Lecture 3

- 2.4 Autonomous Equations and Population Dynamics
- 2.5 Exact Equations
- 2.6 Substitutions

Autonomous Equations and Population Dynamics

Equations of the form

$$\frac{dy}{dt} = f(y) \tag{1}$$

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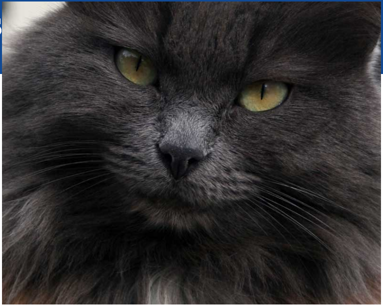

Example (Exponential Growth)

Let $y(t)$ denote the number of cats in İstanbul.
The simplest model is to assume that the rate of change of y is proportional to y.

$$\frac{dy}{dt} = ry$$

for some constant r. We will assume that $r > 0$.

The solution to

$$\begin{cases} y' = ry \\ y(0) = y_0 \end{cases}$$

is

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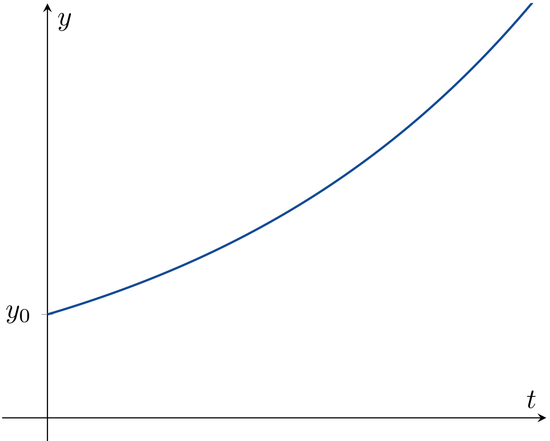
is $y(t) = y_0 e^{rt}$.



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- the food will run out
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- the food will run out
- there will be no space
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So we need a better model.

Example (Logistic Growth)

Now we replace the constant r with a function $h(y)$.

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We want a function h which satisfies

- $h(y) \approx r$ if y is small;
- $h(y)$ decreases as y grows larger; and
- $h(y) < 0$ for large y.

The simplest such h is $h(y) = r - ay$.

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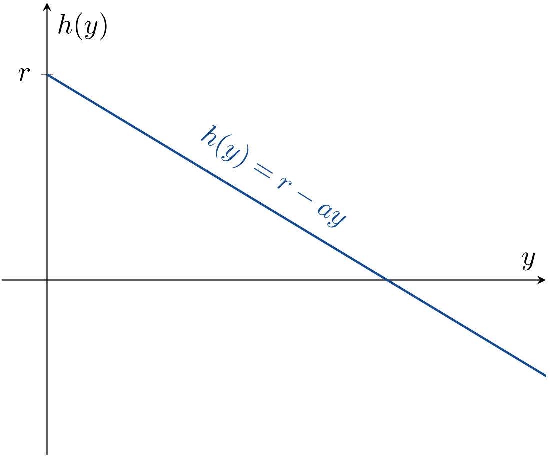

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$$\frac{dy}{dt} = (r - ay)y$$

which we will write as

$$\frac{dy}{dt} = r\left(1 - \frac{y}{K}\right)y$$

for $K = \frac{r}{a}$. This is called the *Logistic Equation*.

First we look for equilibrium solutions – that is solutions with $\frac{dy}{dt} = 0$ for all t.

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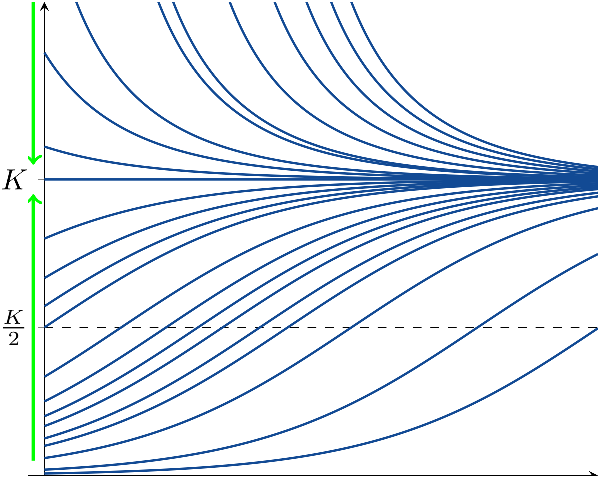
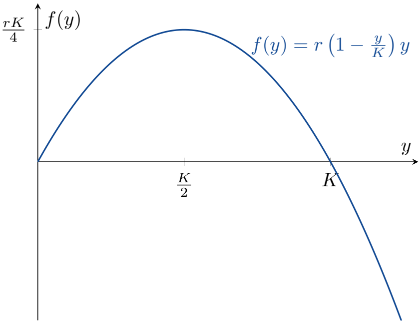
$$0 = \frac{dy}{dt} = r\left(1 - \frac{y}{K}\right)y \qquad \Longrightarrow \qquad y = 0 \quad \text{or} \quad y = K.$$

The equilibrium solutions are important. If we look at some more solutions, we can see that the other solutions converge to $y = K$, but diverge from $y = 0$.


K
$\frac{K}{2}$

To understand this behaviour, we graph $\frac{dy}{dt}$ against y.



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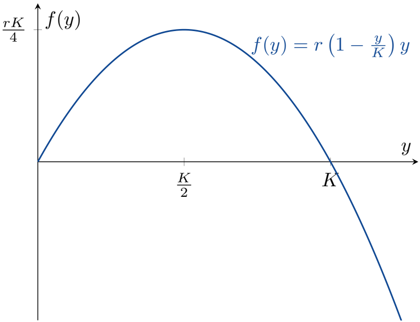

Note that

- $\frac{dy}{dt} > 0 \implies y$ is increasing; and
- $\frac{dy}{dt} < 0 \implies y$ is decreasing; and

We can show this on the graph by drawing green arrows.

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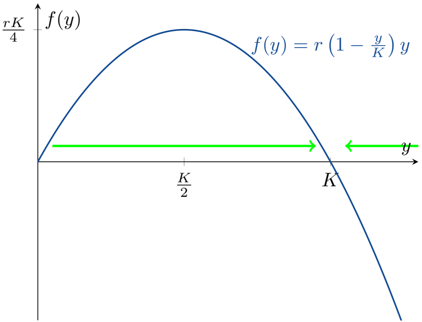

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2.4 Autonomous Equations and Population Dyna

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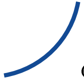
The solution $y(t)$ is concave up ( or ) when $y'' > 0$ (i.e. when both f and f' are both positive or both negative).

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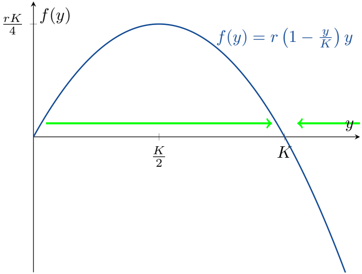
The solution $y(t)$ is concave up (or) when $y'' > 0$ (i.e. when both f and f' are both positive or both negative).

The solution $y(t)$ is concave down (or) when $y'' < 0$ (i.e. when one of f and f' is positive and one is negative).

Look again at the graph of $f(y) = r\left(1 - \frac{y}{K}\right)y$ against y.



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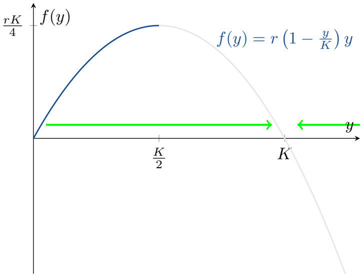

We can see that

- $y \in (0, \frac{K}{2}) \implies f > 0$ and $f' > 0 \implies y(t)$ is increasing and concave up;

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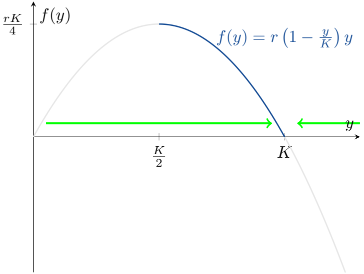

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- $y \in (0, \frac{K}{2}) \implies f > 0$ and $f' > 0 \implies y(t)$ is increasing and concave up;
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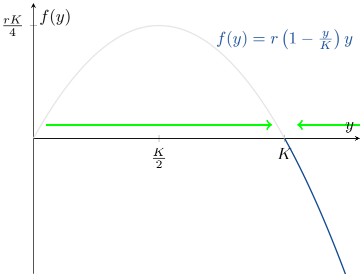

We can see that

- $y \in (0, \frac{K}{2}) \implies f > 0$ and $f' > 0 \implies y(t)$ is increasing and concave up;
- $y \in (\frac{K}{2}, K) \implies f > 0$ and $f' < 0 \implies y(t)$ is increasing and concave down;
- $y \in (K, \infty) \implies f < 0$ and $f' < 0 \implies y(t)$ is decreasing and concave up;

Moreover, remember that a theorem from earlier told us that two solutions can not intersect.

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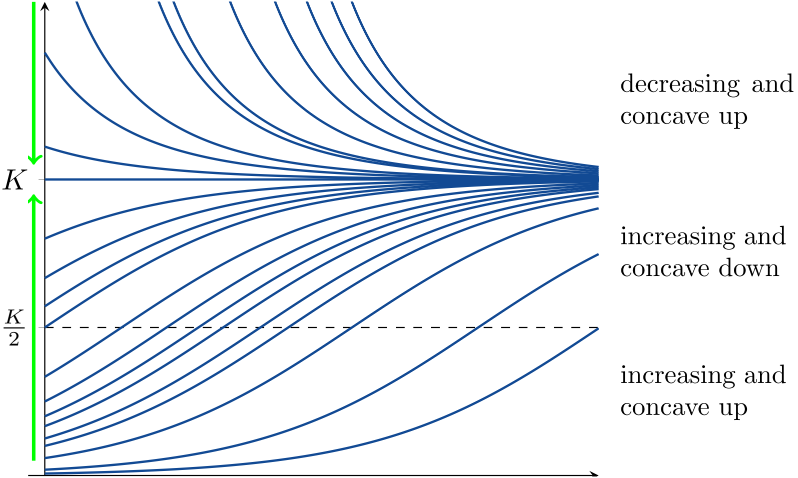

Because solutions converge to $y = K$, we say that $y = K$ is an *asymptotically stable equilibrium solution* or an *asymptotically stable critical point*.

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Because solutions diverge from $y = 0$, we say that $y = 0$ is an *unstable equilibrium solution* or an *unstable critical point*.

Definition

Equilibrium solutions can be

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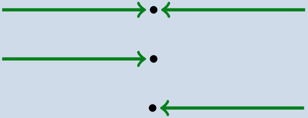
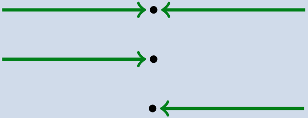
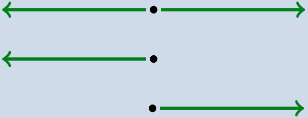
	asymptotically stable

2.4 Autonomous Equations and Population Dyna

Definition

Equilibrium solutions can be

	asymptotically stable
	unstable

2.4 Autonomous Equations and Population Dyna

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Equilibrium solutions can be

$\longrightarrow\bullet\longleftarrow$ $\longrightarrow\bullet$ $\bullet\longleftarrow$	asymptotically stable
$\longleftarrow\bullet\longrightarrow$ $\longleftarrow\bullet$ $\bullet\longrightarrow$	unstable
$\longleftarrow\bullet\longleftarrow$ $\longrightarrow\bullet\longrightarrow$	semistable

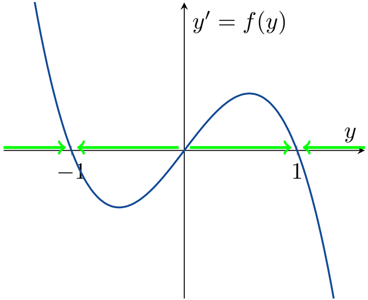
Example

Find all of the critical points of

$$\frac{dy}{dt} = \underbrace{y(1-y^2)}_{f(y)} \qquad (-\infty < y_0 < \infty)$$

and classify each as asymptotically stable, unstable or semistable.

$$\frac{dy}{dt} = y(1 - y^2)$$



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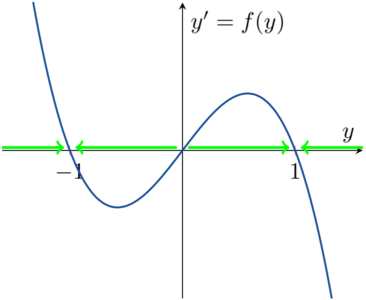
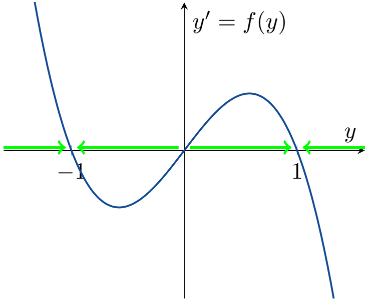


The critical points are $y = -1, 0, 1$.

$$\frac{dy}{dt} = y(1-y^2)$$



The critical points are $y = -1, 0, 1$.

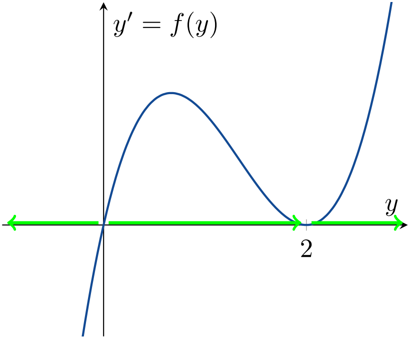
- $y = -1$ is asymptotically stable;
- $y = 0$ is unstable; and
- $y = 1$ is asymptotically stable.

Example

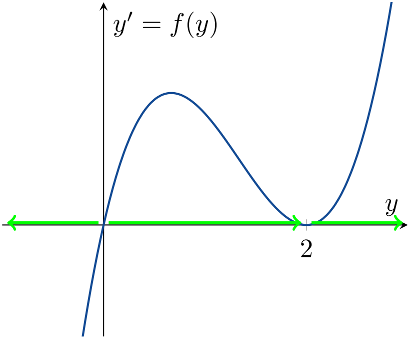
Find all of the critical points of

$$\frac{dy}{dt} = \underbrace{y(y-2)^2}_{f(y)} \qquad (-\infty < y_0 < \infty)$$

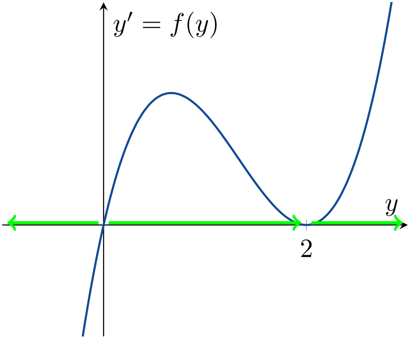
and classify each as asymptotically stable, unstable or semistable.



$y' = f(y)$
y
2



The critical points are $y = 0$ and 2.



The critical points are $y = 0$ and 2.

- $y = 0$ is unstable; and
- $y = 2$ is semistable.

Example

Consider the autonomous differential equation

$$\frac{dy}{dt} = f(y) = y^4 - 5y^3 + 6y^2. \tag{2}$$

1. Find all of the critical points of (2).
2. Sketch the graph of $f(y)$ versus y.
3. Determine whether each critical point is asymptotically stable, unstable or semistable.
4. Sketch 10 (or more) different solutions of (2).

(This is an exam question from 2013: Students had 30 minutes to solve this.)

1

$$\frac{dy}{dt} = f(y) = y^4 - 5y^3 + 6y^2 =$$

1

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1

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The critical points are $y = 0$, $y = 2$ and $y = 3$.

$$\frac{dy}{dt} = f(y) = y^4 - 5y^3 + 6y^2 = y^2(y-2)(y-3)$$

2

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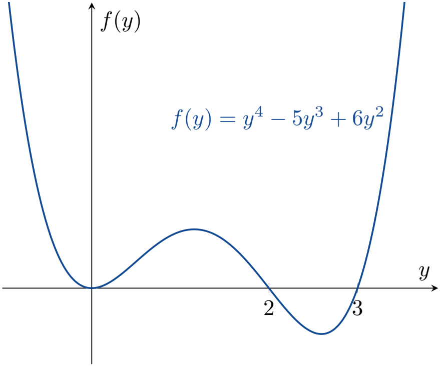
2



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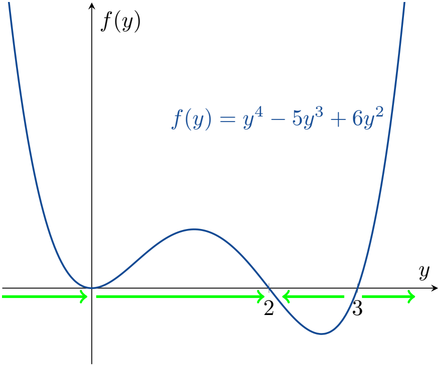

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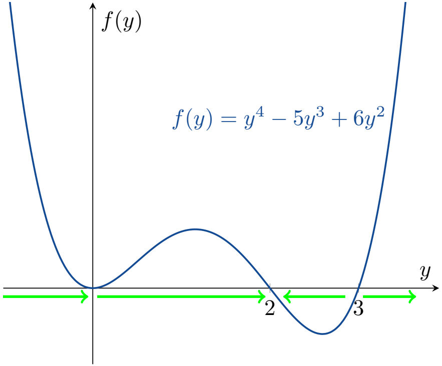
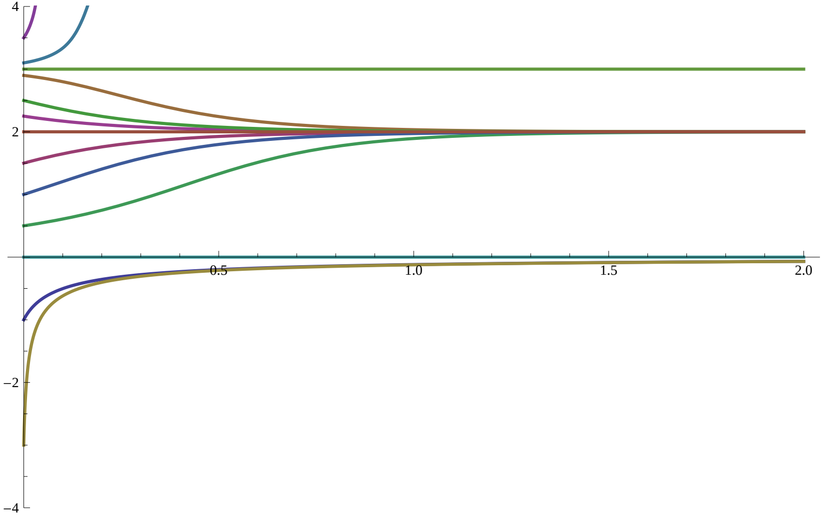
2. 

3. $y = 0$ is semistable, $y = 2$ is asymptotically stable and $y = 3$ is unstable.

4



Example (A Critical Threshold)

Now suppose that we can model the number of cats in İstanbul by

$$\frac{dy}{dt} = -r\left(1 - \frac{y}{T}\right)y$$

where $T > 0$ and $r > 0$.

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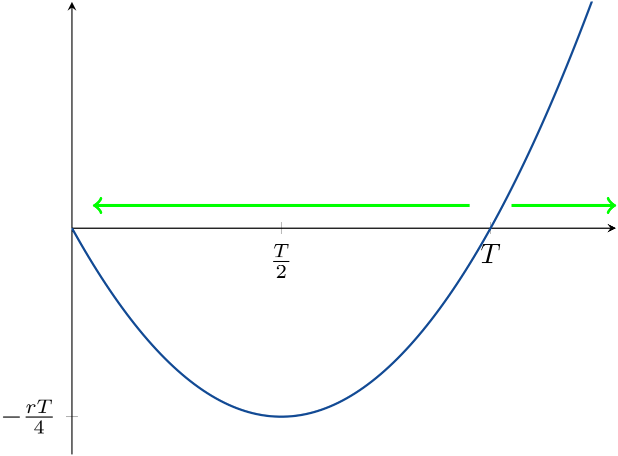
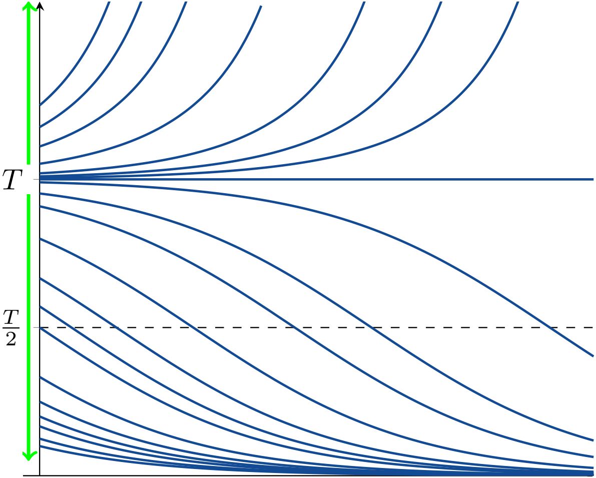


The critical points/equilibrium solutions are $y = 0$ and $y = T$.

- $y = 0$ is asymptotically stable; and
- $y = T$ is unstable.

With this information we can sketch some solutions



T
$\frac{T}{2}$

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The population of some species have the threshold property: If there are not enough individuals, then the species becomes extinct.

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The number T is called a *threshold level*, below which no growth happens.

The population of some species have the threshold property: If there are not enough individuals, then the species becomes extinct.

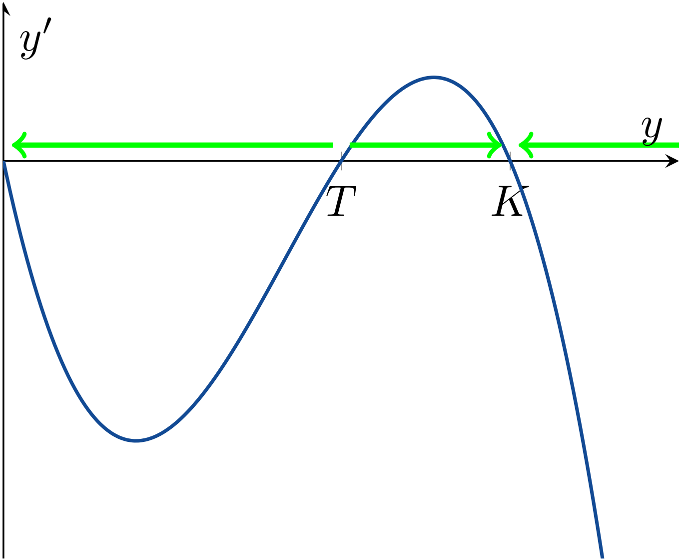
This model predicts that the number of cats in İstanbul will increase to ∞ (if $y_0 > T$), so we need a more advanced model.

Example (Logistic Growth with a Threshold)

Now consider

$$\frac{dy}{dt} = -r\left(1 - \frac{y}{T}\right)\left(1 - \frac{y}{K}\right)y$$

for $0 < T < K$ and $r > 0$.

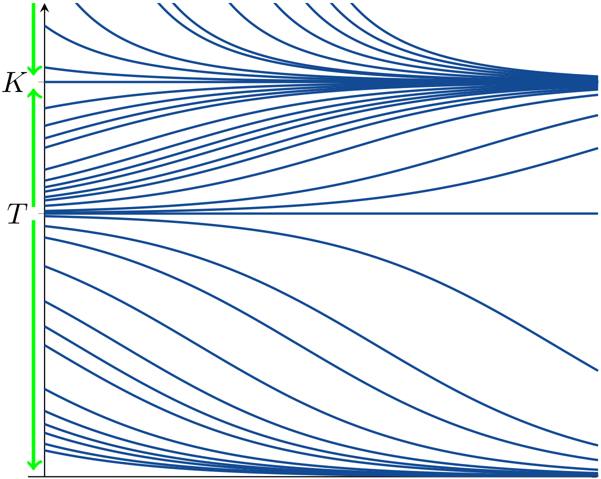


y'
y
T
K

The critical points/equilibrium solutions are $y = 0$, $y = T$ and $y = K$.

- $y = 0$ is asymptotically stable;
- $y = T$ is unstable; and
- $y = K$ is asymptotically stable.

Solutions look like this:



This is an equation which has been used by biologists to model certain populations of animals.

Exact Equations

2.5 Exact Equations

Previously we have looked at linear equations and separable equations. Now we will look at another special type of equation.

Example

Solve $2x + y^2 + 2xyy' = 0$.

This equation is not linear and is not separable.

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Note that if $\psi(x,y) = x^2 + xy^2$, then $\frac{\partial\psi}{\partial x} = 2x + y^2$ and $\frac{\partial\psi}{\partial y} = 2xy$.

Example

Solve $2x + y^2 + 2xyy' = 0$.

This equation is not linear and is not separable.

Note that if $\psi(x,y) = x^2 + xy^2$, then $\frac{\partial\psi}{\partial x} = 2x + y^2$ and $\frac{\partial\psi}{\partial y} = 2xy$. So we can write the ODE as

$$\frac{\partial\psi}{\partial x} + \frac{\partial\psi}{\partial y}\frac{dy}{dx} = 0.$$

Since $y(x)$ is a function of x, we also have that

$$\frac{d}{dx}\Big(\psi(x, y(x))\Big) = \frac{\partial \psi}{\partial x} + \frac{\partial \psi}{\partial y}\frac{dy}{dx}$$

by the Chain Rule.

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by the Chain Rule. So our ODE can be written as

$$\frac{d}{dx}\big(x^2 + xy^2\big) = 0.$$

Therefore

$$x^2 + xy^2 = c.$$

Remark

The key step was finding $\psi(x,y)$.

Now consider

$$M(x, y) + N(x, y)y' = 0. \tag{3}$$

Definition

If we can find a function $\psi(x, y)$ such that

$$\frac{\partial \psi}{\partial x} = M \qquad \text{and} \qquad \frac{\partial \psi}{\partial y} = N,$$

then (3) is called an *exact equation*.

2.5 Exact Equations

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Definition

If we can find a function $\psi(x,y)$ such that

$$\frac{\partial \psi}{\partial x} = M \qquad \text{and} \qquad \frac{\partial \psi}{\partial y} = N,$$

then (3) is called an *exact equation*.

If (3) is exact, then

$$0 = M(x,y)+N(x,y)y' = \frac{\partial \psi}{\partial x}(x,y)+\frac{\partial \psi}{\partial y}(x,y)\frac{dy}{dx} = \frac{d}{dx}\Big(\psi\big(x,y(x)\big)\Big)$$

which has solution

$$\psi(x,y) = c.$$

Remark

To solve an exact equation:

1. Find $\psi(x, y)$;
2. Write $\psi(x, y) = c$.

2.5 Exact Equations

Notation

$$y' = \frac{dy}{dx} \qquad f_x = \frac{\partial f}{\partial x} \qquad f_y = \frac{\partial f}{\partial y}$$

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Theorem

Suppose that M, N, M_y and N_x are continuous on the rectangular region $R = \{(x, y) \ : \ \alpha < x < \beta, \ \gamma < y < \delta\}$.

2.5 Exact Equations

Notation

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Theorem

Suppose that M, N, M_y and N_x are continuous on the rectangular region $R = \{(x,y) \ : \ \alpha < x < \beta, \ \gamma < y < \delta\}$. Then

$$M + Ny' = 0 \text{ is exact} \qquad \Longleftrightarrow \qquad M_y = N_x.$$

Example

Consider

$$(y\cos x + 2xe^y) + (\sin x + x^2e^y - 1)y' = 0.$$

Is this ODE exact? If yes, solve it.

Example

Consider

$$(y\cos x + 2xe^y) + (\sin x + x^2e^y - 1)y' = 0.$$

Is this ODE exact? If yes, solve it.

$$M = y\cos x + 2xe^y \qquad\qquad M_y =$$
$$N = \sin x + x^2e^y - 1 \qquad\qquad N_x =$$

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Yes, the ODE is exact. So $\exists\ \psi$ such that

$$\psi_x = M = y\cos x + 2xe^y$$
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Integrating the first equation (wrt x) gives

$$\psi = \int \psi_x \, dx = y\sin x + x^2e^y + h(y).$$

2.5 Exact Equations

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The solution to the ODE is

$$\boxed{y\sin x + x^2e^y - y = c.}$$

Example

Consider

$$ye^{xy} + e^{xy}y' = 0.$$

Is this ODE exact? If yes, solve it.

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Is this ODE exact? If yes, solve it.

We have

$$\begin{aligned} M &= ye^{xy} & \qquad M_y &= e^{xy} + xye^{xy} \\ N &= e^{xy} & \qquad N_x &= ye^{xy}. \end{aligned}$$

Example

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$$ye^{xy} + e^{xy}y' = 0.$$
Is this ODE exact? If yes, solve it.

We have
$$\begin{aligned} M &= ye^{xy} & \qquad M_y &= e^{xy} + xye^{xy} \\ N &= e^{xy} & \qquad N_x &= ye^{xy}. \end{aligned}$$

Since $M_y \neq N_x$, the ODE is not exact.

Example

Consider

$$\left(4x^3y^3 + \frac{1}{x}\right) + \left(3x^4y^2 + \frac{1}{y}\right) y' = 0.$$

Is this ODE exact? If yes, solve it.

Example

Consider
$$\left(4x^3y^3 + \frac{1}{x}\right) + \left(3x^4y^2 + \frac{1}{y}\right) y' = 0.$$
Is this ODE exact? If yes, solve it.

I leave this one to you to solve. Please check that the solution is

$$\boxed{x^4y^3 + \ln|x| + \ln|y| = c.}$$

Example

Consider

$$1 + \left(1 + 2y + 3y^2\right) y' = 0.$$

Is this ODE exact? If yes, solve it.

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First note that

$$\begin{aligned} M &= 1 & M_y &= 0 \\ N &= 1 + 2y + 3y^2 & N_x &= 0 = M_y \end{aligned}$$

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Yes, the ODE is exact. So $\exists\ \psi$ such that

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Yes, the ODE is exact. So $\exists\ \psi$ such that

$$\begin{aligned} \psi_x &= 1 \\ \psi_y &= 1 + 2y + 3y^2. \end{aligned}$$

We can start with $\psi_x = 1$ or with $\psi_y = 1 + 2y + 3y^2$.

$$\psi_x = 1$$
$$\psi_y = 1 + 2y + 3y^2$$

$$\begin{aligned}
\psi_x &= 1 \\
\psi &= \int 1\,dx = x + h(y) \\
\psi_y &= h'(y) \\
h'(y) &= 1 + 2y + 3y^2 \\
h(y) &= y + y^2 + y^3 \\
\psi &= x + y + y^2 + y^3
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\psi_y &= 1 + 2y + 3y^2 \\
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&= y + y^2 + y^3 + h(x) \\
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Therefore the solution is $\boxed{x + y + y^2 + y^3 = c.}$

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$$(3xy + y^2) + (x^2 + xy)y' = 0.$$

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Since $M_y \neq N_x$, this ODE is not exact. So our method to solve an exact equation *will not work*.

Example

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$$(3xy + y^2) + (x^2 + xy)y' = 0.$$
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First note that

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Since $M_y \neq N_x$, this ODE is not exact. So our method to solve an exact equation *will not work*. But we are going to try our method anyway, so that we can see what goes wrong.

Suppose that $\exists\ \psi(x, y)$ such that

$$\psi_x = 3xy + y^2$$
$$\psi_y = x^2 + xy.$$

2.5 Exact Equations

Suppose that $\exists\ \psi(x, y)$ such that

$$\psi_x = 3xy + y^2$$
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Integrating ψ_x with respect to x gives

$$\psi = \frac{3}{2}x^2y + xy^2 + h(y).$$

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$$x^2 + xy = \psi_y = \frac{\partial}{\partial y}\left(\frac{3}{2}x^2y + xy^2 + h(y)\right) = \frac{3}{2}x^2 + 2xy + h'(y).$$

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So we need h to satisfy

$$h'(y) = -\frac{1}{2}x^2 - xy.$$

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Integrating Factors

It is sometimes possible to convert a differential equation which is not exact into an exact equation by multiplying it by an integrating factor. (Do you remember how we solve linear equations?)

Consider

$$M(x,y)\,dx + N(x,y)\,dy = 0. \tag{4}$$

Suppose that (4) is not exact.

2.5 Exact Equations

Consider

$$M(x,y)\,dx + N(x,y)\,dy = 0. \tag{4}$$

Suppose that (4) is not exact. If we multiply by $\mu(x,y)$, we obtain

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Now

$$\begin{aligned}(\mu M)_y &= (\mu N)_x \\ \mu_y M + \mu M_y &= \mu_x N + \mu N_x \\ M\mu_y - N\mu_x + (M_y - N_x)\mu &= 0. \qquad (6)\end{aligned}$$

If we can find $\mu(x,y)$ which solves (6), then (5) is exact and we know how to solve exact equations.

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$$\boxed{\frac{d\mu}{dx} = \left(\frac{M_y - N_x}{N}\right)\mu.} \tag{7}$$

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$$\boxed{\frac{d\mu}{dx} = \left(\frac{M_y - N_x}{N}\right)\mu.} \tag{7}$$

If $\frac{M_y - N_x}{N}$ is a function only of x, then there is an integrating factor $\mu(x)$. Please note that (7) is both linear and separable.

If instead we looked for $\mu(y)$, we would obtain the ODE

$$\boxed{\frac{d\mu}{dy} = \left(\frac{N_x - M_y}{M} \right) \mu.} \tag{8}$$

Remark

If we were having classroom exams, you would be expected to remember (7) and (8).

Example

Solve

$$(3xy + y^2) + (x^2 + xy)y' = 0.$$

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We know that this equation is not exact. So we will try to find an integrating factor:

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We know that this equation is not exact. So we will try to find an integrating factor: We have that

$$M = 3xy + y^2 \qquad M_y = 3x + 2y$$
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2.5 Exact Equations

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So

$$\frac{M_y - N_x}{N} =$$

and

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$$M = 3xy + y^2 \qquad M_y = 3x + 2y$$
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So

$$\frac{M_y - N_x}{N} = \frac{(3x + 2y) - (2x + y)}{x^2 + xy} = \frac{x + y}{x(x + y)} = \frac{1}{x}$$

and

$$\frac{N_x - M_y}{M} =$$

$$(3xy + y^2) + (x^2 + xy)y' = 0$$

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So

$$\frac{M_y - N_x}{N} = \frac{(3x + 2y) - (2x + y)}{x^2 + xy} = \frac{x + y}{x(x + y)} = \frac{1}{x}$$

and

$$\frac{N_x - M_y}{M} = \frac{(2x + y) - (3x + 2y)}{3xy + y^2} = \frac{-x - y}{y(3x + y)}.$$

Note that $\frac{M_y - N_x}{N}$ is a function only of x – so it is possible to find an integrating factor $\mu(x)$. Moreover note that $\frac{N_x - M_y}{M}$ is *not* a function only of y – so it is *not* possible to find a $\mu(y)$.

We calculate that

$$
\begin{aligned}
\frac{d\mu}{dx} &= \left(\frac{M_y - N_x}{N} \right) \mu \\
\frac{d\mu}{dx} &= \frac{\mu}{x} \\
\frac{d\mu}{\mu} &= \frac{dx}{x} \\
\int \frac{d\mu}{\mu} &= \int \frac{dx}{x} \\
\ln|\mu| &= \ln|x| + C \\
\mu &= cx
\end{aligned}
$$

and we choose $c = 1$ for simplicity. So $\mu(x) = x$.

$$(3xy + y^2) + (x^2 + xy)y' = 0$$

Multiplying our original ODE by $\mu(x) = x$ gives

$$(3x^2y + xy^2) + (x^3 + x^2y)y' = 0.$$

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This ODE is exact ($M_y = 3x^2 + 2xy = N_x$) and we know how to solve exact equations.

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Multiplying our original ODE by $\mu(x) = x$ gives

$$(3x^2y + xy^2) + (x^3 + x^2y)y' = 0.$$

This ODE is exact ($M_y = 3x^2 + 2xy = N_x$) and we know how to solve exact equations. We must find ψ such that

$$\begin{aligned} \psi_x &= 3x^2y + xy^2 \\ \psi_y &= x^3 + x^2y. \end{aligned}$$

$$\psi_x = 3x^2y + xy^2$$
$$\psi_y = x^3 + x^2y$$

Integrating ψ_x wrt x gives

$$\psi = x^3y + \frac{1}{2}x^2y^2 + h(y).$$

$$\psi_x = 3x^2y + xy^2$$
$$\psi_y = x^3 + x^2y$$

Integrating ψ_x wrt x gives

$$\psi = x^3y + \frac{1}{2}x^2y^2 + h(y).$$

Hence

$$x^3 + x^2y = \psi_y = \frac{\partial}{\partial y}\left(x^3y + \frac{1}{2}x^2y^2 + h(y)\right) = x^3 + x^2y + h'(y)$$

$$\psi_x = 3x^2y + xy^2$$
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and we see that we may choose $h(y) = 0$.

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$$\psi = x^3y + \frac{1}{2}x^2y^2.$$

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$$\psi = x^3y + \frac{1}{2}x^2y^2 + h(y).$$

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and we see that we may choose $h(y) = 0$. Therefore

$$\psi = x^3y + \frac{1}{2}x^2y^2.$$

So the solution to the ODE is

$$\boxed{x^3y + \frac{1}{2}x^2y^2 = c.}$$

Example

Solve
$$ye^{xy} + \left(\left(\frac{2}{y} + x \right) e^{xy} \right) y' = 0.$$

This ODE is not exact (you check!).

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$$ye^{xy} + \left(\left(\frac{2}{y} + x\right)e^{xy}\right)y' = 0.$$

This ODE is not exact (you check!).

$$\frac{M_y - N_x}{N} = \frac{e^{xy} + xye^{xy} - e^{xy} - (2 + xy)\,e^{xy}}{\left(\frac{2}{y} + x\right)e^{xy}} = \frac{-2}{\frac{2}{y} + x}$$

$$\frac{N_x - M_y}{M} = \frac{2e^{xy}}{ye^{xy}} = \frac{2}{y}.$$

Since $\frac{N_x - M_y}{M}$ is a function only of y, we look for $\mu(y)$.

$$\frac{d\mu}{dy} = \frac{N_x - M_y}{M}\mu = \frac{2e^{xy}}{ye^{xy}}\mu = \frac{2}{y}\mu$$

$$\frac{d\mu}{dy} = \frac{N_x - M_y}{M}\mu = \frac{2e^{xy}}{ye^{xy}}\mu = \frac{2}{y}\mu$$

$$\vdots$$

(you complete this calculation)

$$\vdots$$

Therefore $\mu(y) = y^2$.

Multiplying our ODE by y^2 gives

$$y^3 e^{xy} + \left(\left(2y + xy^2\right) e^{xy}\right) y' = 0.$$

Multiplying our ODE by y^2 gives

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-
-
- (you complete this calculation)
-
-

Hence the solution is

$$\boxed{y^2 e^{xy} = c.}$$

2.6 Substitutions

Recall how we calculate an integral such as $\int 3x^2 \sin x^3 \; dx$.

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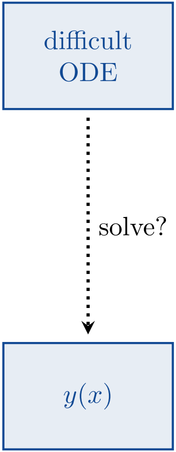
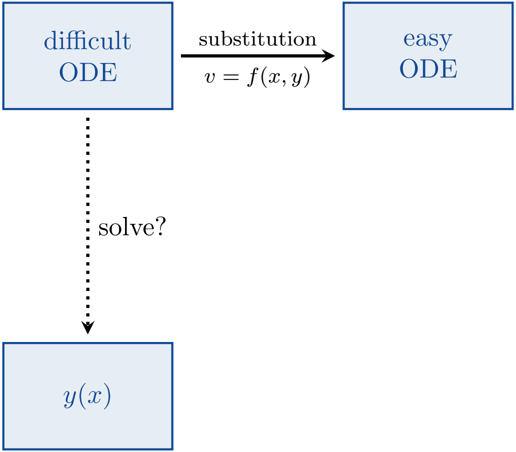
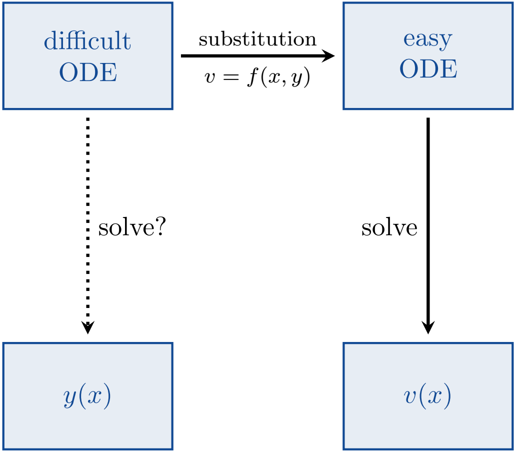
Sometimes we can use the same idea to solve ODEs.


difficult ODE
solve?
$y(x)$

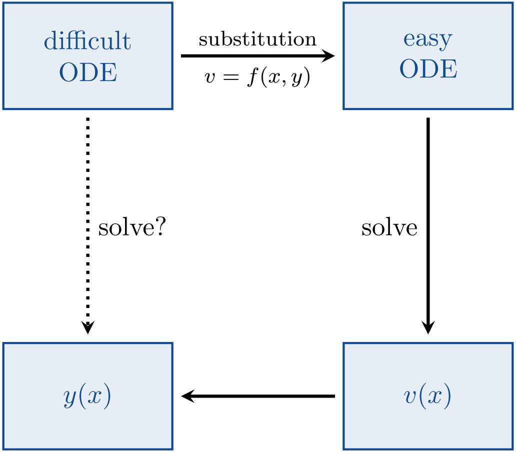
2.6 Substitutions





difficult ODE
substitution
$v = f(x, y)$
easy ODE
solve?
solve
$y(x)$
$v(x)$

2.6 Substitutions



2.6 Substitutions

We will use substitutions to solve two types of first order ODE:

- Homogeneous Equations;
- Bernoulli Equations.

Homogeneous Equations

Definition

The first order ODE $\frac{dy}{dx} = f(x, y)$ is called *homogeneous* iff we can write it as

$$\frac{dy}{dx} = g\left(\frac{y}{x}\right).$$

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For example, the following ODEs are homogeneous:

$$\frac{dy}{dx} = \cos\left(\frac{y}{x}\right) \qquad \frac{dy}{dx} = \left(\frac{y}{x}\right)^3 + \frac{y}{x}$$

$$\frac{dy}{dx} = \cos\left(\frac{x}{y}\right) \qquad \frac{dy}{dx} = \frac{\frac{y}{x} - 4}{1 - \frac{y}{x}}$$

For a homogeneous equation, we use the substitution

$$v(x) = \frac{y}{x}.$$

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Note that $y = xv(x)$ and

$$\frac{dy}{dx} = \frac{d}{dx}\big(xv(x)\big) = v + x\frac{dv}{dx}.$$

Example

Solve $\dfrac{dy}{dx} = \dfrac{y-4x}{x-y}$.

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Note first that

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$$\frac{dy}{dx} = \frac{y-4x}{x-y} = \frac{\frac{y}{x}-4}{1-\frac{y}{x}}.$$

If we substitute in $v = \frac{y}{x}$ we get

$$\frac{dy}{dx} = \frac{v-4}{1-v}.$$

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But remember that $\frac{dy}{dx} = v + x\frac{dv}{dx}$.

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and

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But remember that $\frac{dy}{dx} = v + x\frac{dv}{dx}$. Hence

$$v + x\frac{dv}{dx} = \frac{v-4}{1-v}$$

and

$$x\frac{dv}{dx} = \frac{v-4}{1-v} - v$$

$$\frac{dy}{dx} = \frac{v-4}{1-v}$$

But remember that $\frac{dy}{dx} = v + x\frac{dv}{dx}$. Hence

$$v + x\frac{dv}{dx} = \frac{v-4}{1-v}$$

and

$$x\frac{dv}{dx} = \frac{v-4}{1-v} - v = \frac{v-4}{1-v} - \frac{v-v^2}{1-v} = \frac{v^2-4}{1-v}$$

Note that

$$x\frac{dv}{dx} = \frac{v^2 - 4}{1 - v}$$

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$$x\frac{dv}{dx} = \frac{v^2 - 4}{1 - v}$$

is a separable equation. You know how to solve separable equations – the following should be revision for you. We rearrange to

$$\left(\frac{1 - v}{v^2 - 4}\right) dv = \frac{dx}{x}$$

$$\left(-\frac{3}{4(v + 2)} - \frac{1}{4(v - 2)}\right) dv = \frac{dx}{x}$$

$$\left(-\frac{3}{4(v+2)}-\frac{1}{4(v-2)}\right)dv=\frac{dx}{x}$$

then integrate to find

$$\begin{aligned}
-\frac{3}{4}\ln|v+2|-\frac{1}{4}\ln|v-2| &= \ln|x|+k \\
\ln|v+2|^3+\ln|v-2| &= \ln|x|^{-4}-4k \\
|v+2|^3|v-2| &= c|x|^{-4} \qquad\qquad (c=\pm e^{-4k}) \\
|x|^4|v+2|^3|v-2| &= c \\
|vx+2x|^3|vx-2x| &= c.
\end{aligned}$$

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|x|^4|v+2|^3|v-2| &= c \\
|vx+2x|^3|vx-2x| &= c.
\end{aligned}$$

Now we have an equation for v. The final step is to find an equation for y.

$$|vx + 2x|^3 \, |vx - 2x| = c.$$

If we substitute $y = vx$ into this equation, we find the solution

$$\boxed{|y + 2x|^3 \, |y - 2x| = c.}$$

Remark

To solve a homogeneous equation:

1. Substitute $v = \frac{y}{x}$ (and $\frac{dy}{dx} = v + x\frac{dv}{dx}$);
2. Solve a separable equation;
3. Substitute $y = vx$.

Example

Solve $\dfrac{dy}{dx} = \dfrac{x^2 + 3y^2}{2xy}$.

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First we rearrange

$$\frac{dy}{dx} = \frac{1 + 3\frac{y^2}{x^2}}{2\frac{y}{x}}$$

and substitute $v = \frac{y}{x}$ and $\frac{dy}{dx} = v + x\frac{dv}{dx}$ to get

$$v + x\frac{dv}{dx} = \frac{1 + 3v^2}{2v}.$$

Rearranging gives

$$x\frac{dv}{dx} = \frac{1+3v^2}{2v} - v = \frac{1+3v^2-2v^2}{2v} = \frac{1+v^2}{2v}.$$

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$$x\frac{dv}{dx} = \frac{1+3v^2}{2v} - v = \frac{1+3v^2-2v^2}{2v} = \frac{1+v^2}{2v}.$$

This is a separable equation which we can solve:

$$\begin{aligned}
\frac{2v\,dv}{1+v^2} &= \frac{dx}{x} \\
\int \frac{2v\,dv}{1+v^2} &= \int \frac{dx}{x} \\
\ln\left|1+v^2\right| &= \ln|x| + k \\
1+v^2 &= cx \\
1+v^2-cx &= 0.
\end{aligned}$$

Substituting $v = \frac{y}{x}$ then gives

$$1 + \frac{y^2}{x^2} - cx = 0$$

and

$$\boxed{x^2 + y^2 - cx^3 = 0.}$$

Bernoulli Equations

Definition

An equation of the form

$$y' + p(t)y = q(t)y^{n}$$

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$$\frac{dy}{dx} = \frac{dy}{dv}\frac{dv}{dx} = \frac{1}{2}v^{-\frac{1}{2}}\frac{dv}{dx}.$$

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$$\frac{1}{2}v^{-\frac{1}{2}}\frac{dv}{dx} - \left(\frac{3}{2x}\right) v^{\frac{1}{2}} = 2xv^{-\frac{1}{2}}.$$

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$$\mu(x) = e^{\int -\frac{3}{x}\,dx} = e^{-3\ln|x|} = \ldots = x^{-3}$$

to get

$$x^{-3}\frac{dv}{dx} - 3x^{-4}v = 4x^{-2}$$

which is

$$\frac{d}{dx}\left(x^{-3}v\right) = 4x^{-2}.$$

Integrating gives

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But $v = y^2$, so the solution is

$$\boxed{y^2 = -4x^2 + Cx^3.}$$

Remark

To solve a Bernoulli equation:

1. Substitute $v = y^{1-n}$;
2. Solve a linear equation;
3. Substitute $y^{1-n} = v$.

Example

Solve $x\dfrac{dy}{dx} + 6y = 3xy^{\frac{4}{3}}$.

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Solve $x\dfrac{dy}{dx} + 6y = 3xy^{\frac{4}{3}}$.

Note that this time we have $n = \frac{4}{3}$ and $v = y^{1-n} = y^{-\frac{1}{3}}$. Hence $y = v^{-3}$ and

$$\frac{dy}{dx} = \frac{dy}{dv}\frac{dv}{dx} = -3v^{-4}\frac{dv}{dx}.$$

Thus our ODE becomes

$$\begin{aligned} -3xv^{-4}\frac{dv}{dx} + 6v^{-3} &= 3xv^{-4} \\ -x\frac{dv}{dx} + 2v &= x \\ \frac{dv}{dx} - \frac{2}{x}v &= -1. \end{aligned}$$

Thus our ODE becomes

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Finally we use $v = y^{-\frac{1}{3}}$ to find that

$$\boxed{y = \frac{1}{(x + Cx^2)^3}.}$$

Next Time

- 3.1 Homogeneous Equations with Constant Coefficients
- 3.2 Fundamental Sets of Solutions
- 3.3 Complex Roots of the Characteristic Equation