Lecture 5

- Vector Spaces
- Subspaces
- Null Spaces, Column Spaces, and Linear Transformations
- Linearly Independent Sets; Bases

Vector Spaces

You studied vectors, the dot product, etc. in MATH114. Now it is time to generalise these concepts.

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- If all the scalars k are real numbers, then V is called a *real vector space*.
- If we allow complex numbers k, then V is called a *complex vector space*.
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- $k\mathbf{u} = \mathbf{0} \implies k = 0$ or $\mathbf{u} = \mathbf{0}$.

To Show That a Set with Two Operations Is a Vector Space

Step 1. Identify the set V of objects that will become vectors.

Step 2. Identify the addition and scalar multiplication operations on V.

Step 3. Verify Axioms 1 and 6; that is, adding two vectors in V produces a vector in V, and multiplying a vector in V by a scalar also produces a vector in V. Axiom 1 is called ***closure under addition***, and Axiom 6 is called ***closure under scalar multiplication***.

Step 4. Confirm that Axioms 2, 3, 4, 5, 7, 8, 9, and 10 hold.

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I leave it to you to check that all 10 axioms are satisfied. We call this the *zero vector space*.

Example (The Vector Space of Sequences)

Let V consist of objects of the form

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in which $u_1, u_2, \ldots, u_n, \ldots$ is an infinite sequence of real numbers.

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We define two infinite sequences to be equal if their corresponding components are equal, and we define

$$\begin{aligned} \mathbf{u} + \mathbf{v} &= (u_1, u_2, \ldots, u_n, \ldots) + (v_1, v_2, \ldots, v_n, \ldots) \\ &= (u_1 + v_1, u_2 + v_2, \ldots, u_n + v_n, \ldots) \\ k\mathbf{u} &= (ku_1, ku_2, \ldots, ku_n, \ldots) \end{aligned}$$

Show that V is a vector space.

1. If $\mathbf{u}$ and $\mathbf{v}$ are objects in V, then $\mathbf{u}+\mathbf{v}$ is in V.
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   Clearly $1\mathbf{u} = (1u_1, 1u_2, 1u_3, \ldots) = (u_1, u_2, u_3, \ldots) = \mathbf{u}$ ✓

Therefore V is a vector space.

Example

Let

$$\begin{aligned}\mathbb{P}^n &= \{\text{all polynomials of degree} \leq n\} \\ &= \{\mathbf{p}(t) = a_0 + a_1 t + a_2 t^2 + \ldots + a_n t^n \mid a_j \in \mathbb{R}\}\end{aligned}$$

for $n \in \mathbb{N} \cup \{0\}$. (See page 210, example 4 in your textbook.) Let addition and scalar multiplication be defined in the obvious way.

Show that $\mathbb{P}^n$ is a vector space.

First let's do axioms 1 and 6. If

$$\mathbf{p}(t) = a_0 + a_1 t + a_2 t^2 + \ldots + a_n t^n$$

and

$$\mathbf{q}(t) = b_0 + b_1 t + b_2 t^2 + \ldots + b_n t^n,$$

then

$$\begin{aligned}(\mathbf{p}+\mathbf{q})(t) &= \mathbf{p}(t) + \mathbf{q}(t) \\ &= (a_0 + b_0) + (a_1 + b_1)t + (a_2 + b_2)t^2 + \ldots + (a_n + b_n)t^n\end{aligned}$$

is a polynomial of degree $\leq n$, and

$$k\mathbf{p}(t) = ka_0 + ka_1 t + ka_2 t^2 + \ldots + ka_n t^n$$

is also a polynomial of degree $\leq n$. So 1. and 6. are satisfied.

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5. For each $\mathbf{u}$ in V, there is an object $-\mathbf{u}$ in V such that $\mathbf{u} + (-\mathbf{u}) = \mathbf{0}$.
   For $\mathbf{p}(t)$ as before, just let

$$-\mathbf{p}(t) = -a_0 - a_1 t - a_2 t^2 - \ldots - a_n t^2.$$

Therefore $\mathbb{P}^n$ is a vector space.

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$$(\mathbf{p} + \mathbf{q})(t) = (1 + x + x^2) + (2 - x^2) = 3 + x$$

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V is not closed under addition, so is not a vector space.

EXAMPLE 5 Let V be the set of all real-valued functions defined on a set $\mathbb{D}$. (Typically, $\mathbb{D}$ is the set of real numbers or some interval on the real line.) Functions are added in the usual way: $\mathbf{f}+\mathbf{g}$ is the function whose value at t in the domain $\mathbb{D}$ is $\mathbf{f}(t)+\mathbf{g}(t)$. Likewise, for a scalar c and an $\mathbf{f}$ in V, the scalar multiple $c\mathbf{f}$ is the function whose value at t is $c\mathbf{f}(t)$. For instance, if $\mathbb{D}=\mathbb{R}$, $\mathbf{f}(t)=1+\sin 2t$, and $\mathbf{g}(t)=2+.5t$, then

$$(\mathbf{f}+\mathbf{g})(t)=3+\sin 2t+.5t \quad \text{and} \quad (2\mathbf{g})(t)=4+t$$

Two functions in V are equal if and only if their values are equal for every t in $\mathbb{D}$. Hence the zero vector in V is the function that is identically zero, $\mathbf{f}(t)=0$ for all t, and the negative of $\mathbf{f}$ is $(-1)\mathbf{f}$. Axioms 1 and 6 are obviously true, and the other axioms follow from properties of the real numbers, so V is a vector space. ■

Example

Let

$$\mathbb{R}^{2\times 2} = \{\text{all } 2 \times 2 \text{ matrices with real number entries}\}$$

and let addition and scalar multiplication be the usual operations on matrices: i.e.

$$\mathbf{u} + \mathbf{v} = \begin{bmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{bmatrix} + \begin{bmatrix} v_{11} & v_{12} \\ v_{21} & v_{22} \end{bmatrix} = \begin{bmatrix} u_{11} + v_{11} & u_{12} + v_{12} \\ u_{21} + v_{21} & u_{22} + v_{22} \end{bmatrix}$$

$$k\mathbf{u} = k \begin{bmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{bmatrix} = \begin{bmatrix} ku_{11} & ku_{12} \\ ku_{21} & ku_{22} \end{bmatrix}.$$

Show that $\mathbb{R}^{2\times 2}$ is a vector space.

(Some books use M_{22} instead of $\mathbb{R}^{2\times 2}$.)

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I leave axioms 3,7,8,9 for you to prove.

4. Let $\mathbf{0} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$. Then

$$\mathbf{0} + \mathbf{u} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{bmatrix} = \begin{bmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{bmatrix} = \mathbf{u}$$

and similarly $\mathbf{u} + \mathbf{0} = \mathbf{u}$. ✓

5 Let $-\mathbf{u} = \begin{bmatrix} -u_{11} & -u_{12} \\ -u_{21} & -u_{22} \end{bmatrix}$. Then

$$\mathbf{u} + (-\mathbf{u}) = \begin{bmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{bmatrix} + \begin{bmatrix} -u_{11} & -u_{12} \\ -u_{21} & -u_{22} \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}. \checkmark$$

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This proves that $\mathbb{R}^{2\times 2} = M_{22}$ is a vector space.

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Notation

Let

- $\mathbf{u} \oplus \mathbf{v}$ denote vector addition; and
- $k \odot \mathbf{u}$ denote scalar multiplication.

Example

Let $V = \mathbb{R}^2$ and define $\oplus$ and $\odot$ as follows: If $\mathbf{u} = (u_1, u_2)$ and $\mathbf{v} = (v_1, v_2)$, then we define

$$\mathbf{u} \oplus \mathbf{v} = (u_1 + v_1, u_2 + v_2)$$

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For example, if $\mathbf{u} = (2, 4)$, $\mathbf{v} = (-3, 5)$ and $k = 7$, then

$$\mathbf{u} \oplus \mathbf{v} = (2 + (-3), 4 + 5) = (-1, 9)$$
$$k \odot \mathbf{u} = (7 \cdot 2, 0) = (14, 0).$$

Is V a vector space?

$$k \odot \mathbf{u} = (ku_1, 0)$$

Axioms 1-9 are all true. I leave this for you to check.

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10 However axiom 10 fails. For example, if $\mathbf{u} = (0, 1)$ then

$$1 \odot \mathbf{u} = 1 \odot (0, 1) = (1 \cdot 0, 0) = (0, 0) \neq \mathbf{u}.$$

Therefore V is not a vector space with these operations.

Example (An Unusual Vector Space)

Let V be the set of strictly positive real numbers. Let $\mathbf{u} = u$ and $\mathbf{v} = v$ be any vectors in V (i.e. any real numbers > 0) and let k be a scalar. Define $\oplus$ and $\odot$ by

$$\mathbf{u} \oplus \mathbf{v} = uv$$
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Show that V is a vector space with these operations.

I will prove axioms 4, 5 and 7. I leave the others for you to prove.

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4. The zero vector in this space is the number 1 (i.e. $\mathbf{0} = 1$) since

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$$k \odot (\mathbf{u} \oplus \mathbf{v}) = (uv)^k = u^k v^k = (k \odot \mathbf{u}) \oplus (k \odot \mathbf{v}).$$

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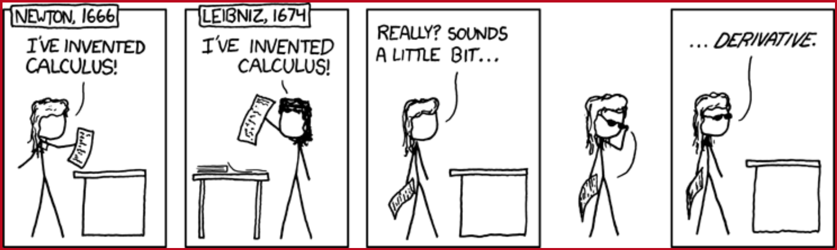
Therefore V is a vector space with these operations.

Subspaces

Null Spaces, Column Spaces, and Linear Transformations

Break

We will continue at 3pm



Linearly Independent Sets; Bases

Next Time

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