

# Lecture 11

- 9.6 Alternating Series and Conditional Convergence
- 9.7 Power Series
- 9.8 Taylor and Maclaurin Series



# Alternating Series and Conditional Convergence



### Alternating Series

Now let's talk about sequences of the form

$$\sum_{n=1}^{\infty} (-1)^{n+1} a_n = a_1 - a_2 + a_3 - a_4 + a_5 - a_6 + a_7 - a_8 + a_9 - a_{10} + \dots$$

where  $a_n > 0 \ \forall n$ .

## 9.6 Alternating Series and Conditional Convergence



$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \dots + \frac{(-1)^{n+1}}{n} + \dots$$

$$1 - 2 + 4 - 8 + 16 - 32 + \dots$$

$$4 - 2 + 1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{8} + \frac{1}{16} - \dots$$

## 9.6 Alternating Series and Conditional Convergence



Theorem (The Alternating Series Test / Alterne Seri Testi)

Let  $(a_n)$  be a sequence such that

- 1  $a_n > 0$  for all  $n$ ;
- 2  $(a_n)$  is decreasing (i.e.  $a_n \geq a_{n+1}$  for all  $n$ ); and
- 3  $a_n \rightarrow 0$  as  $n \rightarrow \infty$ .

## 9.6 Alternating Series and Conditional Convergence



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## 9.6 Alternating Series and Conditional Convergence



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Then  $\sum_{n=1}^{\infty} (-1)^{n+1} a_n$  converges.

Of course we can write condition 2 as “ $(a_n)$  is decreasing eventually (i.e.  $a_n \geq a_{n+1}$  for all  $n > N$  for some  $N \in \mathbb{N}$ )” since we don’t care what happens at the start of a sequence/series.

## 9.6 Alternating Series and Conditional Convergence



Proof.

Let

$$s_n = a_1 - a_2 + a_3 - a_4 + a_5 - a_6 + \dots + (-1)^{n+1} a_n.$$

Then

$$s_{2n} = a_1 - a_2 + a_3 - a_4 + a_5 - a_6 + \dots - a_{2n}.$$

## 9.6 Alternating Series and Conditional Convergence



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Then

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So

$$s_{2n+2} - s_{2n} = a_{2n+1} - a_{2n+2} \geq 0.$$

Therefore the sequence  $(s_{2n})$  is increasing.

## 9.6 Alternating Series and Conditional Convergence



Proof continued.

Moreover, since  $(a_n)$  is positive and decreasing, we have that

$$s_{2n} = a_1 - a_2 + a_3 - a_4 + a_5 - a_6 + \dots - a_{2n-2} + a_{2n-1} - a_{2n}$$

## 9.6 Alternating Series and Conditional Convergence



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$$\begin{aligned}s_{2n} &= a_1 - a_2 + a_3 - a_4 + a_5 - a_6 + \dots - a_{2n-2} + a_{2n-1} - a_{2n} \\&= a_1 - (a_2 - a_3) - (a_4 - a_5) - \dots - (a_{2n-2} - a_{2n-1}) - a_{2n}\end{aligned}$$

## 9.6 Alternating Series and Conditional Convergence



Proof continued.

Moreover, since  $(a_n)$  is positive and decreasing, we have that

$$\begin{aligned}s_{2n} &= a_1 - a_2 + a_3 - a_4 + a_5 - a_6 + \dots - a_{2n-2} + a_{2n-1} - a_{2n} \\&= a_1 - (a_2 - a_3) - (a_4 - a_5) - \dots - (a_{2n-2} - a_{2n-1}) - a_{2n} \\&\leq a_1 - 0 - 0 - \dots - 0 - 0 = a_1.\end{aligned}$$



$$\boxed{\begin{array}{l} a_2 \geq a_3 \\ a_2 - a_3 \geq 0 \\ -(a_2 - a_3) \leq -0 \end{array}}$$

## 9.6 Alternating Series and Conditional Convergence



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So  $(s_{2n})$  is bounded above.

## 9.6 Alternating Series and Conditional Convergence



Proof continued.

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So  $(s_{2n})$  is bounded above.

$$\begin{cases} (s_{2n}) \text{ is increasing} \\ (s_{2n}) \text{ is bounded above} \end{cases} \implies (s_{2n}) \text{ is convergent.}$$

## 9.6 Alternating Series and Conditional Convergence



Proof continued.

Let  $s = \lim_{n \rightarrow \infty} s_{2n}$ . Then  $s_{2n} \rightarrow s$  as  $n \rightarrow \infty$ . Furthermore

$$\begin{aligned}s_{2n+1} &= a_1 - a_2 + a_3 - a_4 + a_5 - a_6 + \dots - a_{2n} + a_{2n+1} \\&= s_{2n} + a_{2n+1} \rightarrow s + 0 = s\end{aligned}$$

as  $n \rightarrow \infty$ .

## 9.6 Alternating Series and Conditional Convergence



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It follows (you prove) that  $s_n \rightarrow s$  as  $n \rightarrow \infty$  also.

## 9.6 Alternating Series and Conditional Convergence



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as  $n \rightarrow \infty$ .

It follows (you prove) that  $s_n \rightarrow s$  as  $n \rightarrow \infty$  also.

Therefore  $\sum_{n=1}^{\infty} (-1)^{n+1} a_n$  is convergent. □

## 9.6 Alternating Series and Conditional Convergence



### Remark

If  $a_n \not\rightarrow 0$  as  $n \rightarrow \infty$ , then  $(-1)^{n+1}a_n \not\rightarrow 0$  as  $n \rightarrow \infty$  and

$$\sum_{n=1}^{\infty} (-1)^{n+1}a_n$$

diverges by the Divergence Test.

## 9.6 Alternating Series and Conditional Convergence



### Example

Let  $a_n = \sin \frac{1}{n}$  for all  $n \in \mathbb{N}$ . Note that  $0 < \frac{1}{n+1} < \frac{1}{n} \leq 1 < \frac{\pi}{2}$  for all  $n \in \mathbb{N}$ . Thus

$$a_n = \sin \frac{1}{n} > 0$$

and

$$a_{n+1} = \sin \frac{1}{n+1} < \sin \frac{1}{n} = a_n.$$

So  $(a_n)$  is a decreasing sequence of positive numbers. Moreover,  $a_n = \sin \frac{1}{n} \rightarrow 0$  as  $n \rightarrow \infty$ .

## 9.6 Alternating Series and Conditional Convergence



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So  $(a_n)$  is a decreasing sequence of positive numbers. Moreover,  $a_n = \sin \frac{1}{n} \rightarrow 0$  as  $n \rightarrow \infty$ . Therefore

$$\sum_{n=1}^{\infty} (-1)^{n+1} \sin \frac{1}{n}$$

converges by the Alternating Series Test.

## 9.6 Alternating Series and Conditional Convergence



### Example

Since  $a_n = \cos \frac{1}{n} \rightarrow 1 \neq 0$  as  $n \rightarrow \infty$ , it follows that

$$\sum_{n=1}^{\infty} (-1)^{n+1} \cos \frac{1}{n}$$

diverges by the Divergence Test.



### Example

Does  $\sum_{n=1}^{\infty} (-1)^{n+1} \sin^2 \frac{1}{n}$  converge or diverge?

## 9.6 Alternating Series and Conditional Convergence



### Example

Does  $\sum_{n=1}^{\infty} (-1)^{n+1} \sin^2 \frac{1}{n}$  converge or diverge?

Let  $a_n = \sin^2 \frac{1}{n}$ . Then  $a_n > a_{n+1} > 0 \ \forall n$  and  $a_n \rightarrow 0$  as  $n \rightarrow \infty$ .

## 9.6 Alternating Series and Conditional Convergence



### Example

Does  $\sum_{n=1}^{\infty} (-1)^{n+1} \sin^2 \frac{1}{n}$  converge or diverge?

Let  $a_n = \sin^2 \frac{1}{n}$ . Then  $a_n > a_{n+1} > 0 \ \forall n$  and  $a_n \rightarrow 0$  as  $n \rightarrow \infty$ .

Therefore

$$\sum_{n=1}^{\infty} (-1)^{n+1} \sin^2 \frac{1}{n}$$

converges by the Alternating Series Test.

## 9.6 Alternating Series and Conditional Convergence



Example

Does  $\sum_{n=1}^{\infty} \frac{(-1)^{n+1} 10n}{n^2 + 16}$  converge or diverge?

## 9.6 Alternating Series and Conditional Convergence



### Example

Does  $\sum_{n=1}^{\infty} \frac{(-1)^{n+1} 10n}{n^2 + 16}$  converge or diverge?

Let  $a_n = \frac{10n}{n^2 + 16}$  and  $f(x) = \frac{10x}{x^2 + 16}$ . Note that  $a_n > 0$  for all  $n$  and  $a_n \rightarrow 0$  as  $n \rightarrow \infty$ .

## 9.6 Alternating Series and Conditional Convergence



### Example

Does  $\sum_{n=1}^{\infty} \frac{(-1)^{n+1} 10n}{n^2 + 16}$  converge or diverge?

Let  $a_n = \frac{10n}{n^2 + 16}$  and  $f(x) = \frac{10x}{x^2 + 16}$ . Note that  $a_n > 0$  for all  $n$  and  $a_n \rightarrow 0$  as  $n \rightarrow \infty$ . Moreover if  $x \geq 4$ , then

$$f'(x) = \frac{10(16 - x^2)}{(x^2 + 16)^2} \leq 0.$$

So  $(a_n)$  is decreasing for  $n \geq 4$ .

## 9.6 Alternating Series and Conditional Convergence



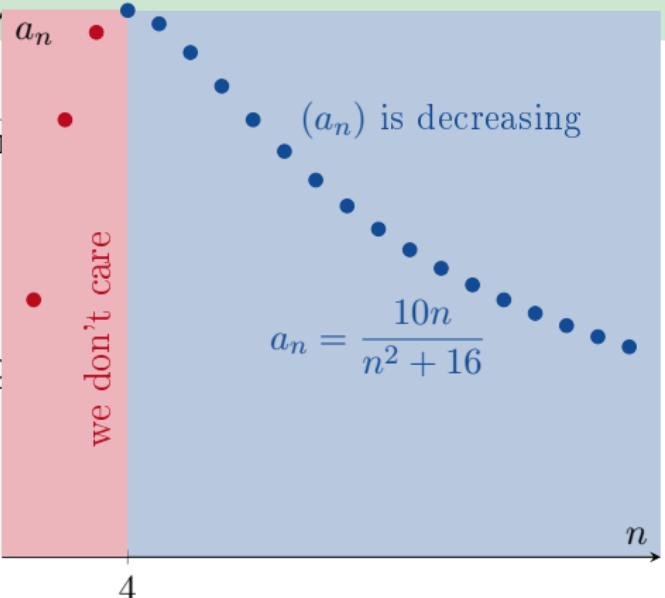
### Example

Does  $\sum_{n=1}^{\infty} \frac{(-1)^{n+1} 10n}{n^2 + 16}$  converge or diverge?

Let  $a_n = \frac{10n}{n^2 + 16}$  and  $f(x) = \frac{1}{x^2 + 16}$   
and  $a_n \rightarrow 0$  as  $n \rightarrow \infty$ . More

$$f'(x) =$$

So  $(a_n)$  is **decreasing** for  $n \geq 1$



## 9.6 Alternating Series and Conditional Convergence



### Example

Does  $\sum_{n=1}^{\infty} \frac{(-1)^{n+1} 10n}{n^2 + 16}$  converge or diverge?

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So  $(a_n)$  is decreasing for  $n \geq 4$ .

Therefore  $\sum_{n=1}^{\infty} \frac{(-1)^{n+1} 10n}{n^2 + 16}$  converges.

## 9.6 Alternating Series and Conditional Convergence



### Definition

If a series  $\sum_{k=1}^{\infty} a_k$  is convergent, but is not absolutely convergent, then we say that it is *conditionally convergent*.

(Equivalently, we can say that the series *converges conditionally*.)

## 9.6 Alternating Series and Conditional Convergence



### Example

$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$  is conditionally convergent.

(because  $\sum_{n=1}^{\infty} \frac{1}{n}$  diverges)

### Example

$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2}$  is absolutely convergent.

## 9.6 Alternating Series and Conditional Convergence



### Remark

We can rearrange an absolutely convergent series without changing its sum.

This is **not true** for conditionally convergent series.

## 9.6 Alternating Series and Conditional Convergence



### Remark

We can rearrange an absolutely convergent series without changing its sum.

This is **not true** for conditionally convergent series.

For example, it is possible (see page 99 of Mary Hart's book) to show that

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \dots = \ln 2$$

and

$$\underbrace{1 + \frac{1}{3} + \frac{1}{5} + \frac{1}{7}}_{\text{4 positive terms}} \underbrace{- \frac{1}{2}}_{\text{negative term}} \underbrace{+ \frac{1}{9} + \frac{1}{11} + \frac{1}{13} + \frac{1}{15}}_{\text{4 positive terms}} \underbrace{- \frac{1}{4}}_{\text{negative term}} + \frac{1}{17} + \frac{1}{19} + \dots = \ln 4.$$



# Power Series

## 9.7 Power Series



Let  $(a_n)_{n=0}^{\infty}$  be a sequence. Then

$$\sum_{n=0}^{\infty} a_n(x - c)^n$$

is a *power series* (kuvvet serisi). This is a function of  $x$ .

## 9.7 Power Series



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“infinite polynomials”

## 9.7 Power Series

### Example

The following are power series:

- $1 + 2x + 3x^2 + 4x^3 + 5x^4 + \dots = \sum_{n=1}^{\infty} nx^{n-1} = \sum_{n=0}^{\infty} (n+1)x^n;$
- $1 + x + x^2 + x^3 + x^4 + \dots = \sum_{n=0}^{\infty} x^n;$
- $1 + x + 2x^2 + 6x^3 + 24x^4 + \dots = \sum_{n=0}^{\infty} n!x^n;$
- $1 + x^2 + x^4 + x^6 + x^8 + \dots = \sum_{n=0}^{\infty} \left(\frac{1+(-1)^2}{2}\right) x^n;$
- $1 + (x-2) + (x-2)^2 + (x-2)^3 + (x-2)^4 + \dots = \sum_{n=0}^{\infty} (x-2)^n.$

## 9.7 Power Series



### Definition

The constant  $c$  is called the *centre of expansion* of the power series  $\sum_{n=0}^{\infty} a_n(x - c)^n$ .

## 9.7 Power Series

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### Remark

To make things easier, we start by looking at power series with  $c = 0$ . So first we will consider

$$\sum_{n=0}^{\infty} a_n x^n,$$

then we will discuss power series with  $c \neq 0$  later.

## 9.7 Power Series



### Remark

We wish to answer the following three questions about power series:

- How does a power series behave?
- Does this depend on  $x$ ?
- Is it possible for a power series to converge for some  $x$ , but diverge for other  $x$ ?

## 9.7 Power Series



### Example

Recall that

$$\sum_{n=0}^{\infty} x^n \begin{cases} \text{converges if } |x| < 1 \\ \text{diverges if } |x| \geq 1. \end{cases}$$

## 9.7 Power Series



### Example

Recall that

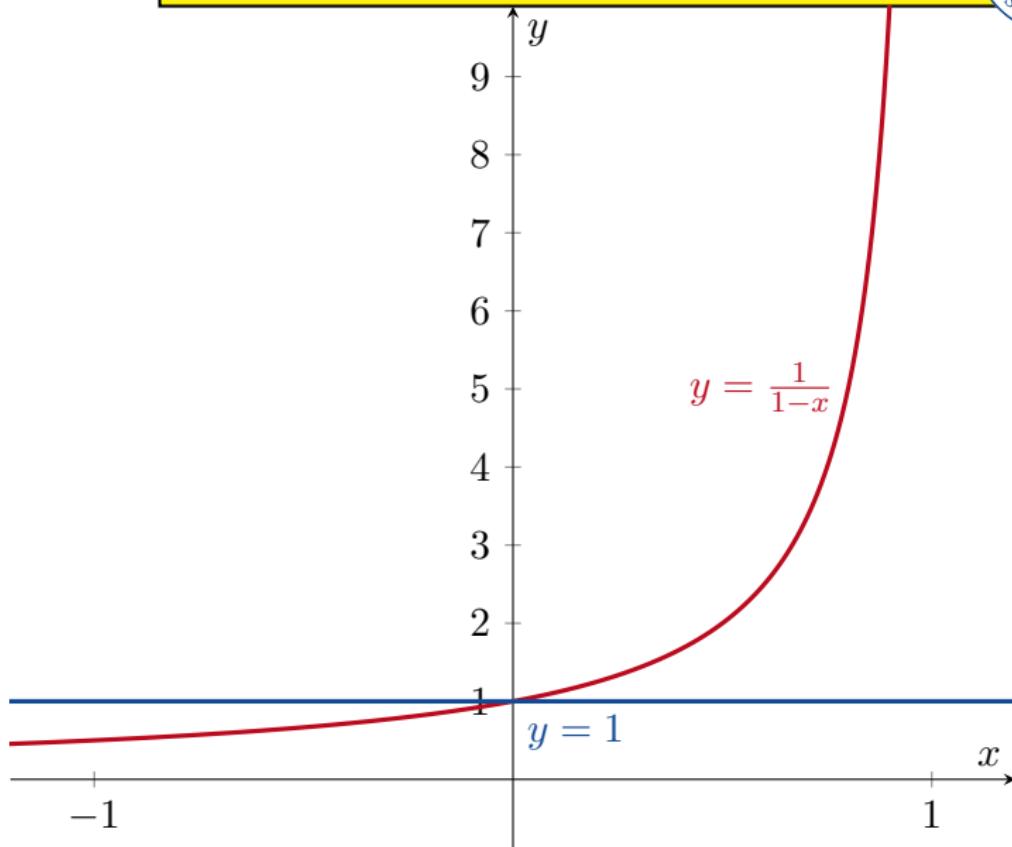
$$\sum_{n=0}^{\infty} x^n \begin{cases} \text{converges if } |x| < 1 \\ \text{diverges if } |x| \geq 1. \end{cases}$$

If  $-1 < x < 1$ , then

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + x^4 + x^5 + x^6 + \dots$$

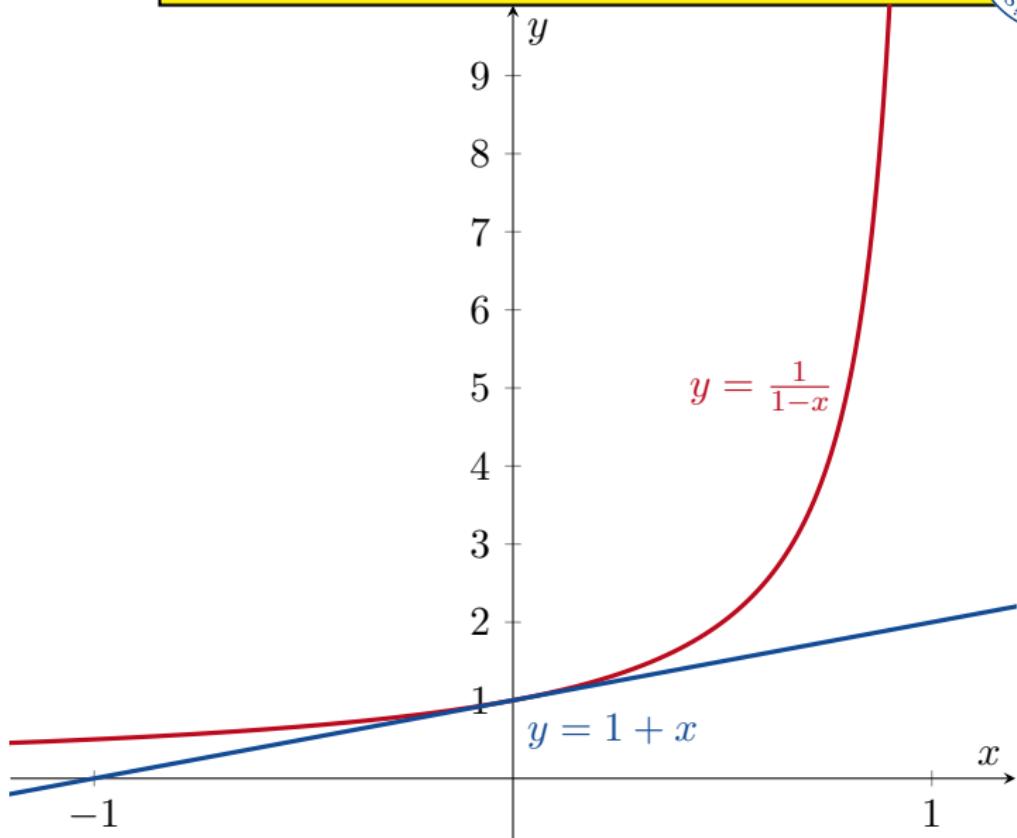
## 9.7 Power Series

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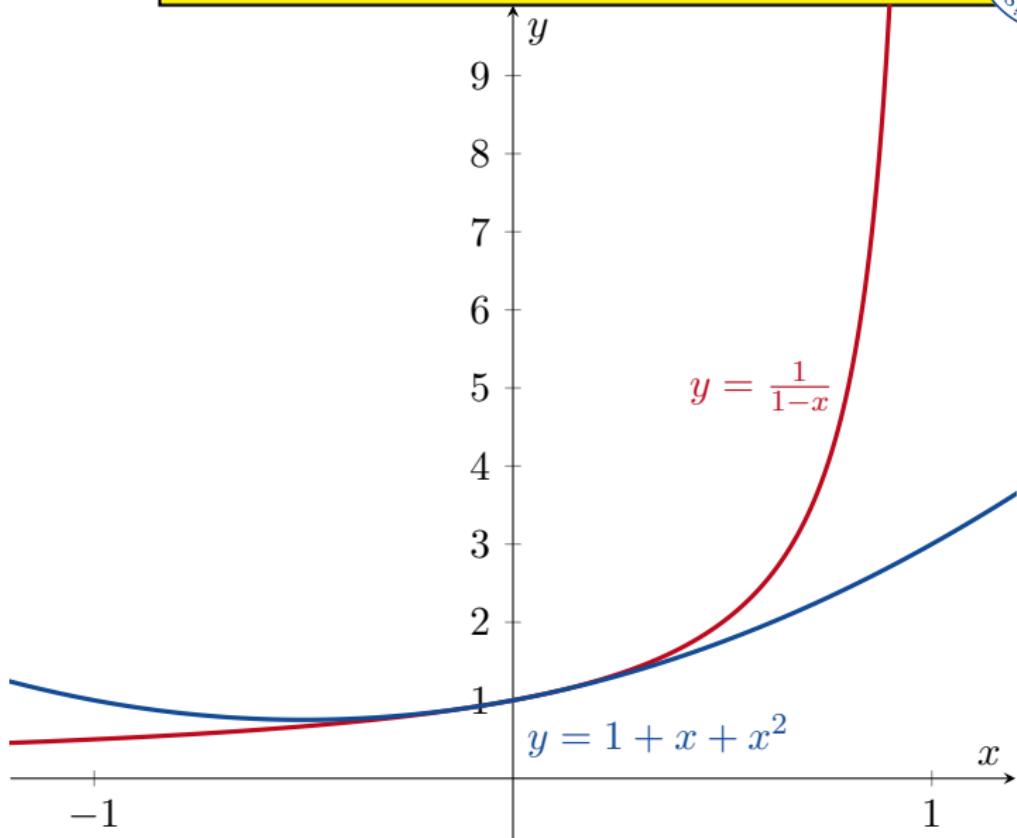
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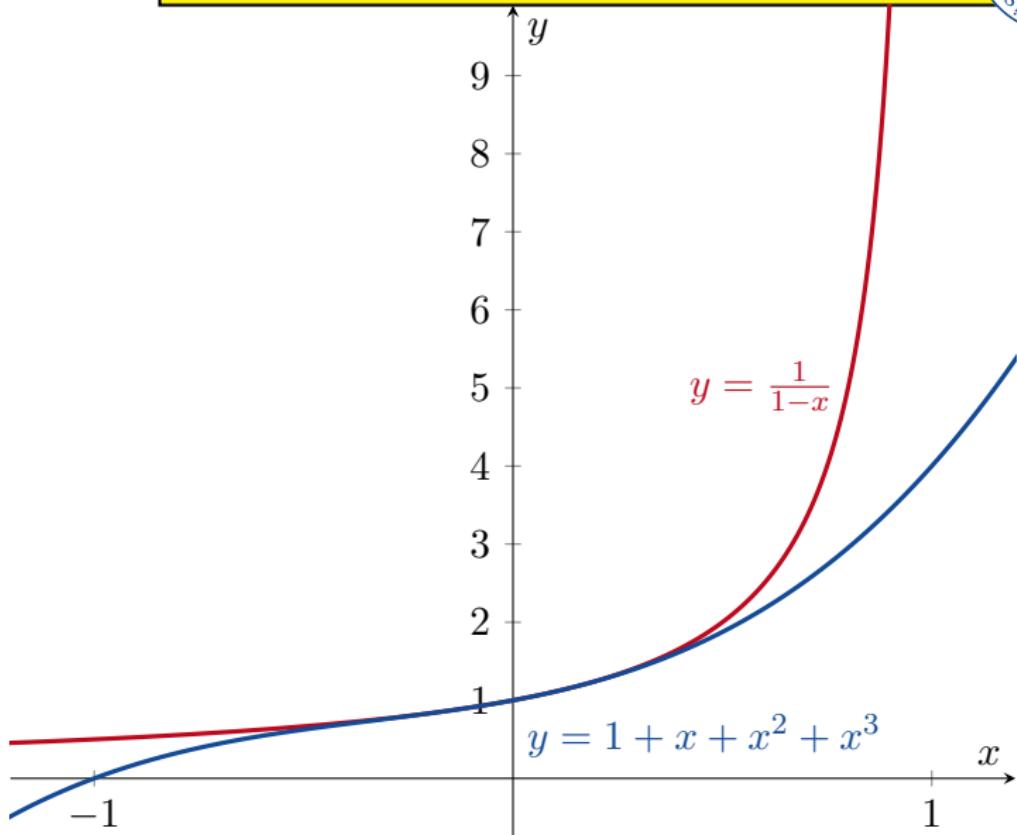
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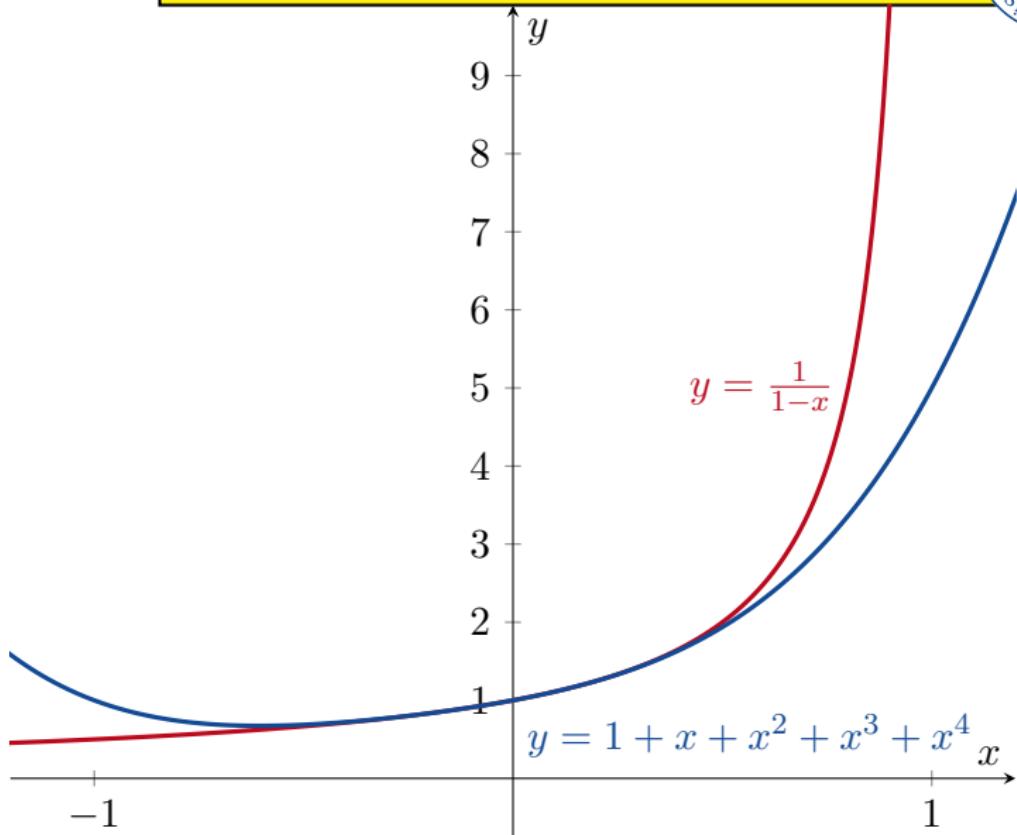
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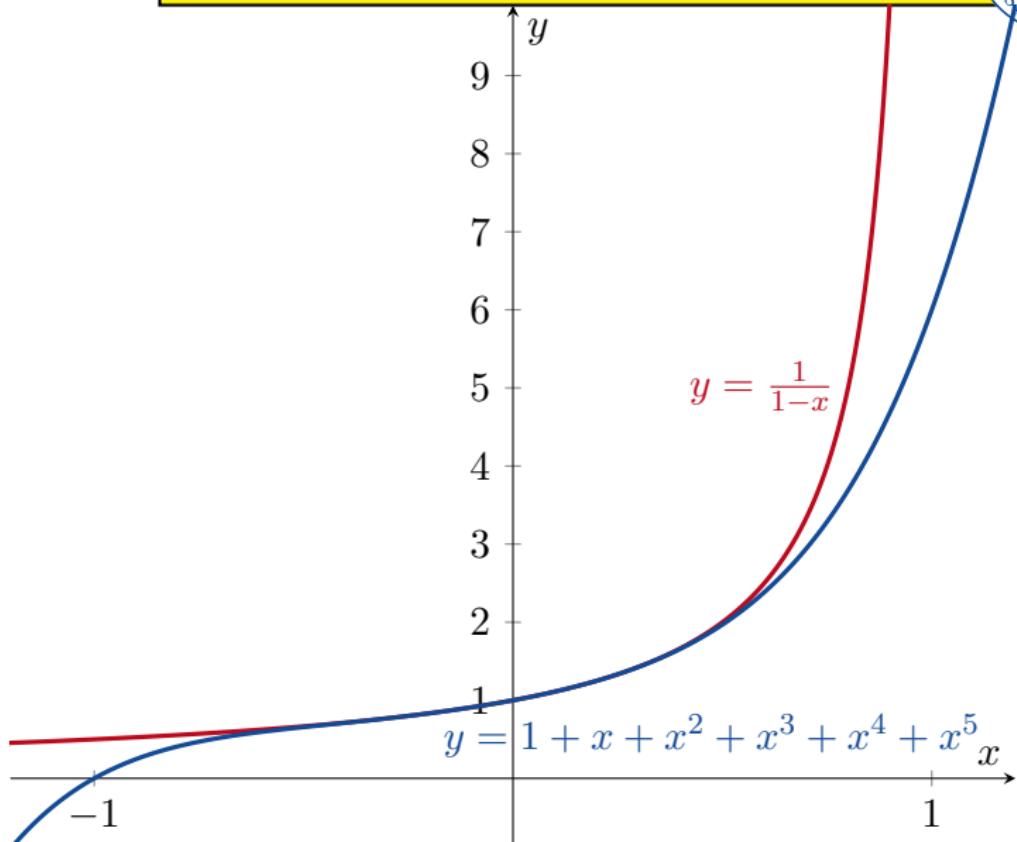
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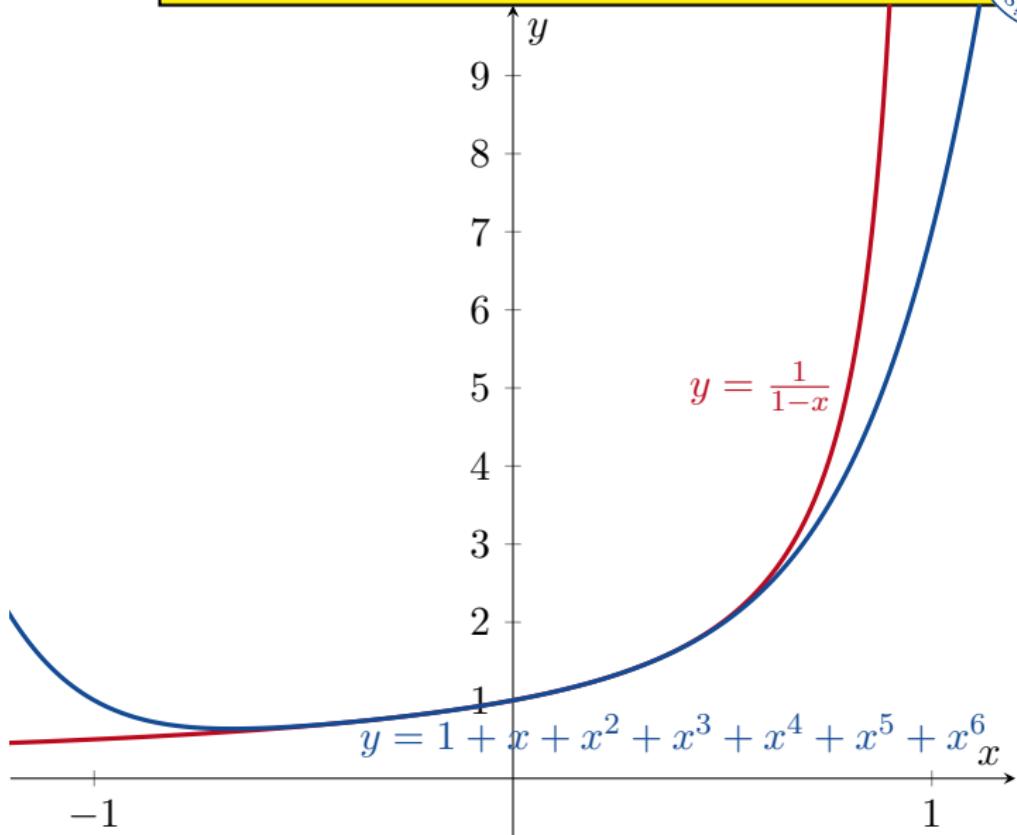
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## 9.7 Power Series

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Consider  $\sum_{n=0}^{\infty} \frac{x^n}{n!}$ .

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Consider  $\sum_{n=0}^{\infty} \frac{x^n}{n!}$ .

- If  $x = 0$ , then  $\sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + 0 + 0 + 0 + 0 + 0 + \dots$  is convergent.

## 9.7 Power Series

### Example

Consider  $\sum_{n=0}^{\infty} \frac{x^n}{n!}$ .

- If  $x = 0$ , then  $\sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + 0 + 0 + 0 + 0 + 0 + \dots$  is convergent.
- Suppose that  $x \neq 0$ . Let  $b_n := \frac{x^n}{n!}$ . Then

$$\left| \frac{b_{n+1}}{b_n} \right| = \left| \frac{x^{n+1}}{(n+1)!} \frac{n!}{x^n} \right| = \frac{|x|}{n+1} \rightarrow 0$$

as  $n \rightarrow \infty$ . Hence  $\sum_{n=0}^{\infty} b_n$  is absolutely convergent by the Ratio Test v2.

## 9.7 Power Series

### Example

Consider  $\sum_{n=0}^{\infty} \frac{x^n}{n!}$ .

- If  $x = 0$ , then  $\sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + 0 + 0 + 0 + 0 + 0 + \dots$  is convergent.
- Suppose that  $x \neq 0$ . Let  $b_n := \frac{x^n}{n!}$ . Then

$$\left| \frac{b_{n+1}}{b_n} \right| = \left| \frac{x^{n+1}}{(n+1)!} \frac{n!}{x^n} \right| = \frac{|x|}{n+1} \rightarrow 0$$

as  $n \rightarrow \infty$ . Hence  $\sum_{n=0}^{\infty} b_n$  is absolutely convergent by the Ratio Test v2.

Therefore  $\sum_{n=0}^{\infty} \frac{x^n}{n!}$  converges  $\forall x \in \mathbb{R}$ .

## 9.7 Power Series



### Example

Consider  $\sum_{n=0}^{\infty} n!x^n$ .

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## 9.7 Power Series



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Consider  $\sum_{n=0}^{\infty} n!x^n$ .

- If  $x = 0$ , then  $\sum_{n=0}^{\infty} n!x^n = 1 + 0 + 0 + 0 + 0 + 0 + \dots$  is convergent.
- Suppose that  $x \neq 0$ . Let  $b_n := n!x^n$  and  $t = \frac{1}{x}$ . Recall that  $\frac{t^n}{n!} \rightarrow 0$  as  $n \rightarrow \infty \forall t \in \mathbb{R}$ .

## 9.7 Power Series



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Consider  $\sum_{n=0}^{\infty} n!x^n$ .

- If  $x = 0$ , then  $\sum_{n=0}^{\infty} n!x^n = 1 + 0 + 0 + 0 + 0 + 0 + \dots$  is convergent.
- Suppose that  $x \neq 0$ . Let  $b_n := n!x^n$  and  $t = \frac{1}{x}$ . Recall that  $\frac{t^n}{n!} \rightarrow 0$  as  $n \rightarrow \infty \ \forall t \in \mathbb{R}$ . So  $|b_n| = |n!x^n| = \left| \frac{n!}{t^n} \right| \rightarrow \infty$  as  $n \rightarrow \infty$ . So  $|b_n| \not\rightarrow 0$  as  $n \rightarrow \infty$ .

## 9.7 Power Series



### Example

Consider  $\sum_{n=0}^{\infty} n!x^n$ .

- If  $x = 0$ , then  $\sum_{n=0}^{\infty} n!x^n = 1 + 0 + 0 + 0 + 0 + 0 + \dots$  is convergent.
- Suppose that  $x \neq 0$ . Let  $b_n := n!x^n$  and  $t = \frac{1}{x}$ . Recall that  $\frac{t^n}{n!} \rightarrow 0$  as  $n \rightarrow \infty \ \forall t \in \mathbb{R}$ . So  $|b_n| = |n!x^n| = \left|\frac{n!}{t^n}\right| \rightarrow \infty$  as  $n \rightarrow \infty$ . So  $|b_n| \not\rightarrow 0$  as  $n \rightarrow \infty$ . By the Divergence Test,  $\sum_{n=0}^{\infty} b_n$  diverges.

## 9.7 Power Series

### Example

Consider  $\sum_{n=0}^{\infty} n!x^n$ .

- If  $x = 0$ , then  $\sum_{n=0}^{\infty} n!x^n = 1 + 0 + 0 + 0 + 0 + 0 + \dots$  is convergent.
- Suppose that  $x \neq 0$ . Let  $b_n := n!x^n$  and  $t = \frac{1}{|x|}$ . Recall that  $\frac{t^n}{n!} \rightarrow 0$  as  $n \rightarrow \infty \forall t \in \mathbb{R}$ . So  $|b_n| = |n!x^n| = \left|\frac{n!}{t^n}\right| \rightarrow \infty$  as  $n \rightarrow \infty$ . So  $|b_n| \not\rightarrow 0$  as  $n \rightarrow \infty$ . By the Divergence Test,  $\sum_{n=0}^{\infty} b_n$  diverges.

Therefore  $\sum_{n=0}^{\infty} \frac{x^n}{n!}$   $\begin{cases} \text{converges if } x = 0 \\ \text{diverges if } x \neq 0. \end{cases}$

## 9.7 Power Series



### Example

Consider  $\sum_{n=1}^{\infty} nx^{n-1} = 1 + 2x + 3x^2 + 4x^3 + 5x^4 + \dots$

## 9.7 Power Series



### Example

Consider  $\sum_{n=1}^{\infty} nx^{n-1} = 1 + 2x + 3x^2 + 4x^3 + 5x^4 + \dots$

- If  $x = 0$ , then  $\sum_{n=1}^{\infty} nx^{n-1} = 1 + 0 + 0 + 0 + 0 + 0 + \dots$  converges.

## 9.7 Power Series

### Example

Consider  $\sum_{n=1}^{\infty} nx^{n-1} = 1 + 2x + 3x^2 + 4x^3 + 5x^4 + \dots$

- Suppose that  $x \neq 0$ . Let  $b_n := nx^{n-1}$ .

## 9.7 Power Series

### Example

Consider  $\sum_{n=1}^{\infty} nx^{n-1} = 1 + 2x + 3x^2 + 4x^3 + 5x^4 + \dots$

- Suppose that  $x \neq 0$ . Let  $b_n := nx^{n-1}$ .

Then

$$\left| \frac{b_{n+1}}{b_n} \right| = \frac{(n+1)|x|^n}{n|x|^{n-1}} = \left(1 + \frac{1}{n}\right)|x| \rightarrow |x|$$

as  $n \rightarrow \infty$ .

## 9.7 Power Series

### Example

Consider  $\sum_{n=1}^{\infty} nx^{n-1} = 1 + 2x + 3x^2 + 4x^3 + 5x^4 + \dots$

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as  $n \rightarrow \infty$ .

By the Ratio Test v2,

$$\sum_{n=1}^{\infty} nx^{n-1} \begin{cases} \text{converges if } 0 < |x| < 1 \\ \text{diverges if } |x| > 1. \end{cases}$$

## 9.7 Power Series



### Example

Consider  $\sum_{n=1}^{\infty} nx^{n-1} = 1 + 2x + 3x^2 + 4x^3 + 5x^4 + \dots$

- Suppose that  $|x| = 1$ .

## 9.7 Power Series



### Example

Consider  $\sum_{n=1}^{\infty} nx^{n-1} = 1 + 2x + 3x^2 + 4x^3 + 5x^4 + \dots$

- Suppose that  $|x| = 1$ . Then  $|nx^{n-1}| = n$  which means that  $nx^{n-1} \not\rightarrow 0$  as  $n \rightarrow \infty$ .

## 9.7 Power Series



### Example

Consider  $\sum_{n=1}^{\infty} nx^{n-1} = 1 + 2x + 3x^2 + 4x^3 + 5x^4 + \dots$

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So  $\sum_{n=1}^{\infty} b_n$  diverges if  $|x| = 1$ .

## 9.7 Power Series



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Consider  $\sum_{n=1}^{\infty} nx^{n-1} = 1 + 2x + 3x^2 + 4x^3 + 5x^4 + \dots$

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So  $\sum_{n=1}^{\infty} b_n$  diverges if  $|x| = 1$ .

Therefore  $\sum_{n=1}^{\infty} nx^{n-1}$   $\begin{cases} \text{converges if } |x| < 1 \\ \text{diverges if } |x| \geq 1. \end{cases}$

## 9.7 Power Series



You can read more examples in the textbook.

## 9.7 Power Series



### Remark

$\sum_{n=0}^{\infty} x^n$  converges for  $|x| < 1$  and diverges for  $|x| \geq 1$ . If we differentiate each term (are we allowed to do this?), we get

$$0 + 1 + 2x + 3x^2 + 4x^3 + 5x^4 + \dots = \sum_{n=1}^{\infty} nx^{n-1}$$

which also converges for  $|x| < 1$  and diverges for  $|x| \geq 1$ .  
Interesting!

## 9.7 Power Series



### Theorem

A power series  $\sum_{n=0}^{\infty} a_n x^n$  satisfies one and only one of the following:

1 It converges absolutely  $\forall x$ ;

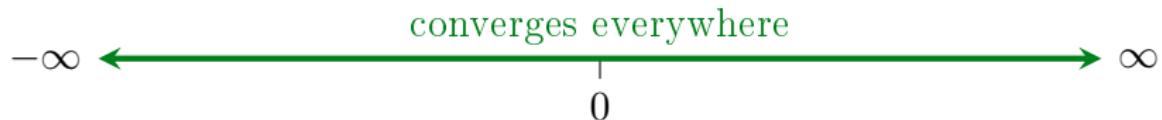
2 It converges for  $x = 0$  and diverges  $\forall x \neq 0$ ; or

3  $\exists R > 0$  such that  $\sum_{n=0}^{\infty} a_n x^n$   $\begin{cases} \text{converges if } |x| < R \\ \text{diverges if } |x| > R. \end{cases}$

## 9.7 Power Series



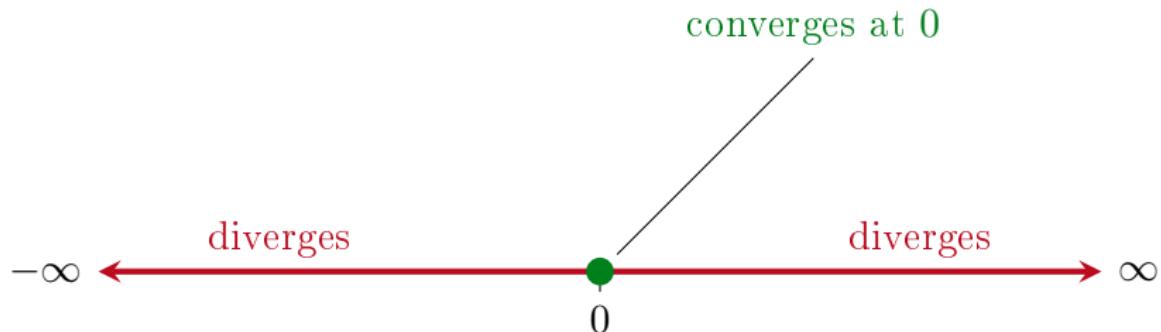
1



## 9.7 Power Series



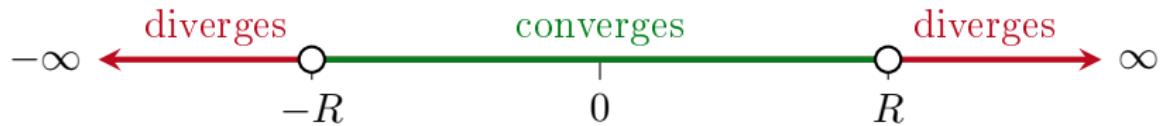
2



## 9.7 Power Series



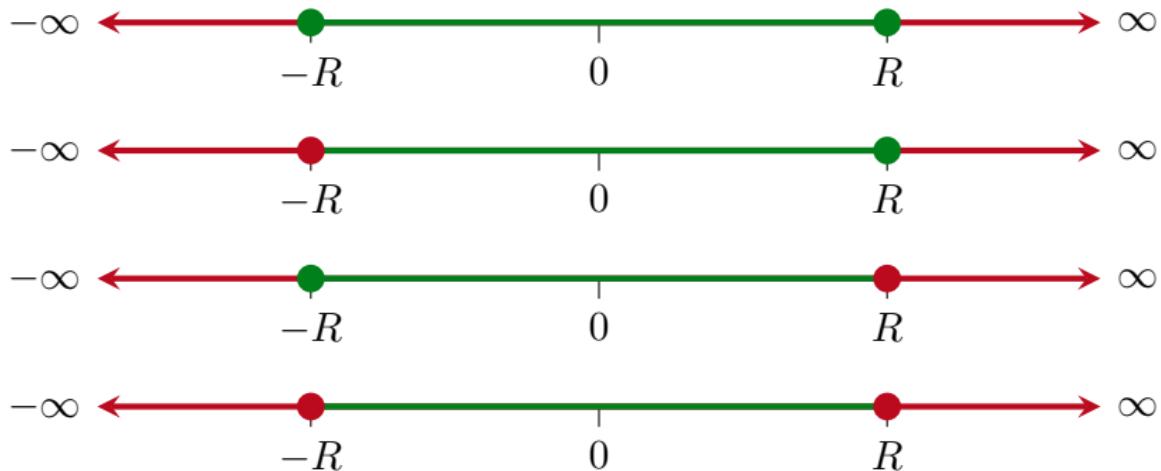
3



## 9.7 Power Series



3



# Radius of Convergence

## Definition

Let  $R \in [0, \infty) \cup \{\infty\}$ .

If  $\sum_{n=0}^{\infty} a_n x^n$  converges  $\forall |x| < R$  and diverges  $\forall |x| > R$ , then  $R$  is called the *radius of convergence* (yakınsaklık yarıçapı) of the power series  $\sum_{n=0}^{\infty} a_n x^n$ .

## 9.7 Power Series

### Definition

If  $R = \infty$ , then we say that  $\sum_{n=0}^{\infty} a_n x^n$  has *infinite radius of convergence*. (This means that  $\sum_{n=0}^{\infty} a_n x^n$  converges  $\forall x$ .)

### Definition

If  $R = 0$ , then we say that  $\sum_{n=0}^{\infty} a_n x^n$  has *zero radius of convergence*. (This means that  $\sum_{n=0}^{\infty} a_n x^n$  converges if  $x = 0$  and diverges  $\forall x \neq 0$ .)

## 9.7 Power Series



### Definition

If  $R > 0$  or  $R = \infty$ , then the open interval  $(-R, R)$  is called the *open interval of convergence* of  $\sum_{n=0}^{\infty} a_n x^n$ .

## 9.7 Power Series



Is there an easy way to find  $R$ ?

## 9.7 Power Series



Is there an easy way to find  $R$ ?

Theorem

Suppose that

$$\left| \frac{a_n}{a_{n+1}} \right| \rightarrow R \in \mathbb{R} \cup \{\infty\}$$

as  $n \rightarrow \infty$ . Then  $\sum_{n=0}^{\infty} a_n x^n$  has radius of convergence  $R$ .

## 9.7 Power Series



### Remark

A power series *always* has a radius of convergence, even if

$$\lim_{n \rightarrow \infty} \left| \frac{a_n}{a_{n+1}} \right| \text{ doesn't exist.}$$

This theorem just gives us an easy way to find  $R$ , if this limit does exist.

If the limit does not exist, then we need to use a different method to find  $R$ .

## 9.7 Power Series

### Remark

A power series *always* has a radius of convergence, even if

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This theorem just gives us an easy way to find  $R$ , if this limit does exist.

If the limit does not exist, then we need to use a different method to find  $R$ .

### Remark

Never, never, never forget to use  $|\cdot|$  when you use this theorem.

## 9.7 Power Series



### Remark

The Ratio Test v2 uses  $\left| \frac{a_{n+1}}{a_n} \right|$ , but this theorem uses  $\left| \frac{a_n}{a_{n+1}} \right|$ .

Don't get these mixed up.

## 9.7 Power Series



We have seen that  $\exists R$  such that

$$\sum_{n=0}^{\infty} a_n x^n \begin{cases} \text{converges if } |x| < R \\ \text{diverges if } |x| > R. \end{cases}$$

Suppose that  $0 < R < \infty$ .

## 9.7 Power Series



We have seen that  $\exists R$  such that

$$\sum_{n=0}^{\infty} a_n x^n \begin{cases} \text{converges if } |x| < R \\ \text{diverges if } |x| > R. \end{cases}$$

Suppose that  $0 < R < \infty$ .

What happens when  $|x| = R$ ?

## 9.7 Power Series

### Example

Consider  $\sum_{n=0}^{\infty} x^n$ .

## 9.7 Power Series

### Example

Consider  $\sum_{n=0}^{\infty} x^n$ .

This is a power series with  $a_n = 1 \ \forall n$ . Since

$$\left| \frac{a_n}{a_{n+1}} \right| = \frac{1}{1} = 1,$$

the radius of convergence is  $R = 1$ .

## 9.7 Power Series

### Example

Consider  $\sum_{n=0}^{\infty} x^n$ .

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the radius of convergence is  $R = 1$ . This means that

$$\sum_{n=0}^{\infty} a_n x^n \begin{cases} \text{converges if } |x| < 1 \\ \text{diverges if } |x| > 1. \end{cases}$$

## 9.7 Power Series

### Example

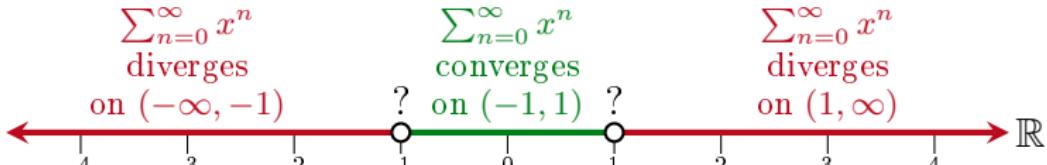
Consider  $\sum_{n=0}^{\infty} x^n$ .

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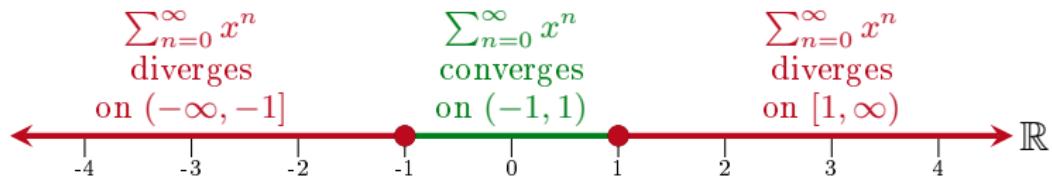
$$\sum_{n=0}^{\infty} a_n x^n \begin{cases} \text{converges if } |x| < 1 \\ \text{diverges if } |x| > 1. \end{cases}$$



## 9.7 Power Series



Previously we saw that  $\sum_{n=0}^{\infty} x^n$  also diverges for  $|x| = 1$ .



For this power series, we have divergence when  $x = \pm R$ .

## 9.7 Power Series



### Example

Now consider  $\sum_{n=0}^{\infty} \frac{x^n}{n+1}$ .

## 9.7 Power Series



### Example

Now consider  $\sum_{n=0}^{\infty} \frac{x^n}{n+1}$ .

For this power series,  $a_n = \frac{1}{n+1} \quad \forall n \in \mathbb{N} \cup \{0\}$  and

$$\left| \frac{a_n}{a_{n+1}} \right| = \frac{n+2}{n+1} = \frac{1 + \frac{2}{n}}{1 + \frac{1}{n}} \rightarrow 1$$

as  $n \rightarrow \infty$ . Thus, the radius of convergence is  $R = 1$  again.

## 9.7 Power Series



### Example

Now consider  $\sum_{n=0}^{\infty} \frac{x^n}{n+1}$ .

For this power series,  $a_n = \frac{1}{n+1} \quad \forall n \in \mathbb{N} \cup \{0\}$  and

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as  $n \rightarrow \infty$ . Thus, the radius of convergence is  $R = 1$  again.

This means that  $\sum_{n=0}^{\infty} \frac{x^n}{n+1} \begin{cases} \text{converges if } |x| < 1 \\ \text{diverges if } |x| > 1. \end{cases}$

## 9.7 Power Series



When  $x = 1$ , the series is

$$\sum_{n=0}^{\infty} \frac{1^n}{n+1} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \dots = \sum_{n=1}^{\infty} \frac{1}{n}$$

which we know diverges.

## 9.7 Power Series

When  $x = 1$ , the series is

$$\sum_{n=0}^{\infty} \frac{1^n}{n+1} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \dots = \sum_{n=1}^{\infty} \frac{1}{n}$$

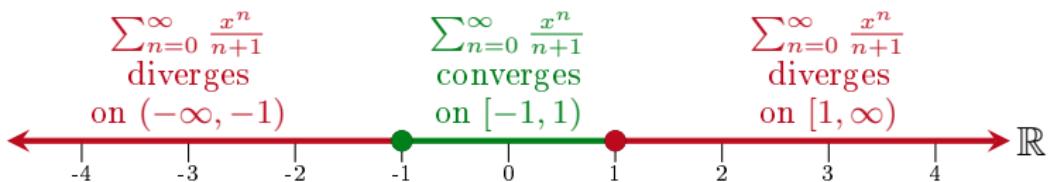
which we know diverges.

When  $x = -1$ , the series is

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{n+1} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \dots$$

which we know converges.

## 9.7 Power Series



For this power series, we have convergence when  $x = -R$  and divergence when  $x = R$ .

## 9.7 Power Series



### Example

Consider  $\sum_{n=0}^{\infty} \frac{x^n}{(n+1)^2}$ .

## 9.7 Power Series



### Example

Consider  $\sum_{n=0}^{\infty} \frac{x^n}{(n+1)^2}$ .

For this power series,  $a_n = \frac{1}{(n+1)^2} \quad \forall n \in \mathbb{N} \cup \{0\}$  and

$$\left| \frac{a_n}{a_{n+1}} \right| = \frac{(n+2)^2}{(n+1)^2} \rightarrow 1$$

as  $n \rightarrow \infty$ . Thus, the radius of convergence is  $R = 1$  again.

## 9.7 Power Series



When  $|x| = R = 1$ ,

$$\sum_{n=0}^{\infty} \frac{|x|^n}{(n+1)^2} = \sum_{n=0}^{\infty} \frac{1}{(n+1)^2} = \sum_{n=1}^{\infty} \frac{1}{n^2}$$

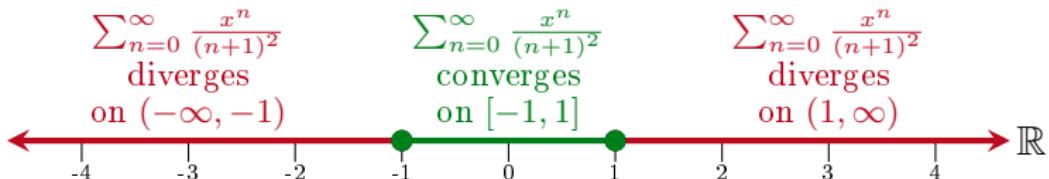
which converges.

## 9.7 Power Series

When  $|x| = R = 1$ ,

$$\sum_{n=0}^{\infty} \frac{|x|^n}{(n+1)^2} = \sum_{n=0}^{\infty} \frac{1}{(n+1)^2} = \sum_{n=1}^{\infty} \frac{1}{n^2}$$

which converges.



For this power series, we have convergence when  $x = \pm R$ .

## 9.7 Power Series



### Remark

The previous three examples show that when  $|x| = R \in (0, \infty)$ , we can have divergence, conditional convergence or absolute convergence.

# Break

We will continue at 3pm

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$$\int \frac{1}{d} \, d\text{house} =$$



## 9.7 Power Series

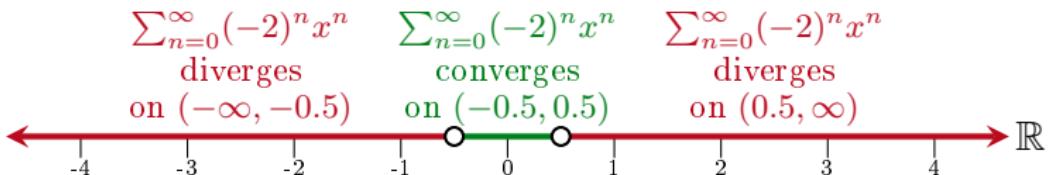
### Example

Consider  $\sum_{n=0}^{\infty} (-2)^n x^n$ .

Since

$$\left| \frac{a_n}{a_{n+1}} \right| = \left| \frac{(-2)^n}{(-2)^{n+1}} \right| = \frac{1}{2},$$

this power series has radius of convergence  $R = \frac{1}{2}$ . The open interval of convergence is  $(-\frac{1}{2}, \frac{1}{2})$ .



## 9.7 Power Series

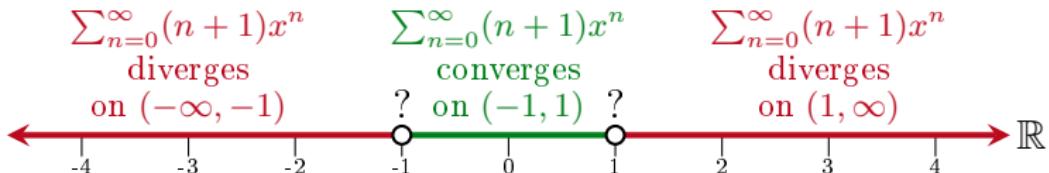
### Example

Consider  $\sum_{n=0}^{\infty} (n+1)x^n$ .

Since

$$\left| \frac{a_n}{a_{n+1}} \right| = \frac{n+1}{n+2} \rightarrow 1$$

as  $n \rightarrow \infty$ , this power series has radius of convergence  $R = 1$ .  
 The open interval of convergence is  $(-1, 1)$ .



## 9.7 Power Series

$$\cosh x = \frac{e^x + e^{-x}}{2}$$



### Example

Consider  $\sum_{n=0}^{\infty} (\cosh n)x^n$ .

Since

$$\left| \frac{a_n}{a_{n+1}} \right| = \left| \frac{\cosh n}{\cosh(n+1)} \right| = \frac{e^n + e^{-n}}{e^{n+1} + e^{-n-1}} = \frac{1 + e^{-2n}}{e + e^{-2n-1}} \rightarrow \frac{1 + 0}{e + 1} = \frac{1}{e}$$

as  $n \rightarrow \infty$ , this power series has radius of convergence  $R = \frac{1}{e}$ .

The open interval of convergence is  $(-\frac{1}{e}, \frac{1}{e})$ .

$$\begin{array}{ccc} \sum_{n=0}^{\infty} (\cosh n)x^n & \sum_{n=0}^{\infty} (\cosh n)x^n & \sum_{n=0}^{\infty} (\cosh n)x^n \\ \text{diverges} & \text{converges} & \text{diverges} \\ \text{on } (-\infty, -1/e) & \text{on } (-1/e, 1/e) & \text{on } (1/e, \infty) \end{array}$$



## 9.7 Power Series



### Example

For the power series  $\sum_{n=0}^{\infty} \frac{x^n}{n!}$ , we have that

$$R = \lim_{n \rightarrow \infty} \left| \frac{a_n}{a_{n+1}} \right| = \infty$$

as  $n \rightarrow \infty$ . The radius of convergence  $R = \infty$ . The open interval of convergence is  $(-\infty, \infty)$ .

$\sum_{n=0}^{\infty} \frac{x^n}{n!}$  converges  $\forall x$





# Term-by-Term Differentiation and Integration

A power series is a function. So can we differentiate it? Can we integrate it?

## 9.7 Power Series



Theorem

Let  $\sum_{n=0}^{\infty} a_n x^n$  be a power series with radius of convergence  $R > 0$ .

If  $|x| < R$ , then

$$\frac{d}{dx} \left( \sum_{n=0}^{\infty} a_n x^n \right) = \sum_{n=0}^{\infty} \left( \frac{d}{dx} a_n x^n \right)$$

and

$$\int \left( \sum_{n=0}^{\infty} a_n x^n \right) dx = \sum_{n=0}^{\infty} \left( \int a_n x^n dx \right).$$

## 9.7 Power Series

**EXAMPLE 4** Find series for  $f'(x)$  and  $f''(x)$  if

$$\begin{aligned}f(x) &= \frac{1}{1-x} = 1 + x + x^2 + x^3 + x^4 + \cdots + x^n + \cdots \\&= \sum_{n=0}^{\infty} x^n, \quad -1 < x < 1.\end{aligned}$$

## 9.7 Power Series

**EXAMPLE 4** Find series for  $f'(x)$  and  $f''(x)$  if

$$\begin{aligned} f(x) &= \frac{1}{1-x} = 1 + x + x^2 + x^3 + x^4 + \cdots + x^n + \cdots \\ &= \sum_{n=0}^{\infty} x^n, \quad -1 < x < 1. \end{aligned}$$

**Solution** We differentiate the power series on the right term by term:

$$\begin{aligned} f'(x) &= \frac{1}{(1-x)^2} = 1 + 2x + 3x^2 + 4x^3 + \cdots + nx^{n-1} + \cdots \\ &= \sum_{n=1}^{\infty} nx^{n-1}, \quad -1 < x < 1; \\ f''(x) &= \frac{2}{(1-x)^3} = 2 + 6x + 12x^2 + \cdots + n(n-1)x^{n-2} + \cdots \\ &= \sum_{n=2}^{\infty} n(n-1)x^{n-2}, \quad -1 < x < 1. \end{aligned}$$



## EXAMPLE 6 The series

$$\frac{1}{1+t} = 1 - t + t^2 - t^3 + \dots$$

converges on the open interval  $-1 < t < 1$ . Therefore,

$$\begin{aligned}\ln(1+x) &= \int_0^x \frac{1}{1+t} dt = t - \frac{t^2}{2} + \frac{t^3}{3} - \frac{t^4}{4} + \dots \Big|_0^x \\ &= x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots\end{aligned}$$

or

$$\ln(1+x) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1} x^n}{n}, \quad -1 < x < 1.$$

### Power Series with Centre of Expansion $c$

The results that we have proved for the power series  $\sum_{n=0}^{\infty} a_n x^n$

are also true for the power series  $\sum_{n=0}^{\infty} a_n (x - c)^n$ .

## 9.7 Power Series



### Example

Recall that  $\sum_{n=0}^{\infty} x^n$  has radius of convergence  $R = 1$ . Therefore  $\sum_{n=0}^{\infty} (x - c)^n$  also has radius of convergence  $R = 1$ .

## 9.7 Power Series



### Example

Recall that  $\sum_{n=0}^{\infty} x^n$  has radius of convergence  $R = 1$ . Therefore  $\sum_{n=0}^{\infty} (x - c)^n$  also has radius of convergence  $R = 1$ . Since

$$\sum_{n=0}^{\infty} x^n \begin{cases} \text{converges absolutely } \forall |x| < 1 \\ \text{diverges } \forall |x| > 1 \end{cases}$$

we have that

$$\sum_{n=0}^{\infty} (x - c)^n \begin{cases} \text{converges absolutely } \forall |x - c| < 1 \\ \text{diverges } \forall |x - c| > 1. \end{cases}$$

## 9.7 Power Series



### Example

Recall that  $\sum_{n=0}^{\infty} x^n$  has radius of convergence  $R = 1$ . Therefore  $\sum_{n=0}^{\infty} (x - c)^n$  also has radius of convergence  $R = 1$ . Since

$$\sum_{n=0}^{\infty} x^n \begin{cases} \text{converges absolutely } \forall |x| < 1 \\ \text{diverges } \forall |x| > 1 \end{cases}$$

we have that

$$\sum_{n=0}^{\infty} (x - c)^n \begin{cases} \text{converges absolutely } \forall |x - c| < 1 \\ \text{diverges } \forall |x - c| > 1. \end{cases}$$

The open interval of convergence for  $\sum_{n=0}^{\infty} (x - c)^n$  is  $(c - 1, c + 1)$ .

## 9.7 Power Series



### Example

Since  $\sum_{n=0}^{\infty} \frac{x^n}{n!}$  has radius of convergence  $R = \infty$ , it follows that

$\sum_{n=0}^{\infty} \frac{(x - c)^n}{n!}$  converges absolutely  $\forall x$ .

The radius of convergence of  $\sum_{n=0}^{\infty} \frac{(x - c)^n}{n!}$  is  $R = \infty$  and the open interval of convergence is  $(-\infty, \infty)$ .

## 9.7 Power Series

### Example

Recall that  $\sum_{n=0}^{\infty} \frac{x^n}{(n+1)^2}$  has radius of convergence  $R = 1$ . So

$$\sum_{n=0}^{\infty} \frac{(x-c)^n}{(n+1)^2} \begin{cases} \text{converges absolutely } \forall |x-c| < 1 \\ \text{diverges } \forall |x-c| > 1. \end{cases}$$

The open interval of convergence of  $\sum_{n=0}^{\infty} \frac{(x-c)^n}{(n+1)^2}$  is  $(c-1, c+1)$ .

## 9.7 Power Series

### Example

Recall that  $\sum_{n=0}^{\infty} \frac{x^n}{(n+1)^2}$  has radius of convergence  $R = 1$ . So

$$\sum_{n=0}^{\infty} \frac{(x-c)^n}{(n+1)^2} \begin{cases} \text{converges absolutely } \forall |x-c| < 1 \\ \text{diverges } \forall |x-c| > 1. \end{cases}$$

The open interval of convergence of  $\sum_{n=0}^{\infty} \frac{(x-c)^n}{(n+1)^2}$  is  $(c-1, c+1)$ .

If  $x \in (c-1, c+1)$ , then

$$\frac{d}{dx} \left( \sum_{n=0}^{\infty} \frac{(x-c)^n}{(n+1)^2} \right) = \sum_{n=0}^{\infty} \frac{d}{dx} \left( \frac{(x-c)^n}{(n+1)^2} \right)$$

and

$$\int \left( \sum_{n=0}^{\infty} \frac{(x-c)^n}{(n+1)^2} \right) dx = \sum_{n=0}^{\infty} \left( \int \frac{(x-c)^n}{(n+1)^2} dx \right).$$



# Taylor and Maclaurin Series

## 9.8 Taylor and Maclaurin Series



Recall Rolle's Theorem and the Mean Value Theorem from MATH113 Mathematics I (see chapter 4 of Thomas' Calculus):

## 9.8 Taylor and Maclaurin Series



Michel Rolle

BORN

21 April 1652

DECEASED

8 November 1719

NATIONALITY

French

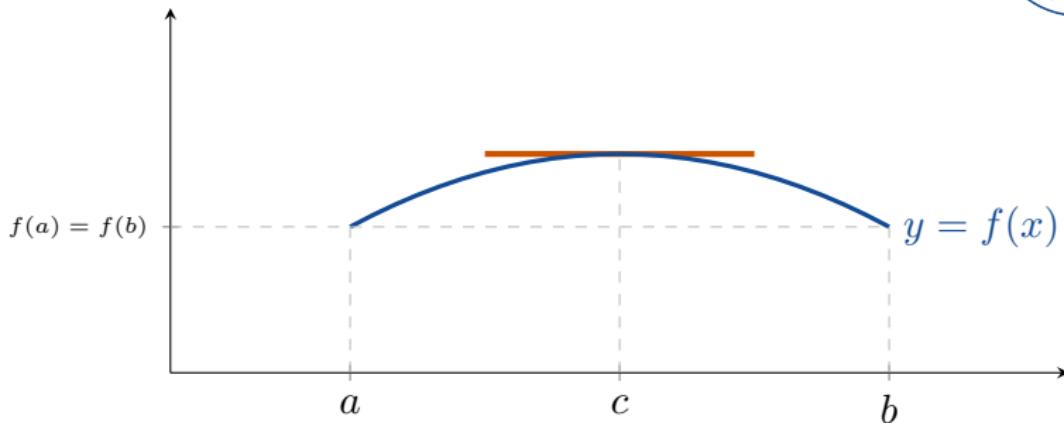
### Theorem (Rolle's Theorem)

Suppose that

- 1  $f : [a, b] \rightarrow \mathbb{R}$  is continuous;
- 2  $f$  is differentiable on  $(a, b)$ ; and
- 3  $f(a) = f(b)$ .

Then  $\exists c \in (a, b)$  such that  $f'(c) = 0$ .

## 9.8 Taylor and Maclaurin Series



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### Augustin-Louis Cauchy

BORN

21 August 1789

DECEASED

23 May 1857

NATIONALITY

French

## Theorem (The Mean Value Theorem)

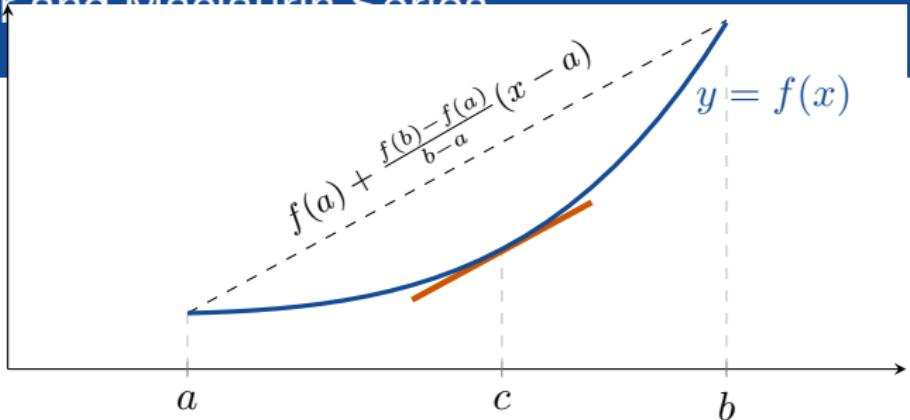
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## 9.8 Taylor and Maclaurin Series



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## 9.8 Taylor and Maclaurin Series



### Remark

In other words,  $\exists c$  such that  $a < c < b$  and

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## 9.8 Taylor and Maclaurin Series

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$$f(b) = f(a) + f'(c)(b - a).$$

Taylor's Theorem takes this formula and expands it to more terms.



**Brook Taylor**

BORN

18 August 1685

DECEASED

29 December 1731

NATIONALITY

British

## 9.8 Taylor and Maclaurin Series



### Theorem (Taylor's Theorem)

Let  $f : [a, b] \rightarrow \mathbb{R}$  be a function. Suppose that

- 1  $f, f', f'', f''', \dots, f^{(n-1)}$  exist and are continuous on  $[a, b]$ ;  
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Then  $\exists c \in (a, b)$  such that

$$\begin{aligned}f(b) &= f(a) + f'(a)(b - a) + \frac{f''(a)}{2!}(b - a)^2 + \dots \\&\quad + \frac{f^{(n-1)}(a)}{(n-1)!}(b - a)^{n-1} + \frac{f^{(n)}(c)}{n!}(b - a)^n.\end{aligned}$$

## 9.8 Taylor and Maclaurin Series



Replacing  $b$  by  $x$  we have

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This is called the *Taylor Series of  $f(x)$  with centre  $a$* .

$$a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \frac{f^{(4)}(a)}{4!}(x-a)^4 + \dots$$

## Definition

The *Taylor Series of  $f(x)$  with centre  $a$*  is the power series

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$$\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!} (x-a)^n.$$

$$R_n(c) = \frac{f^{(n)}(c)}{n!} (x-a)^n = f(x) - \sum_{k=0}^{n-1} \frac{f^{(k)}(a)}{k!} (x-a)^k$$

is called the *remainder term*.

## Remark

The Taylor Series converges to  $f(x)$   $\iff R_n(c) \rightarrow 0$  as  $n \rightarrow \infty$ .

$$a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \frac{f^{(4)}(a)}{4!}(x-a)^4 + \dots$$

## Example

Find the Taylor Series for  $e^x$  centred at 0.

Let  $f(x) = e^x$ . Then  $\frac{d^k f}{dx^k}$  exists and is continuous  $\forall x$  and  $\forall k$ .

Let  $a = 0$  and  $x \neq 0$ . By Taylor's Theorem,

$$\begin{aligned} f(x) &= f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \dots \\ &\quad + \frac{f^{(n-1)}(0)}{(n-1)!}x^{n-1} + \frac{f^{(n)}(c)}{n!}x^n \end{aligned}$$

for some  $c$  between 0 and  $x$  ( $0 < c < x$  or  $x < c < 0$ ).

$$a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \frac{f^{(4)}(a)}{4!}(x-a)^4 + \dots$$

Because  $\frac{d}{dx}e^x = e^x$ , it is easy to see that  $f^{(k)}(0) = 1 \forall k$ .

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Since  $0 < |c| < |x|$ ,

$$0 \leq \left| \frac{e^c}{n!}x^n \right| \leq \frac{e^{|x|}|x|^n}{n!} \rightarrow 0$$

as  $n \rightarrow \infty$ . Hence the remainder term  $R_c(x) = \frac{e^c}{n!}x^n$  tends to zero.

$$a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \frac{f^{(4)}(a)}{4!}(x-a)^4 + \dots$$

Therefore

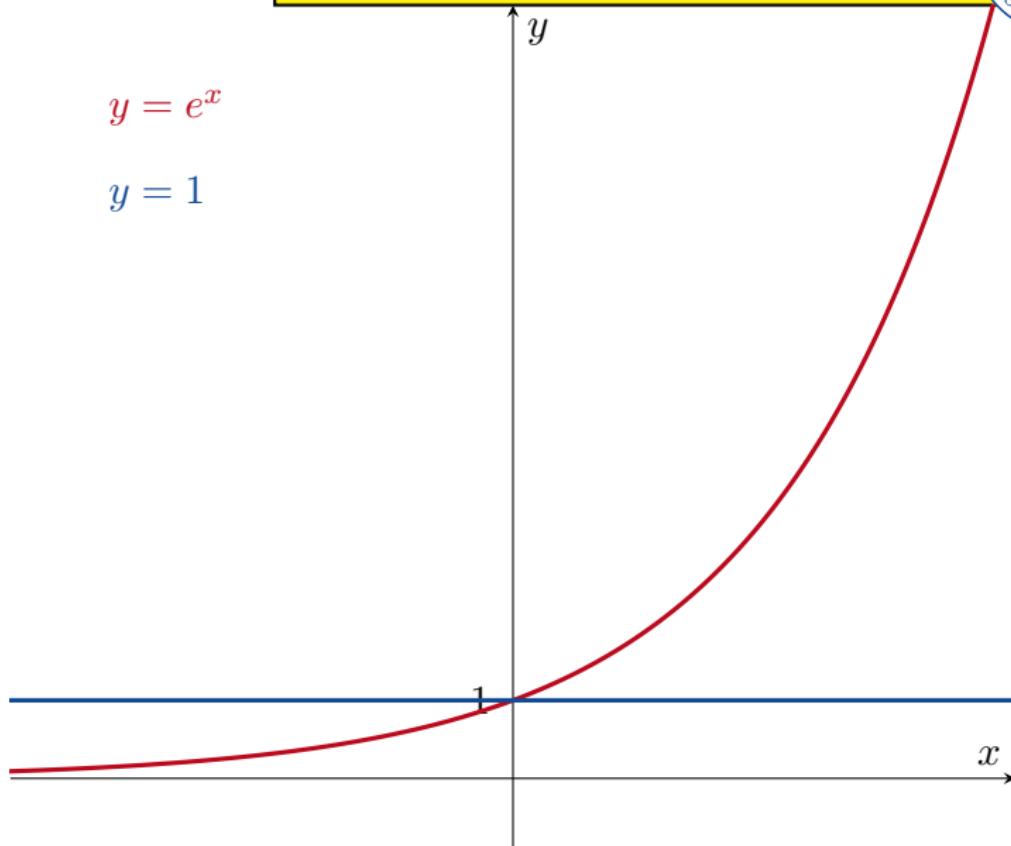
$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

is the Taylor Series of  $e^x$  with centre 0.

(Some people use this as the definition of  $e^x$ , then define  $\ln x$  as the inverse of this.)

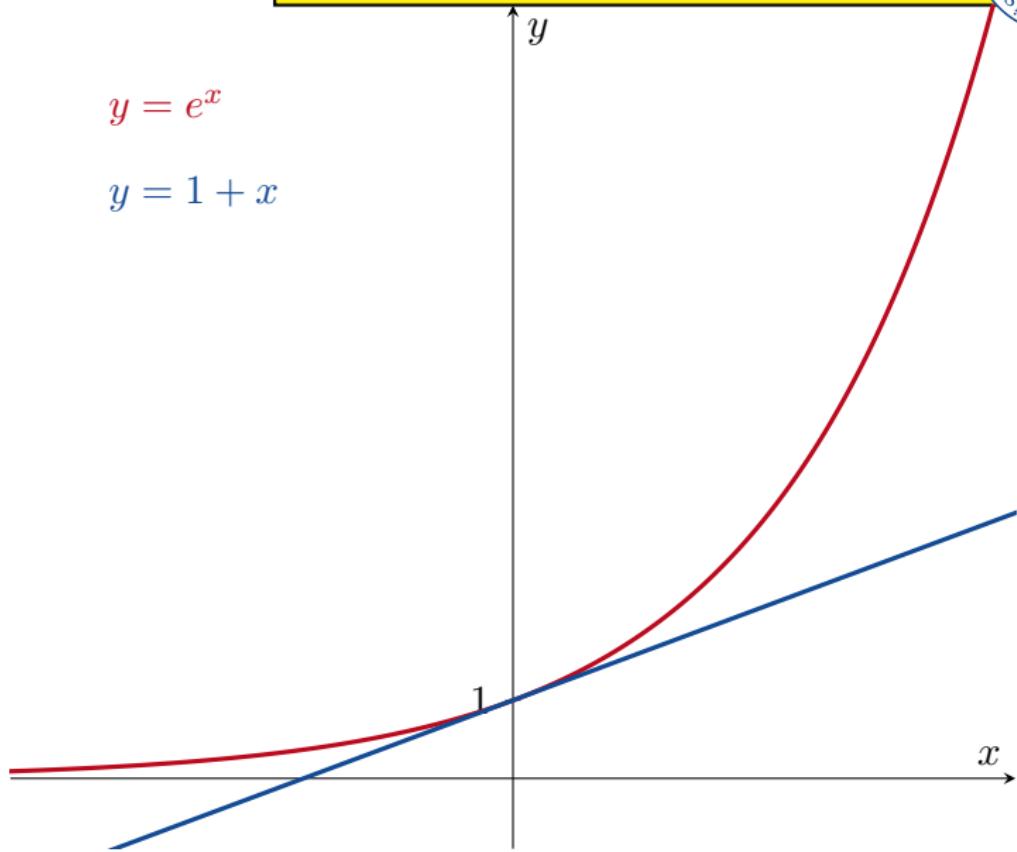
## 9.8 Taylor and Mac

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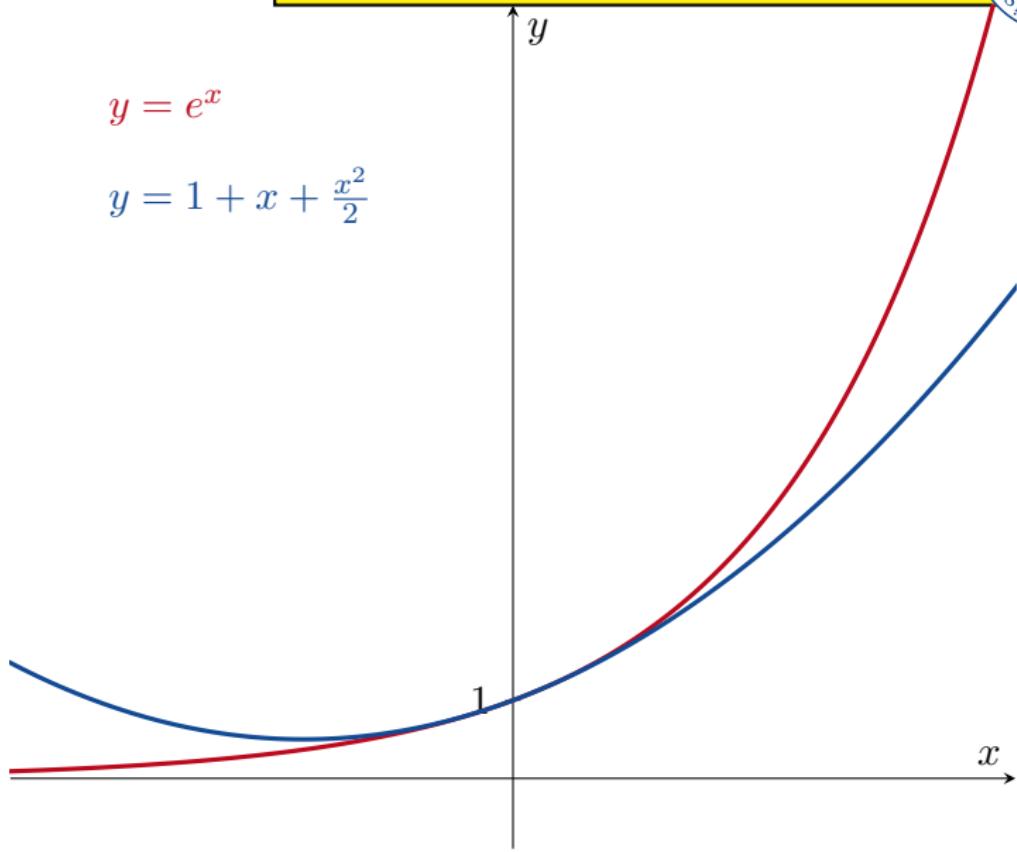
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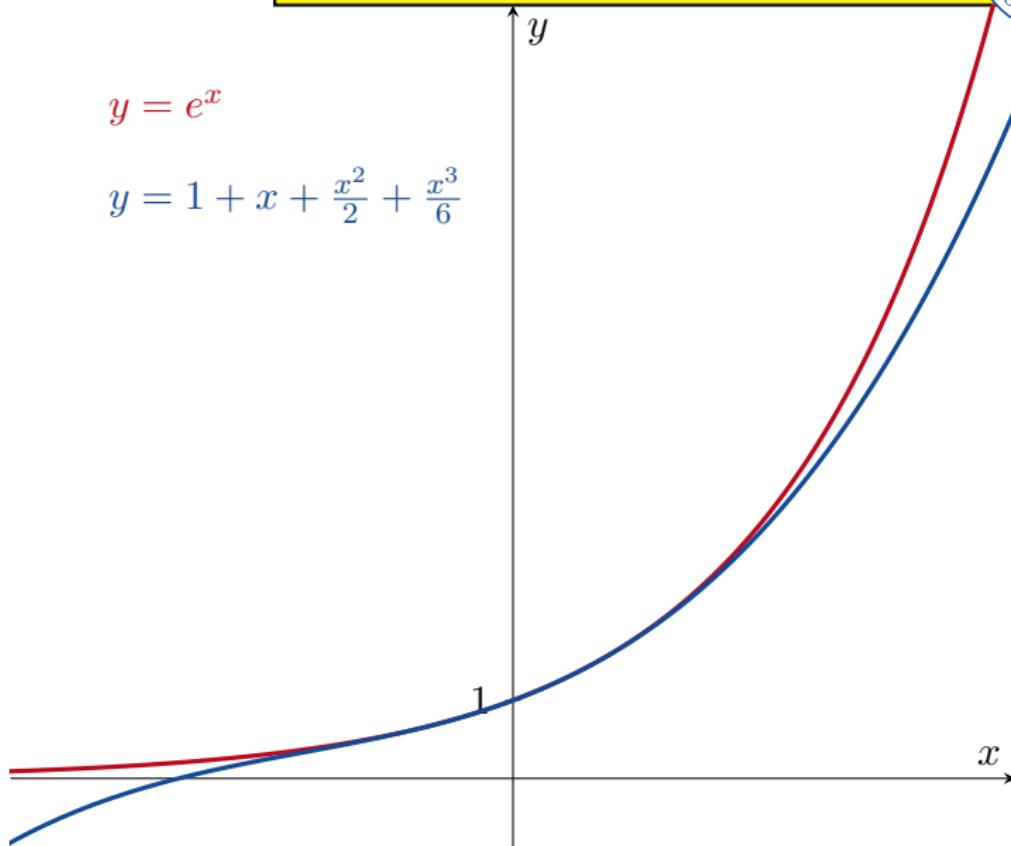
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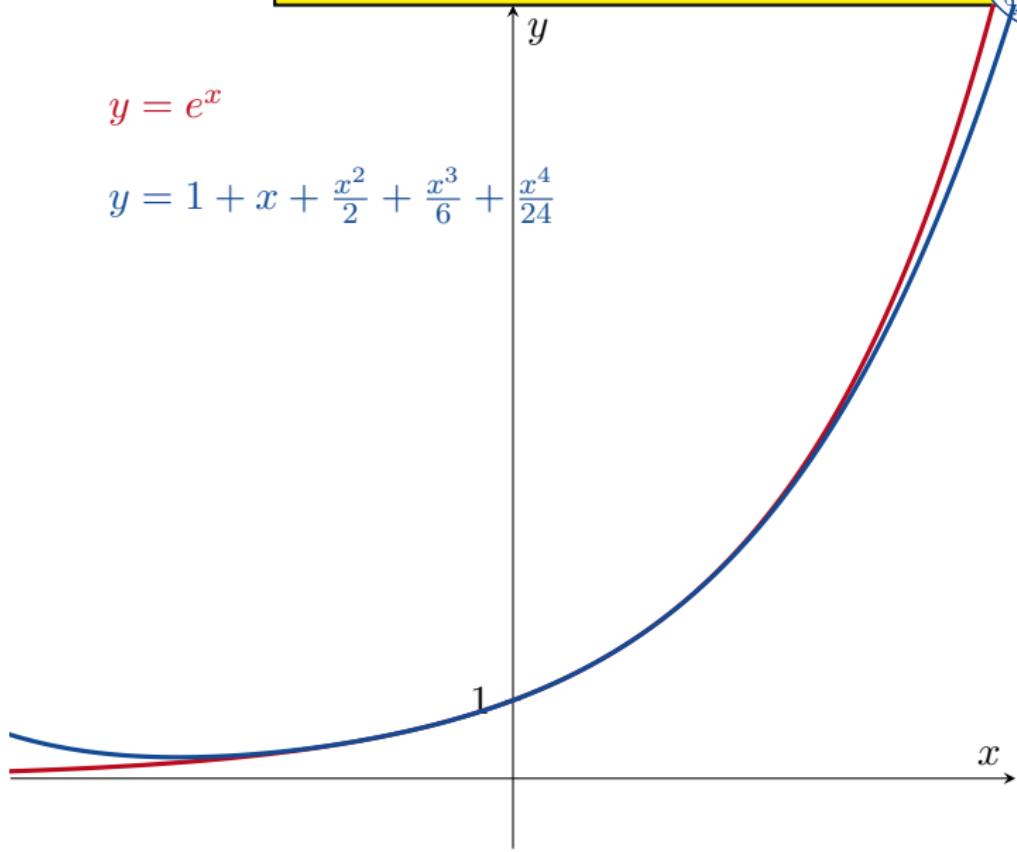
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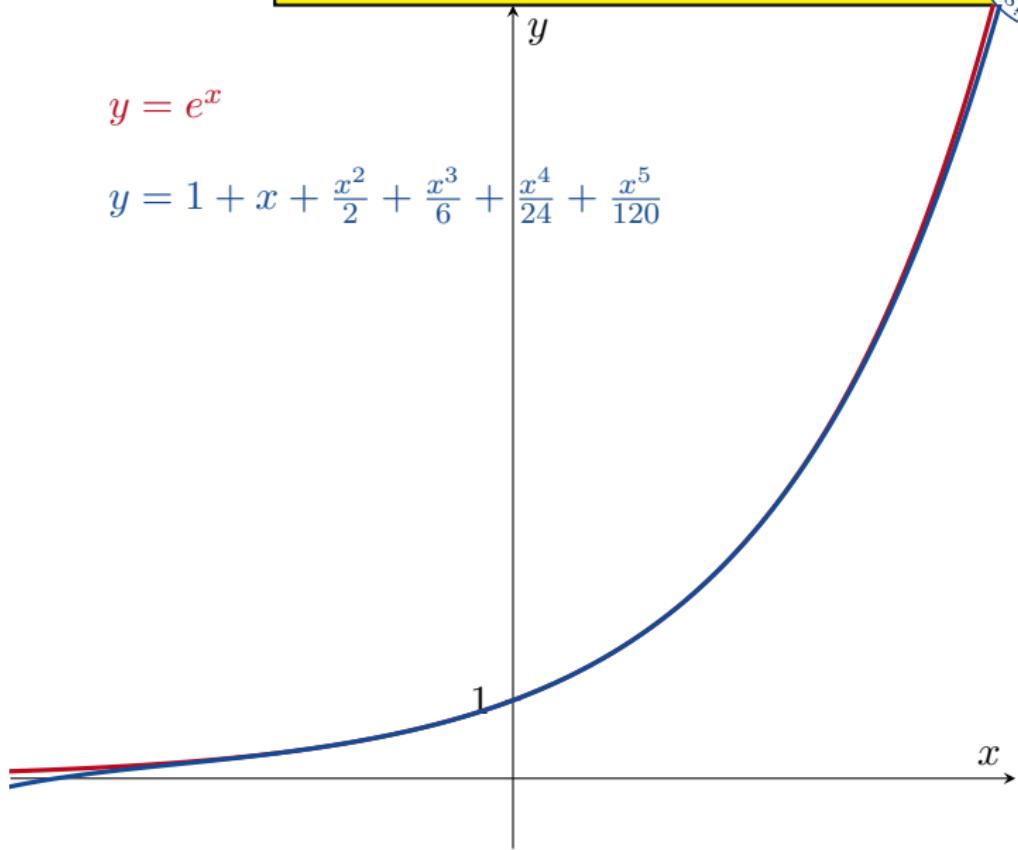
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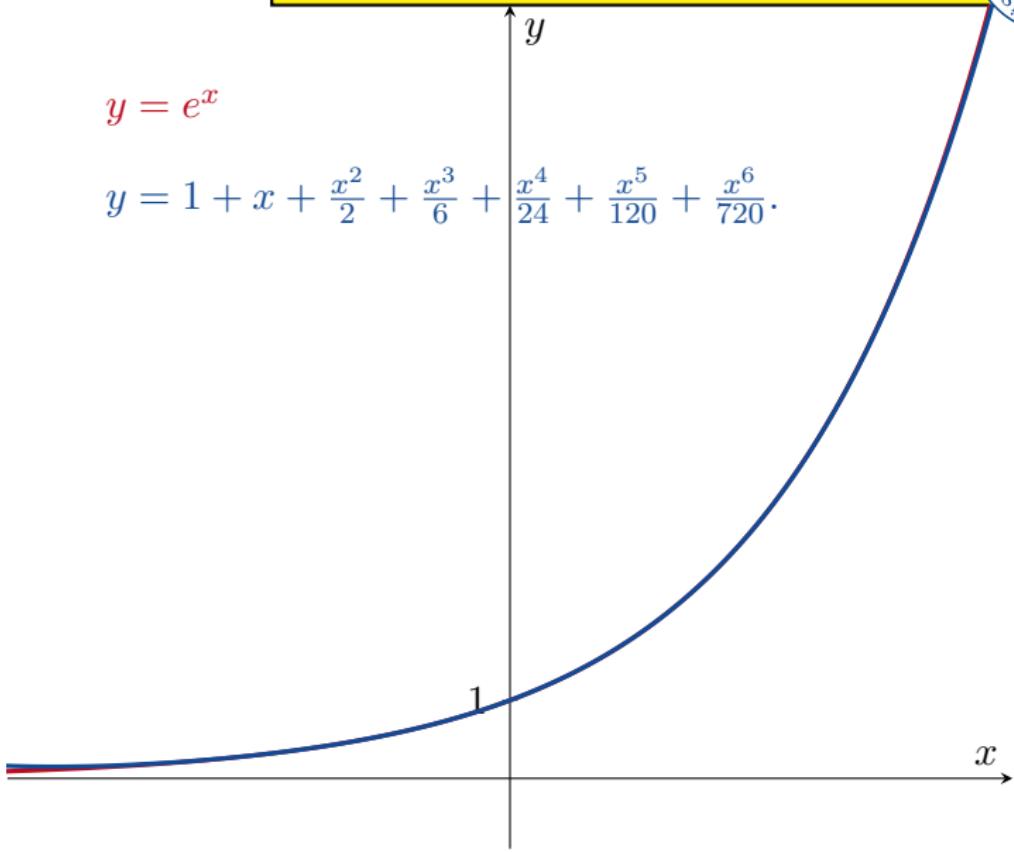
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## Example

Find the Taylor Series for  $\sin x$  centred at 0.

Let  $f(x) = \sin x$ . Then  $\frac{d^k f}{dx^k}$  exists and is continuous  $\forall x$  and  $\forall k$ .

Let  $a = 0$  and  $x \neq 0$ .

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We need

- to find  $\frac{d^k f}{dx^k}$  for all  $k$ ;
- to show that the **remainder term** tends to zero; and
- to calculate  $\frac{d^k f}{dx^k}(0)$  for all  $k$ .

$$a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \frac{f^{(4)}(a)}{4!}(x-a)^4 + \dots$$

First note that

$$\frac{d^k}{dx^k} \sin x = \cos x \quad \text{or} \quad -\sin x \quad \text{or} \quad -\cos x \quad \text{or} \quad \sin x.$$

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Since  $-1 \leq \sin c \leq 1$  and  $-1 \leq \cos c \leq 1 \ \forall c$ , we have that

$$0 \leq \left| \frac{f^{(n)}(c)}{n!} x^n \right| \leq \frac{|x|^n}{n!} \rightarrow 0$$

as  $n \rightarrow \infty$ .

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as  $n \rightarrow \infty$ . Therefore the remainder term  $R_c(x) = \frac{f^{(n)}(c)}{n!} x^n$  tends to zero.

$$a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \frac{f^{(4)}(a)}{4!}(x-a)^4 + \dots$$

I leave it for you to check that

$$f^{(k)}(0) = \begin{cases} 1 & \text{if } k = 1, 5, 9, 13, \dots \\ 0 & \text{if } k = 0, 2, 4, 6, 8, \dots \\ -1 & \text{if } k = 3, 7, 11, 15, \dots \end{cases}$$

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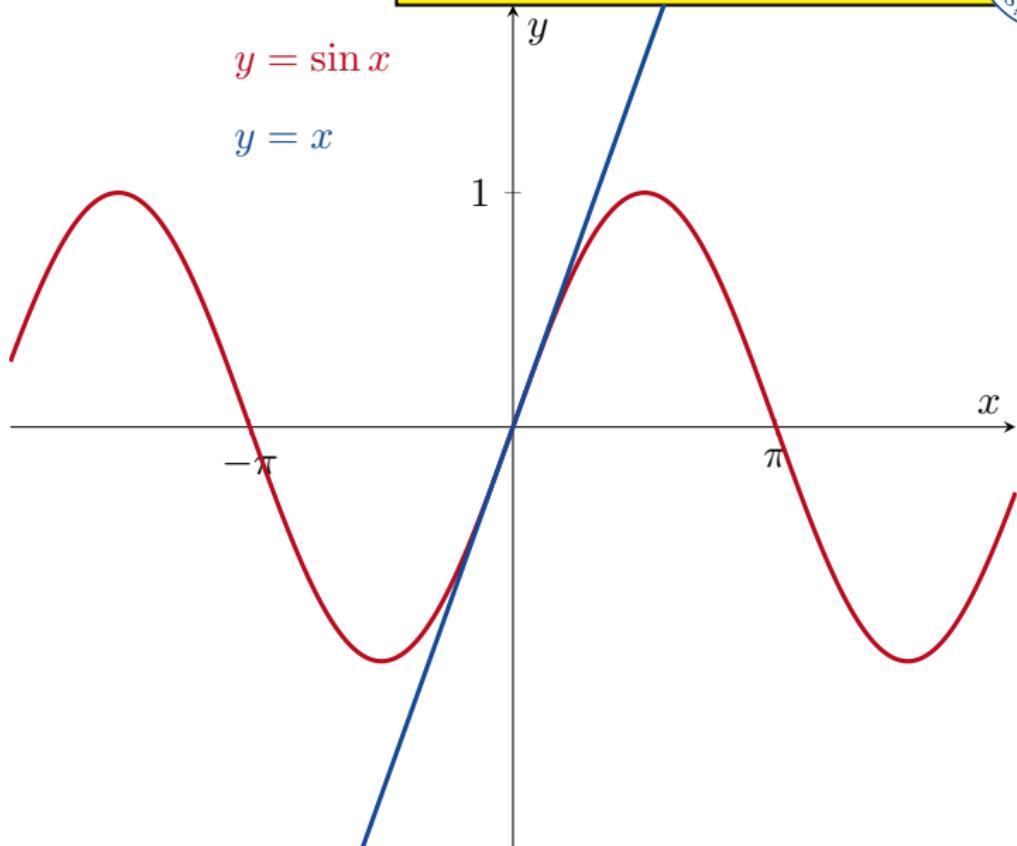
Therefore

$$\begin{aligned} \sin x &= f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \frac{f^{(4)}(0)}{4!}x^4 + \frac{f^{(5)}(0)}{5!}x^5 + \dots \\ &= 0 + x + 0 - \frac{x^3}{3!} + 0 + \frac{x^5}{5!} + \dots \\ &= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots \end{aligned}$$

is the Taylor Series of  $\sin x$  with centre 0.

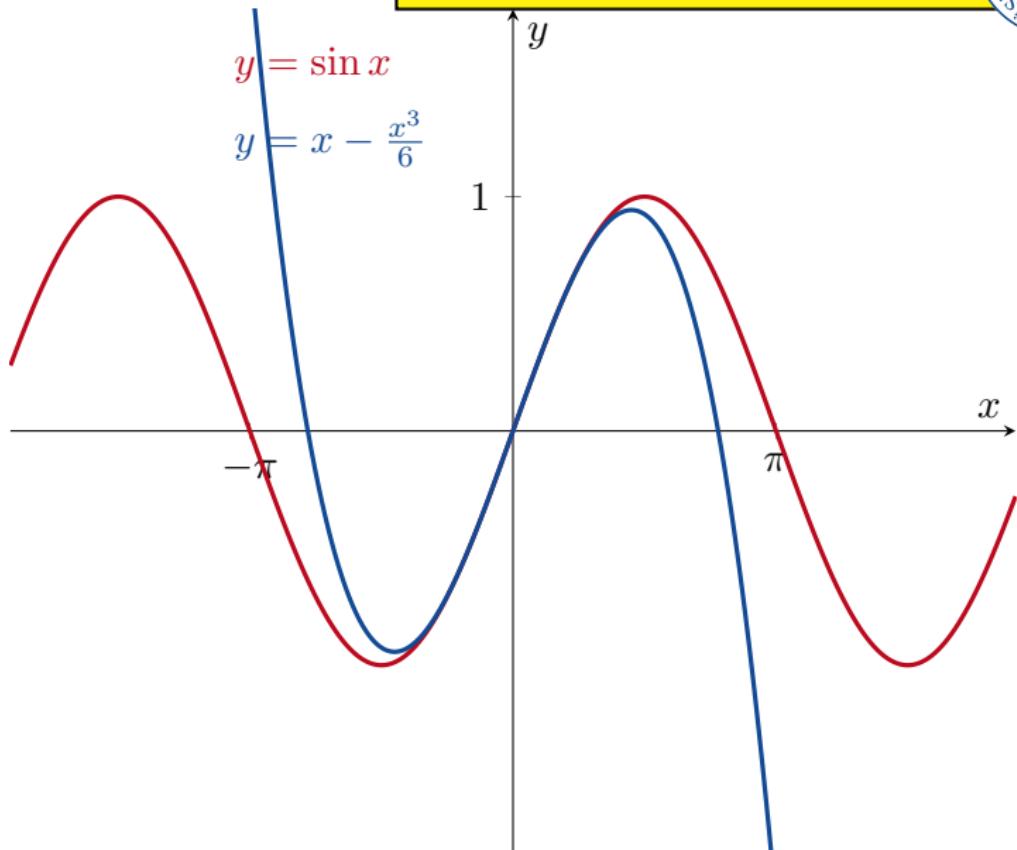
## 9.8 Taylor and Maclaurin

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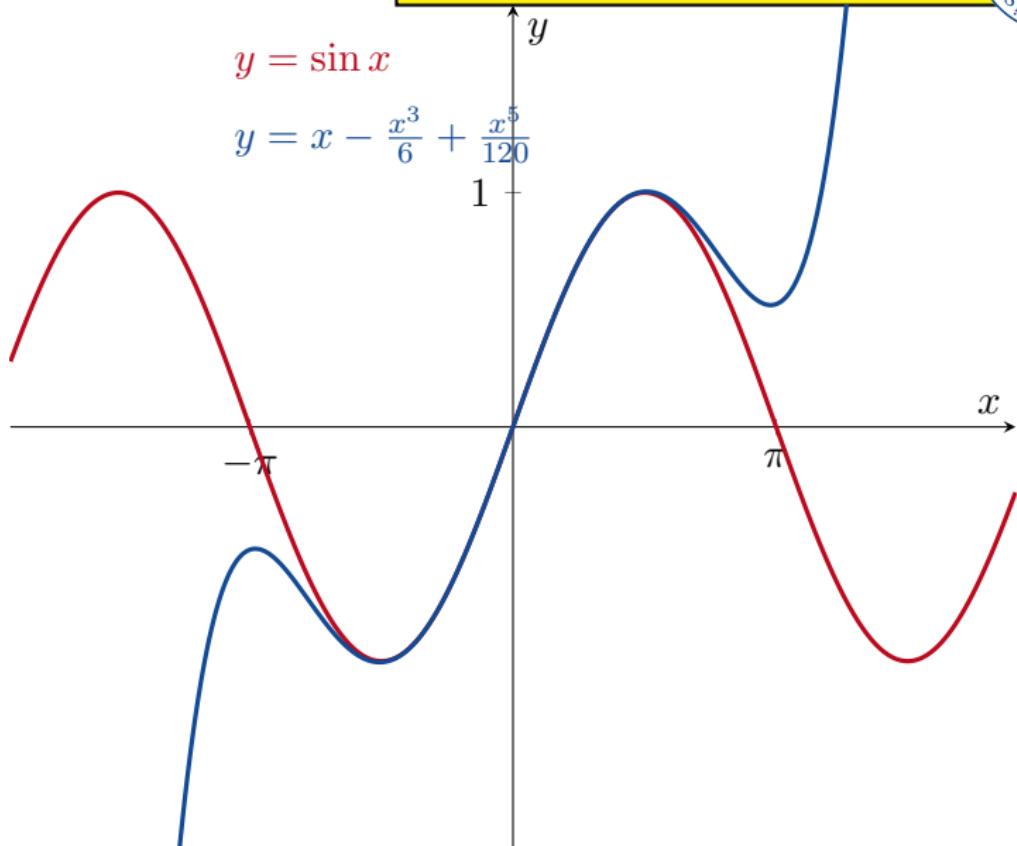
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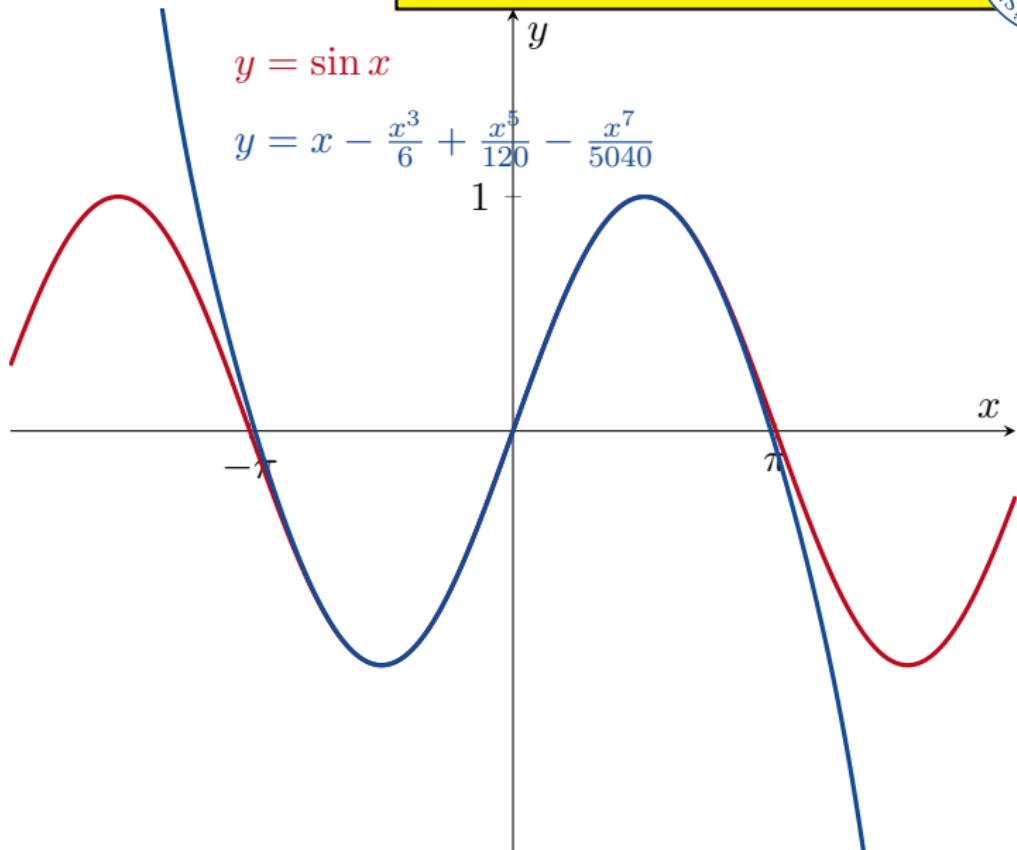
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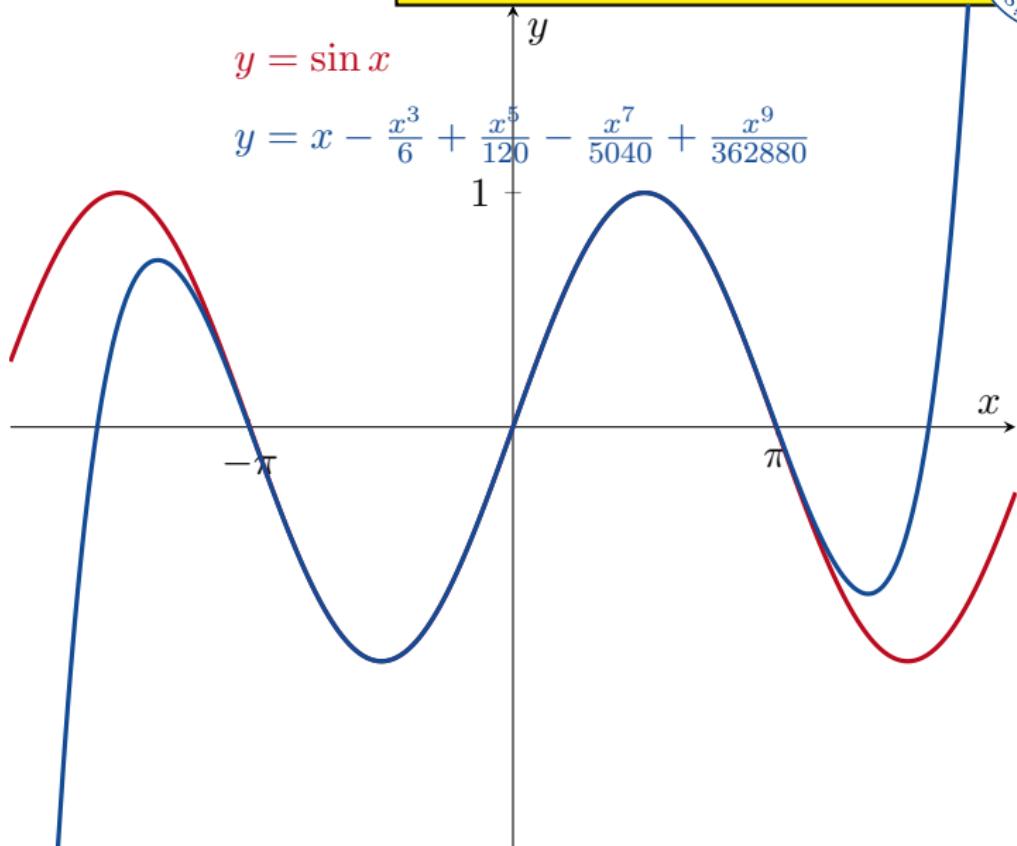
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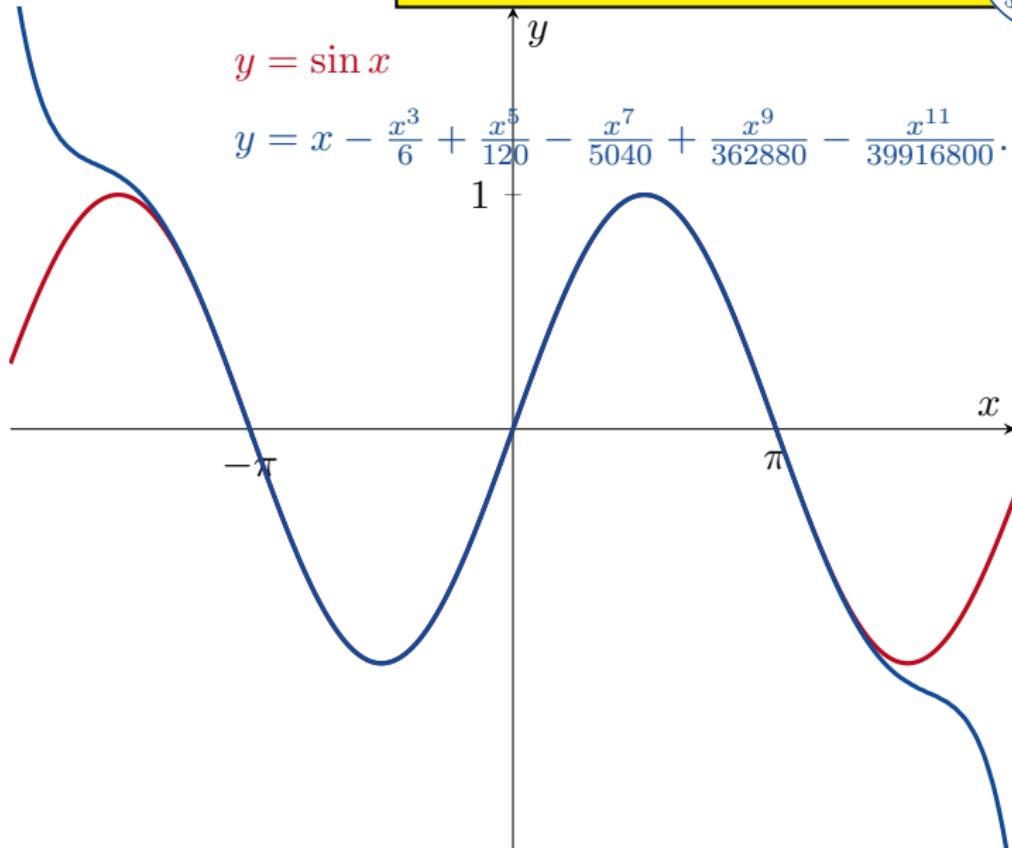
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## Example

Find the Taylor Series for  $\ln(1+x)$  centred at 0.

Let  $f : (-1, \infty) \rightarrow \mathbb{R}$  be defined by  $f(x) = \ln(1+x)$ . Let  $a = 0$  and let  $0 < x \leq 1$ . Then  $f$  and its derivatives exist and are continuous on the closed interval  $[0, x]$ .

$$a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \frac{f^{(4)}(a)}{4!}(x-a)^4 + \dots$$

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I leave it for you to check that

$$f(0) = \ln 1 = 0$$

$$f^{(k)}(x) = \frac{(-1)^{k-1}(k-1)!}{(1+x)^k}$$

$$f^{(k)}(0) = (-1)^{k-1}(k-1)!$$

for  $k \in \mathbb{N}$ .

$$a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \frac{f^{(4)}(a)}{4!}(x-a)^4 + \dots$$

Since  $0 < c < x \leq 1$ , it follows that

$$|R_n(c)| = \left| \frac{f^{(n)}(c)x^n}{n!} \right| = \left| \frac{(n-1)!x^n}{(1+c)^n n!} \right| = \frac{x^n}{(1+c)^n n!} \leq \frac{x^n}{(1+c)^n n} \leq \frac{x^n}{n} \rightarrow 0$$

as  $n \rightarrow \infty$ .

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as  $n \rightarrow \infty$ .

Therefore, if  $0 < x \leq 1$ , then

$$\begin{aligned} \ln(1+x) &= f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \frac{f^{(4)}(0)}{4!}x^4 + \frac{f^{(5)}(0)}{5!}x^5 + \dots \\ &= x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{n} \end{aligned}$$

is the Taylor Series of  $\ln(1+x)$  centred at 0, on the interval  $[0, 1]$ .

$$a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \frac{f^{(4)}(a)}{4!}(x-a)^4 + \dots$$

Since  $0 < c < x \leq 1$ , it follows that

$$|R_n(c)| = \left| \frac{f^{(n)}(c)x^n}{n!} \right| = \left| \frac{(n-1)!x^n}{(1+c)^n n!} \right| = \frac{x^n}{(1+c)^n n!} \leq \frac{x^n}{(1+c)^n n} \leq \frac{x^n}{n} \rightarrow 0$$

as  $n \rightarrow \infty$ .

Therefore, if  $0 < x \leq 1$ , then

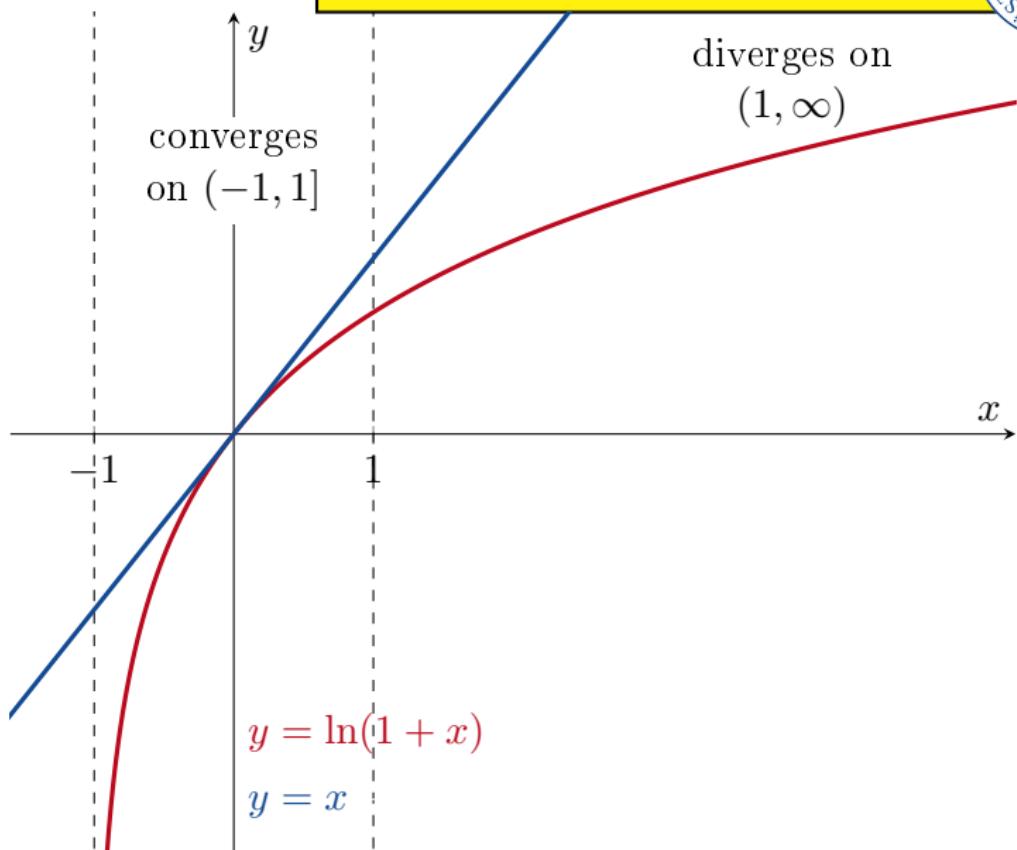
$$\begin{aligned} \ln(1+x) &= f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \frac{f^{(4)}(0)}{4!}x^4 + \frac{f^{(5)}(0)}{5!}x^5 + \dots \\ &= x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{n} \end{aligned}$$

is the Taylor Series of  $\ln(1+x)$  centred at 0, on the interval  $[0, 1]$ .

If can be proved (more difficult) that this series also converges to  $\ln(1+x) \forall x \in (-1, 0)$ . If  $x > 1$ , then the series diverges.

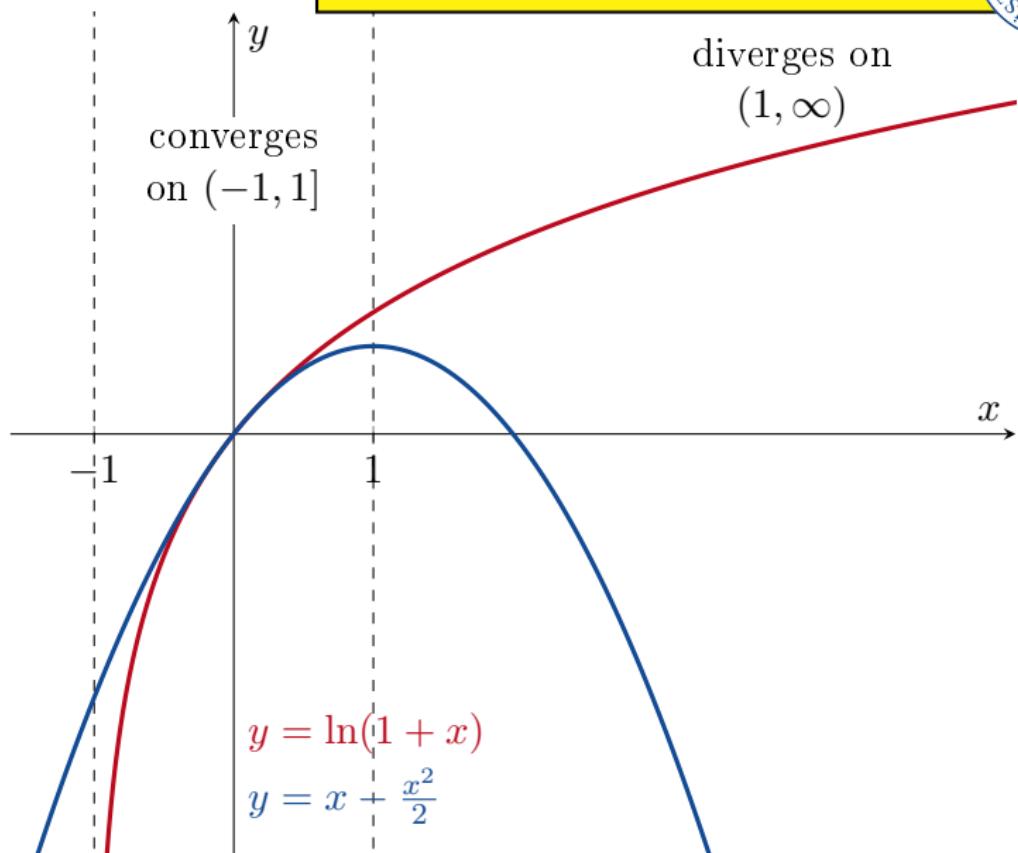
## 9.8 Taylor and Maclaurin Series

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots$$



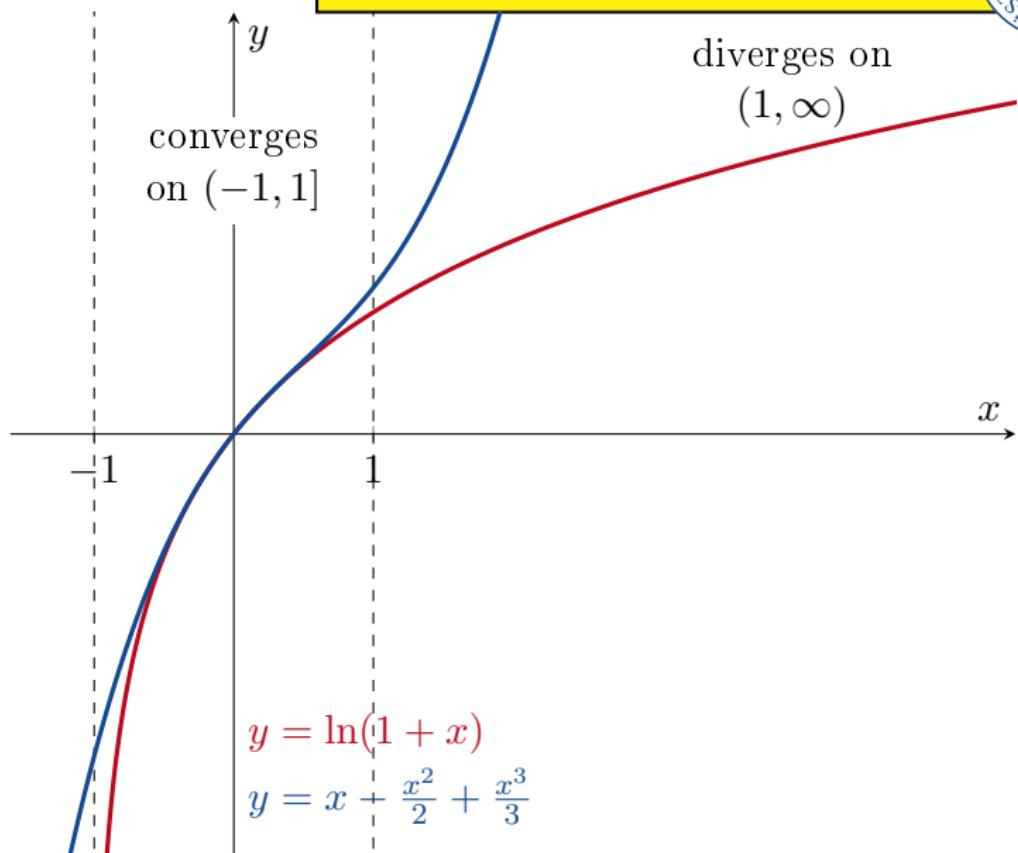
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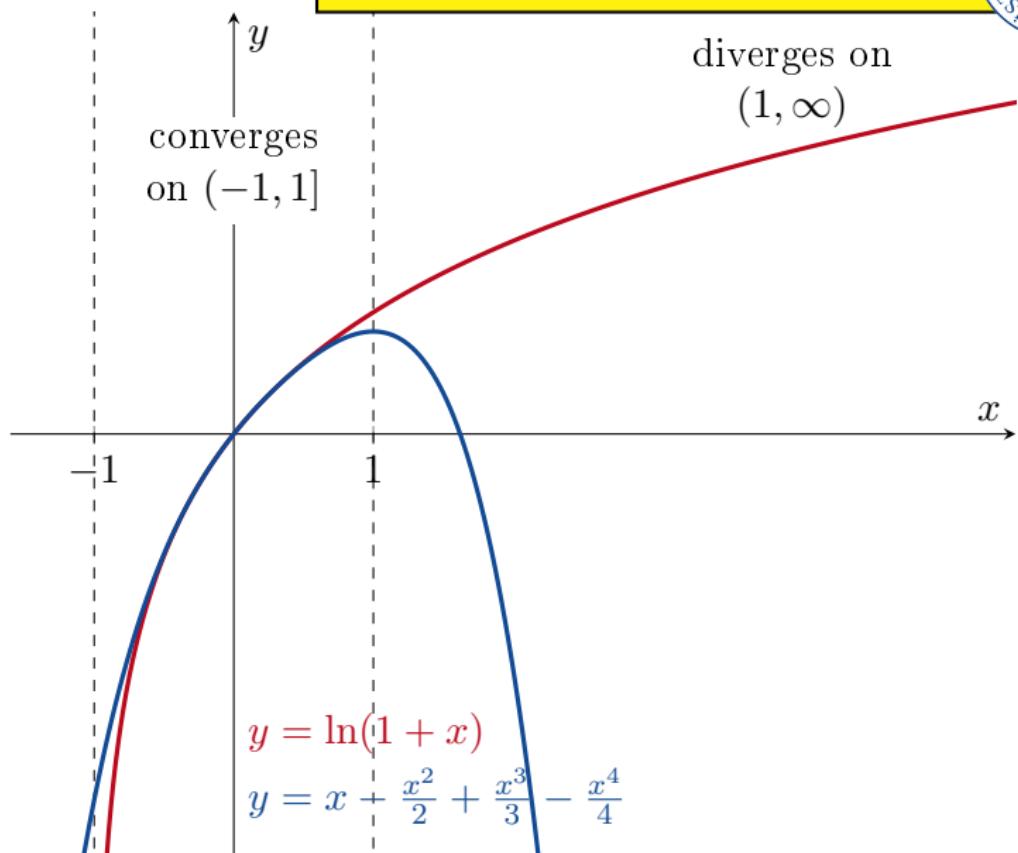
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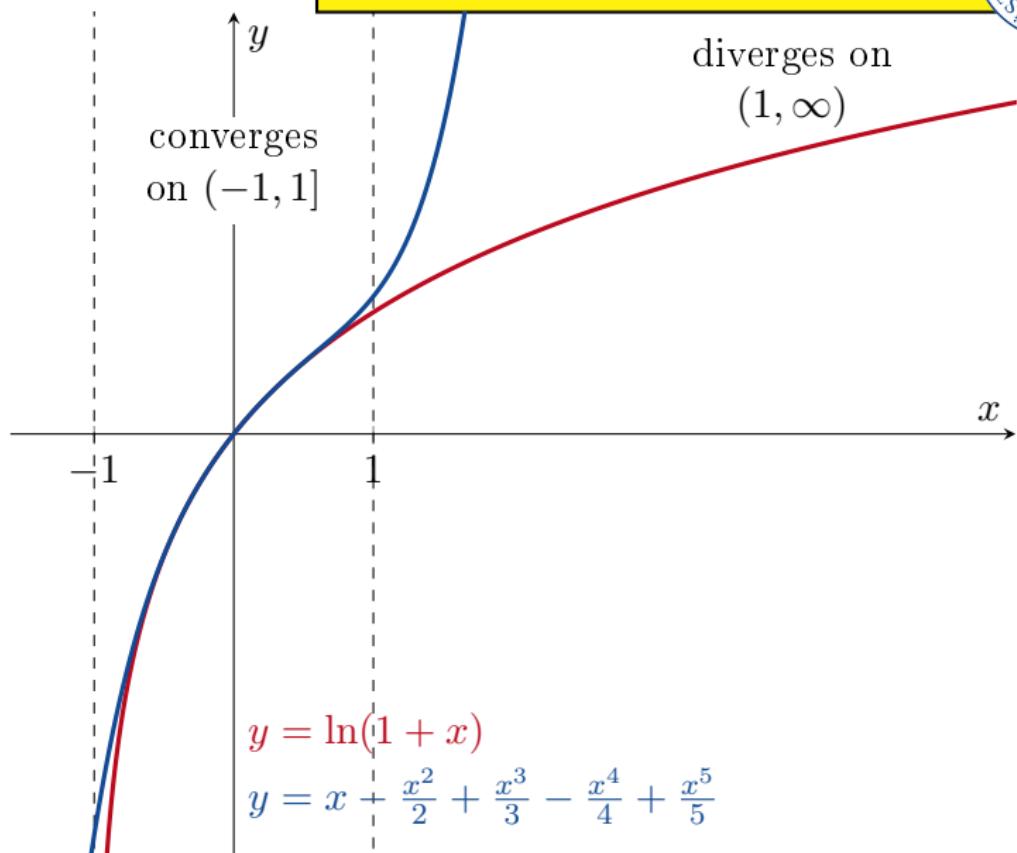
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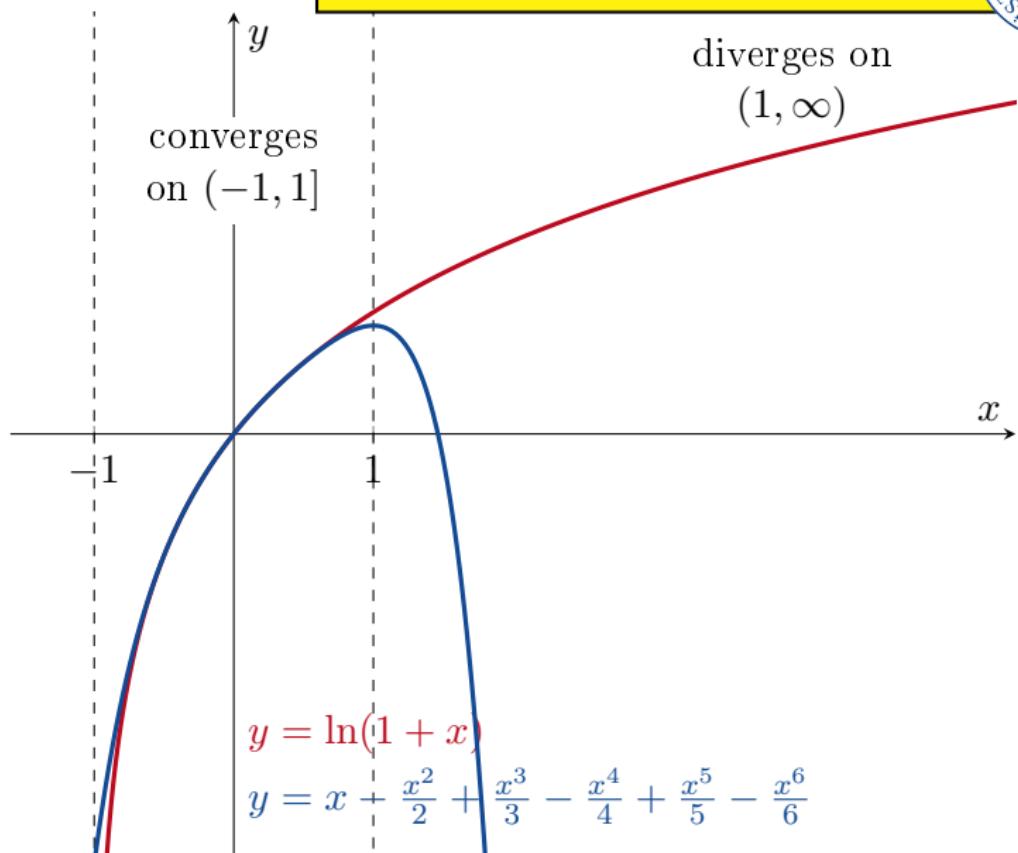
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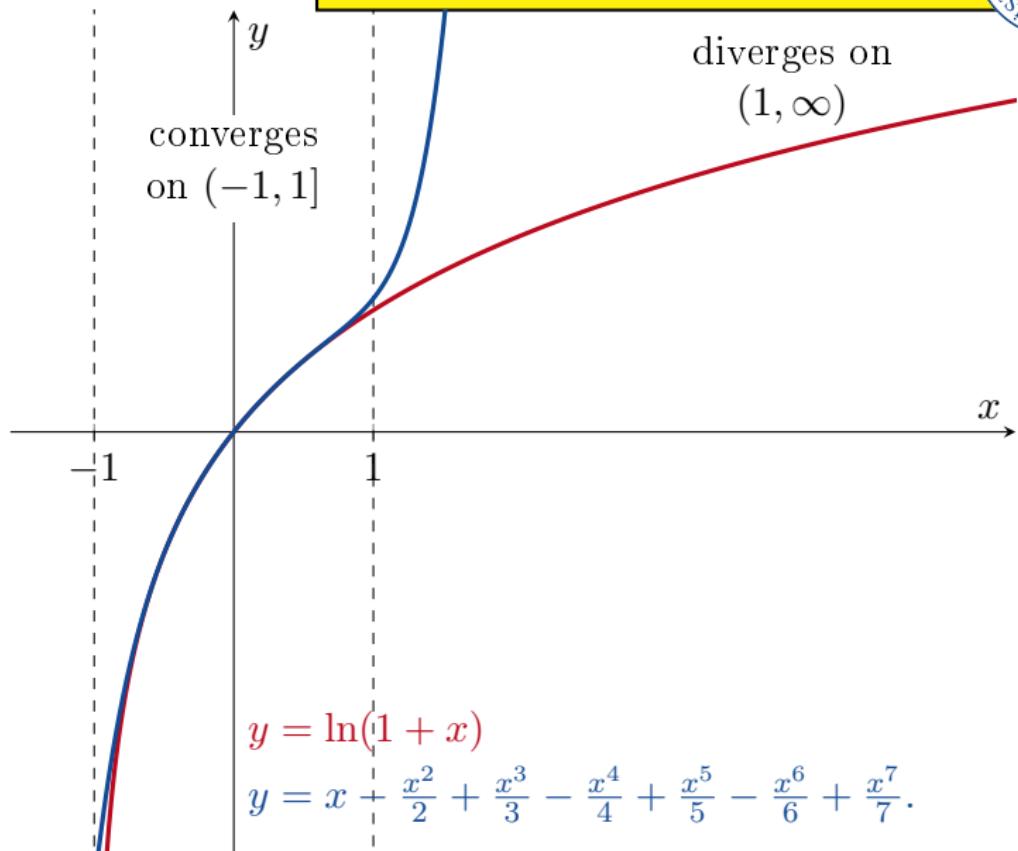
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## 9.8 Taylor and Maclaurin Series

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots$$



### Example

Let  $y = x + 1$ . Then

$$\begin{aligned}\ln y &= (y - 1) - \frac{1}{2}(y - 1)^2 + \frac{1}{3}(y - 1)^3 - \frac{1}{4}(y - 1)^4 + \frac{1}{5}(y - 1)^5 \\ &\quad - \frac{1}{6}(y - 1)^6 + \dots\end{aligned}$$

is the Taylor Series of  $\ln y$  with centre  $a = 1$ . It converges for all  $y \in (0, 2]$ .

## 9.8 Taylor and Maclaurin Series



**Colin Maclaurin**

BORN

February 1698

DECEASED

14 June 1746

NATIONALITY

British

### Definition

A Taylor Series with centre 0 is also called a *Maclaurin Series*.

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n$$

$$f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \frac{f^{(4)}(0)}{4!}x^4 + \dots$$



## Example

Calculate the Maclaurin Series for  $f(x) = \sinh x$ .

$$f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \frac{f^{(4)}(0)}{4!}x^4 + \dots$$



### Example

Calculate the Maclaurin Series for  $f(x) = \sinh x$ .

This is the same as:

### Example

Calculate the Taylor Series for  $f(x) = \sinh x$  centred at 0.

$$f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \frac{f^{(4)}(0)}{4!}x^4 + \dots$$



## Example

Calculate the Maclaurin Series for  $f(x) = \sinh x$ .

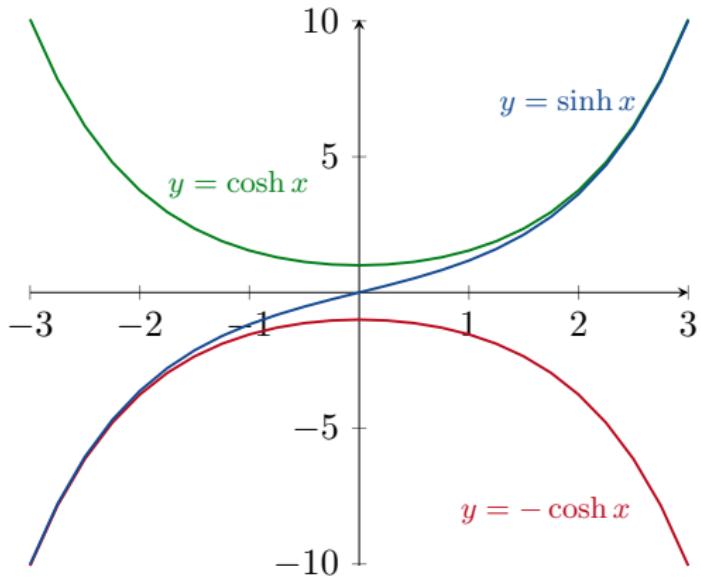
Since  $\frac{d}{dx} \sinh x = \cosh x$  and  $\frac{d}{dx} \cosh x = \sinh x$ , we know that

$$f^{(n)}(x) = \sinh x \quad \text{or} \quad \cosh x$$

for all  $n \in \mathbb{N}$ .

## 9.8 Taylor and

$$f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \frac{f^{(4)}(0)}{4!}x^4 + \dots$$



Note that

$$-\cosh x \leq \sinh x \leq \cosh x$$

for all  $x \in \mathbb{R}$ .

## 9.8 Taylor and Maclaurin Series



$$f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \frac{f^{(4)}(0)}{4!}x^4 + \dots$$

Let  $x \neq 0$  and let  $c$  be between  $0$  and  $x$ . (So  $0 < c < x$  or  $x < c < 0$ .) Then

$$\left|f^{(n)}(c)\right| < \left|f^{(n)}(x)\right| \leq \cosh x.$$

So

$$0 \leq \left| \frac{f^{(n)}(c)x^n}{n!} \right| \leq \cosh x \frac{|x|^n}{n!} \rightarrow 0$$

as  $n \rightarrow \infty$ . By the Sandwich Rule, it follows that

$$R_c(x) = \frac{f^{(n)}(c)x^n}{n!} \rightarrow 0$$

as  $n \rightarrow \infty$ .

## 9.8 Taylor and

$$f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \frac{f^{(4)}(0)}{4!}x^4 + \dots$$



Now, since

$$f^{(n)}(x) = \begin{cases} \sinh x & \text{if } n = 0, 2, 4, 6, 8, \dots \\ \cosh x & \text{if } n = 1, 3, 5, 7, 9, \dots \end{cases}$$

we have that

$$f^{(n)}(0) = \begin{cases} 0 & \text{if } n = 0, 2, 4, 6, 8, \dots \\ 1 & \text{if } n = 1, 3, 5, 7, 9, \dots \end{cases}$$

## 9.8 Taylor and Maclaurin Series



$$f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \frac{f^{(4)}(0)}{4!}x^4 + \dots$$

Now, since

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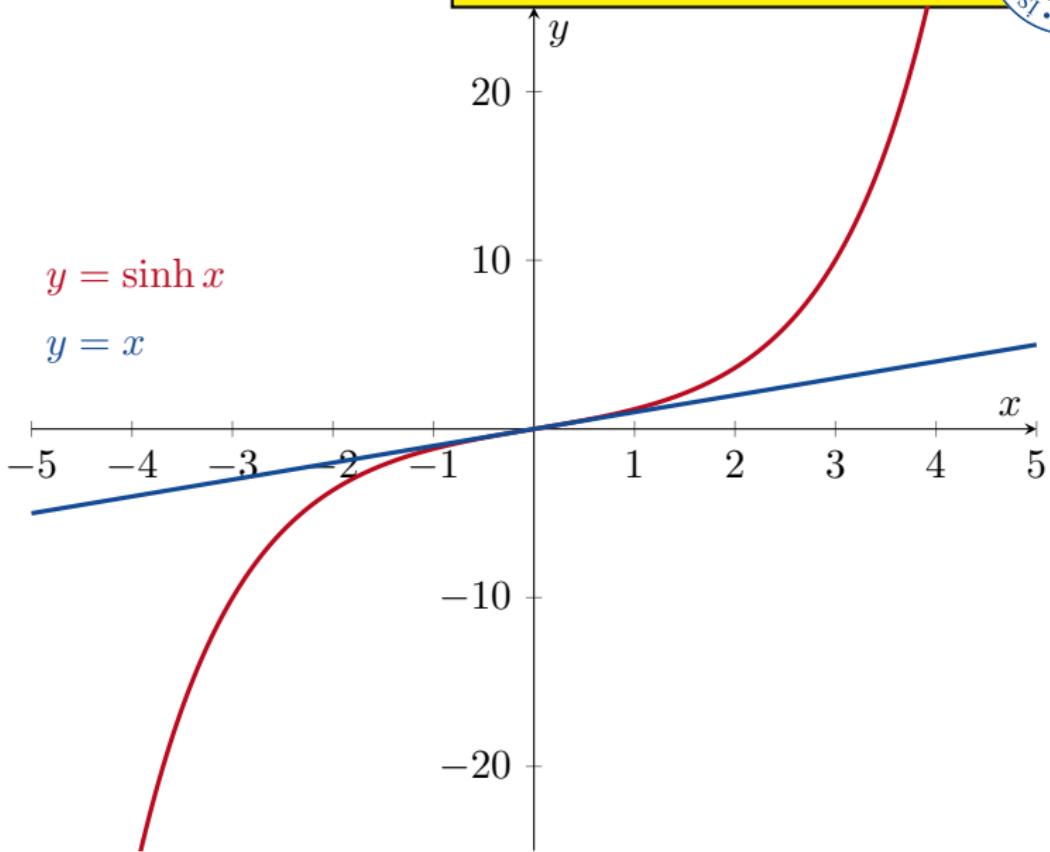
$$f^{(n)}(0) = \begin{cases} 0 & \text{if } n = 0, 2, 4, 6, 8, \dots \\ 1 & \text{if } n = 1, 3, 5, 7, 9, \dots \end{cases}$$

Therefore

$$\begin{aligned} \sinh x &= f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \frac{f^{(4)}(0)}{4!}x^4 \dots \\ &= 0 + 1x + \frac{0}{2!}x^2 + \frac{1}{3!}x^3 + \frac{0}{4!}x^4 \dots \\ &= x + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} + \frac{x^9}{9!} + \dots = \sum_{k=0}^{\infty} \frac{x^{2k+1}}{(2k+1)!}. \end{aligned}$$

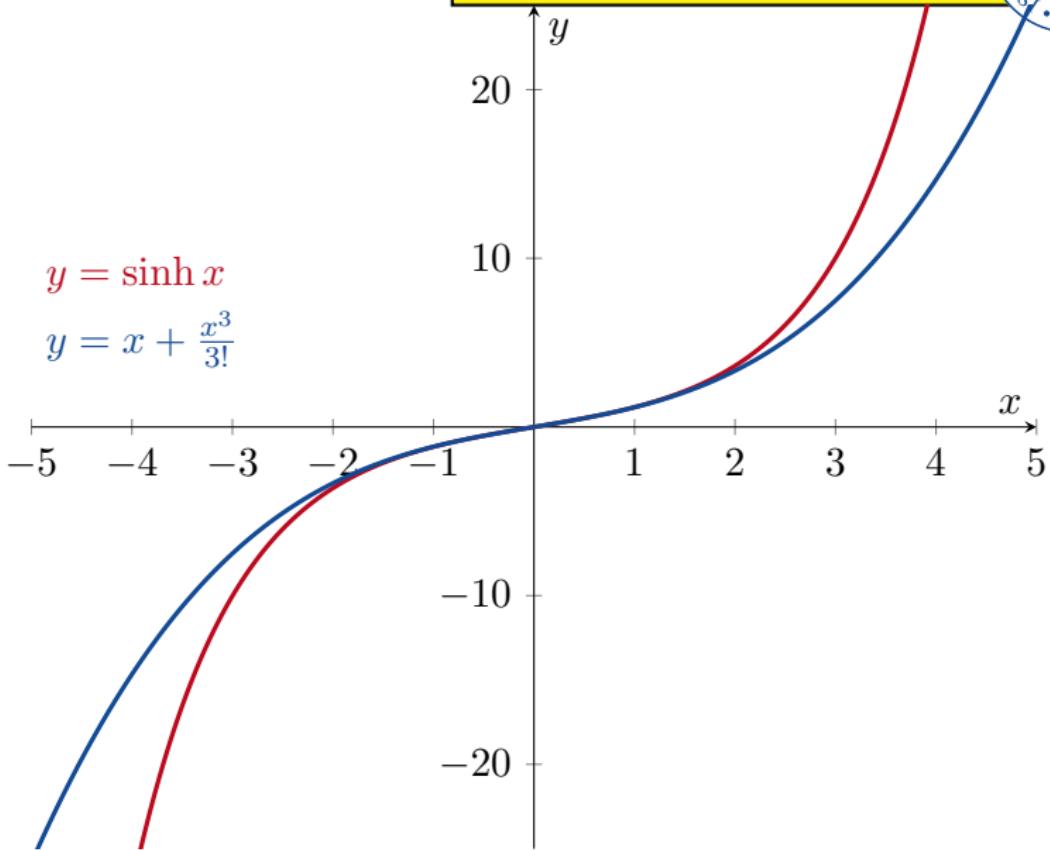
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$$x + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} + \frac{x^9}{9!} + \dots$$



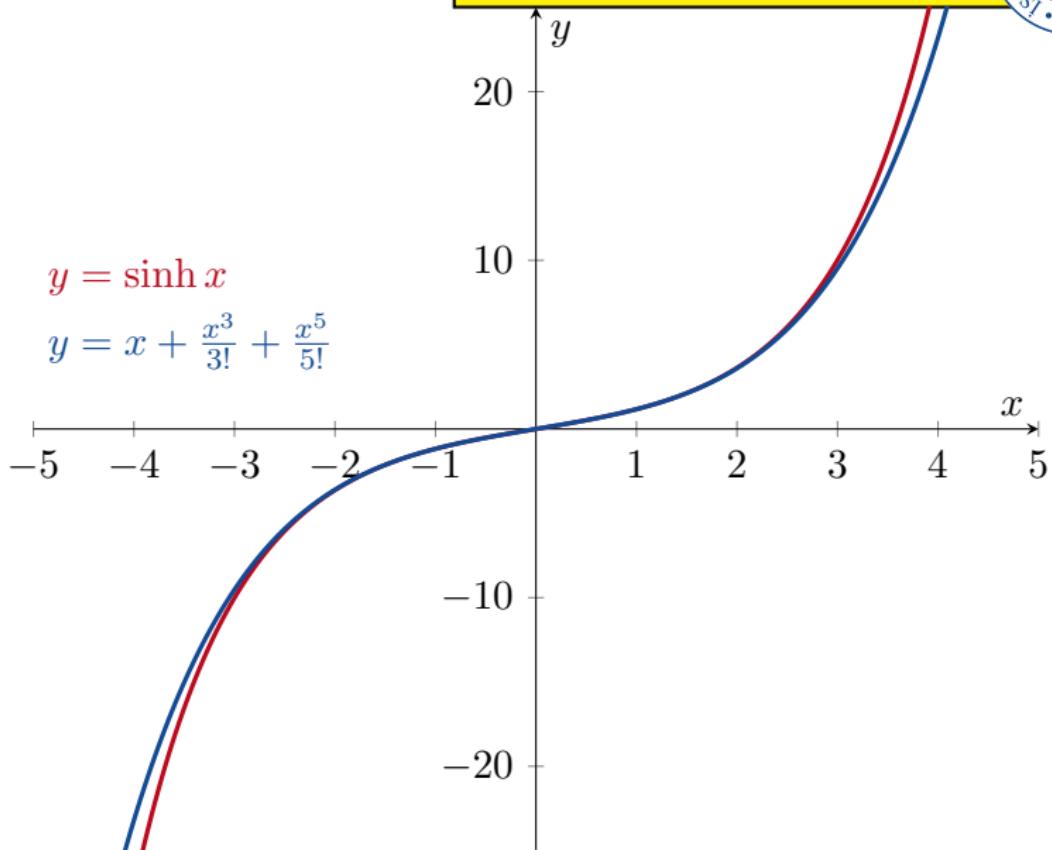
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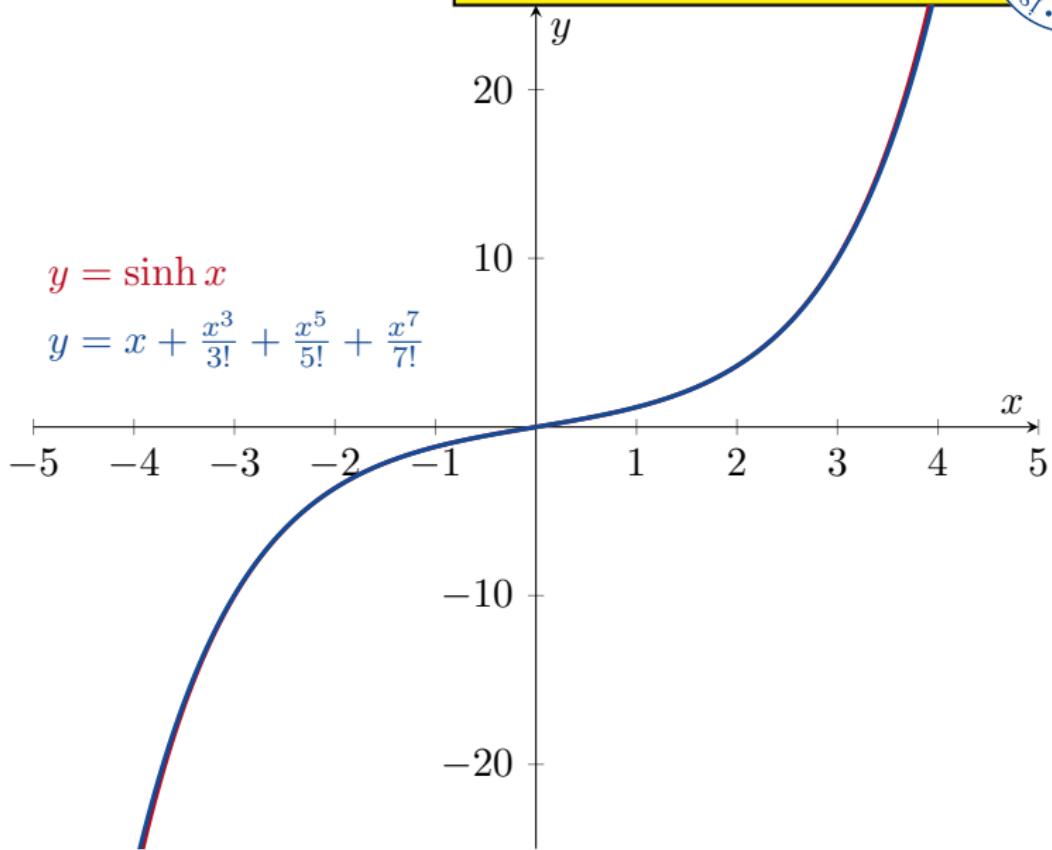
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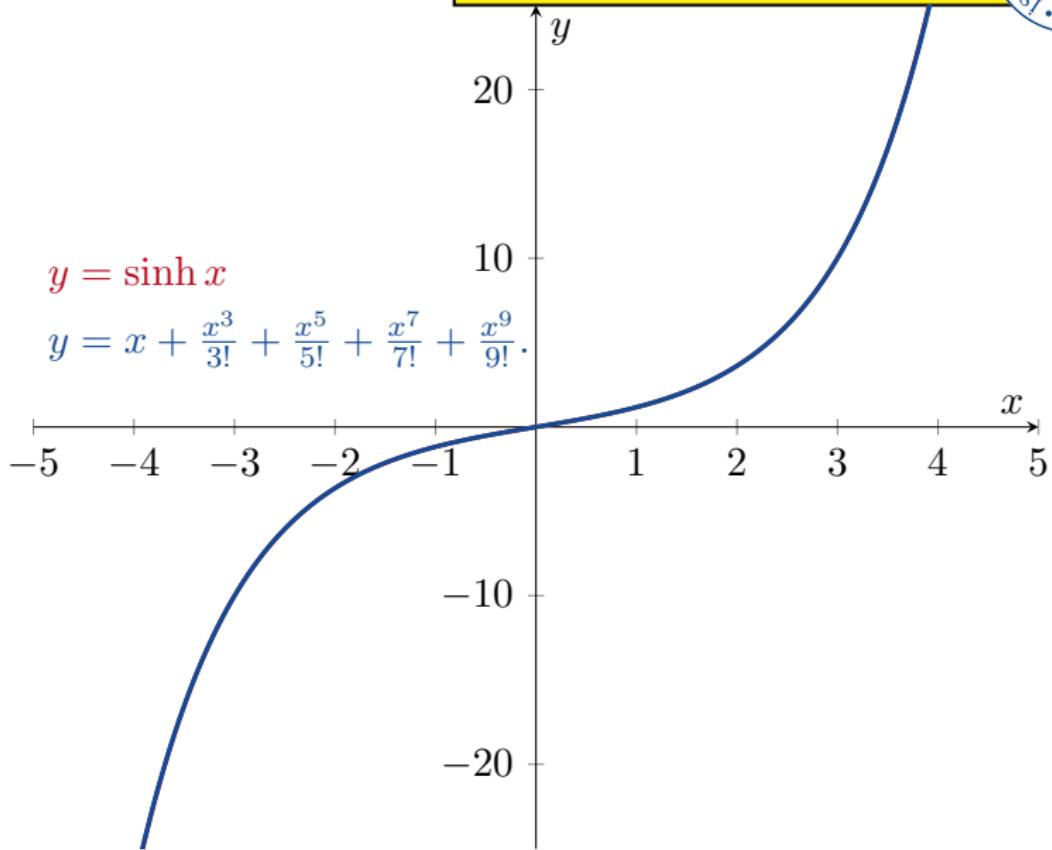
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## 9.8 Taylor and Maclaurin Series



### Example

Calculate the Taylor Series for  $f(x) = \frac{1}{x}$  with centre  $a = 2$ . For which  $x \in \mathbb{R}$  does the series converge?

## 9.8 Taylor and Maclaurin Series

### Example

Calculate the Taylor Series for  $f(x) = \frac{1}{x}$  with centre  $a = 2$ . For which  $x \in \mathbb{R}$  does the series converge?

Since

$$f(x) = x^{-1}$$

$$f(2) = \frac{1}{2}$$

$$f'(x) = -x^{-2}$$

$$f'(2) = -\frac{1}{4}$$

$$f''(x) = 2x^{-3}$$

$$\frac{f''(2)}{2!} = \frac{1}{8}$$

$$f'''(x) = -6x^{-4}$$

$$\frac{f'''(2)}{3!} = -\frac{1}{16}$$

⋮

⋮

$$f^{(n)}(x) = (-1)^n n! x^{-n-1}$$

$$\frac{f^{(n)}(2)}{n!} = \frac{(-1)^n}{2^{n+1}}$$

⋮

⋮

## 9.8 Taylor and Maclaurin Series



the Taylor Series is

$$\begin{aligned}\frac{1}{x} &= f(2) + f'(2)(x - 2) + \frac{f''(2)}{2!}(x - 2)^2 + \frac{f'''(2)}{3!}(x - 2)^3 + \frac{f^{(4)}(2)}{4!}(x - 2)^4 + \dots \\&= \frac{1}{2} - \frac{x - 2}{4} + \frac{(x - 2)^2}{8} - \frac{(x - 2)^3}{16} + \frac{(x - 2)^4}{32} - \dots \\&= \frac{1}{2} (1 + r + r^2 + r^3 + r^4 + \dots)\end{aligned}$$

where  $r = -\frac{x-2}{2}$ .

## 9.8 Taylor and Maclaurin Series



the Taylor Series is

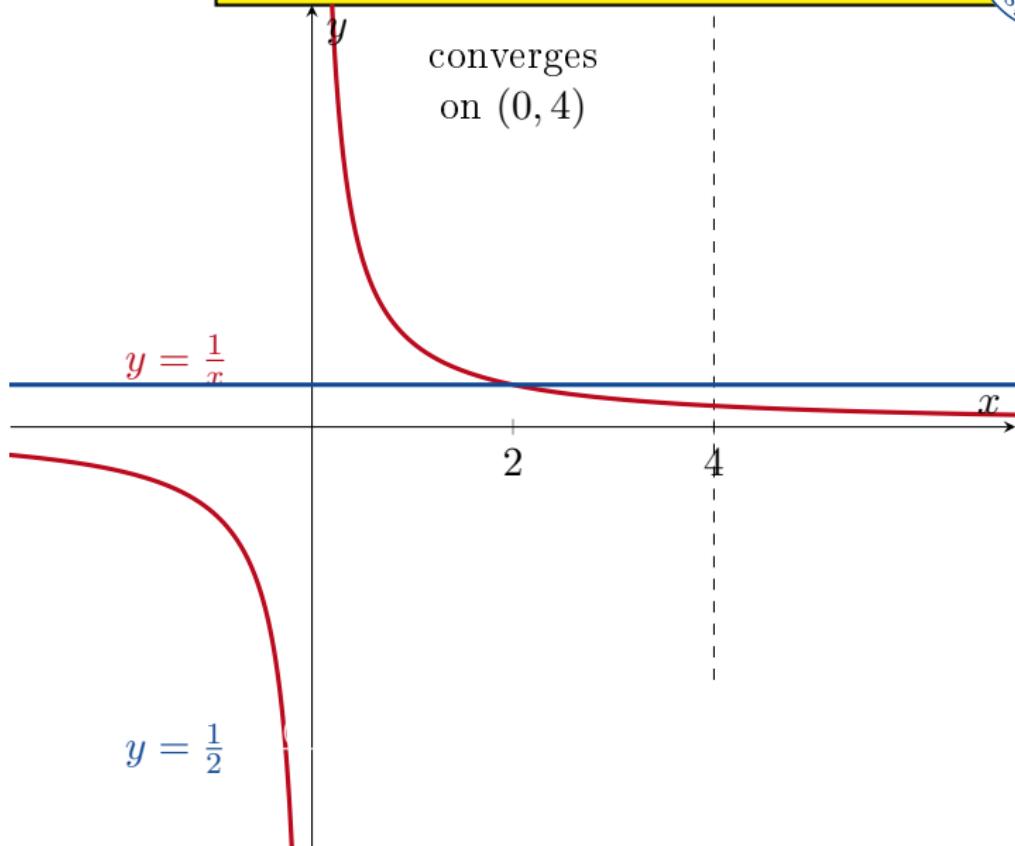
$$\begin{aligned}\frac{1}{x} &= f(2) + f'(2)(x - 2) + \frac{f''(2)}{2!}(x - 2)^2 + \frac{f'''(2)}{3!}(x - 2)^3 + \frac{f^{(4)}(2)}{4!}(x - 2)^4 + \dots \\&= \frac{1}{2} - \frac{x - 2}{4} + \frac{(x - 2)^2}{8} - \frac{(x - 2)^3}{16} + \frac{(x - 2)^4}{32} - \dots \\&= \frac{1}{2} (1 + r + r^2 + r^3 + r^4 + \dots)\end{aligned}$$

where  $r = -\frac{x-2}{2}$ .

This series converges absolutely for  $|r| < 1$  and diverges for  $|r| \geq 1$ . Therefore, the Taylor Series converges for  $0 < x < 4$ .

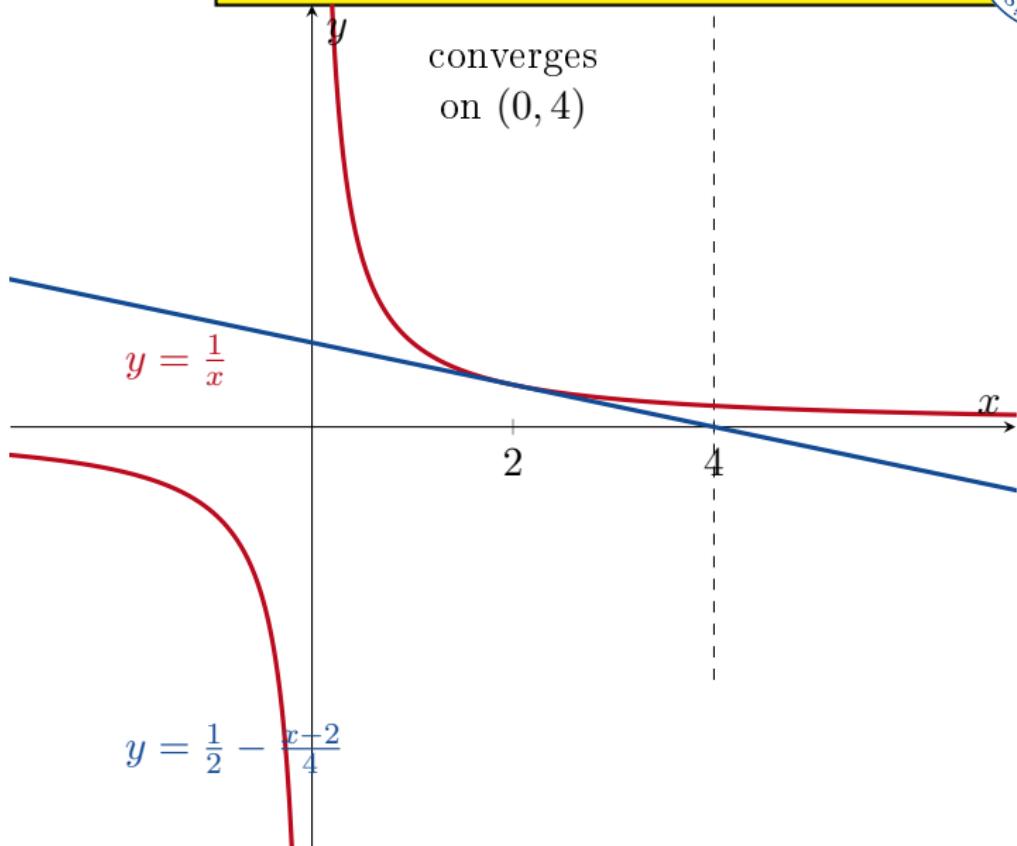
## 9.8 Taylor and M

$$\frac{1}{2} - \frac{x-2}{4} + \frac{(x-2)^2}{8} - \frac{(x-2)^3}{16} + \frac{(x-2)^4}{32} - \dots$$



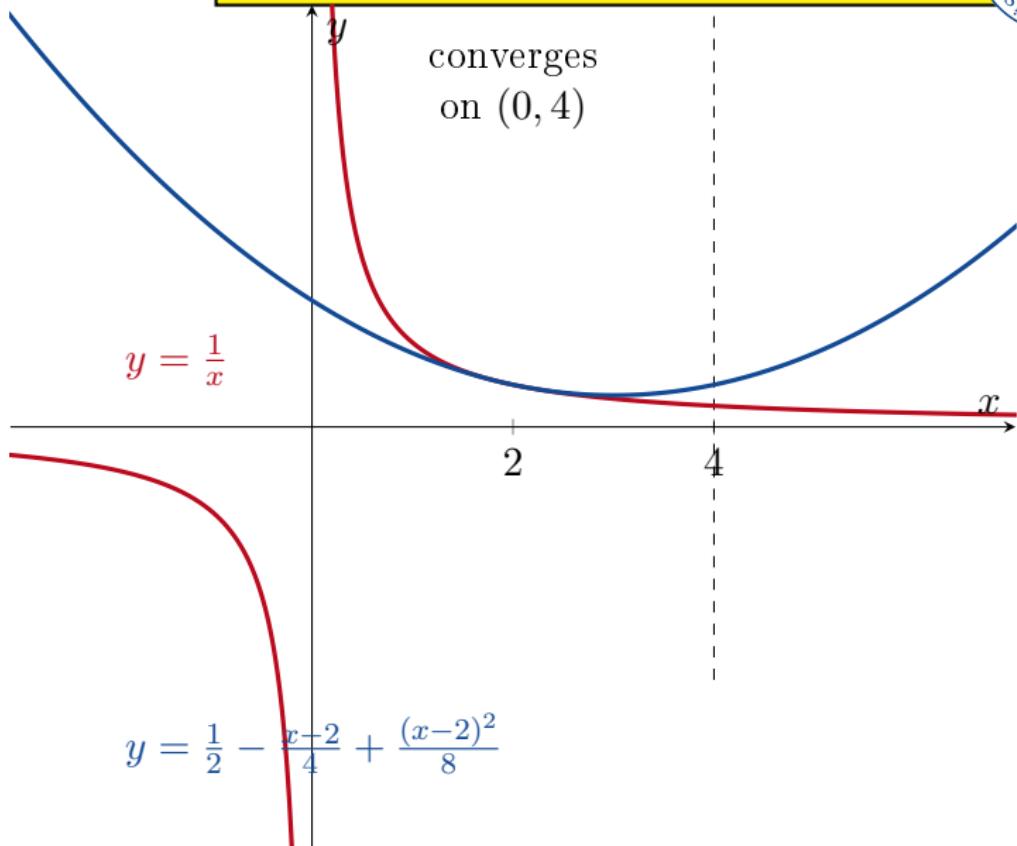
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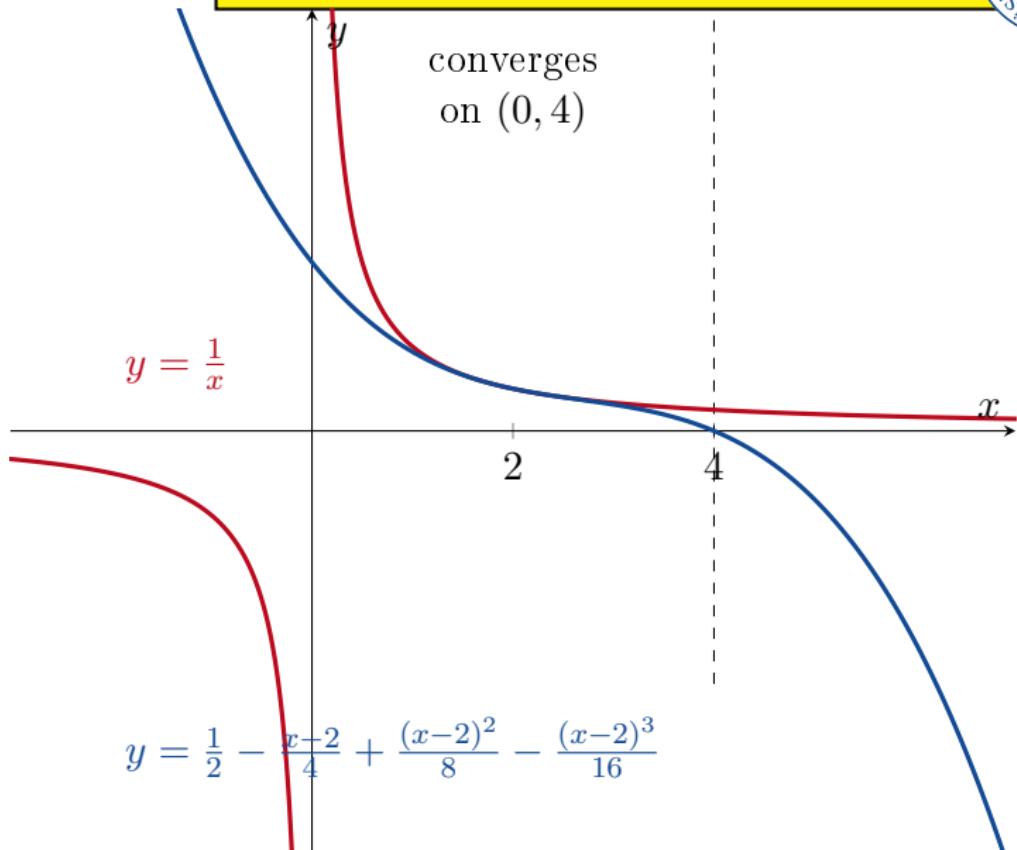
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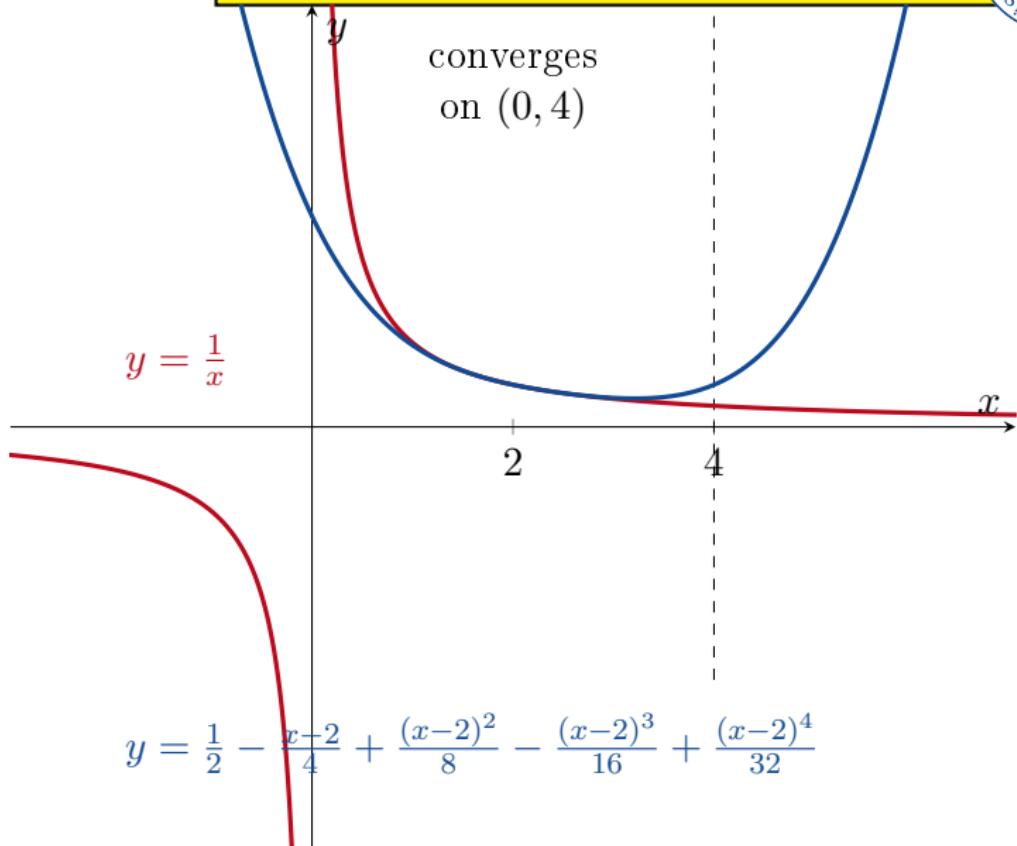


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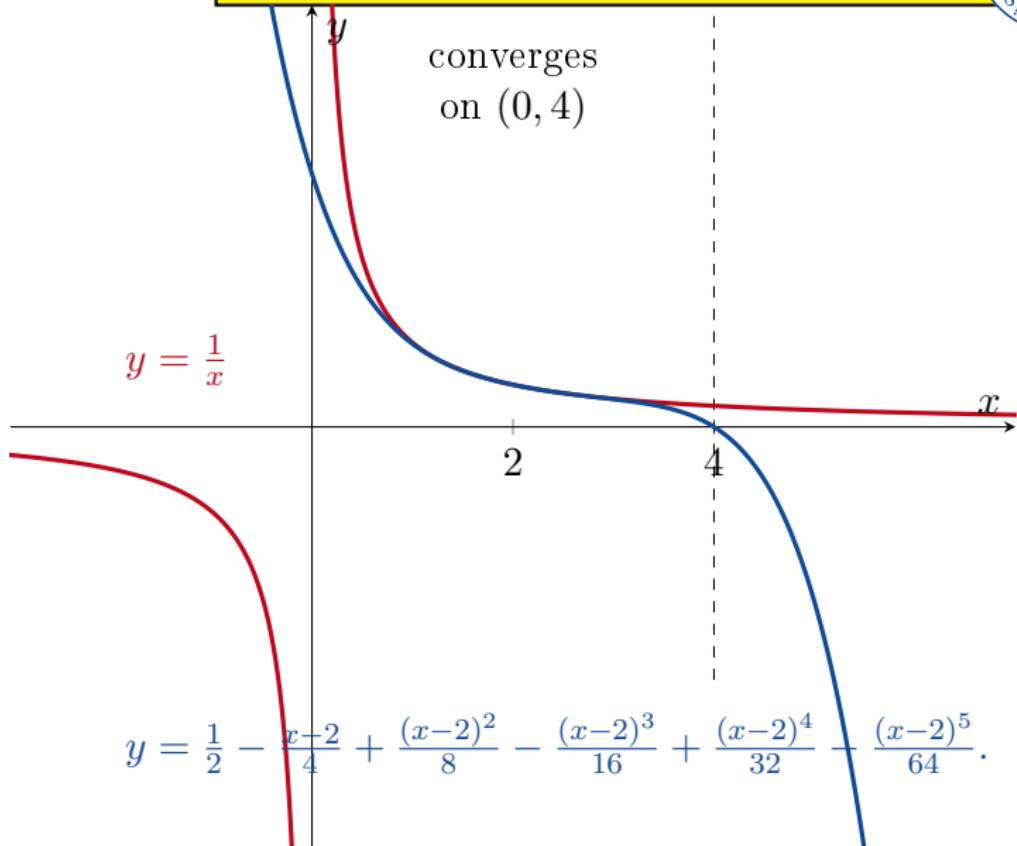


converges  
on  $(0, 4)$



## 9.8 Taylor and M

$$\frac{1}{2} - \frac{x-2}{4} + \frac{(x-2)^2}{8} - \frac{(x-2)^3}{16} + \frac{(x-2)^4}{32} - \dots$$





*The End*

