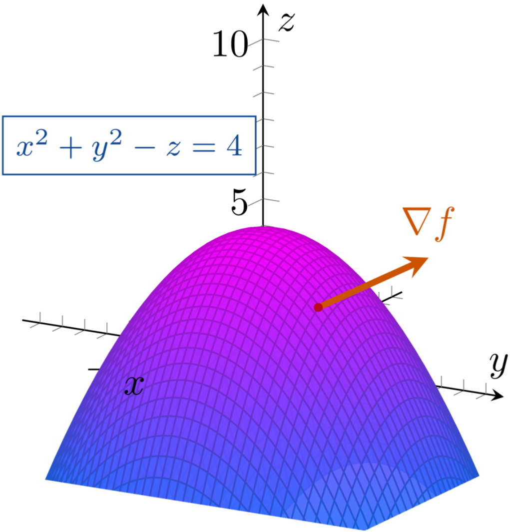
Tangent Planes and Differentials

Tangent Planes and Normal Lines



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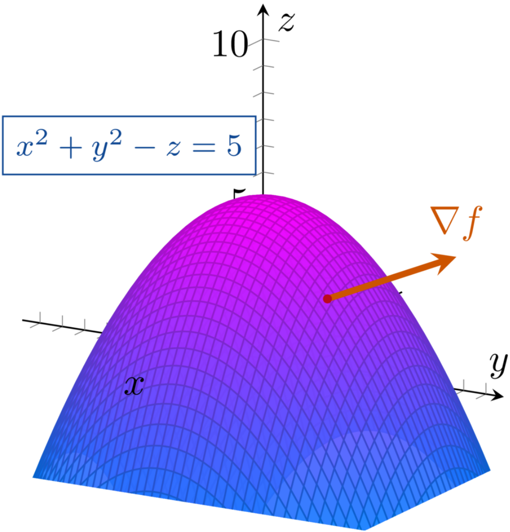

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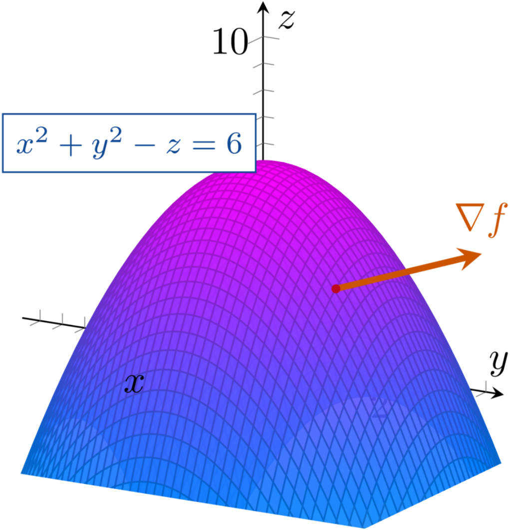

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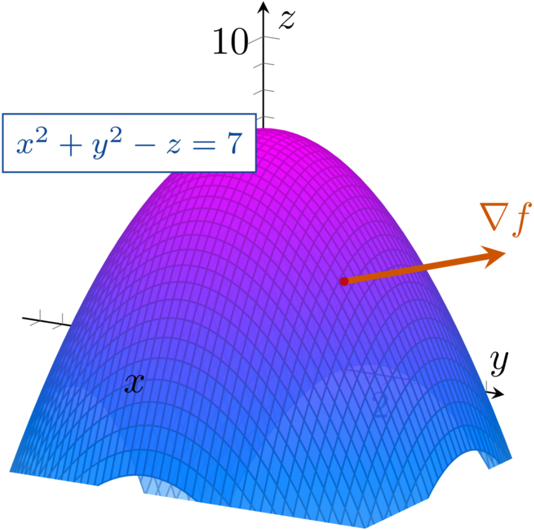

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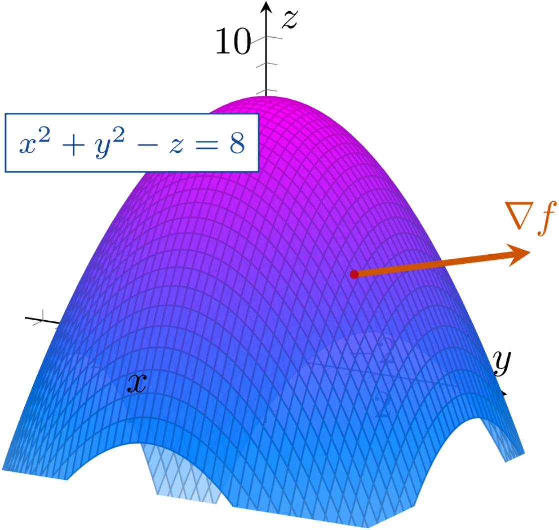

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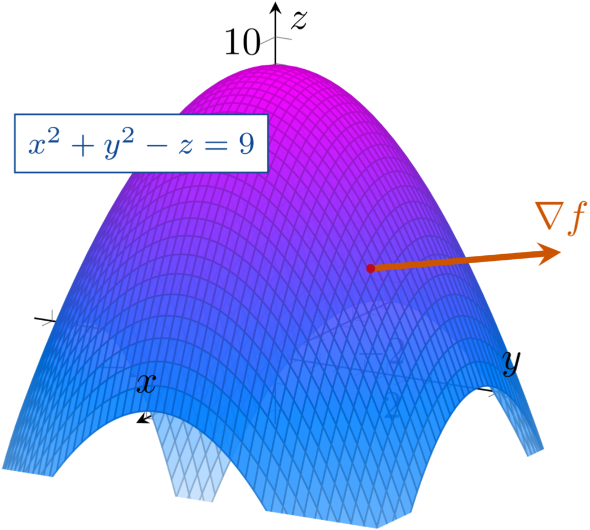

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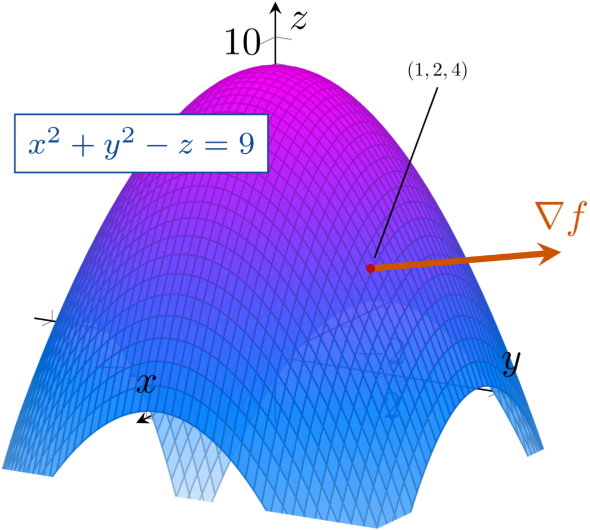

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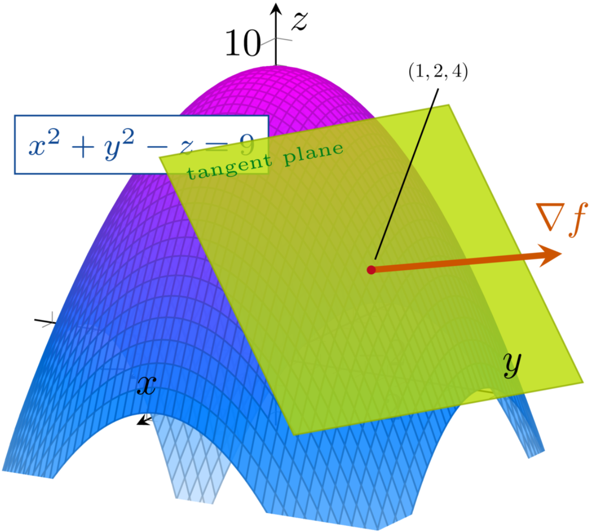

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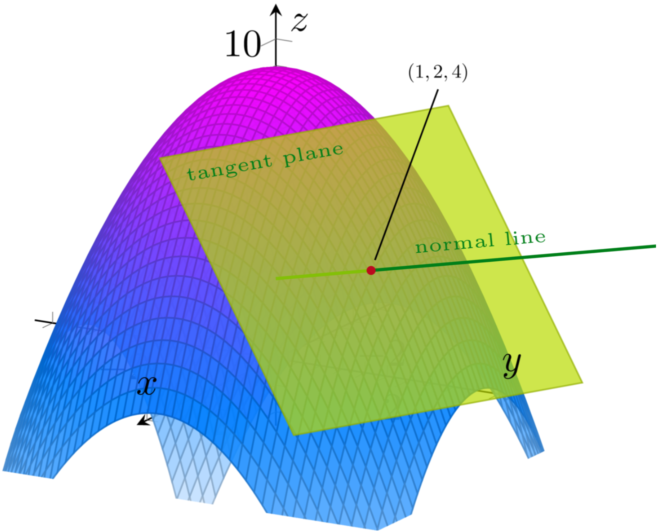

Definition

The *tangent plane* to the surface $f(x, y, z) = c$ at the point $P(x_0, y_0, z_0)$ (where the gradient is not zero) is the plane through P with normal vector $\nabla f|_P$.

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The *normal line* to the surface $f(x, y, z) = c$ at the point P is the line through P parallel to $\nabla f|_P$.

$$x = x_0 + f_x(P)t \qquad y = y_0 + f_y(P)t \qquad z = z_0 + f_z(P)t.$$

EXAMPLE 1 Find the tangent plane and normal line of the level surface

$$f(x, y, z) = x^2 + y^2 + z - 9 = 0 \qquad \text{A circular paraboloid}$$

at the point $P_0(1, 2, 4)$.

Solution The surface is shown in Figure 14.34.

The tangent plane is the plane through P_0 perpendicular to the gradient of f at P_0. The gradient is

$$\nabla f|_{P_0} = (2x\mathbf{i} + 2y\mathbf{j} + \mathbf{k})\Big|_{(1,2,4)} = 2\mathbf{i} + 4\mathbf{j} + \mathbf{k}.$$

The tangent plane is therefore the plane

$$2(x - 1) + 4(y - 2) + (z - 4) = 0, \qquad \text{or} \qquad 2x + 4y + z = 14.$$

The line normal to the surface at P_0 is

$$x = 1 + 2t, \qquad y = 2 + 4t, \qquad z = 4 + t. \qquad \blacksquare$$

Now consider

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Definition

The *tangent plane* to the surface $z = f(x, y)$ at the point $P(x_0, y_0, z_0) = (x_0, y_0, f(x_0, y_0))$ is

$$f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0) - (z - z_0) = 0.$$

EXAMPLE 2 Find the plane tangent to the surface $z = x\cos y - ye^x$ at $(0, 0, 0)$.

Solution We calculate the partial derivatives of $f(x, y) = x\cos y - ye^x$ and use Equation (3):

$$f_x(0, 0) = (\cos y - ye^x)\Big|_{(0,0)} = 1 - 0 \cdot 1 = 1$$

$$f_y(0, 0) = (-x\sin y - e^x)\Big|_{(0,0)} = 0 - 1 = -1.$$

The tangent plane is therefore

$$1 \cdot (x - 0) - 1 \cdot (y - 0) - (z - 0) = 0, \qquad \text{Eq. (3)}$$

or

$$x - y - z = 0.$$

EXAMPLE 3 The surfaces

$$f(x, y, z) = x^2 + y^2 - 2 = 0 \qquad \text{A cylinder}$$

and

$$g(x, y, z) = x + z - 4 = 0 \qquad \text{A plane}$$

meet in an ellipse E (Figure 14.35). Find parametric equations for the line tangent to E at the point $P_0(1, 1, 3)$.

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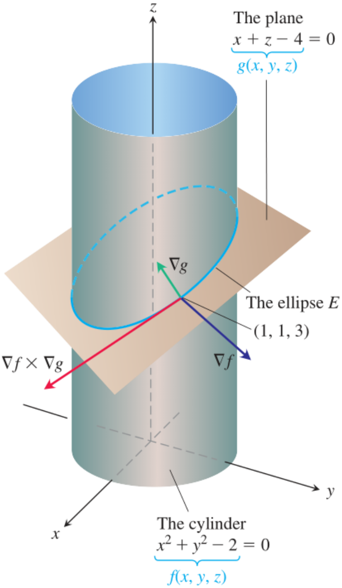

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meet in an ellipse E (Figure 14.35). Find parametric equations for the line tangent to E at the point $P_0(1, 1, 3)$.

Solution The tangent line is orthogonal to both ∇f and ∇g at P_0, and therefore parallel to $\mathbf{v} = \nabla f \times \nabla g$. The components of $\mathbf{v}$ and the coordinates of P_0 give us equations for the line. We have

$$\nabla f\big|_{(1,1,3)} = (2x\mathbf{i} + 2y\mathbf{j})\Big|_{(1,1,3)} = 2\mathbf{i} + 2\mathbf{j}$$

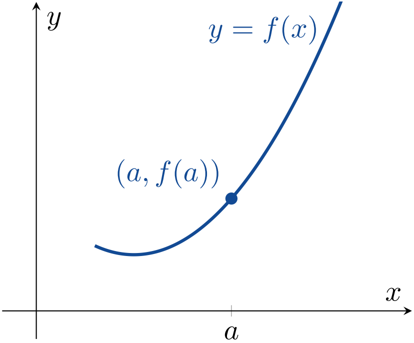
$$\nabla g\big|_{(1,1,3)} = (\mathbf{i} + \mathbf{k})\Big|_{(1,1,3)} = \mathbf{i} + \mathbf{k}$$

$$\mathbf{v} = (2\mathbf{i} + 2\mathbf{j}) \times (\mathbf{i} + \mathbf{k}) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 2 & 0 \\ 1 & 0 & 1 \end{vmatrix} = 2\mathbf{i} - 2\mathbf{j} - 2\mathbf{k}.$$

The tangent line to the ellipse of intersection is

$$x = 1 + 2t, \qquad y = 1 - 2t, \qquad z = 3 - 2t.$$

Linearisation of a Function of One Variable



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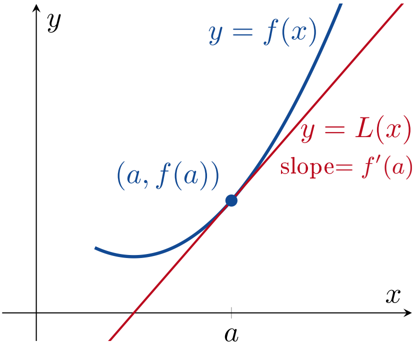

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Linearisation of a Function of Two Variable

Definition

The *linearisation* of a function $f(x, y)$ at a point (x_0, y_0) is

$$L(x, y) = f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0).$$

EXAMPLE 5 Find the linearization of

$$f(x, y) = x^2 - xy + \frac{1}{2}y^2 + 3$$

at the point (3, 2).

Solution We first evaluate f, f_x, and f_y at the point $(x_0, y_0) = (3, 2)$:

$$f(3, 2) = \left(x^2 - xy + \frac{1}{2}y^2 + 3\right)\Big|_{(3,2)} = 8$$

$$f_x(3, 2) = \frac{\partial}{\partial x}\left(x^2 - xy + \frac{1}{2}y^2 + 3\right)\Big|_{(3,2)} = (2x - y)\Big|_{(3,2)} = 4$$

$$f_y(3, 2) = \frac{\partial}{\partial y}\left(x^2 - xy + \frac{1}{2}y^2 + 3\right)\Big|_{(3,2)} = (-x + y)\Big|_{(3,2)} = -1,$$

giving

$$\begin{aligned} L(x, y) &= f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0) \\ &= 8 + (4)(x - 3) + (-1)(y - 2) = 4x - y - 2. \end{aligned}$$

The linearization of f at (3, 2) is $L(x, y) = 4x - y - 2$