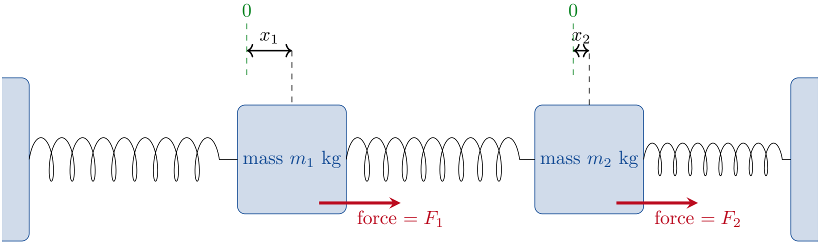
Lecture 10

- 5.1 Introduction
- 5.2 Basic Theory of Systems of First Order Linear Equations
- 5.3 Homogeneous Linear Systems with Constant Coefficients
- 5.4 Complex Eigenvalues

5.1 Introduction

5.1 Introduction



Consider the dynamical system shown above. There are two blocks and three springs. Forces F_1 and F_2 act on the blocks as shown.

See `https://tinyurl.com/wm2ogdh`

We expect that the acceleration of the blocks will depend on

- the displacements x_1 and x_2;
- the forces F_1 and F_2; and
- the masses of the blocks.

We expect that the acceleration of the blocks will depend on

- the displacements x_1 and x_2;
- the forces F_1 and F_2; and
- the masses of the blocks.

So we expect that:

$$\begin{cases} \dfrac{d^2x_1}{dt^2} = f_1(x_1, x_2, F_1, m_1) \\ \dfrac{d^2x_2}{dt^2} = f_2(x_1, x_2, F_2, m_2). \end{cases}$$

5.1 Introduction

We expect that the acceleration of the blocks will depend on

- the displacements x_1 and x_2;
- the forces F_1 and F_2; and
- the masses of the blocks.

So we expect that:

$$\begin{cases} \dfrac{d^2x_1}{dt^2} = f_1(x_1, x_2, F_1, m_1) \\[2ex] \dfrac{d^2x_2}{dt^2} = f_2(x_1, x_2, F_2, m_2). \end{cases}$$

This is a system of two ODEs. To find $x_1(t)$ and $x_2(t)$, we would need to solve these equations at the same time.

The most famous system of ODEs is the system of *Predator-Prey* equations:

$$\begin{cases} \dfrac{dx}{dt} = \alpha x - \beta xy \\[2ex] \dfrac{dy}{dt} = \delta xy - \gamma y \end{cases}$$

where

$$x(t) = \text{number of prey (e.g. mice)}$$
$$y(t) = \text{number of predators (e.g. owls)},$$

which originate circa 1925.

5.1 Introduction

It is possible to convert an nth order linear ODE into a system of n first order linear ODEs. Or vice versa.

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \dots + a_1 y' + a_0 y = g(t) \quad \longleftrightarrow \quad \begin{cases} x_1' = b_{11}x_1 + \ldots + b_{1n}x_n + h_1(t) \\ x_2' = b_{21}x_1 + \ldots + b_{2n}x_n + h_2(t) \\ \vdots \\ x_n' = b_{n1}x_1 + \ldots + b_{nn}x_n + h_n(t) \end{cases}$$

Example

Write

$$u'' + 0.25u' + u = 0$$

as a system of two first order ODEs.

Example

Write
$$u'' + 0.25u' + u = 0$$
as a system of two first order ODEs.

Let $x_1 = u$ and $x_2 = u'$.

Example

Write

$$u'' + 0.25u' + u = 0$$

as a system of two first order ODEs.

Let $x_1 = u$ and $x_2 = u'$. Then clearly $x_1' = u' = x_2$

Example

Write
$$u'' + 0.25u' + u = 0$$
as a system of two first order ODEs.

Let $x_1 = u$ and $x_2 = u'$. Then clearly $x_1' = u' = x_2$ and
$$x_2' = u'' = -0.25u' - u = -0.25x_2 - x_1.$$

Example

Write

$$u'' + 0.25u' + u = 0$$

as a system of two first order ODEs.

Let $x_1 = u$ and $x_2 = u'$. Then clearly $x_1' = u' = x_2$ and

$$x_2' = u'' = -0.25u' - u = -0.25x_2 - x_1.$$

Therefore

$$\begin{cases} x_1' = x_2 \\ x_2' = -x_1 - 0.25x_2. \end{cases}$$

Remark

We will need

- matrices,
- eigenvalues,
- eigenvectors,
- the Wronskian,
- linear independence,
- and more

from MATH215 – please either revise your Linear Algebra lecture notes or read your Linear Algebra book or read §7.2-7.3 in the textbook by Boyce and DiPrima.

Basic Theory of Systems of First Order Linear Equations

5.2

$$\begin{cases} x_1' = p_{11}(t)x_1 + p_{12}(t)x_2 + \ldots + p_{1n}(t)x_n + g_1(t) \\ x_2' = p_{21}(t)x_1 + p_{22}(t)x_2 + \ldots + p_{2n}(t)x_n + g_2(t) \\ \qquad \vdots \\ x_n' = p_{n1}(t)x_1 + p_{n2}(t)x_2 + \ldots + p_{nn}(t)x_n + g_n(t) \end{cases}$$

is a system of n linear ODEs and n variables: x_1, x_2, $\ldots$, x_n.

If we write

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, \ \mathbf{x}' = \begin{bmatrix} x_1' \\ x_2' \\ \vdots \\ x_n' \end{bmatrix}, \ P = \begin{bmatrix} p_{11} & p_{12} & \dots & p_{1n} \\ p_{21} & p_{22} & \dots & p_{2n} \\ \vdots & \vdots & & \vdots \\ p_{n1} & p_{n2} & \dots & p_{nn} \end{bmatrix}, \text{ and } \mathbf{g} = \begin{bmatrix} g_1 \\ g_2 \\ \vdots \\ g_n \end{bmatrix}$$

then we can write this system as

$$\mathbf{x}' = P(t)\mathbf{x} + \mathbf{g}(t).$$

First we will consider the homogeneous system

$$\mathbf{x}' = P(t)\mathbf{x}.$$

In Chapter 3 when we had multiple solutions, we wrote them as $y_1(t)$, $y_2(t)$, $\ldots$. But we are already using x_1, x_2, $\ldots$ to denote coordinates. So we need a new type of notation.

In Chapter 3 when we had multiple solutions, we wrote them as $y_1(t)$, $y_2(t)$, ... But we are already using x_1, x_2, ... to denote coordinates. So we need a new type of notation.

Notation

We use $\mathbf{x}^{(1)}(t)$, $\mathbf{x}^{(2)}(t)$, ... to denote different vector solutions.

Recall from Chapter 3 that if $y_1(t)$ and $y_2(t)$ are both solutions to

$$ay'' + by' + cy = 0,$$

then

$$c_1 y_1 + c_2 y_2$$

is also a solution.

5.2 Basic Theory of Systems of First Order Linear Equations

Theorem

If $\mathbf{x}^{(1)}(t)$ *and* $\mathbf{x}^{(2)}(t)$ *are solutions to* $\mathbf{x}' = P(t)\mathbf{x}$*, then*

$$c_1\mathbf{x}^{(1)} + c_2\mathbf{x}^{(2)}$$

is also a solution for any $c_1, c_2 \in \mathbb{R}$*.*

Example

$\mathbf{x}^{(1)} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} e^{3t}$ and $\mathbf{x}^{(2)} = \begin{bmatrix} 1 \\ -2 \end{bmatrix} e^{-t}$ are both solutions to $\mathbf{x}' = \begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix} \mathbf{x}$ (we will see this later).

Example

$\mathbf{x}^{(1)} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} e^{3t}$ and $\mathbf{x}^{(2)} = \begin{bmatrix} 1 \\ -2 \end{bmatrix} e^{-t}$ are both solutions to $\mathbf{x}' = \begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix} \mathbf{x}$ (we will see this later).Therefore

$$\mathbf{x}(t) = c_1 \mathbf{x}^{(1)} + c_2 \mathbf{x}^{(2)} = c_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} e^{3t} + c_2 \begin{bmatrix} 1 \\ -2 \end{bmatrix} e^{-t}$$

is also a solution to this system.

(Suppose that $P(t)$ is an $n \times n$ matrix.)

Theorem

If $\mathbf{x}^{(1)}(t),\ \mathbf{x}^{(2)}(t),\ \ldots,\ \mathbf{x}^{(n)}(t)$ *are linearly independent solutions to* $\mathbf{x}' = P(t)\mathbf{x}$*, then every solution to this system can be written as*

$$\mathbf{x}(t) = c_1\mathbf{x}^{(1)} + c_2\mathbf{x}^{(2)} + \ldots + c_n\mathbf{x}^{(n)}$$

in exactly one way.

Definition

In this case, we say that $\mathbf{x}^{(1)}(t)$, $\mathbf{x}^{(2)}(t)$, $\dots$, $\mathbf{x}^{(n)}(t)$ form a *fundamental set of solutions* to $\mathbf{x}' = P(t)\mathbf{x}$.

Definition

In this case, we say that $\mathbf{x}^{(1)}(t)$, $\mathbf{x}^{(2)}(t)$, $\ldots$, $\mathbf{x}^{(n)}(t)$ form a *fundamental set of solutions* to $\mathbf{x}' = P(t)\mathbf{x}$.

Definition

In this case,

$$\mathbf{x}(t) = c_1\mathbf{x}^{(1)} + c_2\mathbf{x}^{(2)} + \ldots + c_n\mathbf{x}^{(n)}$$

is called the *general solution* to $\mathbf{x}' = P(t)\mathbf{x}$.

Homogeneous Linear Systems with Constant Coefficients

5.3

Consider

$$\mathbf{x}' = A\mathbf{x}$$

where $A \in \mathbb{R}^{n \times n}$.

5.3 Homogeneous Linear Systems with Constant Coefficients

$$\mathbf{x}' = A\mathbf{x}$$

If $n = 1$, then we just have

$$\frac{dx}{dt} = ax$$

which has general solution $x(t) = ce^{at}$.

$$\mathbf{x}' = A\mathbf{x}$$

If $n = 1$, then we just have

$$\frac{dx}{dt} = ax$$

which has general solution $x(t) = ce^{at}$.

For $n > 1$, we guess that

$$\mathbf{x}(t) = \boldsymbol{\xi} e^{rt}$$

is a solution to $\mathbf{x}' = A\mathbf{x}$, for some number $r \in \mathbb{C}$ and some vector $\boldsymbol{\xi} \in \mathbb{C}^n$.

But if $\mathbf{x}(t) = \boldsymbol{\xi} e^{rt}$, then

$$\mathbf{x}' = A\mathbf{x}$$

But if $\mathbf{x}(t) = \boldsymbol{\xi} e^{rt}$, then

$$r\boldsymbol{\xi} e^{rt} = \mathbf{x}' = A\mathbf{x}$$

But if $\mathbf{x}(t) = \boldsymbol{\xi} e^{rt}$, then

$$r\boldsymbol{\xi} e^{rt} = \mathbf{x}' = A\mathbf{x} = A\boldsymbol{\xi} e^{rt}$$

But if $\mathbf{x}(t) = \boldsymbol{\xi} e^{rt}$, then

$$r\boldsymbol{\xi} e^{rt} = \mathbf{x}' = A\mathbf{x} = A\boldsymbol{\xi} e^{rt}$$
$$r\boldsymbol{\xi} = A\boldsymbol{\xi}$$

But if $\mathbf{x}(t) = \boldsymbol{\xi} e^{rt}$, then

$$\begin{aligned} r\boldsymbol{\xi} e^{rt} = \mathbf{x}' = A\mathbf{x} &= A\boldsymbol{\xi} e^{rt} \\ r\boldsymbol{\xi} &= A\boldsymbol{\xi} \\ (A - rI)\boldsymbol{\xi} &= \mathbf{0} \end{aligned}$$

where I is the identity matrix.

But if $\mathbf{x}(t) = \boldsymbol{\xi} e^{rt}$, then

$$\begin{aligned} r\boldsymbol{\xi} e^{rt} = \mathbf{x}' = A\mathbf{x} = A\boldsymbol{\xi} e^{rt} \\ r\boldsymbol{\xi} = A\boldsymbol{\xi} \\ (A - rI)\boldsymbol{\xi} = \mathbf{0} \end{aligned}$$

where I is the identity matrix. Hence r must be an eigenvalue of A and $\boldsymbol{\xi}$ must be a corresponding eigenvector of A.

Remark

So the idea is:

1. Find the eigenvalues;
2. Find the eigenvectors; then
3. Write $\mathbf{x}^{(j)}(t) = \boldsymbol{\xi}^{(j)} e^{r_j t}$.

Example

Solve
$$\mathbf{x}' = \begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix} \mathbf{x}.$$

Example

Solve

$$\mathbf{x}' = \begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix} \mathbf{x}.$$

First we find the eigenvalues.

Example

Solve

$$\mathbf{x}' = \begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix} \mathbf{x}.$$

First we find the eigenvalues. Since

$$\begin{aligned} 0 = \det(A - rI) &= \begin{vmatrix} 1-r & 1 \\ 4 & 1-r \end{vmatrix} = (1-r)^2 - 4 \\ &= r^2 - 2r - 3 = (r+1)(r-3), \end{aligned}$$

the eigenvalues are $r_1 = 3$ and $r_2 = -1$.

Using the first eigenvalue $r_1 = 3$, we calculate that

$$\mathbf{0} = (A - r_1 I)\boldsymbol{\xi} = \begin{bmatrix} -2 & 1 \\ 4 & -2 \end{bmatrix} \begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix}$$

Using the first eigenvalue $r_1 = 3$, we calculate that

$$\mathbf{0} = (A - r_1 I)\boldsymbol{\xi} = \begin{bmatrix} -2 & 1 \\ 4 & -2 \end{bmatrix} \begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix} \qquad \Longrightarrow \qquad 0 = -2\xi_1 + \xi_2.$$

Using the first eigenvalue $r_1 = 3$, we calculate that

$$\mathbf{0} = (A - r_1 I)\boldsymbol{\xi} = \begin{bmatrix} -2 & 1 \\ 4 & -2 \end{bmatrix} \begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix} \qquad \Longrightarrow \qquad 0 = -2\xi_1 + \xi_2.$$

Hence we can choose $\boldsymbol{\xi}^{(1)} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$.

Using the first eigenvalue $r_1 = 3$, we calculate that

$$\mathbf{0} = (A - r_1 I)\boldsymbol{\xi} = \begin{bmatrix} -2 & 1 \\ 4 & -2 \end{bmatrix} \begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix} \qquad \Longrightarrow \qquad 0 = -2\xi_1 + \xi_2.$$

Hence we can choose $\boldsymbol{\xi}^{(1)} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$. Then using the second eigenvalue $r_2 = -1$, we calculate that

$$\mathbf{0} = (A - r_2 I)\boldsymbol{\xi} = \begin{bmatrix} 2 & 1 \\ 4 & 2 \end{bmatrix} \begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix}$$

Using the first eigenvalue $r_1 = 3$, we calculate that

$$\mathbf{0} = (A - r_1 I)\boldsymbol{\xi} = \begin{bmatrix} -2 & 1 \\ 4 & -2 \end{bmatrix} \begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix} \qquad \Longrightarrow \qquad 0 = -2\xi_1 + \xi_2.$$

Hence we can choose $\boldsymbol{\xi}^{(1)} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$. Then using the second eigenvalue $r_2 = -1$, we calculate that

$$\mathbf{0} = (A - r_2 I)\boldsymbol{\xi} = \begin{bmatrix} 2 & 1 \\ 4 & 2 \end{bmatrix} \begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix} \qquad \Longrightarrow \qquad 0 = 2\xi_1 + \xi_2.$$

Using the first eigenvalue $r_1 = 3$, we calculate that

$$\mathbf{0} = (A - r_1 I)\boldsymbol{\xi} = \begin{bmatrix} -2 & 1 \\ 4 & -2 \end{bmatrix} \begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix} \qquad \Longrightarrow \qquad 0 = -2\xi_1 + \xi_2.$$

Hence we can choose $\boldsymbol{\xi}^{(1)} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$. Then using the second eigenvalue $r_2 = -1$, we calculate that

$$\mathbf{0} = (A - r_2 I)\boldsymbol{\xi} = \begin{bmatrix} 2 & 1 \\ 4 & 2 \end{bmatrix} \begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix} \qquad \Longrightarrow \qquad 0 = 2\xi_1 + \xi_2.$$

Hence we can choose $\boldsymbol{\xi}^{(2)} = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$.

Using the first eigenvalue $r_1 = 3$, we calculate that

$$\mathbf{0} = (A - r_1 I)\boldsymbol{\xi} = \begin{bmatrix} -2 & 1 \\ 4 & -2 \end{bmatrix} \begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix} \qquad \implies \qquad 0 = -2\xi_1 + \xi_2.$$

Hence we can choose $\boldsymbol{\xi}^{(1)} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$. Then using the second eigenvalue $r_2 = -1$, we calculate that

$$\mathbf{0} = (A - r_2 I)\boldsymbol{\xi} = \begin{bmatrix} 2 & 1 \\ 4 & 2 \end{bmatrix} \begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix} \qquad \implies \qquad 0 = 2\xi_1 + \xi_2.$$

Hence we can choose $\boldsymbol{\xi}^{(2)} = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$. This gives us two solutions:

$$\mathbf{x}^{(1)}(t) = \begin{bmatrix} 1 \\ 2 \end{bmatrix} e^{3t} \qquad \text{and} \qquad \mathbf{x}^{(2)}(t) = \begin{bmatrix} 1 \\ -2 \end{bmatrix} e^{-t}.$$

But are these two solutions linearly independent?

But are these two solutions linearly independent? To find out, we calculate the Wronskian of $\mathbf{x}^{(1)}(t)$ and $\mathbf{x}^{(2)}(t)$:

But are these two solutions linearly independent? To find out, we calculate the Wronskian of $\mathbf{x}^{(1)}(t)$ and $\mathbf{x}^{(2)}(t)$:

$$\mathbf{x}^{(1)}(t) = \begin{bmatrix} 1 \\ 2 \end{bmatrix} e^{3t} = \begin{bmatrix} e^{3t} \\ 2e^{3t} \end{bmatrix} \quad \text{and} \quad \mathbf{x}^{(2)}(t) = \begin{bmatrix} 1 \\ -2 \end{bmatrix} e^{-t} = \begin{bmatrix} e^{-t} \\ -2e^{-t} \end{bmatrix}.$$

But are these two solutions linearly independent? To find out, we calculate the Wronskian of $\mathbf{x}^{(1)}(t)$ and $\mathbf{x}^{(2)}(t)$:

$$\mathbf{x}^{(1)}(t) = \begin{bmatrix} 1 \\ 2 \end{bmatrix} e^{3t} = \begin{bmatrix} e^{3t} \\ 2e^{3t} \end{bmatrix} \quad \text{and} \quad \mathbf{x}^{(2)}(t) = \begin{bmatrix} 1 \\ -2 \end{bmatrix} e^{-t} = \begin{bmatrix} e^{-t} \\ -2e^{-t} \end{bmatrix}.$$

$$W\big(\mathbf{x}^{(1)}, \mathbf{x}^{(2)}\big)(t) = \begin{vmatrix} e^{3t} & e^{-t} \\ 2e^{3t} & -2e^{-t} \end{vmatrix} = -4e^{2t} \neq 0.$$

But are these two solutions linearly independent? To find out, we calculate the Wronskian of $\mathbf{x}^{(1)}(t)$ and $\mathbf{x}^{(2)}(t)$:

$$\mathbf{x}^{(1)}(t) = \begin{bmatrix} 1 \\ 2 \end{bmatrix} e^{3t} = \begin{bmatrix} e^{3t} \\ 2e^{3t} \end{bmatrix} \quad \text{and} \quad \mathbf{x}^{(2)}(t) = \begin{bmatrix} 1 \\ -2 \end{bmatrix} e^{-t} = \begin{bmatrix} e^{-t} \\ -2e^{-t} \end{bmatrix}.$$

$$W\big(\mathbf{x}^{(1)}, \mathbf{x}^{(2)}\big)(t) = \begin{vmatrix} e^{3t} & e^{-t} \\ 2e^{3t} & -2e^{-t} \end{vmatrix} = -4e^{2t} \neq 0.$$

Since $W \neq 0$, we have that $\mathbf{x}^{(1)}(t)$ and $\mathbf{x}^{(2)}(t)$ are linearly independent.

But are these two solutions linearly independent? To find out, we calculate the Wronskian of $\mathbf{x}^{(1)}(t)$ and $\mathbf{x}^{(2)}(t)$:

$$\mathbf{x}^{(1)}(t) = \begin{bmatrix} 1 \\ 2 \end{bmatrix} e^{3t} = \begin{bmatrix} e^{3t} \\ 2e^{3t} \end{bmatrix} \quad \text{and} \quad \mathbf{x}^{(2)}(t) = \begin{bmatrix} 1 \\ -2 \end{bmatrix} e^{-t} = \begin{bmatrix} e^{-t} \\ -2e^{-t} \end{bmatrix}.$$

$$W\big(\mathbf{x}^{(1)}, \mathbf{x}^{(2)}\big)(t) = \begin{vmatrix} e^{3t} & e^{-t} \\ 2e^{3t} & -2e^{-t} \end{vmatrix} = -4e^{2t} \neq 0.$$

Since $W \neq 0$, we have that $\mathbf{x}^{(1)}(t)$ and $\mathbf{x}^{(2)}(t)$ are linearly independent. So $\mathbf{x}^{(1)}(t)$ and $\mathbf{x}^{(2)}(t)$ form a fundamental set of solutions.

But are these two solutions linearly independent? To find out, we calculate the Wronskian of $\mathbf{x}^{(1)}(t)$ and $\mathbf{x}^{(2)}(t)$:

$$\mathbf{x}^{(1)}(t) = \begin{bmatrix} 1 \\ 2 \end{bmatrix} e^{3t} = \begin{bmatrix} e^{3t} \\ 2e^{3t} \end{bmatrix} \quad \text{and} \quad \mathbf{x}^{(2)}(t) = \begin{bmatrix} 1 \\ -2 \end{bmatrix} e^{-t} = \begin{bmatrix} e^{-t} \\ -2e^{-t} \end{bmatrix}.$$

$$W\big(\mathbf{x}^{(1)}, \mathbf{x}^{(2)}\big)(t) = \begin{vmatrix} e^{3t} & e^{-t} \\ 2e^{3t} & -2e^{-t} \end{vmatrix} = -4e^{2t} \neq 0.$$

Since $W \neq 0$, we have that $\mathbf{x}^{(1)}(t)$ and $\mathbf{x}^{(2)}(t)$ are linearly independent. So $\mathbf{x}^{(1)}(t)$ and $\mathbf{x}^{(2)}(t)$ form a fundamental set of solutions. Therefore the general solution is

$$\boxed{\mathbf{x}(t) = c_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} e^{3t} + c_2 \begin{bmatrix} 1 \\ -2 \end{bmatrix} e^{-t}.}$$

Example

Solve

$$\begin{cases} \mathbf{x}' = \begin{bmatrix} 8 & -1 \\ 6 & 1 \end{bmatrix} \mathbf{x} \\ \\ \mathbf{x}(0) = \begin{bmatrix} 1 \\ -2 \end{bmatrix}. \end{cases}$$

Example

Solve

$$\begin{cases} \mathbf{x}' = \begin{bmatrix} 8 & -1 \\ 6 & 1 \end{bmatrix} \mathbf{x} \\ \mathbf{x}(0) = \begin{bmatrix} 1 \\ -2 \end{bmatrix}. \end{cases}$$

The eigenvalues are $r_1 = 7$ and $r_2 = 2$.

Example

Solve

$$\begin{cases} \mathbf{x}' = \begin{bmatrix} 8 & -1 \\ 6 & 1 \end{bmatrix} \mathbf{x} \\ \mathbf{x}(0) = \begin{bmatrix} 1 \\ -2 \end{bmatrix}. \end{cases}$$

The eigenvalues are $r_1 = 7$ and $r_2 = 2$. The corresponding eigenvectors are $\boldsymbol{\xi}^{(1)} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\boldsymbol{\xi}^{(2)} = \begin{bmatrix} 1 \\ 6 \end{bmatrix}$.

Example

Solve

$$\begin{cases} \mathbf{x}' = \begin{bmatrix} 8 & -1 \\ 6 & 1 \end{bmatrix} \mathbf{x} \\ \mathbf{x}(0) = \begin{bmatrix} 1 \\ -2 \end{bmatrix}. \end{cases}$$

The eigenvalues are $r_1 = 7$ and $r_2 = 2$. The corresponding eigenvectors are $\boldsymbol{\xi}^{(1)} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\boldsymbol{\xi}^{(2)} = \begin{bmatrix} 1 \\ 6 \end{bmatrix}$. Therefore the general solution to the ODE is

$$\mathbf{x}(t) = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{7t} + c_2 \begin{bmatrix} 1 \\ 6 \end{bmatrix} e^{2t}.$$

$$\mathbf{x}(t) = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{7t} + c_2 \begin{bmatrix} 1 \\ 6 \end{bmatrix} e^{2t}.$$

Setting $t = 0$, we have

$$\begin{bmatrix} 1 \\ -2 \end{bmatrix} = \mathbf{x}(0)$$

$$\mathbf{x}(t) = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{7t} + c_2 \begin{bmatrix} 1 \\ 6 \end{bmatrix} e^{2t}.$$

Setting $t = 0$, we have

$$\begin{bmatrix} 1 \\ -2 \end{bmatrix} = \mathbf{x}(0) = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 6 \end{bmatrix} = \begin{bmatrix} c_1 + c_2 \\ c_1 + 6c_2 \end{bmatrix}$$

$$\mathbf{x}(t) = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{7t} + c_2 \begin{bmatrix} 1 \\ 6 \end{bmatrix} e^{2t}.$$

Setting $t = 0$, we have

$$\begin{bmatrix} 1 \\ -2 \end{bmatrix} = \mathbf{x}(0) = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 6 \end{bmatrix} = \begin{bmatrix} c_1 + c_2 \\ c_1 + 6c_2 \end{bmatrix} \implies \begin{cases} c_1 + c_2 = 1 \\ c_1 + 6c_2 = -2 \end{cases}$$

$$\mathbf{x}(t) = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{7t} + c_2 \begin{bmatrix} 1 \\ 6 \end{bmatrix} e^{2t}.$$

Setting $t = 0$, we have

$$\begin{bmatrix} 1 \\ -2 \end{bmatrix} = \mathbf{x}(0) = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 6 \end{bmatrix} = \begin{bmatrix} c_1 + c_2 \\ c_1 + 6c_2 \end{bmatrix} \implies \begin{cases} c_1 + c_2 = 1 \\ c_1 + 6c_2 = -2 \end{cases}$$

$$\implies \begin{cases} c_1 = \frac{8}{5} \\ c_2 = -\frac{3}{5}. \end{cases}$$

$$\mathbf{x}(t) = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{7t} + c_2 \begin{bmatrix} 1 \\ 6 \end{bmatrix} e^{2t}.$$

Setting $t = 0$, we have

$$\begin{bmatrix} 1 \\ -2 \end{bmatrix} = \mathbf{x}(0) = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 6 \end{bmatrix} = \begin{bmatrix} c_1 + c_2 \\ c_1 + 6c_2 \end{bmatrix} \quad \Longrightarrow \quad \begin{cases} c_1 + c_2 = 1 \\ c_1 + 6c_2 = -2 \end{cases}$$

$$\Longrightarrow \quad \begin{cases} c_1 = \frac{8}{5} \\ c_2 = -\frac{3}{5}. \end{cases}$$

Therefore the solution to the IVP is

$$\boxed{\mathbf{x}(t) = \frac{8}{5} \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{7t} - \frac{3}{5} \begin{bmatrix} 1 \\ 6 \end{bmatrix} e^{2t}.}$$

Example

Solve

$$\mathbf{x}' = \begin{bmatrix} -3 & \sqrt{2} \\ \sqrt{2} & -2 \end{bmatrix} \mathbf{x}.$$

Example

Solve

$$\mathbf{x}' = \begin{bmatrix} -3 & \sqrt{2} \\ \sqrt{2} & -2 \end{bmatrix} \mathbf{x}.$$

The eigenvalues are $r_1 = -1$ and $r_2 = -4$.

Example

Solve

$$\mathbf{x}' = \begin{bmatrix} -3 & \sqrt{2} \\ \sqrt{2} & -2 \end{bmatrix} \mathbf{x}.$$

The eigenvalues are $r_1 = -1$ and $r_2 = -4$. The corresponding eigenvectors are $\boldsymbol{\xi}^{(1)} = \begin{bmatrix} 1 \\ \sqrt{2} \end{bmatrix}$ and $\boldsymbol{\xi}^{(2)} = \begin{bmatrix} -\sqrt{2} \\ 1 \end{bmatrix}$.

Example

Solve

$$\mathbf{x}' = \begin{bmatrix} -3 & \sqrt{2} \\ \sqrt{2} & -2 \end{bmatrix} \mathbf{x}.$$

The eigenvalues are $r_1 = -1$ and $r_2 = -4$. The corresponding eigenvectors are $\boldsymbol{\xi}^{(1)} = \begin{bmatrix} 1 \\ \sqrt{2} \end{bmatrix}$ and $\boldsymbol{\xi}^{(2)} = \begin{bmatrix} -\sqrt{2} \\ 1 \end{bmatrix}$. Hence the general solution is

$$\boxed{\mathbf{x}(t) = c_1 \begin{bmatrix} 1 \\ \sqrt{2} \end{bmatrix} e^{-t} + c_2 \begin{bmatrix} -\sqrt{2} \\ 1 \end{bmatrix} e^{-4t}.}$$

Remark

$$\det(A - rI) = 0$$

There are three possibilities for the eigenvalues of A.

Remark

$$\det(A - rI) = 0$$

There are three possibilities for the eigenvalues of A.

1. All the eigenvalues are real and different;
2. Some eigenvalues occur in complex conjugate pairs;
3. Some eigenvalues are repeated.

If all the eigenvalues are real and different, then the eigenvectors are linearly independent:

If all the eigenvalues are real and different, then the eigenvectors are linearly independent:

So $W\left(\mathbf{x}^{(1)}, \ldots, \mathbf{x}^{(n)}\right)(t) \neq 0$ and $\mathbf{x}^{(1)}(t)$, ..., $\mathbf{x}^{(n)}(t)$ form a fundamental set of solutions.

If some eigenvalues are repeated, *but there are n linearly independent eigenvectors*, then this is also true: $\mathbf{x}^{(1)}(t), \ldots,$ $\mathbf{x}^{(n)}(t)$ form a fundamental set of solutions.

Example

Solve

$$\mathbf{x}' = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} \mathbf{x}.$$

The eigenvalues and eigenvectors are

$$r_1 = 2 \qquad r_2 = -1 \qquad r_3 = -1$$

$$\boldsymbol{\xi}^{(1)} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \qquad \boldsymbol{\xi}^{(2)} = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \qquad \boldsymbol{\xi}^{(3)} = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$$

The eigenvalues and eigenvectors are

$$r_1 = 2 \qquad r_2 = -1 \qquad r_3 = -1$$

$$\boldsymbol{\xi}^{(1)} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \qquad \boldsymbol{\xi}^{(2)} = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \qquad \boldsymbol{\xi}^{(3)} = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$$

which gives us the following three solutions

$$\mathbf{x}^{(1)}(t) = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} e^{2t} \qquad \mathbf{x}^{(2)}(t) = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} e^{-t} \qquad \mathbf{x}^{(3)}(t) = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} e^{-t}.$$

$$\mathbf{x}^{(1)}(t) = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} e^{2t} \qquad \mathbf{x}^{(2)}(t) = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} e^{-t} \qquad \mathbf{x}^{(3)}(t) = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} e^{-t}.$$

You can check that the Wronskian of $\mathbf{x}^{(1)}$, $\mathbf{x}^{(2)}$ and $\mathbf{x}^{(3)}$ is non-zero.

$$\mathbf{x}^{(1)}(t) = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} e^{2t} \qquad \mathbf{x}^{(2)}(t) = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} e^{-t} \qquad \mathbf{x}^{(3)}(t) = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} e^{-t}.$$

You can check that the Wronskian of $\mathbf{x}^{(1)}$, $\mathbf{x}^{(2)}$ and $\mathbf{x}^{(3)}$ is non-zero. Therefore $\mathbf{x}^{(1)}$, $\mathbf{x}^{(2)}$ and $\mathbf{x}^{(3)}$ form a fundamental set of solutions.

$$\mathbf{x}^{(1)}(t) = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} e^{2t} \qquad \mathbf{x}^{(2)}(t) = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} e^{-t} \qquad \mathbf{x}^{(3)}(t) = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} e^{-t}.$$

You can check that the Wronskian of $\mathbf{x}^{(1)}$, $\mathbf{x}^{(2)}$ and $\mathbf{x}^{(3)}$ is non-zero. Therefore $\mathbf{x}^{(1)}$, $\mathbf{x}^{(2)}$ and $\mathbf{x}^{(3)}$ form a fundamental set of solutions. The general solution to the ODE is

$$\boxed{\mathbf{x}(t) = c_1 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} e^{2t} + c_2 \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} e^{-t} + c_3 \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} e^{-t}.}$$

Remark

So if we have repeated eigenvalues with n linearly independent eigenvectors then there is no problem.

In section 5.6 we will study what to do if we do not have enough eigenvectors.

Remark

So if we have repeated eigenvalues with n linearly independent eigenvectors then there is no problem.

In section 5.6 we will study what to do if we do not have enough eigenvectors.

Remark

Next we will study systems with complex eigenvalues.

5.4 Complex Eigenvalues

5.4 Complex Eigenvalues

Consider

$$\mathbf{x}' = A\mathbf{x}$$

where $A \in \mathbb{R}^{n\times n}$.

Consider

$$\mathbf{x}' = A\mathbf{x}$$

where $A \in \mathbb{R}^{n\times n}$.

$$\begin{bmatrix} x_1' \\ x_2' \\ \vdots \\ x_n' \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

Consider

$$\mathbf{x}' = A\mathbf{x}$$

where $A \in \mathbb{R}^{n\times n}$.

$$\begin{cases} x_1' = a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n \\ x_2' = a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n \\ \qquad \vdots \\ x_n' = a_{n1}x_1 + a_{n2}x_2 + \ldots + a_{nn}x_n \end{cases}$$

Any complex eigenvalues of A must occur in complex conjugate pairs: If $r_1 = \lambda + i\mu$ is an eigenvalue of A, then $r_2 = \overline{r}_1 = \lambda - i\mu$ is also an eigenvalue of A.

Moreover, if $\boldsymbol{\xi}^{(1)}$ is an eigenvector of A corresponding to r_1, then $\boldsymbol{\xi}^{(2)} = \overline{\boldsymbol{\xi}^{(1)}}$ is an eigenvector of A corresponding to $r_2 = \bar{r}_1$.

Two solutions of $\mathbf{x}' = A\mathbf{x}$ are

$$\mathbf{x}^{(1)}(t) = \boldsymbol{\xi}^{(1)} e^{r_1 t} \qquad \text{and} \qquad \mathbf{x}^{(2)}(t) = \overline{\boldsymbol{\xi}^{(1)}} e^{\overline{r}_1 t}.$$

Two solutions of $\mathbf{x}' = A\mathbf{x}$ are

$$\mathbf{x}^{(1)}(t) = \boldsymbol{\xi}^{(1)} e^{r_1 t} \qquad \text{and} \qquad \mathbf{x}^{(2)}(t) = \overline{\boldsymbol{\xi}^{(1)}} e^{\overline{r}_1 t}.$$

But $\mathbf{x}^{(1)}, \mathbf{x}^{(2)} : \mathbb{R} \to \mathbb{C}^n$ and we want solutions $: \mathbb{R} \to \mathbb{R}^n$.

If $r_1 = \lambda + i\mu$, and $\boldsymbol{\xi}^{(1)} = \mathbf{a} + i\mathbf{b}$ $(\lambda,\, \mu \in \mathbb{R},\, \mathbf{a},\, \mathbf{b} \in \mathbb{R}^n)$, then

$$\mathbf{x}^{(1)}(t) = (\mathbf{a} + i\mathbf{b})e^{(\lambda+i\mu)t}$$

If $r_1 = \lambda + i\mu$, and $\boldsymbol{\xi}^{(1)} = \mathbf{a} + i\mathbf{b}$ (λ, $\mu \in \mathbb{R}$, $\mathbf{a}$, $\mathbf{b} \in \mathbb{R}^n$), then

$$\begin{aligned}\mathbf{x}^{(1)}(t) &= (\mathbf{a} + i\mathbf{b})e^{(\lambda + i\mu)t} \\ &= (\mathbf{a} + i\mathbf{b})e^{\lambda t}\,(\cos \mu t + i \sin \mu t)\end{aligned}$$

If $r_1 = \lambda + i\mu$, and $\boldsymbol{\xi}^{(1)} = \mathbf{a} + i\mathbf{b}$ (λ, $\mu \in \mathbb{R}$, $\mathbf{a}$, $\mathbf{b} \in \mathbb{R}^n$), then

$$\begin{aligned}
\mathbf{x}^{(1)}(t) &= (\mathbf{a} + i\mathbf{b})e^{(\lambda+i\mu)t} \\
&= (\mathbf{a} + i\mathbf{b})e^{\lambda t}\left(\cos\mu t + i\sin\mu t\right) \\
&= e^{\lambda t}\left(\mathbf{a}\cos\mu t - \mathbf{b}\sin\mu t\right) + ie^{\lambda t}\left(\mathbf{a}\sin\mu t + \mathbf{b}\cos\mu t\right)
\end{aligned}$$

If $r_1 = \lambda + i\mu$, and $\boldsymbol{\xi}^{(1)} = \mathbf{a} + i\mathbf{b}$ (λ, $\mu \in \mathbb{R}$, $\mathbf{a}$, $\mathbf{b} \in \mathbb{R}^n$), then

$$\begin{aligned}
\mathbf{x}^{(1)}(t) &= (\mathbf{a} + i\mathbf{b})e^{(\lambda+i\mu)t} \\
&= (\mathbf{a} + i\mathbf{b})e^{\lambda t}(\cos\mu t + i\sin\mu t) \\
&= e^{\lambda t}(\mathbf{a}\cos\mu t - \mathbf{b}\sin\mu t) + ie^{\lambda t}(\mathbf{a}\sin\mu t + \mathbf{b}\cos\mu t) \\
&= \mathbf{u}(t) + i\mathbf{v}(t).
\end{aligned}$$

Remark

- The functions $\mathbf{u}(t)$ and $\mathbf{v}(t)$ solve the ODE.

Remark

- The functions $\mathbf{u}(t)$ and $\mathbf{v}(t)$ solve the ODE.
- The functions $\mathbf{u}(t)$ and $\mathbf{v}(t)$ will be linearly independent.

Remark

- The functions $\mathbf{u}(t)$ and $\mathbf{v}(t)$ solve the ODE.
- The functions $\mathbf{u}(t)$ and $\mathbf{v}(t)$ will be linearly independent.
- $\operatorname{span}\{\mathbf{u}(t), \mathbf{v}(t)\} = \operatorname{span}\{\mathbf{x}^{(1)}, \mathbf{x}^{(2)}\}$.

Remark

- The functions $\mathbf{u}(t)$ and $\mathbf{v}(t)$ solve the ODE.
- The functions $\mathbf{u}(t)$ and $\mathbf{v}(t)$ will be linearly independent.
- $\mathrm{span}\{\mathbf{u}(t), \mathbf{v}(t)\} = \mathrm{span}\{\mathbf{x}^{(1)}, \mathbf{x}^{(2)}\}$.

So we can include $\mathbf{u}(t)$ and $\mathbf{v}(t)$ in our fundamental set of solutions instead of $\mathbf{x}^{(1)}(t)$ and $\mathbf{x}^{(2)}(t)$.

Example

Solve

$$\mathbf{x}' = \begin{bmatrix} -\frac{1}{2} & 1 \\ -1 & -\frac{1}{2} \end{bmatrix} \mathbf{x}.$$

Example

Solve

$$\mathbf{x}' = \begin{bmatrix} -\frac{1}{2} & 1 \\ -1 & -\frac{1}{2} \end{bmatrix} \mathbf{x}.$$

We calculate that

$$0 = \det(A - rI) = \begin{vmatrix} -\frac{1}{2} - r & 1 \\ -1 & -\frac{1}{2} - r \end{vmatrix} = r^2 + r + \frac{5}{4}$$

and

$$r_{1,2} = \frac{-1 \pm \sqrt{1-5}}{2} = \frac{-1 \pm 2i}{2} = -\frac{1}{2} \pm i.$$

Example

Solve

$$\mathbf{x}' = \begin{bmatrix} -\frac{1}{2} & 1 \\ -1 & -\frac{1}{2} \end{bmatrix} \mathbf{x}.$$

We calculate that

$$0 = \det(A - rI) = \begin{vmatrix} -\frac{1}{2} - r & 1 \\ -1 & -\frac{1}{2} - r \end{vmatrix} = r^2 + r + \frac{5}{4}$$

and

$$r_{1,2} = \frac{-1 \pm \sqrt{1-5}}{2} = \frac{-1 \pm 2i}{2} = -\frac{1}{2} \pm i.$$

So we have $r_1 = -\frac{1}{2} + i$ and $r_2 = -\frac{1}{2} - i$.

Example

Solve
$$\mathbf{x}' = \begin{bmatrix} -\frac{1}{2} & 1 \\ -1 & -\frac{1}{2} \end{bmatrix} \mathbf{x}.$$

We calculate that
$$0 = \det(A - rI) = \begin{vmatrix} -\frac{1}{2} - r & 1 \\ -1 & -\frac{1}{2} - r \end{vmatrix} = r^2 + r + \frac{5}{4}$$

and
$$r_{1,2} = \frac{-1 \pm \sqrt{1-5}}{2} = \frac{-1 \pm 2i}{2} = -\frac{1}{2} \pm i.$$

So we have $r_1 = -\frac{1}{2} + i$ and $r_2 = -\frac{1}{2} - i$. We will use r_1. We do not need r_2.

Since

$$0 = (A - r_1 I)\boldsymbol{\xi}^{(1)} = \begin{bmatrix} -i & 1 \\ -1 & -i \end{bmatrix} \begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix}$$

Since

$$0 = (A - r_1 I)\boldsymbol{\xi}^{(1)} = \begin{bmatrix} -i & 1 \\ -1 & -i \end{bmatrix} \begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix} \quad \Longrightarrow \begin{cases} -i\xi_1 + \xi_2 = 0 \\ -\xi_1 - i\xi_2 = 0 \end{cases}$$

Since

$$0 = (A - r_1 I)\boldsymbol{\xi}^{(1)} = \begin{bmatrix} -i & 1 \\ -1 & -i \end{bmatrix} \begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix} \implies \begin{cases} -i\xi_1 + \xi_2 = 0 \\ -\xi_1 - i\xi_2 = 0 \end{cases}$$

we can choose

$$\boldsymbol{\xi}^{(1)} = \begin{bmatrix} 1 \\ i \end{bmatrix}.$$

Note that we also have

$$\boldsymbol{\xi}^{(2)} = \overline{\boldsymbol{\xi}^{(1)}} = \overline{\begin{bmatrix} 1 \\ i \end{bmatrix}} = \begin{bmatrix} 1 \\ -i \end{bmatrix},$$

Note that we also have

$$\boldsymbol{\xi}^{(2)} = \overline{\boldsymbol{\xi}^{(1)}} = \overline{\begin{bmatrix} 1 \\ i \end{bmatrix}} = \begin{bmatrix} 1 \\ -i \end{bmatrix},$$

but we don't need $\boldsymbol{\xi}^{(2)}$.

Next we look at $\mathbf{x}^{(1)}(t)$:

Next we look at $\mathbf{x}^{(1)}(t)$:

$$\mathbf{x}^{(1)}(t) = \boldsymbol{\xi}^{(1)} e^{r_1 t} = \begin{bmatrix} 1 \\ i \end{bmatrix} e^{-\frac{t}{2}} \left(\cos t + i \sin t\right)$$

$$=$$

$$=$$

Next we look at $\mathbf{x}^{(1)}(t)$:

$$\begin{aligned}\mathbf{x}^{(1)}(t) &= \boldsymbol{\xi}^{(1)} e^{r_1 t} = \begin{bmatrix} 1 \\ i \end{bmatrix} e^{-\frac{t}{2}} (\cos t + i \sin t) \\ &= e^{-\frac{t}{2}} \begin{bmatrix} \cos t + i \sin t \\ i \cos t - \sin t \end{bmatrix} \\ &= \end{aligned}$$

Next we look at $\mathbf{x}^{(1)}(t)$:

$$\begin{aligned}
\mathbf{x}^{(1)}(t) = \boldsymbol{\xi}^{(1)} e^{r_1 t} &= \begin{bmatrix} 1 \\ i \end{bmatrix} e^{-\frac{t}{2}} \left(\cos t + i \sin t\right) \\
&= e^{-\frac{t}{2}} \begin{bmatrix} \cos t + i \sin t \\ i \cos t - \sin t \end{bmatrix} \\
&= e^{-\frac{t}{2}} \begin{bmatrix} \cos t \\ -\sin t \end{bmatrix} + i e^{-\frac{t}{2}} \begin{bmatrix} \sin t \\ \cos t \end{bmatrix}
\end{aligned}$$

Next we look at $\mathbf{x}^{(1)}(t)$:

$$\begin{aligned}
\mathbf{x}^{(1)}(t) &= \boldsymbol{\xi}^{(1)} e^{r_1 t} = \begin{bmatrix} 1 \\ i \end{bmatrix} e^{-\frac{t}{2}} \left(\cos t + i \sin t\right) \\
&= e^{-\frac{t}{2}} \begin{bmatrix} \cos t + i \sin t \\ i \cos t - \sin t \end{bmatrix} \\
&= e^{-\frac{t}{2}} \begin{bmatrix} \cos t \\ -\sin t \end{bmatrix} + i e^{-\frac{t}{2}} \begin{bmatrix} \sin t \\ \cos t \end{bmatrix} \\
&= \mathbf{u}(t) + i \mathbf{v}(t).
\end{aligned}$$

Next we look at $\mathbf{x}^{(1)}(t)$:

$$\begin{aligned}
\mathbf{x}^{(1)}(t) = \boldsymbol{\xi}^{(1)} e^{r_1 t} &= \begin{bmatrix} 1 \\ i \end{bmatrix} e^{-\frac{t}{2}} (\cos t + i \sin t) \\
&= e^{-\frac{t}{2}} \begin{bmatrix} \cos t + i \sin t \\ i \cos t - \sin t \end{bmatrix} \\
&= e^{-\frac{t}{2}} \begin{bmatrix} \cos t \\ -\sin t \end{bmatrix} + i e^{-\frac{t}{2}} \begin{bmatrix} \sin t \\ \cos t \end{bmatrix} \\
&= \mathbf{u}(t) + i\mathbf{v}(t).
\end{aligned}$$

Hence we choose

$$\mathbf{u}(t) = e^{-\frac{t}{2}} \begin{bmatrix} \cos t \\ -\sin t \end{bmatrix} \quad \text{and} \quad \mathbf{v}(t) = e^{-\frac{t}{2}} \begin{bmatrix} \sin t \\ \cos t \end{bmatrix}.$$

$$\mathbf{u}(t) = e^{-\frac{t}{2}} \begin{bmatrix} \cos t \\ -\sin t \end{bmatrix} \quad \text{and} \quad \mathbf{v}(t) = e^{-\frac{t}{2}} \begin{bmatrix} \sin t \\ \cos t \end{bmatrix}.$$

But are $\mathbf{u}(t)$ and $\mathbf{v}(t)$ linearly independent?

$$\mathbf{u}(t) = e^{-\frac{t}{2}} \begin{bmatrix} \cos t \\ -\sin t \end{bmatrix} \quad \text{and} \quad \mathbf{v}(t) = e^{-\frac{t}{2}} \begin{bmatrix} \sin t \\ \cos t \end{bmatrix}.$$

But are $\mathbf{u}(t)$ and $\mathbf{v}(t)$ linearly independent? Since

$$\begin{aligned} W\big(\mathbf{u}(t), \mathbf{v}(t)\big)(t) &= \begin{vmatrix} u_1 & v_1 \\ u_2 & v_2 \end{vmatrix} = \begin{vmatrix} e^{-\frac{t}{2}} \cos t & e^{-\frac{t}{2}} \sin t \\ -e^{-\frac{t}{2}} \sin t & e^{-\frac{t}{2}} \cos t \end{vmatrix} \\ &= e^{-t} \cos^2 t + e^{-t} \sin^2 t = e^{-t} \\ &\neq 0 \end{aligned}$$

the answer is yes.

$$\mathbf{u}(t) = e^{-\frac{t}{2}} \begin{bmatrix} \cos t \\ -\sin t \end{bmatrix} \quad \text{and} \quad \mathbf{v}(t) = e^{-\frac{t}{2}} \begin{bmatrix} \sin t \\ \cos t \end{bmatrix}.$$

But are $\mathbf{u}(t)$ and $\mathbf{v}(t)$ linearly independent? Since

$$\begin{aligned} W\big(\mathbf{u}(t), \mathbf{v}(t)\big)(t) &= \begin{vmatrix} u_1 & v_1 \\ u_2 & v_2 \end{vmatrix} = \begin{vmatrix} e^{-\frac{t}{2}} \cos t & e^{-\frac{t}{2}} \sin t \\ -e^{-\frac{t}{2}} \sin t & e^{-\frac{t}{2}} \cos t \end{vmatrix} \\ &= e^{-t} \cos^2 t + e^{-t} \sin^2 t = e^{-t} \\ &\neq 0 \end{aligned}$$

the answer is yes. Therefore $\mathbf{u}(t)$ and $\mathbf{v}(t)$ form a fundamental set of solutions.

$$\mathbf{u}(t) = e^{-\frac{t}{2}} \begin{bmatrix} \cos t \\ -\sin t \end{bmatrix} \quad \text{and} \quad \mathbf{v}(t) = e^{-\frac{t}{2}} \begin{bmatrix} \sin t \\ \cos t \end{bmatrix}.$$

But are $\mathbf{u}(t)$ and $\mathbf{v}(t)$ linearly independent? Since

$$\begin{aligned} W\big(\mathbf{u}(t), \mathbf{v}(t)\big)(t) &= \begin{vmatrix} u_1 & v_1 \\ u_2 & v_2 \end{vmatrix} = \begin{vmatrix} e^{-\frac{t}{2}} \cos t & e^{-\frac{t}{2}} \sin t \\ -e^{-\frac{t}{2}} \sin t & e^{-\frac{t}{2}} \cos t \end{vmatrix} \\ &= e^{-t} \cos^2 t + e^{-t} \sin^2 t = e^{-t} \\ &\neq 0 \end{aligned}$$

the answer is yes. Therefore $\mathbf{u}(t)$ and $\mathbf{v}(t)$ form a fundamental set of solutions.

Therefore the general solution to $\mathbf{x}' = \begin{bmatrix} -\frac{1}{2} & 1 \\ -1 & -\frac{1}{2} \end{bmatrix} \mathbf{x}$ is

$$\boxed{\mathbf{x}(t) = c_1 \, e^{-\frac{t}{2}} \begin{bmatrix} \cos t \\ -\sin t \end{bmatrix} + c_2 \, e^{-\frac{t}{2}} \begin{bmatrix} \sin t \\ \cos t \end{bmatrix}.}$$

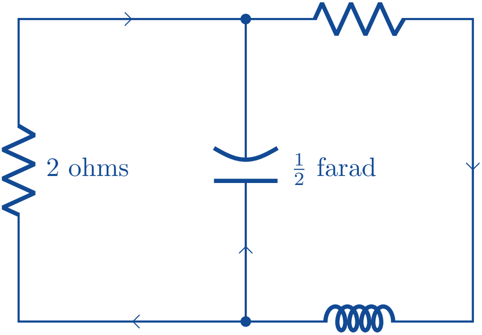
Remark

Our method is

1. Find the eigenvalues;
2. Find the eigenvectors;
3.
   - If r_j is real, just use the solution $\mathbf{x}^{(j)}(t) = \boldsymbol{\xi}^{(j)} e^{r_j t}$;
   - But if r_j is complex, write

$$\mathbf{x}^{(j)}(t) = \boldsymbol{\xi}^{(j)} e^{r_j t} = \begin{pmatrix} \text{real valued} \\ \text{function} \end{pmatrix} + i \begin{pmatrix} \text{real valued} \\ \text{function} \end{pmatrix}$$

and use these two functions.


1 ohm
2 ohms
$\frac{1}{2}$ farad
1 henry

Example

The electric circuit shown above is described by

$$\begin{cases} I' = -I - V \\ V' = 2I - V \end{cases}$$

where

$$I = \text{the current through the inductor}$$
$$V = \text{the voltage drop across the capacitor.}$$

(Ask an Electrical Engineer.)

Example

The electric circuit shown above is described by

$$\begin{cases} I' = -I - V \\ V' = 2I - V \end{cases}$$

where

$$I = \text{the current through the inductor}$$
$$V = \text{the voltage drop across the capacitor.}$$

(Ask an Electrical Engineer.)

Suppose that at time $t = 0$ the current is 2 amperes and the voltage drop is 2 volts. Find $I(t)$ and $V(t)$.

$$\begin{cases} I' = -I - V \\ V' = 2I - V \end{cases}$$

Suppose that at time $t = 0$ the current is 2 amperes and the voltage drop is 2 volts. Find $I(t)$ and $V(t)$.

We must solve the IVP

$$\begin{cases} \dfrac{d}{dt}\begin{bmatrix} I \\ V \end{bmatrix} = \begin{bmatrix} -1 & -1 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} I \\ V \end{bmatrix} \\ \\ \begin{bmatrix} I \\ V \end{bmatrix}(0) = \begin{bmatrix} 2 \\ 2 \end{bmatrix}. \end{cases}$$

The eigenvalues of $\begin{bmatrix} -1 & -1 \\ 2 & -1 \end{bmatrix}$ are $r_1 = -1 + i\sqrt{2}$ and $r_2 = -1 - i\sqrt{2}$ (please check). The corresponding eigenvectors are

$$\boldsymbol{\xi}^{(1)} = \begin{bmatrix} 1 \\ -i\sqrt{2} \end{bmatrix} \qquad \text{and} \qquad \boldsymbol{\xi}^{(2)} = \begin{bmatrix} 1 \\ i\sqrt{2} \end{bmatrix}.$$

The eigenvalues of $\begin{bmatrix} -1 & -1 \\ 2 & -1 \end{bmatrix}$ are $r_1 = -1 + i\sqrt{2}$ and $r_2 = -1 - i\sqrt{2}$ (please check). The corresponding eigenvectors are

$$\boldsymbol{\xi}^{(1)} = \begin{bmatrix} 1 \\ -i\sqrt{2} \end{bmatrix} \qquad \text{and} \qquad \boldsymbol{\xi}^{(2)} = \begin{bmatrix} 1 \\ i\sqrt{2} \end{bmatrix}.$$

Then we calculate that

$$\begin{aligned} \mathbf{x}^{(1)}(t) = \boldsymbol{\xi}^{(1)} e^{r_1 t} &= \begin{bmatrix} 1 \\ -i\sqrt{2} \end{bmatrix} e^{(-1+i\sqrt{2})t} \\ &= \begin{bmatrix} 1 \\ -i\sqrt{2} \end{bmatrix} e^{-t} \big(\cos\sqrt{2}t + i \sin\sqrt{2}t \big) \\ &= e^{-t} \begin{bmatrix} \cos\sqrt{2}t + i\sin\sqrt{2}t \\ -i\sqrt{2}\cos\sqrt{2}t + \sqrt{2}\sin\sqrt{2}t \end{bmatrix} \\ &= e^{-t} \begin{bmatrix} \cos\sqrt{2}t \\ \sqrt{2}\sin\sqrt{2}t \end{bmatrix} + i e^{-t} \begin{bmatrix} \sin\sqrt{2}t \\ -\sqrt{2}\cos\sqrt{2}t \end{bmatrix}. \end{aligned}$$

Hence the general solution to the ODE is

$$\begin{bmatrix} I(t) \\ V(t) \end{bmatrix} = c_1 e^{-t} \begin{bmatrix} \cos\sqrt{2}t \\ \sqrt{2}\sin\sqrt{2}t \end{bmatrix} + c_2 e^{-t} \begin{bmatrix} \sin\sqrt{2}t \\ -\sqrt{2}\cos\sqrt{2}t \end{bmatrix}.$$

Hence the general solution to the ODE is

$$\begin{bmatrix} I(t) \\ V(t) \end{bmatrix} = c_1 e^{-t} \begin{bmatrix} \cos\sqrt{2}t \\ \sqrt{2}\sin\sqrt{2}t \end{bmatrix} + c_2 e^{-t} \begin{bmatrix} \sin\sqrt{2}t \\ -\sqrt{2}\cos\sqrt{2}t \end{bmatrix}.$$

Using the initial condition, we calculate that

$$\begin{bmatrix} 2 \\ 2 \end{bmatrix} = \begin{bmatrix} I(0) \\ V(0) \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ -\sqrt{2} \end{bmatrix} \qquad \Longrightarrow \qquad \begin{cases} c_1 = 2 \\ c_2 = -\sqrt{2}. \end{cases}$$

Hence the general solution to the ODE is

$$\begin{bmatrix} I(t) \\ V(t) \end{bmatrix} = c_1 e^{-t} \begin{bmatrix} \cos\sqrt{2}t \\ \sqrt{2}\sin\sqrt{2}t \end{bmatrix} + c_2 e^{-t} \begin{bmatrix} \sin\sqrt{2}t \\ -\sqrt{2}\cos\sqrt{2}t \end{bmatrix}.$$

Using the initial condition, we calculate that

$$\begin{bmatrix} 2 \\ 2 \end{bmatrix} = \begin{bmatrix} I(0) \\ V(0) \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ -\sqrt{2} \end{bmatrix} \qquad \Longrightarrow \qquad \begin{cases} c_1 = 2 \\ c_2 = -\sqrt{2}. \end{cases}$$

Thus

$$\begin{bmatrix} I(t) \\ V(t) \end{bmatrix} = 2\, e^{-t} \begin{bmatrix} \cos\sqrt{2}t \\ \sqrt{2}\sin\sqrt{2}t \end{bmatrix} - \sqrt{2}\, e^{-t} \begin{bmatrix} \sin\sqrt{2}t \\ -\sqrt{2}\cos\sqrt{2}t \end{bmatrix}.$$

$$\begin{bmatrix} I(t) \\ V(t) \end{bmatrix} = 2\,e^{-t} \begin{bmatrix} \cos\sqrt{2}t \\ \sqrt{2}\sin\sqrt{2}t \end{bmatrix} - \sqrt{2}\,e^{-t} \begin{bmatrix} \sin\sqrt{2}t \\ -\sqrt{2}\cos\sqrt{2}t \end{bmatrix}$$

So the answers to this problem are

$$\boxed{I(t) = \qquad\qquad\qquad\qquad}$$

and

$$\boxed{V(t) = \qquad\qquad\qquad\qquad}$$

$$\begin{bmatrix} I(t) \\ V(t) \end{bmatrix} = 2\,e^{-t} \begin{bmatrix} \cos\sqrt{2}t \\ \sqrt{2}\sin\sqrt{2}t \end{bmatrix} - \sqrt{2}\,e^{-t} \begin{bmatrix} \sin\sqrt{2}t \\ -\sqrt{2}\cos\sqrt{2}t \end{bmatrix}$$

So the answers to this problem are

$$\boxed{I(t) = 2e^{-t}\cos\sqrt{2}t - \sqrt{2}e^{-t}\sin\sqrt{2}t}$$

and

$$\boxed{V(t) = \qquad\qquad\qquad\qquad\qquad\qquad}$$

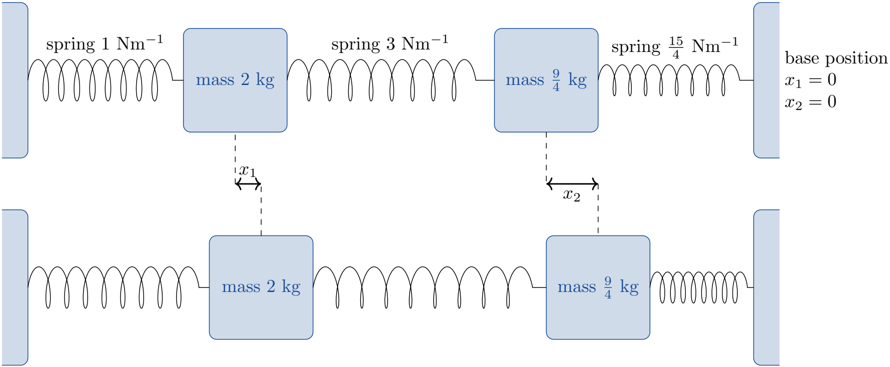
$$\begin{bmatrix} I(t) \\ V(t) \end{bmatrix} = 2\,e^{-t} \begin{bmatrix} \cos\sqrt{2}t \\ \sqrt{2}\sin\sqrt{2}t \end{bmatrix} - \sqrt{2}\,e^{-t} \begin{bmatrix} \sin\sqrt{2}t \\ -\sqrt{2}\cos\sqrt{2}t \end{bmatrix}$$

So the answers to this problem are

$$\boxed{I(t) = 2e^{-t}\cos\sqrt{2}t - \sqrt{2}e^{-t}\sin\sqrt{2}t}$$

and

$$\boxed{V(t) = 2\sqrt{2}e^{-t}\sin\sqrt{2}t + 2e^{-t}\cos\sqrt{2}t.}$$

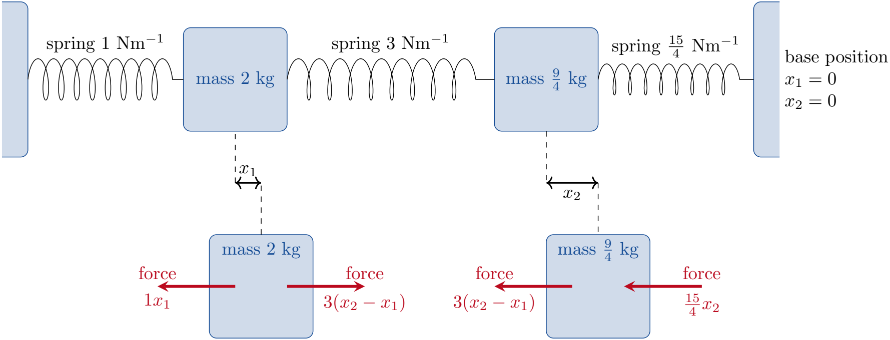


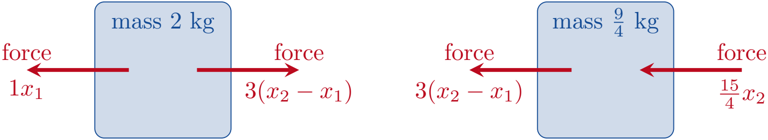
See `https://tinyurl.com/wm2ogdh` for an animated figure.

Example

For the dynamical system shown above, find $x_1(t)$ and $x_2(t)$.

As the springs are stretched and compressed, they apply forces on the blocks as shown below (Hooke's Law).



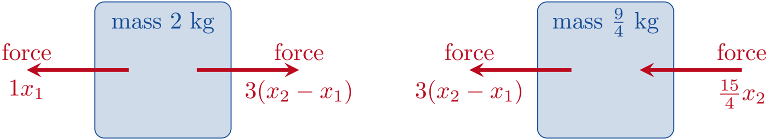


We calculate that

$$\text{mass} \times \text{acceleration} = \text{force}$$

$$\text{mass} \times \text{acceleration} = \text{force}$$

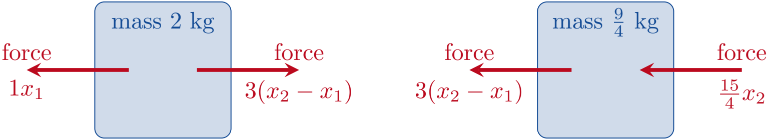
5.4 Complex Eigenvalues



We calculate that

$$2\frac{d^2x_1}{dt^2} = \text{mass} \times \text{acceleration} = \text{force} = -x_1 + 3(x_2 - x_1)$$

$$\text{mass} \times \text{acceleration} = \text{force}$$



We calculate that

$$2\frac{d^2x_1}{dt^2} = \text{mass} \times \text{acceleration} = \text{force} = -x_1 + 3(x_2 - x_1)$$

$$\frac{9}{4}\frac{d^2x_2}{dt^2} = \text{mass} \times \text{acceleration} = \text{force} = -3(x_2 - x_1) - \frac{15}{4}x_2.$$

$$\begin{aligned}
2\frac{d^2x_1}{dt^2} &= -x_1 + 3(x_2 - x_1) \\
\frac{9}{4}\frac{d^2x_2}{dt^2} &= -3(x_2 - x_1) - \frac{15}{4}x_2.
\end{aligned}$$

This is a system of 2 second order ODEs.

$$
\begin{aligned}
2\frac{d^2x_1}{dt^2} &= -x_1 + 3(x_2 - x_1) \\
\frac{9}{4}\frac{d^2x_2}{dt^2} &= -3(x_2 - x_1) - \frac{15}{4}x_2.
\end{aligned}
$$

This is a system of 2 second order ODEs. We want a system of first order ODEs.

$$
\begin{aligned}
2\frac{d^2x_1}{dt^2} &= -x_1 + 3(x_2 - x_1) \\
\frac{9}{4}\frac{d^2x_2}{dt^2} &= -3(x_2 - x_1) - \frac{15}{4}x_2.
\end{aligned}
$$

Now let $y_1 = x_1$, $y_2 = x_2$, $y_3 = x_1'$ and $y_4 = x_2'$.

$$2\frac{d^2x_1}{dt^2} = -x_1 + 3(x_2 - x_1)$$
$$\frac{9}{4}\frac{d^2x_2}{dt^2} = -3(x_2 - x_1) - \frac{15}{4}x_2.$$

Now let $y_1 = x_1$, $y_2 = x_2$, $y_3 = x_1'$ and $y_4 = x_2'$. Then

$$y_1' = x_1' = y_3$$
$$y_2' =$$
$$y_3' =$$
$$y_4' =$$

$$
\begin{aligned}
2\frac{d^2x_1}{dt^2} &= -x_1 + 3(x_2 - x_1) \\
\frac{9}{4}\frac{d^2x_2}{dt^2} &= -3(x_2 - x_1) - \frac{15}{4}x_2.
\end{aligned}
$$

Now let $y_1 = x_1$, $y_2 = x_2$, $y_3 = x_1'$ and $y_4 = x_2'$. Then

$$
\begin{aligned}
y_1' &= x_1' = y_3 \\
y_2' &= x_2' = y_4 \\
y_3' &= \\
y_4' &=
\end{aligned}
$$

$$\begin{aligned} 2\frac{d^2x_1}{dt^2} &= -x_1 + 3(x_2 - x_1) \\ \frac{9}{4}\frac{d^2x_2}{dt^2} &= -3(x_2 - x_1) - \frac{15}{4}x_2. \end{aligned}$$

Now let $y_1 = x_1$, $y_2 = x_2$, $y_3 = x_1'$ and $y_4 = x_2'$. Then

$$\begin{aligned} y_1' &= x_1' = y_3 \\ y_2' &= x_2' = y_4 \\ y_3' &= x_1'' = \frac{1}{2}\Big(- x_1 + 3x_2 - 3x_1\Big) = -2y_1 + \frac{3}{2}y_2 \\ y_4' &= \end{aligned}$$

$$\begin{aligned} 2\frac{d^2x_1}{dt^2} &= -x_1 + 3(x_2 - x_1) \\ \frac{9}{4}\frac{d^2x_2}{dt^2} &= -3(x_2 - x_1) - \frac{15}{4}x_2. \end{aligned}$$

Now let $y_1 = x_1$, $y_2 = x_2$, $y_3 = x_1'$ and $y_4 = x_2'$. Then

$$\begin{aligned} y_1' &= x_1' = y_3 \\ y_2' &= x_2' = y_4 \\ y_3' &= x_1'' = \frac{1}{2}\Big(- x_1 + 3x_2 - 3x_1\Big) = -2y_1 + \frac{3}{2}y_2 \\ y_4' &= x_2'' = \frac{4}{9}\left(-3x_2 + 3x_1 - \frac{15}{4}x_2\right) = \frac{4}{3}y_1 - 3y_2. \end{aligned}$$

So

$$\mathbf{y}' = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -2 & \frac{3}{2} & 0 & 0 \\ \frac{4}{3} & -3 & 0 & 0 \end{bmatrix} \mathbf{y}.$$

The characteristic polynomial of this matrix is

$$0 = r^4 + 5r^2 + 4 = (r^2 + 1)(r^2 + 4).$$

The characteristic polynomial of this matrix is

$$0 = r^4 + 5r^2 + 4 = (r^2 + 1)(r^2 + 4).$$

So $r_1 = i$, $r_2 = -i$, $r_3 = 2i$ and $r_4 = -2i$.

The characteristic polynomial of this matrix is

$$0 = r^4 + 5r^2 + 4 = (r^2 + 1)(r^2 + 4).$$

So $r_1 = i$, $r_2 = -i$, $r_3 = 2i$ and $r_4 = -2i$. We will use r_1 and r_3 (we do not need r_2 and r_4).

The characteristic polynomial of this matrix is

$$0 = r^4 + 5r^2 + 4 = (r^2 + 1)(r^2 + 4).$$

So $r_1 = i$, $r_2 = -i$, $r_3 = 2i$ and $r_4 = -2i$. We will use r_1 and r_3 (we do not need r_2 and r_4).

The corresponding eigenvectors (please check) are

$$\boldsymbol{\xi}^{(1)} = \begin{bmatrix} 3 \\ 2 \\ 3i \\ 2i \end{bmatrix} \qquad \text{and} \qquad \boldsymbol{\xi}^{(3)} = \begin{bmatrix} 3 \\ -4 \\ 6i \\ -8i \end{bmatrix}.$$

It follows that

$$\boldsymbol{\xi}^{(1)} e^{r_1 t} = \begin{bmatrix} 3 \\ 2 \\ 3i \\ 2i \end{bmatrix} (\cos t + i \sin t) = \begin{bmatrix} 3\cos t \\ 2\cos t \\ -3\sin t \\ -2\sin t \end{bmatrix} + i \begin{bmatrix} 3\sin t \\ 2\sin t \\ 3\cos t \\ 2\cos t \end{bmatrix}$$

$$= \mathbf{u}(t) + i\mathbf{v}(t)$$

and

It follows that

$$\boldsymbol{\xi}^{(1)} e^{r_1 t} = \begin{bmatrix} 3 \\ 2 \\ 3i \\ 2i \end{bmatrix} (\cos t + i \sin t) = \begin{bmatrix} 3\cos t \\ 2\cos t \\ -3\sin t \\ -2\sin t \end{bmatrix} + i \begin{bmatrix} 3\sin t \\ 2\sin t \\ 3\cos t \\ 2\cos t \end{bmatrix}$$

$$= \mathbf{u}(t) + i\mathbf{v}(t)$$

and

$$\boldsymbol{\xi}^{(3)} e^{r_3 t} = \begin{bmatrix} 3 \\ -4 \\ 6i \\ -8i \end{bmatrix} (\cos 2t + i \sin 2t) = \begin{bmatrix} 3\cos 2t \\ -4\cos 2t \\ -6\sin 2t \\ +8\sin 2t \end{bmatrix} + i \begin{bmatrix} 3\sin 2t \\ -4\sin 2t \\ 6\cos 2t \\ -8\cos 2t \end{bmatrix}$$

$$= \mathbf{w}(t) + i\mathbf{z}(t)$$

Therefore the general solution is

$$\mathbf{y}(t) = c_1\mathbf{u}(t) + c_2\mathbf{v}(t) + c_3\mathbf{w}(t) + c_4\mathbf{z}(t)$$

Therefore the general solution is

$$\mathbf{y}(t) = c_1\mathbf{u}(t) + c_2\mathbf{v}(t) + c_3\mathbf{w}(t) + c_4\mathbf{z}(t)$$
$$= c_1 \begin{bmatrix} 3\cos t \\ 2\cos t \\ -3\sin t \\ -2\sin t \end{bmatrix} + c_2 \begin{bmatrix} 3\sin t \\ 2\sin t \\ 3\cos t \\ 2\cos t \end{bmatrix} + c_3 \begin{bmatrix} 3\cos 2t \\ -4\cos 2t \\ -6\sin 2t \\ 8\sin 2t \end{bmatrix} + c_4 \begin{bmatrix} 3\sin 2t \\ -4\sin 2t \\ 6\cos 2t \\ -8\cos 2t \end{bmatrix}.$$

Example

Suppose that the above system has initial condition

$$\mathbf{y}(0) = \begin{bmatrix} -1 \\ 4 \\ 1 \\ 1 \end{bmatrix}.$$

Sketch graphs of $y_1(t)$ and $y_2(t)$.

The initial value problem

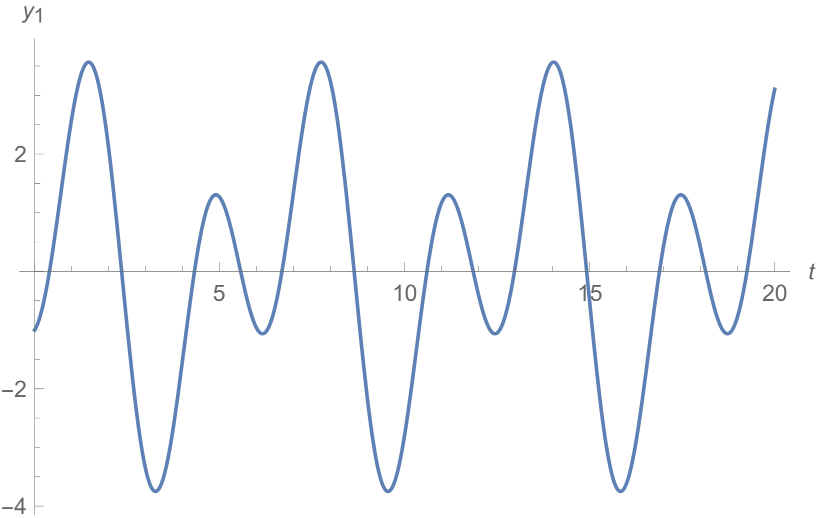
$$\mathbf{y}' = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -2 & \frac{3}{2} & 0 & 0 \\ \frac{4}{3} & -3 & 0 & 0 \end{bmatrix} \mathbf{y}, \qquad \mathbf{y}(0) = \begin{bmatrix} -1 \\ 4 \\ 1 \\ 1 \end{bmatrix}$$

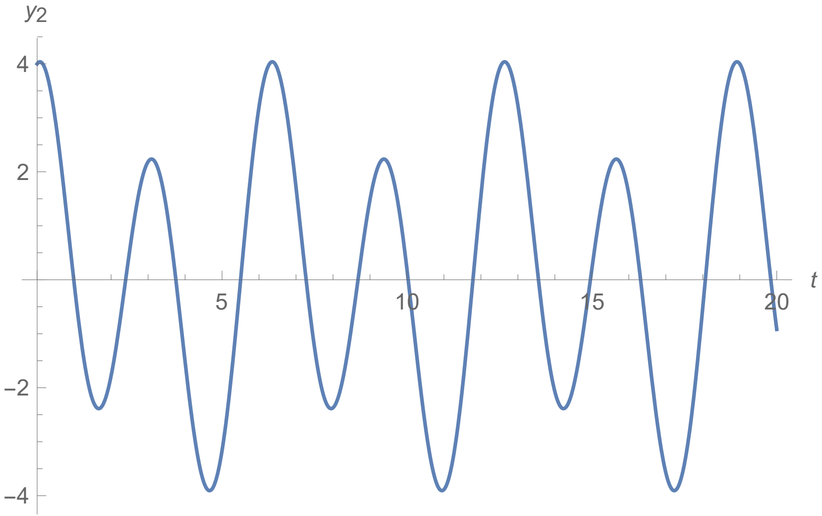
has solution

$$\mathbf{y}(t) = \frac{4}{9} \begin{bmatrix} 3\cos t \\ 2\cos t \\ -3\sin t \\ -2\sin t \end{bmatrix} + \frac{7}{18} \begin{bmatrix} 3\sin t \\ 2\sin t \\ 3\cos t \\ 2\cos t \end{bmatrix} - \frac{7}{9} \begin{bmatrix} 3\cos 2t \\ -4\cos 2t \\ -6\sin 2t \\ 8\sin 2t \end{bmatrix} - \frac{1}{36} \begin{bmatrix} 3\sin 2t \\ -4\sin 2t \\ 6\cos 2t \\ -8\cos 2t \end{bmatrix}.$$

Then we can draw the graphs of y_1 and y_2:

5.4 Complex Eigenvalues





y_2
4
2
−2
−4
5
10
15
20
t

Please see https://tinyurl.com/s7uww7m

Next Time

- 5.5 Fundamental Matrices
- 5.6 Repeated Eigenvalues