

Lecture 5

- 13.1 Functions of Several Variables
- 13.2 Limits and Continuity in Higher Dimensions
- 13.3 Partial Derivatives
- 13.4 The Chain Rule



1 Functions of Several Variables

13.1 Functions of Several Variables



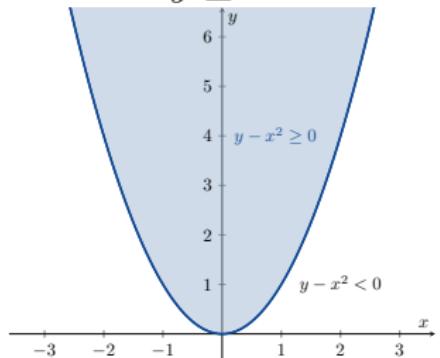
We will be considering functions $f : D \subseteq \mathbb{R}^2 \rightarrow \mathbb{R}$ and functions $g : D \subseteq \mathbb{R}^3 \rightarrow \mathbb{R}$.

13.1 Functions of Several Variables

Example

$$f : D \rightarrow \mathbb{R}, f(x, y) = \sqrt{y - x^2}$$

domain: $y \geq x^2$



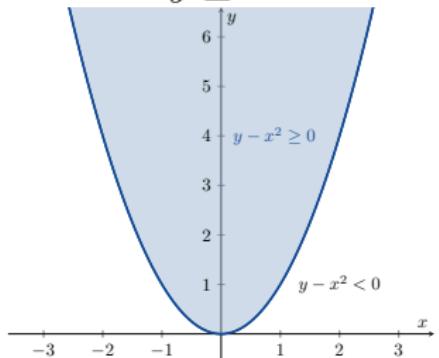
range: $[0, \infty)$

13.1 Functions of Several Variables

Example

$$f : D \rightarrow \mathbb{R}, f(x, y) = \sqrt{y - x^2}$$

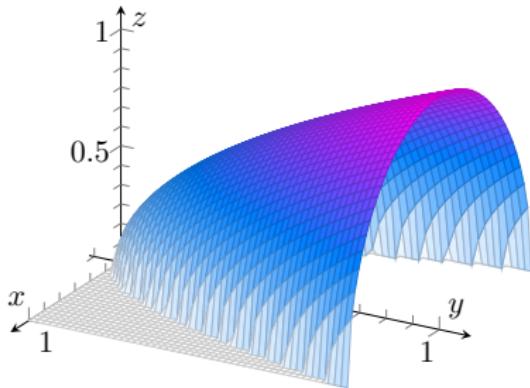
domain: $y \geq x^2$



range: $[0, \infty)$

graph:

$$z = \sqrt{y - x^2}$$





z=sqrt(y-x^2)



Google Search

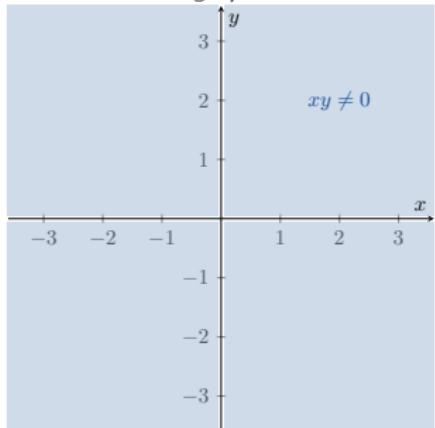
I'm Feeling Lucky

13.1 Functions of Several Variables

Example

$$f : D \rightarrow \mathbb{R}, f(x, y) = \frac{1}{xy}$$

domain: $xy \neq 0$



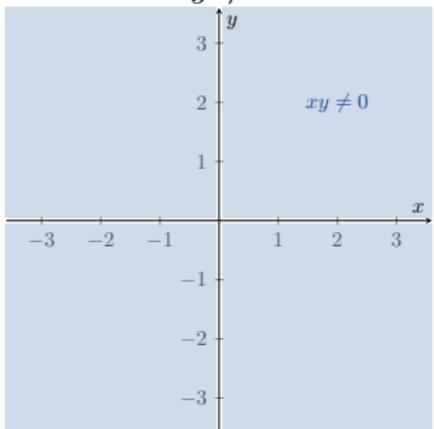
range: $(-\infty, 0) \cup (0, \infty)$

13.1 Functions of Several Variables

Example

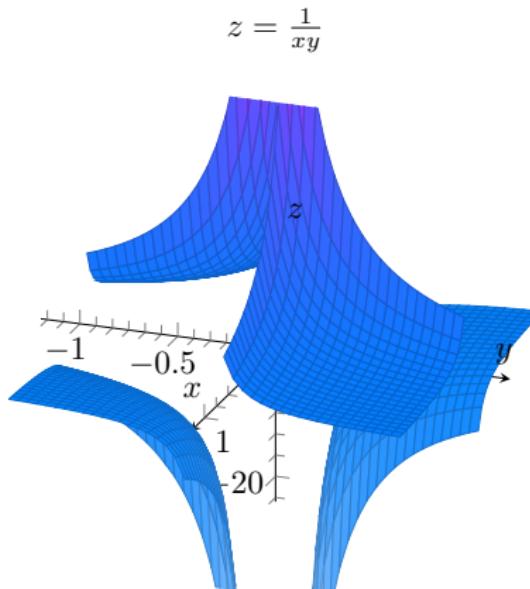
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domain: $xy \neq 0$



range: $(-\infty, 0) \cup (0, \infty)$

graph:

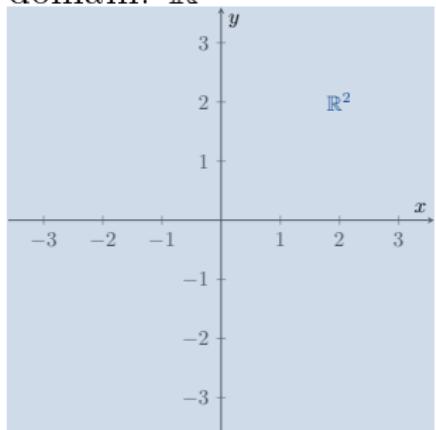


13.1 Functions of Several Variables

Example

$$f : D \rightarrow \mathbb{R}, f(x, y) = \sin xy$$

domain: \mathbb{R}^2



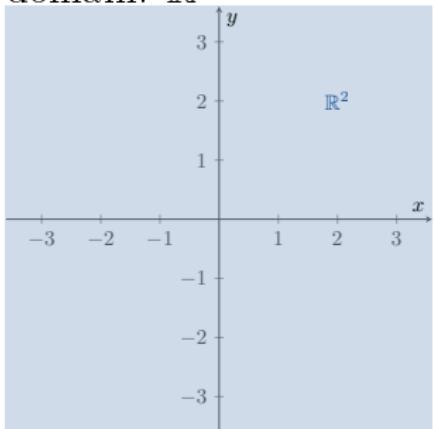
range: $[-1, 1]$

13.1 Functions of Several Variables

Example

$$f : D \rightarrow \mathbb{R}, f(x, y) = \sin xy$$

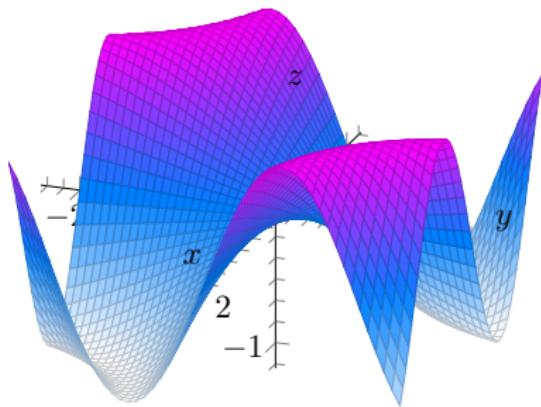
domain: \mathbb{R}^2



range: $[-1, 1]$

graph:

$$z = \sin xy$$



13.1 Functions of Several Variables

Example

$$g : D \rightarrow \mathbb{R}, g(x, y, z) = \sqrt{x^2 + y^2 + z^2}$$

domain: \mathbb{R}^3

range: $[-1, 1]$

13.1 Functions of Several Variables



Example

$$g : D \rightarrow \mathbb{R}, g(x, y, z) = \sqrt{x^2 + y^2 + z^2}$$

domain: \mathbb{R}^3

range: $[-1, 1]$

Example

$$g : D \rightarrow \mathbb{R}, g(x, y, z) = \frac{1}{x^2 + y^2 + z^2}$$

domain: $(x, y, z) \neq (0, 0, 0)$

range: $(0, \infty)$

13.1 Functions of Several Variables

Example

$$g : D \rightarrow \mathbb{R}, g(x, y, z) = \sqrt{x^2 + y^2 + z^2}$$

domain: \mathbb{R}^3

range: $[-1, 1]$

Example

$$g : D \rightarrow \mathbb{R}, g(x, y, z) = \frac{1}{x^2 + y^2 + z^2}$$

domain: $(x, y, z) \neq (0, 0, 0)$

range: $(0, \infty)$

Example

$$g : D \rightarrow \mathbb{R}, g(x, y, z) = xy \ln z$$

domain: $z > 0$

range: \mathbb{R}

13.1 Functions of Several Variables



Topology

In \mathbb{R} we have open intervals (a, b) , closed intervals $[a, b]$ and half-open intervals $[a, b)$, $(a, b]$.

What do we have in \mathbb{R}^2 and \mathbb{R}^3 ?

13.1 Functions of Several Variables



\mathbb{R}^2

\mathbb{R}^3

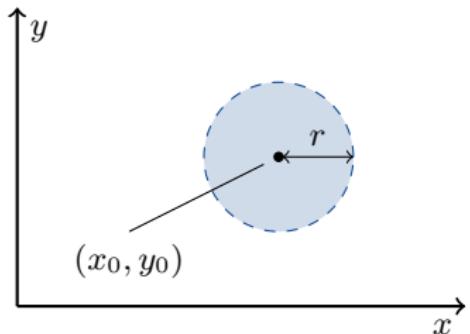
13.1 Functions of Several Variables

\mathbb{R}^2 The set

\mathbb{R}^3

$$\{(x, y) \mid (x - x_0)^2 + (y - y_0)^2 < r^2\}$$

is called an *open disc* of radius r
centred at (x_0, y_0) .



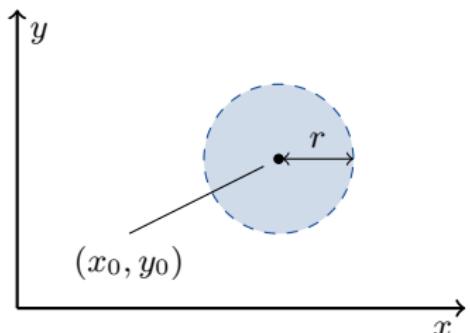
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13.1 Functions of Several Variables

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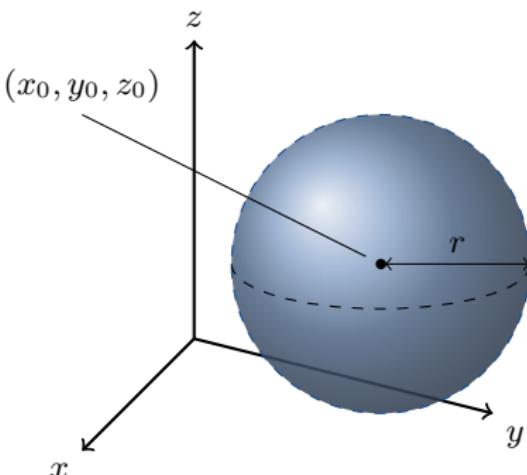


$$r > 0$$

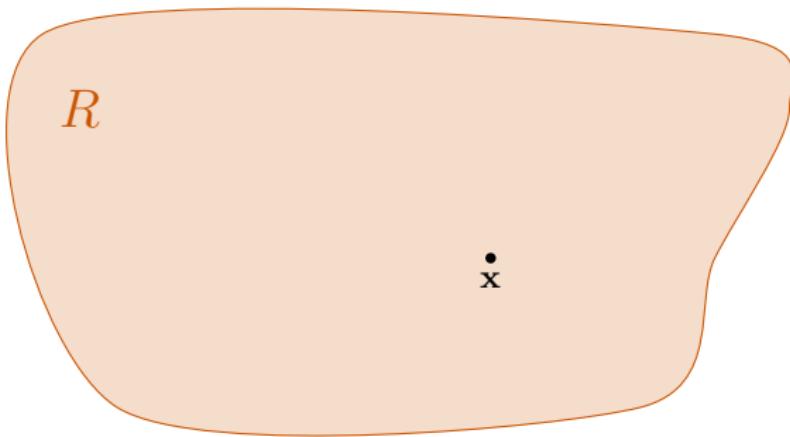
\mathbb{R}^3 The set

$$\{(x, y, z) \mid (x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2 < r^2\}$$

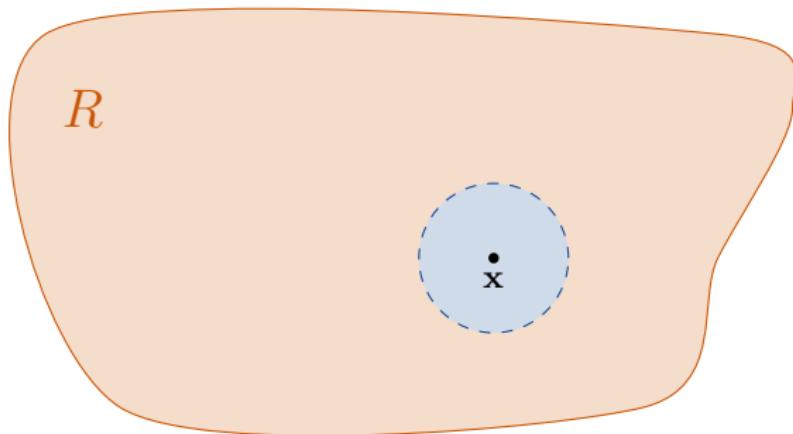
is called an *open ball* of radius r
centred at (x_0, y_0, z_0) .



13.1 Functions of Several Variables



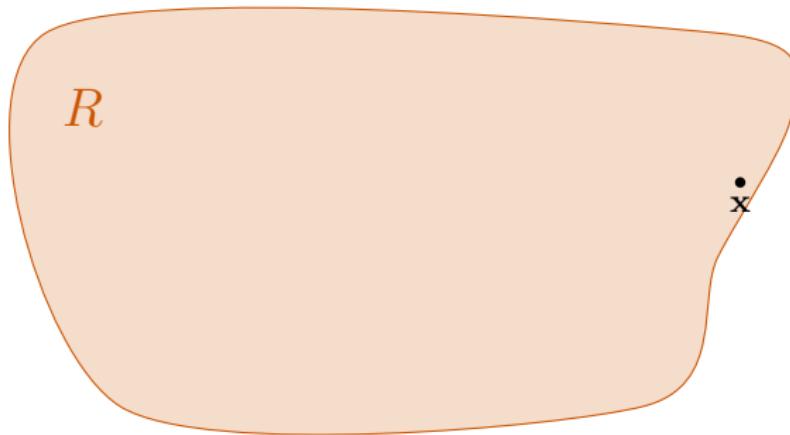
13.1 Functions of Several Variables



Definition

A point \mathbf{x} in R is called an *interior point* of R iff there exists an open disk/ball centred at \mathbf{x} which lies entirely inside R .

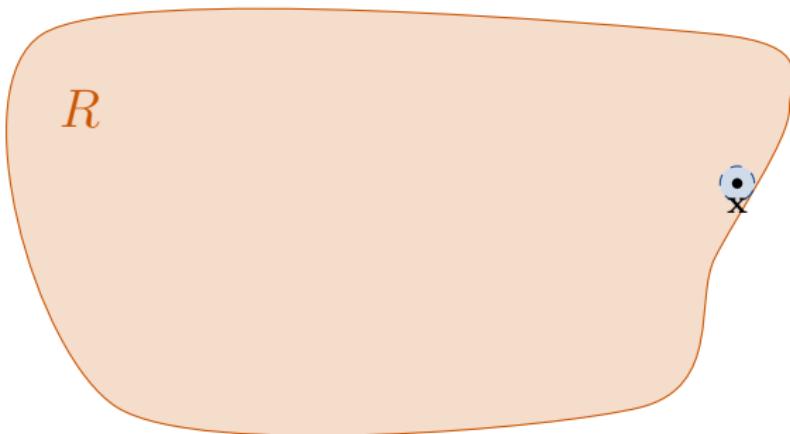
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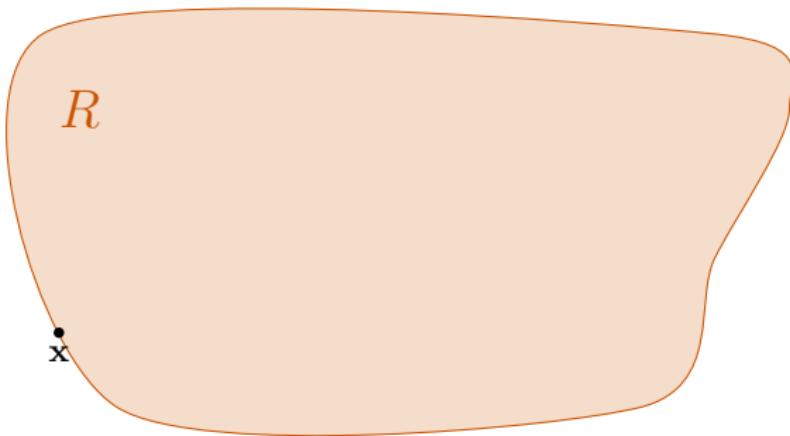
13.1 Functions of Several Variables



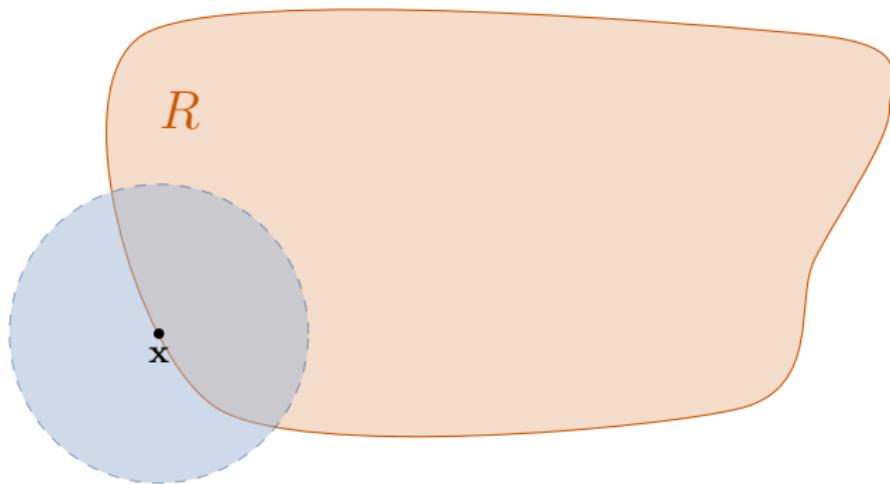
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13.1 Functions of Several Variables



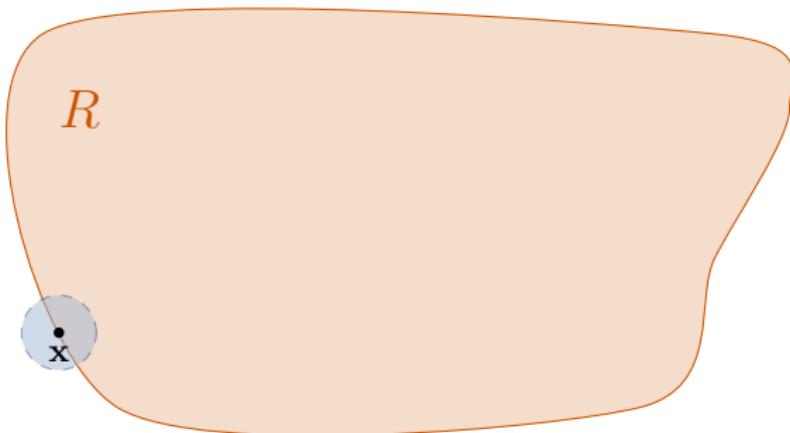
13.1 Functions of Several Variables



Definition

A point \mathbf{x} is called a *boundary point* of R iff every open disk/ball centred at \mathbf{x} which partly inside R and partly outside R .

13.1 Functions of Several Variables



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13.1 Functions of Several Variables



Definition

A set R is called *open* if every point in R is an interior point.

13.1 Functions of Several Variables



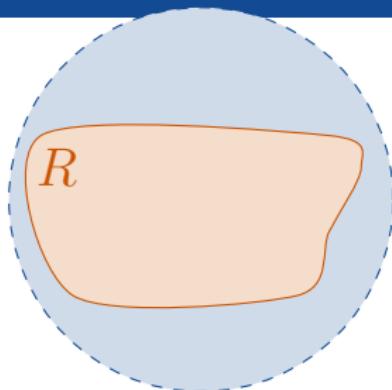
Definition

A set R is called *open* if every point in R is an interior point.

Definition

A set R is called *closed* if it contains all of its boundary points.

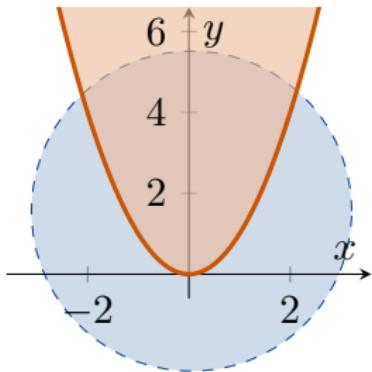
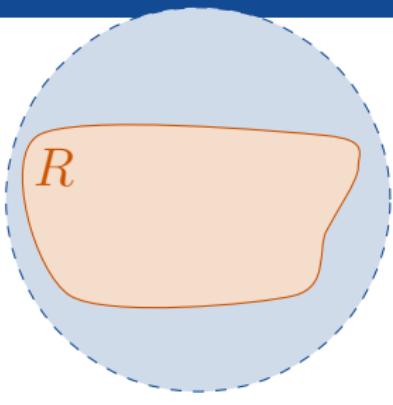
13.1 Functions of Several Variables



Definition

A set R is called *bounded* iff it lies inside a disk/ball.

13.1 Functions of Several Variables



Definition

A set R is called *bounded* iff it lies inside a disk/ball.

Definition

A set R is called *unbounded* iff it is not bounded.



Limits and Continuity in Higher Dimensions



Limits

$$\lim_{x \rightarrow x_0} f(x) = L$$

“If x is close to x_0 (but $x \neq x_0$),
then $f(x)$ is close to L . ”

13.2 Limits and Continuity in Higher Dimensions

Definition

\mathbb{R}

We write $\lim_{x \rightarrow x_0} f(x) = L$ iff for all $\varepsilon > 0$, there exists $\delta = \delta(\varepsilon) > 0$ such that for all $x \in D$

$$0 < |x - x_0| < \delta \implies |f(x) - L| < \varepsilon.$$

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Definition

\mathbb{R}^2

We write $\lim_{(x,y) \rightarrow (x_0,y_0)} f(x) = L$ iff for all $\varepsilon > 0$, there exists $\delta = \delta(\varepsilon) > 0$ such that for all $(x, y) \in D$

$$0 < \|(x, y) - (x_0, y_0)\| < \delta \implies |f(x) - L| < \varepsilon.$$

13.2 Limits and Continuity in Higher Dimensions

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$$0 < \|(x, y) - (x_0, y_0)\| < \delta \implies |f(x) - L| < \varepsilon.$$

$$0 < \sqrt{(x - x_0)^2 + (y - y_0)^2} < \delta$$

13.2 Limits and Continuity in Higher Dimensions

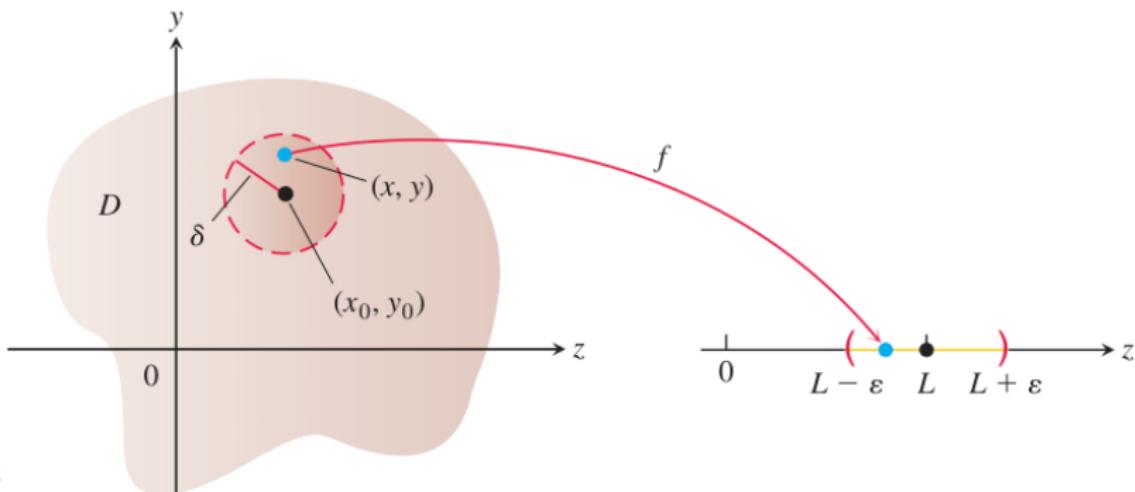


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13.2 Limits and Continuity in Higher Dimensions



Definition

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$$0 < \|(x, y) - (x_0, y_0)\| < \delta \implies |f(x) - L| < \varepsilon.$$

Definition

\mathbb{R}^3 We write $\lim_{(x,y,z) \rightarrow (x_0,y_0,z_0)} f(x) = L$ iff for all $\varepsilon > 0$, there exists $\delta = \delta(\varepsilon) > 0$ such that for all $(x, y, z) \in D$

$$0 < \|(x, y, z) - (x_0, y_0, z_0)\| < \delta \implies |f(x) - L| < \varepsilon.$$

As for functions of a single variable, it can be shown that

$$\lim_{(x,y) \rightarrow (x_0, y_0)} x = x_0$$

$$\lim_{(x,y) \rightarrow (x_0, y_0)} y = y_0$$

$$\lim_{(x,y) \rightarrow (x_0, y_0)} k = k \quad (\text{any number } k).$$

For example, in the first limit statement above, $f(x, y) = x$ and $L = x_0$. Using the definition of limit, suppose that $\varepsilon > 0$ is chosen. If we let δ equal this ε , we see that if

$$0 < \sqrt{(x - x_0)^2 + (y - y_0)^2} < \delta = \varepsilon,$$

then

$$\sqrt{(x - x_0)^2} < \varepsilon \quad (x - x_0)^2 \leq (x - x_0)^2 + (y - y_0)^2$$

$$|x - x_0| < \varepsilon \quad \sqrt{a^2} = |a|$$

$$|f(x, y) - x_0| < \varepsilon. \quad x = f(x, y)$$

That is,

$$|f(x, y) - x_0| < \varepsilon \quad \text{whenever} \quad 0 < \sqrt{(x - x_0)^2 + (y - y_0)^2} < \delta.$$

So a δ has been found satisfying the requirement of the definition, and therefore we have proved that

$$\lim_{(x,y) \rightarrow (x_0, y_0)} f(x, y) = \lim_{(x,y) \rightarrow (x_0, y_0)} x = x_0.$$

THEOREM 1 – Properties of Limits of Functions of Two Variables

The following rules hold if L , M , and k are real numbers and

$$\lim_{(x, y) \rightarrow (x_0, y_0)} f(x, y) = L \quad \text{and} \quad \lim_{(x, y) \rightarrow (x_0, y_0)} g(x, y) = M.$$

- 1. Sum Rule:**

$$\lim_{(x, y) \rightarrow (x_0, y_0)} (f(x, y) + g(x, y)) = L + M$$

- 2. Difference Rule:**

$$\lim_{(x, y) \rightarrow (x_0, y_0)} (f(x, y) - g(x, y)) = L - M$$

- 3. Constant Multiple Rule:**

$$\lim_{(x, y) \rightarrow (x_0, y_0)} kf(x, y) = kL \quad (\text{any number } k)$$

- 4. Product Rule:**

$$\lim_{(x, y) \rightarrow (x_0, y_0)} (f(x, y) \cdot g(x, y)) = L \cdot M$$

- 5. Quotient Rule:**

$$\lim_{(x, y) \rightarrow (x_0, y_0)} \frac{f(x, y)}{g(x, y)} = \frac{L}{M}, \quad M \neq 0$$

- 6. Power Rule:**

$$\lim_{(x, y) \rightarrow (x_0, y_0)} [f(x, y)]^n = L^n, \quad n \text{ a positive integer}$$

- 7. Root Rule:**

$$\lim_{(x, y) \rightarrow (x_0, y_0)} \sqrt[n]{f(x, y)} = \sqrt[n]{L} = L^{1/n},$$

n a positive integer, and if n is even,
we assume that $L > 0$.

EXAMPLE 1 In this example, we can combine the three simple results following the limit definition with the results in Theorem 1 to calculate the limits. We simply substitute the x - and y -values of the point being approached into the functional expression to find the limiting value.

(a) $\lim_{(x, y) \rightarrow (0, 1)} \frac{x - xy + 3}{x^2y + 5xy - y^3} = \frac{0 - (0)(1) + 3}{(0)^2(1) + 5(0)(1) - (1)^3} = -3$

(b) $\lim_{(x, y) \rightarrow (3, -4)} \sqrt{x^2 + y^2} = \sqrt{(3)^2 + (-4)^2} = \sqrt{25} = 5$



13.2 Limits and Continuity in Higher Dimensions



Example

$$\text{Find } \lim_{(x,y) \rightarrow (0,0)} \frac{x^2 - xy}{\sqrt{x} - \sqrt{y}}.$$

13.2 Limits and Continuity in Higher Dimensions



Example

$$\text{Find } \lim_{(x,y) \rightarrow (0,0)} \frac{x^2 - xy}{\sqrt{x} - \sqrt{y}}.$$

We can't use the Quotient Rule here because $\sqrt{x} - \sqrt{y} \rightarrow 0$ as $(x, y) \rightarrow (0, 0)$.

13.2 Limits and Continuity in Higher Dimensions



Example

$$\text{Find } \lim_{(x,y) \rightarrow (0,0)} \frac{x^2 - xy}{\sqrt{x} - \sqrt{y}}.$$

We can't use the Quotient Rule here because $\sqrt{x} - \sqrt{y} \rightarrow 0$ as $(x, y) \rightarrow (0, 0)$. Instead we calculate that

$$\lim_{(x,y) \rightarrow (0,0)} \frac{x^2 - xy}{\sqrt{x} - \sqrt{y}} = \lim_{(x,y) \rightarrow (0,0)} \frac{(x^2 - xy)(\sqrt{x} + \sqrt{y})}{(\sqrt{x} - \sqrt{y})(\sqrt{x} + \sqrt{y})}$$

=

=

=

13.2 Limits and Continuity in Higher Dimensions



Example

Find $\lim_{(x,y) \rightarrow (0,0)} \frac{x^2 - xy}{\sqrt{x} - \sqrt{y}}.$

We can't use the Quotient Rule here because $\sqrt{x} - \sqrt{y} \rightarrow 0$ as $(x, y) \rightarrow (0, 0)$. Instead we calculate that

$$\begin{aligned}\lim_{(x,y) \rightarrow (0,0)} \frac{x^2 - xy}{\sqrt{x} - \sqrt{y}} &= \lim_{(x,y) \rightarrow (0,0)} \frac{(x^2 - xy)(\sqrt{x} + \sqrt{y})}{(\sqrt{x} - \sqrt{y})(\sqrt{x} + \sqrt{y})} \\ &= \lim_{(x,y) \rightarrow (0,0)} \frac{x(x - y)(\sqrt{x} + \sqrt{y})}{x - y} \\ &= \\ &= \end{aligned}$$

13.2 Limits and Continuity in Higher Dimensions



Example

Find $\lim_{(x,y) \rightarrow (0,0)} \frac{x^2 - xy}{\sqrt{x} - \sqrt{y}}.$

We can't use the Quotient Rule here because $\sqrt{x} - \sqrt{y} \rightarrow 0$ as $(x, y) \rightarrow (0, 0)$. Instead we calculate that

$$\begin{aligned}\lim_{(x,y) \rightarrow (0,0)} \frac{x^2 - xy}{\sqrt{x} - \sqrt{y}} &= \lim_{(x,y) \rightarrow (0,0)} \frac{(x^2 - xy)(\sqrt{x} + \sqrt{y})}{(\sqrt{x} - \sqrt{y})(\sqrt{x} + \sqrt{y})} \\ &= \lim_{(x,y) \rightarrow (0,0)} \frac{x(x - y)(\sqrt{x} + \sqrt{y})}{x - y} \\ &= \lim_{(x,y) \rightarrow (0,0)} x(\sqrt{x} + \sqrt{y}) \\ &= \end{aligned}$$

13.2 Limits and Continuity in Higher Dimensions



Example

Find $\lim_{(x,y) \rightarrow (0,0)} \frac{x^2 - xy}{\sqrt{x} - \sqrt{y}}.$

We can't use the Quotient Rule here because $\sqrt{x} - \sqrt{y} \rightarrow 0$ as $(x, y) \rightarrow (0, 0)$. Instead we calculate that

$$\begin{aligned}\lim_{(x,y) \rightarrow (0,0)} \frac{x^2 - xy}{\sqrt{x} - \sqrt{y}} &= \lim_{(x,y) \rightarrow (0,0)} \frac{(x^2 - xy)(\sqrt{x} + \sqrt{y})}{(\sqrt{x} - \sqrt{y})(\sqrt{x} + \sqrt{y})} \\ &= \lim_{(x,y) \rightarrow (0,0)} \frac{x(x - y)(\sqrt{x} + \sqrt{y})}{x - y} \\ &= \lim_{(x,y) \rightarrow (0,0)} x(\sqrt{x} + \sqrt{y}) \\ &= 0(0 + 0) = 0.\end{aligned}$$

13.2 Limits and Continuity in Higher Dimensions



EXAMPLE 3 Find $\lim_{(x,y) \rightarrow (0,0)} \frac{4xy^2}{x^2 + y^2}$ if it exists.

Solution We first observe that along the line $x = 0$, the function always has value 0 when $y \neq 0$. Likewise, along the line $y = 0$, the function has value 0 provided $x \neq 0$. So if the limit does exist as (x, y) approaches $(0, 0)$, the value of the limit must be 0 (see Figure 14.13). To see if this is true, we apply the definition of limit.

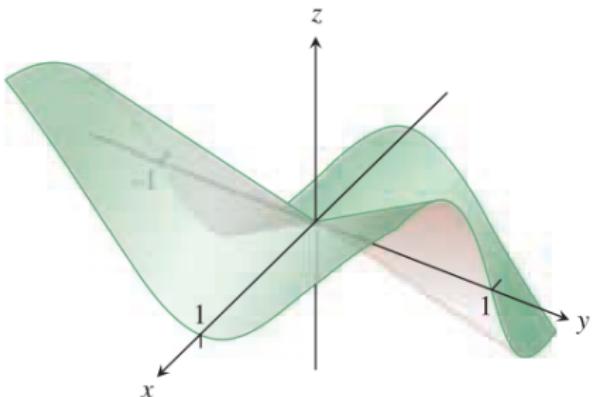


FIGURE 14.13 The surface graph shows the limit of the function in Example 3 must be 0, if it exists.

Let $\varepsilon > 0$ be given, but arbitrary. We want to find a $\delta > 0$ such that

$$\left| \frac{4xy^2}{x^2 + y^2} - 0 \right| < \varepsilon \quad \text{whenever} \quad 0 < \sqrt{x^2 + y^2} < \delta$$

or

$$\frac{4|x|y^2}{x^2 + y^2} < \varepsilon \quad \text{whenever} \quad 0 < \sqrt{x^2 + y^2} < \delta.$$

Since $y^2 \leq x^2 + y^2$ we have that

$$\frac{4|x|y^2}{x^2 + y^2} \leq 4|x| = 4\sqrt{x^2} \leq 4\sqrt{x^2 + y^2}. \quad \frac{y^2}{x^2 + y^2} \leq 1$$

So if we choose $\delta = \varepsilon/4$ and let $0 < \sqrt{x^2 + y^2} < \delta$, we get

$$\left| \frac{4xy^2}{x^2 + y^2} - 0 \right| \leq 4\sqrt{x^2 + y^2} < 4\delta = 4\left(\frac{\varepsilon}{4}\right) = \varepsilon.$$

It follows from the definition that

$$\lim_{(x,y) \rightarrow (0,0)} \frac{4xy^2}{x^2 + y^2} = 0.$$

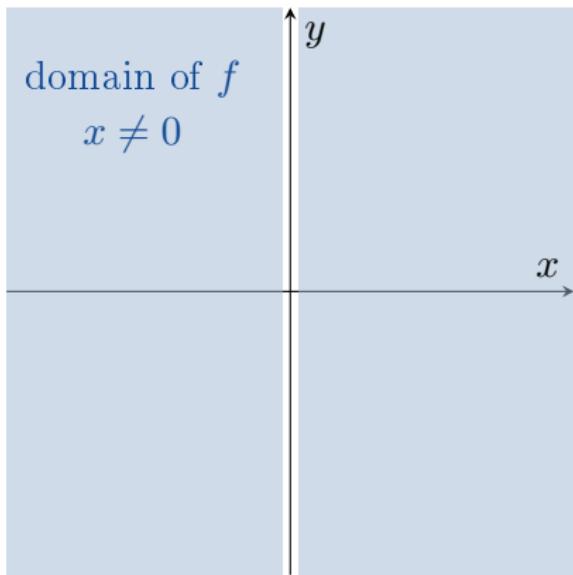


13.2 Limits and Continuity in Higher Dimensions



Example

If $f(x, y) = \frac{y}{x}$, does $\lim_{(x,y) \rightarrow (0,0)} f(x, y)$ exist?

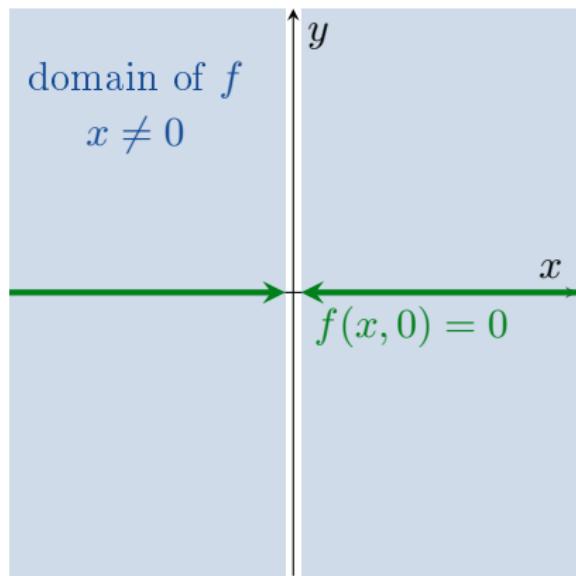


13.2 Limits and Continuity in Higher Dimensions



Example

If $f(x, y) = \frac{y}{x}$, does $\lim_{(x,y) \rightarrow (0,0)} f(x, y)$ exist?



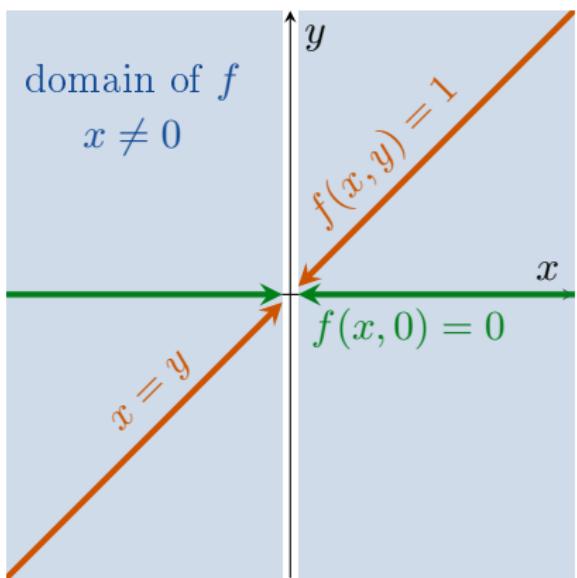
$$\lim_{(x,y) \rightarrow (0,0)} f(x, y) = \lim_{(x,y) \rightarrow (0,0)} 0 = 0$$

13.2 Limits and Continuity in Higher Dimensions



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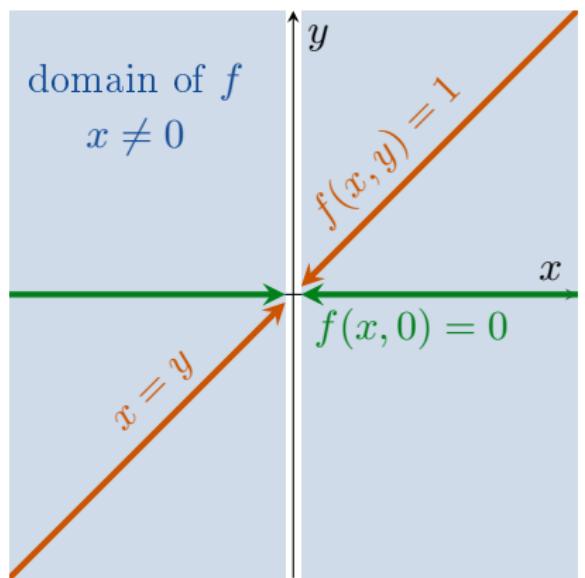
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13.2 Limits and Continuity in Higher Dimensions



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$$\lim_{(x,y) \rightarrow (0,0)} f(x, y) = \lim_{(x,y) \rightarrow (0,0)} 1 = 1$$

$\lim_{(x,y) \rightarrow (0,0)} f(x, y)$ does not exist.



Continuity

DEFINITION A function $f(x, y)$ is **continuous at the point (x_0, y_0)** if

1. f is defined at (x_0, y_0) ,
2. $\lim_{(x, y) \rightarrow (x_0, y_0)} f(x, y)$ exists,
3. $\lim_{(x, y) \rightarrow (x_0, y_0)} f(x, y) = f(x_0, y_0)$.

A function is **continuous** if it is continuous at every point of its domain.

EXAMPLE 5 Show that

$$f(x, y) = \begin{cases} \frac{2xy}{x^2 + y^2}, & (x, y) \neq (0, 0) \\ 0, & (x, y) = (0, 0) \end{cases}$$

is continuous at every point except the origin (Figure 14.14).

Solution The function f is continuous at every point (x, y) except $(0, 0)$ because its values at points other than $(0, 0)$ are given by a rational function of x and y , and therefore at those points the limiting value is simply obtained by substituting the values of x and y into that rational expression.

At $(0, 0)$, the value of f is defined, but f has no limit as $(x, y) \rightarrow (0, 0)$. The reason is that different paths of approach to the origin can lead to different results, as we now see.

For every value of m , the function f has a constant value on the “punctured” line $y = mx, x \neq 0$, because

$$f(x, y) \Big|_{y=mx} = \frac{2xy}{x^2 + y^2} \Big|_{y=mx} = \frac{2x(mx)}{x^2 + (mx)^2} = \frac{2mx^2}{x^2 + m^2x^2} = \frac{2m}{1 + m^2}.$$

Therefore, f has this number as its limit as (x, y) approaches $(0, 0)$ along the line:

$$\lim_{\substack{(x, y) \rightarrow (0, 0) \\ \text{along } y=mx}} f(x, y) = \lim_{(x, y) \rightarrow (0, 0)} \left[f(x, y) \Big|_{y=mx} \right] = \frac{2m}{1 + m^2}.$$

This limit changes with each value of the slope m . There is therefore no single number we may call the limit of f as (x, y) approaches the origin. The limit fails to exist, and the function is not continuous at the origin. ■

13.2 Limits and Continuity in Higher Dimensions



If a function $f(x, y)$ has different limits along two different paths in the domain of f as (x, y) approaches (x_0, y_0) , then

$$\lim_{(x,y) \rightarrow (x_0,y_0)} f(x, y)$$

does not exist.

13.2 Limits and Continuity in Higher Dimensions



EXAMPLE 6 Show that the function

$$f(x, y) = \frac{2x^2y}{x^4 + y^2}$$

(Figure 14.15) has no limit as (x, y) approaches $(0, 0)$.

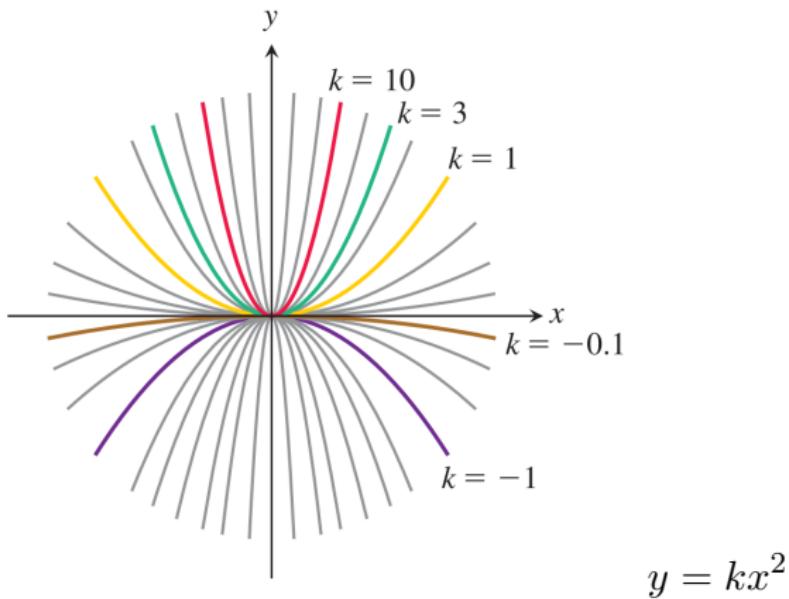
13.2 Limits and Continuity in Higher Dimensions



EXAMPLE 6 Show that the function

$$f(x, y) = \frac{2x^2y}{x^4 + y^2}$$

(Figure 14.15) has no limit as (x, y) approaches $(0, 0)$.



Solution The limit cannot be found by direct substitution, which gives the indeterminate form $0/0$. We examine the values of f along parabolic curves that end at $(0, 0)$. Along the curve $y = kx^2$, $x \neq 0$, the function has the constant value

$$f(x, y) \Big|_{y=kx^2} = \frac{2x^2y}{x^4 + y^2} \Big|_{y=kx^2} = \frac{2x^2(kx^2)}{x^4 + (kx^2)^2} = \frac{2kx^4}{x^4 + k^2x^4} = \frac{2k}{1 + k^2}.$$

Therefore,

$$\lim_{\substack{(x, y) \rightarrow (0, 0) \\ \text{along } y=kx^2}} f(x, y) = \lim_{(x, y) \rightarrow (0, 0)} \left[f(x, y) \Big|_{y=kx^2} \right] = \frac{2k}{1 + k^2}.$$

This limit varies with the path of approach. If (x, y) approaches $(0, 0)$ along the parabola $y = x^2$, for instance, $k = 1$ and the limit is 1. If (x, y) approaches $(0, 0)$ along the x -axis, $k = 0$ and the limit is 0. By the two-path test, f has no limit as (x, y) approaches $(0, 0)$. ■

It can be shown that the function in Example 6 has limit 0 along every straight line path $y = mx$ (Exercise 57). This implies the following observation:

Having the same limit along all straight lines approaching (x_0, y_0) does not imply that a limit exists at (x_0, y_0) .

13.2 Limits and Continuity in Higher Dimensions



Almost everything that we have done here has been for functions of two variables. But the same ideas can be extended to functions of three (or more) variables.

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Example

$$\lim_{(x,y,z) \rightarrow (1,0,-1)} \frac{e^{x+y+z}}{z^2 + \cos \sqrt{xy}} =$$

13.2 Limits and Continuity in Higher Dimensions



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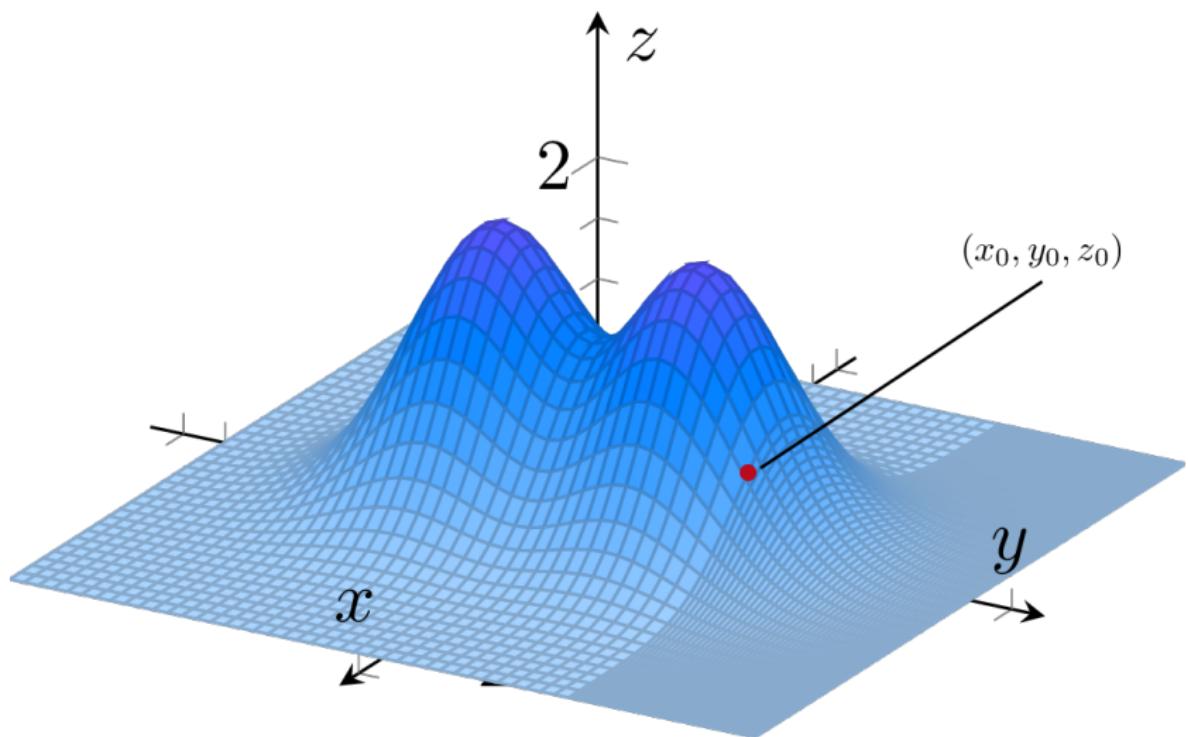
Example

$$\lim_{(x,y,z) \rightarrow (1,0,-1)} \frac{e^{x+y+z}}{z^2 + \cos \sqrt{xy}} = \frac{e^{1+0-1}}{(-1)^2 + \cos 0} = \frac{1}{2}.$$

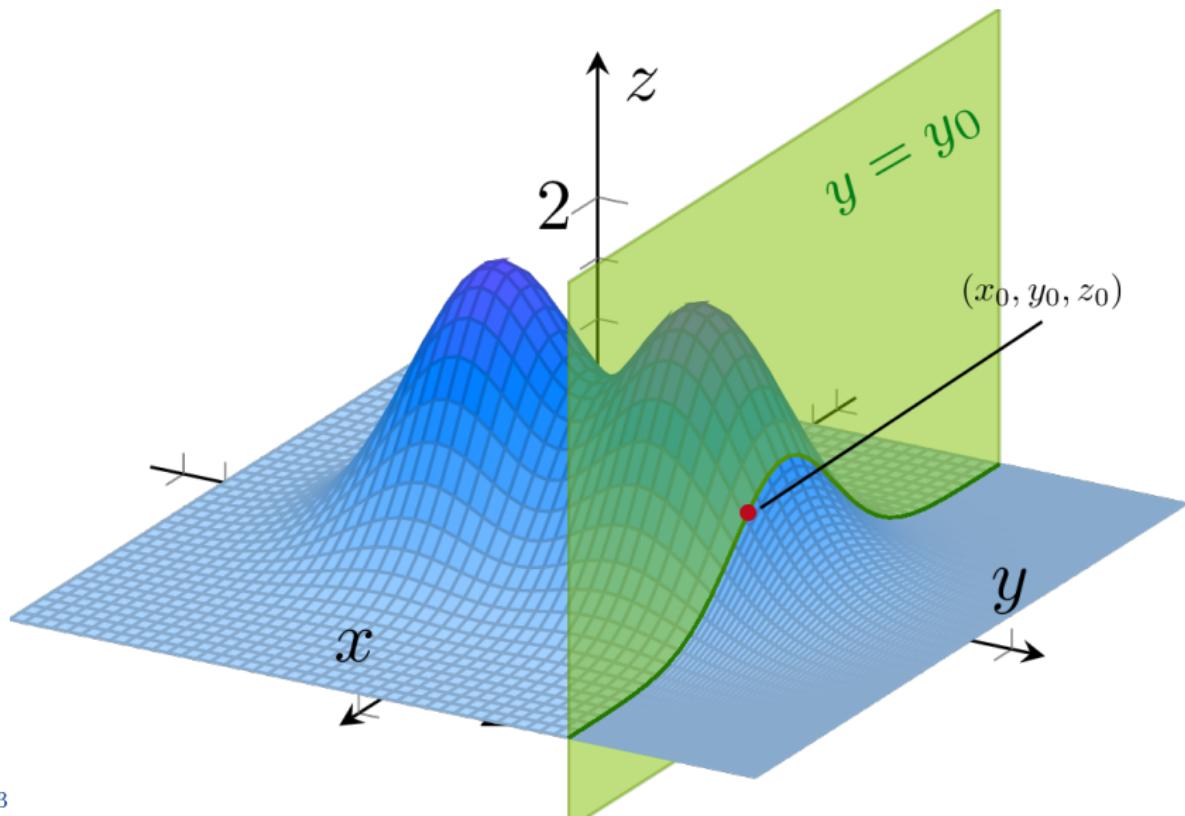


103 Partial Derivatives

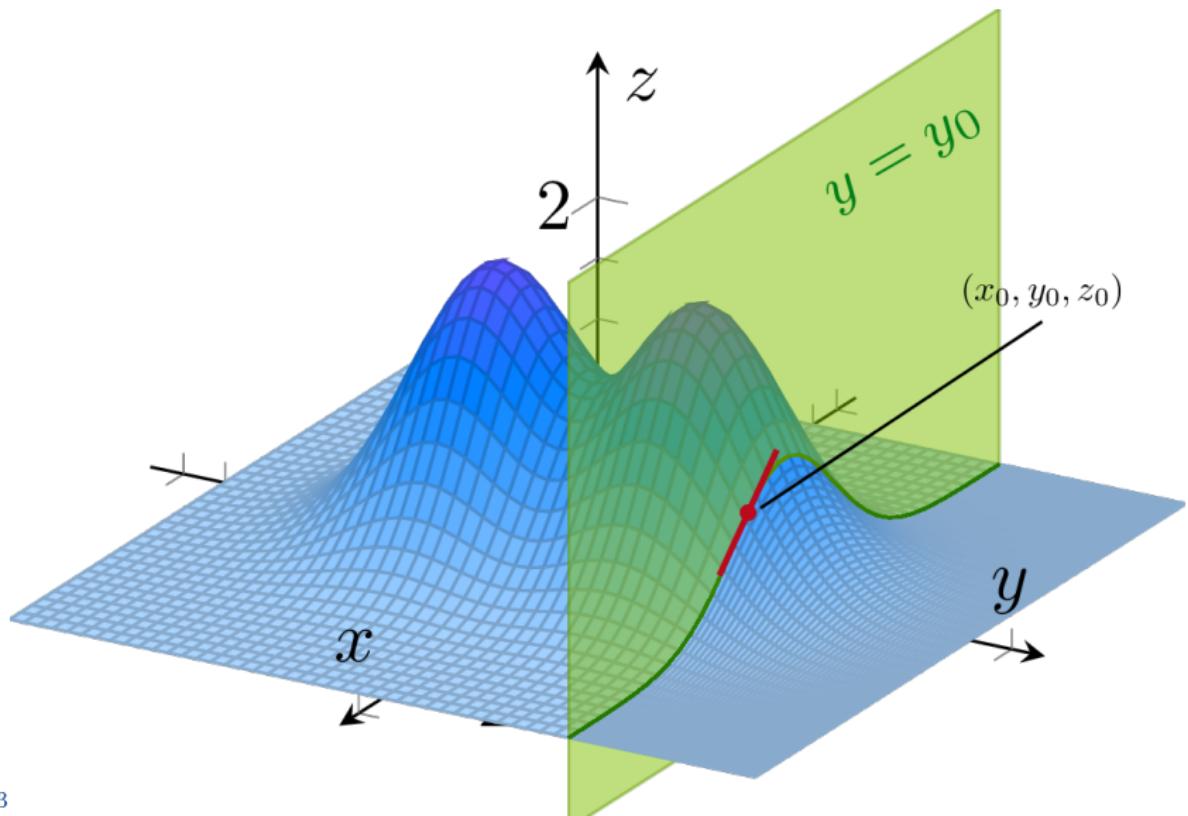
13.3 Partial Derivatives



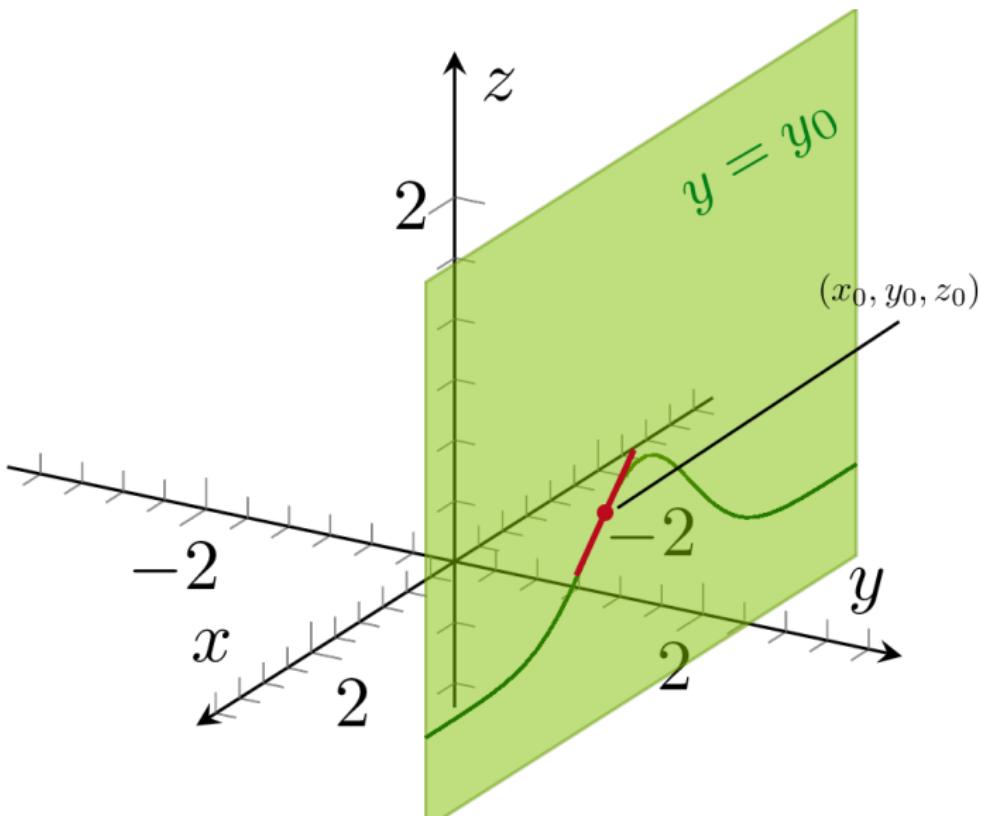
13.3 Partial Derivatives



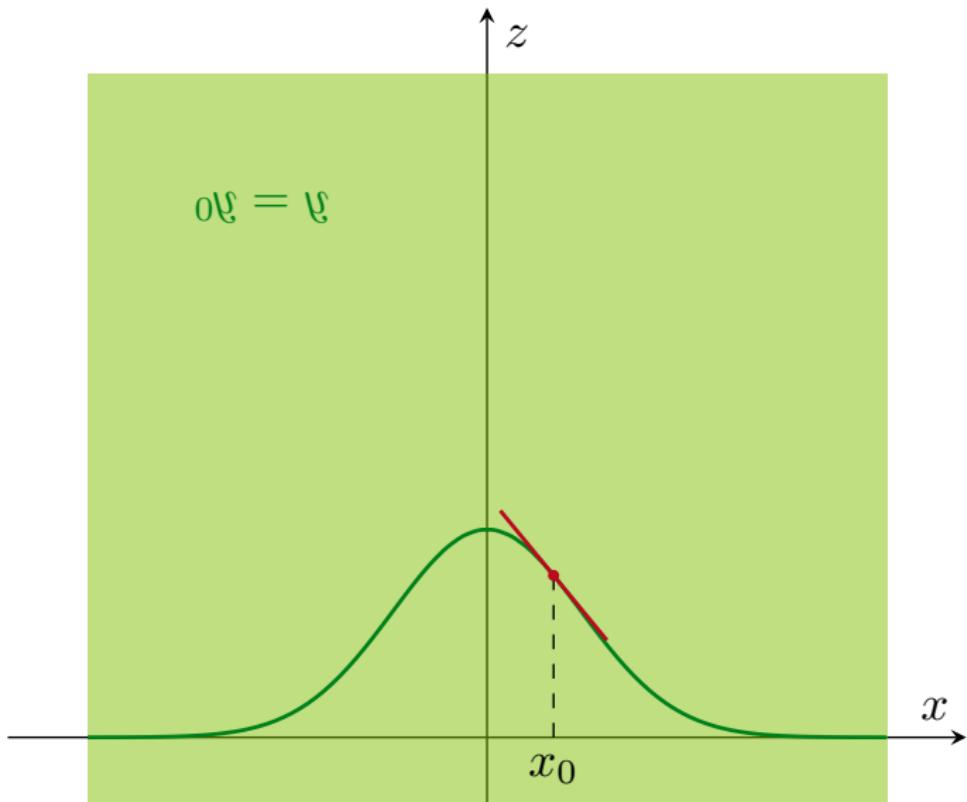
13.3 Partial Derivatives



13.3 Partial Derivatives



13.3 Partial Derivatives



13.3 Partial Derivatives

Definition

The *partial derivative* of $f(x, y)$ with respect to x at the point (x_0, y_0) is

$$\frac{\partial f}{\partial x} \Big|_{(x_0, y_0)} = \lim_{h \rightarrow 0} \frac{f(x_0 + h, y_0) - f(x_0, y_0)}{h}$$

if the limit exists.

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13.3 Partial Derivatives

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13.3 Partial Derivatives

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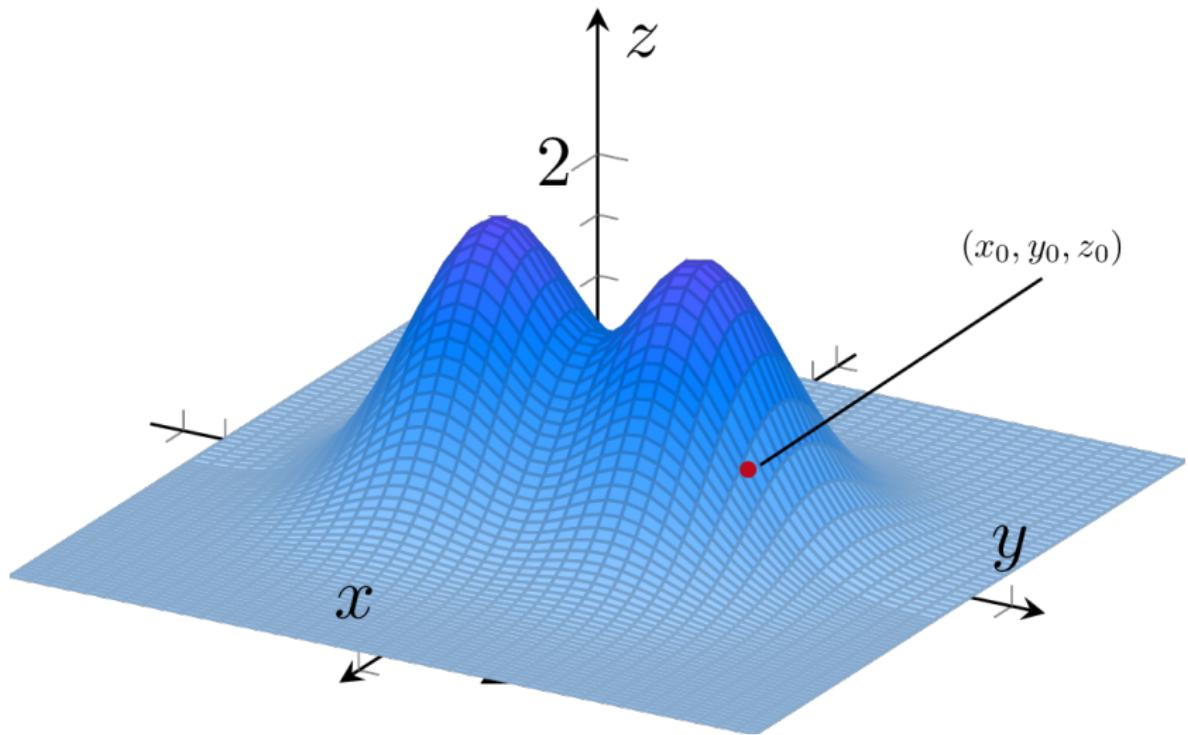
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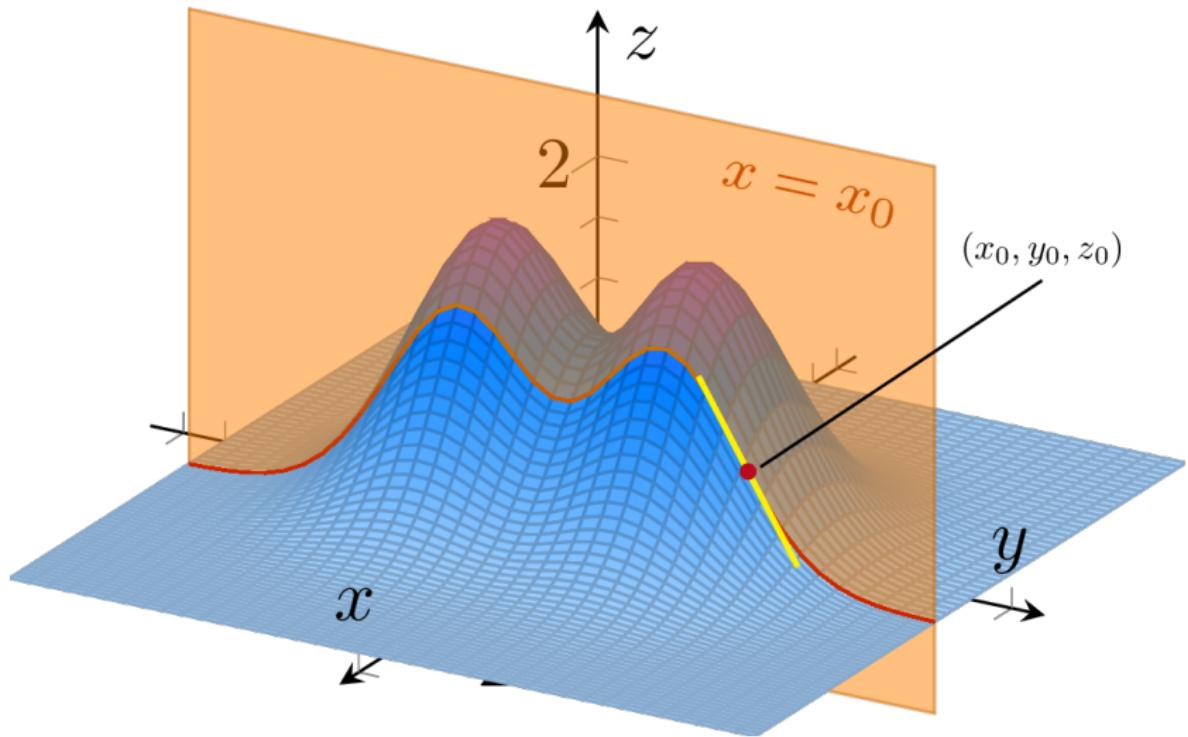
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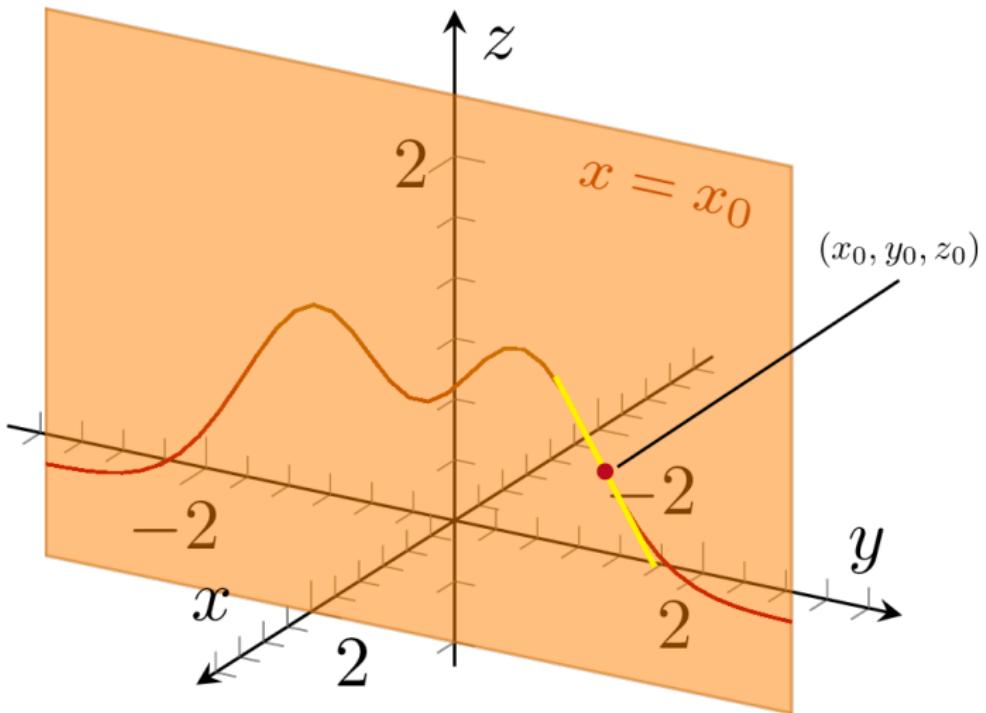
13.3 Partial Derivatives



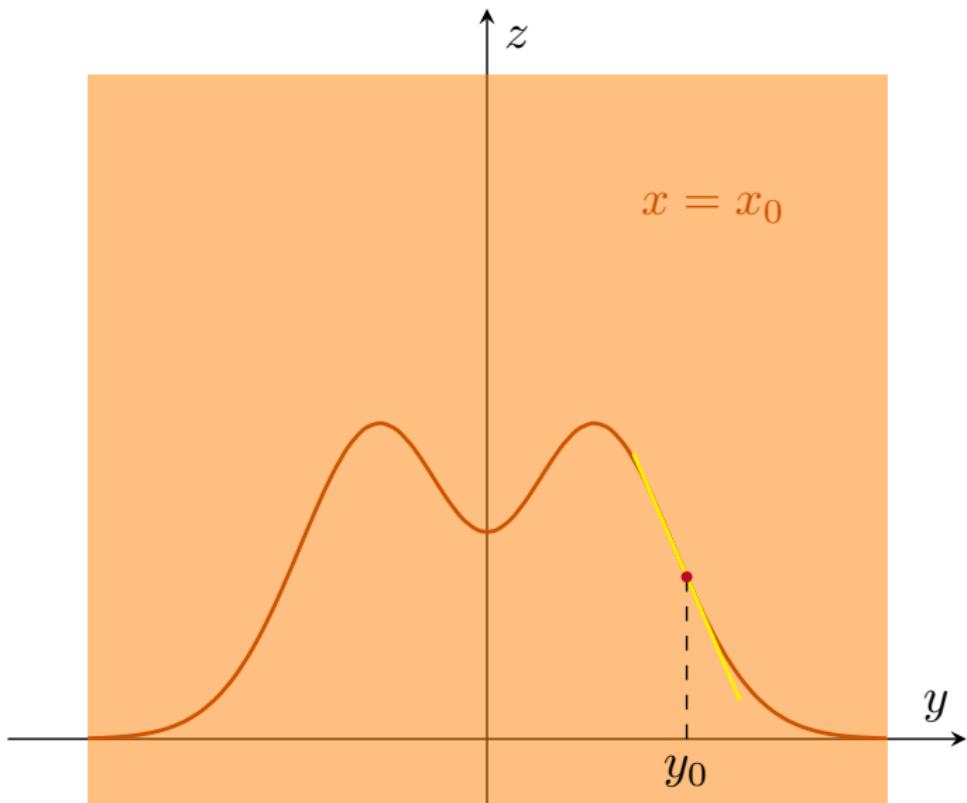
13.3 Partial Derivatives



13.3 Partial Derivatives



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13.3 Partial Derivatives



Notation

$$\frac{df}{dx} = f' \quad \text{ordinary derivative}$$

13.3 Partial Derivatives



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$$\frac{df}{dx} = f' \quad \text{ordinary derivative}$$

$$\frac{\partial f}{\partial x} = f_x \quad \text{partial derivative wrt } x$$

$$\frac{\partial f}{\partial y} = f_y \quad \text{partial derivative wrt } y$$

13.3 Partial Derivatives

Example

Let $f(x) = x^2 + 3xy + y - 1$.

Find $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ at the point $(4, -5)$.

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13.3 Partial Derivatives

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13.3 Partial Derivatives

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13.3 Partial Derivatives

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$$\frac{\partial f}{\partial x} = \frac{\partial}{\partial x} (x^2 + 3xy + y - 1) = 2x + 3y + 0 - 0.$$

13.3 Partial Derivatives

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Let $f(x) = x^2 + 3xy + y - 1$.

Find $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ at the point $(4, -5)$.

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$$\frac{\partial f}{\partial x} = \frac{\partial}{\partial x} (x^2 + 3xy + y - 1) = 2x + 3y + 0 - 0.$$

Therefore

$$\left. \frac{\partial f}{\partial x} \right|_{(4,-5)} = 2x + 3y \Big|_{(4,-5)} = -7.$$

13.3

$$f(x) = x^2 + 3xy + y - 1$$



And to find $\frac{\partial f}{\partial y}$ we pretend that x is a constant.

13.3

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13.3

$$f(x) = x^2 + 3xy + y - 1$$



And to find $\frac{\partial f}{\partial y}$ we pretend that x is a constant. Since

$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y} (x^2 + 3xy + y - 1) = 0 + 3x + 1 - 0$$

we have that

$$\left. \frac{\partial f}{\partial y} \right|_{(4,-5)} = 3x + 1 \Big|_{(4,-5)} = 13.$$



Break

We will continue at 2pm



13.3 Partial Derivatives



Remark

As long as you “pretend” that one of the variables is a constant and only look at the other variable, then you use the tools that you already know.

13.3 Partial Derivatives



Remark

As long as you “pretend” that one of the variables is a constant and only look at the other variable, then you use the tools that you already know.

- the product rule;
- the quotient rule;
- implicit differentiation;
- etc.

EXAMPLE 2 Find $\partial f / \partial y$ as a function if $f(x, y) = y \sin xy$.

Solution We treat x as a constant and f as a product of y and $\sin xy$:

$$\begin{aligned}\frac{\partial f}{\partial y} &= \frac{\partial}{\partial y} (y \sin xy) = y \frac{\partial}{\partial y} \sin xy + (\sin xy) \frac{\partial}{\partial y} (y) \\ &= (y \cos xy) \frac{\partial}{\partial y} (xy) + \sin xy = xy \cos xy + \sin xy.\end{aligned}$$

EXAMPLE 3 Find f_x and f_y as functions if

$$f(x, y) = \frac{2y}{y + \cos x}.$$

Solution We treat f as a quotient. With y held constant, we use the quotient rule to get

$$\begin{aligned} f_x &= \frac{\partial}{\partial x} \left(\frac{2y}{y + \cos x} \right) = \frac{(y + \cos x) \frac{\partial}{\partial x}(2y) - 2y \frac{\partial}{\partial x}(y + \cos x)}{(y + \cos x)^2} \\ &= \frac{(y + \cos x)(0) - 2y(-\sin x)}{(y + \cos x)^2} = \frac{2y \sin x}{(y + \cos x)^2}. \end{aligned}$$

With x held constant and again applying the quotient rule, we get

$$\begin{aligned} f_y &= \frac{\partial}{\partial y} \left(\frac{2y}{y + \cos x} \right) = \frac{(y + \cos x) \frac{\partial}{\partial y}(2y) - 2y \frac{\partial}{\partial y}(y + \cos x)}{(y + \cos x)^2} \\ &= \frac{(y + \cos x)(2) - 2y(1)}{(y + \cos x)^2} = \frac{2 \cos x}{(y + \cos x)^2}. \end{aligned}$$



EXAMPLE 4 Find $\partial z / \partial x$ assuming that the equation

$$yz - \ln z = x + y$$

defines z as a function of the two independent variables x and y and the partial derivative exists.

Solution We differentiate both sides of the equation with respect to x , holding y constant and treating z as a differentiable function of x :

$$\frac{\partial}{\partial x} (yz) - \frac{\partial}{\partial x} \ln z = \frac{\partial x}{\partial x} + \frac{\partial y}{\partial x}$$

$$y \frac{\partial z}{\partial x} - \frac{1}{z} \frac{\partial z}{\partial x} = 1 + 0 \quad \text{With } y \text{ constant, } \frac{\partial}{\partial x} (yz) = y \frac{\partial z}{\partial x}.$$

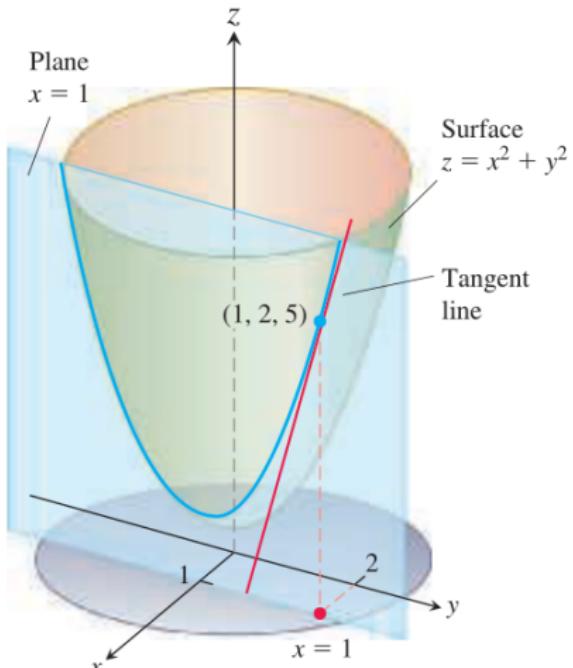
$$\left(y - \frac{1}{z} \right) \frac{\partial z}{\partial x} = 1$$

$$\frac{\partial z}{\partial x} = \frac{z}{yz - 1}.$$



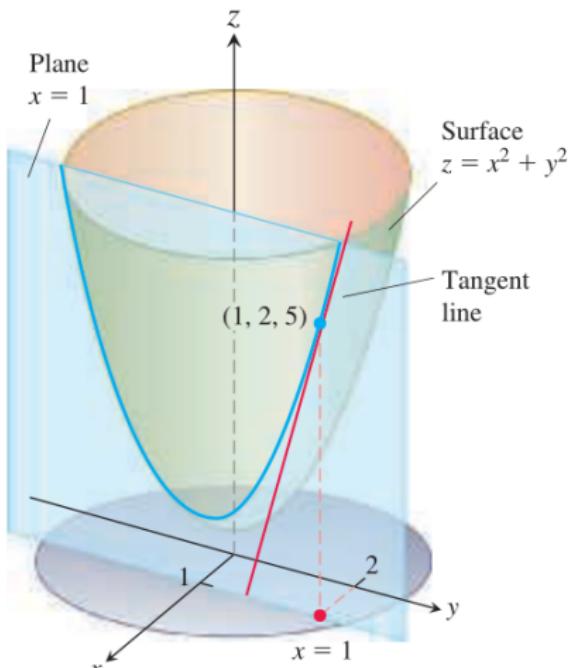
13.3 Partial Derivatives

EXAMPLE 5 The plane $x = 1$ intersects the paraboloid $z = x^2 + y^2$ in a parabola. Find the slope of the tangent to the parabola at $(1, 2, 5)$ (Figure 14.19).



13.3 Partial Derivatives

EXAMPLE 5 The plane $x = 1$ intersects the paraboloid $z = x^2 + y^2$ in a parabola. Find the slope of the tangent to the parabola at $(1, 2, 5)$ (Figure 14.19).



$$\left. \frac{\partial z}{\partial y} \right|_{(1,2)} = ?$$

Solution The parabola lies in a plane parallel to the yz -plane, and the slope is the value of the partial derivative $\partial z / \partial y$ at $(1, 2)$:

$$\frac{\partial z}{\partial y} \Big|_{(1,2)} = \frac{\partial}{\partial y} (x^2 + y^2) \Big|_{(1,2)} = 2y \Big|_{(1,2)} = 2(2) = 4.$$

As a check, we can treat the parabola as the graph of the single-variable function $z = (1)^2 + y^2 = 1 + y^2$ in the plane $x = 1$ and ask for the slope at $y = 2$. The slope, calculated now as an ordinary derivative, is

$$\frac{dz}{dy} \Big|_{y=2} = \frac{d}{dy} (1 + y^2) \Big|_{y=2} = 2y \Big|_{y=2} = 4.$$



13.3 Partial Derivatives



Remark

We can use the same ideas for functions of three (or more) variables.

EXAMPLE 6

If x , y , and z are independent variables and

$$f(x, y, z) = x \sin(y + 3z),$$

then

$$\begin{aligned}\frac{\partial f}{\partial z} &= \frac{\partial}{\partial z} [x \sin(y + 3z)] = x \frac{\partial}{\partial z} \sin(y + 3z) && \text{x held constant} \\ &= x \cos(y + 3z) \frac{\partial}{\partial z}(y + 3z) && \text{Chain rule} \\ &= 3x \cos(y + 3z). && \text{y held constant}\end{aligned}$$

13.3 Partial Derivatives



Please read Example 7 in the textbook.

EXAMPLE 8

Let

$$f(x, y) = \begin{cases} 0, & xy \neq 0 \\ 1, & xy = 0 \end{cases}$$

(Figure 14.21).

- (a) Find the limit of f as (x, y) approaches $(0, 0)$ along the line $y = x$.
- (b) Prove that f is not continuous at the origin.
- (c) Show that both partial derivatives $\partial f / \partial x$ and $\partial f / \partial y$ exist at the origin.

Solution

- (a) Since $f(x, y)$ is constantly zero along the line $y = x$ (except at the origin), we have

$$\lim_{(x, y) \rightarrow (0, 0)} f(x, y) \Big|_{y=x} = \lim_{(x, y) \rightarrow (0, 0)} 0 = 0.$$

- (b)** Since $f(0, 0) = 1$, the limit in part (a) is not equal to $f(0, 0)$, which proves that f is not continuous at $(0, 0)$.
- (c)** To find $\partial f / \partial x$ at $(0, 0)$, we hold y fixed at $y = 0$. Then $f(x, y) = 1$ for all x , and the graph of f is the line L_1 in Figure 14.21. The slope of this line at any x is $\partial f / \partial x = 0$. In particular, $\partial f / \partial x = 0$ at $(0, 0)$. Similarly, $\partial f / \partial y$ is the slope of line L_2 at any y , so $\partial f / \partial y = 0$ at $(0, 0)$. ■

Second-Order Partial Derivatives

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x} \left(\frac{\partial f}{\partial x} \right) = (f_x)_x = f_{xx}$$

Second-Order Partial Derivatives

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13.3 Partial Derivatives



Second-Order Partial Derivatives

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x} \left(\frac{\partial f}{\partial x} \right) = (f_x)_x = f_{xx}$$

first diff. wrt y
then diff. wrt x

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then diff. wrt x

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x} \left(\frac{\partial f}{\partial y} \right) = (f_y)_x = f_{yx}$$

13.3 Partial Derivatives



Second-Order Partial Derivatives

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13.3 Partial Derivatives



Remark

Sometimes $f_{xy} = f_{yx}$ and sometimes $f_{xy} \neq f_{yx}$.

13.3 Partial Derivatives

Example

If $f(x, y) = x \cos y + ye^x$, find the four second order partial derivatives of f .

Since

$$f_x = \quad \text{and} \quad f_y = ,$$

we have that

$$f_{xx} =$$

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13.3 Partial Derivatives

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$$f_x = \cos y + ye^x \quad \text{and} \quad f_y = -x \sin y + e^x,$$

we have that

$$f_{xx} =$$

$$f_{yy} =$$

$$f_{yx} =$$

$$f_{xy} =$$

13.3 Partial Derivatives

Example

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13.3 Partial Derivatives



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$$f_{xy} = \frac{\partial}{\partial y} (f_x) = -\sin y + e^x = f_{yx}.$$

13.3 Partial Derivatives



Alexis Clairaut

BORN

13 May 1713

DECEASED

17 May 1765

NATIONALITY

French

Theorem (Clairaut's Theorem)

If

- R is an open region;
- $(a, b) \in R$;
- f, f_x, f_y, f_{xy} and f_{yx} are all defined on R ; and
- f, f_x, f_y, f_{xy} and f_{yx} are all continuous at (a, b) ;

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- f, f_x, f_y, f_{xy} and f_{yx} are all continuous at (a, b) ;

then

$$f_{xy}(a, b) = f_{yx}(a, b).$$

EXAMPLE 10 Find $\frac{\partial^2 w}{\partial x \partial y}$ if

$$w = xy + \frac{e^y}{y^2 + 1}.$$

Solution The symbol $\partial^2 w / \partial x \partial y$ tells us to differentiate first with respect to y and then with respect to x . However, if we interchange the order of differentiation and differentiate first with respect to x we get the answer more quickly. In two steps,

$$\frac{\partial w}{\partial x} = y \quad \text{and} \quad \frac{\partial^2 w}{\partial y \partial x} = 1.$$

If we differentiate first with respect to y , we obtain $\partial^2 w / \partial x \partial y = 1$ as well. We can differentiate in either order because the conditions of Theorem 2 hold for w at all points (x_0, y_0) .



13.3 Partial Derivatives



Higher Order Partial Derivatives

$$\frac{\partial^3 f}{\partial x \partial y^2} = f_{yyx}$$

$$\frac{\partial^4 f}{\partial x^2 \partial y^2} = f_{yyxx}$$

EXAMPLE 11 Find f_{yxyz} if $f(x, y, z) = 1 - 2xy^2z + x^2y$.

Solution We first differentiate with respect to the variable y , then x , then y again, and finally with respect to z :

$$f_y = -4xyz + x^2$$

$$f_{yx} = -4yz + 2x$$

$$f_{yyx} = -4z$$

$$f_{yxyz} = -4.$$





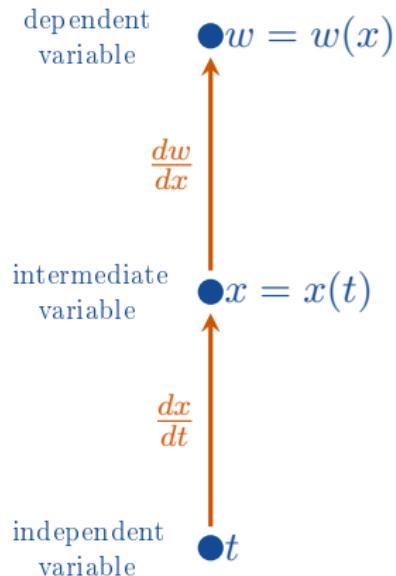
The Chain Rule

13.4 The Chain Rule

Functions of One Variable

Recall that if $w(x)$ and $x(t)$
then

$$\frac{dw}{dt} = \frac{dw}{dx} \frac{dx}{dt}.$$



13.4 The Chain Rule

Functions of Two Variables

But what is $\frac{dw}{dt}$ if $w(x, y)$?

dependent
variable

$$\bullet w = w(x, y)$$

intermediate
variables

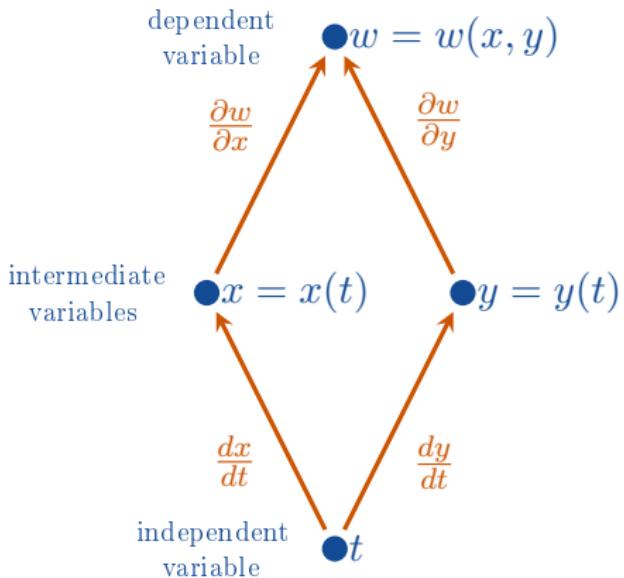
$$\bullet x = x(t) \quad \bullet y = y(t)$$

independent
variable

13.4 The Chain Rule

Functions of Two Variables

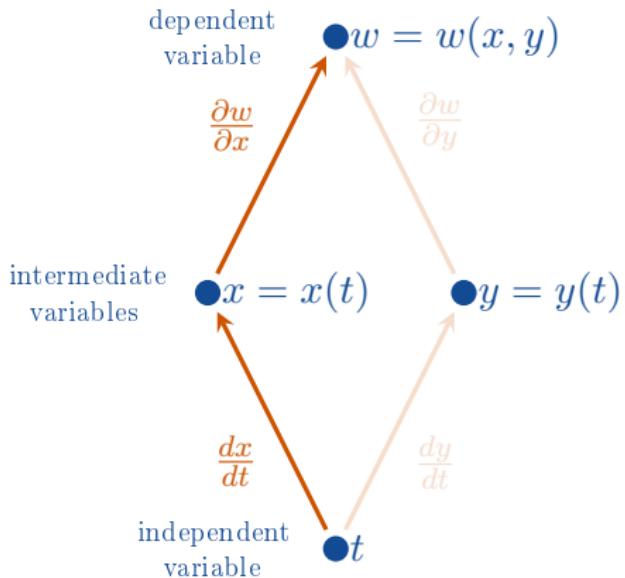
But what is $\frac{dw}{dt}$ if $w(x, y)$?



13.4 The Chain Rule

Functions of Two Variables

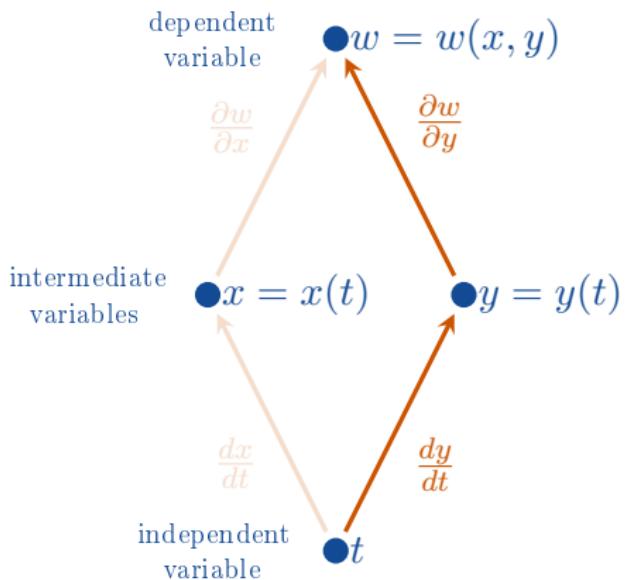
But what is $\frac{dw}{dt}$ if $w(x, y)$?



13.4 The Chain Rule

Functions of Two Variables

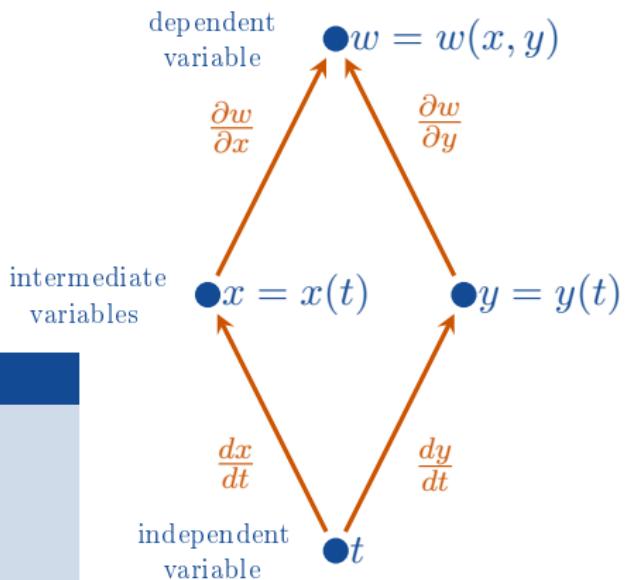
But what is $\frac{dw}{dt}$ if $w(x, y)$?



13.4 The Chain Rule

Functions of Two Variables

But what is $\frac{dw}{dt}$ if $w(x, y)$?



Theorem

$$\frac{dw}{dt} = \frac{\partial w}{\partial x} \frac{dx}{dt} + \frac{\partial w}{\partial y} \frac{dy}{dt}.$$

$$\frac{dw}{dt} = \frac{\partial w}{\partial x} \frac{dx}{dt} + \frac{\partial w}{\partial y} \frac{dy}{dt}$$

Example

Use the Chain Rule to find the derivative of $w = xy$ with respect to t along the path $x = \cos t$, $y = \sin t$. What is the derivative's value at $t = \frac{\pi}{2}$?

$$\frac{dw}{dt} = \frac{\partial w}{\partial x} \frac{dx}{dt} + \frac{\partial w}{\partial y} \frac{dy}{dt}$$

Example

Use the Chain Rule to find the derivative of $w = xy$ with respect to t along the path $x = \cos t$, $y = \sin t$. What is the derivative's value at $t = \frac{\pi}{2}$?

13.4

$$\frac{dw}{dt} = \frac{\partial w}{\partial x} \frac{dx}{dt} + \frac{\partial w}{\partial y} \frac{dy}{dt}$$



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Use the Chain Rule to find the derivative of $w = xy$ with respect to t along the path $x = \cos t$, $y = \sin t$. What is the derivative's value at $t = \frac{\pi}{2}$?

By the Chain Rule

$$\frac{dw}{dt} = \frac{\partial w}{\partial x} \frac{dx}{dt} + \frac{\partial w}{\partial y} \frac{dy}{dt}$$

=

=

=

$$\frac{dw}{dt} = \frac{\partial w}{\partial x} \frac{dx}{dt} + \frac{\partial w}{\partial y} \frac{dy}{dt}$$

Example

Use the Chain Rule to find the derivative of $w = xy$ with respect to t along the path $x = \cos t$, $y = \sin t$. What is the derivative's value at $t = \frac{\pi}{2}$?

By the Chain Rule

$$\begin{aligned}\frac{dw}{dt} &= \frac{\partial w}{\partial x} \frac{dx}{dt} + \frac{\partial w}{\partial y} \frac{dy}{dt} \\ &= (y)(-\sin t) + (x)(\cos t)\end{aligned}$$

=

=

$$\frac{dw}{dt} = \frac{\partial w}{\partial x} \frac{dx}{dt} + \frac{\partial w}{\partial y} \frac{dy}{dt}$$

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$$\begin{aligned}\frac{dw}{dt} &= \frac{\partial w}{\partial x} \frac{dx}{dt} + \frac{\partial w}{\partial y} \frac{dy}{dt} \\&= (y)(-\sin t) + (x)(\cos t) \\&= (\sin t)(-\sin t) + (\cos t)(\cos t) \\&= -\sin^2 t + \cos^2 t = \cos 2t.\end{aligned}$$

$$\frac{dw}{dt} = \frac{\partial w}{\partial x} \frac{dx}{dt} + \frac{\partial w}{\partial y} \frac{dy}{dt}$$

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Use the Chain Rule to find the derivative of $w = xy$ with respect to t along the path $x = \cos t$, $y = \sin t$. What is the derivative's value at $t = \frac{\pi}{2}$?

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$$\begin{aligned}\frac{dw}{dt} &= \frac{\partial w}{\partial x} \frac{dx}{dt} + \frac{\partial w}{\partial y} \frac{dy}{dt} \\ &= (y)(-\sin t) + (x)(\cos t) \\ &= (\sin t)(-\sin t) + (\cos t)(\cos t) \\ &= -\sin^2 t + \cos^2 t = \cos 2t.\end{aligned}$$

Therefore

$$\left. \frac{dw}{dt} \right|_{t=\frac{\pi}{2}} = \cos \left(2 \frac{\pi}{2} \right) = \cos \pi = -1.$$

$$\frac{dw}{dt} = \frac{\partial w}{\partial x} \frac{dx}{dt} + \frac{\partial w}{\partial y} \frac{dy}{dt}$$

Remark

Since the example said “Use the Chain Rule...”, we needed to do it this way.

If it just said “Find the derivative of...”, then we could have just done

$$w = xy = (\cos t)(\sin t) = \frac{1}{2} \sin 2t$$

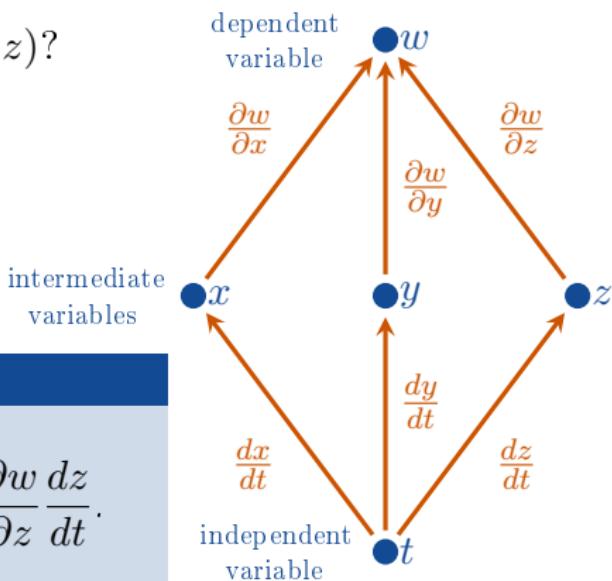
and

$$\frac{dw}{dt} = \frac{d}{dt} \left(\frac{1}{2} \sin 2t \right) = \cos 2t.$$

13.4 The Chain Rule

Functions of Three Variables

And what is $\frac{dw}{dt}$ if $w(x, y, z)$?



Theorem

$$\frac{dw}{dt} = \frac{\partial w}{\partial x} \frac{dx}{dt} + \frac{\partial w}{\partial y} \frac{dy}{dt} + \frac{\partial w}{\partial z} \frac{dz}{dt}.$$

EXAMPLE 2 Find dw/dt if

$$w = xy + z, \quad x = \cos t, \quad y = \sin t, \quad z = t.$$

In this example the values of $w(t)$ are changing along the path of a helix (Section 13.1) as t changes. What is the derivative's value at $t = 0$?

Solution Using the Chain Rule for three intermediate variables, we have

$$\begin{aligned}\frac{dw}{dt} &= \frac{\partial w}{\partial x} \frac{dx}{dt} + \frac{\partial w}{\partial y} \frac{dy}{dt} + \frac{\partial w}{\partial z} \frac{dz}{dt} \\&= (y)(-\sin t) + (x)(\cos t) + (1)(1) \\&= (\sin t)(-\sin t) + (\cos t)(\cos t) + 1 \\&= -\sin^2 t + \cos^2 t + 1 = 1 + \cos 2t,\end{aligned}$$

Substitute for intermediate variables.

so

$$\left. \frac{dw}{dt} \right|_{t=0} = 1 + \cos(0) = 2.$$



13.4 The Chain Rule

Theorem

$$\frac{\partial w}{\partial r} = ,$$

$$\frac{\partial w}{\partial s} = .$$

• w

one dependent variable

• x

• y

• z

three intermediate variables

• r

• s

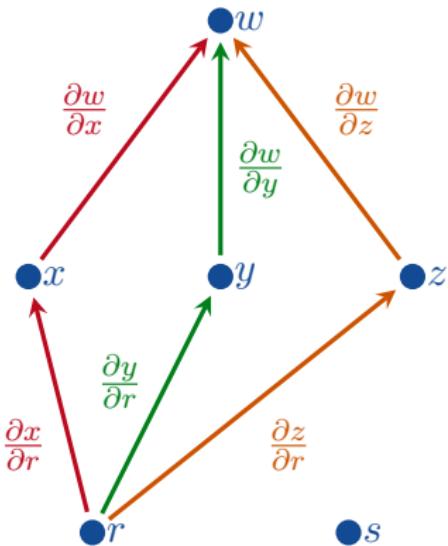
two independent variables

13.4 The Chain Rule

Theorem

$$\frac{\partial w}{\partial r} = \frac{\partial w}{\partial x} \frac{\partial x}{\partial r} + \frac{\partial w}{\partial y} \frac{\partial y}{\partial r} + \frac{\partial w}{\partial z} \frac{\partial z}{\partial r},$$

$$\frac{\partial w}{\partial s} =$$

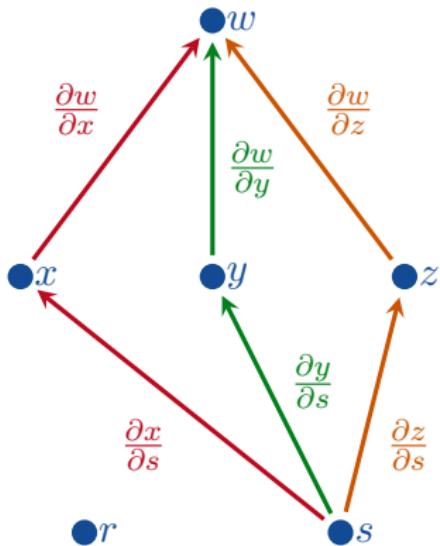


13.4 The Chain Rule

Theorem

$$\frac{\partial w}{\partial r} = \frac{\partial w}{\partial x} \frac{\partial x}{\partial r} + \frac{\partial w}{\partial y} \frac{\partial y}{\partial r} + \frac{\partial w}{\partial z} \frac{\partial z}{\partial r},$$

$$\frac{\partial w}{\partial s} = \frac{\partial w}{\partial x} \frac{\partial x}{\partial s} + \frac{\partial w}{\partial y} \frac{\partial y}{\partial s} + \frac{\partial w}{\partial z} \frac{\partial z}{\partial s}.$$



Once you understand this, you can do the Chain Rule for any number of intermediate and independent variables.

EXAMPLE 3 Express $\partial w/\partial r$ and $\partial w/\partial s$ in terms of r and s if

$$w = x + 2y + z^2, \quad x = \frac{r}{s}, \quad y = r^2 + \ln s, \quad z = 2r.$$

Solution Using the formulas in Theorem 7, we find

$$\begin{aligned}\frac{\partial w}{\partial r} &= \frac{\partial w}{\partial x} \frac{\partial x}{\partial r} + \frac{\partial w}{\partial y} \frac{\partial y}{\partial r} + \frac{\partial w}{\partial z} \frac{\partial z}{\partial r} \\&= (1)\left(\frac{1}{s}\right) + (2)(2r) + (2z)(2) \\&= \frac{1}{s} + 4r + (4r)(2) = \frac{1}{s} + 12r \quad \text{Substitute for intermediate variable } z.\end{aligned}$$

$$\begin{aligned}\frac{\partial w}{\partial s} &= \frac{\partial w}{\partial x} \frac{\partial x}{\partial s} + \frac{\partial w}{\partial y} \frac{\partial y}{\partial s} + \frac{\partial w}{\partial z} \frac{\partial z}{\partial s} \\&= (1)\left(-\frac{r}{s^2}\right) + (2)\left(\frac{1}{s}\right) + (2z)(0) = \frac{2}{s} - \frac{r}{s^2}.\end{aligned}$$



EXAMPLE 4 Express $\partial w/\partial r$ and $\partial w/\partial s$ in terms of r and s if

$$w = x^2 + y^2, \quad x = r - s, \quad y = r + s.$$

Solution The preceding discussion gives the following.

$$\begin{aligned}\frac{\partial w}{\partial r} &= \frac{\partial w}{\partial x} \frac{\partial x}{\partial r} + \frac{\partial w}{\partial y} \frac{\partial y}{\partial r} & \frac{\partial w}{\partial s} &= \frac{\partial w}{\partial x} \frac{\partial x}{\partial s} + \frac{\partial w}{\partial y} \frac{\partial y}{\partial s} \\ &= (2x)(1) + (2y)(1) & &= (2x)(-1) + (2y)(1) \\ &= 2(r - s) + 2(r + s) & &= -2(r - s) + 2(r + s) \\ &= 4r & &= 4s\end{aligned}$$

Substitute
for the
intermediate
variables. ■

If f is a function of x alone, our equations are even simpler.

13.4 The Chain Rule



Implicit Differentiation Revisited

Suppose that $F(x, y) = 0$ and $y = y(x)$.

13.4 The Chain Rule



Implicit Differentiation Revisited

Suppose that $F(x, y) = 0$ and $y = y(x)$. Then

$$0 = F(x, y)$$

13.4 The Chain Rule



Implicit Differentiation Revisited

Suppose that $F(x, y) = 0$ and $y = y(x)$. Then

$$\frac{d}{dx}0 = \frac{d}{dx}F(x, y)$$

13.4 The Chain Rule



Implicit Differentiation Revisited

Suppose that $F(x, y) = 0$ and $y = y(x)$. Then

$$0 = \frac{d}{dx}0 = \frac{d}{dx}F(x, y)$$

13.4 The Chain Rule



Implicit Differentiation Revisited

Suppose that $F(x, y) = 0$ and $y = y(x)$. Then

$$0 = \frac{d}{dx}0 = \frac{d}{dx}F(x, y) = \frac{dF}{dx}\frac{dx}{dx} + \frac{dF}{dy}\frac{dy}{dx}$$

13.4 The Chain Rule



Implicit Differentiation Revisited

Suppose that $F(x, y) = 0$ and $y = y(x)$. Then

$$0 = \frac{d}{dx}0 = \frac{d}{dx}F(x, y) = \frac{dF}{dx}\frac{dx}{dx} + \frac{dF}{dy}\frac{dy}{dx} = F_x \cdot 1 + F_y \frac{dy}{dx}.$$

13.4 The Chain Rule



Implicit Differentiation Revisited

Suppose that $F(x, y) = 0$ and $y = y(x)$. Then

$$0 = \frac{d}{dx}0 = \frac{d}{dx}F(x, y) = \frac{dF}{dx}\frac{dx}{dx} + \frac{dF}{dy}\frac{dy}{dx} = F_x \cdot 1 + F_y \frac{dy}{dx}.$$

Theorem

If $F_y \neq 0$, then

$$\frac{dy}{dx} = -\frac{F_x}{F_y}.$$

$$\frac{dy}{dx} = -\frac{F_x}{F_y}$$

Example

Find $\frac{dy}{dx}$ if $y^2 - x^2 - \sin xy = 0$.

$$\frac{dy}{dx} = -\frac{F_x}{F_y}$$

Example

Find $\frac{dy}{dx}$ if $y^2 - x^2 - \sin xy = 0$.

Let $F(x, y) = y^2 - x^2 - \sin xy$.

$$\frac{dy}{dx} = -\frac{F_x}{F_y}$$

Example

Find $\frac{dy}{dx}$ if $y^2 - x^2 - \sin xy = 0$.

Let $F(x, y) = y^2 - x^2 - \sin xy$. Then

$$\frac{dy}{dx} = -\frac{F_x}{F_y} =$$

$$\frac{dy}{dx} = -\frac{F_x}{F_y}$$

Example

Find $\frac{dy}{dx}$ if $y^2 - x^2 - \sin xy = 0$.

Let $F(x, y) = y^2 - x^2 - \sin xy$. Then

$$\frac{dy}{dx} = -\frac{F_x}{F_y} = -\frac{-2x - y \cos xy}{2y - x \cos xy} = \frac{2xy + \cos xy}{2y - x \cos xy}.$$

13.4 The Chain Rule

If we start with $F(x, y, z) = 0$ and $z = z(x, y)$, then we obtain

$$\begin{aligned} 0 &= \frac{\partial F}{\partial x} \frac{\partial x}{\partial x} + \frac{\partial F}{\partial y} \frac{\partial y}{\partial x} + \frac{\partial F}{\partial z} \frac{\partial z}{\partial x} \\ &= F_x \cdot 1 + F_y \cdot 0 + F_z \frac{\partial z}{\partial x} \quad \implies \quad \frac{\partial z}{\partial x} = -\frac{F_x}{F_z} \end{aligned}$$

13.4 The Chain Rule

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$$\begin{aligned} 0 &= \frac{\partial F}{\partial x} \frac{\partial x}{\partial x} + \frac{\partial F}{\partial y} \frac{\partial y}{\partial x} + \frac{\partial F}{\partial z} \frac{\partial z}{\partial x} \\ &= F_x \cdot 1 + F_y \cdot 0 + F_z \frac{\partial z}{\partial x} \quad \implies \quad \frac{\partial z}{\partial x} = -\frac{F_x}{F_z} \end{aligned}$$

and

$$\begin{aligned} 0 &= \frac{\partial F}{\partial x} \frac{\partial x}{\partial y} + \frac{\partial F}{\partial y} \frac{\partial y}{\partial y} + \frac{\partial F}{\partial z} \frac{\partial z}{\partial y} \\ &= F_x \cdot 0 + F_y \cdot 1 + F_z \frac{\partial z}{\partial y} \quad \implies \quad \frac{\partial z}{\partial y} = -\frac{F_y}{F_z}. \end{aligned}$$

13.4

$$\frac{\partial z}{\partial x} = -\frac{F_x}{F_z} \quad \frac{\partial z}{\partial y} = -\frac{F_y}{F_z}$$



Example

Find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ at $(0, 0, 0)$, if $x^3 + z^2 + ye^{xz} + z \cos y = 0$.

13.4

$$\frac{\partial z}{\partial x} = -\frac{F_x}{F_z} \quad \frac{\partial z}{\partial y} = -\frac{F_y}{F_z}$$



Example

Find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ at $(0, 0, 0)$, if $x^3 + z^2 + ye^{xz} + z \cos y = 0$.

Let $F(x, y, z) = x^3 + z^2 + ye^{xz} + z \cos y$. Then

$$\frac{\partial z}{\partial x} = -\frac{F_x}{F_z} = -\frac{3x^2 + zye^{xz}}{2z + xye^{xz} + \cos y}$$

and

$$\frac{\partial z}{\partial y} = -\frac{F_y}{F_z} = -\frac{e^{xz} - z \sin y}{2z + xye^{xz} + \cos y}.$$

13.4

$$\frac{\partial z}{\partial x} = -\frac{F_x}{F_z} \quad \frac{\partial z}{\partial y} = -\frac{F_y}{F_z}$$



Example

Find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ at $(0, 0, 0)$, if $x^3 + z^2 + ye^{xz} + z \cos y = 0$.

Let $F(x, y, z) = x^3 + z^2 + ye^{xz} + z \cos y$. Then

$$\frac{\partial z}{\partial x} = -\frac{F_x}{F_z} = -\frac{3x^2 + zye^{xz}}{2z + xye^{xz} + \cos y}$$

and

$$\frac{\partial z}{\partial y} = -\frac{F_y}{F_z} = -\frac{e^{xz} - z \sin y}{2z + xye^{xz} + \cos y}.$$

At $(0, 0, 0)$ we have

$$\frac{\partial z}{\partial x} = -\frac{0}{1} = 0 \quad \text{and} \quad \frac{\partial z}{\partial y} = -\frac{1}{1} = -1.$$



Next Time

- 13.5 Directional Derivatives and Gradient Vectors
- 13.6 Tangent Planes and Differentials
- 13.7 Extreme Values and Saddle Points
- 13.8 Lagrange Multipliers