2014.05.29 **MAT234 Matematik IV – Final Sınavın Çözümleri** N. Course

Soru 1 (Supremum and Infinum of a Sequence).

(a) [5p] Let $S \subseteq \mathbb{R}$ be a set. Give the definitions of the *supremum* and *infinum* of S.

The supremum of S, $\sup S$, is the least upper bound for S. The infinum of S, $\inf S$, is the greatest lower bound for S.

Consider the sequence $(a_n)_{n=1}^{\infty}$ defined by

$$a_n := \left(1 + \frac{1}{n}\right) \sin\left(\frac{n\pi}{2}\right) + \frac{(-1)^n}{n}.$$

(b) [4p]Calculate $a_1, a_2, \ldots, a_{10}$.

$$\begin{aligned} a_1 &= 1 & a_6 &= \frac{1}{6} \\ a_2 &= \frac{1}{2} & a_7 &= -\frac{9}{7} \\ a_3 &= -\frac{5}{3} & a_8 &= \frac{1}{8} \\ a_4 &= \frac{1}{4} & a_9 &= 1 \\ a_5 &= 1 & a_{10} &= \frac{1}{10} \end{aligned}$$

$-\frac{1}{2}$ point for each incorrect/absent term.

Define three new sequences $(b_n)_{n=1}^{\infty}$, $(c_n)_{n=1}^{\infty}$ and $(d_n)_{n=1}^{\infty}$ by $b_n := a_{2n}$, $c_n := a_{4n-3}$ and $d_n := a_{4n-1}$. For example, (d_n) is the sequence $a_3, a_7, a_{11}, a_{15}, a_{19}, \ldots$.

Now clearly

$$b_n = a_{2n} = \left(1 + \frac{1}{2n}\right) \sin(n\pi) + \frac{(-1)^{2n}}{2n} = 0 + \frac{1}{2n} = \frac{1}{2n}.$$

Since b_n is a decreasing sequence, $b_1 \geq b_n$ for all $n \in \mathbb{N}$. So $b_1 = \frac{1}{2}$ is an upper bound for (b_n). Furthermore, we can show that $\frac{1}{2}$ is the least upper bound: If $M < \frac{1}{2}$, then $b_1 > M$ and so M is not an upper bound. Therefore we have that $\sup\{b_n : n \in \mathbb{N}\} = \sup\left\{\frac{1}{2}, \frac{1}{4}, \frac{1}{6}, \frac{1}{8}, \frac{1}{10}, \ldots\right\} = \frac{1}{2}$.

(c) [4p] Prove that

$$\inf\{b_n : n \in \mathbb{N}\} = 0.$$

Note first that $b_n = \frac{1}{2n} > 0$ for all n. So 0 is a lower bound.

Let $\varepsilon > 0$. Since $b_n \to 0$ as $n \to \infty$, there exists $N \in \mathbb{N}$ such that $b_N < \varepsilon$. Thus ε is not a lower bound for this sequence.

Hence 0 is the greatest lower bound.

(d) [4p] Find $\inf\{c_n : n \in \mathbb{N}\}$ and $\sup\{c_n : n \in \mathbb{N}\}$.
(You must justify your answer.)

Since $c_n = a_{4n-3} = 1$ for all $n \in \mathbb{N}$, we have that

$$\inf\{c_n : n \in \mathbb{N}\} = 1 = \sup\{c_n : n \in \mathbb{N}\}.$$

(e) [7p] Find $\inf\{d_n : n \in \mathbb{N}\}$ and $\sup\{d_n : n \in \mathbb{N}\}$.
(You must justify your answer.)

Since

$$\begin{aligned} d_n &= a_{4n-1} \\ &= \left(1 + \frac{1}{4n-1}\right) \sin\left(\left(2n - \frac{1}{2}\right)\pi\right) + \frac{(-1)^{4n-1}}{4n-1} \\ &= \left(1 + \frac{1}{4n-1}\right)(-1) - \frac{1}{4n-1} \\ &= -1 - \frac{2}{4n-1}, \end{aligned}$$

(d_n) is an increasing sequence. So $\inf\{d_n : n \in \mathbb{N}\} = d_1 = a_3 = -\frac{5}{3}$.

Moreover, it is clear that $d_n \to -1$ as $n \to \infty$. So $\sup\{d_n : n \in \mathbb{N}\} = -1$ with a similar proof to that used in the answer to part (c).

Soru 2 (Symbolic Logic and Negating a Definition).

(a) [8p] Prove that

$$\Big((P \implies Q) \wedge (P \implies \neg Q)\Big) = \neg P.$$

P	Q	$\neg Q$	$P \implies Q$	$P \implies \neg Q$	$(P \implies Q) \wedge (P \implies \neg Q)$	$\neg P$
T	T	F	T	F	F	F
T	F	T	F	T	F	F
F	T	F	T	T	T	T
F	F	T	T	T	T	T

-1 point for each mistake.

Definition. A sequence (a_n) is a *null sequence* if and only if for all $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that for all $n \in \mathbb{N}$;

$$n > N \implies |a_n| < \varepsilon.$$

(b) [7p] Give the definition of "(a_n) is **not** a null sequence".
[HINT: Negate the definition above.]

A sequence (a_n) is **not** a null sequence if and only if there exists $\varepsilon > 0$ such that for all $N \in \mathbb{N}$, there exists $n \in \mathbb{N}$ such that

$$n > N \qquad \text{and} \qquad |a_n| \geq \varepsilon.$$

Let

$$b_n := \frac{n-2}{n}$$

for all $n \in \mathbb{N}$.

(c) [10p] Show that (b_n) is **not** a null sequence.

Choose $\varepsilon = \frac{1}{2}$. Let $N \in \mathbb{N}$. Choose $n = \max\{N+1, 10\}$.
Then clearly $n > N$. Moreover, $n \geq 10$ which implies that $-\frac{1}{n} \geq -\frac{1}{10}$. Thus

$$|b_n| = \left|\frac{n-2}{n}\right| = 1 - \frac{2}{n} \geq 1 - \frac{2}{10} > 1 - \frac{1}{2} = \frac{1}{2} = \varepsilon.$$

Therefore (b_n) is not a null sequence.

Soru 3 (Series). Decide if each of the following series converges or diverges. Justify (prove) your answers.

(a) [8p] $\displaystyle\sum_{n=1}^{\infty} (-1)^{n+1} \cos^2 \frac{1}{n}$.

(b) [8p] $\displaystyle\sum_{n=1}^{\infty} \frac{3^n\ (n!)^3}{(3n)!}$.

(c) [9p] $\displaystyle\sum_{n=1}^{\infty} \frac{n^2+2}{3n^3+4n}$.

2 pts for "converges/diverges" correct without justification.
2 pts for saying which test is being used (as long as there is some proof given).
Remaining 4/5 pts for accuracy of proof.

If an answer is incorrect, but the proof is well written and contains only a minor error, then a maximum of 5 points (6 points for part (c)) can be awarded.

(a) Since $\cos \frac{1}{n} \to 1 \neq 0$ as $n \to \infty$, it follows that $\sum_{n=1}^{\infty} (-1)^{n+1} \cos^2 \frac{1}{n}$ diverges by the Divergence Test.

(b) Since

$$\begin{aligned}\frac{a_{n+1}}{a_n} &= \frac{3^{n+1}\ ((n+1)!)^3}{(3n+3)!} \cdot \frac{(3n)!}{3^n\ (n!)^3} = \frac{3(n+1)^3}{(3n+3)(3n+2)(3n+1)} \\ &= \frac{3(1+\frac{1}{n})^3}{(3+\frac{3}{n})(3+\frac{2}{n})(3+\frac{1}{n})} \to \frac{3}{9} = \frac{1}{3}\end{aligned}$$

as $n \to \infty$, it follows that $\sum_{n=1}^{\infty} \frac{3^n\ (n!)^3}{(3n)!}$ converges by the Ratio Test.

(c) First note that if $n > 1$, then

$$\frac{n^2+2}{3n^3+4n} \geq \frac{n^2+2}{3n^3} \geq \frac{n^2+n^2}{3n^3} = \frac{2}{3} \times \frac{1}{n}.$$

Since we know that $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges, it follows that $\sum_{n=1}^{\infty} \frac{n^2+2}{3n^3+4n}$ diverges by the Comparison Test.

Soru 4 (Power Series).

(a) [5p] Let $\sum_{n=0}^{\infty} a_n x^n$ be a power series. Give the definition of the *radius of convergence* of $\sum_{n=0}^{\infty} a_n x^n$.

If $\sum_{n=0}^{\infty} a_n x^n$ converges $\forall |x| < R$ and diverges $\forall |x| > R$, then R is called the radius of convergence of $\sum_{n=0}^{\infty} a_n x^n$.

Define the set

$$S := \left\{ x \in \mathbb{R} : \sum_{n=1}^{\infty} (-2)^n x^n \text{ converges} \right\} \subseteq \mathbb{R}.$$

(b) [20p]Find S.

For this power series, $a_n = (-2)^n$ and

$$\left| \frac{a_n}{a_{n+1}} \right| = \frac{(-2)^n}{(-2)^{n+1}} = \frac{1}{2} \qquad \boxed{6} \ \boxed{\text{-1 point if candidate omits absolute value signs}}.$$

By a theorem from the course $\boxed{2}$, the radius of convergence of this power series is $R = \frac{1}{2}$ $\boxed{2}$.

When $x = \frac{1}{2}$, the power series becomes $\sum_{n=1}^{\infty}(-1)^n$, which diverges $\boxed{2}$. When $x = -\frac{1}{2}$, the power series becomes $\sum_{n=1}^{\infty} 1$, which also diverges $\boxed{2}$.

Therefore $\sum_{n=1}^{\infty}(-2)^n x^n$ converges $\forall x \in (-\frac{1}{2}, \frac{1}{2})$ and diverges for all other x $\boxed{2}$. Hence $S = (-\frac{1}{2}, \frac{1}{2})$ $\boxed{4}$.

Soru 5 (Taylor Series).

(a) [10p] Calculate the Taylor Series for $f(x) = \cosh x$, centred at $a = 0$.

[You may assume without proof that $\left| \frac{f^n(c)}{n!} x^n \right| \to 0$ as $n \to \infty$ for all $x \in \mathbb{R}$ and for all c between 0 and x.]

Since

$$\frac{d^n}{dx^n} \cosh x = \begin{cases} \cosh x & n = 0, 2, 4, 6, 8, \dots \\ \sinh x & n = 1, 3, 5, 7, 9, \dots, \end{cases}$$

we can see that

$$f^n(0) = \begin{cases} 1 & n = 0, 2, 4, 6, 8, \dots \\ 0 & n = 1, 3, 5, 7, 9, \dots. \end{cases} \boxed{4}$$

By Taylor's Theorem (and by the hint), we have

$$\begin{aligned} \cos x &= f(0) + f'(0)x + \frac{f''(0)x^2}{2!} + \frac{f'''(0)x^3}{3!} + \frac{f^{(4)}(0)x^4}{4!} + \dots \boxed{4} \\ &= 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \frac{x^8}{8!} + \frac{x^{10}}{10!} + \frac{x^{12}}{12!} + \frac{x^{14}}{14!} + \frac{x^{16}}{16!} + \frac{x^{18}}{18!} + \dots \boxed{5} \end{aligned}$$

(b) [15p] Use your answer to part (a) to calculate $\lim_{t \to 0} \frac{1 - \cosh t + \frac{t^2}{2} + \frac{t^4}{24}}{t^6}$.

By part (a),

$$\cos x = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \frac{x^8}{8!} + \frac{x^{10}}{10!} + \frac{x^{12}}{12!} + \frac{x^{14}}{14!} + \frac{x^{16}}{16!} + \frac{x^{18}}{18!} + \dots.$$

Therefore

$$\begin{aligned} \frac{1 + \frac{t^2}{2} + \frac{t^4}{24} - \cosh t}{t^6} &= \frac{+\frac{t^6}{6!} + \frac{t^8}{8!} + \frac{t^10}{10!} + \dots}{t^6} \qquad \boxed{3} \\ &= \frac{1}{6!} + \frac{t^2}{8!} + \frac{t^4}{10!} + \dots. \qquad \boxed{5} \end{aligned}$$

Hence

$$\lim_{t \to 0} \frac{1 - \cosh t + \frac{t^2}{2} + \frac{t^4}{24}}{t^6} = \lim_{t \to 0} \left(\frac{1}{6!} + \frac{t^2}{8!} + \frac{t^4}{10!} + \dots \right) = \frac{1}{6!} = \frac{1}{720}. \qquad \boxed{7}$$