Lecture 11

- 5.6 Repeated Eigenvalues
- 5.7 Nonhomogeneous Linear Systems

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Example

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Therefore $r_1 = 2 = r_2$. Moreover

$$\mathbf{0} = (A - rI)\boldsymbol{\xi} = \begin{bmatrix} -1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix} \implies \xi_1 + \xi_2 = 0 \implies \boldsymbol{\xi} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

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Note that A has only one linearly independent eigenvector.

Example

Solve
$$\mathbf{x}' = \begin{bmatrix} 1 & -1 \\ 1 & 3 \end{bmatrix} \mathbf{x}.$$

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We know that

$$\mathbf{x}^{(1)}(t) = \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{2t}$$

is a solution. But we need two solutions.

Guess 1: I guess that

$$\mathbf{x}^{(2)}(t) = \boldsymbol{\xi} t e^{2t}$$

for some $\boldsymbol{\xi} \in \mathbb{R}^2$.

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This guess did not work.

Guess 2: Now I guess that

$$\mathbf{x}^{(2)}(t) = \boldsymbol{\xi} t e^{2t} + \boldsymbol{\eta} e^{2t}$$

for some $\boldsymbol{\xi}, \boldsymbol{\eta} \in \mathbb{R}^2$.

Guess 2: Now I guess that

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$$\boldsymbol{\xi} e^{2t} + 2\boldsymbol{\xi} t e^{2t} + 2\boldsymbol{\eta} e^{2t} = \mathbf{x}^{(2)\prime} = A\mathbf{x}^{(2)} = A\left(\boldsymbol{\xi} t e^{2t} + \boldsymbol{\eta} e^{2t}\right)$$

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and

$$\left(2\boldsymbol{\xi} - A\boldsymbol{\xi}\right) t + \left(\boldsymbol{\xi} + 2\boldsymbol{\eta} - A\boldsymbol{\eta}\right) = \mathbf{0}.$$

Since this must be true $\forall\ t$, we must have

$$(A - 2I)\,\boldsymbol{\xi} = \mathbf{0} \qquad \text{and} \qquad (A - 2I)\,\boldsymbol{\eta} = \boldsymbol{\xi}.$$

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Clearly $\boldsymbol{\xi} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$.

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$$\begin{bmatrix} -1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \eta_1 \\ \eta_2 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \implies \eta_1 + \eta_2 = -1$$

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$$\implies \boldsymbol{\eta} = \begin{bmatrix} k \\ -1 - k \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \end{bmatrix} + k \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

for some k.

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for some k. So

$$\mathbf{x}^{(2)}(t) = \boldsymbol{\xi} t e^{2t} + \boldsymbol{\eta} e^{2t} = \begin{bmatrix} 1 \\ -1 \end{bmatrix} t e^{2t} + \begin{bmatrix} 0 \\ -1 \end{bmatrix} e^{2t} + k \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{2t}$$

$$(A-2I)\,\boldsymbol{\xi}=\mathbf{0} \qquad (A-2I)\,\boldsymbol{\eta}=\boldsymbol{\xi}$$

Clearly $\boldsymbol{\xi}=\begin{bmatrix}1\\-1\end{bmatrix}$. Then we calculate that

$$\begin{bmatrix}-1 & -1\\ 1 & 1\end{bmatrix}\begin{bmatrix}\eta_1\\ \eta_2\end{bmatrix}=\begin{bmatrix}1\\-1\end{bmatrix} \implies \eta_1+\eta_2=-1$$

$$\implies \boldsymbol{\eta}=\begin{bmatrix}k\\-1-k\end{bmatrix}=\begin{bmatrix}0\\-1\end{bmatrix}+k\begin{bmatrix}1\\-1\end{bmatrix}$$

for some k. So

$$\begin{aligned}\mathbf{x}^{(2)}(t)=\boldsymbol{\xi}te^{2t}+\boldsymbol{\eta}e^{2t}&=\begin{bmatrix}1\\-1\end{bmatrix}te^{2t}+\begin{bmatrix}0\\-1\end{bmatrix}e^{2t}+k\begin{bmatrix}1\\-1\end{bmatrix}e^{2t}\\&=\begin{bmatrix}1\\-1\end{bmatrix}te^{2t}+\begin{bmatrix}0\\-1\end{bmatrix}e^{2t}+k\mathbf{x}^{(1)}(t).\end{aligned}$$

$$\mathbf{x}^{(2)}(t) = \begin{bmatrix} 1 \\ -1 \end{bmatrix} te^{2t} + \begin{bmatrix} 0 \\ -1 \end{bmatrix} e^{2t} + k\mathbf{x}^{(1)}(t)$$

Because we already have $\mathbf{x}^{(1)}(t)$, we can choose $k = 0$. So

$$\mathbf{x}^{(2)}(t) = \boldsymbol{\xi} te^{2t} + \boldsymbol{\eta} e^{2t} = \begin{bmatrix} 1 \\ -1 \end{bmatrix} te^{2t} + \begin{bmatrix} 0 \\ -1 \end{bmatrix} e^{2t}.$$

The general solution of $\mathbf{x}' = \begin{bmatrix} 1 & -1 \\ 1 & 3 \end{bmatrix} \mathbf{x}$ is therefore

$$\boxed{\mathbf{x}(t) = c_1 \mathbf{x}^{(1)} + c_2 \mathbf{x}^{(2)} = c_1 \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{2t} + c_2 \left(\begin{bmatrix} 1 \\ -1 \end{bmatrix} te^{2t} + \begin{bmatrix} 0 \\ -1 \end{bmatrix} e^{2t} \right).}$$

Example

Find a fundamental matrix for

$$\mathbf{x}' = \begin{bmatrix} 1 & -1 \\ 1 & 3 \end{bmatrix} \mathbf{x}.$$

Then find the special fundamental matrix $\Phi(t)$ which satisfies $\Phi(0) = I$.

Since $\mathbf{x}^{(1)}(t) = \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{2t}$ and $\mathbf{x}^{(2)}(t) = \begin{bmatrix} 1 \\ -1 \end{bmatrix} te^{2t} + \begin{bmatrix} 0 \\ -1 \end{bmatrix} e^{2t}$ we have that

$$\Psi(t) = \begin{bmatrix} \mathbf{x}^{(1)} & \mathbf{x}^{(2)} \end{bmatrix} = \begin{bmatrix} e^{2t} & te^{2t} \\ -e^{2t} & -te^{2t} - e^{2t} \end{bmatrix}$$

is a fundamental matrix for this system.

$$\boxed{\Psi(t) = \begin{bmatrix} e^{2t} & te^{2t} \\ -e^{2t} & -te^{2t} - e^{2t} \end{bmatrix}}$$

Now

$$\Psi(0) = \begin{bmatrix} 1 & 0 \\ -1 & -1 \end{bmatrix} \qquad \text{and} \qquad \Psi^{-1}(0) = \begin{bmatrix} 1 & 0 \\ -1 & -1 \end{bmatrix}.$$

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Therefore

$$\exp(At) = \Phi(t) = \Psi(t)\Psi^{-1}(0)$$

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Therefore

$$\exp(At) = \Phi(t) = \Psi(t)\Psi^{-1}(0) = e^{2t} \begin{bmatrix} 1 & t \\ -1 & -1 - t \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -1 & -1 \end{bmatrix}$$

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Therefore

$$\begin{aligned} \exp(At) = \Phi(t) = \Psi(t)\Psi^{-1}(0) &= e^{2t} \begin{bmatrix} 1 & t \\ -1 & -1-t \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -1 & -1 \end{bmatrix} \\ &= e^{2t} \begin{bmatrix} 1-t & -t \\ t & 1+t \end{bmatrix}. \end{aligned}$$

Remark

$$\mathbf{x}' = A\mathbf{x}$$

For two repeated eigenvalues (but with only one linearly independent eigenvector), the key equations to remember are

$$\boxed{\mathbf{x}^{(2)}(t) = \boldsymbol{\xi} t e^{rt} + \boldsymbol{\eta} e^{rt}} \qquad \text{and} \qquad \boxed{(A - rI)\boldsymbol{\eta} = \boldsymbol{\xi}}.$$

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Definition

η is called a *generalised eigenvector* of A.

Remark

If you have 2 repeated eigenvalues (but with only one linearly independent eigenvector), the method is:

1. Find the eigenvalues and eigenvectors;
2. The first solution is $\mathbf{x}^{(1)}(t) = \boldsymbol{\xi} e^{rt}$;
3. Use $(A - rI)\boldsymbol{\eta} = \boldsymbol{\xi}$ to find a generalised eigenvector $\boldsymbol{\eta}$;
4. The second solution is $\mathbf{x}^{(2)}(t) = \boldsymbol{\xi} t e^{rt} + \boldsymbol{\eta} e^{rt}$.

Example

Solve

$$\begin{cases} \mathbf{x}' = \begin{bmatrix} -\frac{5}{2} & \frac{3}{2} \\ -\frac{3}{2} & \frac{1}{2} \end{bmatrix} \mathbf{x}, \\ \mathbf{x}(0) = \begin{bmatrix} 3 \\ -1 \end{bmatrix}. \end{cases}$$

The only eigenvalue of the matrix is $r = -1$. The corresponding eigenvector is $\boldsymbol{\xi} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. Therefore one solution of the linear system is

$$\mathbf{x}^{(1)}(t) = \boldsymbol{\xi} e^{rt} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{-t}.$$

We need to find a second, linearly independent solution. We will consider the ansatz

$$\mathbf{x}^{(2)}(t) = \boldsymbol{\xi} t e^{-t} + \boldsymbol{\eta} e^{-t}$$

where $\boldsymbol{\xi} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ as above and $\boldsymbol{\eta}$ is a generalised eigenvector solving $(A - rI)\boldsymbol{\eta} = \boldsymbol{\xi}$.

Solving the latter equation,

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$$-\frac{3}{2}\eta_1 + \frac{3}{2}\eta_2 = 1$$

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$$-\frac{3}{2}\eta_1 + \frac{3}{2}\eta_2 = 1$$

$$-\eta_1 + \eta_2 = \frac{2}{3}$$

Solving the latter equation,

$$
\begin{aligned}
(A - rI)\boldsymbol{\eta} &= \boldsymbol{\xi} \\
\begin{bmatrix} -\frac{3}{2} & \frac{3}{2} \\ -\frac{3}{2} & \frac{3}{2} \end{bmatrix} \begin{bmatrix} \eta_1 \\ \eta_2 \end{bmatrix} &= \begin{bmatrix} 1 \\ 1 \end{bmatrix} \\
-\frac{3}{2}\eta_1 + \frac{3}{2}\eta_2 &= 1 \\
-\eta_1 + \eta_2 &= \frac{2}{3}
\end{aligned}
$$

we can choose $\boldsymbol{\eta} = \begin{bmatrix} 0 \\ \frac{2}{3} \end{bmatrix}$.

Note that we don't need to find *every* generalised eigenvector

$$\boldsymbol{\eta} = \begin{bmatrix} k \\ k + \frac{2}{3} \end{bmatrix} = k \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{2}{3} \end{bmatrix} = k\boldsymbol{\xi} + \begin{bmatrix} 0 \\ \frac{2}{3} \end{bmatrix}$$

because we already have $\mathbf{x}^{(1)}(t) = \boldsymbol{\xi} e^{rt}$.

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because we already have $\mathbf{x}^{(1)}(t) = \boldsymbol{\xi} e^{rt}$.

Instead we only need to find *one* generalised eigenvector – that means that we can choose any k that we want.
Hence I have chosen $k = 0$ which gives $\boldsymbol{\eta} = \begin{bmatrix} 0 \\ \frac{2}{3} \end{bmatrix}$.

eigenvector

$$\boldsymbol{\xi} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

generalised eigenvector

$$\boldsymbol{\eta} = \begin{bmatrix} 0 \\ \frac{2}{3} \end{bmatrix}$$

Thus

$$\mathbf{x}^{(1)}(t) = \boldsymbol{\xi} e^{-t} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{-t}$$

and

$$\mathbf{x}^{(2)}(t) = \boldsymbol{\xi} t e^{-t} + \boldsymbol{\eta} e^{-t} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} t e^{-t} + \begin{bmatrix} 0 \\ \frac{2}{3} \end{bmatrix} e^{-t}.$$

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Hence the general solution to the linear system is

$$\mathbf{x}(t) = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{-t} + c_2 \left(\begin{bmatrix} 1 \\ 1 \end{bmatrix} t e^{-t} + \begin{bmatrix} 0 \\ \frac{2}{3} \end{bmatrix} e^{-t} \right).$$

The initial condition gives

$$\begin{bmatrix} 3 \\ -1 \end{bmatrix} = \mathbf{x}(0) = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ \frac{2}{3} \end{bmatrix}$$

which implies that $c_1 = 3$ and $c_2 = -6$.

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Therefore the solution to the IVP is

$$\boxed{\mathbf{x}(t) = 3 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{-t} - 6 \left(\begin{bmatrix} 1 \\ 1 \end{bmatrix} te^{-t} + \begin{bmatrix} 0 \\ \frac{2}{3} \end{bmatrix} e^{-t} \right) = \begin{bmatrix} 3 \\ -1 \end{bmatrix} e^{-t} - 6 \begin{bmatrix} 1 \\ 1 \end{bmatrix} te^{-t}.}$$

Example

Solve

$$\mathbf{x}' = \begin{bmatrix} 1 & -4 \\ 4 & -7 \end{bmatrix} \mathbf{x}, \qquad \mathbf{x}(0) = \begin{bmatrix} 3 \\ 2 \end{bmatrix}.$$

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Example

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$$\mathbf{x}' = \begin{bmatrix} 1 & -4 \\ 4 & -7 \end{bmatrix} \mathbf{x}, \qquad \mathbf{x}(0) = \begin{bmatrix} 3 \\ 2 \end{bmatrix}.$$

The only eigenvalue of the matrix is $r = -3$. The corresponding eigenvector is $\boldsymbol{\xi} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. Therefore one solution of the linear system is

$$\mathbf{x}^{(1)}(t) = \boldsymbol{\xi} e^{rt} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{-3t}.$$

Next we need to find a generalised eigenvector $\boldsymbol{\eta}$.

We calculate that

$$(A - rI)\boldsymbol{\eta} = \boldsymbol{\xi}$$

We calculate that

$$\begin{bmatrix} 4 & -4 \\ 4 & -4 \end{bmatrix} \begin{bmatrix} \eta_1 \\ \eta_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

We calculate that

$$4\eta_1 - 4\eta_2 = 1$$

We calculate that

$$-\eta_1 + \eta_2 = -\frac{1}{4}$$

We calculate that

$$\eta_2 = \eta_1 - \frac{1}{4}.$$

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We can choose any vector $\boldsymbol{\eta}$ that satisfies $\eta_2 = \eta_1 - \frac{1}{4}$.

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We can choose any vector $\boldsymbol{\eta}$ that satisfies $\eta_2 = \eta_1 - \frac{1}{4}$. Thus we may choose $\boldsymbol{\eta} = \begin{bmatrix} 0 \\ -\frac{1}{4} \end{bmatrix}$.

eigenvector

$$\boldsymbol{\xi} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

generalised eigenvector

$$\boldsymbol{\eta} = \begin{bmatrix} 0 \\ -\frac{1}{4} \end{bmatrix}$$

Therefore

$$\mathbf{x}^{(2)}(t) = \begin{bmatrix} 1 \\ 1 \end{bmatrix} te^{-3t} + \begin{bmatrix} 0 \\ -\frac{1}{4} \end{bmatrix} e^{-3t}.$$

Hence the general solution to the ODE is

$$\mathbf{x}(t) = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{-3t} + c_2 \left(\begin{bmatrix} 1 \\ 1 \end{bmatrix} te^{-3t} + \begin{bmatrix} 0 \\ -\frac{1}{4} \end{bmatrix} e^{-3t} \right).$$

The initial condition gives

$$\begin{bmatrix} 3 \\ 2 \end{bmatrix} = \mathbf{x}(0) = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ -\frac{1}{4} \end{bmatrix}$$

which implies that $c_1 = 3$ and $c_2 = 4$.

Therefore the solution to the IVP is

$$\mathbf{x}(t) = 3\begin{bmatrix}1\\1\end{bmatrix}e^{-3t} + 4\left(\begin{bmatrix}1\\1\end{bmatrix}te^{-3t} + \begin{bmatrix}0\\-\frac{1}{4}\end{bmatrix}e^{-3t}\right)$$

$$=$$

$$= \qquad .$$

Therefore the solution to the IVP is

$$\begin{aligned}\mathbf{x}(t) &= 3\begin{bmatrix}1\\1\end{bmatrix}e^{-3t} + 4\left(\begin{bmatrix}1\\1\end{bmatrix}te^{-3t} + \begin{bmatrix}0\\-\frac{1}{4}\end{bmatrix}e^{-3t}\right)\\ &= \begin{bmatrix}3\\2\end{bmatrix}e^{-3t} + 4\begin{bmatrix}1\\1\end{bmatrix}te^{-3t}\\ &= \qquad\qquad\qquad .\end{aligned}$$

Therefore the solution to the IVP is

$$\begin{aligned}
\mathbf{x}(t) &= 3\begin{bmatrix}1\\1\end{bmatrix}e^{-3t} + 4\left(\begin{bmatrix}1\\1\end{bmatrix}te^{-3t} + \begin{bmatrix}0\\-\frac{1}{4}\end{bmatrix}e^{-3t}\right)\\
&= \begin{bmatrix}3\\2\end{bmatrix}e^{-3t} + 4\begin{bmatrix}1\\1\end{bmatrix}te^{-3t}\\
&= \begin{bmatrix}3+4t\\2+4t\end{bmatrix}e^{-3t}.
\end{aligned}$$

5.7 Nonhomogeneous Linear Systems

Consider

$$\mathbf{x}' = P(t)\mathbf{x} + \mathbf{g}(t) \tag{1}$$

where $P(t)$ and $\mathbf{g}(t)$ are continuous for $\alpha < t < \beta$.

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- $c_1\mathbf{x}^{(1)} + c_2\mathbf{x}^{(2)} + \ldots + c_n\mathbf{x}^{(n)}$ is the general solution to the homogeneous system $\mathbf{x}' = P(t)\mathbf{x}$; and

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- $c_1\mathbf{x}^{(1)} + c_2\mathbf{x}^{(2)} + \ldots + c_n\mathbf{x}^{(n)}$ is the general solution to the homogeneous system $\mathbf{x}' = P(t)\mathbf{x}$; and
- $\mathbf{v}(t)$ is a particular solution to (1).

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Remark

We will study four methods to solve (1):

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We will study four methods to solve (1):

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4. The Laplace Transform.

Method 1 – Diagonalisation:

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Suppose that

- $A \in \mathbb{R}^{n\times n}$ is diagonalisable;
- $\mathbf{g} : (\alpha, \beta) \to \mathbb{R}^n$;
- $\boldsymbol{\xi}^{(1)}, \ldots, \boldsymbol{\xi}^{(n)}$ are eigenvectors of A; and
- $T = \begin{bmatrix} \boldsymbol{\xi}^{(1)} & \ldots & \boldsymbol{\xi}^{(n)} \end{bmatrix}$.

Then

$$D = T^{-1}AT = \begin{bmatrix} r_1 & 0 & \cdots & 0 \\ 0 & r_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & r_n \end{bmatrix}.$$

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and

$$\mathbf{y}' = T^{-1}AT\mathbf{y} + T^{-1}\mathbf{g}(t) = D\mathbf{y} + \mathbf{h}(t) \tag{2}$$

where $\mathbf{h} = T^{-1}\mathbf{g}$.

But $\mathbf{y}' = D\mathbf{y} + \mathbf{h}(t)$ is just the system

$$\begin{cases} y_1' = r_1 y_1 + h_1(t) \\ y_2' = r_2 y_2 + h_2(t) \\ \qquad \vdots \\ y_n' = r_n y_n + h_n(t) \end{cases}$$

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$$y_j' - r_j y_j = h_j$$

(see Chapter 2) is

$$y_j(t) = e^{r_j t} \int_{t_0}^{t} e^{-r_j s} h(s)\, ds + c_j e^{r_j t}.$$

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If we know $\mathbf{y}$, then we know $\mathbf{x} = T\mathbf{y}$.

Example

Solve

$$\mathbf{x}' = \begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 2e^{-t} \\ 3t \end{bmatrix} = A\mathbf{x} + \mathbf{g}(t).$$

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The eigenvalues of $A = \begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix}$ are $r_1 = -3$ and $r_2 = -1$.

The eigenvectors are

$$\boldsymbol{\xi}^{(1)} = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \qquad \text{and} \qquad \boldsymbol{\xi}^{(2)} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

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So
$$T = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \qquad \text{and} \qquad T^{-1} = \frac{1}{2} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}.$$

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$$= D\mathbf{y} + \frac{1}{2}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 2e^{-t} \\ 3t \end{bmatrix}$$

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$$
\begin{aligned}
T\mathbf{y}' = \mathbf{x}' = A\mathbf{x} + \begin{bmatrix} 2e^{-t} \\ 3t \end{bmatrix} &= AT\mathbf{y} + \begin{bmatrix} 2e^{-t} \\ 3t \end{bmatrix} \\
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&= D\mathbf{y} + \frac{1}{2}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 2e^{-t} \\ 3t \end{bmatrix} \\
&= \begin{bmatrix} -3 & 0 \\ 0 & -1 \end{bmatrix}\mathbf{y} + \frac{1}{2}\begin{bmatrix} 2e^{-t} - 3t \\ 2e^{-t} + 3t \end{bmatrix}.
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Therefore

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You know how to solve first order linear ODEs. The solutions to these two ODEs are

$$y_1(t) = \frac{1}{2}e^{-t} - \frac{t}{2} + \frac{1}{6} + c_1e^{-3t}$$
$$y_2(t) = te^{-t} + \frac{3t}{2} - \frac{3}{2} + c_2e^{-t}.$$

Finally we calculate that

$$\mathbf{x} = T\mathbf{y} = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

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Example

Solve

$$\mathbf{x}' = \begin{bmatrix} 1 & \sqrt{3} \\ \sqrt{3} & -1 \end{bmatrix} \mathbf{x} + \begin{bmatrix} e^t \\ \sqrt{3}e^{-t} \end{bmatrix}.$$

The eigenvalues of $\begin{bmatrix} 1 & \sqrt{3} \\ \sqrt{3} & -1 \end{bmatrix}$ are $r_1 = -2$ and $r_2 = 2$. The corresponding eigenvectors are $\boldsymbol{\xi}^{(1)} = \begin{bmatrix} 1 \\ -\sqrt{3} \end{bmatrix}$ and $\boldsymbol{\xi}^{(2)} = \begin{bmatrix} \sqrt{3} \\ 1 \end{bmatrix}$. Thus

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Thus

$$T = \begin{bmatrix} 1 & \sqrt{3} \\ -\sqrt{3} & 1 \end{bmatrix},$$

$$T^{-1} = \frac{1}{\det T} \begin{bmatrix} 1 & -\sqrt{3} \\ \sqrt{3} & 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{4} & -\frac{\sqrt{3}}{4} \\ \frac{\sqrt{3}}{4} & \frac{1}{4} \end{bmatrix}$$

and

$$D = T^{-1}AT = \begin{bmatrix} -2 & 0 \\ 0 & 2 \end{bmatrix}.$$

Now we must change variables: Let $\mathbf{y} = T^{-1}\mathbf{x}$.

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$$\mathbf{y}' = D\mathbf{y} + T^{-1}\mathbf{g} = \begin{bmatrix} -2 & 0 \\ 0 & 2 \end{bmatrix} \mathbf{y} + \begin{bmatrix} \frac{1}{4} & -\frac{\sqrt{3}}{4} \\ \frac{\sqrt{3}}{4} & \frac{1}{4} \end{bmatrix} \begin{bmatrix} e^t \\ \sqrt{3}e^{-t} \end{bmatrix}$$

$$= \begin{bmatrix} -2y_1 \\ 2y_2 \end{bmatrix} + \begin{bmatrix} \frac{1}{4}e^t - \frac{3}{4}e^{-t} \\ \frac{\sqrt{3}}{4}e^t + \frac{\sqrt{3}}{4}e^{-t} \end{bmatrix}.$$

We know how to solve

$$y_1' + 2y_1 = \frac{1}{4}e^t - \frac{3}{4}e^{-t}$$

and

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The solutions are

$$y_1(t) = \frac{1}{12}e^t - \frac{3}{4}e^{-t} + c_1e^{-2t}$$

and

$$y_2(t) = -\frac{\sqrt{3}}{4}e^t - \frac{\sqrt{3}}{12}e^{-t} + c_2e^{2t}.$$

So

$$\mathbf{y} = \begin{bmatrix} \frac{1}{12}e^{t} - \frac{3}{4}e^{-t} + c_1 e^{-2t} \\ -\frac{\sqrt{3}}{4}e^{t} - \frac{\sqrt{3}}{12}e^{-t} + c_2 e^{2t} \end{bmatrix}.$$

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Therefore the general solution to the ODE is

$$\mathbf{x} = T\mathbf{y} = \begin{bmatrix} 1 & \sqrt{3} \\ -\sqrt{3} & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{12}e^t - \frac{3}{4}e^{-t} + c_1 e^{-2t} \\ -\frac{\sqrt{3}}{4}e^t - \frac{\sqrt{3}}{12}e^{-t} + c_2 e^{2t} \end{bmatrix} = \dots$$

Method 2 – Undetermined Coefficients:

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(Remember Chapter 3?)
The idea is

1. Find the general solution to $\mathbf{x}' = A\mathbf{x}$.
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The idea is

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3. $1 + 2$.

Example

Solve

$$\mathbf{x}' = \begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 2e^{-t} \\ 3t \end{bmatrix} = A\mathbf{x} + \mathbf{g}(t).$$

1. The solution of $\mathbf{x}' = \begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix} \mathbf{x}$ is

$$\mathbf{x}(t) = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{-t} + c_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{-3t}.$$

2. Since $\mathbf{g}(t) = \begin{bmatrix} 2 \\ 0 \end{bmatrix} e^{-t} + \begin{bmatrix} 0 \\ 3 \end{bmatrix} t$, we try the ansatz

$$\mathbf{x} = \mathbf{v}(t) = \mathbf{a}te^{-t} + \mathbf{b}e^{-t} + \mathbf{c}t + \mathbf{d}.$$

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Then we calculate that

$$\mathbf{x}' = A\mathbf{x} + \mathbf{g}$$

$$\mathbf{a}e^{-t} - \mathbf{a}te^{-t} - \mathbf{b}e^{-t} + \mathbf{c} = A\mathbf{a}te^{-t} + A\mathbf{b}e^{-t} + A\mathbf{c}t + A\mathbf{d} + \begin{bmatrix} 2 \\ 0 \end{bmatrix} e^{-t} + \begin{bmatrix} 0 \\ 3 \end{bmatrix} t.$$

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If we look at the te^{-t} terms, we have

$$-\mathbf{a} = A\mathbf{a}$$

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If we look at the te^{-t} terms, we have

$$-\mathbf{a} = A\mathbf{a} \implies \mathbf{a} \text{ is an eigenvector} \implies \mathbf{a} = \begin{bmatrix} \alpha \\ \alpha \end{bmatrix}$$

for some $\alpha \in \mathbb{R}$.

$$\mathbf{a}e^{-t} - \mathbf{a}te^{-t} - \mathbf{b}e^{-t} + \mathbf{c} = A\mathbf{a}te^{-t} + A\mathbf{b}e^{-t} + A\mathbf{c}t + A\mathbf{d} + \begin{bmatrix} 2 \\ 0 \end{bmatrix} e^{-t} + \begin{bmatrix} 0 \\ 3 \end{bmatrix} t$$

If we look at the e^{-t} terms, we have

$$\mathbf{a} - \mathbf{b} = A\mathbf{b} + \begin{bmatrix} 2 \\ 0 \end{bmatrix}$$

$$\mathbf{a}e^{-t} - \mathbf{a}te^{-t} - \mathbf{b}e^{-t} + \mathbf{c} = A\mathbf{a}te^{-t} + A\mathbf{b}e^{-t} + A\mathbf{c}t + A\mathbf{d} + \begin{bmatrix} 2 \\ 0 \end{bmatrix} e^{-t} + \begin{bmatrix} 0 \\ 3 \end{bmatrix} t$$

If we look at the e^{-t} terms, we have

$$\mathbf{a} - \mathbf{b} = A\mathbf{b} + \begin{bmatrix} 2 \\ 0 \end{bmatrix}$$

which becomes

$$\begin{bmatrix} \alpha - 2 \\ \alpha \end{bmatrix} = \mathbf{a} - \begin{bmatrix} 2 \\ 0 \end{bmatrix} = (A + I)\mathbf{b} = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} = \begin{bmatrix} -b_1 + b_2 \\ b_1 - b_2 \end{bmatrix}.$$

$$\mathbf{a}e^{-t} - \mathbf{a}te^{-t} - \mathbf{b}e^{-t} + \mathbf{c} = A\mathbf{a}te^{-t} + A\mathbf{b}e^{-t} + A\mathbf{c}t + A\mathbf{d} + \begin{bmatrix} 2 \\ 0 \end{bmatrix} e^{-t} + \begin{bmatrix} 0 \\ 3 \end{bmatrix} t$$

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$$\alpha - 2 = -b_1 + b_2 = -(b_1 - b_2) = -\alpha \quad \implies \alpha = 1.$$

So $\mathbf{a} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

Then we have that

$$b_1 - b_2 = 1 \qquad \Longrightarrow \qquad \mathbf{b} = \begin{bmatrix} k \\ k-1 \end{bmatrix}$$

for any k. If we choose $k = 0$, we get $\mathbf{b} = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$.

$$\mathbf{a}e^{-t} - \mathbf{a}te^{-t} - \mathbf{b}e^{-t} + \mathbf{c} = A\mathbf{a}te^{-t} + A\mathbf{b}e^{-t} + A\mathbf{c}t + A\mathbf{d} + \begin{bmatrix} 2 \\ 0 \end{bmatrix} e^{-t} + \begin{bmatrix} 0 \\ 3 \end{bmatrix} t$$

If we look at the t terms, we have

$$0 = A\mathbf{c} + \begin{bmatrix} 0 \\ 3 \end{bmatrix}$$

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If we look at the t terms, we have

$$0 = A\mathbf{c} + \begin{bmatrix} 0 \\ 3 \end{bmatrix} \qquad \Longrightarrow \mathbf{c} = A^{-1} \begin{bmatrix} 0 \\ -3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}.$$

$$\mathbf{a}e^{-t} - \mathbf{a}te^{-t} - \mathbf{b}e^{-t} + \mathbf{c} = A\mathbf{a}te^{-t} + A\mathbf{b}e^{-t} + A\mathbf{c}t + A\mathbf{d} + \begin{bmatrix} 2 \\ 0 \end{bmatrix} e^{-t} + \begin{bmatrix} 0 \\ 3 \end{bmatrix} t$$

Finally, if we look at the 1 terms, we have

$$\mathbf{c} = A\mathbf{d}$$

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Finally, if we look at the 1 terms, we have

$$\mathbf{c} = A\mathbf{d} \quad \Longrightarrow \quad \mathbf{d} = A^{-1}\mathbf{c} = \begin{bmatrix} -\frac{4}{3} \\ -\frac{5}{3} \end{bmatrix}.$$

So

$$\mathbf{v} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} te^{-t} - \begin{bmatrix} 0 \\ 1 \end{bmatrix} e^{-t} + \begin{bmatrix} 1 \\ 2 \end{bmatrix} t - \frac{1}{3} \begin{bmatrix} 4 \\ 5 \end{bmatrix}.$$

3. Therefore the general solution to the ODE is

$$\boxed{\mathbf{x}(t) = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{-t} + c_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{-3t} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} te^{-t} - \begin{bmatrix} 0 \\ 1 \end{bmatrix} e^{-t} + \begin{bmatrix} 1 \\ 2 \end{bmatrix} t - \frac{1}{3} \begin{bmatrix} 4 \\ 5 \end{bmatrix}.}$$

Example

Solve
$$\mathbf{x}' = \begin{bmatrix} 2 & 3 \\ 4 & 1 \end{bmatrix} \mathbf{x} + \begin{bmatrix} e^t \\ -10t - 3 \end{bmatrix}.$$

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3. $\mathbf{x}' = \begin{bmatrix} 2 & 3 \\ 4 & 1 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 0 \\ -10t - 3 \end{bmatrix}$

and then we will add the solutions together.

The matrix $\begin{bmatrix} 2 & 3 \\ 4 & 1 \end{bmatrix}$ has eigenvalues $r_1 = 5$ and $r_2 = -2$ and eigenvectors $\boldsymbol{\xi}^{(1)} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\boldsymbol{\xi}^{(2)} = \begin{bmatrix} -3 \\ 4 \end{bmatrix}$. Hence the general solution of $\mathbf{x}' = \begin{bmatrix} 2 & 3 \\ 4 & 1 \end{bmatrix} \mathbf{x}$ is

$$\mathbf{x}(t) = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{5t} + c_2 \begin{bmatrix} -3 \\ 4 \end{bmatrix} e^{-2t}.$$

Next we need to find a particular solution to

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we try the ansatz $\mathbf{x} = \mathbf{a}e^t$ for some $\mathbf{a} = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} \in \mathbb{R}^2$.

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$$\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} e^t = \mathbf{x}' = \begin{bmatrix} 2 & 3 \\ 4 & 1 \end{bmatrix} \mathbf{x} + \begin{bmatrix} e^t \\ 0 \end{bmatrix} = \begin{bmatrix} 2a_1 + 3a_2 + 1 \\ 4a_1 + a_2 \end{bmatrix} e^t$$

which gives

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which gives

$$\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} 2a_1 + 3a_2 + 1 \\ 4a_1 + a_2 \end{bmatrix}.$$

Hence $a_1 = 0$ and $a_2 = -\frac{1}{3}$. So $\mathbf{x} = \begin{bmatrix} 0 \\ -\frac{1}{3} \end{bmatrix} e^t$.

Then we need to find a particular solution to

$$\mathbf{x}' = \begin{bmatrix} 2 & 3 \\ 4 & 1 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 0 \\ -10t - 3 \end{bmatrix}.$$

Then we need to find a particular solution to $\mathbf{x}' = \begin{bmatrix} 2 & 3 \\ 4 & 1 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 0 \\ -10t - 3 \end{bmatrix}$. We try the ansatz $\mathbf{x} = \mathbf{a}t + \mathbf{b} = \begin{bmatrix} a_1 t + b_1 \\ a_2 t + b_2 \end{bmatrix}$ for some $\mathbf{a}, \mathbf{b} \in \mathbb{R}^2$

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$$\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \mathbf{x}' = \begin{bmatrix} 2 & 3 \\ 4 & 1 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 0 \\ t \end{bmatrix} = \begin{bmatrix} 2a_1 t + 2b_1 + 3a_2 t + 3b_2 \\ 4a_1 t + 4b_1 + a_2 t + b_2 - 10t - 3 \end{bmatrix}$$

which leads to $\begin{cases} 0 = 2a_1 + 3a_2 \\ a_1 = 2b_1 + 3b_2 \\ 0 = 4a_1 + a_2 - 10 \\ a_2 = 4b_1 + b_2 - 3 \end{cases}$.

Then we need to find a particular solution to $\mathbf{x}' = \begin{bmatrix} 2 & 3 \\ 4 & 1 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 0 \\ -10t - 3 \end{bmatrix}$. We try the ansatz $\mathbf{x} = \mathbf{a}t + \mathbf{b} = \begin{bmatrix} a_1 t + b_1 \\ a_2 t + b_2 \end{bmatrix}$ for some $\mathbf{a}, \mathbf{b} \in \mathbb{R}^2$ and calculate that

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which leads to $\begin{cases} 0 = 2a_1 + 3a_2 \\ a_1 = 2b_1 + 3b_2 \\ 0 = 4a_1 + a_2 - 10 \\ a_2 = 4b_1 + b_2 - 3 \end{cases}$ .The solution to this linear system is $\mathbf{a} = \begin{bmatrix} 3 \\ -2 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. Hence $\mathbf{x} = \begin{bmatrix} 3t \\ 1 - 2t \end{bmatrix}$.

Adding all of these together, we find that the general solution to the given ODE is

$$\boxed{\mathbf{x} = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{5t} + c_2 \begin{bmatrix} -3 \\ 4 \end{bmatrix} e^{-2t} + \begin{bmatrix} 0 \\ -\frac{1}{3} \end{bmatrix} e^{t} + \begin{bmatrix} 3t \\ 1 - 2t \end{bmatrix}.}$$

Method 3 – Variation of Parameters:

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Consider

$$\mathbf{x}' = P(t)\mathbf{x} + \mathbf{g}(t) \tag{1}$$

where

- P and $\mathbf{g}$ are continuous for $\alpha < t < \beta$;
- there exists a fundamental matrix $\Psi(t)$ for the homogeneous system $\mathbf{x}' = P(t)\mathbf{x}$.

We know that the general solution to $\mathbf{x}' = P(t)\mathbf{x}$ is $\mathbf{x} = \Psi(t)\mathbf{c}$.

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We guess that

$$\mathbf{x} = \Psi(t)\mathbf{u}(t)$$

is a solution to (1).

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We guess that

$$\mathbf{x} = \Psi(t)\mathbf{u}(t)$$

is a solution to (1). Can we find $\mathbf{u}(t)$?

If $\mathbf{x} = \Psi\mathbf{u}$, we can calculate that

$$\mathbf{x}' = P\mathbf{x} + \mathbf{g} \tag{3}$$

If $\mathbf{x} = \Psi\mathbf{u}$, we can calculate that

$$\Psi'\mathbf{u} + \Psi\mathbf{u}' = \mathbf{x}' = P\mathbf{x} + \mathbf{g} \tag{3}$$

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If $\mathbf{x} = \Psi\mathbf{u}$, we can calculate that

$$\Psi'\mathbf{u} + \Psi\mathbf{u}' = \mathbf{x}' = P\mathbf{x} + \mathbf{g} = P\Psi\mathbf{u} + \mathbf{g}. \tag{3}$$

But remember that

Ψ is a fundamental matrix for $\mathbf{x}' = P(t)\mathbf{x} \implies \Psi$ solves $\Psi' = P\Psi$.

If $\mathbf{x} = \Psi \mathbf{u}$, we can calculate that

$$\cancel{\Psi' \mathbf{u}} + \Psi \mathbf{u}' = \mathbf{x}' = P\mathbf{x} + \mathbf{g} = \cancel{P\Psi \mathbf{u}} + \mathbf{g}. \tag{3}$$

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$$\cancel{\Psi'\mathbf{u}} + \Psi\mathbf{u}' = \mathbf{x}' = P\mathbf{x} + \mathbf{g} = \cancel{P\Psi\mathbf{u}} + \mathbf{g}. \tag{3}$$

But remember that

Ψ is a fundamental matrix for $\mathbf{x}' = P(t)\mathbf{x} \implies \Psi$ solves $\Psi' = P\Psi$.

Hence (3) becomes

$$\Psi\mathbf{u}' = \mathbf{g}.$$

If $\mathbf{x} = \Psi\mathbf{u}$, we can calculate that

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Therefore

$$\mathbf{u}' = \Psi^{-1}\mathbf{g}$$

and

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If $\mathbf{x} = \Psi\mathbf{u}$, we can calculate that

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$$\mathbf{u} = \int \Psi^{-1}\mathbf{g}.$$

Hence

$$\mathbf{x} = \Psi(t)\mathbf{u}(t) = \Psi(t)\int \Psi^{-1}(s)g(s)\,ds.$$

Remark

To solve $\mathbf{x}' = P(t)\mathbf{x} + \mathbf{g}(t)$, the method is

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To solve $\mathbf{x}' = P(t)\mathbf{x} + \mathbf{g}(t)$, the method is

1. Find a fundamental matrix Ψ for $\mathbf{x}' = P(t)\mathbf{x}$;
2. Calculate $\mathbf{x} = \Psi(t) \int \Psi^{-1}(s) g(s)\, ds$.

Example

Solve

$$\mathbf{x}' = \begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 2e^{-t} \\ 3t \end{bmatrix} = A\mathbf{x} + \mathbf{g}(t).$$

Example

Solve

$$\mathbf{x}' = \begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 2e^{-t} \\ 3t \end{bmatrix} = A\mathbf{x} + \mathbf{g}(t).$$

The solution of $\mathbf{x}' = \begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix} \mathbf{x}$ is

$$\mathbf{x}(t) = c_1 \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{-3t} + c_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{-t}.$$

Example

Solve

$$\mathbf{x}' = \begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 2e^{-t} \\ 3t \end{bmatrix} = A\mathbf{x} + \mathbf{g}(t).$$

The solution of $\mathbf{x}' = \begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix} \mathbf{x}$ is

$$\mathbf{x}(t) = c_1 \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{-3t} + c_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{-t}.$$

So

$$\Psi(t) = \begin{bmatrix} e^{-3t} & e^{-t} \\ -e^{-3t} & e^{-t} \end{bmatrix}$$

is a fundamental matrix.

$$\Psi(t) = \begin{bmatrix} e^{-3t} & e^{-t} \\ -e^{-3t} & e^{-t} \end{bmatrix}$$

Then we calculate that

$$\Psi^{-1}(t) = \frac{1}{2e^{-4t}} \begin{bmatrix} e^{-t} & -e^{-t} \\ e^{-3t} & e^{-3t} \end{bmatrix} = \frac{1}{2} e^{4t} \begin{bmatrix} e^{-t} & -e^{-t} \\ e^{-3t} & e^{-3t} \end{bmatrix} = \begin{bmatrix} \frac{1}{2}e^{3t} & -\frac{1}{2}e^{3t} \\ \frac{1}{2}e^{t} & \frac{1}{2}e^{t} \end{bmatrix}$$

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Then we calculate that

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and

$$\begin{aligned} \int \Psi^{-1}(t)\mathbf{g}(t)\,dt &= \int \begin{bmatrix} \frac{1}{2}e^{3t} & -\frac{1}{2}e^{3t} \\ \frac{1}{2}e^{t} & \frac{1}{2}e^{t} \end{bmatrix} \begin{bmatrix} 2e^{-t} \\ 3t \end{bmatrix} dt \\ &= \int \begin{bmatrix} e^{2t} - \frac{3}{2}te^{3t} \\ 1 + \frac{3}{2}te^{t} \end{bmatrix} dt = \begin{bmatrix} \frac{1}{2}e^{2t} - \frac{1}{2}te^{3t} + \frac{1}{6}e^{3t} + c_1 \\ t + \frac{3}{2}te^{t} - \frac{3}{2}e^{t} + c_2 \end{bmatrix}. \end{aligned}$$

Therefore the solution to $\mathbf{x}' = A\mathbf{x} + g$ is

$$\mathbf{x} = \Psi(t) \int \Psi^{-1}(s)\mathbf{g}(s)\,ds$$

$$=$$

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Therefore the solution to $\mathbf{x}' = A\mathbf{x} + g$ is

$$\begin{aligned}
\mathbf{x} &= \Psi(t)\int \Psi^{-1}(s)\mathbf{g}(s)\,ds \\
&= \begin{bmatrix} e^{-3t} & e^{-t} \\ -e^{-3t} & e^{-t} \end{bmatrix} \begin{bmatrix} \frac{1}{2}e^{2t} - \frac{1}{2}te^{3t} + \frac{1}{6}e^{3t} + c_1 \\ t + \frac{3}{2}te^{t} - \frac{3}{2}e^{t} + c_2 \end{bmatrix} \\
&= \qquad .
\end{aligned}$$

Therefore the solution to $\mathbf{x}' = A\mathbf{x} + g$ is

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\mathbf{x} &= \Psi(t) \int \Psi^{-1}(s)\mathbf{g}(s)\,ds \\
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&= c_1 \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{-3t} + c_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{-t} + \frac{1}{2} \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{-t} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} te^{-t} + \begin{bmatrix} 1 \\ 2 \end{bmatrix} t - \frac{1}{3} \begin{bmatrix} 4 \\ 5 \end{bmatrix}.
\end{aligned}$$

Example

Solve

$$\mathbf{x}' = \begin{bmatrix} -4 & 2 \\ 2 & -1 \end{bmatrix} \mathbf{x} + \begin{bmatrix} t^{-1} \\ 2t^{-1} + 4 \end{bmatrix}$$

for $t > 0$.

Example

Solve

$$\mathbf{x}' = \begin{bmatrix} -4 & 2 \\ 2 & -1 \end{bmatrix} \mathbf{x} + \begin{bmatrix} t^{-1} \\ 2t^{-1} + 4 \end{bmatrix}$$

for $t > 0$.

The eigenvalues of $A = \begin{bmatrix} -4 & 2 \\ 2 & -1 \end{bmatrix}$ are $r_1 = 0$ and $r_2 = -5$; and the eigenvectors are $\xi^{(1)} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $\xi^{(2)} = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$. Thus

$$\Psi(t) = \begin{bmatrix} 1 & -2e^{-5t} \\ 2 & e^{-5t} \end{bmatrix}$$

is a fundamental matrix for $\mathbf{x}' = A\mathbf{x}$.

$$\Psi(t) = \begin{bmatrix} 1 & -2e^{-5t} \\ 2 & e^{-5t} \end{bmatrix}$$

Using the formula $\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$ we calculate that

$$\Psi^{-1}(t) = \frac{1}{e^{-5t} + 4e^{-5t}} \begin{bmatrix} e^{-5t} & 2e^{-5t} \\ -2 & 1 \end{bmatrix} = \frac{1}{5} \begin{bmatrix} 1 & 2 \\ -2e^{5t} & e^{5t} \end{bmatrix}.$$

Then

$$\Psi^{-1}(t)\mathbf{g}(t) = \frac{1}{5}\begin{bmatrix} 1 & 2 \\ -2e^{5t} & e^{5t} \end{bmatrix}\begin{bmatrix} t^{-1} \\ 2t^{-1}+4 \end{bmatrix}$$

$$= \frac{1}{5}\begin{bmatrix} t^{-1}+4t^{-1}+8 \\ -2t^{-1}e^{5t}+2t^{-1}e^{5t}+4e^{5t} \end{bmatrix} = \begin{bmatrix} t^{-1}+\frac{8}{5} \\ \frac{4}{5}e^{5t} \end{bmatrix}$$

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and

$$\int \Psi^{-1}(t)\mathbf{g}(t)\,dt = \int \begin{bmatrix} t^{-1}+\frac{8}{5} \\ \frac{4}{5}e^{5t} \end{bmatrix} dt = \begin{bmatrix} \ln t + \frac{8}{5}t + c_1 \\ \frac{4}{25}e^{5t} + c_2 \end{bmatrix}.$$

It follows that

$$\mathbf{x}(t) = \Psi(t) \int \Psi^{-1}(s)\mathbf{g}(s)\, ds$$

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$$\begin{aligned}
\mathbf{x}(t) = \Psi(t) \int \Psi^{-1}(s)\mathbf{g}(s)\, ds &= \begin{bmatrix} 1 & -2e^{-5t} \\ 2 & e^{-5t} \end{bmatrix} \begin{bmatrix} \ln t + \frac{8}{5}t + c_1 \\ \frac{4}{25}e^{5t} + c_2 \end{bmatrix} \\
&= \begin{bmatrix} \ln t + \frac{8}{5}t - \frac{8}{25} + c_1 - 2c_2 e^{-5t} \\ 2\ln t + \frac{16}{5}t + \frac{4}{25} + 2c_1 + c_2 e^{-5t} \end{bmatrix} \\
&= c_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + c_2 \begin{bmatrix} -2 \\ 1 \end{bmatrix} e^{-5t} + \begin{bmatrix} 1 \\ 2 \end{bmatrix} \ln t + \frac{8}{5} \begin{bmatrix} 1 \\ 2 \end{bmatrix} t + \frac{4}{25} \begin{bmatrix} -2 \\ 1 \end{bmatrix}.
\end{aligned}$$

Method 4 – The Laplace Transform:

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First some notation: If $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$, then $\mathbf{X} = \mathcal{L}\left[\mathbf{x}\right] = \begin{bmatrix} \mathcal{L}\left[x_1\right] \\ \mathcal{L}\left[x_2\right] \\ \vdots \\ \mathcal{L}\left[x_n\right] \end{bmatrix}$.

Recall from Chapter 6 that $\mathcal{L}\left[y'\right]$ satisfies

$$\mathcal{L}\left[y'\right](s) = sY(s) - y(0).$$

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It follows that:

Theorem

$$\mathcal{L}\left[\mathbf{x}'\right](s) = s\mathbf{X}(s) - \mathbf{x}(0).$$

Example

Solve

$$\begin{cases} \mathbf{x}' = \begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 2e^{-t} \\ 3t \end{bmatrix} = A\mathbf{x} + \mathbf{g}(t), \\ \mathbf{x}(0) = \mathbf{0}. \end{cases}$$

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Taking Laplace Transforms of the ODE gives

$$s\mathbf{X}(s) - \mathbf{x}(0) = A\mathbf{X}(s) + \mathbf{G}(s)$$

where $\mathbf{G}(s) = \mathcal{L}\left[\mathbf{g}\right](s) = \begin{bmatrix} \frac{2}{s+1} \\ \frac{3}{s^2} \end{bmatrix}$.

$$s\mathbf{X} - \mathbf{x}(0) = A\mathbf{X} + \mathbf{G}$$

Since $\mathbf{x}(0) = \mathbf{0}$ we have that

$$(sI - A)\mathbf{X} = \mathbf{G}$$

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$$(sI - A)\mathbf{X} = \mathbf{G}$$

and

$$\mathbf{X} = (sI - A)^{-1}\mathbf{G}$$

where

$$(sI - A)^{-1} = \begin{bmatrix} s+2 & -1 \\ -1 & s+2 \end{bmatrix}^{-1} = \frac{1}{(s+1)(s+3)} \begin{bmatrix} s+2 & 1 \\ 1 & s+2 \end{bmatrix}.$$

So

$$\begin{aligned}
\mathbf{X} &= (sI - A)^{-1}\mathbf{G} \\
&= \frac{1}{(s+1)(s+3)} \begin{bmatrix} s+2 & 1 \\ 1 & s+2 \end{bmatrix} \begin{bmatrix} \frac{2}{s+1} \\ \frac{3}{s^2} \end{bmatrix} \\
&= \begin{bmatrix} \frac{2(s+2)}{(s+1)^2(s+3)} + \frac{3}{s^2(s+1)(s+3)} \\ \frac{2}{(s+1)^2(s+3)} + \frac{3(s+2)}{s^2(s+1)(s+3)} \end{bmatrix}.
\end{aligned}$$

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$$\begin{aligned}\mathbf{X} &= (sI - A)^{-1}\mathbf{G} \\ &= \frac{1}{(s+1)(s+3)}\begin{bmatrix} s+2 & 1 \\ 1 & s+2 \end{bmatrix}\begin{bmatrix} \frac{2}{s+1} \\ \frac{3}{s^2} \end{bmatrix} \\ &= \begin{bmatrix} \frac{2(s+2)}{(s+1)^2(s+3)} + \frac{3}{s^2(s+1)(s+3)} \\ \frac{2}{(s+1)^2(s+3)} + \frac{3(s+2)}{s^2(s+1)(s+3)} \end{bmatrix}.\end{aligned}$$

When we take the inverse Laplace Transform of this, we find our solution

$$\mathbf{x} = \mathcal{L}^{-1}\left[\mathbf{X}\right] = \begin{bmatrix} 2 \\ 1 \end{bmatrix} e^{-t} - \frac{2}{3}\begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{-3t} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} te^{-t} + \begin{bmatrix} 1 \\ 2 \end{bmatrix} t - \frac{1}{3}\begin{bmatrix} 4 \\ 5 \end{bmatrix}.$$

Example

Solve

$$\begin{cases} 2x' + y' - y - t = 0 \\ x' + y' - t^2 = 0 \\ x(0) = 1 \\ y(0) = 0 \end{cases}$$

Example

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The ODEs above can be written as

$$\begin{cases} x' = y - t^2 + t \\ y' = -y + 2t^2 - t \end{cases}$$

(please check!).

If we write the problem in terms of matrices (using $\mathbf{x} = \begin{bmatrix} x \\ y \end{bmatrix}$) we have

$$\begin{cases} \mathbf{x}' = A\mathbf{x} + \mathbf{g} = \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix} \mathbf{x} + \begin{bmatrix} t - t^2 \\ 2t^2 - t \end{bmatrix} \\ \mathbf{x}(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}. \end{cases}$$

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Taking the Laplace transform of the ODE gives

$$(sI - A)\,\mathbf{X}\,(s) = \mathbf{x}\,(0) + \mathbf{G}(s)$$

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Taking the Laplace transform of the ODE gives

$$(sI - A)\,\mathbf{X}\,(s) = \mathbf{x}\,(0) + \mathbf{G}(s)$$
$$\begin{bmatrix} s & -1 \\ 0 & s+1 \end{bmatrix} \mathbf{X}\,(s) = \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \begin{bmatrix} \frac{1}{s^2} - \frac{2}{s^3} \\ \frac{4}{s^3} - \frac{1}{s^2} \end{bmatrix}.$$

$$\begin{bmatrix} s & -1 \\ 0 & s+1 \end{bmatrix} \mathbf{X}(s) = \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \begin{bmatrix} \frac{1}{s^2} - \frac{2}{s^3} \\ \frac{4}{s^3} - \frac{1}{s^2} \end{bmatrix}$$

Thus

$$\begin{aligned} \mathbf{X}(s) &= \frac{1}{s(s+1)} \begin{bmatrix} s+1 & 1 \\ 0 & s \end{bmatrix} \frac{1}{s^3} \begin{bmatrix} s^3 + s - 2 \\ 4 - s \end{bmatrix} \\ &= \frac{1}{s^4(s+1)} \begin{bmatrix} s^4 + s^3 + s^2 - 2s + 2 \\ 4s - s^2 \end{bmatrix}. \end{aligned}$$

Note that

$$\frac{s^4+s^3+s^2-2s+2}{s^4(s+1)} = \frac{5}{s+1} - 4\frac{1}{s} + 5\frac{1}{s^2} - 4\frac{1}{s^3} + 2\frac{1}{s^4}$$

and

$$\frac{4s-s^2}{s^4(s+1)} = -5\frac{1}{s+1} + 5\frac{1}{s} - 5\frac{1}{s^2} + 4\frac{1}{s^3}$$

(please check!).

It follows that

$$\mathcal{L}^{-1}\left(\frac{s^4+s^3+s^2-2s+2}{s^4(s+1)}\right) = 5e^{-t} - 4 + 5t - 2t^2 + \frac{1}{3}t^3$$

and

$$\mathcal{L}^{-1}\left(\frac{4s-s^2}{s^4(s+1)}\right) = -5e^{-t} + 5 - 5t + 2t^2.$$

It follows that

$$\mathcal{L}^{-1}\left(\frac{s^4+s^3+s^2-2s+2}{s^4\left(s+1\right)}\right)=5e^{-t}-4+5t-2t^2+\frac{1}{3}t^3$$

and

$$\mathcal{L}^{-1}\left(\frac{4s-s^2}{s^4\left(s+1\right)}\right)=-5e^{-t}+5-5t+2t^2.$$

Therefore the solution to the initial value problem is

$$\boxed{\mathbf{x}\left(t\right)=\begin{bmatrix}5e^{-t}-4+5t-2t^2+\frac{1}{3}t^3\\ -5e^{-t}+5-5t+2t^2\end{bmatrix}.}$$

The End