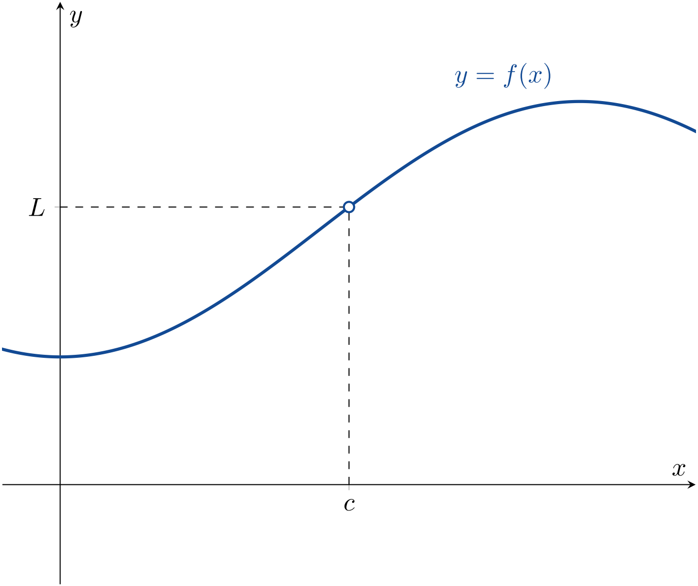
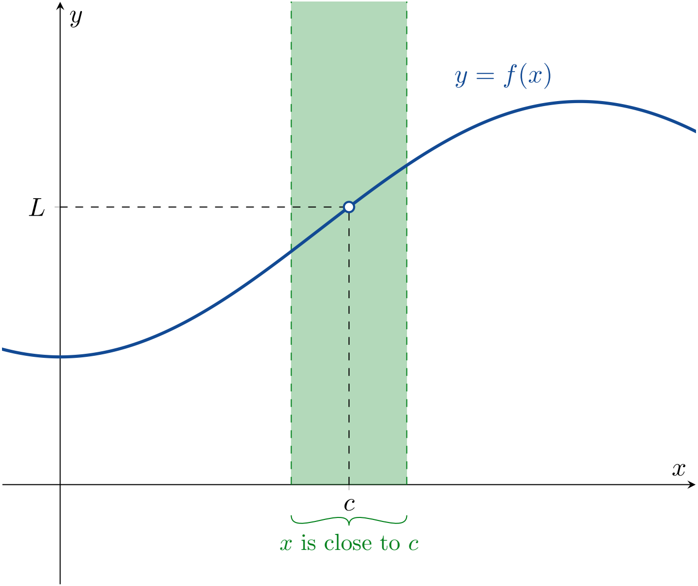
Lecture 3

- 2.4 One-Sided Limits
- 2.5 Continuity
- 2.6 Limits Involving Infinity; Asymptotes of Graphs

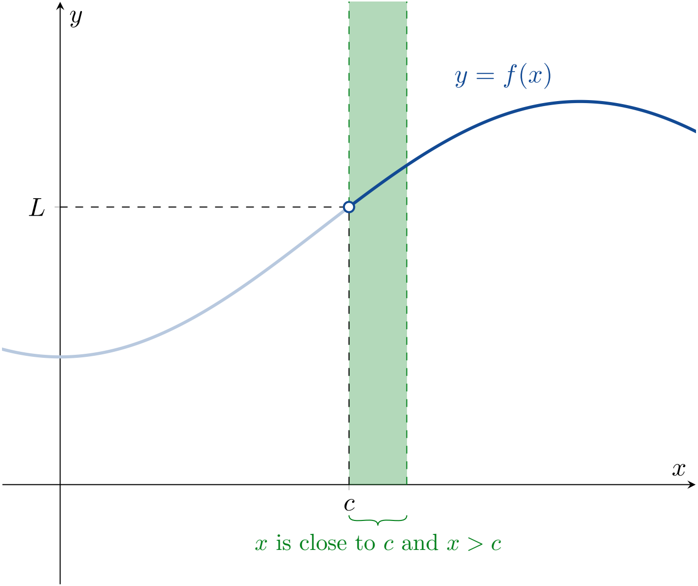
2.4 One-Sided Limits

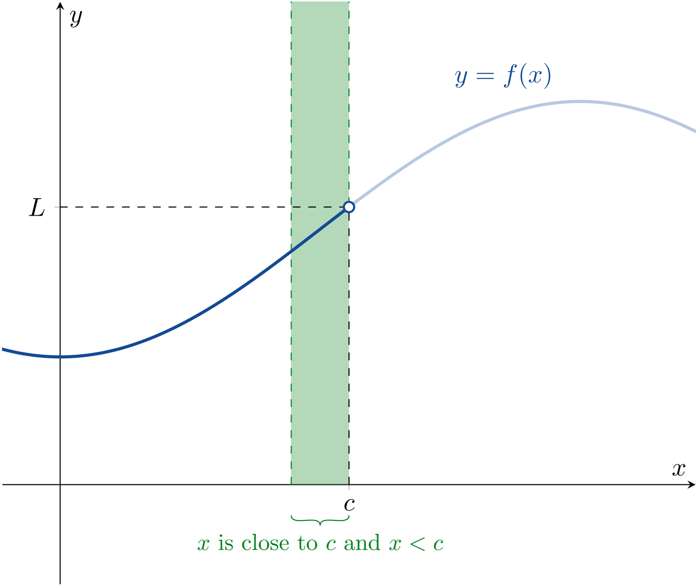

y
$y = f(x)$
L
x
c



y
$y = f(x)$
L
x
c
x is close to c

2.4 One-Sided Limits




y
$y = f(x)$
L
x
c
x is close to c and $x < c$

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$\lim\limits_{x \to c^+} f(x)$ is called the *right-hand limit* of $f(x)$ at c.

This means, the limit of $f(x)$ if we only look at values of x on the right of c $(x > c)$.

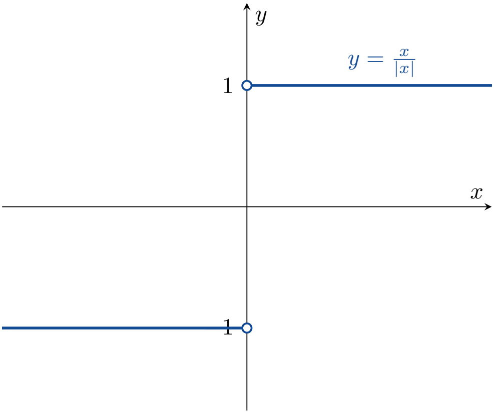
$\lim\limits_{x \to c^-} f(x)$ is called the *left-hand limit* of $f(x)$ at c.

This means, the limit of $f(x)$ if we only look at values of x on the left of c $(x < c)$.

2.4 One-Sided Limits

Example

$$y = \frac{x}{|x|}$$

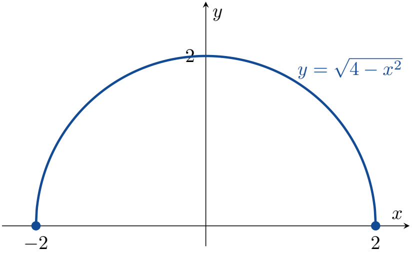


$$\lim_{x \to 0^{+}} \frac{x}{|x|} = 1$$

$$\lim_{x \to 0^{-}} \frac{x}{|x|} = -1$$

Example

$f : [-2, 2] \to \mathbb{R},\ f(x) = \sqrt{4 - x^2}$.



$$\lim_{x \to 2^-} \sqrt{4 - x^2} = 0$$

$$\lim_{x \to 2^+} \sqrt{4 - x^2} \text{ doesn't exist}$$

$$\lim_{x \to -2^+} \sqrt{4 - x^2} = 0$$

$$\lim_{x \to -2^-} \sqrt{4 - x^2} \text{ doesn't exist}$$

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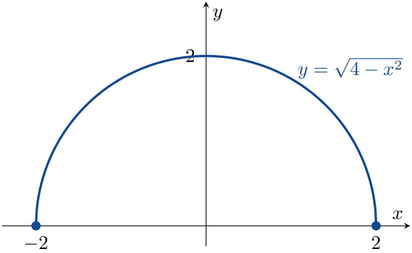

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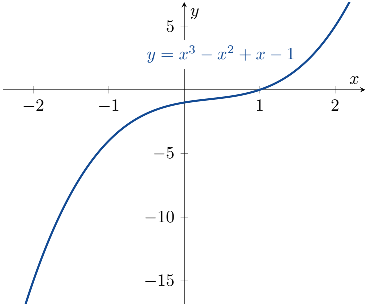
$$\lim_{x \to -2^-} \sqrt{4 - x^2} \text{ doesn't exist}$$

Note that $\lim_{x \to c} \sqrt{4 - x^2}$, $\lim_{x \to c^+} \sqrt{4 - x^2}$ and $\lim_{x \to c^-} \sqrt{4 - x^2}$ all exist for all $c \in (-2, 2)$.

2.4 One-Sided Limits

Example

$y = x^3 - x^2 + x - 1$



Example

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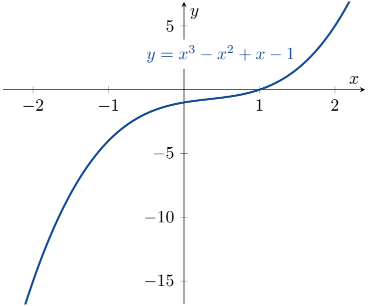
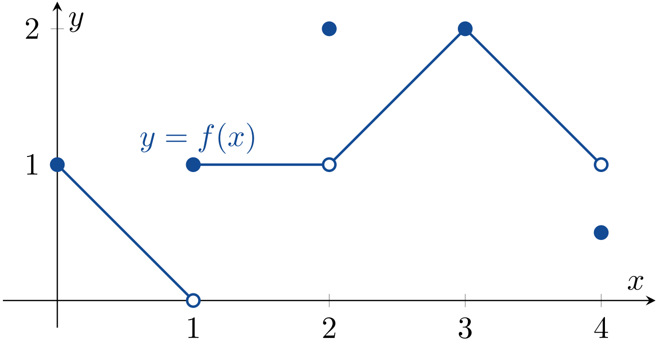


$$\lim_{x\to c}(x^3 - x^2 + x - 1) \text{ exists for all } c \in (-\infty, \infty)$$

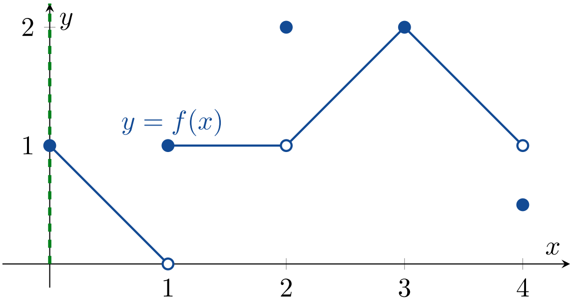
$$\lim_{x\to c^+}(x^3 - x^2 + x - 1) \text{ exists for all } c \in (-\infty, \infty)$$

$$\lim_{x\to c^-}(x^3 - x^2 + x - 1) \text{ exists for all } c \in (-\infty, \infty)$$



Example

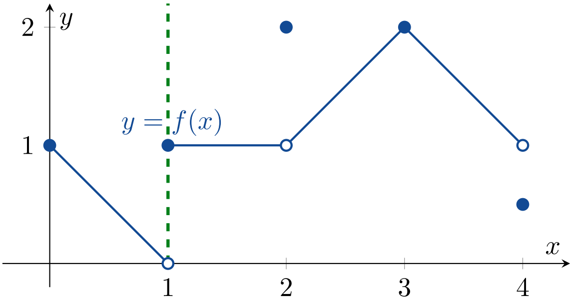
Consider the function $f : [0, 4] \to \mathbb{R}$ with graph shown above.



At $x = 0$: $\lim_{x \to 0^-} f(x)$ does not exist

$$\lim_{x \to 0^+} f(x) = 1$$

$$\lim_{x \to 0} f(x) = 1.$$

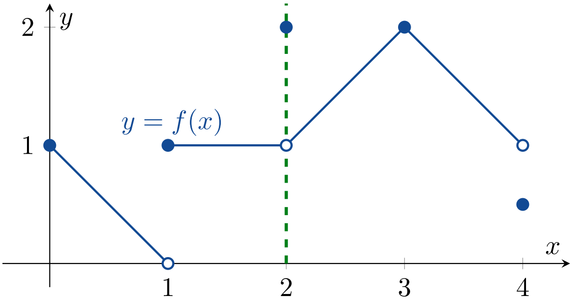


At $x = 1$:

$$\lim_{x \to 1^-} f(x) = 0$$

$$\lim_{x \to 1^+} f(x) = 1$$

$$\lim_{x \to 1} f(x) \text{ does not exist.}$$

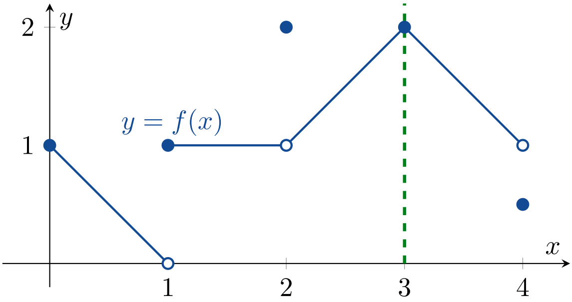


At $x = 2$:

$$\lim_{x \to 2^-} f(x) = 1$$

$$\lim_{x \to 2^+} f(x) = 1$$

$$\lim_{x \to 2} f(x) = 1.$$

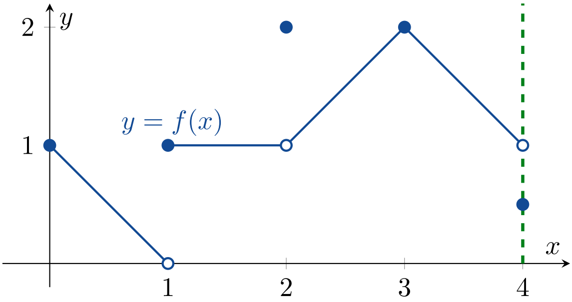


At $x = 3$:

$$\lim_{x \to 3^-} f(x) = 2$$

$$\lim_{x \to 3^+} f(x) = 2$$

$$\lim_{x \to 3} f(x) = 2.$$



At $x = 4$: $$\lim_{x\to 4^-} f(x) = 1$$

$$\lim_{x\to 4^+} f(x) \text{ does not exist}$$

$$\lim_{x\to 4} f(x) = 1.$$

Precise Definitions of One-Sided Limits

Definition (Right-Hand Limit)

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Precise Definitions of One-Sided Limits

Definition (Right-Hand Limit)

We write $\lim_{x\to c^+} f(x) = L$ iff for all $\varepsilon > 0$, there exists $\delta > 0$ such that

$$c < x < c + \delta \qquad \Longrightarrow \qquad |f(x) - L| < \varepsilon.$$

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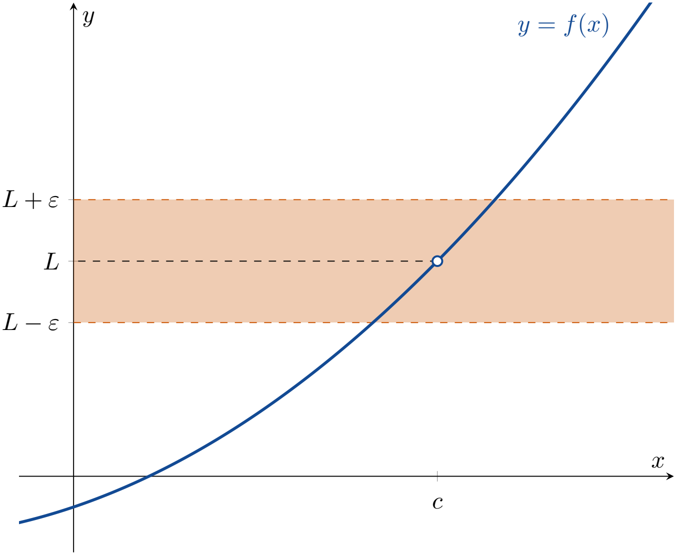
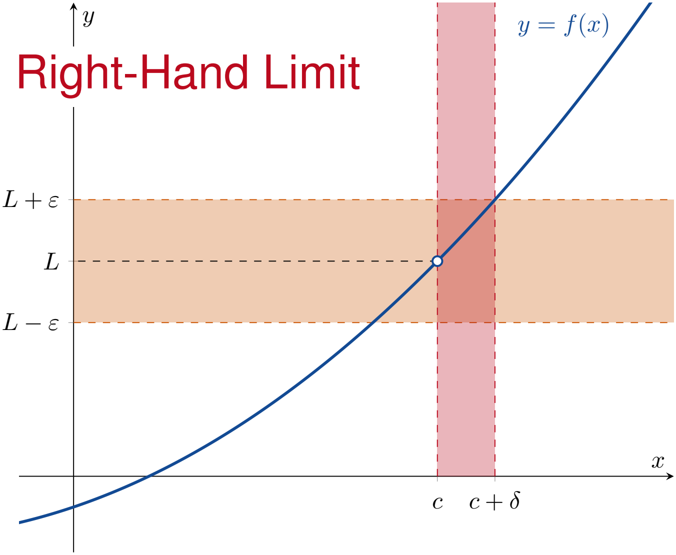


y
y = f(x)
L + ε
L
L − ε
x
c



y
Right-Hand Limit
$y = f(x)$
$L + \varepsilon$
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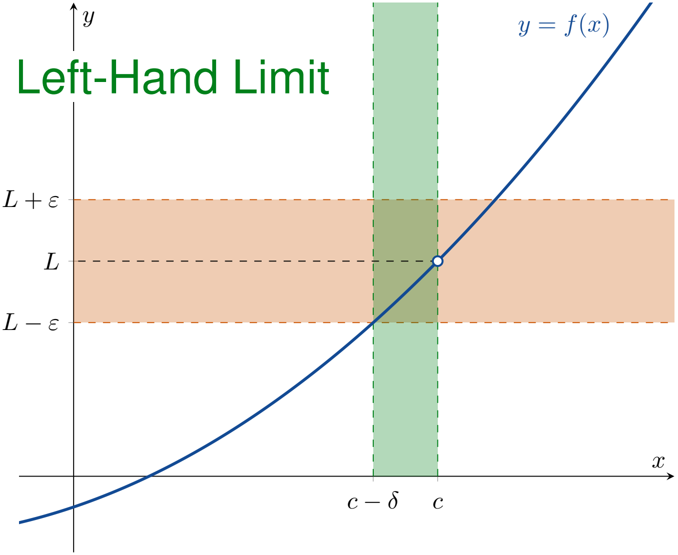
x
c
$c + \delta$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Left-Hand Limit
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x
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$$\forall \varepsilon > 0, \exists \delta > 0 \text{ s.t. } c < x < c + \delta \implies |f(x) - L| < \varepsilon$$

Example

Prove that $\lim_{x \to 0^+} \sqrt{x} = 0$.

Let $\varepsilon > 0$. Choose $\delta =$. Then

$$0 < x < \delta \qquad \implies \qquad \left|\sqrt{x} - 0\right| \qquad .$$

$$\forall \varepsilon > 0, \exists \delta > 0 \text{ s.t. } c < x < c + \delta \implies |f(x) - L| < \varepsilon$$

$$\left|\sqrt{x} - 0\right| < \varepsilon$$

Example

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$$0 < x < \delta \qquad \implies$$

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$$\left|\sqrt{x} - 0\right| < \varepsilon$$

$$-\varepsilon < \sqrt{x} < \varepsilon$$

$$\sqrt{x} < \varepsilon$$

$$x < \varepsilon^2$$

choose $\delta = \varepsilon^2$.

Example

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Let $\varepsilon > 0$. Choose $\delta = \varepsilon^2$. Then

$$0 < x < \delta \qquad \implies \qquad \left|\sqrt{x} - 0\right| = \sqrt{x} < \sqrt{\delta} = \varepsilon.$$

Theorem

$$\lim_{x \to c^-} f(x) = L = \lim_{x \to c^+} f(x) \quad \iff \quad \lim_{x \to c} f(x) = L$$

You prove.

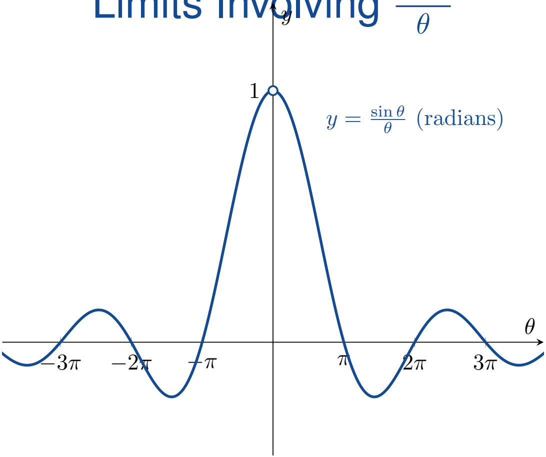
Theorem

$$\lim_{x \to c^-} f(x) = L = \lim_{x \to c^+} f(x) \quad \iff \quad \lim_{x \to c} f(x) = L$$

$$\lim_{x \to c^-} f(x) \neq \lim_{x \to c^+} f(x) \quad \implies \quad \lim_{x \to c} f(x) \textit{ does not exist.}$$

You prove.

Limits Involving $\frac{\sin\theta}{\theta}$



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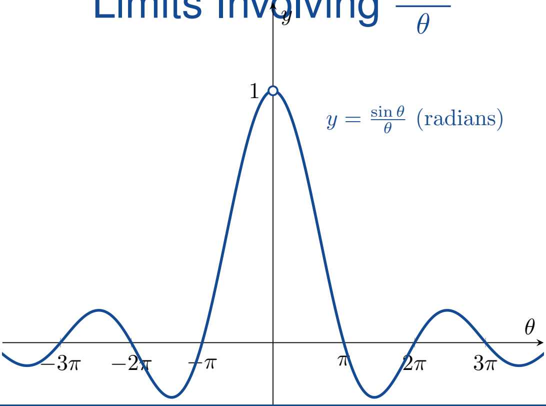
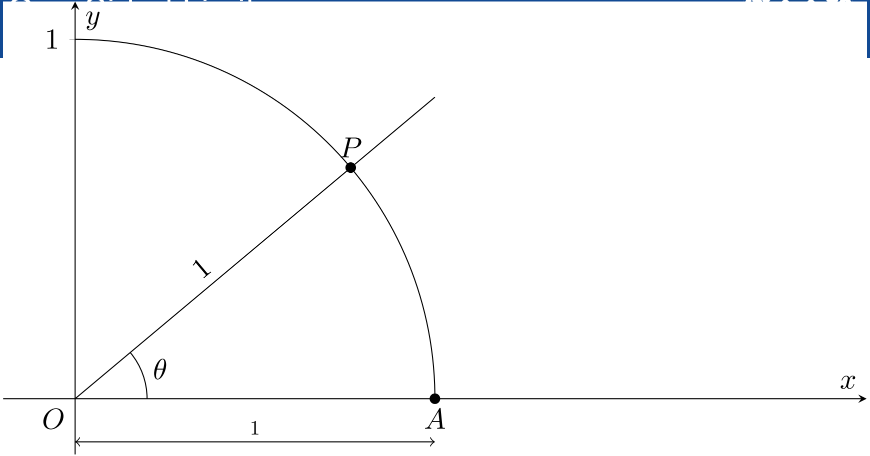


Theorem

$$\lim_{\theta\to 0}\frac{\sin\theta}{\theta} = 1 \qquad (\theta \textit{ in radians})$$

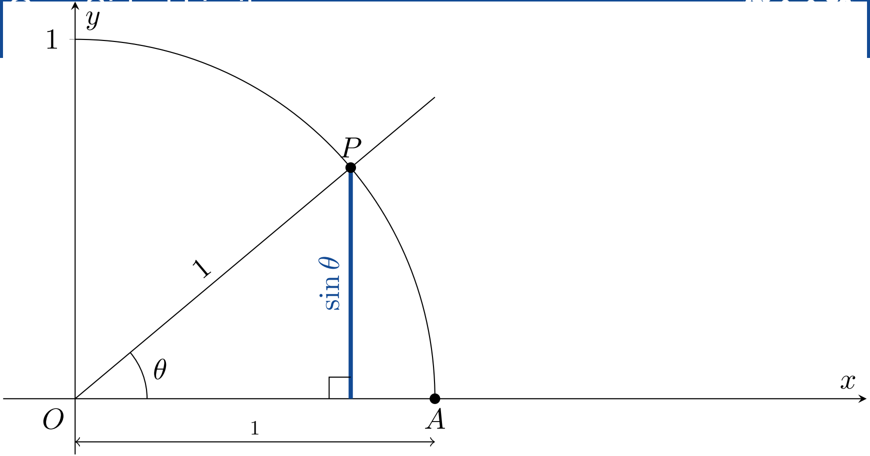
$$\lim_{\theta \to 0} \frac{\sin \theta}{\theta} = 1$$

We will need to use this limit in Lecture 5.



Proof.

We have that



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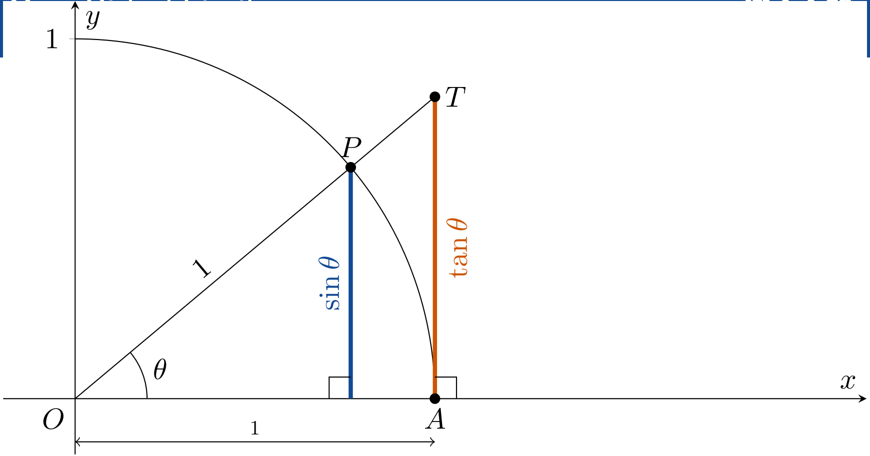

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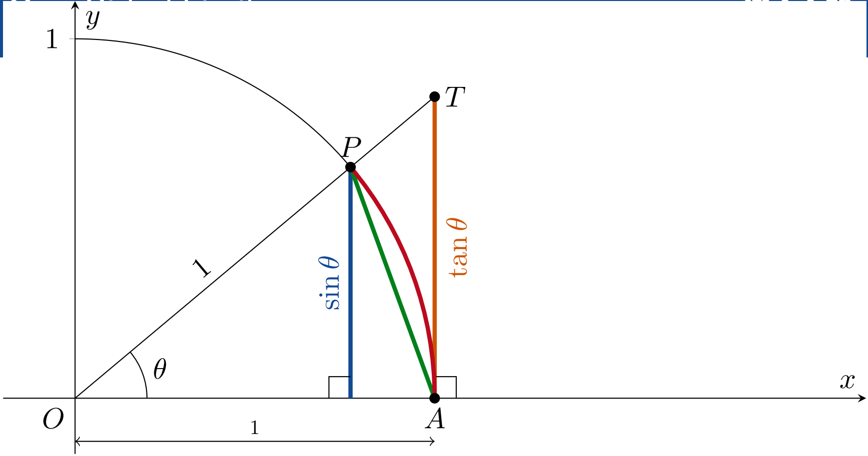
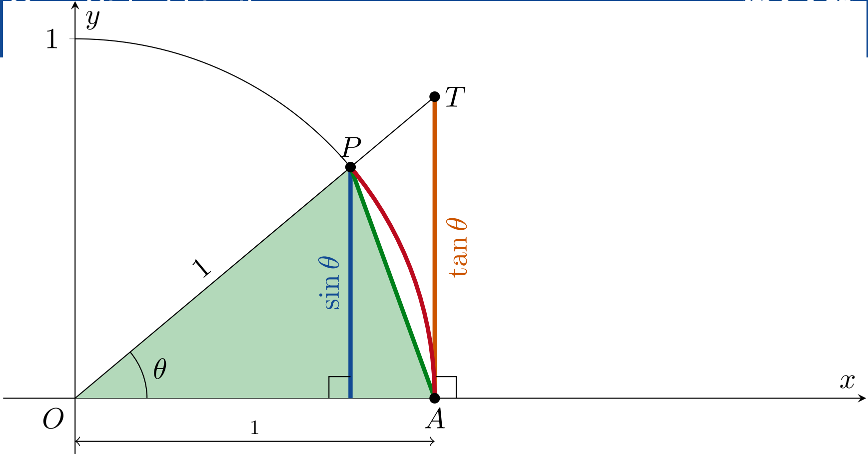


Proof.

We have that

$$\text{area of } \Delta OAP \qquad \text{area of sector } OAP \qquad \text{area of } \Delta OAT.$$



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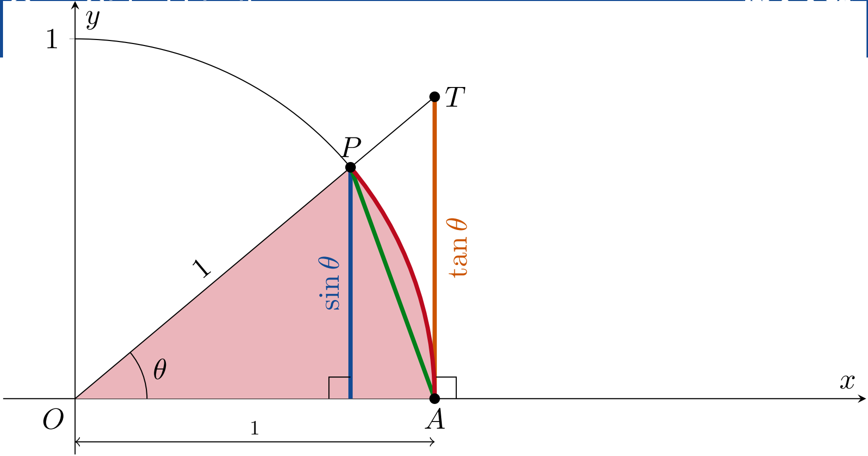

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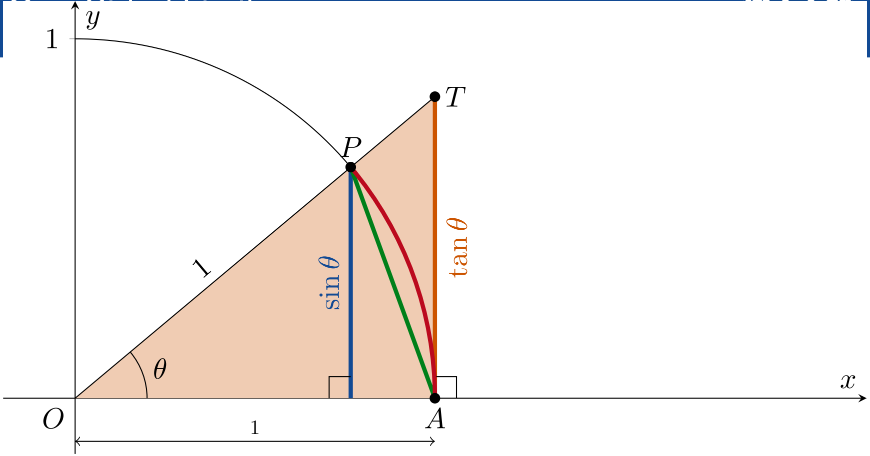

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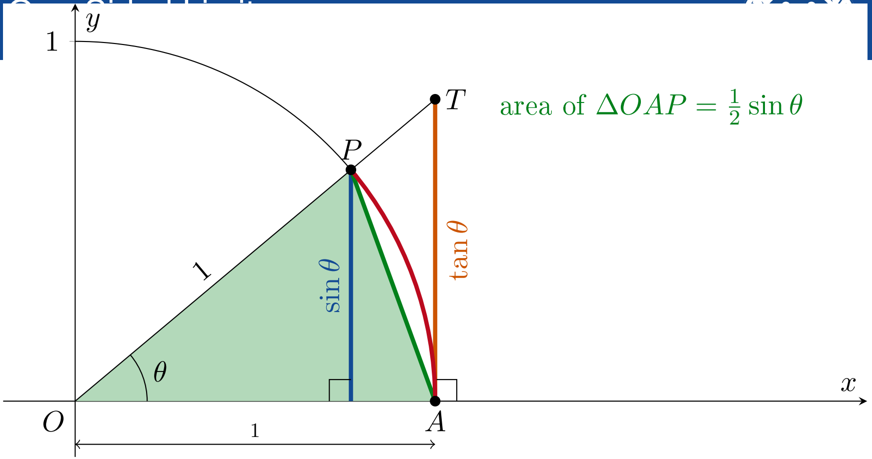

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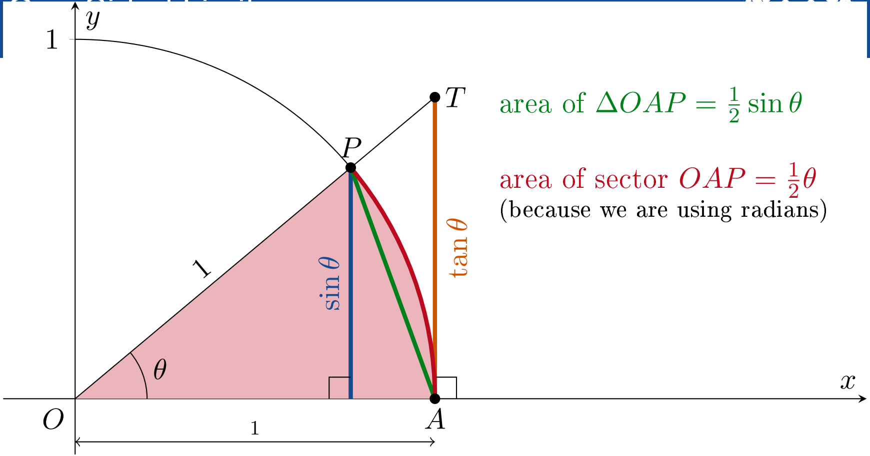


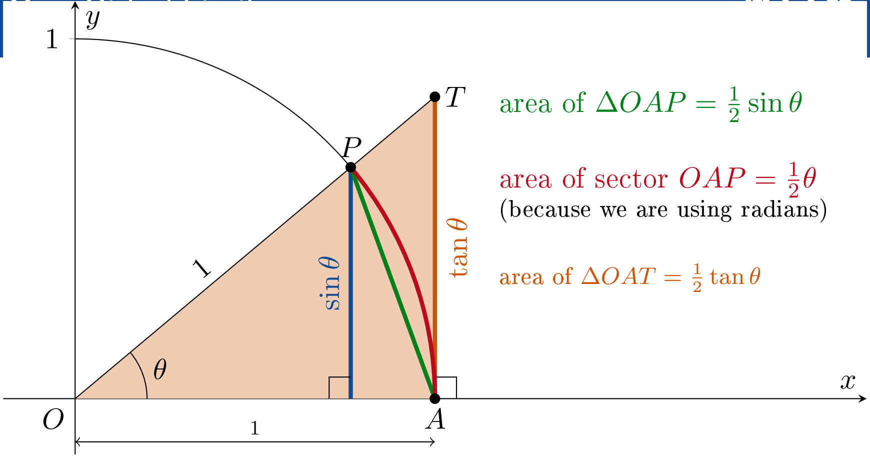
area of $\Delta OAP = \frac{1}{2}\sin\theta$

area of sector $OAP = \frac{1}{2}\theta$
(because we are using radians)

Proof.

We have that

$$\text{area of } \Delta OAP < \text{ area of sector } OAP < \text{ area of } \Delta OAT.$$



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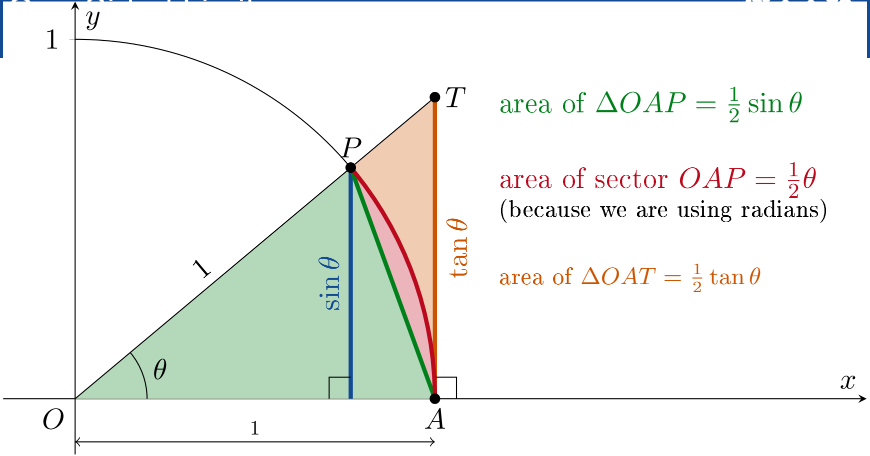
(because we are using radians)

area of $\Delta OAT = \frac{1}{2}\tan\theta$

Proof.

We have that

$$\text{area of } \Delta OAP < \text{ area of sector } OAP < \text{ area of } \Delta OAT.$$



Proof.

We have that

$$\frac{1}{2}\sin\theta < \frac{1}{2}\theta < \frac{1}{2}\tan\theta.$$

Proof continued.

Therefore

$$\frac{1}{2}\sin\theta < \frac{1}{2}\theta < \frac{1}{2}\tan\theta$$

$$\sin\theta < \theta < \tan\theta$$

$$1 < \frac{\theta}{\sin\theta} < \frac{1}{\cos\theta}$$

and

$$1 > \frac{\sin\theta}{\theta} > \cos\theta.$$

Proof continued.

Therefore

$$\frac{1}{2}\sin\theta < \frac{1}{2}\theta < \frac{1}{2}\tan\theta$$

$$\sin\theta < \theta < \tan\theta$$

$$1 < \frac{\theta}{\sin\theta} < \frac{1}{\cos\theta}$$

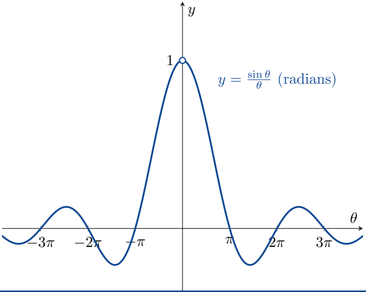
and

$$1 > \frac{\sin\theta}{\theta} > \cos\theta.$$

Since $\lim_{\theta\to 0^+} \cos\theta = 1$, it follows by the Sandwich Theorem that

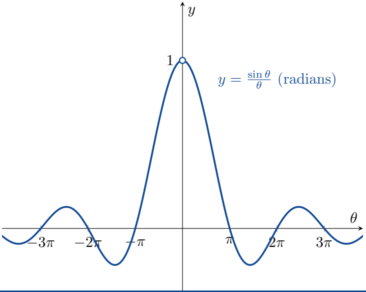
$$\lim_{\theta\to 0^+} \frac{\sin\theta}{\theta} = 1$$

also.



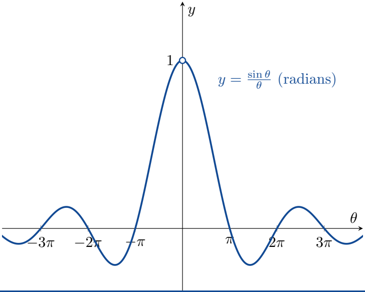
Proof continued.

We also need to do the left-hand limit:



Proof continued.

We also need to do the left-hand limit: Since $\sin\theta$ and θ are both odd functions, it follows that $\frac{\sin\theta}{\theta}$ is an even function.



Proof continued.

We also need to do the left-hand limit: Since $\sin\theta$ and θ are both odd functions, it follows that $\frac{\sin\theta}{\theta}$ is an even function. By symmetry, we must also have that

$$\lim_{\theta\to 0^-} \frac{\sin\theta}{\theta} = 1.$$

Proof continued.

$$\lim_{\theta \to 0^-} \frac{\sin\theta}{\theta} = 1 = \lim_{\theta \to 0^+} \frac{\sin\theta}{\theta}$$

Because the left-hand limit and the right-hand limit are equal, we must also have

$$\lim_{\theta \to 0} \frac{\sin\theta}{\theta} = 1.$$

□

EXAMPLE 5 Show that **(a)** $\lim_{y \to 0} \frac{\cos y - 1}{y} = 0$ and **(b)** $\lim_{x \to 0} \frac{\sin 2x}{5x} = \frac{2}{5}$.

Solution

(a) Using the half-angle formula $\cos y = 1 - 2\sin^2(y/2)$, we calculate

$$\begin{aligned}
\lim_{h \to 0} \frac{\cos y - 1}{y} &= \lim_{h \to 0} -\frac{2\sin^2(y/2)}{y} \\
&= -\lim_{\theta \to 0} \frac{\sin \theta}{\theta} \sin \theta && \text{Let } \theta = y/2. \\
&= -(1)(0) = 0. && \text{Eq. (1) and Example 11a in Section 2.2}
\end{aligned}$$

(b) Equation (1) does not apply to the original fraction. We need a $2x$ in the denominator, not a $5x$. We produce it by multiplying numerator and denominator by $2/5$:

$$\begin{aligned}
\lim_{x \to 0} \frac{\sin 2x}{5x} &= \lim_{x \to 0} \frac{(2/5) \cdot \sin 2x}{(2/5) \cdot 5x} \\
&= \frac{2}{5} \lim_{x \to 0} \frac{\sin 2x}{2x} && \text{Eq. (1) applies with } \theta = 2x. \\
&= \frac{2}{5}(1) = \frac{2}{5}.
\end{aligned}$$

■

EXAMPLE 6 Find $\lim_{t\to 0} \dfrac{\tan t \sec 2t}{3t}$.

Solution From the definition of $\tan t$ and $\sec 2t$, we have

$$\begin{aligned}
\lim_{t\to 0} \frac{\tan t \sec 2t}{3t} &= \lim_{t\to 0} \frac{1}{3} \cdot \frac{1}{t} \cdot \frac{\sin t}{\cos t} \cdot \frac{1}{\cos 2t} \\
&= \frac{1}{3} \lim_{t\to 0} \frac{\sin t}{t} \cdot \frac{1}{\cos t} \cdot \frac{1}{\cos 2t} \\
&= \frac{1}{3}(1)(1)(1) = \frac{1}{3}.
\end{aligned}$$

Eq. (1) and Example 11b in Section 2.2

EXAMPLE 7 Show that for nonzero constants A and B.

$$\lim_{\theta \to 0} \frac{\sin A\theta}{\sin B\theta} = \frac{A}{B}.$$

Solution

$$\lim_{\theta \to 0} \frac{\sin A\theta}{\sin B\theta} = \lim_{\theta \to 0} \frac{\sin A\theta}{A\theta} A\theta \frac{B\theta}{\sin B\theta} \frac{1}{B\theta}$$ Multiply and divide by $A\theta$ and $B\theta$.

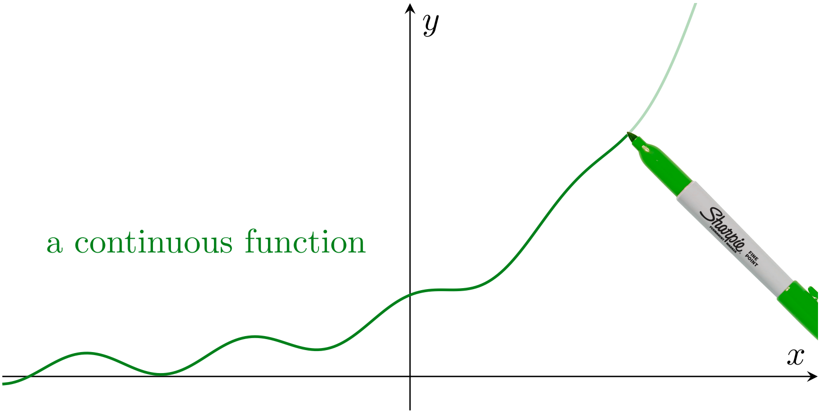
$$= \lim_{\theta \to 0} \frac{\sin A\theta}{A\theta} \frac{B\theta}{\sin B\theta} \frac{A}{B}$$ $\lim_{u \to 0} \frac{\sin u}{u} = 1$, with $u = A\theta$

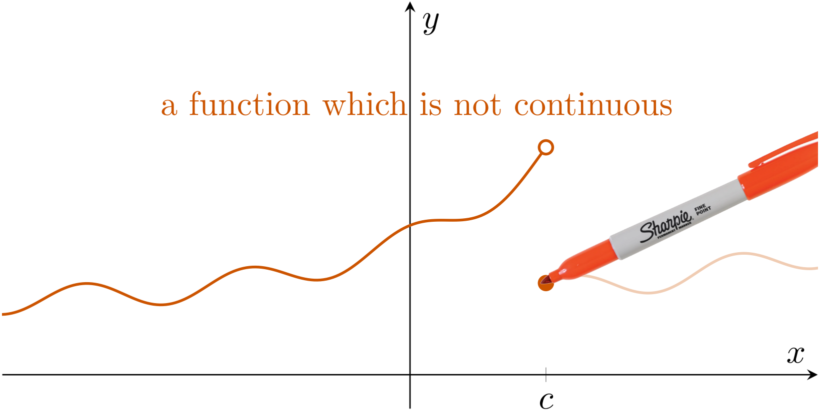
$$= \lim_{\theta \to 0} (1)(1) \frac{A}{B}$$ $\lim_{v \to 0} \frac{v}{\sin v} = 1$, with $v = B\theta$

$$= \frac{A}{B}.$$

■

2.5 Continuity


y
x
a continuous function
Sharpie
FINE POINT



y
a function which is not continuous
Sharpie
FINE POINT
x
c

Definition

The function $f : D \to \mathbb{R}$ is *continuous at* $c \in D$ iff

- $f(c)$ exists;
- $\lim_{x \to c} f(x)$ exists; and
- $\lim_{x \to c} f(x) = f(c)$.

Definition

The function $f : D \to \mathbb{R}$ is *continuous from the right at* $c \in D$ iff

- $f(c)$ exists;
- $\lim\limits_{x \to c+} f(x)$ exists; and
- $\lim\limits_{x \to c+} f(x) = f(c)$.

Definition

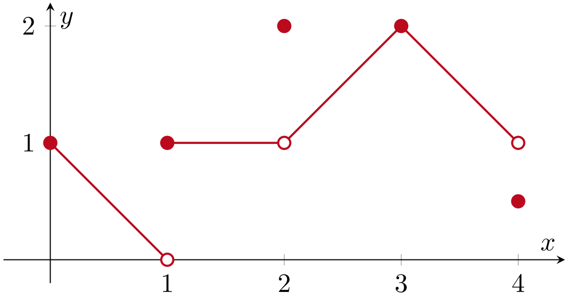
The function $f : D \to \mathbb{R}$ is *continuous from the left at* $c \in D$ iff

- $f(c)$ exists;
- $\lim\limits_{x \to c-} f(x)$ exists; and
- $\lim\limits_{x \to c-} f(x) = f(c)$.

Definition

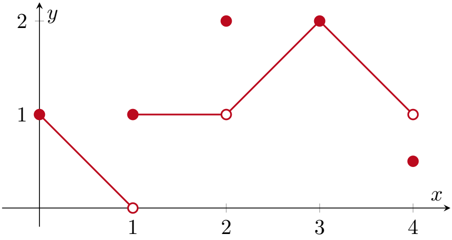
If f is not continuous at c, we say that f is *discontinuous* at c.

We say that c is a *point of discontinuity* of f.

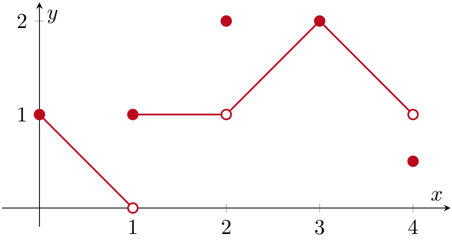


Example

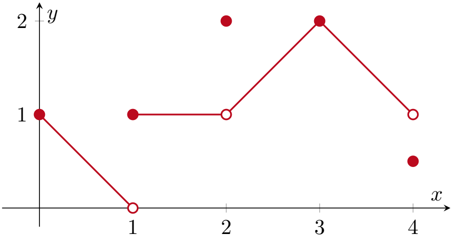
Consider the function $f : [0, 4] \to \mathbb{R}$ above. Where is f continuous? Where is f discontinuous?



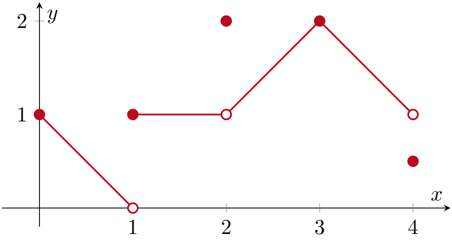
c	Is f continuous at c?	Why?
0		
$(0,1)$		
1		



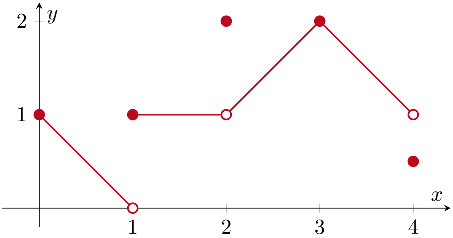
c	Is f continuous at c?	Why?
0	Yes	because $\lim_{x\to 0} f(x) = 1 = f(0)$
$(0,1)$		
1		



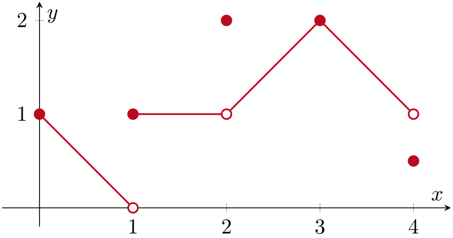
c	Is f continuous at c?	Why?
0	Yes	because $\lim_{x\to 0} f(x) = 1 = f(0)$
$(0,1)$	Yes	because $\lim_{x\to c} f(x) = f(c)$
1		



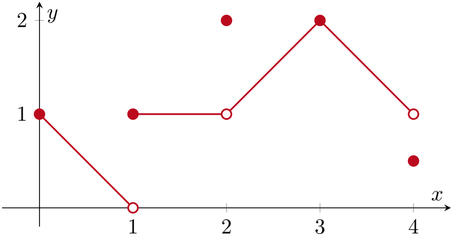
c	Is f continuous at c?	Why?
0	Yes	because $\lim_{x\to 0} f(x) = 1 = f(0)$
$(0,1)$	Yes	because $\lim_{x\to c} f(x) = f(c)$
1	No	because $\lim_{x\to 1} f(x)$ does not exist



c	Is f continuous at c?	Why?
$(1,2)$		
2		
$(2,4)$		
4		



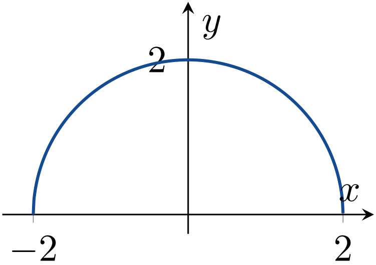
c	Is f continuous at c?	Why?
$(1,2)$	Yes	because $\lim_{x\to c} f(x) = f(c)$
2	No	because $\lim_{x\to 2} f(x) = 1 \neq 2 = f(2)$
$(2,4)$		
4		



c	Is f continuous at c?	Why?
$(1, 2)$	Yes	because $\lim_{x \to c} f(x) = f(c)$
2	No	because $\lim_{x \to 2} f(x) = 1 \neq 2 = f(2)$
$(2, 4)$	Yes	because $\lim_{x \to c} f(x) = f(c)$
4	No	because $\lim_{x \to 4} f(x) = 1 \neq \frac{1}{2} = f(4)$

Example

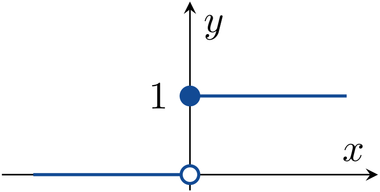
$f : [-2, 2] \to \mathbb{R},\ f(x) = \sqrt{4 - x^2}$



f is continuous at every $c \in [-2, 2]$.

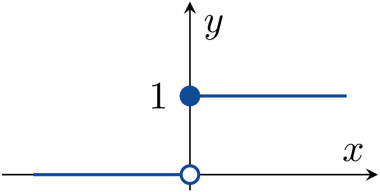
Example

$$g : \mathbb{R} \to \mathbb{R},\ g(x) = \begin{cases} 0 & x < 0 \\ 1 & x \geq 0 \end{cases}$$



Example

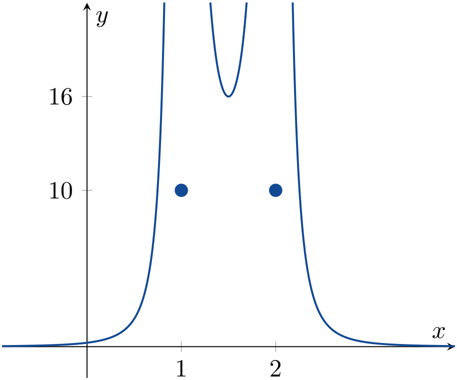
$g : \mathbb{R} \to \mathbb{R}$, $g(x) = \begin{cases} 0 & x < 0 \\ 1 & x \geq 0 \end{cases}$



g has a point of discontinuity at $c = 0$. g is continuous at every point $c \neq 0$.

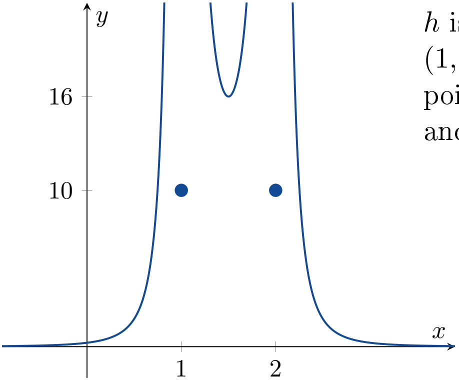
Example

$$h : \mathbb{R} \to \mathbb{R},\ h(x) = \begin{cases} \frac{1}{(x-1)^2(x-2)^2} & x \neq 1 \text{ or } 2 \\ 10 & x = 1 \text{ or } 2 \end{cases}$$

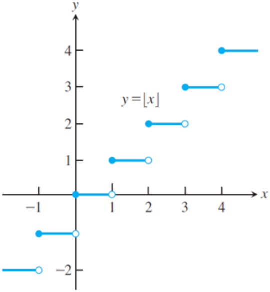


Example

$$h : \mathbb{R} \to \mathbb{R},\ h(x) = \begin{cases} \frac{1}{(x-1)^2(x-2)^2} & x \neq 1 \text{ or } 2 \\ 10 & x = 1 \text{ or } 2 \end{cases}$$



h is continuous on $(-\infty, 1)$, $(1, 2)$ and $(2, \infty)$. h has a points of discontinuity at $c = 1$ and $c = 2$.



EXAMPLE 4 The function $y = \lfloor x \rfloor$ introduced in Section 1.1 is graphed in Figure 2.39. It is discontinuous at every integer n, because the left-hand and right-hand limits are not equal as $x \to n$:

$$\lim_{x \to n^-} \lfloor x \rfloor = n - 1 \quad \text{and} \quad \lim_{x \to n^+} \lfloor x \rfloor = n.$$

Since $\lfloor n \rfloor = n$, the greatest integer function is right-continuous at every integer n (but not left-continuous).

The greatest integer function is continuous at every real number other than the integers. For example,

$$\lim_{x \to 1.5} \lfloor x \rfloor = 1 = \lfloor 1.5 \rfloor.$$

In general, if $n - 1 < c < n$, n an integer, then

$$\lim_{x \to c} \lfloor x \rfloor = n - 1 = \lfloor c \rfloor.$$

Continuous Functions

Definition

$f : D \to \mathbb{R}$ is a *continuous function* if it is continuous at every $c \in D$.

Theorem

Suppose that f and g are continuous at c.

Theorem

Suppose that f and g are continuous at c.

Then $f+g$, $f-g$, kf $(k \in \mathbb{R})$, fg, $\frac{f}{g}$ (if $g(c) \neq 0$) and f^n $(n \in \mathbb{N})$ are all continuous at c.

Theorem

Suppose that f and g are continuous at c.

Then $f+g$, $f-g$, kf ($k \in \mathbb{R}$), fg, $\frac{f}{g}$ (if $g(c) \neq 0$) and f^n ($n \in \mathbb{N}$) are all continuous at c.

If $\sqrt[n]{f}$ is defined on $(c-\delta, c+\delta)$, then $\sqrt[n]{f}$ is also continuous at c ($n \in \mathbb{N}$).

Example

Every polynominal

$$P(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_2 x^2 + a_1 x + a_0$$

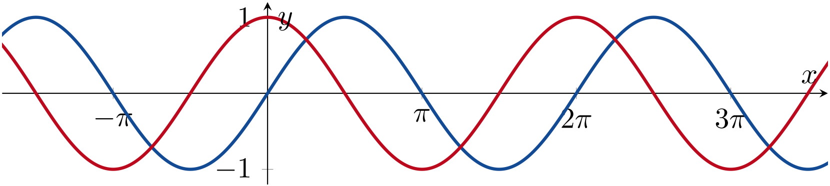
is continuous.

Example

If

- P and Q are polynomials; and
- $Q(c) \neq 0$,

then $\frac{P(x)}{Q(x)}$ is continuous at c.



Example

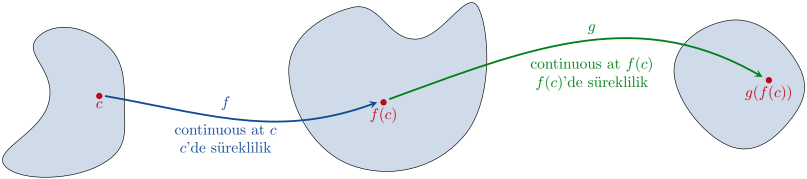
$\sin x$ and $\cos x$ are continuous.

Composites of Continuous Functions

$$g \circ f(x)$$

$g \circ f(x)$ means $g(f(x))$.

2.5 Continuity

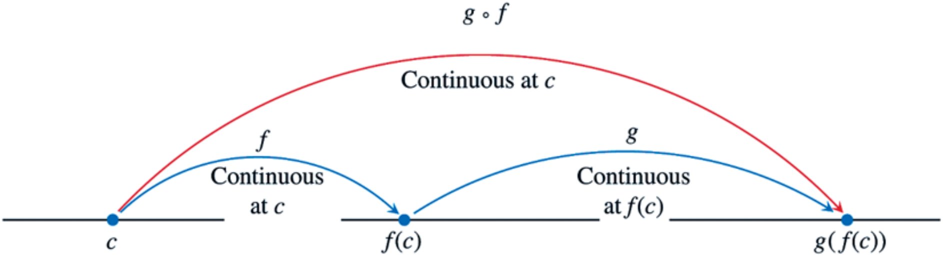


Theorem

If

- f *is continuous at* c*; and*
- g *is continuous at* $f(c)$*,*

then $g \circ f$ *is continuous at* c*.*


$g \circ f$
Continuous at c
f
Continuous
at c
g
Continuous
at $f(c)$
c
$f(c)$
$g(f(c))$

Example

Show that $h(x) = \sqrt{x^2 - 2x - 5}$ is continuous on its natural domain.

Example

Show that $h(x) = \sqrt{x^2 - 2x - 5}$ is continuous on its natural domain.

- The function $g(t) = \sqrt{t}$ is continuous.

Example

Show that $h(x) = \sqrt{x^2 - 2x - 5}$ is continuous on its natural domain.

- The function $g(t) = \sqrt{t}$ is continuous.
- The function $f(x) = x^2 - 2x - 5$ is continuous because all polynomials are continuous.

Example

Show that $h(x) = \sqrt{x^2 - 2x - 5}$ is continuous on its natural domain.

- The function $g(t) = \sqrt{t}$ is continuous.
- The function $f(x) = x^2 - 2x - 5$ is continuous because all polynomials are continuous.

Therefore $h(x) = g \circ f(x)$ is continuous.

Example

Show that $\frac{x^{\frac{2}{3}}}{1+x^4}$ is continuous.

Example

Show that $\frac{x^{\frac{2}{3}}}{1+x^4}$ is continuous.

- $x^{\frac{2}{3}}$ is continuous.
- $1 + x^4$ is continuous.

Example

Show that $\frac{x^{\frac{2}{3}}}{1+x^4}$ is continuous.

- $x^{\frac{2}{3}}$ is continuous.
- $1 + x^4$ is continuous.
- $1 + x^4 \neq 0$ for all x

Example

Show that $\frac{x^{\frac{2}{3}}}{1+x^4}$ is continuous.

- $x^{\frac{2}{3}}$ is continuous.
- $1 + x^4$ is continuous.
- $1 + x^4 \neq 0$ for all x

Hence $\frac{x^{\frac{2}{3}}}{1+x^4}$ is continuous.

Please read part (c) of Example 8 in your textbook.

Example

Where is $h(x) = \left| \dfrac{x \sin x}{x^2 + 2} \right|$ continuous?

Example

Where is $h(x) = \left| \dfrac{x \sin x}{x^2 + 2} \right|$ continuous?

- $x^2 + 2$ is continuous and > 0 everywhere.

Example

Where is $h(x) = \left| \dfrac{x \sin x}{x^2 + 2} \right|$ continuous?

- $x^2 + 2$ is continuous and > 0 everywhere.
- $\sin x$ is continuous everywhere.

Example

Where is $h(x) = \left|\dfrac{x\ \sin x}{x^2+2}\right|$ continuous?

- $x^2 + 2$ is continuous and > 0 everywhere.
- $\sin x$ is continuous everywhere.
- x is continuous everywhere.

Example

Where is $h(x) = \left| \dfrac{x \ \sin x}{x^2 + 2} \right|$ continuous?

- $x^2 + 2$ is continuous and > 0 everywhere.
- $\sin x$ is continuous everywhere.
- x is continuous everywhere.
- $|y|$ is continuous everywhere.

Example

Where is $h(x) = \left|\dfrac{x \sin x}{x^2+2}\right|$ continuous?

- $x^2 + 2$ is continuous and > 0 everywhere.
- $\sin x$ is continuous everywhere.
- x is continuous everywhere.
- $|y|$ is continuous everywhere.

Therefore $h(x)$ is continuous everywhere.

Limits of Continuous Functions

Theorem

If

- $g(x)$ *is continuous at* $x = b$; *and*
- $\lim_{x \to c} f(x) = b$,

then

$$\lim_{x \to c} g(f(x)) = g\left(\lim_{x \to c} f(x)\right).$$

You can read the proof in the textbook.

Limits of Continuous Functions

Theorem

If

- $g(x)$ *is continuous at* $x = b$*; and*
- $\lim_{x \to c} f(x) = b$,

then

$$\lim_{x \to c} g(f(x)) = g\left(\lim_{x \to c} f(x) \right).$$

You can read the proof in the textbook.

Example

$$\lim_{x\to\frac{\pi}{2}} \cos\left[2x + \sin\left(\frac{3\pi}{2} + x\right)\right]$$

$$=$$

$$=$$

$$=$$

$$=$$

Example

$$\begin{aligned}\lim_{x\to\frac{\pi}{2}} \cos\left[2x+\sin\left(\frac{3\pi}{2}+x\right)\right] &= \cos\left[\lim_{x\to\frac{\pi}{2}}\left(2x+\sin\left(\frac{3\pi}{2}+x\right)\right)\right] \\ &= \\ &= \\ &= \end{aligned}$$

Example

$$
\begin{aligned}
\lim_{x\to\frac{\pi}{2}} \cos\left[2x + \sin\left(\frac{3\pi}{2} + x\right)\right]
&= \cos\left[\lim_{x\to\frac{\pi}{2}}\left(2x + \sin\left(\frac{3\pi}{2} + x\right)\right)\right] \\
&= \cos\left[\lim_{x\to\frac{\pi}{2}}(2x) + \lim_{x\to\frac{\pi}{2}}\left(\sin\left(\frac{3\pi}{2} + x\right)\right)\right] \\
&= \cos\left[\pi + \qquad\qquad\qquad\qquad \right] \\
&=
\end{aligned}
$$

Example

$$
\begin{aligned}
\lim_{x\to\frac{\pi}{2}} \cos\left[2x+\sin\left(\frac{3\pi}{2}+x\right)\right] &= \cos\left[\lim_{x\to\frac{\pi}{2}}\left(2x+\sin\left(\frac{3\pi}{2}+x\right)\right)\right] \\
&= \cos\left[\lim_{x\to\frac{\pi}{2}}(2x)+\lim_{x\to\frac{\pi}{2}}\left(\sin\left(\frac{3\pi}{2}+x\right)\right)\right] \\
&= \cos\left[\pi+\sin\left(\lim_{x\to\frac{\pi}{2}}\left(\frac{3\pi}{2}+x\right)\right)\right] \\
&=
\end{aligned}
$$

Example

$$
\begin{aligned}
\lim_{x\to\frac{\pi}{2}} \cos\left[2x + \sin\left(\frac{3\pi}{2} + x\right)\right]
&= \cos\left[\lim_{x\to\frac{\pi}{2}}\left(2x + \sin\left(\frac{3\pi}{2} + x\right)\right)\right] \\
&= \cos\left[\lim_{x\to\frac{\pi}{2}}(2x) + \lim_{x\to\frac{\pi}{2}}\left(\sin\left(\frac{3\pi}{2} + x\right)\right)\right] \\
&= \cos\left[\pi + \sin\left(\lim_{x\to\frac{\pi}{2}}\left(\frac{3\pi}{2} + x\right)\right)\right] \\
&= \cos\left[\pi + \sin 2\pi\right] = \cos\left[\pi + 0\right] = -1.
\end{aligned}
$$

Example

$$\lim_{x\to\frac{\pi}{2}} \tan\left[\frac{5x}{2} - \pi\cos\left(\frac{\pi}{2} - x\right)\right]$$

$$= \tan\left[\lim_{x\to\frac{\pi}{2}}\left(\frac{5x}{2} - \pi\cos\left(\frac{\pi}{2} - x\right)\right)\right]$$

$$=$$

$$=$$

$$=$$

Example

$$\lim_{x\to\frac{\pi}{2}} \tan\left[\frac{5x}{2} - \pi\cos\left(\frac{\pi}{2} - x\right)\right]$$

$$= \tan\left[\lim_{x\to\frac{\pi}{2}}\left(\frac{5x}{2} - \pi\cos\left(\frac{\pi}{2} - x\right)\right)\right]$$

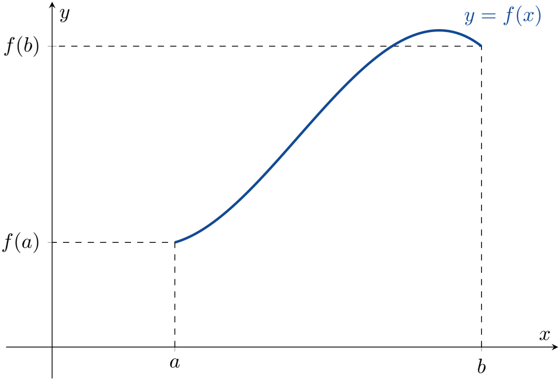
$$= \tan\left[\lim_{x\to\frac{\pi}{2}}\left(\frac{5x}{2}\right) - \pi\lim_{x\to\frac{\pi}{2}}\left(\cos\left(\frac{\pi}{2} - x\right)\right)\right]$$

$$= \tan\left[\frac{5\pi}{4} - \pi\cos\left(\lim_{x\to\frac{\pi}{2}}\left(\frac{\pi}{2} - x\right)\right)\right]$$

$$= \tan\left[\frac{5\pi}{4} - \pi\cos 0\right] = \tan\left[\frac{5\pi}{4} - \pi\right] = \tan\frac{\pi}{4} = 1.$$

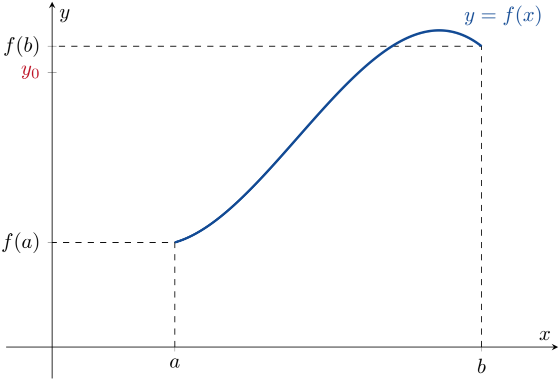
Break

We will continue at 3pm



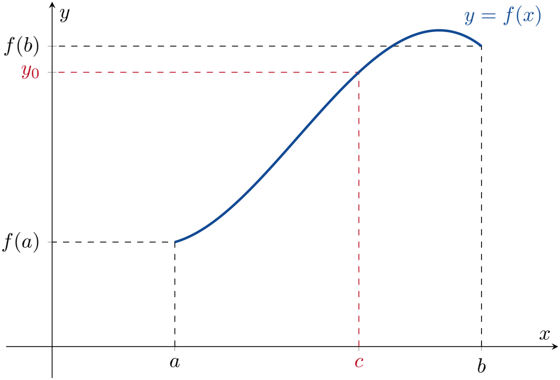
Theorem (The Intermediate Value Theorem)

If f is continuous on a closed interval $[a, b]$,



Theorem (The Intermediate Value Theorem)

If f is continuous on a closed interval $[a, b]$, and if y_0 is any value between $f(a)$ and $f(b)$,

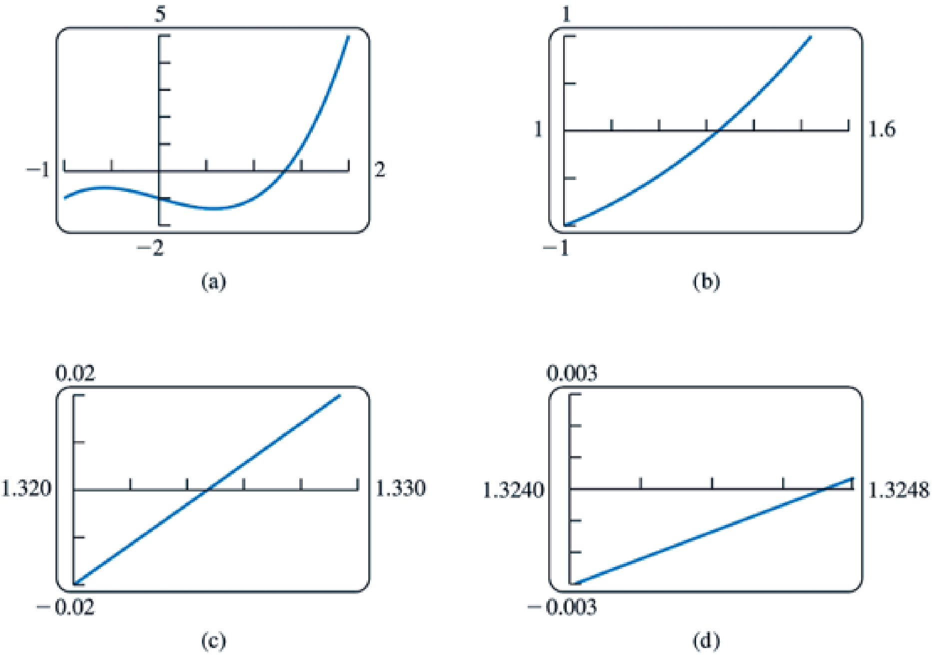


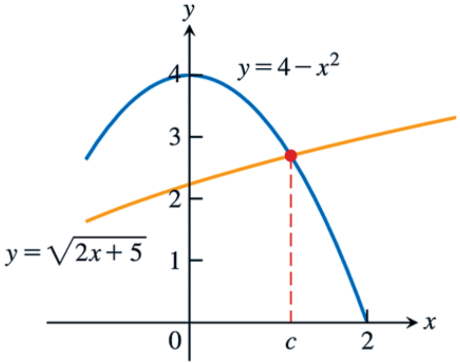
Theorem (The Intermediate Value Theorem)

If f is continuous on a closed interval $[a, b]$, and if y_0 is any value between $f(a)$ and $f(b)$, then there exists $c \in [a, b]$ such that $y_0 = f(c)$.

EXAMPLE 10 Show that there is a root of the equation $x^3 - x - 1 = 0$ between 1 and 2.

Solution Let $f(x) = x^3 - x - 1$. Since $f(1) = 1 - 1 - 1 = -1 < 0$ and $f(2) = 2^3 - 2 - 1 = 5 > 0$, we see that $y_0 = 0$ is a value between $f(1)$ and $f(2)$. Since f is a polynomial, it is continuous, and the Intermediate Value Theorem says there is a zero of f between 1 and 2. Figure 2.45 shows the result of zooming in to locate the root near $x = 1.32$. ■


5
−1
2
−2
(a)
1
1
1.6
−1
(b)
0.02
1.320
1.330
− 0.02
(c)
0.003
1.3240
1.3248
− 0.003
(d)



Example

Use the Intermediate Value Theorem to prove that the equation

$$\sqrt{2x+5} = 4 - x^2$$

has a solution.

This is the same as showing that

$$f(x) = \sqrt{2x+5} + x^2 - 4 = 0$$

has a solution.

This is the same as showing that

$$f(x) = \sqrt{2x+5} + x^2 - 4 = 0$$

has a solution. Note that

- f is continuous on $[-\frac{5}{2}, \infty)$.
- $f(0) = \sqrt{5} + 0 - 4 \approx -1.76$.
- $f(2) = \sqrt{9} + 4 - 4 = 3$.

This is the same as showing that

$$f(x) = \sqrt{2x+5} + x^2 - 4 = 0$$

has a solution. Note that

- f is continuous on $[-\frac{5}{2}, \infty)$.
- $f(0) = \sqrt{5} + 0 - 4 \approx -1.76$.
- $f(2) = \sqrt{9} + 4 - 4 = 3$.

Since $f(0) \leq 0 \leq f(2)$,

This is the same as showing that

$$f(x) = \sqrt{2x+5} + x^2 - 4 = 0$$

has a solution. Note that

- f is continuous on $[-\frac{5}{2}, \infty)$.
- $f(0) = \sqrt{5} + 0 - 4 \approx -1.76$.
- $f(2) = \sqrt{9} + 4 - 4 = 3$.

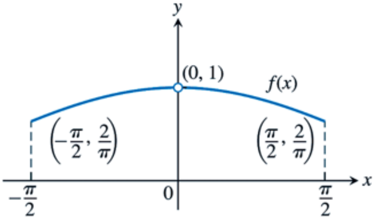
Since $f(0) \leq 0 \leq f(2)$, it follows by the Intermediate Value Theorem that there exists $c \in [0, 2]$ such that $f(c) = 0$.

Continuous Extension to a Point

Continuous Extension to a Point

or “filling in holes”.

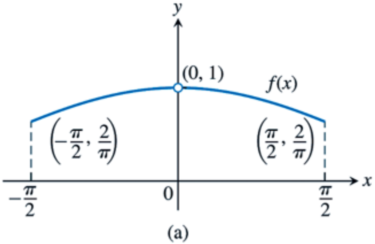
Consider the function $f : (-\infty, 0) \cup (0, \infty) \to \mathbb{R}$, $f(x) = \dfrac{\sin x}{x}$.



(a)

How can we “fill in” the hole at $(0, 1)$?

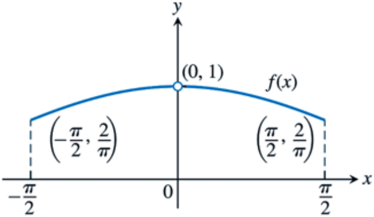
Consider the function $f : (-\infty, 0) \cup (0, \infty) \to \mathbb{R}$, $f(x) = \dfrac{\sin x}{x}$.



How can we “fill in” the hole at $(0, 1)$?

How can we define a new function $F : \mathbb{R} \to \mathbb{R}$ such that

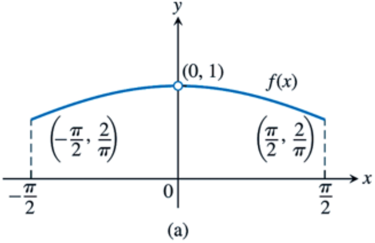
- $F = f$ on the domain of f; and
- F is continuous?



(a)

Since

$$\lim_{x \to 0} f(x) = 1,$$



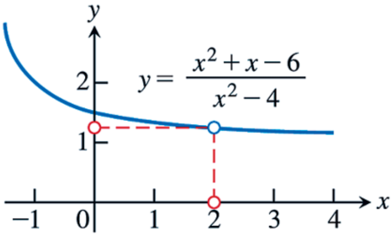
Since

$$\lim_{x \to 0} f(x) = 1,$$

we define

$$F(x) = \begin{cases} \frac{\sin x}{x} & x \neq 0 \\ 1 & x = 0. \end{cases}$$

We say that F is the *continuous extension of* f to $x = 0$.



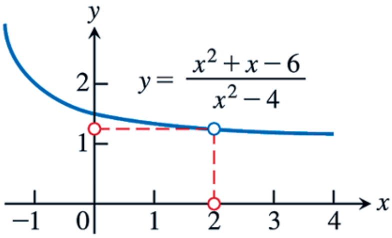
(a)

Example

Show that

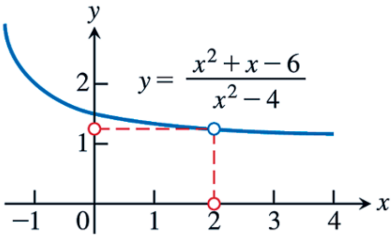
$$f(x) = \frac{x^2 + x - 6}{x^2 - 4}, \qquad x \neq \pm 2$$

has a continuous extension to $x = 2$ and find that extension.



(a)

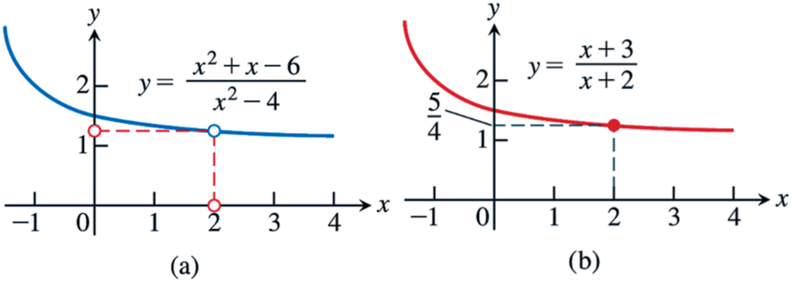
Note that $f(2)$ is not defined.



(a)

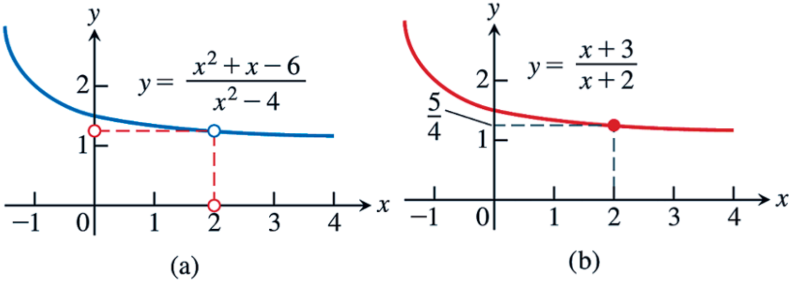
Note that $f(2)$ is not defined. However

$$f(x) = \frac{x^2 + x - 6}{x^2 - 4} = \frac{(x-2)(x+3)}{(x-2)(x+2)} = \frac{x+3}{x+2}.$$



So we define F by

$$F(x) = \frac{x+3}{x+2}.$$

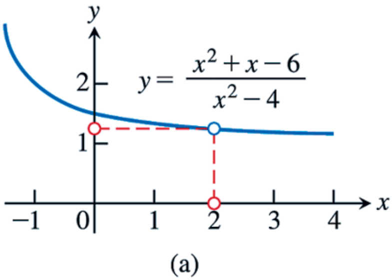


So we define F by

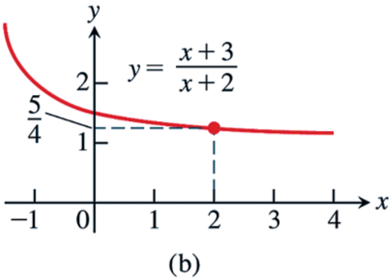
$$F(x) = \frac{x+3}{x+2}.$$

Then

- F is continuous; and
- $F(x) = f(x)$ for all $x \neq \pm 2$.



(a)



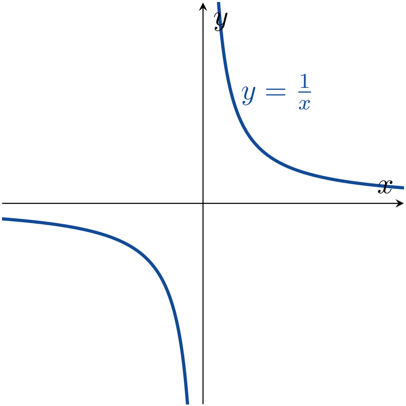
(b)

$F(x) = \dfrac{x+3}{x+2}$ is the continuous extension of $f(x) = \dfrac{x^2+x-6}{x^2-4}$ to $x = 2$.

Limits Involving Infinity; Asymptotes of Graphs

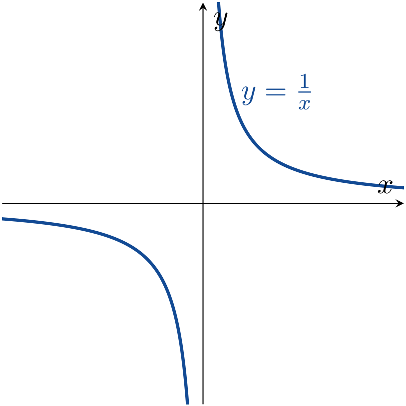
2.6

Finite Limits as $x \to \pm\infty$



Question: If $x > 0$ and x gets bigger and bigger and bigger, what happens to $\frac{1}{x}$?

Finite Limits as $x \to \pm\infty$

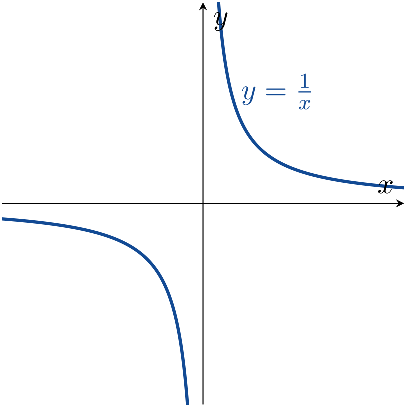


Question: If $x > 0$ and x gets bigger and bigger and bigger, what happens to $\frac{1}{x}$?

Answer: $\frac{1}{x}$ gets closer and closer and closer to 0. We want to write this as

$$\lim_{x \to \infty} \frac{1}{x} = 0.$$

Finite Limits as $x \to \pm\infty$



Question: If $x > 0$ and x gets bigger and bigger and bigger, what happens to $\frac{1}{x}$?

Answer: $\frac{1}{x}$ gets closer and closer and closer to 0. We want to write this as

$$\lim_{x \to \infty} \frac{1}{x} = 0.$$

Let's be more precise.

Definition

We write $\lim_{x\to\infty} f(x) = L$ iff for all $\varepsilon > 0$, there exists a number M such that

$$x > M \qquad \Longrightarrow \qquad |f(x) - L| < \varepsilon.$$

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Definition

We write $\lim_{x\to-\infty} f(x) = L$ iff for all $\varepsilon > 0$, there exists a number N such that

$$x < N \qquad \implies \qquad |f(x) - L| < \varepsilon.$$

Please think

$$\begin{aligned} \varepsilon &= \text{ a very small number} \\ \delta &= \text{ a very small number} \\ M &= \text{ a very large number} \\ N &= \text{ a very large negative number} \end{aligned}$$

Example

Prove that $\lim_{x\to\infty} \frac{1}{x} = 0$ and $\lim_{x\to-\infty} \frac{1}{x} = 0$.

Let $\varepsilon > 0$. Choose $M =$ and $N =$. Then

$$x > M \quad \Longrightarrow$$

and

$$x < N \quad \Longrightarrow$$

Example

Prove that $\lim_{x\to\infty} \frac{1}{x} = 0$ ar

Let $\varepsilon > 0$. Choose $M =$

$$x > M \quad \Longrightarrow$$

and

$$x < N \quad \Longrightarrow$$

If $x > 0$, then

$$\left|\frac{1}{x} - 0\right| < \varepsilon$$

$$\frac{1}{x} < \varepsilon$$

$$\frac{1}{\varepsilon} < x$$

Choose $M = \frac{1}{\varepsilon}$.

Example

Prove that $\lim\limits_{x\to\infty} \dfrac{1}{x} = 0$ ar

Let $\varepsilon > 0$. Choose $M =$

$$x > M \quad \Longrightarrow$$

and

$$x < N \quad \Longrightarrow$$

If $x < 0$, then

$$\left|\frac{1}{x} - 0\right| < \varepsilon$$

$$-\frac{1}{x} < \varepsilon$$

$$-\frac{1}{\varepsilon} > x$$

Choose $N = -\frac{1}{\varepsilon}$.

Example

Prove that $\lim_{x\to\infty} \frac{1}{x} = 0$ and $\lim_{x\to-\infty} \frac{1}{x} = 0$.

Let $\varepsilon > 0$. Choose $M = \frac{1}{\varepsilon}$ and $N = -\frac{1}{\varepsilon}$. Then

$$x > M \quad \Longrightarrow$$

and

$$x < N \quad \Longrightarrow$$

Example

Prove that $\lim_{x\to\infty} \frac{1}{x} = 0$ and $\lim_{x\to-\infty} \frac{1}{x} = 0$.

Let $\varepsilon > 0$. Choose $M = \frac{1}{\varepsilon}$ and $N = -\frac{1}{\varepsilon}$. Then

$$x > M \quad \implies \quad \left|\frac{1}{x} - 0\right| = \frac{1}{x} < \frac{1}{M} = \varepsilon$$

and

$$x < N \quad \implies \quad \left|\frac{1}{x} - 0\right| = -\frac{1}{x} < -\frac{1}{N} = \varepsilon.$$

Example

Prove that $\lim_{x\to\infty} k = k$ for any $k \in \mathbb{R}$.

I leave this for you to do.

Theorem

All of the limit laws (sum rule, difference rule, constant multiple rule, ...) are also true for $\lim\limits_{x\to\infty}$ *and* $\lim\limits_{x\to-\infty}$.

Theorem

All of the limit laws (sum rule, difference rule, constant multiple rule, . . .) are also true for $\lim\limits_{x\to\infty}$ *and* $\lim\limits_{x\to-\infty}$.

Example

$$\lim_{x\to\infty}\left(5+\frac{1}{x}\right)=\lim_{x\to\infty}5+\lim_{x\to\infty}\frac{1}{x}$$

(sum rule)

$$=5+0=5.$$

Example

$$\lim_{x\to-\infty} \frac{\pi\sqrt{3}}{x^2} = \lim_{x\to-\infty} \left(\pi\sqrt{3}\frac{1}{x}\frac{1}{x}\right)$$

$$= \left(\lim_{x\to-\infty} \pi\sqrt{3}\right)\left(\lim_{x\to-\infty} \frac{1}{x}\right)\left(\lim_{x\to-\infty} \frac{1}{x}\right)$$

(product rule)

$$= \pi\sqrt{3} \times 0 \times 0 = 0.$$

Example (Limits at Infinity of Rational Functions)

Find $\displaystyle\lim_{x\to\infty} \frac{5x^2+8x-3}{3x^2+2}$.

Please note that the answer is not "$\frac{\infty}{\infty}$". You can expect to receive zero points in the exam if you write "$\frac{\infty}{\infty}$".

Example (Limits at Infinity of Rational Functions)

Find $\lim\limits_{x\to\infty} \dfrac{5x^2+8x-3}{3x^2+2}$.

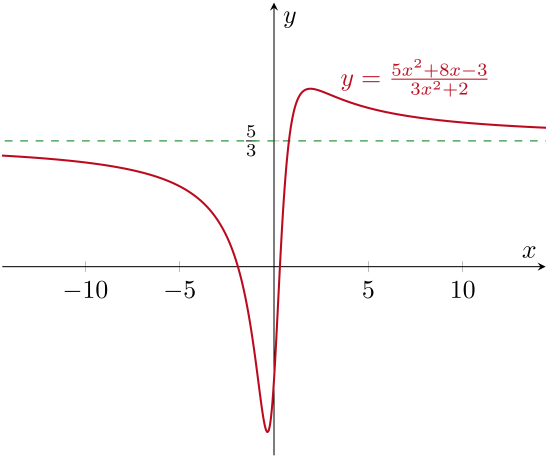
Please note that the answer is not "$\frac{\infty}{\infty}$". You can expect to receive zero points in the exam if you write "$\frac{\infty}{\infty}$".

Instead we calculate that

$$\lim_{x\to\infty} \frac{5x^2+8x-3}{3x^2+2} = \lim_{x\to\infty} \frac{5+\frac{8}{x}-\frac{3}{x^2}}{3+\frac{2}{x^2}} = \frac{5+0-0}{3+0} = \frac{5}{3}.$$

Example (Limits at Infinity of Rational Functions)

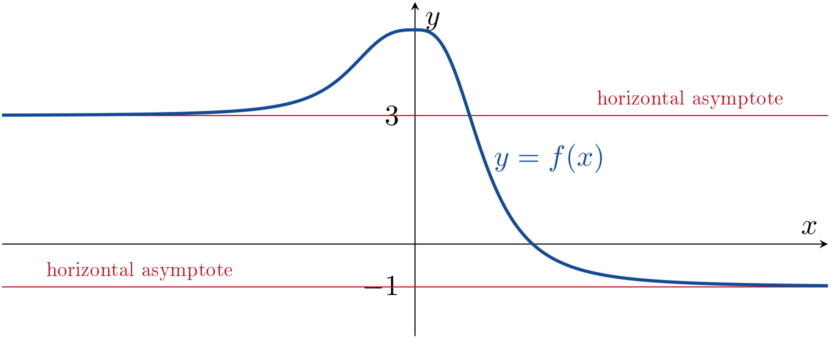
Find $\lim\limits_{x \to \infty} \dfrac{5x^2 + 8x - 3}{3x^2 + 2}$.



Example

$$\lim_{x\to-\infty} \frac{11x+2}{2x^3-1} = \lim_{x\to-\infty} \frac{\frac{11}{x^2}+\frac{2}{x^3}}{2-\frac{1}{x^3}} = \frac{0+0}{2-0} = 0.$$

Horizontal Asymptotes



If $y = f(x)$ gets "closer and closer" to a horizontal line as $x \to \infty$ or $x \to -\infty$, then that line is called a horizontal asymptote of $y = f(x)$.

Definition

A line $y = b$ is a *horizontal asymptote* of $y = f(x)$ if either

$$\lim_{x \to \infty} f(x) = b \qquad \text{or} \qquad \lim_{x \to -\infty} f(x) = b.$$

Definition

A line $y = b$ is a *horizontal asymptote* of $y = f(x)$ if either

$$\lim_{x\to\infty} f(x) = b \qquad \text{or} \qquad \lim_{x\to-\infty} f(x) = b.$$

For example, $y = \frac{5}{2}$ is a horizontal asymptote of $\dfrac{5x^2 + 8x - 3}{3x^2 + 2}$.

Example

Find the horizontal asymptotes of $y = \dfrac{x^3 - 2}{|x|^3 + 1}$.

Example

Find the horizontal asymptotes of $y = \dfrac{x^3 - 2}{|x|^3 + 1}$.

If $x > 0$, then

$$\lim_{x\to\infty} \frac{x^3-2}{|x|^3+1} = \lim_{x\to\infty} \frac{x^3-2}{x^3+1} = \lim_{x\to\infty} \frac{1-\frac{2}{x^3}}{1+\frac{1}{x^3}} = \frac{1-0}{1+0} = 1$$

Example

Find the horizontal asymptotes of $y = \dfrac{x^3 - 2}{|x|^3 + 1}$.

If $x > 0$, then

$$\lim_{x\to\infty} \frac{x^3 - 2}{|x|^3 + 1} = \lim_{x\to\infty} \frac{x^3 - 2}{x^3 + 1} = \lim_{x\to\infty} \frac{1 - \frac{2}{x^3}}{1 + \frac{1}{x^3}} = \frac{1 - 0}{1 + 0} = 1$$

and if $x < 0$ then

$$\lim_{x\to-\infty} \frac{x^3 - 2}{|x|^3 + 1} = \lim_{x\to-\infty} \frac{x^3 - 2}{(-x)^3 + 1} = \lim_{x\to-\infty} \frac{1 - \frac{2}{x^3}}{-1 + \frac{1}{x^3}} = \frac{1 - 0}{-1 + 0} = -1.$$

Example

Find the horizontal asymptotes of $y = \dfrac{x^3 - 2}{|x|^3 + 1}$.

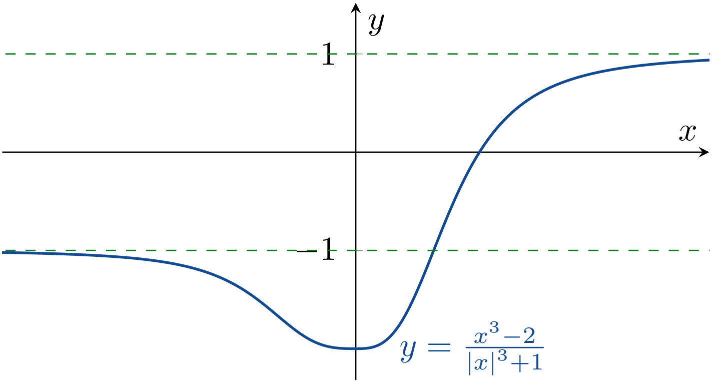
If $x > 0$, then

$$\lim_{x\to\infty} \frac{x^3 - 2}{|x|^3 + 1} = \lim_{x\to\infty} \frac{x^3 - 2}{x^3 + 1} = \lim_{x\to\infty} \frac{1 - \frac{2}{x^3}}{1 + \frac{1}{x^3}} = \frac{1 - 0}{1 + 0} = 1$$

and if $x < 0$ then

$$\lim_{x\to-\infty} \frac{x^3 - 2}{|x|^3 + 1} = \lim_{x\to-\infty} \frac{x^3 - 2}{(-x)^3 + 1} = \lim_{x\to-\infty} \frac{1 - \frac{2}{x^3}}{-1 + \frac{1}{x^3}} = \frac{1 - 0}{-1 + 0} = -1.$$

Therefore $y = 1$ and $y = -1$ are horizontal asymptotes of this function.



y
1
x
-1
$y = \frac{x^3-2}{|x|^3+1}$

Example

Find $\lim_{x\to\infty} \sin\frac{1}{x}$.

Example

Find $\lim_{x\to\infty} x\sin\frac{1}{x}$.

Example

Find $\lim_{x\to\infty} \sin\frac{1}{x}$.

We need to do a substitution here: Let $t = \frac{1}{x}$.

Example

Find $\lim_{x\to\infty} x\sin\frac{1}{x}$.

Example

Find $\lim_{x\to\infty} \sin \dfrac{1}{x}$.

We need to do a substitution here: Let $t = \frac{1}{x}$. Note that

$$x \to \infty \qquad \iff \qquad t \to 0^+.$$

Example

Find $\lim_{x\to\infty} x \sin \dfrac{1}{x}$.

Example

Find $\lim_{x\to\infty} \sin\frac{1}{x}$.

We need to do a substitution here: Let $t = \frac{1}{x}$. Note that

$$x \to \infty \qquad \iff \qquad t \to 0^+.$$

So

$$\lim_{x\to\infty} \sin\frac{1}{x} = \lim_{t\to 0^+} \sin t = 0.$$

Example

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Example

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So

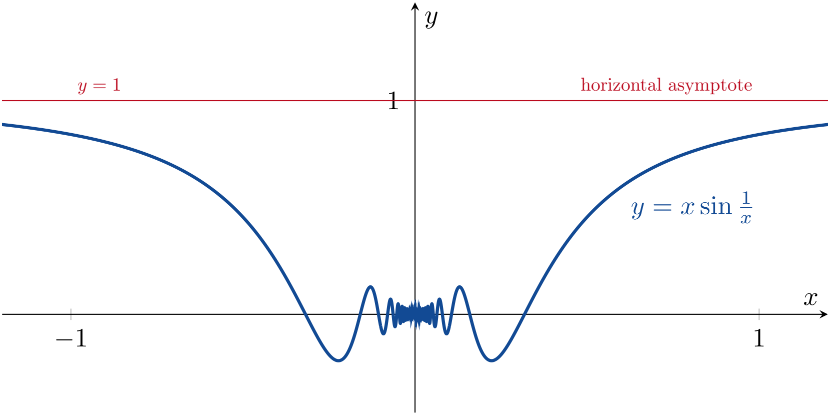
$$\lim_{x\to\infty} \sin\frac{1}{x} = \lim_{t\to 0^+} \sin t = 0.$$

Example

Find $\lim\limits_{x\to\infty} x\sin\frac{1}{x}$.

Again, let $t = \frac{1}{x}$. Then

$$\lim_{x\to\infty} x\sin\frac{1}{x} = \lim_{t\to 0^+} \frac{\sin t}{t} = 1.$$


y
$y = 1$
horizontal asymptote
1
$y = x \sin \frac{1}{x}$
x
-1
1

EXAMPLE 6 Find $\lim_{x\to 0^+} x\left\lfloor \frac{1}{x} \right\rfloor$.

Solution We let $t = 1/x$ so that

$$\lim_{x\to 0^+} x\left\lfloor \frac{1}{x} \right\rfloor = \lim_{t\to\infty} \frac{1}{t}\lfloor t \rfloor$$

From the graph in Figure 2.55, we see that $t - 1 \le \lfloor t \rfloor \le t$, which gives

$$1 - \frac{1}{t} \le \frac{1}{t}\lfloor t \rfloor \le 1 \qquad \text{Multiply inequalities by } \frac{1}{t} > 0.$$

It follows from the Sandwich Theorem that

$$\lim_{t\to\infty} \frac{1}{t}\lfloor t \rfloor = 1,$$

so 1 is the value of the limit we seek.

Example

Use the Sandwich Theorem to calculate

$$\lim_{x\to\infty}\left(2+\frac{\sin x}{x}\right).$$

Since $-1 \leq \sin x \leq 1$, we have that

$$0 \leq \left|\frac{\sin x}{x}\right| \leq \left|\frac{1}{x}\right|.$$

Example

Use the Sandwich Theorem to calculate

$$\lim_{x\to\infty}\left(2+\frac{\sin x}{x}\right).$$

Since $-1 \le \sin x \le 1$, we have that

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Because $\displaystyle\lim_{x\to\infty}\left|\frac{1}{x}\right| = 0$, it follows by the Sandwich Theorem that $\displaystyle\lim_{x\to\infty}\frac{\sin x}{x} = 0.$

Example

Use the Sandwich Theorem to calculate

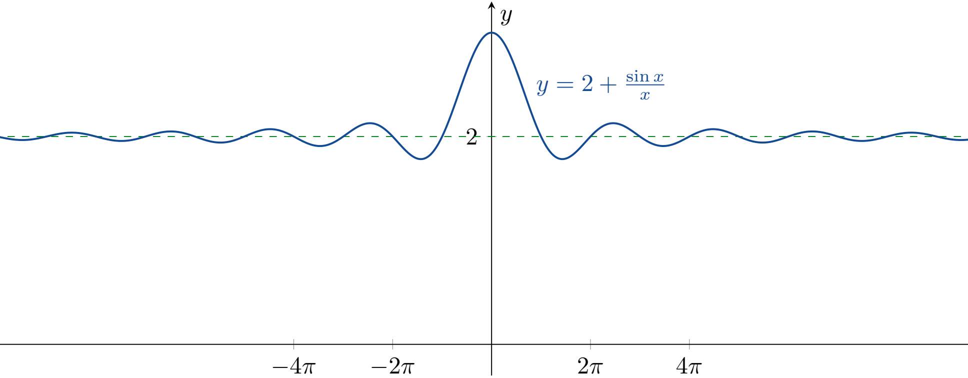
$$\lim_{x\to\infty}\left(2+\frac{\sin x}{x}\right).$$

Since $-1 \leq \sin x \leq 1$, we have that

$$0 \leq \left|\frac{\sin x}{x}\right| \leq \left|\frac{1}{x}\right|.$$

Because $\lim_{x\to\infty}\left|\frac{1}{x}\right| = 0$, it follows by the Sandwich Theorem that $\lim_{x\to\infty}\frac{\sin x}{x} = 0$. Therefore

$$\lim_{x\to\infty}\left(2+\frac{\sin x}{x}\right) = 2+0 = 2.$$



y
$y = 2 + \frac{\sin x}{x}$
2
-4π
-2π
2π
4π

Remark

There is one more trick for limits. Because

$$(a-b)(a+b) = a^2 - b^2,$$

it follows that

$$a - b = \frac{a^2 - b^2}{a+b}.$$

This can be useful if the limit contains a $\sqrt{}$.

Remark

There is one more trick for limits. Because

$$(a-b)(a+b) = a^2 - b^2,$$

it follows that

$$a - b = \frac{a^2 - b^2}{a+b}.$$

This can be useful if the limit contains a $\sqrt{}$.

Example

Calculate $\lim_{x \to \infty} \left(x - \sqrt{x^2 + 16} \right)$.

$$\lim_{x\to\infty}\left(x-\sqrt{x^2+16}\right) = \lim_{x\to\infty}\left(x-\sqrt{x^2+16}\right)\left(\frac{x+\sqrt{x^2+16}}{x+\sqrt{x^2+16}}\right)$$

$$=$$

$$=$$

$$=$$

$$=$$

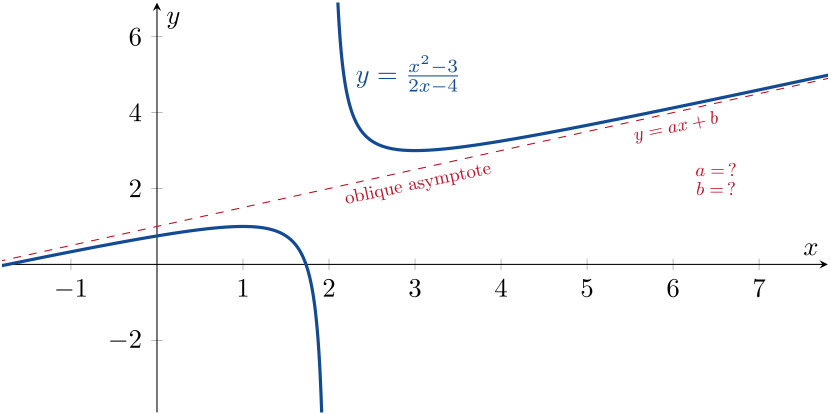
$$\begin{aligned}
\lim_{x\to\infty}\left(x-\sqrt{x^2+16}\right) &= \lim_{x\to\infty}\left(x-\sqrt{x^2+16}\right)\left(\frac{x+\sqrt{x^2+16}}{x+\sqrt{x^2+16}}\right)\\
&= \lim_{x\to\infty}\frac{x^2-(x^2+16)}{x+\sqrt{x^2+16}}\\
&=\\
&=\\
&=
\end{aligned}$$

$$
\begin{aligned}
\lim_{x\to\infty}\left(x-\sqrt{x^2+16}\right) &= \lim_{x\to\infty}\left(x-\sqrt{x^2+16}\right)\left(\frac{x+\sqrt{x^2+16}}{x+\sqrt{x^2+16}}\right) \\
&= \lim_{x\to\infty}\frac{x^2-(x^2+16)}{x+\sqrt{x^2+16}} \\
&= \lim_{x\to\infty}\frac{-16}{x+\sqrt{x^2+16}} \\
&= \\
&=
\end{aligned}
$$

$$\begin{aligned}
\lim_{x\to\infty}\left(x-\sqrt{x^2+16}\right) &= \lim_{x\to\infty}\left(x-\sqrt{x^2+16}\right)\left(\frac{x+\sqrt{x^2+16}}{x+\sqrt{x^2+16}}\right) \\
&= \lim_{x\to\infty}\frac{x^2-(x^2+16)}{x+\sqrt{x^2+16}} \\
&= \lim_{x\to\infty}\frac{-16}{x+\sqrt{x^2+16}} \\
&= \lim_{x\to\infty}\frac{\frac{-16}{x}}{1+\sqrt{1+\frac{16}{x^2}}} \\
&=
\end{aligned}$$

$$\begin{aligned}
\lim_{x\to\infty}\left(x-\sqrt{x^2+16}\right) &= \lim_{x\to\infty}\left(x-\sqrt{x^2+16}\right)\left(\frac{x+\sqrt{x^2+16}}{x+\sqrt{x^2+16}}\right)\\
&= \lim_{x\to\infty}\frac{x^2-(x^2+16)}{x+\sqrt{x^2+16}}\\
&= \lim_{x\to\infty}\frac{-16}{x+\sqrt{x^2+16}}\\
&= \lim_{x\to\infty}\frac{\frac{-16}{x}}{1+\sqrt{1+\frac{16}{x^2}}}\\
&= \frac{0}{1+\sqrt{1+0}} = 0.
\end{aligned}$$

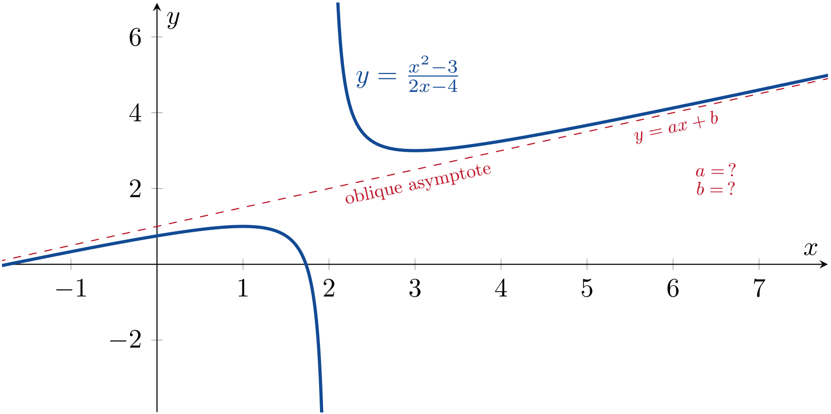
Oblique Asymptotes



Sometimes the graph of a function approaches a sloped line as $x \to \infty$ or $x \to -\infty$. This is called an *oblique asymptote*.

Example

Find the oblique asymptote of $y = \dfrac{x^2 - 3}{2x - 4}$.



Solution 1: We need to find a and b in

$$y = \frac{x^2 - 3}{2x - 4} = (ax + b) + \left(\frac{c}{2x - 4} \right).$$

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$$y = \frac{x^2 - 3}{2x - 4} = (ax + b) + \left(\frac{c}{2x - 4} \right).$$

Note that $\lim\limits_{x \to \pm\infty} \dfrac{c}{2x - 4} = 0$. So $y = \frac{x^2-3}{2x-4}$ and $y = ax + b$ get closer and closer as $x \to \pm\infty$.

So

$$\frac{x^2-3}{2x-4} = (ax+b) + \left(\frac{c}{2x-4}\right)$$

$$=$$

$$=$$

$$=$$

So

$$\frac{x^2-3}{2x-4} = (ax+b) + \left(\frac{c}{2x-4}\right)$$
$$= (ax+b)\left(\frac{2x-4}{2x-4}\right) + \frac{c}{2x-4}$$
$$=$$
$$=$$

So

$$\begin{aligned}\frac{x^2-3}{2x-4} &= (ax+b) + \left(\frac{c}{2x-4}\right) \\ &= (ax+b)\left(\frac{2x-4}{2x-4}\right) + \frac{c}{2x-4} \\ &= \frac{2ax^2-4ax+2bx-4b}{2x-4} + \frac{c}{2x-4} \\ &= \end{aligned}$$

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$$\Downarrow$$

$$\begin{cases}a=\frac{1}{2}\\ b=1\\ c=1\end{cases}$$

So

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$$\Downarrow$$

$$\begin{cases} a = \frac{1}{2} \\ b = 1 \\ c = 1 \end{cases}$$

Therefore $y = \frac{x}{2} + 1$ is the oblique asymptote of $y = \frac{x^2-3}{2x-4}$.

Solution 2: We need to divide $(x^2 - 3)$ by $(2x - 4)$ using long division

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$$
\begin{array}{r|l}
 & \quad \tfrac{1}{2}x + 1 \\
\hline
2x - 4 & \quad x^2 \qquad\quad - 3 \\
 & -x^2 + 2x \\
\cline{2-2}
 & \qquad\quad 2x - 3 \\
 & \qquad -2x + 4 \\
\cline{2-2}
 & \qquad\qquad\quad 1
\end{array}
$$

to obtain the oblique asymptote $y = \frac{1}{2}x + 1$.

In general, we want to write $y = f(x)$ as

$$y = f(x) = \big(\text{a linear function}\big) + \begin{pmatrix}\text{a remainder}\\ \text{function which}\\ \text{has limit=0 as}\\ x \to \pm\infty\end{pmatrix}.$$

Then the linear function will be the oblique asymptote that we require.

Example (page 114, exercise 108)

Find the oblique asymptote of $y = \dfrac{x^3 + 1}{x^2}$.

Example (page 114, exercise 108)

Find the oblique asymptote of $y = \frac{x^3+1}{x^2}$.

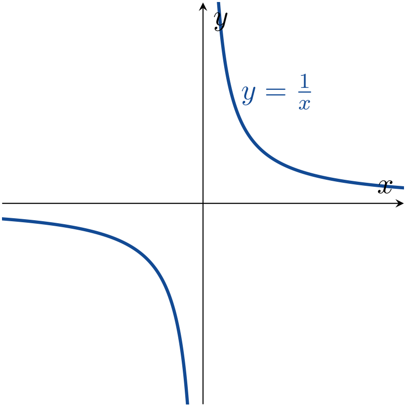
Note that

$$y = \frac{x^3+1}{x^2} = \frac{x^3}{x^2} + \frac{1}{x^2} = x + \frac{1}{x^2}.$$

Since $\lim_{x\to\pm\infty} \frac{1}{x^2} = 0$, the oblique asymptote must be $y = x$.

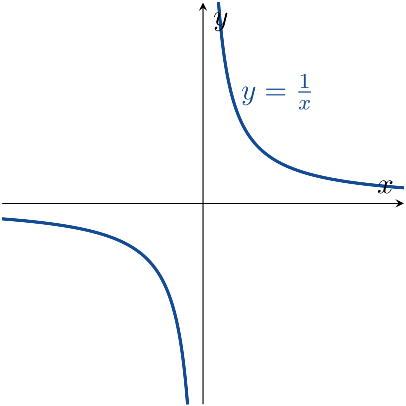
We will do more oblique asymptotes in Lecture 7.

Infinite Limits



Question: When happens to $\frac{1}{x}$ when $x \to 0^+$?

Infinite Limits

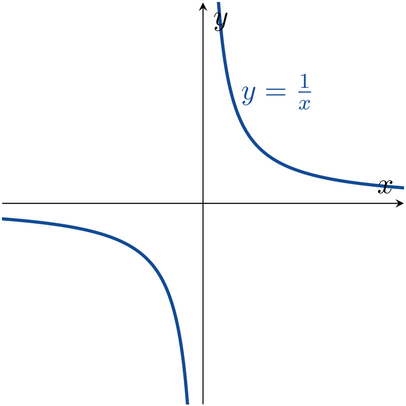


Question: When happens to $\frac{1}{x}$ when $x \to 0^+$?

Answer: $\frac{1}{x}$ gets bigger and bigger. We want to write this as

$$\lim_{x \to 0^+} \frac{1}{x} = \infty.$$

Infinite Limits



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Answer: $\frac{1}{x}$ gets bigger and bigger. We want to write this as

$$\lim_{x \to 0^+} \frac{1}{x} = \infty.$$

Let's be more precise.

Definition

We write $\lim_{x \to c} f(x) = \infty$

Definition

We write $\lim_{x \to c} f(x) = -\infty$

Definition

We write $\lim_{x\to c} f(x) = \infty$ iff for all $B > 0$, there exists $\delta > 0$ such that

$$0 < |x - c| < \delta \qquad \Longrightarrow \qquad f(x) > B.$$

Definition

We write $\lim_{x\to c} f(x) = -\infty$

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We write $\lim_{x\to c} f(x) = \infty$ iff for all $B > 0$, there exists $\delta > 0$ such that

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Definition

We write $\lim_{x\to c} f(x) = -\infty$ iff for all $B > 0$, there exists $\delta > 0$ such that

$$0 < |x - c| < \delta \qquad \Longrightarrow \qquad f(x) < -B.$$

There are similar definitions for $x \to c^+$ and $x \to c^-$: I leave these for you to write down.

Example

Prove that $\lim_{x \to 0} \frac{1}{x^2} = \infty$.

Let $B > 0$.

Example

Prove that $\lim_{x \to 0} \frac{1}{x^2} = \infty$.

Let $B > 0$. Choose $\delta = \frac{1}{\sqrt{B}}$.

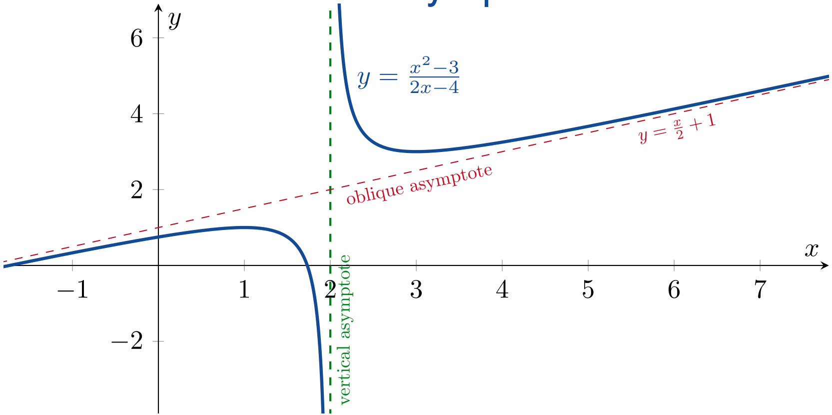
Example

Prove that $\lim_{x\to 0} \frac{1}{x^2} = \infty$.

Let $B > 0$. Choose $\delta = \frac{1}{\sqrt{B}}$. Then

$$0 < |x - 0| < \delta \quad \Longrightarrow \quad x < \delta \quad \Longrightarrow \quad \frac{1}{x^2} > \frac{1}{\delta^2} = B.$$

Vertical Asymptotes



Sometimes the graph of a function approaches the vertical line $x = a$ as $x \to a$ or $x \to a^+$ or $x \to a^-$. This is called an *vertical asymptote*.

Definition

A line $x = a$ is a *vertical asymptote* of $y = f(x)$ if either

$$\lim_{x \to a^+} f(x) = \pm\infty \qquad \text{or} \qquad \lim_{x \to a^-} f(x) = \pm\infty.$$

Example

Find the horizontal and vertical asymptotes of $y = \dfrac{x+3}{x+2}$.

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This function is defined if $x \neq -2$. So we are interested in four limits: $x \to \infty$, $x \to -\infty$, $x \to -2^+$ and $x \to -2^-$.

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I leave it for you to check that

$$\lim_{x\to\infty} \frac{x+3}{x+2} = 1 \qquad \lim_{x\to-2^+} \frac{x+3}{x+2} = \infty$$
$$\lim_{x\to-\infty} \frac{x+3}{x+2} = 1 \qquad \lim_{x\to-2^-} \frac{x+3}{x+2} = -\infty.$$

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This function is defined if $x \neq -2$. So we are interested in four limits: $x \to \infty$, $x \to -\infty$, $x \to -2^+$ and $x \to -2^-$.

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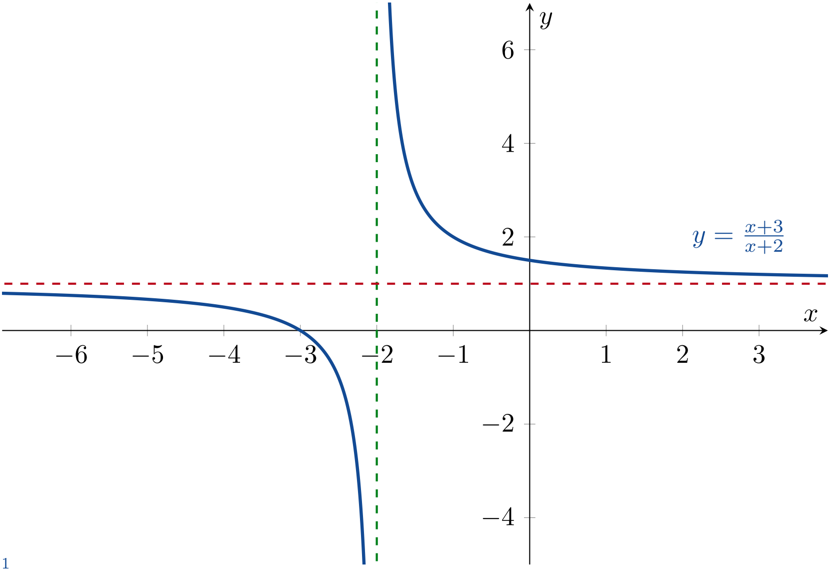
$$\lim_{x\to-\infty} \frac{x+3}{x+2} = 1 \qquad \lim_{x\to -2^-} \frac{x+3}{x+2} = -\infty.$$

Therefore $y = 1$ is a horizontal asymptote and $x = -2$ is a vertical asymptote.


y
x
6
4
2
−2
−4
−6
−5
−4
−3
−2
−1
1
2
3
$y = \frac{x+3}{x+2}$

Example

Find the horizontal and vertical asymptotes of $y = -\dfrac{8}{x^2 - 4}$.

Example

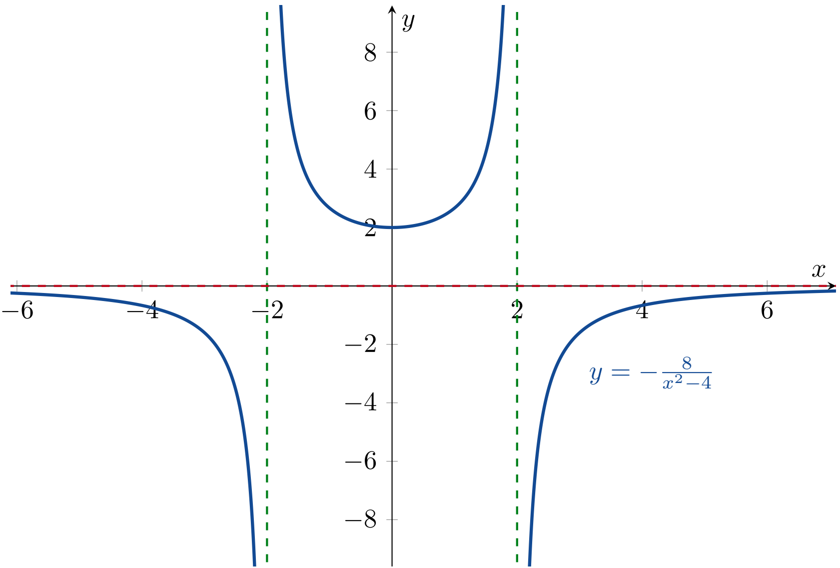
Find the horizontal and vertical asymptotes of $y = -\dfrac{8}{x^2-4}$.

I leave it for you to check that

$$\lim_{x\to\infty} -\frac{8}{x^2-4} = 0 \qquad \lim_{x\to-\infty} -\frac{8}{x^2-4} = 0$$

$$\lim_{x\to-2^+} -\frac{8}{x^2-4} = \infty$$
$$\lim_{x\to-2^-} -\frac{8}{x^2-4} = -\infty$$
$$\lim_{x\to2^+} -\frac{8}{x^2-4} = -\infty$$
$$\lim_{x\to2^-} -\frac{8}{x^2-4} = \infty.$$

Therefore $y = 0$ is a horizontal asymptote. $x = -2$ and $x = 2$ are a vertical asymptotes.


y
8
6
4
2
x
−6
−4
−2
2
4
6
−2
−4
−6
−8
$y = -\frac{8}{x^2-4}$

Please read Example 17 and page 111 in your textbook.

Next Time

- 3.1 Tangents and the Derivative at a Point
- 3.2 The Derivative as a Function
- 3.3 Differentiation Rules
- 3.4 The Derivative as a Rate of Change