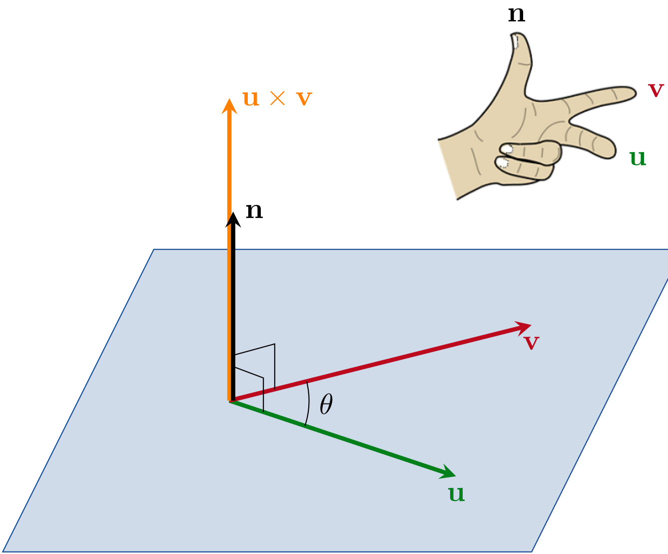
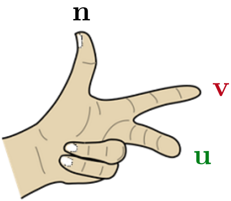
Lecture 4

- 11.4 The Cross Product
- 11.5 Lines and Planes in Space

11.4 The Cross Product


n
v
u
u × v
n
v
θ
u

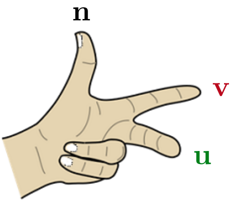
11.4 The Cross Product



Let $\mathbf{n}$ be a unit vector which satisfies

1. $\mathbf{n}$ is orthogonal to $\mathbf{u}$ $\left(\begin{array}{l}\uparrow \mathbf{n}\\ \llcorner\!\longrightarrow \mathbf{u}\end{array}\right)$;
2. $\mathbf{n}$ is orthogonal to $\mathbf{v}$ $\left(\begin{array}{l}\uparrow \mathbf{n}\\ \llcorner\!\longrightarrow \mathbf{v}\end{array}\right)$; and
3. the direction of $\mathbf{n}$ is chosen using the left-hand rule.

11.4 The Cross Product



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3. the direction of $\mathbf{n}$ is chosen using the left-hand rule.

Definition

The *cross product* of $\mathbf{u}$ and $\mathbf{v}$ is

$$\mathbf{u} \times \mathbf{v} = \|\mathbf{u}\| \, \|\mathbf{v}\| \, (\sin\theta)\mathbf{n}.$$

Remark

- $\mathbf{u} \cdot \mathbf{v}$ is a number.
- $\mathbf{u} \times \mathbf{v}$ is a vector.

$$\mathbf{u} \times \mathbf{v} = \|\mathbf{u}\| \, \|\mathbf{v}\| \, (\sin\theta)\mathbf{n}$$

Remark

$$\begin{pmatrix} \mathbf{u} \text{ and } \mathbf{v} \\ \text{are} \\ \text{parallel} \end{pmatrix} \iff \theta = 0^\circ \text{ or } 180^\circ$$
$$\implies \sin\theta = 0 \implies \mathbf{u} \times \mathbf{v} = \mathbf{0}.$$

Properties of the Cross Product

Let $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ be vectors. Let r and s be numbers. Then

1. $(r\mathbf{u}) \times (s\mathbf{v}) = (rs)(\mathbf{u} \times \mathbf{v})$;

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3. $\mathbf{v} \times \mathbf{u} = -\mathbf{u} \times \mathbf{v}$;
4. $(\mathbf{v} + \mathbf{w}) \times \mathbf{u} = (\mathbf{v} \times \mathbf{u}) + (\mathbf{w} \times \mathbf{u})$;

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4. $(\mathbf{v} + \mathbf{w}) \times \mathbf{u} = (\mathbf{v} \times \mathbf{u}) + (\mathbf{w} \times \mathbf{u})$;
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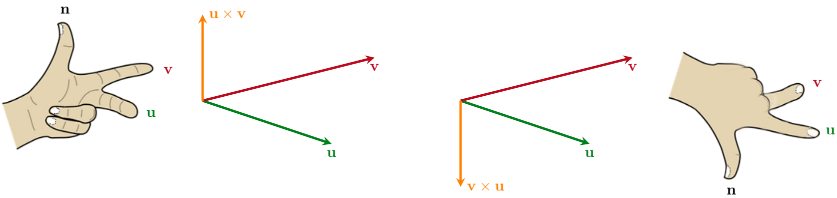
4. $(\mathbf{v} + \mathbf{w}) \times \mathbf{u} = (\mathbf{v} \times \mathbf{u}) + (\mathbf{w} \times \mathbf{u})$;
5. $\mathbf{0} \times \mathbf{u} = \mathbf{0}$; and
6. $\mathbf{u} \times (\mathbf{v} \times \mathbf{w}) = (\mathbf{u} \cdot \mathbf{w})\mathbf{v} - (\mathbf{u} \cdot \mathbf{v})\mathbf{w}$.

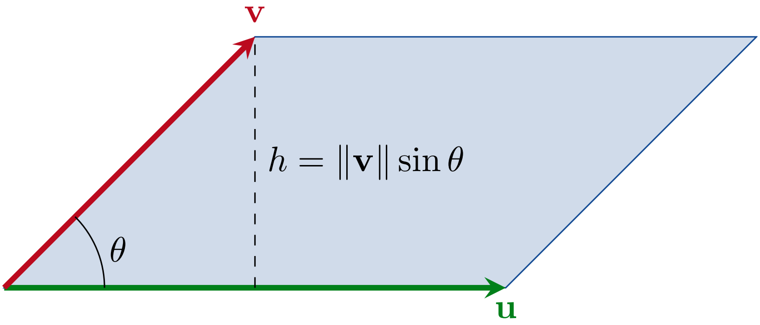
Property (iii)



$$\mathbf{v} \times \mathbf{u} = -\mathbf{u} \times \mathbf{v}$$

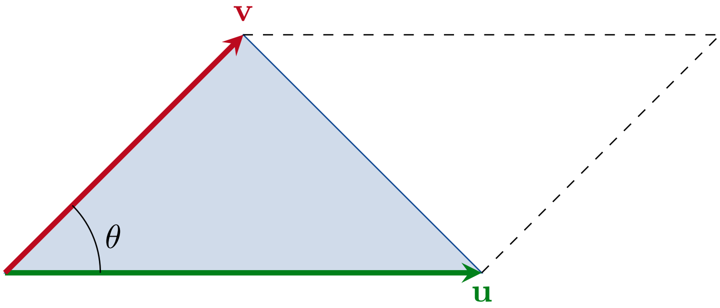
$$\mathbf{u} \times \mathbf{v} = \|\mathbf{u}\| \, \|\mathbf{v}\| \, (\sin\theta)\mathbf{n}$$

Area of a Parallelogram



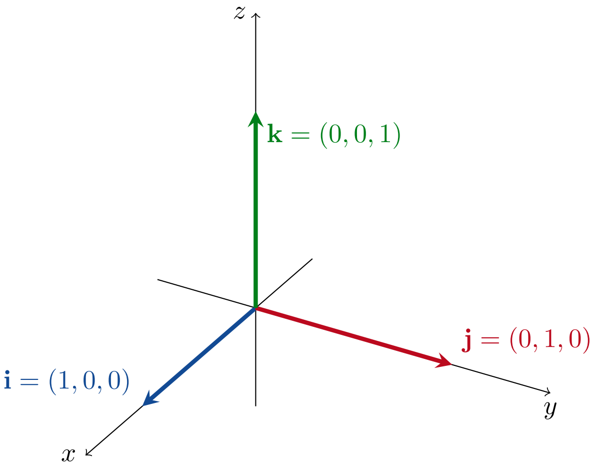
$$\text{area} = (\text{base})\,(\text{height}) = \|\mathbf{u}\|\,\|\mathbf{v}\| \sin\theta = \|\mathbf{u} \times \mathbf{v}\|\,.$$

Area of a Triangle



$$\begin{aligned} \text{area of triangle} &= \frac{1}{2}\,(\text{area of parallelogram}) \\ &= \frac{1}{2}\,\|\mathbf{u}\times\mathbf{v}\|\,. \end{aligned}$$

A Formula for $\mathbf{u} \times \mathbf{v}$



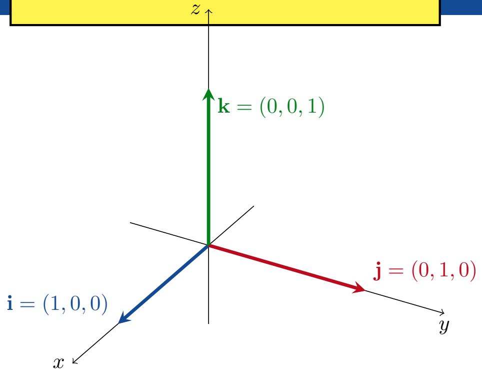
$$\mathbf{u} \times \mathbf{v} = \|\mathbf{u}\| \, \|\mathbf{v}\| \, (\sin\theta)\mathbf{n}$$

Note first that

$$\mathbf{i} \times \mathbf{i} = \|\mathbf{i}\| \, \|\mathbf{i}\| \sin 0^\circ \mathbf{n} = \mathbf{0}.$$

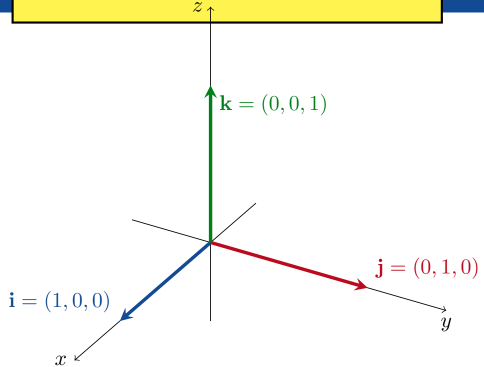
Similarly $\mathbf{j} \times \mathbf{j} = \mathbf{0}$ and $\mathbf{k} \times \mathbf{k} = \mathbf{0}$ also.

$$\mathbf{u} \times \mathbf{v} = \|\mathbf{u}\| \, \|\mathbf{v}\| \, (\sin\theta)\mathbf{n}$$



Next note that $\mathbf{i} \times \mathbf{j}$ must point in the same direction at $\mathbf{k}$ by the left-hand rule.

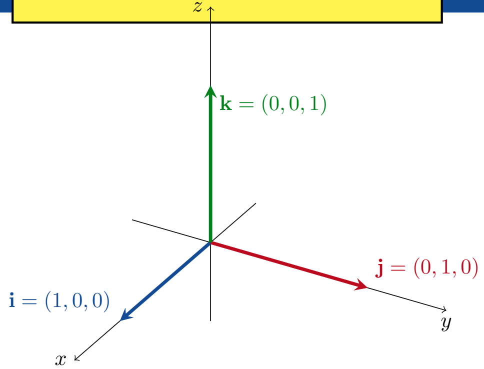
$$\boxed{\mathbf{u} \times \mathbf{v} = \|\mathbf{u}\|\,\|\mathbf{v}\|\,(\sin\theta)\mathbf{n}}$$



Next note that $\mathbf{i} \times \mathbf{j}$ must point in the same direction at $\mathbf{k}$ by the left-hand rule. Thus

$$\mathbf{i} \times \mathbf{j} = \|\mathbf{i}\|\,\|\mathbf{j}\| \sin 90^\circ \mathbf{k} = \mathbf{k}.$$

$$\mathbf{u} \times \mathbf{v} = \|\mathbf{u}\| \, \|\mathbf{v}\| \, (\sin\theta)\mathbf{n}$$



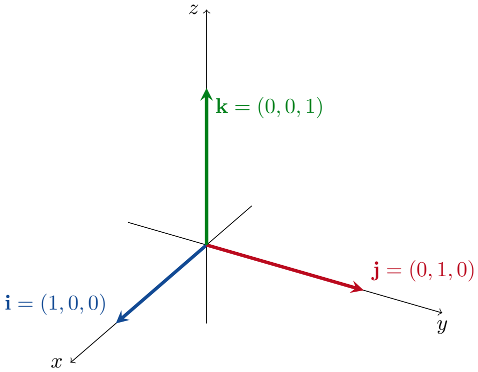
Next note that $\mathbf{i} \times \mathbf{j}$ must point in the same direction at $\mathbf{k}$ by the left-hand rule. Thus

$$\mathbf{i} \times \mathbf{j} = \|\mathbf{i}\| \, \|\mathbf{j}\| \sin 90^\circ \mathbf{k} = \mathbf{k}.$$

We then immediately also have

$$\mathbf{j} \times \mathbf{i} = -(\mathbf{i} \times \mathbf{j}) = -\mathbf{k}.$$

$$\mathbf{u} \times \mathbf{v} = \|\mathbf{u}\| \, \|\mathbf{v}\| \, (\sin\theta)\mathbf{n}$$



I leave it for you to check that

$$\mathbf{j} \times \mathbf{k} = \mathbf{i}, \qquad \mathbf{k} \times \mathbf{j} = -\mathbf{i}, \qquad \mathbf{i} \times \mathbf{k} = -\mathbf{j} \quad \text{and} \quad \mathbf{k} \times \mathbf{i} = \mathbf{j}.$$

Now suppose that $\mathbf{u} = u_1\mathbf{i} + u_2\mathbf{j} + u_3\mathbf{k}$ and $\mathbf{v} = v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}$.

Now suppose that $\mathbf{u} = u_1\mathbf{i} + u_2\mathbf{j} + u_3\mathbf{k}$ and $\mathbf{v} = v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}$. Then we can calculate that

$$
\begin{aligned}
\mathbf{u} \times \mathbf{v} &= (u_1\mathbf{i} + u_2\mathbf{j} + u_3\mathbf{k}) \times (v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}) \\
&= \\
&= \\
&=
\end{aligned}
$$

Now suppose that $\mathbf{u} = u_1\mathbf{i} + u_2\mathbf{j} + u_3\mathbf{k}$ and $\mathbf{v} = v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}$. Then we can calculate that

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\mathbf{u} \times \mathbf{v} &= (u_1\mathbf{i} + u_2\mathbf{j} + u_3\mathbf{k}) \times (v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}) \\
&= u_1v_1\mathbf{i} \times \mathbf{i} + u_1v_2\mathbf{i} \times \mathbf{j} + u_1v_3\mathbf{i} \times \mathbf{k} + u_2v_1\mathbf{j} \times \mathbf{i} + u_2v_2\mathbf{j} \times \mathbf{j} \\
&\quad + u_2v_3\mathbf{j} \times \mathbf{k} + u_3v_1\mathbf{k} \times \mathbf{i} + u_3v_2\mathbf{k} \times \mathbf{j} + u_3v_3\mathbf{k} \times \mathbf{k} \\
&= \\
&=
\end{aligned}$$

$$\begin{aligned}
\mathbf{i} \times \mathbf{j} &= \mathbf{k}\\
\mathbf{j} \times \mathbf{i} &= -\mathbf{k}\\
\mathbf{j} \times \mathbf{k} &= \mathbf{i}\\
\mathbf{k} \times \mathbf{j} &= -\mathbf{i}\\
\mathbf{i} \times \mathbf{k} &= -\mathbf{j}\\
\mathbf{k} \times \mathbf{i} &= \mathbf{j}
\end{aligned}$$

Now suppose that $\mathbf{u} = u_1\mathbf{i} + u_2\mathbf{j} + u_3\mathbf{k}$ and $\mathbf{v} = v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}$. Then we can calculate that

$$\begin{aligned}
\mathbf{u} \times \mathbf{v} &= (u_1\mathbf{i} + u_2\mathbf{j} + u_3\mathbf{k}) \times (v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k})\\
&= u_1v_1\mathbf{i} \times \mathbf{i} + u_1v_2\mathbf{i} \times \mathbf{j} + u_1v_3\mathbf{i} \times \mathbf{k} + u_2v_1\mathbf{j} \times \mathbf{i} + u_2v_2\mathbf{j} \times \mathbf{j}\\
&\quad + u_2v_3\mathbf{j} \times \mathbf{k} + u_3v_1\mathbf{k} \times \mathbf{i} + u_3v_2\mathbf{k} \times \mathbf{j} + u_3v_3\mathbf{k} \times \mathbf{k}\\
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\end{aligned}$$

11.4 The Cross Product

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Now suppose that $\mathbf{u} = u_1\mathbf{i} + u_2\mathbf{j} + u_3\mathbf{k}$ and $\mathbf{v} = v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}$. Then we can calculate that

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\mathbf{u} \times \mathbf{v} &= (u_1\mathbf{i} + u_2\mathbf{j} + u_3\mathbf{k}) \times (v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}) \\
&= u_1v_1\mathbf{i} \times \mathbf{i} + u_1v_2\mathbf{i} \times \mathbf{j} + u_1v_3\mathbf{i} \times \mathbf{k} + u_2v_1\mathbf{j} \times \mathbf{i} + u_2v_2\mathbf{j} \times \mathbf{j} \\
&\quad + u_2v_3\mathbf{j} \times \mathbf{k} + u_3v_1\mathbf{k} \times \mathbf{i} + u_3v_2\mathbf{k} \times \mathbf{j} + u_3v_3\mathbf{k} \times \mathbf{k} \\
&= \mathbf{0} + u_1v_2\mathbf{k} - u_1v_3\mathbf{j} - u_2v_1\mathbf{k} + \mathbf{0} + u_2v_3\mathbf{i} + u_3v_1\mathbf{j} - u_3v_2\mathbf{i} + \mathbf{0} \\
&=
\end{aligned}$$

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&= u_1v_1\mathbf{i} \times \mathbf{i} + u_1v_2\mathbf{i} \times \mathbf{j} + u_1v_3\mathbf{i} \times \mathbf{k} + u_2v_1\mathbf{j} \times \mathbf{i} + u_2v_2\mathbf{j} \times \mathbf{j} \\
&\quad + u_2v_3\mathbf{j} \times \mathbf{k} + u_3v_1\mathbf{k} \times \mathbf{i} + u_3v_2\mathbf{k} \times \mathbf{j} + u_3v_3\mathbf{k} \times \mathbf{k} \\
&= \mathbf{0} + u_1v_2\mathbf{k} - u_1v_3\mathbf{j} - u_2v_1\mathbf{k} + \mathbf{0} + u_2v_3\mathbf{i} + u_3v_1\mathbf{j} - u_3v_2\mathbf{i} + \mathbf{0} \\
&= (u_2v_3 - u_3v_2)\mathbf{i} - (u_1v_3 - u_3v_1)\mathbf{j} + (u_1v_2 - u_2v_1)\mathbf{k}.
\end{aligned}$$

Theorem

If $\mathbf{u} = u_1\mathbf{i} + u_2\mathbf{j} + u_3\mathbf{k}$ *and* $\mathbf{v} = v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}$, *then*

$$\boxed{\mathbf{u} \times \mathbf{v} = (u_2v_3 - u_3v_2)\mathbf{i} - (u_1v_3 - u_3v_1)\mathbf{j} + (u_1v_2 - u_2v_1)\mathbf{k}.}$$

If you studied matrices and determinants at high school, then you may prefer to use the following symbolic determinant formula instead.

$$\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}.$$

$$\mathbf{u} \times \mathbf{v} = (u_2v_3 - u_3v_2)\mathbf{i} - (u_1v_3 - u_3v_1)\mathbf{j} + (u_1v_2 - u_2v_1)\mathbf{k}$$

Example

Find $\mathbf{u} \times \mathbf{v}$ and $\mathbf{v} \times \mathbf{u}$ if $\mathbf{u} = 2\mathbf{i} + \mathbf{j} + \mathbf{k}$ and $\mathbf{v} = -4\mathbf{i} + 3\mathbf{j} + \mathbf{k}$.

$$\mathbf{u} \times \mathbf{v} = (u_2 v_3 - u_3 v_2)\mathbf{i} - (u_1 v_3 - u_3 v_1)\mathbf{j} + (u_1 v_2 - u_2 v_1)\mathbf{k}$$

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$$\mathbf{u} \times \mathbf{v} = (1 - 3)\mathbf{i} - (2 - -4)\mathbf{j} + (6 - -4)\mathbf{k} = -2\mathbf{i} - 6\mathbf{j} + 10\mathbf{k}$$

$$\mathbf{u} \times \mathbf{v} = (u_2v_3 - u_3v_2)\mathbf{i} - (u_1v_3 - u_3v_1)\mathbf{j} + (u_1v_2 - u_2v_1)\mathbf{k}$$

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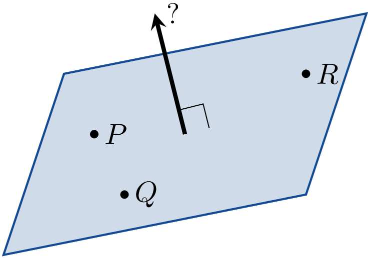
and

$$\mathbf{v} \times \mathbf{u} = -\mathbf{u} \times \mathbf{v} = 2\mathbf{i} + 6\mathbf{j} - 10\mathbf{k}.$$

$$\mathbf{u} \times \mathbf{v} = (u_2v_3 - u_3v_2)\mathbf{i} - (u_1v_3 - u_3v_1)\mathbf{j} + (u_1v_2 - u_2v_1)\mathbf{k}$$

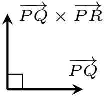
Example

Find a vector perpendicular to the plane containing the three points $P(1, -1, 0)$, $Q(2, 1, -1)$ and $R(-1, 1, 2)$.



$$\mathbf{u}\times\mathbf{v} = (u_2v_3 - u_3v_2)\mathbf{i} - (u_1v_3 - u_3v_1)\mathbf{j} + (u_1v_2 - u_2v_1)\mathbf{k}$$

The vector $\overrightarrow{PQ}\times\overrightarrow{PR}$ is perpendicular to the plane because



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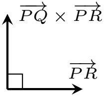

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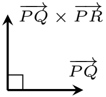

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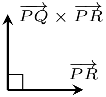

. We calculate that

$$\begin{aligned}
\overrightarrow{PQ} &= Q - P = (2, 1, -1) - (1, -1, 0) \\
&= (2 - 1, 1 + 1, -1 - 0) = \mathbf{i} + 2\mathbf{j} - \mathbf{k} \\
\overrightarrow{PR} &= R - P = (-1, 1, 2) - (1, -1, 0) \\
&= (-1 - 1, 1 + 1, 2 - 0) = -2\mathbf{i} + 2\mathbf{j} + 2\mathbf{k}
\end{aligned}$$

$$\mathbf{u} \times \mathbf{v} = (u_2v_3 - u_3v_2)\mathbf{i} - (u_1v_3 - u_3v_1)\mathbf{j} + (u_1v_2 - u_2v_1)\mathbf{k}$$

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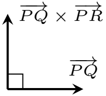

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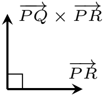

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&= (2 - 1, 1 + 1, -1 - 0) = \mathbf{i} + 2\mathbf{j} - \mathbf{k} \\
\overrightarrow{PR} &= R - P = (-1, 1, 2) - (1, -1, 0) \\
&= (-1 - 1, 1 + 1, 2 - 0) = -2\mathbf{i} + 2\mathbf{j} + 2\mathbf{k}
\end{aligned}$$

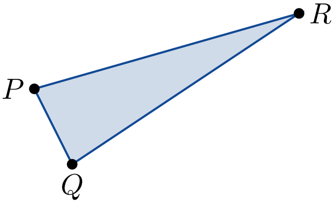
and

$$\overrightarrow{PQ} \times \overrightarrow{PR} = (4 + 2)\mathbf{i} - (2 - 2)\mathbf{j} + (2 + 4)\mathbf{k} = 6\mathbf{i} + 6\mathbf{k}.$$

Example

Find the area of triangle PQR.

$P(1,-1,0)$, $Q(2,1,-1)$ and $R(-1,1,2)$



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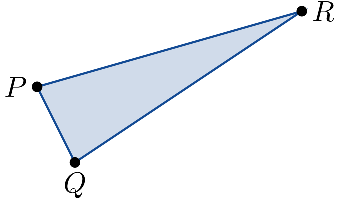

The area of the triangle is

$$\text{area} = \frac{1}{2}\left\|\overrightarrow{PQ}\times\overrightarrow{PR}\right\| = \frac{1}{2}\left\|6\mathbf{i}+6\mathbf{k}\right\|$$
$$= \frac{1}{2}\sqrt{6^2+0^2+6^2} = 3\sqrt{2}.$$

Example

Find a unit vector perpendicular to the plane containing P, Q and R.

$P(1,-1,0)$, $Q(2,1,-1)$ and $R(-1,1,2)$

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We know that $\overrightarrow{PQ} \times \overrightarrow{PR}$ is perpendicular to the plane. We just need to normalise this vector to find a unit vector.

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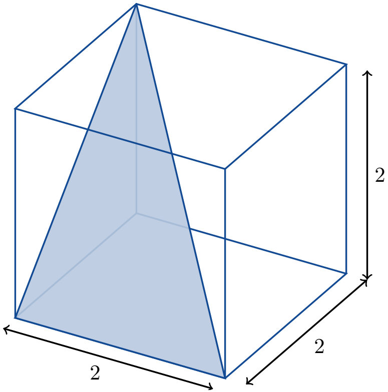
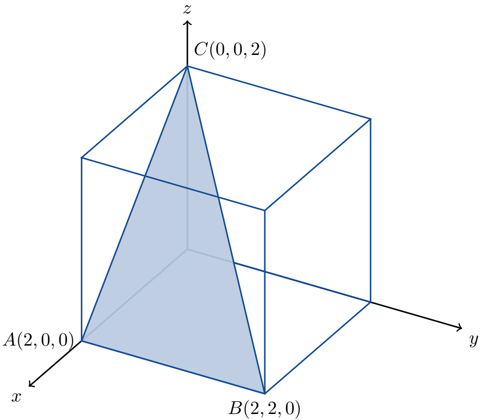
$$\mathbf{n} = \frac{\overrightarrow{PQ} \times \overrightarrow{PR}}{\left\|\overrightarrow{PQ} \times \overrightarrow{PR}\right\|} = \frac{6\mathbf{i} + 6\mathbf{k}}{6\sqrt{2}} = \frac{1}{\sqrt{2}}\mathbf{i} + \frac{1}{\sqrt{2}}\mathbf{k}.$$

Example

A triangle is inscribed inside a cube of side 2 as shown below. Use the cross product to find the area of the triangle.





First we draw coordinate axes and assign coordinates to the vertices of the triangle.

Then we can calculate

$$\overrightarrow{AB} = B - A = (2,2,0) - (2,0,0) = (0,2,0) = 2\mathbf{j}$$

and

$$\overrightarrow{AC} = C - A = (0,0,2) - (2,0,0) = (-2,0,2) = -2\mathbf{i} + 2\mathbf{k}.$$

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and

$$\overrightarrow{AC} = C - A = (0, 0, 2) - (2, 0, 0) = (-2, 0, 2) = -2\mathbf{i} + 2\mathbf{k}.$$

It follows that

$$\begin{aligned}\overrightarrow{AB} \times \overrightarrow{AC} = (2\mathbf{j}) \times (-2I \times 2\mathbf{k}) &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 2 & 0 \\ -2 & 0 & 2 \end{vmatrix} \\ &= \mathbf{i}(4-0) - \mathbf{j}(0-0) + \mathbf{k}(0 - -4) = 4\mathbf{i} + 4\mathbf{k}.\end{aligned}$$

Then we can calculate

$$\overrightarrow{AB} = B - A = (2,2,0) - (2,0,0) = (0,2,0) = 2\mathbf{j}$$

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Therefore

$$\begin{aligned}\text{area of triangle} &= \frac{1}{2}\left\|\overrightarrow{AB} \times \overrightarrow{AC}\right\| = \frac{1}{2}\sqrt{4^2 + 0^2 + 4^2} \\ &= \frac{1}{2}\sqrt{32} = \frac{1}{2}\sqrt{4}\sqrt{8} = \sqrt{8} = 2\sqrt{2}.\end{aligned}$$

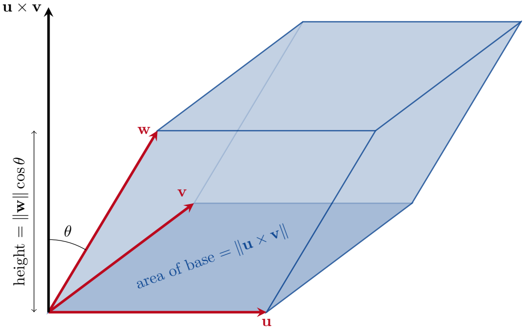
The Triple Scalar Product

Definition

The *triple scalar product* of $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ is

$$(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w}.$$

The Volume of a Parallelepiped



$$\text{volume} = \big(\text{area of base}\big)\big(\text{height}\big) = \|\mathbf{u} \times \mathbf{v}\|\, \|\mathbf{w}\| \cos\theta = |(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w}|$$

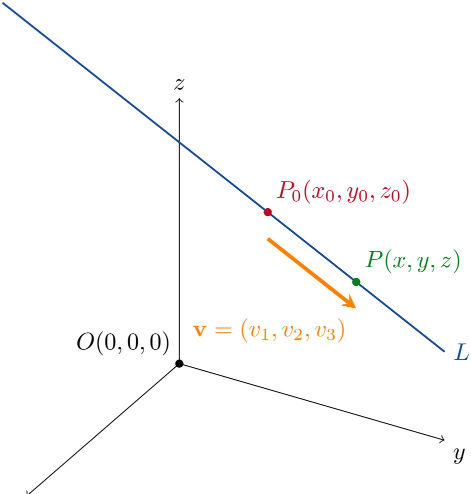
One Final Comment

We can do the dot product in both $\mathbb{R}^2$ and $\mathbb{R}^3$. But we can only do the cross product in $\mathbb{R}^3$. There is no cross product in $\mathbb{R}^2$.

Lines and Planes in Space

To describe a line in $\mathbb{R}^3$, we need

- a point $P_0(x_0, y_0, z_0)$ which the line passes through; and
- a vector $\mathbf{v}$ which gives the direction of the line.



Let $\mathbf{r}_0 = \overrightarrow{OP_0}$ and $\mathbf{r} = \overrightarrow{OP}$.

Definition

The *line* L *passing through* $P_0(x_0, y_0, z_0)$ *parallel to* $\mathbf{v} = (v_1, v_2, v_3)$ has the vector equation

$$\mathbf{r} = \mathbf{r}_0 + t\mathbf{v}, \qquad -\infty < t < \infty.$$

This equation is equivalent to

$$(x, y, z) = (x_0, y_0, z_0) + t(v_1, v_2, v_3)$$

or to the set of three equations

$$x = x_0 + tv_1, \qquad y = y_0 + tv_2, \qquad z = z_0 + tv_3.$$

Definition

The *parametric equations* for the line L passing through $P_0(x_0, y_0, z_0)$ parallel to $\mathbf{v} = (v_1, v_2, v_3)$ are

$$x = x_0 + tv_1, \qquad y = y_0 + tv_2, \qquad z = z_0 + tv_3.$$

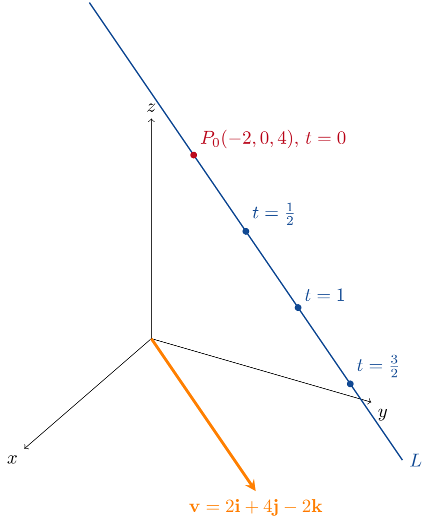
Example

Find parametric equations for the line passing through $P_0(-2, 0, 4)$ parallel to $\mathbf{v} = 2\mathbf{i} + 4\mathbf{j} - 2\mathbf{k}$.

We can write

$$x = -2 + 2t, \qquad y = 4t, \qquad z = 4 - 2t.$$

$$x = -2 + 2t$$
$$y = 4t$$
$$z = 4 - 2t$$



Example

Find parametric equations for the line passing through $P(-3, 2, -3)$ and $Q(1, -1, 4)$.

Choose $P_0 = P$ and $\mathbf{v} = \overrightarrow{PQ} = (4, -3, 7) = 4\mathbf{i} - 3\mathbf{j} + 7\mathbf{k}$. Then we can write

$$x = -3 + 4t, \qquad y = 2 - 3t, \qquad z = -3 + 7t.$$

Definition

The vector equation

$$\mathbf{r} = \mathbf{r}_0 + t\mathbf{v}, \qquad a \leq t \leq b$$

denotes a *line segment*.

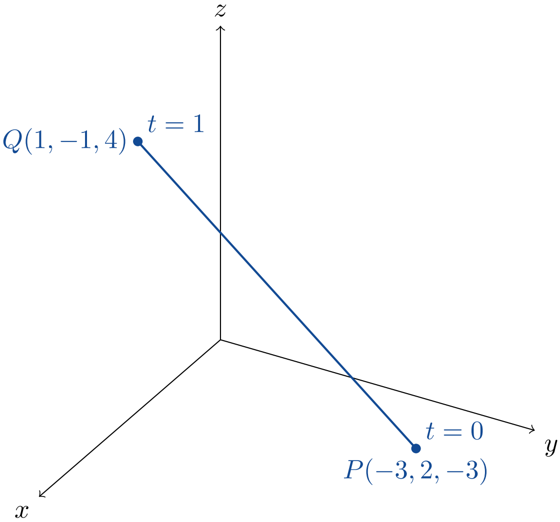
Example

Parametrise the line segment joining $P(-3,2,-3)$ and $Q(1,-1,4)$.

We know that $x=-3+4t$, $y=2-3t$ and $z=-3+7t$. The line passes through P then $t=0$ and passed through Q when $t=1$. Therefore

$$x=-3+4t, \qquad y=2-3t, \qquad z=-3+7t, \qquad 0 \leq t \leq 1$$

denotes the line segment from P to Q.



z
$t = 1$
$Q(1, -1, 4)$
$t = 0$
y
$P(-3, 2, -3)$
x

EXAMPLE 4 A helicopter is to fly directly from a helipad at the origin in the direction of the point (1, 1, 1) at a speed of 60 m/sec. What is the position of the helicopter after 10 sec?

Solution We place the origin at the starting position (helipad) of the helicopter. Then the unit vector

$$\mathbf{u} = \frac{1}{\sqrt{3}}\mathbf{i} + \frac{1}{\sqrt{3}}\mathbf{j} + \frac{1}{\sqrt{3}}\mathbf{k}$$

gives the flight direction of the helicopter. From Equation (4), the position of the helicopter at any time t is

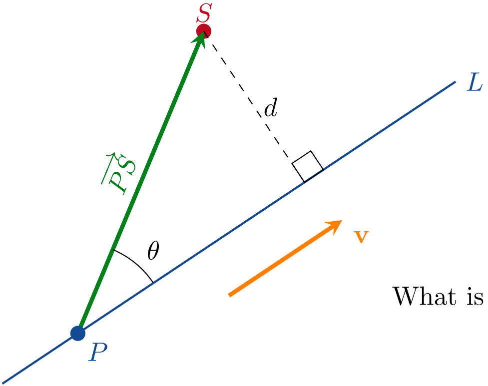
$$\begin{aligned}\mathbf{r}(t) &= \mathbf{r}_0 + t(\text{speed})\mathbf{u} \\ &= \mathbf{0} + t(60)\left(\frac{1}{\sqrt{3}}\mathbf{i} + \frac{1}{\sqrt{3}}\mathbf{j} + \frac{1}{\sqrt{3}}\mathbf{k}\right) \\ &= 20\sqrt{3}t(\mathbf{i} + \mathbf{j} + \mathbf{k}).\end{aligned}$$

When $t = 10$ sec,

$$\begin{aligned}\mathbf{r}(10) &= 200\sqrt{3}\,(\mathbf{i} + \mathbf{j} + \mathbf{k}) \\ &= \left\langle 200\sqrt{3}, 200\sqrt{3}, 200\sqrt{3} \right\rangle.\end{aligned}$$

After 10 sec of flight from the origin toward (1, 1, 1), the helicopter is located at the point $(200\sqrt{3}, 200\sqrt{3}, 200\sqrt{3})$ in space. It has traveled a distance of (60 m/sec)(10 sec) = 600 m, which is the length of the vector $\mathbf{r}(10)$. ■

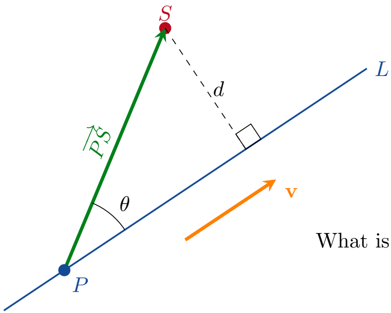
The Distance from a Point to a Line



What is d?

Let d be the shortest distance from the point S to the line L. We can see from this diagram that

$$d = \left\| \overrightarrow{PS} \right\| \sin\theta.$$



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$$d = \left\| \overrightarrow{PS} \right\| \sin\theta.$$

But remember that $\overrightarrow{PS} \times \mathbf{v} = \left\| \overrightarrow{PS} \right\| \|\mathbf{v}\| \sin\theta \ \mathbf{n}$. Therefore

$$\boxed{d = \frac{\left\| \overrightarrow{PS} \times \mathbf{v} \right\|}{\|\mathbf{v}\|}.}$$

Example

Find the distance from the point $S(1,1,5)$ to the line

$$x = 1 + t, \qquad y = 3 - t, \qquad z = 2t.$$

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The line passes through the point $P(1, 3, 0)$ in the direction $\mathbf{v} = \mathbf{i} - \mathbf{j} + 2\mathbf{k}$.

Example

Find the distance from the point $S(1, 1, 5)$ to the line

$$x = 1 + t, \qquad y = 3 - t, \qquad z = 2t.$$

The line passes through the point $P(1, 3, 0)$ in the direction $\mathbf{v} = \mathbf{i} - \mathbf{j} + 2\mathbf{k}$. Thus

$$\overrightarrow{PS} = S - P = (1, 1, 5) - (1, 3, 0) = (0, -2, 5) = -2\mathbf{j} + 5\mathbf{k}$$

and

$$\overrightarrow{PS} \times \mathbf{v} = (-4 + 5)\mathbf{i} - (0 - 5)\mathbf{j} + (0 + 2)\mathbf{k} = \mathbf{i} + 5\mathbf{j} + 2\mathbf{k}.$$

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$$\overrightarrow{PS} \times \mathbf{v} = (-4 + 5)\mathbf{i} - (0 - 5)\mathbf{j} + (0 + 2)\mathbf{k} = \mathbf{i} + 5\mathbf{j} + 2\mathbf{k}.$$

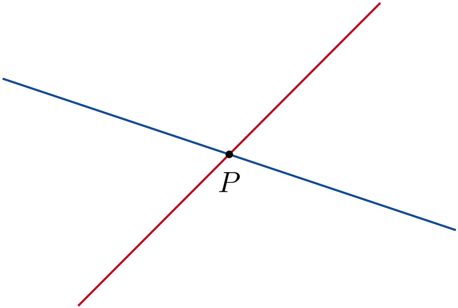
Therefore

$$d = \frac{\left\|\overrightarrow{PS} \times \mathbf{v}\right\|}{\|\mathbf{v}\|} = \frac{\sqrt{1^2 + 5^2 + 2^2}}{\sqrt{1^2 + 1^2 + 2^2}} = \frac{\sqrt{30}}{\sqrt{6}} = \sqrt{5}.$$

Break

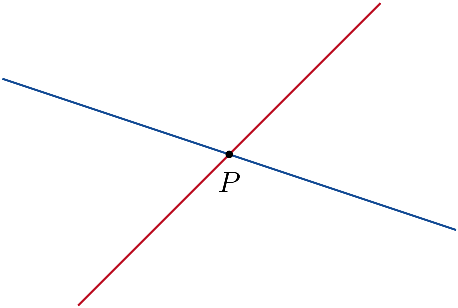
We will continue at 2pm

Intersecting Lines[1]



[1] not in book

Intersecting Lines[1]



Definition

Two lines intersect at a point P if and only if P lies on both lines.

[1] not in book

Example

Do the following two lines intersect? Is yes, where?

1. $x = 7 - t$, $y = 3 + 3t$, $z = 2t$.
2. $x = -1 + 2s$, $y = 3s$, $z = 1 + s$.

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Do the following two lines intersect? Is yes, where?

1. $x = 7 - t$, $y = 3 + 3t$, $z = 2t$.
2. $x = -1 + 2s$, $y = 3s$, $z = 1 + s$.

The two lines intersect if and only if there exist $s, t \in \mathbb{R}$ such that

$$\begin{aligned} 7 - t &= x = -1 + 2s \\ 3 + 3t &= y = 3s \\ 2t &= z = 1 + s \end{aligned}$$

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$$\begin{aligned} 7 - t = x &= -1 + 2s \qquad\qquad \implies t = 8 - 2s \\ 3 + 3t = y &= 3s \\ 2t = z &= 1 + s \end{aligned}$$

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The first equation tells us that $t = 8 - 2s$. Putting this into the second equation gives $s = t + 1 = (8 - 2s) + 1 = 9 - 2s$ which implies that $s = 3$ and $t = 2$. We must check the third equation:

Example

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Example

Do the following two lines intersect? Is yes, where?

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The first equation tells us that $t = 8 - 2s$. Putting this into the second equation gives $s = t + 1 = (8 - 2s) + 1 = 9 - 2s$ which implies that $s = 3$ and $t = 2$. We must check the third equation: $2t = 2 \times 2 = 4 = 1 + 3 = 1 + s$. Because the third equation is also true, we know that the two lines intersect at $P(5, 9, 4)$.

Example

Do the following two lines intersect? If yes, where?

1. $x = 1 + t$, $y = 3t$, $z = 3 + 3t$.
2. $x = -1 + 2s$, $y = 3s$, $z = 1 + s$.

Example

Do the following two lines intersect? If yes, where?

1. $x = 1 + t$, $y = 3t$, $z = 3 + 3t$.
2. $x = -1 + 2s$, $y = 3s$, $z = 1 + s$.

Can we find $s, t \in \mathbb{R}$ such that

$$\begin{aligned} 1 + t &= x = -1 + 2s \\ 3t &= y = 3s \\ 3 + 3t &= z = 1 + s \end{aligned}$$

are all true?

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$$
\begin{aligned}
1 + t = x &= -1 + 2s && \Longrightarrow\ 2 + t = 2t \ \Longrightarrow\ t = 2 \\
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$$\begin{aligned} 1 + t = x = -1 + 2s \quad & \implies 2 + t = 2t \implies t = 2 \\ 3t = y = 3s \quad & \implies s = t \\ 3 + 3t = z = 1 + s \quad & \implies 2 + 2t = 0 \implies t = -2 \neq 2 \end{aligned}$$

are all true?

Example

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are all true?

Therefore it is not possible to find an s and a t. Hence the lines do not intersect.

The Distance Between Two Lines[2]

There are three cases to consider:

[2] not in book

The Distance Between Two Lines[2]

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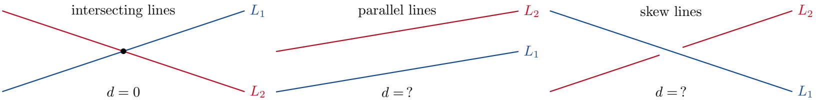
- the lines intersect;

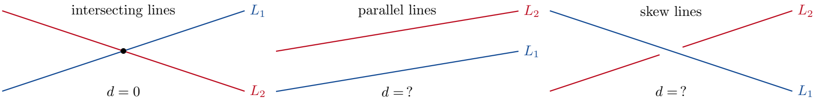


[2] not in book

The Distance Between Two Lines[2]

There are three cases to consider:

- the lines intersect;
- the lines do not intersect and are parallel ($\mathbf{v}_1 = k\mathbf{v}_2$ for some $k \in \mathbb{R}$); or



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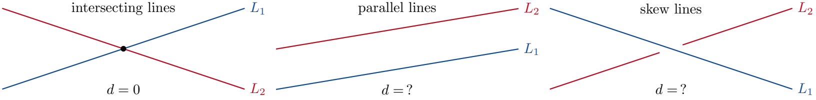
The Distance Between Two Lines[2]

There are three cases to consider:

- the lines intersect;
- the lines do not intersect and are parallel ($\mathbf{v}_1 = k\mathbf{v}_2$ for some $k \in \mathbb{R}$); or
- the lines do not intersect and are skew ($\mathbf{v}_1 \neq k\mathbf{v}_2$ for all $k \in \mathbb{R}$).



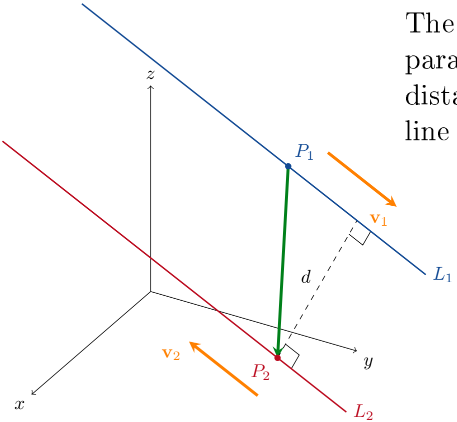
[2] not in book

Intersecting Lines

Clearly the distance between intersecting lines is zero. Hence

$$\boxed{d = 0.}$$

Parallel Lines ($\mathbf{v}_1 \times \mathbf{v}_2 = \mathbf{0}$)



The distance between the two parallel lines is the same as the distance between P_2 and the line L_1.

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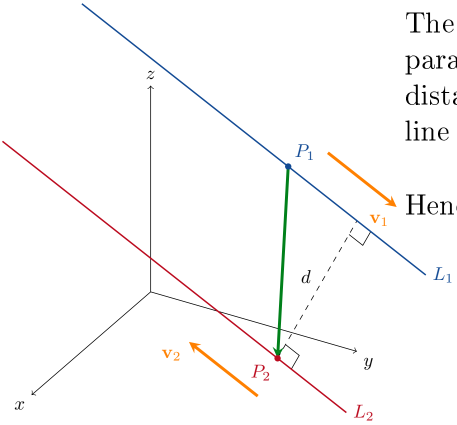
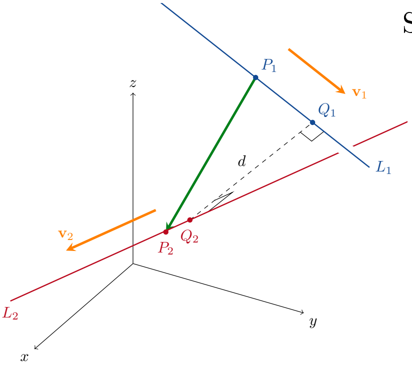


The distance between the two parallel lines is the same as the distance between P_2 and the line L_1.

Hence

$$\boxed{d = \frac{\left\|\overrightarrow{P_1P_2} \times \mathbf{v}_1\right\|}{\|\mathbf{v}_1\|}.}$$

Skew Lines ($\mathbf{v}_1 \times \mathbf{v}_2 \neq \mathbf{0}$)



So

$$d = \left\| \overrightarrow{Q_1 Q_2} \right\|$$

Let $\mathbf{n} = \mathbf{v}_1 \times \mathbf{v}_2$. Then $\mathbf{n}$ is orthogonal to both $\mathbf{v}_1$ and $\mathbf{v}_2$.

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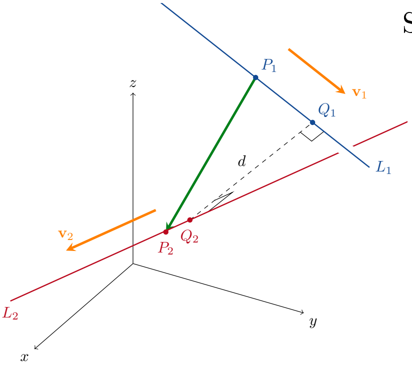

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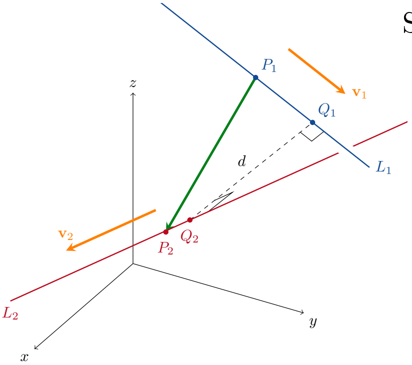

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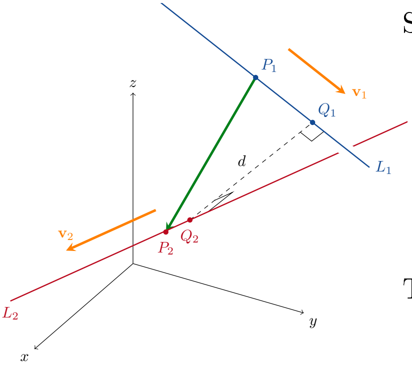

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So

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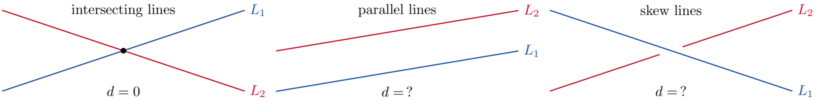
Thus

$$\boxed{d = \frac{\left| \overrightarrow{P_1P_2} \cdot (\mathbf{v}_1 \times \mathbf{v}_2) \right|}{\|\mathbf{v}_1 \times \mathbf{v}_2\|}.}$$



- Intersecting Lines: $d = 0$.
- Parallel Lines ($\mathbf{v}_1 \times \mathbf{v}_2 = \mathbf{0}$): $d = \dfrac{\left\|\overrightarrow{P_1P_2} \times \mathbf{v}_1\right\|}{\|\mathbf{v}_1\|}$.
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Find the distance between the following two lines.
line 1: $x = 0$, $y = -t$, $z = t$,
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$$\mathbf{v}_1 \times \mathbf{v}_2 = 2\mathbf{i} + 2\mathbf{j} + 2\mathbf{k} \neq \mathbf{0},$$

the lines are skew. (Recall that we have $\mathbf{v}_1 \times \mathbf{v}_2 = \mathbf{0}$ for parallel vectors.)

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$$\mathbf{v}_1 \times \mathbf{v}_2 = 2\mathbf{i} + 2\mathbf{j} + 2\mathbf{k} \neq \mathbf{0},$$

the lines are skew. (Recall that we have $\mathbf{v}_1 \times \mathbf{v}_2 = \mathbf{0}$ for parallel vectors.) Moreover note that $\overrightarrow{P_1P_2} = \mathbf{i}$. Then we calculate that

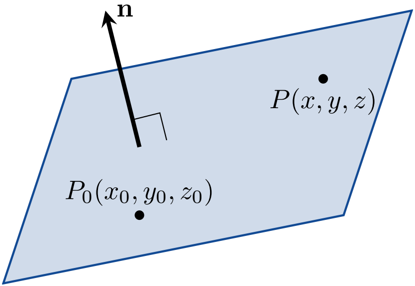
$$d = \frac{\left|\overrightarrow{P_1P_2} \cdot (\mathbf{v}_1 \times \mathbf{v}_2)\right|}{\|\mathbf{v}_1 \times \mathbf{v}_2\|} = \frac{|(\mathbf{i}) \cdot (2\mathbf{i} + 2\mathbf{j} + 2\mathbf{k})|}{\|2\mathbf{i} + 2\mathbf{j} + 2\mathbf{k}\|}$$
$$= \frac{|2 + 0 + 0|}{\sqrt{2^2 + 2^2 + 2^2}} = \frac{1}{\sqrt{3}}.$$

An Equation for a Plane in Space

To describe a plane, we need

- a point $P_0(x_0, y_0, z_0)$ which the plane passes through; and
- a vector $\mathbf{n} = A\mathbf{i} + B\mathbf{j} + C\mathbf{k}$ which is perpendicular to the plane.

The vector $\mathbf{n}$ is said to be *normal* to the plane.



Definition

The plane passing through the point $P_0(x_0, y_0, z_0)$ with normal vector $\mathbf{n} = A\mathbf{i} + B\mathbf{j} + C\mathbf{k}$ has the vector equation

$$\mathbf{n} \cdot \overrightarrow{P_0P} = 0.$$

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Writing this equation in coordinates, we have

$$A(x - x_0) + B(y - y_0) + C(z - z_0) = 0$$

or

$$\boxed{Ax + By + Cz = D}$$

where $D = Ax_0 + By_0 + Cz_0$ is a constant.

Example

Find an equation for the plane passing through $P_0(-3, 0, 7)$ normal to $\mathbf{n} = 5\mathbf{i} + 2\mathbf{j} - \mathbf{k}$.

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$$\begin{aligned} A(x-x_0)+B(y-y_0)+C(z-z_0) &= 0 \\ 5(x-(-3))+2(y-0)+(-1)(z-7) &= 0 \\ 5x-15+2y-z+7 &= 0 \\ 5x+2y-z &= -22. \end{aligned}$$

Remark

The vector $\mathbf{n} = A\mathbf{i} + B\mathbf{j} + C\mathbf{k}$ is normal to the plane $Ax + By + Cz = D$.

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Example

Find a vector normal to the plane $x + 2y + 3z = 4$.

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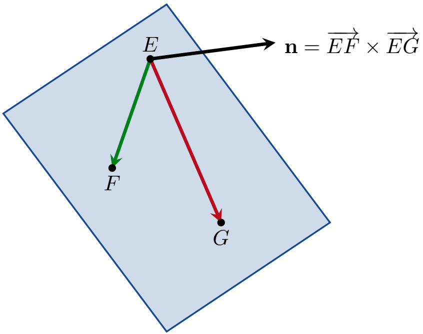
Find a vector normal to the plane $x + 2y + 3z = 4$.

We can immediately write down $\mathbf{n} = \mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$.



Example

Find an equation for the plane containing the points $E(0,0,1)$, $F(2,0,0)$ and $G(0,3,0)$.

First we need to find a vector normal to the plane. Since $\overrightarrow{EF} = 2\mathbf{i} - \mathbf{k}$ and $\overrightarrow{EG} = 3\mathbf{j} - \mathbf{k}$, we have that

$$\begin{aligned}\mathbf{n} &= \overrightarrow{EF} \times \overrightarrow{EG} = (0 - -3)\mathbf{i} - (-2 - 0)\mathbf{j} + (6 - 0)\mathbf{k} \\ &= 3\mathbf{i} + 2\mathbf{j} + 6\mathbf{k}\end{aligned}$$

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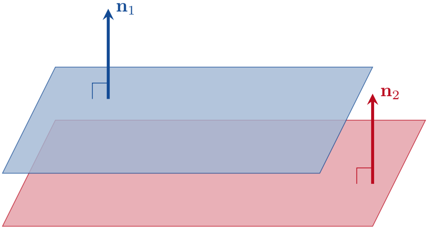
is normal to the plane. Using $P_0 = E(0, 0, 1)$, the equation for the plane is

$$\begin{aligned}3(x - 0) + 2(y - 0) + 6(z - 1) &= 0 \\ 3x + 2y + 6z &= 6.\end{aligned}$$

Lines of Intersection



Two planes are parallel $\iff$ $\mathbf{n}_1 = k\mathbf{n}_2$ for some $k \in \mathbb{R}$.

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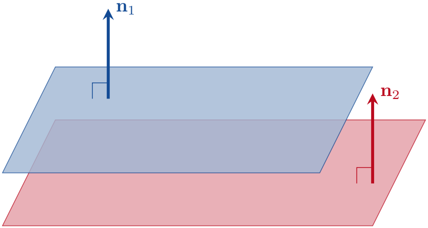

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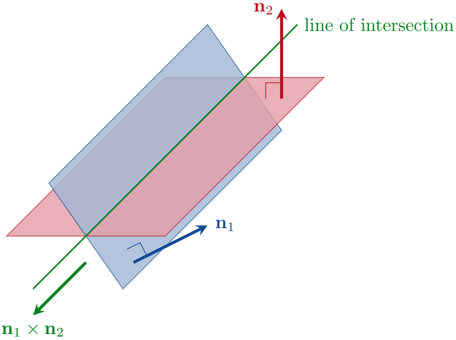

Two planes intersect in a line $\iff$ $\mathbf{n}_1 \neq k\mathbf{n}_2$ for all $k \in \mathbb{R}$.

Example

Find a vector parallel of the line of intersection of the planes $3x - 6y - 2z = 15$ and $2x + y - 2z = 5$.

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Find a vector parallel of the line of intersection of the planes $3x - 6y - 2z = 15$ and $2x + y - 2z = 5$.

We can immediately write down $\mathbf{n}_1 = \mathbf{3i} - \mathbf{6j} - \mathbf{2k}$ and $\mathbf{n}_2 = \mathbf{2i} + \mathbf{j} - \mathbf{2k}$. A vector parallel to the line of intersection is

$$\mathbf{n}_1 \times \mathbf{n}_2 = (12 + 2)\mathbf{i} - (-6 + 4)\mathbf{j} + (3 + 12)\mathbf{k} = 14\mathbf{i} + 2\mathbf{j} + 15\mathbf{k}.$$

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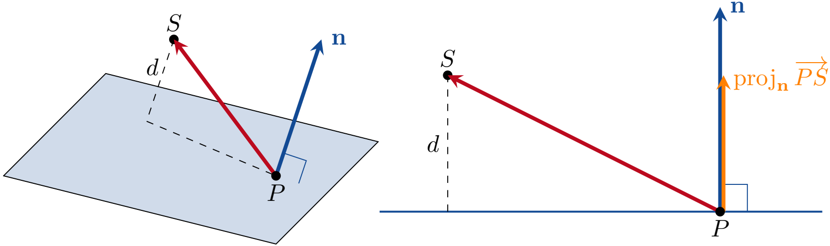
$$P(x,y,z)|_{t=-1} = P\left(\frac{8}{3} + 2t, -2t, 1+t\right)\Big|_{t=-1} = P\left(\frac{2}{3}, 2, 0\right).$$

The Distance from a Point to a Plane



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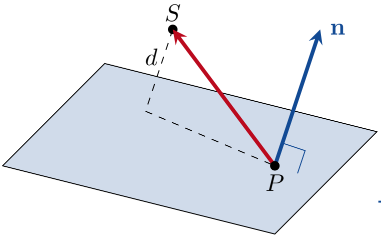

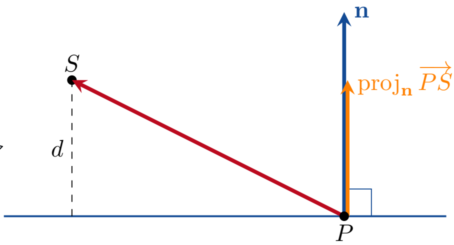

We can see that $d = \left\| \text{proj}_{\mathbf{n}} \overrightarrow{PS} \right\|$.

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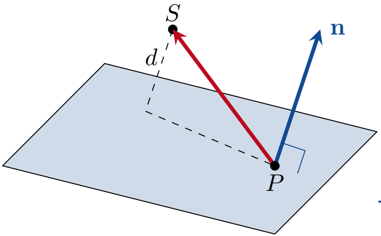

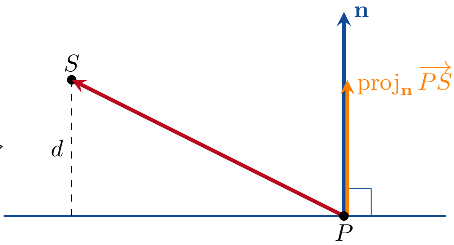

We can see that $d = \left\|\mathrm{proj}_{\mathbf{n}}\,\overrightarrow{PS}\right\|$. Therefore the distance from a point S to a plane with normal $\mathbf{n}$ containing the point P is

$$\boxed{d = \frac{\left|\overrightarrow{PS}\cdot\mathbf{n}\right|}{\|\mathbf{n}\|}.}$$

$$d = \frac{\left|\overrightarrow{PS} \cdot \mathbf{n}\right|}{\|\mathbf{n}\|}$$

Example

Find the distance from the point $S(1,2,3)$ to the plane $x + 2y + 3z = 4$.

First we need a point in the plane.

$$d = \frac{\left|\overrightarrow{PS} \cdot \mathbf{n}\right|}{\|\mathbf{n}\|}$$

Example

Find the distance from the point $S(1,2,3)$ to the plane $x + 2y + 3z = 4$.

First we need a point in the plane. Setting $y = 0$ and $z = 0$ we must have $x = 4 - 2y - 3z = 4$. Therefore $P(4,0,0)$ is in the plane.

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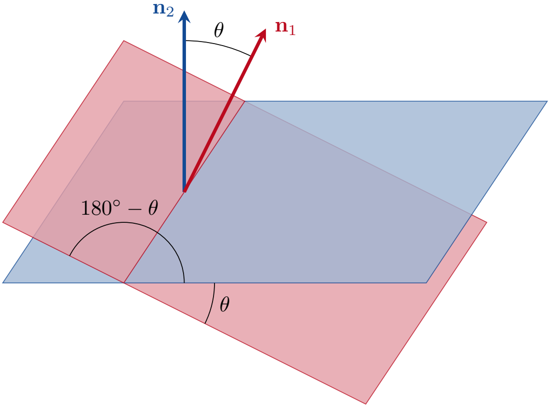
Therefore the required distance is

$$\begin{aligned} d = \frac{\left|\overrightarrow{PS} \cdot \mathbf{n}\right|}{\|\mathbf{n}\|} &= \frac{|(-3\mathbf{i} + 2\mathbf{j} + 3\mathbf{k}) \cdot (\mathbf{i} + 2\mathbf{j} + 3\mathbf{k})|}{\|\mathbf{i} + 2\mathbf{j} + 3\mathbf{k}\|} \\ &= \frac{|-3 + 4 + 9|}{\sqrt{1^2 + 2^2 + 3^2}} = \frac{10}{\sqrt{14}}. \end{aligned}$$

Please read Example 11 in the textbook.

Angles Between Planes

There are two possible angles that can be measured between planes. We are interested in the smaller angle.



Definition

The angle between two planes is defined to be equal to whichever of the following angles is smaller

- the angle between $\mathbf{n}_1$ and $\mathbf{n}_2$;
- 180° minus the angle between $\mathbf{n}_1$ and $\mathbf{n}_2$.

The angle between two planes will always be between 0° and 90°.

Example

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We have normal vectors $\mathbf{n}_1 = 3\mathbf{i} - 6\mathbf{j} - 2\mathbf{k}$ and $\mathbf{n}_2 = -2\mathbf{i} - \mathbf{j} + 2\mathbf{k}$. The angle between $\mathbf{n}_1$ and $\mathbf{n}_2$ is

$$\theta = \cos^{-1}\left(\frac{\mathbf{n}_1 \cdot \mathbf{n}_2}{\|\mathbf{n}_1\|\,\|\mathbf{n}_2\|}\right) = \cos^{-1}\left(\frac{-4}{21}\right) \approx 101^\circ.$$

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Because $101^\circ > 90^\circ$, the angle between the two planes is approximately $180 - 101^\circ = 79^\circ$.

Next Time

- 13.1 Functions of Several Variables
- 13.2 Limits and Continuity in Higher Dimensions
- 13.3 Partial Derivatives
- 13.4 The Chain Rule