Lecture 12

- 7.5 Indeterminate Forms and L'Hôpital's Rule
- 7.6 Inverse Trigonometric Functions
- 7.7 Hyperbolic Functions

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Indeterminate Forms and L'Hôpital's Rule

7.5 Indeterminate Forms and L'Hôpital's Rule

Things like "$\frac{0}{0}$" and "$\frac{\infty}{\infty}$" are not numbers. We call them *indeterminate forms.*

Guillaume de l'Hôpital

BORN
1661

DECEICED
2 February 1704

NATIONALITY
French

Indeterminate Form $\frac{0}{0}$

Theorem (L'Hôpital's Rule)

Suppose that

- $f(a) = g(a) = 0$;
- f *and* g *are differentiable on* $(a - \delta, a + \delta)$ *for some* $\delta > 0$;
- $g(x) \neq 0$ *for all* $x \in (a - \delta, a) \cup (a, a + \delta)$.

Indeterminate Form $\frac{0}{0}$

Theorem (L'Hôpital's Rule)

Suppose that

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- f *and* g *are differentiable on* $(a-\delta, a+\delta)$ *for some* $\delta > 0$;
- $g(x) \neq 0$ *for all* $x \in (a-\delta, a) \cup (a, a+\delta)$.

Then

$$\boxed{\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}}$$

if the limit $\lim_{x\to a} \frac{f'(x)}{g'(x)}$ *exists.*

$$\lim_{x\to a}\frac{f(x)}{g(x)} = \lim_{x\to a}\frac{f'(x)}{g'(x)}$$

Remark

Note that l'Hôpital's Rule says $\dfrac{f'}{g'}$. It does not say $\left(\frac{f}{g}\right)'$.

Remark

The 'H' in l'Hôpital is silent.

$$\lim_{x\to a}\frac{f(x)}{g(x)}=\lim_{x\to a}\frac{f'(x)}{g'(x)}$$

Example

Find $\displaystyle\lim_{x\to 0}\frac{3x-\sin x}{x}$.

If we just replaced x by 0, we would get the indeterminate form "$\frac{0}{0}$". So we use l'Hôpital's Rule.

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$$

Example

Find $\lim_{x \to 0} \frac{3x - \sin x}{x}$.

If we just replaced x by 0, we would get the indeterminate form "$\frac{0}{0}$". So we use l'Hôpital's Rule.

$$\lim_{x \to 0} \frac{3x - \sin x}{x} = \lim_{x \to 0} \frac{(3x - \sin x)'}{(x)'}$$

$$\lim_{x\to a}\frac{f(x)}{g(x)}=\lim_{x\to a}\frac{f'(x)}{g'(x)}$$

Example

Find $\displaystyle\lim_{x\to 0}\frac{3x-\sin x}{x}$.

If we just replaced x by 0, we would get the indeterminate form "$\frac{0}{0}$". So we use l'Hôpital's Rule.

$$\lim_{x\to 0}\frac{3x-\sin x}{x}=\lim_{x\to 0}\frac{(3x-\sin x)'}{(x)'}=\lim_{x\to 0}\frac{3-\cos x}{1}=2.$$

$$\lim_{x\to a}\frac{f(x)}{g(x)} = \lim_{x\to a}\frac{f'(x)}{g'(x)}$$

Example

Find $\displaystyle\lim_{x\to 0}\frac{\sqrt{1+x}-1}{x}$.

$$\lim_{x\to a}\frac{f(x)}{g(x)}=\lim_{x\to a}\frac{f'(x)}{g'(x)}$$

Example

Find $\displaystyle\lim_{x\to 0}\frac{\sqrt{1+x}-1}{x}$.

Again we would get the indeterminate form "$\frac{0}{0}$" if we replaced x by 0, so we use l'Hôpital's Rule to calculate

$$\lim_{x\to 0}\frac{\sqrt{1+x}-1}{x}=\lim_{x\to 0}\frac{(\sqrt{1+x}-1)'}{(x)'}$$

$$\lim_{x\to a}\frac{f(x)}{g(x)} = \lim_{x\to a}\frac{f'(x)}{g'(x)}$$

Example

Find $\displaystyle\lim_{x\to 0}\frac{\sqrt{1+x}-1}{x}$.

Again we would get the indeterminate form "$\frac{0}{0}$" if we replaced x by 0, so we use l'Hôpital's Rule to calculate

$$\lim_{x\to 0}\frac{\sqrt{1+x}-1}{x} = \lim_{x\to 0}\frac{(\sqrt{1+x}-1)'}{(x)'} = \lim_{x\to 0}\frac{\frac{1}{2\sqrt{x+1}}}{1} = \frac{1}{2}.$$

$$\lim_{x\to a}\frac{f(x)}{g(x)} = \lim_{x\to a}\frac{f'(x)}{g'(x)}$$

Example

Find $\displaystyle\lim_{x\to 0}\frac{\sqrt{1+x}-1-\frac{x}{2}}{x^2}$.

Again we would get the indeterminate form "$\frac{0}{0}$" if we replaced x by 0, so we use l'Hôpital's Rule to calculate

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$$

Example

Find $\displaystyle\lim_{x \to 0} \frac{\sqrt{1+x} - 1 - \frac{x}{2}}{x^2}$.

Again we would get the indeterminate form "$\frac{0}{0}$" if we replaced x by 0, so we use l'Hôpital's Rule to calculate

$$\lim_{x \to 0} \frac{\sqrt{1+x} - 1 - \frac{x}{2}}{x^2} = \lim_{x \to 0} \frac{\frac{1}{2}(1+x)^{-\frac{1}{2}} - \frac{1}{2}}{2x}.$$

$$\lim_{x\to a}\frac{f(x)}{g(x)} = \lim_{x\to a}\frac{f'(x)}{g'(x)}$$

Example

Find $\displaystyle\lim_{x\to 0}\frac{\sqrt{1+x}-1-\frac{x}{2}}{x^2}$.

Again we would get the indeterminate form "$\frac{0}{0}$" if we replaced x by 0, so we use l'Hôpital's Rule to calculate

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But again we would get "$\frac{0}{0}$" if we replaced x by 0. So we use l'Hôpital's Rule a second time.

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}$$

Example

Find $\displaystyle\lim_{x\to 0} \frac{\sqrt{1+x} - 1 - \frac{x}{2}}{x^2}$.

Again we would get the indeterminate form "$\frac{0}{0}$" if we replaced x by 0, so we use l'Hôpital's Rule to calculate

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But again we would get "$\frac{0}{0}$" if we replaced x by 0. So we use l'Hôpital's Rule a second time.

$$\begin{aligned}\lim_{x\to 0} \frac{\sqrt{1+x} - 1 - \frac{x}{2}}{x^2} &= \lim_{x\to 0} \frac{\frac{1}{2}(1+x)^{-\frac{1}{2}} - \frac{1}{2}}{2x} \\ &= \lim_{x\to 0} \frac{-\frac{1}{4}(1+x)^{-\frac{3}{2}}}{2} = -\frac{1}{8}.\end{aligned}$$

$$\lim_{x \to 0} \frac{x - \sin x}{x^3}$$ $\frac{0}{0}$; apply l'Hôpital's Rule.

$$= \lim_{x \to 0} \frac{1 - \cos x}{3x^2}$$ Still $\frac{0}{0}$; apply l'Hôpital's Rule again.

$$= \lim_{x \to 0} \frac{\sin x}{6x}$$ Still $\frac{0}{0}$; apply l'Hôpital's Rule again.

$$= \lim_{x \to 0} \frac{\cos x}{6} = \frac{1}{6}$$ Not $\frac{0}{0}$; limit is found.

$$\lim_{x\to a}\frac{f(x)}{g(x)}=\lim_{x\to a}\frac{f'(x)}{g'(x)}$$

Remark

We can only use l'Hôpital's Rule if we have "$\frac{0}{0}$". If we don't have "$\frac{0}{0}$", then we can not use this rule.

$$\lim_{x\to a}\frac{f(x)}{g(x)} = \lim_{x\to a}\frac{f'(x)}{g'(x)}$$

Example

Find $\displaystyle\lim_{x\to 0}\frac{1-\cos x}{x+x^2}$.

$$\lim_{x\to a}\frac{f(x)}{g(x)} = \lim_{x\to a}\frac{f'(x)}{g'(x)}$$

Example

Find $\displaystyle\lim_{x\to 0}\frac{1-\cos x}{x+x^2}$.

Wrong Answer:

$$\lim_{x\to 0}\frac{1-\cos x}{x+x^2} = \lim_{x\to 0}\frac{\sin x}{1+2x} = \lim_{x\to 0}\frac{\cos x}{2} = \frac{1}{2}.$$

Why is this wrong?

$$\lim_{x\to a}\frac{f(x)}{g(x)} = \lim_{x\to a}\frac{f'(x)}{g'(x)}$$

Example

Find $\displaystyle\lim_{x\to 0}\frac{1-\cos x}{x+x^2}$.

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Why is this wrong?

Because $\dfrac{\sin x}{1+2x}$ does not give "$\frac{0}{0}$" if we replace x by 0.

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}$$

Example

Find $\displaystyle\lim_{x\to 0} \frac{1-\cos x}{x+x^2}$.

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Why is this wrong?

Because $\dfrac{\sin x}{1+2x}$ does not give "$\frac{0}{0}$" if we replace x by 0. The correct answer is actually 0. I leave this for you to check.

L'Hôpital's Rule applies to one-sided limits as well.

EXAMPLE 3 In this example the one-sided limits are different.

(a) $\lim_{x \to 0^+} \frac{\sin x}{x^2}$ $\qquad \frac{0}{0}$

$= \lim_{x \to 0^+} \frac{\cos x}{2x} = \infty$ $\qquad$ Positive for $x > 0$

(b) $\lim_{x \to 0^-} \frac{\sin x}{x^2}$ $\qquad \frac{0}{0}$

$= \lim_{x \to 0^-} \frac{\cos x}{2x} = -\infty$ $\qquad$ Negative for $x < 0$

$$\lim_{x\to a}\frac{f(x)}{g(x)} = \lim_{x\to a}\frac{f'(x)}{g'(x)}$$

Indeterminate Form $\frac{\infty}{\infty}$

L'Hôpital's Rule also applies if we would get "$\frac{\infty}{\infty}$".

Indeterminate Form $\frac{\infty}{\infty}$

L'Hôpital's Rule also applies if we would get "$\frac{\infty}{\infty}$".

Theorem (L'Hôpital's Rule)

Let $a \in \mathbb{R}$ or $a = \infty$ or $a = -\infty$.

If $\lim_{x\to a} f(x) = \pm\infty$ and $\lim_{x\to a} g(x) = \pm\infty$, then

$$\boxed{\lim_{x\to a}\frac{f(x)}{g(x)} = \lim_{x\to a}\frac{f'(x)}{g'(x)}}$$

provided the limit $\lim_{x\to a}\frac{f'(x)}{g'(x)}$ exists.

Indeterminate Form $\frac{\infty}{\infty}$

L'Hôpital's Rule also applies if we would get "$\frac{\infty}{\infty}$".

Theorem (L'Hôpital's Rule)

Let $a \in \mathbb{R}$ or $a = \infty$ or $a = -\infty$.

If $\lim\limits_{x\to a} f(x) = \pm\infty$ and $\lim\limits_{x\to a} g(x) = \pm\infty$, then

$$\boxed{\lim_{x\to a}\frac{f(x)}{g(x)} = \lim_{x\to a}\frac{f'(x)}{g'(x)}}$$

provided the limit $\lim\limits_{x\to a}\frac{f'(x)}{g'(x)}$ exists.

This theorem is also true for one sided limits $x \to a^+$ and $x \to a^-$.

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$$

Example

Find $\displaystyle\lim_{x \to \frac{\pi}{2}} \frac{\sec x}{1 + \tan x}$.

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$$

Example

Find $\displaystyle\lim_{x \to \frac{\pi}{2}} \frac{\sec x}{1 + \tan x}$.

Note that this is a "$\frac{\infty}{\infty}$" question.

Since $\sec x$ and $\tan x$ are both discontinuous at $\frac{\pi}{2}$, we need to consider one-sided limits.

$$\lim_{x\to a}\frac{f(x)}{g(x)}=\lim_{x\to a}\frac{f'(x)}{g'(x)}$$

Example

Find $\displaystyle\lim_{x\to\frac{\pi}{2}}\frac{\sec x}{1+\tan x}$.

Note that this is a "$\frac{\infty}{\infty}$" question.

Since $\sec x$ and $\tan x$ are both discontinuous at $\frac{\pi}{2}$, we need to consider one-sided limits.

$$\lim_{x\to\left(\frac{\pi}{2}\right)^-}\frac{\sec x}{1+\tan x}=\lim_{x\to\left(\frac{\pi}{2}\right)^-}\frac{\sec x\tan x}{\sec^2 x}$$

$$=$$

$$\lim_{x\to a}\frac{f(x)}{g(x)} = \lim_{x\to a}\frac{f'(x)}{g'(x)}$$

Example

Find $\displaystyle\lim_{x\to\frac{\pi}{2}} \frac{\sec x}{1+\tan x}$.

Note that this is a "$\frac{\infty}{\infty}$" question.

Since $\sec x$ and $\tan x$ are both discontinuous at $\frac{\pi}{2}$, we need to consider one-sided limits.

$$\begin{aligned}\lim_{x\to\left(\frac{\pi}{2}\right)^-} \frac{\sec x}{1+\tan x} &= \lim_{x\to\left(\frac{\pi}{2}\right)^-} \frac{\sec x\tan x}{\sec^2 x} \\ &= \lim_{x\to\left(\frac{\pi}{2}\right)^-} \frac{\tan x}{\sec x} = \lim_{x\to\left(\frac{\pi}{2}\right)^-} \sin x = 1.\end{aligned}$$

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$$

I leave it to you to check that $\displaystyle\lim_{x \to \left(\frac{\pi}{2}\right)^+} \frac{\sec x}{1 + \tan x} = 1$ also.

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}$$

I leave it to you to check that $\displaystyle\lim_{x\to\left(\frac{\pi}{2}\right)^+} \frac{\sec x}{1+\tan x} = 1$ also.

Therefore

$$\lim_{x\to\frac{\pi}{2}} \frac{\sec x}{1+\tan x} = 1.$$

$$\lim_{x\to a}\frac{f(x)}{g(x)}=\lim_{x\to a}\frac{f'(x)}{g'(x)}$$

Example

$$\lim_{x\to\infty}\frac{\ln x}{2\sqrt{x}}=\lim_{x\to\infty}\frac{\frac{1}{x}}{\frac{1}{\sqrt{x}}}$$

$$\lim_{x\to a}\frac{f(x)}{g(x)}=\lim_{x\to a}\frac{f'(x)}{g'(x)}$$

Example

$$\lim_{x\to\infty}\frac{\ln x}{2\sqrt{x}}=\lim_{x\to\infty}\frac{\frac{1}{x}}{\frac{1}{\sqrt{x}}}=\lim_{x\to\infty}\frac{\sqrt{x}}{x}$$

$$\lim_{x\to a}\frac{f(x)}{g(x)}=\lim_{x\to a}\frac{f'(x)}{g'(x)}$$

Example

$$\lim_{x\to\infty}\frac{\ln x}{2\sqrt{x}}=\lim_{x\to\infty}\frac{\frac{1}{x}}{\frac{1}{\sqrt{x}}}=\lim_{x\to\infty}\frac{\sqrt{x}}{x}=\lim_{x\to\infty}\frac{1}{\sqrt{x}}$$

$$\lim_{x\to a}\frac{f(x)}{g(x)} = \lim_{x\to a}\frac{f'(x)}{g'(x)}$$

Example

$$\lim_{x\to\infty}\frac{\ln x}{2\sqrt{x}} = \lim_{x\to\infty}\frac{\frac{1}{x}}{\frac{1}{\sqrt{x}}} = \lim_{x\to\infty}\frac{\sqrt{x}}{x} = \lim_{x\to\infty}\frac{1}{\sqrt{x}} = 0.$$

$$\lim_{x\to a}\frac{f(x)}{g(x)}=\lim_{x\to a}\frac{f'(x)}{g'(x)}$$

Example

$$\lim_{x\to\infty}\frac{\ln x}{2\sqrt{x}}=\lim_{x\to\infty}\frac{\frac{1}{x}}{\frac{1}{\sqrt{x}}}=\lim_{x\to\infty}\frac{\sqrt{x}}{x}=\lim_{x\to\infty}\frac{1}{\sqrt{x}}=0.$$

Example

$$\lim_{x\to\infty}\frac{e^x}{x^2}=\lim_{x\to\infty}\frac{e^x}{2x}$$

$$\lim_{x\to a}\frac{f(x)}{g(x)} = \lim_{x\to a}\frac{f'(x)}{g'(x)}$$

Example

$$\lim_{x\to\infty}\frac{\ln x}{2\sqrt{x}} = \lim_{x\to\infty}\frac{\frac{1}{x}}{\frac{1}{\sqrt{x}}} = \lim_{x\to\infty}\frac{\sqrt{x}}{x} = \lim_{x\to\infty}\frac{1}{\sqrt{x}} = 0.$$

Example

$$\lim_{x\to\infty}\frac{e^x}{x^2} = \lim_{x\to\infty}\frac{e^x}{2x} = \lim_{x\to\infty}\frac{e^x}{2}$$

$$\lim_{x\to a}\frac{f(x)}{g(x)} = \lim_{x\to a}\frac{f'(x)}{g'(x)}$$

Example

$$\lim_{x\to\infty}\frac{\ln x}{2\sqrt{x}} = \lim_{x\to\infty}\frac{\frac{1}{x}}{\frac{1}{\sqrt{x}}} = \lim_{x\to\infty}\frac{\sqrt{x}}{x} = \lim_{x\to\infty}\frac{1}{\sqrt{x}} = 0.$$

Example

$$\lim_{x\to\infty}\frac{e^x}{x^2} = \lim_{x\to\infty}\frac{e^x}{2x} = \lim_{x\to\infty}\frac{e^x}{2} = \infty.$$

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}$$

Indeterminate Forms $\infty \cdot 0$ and $\infty - \infty$

We don't have a l'Hôpital's Rule for "$\infty \cdot 0$" or "$\infty - \infty$", so we will try to rearrange our problem to either a "$\frac{0}{0}$" problem or a "$\frac{\infty}{\infty}$" problem.

$$\lim_{x\to a}\frac{f(x)}{g(x)} = \lim_{x\to a}\frac{f'(x)}{g'(x)}$$

Example

Find $\displaystyle\lim_{x\to\infty}\left(x\sin\frac{1}{x}\right)$.

This is a "$\infty \cdot 0$" problem.

$$\lim_{x\to a}\frac{f(x)}{g(x)}=\lim_{x\to a}\frac{f'(x)}{g'(x)}$$

Example

Find $\displaystyle\lim_{x\to\infty}\left(x\sin\frac{1}{x}\right)$.

This is a "$\infty\cdot 0$" problem. If we let $h=\frac{1}{x}$, then we can change it into a "$\frac{0}{0}$" problem.

$$\lim_{x\to a}\frac{f(x)}{g(x)} = \lim_{x\to a}\frac{f'(x)}{g'(x)}$$

Example

Find $\displaystyle\lim_{x\to\infty}\left(x\sin\frac{1}{x}\right)$.

This is a "$\infty \cdot 0$" problem. If we let $h = \frac{1}{x}$, then we can change it into a "$\frac{0}{0}$" problem.

$$\lim_{x\to\infty}\underbrace{\left(x\sin\frac{1}{x}\right)}_{\infty\cdot 0} = \lim_{h\to 0^+}\left(\frac{1}{h}\sin h\right) = \lim_{h\to 0^+}\underbrace{\frac{\sin h}{h}}_{\frac{0}{0}}$$

$$\lim_{x\to a}\frac{f(x)}{g(x)} = \lim_{x\to a}\frac{f'(x)}{g'(x)}$$

Example

Find $\displaystyle\lim_{x\to\infty}\left(x\sin\frac{1}{x}\right)$.

This is a "$\infty \cdot 0$" problem. If we let $h = \frac{1}{x}$, then we can change it into a "$\frac{0}{0}$" problem.

$$\lim_{x\to\infty}\underbrace{\left(x\sin\frac{1}{x}\right)}_{\infty\cdot 0} = \lim_{h\to 0^+}\left(\frac{1}{h}\sin h\right) = \lim_{h\to 0^+}\underbrace{\frac{\sin h}{h}}_{\frac{0}{0}} = \lim_{h\to 0^+}\frac{\cos h}{1} = 1.$$

(I didn't need to use l'Hôpital's Rule here because we already know that $\displaystyle\lim_{h\to 0^+}\frac{\sin h}{h} = 1$.)

$$\lim_{x\to a}\frac{f(x)}{g(x)} = \lim_{x\to a}\frac{f'(x)}{g'(x)}$$

Example

Find $\lim\limits_{x\to 0^+} \sqrt{x}\ln x$.

This is another "$\infty \cdot 0$" problem. Actually a "$0 \cdot -\infty$" problem. We are going to change it into a "$\frac{\infty}{\infty}$" problem.

$$\lim_{x\to a}\frac{f(x)}{g(x)} = \lim_{x\to a}\frac{f'(x)}{g'(x)}$$

Example

Find $\lim_{x\to 0^+} \sqrt{x}\ln x$.

This is another "$\infty \cdot 0$" problem. Actually a "$0 \cdot -\infty$" problem. We are going to change it into a "$\frac{\infty}{\infty}$" problem.

$$\lim_{x\to 0^+} \underbrace{\sqrt{x}\ln x}_{0\cdot-\infty} = \lim_{x\to 0^+} \underbrace{\frac{\ln x}{x^{-\frac{1}{2}}}}_{\frac{-\infty}{\infty}}$$

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}$$

Example

Find $\lim\limits_{x\to 0^+} \sqrt{x}\ln x$.

This is another "$\infty \cdot 0$" problem. Actually a "$0 \cdot -\infty$" problem. We are going to change it into a "$\frac{\infty}{\infty}$" problem.

$$\lim_{x\to 0^+} \underbrace{\sqrt{x}\ln x}_{0\cdot-\infty} = \lim_{x\to 0^+} \frac{\ln x}{\underbrace{x^{-\frac{1}{2}}}_{\frac{-\infty}{\infty}}} = \lim_{x\to 0^+} \frac{x^{-1}}{-\frac{1}{2}x^{-\frac{3}{2}}}$$

$$\lim_{x\to a}\frac{f(x)}{g(x)} = \lim_{x\to a}\frac{f'(x)}{g'(x)}$$

Example

Find $\lim_{x\to 0^+} \sqrt{x}\ln x$.

This is another "$\infty \cdot 0$" problem. Actually a "$0 \cdot -\infty$" problem. We are going to change it into a "$\frac{\infty}{\infty}$" problem.

$$\lim_{x\to 0^+} \underbrace{\sqrt{x}\ln x}_{0\cdot -\infty} = \lim_{x\to 0^+} \frac{\ln x}{\underbrace{x^{-\frac{1}{2}}}_{\frac{-\infty}{\infty}}} = \lim_{x\to 0^+} \frac{x^{-1}}{-\frac{1}{2}x^{-\frac{3}{2}}} = \lim_{x\to 0^+} -2\sqrt{x} = 0.$$

$$\lim_{x\to a}\frac{f(x)}{g(x)} = \lim_{x\to a}\frac{f'(x)}{g'(x)}$$

Example

Find $\displaystyle\lim_{x\to 0}\left(\frac{1}{\sin x} - \frac{1}{x}\right)$.

This is a "$\infty - \infty$" problem.

$$\lim_{x\to a}\frac{f(x)}{g(x)} = \lim_{x\to a}\frac{f'(x)}{g'(x)}$$

Example

Find $\displaystyle\lim_{x\to 0}\left(\frac{1}{\sin x} - \frac{1}{x}\right)$.

This is a "$\infty - \infty$" problem. To be more precise:

- If $x \to 0^+$, then $\frac{1}{\sin x} - \frac{1}{x} \to$ "$\infty - \infty$".
- If $x \to 0^-$, then $\frac{1}{\sin x} - \frac{1}{x} \to$ "$-\infty + \infty$".

$$\lim_{x\to a}\frac{f(x)}{g(x)} = \lim_{x\to a}\frac{f'(x)}{g'(x)}$$

Example

Find $\displaystyle\lim_{x\to 0}\left(\frac{1}{\sin x} - \frac{1}{x}\right)$.

This is a "$\infty - \infty$" problem. To be more precise:

- If $x \to 0^+$, then $\frac{1}{\sin x} - \frac{1}{x} \to$ "$\infty - \infty$".
- If $x \to 0^-$, then $\frac{1}{\sin x} - \frac{1}{x} \to$ "$-\infty + \infty$".

We calculate that

$$\lim_{x\to 0}\left(\frac{1}{\sin x} - \frac{1}{x}\right) = \lim_{x\to 0}\underbrace{\frac{x - \sin x}{x\sin x}}_{\frac{0}{0}} =$$

$$= \qquad\qquad =$$

$$\lim_{x\to a}\frac{f(x)}{g(x)} = \lim_{x\to a}\frac{f'(x)}{g'(x)}$$

Example

Find $\displaystyle\lim_{x\to 0}\left(\frac{1}{\sin x} - \frac{1}{x}\right)$.

This is a "$\infty - \infty$" problem. To be more precise:

- If $x \to 0^+$, then $\frac{1}{\sin x} - \frac{1}{x} \to$ "$\infty - \infty$".
- If $x \to 0^-$, then $\frac{1}{\sin x} - \frac{1}{x} \to$ "$-\infty + \infty$".

We calculate that

$$\lim_{x\to 0}\left(\frac{1}{\sin x} - \frac{1}{x}\right) = \lim_{x\to 0}\frac{x - \sin x}{\underbrace{x\sin x}_{\frac{0}{0}}} = \lim_{x\to 0}\frac{1-\cos x}{\underbrace{\sin x + x\cos x}_{\frac{0}{0}}}$$

$$=$$

$$=$$

$$\lim_{x\to a}\frac{f(x)}{g(x)} = \lim_{x\to a}\frac{f'(x)}{g'(x)}$$

Example

Find $\displaystyle\lim_{x\to 0}\left(\frac{1}{\sin x} - \frac{1}{x}\right)$.

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We calculate that

$$\lim_{x\to 0}\left(\frac{1}{\sin x} - \frac{1}{x}\right) = \lim_{x\to 0}\frac{x - \sin x}{\underbrace{x\sin x}_{\frac{0}{0}}} = \lim_{x\to 0}\frac{1-\cos x}{\underbrace{\sin x + x\cos x}_{\frac{0}{0}}}$$

$$= \lim_{x\to 0}\frac{\sin x}{2\cos x - x\sin x} =$$

Example

Find $\displaystyle\lim_{x\to 0}\left(\frac{1}{\sin x} - \frac{1}{x}\right)$.

This is a "$\infty - \infty$" problem. To be more precise:

- If $x \to 0^+$, then $\frac{1}{\sin x} - \frac{1}{x} \to$ "$\infty - \infty$".
- If $x \to 0^-$, then $\frac{1}{\sin x} - \frac{1}{x} \to$ "$-\infty + \infty$".

We calculate that

$$\lim_{x\to 0}\left(\frac{1}{\sin x} - \frac{1}{x}\right) = \lim_{x\to 0}\frac{x - \sin x}{\underbrace{x\sin x}_{\frac{0}{0}}} = \lim_{x\to 0}\frac{1-\cos x}{\underbrace{\sin x + x\cos x}_{\frac{0}{0}}}$$

$$= \lim_{x\to 0}\frac{\sin x}{2\cos x - x\sin x} = \frac{0}{2} = 0.$$

Ask the audience

One of these calculations is correct. The other 3 are wrong. Which one is correct?

1.
$$\lim_{x\to 0^+} x\ln x = 0\cdot(-\infty) = 0$$

3.
$$\lim_{x\to 0^+} x\ln x = 0\cdot(-\infty) = -\infty$$

2.
$$\lim_{x\to 0^+} x\ln x = \lim_{x\to 0^+} \frac{\ln x}{\frac{1}{x}} = \frac{-\infty}{\infty} = -1$$

4.
$$\lim_{x\to 0^+} x\ln x = \lim_{x\to 0^+} \frac{\ln x}{\frac{1}{x}} = \lim_{x\to 0^+} \frac{\frac{1}{x}}{-\frac{1}{x^2}} = \lim_{x\to 0^+} (-x) = 0$$

Ask the audience

One of these calculations is correct. The other 3 are wrong. Which one is correct?

1
$$\lim_{x\to 0^+} x\ln x = 0\cdot(-\infty) = 0$$

2
$$\lim_{x\to 0^+} x\ln x = \lim_{x\to 0^+} \frac{\ln x}{\frac{1}{x}} = \frac{-\infty}{\infty} = -1$$

3
$$\lim_{x\to 0^+} x\ln x = 0\cdot(-\infty) = -\infty$$

4
$$\lim_{x\to 0^+} x\ln x = \lim_{x\to 0^+} \frac{\ln x}{\frac{1}{x}} = \lim_{x\to 0^+} \frac{\frac{1}{x}}{-\frac{1}{x^2}} = \lim_{x\to 0^+} (-x) = 0$$

Indeterminate Powers 1^∞, 0^0 and ∞^0

Theorem

Let $a \in \mathbb{R}$ or $a = \infty$ or $a = -\infty$.

If $\lim\limits_{x \to a} \ln f(x) = L$, then

$$\boxed{\lim_{x \to a} f(x) = \exp\left(\lim_{x \to a} \ln f(x)\right) = e^L.}$$

This theorem is also true for one sided limits $x \to a^+$ and $x \to a^-$.

$$\lim_{x\to a} f(x) = \exp\left(\lim_{x\to a} \ln f(x)\right) = e^L$$

Example

Apply l'Hôpital's Rule to show that $\lim_{x\to 0^+} (1+x)^{\frac{1}{x}} = e$.

This is a "1^∞" problem.

$$\lim_{x\to a} f(x) = \exp\left(\lim_{x\to a} \ln f(x)\right) = e^L$$

Example

Apply l'Hôpital's Rule to show that $\lim\limits_{x\to 0^+} (1+x)^{\frac{1}{x}} = e$.

This is a "1^∞" problem. We will let $f(x) = (1+x)^{\frac{1}{x}}$ and we will find $\lim\limits_{x\to 0^+} \ln f(x)$.

$$\lim_{x \to a} f(x) = \exp\left(\lim_{x \to a} \ln f(x)\right) = e^L$$

Since

$$\ln f(x) = \ln(1+x)^{\frac{1}{x}} = \frac{1}{x}\ln(1+x),$$

$$\lim_{x\to a} f(x) = \exp\left(\lim_{x\to a} \ln f(x)\right) = e^L$$

Since

$$\ln f(x) = \ln(1+x)^{\frac{1}{x}} = \frac{1}{x}\ln(1+x),$$

we can use l'Hôpital's Rule to calculate that

$$\lim_{x\to 0^+} \ln f(x) = \lim_{x\to 0^+} \underbrace{\frac{\ln(1+x)}{x}}_{\frac{0}{0}}$$

$$\lim_{x\to a} f(x) = \exp\left(\lim_{x\to a} \ln f(x)\right) = e^L$$

Since

$$\ln f(x) = \ln(1+x)^{\frac{1}{x}} = \frac{1}{x}\ln(1+x),$$

we can use l'Hôpital's Rule to calculate that

$$\lim_{x\to 0^+} \ln f(x) = \lim_{x\to 0^+} \underbrace{\frac{\ln(1+x)}{x}}_{\frac{0}{0}} = \lim_{x\to 0^+} \frac{\frac{1}{1+x}}{1}$$

$$\lim_{x\to a} f(x) = \exp\left(\lim_{x\to a} \ln f(x)\right) = e^L$$

Since

$$\ln f(x) = \ln(1+x)^{\frac{1}{x}} = \frac{1}{x}\ln(1+x),$$

we can use l'Hôpital's Rule to calculate that

$$\lim_{x\to 0^+} \ln f(x) = \lim_{x\to 0^+} \underbrace{\frac{\ln(1+x)}{x}}_{\frac{0}{0}} = \lim_{x\to 0^+} \frac{\frac{1}{1+x}}{1} = \frac{1}{1} = 1.$$

$$\lim_{x\to a} f(x) = \exp\left(\lim_{x\to a} \ln f(x)\right) = e^L$$

Since

$$\ln f(x) = \ln(1+x)^{\frac{1}{x}} = \frac{1}{x}\ln(1+x),$$

we can use l'Hôpital's Rule to calculate that

$$\lim_{x\to 0^+} \ln f(x) = \lim_{x\to 0^+} \underbrace{\frac{\ln(1+x)}{x}}_{\frac{0}{0}} = \lim_{x\to 0^+} \frac{\frac{1}{1+x}}{1} = \frac{1}{1} = 1.$$

Therefore

$$\lim_{x\to 0^+} (1+x)^{\frac{1}{x}} = \lim_{x\to 0^+} f(x) = \exp\left(\lim_{x\to 0^+} \ln f(x)\right)$$

$$\lim_{x\to a} f(x) = \exp\left(\lim_{x\to a} \ln f(x)\right) = e^L$$

Since

$$\ln f(x) = \ln(1+x)^{\frac{1}{x}} = \frac{1}{x}\ln(1+x),$$

we can use l'Hôpital's Rule to calculate that

$$\lim_{x\to 0^+} \ln f(x) = \lim_{x\to 0^+} \underbrace{\frac{\ln(1+x)}{x}}_{\frac{0}{0}} = \lim_{x\to 0^+} \frac{\frac{1}{1+x}}{1} = \frac{1}{1} = 1.$$

Therefore

$$\lim_{x\to 0^+} (1+x)^{\frac{1}{x}} = \lim_{x\to 0^+} f(x) = \exp\left(\lim_{x\to 0^+} \ln f(x)\right) = e^1 = e.$$

$$\lim_{x\to a} f(x) = \exp\left(\lim_{x\to a} \ln f(x)\right) = e^L$$

Example

Find $\lim_{x\to\infty} x^{\frac{1}{x}}$.

This is an "∞^0" problem.

$$\lim_{x\to a} f(x) = \exp\left(\lim_{x\to a} \ln f(x)\right) = e^L$$

Example

Find $\lim_{x\to\infty} x^{\frac{1}{x}}$.

This is an "∞^0" problem. First we use l'Hôpital's Rule to calculate that

$$\lim_{x\to\infty} \ln x^{\frac{1}{x}}$$

$$\lim_{x \to a} f(x) = \exp\left(\lim_{x \to a} \ln f(x)\right) = e^L$$

Example

Find $\lim_{x \to \infty} x^{\frac{1}{x}}$.

This is an "∞^0" problem. First we use l'Hôpital's Rule to calculate that

$$\lim_{x \to \infty} \ln x^{\frac{1}{x}} = \lim_{x \to \infty} \underbrace{\frac{\ln x}{x}}_{\frac{\infty}{\infty}}$$

$$\lim_{x\to a} f(x) = \exp\left(\lim_{x\to a} \ln f(x)\right) = e^L$$

Example

Find $\lim_{x\to\infty} x^{\frac{1}{x}}$.

This is an "∞^0" problem. First we use l'Hôpital's Rule to calculate that

$$\lim_{x\to\infty} \ln x^{\frac{1}{x}} = \lim_{x\to\infty} \underbrace{\frac{\ln x}{x}}_{\frac{\infty}{\infty}} = \lim_{x\to\infty} \frac{\frac{1}{x}}{1} = \frac{0}{1} = 0.$$

$$\lim_{x\to a} f(x) = \exp\left(\lim_{x\to a} \ln f(x)\right) = e^L$$

Example

Find $\lim\limits_{x\to\infty} x^{\frac{1}{x}}$.

This is an "∞^0" problem. First we use l'Hôpital's Rule to calculate that

$$\lim_{x\to\infty} \ln x^{\frac{1}{x}} = \lim_{x\to\infty} \underbrace{\frac{\ln x}{x}}_{\frac{\infty}{\infty}} = \lim_{x\to\infty} \frac{\frac{1}{x}}{1} = \frac{0}{1} = 0.$$

It follows that

$$\lim_{x\to\infty} x^{\frac{1}{x}} = \exp\left(\lim_{x\to\infty} \ln x^{\frac{1}{x}}\right) = e^0 = 1.$$

Theorem (Cauchy's Mean Value Theorem)

Suppose that

- f *and* g *are continuous on* $[a, b]$;
- f *and* g *are differentiable on* (a, b);
- $g'(x) \neq 0$ *for all* $x \in (a, b)$.

Theorem (Cauchy's Mean Value Theorem)

Suppose that

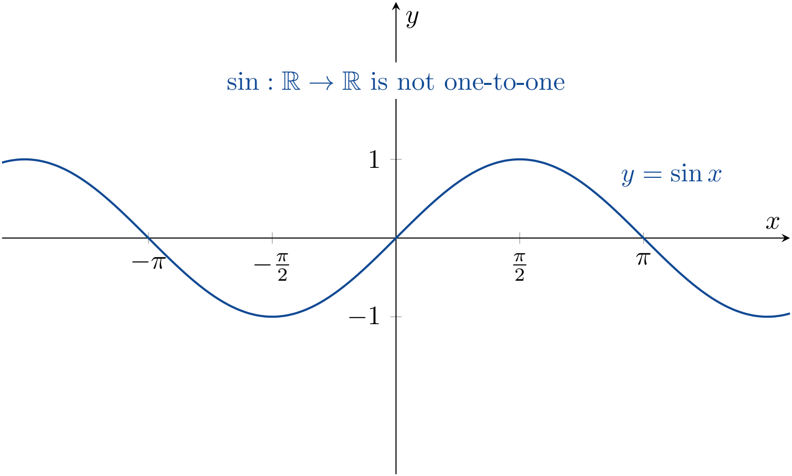
- f *and* g *are continuous on* $[a, b]$*;*
- f *and* g *are differentiable on* (a, b)*;*
- $g'(x) \neq 0$ *for all* $x \in (a, b)$*.*

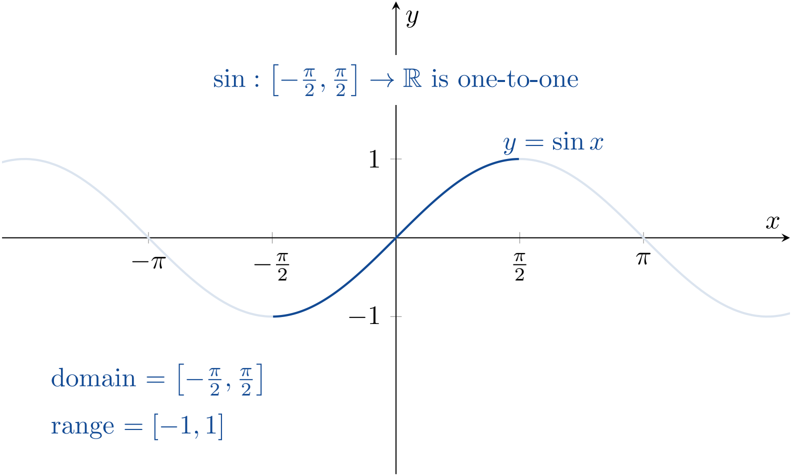
Then there exists $c \in (a, b)$ *such that*

$$\frac{f'(c)}{g'(c)} = \frac{f(b) - f(a)}{g(b) - g(a)}.$$

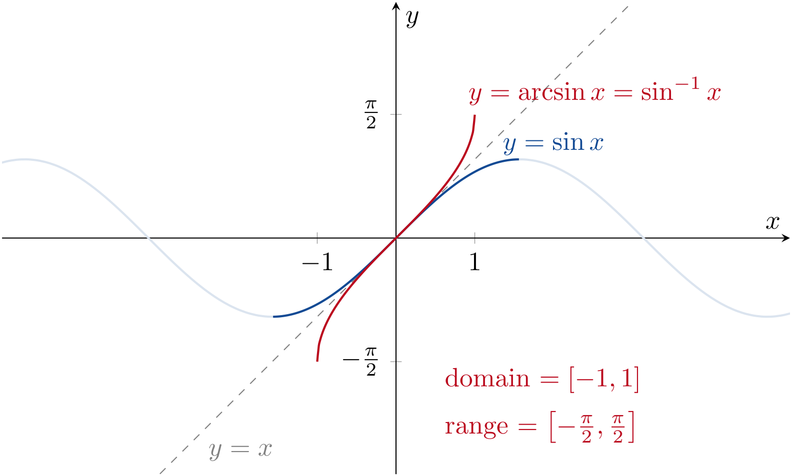
(proof in textbook)

Inverse Trigonometric Functions

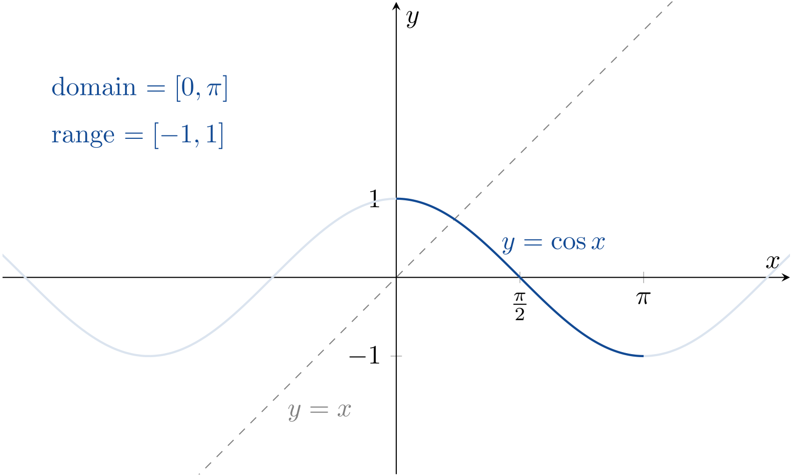

y
$\sin : \mathbb{R} \to \mathbb{R}$ is not one-to-one
1
$y = \sin x$
x
$-\pi$
$-\frac{\pi}{2}$
$\frac{\pi}{2}$
π
-1


y
$\sin : \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \to \mathbb{R}$ is one-to-one
$y = \sin x$
1
x
$-\pi$
$-\frac{\pi}{2}$
$\frac{\pi}{2}$
π
-1
domain $= \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$
range $= [-1, 1]$

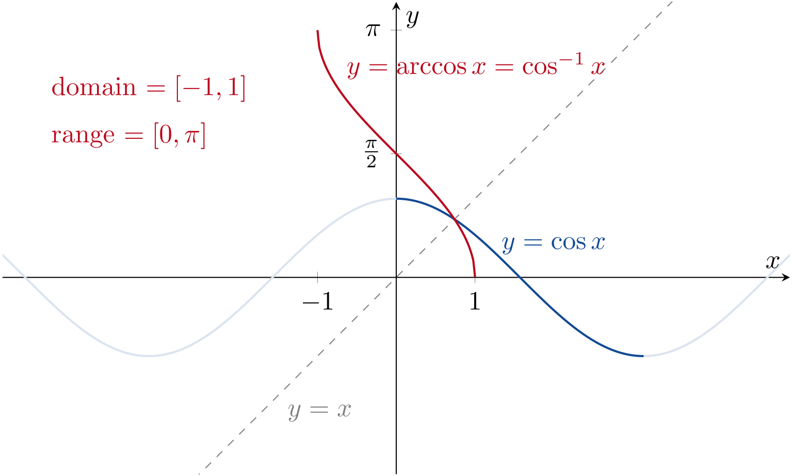
7.6 Inverse Trigonometric Functions



7.6 Inverse Trigonometric Functions



7.6 Inverse Trigonometric Functions

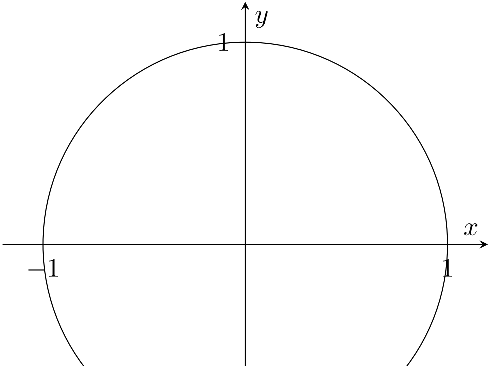


Arcsine and Arccosine

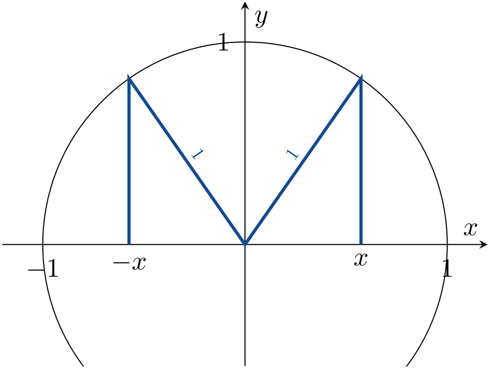
Definition

- $y = \arcsin x$ is the number in $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ for which $\sin y = x$.
- $y = \arccos x$ is the number in $[0, \pi]$ for which $\cos y = x$.

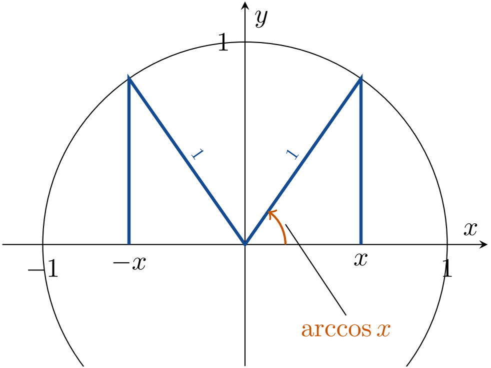
Identities Involving Arcsine and Arccosine



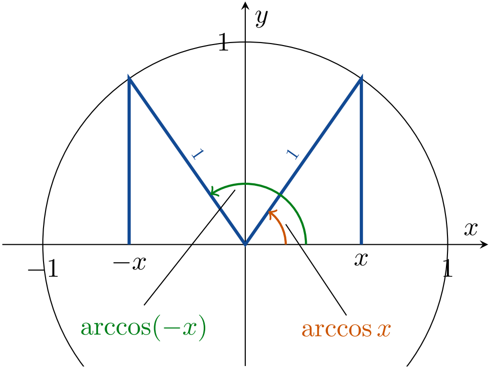
Identities Involving Arcsine and Arccosine



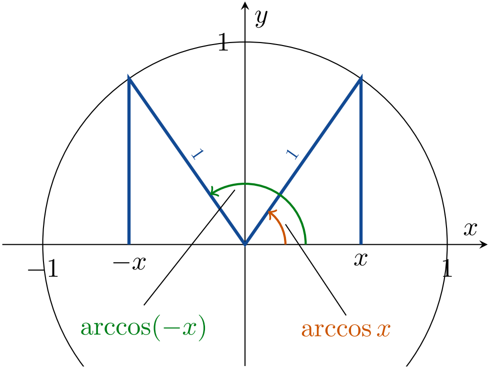
Identities Involving Arcsine and Arccosine



Identities Involving Arcsine and Arccosine

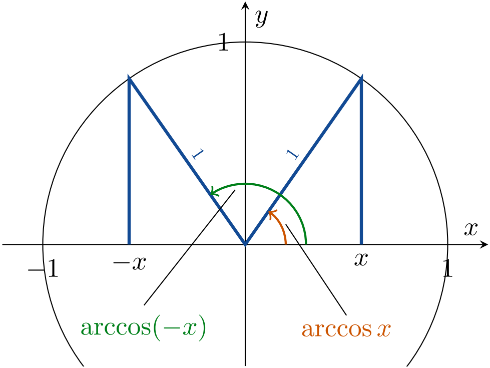


Identities Involving Arcsine and Arccosine



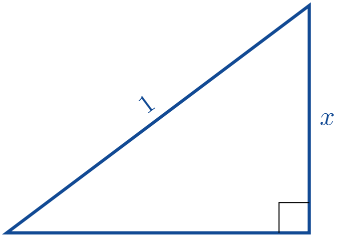
$$\arccos x + \arccos(-x) = \pi$$

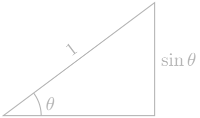
Identities Involving Arcsine and Arccosine

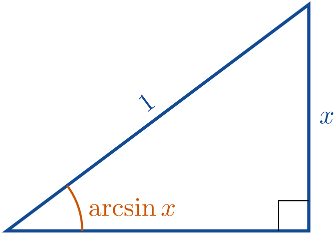


$$\arccos x + \arccos(-x) = \pi$$

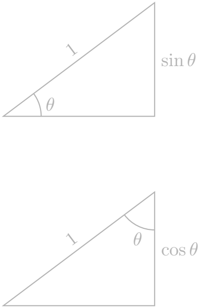
$$\boxed{\arccos(-x) = \pi - \arccos x}$$

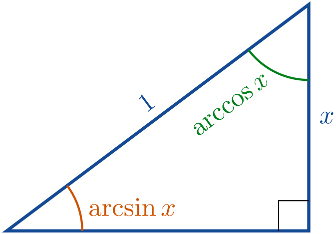

1
x

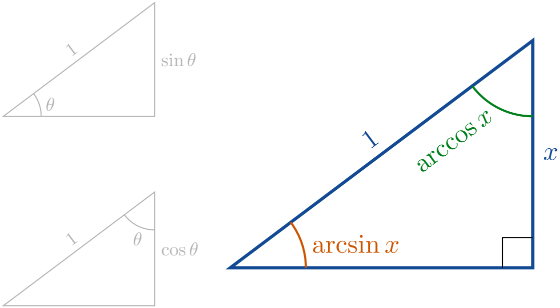

1
$\sin\theta$
θ


1
x
$\arcsin x$

7.6 Inverse Trigonometric Functions



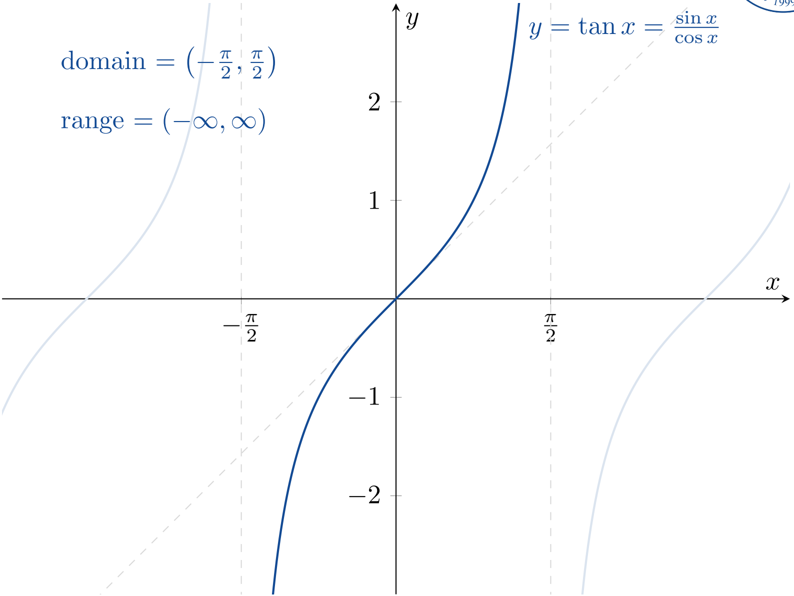




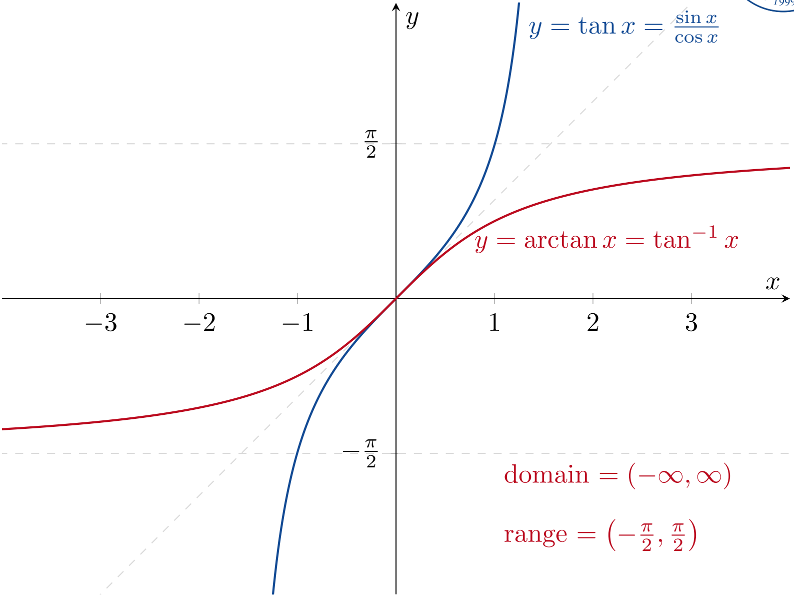
$$\boxed{\arcsin x + \arccos x = \frac{\pi}{2}} \qquad x \in [-1, 1]$$

(From this triangle we can see that this is true for $x \in [0, 1]$.
Using the previous identity, we can prove that it is also true for $x \in [-1, 0)$.)

7.6 Inverse Trigonometric Functions



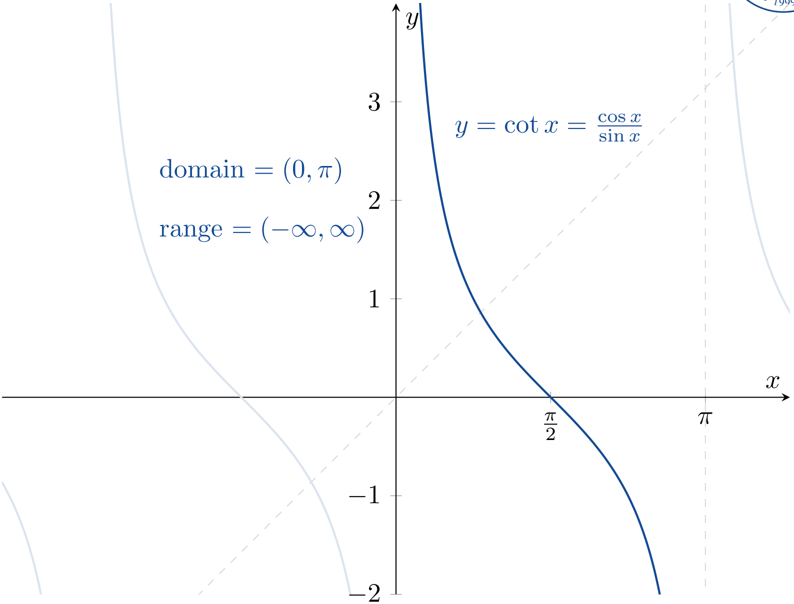
7.6 Inverse Trigonometric Functions



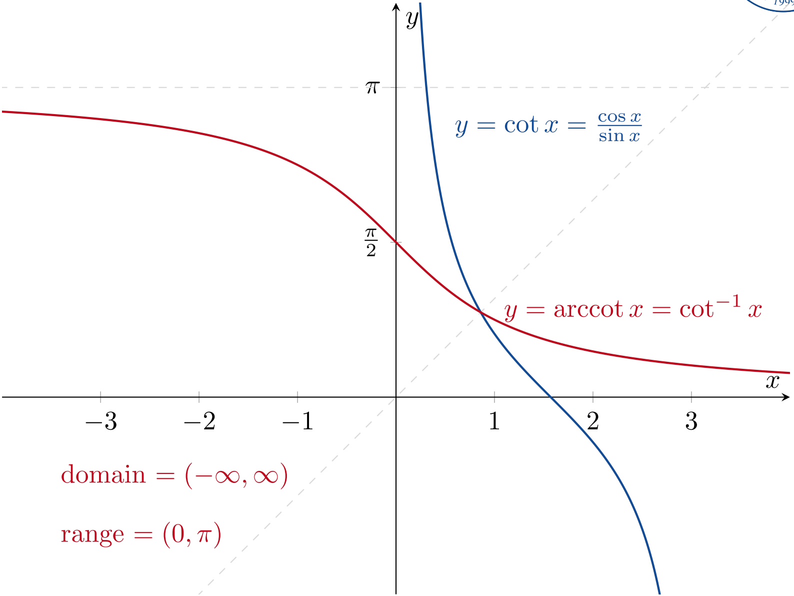
domain $= (-\infty, \infty)$

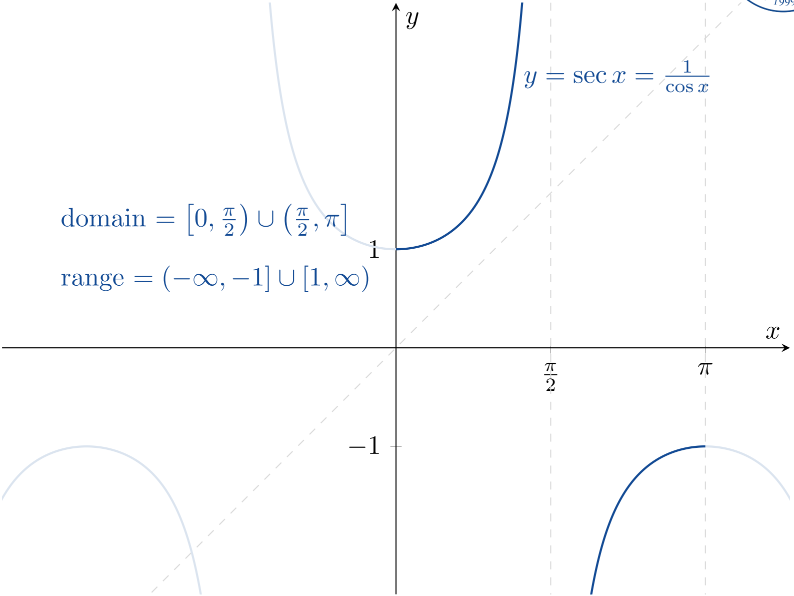
range $= \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

7.6 Inverse Trigonometric Functions



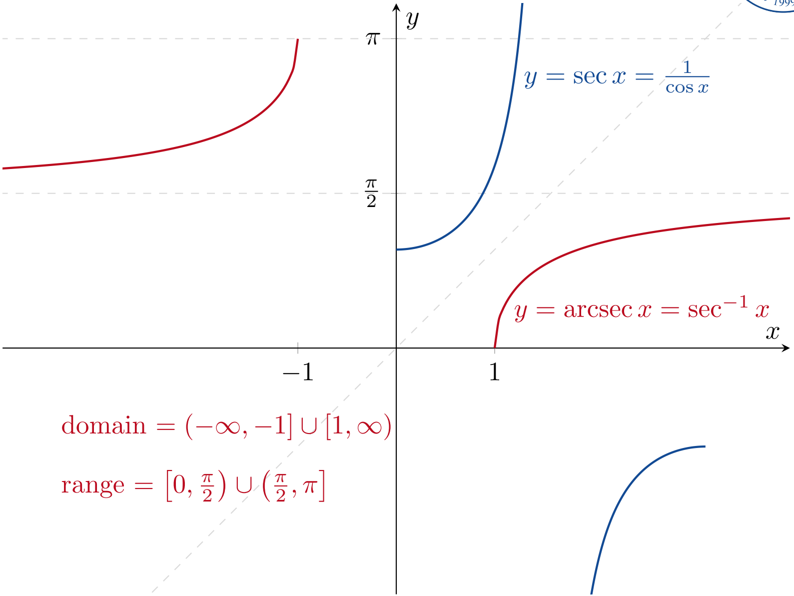
7.6 Inverse Trigonometric Functions





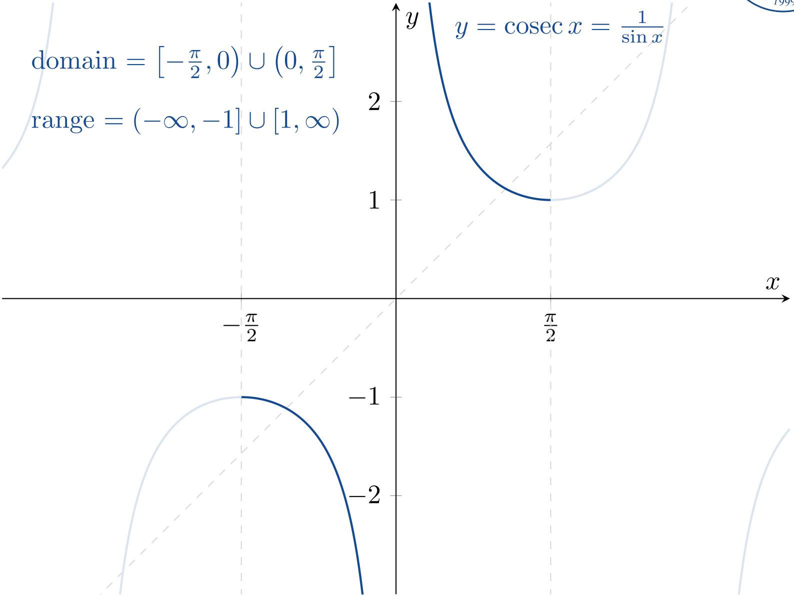
y
$y = \sec x = \frac{1}{\cos x}$
domain $= \left[0, \frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \pi\right]$
range $= (-\infty, -1] \cup [1, \infty)$
1
-1
x
$\frac{\pi}{2}$
π

7.6 Inverse Trigonometric Functions

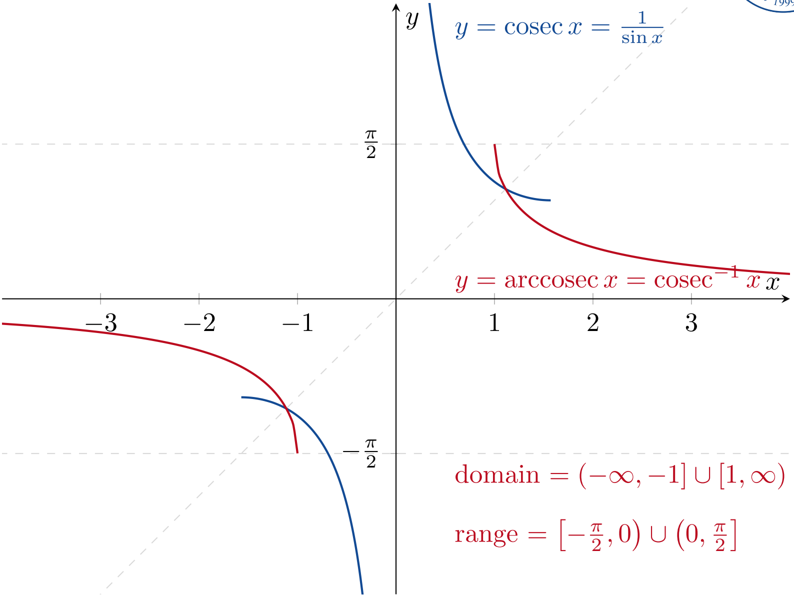


domain $= (-\infty, -1] \cup [1, \infty)$

range $= \left[0, \frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \pi\right]$



domain $= \left[-\frac{\pi}{2}, 0\right) \cup \left(0, \frac{\pi}{2}\right]$
range $= (-\infty, -1] \cup [1, \infty)$
$y = \operatorname{cosec} x = \frac{1}{\sin x}$
y
x
2
1
-1
-2
$-\frac{\pi}{2}$
$\frac{\pi}{2}$



domain $= (-\infty, -1] \cup [1, \infty)$

range $= \left[-\frac{\pi}{2}, 0\right) \cup \left(0, \frac{\pi}{2}\right]$

Arctangent, Arccotangent, Arcsecant and Arccosecant

Definition

- $y = \arctan x$ is the number in $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ for which $\tan y = x$.
- $y = \operatorname{arccot} x$ is the number in $(0, \pi)$ for which $\cot y = x$.
- $y = \operatorname{arcsec} x$ is the number in $\left[0, \frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \pi\right]$ for which $\sec y = x$.
- $y = \operatorname{arccosec} x$ is the number in $\left[-\frac{\pi}{2}, 0\right) \cup \left(0, \frac{\pi}{2}\right]$ for which $\operatorname{cosec} y = x$.

$$\left(f^{-1}\right)'(x) = \frac{1}{f'\left(f^{-1}(x)\right)}$$

The Derivative of $y = \arcsin x$

Let $f(x) = \sin x$ and $f^{-1}(x) = \arcsin x$.

$$\left(f^{-1}\right)'(x) = \frac{1}{f'\left(f^{-1}(x)\right)}$$

The Derivative of $y = \arcsin x$

Let $f(x) = \sin x$ and $f^{-1}(x) = \arcsin x$. Then, if $-1 < x < 1$, we have that

$$\left(f^{-1}\right)'(x) = \frac{1}{f'\left(f^{-1}(x)\right)} = \frac{1}{\cos\left(\arcsin x\right)}$$

$$=$$

$$=$$

$$\left(f^{-1}\right)'(x) = \frac{1}{f'\left(f^{-1}(x)\right)}$$

The Derivative of $y = \arcsin x$

Let $f(x) = \sin x$ and $f^{-1}(x) = \arcsin x$. Then, if $-1 < x < 1$, we have that

$$\begin{aligned}\left(f^{-1}\right)'(x) &= \frac{1}{f'\left(f^{-1}(x)\right)} = \frac{1}{\cos\left(\arcsin x\right)} \\ &= \frac{1}{\sqrt{1-\sin^2\left(\arcsin x\right)}} \\ &\qquad \text{(because } \sin^2\theta + \cos^2\theta = 1\text{)} \\ &= \end{aligned}$$

$$\left(f^{-1}\right)'(x) = \frac{1}{f'\left(f^{-1}(x)\right)}$$

The Derivative of $y = \arcsin x$

Let $f(x) = \sin x$ and $f^{-1}(x) = \arcsin x$. Then, if $-1 < x < 1$, we have that

$$\begin{aligned}\left(f^{-1}\right)'(x) &= \frac{1}{f'\left(f^{-1}(x)\right)} = \frac{1}{\cos\left(\arcsin x\right)} \\ &= \frac{1}{\sqrt{1 - \sin^2\left(\arcsin x\right)}} \\ &\qquad \text{(because } \sin^2\theta + \cos^2\theta = 1\text{)} \\ &= \frac{1}{\sqrt{1-x^2}}. \\ &\qquad \text{(because } \sin\left(\arcsin x\right) = x\text{)}\end{aligned}$$

Theorem

$$\boxed{\frac{d}{dx}\left(\arcsin x\right) = \frac{1}{\sqrt{1-x^2}}, \qquad |x| < 1.}$$

Theorem

$$\boxed{\frac{d}{dx}(\arcsin x) = \frac{1}{\sqrt{1-x^2}}, \qquad |x| < 1.}$$

If $u(x)$ is differentiable and $|u| < 1$, then

$$\frac{d}{dx}(\arcsin u) = \frac{1}{\sqrt{1-u^2}}\frac{du}{dx}.$$

EXAMPLE 4 Using the Chain Rule, we calculate the derivative

$$\frac{d}{dx}(\arcsin x^2) = \frac{1}{\sqrt{1-(x^2)^2}} \cdot \frac{d}{dx}(x^2) = \frac{2x}{\sqrt{1-x^4}}.$$

$$\left(f^{-1}\right)'(x) = \frac{1}{f'\left(f^{-1}(x)\right)}$$

The Derivative of $y = \arctan x$

This time let $f(x) = \tan x$ and $f^{-1}(x) = \arctan x$. Then

$$\left(f^{-1}\right)'(x) = \frac{1}{f'\left(f^{-1}(x)\right)} =$$

$$=$$

$$=$$

$$\boxed{\left(f^{-1}\right)'(x) = \frac{1}{f'\left(f^{-1}(x)\right)}}$$

The Derivative of $y = \arctan x$

This time let $f(x) = \tan x$ and $f^{-1}(x) = \arctan x$. Then

$$\left(f^{-1}\right)'(x) = \frac{1}{f'\left(f^{-1}(x)\right)} = \frac{1}{\sec^2(\arctan x)}$$

$$=$$

$$=$$

$$\left(f^{-1}\right)'(x) = \frac{1}{f'\left(f^{-1}(x)\right)}$$

The Derivative of $y = \arctan x$

This time let $f(x) = \tan x$ and $f^{-1}(x) = \arctan x$. Then

$$\begin{aligned}\left(f^{-1}\right)'(x) &= \frac{1}{f'\left(f^{-1}(x)\right)} = \frac{1}{\sec^2(\arctan x)} \\ &= \frac{1}{1+\tan^2(\arctan x)} \\ &\qquad (\text{because } \sec^2\theta = 1+\tan^2\theta) \\ &= \end{aligned}$$

$$\left(f^{-1}\right)'(x) = \frac{1}{f'\left(f^{-1}(x)\right)}$$

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This time let $f(x) = \tan x$ and $f^{-1}(x) = \arctan x$. Then

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&= \frac{1}{1 + \tan^2(\arctan x)} \\
&\qquad \text{(because } \sec^2\theta = 1 + \tan^2\theta\text{)} \\
&= \frac{1}{1 + x^2}
\end{aligned}$$

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&\qquad (\text{because } \sec^2\theta = 1 + \tan^2\theta) \\
&= \frac{1}{1 + x^2}
\end{aligned}$$

Theorem

$$\boxed{\frac{d}{dx}(\arctan x) = \frac{1}{1 + x^2}.}$$

The Derivative of $y = \operatorname{arcsec} x$

This time we will use Implicit Differentiation.

The Derivative of $y = \operatorname{arcsec} x$

This time we will use Implicit Differentiation.

$$y = \operatorname{arcsec} x$$

$$\sec y = x$$

The Derivative of $y = \operatorname{arcsec} x$

This time we will use Implicit Differentiation.

$$y = \operatorname{arcsec} x$$

$$\frac{d}{dx} \sec y = \frac{d}{dx} x$$

The Derivative of $y = \operatorname{arcsec} x$

This time we will use Implicit Differentiation.

$$
\begin{aligned}
y &= \operatorname{arcsec} x \\
\frac{d}{dx} \sec y &= \frac{d}{dx} x \\
\sec y \tan y \frac{dy}{dx} &= 1
\end{aligned}
$$

The Derivative of $y = \operatorname{arcsec} x$

This time we will use Implicit Differentiation.

$$\begin{aligned} y &= \operatorname{arcsec} x \\ \frac{d}{dx} \sec y &= \frac{d}{dx} x \\ \sec y \tan y \frac{dy}{dx} &= 1 \\ \frac{dy}{dx} &= \frac{1}{\sec y \tan y}. \end{aligned}$$

The Derivative of $y = \operatorname{arcsec} x$

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Next we need to use

$$\sec y = x \qquad \text{and} \qquad \tan y = \pm\sqrt{\sec^2 y - 1} = \pm\sqrt{x^2 - 1}.$$

The Derivative of $y = \operatorname{arcsec} x$

This time we will use Implicit Differentiation.

$$\begin{aligned} y &= \operatorname{arcsec} x \\ \frac{d}{dx} \sec y &= \frac{d}{dx} x \\ \sec y \tan y \frac{dy}{dx} &= 1 \\ \frac{dy}{dx} &= \frac{1}{\sec y \tan y}. \end{aligned}$$

Next we need to use

$$\sec y = x \qquad \text{and} \qquad \tan y = \pm\sqrt{\sec^2 y - 1} = \pm\sqrt{x^2 - 1}.$$

So

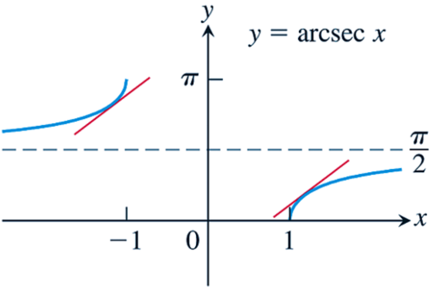
$$\frac{dy}{dx} = \pm\frac{1}{x\sqrt{x^2 - 1}}.$$

$$\frac{d}{dx}\operatorname{arcsec} x = \pm\frac{1}{x\sqrt{x^2-1}}.$$

What can we do about the $\pm$ sign?

$$\frac{d}{dx} \operatorname{arcsec} x = \pm \frac{1}{x\sqrt{x^2 - 1}}.$$

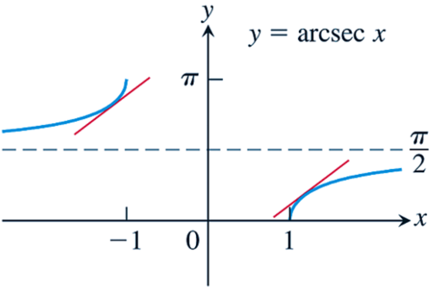
What can we do about the $\pm$ sign?



Note that $\frac{d}{dx} \operatorname{arcsec} x$ is always positive.

$$\frac{d}{dx}\operatorname{arcsec} x = \pm\frac{1}{x\sqrt{x^2-1}}.$$

What can we do about the $\pm$ sign?



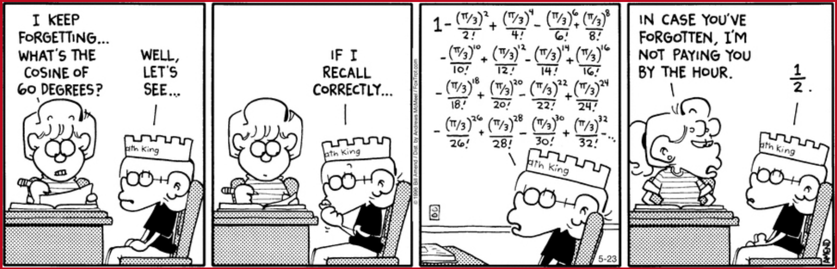
Note that $\frac{d}{dx}\operatorname{arcsec} x$ is always positive. We can replace the $\pm\frac{1}{x}$ by $\frac{1}{|x|}$.

Theorem

$$\boxed{\frac{d}{dx}(\operatorname{arcsec} x) = \frac{1}{|x|\sqrt{x^2-1}}, \qquad |x| > 1.}$$

Break

We will continue at 3pm



Derivatives of the Other Three Inverse Trigonometric Functions

We can find the derivatives of $\arccos x$, $\operatorname{arccot} x$ and $\operatorname{arccosec} x$ by using the identities

$$\begin{aligned} \arccos x &= \frac{\pi}{2} - \arcsin x \\ \operatorname{arccot} x &= \frac{\pi}{2} - \arctan x \\ \operatorname{arccosec} x &= \frac{\pi}{2} - \operatorname{arcsec} x. \end{aligned}$$

(I have proved the first one. The others can be derived in similar ways.)

$$\arccos x = \frac{\pi}{2} - \arcsin x \qquad \frac{d}{dx}(\arcsin x) = \frac{1}{\sqrt{1-x^2}}$$

For example, we can easily calculate that

$$\frac{d}{dx}\arccos x = \frac{d}{dx}\left(\frac{\pi}{2} - \arcsin x\right) = -\frac{1}{\sqrt{1-x^2}}.$$

Derivative Formulae

- $\dfrac{d}{dx}\arcsin x = \dfrac{1}{\sqrt{1-x^2}}, \ |x| < 1$
- $\dfrac{d}{dx}\arccos x = \dfrac{-1}{\sqrt{1-x^2}}, \ |x| < 1$

Derivative Formulae

- $\dfrac{d}{dx} \arcsin x = \dfrac{1}{\sqrt{1-x^2}}, \ |x| < 1$
- $\dfrac{d}{dx} \arccos x = \dfrac{-1}{\sqrt{1-x^2}}, \ |x| < 1$
- $\dfrac{d}{dx} \arctan x = \dfrac{1}{1+x^2}$
- $\dfrac{d}{dx} \operatorname{arccot} x = \dfrac{-1}{1+x^2}$

Derivative Formulae

- $\dfrac{d}{dx}\arcsin x = \dfrac{1}{\sqrt{1-x^2}}, \ |x| < 1$
- $\dfrac{d}{dx}\arccos x = \dfrac{-1}{\sqrt{1-x^2}}, \ |x| < 1$
- $\dfrac{d}{dx}\arctan x = \dfrac{1}{1+x^2}$
- $\dfrac{d}{dx}\operatorname{arccot} x = \dfrac{-1}{1+x^2}$
- $\dfrac{d}{dx}\operatorname{arcsec} x = \dfrac{1}{|x|\sqrt{x^2-1}}, \ |x| > 1$
- $\dfrac{d}{dx}\operatorname{arccosec} x = \dfrac{-1}{|x|\sqrt{x^2-1}}, \ |x| > 1$

7.6 Inverse Trigonometric Functions

Derivative Formulae

- $$\frac{d}{dx}\arcsin x = \frac{1}{\sqrt{1-x^2}},\ |x|<1$$
- $$\frac{d}{dx}\arccos x = \frac{-1}{\sqrt{1-x^2}},\ |x|<1$$
- $$\frac{d}{dx}\arctan x = \frac{1}{1+x^2}$$
- $$\frac{d}{dx}\operatorname{arccot} x = \frac{-1}{1+x^2}$$
- $$\frac{d}{dx}\operatorname{arcsec} x = \frac{1}{|x|\sqrt{x^2-1}},\ |x|>1$$
- $$\frac{d}{dx}\operatorname{arccosec} x = \frac{-1}{|x|\sqrt{x^2-1}},\ |x|>1$$

Integral Formulae

- $$\int \frac{dx}{\sqrt{a^2-x^2}} = \arcsin\left(\frac{x}{a}\right) + C$$

(valid for $x^2 < a^2$)

Derivative Formulae

- $$\frac{d}{dx}\arcsin x = \frac{1}{\sqrt{1-x^2}}, \ |x| < 1$$
- $$\frac{d}{dx}\arccos x = \frac{-1}{\sqrt{1-x^2}}, \ |x| < 1$$
- $$\frac{d}{dx}\arctan x = \frac{1}{1+x^2}$$
- $$\frac{d}{dx}\operatorname{arccot} x = \frac{-1}{1+x^2}$$
- $$\frac{d}{dx}\operatorname{arcsec} x = \frac{1}{|x|\sqrt{x^2-1}}, \ |x| > 1$$
- $$\frac{d}{dx}\operatorname{arccosec} x = \frac{-1}{|x|\sqrt{x^2-1}}, \ |x| > 1$$

Integral Formulae

- $$\int \frac{dx}{\sqrt{a^2-x^2}} = \arcsin\left(\frac{x}{a}\right) + C$$
  (valid for $x^2 < a^2$)
- $$\int \frac{dx}{a^2+x^2} = \frac{1}{a}\arctan\left(\frac{x}{a}\right) + C$$
  (valid for all x)

7.6 Inverse Trigonometric Functions

Derivative Formulae

- $\dfrac{d}{dx}\arcsin x = \dfrac{1}{\sqrt{1-x^2}}, \; |x| < 1$
- $\dfrac{d}{dx}\arccos x = \dfrac{-1}{\sqrt{1-x^2}}, \; |x| < 1$
- $\dfrac{d}{dx}\arctan x = \dfrac{1}{1+x^2}$
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- $\dfrac{d}{dx}\operatorname{arcsec} x = \dfrac{1}{|x|\sqrt{x^2-1}}, \; |x| > 1$
- $\dfrac{d}{dx}\operatorname{arccosec} x = \dfrac{-1}{|x|\sqrt{x^2-1}}, \; |x| > 1$

Integral Formulae

- $\displaystyle\int \frac{dx}{\sqrt{a^2-x^2}} = \arcsin\left(\frac{x}{a}\right) + C$

  (valid for $x^2 < a^2$)
- $\displaystyle\int \frac{dx}{a^2+x^2} = \frac{1}{a}\arctan\left(\frac{x}{a}\right) + C$

  (valid for all x)
- $\displaystyle\int \frac{dx}{|x|\sqrt{x^2-a^2}} = \frac{1}{a}\operatorname{arcsec}\left|\frac{x}{a}\right| + C$

  (valid for $|x| > a > 0$)

$$\int \frac{dx}{\sqrt{a^2 - x^2}} = \arcsin\left(\frac{x}{a}\right) + C$$

Example

Find $\displaystyle\int_{\frac{\sqrt{2}}{2}}^{\frac{\sqrt{3}}{2}} \frac{dx}{\sqrt{1-x^2}}$.

$$\int \frac{dx}{\sqrt{a^2 - x^2}} = \arcsin\left(\frac{x}{a}\right) + C$$

Example

Find $\displaystyle\int_{\frac{\sqrt{2}}{2}}^{\frac{\sqrt{3}}{2}} \frac{dx}{\sqrt{1-x^2}}$.

$$\int_{\frac{\sqrt{2}}{2}}^{\frac{\sqrt{3}}{2}} \frac{dx}{\sqrt{1-x^2}} = \Big[\arcsin x\Big]_{\frac{\sqrt{2}}{2}}^{\frac{\sqrt{3}}{2}} = \ldots = \frac{\pi}{12}.$$

$$\int \frac{dx}{\sqrt{a^2 - x^2}} = \arcsin\left(\frac{x}{a}\right) + C$$

Example

Find $\displaystyle\int \frac{dx}{\sqrt{3 - 4x^2}}$.

$$\int \frac{dx}{\sqrt{a^2 - x^2}} = \arcsin\left(\frac{x}{a}\right) + C$$

Example

Find $\displaystyle\int \frac{dx}{\sqrt{3 - 4x^2}}$.

First we do a substitution: Let $u = 2x$. Then

$$\int \frac{dx}{\sqrt{3 - 4x^2}} = \frac{1}{2} \int \frac{du}{\sqrt{3 - u^2}}.$$

Look at the yellow box at the top: We have $a = \sqrt{3}$.

$$\int \frac{dx}{\sqrt{a^2 - x^2}} = \arcsin\left(\frac{x}{a}\right) + C$$

Example

Find $\displaystyle\int \frac{dx}{\sqrt{3 - 4x^2}}$.

First we do a substitution: Let $u = 2x$. Then

$$\int \frac{dx}{\sqrt{3 - 4x^2}} = \frac{1}{2} \int \frac{du}{\sqrt{3 - u^2}}.$$

Look at the yellow box at the top: We have $a = \sqrt{3}$. So

$$\begin{aligned}\int \frac{dx}{\sqrt{3 - 4x^2}} = \frac{1}{2} \int \frac{du}{\sqrt{3 - u^2}} &= \frac{1}{2} \arcsin\left(\frac{u}{a}\right) + C \\ &= \frac{1}{2} \arcsin\left(\frac{2x}{\sqrt{3}}\right) + C.\end{aligned}$$

$$\int \frac{dx}{|x|\sqrt{x^2-a^2}} = \frac{1}{a}\operatorname{arcsec}\left|\frac{x}{a}\right| + C$$

Example

Find $\displaystyle\int \frac{dx}{\sqrt{e^{2x}-6}}$.

Let $u = e^x$. Then $du = e^x\,dx = u\,dx$ and $dx = \frac{du}{u}$.

$$\int \frac{dx}{|x|\sqrt{x^2-a^2}} = \frac{1}{a}\operatorname{arcsec}\left|\frac{x}{a}\right| + C$$

Example

Find $\displaystyle\int \frac{dx}{\sqrt{e^{2x}-6}}$.

Let $u = e^x$. Then $du = e^x\,dx = u\,dx$ and $dx = \frac{du}{u}$. Therefore

$$\int \frac{dx}{\sqrt{e^{2x}-6}} = \int \frac{\frac{du}{u}}{\sqrt{u^2-6}} = \int \frac{du}{u\sqrt{u^2-6}} \qquad (a=\sqrt{6})$$

$$=$$

$$=$$

$$\int \frac{dx}{|x|\sqrt{x^2-a^2}} = \frac{1}{a}\,\mathrm{arcsec}\left|\frac{x}{a}\right| + C$$

Example

Find $\displaystyle\int \frac{dx}{\sqrt{e^{2x}-6}}$.

Let $u = e^x$. Then $du = e^x\,dx = u\,dx$ and $dx = \frac{du}{u}$. Therefore

$$\int \frac{dx}{\sqrt{e^{2x}-6}} = \int \frac{\frac{du}{u}}{\sqrt{u^2-6}} = \int \frac{du}{u\sqrt{u^2-6}} \qquad (a=\sqrt{6})$$

$$= \frac{1}{\sqrt{6}}\,\mathrm{arcsec}\left|\frac{u}{\sqrt{6}}\right| + C$$

$$=$$

$$\int \frac{dx}{|x|\sqrt{x^2-a^2}} = \frac{1}{a}\,\mathrm{arcsec}\left|\frac{x}{a}\right| + C$$

Example

Find $\displaystyle\int \frac{dx}{\sqrt{e^{2x}-6}}$.

Let $u = e^x$. Then $du = e^x\,dx = u\,dx$ and $dx = \frac{du}{u}$. Therefore

$$\begin{aligned}\int \frac{dx}{\sqrt{e^{2x}-6}} &= \int \frac{\frac{du}{u}}{\sqrt{u^2-6}} = \int \frac{du}{u\sqrt{u^2-6}} \qquad (a=\sqrt{6})\\ &= \frac{1}{\sqrt{6}}\,\mathrm{arcsec}\left|\frac{u}{\sqrt{6}}\right| + C\\ &= \frac{1}{\sqrt{6}}\,\mathrm{arcsec}\left|\frac{e^x}{\sqrt{6}}\right| + C.\end{aligned}$$

$$\int \frac{dx}{\sqrt{a^2 - x^2}} = \arcsin\left(\frac{x}{a}\right) + C$$

Example

Find $\displaystyle\int \frac{dx}{\sqrt{4x - x^2}}$.

Since $\sqrt{4x - x^2}$ doesn't match any of these three integration formulae, we must first rewrite this.

$$\int \frac{dx}{\sqrt{a^2 - x^2}} = \arcsin\left(\frac{x}{a}\right) + C$$

Example

Find $\displaystyle\int \frac{dx}{\sqrt{4x - x^2}}$.

Since $\sqrt{4x - x^2}$ doesn't match any of these three integration formulae, we must first rewrite this.

$$x^2 - 4x = x^2 - 4x + 4 - 4 = x^2 - 4x + 4 - 4 = (x-2)^2 - 4.$$

$$\int \frac{dx}{\sqrt{a^2 - x^2}} = \arcsin\left(\frac{x}{a}\right) + C$$

Example

Find $\displaystyle\int \frac{dx}{\sqrt{4x - x^2}}$.

Since $\sqrt{4x - x^2}$ doesn't match any of these three integration formulae, we must first rewrite this.

$$x^2 - 4x = x^2 - 4x + 4 - 4 = x^2 - 4x + 4 - 4 = (x-2)^2 - 4.$$

So then we have

$$\int \frac{dx}{\sqrt{4x - x^2}} = \int \frac{dx}{\sqrt{4 - (x-2)^2}}$$

$$=$$

$$=$$

$$\int \frac{dx}{\sqrt{a^2 - x^2}} = \arcsin\left(\frac{x}{a}\right) + C$$

Example

Find $\displaystyle\int \frac{dx}{\sqrt{4x - x^2}}$.

Since $\sqrt{4x - x^2}$ doesn't match any of these three integration formulae, we must first rewrite this.

$$x^2 - 4x = x^2 - 4x + 4 - 4 = x^2 - 4x + 4 - 4 = (x-2)^2 - 4.$$

So then we have

$$\begin{aligned} \int \frac{dx}{\sqrt{4x - x^2}} &= \int \frac{dx}{\sqrt{4 - (x-2)^2}} \\ &= \int \frac{du}{\sqrt{a^2 - u^2}} \qquad (u = x - 2,\ a = 2) \\ &= \ldots \end{aligned}$$

$$\int \frac{dx}{a^2+x^2} = \frac{1}{a}\arctan\left(\frac{x}{a}\right) + C$$

Example

Find $\displaystyle\int \frac{dx}{4x^2+4x+2}$.

Again we need to start by completing the square.

$$\int \frac{dx}{a^2+x^2} = \frac{1}{a}\arctan\left(\frac{x}{a}\right) + C$$

Example

Find $\displaystyle\int \frac{dx}{4x^2+4x+2}$.

Again we need to start by completing the square.

$$\begin{aligned} 4x^2+4x+2 = 4(x^2+x)+2 &= 4\left(x^2+x+\frac{1}{4}-\frac{1}{4}\right)+2 \\ &= 4\left(x^2+x+\frac{1}{4}\right)+1 = 4\left(x+\frac{1}{2}\right)^2+1 \\ &= (2x+1)^2+1. \end{aligned}$$

$$\int \frac{dx}{a^2 + x^2} = \frac{1}{a} \arctan\left(\frac{x}{a}\right) + C$$

So we need to calculate

$$\int \frac{dx}{(2x+1)^2 + 1}$$

What does that look like?

$$\int \frac{dx}{a^2+x^2} = \frac{1}{a}\arctan\left(\frac{x}{a}\right) + C$$

So we need to calculate

$$\int \frac{dx}{(2x+1)^2+1}$$

What does that look like? Let $a = 1$ and $u = (2x+1)$. Then we have

$$\int \frac{dx}{(2x+1)^2+1} = \frac{1}{2}\int \frac{du}{u^2+a^2}$$

$$=$$

$$=$$

$$\int \frac{dx}{a^2+x^2} = \frac{1}{a}\arctan\left(\frac{x}{a}\right) + C$$

So we need to calculate

$$\int \frac{dx}{(2x+1)^2+1}$$

What does that look like? Let $a = 1$ and $u = (2x+1)$. Then we have

$$\begin{aligned}\int \frac{dx}{(2x+1)^2+1} &= \frac{1}{2}\int \frac{du}{u^2+a^2} \\ &= \frac{1}{2}\frac{1}{a}\arctan\left(\frac{u}{a}\right) + C \\ &= \end{aligned}$$

$$\int \frac{dx}{a^2+x^2} = \frac{1}{a}\arctan\left(\frac{x}{a}\right) + C$$

So we need to calculate

$$\int \frac{dx}{(2x+1)^2+1}$$

What does that look like? Let $a = 1$ and $u = (2x+1)$. Then we have

$$\begin{aligned}\int \frac{dx}{(2x+1)^2+1} &= \frac{1}{2}\int \frac{du}{u^2+a^2} \\ &= \frac{1}{2}\frac{1}{a}\arctan\left(\frac{u}{a}\right) + C \\ &= \frac{1}{2}\arctan(2x+1) + C.\end{aligned}$$

7.7 Hyperbolic Functions

The hyperbolic sine and hyperbolic cosine functions are defined by

$$\boxed{\sinh x = \frac{e^x - e^{-x}}{2}} \qquad \text{and} \qquad \boxed{\cosh x = \frac{e^x + e^{-x}}{2}}$$

7.7 Hyperbolic Functions

The hyperbolic sine and hyperbolic cosine functions are defined by

$$\boxed{\sinh x = \frac{e^x - e^{-x}}{2}} \qquad \text{and} \qquad \boxed{\cosh x = \frac{e^x + e^{-x}}{2}}$$

If you forget which is which, try to remember

$$\sinh 0 = 0 = \sin 0$$

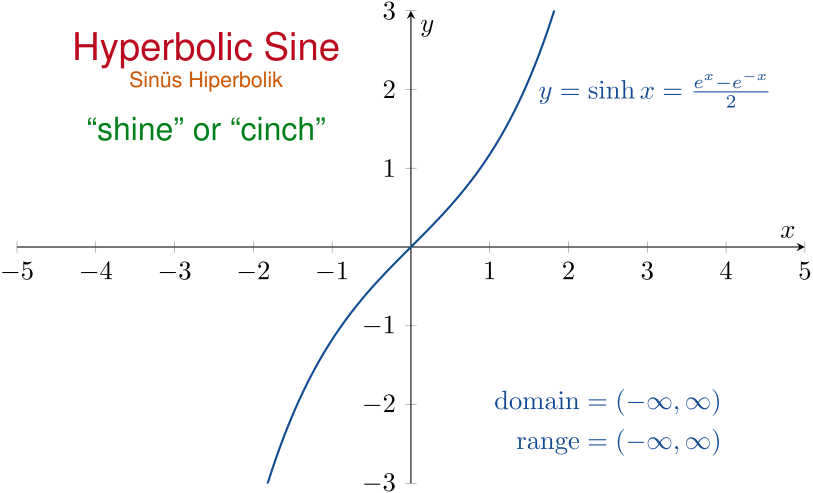
and

$$\cosh 0 = 1 = \cos 0.$$

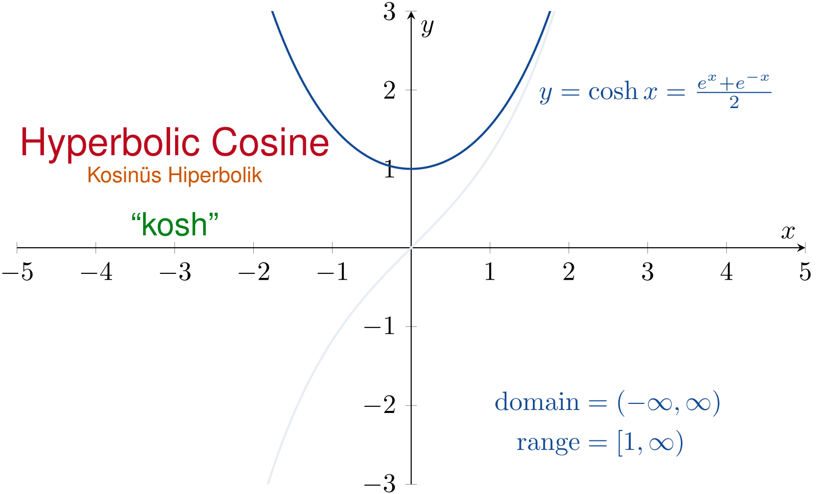
Hyperbolic Sine

Sinüs Hiperbolik

"shine" or "cinch"



7.7 Hyperbolic Functions



Hyperbolic Cosine

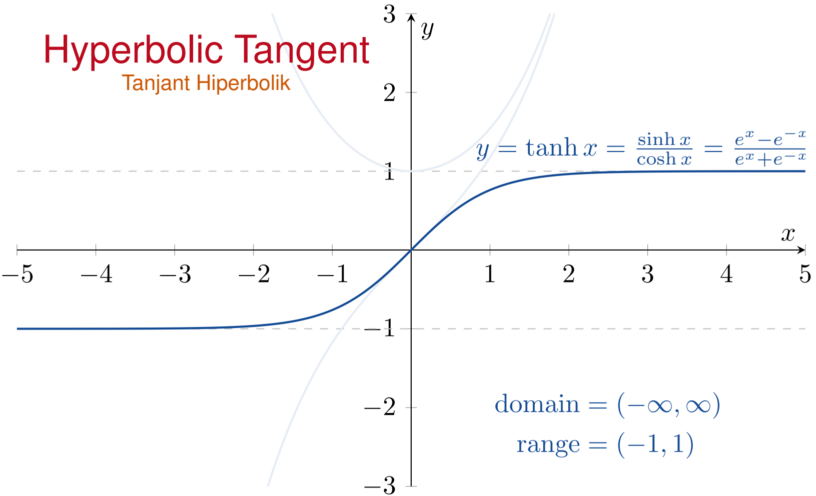
Kosinüs Hiperbolik

"kosh"

$$y = \cosh x = \frac{e^x + e^{-x}}{2}$$

$$\text{domain} = (-\infty, \infty)$$

$$\text{range} = [1, \infty)$$

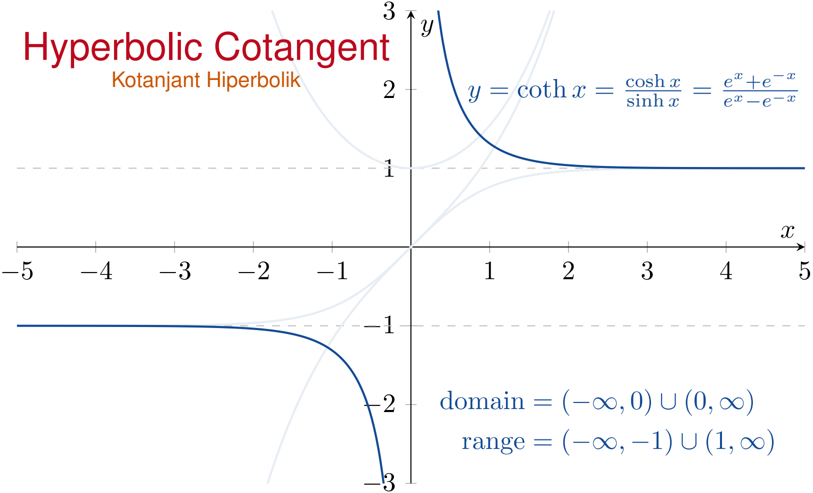


Hyperbolic Tangent
Tanjant Hiperbolik
$y = \tanh x = \frac{\sinh x}{\cosh x} = \frac{e^x - e^{-x}}{e^x + e^{-x}}$
domain $= (-\infty, \infty)$
range $= (-1, 1)$
x
y
−5
−4
−3
−2
−1
1
2
3
4
5
3
2
1
−1
−2
−3

Hyperbolic Cotangent

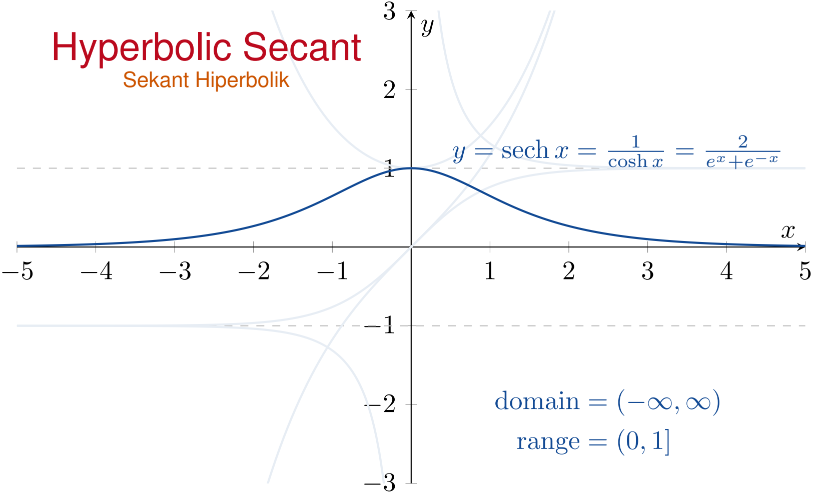
Kotanjant Hiperbolik

$$y = \coth x = \frac{\cosh x}{\sinh x} = \frac{e^x + e^{-x}}{e^x - e^{-x}}$$



$$\text{domain} = (-\infty, 0) \cup (0, \infty)$$

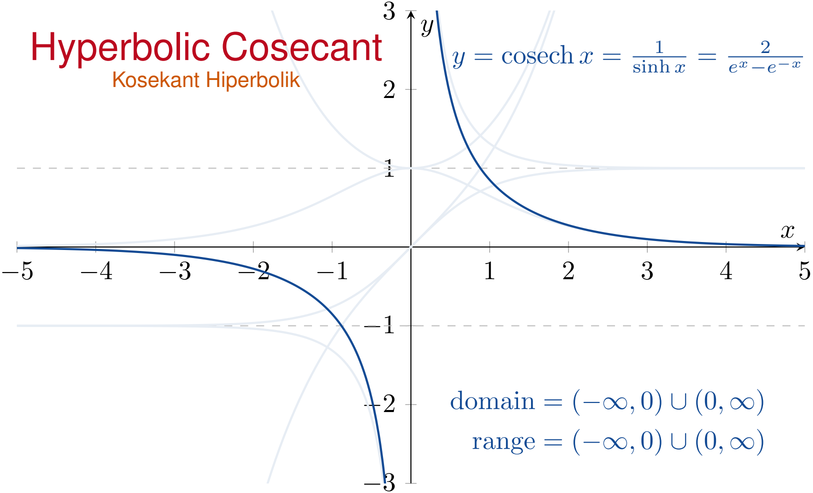
$$\text{range} = (-\infty, -1) \cup (1, \infty)$$


Hyperbolic Secant
Sekant Hiperbolik
$y = \operatorname{sech} x = \frac{1}{\cosh x} = \frac{2}{e^x + e^{-x}}$
domain $= (-\infty, \infty)$
range $= (0, 1]$
x
y
−5 −4 −3 −2 −1 1 2 3 4 5
3 2 1 −1 −2 −3

Hyperbolic Cosecant

Kosekant Hiperbolik

$$y = \operatorname{cosech} x = \frac{1}{\sinh x} = \frac{2}{e^x - e^{-x}}$$



$$\text{domain} = (-\infty, 0) \cup (0, \infty)$$

$$\text{range} = (-\infty, 0) \cup (0, \infty)$$

Identities

$$\cosh^2 x - \sinh^2 u = \left(\frac{e^x + e^{-x}}{2}\right)^2 - \left(\frac{e^x - e^{-x}}{2}\right)^2$$

$$=$$

Identities

$$
\begin{aligned}
\cosh^2 x - \sinh^2 u &= \left(\frac{e^x + e^{-x}}{2}\right)^2 - \left(\frac{e^x - e^{-x}}{2}\right)^2 \\
&= \frac{1}{4}\left(e^{2x} + 2 + e^{-2x} - e^{2x} + 2 - e^{-2x}\right) = 1.
\end{aligned}
$$

TABLE 7.6 Identities for hyperbolic functions

$$\cosh^2 x - \sinh^2 x = 1$$

$$\sinh 2x = 2 \sinh x \cosh x$$

$$\cosh 2x = \cosh^2 x + \sinh^2 x$$

$$\cosh^2 x = \frac{\cosh 2x + 1}{2}$$

$$\sinh^2 x = \frac{\cosh 2x - 1}{2}$$

$$\tanh^2 x = 1 - \operatorname{sech}^2 x$$

$$\coth^2 x = 1 + \operatorname{csch}^2 x$$

Derivatives and Integrals of Hyperbolic Functions

We can calculate that

$$\frac{d}{dx}\sinh x =$$

and

$$\frac{d}{dx}\operatorname{cosech} x =$$

Derivatives and Integrals of Hyperbolic Functions

We can calculate that

$$\frac{d}{dx}\sinh x = \frac{d}{dx}\left(\frac{e^x - e^{-x}}{2}\right)$$

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Derivatives and Integrals of Hyperbolic Functions

We can calculate that

$$\frac{d}{dx}\sinh x = \frac{d}{dx}\left(\frac{e^x - e^{-x}}{2}\right) = \frac{e^x + e^{-x}}{2} = \cosh x$$

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and

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Derivatives and Integrals of Hyperbolic Functions

We can calculate that

$$\frac{d}{dx}\sinh x = \frac{d}{dx}\left(\frac{e^x - e^{-x}}{2}\right) = \frac{e^x + e^{-x}}{2} = \cosh x$$

and

$$\begin{aligned}\frac{d}{dx}\operatorname{cosech} x = \frac{d}{dx}\left(\frac{1}{\sinh x}\right) = -\frac{\cosh x}{\sinh^2 x} = -\frac{1}{\sinh x}\frac{\cosh x}{\sinh x} \\ = \operatorname{cosech} x \coth x.\end{aligned}$$

Derivative Formulae

- $$\frac{d}{dx}\sinh x = \cosh x$$
- $$\frac{d}{dx}\cosh x = \sinh x$$
- $$\frac{d}{dx}\tanh x = \operatorname{sech}^2 x$$
- $$\frac{d}{dx}\coth x = -\operatorname{cosech}^2 x$$
- $$\frac{d}{dx}\operatorname{sech} x = -\operatorname{sech} x \tanh x$$
- $$\frac{d}{dx}\operatorname{cosech} x = -\operatorname{cosech} x \coth x$$

Derivative Formulae

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- $\frac{d}{dx}\operatorname{cosech} x = -\operatorname{cosech} x \coth x$

- $\frac{d}{dx}\sin x = \cos x$
- $\frac{d}{dx}\cos x = -\sin x$
- $\frac{d}{dx}\tan x = \sec^2 x$
- $\frac{d}{dx}\cot x = -\operatorname{cosec}^2 x$
- $\frac{d}{dx}\sec x = +\sec x \tan x$
- $\frac{d}{dx}\operatorname{cosec} x = -\operatorname{cosec} x \cot x$

7.7 Hyperbolic Functions

Derivative Formulae

- $\dfrac{d}{dx}\sinh x = \cosh x$
- $\dfrac{d}{dx}\cosh x = \sinh x$
- $\dfrac{d}{dx}\tanh x = \operatorname{sech}^2 x$
- $\dfrac{d}{dx}\coth x = -\operatorname{cosech}^2 x$
- $\dfrac{d}{dx}\operatorname{sech} x = -\operatorname{sech} x \tanh x$
- $\dfrac{d}{dx}\operatorname{cosech} x = -\operatorname{cosech} x \coth x$

Integral Formulae

- $\displaystyle\int \sinh x \, dx = \cosh x + C$
- $\displaystyle\int \cosh x \, dx = \sinh x + C$
- $\displaystyle\int \operatorname{sech}^2 x \, dx = \tanh x + C$
- $\displaystyle\int \operatorname{cosech}^2 x \, dx = -\coth x + C$
- $\displaystyle\int \operatorname{sech} x \tanh x \, dx = -\operatorname{sech} x + C$
- $\displaystyle\int \operatorname{cosech} x \coth x \, dx = -\operatorname{cosech} x + C$

$$\frac{d}{dx}\tanh x = \operatorname{sech}^2 x$$

Example

Differentiate $\tanh\sqrt{1+t^2}$.

$$\frac{d}{dx}\tanh x = \operatorname{sech}^2 x$$

Example

Differentiate $\tanh\sqrt{1+t^2}$.

$$\begin{aligned}\frac{d}{dt}\tanh\sqrt{1+t^2} &= \operatorname{sech}^2\sqrt{1+t^2}\,\frac{d}{dt}\sqrt{1+t^2}\\ &= \frac{t}{\sqrt{1+t^2}}\operatorname{sech}^2\sqrt{1+t^2}.\end{aligned}$$

(b) $$\int \coth 5x\,dx = \int \frac{\cosh 5x}{\sinh 5x}\,dx = \frac{1}{5}\int \frac{du}{u}$$

$u = \sinh 5x,$
$du = 5\cosh 5x\,dx$

$$= \frac{1}{5}\ln|u| + C = \frac{1}{5}\ln|\sinh 5x| + C$$

(c) $\displaystyle\int_0^1 \sinh^2 x\, dx = \int_0^1 \frac{\cosh 2x - 1}{2} dx$

$$= \frac{1}{2}\int_0^1 (\cosh 2x - 1)\, dx = \frac{1}{2}\left[\frac{\sinh 2x}{2} - x\right]_0^1$$

$$= \frac{\sinh 2}{4} - \frac{1}{2} \approx 0.40672$$

(d) $$\int_0^{\ln 2} 4e^x \sinh x\, dx = \int_0^{\ln 2} 4e^x \frac{e^x - e^{-x}}{2} dx = \int_0^{\ln 2} (2e^{2x} - 2)\, dx$$

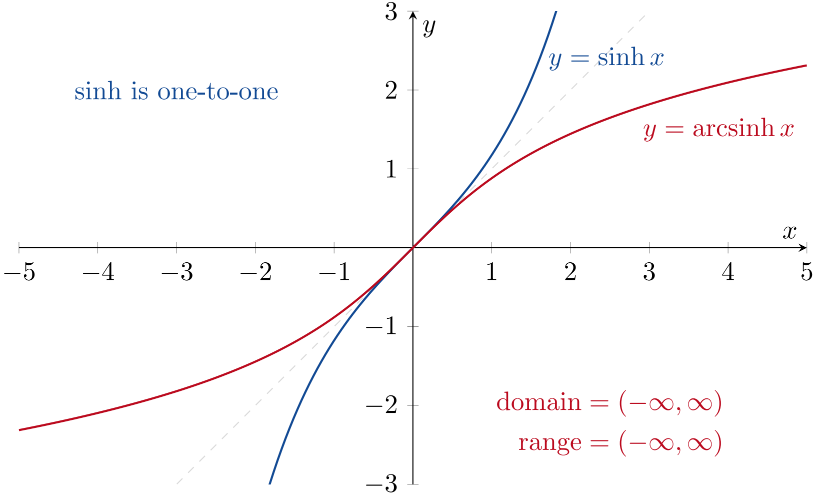
$$= \left[e^{2x} - 2x \right]_0^{\ln 2} = (e^{2 \ln 2} - 2 \ln 2) - (1 - 0)$$

$$= 4 - 2 \ln 2 - 1 \approx 1.6137$$

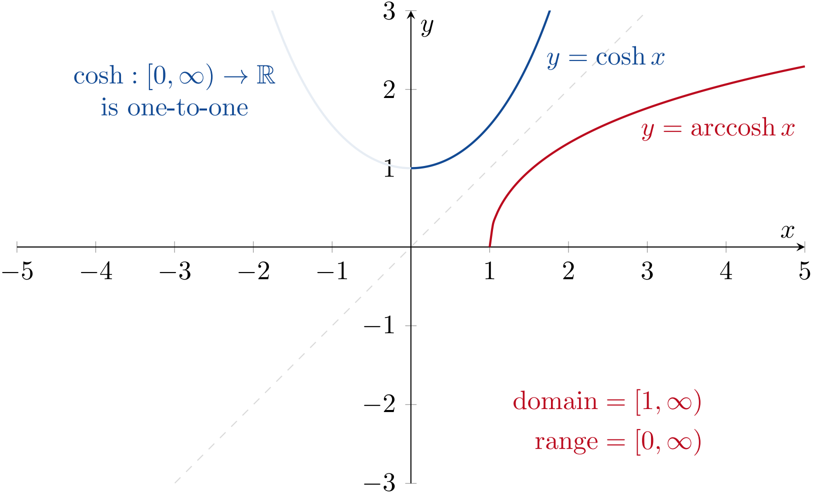
Inverse Hyperbolic Functions

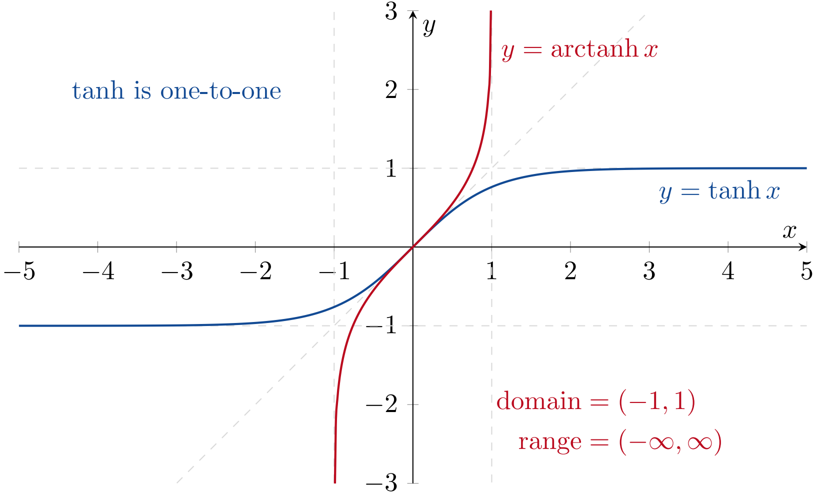
Now it is time to talk about the inverse functions.

sinh, tanh, coth and cosech are all one-to-one functions so have inverses. For cosh and sech we will need to restrict the domain before we can find the inverse.


sinh is one-to-one
$y = \sinh x$
$y = \operatorname{arcsinh} x$
domain $= (-\infty, \infty)$
range $= (-\infty, \infty)$
x
y
−5 −4 −3 −2 −1 1 2 3 4 5
3 2 1 −1 −2 −3

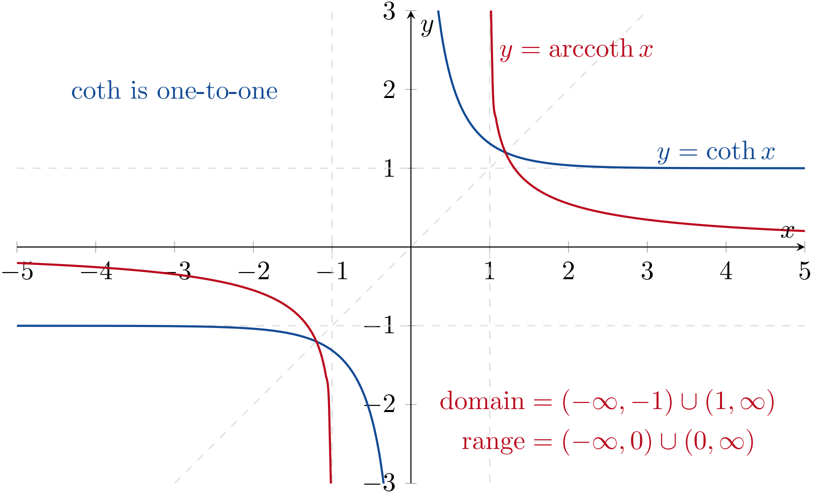
7.7 Hyperbolic Functions

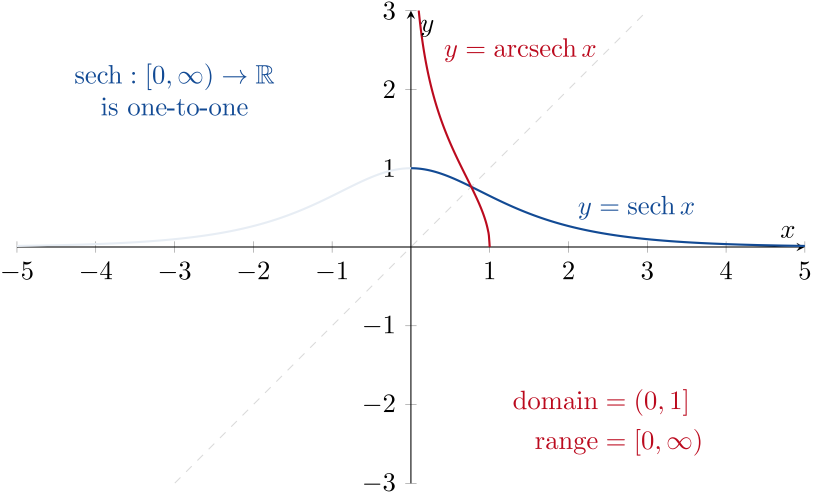




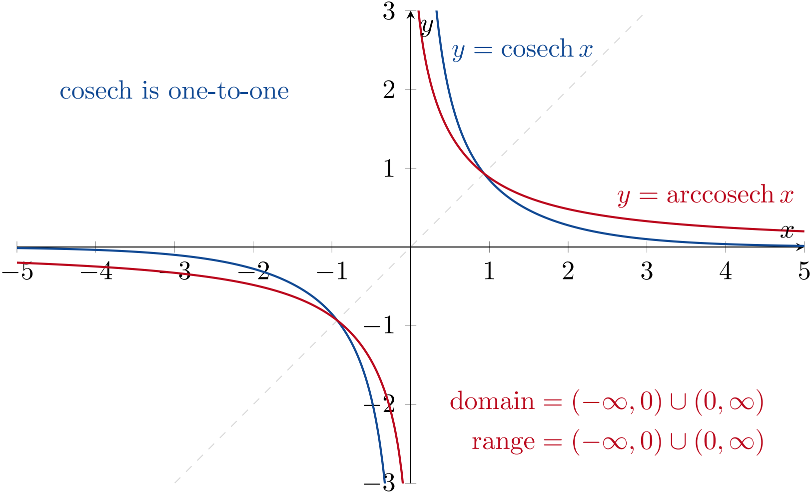
tanh is one-to-one
$y = \operatorname{arctanh} x$
$y = \tanh x$
y
x
3
2
1
−1
−2
−3
−5
−4
−3
−2
−1
1
2
3
4
5
domain $= (-1, 1)$
range $= (-\infty, \infty)$

7.7 Hyperbolic Functions





$\text{sech} : [0, \infty) \to \mathbb{R}$
is one-to-one
$y = \text{arcsech}\, x$
$y = \text{sech}\, x$
x
y
-5 -4 -3 -2 -1 1 2 3 4 5
3 2 1 -1 -2 -3
domain $= (0, 1]$
range $= [0, \infty)$



$y = \operatorname{cosech} x$
cosech is one-to-one
$y = \operatorname{arccosech} x$
domain $= (-\infty, 0) \cup (0, \infty)$
range $= (-\infty, 0) \cup (0, \infty)$

Useful Identities

Note that

$$\operatorname{sech}\left(\cosh^{-1}\left(\frac{1}{x}\right)\right) = \frac{1}{\cosh\left(\cosh^{-1}\left(\frac{1}{x}\right)\right)} = \frac{1}{\left(\frac{1}{x}\right)} = x.$$

Useful Identities

Note that

$$\operatorname{sech}\left(\cosh^{-1}\left(\frac{1}{x}\right)\right) = \frac{1}{\cosh\left(\cosh^{-1}\left(\frac{1}{x}\right)\right)} = \frac{1}{\left(\frac{1}{x}\right)} = x.$$

Taking sech^{-1} of both sides gives

$$\boxed{\cosh^{-1}\left(\frac{1}{x}\right) = \operatorname{sech}^{-1} x.}$$

Useful Identities

Note that

$$\operatorname{sech}\left(\cosh^{-1}\left(\frac{1}{x}\right)\right) = \frac{1}{\cosh\left(\cosh^{-1}\left(\frac{1}{x}\right)\right)} = \frac{1}{\left(\frac{1}{x}\right)} = x.$$

Taking sech^{-1} of both sides gives

$$\boxed{\cosh^{-1}\left(\frac{1}{x}\right) = \operatorname{sech}^{-1} x.}$$

Similarly

$$\boxed{\operatorname{cosech}^{-1} x = \sinh^{-1}\left(\frac{1}{x}\right)} \qquad \text{and} \qquad \boxed{\coth^{-1} x = \tanh^{-1}\left(\frac{1}{x}\right).}$$

(This one is not in the textbook)

Next I want to find a formula for $\operatorname{arcsinh} x = \sinh^{-1} x$.

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$$\begin{aligned} \sinh^{-1} x &= y \\ x &= \sinh y \end{aligned}$$

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Next I want to find a formula for $\operatorname{arcsinh} x = \sinh^{-1} x$.

$$\begin{aligned}\sinh^{-1} x &= y \\ x &= \sinh y \\ x &= \frac{e^y - e^{-y}}{2}\end{aligned}$$

(This one is not in the textbook)

Next I want to find a formula for $\operatorname{arcsinh} x = \sinh^{-1} x$.

$$\begin{aligned} \sinh^{-1} x &= y \\ x &= \sinh y \\ x &= \frac{e^y - e^{-y}}{2} \\ 2x &= e^y - e^{-y} \end{aligned}$$

(This one is not in the textbook)

Next I want to find a formula for $\operatorname{arcsinh} x = \sinh^{-1} x$.

$$\begin{aligned} \sinh^{-1} x &= y \\ x &= \sinh y \\ x &= \frac{e^y - e^{-y}}{2} \\ 2x &= e^y - e^{-y} \\ 2xe^y &= (e^y)^2 - 1 \end{aligned}$$

(This one is not in the textbook)

Next I want to find a formula for $\operatorname{arcsinh} x = \sinh^{-1} x$.

$$\begin{aligned}
\sinh^{-1} x &= y \\
x &= \sinh y \\
x &= \frac{e^y - e^{-y}}{2} \\
2x &= e^y - e^{-y} \\
2xe^y &= (e^y)^2 - 1 \\
0 &= (e^y)^2 - 2xe^y - 1.
\end{aligned}$$

This is a quadratic equation for e^y.

$$\sinh^{-1} x = y \qquad\qquad (e^y)^2 - 2xe^y - 1 = 0$$

We know that the solution to

$$ar^2 + br + c = 0$$

is given by

$$r = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

$$\sinh^{-1} x = y \qquad\qquad (e^y)^2 - 2xe^y - 1 = 0$$

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Therefore

$$e^y = \frac{2x \pm \sqrt{(-2x)^2 + 4}}{2}$$

$$\sinh^{-1} x = y \qquad\qquad (e^y)^2 - 2xe^y - 1 = 0$$

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is given by

$$r = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

Therefore

$$e^y = \frac{2x \pm \sqrt{(-2x)^2 + 4}}{2} = x \pm \sqrt{x^2 + 1}.$$

But do we want "+" or "−" here?

$$\sinh^{-1} x = y \qquad\qquad (e^y)^2 - 2xe^y - 1 = 0$$

We know that the solution to

$$ar^2 + br + c = 0$$

is given by

$$r = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

Therefore

$$e^y = \frac{2x \pm \sqrt{(-2x)^2 + 4}}{2} = x \pm \sqrt{x^2 + 1}.$$

But do we want "+" or "−" here? Remember that e^y is always positive. So we must have "+" here.

$$\sinh^{-1} x = y$$

To finish, we take the natural logarithm of

$$e^y = x + \sqrt{x^2 + 1}$$

to obtain

$$\boxed{\sinh^{-1} x = y = \ln\left(x + \sqrt{x^2 + 1}\right)}$$

$$\sinh^{-1} x = y$$

To finish, we take the natural logarithm of

$$e^y = x + \sqrt{x^2 + 1}$$

to obtain

$$\boxed{\sinh^{-1} x = y = \ln\left(x + \sqrt{x^2 + 1}\right)}$$

Similarly

$$\boxed{\cosh^{-1} x = \ln\left(x + \sqrt{x^2 - 1}\right)}$$

but I leave that for you to prove.

$$\left(f^{-1}\right)'(x) = \frac{1}{f'\left(f^{-1}(x)\right)}$$

Derivatives of Inverse Hyperbolic Functions

We will use the formula in the yellow box with $f(x) = \cosh x$ and $f^{-1}(x) = \operatorname{arccosh} x = \cosh^{-1} x$.

$$\left(f^{-1}\right)'(x) = \frac{1}{f'\left(f^{-1}(x)\right)}$$

Derivatives of Inverse Hyperbolic Functions

We will use the formula in the yellow box with $f(x) = \cosh x$ and $f^{-1}(x) = \operatorname{arccosh} x = \cosh^{-1} x$. Since $\cosh^2 u - \sinh^2 u = 1$, we have that

$$\begin{aligned}\left(\cosh^{-1}\right)'(x) &= \frac{1}{f'\left(f^{-1}(x)\right)} = \frac{1}{\sinh\left(\operatorname{arccosh} x\right)} \\ &= \frac{1}{\sqrt{\cosh^2\left(\operatorname{arccosh} x\right) - 1}} = \frac{1}{\sqrt{x^2 - 1}}.\end{aligned}$$

The other five are similar.

Derivative Formulae

- $\dfrac{d}{dx}\operatorname{arcsinh} x = \dfrac{1}{\sqrt{1+x^2}}$
- $\dfrac{d}{dx}\operatorname{arccosh} x = \dfrac{1}{\sqrt{x^2-1}}, \; x > 1$
- $\dfrac{d}{dx}\operatorname{arctanh} x = \dfrac{1}{1-x^2}, \; |x| < 1$
- $\dfrac{d}{dx}\operatorname{arccoth} x = \dfrac{1}{1-x^2}, \; |x| > 1$
- $\dfrac{d}{dx}\operatorname{sech}^{-1} x = -\dfrac{1}{x\sqrt{1-x^2}},$
  $0 < x < 1$
- $\dfrac{d}{dx}\operatorname{cosech}^{-1} x = -\dfrac{1}{|x|\sqrt{1+x^2}},$
  $x \neq 0$

7.7 Hyperbolic Functions

Derivative Formulae

- $\dfrac{d}{dx}\operatorname{arcsinh} x = \dfrac{1}{\sqrt{1+x^2}}$
- $\dfrac{d}{dx}\operatorname{arccosh} x = \dfrac{1}{\sqrt{x^2-1}}, \ x > 1$
- $\dfrac{d}{dx}\operatorname{arctanh} x = \dfrac{1}{1-x^2}, \ |x| < 1$
- $\dfrac{d}{dx}\operatorname{arccoth} x = \dfrac{1}{1-x^2}, \ |x| > 1$
- $\dfrac{d}{dx}\operatorname{sech}^{-1} x = -\dfrac{1}{x\sqrt{1-x^2}}, \quad 0 < x < 1$
- $\dfrac{d}{dx}\operatorname{cosech}^{-1} x = -\dfrac{1}{|x|\sqrt{1+x^2}}, \quad x \neq 0$

Integral Formulae

- $\displaystyle\int \frac{dx}{\sqrt{a^2+x^2}} = \sinh^{-1}\left(\frac{x}{a}\right) + C, \qquad a > 0$
- $\displaystyle\int \frac{dx}{\sqrt{x^2-a^2}} = \cosh^{-1}\left(\frac{x}{a}\right) + C, \qquad x > a > 0$
- $\displaystyle\int \frac{dx}{a^2-x^2} = \begin{cases} \dfrac{1}{a}\tanh^{-1}\left(\dfrac{x}{a}\right) + C, & x^2 < a^2 \\ \dfrac{1}{a}\coth^{-1}\left(\dfrac{x}{a}\right) + C, & x^2 > a^2 \end{cases}$
- $\displaystyle\int \frac{dx}{x\sqrt{a^2-x^2}} = -\frac{1}{a}\operatorname{sech}^{-1}\left(\frac{x}{a}\right) + C, \quad 0 < x < a$
- $\displaystyle\int \frac{dx}{x\sqrt{a^2+x^2}} = -\frac{1}{a}\operatorname{cosech}^{-1}\left|\frac{x}{a}\right| + C, \quad x \neq 0 \text{ and } a > 0$

$$\int \frac{dx}{\sqrt{a^2 + x^2}} = \sinh^{-1}\left(\frac{x}{a}\right) + C$$

Example (Final example of this course)

Calculate $\displaystyle\int_0^1 \frac{2\,dx}{\sqrt{3 + 4x^2}}$.

$$\int \frac{dx}{\sqrt{a^2 + x^2}} = \sinh^{-1}\left(\frac{x}{a}\right) + C$$

Example (Final example of this course)

Calculate $\displaystyle\int_0^1 \frac{2\,dx}{\sqrt{3 + 4x^2}}$.

We will use the substitution $u = 2x$ and the formula in the yellow box with $a = \sqrt{3}$.

$$\int \frac{dx}{\sqrt{a^2+x^2}} = \sinh^{-1}\left(\frac{x}{a}\right) + C$$

Example (Final example of this course)

Calculate $\displaystyle\int_0^1 \frac{2\,dx}{\sqrt{3+4x^2}}$.

We will use the substitution $u = 2x$ and the formula in the yellow box with $a = \sqrt{3}$. The indefinite integral is

$$\int \frac{2\,dx}{\sqrt{3+4x^2}} = \int \frac{du}{\sqrt{a^2+u^2}} = \sinh^{-1}\left(\frac{u}{a}\right) + C$$

$$= \sinh^{-1}\left(\frac{2x}{\sqrt{3}}\right) + C.$$

$$\int \frac{dx}{\sqrt{a^2+x^2}} = \sinh^{-1}\left(\frac{x}{a}\right) + C$$

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$$\int \frac{2\,dx}{\sqrt{3+4x^2}} = \int \frac{du}{\sqrt{a^2+u^2}} = \sinh^{-1}\left(\frac{u}{a}\right) + C$$
$$= \sinh^{-1}\left(\frac{2x}{\sqrt{3}}\right) + C.$$

Therefore

$$\int_0^1 \frac{2\,dx}{\sqrt{3+4x^2}} = \left[\sinh^{-1}\left(\frac{2x}{\sqrt{3}}\right)\right]_0^1$$
$$= \sinh^{-1}\left(\frac{2}{\sqrt{3}}\right) - \sinh^{-1}(0) = \sinh^{-1}\left(\frac{2}{\sqrt{3}}\right) - 0.$$

The End