2015.03.26 **MAT234 Matematik IV – Ara Sınavın Çözümleri** N. Course

Soru 1 (Convergent Sequences).

(a) [10p] Let (a_n) be a sequence of real numbers and let $l \in \mathbb{R}$. Give the definition of $a_n \to l$ as $n \to \infty$.

We say that (a_n) tends to l ($a_n \to l$ as $n \to \infty$) iff, for all $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that
$$n > N \implies |a_n - l| < \varepsilon.$$

(b) [15p] Let $b_n = \frac{4n-1}{2n+2}$ for all $n \in \mathbb{N}$. **Use the definition** that you wrote in part (a) to prove that $b_n \to 2$ as $n \to \infty$.

Let $\varepsilon > 0$. Choose $N \in \mathbb{N}$ such that $N \geq \frac{5}{2\varepsilon}$. Then
$$n > N \implies |b_n - 2| = \left| \frac{4n-1}{2n+2} - \frac{4n+4}{2n+2} \right|$$
$$= \frac{5}{2n+2} \leq \frac{5}{2n} < \frac{5}{2N} \leq \frac{5}{2\left(\frac{5}{2\varepsilon}\right)} = \varepsilon.$$
Therefore $b_n \to 2$ as $n \to \infty$.

(c) [25p] Suppose that

- $(a_n)_{n=1}^{\infty}$ and $(b_n)_{n=1}^{\infty}$ are sequences;
- $a_n \neq 0$ for all $n \in \mathbb{N}$;
- $a_n \to \infty$ as $n \to \infty$; and
- $b_n = \frac{1}{a_n}$ for all $n \in \mathbb{N}$.

Show that $b_n \to 0$ as $n \to \infty$.

Let $\varepsilon > 0$. Then let $A = \frac{1}{\varepsilon} > 0$. Since $a_n \to \infty$ as $n \to \infty$, $\exists N \in \mathbb{N}$ such that
$$n > N \implies a_n > A.$$
But then
$$n > N \implies 0 < \frac{1}{a_n} < \frac{1}{A} = \varepsilon \implies 0 < b_n < \varepsilon \implies |b_n| < \varepsilon.$$
Therefore $b_n \to 0$ as $n \to \infty$.

Soru 2 (Bounded and Unbounded Sequences).

(a) [10p] Give the definition of a *bounded* sequence

We say that (a_n) is bounded iff, $\exists M > 0$ such that $\forall n \in \mathbb{N}$, $|a_n| \leq M$.

(b) [10p] Give the definition of an *unbounded* sequence.
[HINT: Negate your answer to part (a).]

We say that (a_n) is unbounded iff $\forall M > 0$, $\exists n \in \mathbb{N}$ such that $|a_n| > M$.

(c) Decide if each of the sequences below is bounded or unbounded.

[10p] $c_n = \dfrac{6^n + n!}{n + 7^n}$ [10p] $d_n = \dfrac{6^n + n!}{n + (-7)^n}$ [10p] $e_n = \dfrac{6^n + n!}{n! + (-7)^n}$

[You must prove your answers. You may use any theorem or lemma from the course.]

2pts for correct answering bounded or unbounded | 8pts for reasonable justification.

incorrect answer with incorrect proof can get up to 5pts depending on mistakes in proof

Since

$$|c_n| = \frac{6^n + n!}{n + 7^n} = \frac{\left(\frac{6}{7}\right)^n + \frac{n!}{7^n}}{\frac{n}{7^n} + 1} \geq \frac{\frac{n!}{7^n}}{1+1} = \frac{1}{2}\frac{n!}{7^n} \to \infty$$

as $n \to \infty$, it follows that (c_n) is unbounded.

Note that

$$|d_n| = \left|\frac{6^n + n!}{n + (-7)^n}\right| = \left|\frac{\left(\frac{6}{-7}\right)^n + (-1)^n \frac{n!}{7^n}}{\frac{n}{(-7)^n} + 1}\right| \geq \frac{0 + \frac{1}{2}\frac{n!}{7^n}}{1+1} = \frac{n!}{4 \times 7^n}$$

for sufficiently large n. Since $\frac{n!}{7^n} \to \infty$ as $n \to \infty$, it follows that (d_n) is unbounded.

$$e_n = \frac{6^n + n!}{n! + (-7)^n} = \frac{\frac{6^n}{n!} + 1}{1 + \frac{(-7)^n}{n!}} \to \frac{0+1}{1+0} = 1$$

as $n \to \infty$. Since every convergent sequence is bounded (we proved this), we have that (e_n) is bounded.

Soru 3 (Sequences). Define a sequence of real numbers (a_n) by

$$a_1 = 1 \qquad \text{and} \qquad 20a_{n+1} = a_n^2 + 99.$$

(a) [13p] Show that $0 \leq a_n \leq 9$ for all $n \in \mathbb{N}$.

[HINT: Use proof by induction.].

Since $0 \leq a_1 = 1 \leq 9$, the statement is true for $n = 1$ $\boxed{3}$. Suppose that it is true for $n = k$. Then $0 \leq a_k \leq 9$ $\boxed{2}$. So $20a_{k+1} = a_k^2 + 99 \leq 9^2 + 99 = 180 \implies a_{k+1} \leq 9$ $\boxed{3}$ and $20a_{k+1} = a_k^2 + 99 \geq 0^2 + 99 \geq 0 \implies a_{k+1} \geq 0$ $\boxed{3}$. By the principle of mathematical induction $\boxed{2}$, it follows that $0 \leq a_n \leq 9$ $\forall n \in \mathbb{N}$.

(b) [13p] Show that $a_n \leq a_{n+1}$ for all $n \in \mathbb{N}$.

First note that $a_{n+1} - a_n = \frac{1}{20}(a_n^2 + 99) - a_n = \frac{1}{20}(a_n^2 - 20a_n + 99) = \frac{1}{20}(a_n - 9)(a_n - 11)$ $\boxed{5}$. Since $0 \leq a_n \leq 9$, $(a_n - 9) \leq 0$ and $(a_n - 11) \leq 0$ $\boxed{4}$. Therefore $a_{n+1} - a_n = \frac{1}{20}(a_n - 9)(a_n - 11) \geq 0$. So $a_{n+1} \geq a_n$ $\forall n \in \mathbb{N}$ $\boxed{4}$.

(c) [12p] Show that (a_n) is a convergent sequence.

By a theorem from the course, "every increasing sequence which is bounded above is convergent". In part (a), I proved that (a_n) is bounded above. In part (b), I proved that (a_n) is increasing. Therefore (a_n) is convergent.

(d) [12p] Calculate $\lim_{n\to\infty} a_n$.

Let $a = \lim_{n\to\infty} a_n$. Then $20a \leftarrow 20a_{n+1} = a_n^2 + 99 \to a^2 + 99$ as $n \to \infty$ $\boxed{4}$. Because limits are unique, it follows that $0 = a^2 - 20a + 99 = (a-9)(a-11)$. So $a = 9$ or $a = 11$ $\boxed{4}$. Finally, since (a_n) is bounded above by 9, we must have that $a = 9$ $\boxed{4}$.