Lecture 2

- Solutions of Linear Systems
- Introduction to Matrices
- The Inverse of a Matrix

Solutions of Linear Systems

Definition

Variables which correspond to pivot columns are called *basic variables*.

Other variables are called *free variables*.

Example

Consider

$$\begin{bmatrix} 1 & 0 & -5 & 1 \\ 0 & 1 & 1 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad \begin{cases} x_1 \phantom{{}+x_2} - 5x_3 = 1 \\ \phantom{x_1 +{}} x_2 + x_3 = 4 \\ 0 = 0. \end{cases}$$

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Columns 1 and 2 are the pivot columns.

Therefore x_1 and x_2 are basic variables. x_3 is a free variable.

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Columns 1 and 2 are the pivot columns.

Therefore x_1 and x_2 are basic variables. x_3 is a free variable.

The general solution to the linear system can be written as

$$\begin{cases} x_1 = 1 + 5x_3 \\ x_2 = 4 - x_3 \\ x_3 \text{ is free.} \end{cases}$$

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The statement "x_3 is free" means that you can choose any value for x_3. After you choose x_3, the values of x_1 and x_2 are determined by the formulae.

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- e.g. if you choose $x_3 = 0$, then you get the solution $(1, 4, 0)$;
- e.g. if you choose $x_3 = 1$, then you get $(6, 3, 1)$.

Example

Find the general solution of the linear system whose augmented matrix has been reduced to

$$\begin{bmatrix} 1 & 6 & 2 & -5 & -2 & -4 \\ 0 & 0 & 2 & -8 & -1 & 3 \\ 0 & 0 & 0 & 0 & 1 & 7 \end{bmatrix}.$$

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$$\begin{bmatrix} 1 & 6 & 2 & -5 & -2 & -4 \\ 0 & 0 & 2 & -8 & -1 & 3 \\ 0 & 0 & 0 & 0 & 1 & 7 \end{bmatrix}.$$

This matrix is in REF, but we need it in RREF so we will have to do some row reduction.

$$\begin{bmatrix} 1 & 6 & 2 & -5 & -2 & -4 \\ 0 & 0 & 2 & -8 & -1 & 3 \\ 0 & 0 & 0 & 0 & 1 & 7 \end{bmatrix} \xrightarrow[2R_3+R_1]{R_3+R_2} \begin{bmatrix} 1 & 6 & 2 & -5 & 0 & 10 \\ 0 & 0 & 2 & -8 & 0 & 10 \\ 0 & 0 & 0 & 0 & 1 & 7 \end{bmatrix}$$

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$$\xrightarrow{\frac{1}{2}R_2} \begin{bmatrix} 1 & 6 & 2 & -5 & 0 & 10 \\ 0 & 0 & 1 & -4 & 0 & 5 \\ 0 & 0 & 0 & 0 & 1 & 7 \end{bmatrix}$$

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$$\xrightarrow{\frac{1}{2}R_2} \begin{bmatrix} 1 & 6 & 2 & -5 & 0 & 10 \\ 0 & 0 & 1 & -4 & 0 & 5 \\ 0 & 0 & 0 & 0 & 1 & 7 \end{bmatrix} \xrightarrow{-2R_2+R_1} \begin{bmatrix} 1 & 6 & 0 & 3 & 0 & 0 \\ 0 & 0 & 1 & -4 & 0 & 5 \\ 0 & 0 & 0 & 0 & 1 & 7 \end{bmatrix}.$$

$$\begin{bmatrix} 1 & 6 & 0 & 3 & 0 & 0 \\ 0 & 0 & 1 & -4 & 0 & 5 \\ 0 & 0 & 0 & 0 & 1 & 7 \end{bmatrix}$$

This augmented matrix has 6 columns. That means that the linear system has 5 variables.

$$\begin{cases} x_1 + 6x_2 \qquad + 3x_4 \qquad = 0 \\ \qquad\qquad x_3 - 4x_4 \qquad = 5 \\ \qquad\qquad\qquad\qquad x_5 = 7. \end{cases}$$

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The pivot columns are the first, third and fifth columns. So the basic variables are x_1, x_3 and x_5. The free variables are x_2 and x_4.

$$\begin{cases} x_1 + 6x_2 + 3x_4 = 0 \\ x_3 - 4x_4 = 5 \\ x_5 = 7. \end{cases}$$

The general solution of the linear system is

$$\begin{cases} x_1 = -6x_2 - 3x_4 \\ x_2 \text{ is free} \\ x_3 = 5 + 4x_4 \\ x_4 \text{ is free} \\ x_5 = 7. \end{cases}$$

Theorem (Existence and Uniqueness)

A linear system is consistent if and only if the rightmost column of the augmented matrix is not a pivot column

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A linear system is consistent if and only if the rightmost column of the augmented matrix is not a pivot column - that is, if and only if an echelon form of the augmented matrix has no row that looks like

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for $b \neq 0$.

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A linear system is consistent if and only if the rightmost column of the augmented matrix is not a pivot column - that is, if and only if an echelon form of the augmented matrix has no row that looks like

$$[0 \ \cdots \ 0 \ b]$$

for $b \neq 0$.

If a linear system is consistent, then the solution set contains either

1. *a unique solution (when there are no free variables); or*
2. *infinitely many solutions (when there is at least one free variable).*

USING ROW REDUCTION TO SOLVE A LINEAR SYSTEM

1. Write the augmented matrix of the system.
2. Use the row reduction algorithm to obtain an equivalent augmented matrix in echelon form. Decide whether the system is consistent. If there is no solution, stop; otherwise, go to the next step.
3. Continue row reduction to obtain the reduced echelon form.
4. Write the system of equations corresponding to the matrix obtained in step 3.
5. Rewrite each nonzero equation from step 4 so that its one basic variable is expressed in terms of any free variables appearing in the equation.

Example

How many solutions does the following linear system have?

$$\begin{cases} \phantom{3x_1 - {}} 3x_2 - 6x_3 + 6x_4 + 4x_5 = -5 \\ 3x_1 - 7x_2 + 8x_3 - 5x_4 + 8x_5 = 9 \\ 3x_1 - 9x_2 + 12x_3 - 9x_4 + 6x_5 = 15. \end{cases}$$

Example

How many solutions does the following linear system have?

$$\begin{cases} \quad\quad\;\; 3x_2 - \;\; 6x_3 + 6x_4 + 4x_5 = -5 \\ 3x_1 - 7x_2 + \;\; 8x_3 - 5x_4 + 8x_5 = 9 \\ 3x_1 - 9x_2 + 12x_3 - 9x_4 + 6x_5 = 15. \end{cases}$$

First we write our augmented matrix

$$\begin{bmatrix} 0 & 3 & -6 & 6 & 4 & -5 \\ 3 & -7 & 8 & -5 & 8 & 9 \\ 3 & -9 & 12 & -9 & 6 & 15 \end{bmatrix}.$$

Then we will reduce it to REF to see if the linear system is consistent.

Solutions of Linear Systems

$$\begin{bmatrix} 0 & 3 & -6 & 6 & 4 & -5 \\ 3 & -7 & 8 & -5 & 8 & 9 \\ 3 & -9 & 12 & -9 & 6 & 15 \end{bmatrix} \xrightarrow{R_1 \leftrightarrow R_3} \begin{bmatrix} 3 & -9 & 12 & -9 & 6 & 15 \\ 3 & -7 & 8 & -5 & 8 & 9 \\ 0 & 3 & -6 & 6 & 4 & -5 \end{bmatrix}$$

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$$\xrightarrow{-R_1 + R_2} \begin{bmatrix} 3 & -9 & 12 & -9 & 6 & 15 \\ 0 & 2 & -4 & 4 & 2 & -6 \\ 0 & 3 & -6 & 6 & 4 & -5 \end{bmatrix}$$

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$$\xrightarrow{-\frac{3}{2}R_2 + R_3} \begin{bmatrix} 3 & -9 & 12 & -9 & 6 & 15 \\ 0 & 2 & -4 & 4 & 2 & -6 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{bmatrix}$$

This matrix is now in REF.

$$\begin{bmatrix} 3 & -9 & 12 & -9 & 6 & 15 \\ 0 & 2 & -4 & 4 & 2 & -6 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{bmatrix}$$

- The pivot columns are the first, second and fifth columns.

$$\begin{bmatrix} 3 & -9 & 12 & -9 & 6 & 15 \\ 0 & 2 & -4 & 4 & 2 & -6 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{bmatrix}$$

- The pivot columns are the first, second and fifth columns.
- Since the sixth column is not a pivot column, the linear system is consistent. So either there is one unique solution or there are infinitely many solutions.

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- The basic variables are x_1, x_2 and x_5.
- The free variables are x_3 and x_4.

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- The pivot columns are the first, second and fifth columns.
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- The basic variables are x_1, x_2 and x_5.
- The free variables are x_3 and x_4.
- Since there are free variables, there are infinitely many solutions to the linear system.

Homogeneous Linear Systems

Definition

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$$\left\{\begin{array}{ccccccccccc} a_{11}x_1 & + & a_{12}x_2 & + & a_{13}x_3 & + & \dots & + & a_{1n}x_n & = & 0 \\ a_{21}x_1 & + & a_{22}x_2 & + & a_{23}x_3 & + & \dots & + & a_{2n}x_n & = & 0 \\ \vdots & & \vdots & & \vdots & & & & \vdots & & \vdots \\ a_{m1}x_1 & + & a_{m2}x_2 & + & a_{m3}x_3 & + & \dots & + & a_{mn}x_n & = & 0. \end{array}\right.$$

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Remark

Every homogeneous linear system is consistent because $(0, 0, 0, \dots, 0)$ is always a solution.

Definition

The solution $(0, 0, 0, \ldots, 0)$ is called the *trivial solution*.

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Theorem

A homogeneous linear system has a nontrivial solution if and only if it has atleast one free variable.

Example

Use Gauss-Jordan elimination to solve the homogeneous linear system

$$\begin{cases} x_1 + 3x_2 - 2x_3 \phantom{{}-2x_4} + 2x_5 \phantom{{}-3x_6} = 0 \\ 2x_1 + 6x_2 - 5x_3 - 2x_4 + 4x_5 - 3x_6 = 0 \\ 5x_3 + 10x_4 + 15x_6 = 0 \\ 2x_1 + 6x_2 + 8x_4 + 4x_5 + 18x_6 = 0. \end{cases}$$

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Use Gauss-Jordan elimination to solve the homogeneous linear system

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The augmented matrix is

$$\begin{bmatrix} 1 & 3 & -2 & 0 & 2 & 0 & 0 \\ 2 & 6 & -5 & -2 & 4 & -3 & 0 \\ 0 & 0 & 5 & 10 & 0 & 15 & 0 \\ 2 & 6 & 0 & 8 & 4 & 18 & 0 \end{bmatrix}.$$

The augmented matrix is

$$\begin{bmatrix} 1 & 3 & -2 & 0 & 2 & 0 & 0 \\ 2 & 6 & -5 & -2 & 4 & -3 & 0 \\ 0 & 0 & 5 & 10 & 0 & 15 & 0 \\ 2 & 6 & 0 & 8 & 4 & 18 & 0 \end{bmatrix}$$

can be row-reduced to

$$\begin{bmatrix} 1 & 3 & 0 & 4 & 2 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

I leave it for you to check this (exercise).

The corresponding linear system is

$$\begin{cases} x_1 + 3x_2 \qquad\quad + 4x_4 + 2x_5 \qquad = 0 \\ \qquad\qquad x_3 + 2x_4 \qquad\qquad\quad = 0 \\ \qquad\qquad\qquad\qquad\qquad\quad x_6 = 0 \end{cases}$$

which has solution

$$\begin{cases} x_1 = -3x_2 - 4x_4 - 2x_5 \\ x_2 \text{ is free} \\ x_3 = -2x_4 \\ x_4 \text{ is free} \\ x_5 \text{ is free} \\ x_6 = 0. \end{cases}$$

This linear system has infinitely many solutions.

Theorem

If a homogeneous linear system has n variables, and if the RREF of its augmented matrix has r nonzero rows, then the system has $n - r$ free variables.

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Theorem

A homogeneous linear system with more variables than equations has infinitely many solutions.

Introduction to Matrices

We have been using rectangular arrays of numbers, called *augmented matrices*, to abbreviate linear systems.

Now I want to discuss what a *matrix* is and what we can do with them.

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Definition

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Definition

The numbers in a matrix are called *entries*.

The plural of matrix is *matrices*.

Example

These are matrices:

$$\begin{bmatrix} 1 & 2 \\ 3 & 0 \\ -1 & 4 \end{bmatrix}, \quad \begin{bmatrix} 2 & 1 & 0 & -3 \end{bmatrix}, \quad \begin{bmatrix} e & \pi & -\sqrt{2} \\ 0 & \frac{1}{2} & 1 \\ 0 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 1 \\ 3 \end{bmatrix}, \quad \begin{bmatrix} 4 \end{bmatrix}.$$

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Definition

The *size* of a matrix is

$$\text{number of rows} \times \text{number of column.}$$

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3×2

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1×4

Definition

The *size* of a matrix is

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Example

These are matrices:

$$\begin{bmatrix} 1 & 2 \\ 3 & 0 \\ -1 & 4 \end{bmatrix}, \quad \begin{bmatrix} 2 & 1 & 0 & -3 \end{bmatrix}, \quad \begin{bmatrix} e & \pi & -\sqrt{2} \\ 0 & \frac{1}{2} & 1 \\ 0 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 1 \\ 3 \end{bmatrix}, \quad \begin{bmatrix} 4 \end{bmatrix}.$$

$$3 \times 3$$

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These are matrices:

$$\begin{bmatrix} 1 & 2 \\ 3 & 0 \\ -1 & 4 \end{bmatrix}, \quad \begin{bmatrix} 2 & 1 & 0 & -3 \end{bmatrix}, \quad \begin{bmatrix} e & \pi & -\sqrt{2} \\ 0 & \frac{1}{2} & 1 \\ 0 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 1 \\ 3 \end{bmatrix}, \quad \begin{bmatrix} 4 \end{bmatrix}.$$

2×1

Definition

The *size* of a matrix is

$$\text{number of rows} \times \text{number of column.}$$

Example

These are matrices:

$$\begin{bmatrix} 1 & 2 \\ 3 & 0 \\ -1 & 4 \end{bmatrix}, \quad \begin{bmatrix} 2 & 1 & 0 & -3 \end{bmatrix}, \quad \begin{bmatrix} e & \pi & -\sqrt{2} \\ 0 & \frac{1}{2} & 1 \\ 0 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 1 \\ 3 \end{bmatrix}, \quad \begin{bmatrix} 4 \end{bmatrix}.$$

1×1

Definition

The *size* of a matrix is

$$\text{number of rows} \times \text{number of column.}$$

$$\begin{bmatrix} 2 & 1 & 0 & -3 \end{bmatrix} \quad \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

Definition

A matrix with only one row is called a *row vector*.

Definition

A matrix with only one column is called a *column vector*.

Notation

We will use

- capital letters, $A, B, C, \ldots$ to denote matrices; and
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Notation

The entry with occurs in row i and column j of a matrix A will be denoted by a_{ij}.

Thus a general ${\color{green}3} \times {\color{red}4}$ matrix might be written as

$$\begin{array}{cc} {\color{green}\text{3 rows}} & \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \end{bmatrix} \\ & {\color{red}\text{4 columns}} \end{array}$$

Introduction to Matrices

Thus a general 3×4 matrix might be written as

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \end{bmatrix}$$

row 2
column 4

"a two four"
not "a twenty four"

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and a general $m \times n$ matrix might be written as

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1j} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2j} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots & & \vdots \\ a_{i1} & a_{i2} & \cdots & a_{ij} & \cdots & a_{in} \\ \vdots & \vdots & & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mj} & \cdots & a_{mn} \end{bmatrix}.$$

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and a general $m \times n$ matrix might be written as

$$\begin{array}{rc} \text{row } 1 & \left[\begin{matrix} a_{11} & a_{12} & \cdots & a_{1j} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2j} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots & & \vdots \\ a_{i1} & a_{i2} & \cdots & a_{ij} & \cdots & a_{in} \\ \vdots & \vdots & & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mj} & \cdots & a_{mn} \end{matrix}\right] \end{array}.$$

Thus a general 3 × 4 matrix might be written as

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \end{bmatrix}$$

and a general $m \times n$ matrix might be written as

$$\begin{array}{rc} & \\ \text{row } 2 & \\ & \end{array} \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1j} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2j} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots & & \vdots \\ a_{i1} & a_{i2} & \cdots & a_{ij} & \cdots & a_{in} \\ \vdots & \vdots & & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mj} & \cdots & a_{mn} \end{bmatrix}.$$

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row i (label beside the row $a_{i1} \; a_{i2} \cdots a_{in}$)

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$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \end{bmatrix}$$

and a general $m \times n$ matrix might be written as

$$\begin{array}{r l} & \left[\begin{array}{cccccc} a_{11} & a_{12} & \cdots & a_{1j} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2j} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots & & \vdots \\ a_{i1} & a_{i2} & \cdots & a_{ij} & \cdots & a_{in} \\ \vdots & \vdots & & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mj} & \cdots & a_{mn} \end{array}\right] \\ \text{row } m & \end{array} .$$

Thus a general 3×4 matrix might be written as

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \end{bmatrix}$$

and a general $m \times n$ matrix might be written as

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Introduction to Matrices

Thus a general 3 × 4 matrix might be written as

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \end{bmatrix}$$

and a general $m \times n$ matrix might be written as

$$\begin{array}{cccccc} \multicolumn{6}{c}{\left[\begin{array}{cccccc} a_{11} & a_{12} & \cdots & a_{1j} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2j} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots & & \vdots \\ a_{i1} & a_{i2} & \cdots & a_{ij} & \cdots & a_{in} \\ \vdots & \vdots & & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mj} & \cdots & a_{mn} \end{array}\right]} \\ & \text{column } 2 & & & & \end{array}.$$

Introduction to Matrices

Thus a general 3×4 matrix might be written as

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \end{bmatrix}$$

and a general $m \times n$ matrix might be written as

$$\begin{array}{c} \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1j} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2j} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots & & \vdots \\ a_{i1} & a_{i2} & \cdots & a_{ij} & \cdots & a_{in} \\ \vdots & \vdots & & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mj} & \cdots & a_{mn} \end{bmatrix} . \\ \text{column } j \end{array}$$

Introduction to Matrices

Thus a general 3 × 4 matrix might be written as

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \end{bmatrix}$$

and a general $m \times n$ matrix might be written as

$$\begin{array}{r} \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1j} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2j} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots & & \vdots \\ a_{i1} & a_{i2} & \cdots & a_{ij} & \cdots & a_{in} \\ \vdots & \vdots & & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mj} & \cdots & a_{mn} \end{bmatrix}. \\ \text{column } n \end{array}$$

Thus a general 3×4 matrix might be written as

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \end{bmatrix}$$

and a general $m \times n$ matrix might be written as

$$\begin{array}{cc} & \\ \text{row } i & \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1j} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2j} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots & & \vdots \\ a_{i1} & a_{i2} & \cdots & a_{ij} & \cdots & a_{in} \\ \vdots & \vdots & & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mj} & \cdots & a_{mn} \end{bmatrix} \\ & \text{column } j \end{array}.$$

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1j} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2j} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots & & \vdots \\ a_{i1} & a_{i2} & \cdots & a_{ij} & \cdots & a_{in} \\ \vdots & \vdots & & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mj} & \cdots & a_{mn} \end{bmatrix}.$$

Notation

We can also write $A = [a_{ij}]_{m\times n}$

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1j} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2j} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots & & \vdots \\ a_{i1} & a_{i2} & \cdots & a_{ij} & \cdots & a_{in} \\ \vdots & \vdots & & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mj} & \cdots & a_{mn} \end{bmatrix}.$$

Notation

We can also write $A = [a_{ij}]_{m\times n}$ or just $A = [a_{ij}]$.

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1j} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2j} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots & & \vdots \\ a_{i1} & a_{i2} & \cdots & a_{ij} & \cdots & a_{in} \\ \vdots & \vdots & & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mj} & \cdots & a_{mn} \end{bmatrix}.$$

Notation

We can also write $A = [a_{ij}]_{m \times n}$ or just $A = [a_{ij}]$.

The entry in row i and column j of a matrix A can be denoted by

$$a_{ij} = (A)_{ij}$$

also.

Example

Let $A = \begin{bmatrix} 2 & -3 \\ 7 & 0 \end{bmatrix}$ be a matrix. Then

$$(A)_{11} = 2, \qquad (A)_{12} = -3, \qquad (A)_{21} = 7 \quad \text{and} \quad (A)_{22} = 0.$$

Notation

For vectors, that is $1 \times n$ and $m \times 1$ matrices, it is common practice to denote them by bold lowercase letters instead of capital letters. E.g.

$$\mathbf{a} = \begin{bmatrix} a_1 & a_2 & a_3 & \cdots & a_n \end{bmatrix} \qquad \text{and} \qquad \mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}.$$

Definition

A matrix which has the same number of rows and columns, n say, is called a *square matrix of order* n.

$$\underbrace{\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}}_{n} \Bigg\updownarrow n$$

Definition

A matrix which has the same number of rows and columns, n say, is called a *square matrix of order n*.

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$$

Definition

The entries $a_{11}, a_{22}, \ldots, a_{nn}$ are on the *main diagonal* of a square matrix.

Operations on Matrices

Definition

Two matrices $A = [a_{ij}]$ and $B = [b_{ij}]$, are *equal* if

- they have the same size; and
- the corresponding entries are equal ($a_{ij} = b_{ij}$ for all i and j).

Definition (addition and subtraction)

If A and B have the same size, then we can define the sum $A + B$ and the difference $A - B$ in the obvious way.

Example

Let

$$A = \begin{bmatrix} 2 & 1 & 0 & 3 \\ -1 & 0 & 2 & 4 \\ 4 & -2 & 7 & 0 \end{bmatrix}, \; B = \begin{bmatrix} -4 & 3 & 5 & 1 \\ 2 & 2 & 0 & -1 \\ 3 & 2 & -4 & 5 \end{bmatrix} \text{ and } C = \begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}.$$

Then

$$A+B = \begin{bmatrix} -2 & 4 & 5 & 4 \\ 1 & 2 & 2 & 3 \\ 7 & 0 & 3 & 5 \end{bmatrix} \quad \text{and} \quad A-B = \begin{bmatrix} 6 & -2 & -5 & 2 \\ -3 & -2 & 2 & 5 \\ 1 & -4 & 11 & -5 \end{bmatrix}.$$

Example

Let

$$A = \begin{bmatrix} 2 & 1 & 0 & 3 \\ -1 & 0 & 2 & 4 \\ 4 & -2 & 7 & 0 \end{bmatrix},\ B = \begin{bmatrix} -4 & 3 & 5 & 1 \\ 2 & 2 & 0 & -1 \\ 3 & 2 & -4 & 5 \end{bmatrix} \text{ and } C = \begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}.$$

Then

$$A+B = \begin{bmatrix} -2 & 4 & 5 & 4 \\ 1 & 2 & 2 & 3 \\ 7 & 0 & 3 & 5 \end{bmatrix} \quad \text{and} \quad A-B = \begin{bmatrix} 6 & -2 & -5 & 2 \\ -3 & -2 & 2 & 5 \\ 1 & -4 & 11 & -5 \end{bmatrix}.$$

$A+C$, $B+C$, $A-C$ and $B-C$ are all undefined because the sizes are different.

And we can multiply a matrix by a number in the obvious way.

And we can multiply a matrix by a number in the obvious way.

Example

If

$$A = \begin{bmatrix} 2 & 3 & 4 \\ 1 & 3 & 1 \end{bmatrix}, \ B = \begin{bmatrix} 0 & 2 & 7 \\ -1 & 3 & -5 \end{bmatrix} \text{ and } C = \begin{bmatrix} 9 & -6 & -3 \\ 3 & 0 & 12 \end{bmatrix},$$

then

$$2A = \begin{bmatrix} 4 & 6 & 8 \\ 2 & 6 & 2 \end{bmatrix}, \quad -B = (-1)B = \begin{bmatrix} 0 & -2 & -7 \\ 1 & -3 & 5 \end{bmatrix}$$

and

$$\frac{1}{3}C = \begin{bmatrix} 3 & -2 & -1 \\ 1 & 0 & 4 \end{bmatrix}.$$

The Zero Matrix

Definition

An $m \times n$ matrix whose entries are all zero is called a *zero matrix* and is written as 0 or $0_{m\times n}$.

Example (Some zero matrices)

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \quad \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad \begin{bmatrix} 0 \end{bmatrix}$$

Theorem

Let A, B and C be matrices of the same size. Let r and s be numbers. Then

1. $A + B = B + A$

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3. $A + 0 = A$ *and* $A - 0 = A$

Theorem

Let A, B and C be matrices of the same size. Let r and s be numbers. Then

1. $A + B = B + A$
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4. $r(A + B) = rA + rB$

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1. $A + B = B + A$
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4. $r(A + B) = rA + rB$
5. $(r + s)A = rA + sA$

Theorem

Let A, B and C be matrices of the same size. Let r and s be numbers. Then

1. $A + B = B + A$
2. $(A + B) + C = A + (B + C)$
3. $A + 0 = A$ *and* $A - 0 = A$
4. $r(A + B) = rA + rB$
5. $(r + s)A = rA + sA$
6. $r(sA) = (rs)A.$

Definition

If $A_1, A_2, \ldots, A_r$ are matrices of the same size, and if $c_1, c_2, \ldots, c_r$ are numbers, then an expression of the form

$$c_1 A_1 + c_2 A_2 + \ldots + c_r A_r$$

is called a *linear combination* of $A_1, A_2, \ldots, A_r$ with coefficients $c_1, c_2, \ldots, c_r$.

Matrix Multiplication

Multiplying two matrices together is more complicated.

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$$\underset{m \times r}{A} \cdot \underset{r \times n}{B} = \underset{m \times n}{AB}$$

Definition

If A is an $m \times r$ matrix and B is an $r \times n$ matrix, then AB will be an $m \times n$ matrix.

Matrix Multiplication

Multiplying two matrices together is more complicated.

$$\underset{m \times r}{A} \cdot \underset{r \times n}{B} = \underset{m \times n}{AB}$$

Definition

If A is an $m \times r$ matrix and B is an $r \times n$ matrix, then AB will be an $m \times n$ matrix.

To find the entry in the ith row and jth column of AB, we must multiply each entry from row i of A with the corresponding entry from column j of B, then sum them.

Example

Multiply the following two matrices together:

$$A = \begin{bmatrix} 1 & 2 & 4 \\ 2 & 6 & 0 \end{bmatrix}, \qquad B = \begin{bmatrix} 4 & 1 & 4 & 3 \\ 0 & -1 & 3 & 1 \\ 2 & 7 & 5 & 2 \end{bmatrix}.$$

Example

Multiply the following two matrices together:

$$A = \begin{bmatrix} 1 & 2 & 4 \\ 2 & 6 & 0 \end{bmatrix}, \qquad B = \begin{bmatrix} 4 & 1 & 4 & 3 \\ 0 & -1 & 3 & 1 \\ 2 & 7 & 5 & 2 \end{bmatrix}.$$

A is a 2×3 matrix and B is a 3×4 matrix. So the product AB is a 2×4 matrix.

$$\underset{2\times 3}{\begin{bmatrix} 1 & 2 & 4 \\ 2 & 6 & 0 \end{bmatrix}} \underset{3\times 4}{\begin{bmatrix} 4 & 1 & 4 & 3 \\ 0 & -1 & 3 & 1 \\ 2 & 7 & 5 & 2 \end{bmatrix}} = \underset{2\times 4}{\begin{bmatrix} \boxed{} & \boxed{} & \boxed{} & \boxed{} \\ \boxed{} & \boxed{} & \boxed{} & \boxed{} \end{bmatrix}}$$

Introduction to Matrices

Let's say that I want to calculate the entry in the 2nd row and the 3rd column.

$$\begin{bmatrix} 1 & 2 & 4 \\ 2 & 6 & 0 \end{bmatrix} \begin{bmatrix} 4 & 1 & 4 & 3 \\ 0 & -1 & 3 & 1 \\ 2 & 7 & 5 & 2 \end{bmatrix} = \begin{bmatrix} & & & \\ & & & \end{bmatrix}$$

I need row 2 of the first matrix,

Let's say that I want to calculate the entry in the 2nd row and the 3rd column.

$$\begin{bmatrix} 1 & 2 & 4 \\ 2 & 6 & 0 \end{bmatrix} \begin{bmatrix} 4 & 1 & 4 & 3 \\ 0 & -1 & 3 & 1 \\ 2 & 7 & 5 & 2 \end{bmatrix} = \begin{bmatrix} \square & \square & \square & \square \\ \square & \square & \blacksquare & \square \end{bmatrix}$$

I need row 2 of the first matrix,

and i need column 3 of the second matrix.

Let's say that I want to calculate the entry in the 2nd row and the 3rd column.

$$\begin{bmatrix} 1 & 2 & 4 \\ 2 & 6 & 0 \end{bmatrix} \begin{bmatrix} 4 & 1 & 4 & 3 \\ 0 & -1 & 3 & 1 \\ 2 & 7 & 5 & 2 \end{bmatrix} = \begin{bmatrix} & & & \\ & & & \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & 4 \\ 2 & 6 & 0 \end{bmatrix} \begin{bmatrix} 4 & 1 & 4 & 3 \\ 0 & -1 & 3 & 1 \\ 2 & 7 & 5 & 2 \end{bmatrix} = \begin{bmatrix} & & & \\ & & & \end{bmatrix}$$

$$(2 \cdot 4) + (6 \cdot 3) + (0 \cdot 5) = 26$$

Introduction to Matrices

$$\begin{bmatrix} 1 & 2 & 4 \\ 2 & 6 & 0 \end{bmatrix} \begin{bmatrix} 4 & 1 & 4 & 3 \\ 0 & -1 & 3 & 1 \\ 2 & 7 & 5 & 2 \end{bmatrix} = \begin{bmatrix} & & & \\ & & 26 & \end{bmatrix}$$

$$(2 \cdot 4) + (6 \cdot 3) + (0 \cdot 5) = 26$$

Introduction to Matrices

$$\begin{array}{cccc} & & & \text{row 1} \\ & & & \text{column 4} \end{array}$$

$$\begin{bmatrix} 1 & 2 & 4 \\ 2 & 6 & 0 \end{bmatrix} \begin{bmatrix} 4 & 1 & 4 & 3 \\ 0 & -1 & 3 & 1 \\ 2 & 7 & 5 & 2 \end{bmatrix} = \begin{bmatrix} & & & \\ & & 26 & \end{bmatrix}$$

$$\begin{array}{r}\text{row 1}\\ \end{array}\begin{bmatrix}1 & 2 & 4\\ 2 & 6 & 0\end{bmatrix}\begin{array}{c}\text{column 4}\\ \begin{bmatrix}4 & 1 & 4 & 3\\ 0 & -1 & 3 & 1\\ 2 & 7 & 5 & 2\end{bmatrix}\end{array} = \begin{array}{c}\text{row 1}\\ \text{column 4}\\ \begin{bmatrix} & & & \\ & & 26 & \end{bmatrix}\end{array}$$

$$\begin{bmatrix} 1 & 2 & 4 \\ 2 & 6 & 0 \end{bmatrix} \begin{bmatrix} 4 & 1 & 4 & 3 \\ 0 & -1 & 3 & 1 \\ 2 & 7 & 5 & 2 \end{bmatrix} = \begin{bmatrix} & & & 13 \\ & & 26 & \end{bmatrix}$$

$$(1 \cdot 3) + (2 \cdot 1) + (4 \cdot 2) = 13$$

$$\begin{bmatrix} 1 & 2 & 4 \\ 2 & 6 & 0 \end{bmatrix} \begin{bmatrix} 4 & 1 & 4 & 3 \\ 0 & -1 & 3 & 1 \\ 2 & 7 & 5 & 2 \end{bmatrix} = \begin{bmatrix} 12 & 27 & 30 & 13 \\ 8 & -4 & 26 & 12 \end{bmatrix}$$

I leave the other 6 entries for you to check.

Remember, we can only multiply two matrices together if the inside size numbers are equal.

$$\underset{m \times r}{A} \cdot \underset{r \times n}{B} = AB$$

inside

Introduction to Matrices

Remember, we can only multiply two matrices together if the inside size numbers are equal.

$$\underset{m \times r}{A} \cdot \underset{r \times n}{B} = \underset{m \times n}{AB}$$

inside

outside

Then the outside size numbers give us the size of the product AB.

EXAMPLE 4 If A is a 3×5 matrix and B is a 5×2 matrix, what are the sizes of AB and BA, if they are defined?

EXAMPLE 4 If A is a 3×5 matrix and B is a 5×2 matrix, what are the sizes of AB and BA, if they are defined?

SOLUTION Since A has 5 columns and B has 5 rows, the product AB is defined and is a 3×2 matrix:

$$\underset{3\times 5}{\overset{A}{\begin{bmatrix} * & * & * & * & * \\ * & * & * & * & * \\ * & * & * & * & * \end{bmatrix}}}\underset{5\times 2}{\overset{B}{\begin{bmatrix} * & * \\ * & * \\ * & * \\ * & * \\ * & * \end{bmatrix}}} = \underset{3\times 2}{\overset{AB}{\begin{bmatrix} * & * \\ * & * \\ * & * \end{bmatrix}}}$$

Match

Size of AB

The product BA is *not* defined because the 2 columns of B do not match the 3 rows of A. ∎

Ask the audience 1/3

Suppose that A, B and C are matrices with the following sizes:

$$\begin{array}{ccc} A & B & C \\ 3 \times 4 & 4 \times 7 & 7 \times 3. \end{array}$$

Which of the following is correct?

1. AB is a 3×7 matrix.
   BA is a 4×4 matrix.
2. AB is a 3×7 matrix.
   BA is a 7×3 matrix.
3. AB is a 3×7 matrix.
   BA is undefined.
4. AB is undefined.
   BA is a 4×4 matrix.

Ask the audience 1/3

Suppose that A, B and C are matrices with the following sizes:

A	B	C
3×4	4×7	7×3.

Which of the following is correct?

1. AB is a 3×7 matrix. BA is a 4×4 matrix.
2. AB is a 3×7 matrix. BA is a 7×3 matrix.
3. AB is a 3×7 matrix. BA is undefined.
4. AB is undefined. BA is a 4×4 matrix.

Ask the audience 2/3

Suppose that A, B and C are matrices with the following sizes:

$$\begin{matrix} A & B & C \\ 3 \times 4 & 4 \times 7 & 7 \times 3. \end{matrix}$$

Which of the following is correct?

1. BC is a 4×3 matrix.
   CB is undefined.
2. BC is undefined.
   CB is a 3×4 matrix.
3. BC is undefined.
   CB is undefined.
4. BC is undefined.
   CB is a 7×7 matrix.

Ask the audience 2/3

Suppose that A, B and C are matrices with the following sizes:

$$\begin{array}{ccc} A & B & C \\ 3 \times 4 & 4 \times 7 & 7 \times 3. \end{array}$$

Which of the following is correct?

1. BC is a 4×3 matrix.
   CB is undefined.
2. BC is undefined.
   CB is a 3×4 matrix.
3. BC is undefined.
   CB is undefined.
4. BC is undefined.
   CB is a 7×7 matrix.

Ask the audience 3/3

Suppose that A, B and C are matrices with the following sizes:

$$\begin{array}{ccc} A & B & C \\ 3 \times 4 & 4 \times 7 & 7 \times 3. \end{array}$$

Which of the following is correct?

1. AC is undefined.
   CA is a 7×4 matrix.
2. AC is undefined.
   CA is undefined.
3. AC is a 3×3 matrix.
   CA is a 7×4 matrix.
4. I didn't understand any of this.

Ask the audience 3/3

Suppose that A, B and C are matrices with the following sizes:

$$\begin{array}{ccc} A & B & C \\ 3 \times 4 & 4 \times 7 & 7 \times 3. \end{array}$$

Which of the following is correct?

1. AC is undefined. CA is a 7×4 matrix.
2. AC is undefined. CA is undefined.
3. AC is a 3×3 matrix. CA is a 7×4 matrix.
4. I didn't understand any of this.

In general, if $A = [a_{ij}]$ is an $m \times r$ matrix and $B = [b_{ij}]$ is an $r \times n$ matrix, then, as illustrated by the shading in the following display,

$$AB = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1r} \\ a_{21} & a_{22} & \cdots & a_{2r} \\ \vdots & \vdots & & \vdots \\ a_{i1} & a_{i2} & \cdots & a_{ir} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mr} \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1j} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2j} & \cdots & b_{2n} \\ \vdots & \vdots & & \vdots & & \vdots \\ b_{r1} & b_{r2} & \cdots & b_{rj} & \cdots & b_{rn} \end{bmatrix} \tag{4}$$

the entry $(AB)_{ij}$ in row i and column j of AB is given by

$$(AB)_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + a_{i3}b_{3j} + \cdots + a_{ir}b_{rj} \tag{5}$$

Formula (5) is called the ***row-column rule*** for matrix multiplication.

Break

We will continue at 3pm

The Identity Matrix

Definition

A square matrix with 1's on the main diagonal and 0's everywhere else is called an *identity matrix*, and is denoted by I or I_n.

Example

$$I = I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad I = I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad I = I_4 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Note that if A is a 2×3 matrix, then

$$AI_3 = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Note that if A is a 2×3 matrix, then

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Note that if A is a 2×3 matrix, then

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Note that if A is a 2×3 matrix, then

$$AI_3 = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix} = A$$

and

$$I_2 A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix} = A.$$

Properties of Matrix Multiplication

Theorem

Let A, B and C be matrices and let r be a number. If the sizes of the matrices are correct, then

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Let A, B and C be matrices and let r be a number. If the sizes of the matrices are correct, then

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2. $A(B + C) = AB + AC$
3. $(B + C)A = BA + CA$
4. $r(AB) = (rA)B = A(rB)$
5. $IA = A = AI$

Example

Let $A = \begin{bmatrix} 5 & 1 \\ 3 & -2 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 0 \\ 4 & 3 \end{bmatrix}$. Calculate AB and BA.

I leave it for you to check that

$$AB = \begin{bmatrix} 5 & 1 \\ 3 & -2 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 4 & 3 \end{bmatrix} =$$

and

$$BA = \begin{bmatrix} 2 & 0 \\ 4 & 3 \end{bmatrix} \begin{bmatrix} 5 & 1 \\ 3 & -2 \end{bmatrix} =$$

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Let $A = \begin{bmatrix} 5 & 1 \\ 3 & -2 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 0 \\ 4 & 3 \end{bmatrix}$. Calculate AB and BA.

I leave it for you to check that

$$AB = \begin{bmatrix} 5 & 1 \\ 3 & -2 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 4 & 3 \end{bmatrix} = \begin{bmatrix} 14 & 3 \\ -2 & -6 \end{bmatrix},$$

and

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$$AB = \begin{bmatrix} 5 & 1 \\ 3 & -2 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 4 & 3 \end{bmatrix} = \begin{bmatrix} 14 & 3 \\ -2 & -6 \end{bmatrix},$$

and

$$BA = \begin{bmatrix} 2 & 0 \\ 4 & 3 \end{bmatrix} \begin{bmatrix} 5 & 1 \\ 3 & -2 \end{bmatrix} = \begin{bmatrix} 10 & 2 \\ 29 & -2 \end{bmatrix}.$$

Note that $AB \neq BA$.

numbers	**matrices**
$ab = ba$	in general, $AB \neq BA$

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$ab = ba$	in general, $AB \neq BA$
$ab = ac \implies \begin{array}{c} a = 0 \\ \text{or} \\ b = c \end{array}$	$AB = AC \not\Longrightarrow \begin{array}{c} A = 0 \\ \text{or} \\ B = C \end{array}$

numbers	matrices
$ab = ba$	in general, $AB \neq BA$
$ab = ac \implies a = 0$ or $b = c$	$AB = AC \;\not\Longrightarrow\; A = 0$ or $B = C$
$ab = 0 \implies a = 0$ or $b = 0$	$AB = 0 \;\not\Longrightarrow\; A = 0$ or $B = 0$

Example

Let

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 2 \end{bmatrix}, \qquad B = \begin{bmatrix} 1 & 1 \\ 3 & 4 \end{bmatrix}, \qquad C = \begin{bmatrix} 2 & 5 \\ 3 & 4 \end{bmatrix}.$$

Clearly $B \neq C$ and $A \neq 0$. I leave if for you to check that

$$AB = \begin{bmatrix} 3 & 4 \\ 6 & 8 \end{bmatrix} = AC.$$

Example

Let

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 2 \end{bmatrix}, \qquad B = \begin{bmatrix} 1 & 1 \\ 3 & 4 \end{bmatrix}, \qquad C = \begin{bmatrix} 2 & 5 \\ 3 & 4 \end{bmatrix}.$$

Clearly $B \neq C$ and $A \neq 0$. I leave if for you to check that

$$AB = \begin{bmatrix} 3 & 4 \\ 6 & 8 \end{bmatrix} = AC.$$

Example

Note that

$$\begin{bmatrix} 0 & 1 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 3 & 7 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix},$$

but neither matrix on the left is the zero matrix.

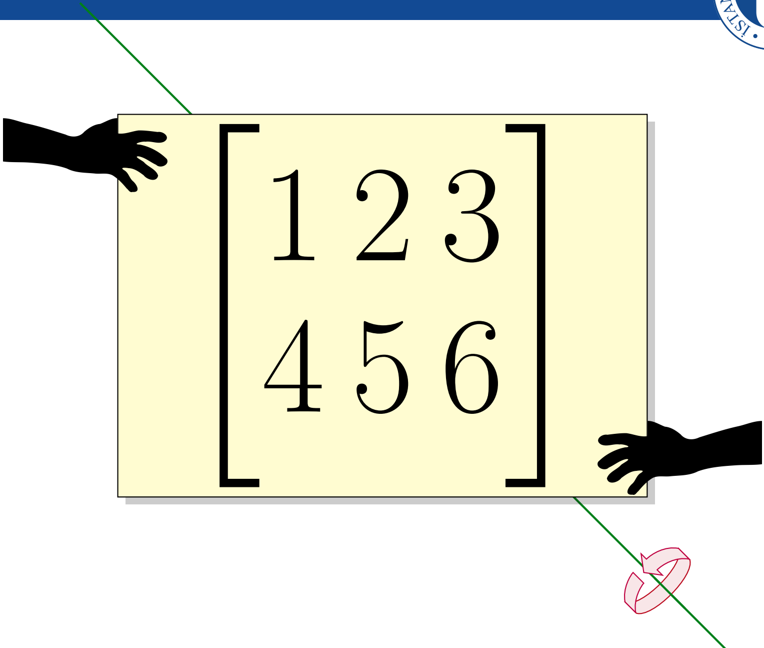
Transpose of a Matrix

Definition

If A is any $m \times n$ matrix, then the transpose of A, denoted by A^T , is defined to be the $n \times m$ matrix that results by interchanging the rows and columns of A; that is, the first column of A^T is the first row of A, the second column of A^T is the second row of A, and so forth.

$$\begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}$$



$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$$

Example

If $A = \begin{bmatrix} 2 & 3 \\ 1 & 4 \\ 5 & 6 \end{bmatrix}$, then $A^T = \begin{bmatrix} 2 & 1 & 5 \\ 3 & 4 & 6 \end{bmatrix}$.

Example

If $C = \begin{bmatrix} 1 & 3 & 5 \end{bmatrix}$, then $C^T = \begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix}$.

Example

If $D = \begin{bmatrix} 4 \end{bmatrix}$, then $D^T = \begin{bmatrix} 4 \end{bmatrix}$.

Remark

Note that since row i of A has the same entries as column i of A^T, and column j of A has the same entries as row j of A^T, we have the formula

$$\left(A^T\right)_{ij} = A_{ji}.$$

Trace of a Matrix

Definition

If A is a square matrix, then the *trace of* A, denoted $\mathrm{tr}(A)$, is defined to be the sum of the entries on the main diagonal.

(The trace of A is not defined if A is not a square matrix.)

Example

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \qquad B = \begin{bmatrix} -1 & 2 & 7 & 0 \\ 9 & 5 & 1 & 1 \\ 0 & 2 & 7 & 0 \\ 4 & 4 & 4 & 0 \end{bmatrix}$$

$$\operatorname{tr}(A) = a_{11} + a_{22} + a_{33} \qquad \operatorname{tr}(B) = -1 + 5 + 7 + 0 = 11$$

The Inverse of a Matrix

In the real numbers, every number $a \neq 0$ has a multiplicative inverse called a^{-1}. For example, if $a = 4$, then we have $4^{-1} = \frac{1}{4}$ which satisfies

$$4 \cdot 4^{-1} = 1 \qquad \text{and} \qquad 4^{-1} \cdot 4 = 1.$$

We want to generalise this idea to <u>square</u> matrices.

The Inverse of a Matrix

Definition

Let A be a square matrix. If there exists a matrix B of the same size which satisfies

$$AB = I \qquad \text{and} \qquad BA = I$$

then we say that A is *invertible* and B is called an *inverse* of A.

If such a B does not exist, then A is called *singular*.

Remark

Note that

$$\begin{array}{c} B \text{ is an} \\ \text{inverse of } A \end{array} \implies AB = I = BA \implies \begin{array}{c} A \text{ is an} \\ \text{inverse of } B \end{array}$$

So A and B are inverses of each other.

If A is invertible, then its inverse B is invertible also.

$$AB = I = BA$$

Example

Let
$$A = \begin{bmatrix} 2 & -5 \\ 1 & 3 \end{bmatrix} \qquad \text{and} \qquad B = \begin{bmatrix} 3 & 5 \\ 1 & 2 \end{bmatrix}.$$

$$AB = I = BA$$

Example

Let

$$A = \begin{bmatrix} 2 & -5 \\ 1 & 3 \end{bmatrix} \qquad \text{and} \qquad B = \begin{bmatrix} 3 & 5 \\ 1 & 2 \end{bmatrix}.$$

Then

$$AB = \begin{bmatrix} 2 & -5 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} 3 & 5 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

and

$$BA = \begin{bmatrix} 3 & 5 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 2 & -5 \\ 1 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

Therefore A and B are invertible and each is the inverse of the other.

$$AB = I = BA$$

A square matrix with a row or column or zeros is always singular.

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Example

Consider $A = \begin{bmatrix} 1 & 4 & 0 \\ 2 & 5 & 0 \\ 3 & 6 & 0 \end{bmatrix}$. I want to show that such a B does not exist.

A square matrix with a row or column or zeros is always singular.

Example

Consider $A = \begin{bmatrix} 1 & 4 & 0 \\ 2 & 5 & 0 \\ 3 & 6 & 0 \end{bmatrix}$. I want to show that such a B does not exist. We calculate that

$$BA = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \begin{bmatrix} 1 & 4 & 0 \\ 2 & 5 & 0 \\ 3 & 6 & 0 \end{bmatrix}$$

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$$BA = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \begin{bmatrix} 1 & 4 & 0 \\ 2 & 5 & 0 \\ 3 & 6 & 0 \end{bmatrix} = \begin{bmatrix} a+2b+3c & 4a+5b+6c & 0 \\ d+2e+3f & 4d+5e+6f & 0 \\ g+2h+3i & 4g+5h+6i & 0 \end{bmatrix}.$$

The 3rd column will always be full of zeros. It is not possible to choose $a, \ldots, i$ to make $(BA)_{33} = 1$.

$$AB = I = BA$$

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The 3rd column will always be full of zeros. It is not possible to choose $a, \ldots, i$ to make $(BA)_{33} = 1$. So A does not have an inverse. Therefore A is singular.

$$AB = I = BA$$

Can an invertible matrix have more than one inverse?

$AB = I = BA$

Can an invertible matrix have more than one inverse? No.

Theorem

If B and C are both inverses of A, then $B = C$.

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Theorem

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Proof.

Suppose that

$$AB = I = BA \qquad \text{and} \qquad AC = I = CA.$$

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Then we have that

$$B = BI$$

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Suppose that

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Then we have that

$$B = BI = B(AC)$$

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$\square$

The Inverse of a Matrix

Definition

~~An~~ The inverse of A is denoted by A^{-1}. Thus

$$\boxed{AA^{-1} = I = A^{-1}A.}$$

The Inverse of a Matrix

Theorem

The matrix

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

is invertible if and only if $ad - bc \neq 0$, *in which case its inverse is given by*

$$A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

We will generalise this to bigger square matrices later in the course.

$$ad - bc \neq 0 \qquad A^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

Definition

The number $ad - bc$ is called the *determinant* of $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and is denoted by

$$\det(A) = ad - bc$$

or by

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc.$$

$$\det(A) = ad - bc \neq 0 \qquad A^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

Example

Is $B = \begin{bmatrix} 6 & 1 \\ 5 & 2 \end{bmatrix}$ invertible? If so, find its inverse.

$$\det(A) = ad - bc \neq 0 \qquad A^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

Example

Is $B = \begin{bmatrix} 6 & 1 \\ 5 & 2 \end{bmatrix}$ invertible? If so, find its inverse.

The determinant of B is

$$\det(B) = 6 \cdot 2 - 1 \cdot 5 = 7 \neq 0.$$

Therefore B is invertible.

$$\det(A) = ad - bc \neq 0 \qquad A^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

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Therefore B is invertible. Its inverse is

$$B^{-1} = \frac{1}{7}\begin{bmatrix} 2 & -1 \\ -5 & 6 \end{bmatrix} = \begin{bmatrix} \frac{2}{7} & -\frac{1}{7} \\ -\frac{5}{7} & \frac{6}{7} \end{bmatrix}.$$

$$\det(A) = ad - bc \neq 0 \qquad A^{-1} = \frac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

Example

Is $C = \begin{bmatrix} -1 & 2 \\ 3 & -6 \end{bmatrix}$ invertible? If so, find its inverse.

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Example

Is $C = \begin{bmatrix} -1 & 2 \\ 3 & -6 \end{bmatrix}$ invertible? If so, find its inverse.

The determinant of C is

$$\det(C) = (-1)\cdot(-6) - 2\cdot 3 = 0.$$

Therefore C is not invertible. C is singular.

$$B = \begin{bmatrix} 6 & 1 \\ 5 & 2 \end{bmatrix} \qquad B^{-1} = \begin{bmatrix} \frac{2}{7} & -\frac{1}{7} \\ -\frac{5}{7} & \frac{6}{7} \end{bmatrix}$$

Inverse matrices can be used to solve linear systems.

Example

Solve $\begin{cases} 6x_1 + x_2 = 2 \\ 5x_1 + 2x_2 = 3. \end{cases}$

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or as

$$\begin{bmatrix} 6 & 1 \\ 5 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}.$$

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Multiplying both sides on the left by B^{-1} gives

$$\begin{bmatrix} \frac{2}{7} & -\frac{1}{7} \\ -\frac{5}{7} & \frac{6}{7} \end{bmatrix} \begin{bmatrix} 6 & 1 \\ 5 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \frac{2}{7} & -\frac{1}{7} \\ -\frac{5}{7} & \frac{6}{7} \end{bmatrix} \begin{bmatrix} 2 \\ 3 \end{bmatrix} =$$

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The Inverse of a Matrix

$$B = \begin{bmatrix} 6 & 1 \\ 5 & 2 \end{bmatrix} \qquad B^{-1} = \begin{bmatrix} \frac{2}{7} & -\frac{1}{7} \\ -\frac{5}{7} & \frac{6}{7} \end{bmatrix}$$

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Remark

In general, the linear system

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots a_{mn}x_n = b_m. \end{cases}$$

can be written as the matrix equation

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

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or just

$$A\mathbf{x} = \mathbf{b}.$$

Remark

Clearly, if A is a square matrix and is invertible, then the solution to

$$A\mathbf{x} = \mathbf{b}$$

is

$$\mathbf{x} = A^{-1}\mathbf{b}.$$

The Inverse of a Matrix

Theorem

If A and B are invertible matrices of the same size, then AB is invertible and

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Proof.

We calculate that

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We calculate that

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Theorem

If A and B are invertible matrices of the same size, then AB is invertible and

$$(AB)^{-1} = B^{-1}A^{-1}.$$

Proof.

We calculate that

$$(AB)(B^{-1}A^{-1}) = A(BB^{-1})A^{-1} = AIA^{-1} = AA^{-1} = I.$$

Similarly

$$(B^{-1}A^{-1})(AB) = B^{-1}(A^{-1}A)B = B^{-1}IB = B^{-1}B = I.$$

Therefore AB is invertible and $B^{-1}A^{-1}$ is the inverse of AB. $\square$

Remark

This is also true if we have more than 2 matrices. For example

$$(ABC)^{-1} = C^{-1}B^{-1}A^{-1},$$

$$(ABCD)^{-1} = D^{-1}C^{-1}B^{-1}A^{-1},$$

$$\vdots$$

Example

Let

$$A = \begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix} \qquad \text{and} \qquad B = \begin{bmatrix} 3 & 2 \\ 2 & 2 \end{bmatrix}.$$

I leave it for you to check the following:

$$AB = \begin{bmatrix} 7 & 6 \\ 9 & 8 \end{bmatrix} \qquad A^{-1} = \begin{bmatrix} 3 & -2 \\ -1 & 1 \end{bmatrix}$$

$$(AB)^{-1} = \begin{bmatrix} 4 & -3 \\ -\frac{9}{2} & \frac{7}{2} \end{bmatrix} \qquad B^{-1} = \begin{bmatrix} 1 & -1 \\ -1 & \frac{3}{2} \end{bmatrix}$$

$$B^{-1}A^{-1} = \begin{bmatrix} 1 & -1 \\ -1 & \frac{3}{2} \end{bmatrix} \begin{bmatrix} 3 & -2 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 4 & -3 \\ -\frac{9}{2} & \frac{7}{2} \end{bmatrix}$$

Powers of a Matrix

Definition

Let A is a square matrix and let $k \in \mathbb{N}$. We define

$$A^0 = I \qquad \text{and} \qquad A^k = \underbrace{A \cdot A \cdot \ldots \cdot A}_{k}.$$

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Note that

$$A^r A^s = A^{r+s} \qquad \text{and} \qquad (A^r)^s = A^{rs}$$

if $r, s \geq 0$ as expected.

Theorem

Let A be an invertible matrix. Let $n \in \mathbb{N} \cup \{0\}$ and $k \neq 0$. Then:

1. *A^{-1} is invertible and $(A^{-1})^{-1} = A$.*

The Inverse of a Matrix

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Let A be an invertible matrix. Let $n \in \mathbb{N} \cup \{0\}$ and $k \neq 0$. Then:

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2. *A^n is invertible and $(A^n)^{-1} = A^{-n} = (A^{-1})^n$.*

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Let A be an invertible matrix. Let $n \in \mathbb{N} \cup \{0\}$ and $k \neq 0$. Then:

1. A^{-1} *is invertible and* $(A^{-1})^{-1} = A$.
2. A^n *is invertible and* $(A^n)^{-1} = A^{-n} = (A^{-1})^n$.
3. kA *is invertible and* $(kA)^{-1} = k^{-1}A^{-1}$.

I will prove part 3. I leave parts 1 and 2 for you to prove.

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I will prove part 3. I leave parts 1 and 2 for you to prove.

Proof of Part 3.

We have that

$$(kA)(k^{-1}A^{-1}) = k^{-1}(kA)A^{-1} = (k^{-1}k)(AA^{-1}) = (1)(I) = I.$$

Similarly $(k^{-1}A^{-1})(kA) = I$. $\square$

Example

Let $A = \begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix}$ and $A^{-1} = \begin{bmatrix} 3 & -2 \\ -1 & 1 \end{bmatrix}$. Then

$$\begin{aligned} A^{-3} = A^{-1}A^{-1}A^{-1} &= \begin{bmatrix} 3 & -2 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 3 & -2 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 3 & -2 \\ -1 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 41 & -30 \\ -15 & 11 \end{bmatrix}. \end{aligned}$$

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Moreover

$$A^3 = AAA = \begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix} = \begin{bmatrix} 11 & 30 \\ 15 & 41 \end{bmatrix}$$

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and

$$(A^3)^{-1} = \frac{1}{(11 \cdot 41 - 30 \cdot 15)} \begin{bmatrix} 41 & -30 \\ -15 & 11 \end{bmatrix} = \begin{bmatrix} 41 & -30 \\ -15 & 11 \end{bmatrix} = (A^{-1})^3$$

as expected.

Remark

If a and b are numbers then $ab = ba$, so we can calculate

$$(a+b)^2 = a^2 + ab + ba + b^2 = a^2 + ab + ab + b^2 = a^2 + 2ab + b^2.$$

Remark

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$(a+b)^2 = a^2 + ab + ba + b^2 = a^2 + ab + ab + b^2 = a^2 + 2ab + b^2.$

But remember that $AB \neq BA$ in general for matrices. So the best that we can do is

$$(A+B)^2 = A^2 + AB + BA + B^2.$$

Properties of the Transpose

Theorem

(If the sizes are correct, then)

1. $(A^T)^T = A$
2. $(A + B)^T = A^T + B^T$
3. $(kA)^T = kA^T$

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Properties of the Transpose

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4. $(AB)^T = B^T A^T$

Theorem

If A is an invertible matrix, then A^T is also invertible and

$$(A^T)^{-1} = (A^{-1})^T.$$

Next Time

- An algorithm for finding A^{-1}
- More About Invertible Matrices
- The Invertible Matrix Theorem
- Diagonal, Triangular, and Symmetric Matrices
- Some Applications of Linear Algebra.