

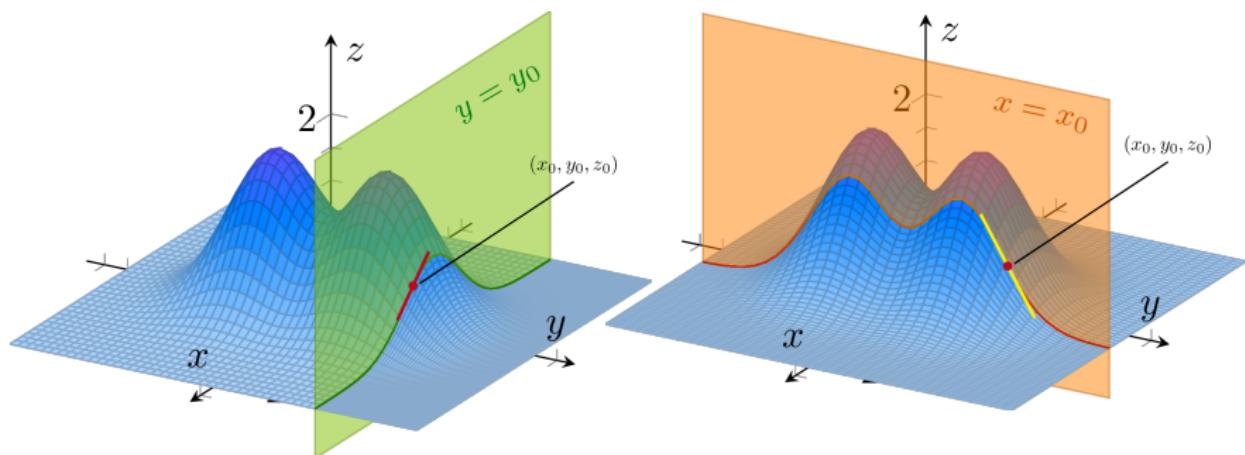
Lecture 6

- 13.5 Directional Derivatives and Gradient Vectors
- 13.6 Tangent Planes and Differentials
- 13.7 Extreme Values and Saddle Points
- 13.8 Lagrange Multipliers

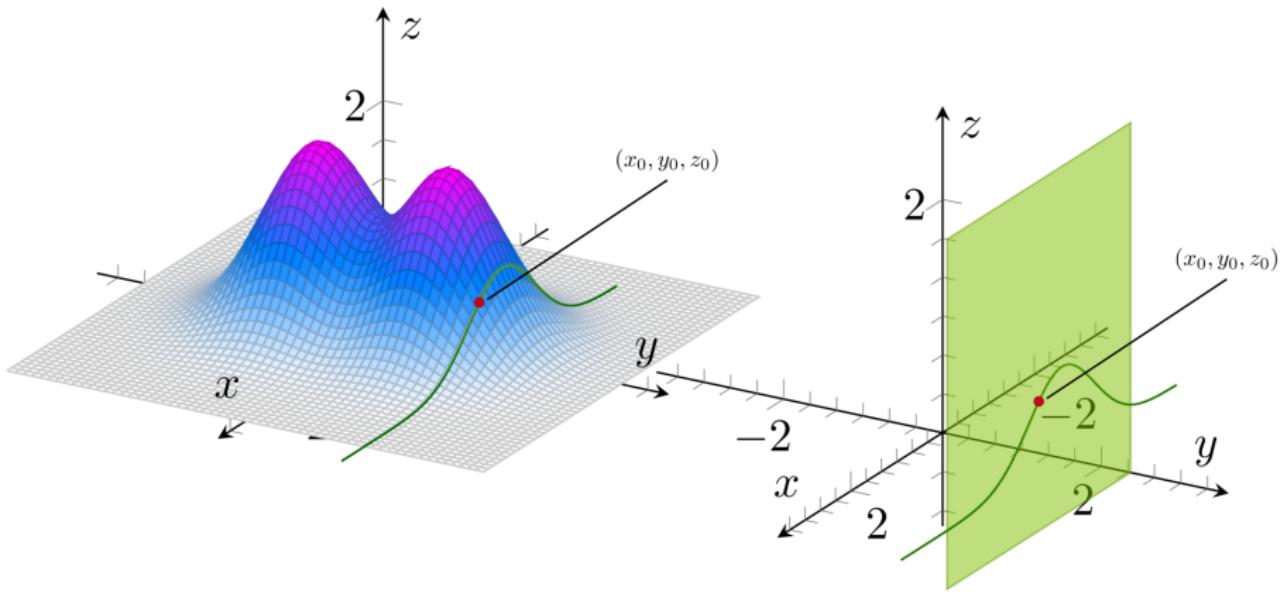


Directional Derivatives and Gradient Vectors

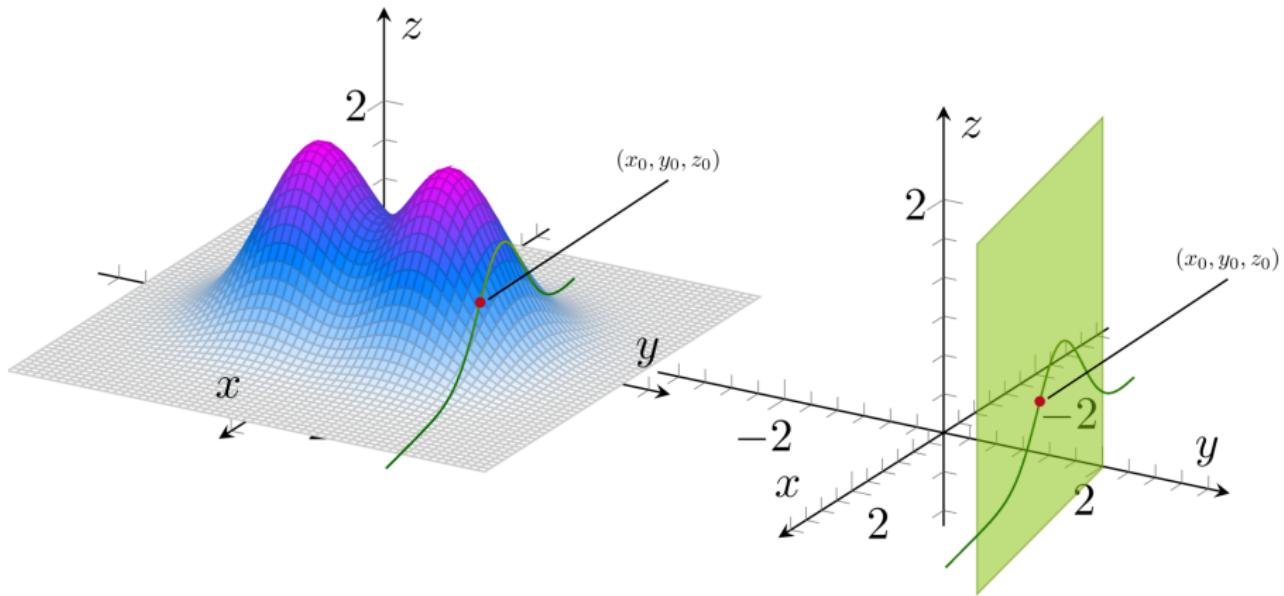
Partial Derivatives (revision)



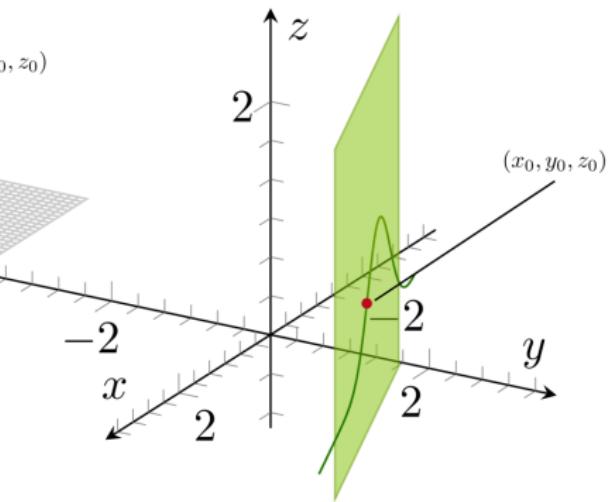
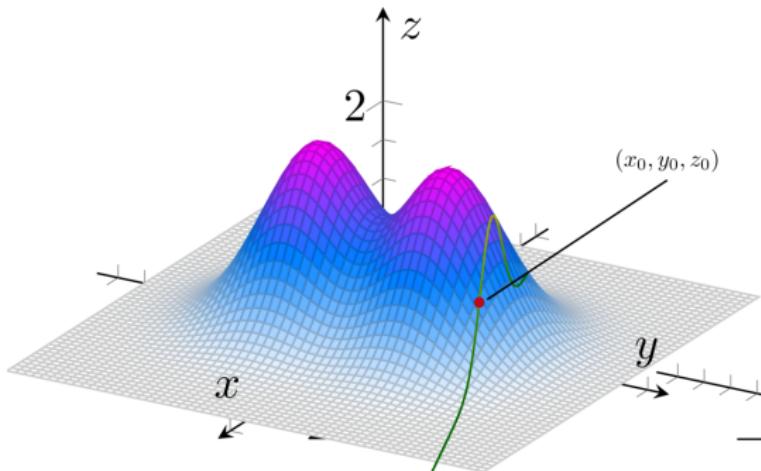
Directional Derivatives



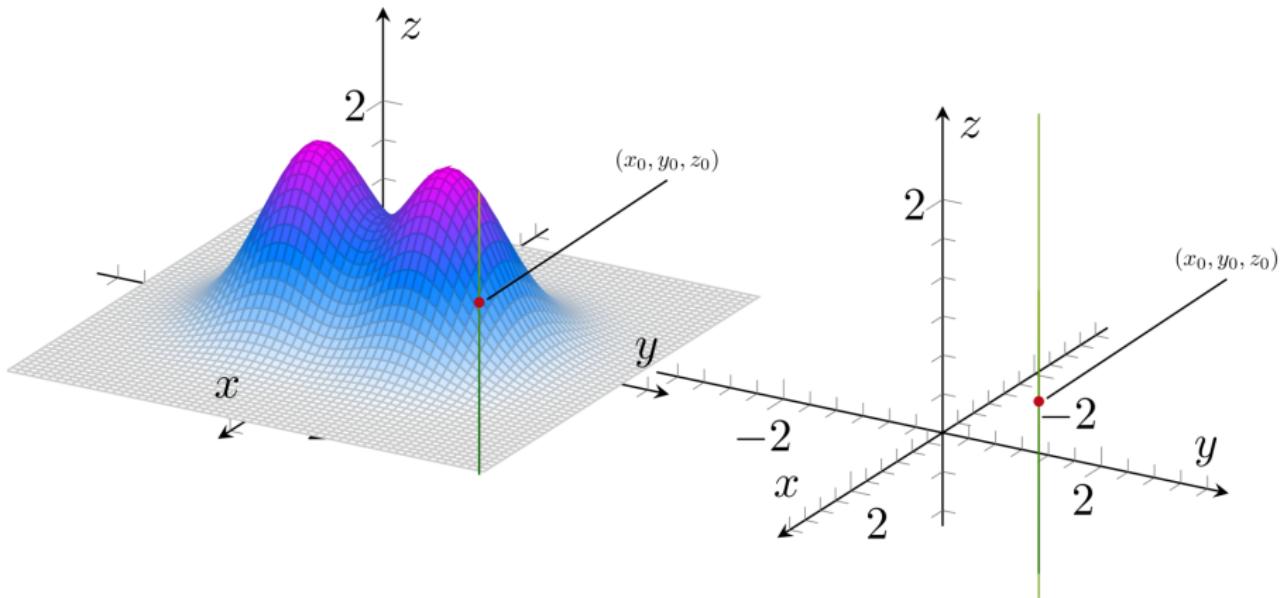
Directional Derivatives



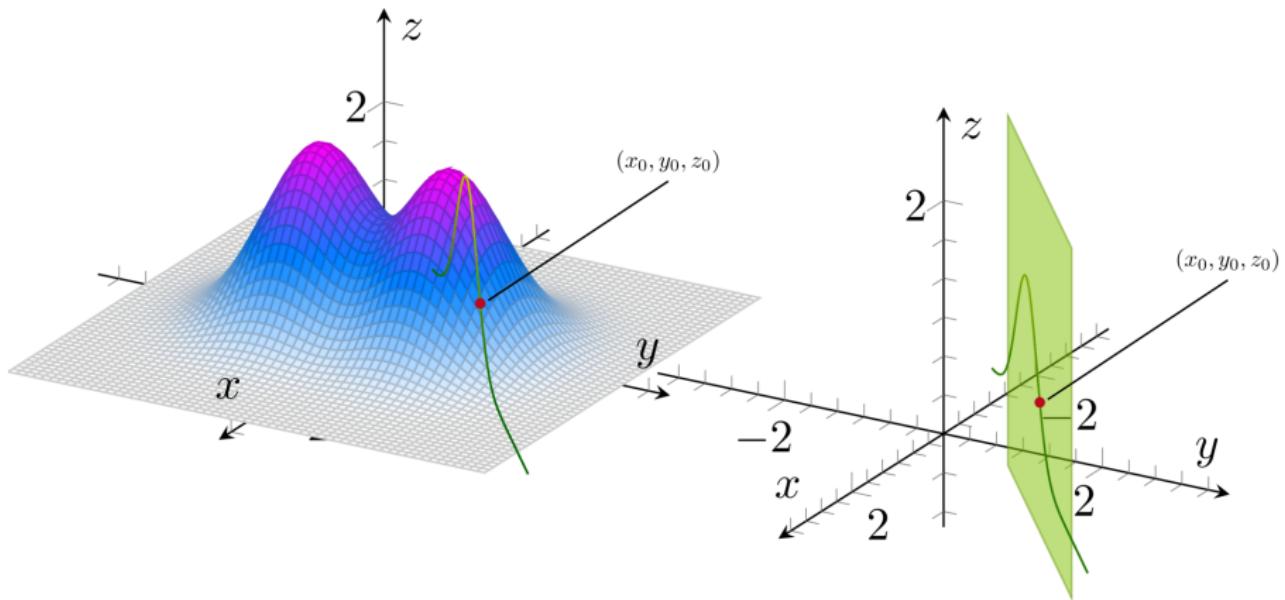
Directional Derivatives



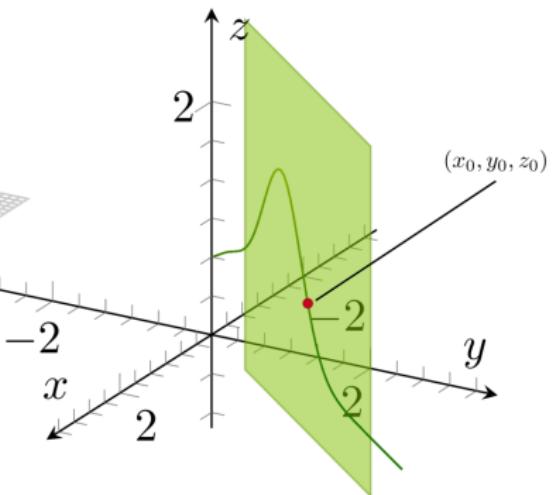
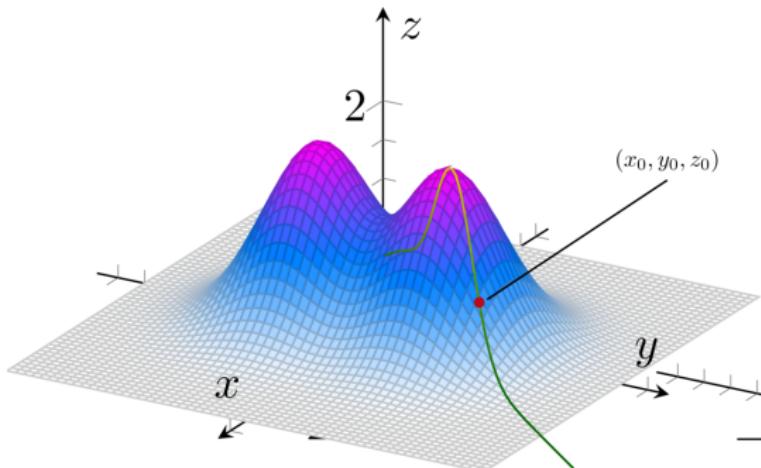
Directional Derivatives



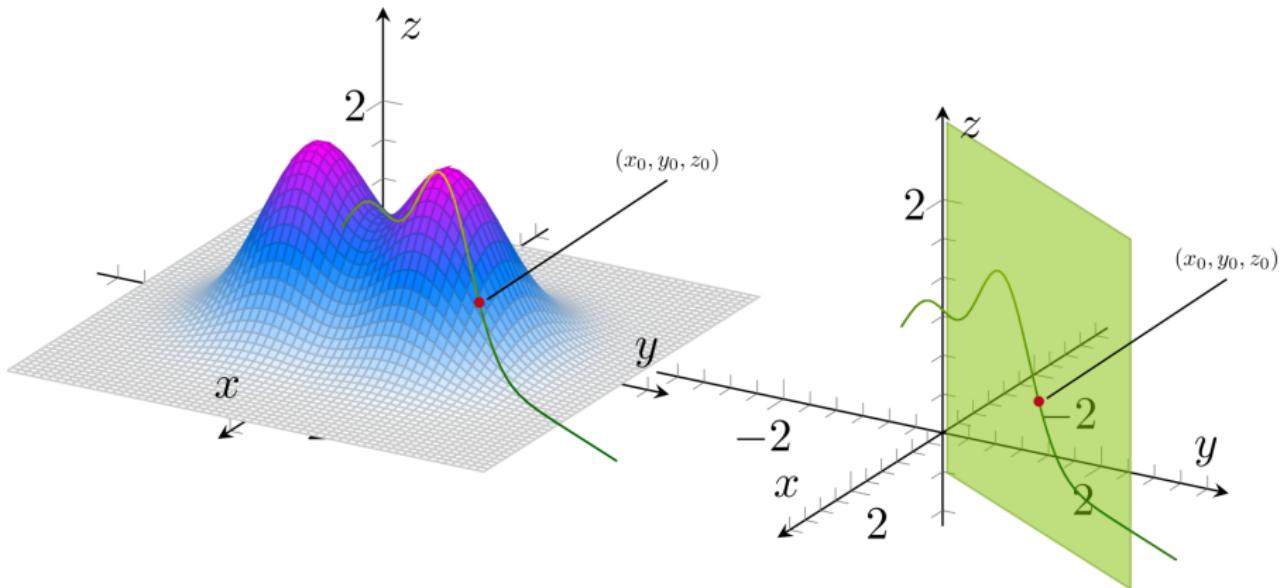
Directional Derivatives



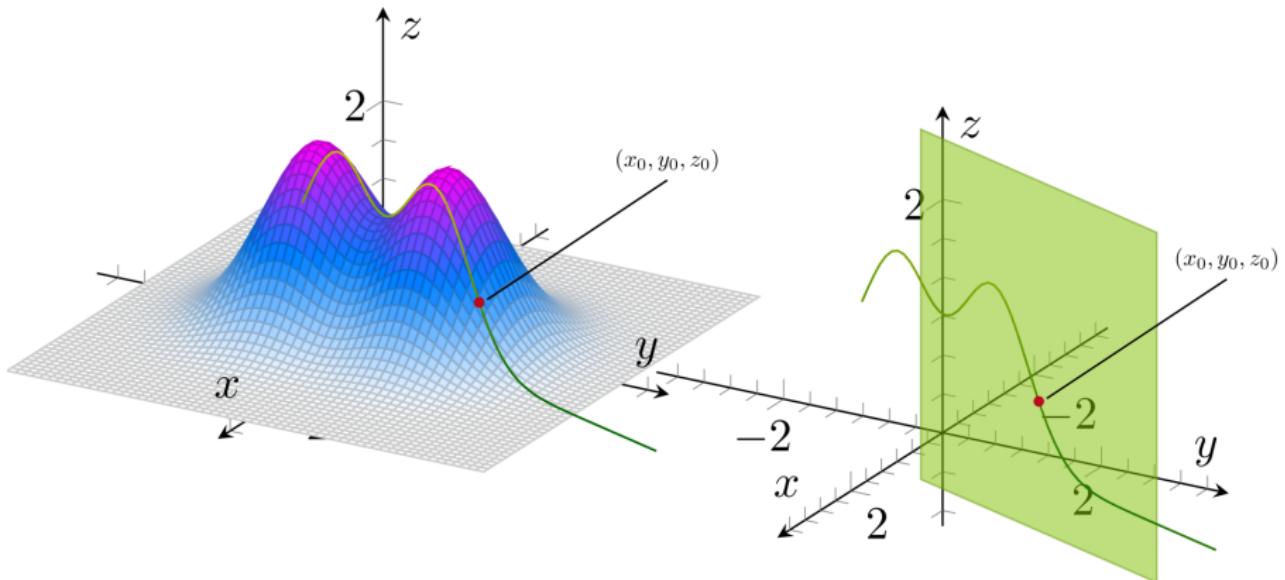
Directional Derivatives



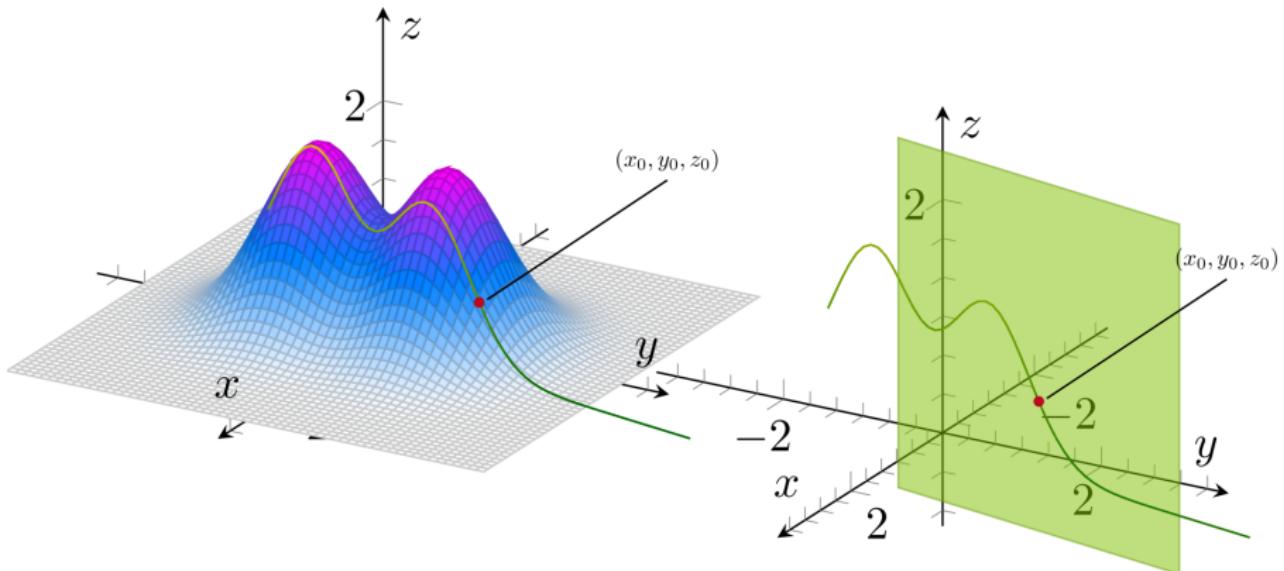
Directional Derivatives



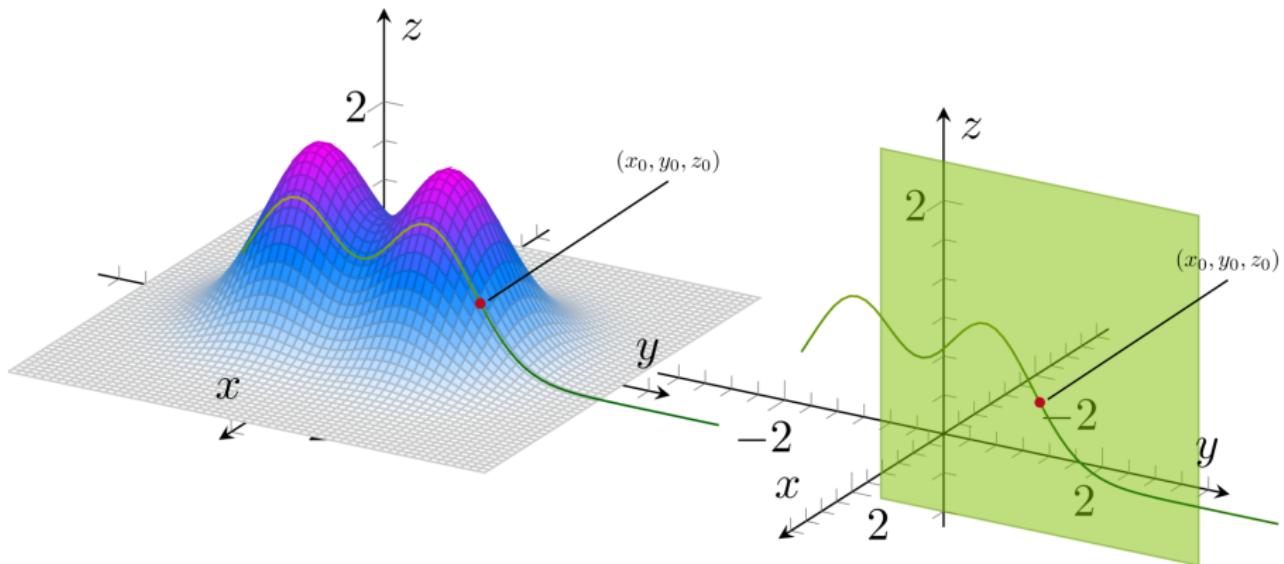
Directional Derivatives



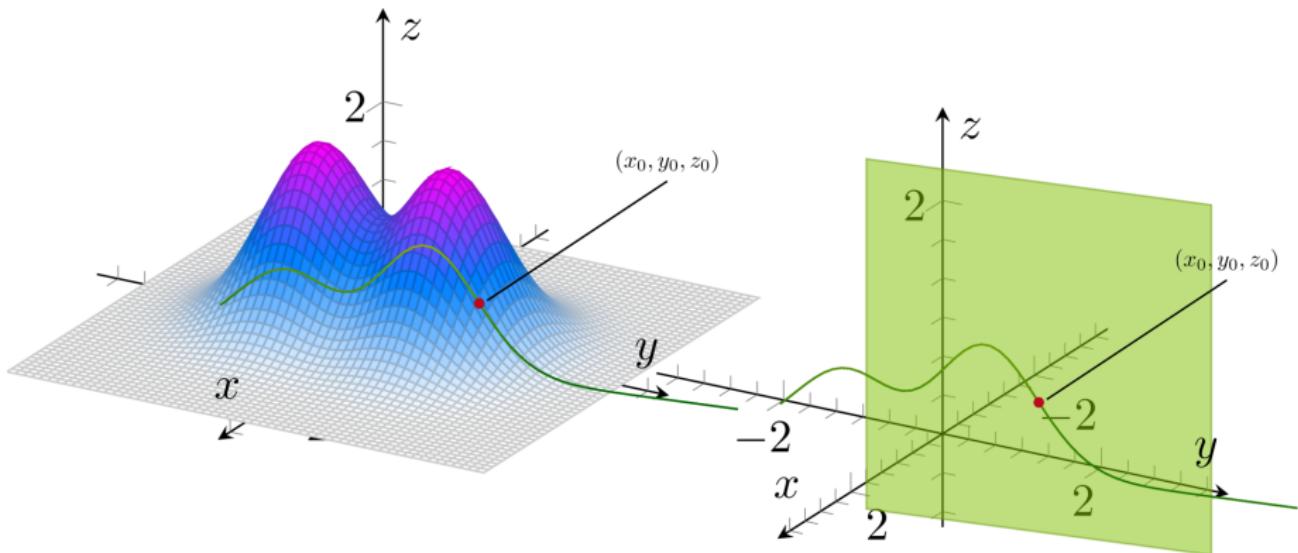
Directional Derivatives



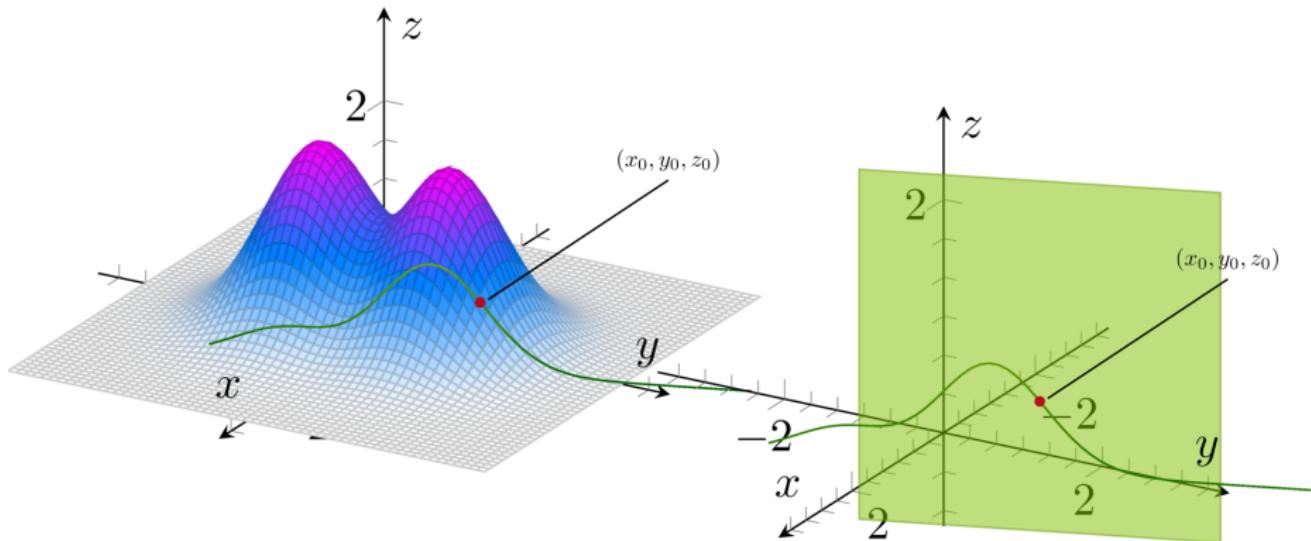
Directional Derivatives



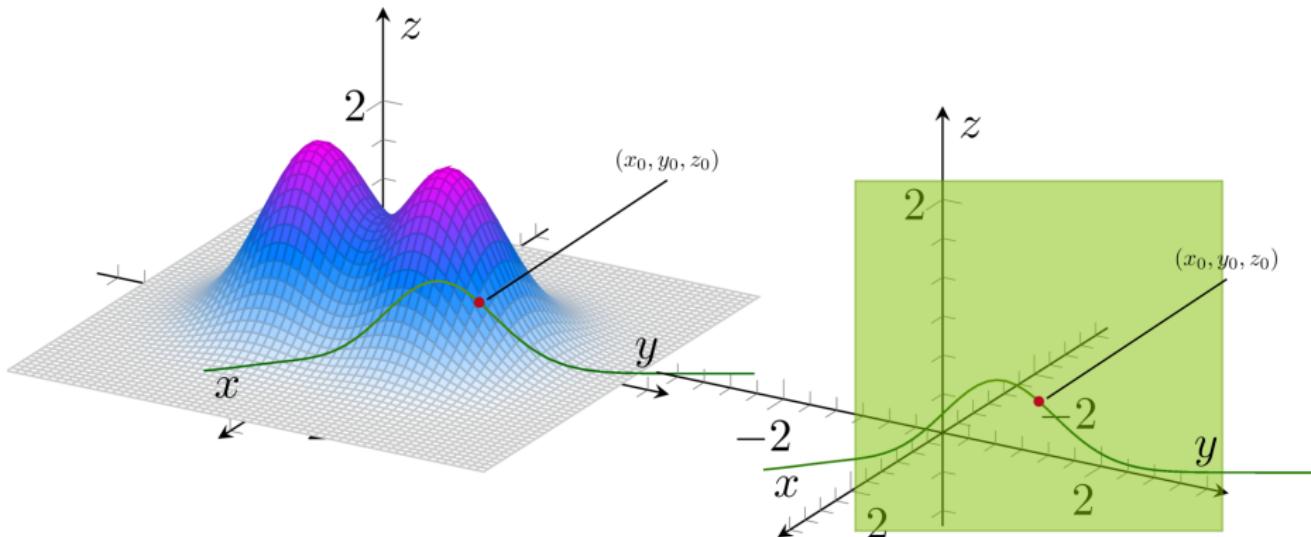
Directional Derivatives



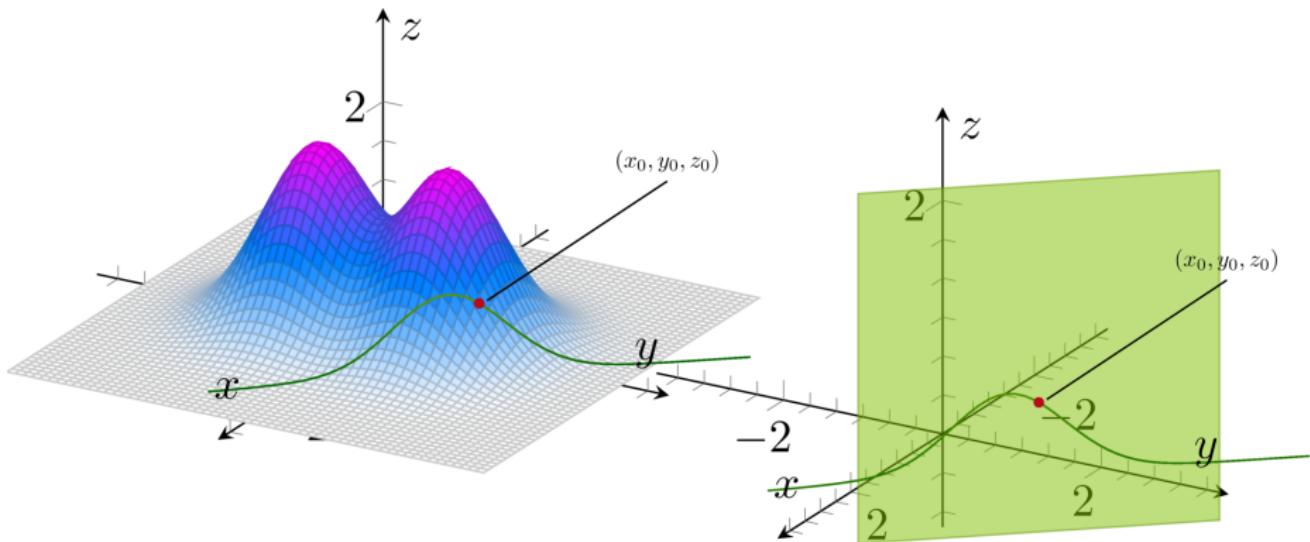
Directional Derivatives



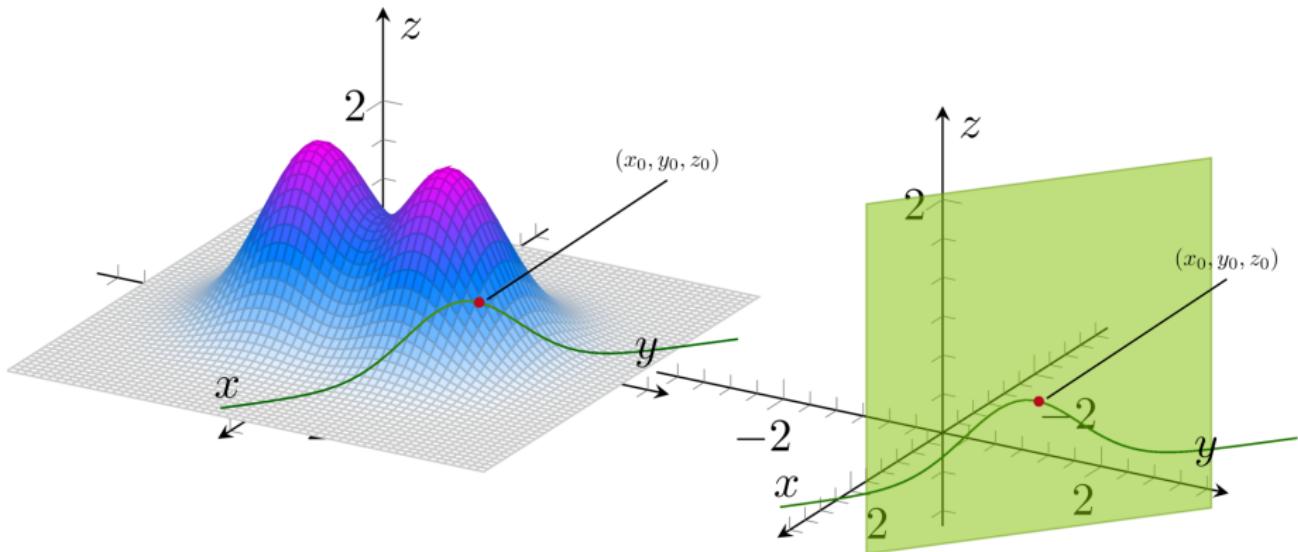
Directional Derivatives



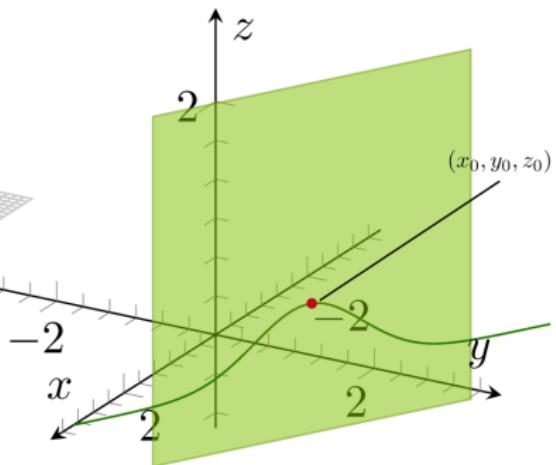
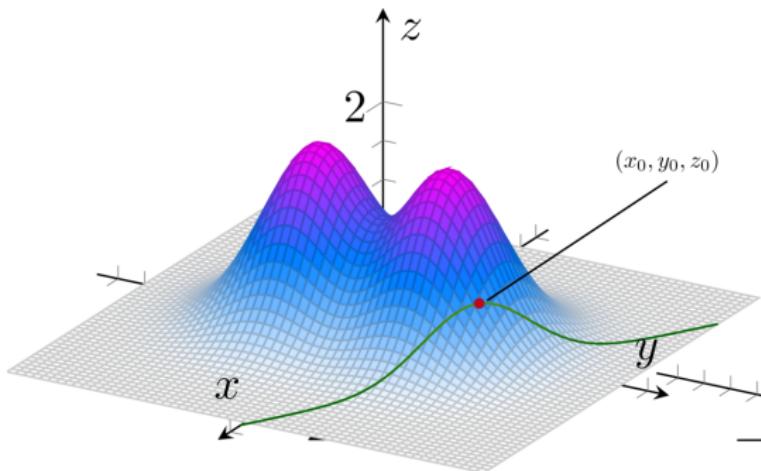
Directional Derivatives



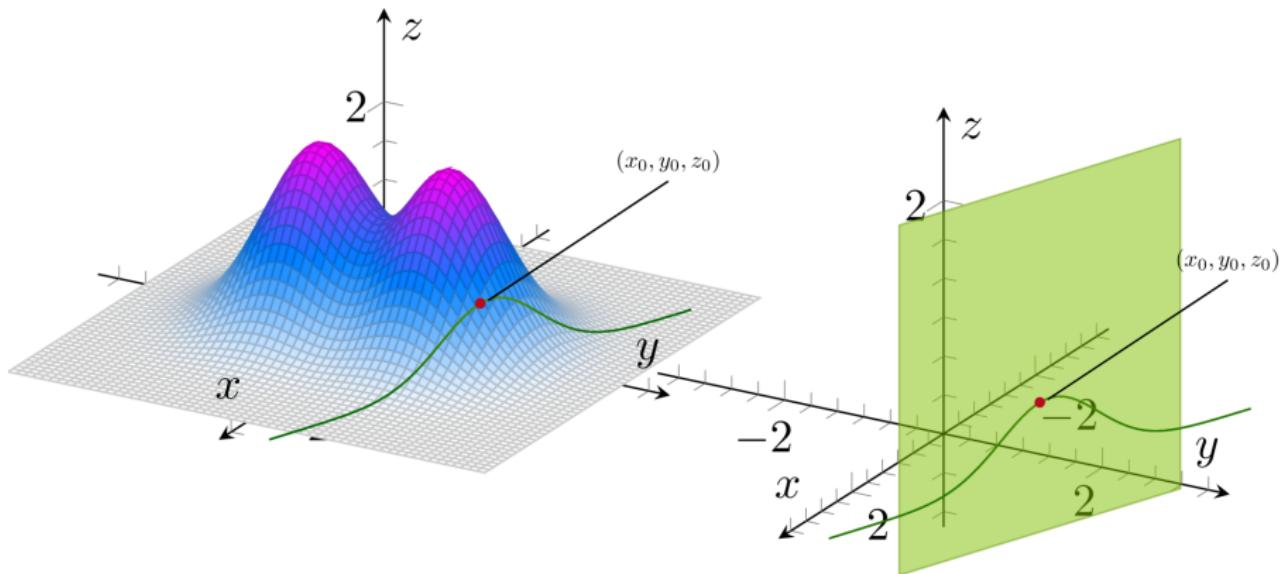
Directional Derivatives



Directional Derivatives



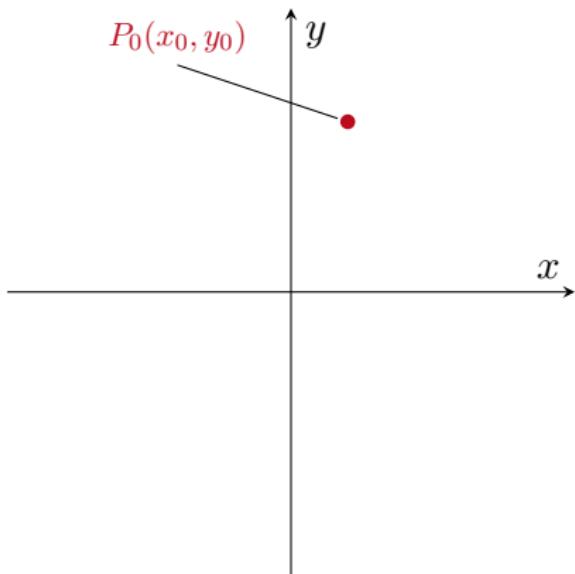
Directional Derivatives



13.5 Directional Derivatives and Gradient Vector



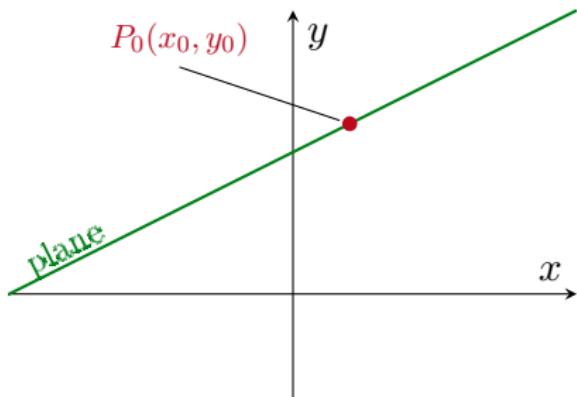
top view



13.5 Directional Derivatives and Gradient Vector



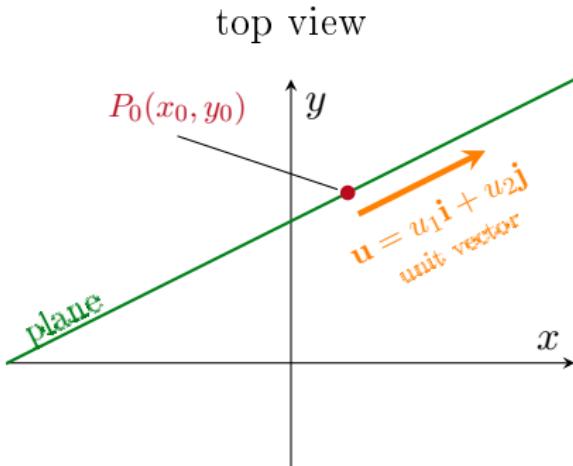
top view



Definition

The *derivative of f at $P_0(x_0, y_0)$*

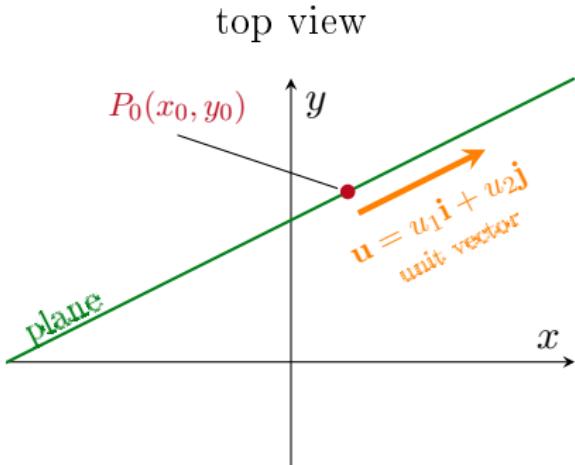
13.5 Directional Derivatives and Gradient Vector



Definition

The derivative of f at $P_0(x_0, y_0)$ in the direction of the unit vector $\mathbf{u} = u_1 \mathbf{i} + u_2 \mathbf{j}$

13.5 Directional Derivatives and Gradient Vector



Definition

The derivative of f at $P_0(x_0, y_0)$ in the direction of the unit vector $\mathbf{u} = u_1 \mathbf{i} + u_2 \mathbf{j}$ is

$$D_{\mathbf{u}} f(P_0) = \lim_{s \rightarrow 0} \frac{f(x_0 + su_1, y_0 + su_2) - f(x_0, y_0)}{s}$$

13.5 Directional Derivatives and Gradient Vector



$$D_{\mathbf{u}} f(P_0) = \left(\frac{df}{ds} \right)_{\mathbf{u}, P_0}$$

EXAMPLE 1 Using the definition, find the derivative of

$$f(x, y) = x^2 + xy$$

at $P_0(1, 2)$ in the direction of the unit vector $\mathbf{u} = (1/\sqrt{2})\mathbf{i} + (1/\sqrt{2})\mathbf{j}$.

Solution Applying the definition in Equation (1), we obtain

$$\begin{aligned}\left(\frac{df}{ds}\right)_{\mathbf{u}, P_0} &= \lim_{s \rightarrow 0} \frac{f(x_0 + su_1, y_0 + su_2) - f(x_0, y_0)}{s} \quad \text{Eq. (1)} \\ &= \lim_{s \rightarrow 0} \frac{f\left(1 + s \cdot \frac{1}{\sqrt{2}}, 2 + s \cdot \frac{1}{\sqrt{2}}\right) - f(1, 2)}{s} \\ &= \lim_{s \rightarrow 0} \frac{\left(1 + \frac{s}{\sqrt{2}}\right)^2 + \left(1 + \frac{s}{\sqrt{2}}\right)\left(2 + \frac{s}{\sqrt{2}}\right) - (1^2 + 1 \cdot 2)}{s} \\ &= \lim_{s \rightarrow 0} \frac{\left(1 + \frac{2s}{\sqrt{2}} + \frac{s^2}{2}\right) + \left(2 + \frac{3s}{\sqrt{2}} + \frac{s^2}{2}\right) - 3}{s} \\ &= \lim_{s \rightarrow 0} \frac{\frac{5s}{\sqrt{2}} + s^2}{s} = \lim_{s \rightarrow 0} \left(\frac{5}{\sqrt{2}} + s\right) = \frac{5}{\sqrt{2}}.\end{aligned}$$

The rate of change of $f(x, y) = x^2 + xy$ at $P_0(1, 2)$ in the direction \mathbf{u} is $5/\sqrt{2}$.

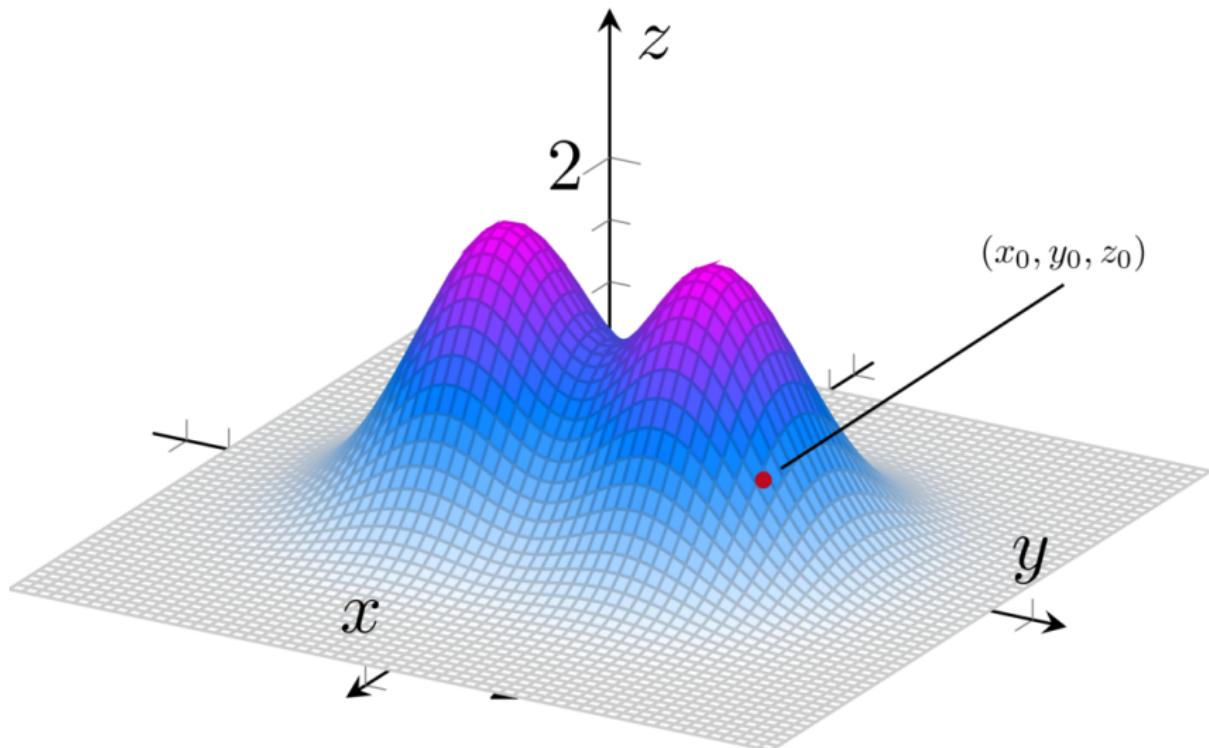
13.5 Directional Derivatives and Gradient Vector



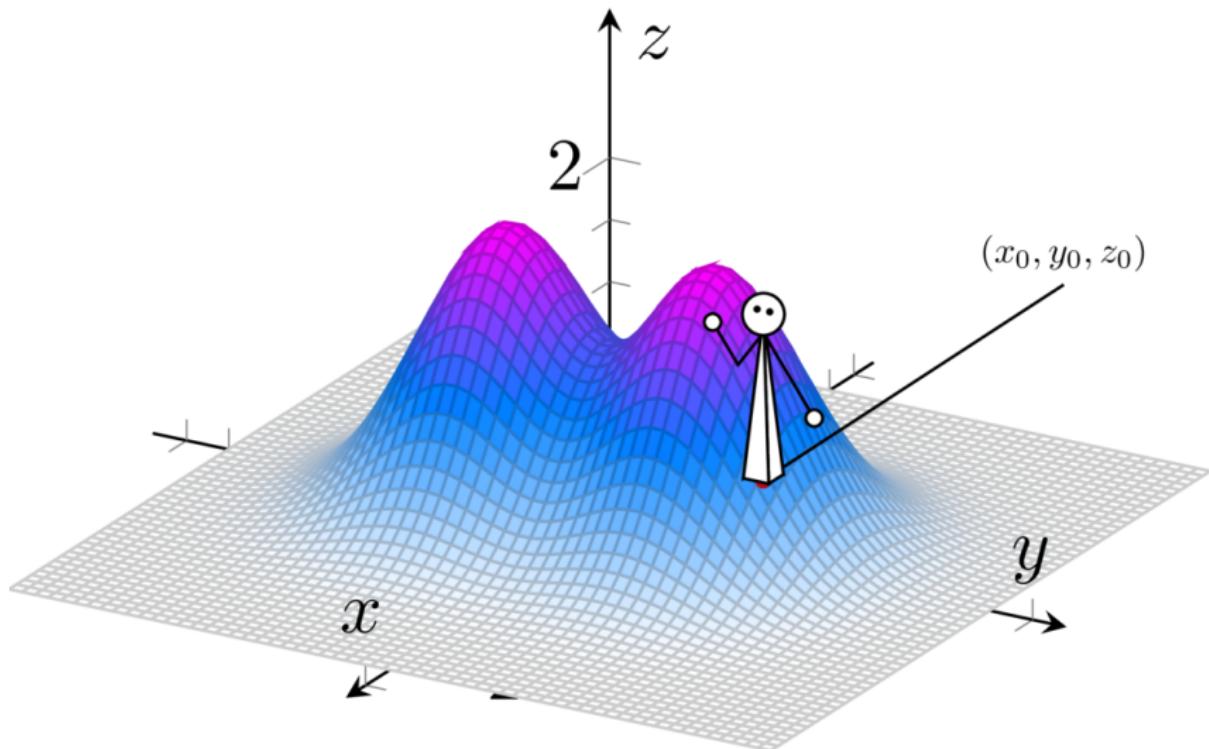
Remark

But it is easier to calculate directional derivatives if we use gradient vectors.

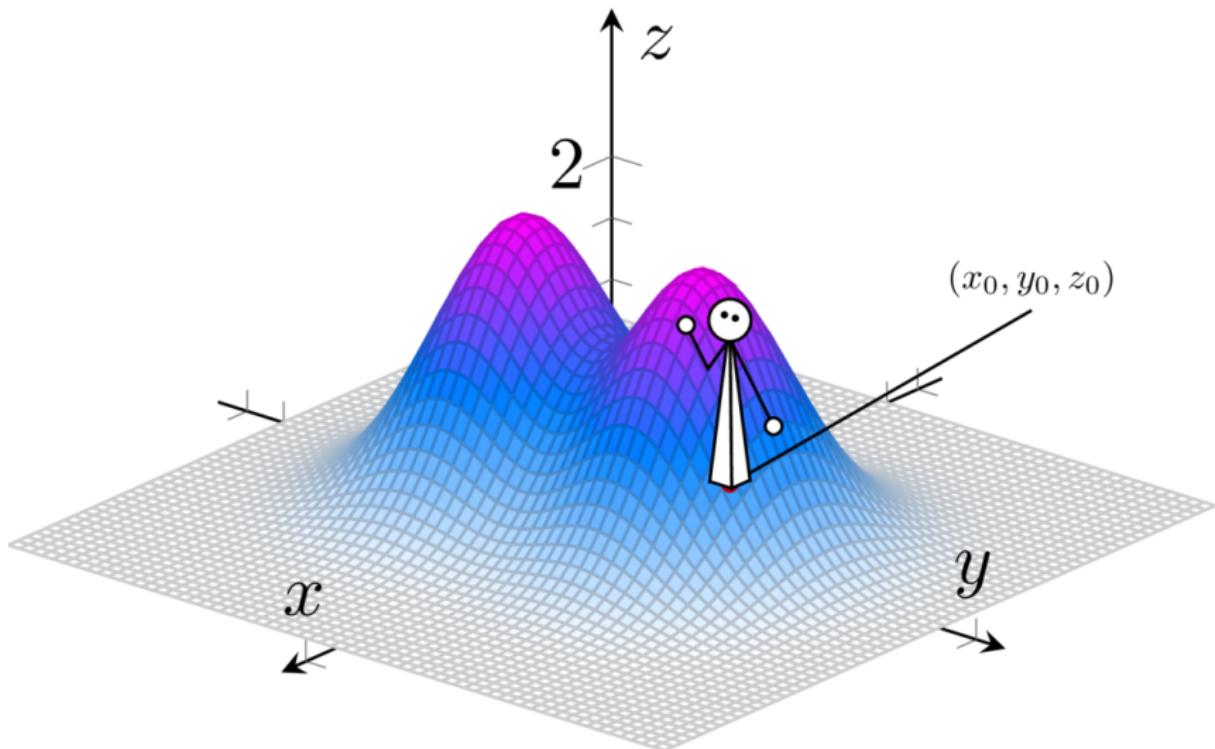
What is a Gradient Vector?



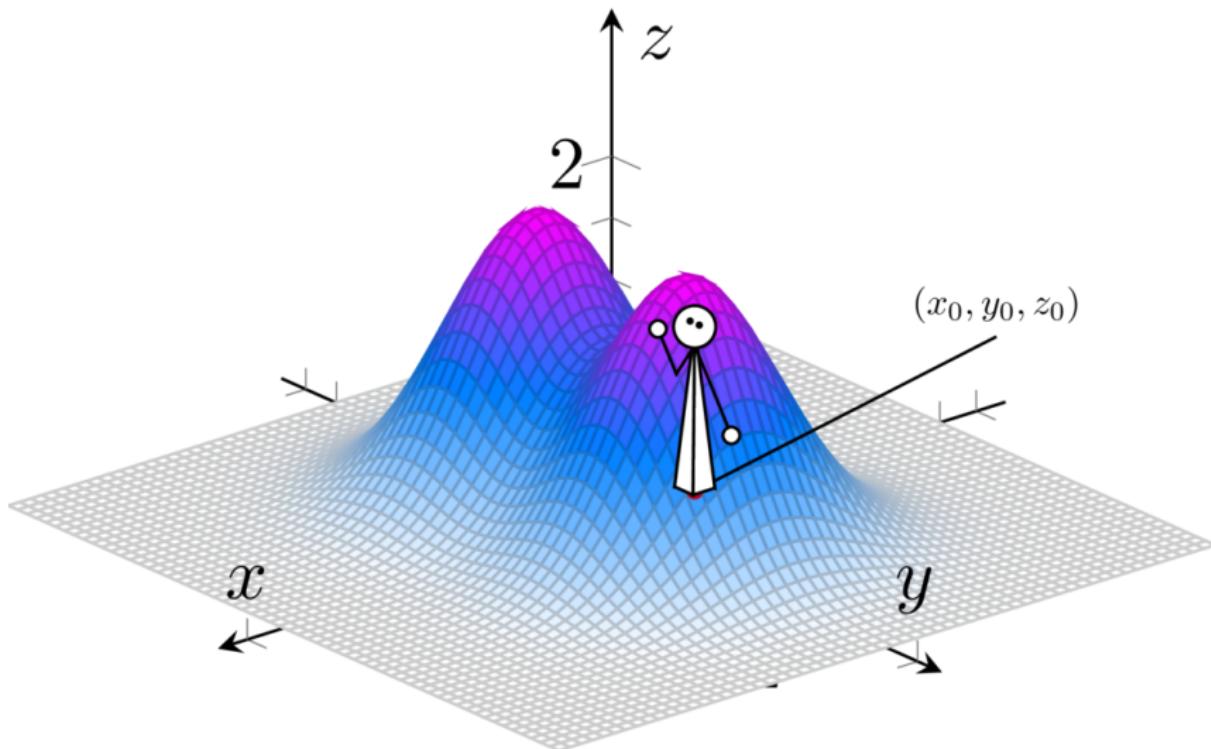
What is a Gradient Vector?



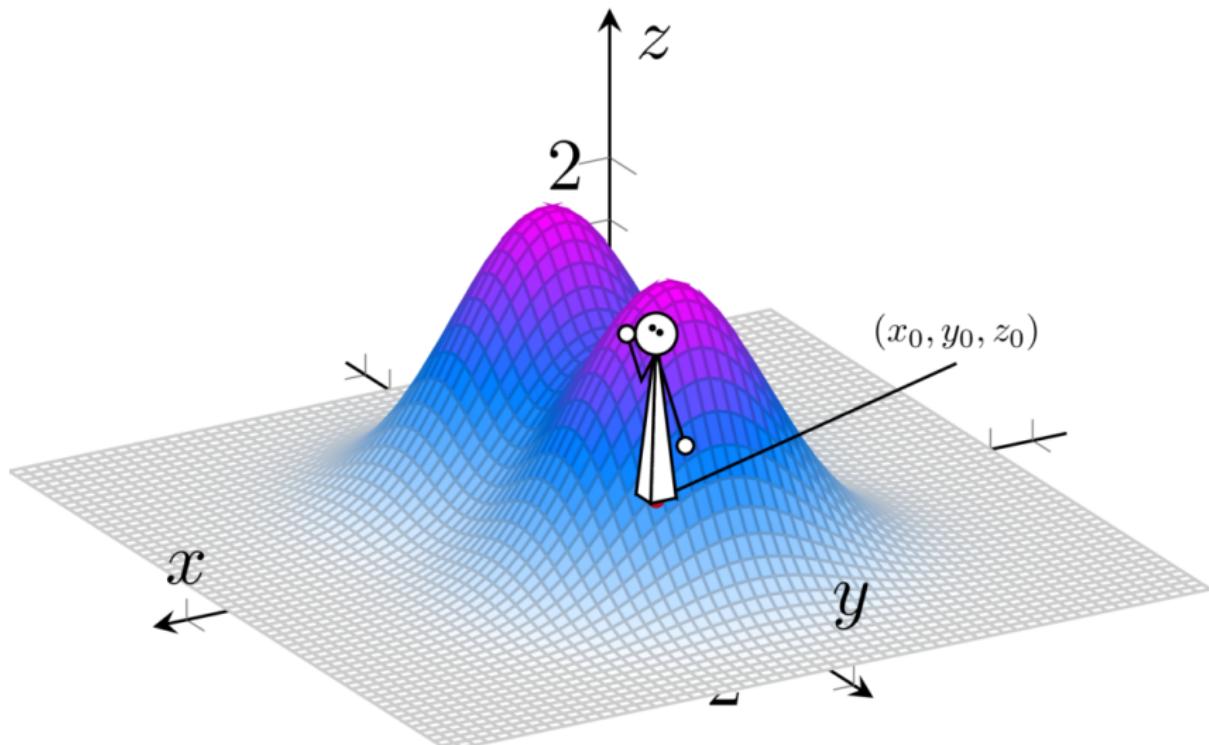
What is a Gradient Vector?



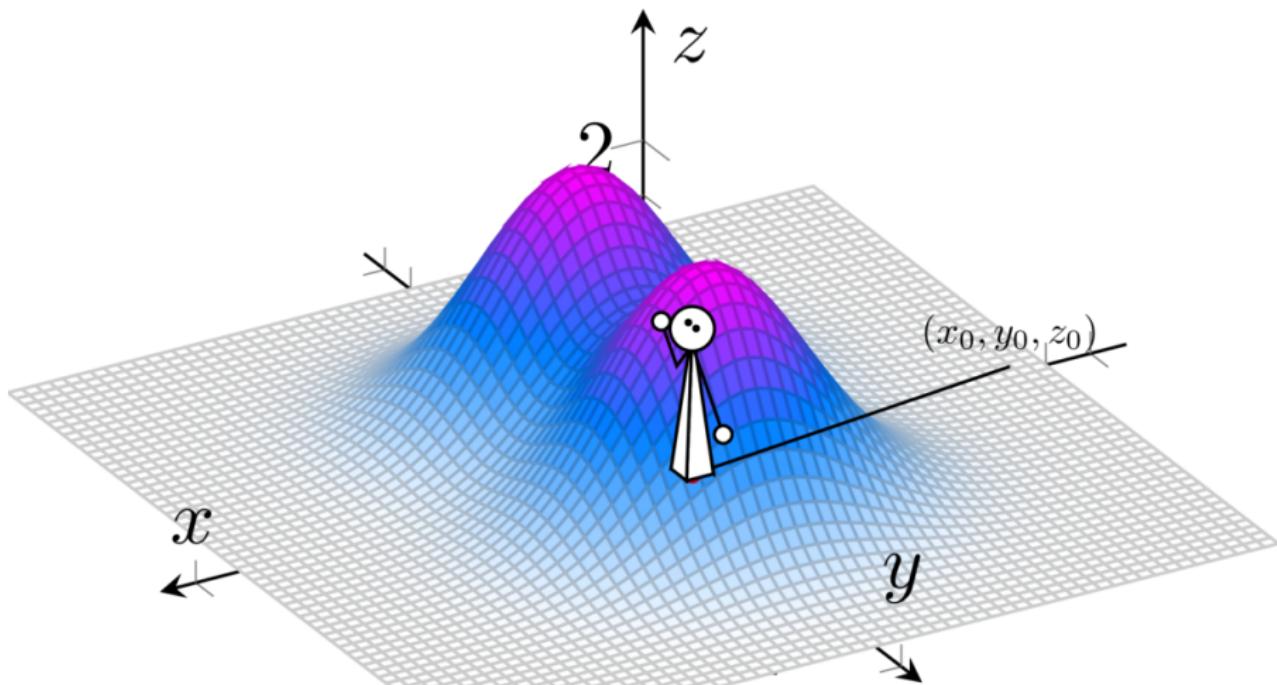
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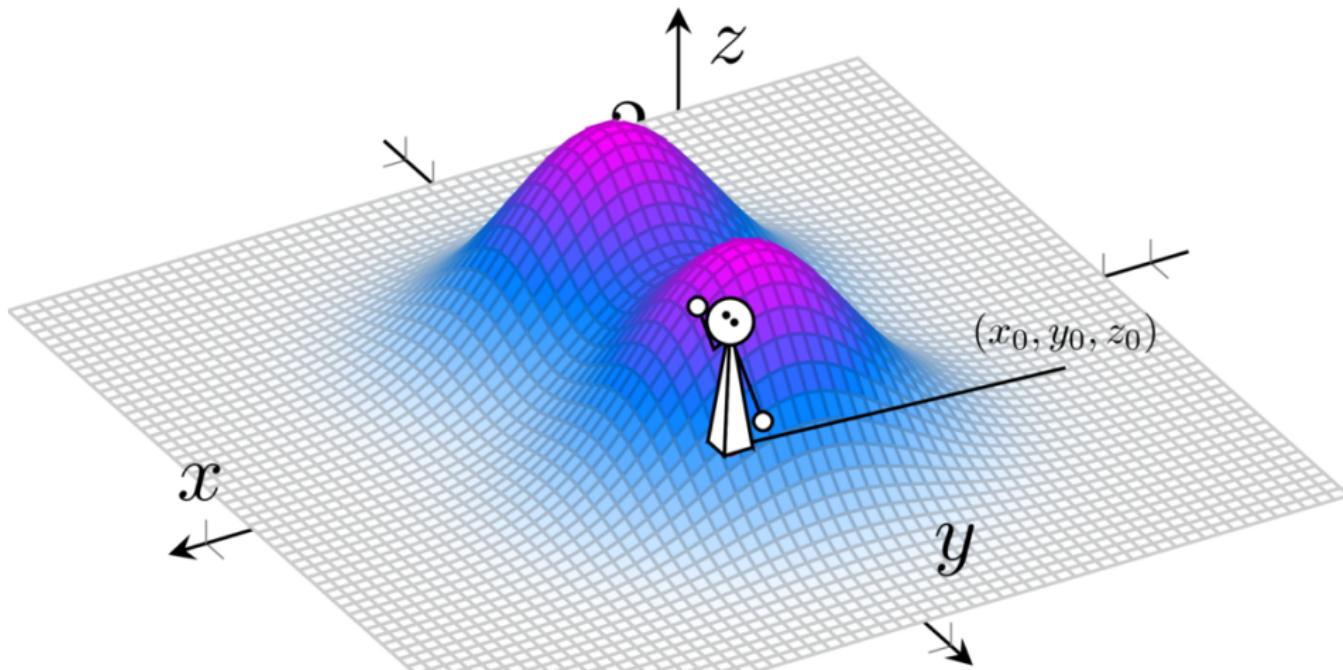
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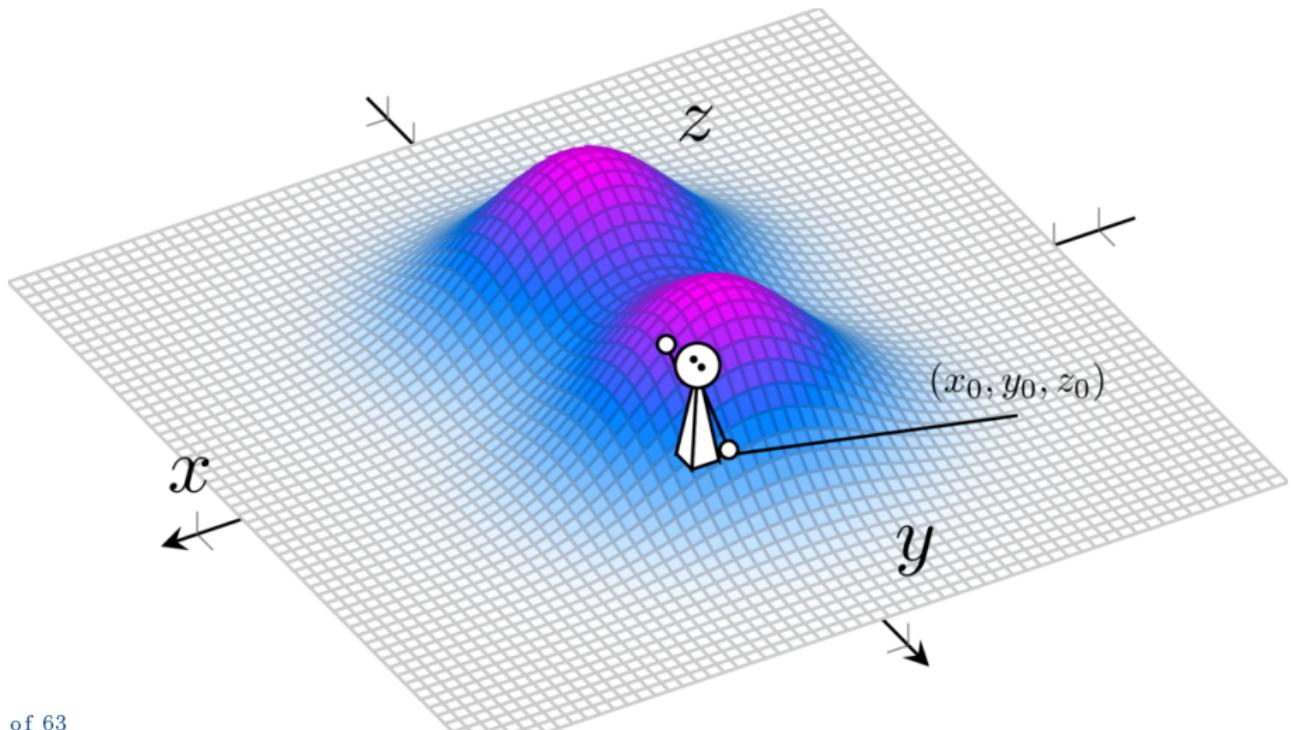
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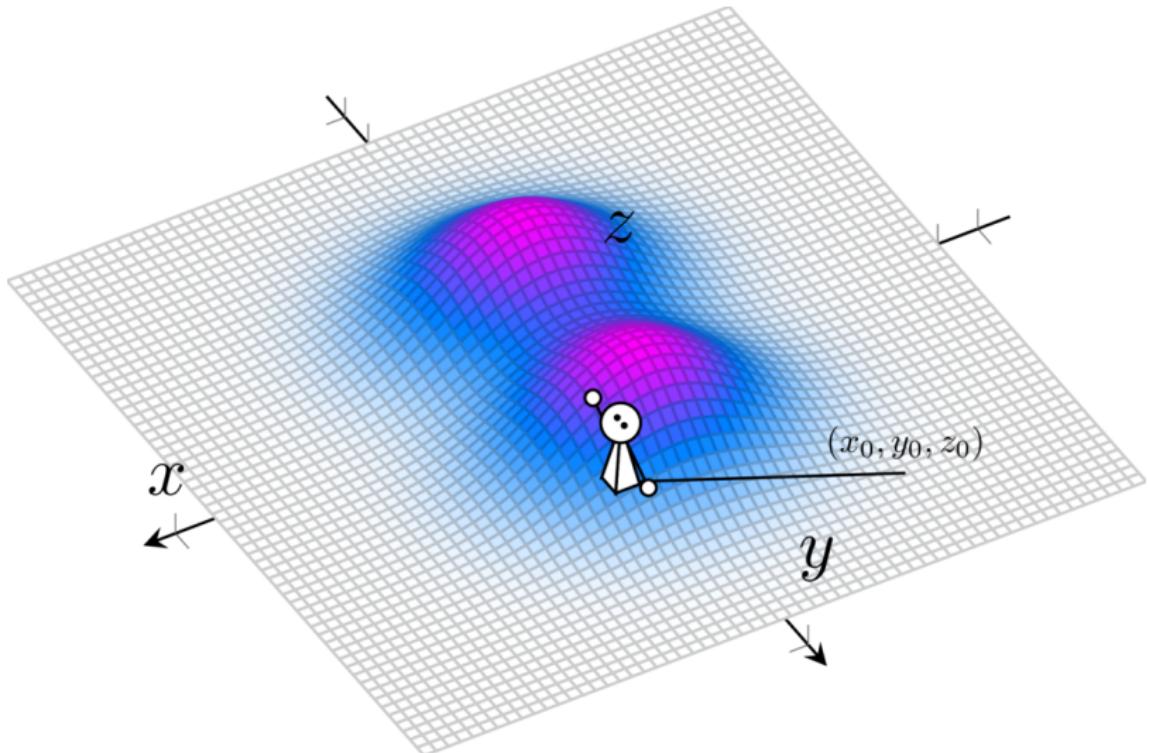
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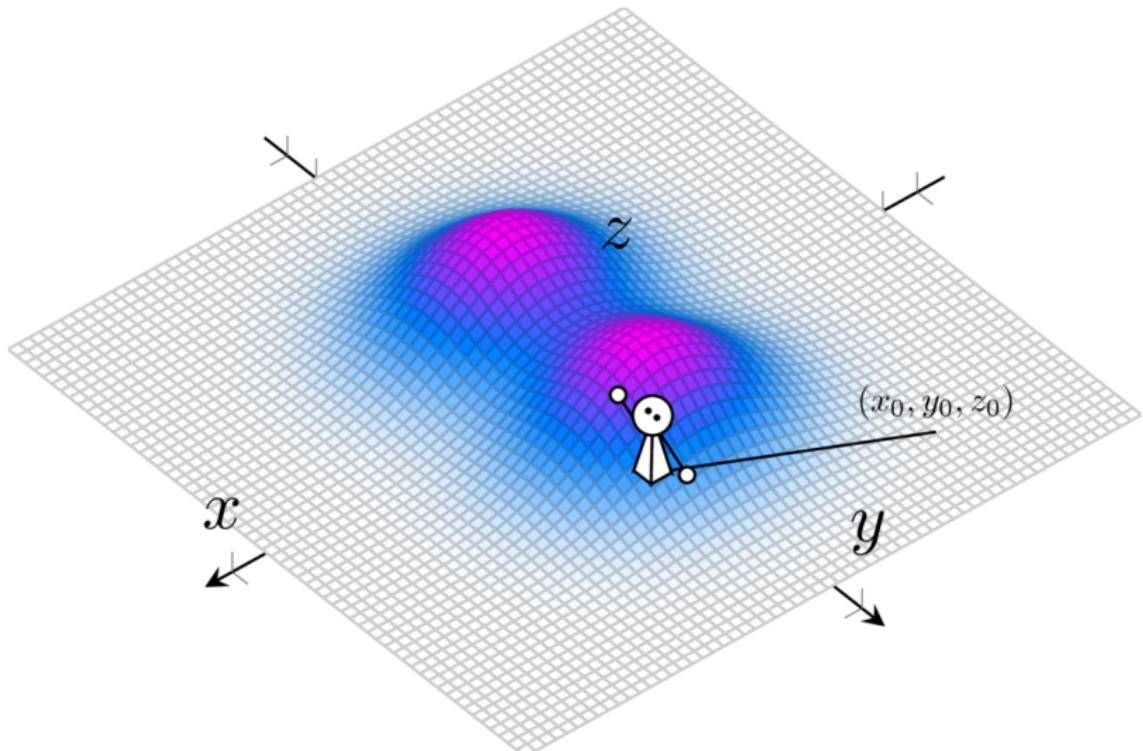
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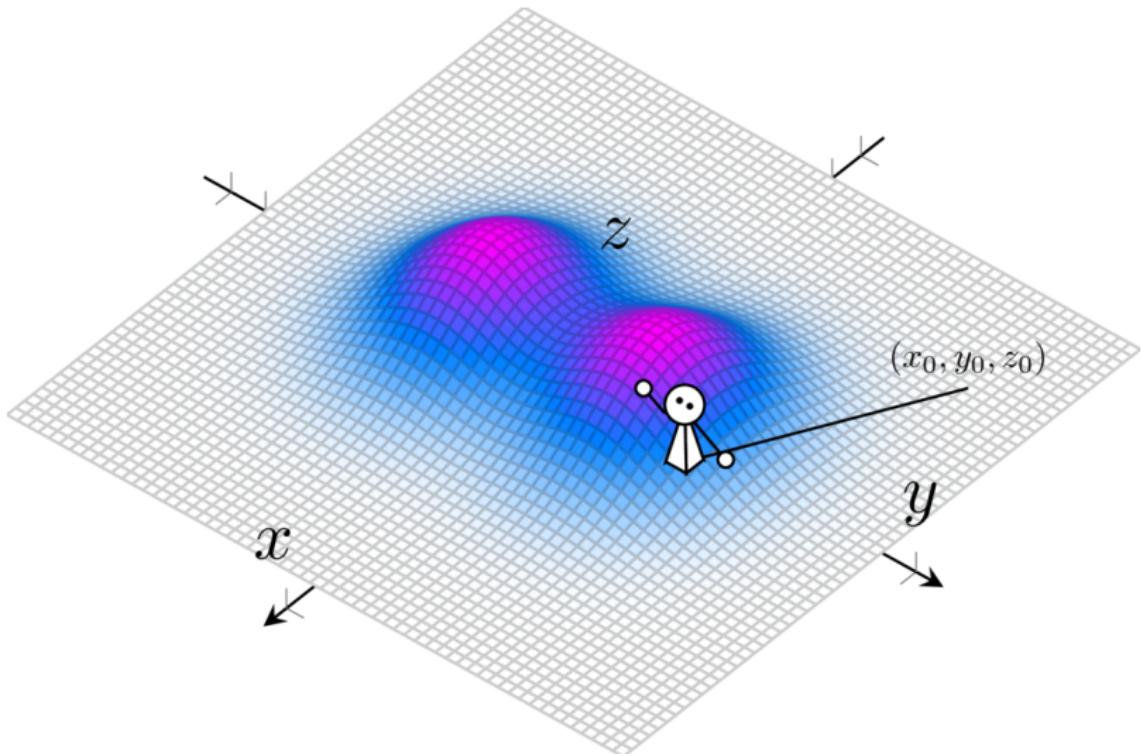
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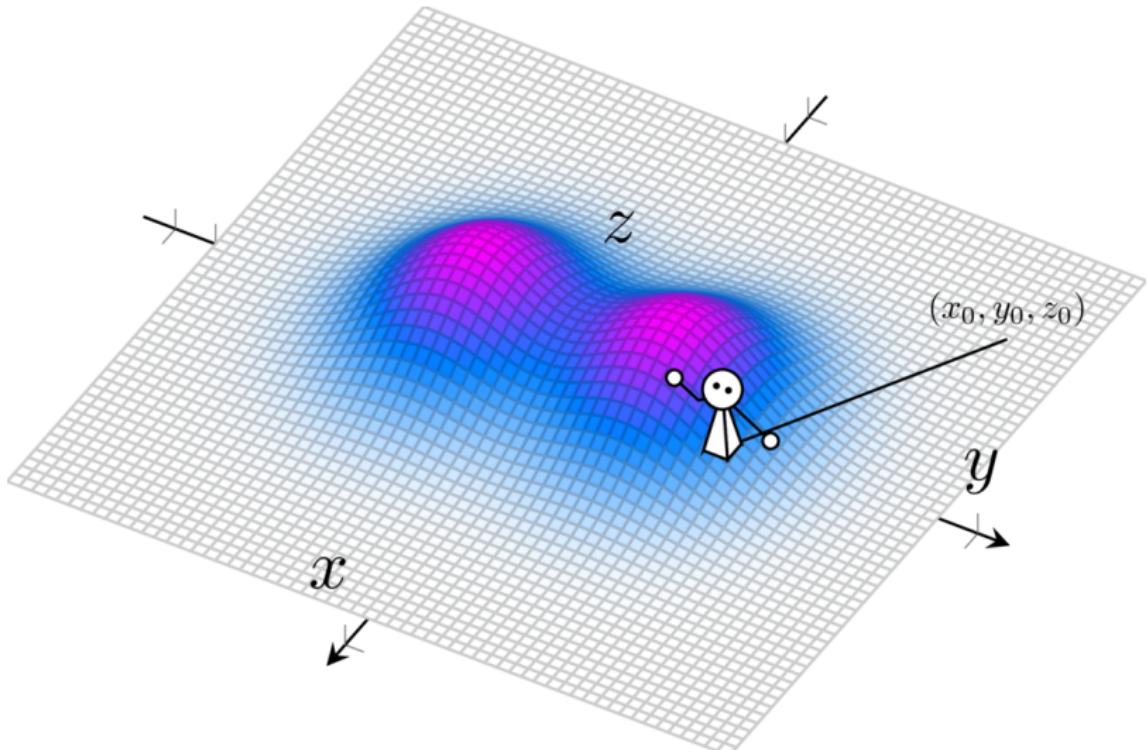
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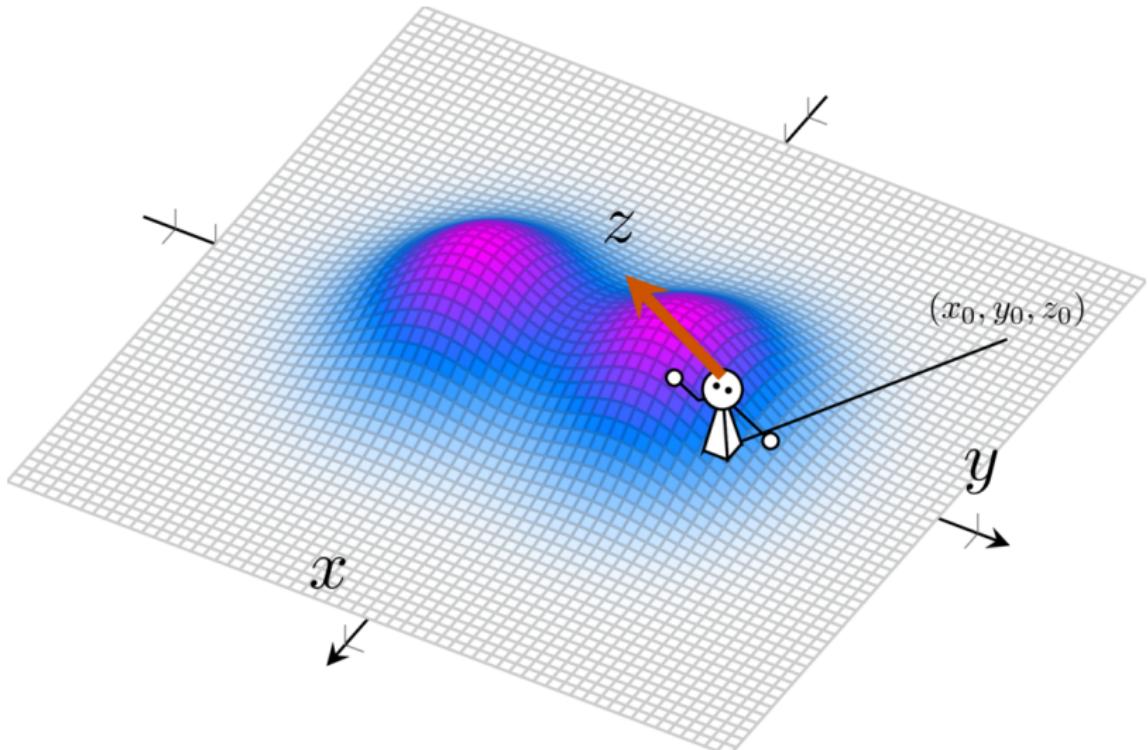
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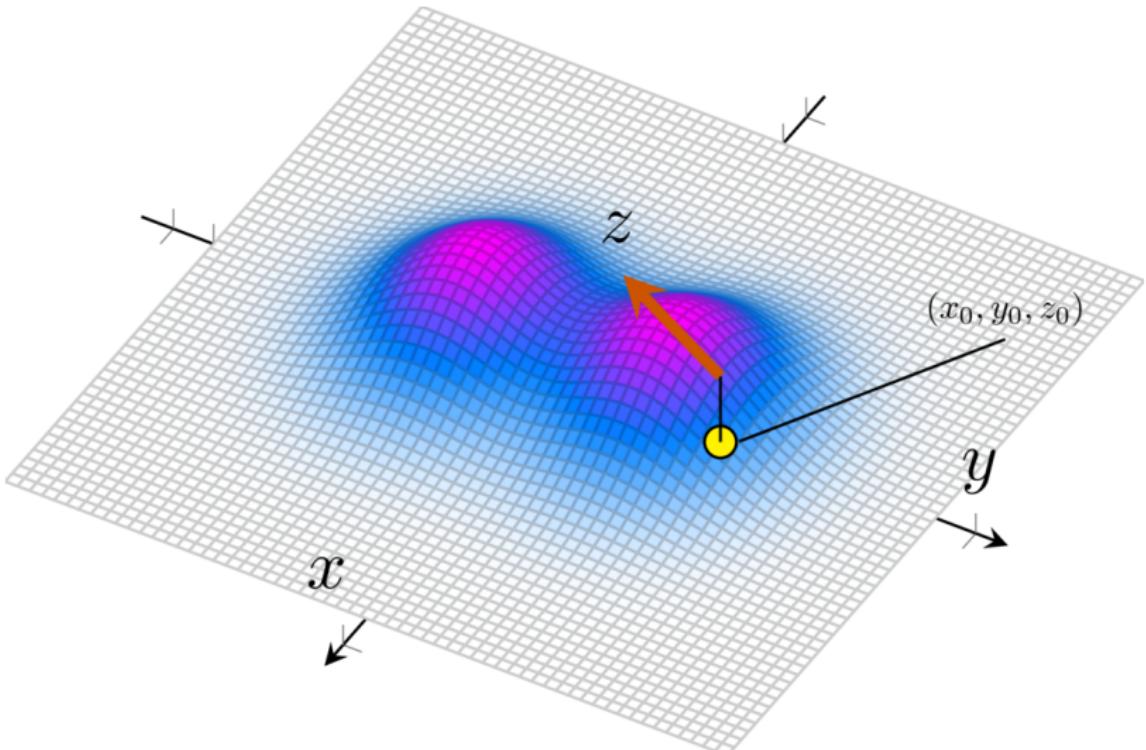
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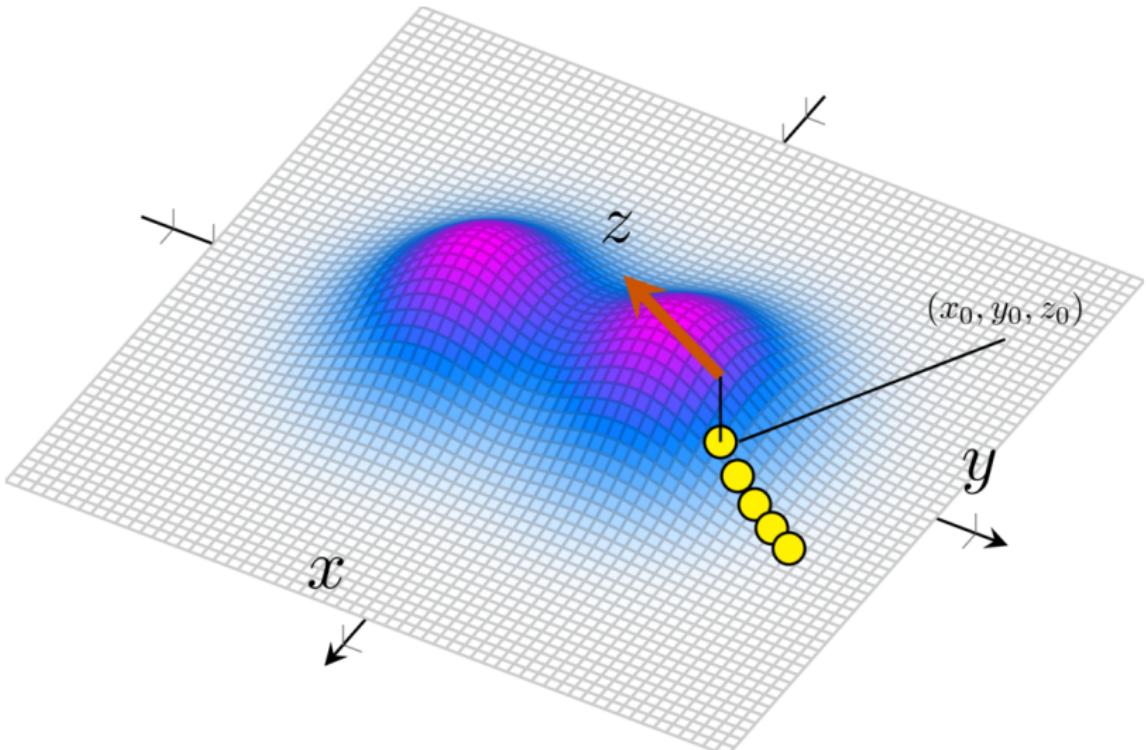
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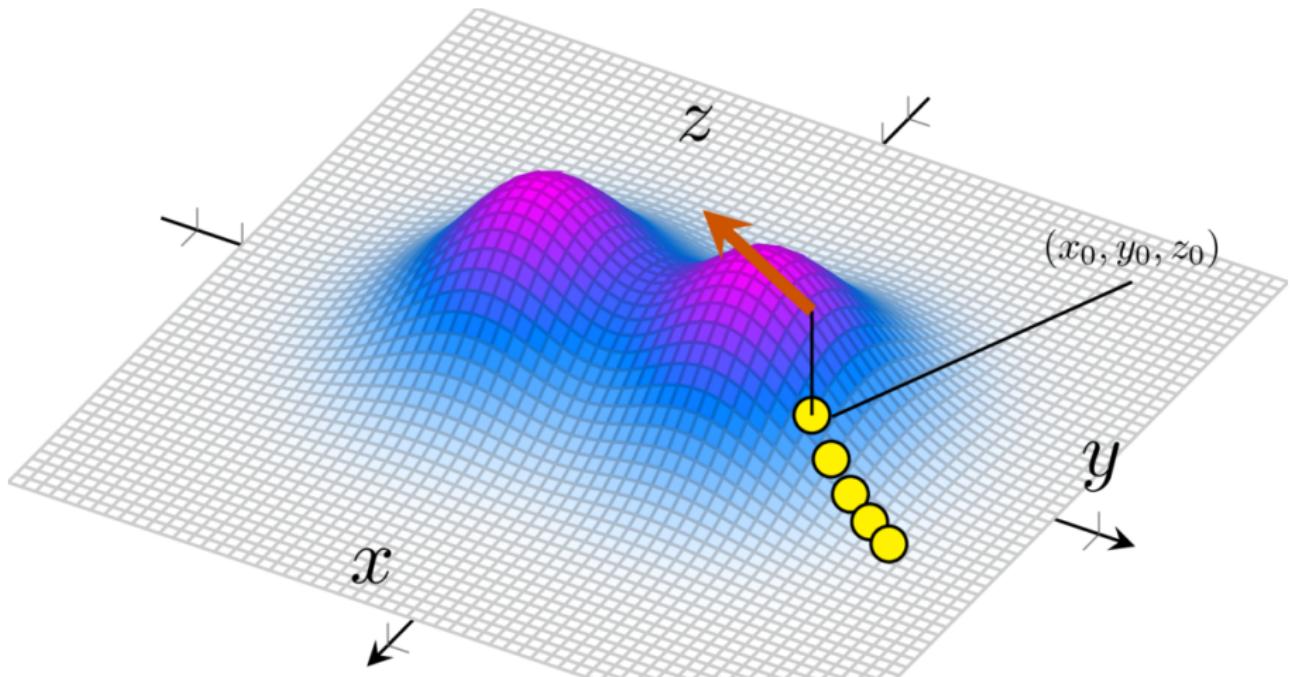
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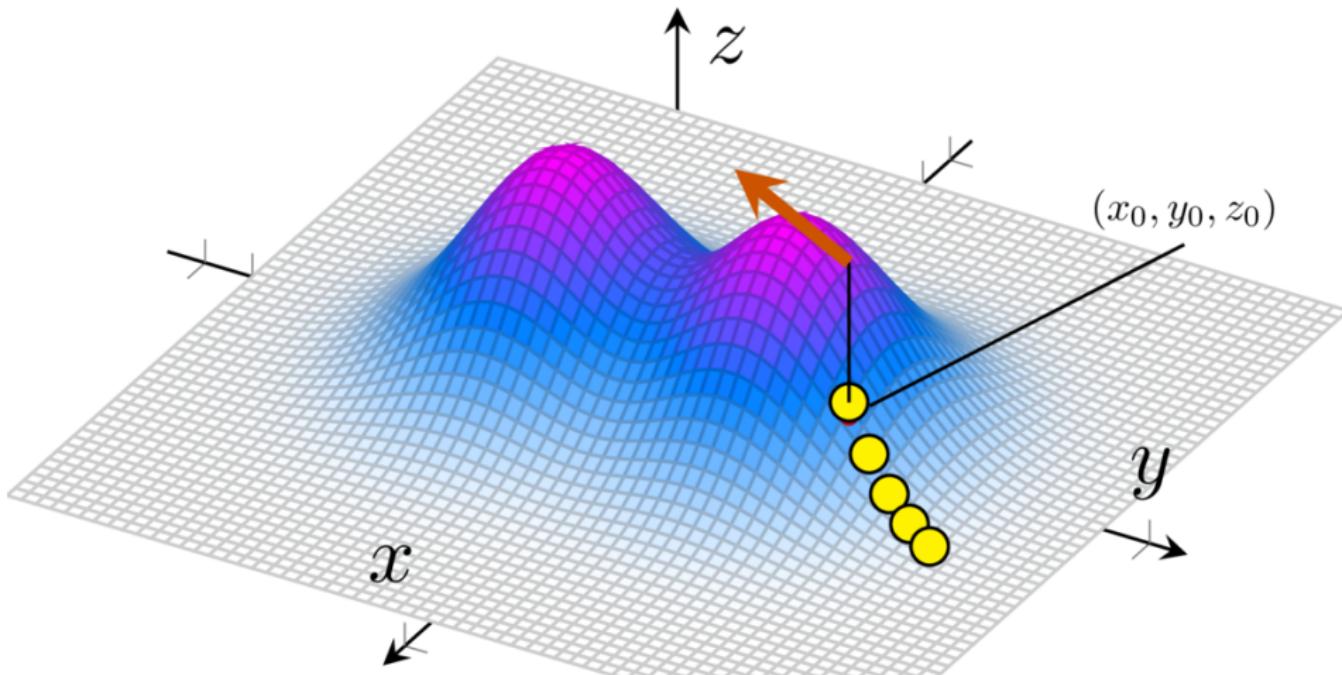
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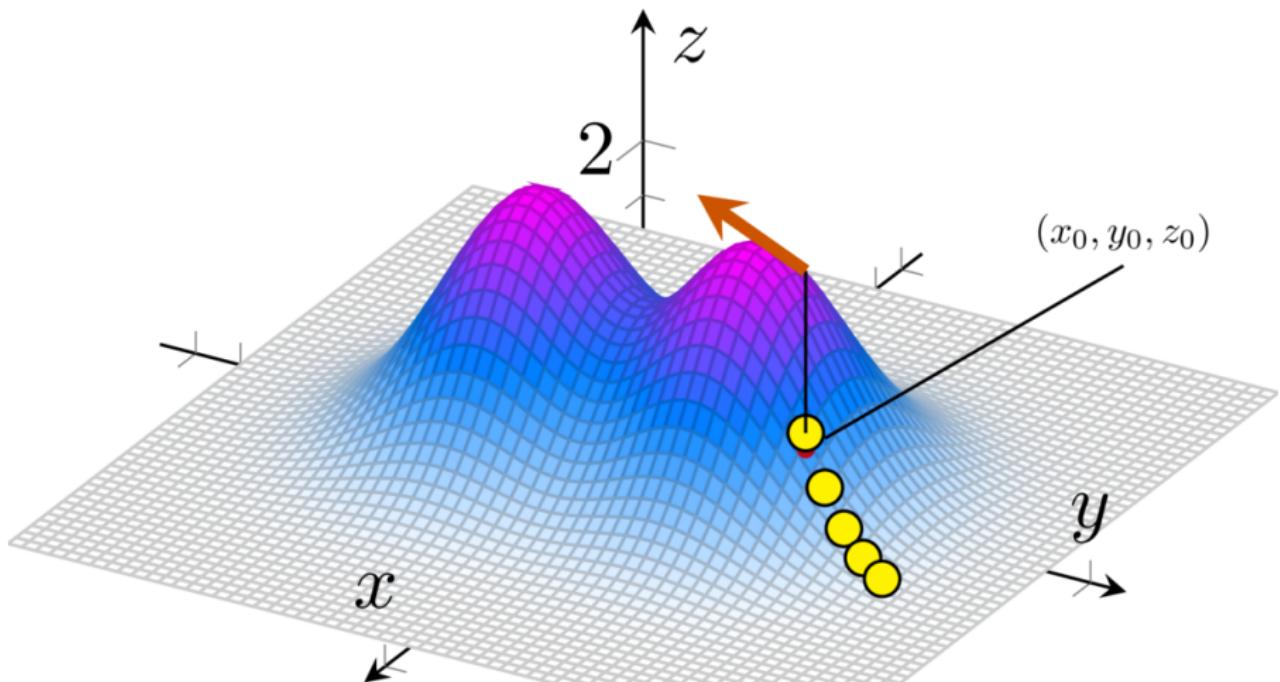
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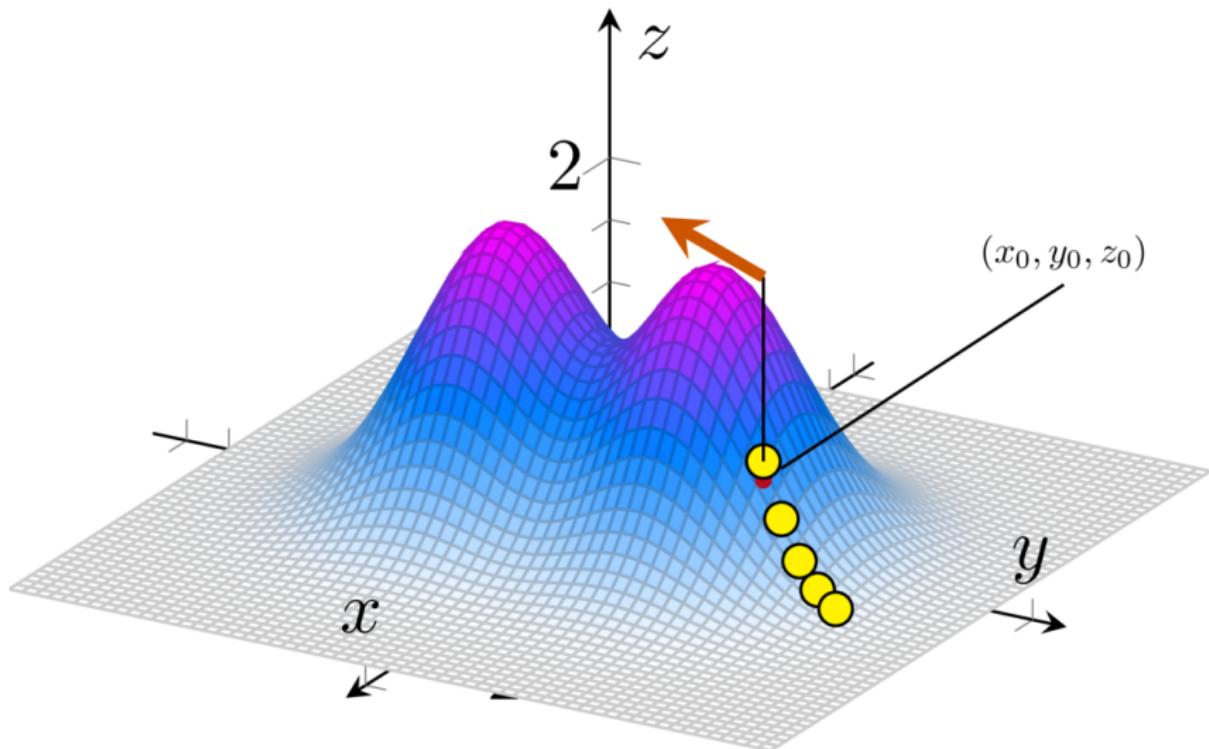
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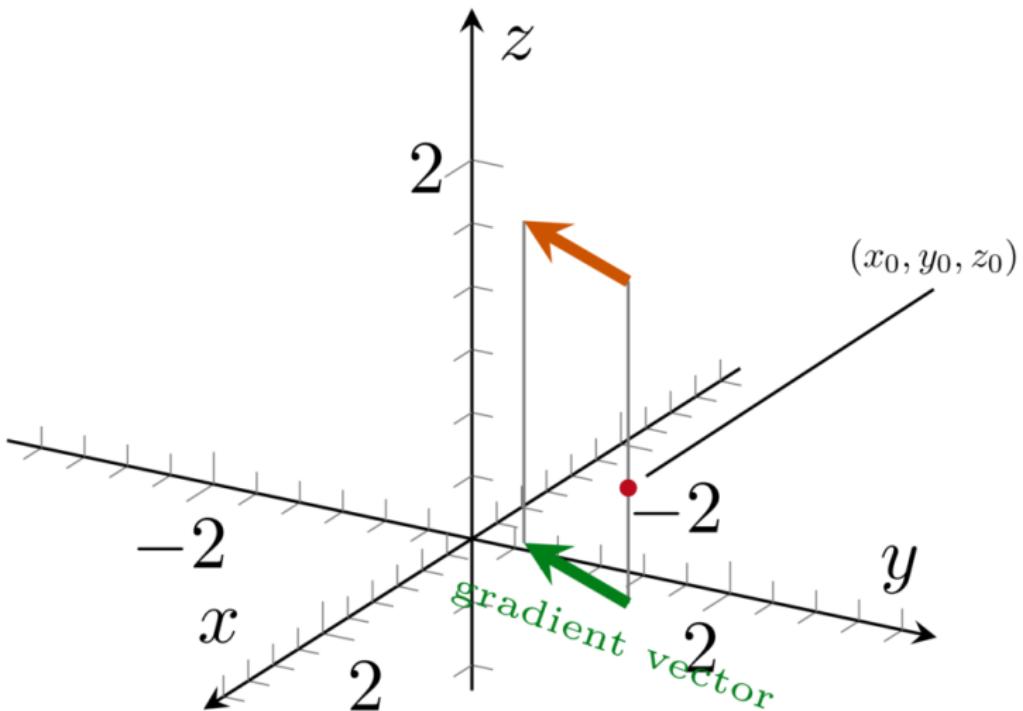
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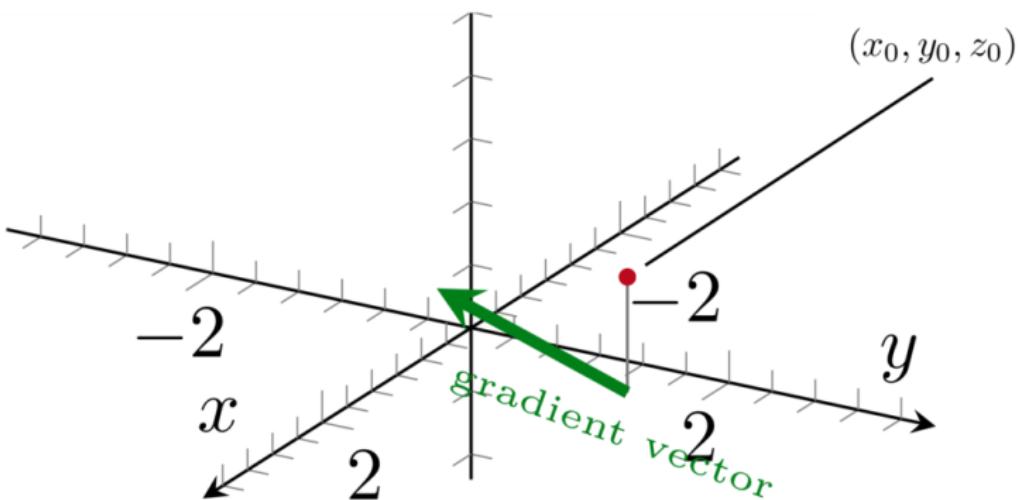
What is a Gradient Vector?



What is a Gradient Vector?

↑
 z

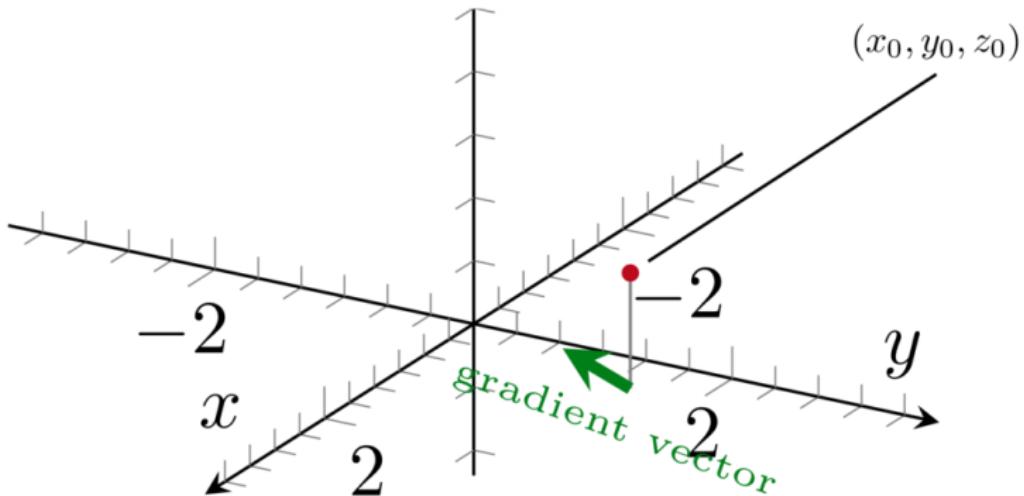
steep slope=long arrow



What is a Gradient Vector?

↑ z

shallow slope=short arrow



13.5 Directional Derivatives and Gradient Vector



Definition

The *gradient vector* of $f(x, y)$ is the vector

$$\nabla f = \frac{\partial f}{\partial x} \mathbf{i} + \frac{\partial f}{\partial y} \mathbf{j}.$$

∇ is pronounced “nabla” or “del”.



Harps, p. 984.

$$\nabla f = \frac{\partial f}{\partial x} \mathbf{i} + \frac{\partial f}{\partial y} \mathbf{j}$$

Example

Find the gradient vector of $f(x, y) = xe^y + \cos(xy)$ at the point $(2, 0)$.

$$\nabla f = \frac{\partial f}{\partial x} \mathbf{i} + \frac{\partial f}{\partial y} \mathbf{j}$$

Example

Find the gradient vector of $f(x, y) = xe^y + \cos(xy)$ at the point $(2, 0)$.

We calculate that

$$f_x(2, 0) =$$

$$f_y(2, 0) =$$

and

$$\nabla f \Big|_{(2,0)} = f_x(2, 0) \mathbf{i} + f_y(2, 0) \mathbf{j} = \quad .$$

$$\nabla f = \frac{\partial f}{\partial x} \mathbf{i} + \frac{\partial f}{\partial y} \mathbf{j}$$

Example

Find the gradient vector of $f(x, y) = xe^y + \cos(xy)$ at the point $(2, 0)$.

We calculate that

$$f_x(2, 0) = e^y - y \sin(xy) \Big|_{(2,0)} = e^0 - 0 = 1,$$

$$f_y(2, 0) =$$

and

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$$f_y(2, 0) = xe^y - x \sin(xy) \Big|_{(2,0)} = 2e^0 - 2 \cdot 0 = 2$$

and

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and

$$\nabla f \Big|_{(2,0)} = f_x(2, 0) \mathbf{i} + f_y(2, 0) \mathbf{j} = \mathbf{i} + 2\mathbf{j}.$$

13.5 Directional Derivatives and Gradient Vector



So how can we use gradient vectors to find directional derivatives?

13.5 Directional Derivatives and Gradient Vector



So how can we use gradient vectors to find directional derivatives?

Theorem

$$D_{\mathbf{u}} f = \nabla f \cdot \mathbf{u}.$$

$$D_{\mathbf{u}} f = \nabla f \cdot \mathbf{u} \quad \nabla f = \frac{\partial f}{\partial x} \mathbf{i} + \frac{\partial f}{\partial y} \mathbf{j}$$

Example

Find the derivative of $f(x, y) = xe^y + \cos(xy)$ at the point $(2, 0)$ in the direction of $\mathbf{v} = 3\mathbf{i} - 4\mathbf{j}$.

13.5

$$D_{\mathbf{u}}f = \nabla f \cdot \mathbf{u} \quad \nabla f = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j}$$



Example

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Recall that $\nabla f|_{(2,0)} = \mathbf{i} + 2\mathbf{j}$.

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Example

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Recall that $\nabla f|_{(2,0)} = \mathbf{i} + 2\mathbf{j}$. We need to find a unit vector \mathbf{u} which points in the same direction as \mathbf{v} ,

$$D_{\mathbf{u}}f = \nabla f \cdot \mathbf{u} \quad \nabla f = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j}$$

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$$\mathbf{u} = \frac{\mathbf{v}}{\|\mathbf{v}\|} = \frac{3\mathbf{i} - 4\mathbf{j}}{\sqrt{3^2 + (-4)^2}} = \frac{3}{5}\mathbf{i} - \frac{4}{5}\mathbf{j}.$$

$$D_{\mathbf{u}}f = \nabla f \cdot \mathbf{u} \quad \nabla f = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j}$$

Example

Find the derivative of $f(x, y) = xe^y + \cos(xy)$ at the point $(2, 0)$ in the direction of $\mathbf{v} = 3\mathbf{i} - 4\mathbf{j}$.

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Therefore

$$D_{\mathbf{u}}f(2, 0) = \nabla f|_{(2,0)} \cdot \mathbf{u} =$$

$$D_{\mathbf{u}}f = \nabla f \cdot \mathbf{u} \quad \nabla f = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j}$$

Example

Find the derivative of $f(x, y) = xe^y + \cos(xy)$ at the point $(2, 0)$ in the direction of $\mathbf{v} = 3\mathbf{i} - 4\mathbf{j}$.

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Therefore

$$D_{\mathbf{u}}f(2, 0) = \nabla f|_{(2,0)} \cdot \mathbf{u} = (\mathbf{i} + 2\mathbf{j}) \cdot \left(\frac{3}{5}\mathbf{i} - \frac{4}{5}\mathbf{j}\right) = \frac{3}{5} - \frac{8}{5} = -1.$$

$$\mathbf{a} \cdot \mathbf{b} = \|\mathbf{a}\| \|\mathbf{b}\| \cos \theta$$

Note that

$$D_{\mathbf{u}} f = \nabla f \cdot \mathbf{u} = \|\nabla f\| \|\mathbf{u}\| \cos \theta = \|\nabla f\| \cos \theta$$

since $\|\mathbf{u}\| = 1$.

$$\mathbf{a} \cdot \mathbf{b} = \|\mathbf{a}\| \|\mathbf{b}\| \cos \theta$$

Note that

$$D_{\mathbf{u}} f = \nabla f \cdot \mathbf{u} = \|\nabla f\| \|\mathbf{u}\| \cos \theta = \|\nabla f\| \cos \theta$$

since $\|\mathbf{u}\| = 1$.

So we must always have

$$-\|\nabla f\| \leq D_{\mathbf{u}} f \leq \|\nabla f\|.$$

$$D_{\mathbf{u}} f = \|\nabla f\| \cos \theta$$

Remark

f increases
mostly rapidly

$$\implies \cos \theta = 1 \implies \theta = 0$$

\mathbf{u} points in the
same direction
as ∇f

∇f points ‘uphill’

13.5

$$D_{\mathbf{u}} f = \|\nabla f\| \cos \theta$$



Remark

f increases mostly rapidly $\implies \cos \theta = 1 \implies \theta = 0 \implies \mathbf{u}$ points in the same direction as ∇f

∇f points ‘uphill’

Remark

f decreases mostly rapidly $\implies \cos \theta = -1 \implies \theta = 180^\circ \implies \mathbf{u}$ points in the opposite direction from ∇f

a ball on a hill rolls in the direction $-\nabla f$

$$D_{\mathbf{u}} f = \|\nabla f\| \cos \theta$$

Remark

f increases mostly rapidly $\implies \cos \theta = 1 \implies \theta = 0 \implies \mathbf{u}$ points in the same direction as ∇f

∇f points ‘uphill’

Remark

f decreases mostly rapidly $\implies \cos \theta = -1 \implies \theta = 180^\circ \implies \mathbf{u}$ points in the opposite direction from ∇f

a ball on a hill rolls in the direction $-\nabla f$

Remark

$$\theta = 90^\circ \implies D_{\mathbf{u}} f = 0.$$

EXAMPLE 3 Find the directions in which $f(x, y) = (x^2/2) + (y^2/2)$

- (a) increases most rapidly at the point $(1, 1)$.
- (b) decreases most rapidly at $(1, 1)$.
- (c) What are the directions of zero change in f at $(1, 1)$?

Solution

- (a) The function increases most rapidly in the direction of ∇f at $(1, 1)$. The gradient there is

$$(\nabla f)_{(1,1)} = (x\mathbf{i} + y\mathbf{j})_{(1,1)} = \mathbf{i} + \mathbf{j}.$$

Its direction is

$$\mathbf{u} = \frac{\mathbf{i} + \mathbf{j}}{|\mathbf{i} + \mathbf{j}|} = \frac{\mathbf{i} + \mathbf{j}}{\sqrt{(1)^2 + (1)^2}} = \frac{1}{\sqrt{2}}\mathbf{i} + \frac{1}{\sqrt{2}}\mathbf{j}.$$

(b) The function decreases most rapidly in the direction of $-\nabla f$ at $(1, 1)$, which is

$$-\mathbf{u} = -\frac{1}{\sqrt{2}}\mathbf{i} - \frac{1}{\sqrt{2}}\mathbf{j}.$$

(c) The directions of zero change at $(1, 1)$ are the directions orthogonal to ∇f :

$$\mathbf{n} = -\frac{1}{\sqrt{2}}\mathbf{i} + \frac{1}{\sqrt{2}}\mathbf{j} \quad \text{and} \quad -\mathbf{n} = \frac{1}{\sqrt{2}}\mathbf{i} - \frac{1}{\sqrt{2}}\mathbf{j}.$$

$$D_{\mathbf{u}} f = \|\nabla f\| \cos \theta$$

Algebra Rules for ∇

Theorem

- 1 *Sum Rule:* $\nabla(f + g) = \nabla f + \nabla g$
- 2 *Difference Rule:* $\nabla(f - g) = \nabla f - \nabla g$
- 3 *Constant Multiple Rule:* $\nabla(kf) = k\nabla f$ (for $k \in \mathbb{R}$)
- 4 *Product Rule:* $\nabla(fg) = g\nabla f + f\nabla g$
- 5 *Quotient Rule:* $\nabla\left(\frac{f}{g}\right) = \frac{g\nabla f - f\nabla g}{g^2}.$

EXAMPLE 5

We illustrate two of the rules with

$$\begin{aligned}f(x, y) &= x - y & g(x, y) &= 3y \\ \nabla f &= \mathbf{i} - \mathbf{j} & \nabla g &= 3\mathbf{j}.\end{aligned}$$

We have

1. $\nabla(f - g) = \nabla(x - 4y) = \mathbf{i} - 4\mathbf{j} = \nabla f - \nabla g$ Rule 2
2. $\nabla(fg) = \nabla(3xy - 3y^2) = 3y\mathbf{i} + (3x - 6y)\mathbf{j}$

and

$$\begin{aligned}f\nabla g + g\nabla f &= (x - y)3\mathbf{j} + 3y(\mathbf{i} - \mathbf{j}) && \text{Substitute.} \\ &= 3y\mathbf{i} + (3x - 6y)\mathbf{j}. && \text{Simplify.}\end{aligned}$$

We have therefore verified for this example that $\nabla(fg) = f\nabla g + g\nabla f$.

$$D_{\mathbf{u}} f = \|\nabla f\| \cos \theta$$



Functions of Three Variables

$$\nabla f = \frac{\partial f}{\partial x} \mathbf{i} + \frac{\partial f}{\partial y} \mathbf{j} + \frac{\partial f}{\partial z} \mathbf{k}$$

$$D_{\mathbf{u}} f = \nabla f \cdot \mathbf{u}.$$

EXAMPLE 6

- (a) Find the derivative of $f(x, y, z) = x^3 - xy^2 - z$ at $P_0(1, 1, 0)$ in the direction of $\mathbf{v} = 2\mathbf{i} - 3\mathbf{j} + 6\mathbf{k}$.
- (b) In what directions does f change most rapidly at P_0 , and what are the rates of change in these directions?

Solution

- (a) The direction of \mathbf{v} is obtained by dividing \mathbf{v} by its length:

$$|\mathbf{v}| = \sqrt{(2)^2 + (-3)^2 + (6)^2} = \sqrt{49} = 7$$

$$\mathbf{u} = \frac{\mathbf{v}}{|\mathbf{v}|} = \frac{2}{7}\mathbf{i} - \frac{3}{7}\mathbf{j} + \frac{6}{7}\mathbf{k}.$$

The partial derivatives of f at P_0 are

$$f_x = (3x^2 - y^2) \Big|_{(1, 1, 0)} = 2, \quad f_y = -2xy \Big|_{(1, 1, 0)} = -2, \quad f_z = -1 \Big|_{(1, 1, 0)} = -1.$$

The gradient of f at P_0 is

$$\nabla f \Big|_{(1, 1, 0)} = 2\mathbf{i} - 2\mathbf{j} - \mathbf{k}.$$

The derivative of f at P_0 in the direction of \mathbf{v} is therefore

$$\begin{aligned}D_{\mathbf{u}}f|_{(1,1,0)} &= \nabla f|_{(1,1,0)} \cdot \mathbf{u} = (2\mathbf{i} - 2\mathbf{j} - \mathbf{k}) \cdot \left(\frac{2}{7}\mathbf{i} - \frac{3}{7}\mathbf{j} + \frac{6}{7}\mathbf{k}\right) \\&= \frac{4}{7} + \frac{6}{7} - \frac{6}{7} = \frac{4}{7}.\end{aligned}$$

- (b) The function increases most rapidly in the direction of $\nabla f = 2\mathbf{i} - 2\mathbf{j} - \mathbf{k}$ and decreases most rapidly in the direction of $-\nabla f$. The rates of change in the directions are, respectively,

$$|\nabla f| = \sqrt{(2)^2 + (-2)^2 + (-1)^2} = \sqrt{9} = 3 \quad \text{and} \quad -|\nabla f| = -3.$$

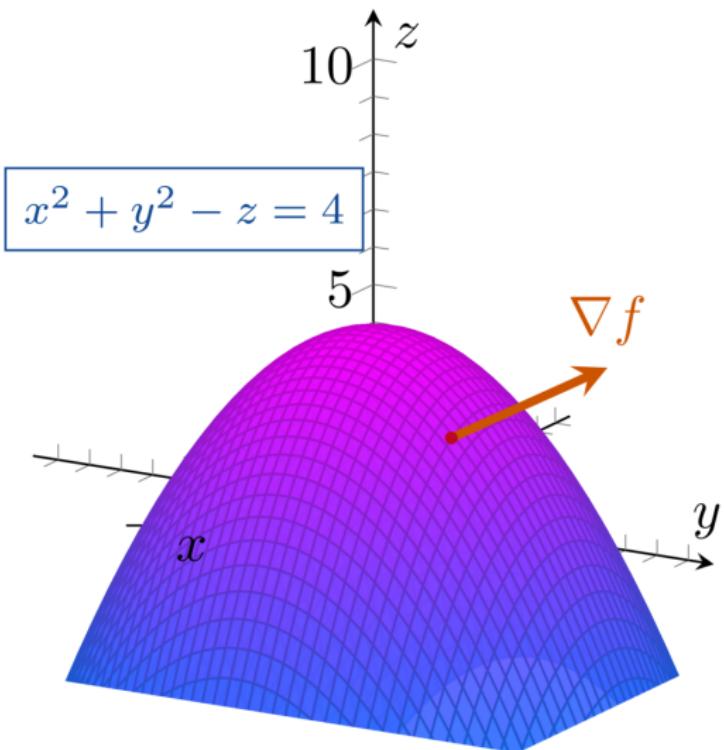




11 Tangent Planes and Differentials

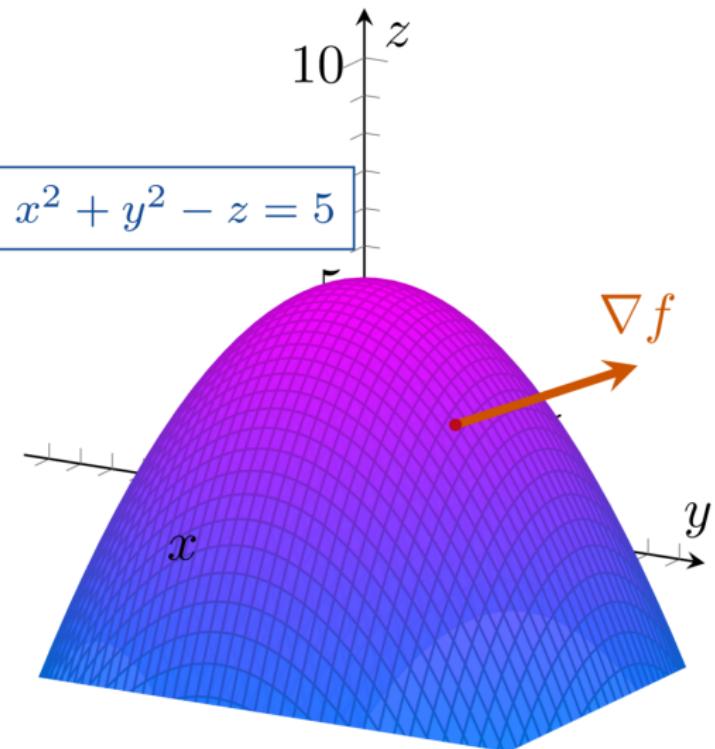
$$D_{\mathbf{u}} f = \|\nabla f\| \cos \theta$$

Tangent Planes and Normal Lines



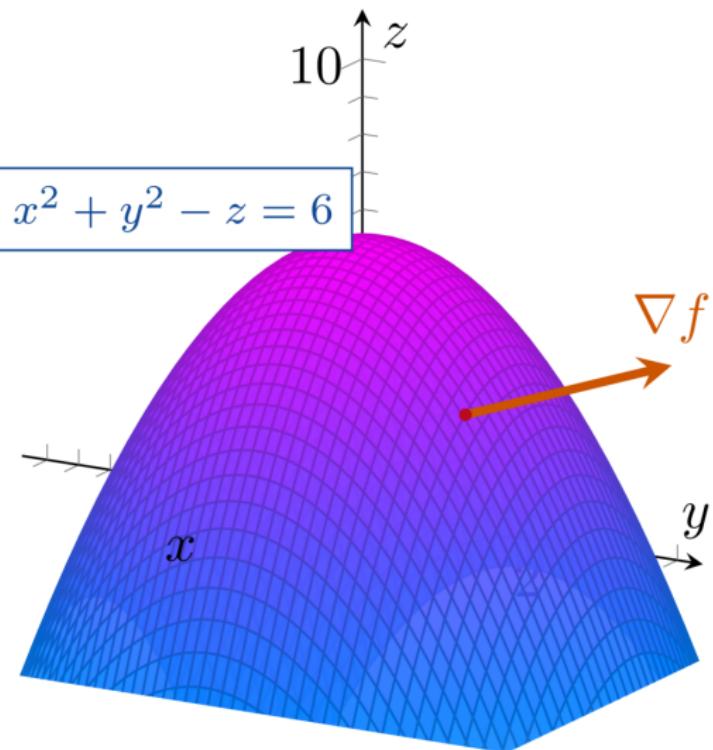
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Tangent Planes and Normal Lines



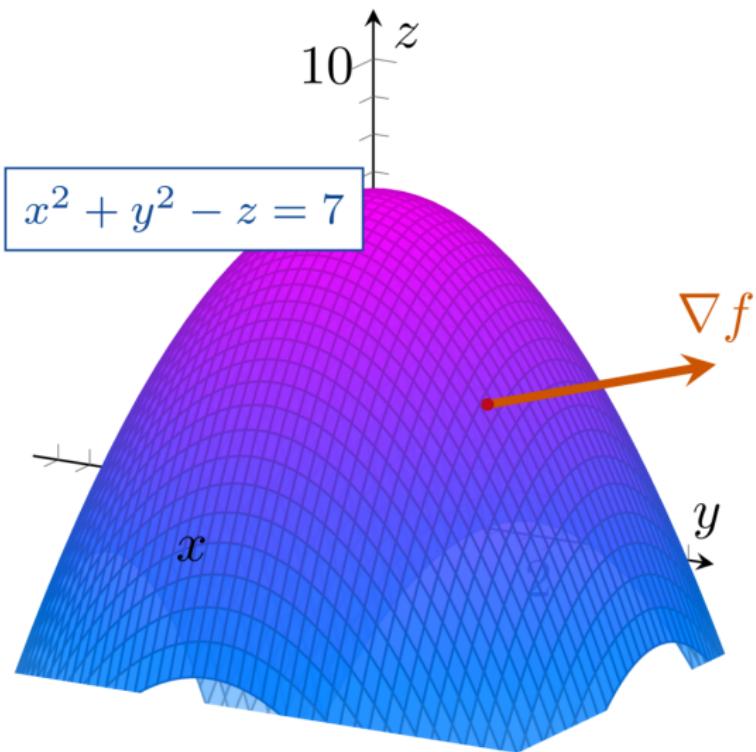
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Tangent Planes and Normal Lines



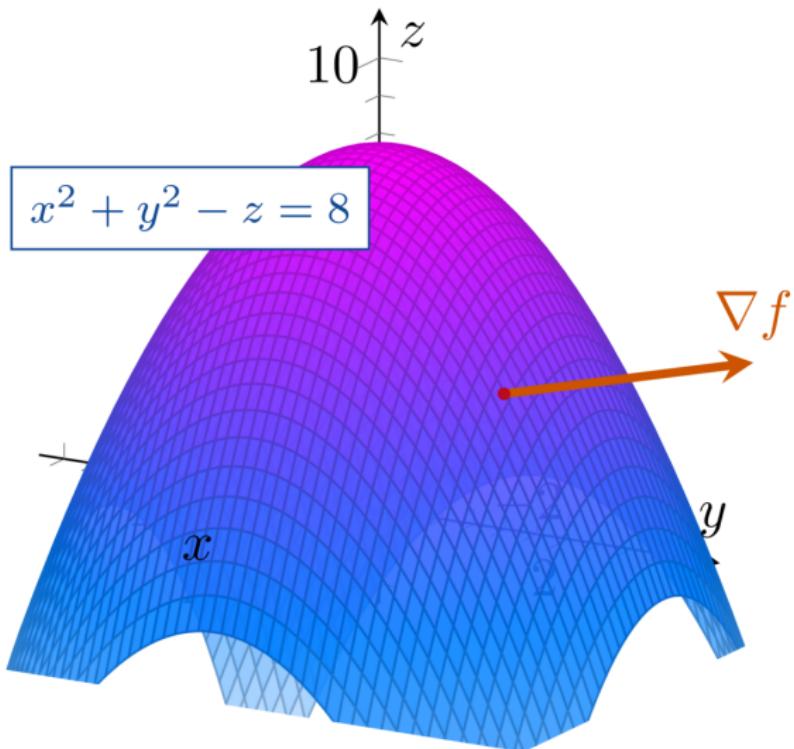
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Tangent Planes and Normal Lines



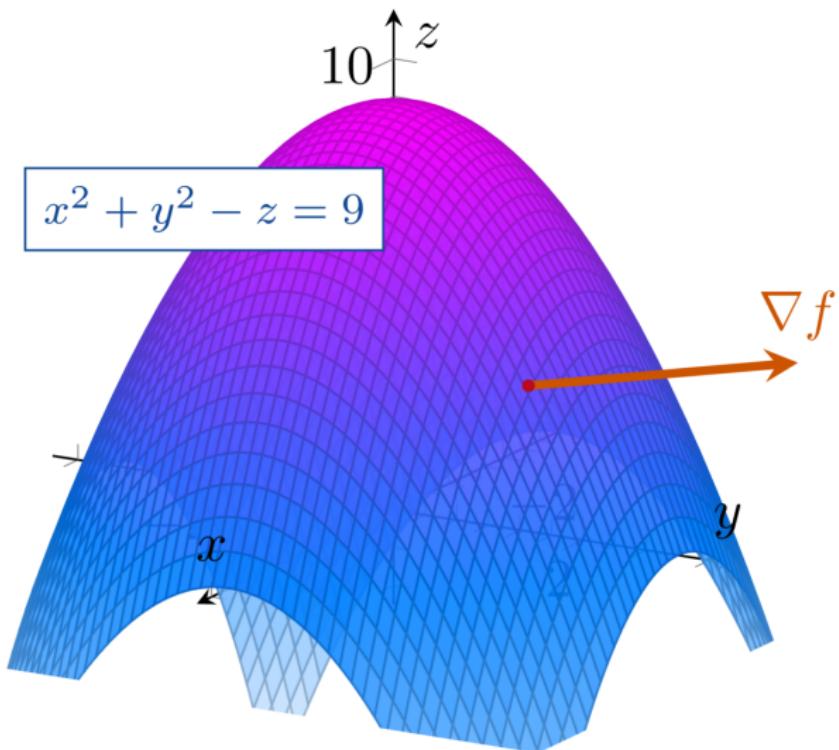
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Tangent Planes and Normal Lines



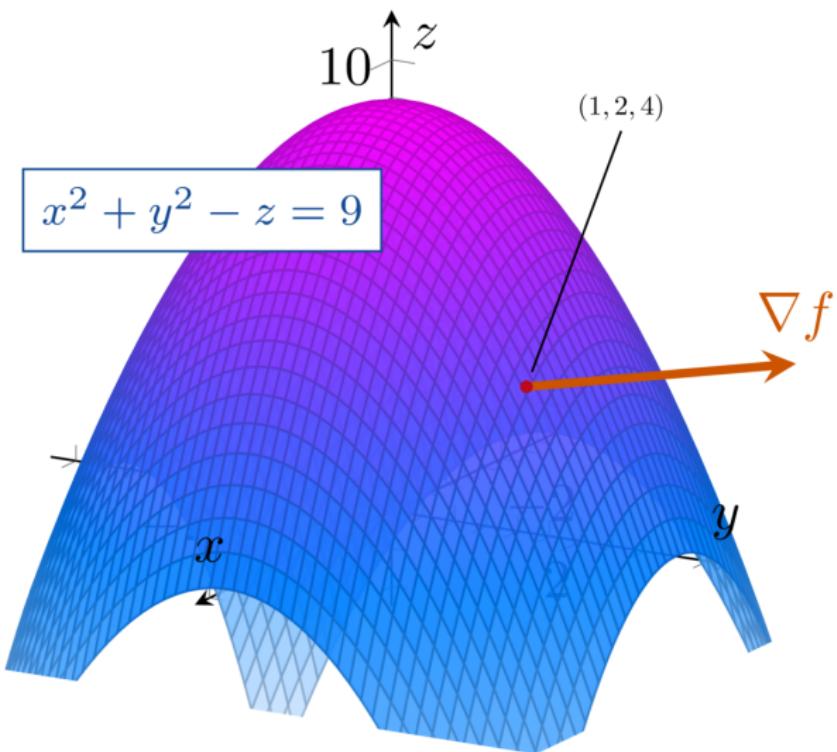
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Tangent Planes and Normal Lines



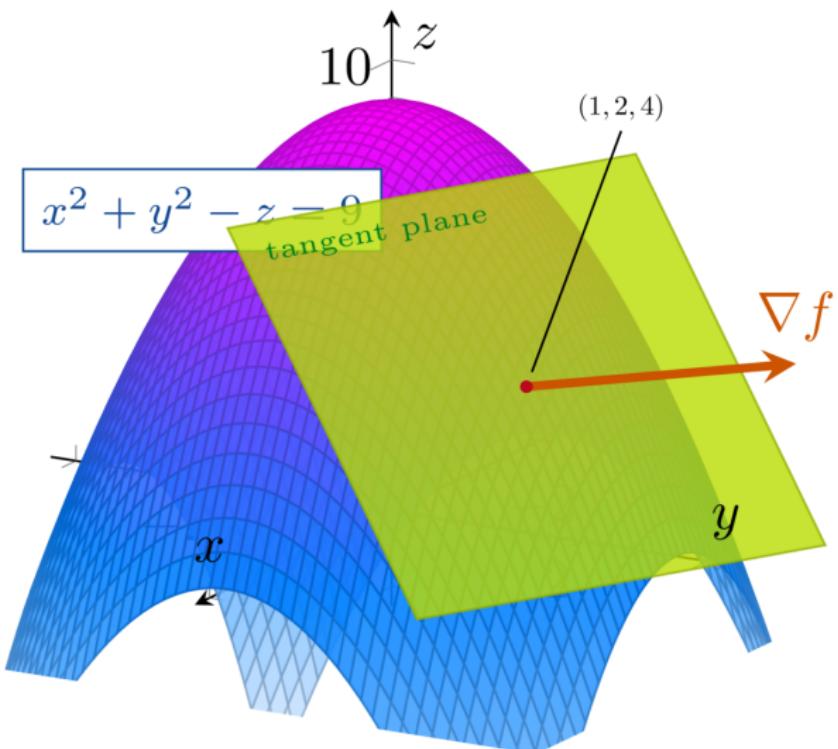
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Tangent Planes and Normal Lines



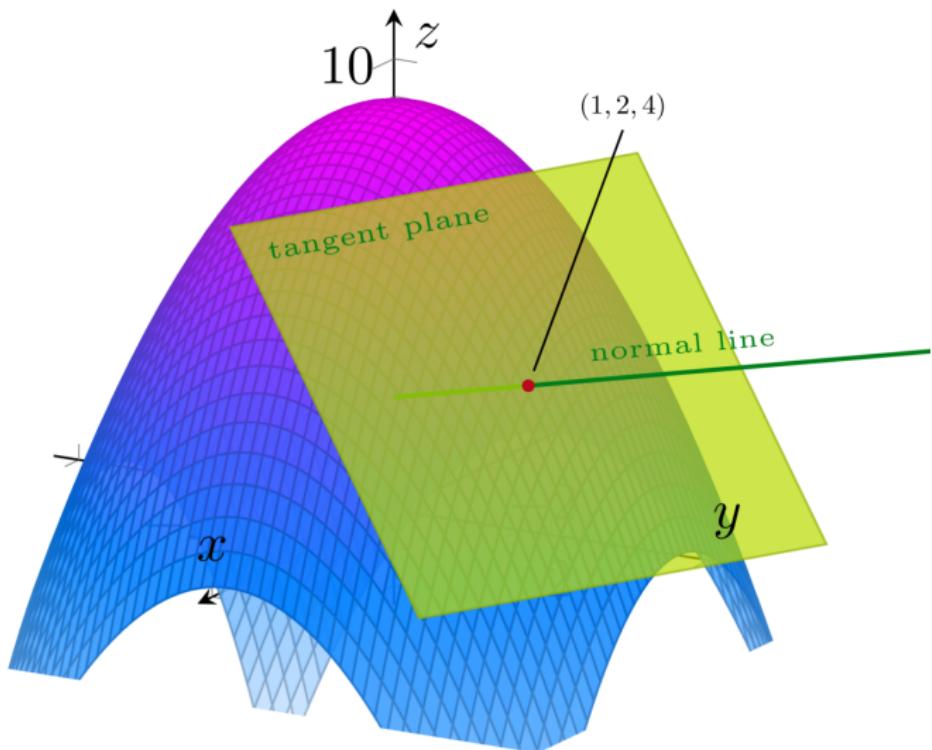
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Tangent Planes and Normal Lines



$$D_{\mathbf{u}} f = \|\nabla f\| \cos \theta$$

Tangent Planes and Normal Lines



$$D_{\mathbf{u}} f = \|\nabla f\| \cos \theta$$

Definition

The *tangent plane* to the surface $f(x, y, z) = c$ at the point $P(x_0, y_0, z_0)$ (where the gradient is not zero) is the plane through P with normal vector $\nabla f|_P$.

$$D_{\mathbf{u}} f = \|\nabla f\| \cos \theta$$

Definition

The *tangent plane* to the surface $f(x, y, z) = c$ at the point $P(x_0, y_0, z_0)$ (where the gradient is not zero) is the plane through P with normal vector $\nabla f|_P$.

$$f_x(P)(x - x_0) + f_y(P)(y - y_0) + f_z(P)(z - z_0) = 0.$$

$$D_{\mathbf{u}} f = \|\nabla f\| \cos \theta$$

Definition

The *tangent plane* to the surface $f(x, y, z) = c$ at the point $P(x_0, y_0, z_0)$ (where the gradient is not zero) is the plane through P with normal vector $\nabla f|_P$.

$$f_x(P)(x - x_0) + f_y(P)(y - y_0) + f_z(P)(z - z_0) = 0.$$

Definition

The *normal line* to the surface $f(x, y, z) = c$ at the point P is the line through P parallel to $\nabla f|_P$.

$$D_{\mathbf{u}} f = \|\nabla f\| \cos \theta$$

Definition

The *tangent plane* to the surface $f(x, y, z) = c$ at the point $P(x_0, y_0, z_0)$ (where the gradient is not zero) is the plane through P with normal vector $\nabla f|_P$.

$$f_x(P)(x - x_0) + f_y(P)(y - y_0) + f_z(P)(z - z_0) = 0.$$

Definition

The *normal line* to the surface $f(x, y, z) = c$ at the point P is the line through P parallel to $\nabla f|_P$.

$$x = x_0 + f_x(P)t \quad y = y_0 + f_y(P)t \quad z = z_0 + f_z(P)t.$$

EXAMPLE 1 Find the tangent plane and normal line of the level surface

$$f(x, y, z) = x^2 + y^2 + z - 9 = 0 \quad \text{A circular paraboloid}$$

at the point $P_0(1, 2, 4)$.

Solution The surface is shown in Figure 14.34.

The tangent plane is the plane through P_0 perpendicular to the gradient of f at P_0 . The gradient is

$$\nabla f|_{P_0} = (2x\mathbf{i} + 2y\mathbf{j} + \mathbf{k}) \Big|_{(1, 2, 4)} = 2\mathbf{i} + 4\mathbf{j} + \mathbf{k}.$$

The tangent plane is therefore the plane

$$2(x - 1) + 4(y - 2) + (z - 4) = 0, \quad \text{or} \quad 2x + 4y + z = 14.$$

The line normal to the surface at P_0 is

$$x = 1 + 2t, \quad y = 2 + 4t, \quad z = 4 + t. \quad \blacksquare$$

13.6

$$D_{\mathbf{u}} f = \|\nabla f\| \cos \theta$$



Now consider

$$z = f(x, y).$$

13.6

$$D_{\mathbf{u}} f = \|\nabla f\| \cos \theta$$



Now consider

$$z = f(x, y).$$

This is equivalent to

$$F(x, y, z) = f(x, y) - z = 0.$$

$$D_{\mathbf{u}} f = \|\nabla f\| \cos \theta$$

Now consider

$$z = f(x, y).$$

This is equivalent to

$$F(x, y, z) = f(x, y) - z = 0.$$

Definition

The *tangent plane* to the surface $z = f(x, y)$ at the point $P(x_0, y_0, z_0) = (x_0, y_0, f(x_0, y_0))$ is

$$f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0) - (z - z_0) = 0.$$

EXAMPLE 2 Find the plane tangent to the surface $z = x \cos y - ye^x$ at $(0, 0, 0)$.

Solution We calculate the partial derivatives of $f(x, y) = x \cos y - ye^x$ and use Equation (3):

$$f_x(0, 0) = (\cos y - ye^y) \Big|_{(0, 0)} = 1 - 0 \cdot 1 = 1$$

$$f_y(0, 0) = (-x \sin y - e^y) \Big|_{(0, 0)} = 0 - 1 = -1.$$

The tangent plane is therefore

$$1 \cdot (x - 0) - 1 \cdot (y - 0) - (z - 0) = 0, \quad \text{Eq. (3)}$$

or

$$x - y - z = 0.$$



$$D_{\mathbf{u}} f = \|\nabla f\| \cos \theta$$

EXAMPLE 3

The surfaces

$$f(x, y, z) = x^2 + y^2 - 2 = 0 \quad \text{A cylinder}$$

and

$$g(x, y, z) = x + z - 4 = 0 \quad \text{A plane}$$

meet in an ellipse E (Figure 14.35). Find parametric equations for the line tangent to E at the point $P_0(1, 1, 3)$.

$$D_{\mathbf{u}} f = \|\nabla f\| \cos \theta$$

EXAMPLE 3

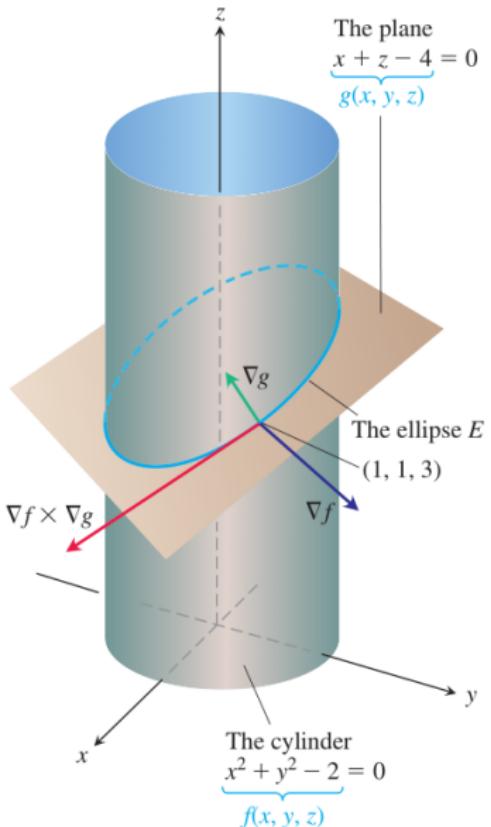
The surfaces

$$f(x, y, z) = x^2 + y^2 - 2$$

and

$$g(x, y, z) = x + z - 4$$

meet in an ellipse E (Figure 14.35). Find parameters of the point $P_0(1, 1, 3)$.



$$D_{\mathbf{u}} f = \|\nabla f\| \cos \theta$$

EXAMPLE 3

The surfaces

$$f(x, y, z) = x^2 + y^2 - 2 = 0 \quad \text{A cylinder}$$

and

$$g(x, y, z) = x + z - 4 = 0 \quad \text{A plane}$$

meet in an ellipse E (Figure 14.35). Find parametric equations for the line tangent to E at the point $P_0(1, 1, 3)$.

Solution The tangent line is orthogonal to both ∇f and ∇g at P_0 , and therefore parallel to $\mathbf{v} = \nabla f \times \nabla g$. The components of \mathbf{v} and the coordinates of P_0 give us equations for the line. We have

$$\nabla f|_{(1, 1, 3)} = (2x\mathbf{i} + 2y\mathbf{j}) \Big|_{(1, 1, 3)} = 2\mathbf{i} + 2\mathbf{j}$$

$$\nabla g|_{(1, 1, 3)} = (\mathbf{i} + \mathbf{k}) \Big|_{(1, 1, 3)} = \mathbf{i} + \mathbf{k}$$

$$\mathbf{v} = (2\mathbf{i} + 2\mathbf{j}) \times (\mathbf{i} + \mathbf{k}) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 2 & 0 \\ 1 & 0 & 1 \end{vmatrix} = 2\mathbf{i} - 2\mathbf{j} - 2\mathbf{k}.$$

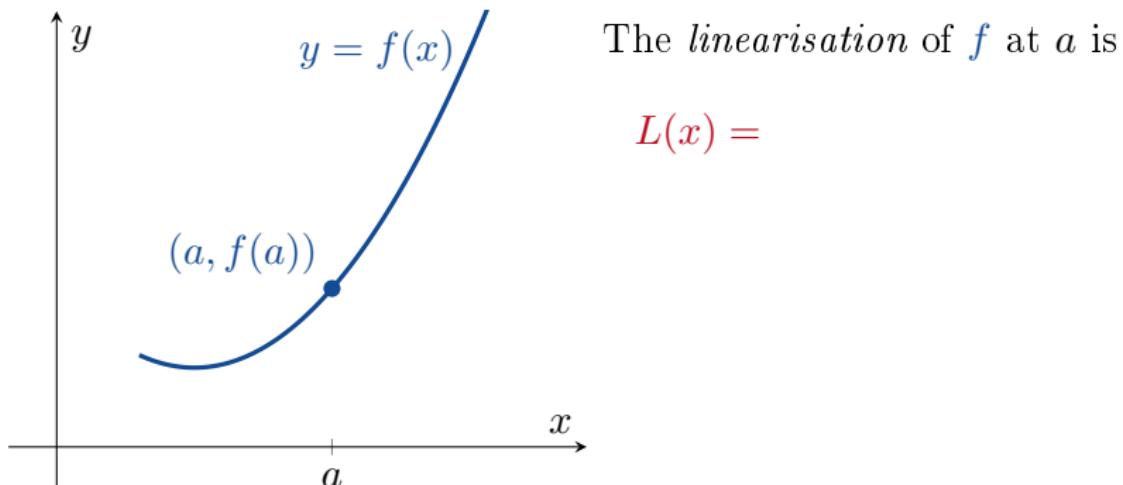
The tangent line to the ellipse of intersection is

$$x = 1 + 2t, \quad y = 1 - 2t, \quad z = 3 - 2t.$$



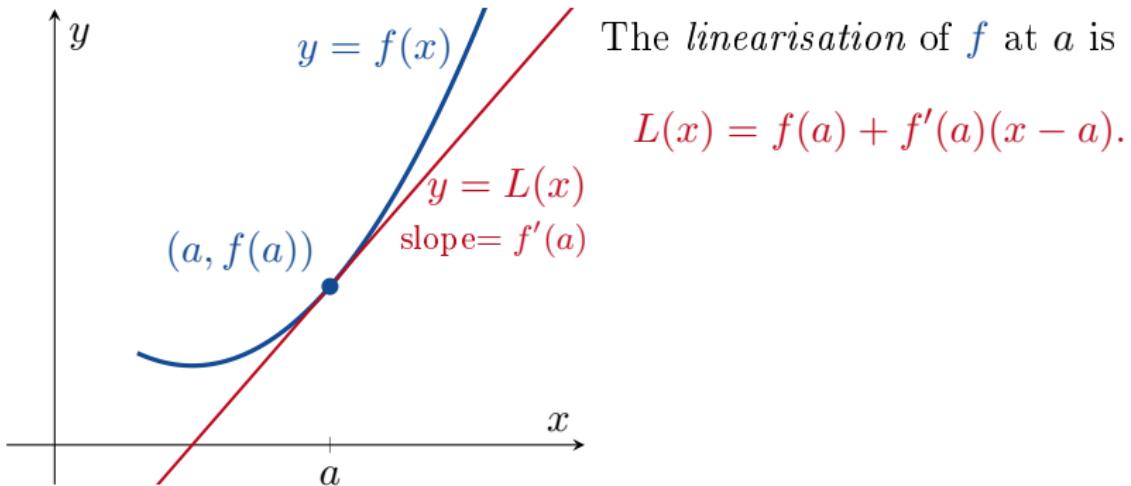
$$D_{\mathbf{u}} f = \|\nabla f\| \cos \theta$$

Linearisation of a Function of One Variable



$$D_{\mathbf{u}} f = \|\nabla f\| \cos \theta$$

Linearisation of a Function of One Variable



$$L(x) = f(a) + f'(a)(x - a)$$

Linearisation of a Function of Two Variables

Definition

The *linearisation* of a function $f(x, y)$ at a point (x_0, y_0) is

$$L(x, y) = f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0).$$

EXAMPLE 5

Find the linearization of

$$f(x, y) = x^2 - xy + \frac{1}{2}y^2 + 3$$

at the point $(3, 2)$.**Solution** We first evaluate f , f_x , and f_y at the point $(x_0, y_0) = (3, 2)$:

$$f(3, 2) = \left. \left(x^2 - xy + \frac{1}{2}y^2 + 3 \right) \right|_{(3, 2)} = 8$$

$$f_x(3, 2) = \left. \frac{\partial}{\partial x} \left(x^2 - xy + \frac{1}{2}y^2 + 3 \right) \right|_{(3, 2)} = \left. (2x - y) \right|_{(3, 2)} = 4$$

$$f_y(3, 2) = \left. \frac{\partial}{\partial y} \left(x^2 - xy + \frac{1}{2}y^2 + 3 \right) \right|_{(3, 2)} = \left. (-x + y) \right|_{(3, 2)} = -1,$$

giving

$$\begin{aligned} L(x, y) &= f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0) \\ &= 8 + (4)(x - 3) + (-1)(y - 2) = 4x - y - 2. \end{aligned}$$

The linearization of f at $(3, 2)$ is $L(x, y) = 4x - y - 2$



Break

We will continue at 2pm



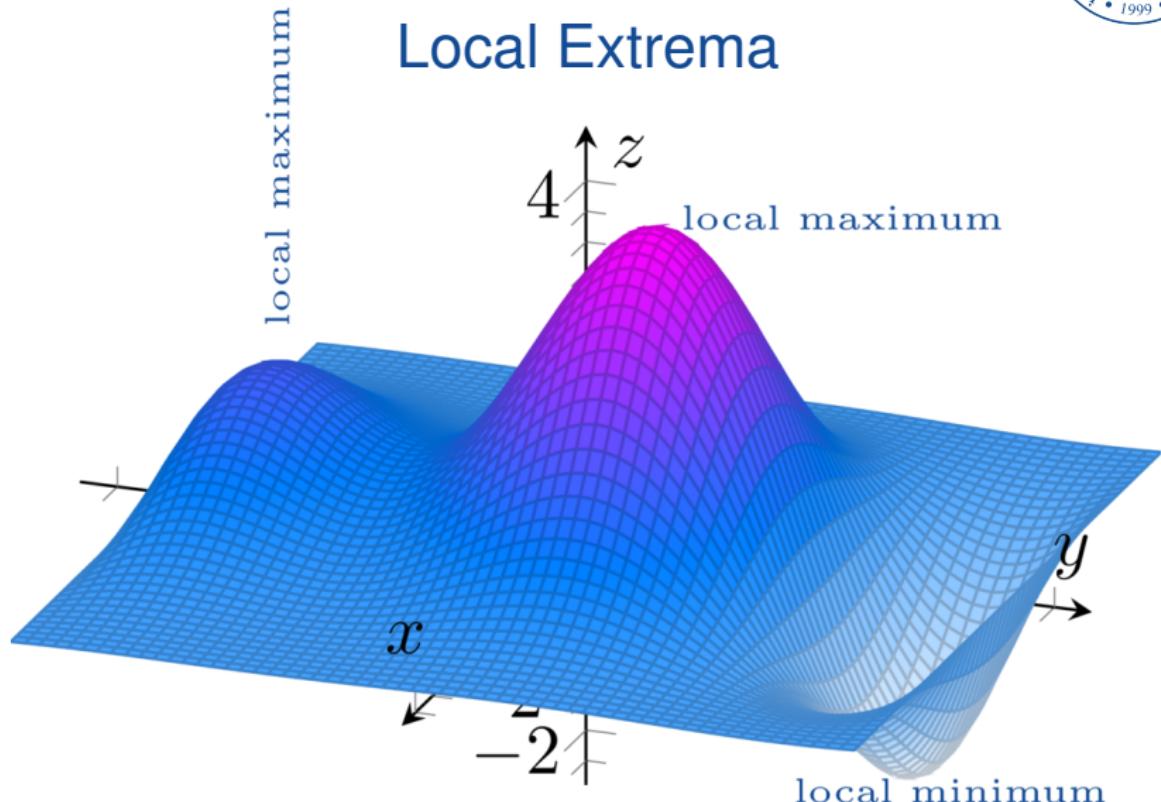


Extreme Values and Saddle Points

13.7 Extreme Values and Saddle Points



Local Extrema



13.7 Extreme Values and Saddle Points



Definition

- 1 $f(a, b)$ is a local maximum value of $f(x, y)$ iff

$$f(a, b) \geq f(x, y)$$

for all (x, y) close to (a, b) .

13.7 Extreme Values and Saddle Points



Definition

- 1 $f(a, b)$ is a local maximum value of $f(x, y)$ iff

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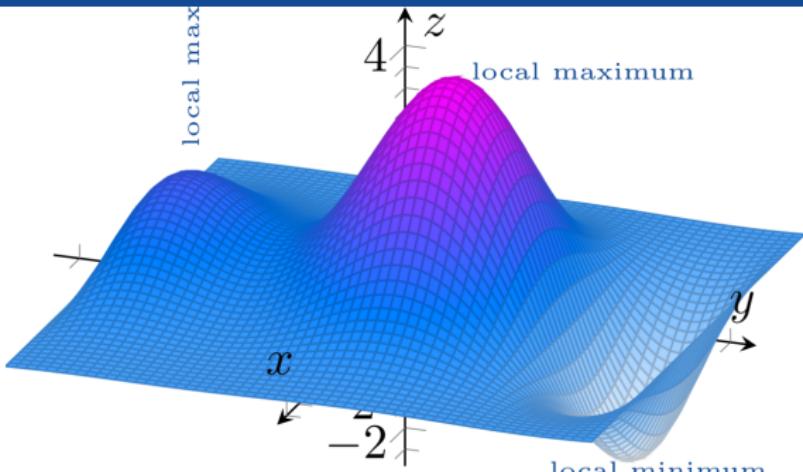
for all (x, y) close to (a, b) .

- 2 $f(a, b)$ is a local **minimum** value of $f(x, y)$ iff

$$f(a, b) \leq f(x, y)$$

for all (x, y) close to (a, b) .

13.7 Extreme Values and Saddle Points



Theorem (First Derivative Test)

$$\left(\begin{array}{l} f(x,y) \text{ has a local} \\ \text{extrema at an interior} \\ \text{point } (a,b) \text{ of its} \\ \text{domain} \end{array} \right) \implies \begin{array}{l} f_x(a,b) = 0 \\ \text{and} \\ f_y(a,b) = 0 \end{array}$$

if $f_x(a,b)$ and $f_y(a,b)$ both exist.

13.7 Extreme Values and Saddle Points



Definition

An interior point of the domain of $f(x, y)$ where either

- 1 $f_x = f_y = 0$;
- 2 f_x does not exist; or
- 3 f_y does not exist

is called a *critical point* of f .

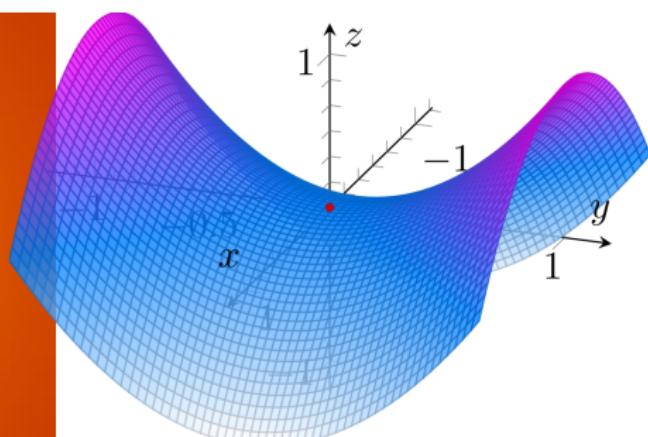
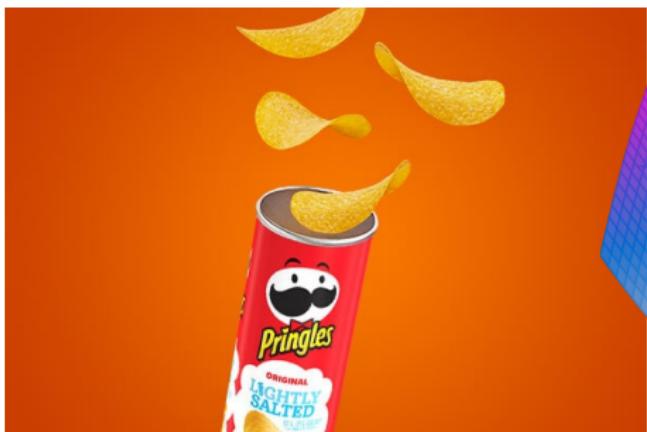
13.7 Extreme Values and Saddle Points



Saddle Points



Saddle Points



The point $(0, 0)$ is a *saddle point* of $z = y^2 - x^2$.

13.7 Extreme Values and Saddle Points



Example

Find the local extrema of $f(x, y) = y^2 - 4y + x^2 + 9$.

domain:

partial derivatives:

$$0 = f_x = \qquad \qquad \qquad \implies \qquad \qquad (x, y) =$$

$$0 = f_y =$$

13.7 Extreme Values and Saddle Points



Example

Find the local extrema of $f(x, y) = y^2 - 4y + x^2 + 9$.

domain: \mathbb{R}^2

partial derivatives:

$$\begin{aligned} 0 &= f_x = 2x \\ 0 &= f_y = 2y - 4 \end{aligned} \qquad \implies \qquad (x, y) = (0, 2)$$

13.7 Extreme Values and Saddle Points



Example

Find the local extrema of $f(x, y) = y^2 - 4y + x^2 + 9$.

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Therefore the only possible place for an extrema is $(0, 2)$, where $f(0, 2) = 5$. Is this a local minimum or a local maximum?

13.7 Extreme Values and Saddle Points



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Find the local extrema of $f(x, y) = y^2 - 4y + x^2 + 9$.

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partial derivatives:

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Therefore the only possible place for an extrema is $(0, 2)$, where $f(0, 2) = 5$. Is this a local minimum or a local maximum?

Since

$$(y - 2)^2 + x^2 + 5 \geq 5$$

for all (x, y) , this must be a local minimum.

13.7 Extreme Values and Saddle Points

Example

Find the local extrema of $f(x, y) = y^2 - 4y - x^2 + 9$.

domain:

partial derivatives:

$$0 = f_x =$$

$$\implies (x, y) =$$

$$0 = f_y =$$

13.7 Extreme Values and Saddle Points



Example

Find the local extrema of $f(x, y) = y^2 - 4y - x^2 + 9$.

domain: \mathbb{R}^2

partial derivatives:

$$\begin{aligned} 0 &= f_x = -2x \\ 0 &= f_y = 2y - 4 \end{aligned} \implies (x, y) = (0, 2)$$

13.7 Extreme Values and Saddle Points

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13.7 Extreme Values and Saddle Points



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domain: \mathbb{R}^2

partial derivatives:

$$\begin{aligned} 0 &= f_x = -2x \\ 0 &= f_y = 2y - 4 \end{aligned} \implies (x, y) = (0, 2)$$

Therefore the only possible place for an extrema is $(0, 2)$, where $f(0, 2) = 5$. Is this a local minimum or a local maximum?

No. Fixing $x = 0$ we have $f(0, y) = (y - 2)^2 + 5$ which curves upwards. But fixing $y = 2$ we have $f(x, 2) = 5 - x^2$ which curves downwards.

So $(0, 2)$ must be a saddle point.

13.7 Extreme Values and Saddle Points

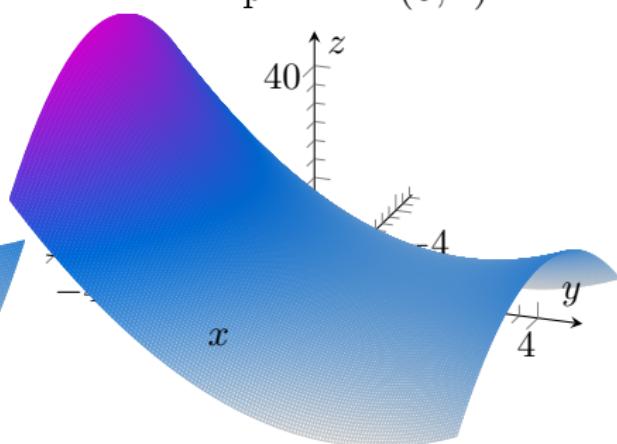
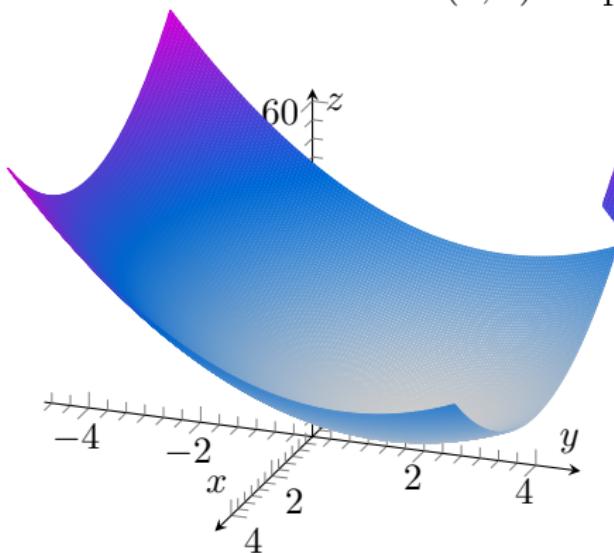


$$z = y^2 - 4y + x^2 + 9$$

has a local minimum at $(0, 2)$.

$$z = y^2 - 4y - x^2 + 9$$

has a saddle point at $(0, 2)$.



13.7 Extreme Values and Saddle Points



Theorem (Second Derivative Test)

Suppose that

- $f(x, y), f_x, f_y, f_{xx}, f_{yy}$ and f_{xy} are all continuous on an open disk centred at (a, b) ; and
- $f_x(a, b) = 0 = f_y(a, b)$.

<i>If at (a, b) we have</i>	<i>then</i>

13.7 Extreme Values and Saddle Points



Theorem (Second Derivative Test)

Suppose that

- $f(x, y), f_x, f_y, f_{xx}, f_{yy}$ and f_{xy} are all continuous on an open disk centred at (a, b) ; and
- $f_x(a, b) = 0 = f_y(a, b)$.

If at (a, b) we have		then
$f_{xx} < 0$	$f_{xx}f_{yy} - f_{xy}^2 > 0$	f has a <i>local maximum</i> at (a, b)

13.7 Extreme Values and Saddle Points



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Suppose that

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- $f_x(a, b) = 0 = f_y(a, b)$.

If at (a, b) we have		then
$f_{xx} < 0$	$f_{xx}f_{yy} - f_{xy}^2 > 0$	f has a local maximum at (a, b)
$f_{xx} > 0$	$f_{xx}f_{yy} - f_{xy}^2 > 0$	f has a local minimum at (a, b)

13.7 Extreme Values and Saddle Points



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Suppose that

- $f(x, y), f_x, f_y, f_{xx}, f_{yy}$ and f_{xy} are all continuous on an open disk centred at (a, b) ; and
- $f_x(a, b) = 0 = f_y(a, b)$.

If at (a, b) we have		then
$f_{xx} < 0$	$f_{xx}f_{yy} - f_{xy}^2 > 0$	f has a local maximum at (a, b)
$f_{xx} > 0$	$f_{xx}f_{yy} - f_{xy}^2 > 0$	f has a local minimum at (a, b)
	$f_{xx}f_{yy} - f_{xy}^2 < 0$	f has a saddle point at (a, b)

13.7 Extreme Values and Saddle Points



Theorem (Second Derivative Test)

Suppose that

- $f(x, y), f_x, f_y, f_{xx}, f_{yy}$ and f_{xy} are all continuous on an open disk centred at (a, b) ; and
- $f_x(a, b) = 0 = f_y(a, b)$.

If at (a, b) we have		then
$f_{xx} < 0$	$f_{xx}f_{yy} - f_{xy}^2 > 0$	f has a local maximum at (a, b)
$f_{xx} > 0$	$f_{xx}f_{yy} - f_{xy}^2 > 0$	f has a local minimum at (a, b)
	$f_{xx}f_{yy} - f_{xy}^2 < 0$	f has a saddle point at (a, b)
	$f_{xx}f_{yy} - f_{xy}^2 = 0$	we don't know

13.7 Extreme Values and Saddle Points



Otto Hesse

BORN

22 April 1811

DECEASED

4 August 1874

NATIONALITY

German

Definition

$f_{xx}f_{yy} - f_{xy}^2$ is called the *Hessian* (or *discriminant*) of f .

EXAMPLE 3 Find the local extreme values of the function

$$f(x, y) = xy - x^2 - y^2 - 2x - 2y + 4.$$

Solution The function is defined and differentiable for all x and y , and its domain has no boundary points. The function therefore has extreme values only at the points where f_x and f_y are simultaneously zero. This leads to

$$f_x = y - 2x - 2 = 0, \quad f_y = x - 2y - 2 = 0,$$

or

$$x = y = -2.$$

Therefore, the point $(-2, -2)$ is the only point where f may take on an extreme value. To see if it does so, we calculate

$$f_{xx} = -2, \quad f_{yy} = -2, \quad f_{xy} = 1.$$

The discriminant of f at $(a, b) = (-2, -2)$ is

$$f_{xx}f_{yy} - f_{xy}^2 = (-2)(-2) - (1)^2 = 4 - 1 = 3.$$

The combination

$$f_{xx} < 0 \quad \text{and} \quad f_{xx}f_{yy} - f_{xy}^2 > 0$$

tells us that f has a local maximum at $(-2, -2)$. The value of f at this point is $f(-2, -2) = 8$. ■

EXAMPLE 4 Find the local extreme values of $f(x, y) = 3y^2 - 2y^3 - 3x^2 + 6xy$.

Solution Since f is differentiable everywhere, it can assume extreme values only where

$$f_x = 6y - 6x = 0 \quad \text{and} \quad f_y = 6y - 6y^2 + 6x = 0.$$

From the first of these equations we find $x = y$, and substitution for y into the second equation then gives

$$6x - 6x^2 + 6x = 0 \quad \text{or} \quad 6x(2 - x) = 0.$$

The two critical points are therefore $(0, 0)$ and $(2, 2)$.

To classify the critical points, we calculate the second derivatives:

$$f_{xx} = -6, \quad f_{yy} = 6 - 12y, \quad f_{xy} = 6.$$

The discriminant is given by

$$f_{xx}f_{yy} - f_{xy}^2 = (-36 + 72y) - 36 = 72(y - 1).$$

At the critical point $(0, 0)$ we see that the value of the discriminant is the negative number -72 , so the function has a saddle point at the origin. At the critical point $(2, 2)$ we see that the discriminant has the positive value 72 . Combining this result with the negative value of the second partial $f_{xx} = -6$, Theorem 11 says that the critical point $(2, 2)$ gives a local maximum value of $f(2, 2) = 12 - 16 - 12 + 24 = 8$. A graph of the surface is shown in Figure 14.48. ■

EXAMPLE 5 Find the critical points of the function $f(x, y) = 10xye^{-(x^2+y^2)}$ and use the Second Derivative Test to classify each point as one where a saddle, local minimum, or local maximum occurs.

Solution First we find the partial derivatives f_x and f_y and set them simultaneously to zero in seeking the critical points:

$$\begin{aligned}f_x &= 10ye^{-(x^2+y^2)} - 20x^2ye^{-(x^2+y^2)} = 10y(1 - 2x^2)e^{-(x^2+y^2)} = 0 \Rightarrow y = 0 \text{ or } 1 - 2x^2 = 0, \\f_y &= 10xe^{-(x^2+y^2)} - 20xy^2e^{-(x^2+y^2)} = 10x(1 - 2y^2)e^{-(x^2+y^2)} = 0 \Rightarrow x = 0 \text{ or } 1 - 2y^2 = 0.\end{aligned}$$

Since both partial derivatives are continuous everywhere, the only critical points are

$$(0, 0), \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right), \left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right), \left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right), \text{ and } \left(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right).$$

Next we calculate the second partial derivatives in order to evaluate the discriminant at each critical point:

$$\begin{aligned}f_{xx} &= -20xy(1 - 2x^2)e^{-(x^2+y^2)} - 40xye^{-(x^2+y^2)} = -20xy(3 - 2x^2)e^{-(x^2+y^2)}, \\f_{xy} = f_{yx} &= 10(1 - 2x^2)e^{-(x^2+y^2)} - 20y^2(1 - 2x^2)e^{-(x^2+y^2)} = 10(1 - 2x^2)(1 - 2y^2)e^{-(x^2+y^2)}, \\f_{yy} &= -20xy(1 - 2y^2)e^{-(x^2+y^2)} - 40xye^{-(x^2+y^2)} = -20xy(3 - 2y^2)e^{-(x^2+y^2)}.\end{aligned}$$

The following table summarizes the values needed by the Second Derivative Test.

Critical Point	f_{xx}	f_{xy}	f_{yy}	Discriminant D
(0, 0)	0	10	0	-100
$\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$	$-\frac{20}{e}$	0	$-\frac{20}{e}$	$\frac{400}{e^2}$
$\left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$	$\frac{20}{e}$	0	$\frac{20}{e}$	$\frac{400}{e^2}$
$\left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)$	$\frac{20}{e}$	0	$\frac{20}{e}$	$\frac{400}{e^2}$
$\left(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)$	$-\frac{20}{e}$	0	$-\frac{20}{e}$	$\frac{400}{e^2}$

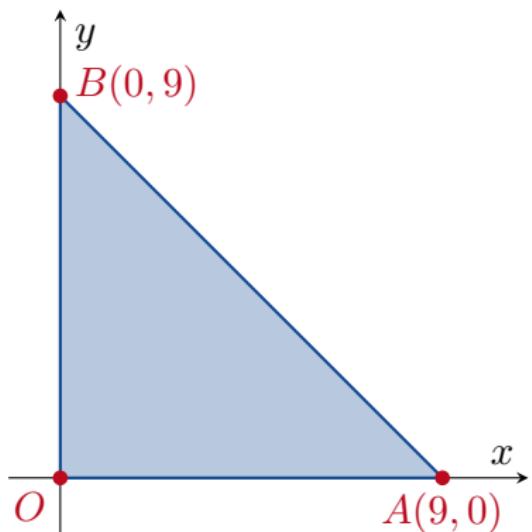
From the table we find that $D < 0$ at the critical point (0, 0), giving a saddle; $D > 0$ and $f_{xx} < 0$ at the critical points $(1/\sqrt{2}, 1/\sqrt{2})$ and $(-1/\sqrt{2}, -1/\sqrt{2})$, giving local maximum values there; and $D > 0$ and $f_{xx} > 0$ at the critical points $(-1/\sqrt{2}, 1/\sqrt{2})$ and $(1/\sqrt{2}, -1/\sqrt{2})$, each giving local minimum values. A graph of the surface is shown in Figure 14.49. ■

Example

Find the absolute maximum and minimum values of

$$f(x, y) = 2 + 2x + 4y - x^2 - y^2$$

on the triangular region in the first quadrant bounded by the lines $x = 0$, $y = 0$ and $y = 9 - x$.

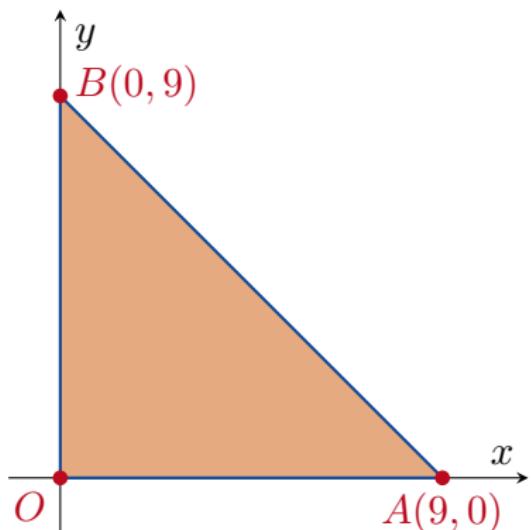


Example

Find the absolute maximum and minimum values of

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on the triangular region in the first quadrant bounded by the lines $x = 0$, $y = 0$ and $y = 9 - x$.



We will look

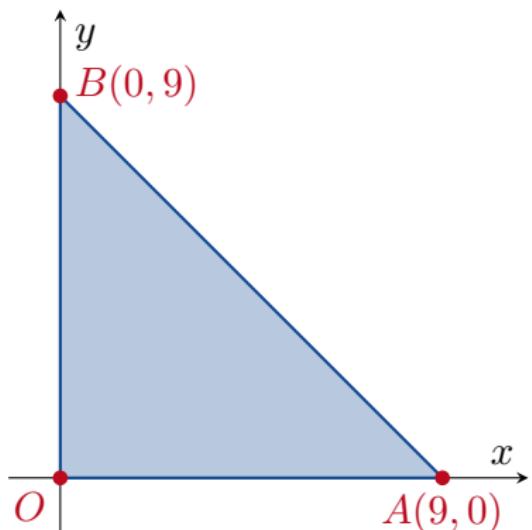
- 1 at the interior of the region;

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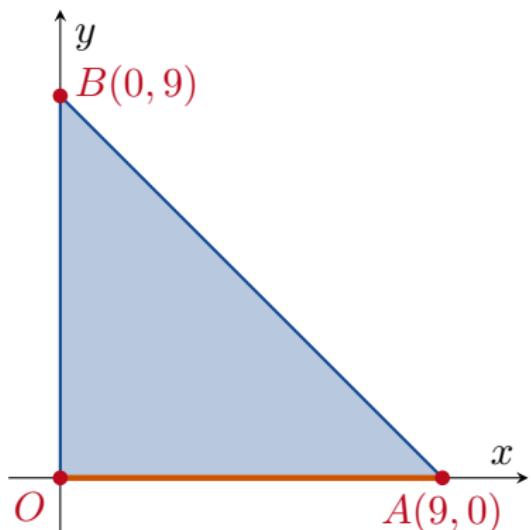
- 1 at the interior of the region;
- 2 at the corners O , A and B ;

Example

Find the absolute maximum and minimum values of

$$f(x, y) = 2 + 2x + 4y - x^2 - y^2$$

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We will look

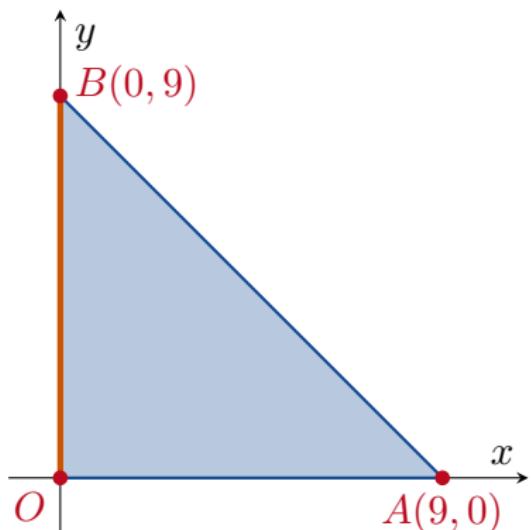
- 1 at the interior of the region;
- 2 at the corners O , A and B ;
- 3 at the line OA ;

Example

Find the absolute maximum and minimum values of

$$f(x, y) = 2 + 2x + 4y - x^2 - y^2$$

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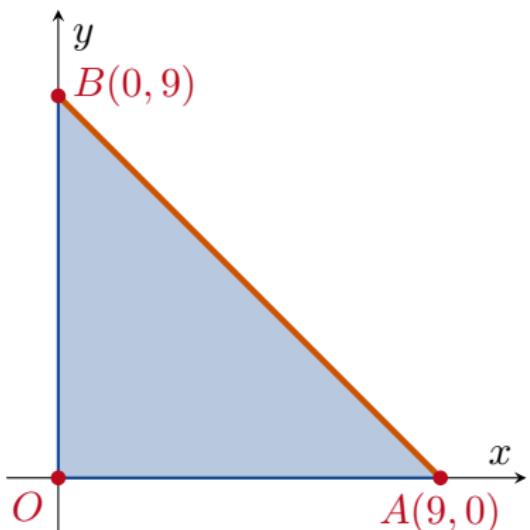
- 1 at the interior of the region;
- 2 at the corners O , A and B ;
- 3 at the line OA ;
- 4 at the line OB ; and

Example

Find the absolute maximum and minimum values of

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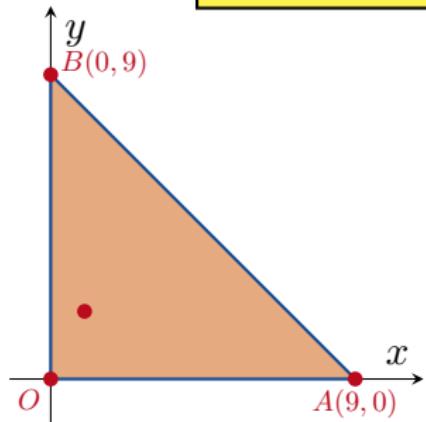


We will look

- 1 at the interior of the region;
- 2 at the corners O , A and B ;
- 3 at the line OA ;
- 4 at the line OB ; and
- 5 at the line AB .

13.7

$$f(x, y) = 2 + 2x + 4y - x^2 - y^2$$



- 1 Consider the interior of the region.
We need to look for the critical points

$$\begin{aligned} 0 &= f_x = 2 - 2x \\ 0 &= f_y = 4 - 2y \end{aligned} \implies (x, y) = (1, 2).$$

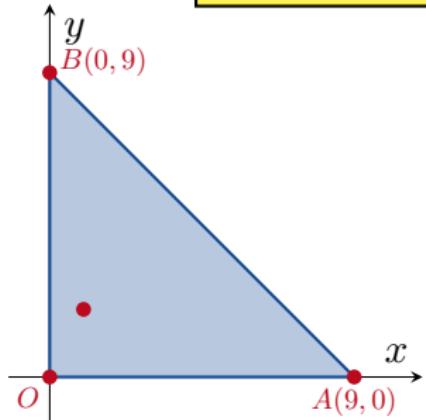
Then we calculate that

$$f(1, 2) = 2 + 2 + 8 - 1 - 4 = 7.$$

13.7

$$f(x, y) = 2 + 2x + 4y - x^2 - y^2$$

$$f(1, 2) = 7$$



2 Consider the corners of the region.

We calculate that

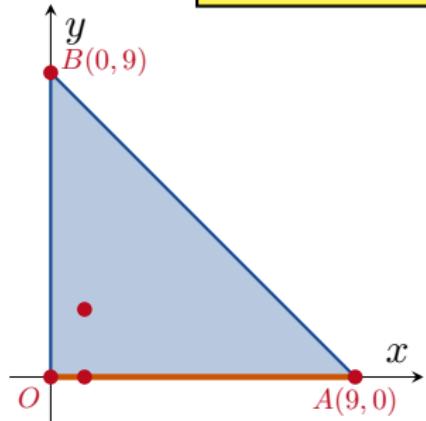
$$f(0, 0) = 2 + 0 + 0 - 0 - 0 = 2$$

$$f(9, 0) = 2 + 18 + 0 - 81 - 0 = -61$$

$$f(0, 9) = 2 + 0 + 36 - 0 - 81 = -43.$$

13.7

$$f(x, y) = 2 + 2x + 4y - x^2 - y^2$$



$f(1, 2) = 7$
$f(0, 0) = 2$
$f(9, 0) = -61$
$f(0, 9) = -43$

3 Consider the line OA .

If we set $y = 0$, then we have a new function

$$g(x) = f(x, 0) = 2 + 2x - x^2.$$

Since

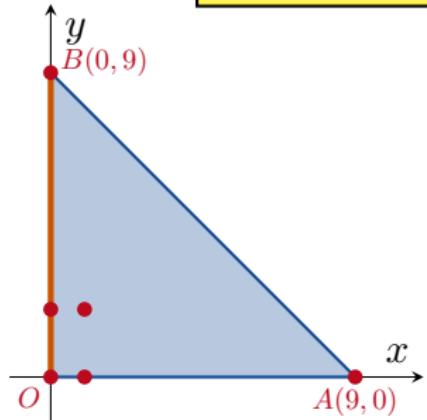
$$0 = g'(x) = 2 - 2x \implies x = 1$$

we calculate

$$g(1) = f(1, 0) = 2 + 2 - 1 = 3.$$

13.7

$$f(x, y) = 2 + 2x + 4y - x^2 - y^2$$



$f(1, 2) = 7$
$f(0, 0) = 2$
$f(9, 0) = -61$
$f(0, 9) = -43$
$f(1, 0) = 3$

4 Consider the line OB .

If we set $x = 0$, then we have a new function

$$h(y) = f(0, y) = 2 + 4y - y^2.$$

Since

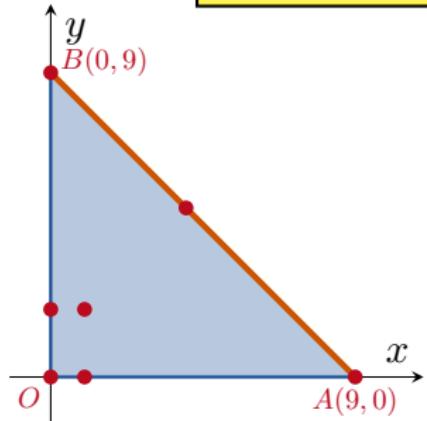
$$0 = h'(x) = 4 - 2y \implies y = 2$$

we calculate

$$h(2) = f(0, 2) = 2 + 8 - 4 = 6.$$

13.7

$$f(x, y) = 2 + 2x + 4y - x^2 - y^2$$



$f(1, 2) = 7$
$f(0, 0) = 2$
$f(9, 0) = -61$
$f(0, 9) = -43$
$f(1, 0) = 3$
$f(0, 2) = 6$

5 Finally consider the line AB .

If we set $y = 9 - x$, then we have a new function

$$k(x) = f(x, 9-x) = 2+2x+4(9-x)-x^2-(9-x)^2 = -43+16x-2x^2.$$

Since

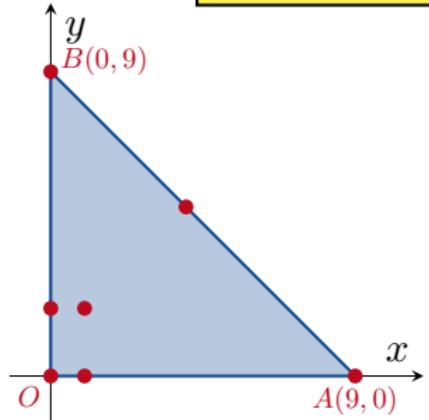
$$0 = k'(x) = 16 - 4x \implies x = 4$$

we calculate

$$k(4) = f(4, 5) = -43 + 64 - 32 = -11.$$

13.7

$$f(x, y) = 2 + 2x + 4y - x^2 - y^2$$



$f(1, 2) = 7$
$f(0, 0) = 2$
$f(9, 0) = -61$
$f(0, 9) = -43$
$f(1, 0) = 3$
$f(0, 2) = 6$
$f(4, 5) = -11$

We have found the values

$$7, 2, -61, -43, 3, 6, -11.$$

The biggest of these numbers is 7 and the least is -61.

Therefore the absolute maximum value of f on this region is 7 and the absolute minimum value of f on this region is -61.

13.7 Extreme Values and Saddle Points



Please read Example 7 in the textbook.



123 Lagrange Multipliers

13.8 Lagrange Multipliers

Example

Find the point $P(x, y, z)$ on the plane $2x + y - z - 5 = 0$ that is closest to the origin.

13.8 Lagrange Multipliers

Example

Find the point $P(x, y, z)$ on the plane $2x + y - z - 5 = 0$ that is closest to the origin.

We need to find the minimum of

$$\|\overrightarrow{OP}\| = \sqrt{x^2 + y^2 + z^2}$$

subject to the constraint that

$$2x + y - z - 5 = 0.$$

13.8 Lagrange Multipliers

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subject to the constraint that

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Let $f(x, y) = x^2 + y^2 + z^2$. We will study f instead of $\|\overrightarrow{OP}\|$.

13.8 Lagrange Multipliers



We will let x and y be the independent variables and write

$$z = 2x + y - 5.$$

13.8 Lagrange Multipliers

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So we want to find the minimum of

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$$0 = h_x$$

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We will use the First Derivative Test:

$$0 = h_x = 2x + 2(2x + y - 5)(2)$$

$$0 = h_y = 2y + 2(2x + y - 5)$$

13.8 Lagrange Multipliers

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We will use the First Derivative Test:

$$\begin{aligned} 0 &= h_x = 2x + 2(2x + y - 5)(2) \implies 10x + 4y = 20 \\ 0 &= h_y = 2y + 2(2x + y - 5) \implies 4x + 4y = 10 \end{aligned}$$

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Then we have

$$z = 2x + y - 5 = \frac{10}{3} + \frac{5}{6} - 5 = -\frac{5}{6}.$$

13.8 Lagrange Multipliers

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$$z = 2x + y - 5.$$

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Then we have

$$z = 2x + y - 5 = \frac{10}{3} + \frac{5}{6} - 5 = -\frac{5}{6}.$$

The point on this plane which is closest to the origin is

$$P\left(\frac{5}{3}, \frac{5}{6}, -\frac{5}{6}\right).$$

The Method of Lagrange Multipliers

Suppose that we want to find the maximum/minimum of

$$f(x, y, z)$$

subject to the constraint that

$$g(x, y, z) = 0.$$

The Method of Lagrange Multipliers

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Theorem (The Method of Lagrange Multipliers)

We only need to find x, y, z and λ which satisfy

$$\nabla f = \lambda \nabla g \quad \text{and} \quad g(x, y, z) = 0.$$

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13.8

$$\nabla f = \lambda \nabla g \quad g(x, y, z) = 0.$$



Example (repeat)

Find the point $P(x, y, z)$ on the plane $2x + y - z - 5 = 0$ that is closest to the origin.

13.8

$$\nabla f = \lambda \nabla g \quad g(x, y, z) = 0.$$



Example (repeat)

Find the point $P(x, y, z)$ on the plane $2x + y - z - 5 = 0$ that is closest to the origin.

Let $f(x, y, z) = x^2 + y^2 + z^2$ and $g(x, y, z) = 2x + y - z - 5$.

13.8

$$\nabla f = \lambda \nabla g \quad g(x, y, z) = 0.$$



Example (repeat)

Find the point $P(x, y, z)$ on the plane $2x + y - z - 5 = 0$ that is closest to the origin.

Let $f(x, y, z) = x^2 + y^2 + z^2$ and $g(x, y, z) = 2x + y - z - 5$. Then

$$= \nabla f = \lambda \nabla g =$$

13.8

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Example (repeat)

Find the point $P(x, y, z)$ on the plane $2x + y - z - 5 = 0$ that is closest to the origin.

Let $f(x, y, z) = x^2 + y^2 + z^2$ and $g(x, y, z) = 2x + y - z - 5$. Then

$$2x\mathbf{i} + 2y\mathbf{j} + 2z\mathbf{k} = \nabla f = \lambda \nabla g =$$

13.8

$$\nabla f = \lambda \nabla g \quad g(x, y, z) = 0.$$



Example (repeat)

Find the point $P(x, y, z)$ on the plane $2x + y - z - 5 = 0$ that is closest to the origin.

Let $f(x, y, z) = x^2 + y^2 + z^2$ and $g(x, y, z) = 2x + y - z - 5$. Then

$$2x\mathbf{i} + 2y\mathbf{j} + 2z\mathbf{k} = \nabla f = \lambda \nabla g = 2\lambda\mathbf{i} + \mathbf{j} - \lambda\mathbf{k}$$

13.8

$$\nabla f = \lambda \nabla g \quad g(x, y, z) = 0.$$



Example (repeat)

Find the point $P(x, y, z)$ on the plane $2x + y - z - 5 = 0$ that is closest to the origin.

Let $f(x, y, z) = x^2 + y^2 + z^2$ and $g(x, y, z) = 2x + y - z - 5$. Then

$$2x\mathbf{i} + 2y\mathbf{j} + 2z\mathbf{k} = \nabla f = \lambda \nabla g = 2\lambda\mathbf{i} + \lambda\mathbf{j} - \lambda\mathbf{k} \implies \begin{cases} x = \lambda \\ y = \frac{\lambda}{2} \\ z = -\frac{\lambda}{2}. \end{cases}$$

13.8

$$\nabla f = \lambda \nabla g \quad g(x, y, z) = 0.$$



Example (repeat)

Find the point $P(x, y, z)$ on the plane $2x + y - z - 5 = 0$ that is closest to the origin.

Let $f(x, y, z) = x^2 + y^2 + z^2$ and $g(x, y, z) = 2x + y - z - 5$. Then

$$2x\mathbf{i} + 2y\mathbf{j} + 2z\mathbf{k} = \nabla f = \lambda \nabla g = 2\lambda\mathbf{i} + \lambda\mathbf{j} - \lambda\mathbf{k} \implies \begin{cases} x = \lambda \\ y = \frac{\lambda}{2} \\ z = -\frac{\lambda}{2}. \end{cases}$$

Hence

$$0 = 2x + y - z - 5 =$$

13.8

$$\nabla f = \lambda \nabla g \quad g(x, y, z) = 0.$$



Example (repeat)

Find the point $P(x, y, z)$ on the plane $2x + y - z - 5 = 0$ that is closest to the origin.

Let $f(x, y, z) = x^2 + y^2 + z^2$ and $g(x, y, z) = 2x + y - z - 5$. Then

$$2x\mathbf{i} + 2y\mathbf{j} + 2z\mathbf{k} = \nabla f = \lambda \nabla g = 2\lambda\mathbf{i} + \lambda\mathbf{j} - \lambda\mathbf{k} \implies \begin{cases} x = \lambda \\ y = \frac{\lambda}{2} \\ z = -\frac{\lambda}{2}. \end{cases}$$

Hence

$$0 = 2x + y - z - 5 = 2\lambda + \frac{\lambda}{2} - \left(-\frac{\lambda}{2}\right) - 5 = 3\lambda - 5$$

13.8

$$\nabla f = \lambda \nabla g \quad g(x, y, z) = 0.$$



Example (repeat)

Find the point $P(x, y, z)$ on the plane $2x + y - z - 5 = 0$ that is closest to the origin.

Let $f(x, y, z) = x^2 + y^2 + z^2$ and $g(x, y, z) = 2x + y - z - 5$. Then

$$2x\mathbf{i} + 2y\mathbf{j} + 2z\mathbf{k} = \nabla f = \lambda \nabla g = 2\lambda\mathbf{i} + \lambda\mathbf{j} - \lambda\mathbf{k} \implies \begin{cases} x = \lambda \\ y = \frac{\lambda}{2} \\ z = -\frac{\lambda}{2}. \end{cases}$$

Hence

$$0 = 2x + y - z - 5 = 2\lambda + \frac{\lambda}{2} - \left(-\frac{\lambda}{2}\right) - 5 = 3\lambda - 5 \implies \lambda = \frac{5}{3}.$$

$$\nabla f = \lambda \nabla g \quad g(x, y, z) = 0.$$

Example (repeat)

Find the point $P(x, y, z)$ on the plane $2x + y - z - 5 = 0$ that is closest to the origin.

Let $f(x, y, z) = x^2 + y^2 + z^2$ and $g(x, y, z) = 2x + y - z - 5$. Then

$$2x\mathbf{i} + 2y\mathbf{j} + 2z\mathbf{k} = \nabla f = \lambda \nabla g = 2\lambda\mathbf{i} + \mathbf{j} - \lambda\mathbf{k} \implies \begin{cases} x = \lambda \\ y = \frac{\lambda}{2} \\ z = -\frac{\lambda}{2}. \end{cases}$$

Hence

$$0 = 2x + y - z - 5 = 2\lambda + \frac{\lambda}{2} - \left(-\frac{\lambda}{2}\right) - 5 = 3\lambda - 5 \implies \lambda = \frac{5}{3}.$$

Therefore

$$P(x, y, z) = \left(\lambda, \frac{\lambda}{2}, -\frac{\lambda}{2}\right) = \left(\frac{5}{3}, \frac{5}{6}, -\frac{5}{6}\right).$$

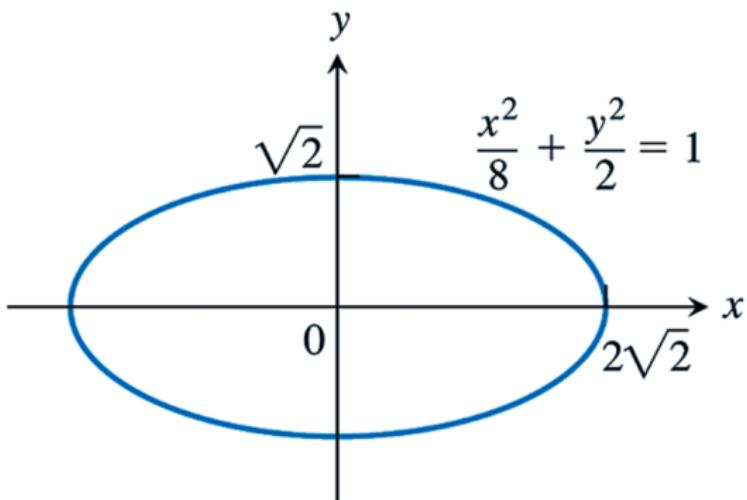
13.8 Lagrange Multipliers

EXAMPLE 3 Find the greatest and smallest values that the function

$$f(x, y) = xy$$

takes on the ellipse (Figure 14.55)

$$\frac{x^2}{8} + \frac{y^2}{2} = 1.$$



Solution We want to find the extreme values of $f(x, y) = xy$ subject to the constraint

$$g(x, y) = \frac{x^2}{8} + \frac{y^2}{2} - 1 = 0.$$

To do so, we first find the values of x , y , and λ for which

$$\nabla f = \lambda \nabla g \quad \text{and} \quad g(x, y) = 0.$$

The gradient equation in Equations (1) gives

$$y\mathbf{i} + x\mathbf{j} = \frac{\lambda}{4}x\mathbf{i} + \lambda y\mathbf{j},$$

from which we find

$$y = \frac{\lambda}{4}x, \quad x = \lambda y, \quad \text{and} \quad y = \frac{\lambda}{4}(\lambda y) = \frac{\lambda^2}{4}y,$$

so that $y = 0$ or $\lambda = \pm 2$. We now consider these two cases.

13.8 Lagrange Multipliers

Case 1: If $y = 0$, then $x = y = 0$. But $(0, 0)$ is not on the ellipse. Hence, $y \neq 0$.

Case 2: If $y \neq 0$, then $\lambda = \pm 2$ and $x = \pm 2y$. Substituting this in the equation $g(x, y) = 0$ gives

$$\frac{(\pm 2y)^2}{8} + \frac{y^2}{2} = 1, \quad 4y^2 + 4y^2 = 8 \quad \text{and} \quad y = \pm 1.$$

The function $f(x, y) = xy$ therefore takes on its extreme values on the ellipse at the four points $(\pm 2, 1), (\pm 2, -1)$. The extreme values are $xy = 2$ and $xy = -2$.

EXAMPLE 4 Find the maximum and minimum values of the function $f(x, y) = 3x + 4y$ on the circle $x^2 + y^2 = 1$.

Solution We model this as a Lagrange multiplier problem with

$$f(x, y) = 3x + 4y, \quad g(x, y) = x^2 + y^2 - 1$$

and look for the values of x , y , and λ that satisfy the equations

$$\begin{aligned}\nabla f = \lambda \nabla g: \quad & 3\mathbf{i} + 4\mathbf{j} = 2x\lambda\mathbf{i} + 2y\lambda\mathbf{j} \\ g(x, y) = 0: \quad & x^2 + y^2 - 1 = 0.\end{aligned}$$

The gradient equation in Equations (1) implies that $\lambda \neq 0$ and gives

$$x = \frac{3}{2\lambda}, \quad y = \frac{2}{\lambda}.$$

These equations tell us, among other things, that x and y have the same sign. With these values for x and y , the equation $g(x, y) = 0$ gives

$$\left(\frac{3}{2\lambda}\right)^2 + \left(\frac{2}{\lambda}\right)^2 - 1 = 0,$$

so

$$\frac{9}{4\lambda^2} + \frac{4}{\lambda^2} = 1, \quad 9 + 16 = 4\lambda^2, \quad 4\lambda^2 = 25, \quad \text{and} \quad \lambda = \pm \frac{5}{2}.$$

Thus,

$$x = \frac{3}{2\lambda} = \pm \frac{3}{5}, \quad y = \frac{2}{\lambda} = \pm \frac{4}{5},$$

and $f(x, y) = 3x + 4y$ has extreme values at $(x, y) = \pm(3/5, 4/5)$.

By calculating the value of $3x + 4y$ at the points $\pm(3/5, 4/5)$, we see that its maximum and minimum values on the circle $x^2 + y^2 = 1$ are

$$3\left(\frac{3}{5}\right) + 4\left(\frac{4}{5}\right) = \frac{25}{5} = 5 \quad \text{and} \quad 3\left(-\frac{3}{5}\right) + 4\left(-\frac{4}{5}\right) = -\frac{25}{5} = -5.$$



Next Time

- 14.1 Double and Iterated Integrals over Rectangles
- 14.2 Double Integrals over General Regions
- 14.3 Area by Double Integration
- 10.3 Polar Coordinates