

Lecture 8

- Matrices for Linear Transformations
- Similarity
- Complex Numbers



Matrices for Linear Transformations

Matrices for Linear Transformations



Example

Suppose that $T : \mathbb{R}^2 \rightarrow \mathbb{R}^3$ is a linear transformation which satisfies

$$T(\mathbf{e}_1) = \begin{bmatrix} 5 \\ -7 \\ 2 \end{bmatrix} \quad \text{and} \quad T(\mathbf{e}_2) = \begin{bmatrix} -3 \\ 8 \\ 0 \end{bmatrix}$$

where $\mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\mathbf{e}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. Find a formula for $T(\mathbf{x})$.

Matrices for Linear T

$$T(\mathbf{e}_1) = \begin{bmatrix} 5 \\ -7 \\ 2 \end{bmatrix} \quad T(\mathbf{e}_2) = \begin{bmatrix} -3 \\ 8 \\ 0 \end{bmatrix}$$



Because T is linear and because

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix} = x_1 \mathbf{e}_1 + x_2 \mathbf{e}_2$$

Matrices for Linear Transformations

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we have that

$$T(\mathbf{x}) = T(x_1 \mathbf{e}_1 + x_2 \mathbf{e}_2) = x_1 T(\mathbf{e}_1) + x_2 T(\mathbf{e}_2)$$

=

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Matrices for Linear T

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$$T(\mathbf{x}) = T(x_1 \mathbf{e}_1 + x_2 \mathbf{e}_2) = x_1 T(\mathbf{e}_1) + x_2 T(\mathbf{e}_2)$$

$$= x_1 \begin{bmatrix} 5 \\ -7 \\ 2 \end{bmatrix} + x_2 \begin{bmatrix} -3 \\ 8 \\ 0 \end{bmatrix} = \begin{bmatrix} 5x_1 - 3x_2 \\ -7x_1 + 8x_2 \\ 2x_1 \end{bmatrix}$$

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Matrices for Linear T

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$$= \begin{bmatrix} 5 & -3 \\ -7 & 8 \\ 2 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} =$$

Matrices for Linear Transformations

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we have that

$$\begin{aligned} T(\mathbf{x}) &= T(x_1 \mathbf{e}_1 + x_2 \mathbf{e}_2) = x_1 T(\mathbf{e}_1) + x_2 T(\mathbf{e}_2) \\ &= x_1 \begin{bmatrix} 5 \\ -7 \\ 2 \end{bmatrix} + x_2 \begin{bmatrix} -3 \\ 8 \\ 0 \end{bmatrix} = \begin{bmatrix} 5x_1 - 3x_2 \\ -7x_1 + 8x_2 \\ 2x_1 \end{bmatrix} \\ &= \begin{bmatrix} 5 & -3 \\ -7 & 8 \\ 2 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} T(\mathbf{e}_1) & T(\mathbf{e}_2) \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = A\mathbf{x}. \end{aligned}$$

Matrices for Linear Transformations



Theorem

Let $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$ be a linear transformation. Then there exists a unique matrix A such that

$$T(\mathbf{x}) = A\mathbf{x}$$

for all $\mathbf{x} \in \mathbb{R}^n$.

Matrices for Linear Transformations



Theorem

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$$T(\mathbf{x}) = A\mathbf{x}$$

for all $\mathbf{x} \in \mathbb{R}^n$.

In fact, A is the matrix

$$A = \begin{bmatrix} T(\mathbf{e}_1) & T(\mathbf{e}_2) & \cdots & T(\mathbf{e}_n) \end{bmatrix}$$

where \mathbf{e}_j is the j th column of the identity matrix I_n .

Matrices for Linear Transformations



Proof.

First suppose that $\mathbf{x} = x_1\mathbf{e}_1 + x_2\mathbf{e}_2 + \dots + x_n\mathbf{e}_n$. Then

$$\begin{aligned} T(\mathbf{x}) &= T(x_1\mathbf{e}_1 + x_2\mathbf{e}_2 + \dots + x_n\mathbf{e}_n) \\ &= x_1T(\mathbf{e}_1) + x_2T(\mathbf{e}_2) + \dots + x_nT(\mathbf{e}_n) \\ &= \begin{bmatrix} T(\mathbf{e}_1) & T(\mathbf{e}_2) & \cdots & T(\mathbf{e}_n) \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = A\mathbf{x}. \end{aligned}$$

This proves the existence part of the theorem.

Matrices for Linear Transformations



Proof continued.

Now we need to prove that such a matrix is unique: Suppose that $T(\mathbf{x}) = A\mathbf{x}$ and $T(\mathbf{x}) = B\mathbf{x}$.

Matrices for Linear Transformations



Proof continued.

Now we need to prove that such a matrix is unique: Suppose that $T(\mathbf{x}) = A\mathbf{x}$ and $T(\mathbf{x}) = B\mathbf{x}$. Then for each j we have

$$A\mathbf{e}_j = T(\mathbf{e}_j) = B\mathbf{e}_j =$$

Matrices for Linear Transformations



Proof continued.

Now we need to prove that such a matrix is unique: Suppose that $T(\mathbf{x}) = A\mathbf{x}$ and $T(\mathbf{x}) = B\mathbf{x}$. Then for each j we have

$$\begin{array}{ccc} \text{the } j\text{th} & & \text{the } j\text{th} \\ \text{column of } & = A\mathbf{e}_j = T(\mathbf{e}_j) = B\mathbf{e}_j = & \text{column of } \\ A & & B \end{array}$$

Matrices for Linear Transformations



Proof continued.

Now we need to prove that such a matrix is unique: Suppose that $T(\mathbf{x}) = A\mathbf{x}$ and $T(\mathbf{x}) = B\mathbf{x}$. Then for each j we have

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Hence $A = B$ and we are finished. □

Matrices for Linear Transformations



Proof continued.

Now we need to prove that such a matrix is unique: Suppose that $T(\mathbf{x}) = A\mathbf{x}$ and $T(\mathbf{x}) = B\mathbf{x}$. Then for each j we have

$$\begin{array}{ll} \text{the } j\text{th} & \text{the } j\text{th} \\ \text{column of } & = A\mathbf{e}_j = T(\mathbf{e}_j) = B\mathbf{e}_j = \text{column of} \\ A & B \end{array}$$

Hence $A = B$ and we are finished. □

Definition

The matrix A is called the *standard matrix for T* and is written $[T]$.

EXAMPLE 2 Find the standard matrix A for the dilation transformation $T(\mathbf{x}) = 3\mathbf{x}$, for \mathbf{x} in \mathbb{R}^2 .

SOLUTION Write

$$T(\mathbf{e}_1) = 3\mathbf{e}_1 = \begin{bmatrix} 3 \\ 0 \end{bmatrix} \quad \text{and} \quad T(\mathbf{e}_2) = 3\mathbf{e}_2 = \begin{bmatrix} 0 \\ 3 \end{bmatrix}$$

The diagram illustrates the mapping of the standard basis vectors \mathbf{e}_1 and \mathbf{e}_2 under the dilation transformation T . Two blue arrows point from the vectors $T(\mathbf{e}_1)$ and $T(\mathbf{e}_2)$ down to their respective entries in the columns of matrix A . The matrix A is shown below:

$$A = \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}$$

EXAMPLE 3 Let $T : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ be the transformation that rotates each point in \mathbb{R}^2 about the origin through an angle φ , with counterclockwise rotation for a positive angle. We could show geometrically that such a transformation is linear. (See Fig. 6 in Section 1.8.) Find the standard matrix A of this transformation.

SOLUTION $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ rotates into $\begin{bmatrix} \cos \varphi \\ \sin \varphi \end{bmatrix}$, and $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ rotates into $\begin{bmatrix} -\sin \varphi \\ \cos \varphi \end{bmatrix}$. See Fig. 1.

By Theorem 10,

$$A = \begin{bmatrix} \cos \varphi & -\sin \varphi \\ \sin \varphi & \cos \varphi \end{bmatrix}$$

Example 5 in Section 1.8 is a special case of this transformation, with $\varphi = \pi/2$. ■

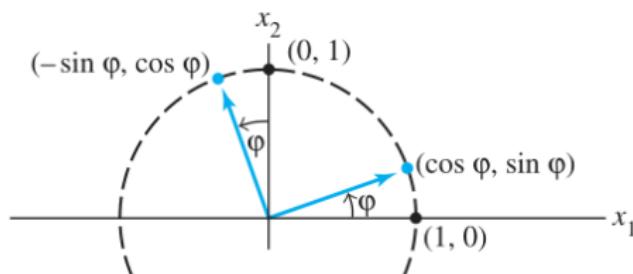
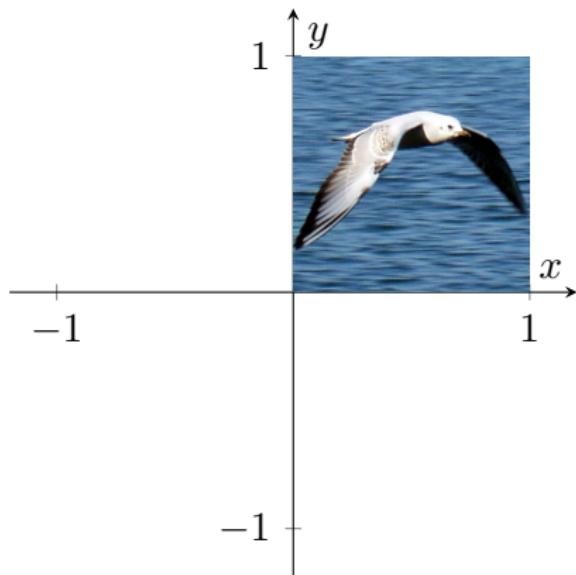
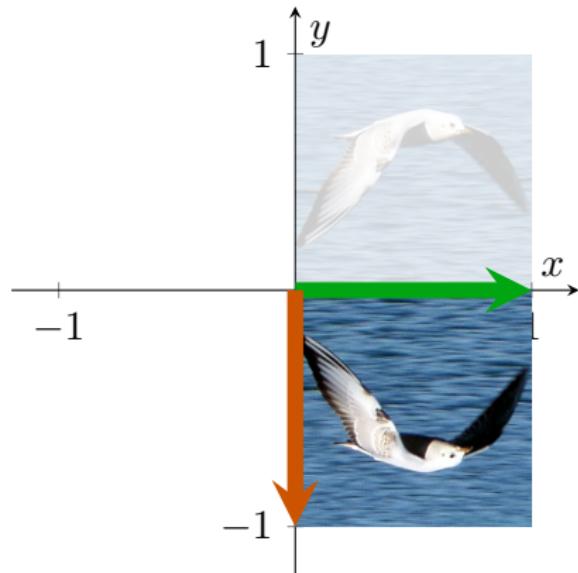


FIGURE 1 A rotation transformation.

Reflections



Reflections



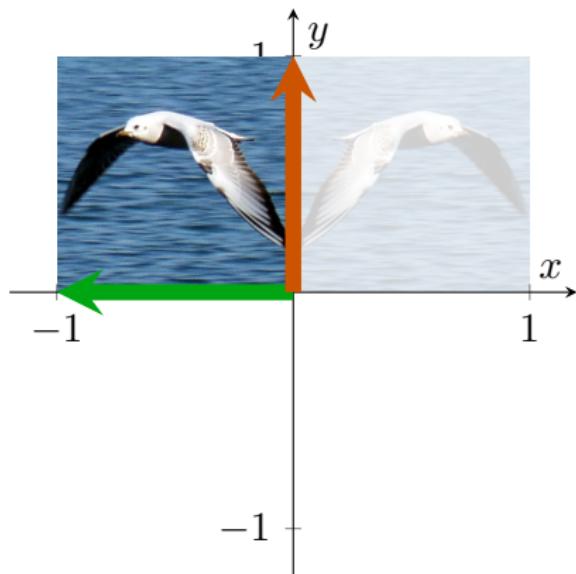
Reflection about the x -axis

$$T(\mathbf{e}_1) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$T(\mathbf{e}_2) = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$$

$$[T] = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

Reflections



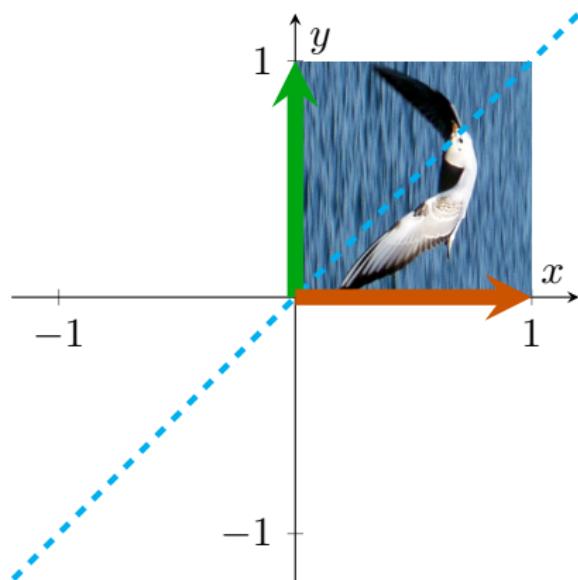
Reflection about the y -axis

$$T(\mathbf{e}_1) = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$$

$$T(\mathbf{e}_2) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$[T] = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$$

Reflections



Reflection about the line

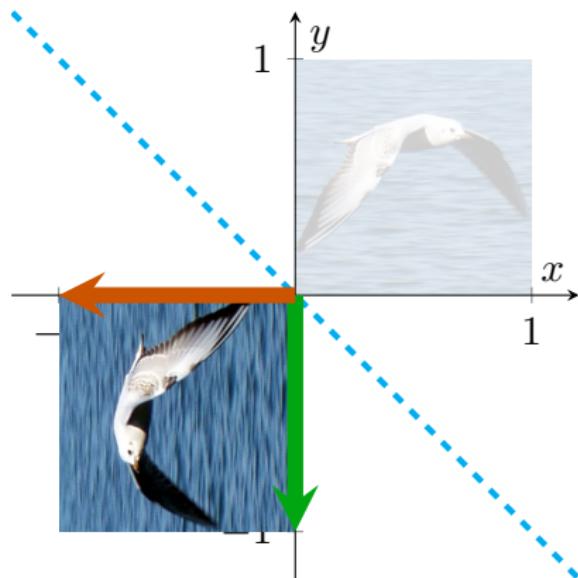
$$y = x$$

$$T(\mathbf{e}_1) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$T(\mathbf{e}_2) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$[T] = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

Reflections



Reflection about the line

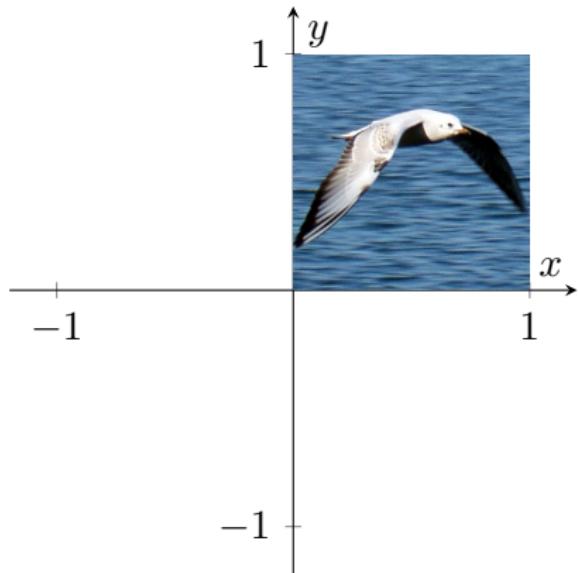
$$y = -x$$

$$T(\mathbf{e}_1) = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$$

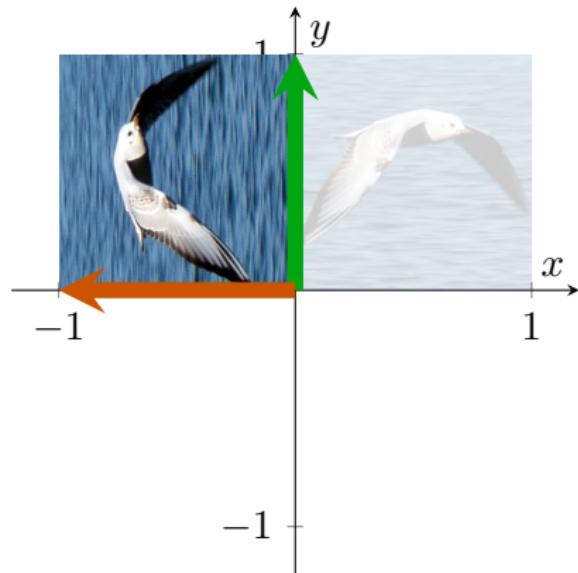
$$T(\mathbf{e}_2) = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$$

$$[T] = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$$

Rotations



Rotations



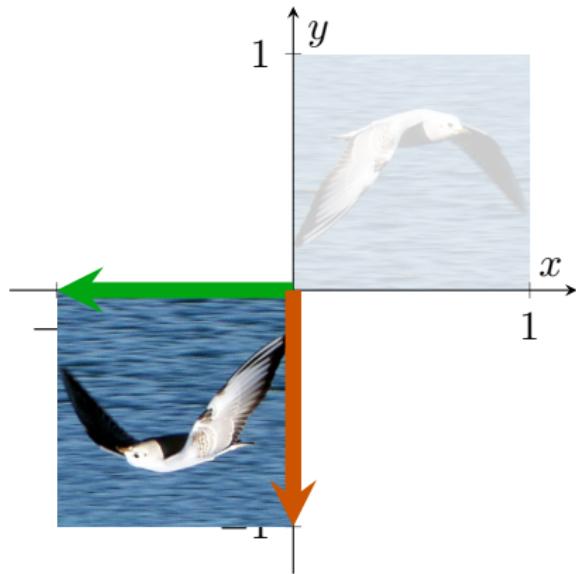
Rotate by 90°

$$T(\mathbf{e}_1) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$T(\mathbf{e}_2) = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$$

$$[T] = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

Rotations



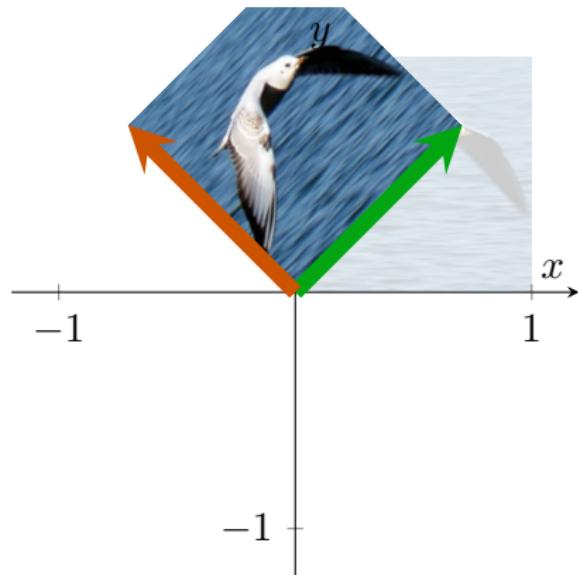
Rotate by 180°

$$T(\mathbf{e}_1) = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$$

$$T(\mathbf{e}_2) = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$$

$$[T] = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$

Rotations



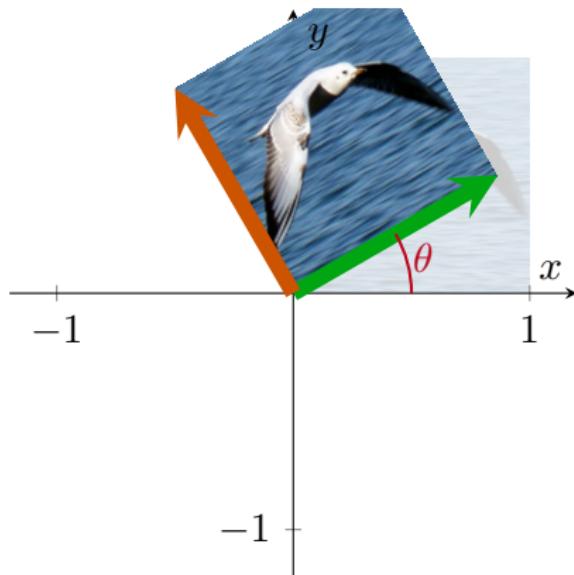
Rotate by 45°

$$T(\mathbf{e}_1) = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}$$

$$T(\mathbf{e}_2) = \begin{bmatrix} -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}$$

$$[T] = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$$

Rotations



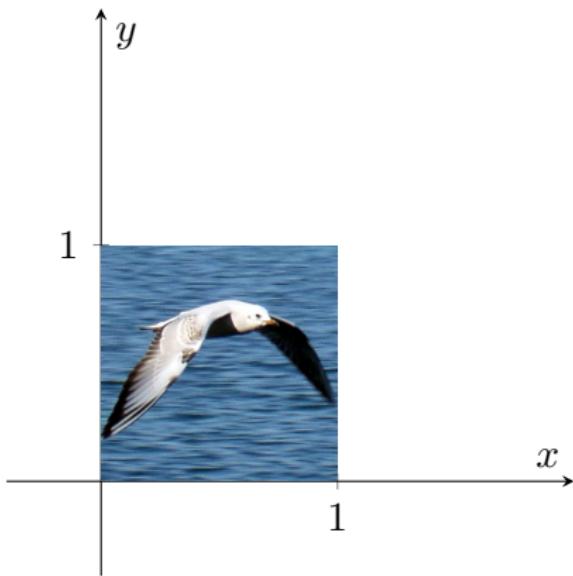
Rotate by θ

$$T(\mathbf{e}_1) = \begin{bmatrix} \cos \theta \\ \sin \theta \end{bmatrix}$$

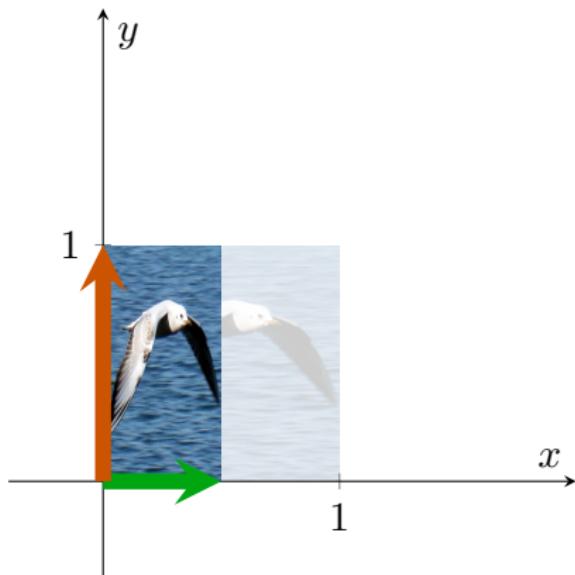
$$T(\mathbf{e}_2) = \begin{bmatrix} -\sin \theta \\ \cos \theta \end{bmatrix}$$

$$[T] = \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix}$$

Contractions and Expansions



Contractions and Expansions



Horizontal Contraction

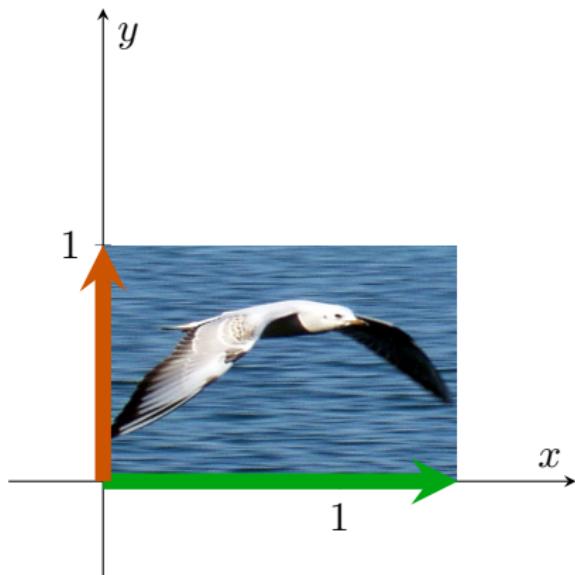
$$T(\mathbf{e}_1) = \begin{bmatrix} k \\ 0 \end{bmatrix}$$

$$T(\mathbf{e}_2) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$[T] = \begin{bmatrix} k & 0 \\ 0 & 1 \end{bmatrix}$$

$$0 < k < 1$$

Contractions and Expansions



Horizontal Expansion

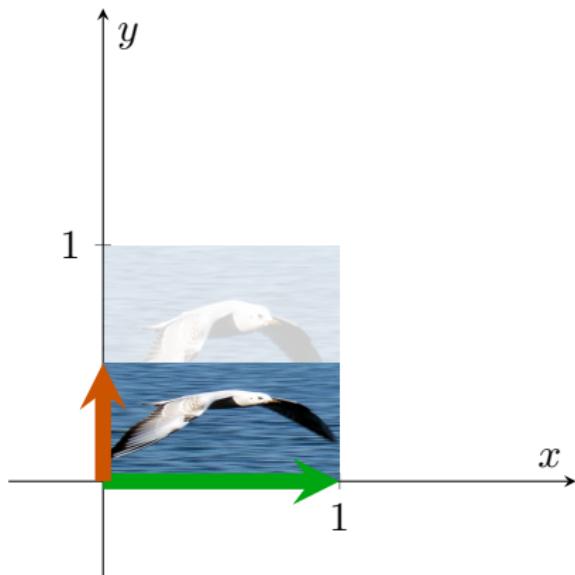
$$T(\mathbf{e}_1) = \begin{bmatrix} k \\ 0 \end{bmatrix}$$

$$T(\mathbf{e}_2) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$[T] = \begin{bmatrix} k & 0 \\ 0 & 1 \end{bmatrix}$$

$$k > 1$$

Contractions and Expansions



Vertical Contraction

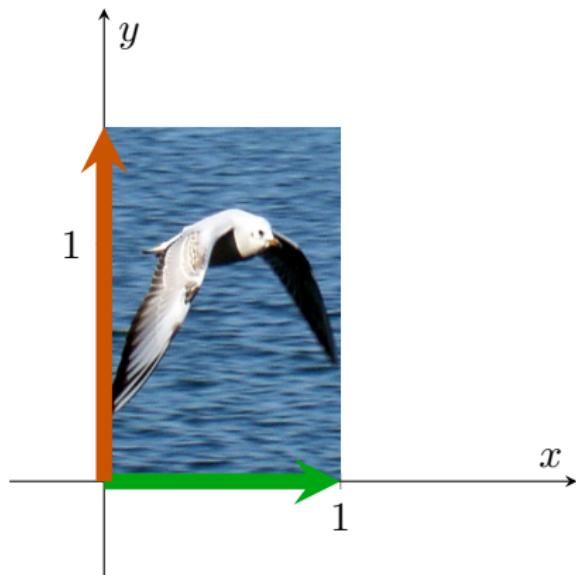
$$T(\mathbf{e}_1) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$T(\mathbf{e}_2) = \begin{bmatrix} 0 \\ k \end{bmatrix}$$

$$[T] = \begin{bmatrix} 1 & 0 \\ 0 & k \end{bmatrix}$$

$$0 < k < 1$$

Contractions and Expansions



Vertical Expansion

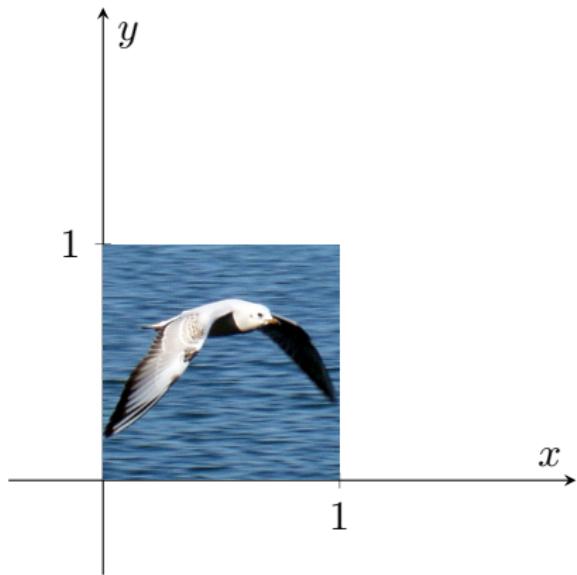
$$T(\mathbf{e}_1) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$T(\mathbf{e}_2) = \begin{bmatrix} 0 \\ k \end{bmatrix}$$

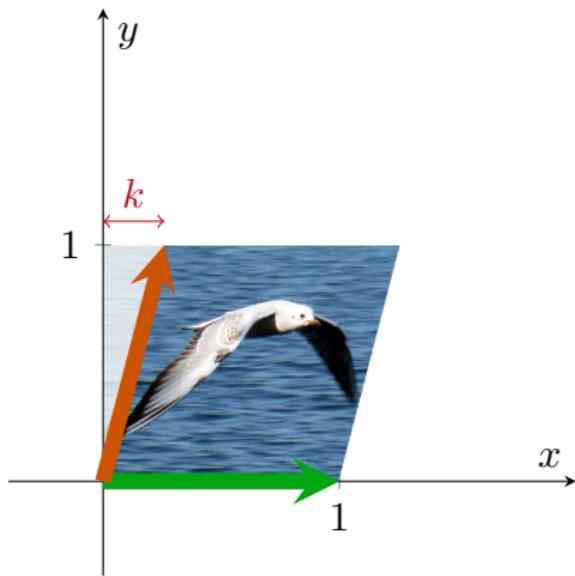
$$[T] = \begin{bmatrix} 1 & 0 \\ 0 & k \end{bmatrix}$$

$$k > 1$$

Shears



Shears



Horizontal Shear

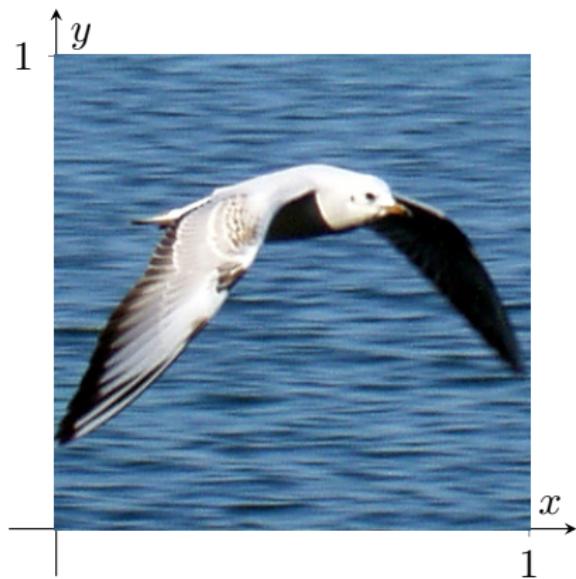
$$T(\mathbf{e}_1) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$T(\mathbf{e}_2) = \begin{bmatrix} k \\ 1 \end{bmatrix}$$

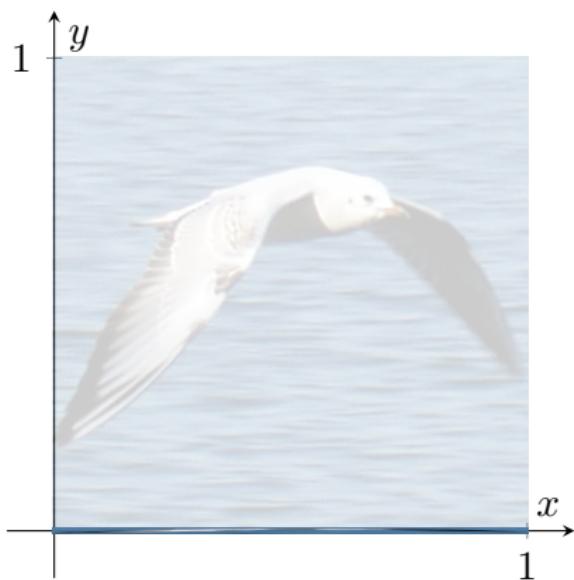
$$[T] = \begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix}$$

$$k > 0$$

Orthogonal Projections



Orthogonal Projections



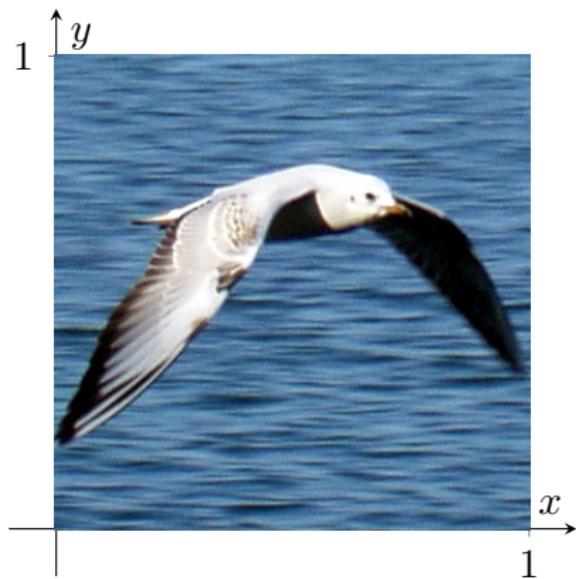
Projection onto the x -axis

$$T(\mathbf{e}_1) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

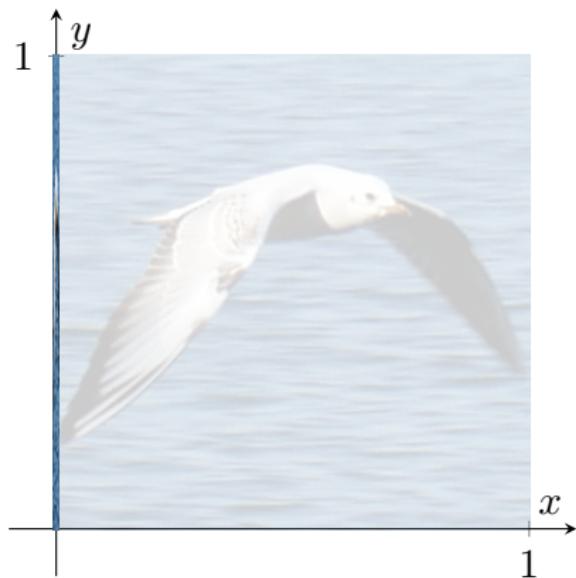
$$T(\mathbf{e}_2) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$[T] = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

Orthogonal Projections



Orthogonal Projections



Projection onto the y -axis

$$T(\mathbf{e}_1) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$T(\mathbf{e}_2) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$[T] = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

A Matrix for T relative to two bases

Let

- V be an n -dimensional vector space;
- \mathcal{B} be a basis for V ;
- W be an m -dimensional vector space;
- \mathcal{C} is a basis for W ; and
- $T : V \rightarrow W$ be a linear transformation.

Matrices for Linear Transformations

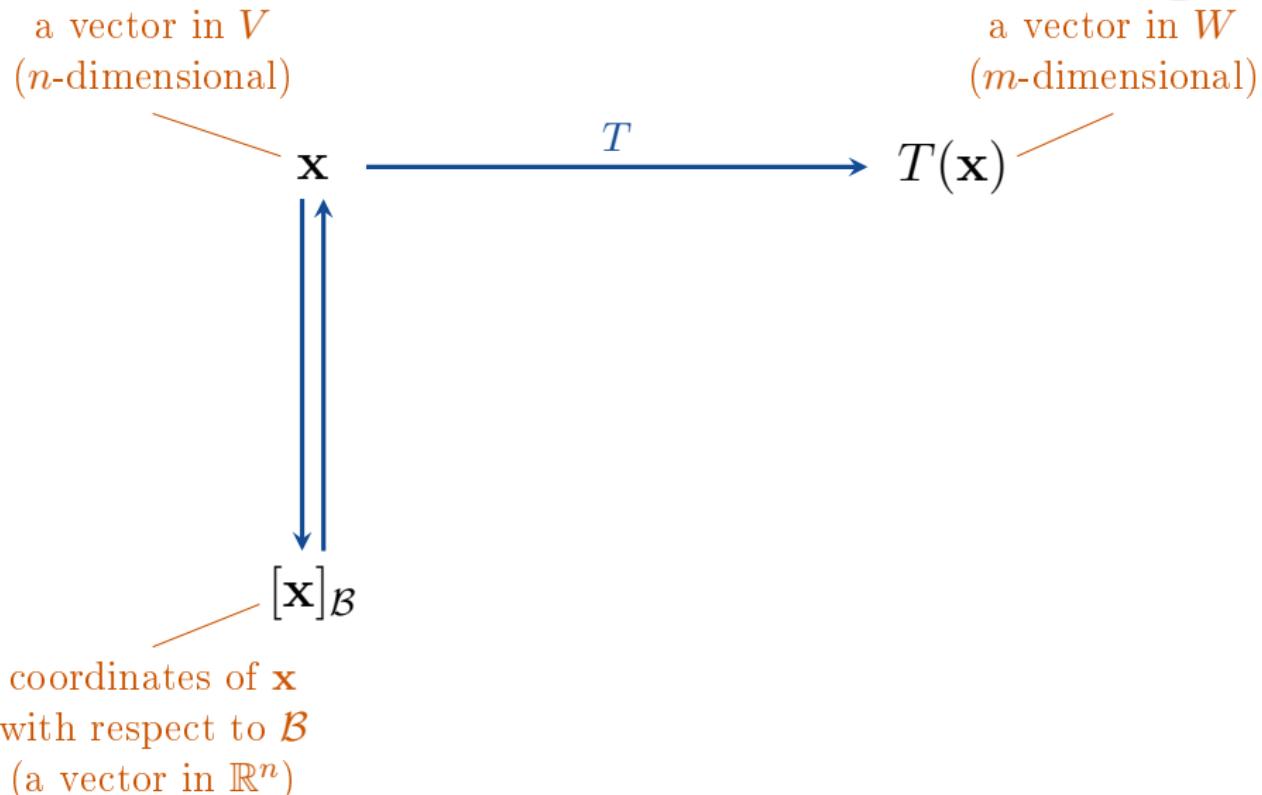


a vector in V
(n -dimensional)

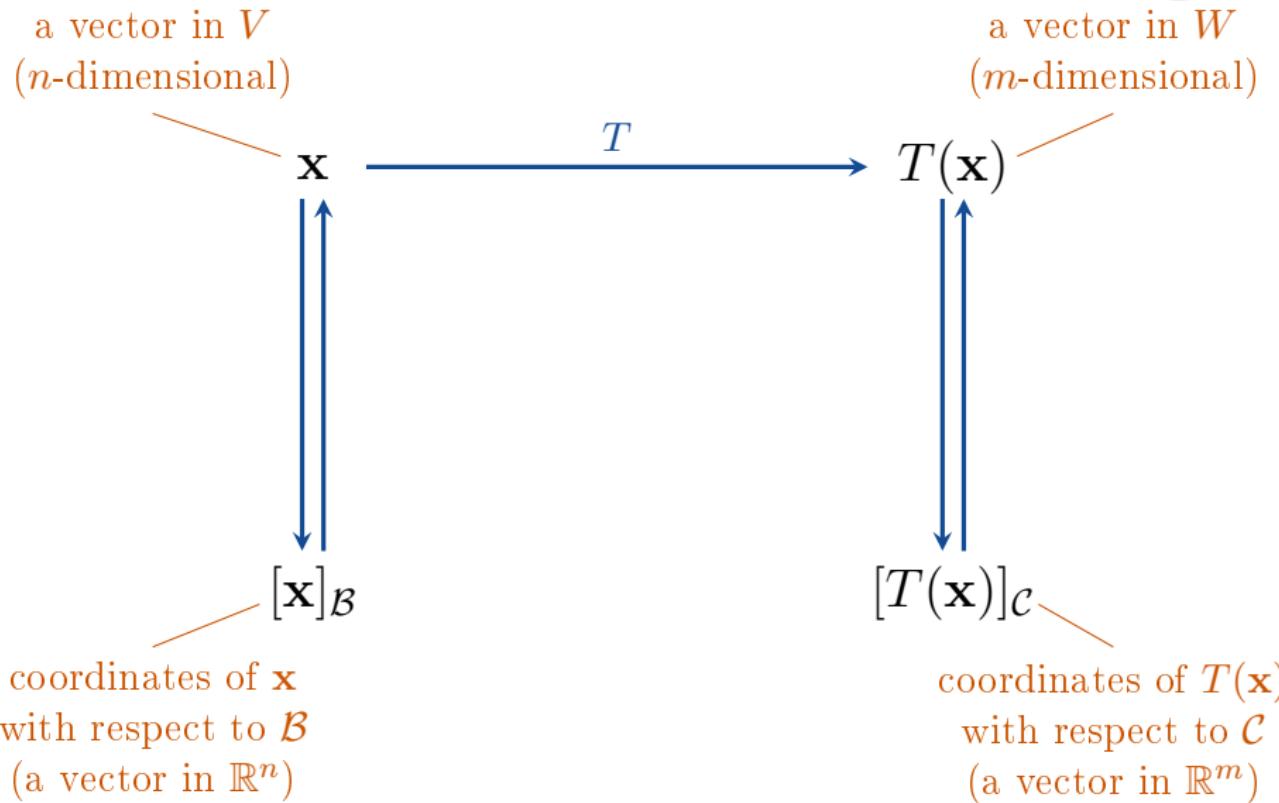


a vector in W
(m -dimensional)

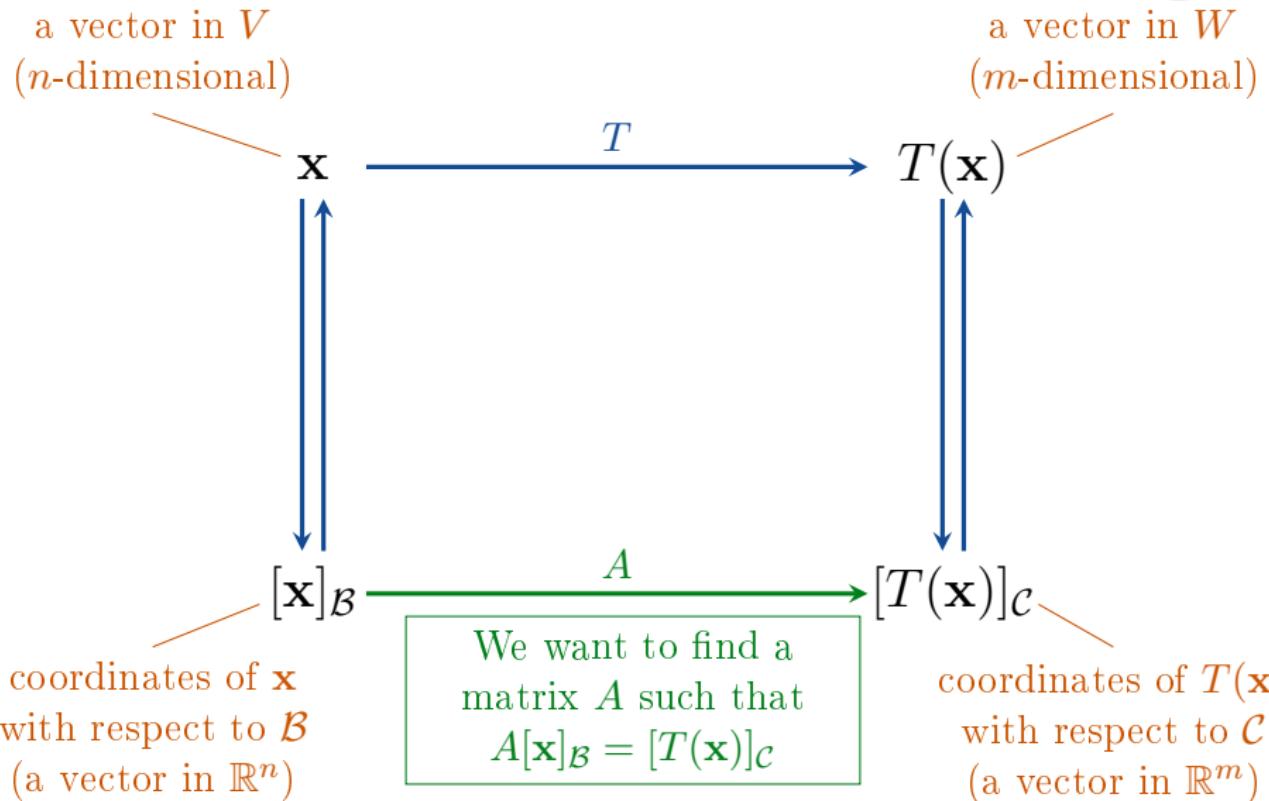
Matrices for Linear Transformations



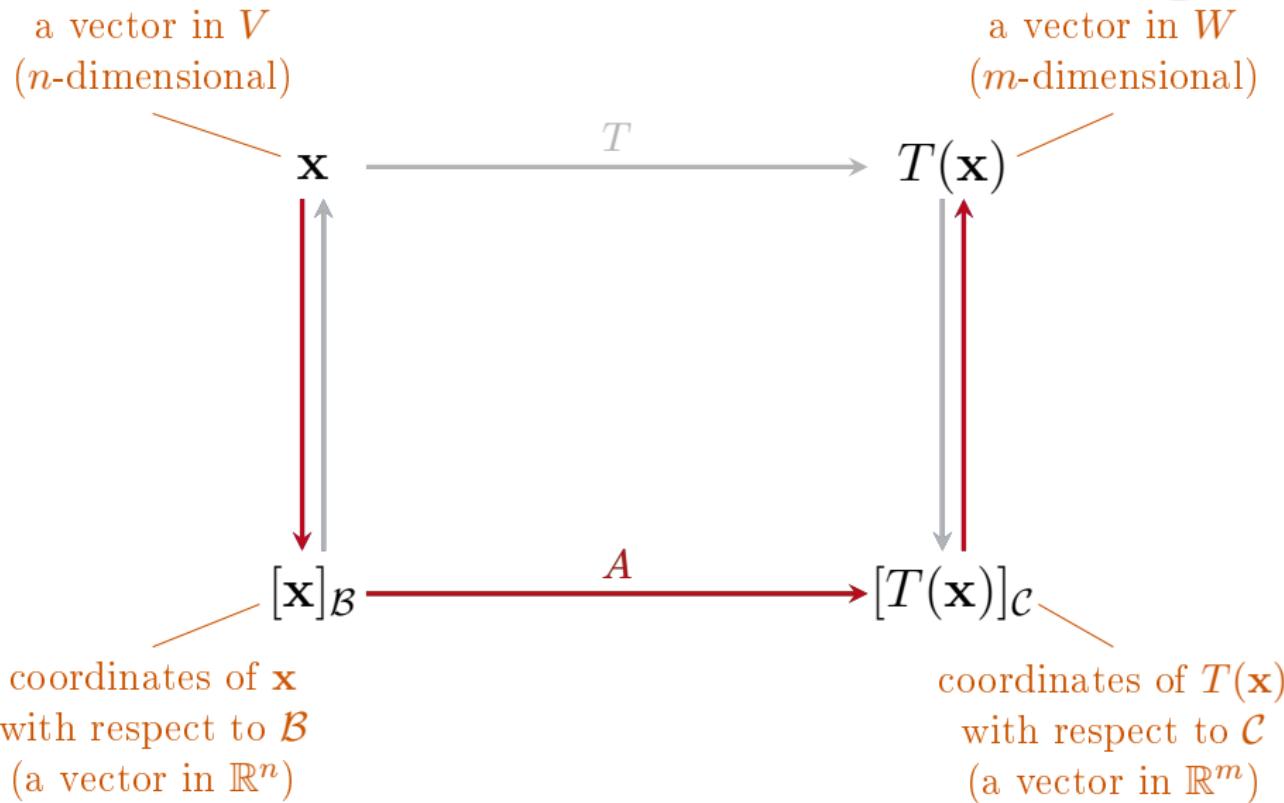
Matrices for Linear Transformations



Matrices for Linear Transformations



Matrices for Linear Transformations



Matrices for Linear Transformations



$$T : V \rightarrow W$$

We want to find an $m \times n$ matrix A such that

$$A[\mathbf{x}]_{\mathcal{B}} = [T(\mathbf{x})]_{\mathcal{C}}.$$

Matrices for Linear Transformations



$$T : V \rightarrow W$$

We want to find an $m \times n$ matrix A such that

$$A[\mathbf{x}]_{\mathcal{B}} = [T(\mathbf{x})]_{\mathcal{C}}.$$

Let

- $\mathcal{B} = \{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\}$ be a basis for the n -dimensional vector space V ; and
- $\mathcal{C} = \{\mathbf{w}_1, \mathbf{w}_2, \dots, \mathbf{w}_m\}$ be a basis for the m -dimensional vector space W .

(This is on page 307 of your textbook.)

$$A[\mathbf{x}]_{\mathcal{B}} = [T(\mathbf{x})]_{\mathcal{C}}$$



We need A to satisfy

$$A[\mathbf{v}_1]_{\mathcal{B}} = [T(\mathbf{v}_1)]_{\mathcal{C}}, \quad A[\mathbf{v}_2]_{\mathcal{B}} = [T(\mathbf{v}_2)]_{\mathcal{C}}, \quad \dots, \quad A[\mathbf{v}_n]_{\mathcal{B}} = [T(\mathbf{v}_n)]_{\mathcal{C}}.$$

$$A[\mathbf{x}]_{\mathcal{B}} = [T(\mathbf{x})]_{\mathcal{C}}$$



We need A to satisfy

$$A[\mathbf{v}_1]_{\mathcal{B}} = [T(\mathbf{v}_1)]_{\mathcal{C}}, \quad A[\mathbf{v}_2]_{\mathcal{B}} = [T(\mathbf{v}_2)]_{\mathcal{C}}, \quad \dots, \quad A[\mathbf{v}_n]_{\mathcal{B}} = [T(\mathbf{v}_n)]_{\mathcal{C}}.$$

But $\mathbf{v}_1 = 1\mathbf{v}_1 + 0\mathbf{v}_2 + 0\mathbf{v}_3 + \dots + 0\mathbf{v}_n$ so

$$[\mathbf{v}_1]_{\mathcal{B}} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

We need A to satisfy

$$A[\mathbf{v}_1]_{\mathcal{B}} = [T(\mathbf{v}_1)]_{\mathcal{C}}, \quad A[\mathbf{v}_2]_{\mathcal{B}} = [T(\mathbf{v}_2)]_{\mathcal{C}}, \quad \dots, \quad A[\mathbf{v}_n]_{\mathcal{B}} = [T(\mathbf{v}_n)]_{\mathcal{C}}.$$

But $\mathbf{v}_1 = 1\mathbf{v}_1 + 0\mathbf{v}_2 + 0\mathbf{v}_3 + \dots + 0\mathbf{v}_n$ so

$$[\mathbf{v}_1]_{\mathcal{B}} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

which implies that

$$[T(\mathbf{v}_1)]_{\mathcal{C}} = A[\mathbf{v}_1]_{\mathcal{B}} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} =$$

We need A to satisfy

$$A[\mathbf{v}_1]_{\mathcal{B}} = [T(\mathbf{v}_1)]_{\mathcal{C}}, \quad A[\mathbf{v}_2]_{\mathcal{B}} = [T(\mathbf{v}_2)]_{\mathcal{C}}, \quad \dots, \quad A[\mathbf{v}_n]_{\mathcal{B}} = [T(\mathbf{v}_n)]_{\mathcal{C}}.$$

But $\mathbf{v}_1 = 1\mathbf{v}_1 + 0\mathbf{v}_2 + 0\mathbf{v}_3 + \dots + 0\mathbf{v}_n$ so

$$[\mathbf{v}_1]_{\mathcal{B}} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

which implies that

$$[T(\mathbf{v}_1)]_{\mathcal{C}} = A[\mathbf{v}_1]_{\mathcal{B}} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} = \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix}.$$

$$A[\mathbf{x}]_{\mathcal{B}} = [T(\mathbf{x})]_{\mathcal{C}}$$



Similarly

$$[T(\mathbf{v}_2)]_{\mathcal{C}} = A[\mathbf{v}_2]_{\mathcal{B}} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} = \begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{bmatrix}.$$

⋮

$$[T(\mathbf{v}_n)]_{\mathcal{C}} = A[\mathbf{v}_n]_{\mathcal{B}} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} = \begin{bmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{bmatrix}.$$

$$A[\mathbf{x}]_{\mathcal{B}} = [T(\mathbf{x})]_{\mathcal{C}}$$



So the $m \times n$ matrix that we want is

$$A = \begin{bmatrix} [T(\mathbf{v}_1)]_{\mathcal{C}} & [T(\mathbf{v}_2)]_{\mathcal{C}} & \cdots & [T(\mathbf{v}_n)]_{\mathcal{C}} \end{bmatrix}.$$

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So the $m \times n$ matrix that we want is

$$A = \begin{bmatrix} [T(\mathbf{v}_1)]_{\mathcal{C}} & [T(\mathbf{v}_2)]_{\mathcal{C}} & \cdots & [T(\mathbf{v}_n)]_{\mathcal{C}} \end{bmatrix}.$$

Definition

This matrix is called the *matrix for T relative to the bases \mathcal{B} and \mathcal{C}* and is denoted by $[T]_{\mathcal{C}, \mathcal{B}}$.

Matrices for

$$[T]_{\mathcal{C}, \mathcal{B}} = [[T(\mathbf{v}_1)]_{\mathcal{C}} \quad [T(\mathbf{v}_2)]_{\mathcal{C}} \quad \cdots \quad [T(\mathbf{v}_n)]_{\mathcal{C}}]$$



Remark

The linear transformation

$$\begin{array}{ccc} T : & V & \rightarrow W \\ & \text{basis } \mathcal{B} & \text{basis } \mathcal{C} \end{array}$$

has matrix

$$[T]_{\mathcal{C}, \mathcal{B}}.$$

Matrices for

$$[T]_{\mathcal{C}, \mathcal{B}} = [[T(\mathbf{v}_1)]_{\mathcal{C}} \quad [T(\mathbf{v}_2)]_{\mathcal{C}} \quad \cdots \quad [T(\mathbf{v}_n)]_{\mathcal{C}}]$$

Remark

The linear transformation

$$\begin{array}{ccc} T : & V & \rightarrow W \\ & \text{basis } \mathcal{B} & \text{basis } \mathcal{C} \end{array}$$

has matrix

$$[T]_{\mathcal{C}, \mathcal{B}}.$$

We will use the formula

$$[T]_{\mathcal{C}, \mathcal{B}}[\mathbf{x}]_{\mathcal{B}} = [T(\mathbf{x})]_{\mathcal{C}}.$$

$$[T]_{\mathcal{C}, \mathcal{B}}[\mathbf{x}]_{\mathcal{B}} = [T(\mathbf{x})]_{\mathcal{C}}, \quad [T]_{\mathcal{C}, \mathcal{B}} = \begin{bmatrix} [T(\mathbf{v}_1)]_{\mathcal{C}} & [T(\mathbf{v}_2)]_{\mathcal{C}} & \cdots & [T(\mathbf{v}_n)]_{\mathcal{C}} \end{bmatrix}$$

Example

I leave it for you to prove that if $T_A : \mathbb{R}^n \rightarrow \mathbb{R}^m$ is a matrix transformation ($T_A(\mathbf{x}) = A\mathbf{x}$) and if \mathcal{B} and \mathcal{C} are the standard bases, then

$$[T_A]_{\mathcal{C}, \mathcal{B}} = A.$$

$$[T]_{\mathcal{C}, \mathcal{B}}[\mathbf{x}]_{\mathcal{B}} = [T(\mathbf{x})]_{\mathcal{C}}, \quad [T]_{\mathcal{C}, \mathcal{B}} = \begin{bmatrix} [T(\mathbf{v}_1)]_{\mathcal{C}} & [T(\mathbf{v}_2)]_{\mathcal{C}} & \cdots & [T(\mathbf{v}_n)]_{\mathcal{C}} \end{bmatrix}$$

Example

I leave it for you to prove that if $T_A : \mathbb{R}^n \rightarrow \mathbb{R}^m$ is a matrix transformation ($T_A(\mathbf{x}) = A\mathbf{x}$) and if \mathcal{B} and \mathcal{C} are the standard bases, then

$$[T_A]_{\mathcal{C}, \mathcal{B}} = A.$$

Remark

If we are using the standard bases on both V and W , then we can write the matrix for T as just

$$[T].$$

This is called the *standard matrix* for T .

$$[T]_{\mathcal{C}, \mathcal{B}}[\mathbf{x}]_{\mathcal{B}} = [T(\mathbf{x})]_{\mathcal{C}}, \quad [T]_{\mathcal{C}, \mathcal{B}} = \begin{bmatrix} [T(\mathbf{v}_1)]_{\mathcal{C}} & [T(\mathbf{v}_2)]_{\mathcal{C}} & \cdots & [T(\mathbf{v}_n)]_{\mathcal{C}} \end{bmatrix}$$

Example

Let $T : \mathbb{P}^1 \rightarrow \mathbb{P}^2$ be the linear transformation defined by

$$T(\mathbf{p}) = x\mathbf{p}.$$

(For example, if $\mathbf{p} = 1 + x$, then $T(\mathbf{p}) = x + x^2$.)

Find the matrix for T with respect to the bases

$$\mathcal{B} = \{\mathbf{v}_1 = 1, \mathbf{v}_2 = x\} \quad \text{and} \quad \mathcal{C} = \{\mathbf{w}_1 = 1, \mathbf{w}_2 = x, \mathbf{w}_3 = x^2\}.$$

$$[T]_{\mathcal{C}, \mathcal{B}}[\mathbf{x}]_{\mathcal{B}} = [T(\mathbf{x})]_{\mathcal{C}}, \quad [T]_{\mathcal{C}, \mathcal{B}} = \begin{bmatrix} [T(\mathbf{v}_1)]_{\mathcal{C}} & [T(\mathbf{v}_2)]_{\mathcal{C}} & \cdots & [T(\mathbf{v}_n)]_{\mathcal{C}} \end{bmatrix}$$

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Find the matrix for T with respect to the bases

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First we calculate that

$$T(\mathbf{v}_1) = T(1) = (x)(1) = x = 0\mathbf{w}_1 + 1\mathbf{w}_2 + 0\mathbf{w}_3$$

$$T(\mathbf{v}_2) = T(x) = (x)(x) = x^2 = 0\mathbf{w}_1 + 0\mathbf{w}_2 + 1\mathbf{w}_3.$$

$$[T]_{\mathcal{C}, \mathcal{B}}[\mathbf{x}]_{\mathcal{B}} = [T(\mathbf{x})]_{\mathcal{C}}, \quad [T]_{\mathcal{C}, \mathcal{B}} = \begin{bmatrix} [T(\mathbf{v}_1)]_{\mathcal{C}} & [T(\mathbf{v}_2)]_{\mathcal{C}} & \cdots & [T(\mathbf{v}_n)]_{\mathcal{C}} \end{bmatrix}$$

$$\begin{aligned} T(\mathbf{v}_1) &= T(1) = (x)(1) = x = 0\mathbf{w}_1 + 1\mathbf{w}_2 + 0\mathbf{w}_3 \\ T(\mathbf{v}_2) &= T(x) = (x)(x) = x^2 = 0\mathbf{w}_1 + 0\mathbf{w}_2 + 1\mathbf{w}_3. \end{aligned}$$

Therefore the coordinates of $T(\mathbf{v}_1)$ and $T(\mathbf{v}_2)$ with respect to \mathcal{C} are

$$[T(\mathbf{v}_1)]_{\mathcal{C}} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \quad \text{and} \quad [T(\mathbf{v}_2)]_{\mathcal{C}} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

¹with respect to

$$[T]_{\mathcal{C}, \mathcal{B}}[\mathbf{x}]_{\mathcal{B}} = [T(\mathbf{x})]_{\mathcal{C}}, \quad [T]_{\mathcal{C}, \mathcal{B}} = \begin{bmatrix} [T(\mathbf{v}_1)]_{\mathcal{C}} & [T(\mathbf{v}_2)]_{\mathcal{C}} & \cdots & [T(\mathbf{v}_n)]_{\mathcal{C}} \end{bmatrix}$$

$$\begin{aligned} T(\mathbf{v}_1) &= T(1) = (x)(1) = x = 0\mathbf{w}_1 + 1\mathbf{w}_2 + 0\mathbf{w}_3 \\ T(\mathbf{v}_2) &= T(x) = (x)(x) = x^2 = 0\mathbf{w}_1 + 0\mathbf{w}_2 + 1\mathbf{w}_3. \end{aligned}$$

Therefore the coordinates of $T(\mathbf{v}_1)$ and $T(\mathbf{v}_2)$ with respect to \mathcal{C} are

$$[T(\mathbf{v}_1)]_{\mathcal{C}} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \quad \text{and} \quad [T(\mathbf{v}_2)]_{\mathcal{C}} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

Hence the matrix for T wrt¹ \mathcal{B} and \mathcal{C} is

$$[T]_{\mathcal{C}, \mathcal{B}} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

¹with respect to

$$[T]_{\mathcal{C}, \mathcal{B}}[\mathbf{x}]_{\mathcal{B}} = [T(\mathbf{x})]_{\mathcal{C}}, \quad [T]_{\mathcal{C}, \mathcal{B}} = \begin{bmatrix} [T(\mathbf{v}_1)]_{\mathcal{C}} & [T(\mathbf{v}_2)]_{\mathcal{C}} & \cdots & [T(\mathbf{v}_n)]_{\mathcal{C}} \end{bmatrix}$$

Example

Let $T : \mathbb{R}^2 \rightarrow \mathbb{R}^3$ be the linear transformation defined by

$$T \left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \right) = \begin{bmatrix} x_2 \\ -5x_1 + 13x_2 \\ -7x_1 + 16x_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -5 & 13 \\ -7 & 16 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}.$$

Find the matrix for T wrt the bases $\mathcal{B} = \{\mathbf{v}_1, \mathbf{v}_2\}$ for \mathbb{R}^2 , and $\mathcal{C} = \{\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3\}$ for \mathbb{R}^3 , where

$$\mathbf{v}_1 = \begin{bmatrix} 3 \\ 1 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 5 \\ 2 \end{bmatrix}, \quad \mathbf{w}_1 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \quad \mathbf{w}_2 = \begin{bmatrix} -1 \\ 2 \\ 2 \end{bmatrix}, \quad \mathbf{w}_3 = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}.$$

$$[T]_{\mathcal{C}, \mathcal{B}}[\mathbf{x}]_{\mathcal{B}} = [T(\mathbf{x})]_{\mathcal{C}}, \quad [T]_{\mathcal{C}, \mathcal{B}} = \begin{bmatrix} [T(\mathbf{v}_1)]_{\mathcal{C}} & [T(\mathbf{v}_2)]_{\mathcal{C}} & \cdots & [T(\mathbf{v}_n)]_{\mathcal{C}} \end{bmatrix}$$

$$T \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{bmatrix} x_2 \\ -5x_1 + 13x_2 \\ -7x_1 + 16x_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -5 & 13 \\ -7 & 16 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Note, if we were using the standard bases on \mathbb{R}^2 and \mathbb{R}^3 , then the answer would just be the standard matrix

$$[T] = \begin{bmatrix} 0 & 1 \\ -5 & 13 \\ -7 & 16 \end{bmatrix}.$$

But we are using different bases, so this is not the answer.

$$[T]_{\mathcal{C}, \mathcal{B}}[\mathbf{x}]_{\mathcal{B}} = [T(\mathbf{x})]_{\mathcal{C}}, \quad [T]_{\mathcal{C}, \mathcal{B}} = \begin{bmatrix} [T(\mathbf{v}_1)]_{\mathcal{C}} & [T(\mathbf{v}_2)]_{\mathcal{C}} & \cdots & [T(\mathbf{v}_n)]_{\mathcal{C}} \end{bmatrix}$$

$$T \left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \right) = \begin{bmatrix} x_2 \\ -5x_1 + 13x_2 \\ -7x_1 + 16x_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -5 & 13 \\ -7 & 16 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$\mathbf{u}_1 = \begin{bmatrix} 3 \\ 1 \end{bmatrix}, \quad \mathbf{u}_2 = \begin{bmatrix} 5 \\ 2 \end{bmatrix}, \quad \mathbf{w}_1 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \quad \mathbf{w}_2 = \begin{bmatrix} -1 \\ 2 \\ 2 \end{bmatrix}, \quad \mathbf{w}_3 = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}$$

We start by calculating

$$T(\mathbf{v}_1) = \begin{bmatrix} 0 & 1 \\ -5 & 13 \\ -7 & 16 \end{bmatrix} \begin{bmatrix} 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \\ -5 \end{bmatrix} = \mathbf{w}_1 - 2\mathbf{w}_3$$

$$T(\mathbf{v}_2) = \begin{bmatrix} 0 & 1 \\ -5 & 13 \\ -7 & 16 \end{bmatrix} \begin{bmatrix} 5 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ -3 \end{bmatrix} = 3\mathbf{w}_1 + \mathbf{w}_2 - \mathbf{w}_3$$

(please check).

$$[T]_{\mathcal{C}, \mathcal{B}}[\mathbf{x}]_{\mathcal{B}} = [T(\mathbf{x})]_{\mathcal{C}}, \quad [T]_{\mathcal{C}, \mathcal{B}} = \begin{bmatrix} [T(\mathbf{v}_1)]_{\mathcal{C}} & [T(\mathbf{v}_2)]_{\mathcal{C}} & \cdots & [T(\mathbf{v}_n)]_{\mathcal{C}} \end{bmatrix}$$

$$T(\mathbf{v}_1) = \mathbf{w}_1 - 2\mathbf{w}_3 \qquad \qquad T(\mathbf{v}_2) = 3\mathbf{w}_1 + \mathbf{w}_2 - \mathbf{w}_3$$

Therefore

$$[T(\mathbf{v}_1)]_{\mathcal{C}} = \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix} \quad \text{and} \quad [T(\mathbf{v}_2)]_{\mathcal{C}} = \begin{bmatrix} 3 \\ 1 \\ -1 \end{bmatrix}.$$

$$[T]_{\mathcal{C}, \mathcal{B}}[\mathbf{x}]_{\mathcal{B}} = [T(\mathbf{x})]_{\mathcal{C}}, \quad [T]_{\mathcal{C}, \mathcal{B}} = \begin{bmatrix} [T(\mathbf{v}_1)]_{\mathcal{C}} & [T(\mathbf{v}_2)]_{\mathcal{C}} & \cdots & [T(\mathbf{v}_n)]_{\mathcal{C}} \end{bmatrix}$$

$$T(\mathbf{v}_1) = \mathbf{w}_1 - 2\mathbf{w}_3 \qquad \qquad T(\mathbf{v}_2) = 3\mathbf{w}_1 + \mathbf{w}_2 - \mathbf{w}_3$$

Therefore

$$[T(\mathbf{v}_1)]_{\mathcal{C}} = \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix} \quad \text{and} \quad [T(\mathbf{v}_2)]_{\mathcal{C}} = \begin{bmatrix} 3 \\ 1 \\ -1 \end{bmatrix}.$$

Hence the matrix for T wrt \mathcal{B} and \mathcal{C} is

$$[T]_{\mathcal{C}, \mathcal{B}} = \begin{bmatrix} [T(\mathbf{v}_1)]_{\mathcal{C}} & [T(\mathbf{v}_2)]_{\mathcal{C}} \end{bmatrix} = \begin{bmatrix} 1 & 3 \\ 0 & 1 \\ -2 & -1 \end{bmatrix}.$$

Matrices for Linear Transformations



Remark

$$[T] = \begin{bmatrix} 0 & 1 \\ -5 & 13 \\ -7 & 16 \end{bmatrix} \quad \text{and} \quad [T]_{\mathcal{C}, \mathcal{B}} = \begin{bmatrix} 1 & 3 \\ 0 & 1 \\ -2 & -1 \end{bmatrix}$$

are two different matrices for the same linear transformation.

Linear Operators

Recall that if $V = W$, then the linear transformation $T : V \rightarrow V$ is called a linear operator.

(If we are using the same basis \mathcal{B} on both the domain and the target of T , then) instead of writing $[T]_{\mathcal{B}, \mathcal{B}}$ we just write

$$[T]_{\mathcal{B}}.$$

Linear Operators

Recall that if $V = W$, then the linear transformation $T : V \rightarrow V$ is called a linear operator.

(If we are using the same basis \mathcal{B} on both the domain and the target of T , then) instead of writing $[T]_{\mathcal{B}, \mathcal{B}}$ we just write

$$[T]_{\mathcal{B}}.$$

The formulae are then

$$[T]_{\mathcal{B}} = \begin{bmatrix} [T(\mathbf{v}_1)]_{\mathcal{B}} & [T(\mathbf{v}_2)]_{\mathcal{B}} & \cdots & [T(\mathbf{v}_n)]_{\mathcal{B}} \end{bmatrix}$$

and

$$[T]_{\mathcal{B}}[\mathbf{x}]_{\mathcal{B}} = [T(\mathbf{x})]_{\mathcal{B}}.$$

Identity Operators

Example

Let V be a finite dimensional vector space and let $\mathcal{B} = \{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\}$ be a basis for V . Consider the identity operator $I : V \rightarrow V$, $I(\mathbf{x}) = \mathbf{x}$.

Identity Operators

Example

Let V be a finite dimensional vector space and let $\mathcal{B} = \{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\}$ be a basis for V . Consider the identity operator $I : V \rightarrow V$, $I(\mathbf{x}) = \mathbf{x}$. Note that

$$I(\mathbf{v}_1) = \mathbf{v}_1, \quad I(\mathbf{v}_2) = \mathbf{v}_2, \quad \dots, \quad I(\mathbf{v}_n) = \mathbf{v}_n.$$

Identity Operators

Example

Let V be a finite dimensional vector space and let $\mathcal{B} = \{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\}$ be a basis for V . Consider the identity operator $I : V \rightarrow V$, $I(\mathbf{x}) = \mathbf{x}$. Note that

$$I(\mathbf{v}_1) = \mathbf{v}_1, \quad I(\mathbf{v}_2) = \mathbf{v}_2, \quad \dots, \quad I(\mathbf{v}_n) = \mathbf{v}_n.$$

Hence

$$[I]_{\mathcal{B}} = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix} = I_n.$$

Matrices for Linear Transformations



Example

Let $T : \mathbb{P}^2 \rightarrow \mathbb{P}^2$ be the linear operator defined by

$$T(\mathbf{p}) = \mathbf{p}(3x - 5).$$

For example, if $\mathbf{p} = 2x + x^2$, then

$$T(\mathbf{p}) = \mathbf{p}(3x - 5) = 2(3x - 5) + (3x - 5)^2.$$

Matrices for Linear Transformations



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- 1 Find $[T]_{\mathcal{B}}$ where $\mathcal{B} = \{1, x, x^2\}$.
- 2 Use this to calculate $T(1 + 2x + 3x^2)$.

Matrices for Linear Transformations



1 Since

$$T(1) = 1, \quad T(x) = (3x-5), \quad T(x^2) = (3x-5)^2 = 9x^2 - 30x + 25$$

we have

$$[T(1)]_{\mathcal{B}} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad [T(x)]_{\mathcal{B}} = \begin{bmatrix} -5 \\ 3 \\ 0 \end{bmatrix}, \quad [T(x^2)]_{\mathcal{B}} = \begin{bmatrix} 25 \\ -30 \\ 9 \end{bmatrix}.$$

Matrices for Linear Transformations



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Hence

$$[T]_{\mathcal{B}} = \begin{bmatrix} 1 & -5 & 25 \\ 0 & 3 & -30 \\ 0 & 0 & 9 \end{bmatrix}.$$

$$[T]_{\mathcal{B}} = \begin{bmatrix} 1 & -5 & 25 \\ 0 & 3 & -30 \\ 0 & 0 & 9 \end{bmatrix}$$



- 2 The vector $\mathbf{p} = 1 + 2x + 3x^2$ has coordinate vector

$$[\mathbf{p}]_{\mathcal{B}} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}.$$

$$[T]_{\mathcal{B}} = \begin{bmatrix} 1 & -5 & 25 \\ 0 & 3 & -30 \\ 0 & 0 & 9 \end{bmatrix}$$



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$$[\mathbf{p}]_{\mathcal{B}} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}.$$

Thus

$$[T(\mathbf{p})]_{\mathcal{B}} = [T]_{\mathcal{B}} [\mathbf{p}]_{\mathcal{B}} = \begin{bmatrix} 1 & -5 & 25 \\ 0 & 3 & -30 \\ 0 & 0 & 9 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 66 \\ -84 \\ 27 \end{bmatrix}.$$

Matrices for Linear Transformations

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Therefore

$$T(1 + 2x + 3x^2) = 66 - 84x + 27x^2.$$

Compositions and Inverses

Theorem

If $T_1 : U \rightarrow V$ and $T_2 : V \rightarrow W$ are linear transformations, and if \mathcal{A} , \mathcal{B} and \mathcal{C} are bases for U , V , and W , respectively, then

$$[T_2 \circ T_1]_{\mathcal{C}, \mathcal{A}} = [T_2]_{\mathcal{C}, \mathcal{B}} [T_1]_{\mathcal{B}, \mathcal{A}}.$$

Theorem

If $T : V \rightarrow V$ is a linear operator, and if \mathcal{B} is a basis for V , then the following are equivalent:

- 1 T is one-to-one.
- 2 $[T]_{\mathcal{B}}$ is invertible.

Theorem

If $T : V \rightarrow V$ is a linear operator, and if \mathcal{B} is a basis for V , then the following are equivalent:

- 1 T is one-to-one.
- 2 $[T]_{\mathcal{B}}$ is invertible.

Moreover, when these equivalent conditions hold,

$$[T^{-1}]_{\mathcal{B}} = [T]_{\mathcal{B}}^{-1}$$

Matrices for Linear Transformations



Example (Composition)

Let $T_1 : \mathbb{P}^1 \rightarrow \mathbb{P}^2$ and $T_2 : \mathbb{P}^2 \rightarrow \mathbb{P}^2$ be defined by

$$T_1(\mathbf{p}) = x\mathbf{p} \quad \text{and} \quad T_2(\mathbf{p}) = \mathbf{p}(3x - 5).$$

Find the matrix of $T_2 \circ T_1$ wrt to the standard bases on \mathbb{P}^1 and \mathbb{P}^2 .

Let

$$\mathcal{B} = \{1, x\} \quad \text{and} \quad \mathcal{C} = \{1, x, x^2\}.$$

Matrices for Linear Transformations



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Let $T_1 : \mathbb{P}^1 \rightarrow \mathbb{P}^2$ and $T_2 : \mathbb{P}^2 \rightarrow \mathbb{P}^2$ be defined by

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Find the matrix of $T_2 \circ T_1$ wrt to the standard bases on \mathbb{P}^1 and \mathbb{P}^2 .

Let

$$\mathcal{B} = \{1, x\} \quad \text{and} \quad \mathcal{C} = \{1, x, x^2\}.$$

We have seen today that

$$[T_1]_{\mathcal{C}, \mathcal{B}} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad [T_2]_{\mathcal{C}} = \begin{bmatrix} 1 & -5 & 25 \\ 0 & 3 & -30 \\ 0 & 0 & 9 \end{bmatrix}.$$

Matrices for Linear Transformations



It follows that

$$[T_2 \circ T_1]_{\mathcal{C}, \mathcal{B}} = [T_2]_{\mathcal{C}} [T_1]_{\mathcal{C}, \mathcal{B}} = \begin{bmatrix} 1 & -5 & 25 \\ 0 & 3 & -30 \\ 0 & 0 & 9 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} -5 & 25 \\ 3 & -30 \\ 0 & 9 \end{bmatrix}$$



Break

We will continue at 3pm





Similarity

Similarity

Consider the matrix operator $T_A : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ where

$$A = \begin{bmatrix} 1 & 1 \\ -2 & 4 \end{bmatrix}.$$

The matrix for T_A relative to the standard basis $\mathcal{B} = \{\mathbf{e}_1, \mathbf{e}_2\}$ for \mathbb{R}^2 is

$$[T_A] = [T_A]_{\mathcal{B}} = A = \begin{bmatrix} 1 & 1 \\ -2 & 4 \end{bmatrix}.$$

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$$[T_A] = [T_A]_{\mathcal{B}} = A = \begin{bmatrix} 1 & 1 \\ -2 & 4 \end{bmatrix}.$$

I want to calculate the matrix for the same operator T_A relative to a different basis.

Similarity

$$A = \begin{bmatrix} 1 & 1 \\ -2 & 4 \end{bmatrix}$$



Namely, the basis $\mathcal{C} = \{\mathbf{v}_1, \mathbf{v}_2\}$ for \mathbb{R}^2 where

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad \text{and} \quad \mathbf{v}_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}.$$

Similarity

$$A = \begin{bmatrix} 1 & 1 \\ -2 & 4 \end{bmatrix}$$



Namely, the basis $\mathcal{C} = \{\mathbf{v}_1, \mathbf{v}_2\}$ for \mathbb{R}^2 where

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad \text{and} \quad \mathbf{v}_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}.$$

Since

$$T_A(\mathbf{v}_1) = A\mathbf{v}_1 = \begin{bmatrix} 1 & 1 \\ -2 & 4 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \end{bmatrix} = 2\mathbf{v}_1$$

and

$$T_A(\mathbf{v}_2) = A\mathbf{v}_2 = \begin{bmatrix} 1 & 1 \\ -2 & 4 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 3 \\ 6 \end{bmatrix} = 3\mathbf{v}_2$$

Similarity

$$A = \begin{bmatrix} 1 & 1 \\ -2 & 4 \end{bmatrix}$$



Namely, the basis $\mathcal{C} = \{\mathbf{v}_1, \mathbf{v}_2\}$ for \mathbb{R}^2 where

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and

$$T_A(\mathbf{v}_2) = A\mathbf{v}_2 = \begin{bmatrix} 1 & 1 \\ -2 & 4 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 3 \\ 6 \end{bmatrix} = 3\mathbf{v}_2$$

it follows that

$$[T_A(\mathbf{v}_1)]_{\mathcal{C}} = \begin{bmatrix} 2 \\ 0 \end{bmatrix} \quad \text{and} \quad [T_A(\mathbf{v}_2)]_{\mathcal{C}} = \begin{bmatrix} 0 \\ 3 \end{bmatrix}.$$

Therefore the matrix for T_A relative to the basis \mathcal{C} is

$$[T_A]_{\mathcal{C}} = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}.$$

Compare these two matrices

$$[T_A]_{\mathcal{B}} = \begin{bmatrix} 1 & 1 \\ -2 & 4 \end{bmatrix} \quad [T_A]_{\mathcal{C}} = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}.$$

- The determinant of both matrices is 6.
- Both matrices are invertible.
- The trace of both matrices is 5.
- The latter matrix is diagonal and diagonal matrices are easier to deal with.

Definition

If A and B are square matrices, then we say that B is *similar* to A iff there exists an invertible matrix P such that

$$B = P^{-1}AP.$$

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Suppose that $B = P^{-1}AP$. Let $Q = P^{-1}$. Then $B = QAQ^{-1}$ which rearranges to $A = Q^{-1}BQ$. Hence A is similar to B . \square

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If A and B are similar matrices, then

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$$\det(B) = \det(P^{-1}AP) = \det(P^{-1}) \det(A) \det(P)$$

=

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Theorem

If A and B are similar matrices, then

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Proof.

If $B = P^{-1}AP$, then

$$\begin{aligned}\det(B) &= \det(P^{-1}AP) = \det(P^{-1})\det(A)\det(P) \\ &= \frac{1}{\det(P)}\det(A)\det(P) = \det(A).\end{aligned}$$



Lemma

If A and B are square matrices of the same size, then

$$\text{tr}(AB) = \text{tr}(BA)$$

Similarity

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2n} \\ \vdots & \vdots & & \vdots \\ b_{n1} & b_{n2} & \cdots & b_{nn} \end{bmatrix} = \begin{bmatrix} (AB)_{11} & ? & \cdots & ? \\ ? & (AB)_{22} & \cdots & ? \\ \vdots & \vdots & & \vdots \\ ? & ? & \cdots & (AB)_{nn} \end{bmatrix}$$

Proof.

Recall that $\text{tr}(AB)$ is the sum of the entries on the main diagonal of AB . If A and B are $n \times n$ matrices, then

$$\begin{aligned} \text{tr}(AB) &= (AB)_{11} + (AB)_{22} + (AB)_{33} + \dots + (AB)_{nn} \\ &= \sum_{i=1}^n (AB)_{ii} \\ &= \end{aligned}$$

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Similarity

$$\text{tr}(AB) = \sum_{i=1}^n \sum_{j=1}^n a_{ij} b_{ji}$$



Proof continued.

Similarly we have that

$$\text{tr}(BA) = \sum_{i=1}^n (BA)_{ii} = \sum_{i=1}^n \sum_{j=1}^n b_{ij} a_{ji} = \sum_{i=1}^n \sum_{j=1}^n a_{ji} b_{ij}$$

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$$= \sum_{i=1}^n \sum_{\mathbf{k}=1}^n a_{\mathbf{k}i} b_{i\mathbf{k}} =$$

=



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$$\text{tr}(\textcolor{red}{X}\textcolor{blue}{Y}) = \text{tr}(\textcolor{blue}{Y}\textcolor{red}{X})$$



Theorem

If A and B are similar matrices, then

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A New View of Transition Matrices

Suppose that $\mathcal{B} = \{\mathbf{b}_1, \dots, \mathbf{b}_n\}$ and $\mathcal{C} = \{\mathbf{c}_1, \dots, \mathbf{c}_n\}$ are two bases for a vector space V . Recall that

- the *change-of-coordinates matrix* or *transition matrix* from \mathcal{B} to \mathcal{C} is the matrix

$$P_{\mathcal{C} \leftarrow \mathcal{B}} = \begin{bmatrix} [\mathbf{b}_1]_{\mathcal{C}} & [\mathbf{b}_2]_{\mathcal{C}} & \cdots & [\mathbf{b}_n]_{\mathcal{C}} \end{bmatrix}$$

Similarity

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- these two matrices are inverses of each other

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- these two matrices are inverses of each other

$$P_{\mathcal{C} \leftarrow \mathcal{B}} [\mathbf{v}]_{\mathcal{B}} = [\mathbf{v}]_{\mathcal{C}}$$

- the *change-of-coordinates matrix* or *transition matrix* from \mathcal{C} to \mathcal{B} is the matrix

$$P_{\mathcal{B} \leftarrow \mathcal{C}} = \begin{bmatrix} [\mathbf{c}_1]_{\mathcal{B}} & [\mathbf{c}_2]_{\mathcal{B}} & \cdots & [\mathbf{c}_n]_{\mathcal{B}} \end{bmatrix}$$

- $P_{\mathcal{B} \leftarrow \mathcal{C}} [\mathbf{v}]_{\mathcal{C}} = [\mathbf{v}]_{\mathcal{B}}$

Theorem

If $\mathcal{B} = \{\mathbf{b}_1, \dots, \mathbf{b}_n\}$ and $\mathcal{C} = \{\mathbf{c}_1, \dots, \mathbf{c}_n\}$ are bases for a finite-dimensional vector space V , and if $I : V \rightarrow V$ is the identity operator on V , then

$$P_{\mathcal{C} \leftarrow \mathcal{B}} = [I]_{\mathcal{C}, \mathcal{B}} \quad \text{and} \quad P_{\mathcal{B} \leftarrow \mathcal{C}} = [I]_{\mathcal{B}, \mathcal{C}}.$$

Similarity

Proof.

Since $I(\mathbf{v}) = \mathbf{v}$ for all $\mathbf{v} \in V$, we have that

$$[I]_{\mathcal{C}, \mathcal{B}} = \begin{bmatrix} [I(\mathbf{b}_1)]_{\mathcal{C}} & [I(\mathbf{b}_2)]_{\mathcal{C}} & \cdots & [I(\mathbf{b}_n)]_{\mathcal{C}} \end{bmatrix}$$

=

=

Similarity

Proof.

Since $I(\mathbf{v}) = \mathbf{v}$ for all $\mathbf{v} \in V$, we have that

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Since $I(\mathbf{v}) = \mathbf{v}$ for all $\mathbf{v} \in V$, we have that

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Similarly ${}_{\mathcal{B} \leftarrow \mathcal{C}}^P = [I]_{\mathcal{B}, \mathcal{C}}$.

□

The Effect of Changing Bases on Matrices of Linear Operators

Question: If \mathcal{B} and \mathcal{C} are two bases for a finite-dimensional vector space V , and if $T : V \rightarrow V$ is a linear operator, what relationship, if any, exists between the matrices $[T]_{\mathcal{B}}$ and $[T]_{\mathcal{C}}$?

Similarity



basis \mathcal{C}



basis \mathcal{C}



V

basis \mathcal{B}

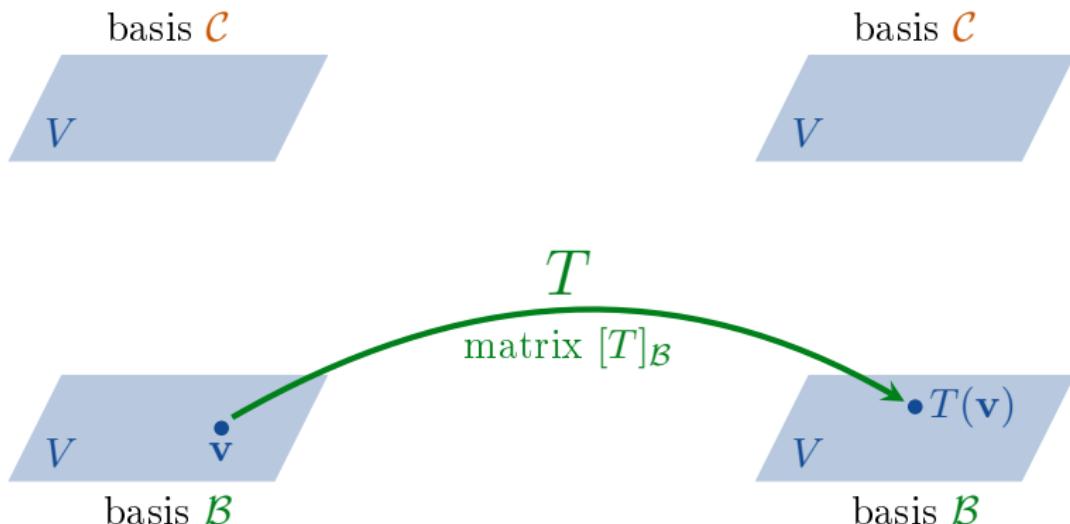


V

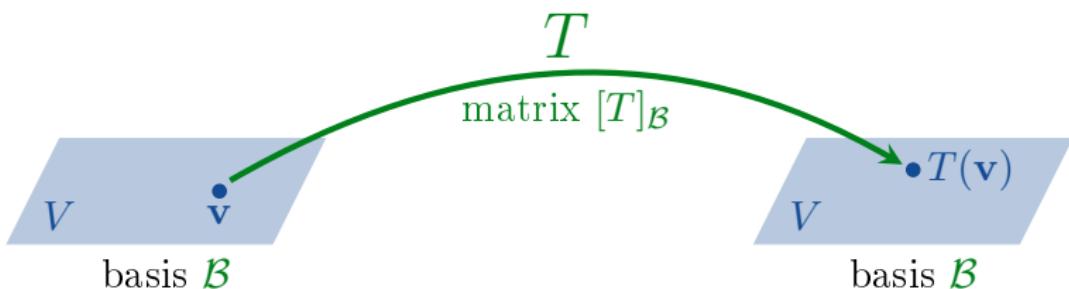
basis \mathcal{B}



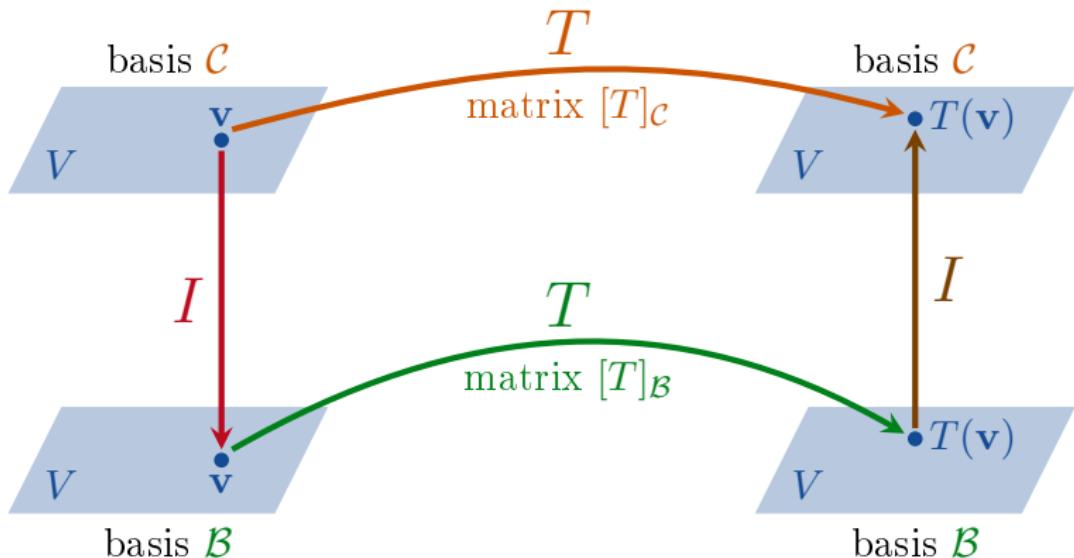
Similarity



Similarity

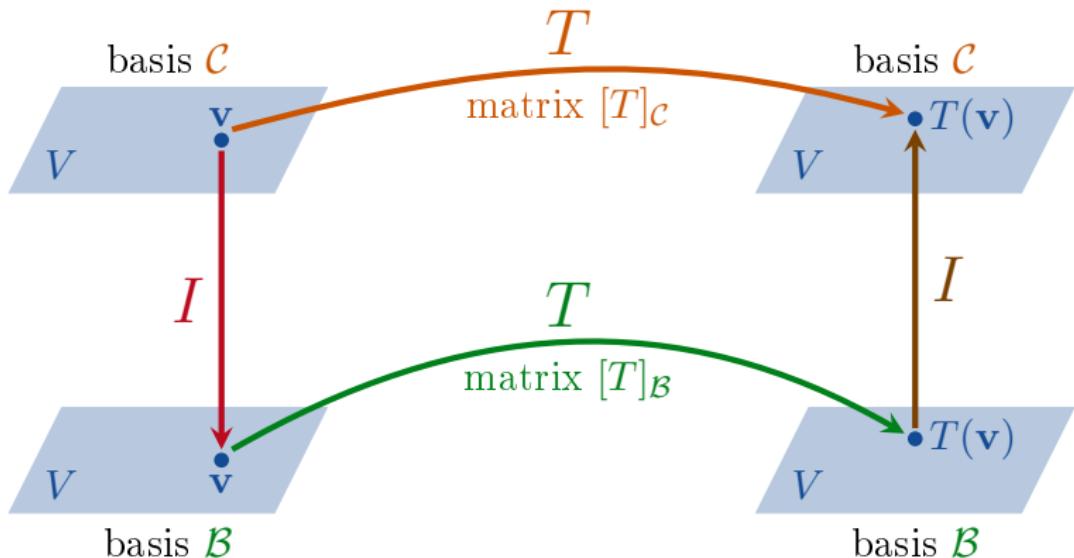


Similarity



Note that $T = I \circ T \circ I$.

Similarity



Note that $\textcolor{brown}{T} = \textcolor{red}{I} \circ \textcolor{green}{T} \circ \textcolor{red}{I}$. It follows that

$$[T]_{\mathcal{C}} = [\textcolor{red}{I} \circ \textcolor{green}{T} \circ \textcolor{red}{I}]_{\mathcal{C}} = [\textcolor{red}{I}]_{\mathcal{C}, \mathcal{B}} [T]_{\mathcal{B}} [\textcolor{red}{I}]_{\mathcal{B}, \mathcal{C}}.$$

Similarity

$$[T]_{\mathcal{C}} = [I \circ T \circ I]_{\mathcal{C}} = [I]_{\mathcal{C}, \mathcal{B}} [T]_{\mathcal{B}} [I]_{\mathcal{B}, \mathcal{C}}$$

Since

$$\underset{\mathcal{C} \leftarrow \mathcal{B}}{P} = [I]_{\mathcal{C}, \mathcal{B}} \quad \text{and} \quad \underset{\mathcal{B} \leftarrow \mathcal{C}}{P} = [I]_{\mathcal{B}, \mathcal{C}}$$

we have

Theorem

Let $T : V \rightarrow V$ be a linear operator on a finite-dimensional vector space V , and let \mathcal{B} and \mathcal{C} be bases for V . Then

$$[T]_{\mathcal{C}} = P^{-1} [T]_{\mathcal{B}} P$$

where $P = \underset{\mathcal{B} \leftarrow \mathcal{C}}{P}$ and $P^{-1} = \underset{\mathcal{C} \leftarrow \mathcal{B}}{P}$.

Remark

Therefore, two matrices representing the same linear operator, must be similar.

Similarity

Remark

Therefore, two matrices representing the same linear operator, must be similar.

Theorem

Two matrices A and B

*represent the same
linear operator*

$$\iff$$

A and B are similar.



Complex Numbers

The Natural Numbers

The set

$$\mathbb{N} = \{1, 2, 3, 4, 5, 6, \dots\}$$

is called the set of *natural numbers*. These are the first numbers that children learn. For example

$2 \in \mathbb{N}$ means “2 is a natural number”

$7 \in \mathbb{N}$ means “7 is a natural number”

$\frac{1}{2} \notin \mathbb{N}$ means “ $\frac{1}{2}$ is **not** a natural number”

$0 \notin \mathbb{N}$ means “0 is **not** a natural number”

$-5 \notin \mathbb{N}$ means “−5 is **not** a natural number”

Complex Numbers



In the natural numbers, we can do “+” and “ \times ”

$$2 + 7 = 9 \in \mathbb{N}, \quad 2 \times 7 = 14 \in \mathbb{N}.$$

However we can not do “−” because

$$2 - 7 \notin \mathbb{N}.$$

Complex Numbers



In the natural numbers, we can do “+” and “ \times ”

$$2 + 7 = 9 \in \mathbb{N}, \quad 2 \times 7 = 14 \in \mathbb{N}.$$

However we can not do “−” because

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So we invent new numbers!

The Integers

The set

$$\mathbb{Z} = \{\dots, -4, -3, -2, -1, 0, 1, 2, 3, 4, \dots\}$$

is called the set of *integers*. We use a \mathbb{Z} for the German word 'zahlen' (numbers).

The Integers

the natural
numbers “exist”
in the world

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zero and the negative numbers were invented by humans

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In \mathbb{Z} , we can do “+”, “−” and “ \times ” but we can not do “ \div ”. For example $3 \in \mathbb{Z}$, $4 \in \mathbb{Z}$, $-5 \in \mathbb{Z}$ and

$$3 + 4 \in \mathbb{Z}, \quad 3 - 4 \in \mathbb{Z}, \quad 3 \times 4 \in \mathbb{Z}, \quad 3 \div 4 \notin \mathbb{Z},$$

$$3 + (-5) \in \mathbb{Z}, \quad 3 - (-5) \in \mathbb{Z}, \quad 3 \times (-5) \in \mathbb{Z}, \quad 3 \div (-5) \notin \mathbb{Z}.$$

The Integers

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So we invent new numbers!

The Rational Numbers

The set

$$\mathbb{Q} = \{\text{all fractions}\} = \left\{ \frac{a}{b} \mid a, b \in \mathbb{Z} \text{ and } b \neq 0 \right\}$$

is called the set of *rational numbers*. We use a \mathbb{Q} for the word ‘quotient’.

The Rational Numbers

The set

$$\mathbb{Q} = \{\text{all fractions}\} = \left\{ \frac{a}{b} \mid a, b \in \mathbb{Z} \text{ and } b \neq 0 \right\}$$

is called the set of *rational numbers*. We use a \mathbb{Q} for the word ‘quotient’. For example

$$0 = \frac{0}{1} \in \mathbb{Q}$$

$$\frac{100}{13} \in \mathbb{Q}$$

$$1 = \frac{1}{1} \in \mathbb{Q}$$

$$\sqrt{2} \notin \mathbb{Q}$$

$$\frac{3}{4} \in \mathbb{Q}$$

$$-4 = \frac{8}{-2} \in \mathbb{Q}$$

$$\pi \notin \mathbb{Q}$$

$$0.12345 = \frac{12345}{100000} \in \mathbb{Q}.$$

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In \mathbb{Q} we can do “+”, “-”, “ \times ” and “ \div (by a number $\neq 0$)”.

Complex Numbers



Are we happy now?

Complex Numbers



Are we happy now?

No!



Are we happy now?

No!

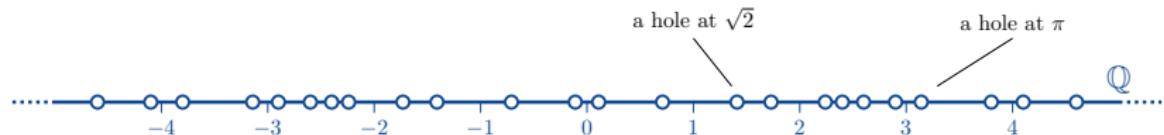
Why?

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Why?

Because if we draw all the rational numbers in a line, then the line has lots of holes in it. In fact, \mathbb{Q} has ∞ many holes in it.

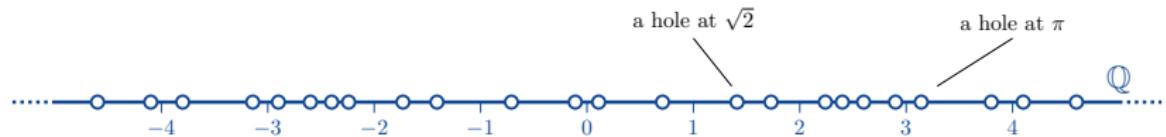


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So we invent new numbers!

The Real Numbers

The set

$$\mathbb{R} = \{\text{all numbers which can be written as a decimal}\}$$

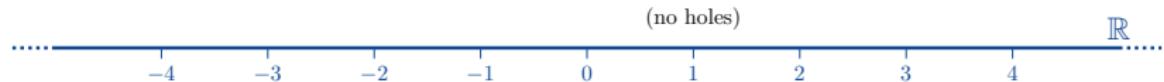
is called the set of *real numbers*. For example

$$\begin{array}{ll} 0 = 0.0 \in \mathbb{R} & \frac{100}{13} = 7.692307\dots \in \mathbb{R} \\ \frac{23}{99} = 0.232323\dots \in \mathbb{R} & \sqrt{2} = 1.414213\dots \in \mathbb{R} \\ \frac{3}{4} = 0.75 \in \mathbb{R} & \frac{123}{999} = 0.123123\dots \in \mathbb{R} \\ \pi = 3.141592\dots \in \mathbb{R} & \frac{12345}{100000} = 0.12345 \in \mathbb{R}. \end{array}$$

Complex Numbers



The real numbers are complete – this means that if we draw all the real numbers in a line, then there are no holes in the line.



Complex Numbers



Are we happy now?

Complex Numbers



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No!

Are we happy now?

No!

Why?

Are we happy now?

No!

Why?

Because not every polynomial has a root in \mathbb{R} . For example,
there is no solution to

$$x^2 + 1 = 0$$

in \mathbb{R} .

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Complex Numbers

We invent a new number called i which we say is the solution to

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Complex Numbers

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Or

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Complex Numbers



i is a made up number. It was invented by humans.

Complex Numbers



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But so are 0 and -3 and $\frac{7}{8}$ and π . Zero, negative numbers, fractions and irrational numbers are all inventions of humans.

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i is also useful to us.

Complex Numbers



Now that we have i , we can start working with numbers like $3 + 4i$, $2 + 7i$, $-12i$, etc.

Complex Numbers



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Definition

A *complex number* is any number that can be written as

$$a + bi$$

where i solves $i^2 + 1 = 0$ and $a, b \in \mathbb{R}$.

Definition

The set of all complex numbers is denoted by \mathbb{C} .

Complex Numbers



$$z = \cancel{a} + \cancel{bi}$$

real part imaginary part

Definition

$\operatorname{Re}(z) = a$ is called the *real* part of z , and $\operatorname{Im}(z) = b$ is called the *imaginary* part of z .

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Example

- $-2 + 7i$ has real part -2 and imaginary part 7 ;
- $7i$ has real part 0 and imaginary part 7 ; and
- -2 has real part -2 and imaginary part 0 .

Complex Numbers

Definition

An *imaginary number* is a complex number $a + bi$ where $a = 0$.

A *real number* is a complex number $a + bi$ where $b = 0$.

$$4 + 2i \quad -3 - \sqrt{5}i \quad 1 - i$$

$$3 \quad \sqrt{5} \quad -12.2$$

$$-12.2i \quad \sqrt{2}i \quad 7i$$

Complex Numbers



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complex numbers

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real numbers

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imaginary numbers

$-12.2i$ $\sqrt{2}i$ $7i$

Complex Numbers

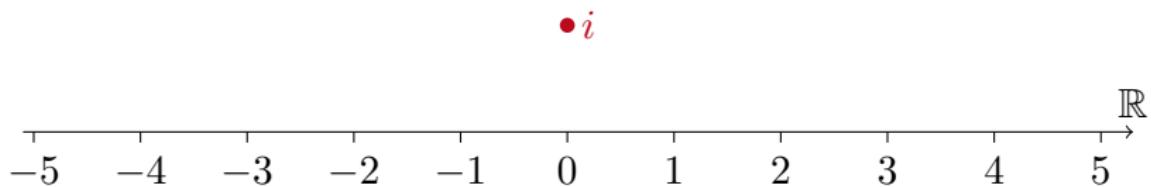


Remark

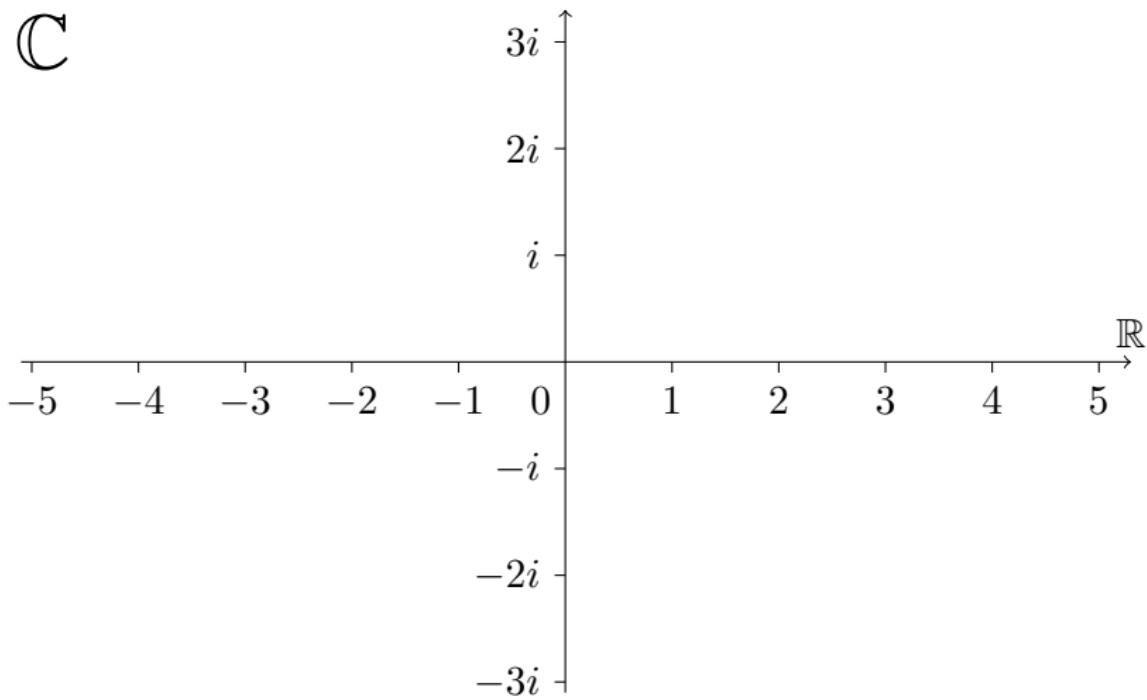
The names “real number” and “imaginary number” come from the 17th century.

These are really bad names because real numbers are just as imaginary, or made up, as imaginary numbers.

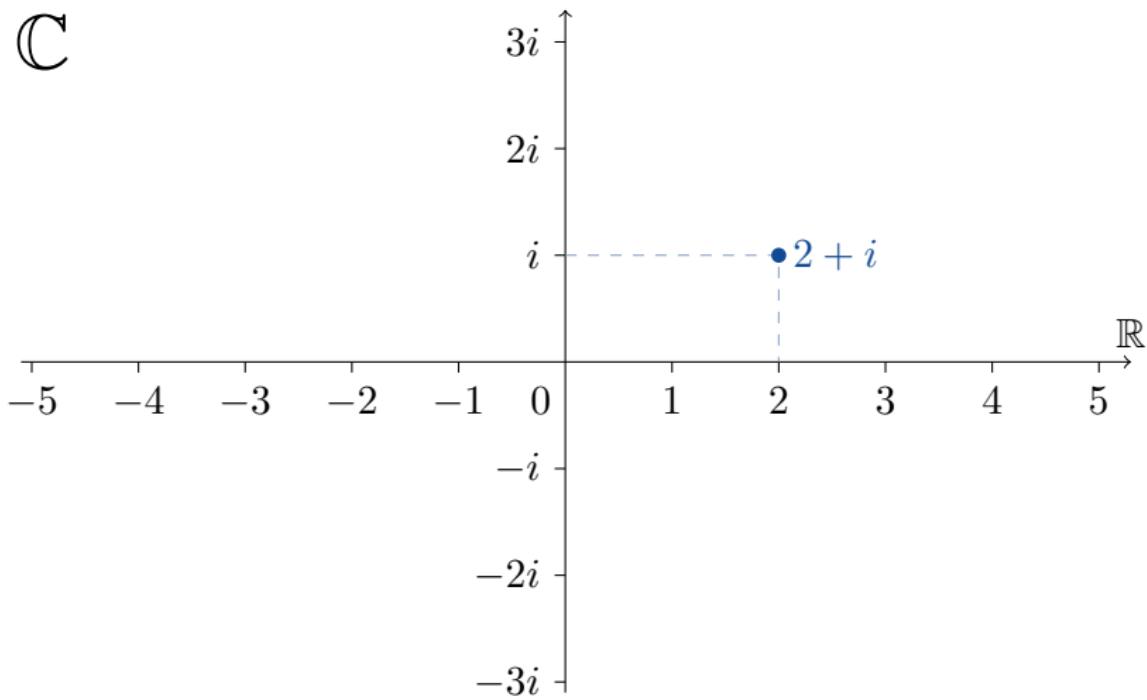
Complex Numbers



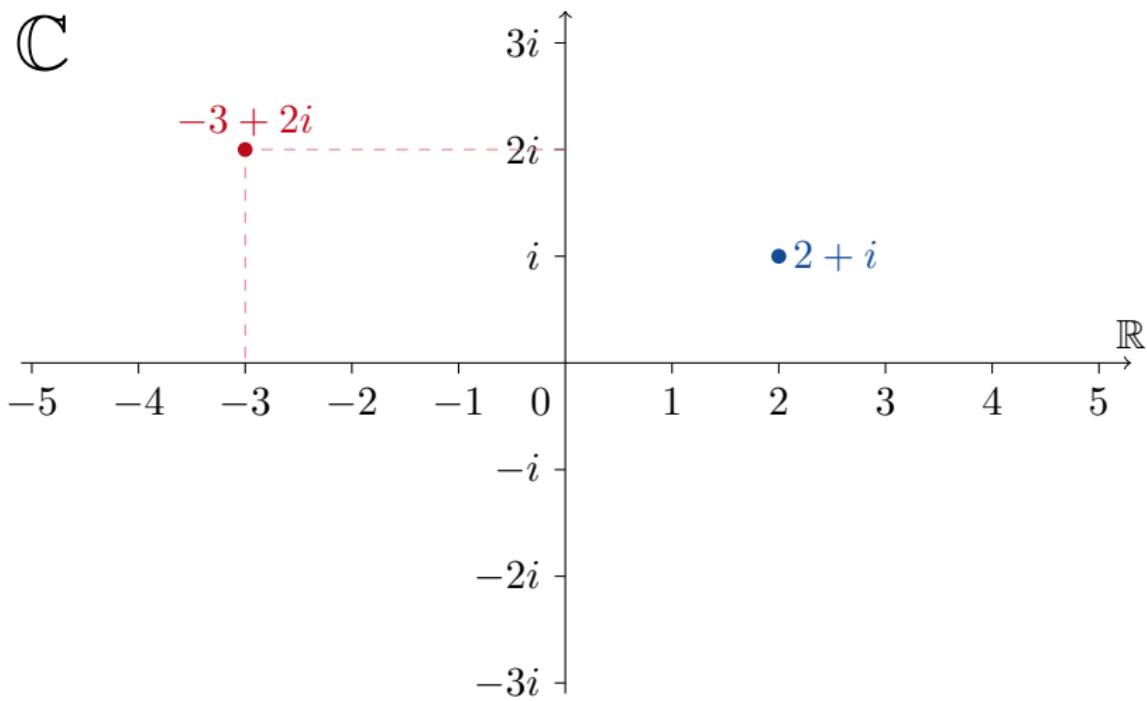
Complex Numbers



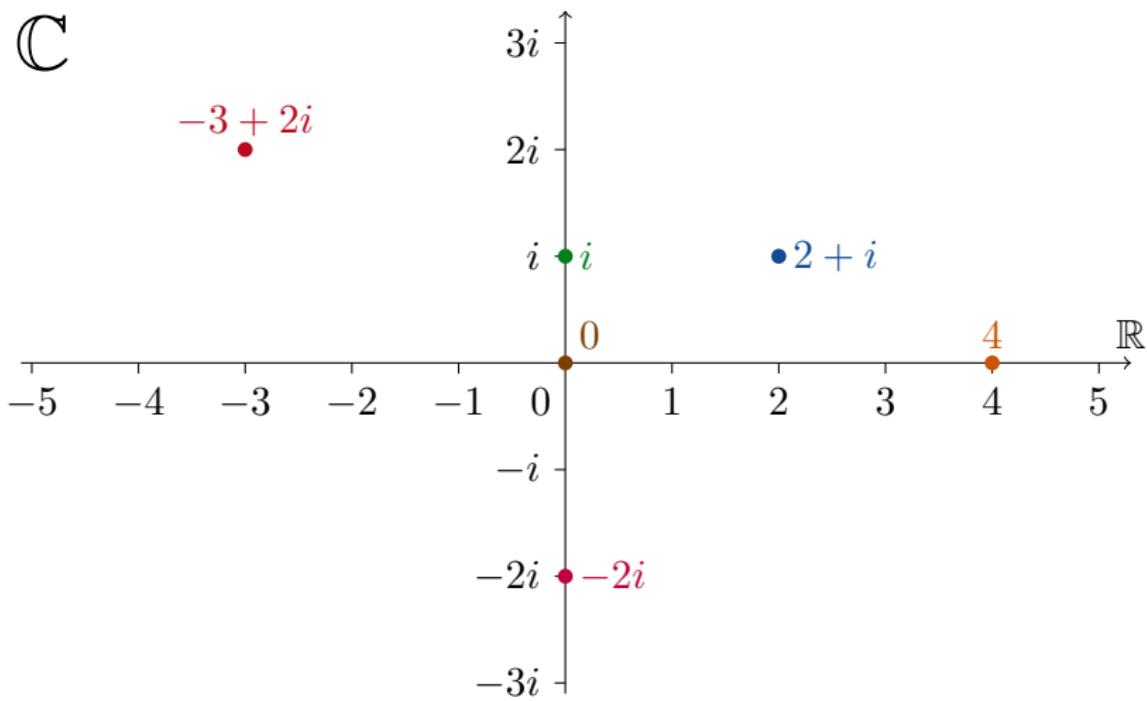
Complex Numbers



Complex Numbers



Complex Numbers



Complex Numbers



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Complex Numbers



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Complex Numbers



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$$\begin{aligned}(2 - i)(3 + 4i) &= (2)(3) + (2)(4i) + (-i)(3) + (-i)(4i) \\&= 6 + 8i - 3i - 4i^2 \\&= 6 + 8i - 3i - 4(-1) \\&= 6 + 5i + 4 \\&= 10 + 5i\end{aligned}$$

Complex Numbers



$$\frac{3-i}{2+i} =$$

Complex Numbers



$$\frac{3-i}{2+i} = \frac{3-i}{2+i} \cdot \frac{\textcolor{brown}{2-i}}{\textcolor{brown}{2-i}}$$

Complex Numbers



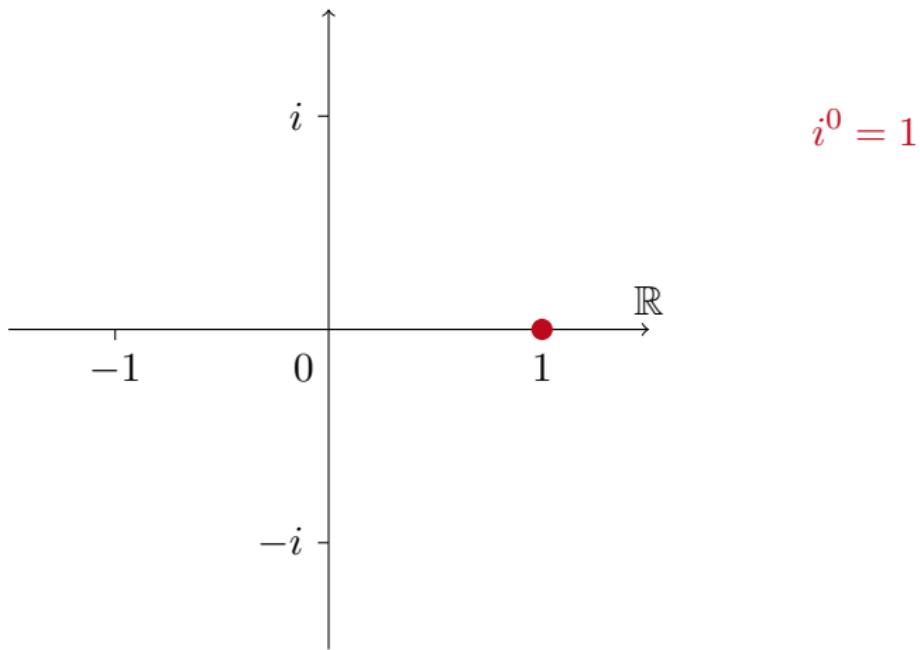
$$\frac{3-i}{2+i} = \frac{3-i}{2+i} \cdot \frac{\textcolor{brown}{2-i}}{\textcolor{brown}{2-i}} = \frac{6-3i-2i-1}{4-2i+2i+1}$$

Complex Numbers

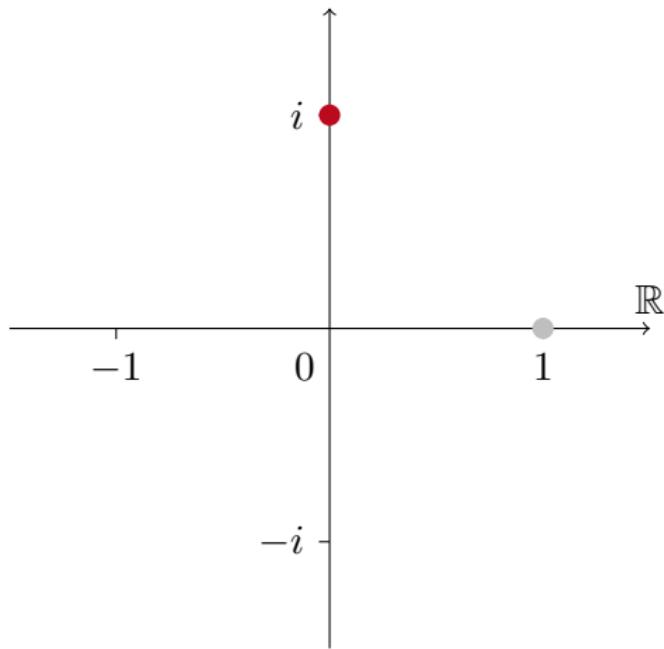


$$\frac{3-i}{2+i} = \frac{3-i}{2+i} \cdot \frac{\textcolor{brown}{2-i}}{\textcolor{brown}{2-i}} = \frac{6-3i-2i-1}{4-2i+2i+1} = \frac{5-5i}{5} = 1-i.$$

Complex Numbers



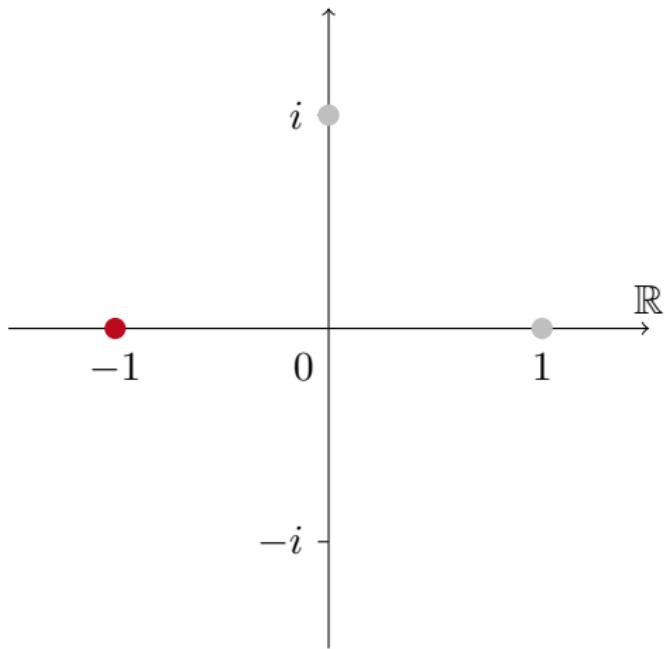
Complex Numbers



$$i^0 = 1$$

$$i^1 = i$$

Complex Numbers

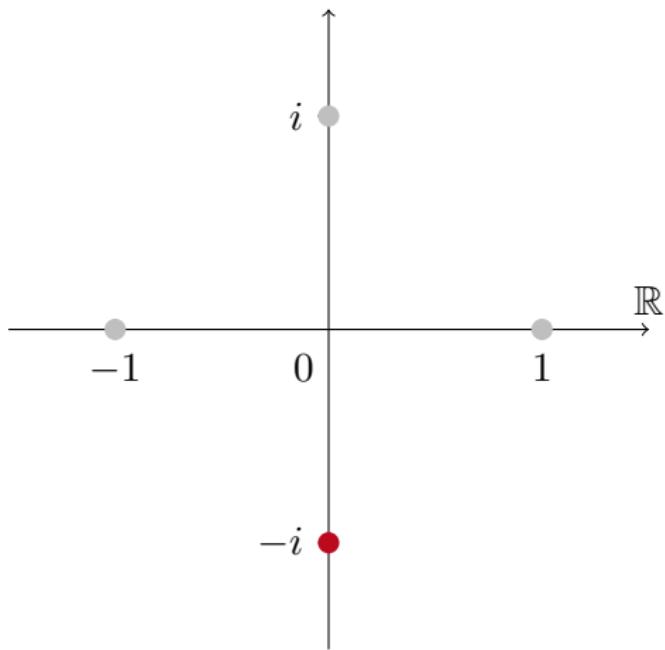


$$i^0 = 1$$

$$i^1 = i$$

$$i^2 = -1$$

Complex Numbers



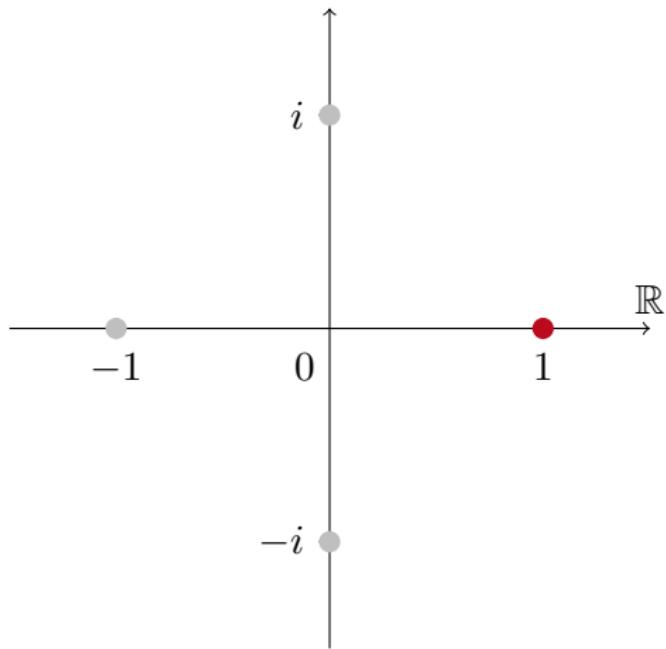
$$i^0 = 1$$

$$i^1 = i$$

$$i^2 = -1$$

$$i^3 = -i$$

Complex Numbers



$$i^0 = 1$$

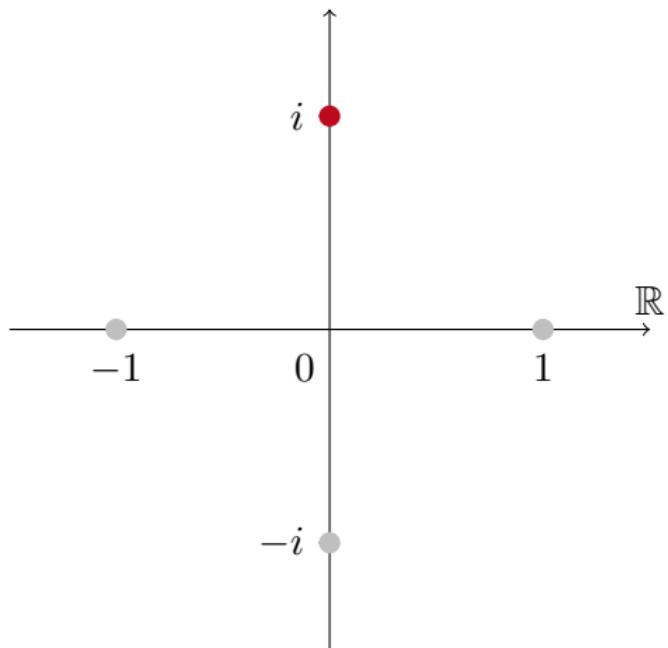
$$i^1 = i$$

$$i^2 = -1$$

$$i^3 = -i$$

$$i^4 = 1$$

Complex Numbers



$$i^0 = 1$$

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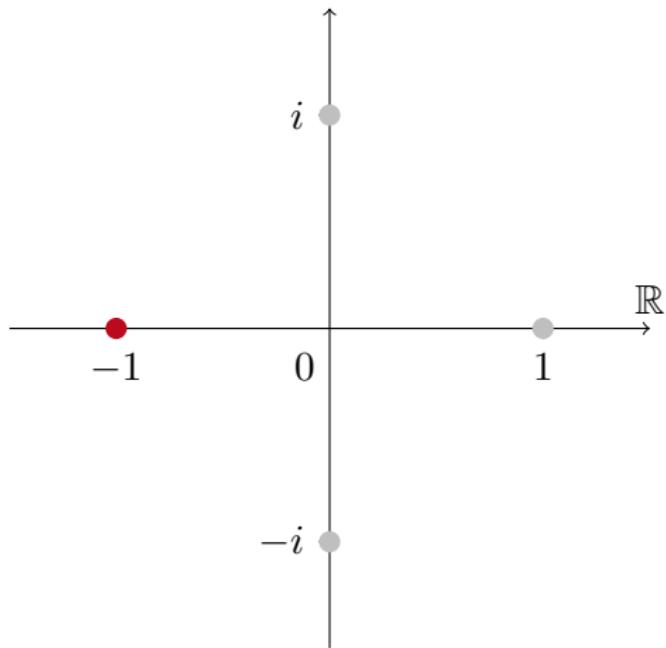
$$i^2 = -1$$

$$i^3 = -i$$

$$i^4 = 1$$

$$i^5 = i$$

Complex Numbers



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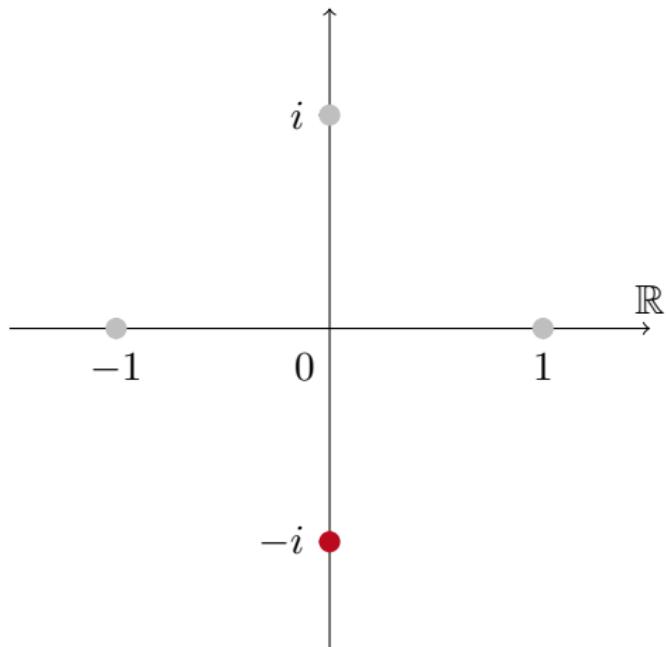
$$i^3 = -i$$

$$i^4 = 1$$

$$i^5 = i$$

$$i^6 = -1$$

Complex Numbers



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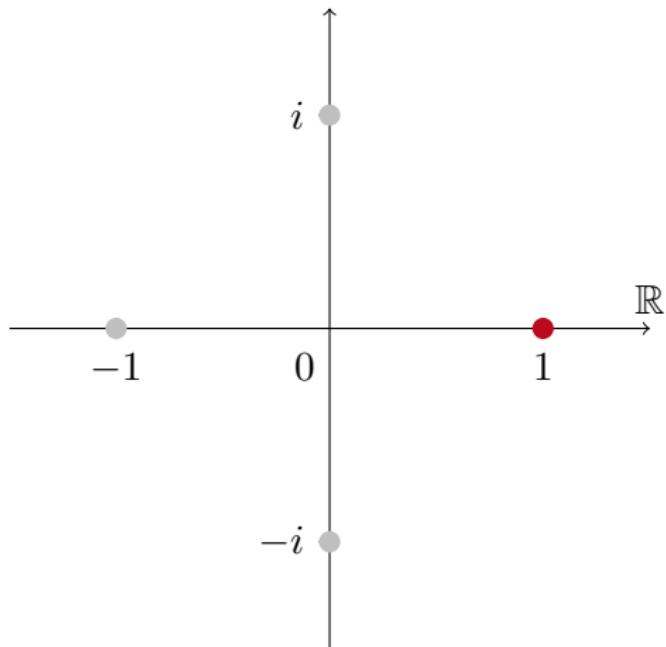
$$i^4 = 1$$

$$i^5 = i$$

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$$i^7 = -i$$

Complex Numbers



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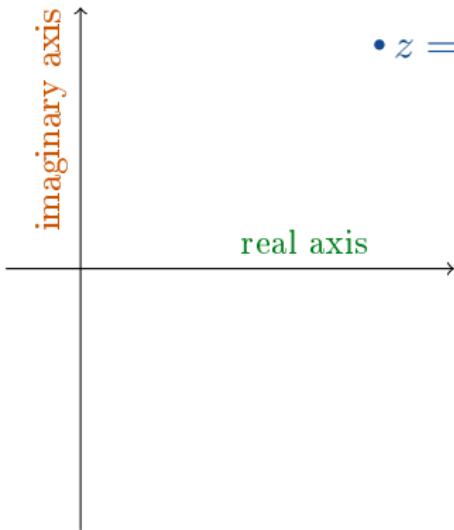
$$i^5 = i$$

$$i^6 = -1$$

$$i^7 = -i$$

$$i^8 = 1$$

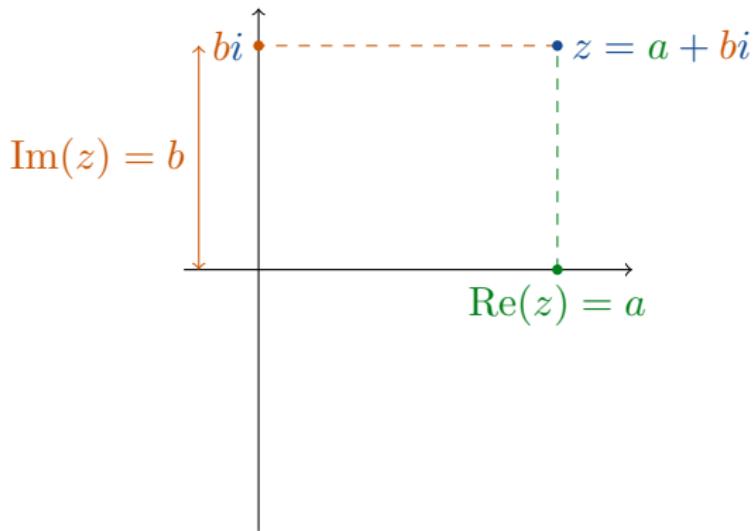
Complex Numbers



$$\bullet z = a + bi$$

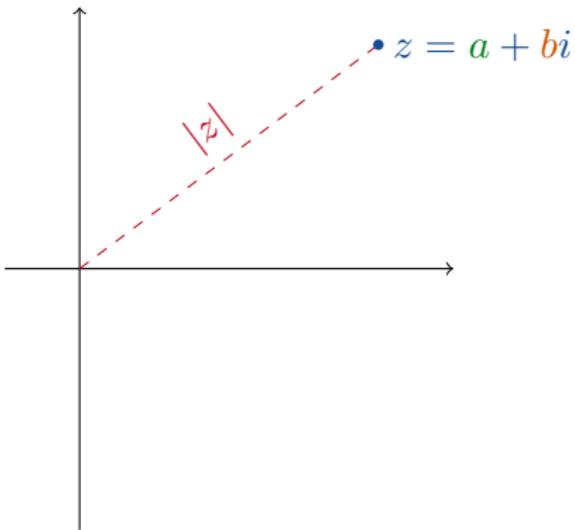
Definition

Complex Numbers



Definition

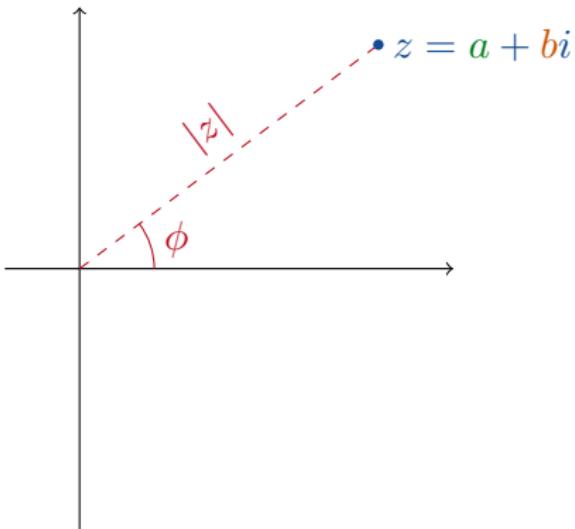
Complex Numbers



Definition

$|z| = \sqrt{a^2 + b^2}$ is called the *modulus* of z .

Complex Numbers

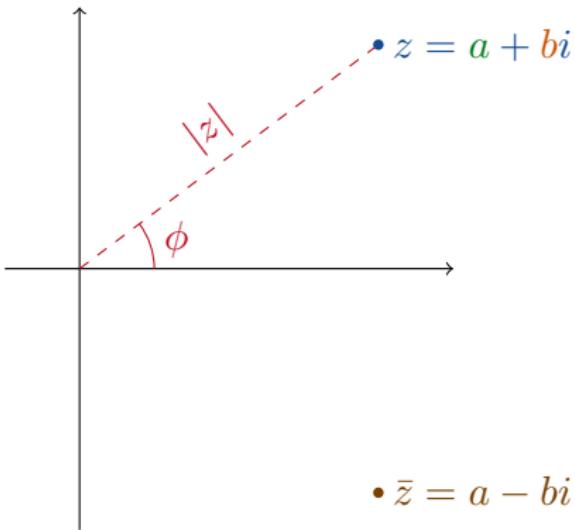


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The angle $\arg(z) = \phi$ is called the *argument* of z .

Complex Numbers



Definition

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The angle $\arg(z) = \phi$ is called the *argument* of z .

$\bar{z} = a - bi$ is called the *complex conjugate* of z .

Complex Numbers



Remark

Note that

$$z\bar{z} = (a + bi)(a - bi) = a^2 + b^2 = |z|^2.$$

Complex Numbers

Remark

Note that

$$z\bar{z} = (a + bi)(a - bi) = a^2 + b^2 = |z|^2.$$

Remark

$$z = \bar{z} \iff z \in \mathbb{R}$$

because

$$a + bi = a - bi \iff b = 0.$$

Complex Numbers



Theorem

$$\overline{zw} = \bar{z}\bar{w}.$$

(you prove)

Roots of polynomials

equation	solution	set
$x - 5 = 0$	$x = 5$	\mathbb{N}

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Question: Are there any more numbers after \mathbb{C} ?

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Question: Are there any more numbers after \mathbb{C} ?

Answer: No.

Complex Numbers



Theorem (The Fundamental Theorem of Algebra)

Any polynomial with complex coefficients will only have complex roots.

In other words: Using the five operations $+$, $-$, \times , \div and a^b , we can not make any more numbers. \mathbb{C} is the end of the line.

Next Time

- Eigenvalues and Eigenvectors
- Diagonalisation
- Complex Vector Spaces