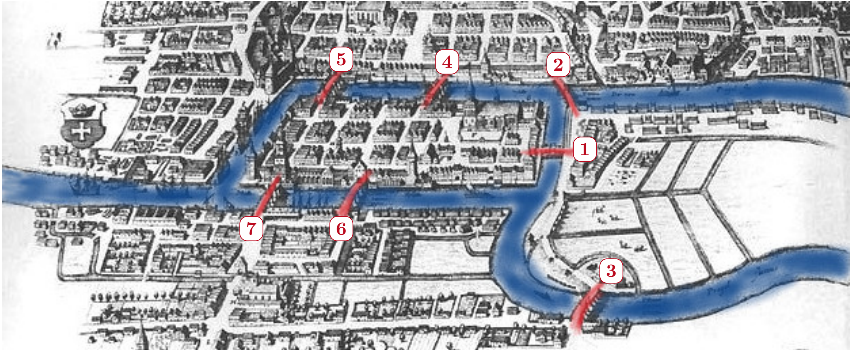
Week 8

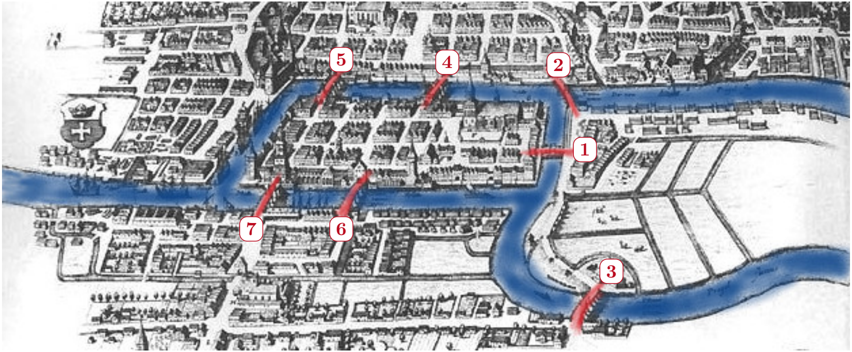
- 23. Graph Theory

Graph Theory

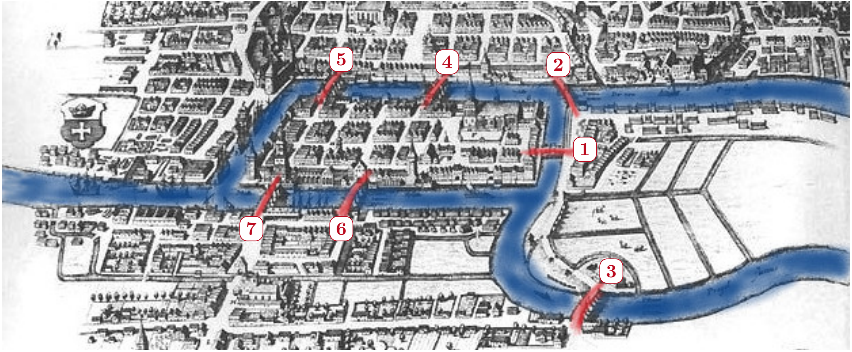
Often when analysing theoretical problems, it is useful to transform the problem into a collection of vertices joined by lines.



When visiting the city of Königsberg in 1736, the Swiss Mathematician Leonhard Euler (1707-1783) was set a problem by the inhabitants. To solve this problem, he invented a type of Mathematics called Graph Theory.



The town of Königsberg in Prussia (now Kaliningrad, Russia) was divided into four landmasses by the Pregel river. There were 7 separate bridges between these landmasses as shown above.

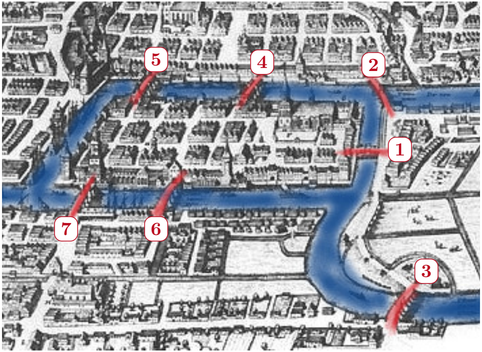


Visitors were often asked the following problem by the locals:

Can a person walk around the town and cross each bridge once and only once?

Euler was the first person to solve this problem.

Leonhard Euler
1707–1783
Switzerland


5
4
2
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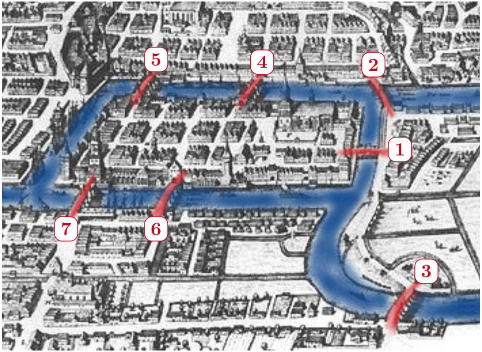
5
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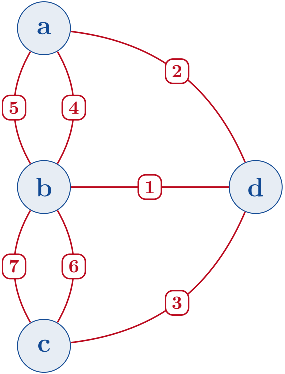
a
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7

Graph Theory, which has been used by mathematicians for many years to solve interesting riddles and puzzles, is nowadays in computing (algorithm design, telecommunication and GPS), physics (atomic structures), neurology (brain-like structures), chemistry (molecular structures), and many other disciplines.

Definition

A graph is formed by points called *vertices* (or *nodes*), and lines called *edges*.

Notation

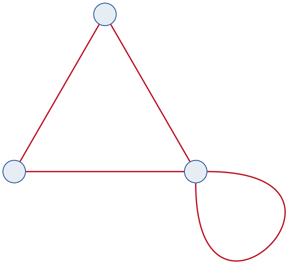
Vertices are denoted by lowercase letters: a, b, c, ... The edge from vertex u to vertex v is denoted by $e = (u, v)$. u and v are called *endpoints* of e.

Definition

A non-empty set of vertices V together with a set of edges E is called a *graph* and is denoted by $G(V, E)$

Definition

An edge which starts and finishes at the same vertex is called a *loop*.



Example

The graph above has 3 vertices and 4 edges. One of the edges is a loop.

Example

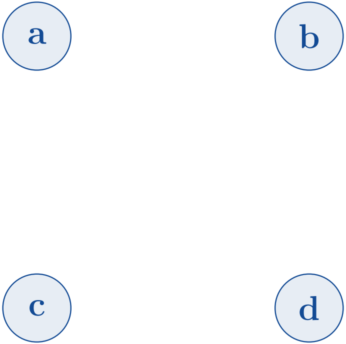
Let $V = \{a, b, c, d\}$ and $E = \{e_1, e_2, e_3, e_4, e_5\}$ where $e_1 = (a, b)$, $e_2 = (b, c)$, $e_3 = (c, d)$, $e_4 = (a, c)$ and $e_5 = (b, d)$. Draw the graph $G = (V, E)$.

solution:

Example

Let $V = \{a, b, c, d\}$ and $E = \{e_1, e_2, e_3, e_4, e_5\}$ where $e_1 = (a, b)$, $e_2 = (b, c)$, $e_3 = (c, d)$, $e_4 = (a, c)$ and $e_5 = (b, d)$. Draw the graph $G = (V, E)$.

solution:



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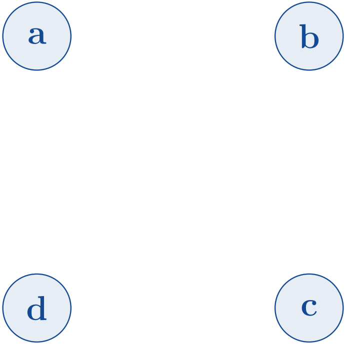

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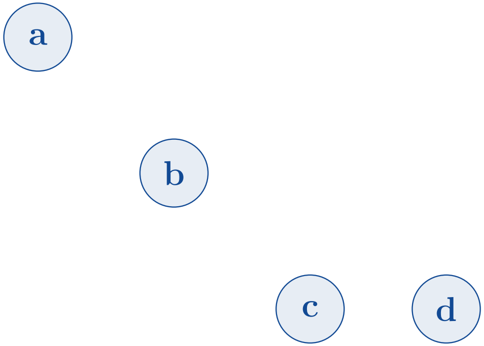
solution:



Example

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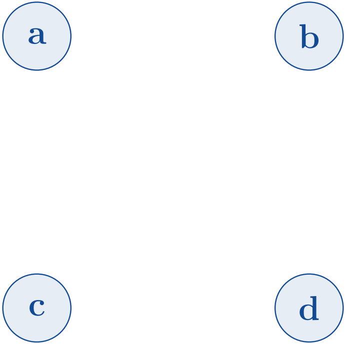

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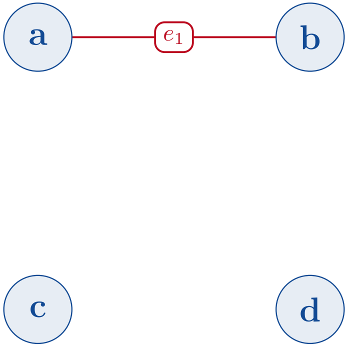

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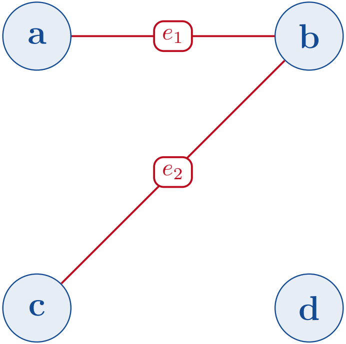

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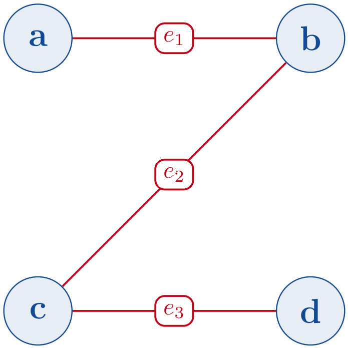

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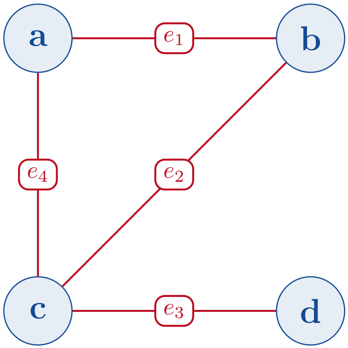

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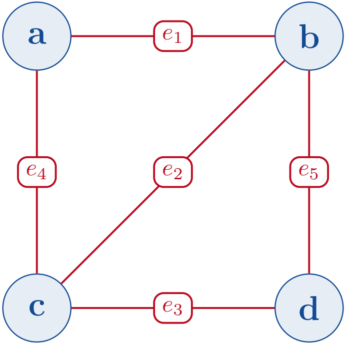
solution:



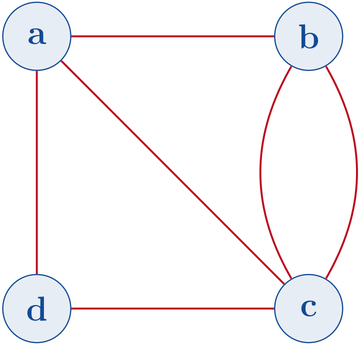
Definition

Two edges with the same endpoints are called *parallel edges*.

Types of Graph

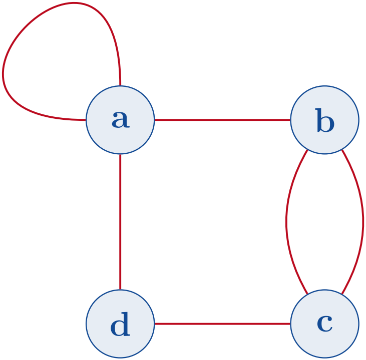
Definition

A *simple graph* is a graph without parallel edges or loops.



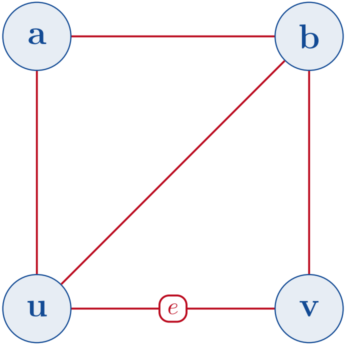
Definition

If a graph contains parallel edges, it is called a *multigraph*.



Definition

A *pseudograph* is a non-simple graph in which both loops and parallel edges are permitted.



Definition

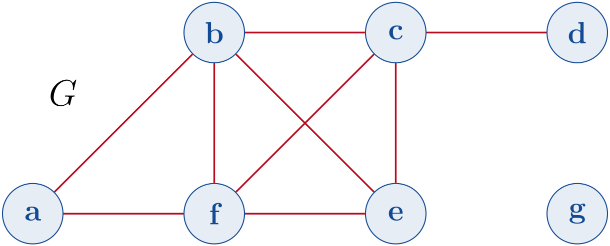
If $e = (u, v)$ is an edge of a undirected graph G, then the vertices u and v are called *neighbours* in G.

Definition

The *degree* of a vertex v in a(n undirected) graph is

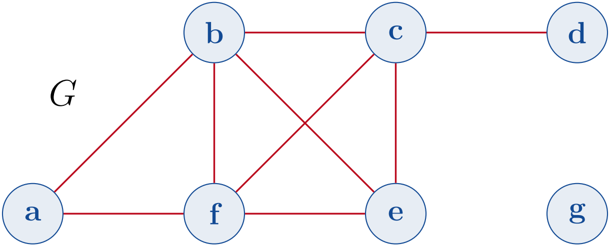
$$\deg(v) = \text{the number of edges connected to } v.$$

When we calculate the degree of a vertex, a loop is counted as 2.



Example

Give the degrees of all the vertices in graph G.



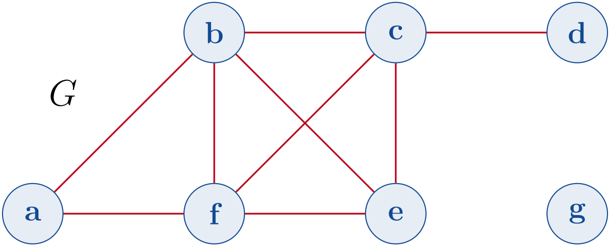
Example

Give the degrees of all the vertices in graph G.

solution:

$$\deg(a) =$$
$$\deg(b) =$$
$$\deg(c) =$$
$$\deg(d) =$$

$$\deg(e) =$$
$$\deg(f) =$$
$$\deg(g) =$$

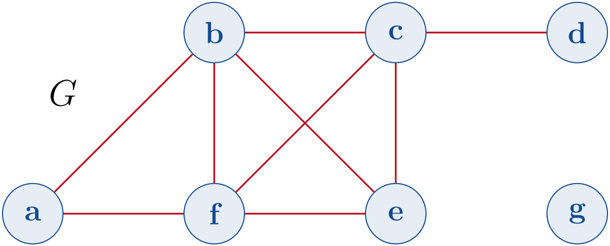


Example

Give the degrees of all the vertices in graph G.

solution:

$$\deg(a) = 2$$
$$\deg(b) =$$
$$\deg(c) =$$
$$\deg(d) =$$
$$\deg(e) =$$
$$\deg(f) =$$
$$\deg(g) =$$



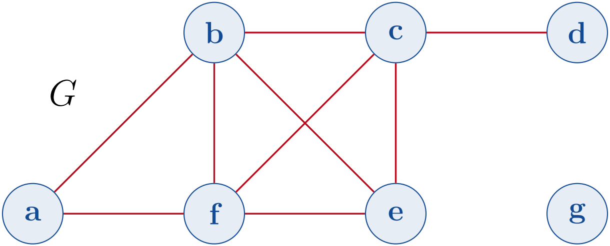
Example

Give the degrees of all the vertices in graph G.

solution:

$$\deg(a) = 2$$
$$\deg(b) = 4$$
$$\deg(c) =$$
$$\deg(d) =$$

$$\deg(e) =$$
$$\deg(f) =$$
$$\deg(g) =$$



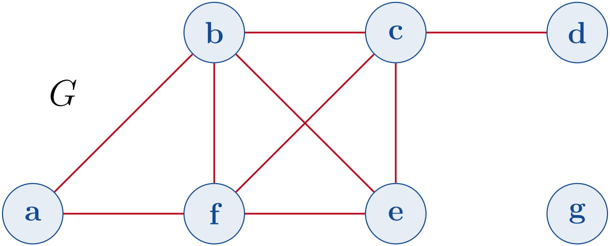
Example

Give the degrees of all the vertices in graph G.

solution:

$$\deg(a) = 2$$
$$\deg(b) = 4$$
$$\deg(c) = 4$$
$$\deg(d) =$$

$$\deg(e) =$$
$$\deg(f) =$$
$$\deg(g) =$$

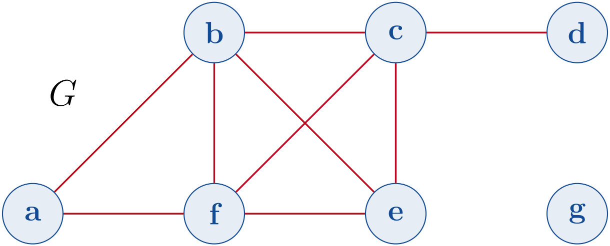


Example

Give the degrees of all the vertices in graph G.

solution:

$$\deg(a) = 2 \qquad \deg(e) =$$
$$\deg(b) = 4 \qquad \deg(f) =$$
$$\deg(c) = 4 \qquad \deg(g) =$$
$$\deg(d) = 1$$

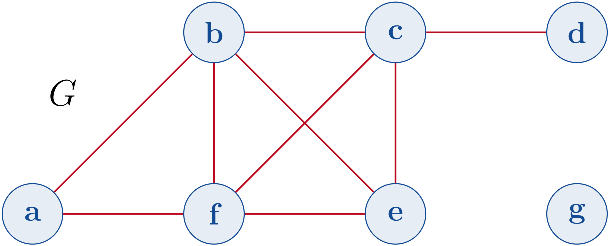


Example

Give the degrees of all the vertices in graph G.

solution:

$$\deg(a) = 2$$
$$\deg(b) = 4$$
$$\deg(c) = 4$$
$$\deg(d) = 1$$
$$\deg(e) = 3$$
$$\deg(f) =$$
$$\deg(g) =$$

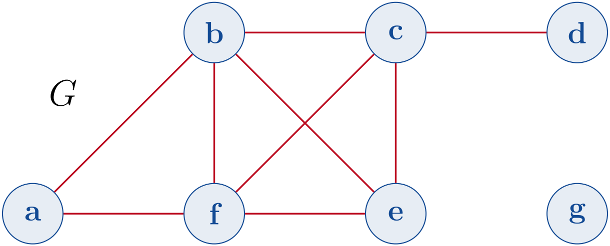


Example

Give the degrees of all the vertices in graph G.

solution:

$$\deg(a) = 2 \qquad \deg(e) = 3$$
$$\deg(b) = 4 \qquad \deg(f) = 4$$
$$\deg(c) = 4 \qquad \deg(g) =$$
$$\deg(d) = 1$$



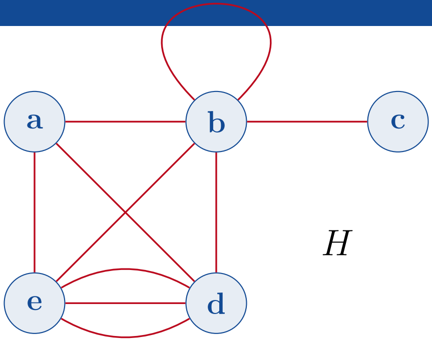
Example

Give the degrees of all the vertices in graph G.

solution:

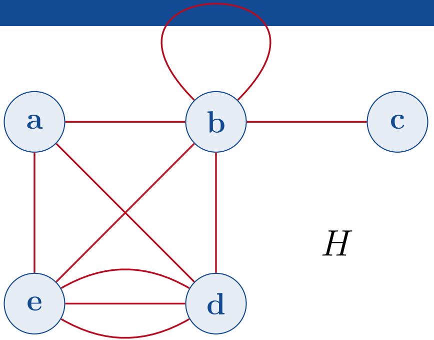
$$\deg(a) = 2 \qquad \deg(b) = 4 \qquad \deg(c) = 4 \qquad \deg(d) = 1$$

$$\deg(e) = 3 \qquad \deg(f) = 4 \qquad \deg(g) = 0$$



Example

Give the degrees of all the vertices in graph H.



Example

Give the degrees of all the vertices in graph H.

solution:

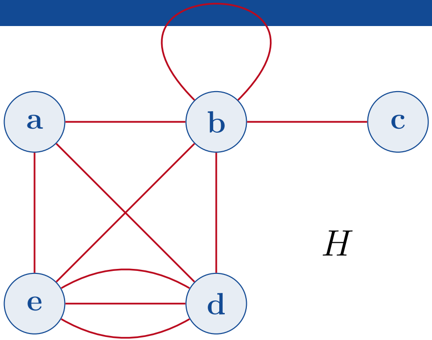
$\deg(a) =$

$\deg(b) =$

$\deg(c) =$

$\deg(d) =$

$\deg(e) =$



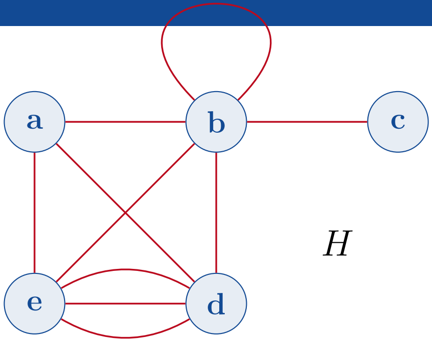
H

Example

Give the degrees of all the vertices in graph H.

solution:

$$\deg(a) = 3$$
$$\deg(b) =$$
$$\deg(c) =$$
$$\deg(d) =$$
$$\deg(e) =$$

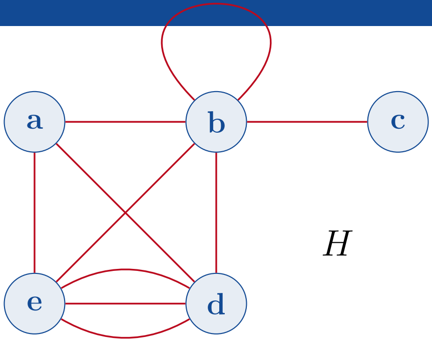


Example

Give the degrees of all the vertices in graph H.

solution:

$$\deg(a) = 3$$
$$\deg(b) = 6$$
$$\deg(c) =$$
$$\deg(d) =$$
$$\deg(e) =$$



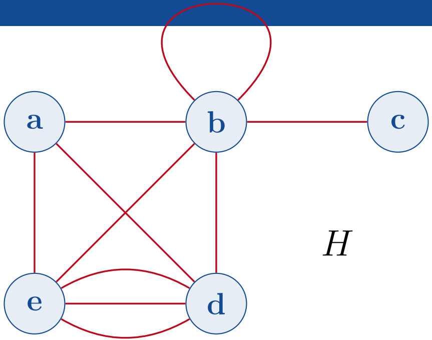
Example

Give the degrees of all the vertices in graph H.

solution:

$$\deg(a) = 3 \qquad \deg(b) = 6 \qquad \deg(c) = 1$$

$$\deg(d) = \qquad \deg(e) =$$



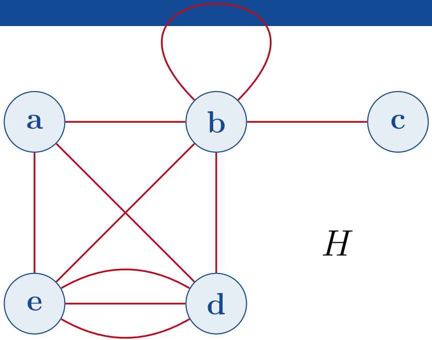
Example

Give the degrees of all the vertices in graph H.

solution:

$$\deg(a) = 3$$
$$\deg(b) = 6$$
$$\deg(c) = 1$$

$$\deg(d) = 5$$
$$\deg(e) =$$



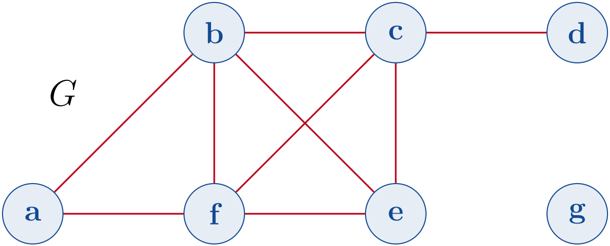
Example

Give the degrees of all the vertices in graph H.

solution:

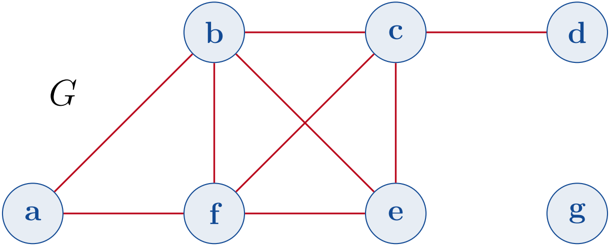
$$\deg(a) = 3 \qquad \deg(b) = 6 \qquad \deg(c) = 1$$

$$\deg(d) = 5 \qquad \deg(e) = 5$$



Definition

A vertex without an edge (degree=0) is called an *isolated vertex*.



Definition

A vertex of degree 1 is called a *pendant*. A pendant has only one edge.

Theorem

Let $G = (V, E)$ be a pseudograph and let $n(E)$ denote the number of edges in G. Then

$$n(E) = \frac{1}{2} \sum_{v \in V} \deg(v).$$

Example

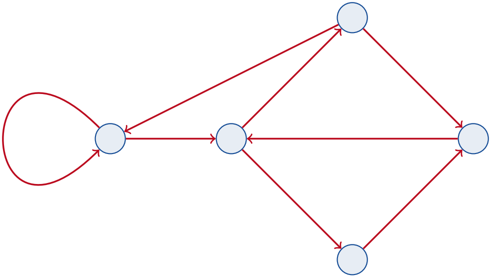
A graph has 10 vertices, each of degree 4. How many edges does this graph have?

solution:

Suppose that $V = \{v_1, \dots, v_{10}\}$. Then by Theorem 16, we have that

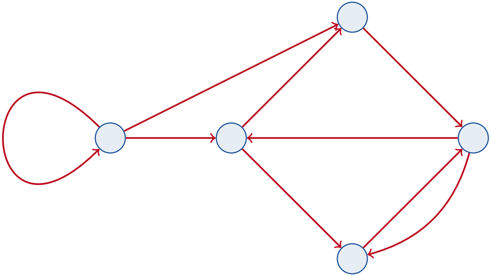
$$\begin{aligned} n(E) &= \frac{1}{2}\sum_{i=1}^{10} \deg(v_i) \\ &= \frac{1}{2}\big(\deg(v_1) + \deg(v_2) + \ldots + \deg(v_{10})\big) \\ &= \frac{1}{2}(4 + 4 + \ldots + 4) = \frac{1}{2}(40) = 20. \end{aligned}$$

Therefore this graph has 20 edges.



Definition

If the edges in a graph have directions, the graph is called a *directed graph* (or *digraph*). This direction indicates where the connection starts and ends.



Definition

If a directed graph has parallel edges, it is called a *directed multigraph* (or *multidigraph*).

Remark

In a graph, all edges are the same type. So either all edges are directed or all edges are undirected.

Notation

Let G be a directed graph. Now when we write $e = (u, v)$, the order of u and v is important. The edge $e = (u, v)$ starts at u and finishes at v.

Definition

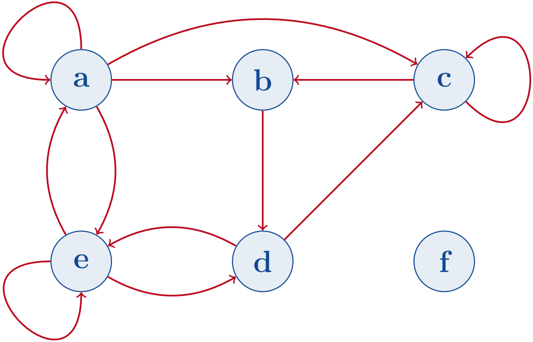
Let G be a directed graph. The *indegree* of a number vertex v is

$$\deg^-(v) = \text{the number of edges coming } \textit{into } v$$

and the *outdegree* of v is

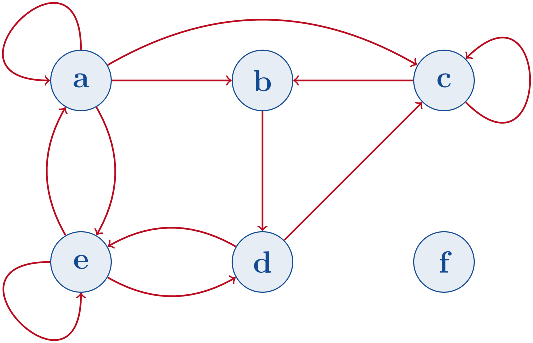
$$\deg^+(v) = \text{the number of edges coming } \textit{out} \text{ of } v.$$

A loop is counted as 1 for both $deg^-(v)$ and $deg^+(v)$.



Example

Find the indegree and outdegree of each vertex in this graph.



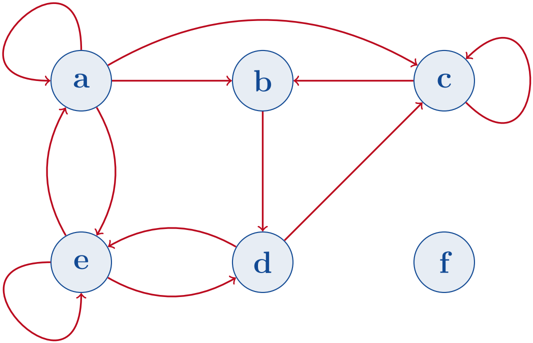
solution:

The indegrees are:

$$\deg^{-}(a) = \quad , \quad \deg^{-}(b) = \quad , \quad \deg^{-}(c) =$$
$$\deg^{-}(d) = \quad , \quad \deg^{-}(e) = \quad , \quad \deg^{-}(f) =$$

The outdegrees are:

$$\deg^{+}(a) = \quad , \quad \deg^{+}(b) = \quad , \quad \deg^{+}(c) =$$
$$\deg^{+}(d) = \quad , \quad \deg^{+}(e) = \quad , \quad \deg^{+}(f) = \quad .$$



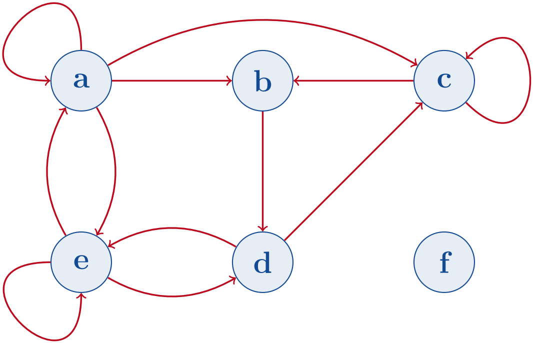
solution:

The indegrees are:

$$\deg^-(a) = 2, \quad \deg^-(b) = 2, \quad \deg^-(c) = 3$$
$$\deg^-(d) = 2, \quad \deg^-(e) = 3, \quad \deg^-(f) = 0$$

The outdegrees are:

$$\deg^+(a) = \quad , \quad \deg^+(b) = \quad , \quad \deg^+(c) =$$
$$\deg^+(d) = \quad , \quad \deg^+(e) = \quad , \quad \deg^+(f) = \quad .$$



solution:

The indegrees are:

$$\deg^-(a) = 2, \quad \deg^-(b) = 2, \quad \deg^-(c) = 3$$
$$\deg^-(d) = 2, \quad \deg^-(e) = 3, \quad \deg^-(f) = 0$$

The outdegrees are:

$$\deg^+(a) = 4, \quad \deg^+(b) = 1, \quad \deg^+(c) = 2$$
$$\deg^+(d) = 2, \quad \deg^+(e) = 3, \quad \deg^+(f) = 0.$$

Theorem

Let $G = (V, E)$ be a directed graph and let $n(E)$ denote the number of edges in G. Then

$$n(E) = \sum_{v \in V} \deg^-(v) = \sum_{v \in V} \deg^+(v).$$

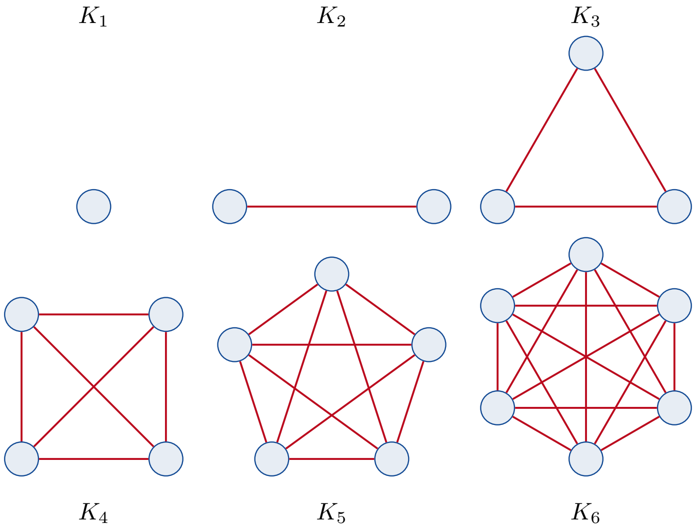
Definition

A simple graph which includes all possible edges is called a *complete graph*.

Remark

In a complete graph, every vertex has an edge with every other vertex. A complete graph with n vertices is denoted by K_n.

The degree of every vertex in K_n is $(n-1)$. Thus K_n has a total of $\frac{n(n-1)}{2}$ edges.


K_1
K_2
K_3
K_4
K_5
K_6

Definition

A *planar graph* is a graph that can be drawn on the plane in such a way that its edges intersect only at their endpoints.

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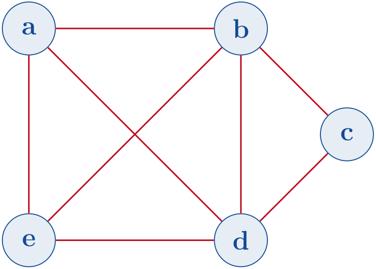
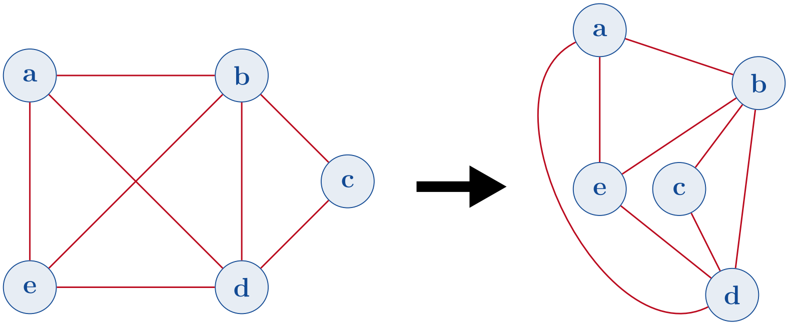


Definition

A *planar graph* is a graph that can be drawn on the plane in such a way that its edges intersect only at their endpoints.



Definition

A graph that is not a planar graph, and can only be drawn without intersecting edges in three-dimensional space is called a *three-dimensional graph*.

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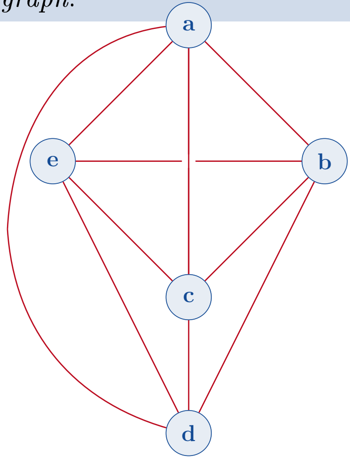
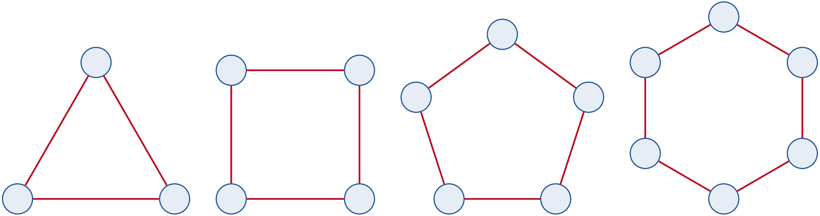


Definition

Suppose that $n \geq 3$, that $V = \{v_1, v_2, \ldots, v_n\}$ and that $E = \{(v_1, v_2), (v_2, v_3), \ldots, (v_{n-1}, v_n), (v_n, v_1)\}$. Then the graph $C_n = (V, E)$ is called a *cycle graph*.

Definition

Suppose that $n \geq 3$, that $V = \{v_1, v_2, \ldots, v_n\}$ and that $E = \{(v_1, v_2), (v_2, v_3), \ldots, (v_{n-1}, v_n), (v_n, v_1)\}$. Then the graph $C_n = (V, E)$ is called a *cycle graph*.



C_3 C_4 C_5 C_6

Definition

If we take C_n and add a new vertex which is attached to all the other vertices, we get the *wheel graph* W_n.

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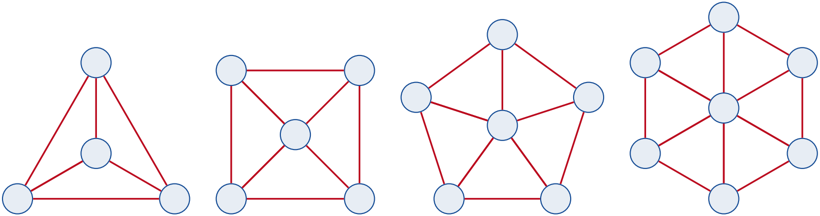


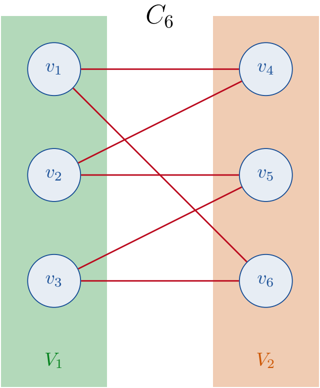
W_3 W_4 W_5 W_6

Definition

A *bipartite graph* is a graph where the set V of vertices can be divided into two distinct subsets V_1 and V_2 such that every edge connects a vertex in V_1 with a vertex in V_2.

Remark

In a bipartite graph, there are no edges going from V_1 to V_1, or from V_2 to V_2.

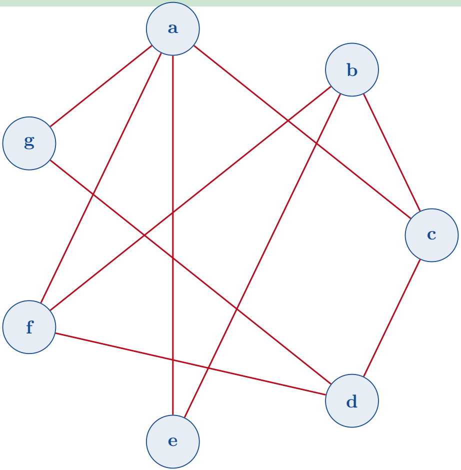


Example

Note that the graph C_6 is a bipartite graph because if we let $V_1 = \{v_1, v_2, v_3\}$ and $V_2 = \{v_4, v_5, v_6\}$, then every edge goes from V_1 to V_2.

Example

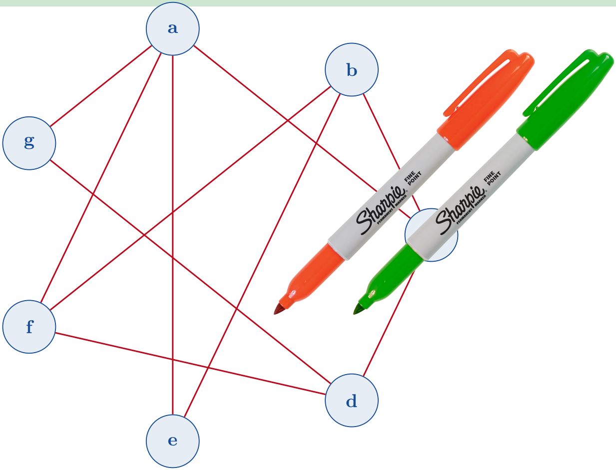
Is G a bipartite graph?



G

Example

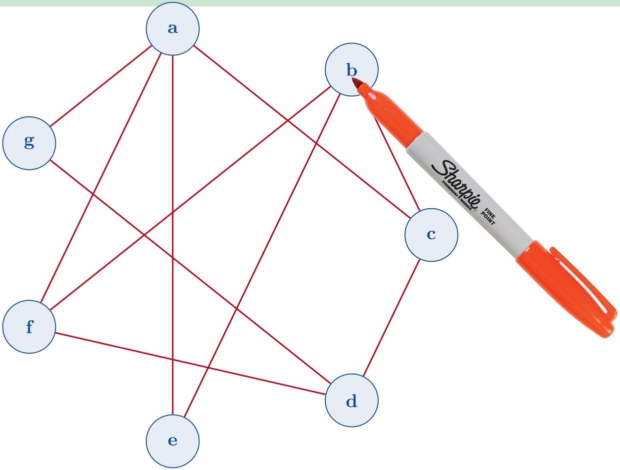
Is G a bipartite graph?



G

Example

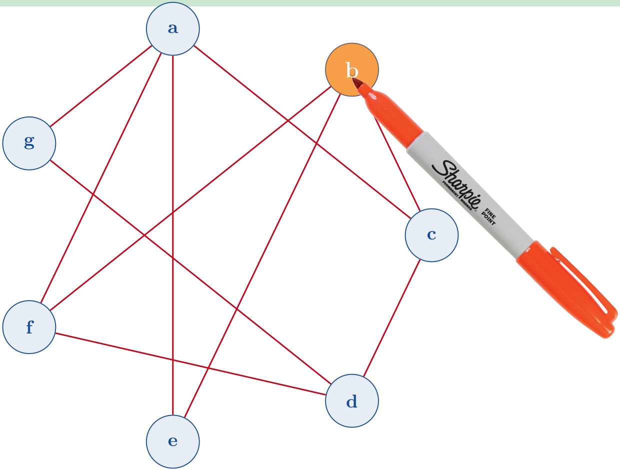
Is G a bipartite graph?



G

Example

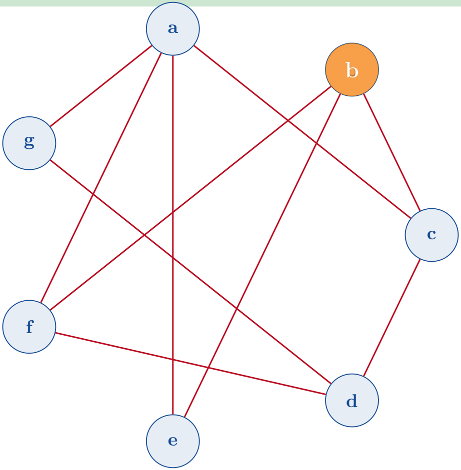
Is G a bipartite graph?



G

Example

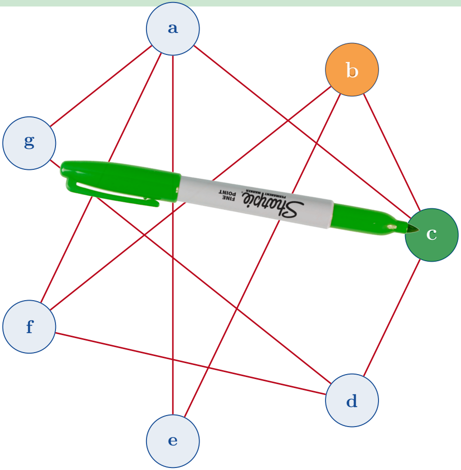
Is G a bipartite graph?



G

Example

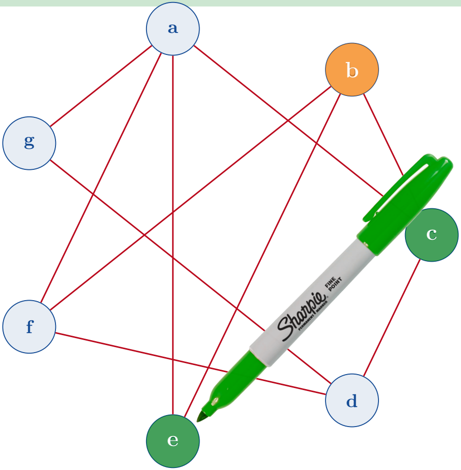
Is G a bipartite graph?



G

Example

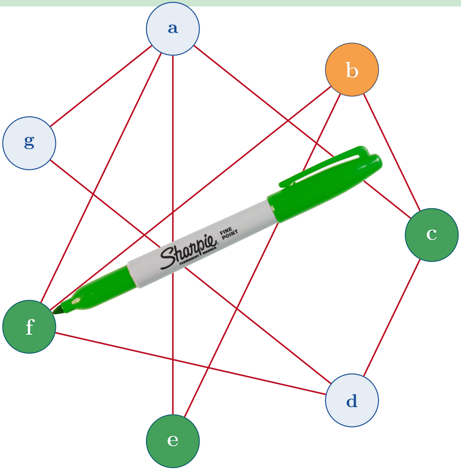
Is G a bipartite graph?



G

Example

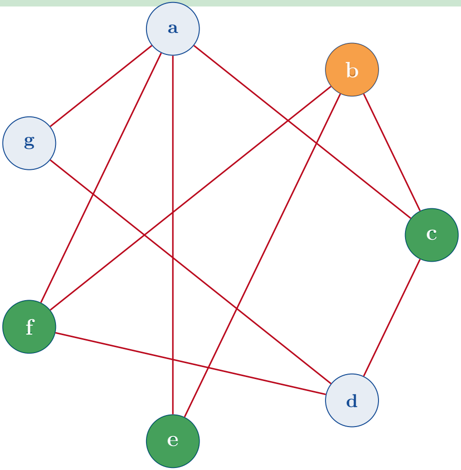
Is G a bipartite graph?



G

Example

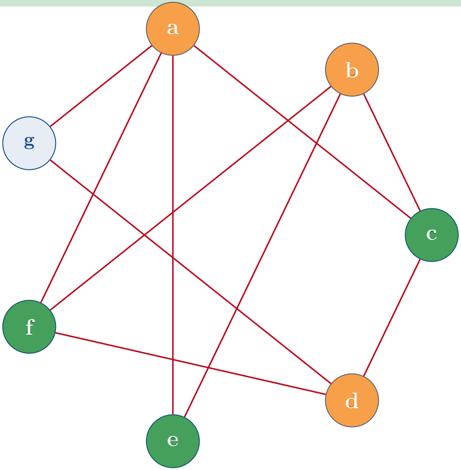
Is G a bipartite graph?



G

Example

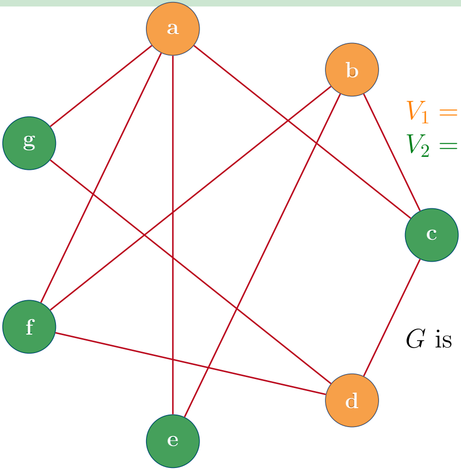
Is G a bipartite graph?



G

Example

Is G a bipartite graph?



$V_1 = \{a, b, d\}$

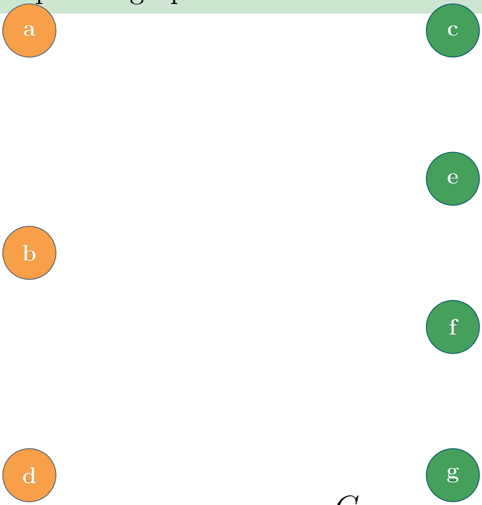
$V_2 = \{c, e, f, g\}$

G is bipartite

G

Example

Is G a bipartite graph?



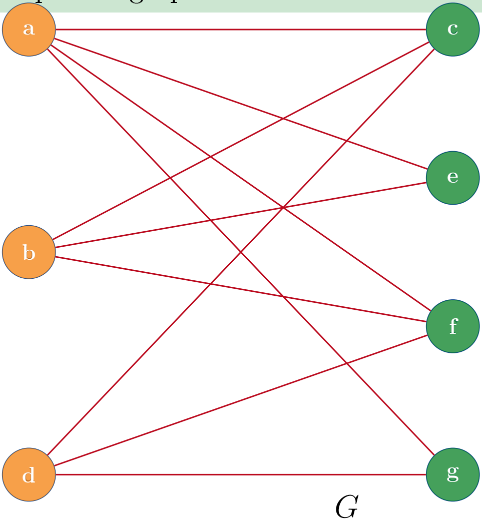
G

$V_1 = \{a, b, d\}$
$V_2 = \{c, e, f, g\}$

G is bipartite

Example

Is G a bipartite graph?

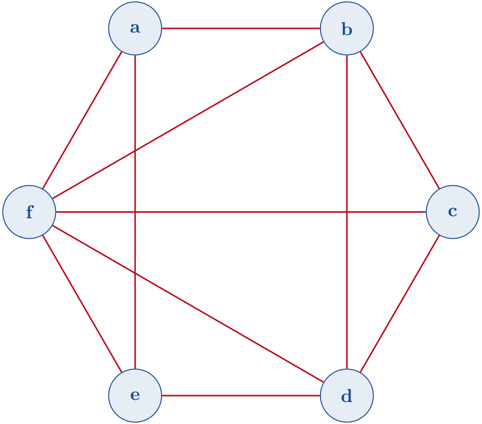


$V_1 = \{a, b, d\}$
$V_2 = \{c, e, f, g\}$

G is bipartite

Example

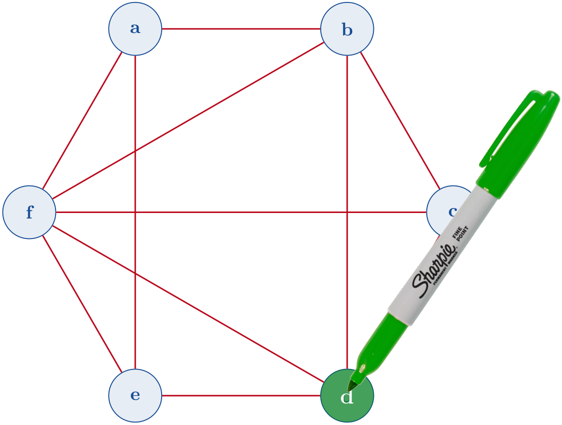
Is H a bipartite graph?



H

Example

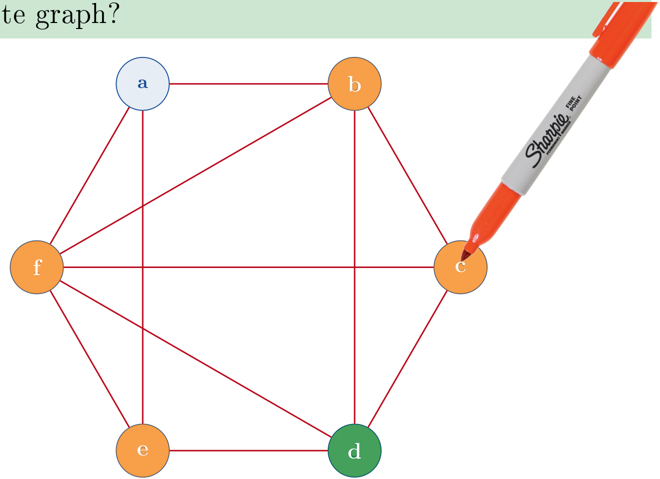
Is H a bipartite graph?



H

Example

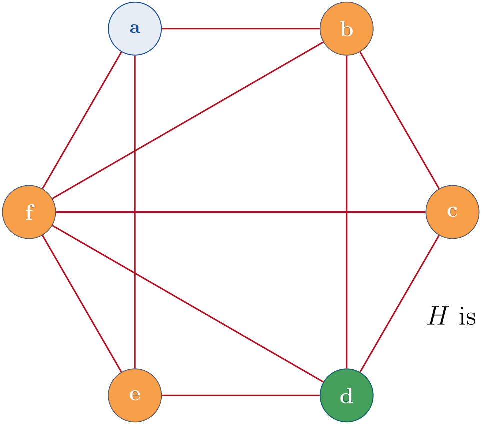
Is H a bipartite graph?



H

Example

Is H a bipartite graph?



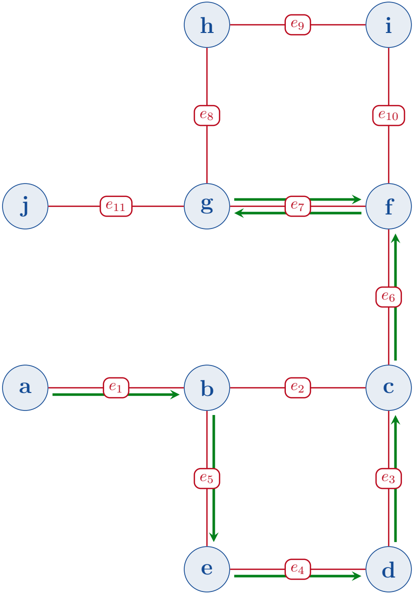
H is not bipartite

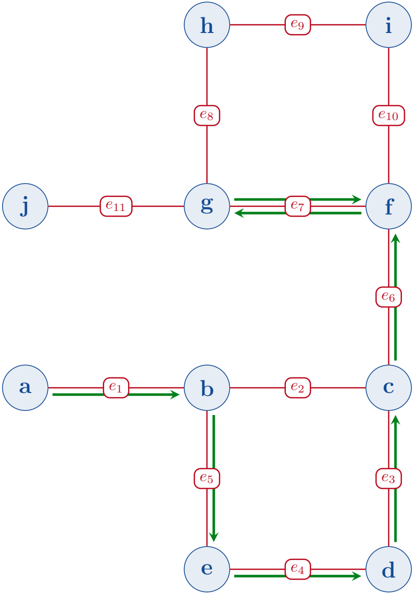
H

Walks

Definition

A *walk* is a list $v_0, e_1, v_1, \ldots, e_k, v_k$ of vertices and edges such that for $1 \leq i \leq k$, the edge e_i has endpoints v_{i-1} and v_i.


h
e_9
i
e_8
e_{10}
j
e_{11}
g
e_7
f
e_6
a
e_1
b
e_2
c
e_5
e_3
e
e_4
d



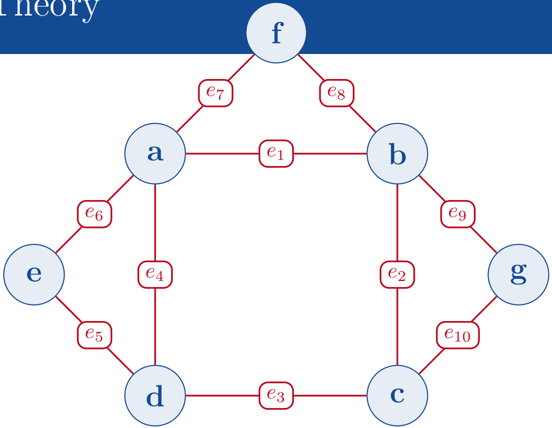
Example

$a, e_1, b, e_5, e, e_4, d, e_3, c, e_6, f,$
e_7, g, e_7, f is a walk in this graph.

Definition

An *Eulerian trail* is a walk such that

1. no edge is repeated; and
2. every edge is included.



Example

The walk

$$d, e_5, e, e_6, a, e_7, f, e_8, b, e_1, a, e_4, d, e_3, c, e_{10}, g, e_9, b, e_2, c$$

is an Eulerian trail. Each of the ten edges appears once and only once in this list.

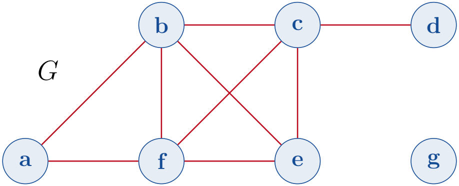
Remark

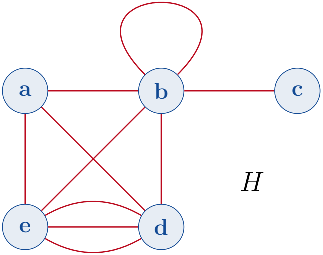
The Königsberg bridge problem can be rephrased as:

Does there exist an Eulerian trail in Königsberg?

Definition

A graph is *connected* if there exists a walk between every pair of vertices.





Example

Graph H is connected, but graph G is not connected.

Theorem

Let G be a connected graph.

Theorem

Let G be a connected graph.

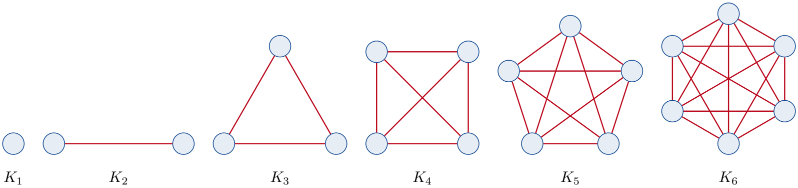
Then there exists an Eulerian trail if and only if the number of vertices of odd degree is either 0 or 2.

Theorem

Let G be a connected graph.

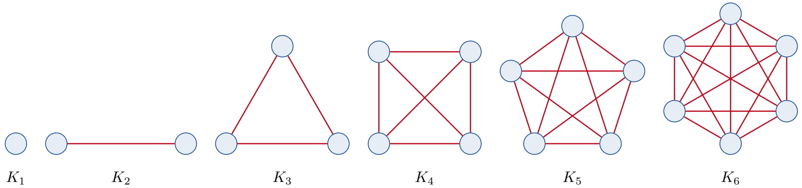
Then there exists an Eulerian trail if and only if the number of vertices of odd degree is either 0 or 2.

Furthermore, if G has 2 vertices of odd degree, then the Eulerian trail must start and finish at these two vertices.



Example

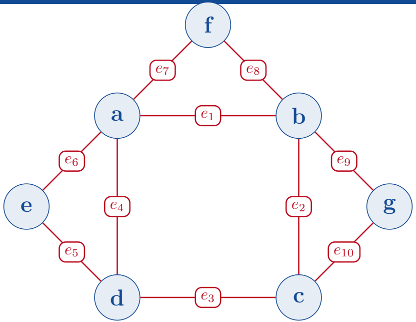
Note that in K_3 and K_5, every vertex has even degree. So there is an Eulerian trail in K_3 and in K_5.



Example

Note that in K_3 and K_5, every vertex has even degree. So there is an Eulerian trail in K_3 and in K_5.

Notice further that all four vertices in K_4 are of odd degree. This means that K_4 does not contain an Eulerian trail.

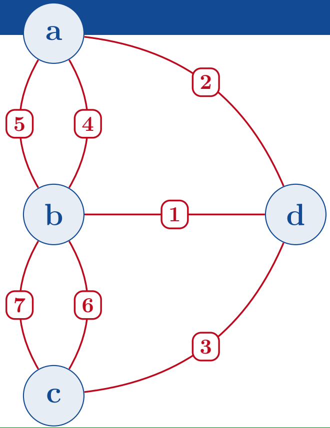


Example

Note that

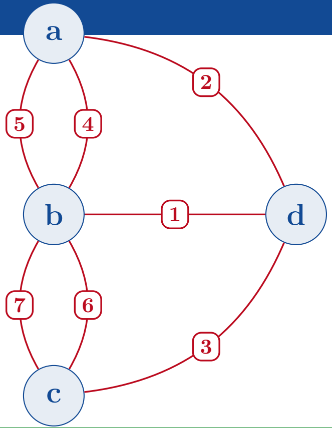
$$\deg(a) = 4,\ \deg(b) = 4,\ \deg(c) = 3,\ \deg(d) = 3,$$
$$\deg(e) = 2,\ \deg(f) = 2,\ \deg(g) = 2.$$

Two of the vertices have odd degree, c and d. So there must exist an Eulerian trail and it must start and end at c and d. (We have already found this trail.)



Example (The Königsberg Bridge Problem)

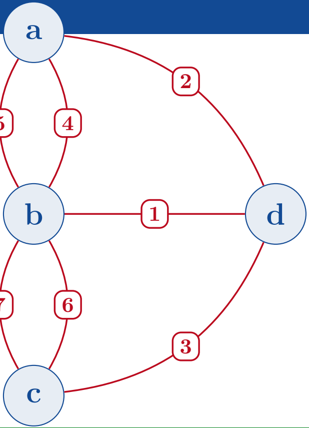
Now consider Königsberg as shown above.



Example (The Königsberg Bridge Problem)

Now consider Königsberg as shown above. Note that

$$\deg(a) = 3, \qquad \deg(b) = 5, \qquad \deg(c) = 3, \qquad \deg(d) = 3.$$

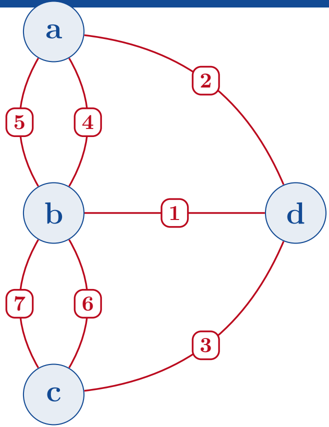


Example (The Königsberg Bridge Problem)

Now consider Königsberg as shown above. Note that

$$\deg(a) = 3, \qquad \deg(b) = 5, \qquad \deg(c) = 3, \qquad \deg(d) = 3.$$

Since all four vertices have odd degree, there does not exist an Eulerian trail in Königsberg.



Example (The Königsberg Bridge Problem)

Therefore it was not possible to walk around the city of Königsberg and cross each bridge once.

Euler's Formula for Polyhedra

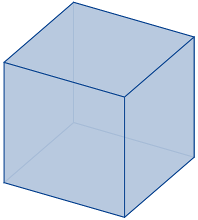
Euler's formula is

$$n(V) - n(E) + n(F)$$

where

$$\begin{aligned} n(V) &= \text{number of vertices} \\ n(E) &= \text{number of edges} \\ n(F) &= \text{number of faces.} \end{aligned}$$

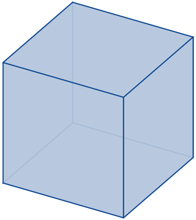
cube



$$n(V)$$
$$n(E)$$
$$n(F)$$

$$n(V) - n(E) + n(F)$$

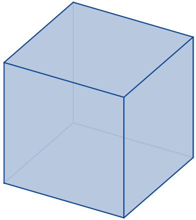
cube



$n(V)$	8
$n(E)$	12
$n(F)$	6

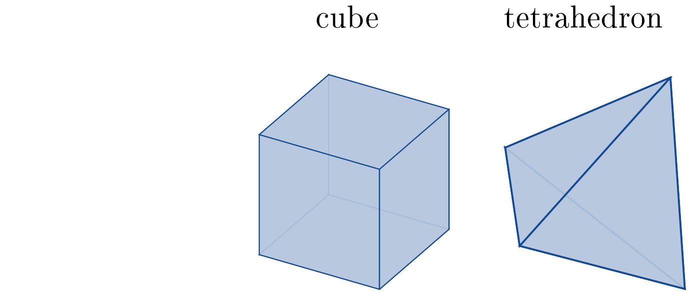
$$n(V) - n(E) + n(F)$$

cube

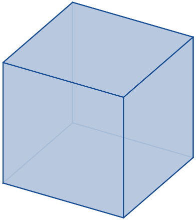
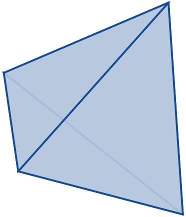


$n(V)$	8
$n(E)$	12
$n(F)$	6

$n(V) - n(E) + n(F)$ $\quad 8 - 12 + 6 = 2$

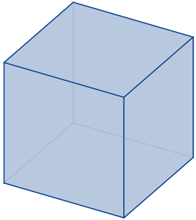


	cube	tetrahedron
$n(V)$	8	
$n(E)$	12	
$n(F)$	6	
$n(V) - n(E) + n(F)$	$8 - 12 + 6 = 2$	

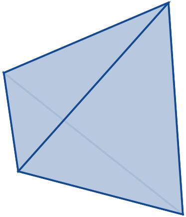
	cube	tetrahedron
		
$n(V)$	8	4
$n(E)$	12	6
$n(F)$	6	4

$n(V) - n(E) + n(F) \qquad 8 - 12 + 6 = 2$

cube

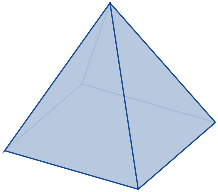


tetrahedron



	cube	tetrahedron
$n(V)$	8	4
$n(E)$	12	6
$n(F)$	6	4
$n(V) - n(E) + n(F)$	$8 - 12 + 6 = 2$	$4 - 6 + 4 = 2$

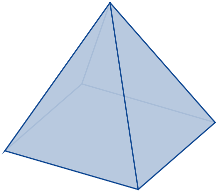
pyramid



$$n(V)$$
$$n(E)$$
$$n(F)$$

$$n(V) - n(E) + n(F)$$

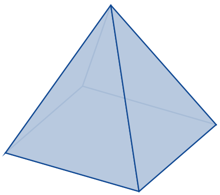
pyramid



$n(V)$	5
$n(E)$	8
$n(F)$	5

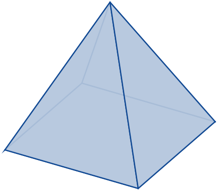
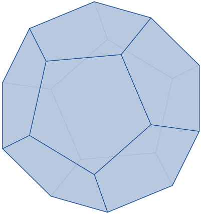
$n(V) - n(E) + n(F)$

pyramid

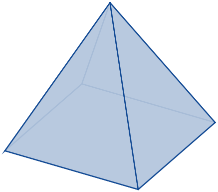
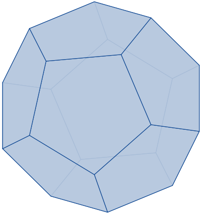


$n(V)$	5
$n(E)$	8
$n(F)$	5

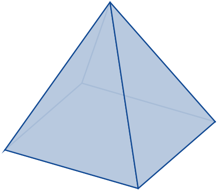
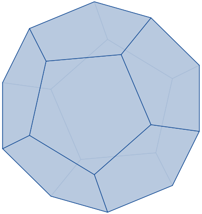
$n(V) - n(E) + n(F)$ $\quad 5 - 8 + 5 = 2$

	pyramid	dodecahedron
		
$n(V)$	5	
$n(E)$	8	
$n(F)$	5	

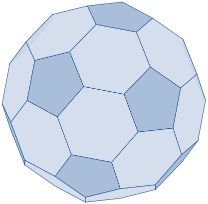
$n(V) - n(E) + n(F)$ $\quad$ $5 - 8 + 5 = 2$

	pyramid	dodecahedron
		
$n(V)$	5	20
$n(E)$	8	30
$n(F)$	5	12

$n(V) - n(E) + n(F)$ $\quad$ $5 - 8 + 5 = 2$

	pyramid	dodecahedron
		
$n(V)$	5	20
$n(E)$	8	30
$n(F)$	5	12
$n(V) - n(E) + n(F)$	$5 - 8 + 5 = 2$	$20 - 30 + 12 = 2$

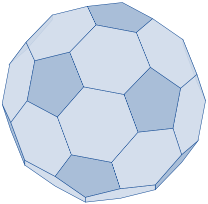
football
(12 pentagons &
20 hexagons)



$$n(V)$$
$$n(E)$$
$$n(F)$$

$$n(V) - n(E) + n(F)$$

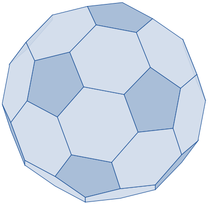
football
(12 pentagons &
20 hexagons)



$n(V)$	60
$n(E)$	90
$n(F)$	32

$$n(V) - n(E) + n(F)$$

football
(12 pentagons &
20 hexagons)

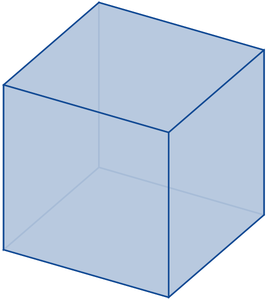


$n(V)$	60
$n(E)$	90
$n(F)$	32

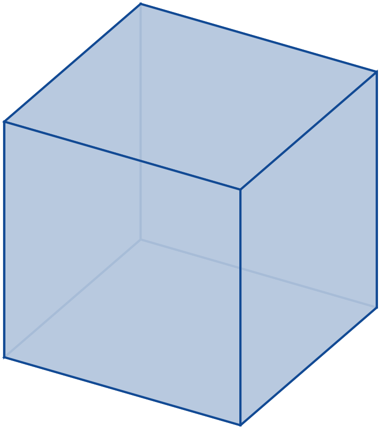
$n(V) - n(E) + n(F) \quad 60 - 90 + 32 = 2$

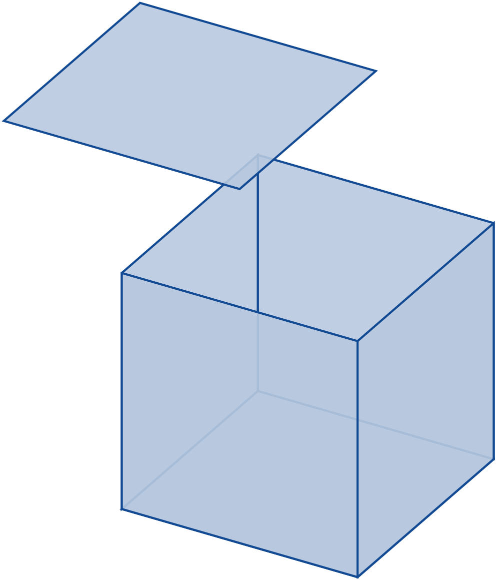
Remark

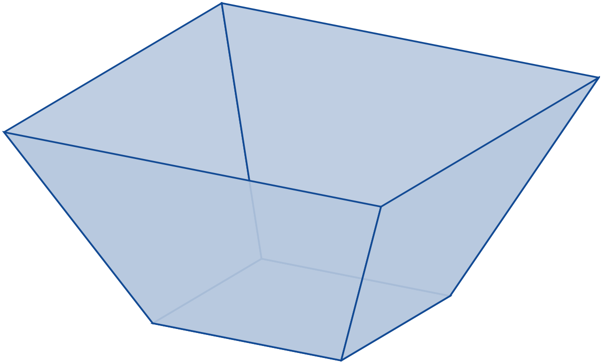
If we have a polyhedron without any holes in it, is Euler's formula always equal to 2? And if so, how can we prove it?

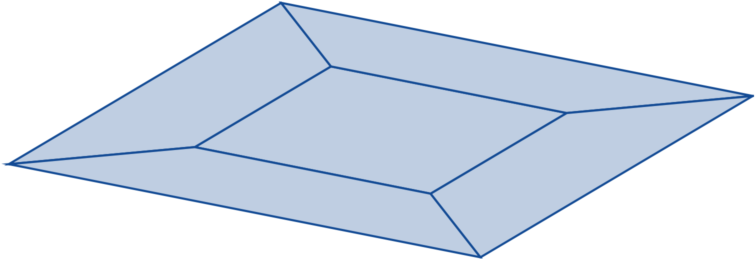


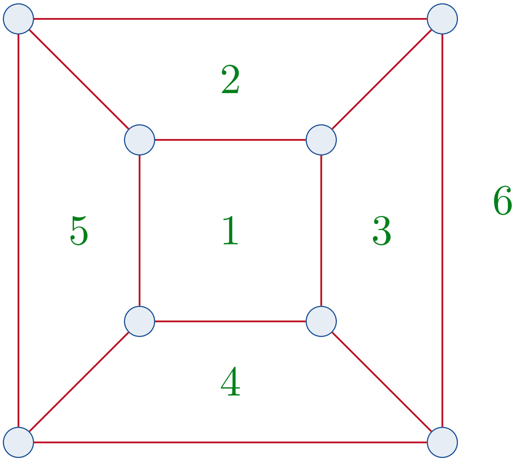
$$n(V) = 8$$
$$n(E) = 12$$
$$n(F) = 6$$







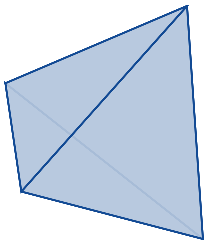




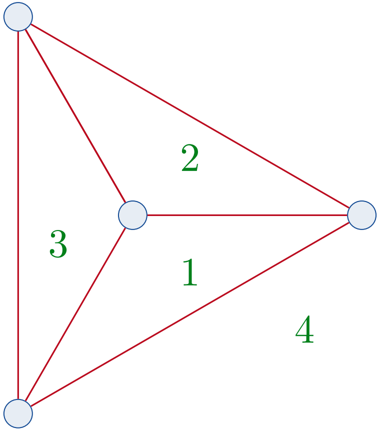
$n(V) = 8$

$n(E) = 12$

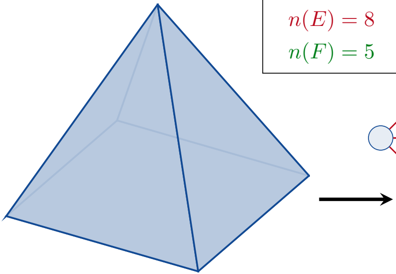
$n(F) = 6$

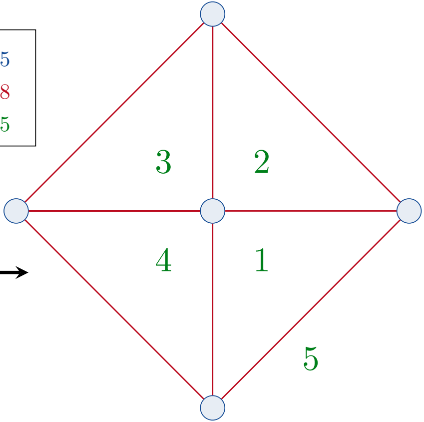


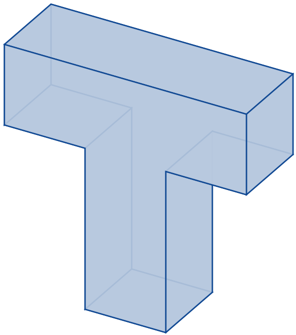
$$n(V) = 4$$
$$n(E) = 6$$
$$n(F) = 4$$



$$n(V) = 5$$
$$n(E) = 8$$
$$n(F) = 5$$



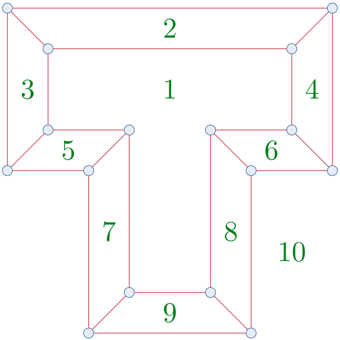




$n(V) = 16$

$n(E) = 24$

$n(F) = 10$



Every three dimensional polyhedron is equivalent to a connected, planar, simple graph. So if we know something about these graphs, then we also know it about polyhedra.

Every three dimensional polyhedron is equivalent to a connected, planar, simple graph. So if we know something about these graphs, then we also know it about polyhedra.

What do we know about such graphs?

Let us start with the first complete graph:

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$n(V) = 1$
$n(E) = 0$
$n(F) = 1$

This graph, K_1, is called the *trivial graph*. It has one vertex, zero edges and one face.

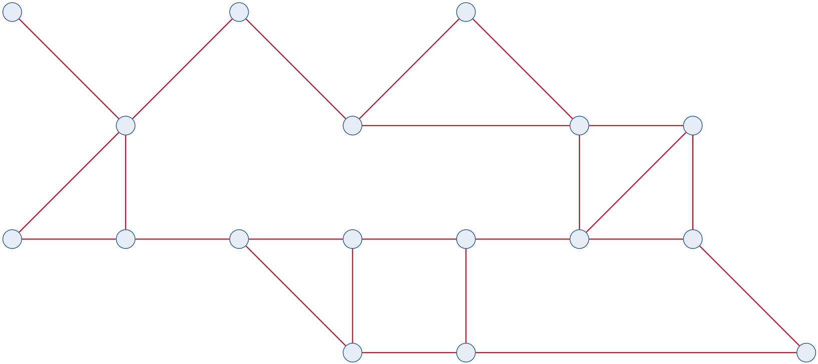
Let us start with the first complete graph:

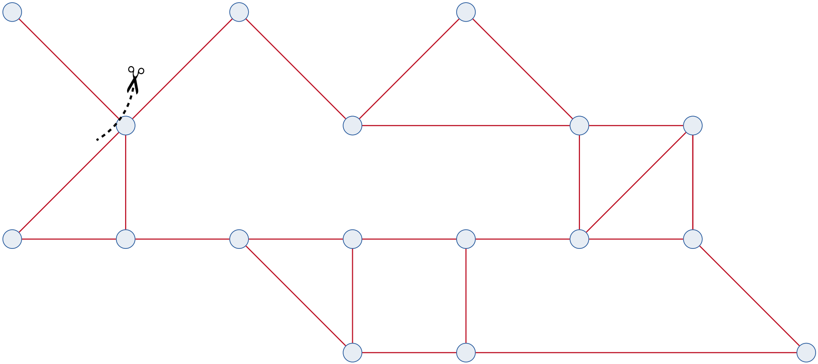
$n(V) = 1$
$n(E) = 0$
$n(F) = 1$

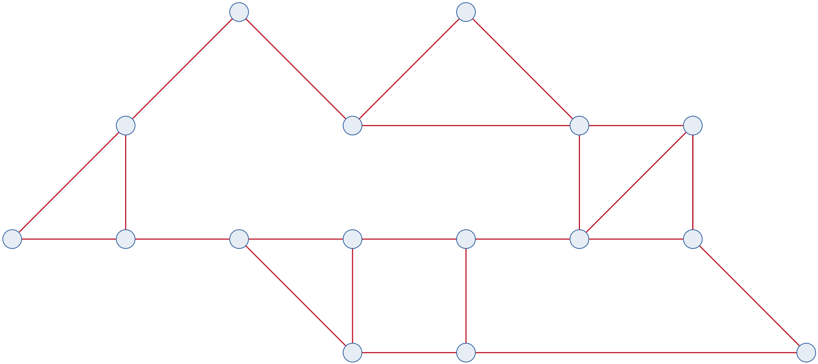
This graph, K_1, is called the *trivial graph*. It has one vertex, zero edges and one face. So

$$n(V) - n(E) + n(F) = 1 - 0 + 1 = 2.$$

Now let us take any connected, planar, simple graph.

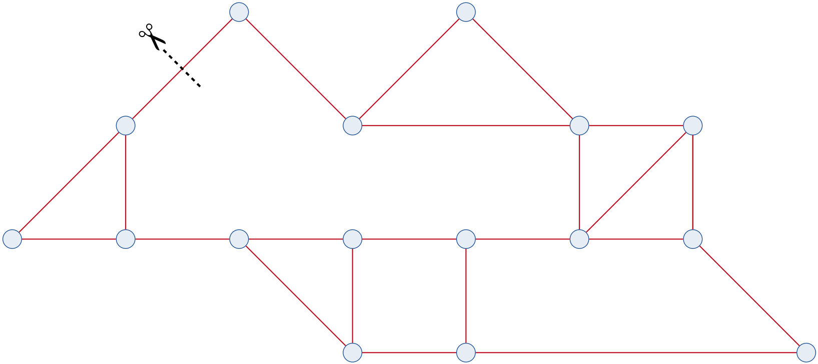


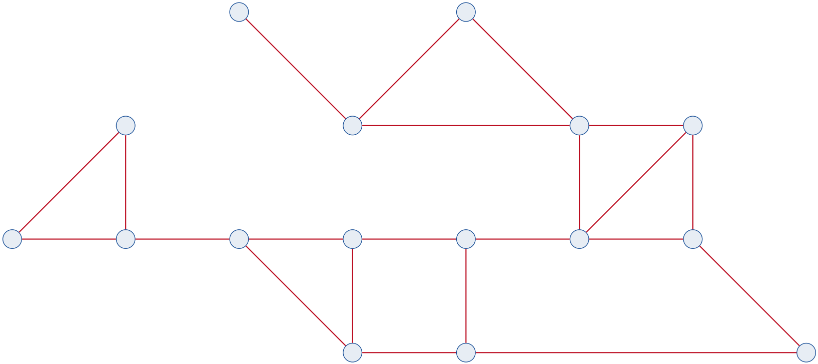




$n(V)$ decreases by 1, $n(E)$ decreases by 1, $n(F)$ stays the same,

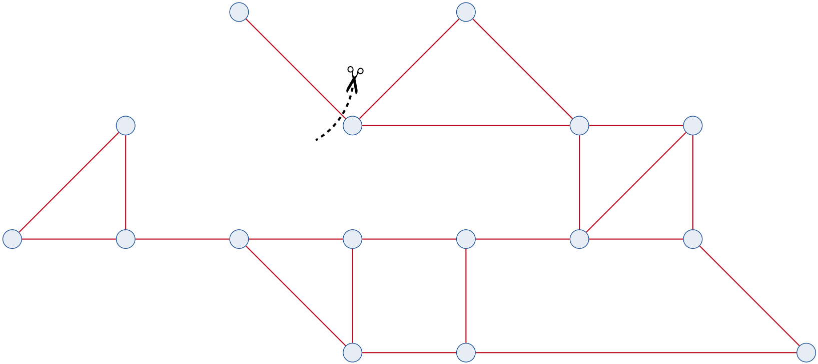
$n(V) - n(E) + n(F)$ stays the same

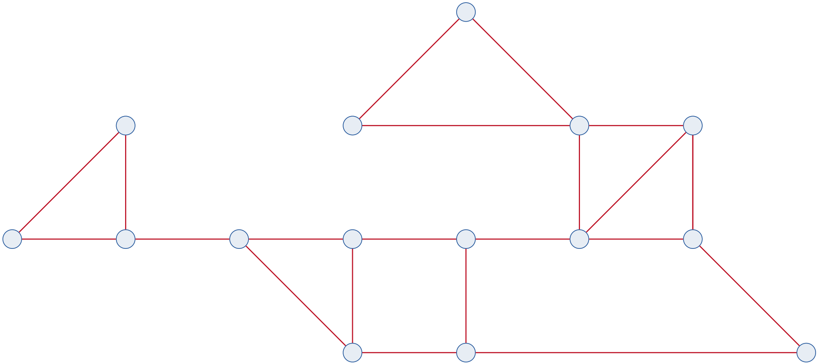


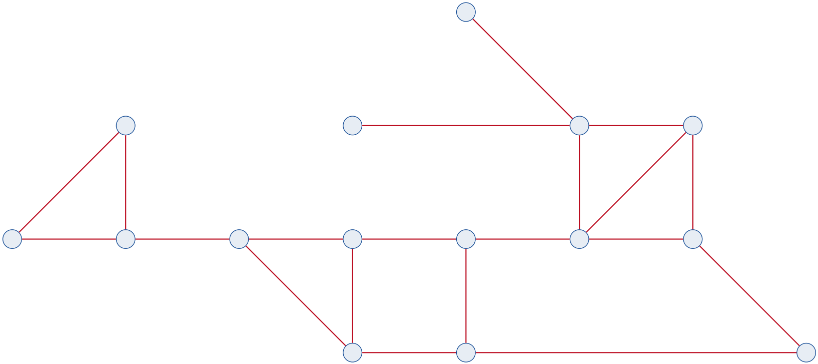


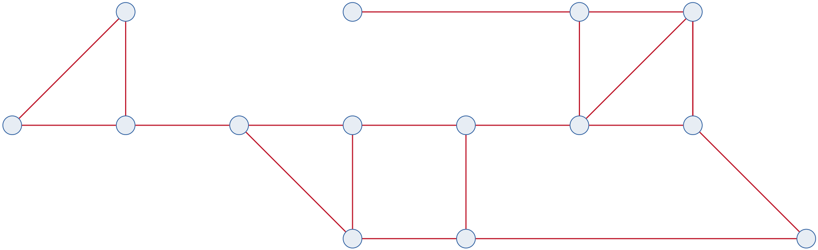
$n(V)$ stays the same, $n(E)$ decreases by 1, $n(F)$ decreases by 1,

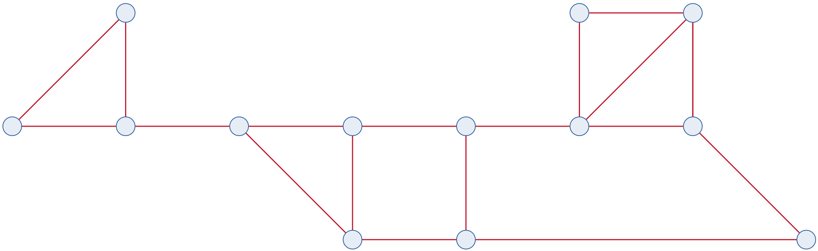
$n(V) - n(E) + n(F)$ stays the same

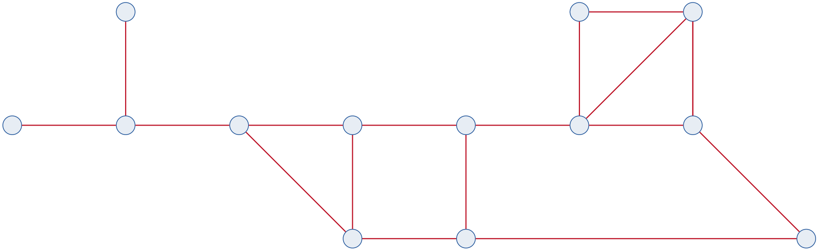


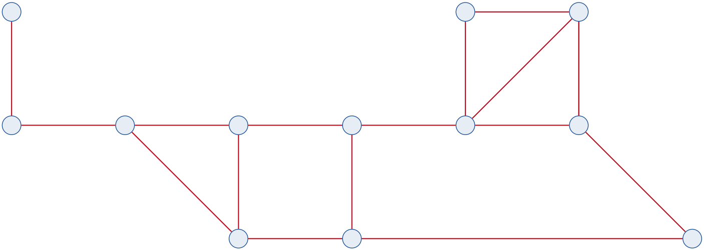


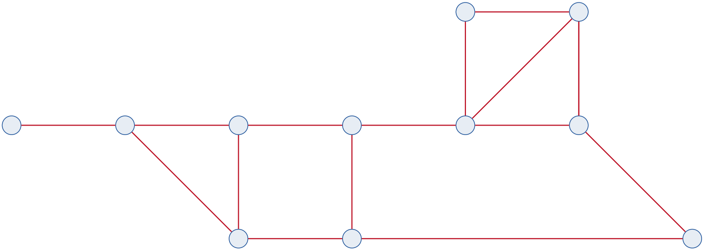


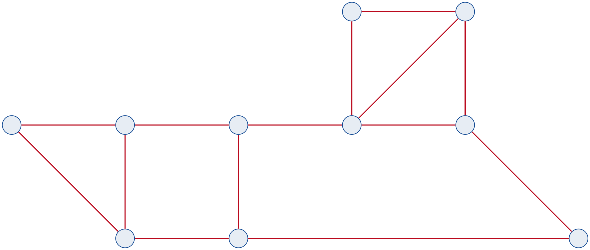


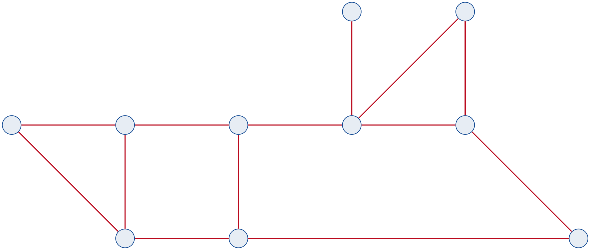


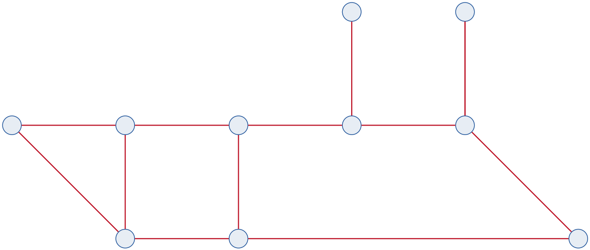


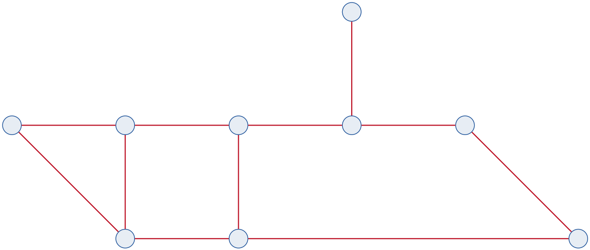


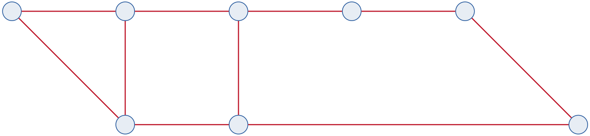


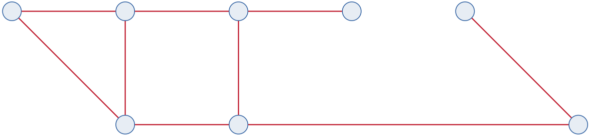


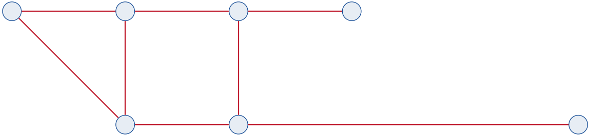


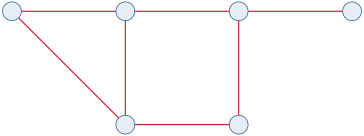


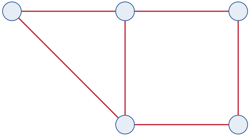


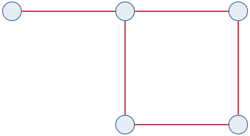


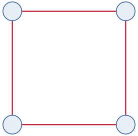


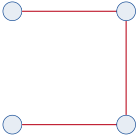


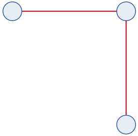












23. Graph Theory

Theorem

If G is a connected, planar, simple graph, then

$$n(V) - n(E) + n(F) = 2.$$

Theorem

If G is a connected, planar, simple graph, then

$$n(V) - n(E) + n(F) = 2.$$

. . . and the same is true for polyhedra without holes.

Next Week

- Midterm Exam (Chapters 1-23)