Week 5

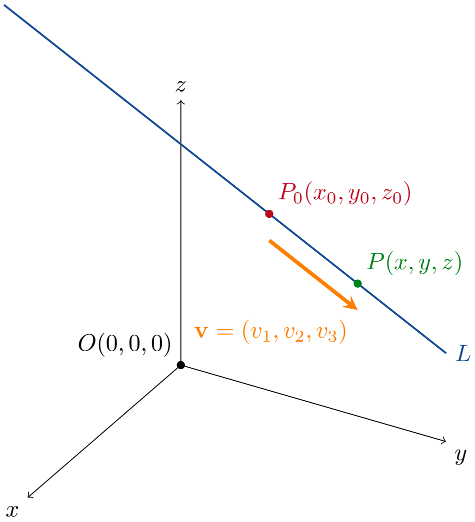
- 14. Lines
- 15. Planes
- 16. Projections

14

Lines

To describe a line in $\mathbb{R}^3$, we need

- a point $P_0(x_0, y_0, z_0)$ which the line passes through; and
- a vector $\mathbf{v}$ which gives the direction of the line.



z
$P_0(x_0, y_0, z_0)$
$P(x, y, z)$
$\mathbf{v} = (v_1, v_2, v_3)$
$O(0,0,0)$
L
y
x

Let $\mathbf{r}_0 = \overrightarrow{OP_0}$ and $\mathbf{r} = \overrightarrow{OP}$.

Definition

The *line* L *passing through* $P_0(x_0, y_0, z_0)$ *parallel to* $\mathbf{v} = (v_1, v_2, v_3)$ has the vector equation

$$\mathbf{r} = \mathbf{r}_0 + t\mathbf{v}, \qquad -\infty < t < \infty.$$

This equation is equivalent to

$$(x, y, z) = (x_0, y_0, z_0) + t(v_1, v_2, v_3)$$

or to the set of three equations

$$x = x_0 + tv_1, \qquad y = y_0 + tv_2, \qquad z = z_0 + tv_3.$$

Definition

The *parametric equations* for the line L passing through $P_0(x_0, y_0, z_0)$ parallel to $\mathbf{v} = (v_1, v_2, v_3)$ are

$$x = x_0 + tv_1, \qquad y = y_0 + tv_2, \qquad z = z_0 + tv_3.$$

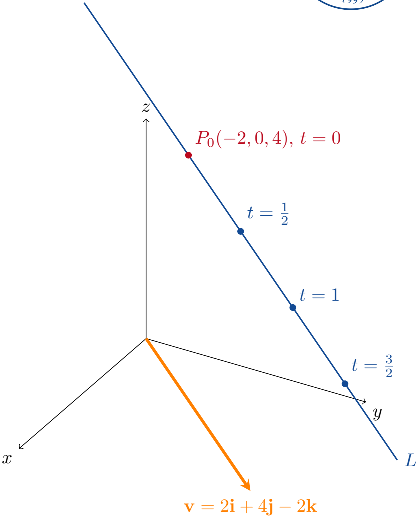
Example

Find parametric equations for the line passing through $P_0(-2, 0, 4)$ parallel to $\mathbf{v} = 2\mathbf{i} + 4\mathbf{j} - 2\mathbf{k}$.

solution: We can write

$$x = -2 + 2t, \qquad y = 4t, \qquad z = 4 - 2t.$$

$$x = -2 + 2t$$
$$y = 4t$$
$$z = 4 - 2t$$



Example

Find parametric equations for the line passing through $P(-3, 2, -3)$ and $Q(1, -1, 4)$.

solution: Choose $P_0 = P$ and $\mathbf{v} = \overrightarrow{PQ} = (4, -3, 7) = 4\mathbf{i} - 3\mathbf{j} + 7\mathbf{k}$. Then we can write

$$x = -3 + 4t, \qquad y = 2 - 3t, \qquad z = -3 + 7t.$$

Definition

The vector equation

$$\mathbf{r} = \mathbf{r}_0 + t\mathbf{v}, \qquad a \leq t \leq b$$

denotes a *line segment*.

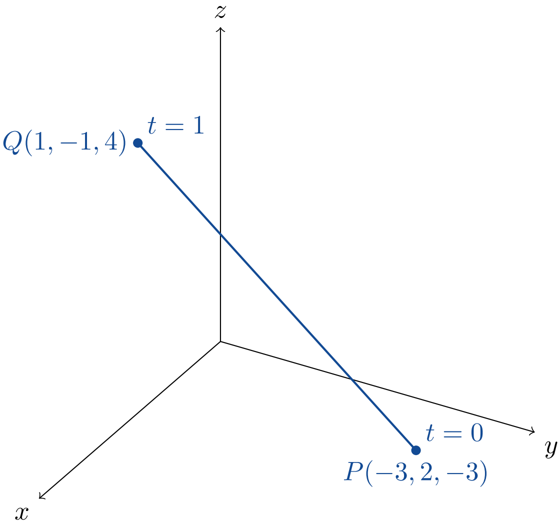
Example

Parametrise the line segment joining $P(-3, 2, -3)$ and $Q(1, -1, 4)$.

solution: We know that $x = -3 + 4t$, $y = 2 - 3t$ and $z = -3 + 7t$. The line passes through P then $t = 0$ and passed through Q when $t = 1$. Therefore

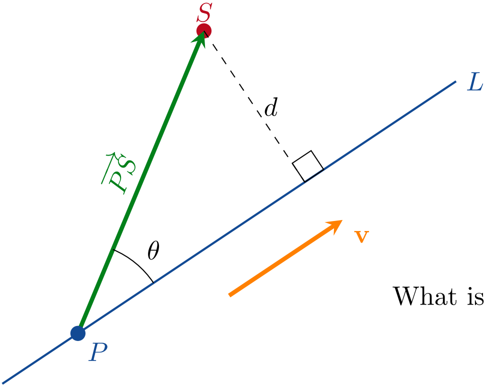
$$x = -3 + 4t, \qquad y = 2 - 3t, \qquad z = -3 + 7t, \qquad 0 \leq t \leq 1$$

denotes the line segment from P to Q.



z
$t = 1$
$Q(1, -1, 4)$
$t = 0$
y
$P(-3, 2, -3)$
x

The Distance from a Point to a Line



What is d?

Let d be the shortest distance from the point S to the line L. We can see from this diagram that

$$d = \left\| \overrightarrow{PS} \right\| \sin\theta.$$

But remember that $\overrightarrow{PS} \times \mathbf{v} = \left\| \overrightarrow{PS} \right\| \|\mathbf{v}\| \sin\theta \; \mathbf{n}$. Therefore

$$\boxed{d = \frac{\left\| \overrightarrow{PS} \times \mathbf{v} \right\|}{\|\mathbf{v}\|}.}$$

Example

Find the distance from the point $S(1,1,5)$ to the line

$$x = 1 + t, \qquad y = 3 - t, \qquad z = 2t.$$

solution: The line passes through the point $P(1,3,0)$ in the direction $\mathbf{v} = \mathbf{i} - \mathbf{j} + 2\mathbf{k}$. Thus

$$\overrightarrow{PS} = S - P = (1,1,5) - (1,3,0) = (0,-2,5) = -2\mathbf{j} + 5\mathbf{k}$$

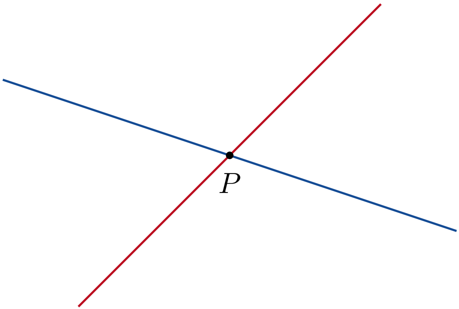
and

$$\overrightarrow{PS} \times \mathbf{v} = (-4+5)\mathbf{i} - (0-5)\mathbf{j} + (0+2)\mathbf{k} = \mathbf{i} + 5\mathbf{j} + 2\mathbf{k}.$$

Therefore

$$d = \frac{\left\|\overrightarrow{PS} \times \mathbf{v}\right\|}{\|\mathbf{v}\|} = \frac{\sqrt{1^2 + 5^2 + 2^2}}{\sqrt{1^2 + 1^2 + 2^2}} = \frac{\sqrt{30}}{\sqrt{6}} = \sqrt{5}.$$

Intersecting Lines



Definition

Two lines intersect at a point P if and only if P lies on both lines.

Example

Do the following two lines intersect? Is yes, where?

1. $x = 7 - t$, $y = 3 + 3t$, $z = 2t$.
2. $x = -1 + 2s$, $y = 3s$, $z = 1 + s$.

solution: The two lines intersect if and only if there exist $s, t \in \mathbb{R}$ such that

$$\begin{aligned} 7 - t = x &= -1 + 2s && \implies t = 8 - 2s \\ 3 + 3t = y &= 3s && \implies s = t + 1 \\ 2t = z &= 1 + s \end{aligned}$$

The first equation tells us that $t = 8 - 2s$. Putting this into the second equation gives $s = t + 1 = (8 - 2s) + 1 = 9 - 2s$ which implies that $s = 3$ and $t = 2$. We must check the third equation:

$2t = 2 \times 2 = 4 = 1 + 3 = 1 + s$. Because the third equation is also true, we know that they two lines intersect at $P(5, 9, 4)$.

Example

Do the following two lines intersect? If yes, where?

1. $x = 1 + t$, $y = 3t$, $z = 3 + 3t$.
2. $x = -1 + 2s$, $y = 3s$, $z = 1 + s$.

solution: Can we find $s, t \in \mathbb{R}$ such that

$$\left.\begin{aligned} 1 + t &= x = -1 + 2s \\ 3t &= y = 3s \\ 3 + 3t &= z = 1 + s \end{aligned}\right. \qquad \implies \quad s = t$$

are all true?

The second equation gives $s = t$. Thus

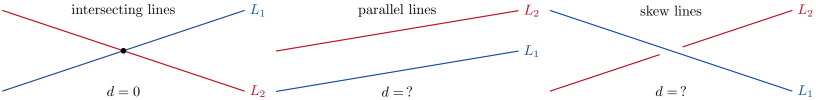
$1 + t = -1 + 2t \implies 2 + t = 2t \implies t = 2$. However

$3 + 3t = 1 + t \implies 2 + 2t = 0 \implies t = -2 \neq 2$. Therefore it is not possible to find an s and a t. Hence the lines do not intersect.

The Distance Between Two Lines

There are three cases to consider:

- the lines intersect;
- the lines do not intersect and are parallel ($\mathbf{v}_1 = k\mathbf{v}_2$ for some $k \in \mathbb{R}$); or
- the lines do not intersect and are skew ($\mathbf{v}_1 \neq k\mathbf{v}_2$ for all $k \in \mathbb{R}$).

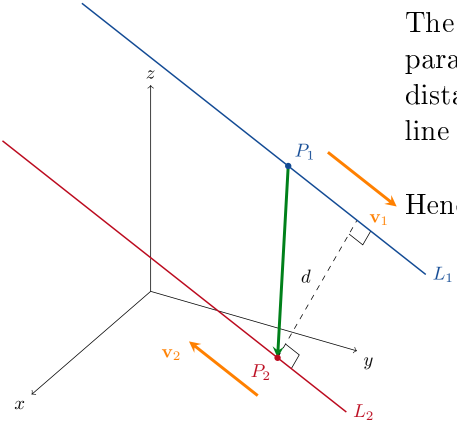


Intersecting Lines

Clearly the distance between intersecting lines is zero. Hence

$$\boxed{d = 0.}$$

Parallel Lines ($\mathbf{v}_1 \times \mathbf{v}_2 = \mathbf{0}$)

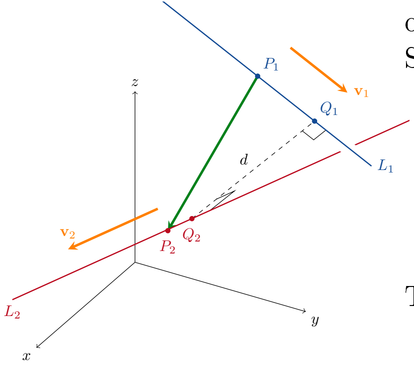


The distance between the two parallel lines is the same as the distance between P_2 and the line L_1.

Hence

$$\boxed{d = \frac{\left\|\overrightarrow{P_1P_2} \times \mathbf{v}_1\right\|}{\|\mathbf{v}_1\|}.}$$

Skew Lines ($\mathbf{v}_1 \times \mathbf{v}_2 \neq \mathbf{0}$)



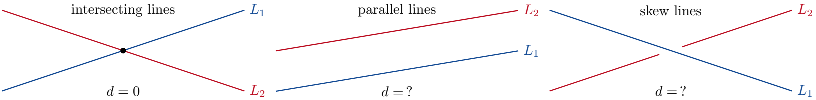
Let $\mathbf{n} = \mathbf{v}_1 \times \mathbf{v}_2$. Then $\mathbf{n}$ is orthogonal to both $\mathbf{v}_1$ and $\mathbf{v}_2$. So

$$d = \left\| \overrightarrow{Q_1Q_2} \right\| = \left\| \operatorname{proj}_{\mathbf{n}} \overrightarrow{P_1P_2} \right\| = \frac{\left| \overrightarrow{P_1P_2} \bullet \mathbf{n} \right|}{\|\mathbf{n}\|}.$$

Thus

$$\boxed{d = \frac{\left| \overrightarrow{P_1P_2} \bullet (\mathbf{v}_1 \times \mathbf{v}_2) \right|}{\|\mathbf{v}_1 \times \mathbf{v}_2\|}.}$$

14. Lines



- Intersecting Lines: $d = 0$.
- Parallel Lines ($\mathbf{v}_1 \times \mathbf{v}_2 = \mathbf{0}$): $d = \dfrac{\left\|\overrightarrow{P_1P_2} \times \mathbf{v}_1\right\|}{\|\mathbf{v}_1\|}$.
- Skew Lines ($\mathbf{v}_1 \times \mathbf{v}_2 \neq \mathbf{0}$): $d = \dfrac{\left|\overrightarrow{P_1P_2} \bullet (\mathbf{v}_1 \times \mathbf{v}_2)\right|}{\|\mathbf{v}_1 \times \mathbf{v}_2\|}$.

Example

Find the distance between the following two lines.
line 1: $x = 0$, $y = -t$, $z = t$,
line 2: $x = 1 + 2s$, $y = s$, $z = -3s$.

solution: We have that $P_1(0,0,0)$, $\mathbf{v}_1 = -\mathbf{j} + \mathbf{k}$, $P_2(1,0,0)$ and $\mathbf{v}_2 = 2\mathbf{i} + \mathbf{j} - 3\mathbf{k}$. Since

$$\mathbf{v}_1 \times \mathbf{v}_2 = 2\mathbf{i} + 2\mathbf{j} + 2\mathbf{k} \neq \mathbf{0},$$

the lines are skew. (Recall that we have $\mathbf{v}_1 \times \mathbf{v}_2 = \mathbf{0}$ for parallel vectors.) Moreover note that $\overrightarrow{P_1P_2} = \mathbf{i}$. Then we calculate that

$$d = \frac{\left|\overrightarrow{P_1P_2} \cdot (\mathbf{v}_1 \times \mathbf{v}_2)\right|}{\|\mathbf{v}_1 \times \mathbf{v}_2\|} = \frac{|(\mathbf{i}) \cdot (2\mathbf{i} + 2\mathbf{j} + 2\mathbf{k})|}{\|2\mathbf{i} + 2\mathbf{j} + 2\mathbf{k}\|}$$
$$= \frac{|2 + 0 + 0|}{\sqrt{2^2 + 2^2 + 2^2}} = \frac{1}{\sqrt{3}}.$$

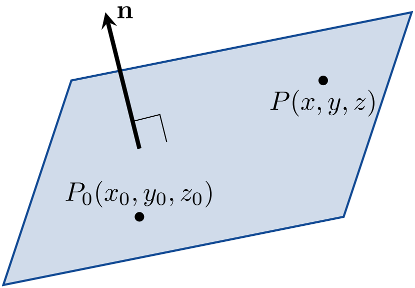
15

Planes

To describe a plane, we need

- a point $P_0(x_0, y_0, z_0)$ which the plane passes through; and
- a vector $\mathbf{n} = A\mathbf{i} + B\mathbf{j} + C\mathbf{k}$ which is perpendicular to the plane.

The vector $\mathbf{n}$ is said to be *normal* to the plane.



Definition

The plane passing through the point $P_0(x_0, y_0, z_0)$ with normal vector $\mathbf{n} = A\mathbf{i} + B\mathbf{j} + C\mathbf{k}$ has the vector equation

$$\mathbf{n} \cdot \overrightarrow{P_0P} = 0.$$

Writing this equation in coordinates, we have

$$A(x - x_0) + B(y - y_0) + C(z - z_0) = 0$$

or

$$\boxed{Ax + By + Cz = D}$$

where $D = Ax_0 + By_0 + Cz_0$ is a constant.

Example

Find an equation for the plane passing through $P_0(-3, 0, 7)$ normal to $\mathbf{n} = 5\mathbf{i} + 2\mathbf{j} - \mathbf{k}$.

solution:

$$\begin{aligned}
A(x - x_0) + B(y - y_0) + C(z - z_0) &= 0 \\
5(x - (-3)) + 2(y - 0) + (-1)(z - 7) &= 0 \\
5x - 15 + 2y - z + 7 &= 0 \\
5x + 2y - z &= -22.
\end{aligned}$$

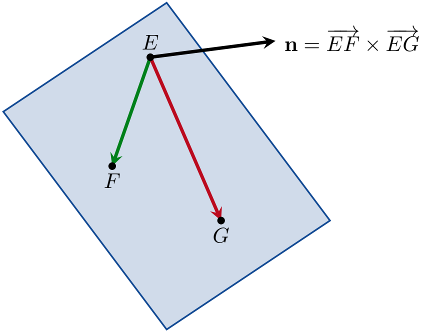
Remark

The vector $\mathbf{n} = A\mathbf{i} + B\mathbf{j} + C\mathbf{k}$ is normal to the plane $Ax + By + Cz = D$.

Example

Find a vector normal to the plane $x + 2y + 3z = 4$.

solution: We can immediately write down $\mathbf{n} = \mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$.



Example

Find an equation for the plane containing the points $E(0,0,1)$, $F(2,0,0)$ and $G(0,3,0)$.

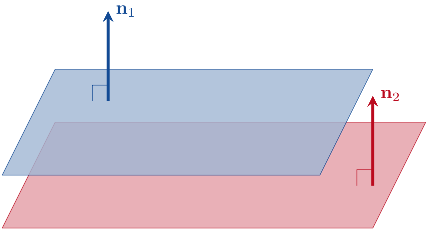
solution: First we need to find a vector normal to the plane. Since $\overrightarrow{EF} = 2\mathbf{i} - \mathbf{k}$ and $\overrightarrow{EG} = 3\mathbf{j} - \mathbf{k}$, we have that

$$\begin{aligned}\mathbf{n} &= \overrightarrow{EF} \times \overrightarrow{EG} = (0 - -3)\mathbf{i} - (-2 - 0)\mathbf{j} + (6 - 0)\mathbf{k} \\ &= 3\mathbf{i} + 2\mathbf{j} + 6\mathbf{k}\end{aligned}$$

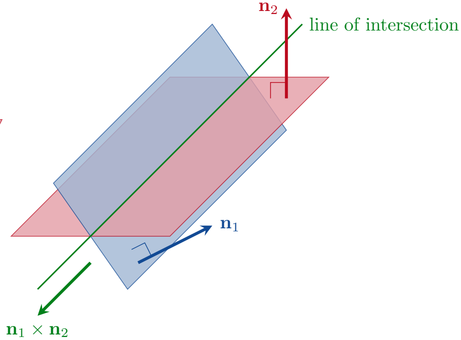
is normal to the plane. Using $P_0 = E(0, 0, 1)$, the equation for the plane is

$$\begin{aligned}3(x - 0) + 2(y - 0) + 6(z - 1) &= 0 \\ 3x + 2y + 6z &= 6.\end{aligned}$$

Lines of Intersection



Two planes are parallel $\iff$ $\mathbf{n}_1 = k\mathbf{n}_2$ for some $k \in \mathbb{R}$.



Two planes intersect in a line $\iff$ $\mathbf{n}_1 \neq k\mathbf{n}_2$ for all $k \in \mathbb{R}$.

Example

Find a vector parallel of the line of intersection of the planes $3x - 6y - 2z = 15$ and $2x + y - 2z = 5$.

solution: We can immediately write down $\mathbf{n}_1 = 3\mathbf{i} - 6\mathbf{j} - 2\mathbf{k}$ and $\mathbf{n}_2 = 2\mathbf{i} + \mathbf{j} - 2\mathbf{k}$. A vector parallel to the line of intersection is

$$\mathbf{n}_1 \times \mathbf{n}_2 = (12 + 2)\mathbf{i} - (-6 + 4)\mathbf{j} + (3 + 12)\mathbf{k} = 14\mathbf{i} + 2\mathbf{j} + 15\mathbf{k}.$$

Example

Find the point where the line $x = \frac{8}{3} + 2t$, $y = -2t$, $z = 1 + t$ intersects the plane $3x + 2y + 6z = 6$.

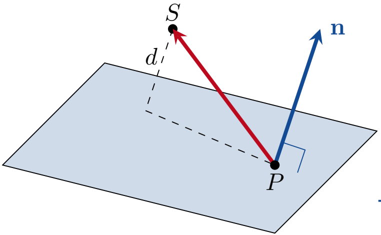
solution: We calculate that

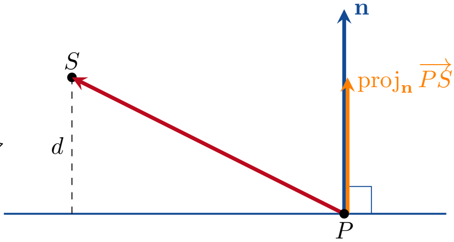
$$\begin{aligned} 3x + 2y + 6z &= 6 \\ 3\left(\tfrac{8}{3} + 2t\right) + 2(-2t) + 6(1+t) &= 6 \\ 8 + 6t - 4t + 6 + 6t &= 6 \\ 8t &= -8 \\ t &= -1. \end{aligned}$$

The point of intersection is

$$P(x,y,z)\big|_{t=-1} = P\left(\frac{8}{3} + 2t, -2t, 1+t\right)\bigg|_{t=-1} = P\left(\frac{2}{3}, 2, 0\right).$$

The Distance from a Point to a Plane





We can see that $d = \left\| \mathrm{proj}_{\mathbf{n}}\,\overrightarrow{PS} \right\|$. Therefore the distance from a point S to a plane containing the point P is

$$\boxed{d = \frac{\left|\overrightarrow{PS} \cdot \mathbf{n}\right|}{\|\mathbf{n}\|}.}$$

Example

Find the distance from the point $S(1,2,3)$ to the plane $x + 2y + 3z = 4$.

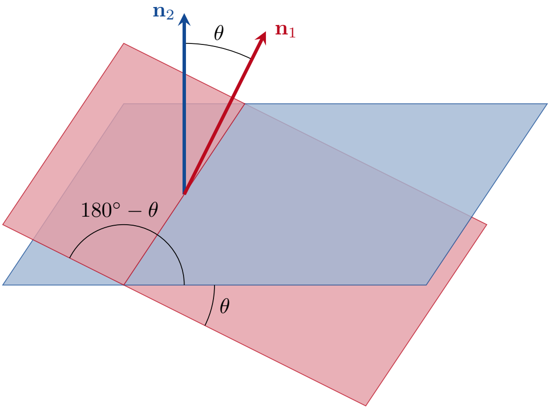
solution: First we need a point in the plane. Setting $y = 0$ and $z = 0$ we must have $x = 4 - 2y - 3z = 4$. Therefore $P(4,0,0)$ is in the plane. Clearly $\mathbf{n} = \mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$.
Therefore the required distance is

$$d = \frac{\left|\overrightarrow{PS} \cdot \mathbf{n}\right|}{\|\mathbf{n}\|} = \frac{|(-3\mathbf{i} + 2\mathbf{j} + 3\mathbf{k}) \cdot (\mathbf{i} + 2\mathbf{j} + 3\mathbf{k})|}{\|\mathbf{i} + 2\mathbf{j} + 3\mathbf{k}\|}$$

$$= \frac{|-3 + 4 + 9|}{\sqrt{1^2 + 2^2 + 3^2}} = \frac{10}{\sqrt{14}}.$$

Angles Between Planes

There are two possible angles that can be measured between planes. We are interested in the smaller angle.



Definition

The angle between two planes is defined to be equal to whichever of the following angles is smaller

- the angle between $\mathbf{n}_1$ and $\mathbf{n}_2$;
- 180° minus the angle between $\mathbf{n}_1$ and $\mathbf{n}_2$.

The angle between two planes will always be between 0° and 90°.

Example

Find the angle between the planes $3x - 6y - 2z = 15$ and $-2x - y + 2z = 5$.

solution: We have normal vectors $\mathbf{n}_1 = 3\mathbf{i} - 6\mathbf{j} - 2\mathbf{k}$ and $\mathbf{n}_2 = -2\mathbf{i} - \mathbf{j} + 2\mathbf{k}$. The angle between $\mathbf{n}_1$ and $\mathbf{n}_2$ is

$$\theta = \cos^{-1}\left(\frac{\mathbf{n}_1 \cdot \mathbf{n}_2}{\|\mathbf{n}_1\|\,\|\mathbf{n}_2\|}\right) = \cos^{-1}\left(\frac{-4}{21}\right) \approx 101^\circ.$$

Because $101^\circ > 90^\circ$, the angle between the two planes is approximately $180 - 101^\circ = 79^\circ$.

16

Projections

Recall that last week we defined the projection of a vector $\mathbf{u}$ onto a vector $\mathbf{v}$ to be

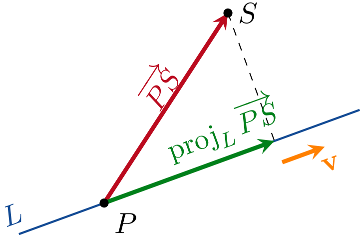
$$\operatorname{proj}_{\mathbf{v}} \mathbf{u} = \left(\frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{v}\|^2} \right) \mathbf{v}.$$

Projection of a Vector onto a Line

Definition

Let L be the line passing through the point P in the direction $\mathbf{v}$. The projection of a vector $\mathbf{u}$ onto the line L is

$$\operatorname{proj}_L \mathbf{u} = \operatorname{proj}_{\mathbf{v}} \mathbf{u}.$$



Example

Find the projection of the vector $\mathbf{u} = 2\mathbf{i} - \mathbf{j} + 3\mathbf{k}$ onto the line $x = 1 + 2t$, $y = 2 - t$, $z = 4 - 4t$.

solution: Clearly $\mathbf{v} = 2\mathbf{i} - \mathbf{j} - 4\mathbf{k}$ is parallel to the line. Thus

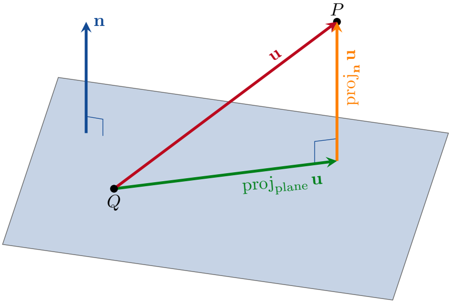
$$\begin{aligned}
\operatorname{proj}_L \mathbf{u} = \operatorname{proj}_{\mathbf{v}} \mathbf{u} &= \left(\frac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{v}\|^2}\right)\mathbf{v} \\
&= \left(\frac{4+1-12}{2^2+(-1)^2+(-4)^2}\right)(2\mathbf{i} - \mathbf{j} - 4\mathbf{k}) \\
&= \left(\frac{-7}{21}\right)(2\mathbf{i} - \mathbf{j} - 4\mathbf{k}) \\
&= -\frac{1}{3}(2\mathbf{i} - \mathbf{j} - 4\mathbf{k}) \\
&= -\frac{2}{3}\mathbf{i} + \frac{1}{3}\mathbf{j} + \frac{4}{3}\mathbf{k}.
\end{aligned}$$

Projection of a Vector onto a Plane

Definition

The *projection* of a vector $\mathbf{u}$ onto a plane with normal vector $\mathbf{n}$ is

$$\operatorname{proj}_{\text{plane}} \mathbf{u} = \mathbf{u} - \operatorname{proj}_{\mathbf{n}} \mathbf{u} = \mathbf{u} - \left(\frac{\mathbf{u} \cdot \mathbf{n}}{\|\mathbf{n}\|^2} \right) \mathbf{n}.$$



$\mathbf{n}$
P
$\mathbf{u}$
$\mathrm{proj}_{\mathbf{n}}\,\mathbf{u}$
$\mathrm{proj}_{\mathrm{plane}}\,\mathbf{u}$
Q

Example

Find the projection of the vector $\mathbf{u} = \mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$ onto the plane $3x - y + 2z = 7$.

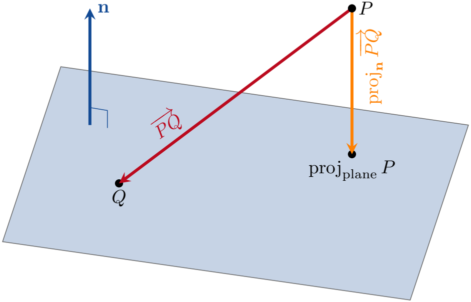
solution: Clearly $\mathbf{n} = 3\mathbf{i} - \mathbf{j} + 2\mathbf{k}$ and

$$\begin{aligned}\operatorname{proj}_{\mathbf{n}} \mathbf{u} &= \left(\frac{\mathbf{u} \cdot \mathbf{n}}{\|\mathbf{n}\|^2}\right)\mathbf{n} = \left(\frac{3-2+6}{3^2+(-1)^2+2^2}\right)(3\mathbf{i} - \mathbf{j} + 2\mathbf{k}) \\ &= \frac{1}{2}(3\mathbf{i} - \mathbf{j} + 2\mathbf{k}) = \frac{3}{2}\mathbf{i} - \frac{1}{2}\mathbf{j} + \mathbf{k}.\end{aligned}$$

Therefore

$$\begin{aligned}\operatorname{proj}_{\text{plane}} \mathbf{u} &= \mathbf{u} - \operatorname{proj}_{\mathbf{n}} \mathbf{u} \\ &= (\mathbf{i} + 2\mathbf{j} + 3\mathbf{k}) - \left(\frac{3}{2}\mathbf{i} - \frac{1}{2}\mathbf{j} + \mathbf{k}\right) \\ &= -\frac{1}{2}\mathbf{i} + \frac{5}{2}\mathbf{j} + 2\mathbf{k}.\end{aligned}$$

Projection of a Point onto a Plane



Definition

Let P be a point and let $Ax + By + Cz = D$ be a plane. Let Q be a point on the plane and let $\mathbf{n} = A\mathbf{i} + B\mathbf{j} + C\mathbf{k}$ denote a vector normal to the plane.
The projection of the point P onto this plane is

$$\operatorname{proj}_{\text{plane}} P = P + \operatorname{proj}_{\mathbf{n}} \overrightarrow{PQ}.$$

Example

Find the projection of the point $P(1,2,-4)$ on the plane $2x+y+4z=2$.

solution: Note first that $\mathbf{n}=2\mathbf{i}+\mathbf{j}+4\mathbf{k}$ and that the point $Q(1,0,0)$ lies on the plane. Since

$$\overrightarrow{PQ}=Q-P=(1,0,0)-(1,2,-4)=(0,-2,4)=-2\mathbf{j}+4\mathbf{k},$$

we have

$$\begin{aligned}\operatorname{proj}_{\mathbf{n}}\overrightarrow{PQ}&=\left(\frac{\overrightarrow{PQ}\cdot\mathbf{n}}{\|\mathbf{n}\|^2}\right)\mathbf{n}=\left(\frac{0-2+16}{2^2+1^2+4^2}\right)(2\mathbf{i}+\mathbf{j}+4\mathbf{k})\\&=\left(\frac{14}{21}\right)(2\mathbf{i}+\mathbf{j}+4\mathbf{k})=\frac{2}{3}(2\mathbf{i}+\mathbf{j}+4\mathbf{k})\\&=\frac{4}{3}\mathbf{i}+\frac{2}{3}\mathbf{j}+\frac{8}{3}\mathbf{k}.\end{aligned}$$

Therefore

$$\begin{aligned}\operatorname{proj}_{\text{plane}} P &= P + \operatorname{proj}_{\mathbf{n}} \overrightarrow{PQ} \\ &= (1, 2, -4) + \left(\frac{4}{3}, \frac{2}{3}, \frac{8}{3}\right) \\ &= \left(\frac{7}{3}, \frac{8}{3}, -\frac{4}{3}\right).\end{aligned}$$

We should double check that the point $\left(\frac{7}{3}, \frac{8}{3}, -\frac{4}{3}\right)$ is on the plane $2x + y + 4z = 2$.

$$2x+y+4z = 2\left(\frac{7}{3}\right)+\left(\frac{8}{3}\right)+4\left(-\frac{4}{3}\right) = \frac{14}{3}+\frac{8}{3}-\frac{16}{3} = \frac{6}{3} = 2 \checkmark$$

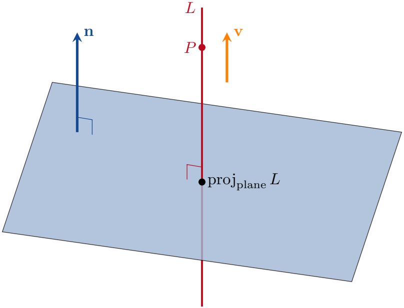
Projection of a Line onto a Plane

Let L be a line passing through the point P in the direction $\mathbf{v}$. Let $Ax + By + Cz = D$ be a plane with normal vector $\mathbf{n} = A\mathbf{i} + B\mathbf{j} + C\mathbf{k}$.

There are three cases to consider:

1. The line is orthogonal to the plane ($\mathbf{v} \times \mathbf{n} = \mathbf{0}$);
2. The line is parallel to the plane ($\mathbf{v} \cdot \mathbf{n} = 0$); and
3. The line is not parallel to the plane and is not orthogonal to the plane ($\mathbf{v} \cdot \mathbf{n} \neq 0$ and $\mathbf{v} \times \mathbf{n} \neq \mathbf{0}$).

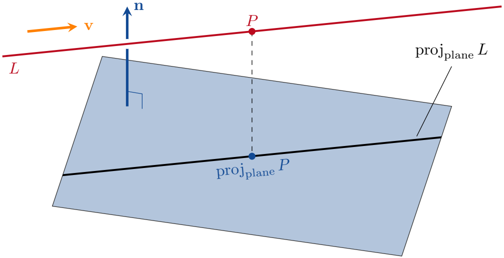
A Line Orthogonal to a Plane ($\mathbf{v} \times \mathbf{n} = \mathbf{0}$)



This is the easiest case: The projection of the line onto the plane is just the point where they intersect. Therefore

$$\mathrm{proj}_{\mathrm{plane}}\, L = \mathrm{proj}_{\mathrm{plane}}\, P.$$

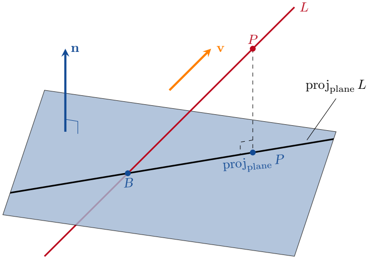
A Line Parallel to a Plane ($\mathbf{v} \bullet \mathbf{n} = 0$)



We can see that

$$\mathrm{proj}_{\mathrm{plane}}\, L = \left(\begin{array}{c} \text{the line passing through the} \\ \text{point } \mathrm{proj}_{\mathrm{plane}}\, P \text{ in the direction} \\ \mathbf{v}. \end{array} \right)$$

A Line which is Neither Parallel to nor Orthogonal to the Plane



If $\mathbf{v} \bullet \mathbf{n} \neq 0$, then the line must intersect the plane at some point B. Assuming $B \neq P$, we have

$$\operatorname{proj}_{\text{plane}} L = \left(\begin{array}{c} \text{the line passing through} \\ \text{the points } B \text{ and} \\ \operatorname{proj}_{\text{plane}} P. \end{array} \right)$$

Example

Find the projection of the line $x = 7 + 6t$, $y = -3 + 15t$, $z = 10 - 12t$ onto the plane $2x + 5y - 4z = 13$.

solution:

1. Find $\mathbf{v}$ and $\mathbf{n}$.

$$\mathbf{v} = 6\mathbf{i} + 15\mathbf{j} - 12\mathbf{k}$$
$$\mathbf{n} = 2\mathbf{i} + 5\mathbf{j} - 4\mathbf{k}$$

2. Does the line intersect the plane?

Since

$$\mathbf{v} \bullet \mathbf{n} = 12 + 75 + 48 = 135 \neq 0,$$

the answer is yes, the line does intersect the plane.

3. Find the point of intersection.
We calculate that

$$
\begin{aligned}
13 &= 2x + 5y - 4z \\
&= 2(7 + 6t) + 5(-3 + 15t) - 4(10 - 12t) \\
&= 14 + 12t - 15 + 75t - 40 + 48t \\
&= -41 + 135t \\
54 &= 135t \\
2 &= 5t \\
\frac{2}{5} &= t.
\end{aligned}
$$

Hence the point of intersection is

$$
\begin{aligned}
B(x, y, z)\big|_{t=\frac{2}{5}} &= B(7 + 6t, -3 + 15t, 10 - 12t)\big|_{t=\frac{2}{5}} \\
&= B(9.4, 3, 5.2)
\end{aligned}
$$

4. Is the line orthogonal to the plane?

Since

$$\mathbf{v} \times \mathbf{n} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 6 & 15 & -12 \\ 2 & 5 & -4 \end{vmatrix} = 0\mathbf{i} + 0\mathbf{j} + 0\mathbf{k} = \mathbf{0},$$

the answer is yes, the line is orthogonal to the plane.

5 Find $\mathrm{proj}_{\mathrm{plane}}\, L$.

The projection of the line on the plane is the point

$$\mathrm{proj}_{\mathrm{plane}}\, L = B(9.4, 3, 5.2).$$

Example

Find the projection of the line $x = 1 + 4t$, $y = 2 + 4t$, $z = 3 + 4t$ onto the plane $3x + 4y - 7z = 27$.

solution:

1. Find $\mathbf{v}$ and $\mathbf{n}$.

$$\mathbf{v} = 4\mathbf{i} + 4\mathbf{j} + 4\mathbf{k}$$

$$\mathbf{n} = 3\mathbf{i} + 4\mathbf{j} - 7\mathbf{k}$$

2. Does the line intersect the plane?

   Since

$$\mathbf{v} \bullet \mathbf{n} = 12 + 16 - 28 = 0,$$

   the line does not intersect the plane. Therefore the line is parallel to the plane.

3. Find a point on $\operatorname{proj}_{\text{plane}} L$.

$P(1,2,3)$ lies on the original line and $Q(9,0,0)$ lies on the plane. So

$$\begin{aligned}\overrightarrow{PQ} &= Q - P = (9,0,0) - (1,2,3) = (8,-2,-3)\\ &= 8\mathbf{i} - 2\mathbf{j} - 3\mathbf{k}\end{aligned}$$

and

$$\begin{aligned}\operatorname{proj}_{\mathbf{n}} \overrightarrow{PQ} &= \left(\frac{\overrightarrow{PQ} \cdot \mathbf{n}}{\|\mathbf{n}\|^2}\right)\mathbf{n} = \left(\frac{24 - 8 + 21}{9 + 16 + 49}\right)\mathbf{n}\\ &= \left(\frac{37}{74}\right)\mathbf{n} = \frac{1}{2}\mathbf{n}.\end{aligned}$$

Therefore

$$\begin{aligned}\operatorname{proj}_{\text{plane}} P &= P + \operatorname{proj}_{\mathbf{n}} \overrightarrow{PQ} \\ &= (1,2,3) + \left(\frac{3}{2}, 2, -\frac{7}{2}\right) \\ &= \left(\frac{5}{2}, 4, -\frac{1}{2}\right).\end{aligned}$$

We should quickly double check that our $\operatorname{proj}_{\text{plane}} P$ really is on the plane:

$$\begin{aligned}3x + 4y - 7z &= 3\left(\frac{5}{2}\right) + 4(4) - 7\left(-\frac{1}{2}\right) \\ &= \frac{15}{2} + 16 + \frac{7}{2} = 27. \checkmark\end{aligned}$$

4. Find $\text{proj}_{\text{plane}}\, L$.

The projection of the original line on the plane is the line passing through the point $\text{proj}_{\text{plane}}\, P = \left(\frac{5}{2}, 4, -\frac{1}{2}\right)$ in the direction $\mathbf{v} = 4\mathbf{i} + 4\mathbf{j} + 4\mathbf{k}$, which has parametrised equations

$$x = \frac{5}{2} + 4t, \qquad y = 4 + 4t, \qquad z = -\frac{1}{2} + 4t.$$

Example

Find the projection of the line $x = 15 + 15t$, $y = -12 - 15t$, $z = 17 + 11t$ on the plane $13x - 9y + 16z = 69$.

solution:

1. Find $\mathbf{v}$ and $\mathbf{n}$.

$$\mathbf{v} = 15\mathbf{i} - 15\mathbf{j} + 11\mathbf{k}$$
$$\mathbf{n} = 13\mathbf{i} - 9\mathbf{j} + 16\mathbf{k}$$

2. Does the line intersect the plane?

   Since

$$\mathbf{v} \bullet \mathbf{n} = 506 \neq 0,$$

   the line intersects the plane.

3 Find the point of intersection.

We calculate that

$$\begin{aligned} 69 &= 13x - 9y + 16z \\ &= 13(15 + 15t) - 9(-12 - 15) + 16(17 + 11t) \\ &= 195 + 195t + 108 + 135t + 272 + 176t \\ &= 575 + 506t \\ -506 &= 506t \\ -1 &= t. \end{aligned}$$

Thus the line intersects the plane at

$$\begin{aligned} B(x, y, z)\big|_{t=-1} &= B(15 + 15t, -12 - 15t, 17 + 11t)\big|_{t=-1} \\ &= B(0, 3, 6). \end{aligned}$$

4. Is the line orthogonal to the plane?

Since

$$\mathbf{v} \times \mathbf{n} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 15 & -15 & 11 \\ 13 & -9 & 16 \end{vmatrix} = -141\mathbf{i} - 97\mathbf{j} + 60\mathbf{k} \neq \mathbf{0},$$

the line is not orthogonal to the plane.

5. Find another point on $\operatorname{proj}_{\text{plane}} L$.

The point $P(15, -12, 17)$ lies on the original line. Since $\overrightarrow{PB} = (-15, 15, -11)$ and

$$\operatorname{proj}_{\mathbf{n}} \overrightarrow{PB} = \left(\frac{\overrightarrow{PB} \bullet \mathbf{n}}{\|\mathbf{n}\|^2} \right) \mathbf{n} \quad = \left(\frac{-506}{506} \right) \mathbf{n} = -\mathbf{n}$$

we have that

$$\begin{aligned} \operatorname{proj}_{\text{plane}} P &= P + \operatorname{proj}_{\mathbf{n}} \overrightarrow{PB} \\ &= (15, -12, 17) + (-13, 9, -16) = (2, -3, 1). \end{aligned}$$

6 Find $\mathrm{proj}_{\mathrm{plane}}\, L$.

Let

$$\mathbf{v}_2 = \text{the vector from } B \text{ to } \mathrm{proj}_{\mathrm{plane}}\, P = 2\mathbf{i} - 6\mathbf{j} - 5\mathbf{k}.$$

Then $\mathrm{proj}_{\mathrm{plane}}\, L$ is the line passing through $B(0, 3, 6)$ in the direction $\mathbf{v}_2 = 2\mathbf{i} - 6\mathbf{j} - 5\mathbf{k}$ which has parametrised equations

$$x = 2t, \qquad y = 3 - 6t, \qquad z = 6 - 5t.$$

Next Week

- 17. Combinatorics: Basic Counting Principles
- 18. Combinatorics: Permutations and Combinations
- 19. Introduction to Probability