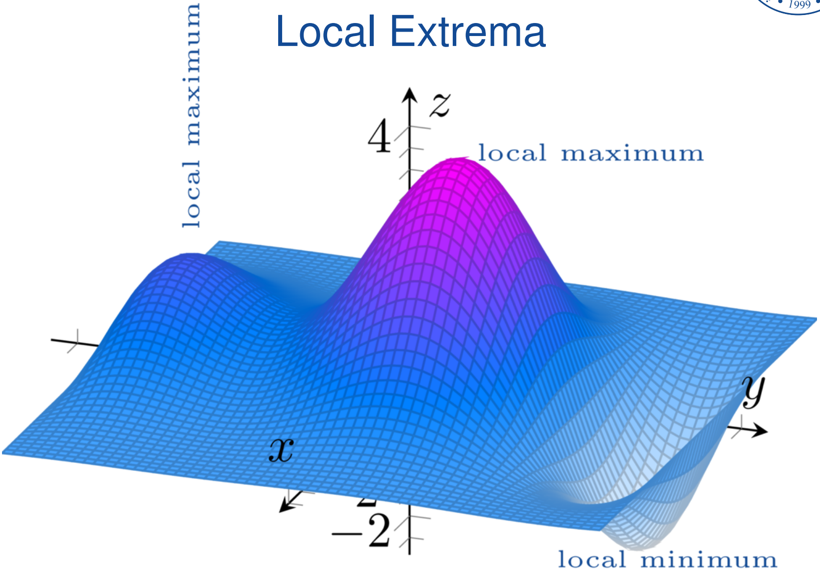
13.7

Extreme Values and Saddle Points

Local Extrema



Definition

1. $f(a,b)$ is a local maximum value of $f(x,y)$ iff

$$f(a,b) \geq f(x,y)$$

for all (x,y) close to (a,b).

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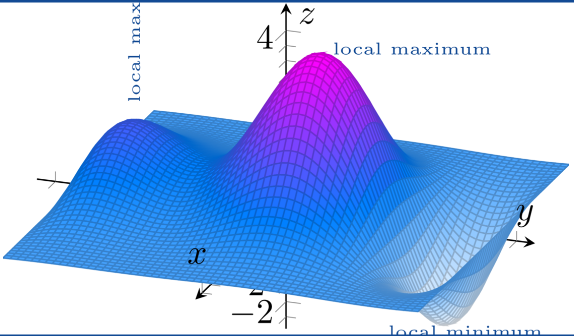
13.7 Extreme Values and Saddle Points



Theorem (First Derivative Test)

$$\begin{pmatrix} f(x,y) \text{ has a local} \\ \text{extrema at an interior} \\ \text{point } (a,b) \text{ of its} \\ \text{domain} \end{pmatrix} \implies \begin{matrix} f_x(a,b) = 0 \\ \text{and} \\ f_y(a,b) = 0 \end{matrix}$$

if $f_x(a,b)$ and $f_y(a,b)$ both exist.

Definition

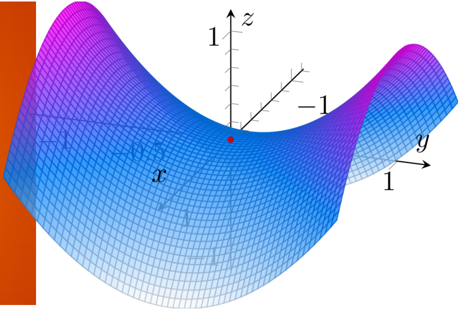
An interior point of the domain of $f(x, y)$ where either

1. $f_x = f_y = 0$;
2. f_x does not exist; or
3. f_y does not exist

is called a *critical point* of f.

Saddle Points

Saddle Points



The point $(0, 0)$ is a *saddle point* of $z = y^2 - x^2$.

Example

Find the local extrema of $f(x, y) = y^2 - 4y + x^2 + 9$.

domain:

partial derivatives:

$$\begin{aligned} 0 &= f_x = \\ 0 &= f_y = \end{aligned} \qquad \Longrightarrow \qquad (x, y) =$$

Example

Find the local extrema of $f(x, y) = y^2 - 4y + x^2 + 9$.

domain: $\mathbb{R}^2$

partial derivatives:

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Since

$$(y - 2)^2 + x^2 + 5 \geq 5$$

for all (x, y), this must be a local minimum.

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Find the local extrema of $f(x, y) = y^2 - 4y - x^2 + 9$.

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Therefore the only possible place for an extrema is $(0, 2)$, where $f(0, 2) = 5$. Is this a local minimum or a local maximum?

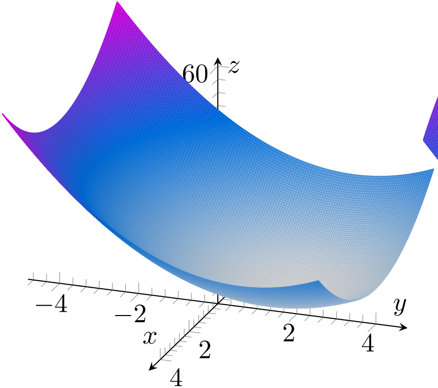
No. Fixing $x = 0$ we have $f(0, y) = (y - 2)^2 + 5$ which curves upwards. But fixing $y = 2$ we have $f(x, 2) = 5 - x^2$ which curves downwards.

So $(0, 2)$ must be a saddle point.

13.7 Extreme Values and Saddle Points

$$z = y^2 - 4y + x^2 + 9$$

has a local minimum at $(0, 2)$.



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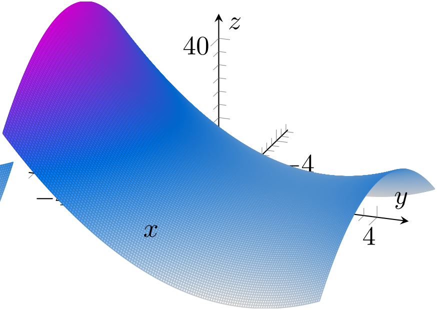

Theorem (Second Derivative Test)

Suppose that

- $f(x,y)$, f_x, f_y, f_{xx}, f_{yy} *and* f_{xy} *are all continuous on an open disk centred at* (a,b); *and*
- $f_x(a,b) = 0 = f_y(a,b)$.

If at (a,b) *we have*		*then*

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$f_{xx} < 0$	$f_{xx}f_{yy} - f_{xy}^2 > 0$	f *has a local maximum at* (a, b)

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	$f_{xx}f_{yy} - f_{xy}^2 < 0$	f *has a* *saddle point* *at* (a, b)

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	$f_{xx}f_{yy} - f_{xy}^2 < 0$	f *has a saddle point at* (a, b)
	$f_{xx}f_{yy} - f_{xy}^2 = 0$	*we don't know*

Otto Hesse

BORN
22 April 1811

DECEICED
4 August 1874

NATIONALITY
German

Definition

$f_{xx}f_{yy} - f_{xy}^2$ is called the *Hessian* (or *discriminant*) of f.

EXAMPLE 3 Find the local extreme values of the function

$$f(x, y) = xy - x^2 - y^2 - 2x - 2y + 4.$$

Solution The function is defined and differentiable for all x and y, and its domain has no boundary points. The function therefore has extreme values only at the points where f_x and f_y are simultaneously zero. This leads to

$$f_x = y - 2x - 2 = 0, \qquad f_y = x - 2y - 2 = 0,$$

or

$$x = y = -2.$$

Therefore, the point $(-2, -2)$ is the only point where f may take on an extreme value. To see if it does so, we calculate

$$f_{xx} = -2, \qquad f_{yy} = -2, \qquad f_{xy} = 1.$$

The discriminant of f at $(a, b) = (-2, -2)$ is

$$f_{xx}f_{yy} - {f_{xy}}^2 = (-2)(-2) - (1)^2 = 4 - 1 = 3.$$

The combination

$$f_{xx} < 0 \qquad \text{and} \qquad f_{xx}f_{yy} - {f_{xy}}^2 > 0$$

tells us that f has a local maximum at $(-2, -2)$. The value of f at this point is $f(-2, -2) = 8$. ■

EXAMPLE 4 Find the local extreme values of $f(x, y) = 3y^2 - 2y^3 - 3x^2 + 6xy$.

Solution Since f is differentiable everywhere, it can assume extreme values only where

$$f_x = 6y - 6x = 0 \qquad \text{and} \qquad f_y = 6y - 6y^2 + 6x = 0.$$

From the first of these equations we find $x = y$, and substitution for y into the second equation then gives

$$6x - 6x^2 + 6x = 0 \qquad \text{or} \qquad 6x(2 - x) = 0.$$

The two critical points are therefore (0, 0) and (2, 2).

To classify the critical points, we calculate the second derivatives:

$$f_{xx} = -6, \qquad f_{yy} = 6 - 12y, \qquad f_{xy} = 6.$$

The discriminant is given by

$$f_{xx}f_{yy} - f_{xy}{}^2 = (-36 + 72y) - 36 = 72(y - 1).$$

At the critical point (0, 0) we see that the value of the discriminant is the negative number -72, so the function has a saddle point at the origin. At the critical point (2, 2) we see that the discriminant has the positive value 72. Combining this result with the negative value of the second partial $f_{xx} = -6$, Theorem 11 says that the critical point (2, 2) gives a local maximum value of $f(2, 2) = 12 - 16 - 12 + 24 = 8$. A graph of the surface is shown in Figure 14.48. ■

EXAMPLE 5 Find the critical points of the function $f(x, y) = 10xye^{-(x^2+y^2)}$ and use the Second Derivative Test to classify each point as one where a saddle, local minimum, or local maximum occurs.

Solution First we find the partial derivatives f_x and f_y and set them simultaneously to zero in seeking the critical points:

$$f_x = 10ye^{-(x^2+y^2)} - 20x^2ye^{-(x^2+y^2)} = 10y(1-2x^2)e^{-(x^2+y^2)} = 0 \Rightarrow y = 0 \text{ or } 1 - 2x^2 = 0,$$
$$f_y = 10xe^{-(x^2+y^2)} - 20xy^2e^{-(x^2+y^2)} = 10x(1-2y^2)e^{-(x^2+y^2)} = 0 \Rightarrow x = 0 \text{ or } 1 - 2y^2 = 0.$$

Since both partial derivatives are continuous everywhere, the only critical points are

$$(0, 0), \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right), \left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right), \left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right), \text{ and } \left(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right).$$

Next we calculate the second partial derivatives in order to evaluate the discriminant at each critical point:

$$f_{xx} = -20xy(1-2x^2)e^{-(x^2+y^2)} - 40xye^{-(x^2+y^2)} = -20xy(3-2x^2)e^{-(x^2+y^2)},$$
$$f_{xy} = f_{yx} = 10(1-2x^2)e^{-(x^2+y^2)} - 20y^2(1-2x^2)e^{-(x^2+y^2)} = 10(1-2x^2)(1-2y^2)e^{-(x^2+y^2)},$$
$$f_{yy} = -20xy(1-2y^2)e^{-(x^2+y^2)} - 40xye^{-(x^2+y^2)} = -20xy(3-2y^2)e^{-(x^2+y^2)}.$$

The following table summarizes the values needed by the Second Derivative Test.

Critical Point	f_{xx}	f_{xy}	f_{yy}	**Discriminant D**
$(0, 0)$	0	10	0	-100
$\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$	$-\frac{20}{e}$	0	$-\frac{20}{e}$	$\frac{400}{e^2}$
$\left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$	$\frac{20}{e}$	0	$\frac{20}{e}$	$\frac{400}{e^2}$
$\left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)$	$\frac{20}{e}$	0	$\frac{20}{e}$	$\frac{400}{e^2}$
$\left(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)$	$-\frac{20}{e}$	0	$-\frac{20}{e}$	$\frac{400}{e^2}$

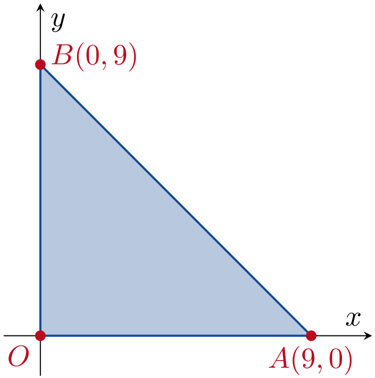
From the table we find that $D < 0$ at the critical point $(0, 0)$, giving a saddle; $D > 0$ and $f_{xx} < 0$ at the critical points $\left(1/\sqrt{2}, 1/\sqrt{2}\right)$ and $\left(-1/\sqrt{2}, -1/\sqrt{2}\right)$, giving local maximum values there; and $D > 0$ and $f_{xx} > 0$ at the critical points $\left(-1/\sqrt{2}, 1/\sqrt{2}\right)$ and $\left(1/\sqrt{2}, -1/\sqrt{2}\right)$, each giving local minimum values. A graph of the surface is shown in Figure 14.49. ■

Example

Find the absolute maximum and minimum values of

$$f(x, y) = 2 + 2x + 4y - x^2 - y^2$$

on the triangular region in the first quadrant bounded by the lines $x = 0$, $y = 0$ and $y = 9 - x$.



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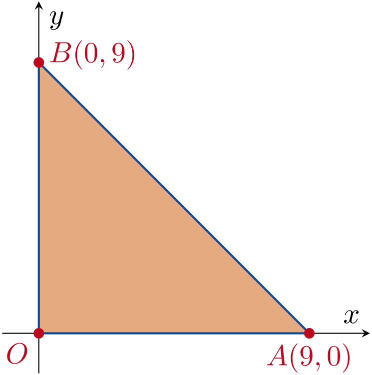

We will look

1. at the interior of the region;

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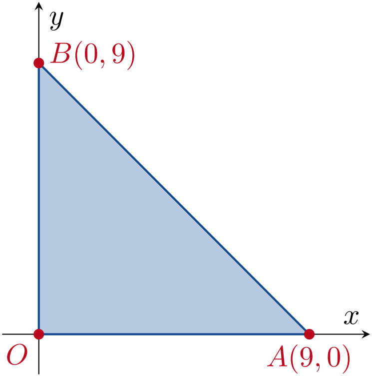

We will look

1. at the interior of the region;
2. at the corners O, A and B;

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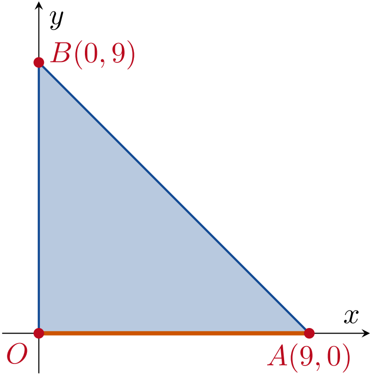

We will look

1. at the interior of the region;
2. at the corners O, A and B;
3. at the line OA;

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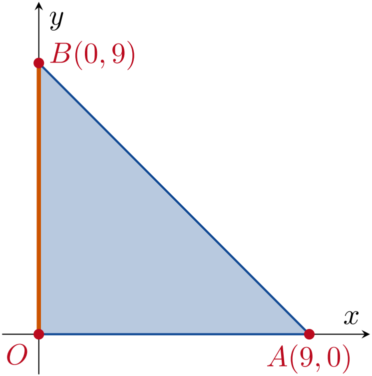

We will look

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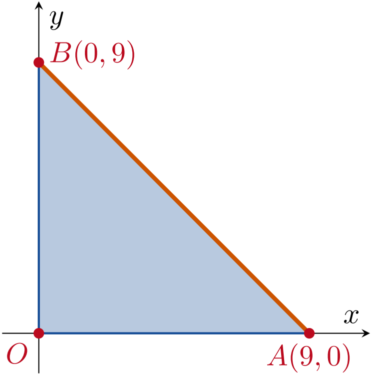
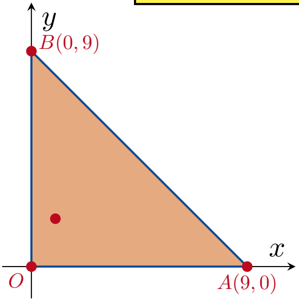


We will look

1. at the interior of the region;
2. at the corners O, A and B;
3. at the line OA;
4. at the line OB; and
5. at the line AB.

$$f(x,y) = 2 + 2x + 4y - x^2 - y^2$$



1 Consider the interior of the region.

We need to look for the critical points

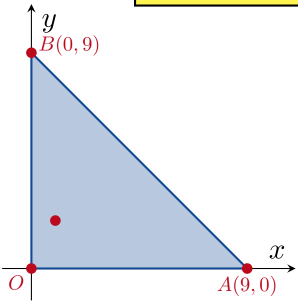
$$\begin{aligned} 0 &= f_x = 2 - 2x \\ 0 &= f_y = 4 - 2y \end{aligned} \qquad \Longrightarrow \qquad (x,y) = (1,2).$$

Then we calculate that

$$f(1,2) = 2 + 2 + 8 - 1 - 4 = 7.$$

$$f(x,y) = 2 + 2x + 4y - x^2 - y^2$$

$f(1,2) = 7$



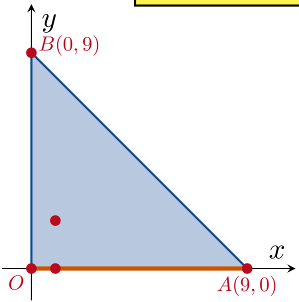
2 Consider the corners of the region.

We calculate that

$$f(0,0) = 2 + 0 + 0 - 0 - 0 = 2$$
$$f(9,0) = 2 + 18 + 0 - 81 - 0 = -61$$
$$f(0,9) = 2 + 0 + 36 - 0 - 81 = -43.$$

$$f(x,y) = 2 + 2x + 4y - x^2 - y^2$$

$f(1,2) = 7$
$f(0,0) = 2$
$f(9,0) = -61$
$f(0,9) = -43$



3 Consider the line OA.

If we set $y = 0$, then we have a new function

$$g(x) = f(x,0) = 2 + 2x - x^2.$$

Since

$$0 = g'(x) = 2 - 2x \qquad \Longrightarrow \qquad x = 1$$

we calculate

$$g(1) = f(1,0) = 2 + 2 - 1 = 3.$$

$$f(x,y) = 2 + 2x + 4y - x^2 - y^2$$

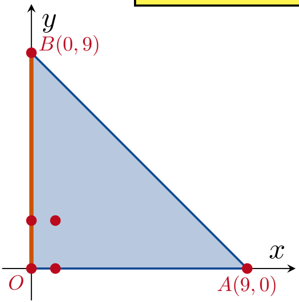
$f(1,2) = 7$

$f(0,0) = 2$

$f(9,0) = -61$

$f(0,9) = -43$

$f(1,0) = 3$



4 Consider the line OB.

If we set $x = 0$, then we have a new function

$$h(y) = f(0,y) = 2 + 4y - y^2.$$

Since

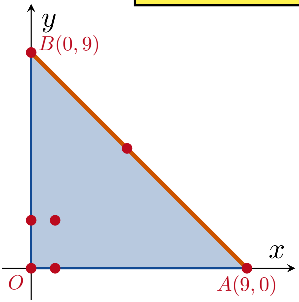
$$0 = h'(x) = 4 - 2y \qquad \Longrightarrow \qquad y = 2$$

we calculate

$$h(2) = f(0,2) = 2 + 8 - 4 = 6.$$

$$f(x,y) = 2 + 2x + 4y - x^2 - y^2$$

$f(1,2) = 7$
$f(0,0) = 2$
$f(9,0) = -61$
$f(0,9) = -43$
$f(1,0) = 3$
$f(0,2) = 6$



5 Finally consider the line AB.

If we set $y = 9 - x$, then we have a new function

$$k(x) = f(x, 9-x) = 2+2x+4(9-x)-x^2-(9-x)^2 = -43+16x-2x^2.$$

Since

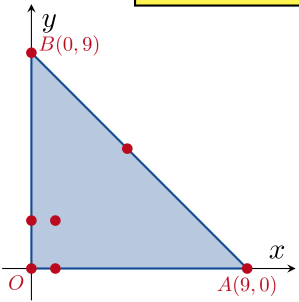
$$0 = k'(x) = 16 - 4x \qquad \Longrightarrow \qquad x = 4$$

we calculate

$$k(4) = f(4,5) = -43 + 64 - 32 = -11.$$

$$f(x, y) = 2 + 2x + 4y - x^2 - y^2$$

$f(1, 2) = 7$
$f(0, 0) = 2$
$f(9, 0) = -61$
$f(0, 9) = -43$
$f(1, 0) = 3$
$f(0, 2) = 6$
$f(4, 5) = -11$



We have found the values

$$7,\ 2,\ -61,\ -43,\ 3,\ 6,\ -11.$$

The biggest of these numbers is 7 and the least is -61.

Therefore the absolute maximum value of f on this region is 7 and the absolute minimum value of f on this region is -61.

Please read Example 7 in the textbook.

13.8

Lagrange Multipliers

Example

Find the point $P(x, y, z)$ on the plane $2x + y - z - 5 = 0$ that is closest to the origin.

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We need to find the minimum of

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subject to the constraint that

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subject to the constraint that

$$2x + y - z - 5 = 0.$$

Let $f(x, y) = x^2 + y^2 + z^2$. We will study f instead of $\left\|\overrightarrow{OP}\right\|$.

We will let x and y be the independent variables and write

$$z = 2x + y - 5.$$

We will let x and y be the independent variables and write

$$z = 2x + y - 5.$$

So we want to find the minimum of

$$h(x, y) = f(x, y, 2x + y - 5) = x^2 + y^2 + (2x + y - 5)^2.$$

We will let x and y be the independent variables and write

$$z = 2x + y - 5.$$

So we want to find the minimum of

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The point on this plane which is closest to the origin is

$$P\left(\frac{5}{3}, \frac{5}{6}, -\frac{5}{6}\right).$$

The Method of Lagrange Multipliers

Suppose that we want to find the maximum/minimum of

$$f(x, y, z)$$

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Example (repeat)

Find the point $P(x, y, z)$ on the plane $2x + y - z - 5 = 0$ that is closest to the origin.

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Example (repeat)

Find the point $P(x, y, z)$ on the plane $2x + y - z - 5 = 0$ that is closest to the origin.

Let $f(x, y, z) = x^2 + y^2 + z^2$ and $g(x, y, z) = 2x + y - z - 5$.

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Example (repeat)

Find the point $P(x,y,z)$ on the plane $2x + y - z - 5 = 0$ that is closest to the origin.

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Hence

$$0 = 2x + y - z - 5 = 2\lambda + \frac{\lambda}{2} - \left(-\frac{\lambda}{2}\right) - 5 = 3\lambda - 5$$

$$\nabla f = \lambda \nabla g \qquad g(x,y,z) = 0.$$

Example (repeat)

Find the point $P(x,y,z)$ on the plane $2x + y - z - 5 = 0$ that is closest to the origin.

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Find the point $P(x, y, z)$ on the plane $2x + y - z - 5 = 0$ that is closest to the origin.

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Hence

$$0 = 2x + y - z - 5 = 2\lambda + \frac{\lambda}{2} - \left(-\frac{\lambda}{2}\right) - 5 = 3\lambda - 5 \implies \lambda = \frac{5}{3}.$$

Therefore

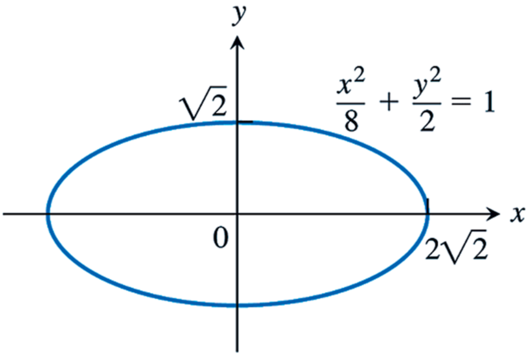
$$P(x, y, z) = \left(\lambda, \frac{\lambda}{2}, -\frac{\lambda}{2}\right) = \left(\frac{5}{3}, \frac{5}{6}, -\frac{5}{6}\right).$$

EXAMPLE 3 Find the greatest and smallest values that the function

$$f(x, y) = xy$$

takes on the ellipse (Figure 14.55)

$$\frac{x^2}{8} + \frac{y^2}{2} = 1.$$



Solution We want to find the extreme values of $f(x, y) = xy$ subject to the constraint

$$g(x, y) = \frac{x^2}{8} + \frac{y^2}{2} - 1 = 0.$$

To do so, we first find the values of x, y, and λ for which

$$\nabla f = \lambda \nabla g \qquad \text{and} \qquad g(x, y) = 0.$$

The gradient equation in Equations (1) gives

$$y\mathbf{i} + x\mathbf{j} = \frac{\lambda}{4} x\mathbf{i} + \lambda y\mathbf{j},$$

from which we find

$$y = \frac{\lambda}{4} x, \qquad x = \lambda y, \qquad \text{and} \qquad y = \frac{\lambda}{4}(\lambda y) = \frac{\lambda^2}{4} y,$$

so that $y = 0$ or $\lambda = \pm 2$. We now consider these two cases.

Case 1: If $y = 0$, then $x = y = 0$. But $(0, 0)$ is not on the ellipse. Hence, $y \neq 0$.

Case 2: If $y \neq 0$, then $\lambda = \pm 2$ and $x = \pm 2y$. Substituting this in the equation $g(x, y) = 0$ gives

$$\frac{(\pm 2y)^2}{8} + \frac{y^2}{2} = 1, \qquad 4y^2 + 4y^2 = 8 \qquad \text{and} \qquad y = \pm 1.$$

The function $f(x, y) = xy$ therefore takes on its extreme values on the ellipse at the four points $(\pm 2, 1), (\pm 2, -1)$. The extreme values are $xy = 2$ and $xy = -2$.

EXAMPLE 4 Find the maximum and minimum values of the function $f(x, y) = 3x + 4y$ on the circle $x^2 + y^2 = 1$.

Solution We model this as a Lagrange multiplier problem with

$$f(x, y) = 3x + 4y, \qquad g(x, y) = x^2 + y^2 - 1$$

and look for the values of x, y, and λ that satisfy the equations

$$\begin{aligned} \nabla f = \lambda \nabla g: &\quad 3\mathbf{i} + 4\mathbf{j} = 2x\lambda\mathbf{i} + 2y\lambda\mathbf{j} \\ g(x, y) = 0: &\quad x^2 + y^2 - 1 = 0. \end{aligned}$$

The gradient equation in Equations (1) implies that $\lambda \neq 0$ and gives

$$x = \frac{3}{2\lambda}, \qquad y = \frac{2}{\lambda}.$$

These equations tell us, among other things, that x and y have the same sign. With these values for x and y, the equation $g(x, y) = 0$ gives

$$\left(\frac{3}{2\lambda}\right)^2 + \left(\frac{2}{\lambda}\right)^2 - 1 = 0,$$

so

$$\frac{9}{4\lambda^2} + \frac{4}{\lambda^2} = 1, \qquad 9 + 16 = 4\lambda^2, \qquad 4\lambda^2 = 25, \qquad \text{and} \qquad \lambda = \pm\frac{5}{2}.$$

Thus,

$$x = \frac{3}{2\lambda} = \pm\frac{3}{5}, \qquad y = \frac{2}{\lambda} = \pm\frac{4}{5},$$

and $f(x, y) = 3x + 4y$ has extreme values at $(x, y) = \pm(3/5, 4/5)$.

By calculating the value of $3x + 4y$ at the points $\pm(3/5, 4/5)$, we see that its maximum and minimum values on the circle $x^2 + y^2 = 1$ are

$$3\left(\frac{3}{5}\right) + 4\left(\frac{4}{5}\right) = \frac{25}{5} = 5 \qquad \text{and} \qquad 3\left(-\frac{3}{5}\right) + 4\left(-\frac{4}{5}\right) = -\frac{25}{5} = -5.$$

Next Time

- 14.1 Double and Iterated Integrals over Rectangles
- 14.2 Double Integrals over General Regions
- 14.3 Area by Double Integration
- 10.3 Polar Coordinates