

# Lecture 6

- Coordinates and Basis
- Dimension
- Change of Basis
- Row Space, Column Space, and Null Space



# Coordinates and Basis

## Definition

A vector space  $V$  is said to be *finite-dimensional* if there is a finite set of vectors in  $V$  that spans  $V$  and is said to be *infinite-dimensional* if no such set exists.

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If  $S = \{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\}$  is a set of vectors in a finite-dimensional vector space  $V$ , then  $S$  is called a *basis* for  $V$  if:

- 1  $S$  spans  $V$ ;<sup>1</sup> and
- 2  $S$  is linearly independent.

(The plural of basis is bases.)

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<sup>1</sup>i.e. if  $\text{span } S = V$ .

$$\mathbf{i} = (1, 0, 0) \quad \mathbf{j} = (0, 1, 0) \quad \mathbf{k} = (0, 0, 1)$$



## Example

Recall from last week that the standard unit vectors

$$\mathbf{e}_1 = (1, 0, 0, \dots, 0), \mathbf{e}_2 = (0, 1, 0, \dots, 0), \dots, \mathbf{e}_n = (0, 0, 0, \dots, 1).$$

span  $\mathbb{R}^n$ . These vectors are linearly independent (I only proved it for  $n = 3$ .)

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Therefore these vectors form a basis for  $\mathbb{R}^n$  that we call the *standard basis for  $\mathbb{R}^n$* .

## Example

$S = \{\mathbf{i}, \mathbf{j}, \mathbf{k}\}$  is called the *standard basis for  $\mathbb{R}^3$* .

# Coordinates and Basis



Example (The standard basis for  $\mathbb{P}^n$ )

Show that  $S = \{1, x, x^2, \dots, x^n\}$  is a basis for the vector space  $\mathbf{P}^n$  of polynomials of degree  $n$  or less.

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Last week we showed that  $\text{span } S = \mathbb{P}^n$  and that  $S$  is linearly independent. Therefore  $S$  is a basis for  $\mathbb{P}^n$ .

# Coordinates and Basis

## Example (Another Basis for R3)

Show that the vectors  $\mathbf{v}_1 = (1, 2, 1)$ ,  $\mathbf{v}_2 = (2, 9, 0)$ , and  $\mathbf{v}_3 = (3, 3, 4)$  form a basis for  $\mathbb{R}^3$ .

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For 1. we must show that

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3 = \mathbf{0}$$

has only the trivial solution.

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has only the trivial solution. For 2., we must show that

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3 = \mathbf{b}$$

is consistent for every  $\mathbf{b} \in \mathbb{R}^3$ .

# Coordinates and Basis



So we have two linear systems to consider:

$$\begin{cases} c_1 + 2c_2 + 3c_3 = 0 \\ 2c_1 + 9c_2 + 3c_3 = 0 \\ c_1 + 4c_3 = 0 \end{cases} \quad \text{and} \quad \begin{cases} c_1 + 2c_2 + 3c_3 = b_1 \\ 2c_1 + 9c_2 + 3c_3 = b_2 \\ c_1 + 4c_3 = b_3. \end{cases}$$

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These two linear systems have the same coefficient matrix

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 9 & 3 \\ 1 & 0 & 4 \end{bmatrix}.$$

If we can show that  $\det(A) \neq 0$ , then we can prove both things at the same time.

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If  $\det(A) \neq 0$ , then

$$A\mathbf{c} = \mathbf{0} \implies \mathbf{c} = A^{-1}\mathbf{0} = \mathbf{0}$$

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I leave it for you to prove that  $\det(A) = -1$ . Hence  $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$  form a basis for  $\mathbb{R}^3$ .

# Coordinates and Basis

Example (The Standard Basis for  $\mathbb{R}^{m \times n} = M_{mn}$ )

Show that the matrices

$$M_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad M_2 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad M_3 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \quad M_4 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

form a basis for the vector space  $\mathbb{R}^{2 \times 2}$  of  $2 \times 2$  matrices.

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Hence

$$c_1 = c_2 = c_3 = c_4 = 0$$

which proves that these four matrices are linearly independent.

# Coordinates and Basis

Let  $B = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$  be any matrix in  $\mathbb{R}^{2 \times 2}$ .

$$c_1 M_1 + c_2 M_2 + c_3 M_3 + c_4 M_4 = B$$

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Hence

$$\begin{cases} c_1 = a \\ c_2 = b \\ c_3 = c \\ c_4 = d. \end{cases}$$

This proves that these four matrices span  $\mathbb{R}^{2 \times 2}$ .

Therefore  $M_1, M_2, M_3$  and  $M_4$  for a basis for  $\mathbb{R}^{2 \times 2}$  called the *standard basis for  $\mathbb{R}^{2 \times 2} = M_{22}$* .

# Coordinates and Basis

## Example

Show that the vector space  $\mathbb{P}$  is infinite dimensional<sup>2</sup>.

We need to show that  $\mathbb{P}$  does not have a finite spanning set.

---

<sup>2</sup> $\mathbb{P}$  = {all polynomials with real coefficients}

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Show that the vector space  $\mathbb{P}$  is infinite dimensional<sup>2</sup>.

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If there were a finite spanning set, say  $S = \{\mathbf{p}_1, \mathbf{p}_2, \dots, \mathbf{p}_r\}$ , then the degrees of the polynomials in  $S$  would have a maximum value, say  $n$ ;

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Thus, there would be no way to express the polynomial  $x^{n+1}$  as a linear combination of the polynomials in  $S$ , contradicting the fact that the vectors in  $S$  span  $\mathbb{P}$ .

---

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# Coordinates and Basis

## Example

Some other infinite dimensional vector spaces are

$$\mathbb{R}^{\mathbb{N}} = \{\text{all sequences of real numbers}\}$$

$$F(-\infty, \infty) = \{\text{all functions } f : \mathbb{R} \rightarrow \mathbb{R}\}$$

$$C(-\infty, \infty) = \{\text{all continuous functions } f : \mathbb{R} \rightarrow \mathbb{R}\}$$

$$C^k(-\infty, \infty) = \left\{ \begin{array}{l} \text{all functions } f : \mathbb{R} \rightarrow \mathbb{R} \text{ which are} \\ \text{continuously differentiable } k \text{ times} \end{array} \right\}$$

$$C^\infty(-\infty, \infty) = \left\{ \begin{array}{l} \text{all functions } f : \mathbb{R} \rightarrow \mathbb{R} \text{ which} \\ \text{can be differentiated } \infty \text{ times} \end{array} \right\}$$

## Coordinates Relative to a Basis

### Theorem

*If  $S = \{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\}$  is a basis for a vector space  $V$ , then every vector  $\mathbf{v}$  in  $V$  can be expressed in the form*

$$\mathbf{v} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \dots + c_n\mathbf{v}_n$$

*in exactly one way.*

Now we can start talking about coordinates in a general vector space  $V$ .

# Coordinates and Basis

## Definition

If  $S = \{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\}$  is a basis for a vector space  $V$ , and

$$\mathbf{v} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \dots + c_n\mathbf{v}_n$$

is the expression for a vector  $\mathbf{v}$  in terms of the basis  $S$ , then the scalars  $c_1, c_2, \dots, c_n$  are called the *coordinates* of  $\mathbf{v}$  relative to the basis  $S$ .

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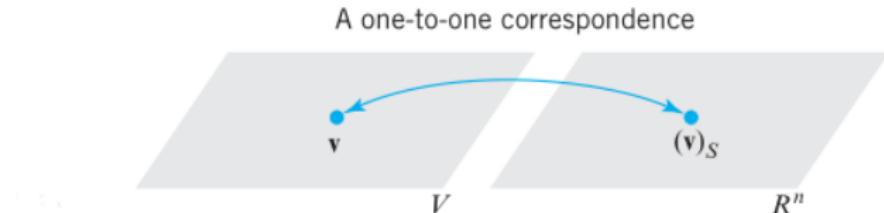
is the expression for a vector  $\mathbf{v}$  in terms of the basis  $S$ , then the scalars  $c_1, c_2, \dots, c_n$  are called the *coordinates* of  $\mathbf{v}$  relative to the basis  $S$ . The vector  $(c_1, c_2, \dots, c_n)$  in  $\mathbb{R}^n$  constructed from these coordinates is called the *coordinate vector of  $\mathbf{v}$  relative to  $S$* ; it is denoted by

$$(\mathbf{v})_S = (c_1, c_2, \dots, c_n).$$

## Remark

When we do coordinate, the order of the vectors in  $S$  is important. Some books use the term *ordered basis* which means a basis in which the order of the vectors is fixed.

Observe that  $(\mathbf{v})_S$  is a vector in  $\mathbb{R}^n$ , so that once an ordered basis  $S$  is given for a vector space  $V$ , Theorem 4.4.1 establishes a one-to-one correspondence between vectors in  $V$  and vectors in  $\mathbb{R}^n$  (Figure 4.4.6).



$$\mathbf{i} = (1, 0, 0) \quad \mathbf{j} = (0, 1, 0) \quad \mathbf{k} = (0, 0, 1)$$



## Example

$V = \mathbb{R}^3$ . Standard basis  $\mathbf{i}, \mathbf{j}, \mathbf{k}$ .

If  $\mathbf{v} = (a, b, c)$  is any vector in  $\mathbb{R}^3$ , then

$$\mathbf{v} = a\mathbf{i} + b\mathbf{j} + c\mathbf{k} \implies (\mathbf{v})_S = (a, b, c).$$

# Coordinates and Basis



## Example

Find the coordinate vector for the polynomial

$$\mathbf{p}(x) = c_0 + c_1x + c_2x^2 + \dots + c_nx^n$$

relative to the standard basis for the vector space  $\mathbb{P}^n$ .

The standard basis for  $\mathbb{P}^n$  is  $S = \{1, x, x^2, \dots, x^n\}$ . So clearly

$$(\mathbf{p})_S = (c_0, c_1, c_2, \dots, c_n).$$

# Coordinates and Basis

## Example

Find the coordinate vector of

$$B = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

relative to the standard basis for  $\mathbb{R}^{2 \times 2}$ .

Since

$$B = \begin{bmatrix} a & b \\ c & d \end{bmatrix} = a \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + b \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + c \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + d \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

we have

$$(B)_S = (a, b, c, d).$$

# Coordinates and Basis

## Example (Coordinates in $\mathbb{R}^3$ )

We showed in a previous example that the vectors

$$\mathbf{v}_1 = (1, 2, 1), \quad \mathbf{v}_2 = (2, 9, 0), \quad \mathbf{v}_3 = (3, 3, 4)$$

form a basis for  $\mathbb{R}^3$ .

- 1 Find the coordinate vector of  $\mathbf{v} = (5, -1, 9)$  relative to the basis  $S = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ .
- 2 Find the vector  $\mathbf{u}$  in  $\mathbb{R}^3$  whose coordinate vector relative to  $S$  is  $(\mathbf{u})_S = (-1, 3, 2)$ .

## Coordinate

$$\mathbf{v}_1 = (1, 2, 1), \quad \mathbf{v}_2 = (2, 9, 0), \quad \mathbf{v}_3 = (3, 3, 4)$$

- 1 Find the coordinate vector of  $\mathbf{v} = (5, -1, 9)$  relative to the basis  $S = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ .

We need to find scalars  $c_1, c_2, c_3$  such that

$$\mathbf{v} = c_1 \mathbf{v}_1 + c_2 \mathbf{v}_2 + c_3 \mathbf{v}_3.$$

Coordinate basis  $\mathbf{v}_1 = (1, 2, 1), \mathbf{v}_2 = (2, 9, 0), \mathbf{v}_3 = (3, 3, 4)$

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I.e. such that

$$(5, -1, 9) = c_1(1, 2, 1) + c_2(2, 9, 0) + c_3(3, 3, 4).$$

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So we need to solve the linear system

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I leave it to you to check that the solution is  $c_1 = 1, c_2 = -1$  and  $c_3 = 2$ . Therefore

$$(\mathbf{v})_S = (1, -1, 2).$$

## Coordinate

$$\mathbf{v}_1 = (1, 2, 1), \quad \mathbf{v}_2 = (2, 9, 0), \quad \mathbf{v}_3 = (3, 3, 4)$$

- 2 Find the vector  $\mathbf{u}$  in  $\mathbb{R}^3$  whose coordinate vector relative to  $S$  is  $(\mathbf{u})_S = (-1, 3, 2)$ .

This one is easier.

$$\mathbf{u} = (-1)\mathbf{v}_1 + 3\mathbf{v}_2 + 2\mathbf{v}_3$$

=

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This one is easier.

$$\begin{aligned}\mathbf{u} &= (-1)\mathbf{v}_1 + 3\mathbf{v}_2 + 2\mathbf{v}_3 \\ &= (-1)(1, 2, 1) + 3(2, 9, 0) + 2(3, 3, 4) \\ &= (11, 31, 7).\end{aligned}$$



# Dimension

We have seen two bases for  $\mathbb{R}^3$ .

$$\mathbf{i} = (1, 0, 0), \quad \mathbf{j} = (0, 1, 0), \quad \mathbf{k} = (0, 0, 1)$$

$$\mathbf{v}_1 = (1, 2, 1), \quad \mathbf{v}_2 = (2, 9, 0), \quad \mathbf{v}_3 = (3, 3, 4)$$

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## Theorem

*All bases for a finite-dimensional vector space have the same number of vectors.*

## Definition

The dimension of a finite-dimensional vector space  $V$ , denoted by  $\dim(V)$ , is defined to be the number of vectors in a basis for  $V$ .

The dimension of the zero vector space  $\{\mathbf{0}\}$  is defined to be 0.

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Engineers often use the term *degrees of freedom* as a synonym for dimension.

## Theorem

Let  $V$  be an  $n$ -dimensional vector space, and let  $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\}$  be any basis.

- 1 If a set in  $V$  has more than  $n$  vectors, then it is linearly dependent.
- 2 If a set in  $V$  has fewer than  $n$  vectors, then it does not span  $V$ .

## Example

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- $\dim(\mathbb{R}^{2 \times 2}) = 4$  because the standard basis contains the following four vectors

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}.$$

# Dimension

## Example

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- Similarly  $\dim(\mathbb{R}^{m \times n}) = mn$ .

## ► EXAMPLE 2 Dimension of $\text{Span}(S)$

If  $S = \{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_r\}$  then every vector in  $\text{span}(S)$  is expressible as a linear combination of the vectors in  $S$ . Thus, if the vectors in  $S$  are *linearly independent*, they automatically form a basis for  $\text{span}(S)$ , from which we can conclude that

$$\dim[\text{span}\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_r\}] = r$$

In words, the dimension of the space spanned by a linearly independent set of vectors is equal to the number of vectors in that set.

## Example (Dimension of a Solution Space)

Find a basis for and the dimension of the solution space of the homogeneous linear system

$$\begin{cases} x_1 + 3x_2 - 2x_3 + 2x_5 = 0 \\ 2x_1 + 6x_2 - 5x_3 - 2x_4 + 4x_5 - 3x_6 = 0 \\ 5x_3 + 10x_4 + 15x_6 = 0 \\ 2x_1 + 6x_2 + 8x_4 + 4x_5 + 18x_6 = 0. \end{cases}$$

# Dimension

I leave it to you to check that the solution is

$$\begin{cases} x_1 = -3x_2 - 4x_4 - 2x_5 \\ x_2 \text{ is free} \\ x_3 = -2x_4 \\ x_4 \text{ is free} \\ x_5 \text{ is free} \\ x_6 = 0. \end{cases}$$

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We can write this as

$$\begin{aligned} (x_1, x_2, x_3, x_4, x_5, x_6) &= (-3x_2 - 4x_4 - 2x_5, x_2, -2x_4, x_4, x_5, 0) \\ &= x_2(-3, 1, 0, 0, 0, 0) + x_4(-4, 0, -2, 1, 0, 0) \\ &\quad + x_5(-2, 0, 0, 0, 1, 0). \end{aligned}$$

# Dimension



This shows that the vectors

$$\mathbf{v}_1 = (-3, 1, 0, 0, 0, 0), \mathbf{v}_2 = (-4, 0, -2, 1, 0, 0), \mathbf{v}_3 = (-2, 0, 0, 0, 1, 0)$$

span the solution space.

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Therefore the solution space has dimension 3.

## Some More Theorems

Theorem (The Basis Theorem<sup>3</sup>)

Let  $V$  be an  $n$ -dimensional vector space, and let  $S$  be a set in  $V$  with exactly  $n$  vectors. Then the following are equivalent:

- 1  $S$  is a basis for  $V$ ;
- 2  $S$  spans  $V$ ;
- 3  $S$  is linearly independent.

---

<sup>3</sup>page 245 in your textbook

## ► EXAMPLE 5 Bases by Inspection

- (a) Explain why the vectors  $\mathbf{v}_1 = (-3, 7)$  and  $\mathbf{v}_2 = (5, 5)$  form a basis for  $R^2$ .
- (b) Explain why the vectors  $\mathbf{v}_1 = (2, 0, -1)$ ,  $\mathbf{v}_2 = (4, 0, 7)$ , and  $\mathbf{v}_3 = (-1, 1, 4)$  form a basis for  $R^3$ .

**Solution (a)** Since neither vector is a scalar multiple of the other, the two vectors form a linearly independent set in the two-dimensional space  $R^2$ , and hence they form a basis by Theorem 4.5.4.

**Solution (b)** The vectors  $\mathbf{v}_1$  and  $\mathbf{v}_2$  form a linearly independent set in the  $xz$ -plane (why?). The vector  $\mathbf{v}_3$  is outside of the  $xz$ -plane, so the set  $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$  is also linearly independent. Since  $R^3$  is three-dimensional, Theorem 4.5.4 implies that  $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$  is a basis for the vector space  $R^3$ . ◀

## Theorem

Let  $S$  be a finite set of vectors in a finite-dimensional vector space  $V$ .

- 1 If  $S$  spans  $V$  but is not a basis for  $V$ , then  $S$  can be reduced to a basis for  $V$  by removing appropriate vectors from  $S$ .
- 2 If  $S$  is a linearly independent set that is not already a basis for  $V$ , then  $S$  can be enlarged to a basis for  $V$  by inserting appropriate vectors into  $S$ .

## Theorem

*If  $W$  is a subspace of a finite-dimensional vector space  $V$ , then:*

- 1**  $W$  is finite-dimensional;
- 2**  $\dim(W) \leq \dim(V)$ ; and
- 3**  $W = V$  if and only if  $\dim(W) = \dim(V)$ .



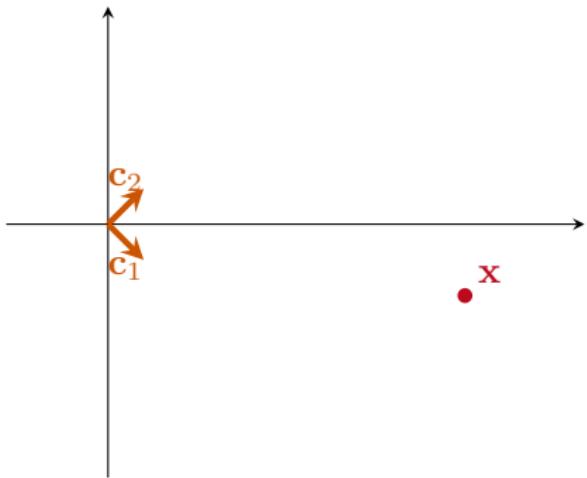
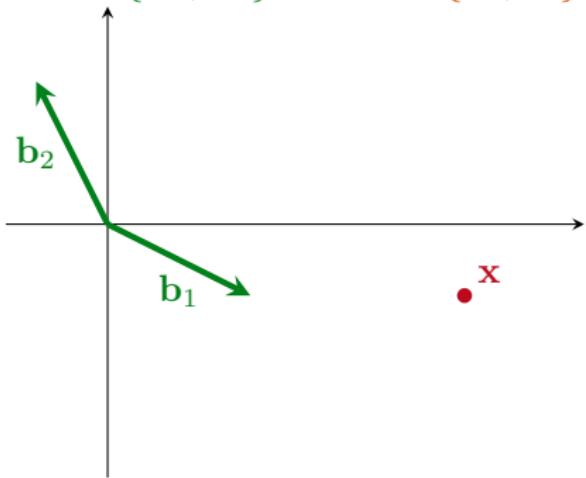
# Change of Basis

# Change of Basis

Consider the vector space  $V = \mathbb{R}^2$ . Suppose that  $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2\}$  and  $\mathcal{C} = \{\mathbf{c}_1, \mathbf{c}_2\}$  are two bases for  $V$ .

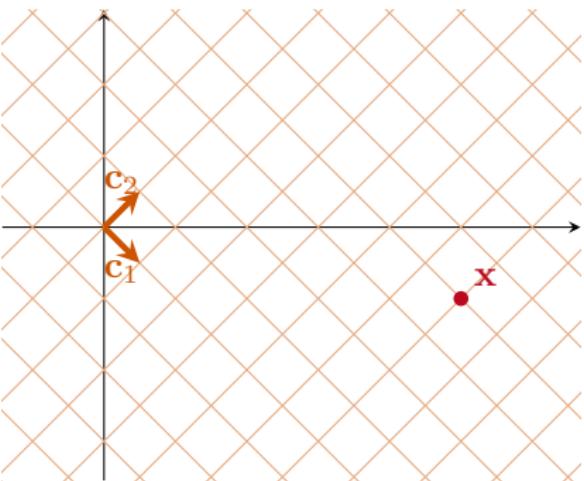
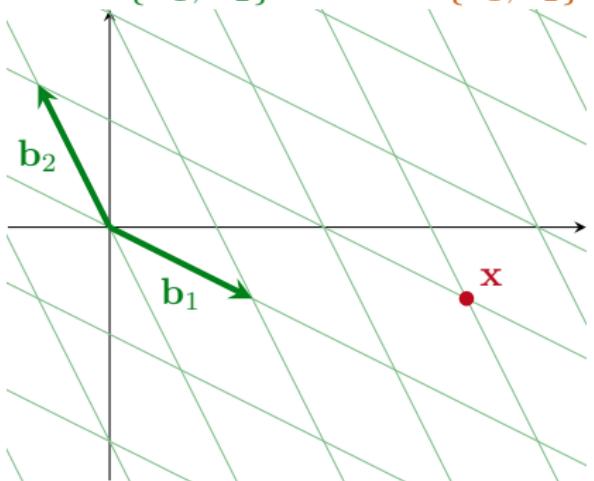
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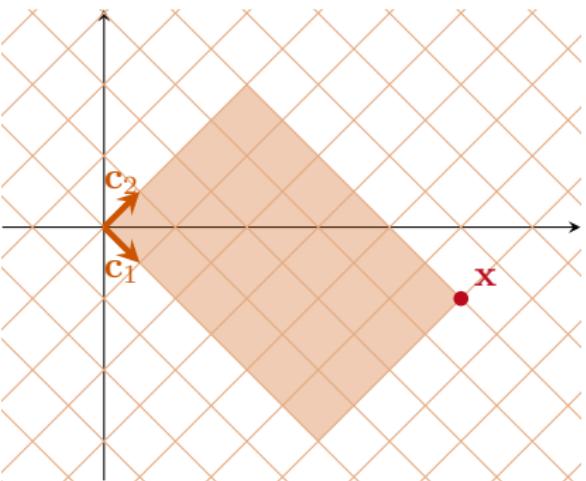
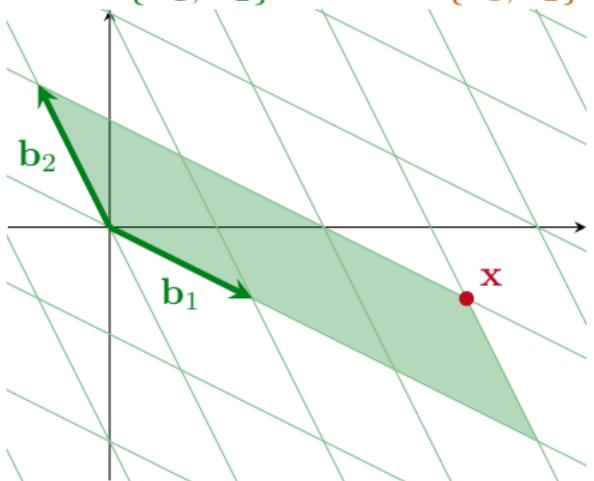
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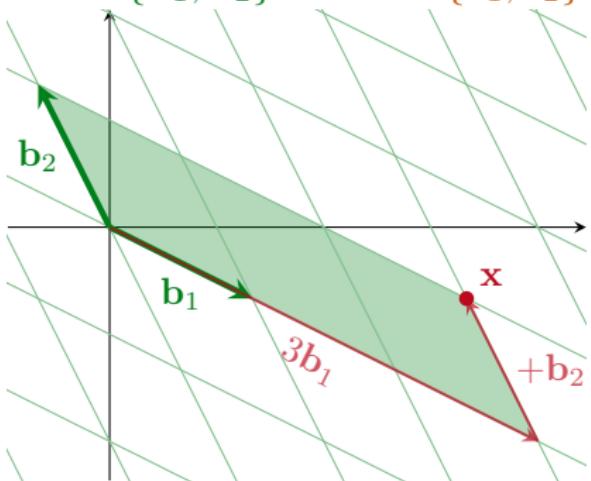
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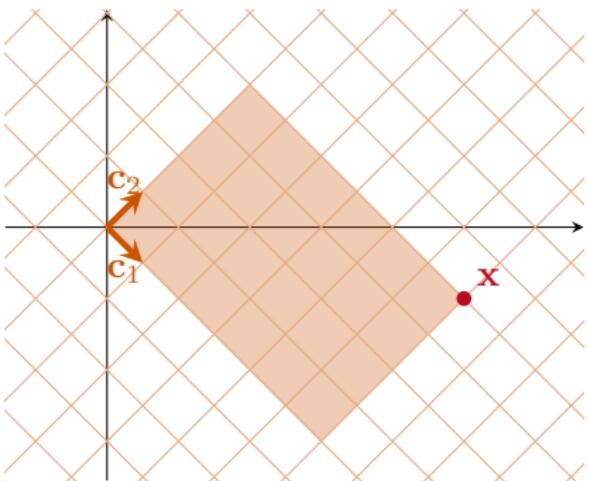
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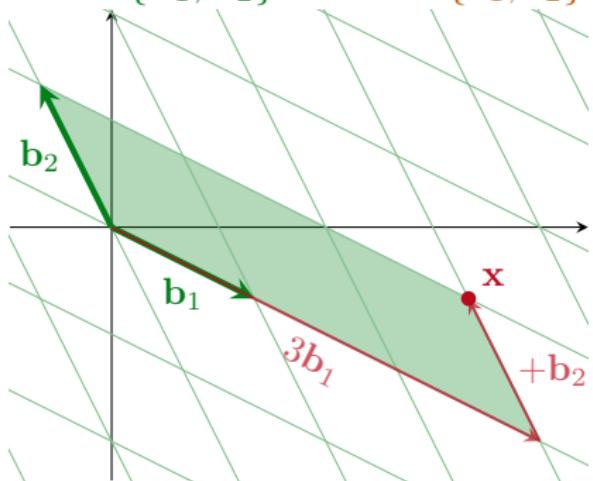
$$\mathbf{x} = 3\mathbf{b}_1 + \mathbf{b}_2$$

$$(\mathbf{x})_{\mathcal{B}} = (3, 1) \quad [\mathbf{x}]_{\mathcal{B}} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$



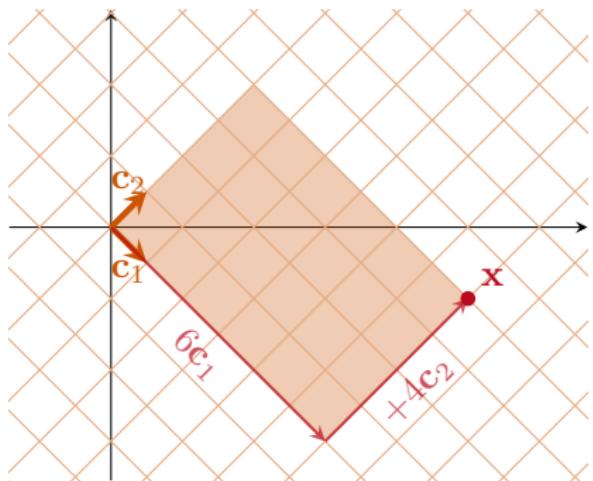
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$$(\mathbf{x})_{\mathcal{B}} = (3, 1) \quad [\mathbf{x}]_{\mathcal{B}} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$



$$\mathbf{x} = 6\mathbf{c}_1 + 4\mathbf{c}_2$$

$$(\mathbf{x})_{\mathcal{C}} = (6, 4) \quad [\mathbf{x}]_{\mathcal{C}} = \begin{bmatrix} 6 \\ 4 \end{bmatrix}$$

# Change of Basis



$$[\mathbf{x}]_{\mathcal{B}} = \begin{bmatrix} 3 \\ 1 \end{bmatrix} \xrightarrow{\text{HOW?}} [\mathbf{x}]_{\mathcal{C}} = \begin{bmatrix} 6 \\ 4 \end{bmatrix}$$

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$$[\mathbf{x}]_{\mathcal{B}} = \begin{bmatrix} 3 \\ 1 \end{bmatrix} \xrightarrow{\text{HOW?}} [\mathbf{x}]_{\mathcal{C}} = \begin{bmatrix} 6 \\ 4 \end{bmatrix}$$

## Example

Suppose that  $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2\}$  and  $\mathcal{C} = \{\mathbf{c}_1, \mathbf{c}_2\}$  are two bases for a vector space  $V$ . Suppose you know that

$$\mathbf{b}_1 = 4\mathbf{c}_1 + \mathbf{c}_2, \quad \mathbf{b}_2 = -6\mathbf{c}_1 + \mathbf{c}_2$$

and that the vector  $\mathbf{x}$  has coordinates  $[\mathbf{x}]_{\mathcal{B}} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$ . Find  $[\mathbf{x}]_{\mathcal{C}}$ .

# Change of Basis

Note that

$$[\mathbf{x}]_{\mathcal{B}} = \begin{bmatrix} 3 \\ 1 \end{bmatrix} \implies \mathbf{x} = 3\mathbf{b}_1 + \mathbf{b}_2.$$

Hence

$$[\mathbf{x}]_{\mathcal{C}} = [3\mathbf{b}_1 + \mathbf{b}_2]_{\mathcal{C}} = 3 [\mathbf{b}_1]_{\mathcal{C}} + [\mathbf{b}_2]_{\mathcal{C}}$$

=

# Change of Basis



Note that

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So what are  $[\mathbf{b}_1]_{\mathcal{C}}$  and  $[\mathbf{b}_2]_{\mathcal{C}}$ ?

## Change of Basis

The question told us  $\mathbf{b}_1 = 4\mathbf{c}_1 + \mathbf{c}_2$  and  $\mathbf{b}_2 = -6\mathbf{c}_1 + \mathbf{c}_2$ . Hence

$$[\mathbf{b}_1]_{\mathcal{C}} = \begin{bmatrix} 4 \\ 1 \end{bmatrix} \quad \text{and} \quad [\mathbf{b}_2]_{\mathcal{C}} = \begin{bmatrix} -6 \\ 1 \end{bmatrix}.$$

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Therefore

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Therefore

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## Remark

We did

$$[\mathbf{x}]_{\mathcal{C}} = \begin{bmatrix} 4 & -6 \\ 1 & 1 \end{bmatrix} [\mathbf{x}]_{\mathcal{B}}.$$

# Change of Basis

## Theorem

Let  $\mathcal{B} = \{\mathbf{b}_1, \dots, \mathbf{b}_n\}$  and  $\mathcal{C} = \{C_1, \dots, \mathbf{c}_n\}$  be bases of a vector space  $V$ . Then there is a unique  $n \times n$  matrix  $P_{\mathcal{C} \leftarrow \mathcal{B}}$  such that

$$[\mathbf{x}]_{\mathcal{C}} = P_{\mathcal{C} \leftarrow \mathcal{B}} [\mathbf{x}]_{\mathcal{B}}.$$

The columns of  $P_{\mathcal{C} \leftarrow \mathcal{B}}$  are the  $\mathcal{C}$ -coordinate vectors of the vectors in the basis  $\mathcal{B}$ . That is,

$$P_{\mathcal{C} \leftarrow \mathcal{B}} = \begin{bmatrix} [\mathbf{b}_1]_{\mathcal{C}} & [\mathbf{b}_2]_{\mathcal{C}} & \cdots & [\mathbf{b}_n]_{\mathcal{C}} \end{bmatrix}.$$

# Change of Basis



## Remark

Some books write  $P_{\mathcal{C} \leftarrow \mathcal{B}}$ , other books write  $P_{\mathcal{B} \rightarrow \mathcal{C}}$ .

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## Definition

The matrix  $P_{\mathcal{C} \leftarrow \mathcal{B}}$  is called the *change-of-coordinates matrix from  $\mathcal{B}$  to  $\mathcal{C}$*  or the *transition matrix from  $\mathcal{B}$  to  $\mathcal{C}$* .

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## Theorem

The change-of-coordinates matrix  $P_{\mathcal{C} \leftarrow \mathcal{B}}$  is invertible and

$$\left( P_{\mathcal{C} \leftarrow \mathcal{B}} \right)^{-1} = P_{\mathcal{B} \leftarrow \mathcal{C}}.$$



# Break

We will continue at 3pm



# Change of Basis



## Example

Consider  $V = \mathbb{R}^2$  with the bases  $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2\}$  and  $\mathcal{C} = \{\mathbf{c}_1, \mathbf{c}_2\}$ , where

$$\mathbf{b}_1 = \begin{bmatrix} -9 \\ 1 \end{bmatrix}, \quad \mathbf{b}_2 = \begin{bmatrix} -5 \\ -1 \end{bmatrix}, \quad \mathbf{c}_1 = \begin{bmatrix} 1 \\ -4 \end{bmatrix} \text{ and } \mathbf{c}_2 = \begin{bmatrix} 3 \\ -5 \end{bmatrix}.$$

Find the change-of-coordinates matrix from  $\mathcal{B}$  to  $\mathcal{C}$ .

# Change of Basis

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Find the change-of-coordinates matrix from  $\mathcal{B}$  to  $\mathcal{C}$ .

## Theorem

$$\begin{bmatrix} \mathbf{c}_1 & \mathbf{c}_2 & \mathbf{b}_1 & \mathbf{b}_2 \end{bmatrix} \sim \begin{bmatrix} I & P_{\mathcal{C} \leftarrow \mathcal{B}} \end{bmatrix}.$$

# Change of Basis



$$\begin{bmatrix} \mathbf{c}_1 & \mathbf{c}_2 & \mathbf{b}_1 & \mathbf{b}_2 \end{bmatrix} = \begin{bmatrix} 1 & 3 & -9 & -5 \\ -4 & -5 & 1 & -1 \end{bmatrix}$$

# Change of Basis

$$\begin{aligned}
 \begin{bmatrix} \mathbf{c}_1 & \mathbf{c}_2 & \mathbf{b}_1 & \mathbf{b}_2 \end{bmatrix} &= \begin{bmatrix} 1 & 3 & -9 & -5 \\ -4 & -5 & 1 & -1 \end{bmatrix} \\
 &\sim \begin{bmatrix} 1 & 3 & -9 & -5 \\ 0 & 7 & -35 & -21 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 & -9 & -5 \\ 0 & 1 & -5 & -3 \end{bmatrix} \\
 &\sim \begin{bmatrix} 1 & 0 & 6 & 4 \\ 0 & 1 & -5 & -3 \end{bmatrix} = \begin{bmatrix} I & P \\ \mathcal{C} \leftarrow \mathcal{B} \end{bmatrix}.
 \end{aligned}$$

# Change of Basis

$$\begin{aligned}
 \begin{bmatrix} \mathbf{c}_1 & \mathbf{c}_2 & \mathbf{b}_1 & \mathbf{b}_2 \end{bmatrix} &= \begin{bmatrix} 1 & 3 & -9 & -5 \\ -4 & -5 & 1 & -1 \end{bmatrix} \\
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 &\sim \begin{bmatrix} 1 & 0 & 6 & 4 \\ 0 & 1 & -5 & -3 \end{bmatrix} = \begin{bmatrix} I & P_{\mathcal{C} \leftarrow \mathcal{B}} \end{bmatrix}.
 \end{aligned}$$

Therefore

$$P_{\mathcal{C} \leftarrow \mathcal{B}} = \begin{bmatrix} 6 & 4 \\ -5 & -3 \end{bmatrix}.$$

# Change of Basis



## Example

Let  $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2\}$  and  $\mathcal{C} = \{\mathbf{c}_1, \mathbf{c}_2\}$  be two bases for  $V = \mathbb{R}^2$ , where

$$\mathbf{b}_1 = \begin{bmatrix} 1 \\ 3 \end{bmatrix}, \quad \mathbf{b}_2 = \begin{bmatrix} -2 \\ 4 \end{bmatrix}, \quad \mathbf{c}_1 = \begin{bmatrix} -7 \\ 9 \end{bmatrix} \quad \text{and} \quad \mathbf{c}_2 = \begin{bmatrix} -5 \\ 7 \end{bmatrix}.$$

- 1** Find the change-of-coordinates matrix from  $\mathcal{C}$  to  $\mathcal{B}$ .
- 2** Find the change-of-coordinates matrix from  $\mathcal{B}$  to  $\mathcal{C}$ .

$$\mathbf{b}_1 = \begin{bmatrix} 1 \\ 3 \end{bmatrix}, \quad \mathbf{b}_2 = \begin{bmatrix} -2 \\ 4 \end{bmatrix}, \quad \mathbf{c}_1 = \begin{bmatrix} -7 \\ 9 \end{bmatrix}, \quad \mathbf{c}_2 = \begin{bmatrix} -5 \\ 7 \end{bmatrix}.$$



- 1 Note that we need to find  $P_{\mathcal{B} \leftarrow \mathcal{C}}$  this time.

$$\mathbf{b}_1 = \begin{bmatrix} 1 \\ 3 \end{bmatrix}, \quad \mathbf{b}_2 = \begin{bmatrix} -2 \\ 4 \end{bmatrix}, \quad \mathbf{c}_1 = \begin{bmatrix} -7 \\ 9 \end{bmatrix}, \quad \mathbf{c}_2 = \begin{bmatrix} -5 \\ 7 \end{bmatrix}.$$

- 1 Note that we need to find  $P_{\mathcal{B} \leftarrow \mathcal{C}}$  this time. Since

$$\begin{bmatrix} \mathbf{b}_1 & \mathbf{b}_2 & \mathbf{c}_1 & \mathbf{c}_2 \end{bmatrix} = \begin{bmatrix} 1 & -2 & -7 & -5 \\ 3 & 4 & 9 & 7 \end{bmatrix}$$

$$\mathbf{b}_1 = \begin{bmatrix} 1 \\ 3 \end{bmatrix}, \quad \mathbf{b}_2 = \begin{bmatrix} -2 \\ 4 \end{bmatrix}, \quad \mathbf{c}_1 = \begin{bmatrix} -7 \\ 9 \end{bmatrix}, \quad \mathbf{c}_2 = \begin{bmatrix} -5 \\ 7 \end{bmatrix}.$$

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$$\mathbf{b}_1 = \begin{bmatrix} 1 \\ 3 \end{bmatrix}, \quad \mathbf{b}_2 = \begin{bmatrix} -2 \\ 4 \end{bmatrix}, \quad \mathbf{c}_1 = \begin{bmatrix} -7 \\ 9 \end{bmatrix}, \quad \mathbf{c}_2 = \begin{bmatrix} -5 \\ 7 \end{bmatrix}.$$

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we have that

$$P_{\mathcal{B} \leftarrow \mathcal{C}} = \begin{bmatrix} 5 & 3 \\ 6 & 4 \end{bmatrix}.$$

$$\mathbf{b}_1 = \begin{bmatrix} 1 \\ 3 \end{bmatrix}, \quad \mathbf{b}_2 = \begin{bmatrix} -2 \\ 4 \end{bmatrix}, \quad \mathbf{c}_1 = \begin{bmatrix} -7 \\ 9 \end{bmatrix}, \quad \mathbf{c}_2 = \begin{bmatrix} -5 \\ 7 \end{bmatrix}.$$

- 1 Note that we need to find  $P_{\mathcal{B} \leftarrow \mathcal{C}}$  this time. Since

$$\begin{bmatrix} \mathbf{b}_1 & \mathbf{b}_2 & \mathbf{c}_1 & \mathbf{c}_2 \end{bmatrix} = \begin{bmatrix} 1 & -2 & -7 & -5 \\ 3 & 4 & 9 & 7 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 5 & 3 \\ 0 & 1 & 6 & 4 \end{bmatrix}$$

we have that

$$P_{\mathcal{B} \leftarrow \mathcal{C}} = \begin{bmatrix} 5 & 3 \\ 6 & 4 \end{bmatrix}.$$

- 2 We calculate that

$$P_{\mathcal{C} \leftarrow \mathcal{B}} = \left( P_{\mathcal{B} \leftarrow \mathcal{C}} \right)^{-1} =$$

$$\mathbf{b}_1 = \begin{bmatrix} 1 \\ 3 \end{bmatrix}, \quad \mathbf{b}_2 = \begin{bmatrix} -2 \\ 4 \end{bmatrix}, \quad \mathbf{c}_1 = \begin{bmatrix} -7 \\ 9 \end{bmatrix}, \quad \mathbf{c}_2 = \begin{bmatrix} -5 \\ 7 \end{bmatrix}.$$

- 1 Note that we need to find  $P_{\mathcal{B} \leftarrow \mathcal{C}}$  this time. Since

$$\begin{bmatrix} \mathbf{b}_1 & \mathbf{b}_2 & \mathbf{c}_1 & \mathbf{c}_2 \end{bmatrix} = \begin{bmatrix} 1 & -2 & -7 & -5 \\ 3 & 4 & 9 & 7 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 5 & 3 \\ 0 & 1 & 6 & 4 \end{bmatrix}$$

we have that

$$P_{\mathcal{B} \leftarrow \mathcal{C}} = \begin{bmatrix} 5 & 3 \\ 6 & 4 \end{bmatrix}.$$

- 2 We calculate that

$$P_{\mathcal{C} \leftarrow \mathcal{B}} = \left( P_{\mathcal{B} \leftarrow \mathcal{C}} \right)^{-1} = \begin{bmatrix} 5 & 3 \\ 6 & 4 \end{bmatrix}^{-1} =$$

$$\mathbf{b}_1 = \begin{bmatrix} 1 \\ 3 \end{bmatrix}, \quad \mathbf{b}_2 = \begin{bmatrix} -2 \\ 4 \end{bmatrix}, \quad \mathbf{c}_1 = \begin{bmatrix} -7 \\ 9 \end{bmatrix}, \quad \mathbf{c}_2 = \begin{bmatrix} -5 \\ 7 \end{bmatrix}.$$

- 1 Note that we need to find  $P_{\mathcal{B} \leftarrow \mathcal{C}}$  this time. Since

$$\begin{bmatrix} \mathbf{b}_1 & \mathbf{b}_2 & \mathbf{c}_1 & \mathbf{c}_2 \end{bmatrix} = \begin{bmatrix} 1 & -2 & -7 & -5 \\ 3 & 4 & 9 & 7 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 5 & 3 \\ 0 & 1 & 6 & 4 \end{bmatrix}$$

we have that

$$P_{\mathcal{B} \leftarrow \mathcal{C}} = \begin{bmatrix} 5 & 3 \\ 6 & 4 \end{bmatrix}.$$

- 2 We calculate that

$$P_{\mathcal{C} \leftarrow \mathcal{B}} = \left( P_{\mathcal{B} \leftarrow \mathcal{C}} \right)^{-1} = \begin{bmatrix} 5 & 3 \\ 6 & 4 \end{bmatrix}^{-1} = \frac{1}{2} \begin{bmatrix} 4 & -3 \\ -6 & 5 \end{bmatrix} = \begin{bmatrix} 2 & -\frac{3}{2} \\ -3 & \frac{5}{2} \end{bmatrix}.$$



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3. [25 points] Let  $\mathcal{A} = \{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3\}$  and  $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3\}$  be bases for the vector space  $V$ , and suppose  $\mathbf{a}_1 = 4\mathbf{b}_1 - \mathbf{b}_2$ ,  $\mathbf{a}_2 = -\mathbf{b}_1 + \mathbf{b}_2 + \mathbf{b}_3$ , and  $\mathbf{a}_3 = \mathbf{b}_2 - 2\mathbf{b}_3$ .

- (a) Find the change of coordinates matrix from  $\mathcal{A}$  to  $\mathcal{B}$ .

**Solution:**

$$[\mathbf{a}_1]_{\mathcal{B}} = \begin{bmatrix} 4 \\ -1 \\ 0 \end{bmatrix}, \quad [\mathbf{a}_2]_{\mathcal{B}} = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}, \quad [\mathbf{a}_3]_{\mathcal{B}} = \begin{bmatrix} 0 \\ 1 \\ -2 \end{bmatrix} \Rightarrow P = \begin{bmatrix} 4 & -1 & 0 \\ -1 & 1 & 1 \\ 0 & 1 & -2 \end{bmatrix}$$

- (b) Find  $[\mathbf{x}]_{\mathcal{B}}$  for  $\mathbf{x} = 3\mathbf{a}_1 + 4\mathbf{a}_2 + \mathbf{a}_3$ .

**Solution:**

$$[\mathbf{x}]_{\mathcal{A}} = \begin{bmatrix} 3 \\ 4 \\ 1 \end{bmatrix} \Rightarrow [\mathbf{x}]_{\mathcal{B}} = P[\mathbf{x}]_{\mathcal{A}} = \begin{bmatrix} 4 & -1 & 0 \\ -1 & 1 & 1 \\ 0 & 1 & -2 \end{bmatrix} \begin{bmatrix} 3 \\ 4 \\ 1 \end{bmatrix} = \begin{bmatrix} 8 \\ 2 \\ 2 \end{bmatrix}$$

# Change of Basis



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3. Let  $V = \left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ -3 \\ -3 \end{bmatrix}, \begin{bmatrix} 1 \\ -3 \\ -5 \end{bmatrix} \right\}$  and  $W = \left\{ \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ 6 \end{bmatrix} \right\}$  be two bases for  $\mathbb{R}^3$ .

(a) 10 points Find the coordinates of  $\mathbf{v} = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$  relative to the basis  $V$ .

# Change of Basis



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5 December 2018 [16:00-17:10]

MATH215, Second Exam

3. Let  $V = \left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ -3 \\ -3 \end{bmatrix}, \begin{bmatrix} 1 \\ -3 \\ -5 \end{bmatrix} \right\}$  and  $W = \left\{ \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ 6 \end{bmatrix} \right\}$  be two bases for  $\mathbb{R}^3$ .

(a) 10 points Find the coordinates of  $\mathbf{v} = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$  relative to the basis  $V$ .

$$\begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix} = c_1 \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 2 \\ -3 \\ -3 \end{bmatrix} + c_3 \begin{bmatrix} 1 \\ -3 \\ -5 \end{bmatrix} \Rightarrow \begin{bmatrix} -1 & 2 & 1 \\ 1 & -3 & -3 \\ 0 & -3 & -5 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$$

$$\left[ \begin{array}{ccc|c} -1 & 2 & 1 & 2 \\ 1 & -3 & -3 & -1 \\ 0 & -3 & -5 & 1 \end{array} \right] \sim \left[ \begin{array}{ccc|c} 1 & -2 & -1 & -2 \\ 0 & 1 & 2 & -1 \\ 0 & -3 & -5 & 1 \end{array} \right] \sim \left[ \begin{array}{ccc|c} 1 & 0 & 3 & -4 \\ 0 & 1 & 2 & -1 \\ 0 & 0 & 1 & -2 \end{array} \right] \sim \left[ \begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & -2 \end{array} \right]$$

$$[\mathbf{v}]_V = \begin{bmatrix} 2 \\ 3 \\ -2 \end{bmatrix}$$

# Change of Basis



- (b) 10 points Find the change of coordinates matrix  $P_{W \leftarrow V}$  from  $V$  to  $W$ .

# Change of Basis



- (b) 10 points Find the change of coordinates matrix  $P_{W \leftarrow V}$  from  $V$  to  $W$ .

**Solution:** We can use  $P_{W \leftarrow V} = W^{-1}V$ ,  $[W|V] \sim \left[ I \mid P_{W \leftarrow V} \right]$  or  $P_{W \leftarrow V} = [[\mathbf{v}_1]_W \quad [\mathbf{v}_2]_W \quad [\mathbf{v}_3]_W]$

$$[W|V] \sim \left[ I \mid P_{W \leftarrow V} \right] \Rightarrow \left[ \begin{array}{ccc|ccc} 1 & 1 & 2 & -1 & 2 & 1 \\ 2 & 2 & 3 & 1 & -3 & -3 \\ 4 & 3 & 6 & 0 & -3 & -5 \end{array} \right] \sim \left[ \begin{array}{ccc|ccc} 1 & 1 & 2 & -1 & 2 & 1 \\ 0 & 0 & -1 & 3 & -7 & -5 \\ 0 & -1 & -2 & 4 & -11 & -9 \end{array} \right]$$
$$\sim \left[ \begin{array}{ccc|ccc} 1 & 1 & 2 & -1 & 2 & 1 \\ 0 & 1 & 2 & -4 & 11 & 9 \\ 0 & 0 & 1 & -3 & 7 & 5 \end{array} \right] \sim \left[ \begin{array}{ccc|ccc} 1 & 0 & 0 & 3 & -9 & -8 \\ 0 & 1 & 2 & -4 & 11 & 9 \\ 0 & 0 & 1 & -3 & 7 & 5 \end{array} \right] \sim \left[ \begin{array}{ccc|ccc} 1 & 0 & 0 & 3 & -9 & -8 \\ 0 & 1 & 0 & 2 & -3 & -1 \\ 0 & 0 & 1 & -3 & 7 & 5 \end{array} \right]$$
$$P_{W \leftarrow V} = \begin{bmatrix} 3 & -9 & -8 \\ 2 & -3 & -1 \\ -3 & 7 & 5 \end{bmatrix}$$

# Change of Basis



- (c) 5 points Find the coordinates of  $\mathbf{v}$  relative to  $W$  by using  $P_{W \leftarrow V}$ .

**Solution:**

$$[\mathbf{v}]_W = P_{W \leftarrow V} [\mathbf{v}]_V = \begin{bmatrix} 3 & -9 & -8 \\ 2 & -3 & -1 \\ -3 & 7 & 5 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \\ -2 \end{bmatrix} = \begin{bmatrix} -5 \\ -3 \\ 5 \end{bmatrix}$$

# Change of Basis



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3. Let  $V = \left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ -3 \\ -3 \end{bmatrix}, \begin{bmatrix} 1 \\ -3 \\ -5 \end{bmatrix} \right\}$  and  $W = \left\{ \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ 6 \end{bmatrix} \right\}$  be two bases for  $\mathbb{R}^3$ .

(a) 10 points Find the coordinates of  $\mathbf{v} = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$  relative to the basis  $V$ .

## Remark

I think that the 2019 question (below) is easier than the 2018 question (above), but I remember that the 2019 marks were lower than the 2018 marks.



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3. 25 points Let  $\mathcal{A} = \{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3\}$  and  $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3\}$  be bases for the vector space  $V$ , and suppose  $\mathbf{a}_1 = 4\mathbf{b}_1 - \mathbf{b}_2$ ,  $\mathbf{a}_2 = -\mathbf{b}_1 + \mathbf{b}_2 + \mathbf{b}_3$ , and  $\mathbf{a}_3 = \mathbf{b}_2 - 2\mathbf{b}_3$ .
- (a) Find the change of coordinates matrix from  $\mathcal{A}$  to  $\mathcal{B}$ .



# Row Space, Column Space, and Null Space

# Row Space, Column Space, and Null Space



Consider an  $m \times n$  matrix

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}.$$

# Row Space, Column Space, and Null Space



Consider an  $m \times n$  matrix

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}.$$

## Definition

The *row vectors* of  $A$  are the vectors

$$\mathbf{r}_1 = [a_{11} \quad a_{12} \quad \cdots \quad a_{1n}]$$

$$\mathbf{r}_2 = [a_{21} \quad a_{22} \quad \cdots \quad a_{2n}]$$

⋮

$$\mathbf{r}_m = [a_{m1} \quad a_{m2} \quad \cdots \quad a_{mn}]$$

Note that  $\mathbf{r}_k \in \mathbb{R}^n$  for each  $1 \leq k \leq m$ .

# Row Space, Column Space, and Null Space



Consider an  $m \times n$  matrix

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}.$$

## Definition

The *column vectors of A* are the vectors

$$\mathbf{c}_1 = \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix}, \quad \mathbf{c}_2 = \begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{bmatrix}, \quad \dots, \quad \mathbf{c}_m = \begin{bmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{bmatrix}$$

Note that  $\mathbf{c}_k \in \mathbb{R}^m$  for each  $1 \leq k \leq n$ .

# Row Space, Column Space, and Null Space



## Example

Find the row and column vectors of

$$A = \begin{bmatrix} 2 & 1 & 0 \\ 3 & -1 & 4 \end{bmatrix}.$$

## Example

Find the row and column vectors of

$$A = \begin{bmatrix} 2 & 1 & 0 \\ 3 & -1 & 4 \end{bmatrix}.$$

The row vectors of  $A$  are

$$\mathbf{r}_1 = [2 \ 1 \ 1] \quad \text{and} \quad \mathbf{r}_2 = [3 \ -1 \ 4].$$

The column vectors of  $A$  are

$$\mathbf{c}_1 = \begin{bmatrix} 2 \\ 3 \end{bmatrix}, \quad \mathbf{c}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \quad \text{and} \quad \mathbf{c}_3 = \begin{bmatrix} 0 \\ 4 \end{bmatrix}.$$

# Row Space, Column Space, and Null Space



Let  $A$  be an  $m \times n$  matrix.

## Definition

The subspace of  $\mathbb{R}^n$  spanned by the row vectors of  $A$  is called the *row space* of  $A$ . This is denoted by  $\text{Row } A$ .

## Definition

## Definition

# Row Space, Column Space, and Null Space



Let  $A$  be an  $m \times n$  matrix.

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The subspace of  $\mathbb{R}^n$  spanned by the row vectors of  $A$  is called the *row space* of  $A$ . This is denoted by  $\text{Row } A$ .

## Definition

The subspace of  $\mathbb{R}^m$  spanned by the column vectors of  $A$  is called the *column space* of  $A$ . This is denoted by  $\text{Col } A$ .

## Definition

# Row Space, Column Space, and Null Space



Let  $A$  be an  $m \times n$  matrix.

## Definition

The subspace of  $\mathbb{R}^n$  spanned by the row vectors of  $A$  is called the *row space* of  $A$ . This is denoted by  $\text{Row } A$ .

## Definition

The subspace of  $\mathbb{R}^m$  spanned by the column vectors of  $A$  is called the *column space* of  $A$ . This is denoted by  $\text{Col } A$ .

## Definition

The *null space* of  $A$  is the subspace

$$\text{Nul } A = \{\mathbf{x} \in \mathbb{R}^n \mid A\mathbf{x} = \mathbf{0}\} \subseteq \mathbb{R}^n.$$

# Row Space, Column Space, and Null Space

Consider  $A\mathbf{x} = \mathbf{b}$  where

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \quad \text{and} \quad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}.$$

Note that

$$\mathbf{b} = A\mathbf{x} = x_1\mathbf{c}_1 + x_2\mathbf{c}_2 + \dots + x_n\mathbf{c}_n$$

where  $\mathbf{c}_1, \mathbf{c}_2, \dots, \mathbf{c}_n$  are the column vectors of  $A$ .

# Row Space, Column Space, and Null Space



Consider  $A\mathbf{x} = \mathbf{b}$  where

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \quad \text{and} \quad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}.$$

Note that

$$\mathbf{b} = A\mathbf{x} = x_1\mathbf{c}_1 + x_2\mathbf{c}_2 + \dots + x_n\mathbf{c}_n$$

where  $\mathbf{c}_1, \mathbf{c}_2, \dots, \mathbf{c}_n$  are the column vectors of  $A$ . So  $A\mathbf{x} = \mathbf{b}$  is consistent iff  $\mathbf{b}$  is a linear combination of the column vectors of  $A$ .

# Row Space, Column Space, and Null Space



Consider  $A\mathbf{x} = \mathbf{b}$  where

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \quad \text{and} \quad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}.$$

Note that

$$\mathbf{b} = A\mathbf{x} = x_1\mathbf{c}_1 + x_2\mathbf{c}_2 + \dots + x_n\mathbf{c}_n$$

where  $\mathbf{c}_1, \mathbf{c}_2, \dots, \mathbf{c}_n$  are the column vectors of  $A$ . So  $A\mathbf{x} = \mathbf{b}$  is consistent iff  $\mathbf{b}$  is a linear combination of the column vectors of  $A$ .

Theorem

$$A\mathbf{x} = \mathbf{b} \text{ is consistent} \iff \mathbf{b} \in \text{Col } A$$

## ► EXAMPLE 2 A Vector $\mathbf{b}$ in the Column Space of $A$

Let  $A\mathbf{x} = \mathbf{b}$  be the linear system

$$\begin{bmatrix} -1 & 3 & 2 \\ 1 & 2 & -3 \\ 2 & 1 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ -9 \\ -3 \end{bmatrix}$$

Show that  $\mathbf{b}$  is in the column space of  $A$  by expressing it as a linear combination of the column vectors of  $A$ .

**Solution** Solving the system by Gaussian elimination yields (verify)

$$x_1 = 2, \quad x_2 = -1, \quad x_3 = 3$$

It follows from this and Formula (2) that

$$2 \begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix} - \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} + 3 \begin{bmatrix} 2 \\ -3 \\ -2 \end{bmatrix} = \begin{bmatrix} 1 \\ -9 \\ -3 \end{bmatrix} \quad \blacktriangleleft$$

# Row Space, Column Space, and Null Space



## Example

Let

$$A = \begin{bmatrix} 2 & 4 & -2 & 1 \\ -2 & -5 & 7 & 3 \\ 3 & 7 & -8 & 6 \end{bmatrix}, \quad \mathbf{u} = \begin{bmatrix} 3 \\ -2 \\ -1 \\ 0 \end{bmatrix}, \quad \text{and} \quad \mathbf{v} = \begin{bmatrix} 3 \\ -1 \\ 3 \end{bmatrix}.$$

- 1 Is  $\mathbf{u}$  in  $\text{Nul } A$ ?
- 3 Is  $\mathbf{v}$  in  $\text{Nul } A$ ?
- 2 Is  $\mathbf{u}$  in  $\text{Col } A$ ?
- 4 Is  $\mathbf{v}$  in  $\text{Col } A$ ?

# Row Space, Column Space, and Null Space



$$A = \begin{bmatrix} 2 & 4 & -2 & 1 \\ -2 & -5 & 7 & 3 \\ 3 & 7 & -8 & 6 \end{bmatrix}, \quad \mathbf{u} = \begin{bmatrix} 3 \\ -2 \\ -1 \\ 0 \end{bmatrix}, \text{ and } \mathbf{v} = \begin{bmatrix} 3 \\ -1 \\ 3 \end{bmatrix}.$$

1 Is  $\mathbf{u}$  in  $\text{Nul } A$ ?

Recall that

$$\text{Nul } A = \{\mathbf{x} \in \mathbb{R}^4 \mid A\mathbf{x} = \mathbf{0}\} \subseteq \mathbb{R}^4$$

since  $A$  is a  $3 \times 4$  matrix. So  $\mathbf{u} \in \text{Nul } A$  if and only if  $A\mathbf{u} = \mathbf{0}$ .

# Row Space, Column Space, and Null Space



$$A = \begin{bmatrix} 2 & 4 & -2 & 1 \\ -2 & -5 & 7 & 3 \\ 3 & 7 & -8 & 6 \end{bmatrix}, \quad \mathbf{u} = \begin{bmatrix} 3 \\ -2 \\ -1 \\ 0 \end{bmatrix}, \text{ and } \mathbf{v} = \begin{bmatrix} 3 \\ -1 \\ 3 \end{bmatrix}.$$

1 Is  $\mathbf{u}$  in  $\text{Nul } A$ ?

Recall that

$$\text{Nul } A = \{\mathbf{x} \in \mathbb{R}^4 \mid A\mathbf{x} = \mathbf{0}\} \subseteq \mathbb{R}^4$$

since  $A$  is a  $3 \times 4$  matrix. So  $\mathbf{u} \in \text{Nul } A$  if and only if  $A\mathbf{u} = \mathbf{0}$ .

We calculate that

$$A\mathbf{u} = \begin{bmatrix} 2 & 4 & -2 & 1 \\ -2 & -5 & 7 & 3 \\ 3 & 7 & -8 & 6 \end{bmatrix} \begin{bmatrix} 3 \\ -2 \\ -1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

Therefore the answer is YES.

# Row Space, Column Space, and Null Space



$$A = \begin{bmatrix} 2 & 4 & -2 & 1 \\ -2 & -5 & 7 & 3 \\ 3 & 7 & -8 & 6 \end{bmatrix}, \quad \mathbf{u} = \begin{bmatrix} 3 \\ -2 \\ -1 \\ 0 \end{bmatrix}, \quad \text{and} \quad \mathbf{v} = \begin{bmatrix} 3 \\ -1 \\ 3 \end{bmatrix}.$$

2 Is  $\mathbf{u}$  in  $\text{Col } A$ ?

Since  $A$  is a  $3 \times 4$  matrix,  $\text{Col } A$  is a subspace of  $\mathbb{R}^3$ .

# Row Space, Column Space, and Null Space



$$A = \begin{bmatrix} 2 & 4 & -2 & 1 \\ -2 & -5 & 7 & 3 \\ 3 & 7 & -8 & 6 \end{bmatrix}, \quad \mathbf{u} = \begin{bmatrix} 3 \\ -2 \\ -1 \\ 0 \end{bmatrix}, \text{ and } \mathbf{v} = \begin{bmatrix} 3 \\ -1 \\ 3 \end{bmatrix}.$$

2 Is  $\mathbf{u}$  in  $\text{Col } A$ ?

Since  $A$  is a  $3 \times 4$  matrix,  $\text{Col } A$  is a subspace of  $\mathbb{R}^3$ . But  $\mathbf{u} \in \mathbb{R}^4$ . Therefore the answer is NO.

# Row Space, Column Space, and Null Space



$$A = \begin{bmatrix} 2 & 4 & -2 & 1 \\ -2 & -5 & 7 & 3 \\ 3 & 7 & -8 & 6 \end{bmatrix}, \quad \mathbf{u} = \begin{bmatrix} 3 \\ -2 \\ -1 \\ 0 \end{bmatrix}, \quad \text{and} \quad \mathbf{v} = \begin{bmatrix} 3 \\ -1 \\ 3 \end{bmatrix}.$$

3 Is  $\mathbf{v}$  in  $\text{Nul } A$ ?

NO, because  $\mathbf{v} \in \mathbb{R}^3$ , but  $\text{Nul } A \subseteq \mathbb{R}^4$ .

# Row Space, Column Space, and Null Space



$$A = \begin{bmatrix} 2 & 4 & -2 & 1 \\ -2 & -5 & 7 & 3 \\ 3 & 7 & -8 & 6 \end{bmatrix}, \quad \mathbf{u} = \begin{bmatrix} 3 \\ -2 \\ -1 \\ 0 \end{bmatrix}, \quad \text{and} \quad \mathbf{v} = \begin{bmatrix} 3 \\ -1 \\ 3 \end{bmatrix}.$$

4 Is  $\mathbf{v}$  in  $\text{Col } A$ ?

To answer this, we need to reduce  $[A \ \mathbf{v}]$  to REF.

# Row Space, Column Space, and Null Space



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# Row Space, Column Space, and Null Space



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We can now see that the linear system  $A\mathbf{x} = \mathbf{v}$  is consistent. Therefore the answer to this question is YES.

## Theorem

Suppose that

- $A\mathbf{x} = \mathbf{b}$  is consistent;
- $\mathbf{x}_0$  is any solution of  $A\mathbf{x} = \mathbf{b}$ ;
- $S = \{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k\}$  is a basis for  $\text{Nul } A$ .

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Then every solution of  $A\mathbf{x} = \mathbf{b}$  can be written in the form

$$\mathbf{x} = \mathbf{x}_0 + c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \dots + c_k\mathbf{v}_k.$$

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Conversely, the vector  $\mathbf{x}$  in this formula is a solution of  $A\mathbf{x} = \mathbf{b}$  for any choice of scalars  $c_j \in \mathbb{R}$ .

# Row Space, Column Space, and Null Space



$$\mathbf{x} = \mathbf{x}_0 + c_1 \mathbf{v}_1 + c_2 \mathbf{v}_2 + \dots + c_k \mathbf{v}_k$$

# Row Space, Column Space, and Null Space



$$\mathbf{x} = \underbrace{\mathbf{x}_0}_{\text{a particular solution of }} + c_1 \mathbf{v}_1 + c_2 \mathbf{v}_2 + \dots + c_k \mathbf{v}_k$$

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# Row Space, Column Space, and Null Space



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the general solution of  $A\mathbf{x} = \mathbf{b}$



## Bases for Row Spaces, Column Spaces, and Null Spaces

### Theorem

*Elementary row operations do not change the null space of a matrix.*

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# Row Space, Column Space, and Null Space



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For example, consider

$$A = \begin{bmatrix} 1 & 3 \\ 2 & 6 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 & 3 \\ 0 & 0 \end{bmatrix}.$$

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and

$$\text{Col } B = \text{span} \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\} \neq \text{Col } A.$$

# Row Space, Column Space, and Null Space



## Example

Find a basis the null space of the matrix

$$A = \begin{bmatrix} 1 & 3 & -2 & 0 & 2 & 0 \\ 2 & 6 & -5 & -2 & 4 & -3 \\ 0 & 0 & 5 & 10 & 0 & 15 \\ 2 & 6 & 0 & 8 & 4 & 18 \end{bmatrix}.$$

# Row Space, Column Space, and Null Space



## Example

Find a basis the null space of the matrix

$$A = \begin{bmatrix} 1 & 3 & -2 & 0 & 2 & 0 \\ 2 & 6 & -5 & -2 & 4 & -3 \\ 0 & 0 & 5 & 10 & 0 & 15 \\ 2 & 6 & 0 & 8 & 4 & 18 \end{bmatrix}.$$

We need to find the solution space of  $A\mathbf{x} = \mathbf{0}$ . I leave it to you to check that

$$\left[ \begin{array}{cccccc|c} 1 & 3 & -2 & 0 & 2 & 0 & 0 \\ 2 & 6 & -5 & -2 & 4 & -3 & 0 \\ 0 & 0 & 5 & 10 & 0 & 15 & 0 \\ 2 & 6 & 0 & 8 & 4 & 18 & 0 \end{array} \right] \sim \left[ \begin{array}{cccccc|c} 1 & 3 & 0 & 4 & 2 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array} \right].$$

# Row Space, Column Space, and Null Space



Therefore the solution of  $A\mathbf{x} = \mathbf{0}$  is

$$\begin{cases} x_1 = -3x_2 - 4x_4 - 2x_5 \\ x_2 \text{ is free} \\ x_3 = -2x_4 \\ x_4 \text{ is free} \\ x_5 \text{ is free} \\ x_6 = 0 \end{cases}$$

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or in vectors,

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} = \begin{bmatrix} -3x_2 - 4x_4 - 2x_5 \\ x_2 \\ -2x_4 \\ x_4 \\ x_5 \\ 0 \end{bmatrix} = x_2 \begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -4 \\ 0 \\ -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} -2 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}.$$

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# Row Space, Column Space, and Null Space



Therefore

$$\left\{ \mathbf{v}_1 = \begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} -4 \\ 0 \\ -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \mathbf{v}_3 = \begin{bmatrix} -2 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \right\}$$

is a basis for  $\text{Nul } A$ .

# Row Space, Column Space, and Null Space



## Remark

In the previous example we found  
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$$\begin{cases} x_1 = -3x_2 - 4x_4 - 2x_5 \\ x_2 \text{ is free} \\ x_3 = -2x_4 \\ x_4 \text{ is free} \\ x_5 \text{ is free} \\ x_6 = 0 \end{cases}$$

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is the general solution of  $A\mathbf{x} = \mathbf{0}$ .

Note that if we set  $x_2 = 1$  and  $x_4 = x_5 = 0$  then we get

$$\mathbf{v}_1 = \begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

which is the first vector in  $\text{Nul } A$ .

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which is the first vector in  $\text{Nul } A$ .

Setting  $x_4 = 1$  and then  $x_5 = 1$  (and the other two to 0) gives us the other two vectors in  $\text{Nul } A$ .

## Theorem

Suppose that a matrix  $R$  is in row echelon form. Then

- the row vectors with the pivots (i.e. the nonzero row vectors) form a basis for the row space of  $R$ ; and
- the column vectors with the pivots (i.e. the pivot columns) form a basis for the column space of  $R$ .

# Row Space, Column Space, and Null Space



## Example

Find bases for the row and column spaces of the matrix

$$R = \begin{bmatrix} 1 & -2 & 5 & 0 & 3 \\ 0 & 1 & 3 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

This matrix is in REF, so we can use the previous theorem.

# Row Space, Column Space, and Null Space



## Example

Find bases for the row and column spaces of the matrix

$$R = \begin{bmatrix} 1 & -2 & 5 & 0 & 3 \\ 0 & 1 & 3 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

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There are three pivots (shown in green).

# Row Space, Column Space, and Null Space



## Example

Find bases for the row and column spaces of the matrix

$$R = \begin{bmatrix} 1 & -2 & 5 & 0 & 3 \\ 0 & 1 & 3 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

This matrix is in REF, so we can use the previous theorem.

There are three pivots (shown in green).

- $\{\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3\}$  is a basis for Row  $R$  since the pivots are in rows 1, 2 and 3.

# Row Space, Column Space, and Null Space



## Example

Find bases for the row and column spaces of the matrix

$$R = \begin{bmatrix} 1 & -2 & 5 & 0 & 3 \\ 0 & 1 & 3 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

This matrix is in REF, so we can use the previous theorem.

There are three pivots (shown in green).

- $\{\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3\}$  is a basis for Row  $R$  since the pivots are in rows 1, 2 and 3.
- $\{\mathbf{c}_1, \mathbf{c}_2, \mathbf{c}_4\}$  is a basis for Col  $R$  since the pivots are in columns 1, 2 and 4.

# Row Space, Column Space, and Null Space



## Example

Find a basis for the row space of the matrix

$$A = \begin{bmatrix} 1 & -3 & 4 & -2 & 5 & 4 \\ 2 & -6 & 9 & -1 & 8 & 2 \\ 2 & -6 & 9 & -1 & 9 & 7 \\ -1 & 3 & -4 & 2 & -5 & -4 \end{bmatrix}.$$

Recall that elementary row operations do not change the row space of a matrix.

## Example

Find a basis for the row space of the matrix

$$A = \begin{bmatrix} 1 & -3 & 4 & -2 & 5 & 4 \\ 2 & -6 & 9 & -1 & 8 & 2 \\ 2 & -6 & 9 & -1 & 9 & 7 \\ -1 & 3 & -4 & 2 & -5 & -4 \end{bmatrix}.$$

Recall that elementary row operations do not change the row space of a matrix. So our method is:

- 1 Reduce  $A$  to REF. Call this new matrix  $R$ ;
- 2 Take the row vectors of  $R$  which contain a pivot.

# Row Space, Column Space, and Null Space



As always, I leave it for you to check that

$$A = \begin{bmatrix} 1 & -3 & 4 & -2 & 5 & 4 \\ 2 & -6 & 9 & -1 & 8 & 2 \\ 2 & -6 & 9 & -1 & 9 & 7 \\ -1 & 3 & -4 & 2 & -5 & -4 \end{bmatrix} \sim R = \begin{bmatrix} 1 & -3 & 4 & -2 & 5 & 4 \\ 0 & 0 & 1 & 3 & -2 & -6 \\ 0 & 0 & 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Therefore

$$\mathbf{r}_1 = [ \ 1 \ -3 \ 4 \ -2 \ 5 \ 4 \ ]$$

$$\mathbf{r}_2 = [ \ 0 \ 0 \ 1 \ 3 \ -2 \ -6 \ ]$$

$$\mathbf{r}_3 = [ \ 0 \ 0 \ 0 \ 0 \ 1 \ 5 \ ]$$

form a basis for  $\text{Row } A = \text{Row } R$ .



## Finding a Basis for Col $A$

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### Theorem

*The pivot columns of  $A$  form a basis of  $\text{Col } A$ .*

## Finding a Basis for $\text{Col } A$

Elementary row operations can change the column space of a matrix  $A$ , but they do not change *which columns* of  $A$  we want in our basis.

### Theorem

*The pivot columns of  $A$  form a basis of  $\text{Col } A$ .*

So our method is:

- 1 Reduce  $A$  to REF. Call this new matrix  $R$ ;
- 2 Find the pivot columns of  $R$ ;
- 3 Go back to looking at the original matrix  $A$ ;
- 4 Take the pivot columns of  $A$ .

# Row Space, Column Space, and Null Space



## Example

Find a basis for the column space of the matrix

$$A = \begin{bmatrix} 1 & -3 & 4 & -2 & 5 & 4 \\ 2 & -6 & 9 & -1 & 8 & 2 \\ 2 & -6 & 9 & -1 & 9 & 7 \\ -1 & 3 & -4 & 2 & -5 & -4 \end{bmatrix}.$$

# Row Space, Column Space, and Null Space



Recall that

$$A = \begin{bmatrix} 1 & -3 & 4 & -2 & 5 & 4 \\ 2 & -6 & 9 & -1 & 8 & 2 \\ 2 & -6 & 9 & -1 & 9 & 7 \\ -1 & 3 & -4 & 2 & -5 & -4 \end{bmatrix} \sim R = \begin{bmatrix} 1 & -3 & 4 & -2 & 5 & 4 \\ 0 & 0 & 1 & 3 & -2 & -6 \\ 0 & 0 & 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Look at  $R$ . The pivot columns of  $R$  are the first, third and fifth columns.

# Row Space, Column Space, and Null Space



Recall that

$$A = \begin{bmatrix} 1 & -3 & 4 & -2 & 5 & 4 \\ 2 & -6 & 9 & -1 & 8 & 2 \\ 2 & -6 & 9 & -1 & 9 & 7 \\ -1 & 3 & -4 & 2 & -5 & -4 \end{bmatrix} \sim R = \begin{bmatrix} 1 & -3 & 4 & -2 & 5 & 4 \\ 0 & 0 & 1 & 3 & -2 & -6 \\ 0 & 0 & 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Look at  $R$ . The pivot columns of  $R$  are the first, third and fifth columns. Therefore the pivot columns of  $A$  are also the **first, third and fifth** columns.

# Row Space, Column Space, and Null Space



Recall that

$$A = \begin{bmatrix} 1 & -3 & 4 & -2 & 5 & 4 \\ 2 & -6 & 9 & -1 & 8 & 2 \\ 2 & -6 & 9 & -1 & 9 & 7 \\ -1 & 3 & -4 & 2 & -5 & -4 \end{bmatrix} \sim R = \begin{bmatrix} 1 & -3 & 4 & -2 & 5 & 4 \\ 0 & 0 & 1 & 3 & -2 & -6 \\ 0 & 0 & 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Look at  $R$ . The pivot columns of  $R$  are the first, third and fifth columns. Therefore the pivot columns of  $A$  are also the **first, third and fifth** columns. Hence

$$\left\{ \mathbf{c}_1 = \begin{bmatrix} 1 \\ 2 \\ 2 \\ -1 \end{bmatrix}, \mathbf{c}_3 = \begin{bmatrix} 4 \\ 9 \\ 9 \\ -4 \end{bmatrix}, \mathbf{c}_5 = \begin{bmatrix} 5 \\ 8 \\ 9 \\ -5 \end{bmatrix} \right\}$$

is a basis for  $\text{Col } A$ .

# Row Space, Column Space, and Null Space



## Example

Find a subset of the vectors

$$\mathbf{v}_1 = (1, -2, 0, 3), \quad \mathbf{v}_2 = (2, -5, -3, 6), \quad \mathbf{v}_3 = (0, 1, 3, 0),$$

$$\mathbf{v}_4 = (2, -1, 4, -7), \quad \mathbf{v}_5 = (5, -8, 1, 2)$$

that forms a basis for  $\text{span}\{\mathbf{v}_1, \dots, \mathbf{v}_5\}$ .

## Example

Find a subset of the vectors

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that forms a basis for  $\text{span}\{\mathbf{v}_1, \dots, \mathbf{v}_5\}$ .

If we write these vectors as columns of a matrix

$$A = \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3 & \mathbf{v}_4 & \mathbf{v}_5 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 0 & 2 & 5 \\ -2 & -5 & 1 & -1 & -8 \\ 0 & -3 & 3 & 4 & 1 \\ 3 & 6 & 0 & -7 & 2 \end{bmatrix}$$

then we just need to find a basis for  $\text{Col } A$ .

# Row Space, Column Space, and Null Space



Since

$$A \sim \begin{bmatrix} 1 & 0 & 2 & 0 & 1 \\ 0 & 1 & -1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

(please check) we can see that the pivot columns are the first, second and fourth columns.

# Row Space, Column Space, and Null Space



Since

$$A \sim \begin{bmatrix} 1 & 0 & 2 & 0 & 1 \\ 0 & 1 & -1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

(please check) we can see that the pivot columns are the first, second and fourth columns.

Therefore  $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_4\}$  is a basis for  $\text{span}\{\mathbf{v}_1, \dots, \mathbf{v}_5\}$ .



# Next Time

- Rank and Nullity
- The Fundamental Matrix Spaces
- Linear Transformations
- Composition and Inverse Transformations
- Isomorphisms