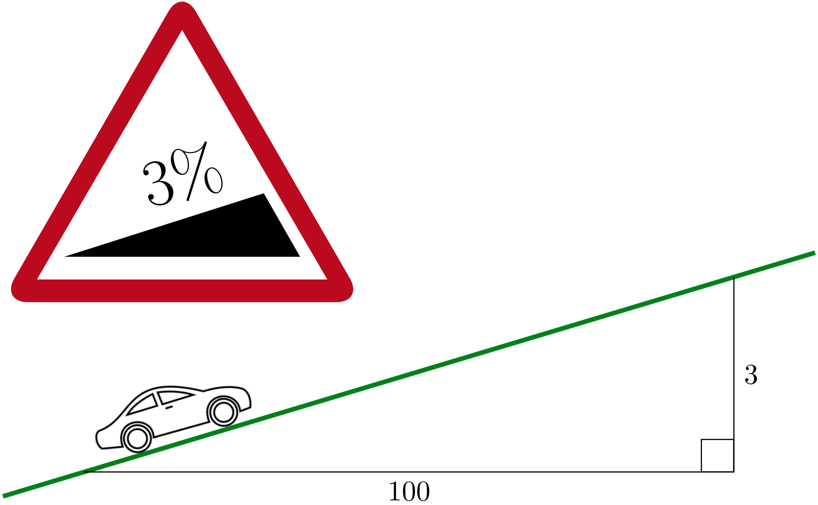
Lecture 4

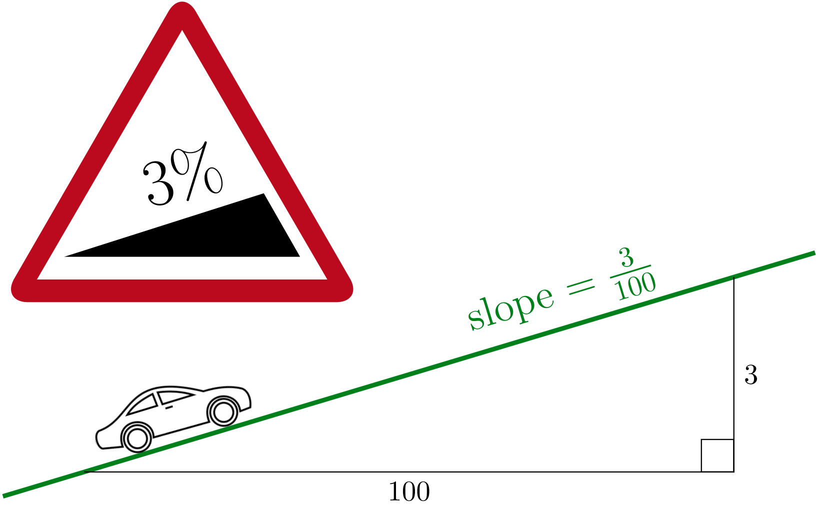
- 3.1 Tangents and the Derivative at a Point
- 3.2 The Derivative as a Function
- 3.3 Differentiation Rules
- 3.4 The Derivative as a Rate of Change

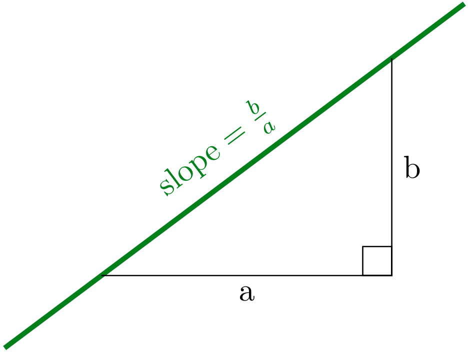
Tangents and the Derivative at a Point

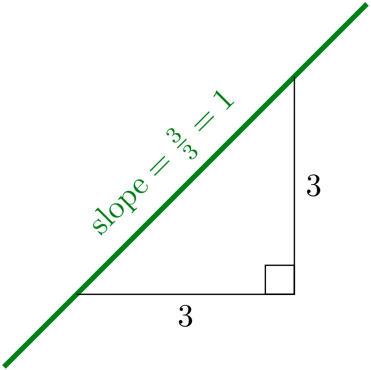
3.1 Tangents and the Derivative at a Point

3.1 Tangents and the Derivative at a Point

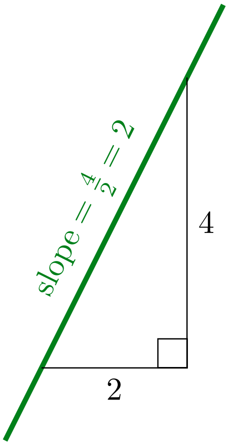


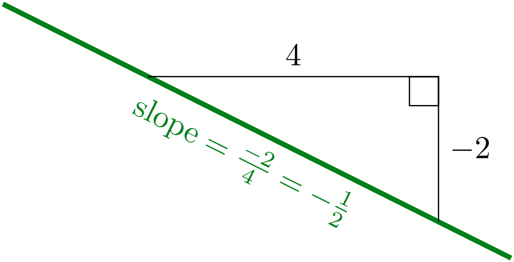

3%
slope = $\frac{3}{100}$
3
100


slope = $\frac{b}{a}$
b
a

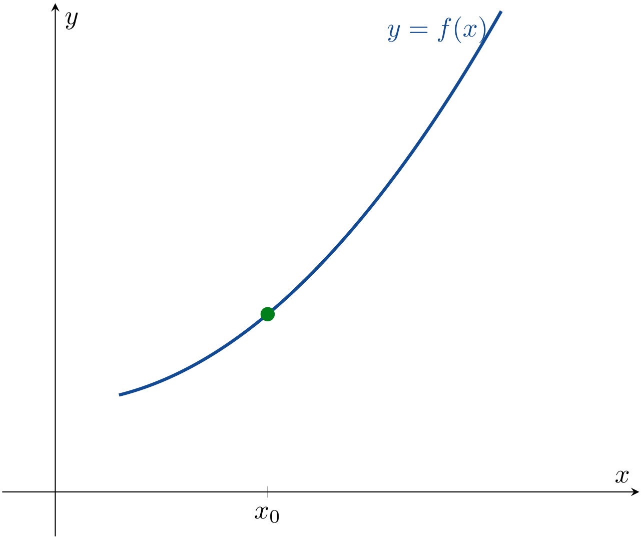

slope = 3/3 = 1
3
3

3.1 Tangents and the Derivative at a Point

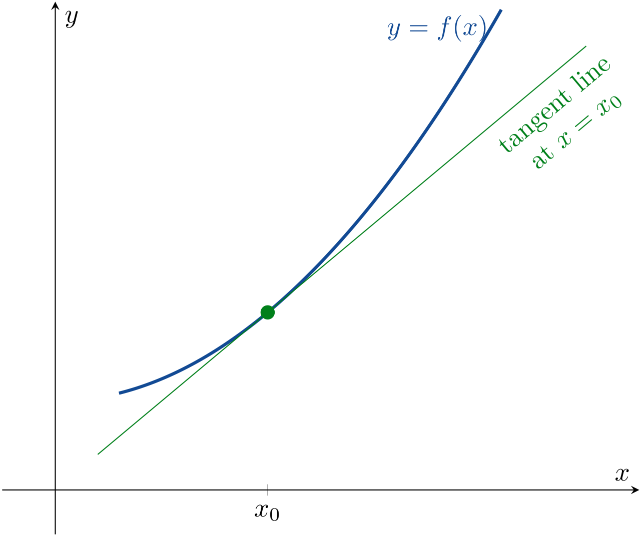


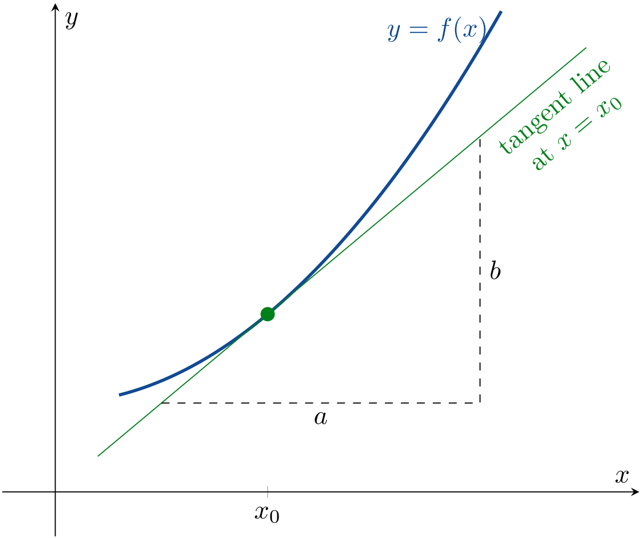

4
-2
slope $= \frac{-2}{4} = -\frac{1}{2}$

3.1 Tangents and the Derivative at a Point

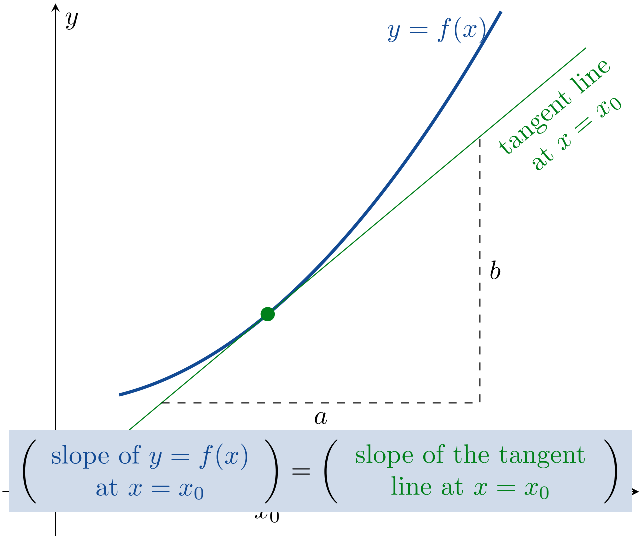


3.1 Tangents and the Derivative at a Point




y
$y = f(x)$
tangent line
at $x = x_0$
b
a
x
x_0

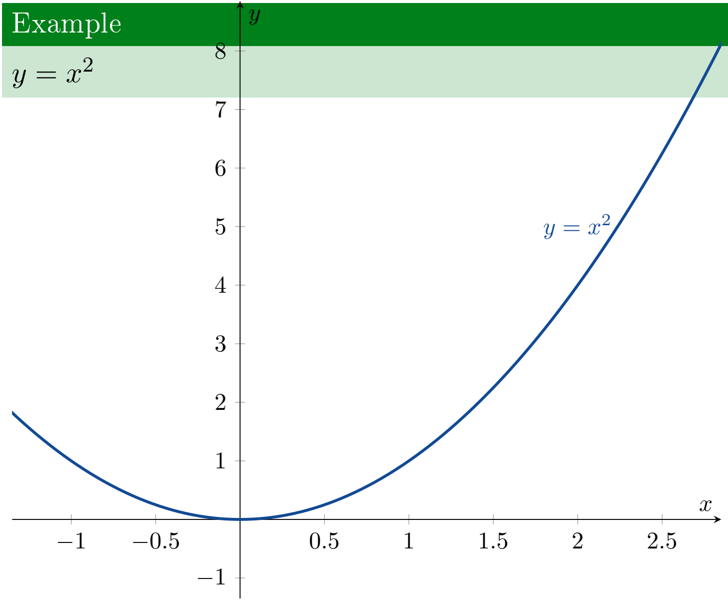
3.1 Tangents and the Derivative at a Point



$$\left(\begin{array}{c} \text{slope of } y = f(x) \\ \text{at } x = x_0 \end{array} \right) = \left(\begin{array}{c} \text{slope of the tangent} \\ \text{line at } x = x_0 \end{array} \right)$$

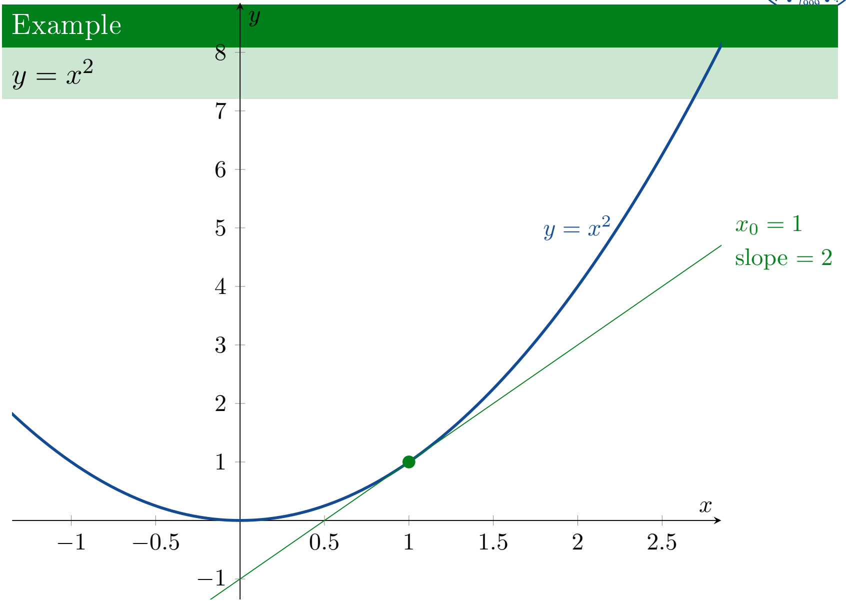
Example

$y = x^2$



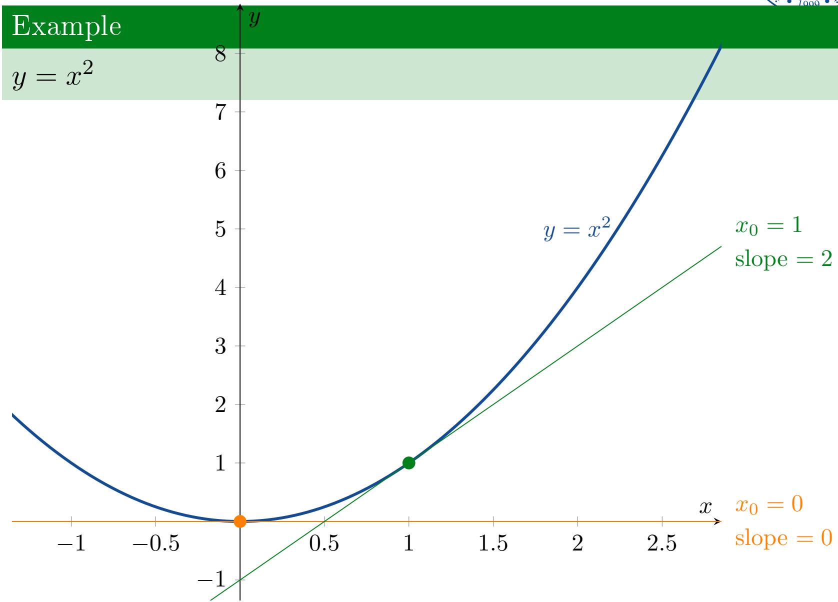
Example

$y = x^2$



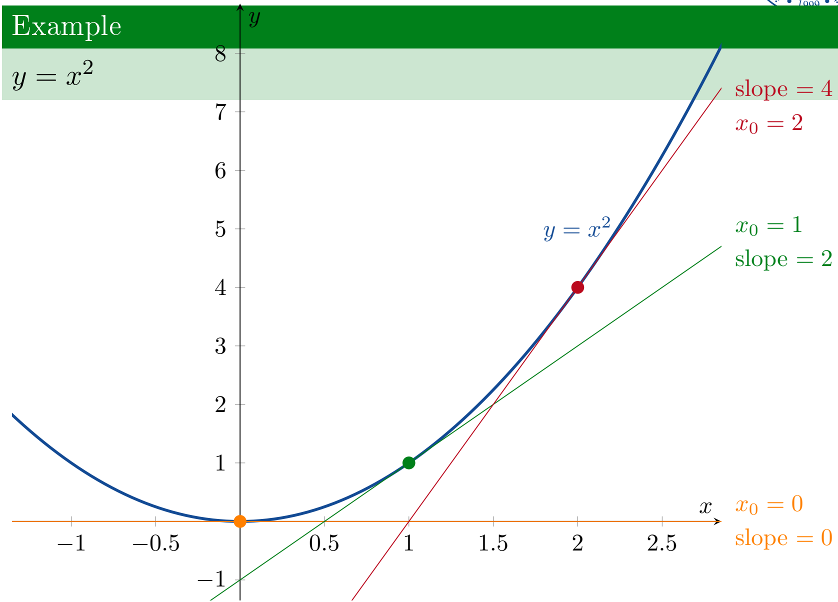
Example

$y = x^2$



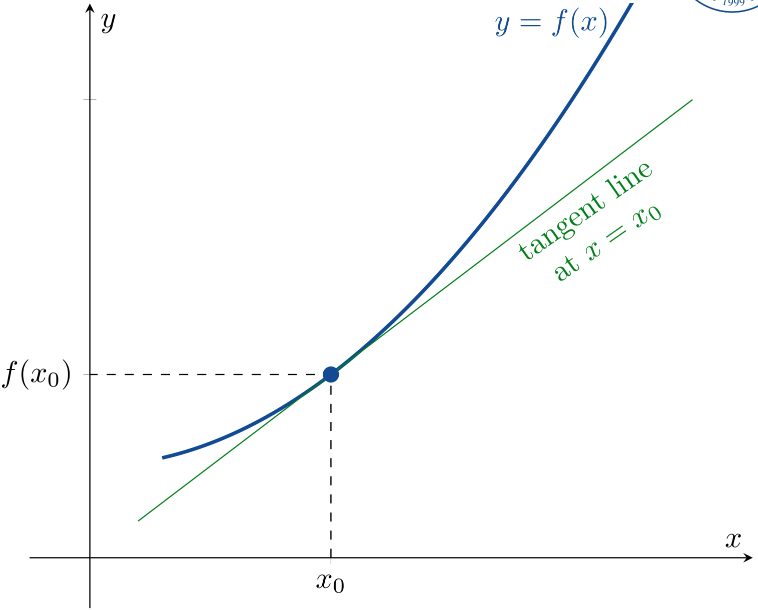
Example

$y = x^2$

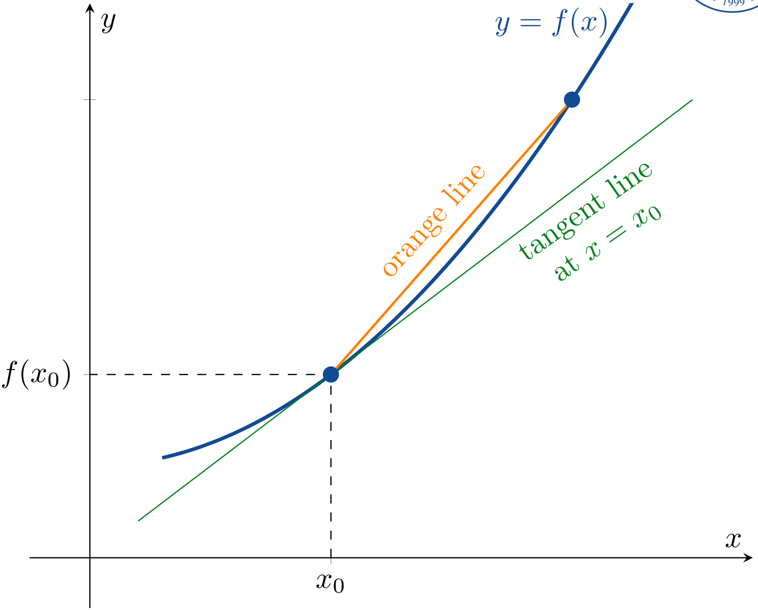


y
x
8
7
6
5
4
3
2
1
−1
−1
−0.5
0.5
1
1.5
2
2.5
$y = x^2$
slope = 4
$x_0 = 2$
$x_0 = 1$
slope = 2
$x_0 = 0$
slope = 0

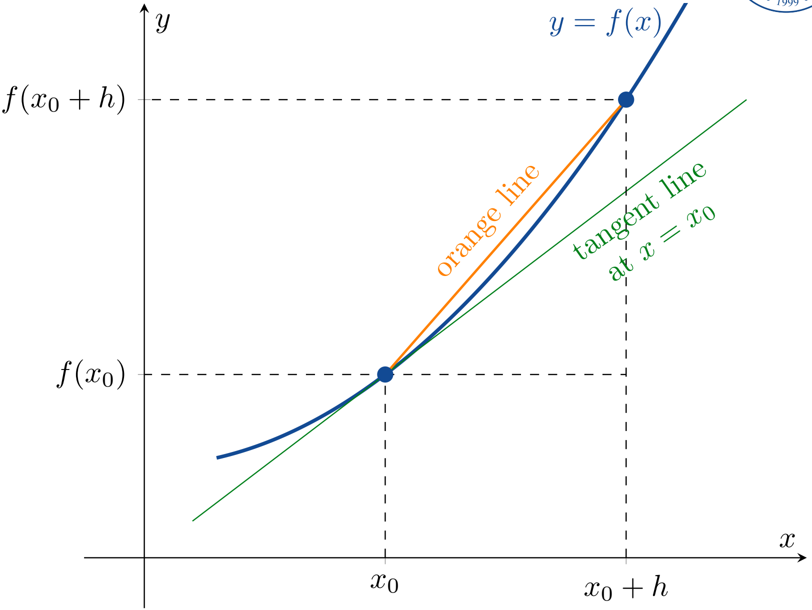
How can we calculate the slope of the tangent line?



y
$y = f(x)$
tangent line
at $x = x_0$
$f(x_0)$
x
x_0


y
$y = f(x)$
orange line
tangent line
at $x = x_0$
$f(x_0)$
x
x_0

3.1 Tangents and the Derivative at a Point



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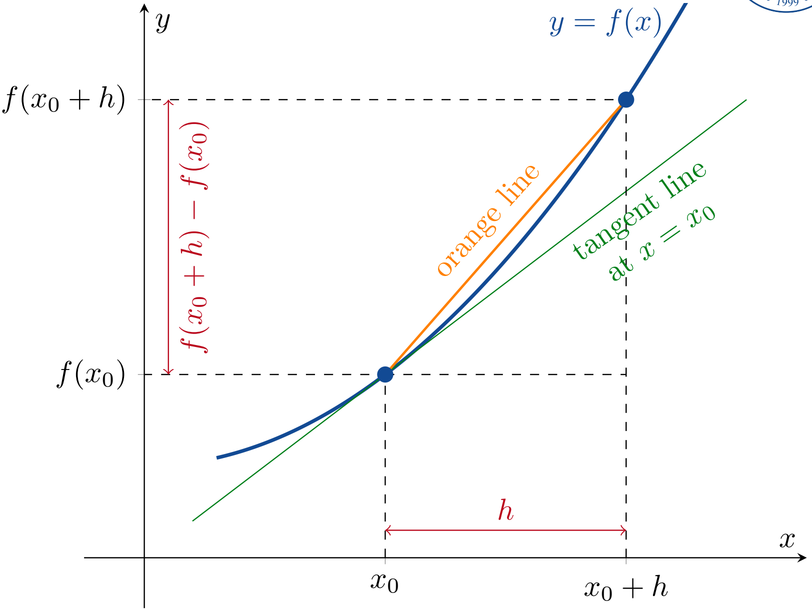
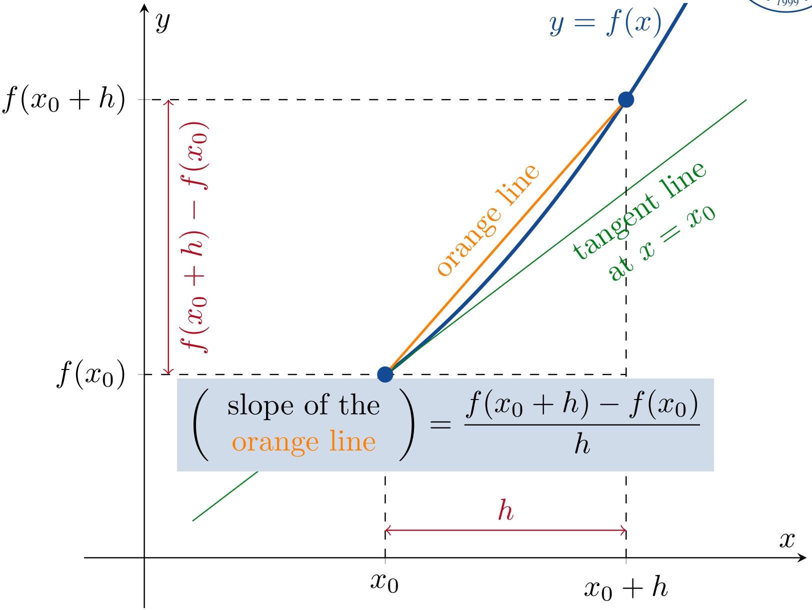
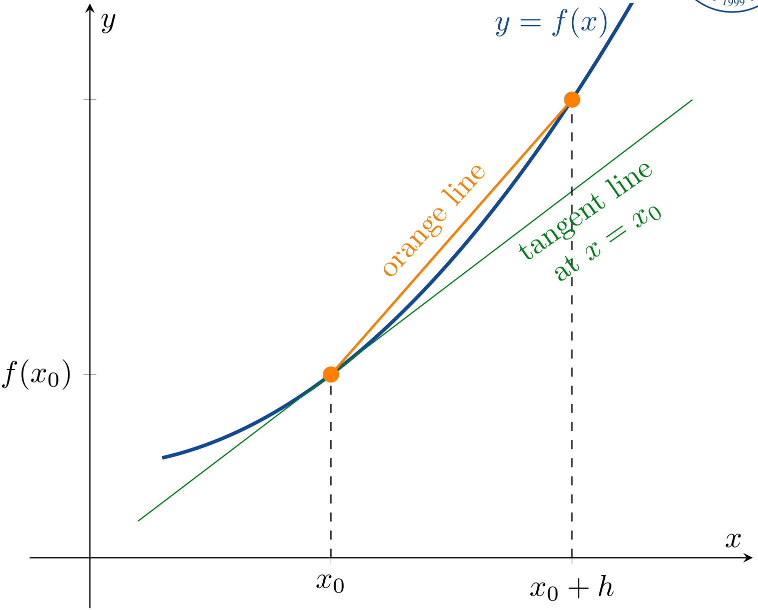
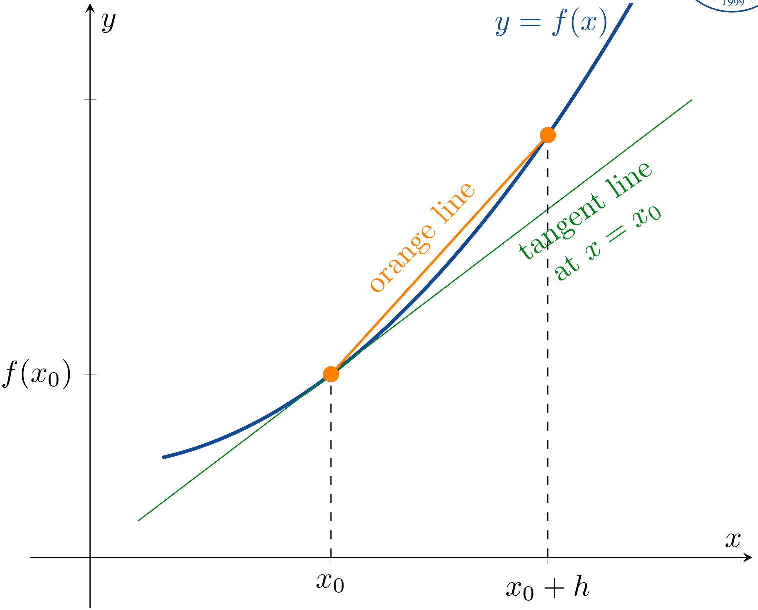


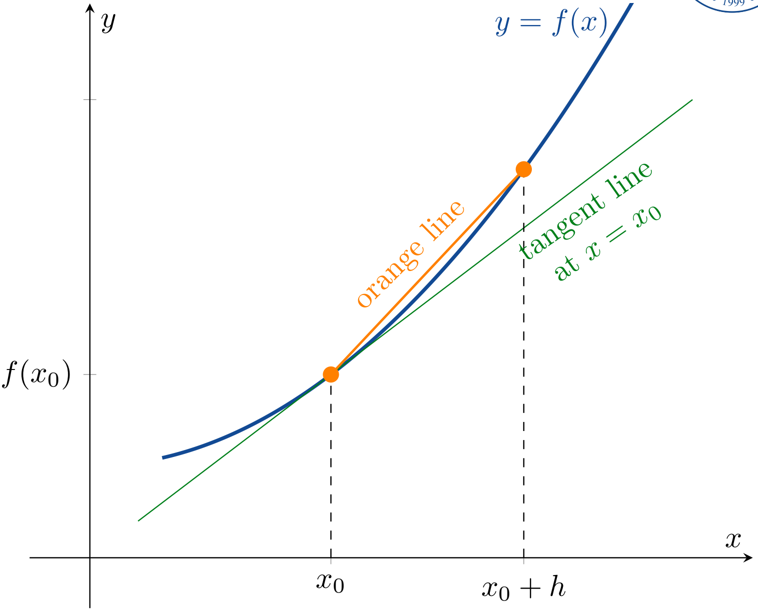


$$\left(\begin{array}{c}\text{slope of the}\\ \text{orange line}\end{array}\right) = \frac{f(x_0+h) - f(x_0)}{h}$$

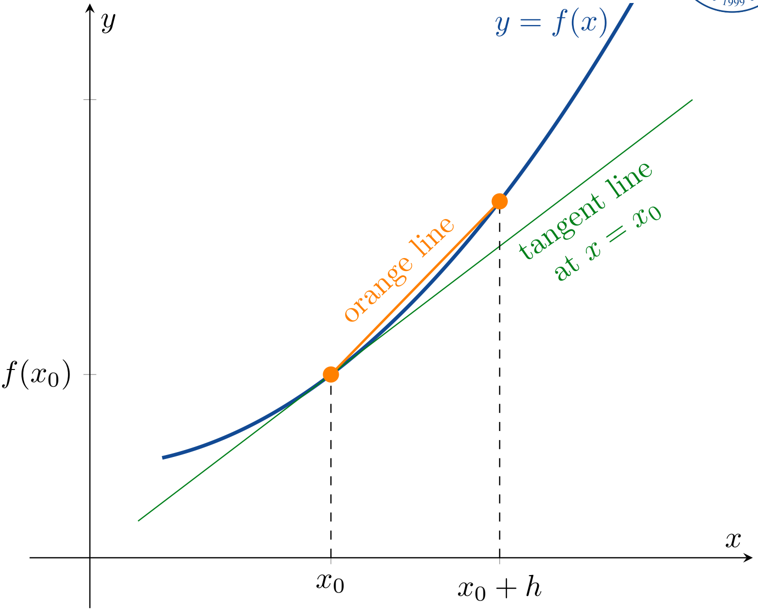
3.1 Tangents and the Derivative at a Point




y
y = f(x)
orange line
tangent line
at x = x0
f(x0)
x
x0
x0 + h



y
$y = f(x)$
orange line
tangent line
at $x = x_0$
$f(x_0)$
x
x_0
$x_0 + h$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orange line
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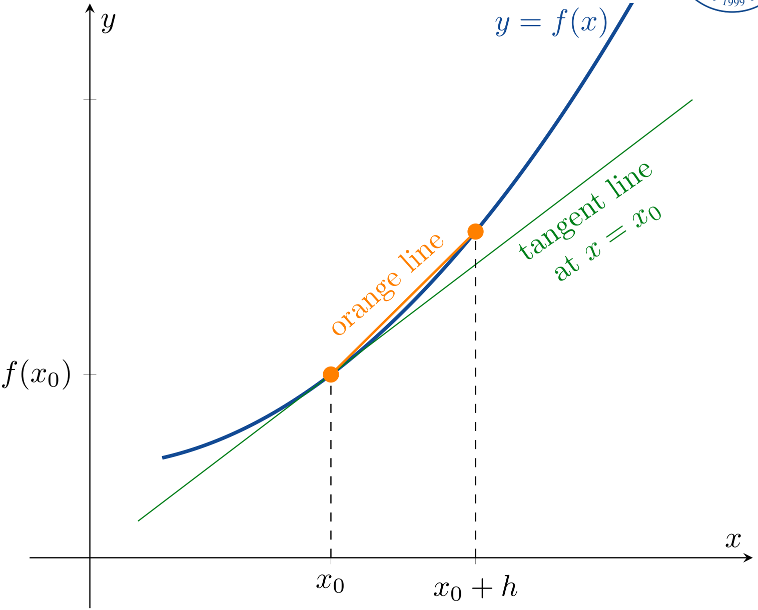
$f(x_0)$
x
x_0
$x_0 + h$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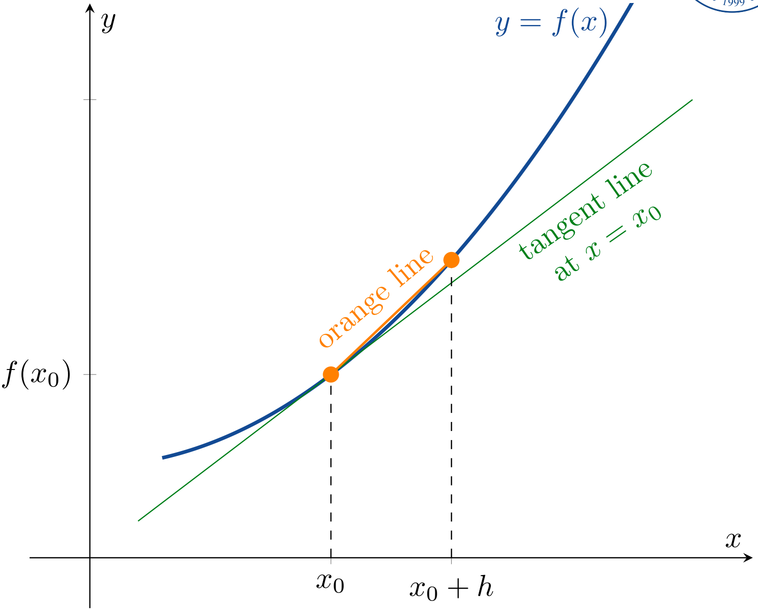
orange line
$f(x_0)$
x
x_0
$x_0 + h$



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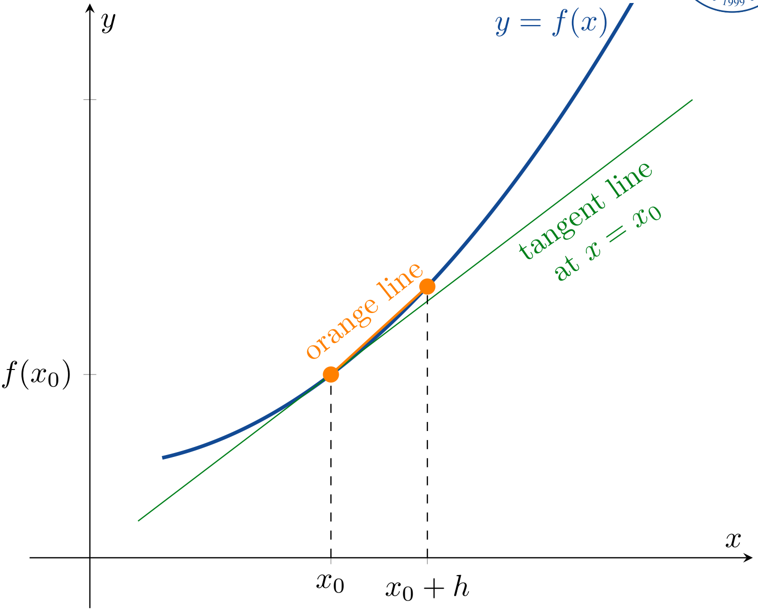
$f(x_0)$
x
x_0
$x_0 + h$



y
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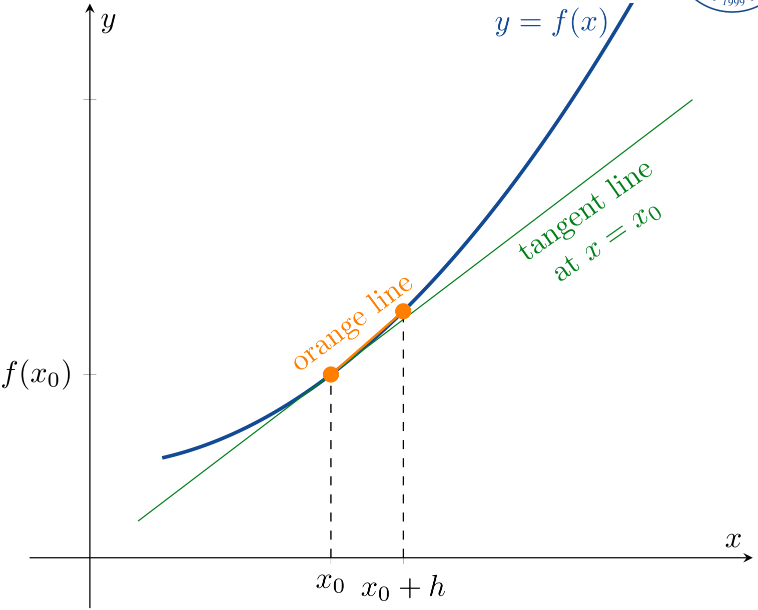
orange line
$f(x_0)$
x
x_0
$x_0 + h$

3.1 Tangents and the Derivative at a Point



If h is very very small, then

$$\left(\begin{array}{c} \text{slope of the} \\ \text{tangent line} \end{array} \right) \approx \left(\begin{array}{c} \text{slope of the} \\ \text{orange line} \end{array} \right) = \frac{f(x_0 + h) - f(x_0)}{h}$$

The Derivative of f

Definition

The *derivative of a function* f *at a point* x_0 is

$$f'(x_0) = \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h}$$

if the limit exists.

(f' is pronounced "f prime")

Example

Find the derivative of the function $g(x) = \frac{1}{x}$, $x \neq 0$.

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If $x_0 \neq 0$, then

$$g'(x_0) = \lim_{h \to 0} \frac{g(x_0 + h) - g(x_0)}{h}$$

$$=$$

$$=$$

$$=$$

$$=$$

Example

Find the derivative of the function $g(x) = \frac{1}{x}$, $x \neq 0$.

If $x_0 \neq 0$, then

$$
\begin{aligned}
g'(x_0) &= \lim_{h \to 0} \frac{g(x_0 + h) - g(x_0)}{h} \\
&= \lim_{h \to 0} \frac{\frac{1}{x_0+h} - \frac{1}{x_0}}{h} \\
&= \\
&= \\
&=
\end{aligned}
$$

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&= \lim_{h \to 0} \frac{\frac{1}{x_0+h} - \frac{1}{x_0}}{h} \\
&= \lim_{h \to 0} \frac{\left(\frac{x_0}{x_0(x_0+h)} - \frac{x_0+h}{x_0(x_0+h)} \right)}{h} \\
&= \\
&=
\end{aligned}
$$

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&= \lim_{h\to 0} \frac{\frac{1}{x_0+h} - \frac{1}{x_0}}{h} \\
&= \lim_{h\to 0} \frac{\left(\frac{x_0}{x_0(x_0+h)} - \frac{x_0+h}{x_0(x_0+h)}\right)}{h} \\
&= \lim_{h\to 0} \frac{x_0 - x_0 - h}{hx_0(x_0+h)} \\
&=
\end{aligned}
$$

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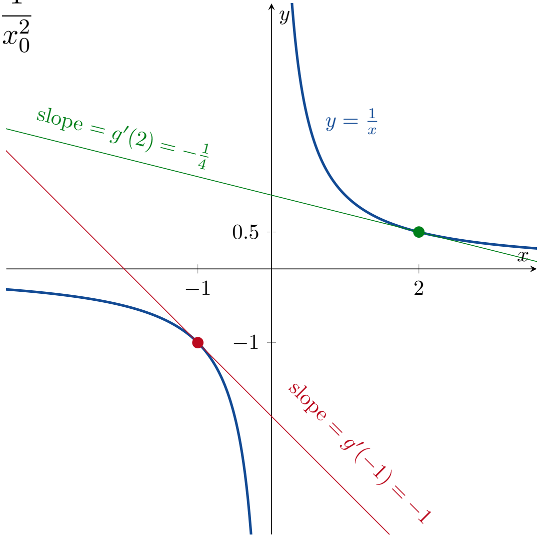
&= \lim_{h\to 0} \frac{\left(\frac{x_0}{x_0(x_0+h)} - \frac{x_0+h}{x_0(x_0+h)}\right)}{h} \\
&= \lim_{h\to 0} \frac{x_0 - x_0 - h}{hx_0(x_0+h)} \\
&= \lim_{h\to 0} \frac{-1}{x_0(x_0+h)} = -\frac{1}{x_0^2}.
\end{aligned}$$

$$g'(x_0) = -\frac{1}{x_0^2}$$



The Derivative as a Function

Definition

If $f'(x_0)$ exists, we say that f *is differentiable at* x_0.

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Let $f : D \to \mathbb{R}$ be a function. If f is differentiable at every $x_0 \in D$, we say that f is *differentiable*.

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Definition

Let $f : D \to \mathbb{R}$ be a function. If f is differentiable at every $x_0 \in D$, we say that f is *differentiable*.

If $f : D \to \mathbb{R}$ is differentiable, then we have a new function $f' : D \to \mathbb{R}$.

3.2 The Derivative as a Function

Definition

If $f'(x_0)$ exists, we say that f *is differentiable at* x_0.

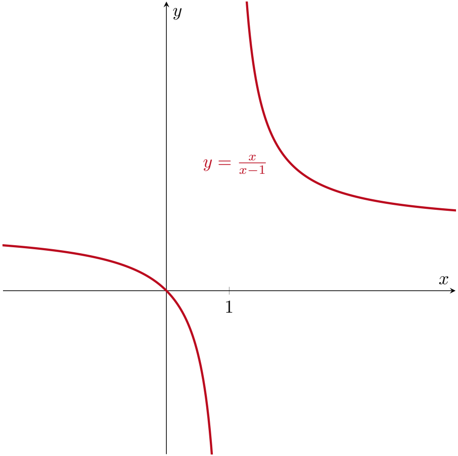
Definition

Let $f : D \to \mathbb{R}$ be a function. If f is differentiable at every $x_0 \in D$, we say that f is *differentiable*.

If $f : D \to \mathbb{R}$ is differentiable, then we have a new function $f' : D \to \mathbb{R}$.

Definition

f' is called the *derivative* of f.



Example

Differentiate $f(x) = \dfrac{x}{x-1}$.

First note that if $f(x) = \dfrac{x}{x-1}$ then $f(x+h) = \dfrac{x+h}{x+h-1}$.

Therefore

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h\to 0} \frac{\frac{x+h}{x+h-1} - \frac{x}{x-1}}{h} \\ &= \\ &= \\ &= \\ &= \end{aligned}$$

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Therefore

$$\begin{aligned}
f'(x) &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \\
&= \lim_{h\to 0} \frac{\frac{x+h}{x+h-1} - \frac{x}{x-1}}{h} \\
&= \lim_{h\to 0} \frac{1}{h}\left(\frac{(x+h)(x-1) - x(x+h-1)}{(x-1)(x+h-1)} \right) \\
&= \\
&= \\
&=
\end{aligned}$$

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Therefore

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&= \lim_{h\to 0} \frac{1}{h} \left(\frac{(x+h)(x-1) - x(x+h-1)}{(x-1)(x+h-1)} \right) \\
&= \lim_{h\to 0} \frac{1}{h} \left(\frac{x^2 - x + hx - h - x^2 - hx + x}{(x-1)(x+h-1)} \right) \\
&= \\
&=
\end{aligned}
$$

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&= \lim_{h\to 0} \frac{1}{h}\left(\frac{(x+h)(x-1) - x(x+h-1)}{(x-1)(x+h-1)}\right) \\
&= \lim_{h\to 0} \frac{1}{h}\left(\frac{x^2 - x + hx - h - x^2 - hx + x}{(x-1)(x+h-1)}\right) \\
&= \lim_{h\to 0} \frac{1}{h}\left(\frac{-h}{(x-1)(x+h-1)}\right) = \lim_{h\to 0} \frac{-1}{(x-1)(x+h-1)} \\
&=
\end{aligned}
$$

First note that if $f(x) = \dfrac{x}{x-1}$ then $f(x+h) = \dfrac{x+h}{x+h-1}$.

Therefore

$$\begin{aligned}
f'(x) &= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} \\
&= \lim_{h\to 0} \frac{\frac{x+h}{x+h-1} - \frac{x}{x-1}}{h} \\
&= \lim_{h\to 0} \frac{1}{h}\left(\frac{(x+h)(x-1)-x(x+h-1)}{(x-1)(x+h-1)}\right) \\
&= \lim_{h\to 0} \frac{1}{h}\left(\frac{x^2-x+hx-h-x^2-hx+x}{(x-1)(x+h-1)}\right) \\
&= \lim_{h\to 0} \frac{1}{h}\left(\frac{-h}{(x-1)(x+h-1)}\right) = \lim_{h\to 0} \frac{-1}{(x-1)(x+h-1)} \\
&= \frac{-1}{(x-1)(x+0-1)} = \frac{-1}{(x-1)^2}.
\end{aligned}$$

The derivative of $f(x) = \dfrac{x}{x-1}$ is $f'(x) = \dfrac{-1}{(x-1)^2}$.

Sometimes it is easier to use an alternative formula for the derivative. If we take

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

and substitute in $z = x + h$, then we get:

Sometimes it is easier to use an alternative formula for the derivative. If we take

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

and substitute in $z = x + h$, then we get:

Definition

$$f'(x) = \lim_{z \to x} \frac{f(z) - f(x)}{z - x}.$$

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Example

Find the derivative of $f(x) = \sqrt{x}$ for $x > 0$.

$$f'(x) = \lim_{z\to x} \frac{f(z)-f(x)}{z-x}$$

Example

Find the derivative of $f(x) = \sqrt{x}$ for $x > 0$.

$$f'(x) = \lim_{z\to x} \frac{f(z)-f(x)}{z-x} = \lim_{z\to x} \frac{\sqrt{z}-\sqrt{x}}{z-x}$$

$$=$$

$$=$$

$$=$$

$$f'(x) = \lim_{z \to x} \frac{f(z) - f(x)}{z - x}$$

Example

Find the derivative of $f(x) = \sqrt{x}$ for $x > 0$.

$$\begin{aligned} f'(x) = \lim_{z \to x} \frac{f(z) - f(x)}{z - x} &= \lim_{z \to x} \frac{\sqrt{z} - \sqrt{x}}{z - x} \\ &= \lim_{z \to x} \frac{\sqrt{z} - \sqrt{x}}{(\sqrt{z} - \sqrt{x})(\sqrt{z} + \sqrt{x})} \\ &= \\ &= \end{aligned}$$

$$f'(x) = \lim_{z \to x} \frac{f(z) - f(x)}{z - x}$$

Example

Find the derivative of $f(x) = \sqrt{x}$ for $x > 0$.

$$\begin{aligned} f'(x) &= \lim_{z \to x} \frac{f(z) - f(x)}{z - x} = \lim_{z \to x} \frac{\sqrt{z} - \sqrt{x}}{z - x} \\ &= \lim_{z \to x} \frac{\sqrt{z} - \sqrt{x}}{(\sqrt{z} - \sqrt{x})(\sqrt{z} + \sqrt{x})} \\ &= \lim_{z \to x} \frac{1}{\sqrt{z} + \sqrt{x}} \\ &= \end{aligned}$$

$$f'(x) = \lim_{z \to x} \frac{f(z) - f(x)}{z - x}$$

Example

Find the derivative of $f(x) = \sqrt{x}$ for $x > 0$.

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f'(x) = \lim_{z \to x} \frac{f(z) - f(x)}{z - x} &= \lim_{z \to x} \frac{\sqrt{z} - \sqrt{x}}{z - x} \\
&= \lim_{z \to x} \frac{\sqrt{z} - \sqrt{x}}{(\sqrt{z} - \sqrt{x})(\sqrt{z} + \sqrt{x})} \\
&= \lim_{z \to x} \frac{1}{\sqrt{z} + \sqrt{x}} \\
&= \frac{1}{2\sqrt{x}}.
\end{aligned}$$

Remark

The line passing through the point (x_0, y_0) with slope k has the equation
$$y = y_0 + k(x - x_0).$$

Example

Find the tangent line to the curve $y = \sqrt{x}$ at $x = 4$.

We know that the derivative of $f(x) = \sqrt{x}$ is $f'(x) = \frac{1}{2\sqrt{x}}$.

Example

Find the tangent line to the curve $y = \sqrt{x}$ at $x = 4$.

We know that the derivative of $f(x) = \sqrt{x}$ is $f'(x) = \frac{1}{2\sqrt{x}}$.
Therefore

$$f'(4) = \frac{1}{2\sqrt{4}} = \frac{1}{4}.$$

Example

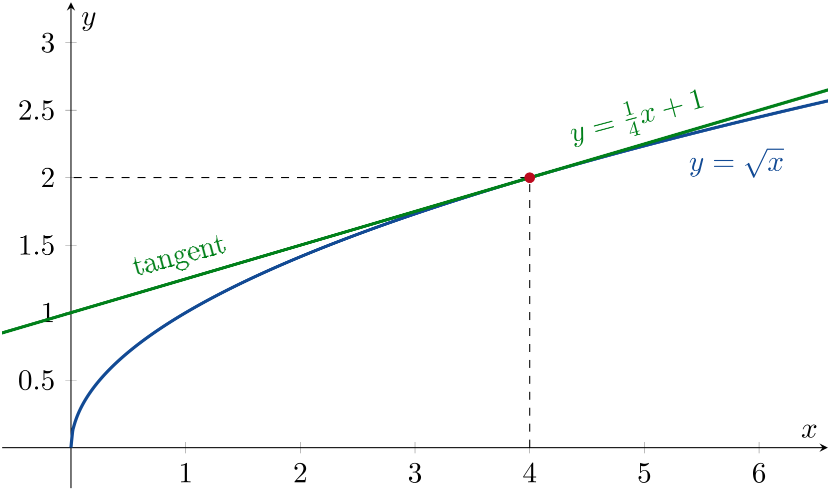
Find the tangent line to the curve $y = \sqrt{x}$ at $x = 4$.

We know that the derivative of $f(x) = \sqrt{x}$ is $f'(x) = \frac{1}{2\sqrt{x}}$. Therefore

$$f'(4) = \frac{1}{2\sqrt{4}} = \frac{1}{4}.$$

The tangent is the line passing through the point $(4, 2)$ with slope $\frac{1}{4}$. So the tangent is

$$y = 2 + \frac{1}{4}(x - 4) = \frac{1}{4}x + 1.$$



y
x
3
2.5
2
1.5
1
0.5
1
2
3
4
5
6
$y = \frac{1}{4}x + 1$
$y = \sqrt{x}$
tangent

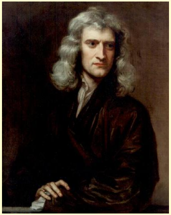
Notation

There are many ways to write the derivative of $y = f(x)$.

$$f'(x) = y' = \frac{dy}{dx} = \frac{df}{dx} = \frac{d}{dx}f(x) = \dot{y} = \dot{f}(x)$$

"the derivative of y with respect to x"

Calculus was started by two men who each claimed to have invented it, then grew to hate each other:



Sir Isaac Newton

BORN
4 January 1643

DECEICED
31 March 1727 (aged 84)

NATIONALITY
British

Calculus was started by two men who each claimed to have invented it, then grew to hate each other: Sir Isaac Newton used $\dot{f}$ and $\dot{y}$.

3.2 The Derivative as a Function

Sir Isaac Newton

Gottfried Leibniz

BORN
1 July 1646

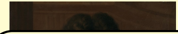
DECEICED
14 November 1716 (aged 70)

NATIONALITY
German

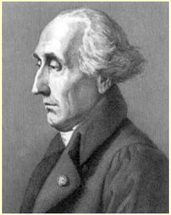
Calculus was started by two men who each claimed to have invented it, then grew to hate each other: Sir Isaac Newton used $\dot{f}$ and $\dot{y}$. Gottfried Leibniz used $\frac{df}{dx}$ and $\frac{dy}{dx}$.

3.2 The Derivative as a Function

Sir Isaac Newton



Gottfried Leibniz



Joseph-Louis Lagrange

BORN
25 January 1736

DECEICED
10 April 1813 (aged 77)

NATIONALITY
Italian

Calculus was started by two men who each claimed to have invented it, then grew to hate each other: Sir Isaac Newton used $\dot{f}$ and $\dot{y}$. Gottfried Leibniz used $\frac{df}{dx}$ and $\frac{dy}{dx}$.

The f' and y' notation came later from Lagrange.

If we want the derivative of $y = f(x)$ at the point $x = x_0$, we can write

$$f'(x_0) = \left.\frac{dy}{dx}\right|_{x=x_0} = \left.\frac{df}{dx}\right|_{x=x_0} = \left.\frac{d}{dx}f(x)\right|_{x=x_0}$$

"the derivative of y with respect to x at $x = x_0$"

If we want the derivative of $y = f(x)$ at the point $x = x_0$, we can write

$$f'(x_0) = \left.\frac{dy}{dx}\right|_{x=x_0} = \left.\frac{df}{dx}\right|_{x=x_0} = \left.\frac{d}{dx}f(x)\right|_{x=x_0}$$

"the derivative of y with respect to x at $x = x_0$"

For example, if $u(x) = \frac{1}{x}$, then

$$u'(4) = \left.\frac{d}{dx}\left(\frac{1}{x}\right)\right|_{x=4} = \left.\frac{-1}{x^2}\right|_{x=4} = \frac{-1}{4^2} = \frac{-1}{16}.$$

Example

Show that $f(x) = |x|$ is not differentiable at $x = 0$.

Example

Show that $f(x) = |x|$ is not differentiable at $x = 0$.

One-Sided Derivatives

Just as we had one-sided limits, we can have one-sided derivatives.

$$\begin{array}{c}\text{Right-hand derivative}\\ \text{of } f \text{ at } a\end{array} = \lim_{h\to 0^+} \frac{f(a+h)-f(a)}{h}$$

$$\begin{array}{c}\text{Left-hand derivative}\\ \text{of } f \text{ at } b\end{array} = \lim_{h\to 0^-} \frac{f(b+h)-f(b)}{h}$$

Example

Show that $f(x) = |x|$ is not differentiable at $x = 0$.

Right-hand derivative

$$\lim_{h\to 0^+} \frac{|0+h| - 0}{h} = \lim_{h\to 0^+} \frac{h}{h} = \lim_{h\to 0^+} 1 = 1.$$

Left-hand derivative

$$\lim_{h\to 0^-} \frac{|0+h| - 0}{h} = \lim_{h\to 0^-} \frac{-h}{h} = \lim_{h\to 0^-} -1 = -1.$$

Example

Show that $f(x) = |x|$ is not differentiable at $x = 0$.

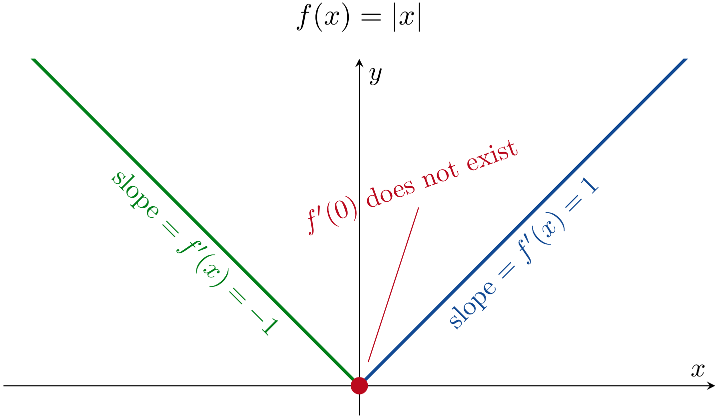
Right-hand derivative

$$\lim_{h\to 0^+} \frac{|0+h|-0}{h} = \lim_{h\to 0^+} \frac{h}{h} = \lim_{h\to 0^+} 1 = 1.$$

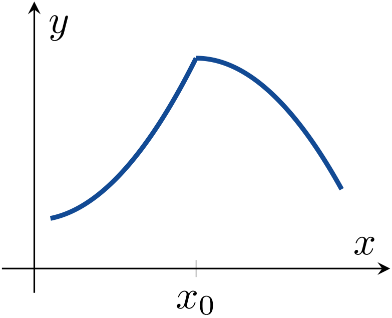
Left-hand derivative

$$\lim_{h\to 0^-} \frac{|0+h|-0}{h} = \lim_{h\to 0^-} \frac{-h}{h} = \lim_{h\to 0^-} -1 = -1.$$

Since the right-hand derivative and left-hand derivative at $x = 0$ are not equal, $|x|$ is not differentiable at $x = 0$.


$f(x) = |x|$
y
x
slope $= f'(x) = -1$
$f'(0)$ does not exist
slope $= f'(x) = 1$

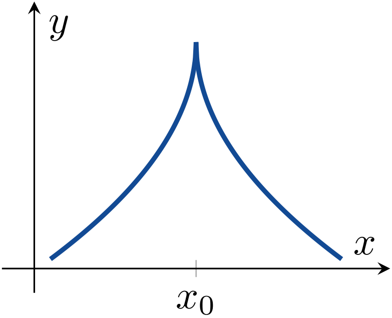
When Does a Function Not Have a Derivative at a Point?



a corner

$f'(x_0)$ does not exist

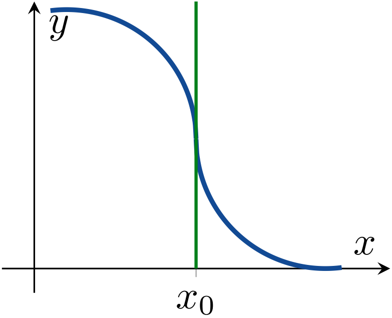
When Does a Function Not Have a Derivative at a Point?



a cusp

$f'(x_0)$ does not exist

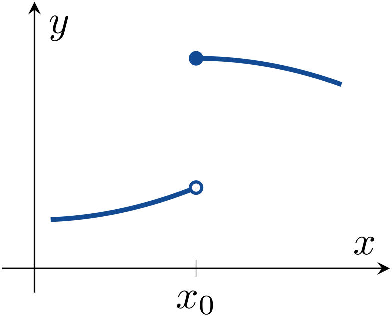
When Does a Function Not Have a Derivative at a Point?



a vertical tangent

$f'(x_0)$ does not exist

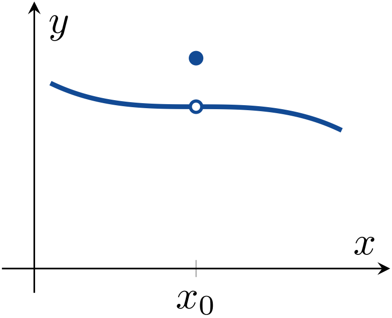
When Does a Function Not Have a Derivative at a Point?



a discontinuity

$f'(x_0)$ does not exist

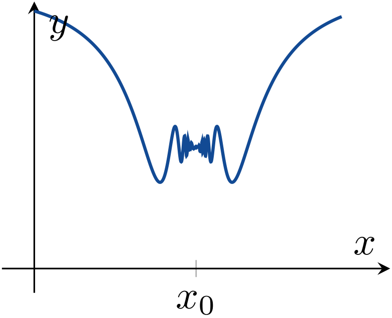
When Does a Function Not Have a Derivative at a Point?



a discontinuity

$f'(x_0)$ does not exist

When Does a Function Not Have a Derivative at a Point?



wild oscillation

$f'(x_0)$ does not exist

Theorem

$$\begin{pmatrix} f \ \ has \ \ a \ \ derivative \\ at \ x = c \end{pmatrix} \implies \begin{pmatrix} f \ \ is \ continuous \\ at \ x = c \end{pmatrix}$$

Theorem

$$\begin{pmatrix} f \text{ has a derivative} \\ \text{at } x = c \end{pmatrix} \implies \begin{pmatrix} f \text{ is continuous} \\ \text{at } x = c \end{pmatrix}$$

Proof.

We know that $f'(c)$ exists.

Theorem

$$\begin{pmatrix} f \text{ has a derivative} \\ \text{at } x = c \end{pmatrix} \implies \begin{pmatrix} f \text{ is continuous} \\ \text{at } x = c \end{pmatrix}$$

Proof.

We know that $f'(c)$ exists. Writing $h = c - x$, we have

$$\lim_{x \to c} f(x) = \lim_{h \to 0} f(c+h) =$$

$$=$$

$$=$$

$$=$$

$\square$

Theorem

$$\begin{pmatrix} f \text{ has a derivative} \\ \text{at } x = c \end{pmatrix} \implies \begin{pmatrix} f \text{ is continuous} \\ \text{at } x = c \end{pmatrix}$$

Proof.

We know that $f'(c)$ exists. Writing $h = c - x$, we have

$$\lim_{x \to c} f(x) = \lim_{h \to 0} f(c+h) = \lim_{h \to 0} \Big(f(c) + f(c+h) - f(c) \Big)$$

$$=$$

$$=$$

$$=$$

$\square$

Theorem

$$\begin{pmatrix} f \text{ has a derivative} \\ \text{at } x = c \end{pmatrix} \implies \begin{pmatrix} f \text{ is continuous} \\ \text{at } x = c \end{pmatrix}$$

Proof.

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$$\lim_{x \to c} f(x) = \lim_{h \to 0} f(c+h) = \lim_{h \to 0} \Big(f(c) + f(c+h) - f(c) \Big)$$

$$= \lim_{h \to 0} \left(f(c) + \frac{f(c+h) - f(c)}{h} h \right)$$

$$=$$

$$=$$

□

Theorem

$$\begin{pmatrix} f \text{ has a derivative} \\ \text{at } x = c \end{pmatrix} \implies \begin{pmatrix} f \text{ is continuous} \\ \text{at } x = c \end{pmatrix}$$

Proof.

We know that $f'(c)$ exists. Writing $h = c - x$, we have

$$\begin{aligned} \lim_{x \to c} f(x) = \lim_{h \to 0} f(c+h) &= \lim_{h \to 0} \Big(f(c) + f(c+h) - f(c) \Big) \\ &= \lim_{h \to 0} \left(f(c) + \frac{f(c+h) - f(c)}{h} h \right) \\ &= \lim_{h \to 0} f(c) + \left(\lim_{h \to 0} \frac{f(c+h) - f(c)}{h} \right) \left(\lim_{h \to 0} h \right) \\ &= \end{aligned}$$

$\square$

Theorem

$$\begin{pmatrix} f \text{ has a derivative} \\ \text{at } x = c \end{pmatrix} \implies \begin{pmatrix} f \text{ is continuous} \\ \text{at } x = c \end{pmatrix}$$

Proof.

We know that $f'(c)$ exists. Writing $h = c - x$, we have

$$\begin{aligned}
\lim_{x \to c} f(x) = \lim_{h \to 0} f(c+h) &= \lim_{h \to 0} \Big(f(c) + f(c+h) - f(c) \Big) \\
&= \lim_{h \to 0} \left(f(c) + \frac{f(c+h) - f(c)}{h} h \right) \\
&= \lim_{h \to 0} f(c) + \left(\lim_{h \to 0} \frac{f(c+h) - f(c)}{h} \right) \left(\lim_{h \to 0} h \right) \\
&= f(c) + f'(c) \cdot 0 = f(c).
\end{aligned}$$

Therefore f is continuous at c. $\square$

Remark

$$\begin{pmatrix} f \text{ has a derivative} \\ \text{at } x = c \end{pmatrix} \implies \begin{pmatrix} f \text{ is continuous} \\ \text{at } x = c \end{pmatrix}$$

but

$$\begin{pmatrix} f \text{ has a derivative} \\ \text{at } x = c \end{pmatrix} \not\Longleftarrow \begin{pmatrix} f \text{ is continuous} \\ \text{at } x = c \end{pmatrix}.$$

For example $f(x) = |x|$ is continuous at $x = 0$, but is not differentiable at $x = 0$.

Break

We will continue at 3pm

Differentiation Rules

Constant Function

If $k \in \mathbb{R}$, then

$$\boxed{\frac{d}{dx}(k) = 0.}$$

(You can read the proof in your textbook.)

Power Function

If $n \in \mathbb{R}$, then

$$\boxed{\frac{d}{dx}\left(x^n\right) = nx^{n-1}.}$$

(You can read the proof in your textbook.)

Power Function

If $n \in \mathbb{R}$, then

$$\boxed{\frac{d}{dx}\left(x^n\right) = nx^{n-1}.}$$

(You can read the proof in your textbook.)

Example

$$\frac{d}{dx}\left(x^3\right) = 3x^{3-1} = 3x^2$$

Example

$$\frac{d}{dx}\left(\sqrt{x}\right) = \frac{d}{dx}\left(x^{\frac{1}{2}}\right) = \frac{1}{2}x^{\frac{1}{2}-1} = \frac{1}{2}x^{-\frac{1}{2}} = \frac{1}{2\sqrt{x}}$$

Example

$$\frac{d}{dx}\left(\frac{1}{x^4}\right) = \frac{d}{dx}\left(x^{-4}\right) = -4x^{-4-1} = -4x^{-5} = -\frac{4}{x^5}$$

(You can read more examples in the textbook.)

The Constant Multiple Rule

If $u(x)$ is differentiable and $k \in \mathbb{R}$, then

$$\boxed{\frac{d}{dx}(ku) = k\frac{du}{dx}.}$$

The Constant Multiple Rule

If $u(x)$ is differentiable and $k \in \mathbb{R}$, then

$$\boxed{\frac{d}{dx}(ku) = k\frac{du}{dx}.}$$

Proof.

$$\begin{aligned}
\frac{d}{dx}(ku) &= \lim_{h \to 0} \frac{ku(x+h) - ku(x)}{h} \\
&= \lim_{h \to 0} k \left(\frac{u(x+h) - u(x)}{h} \right) \\
&= \qquad\qquad\qquad\qquad =
\end{aligned}$$

□

The Constant Multiple Rule

If $u(x)$ is differentiable and $k \in \mathbb{R}$, then

$$\boxed{\frac{d}{dx}(ku) = k\frac{du}{dx}.}$$

Proof.

$$\begin{aligned}
\frac{d}{dx}(ku) &= \lim_{h \to 0} \frac{ku(x+h) - ku(x)}{h} \\
&= \lim_{h \to 0} k\left(\frac{u(x+h) - u(x)}{h}\right) \\
&= k \lim_{h \to 0} \frac{u(x+h) - u(x)}{h} =
\end{aligned}$$

$\square$

The Constant Multiple Rule

If $u(x)$ is differentiable and $k \in \mathbb{R}$, then

$$\boxed{\frac{d}{dx}(ku) = k\frac{du}{dx}.}$$

Proof.

$$\begin{aligned}
\frac{d}{dx}(ku) &= \lim_{h\to 0} \frac{ku(x+h) - ku(x)}{h} \\
&= \lim_{h\to 0} k\left(\frac{u(x+h) - u(x)}{h}\right) \\
&= k \lim_{h\to 0} \frac{u(x+h) - u(x)}{h} = k\frac{du}{dx}
\end{aligned}$$

□

Example

$$\frac{d}{dx}\left(3x^2\right) = 3\frac{d}{dx}\left(x^2\right) = 3 \times 2x = 6x$$

Example

$$\frac{d}{dx}\left(-u\right) = \frac{d}{dx}\left(-1 \times u\right) = -1 \times \frac{du}{dx} = -\frac{du}{dx}$$

The Sum Rule

If $u(x)$ and $v(x)$ are differentiable at x_0, then $u+v$ is also differentiable at x_0 and

$$\boxed{\frac{d}{dx}(u+v) = \frac{du}{dx} + \frac{dv}{dx}.}$$

(You can read the proof in your textbook.)

Example

Differentiate $y = x^3 + \frac{4}{3}x^2 - 5x + 1$.

$$\begin{aligned}\frac{dy}{dx} &= \frac{d}{dx}\left(x^3 + \frac{4}{3}x^2 - 5x + 1\right) \\ &= \frac{d}{dx}\left(x^3\right) + \frac{d}{dx}\left(\frac{4}{3}x^2\right) - \frac{d}{dx}\left(5x\right) + \frac{d}{dx}\left(1\right) \\ &= 3x^2 + \frac{8}{3}x - 5 + 0.\end{aligned}$$

Example

Does the curve $y = x^4 - 2x^2 + 2$ have any points where $\frac{dy}{dx} = 0$? If so, where?

Example

Does the curve $y = x^4 - 2x^2 + 2$ have any points where $\frac{dy}{dx} = 0$? If so, where?

Since

$$\frac{dy}{dx} = 4x^3 - 4x = 4x(x^2 - 1) = 4x(x-1)(x+1),$$

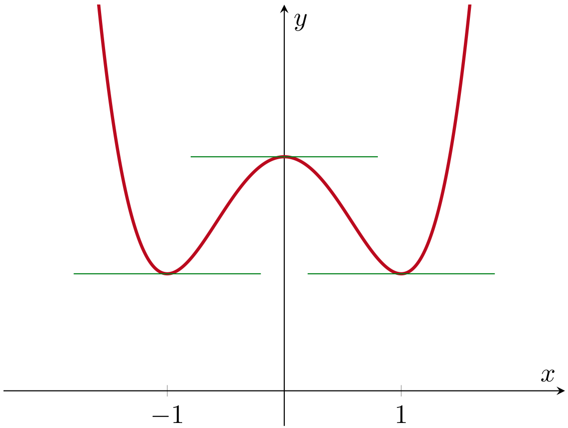
Example

Does the curve $y = x^4 - 2x^2 + 2$ have any points where $\frac{dy}{dx} = 0$? If so, where?

Since

$$\frac{dy}{dx} = 4x^3 - 4x = 4x(x^2 - 1) = 4x(x - 1)(x + 1),$$

we can see that $\frac{dy}{dx} = 0$ if and only if $x = -1$, 0 or 1.



y
x
−1
1

The Product Rule

If $u(x)$ and $v(x)$ are differentiable at x_0, then $u(x)v(x)$ is also differentiable at x_0 and

$$\boxed{\frac{d}{dx}(uv) = \frac{du}{dx}v + u\frac{dv}{dx}.}$$

(You can read the proof in your textbook.)

The Product Rule

If $u(x)$ and $v(x)$ are differentiable at x_0, then $u(x)v(x)$ is also differentiable at x_0 and

$$\boxed{\frac{d}{dx}(uv) = \frac{du}{dx}v + u\frac{dv}{dx}.}$$

Using prime notation, the product rule is

$$\boxed{(uv)' = u'v + uv'.}$$

(You can read the proof in your textbook.)

$$(uv)' = u'v + uv'$$

Example

Differentiate $y = (x^2 + 1)(x^3 + 3)$.

solution 1: We have $y = uv$ with $u = x^2 + 1$ and $v = x^3 + 3$.

$$(uv)' = u'v + uv'$$

Example

Differentiate $y = (x^2 + 1)(x^3 + 3)$.

solution 1: We have $y = uv$ with $u = x^2 + 1$ and $v = x^3 + 3$. So

$$\begin{aligned}
\frac{dy}{dx} &= u'v + uv' \\
&= (x^2 + 1)'(x^3 + 3) + (x^2 + 1)(x^3 + 3)' \\
&= \\
&= \\
&=
\end{aligned}$$

$$(uv)' = u'v + uv'$$

Example

Differentiate $y = (x^2+1)(x^3+3)$.

solution 1: We have $y = uv$ with $u = x^2+1$ and $v = x^3+3$. So

$$\begin{aligned}
\frac{dy}{dx} &= u'v + uv' \\
&= (x^2+1)'(x^3+3) + (x^2+1)(x^3+3)' \\
&= (2x+0)(x^3+3) + (x^2+1)(3x^2+0) \\
&= \\
&=
\end{aligned}$$

$$(uv)' = u'v + uv'$$

Example

Differentiate $y = (x^2 + 1)(x^3 + 3)$.

solution 1: We have $y = uv$ with $u = x^2 + 1$ and $v = x^3 + 3$. So

$$\begin{aligned}
\frac{dy}{dx} &= u'v + uv' \\
&= (x^2 + 1)'(x^3 + 3) + (x^2 + 1)(x^3 + 3)' \\
&= (2x + 0)(x^3 + 3) + (x^2 + 1)(3x^2 + 0) \\
&= 2x^4 + 6x + 3x^4 + 3x^2 \\
&= 5x^4 + 3x^2 + 6x.
\end{aligned}$$

Example

Differentiate $y = (x^2 + 1)(x^3 + 3)$.

solution 2: Since

$$y = (x^2 + 1)(x^3 + 3) = x^5 + x^3 + 3x^2 + 3,$$

Example

Differentiate $y = (x^2 + 1)(x^3 + 3)$.

solution 2: Since

$$y = (x^2 + 1)(x^3 + 3) = x^5 + x^3 + 3x^2 + 3,$$

we have that

$$\frac{dy}{dx} = 5x^4 + 3x^2 + 6x + 0.$$

The Quotient Rule

If $u(x)$ and $v(x)$ are differentiable at x_0 and if $v(x_0) \neq 0$, then $\frac{u}{v}$ is also differentiable at x_0 and

$$\boxed{\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{u'v - uv'}{v^2}.}$$

(You can read the proof in your textbook.)

$$\left(\frac{u}{v}\right)' = \frac{u'v - uv'}{v^2}$$

Example

Differentiate $y = \frac{t^2-1}{t^3+1}$.

We have $y = \dfrac{u}{v}$ with $u = t^2 - 1$ and $v = t^3 + 1$.

$$\left(\frac{u}{v}\right)' = \frac{u'v - uv'}{v^2}$$

Example

Differentiate $y = \frac{t^2-1}{t^3+1}$.

We have $y = \dfrac{u}{v}$ with $u = t^2 - 1$ and $v = t^3 + 1$. Therefore

$$\begin{aligned}
\frac{dy}{dt} &= \frac{u'v - uv'}{v^2} \\
&= \frac{(t^2-1)'(t^3+1) - (t^2-1)(t^3+1)'}{(t^3+1)^2} \\
&= \\
&= \\
&=
\end{aligned}$$

$$\left(\frac{u}{v}\right)' = \frac{u'v - uv'}{v^2}$$

Example

Differentiate $y = \frac{t^2-1}{t^3+1}$.

We have $y = \dfrac{u}{v}$ with $u = t^2 - 1$ and $v = t^3 + 1$. Therefore

$$\begin{aligned}
\frac{dy}{dt} &= \frac{u'v - uv'}{v^2} \\
&= \frac{(t^2-1)'(t^3+1) - (t^2-1)(t^3+1)'}{(t^3+1)^2} \\
&= \frac{(2t)(t^3+1) - (t^2-1)(3t^2)}{(t^3+1)^2} \\
&= \\
&=
\end{aligned}$$

$$\left(\frac{u}{v}\right)' = \frac{u'v - uv'}{v^2}$$

Example

Differentiate $y = \frac{t^2-1}{t^3+1}$.

We have $y = \dfrac{u}{v}$ with $u = t^2 - 1$ and $v = t^3 + 1$. Therefore

$$\begin{aligned}
\frac{dy}{dt} &= \frac{u'v - uv'}{v^2} \\
&= \frac{(t^2-1)'(t^3+1) - (t^2-1)(t^3+1)'}{(t^3+1)^2} \\
&= \frac{(2t)(t^3+1) - (t^2-1)(3t^2)}{(t^3+1)^2} \\
&= \frac{2t^4 + 2t - 3t^4 + 3t^2}{(t^3+1)^2} \\
&= \frac{-t^4 + 3t^2 + 2t}{(t^3+1)^2}.
\end{aligned}$$

$$\left(\frac{u}{v}\right)' = \frac{u'v - uv'}{v^2}$$

Example

Differentiate $f(s) = \frac{\sqrt{s}-1}{\sqrt{s}+1}$.

$$\left(\frac{u}{v}\right)' = \frac{u'v - uv'}{v^2}$$

Example

Differentiate $f(s) = \frac{\sqrt{s}-1}{\sqrt{s}+1}$.

We have $f(s) = \dfrac{u}{v}$ with $u = \sqrt{s} - 1$ and $v = \sqrt{s} + 1$. Remember that $\frac{d}{ds}\left(\sqrt{s}\right) = \frac{1}{2\sqrt{s}}$.

$$\left(\frac{u}{v}\right)' = \frac{u'v - uv'}{v^2}$$

Example

Differentiate $f(s) = \frac{\sqrt{s}-1}{\sqrt{s}+1}$.

We have $f(s) = \dfrac{u}{v}$ with $u = \sqrt{s} - 1$ and $v = \sqrt{s} + 1$. Remember that $\frac{d}{ds}\left(\sqrt{s}\right) = \frac{1}{2\sqrt{s}}$. Therefore

$$\begin{aligned}
\frac{df}{ds} &= \frac{u'v - uv'}{v^2} \\
&= \frac{(\sqrt{s}-1)'(\sqrt{s}+1) - (\sqrt{s}-1)(\sqrt{s}+1)'}{\left(\sqrt{s}+1\right)^2} \\
&= \frac{\left(\frac{1}{2\sqrt{s}}\right)(\sqrt{s}+1) - (\sqrt{s}-1)\left(\frac{1}{2\sqrt{s}}\right)}{\left(\sqrt{s}+1\right)^2} \\
&= \frac{\frac{1}{2} + \frac{1}{2\sqrt{s}} - \frac{1}{2} + \frac{1}{2\sqrt{s}}}{\left(\sqrt{s}+1\right)^2} = \frac{1}{\sqrt{s}\left(\sqrt{s}+1\right)^2}.
\end{aligned}$$

Example

Differentiate $y = \frac{(x-1)(x^2-2x)}{x^4}$.

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Thus

$$y' = -x^{-2} - 3(-2)x^{-3} + 2(-3)x^{-4} = -\frac{1}{x^2} + \frac{6}{x^3} - \frac{6}{x^4}.$$

Second Order Derivatives

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f'' is called the *second derivative* of f.

We can write

$$f''(x) = \frac{d}{dx}f'(x) = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d^2y}{dx^2} = \frac{dy'}{dx} = y''$$

"d squared y, dx squared"

Example

If $y = x^6$, then

$$y' = \frac{d}{dx}\left(x^6\right) = 6x^5$$

and

$$y'' = \frac{d}{dx}\left(y'\right) = \frac{d}{dx}\left(6x^5\right) = 30x^4.$$

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Equivalently, we can write

$$\frac{d^2}{dx^2}\left(x^6\right) = \frac{d}{dx}\left(\frac{d}{dx}\left(x^6\right)\right) = \frac{d}{dx}\left(6x^5\right) = 30x^4.$$

Higher Order Derivatives

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If $f^{(4)}$ is differentiable, then its derivative $f^{(5)} = \frac{d^5 f}{dx^5}$ is the *fifth derivative* of f.

$$\vdots$$

If $f^{(n-1)}$ is differentiable, then its derivative $f^{(n)} = \frac{d^n f}{dx^n}$ is the *nth derivative* of f.

Remark

For the 13^{th} derivative of $f(x)$, writing

$$f'''''''''''''(x)$$

would just be silly. So we write

$$f^{(13)}(x)$$

instead.

Example

Find the first four derivatives of $y = x^3 - 3x^2 + 2$.

First derivative: $y' = 3x^2 - 6x$

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First derivative: $y' = 3x^2 - 6x$

Second derivative: $y'' = 6x - 6$

Third derivative: $y''' = 6$

Fourth derivative: $y^{(4)} = 0$.

(Note that since $\frac{d}{dx}(0) = 0$, if $n \geq 4$ then $y^{(n)} = 0$ also.)

The Derivative as a Rate of Change

3.4 The Derivative as a Rate of Change

Definition

The *rate of change* of f with respect to x at x_0 is the derivative

$$f'(x_0) = \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h}$$

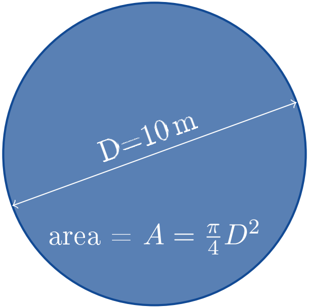
provided the limit exists.

Example

The area A of a circle is related to its diameter by the equation

$$A = \frac{\pi}{4} D^2.$$

How fast does the area change with respect to the diameter when the diameter is $10\,\text{m}$?



Example

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The rate of change is

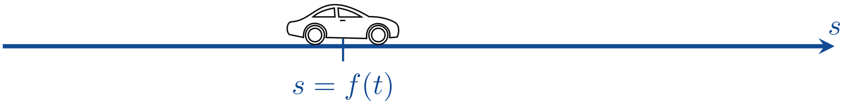
$$\frac{dA}{dD} = \frac{\pi}{4} \cdot 2D = \frac{\pi D}{2}.$$

When $D = 10\,\mathrm{m}$, this is $5\pi\,\mathrm{m}^2/\mathrm{m}$.

Motion Along a Line; Displacement, Velocity, Speed, Acceleration and Jerk

Suppose that an object is moving along a line.

Motion Along a Line; Displacement, Velocity, Speed, Acceleration and Jerk



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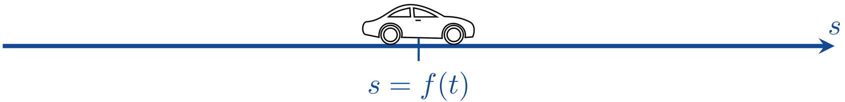
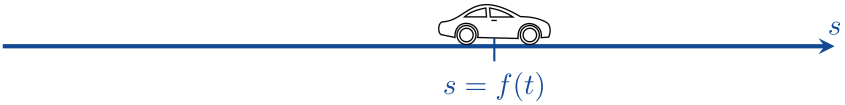


Suppose that an object is moving along a line. Let

$$s = f(t)$$

be its position at time t.

Motion Along a Line; Displacement, Velocity, Speed, Acceleration and Jerk



Suppose that an object is moving along a line. Let

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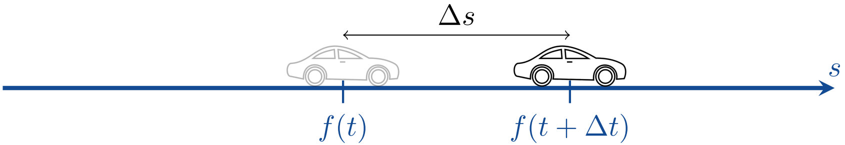
Motion Along a Line; Displacement, Velocity, Speed, Acceleration and Jerk

The *displacement* of the object over the time interval from t to $t + \Delta t$ is

Motion Along a Line; Displacement, Velocity, Speed, Acceleration and Jerk



The *displacement* of the object over the time interval from t to $t + \Delta t$ is

$$\Delta s = f(t + \Delta t) - f(t).$$

The *average velocity* of the object between time t and time $t + \Delta t$ is

$$\frac{\text{displacement}}{\text{travel time}} = \frac{\Delta s}{\Delta t} = \frac{f(t + \Delta t) - f(t)}{\Delta t}.$$

Taking the limit as $\Delta t \to 0$, we obtain:

3.4 The Derivative as a Rate of Change

Definition

$$\text{velocity} = \frac{d}{dt}\,(\text{position})$$

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Definition

$$\text{speed} = |\text{velocity}| = \left|\frac{ds}{dt}\right|.$$

$$\text{velocity} = v(t) = \frac{d}{dt}\,(\text{position}) = \frac{ds}{dt}.$$

Definition

$$\text{acceleration} = a(t) = \frac{d}{dt}\,(\text{velocity}) = \frac{dv}{dt}.$$

$$\text{velocity} = v(t) = \frac{d}{dt}(\text{position}) = \frac{ds}{dt}.$$

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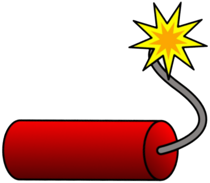
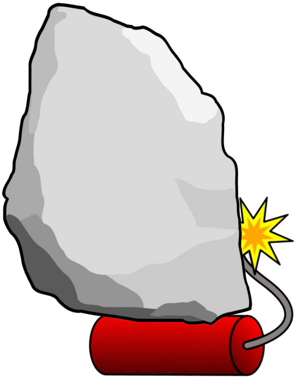
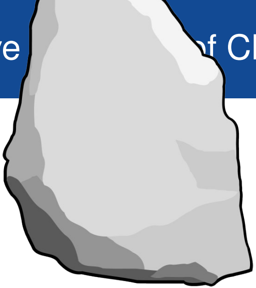
$$\text{acceleration} = a(t) = \frac{d}{dt}(\text{velocity}) = \frac{dv}{dt}.$$

Definition

$$\text{jerk} = j(t) = \frac{d}{dt}(\text{acceleration}) = \frac{da}{dt}.$$







Example

A dynamite blast blows a heavy rock straight upwards with a launch velocity of $49\,\mathrm{m/s}$. It reaches a height of

$$s = 49t - 4.9t^2\,\mathrm{m}$$

after t seconds.

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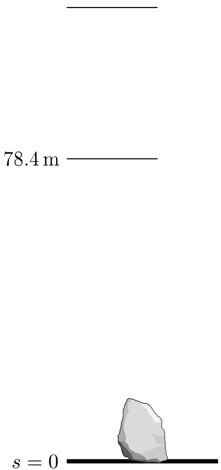
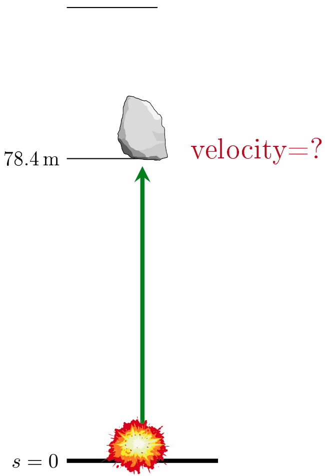
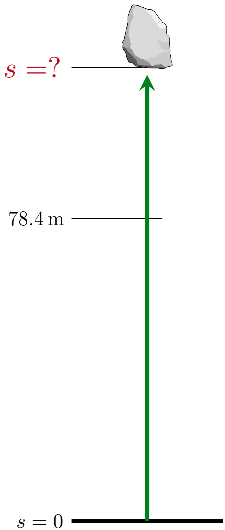
1. How high does the rock go?
2. What are the velocity and speed of the rock when it is 78.4 m above the ground on the way up? On the way down?
3. What is the acceleration of the rock at time t during its flight (after the blast)?
4. When does the rock hit the ground?

3.4 The Derivative as a Rate of Change





78.4 m
velocity=?
$s=0$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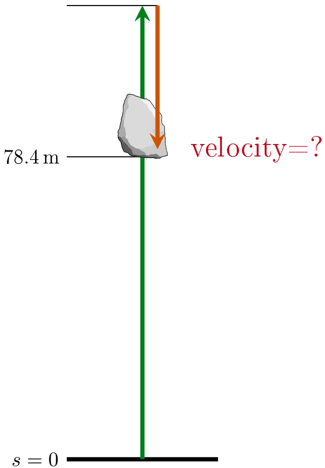
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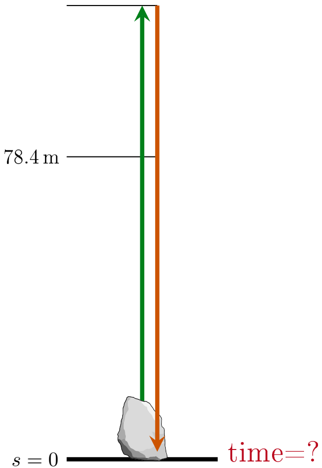
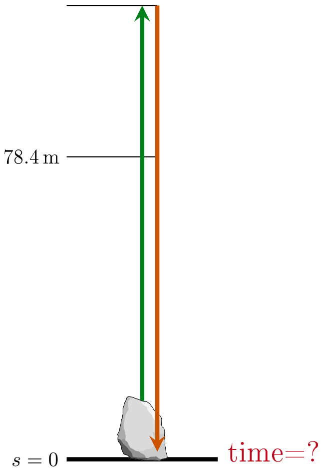
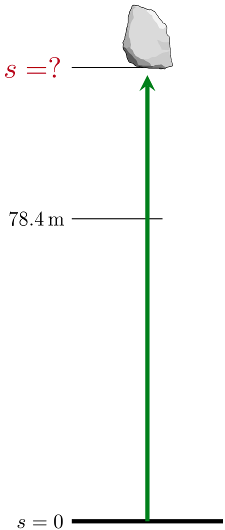

78.4 m
$s = 0$
time=?



Since $s(t) = 49t - 4.9t^2\,m$, we have

$$v(t) = s'(t) = 49 - 9.8t\,m/s.$$



Since $s(t) = 49t - 4.9t^2\,m$, we have

$$v(t) = s'(t) = 49 - 9.8t\,m/s.$$

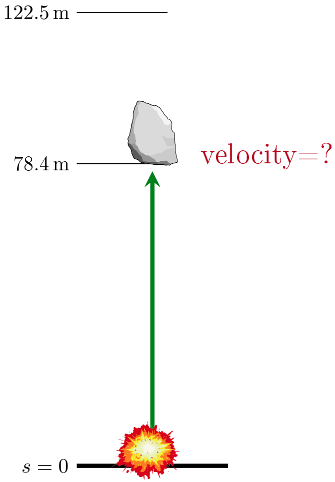
At the top, we have

$$0 = v(t) = 49 - 9.8t \implies t = 5\,s.$$

So the rock goes

$$s(5) = 49(5) - 4.9(25) = 122.5\,m$$

high.



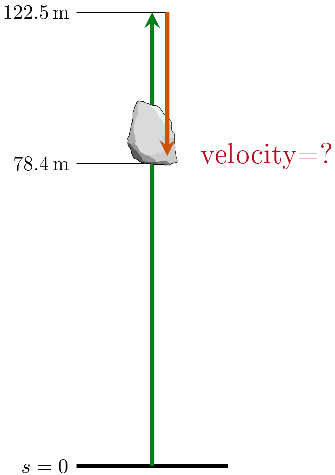
Now

$$78.4 = s(t) = 49t - 4.9t^2$$

$$4.9(t^2 - 10t + 16) = 0$$

$$4.9(t-2)(t-8) = 0$$

implies that $t = 2$ or $t = 8$.



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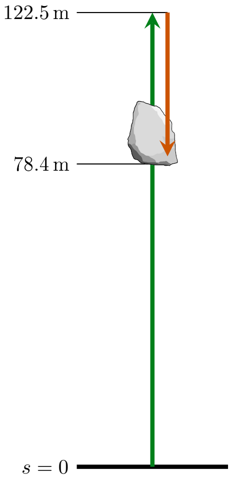
We have that

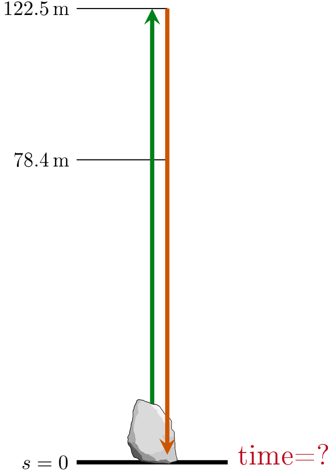
$$v(2) = 49 - 9.8(2) = 29.4\,\mathrm{m/s}$$
$$v(8) = 49 - 9.8(8) = -29.4\,\mathrm{m/s}.$$

The speed at both points is $29.4\,\mathrm{m/s}$.



Since $v(t) = 49 - 9.8t\,\mathrm{m/s}$, the acceleration is

$$a(t) = v'(t) = -9.8\,\mathrm{m/s}^2.$$



Since

$$\begin{aligned} 0 = s(t) &= 49t - 4.9t^2 \\ &= 4.9(t)(10 - t), \end{aligned}$$

the rock hits the ground after 10 seconds.

Derivatives in Economics

Suppose that

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Derivatives in Economics

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It costs more to produce $x + h$ tonnes of steel, and the cost difference divided by h is:

$$\text{average cost of producing each addition tonne} = \frac{c(x+h) - c(x)}{h}.$$

Taking the limit as $h \to 0$, we get the *marginal cost* of producing more steel per week when the current weekly production is x tonnes:

$$\textbf{marginal cost of production} = \frac{dc}{dx} = \lim_{h \to 0} \frac{c(x+h) - c(x)}{h}.$$

Example

Suppose that it costs

$$c(x) = x^3 - 6x^2 + 15x$$

lira to produce x radiators per day (when $x \in [8, 30]$) and that you can earn

$$r(x) = x^3 - 3x^2 + 12x$$

lira from selling x radiators. Your factory currently makes 10 radiators per day.

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lira from selling x radiators. Your factory currently makes 10 radiators per day.

1. How much extra will it cost to produce one more radiator per day?
2. How much more profit can you make by producing 11 radiators per day instead of 10?

$$c(x) = x^3 - 6x^2 + 15x \qquad r(x) = x^3 - 3x^2 + 12x$$

The marginal cost of production is

$$c'(x) = 3x^2 - 12x + 15$$
$$c'(10) = 3(100) - 12(10) + 15 = 195.$$

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The marginal cost of production is

$$c'(x) = 3x^2 - 12x + 15$$
$$c'(10) = 3(100) - 12(10) + 15 = 195.$$

The marginal revenue is

$$r'(x) = 3x^2 - 6x + 12$$
$$r'(10) = 3(100) - 6(10) + 12 = 252.$$

$$c(x) = x^3 - 6x^2 + 15x \qquad r(x) = x^3 - 3x^2 + 12x$$

The marginal cost of production is

$$c'(x) = 3x^2 - 12x + 15$$
$$c'(10) = 3(100) - 12(10) + 15 = 195.$$

The marginal revenue is

$$r'(x) = 3x^2 - 6x + 12$$
$$r'(10) = 3(100) - 6(10) + 12 = 252.$$

So producing 11 radiators per day instead of 10 will cost approximately 195 liras extra, but will bring in approximately 252 liras more.

We would expect your profit to increase by approximately $252 - 195 = 57$ liras.

Next Time

- 3.5 Derivatives of Trigonometric Functions
- 3.6 The Chain Rule
- 3.7 Implicit Differentiation