Lecture 9

- 5.4 The Fundamental Theorem of Calculus
- 5.5 Indefinite Integrals and the Substitution Method
- 5.6 Substitution and Area Between Curves

The Fundamental Theorem of Calculus

5.4

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Remark

The following theorem is the most important theorem in Calculus. If you can only memorise one theorem for the exams, it should be this one.

Theorem (The Fundamental Theorem of Calculus)

Suppose that $f : [a, b] \to \mathbb{R}$ *is a continuous function.*

1.
2.

Theorem (The Fundamental Theorem of Calculus)

Suppose that $f : [a, b] \to \mathbb{R}$ is a continuous function.

1. *Then the function $F : [a, b] \to \mathbb{R}$ defined by*

$$F(x) = \int_a^x f(t)\ dt$$

is continuous on $[a, b]$; differentiable on (a, b); and its derivative is

$$\boxed{F'(x) = \frac{d}{dx} \int_a^x f(t)\ dt = f(x).}$$

2.

Theorem (The Fundamental Theorem of Calculus)

Suppose that $f : [a, b] \to \mathbb{R}$ is a continuous function.

1. *Then the function...*
2. *If F is any antiderivative of f on $[a, b]$, then*

$$\boxed{\int_a^b f(x)\,dx = F(b) - F(a).}$$

Remark

Part 1 of the theorem tells how to differentiate $\int_a^x f(t)\, dt$.

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Example

Find $\frac{dy}{dx}$ if $y = \int_a^x (t^3+1)\,dt$.

$$\frac{dy}{dx} = \frac{d}{dx}\int_a^x (t^3+1)\,dt = x^3+1.$$

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$$\frac{dy}{dx} = \frac{d}{dx}\int_a^x (t^3 + 1)\,dt = x^3 + 1.$$

Example

Find $\dfrac{dy}{dx}$ if $y = \int_1^x \sin t\,dt$.

$$\frac{dy}{dx} = \frac{d}{dx}\int_1^x \sin t\,dt = \sin x.$$

Example

Find $\dfrac{dy}{dx}$ if $y = \displaystyle\int_0^x \sin \ln \tan e^{t^2} \; dt$.

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$$\frac{dy}{dx} = \frac{d}{dx} \int_0^x \sin \ln \tan e^{t^2} \, dt = \sin \ln \tan e^{x^2}.$$

Example

Find $\dfrac{dy}{dx}$ if $y = \displaystyle\int_x^5 3t \sin t \; dt$.

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$$\begin{aligned}\frac{dy}{dx} &= \frac{d}{dx} \int_x^5 3t \sin t \; dt \\ &= \frac{d}{dx} \left(- \int_5^x 3t \sin t \; dt \right)\end{aligned}$$

Example

Find $\dfrac{dy}{dx}$ if $y = \displaystyle\int_x^5 3t \sin t \; dt$.

$$\begin{aligned}\frac{dy}{dx} &= \frac{d}{dx} \int_x^5 3t \sin t \; dt \\ &= \frac{d}{dx} \left(- \int_5^x 3t \sin t \; dt \right) \\ &= -3x \sin x.\end{aligned}$$

Example

Find $\dfrac{dy}{dx}$ if $y = \displaystyle\int_1^{x^2} \cos t \, dt$.

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$$\begin{aligned} \frac{dy}{dx} &= \frac{dy}{du}\frac{du}{dx} \\ &= \left(\frac{d}{du} \int_1^u \cos t \; dt \right) \left(\frac{d}{dx} x^2 \right) \end{aligned}$$

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$$\begin{aligned}
\frac{dy}{dx} &= \frac{dy}{du}\frac{du}{dx} \\
&= \left(\frac{d}{du}\int_1^u \cos t \; dt\right)\left(\frac{d}{dx}x^2\right) \\
&= (\cos u)\,(2x) = 2x\cos x^2.
\end{aligned}$$

Remark

Part 2 of the theorem tells us how to calculate the definite integral of f over $[a, b]$:

1. Find an antiderivative F of f.
2. Calculate $F(b) - F(a)$.

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Notation

We will write

$$\Big[F(x)\Big]_a^b = F(b) - F(a).$$

Example

$$\int_0^{\pi} \cos x \; dx = \Big[\sin x \Big]_0^{\pi}$$

$$\left(\text{because } \frac{d}{dx} \sin x = \cos x\right)$$

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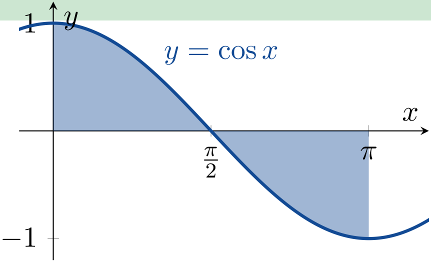
$$= \sin\pi - \sin 0$$

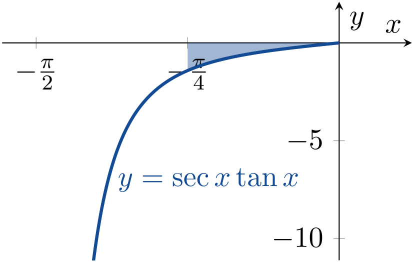
$$= 0 - 0$$

$$= 0$$

Example

$$\begin{aligned}\int_0^{\pi} \cos x \; dx &= \Big[\sin x\Big]_0^{\pi} \\ &\qquad \left(\text{because } \tfrac{d}{dx}\sin x = \cos x\right) \\ &= \sin \pi - \sin 0 \\ &= 0 - 0 \\ &= 0\end{aligned}$$

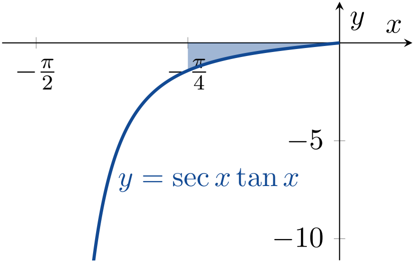




Example

$$\int_{-\frac{\pi}{4}}^{0} \sec x \tan x \, dx = \Big[\sec x \Big]_{-\frac{\pi}{4}}^{0}$$

$$\left(\text{because } \frac{d}{dx} \sec x = \sec x \tan x\right)$$



Example

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$$\left(\text{because } \tfrac{d}{dx}\sec x = \sec x \tan x\right)$$

$$= \sec 0 - \sec\left(-\frac{\pi}{4}\right)$$

$$= 1 - \sqrt{2}.$$

Example

$$\int_1^4 \left(\frac{3}{2}\sqrt{x} - \frac{4}{x^2}\right)\, dx = \left[x^{\frac{3}{2}} + \frac{4}{x}\right]_1^4$$

$$\left(\text{because } \tfrac{d}{dx}\left(x^{\frac{3}{2}} + \frac{4}{x}\right) = \frac{3}{2}\sqrt{x} - \frac{4}{x^2}\right)$$

Example

$$\int_1^4 \left(\frac{3}{2}\sqrt{x} - \frac{4}{x^2}\right)\, dx = \left[x^{\frac{3}{2}} + \frac{4}{x}\right]_1^4$$

$$\left(\text{because } \tfrac{d}{dx}\left(x^{\frac{3}{2}} + \frac{4}{x}\right) = \frac{3}{2}\sqrt{x} - \frac{4}{x^2}\right)$$

$$= \left(4^{\frac{3}{2}} + \frac{4}{4}\right) - \left(1^{\frac{3}{2}} + \frac{4}{1}\right)$$

$$= (8+1) - (1+4)$$

$$= 4.$$

Summary

Remark

Part 1 of the *Fundamental Theorem of Calculus* says

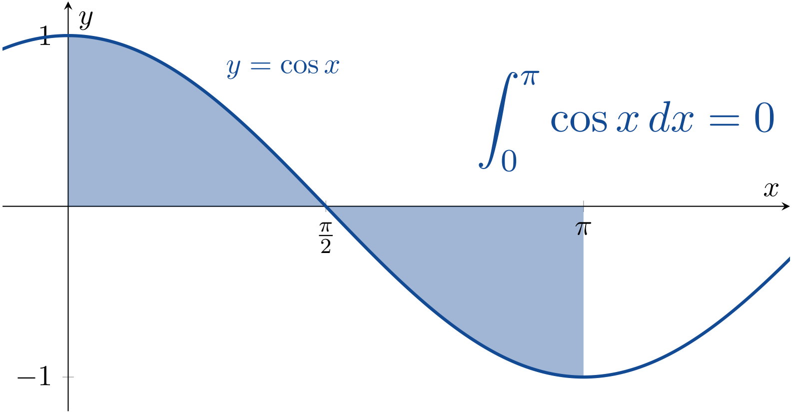
$$\boxed{\frac{d}{dx}\int_a^{x} f(t)\,dt = f(x)}$$

Remark

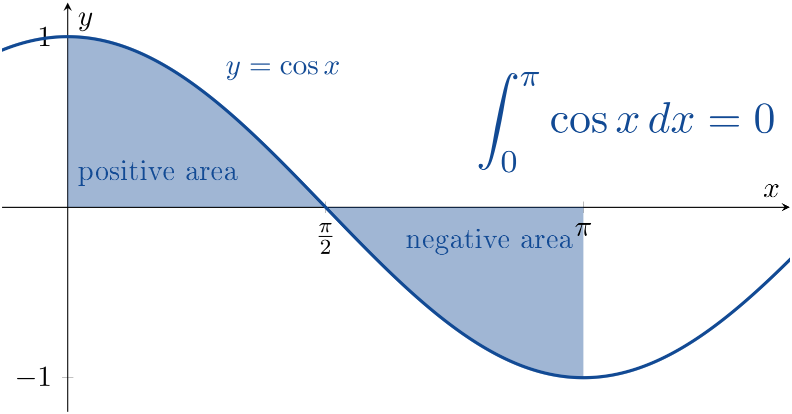
Part 2 of the *Fundamental Theorem of Calculus* says

$$\boxed{\int_a^b f(x)\,dx = \Big[F(x)\Big]_a^b}$$

Total Area

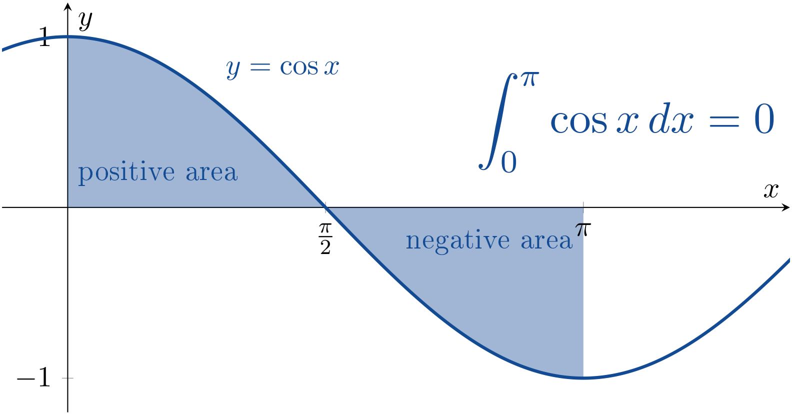


Total Area



$$\int_0^{\pi} \cos x \, dx = 0$$

Total Area



But what if we say that all area is "positive area"?

Example

Let $f(x) = x^2 - 4$. We have that

$$\begin{aligned}\int_{-2}^{2} f(x)\ dx = \int_{-2}^{2} (x^2 - 4)\ dx &= \left[\frac{x^3}{3} - 4x\right]_{-2}^{2} \\ &= \left(\frac{8}{3} - 8\right) - \left(\frac{-8}{3} + 8\right) = -\frac{32}{3}.\end{aligned}$$

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The total area between the graph of $y = f(x)$ and the x-axis, over $[-2, 2]$, is $\left|-\frac{32}{3}\right| = \frac{32}{3}$.

Example

Let $g(x) = 4 - x^2$. We have that

$$\int_{-2}^{2} g(x)\ dx = \int_{-2}^{2} (4 - x^2)\ dx = \left[4x - \frac{x^3}{3}\right]_{-2}^{2}$$
$$= \left(8 - \frac{8}{3}\right) - \left(8 + \frac{-8}{3}\right) = \frac{32}{3}.$$

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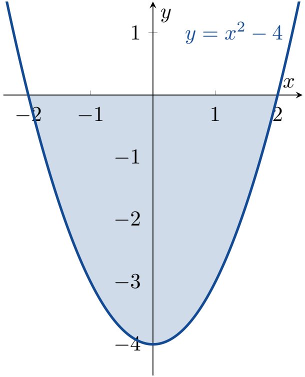

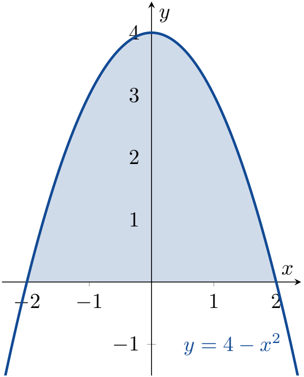

$\text{integral} = -\frac{32}{3}$
$\text{total area} = \frac{32}{3}$

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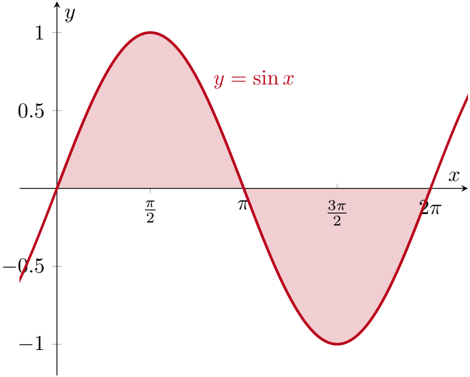
Example

Let $f(x) = \sin x$. Calculate

1. the definite integral of f over $[0, 2\pi]$; and
2. the total area between the graph of $y = f(x)$ and the x-axis over $[0, 2\pi]$.



y
1
0.5
$y = \sin x$
x
$\frac{\pi}{2}$
π
$\frac{3\pi}{2}$
2π
-0.5
-1

1

$$\int_0^{2\pi} \sin x \, dx = \Big[-\cos x \Big]_0^{2\pi} = -\cos 2\pi + \cos 0$$
$$= -1 + 1 = 0.$$

1

$$\begin{aligned}\int_0^{2\pi} \sin x \, dx &= \Big[-\cos x \Big]_0^{2\pi} = -\cos 2\pi + \cos 0 \\ &= -1 + 1 = 0.\end{aligned}$$

2

$$\text{total area} = \left| \int_0^{\pi} \sin x \, dx \right| + \left| \int_{\pi}^{2\pi} \sin x \, dx \right|$$

1

$$\int_0^{2\pi} \sin x \, dx = \Big[-\cos x \Big]_0^{2\pi} = -\cos 2\pi + \cos 0$$
$$= -1 + 1 = 0.$$

2

$$\begin{aligned}
\text{total area} &= \left| \int_0^{\pi} \sin x \, dx \right| + \left| \int_{\pi}^{2\pi} \sin x \, dx \right| \\
&= \left| \Big[-\cos x \Big]_0^{\pi} \right| + \left| \Big[-\cos x \Big]_{\pi}^{2\pi} \right| \\
&= \Big| -\cos \pi + \cos 0 \Big| + \Big| -\cos 2\pi + \cos \pi \Big| \\
&= \Big| -(-1) + 1 \Big| + \Big| -1 + (-1) \Big| = 4.
\end{aligned}$$

Summary

To find the *total area* between the graph of $y = f(x)$ and the x-axis over $[a, b]$:

1. Divide $[a, b]$ at the zeroes of f.
2. Integrate f over each subinterval.
3. Add the absolute values of the integrals.

Example

Find the total area between the graph of $y = x^3 - x^2 - 2x$ and the x-axis for $-1 \leq x \leq 2$.

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1. Let $f(x) = x^3 - x^2 - 2x$.

   Since

$$\begin{aligned} 0 &= f(x) \\ &= x^3 - x^2 - 2x \\ &= x(x+1)(x-2) \end{aligned}$$

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   we divide $[-1, 2]$ into $[-1, 0]$ and $[0, 2]$.

2 We calculate that

$$\begin{aligned}\int_{-1}^{0}(x^3 - x^2 - 2x)\ dx &= \left[\frac{x^4}{4} - \frac{x^3}{3} - x^2\right]_{-1}^{0} \\ &= (0 - 0 - 0) - \left(\frac{1}{4} + \frac{1}{3} - 1\right) \\ &= \frac{5}{12}\end{aligned}$$

2 We calculate that

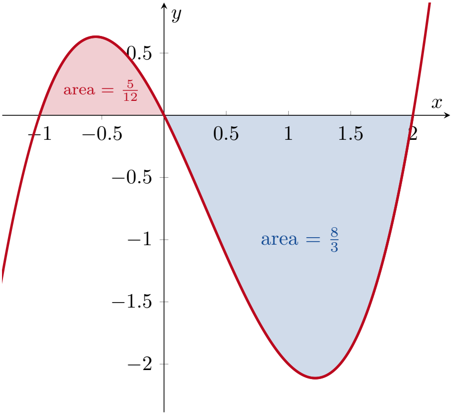
$$\int_{-1}^{0} (x^3 - x^2 - 2x)\ dx = \left[\frac{x^4}{4} - \frac{x^3}{3} - x^2\right]_{-1}^{0}$$
$$= (0 - 0 - 0) - \left(\frac{1}{4} + \frac{1}{3} - 1\right)$$
$$= \frac{5}{12}$$

and

$$\int_{0}^{2} (x^3 - x^2 - 2x)\ dx = \left[\frac{x^4}{4} - \frac{x^3}{3} - x^2\right]_{0}^{2}$$
$$= \left(\frac{16}{4} - \frac{8}{3} - 4\right) - (0 - 0 - 0)$$
$$= -\frac{8}{3}.$$

3. Therefore

$$\text{total area} = \left|\frac{5}{12}\right| + \left|-\frac{8}{3}\right| = \frac{37}{12}.$$



The Average Value of a Continuous Function

The average of $\{1, 2, 2, 6, 9\}$ is $\frac{1+2+2+6+9}{5} = \frac{20}{5} = 4$. We can also calculate the average value of a continuous function.

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Definition

If f is integrable on $[a, b]$, then the *average value of* f *on* $[a, b]$ is

$$\boxed{\operatorname{av}(f) = \frac{1}{b-a} \int_a^b f(x)\ dx.}$$

$$\mathrm{av}(f) = \frac{1}{b-a}\int_a^b f(x)\ d$$

Example

Find the average value of $f(x) = \sqrt{4-x^2}$ on $[-2, 2]$.

$$\text{av}(f) = \frac{1}{b-a}\int_a^b f(x)\ d$$

Example

Find the average value of $f(x) = \sqrt{4-x^2}$ on $[-2,2]$.

Since

$$\begin{aligned}\int_{-2}^{2} f(x)\ dx &= \frac{1}{2} \times \text{the area of a circle of radius 2} \\ &= \frac{1}{2}\pi 2^2 = 2\pi,\end{aligned}$$

$$\mathrm{av}(f) = \frac{1}{b-a}\int_a^b f(x)\ d$$

Example

Find the average value of $f(x) = \sqrt{4 - x^2}$ on $[-2, 2]$.

Since

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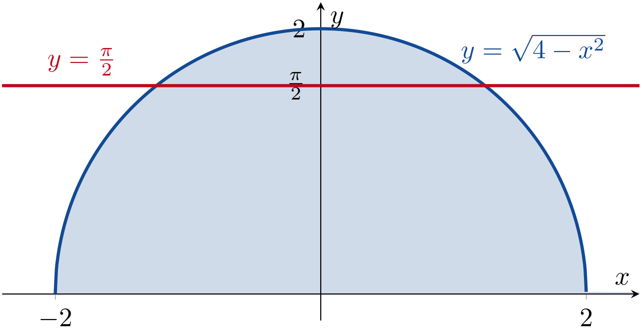
we have that

$$\mathrm{av}(f) = \frac{1}{2-(-2)}\int_{-2}^{2} f(x)\ dx = \frac{2\pi}{4} = \frac{\pi}{2}.$$



y
2
$y = \sqrt{4-x^2}$
$y = \frac{\pi}{2}$
$\frac{\pi}{2}$
x
-2
2

$$\mathrm{av}(f) = \frac{1}{b-a}\int_a^b f(x)\ dx$$

Example

Find the average value of $g(x) = x^3 - x$ on $[0, 1]$.

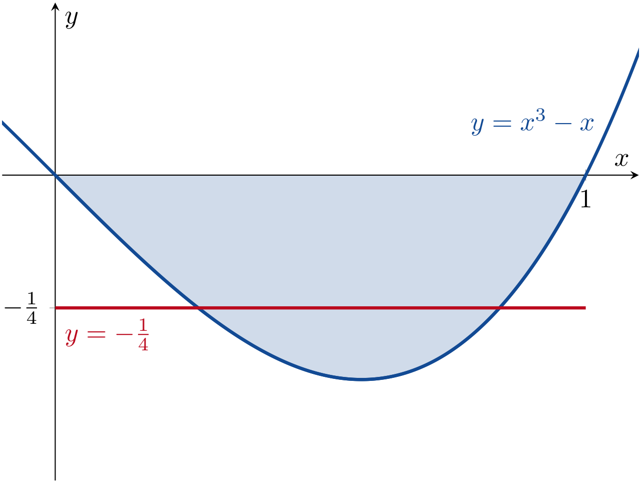
$$\mathrm{av}(g) = \frac{1}{1-0}\int_0^1 g(x)\ dx$$

$$\mathrm{av}(f) = \frac{1}{b-a}\int_a^b f(x)\ d$$

Example

Find the average value of $g(x) = x^3 - x$ on $[0, 1]$.

$$\begin{aligned}
\mathrm{av}(g) &= \frac{1}{1-0}\int_0^1 g(x)\ dx \\
&= \int_0^1 (x^3 - x)\ dx \\
&= \left[\frac{x^4}{4} - \frac{x^2}{2}\right]_0^1 \\
&= \frac{1}{4} - \frac{1}{2} = -\frac{1}{4}.
\end{aligned}$$



y
y = x^3 − x
x
1
−1/4
y = −1/4

Indefinite Integrals & Definite Integrals

Remember that

$\int f(x)\,dx$ is a function.

Remember that

$\int_a^b f(x)\,dx$ is a number.

Indefinite Integrals & Definite Integrals

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$\int f(x)\ dx$ is a function.

For example

$$\int x\ dx = \frac{x^2}{2} + C$$

and

$\int \cos x\ dx = \sin x + C.$

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$$\int x\ dx = \frac{x^2}{2} + C$$

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Remember that

$$\int_a^b f(x)\ dx \text{ is a number.}$$

For example

$$\int_0^1 x\ dx = \frac{1}{2}$$

and

$$\int_0^{\frac{\pi}{2}} \cos x\ dx = 1.$$

Indefinite Integrals and the Substitution Method

By the Chain rule,

$$\frac{d}{dx}\left(\frac{u^{n+1}}{n+1}\right) = u^n \frac{du}{dx}.$$

So

$$\int u^n \frac{du}{dx}\ dx = \frac{u^{n+1}}{n+1} + C.$$

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So

$$\int u^n \frac{du}{dx}\ dx = \frac{u^{n+1}}{n+1} + C.$$

But we know that

$$\int u^n\ du = \frac{u^{n+1}}{n+1} + C$$

also.

By the Chain rule,

$$\frac{d}{dx}\left(\frac{u^{n+1}}{n+1}\right) = u^n \frac{du}{dx}.$$

So

$$\int u^n \frac{du}{dx}\ dx = \frac{u^{n+1}}{n+1} + C.$$

But we know that

$$\int u^n\ du = \frac{u^{n+1}}{n+1} + C$$

also. So it looks like

$$\boxed{du = \frac{du}{dx}\ dx.}$$

Example

Find $\displaystyle\int (x^3 + x)^5 (3x^2 + 1)\; dx.$

Example

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Let $u = x^3 + x$.

Example

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Let $u = x^3 + x$. Then $du = \frac{du}{dx}\ dx = (3x^2 + 1)\ dx$.

Example

Find $\displaystyle\int (x^3 + x)^5 (3x^2 + 1)\ dx$.

Let $u = x^3 + x$. Then $du = \frac{du}{dx}\ dx = (3x^2 + 1)\ dx$. By substitution, we have that

$$\int (x^3 + x)^5 (3x^2 + 1)\, dx = \int u^5\, du$$

Example

Find $\int (x^3+x)^5(3x^2+1)\,dx$.

Let $u = x^3 + x$. Then $du = \frac{du}{dx}\,dx = (3x^2+1)\,dx$. By substitution, we have that

$$\begin{aligned}\int (x^3+x)^5(3x^2+1)\,dx &= \int u^5\,du \\ &= \frac{u^6}{6} + C \\ &= \frac{1}{6}(x^3+x)^6 + C.\end{aligned}$$

$$\begin{pmatrix} \text{The} \\ \text{substitution} \\ \text{method} \end{pmatrix} = \begin{pmatrix} \text{doing the} \\ \text{Chain Rule} \\ \text{backwards.} \end{pmatrix}$$

Example

Find $\displaystyle\int \sqrt{2x+1}\, dx$.

Let $u = 2x + 1$.

Example

Find $\int \sqrt{2x+1}\, dx$.

Let $u = 2x + 1$. Then $du = \frac{du}{dx}\, dx = 2dx$. So $dx = \frac{1}{2}\, du$.

Example

Find $\int \sqrt{2x+1}\, dx$.

Let $u = 2x+1$. Then $du = \frac{du}{dx}\, dx = 2dx$. So $dx = \frac{1}{2}\, du$.
Therefore

$$\begin{aligned}\int \sqrt{2x+1}\, dx &= \int u^{\frac{1}{2}} \left(\tfrac{1}{2} du\right) = \frac{1}{2}\int u^{\frac{1}{2}}\, du \\ &= \frac{1}{2}\left(\frac{u^{\frac{3}{2}}}{\frac{3}{2}}\right) + C = \frac{1}{3}(2x+1)^{\frac{3}{2}} + C.\end{aligned}$$

Theorem (The Substitution Method)

If

- $u = g(x)$ *is differentiable;*
- $g : \mathbb{R} \to I$*; and*
- $f : I \to \mathbb{R}$ *is continuous,*

then

$$\int f(g(x))g'(x)\ dx = \int f(u)\ du.$$

Example

Find $\displaystyle\int 5\sec^2(5t+1)\ dt$.

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Let $u = 5t+1$. Then $du = \frac{du}{dt}\ dt = 5dt$. So

$$\begin{aligned}\int 5\sec^2(5t+1)\ dt &= \int \sec^2 u\ du \\ &= \tan u + C \\ &\quad \left(\text{because } \frac{d}{du}\tan u = \sec^2 u\right) \\ &= \tan(5t+1) + C.\end{aligned}$$

Example

Find $\displaystyle\int \cos(7\theta + 3)\ d\theta$.

Let $u = 7\theta + 3$. Then $du = \frac{du}{d\theta}\ d\theta = 7d\theta$. So $d\theta = \frac{1}{7}du$

Example

Find $\int \cos(7\theta + 3)\ d\theta$.

Let $u = 7\theta + 3$. Then $du = \frac{du}{d\theta}\ d\theta = 7d\theta$. So $d\theta = \frac{1}{7}du$ and

$$\begin{aligned}\int \cos(7\theta + 3)\ d\theta &= \frac{1}{7}\int \cos u\ du \\ &= \frac{1}{7}\sin u + C = \frac{1}{7}\sin(7\theta + 3) + C.\end{aligned}$$

Example

Find $\displaystyle\int x^2 \sin(x^3)\,dx$.

There are both x^2 and x^3 in the integrand. Which should we choose to be equal to u?

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There are both x^2 and x^3 in the integrand. Which should we choose to be equal to u?

Let $u = x^3$. Then $du = \frac{du}{dx}\,dx = 3x^2\,dx$. So $\frac{1}{3}du = x^2\,dx$ and

$$\begin{aligned}\int x^2 \sin(x^3)\,dx = \int \frac{1}{3}\sin u\,du &= -\frac{1}{3}\cos u + C \\ &= -\frac{1}{3}\cos(x^3) + C.\end{aligned}$$

Example

Find $\displaystyle\int x\sqrt{2x+1}\,dx$.

Let $u = 2x+1$. Then $du = \frac{du}{dx}\,dx = 2\,dx$. So $dx = \frac{1}{2}\,du$ and

$$\int x\sqrt{2x+1}\,dx = \int x\sqrt{u}\,\frac{1}{2}\,du.$$

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$$\int x\sqrt{2x+1}\,dx = \int x\sqrt{u}\,\frac{1}{2}\,du.$$

But we still have an x here. We can't integrate until we change all the x terms to u terms. Note that

$$u = 2x+1 \implies u-1 = 2x \implies \frac{1}{2}(u-1) = x.$$

Therefore

$$
\begin{aligned}
\int x\sqrt{2x+1}\;dx &= \int x\sqrt{u}\,\frac{1}{2}\,du \\
&= \int \frac{1}{2}(u-1)\sqrt{u}\;\frac{1}{2}du
\end{aligned}
$$

Therefore

$$\begin{aligned}
\int x\sqrt{2x+1}\,dx &= \int x\sqrt{u}\,\frac{1}{2}\,du \\
&= \int \frac{1}{2}(u-1)\sqrt{u}\;\frac{1}{2}du \\
&= \frac{1}{4}\int u^{\frac{3}{2}} - u^{\frac{1}{2}}\;du \\
&= \frac{1}{4}\left(\frac{2}{5}u^{\frac{5}{2}} - \frac{2}{3}u^{\frac{3}{2}}\right) + C \\
&= \frac{1}{10}u^{\frac{5}{2}} - \frac{1}{6}u^{\frac{3}{2}} + C \\
&= \frac{1}{10}(2x+1)^{\frac{5}{2}} - \frac{1}{6}(2x+1)^{\frac{3}{2}} + C.
\end{aligned}$$

Example

Find $\displaystyle\int \sin^2 x \; dx.$

We use the identity

$$\sin^2 x = \frac{1-\cos 2x}{2}$$

Example

Find $\displaystyle\int \sin^2 x \; dx$.

We use the identity

$$\sin^2 x = \frac{1 - \cos 2x}{2}$$

to calculate that

$$\begin{aligned} \int \sin^2 x \; dx &= \int \frac{1 - \cos 2x}{2} \; dx \\ &= \frac{1}{2} \int (1 - \cos 2x) \; dx \\ &= \frac{1}{2} \left(x - \frac{1}{2} \sin 2x \right) + C \\ &= \frac{1}{2} x - \frac{1}{4} \sin 2x + C. \end{aligned}$$

Example

Similarly

$$\int \cos^2 x \; dx = \int \frac{1 + \cos 2x}{2} \; dx = \frac{1}{2}x + \frac{1}{4}\sin 2x + C.$$

(c) $\displaystyle\int (1 - 2\sin^2 x)\sin 2x\,dx = \int (\cos^2 x - \sin^2 x)\sin 2x\,dx$

$$= \int \cos 2x \sin 2x\,dx \qquad \cos 2x = \cos^2 x - \sin^2 x$$

$$= \int \frac{1}{2}\sin 4x\,dx = \int \frac{1}{8}\sin u\,du \qquad u = 4x,\ du = 4x\,dx$$

$$= -\cos 4x + C.$$

Sometimes there are more than one choice of substitution that we can use.

EXAMPLE 8 Evaluate $\int \frac{2z\,dz}{\sqrt[3]{z^2+1}}$.

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EXAMPLE 8 Evaluate $\int \frac{2z\,dz}{\sqrt[3]{z^2+1}}$.

Method 1: Substitute $u = z^2 + 1$.

$$\int \frac{2z\,dz}{\sqrt[3]{z^2+1}} = \int \frac{du}{u^{1/3}} \qquad \text{Let } u = z^2 + 1,\ du = 2z\,dz.$$

$$= \int u^{-1/3}\,du \qquad \text{In the form } \int u^n\,du$$

$$= \frac{u^{2/3}}{2/3} + C \qquad \text{Integrate.}$$

$$= \frac{3}{2}u^{2/3} + C$$

$$= \frac{3}{2}(z^2+1)^{2/3} + C \qquad \text{Replace } u \text{ by } z^2 + 1.$$

Sometimes there are more than one choice of substitution that we can use.

EXAMPLE 8 Evaluate $\int \frac{2z\,dz}{\sqrt[3]{z^2+1}}$.

Method 2: Substitute $u = \sqrt[3]{z^2+1}$ instead.

$$\int \frac{2z\,dz}{\sqrt[3]{z^2+1}} = \int \frac{3u^2\,du}{u}$$ Let $u = \sqrt[3]{z^2+1}$, $u^3 = z^2+1$, $3u^2\,du = 2z\,dz$.

$$= 3\int u\,du$$

$$= 3 \cdot \frac{u^2}{2} + C$$ Integrate.

$$= \frac{3}{2}(z^2+1)^{2/3} + C$$ Replace u by $(z^2+1)^{1/3}$. ■

Break

We will continue at 2pm

5.6 Substitution and Area Between Curves

Theorem (The Substitution Method)

If

- $u = g(x)$ *is differentiable on* $[a, b]$;
- g' *is continuous on* $[a, b]$; *and*
- f *is continous on the range of* g,

then

$$\int_a^b f(g(x))g'(x)\ dx = \int_{g(a)}^{g(b)} f(u)\ du.$$

Example

Calculate $\displaystyle\int_{-1}^{1} 3x^2\sqrt{x^3+1}\,dx$.

solution 1. We can use the previous theorem to solve this example.

Let $u = x^3 + 1$. Then $du = 3x^2\,dx$.

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Moreover $x = -1 \implies u = 0$ and $x = 1 \implies u = 2$.

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Calculate $\displaystyle\int_{-1}^{1} 3x^2\sqrt{x^3+1}\,dx$.

solution 1. We can use the previous theorem to solve this example.

Let $u = x^3 + 1$. Then $du = 3x^2\,dx$.
Moreover $x = -1 \implies u = 0$ and $x = 1 \implies u = 2$. So

$$\int_{x=-1}^{x=1} 3x^2\sqrt{x^3+1}\,dx = \int_{u=0}^{u=2} \sqrt{u}\,du$$

Example

Calculate $\displaystyle\int_{-1}^{1} 3x^2\sqrt{x^3+1}\,dx$.

solution 1. We can use the previous theorem to solve this example.

Let $u = x^3 + 1$. Then $du = 3x^2\,dx$.
Moreover $x = -1 \implies u = 0$ and $x = 1 \implies u = 2$. So

$$\begin{aligned}\int_{x=-1}^{x=1} 3x^2\sqrt{x^3+1}\,dx &= \int_{u=0}^{u=2} \sqrt{u}\,du \\ &= \left[\frac{2}{3}u^{\frac{3}{2}}\right]_0^2 \\ &= \frac{2}{3}\left(2^{\frac{3}{2}} - 0^{\frac{3}{2}}\right) = \frac{2}{3}2\sqrt{2} = \frac{4\sqrt{2}}{3}.\end{aligned}$$

solution 2. Alternately, we can first find the indefinite integral, then find the required definite integral.

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Let $u = x^3 + 1$. Then $du = 3x^2\ dx$. So

$$\int 3x^2\sqrt{x^3+1}\ dx = \int \sqrt{u}\ du = \frac{2}{3}u^{\frac{3}{2}} + C = \frac{2}{3}(x^3+1)^{\frac{3}{2}} + C.$$

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Let $u = x^3 + 1$. Then $du = 3x^2\ dx$. So

$$\int 3x^2 \sqrt{x^3+1}\ dx = \int \sqrt{u}\ du = \frac{2}{3}u^{\frac{3}{2}} + C = \frac{2}{3}(x^3+1)^{\frac{3}{2}} + C.$$

Therefore

$$\begin{aligned} \int_{-1}^{1} 3x^2 \sqrt{x^3+1}\ dx &= \left[\frac{2}{3}(x^3+1)^{\frac{3}{2}}\right]_{-1}^{1} \\ &= \left(\frac{2}{3}(1+1)^{\frac{3}{2}}\right) - \left(\frac{2}{3}(-1+1)^{\frac{3}{2}}\right) \\ &= \frac{2}{3} \times 2^{\frac{3}{2}} = \frac{4\sqrt{2}}{3}. \end{aligned}$$

Example

Calculate $\displaystyle\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \cot\theta \operatorname{cosec}^2\theta \; d\theta$.

Let $u = \cot\theta$. Then $du = \frac{du}{d\theta}\; d\theta = -\operatorname{cosec}^2\theta \; d\theta$. So $-du = \operatorname{cosec}^2\theta \; d\theta$. Moreover $\theta = \frac{\pi}{4} \implies u = \cot\frac{\pi}{4} = 1$ and $\theta = \frac{\pi}{2} \implies u = \cot\frac{\pi}{2} = 0$.

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$$\int_{\theta=\frac{\pi}{4}}^{\theta=\frac{\pi}{2}} \cot\theta \operatorname{cosec}^2\theta \; d\theta = \int_{u=1}^{u=0} u\;(-du) = -\int_1^0 u\; du$$

Example

Calculate $\displaystyle\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \cot\theta \operatorname{cosec}^2\theta \; d\theta$.

Let $u = \cot\theta$. Then $du = \frac{du}{d\theta}\; d\theta = -\operatorname{cosec}^2\theta \; d\theta$. So $-du = \operatorname{cosec}^2\theta \; d\theta$. Moreover $\theta = \frac{\pi}{4} \implies u = \cot\frac{\pi}{4} = 1$ and $\theta = \frac{\pi}{2} \implies u = \cot\frac{\pi}{2} = 0$. Hence

$$\begin{aligned}\int_{\theta=\frac{\pi}{4}}^{\theta=\frac{\pi}{2}} \cot\theta \operatorname{cosec}^2\theta \; d\theta &= \int_{u=1}^{u=0} u \; (-du) = -\int_1^0 u \; du \\ &= -\left[\frac{u^2}{2}\right]_1^0 = -\left(\frac{0^2}{2} - \frac{1^2}{2}\right) = \frac{1}{2}.\end{aligned}$$

Remark

You don't need to write "$x =$" and "$u =$" if you understand which is which.

$$\int_{u=0}^{u=1} u\,du = \int_0^1 u\,du$$

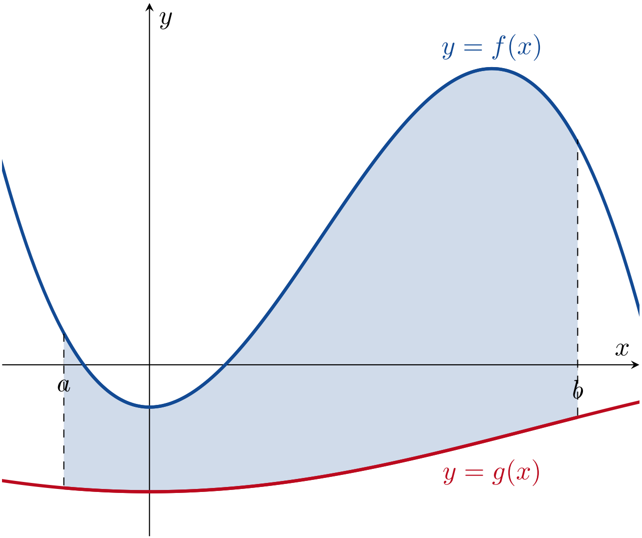
(b) $$\int_0^{\pi/2} \frac{2 \sin x \cos x}{(1 + \sin^2 x)^3}\, dx = \int_1^2 \frac{1}{u^3}\, du$$

Let $u = 1 + \sin^2 x$, $du = 2 \sin x \cos x\, dx$.
When $x = 0$, $u = 1$.
When $x = \pi/2$, $u = 2$.

$$= -\frac{1}{2u^2}\Big]_1^2$$

$$= -\frac{1}{8} - \left(-\frac{1}{2}\right) = \frac{3}{8}$$

Area Between Curves



Definition

If

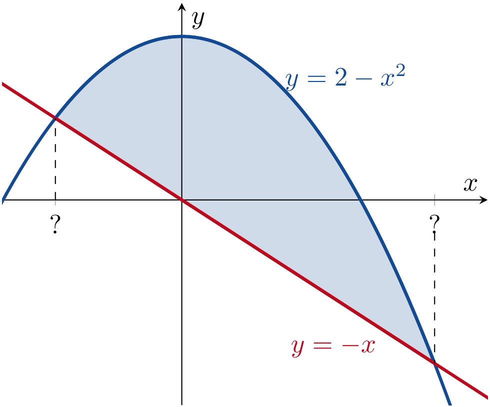
- f is continous;
- g is continous; and
- $f(x) \geq g(x)$ on $[a, b]$,

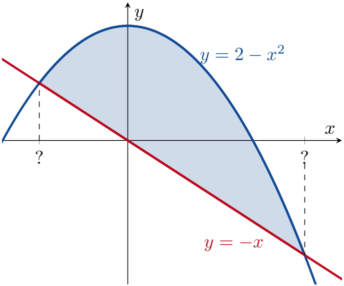
then the *area of the region between the curves* $y = f(x)$ *and* $y = g(x)$ *for* $a \leq x \leq b$ is

$$\boxed{\text{area} = \int_a^b \Big(f(x) - g(x) \Big) \, dx.}$$

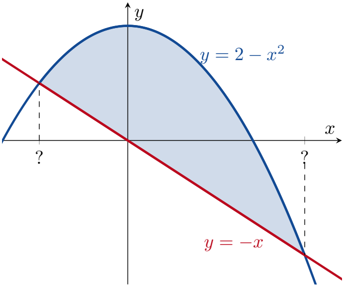
Example

Find the area between $y = 2 - x^2$ and $y = -x$.





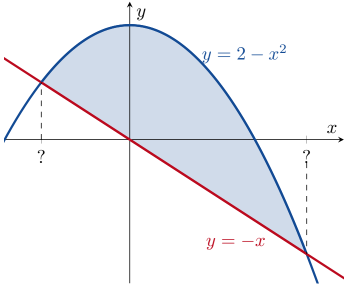
First we need to find the limits of integration:



First we need to find the limits of integration:

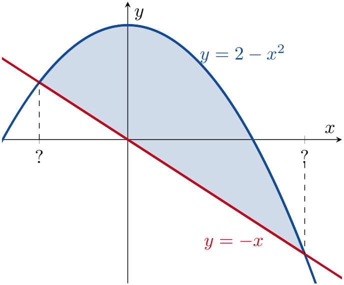
$$\begin{aligned} 2 - x^2 &= -x \\ 0 &= x^2 - x - 2 \\ 0 &= (x+1)(x-2) \implies x = -1 \text{ or } 2. \end{aligned}$$

We need to integrate from $x = -1$ to $x = 2$.



Therefore

$$\text{area} = \int_{-1}^{2} \Big((2 - x^2) - (-x) \Big) \, dx$$

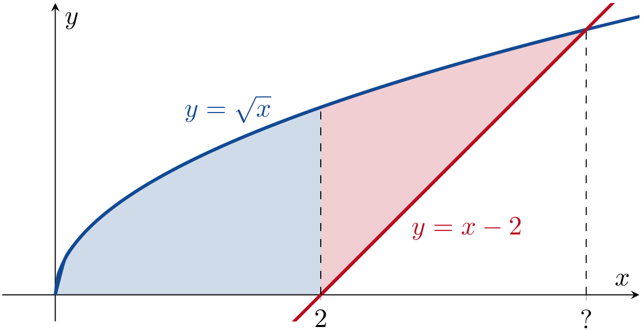


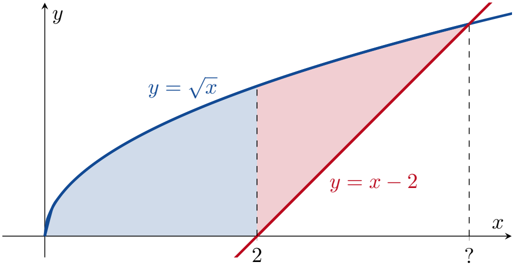
Therefore

$$\begin{aligned}\text{area} &= \int_{-1}^{2} \Big((2 - x^2) - (-x) \Big)\, dx \\ &= \int_{-1}^{2} (2 + x - x^2)\, dx = \Big[2x + \tfrac{1}{2}x^2 - \tfrac{1}{3}x^3 \Big]_{-1}^{2} \\ &= \left(4 + \frac{4}{2} - \frac{8}{3} \right) - \left(-2 + \frac{1}{2} + \frac{1}{3} \right) = \frac{9}{2}.\end{aligned}$$

Example

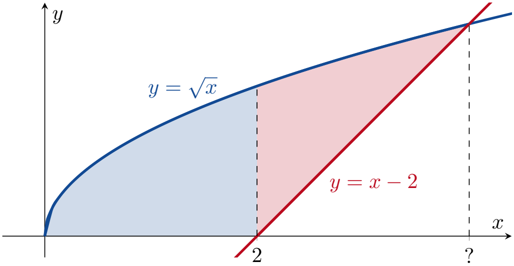
Find the area bounded by $y = \sqrt{x}$, $y = x - 2$ and the x-axis, for $x \geq 0$ and $y \geq 0$.





First we calculate that

$$\begin{aligned}\sqrt{x} &= x - 2 \\ x &= (x-2)^2 = x^2 - 4x + 4 \\ 0 &= x^2 - 5x + 4 = (x-1)(x-4) \implies x = 1 \text{ or } 4.\end{aligned}$$



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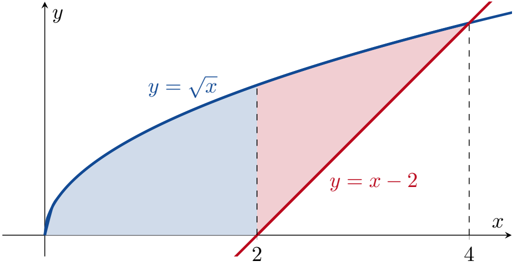
\sqrt{x} &= x - 2 \\
x &= (x-2)^2 = x^2 - 4x + 4 \\
0 &= x^2 - 5x + 4 = (x-1)(x-4) \implies x = 1 \text{ or } 4.
\end{aligned}
$$

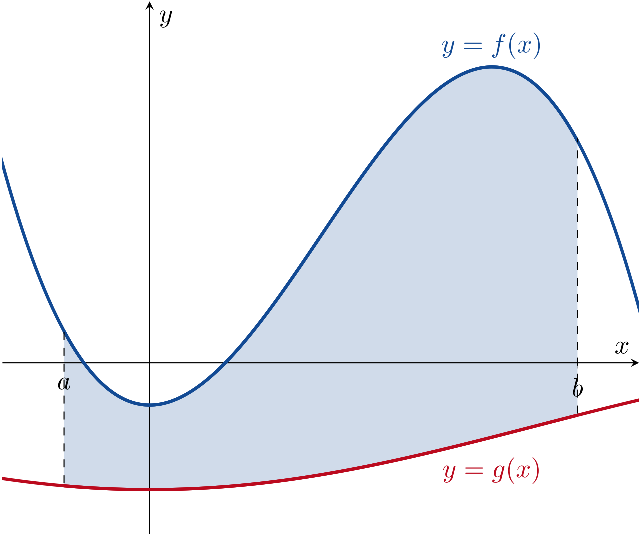
Since $\sqrt{1} \neq 1 - 2$, we must have $x = 4$.



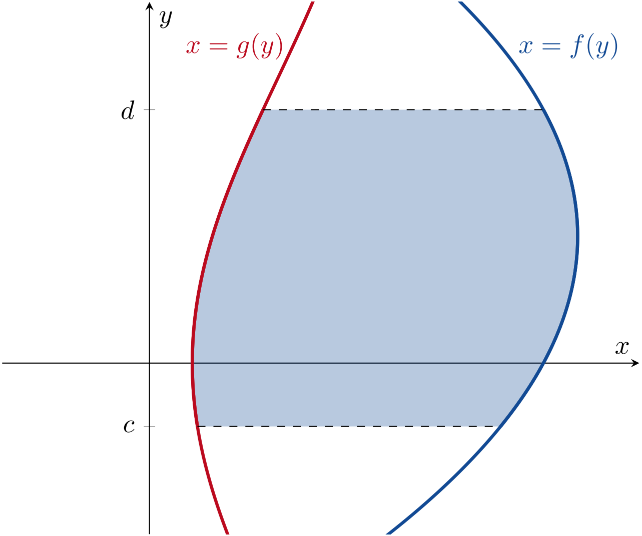
Therefore

$$\begin{aligned} \text{area} &= \text{blue area} + \text{red area} \\ &= \int_0^2 \left(\sqrt{x} - 0\right)\ dx + \int_2^4 \Big(\sqrt{x} - (x-2)\Big)\ dx \\ &\ \vdots \\ &= \frac{10}{3}. \end{aligned}$$

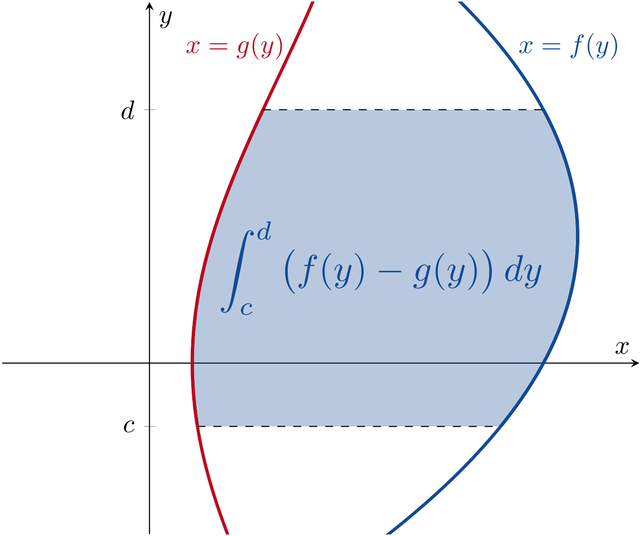
Integration With Respect To y



Integration With Respect To y



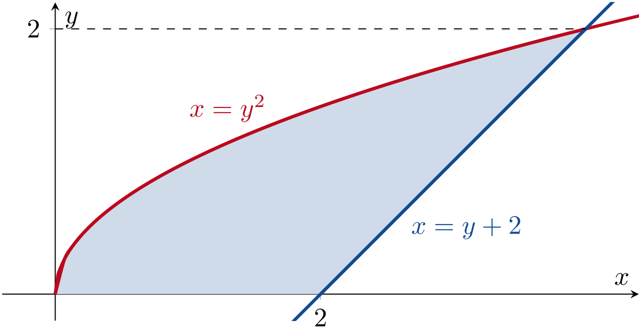
Integration With Respect To y



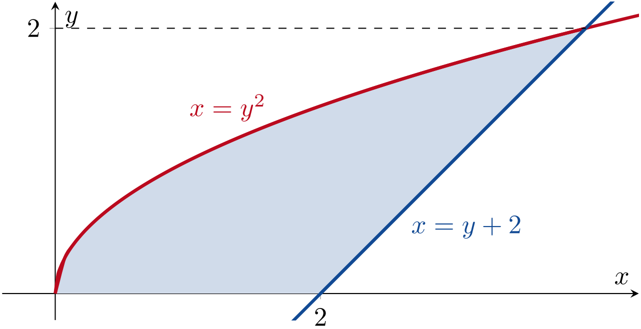
wrt = with respect to

Example

Find the area bounded by $y = \sqrt{x}$, $y = x - 2$ and the x-axis, for $x \geq 0$ and $y \geq 0$, by integrating wrt y.



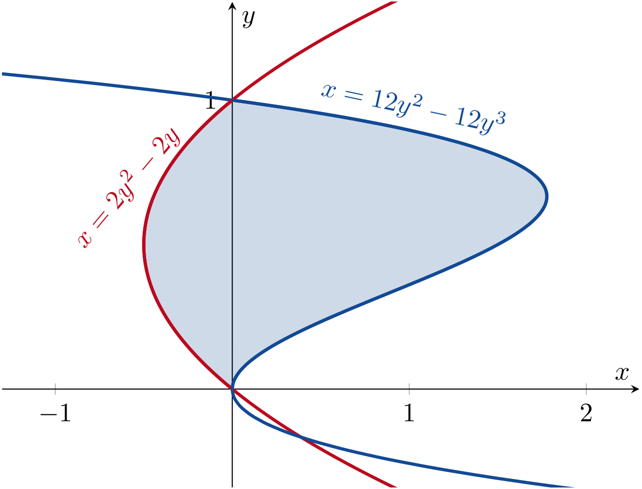
We will integrate between $y = 0$ and $y = 2$.

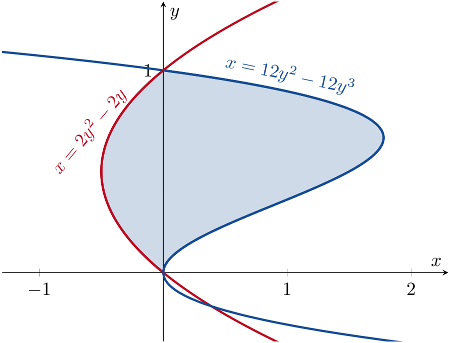


$$\text{area} = \int_c^d f(y) - g(y)\,dy = \int_0^2 (y+2) - (y^2)\,dy = \ldots = \frac{10}{3}.$$

Example (page 322, exercise 33)

Find the area of the region shown below.

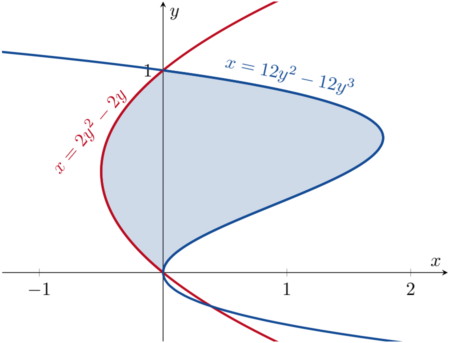




$$\text{area} = \int_0^1 \left(\begin{matrix} \text{function} \\ \text{on right} \end{matrix} \right) - \left(\begin{matrix} \text{function} \\ \text{on left} \end{matrix} \right) dy$$

$$= \qquad\qquad\qquad\qquad =$$

$$= \qquad\qquad\qquad = \qquad\qquad\qquad = \quad .$$



$$\text{area} = \int_0^1 \begin{pmatrix} \text{function} \\ \text{on right} \end{pmatrix} - \begin{pmatrix} \text{function} \\ \text{on left} \end{pmatrix} dy$$

$$= \int_0^1 \left(12y^2 - 12y^3\right) - \left(2y^2 - 2y\right)\, dy = \int_0^1 10y^2 - 12y^3 + 2y\, dy$$

$$= \left[\frac{10}{3}y^3 - 3y^4 + y^2\right]_0^1 = \left(\frac{10}{3} - 3 + 1\right) - (0 - 0 + 0) = \frac{4}{3}.$$

Next Time

- 6.1 Volumes Using Cross-Sections
- 6.2 Volumes Using Cylindrical Shells
- 6.3 Arc Length
- 6.4 Areas of Surfaces of Revolution