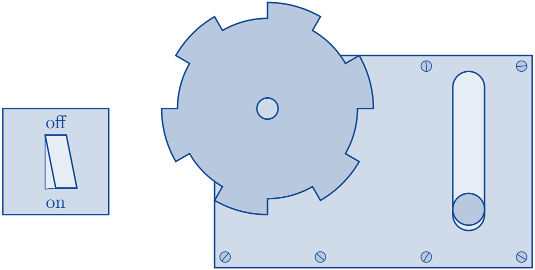
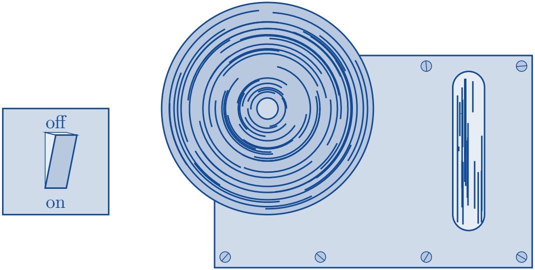
Lecture 9

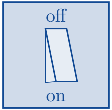
- 4.5 ODEs with Discontinuous Forcing Functions
- 4.6 The Convolution Integral
- 5.1 Introduction
- 5.2 Basic Theory of Systems of First Order Linear Equations

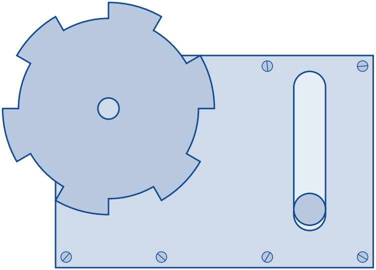
ODEs with Discontinuous Forcing Functions

4.5

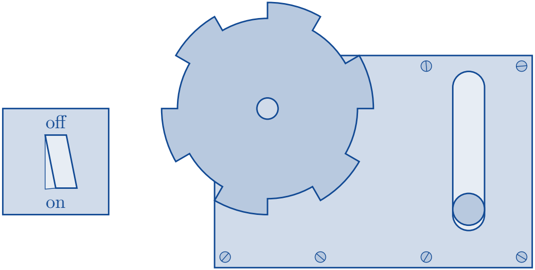

off
on

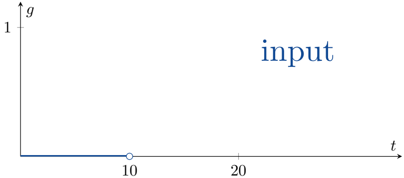

off
on

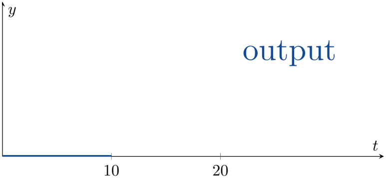

off
on



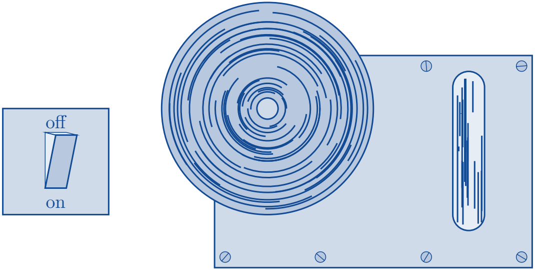
4.5 ODEs with Discontinuous Forcing Functions

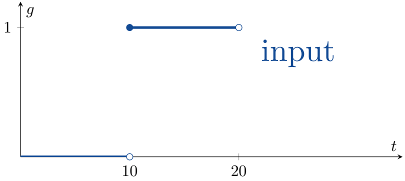


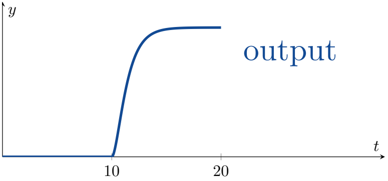


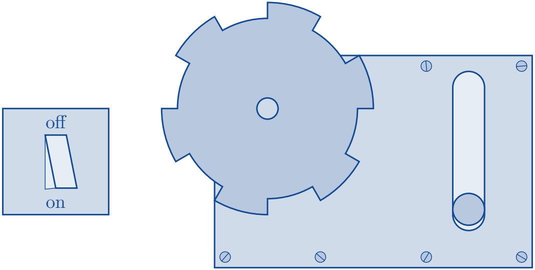


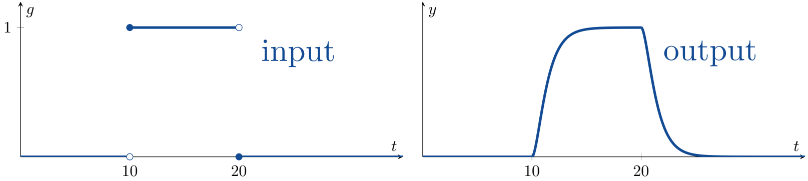
4.5 ODEs with Discontinuous Forcing Functions








off
on


g
1
input
t
10
20
y
output
t
10
20

Example

Solve

$$\begin{cases} y'' + 4y = f(t) = \begin{cases} 0 & 0 \leq t < 5 \\ \frac{1}{5}(t-5) & 5 \leq t < 10 \\ 1 & 10 \leq t \end{cases} \\ y(0) = 0 \\ y'(0) = 0. \end{cases}$$

Example

Solve

$$\begin{cases} y'' + 4y = f(t) = \begin{cases} 0 & 0 \leq t < 5 \\ \frac{1}{5}(t-5) & 5 \leq t < 10 \\ 1 & 10 \leq t \end{cases} \\ y(0) = 0 \\ y'(0) = 0. \end{cases}$$

Note that

$$f(t) = 0 + \left(\frac{1}{5}(t-5) - 0\right) u_5(t) + \left(1 - \frac{1}{5}(t-5)\right) u_{10}(t)$$

$$=$$

Example

Solve

$$\begin{cases} y'' + 4y = f(t) = \begin{cases} 0 & 0 \le t < 5 \\ \frac{1}{5}(t-5) & 5 \le t < 10 \\ 1 & 10 \le t \end{cases} \\ y(0) = 0 \\ y'(0) = 0. \end{cases}$$

Note that

$$\begin{aligned} f(t) &= 0 + \left(\frac{1}{5}(t-5) - 0\right) u_5(t) + \left(1 - \frac{1}{5}(t-5)\right) u_{10}(t) \\ &= \frac{1}{5}\Big(u_5(t)(t-5) - u_{10}(t)(t-10)\Big). \end{aligned}$$

$$\mathcal{L}\left[u_c(t)f(t-c)\right](s) = e^{-cs}F(s) \qquad \mathcal{L}\left[t\right] = \frac{1}{s^2}$$

So our IVP is

$$\begin{cases} y'' + 4y = \frac{1}{5}\Big(u_5(t)(t-5) - u_{10}(t)(t-10)\Big) \\ y(0) = 0 \\ y'(0) = 0. \end{cases}$$

$$\mathcal{L}\left[u_c(t)f(t-c)\right](s) = e^{-cs}F(s) \qquad \mathcal{L}\left[t\right] = \frac{1}{s^2}$$

So our IVP is

$$\begin{cases} y'' + 4y = \frac{1}{5}\Big(u_5(t)(t-5) - u_{10}(t)(t-10)\Big) \\ y(0) = 0 \\ y'(0) = 0. \end{cases}$$

Taking the Laplace transform of the ODE gives

$$(s^2+4)Y = \frac{1}{5}\frac{e^{-5s} - e^{-10s}}{s^2}$$

$$\mathcal{L}\left[u_c(t)f(t-c)\right](s) = e^{-cs}F(s) \qquad \mathcal{L}\left[t\right] = \frac{1}{s^2}$$

So our IVP is

$$\begin{cases} y'' + 4y = \frac{1}{5}\Big(u_5(t)(t-5) - u_{10}(t)(t-10)\Big) \\ y(0) = 0 \\ y'(0) = 0. \end{cases}$$

Taking the Laplace transform of the ODE gives

$$(s^2+4)Y = \frac{1}{5}\frac{e^{-5s} - e^{-10s}}{s^2}$$

and

$$Y = \frac{1}{5}\frac{e^{-5s} - e^{-10s}}{s^2(s^2+4)}.$$

$$Y = \frac{1}{5}\frac{e^{-5s} - e^{-10s}}{s^2(s^2+4)}$$

Let

$$H(s) = \frac{1}{s^2(s^2+4)}.$$

Then

$$Y(s) = \frac{1}{5}e^{-5s}H(s) - \frac{1}{5}e^{-10s}H(s).$$

$$Y(s) = \frac{1}{5}e^{-5s}H(s) - \frac{1}{5}e^{-10s}H(s)$$

Since

$$\mathcal{L}\left[u_c(t)h(t-c)\right](s) = e^{-cs}H(s)$$

$$Y(s) = \frac{1}{5}e^{-5s}H(s) - \frac{1}{5}e^{-10s}H(s)$$

Since

$$\mathcal{L}\left[u_c(t)h(t-c)\right](s) = e^{-cs}H(s)$$

we have that

$$u_c(t)h(t-c) = \mathcal{L}^{-1}\left[e^{-cs}H(s)\right](t).$$

$$Y(s) = \frac{1}{5}e^{-5s}H(s) - \frac{1}{5}e^{-10s}H(s)$$

Since

$$\mathcal{L}\left[u_c(t)h(t-c)\right](s) = e^{-cs}H(s)$$

we have that

$$u_c(t)h(t-c) = \mathcal{L}^{-1}\left[e^{-cs}H(s)\right](t).$$

If we can find $h(t)$, then we can find $y(t)$.

Using partial fractions, we calculate (please check!) that

$$\begin{aligned} H(s) &= \frac{1}{s^2(s^2+4)} = \frac{As+B}{s^2} + \frac{Cs+D}{s^2+4} \\ &= \frac{As^3+Bs^2+4As+4B+Cs^3+Ds^2}{s^2(s^2+4)} \\ &= \frac{0s+\frac{1}{4}}{s^2} + \frac{0s-\frac{1}{4}}{s^2+4} = \frac{\frac{1}{4}}{s^2} - \frac{\frac{1}{4}}{s^2+4}. \end{aligned}$$

Using partial fractions, we calculate (please check!) that

$$
\begin{aligned}
H(s) = \frac{1}{s^2(s^2+4)} &= \frac{As+B}{s^2} + \frac{Cs+D}{s^2+4} \\
&= \frac{As^3 + Bs^2 + 4As + 4B + Cs^3 + Ds^2}{s^2(s^2+4)} \\
&= \frac{0s + \frac{1}{4}}{s^2} + \frac{0s - \frac{1}{4}}{s^2+4} = \frac{\frac{1}{4}}{s^2} - \frac{\frac{1}{4}}{s^2+4}.
\end{aligned}
$$

Hence

$$
h(t) = \frac{1}{4}\mathcal{L}^{-1}\left[\frac{1}{s^2}\right] - \frac{1}{8}\mathcal{L}^{-1}\left[\frac{2}{s^2+4}\right] = \qquad\qquad\qquad .
$$

Using partial fractions, we calculate (please check!) that

$$H(s) = \frac{1}{s^2(s^2+4)} = \frac{As+B}{s^2} + \frac{Cs+D}{s^2+4}$$
$$= \frac{As^3 + Bs^2 + 4As + 4B + Cs^3 + Ds^2}{s^2(s^2+4)}$$
$$= \frac{0s + \frac{1}{4}}{s^2} + \frac{0s - \frac{1}{4}}{s^2+4} = \frac{\frac{1}{4}}{s^2} - \frac{\frac{1}{4}}{s^2+4}.$$

Hence

$$h(t) = \frac{1}{4}\mathcal{L}^{-1}\left[\frac{1}{s^2}\right] - \frac{1}{8}\mathcal{L}^{-1}\left[\frac{2}{s^2+4}\right] = \frac{t}{4} - \frac{1}{8}\sin 2t.$$

$$u_c(t)h(t-c)(s) = \mathcal{L}^{-1}\left[e^{-cs}H(s)\right]$$

Therefore

$$y(t) = \mathcal{L}^{-1}\left[\frac{1}{5}e^{-5s}H(s) - \frac{1}{5}e^{-10s}H(s)\right]$$

$$=$$

$$=$$

.

$$u_c(t)h(t-c)(s) = \mathcal{L}^{-1}\left[e^{-cs}H(s)\right]$$

Therefore

$$\begin{aligned} y(t) &= \mathcal{L}^{-1}\left[\frac{1}{5}e^{-5s}H(s) - \frac{1}{5}e^{-10s}H(s)\right] \\ &= \frac{1}{5}u_5(t)h(t-5) - \frac{1}{5}u_{10}(t)h(t-10) \\ &= \end{aligned}$$

.

$$u_c(t)h(t-c)(s) = \mathcal{L}^{-1}\left[e^{-cs}H(s)\right]$$

Therefore

$$\begin{aligned}
y(t) &= \mathcal{L}^{-1}\left[\frac{1}{5}e^{-5s}H(s) - \frac{1}{5}e^{-10s}H(s)\right] \\
&= \frac{1}{5}u_5(t)h(t-5) - \frac{1}{5}u_{10}(t)h(t-10) \\
&= u_5(t)\left(\frac{t-5}{20} - \frac{1}{40}\sin(2t-10)\right) \\
&\qquad - u_{10}(t)\left(\frac{t-10}{20} - \frac{1}{40}\sin(2t-20)\right).
\end{aligned}$$

Example

Solve

$$\begin{cases} y'' + 3y' + 2y = f(t) = \begin{cases} 1 & 0 \le t < 10 \\ 0 & 10 \le t \end{cases} \\ y(0) = 1 \\ y'(0) = 0. \end{cases}$$

Example

Solve

$$\begin{cases} y'' + 3y' + 2y = f(t) = \begin{cases} 1 & 0 \leq t < 10 \\ 0 & 10 \leq t \end{cases} \\ y(0) = 1 \\ y'(0) = 0. \end{cases}$$

Since $f(t) = 1 - u_{10}(t)$, the Laplace Transform of the ODE is

$$(s^2 + 3s + 2)Y - (s + 3) = \frac{1 - e^{-10s}}{s}.$$

Thus

$$
\begin{aligned}
Y(s) &= \frac{1 - e^{-10s}}{s(s^2 + 3s + 2)} + \frac{s + 3}{s^2 + 3s + 2} \\
&= \frac{(s^2 + 3s + 1) - e^{-10s}}{s(s^2 + 3s + 2)}.
\end{aligned}
$$

Thus

$$Y(s) = \frac{1 - e^{-10s}}{s(s^2 + 3s + 2)} + \frac{s+3}{s^2 + 3s + 2}$$
$$= \frac{(s^2 + 3s + 1) - e^{-10s}}{s(s^2 + 3s + 2)}.$$

Let

$$G(s) = \frac{s^2 + 3s + 1}{s(s^2 + 3s + 2)} \qquad \text{and} \qquad H(s) = \frac{1}{s(s^2 + 3s + 2)}.$$

Thus

$$Y(s) = \frac{1 - e^{-10s}}{s(s^2 + 3s + 2)} + \frac{s + 3}{s^2 + 3s + 2}$$
$$= \frac{(s^2 + 3s + 1) - e^{-10s}}{s(s^2 + 3s + 2)}.$$

Let

$$G(s) = \frac{s^2 + 3s + 1}{s(s^2 + 3s + 2)} \qquad \text{and} \qquad H(s) = \frac{1}{s(s^2 + 3s + 2)}.$$

Then $Y = G(s) - e^{-10s} H(s)$.

Thus

$$\begin{aligned} Y(s) &= \frac{1 - e^{-10s}}{s(s^2+3s+2)} + \frac{s+3}{s^2+3s+2} \\ &= \frac{(s^2+3s+1) - e^{-10s}}{s(s^2+3s+2)}. \end{aligned}$$

Let

$$G(s) = \frac{s^2+3s+1}{s(s^2+3s+2)} \qquad \text{and} \qquad H(s) = \frac{1}{s(s^2+3s+2)}.$$

Then $Y = G(s) - e^{-10s}H(s)$. If we can find $g(t)$ and $h(t)$, then we can find $y(t)$.

Using partial fractions we get

$$G(s) = \frac{A}{s} + \frac{B}{s+1} + \frac{C}{s+2} = \frac{\frac{1}{2}}{s} + \frac{1}{s+1} - \frac{\frac{1}{2}}{s+2}$$

and

$$H(s) = \frac{D}{s} + \frac{E}{s+1} + \frac{F}{s+2} = \frac{\frac{1}{2}}{s} - \frac{1}{s+1} + \frac{\frac{1}{2}}{s+2}$$

(please check!).

Using partial fractions we get

$$G(s) = \frac{A}{s} + \frac{B}{s+1} + \frac{C}{s+2} = \frac{\frac{1}{2}}{s} + \frac{1}{s+1} - \frac{\frac{1}{2}}{s+2}$$

and

$$H(s) = \frac{D}{s} + \frac{E}{s+1} + \frac{F}{s+2} = \frac{\frac{1}{2}}{s} - \frac{1}{s+1} + \frac{\frac{1}{2}}{s+2}$$

(please check!). It follows that

$$g(t) = \frac{1}{2}\left(1 + 2e^{-t} - e^{-2t}\right) \qquad \text{and} \qquad h(t) = \frac{1}{2}\left(1 - 2e^{-t} + e^{-2t}\right).$$

$$g(t) = \frac{1}{2}\left(1 + 2e^{-t} - e^{-2t}\right) \qquad h(t) = \frac{1}{2}\left(1 - 2e^{-t} + e^{-2t}\right)$$

Therefore

$$\begin{aligned} y(t) &= \mathcal{L}^{-1}\left[Y\right] \\ &= \\ &= \\ &= \qquad . \end{aligned}$$

$$g(t) = \frac{1}{2}\left(1 + 2e^{-t} - e^{-2t}\right) \qquad h(t) = \frac{1}{2}\left(1 - 2e^{-t} + e^{-2t}\right)$$

Therefore

$$\begin{aligned} y(t) &= \mathcal{L}^{-1}\left[Y\right] \\ &= \mathcal{L}^{-1}\left[G(s) - e^{-10s}H(s)\right] \\ &= \\ &= \qquad . \end{aligned}$$

$$g(t) = \frac{1}{2}\left(1 + 2e^{-t} - e^{-2t}\right) \qquad h(t) = \frac{1}{2}\left(1 - 2e^{-t} + e^{-2t}\right)$$

Therefore

$$\begin{aligned} y(t) &= \mathcal{L}^{-1}\left[Y\right] \\ &= \mathcal{L}^{-1}\left[G(s) - e^{-10s}H(s)\right] \\ &= g(t) - u_{10}(t)h(t-10) \\ &= \qquad . \end{aligned}$$

$$g(t) = \frac{1}{2}\left(1 + 2e^{-t} - e^{-2t}\right) \qquad h(t) = \frac{1}{2}\left(1 - 2e^{-t} + e^{-2t}\right)$$

Therefore

$$\begin{aligned} y(t) &= \mathcal{L}^{-1}\left[Y\right] \\ &= \mathcal{L}^{-1}\left[G(s) - e^{-10s}H(s)\right] \\ &= g(t) - u_{10}(t)h(t-10) \\ &= \frac{1}{2}\left(1 + 2e^{-t} - e^{-2t}\right) - \frac{1}{2}u_{10}(t)\left(1 - 2e^{-(t-10)} + e^{-2(t-10)}\right). \end{aligned}$$

Example

Solve

$$\begin{cases} y'' + 4y = u_\pi(t) - u_{3\pi}(t) \\ y(0) = 0 \\ y'(0) = 0. \end{cases}$$

Example

Solve

$$\begin{cases} y'' + 4y = u_\pi(t) - u_{3\pi}(t) \\ y(0) = 0 \\ y'(0) = 0. \end{cases}$$

Taking the Laplace Transform of the ODE gives

$$(s^2+4)Y(s) = \frac{e^{-\pi s} - e^{-3\pi s}}{s}.$$

Example

Solve

$$\begin{cases} y'' + 4y = u_\pi(t) - u_{3\pi}(t) \\ y(0) = 0 \\ y'(0) = 0. \end{cases}$$

Taking the Laplace Transform of the ODE gives

$$(s^2+4)Y(s) = \frac{e^{-\pi s} - e^{-3\pi s}}{s}.$$

Thus

$$Y(s) = \frac{e^{-\pi s} - e^{-3\pi s}}{s(s^2+4)}.$$

Example

Solve

$$\begin{cases} y'' + 4y = u_\pi(t) - u_{3\pi}(t) \\ y(0) = 0 \\ y'(0) = 0. \end{cases}$$

Taking the Laplace Transform of the ODE gives

$$(s^2+4)Y(s) = \frac{e^{-\pi s} - e^{-3\pi s}}{s}.$$

Thus

$$Y(s) = \frac{e^{-\pi s} - e^{-3\pi s}}{s(s^2+4)}.$$

Let

$$H(s) = \frac{1}{s(s^2+4)}.$$

Using partial fractions, we calculate that

$$H(s) = \frac{1}{s(s^2+4)} = \frac{A}{s} + \frac{Bs+C}{s^2+4} = \frac{\frac{1}{4}}{s} + \frac{-\frac{1}{4}s+0}{s^2+4}$$
$$= \frac{1}{4}\left(\frac{1}{s}\right) - \frac{1}{4}\left(\frac{s}{s^2+4}\right) = \frac{1}{4}\mathcal{L}\left[1\right] - \frac{1}{4}\mathcal{L}\left[\cos 2t\right].$$

Using partial fractions, we calculate that

$$\begin{aligned} H(s) &= \frac{1}{s(s^2+4)} = \frac{A}{s} + \frac{Bs+C}{s^2+4} = \frac{\frac{1}{4}}{s} + \frac{-\frac{1}{4}s+0}{s^2+4} \\ &= \frac{1}{4}\left(\frac{1}{s}\right) - \frac{1}{4}\left(\frac{s}{s^2+4}\right) = \frac{1}{4}\mathcal{L}\left[1\right] - \frac{1}{4}\mathcal{L}\left[\cos 2t\right]. \end{aligned}$$

It follows that

$$h(t) = \frac{1}{4} - \frac{1}{4}\cos 2t$$

Using partial fractions, we calculate that

$$H(s) = \frac{1}{s(s^2+4)} = \frac{A}{s} + \frac{Bs+C}{s^2+4} = \frac{\frac{1}{4}}{s} + \frac{-\frac{1}{4}s+0}{s^2+4}$$
$$= \frac{1}{4}\left(\frac{1}{s}\right) - \frac{1}{4}\left(\frac{s}{s^2+4}\right) = \frac{1}{4}\mathcal{L}\left[1\right] - \frac{1}{4}\mathcal{L}\left[\,\cos 2t\right].$$

It follows that

$$h(t) = \frac{1}{4} - \frac{1}{4}\cos 2t$$

and the solution to the IVP is

$$y(t) = \mathcal{L}^{-1}\left[e^{-\pi s}H(s)\right] - \mathcal{L}^{-1}\left[e^{-3\pi s}H(s)\right]$$
$$=$$
$$= \qquad .$$

Using partial fractions, we calculate that

$$H(s) = \frac{1}{s(s^2+4)} = \frac{A}{s} + \frac{Bs+C}{s^2+4} = \frac{\frac{1}{4}}{s} + \frac{-\frac{1}{4}s+0}{s^2+4}$$
$$= \frac{1}{4}\left(\frac{1}{s}\right) - \frac{1}{4}\left(\frac{s}{s^2+4}\right) = \frac{1}{4}\mathcal{L}\left[1\right] - \frac{1}{4}\mathcal{L}\left[\cos 2t\right].$$

It follows that

$$h(t) = \frac{1}{4} - \frac{1}{4}\cos 2t$$

and the solution to the IVP is

$$\begin{aligned} y(t) &= \mathcal{L}^{-1}\left[e^{-\pi s}H(s)\right] - \mathcal{L}^{-1}\left[e^{-3\pi s}H(s)\right] \\ &= u_\pi(t)h(t-\pi) - u_{3\pi}(t)h(t-3\pi) \\ &= \qquad\qquad . \end{aligned}$$

Using partial fractions, we calculate that

$$H(s) = \frac{1}{s(s^2+4)} = \frac{A}{s} + \frac{Bs+C}{s^2+4} = \frac{\frac{1}{4}}{s} + \frac{-\frac{1}{4}s+0}{s^2+4}$$
$$= \frac{1}{4}\left(\frac{1}{s}\right) - \frac{1}{4}\left(\frac{s}{s^2+4}\right) = \frac{1}{4}\mathcal{L}\left[1\right] - \frac{1}{4}\mathcal{L}\left[\cos 2t\right].$$

It follows that

$$h(t) = \frac{1}{4} - \frac{1}{4}\cos 2t$$

and the solution to the IVP is

$$\begin{aligned} y(t) &= \mathcal{L}^{-1}\left[e^{-\pi s}H(s)\right] - \mathcal{L}^{-1}\left[e^{-3\pi s}H(s)\right] \\ &= u_\pi(t)h(t-\pi) - u_{3\pi}(t)h(t-3\pi) \\ &= \frac{1}{4}u_\pi(t)\big(1-\cos(2t-2\pi)\big) - \frac{1}{4}u_{3\pi}(t)\big(1-\cos(2t-6\pi)\big). \end{aligned}$$

4.6 The Convolution Integral

Let $f : [0, \infty) \to \mathbb{R}$ and $g : [0, \infty) \to \mathbb{R}$ be piecewise continuous functions.

Definition

The *convolution* of f and g is

$$\big(f * g\big)(t) = \int_0^t f(\tau)g(t-\tau)\,d\tau.$$

4.6 $\big(f * g\big)(t) = \int_0^t f(\tau)g(t-\tau)\,d\tau$

Theorem (Properties)

- $f * g = g * f$
- $f * (g * h) = (f * g) * h$
- $f*(g+h) = (f*g)+(f*h)$
- $f * 0 = 0 = 0 * f$

Theorem (Properties)

- $f * g = g * f$
- $f*(g+h) = (f*g)+(f*h)$
- $f * (g * h) = (f * g) * h$
- $f * 0 = 0 = 0 * f$

Example

$$\big(\cos * 1\big)(t) = \int_0^t \cos\tau \cdot 1\,d\tau = [\sin\tau]_0^t = \sin t - \sin 0 = \sin t$$

$$\big(1 * \cos\big)(t) =$$

Theorem (Properties)

- $f * g = g * f$
- $f*(g+h) = (f*g)+(f*h)$
- $f * (g * h) = (f * g) * h$
- $f * 0 = 0 = 0 * f$

Example

$$\big(\cos * 1\big)(t) = \int_0^t \cos\tau \cdot 1\,d\tau = [\sin\tau]_0^t = \sin t - \sin 0 = \sin t$$

$$\begin{aligned}\big(1 * \cos\big)(t) &= \int_0^t 1 \cdot \cos(t-\tau)\,d\tau = [-\sin(t-\tau)]_0^t \\ &= -\sin 0 + \sin t = \sin t\end{aligned}$$

Theorem (Properties)

- $f * g = g * f$
- $f * (g * h) = (f * g) * h$
- $f*(g+h) = (f*g)+(f*h)$
- $f * 0 = 0 = 0 * f$

Example

$$\big(\cos * 1\big)(t) = \int_0^t \cos\tau \cdot 1\,d\tau = [\sin\tau]_0^t = \sin t - \sin 0 = \sin t$$

$$\begin{aligned}\big(1 * \cos\big)(t) &= \int_0^t 1 \cdot \cos(t-\tau)\,d\tau = [-\sin(t-\tau)]_0^t \\ &= -\sin 0 + \sin t = \sin t\end{aligned}$$

Note that $f * 1 \neq f$ in general.

Example

$$
\begin{aligned}
\big(\sin * \sin\big)(t) &= \int_0^t \sin\tau \sin(t-\tau)\,d\tau \\
&= \\
&= \\
&= \\
&= \\
&=
\end{aligned}
$$

Example

$$
\begin{aligned}
\big(\sin * \sin\big)(t) &= \int_0^t \sin\tau \sin(t-\tau)\,d\tau \\
&= \int_0^t \sin\tau\,(\sin t\cos\tau - \cos t\sin\tau)\,\,d\tau \\
&= \\
&= \\
&= \\
&=
\end{aligned}
$$

Example

$$\begin{aligned}
\big(\sin * \sin\big)(t) &= \int_0^t \sin\tau \sin(t-\tau)\,d\tau \\
&= \int_0^t \sin\tau\left(\sin t\cos\tau - \cos t\sin\tau\right)\,d\tau \\
&= \sin t\int_0^t \sin\tau\cos\tau\,d\tau - \cos t\int_0^t \sin^2\tau\,d\tau \\
&= \\
&= \\
&=
\end{aligned}$$

Example

$$\begin{aligned}
\big(\sin * \sin\big)(t) &= \int_0^t \sin\tau\,\sin(t-\tau)\,d\tau \\
&= \int_0^t \sin\tau\left(\sin t\cos\tau - \cos t\sin\tau\right)\,d\tau \\
&= \sin t\int_0^t \sin\tau\cos\tau\,d\tau - \cos t\int_0^t \sin^2\tau\,d\tau \\
&= \sin t\left[-\frac{1}{2}\cos^2\tau\right]_0^t - \cos t\left[\frac{1}{2}\left(\tau - \sin\tau\cos\tau\right)\right]_0^t \\
&= \\
&=
\end{aligned}$$

Example

$$
\begin{aligned}
\big(\sin * \sin\big)(t) &= \int_0^t \sin\tau \sin(t-\tau)\,d\tau \\
&= \int_0^t \sin\tau \left(\sin t\cos\tau - \cos t\sin\tau\right)\,d\tau \\
&= \sin t \int_0^t \sin\tau\cos\tau\,d\tau - \cos t \int_0^t \sin^2\tau\,d\tau \\
&= \sin t \left[-\frac{1}{2}\cos^2\tau\right]_0^t - \cos t \left[\frac{1}{2}\left(\tau - \sin\tau\cos\tau\right)\right]_0^t \\
&= \frac{1}{2}\sin t\left(1-\cos^2 t\right) - \frac{1}{2}\cos t\left(t - \sin t\cos t\right) \\
&=
\end{aligned}
$$

Example

$$
\begin{aligned}
\big(\sin * \sin\big)(t) &= \int_0^t \sin\tau \sin(t-\tau)\,d\tau \\
&= \int_0^t \sin\tau \left(\sin t\cos\tau - \cos t\sin\tau\right)\, d\tau \\
&= \sin t \int_0^t \sin\tau\cos\tau\,d\tau - \cos t \int_0^t \sin^2\tau\,d\tau \\
&= \sin t \left[-\frac{1}{2}\cos^2\tau\right]_0^t - \cos t \left[\frac{1}{2}\left(\tau - \sin\tau\cos\tau\right)\right]_0^t \\
&= \frac{1}{2}\sin t\left(1-\cos^2 t\right) - \frac{1}{2}\cos t\left(t - \sin t\cos t\right) \\
&= \frac{1}{2}\sin t - \frac{t}{2}\cos t.
\end{aligned}
$$

Example

$$
\begin{aligned}
\big(\sin * \sin\big)(t) &= \int_0^t \sin\tau\sin(t-\tau)\,d\tau \\
&= \int_0^t \sin\tau\left(\sin t\cos\tau - \cos t\sin\tau\right)\,d\tau \\
&= \sin t\int_0^t \sin\tau\cos\tau\,d\tau - \cos t\int_0^t \sin^2\tau\,d\tau \\
&= \sin t\left[-\frac{1}{2}\cos^2\tau\right]_0^t - \cos t\left[\frac{1}{2}\left(\tau - \sin\tau\cos\tau\right)\right]_0^t \\
&= \frac{1}{2}\sin t\left(1-\cos^2 t\right) - \frac{1}{2}\cos t\left(t - \sin t\cos t\right) \\
&= \frac{1}{2}\sin t - \frac{t}{2}\cos t.
\end{aligned}
$$

Note that $f * f \geq 0$ is *not* true in general.

Theorem

$$\mathcal{L}\left[f * g\right](s) = F(s)G(s)$$

Theorem

$$\mathcal{L}\left[f * g\right](s) = F(s)G(s)$$

This means that $\mathcal{L}^{-1}\left[FG\right] = f * g$.

Example

Find the inverse Laplace Transform of $H(s) = \dfrac{a}{s^2(s^2+a^2)}$.

Example

Find the inverse Laplace Transform of $H(s) = \dfrac{a}{s^2(s^2+a^2)}$.

Note that $H(s) = \left(\dfrac{1}{s^2}\right)\left(\dfrac{a}{s^2+a^2}\right)$.

Example

Find the inverse Laplace Transform of $H(s) = \dfrac{a}{s^2(s^2+a^2)}$.

Note that $H(s) = \left(\dfrac{1}{s^2}\right)\left(\dfrac{a}{s^2+a^2}\right)$. We know that $\mathcal{L}\left[t\right] = \frac{1}{s^2}$ and $\mathcal{L}\left[\sin at\right] = \frac{a}{s^2+a^2}$.

Example

Find the inverse Laplace Transform of $H(s) = \dfrac{a}{s^2(s^2+a^2)}$.

Note that $H(s) = \left(\dfrac{1}{s^2}\right)\left(\dfrac{a}{s^2+a^2}\right)$. We know that $\mathcal{L}\left[t\right] = \frac{1}{s^2}$ and $\mathcal{L}\left[\sin at\right] = \frac{a}{s^2+a^2}$. So

$$h(t) = \mathcal{L}^{-1}\left[\left(\frac{1}{s^2}\right)\left(\frac{a}{s^2+a^2}\right)\right] =$$

$$=$$

$$=$$

Example

Find the inverse Laplace Transform of $H(s) = \dfrac{a}{s^2(s^2+a^2)}$.

Note that $H(s) = \left(\dfrac{1}{s^2}\right)\left(\dfrac{a}{s^2+a^2}\right)$. We know that $\mathcal{L}\left[t\right] = \frac{1}{s^2}$ and $\mathcal{L}\left[\sin at\right] = \frac{a}{s^2+a^2}$. So

$$h(t) = \mathcal{L}^{-1}\left[\left(\frac{1}{s^2}\right)\left(\frac{a}{s^2+a^2}\right)\right] = \mathcal{L}^{-1}\left[\frac{1}{s^2}\right] * \mathcal{L}^{-1}\left[\frac{a}{s^2+a^2}\right]$$

$$=$$

$$=$$

Example

Find the inverse Laplace Transform of $H(s) = \dfrac{a}{s^2(s^2+a^2)}$.

Note that $H(s) = \left(\dfrac{1}{s^2}\right)\left(\dfrac{a}{s^2+a^2}\right)$. We know that $\mathcal{L}\left[t\right] = \frac{1}{s^2}$ and $\mathcal{L}\left[\sin at\right] = \frac{a}{s^2+a^2}$. So

$$\begin{aligned} h(t) &= \mathcal{L}^{-1}\left[\left(\frac{1}{s^2}\right)\left(\frac{a}{s^2+a^2}\right)\right] = \mathcal{L}^{-1}\left[\frac{1}{s^2}\right] * \mathcal{L}^{-1}\left[\frac{a}{s^2+a^2}\right] \\ &= t * \sin at = \int_0^t \tau \sin a(t-\tau)\,d\tau \\ &= \end{aligned}$$

Example

Find the inverse Laplace Transform of $H(s) = \dfrac{a}{s^2(s^2+a^2)}$.

Note that $H(s) = \left(\dfrac{1}{s^2}\right)\left(\dfrac{a}{s^2+a^2}\right)$. We know that $\mathcal{L}\left[t\right] = \frac{1}{s^2}$ and $\mathcal{L}\left[\sin at\right] = \frac{a}{s^2+a^2}$. So

$$
\begin{aligned}
h(t) &= \mathcal{L}^{-1}\left[\left(\frac{1}{s^2}\right)\left(\frac{a}{s^2+a^2}\right)\right] = \mathcal{L}^{-1}\left[\frac{1}{s^2}\right] * \mathcal{L}^{-1}\left[\frac{a}{s^2+a^2}\right] \\
&= t * \sin at = \int_0^t \tau \sin a(t-\tau)\,d\tau \\
&= \frac{at - \sin at}{a^2}.
\end{aligned}
$$

Example

Solve

$$\begin{cases} y'' + 4y = g(t) \\ y(0) = 3 \\ y'(0) = -1. \end{cases}$$

Example

Solve

$$\begin{cases} y'' + 4y = g(t) \\ y(0) = 3 \\ y'(0) = -1. \end{cases}$$

Taking the Laplace Transform of the ODE gives

$$(s^2Y - 3s + 1) + 4Y = G(s)$$

Example

Solve

$$\begin{cases} y'' + 4y = g(t) \\ y(0) = 3 \\ y'(0) = -1. \end{cases}$$

Taking the Laplace Transform of the ODE gives

$$(s^2Y - 3s + 1) + 4Y = G(s)$$

which rearranges to

$$\begin{aligned} Y(s) &= \frac{3s-1}{s^2+4} + \frac{G(s)}{s^2+4} \\ &= 3\left(\frac{s}{s^2+4}\right) - \frac{1}{2}\left(\frac{2}{s^2+4}\right) + \frac{1}{2}\left(\frac{2}{s^2+4}\right)G(s). \end{aligned}$$

$$Y(s) = 3\left(\frac{s}{s^2+4}\right) - \frac{1}{2}\left(\frac{2}{s^2+4}\right) + \frac{1}{2}\left(\frac{2}{s^2+4}\right)G(s)$$

Hence the solution to the IVP is

$$y(t) = 3\mathcal{L}^{-1}\left[\frac{s}{s^2+4}\right] - \frac{1}{2}\mathcal{L}^{-1}\left[\frac{2}{s^2+4}\right] + \frac{1}{2}\mathcal{L}^{-1}\left[\left(\frac{2}{s^2+4}\right)G(s)\right]$$

$$=$$

$$=$$

$$Y(s) = 3\left(\frac{s}{s^2+4}\right) - \frac{1}{2}\left(\frac{2}{s^2+4}\right) + \frac{1}{2}\left(\frac{2}{s^2+4}\right)G(s)$$

Hence the solution to the IVP is

$$\begin{aligned} y(t) &= 3\mathcal{L}^{-1}\left[\frac{s}{s^2+4}\right] - \frac{1}{2}\mathcal{L}^{-1}\left[\frac{2}{s^2+4}\right] + \frac{1}{2}\mathcal{L}^{-1}\left[\left(\frac{2}{s^2+4}\right)G(s)\right] \\ &= 3\cos 2t - \frac{1}{2}\sin 2t + \frac{1}{2}\sin 2t * g(t) \\ &= \end{aligned}$$

$$Y(s) = 3\left(\frac{s}{s^2+4}\right) - \frac{1}{2}\left(\frac{2}{s^2+4}\right) + \frac{1}{2}\left(\frac{2}{s^2+4}\right)G(s)$$

Hence the solution to the IVP is

$$\begin{aligned} y(t) &= 3\mathcal{L}^{-1}\left[\frac{s}{s^2+4}\right] - \frac{1}{2}\mathcal{L}^{-1}\left[\frac{2}{s^2+4}\right] + \frac{1}{2}\mathcal{L}^{-1}\left[\left(\frac{2}{s^2+4}\right)G(s)\right] \\ &= 3\cos 2t - \frac{1}{2}\sin 2t + \frac{1}{2}\sin 2t * g(t) \\ &= 3\cos 2t - \frac{1}{2}\sin 2t + \frac{1}{2}\int_0^t \sin 2(t-\tau)g(\tau)\,d\tau. \end{aligned}$$

Example

Find the inverse Laplace Transform of $\dfrac{2}{(s-1)(s^2+4)}$.

Example

Find the inverse Laplace Transform of $\dfrac{2}{(s-1)(s^2+4)}$.

$$\mathcal{L}^{-1}\left[\frac{2}{(s-1)(s^2+4)}\right] = \mathcal{L}^{-1}\left[\left(\frac{2}{s^2+4}\right)\left(\frac{1}{s-1}\right)\right]$$

Example

Find the inverse Laplace Transform of $\dfrac{2}{(s-1)(s^2+4)}$.

$$\mathcal{L}^{-1}\left[\frac{2}{(s-1)(s^2+4)}\right] = \mathcal{L}^{-1}\left[\left(\frac{2}{s^2+4}\right)\left(\frac{1}{s-1}\right)\right] = \sin 2t * e^t$$

Example

Find the inverse Laplace Transform of $\dfrac{2}{(s-1)(s^2+4)}$.

$$\mathcal{L}^{-1}\left[\frac{2}{(s-1)(s^2+4)}\right] = \mathcal{L}^{-1}\left[\left(\frac{2}{s^2+4}\right)\left(\frac{1}{s-1}\right)\right] = \sin 2t * e^t$$

$$= \int_0^t e^{t-\tau}\sin 2\tau\,d\tau$$

Example

Find the inverse Laplace Transform of $\dfrac{2}{(s-1)(s^2+4)}$.

$$\mathcal{L}^{-1}\left[\frac{2}{(s-1)(s^2+4)}\right] = \mathcal{L}^{-1}\left[\left(\frac{2}{s^2+4}\right)\left(\frac{1}{s-1}\right)\right] = \sin 2t * e^t$$

$$= \int_0^t e^{t-\tau}\sin 2\tau\,d\tau = e^t\int_0^t e^{-\tau}\sin 2\tau\,d\tau$$

Example

Find the inverse Laplace Transform of $\dfrac{2}{(s-1)(s^2+4)}$.

$$\begin{aligned}
\mathcal{L}^{-1}\left[\frac{2}{(s-1)(s^2+4)}\right] &= \mathcal{L}^{-1}\left[\left(\frac{2}{s^2+4}\right)\left(\frac{1}{s-1}\right)\right] = \sin 2t * e^t \\
&= \int_0^t e^{t-\tau}\sin 2\tau\,d\tau = e^t\int_0^t e^{-\tau}\sin 2\tau\,d\tau \\
&= e^t\left[\frac{e^{-\tau}}{5}\left(-\sin 2\tau - 2\cos 2\tau\right)\right]_0^t
\end{aligned}$$

Example

Find the inverse Laplace Transform of $\dfrac{2}{(s-1)(s^2+4)}$.

$$\begin{aligned}
\mathcal{L}^{-1}\left[\frac{2}{(s-1)(s^2+4)}\right] &= \mathcal{L}^{-1}\left[\left(\frac{2}{s^2+4}\right)\left(\frac{1}{s-1}\right)\right] = \sin 2t * e^t \\
&= \int_0^t e^{t-\tau}\sin 2\tau\,d\tau = e^t\int_0^t e^{-\tau}\sin 2\tau\,d\tau \\
&= e^t\left[\frac{e^{-\tau}}{5}\left(-\sin 2\tau - 2\cos 2\tau\right)\right]_0^t \\
&= \frac{2}{5}e^t - \frac{1}{5}\sin 2t - \frac{2}{5}\cos 2t.
\end{aligned}$$

Example

Solve

$$\begin{cases} 4y'' + y = g(t) \\ y(0) = 3 \\ y'(0) = -7. \end{cases}$$

Example

Solve

$$\begin{cases} 4y'' + y = g(t) \\ y(0) = 3 \\ y'(0) = -7. \end{cases}$$

$$4y'' + y \; = \quad g(t)$$

Example

Solve

$$\begin{cases} 4y'' + y = g(t) \\ y(0) = 3 \\ y'(0) = -7. \end{cases}$$

$$\mathcal{L}\left[4y'' + y\right] = \mathcal{L}\left[g(t)\right]$$

Example

Solve

$$\begin{cases} 4y'' + y = g(t) \\ y(0) = 3 \\ y'(0) = -7. \end{cases}$$

$$\mathcal{L}\left[4y'' + y\right] = \mathcal{L}\left[g(t)\right]$$

$$4(s^2Y - sy(0) - y'(0)) + Y = G(s)$$

Example

Solve

$$\begin{cases} 4y'' + y = g(t) \\ y(0) = 3 \\ y'(0) = -7. \end{cases}$$

$$\mathcal{L}\left[4y'' + y\right] = \mathcal{L}\left[g(t)\right]$$

$$4(s^2Y - 3s + 7) + Y = G(s)$$

Example

Solve

$$\begin{cases} 4y'' + y = g(t) \\ y(0) = 3 \\ y'(0) = -7. \end{cases}$$

$$\mathcal{L}\left[4y'' + y\right] = \mathcal{L}\left[g(t)\right]$$

$$(4s^2 + 1)Y - 12s + 28 = G(s)$$

Example

Solve

$$\begin{cases} 4y'' + y = g(t) \\ y(0) = 3 \\ y'(0) = -7. \end{cases}$$

$$\mathcal{L}\left[4y'' + y\right] = \mathcal{L}\left[g(t)\right]$$

$$(4s^2 + 1)Y = 12s - 28 + G(s)$$

Example

Solve

$$\begin{cases} 4y'' + y = g(t) \\ y(0) = 3 \\ y'(0) = -7. \end{cases}$$

$$\mathcal{L}\left[4y'' + y\right] = \mathcal{L}\left[g(t)\right]$$

$$4\left(s^2 + \frac{1}{4}\right)Y = 12s - 28 + G(s)$$

Example

Solve

$$\begin{cases} 4y'' + y = g(t) \\ y(0) = 3 \\ y'(0) = -7. \end{cases}$$

$$\mathcal{L}\left[4y'' + y\right] = \mathcal{L}\left[g(t)\right]$$

$$Y = \frac{12s}{4(s^2+\frac{1}{4})} - \frac{28}{4(s^2+\frac{1}{4})} + \frac{G(s)}{4(s^2+\frac{1}{4})}$$

Example

Solve

$$\begin{cases} 4y'' + y = g(t) \\ y(0) = 3 \\ y'(0) = -7. \end{cases}$$

$$Y = \frac{3s}{s^2 + \frac{1}{4}} - \frac{7}{s^2 + \frac{1}{4}} + G(s)\frac{\frac{1}{4}}{s^2 + \frac{1}{4}}$$

Example

Solve

$$\begin{cases} 4y'' + y = g(t) \\ y(0) = 3 \\ y'(0) = -7. \end{cases}$$

$$Y = 3\left(\frac{s}{s^2 + \frac{1}{4}}\right) - 14\left(\frac{\frac{1}{2}}{s^2 + \frac{1}{4}}\right) + \frac{1}{2}G(s)\left(\frac{\frac{1}{2}}{s^2 + \frac{1}{4}}\right)$$

Example

Solve

$$\begin{cases} 4y'' + y = g(t) \\ y(0) = 3 \\ y'(0) = -7. \end{cases}$$

$$Y = 3\mathcal{L}\left[\cos\frac{t}{2}\right] - 14\left(\frac{\frac{1}{2}}{s^2 + \frac{1}{4}}\right) + \frac{1}{2}G(s)\left(\frac{\frac{1}{2}}{s^2 + \frac{1}{4}}\right)$$

Example

Solve

$$\begin{cases} 4y'' + y = g(t) \\ y(0) = 3 \\ y'(0) = -7. \end{cases}$$

$$Y = 3\mathcal{L}\left[\cos\frac{t}{2}\right] - 14\mathcal{L}\left[\sin\frac{t}{2}\right] + \frac{1}{2}G(s)\mathcal{L}\left[\sin\frac{t}{2}\right]$$

Example

Solve

$$\begin{cases} 4y'' + y = g(t) \\ y(0) = 3 \\ y'(0) = -7. \end{cases}$$

$$Y = 3\mathcal{L}\left[\cos\frac{t}{2}\right] - 14\mathcal{L}\left[\sin\frac{t}{2}\right] + \frac{1}{2}G(s)\mathcal{L}\left[\sin\frac{t}{2}\right]$$

$$\boxed{y(t) = \qquad\qquad\qquad\qquad}$$

Example

Solve

$$\begin{cases} 4y'' + y = g(t) \\ y(0) = 3 \\ y'(0) = -7. \end{cases}$$

$$Y = 3\mathcal{L}\left[\cos\frac{t}{2}\right] - 14\mathcal{L}\left[\sin\frac{t}{2}\right] + \frac{1}{2}G(s)\mathcal{L}\left[\sin\frac{t}{2}\right]$$

$$\boxed{y(t) = 3\cos\frac{t}{2}}$$

Example

Solve

$$\begin{cases} 4y'' + y = g(t) \\ y(0) = 3 \\ y'(0) = -7. \end{cases}$$

$$Y = 3\mathcal{L}\left[\cos\frac{t}{2}\right] - 14\mathcal{L}\left[\sin\frac{t}{2}\right] + \frac{1}{2}G(s)\mathcal{L}\left[\sin\frac{t}{2}\right]$$

$$\boxed{y(t) = 3\cos\frac{t}{2} - 14\sin\frac{t}{2}}$$

Example

Solve

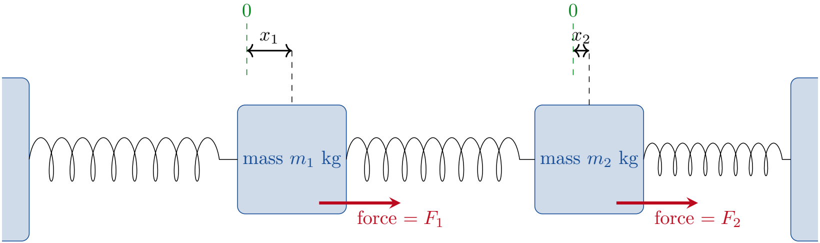
$$\begin{cases} 4y'' + y = g(t) \\ y(0) = 3 \\ y'(0) = -7. \end{cases}$$

$$Y = 3\mathcal{L}\left[\cos\frac{t}{2}\right] - 14\mathcal{L}\left[\sin\frac{t}{2}\right] + \frac{1}{2}G(s)\mathcal{L}\left[\sin\frac{t}{2}\right]$$

$$\boxed{y(t) = 3\cos\frac{t}{2} - 14\sin\frac{t}{2} + \frac{1}{2}g(t) * \sin\frac{t}{2}.}$$

Systems of First Order Linear Equations

5.1 Introduction



Consider the dynamical system shown above. There are two blocks and three springs. Forces F_1 and F_2 act on the blocks as shown.

See `https://tinyurl.com/wm2ogdh`

We expect that the acceleration of the blocks will depend on

- the displacements x_1 and x_2;
- the forces F_1 and F_2; and
- the masses of the blocks.

5.1 Introduction

We expect that the acceleration of the blocks will depend on

- the displacements x_1 and x_2;
- the forces F_1 and F_2; and
- the masses of the blocks.

So we expect that:

$$\begin{cases} \dfrac{d^2x_1}{dt^2} = f_1(x_1, x_2, F_1, m_1) \\ \dfrac{d^2x_2}{dt^2} = f_2(x_1, x_2, F_2, m_2). \end{cases}$$

5.1 Introduction

We expect that the acceleration of the blocks will depend on

- the displacements x_1 and x_2;
- the forces F_1 and F_2; and
- the masses of the blocks.

So we expect that:

$$\begin{cases} \dfrac{d^2x_1}{dt^2} = f_1(x_1, x_2, F_1, m_1) \\ \dfrac{d^2x_2}{dt^2} = f_2(x_1, x_2, F_2, m_2). \end{cases}$$

This is a system of two ODEs. To find $x_1(t)$ and $x_2(t)$, we would need to solve these equations at the same time.

5.1 Introduction

The most famous system of ODEs is the system of *Predator-Prey* equations:

$$\begin{cases} \dfrac{dx}{dt} = \alpha x - \beta xy \\[2ex] \dfrac{dy}{dt} = \delta xy - \gamma y \end{cases}$$

where

$$\begin{aligned} x(t) &= \text{number of prey (e.g. mice)} \\ y(t) &= \text{number of predators (e.g. owls),} \end{aligned}$$

which originate circa 1925.

It is possible to convert an nth order linear ODE into a system of n first order linear ODEs. Or vice versa.

$$
\begin{aligned}
a_n y^{(n)} + a_{n-1} y^{(n-1)} + \dots \\
+ a_1 y' + a_0 y = g(t)
\end{aligned}
\quad \longleftrightarrow \quad
\begin{cases}
x_1' = b_{11} x_1 + \ldots + b_{1n} x_n + h_1(t) \\
x_2' = b_{21} x_1 + \ldots + b_{2n} x_n + h_2(t) \\
\vdots \\
x_n' = b_{n1} x_1 + \ldots + b_{nn} x_n + h_n(t)
\end{cases}
$$

Example

Write

$$u'' + 0.25u' + u = 0$$

as a system of two first order ODEs.

Example

Write

$$u'' + 0.25u' + u = 0$$

as a system of two first order ODEs.

Let $x_1 = u$ and $x_2 = u'$.

Example

Write

$$u'' + 0.25u' + u = 0$$

as a system of two first order ODEs.

Let $x_1 = u$ and $x_2 = u'$. Then clearly $x_1' = u' = x_2$

Example

Write

$$u'' + 0.25u' + u = 0$$

as a system of two first order ODEs.

Let $x_1 = u$ and $x_2 = u'$. Then clearly $x_1' = u' = x_2$ and

$$x_2' = u'' = -0.25u' - u = -0.25x_2 - x_1.$$

Example

Write
$$u'' + 0.25u' + u = 0$$
as a system of two first order ODEs.

Let $x_1 = u$ and $x_2 = u'$. Then clearly $x_1' = u' = x_2$ and
$$x_2' = u'' = -0.25u' - u = -0.25x_2 - x_1.$$

Therefore
$$\begin{cases} x_1' = x_2 \\ x_2' = -x_1 - 0.25x_2. \end{cases}$$

5.1 Introduction

Remark

We will need

- matrices,
- eigenvalues,
- eigenvectors,
- the Wronskian,
- linear independence,
- and more

from MATH215 – please either revise your Linear Algebra lecture notes or read your Linear Algebra book or read §7.2-7.3 in the textbook by Boyce and DiPrima.

Basic Theory of Systems of First Order Linear Equations

$$\begin{cases} x_1' = p_{11}(t)x_1 + p_{12}(t)x_2 + \ldots + p_{1n}(t)x_n + g_1(t) \\ x_2' = p_{21}(t)x_1 + p_{22}(t)x_2 + \ldots + p_{2n}(t)x_n + g_2(t) \\ \qquad \vdots \\ x_n' = p_{n1}(t)x_1 + p_{n2}(t)x_2 + \ldots + p_{nn}(t)x_n + g_n(t) \end{cases}$$

is a system of n linear ODEs and n variables: $x_1, x_2, \ldots, x_n$.

If we write

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, \ \mathbf{x}' = \begin{bmatrix} x_1' \\ x_2' \\ \vdots \\ x_n' \end{bmatrix}, \ P = \begin{bmatrix} p_{11} & p_{12} & \dots & p_{1n} \\ p_{21} & p_{22} & \dots & p_{2n} \\ \vdots & \vdots & & \vdots \\ p_{n1} & p_{n2} & \dots & p_{nn} \end{bmatrix}, \text{ and } \mathbf{g} = \begin{bmatrix} g_1 \\ g_2 \\ \vdots \\ g_n \end{bmatrix}$$

then we can write this system as

$$\mathbf{x}' = P(t)\mathbf{x} + \mathbf{g}(t).$$

First we will consider the homogeneous system

$$\mathbf{x}' = P(t)\mathbf{x}.$$

In Chapters 3 and 4 when we had multiple solutions, we wrote them as $y_1(t)$, $y_2(t)$, ... But we are already using x_1, x_2, ... to denote coordinates. So we need a new type of notation.

5.2 Basic Theory of Systems of First Order Linear Equations

In Chapters 3 and 4 when we had multiple solutions, we wrote them as $y_1(t)$, $y_2(t)$, ... But we are already using x_1, x_2, ... to denote coordinates. So we need a new type of notation.

Notation

We use $\mathbf{x}^{(1)}(t)$, $\mathbf{x}^{(2)}(t)$, ... to denote different vector solutions.

Recall from Chapter 3 that if $y_1(t)$ and $y_2(t)$ are both solutions to

$$ay'' + by' + cy = 0,$$

then

$$c_1 y_1 + c_2 y_2$$

is also a solution.

Theorem

If $\mathbf{x}^{(1)}(t)$ *and* $\mathbf{x}^{(2)}(t)$ *are solutions to* $\mathbf{x}' = P(t)\mathbf{x}$, *then*

$$c_1\mathbf{x}^{(1)} + c_2\mathbf{x}^{(2)}$$

is also a solution for any $c_1, c_2 \in \mathbb{R}$.

Example

$\mathbf{x}^{(1)} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} e^{3t}$ and $\mathbf{x}^{(2)} = \begin{bmatrix} 1 \\ -2 \end{bmatrix} e^{-t}$ are both solutions to $\mathbf{x}' = \begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix} \mathbf{x}$ (we will see this later).

Example

$\mathbf{x}^{(1)} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} e^{3t}$ and $\mathbf{x}^{(2)} = \begin{bmatrix} 1 \\ -2 \end{bmatrix} e^{-t}$ are both solutions to $\mathbf{x}' = \begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix} \mathbf{x}$ (we will see this later).Therefore

$$\mathbf{x}(t) = c_1\mathbf{x}^{(1)} + c_2\mathbf{x}^{(2)} = c_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} e^{3t} + c_2 \begin{bmatrix} 1 \\ -2 \end{bmatrix} e^{-t}$$

is also a solution to this system.

(Suppose that $P(t)$ is an $n \times n$ matrix.)

Theorem

If $\mathbf{x}^{(1)}(t)$, $\mathbf{x}^{(2)}(t)$, $\ldots$, $\mathbf{x}^{(n)}(t)$ *are linearly independent solutions to* $\mathbf{x}' = P(t)\mathbf{x}$, *then every solution to this system can be written as*

$$\mathbf{x}(t) = c_1\mathbf{x}^{(1)} + c_2\mathbf{x}^{(2)} + \ldots + c_n\mathbf{x}^{(n)}$$

in exactly one way.

Definition

In this case, we say that $\mathbf{x}^{(1)}(t)$, $\mathbf{x}^{(2)}(t)$, $\ldots$, $\mathbf{x}^{(n)}(t)$ form a *fundamental set of solutions* to $\mathbf{x}' = P(t)\mathbf{x}$.

Definition

In this case, we say that $\mathbf{x}^{(1)}(t)$, $\mathbf{x}^{(2)}(t)$, $\ldots$, $\mathbf{x}^{(n)}(t)$ form a *fundamental set of solutions* to $\mathbf{x}' = P(t)\mathbf{x}$.

Definition

In this case,

$$\mathbf{x}(t) = c_1\mathbf{x}^{(1)} + c_2\mathbf{x}^{(2)} + \ldots + c_n\mathbf{x}^{(n)}$$

is called the *general solution* to $\mathbf{x}' = P(t)\mathbf{x}$.

Next Time

- 5.3 Homogeneous Linear Systems with Constant Coefficients
- 5.4 Complex Eigenvalues
- 5.5 Fundamental Matrices