Lecture 7

- Rank and Nullity
- The Fundamental Matrix Spaces
- Linear Transformations
- Composition and Inverse Transformations
- Isomorphisms

Rank and Nullity

Recall, to find a basis for the row space of a matrix A, we reduce it to REF, then take the rows with a pivot.

$$A = \begin{bmatrix} 1 & -3 & 4 & -2 & 5 & 4 \\ 2 & -6 & 9 & -1 & 8 & 2 \\ 2 & -6 & 9 & -1 & 9 & 7 \\ -1 & 3 & -4 & 2 & -5 & -4 \end{bmatrix} \sim R = \begin{bmatrix} 1 & -3 & 4 & -2 & 5 & 4 \\ 0 & 0 & 1 & 3 & -2 & -6 \\ 0 & 0 & 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

For example

$$\begin{aligned} \mathbf{r}_1 &= \begin{bmatrix} 1 & -3 & 4 & -2 & 5 & 4 \end{bmatrix} \\ \mathbf{r}_2 &= \begin{bmatrix} 0 & 0 & 1 & 3 & -2 & -6 \end{bmatrix} \\ \mathbf{r}_3 &= \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 5 \end{bmatrix} \end{aligned}$$

form a basis for Row A = Row R. Since R has three pivots, the dimension of Row A is 3.

And to find a basis for a column space, we take the pivot columns.

$$A = \begin{bmatrix} 1 & -3 & 4 & -2 & 5 & 4 \\ 2 & -6 & 9 & -1 & 8 & 2 \\ 2 & -6 & 9 & -1 & 9 & 7 \\ -1 & 3 & -4 & 2 & -5 & -4 \end{bmatrix} \sim R = \begin{bmatrix} 1 & -3 & 4 & -2 & 5 & 4 \\ 0 & 0 & 1 & 3 & -2 & -6 \\ 0 & 0 & 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

The the pivot columns of A are also the first, third and fifth columns. Hence

$$\left\{ \mathbf{c}_1 = \begin{bmatrix} 1 \\ 2 \\ 2 \\ -1 \end{bmatrix}, \ \mathbf{c}_3 = \begin{bmatrix} 4 \\ 9 \\ 9 \\ -4 \end{bmatrix}, \ \mathbf{c}_5 = \begin{bmatrix} 5 \\ 8 \\ 9 \\ -5 \end{bmatrix} \right\}$$

is a basis for $\operatorname{Col} A$. Since A has three pivot columns, the dimension of $\operatorname{Col} A$ is 3.

Theorem

The row space and the column space of a matrix A have the same dimension.

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Rank and Nullity

Definition

The *rank* of a matrix A is

$$\operatorname{rank} A = \dim(\operatorname{Col} A) = \dim(\operatorname{Row} A).$$

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$$\operatorname{rank} A = \dim(\operatorname{Col} A) = \dim(\operatorname{Row} A).$$

Definition

The *nullity* of A is

$$\operatorname{nullity} A = \dim(\operatorname{Nul} A).$$

Example

Find the rank and nullity of the matrix

$$A = \begin{bmatrix} -1 & 2 & 0 & 4 & 5 & -3 \\ 3 & -7 & 2 & 0 & 1 & 4 \\ 2 & -5 & 2 & 4 & 6 & 1 \\ 4 & -9 & 2 & -4 & -4 & 7 \end{bmatrix}.$$

Example

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Please check that A is row equivalent to the RREF matrix

$$R = \begin{bmatrix} 1 & 0 & -4 & -28 & -37 & 13 \\ 0 & 1 & -2 & -12 & -16 & 5 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

The matrix

$$R = \begin{bmatrix} 1 & 0 & -4 & -28 & -37 & 13 \\ 0 & 1 & -2 & -12 & -16 & 5 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

has two pivots. So both Row A and Col A are two dimensional. Hence rank $A = 2$.

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has two pivots. So both $\mathrm{Row}\,A$ and $\mathrm{Col}\,A$ are two dimensional. Hence $\mathrm{rank}\,A = 2$.

To find the nullity of A, we need to find the dimension of the solution space of $A\mathbf{x} = \mathbf{0}$.

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has two pivots. So both $\mathrm{Row}\, A$ and $\mathrm{Col}\, A$ are two dimensional. Hence $\mathrm{rank}\, A = 2$.

To find the nullity of A, we need to find the dimension of the solution space of $A\mathbf{x} = \mathbf{0}$. We can see from R that the general solution of $A\mathbf{x} = \mathbf{0}$ is

$$\begin{cases} x_1 = 4x_3 + 28x_4 + 37x_5 - 13x_6 \\ x_2 = 2x_3 + 12x_4 + 16x_5 - 5x_6 \\ x_3, x_4, x_5, x_6 \text{ are free.} \end{cases}$$

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In vector form, the general solution is

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} = r \begin{bmatrix} 4 \\ 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + s \begin{bmatrix} 28 \\ 12 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} 37 \\ 16 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} + u \begin{bmatrix} -13 \\ -5 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

for any $r, s, t, u \in \mathbb{R}$.

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for any $r, s, t, u \in \mathbb{R}$. These four vectors (on the right hand side) form a basis for $\operatorname{Nul} A$. Hence nullity $A = 4$.

Rank and Nullity

Note that

$$\operatorname{rank} A = \text{number of pivot positions}$$

and

$$\text{nullity}\, A = \text{number of free variables.}$$

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Theorem

If A is an $m \times n$ matrix (i.e. with n columns), then

$$\operatorname{rank} A + \operatorname{nullity} A = n.$$

$$\text{rank}\, A + \text{nullity}\, A = \text{number of columns}$$

Example

The matrix

$$A = \begin{bmatrix} -1 & 2 & 0 & 4 & 5 & -3 \\ 3 & -7 & 2 & 0 & 1 & 4 \\ 2 & -5 & 2 & 4 & 6 & 1 \\ 4 & -9 & 2 & -4 & -4 & 7 \end{bmatrix}$$

has 6 columns. Therefore

$$\text{rank}\, A + \text{nullity}\, A = 6.$$

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$$\text{rank } A + \text{nullity } A = 6.$$

In the previous example we showed that $\text{rank } A = 2$ and $\text{nullity } A = 4$.

$$\text{rank}\, A + \text{nullity}\, A = \text{number of columns}$$

Ask the audience 1/2

Suppose that A is a 5×7 matrix of rank 3. Find the nullity of A.

1. nullity $A = 1$
2. nullity $A = 2$
3. nullity $A = 3$
4. nullity $A = 4$

$$\text{rank } A + \text{nullity } A = \text{number of columns}$$

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$\text{rank}\, A + \text{nullity}\, A = \text{number of columns}$

Ask the audience 2/2

Suppose that the solution space of $B\mathbf{x} = \mathbf{0}$ is two dimensional. If B is a 5×7, find the rank of B.

1. $\text{rank}\, B = 4$
2. $\text{rank}\, B = 5$
3. $\text{rank}\, B = 6$
4. $\text{rank}\, B = 7$

rank A + nullity A = number of columns

Ask the audience 2/2

Suppose that the solution space of $B\mathbf{x} = \mathbf{0}$ is two dimensional. If B is a 5×7, find the rank of B.

1. rank $B = 4$
2. rank $B = 5$
3. rank $B = 6$
4. rank $B = 7$

Theorem

If $A\mathbf{x} = \mathbf{b}$ is a consistent linear system of m equations in n unknowns, and if A has rank r, then the general solution of the system contains $n - r$ free variables (parameters).

The Fundamental Matrix Spaces

Consider the following six vector spaces:

- the row space of A
- the column space of A
- the null space of A
- the row space of A^T
- the column space of A^T
- the null space of A^T

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- ~~the column space of A^T~~
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When we take the transpose of a matrix, rows become columns and columns become rows. That means that

$$\mathrm{Row}\, A = \mathrm{Col}\, A^T$$

and

$$\mathrm{Col}\, A = \mathrm{Row}\, A^T.$$

Definition

The *fundamental spaces* of a matrix A are

1. the row space of A
2. the column space of A
3. the null space of A
4. the null space of A^T.

We will talk about how these four vector spaces are related.

Theorem

If A is any matrix, then

$$\operatorname{rank} A = \operatorname{rank} A^T.$$

The Fundamental Matrix Spaces

Theorem

If A is any matrix, then

$$\operatorname{rank} A = \operatorname{rank} A^T.$$

Proof.

$$\begin{aligned}\operatorname{rank} A &= \dim(\text{row space of } A)\\ &= \dim(\text{column space of } A^T) = \operatorname{rank} A^T.\end{aligned}$$

$\square$

If A is an $m \times n$ matrix, then A^T is an $n \times m$ matrix.

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If $\operatorname{rank} A = r$, then

$$\dim(\operatorname{Row} A) = r \qquad \dim(\operatorname{Col} A) = r$$
$$\dim(\operatorname{Nul} A) = n - r \qquad \dim\left(\operatorname{Nul} A^T\right) = m - r.$$

Now suppose that **x** is a solution of

$$\begin{bmatrix} 1 & 3 & -2 & 0 & 2 & 0 \\ 2 & 6 & -5 & -2 & 4 & -3 \\ 0 & 0 & 5 & 10 & 0 & 15 \\ 2 & 6 & 0 & 8 & 4 & 18 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

Now suppose that $\mathbf{x}$ is a solution of

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Note that

$$\mathbf{r}_1 \cdot \mathbf{x} = (1, 3, -2, 0, 2, 0) \cdot (x_1, x_2, x_3, x_4, x_5, x_6)$$

$$=$$

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So

$$\mathbf{x} \text{ solves } A\mathbf{x} = \mathbf{0} \quad \Longrightarrow \quad \mathbf{x} \text{ is orthogonal to } \mathbf{r}_1.$$

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Similarly $\mathbf{r}_2 \cdot \mathbf{x} = 0$, $\mathbf{r}_3 \cdot \mathbf{x} = 0$ and $\mathbf{r}_4 \cdot \mathbf{x} = 0$.

The Fundamental Matrix Spaces

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Similarly $\mathbf{r}_2 \cdot \mathbf{x} = 0$, $\mathbf{r}_3 \cdot \mathbf{x} = 0$ and $\mathbf{r}_4 \cdot \mathbf{x} = 0$.

Theorem

If A is an $m \times n$ matrix, then the null space of A (i.e. the solution space of $A\mathbf{x} = \mathbf{0}$) consists of all vectors in $\mathbb{R}^n$ that are orthogonal to every row vector of A.

Definition

If W is a subspace of $\mathbb{R}^n$, then the set of all vectors in $\mathbb{R}^n$ that are orthogonal to every vector in W is called the orthogonal complement of W and is denoted by the symbol $W^\perp$.

In other words,

$$W^\perp = \{\mathbf{v} \in V \mid \mathbf{v} \bullet \mathbf{w} = 0 \text{ for all } \mathbf{w} \in W\}.$$

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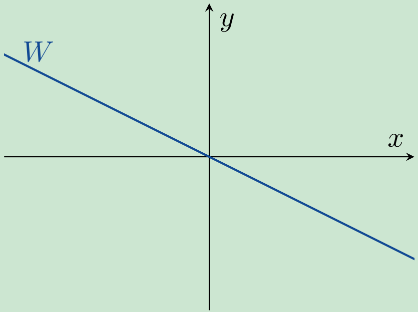
Theorem

If W is a subspace of $\mathbb{R}^n$, then:

1. $W^\perp$ *is also a subspace of* $\mathbb{R}^n$.
2. $W \cap W^\perp = \{\mathbf{0}\}$.
3. $\left(W^\perp\right)^\perp = W$.

Example

In $\mathbb{R}^2$:



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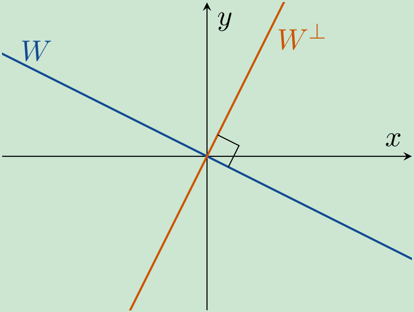
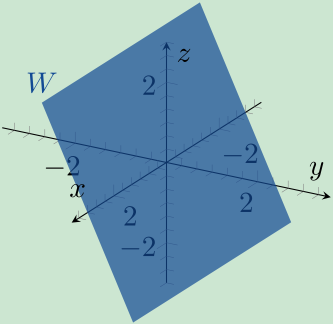


Example

In $\mathbb{R}^3$:



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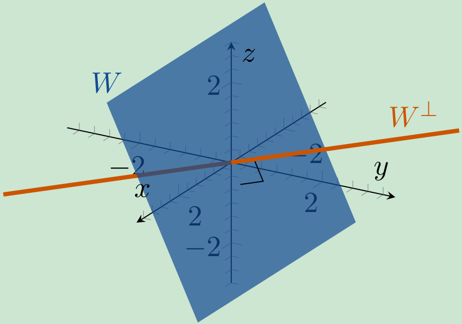

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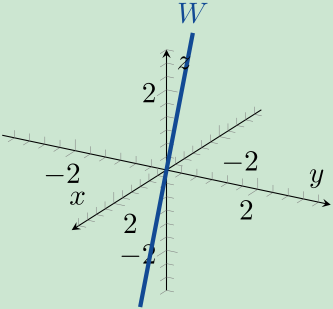

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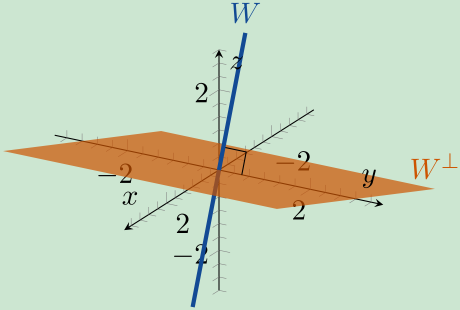

Remark

Note that

$$V^{\perp} = \{\mathbf{0}\}$$

because only the zero vector is orthogonal to every vector in V; and hence

$$\{\mathbf{0}\}^{\perp} = V.$$

Theorem

Let A be an $m \times n$ matrix. Then

1. *The null space of A is the orthogonal complement of the row space of A, in $\mathbb{R}^n$.*

$$\operatorname{Nul} A = (\operatorname{Row} A)^\perp \qquad \textit{and} \qquad \operatorname{Row} A = (\operatorname{Nul} A)^\perp .$$

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2. *The null space of A^T is the orthogonal complement of the column space of A, in $\mathbb{R}^m$.*

$$\operatorname{Nul} A^T = (\operatorname{Col} A)^{\perp} \qquad \text{and} \qquad \operatorname{Col} A = \left(\operatorname{Nul} A^T\right)^{\perp}.$$

Theorem

Let A be an $n \times n$ matrix. The following statements are equivalent:

1. *A is invertible.*
2. *$A\mathbf{x} = \mathbf{0}$ has only the trivial solution.*
3. *The reduced row echelon form of A is I_n.*
4. *A is expressible as a product of elementary matrices.*
5. *$A\mathbf{x} = \mathbf{b}$ is consistent for every $n \times 1$ matrix $\mathbf{b}$.*
6. *$A\mathbf{x} = \mathbf{b}$ has exactly one solution for every $n \times 1$ matrix $\mathbf{b}$.*
7. $\det(A) \neq 0$.
8.
9.
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The Fundamental Matrix Spaces

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7. $\det(A) \neq 0$.
8. *The column vectors of A are linearly independent.*
9. *The row vectors of A are linearly independent.*
10. *The column vectors of A span $\mathbb{R}^n$.*
11. *The row vectors of A span $\mathbb{R}^n$.*
12. *The column vectors of A form a basis for $\mathbb{R}^n$.*
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14. $\operatorname{rank} A = n$.
15. $\operatorname{nullity} A = 0$.
16. $(\operatorname{Nul} A)^\perp = \mathbb{R}^n$.
17. $(\operatorname{Row} A)^\perp = \{\mathbf{0}\}$.

Linear Transformations

Definition

Let V and W be vector spaces. A function $T : V \to W$ is called a *linear transformation* iff

1. $T(\mathbf{u} + \mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v})$
2. $T(k\mathbf{u}) = kT(\mathbf{u})$; and

for all vectors $\mathbf{u}, \mathbf{v} \in V$ and all scalars k.

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Definition

If $V = W$, then the linear transformation $T : V \to V$ is called a *linear operator* on V.

$$T(\mathbf{u}+\mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v}) \text{ and } T(k\mathbf{u}) = kT(\mathbf{u})$$

Remark

I have tried to avoid any mention of linear transformations up to now. Your textbook introduces them in section 1.8.

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"Transformation" is just another word for "function". The important word in "linear transformation" is "linear".

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Remark

We can combine rules 1 and 2 to show that

$$T(k_1\mathbf{v}_1 + k_2\mathbf{v}_2) = k_1T(\mathbf{v}_1) + k_2T(\mathbf{v}_2)$$

$T(\mathbf{u}+\mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v})$ and $T(k\mathbf{u}) = kT(\mathbf{u})$

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Remark

We can combine rules 1 and 2 to show that

$$T(k_1\mathbf{v}_1 + k_2\mathbf{v}_2) = k_1T(\mathbf{v}_1) + k_2T(\mathbf{v}_2)$$

or more generally that

$$T(k_1\mathbf{v}_1 + k_2\mathbf{v}_2 + \ldots + k_r\mathbf{v}_r) = k_1T(\mathbf{v}_1) + k_2T(\mathbf{v}_2) + \ldots + k_rT(\mathbf{v}_r).$$

$T(\mathbf{u}+\mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v})$ and $T(k\mathbf{u}) = kT(\mathbf{u})$

Theorem

If $T : V \to W$ is a linear transformation, then

$$T(\mathbf{0}) = \mathbf{0}.$$

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$$T(\mathbf{0}) = \mathbf{0}.$$

Proof.

Let $\mathbf{u} \in V$ be any vector. Since $0\mathbf{u} = \mathbf{0}$, it follows that

$$T(\mathbf{0}) = T(0\mathbf{u}) = 0T(\mathbf{u}) = \mathbf{0}.$$

□

$T(\mathbf{u}+\mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v})$ and $T(k\mathbf{u}) = kT(\mathbf{u})$

Example (Matrix Transformations)

Let A be an $m \times n$ matrix. We can define a function $T_A : \mathbb{R}^n \to \mathbb{R}^m$ by

$$T_A(\mathbf{x}) = A\mathbf{x}.$$

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Note that if $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$ are any vectors, and k is any number, then

$$T_A(\mathbf{u}+\mathbf{v}) = A(\mathbf{u}+\mathbf{v}) = A\mathbf{u} + A\mathbf{v} = T_A(\mathbf{u}) + T_A(\mathbf{v})$$

and

$$T_A(k\mathbf{u}) = A(k\mathbf{u}) = k(A\mathbf{u}) = kT_A(\mathbf{u}).$$

Therefore T_A is a linear transformation.

$T(\mathbf{u}+\mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v})$ and $T(k\mathbf{u}) = kT(\mathbf{u})$

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Therefore T_A is a linear transformation.

Functions of the form $T_A : \mathbb{R}^n \to \mathbb{R}^m$ are called *matrix transformations*.

$T(\mathbf{u}+\mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v})$ and $T(k\mathbf{u}) = kT(\mathbf{u})$

Example (The Zero Transformation)

Let V and W be any two vector spaces. Define a function $T : V \to W$ by $T(\mathbf{v}) = \mathbf{0}$ for all $\mathbf{v} \in V$.

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for all $\mathbf{u}$, $\mathbf{v}$ and k. Hence T is linear.

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for all $\mathbf{u}$, $\mathbf{v}$ and k. Hence T is linear.

Example (The Identity Operator)

Let V be any vector space. The function $I : V \to V$ defined by $I(\mathbf{v}) = \mathbf{v}$ is called the *identity operator* on V. I leave it to you to show that I is linear.

$T(\mathbf{u}+\mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v})$ and $T(k\mathbf{u}) = kT(\mathbf{u})$

Example (A Linear Transformation from $\mathbb{P}^n$ to $\mathbb{P}^{n+1}$)

Let $\mathbf{p} = a_0 + a_1 x + \ldots + a_n x^n$ be a polynomial in $\mathbb{P}^n$. Define $T : \mathbb{P}^n \to \mathbb{P}^{n+1}$ by $T(\mathbf{p}) = x\mathbf{p}$. That is

$$T(\mathbf{p}) = a_0 x + a_1 x^2 + \ldots + a_n x^{n+1}.$$

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$$T(\mathbf{p}) = a_0 x + a_1 x^2 + \ldots + a_n x^{n+1}.$$

This function is a linear transformation because

$$T(\mathbf{p}_1 + \mathbf{p}_2) = x(\mathbf{p}_1 + \mathbf{p}_2) = x\mathbf{p}_1 + x\mathbf{p}_2 = T(\mathbf{p}_1) + T(\mathbf{p}_2)$$

and

$$T(k\mathbf{p}) = x(k\mathbf{p}) = k(x\mathbf{p}) = kT(\mathbf{p}).$$

$T(\mathbf{u}+\mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v})$ and $T(k\mathbf{u}) = kT(\mathbf{u})$

Example

Let $\mathbb{R}^{n\times n} = M_{nn}$ be the vector space of $n \times n$ matrices.

1. Is $T_1 : \mathbb{R}^{n\times n} \to \mathbb{R}^{n\times n}$, $T_1(A) = A^T$ linear?
2. Is $T_2 : \mathbb{R}^{n\times n} \to \mathbb{R}$, $T_2(A) = \det(A)$ linear?

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1. Since

$$T_1(A+B) = (A+B)^T = A^T + B^T = T_1(A) + T_1(B)$$

and

$$T_2(kA) = (kA)^T = kA^T = kT_2(A)$$

the answer is YES.

$T(\mathbf{u}+\mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v})$ and $T(k\mathbf{u}) = kT(\mathbf{u})$

Example

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1. Since
$$T_1(A+B) = (A+B)^T = A^T + B^T = T_1(A) + T_1(B)$$
and
$$T_2(kA) = (kA)^T = kA^T = kT_2(A)$$
the answer is YES.
2. Recall from Lecture 4 that
$$T_2(kA) = \det(kA) = k^n \det(A) = k^n T_2(A).$$
If $n \geq 2$, then we have $T_2(kA) \neq kT_2(A)$. So the answer is NO (if $n \geq 2$).

$T(\mathbf{u}+\mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v})$ and $T(k\mathbf{u}) = kT(\mathbf{u})$

Recall that Linear Transformations always have $T(\mathbf{0}) = \mathbf{0}$.

Example (A Nonlinear Transformation)

Let $\mathbf{b} \in \mathbb{R}^2$ such that $\mathbf{b} \neq \mathbf{0}$. Define $T : \mathbb{R}^2 \to \mathbb{R}^2$ by $T(\mathbf{x}) = \mathbf{x} + \mathbf{b}$. Note that T is not linear because

$$T(\mathbf{0}) = \mathbf{b} \neq \mathbf{0}.$$

$T(\mathbf{u}+\mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v})$ and $T(k\mathbf{u}) = kT(\mathbf{u})$

Example

$\frac{d}{dx} : C^1(-\infty,\infty) \to F(-\infty,\infty)$ is a linear operator because

$$\frac{d}{dx}(f+g) = \frac{d}{dx}(f) + \frac{d}{dx}(g)$$

and

$$\frac{d}{dx}(kf) = k\frac{d}{dx}(f).$$

Finding Linear Transformations from Images of Basis Vectors

Theorem

Let $T : V \to W$ be a linear transformation, where V is finite-dimensional. If $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is a basis for V, then the image of any vector $\mathbf{v} \in V$ can be expressed as

$$T(\mathbf{v}) = c_1 T(\mathbf{v}_1) + c_2 T(\mathbf{v}_2) + \ldots + c_n T(\mathbf{v}_n)$$

where $c_1, c_2, \ldots, c_n$ are the coefficients required to express $\mathbf{v}$ as a linear combination of the vectors in the basis S.

▶ EXAMPLE 10 Computing with Images of Basis Vectors

Consider the basis $S = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ for R^3, where

$$\mathbf{v}_1 = (1, 1, 1), \quad \mathbf{v}_2 = (1, 1, 0), \quad \mathbf{v}_3 = (1, 0, 0)$$

Let $T : R^3 \rightarrow R^2$ be the linear transformation for which

$$T(\mathbf{v}_1) = (1, 0), \quad T(\mathbf{v}_2) = (2, -1), \quad T(\mathbf{v}_3) = (4, 3)$$

Find a formula for $T(x_1, x_2, x_3)$, and then use that formula to compute $T(2, -3, 5)$.

Solution We first need to express $\mathbf{x} = (x_1, x_2, x_3)$ as a linear combination of $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$. If we write

$$(x_1, x_2, x_3) = c_1(1, 1, 1) + c_2(1, 1, 0) + c_3(1, 0, 0)$$

then on equating corresponding components, we obtain

$$\begin{aligned} c_1 + c_2 + c_3 &= x_1 \\ c_1 + c_2 &= x_2 \\ c_1 &= x_3 \end{aligned}$$

which yields $c_1 = x_3$, $c_2 = x_2 - x_3$, $c_3 = x_1 - x_2$, so

$$\begin{aligned} (x_1, x_2, x_3) &= x_3(1, 1, 1) + (x_2 - x_3)(1, 1, 0) + (x_1 - x_2)(1, 0, 0) \\ &= x_3\mathbf{v}_1 + (x_2 - x_3)\mathbf{v}_2 + (x_1 - x_2)\mathbf{v}_3 \end{aligned}$$

Thus

$$\begin{aligned} T(x_1, x_2, x_3) &= x_3 T(\mathbf{v}_1) + (x_2 - x_3)T(\mathbf{v}_2) + (x_1 - x_2)T(\mathbf{v}_3) \\ &= x_3(1, 0) + (x_2 - x_3)(2, -1) + (x_1 - x_2)(4, 3) \\ &= (4x_1 - 2x_2 - x_3, 3x_1 - 4x_2 + x_3) \end{aligned}$$

From this formula we obtain

$$T(2, -3, 5) = (9, 23)$$

Kernal and Range

Definition

If $T : V \to W$ is a linear transformation, then the set of vectors in V that T maps to $\mathbf{0}$ is called the kernel of T and is denoted by $\ker(T)$.

$$\ker(T) = \{\mathbf{v} \in V \mid T(\mathbf{v}) = \mathbf{0}\}$$

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$$\ker(T) = \{\mathbf{v} \in V \mid T(\mathbf{v}) = \mathbf{0}\}$$

Definition

The set of all vectors in W that are images under T of at least one vector in V is called the range of T and is denoted by $R(T)$.

$$R(T) = \{\mathbf{w} \in W \mid \exists\ \mathbf{v} \in V \text{ such that } T(\mathbf{v}) = \mathbf{w}\}$$

Example (Matrix Transformations)

If $T_A : \mathbb{R}^n \to \mathbb{R}^m$ is a matrix transformation (i.e. $T_A(\mathbf{x}) = A\mathbf{x}$), then

$$\ker(T_A) = \operatorname{Nul} A \quad \text{and} \quad R(T_A) = \operatorname{Col}(A).$$

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$$\ker(T_A) = \operatorname{Nul} A \quad \text{and} \quad R(T_A) = \operatorname{Col}(A).$$

Example (Zero Transformation)

Let $T : V \to W$ be the zero transformation. Since T maps every vector in V to $\mathbf{0}$, it follows that

$$\ker(T) = V \quad \text{and} \quad R(T) = \{\mathbf{0}\}.$$

Example (Kernel and Range of the Identity Operator)

Let $I : V \to V$ be the identity operator. Since $I(\mathbf{v}) = \mathbf{v}$ for all vectors in V, every vector in V is the image of some vector (namely, itself); thus

$$R(I) = V.$$

Since the only vector that I maps to $\mathbf{0}$ is $\mathbf{0}$ itself, it follows that

$$\ker(I) = \{\mathbf{0}\}.$$

Example

Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be the orthogonal projection onto the xy-plane (i.e. $T(x, y, z) = (x, y, 0)$).

Example

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$$\begin{aligned}(x, y, z) \in \ker(T) &\implies (0, 0, 0) = T(x, y, z) = (x, y, 0) \\ &\implies x = y = 0.\end{aligned}$$

Example

Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be the orthogonal projection onto the xy-plane (i.e. $T(x, y, z) = (x, y, 0)$). Then

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Therefore

$$\ker(T) = \{(0, 0, z) \mid z \in \mathbb{R}\}.$$

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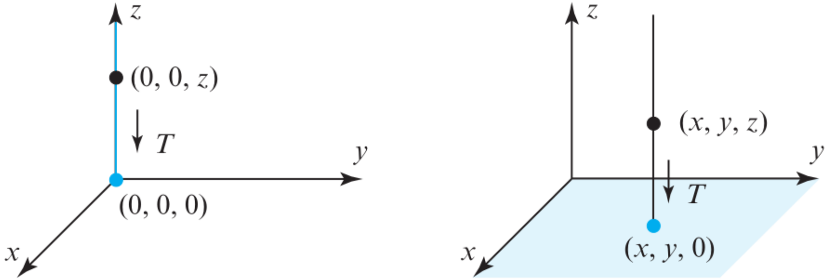
Therefore

$$\ker(T) = \{(0, 0, z) \mid z \in \mathbb{R}\}.$$

Moreover $R(T)$ is the set of points of the form $(x, y, 0)$.

Example

Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be the orthogonal projection onto the xy-plane (i.e. $T(x, y, z) = (x, y, 0)$). Then



(a) $\ker(T)$ is the z-axis. (b) $R(T)$ is the entire xy-plane

▶ EXAMPLE 17 Kernel and Range of a Rotation

Let $T: R^2 \rightarrow R^2$ be the linear operator that rotates each vector in the xy-plane through the angle θ (Figure 8.1.3). Since *every* vector in the xy-plane can be obtained by rotating some vector through the angle θ, it follows that $R(T) = R^2$. Moreover, the only vector that rotates into $\mathbf{0}$ is $\mathbf{0}$, so $\ker(T) = \{\mathbf{0}\}$.

Properties of Kernal and Range

Theorem

If $T : V \to W$ is a linear transformation, then:

1. *The kernel of T is a subspace of V.*
2. *The range of T is a subspace of W.*

Rank and Nullity

Let $T : V \to W$ be a linear transformation.

Definition

If the range of T is finite-dimensional, then its dimension is called the *rank of* T.

Definition

If the kernel of T is finite-dimensional, then its dimension is called the nullity of T.

Theorem

If $T : V \to W$ is a linear transformation from a finite-dimensional vector space V to a vector space W, then the range of T is finite-dimensional, and

$$\operatorname{rank} T + \operatorname{nullity} T = \dim(V).$$

Break

We will continue at 3pm

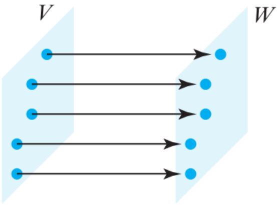
"The next part of this recipe will involve some calculus."

Composition and Inverse Transformations

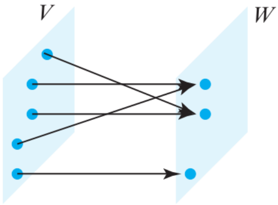
One-to-One and Onto

Definition

If $T : V \to W$ is a linear transformation from a vector space V to a vector space W, then T is said to be *one-to-one* if T maps distinct vectors in V into distinct vectors in W.



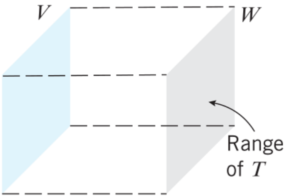
One-to-one. Distinct vectors in V have distinct images in W.



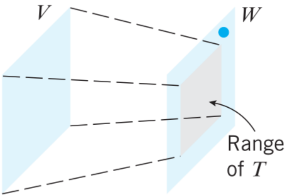
Not one-to-one. There exist distinct vectors in V with the same image.

Definition

If $T : V \to W$ is a linear transformation from a vector space V to a vector space W, then T is said to be *onto* if every vector in W is the image of at least one vector in V.



Onto W. Every vector in W is the image of some vector in V.



Not onto W. Not every vector in W is the image of some vector in V.

Theorem

If $T : V \to W$ is a linear transformation, then the following statements are equivalent:

1. *T is one-to-one.*
2. $\ker(T) = \{\mathbf{0}\}$.

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If $T : V \to W$ is a linear transformation, then the following statements are equivalent:

1. *T is one-to-one.*
2. $\ker(T) = \{\mathbf{0}\}$.

Proof.

1 $\Longrightarrow$ 2: We know that $T(\mathbf{0}) = \mathbf{0}$ because T is linear. But if T is one-to-one, then $\mathbf{0}$ is the only vector that maps to $\mathbf{0}$. Hence $\ker(T) = \{\mathbf{0}\}$.

Composition and Inverse Transformations

Theorem

If $T : V \to W$ is a linear transformation, then the following statements are equivalent:

1. *T is one-to-one.*
2. $\ker(T) = \{\mathbf{0}\}$.

Proof.

1 $\implies$ 2: We know that $T(\mathbf{0}) = \mathbf{0}$ because T is linear. But if T is one-to-one, then $\mathbf{0}$ is the only vector that maps to $\mathbf{0}$. Hence $\ker(T) = \{\mathbf{0}\}$.

1 $\impliedby$ 2: Suppose that $\ker(T) = \{\mathbf{0}\}$. Let $\mathbf{u}, \mathbf{v}$ be two different vectors in V. So $\mathbf{u} - \mathbf{v} \neq \mathbf{0}$ which implies that $T(\mathbf{u} - \mathbf{v}) \neq \mathbf{0}$.

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Proof.

1 $\Longrightarrow$ 2: We know that $T(\mathbf{0}) = \mathbf{0}$ because T is linear. But if T is one-to-one, then $\mathbf{0}$ is the only vector that maps to $\mathbf{0}$. Hence $\ker(T) = \{\mathbf{0}\}$.

1 $\Longleftarrow$ 2: Suppose that $\ker(T) = \{\mathbf{0}\}$. Let $\mathbf{u}, \mathbf{v}$ be two different vectors in V. So $\mathbf{u} - \mathbf{v} \neq \mathbf{0}$ which implies that $T(\mathbf{u} - \mathbf{v}) \neq \mathbf{0}$. But then

$$T(\mathbf{u}) - T(\mathbf{v}) = T(\mathbf{u} - \mathbf{v}) \neq \mathbf{0}$$

Theorem

If $T : V \to W$ is a linear transformation, then the following statements are equivalent:

1. *T is one-to-one.*
2. $\ker(T) = \{\mathbf{0}\}$.

Proof.

1 $\implies$ 2: We know that $T(\mathbf{0}) = \mathbf{0}$ because T is linear. But if T is one-to-one, then $\mathbf{0}$ is the only vector that maps to $\mathbf{0}$. Hence $\ker(T) = \{\mathbf{0}\}$.

1 $\impliedby$ 2: Suppose that $\ker(T) = \{\mathbf{0}\}$. Let $\mathbf{u}, \mathbf{v}$ be two different vectors in V. So $\mathbf{u} - \mathbf{v} \neq \mathbf{0}$ which implies that $T(\mathbf{u} - \mathbf{v}) \neq \mathbf{0}$. But then

$$T(\mathbf{u}) - T(\mathbf{v}) = T(\mathbf{u} - \mathbf{v}) \neq \mathbf{0} \quad \implies \quad T(\mathbf{u}) \neq T(\mathbf{v}).$$

□

Theorem

If V and W are finite-dimensional vector spaces with the same dimension, and if $T : V \to W$ is a linear transformation, then the following statements are equivalent:

1. *T is one-to-one.*
2. $\ker(T) = \{\mathbf{0}\}$.
3. *T is onto [i.e., $R(T) = W$].*

(proof omitted.)

Example (Matrix Transformations)

Consider $T_A : \mathbb{R}^n \to \mathbb{R}^m$, $T_A(\mathbf{x}) = A\mathbf{x}$ where A is an $m \times n$ matrix.

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- If $m < n$, then T_A is not one-to-one.

Example (Matrix Transformations)

Consider $T_A : \mathbb{R}^n \to \mathbb{R}^m$, $T_A(\mathbf{x}) = A\mathbf{x}$ where A is an $m \times n$ matrix.

- If $m < n$, then T_A is not one-to-one.
- If $m > n$, then T_A is not onto.

Example (Matrix Transformations)

Consider $T_A : \mathbb{R}^n \to \mathbb{R}^m$, $T_A(\mathbf{x}) = A\mathbf{x}$ where A is an $m \times n$ matrix.

- If $m < n$, then T_A is not one-to-one.
- If $m > n$, then T_A is not onto.
- If $m = n$, then

$$T_A \text{ is both one-to-one and onto} \quad \Longleftrightarrow \quad A \text{ is invertible}$$

Example

The linear transformation $T_1 : \mathbb{P}^3 \to \mathbb{R}^4$,

$$T_1(a + bx + cx^2 + dx^3) = (a, b, c, d)$$

is both one-to-one and onto because $\ker(T_1) = \{\mathbf{0}\}$ (you prove).

Example

The linear transformation $T_1 : \mathbb{P}^3 \to \mathbb{R}^4$,

$$T_1(a + bx + cx^2 + dx^3) = (a, b, c, d)$$

is both one-to-one and onto because $\ker(T_1) = \{\mathbf{0}\}$ (you prove).

Example

The linear transformation $T_2 : \mathbb{R}^{2\times 2} \to \mathbb{R}^4$,

$$T_2\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = (a, b, c, d)$$

is both one-to-one and onto because $\ker(T_2) = \{\mathbf{0}\}$ (you prove).

Example

Let $T : \mathbb{P}^n \to \mathbb{P}^{n+1}$ be the linear transformation

$$T(\mathbf{p}) = x\mathbf{p}.$$

If $\mathbf{p} \neq \mathbf{q}$, then $x\mathbf{p} \neq x\mathbf{q}$. Therefore T is one-to-one.

Example

Let $T : \mathbb{P}^n \to \mathbb{P}^{n+1}$ be the linear transformation

$$T(\mathbf{p}) = x\mathbf{p}.$$

If $\mathbf{p} \neq \mathbf{q}$, then $x\mathbf{p} \neq x\mathbf{q}$. Therefore T is one-to-one.

However, T is not onto because all images under T have a zero constant term. Thus, for example, there does not exist a polynomial $\mathbf{p}$ in $\mathbb{P}^n$ such that $T(\mathbf{p}) = 1$.

Example

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Example

$\frac{d}{dx} : C^1(-\infty, \infty) \to F(-\infty, \infty)$ is not one-to-one because, for example,

$$\frac{d}{dx}(x^2) = \frac{d}{dx}(x^2 + 1).$$

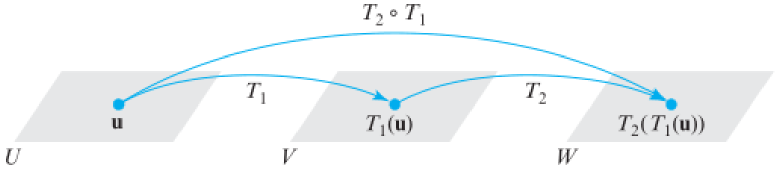
Compositions of Linear Transformations

Definition

If $T_1 : U \to V$ and $T_2 : V \to W$ are linear transformations, then the composition of T_2 with T_1, denoted by $T_2 \circ T_1$, is the function defined by the formula

$$(T_2 \circ T_1)(\mathbf{u}) = T_2\big(T_1(\mathbf{u})\big)$$

where $\mathbf{u}$ is a vector in U.



Composition and Inverse Transformations

Theorem

If $T_1 : U \to V$ *and* $T_2 : V \to W$ *are linear transformations, then* $(T_2 \circ T_1) : U \to W$ *is also a linear transformation.*

EXAMPLE 6 Composition of Linear Transformations

Let $T_1\colon P_1 \to P_2$ and $T_2\colon P_2 \to P_2$ be the linear transformations given by the formulas

$$T_1(p(x)) = xp(x) \quad \text{and} \quad T_2(p(x)) = p(2x+4)$$

Then the composition $(T_2 \circ T_1)\colon P_1 \to P_2$ is given by the formula

$$(T_2 \circ T_1)(p(x)) = T_2(T_1(p(x))) = T_2(xp(x)) = (2x+4)p(2x+4)$$

In particular, if $p(x) = c_0 + c_1 x$, then

$$\begin{aligned}(T_2 \circ T_1)(p(x)) = (T_2 \circ T_1)(c_0 + c_1 x) &= (2x+4)(c_0 + c_1(2x+4)) \\ &= c_0(2x+4) + c_1(2x+4)^2\end{aligned}$$

EXAMPLE 7 Composition with the Identity Operator

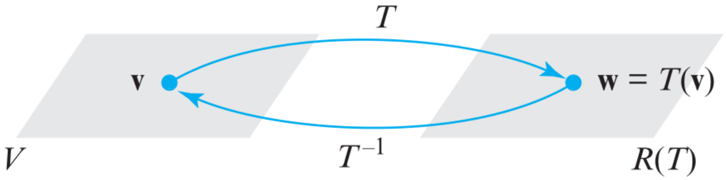
If $T: V \rightarrow V$ is any linear operator, and if $I: V \rightarrow V$ is the identity operator (Example 3 of Section 8.1), then for all vectors $\mathbf{v}$ in V, we have

$$(T \circ I)(\mathbf{v}) = T(I(\mathbf{v})) = T(\mathbf{v})$$
$$(I \circ T)(\mathbf{v}) = I(T(\mathbf{v})) = T(\mathbf{v})$$

It follows that $T \circ I$ and $I \circ T$ are the same as T; that is,

$$\boxed{T \circ I = T} \quad \text{and} \quad \boxed{I \circ T = T} \tag{2}$$

Inverse Linear Transformations



If $T : V \to W$ is a linear transformation, then its *inverse* $T^{-1} : R(T) \to V$ is also a linear transformation.

$$T(\mathbf{v}) = \mathbf{w} \qquad \Longleftrightarrow \qquad \mathbf{v} = T^{-1}(\mathbf{w}).$$

EXAMPLE 8 An Inverse Transformation

We showed in Example 3 of this section that the linear transformation $T: P_n \to P_{n+1}$ given by

$$T(\mathbf{p}) = T(p(x)) = xp(x)$$

is one-to-one but not onto. The fact that it is not onto can be seen explicitly from the formula

$$T(c_0 + c_1 x + \cdots + c_n x^n) = c_0 x + c_1 x^2 + \cdots + c_n x^{n+1} \tag{6}$$

which makes it clear that the range of T consists of all polynomials in P_{n+1} that have zero constant term. Since T is one-to-one it has an inverse, and from (6) this inverse is given by the formula

$$T^{-1}(c_0 x + c_1 x^2 + \cdots + c_n x^{n+1}) = c_0 + c_1 x + \cdots + c_n x^n$$

For example, in the case where $n \geq 3$,

$$T^{-1}(2x - x^2 + 5x^3 + 3x^4) = 2 - x + 5x^2 + 3x^3$$

Theorem

If $T_1 : U \to V$ and $T_2 : V \to W$ are one-to-one linear transformations, then:

1. *$T_2 \circ T_1$ is one-to-one.*
2. *$(T_2 \circ T_1)^{-1} = T_1^{-1} \circ T_2^{-1}$.*

Isomorphisms

Isomorphisms

Definition

A linear transformation $T : V \to W$ that is both one-to-one and onto is called an *isomorphism*.

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A linear transformation $T : V \to W$ that is both one-to-one and onto is called an *isomorphism*.

Definition

If there exists an isomorphism $V \to W$, then we say that V and W are *isomorphic*.

Remark

$T : V \to W$ is an isomorphism $\Longrightarrow$ $T^{-1} : W \to V$ is also an isomorphism

(you prove)

What does isomorphic really mean?

Consider the following isomorphism $T : \mathbb{P}^2 \to \mathbb{R}^3$

$$a_0 + a_1 x + a_2 x^2 \quad \underset{T^{-1}}{\overset{T}{\rightleftarrows}} \quad (a_0, a_1, a_2)$$

What does isomorphic really mean?

Consider the following isomorphism $T : \mathbb{P}^2 \to \mathbb{R}^3$

$$a_0 + a_1 x + a_2 x^2 \quad \underset{T^{-1}}{\overset{T}{\rightleftarrows}} \quad (a_0, a_1, a_2)$$

On the left we have a polynomial function. On the right we have a vector. But the numbers a_0, a_1 and a_2 are the same.

The behaviour under vector addition and scalar multipliation are also the same:

$$(a_0 + a_1 x + a_2 x^2) + (b_0 + b_1 x + b_2 x^2) \quad \underset{T^{-1}}{\overset{T}{\rightleftarrows}} \quad (a_0, a_1, a_2) + (b_0, b_1, b_2)$$

$$(a_0 + b_0) + (a_1 + b_1)x + (a_2 + b_2)x^2 \quad \underset{T^{-1}}{\overset{T}{\rightleftarrows}} \quad (a_0 + b_0, a_1 + b_1, a_2 + b_2)$$

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$$(a_0 + a_1x + a_2x^2) + (b_0 + b_1x + b_2x^2) \quad \underset{T^{-1}}{\overset{T}{\rightleftarrows}} \quad (a_0, a_1, a_2) + (b_0, b_1, b_2)$$

$$(a_0 + b_0) + (a_1 + b_1)x + (a_2 + b_2)x^2 \quad \underset{T^{-1}}{\overset{T}{\rightleftarrows}} \quad (a_0 + b_0, a_1 + b_1, a_2 + b_2)$$

and

$$k(a_0 + a_1x + a_2x^2) \quad \underset{T^{-1}}{\overset{T}{\rightleftarrows}} \quad k(a_0, a_1, a_2)$$

$$ka_0 + ka_1x + ka_2x^2 \quad \underset{T^{-1}}{\overset{T}{\rightleftarrows}} \quad (ka_0, ka_1, ka_2)$$

Isomorphisms

Remark

In the Greek language,

- 'morphe' means 'form'; and
- 'iso' means 'identical'.

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An Important Theorem

Theorem ($\star\ \star\ \star\ \star\ \star$)

Every real n-dimensional vector space is isomorphic to $\mathbb{R}^n$.

Isomorphisms

Theorem

If $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is a basis for a vector space V, then the coordinate map

$$T : \mathbf{u} \mapsto (\mathbf{u})_S$$

is an isomorphism between V and $\mathbb{R}^n$.

Theorem

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$$T : \mathbf{u} \mapsto (\mathbf{u})_S$$

is an isomorphism between V and $\mathbb{R}^n$.

Example (The Natural Isomorphism Between $\mathbb{P}^{n-1}$ and $\mathbb{R}^n$)

The linear transformation $T : \mathbb{P}^{n-1} \to \mathbb{R}^n$,

$$T : a_0 + a_1 x + a_2 X^2 + \ldots + a_{n-1} x^{n-1} \mapsto (a_0, a_1, a_2, \ldots, a_{n-1})$$

is an isomorphism between $\mathbb{P}^{n-1}$ and $\mathbb{R}^n$. This is called the *natural isomorphism* between these vector spaces.

Example (The Natural Isomorphism Between $\mathbb{R}^{2\times 2}$ and $\mathbb{R}^4$)

The coordinate map

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \mapsto (a, b, c, d)$$

defines an isomorphism between $\mathbb{R}^{2\times 2} = M_{22}$ and $\mathbb{R}^4$. We call this the *natural isomorphism between* $\mathbb{R}^{2\times 2}$ *and* $\mathbb{R}^4$.

Working with Isomorphisms

Example

Are the following polynomials linearly independent?

$$\mathbf{p}_1 = 1 + 2x - 3x^2 + 4x^3 + x^5$$
$$\mathbf{p}_2 = 1 + 3x - 4x^2 + 6x^3 + 5x^4 + 4x^5$$
$$\mathbf{p}_3 = 3 + 8x - 11x^2 - 16x^3 + 10x^4 + 9x^5$$

Isomorphisms

We will use the fact that $\mathbb{P}^5$ is isomorphic to $\mathbb{R}^6$.

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The following two questions are equivalent: Are the following polynomials linearly independent?

$$\begin{array}{ccccccccccccc}
\mathbf{p}_1 & = & 1 & + & 2x & - & 3x^2 & + & 4x^3 & & & + & x^5 \\
\mathbf{p}_2 & = & 1 & + & 3x & - & 4x^2 & + & 6x^3 & + & 5x^4 & + & 4x^5 \\
\mathbf{p}_3 & = & 3 & + & 8x & - & 11x^2 & - & 16x^3 & + & 10x^4 & + & 9x^5
\end{array}$$

Are the following vectors linearly independent?

$$\begin{array}{cccccccc}
(& 1, & 2, & -3, & 4, & 0 & 1 &) \\
(& 1, & 3, & -4, & 6, & 5, & 4 &) \\
(& 3, & 8, & -11, & -16, & 10, & 9 &)
\end{array}$$

Isomorphisms

Another equivalent question is: Is the row space of the following matrix 3 dimensional?

$$A = \begin{bmatrix} 1 & 2 & -3 & 4 & 0 & 1 \\ 1 & 3 & -4 & 6 & 5 & 4 \\ 3 & 8 & -11 & 16 & 10 & 9 \end{bmatrix}$$

Another equivalent question is: Is the row space of the following matrix 3 dimensional?

$$A = \begin{bmatrix} 1 & 2 & -3 & 4 & 0 & 1 \\ 1 & 3 & -4 & 6 & 5 & 4 \\ 3 & 8 & -11 & 16 & 10 & 9 \end{bmatrix}$$

I leave it to you to check that A is row equivalent to

$$R = \begin{bmatrix} 1 & 2 & -3 & 4 & 0 & 1 \\ 0 & 1 & -1 & 2 & 5 & 3 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

We can see that the answer to all three questions is **NO**.

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4. (a) $\boxed{\text{10 points}}$ Let $L : \mathbb{R}^3 \to \mathbb{R}^4$ be a linear transformation defined by $L\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} 2y+3z \\ x+y-2z \\ 4x+y \\ 3x-y-z \end{bmatrix}$. Find the matrix representation of L.

(b) $\boxed{\text{15 points}}$ Define $T : \mathbb{P}_2 \to \mathbb{R}^2$ by $T(\mathbf{p}) = \begin{bmatrix} \mathbf{p}(0) \\ \mathbf{p}(1) \end{bmatrix}$. Find a polynomial $\mathbf{p} \in \mathbb{P}_2$ which is a basis for kernel of T.

Solution:

$$\ker T = \{\mathbf{p} : \mathbf{p} \in \mathbb{P}_2 \text{ and } T(\mathbf{p}) = \mathbf{0}\}$$

$$\mathbf{p}(t) = a + bt + ct^2 \Rightarrow T(\mathbf{p}(t)) = \begin{bmatrix} \mathbf{p}(0) \\ \mathbf{p}(1) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} \mathbf{p}(0) \\ \mathbf{p}(1) \end{bmatrix} = \begin{bmatrix} a \\ a+b+c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{array}{c} a = 0 \\ b + c = 0 \end{array} \Rightarrow \mathbf{p}(t) = -ct + ct^2$$

$$\ker T = \left\{\mathbf{p} : \mathbf{p}(t) = \left(-t + t^2\right) c, c \in \mathbb{R}\right\} = \operatorname{Span}\left\{-t + t^2\right\}$$

$$\mathbf{p}(t) = -t + t^2$$

Next Time

- Matrices for Linear Transformations
- Similarity
- Complex Numbers