OKAN ÜNİVERSİTESİ
FEN EDEBİYAT FAKÜLTESİ
MATEMATİK BÖLÜMÜ

24.05.2012 **MAT 234 – Matematik IV – Yarıyıl Sonu Sınavı Çözümleri** N. Course

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Question 1 (Absolute Convergence)**.**

(a) [5 pts] Give the definition of an *absolutely convergent* series.

A series $\sum_{n=1}^{\infty} a_n$ is called absolutely convergent iff the series $\sum_{n=1}^{\infty} |a_n|$ converges.

For parts (b) – (e), suppose that

- $y \neq 0$;
- $\sum_{n=0}^{\infty} a_n y^n$ converges;
- $|x| < |y|$.

(b) [5 pts] Show that $(a_n y^n)$ is bounded.
[In other words: Show that $\exists K$ such that $|a_n y^n| < K \ \forall n$.]

$\sum_{n=0}^{\infty} a_n y^n$ converges $\implies$ $a_n y^n \to 0$ as $n \to \infty$ (by the Divergence Test) $\implies$ $(a_n y^n)$ is convergent. By a theorem from the course, every convergent sequence is bounded. Therefore $(a_n y^n)$ is bounded.

(c) [5 pts] Show that

$$|a_n x^n| \leq K \left(\frac{|x|}{|y|} \right)^n$$

for some constant K.

Suppose $|a_n y^n| < K \ \forall n \in \mathbb{N}$. Then, since $y \neq 0$,

$$|a_n x^n| = \left| a_n y^n \frac{x^n}{y^n} \right| = |a_n y^n| \frac{|x|^n}{|y|^n} \leq K \left(\frac{|x|}{|y|} \right)^n .$$

(d) [5 pts] Let $b_n = \left(\frac{|x|}{|y|} \right)^n$. Show that $\sum_{n=0}^{\infty} b_n$ converges.

If $x = 0$, then $b_n = 0 \ \forall n$ so $\sum_{n=0}^{\infty} b_n$ converges.
If $0 < |x| < |y|$, then $\frac{|x|}{|y|} < 1$. So

$$\frac{b_{n+1}}{b_n} = \frac{\left(\frac{|x|}{|y|} \right)^{n+1}}{\left(\frac{|x|}{|y|} \right)^n} = \frac{|x|}{|y|} \to \frac{|x|}{|y|} < 1$$

as $n \to \infty$. By the Ratio Test, $\sum_{n=0}^{\infty} b_n$ converges.

(e) [5 pts] Show that $\sum_{n=0}^{\infty} a_n x^n$ is absolutely convergent.

Since $0 \leq |a_n x^n| \leq K \left(\frac{|x|}{|y|} \right)^n$, it follows by the Comparison Test, and by part (d), that $\sum_{n=0}^{\infty} |a_n x^n|$ converges. So $\sum_{n=0}^{\infty} a_n x^n$ is absolutely convergent.

Question 2 (Sequences). Define a sequence of real numbers (a_n) by

$$a_1 = 1 \qquad \text{and} \qquad 7a_{n+1} = a_n^2 + 12. \tag{1}$$

(a) [7 pts] Show that $0 \le a_n \le 3$ for all $n \in \mathbb{N}$.
[HINT: Use proof by induction.].

Since $0 \le a_1 = 1 \le 3$, the statement is true for $n = 1$ $\boxed{1}$. Suppose that it is true for $n = k$. Then $0 \le a_k \le 3$ $\boxed{1}$. So $7a_{k+1} = a_k^2 + 12 \le 3^2 + 12 = 21 \implies a_{k+1} \le 3$ $\boxed{2}$ and $7a_{k+1} = a_k^2 + 12 \ge 0^2 + 12 \ge 0 \implies a_{k+1} \ge 0$ $\boxed{2}$. By the principle of mathematical induction, it follows that $0 \le a_n \le 3$ $\forall n \in \mathbb{N}$ $\boxed{1}$.

(b) [6 pts] Show that $a_n \le a_{n+1}$ for all $n \in \mathbb{N}$.

First note that $a_{n+1} - a_n = \frac{1}{7}(a_n^2 + 12) - a_n = \frac{1}{7}(a_n^2 - 7a_n + 12) = \frac{1}{7}(a_n - 3)(a_n - 4)$ $\boxed{2}$. Since $0 \le a_n \le 3$, $(a_n - 3) \le 0$ and $(a_n - 4) \le 0$ $\boxed{2}$. Therefore $a_{n+1} - a_n = \frac{1}{7}(a_n - 3)(a_n - 4) \ge 0$. So $a_{n+1} \ge a_n$ $\forall n \in \mathbb{N}$ $\boxed{2}$.

(c) [6 pts] Show that (a_n) is a convergent sequence.

By a theorem from the course, "every increasing sequence which is bounded above is convergent". In part (a), I proved that (a_n) is bounded above. In part (b), I proved that (a_n) is increasing. Therefore (a_n) is convergent.

(d) [6 pts] Calculate $\lim_{n\to\infty} a_n$.

Let $a = \lim_{n\to\infty} a_n$. Then $7a \leftarrow 7a_{n+1} = a_n^2 + 12 \to a^2 + 12$ as $n \to \infty$ $\boxed{2}$. So $0 = a^2 - 7a + 12 = (a-3)(a-4)$. So $a = 3$ or $a = 4$ $\boxed{2}$. Finally, since $a_n \le 3$ $\forall n \in \mathbb{N}$, we must have that $a = 3$ $\boxed{2}$.

Question 3 (Power Series).

(a) [5 pts] Let $\sum_{n=0}^{\infty} a_n x^n$ be a power series. Give the definition of the *radius of convergence* of $\sum_{n=0}^{\infty} a_n x^n$.

If $\sum_{n=0}^{\infty} a_n x^n$ converges $\forall\, |x| < R$ and diverges $\forall\, |x| > R$, then R is called the radius of convergence of $\sum_{n=0}^{\infty} a_n x^n$.

Consider the power series

$$\sum_{n=0}^{\infty} \frac{x^n}{n+2}. \tag{2}$$

(b) [7 pts] Find the radius of convergence of (2).

For this power series, $a_n = \frac{1}{n+2}$ and

$$\left|\frac{a_n}{a_{n+1}}\right| = \frac{n+3}{n+2} = \frac{1+\frac{3}{n}}{1+\frac{2}{n}} \to \frac{1+0}{1+0} = 1$$

as $n \to \infty$ $\boxed{4}$. By a theorem from the course $\boxed{1}$, the radius of convergence of (2) is $R = 1$ $\boxed{2}$.

(c) [1 pts] What is the open interval of convergence of (2)?

$(-1, 1)$

Let R be the radius of convergence of (2), that you calculated in part (b).

(d) [6 pts] If $x = R$, does (2) converge or diverge?

If $x = R = 1$, then (2) is

$$\sum_{n=0}^{\infty} \frac{1^n}{n+2} = \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \ldots = \sum_{n=2}^{\infty} \frac{1}{n}$$

which diverges because $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges.

(e) [6 pts] If $x = -R$, does (2) converge or diverge?

If $x = -R = -1$, then (2) is

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{n+2} = \sum_{n=2}^{\infty} \frac{(-1)^n}{n}$$

which converges by the Alternating Series Test.

(d) & (e): 2 pts for "converges/diverges". 4pts for proof.

Question 4 (Taylor Series)**.** Let $f(x) = \sin x$.

(a) [7 pts] Let $x \in \mathbb{R}$, $x \neq 0$ and let c be between 0 and x [so either $0 < c < x$, or $x < c < 0$]. Let

$$R_n = \frac{f^{(n)}(c)\ x^n}{n!}$$

where $f^{(n)} = \frac{d^n f}{dx^n}$.

Show that $R_n \to 0$ as $n \to \infty$.

Since $f^{(n)}(x) = \sin x$ or $\cos x$ or $-\sin x$ or $-\cos x$ for all n, it follows that $\left|f^{(n)}(x)\right| \leq 1$ $\forall x \in \mathbb{R}$ and $\forall n \in \mathbb{N}$ $\boxed{2}$.
Therefore

$$0 \leq |R_n| \leq \frac{|x|^2}{n!} \to 0$$

as $n \to \infty$ $\forall x \in \mathbb{R}$ $\boxed{3}$. It follows by the Sandwich Rule $\boxed{2}$ that $R_n \to 0$ as $n \to \infty$ $\forall x \in \mathbb{R}$.

(b) [18 pts] Calculate the Taylor Series for $f(x) = \sin x$, centred at 0.

Since

$$f^{(n)}(x) = \begin{cases} \sin x & n = 0, 4, 8, 12, \ldots \\ \cos x & n = 1, 5, 9, 13, \ldots \\ -\sin x & n = 2, 6, 10, 14, \ldots \\ -\cos x & n = 3, 7, 11, 15, \ldots \end{cases}$$

we have that

$$f^{(n)}(0) = \begin{cases} 0 & n \text{ is even} \\ 1 & n = 1, 5, 9, 13, \ldots \\ -1 & n = 3, 7, 11, 15, \ldots \quad \boxed{6}. \end{cases}$$

Therefore (by part (a)), it follows that

$$\begin{aligned} \sin x = f(x) &= f(0) + f'(0)x + \frac{f''(0)x^2}{2!} + \frac{f'''(0)x^3}{3!} + \frac{f^{(4)}(0)x^4}{4!} + \ldots \boxed{6} \\ &= 0 + x + \frac{0x^2}{2!} + \frac{(-1)x^3}{3!} + \frac{0x^4}{4!} + \frac{1x^5}{5!} + \ldots \\ &= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \ldots \boxed{6} \\ &= \sum_{k=0}^{\infty} \frac{(-1)^k x^{1+2k}}{(1+2k)!} \quad \boxed{\text{optional}}. \end{aligned}$$

Question 5 (Series)**.** Decide if each of the following series converges or diverges. Justify (explain) your answers.

(a) [9 pts] $\sum_{n=1}^{\infty} \frac{(2n)!}{(n!)^2}$.

(b) [8 pts] $\sum_{n=1}^{\infty} \frac{\sin n}{n^2}$.

(c) [8 pts] $\sum_{n=1}^{\infty} \frac{(-1)^n n^2}{n^3+1}$.

2 pts for "converges/diverges" without justification.
2 pts for saying which test is being used (as long as there is some proof given).
Remaining 4/5 pts for accuracy of proof.

[You may use any theorem/lemma/test/example/etc. from the course, but you must say which one you are using.]

(a) Let $a_n = \frac{(2n)!}{(n!)^2}$. Then

$$\frac{a_{n+1}}{a_n} = \frac{(2n+2)!}{((n+1)!)^2} \frac{(n!)^2}{(2n)!} = \frac{(2n+2)(2n+1)}{(n+1)(n+1)} = \frac{(2+\frac{2}{n})(2+\frac{1}{n})}{(1+\frac{1}{n})(1+\frac{1}{n})} \to \frac{(2+0)(2+0)}{(1+0)(1+0)} = 4 > 1$$

as $n \to \infty$. It follows that $\sum_{n=1}^{\infty} \frac{(2n)!}{(n!)^2}$ diverges by the Ratio Test.

(b) Since $0 \leq \left|\frac{\sin n}{n^2}\right| \leq \frac{1}{n^2}$ $\forall n$, and since $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges, it follows by the Comparison Test that $\sum_{n=1}^{\infty} \frac{\sin n}{n^2}$ converges.

(c) Since

$$a_n = \frac{n^2}{n^3+1} = \frac{1}{n+\frac{1}{n^2}} \to 0$$

as $n \to \infty$, since (a_n) is decreasing, and since $a_n > 0$ $\forall n$, it follows by the Alternating Series Test that $\sum_{n=1}^{\infty} \frac{(-1)^n n^2}{n^3+1}$ converges.