

# Lecture 8

## ■ 23. Graph Theory



Happy Easter Friday





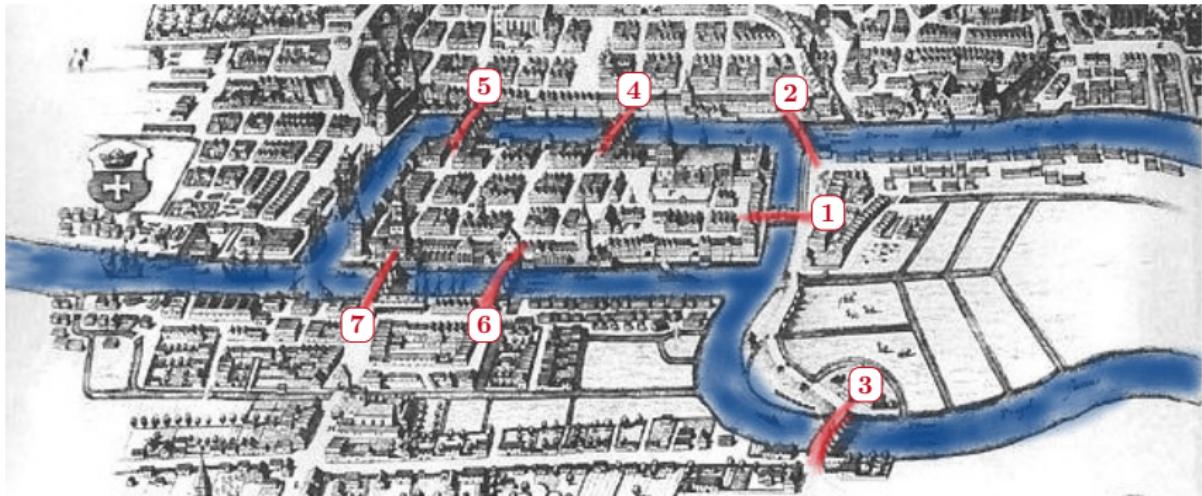
# Graph Theory

## 23. Graph Theory



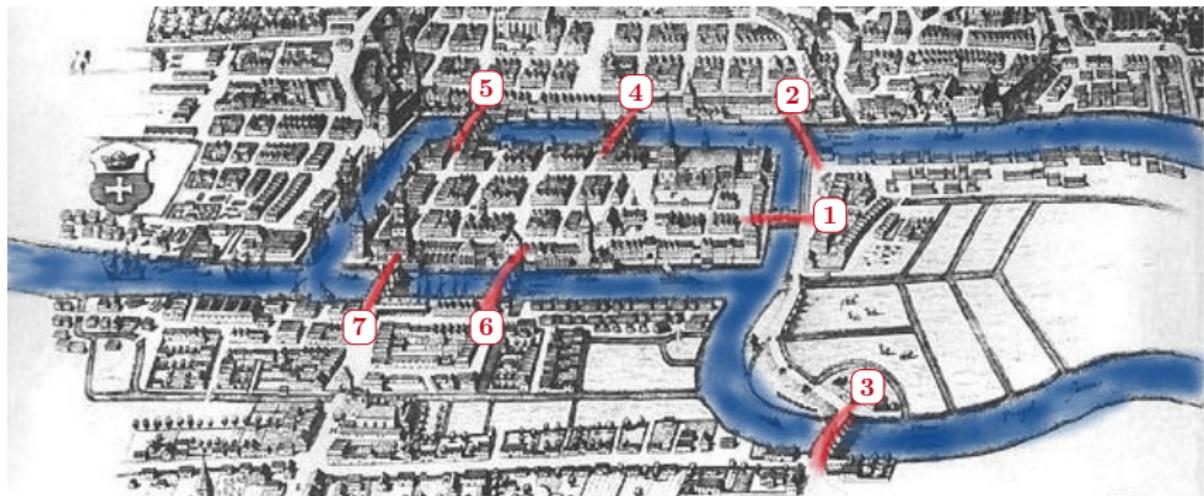
Often when analysing theoretical problems, it is useful to transform the problem into a collection of vertices joined by lines.

## 23. Graph Theory



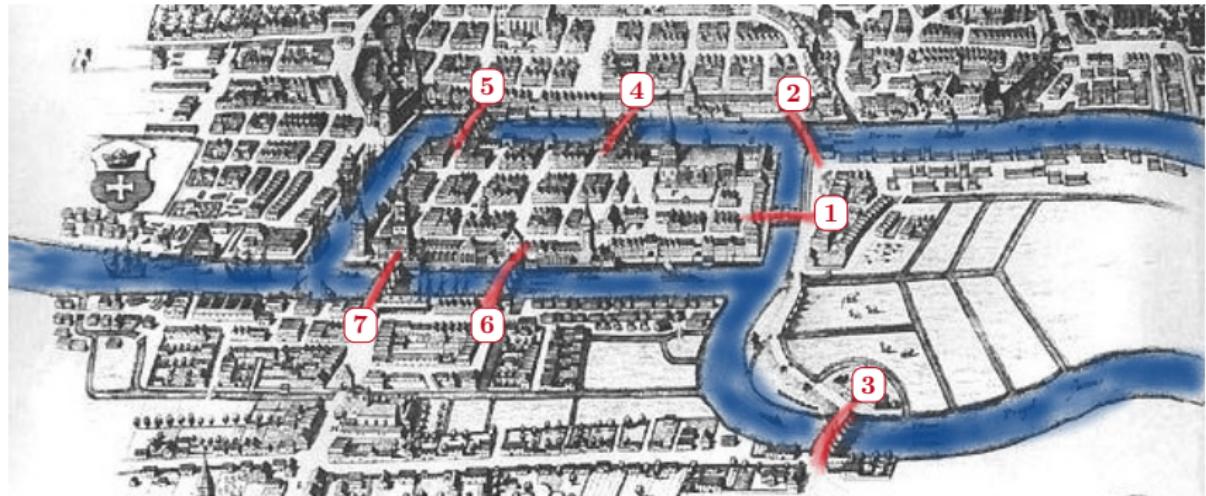
When visiting the city of Königsberg in 1736, the Swiss Mathematician Leonhard Euler (1707-1783) was set a problem by the inhabitants. To solve this problem, he invented a type of Mathematics called Graph Theory.

## 23. Graph Theory



The town of Königsberg in Prussia (now Kaliningrad, Russia) was divided into four landmasses by the Pregel river. There were 7 separate bridges between these landmasses as shown above.

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Visitors were often asked the following problem by the locals:

*Can a person walk around the town and cross each bridge once and only once?*

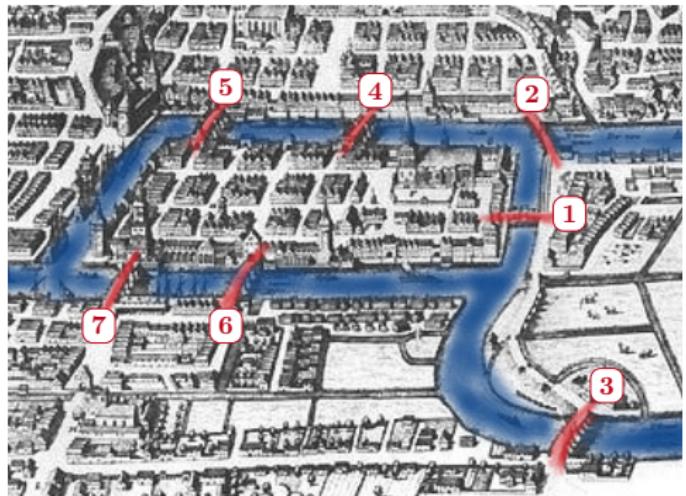
Euler was the first person to solve this problem.

## 23. Graph Theory

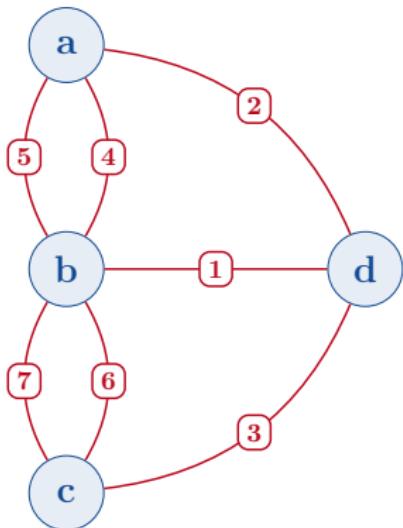
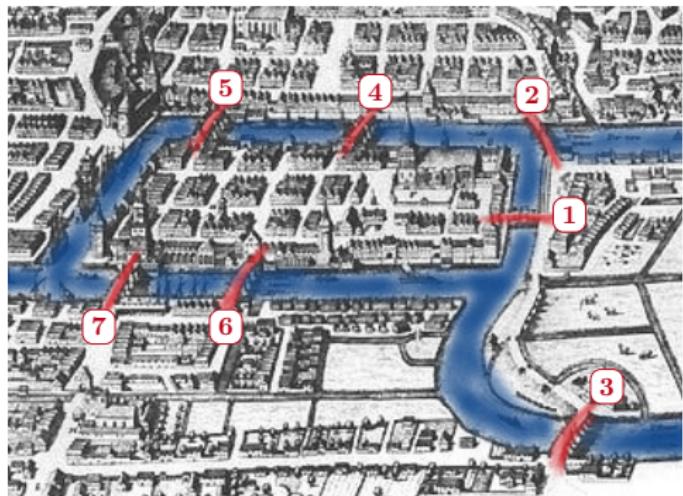


Leonhard Euler  
1707–1783  
Switzerland

## 23. Graph Theory



## 23. Graph Theory



## 23. Graph Theory



Graph Theory, which has been used by mathematicians for many years to solve interesting riddles and puzzles, is nowadays in computing (algorithm design, telecommunication and GPS), physics (atomic structures), neurology (brain-like structures), chemistry (molecular structures), and many other disciplines.

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### Definition

A graph is formed by points called *vertices* (or *nodes*), and lines called *edges*.

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### Notation

Vertices are denoted by lowercase letters:  $a, b, c, \dots$ . The edge from vertex  $u$  to vertex  $v$  is denoted by  $e = (u, v)$ .  $u$  and  $v$  are called *endpoints* of  $e$ .

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### Definition

A non-empty set of vertices  $V$  together with a set of edges  $E$  is called a *graph* and is denoted by  $G(V, E)$

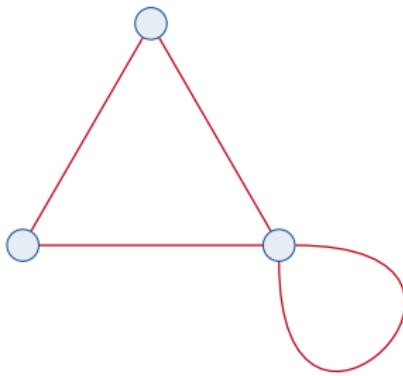
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### Definition

An edge which starts and finishes at the same vertex is called a *loop*.

## 23. Graph Theory



### Example

The graph above has 3 vertices and 4 edges. One of the edges is a loop.

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### Example

Let  $V = \{a, b, c, d\}$  and  $E = \{e_1, e_2, e_3, e_4, e_5\}$  where  $e_1 = (a, b)$ ,  $e_2 = (b, c)$ ,  $e_3 = (c, d)$ ,  $e_4 = (a, c)$  and  $e_5 = (b, d)$ . Draw the graph  $G = (V, E)$ .

*solution:*

## 23. Graph Theory



### Example

Let  $V = \{a, b, c, d\}$  and  $E = \{e_1, e_2, e_3, e_4, e_5\}$  where  $e_1 = (a, b)$ ,  $e_2 = (b, c)$ ,  $e_3 = (c, d)$ ,  $e_4 = (a, c)$  and  $e_5 = (b, d)$ . Draw the graph  $G = (V, E)$ .

*solution:*

a

b

c

d

## 23. Graph Theory



### Example

Let  $V = \{a, b, c, d\}$  and  $E = \{e_1, e_2, e_3, e_4, e_5\}$  where  $e_1 = (a, b)$ ,  $e_2 = (b, c)$ ,  $e_3 = (c, d)$ ,  $e_4 = (a, c)$  and  $e_5 = (b, d)$ . Draw the graph  $G = (V, E)$ .

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a

b

d

c

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### Example

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*solution:*

a

b

c

d

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### Example

Let  $V = \{a, b, c, d\}$  and  $E = \{e_1, e_2, e_3, e_4, e_5\}$  where  $e_1 = (a, b)$ ,  $e_2 = (b, c)$ ,  $e_3 = (c, d)$ ,  $e_4 = (a, c)$  and  $e_5 = (b, d)$ . Draw the graph  $G = (V, E)$ .

*solution:*



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### Example

Let  $V = \{a, b, c, d\}$  and  $E = \{e_1, e_2, e_3, e_4, e_5\}$  where  $e_1 = (a, b)$ ,  $e_2 = (b, c)$ ,  $e_3 = (c, d)$ ,  $e_4 = (a, c)$  and  $e_5 = (b, d)$ . Draw the graph  $G = (V, E)$ .

*solution:*

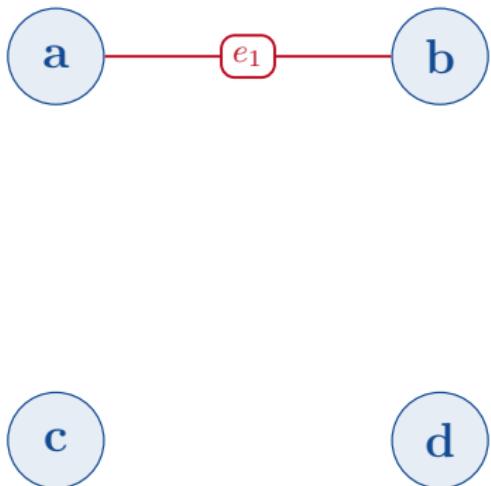


## 23. Graph Theory

### Example

Let  $V = \{a, b, c, d\}$  and  $E = \{e_1, e_2, e_3, e_4, e_5\}$  where  $e_1 = (a, b)$ ,  $e_2 = (b, c)$ ,  $e_3 = (c, d)$ ,  $e_4 = (a, c)$  and  $e_5 = (b, d)$ . Draw the graph  $G = (V, E)$ .

*solution:*

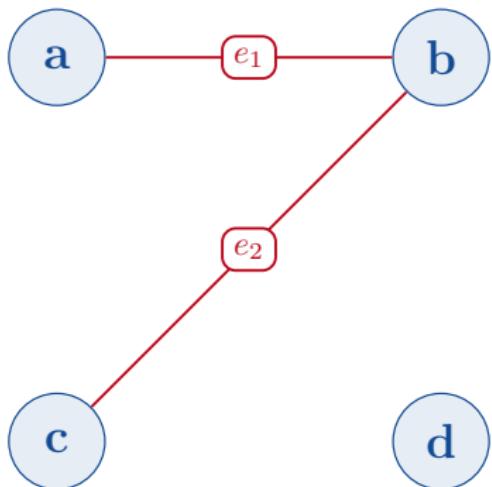


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### Example

Let  $V = \{a, b, c, d\}$  and  $E = \{e_1, e_2, e_3, e_4, e_5\}$  where  $e_1 = (a, b)$ ,  $e_2 = (b, c)$ ,  $e_3 = (c, d)$ ,  $e_4 = (a, c)$  and  $e_5 = (b, d)$ . Draw the graph  $G = (V, E)$ .

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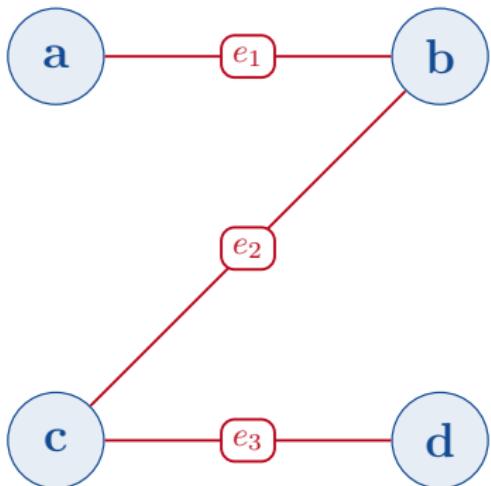


## 23. Graph Theory

### Example

Let  $V = \{a, b, c, d\}$  and  $E = \{e_1, e_2, e_3, e_4, e_5\}$  where  $e_1 = (a, b)$ ,  $e_2 = (b, c)$ ,  $e_3 = (c, d)$ ,  $e_4 = (a, c)$  and  $e_5 = (b, d)$ . Draw the graph  $G = (V, E)$ .

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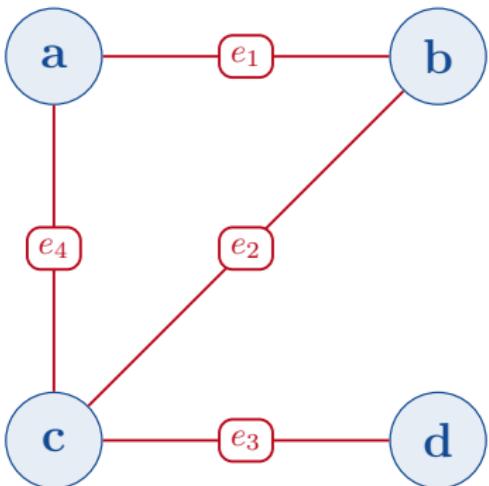


## 23. Graph Theory

### Example

Let  $V = \{a, b, c, d\}$  and  $E = \{e_1, e_2, e_3, e_4, e_5\}$  where  $e_1 = (a, b)$ ,  $e_2 = (b, c)$ ,  $e_3 = (c, d)$ ,  $e_4 = (a, c)$  and  $e_5 = (b, d)$ . Draw the graph  $G = (V, E)$ .

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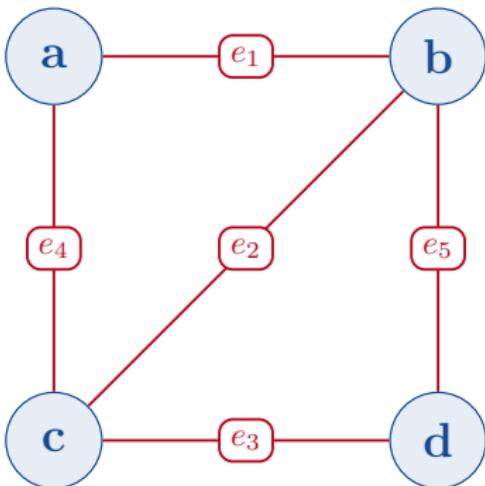


## 23. Graph Theory

### Example

Let  $V = \{a, b, c, d\}$  and  $E = \{e_1, e_2, e_3, e_4, e_5\}$  where  $e_1 = (a, b)$ ,  $e_2 = (b, c)$ ,  $e_3 = (c, d)$ ,  $e_4 = (a, c)$  and  $e_5 = (b, d)$ . Draw the graph  $G = (V, E)$ .

*solution:*



## 23. Graph Theory



### Definition

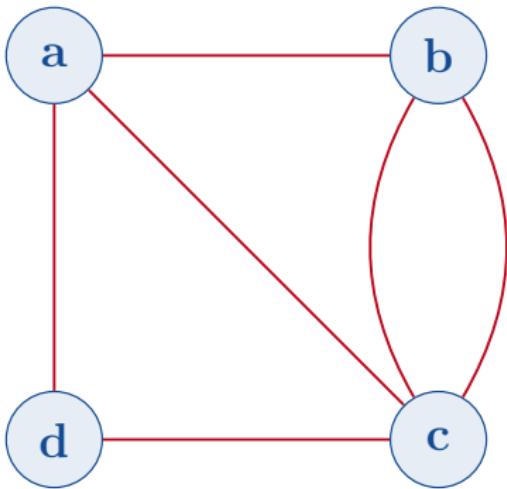
Two edges with the same endpoints are called *parallel edges*.

# Types of Graph

## Definition

A *simple graph* is a graph without parallel edges or loops.

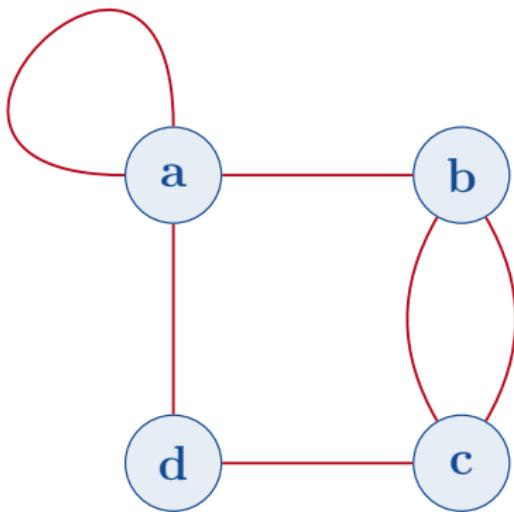
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### Definition

If a graph contains parallel edges, it is called a *multigraph*.

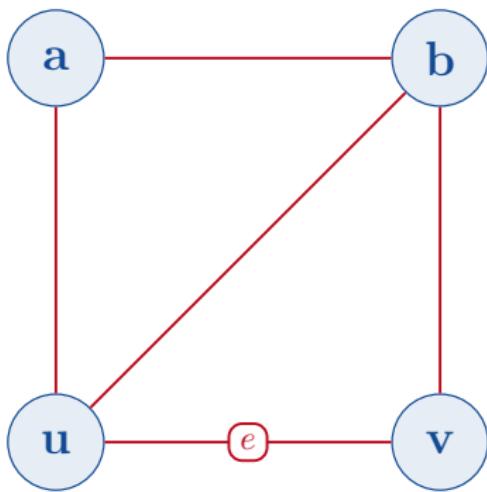
## 23. Graph Theory



### Definition

A *pseudograph* is a non-simple graph in which both loops and parallel edges are permitted.

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### Definition

If  $e = (u, v)$  is an edge of a undirected graph  $G$ , then the vertices  $u$  and  $v$  are called *neighbours* in  $G$ .

## 23. Graph Theory



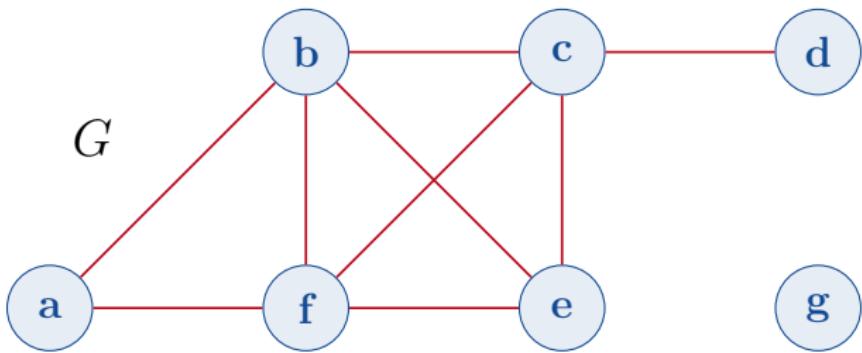
### Definition

The *degree* of a vertex  $v$  in a(n undirected) graph is

$$\deg(v) = \text{the number of edges connected to } v.$$

When we calculate the degree of a vertex, a loop is counted as 2.

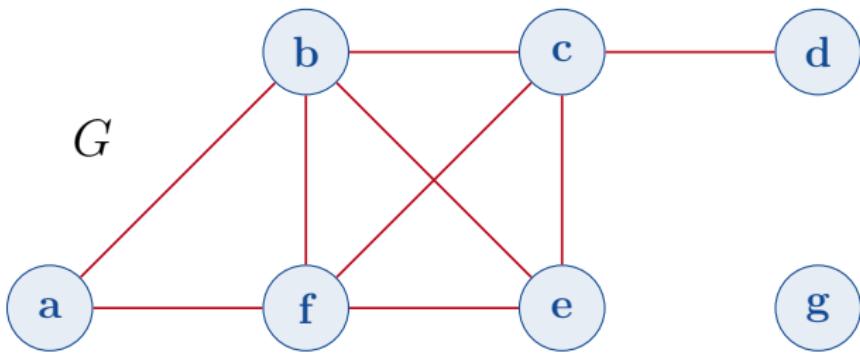
## 23. Graph Theory



Example

Give the degrees of all the vertices in graph  $G$ .

## 23. Graph Theory



Example

Give the degrees of all the vertices in graph  $G$ .

*solution:*

$$\deg(a) =$$

$$\deg(b) =$$

$$\deg(c) =$$

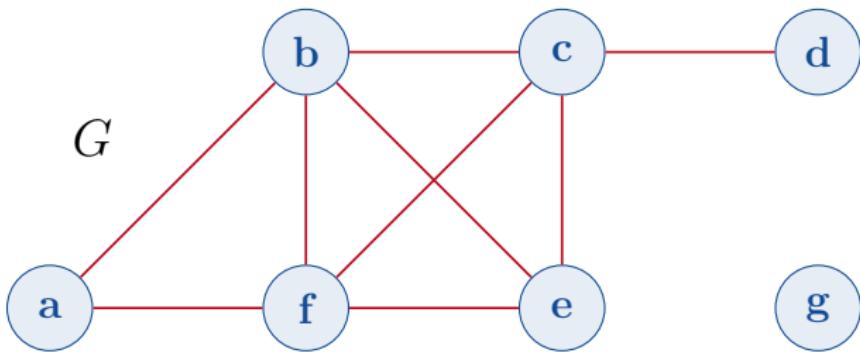
$$\deg(d) =$$

$$\deg(e) =$$

$$\deg(f) =$$

$$\deg(g) =$$

## 23. Graph Theory



Example

Give the degrees of all the vertices in graph  $G$ .

*solution:*

$$\deg(a) = 2$$

$$\deg(b) =$$

$$\deg(c) =$$

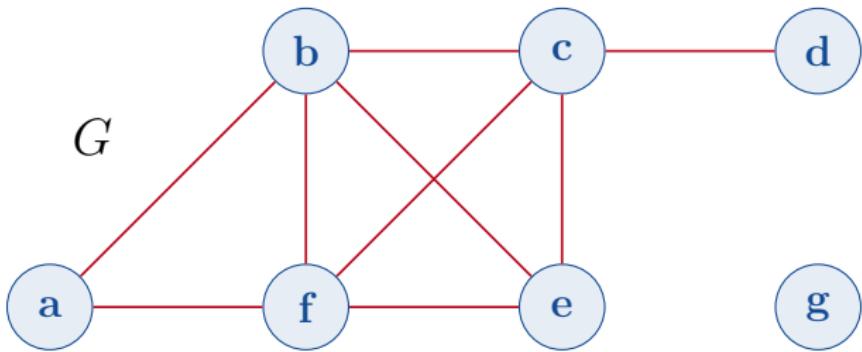
$$\deg(d) =$$

$$\deg(e) =$$

$$\deg(f) =$$

$$\deg(g) =$$

## 23. Graph Theory



Example

Give the degrees of all the vertices in graph  $G$ .

*solution:*

$$\deg(a) = 2$$

$$\deg(b) = 4$$

$$\deg(c) =$$

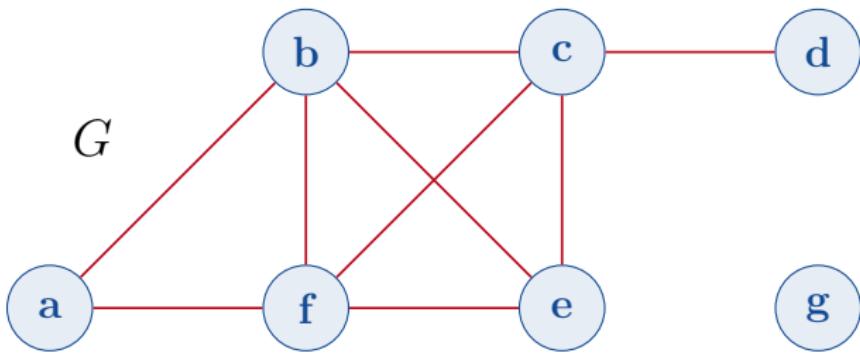
$$\deg(d) =$$

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## 23. Graph Theory



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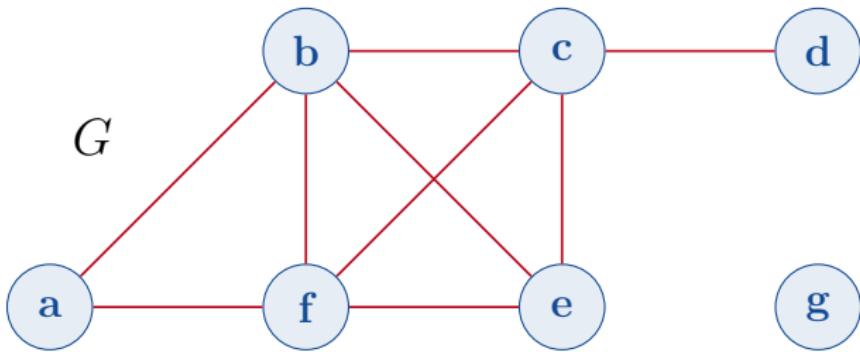
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## 23. Graph Theory



Example

Give the degrees of all the vertices in graph  $G$ .

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$$\deg(a) = 2$$

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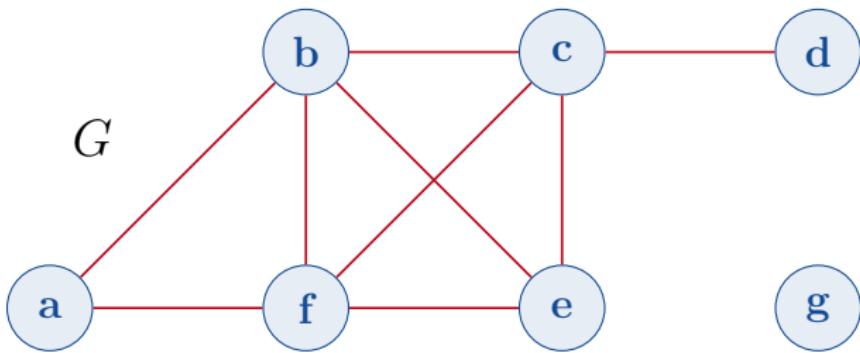
$$\deg(d) = 1$$

$$\deg(e) =$$

$$\deg(f) =$$

$$\deg(g) =$$

## 23. Graph Theory



Example

Give the degrees of all the vertices in graph  $G$ .

*solution:*

$$\deg(a) = 2$$

$$\deg(b) = 4$$

$$\deg(c) = 4$$

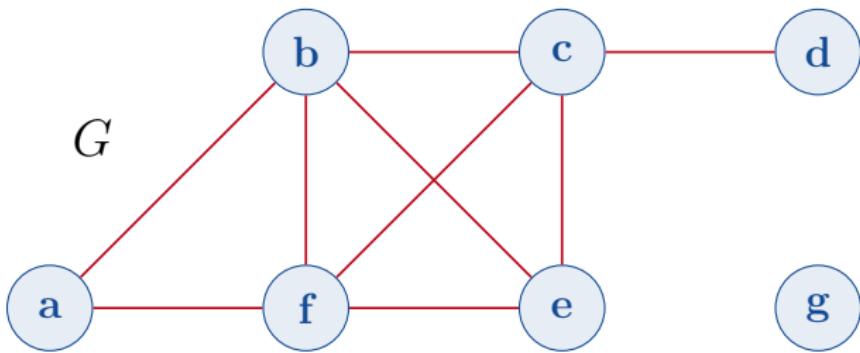
$$\deg(d) = 1$$

$$\deg(e) = 3$$

$$\deg(f) =$$

$$\deg(g) =$$

## 23. Graph Theory



Example

Give the degrees of all the vertices in graph  $G$ .

*solution:*

$$\deg(a) = 2$$

$$\deg(b) = 4$$

$$\deg(c) = 4$$

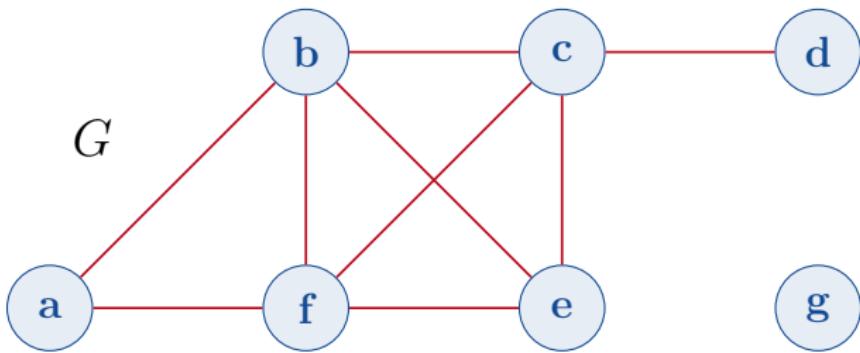
$$\deg(d) = 1$$

$$\deg(e) = 3$$

$$\deg(f) = 4$$

$$\deg(g) =$$

## 23. Graph Theory



Example

Give the degrees of all the vertices in graph  $G$ .

*solution:*

$$\deg(a) = 2$$

$$\deg(b) = 4$$

$$\deg(c) = 4$$

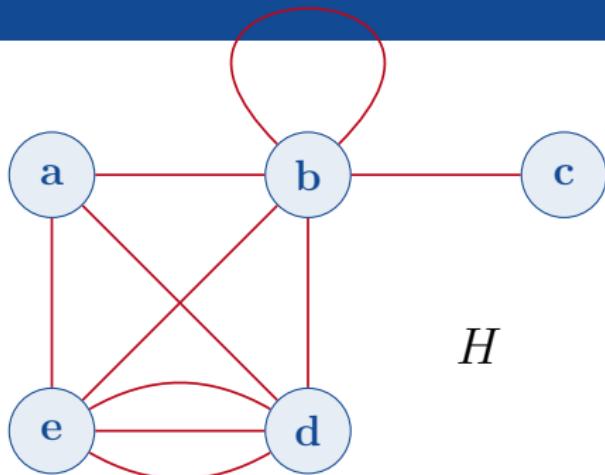
$$\deg(d) = 1$$

$$\deg(e) = 3$$

$$\deg(f) = 4$$

$$\deg(g) = 0$$

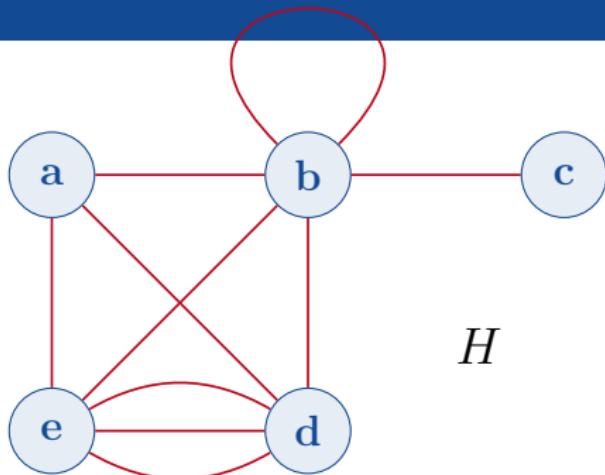
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Example

Give the degrees of all the vertices in graph  $H$ .

## 23. Graph Theory



Example

Give the degrees of all the vertices in graph  $H$ .

*solution:*

$$\deg(a) =$$

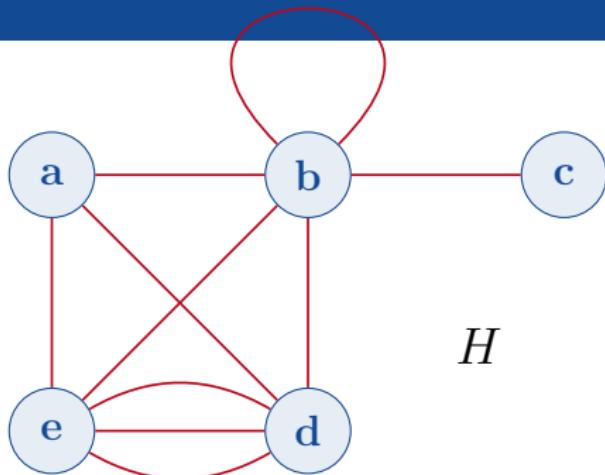
$$\deg(b) =$$

$$\deg(c) =$$

$$\deg(d) =$$

$$\deg(e) =$$

## 23. Graph Theory



Example

Give the degrees of all the vertices in graph  $H$ .

*solution:*

$$\deg(a) = 3$$

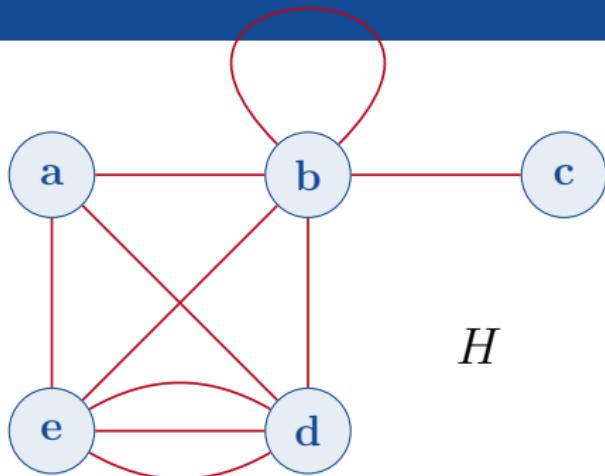
$$\deg(b) =$$

$$\deg(c) =$$

$$\deg(d) =$$

$$\deg(e) =$$

## 23. Graph Theory



Example

Give the degrees of all the vertices in graph  $H$ .

*solution:*

$$\deg(a) = 3$$

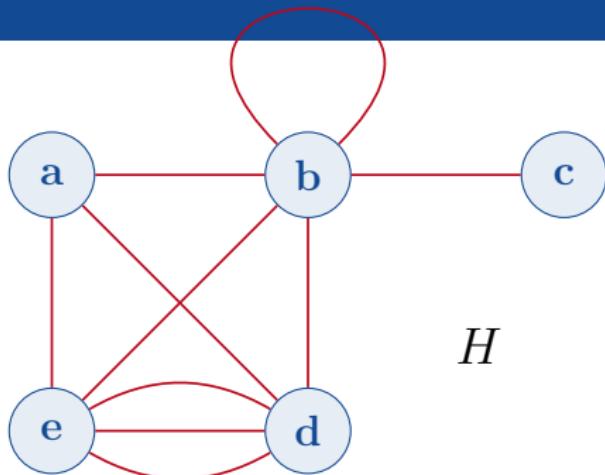
$$\deg(b) = 6$$

$$\deg(c) =$$

$$\deg(d) =$$

$$\deg(e) =$$

## 23. Graph Theory



Example

Give the degrees of all the vertices in graph  $H$ .

*solution:*

$$\deg(a) = 3$$

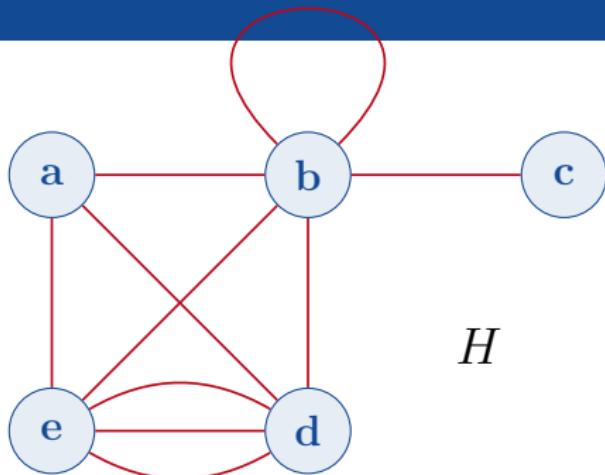
$$\deg(b) = 6$$

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## 23. Graph Theory



Example

Give the degrees of all the vertices in graph  $H$ .

*solution:*

$$\deg(a) = 3$$

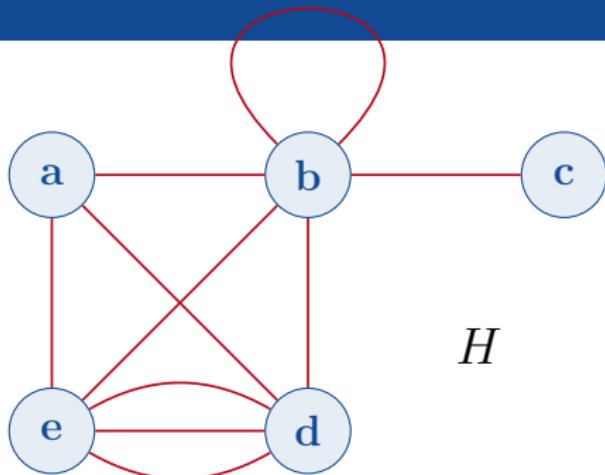
$$\deg(b) = 6$$

$$\deg(c) = 1$$

$$\deg(d) = 5$$

$$\deg(e) =$$

## 23. Graph Theory



Example

Give the degrees of all the vertices in graph  $H$ .

*solution:*

$$\deg(a) = 3$$

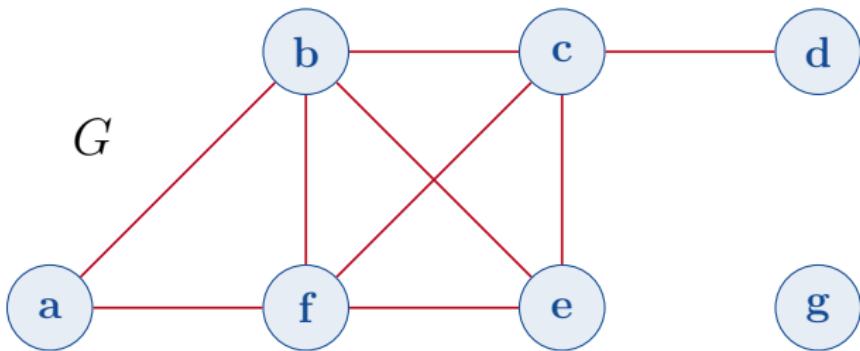
$$\deg(b) = 6$$

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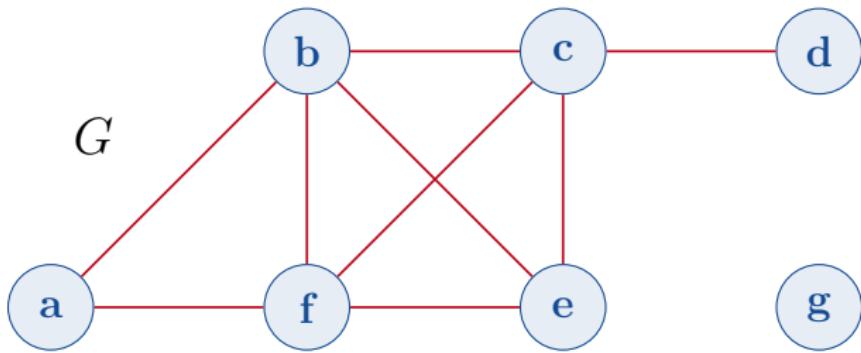
## 23. Graph Theory



### Definition

A vertex without an edge (degree=0) is called an *isolated vertex*.

## 23. Graph Theory



### Definition

A vertex of degree 1 is called a *pendant*. A pendant has only one edge.

## 23. Graph Theory



### Theorem

Let  $G = (V, E)$  be a pseudograph and let  $n(E)$  denote the number of edges in  $G$ . Then

$$n(E) = \frac{1}{2} \sum_{v \in V} \deg(v).$$

## 23. Graph Theory



### Example

A graph has 10 vertices, each of degree 4. How many edges does this graph have?

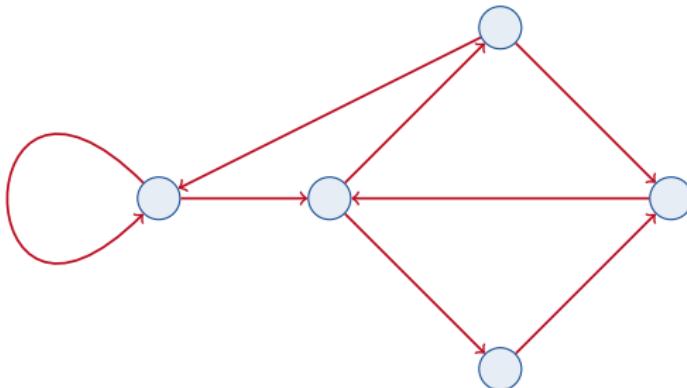
*solution:*

Suppose that  $V = \{v_1, \dots, v_{10}\}$ . Then by Theorem 16, we have that

$$\begin{aligned} n(E) &= \frac{1}{2} \sum_{i=1}^{10} \deg(v_i) \\ &= \frac{1}{2} (\deg(v_1) + \deg(v_2) + \dots + \deg(v_{10})) \\ &= \frac{1}{2} (4 + 4 + \dots + 4) = \frac{1}{2} (40) = 20. \end{aligned}$$

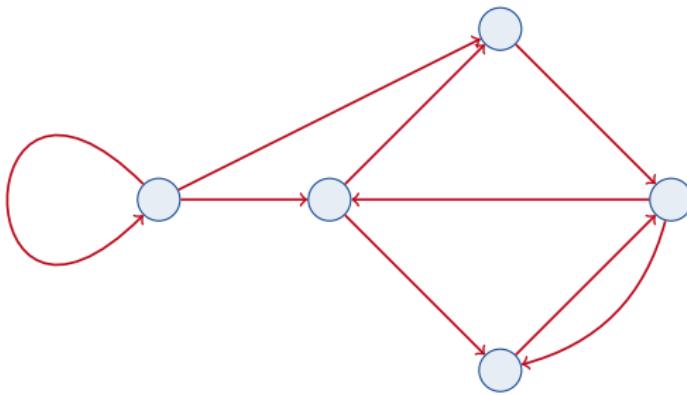
Therefore this graph has 20 edges.

## 23. Graph Theory



### Definition

If the edges in a graph have directions, the graph is called a *directed graph* (or *digraph*). This direction indicates where the connection starts and ends.



### Definition

If a directed graph has parallel edges, it is called a *directed multigraph* (or *multidigraph*).

## 23. Graph Theory



### Remark

In a graph, all edges are the same type. So either all edges are directed or all edges are undirected.

## 23. Graph Theory



### Notation

Let  $G$  be a directed graph. Now when we write  $e = (u, v)$ , the order of  $u$  and  $v$  is important. The edge  $e = (u, v)$  starts at  $u$  and finishes at  $v$ .

### Definition

Let  $G$  be a directed graph. The *indegree* of a number vertex  $v$  is

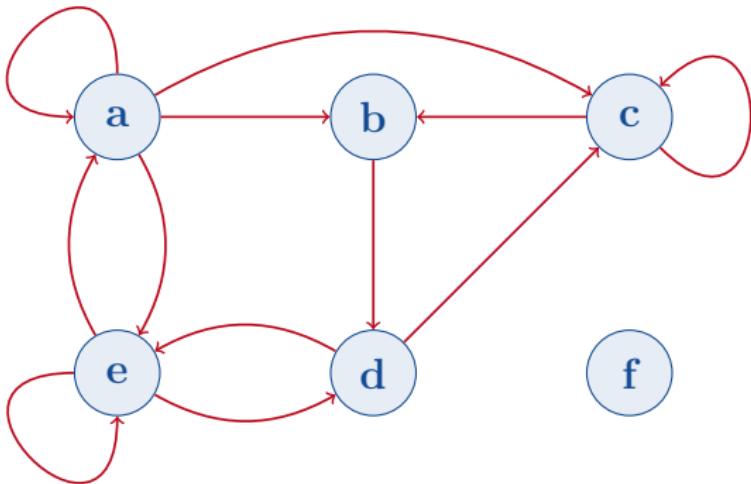
$\deg^-(v) =$  the number of edges coming *into*  $v$

and the *outdegree* of  $v$  is

$\deg^+(v) =$  the number of edges coming *out* of  $v$ .

A loop is counted as 1 for both  $\deg^-(v)$  and  $\deg^+(v)$ .

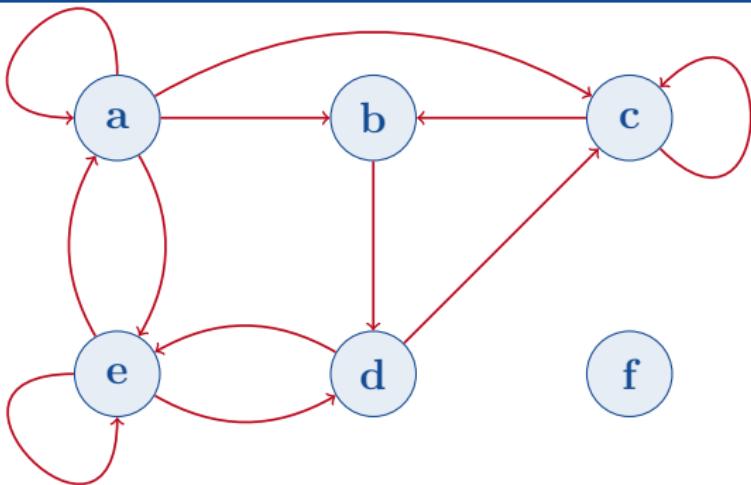
## 23. Graph Theory



Example

Find the indegree and outdegree of each vertex in this graph.

## 23. Graph Theory



*solution:*

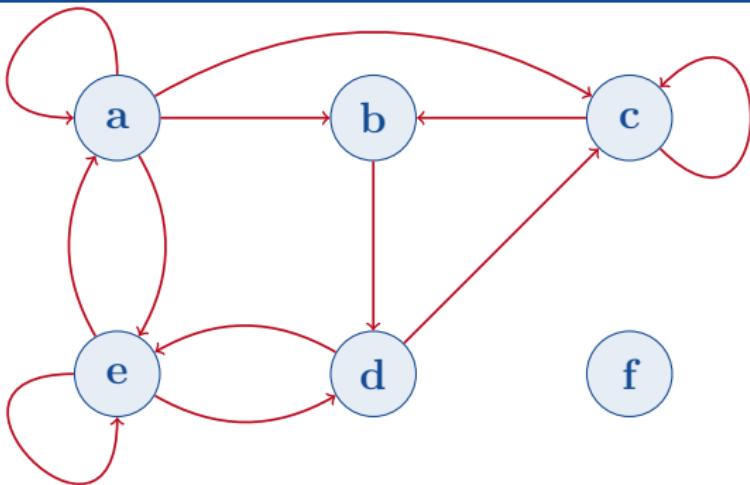
The indegrees are:

$$\deg^-(a) = , \quad \deg^-(b) = , \quad \deg^-(c) = \\ \deg^-(d) = , \quad \deg^-(e) = , \quad \deg^-(f) =$$

The outdegrees are:

$$\deg^+(a) = , \quad \deg^+(b) = , \quad \deg^+(c) = \\ \deg^+(d) = , \quad \deg^+(e) = , \quad \deg^+(f) = .$$

## 23. Graph Theory



*solution:*

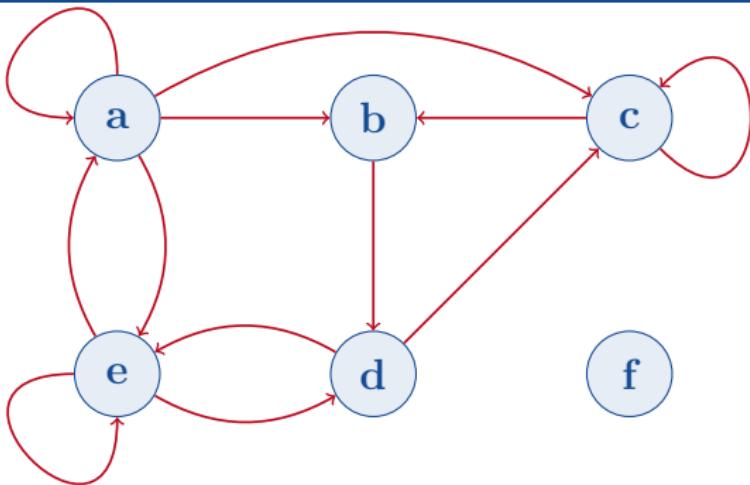
The indegrees are:

$$\deg^-(a) = 2, \quad \deg^-(b) = 2, \quad \deg^-(c) = 3 \\ \deg^-(d) = 2, \quad \deg^-(e) = 3, \quad \deg^-(f) = 0$$

The outdegrees are:

$$\deg^+(a) = , \quad \deg^+(b) = , \quad \deg^+(c) = \\ \deg^+(d) = , \quad \deg^+(e) = , \quad \deg^+(f) = .$$

## 23. Graph Theory



*solution:*

The indegrees are:

$$\begin{aligned}\deg^-(a) &= 2, & \deg^-(b) &= 2, & \deg^-(c) &= 3 \\ \deg^-(d) &= 2, & \deg^-(e) &= 3, & \deg^-(f) &= 0\end{aligned}$$

The outdegrees are:

$$\begin{aligned}\deg^+(a) &= 4, & \deg^+(b) &= 1, & \deg^+(c) &= 2 \\ \deg^+(d) &= 2, & \deg^+(e) &= 3, & \deg^+(f) &= 0.\end{aligned}$$

## 23. Graph Theory



### Theorem

Let  $G = (V, E)$  be a directed graph and let  $n(E)$  denote the number of edges in  $G$ . Then

$$n(E) = \sum_{v \in V} \deg^-(v) = \sum_{v \in V} \deg^+(v).$$

## 23. Graph Theory



### Definition

A simple graph which includes all possible edges is called a *complete graph*.

### Remark

In a complete graph, every vertex has an edge with every other vertex. A complete graph with  $n$  vertices is denoted by  $K_n$ .

The degree of every vertex in  $K_n$  is  $(n - 1)$ . Thus  $K_n$  has a total of  $\frac{n(n-1)}{2}$  edges.

## 23. Graph Theory



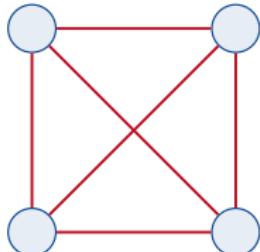
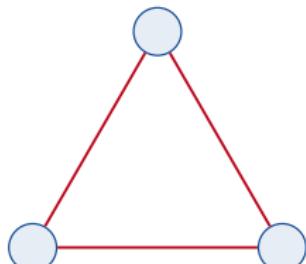
$K_1$



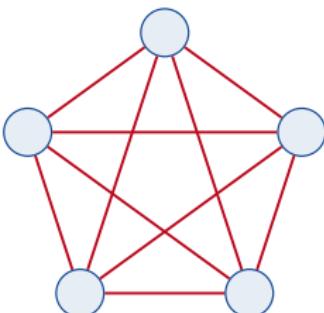
$K_2$



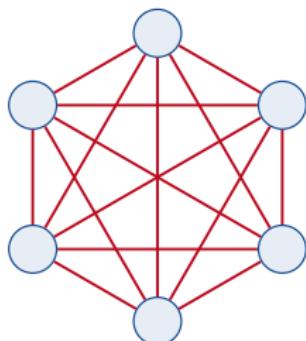
$K_3$



$K_4$



$K_5$



$K_6$

## 23. Graph Theory



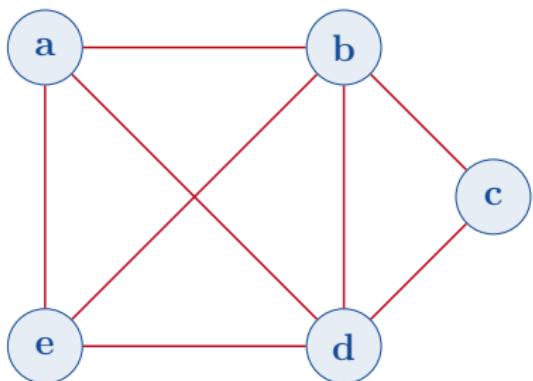
### Definition

A *planar graph* is a graph that can be drawn on the plane in such a way that its edges intersect only at their endpoints.

## 23. Graph Theory

### Definition

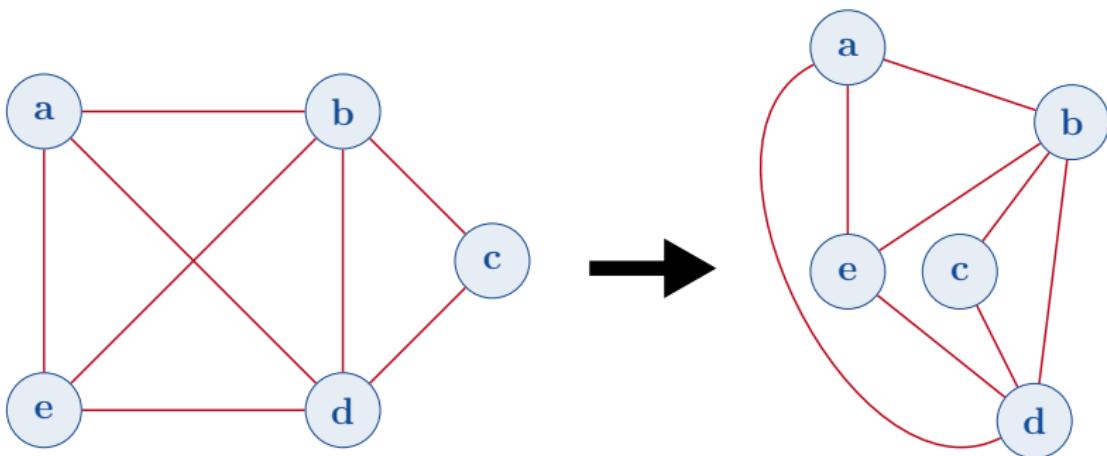
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## 23. Graph Theory

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## 23. Graph Theory

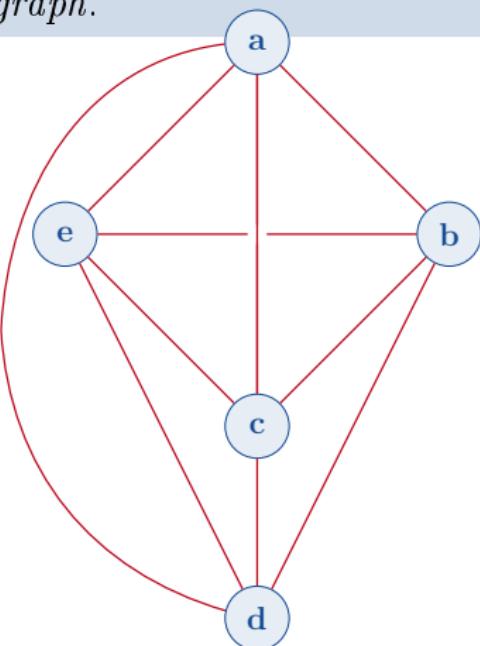
### Definition

A graph that is not a planar graph, and can only be drawn without intersecting edges in three-dimensional space is called a *three-dimensional graph*.

## 23. Graph Theory

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## 23. Graph Theory



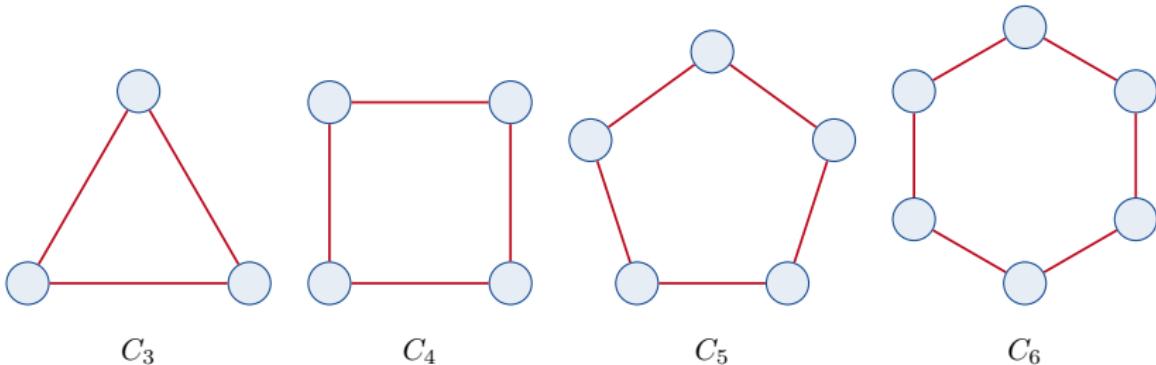
### Definition

Suppose that  $n \geq 3$ , that  $V = \{v_1, v_2, \dots, v_n\}$  and that  $E = \{(v_1, v_2), (v_2, v_3), \dots, (v_{n-1}, v_n), (v_n, v_1)\}$ . Then the graph  $C_n = (V, E)$  is called a *cycle graph*.

## 23. Graph Theory

### Definition

Suppose that  $n \geq 3$ , that  $V = \{v_1, v_2, \dots, v_n\}$  and that  $E = \{(v_1, v_2), (v_2, v_3), \dots, (v_{n-1}, v_n), (v_n, v_1)\}$ . Then the graph  $C_n = (V, E)$  is called a *cycle graph*.



## 23. Graph Theory



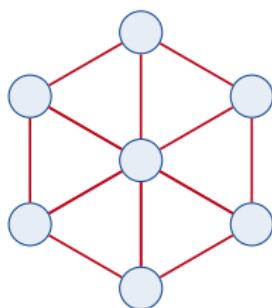
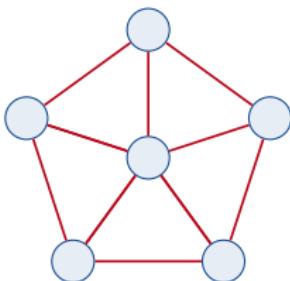
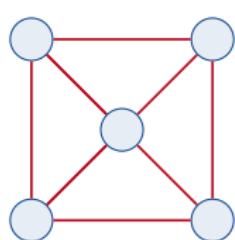
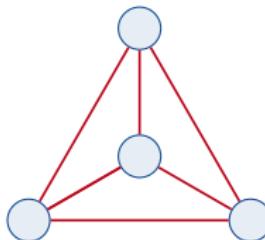
### Definition

If we take  $C_n$  and add a new vertex which is attached to all the other vertices, we get the *wheel graph*  $W_n$ .

## 23. Graph Theory

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If we take  $C_n$  and add a new vertex which is attached to all the other vertices, we get the *wheel graph*  $W_n$ .



$W_3$

$W_4$

$W_5$

$W_6$

## 23. Graph Theory



### Definition

A *bipartite graph* is a graph where the set  $V$  of vertices can be divided into two distinct subsets  $V_1$  and  $V_2$  such that every edge connects a vertex in  $V_1$  with a vertex in  $V_2$ .

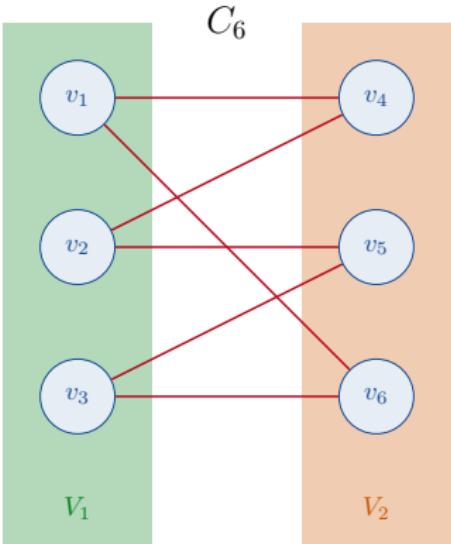
## 23. Graph Theory



### Remark

In a bipartite graph, there are no edges going from  $V_1$  to  $V_1$ , or from  $V_2$  to  $V_2$ .

## 23. Graph Theory



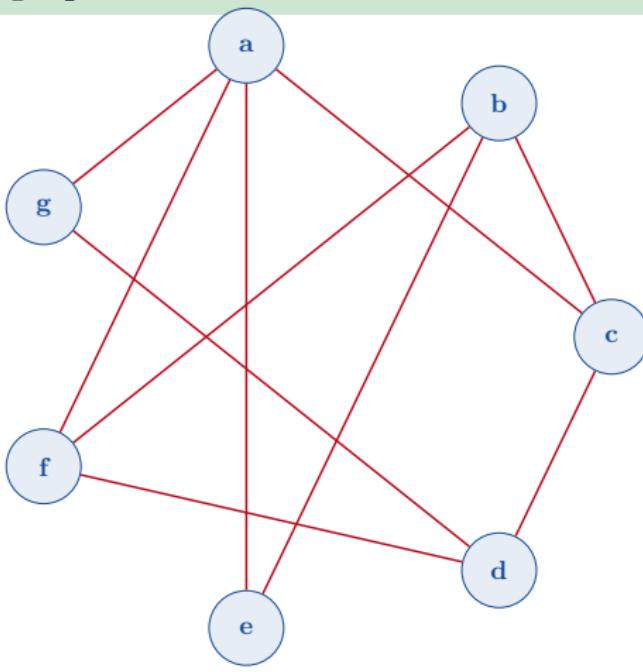
### Example

Note that the graph  $C_6$  is a bipartite graph because if we let  $V_1 = \{v_1, v_2, v_3\}$  and  $V_2 = \{v_4, v_5, v_6\}$ , then every edge goes from  $V_1$  to  $V_2$ .

## 23. Graph Theory

### Example

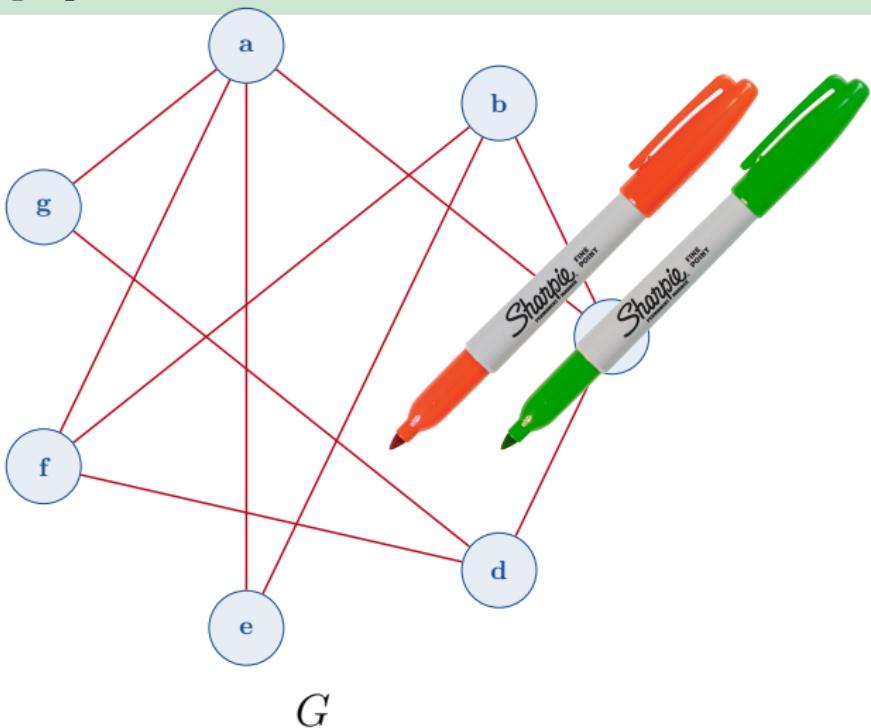
Is  $G$  a bipartite graph?



## 23. Graph Theory

### Example

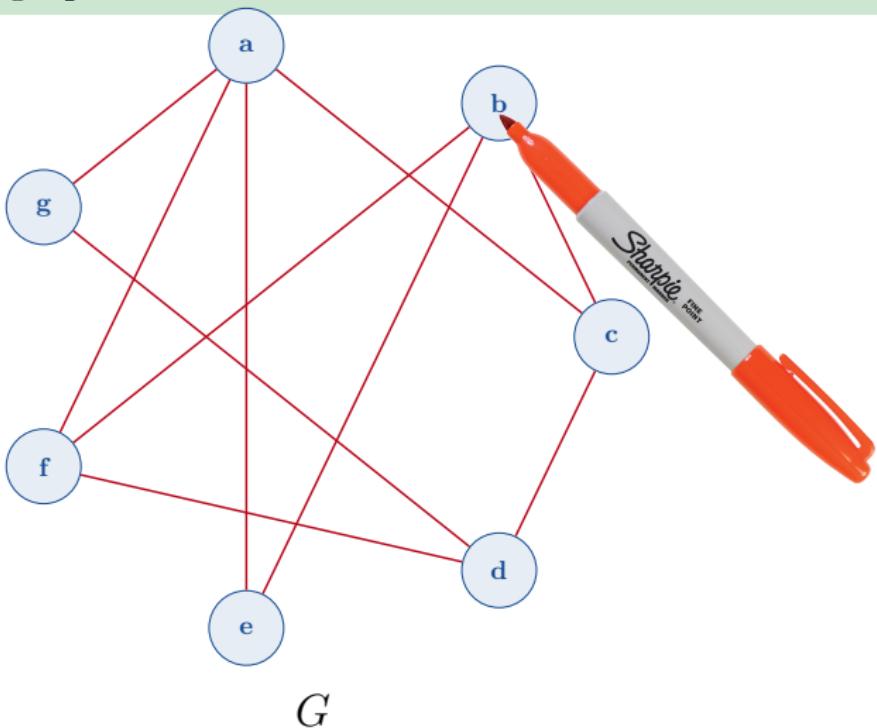
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## 23. Graph Theory

### Example

Is  $G$  a bipartite graph?

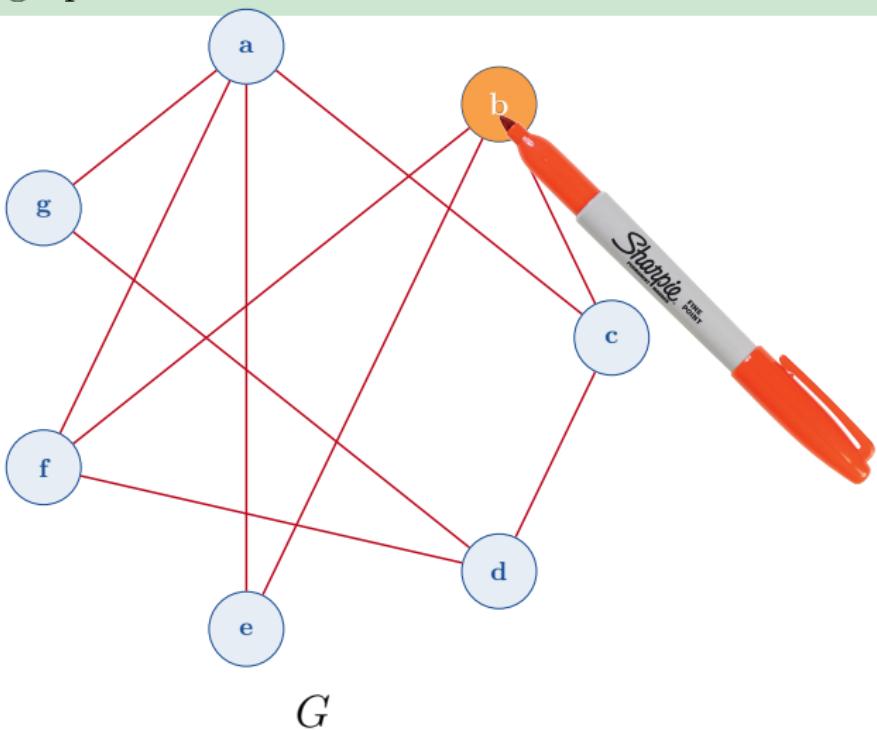


$G$

## 23. Graph Theory

### Example

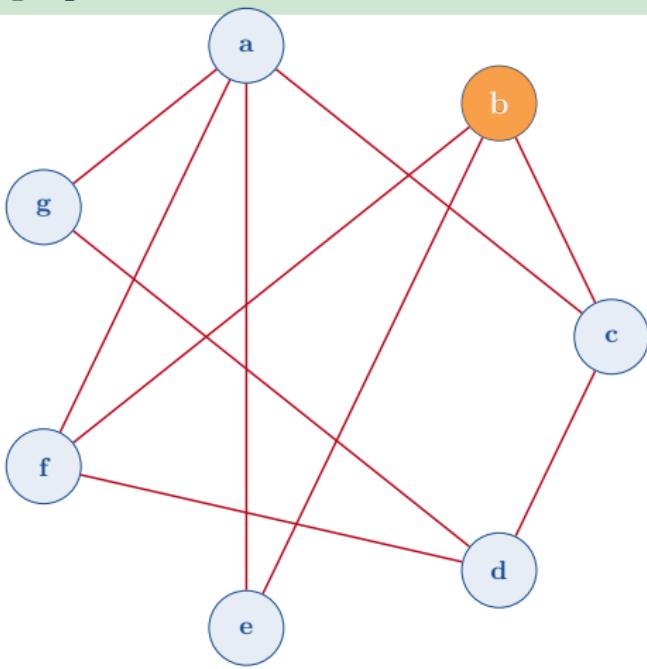
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## 23. Graph Theory

### Example

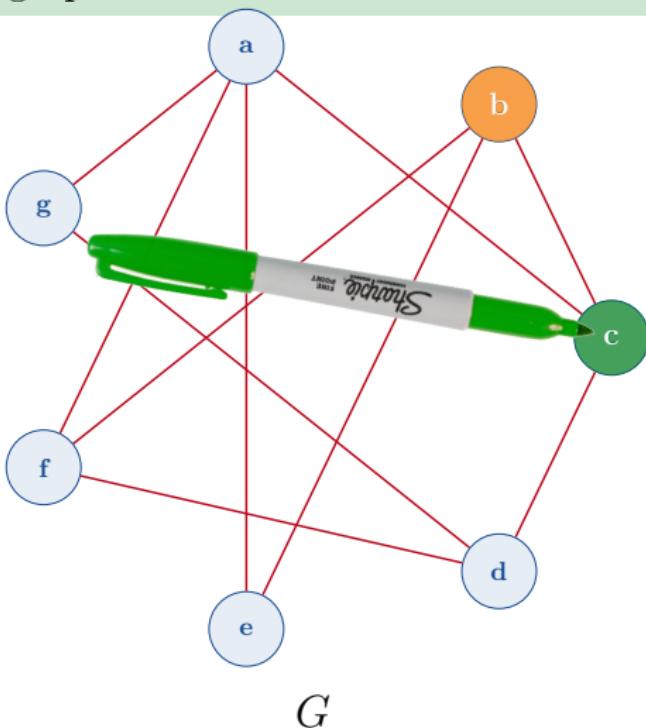
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## 23. Graph Theory

### Example

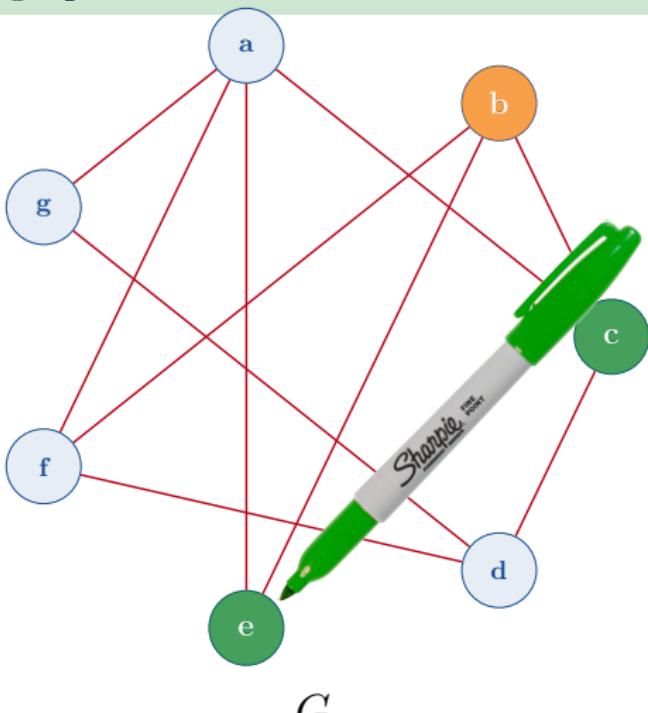
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## 23. Graph Theory

### Example

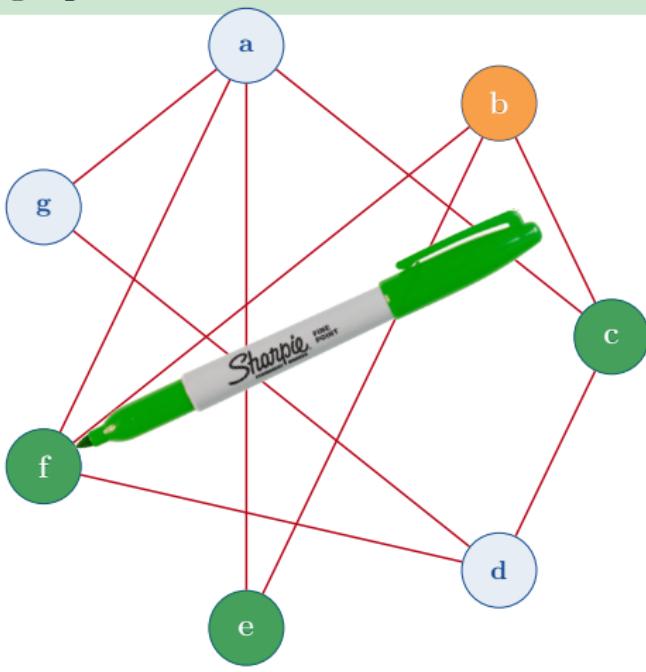
Is  $G$  a bipartite graph?



## 23. Graph Theory

### Example

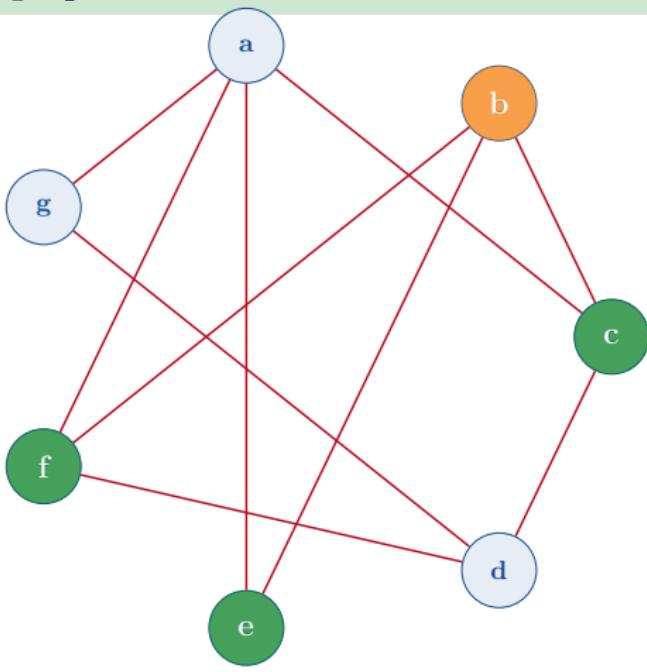
Is  $G$  a bipartite graph?



## 23. Graph Theory

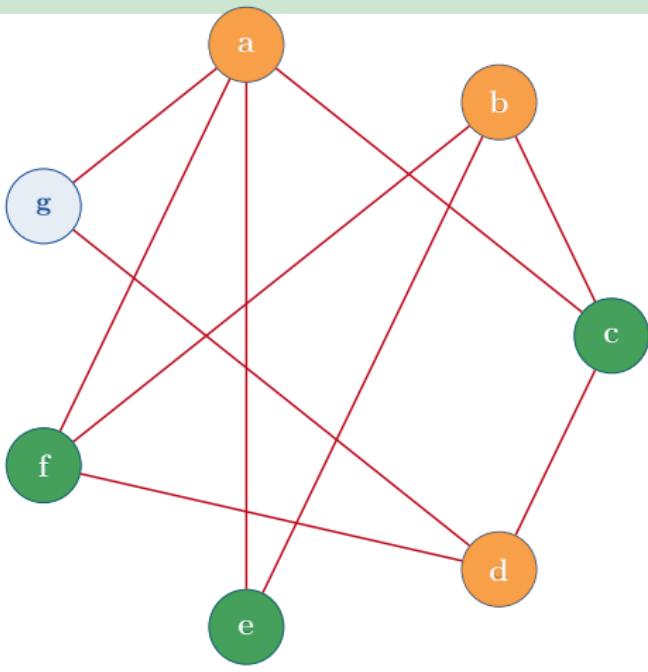
### Example

Is  $G$  a bipartite graph?



## 23. Graph Theory

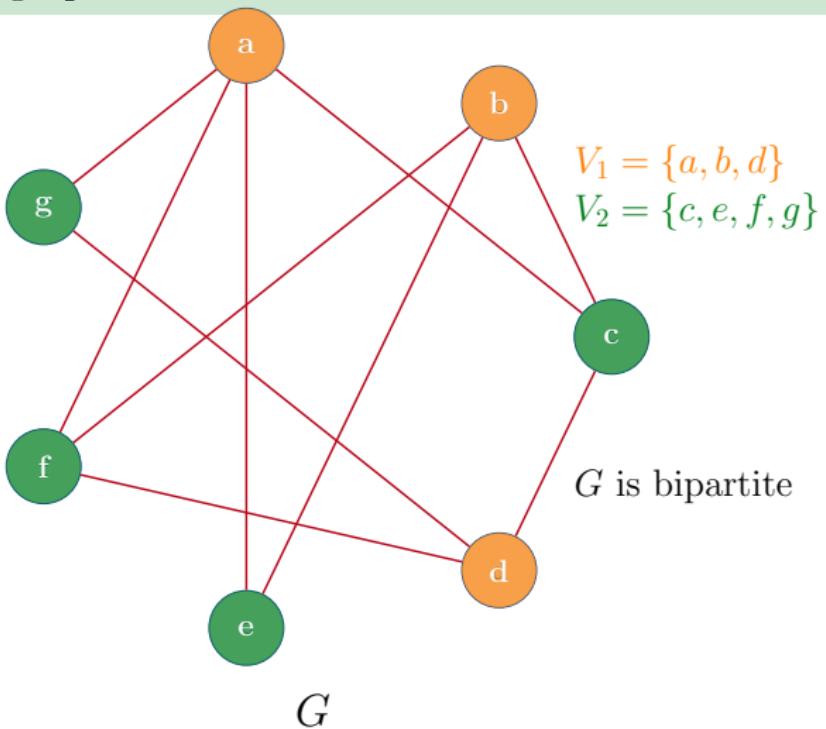
Example

Is  $G$  a bipartite graph?

## 23. Graph Theory

### Example

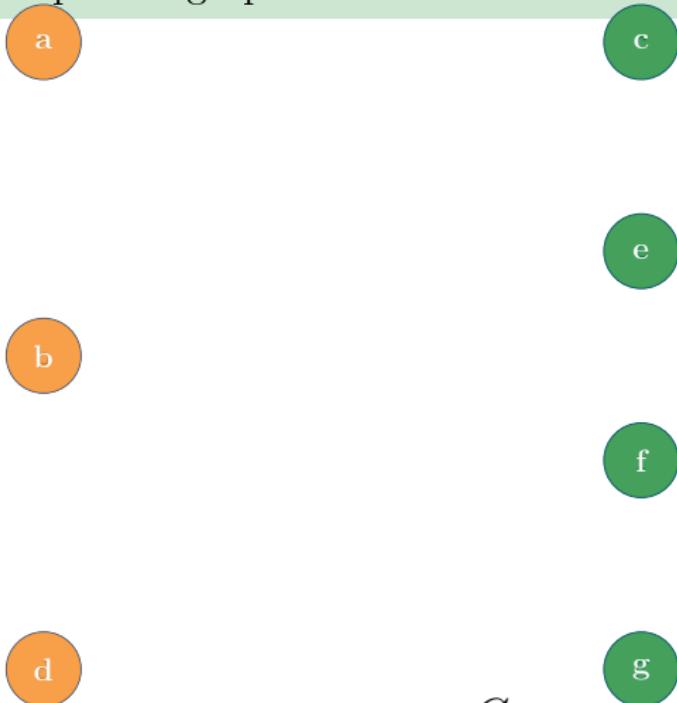
Is  $G$  a bipartite graph?



## 23. Graph Theory

### Example

Is  $G$  a bipartite graph?

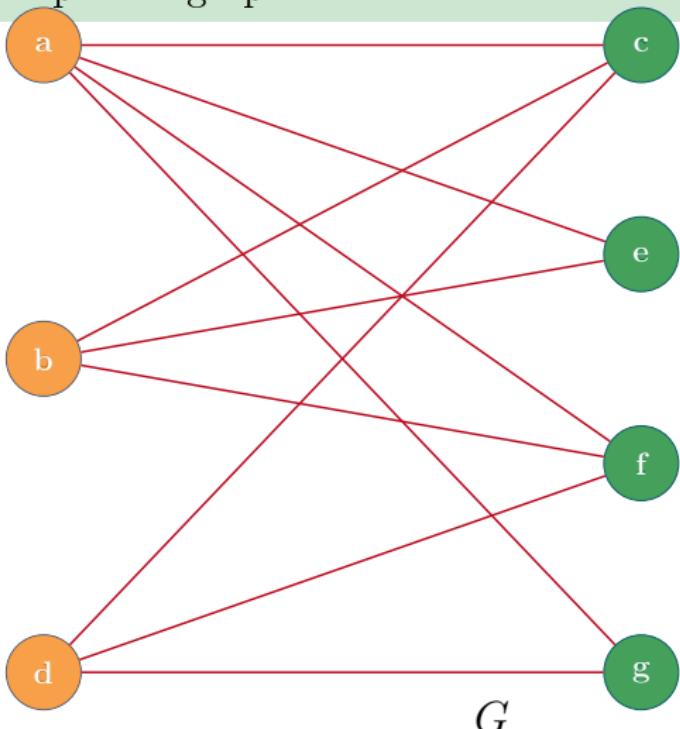


$$V_1 = \{a, b, d\}$$
$$V_2 = \{c, e, f, g\}$$

$G$  is bipartite

## 23. Graph Theory

Example

Is  $G$  a bipartite graph?

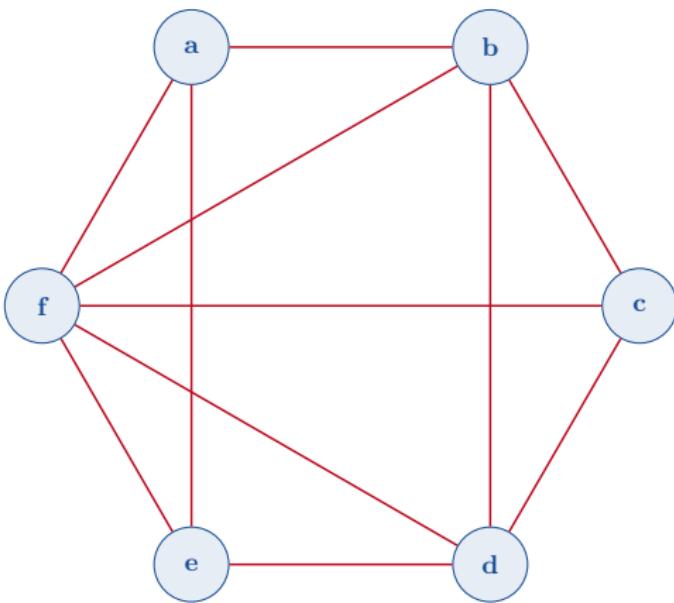
$$V_1 = \{a, b, d\}$$
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 $G$  is bipartite

## 23. Graph Theory

### Example

Is  $H$  a bipartite graph?

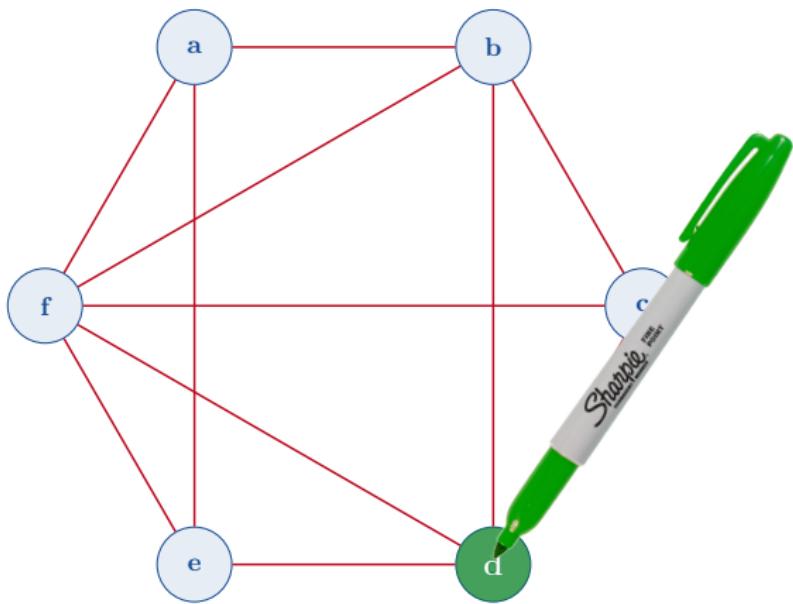


$H$

## 23. Graph Theory

### Example

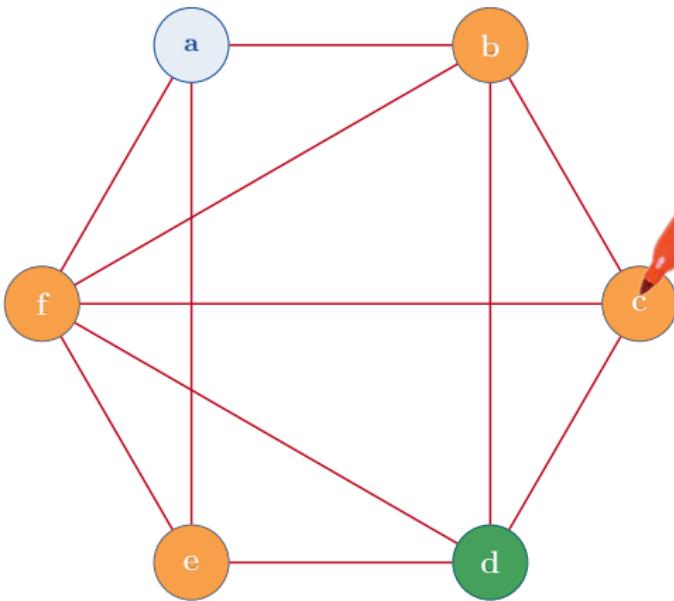
Is  $H$  a bipartite graph?



## 23. Graph Theory

### Example

Is  $H$  a bipartite graph?

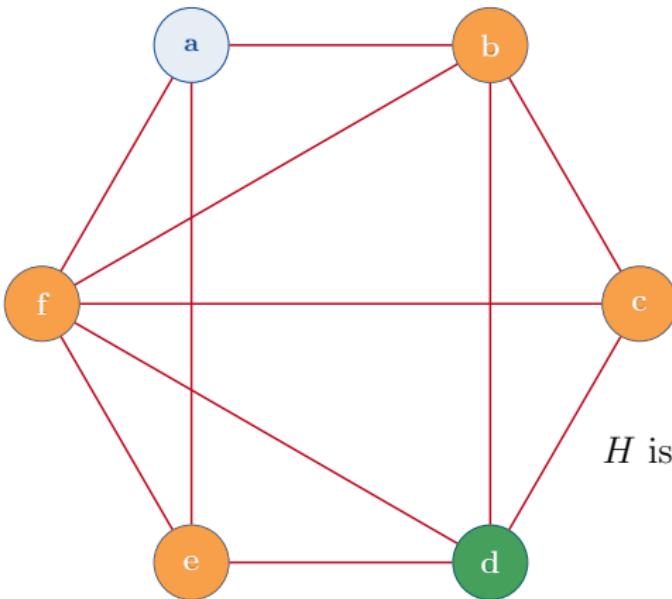


$H$

## 23. Graph Theory

### Example

Is  $H$  a bipartite graph?



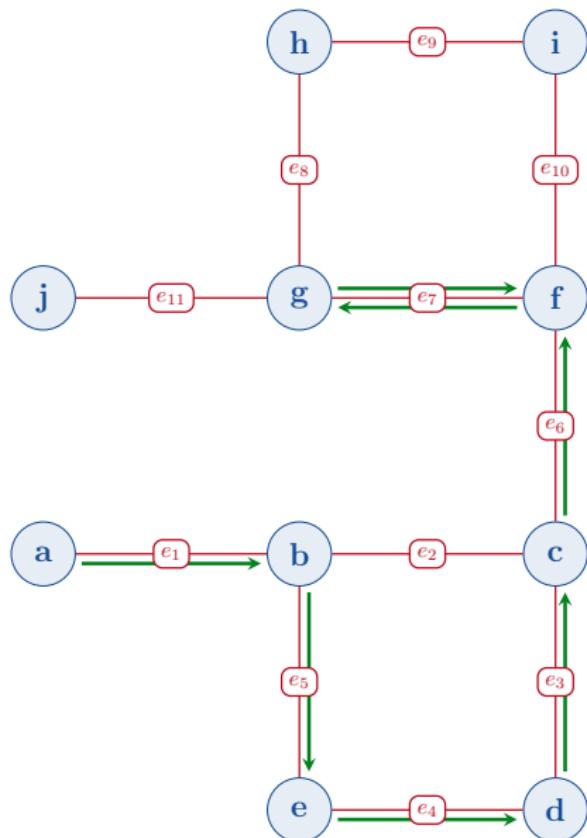
$H$  is not bipartite

# Walks

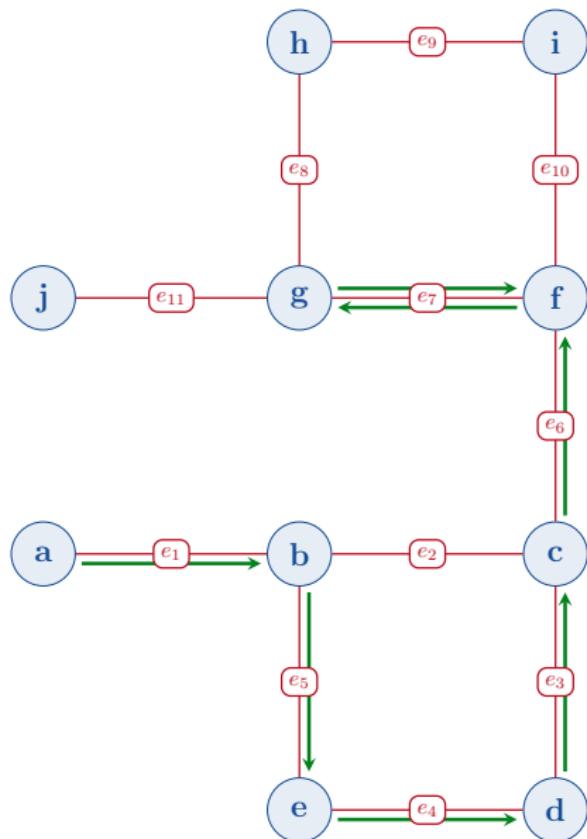
## Definition

A *walk* is a list  $v_0, e_1, v_1, \dots, e_k, v_k$  of vertices and edges such that for  $1 \leq i \leq k$ , the edge  $e_i$  has endpoints  $v_{i-1}$  and  $v_i$ .

## 23. Graph Theory



## 23. Graph Theory



Example

$a, e_1, b, e_5, e, e_4, d, e_3, c, e_6, f, e_7, g, e_7, f$  is a walk in this graph.

## 23. Graph Theory

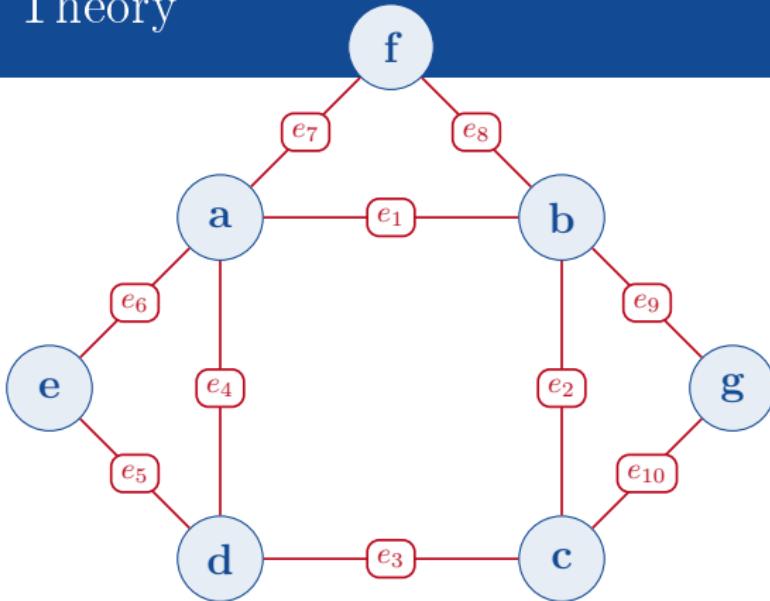


### Definition

An *Eulerian trail* is a walk such that

- 1 no edge is repeated; and
- 2 every edge is included.

## 23. Graph Theory



Example

The walk

$d, e_5, e, e_6, a, e_7, f, e_8, b, e_1, a, e_4, d, e_3, c, e_{10}, g, e_9, b, e_2, c$

is an Eulerian trail. Each of the ten edges appears once and only once in this list.

## 23. Graph Theory



### Remark

The Königsberg bridge problem can be rephrased as:

*Does there exist an Eulerian trail in Königsberg?*

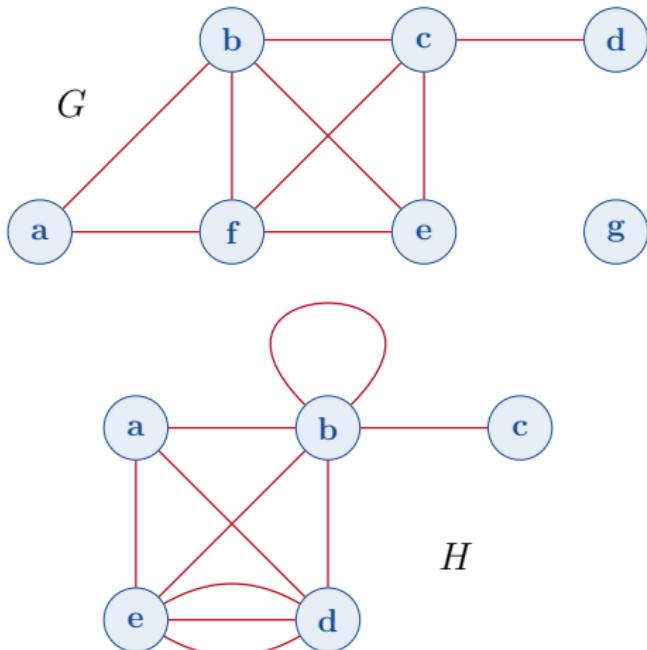
## 23. Graph Theory



### Definition

A graph is *connected* if there exists a walk between every pair of vertices.

## 23. Graph Theory



Example

Graph  $H$  is connected, but graph  $G$  is not connected.

## 23. Graph Theory



### Theorem

*Let  $G$  be a connected graph.*

## 23. Graph Theory



### Theorem

*Let  $G$  be a connected graph.*

*Then there exists an Eulerian trail if and only if the number of vertices of odd degree is either 0 or 2.*

## 23. Graph Theory



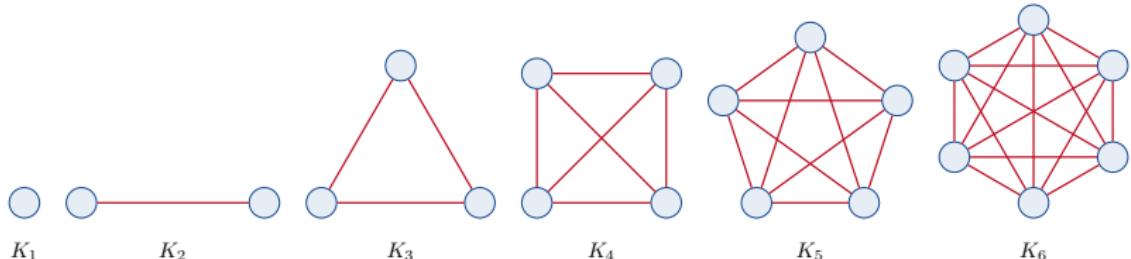
### Theorem

*Let  $G$  be a connected graph.*

*Then there exists an Eulerian trail if and only if the number of vertices of odd degree is either 0 or 2.*

*Furthermore, if  $G$  has 2 vertices of odd degree, then the Eulerian trail must start and finish at these two vertices.*

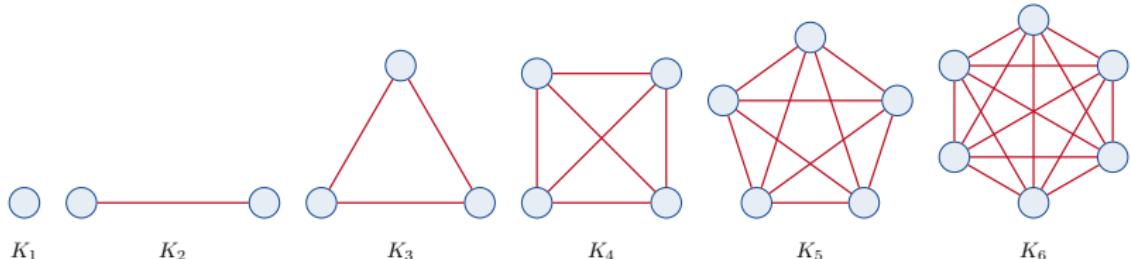
## 23. Graph Theory



### Example

Note that in  $K_3$  and  $K_5$ , every vertex has even degree. So there is an Eulerian trail in  $K_3$  and in  $K_5$ .

## 23. Graph Theory

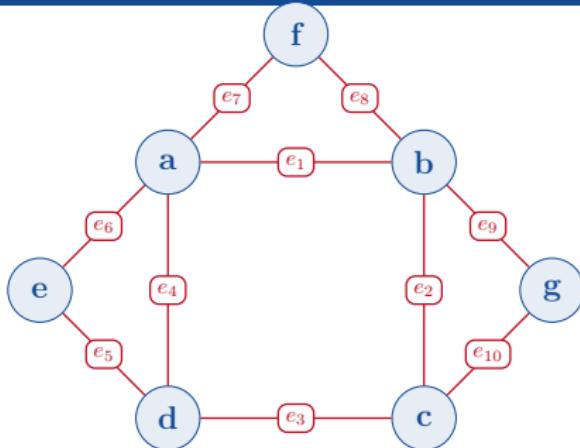


### Example

Note that in  $K_3$  and  $K_5$ , every vertex has even degree. So there is an Eulerian trail in  $K_3$  and in  $K_5$ .

Notice further that all four vertices in  $K_4$  are of odd degree. This means that  $K_4$  does not contain an Eulerian trail.

## 23. Graph Theory



Example

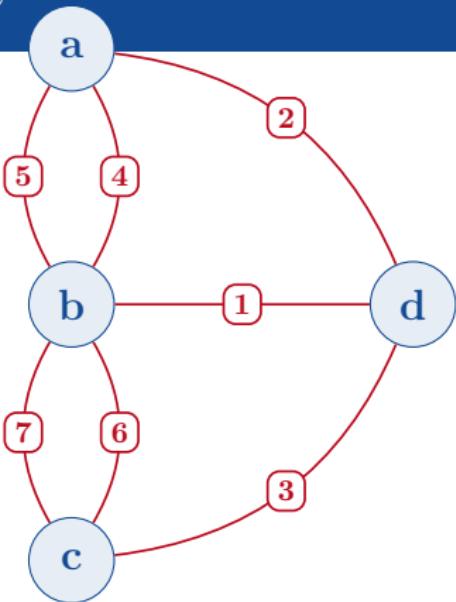
Note that

$$\deg(a) = 4, \deg(b) = 4, \deg(c) = 3, \deg(d) = 3,$$

$$\deg(e) = 2, \deg(f) = 2, \deg(g) = 2.$$

Two of the vertices have odd degree,  $c$  and  $d$ . So there must exist an Eulerian trail and it must start and end at  $c$  and  $d$ . (We have already found this trail.)

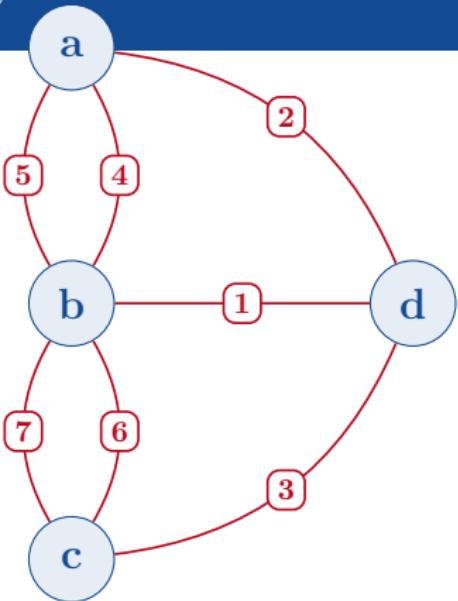
## 23. Graph Theory



Example (The Königsberg Bridge Problem)

Now consider Königsberg as shown above.

## 23. Graph Theory

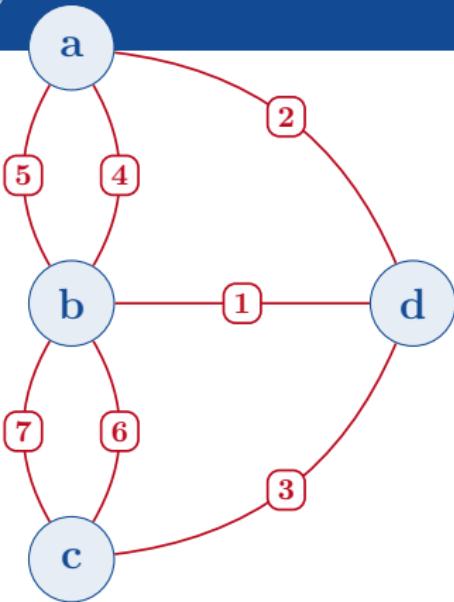


Example (The Königsberg Bridge Problem)

Now consider Königsberg as shown above. Note that

$$\deg(a) = 3, \quad \deg(b) = 5, \quad \deg(c) = 3, \quad \deg(d) = 3.$$

## 23. Graph Theory



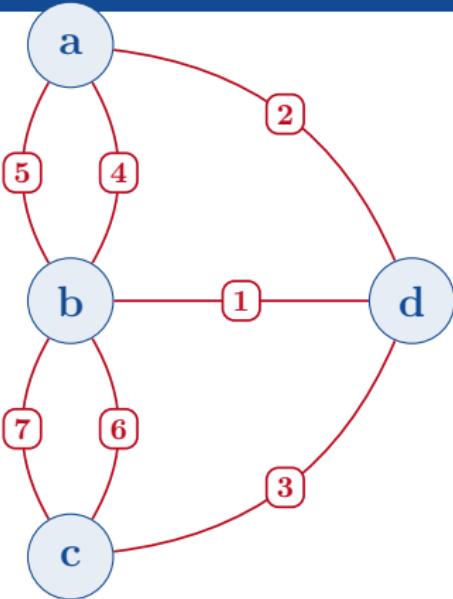
Example (The Königsberg Bridge Problem)

Now consider Königsberg as shown above. Note that

$$\deg(a) = 3, \quad \deg(b) = 5, \quad \deg(c) = 3, \quad \deg(d) = 3.$$

Since all four vertices have odd degree, there does not exist an Eulerian trail in Königsberg.

## 23. Graph Theory



Example (The Königsberg Bridge Problem)

Therefore it was not possible to walk around the city of Königsberg and cross each bridge once.

## Euler's Formula for Polyhedra

*Euler's formula* is

$$n(V) - n(E) + n(F)$$

where

$n(V)$  = number of vertices

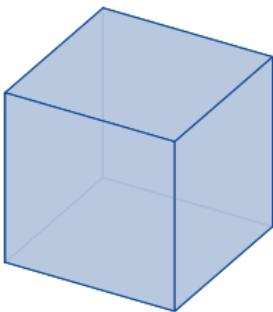
$n(E)$  = number of edges

$n(F)$  = number of faces.

## 23. Graph Theory



cube



$$n(V)$$

$$n(E)$$

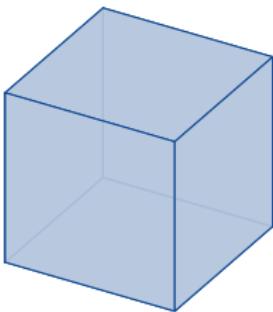
$$n(F)$$

$$n(V) - n(E) + n(F)$$

## 23. Graph Theory



cube



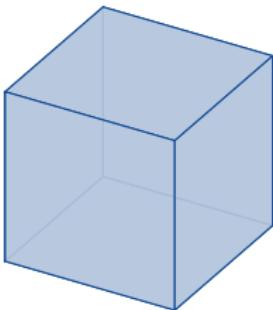
$n(V)$	8
$n(E)$	12
$n(F)$	6

$$n(V) - n(E) + n(F)$$

## 23. Graph Theory



cube



$$n(V) = 8$$

$$n(E) = 12$$

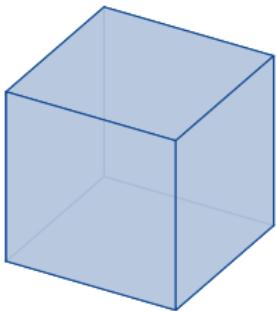
$$n(F) = 6$$

$$n(V) - n(E) + n(F) = 8 - 12 + 6 = \mathbf{2}$$

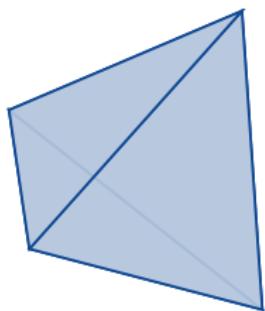
## 23. Graph Theory



cube



tetrahedron

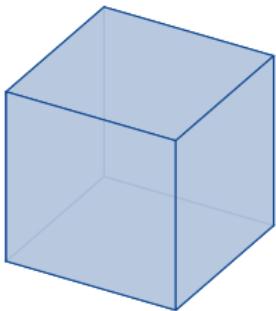


$n(V)$	8
$n(E)$	12
$n(F)$	6

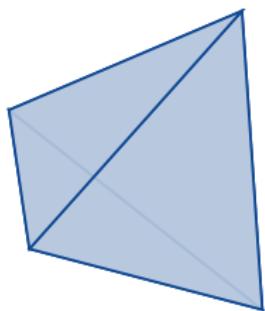
$$n(V) - n(E) + n(F) \quad 8 - 12 + 6 = \mathbf{2}$$

## 23. Graph Theory

cube



tetrahedron

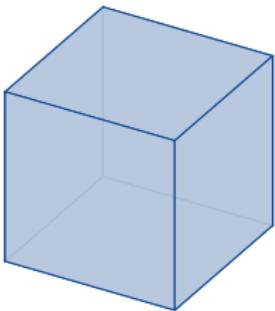


$n(V)$	8	4
$n(E)$	12	6
$n(F)$	6	4

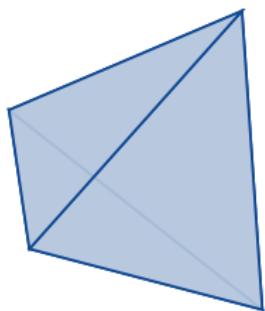
$$n(V) - n(E) + n(F) \quad 8 - 12 + 6 = 2$$

## 23. Graph Theory

cube



tetrahedron



$$n(V)$$

$$8$$

$$4$$

$$n(E)$$

$$12$$

$$6$$

$$n(F)$$

$$6$$

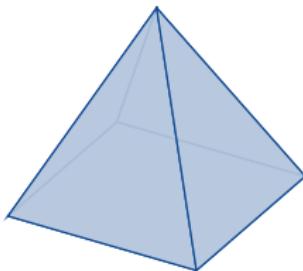
$$4$$

$$n(V) - n(E) + n(F) \quad 8 - 12 + 6 = 2 \quad 4 - 6 + 4 = 2$$

## 23. Graph Theory



pyramid



$$n(V)$$

$$n(E)$$

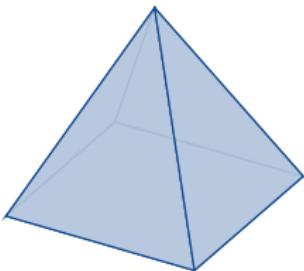
$$n(F)$$

$$n(V) - n(E) + n(F)$$

## 23. Graph Theory



pyramid



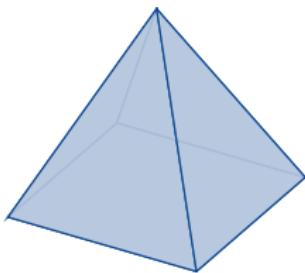
$n(V)$	5
$n(E)$	8
$n(F)$	5

$$n(V) - n(E) + n(F)$$

## 23. Graph Theory



pyramid



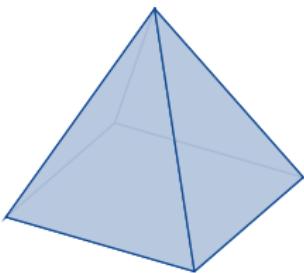
$n(V)$	5
$n(E)$	8
$n(F)$	5

$$n(V) - n(E) + n(F) \quad 5 - 8 + 5 = 2$$

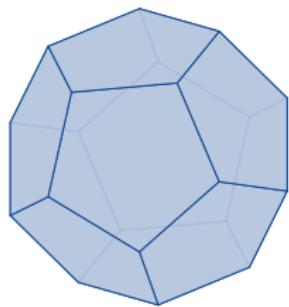
## 23. Graph Theory



pyramid



dodecahedron



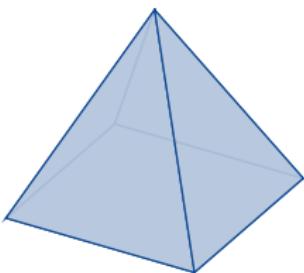
$n(V)$	5
$n(E)$	8
$n(F)$	5

$$n(V) - n(E) + n(F) \quad 5 - 8 + 5 = 2$$

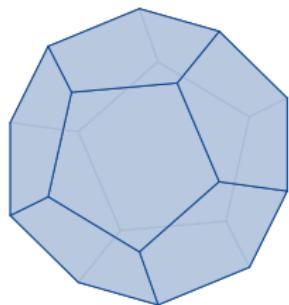
## 23. Graph Theory



pyramid



dodecahedron



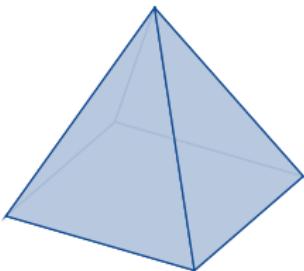
$n(V)$	5	20
$n(E)$	8	30
$n(F)$	5	12

$$n(V) - n(E) + n(F) \quad 5 - 8 + 5 = 2$$

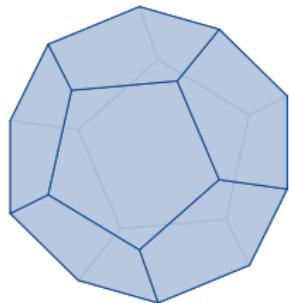
## 23. Graph Theory



pyramid



dodecahedron



$$n(V)$$

$$5$$

$$20$$

$$n(E)$$

$$8$$

$$30$$

$$n(F)$$

$$5$$

$$12$$

$$n(V) - n(E) + n(F)$$

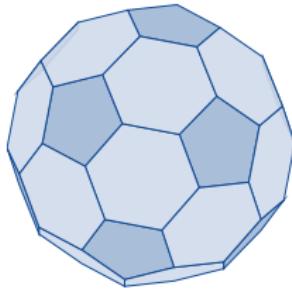
$$5 - 8 + 5 = \mathbf{2}$$

$$20 - 30 + 12 = \mathbf{2}$$

## 23. Graph Theory



football  
(12 pentagons &  
20 hexagons)



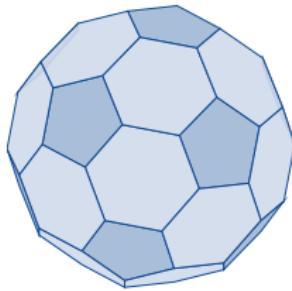
$$\begin{aligned}n(V) \\ n(E) \\ n(F)\end{aligned}$$

$$n(V) - n(E) + n(F)$$

## 23. Graph Theory



football  
(12 pentagons &  
20 hexagons)



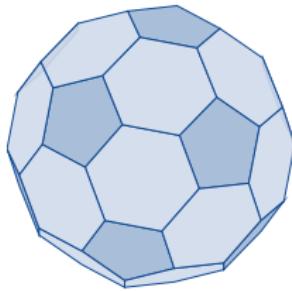
$n(V)$	60
$n(E)$	90
$n(F)$	32

$$n(V) - n(E) + n(F)$$

## 23. Graph Theory



football  
(12 pentagons &  
20 hexagons)



$n(V)$	60
$n(E)$	90
$n(F)$	32

$$n(V) - n(E) + n(F) \quad 60 - 90 + 32 = 2$$

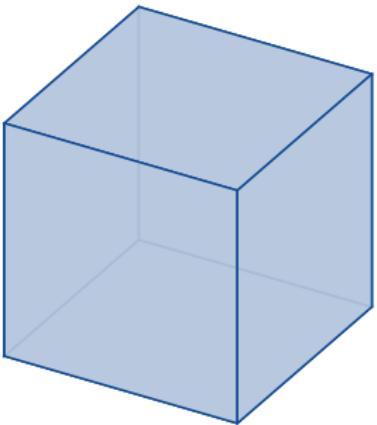
## 23. Graph Theory



### Remark

If we have a polyhedron without any holes in it, is Euler's formula always equal to 2? And if so, how can we prove it?

## 23. Graph Theory

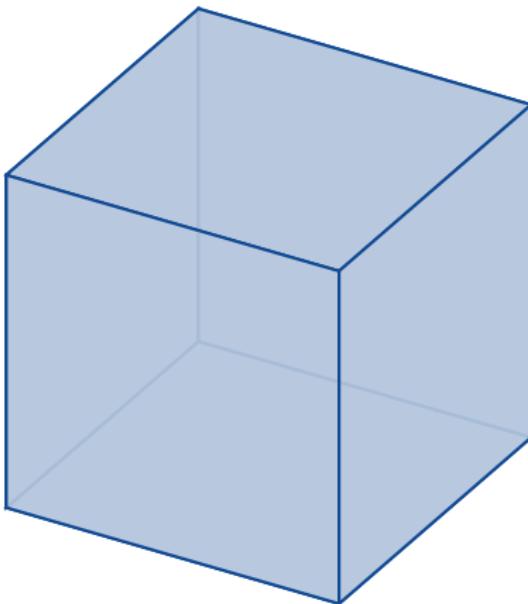


$$n(V) = 8$$

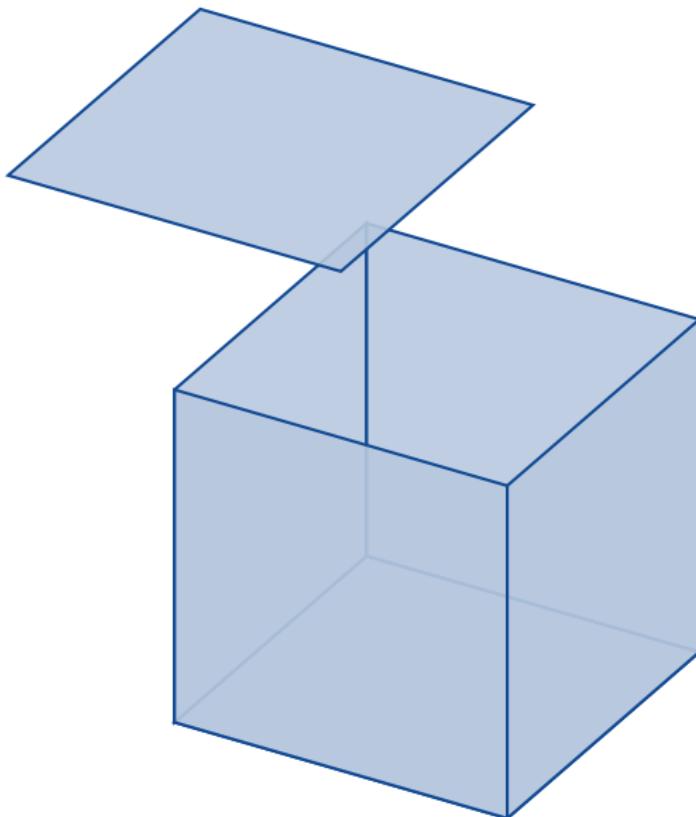
$$n(E) = 12$$

$$n(F) = 6$$

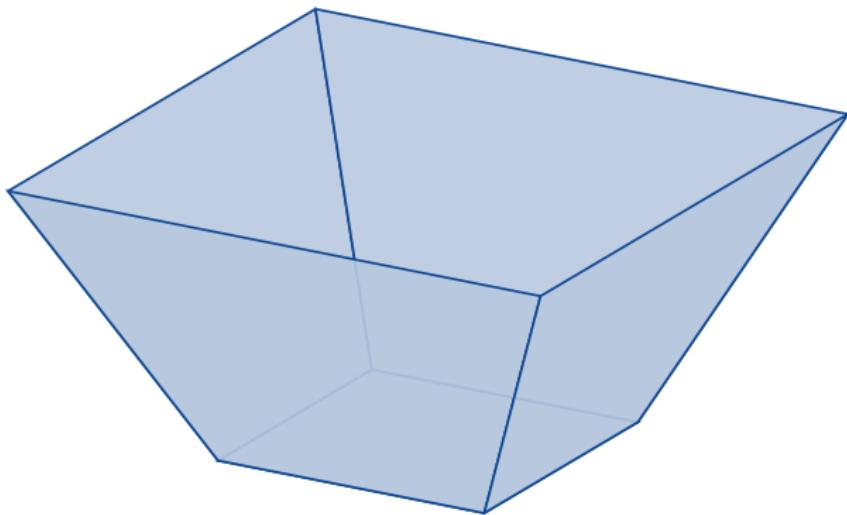
## 23. Graph Theory



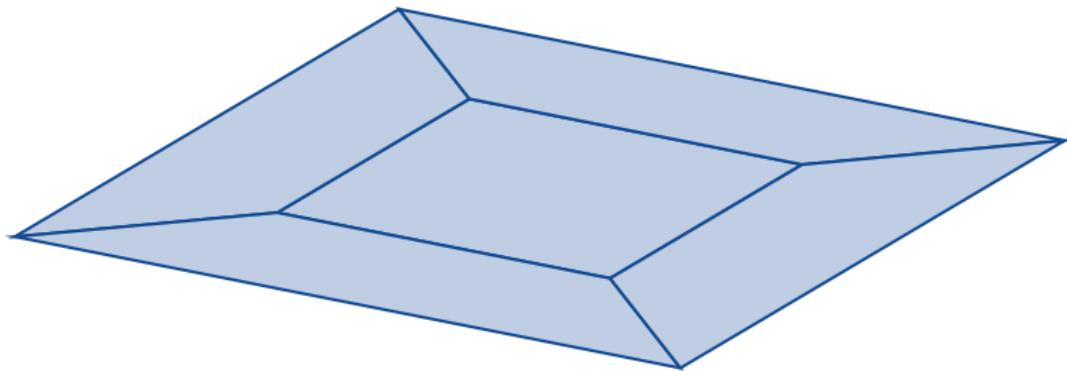
## 23. Graph Theory



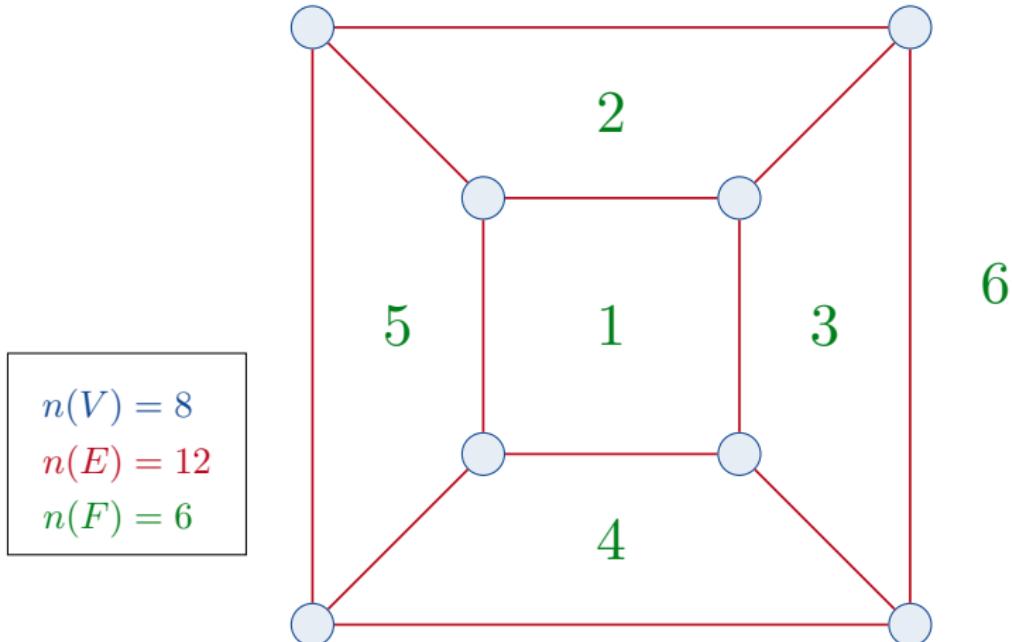
## 23. Graph Theory



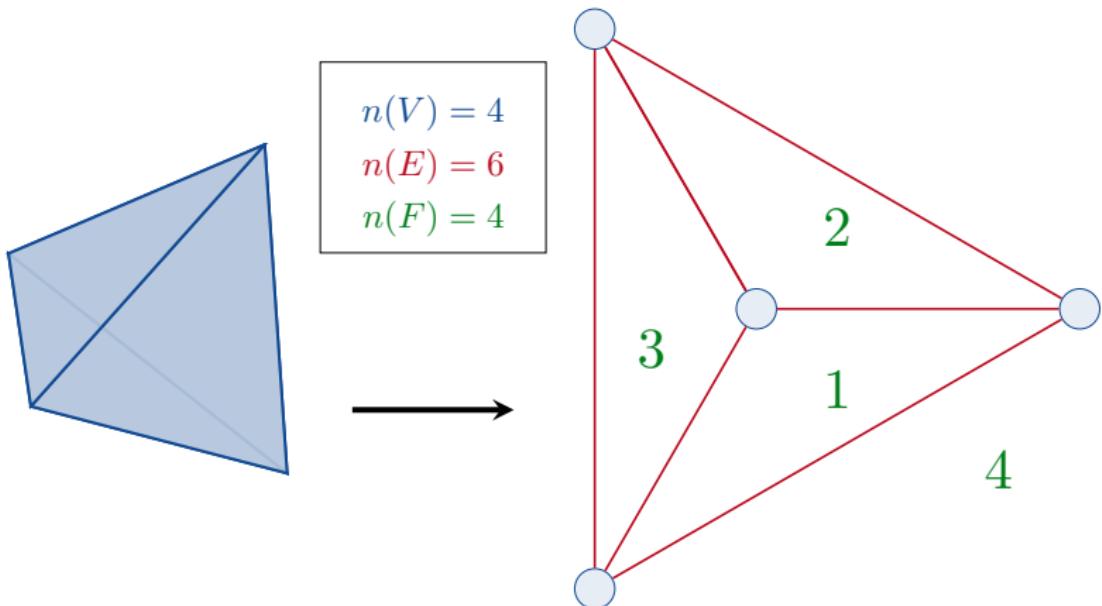
## 23. Graph Theory



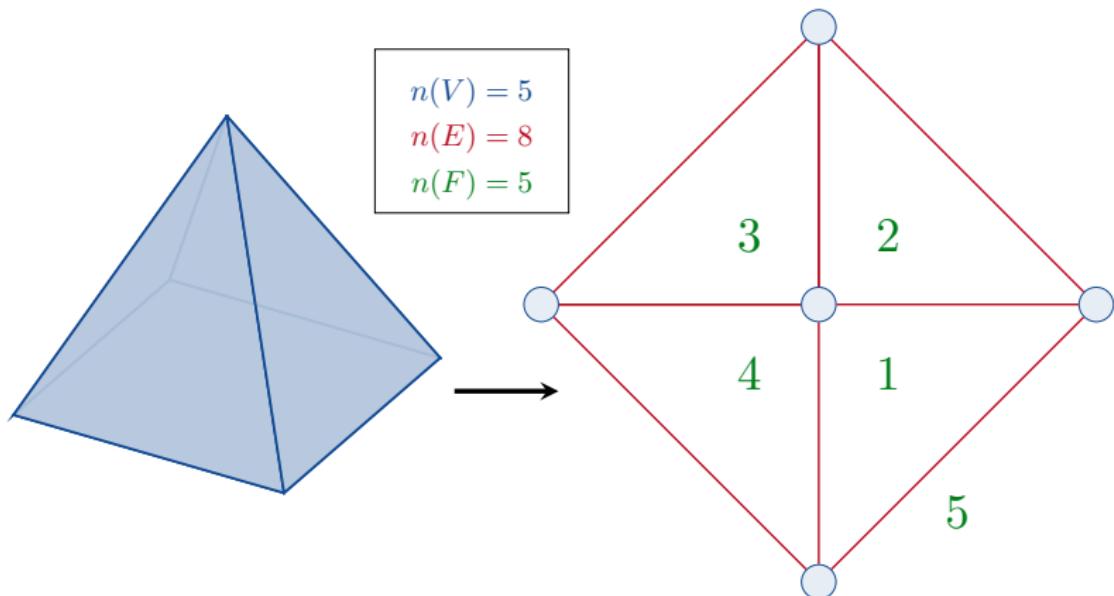
## 23. Graph Theory



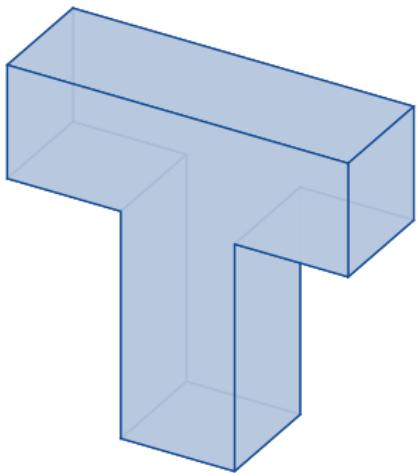
## 23. Graph Theory



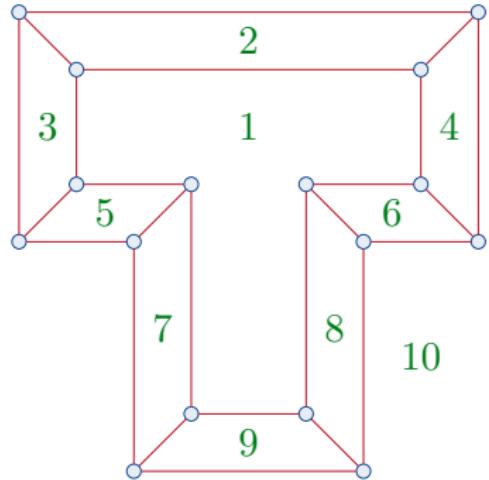
## 23. Graph Theory



## 23. Graph Theory



$$\begin{aligned}n(V) &= 16 \\n(E) &= 24 \\n(F) &= 10\end{aligned}$$



## 23. Graph Theory



Every three dimensional polyhedron is equivalent to a connected, planar, simple graph. So if we know something about these graphs, then we also know it about polyhedra.

## 23. Graph Theory



Every three dimensional polyhedron is equivalent to a connected, planar, simple graph. So if we know something about these graphs, then we also know it about polyhedra.

What do we know about such graphs?

## 23. Graph Theory



Let us start with the first complete graph:

## 23. Graph Theory



Let us start with the first complete graph:

1



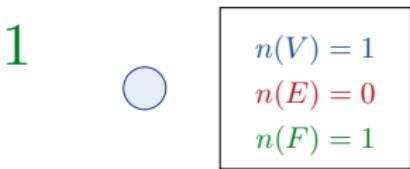
$$\begin{aligned} n(V) &= 1 \\ n(E) &= 0 \\ n(F) &= 1 \end{aligned}$$

This graph,  $K_1$ , is called the *trivial graph*. It has one vertex, zero edges and one face.

## 23. Graph Theory



Let us start with the first complete graph:



This graph,  $K_1$ , is called the *trivial graph*. It has one vertex, zero edges and one face. So

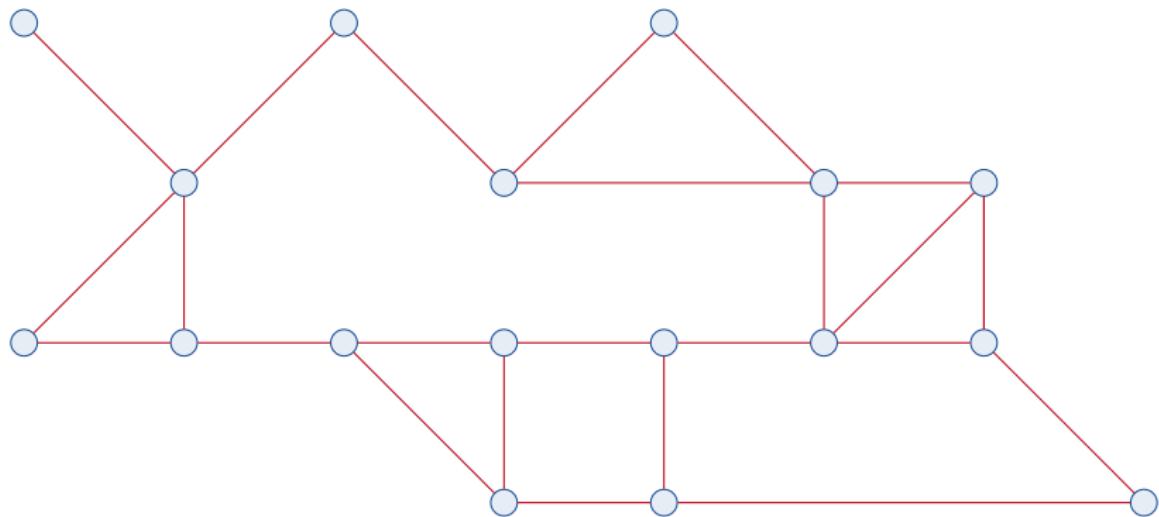
$$n(V) - n(E) + n(F) = 1 - 0 + 1 = 2.$$

## 23. Graph Theory

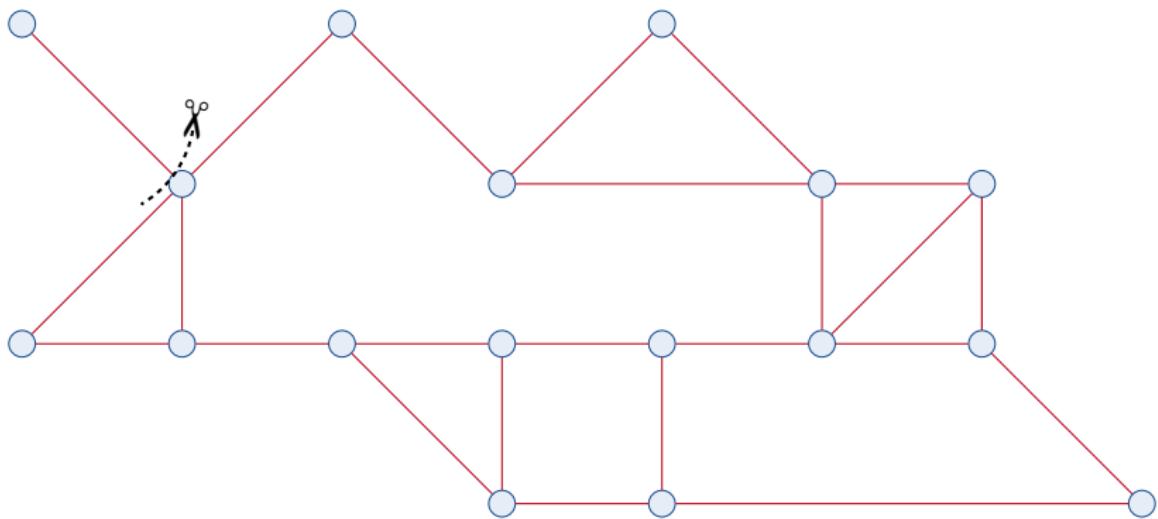


Now let us take any connected, planar, simple graph.

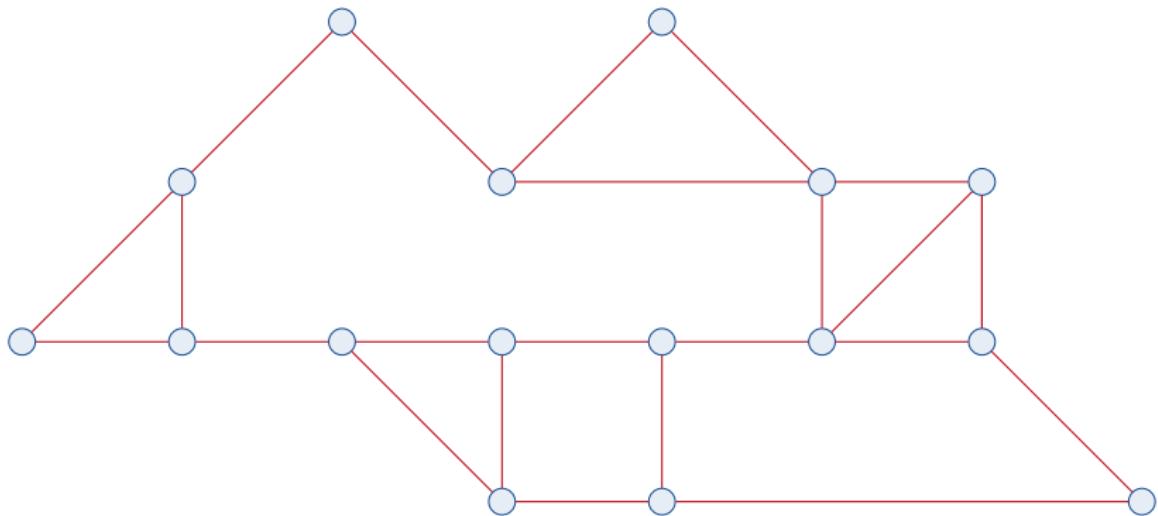
## 23. Graph Theory



## 23. Graph Theory



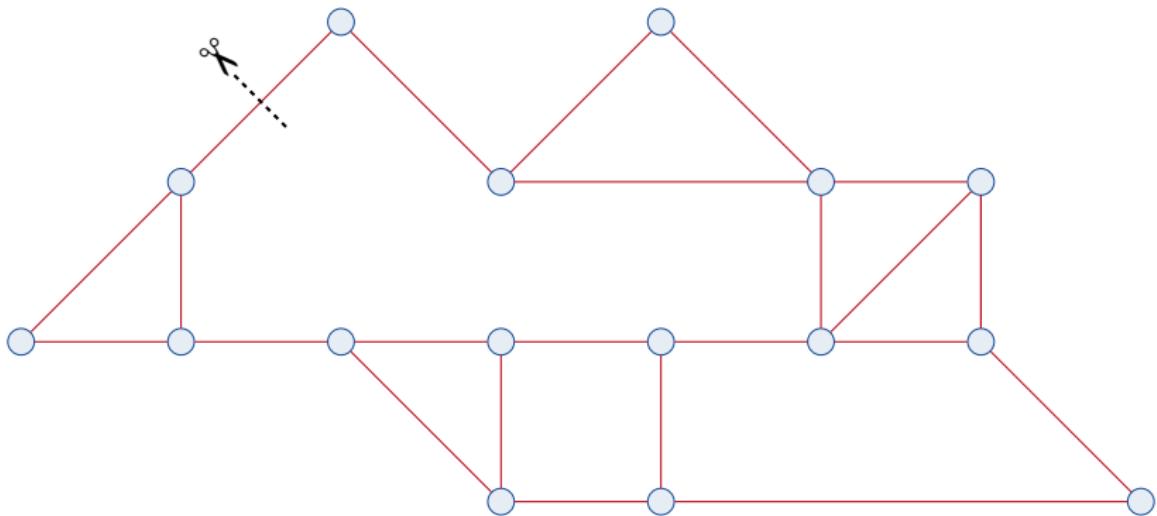
## 23. Graph Theory



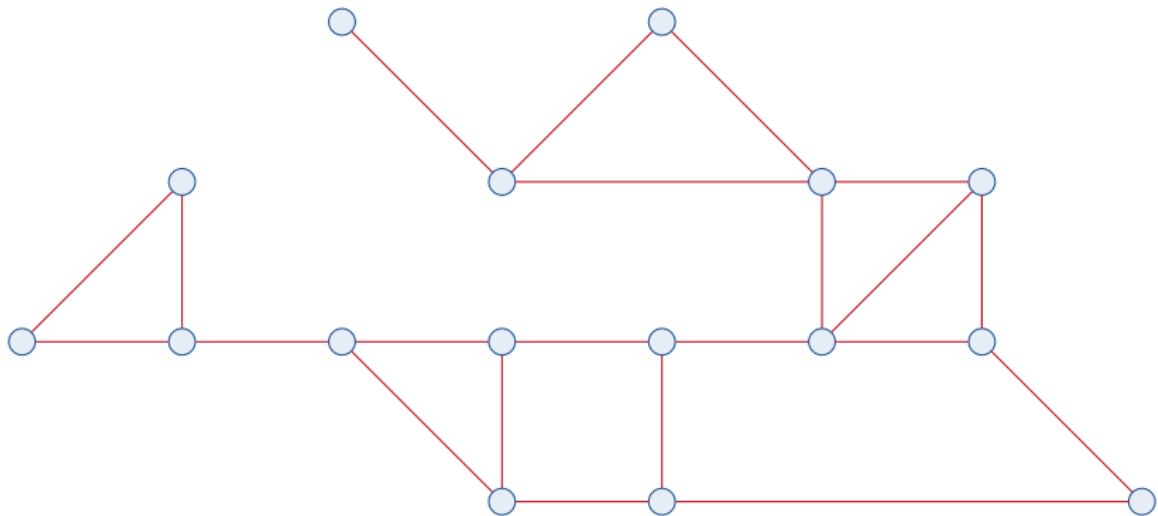
$n(V)$  decreases by 1,       $n(E)$  decreases by 1,       $n(F)$  stays the same,

$n(V) - n(E) + n(F)$  stays the same

## 23. Graph Theory

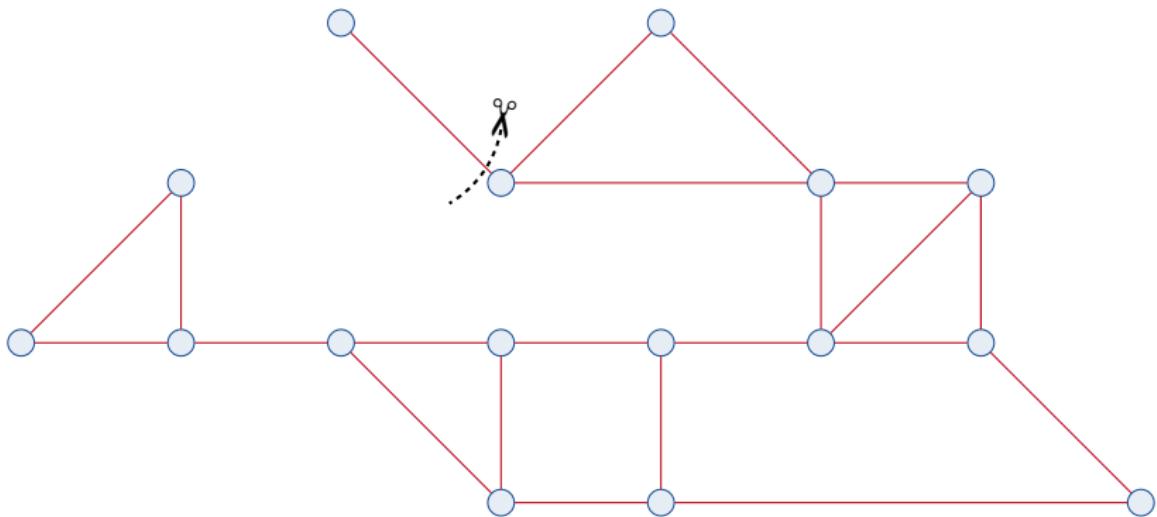


## 23. Graph Theory

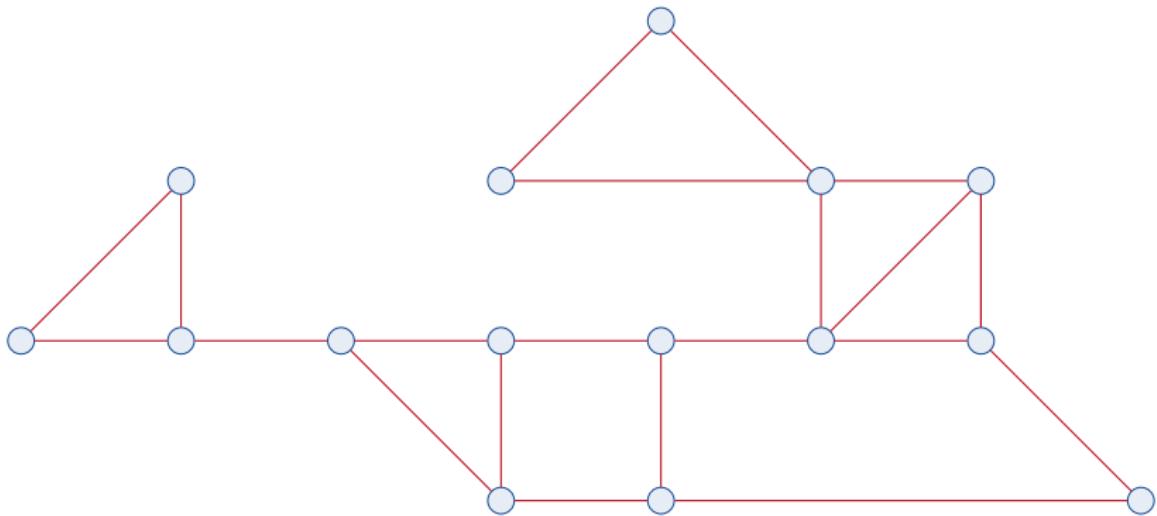


$n(V)$  stays the same,  
 $n(E)$  decreases by 1,  
 $n(F)$  decreases by 1,  
 $n(V) - n(E) + n(F)$  stays the same

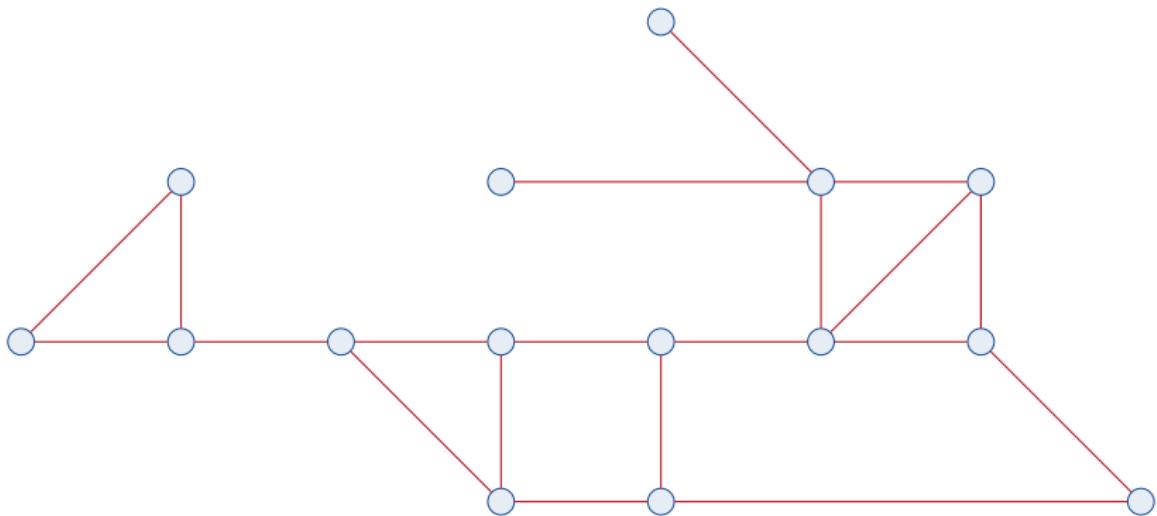
## 23. Graph Theory



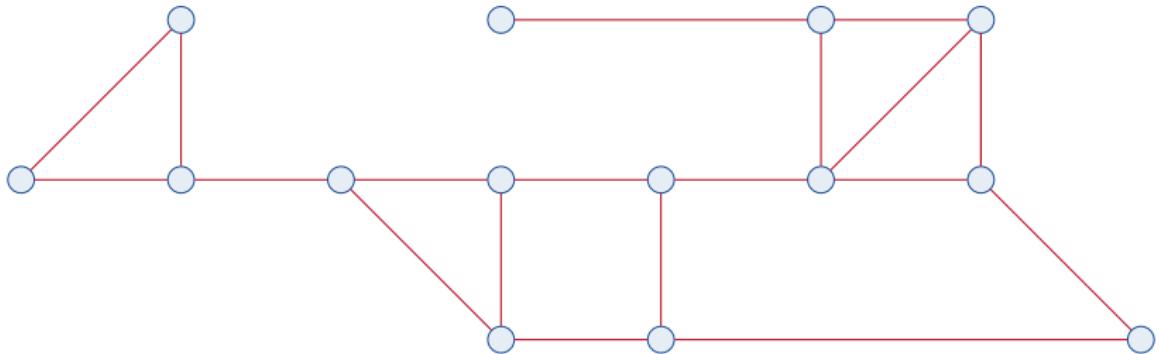
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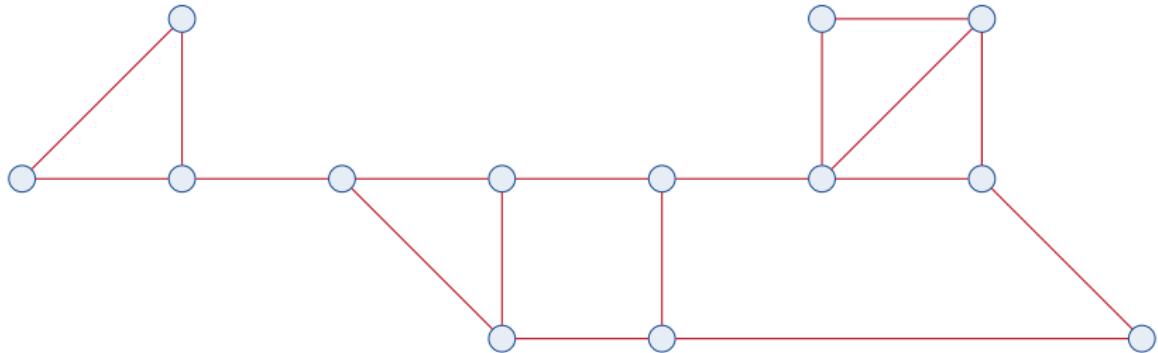
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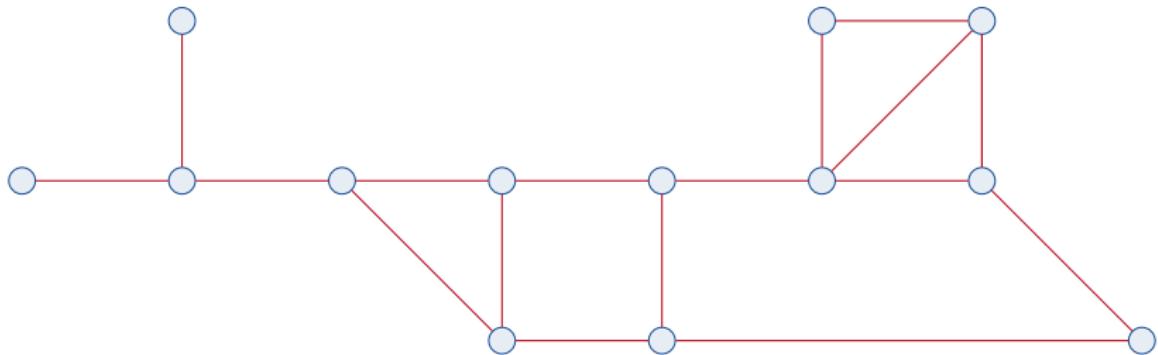
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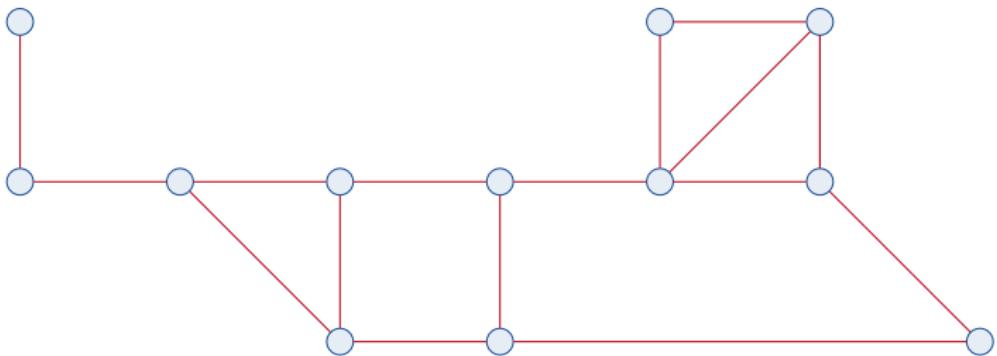
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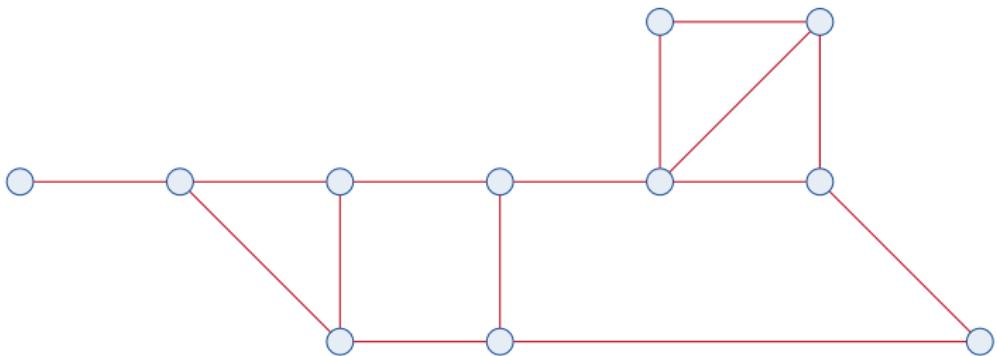
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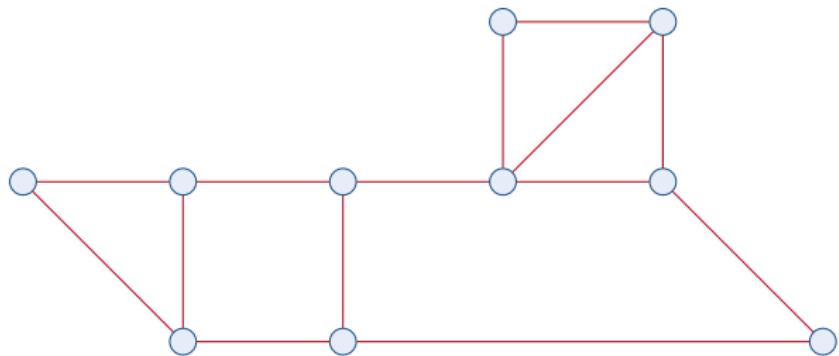
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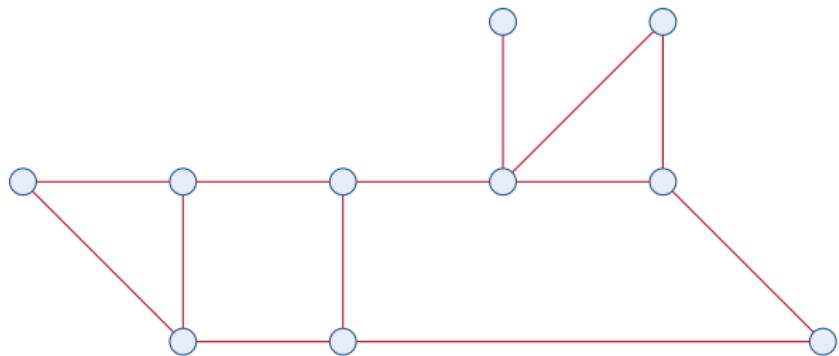
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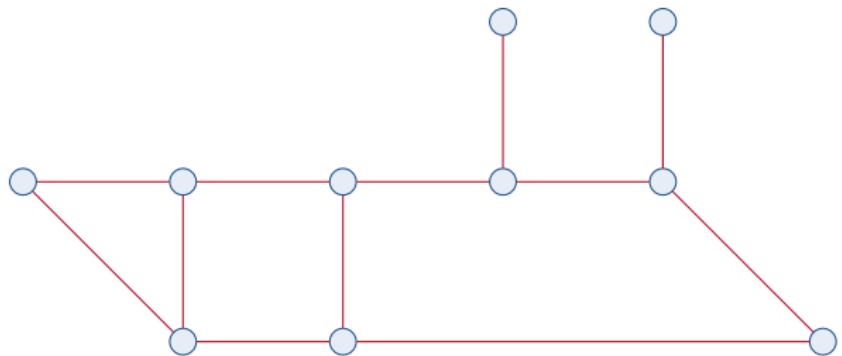
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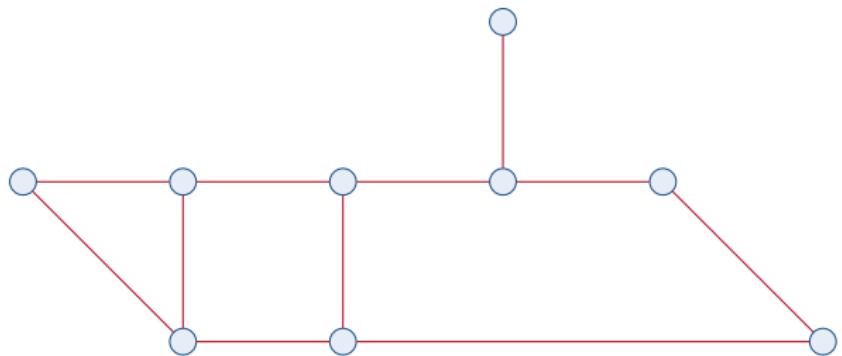
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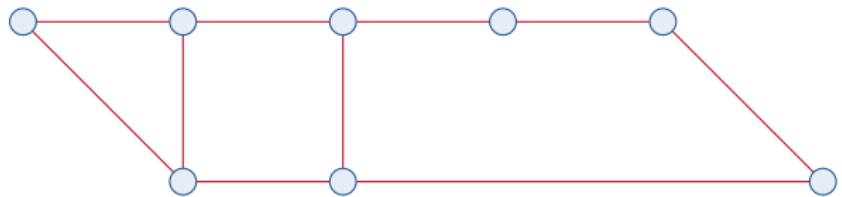
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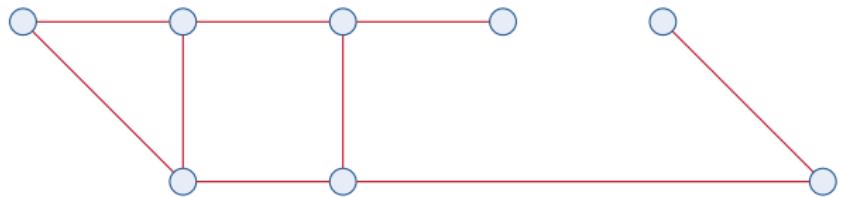
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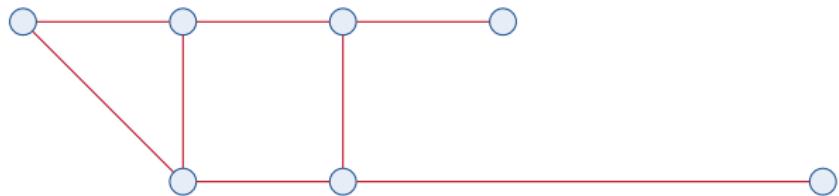
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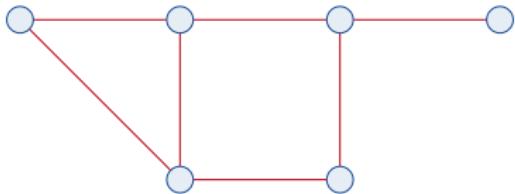
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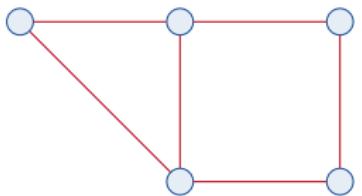
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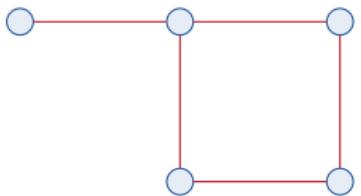
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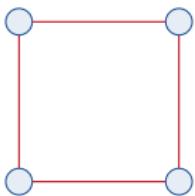
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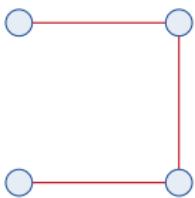
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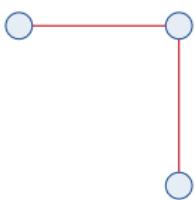
## 23. Graph Theory



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## 23. Graph Theory



### Theorem

*If  $G$  is a connected, planar, simple graph, then*

$$n(V) - n(E) + n(F) = 2.$$

## 23. Graph Theory



### Theorem

*If  $G$  is a connected, planar, simple graph, then*

$$n(V) - n(E) + n(F) = 2.$$

. . . and the same is true for polyhedra without holes.



# Next Time

- 24. Limits
- 25. Continuity
- 26. Differentiation