Lecture 6

- 3.8 Related Rates
- 3.9 Linearisation and Differentials
- 4.1 Extreme Values of Functions on Closed Intervals
- 4.2 The Mean Value Theorem

3.8 Related Rates

$$A \leftrightarrow B$$

Sometimes the rate of change of one quantity affects the rate of change of a different quantity.

$$V = \frac{4}{3}\pi r^3$$

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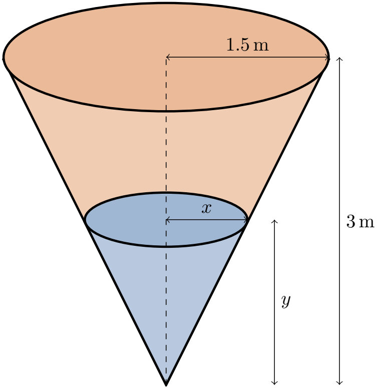
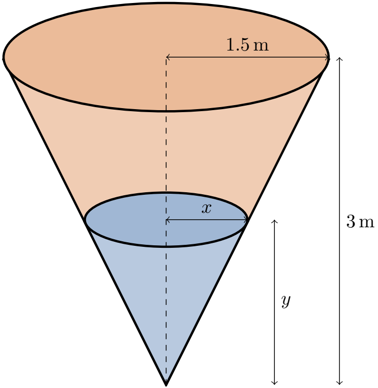
If we know r and $\frac{dV}{dt}$, then we can find $\frac{dr}{dt}$.

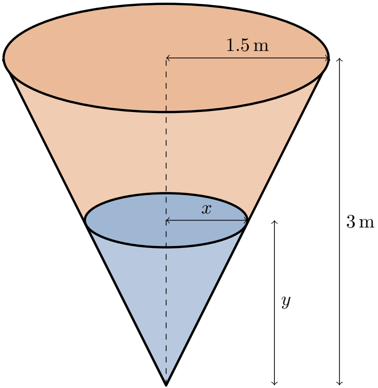


Example

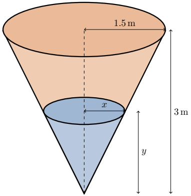
Water runs into a conical tank at the rate of $0.25\,\mathrm{m}^3/\mathrm{min}$. The tank stands point down and has a height of $3\,\mathrm{m}$ and a base radius of $1.5\,\mathrm{m}$. How fast is the water level rising when the water is $1.8\,\mathrm{m}$ deep?



$$t = \text{time (min)}$$
$$V = \text{volume } (\text{m}^3) \text{ of the water in the tank at time}$$
$$x = \text{radius (m) of the water at time } t$$
$$y = \text{height (m) of the water at time } t.$$

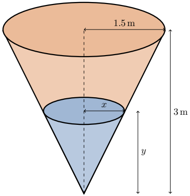


We know $y = 1.8\,\text{m}$ and $\frac{dV}{dt} = 0.25\,\text{m}^3/\text{min}$. We are asked to find $\frac{dy}{dt}$.



Since $x = \frac{y}{2}$, the volume of the water is

$$V = \frac{1}{3}\pi x^2 y = \frac{1}{3}\pi \left(\frac{y}{2}\right)^2 y = \frac{\pi}{12} y^3.$$



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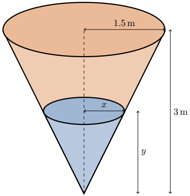
$$\frac{dV}{dt} = \frac{\pi}{4} y^2 \frac{dy}{dt}$$



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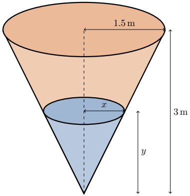
$$0.25 = \frac{dV}{dt} = \frac{\pi}{4}y^2\frac{dy}{dt} = \frac{\pi}{4}(1.8)^2\frac{dy}{dt}.$$



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Differentiating gives

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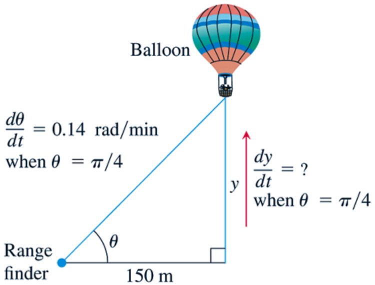
It follows that

$$\frac{dy}{dt} = \frac{1}{3.24\pi} \approx 0.098\,\mathrm{m/min}.$$

Related Rates Problem Strategy

1. *Draw a picture and name the variables and constants*. Use t for time. Assume that all variables are differentiable functions of t.
2. *Write down the numerical information* (in terms of the symbols you have chosen).
3. *Write down what you are asked to find* (usually a rate, expressed as a derivative).
4. *Write an equation that relates the variables*. You may have to combine two or more equations to get a single equation that relates the variable whose rate you want to the variables whose rates you know.
5. *Differentiate with respect to t*. Then express the rate you want in terms of the rates and variables whose values you know.
6. *Evaluate*. Use known values to find the unknown rate.



EXAMPLE 2 A hot air balloon rising straight up from a level field is tracked by a range finder 150 m from the liftoff point. At the moment the range finder's elevation angle is $\pi/4$, the angle is increasing at the rate of 0.14 rad/min. How fast is the balloon rising at that moment?

Solution We answer the question in the six strategy steps.

1. *Draw a picture and name the variables and constants* (Figure 3.34). The variables in the picture are

$$\theta = \text{the angle in radians the range finder makes with the ground.}$$
$$y = \text{the height in meters of the balloon above the ground.}$$

We let t represent time in minutes and assume that θ and y are differentiable functions of t.

The one constant in the picture is the distance from the range finder to the liftoff point (150 m). There is no need to give it a special symbol.

2. *Write down the additional numerical information.*

$$\frac{d\theta}{dt} = 0.14 \text{ rad/min} \qquad \text{when} \qquad \theta = \frac{\pi}{4}$$

3. *Write down what we are to find.* We want dy/dt when $\theta = \pi/4$.

4. *Write an equation that relates the variables y and θ.*

$$\frac{y}{150} = \tan\theta \qquad \text{or} \qquad y = 150\tan\theta$$

5. *Differentiate with respect to* t *using the Chain Rule*. The result tells how dy/dt (which we want) is related to $d\theta/dt$ (which we know).

$$\frac{dy}{dt} = 150\,(\sec^2\theta)\frac{d\theta}{dt}$$

6. *Evaluate with $\theta = \pi/4$ and $d\theta/dt = 0.14$ to find dy/dt.*

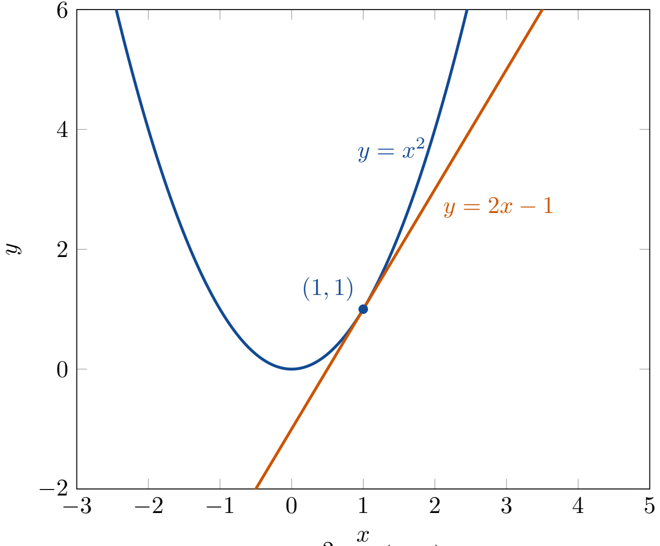
$$\frac{dy}{dt} = 150(\sqrt{2})^2(0.14) = 42 \qquad \sec\frac{\pi}{4} = \sqrt{2}$$

At the moment in question, the balloon is rising at the rate of 42 m/min. ■

3.8 Related Rates

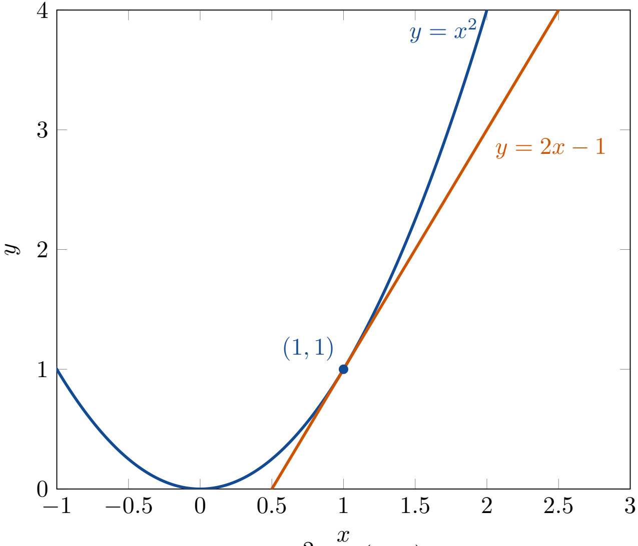
Please read examples 3-6 in your textbook.

Linearisation and Differentials



Consider the tangent to $y = x^2$ at $(1, 1)$. Let's zoom in.

3.9 Linearisation and Differentials



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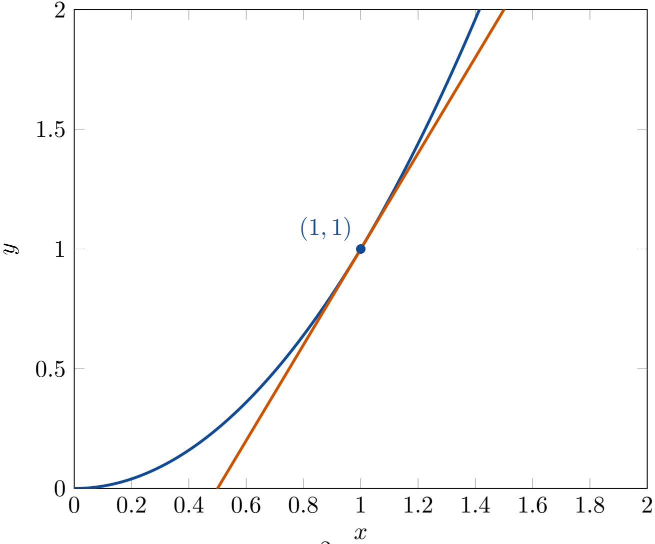
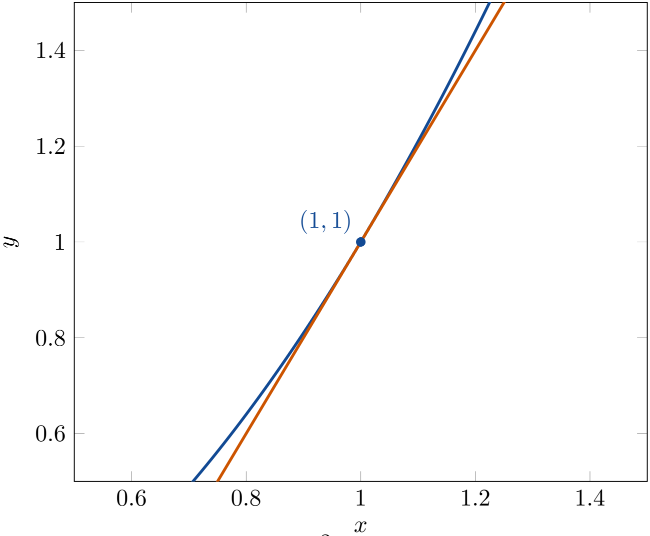


Consider the tangent to $y = x^2$ at $(1, 1)$. Let's zoom in.



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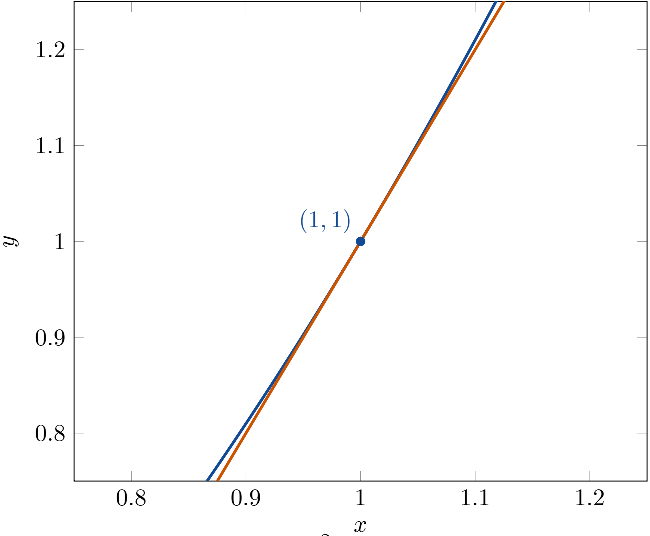

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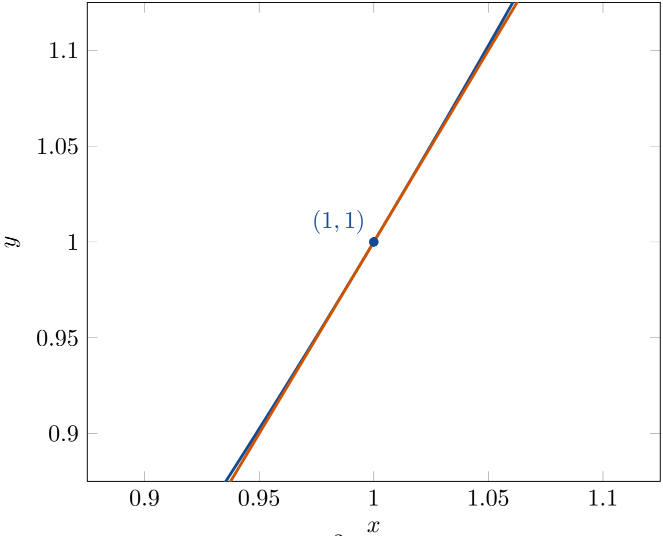

Consider the tangent to $y = x^2$ at $(1,1)$. Let's zoom in. If x is close to 1, then $x^2 \approx 2x - 1$.

Definition

If f is differentiable at $x = a$, then the approximating function

$$L(x) = f(a) + f'(a)(x - a)$$

is the *linearisation* of f at a.

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The approximation

$$f(x) \approx L(x)$$

of f by L is called the *standard linear approximation* of f at a. The point $x = a$ is the *centre* of the approximation.

EXAMPLE 1 Find the linearization of $f(x) = \sqrt{1+x}$ at $x = 0$ (Figure 3.41).

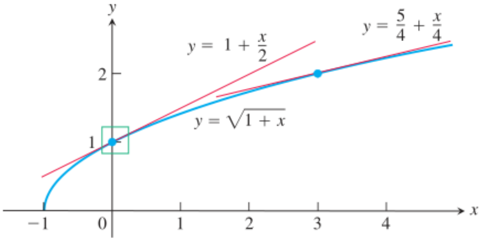


FIGURE 3.41 The graph of $y = \sqrt{1+x}$ and its linearizations at $x = 0$ and $x = 3$. Figure 3.42 shows a magnified view of the small window about 1 on the y-axis.

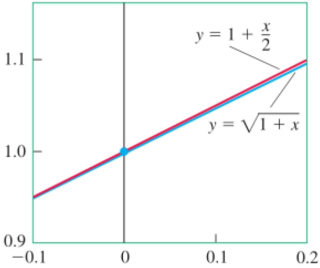


FIGURE 3.42 Magnified view of the window in Figure 3.41.

Solution Since

$$f'(x) = \frac{1}{2}(1 + x)^{-1/2},$$

we have $f(0) = 1$ and $f'(0) = 1/2$, giving the linearization

$$L(x) = f(a) + f'(a)(x - a) = 1 + \frac{1}{2}(x - 0) = 1 + \frac{x}{2}.$$

See Figure 3.42. ■

The following table shows how accurate the approximation $\sqrt{1+x} \approx 1 + (x/2)$ from Example 1 is for some values of x near 0. As we move away from zero, we lose accuracy. For example, for $x = 2$, the linearization gives 2 as the approximation for $\sqrt{3}$, which is not even accurate to one decimal place.

Approximation	**True value**	**$\lvert$True value $-$ approximation$\rvert$**
$\sqrt{1.005} \approx 1 + \dfrac{0.005}{2} = 1.00250$	1.002497	$0.000003 < 10^{-5}$
$\sqrt{1.05} \approx 1 + \dfrac{0.05}{2} = 1.025$	1.024695	$0.000305 < 10^{-3}$
$\sqrt{1.2} \approx 1 + \dfrac{0.2}{2} = 1.10$	1.095445	$0.004555 < 10^{-2}$

EXAMPLE 2 Find the linearization of $f(x) = \sqrt{1 + x}$ at $x = 3$. (See Figure 3.41.)

Solution We evaluate the equation defining $L(x)$ at $a = 3$. With

$$f(3) = 2, \qquad f'(3) = \frac{1}{2}(1 + x)^{-1/2}\bigg|_{x=3} = \frac{1}{4},$$

we have

$$L(x) = 2 + \frac{1}{4}(x - 3) = \frac{5}{4} + \frac{x}{4}.$$

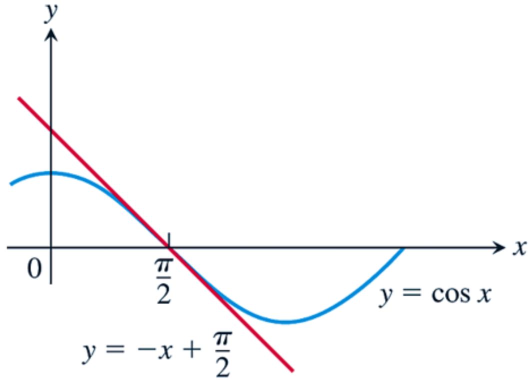
At $x = 3.2$, the linearization in Example 2 gives

$$\sqrt{1 + x} = \sqrt{1 + 3.2} \approx \frac{5}{4} + \frac{3.2}{4} = 1.250 + 0.800 = 2.050,$$

which differs from the true value $\sqrt{4.2} \approx 2.04939$ by less than one one-thousandth. The linearization in Example 1 gives

$$\sqrt{1 + x} = \sqrt{1 + 3.2} \approx 1 + \frac{3.2}{2} = 1 + 1.6 = 2.6,$$

a result that is off by more than 25%. ■



EXAMPLE 3 Find the linearization of $f(x) = \cos x$ at $x = \pi/2$ (Figure 3.43).

Solution Since $f(\pi/2) = \cos(\pi/2) = 0$, $f'(x) = -\sin x$, and $f'(\pi/2) = -\sin(\pi/2) = -1$, we find the linearization at $a = \pi/2$ to be

$$\begin{aligned} L(x) &= f(a) + f'(a)(x - a) \\ &= 0 + (-1)\left(x - \frac{\pi}{2}\right) \\ &= -x + \frac{\pi}{2}. \end{aligned}$$

$$L(x) = f(a) + f'(a)(x - a)$$

Now consider

$$f(x) = (1 + x)^k$$

close to $x = 0$ for any number k.

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close to $x = 0$ for any number k.
Since

$$f'(x) = k(1+x)^{k-1}$$
$$f'(0) = k(1+0)^{k-1} = k,$$

we have that

$$L(x) = f(0) + f'(0)(x-0) = 1 + kx.$$

$$(1+x)^k \approx 1+kx$$

So for example,

$$\sqrt{1+x} = (1+x)^{\frac{1}{2}} \approx 1 + \frac{1}{2}x$$

$$\frac{1}{1-x} = (1+(-x))^{-1} \approx 1 + (-1)(-x) = 1+x$$

$$\sqrt[3]{1+5x^4} = \left(1+5x^4\right)^{\frac{1}{3}} \approx 1 + \frac{1}{3}\left(5x^4\right) = 1 + \frac{5}{3}x^4$$

$$\frac{1}{\sqrt{1-x^2}} = \left(1-x^2\right)^{-\frac{1}{2}} \approx 1 + \left(-\frac{1}{2}\right)\left(-x^2\right) = 1 + \frac{1}{2}x^2$$

Differentials

Recall that

$$\frac{dy}{dx}$$

means the derivative of y. It does not mean dy divided by dx.

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Unless we create two new variables dy and dx such that dy divided by dx is equal to the derivative of y.

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The *differential* dx is an independent variable.

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$$dy = f'(x)\,dx.$$

Then

$$dy \div dx = \frac{f'(x)\,dx}{dx} = f'(x) = \frac{dy}{dx}.$$

.

EXAMPLE 4

(a) Find dy if $y = x^5 + 37x$.

(b) Find the value of dy when $x = 1$ and $dx = 0.2$.

Solution

(a) $dy = (5x^4 + 37)\,dx$

(b) Substituting $x = 1$ and $dx = 0.2$ in the expression for dy, we have

$$dy = (5 \cdot 1^4 + 37)\,0.2 = 8.4.$$

If $f(x) = 3x^2 - 6$, then the *differenial of f* is

$$df = d\left(3x^2 - 6\right) = 6x\,dx.$$

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Every differential formula like

$$\frac{d}{dx}(u+v) = \frac{du}{dx} + \frac{dv}{dx} \qquad \text{or} \qquad \frac{d}{dx}(\sin u) = \cos u \frac{du}{dx}$$

has a corresponding differential form like

$$d(u+v) = du + dv \qquad \text{or} \qquad d(\sin u) = \cos u\,du.$$

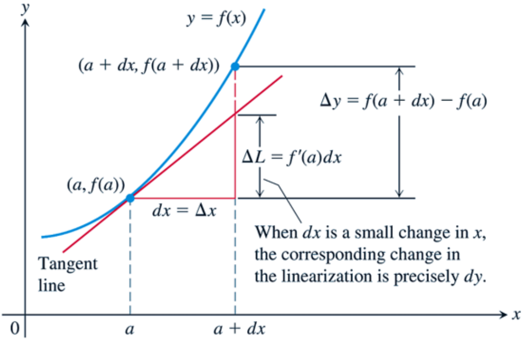
EXAMPLE 5 We can use the Chain Rule and other differentiation rules to find differentials of functions.

(a) $d(\tan 2x) = \sec^2(2x)\, d(2x) = 2\sec^2 2x\, dx$

(b) $$d\left(\frac{x}{x+1}\right) = \frac{(x+1)\,dx - x\,d(x+1)}{(x+1)^2} = \frac{x\,dx + dx - x\,dx}{(x+1)^2} = \frac{dx}{(x+1)^2}$$ ■

$$dy = f'(x)\,dx$$

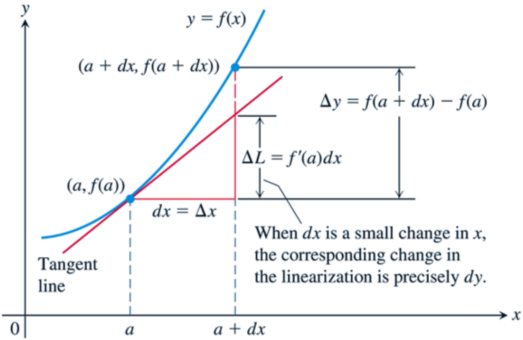
Estimating with Differentials



Suppose that we know the value of $f(x)$ at $x = a$ and we want to estimate how much this value will change if we move to $x = a + dx$.

$$dy = f'(x)\,dx$$

Estimating with Differentials



If dx is small, then

$$f(a + dx) \approx f(a) + dy.$$

$$f(a+dx) \approx f(a) + dy \qquad dy = f'(x)\,dx$$

Example

Use differentials to estimate $7.97^{\frac{1}{3}}$.

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Use differentials to estimate $7.97^{\frac{1}{3}}$.

Note first that if $y = f(x) = x^{\frac{1}{3}}$, then

$$dy = f'(x)\,dx = \frac{1}{3x^{\frac{2}{3}}}dx.$$

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Note first that if $y = f(x) = x^{\frac{1}{3}}$, then

$$dy = f'(x)\,dx = \frac{1}{3x^{\frac{2}{3}}}dx.$$

We are close to $a = 8$ $(7.97 = a - .03)$. Using $dx = -.03$,

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$$\begin{aligned} f(7.97) = f(a+dx) &\approx f(a) + dy \\ &= 8^{\frac{1}{3}} + \frac{1}{3(8)^{\frac{2}{3}}}(-0.03) \\ &= 2 + \frac{1}{12}(-0.03) = 1.9975. \end{aligned}$$

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The true value is $7.97^{\frac{1}{3}} = 1.99749686847326240\ldots$.

EXAMPLE 6 The radius r of a circle increases from $a = 10$ m to 10.1 m (Figure 3.45). Use dA to estimate the increase in the circle's area A. Estimate the area of the enlarged circle and compare your estimate to the true area found by direct calculation.

Solution Since $A = \pi r^2$, the estimated increase is

$$dA = A'(a)\,dr = 2\pi a\,dr = 2\pi(10)(0.1) = 2\pi \text{ m}^2.$$

Thus, since $A(r + \Delta r) \approx A(r) + dA$, we have

$$\begin{aligned} A(10 + 0.1) &\approx A(10) + 2\pi \\ &= \pi(10)^2 + 2\pi = 102\pi. \end{aligned}$$

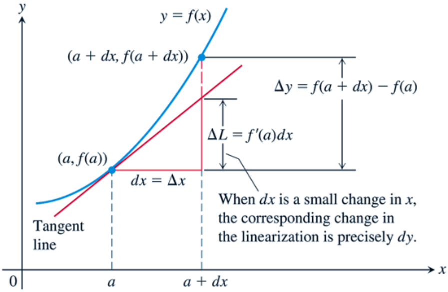
The area of a circle of radius 10.1 m is approximately $102\pi \text{ m}^2$.

The true area is

$$\begin{aligned} A(10.1) &= \pi(10.1)^2 \\ &= 102.01\pi \text{ m}^2. \end{aligned}$$

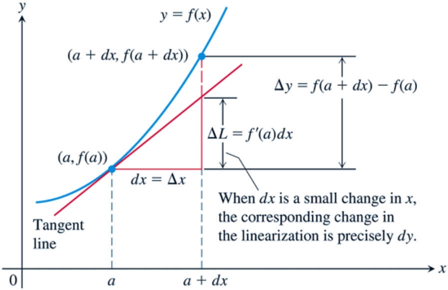
The error in our estimate is $0.01\pi \text{ m}^2$, which is the difference $\Delta A - dA$. ■

Error in Differential Approximation



How well does $f(a) + dy$ approximate $f(a + dx)$?

Error in Differential Approximation



How well does $f(a) + dy$ approximate $f(a + dx)$?

$$\text{true change:} \qquad \Delta y = f(a + dx) - f(a)$$

$$\text{approximation change:} \qquad dy = f'(a)\, dx$$

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$$\begin{aligned} \text{error} &= \Delta y - dy \\ &= f(a + dx) - f(a) - f'(a)\,dx \\ &= \\ &= \end{aligned}$$

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$$\begin{aligned} \text{error} &= \Delta y - dy \\ &= f(a + dx) - f(a) - f'(a)\, dx \\ &= \left(\frac{f(a + dx) - f(a)}{dx} - f'(a) \right) dx \\ &= \varepsilon\, dx. \end{aligned}$$

Error in Differential Approximation

How well does $f(a) + dy$ approximate $f(a + dx)$?

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Therefore

$$\boxed{\Delta y = f'(a)\, dx + \varepsilon\, dx.}$$

In Example 6 we found that

$$\Delta A = \pi(10.1)^2 - \pi(10)^2 = (102.01 - 100)\pi = (\underbrace{2\pi}_{dA} + \underbrace{0.01\pi}_{\text{error}})\ \mathrm{m}^2$$

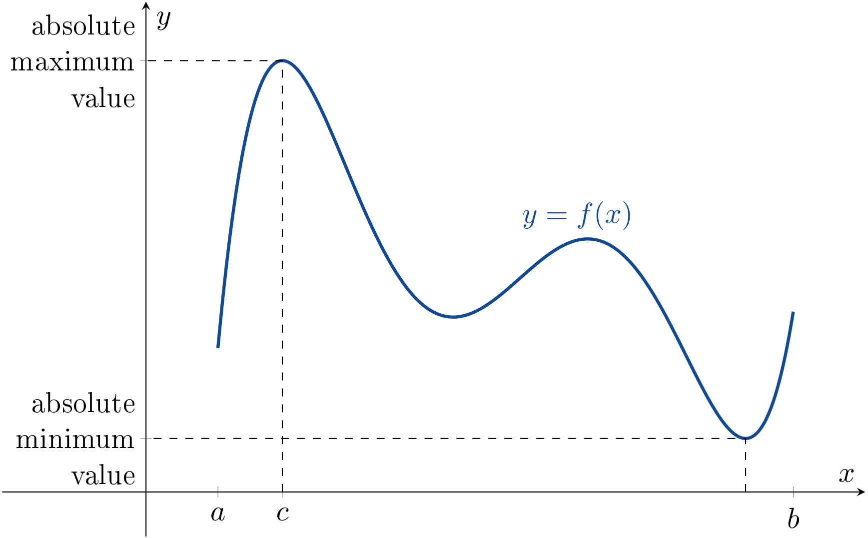
so the approximation error is $\Delta A - dA = \varepsilon \Delta r = 0.01\pi$ and $\varepsilon = 0.01\pi/\Delta r = 0.01\pi/0.1 = 0.1\pi$ m.

Break

We will continue at 2pm

Extreme Values of Functions on Closed Intervals

4.1 Extreme Values of Functions on Closed Intervals



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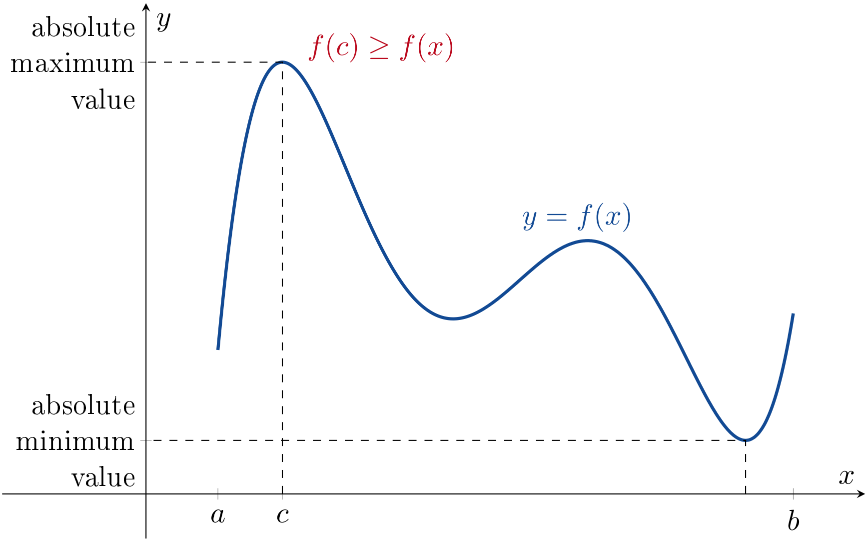

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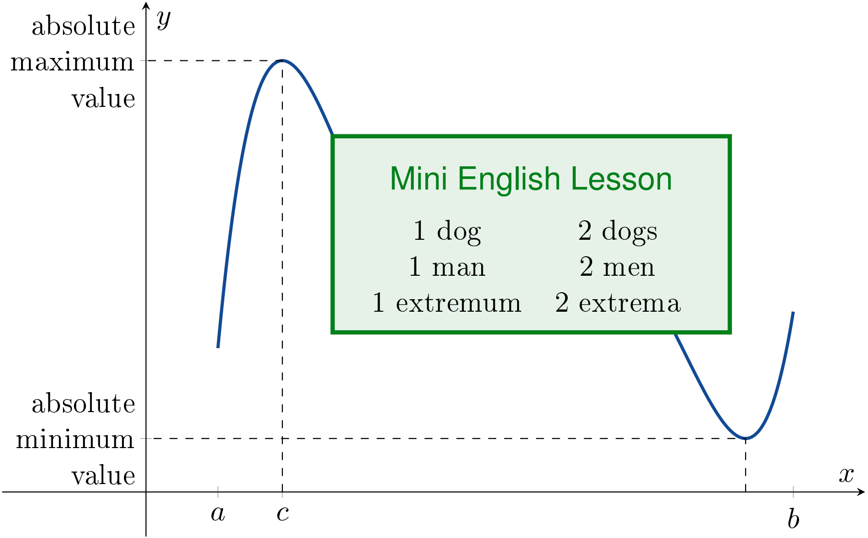


Mini English Lesson

1 dog	2 dogs
1 man	2 men
1 extremum	2 extrema

4.1 Extreme Values of Functions on Closed Intervals

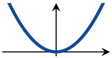
Definition

Let $f : D \to \mathbb{R}$ be a function.

- f has an *absolute maximum value* on D at a point c if
$$f(x) \leq f(c)$$
for all $x \in D$.
- f has an *absolute minimum value* on D at a point c if
$$f(x) \geq f(c)$$
for all $x \in D$.

Maximum and minimum values are called *extrema*/*extreme values*.

Example

function	domain, D	graph	absolute extrema on D
$y = x^2$	$(-\infty, \infty)$		No absolute maximum. Absolute minimum of 0 at $x = 0$.
$y = x^2$	$[0, 2]$		
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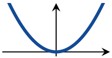
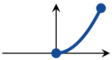
$y = x^2$	$(-\infty, \infty)$		No absolute maximum. Absolute minimum of 0 at $x = 0$.
$y = x^2$	$[0, 2]$		Absolute maximum of 4 at $x = 2$. Absolute minimum of 0 at $x = 0$.
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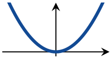
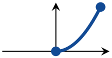
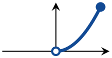
$y = x^2$	$(-\infty, \infty)$		No absolute maximum. Absolute minimum of 0 at $x = 0$.
$y = x^2$	$[0, 2]$		Absolute maximum of 4 at $x = 2$. Absolute minimum of 0 at $x = 0$.
$y = x^2$	$(0, 2]$		Absolute maximum of 4 at $x = 2$. No absolute minimum.
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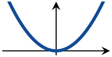
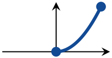
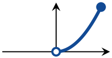
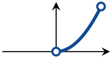
$y = x^2$	$(-\infty, \infty)$		No absolute maximum. Absolute minimum of 0 at $x = 0$.
$y = x^2$	$[0, 2]$		Absolute maximum of 4 at $x = 2$. Absolute minimum of 0 at $x = 0$.
$y = x^2$	$(0, 2]$		Absolute maximum of 4 at $x = 2$. No absolute minimum.
$y = x^2$	$(0, 2)$		No absolute extrema.

Theorem (The Extreme Value Theorem)

Suppose that

- $f : D \to \mathbb{R}$ *is continuous; and*
- $D = [a, b]$ *is a closed interval.*

Then f *attains both an absolute maximum value* M *and an absolute minimum value* m *in* $[a, b]$.

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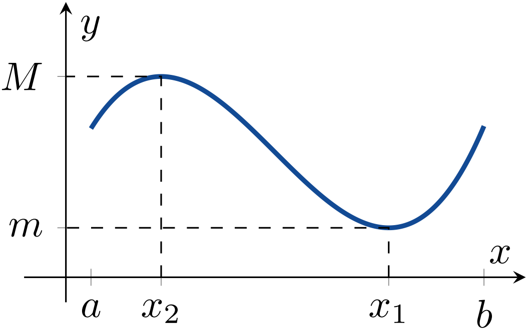
Then f *attains both an absolute maximum value* M *and an absolute minimum value* m *in* $[a, b]$.

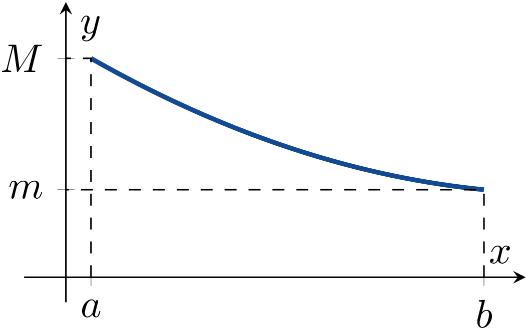
Remark

The Extreme Value Theorem says that there are numbers $x_1, x_2 \in [a, b]$ such that

- $f(x_1) = m$;
- $f(x_2) = M$; and
- $m \leq f(x) \leq M$ for all $x \in [a, b]$.



The absolute maximum and absolute minimum are at interior points.



The absolute maximum and absolute minimum are at endpoints.

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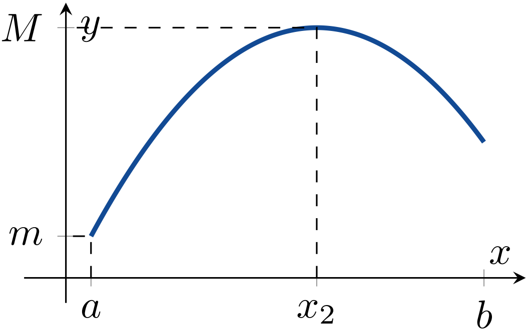
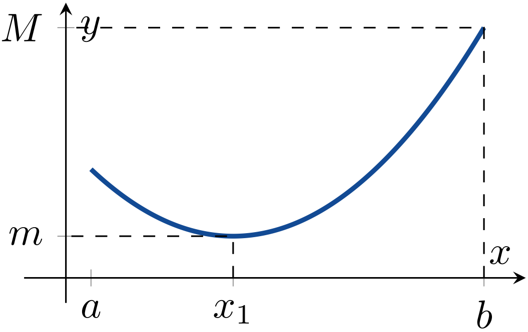
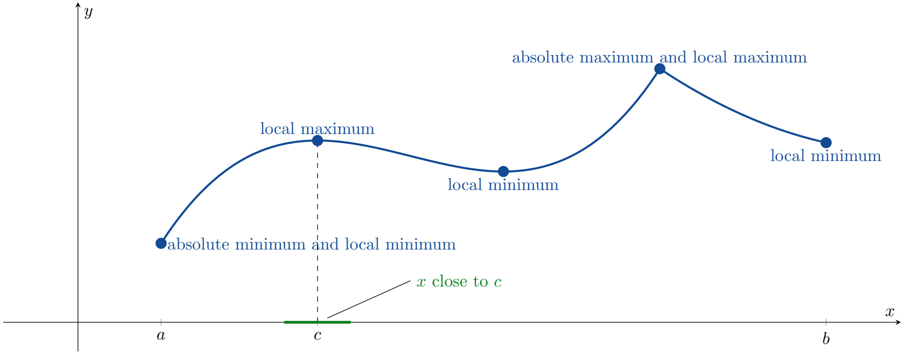


The absolute maximum is at an interior point. The absolute minimum is at an endpoint.



The absolute maximum is at an endpoint. The absolute minimum is at an interior point.

Local Extreme Values



Definition

Let $f : D \to \mathbb{R}$ be a function.

- f has a *local maximum value* at a point $c \in D$ if
$$f(x) \leq f(c)$$
for all x close to c.
- f has a *local minimum value* at a point $c \in D$ if
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for all x close to c.

An absolute maximum is always a local maximum too. An absolute minimum is always a local minimum too.

Theorem (The First Derivative Test for Local Extrema)

Suppose that

- f *has a local maximum/minimum value at an interior point* $c \in D$*; and*
- $f'(c)$ *exists.*

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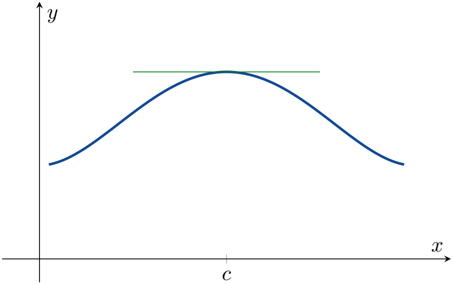

Theorem (The First Derivative Test for Local Extrema)

Suppose that

- f *has a local maximum/minimum value at an interior point* $c \in D$; *and*
- $f'(c)$ *exists.*

Then $f'(c) = 0$.

Remark

The First Derivative Test tells us that the only places where $f : D \to \mathbb{R}$ can have an extreme value are

- interior points where $f'(c) = 0$;
- interior points where $f'(c)$ does not exist; and
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- interior points where $f'(c)$ does not exist; and
- endpoints of D.

Definition

An interior point of D where either

- $f' = 0$; or
- f' does not exist,

is called a *critical point* of f.

How to find the absolute extrema of a continuous function $f : [a, b] \to \mathbb{R}$

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1. Find the critical points of f.
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3. Calculate $f(a)$ and $f(b)$.
4. Take the largest and smallest values.

Example

Find the absolute maximum and absolute minimum values of $f(x) = x^2$ on $[-2, 1]$.

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1. We know that $f(x) = x^2$ is differentiable on $[-2, 1]$. So $f'(x)$ exists for all interior points $x \in (-2, 1)$. The only critical point is

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2. $f(0) = 0$.
3. $f(-2) = 4$ and $f(1) = 1$.
4. The largest and smallest numbers in $\{0, 1, 4\}$ are 4 and 0. Therefore the absolute maximum value of $f(x) = x^2$ on $[-2, 1]$ is 4 and the absolute minimum value of f on $[-2, 1]$ is 0.

Example

Find the absolute maximum and absolute minimum values of $g(t) = 8t - t^4$ on $[-2, 1]$.

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1. $g'(t)$ exists for all $t \in (-2, 1)$. Since

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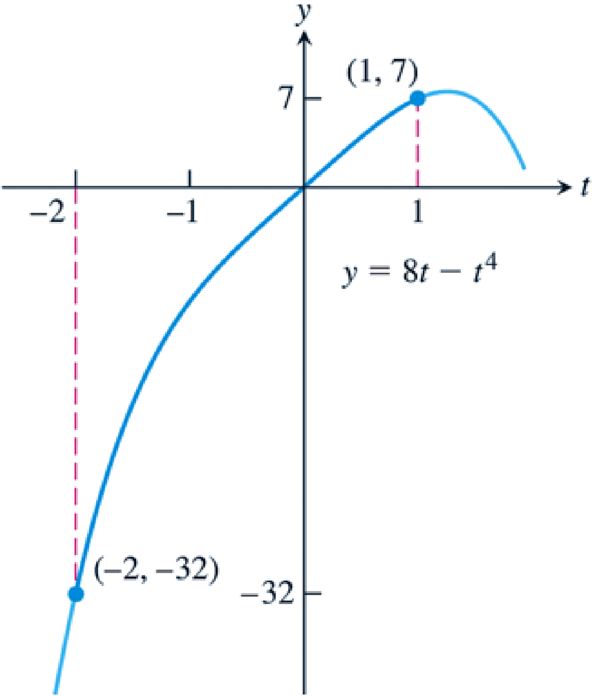
2.
3. $g(-2) = -32$ and $g(1) = 7$.
4. Therefore

$$\max_{t\in[-2,1]} g(t) = 7 \qquad \text{and} \qquad \min_{t\in[-2,1]} g(t) = -32.$$



y
t
(1, 7)
7
−2
−1
1
$y = 8t - t^4$
(−2, −32)
−32

Example

Find the absolute maximum and absolute minimum values of $h(x) = x^{\frac{2}{3}}$ on $[-2, 3]$.

1. We calculate that

$$h'(x) = \frac{d}{dx}\left(x^{\frac{2}{3}}\right) = \frac{2}{3}x^{-\frac{1}{3}} = \frac{2}{3\sqrt[3]{x}}.$$

   Hence h' does not exist if $x = 0$. We can also see that $h'(x) \neq 0$ if $x \in [-2, 0)$ or $x \in (0, 3]$. The only critical point is $x = 0$

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2. $h(0) = 0$.
3. $h(-2) = (-2)^{\frac{2}{3}} = (4)^{\frac{1}{3}} = \sqrt[3]{4}$ and $h(3) = (3)^{\frac{2}{3}} = (9)^{\frac{1}{3}} = \sqrt[3]{9}$.

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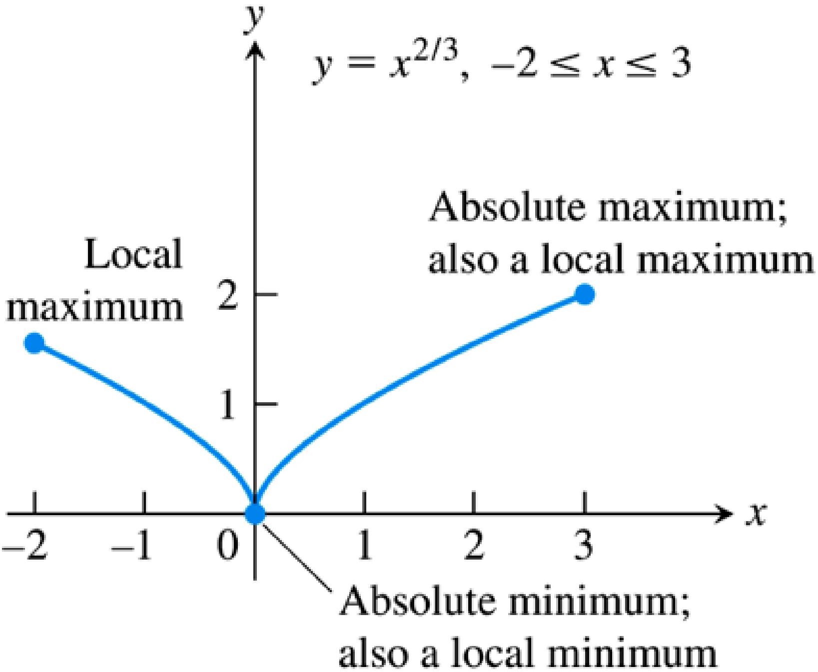
3. $h(-2) = (-2)^{\frac{2}{3}} = (4)^{\frac{1}{3}} = \sqrt[3]{4}$ and $h(3) = (3)^{\frac{2}{3}} = (9)^{\frac{1}{3}} = \sqrt[3]{9}$.
4. Therefore
$$\max_{x\in[-2,3]} h(x) = \sqrt[3]{9} \approx 2.08 \qquad \text{and} \qquad \min_{x\in[-2,3]} h(t) = 0.$$


y
$y = x^{2/3}, \ -2 \le x \le 3$
Absolute maximum;
also a local maximum
Local
maximum
2
1
x
–2
–1
0
1
2
3
Absolute minimum;
also a local minimum

4.2 The Mean Value Theorem

Michel Rolle

BORN
21 April 1652

DECEICED
8 November 1719

NATIONALITY
French

Theorem (Rolle's Theorem)

Suppose that

1. $f : [a, b] \to \mathbb{R}$ *is continuous;*
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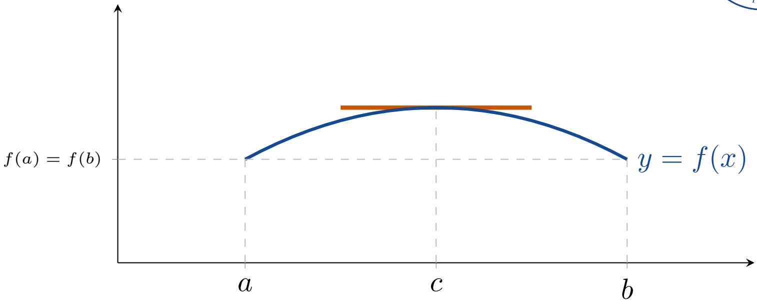
4.2 The Mean Value Theorem



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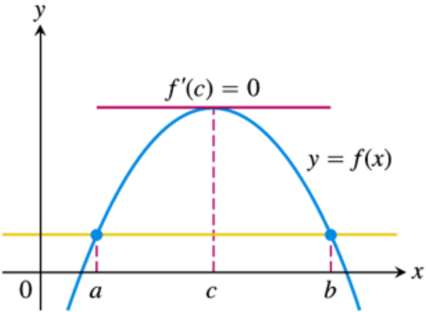
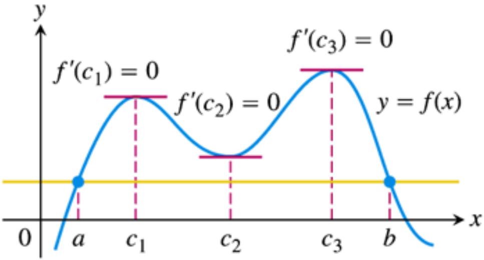
Then $\exists\ c \in (a, b)$ *such that* $f'(c) = 0$.



(a)



(b)

Augustin-Louis Cauchy

BORN
21 August 1789

DECEICED
23 May 1857

NATIONALITY
French

Theorem (The Mean Value Theorem)

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1. $f : [a, b] \to \mathbb{R}$ *is continuous; and*
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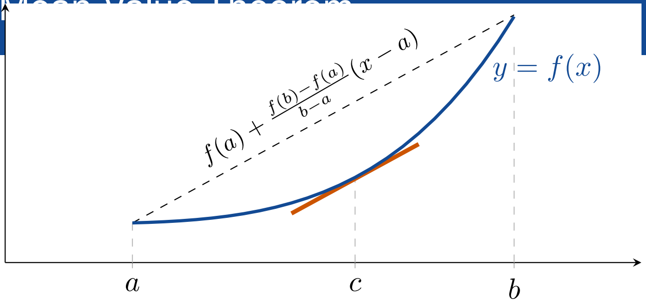
Then $\exists\ c \in (a, b)$ *such that*

$$\frac{f(b) - f(a)}{b - a} = f'(c).$$



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Note that $f(x) = x^2$ is

- continuous on $[0, 2]$;
- differentiable on $(0, 2)$;
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Note that $f(x) = x^2$ is

- continuous on $[0, 2]$;
- differentiable on $(0, 2)$;
- $f(0) = 0$ and $f(2) = 4$.

By the Mean Value Theorem, there exists $c \in (0, 2)$ at which

$$f'(c) = \frac{f(2) - f(0)}{2 - 0} = \frac{4 - 0}{2 - 0} = 2.$$

Mathematical Consequences

Corollary

If $f'(x) = 0$ at each point $x \in (a, b)$, then $f(x) = C$ for all $x \in (a, b)$, where C is a constant.

Corollary

If $f'(x) = g'(x)$ at each point $x \in (a, b)$, then $f(x) = g(x) + C$ for all $x \in (a, b)$, where C is a constant.

Next Time

- 4.3 Monotonic Functions and the First Derivative Test
- 4.4 Concavity and Curve Sketching
- 4.5 Applied Optimisation
- 4.7 Antiderivatives