

# Lecture 4

- 3.1 Tangents and the Derivative at a Point
- 3.2 The Derivative as a Function
- 3.3 Differentiation Rules
- 3.4 The Derivative as a Rate of Change

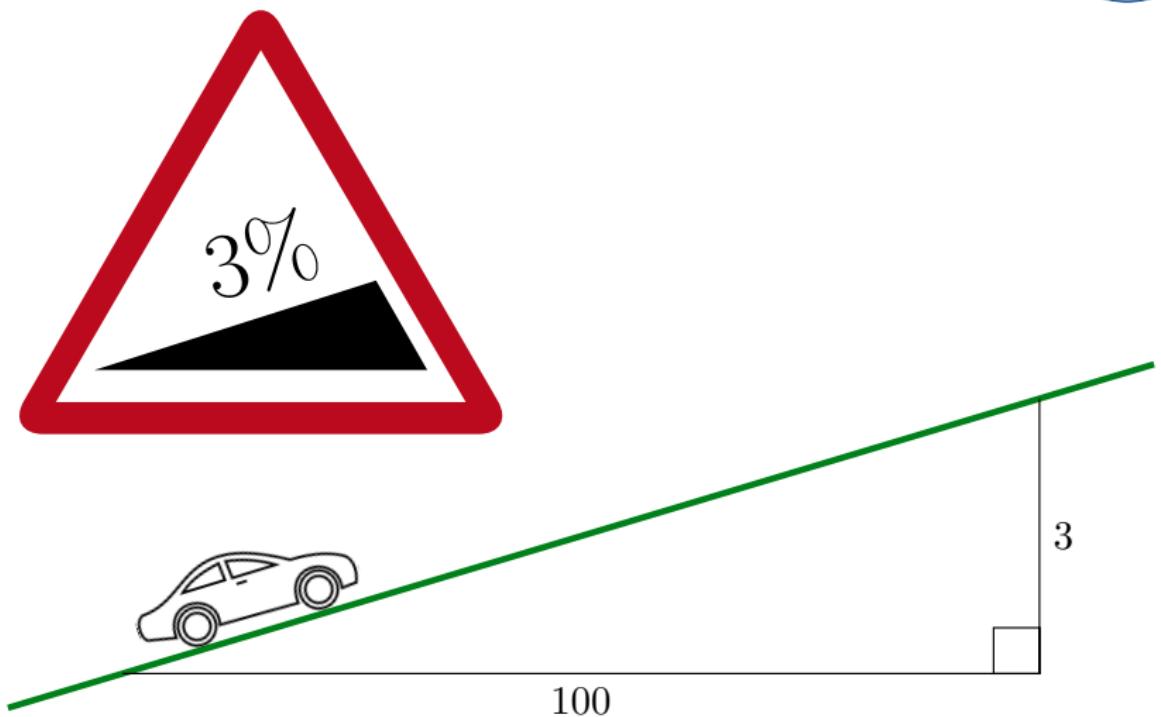


# Tangents and the Derivative at a Point

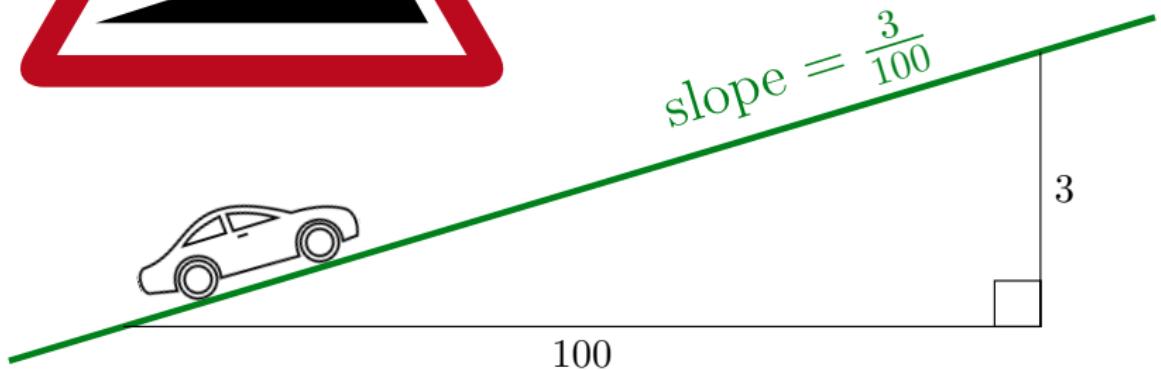
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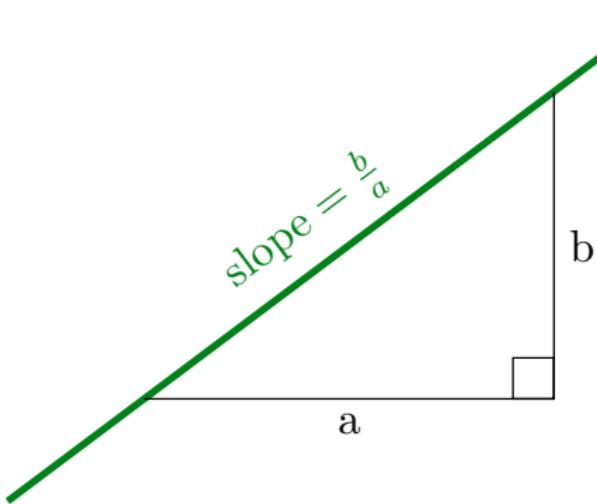
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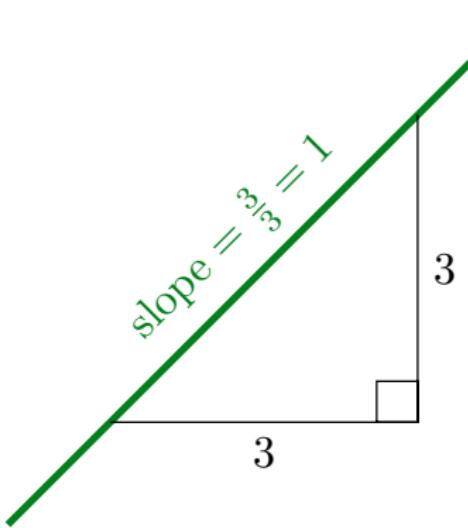
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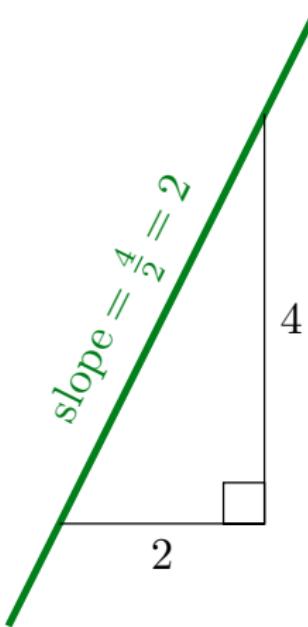
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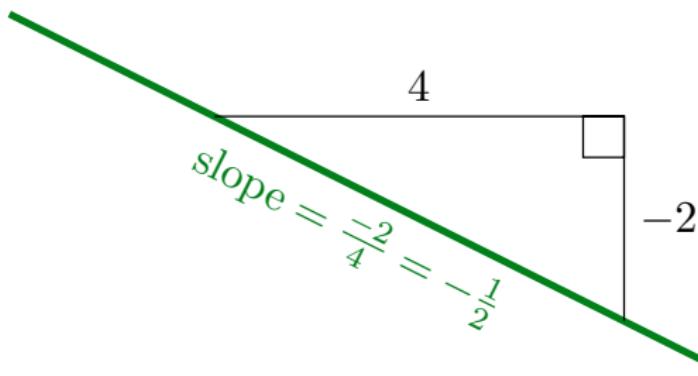
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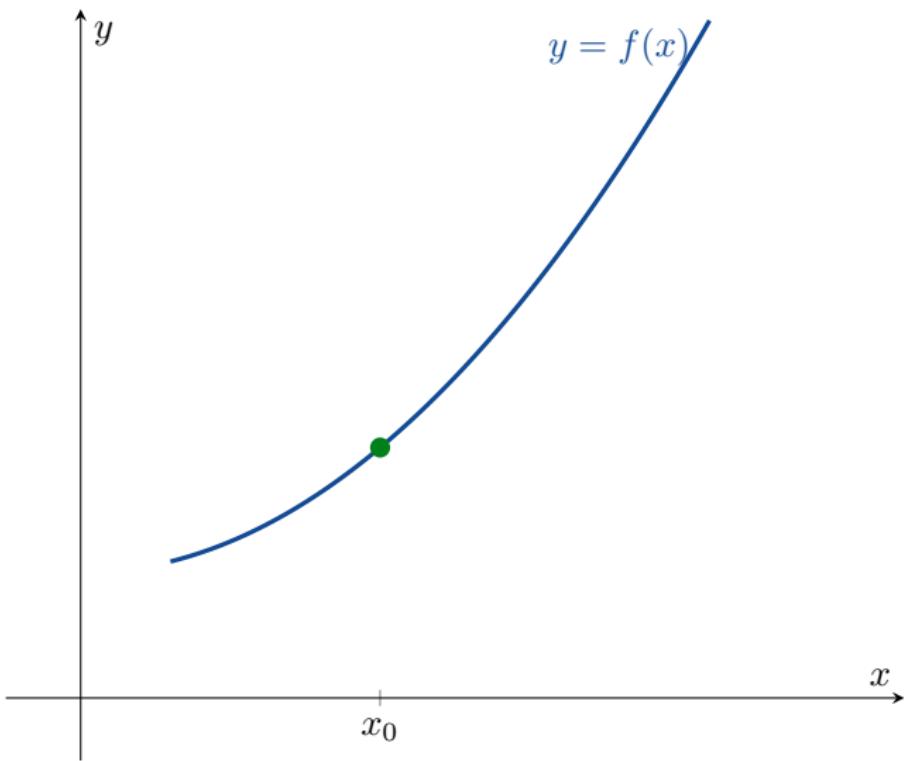
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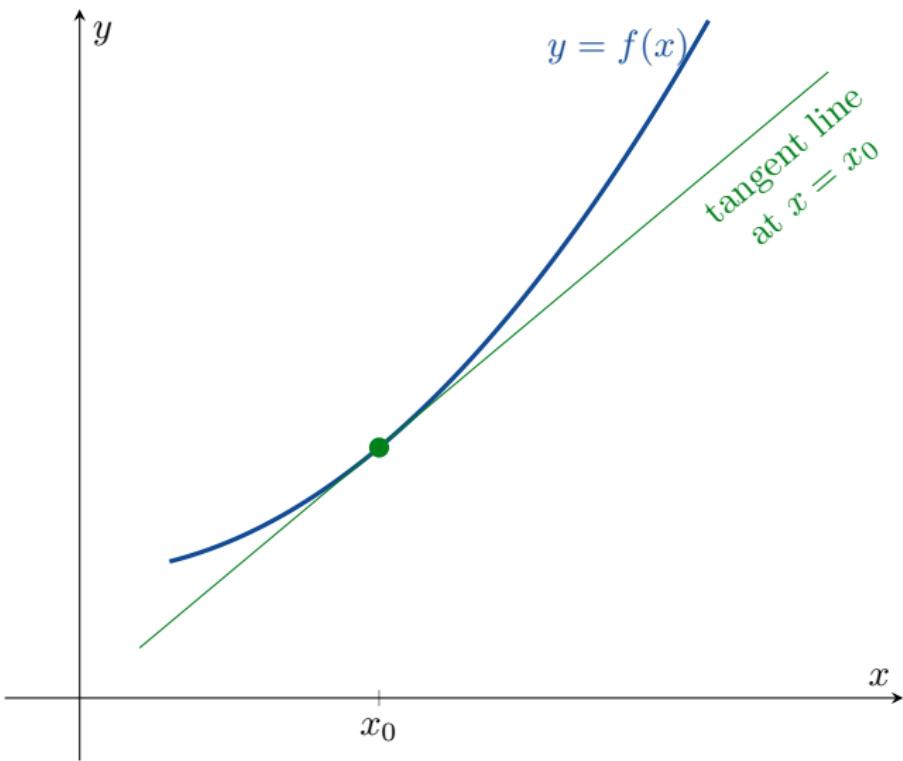
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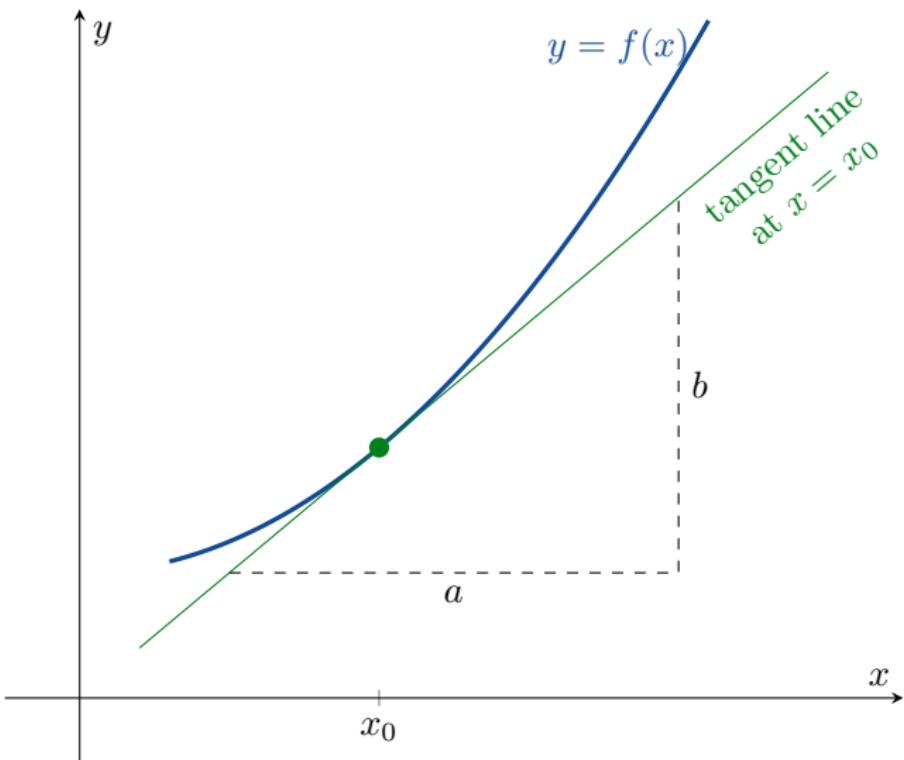
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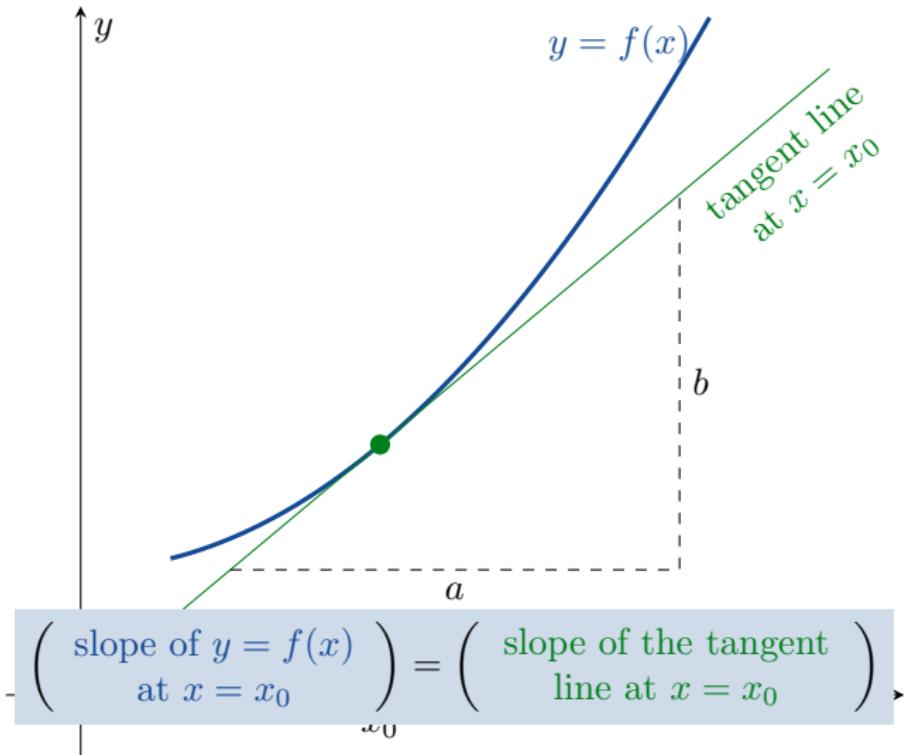
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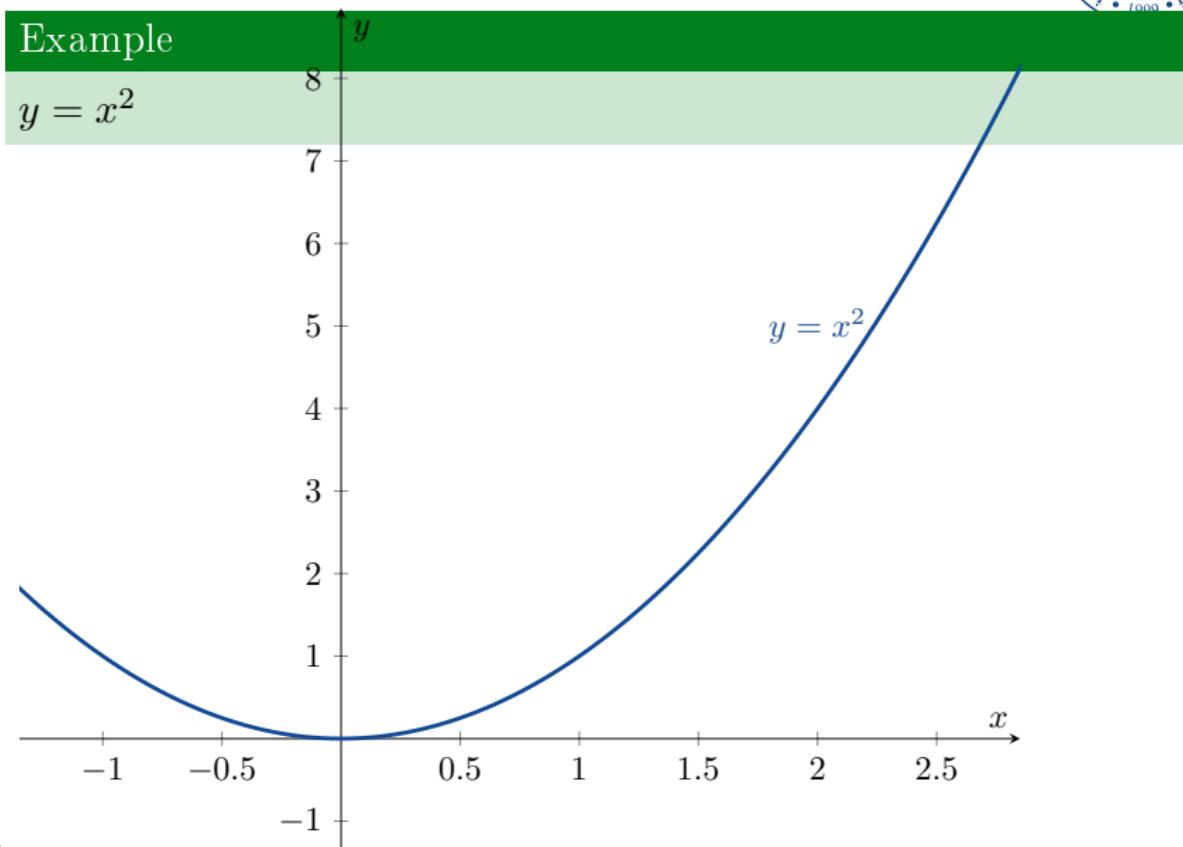
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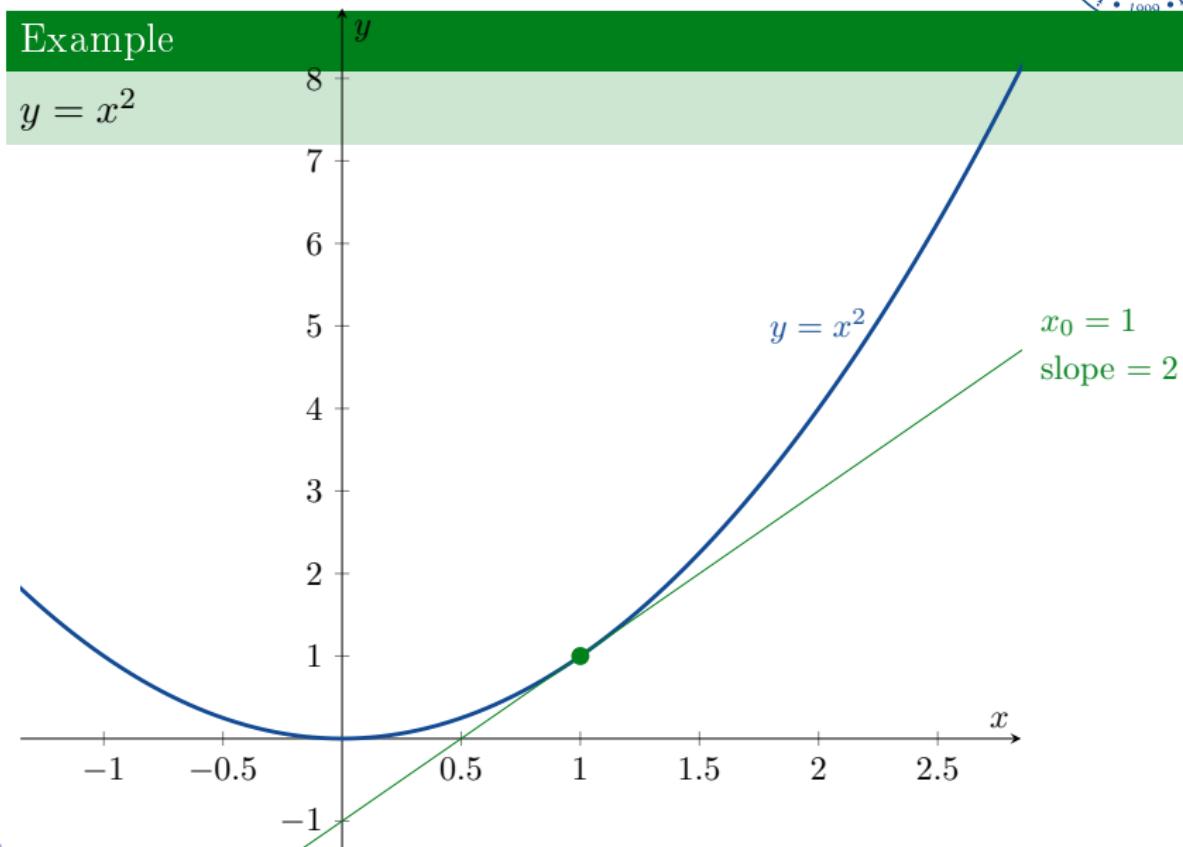
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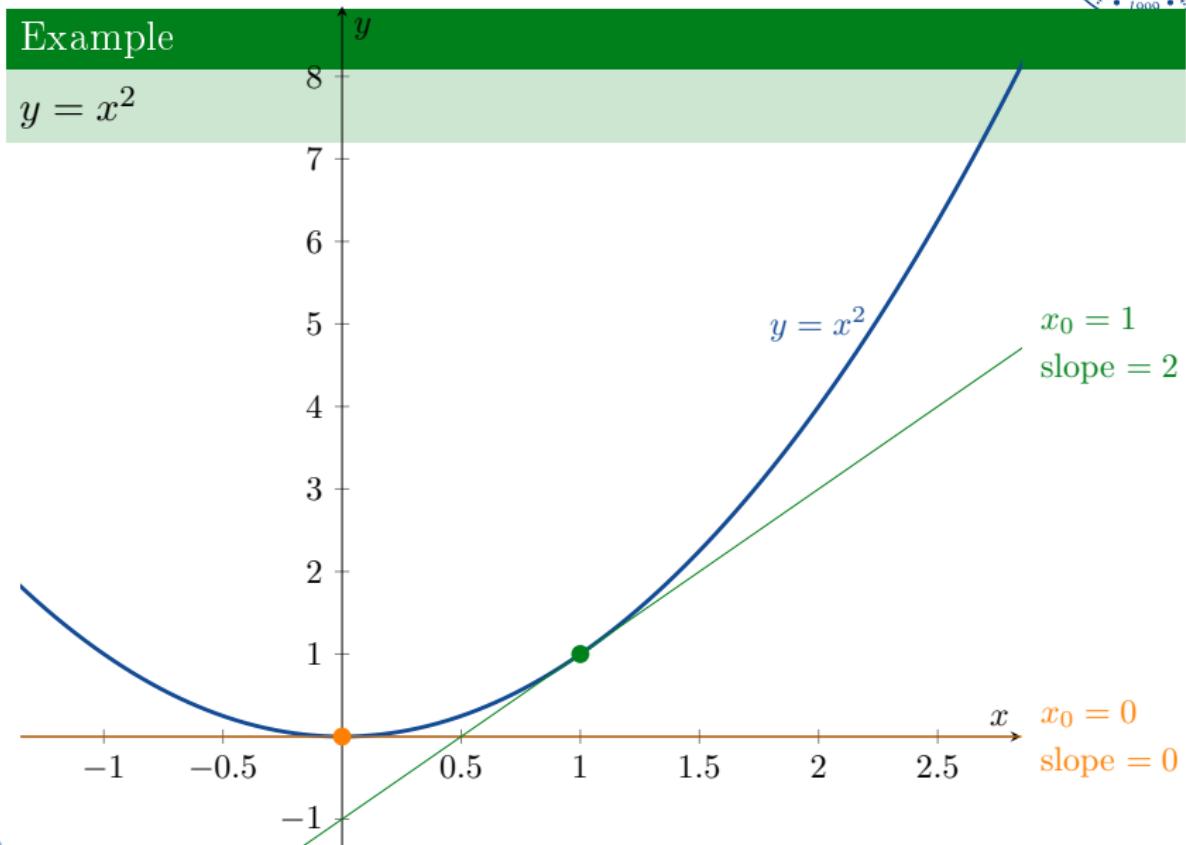
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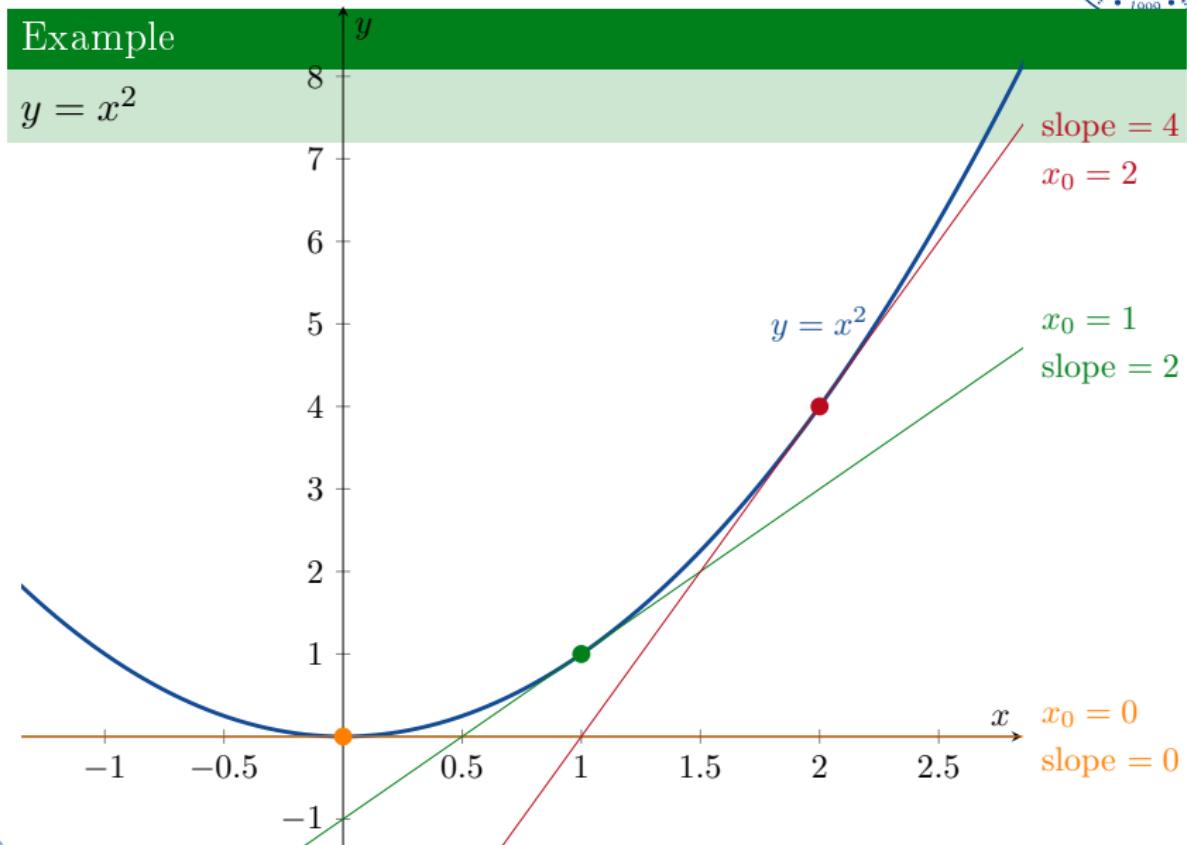
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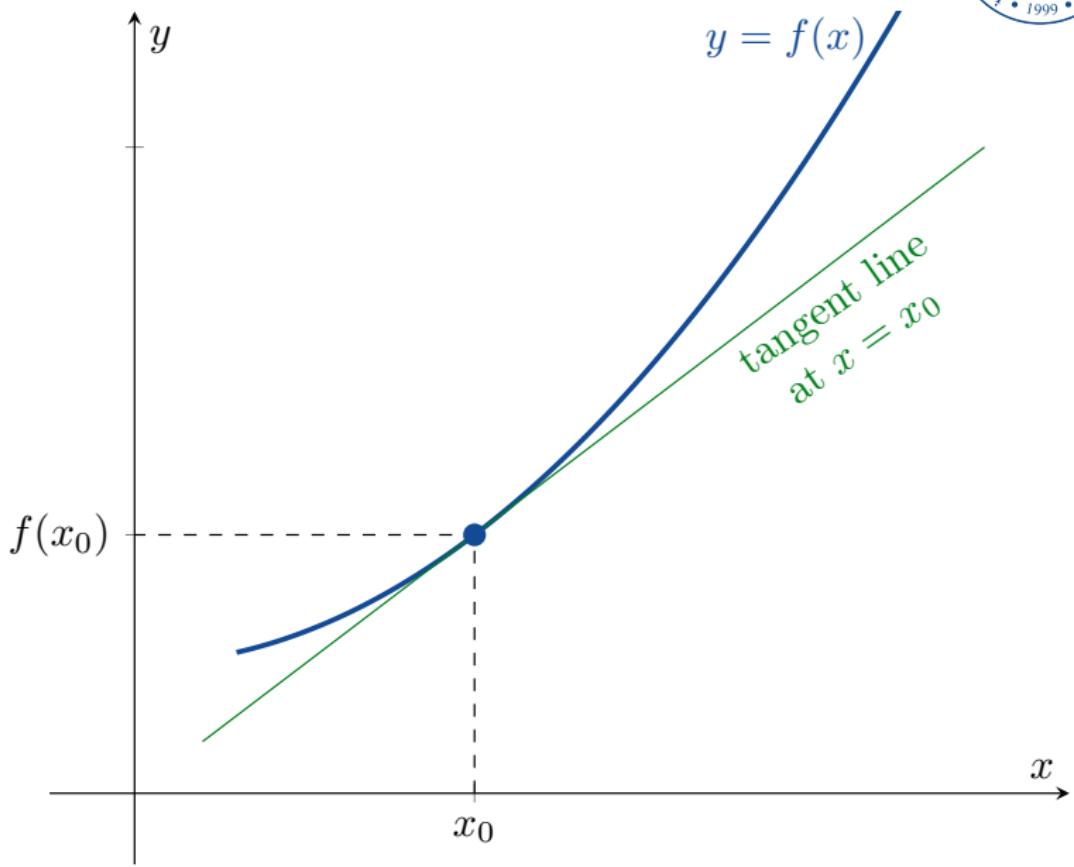


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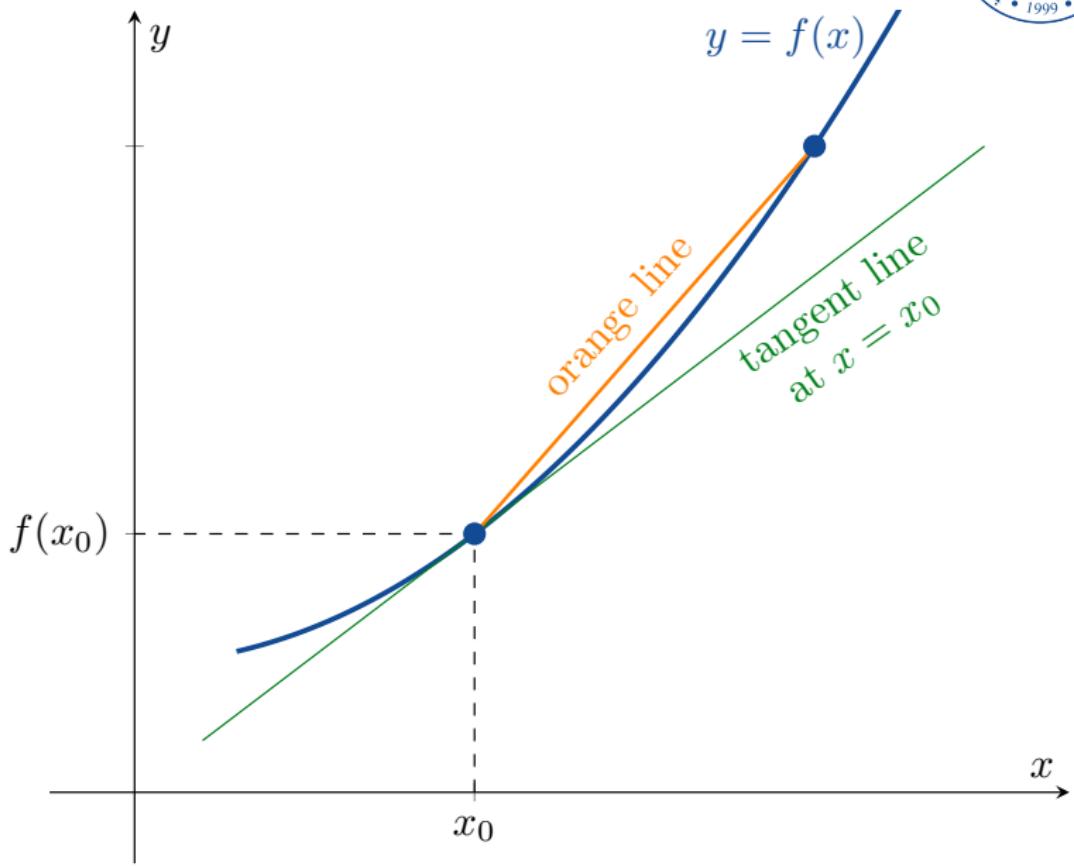


How can we calculate the slope of the tangent line?

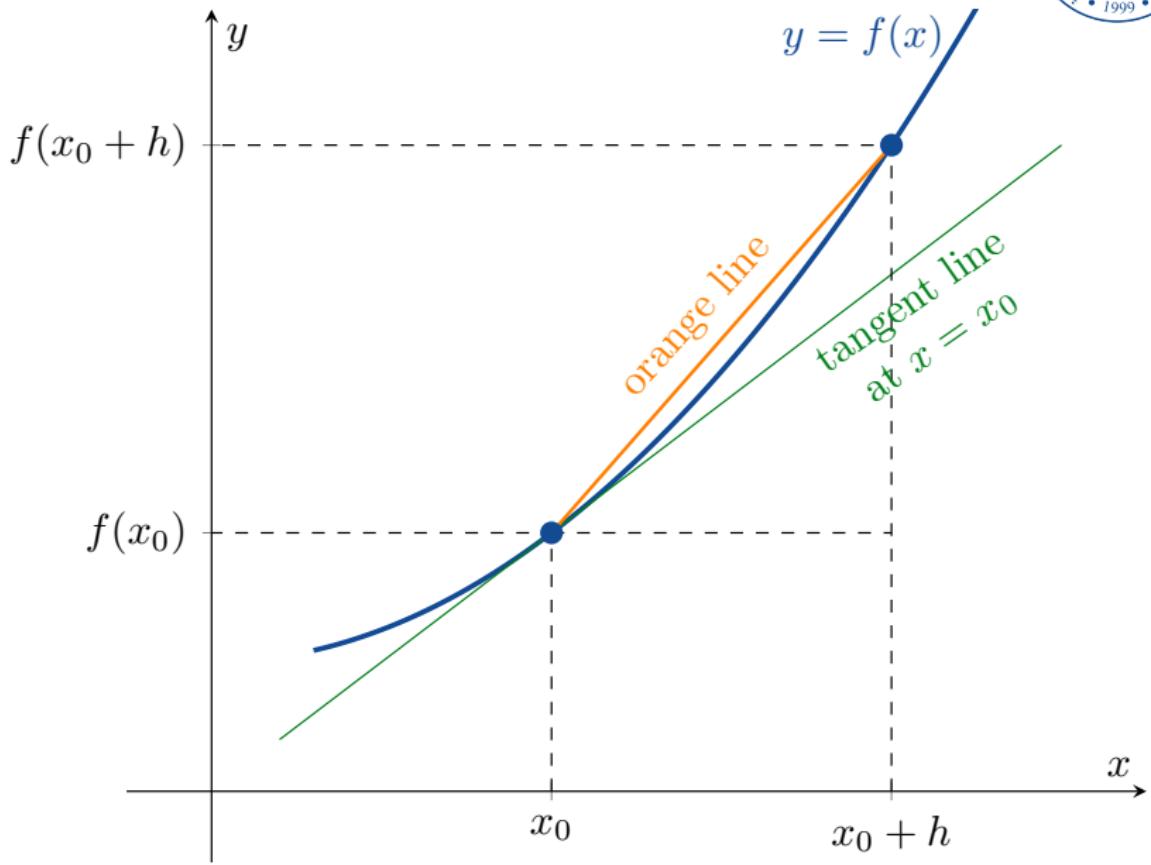
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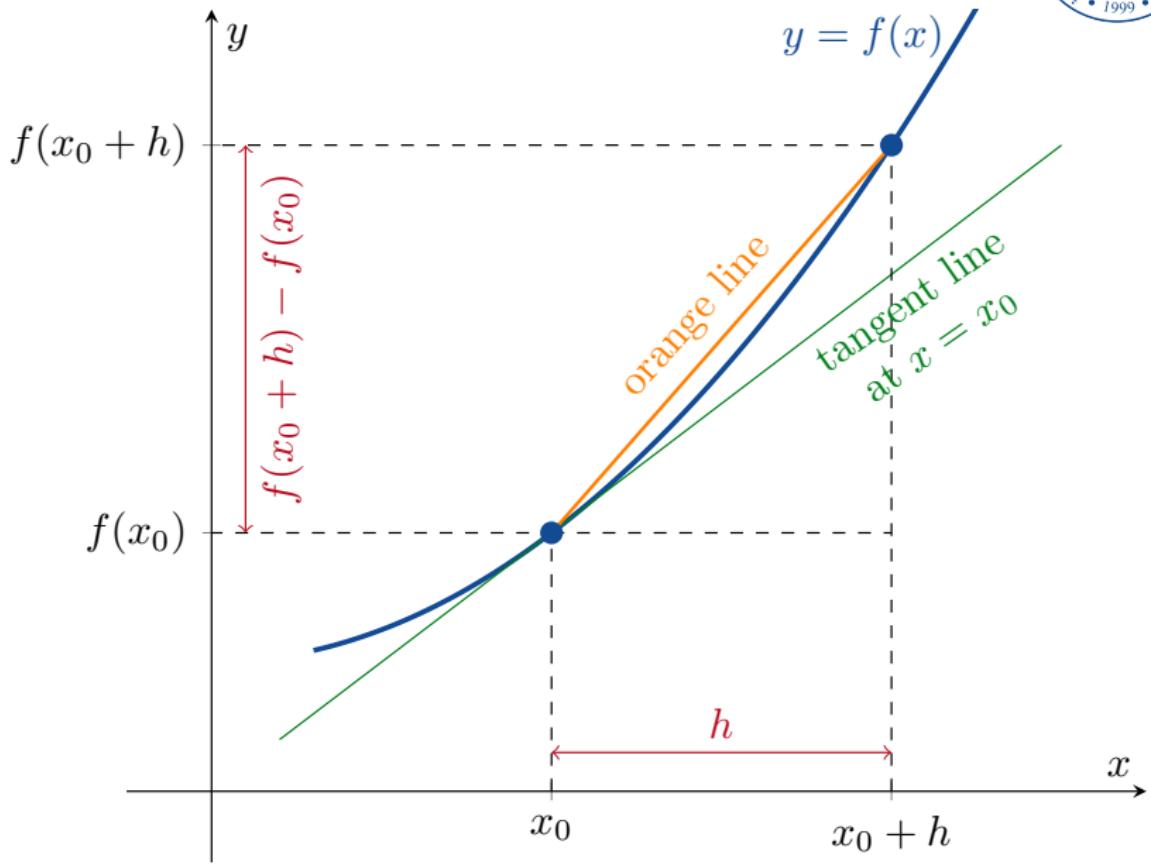
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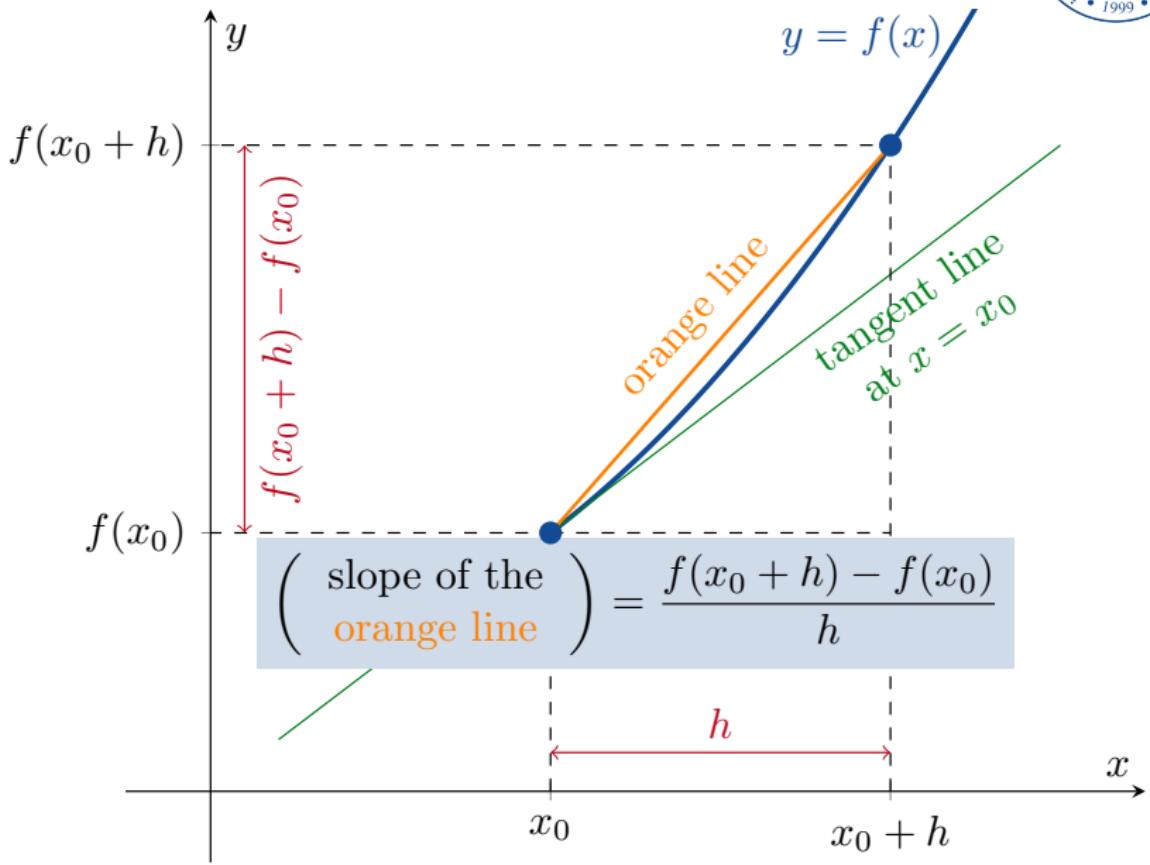
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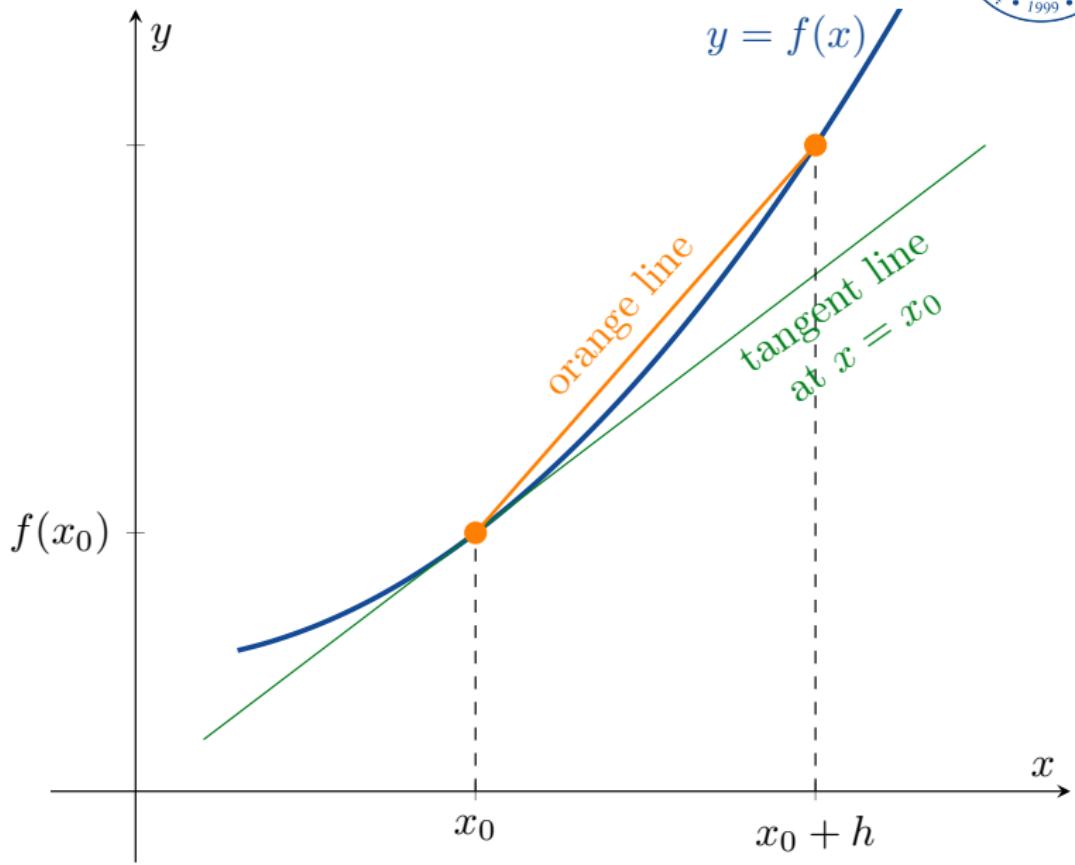
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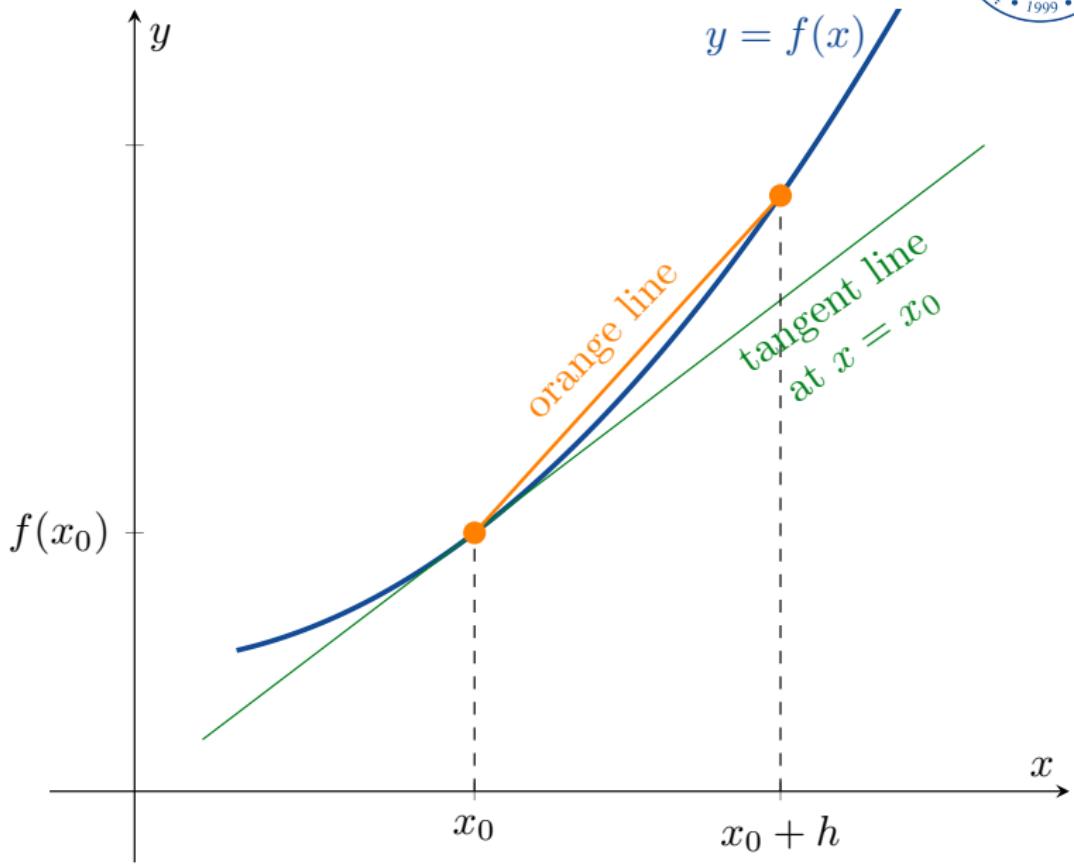
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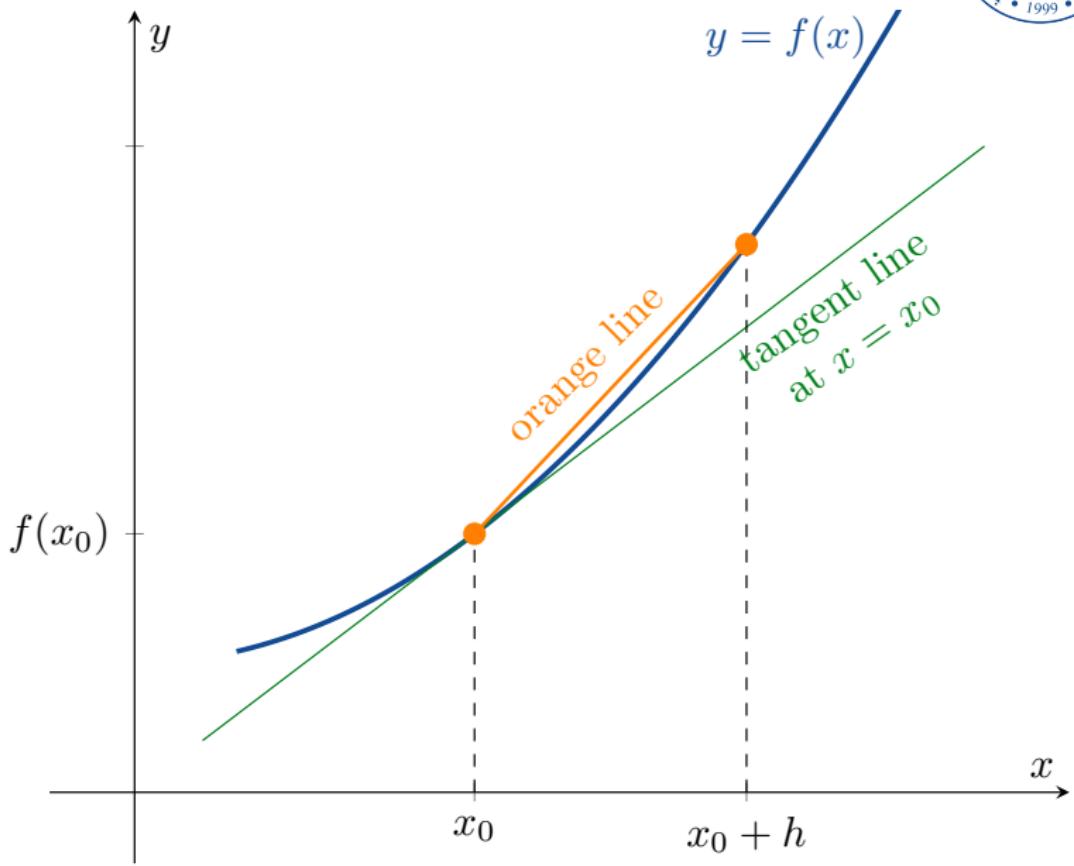
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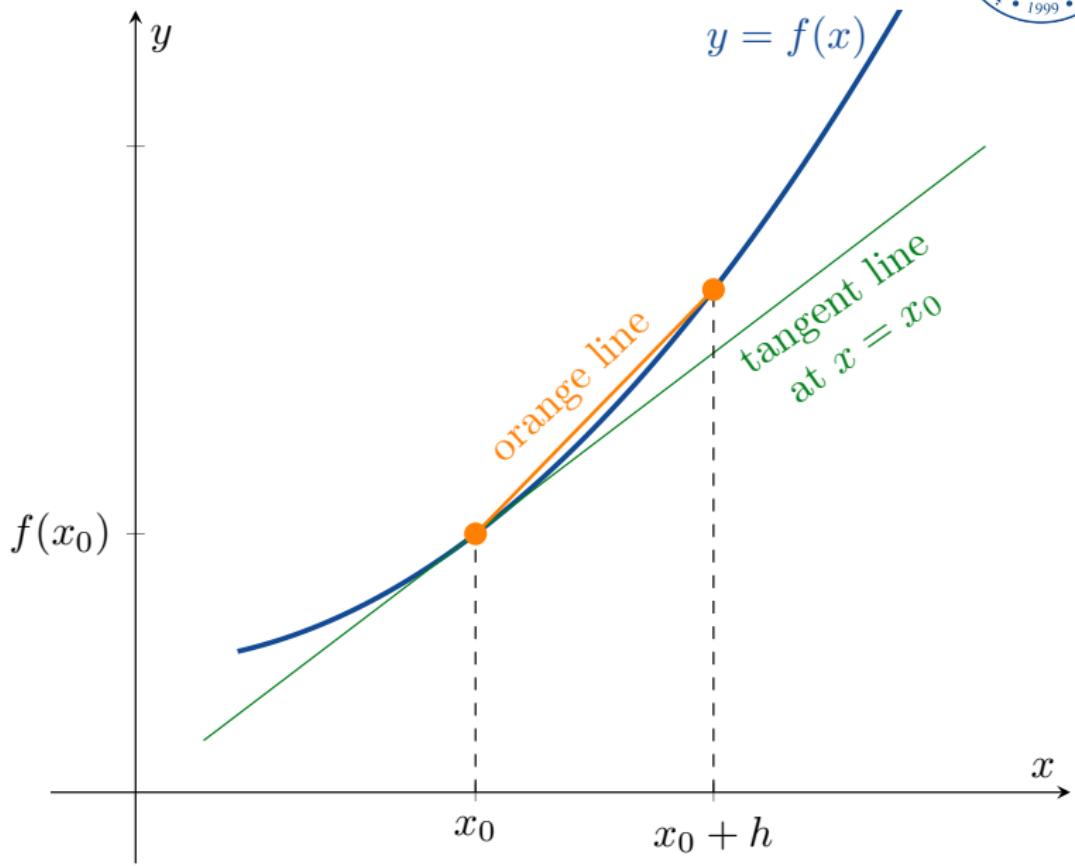
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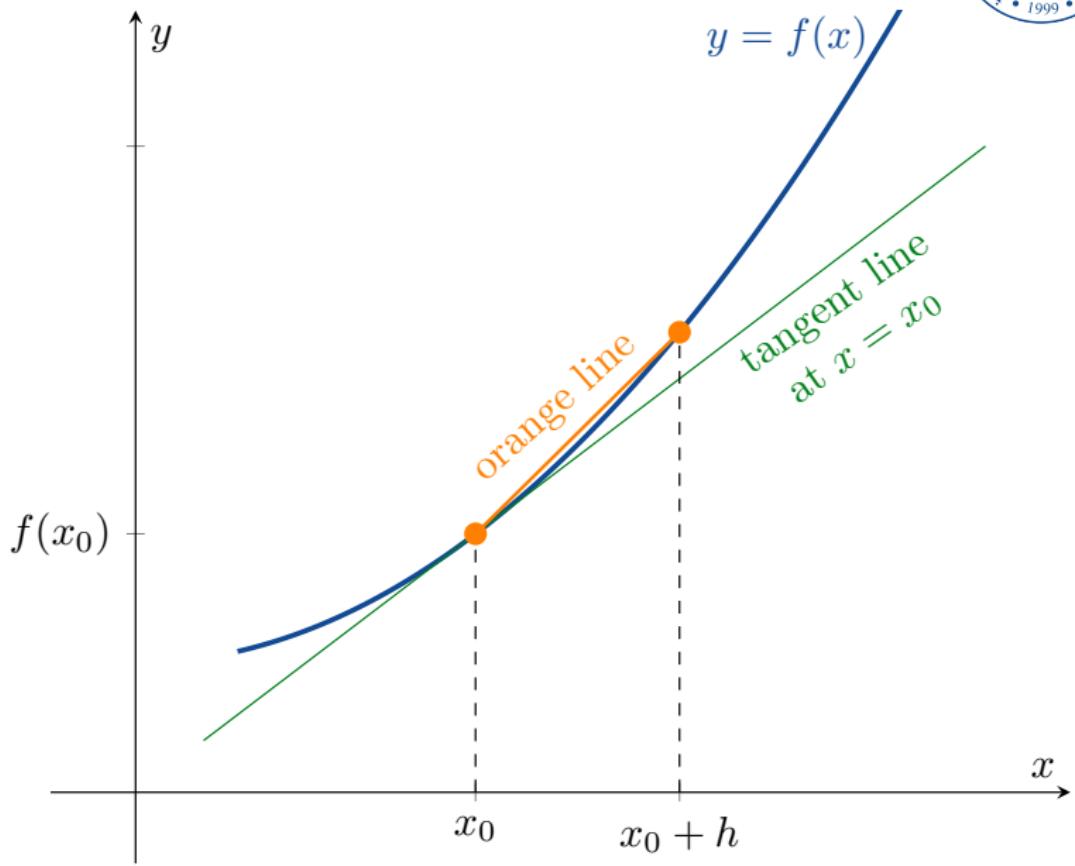
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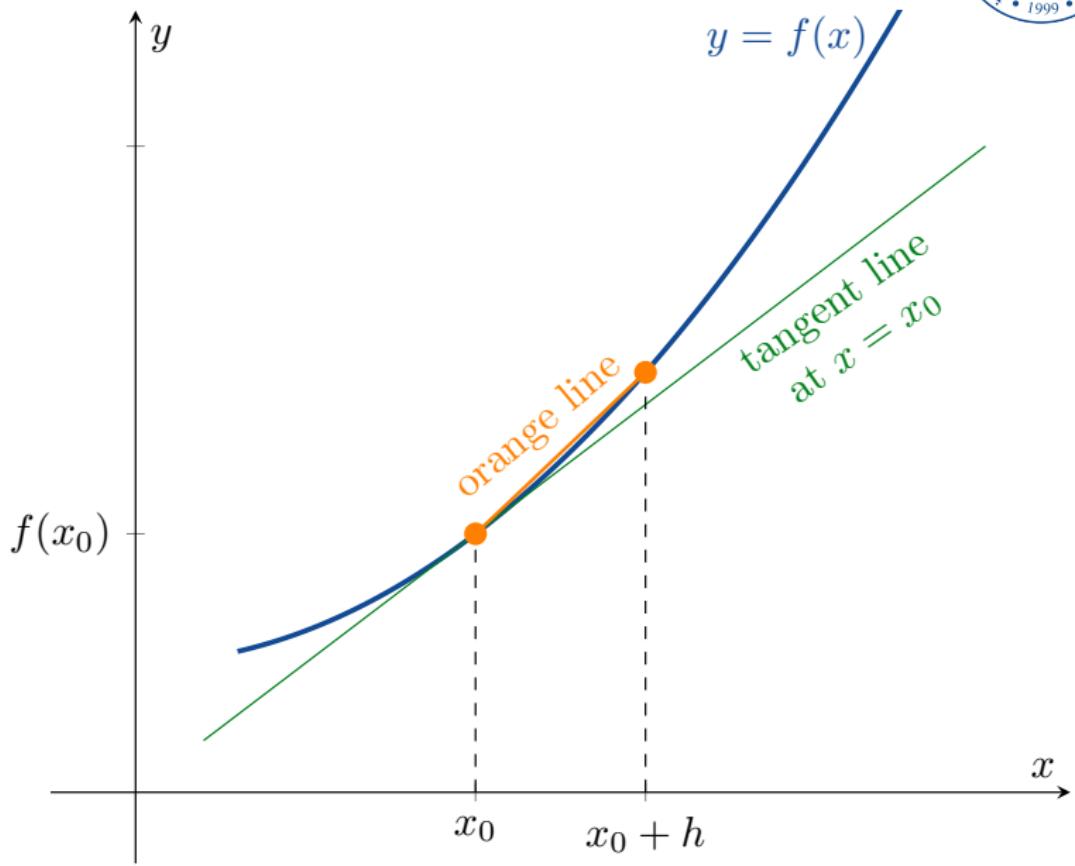
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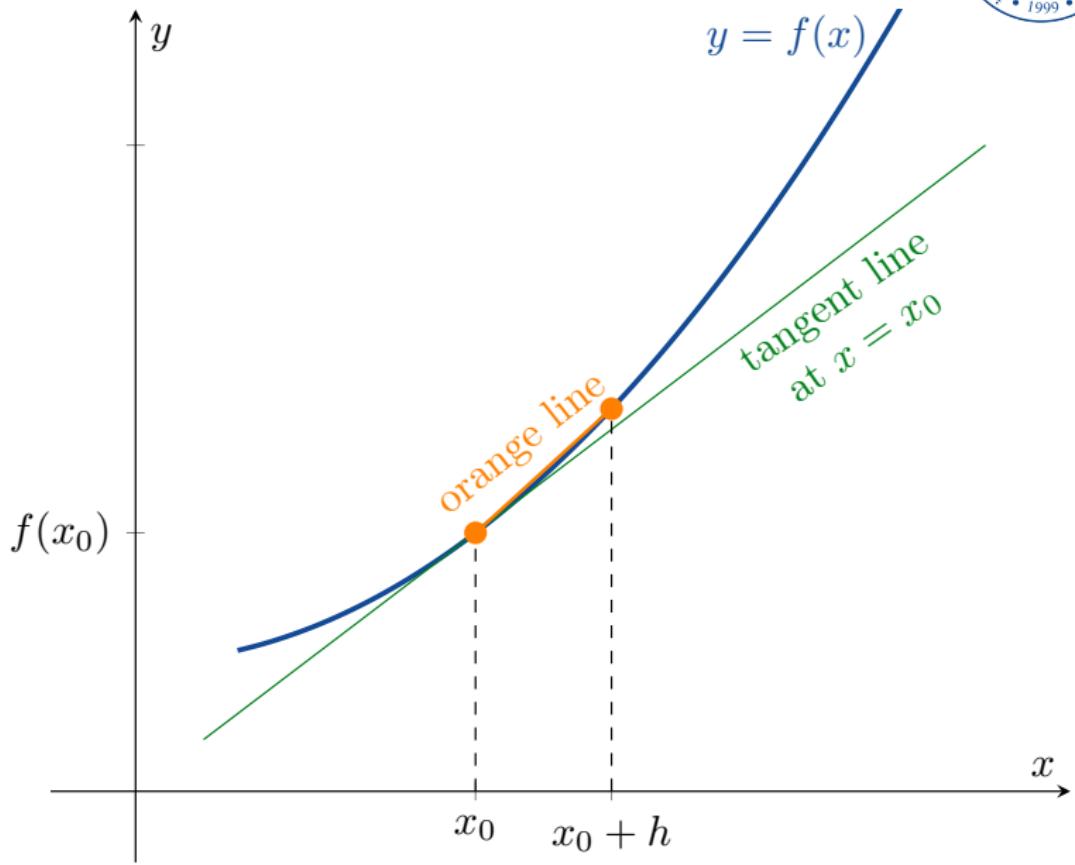
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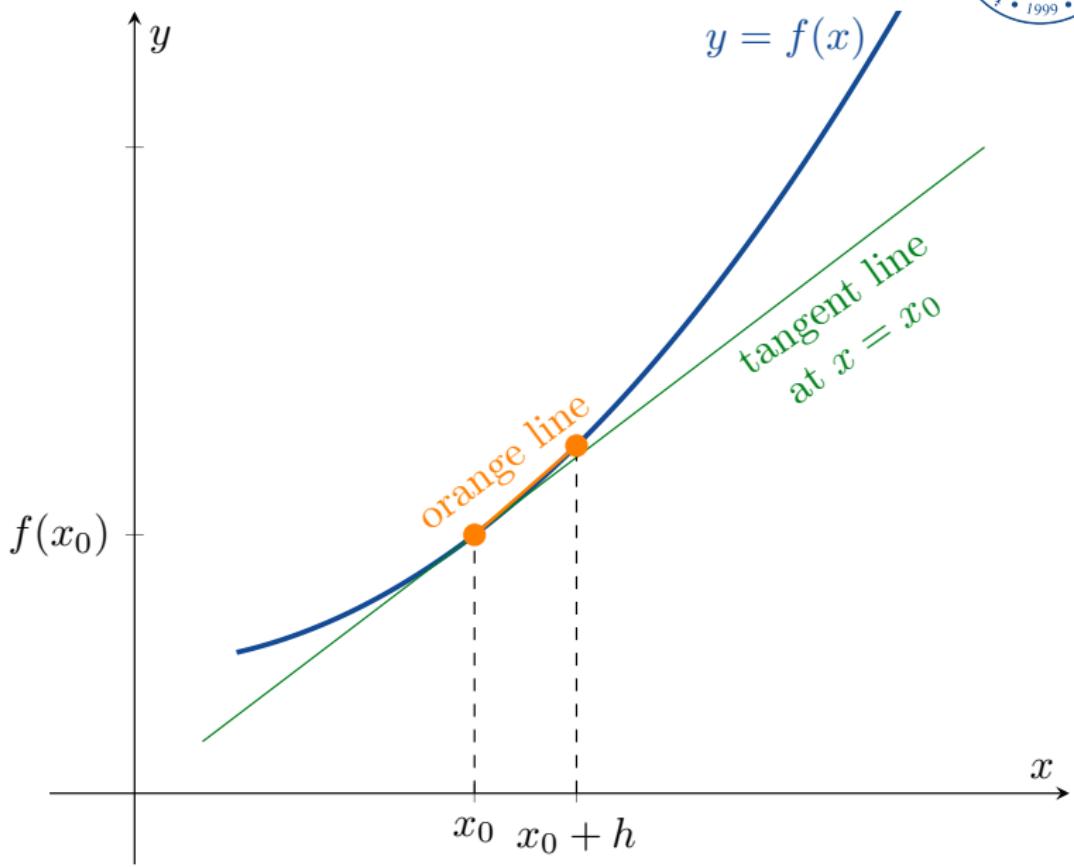
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If  $h$  is very very small, then

$$\left( \begin{array}{l} \text{slope of the} \\ \text{tangent line} \end{array} \right) \approx \left( \begin{array}{l} \text{slope of the} \\ \text{orange line} \end{array} \right) = \frac{f(x_0 + h) - f(x_0)}{h}$$

### 3.1 Tangents and the Derivative at a Point



## The Derivative of $f$

### Definition

The *derivative of a function f at a point  $x_0$*  is

$$f'(x_0) = \lim_{h \rightarrow 0} \frac{f(x_0 + h) - f(x_0)}{h}$$

if the limit exists.

( $f'$  is pronounced “ $f$  prime”)

### 3.1 Tangents and the Derivative at a Point



#### Example

Find the derivative of the function  $g(x) = \frac{1}{x}$ ,  $x \neq 0$ .

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### 3.1 Tangents and the Derivative at a Point

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### 3.1 Tangents and the Derivative at a Point



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### 3.1 Tangents and the Derivative at a Point



#### Example

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### 3.1 Tangents and the Derivative at a Point



#### Example

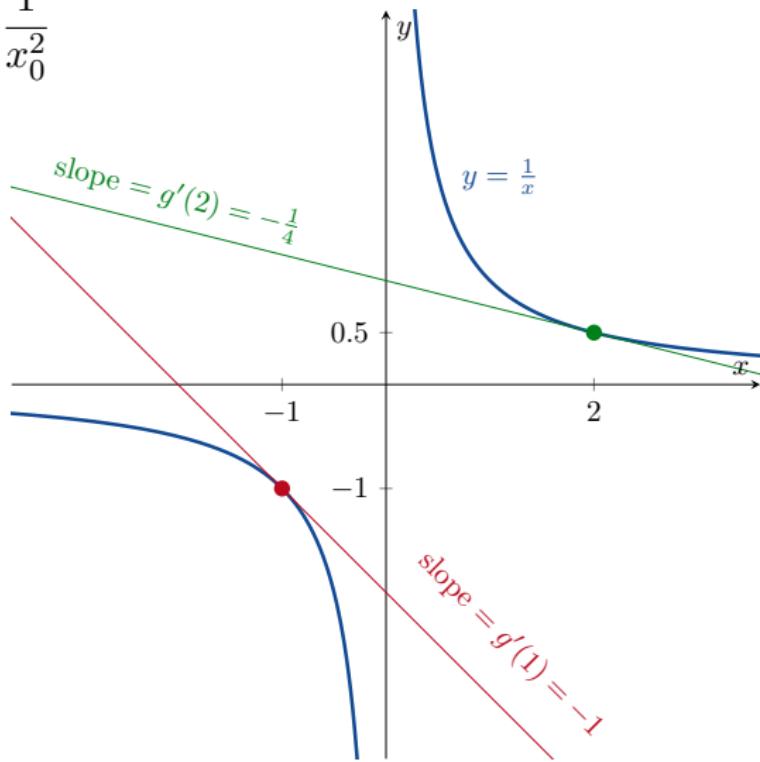
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### 3.1 Tangents and the Derivative at a Point

$$g'(x_0) = -\frac{1}{x_0^2}$$





# The Derivative as a Function

## 3.2 The Derivative as a Function



### Definition

If  $f'(x_0)$  exists, we say that  $f$  is *differentiable at  $x_0$* .

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Let  $f : D \rightarrow \mathbb{R}$  be a function. If  $f$  is differentiable at every  $x_0 \in D$ , we say that  $f$  is *differentiable*.

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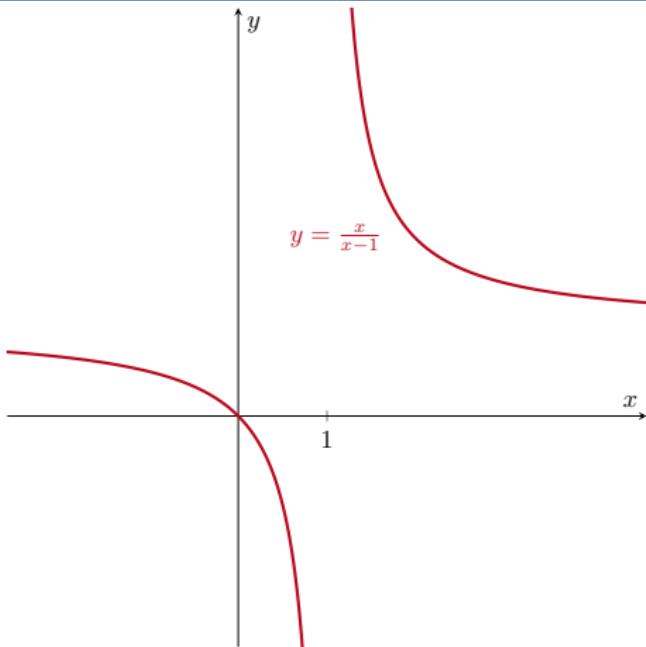
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If  $f : D \rightarrow \mathbb{R}$  is differentiable, then we have a new function  $f' : D \rightarrow \mathbb{R}$ .

### Definition

$f'$  is called the *derivative* of  $f$ .

## 3.2 The Derivative as a Function



Example

Differentiate  $f(x) = \frac{x}{x-1}$ .

## 3.2 The Derivative as a Function

First note that if  $f(x) = \frac{x}{x-1}$  then  $f(x+h) = \frac{x+h}{x+h-1}$ .

Therefore

$$f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \rightarrow 0} \frac{\frac{x+h}{x+h-1} - \frac{x}{x-1}}{h}$$

=

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 &= \lim_{h \rightarrow 0} \frac{\frac{x+h}{x+h-1} - \frac{x}{x-1}}{h} \\
 &= \lim_{h \rightarrow 0} \frac{1}{h} \left( \frac{(x+h)(x-1) - x(x+h-1)}{(x-1)(x+h-1)} \right)
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## 3.2 The Derivative as a Function

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 &= \frac{-1}{(x-1)(x+0-1)} = \frac{-1}{(x-1)^2}.
 \end{aligned}$$

## 3.2 The Derivative as a Function



The derivative of  $f(x) = \frac{x}{x-1}$  is  $f'(x) = \frac{-1}{(x-1)^2}$ .

## 3.2 The Derivative as a Function



Sometimes it is easier to use an alternative formula for the derivative. If we take

$$f'(x) = \lim_{h \rightarrow 0} \frac{f(x + h) - f(x)}{h}$$

and substitute in  $z = x + h$ , then we get:

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### Definition

$$f'(x) = \lim_{z \rightarrow x} \frac{f(z) - f(x)}{z - x}.$$

### 3.2 The Derivative as a Function

$$f'(x) = \lim_{z \rightarrow x} \frac{f(z) - f(x)}{z - x}$$


#### Example

Find the derivative of  $f(x) = \sqrt{x}$  for  $x > 0$ .

### 3.2 The Derivative as a Function

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### 3.2 The Derivative as a Function

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$$= \lim_{z \rightarrow x} \frac{\sqrt{z} - \sqrt{x}}{(\sqrt{z} - \sqrt{x})(\sqrt{z} + \sqrt{x})}$$

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## 3.2 The Derivative as a Function



### Remark

The line passing through the point  $(x_0, y_0)$  with slope  $k$  has the equation

$$y = y_0 + k(x - x_0).$$

## 3.2 The Derivative as a Function



### Example

Find the tangent line to the curve  $y = \sqrt{x}$  at  $x = 4$ .

We know that the derivative of  $f(x) = \sqrt{x}$  is  $f'(x) = \frac{1}{2\sqrt{x}}$ .

## 3.2 The Derivative as a Function

### Example

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We know that the derivative of  $f(x) = \sqrt{x}$  is  $f'(x) = \frac{1}{2\sqrt{x}}$ .

Therefore

$$f'(4) = \frac{1}{2\sqrt{4}} = \frac{1}{4}.$$

## 3.2 The Derivative as a Function



### Example

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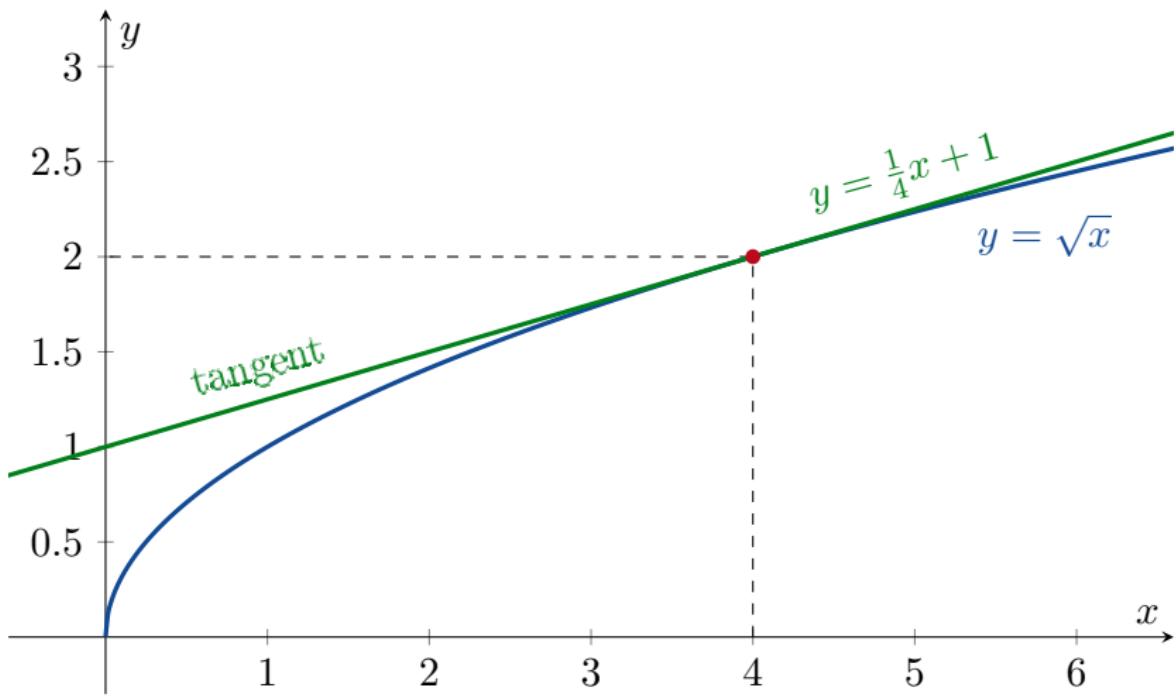
Therefore

$$f'(4) = \frac{1}{2\sqrt{4}} = \frac{1}{4}.$$

The tangent is the line passing through the point  $(4, 2)$  with slope  $\frac{1}{4}$ . So the tangent is

$$y = 2 + \frac{1}{4}(x - 4) = \frac{1}{4}x + 1.$$

## 3.2 The Derivative as a Function



## 3.2 The Derivative as a Function



### Notation

There are many ways to write the derivative of  $y = f(x)$ .

$$f'(x) = y' = \frac{dy}{dx} = \frac{df}{dx} = \frac{d}{dx} f(x) = \dot{y} = \dot{f}(x)$$

“the derivative of  $y$  with respect to  $x$ ”

## 3.2 The Derivative as a Function



Calculus was started by two men who each claimed to have invented it, then grew to hate each other:

## 3.2 The Derivative as a Function



### Sir Isaac Newton

BORN

4 January 1643

DECEASED

31 March 1727 (aged 84)

NATIONALITY

British

Calculus was started by two men who each claimed to have invented it, then grew to hate each other: Sir Isaac Newton used  $\dot{f}$  and  $\dot{y}$ .

## 3.2 The Derivative as a Function



Sir Isaac Newton



Gottfried Leibniz

BORN

1 July 1646

DECEASED

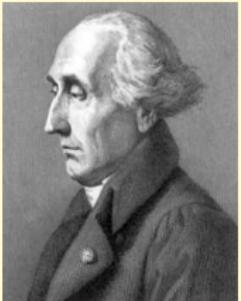
14 November 1716 (aged 70)

NATIONALITY

German

Calculus was started by two men who each claimed to have invented it, then grew to hate each other: Sir Isaac Newton used  $\dot{f}$  and  $\dot{y}$ . Gottfried Leibniz used  $\frac{df}{dx}$  and  $\frac{dy}{dx}$ .

## 3.2 The Derivative as a Function



Sir Isaac Newton

Gottfried Leibniz

Joseph-Louis Lagrange

BORN  
25 January 1736

DECEASED  
10 April 1813 (aged 77)

NATIONALITY  
Italian

A card-style graphic featuring three historical figures. At the top, there are two smaller cards for Sir Isaac Newton and Gottfried Leibniz, each with a portrait and the name. Below them is a larger card for Joseph-Louis Lagrange, which includes his portrait on the left and his birth and death details on the right. The text is in a serif font, and the background is a light yellow color.

Calculus was started by two men who each claimed to have invented it, then grew to hate each other: Sir Isaac Newton used  $\dot{f}$  and  $\dot{y}$ . Gottfried Leibniz used  $\frac{df}{dx}$  and  $\frac{dy}{dx}$ .

The  $f'$  and  $y'$  notation came later from Lagrange.

## 3.2 The Derivative as a Function



If we want the derivative of  $y = f(x)$  at the point  $x = x_0$ , we can write

$$f'(x_0) = \frac{dy}{dx} \Big|_{x=x_0} = \frac{df}{dx} \Big|_{x=x_0} = \frac{d}{dx} f(x) \Big|_{x=x_0}$$

“the derivative of  $y$  with respect to  $x$  at  $x = x_0$ ”

## 3.2 The Derivative as a Function



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“the derivative of  $y$  with respect to  $x$  at  $x = x_0$ ”

For example, if  $u(x) = \frac{1}{x}$ , then

$$u'(4) = \frac{d}{dx} \left( \frac{1}{x} \right) \Big|_{x=4} = \frac{-1}{x^2} \Big|_{x=4} = \frac{-1}{4^2} = \frac{-1}{16}.$$

## 3.2 The Derivative as a Function

### Example

Show that  $f(x) = |x|$  is not differentiable at  $x = 0$ .

### 3.2 The Derivative as a Function



#### Example

Show that  $f(x) = |x|$  is not differentiable at  $x = 0$ .

## One-Sided Derivatives

Just as we had one-sided limits, we can have one-sided derivatives.

$$\begin{array}{l} \text{Right-hand derivative} \\ \text{of } f \text{ at } a \end{array} = \lim_{h \rightarrow 0^+} \frac{f(a + h) - f(a)}{h}$$

$$\begin{array}{l} \text{Left-hand derivative} \\ \text{of } f \text{ at } b \end{array} = \lim_{h \rightarrow 0^-} \frac{f(b + h) - f(b)}{h}$$

### 3.2 The Derivative as a Function



#### Example

Show that  $f(x) = |x|$  is not differentiable at  $x = 0$ .

Right-hand derivative

$$\lim_{h \rightarrow 0^+} \frac{|0 + h| - 0}{h} = \lim_{h \rightarrow 0^+} \frac{h}{h} = \lim_{h \rightarrow 0^+} 1 = 1.$$

Left-hand derivative

$$\lim_{h \rightarrow 0^-} \frac{|0 + h| - 0}{h} = \lim_{h \rightarrow 0^+} \frac{-h}{h} = \lim_{h \rightarrow 0^-} -1 = -1.$$

### 3.2 The Derivative as a Function



#### Example

Show that  $f(x) = |x|$  is not differentiable at  $x = 0$ .

Right-hand derivative

$$\lim_{h \rightarrow 0^+} \frac{|0 + h| - 0}{h} = \lim_{h \rightarrow 0^+} \frac{h}{h} = \lim_{h \rightarrow 0^+} 1 = 1.$$

Left-hand derivative

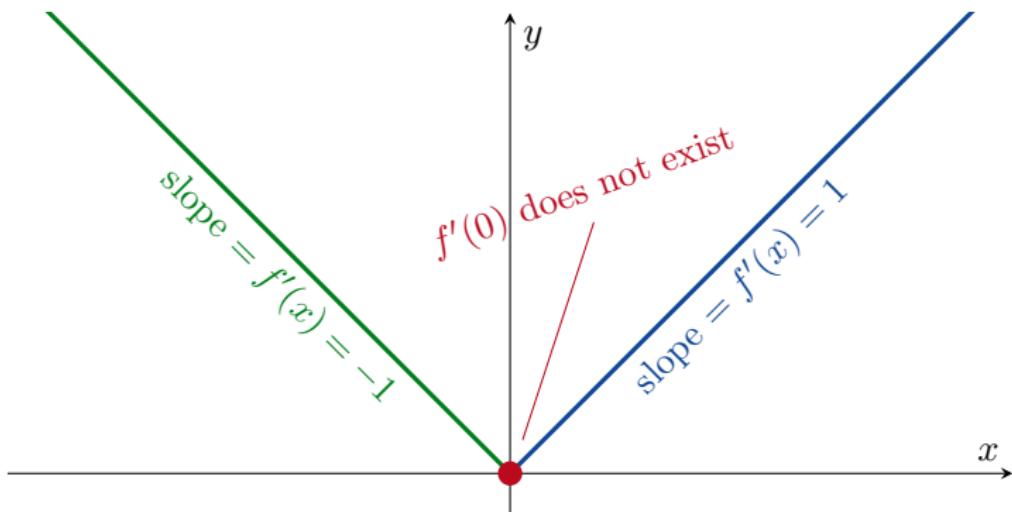
$$\lim_{h \rightarrow 0^-} \frac{|0 + h| - 0}{h} = \lim_{h \rightarrow 0^+} \frac{-h}{h} = \lim_{h \rightarrow 0^-} -1 = -1.$$

Since the right-hand derivative and left-hand derivative at  $x = 0$  are not equal,  $|x|$  is not differentiable at  $x = 0$ .

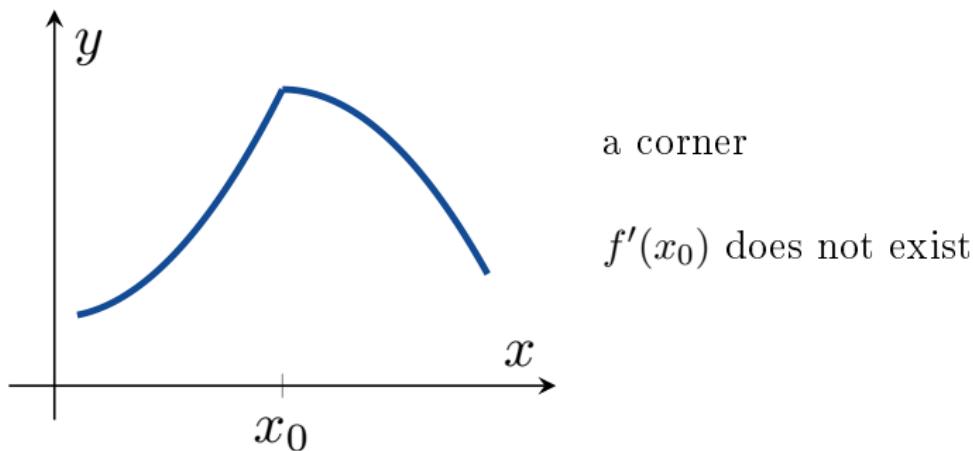
## 3.2 The Derivative as a Function



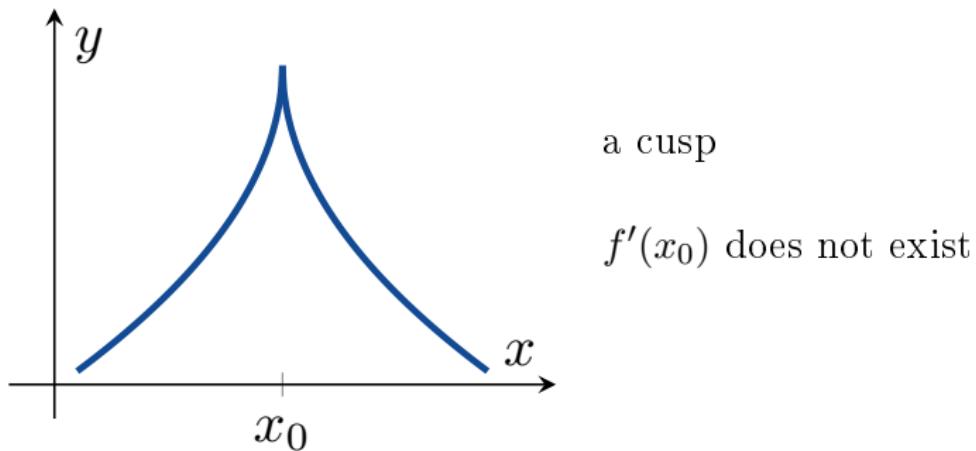
$$f(x) = |x|$$



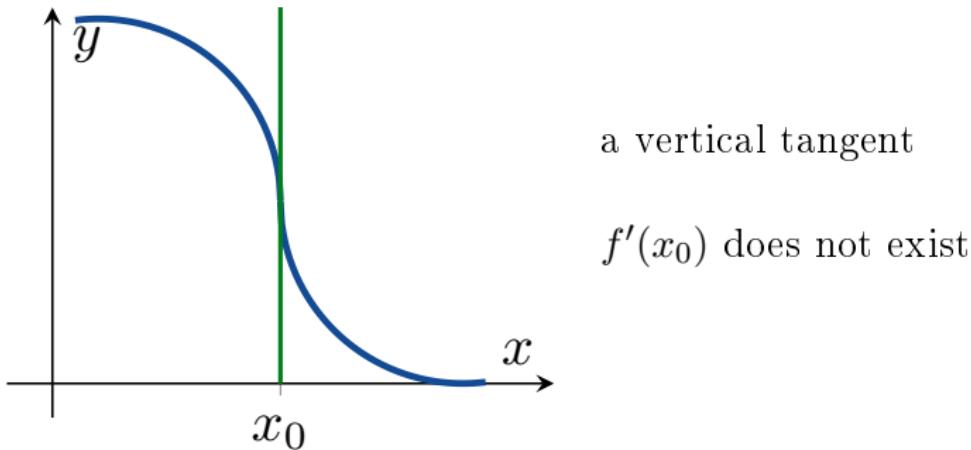
## When Does a Function Not Have a Derivative at a Point?



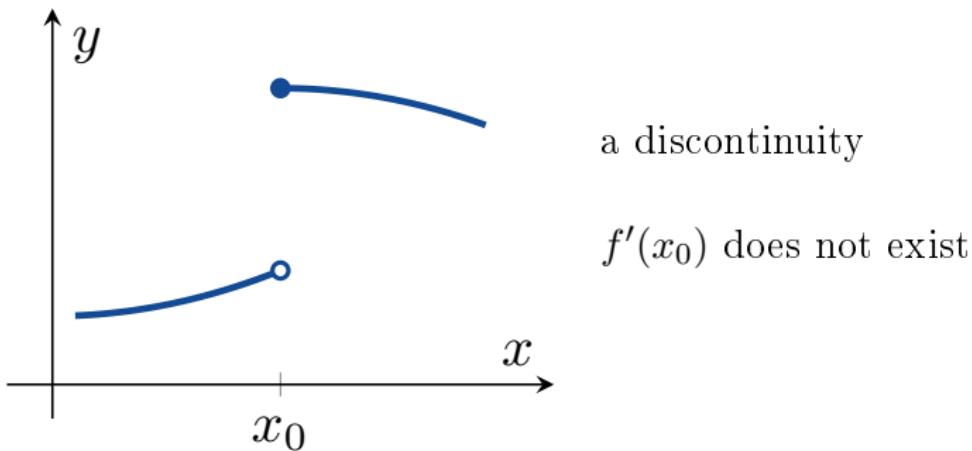
### When Does a Function Not Have a Derivative at a Point?



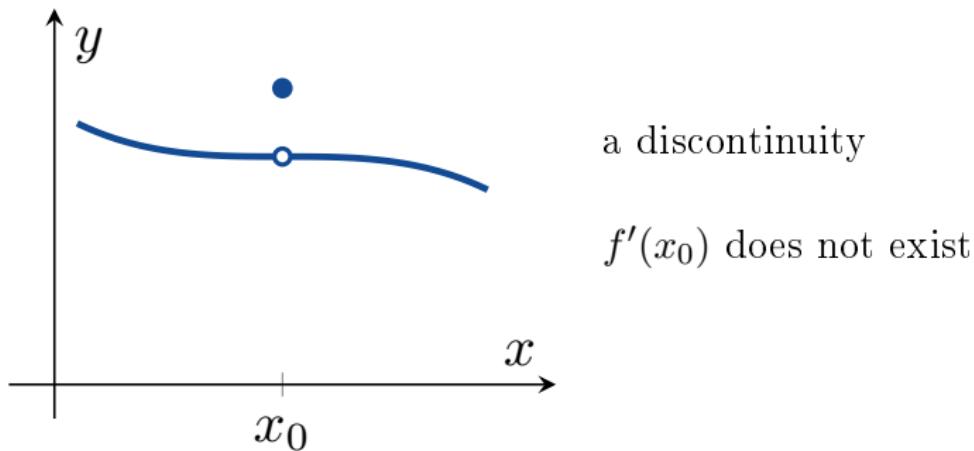
### When Does a Function Not Have a Derivative at a Point?



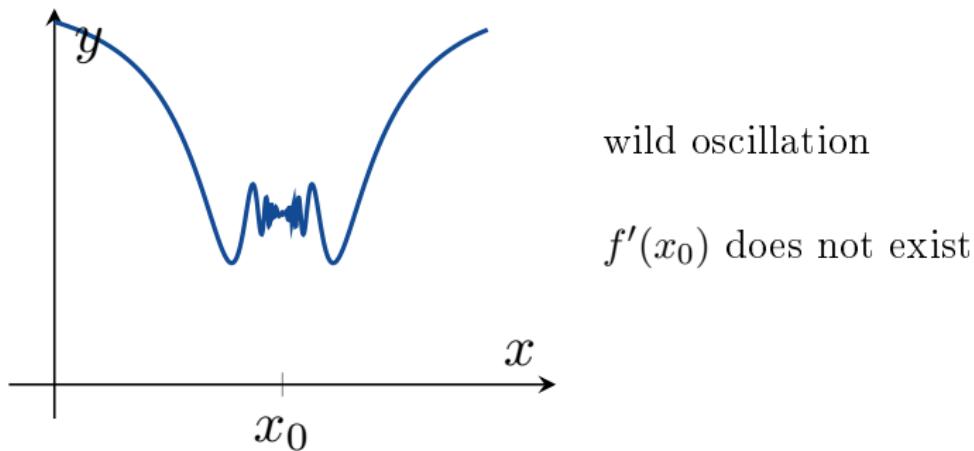
## When Does a Function Not Have a Derivative at a Point?



## When Does a Function Not Have a Derivative at a Point?



## When Does a Function Not Have a Derivative at a Point?



## 3.2 The Derivative as a Function



Theorem

$$\left( \begin{array}{c} f \text{ has a derivative} \\ \text{at } x = c \end{array} \right) \implies \left( \begin{array}{c} f \text{ is continuous} \\ \text{at } x = c \end{array} \right)$$

## 3.2 The Derivative as a Function



### Theorem

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### Proof.

We know that  $f'(c)$  exists.

## 3.2 The Derivative as a Function



### Theorem

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### Proof.

We know that  $f'(c)$  exists. Writing  $h = c - x$ , we have

$$\lim_{x \rightarrow c} f(x) = \lim_{h \rightarrow 0} f(c + h) =$$

=

=

=



## 3.2 The Derivative as a Function



### Theorem

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We know that  $f'(c)$  exists. Writing  $h = c - x$ , we have

$$\lim_{x \rightarrow c} f(x) = \lim_{h \rightarrow 0} f(c + h) = \lim_{h \rightarrow 0} (f(c) + f(c + h) - f(c))$$

=

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=



## 3.2 The Derivative as a Function



### Theorem

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$$= \lim_{h \rightarrow 0} \left( f(c) + \frac{f(c + h) - f(c)}{h} h \right)$$

=

=



## 3.2 The Derivative as a Function



Theorem

$$\left( \begin{array}{c} f \text{ has a derivative} \\ \text{at } x = c \end{array} \right) \implies \left( \begin{array}{c} f \text{ is continuous} \\ \text{at } x = c \end{array} \right)$$

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$$\begin{aligned}\lim_{x \rightarrow c} f(x) &= \lim_{h \rightarrow 0} f(c + h) = \lim_{h \rightarrow 0} (f(c) + f(c + h) - f(c)) \\&= \lim_{h \rightarrow 0} \left( f(c) + \frac{f(c + h) - f(c)}{h} h \right) \\&= \lim_{h \rightarrow 0} f(c) + \left( \lim_{h \rightarrow 0} \frac{f(c + h) - f(c)}{h} \right) \left( \lim_{h \rightarrow 0} h \right) \\&= \end{aligned}$$



## 3.2 The Derivative as a Function



Theorem

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Therefore  $f$  is continuous at  $c$ .



### 3.2 The Derivative as a Function



#### Remark

$$\left( \begin{array}{l} f \text{ has a derivative} \\ \text{at } x = c \end{array} \right) \implies \left( \begin{array}{l} f \text{ is continuous} \\ \text{at } x = c \end{array} \right)$$

but

$$\left( \begin{array}{l} f \text{ has a derivative} \\ \text{at } x = c \end{array} \right) \not\iff \left( \begin{array}{l} f \text{ is continuous} \\ \text{at } x = c \end{array} \right).$$

For example  $f(x) = |x|$  is continuous at  $x = 0$ , but is not differentiable at  $x = 0$ .

# Break

We will continue at 2pm





# Differentiation Rules

### 3.3 Differentiation Rules



## Constant Function

If  $k \in \mathbb{R}$ , then

$$\frac{d}{dx}(k) = 0.$$

(You can read the proof in your textbook.)

### 3.3 Differentiation Rules



## Power Function

If  $n \in \mathbb{R}$ , then

$$\frac{d}{dx} (x^n) = nx^{n-1}.$$

(You can read the proof in your textbook.)

### 3.3 Differentiation Rules



## Power Function

If  $n \in \mathbb{R}$ , then

$$\frac{d}{dx} (x^n) = nx^{n-1}.$$

(You can read the proof in your textbook.)

#### Example

$$\frac{d}{dx} (x^3) = 3x^{3-1} = 3x^2$$

### 3.3 Differentiation Rules



Example

$$\frac{d}{dx} (\sqrt{x}) = \frac{d}{dx} (x^{\frac{1}{2}}) = \frac{1}{2}x^{\frac{1}{2}-1} = \frac{1}{2}x^{-\frac{1}{2}} = \frac{1}{2\sqrt{x}}$$

Example

$$\frac{d}{dx} \left( \frac{1}{x^4} \right) = \frac{d}{dx} (x^{-4}) = -4x^{-4-1} = -4x^{-5} = -\frac{4}{x^5}$$

(You can read more examples in the textbook.)

### 3.3 Differentiation Rules



## The Constant Multiple Rule

If  $u(x)$  is differentiable and  $k \in \mathbb{R}$ , then

$$\frac{d}{dx} (ku) = k \frac{du}{dx}.$$

### 3.3 Differentiation Rules



## The Constant Multiple Rule

If  $u(x)$  is differentiable and  $k \in \mathbb{R}$ , then

$$\boxed{\frac{d}{dx} (ku) = k \frac{du}{dx}.}$$

Proof.

$$\begin{aligned}\frac{d}{dx} (ku) &= \lim_{h \rightarrow 0} \frac{ku(x+h) - ku(x)}{h} \\&= \lim_{h \rightarrow 0} k \left( \frac{u(x+h) - u(x)}{h} \right)\end{aligned}$$

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### 3.3 Differentiation Rules



## The Constant Multiple Rule

If  $u(x)$  is differentiable and  $k \in \mathbb{R}$ , then

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### 3.3 Differentiation Rules



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### 3.3 Differentiation Rules

#### Example

$$\frac{d}{dx} (3x^2) = 3 \frac{d}{dx} (x^2) = 3 \times 2x = 6x$$

#### Example

$$\frac{d}{dx} (-u) = \frac{d}{dx} (-1 \times u) = -1 \times \frac{du}{dx} = -\frac{du}{dx}$$

### 3.3 Differentiation Rules



#### The Sum Rule

If  $u(x)$  and  $v(x)$  are differentiable at  $x_0$ , then  $u + v$  is also differentiable at  $x_0$  and

$$\frac{d}{dx} (u + v) = \frac{du}{dx} + \frac{dv}{dx}.$$

(You can read the proof in your textbook.)

### 3.3 Differentiation Rules



#### Example

Differentiate  $y = x^3 + \frac{4}{3}x^2 - 5x + 1$ .

$$\begin{aligned}\frac{dy}{dx} &= \frac{d}{dx} \left( x^3 + \frac{4}{3}x^2 - 5x + 1 \right) \\&= \frac{d}{dx} (x^3) + \frac{d}{dx} \left( \frac{4}{3}x^2 \right) - \frac{d}{dx} (5x) + \frac{d}{dx} (1) \\&= 3x^2 + \frac{8}{3}x - 5 + 0.\end{aligned}$$

### 3.3 Differentiation Rules



#### Example

Does the curve  $y = x^4 - 2x^2 + 2$  have any points where  $\frac{dy}{dx} = 0$ ?  
If so, where?

### 3.3 Differentiation Rules



#### Example

Does the curve  $y = x^4 - 2x^2 + 2$  have any points where  $\frac{dy}{dx} = 0$ ?  
If so, where?

Since

$$\frac{dy}{dx} = 4x^3 - 4x = 4x(x^2 - 1) = 4x(x - 1)(x + 1),$$

### 3.3 Differentiation Rules



#### Example

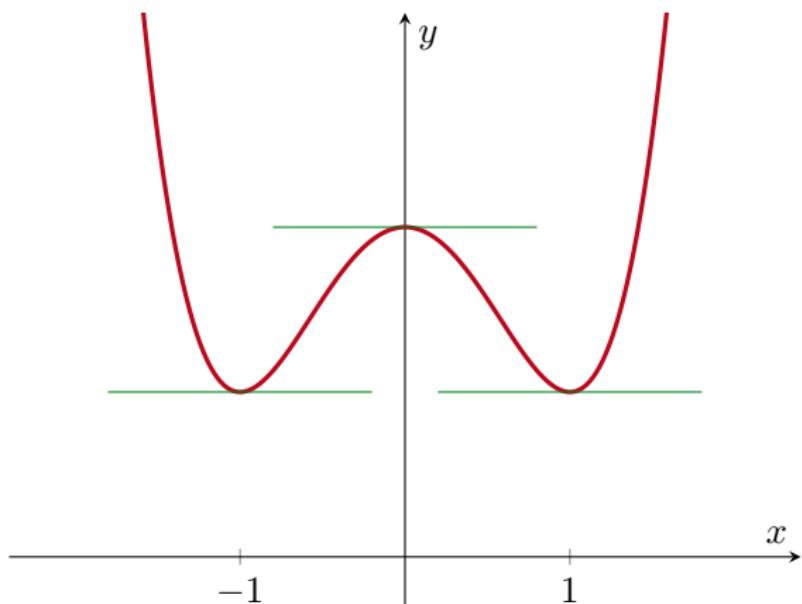
Does the curve  $y = x^4 - 2x^2 + 2$  have any points where  $\frac{dy}{dx} = 0$ ?  
If so, where?

Since

$$\frac{dy}{dx} = 4x^3 - 4x = 4x(x^2 - 1) = 4x(x - 1)(x + 1),$$

we can see that  $\frac{dy}{dx} = 0$  if and only if  $x = -1, 0$  or  $1$ .

### 3.3 Differentiation Rules



### 3.3 Differentiation Rules



## The Product Rule

If  $u(x)$  and  $v(x)$  are differentiable at  $x_0$ , then  $u(x)v(x)$  is also differentiable at  $x_0$  and

$$\frac{d}{dx} (uv) = \frac{du}{dx}v + u\frac{dv}{dx}.$$

(You can read the proof in your textbook.)

### 3.3 Differentiation Rules



## The Product Rule

If  $u(x)$  and  $v(x)$  are differentiable at  $x_0$ , then  $u(x)v(x)$  is also differentiable at  $x_0$  and

$$\frac{d}{dx}(uv) = \frac{du}{dx}v + u\frac{dv}{dx}.$$

Using prime notation, the product rule is

$$(uv)' = u'v + uv'.$$

(You can read the proof in your textbook.)

### 3.3 Differentiation Rules

$$(uv)' = u'v + uv'$$



#### Example

Differentiate  $y = (x^2 + 1)(x^3 + 3)$ .

*solution 1:* We have  $y = uv$  with  $u = x^2 + 1$  and  $v = x^3 + 3$ .

### 3.3 Differentiation Rules

$$(uv)' = u'v + uv'$$



#### Example

Differentiate  $y = (x^2 + 1)(x^3 + 3)$ .

*solution 1:* We have  $y = uv$  with  $u = x^2 + 1$  and  $v = x^3 + 3$ . So

$$\frac{dy}{dx} = u'v + uv'$$

$$= (x^2 + 1)'(x^3 + 3) + (x^2 + 1)(x^3 + 3)'$$

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### 3.3 Differentiation Rules

$$(uv)' = u'v + uv'$$



#### Example

Differentiate  $y = (x^2 + 1)(x^3 + 3)$ .

*solution 1:* We have  $y = uv$  with  $u = x^2 + 1$  and  $v = x^3 + 3$ . So

$$\begin{aligned}\frac{dy}{dx} &= u'v + uv' \\&= (x^2 + 1)'(x^3 + 3) + (x^2 + 1)(x^3 + 3)' \\&= (2x + 0)(x^3 + 3) + (x^2 + 1)(3x^2 + 0) \\&= \\&= \end{aligned}$$

### 3.3 Differentiation Rules

$$(uv)' = u'v + uv'$$



#### Example

Differentiate  $y = (x^2 + 1)(x^3 + 3)$ .

*solution 1:* We have  $y = uv$  with  $u = x^2 + 1$  and  $v = x^3 + 3$ . So

$$\begin{aligned}\frac{dy}{dx} &= u'v + uv' \\&= (x^2 + 1)'(x^3 + 3) + (x^2 + 1)(x^3 + 3)' \\&= (2x + 0)(x^3 + 3) + (x^2 + 1)(3x^2 + 0) \\&= 2x^4 + 6x + 3x^4 + 3x^2 \\&= 5x^4 + 3x^2 + 6x.\end{aligned}$$

### 3.3 Differentiation Rules



#### Example

Differentiate  $y = (x^2 + 1)(x^3 + 3)$ .

*solution 2:* Since

$$y = (x^2 + 1)(x^3 + 3) = x^5 + x^3 + 3x^2 + 3,$$

### 3.3 Differentiation Rules



#### Example

Differentiate  $y = (x^2 + 1)(x^3 + 3)$ .

*solution 2:* Since

$$y = (x^2 + 1)(x^3 + 3) = x^5 + x^3 + 3x^2 + 3,$$

we have that

$$\frac{dy}{dx} = 5x^4 + 3x^2 + 6x + 0.$$

## The Quotient Rule

If  $u(x)$  and  $v(x)$  are differentiable at  $x_0$  and if  $v(x_0) \neq 0$ , then  $\frac{u}{v}$  is also differentiable at  $x_0$  and

$$\boxed{\frac{d}{dx} \left( \frac{u}{v} \right) = \frac{u'v - uv'}{v^2}.}$$

(You can read the proof in your textbook.)

### 3.3 Differentiation Rules

$$\left(\frac{u}{v}\right)' = \frac{u'v - uv'}{v^2}$$



#### Example

Differentiate  $y = \frac{t^2-1}{t^3+1}$ .

We have  $y = \frac{u}{v}$  with  $u = t^2 - 1$  and  $v = t^3 + 1$ .

### 3.3 Differentiation Rules

$$\left(\frac{u}{v}\right)' = \frac{u'v - uv'}{v^2}$$


#### Example

Differentiate  $y = \frac{t^2 - 1}{t^3 + 1}$ .

We have  $y = \frac{u}{v}$  with  $u = t^2 - 1$  and  $v = t^3 + 1$ . Therefore

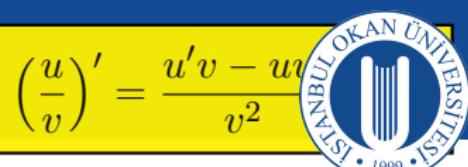
$$\begin{aligned}\frac{dy}{dt} &= \frac{u'v - uv'}{v^2} \\ &= \frac{(t^2 - 1)'(t^3 + 1) - (t^2 - 1)(t^3 + 1)'}{(t^3 + 1)^2}\end{aligned}$$

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### 3.3 Differentiation Rules



#### Example

Differentiate  $y = \frac{t^2 - 1}{t^3 + 1}$ .

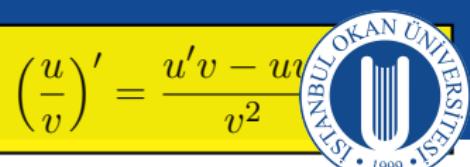
We have  $y = \frac{u}{v}$  with  $u = t^2 - 1$  and  $v = t^3 + 1$ . Therefore

$$\begin{aligned}\frac{dy}{dt} &= \frac{u'v - uv'}{v^2} \\&= \frac{(t^2 - 1)'(t^3 + 1) - (t^2 - 1)(t^3 + 1)'}{(t^3 + 1)^2} \\&= \frac{(2t)(t^3 + 1) - (t^2 - 1)(3t^2)}{(t^3 + 1)^2}\end{aligned}$$

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### 3.3 Differentiation Rules



#### Example

Differentiate  $y = \frac{t^2 - 1}{t^3 + 1}$ .

We have  $y = \frac{u}{v}$  with  $u = t^2 - 1$  and  $v = t^3 + 1$ . Therefore

$$\begin{aligned}\frac{dy}{dt} &= \frac{u'v - uv'}{v^2} \\&= \frac{(t^2 - 1)'(t^3 + 1) - (t^2 - 1)(t^3 + 1)'}{(t^3 + 1)^2} \\&= \frac{(2t)(t^3 + 1) - (t^2 - 1)(3t^2)}{(t^3 + 1)^2} \\&= \frac{2t^4 + 2t - 3t^4 + 3t^2}{(t^3 + 1)^2} \\&= \frac{-t^4 + 3t^2 + 2t}{(t^3 + 1)^2}.\end{aligned}$$

### 3.3 Differentiation Rules

$$\left(\frac{u}{v}\right)' = \frac{u'v - uv'}{v^2}$$



#### Example

Differentiate  $f(s) = \frac{\sqrt{s}-1}{\sqrt{s}+1}$ .

### 3.3 Differentiation Rules

$$\left(\frac{u}{v}\right)' = \frac{u'v - uv'}{v^2}$$



#### Example

Differentiate  $f(s) = \frac{\sqrt{s}-1}{\sqrt{s}+1}$ .

We have  $f(s) = \frac{u}{v}$  with  $u = \sqrt{s} - 1$  and  $v = \sqrt{s} + 1$ . Remember that  $\frac{d}{ds}(\sqrt{s}) = \frac{1}{2\sqrt{s}}$ .

### 3.3 Differentiation Rules

$$\left(\frac{u}{v}\right)' = \frac{u'v - uv'}{v^2}$$


#### Example

Differentiate  $f(s) = \frac{\sqrt{s}-1}{\sqrt{s}+1}$ .

We have  $f(s) = \frac{u}{v}$  with  $u = \sqrt{s} - 1$  and  $v = \sqrt{s} + 1$ . Remember that  $\frac{d}{ds}(\sqrt{s}) = \frac{1}{2\sqrt{s}}$ . Therefore

$$\begin{aligned} \frac{df}{ds} &= \frac{u'v - uv'}{v^2} \\ &= \frac{(\sqrt{s}-1)'(\sqrt{s}+1) - (\sqrt{s}-1)(\sqrt{s}+1)'}{(\sqrt{s}+1)^2} \\ &= \frac{\left(\frac{1}{2\sqrt{s}}\right)(\sqrt{s}+1) - (\sqrt{s}-1)\left(\frac{1}{2\sqrt{s}}\right)}{(\sqrt{s}+1)^2} \\ &= \frac{\frac{1}{2} + \frac{1}{2\sqrt{s}} - \frac{1}{2} + \frac{1}{2\sqrt{s}}}{(\sqrt{s}+1)^2} = \frac{1}{\sqrt{s}(\sqrt{s}+1)^2}. \end{aligned}$$

### 3.3 Differentiation Rules

#### Example

Differentiate  $y = \frac{(x-1)(x^2-2x)}{x^4}$ .

It is possible to use the Quotient Rule to answer this question, but there is an easier way.

### 3.3 Differentiation Rules

#### Example

Differentiate  $y = \frac{(x-1)(x^2-2x)}{x^4}$ .

It is possible to use the Quotient Rule to answer this question, but there is an easier way. Note that

$$y = \frac{(x-1)(x^2-2x)}{x^4} = \frac{x^3 - 3x^2 + 2x}{x^4} = x^{-1} - 3x^{-2} + 2x^{-3}.$$

### 3.3 Differentiation Rules

#### Example

Differentiate  $y = \frac{(x-1)(x^2-2x)}{x^4}$ .

It is possible to use the Quotient Rule to answer this question, but there is an easier way. Note that

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Thus

$$y' = -x^{-2} - 3(-2)x^{-3} + 2(-3)x^{-4} = -\frac{1}{x^2} + \frac{6}{x^3} - \frac{6}{x^4}.$$

### 3.3 Differentiation Rules



## Second Order Derivatives

If  $y = f(x)$  is a differentiable function, then  $f'(x)$  is also a function.

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We can write

$$f''(x) = \frac{d}{dx} f'(x) = \frac{d}{dx} \left( \frac{dy}{dx} \right) = \frac{d^2y}{dx^2} = \frac{dy'}{dx} = y''$$

↗  
“ $d$  squared  $y$ ,  $dx$  squared”

### 3.3 Differentiation Rules

#### Example

If  $y = x^6$ , then

$$y' = \frac{d}{dx} (x^6) = 6x^5$$

and

$$y'' = \frac{d}{dx} (y') = \frac{d}{dx} (6x^5) = 30x^4.$$

### 3.3 Differentiation Rules

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and

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Equivalently, we can write

$$\frac{d^2}{dx^2} (x^6) = \frac{d}{dx} \left( \frac{d}{dx} (x^6) \right) = \frac{d}{dx} (6x^5) = 30x^4.$$

## Higher Order Derivatives

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### 3.3 Differentiation Rules



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⋮

If  $f^{(n-1)}$  is differentiable, then its derivative  $f^{(n)} = \frac{d^n f}{dx^n}$  is the  *$n$ th derivative* of  $f$ .

### 3.3 Differentiation Rules



#### Remark

For the 13<sup>th</sup> derivative of  $f(x)$ , writing

$$f''''''''''''(x)$$

would just be silly. So we write

$$f^{(13)}(x)$$

instead.

### 3.3 Differentiation Rules



#### Example

Find the first four derivatives of  $y = x^3 - 3x^2 + 2$ .

First derivative:  $y' = 3x^2 - 6x$

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Fourth derivative:  $y^{(4)} = 0$ .

### 3.3 Differentiation Rules



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Find the first four derivatives of  $y = x^3 - 3x^2 + 2$ .

First derivative:  $y' = 3x^2 - 6x$

Second derivative:  $y'' = 6x - 6$

Third derivative:  $y''' = 6$

Fourth derivative:  $y^{(4)} = 0$ .

(Note that since  $\frac{d}{dx}(0) = 0$ , if  $n \geq 4$  then  $y^{(n)} = 0$  also.)



# The Derivative as a Rate of Change

### 3.4 The Derivative as a Rate of Change



#### Definition

The *rate of change* of  $f$  with respect to  $x$  at  $x_0$  is the derivative

$$f'(x_0) = \lim_{h \rightarrow 0} \frac{f(x_0 + h) - f(x_0)}{h}$$

provided the limit exists.

### 3.4 The Derivative as a Rate of Change

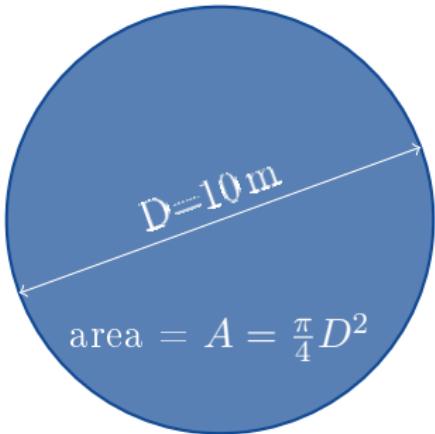


#### Example

The area  $A$  of a circle is related to its diameter by the equation

$$A = \frac{\pi}{4}D^2.$$

How fast does the area change with respect to the diameter when the diameter is 10 m?



### 3.4 The Derivative as a Rate of Change



#### Example

The area  $A$  of a circle is related to its diameter by the equation

$$A = \frac{\pi}{4}D^2.$$

How fast does the area change with respect to the diameter when the diameter is 10 m?

The rate of change is

$$\frac{dA}{dD} = \frac{\pi}{4} \cdot 2D = \frac{\pi D}{2}.$$

When  $D = 10$  m, this is  $5\pi$  m<sup>2</sup>/m.

### 3.4 The Derivative as a Rate of Change



## Motion Along a Line; Displacement, Velocity, Speed, Acceleration and Jerk

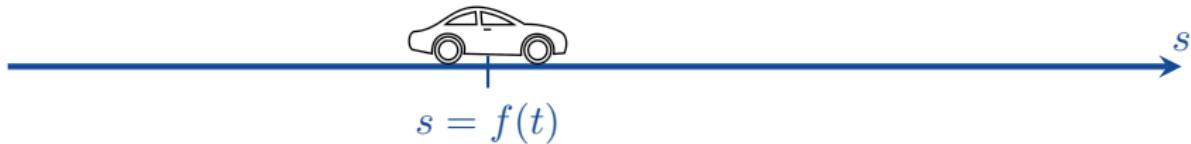


Suppose that an object is moving along a line.

### 3.4 The Derivative as a Rate of Change



## Motion Along a Line; Displacement, Velocity, Speed, Acceleration and Jerk

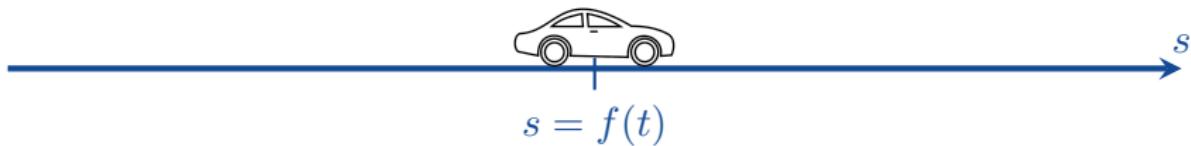


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### 3.4 The Derivative as a Rate of Change



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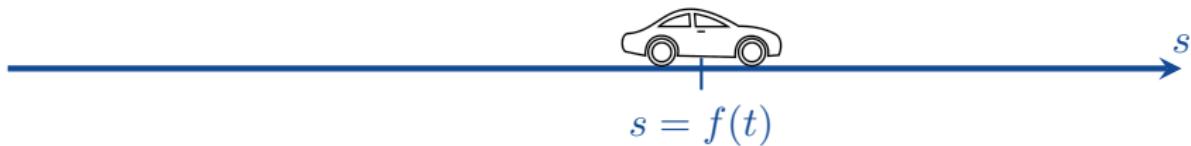
$$s = f(t)$$

be its position at time  $t$ .

### 3.4 The Derivative as a Rate of Change



## Motion Along a Line; Displacement, Velocity, Speed, Acceleration and Jerk



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### 3.4 The Derivative as a Rate of Change



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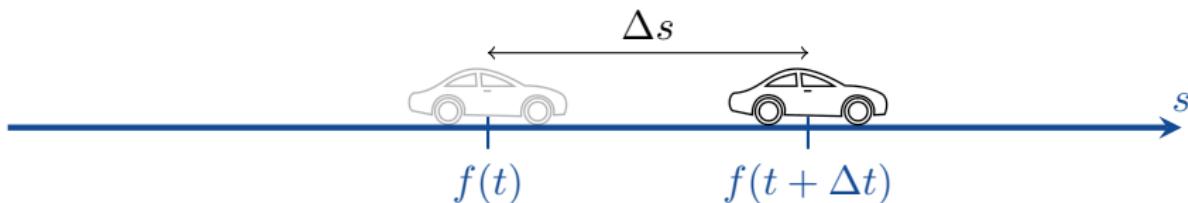


The *displacement* of the object over the time interval from  $t$  to  $t + \Delta t$  is

### 3.4 The Derivative as a Rate of Change



## Motion Along a Line; Displacement, Velocity, Speed, Acceleration and Jerk



The *displacement* of the object over the time interval from  $t$  to  $t + \Delta t$  is

$$\Delta s = f(t + \Delta t) - f(t).$$

### 3.4 The Derivative as a Rate of Change



The *average velocity* of the object between time  $t$  and time  $t + \Delta t$  is

$$\frac{\text{displacement}}{\text{travel time}} = \frac{\Delta s}{\Delta t} = \frac{f(t + \Delta t) - f(t)}{\Delta t}.$$

Taking the limit as  $\Delta t \rightarrow 0$ , we obtain:

## 3.4 The Derivative as a Rate of Change



### Definition

$$\text{velocity} = \frac{d}{dt} (\text{position})$$

### 3.4 The Derivative as a Rate of Change



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#### Definition

$$\text{speed} = |\text{velocity}| = \left| \frac{ds}{dt} \right|.$$

### 3.4 The Derivative as a Rate of Change



$$\text{velocity} = v(t) = \frac{d}{dt} (\text{position}) = \frac{ds}{dt}.$$

#### Definition

$$\text{acceleration} = a(t) = \frac{d}{dt} (\text{velocity}) = \frac{dv}{dt}.$$

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Definition

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Definition

$$\text{jerk} = j(t) = \frac{d}{dt} (\text{acceleration}) = \frac{da}{dt}.$$

## 3.4 The Derivative as a Rate of Change

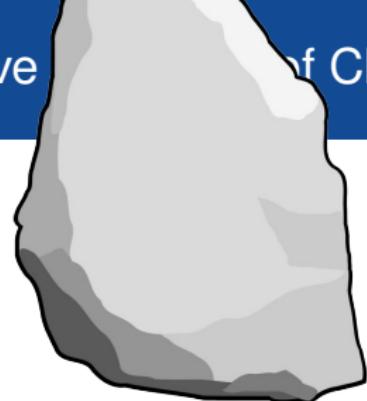


## 3.4 The Derivative as a Rate of Change



### 3.4 The Derivative

### of Change



### 3.4 The Derivative as a Rate of Change



#### Example

A dynamite blast blows a heavy rock straight upwards with a launch velocity of 49 m/s. It reaches a height of

$$s = 49t - 4.9t^2 \text{ m}$$

after  $t$  seconds.

### 3.4 The Derivative as a Rate of Change



#### Example

A dynamite blast blows a heavy rock straight upwards with a launch velocity of 49 m/s. It reaches a height of

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after  $t$  seconds.

- 1 How high does the rock go?
- 2 What are the velocity and speed of the rock when it is 78.4 m above the ground on the way up? On the way down?
- 3 What is the acceleration of the rock at time  $t$  during its flight (after the blast)?
- 4 When does the rock hit the ground?

### 3.4 The Derivative as a Rate of Change

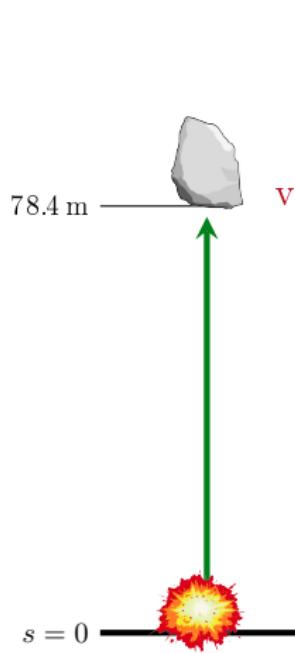


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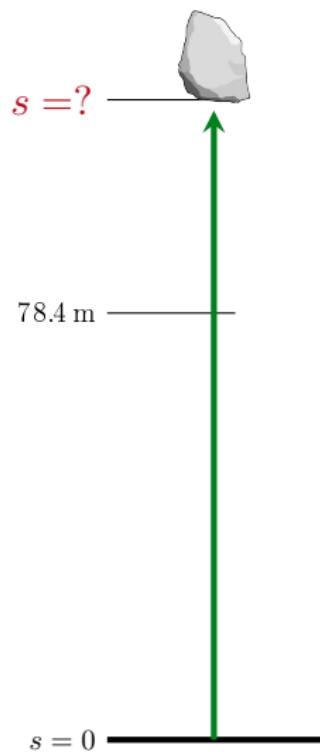
78.4 m —



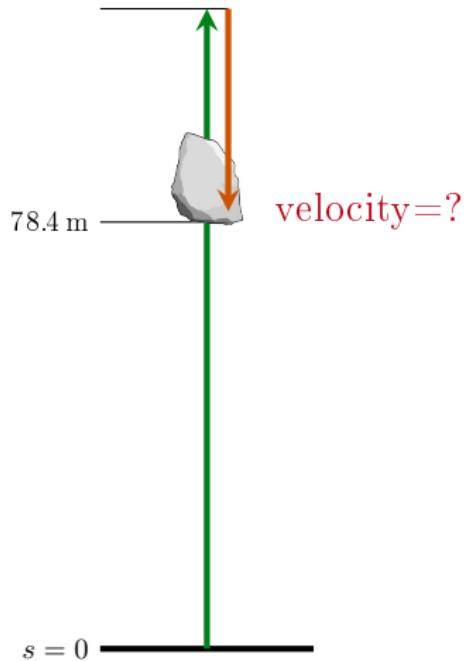
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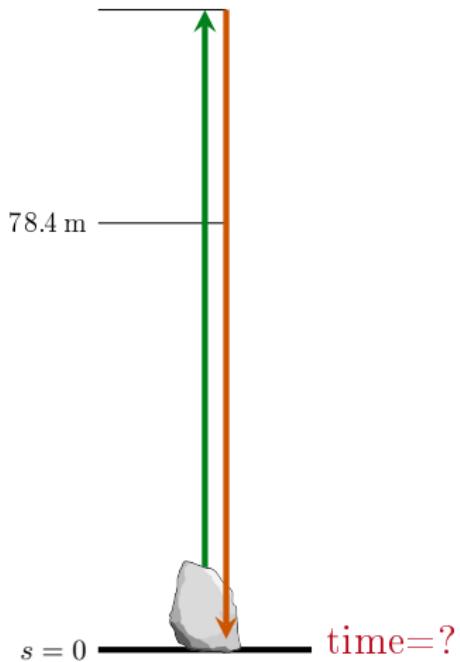
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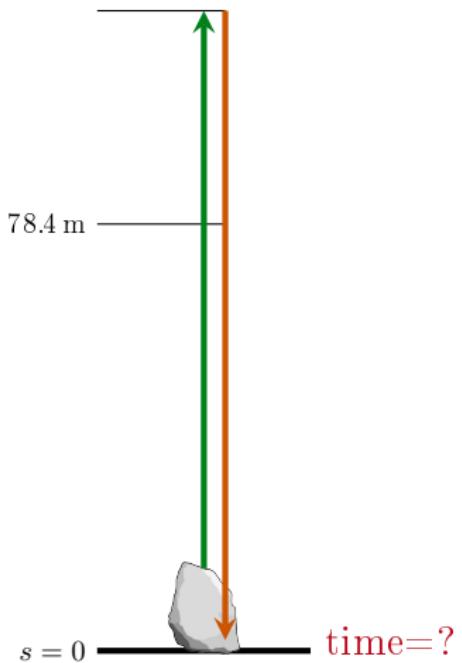


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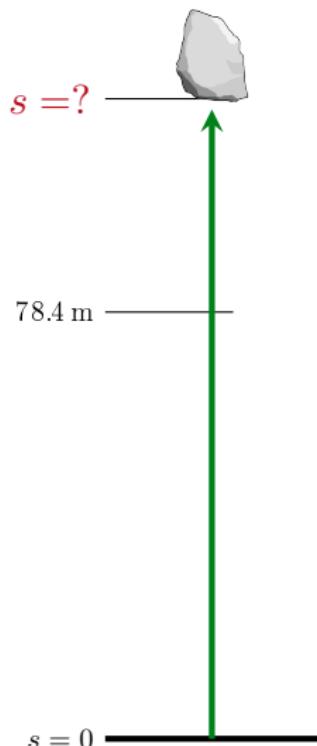


Since  $s(t) = 49t - 4.9t^2$  m, we have

$$v(t) = s'(t) = 49 - 9.8t \text{ m/s}.$$



### 3.4 The Derivative as a Rate of Change



Since  $s(t) = 49t - 4.9t^2$  m, we have

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At the top, we have

$$0 = v(t) = 49 - 9.8t \implies t = 5 \text{ s.}$$

So the rock goes

$$s(5) = 49(5) - 4.9(25) = 122.5 \text{ m}$$

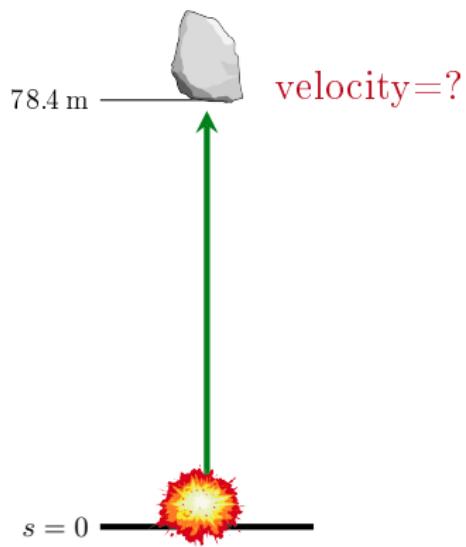
high.

### 3.4 The Derivative as a Rate of Change



Now

122.5 m —————



$$78.4 = s(t) = 49t - 4.9t^2$$

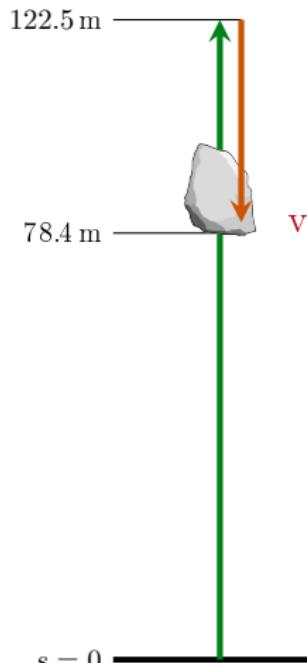
$$4.9(t^2 - 10t + 16) = 0$$

$$4.9(t - 2)(t - 8) = 0$$

implies that  $t = 2$  or  $t = 8$ .

### 3.4 The Derivative as a Rate of Change

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We have that

$$v(2) = 49 - 9.8(2) = 29.4 \text{ m/s}$$

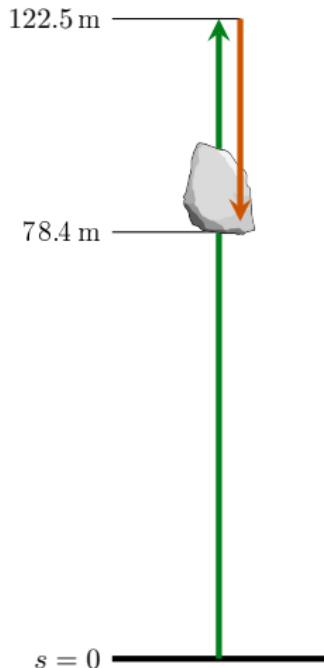
$$v(8) = 49 - 9.8(8) = -29.4 \text{ m/s}.$$

The speed at both points is 29.4 m/s.

### 3.4 The Derivative as a Rate of Change

Since  $v(t) = 49 - 9.8t$  m/s, the acceleration is

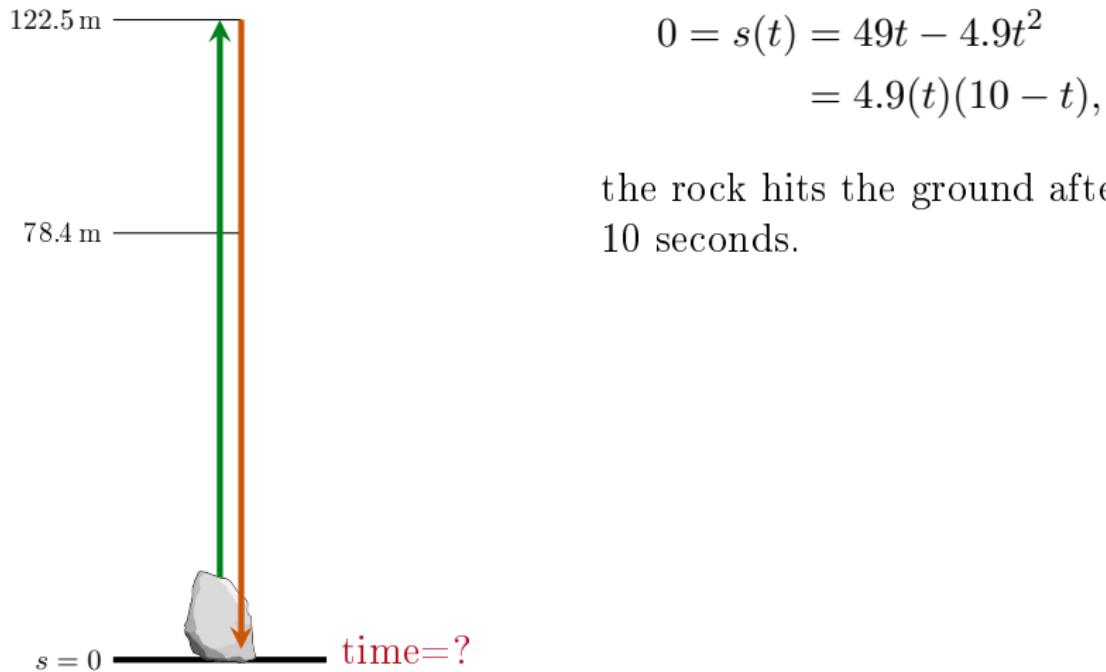
$$a(t) = v'(t) = -9.8 \text{ m/s}^2.$$



### 3.4 The Derivative as a Rate of Change



Since



## Derivatives in Economics

Suppose that

$c(x)$  = the cost to produce  $x$  tonnes of steel in one week.

### 3.4 The Derivative as a Rate of Change



## Derivatives in Economics

Suppose that

$c(x)$  = the cost to produce  $x$  tonnes of steel in one week.

It costs more to produce  $x + h$  tonnes of steel, and the cost difference divided by  $h$  is:

$$\text{average cost of producing each addition tonne} = \frac{c(x + h) - c(x)}{h}.$$

Taking the limit as  $h \rightarrow 0$ , we get the *marginal cost* of producing more steel per week when the current weekly production is  $x$  tonnes:

$$\text{marginal cost of production} = \frac{dc}{dx} = \lim_{h \rightarrow 0} \frac{c(x + h) - c(x)}{h}.$$

### 3.4 The Derivative as a Rate of Change



#### Example

Suppose that it costs

$$c(x) = x^3 - 6x^2 + 15x$$

lira to produce  $x$  radiators per day (when  $x \in [8, 30]$ ) and that you can earn

$$r(x) = x^3 - 3x^2 + 12x$$

lira from selling  $x$  radiators. Your factory currently makes 10 radiators per day.

### 3.4 The Derivative as a Rate of Change



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- 1 How much extra will it cost to produce one more radiator per day?
- 2 How much more profit can you make by producing 11 radiators per day instead of 10?

### 3.4 The Derivative of Change

$$c(x) = x^3 - 6x^2 + 15x \quad r(x) = x^3 - 3x^2 + 12x$$



The marginal cost of production is

$$c'(x) = 3x^2 - 12x + 15$$

$$c'(1) = 3(100) - 12(10) + 15 = 195.$$

### 3.4 The De

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The marginal cost of production is

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$$c'(1) = 3(100) - 12(10) + 15 = 195.$$

The marginal revenue is

$$r'(x) = 3x^2 - 6x + 12$$

$$r'(10) = 3(100) - 6(10) + 12 = 252.$$

### 3.4 The Derivative of Change

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The marginal cost of production is

$$c'(x) = 3x^2 - 12x + 15$$

$$c'(1) = 3(100) - 12(10) + 15 = 195.$$

The marginal revenue is

$$r'(x) = 3x^2 - 6x + 12$$

$$r'(10) = 3(100) - 6(10) + 12 = 252.$$

So producing 11 radiators per day instead of 10 will cost approximately 195 liras extra, but will bring in approximately 252 liras more.

We would expect your profit to increase by approximately  $252 - 195 = 57$  liras.



# Next Time

- 3.5 Derivatives of Trigonometric Functions
- 3.6 The Chain Rule
- 3.7 Implicit Differentiation