Week 6

- 3.4 Repeated Roots of the Characteristic Equation
- 3.5 Reduction of Order
- 3.6 Nonhomogeneous Equations
- 3.7 The Method of Undetermined Coefficients

Summary

To solve

$$ay'' + by' + cy = 0$$

we need to find two linearly independent solutions, $y_1(t)$ and $y_2(t)$. Then the general solution to the ODE is

$$y(t) = c_1 y_1(t) + c_2 y_2(t).$$

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First we solve the characteristic equation

$$ar^2 + br + c = 0$$

and find the roots r_1 and r_2.

Summary

1. If $r_1, r_2 \in \mathbb{R}$ and $r_1 \neq r_2$, then
$$y_1(t) = e^{r_1 t} \qquad \text{and} \qquad y_2(t) = e^{r_2 t};$$
2. If $r_{1,2} = \lambda \pm i\mu$ $(\lambda, \mu \in \mathbb{R})$, then
$$y_1(t) = e^{\lambda t} \cos \mu t \qquad \text{and} \qquad y_2(t) = e^{\lambda t} \sin \mu t;$$
3. If the roots are repeated, then ?????????????

Repeated Roots of the Characteristic Equation

3.4

Now consider

$$ay'' + by' + cy = 0 \tag{1}$$

where $b^2 - 4ac = 0$. Then the only root of

$$ar^2 + br + c = 0$$

is

$$r = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-b \pm \sqrt{0}}{2a} = -\frac{b}{2a}.$$

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We know that $y_1(t) = e^{-\frac{bt}{2a}}$ is a solution of (1), but how do we find a linearly independent second solution?

Example

Solve $y'' + 4y' + 4y = 0$.

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The characteristic equation

$$0 = r^2 + 4r + 4 = (r+2)^2$$

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The idea is:

- We know that $y_1(t)$ is a solution;
- So $cy_1(t)$ is a solution for any $c \in \mathbb{R}$;
- Maybe $v(t)y_1(t)$ is a solution for some non-constant function $v(t)$.

3.4 Repeated Roots of the Characteristic Equation

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and that

$$\begin{aligned} 0 &= y_2'' + 4y_2' + 4y_2 \\ &= \left(v''e^{-2t} - 4ve^{-2t} + 4ve^{-2t}\right) + 4\left(v'e^{-2t} - 2ve^{-2t}\right) + 4\left(ve^{-2t}\right) \\ &= e^{-2t}\left[v'' - 4v' + 4v + 4v' - 8v + 4v\right] \\ &= v''e^{-2t}. \end{aligned}$$

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$$y_2(t) = te^{-2t}.$$

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$$y_2(t) = te^{-2t}.$$

But are $y_1(t)$ and $y_2(t)$ linearly independent? Since

$$W\left(y_1, y_2\right)(t) = \begin{vmatrix} e^{-2t} & te^{-2t} \\ -2e^{-2t} & (1-2t)e^{-2t} \end{vmatrix} = e^{-4t} \neq 0,$$

the answer is YES.

Therefore $y_1(t) = e^{-2t}$ and $y_2(t) = te^{-2t}$ form a fundamental set of solutions and the general solution is

$$y(t) = c_1 e^{-2t} + c_2 t e^{-2t}.$$

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$$0 = ay_2'' + by_2' + cy_2 = \ldots = ae^{-\frac{bt}{2a}}v''.$$

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$$0 = ay_2'' + by_2' + cy_2 = \ldots = ae^{-\frac{bt}{2a}}v''.$$

So again we want $v'' = 0$ and we can choose $v(t) = t$. Thus $y_2(t) = te^{rt} = te^{-\frac{bt}{2a}}$.

I leave it for you to calculate that $W(e^{rt}, te^{rt}) \neq 0$. Thus e^{rt} and te^{rt} form a fundamental set of solutions to (1).

Example

Solve

$$\begin{cases} y'' - y' + \frac{1}{4}y = 0 \\ y(0) = 2 \\ y'(0) = \frac{1}{3}. \end{cases}$$

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The characteristic equation

$$0 = r^2 - r + \frac{1}{4} = \left(r - \frac{1}{2}\right)^2$$

has repeated root $r = \frac{1}{2}$.

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Solve

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The characteristic equation

$$0 = r^2 - r + \frac{1}{4} = \left(r - \frac{1}{2}\right)^2$$

has repeated root $r = \frac{1}{2}$. So we know that the general solution to the ODE is

$$y(t) = c_1 e^{\frac{t}{2}} + c_2 t e^{\frac{t}{2}}.$$

Next we need to look at the initial conditions: Since $y'(t) = \frac{1}{2}c_1 e^{\frac{t}{2}} + c_2 e^{\frac{t}{2}} + \frac{1}{2}c_2 t e^{\frac{t}{2}}$, we have that

$$\begin{aligned} 2 = y(0) &= c_1 + 0 && \implies && c_1 = 2 \\ \frac{1}{3} = y'(0) &= \frac{1}{2}c_1 + c_2 + 0 && \implies && c_2 = -\frac{2}{3}. \end{aligned}$$

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Therefore the solution to the IVP is

$$y = 2e^{\frac{t}{2}} - \tfrac{2}{3}te^{\frac{t}{2}}.$$

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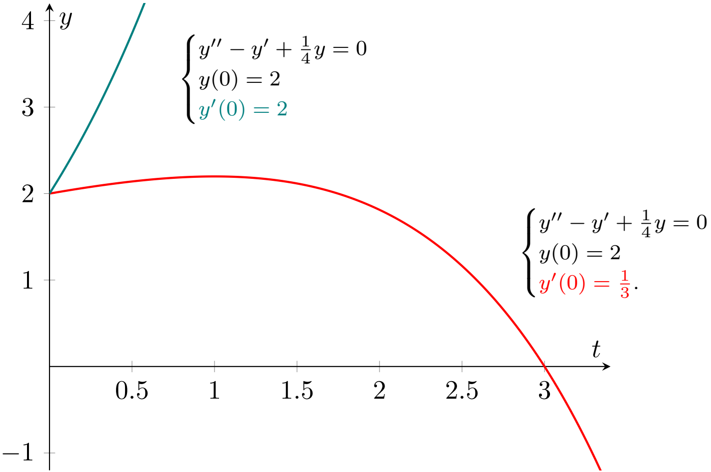
You can check that the solution is

$$y = 2e^{\frac{t}{2}} + te^{\frac{t}{2}}.$$

The graph of this solution, and the solution to the previous example, are shown on the next slide.

3.4 Repeated Roots of the Characteristic Equation



Note that even though these two functions share the same $y(0)$ value, and that their $y'(0)$ value does not differ by much, their behaviour as $t \to \infty$ is very different.

3.4 Repeated Roots of the Characteristic Equation

Summary

To solve

$$ay'' + by' + cy = 0$$

we need to find two linearly independent solutions.

1. If $r_1, r_2 \in \mathbb{R}$ and $r_1 \neq r_2$, then
$$y_1(t) = e^{r_1 t} \qquad \text{and} \qquad y_2(t) = e^{r_2 t};$$
2. If $r_{1,2} = \lambda \pm i\mu$ $(\lambda, \mu \in \mathbb{R})$, then
$$y_1(t) = e^{\lambda t} \cos \mu t \qquad \text{and} \qquad y_2(t) = e^{\lambda t} \sin \mu t;$$
3. If $r_1, r_2 \in \mathbb{R}$ but $r_1 = r_2$, then
$$y_1(t) = e^{r_1 t} \qquad \text{and} \qquad y_2(t) = te^{r_1 t}.$$

3.5 Reduction of Order

Consider

$$y'' + p(t)y' + q(t)y = 0. \qquad (2)$$

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Suppose that we know that $y_1(t)$ is a solution to (2) and suppose that we want to find a second, linearly independent solution.

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Suppose that we know that $y_1(t)$ is a solution to (2) and suppose that we want to find a second, linearly independent solution.

The main idea in this section is that we guess that

$$y_2(t) = v(t)y_1(t)$$

for some non-constant function $v(t)$. If we can find $v(t)$, then we have our $y_2(t)$.

Then we calculate that

$$\begin{aligned} y_2 &= vy_1 \\ y_2' &= v'y_1 + vy_1' \\ y_2'' &= v''y_1 + 2v'y_1' + vy_1'' \end{aligned}$$

and

$$\begin{aligned} 0 &= y_2'' + py_2' + qy_2 \\ &= \\ &= \\ &= \qquad . \end{aligned}$$

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Then we calculate that

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\begin{aligned}
y_2 &= vy_1 \\
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&= v''y_1 + v'\left(2y_1' + py_1\right).
\end{aligned}
$$

$$0 = v''y_1 + v'\left(2y_1' + py_1\right) + 0v$$

Remark

Note that since y_1 solves the ODE, we must always get "$0v$" here. We can have v' and v'' terms, but if you do a reduction of order calculation and still have v terms, then you have made a mistake.

Remark

$$v''y_1 + v'\left(2y_1' + py_1\right) = 0 \tag{3}$$

is actually a first order ODE for v'.

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If we can find $u(t)$, then we can find $v(t) = \int u(t)\,dt$ and $y_2(t) = v(t)y_1(t)$.

Remark

Instead of having to solve a second order ODE to find y_2, we only need to solve a first order ODE to find $u(t)$. Hence the name "Reduction of Order".

Remark

The method is

1. Guess $y_2 = vy_1$.
2. Put this into your ODE and find an equation for v;
3. Set $u = v'$;
4. Find u;
5. Integrate to find v;
6. Then $y_2(t) = v(t)y_1(t)$.

Example

Given that $y_1(t) = \frac{1}{t}$ is a solution of

$$2t^2y'' + 3ty' - y = 0, \qquad t > 0$$

find a linearly independent second solution.

$$y_1(t) = \frac{1}{t}$$

Let $y_2(t) = v(t)y_1(t)$. Then we have

$$\begin{aligned} y_2 &= vt^{-1} \\ y_2' &= v't^{-1} - vt^{-2} \\ y_2'' &= v''t^{-1} - 2v't^{-2} + 2vt^{-3} \end{aligned}$$

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and

$$\begin{aligned}
0 &= 2t^2 y_2'' + 3ty_2' - y_2 \\
&= 2t^2 \left(v''t^{-1} - 2v't^{-2} + 2vt^{-3}\right) + 3t\left(v't^{-1} - vt^{-2}\right) - vt^{-1} \\
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This equation is both linear and separable, so we know 2 ways to solve it.

$$
\begin{aligned}
2t\frac{du}{dt} &= u \\
\frac{du}{u} &= \frac{1}{2}\frac{dt}{t} \\
\int \frac{du}{u} &= \int \frac{1}{2}\frac{dt}{t} \\
\ln|u| &= \frac{1}{2}\ln|t| + C \\
e^{\ln|u|} &= e^{\ln|t|^{\frac{1}{2}}}e^{C} \\
|u| &= |t|^{\frac{1}{2}}\,e^{C} \\
u &= \pm e^{C}t^{\frac{1}{2}} = ct^{\frac{1}{2}}.
\end{aligned}
$$

$$u(t) = ct^{\frac{1}{2}}$$

Then we have

$$v(t) = \int u(t)\,dt = \int ct^{\frac{1}{2}}\,dt = \frac{2}{3}ct^{\frac{3}{2}} + k$$

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Remember that we are trying to find a solution that is linearly independent from $y_1(t) = t^{-1}$. The second term in $y_2(t) = \frac{2}{3}ct^{\frac{1}{2}} + kt^{-1}$ is just a multiple of $y_1(t)$ – we don't need this. So it is ok to choose $k = 0$.

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Finally, since I like simple functions I choose $c = \frac{3}{2}$ to get

$$\boxed{y_2(t) = t^{\frac{1}{2}}.}$$

I leave it to you to check that $W(t^{-1}, t^{\frac{1}{2}})$ is not always zero.

Example

Given that $y_1(t) = t$ solves

$$t^2y'' + 2ty' - 2y = 0, \quad t > 0,$$

find a second linearly independent solution $y_2(t)$.

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find a second linearly independent solution $y_2(t)$.

We start with $y_2(t) = v(t)y_1(t) = v(t)t$. Then $y_2' = v't + v$ and $y_2'' = v''t + 2v'$. Substituting into the ODE, we calculate that

$$\begin{aligned} 0 &= t^2y_2'' + 2ty_2' - 2y_2 \\ &= t^2(v''t + 2v') + 2t(v't + v) - 2vt \\ &= t^3v'' + v'(2t^2 + 2t^2) + v(2t - 2t) \\ &= t^3v'' + 4t^2v' \\ &= t^2(tv'' + 4v'). \end{aligned}$$

$$t^2(tv'' + 4v') = 0$$

Letting $u = v'$, we obtain the first order ODE

$$t\frac{du}{dt} + 4u = 0.$$

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$$t\frac{du}{dt} + 4u = 0.$$

We calculate that

$$\begin{aligned} t\frac{du}{dt} &= -4u \\ \frac{du}{u} &= -4\frac{dt}{t} \\ \int \frac{du}{u} &= -4\int \frac{dt}{t} \\ \ln|u| &= -4\ln|t| + C \\ u &= \pm e^C t^{-4} = ct^{-4} \end{aligned}$$

$$y_1(t) = t \qquad v' = u \qquad u = ct^{-4}$$

and

$$\begin{aligned} v &= \int u \ dt = \int ct^{-4} \ dt \\ &= -\frac{1}{3}ct^{-3} + k. \end{aligned}$$

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Thus

$$y_2(t) = v(t)t = -\frac{1}{3}ct^{-2} + kt.$$

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Thus

$$y_2(t) = v(t)t = -\frac{1}{3}ct^{-2} + kt.$$

Choosing $c = -3$ and $k = 0$, we obtain the solution

$$y_2(t) = t^{-2}.$$

Does $y_2(t) = t^{-2}$ really solve $t^2y'' + 2ty' - 2y = 0$?

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Since $y_2' = -2t^{-3}$ and $y_2'' = 6t^{-4}$, we have that

$$\begin{aligned} t^2y_2'' + 2ty_2' - 2y_2 &= t^2(6t^{-4}) + 2t(-2t^{-3}) - 2t^{-2} \\ &= 6t^{-2} - 4t^{-2} - 2t^{-2} \\ &= 0. \end{aligned}$$

The answer is YES!!

Are $y_1(t) = t$ and $y_2(t) = t^{-2}$ linearly independent?

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We have that

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} t & t^{-2} \\ 1 & -2t^{-3} \end{vmatrix} = -2t^{-2} - t^{-2} = -3t^{-2} \neq 0$$

since $t > 0$. Therefore y_1 and y_2 are linearly independent.

Nonhomogeneous Equations

3.6

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Consider

$$y'' + p(t)y' + q(t)y = g(t). \tag{5}$$

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The equation

$$y'' + p(t)y' + q(t)y = 0 \tag{6}$$

is called the *homogeneous equation corresponding to* (5).

3.6 Nonhomogeneous Equations

$$y'' + p(t)y' + q(t)y = g(t) \tag{5}$$

Theorem

The general solution to (5) can be written in the form

$$y(t) = c_1 y_1(t) + c_2 y_2(t) + Y(t)$$

where

$$y'' + p(t)y' + q(t)y = g(t) \tag{5}$$

Theorem

The general solution to (5) can be written in the form

$$y(t) = c_1 y_1(t) + c_2 y_2(t) + Y(t)$$

where

- y_1 *and* y_2 *form a fundamental set of solutions to the homogeneous equation corresponding to (5);*
- c_1 *and* c_2 *are constants; and*
- Y *is a particular solution to (5).*

To solve $L[y] = g$

1. Find the general solution to $L[y] = 0$;
2. Find a particular solution to $L[y] = g$;
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We will study 2 methods to do step 2. One method this week and one method next week.

The Method of Undetermined Coefficients

$$y'' + p(t)y' + q(t)y = g(t) \tag{5}$$

The idea is:

1. Look at $g(t)$
2. Make a guess with constants
3. Try to find the constants

Example

Find a particular solution to $y'' - 3y' - 4y = 3e^{2t}$.

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We must try to find A. We calculate that

$$Y(t) = Ae^{2t} \qquad Y'(t) = 2Ae^{2t} \qquad Y''(t) = 4Ae^{2t}$$

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and

$$\begin{aligned} 3e^{2t} &= Y'' - 3Y' - 4Y = 4Ae^{2t} - 3\left(2Ae^{2t}\right) - 4\left(Ae^{2t}\right) \\ &= -6Ae^{2t}. \end{aligned}$$

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We must have $A = -\frac{1}{2}$. Therefore a particular solution is

$$\boxed{Y(t) = -\frac{1}{2}e^{2t}.}$$

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Find a particular solution to $y'' - 3y' - 4y = 4t^2 - 1$.

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Since $g(t) = 4t^2 - 1$ is a 2nd degree polynomial, we guess that Y is also a second degree polynomial. So we try the ansatz

$$Y(t) = At^2 + Bt + C.$$

I will leave this example for you to finish.

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(ansatz = a mathematical guess)

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First guess: $Y(t) = A\sin t$. Then $Y' = A\cos t$ and $Y'' = -A\sin t$. Hence

$$\begin{aligned} 2\sin t &= Y'' - 3Y' - 4Y \\ &= (-A\sin t) - 3(A\cos t) - 4(A\sin t) = -5A\sin t - 3A\cos t. \end{aligned}$$

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$$\begin{cases} -5A = 2 \\ -3A = 0. \end{cases}$$

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We can see that we must have

$$\begin{cases} -5A = 2 \\ -3A = 0. \end{cases}$$

This linear system is inconsistent: It not possible to find a constant A which satisfies both of these equations. Our first guess failed.

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$$Y' = A\cos t - B\sin t, \qquad\qquad Y'' = -A\sin t - B\cos t$$

and

$$\begin{aligned} 2\sin t &= Y'' - 3Y' - 4Y \\ &= (-A\sin t - B\cos t) - 3(A\cos t - B\sin t) - 4(A\sin t + B\cos t) \\ &= (-5A + 3B)\sin t + (-3A - 5B)\cos t. \end{aligned}$$

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So we need A and B to satisfy

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Please check that the solution to this linear system is $A = -\frac{5}{17}$ and $B = \frac{3}{17}$.

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$$\boxed{Y(t) = -\frac{5}{17}\sin t + \frac{3}{17}\cos t.}$$

Remark

sin and cos are friends! They always go together. If you see either sin or cos in $g(t)$, then your ansatz needs to contain *both* sin and cos.
Likewise sinh and cosh always go together.

Example

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$$Y(t) = Ae^t \cos 2t + Be^t \sin 2t.$$

Then

$$\begin{aligned} Y'(t) &= Ae^t \cos 2t - 2Ae^t \sin 2t + Be^t \sin 2t + 2Be^t \cos 2t \\ &= (A + 2B)e^t \cos 2t + (B - 2A)e^t \sin 2t, \\ Y''(t) &= (A + 2B)e^t \cos 2t - 2(A + 2B)e^t \sin 2t + (B - 2A)e^t \sin 2t \\ &\qquad + 2(B - 2A)e^t \cos 2t \\ &= (-3A + 4B)e^t \cos 2t + (-4A - 3B)e^t \sin 2t \end{aligned}$$

and

$$
\begin{aligned}
-8e^t\cos 2t &= Y''-3Y'-4Y \\
&= (-3A+4B)e^t\cos 2t + (-4A-3B)e^t\sin 2t \\
&\quad + (-3A-6B)e^t\cos 2t + (-3B+6A)e^t\sin 2t \\
&\quad + (-4A)e^t\cos 2t + (-4B)e^t\sin 2t \\
&= (-10A-2B)e^t\cos 2t + (2A-10B)e^t\sin 2t.
\end{aligned}
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\end{aligned}
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Thus we must solve

$$
\begin{cases}
10A + 2B = 8 \\
2A - 10B = 0.
\end{cases}
$$

and

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\begin{cases} 10A + 2B = 8 \\ 2A - 10B = 0. \end{cases}
$$

Please check that the solution to this linear system is $A = \frac{10}{13}$ and $B = \frac{2}{13}$. Therefore a particular solution is

$$
\boxed{Y(t) = \frac{10}{13}e^t \cos 2t + \frac{2}{13}e^t \sin 2t.}
$$

Theorem

$$\left.\begin{array}{c} Y_1 \ \ solves \\ ay'' + by' + cy = g_1(t) \\ \\ Y_2 \ \ solves \\ ay'' + by' + cy = g_2(t) \end{array}\right\} \implies \begin{array}{c} Y_1 + Y_2 \ \ solves \\ ay'' + by' + cy = g_1(t) + g_2(t) \end{array}$$

Example

Find a particular solution to

$$y'' - 3y' - 4y = 3e^{2t} + 2\sin t - 8e^{t}\cos 2t. \tag{7}$$

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We can split this problem up into three easier problems:

$$\begin{aligned} y'' - 3y' - 4y &= 3e^{2t} \\ y'' - 3y' - 4y &= 2\sin t \\ y'' - 3y' - 4y &= -8e^{t}\cos 2t \end{aligned}$$

We know particular solutions to these three ODEs.

Example

Find a particular solution to

$$y'' - 3y' - 4y = 3e^{2t} + 2\sin t - 8e^{t}\cos 2t. \tag{7}$$

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We know particular solutions to these three ODEs. Therefore

$$\boxed{Y(t) = -\frac{1}{2}e^{2t} - \frac{5}{17}\sin t + \frac{3}{17}\cos t + \frac{10}{13}e^{t}\cos 2t + \frac{2}{13}e^{t}\sin 2t.}$$

is a particular solution to (7).

Remark

To find a particular solution to $ay'' + by' + cy = g(t)$, we have been looking at $g(t)$ and choosing a similar function for $Y(t)$.

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To find a particular solution to $ay'' + by' + cy = g(t)$, we have been looking at $g(t)$ and choosing a similar function for $Y(t)$.

This method doesn't always work: There is a difficulty that can occur as we shall see in the next example.

Example

Find a particular solution to $y'' + 4y = 3\cos 2t$.

First guess: $Y(t) = A\cos 2t + B\sin 2t$.

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First guess: $Y(t) = A\cos 2t + B\sin 2t$.
Then we have that

$$\begin{aligned} Y' &= -2A\sin 2t + 2B\cos 2t \\ Y'' &= -4A\cos 2t - 4B\sin 2t \end{aligned}$$

and

$$\begin{aligned} 3\cos 2t &= Y'' + 4Y \\ &= (-4A\cos 2t - 4B\sin 2t) + 4(A\cos 2t + B\sin 2t) = 0. \end{aligned}$$

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$$\begin{aligned} 3\cos 2t &= Y'' + 4Y \\ &= (-4A\cos 2t - 4B\sin 2t) + 4(A\cos 2t + B\sin 2t) = 0. \end{aligned}$$

This is a FAILURE!!! It not possible to choose A and B such that

$$3\cos 2t = 0$$

for all t.

Why did this happen? Why didn't our usual method work?

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$$0 = r^2 + 4 = (r + 2i)(r - 2i).$$

So $r = \pm 2i$.

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RULE: If in doubt, multiply by t.

We need two functions which, when differentiated, give us $\cos 2t$ and $\sin 2t$. We will try $t \cos 2t$ and $t \sin 2t$ because $\frac{d}{dt} t \cos 2t = \cos 2t - 2t \sin 2t$ and $\frac{d}{dt} t \sin 2t = \sin 2t + 2t \cos 2t$.

Second guess: $Y(t) = At\cos 2t + Bt\sin 2t$.

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We have that

$$\begin{aligned}
Y' &= A\cos 2t - 2At\sin 2t + B\sin 2t + 2Bt\cos 2t \\
&= (A + 2Bt)\cos 2t + (B - 2At)\sin 2t, \\
Y'' &= 2B\cos 2t - 2(A + 2Bt)\sin 2t - 2A\sin 2t + 2(B - 2At)\cos 2t \\
&= (4B - 4At)\cos 2t + (-4A - 4Bt)\sin 2t
\end{aligned}$$

and

$$\begin{aligned}
3\cos 2t &= Y'' + 4Y \\
&= (4B - 4At)\cos 2t + (-4A - 4Bt)\sin 2t \\
&\quad + 4At\cos 2t + 4Bt\sin 2t \\
&= 4B\cos 2t - 4A\sin 2t.
\end{aligned}$$

$$3\cos 2t = 4B\cos 2t - 4A\sin 2t$$

Thus

$$\begin{cases} -4A = 0 \\ 4B = 3 \end{cases}$$

which has solution $A = 0$ and $B = \frac{3}{4}$.

$$3\cos 2t = 4B\cos 2t - 4A\sin 2t$$

Thus

$$\begin{cases} -4A = 0 \\ 4B = 3 \end{cases}$$

which has solution $A = 0$ and $B = \frac{3}{4}$. Therefore a particular solution is

$$\boxed{Y(t) = \frac{3}{4}t\sin 2t.}$$

Next Week

- 3.8 Solving Initial Value Problems
- 3.9 The Method of Variation of Parameters
- 3.10 Higher Order Linear ODEs