4 & 5 & -9 & -7 \\ 0 & 2 & 4 & -6 & -6 \\ 0 & 5 & 10 & -15 & -15 \\ 0 & -3 & -6 & 4 & 9 \end{bmatrix}$$

- **Step 4:** Ignore the row and column containing the pivot. Goto step 1.

$$\begin{bmatrix} 1 & 4 & 5 & -9 & -7 \\ 0 & 2 & 4 & -6 & -6 \\ 0 & 5 & 10 & -15 & -15 \\ 0 & -3 & -6 & 4 & 9 \end{bmatrix}$$

- **Step 4:** Ignore the row and column containing the pivot. Goto step 1.

$$\begin{bmatrix} 1 & 4 & 5 & -9 & -7 \\ 0 & 2 & 4 & -6 & -6 \\ 0 & 5 & 10 & -15 & -15 \\ 0 & -3 & -6 & 4 & 9 \end{bmatrix}$$

ignore

- **Step 4:** Ignore the row and column containing the pivot. Goto step 1.

$$\begin{bmatrix} 1 & 4 & 5 & -9 & -7 \\ 0 & 2 & 4 & -6 & -6 \\ 0 & 5 & 10 & -15 & -15 \\ 0 & -3 & -6 & 4 & 9 \end{bmatrix}$$

ignore

ignore

- **Step 4:** Ignore the row and column containing the pivot. Goto step 1.

$$\begin{bmatrix} 1 & 4 & 5 & -9 & -7 \\ 0 & 2 & 4 & -6 & -6 \\ 0 & 5 & 10 & -15 & -15 \\ 0 & -3 & -6 & 4 & 9 \end{bmatrix}$$

ignore

ignore

$$\begin{bmatrix} 1 & 4 & 5 & -9 & -7 \\ 0 & 2 & 4 & -6 & -6 \\ 0 & 5 & 10 & -15 & -15 \\ 0 & -3 & -6 & 4 & 9 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 4 & 5 & -9 & -7 \\ 0 & 2 & 4 & -6 & -6 \\ 0 & 5 & 10 & -15 & -15 \\ 0 & -3 & -6 & 4 & 9 \end{bmatrix}$$

This column is not all zeros.

I don't want 0 here.

$$\begin{bmatrix} 1 & 4 & 5 & -9 & -7 \\ 0 & 2 & 4 & -6 & -6 \\ 0 & 5 & 10 & -15 & -15 \\ 0 & -3 & -6 & 4 & 9 \end{bmatrix}$$

I want $\begin{matrix}0\\0\end{matrix}$ here.

$$\begin{bmatrix} 1 & 4 & 5 & -9 & -7 \\ 0 & 2 & 4 & -6 & -6 \\ 0 & 5 & 10 & -15 & -15 \\ 0 & -3 & -6 & 4 & 9 \end{bmatrix}$$

I want $\begin{matrix} 0 \\ 0 \end{matrix}$ here.

$$\begin{bmatrix} 1 & 4 & 5 & -9 & -7 \\ 0 & 2 & 4 & -6 & -6 \\ 0 & 5 & 10 & -15 & -15 \\ 0 & -3 & -6 & 4 & 9 \end{bmatrix}$$

$-\frac{5}{2}R_2 + R_3 \to R_3$
$$\begin{bmatrix} 1 & 4 & 5 & -9 & -7 \\ 0 & 2 & 4 & -6 & -6 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & -3 & -6 & 4 & 9 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 4 & 5 & -9 & -7 \\ 0 & 2 & 4 & -6 & -6 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & -3 & -6 & 4 & 9 \end{bmatrix}$$

$\frac{3}{2}R_2 + R_4 \to R_4$

$$\begin{bmatrix} 1 & 4 & 5 & -9 & -7 \\ 0 & 2 & 4 & -6 & -6 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -5 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 4 & 5 & -9 & -7 \\ 0 & 2 & 4 & -6 & -6 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -5 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 4 & 5 & -9 & -7 \\ 0 & 2 & 4 & -6 & -6 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -5 & 0 \end{bmatrix}$$

Ignore R_2 and column 2. Find the first non-zero column.

$$\begin{bmatrix} 1 & 4 & 5 & -9 & -7 \\ 0 & 2 & 4 & -6 & -6 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -5 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 4 & 5 & -9 & -7 \\ 0 & 2 & 4 & -6 & -6 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -5 & 0 \end{bmatrix}$$

Ignore R_2 and column 2. Find the first non-zero column.

$$\begin{bmatrix} 1 & 4 & 5 & -9 & -7 \\ 0 & 2 & 4 & -6 & -6 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -5 & 0 \end{bmatrix}$$

This column is all zeros.

$$\begin{bmatrix} 1 & 4 & 5 & -9 & -7 \\ 0 & 2 & 4 & -6 & -6 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -5 & 0 \end{bmatrix}$$

Ignore R_2 and column 2. Find the first non-zero column.

$$\begin{bmatrix} 1 & 4 & 5 & -9 & -7 \\ 0 & 2 & 4 & -6 & -6 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -5 & 0 \end{bmatrix}$$

This column is not all zeros.

$$\begin{bmatrix} 1 & 4 & 5 & -9 & -7 \\ 0 & 2 & 4 & -6 & -6 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -5 & 0 \end{bmatrix}$$

Ignore R_2 and column 2. Find the first non-zero column.

I don't want 0 here.

$$\begin{bmatrix} 1 & 4 & 5 & -9 & -7 \\ 0 & 2 & 4 & -6 & -6 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -5 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 4 & 5 & -9 & -7 \\ 0 & 2 & 4 & -6 & -6 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -5 & 0 \end{bmatrix}$$

$R_3 \leftrightarrow R_4$

$$\begin{bmatrix} 1 & 4 & 5 & -9 & -7 \\ 0 & 2 & 4 & -6 & -6 \\ 0 & 0 & 0 & -5 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

[1] row echelon form

$$\begin{bmatrix} 1 & 4 & 5 & -9 & -7 \\ 0 & 2 & 4 & -6 & -6 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -5 & 0 \end{bmatrix}$$

$R_3 \leftrightarrow R_4$

$$\begin{bmatrix} 1 & 4 & 5 & -9 & -7 \\ 0 & 2 & 4 & -6 & -6 \\ 0 & 0 & 0 & -5 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

The matrix is now in REF[1].

[1] row echelon form

- **Step 5:** Make every pivot a 1 by multiplying rows by constants.

$$\begin{bmatrix} 1 & 4 & 5 & -9 & -7 \\ 0 & 2 & 4 & -6 & -6 \\ 0 & 0 & 0 & -5 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

- **Step 5:** Make every pivot a 1 by multiplying rows by constants.

pivot

$$\begin{bmatrix} 1 & 4 & 5 & -9 & -7 \\ 0 & 2 & 4 & -6 & -6 \\ 0 & 0 & 0 & -5 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

- **Step 5:** Make every pivot a 1 by multiplying rows by constants.

$$\begin{bmatrix} 1 & 4 & 5 & -9 & -7 \\ 0 & 2 & 4 & -6 & -6 \\ 0 & 0 & 0 & -5 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

pivot columns

- **Step 5:** Make every pivot a 1 by multiplying rows by constants.

$$\begin{bmatrix} 1 & 4 & 5 & -9 & -7 \\ 0 & 2 & 4 & -6 & -6 \\ 0 & 0 & 0 & -5 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{array}{r} \frac{1}{2}R_2 \to R_2 \\ -\frac{1}{5}R_3 \to R_3 \end{array} \qquad \begin{bmatrix} 1 & 4 & 5 & -9 & -7 \\ 0 & 1 & 2 & -3 & -3 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

- **Step 5:** Make every pivot a 1 by multiplying rows by constants.

$$\begin{bmatrix} 1 & 4 & 5 & -9 & -7 \\ 0 & 2 & 4 & -6 & -6 \\ 0 & 0 & 0 & -5 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

WARNING

Be very careful if you do 2 row operations in the same step.

Do not change the same row twice in one step.

$$\begin{array}{l} \frac{1}{2}R_2 \to R_2 \\ -\frac{1}{5}R_3 \to R_3 \end{array} \quad \begin{bmatrix} 1 & 4 & 5 & -9 & -7 \\ 0 & 1 & 2 & -3 & -3 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

- **Step 6:** Starting with the bottom-rightmost pivot and working upwards, create zeros <u>above</u> each pivot.

$$\begin{bmatrix} 1 & 4 & 5 & -9 & -7 \\ 0 & 1 & 2 & -3 & -3 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

- **Step 6:** Starting with the bottom-rightmost pivot and working upwards, create zeros above each pivot.

first make this 0

$$\begin{bmatrix} 1 & 4 & 5 & -9 & -7 \\ 0 & 1 & 2 & -3 & -3 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

- **Step 6:** Starting with the bottom-rightmost pivot and working upwards, create zeros above each pivot.

then make this 0

$$\begin{bmatrix} 1 & 4 & 5 & -9 & -7 \\ 0 & 1 & 2 & -3 & -3 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

- **Step 6:** Starting with the bottom-rightmost pivot and working upwards, create zeros above each pivot.

finally make this 0

$$\begin{bmatrix} 1 & 4 & 5 & -9 & -7 \\ 0 & 1 & 2 & -3 & -3 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

- **Step 6:** Starting with the bottom-rightmost pivot and working upwards, create zeros above each pivot.

$$\begin{bmatrix} 1 & 4 & 5 & -9 & -7 \\ 0 & 1 & 2 & -3 & -3 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$3R_3 + R_2 \to R_2$

$$\begin{bmatrix} 1 & 4 & 5 & -9 & -7 \\ 0 & 1 & 2 & 0 & -3 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 4 & 5 & -9 & -7 \\ 0 & 1 & 2 & 0 & -3 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$9R_3 + R_1 \to R_1$

$$\begin{bmatrix} 1 & 4 & 5 & 0 & -7 \\ 0 & 1 & 2 & 0 & -3 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 4 & 5 & -9 & -7 \\ 0 & 1 & 2 & 0 & -3 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$9R_3 + R_1 \to R_1$

$$\begin{bmatrix} 1 & 4 & 5 & 0 & -7 \\ 0 & 1 & 2 & 0 & -3 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$-4R_2 + R_1 \to R_1$

$$\begin{bmatrix} 1 & 0 & -3 & 0 & 5 \\ 0 & 1 & 2 & 0 & -3 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & -3 & 0 & 5 \\ 0 & 1 & 2 & 0 & -3 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

The matrix is now in RREF[2].

[2] reduced row echelon form

$$\underset{\text{original matrix}}{\begin{bmatrix} 0 & -3 & -6 & 4 & 9 \\ -1 & -2 & -1 & 3 & 1 \\ -2 & -3 & 0 & 3 & -1 \\ 1 & 4 & 5 & -9 & -7 \end{bmatrix}} \xrightarrow[\text{operations}]{\text{row}} \underset{\text{RREF matrix}}{\begin{bmatrix} 1 & 0 & -3 & 0 & 5 \\ 0 & 1 & 2 & 0 & -3 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}}$$

pivots

$$\begin{bmatrix} 0 & -3 & -6 & 4 & 9 \\ -1 & -2 & -1 & 3 & 1 \\ -2 & -3 & 0 & 3 & -1 \\ 1 & 4 & 5 & -9 & -7 \end{bmatrix} \xrightarrow[\text{operations}]{\text{row}} \begin{bmatrix} 1 & 0 & -3 & 0 & 5 \\ 0 & 1 & 2 & 0 & -3 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{array}{c} \text{pivot positions} \\ \begin{bmatrix} 0 & -3 & -6 & 4 & 9 \\ -1 & -2 & -1 & 3 & 1 \\ -2 & -3 & 0 & 3 & -1 \\ 1 & 4 & 5 & -9 & -7 \end{bmatrix} \end{array} \xrightarrow[\text{operations}]{\text{row}} \begin{array}{c} \text{pivots} \\ \begin{bmatrix} 1 & 0 & -3 & 0 & 5 \\ 0 & 1 & 2 & 0 & -3 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

1.2 Row Reduction and Echelon Forms

pivots

pivot positions

$$\begin{bmatrix} 0 & -3 & -6 & 4 & 9 \\ -1 & -2 & -1 & 3 & 1 \\ -2 & -3 & 0 & 3 & -1 \\ 1 & 4 & 5 & -9 & -7 \end{bmatrix} \xrightarrow[\text{operations}]{\text{row}} \begin{bmatrix} 1 & 0 & -3 & 0 & 5 \\ 0 & 1 & 2 & 0 & -3 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

pivot columns

Solutions of Linear Systems

Definition

Variables which correspond to pivot columns are called *basic variables*.

Other variables are called *free variables*.

Example

Consider

$$\begin{bmatrix} 1 & 0 & -5 & 1 \\ 0 & 1 & 1 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad \begin{cases} x_1 \phantom{{}+x_2} - 5x_3 = 1 \\ \phantom{x_1 - {}} x_2 + x_3 = 4 \\ 0 = 0. \end{cases}$$

Example

Consider

$$\begin{bmatrix} 1 & 0 & -5 & 1 \\ 0 & 1 & 1 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad \begin{cases} x_1 \quad\quad - 5x_3 = 1 \\ \quad\quad x_2 + \;\; x_3 = 4 \\ \quad\quad\quad\quad\quad\; 0 = 0. \end{cases}$$

Columns 1 and 2 are the pivot columns.

Therefore x_1 and x_2 are basic variables. x_3 is a free variable.

Example

Consider

$$\begin{bmatrix} 1 & 0 & -5 & 1 \\ 0 & 1 & 1 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad \begin{cases} x_1 \phantom{{}+x_2} - 5x_3 = 1 \\ x_2 + x_3 = 4 \\ 0 = 0. \end{cases}$$

Columns 1 and 2 are the pivot columns.
Therefore x_1 and x_2 are basic variables. x_3 is a free variable.
The general solution to the linear system can be written as

$$\begin{cases} x_1 = 1 + 5x_3 \\ x_2 = 4 - x_3 \\ x_3 \text{ is free.} \end{cases}$$

$$\begin{cases} x_1 = 1 + 5x_3 \\ x_2 = 4 - x_3 \\ x_3 \text{ is free.} \end{cases}$$

The statement "x_3 is free" means that you can choose any value for x_3. After you choose x_3, the values of x_1 and x_2 are determined by the formulae.

$$\begin{cases} x_1 = 1 + 5x_3 \\ x_2 = 4 - x_3 \\ x_3 \text{ is free.} \end{cases}$$

The statement "x_3 is free" means that you can choose any value for x_3. After you choose x_3, the values of x_1 and x_2 are determined by the formulae.

- e.g. if you choose $x_3 = 0$, then you get the solution $(1, 4, 0)$;
- e.g. if you choose $x_3 = 1$, then you get $(6, 3, 1)$.

Example

Find the general solution of the linear system whose augmented matrix has been reduced to

$$\begin{bmatrix} 1 & 6 & 2 & -5 & -2 & -4 \\ 0 & 0 & 2 & -8 & -1 & 3 \\ 0 & 0 & 0 & 0 & 1 & 7 \end{bmatrix}.$$

Example

Find the general solution of the linear system whose augmented matrix has been reduced to

$$\begin{bmatrix} 1 & 6 & 2 & -5 & -2 & -4 \\ 0 & 0 & 2 & -8 & -1 & 3 \\ 0 & 0 & 0 & 0 & 1 & 7 \end{bmatrix}.$$

This matrix is in REF, but we need it in RREF so we will have to do some row reduction.

$$\begin{bmatrix} 1 & 6 & 2 & -5 & -2 & -4 \\ 0 & 0 & 2 & -8 & -1 & 3 \\ 0 & 0 & 0 & 0 & 1 & 7 \end{bmatrix} \xrightarrow[2R_3+R_1]{R_3+R_2} \begin{bmatrix} 1 & 6 & 2 & -5 & 0 & 10 \\ 0 & 0 & 2 & -8 & 0 & 10 \\ 0 & 0 & 0 & 0 & 1 & 7 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 6 & 2 & -5 & -2 & -4 \\ 0 & 0 & 2 & -8 & -1 & 3 \\ 0 & 0 & 0 & 0 & 1 & 7 \end{bmatrix} \xrightarrow[2R_3+R_1]{R_3+R_2} \begin{bmatrix} 1 & 6 & 2 & -5 & 0 & 10 \\ 0 & 0 & 2 & -8 & 0 & 10 \\ 0 & 0 & 0 & 0 & 1 & 7 \end{bmatrix}$$

$$\xrightarrow{\frac{1}{2}R_2} \begin{bmatrix} 1 & 6 & 2 & -5 & 0 & 10 \\ 0 & 0 & 1 & -4 & 0 & 5 \\ 0 & 0 & 0 & 0 & 1 & 7 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 6 & 2 & -5 & -2 & -4 \\ 0 & 0 & 2 & -8 & -1 & 3 \\ 0 & 0 & 0 & 0 & 1 & 7 \end{bmatrix} \xrightarrow[2R_3+R_1]{R_3+R_2} \begin{bmatrix} 1 & 6 & 2 & -5 & 0 & 10 \\ 0 & 0 & 2 & -8 & 0 & 10 \\ 0 & 0 & 0 & 0 & 1 & 7 \end{bmatrix}$$

$$\xrightarrow{\frac{1}{2}R_2} \begin{bmatrix} 1 & 6 & 2 & -5 & 0 & 10 \\ 0 & 0 & 1 & -4 & 0 & 5 \\ 0 & 0 & 0 & 0 & 1 & 7 \end{bmatrix} \xrightarrow{-2R_2+R_1} \begin{bmatrix} 1 & 6 & 0 & 3 & 0 & 0 \\ 0 & 0 & 1 & -4 & 0 & 5 \\ 0 & 0 & 0 & 0 & 1 & 7 \end{bmatrix}.$$

$$\begin{bmatrix} 1 & 6 & 0 & 3 & 0 & 0 \\ 0 & 0 & 1 & -4 & 0 & 5 \\ 0 & 0 & 0 & 0 & 1 & 7 \end{bmatrix}$$

This augmented matrix has 6 columns. That means that the linear system has 5 variables.

$$\begin{cases} x_1 + 6x_2 \qquad + 3x_4 \qquad = 0 \\ \qquad\qquad x_3 - 4x_4 \qquad = 5 \\ \qquad\qquad\qquad\qquad x_5 = 7. \end{cases}$$

$$\begin{bmatrix} 1 & 6 & 0 & 3 & 0 & 0 \\ 0 & 0 & 1 & -4 & 0 & 5 \\ 0 & 0 & 0 & 0 & 1 & 7 \end{bmatrix}$$

This augmented matrix has 6 columns. That means that the linear system has 5 variables.

$$\begin{cases} x_1 + 6x_2 \qquad + 3x_4 \qquad = 0 \\ \qquad\qquad x_3 - 4x_4 \qquad = 5 \\ \qquad\qquad\qquad\qquad x_5 = 7. \end{cases}$$

The pivot columns are the first, third and fifth columns. So the basic variables are x_1, x_3 and x_5. The free variables are x_2 and x_4.

$$\begin{cases} x_1 + 6x_2 + 3x_4 = 0 \\ x_3 - 4x_4 = 5 \\ x_5 = 7. \end{cases}$$

The general solution of the linear system is

$$\begin{cases} x_1 = -6x_2 - 3x_4 \\ x_2 \text{ is free} \\ x_3 = 5 + 4x_4 \\ x_4 \text{ is free} \\ x_5 = 7. \end{cases}$$

Theorem (Existence and Uniqueness)

A linear system is consistent if and only if the rightmost column of the augmented matrix is not a pivot column

Theorem (Existence and Uniqueness)

A linear system is consistent if and only if the rightmost column of the augmented matrix is not a pivot column - that is, if and only if an echelon form of the augmented matrix has no row that looks like

$$[0 \ \cdots \ 0 \ b]$$

for $b \neq 0$.

Theorem (Existence and Uniqueness)

A linear system is consistent if and only if the rightmost column of the augmented matrix is not a pivot column - that is, if and only if an echelon form of the augmented matrix has no row that looks like

$$[0 \ \cdots \ 0 \ b]$$

for $b \neq 0$.

If a linear system is consistent, then the solution set contains either

1. *a unique solution (when there are no free variables); or*
2. *infinitely many solutions (when there is at least on free variable).*

USING ROW REDUCTION TO SOLVE A LINEAR SYSTEM

1. Write the augmented matrix of the system.
2. Use the row reduction algorithm to obtain an equivalent augmented matrix in echelon form. Decide whether the system is consistent. If there is no solution, stop; otherwise, go to the next step.
3. Continue row reduction to obtain the reduced echelon form.
4. Write the system of equations corresponding to the matrix obtained in step 3.
5. Rewrite each nonzero equation from step 4 so that its one basic variable is expressed in terms of any free variables appearing in the equation.

Example

How many solutions does the following linear system have?

$$\begin{cases} \phantom{3x_1 - {}} 3x_2 - 6x_3 + 6x_4 + 4x_5 = -5 \\ 3x_1 - 7x_2 + 8x_3 - 5x_4 + 8x_5 = 9 \\ 3x_1 - 9x_2 + 12x_3 - 9x_4 + 6x_5 = 15. \end{cases}$$

Example

How many solutions does the following linear system have?

$$\begin{cases} \qquad\quad 3x_2 - \ \ 6x_3 + 6x_4 + 4x_5 = -5 \\ 3x_1 - 7x_2 + \ \ 8x_3 - 5x_4 + 8x_5 = 9 \\ 3x_1 - 9x_2 + 12x_3 - 9x_4 + 6x_5 = 15. \end{cases}$$

First we write our augmented matrix

$$\begin{bmatrix} 0 & 3 & -6 & 6 & 4 & -5 \\ 3 & -7 & 8 & -5 & 8 & 9 \\ 3 & -9 & 12 & -9 & 6 & 15 \end{bmatrix}.$$

Then we will reduce it to REF to see if the linear system is consistent.

$$\begin{bmatrix} 0 & 3 & -6 & 6 & 4 & -5 \\ 3 & -7 & 8 & -5 & 8 & 9 \\ 3 & -9 & 12 & -9 & 6 & 15 \end{bmatrix} \xrightarrow{R_1 \leftrightarrow R_3} \begin{bmatrix} 3 & -9 & 12 & -9 & 6 & 15 \\ 3 & -7 & 8 & -5 & 8 & 9 \\ 0 & 3 & -6 & 6 & 4 & -5 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 3 & -6 & 6 & 4 & -5 \\ 3 & -7 & 8 & -5 & 8 & 9 \\ 3 & -9 & 12 & -9 & 6 & 15 \end{bmatrix} \xrightarrow{R_1 \leftrightarrow R_3} \begin{bmatrix} 3 & -9 & 12 & -9 & 6 & 15 \\ 3 & -7 & 8 & -5 & 8 & 9 \\ 0 & 3 & -6 & 6 & 4 & -5 \end{bmatrix}$$

$$\xrightarrow{-R_1 + R_2} \begin{bmatrix} 3 & -9 & 12 & -9 & 6 & 15 \\ 0 & 2 & -4 & 4 & 2 & -6 \\ 0 & 3 & -6 & 6 & 4 & -5 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 3 & -6 & 6 & 4 & -5 \\ 3 & -7 & 8 & -5 & 8 & 9 \\ 3 & -9 & 12 & -9 & 6 & 15 \end{bmatrix} \xrightarrow{R_1 \leftrightarrow R_3} \begin{bmatrix} 3 & -9 & 12 & -9 & 6 & 15 \\ 3 & -7 & 8 & -5 & 8 & 9 \\ 0 & 3 & -6 & 6 & 4 & -5 \end{bmatrix}$$

$$\xrightarrow{-R_1 + R_2} \begin{bmatrix} 3 & -9 & 12 & -9 & 6 & 15 \\ 0 & 2 & -4 & 4 & 2 & -6 \\ 0 & 3 & -6 & 6 & 4 & -5 \end{bmatrix}$$

$$\xrightarrow{-\frac{3}{2} R_2 + R_3} \begin{bmatrix} 3 & -9 & 12 & -9 & 6 & 15 \\ 0 & 2 & -4 & 4 & 2 & -6 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{bmatrix}$$

This matrix is now in REF.

$$\begin{bmatrix} 3 & -9 & 12 & -9 & 6 & 15 \\ 0 & 2 & -4 & 4 & 2 & -6 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{bmatrix}$$

- The pivot columns are the first, second and fifth columns.

$$\begin{bmatrix} 3 & -9 & 12 & -9 & 6 & 15 \\ 0 & 2 & -4 & 4 & 2 & -6 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{bmatrix}$$

- The pivot columns are the first, second and fifth columns.
- Since the sixth column is not a pivot column, the linear system is consistent. So either there is one unique solution or there are infinitely many solutions.

$$\begin{bmatrix} 3 & -9 & 12 & -9 & 6 & 15 \\ 0 & 2 & -4 & 4 & 2 & -6 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{bmatrix}$$

- The pivot columns are the first, second and fifth columns.
- Since the sixth column is not a pivot column, the linear system is consistent. So either there is one unique solution or there are infinitely many solutions.
- The basic variables are x_1, x_2 and x_5.

$$\begin{bmatrix} 3 & -9 & 12 & -9 & 6 & 15 \\ 0 & 2 & -4 & 4 & 2 & -6 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{bmatrix}$$

- The pivot columns are the first, second and fifth columns.
- Since the sixth column is not a pivot column, the linear system is consistent. So either there is one unique solution or there are infinitely many solutions.
- The basic variables are x_1, x_2 and x_5.
- The free variables are x_3 and x_4.

$$\begin{bmatrix} 3 & -9 & 12 & -9 & 6 & 15 \\ 0 & 2 & -4 & 4 & 2 & -6 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{bmatrix}$$

- The pivot columns are the first, second and fifth columns.
- Since the sixth column is not a pivot column, the linear system is consistent. So either there is one unique solution or there are infinitely many solutions.
- The basic variables are x_1, x_2 and x_5.
- The free variables are x_3 and x_4.
- Since there are free variables, there are infinitely many solutions to the linear system.

Next Time

- 1.5 Solution Sets of Linear Systems
- 2.1 Matrix Operations
- 2.2 The Inverse of a Matrix

This is the end of the lecture. Please ask your questions now.
You are welcome to leave whenever you wish.