

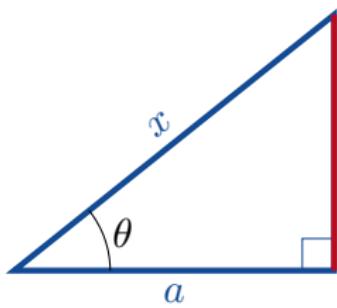
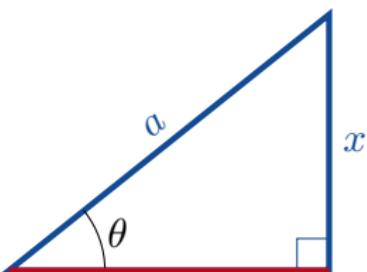
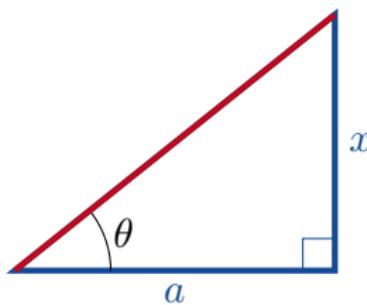
# Lecture 2

- 8.4 Trigonometric Substitutions
- 8.5 Integration of Rational Functions by Partial Fractions
- 8.8 Improper Integrals



# Trigonometric Substitutions

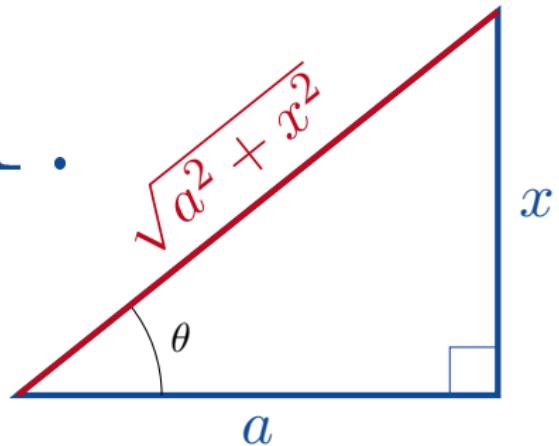
## 8.4 Trigonometric Substitutions



## 8.4 Trigonometric Substitutions



1.



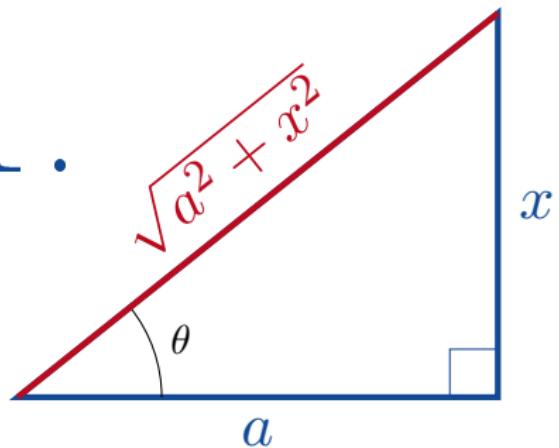
$$x = a \tan \theta$$

$$a^2 + x^2 = \quad = \quad = .$$

## 8.4 Trigonometric Substitutions



1.



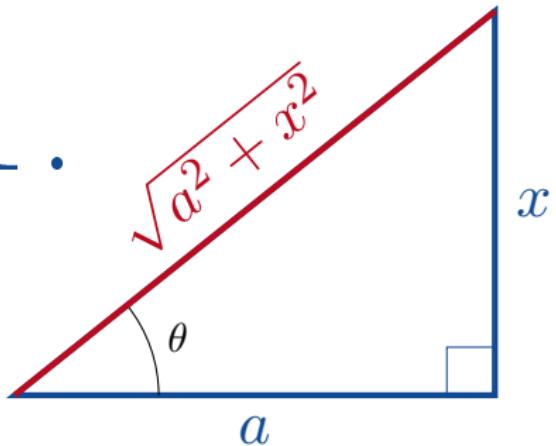
$$x = a \tan \theta$$

$$a^2 + x^2 = a^2 + a^2 \tan^2 \theta = \quad = \quad .$$

## 8.4 Trigonometric Substitutions



1.



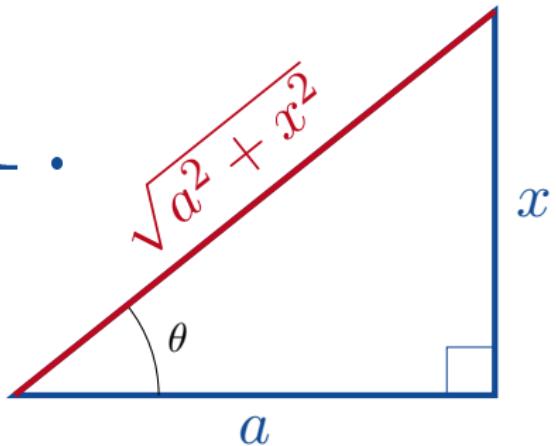
$$x = a \tan \theta$$

$$a^2 + x^2 = a^2 + a^2 \tan^2 \theta = a^2(1 + \tan^2 \theta) = .$$

## 8.4 Trigonometric Substitutions



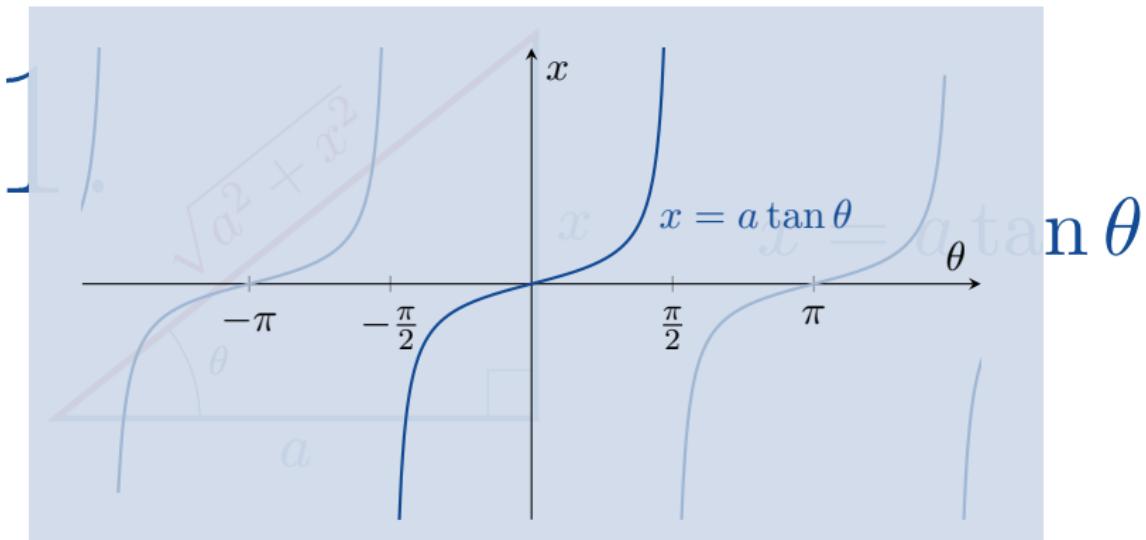
1.



$$x = a \tan \theta$$

$$a^2 + x^2 = a^2 + a^2 \tan^2 \theta = a^2(1 + \tan^2 \theta) = a^2 \sec^2 \theta.$$

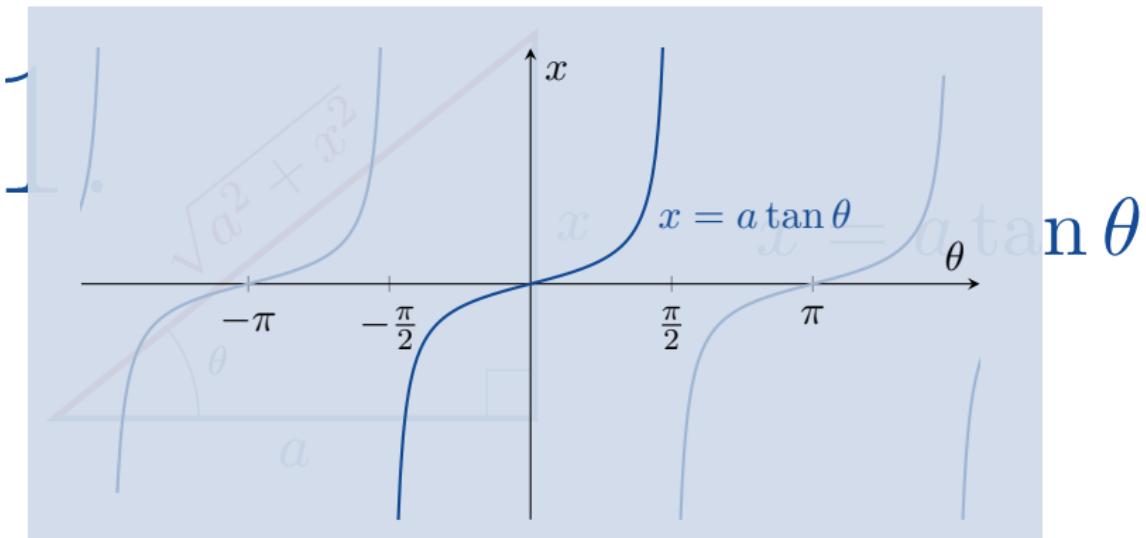
## 8.4 Trigonometric Substitutions



$$a^2 + x^2 = a^2 + a^2 \tan^2 \theta = a^2(1 + \tan^2 \theta) = a^2 \sec^2 \theta.$$



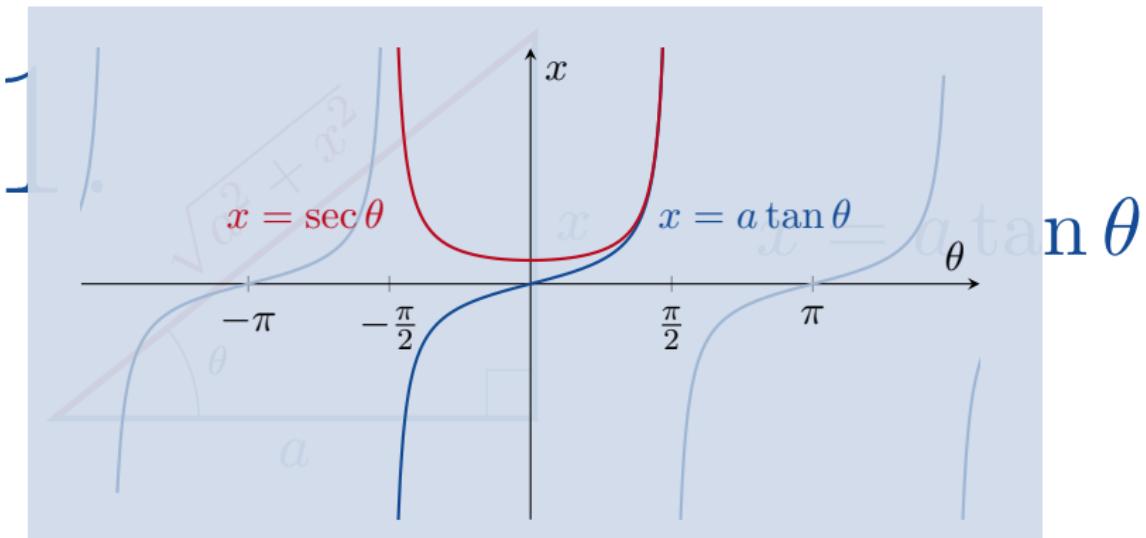
## 8.4 Trigonometric Substitutions



$$a^2 + x^2 = a^2 + a^2 \tan^2 \theta = a^2(1 + \tan^2 \theta) = a^2 \sec^2 \theta.$$

$$-\frac{\pi}{2} < \theta < \frac{\pi}{2}.$$

## 8.4 Trigonometric Substitutions



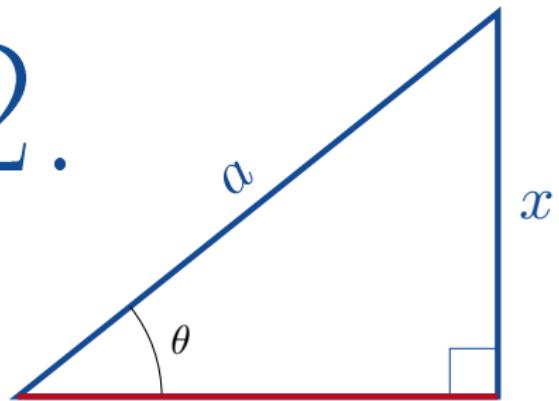
$$a^2 + x^2 = a^2 + a^2 \tan^2 \theta = a^2(1 + \tan^2 \theta) = a^2 \sec^2 \theta.$$

$$\boxed{\sqrt{a^2 + x^2} = a \sec \theta \quad -\frac{\pi}{2} < \theta < \frac{\pi}{2}.}$$

## 8.4 Trigonometric Substitutions



2.



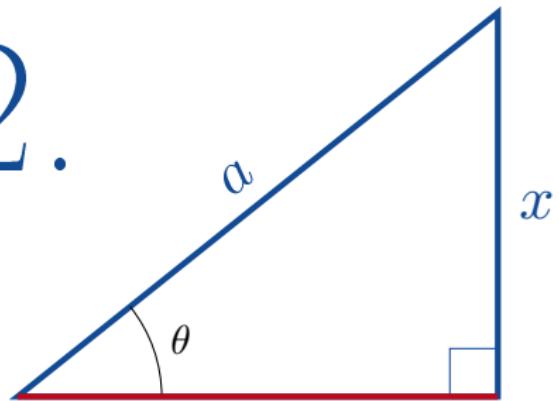
$$x = a \sin \theta$$

$$a^2 - x^2 = \quad = \quad .$$

## 8.4 Trigonometric Substitutions



2.



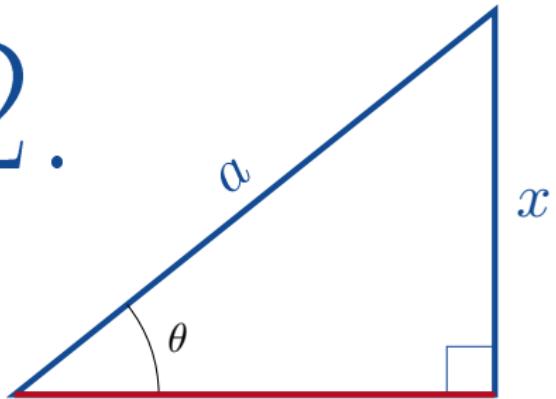
$$x = a \sin \theta$$

$$a^2 - x^2 = a^2 - a^2 \sin^2 \theta = \quad = \quad .$$

## 8.4 Trigonometric Substitutions



2.



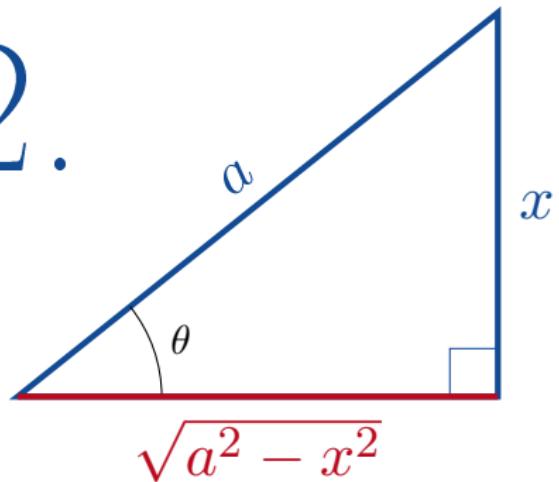
$$x = a \sin \theta$$

$$a^2 - x^2 = a^2 - a^2 \sin^2 \theta = a^2(1 - \sin^2 \theta) = .$$

## 8.4 Trigonometric Substitutions



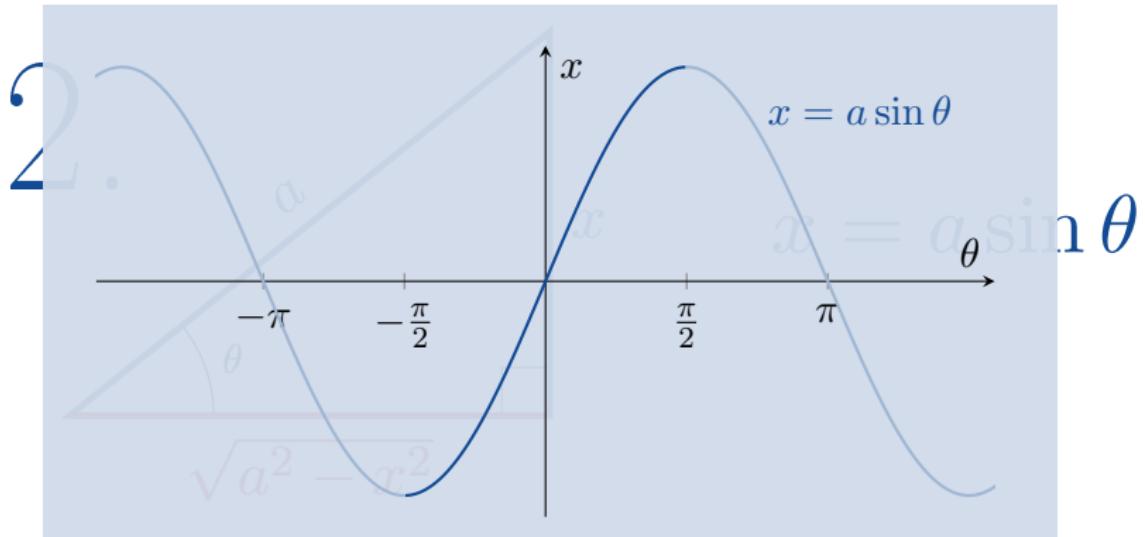
2.



$$x = a \sin \theta$$

$$a^2 - x^2 = a^2 - a^2 \sin^2 \theta = a^2(1 - \sin^2 \theta) = a^2 \cos^2 \theta.$$

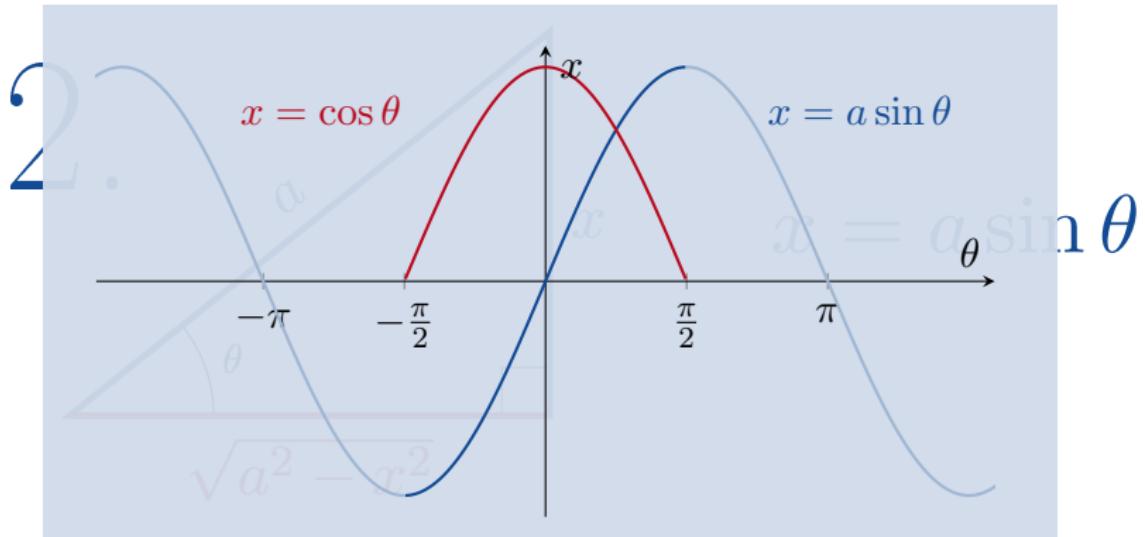
## 8.4 Trigonometric Substitutions



$$a^2 - x^2 = a^2 - a^2 \sin^2 \theta = a^2(1 - \sin^2 \theta) = a^2 \cos^2 \theta.$$

$$-\frac{\pi}{2} < \theta < \frac{\pi}{2}.$$

## 8.4 Trigonometric Substitutions



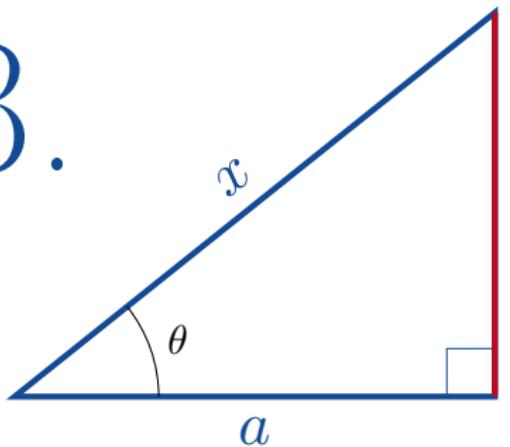
$$a^2 - x^2 = a^2 - a^2 \sin^2 \theta = a^2(1 - \sin^2 \theta) = a^2 \cos^2 \theta.$$

$$\boxed{\sqrt{a^2 - x^2} = a \cos \theta \quad -\frac{\pi}{2} < \theta < \frac{\pi}{2}.}$$

## 8.4 Trigonometric Substitutions



3.



$$\sqrt{x^2 - a^2}$$

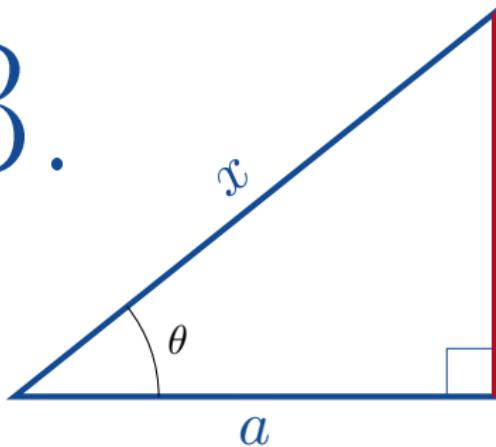
$$x = a \sec \theta$$

$$x^2 - a^2 = \quad = \quad .$$

## 8.4 Trigonometric Substitutions



3.



$$\sqrt{x^2 - a^2}$$

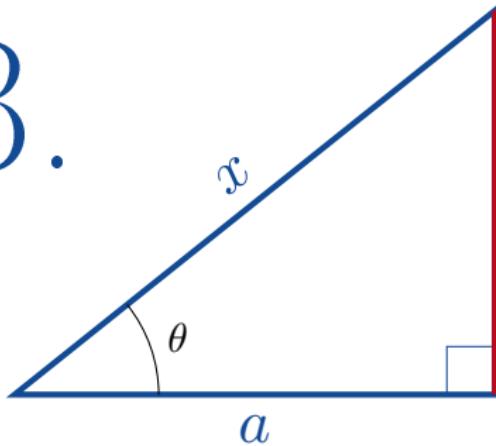
$$x = a \sec \theta$$

$$x^2 - a^2 = a^2 \sec^2 \theta - a^2 = \quad = \quad .$$

## 8.4 Trigonometric Substitutions



3.



$$\sqrt{x^2 - a^2}$$

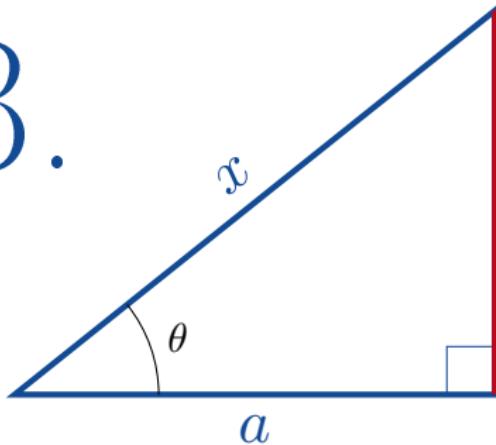
$$x = a \sec \theta$$

$$x^2 - a^2 = a^2 \sec^2 \theta - a^2 = a^2(\sec^2 \theta - 1) = .$$

## 8.4 Trigonometric Substitutions



3.

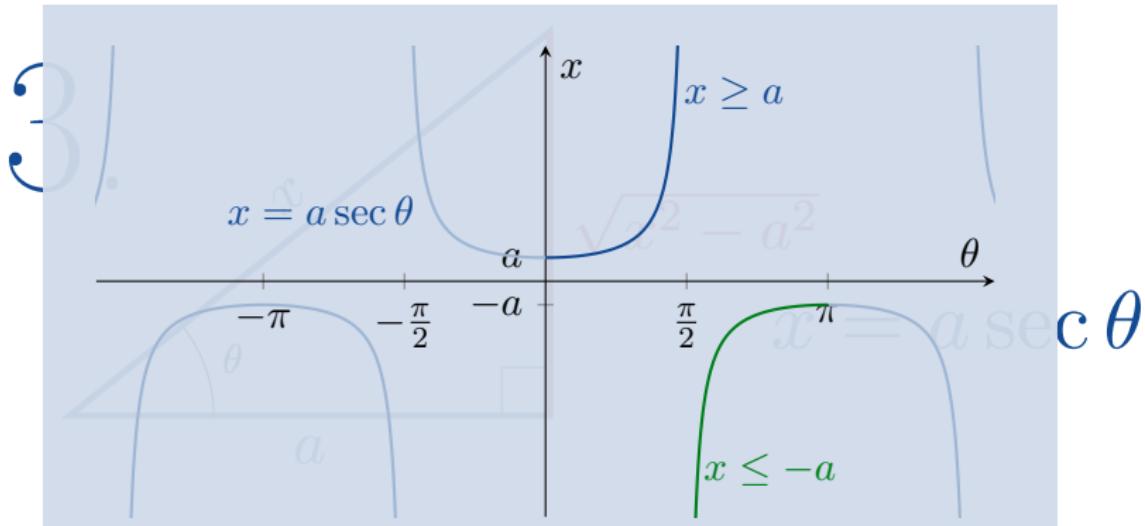


$$\sqrt{x^2 - a^2}$$

$$x = a \sec \theta$$

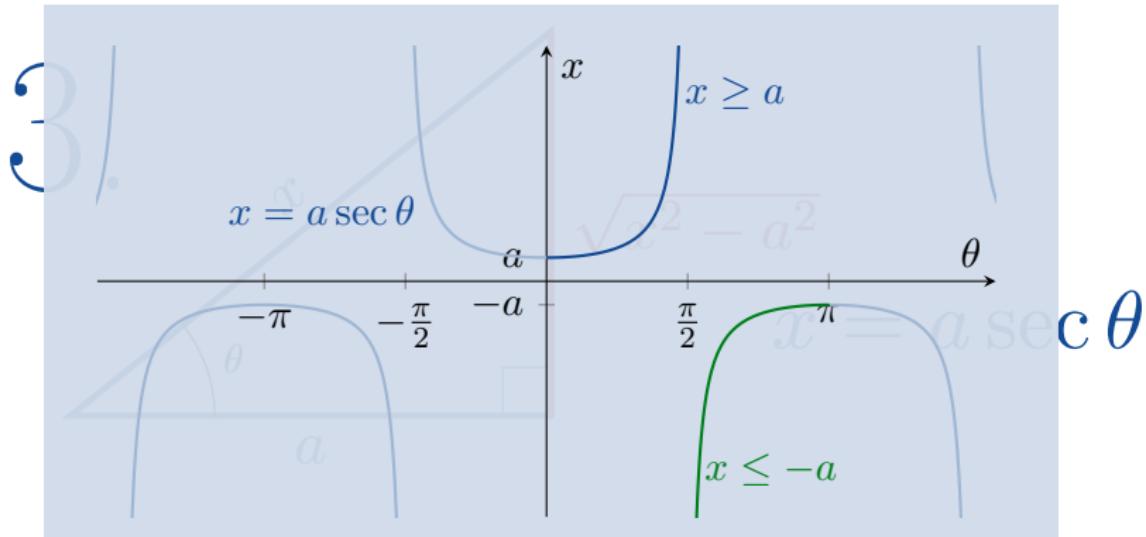
$$x^2 - a^2 = a^2 \sec^2 \theta - a^2 = a^2(\sec^2 \theta - 1) = a^2 \tan^2 \theta.$$

## 8.4 Trigonometric Substitutions



$$x^2 - a^2 = a^2 \sec^2 \theta - a^2 = a^2 (\sec^2 \theta - 1) = a^2 \tan^2 \theta.$$

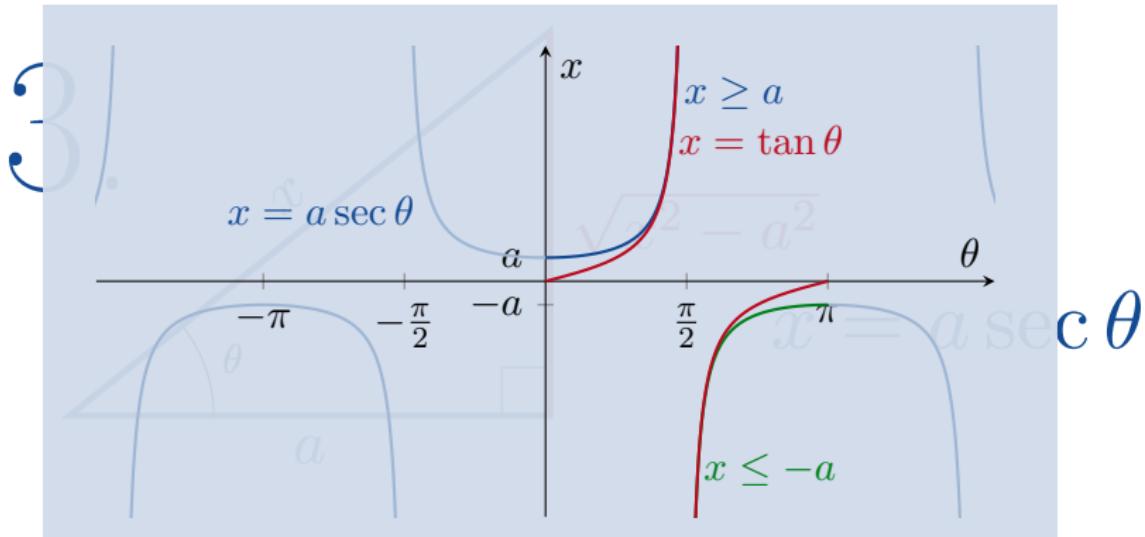
## 8.4 Trigonometric Substitutions



$$x^2 - a^2 = a^2 \sec^2 \theta - a^2 = a^2(\sec^2 \theta - 1) = a^2 \tan^2 \theta.$$

$$\begin{cases} 0 \leq \theta < \frac{\pi}{2} & x \geq a \\ \frac{\pi}{2} < \theta \leq \pi & x \leq -a \end{cases}.$$

## 8.4 Trigonometric Substitutions



$$x^2 - a^2 = a^2 \sec^2 \theta - a^2 = a^2 (\sec^2 \theta - 1) = a^2 \tan^2 \theta.$$

$$\sqrt{x^2 - a^2} = a |\tan x| \quad \begin{cases} 0 \leq \theta < \frac{\pi}{2} & x \geq a \\ \frac{\pi}{2} < \theta \leq \pi & x \leq -a \end{cases}$$

## 8.4 Tri

$$\begin{array}{lll} x = a \tan \theta & x = a \sin \theta & x = a \sec \theta \\ \sqrt{a^2 + x^2} = a \sec \theta & \sqrt{a^2 - x^2} = a \cos \theta & \sqrt{x^2 - a^2} = a |\tan \theta| \\ -\frac{\pi}{2} < \theta < \frac{\pi}{2} & -\frac{\pi}{2} < \theta < \frac{\pi}{2} & \begin{cases} 0 \leq \theta < \frac{\pi}{2} & x \geq a \\ \frac{\pi}{2} < \theta \leq \pi & x \leq -a \end{cases} \end{array}$$



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### Example

Find  $\int \frac{dx}{\sqrt{4+x^2}}$ .

Let  $x =$  .

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### Example

Find  $\int \frac{dx}{\sqrt{4+x^2}}$ .

Let  $x = 2 \tan \theta$ .

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### Example

Find  $\int \frac{dx}{\sqrt{4+x^2}}$ .

Let  $x = 2 \tan \theta$ . Then  $dx = 2 \sec^2 \theta d\theta$  and  $\sqrt{4+x^2} = 2 \sec \theta$ .

## 8.4 Tri

$$\begin{array}{lll} x = a \tan \theta & x = a \sin \theta & x = a \sec \theta \\ \sqrt{a^2 + x^2} = a \sec \theta & \sqrt{a^2 - x^2} = a \cos \theta & \sqrt{x^2 - a^2} = a |\tan \theta| \\ -\frac{\pi}{2} < \theta < \frac{\pi}{2} & -\frac{\pi}{2} < \theta < \frac{\pi}{2} & \begin{cases} 0 \leq \theta < \frac{\pi}{2} & x \geq a \\ \frac{\pi}{2} < \theta \leq \pi & x \leq -a \end{cases} \end{array}$$



## Example

Find  $\int \frac{dx}{\sqrt{4+x^2}}$ .

Let  $x = 2 \tan \theta$ . Then  $dx = 2 \sec^2 \theta d\theta$  and  $\sqrt{4+x^2} = 2 \sec \theta$ .

Therefore

$$\int \frac{dx}{\sqrt{4+x^2}} = \int \frac{2 \sec^2 \theta d\theta}{2 \sec \theta}$$

=

=

=

.

## 8.4 Tri

$x = a \tan \theta$	$x = a \sin \theta$	$x = a \sec \theta$
$\sqrt{a^2 + x^2} = a \sec \theta$	$\sqrt{a^2 - x^2} = a \cos \theta$	$\sqrt{x^2 - a^2} = a  \tan \theta $
$-\frac{\pi}{2} < \theta < \frac{\pi}{2}$	$-\frac{\pi}{2} < \theta < \frac{\pi}{2}$	$\begin{cases} 0 \leq \theta < \frac{\pi}{2} & x \geq a \\ \frac{\pi}{2} < \theta \leq \pi & x \leq -a \end{cases}$



## Example

Find  $\int \frac{dx}{\sqrt{4+x^2}}$ .

Let  $x = 2 \tan \theta$ . Then  $dx = 2 \sec^2 \theta d\theta$  and  $\sqrt{4+x^2} = 2 \sec \theta$ .

Therefore

$$\int \frac{dx}{\sqrt{4+x^2}} = \int \frac{2 \sec^2 \theta d\theta}{2 \sec \theta}$$

$$= \int \sec \theta d\theta$$

=

=

.

## 8.4 Tri

$x = a \tan \theta$	$x = a \sin \theta$	$x = a \sec \theta$
$\sqrt{a^2 + x^2} = a \sec \theta$	$\sqrt{a^2 - x^2} = a \cos \theta$	$\sqrt{x^2 - a^2} = a  \tan \theta $
$-\frac{\pi}{2} < \theta < \frac{\pi}{2}$	$-\frac{\pi}{2} < \theta < \frac{\pi}{2}$	$\begin{cases} 0 \leq \theta < \frac{\pi}{2} & x \geq a \\ \frac{\pi}{2} < \theta \leq \pi & x \leq -a \end{cases}$



## Example

Find  $\int \frac{dx}{\sqrt{4+x^2}}$ .

Let  $x = 2 \tan \theta$ . Then  $dx = 2 \sec^2 \theta d\theta$  and  $\sqrt{4+x^2} = 2 \sec \theta$ .  
 Therefore

$$\begin{aligned} \int \frac{dx}{\sqrt{4+x^2}} &= \int \frac{2 \sec^2 \theta d\theta}{2 \sec \theta} \\ &= \int \sec \theta d\theta \\ &= \ln |\sec \theta + \tan \theta| + C \end{aligned}$$

=

.

## 8.4 Tri

$x = a \tan \theta$	$x = a \sin \theta$	$x = a \sec \theta$
$\sqrt{a^2 + x^2} = a \sec \theta$	$\sqrt{a^2 - x^2} = a \cos \theta$	$\sqrt{x^2 - a^2} = a  \tan \theta $
$-\frac{\pi}{2} < \theta < \frac{\pi}{2}$	$-\frac{\pi}{2} < \theta < \frac{\pi}{2}$	$\begin{cases} 0 \leq \theta < \frac{\pi}{2} & x \geq a \\ \frac{\pi}{2} < \theta \leq \pi & x \leq -a \end{cases}$



## Example

Find  $\int \frac{dx}{\sqrt{4+x^2}}$ .

Let  $x = 2 \tan \theta$ . Then  $dx = 2 \sec^2 \theta d\theta$  and  $\sqrt{4+x^2} = 2 \sec \theta$ .

Therefore

$$\begin{aligned}
 \int \frac{dx}{\sqrt{4+x^2}} &= \int \frac{2 \sec^2 \theta d\theta}{2 \sec \theta} \\
 &= \int \sec \theta d\theta \\
 &= \ln |\sec \theta + \tan \theta| + C \\
 &= \ln \left| \frac{\sqrt{4+x^2}}{2} + \frac{x}{2} \right| + C.
 \end{aligned}$$

## 8.4 Tri

$$\begin{array}{lll} x = a \tan \theta & x = a \sin \theta & x = a \sec \theta \\ \sqrt{a^2 + x^2} = a \sec \theta & \sqrt{a^2 - x^2} = a \cos \theta & \sqrt{x^2 - a^2} = a |\tan \theta| \\ -\frac{\pi}{2} < \theta < \frac{\pi}{2} & -\frac{\pi}{2} < \theta < \frac{\pi}{2} & \begin{cases} 0 \leq \theta < \frac{\pi}{2} & x \geq a \\ \frac{\pi}{2} < \theta \leq \pi & x \leq -a \end{cases} \end{array}$$



Example

Calculate  $\int_0^{\frac{3}{2}} \frac{dx}{\sqrt{9 - x^2}}$ .

Let  $x =$  .

## 8.4 Tri

$$\begin{array}{lll} x = a \tan \theta & x = a \sin \theta & x = a \sec \theta \\ \sqrt{a^2 + x^2} = a \sec \theta & \sqrt{a^2 - x^2} = a \cos \theta & \sqrt{x^2 - a^2} = a |\tan \theta| \\ -\frac{\pi}{2} < \theta < \frac{\pi}{2} & -\frac{\pi}{2} < \theta < \frac{\pi}{2} & \begin{cases} 0 \leq \theta < \frac{\pi}{2} & x \geq a \\ \frac{\pi}{2} < \theta \leq \pi & x \leq -a \end{cases} \end{array}$$



Example

Calculate  $\int_0^{\frac{3}{2}} \frac{dx}{\sqrt{9 - x^2}}$ .

Let  $x = 3 \sin \theta$ .

## 8.4 Tri

$$\begin{array}{lll} x = a \tan \theta & x = a \sin \theta & x = a \sec \theta \\ \sqrt{a^2 + x^2} = a \sec \theta & \sqrt{a^2 - x^2} = a \cos \theta & \sqrt{x^2 - a^2} = a |\tan \theta| \\ -\frac{\pi}{2} < \theta < \frac{\pi}{2} & -\frac{\pi}{2} < \theta < \frac{\pi}{2} & \begin{cases} 0 \leq \theta < \frac{\pi}{2} & x \geq a \\ \frac{\pi}{2} < \theta \leq \pi & x \leq -a \end{cases} \end{array}$$



### Example

Calculate  $\int_0^{\frac{3}{2}} \frac{dx}{\sqrt{9 - x^2}}$ .

Let  $x = 3 \sin \theta$ . Then  $dx = 3 \cos \theta d\theta$  and  $\sqrt{9 - x^2} = 3 \cos \theta$ .

## 8.4 Tri

$$\begin{array}{lll} x = a \tan \theta & x = a \sin \theta & x = a \sec \theta \\ \sqrt{a^2 + x^2} = a \sec \theta & \sqrt{a^2 - x^2} = a \cos \theta & \sqrt{x^2 - a^2} = a |\tan \theta| \\ -\frac{\pi}{2} < \theta < \frac{\pi}{2} & -\frac{\pi}{2} < \theta < \frac{\pi}{2} & \begin{cases} 0 \leq \theta < \frac{\pi}{2} & x \geq a \\ \frac{\pi}{2} < \theta \leq \pi & x \leq -a \end{cases} \end{array}$$



Example

Calculate  $\int_0^{\frac{3}{2}} \frac{dx}{\sqrt{9 - x^2}}$ .

Let  $x = 3 \sin \theta$ . Then  $dx = 3 \cos \theta d\theta$  and  $\sqrt{9 - x^2} = 3 \cos \theta$ .

Moreover

$$x = \frac{3}{2} \implies \theta = \sin^{-1} \frac{x}{3} = \sin^{-1} \frac{1}{2} = \frac{\pi}{6}$$

$$x = 0 \implies \theta = \sin^{-1} \frac{x}{3} = \sin^{-1} 0 = 0$$

## 8.4 Tri

$$\begin{array}{lll} x = a \tan \theta & x = a \sin \theta & x = a \sec \theta \\ \sqrt{a^2 + x^2} = a \sec \theta & \sqrt{a^2 - x^2} = a \cos \theta & \sqrt{x^2 - a^2} = a |\tan \theta| \\ -\frac{\pi}{2} < \theta < \frac{\pi}{2} & -\frac{\pi}{2} < \theta < \frac{\pi}{2} & \begin{cases} 0 \leq \theta < \frac{\pi}{2} & x \geq a \\ \frac{\pi}{2} < \theta \leq \pi & x \leq -a \end{cases} \end{array}$$



Example

Calculate  $\int_0^{\frac{3}{2}} \frac{dx}{\sqrt{9 - x^2}}$ .

Let  $x = 3 \sin \theta$ . Then  $dx = 3 \cos \theta d\theta$  and  $\sqrt{9 - x^2} = 3 \cos \theta$ .

Moreover

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$$x = 0 \implies \theta = \sin^{-1} \frac{x}{3} = \sin^{-1} 0 = 0$$

Therefore

$$\int_0^{\frac{3}{2}} \frac{dx}{\sqrt{9 - x^2}} = \int_0^{\frac{\pi}{6}} \frac{3 \cos \theta d\theta}{3 \cos \theta} = \int_0^{\frac{\pi}{6}} d\theta = \frac{\pi}{6}.$$

## 8.4 Tri

$$\begin{array}{lll} x = a \tan \theta & x = a \sin \theta & x = a \sec \theta \\ \sqrt{a^2 + x^2} = a \sec \theta & \sqrt{a^2 - x^2} = a \cos \theta & \sqrt{x^2 - a^2} = a |\tan \theta| \\ -\frac{\pi}{2} < \theta < \frac{\pi}{2} & -\frac{\pi}{2} < \theta < \frac{\pi}{2} & \begin{cases} 0 \leq \theta < \frac{\pi}{2} & x \geq a \\ \frac{\pi}{2} < \theta \leq \pi & x \leq -a \end{cases} \end{array}$$



Example

Calculate  $\int_0^{\frac{3}{2}} \frac{dx}{\sqrt{9 - x^2}}$ .

Let  $x = 3 \sin \theta$ . Then  $dx = 3 \cos \theta d\theta$  and  $\sqrt{9 - x^2} = 3 \cos \theta$ .

Moreover

$$x = \frac{3}{2} \implies \theta = \sin^{-1} \frac{x}{3} = \sin^{-1} \frac{1}{2} = \frac{\pi}{6}$$

$$x = 0 \implies \theta = \sin^{-1} \frac{x}{3} = \sin^{-1} 0 = 0$$

Therefore

$$\int_0^{\frac{3}{2}} \frac{dx}{\sqrt{9 - x^2}} = \int_0^{\frac{\pi}{6}} \frac{3 \cos \theta d\theta}{3 \cos \theta} = \int_0^{\frac{\pi}{6}} d\theta = \frac{\pi}{6}.$$

Or

$$\int_0^{\frac{3}{2}} \frac{dx}{\sqrt{9 - x^2}} = \left[ \sin^{-1} \frac{x}{3} \right]_0^{\frac{3}{2}} = \frac{\pi}{6} - 0 = \frac{\pi}{6}.$$

## 8.4 Tri

$$\begin{array}{lll} x = a \tan \theta & x = a \sin \theta & x = a \sec \theta \\ \sqrt{a^2 + x^2} = a \sec \theta & \sqrt{a^2 - x^2} = a \cos \theta & \sqrt{x^2 - a^2} = a |\tan \theta| \\ -\frac{\pi}{2} < \theta < \frac{\pi}{2} & -\frac{\pi}{2} < \theta < \frac{\pi}{2} & \begin{cases} 0 \leq \theta < \frac{\pi}{2} & x \geq a \\ \frac{\pi}{2} < \theta \leq \pi & x \leq -a \end{cases} \end{array}$$



Example

Calculate  $\int \frac{x^2 dx}{\sqrt{9 - x^2}}$ .

Let  $x = \dots$ .

## 8.4 Tri

$$\begin{array}{lll} x = a \tan \theta & x = a \sin \theta & x = a \sec \theta \\ \sqrt{a^2 + x^2} = a \sec \theta & \sqrt{a^2 - x^2} = a \cos \theta & \sqrt{x^2 - a^2} = a |\tan \theta| \\ -\frac{\pi}{2} < \theta < \frac{\pi}{2} & -\frac{\pi}{2} < \theta < \frac{\pi}{2} & \begin{cases} 0 \leq \theta < \frac{\pi}{2} & x \geq a \\ \frac{\pi}{2} < \theta \leq \pi & x \leq -a \end{cases} \end{array}$$



### Example

Calculate  $\int \frac{x^2 dx}{\sqrt{9 - x^2}}$ .

Let  $x = 3 \sin \theta$ .

## 8.4 Tri

$$\begin{array}{lll} x = a \tan \theta & x = a \sin \theta & x = a \sec \theta \\ \sqrt{a^2 + x^2} = a \sec \theta & \sqrt{a^2 - x^2} = a \cos \theta & \sqrt{x^2 - a^2} = a |\tan \theta| \\ -\frac{\pi}{2} < \theta < \frac{\pi}{2} & -\frac{\pi}{2} < \theta < \frac{\pi}{2} & \begin{cases} 0 \leq \theta < \frac{\pi}{2} & x \geq a \\ \frac{\pi}{2} < \theta \leq \pi & x \leq -a \end{cases} \end{array}$$



### Example

Calculate  $\int \frac{x^2 dx}{\sqrt{9-x^2}}$ .

Let  $x = 3 \sin \theta$ . Then  $dx = 3 \cos \theta d\theta$  and  $\sqrt{9-x^2} = 3 \cos \theta$ .

## 8.4 Tri

$$\begin{array}{lll} x = a \tan \theta & x = a \sin \theta & x = a \sec \theta \\ \sqrt{a^2 + x^2} = a \sec \theta & \sqrt{a^2 - x^2} = a \cos \theta & \sqrt{x^2 - a^2} = a |\tan \theta| \\ -\frac{\pi}{2} < \theta < \frac{\pi}{2} & -\frac{\pi}{2} < \theta < \frac{\pi}{2} & \begin{cases} 0 \leq \theta < \frac{\pi}{2} & x \geq a \\ \frac{\pi}{2} < \theta \leq \pi & x \leq -a \end{cases} \end{array}$$



### Example

Calculate  $\int \frac{x^2 dx}{\sqrt{9-x^2}}$ .

Let  $x = 3 \sin \theta$ . Then  $dx = 3 \cos \theta d\theta$  and  $\sqrt{9-x^2} = 3 \cos \theta$ .  
Hence

$$\begin{aligned} \int \frac{x^2 dx}{\sqrt{9-x^2}} &= \int \frac{9 \sin^2 \theta \cdot 3 \cos \theta d\theta}{3 \cos \theta} \\ &= 9 \int \sin^2 \theta d\theta = \dots \end{aligned}$$

## 8.4 Tri

$$\begin{array}{lll} x = a \tan \theta & x = a \sin \theta & x = a \sec \theta \\ \sqrt{a^2 + x^2} = a \sec \theta & \sqrt{a^2 - x^2} = a \cos \theta & \sqrt{x^2 - a^2} = a |\tan \theta| \\ -\frac{\pi}{2} < \theta < \frac{\pi}{2} & -\frac{\pi}{2} < \theta < \frac{\pi}{2} & \begin{cases} 0 \leq \theta < \frac{\pi}{2} & x \geq a \\ \frac{\pi}{2} < \theta \leq \pi & x \leq -a \end{cases} \end{array}$$



### Example

Use  $\int \frac{dx}{\sqrt{a^2 + x^2}}$  to find an expression for  $\sinh^{-1} \frac{x}{a}$  in terms of the natural logarithm.

## 8.4 Tri

$$\begin{array}{lll} x = a \tan \theta & x = a \sin \theta & x = a \sec \theta \\ \sqrt{a^2 + x^2} = a \sec \theta & \sqrt{a^2 - x^2} = a \cos \theta & \sqrt{x^2 - a^2} = a |\tan \theta| \\ -\frac{\pi}{2} < \theta < \frac{\pi}{2} & -\frac{\pi}{2} < \theta < \frac{\pi}{2} & \begin{cases} 0 \leq \theta < \frac{\pi}{2} & x \geq a \\ \frac{\pi}{2} < \theta \leq \pi & x \leq -a \end{cases} \end{array}$$



## Example

Use  $\int \frac{dx}{\sqrt{a^2 + x^2}}$  to find an expression for  $\sinh^{-1} \frac{x}{a}$  in terms of the natural logarithm.

This time we use the substitution  $x = a \tan \theta$  with  $dx = a \sec^2 \theta d\theta$  and  $\sqrt{a^2 + x^2} = a \sec \theta$  to calculate

$$\begin{aligned} \int \frac{dx}{\sqrt{a^2 + x^2}} &= \int \frac{a \sec^2 \theta d\theta}{a \sec \theta} = \int \sec \theta d\theta \\ &= \ln |\sec \theta + \tan \theta| + C \\ &= \ln \left| \frac{\sqrt{a^2 + x^2}}{a} + \frac{x}{a} \right| + C. \end{aligned}$$

## 8.4 Tri

$$\begin{array}{lll} x = a \tan \theta & x = a \sin \theta & x = a \sec \theta \\ \sqrt{a^2 + x^2} = a \sec \theta & \sqrt{a^2 - x^2} = a \cos \theta & \sqrt{x^2 - a^2} = a |\tan \theta| \\ -\frac{\pi}{2} < \theta < \frac{\pi}{2} & -\frac{\pi}{2} < \theta < \frac{\pi}{2} & \begin{cases} 0 \leq \theta < \frac{\pi}{2} & x \geq a \\ \frac{\pi}{2} < \theta \leq \pi & x \leq -a \end{cases} \end{array}$$



## Example

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$$\begin{aligned} \int \frac{dx}{\sqrt{a^2 + x^2}} &= \int \frac{a \sec^2 \theta d\theta}{a \sec \theta} = \int \sec \theta d\theta \\ &= \ln |\sec \theta + \tan \theta| + C \\ &= \ln \left| \frac{\sqrt{a^2 + x^2}}{a} + \frac{x}{a} \right| + C. \end{aligned}$$

But we know from MATH113 that  $\sinh^{-1} \frac{x}{a}$  is also an antiderivative of  $\frac{1}{\sqrt{a^2+x^2}}$ .

## 8.4 Trigonometric Substitutions



Recall that if  $F(x)$  and  $G(x)$  are both antiderivatives of  $f(x)$  then we must have

$$F(x) = G(x) + C.$$

## 8.4 Trigonometric Substitutions



Recall that if  $F(x)$  and  $G(x)$  are both antiderivatives of  $f(x)$  then we must have

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Hence

$$\sinh^{-1} \frac{x}{a} = \ln \left| \frac{\sqrt{a^2 + x^2}}{a} + \frac{x}{a} \right| + C.$$

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Recall that if  $F(x)$  and  $G(x)$  are both antiderivatives of  $f(x)$  then we must have

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Hence

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What is  $C$ ?

## 8.4 Trigonometric Substitutions



Recall that if  $F(x)$  and  $G(x)$  are both antiderivatives of  $f(x)$  then we must have

$$F(x) = G(x) + C.$$

Hence

$$\sinh^{-1} \frac{x}{a} = \ln \left| \frac{\sqrt{a^2 + x^2}}{a} + \frac{x}{a} \right| + C.$$

What is  $C$ ? If we put  $x = 0$  then we get

$$0 = \sinh^{-1} 0 = \ln \left| \frac{\sqrt{a^2 + 0^2}}{a} + 0 \right| + C = \ln |1| + C = 0 + C.$$

Hence  $C = 0$ .

## 8.4 Trigonometric Substitutions



$$\sinh^{-1} \frac{x}{a} = \ln \left| \frac{\sqrt{a^2 + x^2}}{a} + \frac{x}{a} \right| + C$$

Since  $\sqrt{a^2 + x^2} > |x|$ , we conclude that

$$\boxed{\sinh^{-1} \frac{x}{a} = \ln \left( \frac{\sqrt{a^2 + x^2}}{a} + \frac{x}{a} \right)}.$$

## 8.4 Tri

$$\begin{array}{lll} x = a \tan \theta & x = a \sin \theta & x = a \sec \theta \\ \sqrt{a^2 + x^2} = a \sec \theta & \sqrt{a^2 - x^2} = a \cos \theta & \sqrt{x^2 - a^2} = a |\tan \theta| \\ -\frac{\pi}{2} < \theta < \frac{\pi}{2} & -\frac{\pi}{2} < \theta < \frac{\pi}{2} & \begin{cases} 0 \leq \theta < \frac{\pi}{2} & x \geq a \\ \frac{\pi}{2} < \theta \leq \pi & x \leq -a \end{cases} \end{array}$$



### Example

Calculate  $\int \frac{dx}{\sqrt{25x^2 - 4}}$  for  $x > \frac{2}{5}$ .

## 8.4 Trigonometric Substitutions

$x = a \tan \theta$ $\sqrt{a^2 + x^2} = a \sec \theta$ $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$	$x = a \sin \theta$ $\sqrt{a^2 - x^2} = a \cos \theta$ $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$	$x = a \sec \theta$ $\sqrt{x^2 - a^2} = a  \tan \theta $ $\begin{cases} 0 \leq \theta < \frac{\pi}{2} & x \geq a \\ \frac{\pi}{2} < \theta \leq \pi & x \leq -a \end{cases}$
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### Example

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First note that

$$\sqrt{25x^2 - 4} = \sqrt{25 \left( x^2 - \frac{4}{25} \right)} = 5 \sqrt{x^2 - \left( \frac{2}{5} \right)^2}.$$

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So we want to integrate  $\frac{1}{5} \int \frac{dx}{\sqrt{x^2 - a^2}}$  with  $a = \frac{2}{5}$ .

## 8.4 Tri

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So we want to integrate  $\frac{1}{5} \int \frac{dx}{\sqrt{x^2 - a^2}}$  with  $a = \frac{2}{5}$ .

Let  $x = \frac{2}{5} \sec \theta$ .

## 8.4 Tri

$$\begin{array}{lll} x = a \tan \theta & x = a \sin \theta & x = a \sec \theta \\ \sqrt{a^2 + x^2} = a \sec \theta & \sqrt{a^2 - x^2} = a \cos \theta & \sqrt{x^2 - a^2} = a |\tan \theta| \\ -\frac{\pi}{2} < \theta < \frac{\pi}{2} & -\frac{\pi}{2} < \theta < \frac{\pi}{2} & \begin{cases} 0 \leq \theta < \frac{\pi}{2} & x \geq a \\ \frac{\pi}{2} < \theta \leq \pi & x \leq -a \end{cases} \end{array}$$



$$\int \frac{dx}{\sqrt{25x^2 - 4}} = \frac{1}{5} \int \frac{dx}{\sqrt{x^2 - \left(\frac{2}{5}\right)^2}}$$

Let  $x = \frac{2}{5} \sec \theta$ .

## 8.4 Tri

$x = a \tan \theta$ $\sqrt{a^2 + x^2} = a \sec \theta$ $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$	$x = a \sin \theta$ $\sqrt{a^2 - x^2} = a \cos \theta$ $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$	$x = a \sec \theta$ $\sqrt{x^2 - a^2} = a  \tan \theta $ $\begin{cases} 0 \leq \theta < \frac{\pi}{2} & x \geq a \\ \frac{\pi}{2} < \theta \leq \pi & x \leq -a \end{cases}$
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Let  $x = \frac{2}{5} \sec \theta$ . Then

$$\frac{1}{5} \int \frac{dx}{\sqrt{x^2 - \left(\frac{2}{5}\right)^2}} = \frac{1}{5} \int \frac{\frac{2}{5} \sec \theta \tan \theta d\theta}{\frac{2}{5} |\tan \theta|}.$$

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$x = a \tan \theta$	$x = a \sin \theta$	$x = a \sec \theta$
$\sqrt{a^2 + x^2} = a \sec \theta$	$\sqrt{a^2 - x^2} = a \cos \theta$	$\sqrt{x^2 - a^2} = a  \tan \theta $
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But

$$x > \frac{2}{5} \implies 0 \leq \theta < \frac{\pi}{2}$$

## 8.4 Trigonometric Substitutions

$x = a \tan \theta$	$x = a \sin \theta$	$x = a \sec \theta$
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$$\int \frac{dx}{\sqrt{25x^2 - 4}} = \frac{1}{5} \int \frac{dx}{\sqrt{x^2 - (\frac{2}{5})^2}}$$

Let  $x = \frac{2}{5} \sec \theta$ . Then

$$\frac{1}{5} \int \frac{dx}{\sqrt{x^2 - (\frac{2}{5})^2}} = \frac{1}{5} \int \frac{\frac{2}{5} \sec \theta \tan \theta d\theta}{\frac{2}{5} |\tan \theta|}.$$

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$$x > \frac{2}{5} \implies 0 \leq \theta < \frac{\pi}{2} \implies \tan \theta > 0.$$

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$$\int \frac{dx}{\sqrt{25x^2 - 4}} = \frac{1}{5} \int \frac{dx}{\sqrt{x^2 - \left(\frac{2}{5}\right)^2}}$$

Let  $x = \frac{2}{5} \sec \theta$ . Then

$$\frac{1}{5} \int \frac{dx}{\sqrt{x^2 - \left(\frac{2}{5}\right)^2}} = \frac{1}{5} \int \frac{\frac{2}{5} \sec \theta \tan \theta d\theta}{\frac{2}{5} |\tan \theta|}.$$

But

$$x > \frac{2}{5} \implies 0 \leq \theta < \frac{\pi}{2} \implies \tan \theta > 0.$$

Hence  $|\tan \theta| = \tan \theta$ .

## 8.4 Tri

$x = a \tan \theta$	$x = a \sin \theta$	$x = a \sec \theta$
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Thus

$$\frac{1}{5} \int \frac{\frac{2}{5} \sec \theta \tan \theta d\theta}{\frac{2}{5} |\tan \theta|} = \frac{1}{5} \int \frac{\frac{2}{5} \sec \theta \tan \theta d\theta}{\frac{2}{5} \tan \theta}$$

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## 8.4 Tri

$$\begin{array}{lll}
 x = a \tan \theta & x = a \sin \theta & x = a \sec \theta \\
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 \end{array}$$



Thus

$$\begin{aligned}
 \frac{1}{5} \int \frac{\frac{2}{5} \sec \theta \tan \theta d\theta}{\frac{2}{5} |\tan \theta|} &= \frac{1}{5} \int \frac{\frac{2}{5} \sec \theta \tan \theta d\theta}{\frac{2}{5} \tan \theta} \\
 &= \frac{1}{5} \int \sec \theta d\theta
 \end{aligned}$$

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=

## 8.4 Tri

$x = a \tan \theta$	$x = a \sin \theta$	$x = a \sec \theta$
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Thus

$$\begin{aligned}
 \frac{1}{5} \int \frac{\frac{2}{5} \sec \theta \tan \theta d\theta}{\frac{2}{5} |\tan \theta|} &= \frac{1}{5} \int \frac{\frac{2}{5} \sec \theta \tan \theta d\theta}{\frac{2}{5} \tan \theta} \\
 &= \frac{1}{5} \int \sec \theta d\theta \\
 &= \frac{1}{5} \ln |\sec \theta + \tan \theta| + C
 \end{aligned}$$

=

## 8.4 Tri

$x = a \tan \theta$	$x = a \sin \theta$	$x = a \sec \theta$
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Thus

$$\begin{aligned}
 \frac{1}{5} \int \frac{\frac{2}{5} \sec \theta \tan \theta d\theta}{\frac{2}{5} |\tan \theta|} &= \frac{1}{5} \int \frac{\frac{2}{5} \sec \theta \tan \theta d\theta}{\frac{2}{5} \tan \theta} \\
 &= \frac{1}{5} \int \sec \theta d\theta \\
 &= \frac{1}{5} \ln |\sec \theta + \tan \theta| + C \\
 &= \frac{1}{5} \ln \left| \frac{5x}{2} + \frac{\sqrt{25x^2 - 4}}{2} \right| + C
 \end{aligned}$$

since we used  $x = \frac{2}{5} \sec \theta$  and  $\sqrt{25x^2 - 4} = 2 \tan \theta$ .



# Integration of Rational Functions by Partial Fractions

## 8.5 Integration of Rational Functions by Partial Fractions



$$\frac{2}{x+1} + \frac{3}{x-3} = \frac{\text{something?}}{\text{something?}}$$

## 8.5 Integration of Rational Functions by Partial Fractions



$$\frac{2}{x+1} + \frac{3}{x-3} = \frac{2}{(x+1)} + \frac{3}{(x-3)}$$

## 8.5 Integration of Rational Functions by Partial Fractions



$$\frac{2}{x+1} + \frac{3}{x-3} = \frac{2(x-3)}{(x+1)(x-3)} + \frac{3}{(x-3)}$$

## 8.5 Integration of Rational Functions by Partial Fractions



$$\frac{2}{x+1} + \frac{3}{x-3} = \frac{2(x-3)}{(x+1)(x-3)} + \frac{3(x+1)}{(x-3)(x+1)}$$

## 8.5 Integration of Rational Functions by Partial Fractions



$$\begin{aligned}\frac{2}{x+1} + \frac{3}{x-3} &= \frac{2(x-3)}{(x+1)(x-3)} + \frac{3(x+1)}{(x-3)(x+1)} \\ &= \frac{2x-6}{x^2-2x-3} + \frac{3x+3}{x^2-2x-3}\end{aligned}$$

## 8.5 Integration of Rational Functions by Partial Fractions



$$\begin{aligned}\frac{2}{x+1} + \frac{3}{x-3} &= \frac{2(x-3)}{(x+1)(x-3)} + \frac{3(x+1)}{(x-3)(x+1)} \\&= \frac{2x-6}{x^2-2x-3} + \frac{3x+3}{x^2-2x-3} \\&= \frac{5x-3}{x^2-2x-3}.\end{aligned}$$

## 8.5 Integration of Rational Functions by Partial Fractions



$$\begin{aligned}\frac{2}{x+1} + \frac{3}{x-3} &= \frac{2(x-3)}{(x+1)(x-3)} + \frac{3(x+1)}{(x-3)(x+1)} \\&= \frac{2x-6}{x^2-2x-3} + \frac{3x+3}{x^2-2x-3} \\&= \frac{5x-3}{x^2-2x-3}.\end{aligned}$$

But how do we do the opposite?

$$\frac{13x+1}{x^2-9} = \frac{\text{something?}}{x-3} + \frac{\text{something?}}{x+3}.$$

## 8.5 Integration of Rational Functions by Partial Fractions



$$\frac{5x - 3}{x^2 - 2x - 3} = \frac{A}{x + 1} + \frac{B}{x - 3}.$$

Pretend for a moment that we don't know that  $A = 2$  and  $B = 3$ . How can we find  $A$  and  $B$ ?

## 8.5 Integration of Rational Functions by Partial Fractions



$$\frac{5x - 3}{x^2 - 2x - 3} = \frac{A}{x + 1} + \frac{B}{x - 3}.$$

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$$\frac{5x - 3}{x^2 - 2x - 3} = \frac{A}{(x + 1)} + \frac{B}{(x - 3)}$$

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## 8.5 Integration of Rational Functions by Partial Fractions



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$$\frac{5x - 3}{x^2 - 2x - 3} = \frac{A(x - 3)}{(x + 1)(x - 3)} + \frac{B(x + 1)}{(x - 3)(x + 1)}$$

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## 8.5 Integration of Rational Functions by Partial Fractions



$$\frac{5x - 3}{x^2 - 2x - 3} = \frac{A}{x + 1} + \frac{B}{x - 3}.$$

Pretend for a moment that we don't know that  $A = 2$  and  $B = 3$ . How can we find  $A$  and  $B$ ?

$$\begin{aligned}\frac{5x - 3}{x^2 - 2x - 3} &= \frac{A(x - 3)}{(x + 1)(x - 3)} + \frac{B(x + 1)}{(x - 3)(x + 1)} \\&= \frac{Ax - 3A}{x^2 - 2x - 3} + \frac{Bx + B}{x^2 - 2x - 3} \\&= \end{aligned}$$

## 8.5 Integration of Rational Functions by Partial Fractions



$$\frac{5x - 3}{x^2 - 2x - 3} = \frac{A}{x + 1} + \frac{B}{x - 3}.$$

Pretend for a moment that we don't know that  $A = 2$  and  $B = 3$ . How can we find  $A$  and  $B$ ?

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## 8.5 Integration of Rational Functions by Partial Fractions



$$\frac{5x - 3}{x^2 - 2x - 3} = \frac{A}{x + 1} + \frac{B}{x - 3}.$$

Pretend for a moment that we don't know that  $A = 2$  and  $B = 3$ . How can we find  $A$  and  $B$ ?

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## 8.5 Integration of Rational Functions by Partial Fractions



$$\frac{5x - 3}{x^2 - 2x - 3} = \frac{A}{x + 1} + \frac{B}{x - 3}.$$

Pretend for a moment that we don't know that  $A = 2$  and  $B = 3$ . How can we find  $A$  and  $B$ ?

$$\begin{aligned}\frac{5x - 3}{x^2 - 2x - 3} &= \frac{A(x - 3)}{(x + 1)(x - 3)} + \frac{B(x + 1)}{(x - 3)(x + 1)} \\&= \frac{Ax - 3A}{x^2 - 2x - 3} + \frac{Bx + B}{x^2 - 2x - 3} \\&= \frac{(A + B)x + (B - 3A)}{x^2 - 2x - 3}.\end{aligned}$$

Hence

$$\begin{cases} A + B = 5 \\ B - 3A = -3 \end{cases}$$

## 8.5 Integration of Rational Functions by Partial Fractions



$$\frac{5x - 3}{x^2 - 2x - 3} = \frac{A}{x + 1} + \frac{B}{x - 3}.$$

Pretend for a moment that we don't know that  $A = 2$  and  $B = 3$ . How can we find  $A$  and  $B$ ?

$$\begin{aligned}\frac{5x - 3}{x^2 - 2x - 3} &= \frac{A(x - 3)}{(x + 1)(x - 3)} + \frac{B(x + 1)}{(x - 3)(x + 1)} \\&= \frac{Ax - 3A}{x^2 - 2x - 3} + \frac{Bx + B}{x^2 - 2x - 3} \\&= \frac{(A + B)x + (B - 3A)}{x^2 - 2x - 3}.\end{aligned}$$

Hence

$$\begin{cases} A + B = 5 \\ B - 3A = -3 \end{cases} \implies \begin{cases} A = 2 \\ B = 3. \end{cases}$$

## 8.5 Integration of Rational Functions by Partial Fractions



We can use *partial fractions* on  $\frac{f(x)}{g(x)}$

## 8.5 Integration of Rational Functions by Partial Fractions



We can use *partial fractions* on  $\frac{f(x)}{g(x)}$  if

- $\left( \begin{array}{c} \text{the degree} \\ \text{of } f(x) \end{array} \right) < \left( \begin{array}{c} \text{the degree} \\ \text{of } g(x) \end{array} \right);$

## 8.5 Integration of Rational Functions by Partial Fractions



We can use *partial fractions* on  $\frac{f(x)}{g(x)}$  if

- $\left( \begin{array}{c} \text{the degree} \\ \text{of } f(x) \end{array} \right) < \left( \begin{array}{c} \text{the degree} \\ \text{of } g(x) \end{array} \right)$ ; and
- we can factorise  $g(x)$ .

## 8.5 Integration of Rational Functions by Partial Fractions



### Example

Write  $\frac{x^2 + x + 2}{(x + 1)(x^2 + 1)}$  in partial fractions.

## 8.5 Integration of Rational Functions by Partial Fractions



### Example

Write  $\frac{x^2 + x + 2}{(x + 1)(x^2 + 1)}$  in partial fractions.

$$\frac{x^2 + x + 2}{(x + 1)(x^2 + 1)} = \frac{A}{x + 1} + \frac{Bx + C}{x^2 + 1}$$

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## 8.5 Integration of Rational Functions by Partial Fractions



### Example

Write  $\frac{x^2 + x + 2}{(x + 1)(x^2 + 1)}$  in partial fractions.

$$\begin{aligned}\frac{x^2 + x + 2}{(x + 1)(x^2 + 1)} &= \frac{A}{x + 1} + \frac{Bx + C}{x^2 + 1} \\&= \frac{A(x^2 + 1)}{(x + 1)(x^2 + 1)} + \frac{(Bx + C)(x + 1)}{(x^2 + 1)(x + 1)} \\&= \\&= \\&= .\end{aligned}$$

## 8.5 Integration of Rational Functions by Partial Fractions



### Example

Write  $\frac{x^2 + x + 2}{(x + 1)(x^2 + 1)}$  in partial fractions.

$$\begin{aligned}\frac{x^2 + x + 2}{(x + 1)(x^2 + 1)} &= \frac{A}{x + 1} + \frac{Bx + C}{x^2 + 1} \\&= \frac{A(x^2 + 1)}{(x + 1)(x^2 + 1)} + \frac{(Bx + C)(x + 1)}{(x^2 + 1)(x + 1)} \\&= \frac{Ax^2 + A + Bx^2 + Bx + Cx + C}{(x + 1)(x^2 + 1)}\end{aligned}$$

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## 8.5 Integration of Rational Functions by Partial Fractions



### Example

Write  $\frac{x^2 + x + 2}{(x + 1)(x^2 + 1)}$  in partial fractions.

$$\begin{aligned}\frac{x^2 + x + 2}{(x + 1)(x^2 + 1)} &= \frac{A}{x + 1} + \frac{Bx + C}{x^2 + 1} \\&= \frac{A(x^2 + 1)}{(x + 1)(x^2 + 1)} + \frac{(Bx + C)(x + 1)}{(x^2 + 1)(x + 1)} \\&= \frac{Ax^2 + A + Bx^2 + Bx + Cx + C}{(x + 1)(x^2 + 1)} \\&= \frac{(A + B)x^2 + (B + C)x + (A + C)}{(x + 1)(x^2 + 1)} \\&= \end{aligned}$$

## 8.5 Integration of Rational Functions by Partial Fractions



### Example

Write  $\frac{x^2 + x + 2}{(x + 1)(x^2 + 1)}$  in partial fractions.

$$\begin{aligned}\frac{x^2 + x + 2}{(x + 1)(x^2 + 1)} &= \frac{A}{x + 1} + \frac{Bx + C}{x^2 + 1} \\&= \frac{A}{(x + 1)} + \frac{(Bx + C)}{(x^2 + 1)} \\&= \frac{Ax^2 + A + Bx^2 + Bx + Cx + C}{(x + 1)(x^2 + 1)} \\&= \frac{(A + B)x^2 + (B + C)x + (A + C)}{(x + 1)(x^2 + 1)} \\&= \end{aligned}$$

$$\begin{aligned}A + B &= 1 \\B + C &= 1 \\A + C &= 2\end{aligned}$$

$$\frac{(x + C)(x + 1)}{(x^2 + 1)(x + 1)}$$

## 8.5 Integration of Rational Functions by Partial Fractions



Example

Write  $\frac{x^2 + x + 2}{(x + 1)(x^2 + 1)}$  in partial fractions.

$$\begin{aligned}\frac{x^2 + x + 2}{(x + 1)(x^2 + 1)} &= \frac{A}{x + 1} + \frac{Bx + C}{x^2 + 1} \\&= \frac{A}{(x + 1)} + \frac{Bx + C}{x^2 + 1} \\&= \frac{Ax^2 + A + Bx^2 + Cx}{(x + 1)(x^2 + 1)} \\&= \frac{(A + B)x^2 + (B + C)x + A + C}{(x + 1)(x^2 + 1)} \\&= \end{aligned}$$

A + B = 1  
B + C = 1  
A + C = 2

A = 1  
B = 0  
C = 1

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## 8.5 Integration of Rational Functions by Partial Fractions



### Example

Write  $\frac{x^2 + x + 2}{(x + 1)(x^2 + 1)}$  in partial fractions.

$$\begin{aligned}\frac{x^2 + x + 2}{(x + 1)(x^2 + 1)} &= \frac{A}{x + 1} + \frac{Bx + C}{x^2 + 1} \\&= \frac{A(x^2 + 1)}{(x + 1)(x^2 + 1)} + \left| \begin{array}{l} A = 1 \\ B = 0 \\ C = 1 \end{array} \right| \\&= \frac{Ax^2 + A + Bx^2 + }{(x + 1)(x^2 + 1)} \\&= \frac{(A + B)x^2 + (B + C)x + (A + C)}{(x + 1)(x^2 + 1)} \\&= \frac{1}{x + 1} + \frac{1}{x^2 + 1}.\end{aligned}$$

## 8.5 Integration of Rational Functions by Partial Fractions



### Example

Write  $\frac{3x^3 + 2x^2 + x}{(x + 3)^4}$  in partial fractions.

## 8.5 Integration of Rational Functions by Partial Fractions



### Example

Write  $\frac{3x^3 + 2x^2 + x}{(x + 3)^4}$  in partial fractions.

$$\frac{3x^3 + 2x^2 + x}{(x + 3)^4} = \frac{A}{x + 3} + \frac{B}{(x + 3)^2} + \frac{C}{(x + 3)^3} + \frac{D}{(x + 3)^4} = \dots$$

## 8.5 Integration of Rational Functions by Partial Fractions



Example

Write  $\frac{3x^3 + 2x^2 + x}{(x + 3)^4}$  in partial fractions.

$$\frac{3x^3 + 2x^2 + x}{(x + 3)^4} = \frac{A}{x + 3} + \frac{B}{(x + 3)^2} + \frac{C}{(x + 3)^3} + \frac{D}{(x + 3)^4} = \dots$$

Example

Write  $\frac{71}{(x + 3)(x^2 + 2x + 3)^2}$  in partial fractions.

## 8.5 Integration of Rational Functions by Partial Fractions



Example

Write  $\frac{3x^3 + 2x^2 + x}{(x + 3)^4}$  in partial fractions.

$$\frac{3x^3 + 2x^2 + x}{(x + 3)^4} = \frac{A}{x + 3} + \frac{B}{(x + 3)^2} + \frac{C}{(x + 3)^3} + \frac{D}{(x + 3)^4} = \dots$$

Example

Write  $\frac{71}{(x + 3)(x^2 + 2x + 3)^2}$  in partial fractions.

$$\begin{aligned}\frac{71}{(x + 3)(x^2 + 2x + 3)^2} &= \frac{A}{x + 3} + \frac{Bx + C}{(x^2 + 2x + 3)} + \frac{Dx + E}{(x^2 + 2x + 3)^2} \\ &= \dots\end{aligned}$$

## Method of Partial Fractions When $f(x)/g(x)$ Is Proper

- Let  $x - r$  be a linear factor of  $g(x)$ . Suppose that  $(x - r)^m$  is the highest power of  $x - r$  that divides  $g(x)$ . Then, to this factor, assign the sum of the  $m$  partial fractions:

$$\frac{A_1}{(x - r)} + \frac{A_2}{(x - r)^2} + \cdots + \frac{A_m}{(x - r)^m}.$$

Do this for each distinct linear factor of  $g(x)$ .

- Let  $x^2 + px + q$  be an irreducible quadratic factor of  $g(x)$  so that  $x^2 + px + q$  has no real roots. Suppose that  $(x^2 + px + q)^n$  is the highest power of this factor that divides  $g(x)$ . Then, to this factor, assign the sum of the  $n$  partial fractions:

$$\frac{B_1x + C_1}{(x^2 + px + q)} + \frac{B_2x + C_2}{(x^2 + px + q)^2} + \cdots + \frac{B_nx + C_n}{(x^2 + px + q)^n}.$$

Do this for each distinct quadratic factor of  $g(x)$ .

- Set the original fraction  $f(x)/g(x)$  equal to the sum of all these partial fractions. Clear the resulting equation of fractions and arrange the terms in decreasing powers of  $x$ .
- Equate the coefficients of corresponding powers of  $x$  and solve the resulting equations for the undetermined coefficients.

## 8.5 Integration of Rational Functions by Partial Fractions



### Example

Use partial fractions to find  $\int \frac{x^2 + 4x + 1}{(x - 1)(x + 1)(x + 3)} dx$ .

## 8.5 Integration of Rational Functions by Partial Fractions



Since

$$\frac{x^2 + 4x + 1}{(x - 1)(x + 1)(x + 3)} = \frac{A}{x - 1} + \frac{B}{x + 1} + \frac{C}{x + 3}$$

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## 8.5 Integration of Rational Functions by Partial Fractions



Since

$$\begin{aligned}& \frac{x^2 + 4x + 1}{(x - 1)(x + 1)(x + 3)} \\&= \frac{A}{x - 1} + \frac{B}{x + 1} + \frac{C}{x + 3} \\&= \frac{A(x + 1)(x + 3) + B(x - 1)(x + 3) + C(x - 1)(x + 1)}{(x - 1)(x + 1)(x + 3)}\end{aligned}$$

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## 8.5 Integration of Rational Functions by Partial Fractions



Since

$$\begin{aligned}& \frac{x^2 + 4x + 1}{(x - 1)(x + 1)(x + 3)} \\&= \frac{A}{x - 1} + \frac{B}{x + 1} + \frac{C}{x + 3} \\&= \frac{A(x + 1)(x + 3) + B(x - 1)(x + 3) + C(x - 1)(x + 1)}{(x - 1)(x + 1)(x + 3)} \\&= \frac{A(x^2 + 4x + 3) + B(x^2 + 2x - 3) + C(x^2 - 1)}{(x - 1)(x + 1)(x + 3)}\end{aligned}$$

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## 8.5 Integration of Rational Functions by Partial Fractions



Since

$$\begin{aligned} & \frac{x^2 + 4x + 1}{(x - 1)(x + 1)(x + 3)} \\ &= \frac{A}{x - 1} + \frac{B}{x + 1} + \frac{C}{x + 3} \\ &= \frac{A(x + 1)(x + 3) + B(x - 1)(x + 3) + C(x - 1)(x + 1)}{(x - 1)(x + 1)(x + 3)} \\ &= \frac{A(x^2 + 4x + 3) + B(x^2 + 2x - 3) + C(x^2 - 1)}{(x - 1)(x + 1)(x + 3)} \\ &= \frac{(A + B + C)x^2 + (4A + 2B)x + (3A - 3B - C)}{(x - 1)(x + 1)(x + 3)} \\ &= \end{aligned}$$

## 8.5 Integration of Rational Functions by Partial Fractions



Since

$$\begin{aligned} & \frac{x^2 + 4x + 1}{(x - 1)(x + 1)(x + 3)} \\ &= \frac{A}{x - 1} + \frac{B}{x + 1} + \boxed{\begin{array}{l} A + B + C = 1 \\ 4A + 2B = 4 \\ 3A - 3B - C = 1 \end{array}} \frac{(x - 1)(x + 1)}{x^2 - 1} \\ &= \frac{A(x^2 + 4x + 3)}{(x - 1)(x + 1)(x + 3)} \\ &= \frac{(A + B + C)x^2 + (4A + 2B)x + (3A - 3B - C)}{(x - 1)(x + 1)(x + 3)} \end{aligned}$$

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## 8.5 Integration of Rational Functions by Partial Fractions



Since

$$\begin{aligned} & \frac{x^2 + 4x + 1}{(x - 1)(x + 1)(x + 3)} \\ &= \frac{A}{x - 1} + \frac{B}{x + 1} + \boxed{\begin{array}{l} A + B + C = 1 \\ 4A + 2B = 4 \\ 3A - 3B - C = 1 \end{array}} \\ &= \frac{A(x + 1)(x + 3)}{(x - 1)(x + 1)(x + 3)} \\ &= \frac{A(x^2 + 4x + 3)}{(x - 1)(x + 1)(x + 3)} \\ &= \frac{(A + B + C)x^2 + (4A + 2B)x + (3A - 3B - C)}{(x - 1)(x + 1)(x + 3)} \end{aligned}$$

$A = \frac{3}{4}$   
 $B = \frac{1}{2}$   
 $C = -\frac{1}{4}$

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## 8.5 Integration of Rational Functions by Partial Fractions



Since

$$\begin{aligned} & \frac{x^2 + 4x + 1}{(x - 1)(x + 1)(x + 3)} \\ &= \frac{A}{x - 1} + \frac{B}{x + 1} + \frac{C}{x + 3} \\ &= \frac{A(x + 1)(x + 3) + B(x - 1)(x + 3) + C(x - 1)}{(x - 1)(x + 1)(x + 3)} \\ &= \frac{A(x^2 + 4x + 3) + B(x^2 + 2x - 3) + Cx - C}{(x - 1)(x + 1)(x + 3)} \\ &= \frac{(A + B + C)x^2 + (4A + 2B)x + (3A - 3B - C)}{(x - 1)(x + 1)(x + 3)} \\ &= \frac{\frac{3}{4}}{x - 1} + \frac{\frac{1}{2}}{x + 1} + \frac{-\frac{1}{4}}{x + 3} \end{aligned}$$

$A = \frac{3}{4}$   
 $B = \frac{1}{2}$   
 $C = -\frac{1}{4}$

1)

## 8.5 Integration of Rational Functions by Partial Fractions

$$\frac{x^2+4x+1}{(x-1)(x+1)(x+3)} = \frac{\frac{3}{4}}{x-1} + \frac{\frac{1}{2}}{x+1} + \frac{-\frac{1}{4}}{x+3}$$



We have that

$$\begin{aligned}\int \frac{x^2+4x+1}{(x-1)(x+1)(x+3)} dx \\ &= \int \frac{\frac{3}{4}}{x-1} + \frac{\frac{1}{2}}{x+1} + \frac{-\frac{1}{4}}{x+3} dx\end{aligned}$$

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## 8.5 Integration of Rational Functions by Partial Fractions

$$\frac{x^2+4x+1}{(x-1)(x+1)(x+3)} = \frac{\frac{3}{4}}{x-1} + \frac{\frac{1}{2}}{x+1} + \frac{-\frac{1}{4}}{x+3}$$



We have that

$$\begin{aligned} & \int \frac{x^2+4x+1}{(x-1)(x+1)(x+3)} dx \\ &= \int \frac{\frac{3}{4}}{x-1} + \frac{\frac{1}{2}}{x+1} + \frac{-\frac{1}{4}}{x+3} dx \\ &= \frac{3}{4} \int \frac{dx}{x-1} + \frac{1}{2} \int \frac{dx}{x+1} - \frac{1}{4} \int \frac{dx}{x+3} \\ &= \end{aligned}$$

## 8.5 Integration of Rational Functions by Partial Fractions

$$\frac{x^2+4x+1}{(x-1)(x+1)(x+3)} = \frac{\frac{3}{4}}{x-1} + \frac{\frac{1}{2}}{x+1} + \frac{-\frac{1}{4}}{x+3}$$



We have that

$$\begin{aligned} & \int \frac{x^2+4x+1}{(x-1)(x+1)(x+3)} dx \\ &= \int \frac{\frac{3}{4}}{x-1} + \frac{\frac{1}{2}}{x+1} + \frac{-\frac{1}{4}}{x+3} dx \\ &= \frac{3}{4} \int \frac{dx}{x-1} + \frac{1}{2} \int \frac{dx}{x+1} - \frac{1}{4} \int \frac{dx}{x+3} \\ &= \frac{3}{4} \ln|x-1| + \frac{1}{2} \ln|x+1| - \frac{1}{4} \ln|x+3| + K. \end{aligned}$$

(I already used  $C$ )

## 8.5 Integration of Rational Functions by Partial Fractions



### Example

Use partial fractions to find  $\int \frac{6x + 7}{(x + 2)^2} dx$ .

## 8.5 Integration of Rational Functions by Partial Fractions



### Example

Use partial fractions to find  $\int \frac{6x + 7}{(x + 2)^2} dx$ .

Because we have  $(x + 2)^2$  in the denominator, our partial sum will look like

$$\frac{6x + 7}{(x + 2)^2} = \frac{\text{something}}{x + 2} + \frac{\text{something}}{(x + 2)^2}$$

## 8.5 Integration of Rational Functions by Partial Fractions



### Example

Use partial fractions to find  $\int \frac{6x + 7}{(x + 2)^2} dx$ .

Because we have  $(x + 2)^2$  in the denominator, our partial sum will look like

$$\frac{6x + 7}{(x + 2)^2} = \frac{A}{x + 2} + \frac{B}{(x + 2)^2}$$

## 8.5 Integration of Rational Functions by Partial Fractions



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Use partial fractions to find  $\int \frac{6x + 7}{(x + 2)^2} dx$ .

Because we have  $(x + 2)^2$  in the denominator, our partial sum will look like

$$\frac{6x + 7}{(x + 2)^2} = \frac{A}{x + 2} + \frac{B}{(x + 2)^2}$$

Multiplying by  $(x + 2)^2$  gives

$$6x + 7 = A(x + 2) + B = Ax + (2A + B).$$

## 8.5 Integration of Rational Functions by Partial Fractions



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Multiplying by  $(x + 2)^2$  gives

$$6x + 7 = A(x + 2) + B = Ax + (2A + B).$$

So  $A = 6$  and  $B = 7 - 2A = 7 - 12 = -5$ .

## 8.5 Integration of Rational Functions by Partial Fractions



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Use partial fractions to find  $\int \frac{6x + 7}{(x + 2)^2} dx$ .

Because we have  $(x + 2)^2$  in the denominator, our partial sum will look like

$$\frac{6x + 7}{(x + 2)^2} = \frac{A}{x + 2} + \frac{B}{(x + 2)^2}$$

Multiplying by  $(x + 2)^2$  gives

$$6x + 7 = A(x + 2) + B = Ax + (2A + B).$$

So  $A = 6$  and  $B = 7 - 2A = 7 - 12 = -5$ . Thus

$$\frac{6x + 7}{(x + 2)^2} = \frac{6}{x + 2} + \frac{-5}{(x + 2)^2}.$$

Now we can integrate.

## 8.5 Integration of Rational Functions by Partial Fractions



$$\begin{aligned}\int \frac{6x + 7}{(x + 2)^2} dx &= \int \frac{6}{x + 2} + \frac{-5}{(x + 2)^2} dx \\&= 6 \int \frac{dx}{x + 2} - 5 \int (x + 2)^{-2} dx\end{aligned}$$

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## 8.5 Integration of Rational Functions by Partial Fractions



$$\begin{aligned}\int \frac{6x + 7}{(x + 2)^2} dx &= \int \frac{6}{x + 2} + \frac{-5}{(x + 2)^2} dx \\&= 6 \int \frac{dx}{x + 2} - 5 \int (x + 2)^{-2} dx \\&= 6 \ln|x + 2| + 5(x + 2)^{-1} + C \\&= 6 \ln|x + 2| + \frac{5}{x + 2} + C.\end{aligned}$$

## 8.5 Integration of Rational Functions by Partial Fractions



### Example

Use partial fractions to find  $\int \frac{2x^3 - 4x^2 - x - 3}{x^2 - 2x - 3} dx$ .

## 8.5 Integration of Rational Functions by Partial Fractions



### Example

Use partial fractions to find  $\int \frac{2x^3 - 4x^2 - x - 3}{x^2 - 2x - 3} dx$ .

Here the numerator is a 3rd order polynomial but the denominator is only a 2nd order polynomial.

## 8.5 Integration of Rational Functions by Partial Fractions



### Example

Use partial fractions to find  $\int \frac{2x^3 - 4x^2 - x - 3}{x^2 - 2x - 3} dx$ .

Here the numerator is a 3rd order polynomial but the denominator is only a 2nd order polynomial. Remember that we can only use partial fractions on  $\frac{f(x)}{g(x)}$  if

$$\left( \begin{array}{c} \text{the degree} \\ \text{of } f(x) \end{array} \right) < \left( \begin{array}{c} \text{the degree} \\ \text{of } g(x) \end{array} \right).$$

## 8.5 Integration of Rational Functions by Partial Fractions



### Example

Use partial fractions to find  $\int \frac{2x^3 - 4x^2 - x - 3}{x^2 - 2x - 3} dx.$

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So what do we do?

## 8.5 Integration of Rational Functions by Partial Fractions



### Example

Use partial fractions to find  $\int \frac{2x^3 - 4x^2 - x - 3}{x^2 - 2x - 3} dx$ .

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So what do we do? First we need to write our integrand as

$$\frac{2x^3 - 4x^2 - x - 3}{x^2 - 2x - 3} = (\text{something}) + \frac{\text{something}}{x^2 - 2x - 3}.$$

## 8.5 Integration of Rational Functions by Partial Fractions



### Example

Use partial fractions to find  $\int \frac{2x^3 - 4x^2 - x - 3}{x^2 - 2x - 3} dx$ .

Here the numerator is a 3rd order polynomial but the denominator is only a 2nd order polynomial. Remember that we can only use partial fractions on  $\frac{f(x)}{g(x)}$  if

$$\left( \begin{array}{c} \text{the degree} \\ \text{of } f(x) \end{array} \right) < \left( \begin{array}{c} \text{the degree} \\ \text{of } g(x) \end{array} \right).$$

So what do we do? First we need to write our integrand as

$$\frac{2x^3 - 4x^2 - x - 3}{x^2 - 2x - 3} = (\text{linear function}) + \frac{\text{something}}{x^2 - 2x - 3}.$$

## 8.5 Integration of Rational Functions by Partial Fractions



$$\frac{2x^3 - 4x^2 - x - 3}{x^2 - 2x - 3} = (\text{linear function}) + \frac{\text{something}}{x^2 - 2x - 3}.$$

Your textbook really likes “long division” and does this:

$$\begin{array}{r} 2x \\ \hline x^2 - 2x - 3 \overline{)2x^3 - 4x^2 - x - 3} \\ \underline{2x^3 - 4x^2 - 6x} \\ 5x - 3 \end{array}$$

If you understand this, then great do this.

## 8.5 Integration of Rational Functions by Partial Fractions



$$\frac{2x^3 - 4x^2 - x - 3}{x^2 - 2x - 3} = (\text{linear function}) + \frac{\text{something}}{x^2 - 2x - 3}.$$

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If you understand this, then great do this. As you know, I prefer to do this a different way.

## 8.5 Integration of Rational Functions by Partial Fractions



$$\frac{2x^3 - 4x^2 - x - 3}{x^2 - 2x - 3} = \text{(linear function)} + \frac{\text{something}}{x^2 - 2x - 3}$$

## 8.5 Integration of Rational Functions by Partial Fractions



$$\frac{2x^3 - 4x^2 - x - 3}{x^2 - 2x - 3} = (Ax + B) + \frac{Cx + D}{x^2 - 2x - 3}$$

## 8.5 Integration of Rational Functions by Partial Fractions



$$\frac{2x^3 - 4x^2 - x - 3}{x^2 - 2x - 3} = (\textcolor{red}{Ax + B}) + \frac{\textcolor{green}{Cx + D}}{x^2 - 2x - 3}$$

Multiplying both sides by  $(x^2 - 2x - 3)$  gives

$$\begin{aligned} 2x^3 - 4x^2 - x - 3 &= (\textcolor{red}{Ax + B})(x^2 - 2x - 3) + (\textcolor{green}{Cx + D}) \\ &= Ax^3 - 2Ax^2 - 3Ax + Bx^2 - 2Bx - 3B + Cx + D \\ &= Ax^3 + (-2A + B)x^2 + (-3A - 2B + C)x + D \end{aligned}$$

which implies  $A = 2$ ,  $B = 0$ ,  $C = 5$  and  $D = -3$  (please check!).

## 8.5 Integration of Rational Functions by Partial Fractions



$$\frac{2x^3 - 4x^2 - x - 3}{x^2 - 2x - 3} = (\textcolor{red}{Ax + B}) + \frac{\textcolor{green}{Cx + D}}{x^2 - 2x - 3}$$

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$$\begin{aligned} 2x^3 - 4x^2 - x - 3 &= (\textcolor{red}{Ax + B})(x^2 - 2x - 3) + (\textcolor{green}{Cx + D}) \\ &= Ax^3 - 2Ax^2 - 3Ax + Bx^2 - 2Bx - 3B + Cx + D \\ &= Ax^3 + (-2A + B)x^2 + (-3A - 2B + C)x + D \end{aligned}$$

which implies  $A = 2$ ,  $B = 0$ ,  $C = 5$  and  $D = -3$  (please check!).

Thus our integrand is

$$\frac{2x^3 - 4x^2 - x - 3}{x^2 - 2x - 3} = \textcolor{red}{2x} + \frac{\textcolor{green}{5x - 3}}{x^2 - 2x - 3}.$$

## 8.5 Integration of Rational Functions by Partial Fractions



$$\frac{2x^3 - 4x^2 - x - 3}{x^2 - 2x - 3} = \underline{2x} + \frac{\textcolor{green}{5x - 3}}{x^2 - 2x - 3}$$

At the start of this section, we found that

$$\frac{\textcolor{green}{5x - 3}}{x^2 - 2x - 3} = \frac{2}{x + 2} + \frac{3}{x - 1}.$$

## 8.5 Integration of Rational Functions by Partial Fractions



$$\frac{2x^3 - 4x^2 - x - 3}{x^2 - 2x - 3} = \cancel{2x} + \frac{\cancel{5x} - 3}{x^2 - 2x - 3}$$

At the start of this section, we found that

$$\frac{\cancel{5x} - 3}{x^2 - 2x - 3} = \frac{2}{x+2} + \frac{3}{x-1}.$$

Therefore

$$\begin{aligned}\frac{2x^3 - 4x^2 - x - 3}{x^2 - 2x - 3} &= \int \cancel{2x} \, dx + \int \frac{2}{x+2} \, dx + \int \frac{3}{x-1} \, dx \\ &= \end{aligned}$$

## 8.5 Integration of Rational Functions by Partial Fractions



$$\frac{2x^3 - 4x^2 - x - 3}{x^2 - 2x - 3} = \cancel{2x} + \frac{\cancel{5x} - 3}{x^2 - 2x - 3}$$

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$$\begin{aligned}\frac{2x^3 - 4x^2 - x - 3}{x^2 - 2x - 3} &= \int \cancel{2x} \, dx + \int \frac{2}{x+2} \, dx + \int \frac{3}{x-1} \, dx \\ &= x^2 + 2 \ln|x+1| + 3 \ln|x-1| + C.\end{aligned}$$

## 8.5 Integration of Rational Functions by Partial Fractions



### Example

Use partial fractions to find  $\int \frac{-2x + 4}{(x^2 + 1)(x - 1)^2} dx$ .

## 8.5 Integration of Rational Functions by Partial Fractions



### Example

Use partial fractions to find  $\int \frac{-2x + 4}{(x^2 + 1)(x - 1)^2} dx$ .

We want

$$\frac{-2x + 4}{(x^2 + 1)(x - 1)^2} = \frac{\text{something}}{x^2 + 1} + \frac{\text{something}}{x - 1} + \frac{\text{something}}{(x - 1)^2}$$

## 8.5 Integration of Rational Functions by Partial Fractions



### Example

Use partial fractions to find  $\int \frac{-2x + 4}{(x^2 + 1)(x - 1)^2} dx$ .

We want

$$\frac{-2x + 4}{(x^2 + 1)(x - 1)^2} = \frac{Ax + B}{x^2 + 1} + \frac{\text{something}}{x - 1} + \frac{\text{something}}{(x - 1)^2}$$

## 8.5 Integration of Rational Functions by Partial Fractions



### Example

Use partial fractions to find  $\int \frac{-2x + 4}{(x^2 + 1)(x - 1)^2} dx$ .

We want

$$\frac{-2x + 4}{(x^2 + 1)(x - 1)^2} = \frac{Ax + B}{x^2 + 1} + \frac{C}{x - 1} + \frac{D}{(x - 1)^2}$$

## 8.5 Integration of Rational Functions by Partial Fractions



### Example

Use partial fractions to find  $\int \frac{-2x + 4}{(x^2 + 1)(x - 1)^2} dx$ .

We want

$$\frac{-2x + 4}{(x^2 + 1)(x - 1)^2} = \frac{Ax + B}{x^2 + 1} + \frac{C}{x - 1} + \frac{D}{(x - 1)^2}$$

Please check that multiplying by  $(x^2 + 1)(x - 1)^2$  gives

$$\begin{aligned}-2x + 4 &= (Ax + B)(x - 1)^2 + C(x^2 + 1)(x - 1) + D(x^2 + 1) \\&= (A + C)x^3 + (-2A + B - C + D)x^2 + (A - 2B + C)x \\&\quad + (B - C + D).\end{aligned}$$

## 8.5 Integration of Rational Functions by Partial Fractions



$$0x^3 + 0x^2 - 2x + 4 = (A + C)x^3 + (-2A + B - C + D)x^2 + (A - 2B + C)x + (B - C + D).$$

Equating coefficients gives

$$\text{coefficient of } x^3: \quad 0 = A + C$$

$$\text{coefficient of } x^2: \quad 0 = -2A + B - C + D$$

$$\text{coefficient of } x^1: \quad -2 = A - 2B + C$$

$$\text{coefficient of } x^0: \quad 4 = B - C + D.$$

## 8.5 Integration of Rational Functions by Partial Fractions



$$0x^3 + \textcolor{brown}{0}x^2 - 2x + 4 = (A + C)x^3 + (\textcolor{brown}{-2}A + \textcolor{brown}{B} - \textcolor{brown}{C} + \textcolor{brown}{D})x^2 \\ + (A - 2B + C)x + (B - C + D).$$

Equating coefficients gives

$$\text{coefficient of } x^3: \quad 0 = A + C$$

$$\text{coefficient of } x^2: \quad \textcolor{brown}{0} = \textcolor{brown}{-2}A + \textcolor{brown}{B} - \textcolor{brown}{C} + \textcolor{brown}{D}$$

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## 8.5 Integration of Rational Functions by Partial Fractions



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## 8.5 Integration of Rational Functions by Partial Fractions



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$$\text{coefficient of } x^1: \quad -2 = A - 2B + C$$

$$\text{coefficient of } x^0: \quad 4 = B - C + D.$$

## 8.5 Integration of Rational Functions by Partial Fractions



$$\begin{cases} A + C = 0 \\ -2A + B - C + D = 0 \\ A - 2B + C = -2 \\ B - C + D = 4 \end{cases}$$

is a system of four linear equations. You will study how to solve systems like this (and much bigger) in your MATH215 course.

## 8.5 Integration of Rational Functions by Partial Fractions



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We can solve it as follows:

$$\begin{array}{r} [\text{equation 2}] \\ - [\text{equation 4}] \\ \hline \end{array}$$

## 8.5 Integration of Rational Functions by Partial Fractions



$$\begin{cases} A + C = 0 \\ -2A + B - C + D = 0 \\ A - 2B + C = -2 \\ B - C + D = 4 \end{cases}$$

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We can solve it as follows:

$$\begin{array}{r} [-2A + B - C + D = 0] \\ - [ \quad \quad \quad B - C + D = 4] \\ \hline \end{array}$$

## 8.5 Integration of Rational Functions by Partial Fractions



$$\begin{cases} A + C = 0 \\ -2A + B - C + D = 0 \\ A - 2B + C = -2 \\ B - C + D = 4 \end{cases}$$

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We can solve it as follows:

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## 8.5 Integration of Rational Functions by Partial Fractions



$$\begin{cases} A + C = 0 \\ -2A + B - C + D = 0 \\ A - 2B + C = -2 \\ B - C + D = 4 \end{cases}$$

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We can solve it as follows:

$$\begin{array}{r} [-2A + B - C + D = 0] \\ - [ \quad B - C + D = 4] \\ \hline -2A & = -4 \end{array}$$

So  $A = 2$ .

## 8.5 Integration of Rational Functions by Partial Fractions



$$\begin{cases} A + C = 0 \\ -2A + B - C + D = 0 \\ A - 2B + C = -2 \\ B - C + D = 4 \end{cases}$$

has solution

$$\begin{cases} A = 2 \\ B = \\ C = \\ D = \end{cases}$$

## 8.5 Integration of Rational Functions by Partial Fractions



$$\begin{cases} A + C = 0 \\ -2A + B - C + D = 0 \\ A - 2B + C = -2 \\ B - C + D = 4 \end{cases}$$

has solution

$$\begin{cases} A = 2 \\ B = \\ C = -2 \\ D = \end{cases}$$

## 8.5 Integration of Rational Functions by Partial Fractions



$$\begin{cases} A + C = 0 \\ -2A + B - C + D = 0 \\ A - 2B + C = -2 \\ B - C + D = 4 \end{cases}$$

has solution

$$\begin{cases} A = 2 \\ B = 1 \\ C = -2 \\ D = \end{cases}$$

## 8.5 Integration of Rational Functions by Partial Fractions



$$\begin{cases} A + C = 0 \\ -2A + B - C + D = 0 \\ A - 2B + C = -2 \\ B - C + D = 4 \end{cases}$$

has solution

$$\begin{cases} A = 2 \\ B = 1 \\ C = -2 \\ D = 1. \end{cases}$$

## 8.5 Integration of Rational Functions by Partial Fractions



$$\begin{cases} A + C = 0 \\ -2A + B - C + D = 0 \\ A - 2B + C = -2 \\ B - C + D = 4 \end{cases}$$

has solution

$$\begin{cases} A = 2 \\ B = 1 \\ C = -2 \\ D = 1. \end{cases}$$

Hence our integrand is

$$\frac{-2x + 4}{(x^2 + 1)(x - 1)^2} = \frac{2x + 1}{x^2 + 1} + \frac{-2}{x - 1} + \frac{1}{(x - 1)^2}.$$

## 8.5 Integration of Rational Functions by Partial Fractions



Now we can integrate

$$\begin{aligned} & \int \frac{-2x + 4}{(x^2 + 1)(x - 1)^2} dx \\ &= \int \frac{2x + 1}{x^2 + 1} dx + \int \frac{-2}{x - 1} dx + \int \frac{1}{(x - 1)^2} dx \end{aligned}$$

=

=

## 8.5 Integration of Rational Functions by Partial Fractions



Now we can integrate

$$\begin{aligned} & \int \frac{-2x + 4}{(x^2 + 1)(x - 1)^2} dx \\ &= \int \frac{2x + 1}{x^2 + 1} dx + \int \frac{-2}{x - 1} dx + \int \frac{1}{(x - 1)^2} dx \\ &= \int \frac{2x}{x^2 + 1} dx + \int \frac{1}{x^2 + 1} dx - \int \frac{2}{x - 1} dx + \int \frac{1}{(x - 1)^2} dx \\ &= \end{aligned}$$

## 8.5 Integration of Rational Functions by Partial Fractions



Now we can integrate

$$\begin{aligned} & \int \frac{-2x + 4}{(x^2 + 1)(x - 1)^2} dx \\ &= \int \frac{2x + 1}{x^2 + 1} dx + \int \frac{-2}{x - 1} dx + \int \frac{1}{(x - 1)^2} dx \\ &= \int \frac{2x}{x^2 + 1} dx + \int \frac{1}{x^2 + 1} dx - \int \frac{2}{x - 1} dx + \int \frac{1}{(x - 1)^2} dx \\ &= \ln(x^2 + 1) + \tan^{-1} x - 2 \ln|x - 1| - \frac{1}{x - 1} + C. \end{aligned}$$

**EXAMPLE 5** Use partial fractions to evaluate

$$\int \frac{dx}{x(x^2 + 1)^2}.$$

**Solution** The form of the partial fraction decomposition is

$$\frac{1}{x(x^2 + 1)^2} = \frac{A}{x} + \frac{Bx + C}{x^2 + 1} + \frac{Dx + E}{(x^2 + 1)^2}.$$

Multiplying by  $x(x^2 + 1)^2$ , we have

$$\begin{aligned} 1 &= A(x^2 + 1)^2 + (Bx + C)x(x^2 + 1) + (Dx + E)x \\ &= A(x^4 + 2x^2 + 1) + B(x^4 + x^2) + C(x^3 + x) + Dx^2 + Ex \\ &= (A + B)x^4 + Cx^3 + (2A + B + D)x^2 + (C + E)x + A. \end{aligned}$$

If we equate coefficients, we get the system

$$A + B = 0, \quad C = 0, \quad 2A + B + D = 0, \quad C + E = 0, \quad A = 1.$$

Solving this system gives  $A = 1$ ,  $B = -1$ ,  $C = 0$ ,  $D = -1$ , and  $E = 0$ . Thus,

Solving this system gives  $A = 1$ ,  $B = -1$ ,  $C = 0$ ,  $D = -1$ , and  $E = 0$ . Thus,

$$\begin{aligned}\int \frac{dx}{x(x^2 + 1)^2} &= \int \left[ \frac{1}{x} + \frac{-x}{x^2 + 1} + \frac{-x}{(x^2 + 1)^2} \right] dx \\&= \int \frac{dx}{x} - \int \frac{x \, dx}{x^2 + 1} - \int \frac{x \, dx}{(x^2 + 1)^2} \\&= \int \frac{dx}{x} - \frac{1}{2} \int \frac{du}{u} - \frac{1}{2} \int \frac{du}{u^2} && u = x^2 + 1, \\&= \ln |x| - \frac{1}{2} \ln |u| + \frac{1}{2u} + K \\&= \ln |x| - \frac{1}{2} \ln (x^2 + 1) + \frac{1}{2(x^2 + 1)} + K \\&= \ln \frac{|x|}{\sqrt{x^2 + 1}} + \frac{1}{2(x^2 + 1)} + K.\end{aligned}$$



## 8.5 Integration of Rational Functions by Partial Fractions



### Remark

When we have

$$\frac{f(x)}{(x - r_1)(x - r_2) \cdots (x - r_n)},$$

where  $r_1, r_2, \dots, r_n$  are all different, there is a quicker way to find partial fractions.

## 8.5 Integration of Rational Functions by Partial Fractions



Example

$$\text{Find } \int \frac{x+4}{x^3 + 3x^2 - 10x} dx.$$

## 8.5 Integration of Rational Functions by Partial Fractions



### Example

Find  $\int \frac{x+4}{x^3 + 3x^2 - 10x} dx.$

First we have

$$\frac{x+4}{x^3 + 3x^2 - 10x} = \frac{x+4}{x(x-2)(x+5)}$$

=

## 8.5 Integration of Rational Functions by Partial Fractions



Example

$$\text{Find } \int \frac{x+4}{x^3 + 3x^2 - 10x} dx.$$

First we have

$$\begin{aligned}\frac{x+4}{x^3 + 3x^2 - 10x} &= \frac{x+4}{x(x-2)(x+5)} \\ &= \frac{A}{x} + \frac{B}{x-2} + \frac{C}{x+5}.\end{aligned}$$

## 8.5 Integration of Rational Functions by Partial Fractions



### Example

Find  $\int \frac{x+4}{x^3 + 3x^2 - 10x} dx.$

First we have

$$\begin{aligned}\frac{x+4}{x^3 + 3x^2 - 10x} &= \frac{x+4}{x(x-2)(x+5)} \\ &= \frac{A}{x} + \frac{B}{x-2} + \frac{C}{x+5}.\end{aligned}$$

- 1 multiply by  $x$ , then set  $x = 0$ ;
- 2 multiply by  $(x - 2)$ , then set  $x = 2$ ;
- 3 multiply by  $(x + 5)$ , then set  $x = -5$ .

## 8.5 Integration of Fractions

$$\frac{x+4}{x(x-2)(x+5)} = \frac{A}{x} + \frac{B}{x-2} + \frac{C}{x+5}$$



- 1 multiply by  $x$ , then set  $x = 0$ ;

$$\frac{x+4}{(x-2)(x+5)} = A + \frac{Bx}{x-2} + \frac{Cx}{x+5}$$

$$\frac{4}{(-2)(5)} = A + 0 + 0$$

$$-\frac{2}{5} = A$$

## 8.5 Integration of Fractions

$$\frac{x+4}{x(x-2)(x+5)} = \frac{A}{x} + \frac{B}{x-2} + \frac{C}{x+5}$$



2 multiply by  $(x - 2)$ , then set  $x = 2$ ;

$$\frac{x+4}{x(x+5)} = \frac{A(x-2)}{x} + B + \frac{C(x-2)}{x+5}$$

$$\frac{2+4}{(2)(7)} = 0 + B + 0$$

$$\frac{3}{7} = B$$

## 8.5 Integration of Fractions

$$\frac{x+4}{x(x-2)(x+5)} = \frac{A}{x} + \frac{B}{x-2} + \frac{C}{x+5}$$



- 3 multiply by  $(x + 5)$ , then set  $x = -5$ .

$$\frac{x+4}{x(x-2)} = \frac{A(x+5)}{x} + \frac{B(x+5)}{x-2} + C$$

$$\frac{-5+4}{(-5)(-7)} = 0 + 0 + C$$

$$-\frac{1}{35} = C$$

## 8.5 Integration of Rational Functions by Partial Fractions



Therefore

$$\frac{x+4}{x(x-2)(x+5)} = \frac{-\frac{2}{5}}{x} + \frac{\frac{3}{7}}{x-2} + \frac{-\frac{1}{35}}{x+5}$$

## 8.5 Integration of Rational Functions by Partial Fractions



Therefore

$$\frac{x+4}{x(x-2)(x+5)} = \frac{-\frac{2}{5}}{x} + \frac{\frac{3}{7}}{x-2} + \frac{-\frac{1}{35}}{x+5}$$

and thus

$$\int \frac{x+4}{x(x-2)(x+5)} dx = -\frac{2}{5} \ln|x| + \frac{3}{7} \ln|x-2| - \frac{1}{35} \ln|x+5| + C.$$

**EXAMPLE 6**

Find  $A$ ,  $B$ , and  $C$  in the partial fraction expansion

$$\frac{x^2 + 1}{(x - 1)(x - 2)(x - 3)} = \frac{A}{x - 1} + \frac{B}{x - 2} + \frac{C}{x - 3}. \quad (3)$$

**Solution** If we multiply both sides of Equation (3) by  $(x - 1)$  to get

$$\frac{x^2 + 1}{(x - 2)(x - 3)} = A + \frac{B(x - 1)}{x - 2} + \frac{C(x - 1)}{x - 3}$$

and set  $x = 1$ , the resulting equation gives the value of  $A$ :

$$\begin{aligned}\frac{(1)^2 + 1}{(1 - 2)(1 - 3)} &= A + 0 + 0, \\ A &= 1.\end{aligned}$$

In exactly the same way, we can multiply both sides by  $(x - 2)$  and then substitute in  $x = 2$ . This gives

$$\frac{(2)^2 + 1}{(2 - 1)(2 - 3)} = B.$$

So  $B = -5$ . Finally, we multiply both sides by  $(x - 3)$  and then substitute in  $x = 3$ , which yields

$$\frac{(3)^2 + 1}{(3 - 1)(3 - 2)} = C,$$

and  $C = 5$ .



# Break

We will continue at 3pm



## 8.5 Integration of Rational Functions by Partial Fractions



### Remark

We can also use differentiation to find partial fractions.

## 8.5 Integration of Rational Functions by Partial Fractions



### Example

Find  $A$ ,  $B$  and  $C$  in the equation

$$\frac{x - 1}{(x + 1)^3} = \frac{A}{x + 1} + \frac{B}{(x + 1)^2} + \frac{C}{(x + 1)^3}$$

by clearing fractions, differentiating and setting  $x = -1$ .

## 8.5 Integration of Rational Functions by Partial Fractions



### Example

Find  $A$ ,  $B$  and  $C$  in the equation

$$\frac{x - 1}{(x + 1)^3} = \frac{A}{x + 1} + \frac{B}{(x + 1)^2} + \frac{C}{(x + 1)^3}$$

by clearing fractions, differentiating and setting  $x = -1$ .

First we clear the fractions: Multiplying by  $(x + 1)^3$  gives

$$x - 1 = A(x + 1)^2 + B(x + 1) + C.$$

## 8.5 Integration of Rational Functions by Partial Fractions



### Example

Find  $A$ ,  $B$  and  $C$  in the equation

$$\frac{x - 1}{(x + 1)^3} = \frac{A}{x + 1} + \frac{B}{(x + 1)^2} + \frac{C}{(x + 1)^3}$$

by clearing fractions, differentiating and setting  $x = -1$ .

First we clear the fractions: Multiplying by  $(x + 1)^3$  gives

$$x - 1 = A(x + 1)^2 + B(x + 1) + C.$$

Putting in  $x = -1$  gives

$$-2 = 0 + 0 + C.$$

## 8.5 Integration of Rational Functions by Partial Fractions



### Example

Find  $A$ ,  $B$  and  $C$  in the equation

$$\frac{x - 1}{(x + 1)^3} = \frac{A}{x + 1} + \frac{B}{(x + 1)^2} + \frac{C}{(x + 1)^3}$$

by clearing fractions, differentiating and setting  $x = -1$ .

First we clear the fractions: Multiplying by  $(x + 1)^3$  gives

$$x - 1 = A(x + 1)^2 + B(x + 1) + C.$$

Putting in  $x = -1$  gives

$$-2 = 0 + 0 + C.$$

Now differentiate the green equation with respect to  $x$  to obtain

$$1 = 2A(x + 1) + B + 0.$$

## 8.5 Integration of Rational Functions by Partial Fractions



$$1 = 2A(x + 1) + B$$

Putting in  $x = -1$  gives

$$1 = 0 + B.$$

## 8.5 Integration of Rational Functions by Partial Fractions



$$1 = 2A(x + 1) + B$$

Putting in  $x = -1$  gives

$$1 = 0 + B.$$

Finally differentiate the orange equation to obtain

$$0 = 2A + 0.$$

## 8.5 Integration of Rational Functions by Partial Fractions



$$1 = 2A(x + 1) + B$$

Putting in  $x = -1$  gives

$$1 = 0 + B.$$

Finally differentiate the orange equation to obtain

$$0 = 2A + 0.$$

Therefore

$$\begin{cases} A = 0 \\ B = 1 \\ C = -2 \end{cases}$$

is the answer.

## 8.5 Integration of Rational Functions by Partial Fractions



### Remark

Sometimes we can just try putting in small numbers  $x = 0$ ,  $x = \pm 1$ ,  $x = \pm 2$ , etc. to find the coefficients  $A, B, C, \dots$

## 8.5 Integration of Rational Functions by Partial Fractions



### Example

Find  $A$ ,  $B$  and  $C$  in the equation

$$\frac{x^2 + 1}{(x - 1)(x - 2)(x - 3)}$$

by assigning numerical values to  $x$ .

## 8.5 Integration of Rational Functions by Partial Fractions



### Example

Find  $A$ ,  $B$  and  $C$  in the equation

$$\frac{x^2 + 1}{(x - 1)(x - 2)(x - 3)}$$

by assigning numerical values to  $x$ .

First we clear fractions to get

$$x^2 + 1 = A(x - 2)(x - 3) + B(x - 1)(x - 3) + C(x - 1)(x - 2)$$

## 8.5 Integration of Rational Functions by Partial Fractions



### Example

Find  $A$ ,  $B$  and  $C$  in the equation

$$\frac{x^2 + 1}{(x - 1)(x - 2)(x - 3)}$$

by assigning numerical values to  $x$ .

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Clearly the important numbers here are 1, 2 and 3.

## 8.5 Integration of Rational Functions by Partial Fractions



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Clearly the important numbers here are 1, 2 and 3. So we are going to try  $x = 1$ ,  $x = 2$  and  $x = 3$ .

## 8.5 Integration of Rational Functions by Partial Fractions



$$x^2 + 1 = A(x - 2)(x - 3) + B(x - 1)(x - 3) + C(x - 1)(x - 2)$$

$$\begin{aligned}x &= 1 & (1)^2 + 1 &= A(-1)(-2) + B(0) + C(0) \\&& 2 &= 2A \\&& A &= 1\end{aligned}$$

## 8.5 Integration of Rational Functions by Partial Fractions



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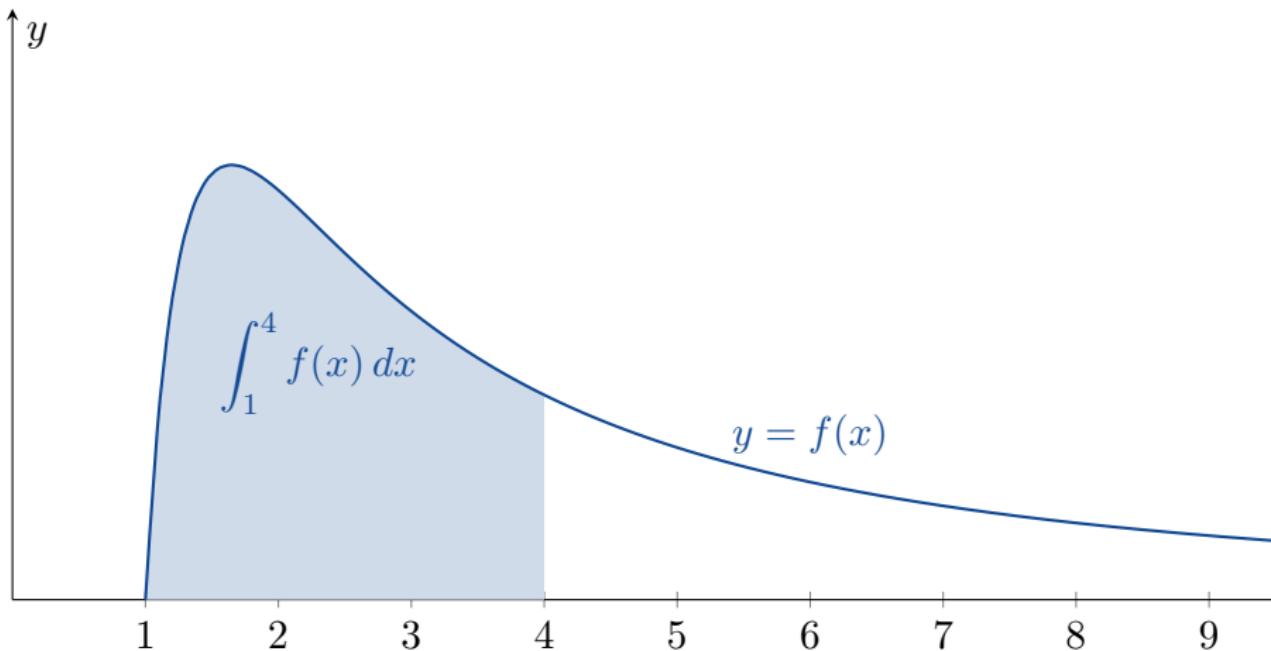
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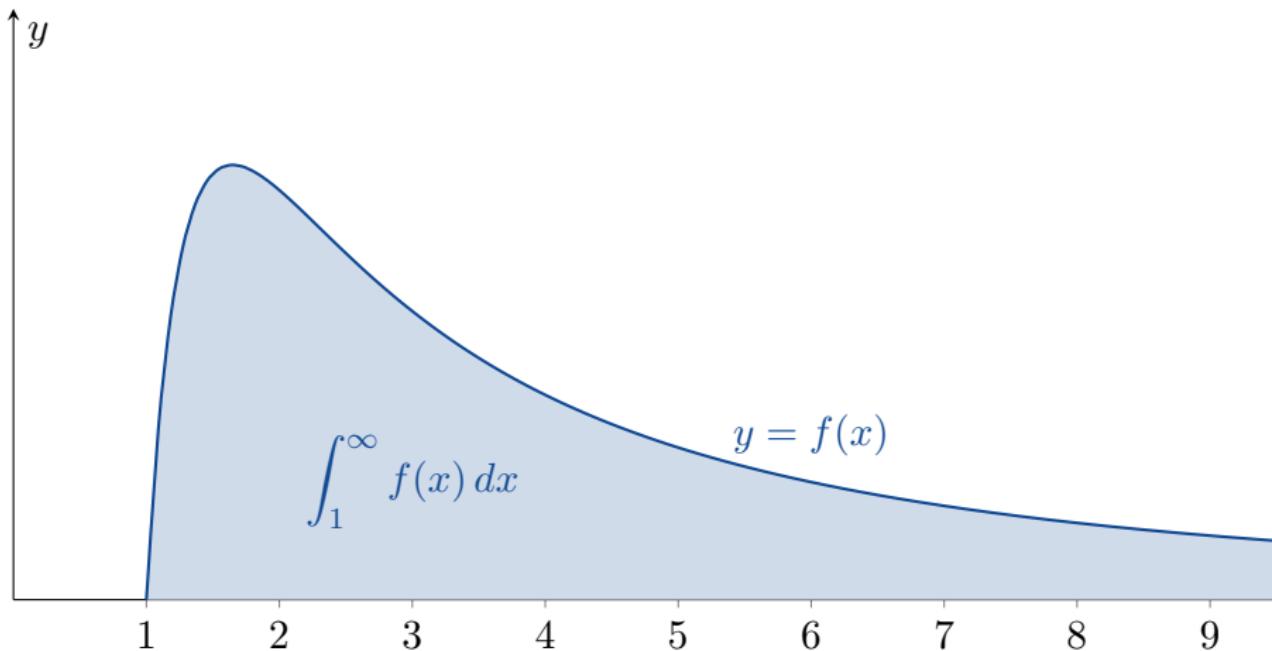


# Improper Integrals

## 8.8 Improper Integrals



## 8.8 Improper Integrals



## 8.8 Improper Integrals



We need to use limits.

## 8.8 Improper Integrals

Example

Calculate  $\int_0^\infty e^{-\frac{x}{2}} dx.$

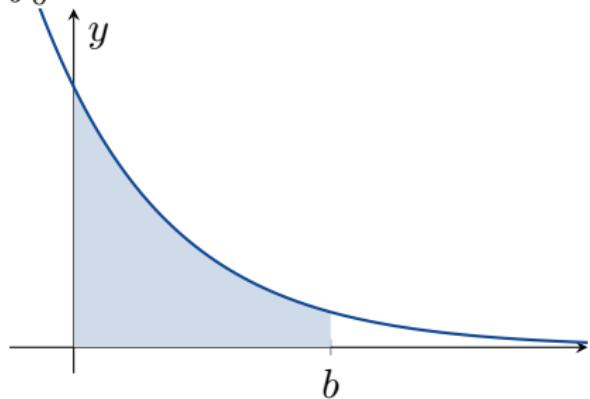
## 8.8 Improper Integrals

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Calculate  $\int_0^\infty e^{-\frac{x}{2}} dx.$

Step 1:

$$\int_0^b e^{-\frac{x}{2}} dx = ?$$



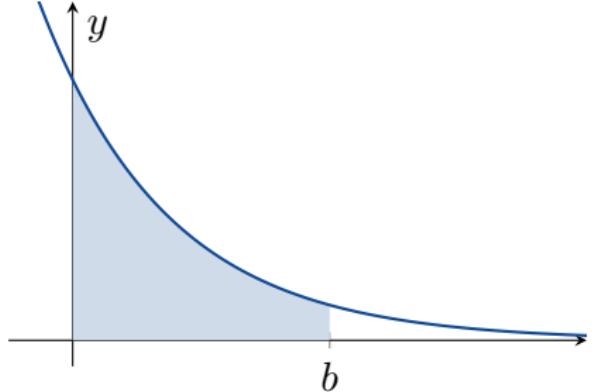
## 8.8 Improper Integrals

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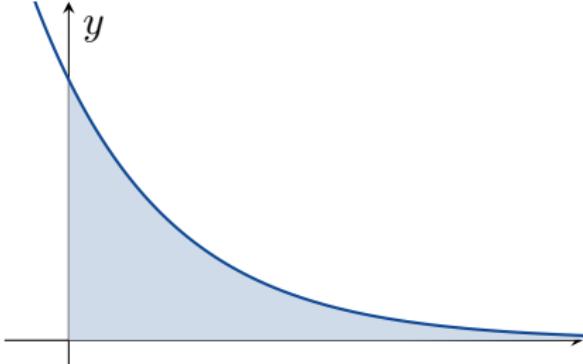
Step 1:

$$\int_0^b e^{-\frac{x}{2}} dx = ?$$



Step 2:

$$\int_0^\infty e^{-\frac{x}{2}} dx = \lim_{b \rightarrow \infty} \int_0^b e^{-\frac{x}{2}} dx$$



## 8.8 Improper Integrals



Since

$$\int_0^b e^{-\frac{x}{2}} dx = \left[ -2e^{-\frac{x}{2}} \right]_0^b = -2e^{-\frac{b}{2}} + 2,$$

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## 8.8 Improper Integrals

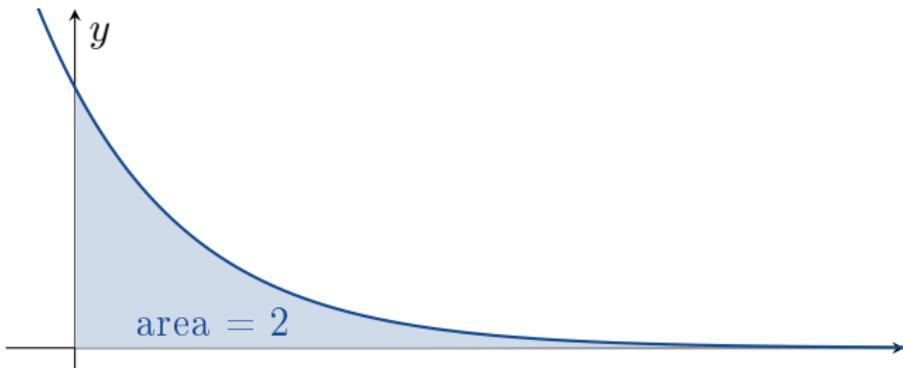


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**DEFINITION** Integrals with infinite limits of integration are **improper integrals of Type I**.

1. If  $f(x)$  is continuous on  $[a, \infty)$ , then

$$\int_a^{\infty} f(x) dx = \lim_{b \rightarrow \infty} \int_a^b f(x) dx.$$

2. If  $f(x)$  is continuous on  $(-\infty, b]$ , then

$$\int_{-\infty}^b f(x) dx = \lim_{a \rightarrow -\infty} \int_a^b f(x) dx.$$

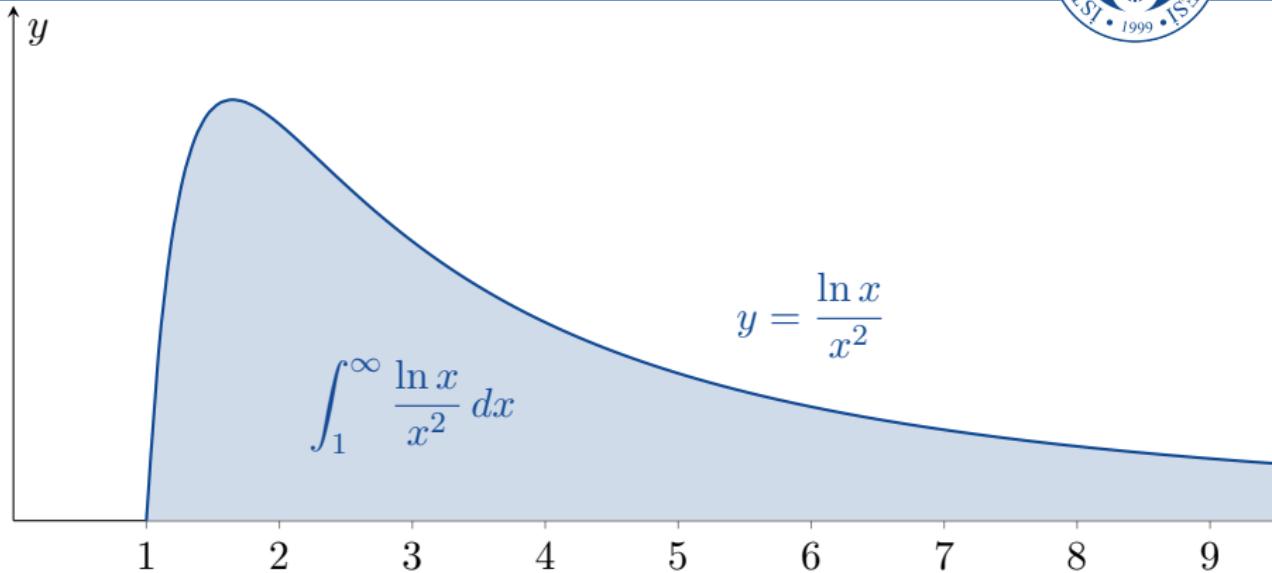
3. If  $f(x)$  is continuous on  $(-\infty, \infty)$ , then

$$\int_{-\infty}^{\infty} f(x) dx = \int_{-\infty}^c f(x) dx + \int_c^{\infty} f(x) dx,$$

where  $c$  is any real number.

In each case, if the limit exists and is finite, we say that the improper integral **converges** and that the limit is the **value** of the improper integral. If the limit fails to exist, the improper integral **diverges**.

## 8.8 Improper Integrals



### Example

Is the area under the curve  $y = \frac{\ln x}{x^2}$ , from  $x = 1$  to  $x = \infty$ , finite? Is so, what is its value?

## 8.8 Improper Integrals

Since

$$\begin{aligned}\int_1^b \frac{\ln x}{x^2} dx &= \left[ (\ln x) \left( -\frac{1}{x} \right) \right]_1^b - \int_1^b \left( -\frac{1}{x} \right) \left( \frac{1}{x} \right) dx \\ &= -\frac{\ln b}{b} - \left[ \frac{1}{x} \right]_1^b = -\frac{\ln b}{b} - \frac{1}{b} + 1,\end{aligned}$$

## 8.8 Improper Integrals

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## 8.8 Improper Integrals

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## 8.8 Improper Integrals

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## 8.8 Improper Integrals

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Therefore the integral converges and the area has finite value 1.

**EXAMPLE 2** Evaluate

$$\int_{-\infty}^{\infty} \frac{dx}{1+x^2}.$$

**Solution** According to the definition (Part 3), we can choose  $c = 0$  and write

$$\int_{-\infty}^{\infty} \frac{dx}{1+x^2} = \int_{-\infty}^0 \frac{dx}{1+x^2} + \int_0^{\infty} \frac{dx}{1+x^2}.$$

Next we evaluate each improper integral on the right side of the equation above.

$$\begin{aligned}\int_{-\infty}^0 \frac{dx}{1+x^2} &= \lim_{a \rightarrow -\infty} \int_a^0 \frac{dx}{1+x^2} \\&= \lim_{a \rightarrow -\infty} \left[ \tan^{-1} x \right]_a^0 \\&= \lim_{a \rightarrow -\infty} (\tan^{-1} 0 - \tan^{-1} a) = 0 - \left( -\frac{\pi}{2} \right) = \frac{\pi}{2}\end{aligned}$$

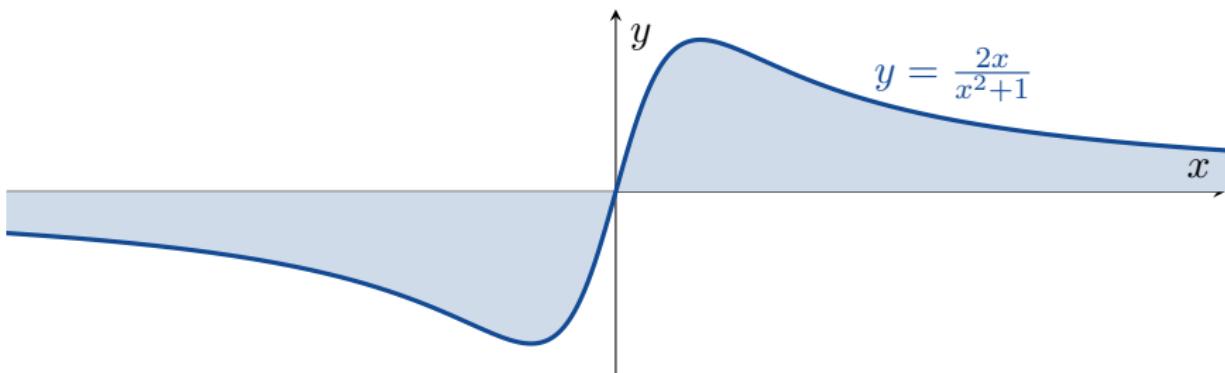
$$\begin{aligned}
 \int_0^\infty \frac{dx}{1+x^2} &= \lim_{b \rightarrow \infty} \int_0^b \frac{dx}{1+x^2} \\
 &= \lim_{b \rightarrow \infty} \left[ \tan^{-1} x \right]_0^b \\
 &= \lim_{b \rightarrow \infty} (\tan^{-1} b - \tan^{-1} 0) = \frac{\pi}{2} - 0 = \frac{\pi}{2}
 \end{aligned}$$

Thus,

$$\int_{-\infty}^{\infty} \frac{dx}{1+x^2} = \frac{\pi}{2} + \frac{\pi}{2} = \pi.$$

Since  $1/(1+x^2) > 0$ , the improper integral can be interpreted as the (finite) area beneath the curve and above the  $x$ -axis (Figure 8.15). ■

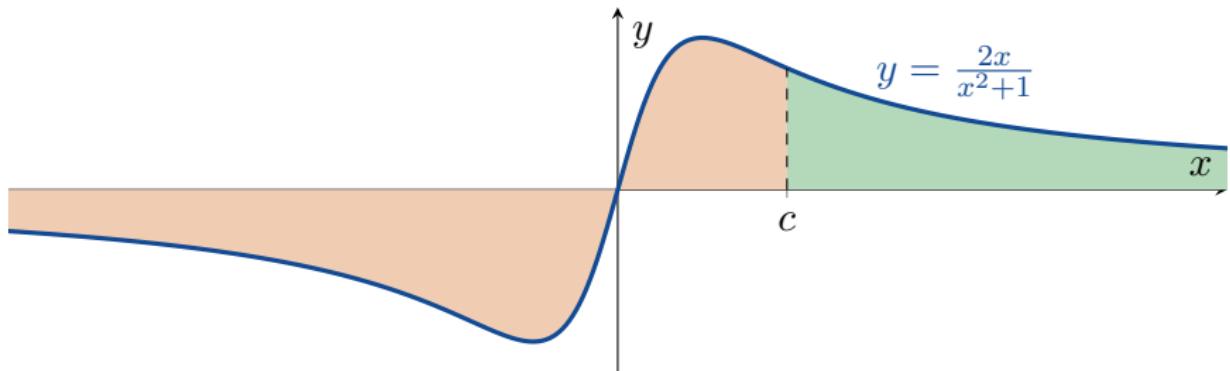
## 8.8 Improper Integrals



Remark

$$\int_{-\infty}^{\infty} f(x) dx$$

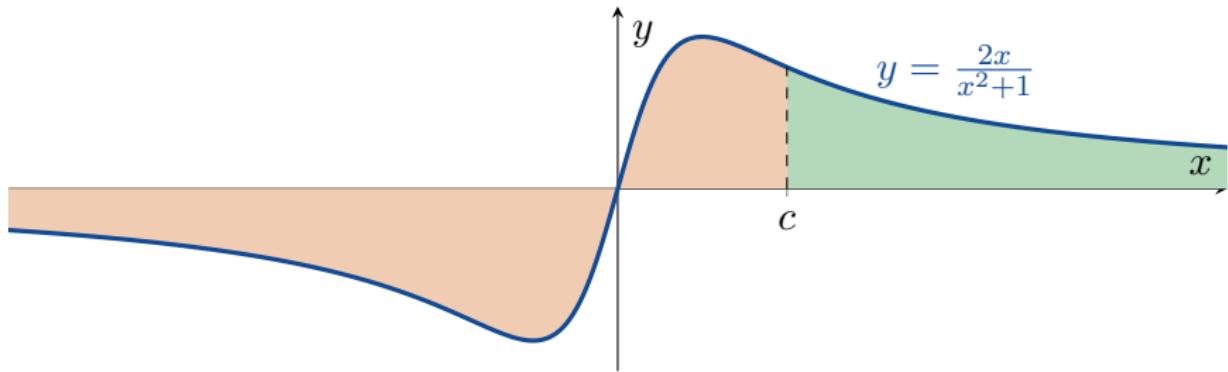
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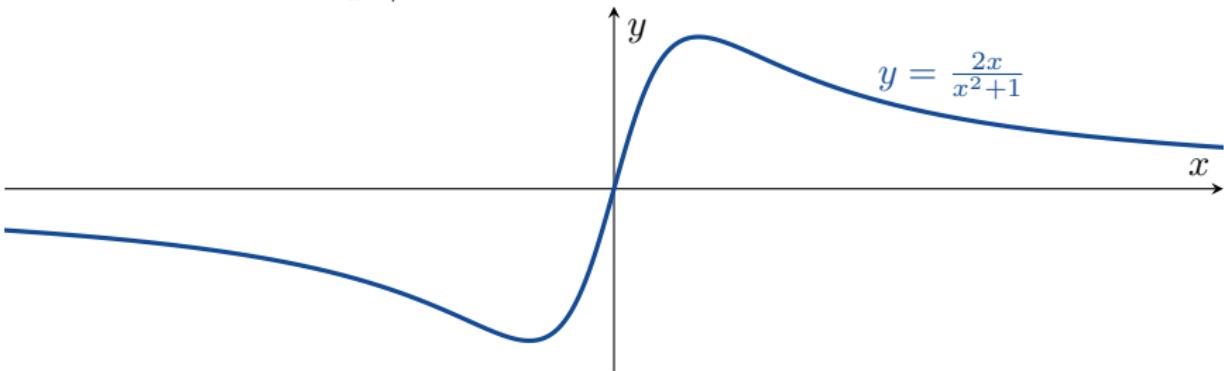
$$\int_{-\infty}^{\infty} f(x) dx = \int_{-\infty}^c f(x) dx + \int_c^{\infty} f(x) dx.$$

!!! This is not the same as  $\lim_{b \rightarrow \infty} \int_{-b}^b f(x) dx$  !!!

## 8.8 Improper Integrals



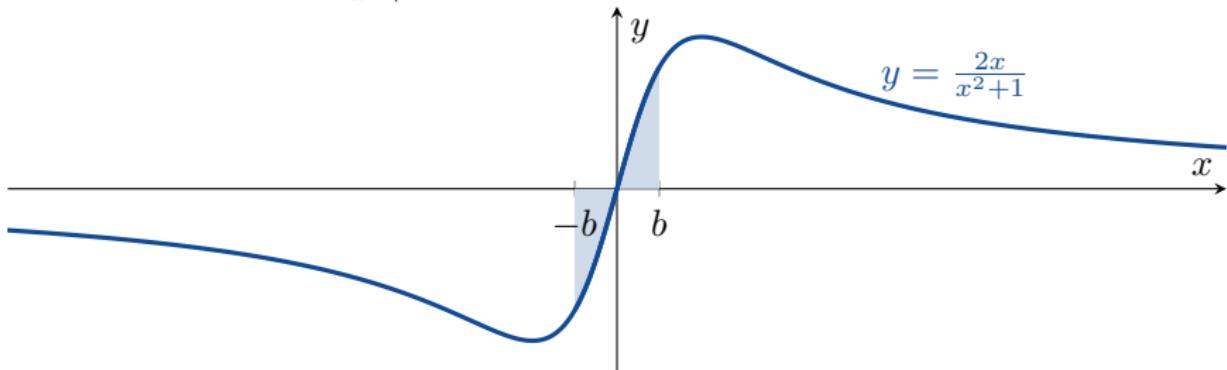
Consider  $f(x) = \frac{2x}{x^2+1}$ .



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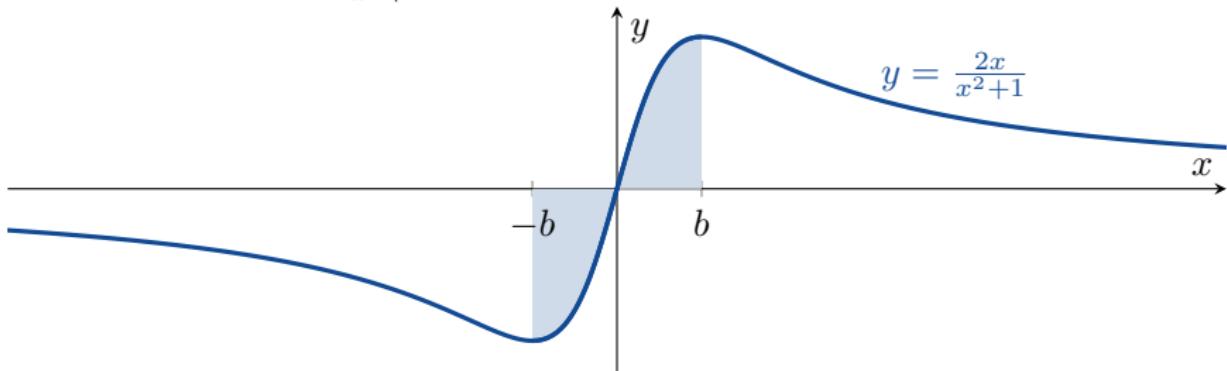


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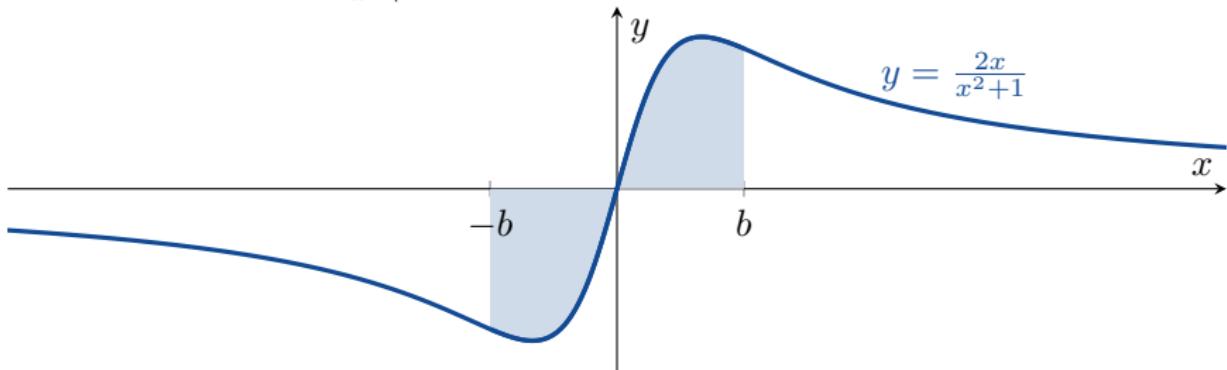
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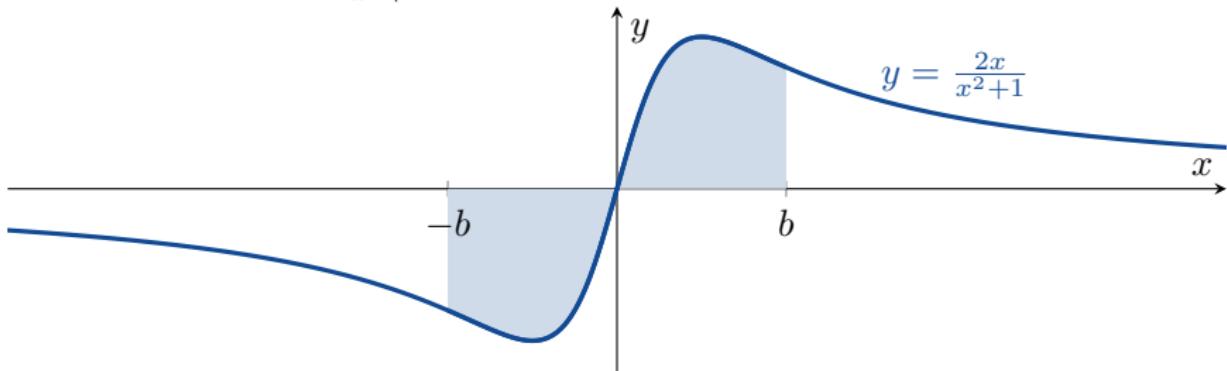
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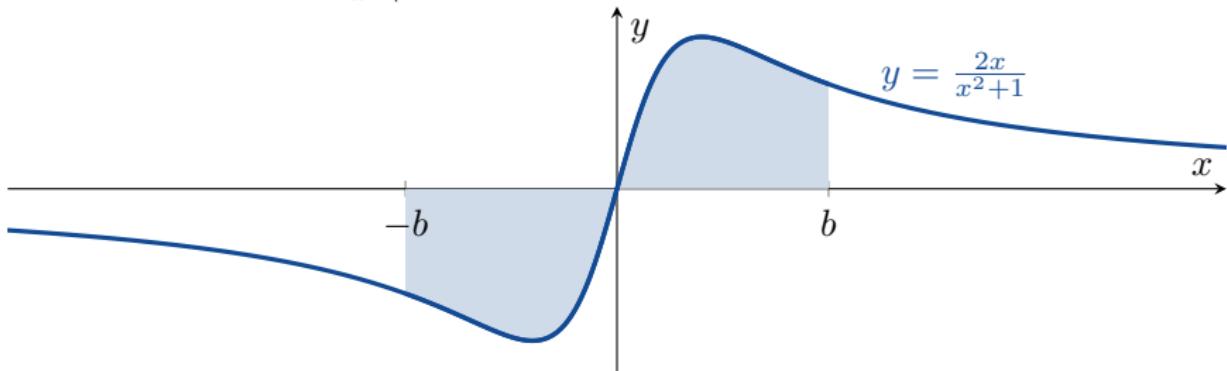


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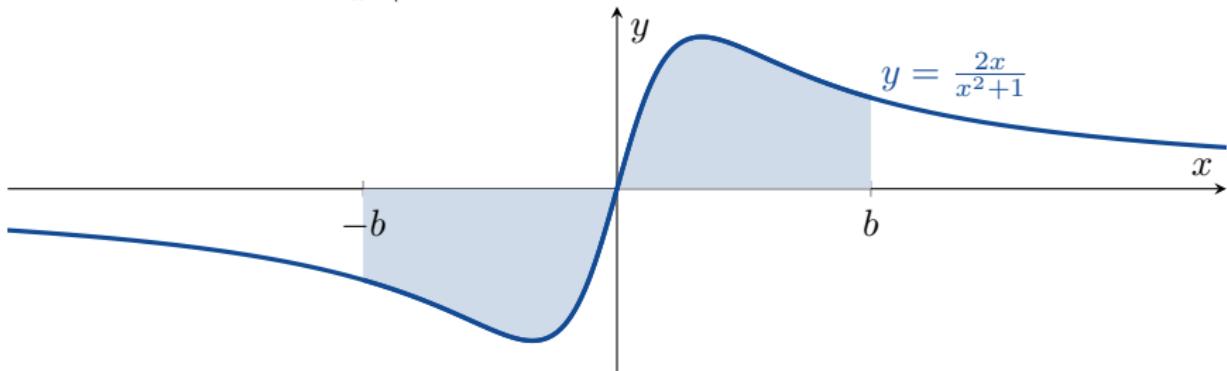


Since  $f$  is an odd function,

$$\lim_{b \rightarrow \infty} \int_{-b}^b \frac{2x}{x^2+1} dx = \lim_{b \rightarrow \infty} 0 = 0.$$

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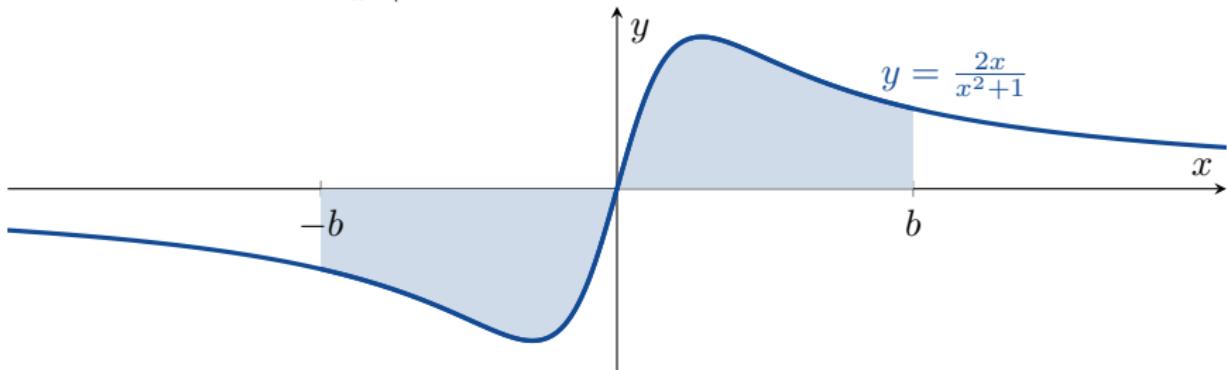


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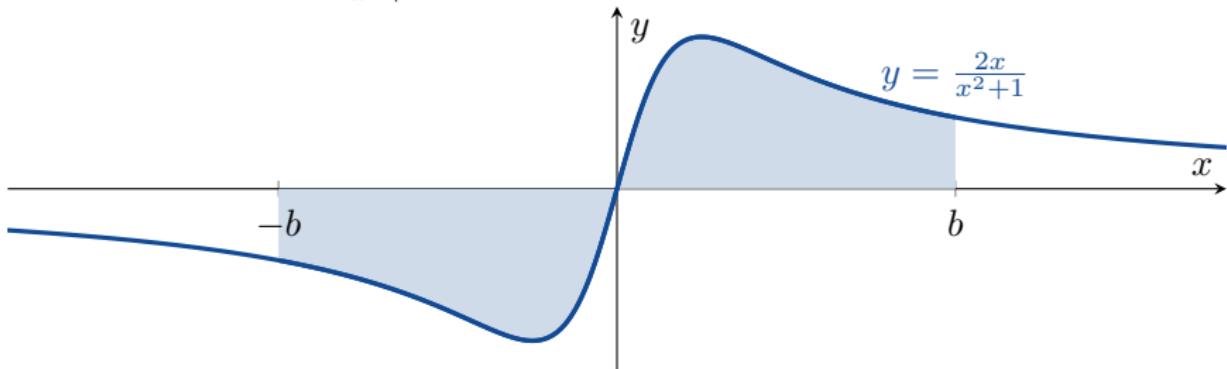


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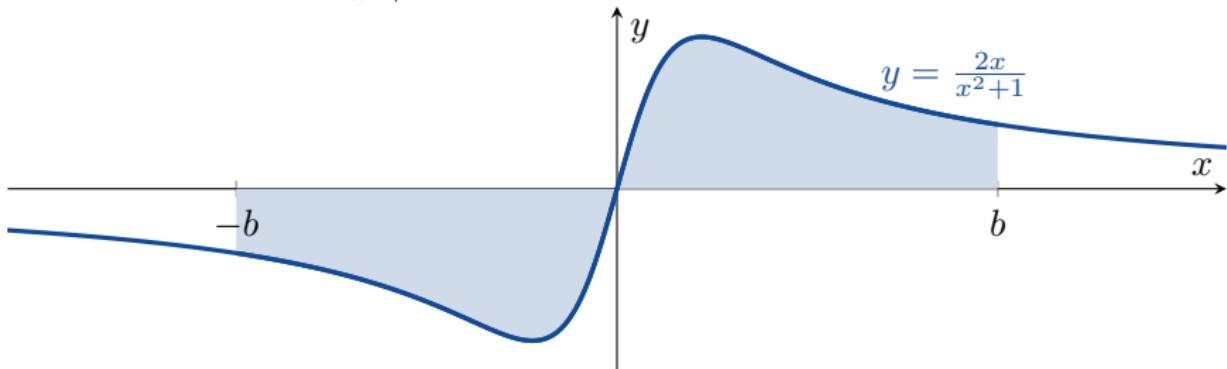
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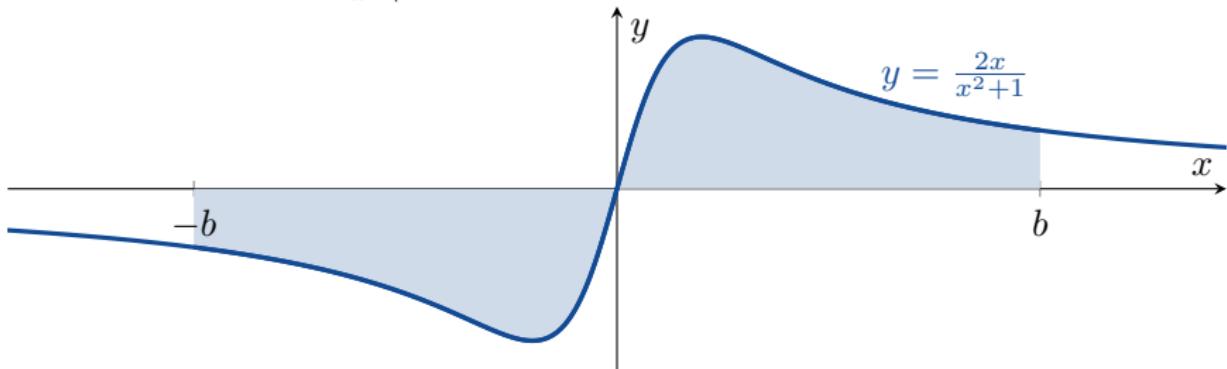
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So we must have  $\int_{-\infty}^{\infty} \frac{2x}{x^2+1} dx = 0$  right?

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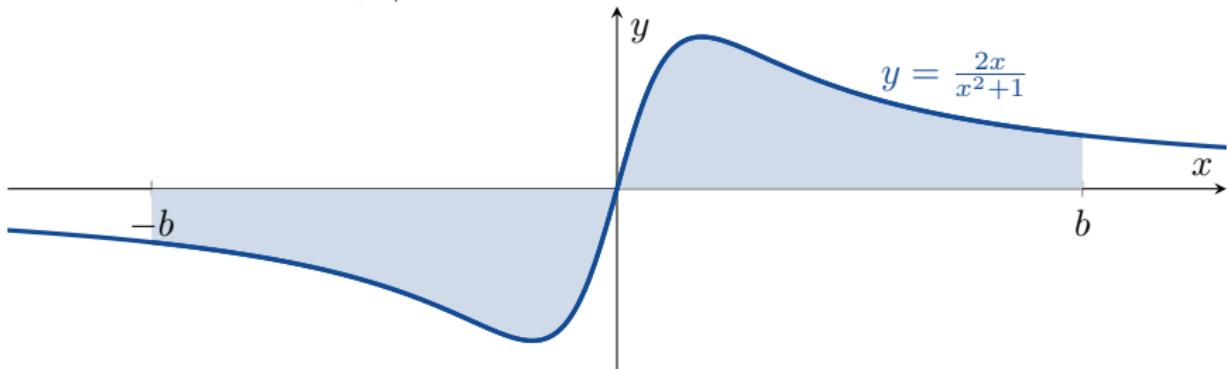
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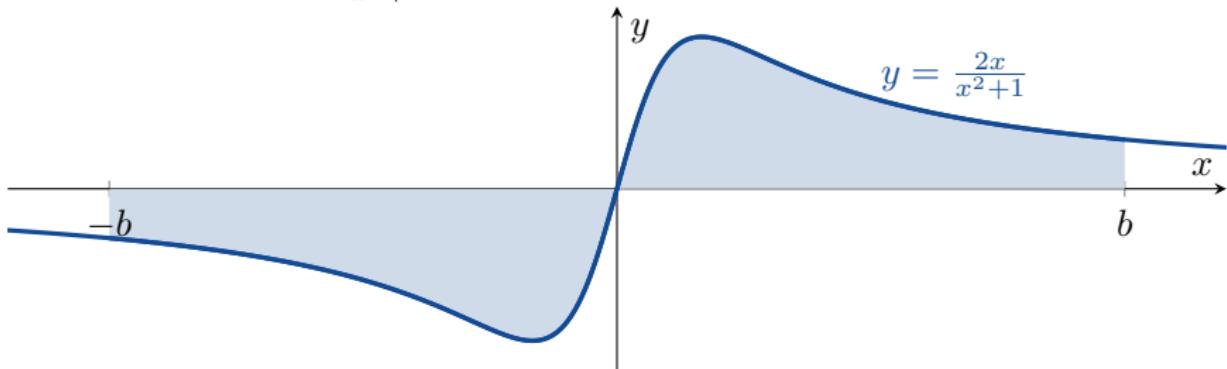
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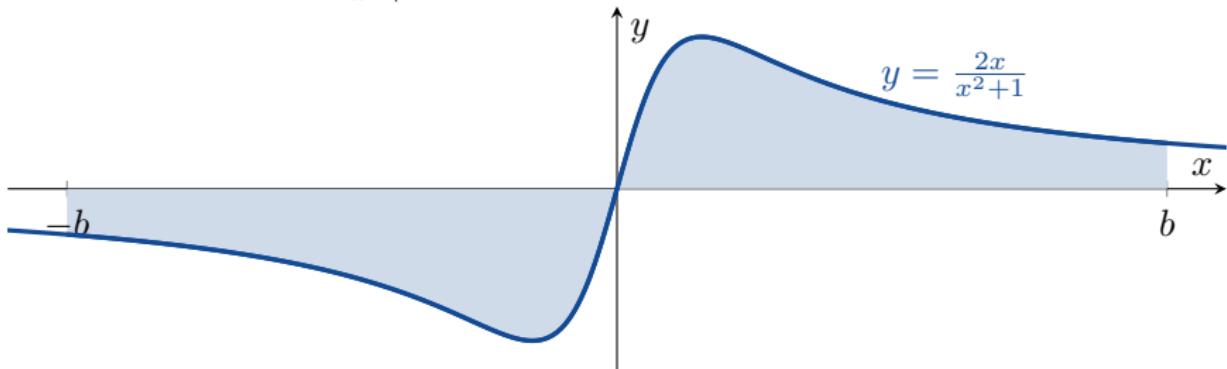
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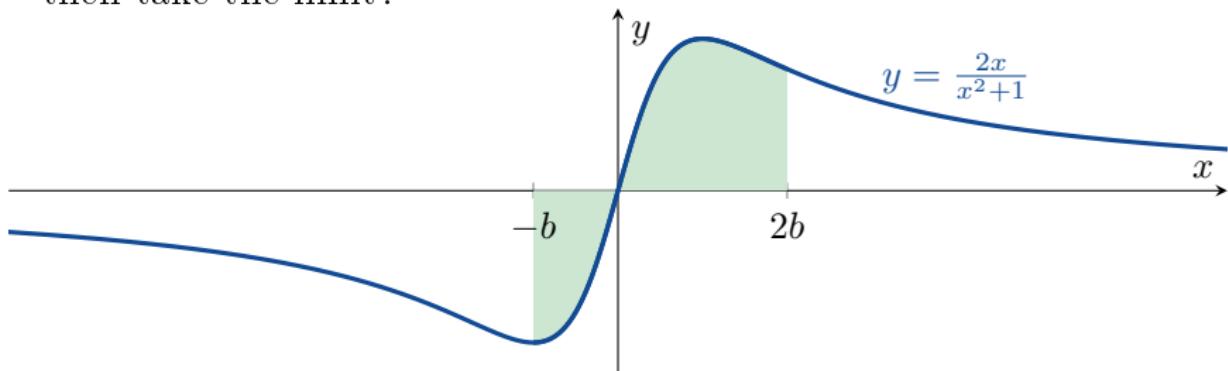
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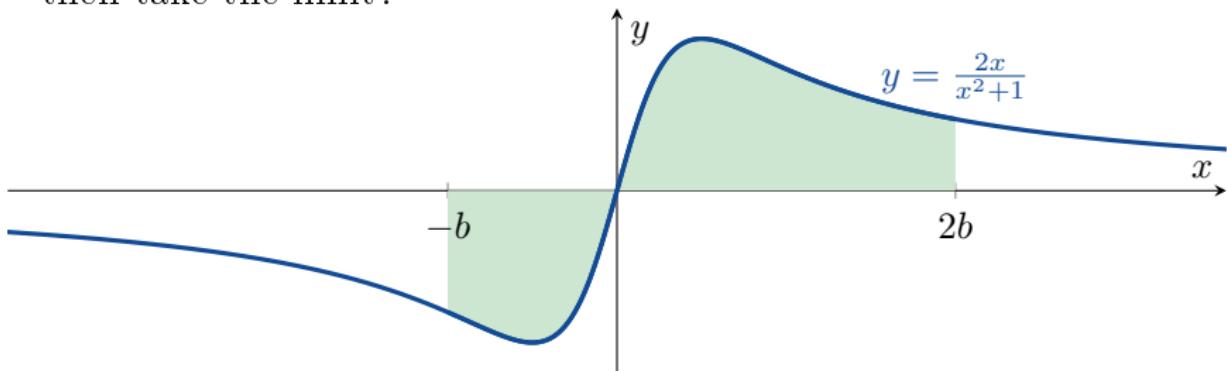
## 8.8 Improper Integrals

Let's try again. What do we get if we integrate over  $[-b, 2b]$  then take the limit?



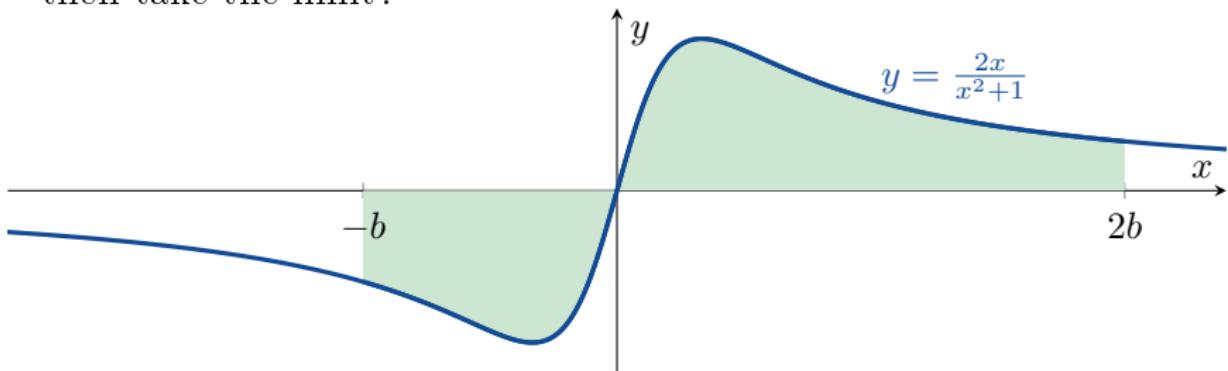
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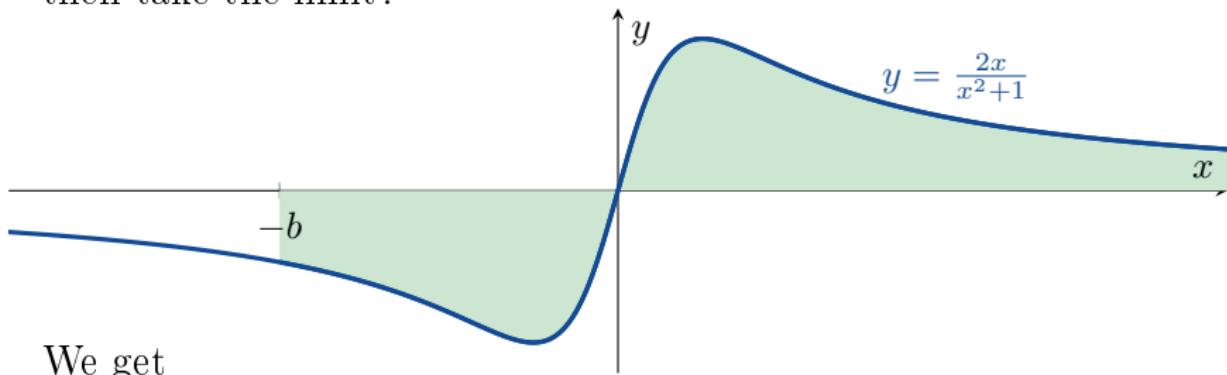
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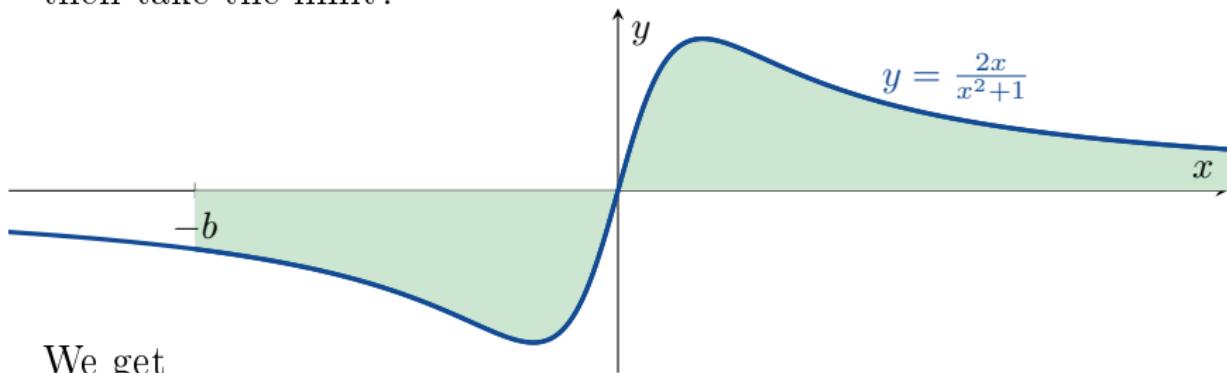


We get

$$\begin{aligned} \lim_{b \rightarrow \infty} \int_{-b}^{2b} \frac{2x}{x^2 + 1} dx &= \lim_{b \rightarrow \infty} \left[ \ln(x^2 + 1) \right]_{-b}^{2b} \\ &= \lim_{b \rightarrow \infty} \ln \left( \frac{4b^2 + 1}{b^2 + 1} \right) = \ln 4. \end{aligned}$$

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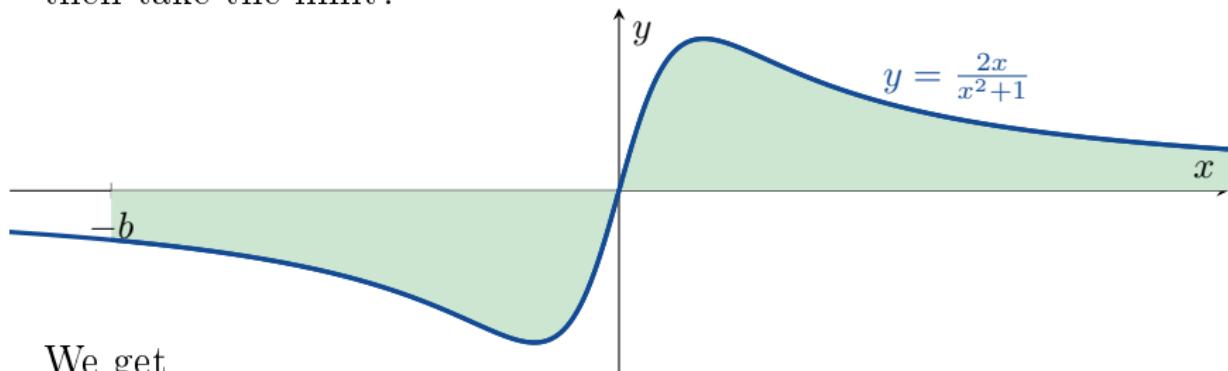
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So we must have  $\int_{-\infty}^{\infty} \frac{2x}{x^2+1} dx = \ln 4$  right? **NO!!!**

## 8.8 Improper Integrals



The correct answer is that since

$$\int_0^\infty \frac{2x \, dx}{x^2 + 1} = \infty,$$

and

$$\int_{-\infty}^0 \frac{2x \, dx}{x^2 + 1} = -\infty$$

(you prove)

## 8.8 Improper Integrals



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$$\int_0^\infty \frac{2x \, dx}{x^2 + 1} = \infty,$$

and

$$\int_{-\infty}^0 \frac{2x \, dx}{x^2 + 1} = -\infty$$

(you prove) it follows that

$$\int_{-\infty}^\infty \frac{2x \, dx}{x^2 + 1} = \int_{-\infty}^0 \frac{2x \, dx}{x^2 + 1} + \int_0^\infty \frac{2x \, dx}{x^2 + 1} = \text{“} \underbrace{\infty - \infty}_{\text{indeterminate form}} \text{”}$$

diverges.

## 8.8 Improper Integrals

### Example

For what values of  $p$  does the integral  $\int_1^\infty \frac{dx}{x^p}$  converge? When the integral does converge, what is its value?

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## 8.8 Improper Integrals

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Thus

$$\begin{aligned} \int_1^\infty x^{-p} dx &= \lim_{b \rightarrow \infty} \int_1^b x^{-p} dx = \frac{1}{1-p} \lim_{b \rightarrow \infty} \left( \frac{1}{b^{p-1}} - 1 \right) \\ &= \begin{cases} \frac{1}{p-1} & p > 1 \\ \infty & p < 1. \end{cases} \end{aligned}$$

## 8.8 Improper Integrals

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So the integral converges to  $\frac{1}{p-1}$  if  $p > 1$ , but diverges if  $p < 1$ .

## 8.8 Improper Integrals



What happens if  $p = 1$ ?

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Therefore the integral also diverges for  $p = 1$ .

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Theorem

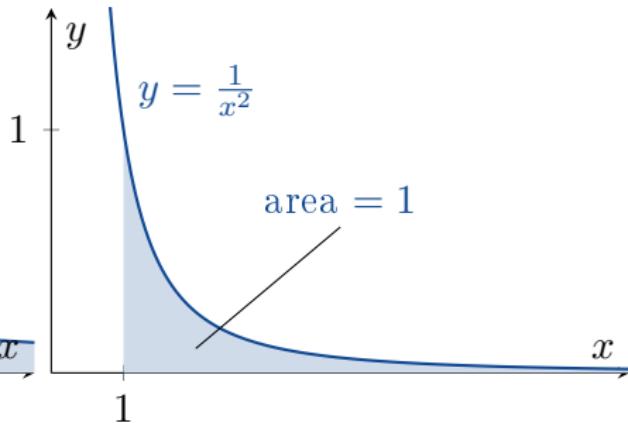
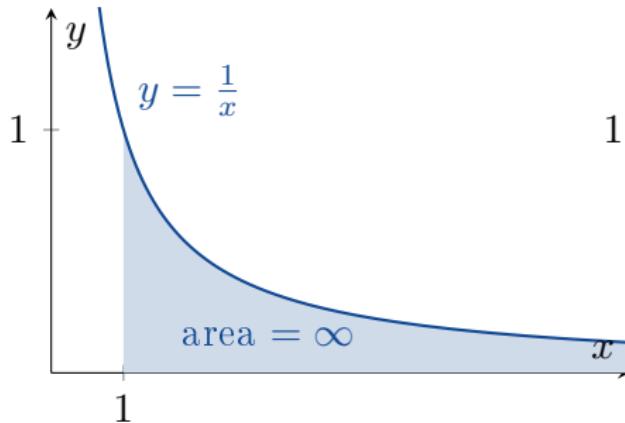
$$\int_1^{\infty} \frac{dx}{x^p} \quad \begin{cases} \text{converges if } p > 1, \\ \text{diverges if } p \leq 1. \end{cases}$$

## 8.8 Improper Integrals

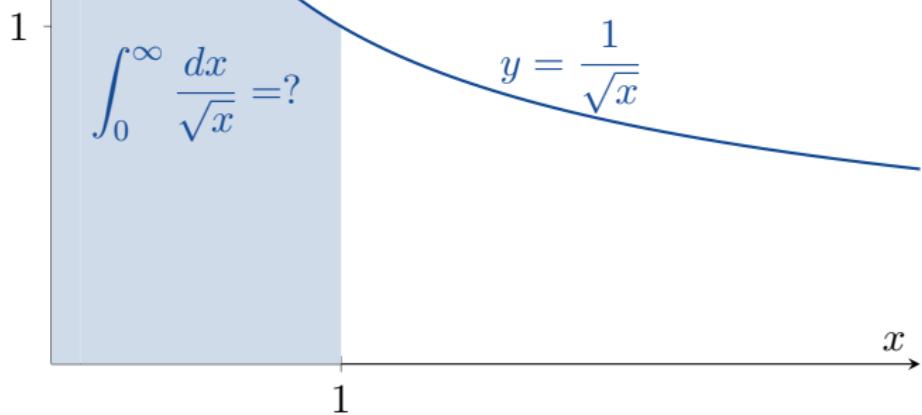
### Remark

In particular, please remember that

$$\int_1^{\infty} \frac{dx}{x} \quad \text{diverges} \quad \text{and} \quad \int_1^{\infty} \frac{dx}{x^2} \quad \text{converges.}$$



# Integrands with Vertical Asymptotes

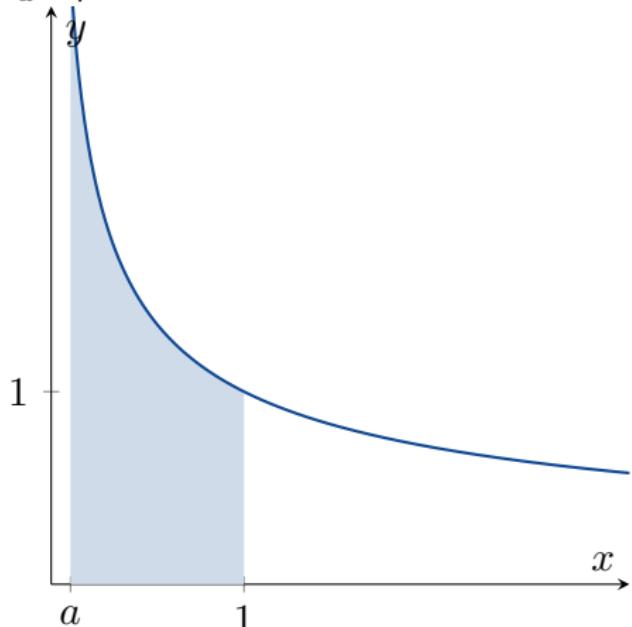


## 8.8 Improper Integrals



Step 1:

$$\int_a^1 \frac{dx}{\sqrt{x}} = ?$$

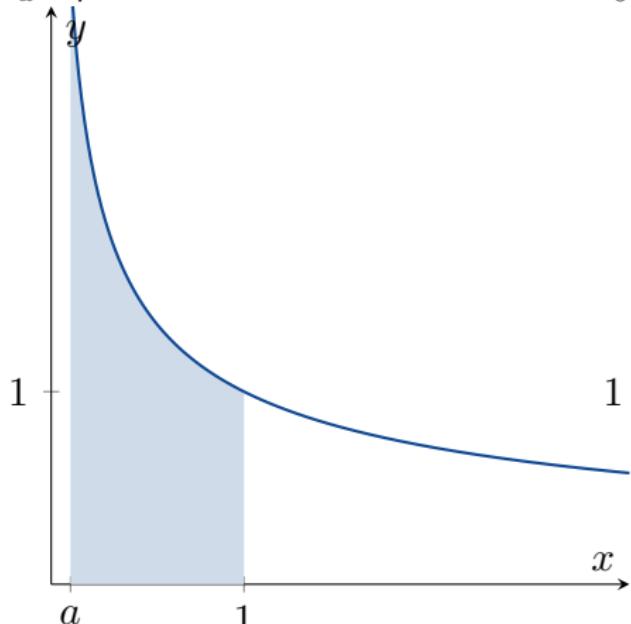


## 8.8 Improper Integrals



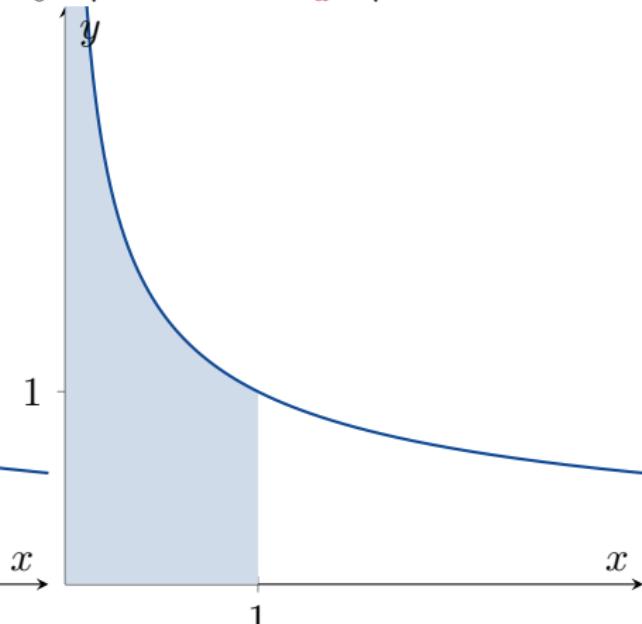
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$$\int_a^1 \frac{dx}{\sqrt{x}} = ?$$



Step 2:

$$\int_0^1 \frac{dx}{\sqrt{x}} = \lim_{a \rightarrow 0^+} \int_a^1 \frac{dx}{\sqrt{x}}$$



## 8.8 Improper Integrals



Since

$$\int_a^1 \frac{dx}{\sqrt{x}} = \left[ 2\sqrt{x} \right]_a^1 = 2 - 2\sqrt{a},$$

## 8.8 Improper Integrals



Since

$$\int_a^1 \frac{dx}{\sqrt{x}} = \left[ 2\sqrt{x} \right]_a^1 = 2 - 2\sqrt{a},$$

we have that

$$\int_0^1 \frac{dx}{\sqrt{x}} = \lim_{a \rightarrow 0^+} \int_a^1 \frac{dx}{\sqrt{x}} = \lim_{a \rightarrow 0^+} (2 - 2\sqrt{a}) = 2.$$

**DEFINITION** Integrals of functions that become infinite at a point within the interval of integration are **improper integrals of Type II**.

1. If  $f(x)$  is continuous on  $(a, b]$  and discontinuous at  $a$ , then

$$\int_a^b f(x) dx = \lim_{c \rightarrow a^+} \int_c^b f(x) dx.$$

2. If  $f(x)$  is continuous on  $[a, b)$  and discontinuous at  $b$ , then

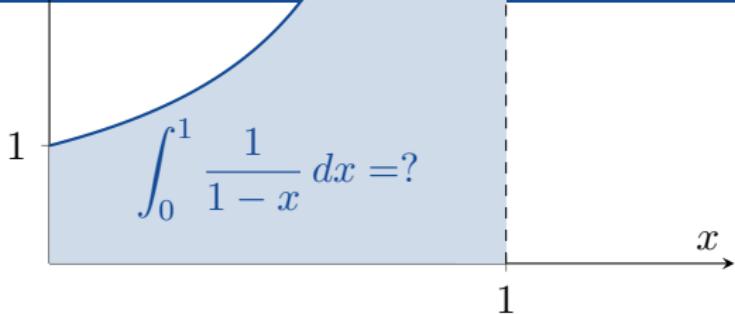
$$\int_a^b f(x) dx = \lim_{c \rightarrow b^-} \int_a^c f(x) dx.$$

3. If  $f(x)$  is discontinuous at  $c$ , where  $a < c < b$ , and continuous on  $[a, c) \cup (c, b]$ , then

$$\int_a^b f(x) dx = \int_a^c f(x) dx + \int_c^b f(x) dx.$$

In each case, if the limit exists and is finite, we say the improper integral **converges** and that the limit is the **value** of the improper integral. If the limit does not exist, the integral **diverges**.

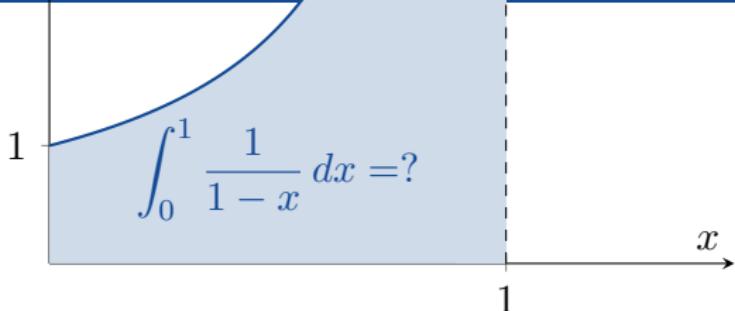
## 8.8 Improper Integrals



Example

Investigate the convergence of  $\int_0^1 \frac{1}{1-x} dx$ .

## 8.8 Improper Integrals

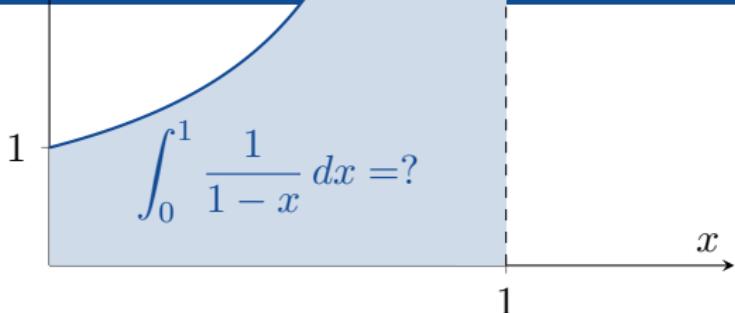


Example

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$$\int_0^1 \frac{1}{1-x} dx = \lim_{b \rightarrow 1^-} \int_0^b \frac{1}{1-x} dx = \\ =$$

## 8.8 Improper Integrals

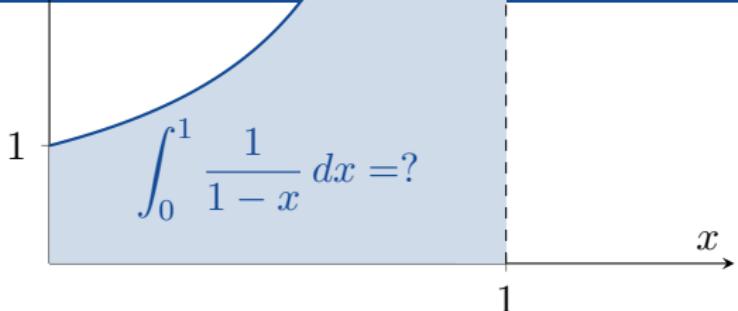


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$$\begin{aligned}\int_0^1 \frac{1}{1-x} dx &= \lim_{b \rightarrow 1^-} \int_0^b \frac{1}{1-x} dx = \lim_{b \rightarrow 1^-} \left[ -\ln|1-x| \right]_0^b \\ &= \end{aligned}$$

## 8.8 Improper Integrals



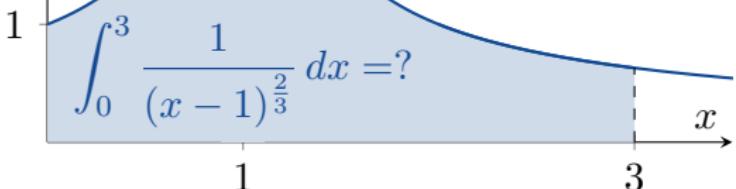
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This integral diverges.

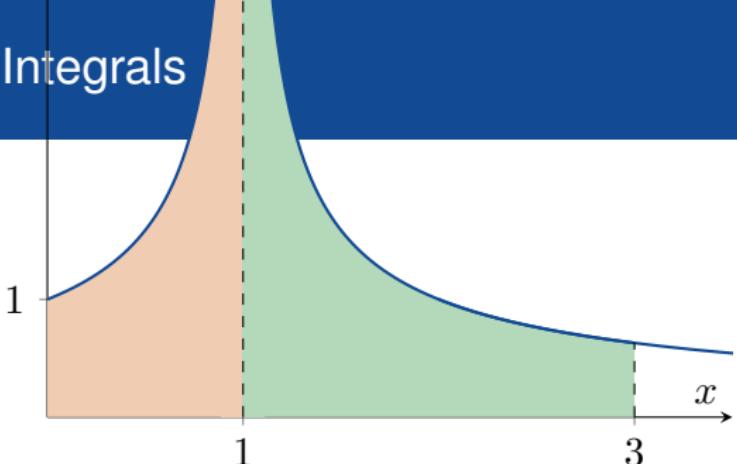
## 8.8 Improper Integrals



Example

Calculate  $\int_0^3 \frac{dx}{(x-1)^{\frac{2}{3}}}$ .

## 8.8 Improper Integrals



Example

Calculate  $\int_0^3 \frac{dx}{(x-1)^{\frac{2}{3}}}.$

Because we have a discontinuity at  $x = 1$ , we need to consider

$$\int_0^3 \frac{dx}{(x-1)^{\frac{2}{3}}} = \int_0^1 \frac{dx}{(x-1)^{\frac{2}{3}}} + \int_1^3 \frac{dx}{(x-1)^{\frac{2}{3}}}$$

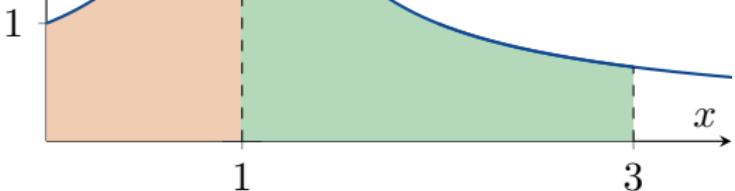
## 8.8 Improper Integrals



Since

$$\begin{aligned}\int_0^1 \frac{dx}{(x-1)^{\frac{2}{3}}} &= \lim_{b \rightarrow 1^-} \int_0^b \frac{dx}{(x-1)^{\frac{2}{3}}} = \lim_{b \rightarrow 1^-} \left[ 3(x-1)^{\frac{1}{3}} \right]_0^b \\ &= \lim_{b \rightarrow 1^-} \left( 3(b-1)^{\frac{1}{3}} + 3 \right) = 3\end{aligned}$$

## 8.8 Improper Integrals



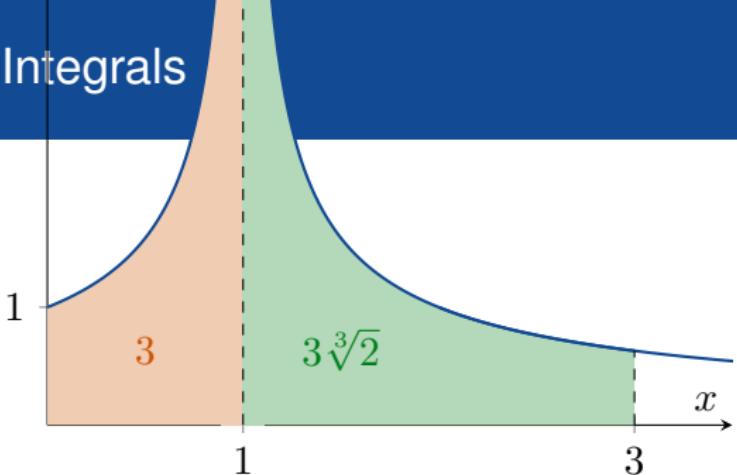
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and

$$\begin{aligned}\int_1^3 \frac{dx}{(x-1)^{\frac{2}{3}}} &= \lim_{b \rightarrow 1^+} \int_b^3 \frac{dx}{(x-1)^{\frac{2}{3}}} = \lim_{b \rightarrow 1^+} \left[ 3(x-1)^{\frac{1}{3}} \right]_b^3 \\ &= \lim_{b \rightarrow 1^+} \left( 3(3-1)^{\frac{1}{3}} - 3(b-1)^{\frac{1}{3}} \right) = 3\sqrt[3]{2}\end{aligned}$$

## 8.8 Improper Integrals



we have that

$$\int_0^3 \frac{dx}{(x-1)^{\frac{2}{3}}} = \int_0^1 \frac{dx}{(x-1)^{\frac{2}{3}}} + \int_1^3 \frac{dx}{(x-1)^{\frac{2}{3}}} = 3 + 3\sqrt[3]{2}.$$

## 8.8 Improper Integrals



### Remark

Sometimes we cannot evaluate an improper integral, but we can still determine whether it converges or diverges.

## 8.8 Improper Integrals



### Remark

Sometimes we cannot evaluate an improper integral, but we can still determine whether it converges or diverges.

### Example

Does  $\int_1^{\infty} e^{-x^2} dx$  converge or diverge?

We can not calculate  $\int_1^b e^{-x^2} dx$  because it is nonelementary.  
But we can answer this example another way.

## 8.8 Improper Integrals



Since  $e^{-x^2} > 0$ , we know that  $I(b) = \int_1^b e^{-x^2} dx$  is an increasing function of  $b$ .

## 8.8 Improper Integrals



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So either

- $\lim_{b \rightarrow \infty} \int_1^b e^{-x^2} dx = \infty$ ; or
- $\lim_{b \rightarrow \infty} \int_1^b e^{-x^2} dx$  is a finite number.

## 8.8 Improper Integrals



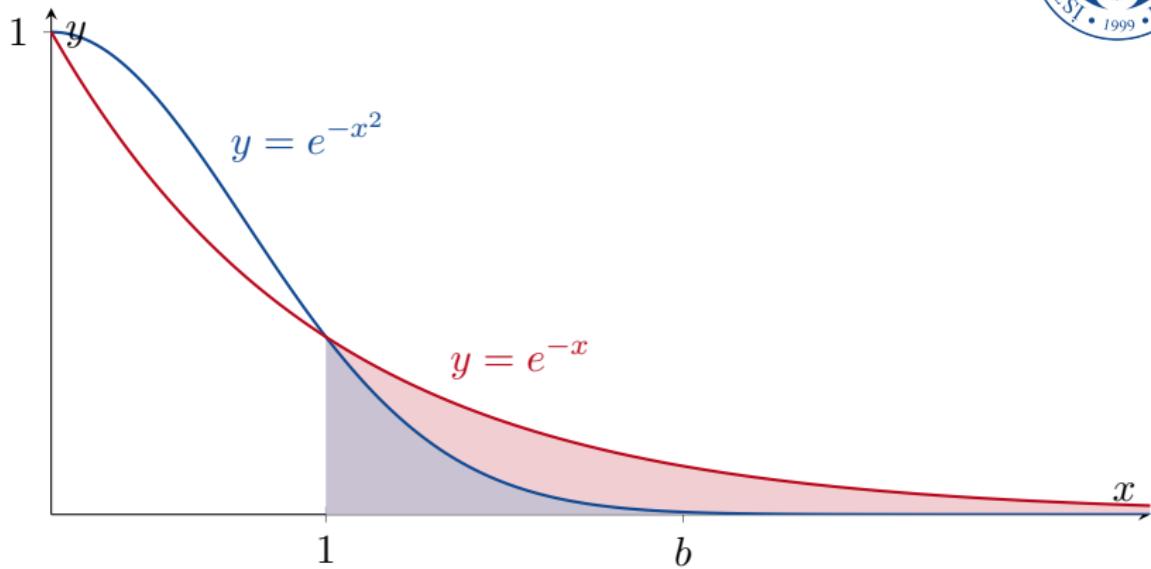
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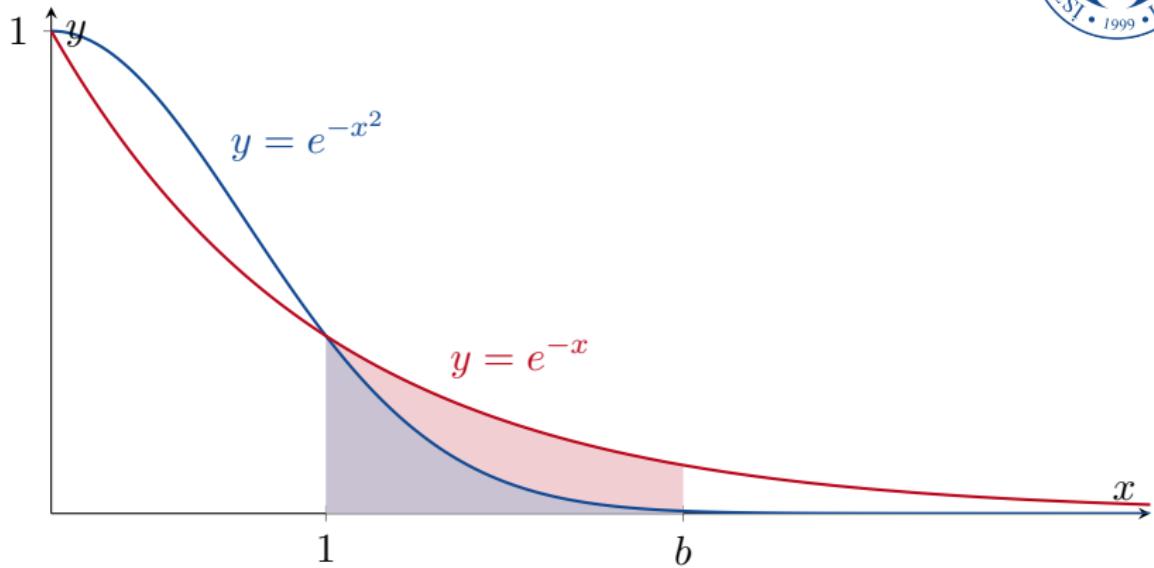
I am going to prove to you that  $\lim_{b \rightarrow \infty} \int_1^b e^{-x^2} dx$  is finite.

## 8.8 Improper Integrals



Note that  $e^{-x^2} \leq e^{-x}$  for all  $x \geq 1$ .

## 8.8 Improper Integrals

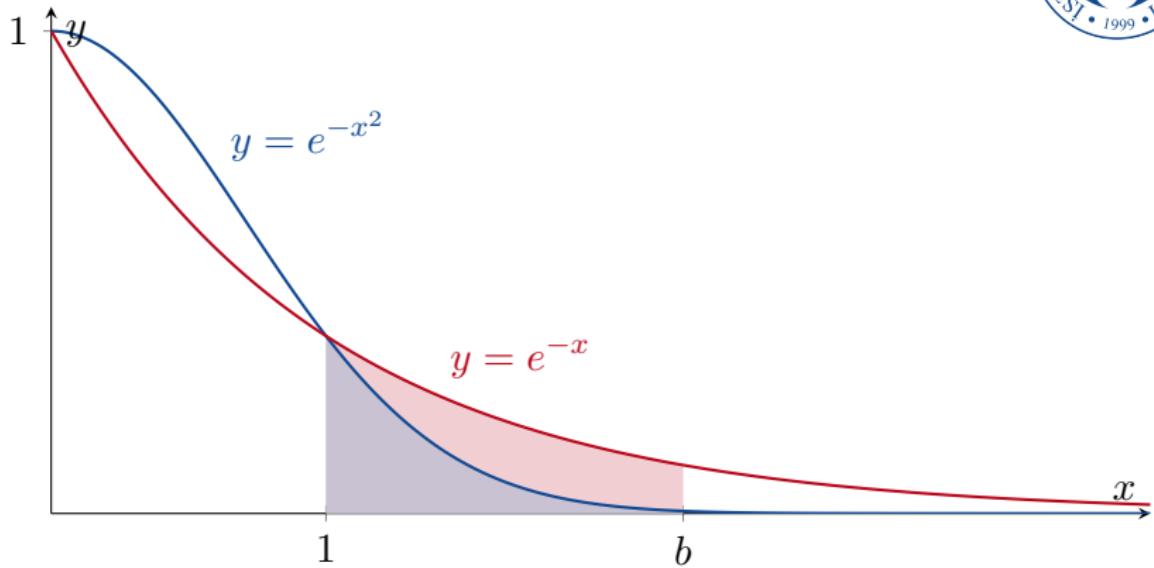


Note that  $e^{-x^2} \leq e^{-x}$  for all  $x \geq 1$ . So

$$\int_1^b e^{-x^2} dx \leq \int_1^b e^{-x} dx$$

for any  $b > 1$ .

## 8.8 Improper Integrals



Note that  $e^{-x^2} \leq e^{-x}$  for all  $x \geq 1$ . So

$$\int_1^b e^{-x^2} dx \leq \int_1^b e^{-x} dx = -e^{-b} + e^{-1} < e^{-1} \approx 0.36788$$

for any  $b > 1$ .

## 8.8 Improper Integrals



Therefore

$$\int_1^{\infty} e^{-x^2} dx = \lim_{b \rightarrow \infty} \int_1^b e^{-x^2} dx$$

converges to a finite value.

## 8.8 Improper Integrals



Theorem (Direct Comparison Test)

Let  $f : [a, \infty) \rightarrow \mathbb{R}$  and  $g : [a, \infty) \rightarrow \mathbb{R}$  be continuous functions.  
Suppose that

$$0 \leq f(x) \leq g(x)$$

for all  $x \in [a, \infty)$ .

## 8.8 Improper Integrals



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**1**  $\int_a^\infty g(x) dx$  converges  $\implies \int_a^\infty f(x) dx$  converges;

## 8.8 Improper Integrals



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1  $\int_a^\infty g(x) dx$  converges  $\implies \int_a^\infty f(x) dx$  converges;

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## 8.8 Improper Integrals



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2  $\int_a^\infty f(x) dx$  diverges  $\implies \int_a^\infty g(x) dx$  diverges.

(you can read the proof in the book)

**EXAMPLE 7** These examples illustrate how we use Theorem 2.

(a)  $\int_1^\infty \frac{\sin^2 x}{x^2} dx$  converges because

$$0 \leq \frac{\sin^2 x}{x^2} \leq \frac{1}{x^2} \text{ on } [1, \infty) \text{ and } \int_1^\infty \frac{1}{x^2} dx \text{ converges.}$$

Example 3

(b)  $\int_1^\infty \frac{1}{\sqrt{x^2 - 0.1}} dx$  diverges because

$$\frac{1}{\sqrt{x^2 - 0.1}} \geq \frac{1}{x} \text{ on } [1, \infty) \text{ and } \int_1^\infty \frac{1}{x} dx \text{ diverges.}$$

Example 3

(c)  $\int_0^{\pi/2} \frac{\cos x}{\sqrt{x}} dx$  converges because

$$0 \leq \frac{\cos x}{\sqrt{x}} \leq \frac{1}{\sqrt{x}} \text{ on } \left[0, \frac{\pi}{2}\right], \quad 0 \leq \cos x \leq 1 \text{ on } \left[0, \frac{\pi}{2}\right]$$

and

$$\begin{aligned} \int_0^{\pi/2} \frac{dx}{\sqrt{x}} &= \lim_{a \rightarrow 0^+} \int_a^{\pi/2} \frac{dx}{\sqrt{x}} \\ &= \lim_{a \rightarrow 0^+} \left[ \sqrt{4x} \right]_a^{\pi/2} \quad 2\sqrt{x} = \sqrt{4x} \\ &= \lim_{a \rightarrow 0^+} (\sqrt{2\pi} - \sqrt{4a}) = \sqrt{2\pi} \quad \text{converges.} \end{aligned}$$



## 8.8 Improper Integrals



### Theorem (Limit Comparison Test)

Suppose that

- $f : [a, \infty) \rightarrow \mathbb{R}$  and  $g : [a, \infty) \rightarrow \mathbb{R}$  are continuous;
- $f > 0$  and  $g > 0$ ;
- $\lim_{x \rightarrow \infty} \frac{f(x)}{g(x)}$  exists and  $0 < \lim_{x \rightarrow \infty} \frac{f(x)}{g(x)} < \infty$ .

## 8.8 Improper Integrals



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## 8.8 Improper Integrals



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Then

$$\int_a^{\infty} f(x) dx \quad \text{and} \quad \int_a^{\infty} g(x) dx$$

both converge or both diverge.

## 8.8 Improper Integrals

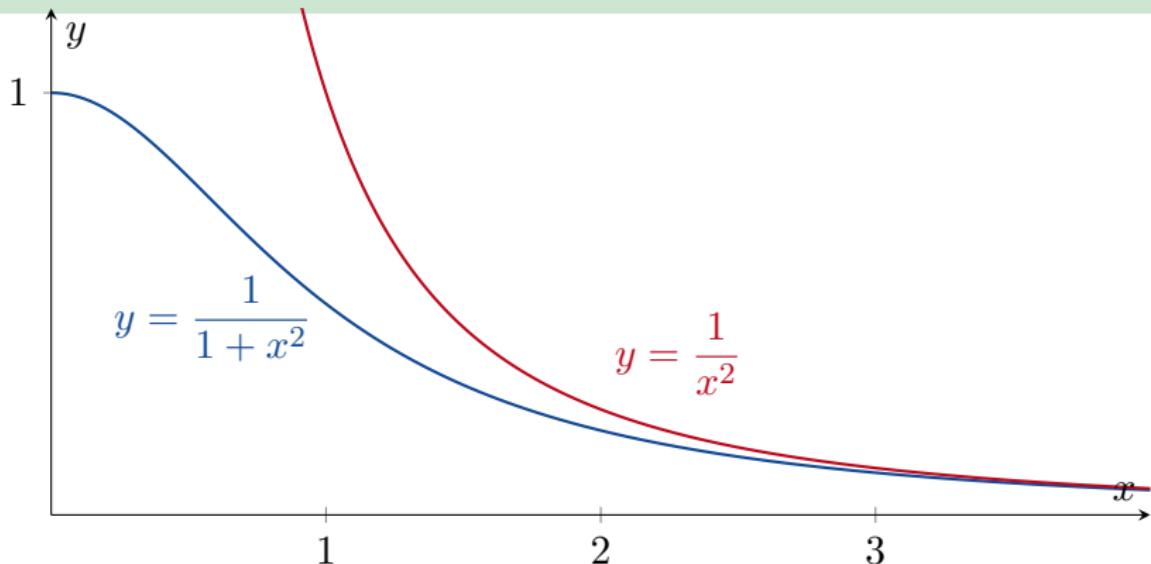
### Example

Show that the integral  $\int_1^\infty \frac{dx}{1+x^2}$  converges, by comparing it with the integral  $\int_1^\infty \frac{1}{x^2} dx$ .

## 8.8 Improper Integrals

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## 8.8 Improper Integrals



Note first that the functions  $f(x) = \frac{1}{x^2}$  and  $g(x) = \frac{1}{1+x^2}$  are both

- continuous on  $[1, \infty)$ ; and
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## 8.8 Improper Integrals

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Moreover

$$\lim_{x \rightarrow \infty} \frac{f(x)}{g(x)} = \lim_{x \rightarrow \infty} \frac{\frac{1}{x^2}}{\frac{1}{1+x^2}} = \lim_{x \rightarrow \infty} \frac{1+x^2}{x^2} = \lim_{x \rightarrow \infty} \left( \frac{1}{x^2} + 1 \right) = 0 + 1 = 1 > 0.$$

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This means that the integrals  $\int_1^\infty \frac{dx}{1+x^2}$  and  $\int_1^\infty \frac{dx}{x^2}$  both converge, or both diverge.

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This means that the integrals  $\int_1^\infty \frac{dx}{1+x^2}$  and  $\int_1^\infty \frac{dx}{x^2}$  both converge, or both diverge.

Since we already know that  $\int_1^\infty \frac{dx}{x^2}$  converges, we must have that

$\int_1^\infty \frac{dx}{1+x^2}$  converges as well.

## 8.8 Improper Integrals



### Remark

!!! This does not mean that these two integrals converge to the same value. !!!

## 8.8 Improper Integrals



### Remark

!!! This does not mean that these two integrals converge to the same value. !!!

Actually,

$$\int_1^\infty \frac{dx}{x^2} = 1$$

and

$$\int_1^\infty \frac{dx}{1+x^2} = \frac{\pi}{4}.$$

**EXAMPLE 9** Investigate the convergence of  $\int_1^\infty \frac{1 - e^{-x}}{x} dx$ .

**Solution** The integrand suggests a comparison of  $f(x) = (1 - e^{-x})/x$  with  $g(x) = 1/x$ . However, we cannot use the Direct Comparison Test because  $f(x) \leq g(x)$  and the integral of  $g(x)$  diverges. On the other hand, using the Limit Comparison Test we find that

$$\lim_{x \rightarrow \infty} \frac{f(x)}{g(x)} = \lim_{x \rightarrow \infty} \left( \frac{1 - e^{-x}}{x} \right) \left( \frac{x}{1} \right) = \lim_{x \rightarrow \infty} (1 - e^{-x}) = 1,$$

which is a positive finite limit. Therefore,  $\int_1^\infty \frac{1 - e^{-x}}{x} dx$  diverges because  $\int_1^\infty \frac{dx}{x}$  diverges. Approximations to the improper integral are given in Table 8.5. Note that the values do not appear to approach any fixed limiting value as  $b \rightarrow \infty$ . ■

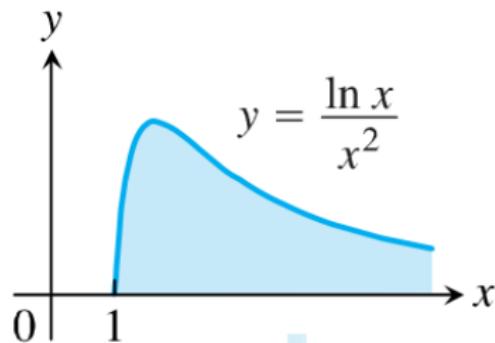
## 8.8 Improper Integrals



Let's finish with a quick recap.

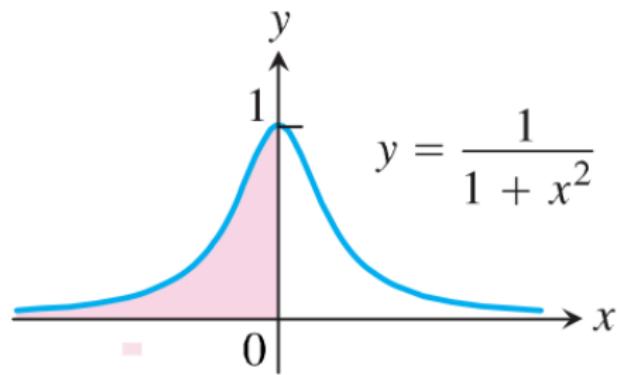
## Upper limit

$$\int_1^{\infty} \frac{\ln x}{x^2} dx = \lim_{b \rightarrow \infty} \int_1^b \frac{\ln x}{x^2} dx$$



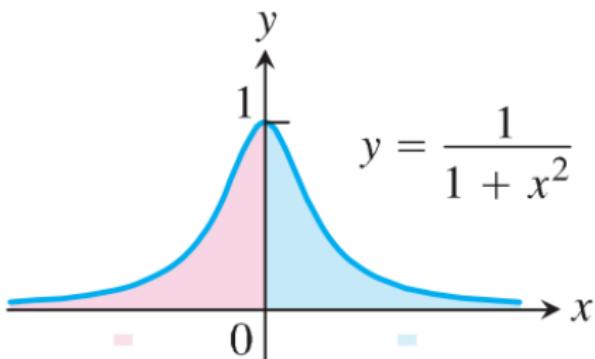
## Lower limit

$$\int_{-\infty}^0 \frac{dx}{1+x^2} = \lim_{a \rightarrow -\infty} \int_a^0 \frac{dx}{1+x^2}$$



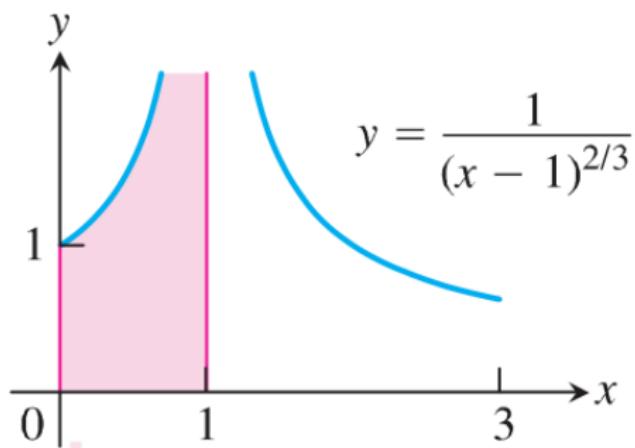
## Both limits

$$\int_{-\infty}^{\infty} \frac{dx}{1+x^2} = \lim_{b \rightarrow -\infty} \int_b^0 \frac{dx}{1+x^2} + \lim_{c \rightarrow \infty} \int_0^c \frac{dx}{1+x^2}$$



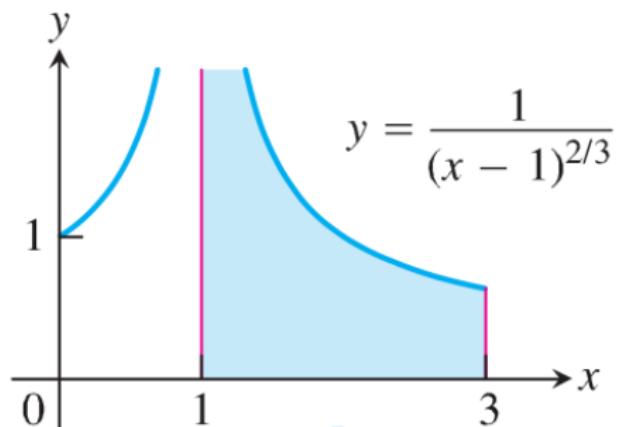
## Upper endpoint

$$\int_0^1 \frac{dx}{(x - 1)^{2/3}} = \lim_{b \rightarrow 1^-} \int_0^b \frac{dx}{(x - 1)^{2/3}}$$



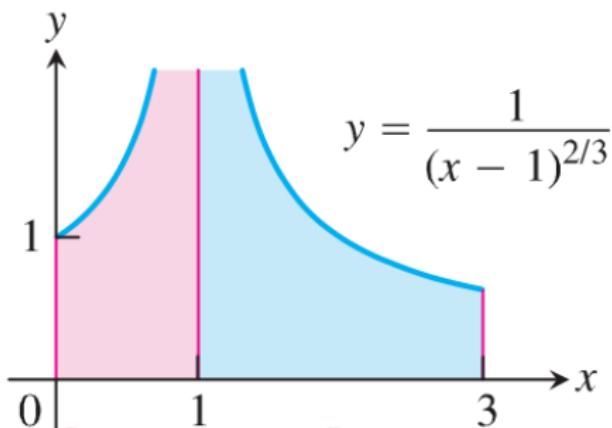
## Lower endpoint

$$\int_1^3 \frac{dx}{(x - 1)^{2/3}} = \lim_{d \rightarrow 1^+} \int_d^3 \frac{dx}{(x - 1)^{2/3}}$$



## Interior point

$$\int_0^3 \frac{dx}{(x - 1)^{2/3}} = \int_0^1 \frac{dx}{(x - 1)^{2/3}} + \int_1^3 \frac{dx}{(x - 1)^{2/3}}$$





# Next Time

- 11.1 Three-Dimensional Coordinate Systems
- 11.2 Vectors
- 11.3 The Dot Product