

# Lecture 5

- 13.1 Functions of Several Variables
- 13.2 Limits and Continuity in Higher Dimensions
- 13.3 Partial Derivatives
- 13.4 The Chain Rule



# 1 Functions of Several Variables

## 13.1 Functions of Several Variables



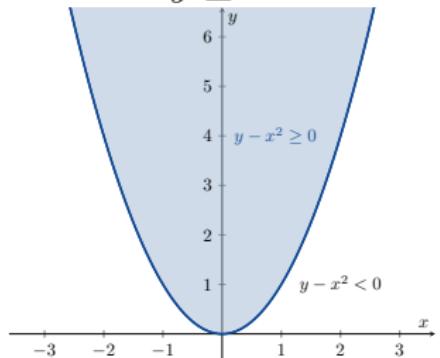
We will be considering functions  $f : D \subseteq \mathbb{R}^2 \rightarrow \mathbb{R}$  and functions  $g : D \subseteq \mathbb{R}^3 \rightarrow \mathbb{R}$ .

## 13.1 Functions of Several Variables

### Example

$$f : D \rightarrow \mathbb{R}, f(x, y) = \sqrt{y - x^2}$$

domain:  $y \geq x^2$



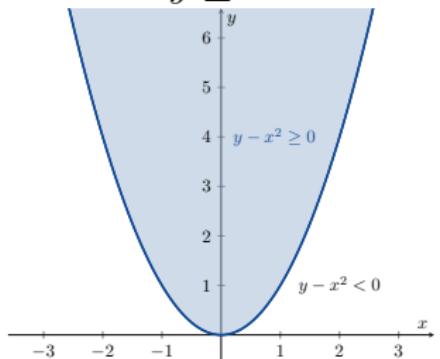
range:  $[0, \infty)$

## 13.1 Functions of Several Variables

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$$f : D \rightarrow \mathbb{R}, f(x, y) = \sqrt{y - x^2}$$

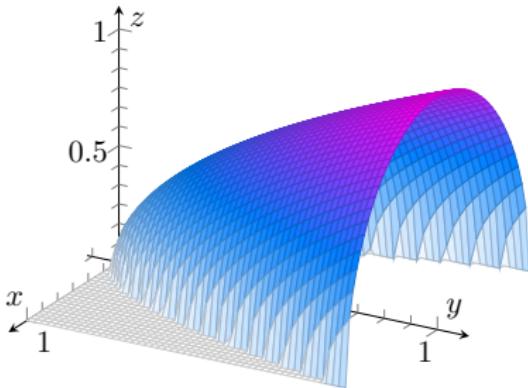
domain:  $y \geq x^2$



range:  $[0, \infty)$

graph:

$$z = \sqrt{y - x^2}$$





z=sqrt(y-x^2)



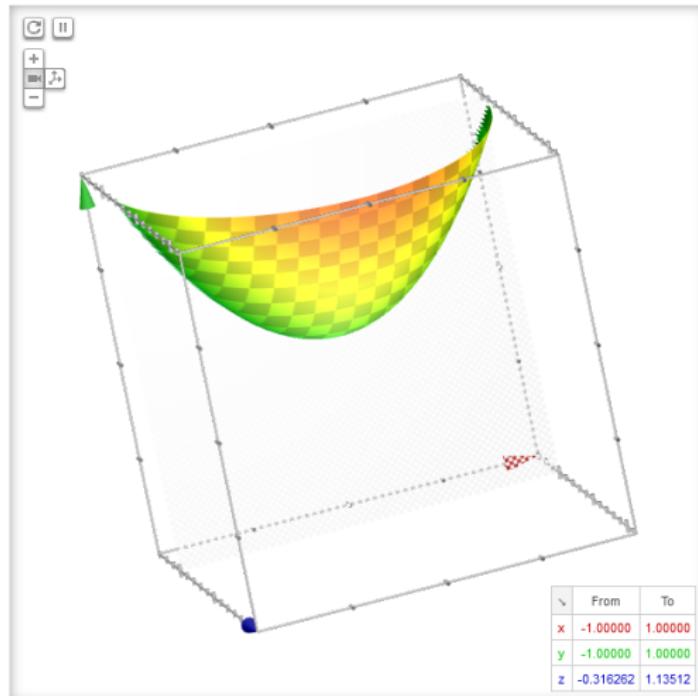
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## Graph for $\sqrt{y-x^2}$

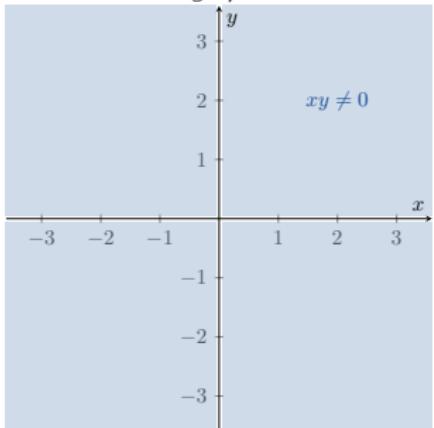


## 13.1 Functions of Several Variables

Example

$$f : D \rightarrow \mathbb{R}, f(x, y) = \frac{1}{xy}$$

domain:  $xy \neq 0$



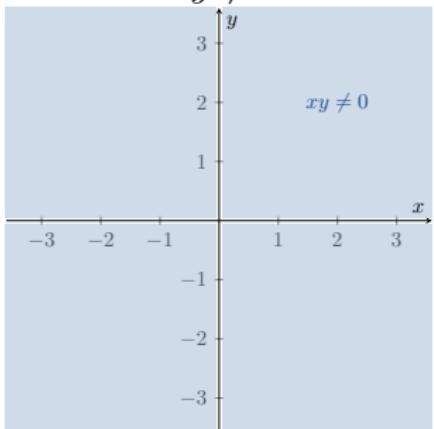
range:  $(-\infty, 0) \cup (0, \infty)$

## 13.1 Functions of Several Variables

### Example

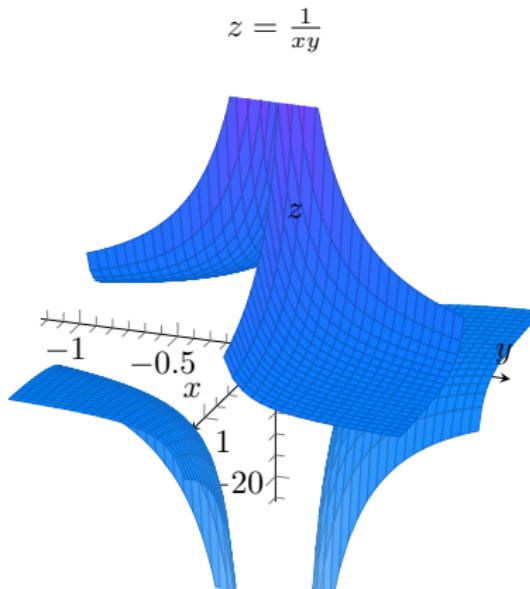
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graph:



## 13.1 Functions of Several Variables



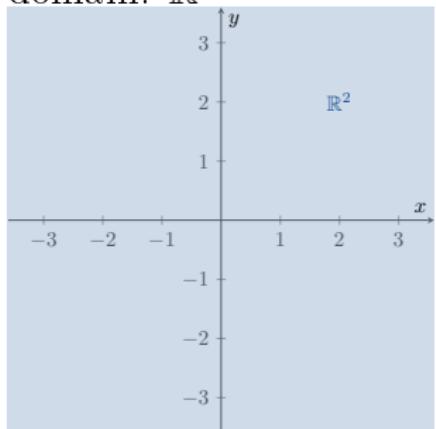
Try  $z=1/(x*y)$  on Google.

## 13.1 Functions of Several Variables

### Example

$$f : D \rightarrow \mathbb{R}, f(x, y) = \sin xy$$

domain:  $\mathbb{R}^2$



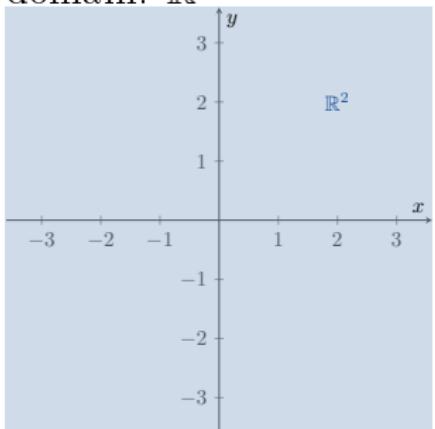
range:  $[-1, 1]$

## 13.1 Functions of Several Variables

### Example

$$f : D \rightarrow \mathbb{R}, f(x, y) = \sin xy$$

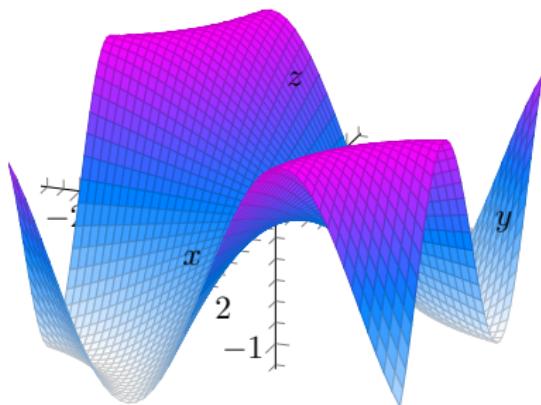
domain:  $\mathbb{R}^2$



range:  $[-1, 1]$

graph:

$$z = \sin xy$$

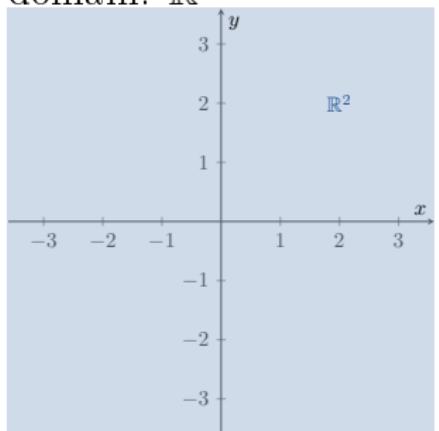


## 13.1 Functions of Several Variables

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$$f : D \rightarrow \mathbb{R}, f(x, y) = \sin xy$$

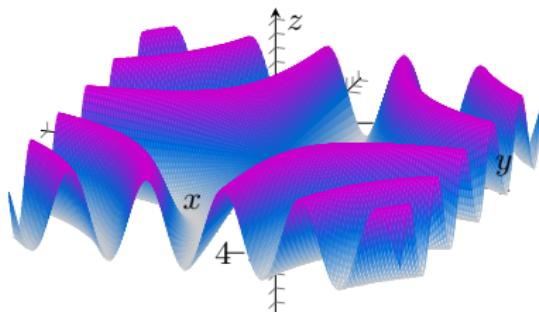
domain:  $\mathbb{R}^2$



range:  $[-1, 1]$

graph:

$$z = \sin xy$$



## 13.1 Functions of Several Variables

### Example

$$g : D \rightarrow \mathbb{R}, g(x, y, z) = \sqrt{x^2 + y^2 + z^2}$$

domain:  $\mathbb{R}^3$

range:  $[0, \infty)$

## 13.1 Functions of Several Variables



Example

$$g : D \rightarrow \mathbb{R}, g(x, y, z) = \sqrt{x^2 + y^2 + z^2}$$

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Example

$$g : D \rightarrow \mathbb{R}, g(x, y, z) = \frac{1}{x^2 + y^2 + z^2}$$

domain:  $(x, y, z) \neq (0, 0, 0)$

range:  $(0, \infty)$

## 13.1 Functions of Several Variables

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domain:  $(x, y, z) \neq (0, 0, 0)$

range:  $(0, \infty)$

Example

$$g : D \rightarrow \mathbb{R}, g(x, y, z) = xy \ln z$$

domain:  $z > 0$

range:  $\mathbb{R}$



# Topology

In  $\mathbb{R}$  we have open intervals  $(a, b)$ , closed intervals  $[a, b]$  and half-open intervals  $[a, b)$ ,  $(a, b]$ .

What do we have in  $\mathbb{R}^2$  and  $\mathbb{R}^3$ ?

## 13.1 Functions of Several Variables

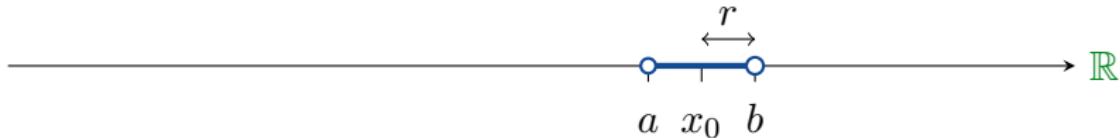


### Topology

In  $\mathbb{R}$  we have open intervals  $(a, b)$ , closed intervals  $[a, b]$  and half-open intervals  $[a, b)$ ,  $(a, b]$ .

What do we have in  $\mathbb{R}^2$  and  $\mathbb{R}^3$ ?

We can think of  $(a, b)$  as  $(x_0 - r, x_0 + r) = \{x \mid |x - x_0| < r\}$ .



## 13.1 Functions of Several Variables



$\mathbb{R}^2$

$\mathbb{R}^3$

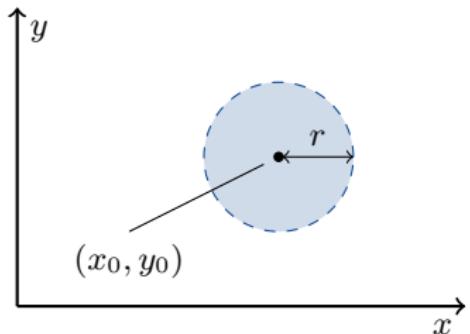
## 13.1 Functions of Several Variables

$\mathbb{R}^2$  The set

$\mathbb{R}^3$

$$\{(x, y) \mid (x - x_0)^2 + (y - y_0)^2 < r^2\}$$

is called an *open disc* of radius  $r$   
centred at  $(x_0, y_0)$ .



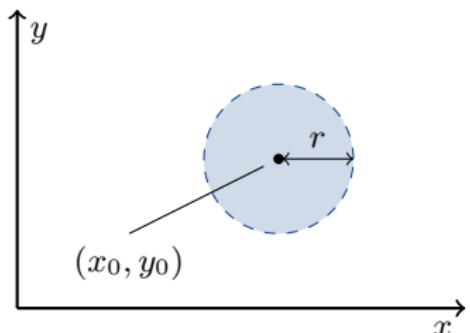
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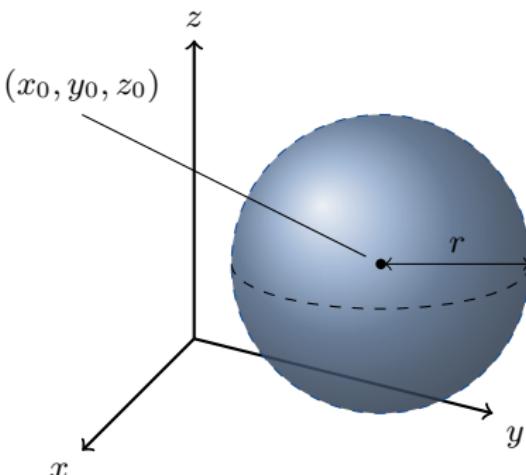


$$r > 0$$

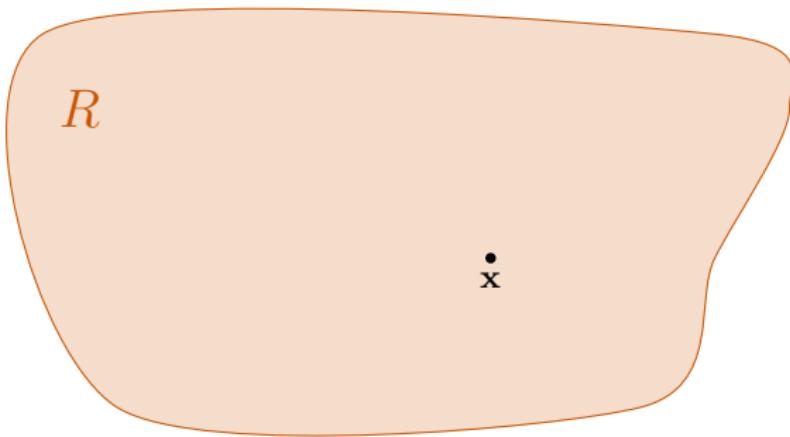
$\mathbb{R}^3$  The set

$$\{(x, y, z) \mid (x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2 < r^2\}$$

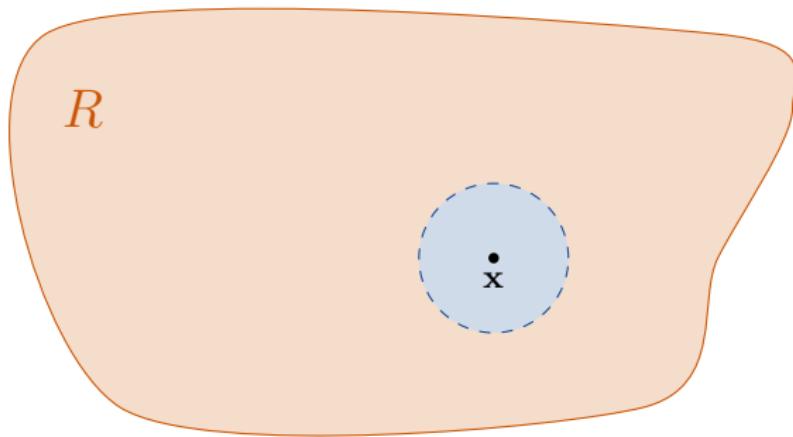
is called an *open ball* of radius  $r$   
centred at  $(x_0, y_0, z_0)$ .



## 13.1 Functions of Several Variables



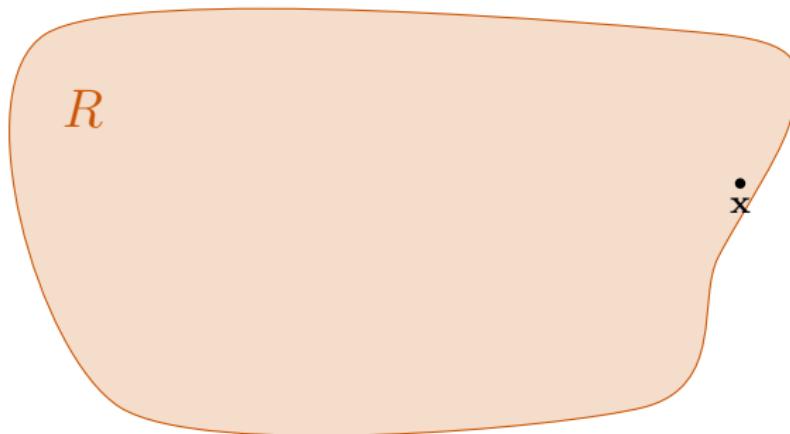
## 13.1 Functions of Several Variables



### Definition

A point  $\mathbf{x}$  in  $R$  is called an *interior point* of  $R$  iff there exists an open disk/ball centred at  $\mathbf{x}$  which lies entirely inside  $R$ .

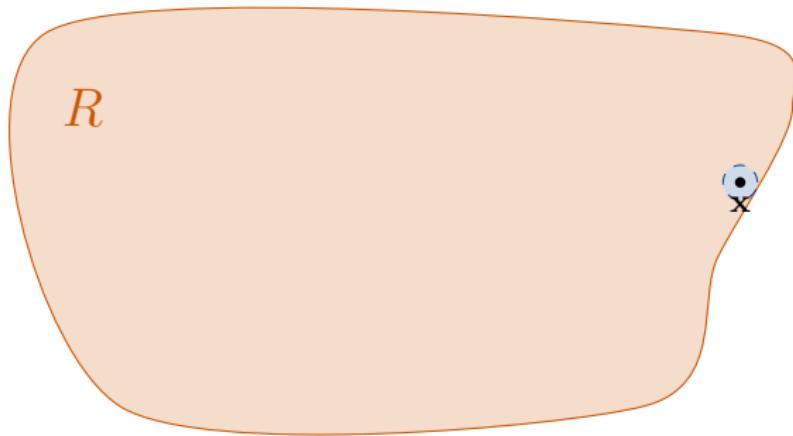
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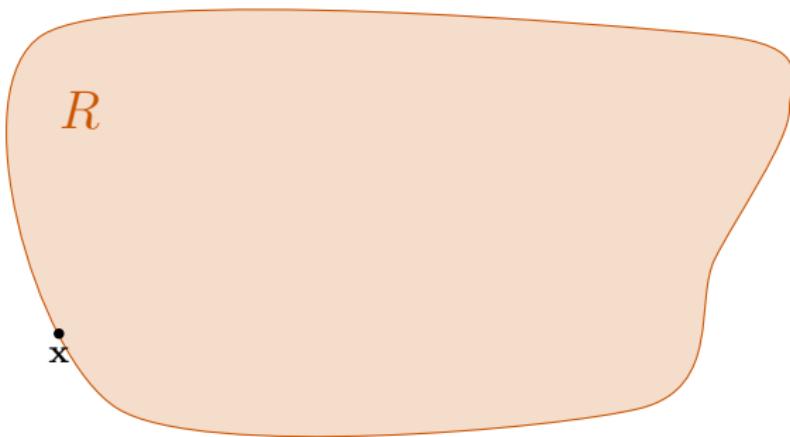
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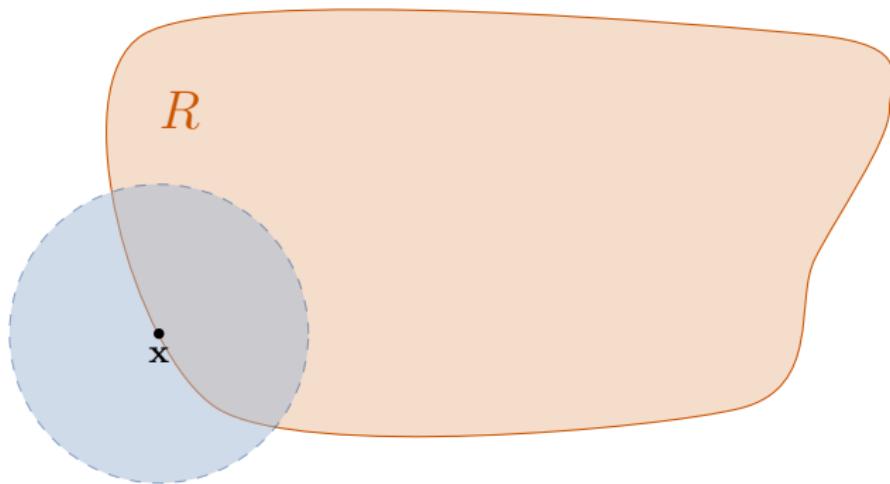
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## 13.1 Functions of Several Variables



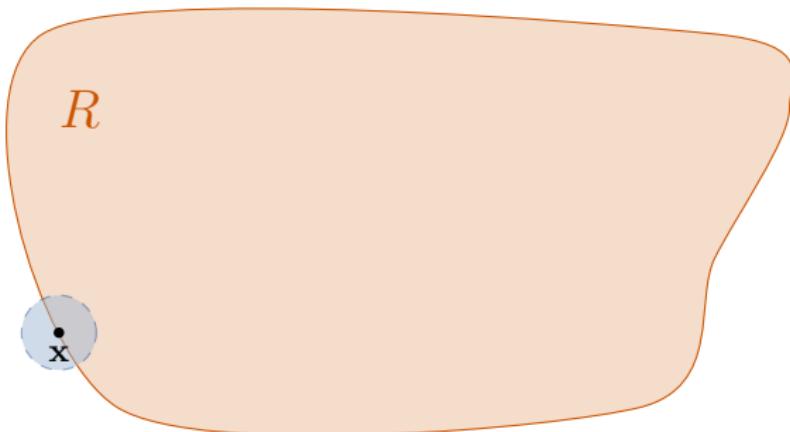
## 13.1 Functions of Several Variables



### Definition

A point  $\mathbf{x}$  is called a *boundary point* of  $R$  iff every open disk/ball centred at  $\mathbf{x}$  which partly inside  $R$  and partly outside  $R$ .

## 13.1 Functions of Several Variables



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## 13.1 Functions of Several Variables



### Definition

A set  $R$  is called *open* if every point in  $R$  is an interior point.

## 13.1 Functions of Several Variables



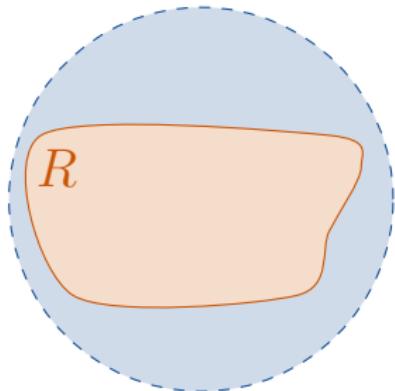
### Definition

A set  $R$  is called *open* if every point in  $R$  is an interior point.

### Definition

A set  $R$  is called *closed* if it contains all of its boundary points.

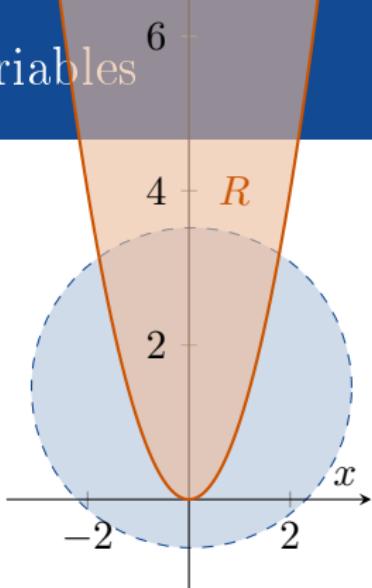
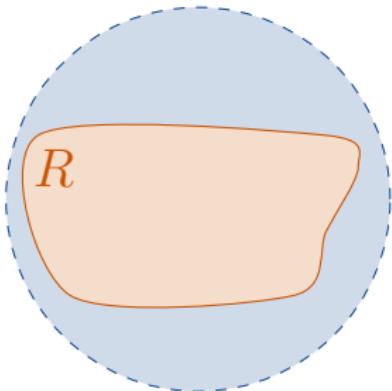
## 13.1 Functions of Several Variables



### Definition

A set  $R$  is called *bounded* iff it lies inside a disk/ball.

## 13.1 Functions of Several Variables



### Definition

A set  $R$  is called *bounded* iff it lies inside a disk/ball.

### Definition

A set  $R$  is called *unbounded* iff it is not bounded.

# Limits and Continuity in Higher Dimensions



## Limits

$$\lim_{x \rightarrow x_0} f(x) = L$$

“If  $x$  is close to  $x_0$  (but  $x \neq x_0$ ),  
then  $f(x)$  is close to  $L$ . ”

## 13.2 Limits and Continuity in Higher Dimensions

### Definition

$\mathbb{R}$

We write  $\lim_{x \rightarrow x_0} f(x) = L$  iff for all  $\varepsilon > 0$ , there exists  $\delta = \delta(\varepsilon) > 0$  such that for all  $x \in D$

$$0 < |x - x_0| < \delta \implies |f(x) - L| < \varepsilon.$$

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### Definition

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We write  $\lim_{(x,y) \rightarrow (x_0,y_0)} f(x) = L$  iff for all  $\varepsilon > 0$ , there exists  $\delta = \delta(\varepsilon) > 0$  such that for all  $(x, y) \in D$

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$$0 < \|(x, y) - (x_0, y_0)\| < \delta \implies |f(x) - L| < \varepsilon.$$

$$0 < \sqrt{(x - x_0)^2 + (y - y_0)^2} < \delta$$

## 13.2 Limits and Continuity in Higher Dimensions

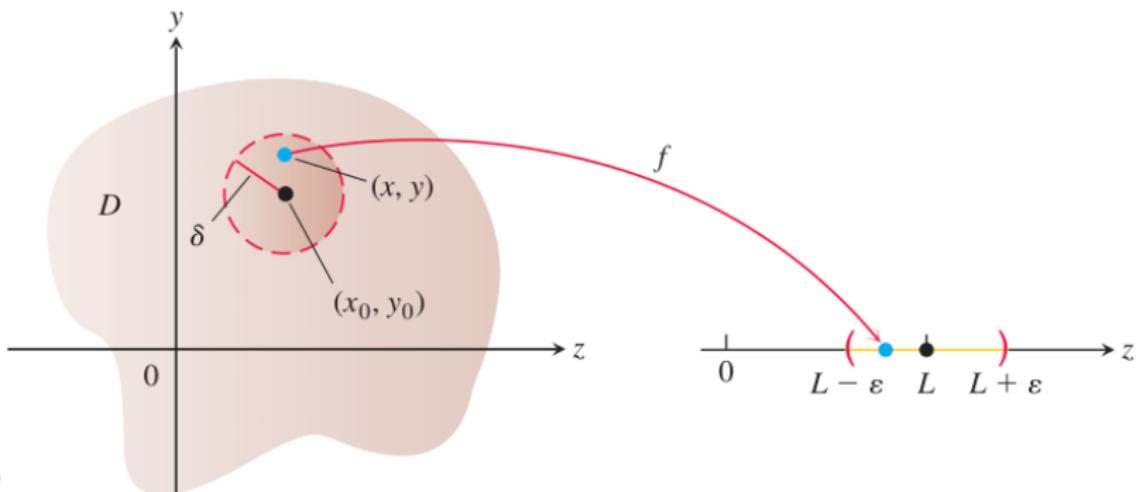


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## 13.2 Limits and Continuity in Higher Dimensions



### Definition

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### Definition

$\mathbb{R}^3$  We write  $\lim_{(x,y,z) \rightarrow (x_0,y_0,z_0)} f(x) = L$  iff for all  $\varepsilon > 0$ , there exists  $\delta = \delta(\varepsilon) > 0$  such that for all  $(x, y, z) \in D$

$$0 < \|(x, y, z) - (x_0, y_0, z_0)\| < \delta \implies |f(x) - L| < \varepsilon.$$

As for functions of a single variable, it can be shown that

$$\lim_{(x, y) \rightarrow (x_0, y_0)} x = x_0$$

$$\lim_{(x, y) \rightarrow (x_0, y_0)} y = y_0$$

$$\lim_{(x, y) \rightarrow (x_0, y_0)} k = k \quad (\text{any number } k).$$

For example, in the first limit statement above,  $f(x, y) = x$  and  $L = x_0$ . Using the definition of limit, suppose that  $\varepsilon > 0$  is chosen. If we let  $\delta$  equal this  $\varepsilon$ , we see that if

$$0 < \sqrt{(x - x_0)^2 + (y - y_0)^2} < \delta = \varepsilon,$$

then

$$\sqrt{(x - x_0)^2} < \varepsilon \quad (x - x_0)^2 \leq (x - x_0)^2 + (y - y_0)^2$$

$$|x - x_0| < \varepsilon \quad \sqrt{a^2} = |a|$$

$$|f(x, y) - x_0| < \varepsilon. \quad x = f(x, y)$$

That is,

$$|f(x, y) - x_0| < \varepsilon \quad \text{whenever} \quad 0 < \sqrt{(x - x_0)^2 + (y - y_0)^2} < \delta.$$

So a  $\delta$  has been found satisfying the requirement of the definition, and therefore we have proved that

$$\lim_{(x, y) \rightarrow (x_0, y_0)} f(x, y) = \lim_{(x, y) \rightarrow (x_0, y_0)} x = x_0.$$

## THEOREM 1 – Properties of Limits of Functions of Two Variables

The following rules hold if  $L$ ,  $M$ , and  $k$  are real numbers and

$$\lim_{(x, y) \rightarrow (x_0, y_0)} f(x, y) = L \quad \text{and} \quad \lim_{(x, y) \rightarrow (x_0, y_0)} g(x, y) = M.$$

- 1. Sum Rule:**

$$\lim_{(x, y) \rightarrow (x_0, y_0)} (f(x, y) + g(x, y)) = L + M$$

- 2. Difference Rule:**

$$\lim_{(x, y) \rightarrow (x_0, y_0)} (f(x, y) - g(x, y)) = L - M$$

- 3. Constant Multiple Rule:**

$$\lim_{(x, y) \rightarrow (x_0, y_0)} kf(x, y) = kL \quad (\text{any number } k)$$

- 4. Product Rule:**

$$\lim_{(x, y) \rightarrow (x_0, y_0)} (f(x, y) \cdot g(x, y)) = L \cdot M$$

- 5. Quotient Rule:**

$$\lim_{(x, y) \rightarrow (x_0, y_0)} \frac{f(x, y)}{g(x, y)} = \frac{L}{M}, \quad M \neq 0$$

- 6. Power Rule:**

$$\lim_{(x, y) \rightarrow (x_0, y_0)} [f(x, y)]^n = L^n, \quad n \text{ a positive integer}$$

- 7. Root Rule:**

$$\lim_{(x, y) \rightarrow (x_0, y_0)} \sqrt[n]{f(x, y)} = \sqrt[n]{L} = L^{1/n},$$

$n$  a positive integer, and if  $n$  is even,  
we assume that  $L > 0$ .

**EXAMPLE 1** In this example, we can combine the three simple results following the limit definition with the results in Theorem 1 to calculate the limits. We simply substitute the  $x$ - and  $y$ -values of the point being approached into the functional expression to find the limiting value.

(a)  $\lim_{(x, y) \rightarrow (0, 1)} \frac{x - xy + 3}{x^2y + 5xy - y^3} = \frac{0 - (0)(1) + 3}{(0)^2(1) + 5(0)(1) - (1)^3} = -3$

(b)  $\lim_{(x, y) \rightarrow (3, -4)} \sqrt{x^2 + y^2} = \sqrt{(3)^2 + (-4)^2} = \sqrt{25} = 5$



## 13.2 Limits and Continuity in Higher Dimensions



### Example

$$\text{Find } \lim_{(x,y) \rightarrow (0,0)} \frac{x^2 - xy}{\sqrt{x} - \sqrt{y}}.$$

## 13.2 Limits and Continuity in Higher Dimensions



### Example

$$\text{Find } \lim_{(x,y) \rightarrow (0,0)} \frac{x^2 - xy}{\sqrt{x} - \sqrt{y}}.$$

We can't use the Quotient Rule here because  $\sqrt{x} - \sqrt{y} \rightarrow 0$  as  $(x, y) \rightarrow (0, 0)$ .

## 13.2 Limits and Continuity in Higher Dimensions



### Example

$$\text{Find } \lim_{(x,y) \rightarrow (0,0)} \frac{x^2 - xy}{\sqrt{x} - \sqrt{y}}.$$

We can't use the Quotient Rule here because  $\sqrt{x} - \sqrt{y} \rightarrow 0$  as  $(x, y) \rightarrow (0, 0)$ . Instead we calculate that

$$\lim_{(x,y) \rightarrow (0,0)} \frac{x^2 - xy}{\sqrt{x} - \sqrt{y}} = \lim_{(x,y) \rightarrow (0,0)} \frac{(x^2 - xy)(\sqrt{x} + \sqrt{y})}{(\sqrt{x} - \sqrt{y})(\sqrt{x} + \sqrt{y})}$$

=

=

=

## 13.2 Limits and Continuity in Higher Dimensions



### Example

$$\text{Find } \lim_{(x,y) \rightarrow (0,0)} \frac{x^2 - xy}{\sqrt{x} - \sqrt{y}}.$$

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$$= \lim_{(x,y) \rightarrow (0,0)} \frac{x(x - y)(\sqrt{x} + \sqrt{y})}{x - y}$$

=

=

## 13.2 Limits and Continuity in Higher Dimensions



### Example

Find  $\lim_{(x,y) \rightarrow (0,0)} \frac{x^2 - xy}{\sqrt{x} - \sqrt{y}}.$

We can't use the Quotient Rule here because  $\sqrt{x} - \sqrt{y} \rightarrow 0$  as  $(x, y) \rightarrow (0, 0)$ . Instead we calculate that

$$\begin{aligned}\lim_{(x,y) \rightarrow (0,0)} \frac{x^2 - xy}{\sqrt{x} - \sqrt{y}} &= \lim_{(x,y) \rightarrow (0,0)} \frac{(x^2 - xy)(\sqrt{x} + \sqrt{y})}{(\sqrt{x} - \sqrt{y})(\sqrt{x} + \sqrt{y})} \\ &= \lim_{(x,y) \rightarrow (0,0)} \frac{x(x - y)(\sqrt{x} + \sqrt{y})}{x - y} \\ &= \lim_{(x,y) \rightarrow (0,0)} x(\sqrt{x} + \sqrt{y}) \\ &= \end{aligned}$$

## 13.2 Limits and Continuity in Higher Dimensions



### Example

Find  $\lim_{(x,y) \rightarrow (0,0)} \frac{x^2 - xy}{\sqrt{x} - \sqrt{y}}.$

We can't use the Quotient Rule here because  $\sqrt{x} - \sqrt{y} \rightarrow 0$  as  $(x, y) \rightarrow (0, 0)$ . Instead we calculate that

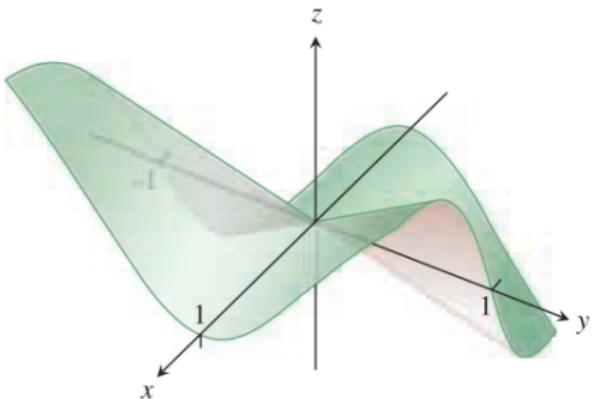
$$\begin{aligned}\lim_{(x,y) \rightarrow (0,0)} \frac{x^2 - xy}{\sqrt{x} - \sqrt{y}} &= \lim_{(x,y) \rightarrow (0,0)} \frac{(x^2 - xy)(\sqrt{x} + \sqrt{y})}{(\sqrt{x} - \sqrt{y})(\sqrt{x} + \sqrt{y})} \\ &= \lim_{(x,y) \rightarrow (0,0)} \frac{x(x - y)(\sqrt{x} + \sqrt{y})}{x - y} \\ &= \lim_{(x,y) \rightarrow (0,0)} x(\sqrt{x} + \sqrt{y}) \\ &= 0(0 + 0) = 0.\end{aligned}$$

## 13.2 Limits and Continuity in Higher Dimensions



**EXAMPLE 3** Find  $\lim_{(x,y) \rightarrow (0,0)} \frac{4xy^2}{x^2 + y^2}$  if it exists.

**Solution** We first observe that along the line  $x = 0$ , the function always has value 0 when  $y \neq 0$ . Likewise, along the line  $y = 0$ , the function has value 0 provided  $x \neq 0$ . So if the limit does exist as  $(x, y)$  approaches  $(0, 0)$ , the value of the limit must be 0 (see Figure 14.13). To see if this is true, we apply the definition of limit.



**FIGURE 14.13** The surface graph shows the limit of the function in Example 3 must be 0, if it exists.

Let  $\varepsilon > 0$  be given, but arbitrary. We want to find a  $\delta > 0$  such that

$$\left| \frac{4xy^2}{x^2 + y^2} - 0 \right| < \varepsilon \quad \text{whenever} \quad 0 < \sqrt{x^2 + y^2} < \delta$$

or

$$\frac{4|x|y^2}{x^2 + y^2} < \varepsilon \quad \text{whenever} \quad 0 < \sqrt{x^2 + y^2} < \delta.$$

Since  $y^2 \leq x^2 + y^2$  we have that

$$\frac{4|x|y^2}{x^2 + y^2} \leq 4|x| = 4\sqrt{x^2} \leq 4\sqrt{x^2 + y^2}. \quad \frac{y^2}{x^2 + y^2} \leq 1$$

So if we choose  $\delta = \varepsilon/4$  and let  $0 < \sqrt{x^2 + y^2} < \delta$ , we get

$$\left| \frac{4xy^2}{x^2 + y^2} - 0 \right| \leq 4\sqrt{x^2 + y^2} < 4\delta = 4\left(\frac{\varepsilon}{4}\right) = \varepsilon.$$

It follows from the definition that

$$\lim_{(x,y) \rightarrow (0,0)} \frac{4xy^2}{x^2 + y^2} = 0.$$

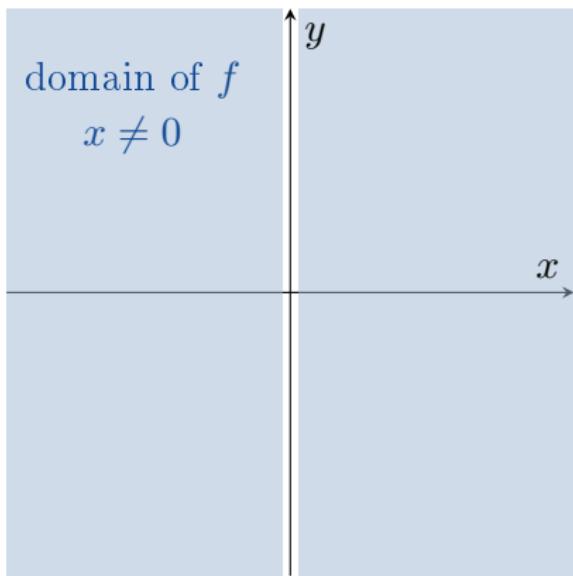


## 13.2 Limits and Continuity in Higher Dimensions



### Example

If  $f(x, y) = \frac{y}{x}$ , does  $\lim_{(x,y) \rightarrow (0,0)} f(x, y)$  exist?

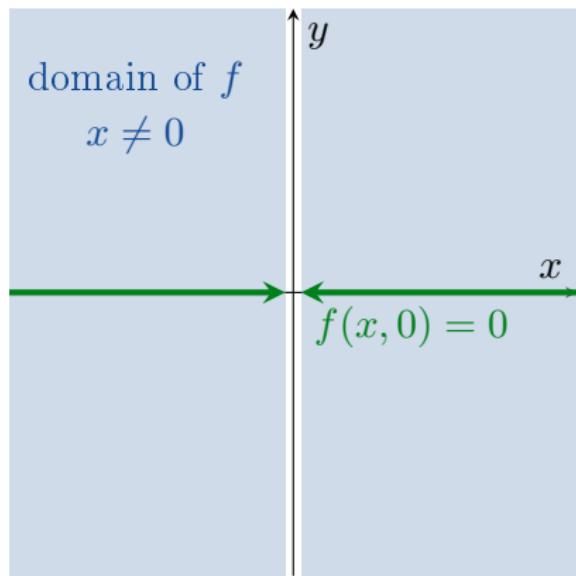


## 13.2 Limits and Continuity in Higher Dimensions



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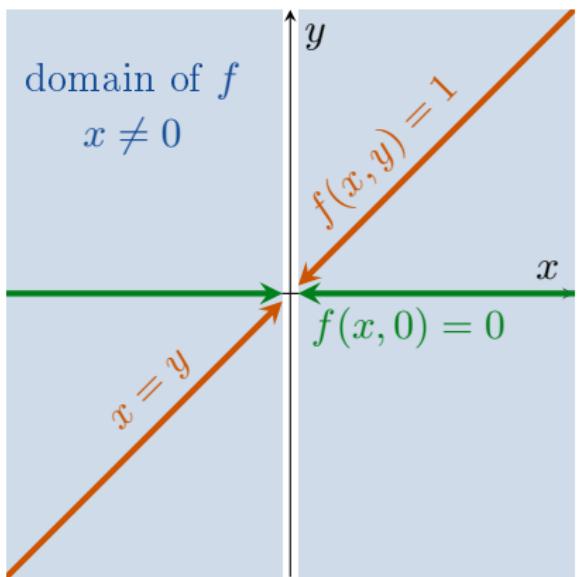
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## 13.2 Limits and Continuity in Higher Dimensions



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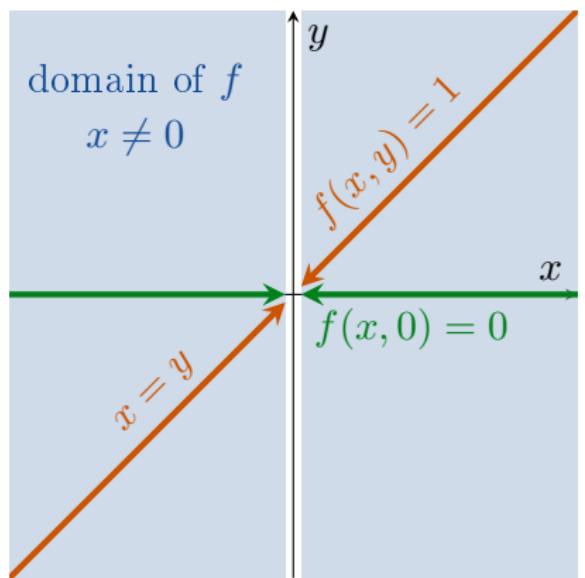
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## 13.2 Limits and Continuity in Higher Dimensions



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$$\lim_{(x,y) \rightarrow (0,0)} f(x, y) = \lim_{(x,y) \rightarrow (0,0)} 0 = 0$$

$$\lim_{(x,y) \rightarrow (0,0)} f(x, y) = \lim_{(x,y) \rightarrow (0,0)} 1 = 1$$

$\lim_{(x,y) \rightarrow (0,0)} f(x, y)$  does not exist.



## Continuity

**DEFINITION** A function  $f(x, y)$  is **continuous at the point  $(x_0, y_0)$**  if

1.  $f$  is defined at  $(x_0, y_0)$ ,
2.  $\lim_{(x, y) \rightarrow (x_0, y_0)} f(x, y)$  exists,
3.  $\lim_{(x, y) \rightarrow (x_0, y_0)} f(x, y) = f(x_0, y_0)$ .

A function is **continuous** if it is continuous at every point of its domain.

## 13.2 Limits and Continuity in Higher Dimensions



### Example

Consider  $f(x, y) = \begin{cases} \frac{2xy}{x^2+y^2} & \text{if } (x, y) \neq (0, 0) \\ 0 & \text{if } (x, y) = (0, 0) \end{cases}$ .

Clearly this function is continuous away from the origin. But is it also continuous at the origin?

## 13.2 Limits and Continuity in Higher Dimensions

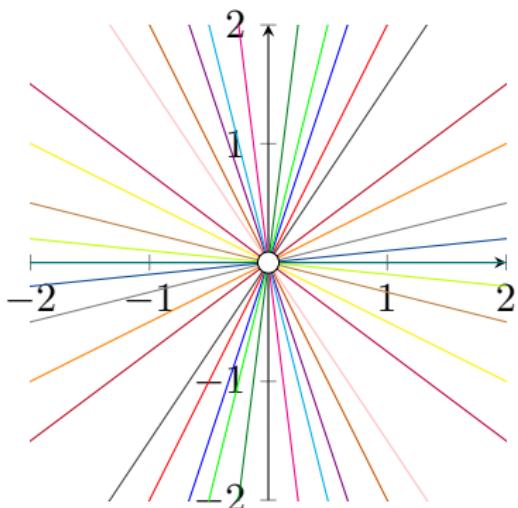


### Example

Consider  $f(x, y) = \begin{cases} \frac{2xy}{x^2+y^2} & \text{if } (x, y) \neq (0, 0) \\ 0 & \text{if } (x, y) = (0, 0) \end{cases}$ .

Clearly this function is continuous away from the origin. But is it also continuous at the origin?

We will consider the behaviour of  $f(x, y)$  along the lines  $y = mx$  ( $x \neq 0$ ).



## 13.2 Limits and Continuity in Higher Dimensions



Note that

$$f(x, y) \Big|_{y=mx} = \frac{2xy}{x^2 + y^2} \Big|_{y=mx} = \frac{2mx^2}{x^2 + m^2x^2} = \frac{2m}{1 + m^2}$$

## 13.2 Limits and Continuity in Higher Dimensions



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and

$$\lim_{\substack{(x,y) \rightarrow (0,0) \\ \text{along } y=mx}} f(x, y) = \frac{2m}{1 + m^2}.$$

This limit depends on  $m$ . So if we approach  $(0, 0)$  from different directions, then we would get a different value for the limit.

## 13.2 Limits and Continuity in Higher Dimensions



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Therefore the limit  $\lim_{(x,y) \rightarrow (0,0)} f(x, y)$  does not exist.

## 13.2 Limits and Continuity in Higher Dimensions



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This limit depends on  $m$ . So if we approach  $(0, 0)$  from different directions, then we would get a different value for the limit.

Therefore the limit  $\lim_{(x,y) \rightarrow (0,0)} f(x, y)$  does not exist.

Therefore  $f(x, y)$  is NOT continuous at  $(0, 0)$ .

## 13.2 Limits and Continuity in Higher Dimensions



### Remark

If a function  $f(x, y)$  has different limits along two different paths in the domain of  $f$  as  $(x, y)$  approaches  $(x_0, y_0)$ , then

$$\lim_{(x,y) \rightarrow (x_0,y_0)} f(x, y)$$

does not exist.

## 13.2 Limits and Continuity in Higher Dimensions



**EXAMPLE 6** Show that the function

$$f(x, y) = \frac{2x^2y}{x^4 + y^2}$$

(Figure 14.15) has no limit as  $(x, y)$  approaches  $(0, 0)$ .

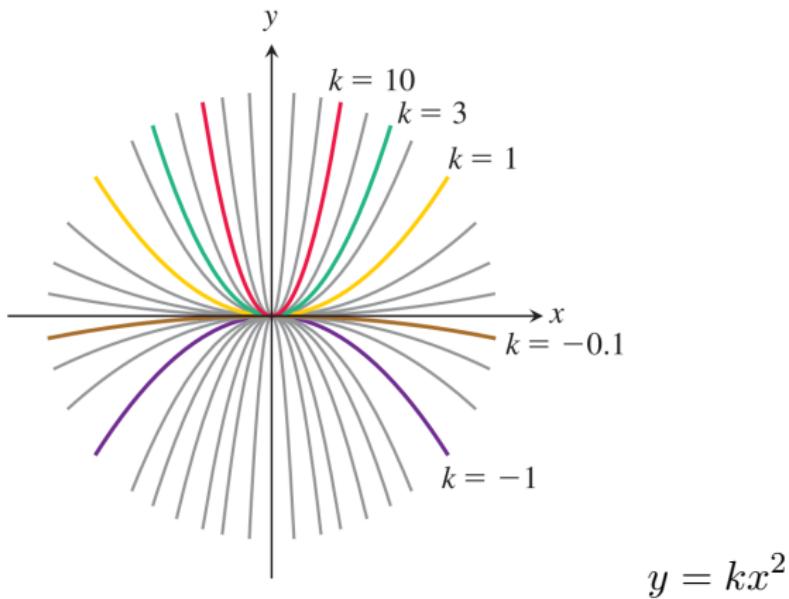
## 13.2 Limits and Continuity in Higher Dimensions



**EXAMPLE 6** Show that the function

$$f(x, y) = \frac{2x^2y}{x^4 + y^2}$$

(Figure 14.15) has no limit as  $(x, y)$  approaches  $(0, 0)$ .



**Solution** The limit cannot be found by direct substitution, which gives the indeterminate form  $0/0$ . We examine the values of  $f$  along parabolic curves that end at  $(0, 0)$ . Along the curve  $y = kx^2$ ,  $x \neq 0$ , the function has the constant value

$$f(x, y) \Big|_{y=kx^2} = \frac{2x^2y}{x^4 + y^2} \Big|_{y=kx^2} = \frac{2x^2(kx^2)}{x^4 + (kx^2)^2} = \frac{2kx^4}{x^4 + k^2x^4} = \frac{2k}{1 + k^2}.$$

Therefore,

$$\lim_{\substack{(x, y) \rightarrow (0, 0) \\ \text{along } y=kx^2}} f(x, y) = \lim_{(x, y) \rightarrow (0, 0)} \left[ f(x, y) \Big|_{y=kx^2} \right] = \frac{2k}{1 + k^2}.$$

This limit varies with the path of approach. If  $(x, y)$  approaches  $(0, 0)$  along the parabola  $y = x^2$ , for instance,  $k = 1$  and the limit is 1. If  $(x, y)$  approaches  $(0, 0)$  along the  $x$ -axis,  $k = 0$  and the limit is 0. By the two-path test,  $f$  has no limit as  $(x, y)$  approaches  $(0, 0)$ . ■

It can be shown that the function in Example 6 has limit 0 along every straight line path  $y = mx$  (Exercise 57). This implies the following observation:

Having the same limit along all straight lines approaching  $(x_0, y_0)$  does not imply that a limit exists at  $(x_0, y_0)$ .

## 13.2 Limits and Continuity in Higher Dimensions



Almost everything that we have done here has been for functions of two variables. But the same ideas can be extended to functions of three (or more) variables.

## 13.2 Limits and Continuity in Higher Dimensions



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### Example

$$\lim_{(x,y,z) \rightarrow (1,0,-1)} \frac{e^{x+y+z}}{z^2 + \cos \sqrt{xy}} =$$

## 13.2 Limits and Continuity in Higher Dimensions



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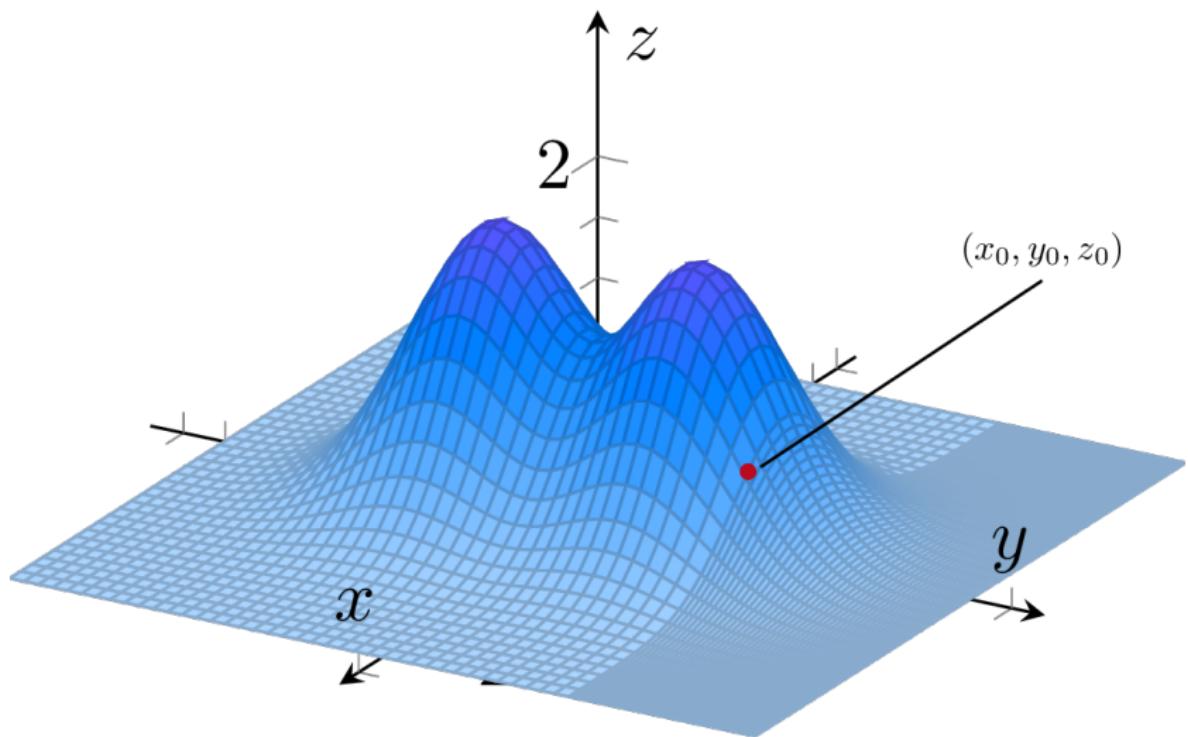
### Example

$$\lim_{(x,y,z) \rightarrow (1,0,-1)} \frac{e^{x+y+z}}{z^2 + \cos \sqrt{xy}} = \frac{e^{1+0-1}}{(-1)^2 + \cos 0} = \frac{1}{2}.$$

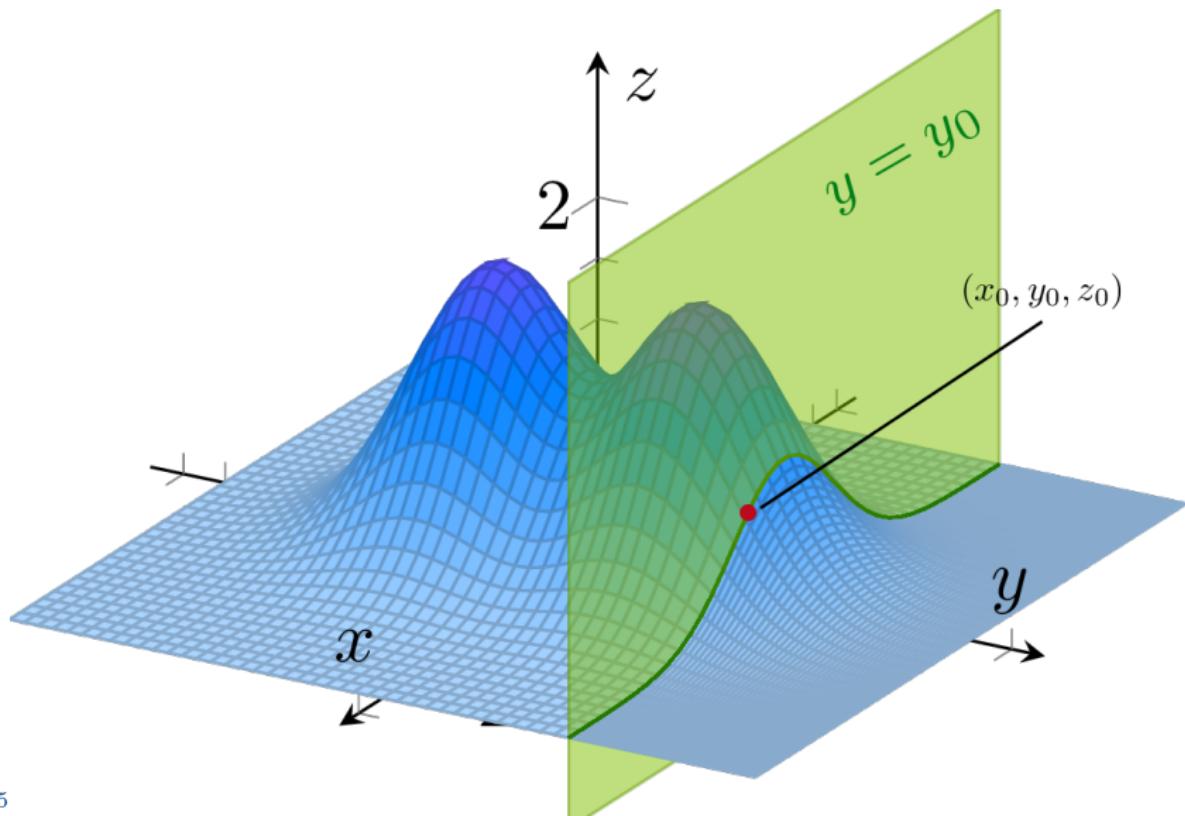


# 103 Partial Derivatives

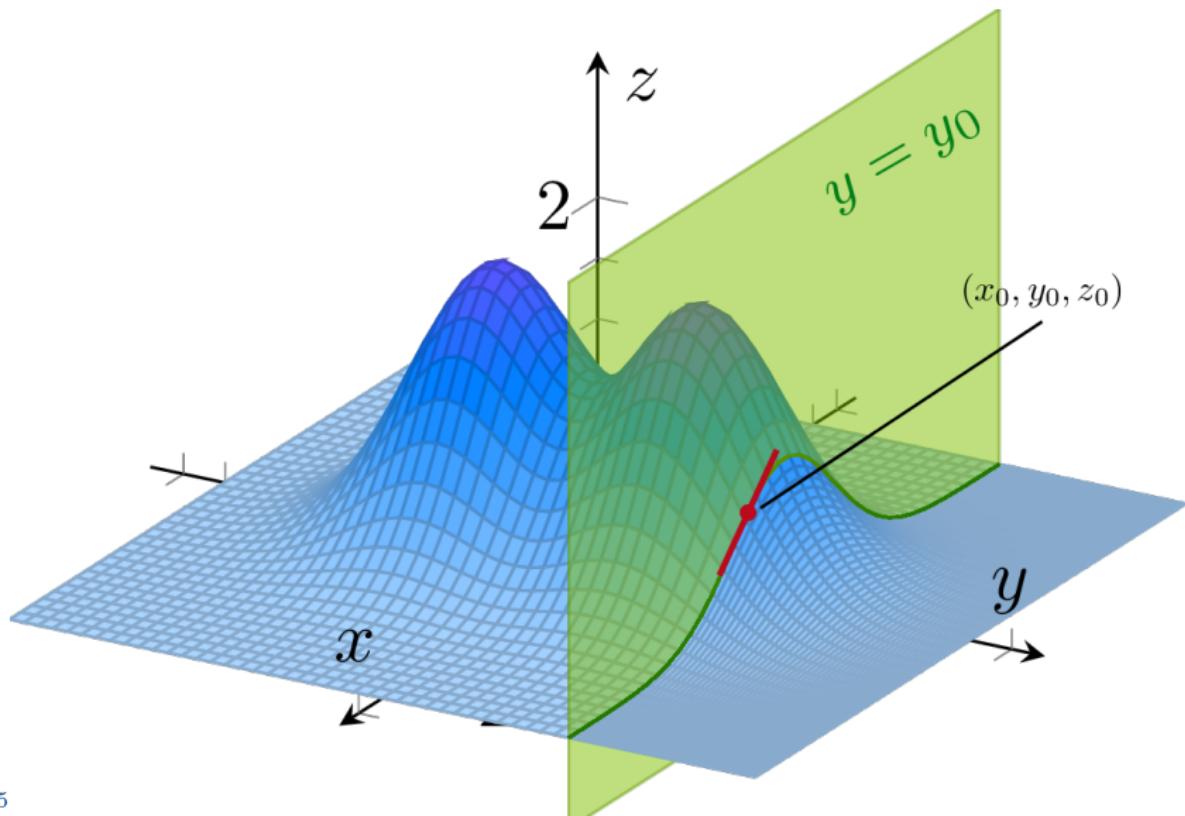
### 13.3 Partial Derivatives



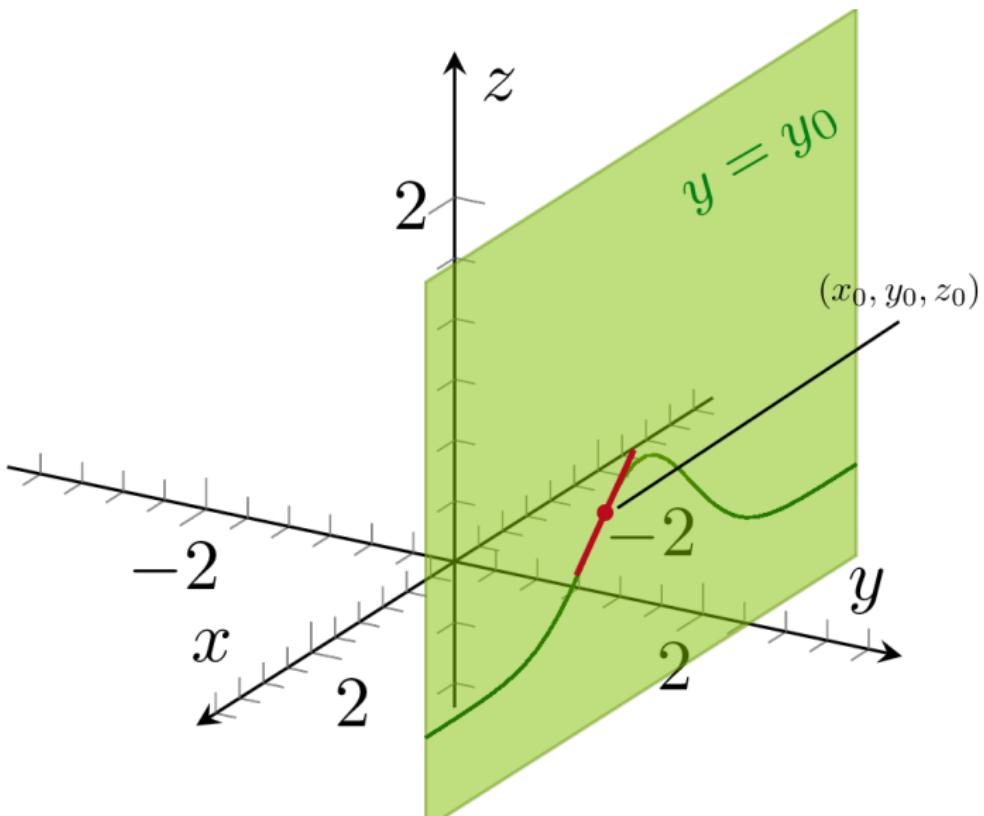
### 13.3 Partial Derivatives



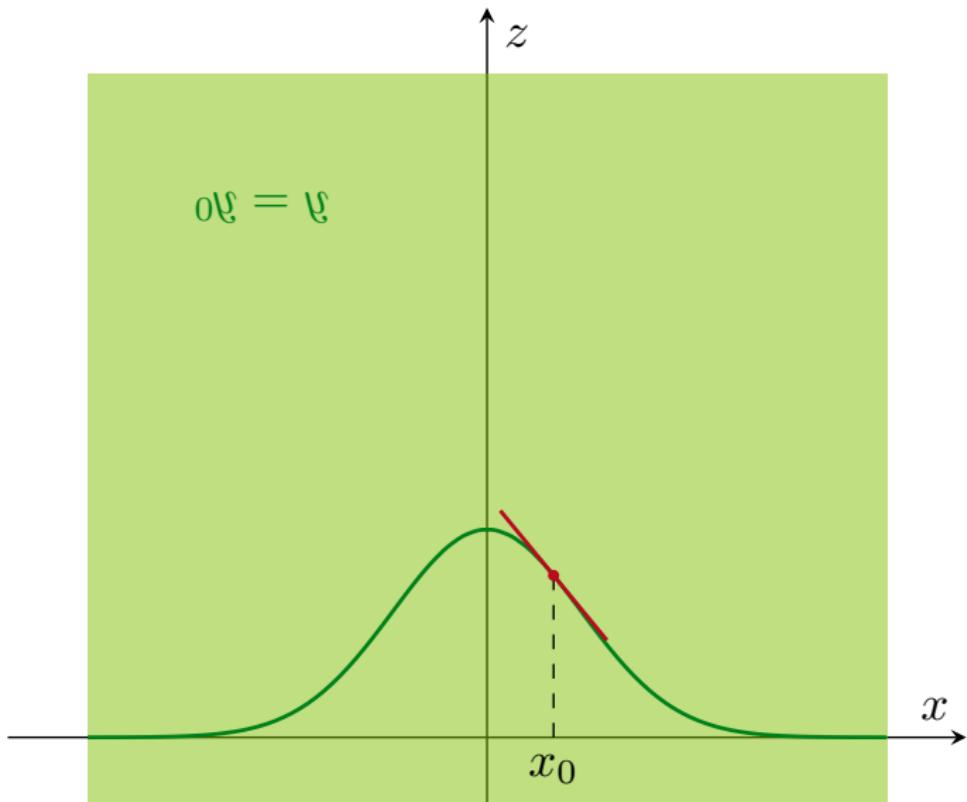
### 13.3 Partial Derivatives



### 13.3 Partial Derivatives



### 13.3 Partial Derivatives



### 13.3 Partial Derivatives

#### Definition

The *partial derivative* of  $f(x, y)$  with respect to  $x$  at the point  $(x_0, y_0)$  is

$$\frac{\partial f}{\partial x} \Big|_{(x_0, y_0)} = \lim_{h \rightarrow 0} \frac{f(x_0 + h, y_0) - f(x_0, y_0)}{h}$$

if the limit exists.

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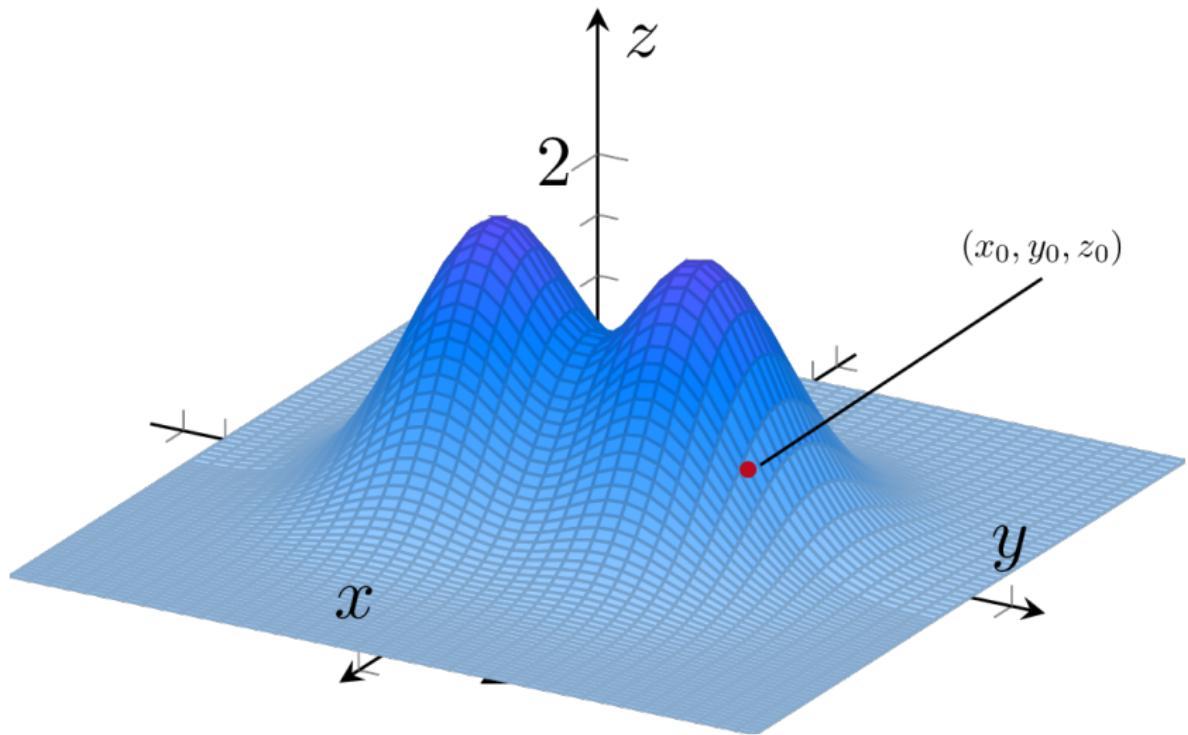
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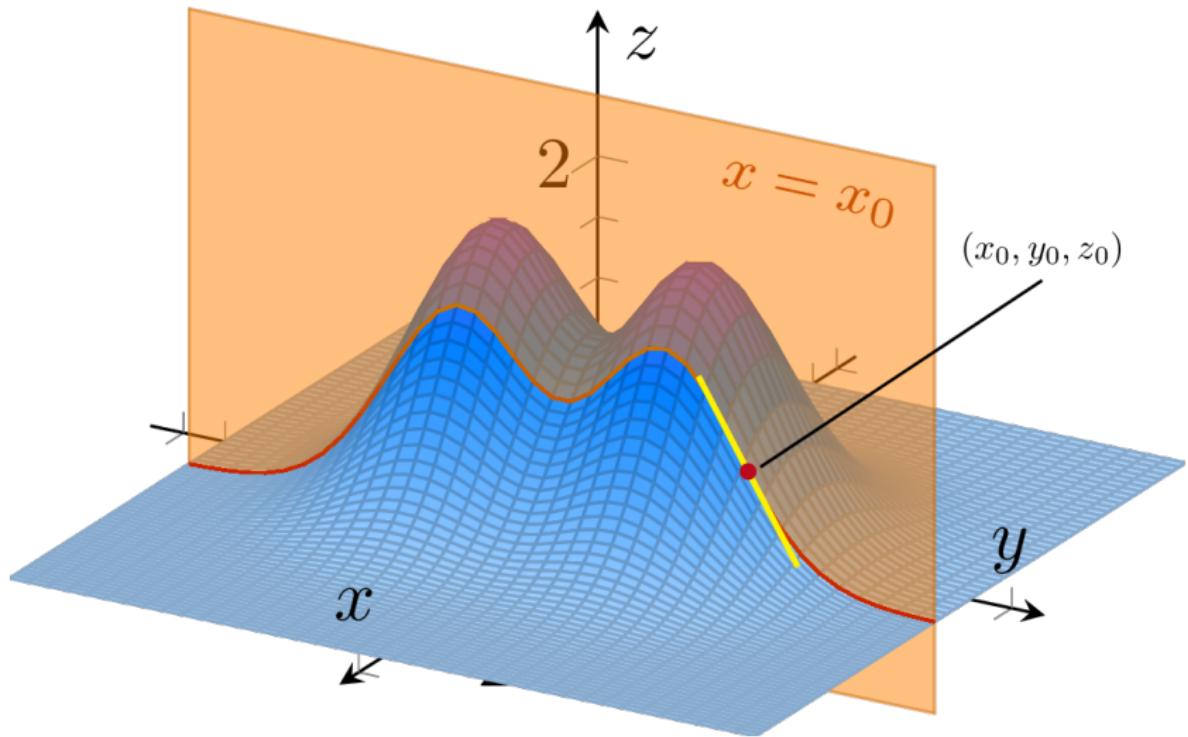
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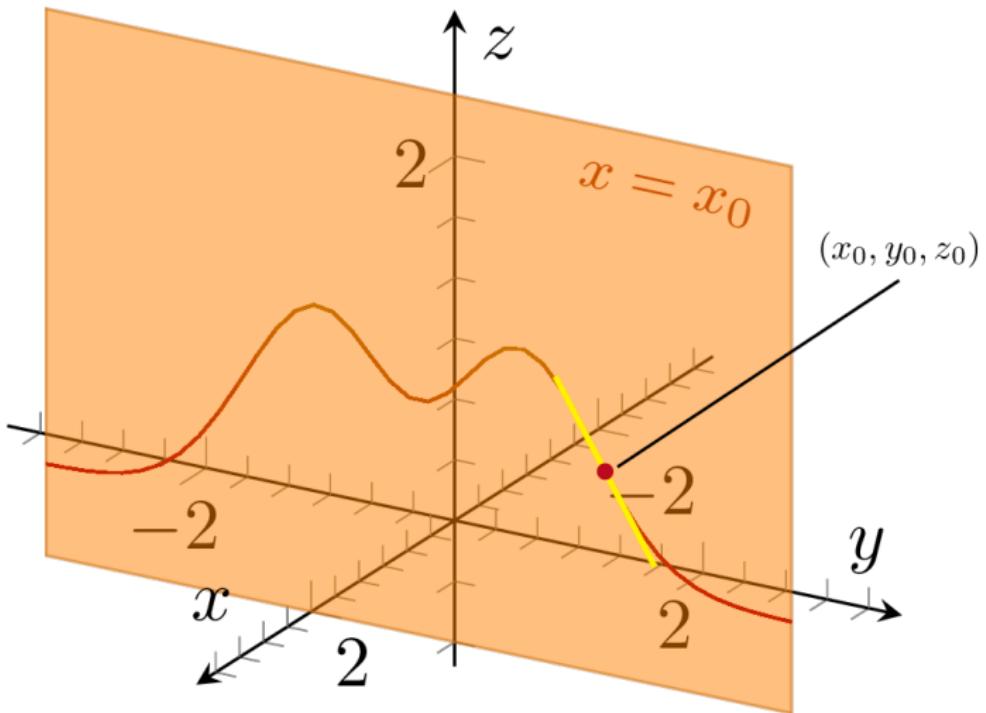
### 13.3 Partial Derivatives



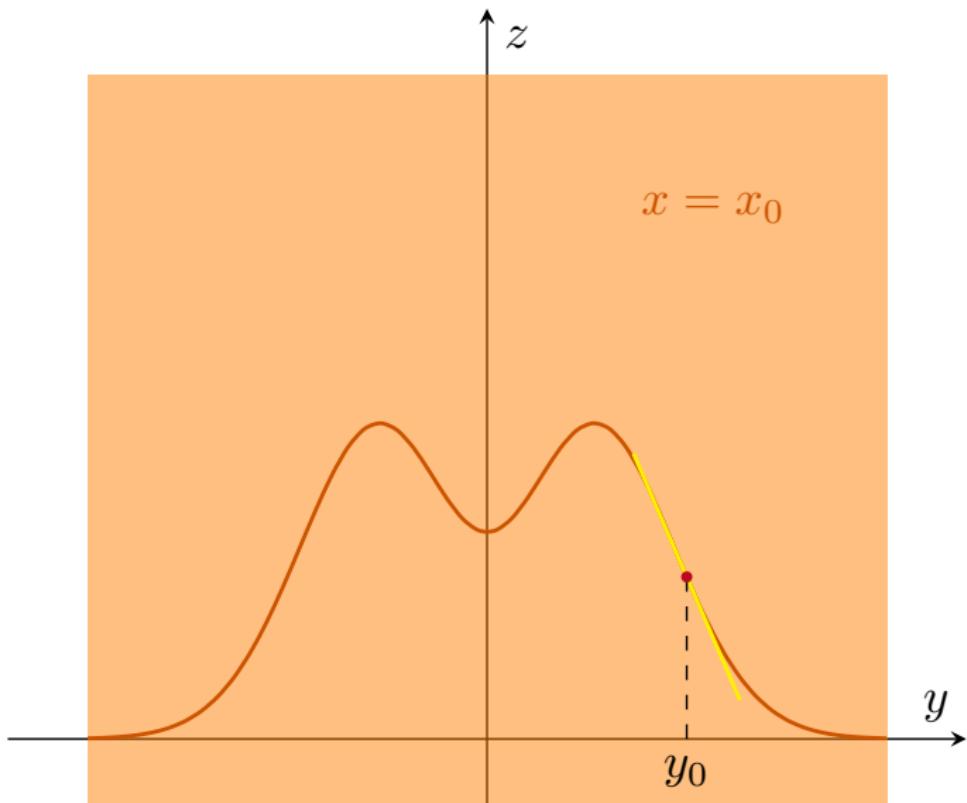
### 13.3 Partial Derivatives



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## Notation

$$\frac{df}{dx} = f' \quad \text{ordinary derivative}$$

### 13.3 Partial Derivatives



## Notation

$$\frac{df}{dx} = f' \quad \text{ordinary derivative}$$

$$\frac{\partial f}{\partial x} = f_x \quad \text{partial derivative wrt } x$$

$$\frac{\partial f}{\partial y} = f_y \quad \text{partial derivative wrt } y$$

### 13.3 Partial Derivatives

#### Example

Let  $f(x) = x^2 + 3xy + y - 1$ .

Find  $\frac{\partial f}{\partial x}$  and  $\frac{\partial f}{\partial y}$  at the point  $(4, -5)$ .

### 13.3 Partial Derivatives

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To find  $\frac{\partial f}{\partial x}$  we pretend that  $y$  is a constant.

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To find  $\frac{\partial f}{\partial x}$  we pretend that  $y$  is a constant. We pretend that we are differentiating  $g(x) = x^2 + 3x\textcolor{brown}{c} + \textcolor{brown}{c} - 1$ .

### 13.3 Partial Derivatives

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### 13.3 Partial Derivatives

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$$\frac{\partial f}{\partial x} = \frac{\partial}{\partial x} (x^2 + 3xy + y - 1) = 2x + 3y + 0 - 0.$$

### 13.3 Partial Derivatives

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$$\frac{\partial f}{\partial x} = \frac{\partial}{\partial x} (x^2 + 3xy + y - 1) = 2x + 3y + 0 - 0.$$

Therefore

$$\left. \frac{\partial f}{\partial x} \right|_{(4,-5)} = 2x + 3y \Big|_{(4,-5)} = -7.$$

13.3

$$f(x) = x^2 + 3xy + y - 1$$



And to find  $\frac{\partial f}{\partial y}$  we pretend that  $x$  is a constant.

13.3

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$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y} (x^2 + 3xy + y - 1) = 0 + 3x + 1 - 0$$

13.3

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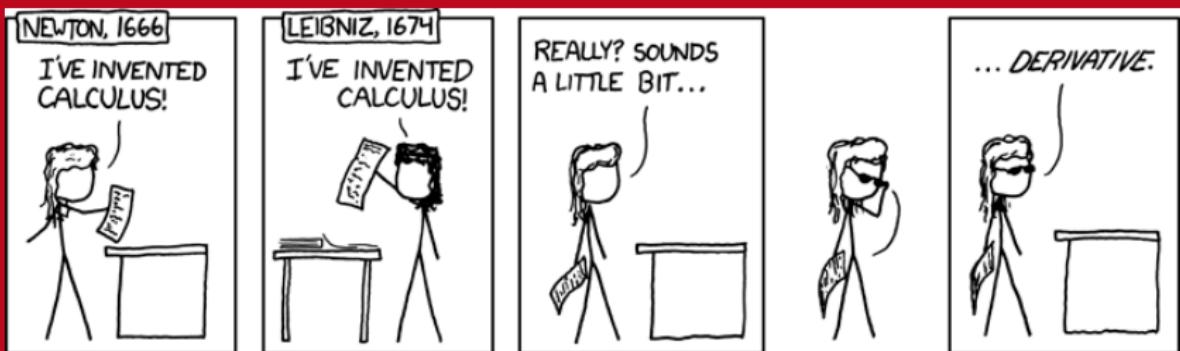
$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y} (x^2 + 3xy + y - 1) = 0 + 3x + 1 - 0$$

we have that

$$\left. \frac{\partial f}{\partial y} \right|_{(4,-5)} = 3x + 1 \Big|_{(4,-5)} = 13.$$

# Break

We will continue at 2pm



### 13.3 Partial Derivatives



#### Remark

As long as you “pretend” that one of the variables is a constant and only look at the other variable, then you use the tools that you already know.

### 13.3 Partial Derivatives



#### Remark

As long as you “pretend” that one of the variables is a constant and only look at the other variable, then you use the tools that you already know.

- the product rule;
- the quotient rule;
- implicit differentiation;
- etc.

**EXAMPLE 2** Find  $\partial f / \partial y$  as a function if  $f(x, y) = y \sin xy$ .

**Solution** We treat  $x$  as a constant and  $f$  as a product of  $y$  and  $\sin xy$ :

$$\begin{aligned}\frac{\partial f}{\partial y} &= \frac{\partial}{\partial y} (y \sin xy) = y \frac{\partial}{\partial y} \sin xy + (\sin xy) \frac{\partial}{\partial y} (y) \\ &= (y \cos xy) \frac{\partial}{\partial y} (xy) + \sin xy = xy \cos xy + \sin xy.\end{aligned}$$

**EXAMPLE 3** Find  $f_x$  and  $f_y$  as functions if

$$f(x, y) = \frac{2y}{y + \cos x}.$$

**Solution** We treat  $f$  as a quotient. With  $y$  held constant, we use the quotient rule to get

$$\begin{aligned} f_x &= \frac{\partial}{\partial x} \left( \frac{2y}{y + \cos x} \right) = \frac{(y + \cos x) \frac{\partial}{\partial x}(2y) - 2y \frac{\partial}{\partial x}(y + \cos x)}{(y + \cos x)^2} \\ &= \frac{(y + \cos x)(0) - 2y(-\sin x)}{(y + \cos x)^2} = \frac{2y \sin x}{(y + \cos x)^2}. \end{aligned}$$

With  $x$  held constant and again applying the quotient rule, we get

$$\begin{aligned} f_y &= \frac{\partial}{\partial y} \left( \frac{2y}{y + \cos x} \right) = \frac{(y + \cos x) \frac{\partial}{\partial y}(2y) - 2y \frac{\partial}{\partial y}(y + \cos x)}{(y + \cos x)^2} \\ &= \frac{(y + \cos x)(2) - 2y(1)}{(y + \cos x)^2} = \frac{2 \cos x}{(y + \cos x)^2}. \end{aligned}$$



**EXAMPLE 4** Find  $\partial z / \partial x$  assuming that the equation

$$yz - \ln z = x + y$$

defines  $z$  as a function of the two independent variables  $x$  and  $y$  and the partial derivative exists.

**Solution** We differentiate both sides of the equation with respect to  $x$ , holding  $y$  constant and treating  $z$  as a differentiable function of  $x$ :

$$\frac{\partial}{\partial x} (yz) - \frac{\partial}{\partial x} \ln z = \frac{\partial x}{\partial x} + \frac{\partial y}{\partial x}$$

$$y \frac{\partial z}{\partial x} - \frac{1}{z} \frac{\partial z}{\partial x} = 1 + 0 \quad \text{With } y \text{ constant, } \frac{\partial}{\partial x} (yz) = y \frac{\partial z}{\partial x}.$$

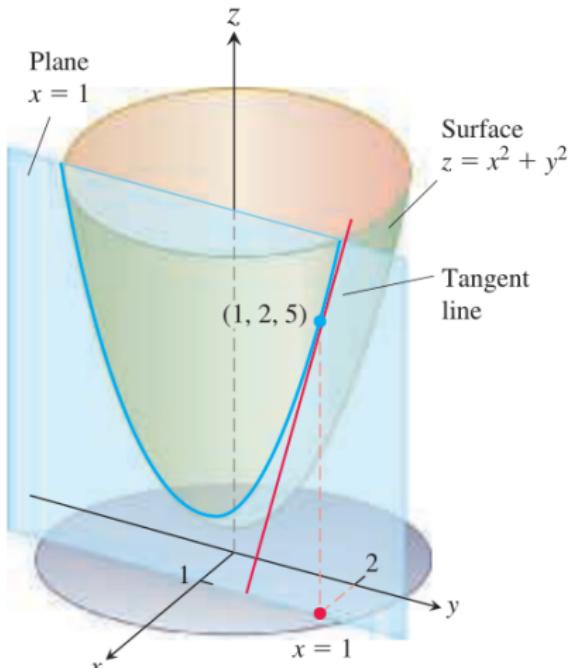
$$\left( y - \frac{1}{z} \right) \frac{\partial z}{\partial x} = 1$$

$$\frac{\partial z}{\partial x} = \frac{z}{yz - 1}.$$



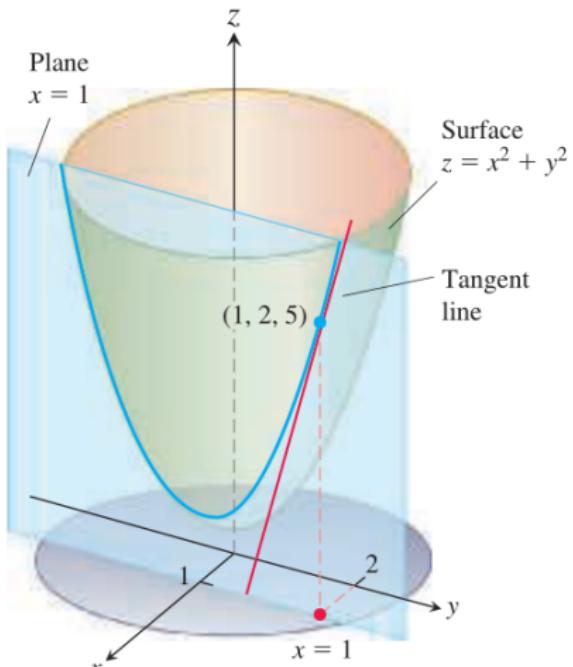
### 13.3 Partial Derivatives

**EXAMPLE 5** The plane  $x = 1$  intersects the paraboloid  $z = x^2 + y^2$  in a parabola. Find the slope of the tangent to the parabola at  $(1, 2, 5)$  (Figure 14.19).



### 13.3 Partial Derivatives

**EXAMPLE 5** The plane  $x = 1$  intersects the paraboloid  $z = x^2 + y^2$  in a parabola. Find the slope of the tangent to the parabola at  $(1, 2, 5)$  (Figure 14.19).



$$\left. \frac{\partial z}{\partial y} \right|_{(1,2)} = ?$$

**Solution** The parabola lies in a plane parallel to the  $yz$ -plane, and the slope is the value of the partial derivative  $\partial z / \partial y$  at  $(1, 2)$ :

$$\frac{\partial z}{\partial y} \Big|_{(1,2)} = \frac{\partial}{\partial y} (x^2 + y^2) \Big|_{(1,2)} = 2y \Big|_{(1,2)} = 2(2) = 4.$$

As a check, we can treat the parabola as the graph of the single-variable function  $z = (1)^2 + y^2 = 1 + y^2$  in the plane  $x = 1$  and ask for the slope at  $y = 2$ . The slope, calculated now as an ordinary derivative, is

$$\frac{dz}{dy} \Big|_{y=2} = \frac{d}{dy} (1 + y^2) \Big|_{y=2} = 2y \Big|_{y=2} = 4.$$



## 13.3 Partial Derivatives



### Remark

We can use the same ideas for functions of three (or more) variables.

**EXAMPLE 6**

If  $x$ ,  $y$ , and  $z$  are independent variables and

$$f(x, y, z) = x \sin(y + 3z),$$

then

$$\begin{aligned}\frac{\partial f}{\partial z} &= \frac{\partial}{\partial z} [x \sin(y + 3z)] = x \frac{\partial}{\partial z} \sin(y + 3z) && x \text{ held constant} \\ &= x \cos(y + 3z) \frac{\partial}{\partial z}(y + 3z) && \text{Chain rule} \\ &= 3x \cos(y + 3z). && y \text{ held constant}\end{aligned}$$

### 13.3 Partial Derivatives



Please read Example 7 in the textbook.

### EXAMPLE 8

Let

$$f(x, y) = \begin{cases} 0, & xy \neq 0 \\ 1, & xy = 0 \end{cases}$$

(Figure 14.21).

- (a) Find the limit of  $f$  as  $(x, y)$  approaches  $(0, 0)$  along the line  $y = x$ .
- (b) Prove that  $f$  is not continuous at the origin.
- (c) Show that both partial derivatives  $\partial f / \partial x$  and  $\partial f / \partial y$  exist at the origin.

### Solution

- (a) Since  $f(x, y)$  is constantly zero along the line  $y = x$  (except at the origin), we have

$$\lim_{(x, y) \rightarrow (0, 0)} f(x, y) \Big|_{y=x} = \lim_{(x, y) \rightarrow (0, 0)} 0 = 0.$$

- (b)** Since  $f(0, 0) = 1$ , the limit in part (a) is not equal to  $f(0, 0)$ , which proves that  $f$  is not continuous at  $(0, 0)$ .
- (c)** To find  $\partial f / \partial x$  at  $(0, 0)$ , we hold  $y$  fixed at  $y = 0$ . Then  $f(x, y) = 1$  for all  $x$ , and the graph of  $f$  is the line  $L_1$  in Figure 14.21. The slope of this line at any  $x$  is  $\partial f / \partial x = 0$ . In particular,  $\partial f / \partial x = 0$  at  $(0, 0)$ . Similarly,  $\partial f / \partial y$  is the slope of line  $L_2$  at any  $y$ , so  $\partial f / \partial y = 0$  at  $(0, 0)$ . ■

### Second-Order Partial Derivatives

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x} \left( \frac{\partial f}{\partial x} \right) = (f_x)_x = f_{xx}$$

## Second-Order Partial Derivatives

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$$\frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y} \left( \frac{\partial f}{\partial y} \right) = (f_y)_y = f_{yy}$$

### 13.3 Partial Derivatives



## Second-Order Partial Derivatives

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x} \left( \frac{\partial f}{\partial x} \right) = (f_x)_x = f_{xx}$$

first diff. wrt  $y$   
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$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x} \left( \frac{\partial f}{\partial y} \right) = (f_y)_x = f_{yx}$$

### 13.3 Partial Derivatives



## Second-Order Partial Derivatives

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$$\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y} \left( \frac{\partial f}{\partial x} \right) = (f_x)_y = f_{xy}$$

### 13.3 Partial Derivatives



#### Remark

Sometimes  $f_{xy} = f_{yx}$  and sometimes  $f_{xy} \neq f_{yx}$ .

### 13.3 Partial Derivatives

#### Example

If  $f(x, y) = x \cos y + ye^x$ , find the four second order partial derivatives of  $f$ .

Since

$$f_x = \quad \text{and} \quad f_y = ,$$

we have that

$$f_{xx} =$$

$$f_{yy} =$$

$$f_{yx} =$$

$$f_{xy} =$$

### 13.3 Partial Derivatives

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$$f_{yx} =$$

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### 13.3 Partial Derivatives

#### Example

If  $f(x, y) = x \cos y + ye^x$ , find the four second order partial derivatives of  $f$ .

Since

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$$f_{yy} = \frac{\partial}{\partial y} (f_y) = -x \cos y$$

$$f_{yx} = \frac{\partial}{\partial x} (f_y) = -\sin y + e^x$$

$$f_{xy} = \frac{\partial}{\partial y} (f_x) = -\sin y + e^x = f_{yx}.$$

### 13.3 Partial Derivatives



Alexis Clairaut

BORN

13 May 1713

DECEASED

17 May 1765

NATIONALITY

French

#### Theorem (Clairaut's Theorem)

If

- $R$  is an open region;
- $(a, b) \in R$ ;
- $f, f_x, f_y, f_{xy}$  and  $f_{yx}$  are all defined on  $R$ ; and
- $f, f_x, f_y, f_{xy}$  and  $f_{yx}$  are all continuous at  $(a, b)$ ;

### 13.3 Partial Derivatives



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- $f, f_x, f_y, f_{xy}$  and  $f_{yx}$  are all continuous at  $(a, b)$ ;

then

$$f_{xy}(a, b) = f_{yx}(a, b).$$

**EXAMPLE 10** Find  $\frac{\partial^2 w}{\partial x \partial y}$  if

$$w = xy + \frac{e^y}{y^2 + 1}.$$

**Solution** The symbol  $\partial^2 w / \partial x \partial y$  tells us to differentiate first with respect to  $y$  and then with respect to  $x$ . However, if we interchange the order of differentiation and differentiate first with respect to  $x$  we get the answer more quickly. In two steps,

$$\frac{\partial w}{\partial x} = y \quad \text{and} \quad \frac{\partial^2 w}{\partial y \partial x} = 1.$$

If we differentiate first with respect to  $y$ , we obtain  $\partial^2 w / \partial x \partial y = 1$  as well. We can differentiate in either order because the conditions of Theorem 2 hold for  $w$  at all points  $(x_0, y_0)$ .



### 13.3 Partial Derivatives



## Higher Order Partial Derivatives

$$\frac{\partial^3 f}{\partial x \partial y^2} = f_{yyx}$$

$$\frac{\partial^4 f}{\partial x^2 \partial y^2} = f_{yyxx}$$

**EXAMPLE 11** Find  $f_{yxyz}$  if  $f(x, y, z) = 1 - 2xy^2z + x^2y$ .

**Solution** We first differentiate with respect to the variable  $y$ , then  $x$ , then  $y$  again, and finally with respect to  $z$ :

$$f_y = -4xyz + x^2$$

$$f_{yx} = -4yz + 2x$$

$$f_{yxy} = -4z$$

$$f_{yxyz} = -4.$$





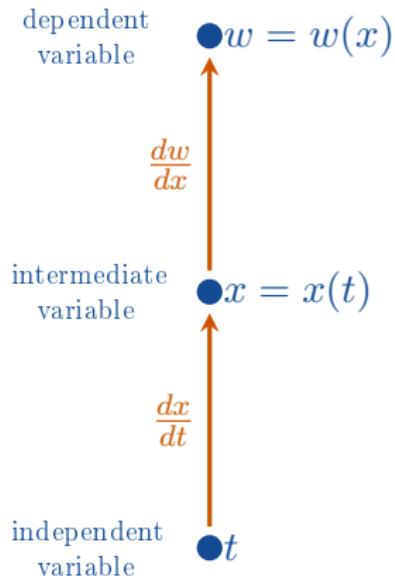
# The Chain Rule

## 13.4 The Chain Rule

### Functions of One Variable

Recall that if  $w(x)$  and  $x(t)$   
then

$$\frac{dw}{dt} = \frac{dw}{dx} \frac{dx}{dt}.$$



## 13.4 The Chain Rule



# Functions of Two Variables

But what is  $\frac{dw}{dt}$  if  $w(x, y)$ ?

dependent  
variable

$$\bullet w = w(x, y)$$

intermediate  
variables

$$\bullet x = x(t)$$

$$\bullet y = y(t)$$

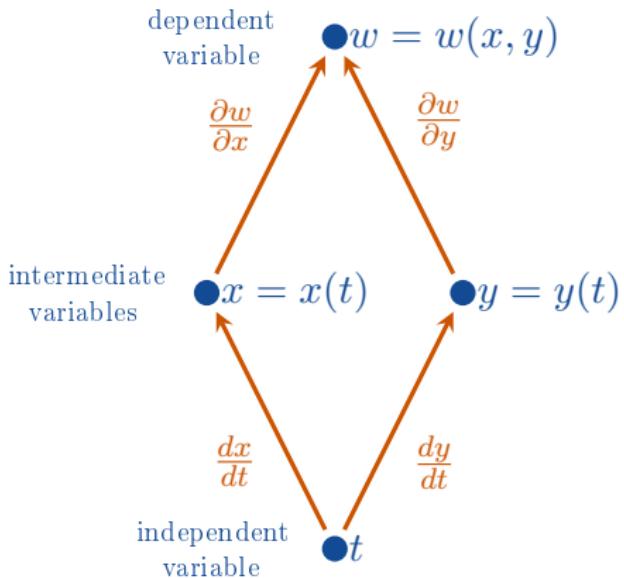
independent  
variable

$$\bullet t$$

## 13.4 The Chain Rule

### Functions of Two Variables

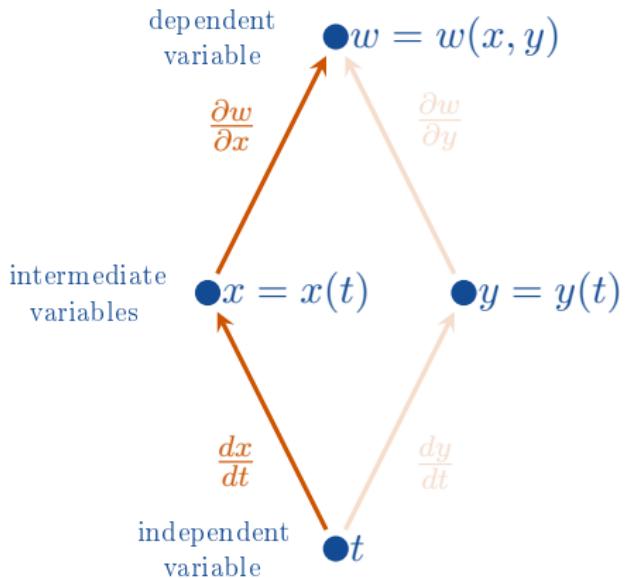
But what is  $\frac{dw}{dt}$  if  $w(x, y)$ ?



## 13.4 The Chain Rule

### Functions of Two Variables

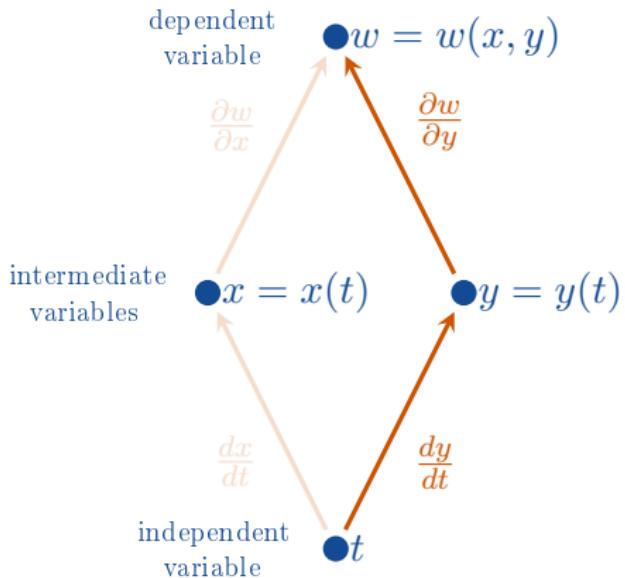
But what is  $\frac{dw}{dt}$  if  $w(x, y)$ ?



## 13.4 The Chain Rule

### Functions of Two Variables

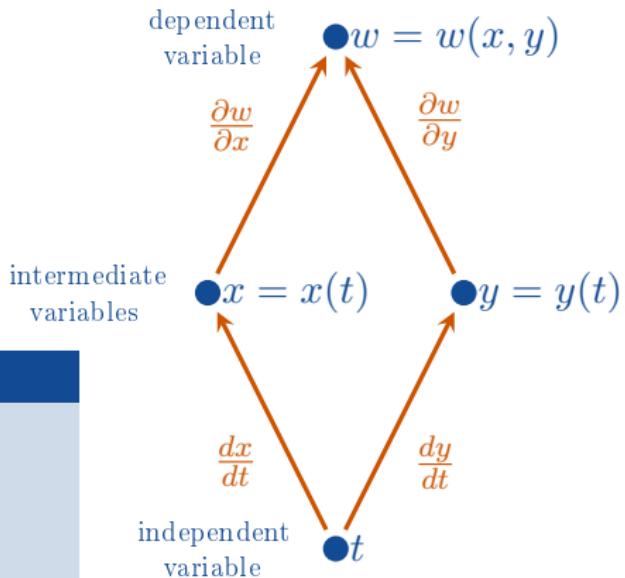
But what is  $\frac{dw}{dt}$  if  $w(x, y)$ ?



## 13.4 The Chain Rule

### Functions of Two Variables

But what is  $\frac{dw}{dt}$  if  $w(x, y)$ ?



#### Theorem

$$\frac{dw}{dt} = \frac{\partial w}{\partial x} \frac{dx}{dt} + \frac{\partial w}{\partial y} \frac{dy}{dt}.$$

13.4

$$\frac{dw}{dt} = \frac{\partial w}{\partial x} \frac{dx}{dt} + \frac{\partial w}{\partial y} \frac{dy}{dt}$$



### Example

Use the Chain Rule to find the derivative of  $w = xy$  with respect to  $t$  along the path  $x = \cos t$ ,  $y = \sin t$ . What is the derivative's value at  $t = \frac{\pi}{2}$ ?

$$\frac{dw}{dt} = \frac{\partial w}{\partial x} \frac{dx}{dt} + \frac{\partial w}{\partial y} \frac{dy}{dt}$$

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13.4

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By the Chain Rule

$$\frac{dw}{dt} = \frac{\partial w}{\partial x} \frac{dx}{dt} + \frac{\partial w}{\partial y} \frac{dy}{dt}$$

=

=

=

$$\frac{dw}{dt} = \frac{\partial w}{\partial x} \frac{dx}{dt} + \frac{\partial w}{\partial y} \frac{dy}{dt}$$

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Use the Chain Rule to find the derivative of  $w = xy$  with respect to  $t$  along the path  $x = \cos t$ ,  $y = \sin t$ . What is the derivative's value at  $t = \frac{\pi}{2}$ ?

By the Chain Rule

$$\begin{aligned}\frac{dw}{dt} &= \frac{\partial w}{\partial x} \frac{dx}{dt} + \frac{\partial w}{\partial y} \frac{dy}{dt} \\ &= (y)(-\sin t) + (x)(\cos t)\end{aligned}$$

=

=

$$\frac{dw}{dt} = \frac{\partial w}{\partial x} \frac{dx}{dt} + \frac{\partial w}{\partial y} \frac{dy}{dt}$$

## Example

Use the Chain Rule to find the derivative of  $w = xy$  with respect to  $t$  along the path  $x = \cos t$ ,  $y = \sin t$ . What is the derivative's value at  $t = \frac{\pi}{2}$ ?

By the Chain Rule

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$$\frac{dw}{dt} = \frac{\partial w}{\partial x} \frac{dx}{dt} + \frac{\partial w}{\partial y} \frac{dy}{dt}$$

### Example

Use the Chain Rule to find the derivative of  $w = xy$  with respect to  $t$  along the path  $x = \cos t$ ,  $y = \sin t$ . What is the derivative's value at  $t = \frac{\pi}{2}$ ?

By the Chain Rule

$$\begin{aligned}\frac{dw}{dt} &= \frac{\partial w}{\partial x} \frac{dx}{dt} + \frac{\partial w}{\partial y} \frac{dy}{dt} \\ &= (y)(-\sin t) + (x)(\cos t) \\ &= (\sin t)(-\sin t) + (\cos t)(\cos t) \\ &= -\sin^2 t + \cos^2 t = \cos 2t.\end{aligned}$$

Therefore

$$\left. \frac{dw}{dt} \right|_{t=\frac{\pi}{2}} = \cos \left( 2 \frac{\pi}{2} \right) = \cos \pi = -1.$$

$$\frac{dw}{dt} = \frac{\partial w}{\partial x} \frac{dx}{dt} + \frac{\partial w}{\partial y} \frac{dy}{dt}$$

### Remark

Since the example said “Use the Chain Rule...”, we needed to do it this way.

If it just said “Find the derivative of...”, then we could have just done

$$w = xy = (\cos t)(\sin t) = \frac{1}{2} \sin 2t$$

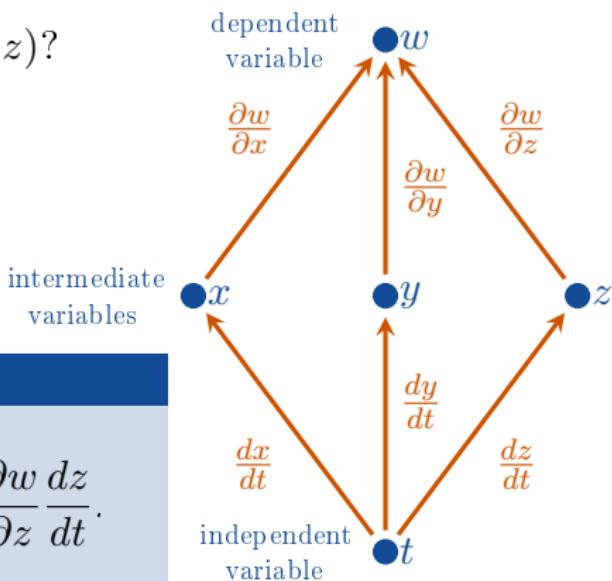
and

$$\frac{dw}{dt} = \frac{d}{dt} \left( \frac{1}{2} \sin 2t \right) = \cos 2t.$$

## 13.4 The Chain Rule

### Functions of Three Variables

And what is  $\frac{dw}{dt}$  if  $w(x, y, z)$ ?



#### Theorem

$$\frac{dw}{dt} = \frac{\partial w}{\partial x} \frac{dx}{dt} + \frac{\partial w}{\partial y} \frac{dy}{dt} + \frac{\partial w}{\partial z} \frac{dz}{dt}.$$

**EXAMPLE 2** Find  $dw/dt$  if

$$w = xy + z, \quad x = \cos t, \quad y = \sin t, \quad z = t.$$

In this example the values of  $w(t)$  are changing along the path of a helix (Section 13.1) as  $t$  changes. What is the derivative's value at  $t = 0$ ?

**Solution** Using the Chain Rule for three intermediate variables, we have

$$\begin{aligned}\frac{dw}{dt} &= \frac{\partial w}{\partial x} \frac{dx}{dt} + \frac{\partial w}{\partial y} \frac{dy}{dt} + \frac{\partial w}{\partial z} \frac{dz}{dt} \\&= (y)(-\sin t) + (x)(\cos t) + (1)(1) \\&= (\sin t)(-\sin t) + (\cos t)(\cos t) + 1 \\&= -\sin^2 t + \cos^2 t + 1 = 1 + \cos 2t,\end{aligned}$$

Substitute for intermediate variables.

so

$$\left. \frac{dw}{dt} \right|_{t=0} = 1 + \cos(0) = 2.$$



## 13.4 The Chain Rule

### Theorem

$$\frac{\partial w}{\partial r} = ,$$

$$\frac{\partial w}{\partial s} = .$$

•  $w$

one dependent variable

•  $x$

•  $y$

•  $z$

three intermediate variables

•  $r$

•  $s$

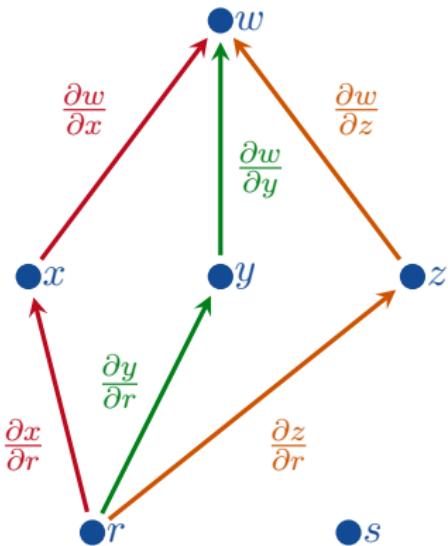
two independent variables

## 13.4 The Chain Rule

Theorem

$$\frac{\partial w}{\partial r} = \frac{\partial w}{\partial x} \frac{\partial x}{\partial r} + \frac{\partial w}{\partial y} \frac{\partial y}{\partial r} + \frac{\partial w}{\partial z} \frac{\partial z}{\partial r},$$

$$\frac{\partial w}{\partial s} =$$



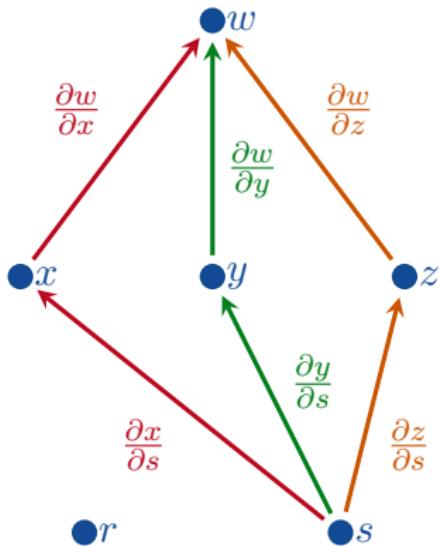
## 13.4 The Chain Rule

### Theorem

$$\frac{\partial w}{\partial r} = \frac{\partial w}{\partial x} \frac{\partial x}{\partial r} + \frac{\partial w}{\partial y} \frac{\partial y}{\partial r} + \frac{\partial w}{\partial z} \frac{\partial z}{\partial r},$$

$$\frac{\partial w}{\partial s} = \frac{\partial w}{\partial x} \frac{\partial x}{\partial s} + \frac{\partial w}{\partial y} \frac{\partial y}{\partial s} + \frac{\partial w}{\partial z} \frac{\partial z}{\partial s}.$$

Once you understand this, you can do the Chain Rule for any number of intermediate and independent variables.



**EXAMPLE 3** Express  $\partial w/\partial r$  and  $\partial w/\partial s$  in terms of  $r$  and  $s$  if

$$w = x + 2y + z^2, \quad x = \frac{r}{s}, \quad y = r^2 + \ln s, \quad z = 2r.$$

**Solution** Using the formulas in Theorem 7, we find

$$\begin{aligned}\frac{\partial w}{\partial r} &= \frac{\partial w}{\partial x} \frac{\partial x}{\partial r} + \frac{\partial w}{\partial y} \frac{\partial y}{\partial r} + \frac{\partial w}{\partial z} \frac{\partial z}{\partial r} \\&= (1)\left(\frac{1}{s}\right) + (2)(2r) + (2z)(2) \\&= \frac{1}{s} + 4r + (4r)(2) = \frac{1}{s} + 12r \quad \text{Substitute for intermediate variable } z.\end{aligned}$$

$$\begin{aligned}\frac{\partial w}{\partial s} &= \frac{\partial w}{\partial x} \frac{\partial x}{\partial s} + \frac{\partial w}{\partial y} \frac{\partial y}{\partial s} + \frac{\partial w}{\partial z} \frac{\partial z}{\partial s} \\&= (1)\left(-\frac{r}{s^2}\right) + (2)\left(\frac{1}{s}\right) + (2z)(0) = \frac{2}{s} - \frac{r}{s^2}.\end{aligned}$$



**EXAMPLE 4** Express  $\partial w/\partial r$  and  $\partial w/\partial s$  in terms of  $r$  and  $s$  if

$$w = x^2 + y^2, \quad x = r - s, \quad y = r + s.$$

**Solution** The preceding discussion gives the following.

$$\begin{aligned}\frac{\partial w}{\partial r} &= \frac{\partial w}{\partial x} \frac{\partial x}{\partial r} + \frac{\partial w}{\partial y} \frac{\partial y}{\partial r} & \frac{\partial w}{\partial s} &= \frac{\partial w}{\partial x} \frac{\partial x}{\partial s} + \frac{\partial w}{\partial y} \frac{\partial y}{\partial s} \\ &= (2x)(1) + (2y)(1) & &= (2x)(-1) + (2y)(1) \\ &= 2(r - s) + 2(r + s) & &= -2(r - s) + 2(r + s) \\ &= 4r & &= 4s\end{aligned}$$

Substitute  
for the  
intermediate  
variables. ■

If  $f$  is a function of  $x$  alone, our equations are even simpler.

## 13.4 The Chain Rule



# Implicit Differentiation Revisited

Suppose that  $F(x, y) = 0$  and  $y = y(x)$ .

## 13.4 The Chain Rule



### Implicit Differentiation Revisited

Suppose that  $F(x, y) = 0$  and  $y = y(x)$ . Then

$$0 = F(x, y)$$

## 13.4 The Chain Rule



### Implicit Differentiation Revisited

Suppose that  $F(x, y) = 0$  and  $y = y(x)$ . Then

$$\frac{d}{dx}0 = \frac{d}{dx}F(x, y)$$

## 13.4 The Chain Rule



### Implicit Differentiation Revisited

Suppose that  $F(x, y) = 0$  and  $y = y(x)$ . Then

$$0 = \frac{d}{dx}0 = \frac{d}{dx}F(x, y)$$

## 13.4 The Chain Rule



### Implicit Differentiation Revisited

Suppose that  $F(x, y) = 0$  and  $y = y(x)$ . Then

$$0 = \frac{d}{dx}0 = \frac{d}{dx}F(x, y) = \frac{dF}{dx}\frac{dx}{dx} + \frac{dF}{dy}\frac{dy}{dx}$$

## 13.4 The Chain Rule



### Implicit Differentiation Revisited

Suppose that  $F(x, y) = 0$  and  $y = y(x)$ . Then

$$0 = \frac{d}{dx}0 = \frac{d}{dx}F(x, y) = \frac{dF}{dx}\frac{dx}{dx} + \frac{dF}{dy}\frac{dy}{dx} = F_x \cdot 1 + F_y \frac{dy}{dx}.$$

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#### Theorem

If  $F_y \neq 0$ , then

$$\frac{dy}{dx} = -\frac{F_x}{F_y}.$$

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### Example

Find  $\frac{dy}{dx}$  if  $y^2 - x^2 - \sin xy = 0$ .

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Let  $F(x, y) = y^2 - x^2 - \sin xy$ . Then

$$\frac{dy}{dx} = -\frac{F_x}{F_y} = -\frac{-2x - y \cos xy}{2y - x \cos xy} = \frac{2xy + \cos xy}{2y - x \cos xy}.$$

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If we start with  $F(x, y, z) = 0$  and  $z = z(x, y)$ , then we obtain

$$\begin{aligned} 0 &= \frac{\partial F}{\partial x} \frac{\partial x}{\partial x} + \frac{\partial F}{\partial y} \frac{\partial y}{\partial x} + \frac{\partial F}{\partial z} \frac{\partial z}{\partial x} \\ &= F_x \cdot 1 + F_y \cdot 0 + F_z \frac{\partial z}{\partial x} \quad \implies \quad \frac{\partial z}{\partial x} = -\frac{F_x}{F_z} \end{aligned}$$

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### Example

Find  $\frac{\partial z}{\partial x}$  and  $\frac{\partial z}{\partial y}$  at  $(0, 0, 0)$ , if  $x^3 + z^2 + ye^{xz} + z \cos y = 0$ .

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At  $(0, 0, 0)$  we have

$$\frac{\partial z}{\partial x} = -\frac{0}{1} = 0 \quad \text{and} \quad \frac{\partial z}{\partial y} = -\frac{1}{1} = -1.$$



# Next Time

- 13.5 Directional Derivatives and Gradient Vectors
- 13.6 Tangent Planes and Differentials
- 13.7 Extreme Values and Saddle Points
- 13.8 Lagrange Multipliers