Lecture 2

- 1.5 Solution Sets of Linear Systems
- 2.1 Matrix Operations
- 2.2 The Inverse of a Matrix

Solution Sets of Linear Systems

Definition

A system of linear equations is said to be *homogeneous* if the constant terms are all zero;

Definition

A system of linear equations is said to be *homogeneous* if the constant terms are all zero; that is, if the system has the form

$$\begin{cases} a_{11}x_1 & + & a_{12}x_2 & + & a_{13}x_3 & + & \dots & + & a_{1n}x_n & = & 0 \\ a_{21}x_1 & + & a_{22}x_2 & + & a_{23}x_3 & + & \dots & + & a_{2n}x_n & = & 0 \\ \vdots & & \vdots & & \vdots & & & & \vdots & & \vdots \\ a_{m1}x_1 & + & a_{m2}x_2 & + & a_{m3}x_3 & + & \dots & + & a_{mn}x_n & = & 0. \end{cases}$$

Definition

A system of linear equations is said to be *homogeneous* if the constant terms are all zero; that is, if the system has the form

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \dots + a_{1n}x_n = 0 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \dots + a_{2n}x_n = 0 \\ \quad\vdots \qquad\qquad \vdots \qquad\qquad \vdots \qquad\qquad\qquad \vdots \qquad\quad \vdots \\ a_{m1}x_1 + a_{m2}x_2 + a_{m3}x_3 + \dots + a_{mn}x_n = 0. \end{cases}$$

Remark

Every homogeneous linear system is consistent because $(0, 0, 0, \dots, 0)$ is always a solution.

Definition

The solution $(0, 0, 0, \ldots, 0)$ is called the *trivial solution*.

Definition

The solution $(0, 0, 0, \ldots, 0)$ is called the *trivial solution*.

If there are other solutions, they are called *nontrivial solutions*.

Definition

The solution $(0, 0, 0, \ldots, 0)$ is called the *trivial solution*.

If there are other solutions, they are called *nontrivial solutions*.

Theorem

A homogeneous linear system has a nontrivial solution if and only if it has atleast one free variable.

Example

Use Gauss-Jordan elimination to solve the homogeneous linear system

$$\begin{cases} x_1 + 3x_2 - 2x_3 + 2x_5 = 0 \\ 2x_1 + 6x_2 - 5x_3 - 2x_4 + 4x_5 - 3x_6 = 0 \\ 5x_3 + 10x_4 + 15x_6 = 0 \\ 2x_1 + 6x_2 + 8x_4 + 4x_5 + 18x_6 = 0. \end{cases}$$

Example

Use Gauss-Jordan elimination to solve the homogeneous linear system

$$\begin{cases} x_1 + 3x_2 - 2x_3 \qquad\qquad + 2x_5 \qquad\qquad = 0 \\ 2x_1 + 6x_2 - 5x_3 - 2x_4 + 4x_5 - 3x_6 = 0 \\ \qquad\qquad 5x_3 + 10x_4 \qquad\quad + 15x_6 = 0 \\ 2x_1 + 6x_2 \qquad\quad + 8x_4 + 4x_5 + 18x_6 = 0. \end{cases}$$

The augmented matrix is

$$\begin{bmatrix} 1 & 3 & -2 & 0 & 2 & 0 & 0 \\ 2 & 6 & -5 & -2 & 4 & -3 & 0 \\ 0 & 0 & 5 & 10 & 0 & 15 & 0 \\ 2 & 6 & 0 & 8 & 4 & 18 & 0 \end{bmatrix}.$$

The augmented matrix is

$$\begin{bmatrix} 1 & 3 & -2 & 0 & 2 & 0 & 0 \\ 2 & 6 & -5 & -2 & 4 & -3 & 0 \\ 0 & 0 & 5 & 10 & 0 & 15 & 0 \\ 2 & 6 & 0 & 8 & 4 & 18 & 0 \end{bmatrix}$$

can be row-reduced to

$$\begin{bmatrix} 1 & 3 & 0 & 4 & 2 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

I leave it for you to check this (exercise).

The corresponding linear system is

$$\begin{cases} x_1 + 3x_2 \qquad\quad + 4x_4 + 2x_5 \qquad = 0 \\ \qquad\qquad\quad x_3 + 2x_4 \qquad\qquad\quad = 0 \\ \qquad\qquad\qquad\qquad\qquad\qquad x_6 = 0 \end{cases}$$

which has solution

$$\begin{cases} x_1 = -3x_2 - 4x_4 - 2x_5 \\ x_2 \text{ is free} \\ x_3 = -2x_4 \\ x_4 \text{ is free} \\ x_5 \text{ is free} \\ x_6 = 0. \end{cases}$$

This linear system has infinitely many solutions.

Theorem

If a homogeneous linear system has n variables, and if the RREF of its augmented matrix has r nonzero rows, then the system has $n - r$ free variables.

Theorem

If a homogeneous linear system has n variables, and if the RREF of its augmented matrix has r nonzero rows, then the system has $n - r$ free variables.

Theorem

A homogeneous linear system with more variables than equations has infinitely many solutions.

2.1 Matrix Operations

2.1 Matrix Operations

Last week we used rectangular arrays of numbers, called *augmented matrices*, to abbreviate linear systems.

This week I want to discuss what a *matrix* is and what we can do with them.

Last week we used rectangular arrays of numbers, called *augmented matrices*, to abbreviate linear systems.

This week I want to discuss what a *matrix* is and what we can do with them.

Definition

A *matrix* is a rectangular array of numbers.

Last week we used rectangular arrays of numbers, called *augmented matrices*, to abbreviate linear systems.

This week I want to discuss what a *matrix* is and what we can do with them.

Definition

A *matrix* is a rectangular array of numbers.

Definition

The numbers in a matrix are called *entries*.

Last week we used rectangular arrays of numbers, called *augmented matrices*, to abbreviate linear systems.

This week I want to discuss what a *matrix* is and what we can do with them.

Definition

A *matrix* is a rectangular array of numbers.

Definition

The numbers in a matrix are called *entries*.

The plural of matrix is *matrices*.

Example

These are matrices:

$$\begin{bmatrix} 1 & 2 \\ 3 & 0 \\ -1 & 4 \end{bmatrix}, \quad \begin{bmatrix} 2 & 1 & 0 & -3 \end{bmatrix}, \quad \begin{bmatrix} e & \pi & -\sqrt{2} \\ 0 & \frac{1}{2} & 1 \\ 0 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 1 \\ 3 \end{bmatrix}, \quad \begin{bmatrix} 4 \end{bmatrix}.$$

Example

These are matrices:

$$\begin{bmatrix} 1 & 2 \\ 3 & 0 \\ -1 & 4 \end{bmatrix}, \quad \begin{bmatrix} 2 & 1 & 0 & -3 \end{bmatrix}, \quad \begin{bmatrix} e & \pi & -\sqrt{2} \\ 0 & \frac{1}{2} & 1 \\ 0 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 1 \\ 3 \end{bmatrix}, \quad \begin{bmatrix} 4 \end{bmatrix}.$$

Definition

The *size* of a matrix is

$$\text{number of rows} \times \text{number of column}.$$

Example

These are matrices:

$$\begin{bmatrix} 1 & 2 \\ 3 & 0 \\ -1 & 4 \end{bmatrix}, \quad \begin{bmatrix} 2 & 1 & 0 & -3 \end{bmatrix}, \quad \begin{bmatrix} e & \pi & -\sqrt{2} \\ 0 & \frac{1}{2} & 1 \\ 0 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 1 \\ 3 \end{bmatrix}, \quad \begin{bmatrix} 4 \end{bmatrix}.$$

3×2

Definition

The *size* of a matrix is

$$\text{number of rows} \times \text{number of column.}$$

Example

These are matrices:

$$\begin{bmatrix} 1 & 2 \\ 3 & 0 \\ -1 & 4 \end{bmatrix}, \quad \begin{bmatrix} 2 & 1 & 0 & -3 \end{bmatrix}, \quad \begin{bmatrix} e & \pi & -\sqrt{2} \\ 0 & \frac{1}{2} & 1 \\ 0 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 1 \\ 3 \end{bmatrix}, \quad \begin{bmatrix} 4 \end{bmatrix}.$$

1×4

Definition

The *size* of a matrix is

$$\text{number of rows} \times \text{number of column.}$$

Example

These are matrices:

$$\begin{bmatrix} 1 & 2 \\ 3 & 0 \\ -1 & 4 \end{bmatrix}, \quad \begin{bmatrix} 2 & 1 & 0 & -3 \end{bmatrix}, \quad \begin{bmatrix} e & \pi & -\sqrt{2} \\ 0 & \frac{1}{2} & 1 \\ 0 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 1 \\ 3 \end{bmatrix}, \quad \begin{bmatrix} 4 \end{bmatrix}.$$

$$3 \times 3$$

Definition

The *size* of a matrix is

$$\text{number of rows} \times \text{number of column.}$$

Example

These are matrices:

$$\begin{bmatrix} 1 & 2 \\ 3 & 0 \\ -1 & 4 \end{bmatrix}, \quad \begin{bmatrix} 2 & 1 & 0 & -3 \end{bmatrix}, \quad \begin{bmatrix} e & \pi & -\sqrt{2} \\ 0 & \frac{1}{2} & 1 \\ 0 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 1 \\ 3 \end{bmatrix}, \quad \begin{bmatrix} 4 \end{bmatrix}.$$

2×1

Definition

The *size* of a matrix is

$$\text{number of rows} \times \text{number of column.}$$

Example

These are matrices:

$$\begin{bmatrix} 1 & 2 \\ 3 & 0 \\ -1 & 4 \end{bmatrix}, \quad \begin{bmatrix} 2 & 1 & 0 & -3 \end{bmatrix}, \quad \begin{bmatrix} e & \pi & -\sqrt{2} \\ 0 & \frac{1}{2} & 1 \\ 0 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 1 \\ 3 \end{bmatrix}, \quad \begin{bmatrix} 4 \end{bmatrix}.$$

1×1

Definition

The *size* of a matrix is

$$\text{number of rows} \times \text{number of column.}$$

$$\begin{bmatrix} 2 & 1 & 0 & -3 \end{bmatrix} \quad \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

Definition

A matrix with only one row is called a *row vector*.

Definition

A matrix with only one column is called a *column vector*.

Notation

We will use

- capital letters, $A, B, C, \ldots$ to denote matrices; and
- lowercase letters, $a, b, c, \ldots$ to denote numbers.

Notation

We will use

- capital letters, $A, B, C, \ldots$ to denote matrices; and
- lowercase letters, $a, b, c, \ldots$ to denote numbers.

Notation

The entry with occurs in row i and column j of a matrix A will be denoted by a_{ij}.

2.1 Matrix Operations

Thus a general 3×4 matrix might be written as

$$\text{3 rows} \quad \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \end{bmatrix}$$

$$\text{4 columns}$$

Thus a general 3×4 matrix might be written as

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \end{bmatrix}$$

row 2
column 4

"a two four"
not "a twenty four"

Thus a general 3×4 matrix might be written as

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \end{bmatrix}$$

and a general $m \times n$ matrix might be written as

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1j} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2j} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots & & \vdots \\ a_{i1} & a_{i2} & \cdots & a_{ij} & \cdots & a_{in} \\ \vdots & \vdots & & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mj} & \cdots & a_{mn} \end{bmatrix}.$$

Thus a general 3×4 matrix might be written as

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \end{bmatrix}$$

and a general $m \times n$ matrix might be written as

$$\begin{array}{rc} \text{row } 1 & \\ & \end{array} \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1j} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2j} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots & & \vdots \\ a_{i1} & a_{i2} & \cdots & a_{ij} & \cdots & a_{in} \\ \vdots & \vdots & & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mj} & \cdots & a_{mn} \end{bmatrix}.$$

Thus a general 3×4 matrix might be written as

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \end{bmatrix}$$

and a general $m \times n$ matrix might be written as

$$\begin{array}{c} \\ \text{row } 2 \\ \\ \\ \\ \\ \end{array} \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1j} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2j} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots & & \vdots \\ a_{i1} & a_{i2} & \cdots & a_{ij} & \cdots & a_{in} \\ \vdots & \vdots & & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mj} & \cdots & a_{mn} \end{bmatrix}.$$

Thus a general 3×4 matrix might be written as

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \end{bmatrix}$$

and a general $m \times n$ matrix might be written as

$$\begin{array}{r c} & \\ \text{row } i & \end{array} \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1j} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2j} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots & & \vdots \\ a_{i1} & a_{i2} & \cdots & a_{ij} & \cdots & a_{in} \\ \vdots & \vdots & & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mj} & \cdots & a_{mn} \end{bmatrix}.$$

Thus a general 3×4 matrix might be written as

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \end{bmatrix}$$

and a general $m \times n$ matrix might be written as

$$\begin{array}{r c} & \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1j} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2j} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots & & \vdots \\ a_{i1} & a_{i2} & \cdots & a_{ij} & \cdots & a_{in} \\ \vdots & \vdots & & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mj} & \cdots & a_{mn} \end{bmatrix} \\ \text{row } m & \end{array}.$$

2.1 Matrix Operations

Thus a general 3×4 matrix might be written as

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \end{bmatrix}$$

and a general $m \times n$ matrix might be written as

$$\begin{array}{l} \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1j} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2j} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots & & \vdots \\ a_{i1} & a_{i2} & \cdots & a_{ij} & \cdots & a_{in} \\ \vdots & \vdots & & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mj} & \cdots & a_{mn} \end{bmatrix}. \\ \text{column 1} \end{array}$$

2.1 Matrix Operations

Thus a general 3×4 matrix might be written as

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \end{bmatrix}$$

and a general $m \times n$ matrix might be written as

$$\begin{array}{c} \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1j} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2j} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots & & \vdots \\ a_{i1} & a_{i2} & \cdots & a_{ij} & \cdots & a_{in} \\ \vdots & \vdots & & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mj} & \cdots & a_{mn} \end{bmatrix} . \\ \text{column 2} \end{array}$$

Thus a general 3×4 matrix might be written as

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \end{bmatrix}$$

and a general $m \times n$ matrix might be written as

$$\begin{array}{c} \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1j} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2j} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots & & \vdots \\ a_{i1} & a_{i2} & \cdots & a_{ij} & \cdots & a_{in} \\ \vdots & \vdots & & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mj} & \cdots & a_{mn} \end{bmatrix} . \\ \text{column } j \end{array}$$

2.1 Matrix Operations

Thus a general 3×4 matrix might be written as

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \end{bmatrix}$$

and a general $m \times n$ matrix might be written as

$$\begin{array}{r} \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1j} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2j} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots & & \vdots \\ a_{i1} & a_{i2} & \cdots & a_{ij} & \cdots & a_{in} \\ \vdots & \vdots & & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mj} & \cdots & a_{mn} \end{bmatrix}. \\ \text{column } n \end{array}$$

2.1 Matrix Operations

Thus a general 3×4 matrix might be written as

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \end{bmatrix}$$

and a general $m \times n$ matrix might be written as

$$\begin{array}{rc} \begin{array}{r} \\ \\ \\ \text{row } i \\ \\ \\ \end{array} & \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1j} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2j} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots & & \vdots \\ a_{i1} & a_{i2} & \cdots & a_{ij} & \cdots & a_{in} \\ \vdots & \vdots & & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mj} & \cdots & a_{mn} \end{bmatrix} \\ & \text{column } j \end{array}.$$

2.1 Matrix Operations

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1j} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2j} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots & & \vdots \\ a_{i1} & a_{i2} & \cdots & a_{ij} & \cdots & a_{in} \\ \vdots & \vdots & & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mj} & \cdots & a_{mn} \end{bmatrix}.$$

Notation

We can also write $A = [a_{ij}]_{m\times n}$

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1j} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2j} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots & & \vdots \\ a_{i1} & a_{i2} & \cdots & a_{ij} & \cdots & a_{in} \\ \vdots & \vdots & & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mj} & \cdots & a_{mn} \end{bmatrix}.$$

Notation

We can also write $A = [a_{ij}]_{m \times n}$ or just $A = [a_{ij}]$.

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1j} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2j} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots & & \vdots \\ a_{i1} & a_{i2} & \cdots & a_{ij} & \cdots & a_{in} \\ \vdots & \vdots & & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mj} & \cdots & a_{mn} \end{bmatrix}.$$

Notation

We can also write $A = [a_{ij}]_{m\times n}$ or just $A = [a_{ij}]$.

The entry in row i and column j of a matrix A can be denoted by

$$a_{ij} = (A)_{ij}$$

also.

Example

Let $A = \begin{bmatrix} 2 & -3 \\ 7 & 0 \end{bmatrix}$ be a matrix. Then

$$(A)_{11} = 2, \qquad (A)_{12} = -3, \qquad (A)_{21} = 7 \quad \text{and} \quad (A)_{22} = 0.$$

Notation

For vectors, that is $1 \times n$ and $m \times 1$ matrices, it is common practice to denote them by bold lowercase letters instead of capital letters. E.g.

$$\mathbf{a} = \begin{bmatrix} a_1 & a_2 & a_3 & \cdots & a_n \end{bmatrix} \qquad \text{and} \qquad \mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}.$$

Definition

A matrix which has the same number of rows and columns, n say, is called a *square matrix of order n*.

$$\begin{array}{cc} \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} & \updownarrow n \\ \overset{\longleftrightarrow}{n} & \end{array}$$

Definition

A matrix which has the same number of rows and columns, n say, is called a *square matrix of order n*.

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$$

Definition

The entries $a_{11}, a_{22}, \ldots, a_{nn}$ are on the *main diagonal* of a square matrix.

Operations on Matrices

Definition

Two matrices $A = [a_{ij}]$ and $B = [b_{ij}]$, are *equal* if

- they have the same size; and
- the corresponding entries are equal ($a_{ij} = b_{ij}$ for all i and j).

Definition (addition and subtraction)

If A and B have the same size, then we can define the sum $A + B$ and the difference $A - B$ in the obvious way.

Example

Let

$$A = \begin{bmatrix} 2 & 1 & 0 & 3 \\ -1 & 0 & 2 & 4 \\ 4 & -2 & 7 & 0 \end{bmatrix},\ B = \begin{bmatrix} -4 & 3 & 5 & 1 \\ 2 & 2 & 0 & -1 \\ 3 & 2 & -4 & 5 \end{bmatrix} \text{ and } C = \begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}.$$

Then

$$A + B = \begin{bmatrix} -2 & 4 & 5 & 4 \\ 1 & 2 & 2 & 3 \\ 7 & 0 & 3 & 5 \end{bmatrix} \quad \text{and} \quad A - B = \begin{bmatrix} 6 & -2 & -5 & 2 \\ -3 & -2 & 2 & 5 \\ 1 & -4 & 11 & -5 \end{bmatrix}.$$

Example

Let

$$A = \begin{bmatrix} 2 & 1 & 0 & 3 \\ -1 & 0 & 2 & 4 \\ 4 & -2 & 7 & 0 \end{bmatrix}, \; B = \begin{bmatrix} -4 & 3 & 5 & 1 \\ 2 & 2 & 0 & -1 \\ 3 & 2 & -4 & 5 \end{bmatrix} \text{ and } C = \begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}.$$

Then

$$A+B = \begin{bmatrix} -2 & 4 & 5 & 4 \\ 1 & 2 & 2 & 3 \\ 7 & 0 & 3 & 5 \end{bmatrix} \quad \text{and} \quad A-B = \begin{bmatrix} 6 & -2 & -5 & 2 \\ -3 & -2 & 2 & 5 \\ 1 & -4 & 11 & -5 \end{bmatrix}.$$

$A+C$, $B+C$, $A-C$ and $B-C$ are all undefined because the sizes are different.

And we can multiply a matrix by a number in the obvious way.

And we can multiply a matrix by a number in the obvious way.

Example

If

$$A = \begin{bmatrix} 2 & 3 & 4 \\ 1 & 3 & 1 \end{bmatrix}, \; B = \begin{bmatrix} 0 & 2 & 7 \\ -1 & 3 & -5 \end{bmatrix} \text{ and } C = \begin{bmatrix} 9 & -6 & -3 \\ 3 & 0 & 12 \end{bmatrix},$$

then

$$2A = \begin{bmatrix} 4 & 6 & 8 \\ 2 & 6 & 2 \end{bmatrix}, \quad -B = (-1)B = \begin{bmatrix} 0 & -2 & -7 \\ 1 & -3 & 5 \end{bmatrix}$$

and

$$\frac{1}{3}C = \begin{bmatrix} 3 & -2 & -1 \\ 1 & 0 & 4 \end{bmatrix}.$$

The Zero Matrix

Definition

An $m \times n$ matrix whose entries are all zero is called a *zero matrix* and is written as 0 or $0_{m \times n}$.

Example (Some zero matrices)

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \quad \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad \begin{bmatrix} 0 \end{bmatrix}$$

Let A, B, and C be matrices of the same size, and let r and s be scalars.

a. $A + B = B + A$
b. $(A + B) + C = A + (B + C)$
c. $A + 0 = A$
d. $r(A + B) = rA + rB$
e. $(r + s)A = rA + sA$
f. $r(sA) = (rs)A$

Theorem

Let A, B and C be matrices of the same size. Let r and s be numbers. Then

1. $A + B = B + A$

Theorem

Let A, B and C be matrices of the same size. Let r and s be numbers. Then

1. $A + B = B + A$
2. $(A + B) + C = A + (B + C)$

Theorem

Let A, B and C be matrices of the same size. Let r and s be numbers. Then

1. $A + B = B + A$
2. $(A + B) + C = A + (B + C)$
3. $A + 0 = A$ *and* $A - 0 = A$

Theorem

Let A, B and C be matrices of the same size. Let r and s be numbers. Then

1. $A + B = B + A$
2. $(A + B) + C = A + (B + C)$
3. $A + 0 = A$ *and* $A - 0 = A$
4. $r(A + B) = rA + rB$

Theorem

Let A, B and C be matrices of the same size. Let r and s be numbers. Then

1. $A + B = B + A$
2. $(A + B) + C = A + (B + C)$
3. $A + 0 = A$ *and* $A - 0 = A$
4. $r(A + B) = rA + rB$
5. $(r + s)A = rA + sA$

Theorem

Let A, B and C be matrices of the same size. Let r and s be numbers. Then

1. $A + B = B + A$
2. $(A + B) + C = A + (B + C)$
3. $A + 0 = A$ *and* $A - 0 = A$
4. $r(A + B) = rA + rB$
5. $(r + s)A = rA + sA$
6. $r(sA) = (rs)A.$

Definition

If $A_1, A_2, \ldots, A_r$ are matrices of the same size, and if $c_1, c_2, \ldots, c_r$ are numbers, then an expression of the form

$$c_1A_1 + c_2A_2 + \ldots + c_rA_r$$

is called a *linear combination* of $A_1, A_2, \ldots, A_r$ with coefficients $c_1, c_2, \ldots, c_r$.

Matrix Multiplication

Multiplying two matrices together is more complicated.

Matrix Multiplication

Multiplying two matrices together is more complicated.

$$\underset{m \times r}{A} \cdot \underset{r \times n}{B} = \underset{m \times n}{AB}$$

Definition

If A is an $m \times r$ matrix and B is an $r \times n$ matrix, then AB will be an $m \times n$ matrix.

Matrix Multiplication

Multiplying two matrices together is more complicated.

$$\underset{m \times r}{A} \cdot \underset{r \times n}{B} = \underset{m \times n}{AB}$$

Definition

If A is an $m \times r$ matrix and B is an $r \times n$ matrix, then AB will be an $m \times n$ matrix.

To find the entry in the ith row and jth column of AB, we must multiply each entry from row i of A with the corresponding entry from column j of B, then sum them.

Example

Multiply the following two matrices together:

$$A = \begin{bmatrix} 1 & 2 & 4 \\ 2 & 6 & 0 \end{bmatrix}, \qquad B = \begin{bmatrix} 4 & 1 & 4 & 3 \\ 0 & -1 & 3 & 1 \\ 2 & 7 & 5 & 2 \end{bmatrix}.$$

Example

Multiply the following two matrices together:

$$A = \begin{bmatrix} 1 & 2 & 4 \\ 2 & 6 & 0 \end{bmatrix}, \qquad B = \begin{bmatrix} 4 & 1 & 4 & 3 \\ 0 & -1 & 3 & 1 \\ 2 & 7 & 5 & 2 \end{bmatrix}.$$

A is a 2×3 matrix and B is a 3×4 matrix. So the product AB is a 2×4 matrix.

2.1 Matrix Operations

$$\begin{bmatrix} 1 & 2 & 4 \\ 2 & 6 & 0 \end{bmatrix} \begin{bmatrix} 4 & 1 & 4 & 3 \\ 0 & -1 & 3 & 1 \\ 2 & 7 & 5 & 2 \end{bmatrix} = \begin{bmatrix} \quad & \quad & \quad & \quad \\ \quad & \quad & \quad & \quad \end{bmatrix}$$

$$2 \times 3 \qquad 3 \times 4 \qquad 2 \times 4$$

Let's say that I want to calculate the entry in the 2nd row and the 3rd column.

$$\begin{bmatrix} 1 & 2 & 4 \\ 2 & 6 & 0 \end{bmatrix} \begin{bmatrix} 4 & 1 & 4 & 3 \\ 0 & -1 & 3 & 1 \\ 2 & 7 & 5 & 2 \end{bmatrix} = \begin{bmatrix} & & & \\ & & & \end{bmatrix}$$

2.1 Matrix Operations

I need row 2 of the first matrix,

Let's say that I want to calculate the entry in the 2nd row and the 3rd column.

$$\begin{bmatrix} 1 & 2 & 4 \\ 2 & 6 & 0 \end{bmatrix} \begin{bmatrix} 4 & 1 & 4 & 3 \\ 0 & -1 & 3 & 1 \\ 2 & 7 & 5 & 2 \end{bmatrix} = \begin{bmatrix} \quad & \quad & \quad & \quad \\ \quad & \quad & \quad & \quad \end{bmatrix}$$

2.1 Matrix Operations

I need row 2 of the first matrix,

and i need column 3 of the second matrix.

Let's say that I want to calculate the entry in the 2nd row and the 3rd column.

$$\begin{bmatrix}1 & 2 & 4\\ 2 & 6 & 0\end{bmatrix}\begin{bmatrix}4 & 1 & 4 & 3\\ 0 & -1 & 3 & 1\\ 2 & 7 & 5 & 2\end{bmatrix} = \begin{bmatrix} & & & \\ & & & \end{bmatrix}$$

2.1 Matrix Operations

$$\begin{bmatrix} 1 & 2 & 4 \\ 2 & 6 & 0 \end{bmatrix} \begin{bmatrix} 4 & 1 & 4 & 3 \\ 0 & -1 & 3 & 1 \\ 2 & 7 & 5 & 2 \end{bmatrix} = \begin{bmatrix} \square & \square & \square & \square \\ \square & \square & \blacksquare & \square \end{bmatrix}$$

$$(2 \cdot 4) + (6 \cdot 3) + (0 \cdot 5) = 26$$

2.1 Matrix Operations

$$\begin{bmatrix} 1 & 2 & 4 \\ 2 & 6 & 0 \end{bmatrix} \begin{bmatrix} 4 & 1 & 4 & 3 \\ 0 & -1 & 3 & 1 \\ 2 & 7 & 5 & 2 \end{bmatrix} = \begin{bmatrix} & & & \\ & & 26 & \end{bmatrix}$$

$$(2 \cdot 4) + (6 \cdot 3) + (0 \cdot 5) = 26$$

$$\begin{bmatrix} 1 & 2 & 4 \\ 2 & 6 & 0 \end{bmatrix} \begin{bmatrix} 4 & 1 & 4 & 3 \\ 0 & -1 & 3 & 1 \\ 2 & 7 & 5 & 2 \end{bmatrix} = \begin{array}{c} \hspace{10em}\text{row 1} \\ \hspace{10em}\text{column 4} \\ \begin{bmatrix} & & & \\ & & 26 & \end{bmatrix} \end{array}$$

$$\begin{array}{cc} \text{row } 1 & \begin{bmatrix}1 & 2 & 4\\ 2 & 6 & 0\end{bmatrix}\end{array}\begin{array}{cccc} & & & \text{column } 4\\ \multicolumn{4}{c}{\begin{bmatrix}4 & 1 & 4 & 3\\ 0 & -1 & 3 & 1\\ 2 & 7 & 5 & 2\end{bmatrix}}\end{array} = \begin{array}{cccc} & & & \text{row } 1\\ & & & \text{column } 4\\ \multicolumn{4}{c}{\begin{bmatrix}\square & \square & \square & \square\\ \square & \square & 26 & \square\end{bmatrix}}\end{array}$$

$$\begin{bmatrix} 1 & 2 & 4 \\ 2 & 6 & 0 \end{bmatrix} \begin{bmatrix} 4 & 1 & 4 & 3 \\ 0 & -1 & 3 & 1 \\ 2 & 7 & 5 & 2 \end{bmatrix} = \begin{bmatrix} \square & \square & \square & 13 \\ \square & \square & 26 & \square \end{bmatrix}$$

$$(1 \cdot 3) + (2 \cdot 1) + (4 \cdot 2) = 13$$

$$\begin{bmatrix}1 & 2 & 4\\2 & 6 & 0\end{bmatrix}\begin{bmatrix}4 & 1 & 4 & 3\\0 & -1 & 3 & 1\\2 & 7 & 5 & 2\end{bmatrix} = \begin{bmatrix}12 & 27 & 30 & 13\\8 & -4 & 26 & 12\end{bmatrix}$$

I leave the other 6 entries for you to check.

Remember, we can only multiply two matrices together if the inside size numbers are equal.

$$\underset{m \times r}{A} \cdot \underset{r \times n}{B} = AB$$

inside

2.1 Matrix Operations

Remember, we can only multiply two matrices together if the inside size numbers are equal.

$$\underset{m \times r}{A} \cdot \underset{r \times n}{B} = \underset{m \times n}{AB}$$

inside

outside

Then the outside size numbers give us the size of the product AB.

EXAMPLE 4 If A is a 3×5 matrix and B is a 5×2 matrix, what are the sizes of AB and BA, if they are defined?

EXAMPLE 4 If A is a 3×5 matrix and B is a 5×2 matrix, what are the sizes of AB and BA, if they are defined?

SOLUTION Since A has 5 columns and B has 5 rows, the product AB is defined and is a 3×2 matrix:

$$\underset{3 \times 5}{\overset{A}{\begin{bmatrix} * & * & * & * & * \\ * & * & * & * & * \\ * & * & * & * & * \end{bmatrix}}} \underset{5 \times 2}{\overset{B}{\begin{bmatrix} * & * \\ * & * \\ * & * \\ * & * \\ * & * \end{bmatrix}}} = \underset{3 \times 2}{\overset{AB}{\begin{bmatrix} * & * \\ * & * \\ * & * \end{bmatrix}}}$$

Match

Size of AB

The product BA is *not* defined because the 2 columns of B do not match the 3 rows of A. ■

Ask the audience 1/3

Suppose that A, B and C are matrices with the following sizes:

$$\begin{array}{ccc} A & B & C \\ 3 \times 4 & 4 \times 7 & 7 \times 3. \end{array}$$

Which of the following is correct?

1. AB is a 3×7 matrix.
   BA is a 4×4 matrix.
2. AB is a 3×7 matrix.
   BA is a 7×3 matrix.
3. AB is a 3×7 matrix.
   BA is undefined.
4. AB is undefined.
   BA is a 4×4 matrix.

Ask the audience 1/3

Suppose that A, B and C are matrices with the following sizes:

$$\begin{array}{ccc} A & B & C \\ 3 \times 4 & 4 \times 7 & 7 \times 3. \end{array}$$

Which of the following is correct?

1. AB is a 3×7 matrix. BA is a 4×4 matrix.
2. AB is a 3×7 matrix. BA is a 7×3 matrix.
3. AB is a 3×7 matrix. BA is undefined.
4. AB is undefined. BA is a 4×4 matrix.

Ask the audience 2/3

Suppose that A, B and C are matrices with the following sizes:

$$\begin{array}{ccc} A & B & C \\ 3 \times 4 & 4 \times 7 & 7 \times 3. \end{array}$$

Which of the following is correct?

1. BC is a 4×3 matrix.
   CB is undefined.
2. BC is undefined.
   CB is a 3×4 matrix.
3. BC is undefined.
   CB is undefined.
4. BC is undefined.
   CB is a 7×7 matrix.

Ask the audience 2/3

Suppose that A, B and C are matrices with the following sizes:

$$\begin{array}{ccc} A & B & C \\ 3 \times 4 & 4 \times 7 & 7 \times 3. \end{array}$$

Which of the following is correct?

1. BC is a 4×3 matrix.
   CB is undefined.
2. BC is undefined.
   CB is a 3×4 matrix.
3. BC is undefined.
   CB is undefined.
4. BC is undefined.
   CB is a 7×7 matrix.

Ask the audience 3/3

Suppose that A, B and C are matrices with the following sizes:

$$\begin{array}{ccc} A & B & C \\ 3 \times 4 & 4 \times 7 & 7 \times 3. \end{array}$$

Which of the following is correct?

1. AC is undefined.
   CA is a 7×4 matrix.
2. AC is undefined.
   CA is undefined.
3. AC is a 3×3 matrix.
   CA is a 7×4 matrix.
4. I didn't understand any of this.

Ask the audience 3/3

Suppose that A, B and C are matrices with the following sizes:

$$\begin{array}{ccc} A & B & C \\ 3 \times 4 & 4 \times 7 & 7 \times 3. \end{array}$$

Which of the following is correct?

1. AC is undefined. CA is a 7×4 matrix.
2. AC is undefined. CA is undefined.
3. AC is a 3×3 matrix. CA is a 7×4 matrix.
4. I didn't understand any of this.

In general, if $A = [a_{ij}]$ is an $m \times r$ matrix and $B = [b_{ij}]$ is an $r \times n$ matrix, then, as illustrated by the shading in the following display,

$$AB = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1r} \\ a_{21} & a_{22} & \cdots & a_{2r} \\ \vdots & \vdots & & \vdots \\ a_{i1} & a_{i2} & \cdots & a_{ir} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mr} \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1j} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2j} & \cdots & b_{2n} \\ \vdots & \vdots & & \vdots & & \vdots \\ b_{r1} & b_{r2} & \cdots & b_{rj} & \cdots & b_{rn} \end{bmatrix} \tag{4}$$

the entry $(AB)_{ij}$ in row i and column j of AB is given by

$$(AB)_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + a_{i3}b_{3j} + \cdots + a_{ir}b_{rj} \tag{5}$$

Formula (5) is called the ***row-column rule*** for matrix multiplication.

The Identity Matrix

Definition

A square matrix with 1's on the main diagonal and 0's everywhere else is called an *identity matrix*, and is denoted by I or I_n.

Example

$$I = I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad I = I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad I = I_4 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Note that if A is a 3×2 matrix, then

$$AI_3 = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Note that if A is a 3×2 matrix, then

$$AI_3 = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix}$$

Note that if A is a 3×2 matrix, then

$$AI_3 = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix} = A$$

Note that if A is a 3×2 matrix, then

$$AI_3 = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix} = A$$

and

$$I_2 A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix} = A.$$

Properties of Matrix Multiplication

Theorem

Let A, B and C be matrices and let r be a number. If the sizes of the matrices are correct, then

1. $A(BC) = (AB)C$

Properties of Matrix Multiplication

Theorem

Let A, B and C be matrices and let r be a number. If the sizes of the matrices are correct, then

1. $A(BC) = (AB)C$
2. $A(B + C) = AB + AC$

Properties of Matrix Multiplication

Theorem

Let A, B and C be matrices and let r be a number. If the sizes of the matrices are correct, then

1. $A(BC) = (AB)C$
2. $A(B + C) = AB + AC$
3. $(B + C)A = BA + CA$

Properties of Matrix Multiplication

Theorem

Let A, B and C be matrices and let r be a number. If the sizes of the matrices are correct, then

1. $A(BC) = (AB)C$
2. $A(B + C) = AB + AC$
3. $(B + C)A = BA + CA$
4. $r(AB) = (rA)B = A(rB)$

Properties of Matrix Multiplication

Theorem

Let A, B and C be matrices and let r be a number. If the sizes of the matrices are correct, then

1. $A(BC) = (AB)C$
2. $A(B + C) = AB + AC$
3. $(B + C)A = BA + CA$
4. $r(AB) = (rA)B = A(rB)$
5. $IA = A = AI$

Example

Let $A = \begin{bmatrix} 5 & 1 \\ 3 & -2 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 0 \\ 4 & 3 \end{bmatrix}$. Calculate AB and BA.

I leave it for you to check that

$$AB = \begin{bmatrix} 5 & 1 \\ 3 & -2 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 4 & 3 \end{bmatrix} =$$

and

$$BA = \begin{bmatrix} 2 & 0 \\ 4 & 3 \end{bmatrix} \begin{bmatrix} 5 & 1 \\ 3 & -2 \end{bmatrix} =$$

Example

Let $A = \begin{bmatrix} 5 & 1 \\ 3 & -2 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 0 \\ 4 & 3 \end{bmatrix}$. Calculate AB and BA.

I leave it for you to check that

$$AB = \begin{bmatrix} 5 & 1 \\ 3 & -2 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 4 & 3 \end{bmatrix} = \begin{bmatrix} 14 & 3 \\ -2 & -6 \end{bmatrix},$$

and

$$BA = \begin{bmatrix} 2 & 0 \\ 4 & 3 \end{bmatrix} \begin{bmatrix} 5 & 1 \\ 3 & -2 \end{bmatrix} =$$

Example

Let $A = \begin{bmatrix} 5 & 1 \\ 3 & -2 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 0 \\ 4 & 3 \end{bmatrix}$. Calculate AB and BA.

I leave it for you to check that

$$AB = \begin{bmatrix} 5 & 1 \\ 3 & -2 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 4 & 3 \end{bmatrix} = \begin{bmatrix} 14 & 3 \\ -2 & -6 \end{bmatrix},$$

and

$$BA = \begin{bmatrix} 2 & 0 \\ 4 & 3 \end{bmatrix} \begin{bmatrix} 5 & 1 \\ 3 & -2 \end{bmatrix} = \begin{bmatrix} 10 & 2 \\ 29 & -2 \end{bmatrix}.$$

Note that $AB \neq BA$.

numbers	matrices
$ab = ba$	in general, $AB \neq BA$

numbers	matrices
$ab = ba$	in general, $AB \neq BA$
$ab = ac \implies \begin{array}{c} a = 0 \\ \text{or} \\ b = c \end{array}$	$AB = AC \not\Longrightarrow \begin{array}{c} A = 0 \\ \text{or} \\ B = C \end{array}$

numbers	matrices
$ab = ba$	in general, $AB \neq BA$
$ab = ac \implies \begin{matrix} a = 0 \\ \text{or} \\ b = c \end{matrix}$	$AB = AC \;\not\Longrightarrow\; \begin{matrix} A = 0 \\ \text{or} \\ B = C \end{matrix}$
$ab = 0 \implies \begin{matrix} a = 0 \\ \text{or} \\ b = 0 \end{matrix}$	$AB = 0 \;\not\Longrightarrow\; \begin{matrix} A = 0 \\ \text{or} \\ B = 0 \end{matrix}$

Example

Let

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 2 \end{bmatrix}, \qquad B = \begin{bmatrix} 1 & 1 \\ 3 & 4 \end{bmatrix}, \qquad C = \begin{bmatrix} 2 & 5 \\ 3 & 4 \end{bmatrix}.$$

Clearly $B \neq C$ and $A \neq 0$. I leave if for you to check that

$$AB = \begin{bmatrix} 3 & 4 \\ 6 & 8 \end{bmatrix} = AC.$$

Example

Let

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 2 \end{bmatrix}, \qquad B = \begin{bmatrix} 1 & 1 \\ 3 & 4 \end{bmatrix}, \qquad C = \begin{bmatrix} 2 & 5 \\ 3 & 4 \end{bmatrix}.$$

Clearly $B \neq C$ and $A \neq 0$. I leave if for you to check that

$$AB = \begin{bmatrix} 3 & 4 \\ 6 & 8 \end{bmatrix} = AC.$$

Example

Note that

$$\begin{bmatrix} 0 & 1 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 3 & 7 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix},$$

but neither matrix on the left is the zero matrix.

Transpose of a Matrix

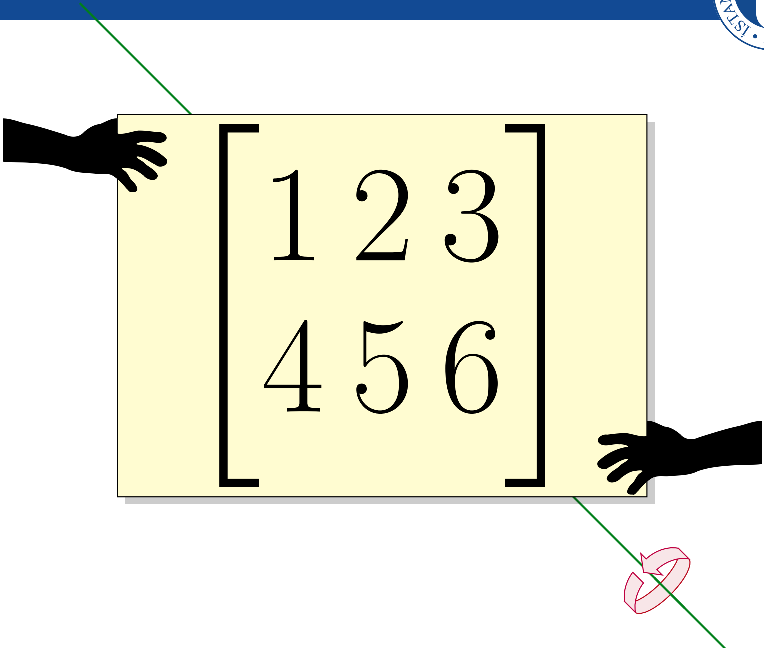
Definition

If A is any $m \times n$ matrix, then the transpose of A, denoted by A^T , is defined to be the $n \times m$ matrix that results by interchanging the rows and columns of A; that is, the first column of A^T is the first row of A, the second column of A^T is the second row of A, and so forth.

$$\begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}$$

2.1 Matrix Operations



Example

If $A = \begin{bmatrix} 2 & 3 \\ 1 & 4 \\ 5 & 6 \end{bmatrix}$, then $A^T = \begin{bmatrix} 2 & 1 & 5 \\ 3 & 4 & 6 \end{bmatrix}$.

Example

If $C = \begin{bmatrix} 1 & 3 & 5 \end{bmatrix}$, then $C^T = \begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix}$.

Example

If $D = \begin{bmatrix} 4 \end{bmatrix}$, then $D^T = \begin{bmatrix} 4 \end{bmatrix}$.

Remark

Note that since row i of A has the same entries as column i of A^T, and column j of A has the same entries as row j of A^T, we have the formula

$$\left(A^T\right)_{ij} = A_{ji}.$$

Trace of a Matrix

Definition

If A is a square matrix, then the *trace of* A, denoted $\mathrm{tr}(A)$, is defined to be the sum of the entries on the main diagonal.

(The trace of A is not defined if A is not a square matrix.)

Example

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \qquad B = \begin{bmatrix} -1 & 2 & 7 & 0 \\ 9 & 5 & 1 & 1 \\ 0 & 2 & 7 & 0 \\ 4 & 4 & 4 & 0 \end{bmatrix}$$

$$\operatorname{tr}(A) = a_{11} + a_{22} + a_{33} \qquad \operatorname{tr}(B) = -1 + 5 + 7 + 0 = 11$$

Break

We will continue at 3pm

The Inverse of a Matrix

2.2 The Inverse of a Matrix

In the real numbers, every number $a \neq 0$ has a multiplicative inverse called a^{-1}. For example, if $a = 4$, then we have $4^{-1} = \frac{1}{4}$ which satisfies

$$4 \cdot 4^{-1} = 1 \qquad \text{and} \qquad 4^{-1} \cdot 4 = 1.$$

We want to generalise this idea to <u>square</u> matrices.

Definition

Let A be a square matrix. If there exists a matrix B of the same size which satisfies

$$AB = I \qquad \text{and} \qquad BA = I$$

then we say that A is *invertible* and B is called the *inverse* of A.

If such a B does not exist, then A is called *singular*.

Remark

Note that

B is the inverse of A	$\Longrightarrow$	$AB = I = BA$	$\Longrightarrow$	A is the inverse of B

So A and B are inverses of each other.

If A is invertible, then its inverse B is invertible also.

$AB = I = B$

Example

Let

$$A = \begin{bmatrix} 2 & -5 \\ 1 & 3 \end{bmatrix} \qquad \text{and} \qquad B = \begin{bmatrix} 3 & 5 \\ 1 & 2 \end{bmatrix}.$$

$AB = I = BA$

Example

Let

$$A = \begin{bmatrix} 2 & -5 \\ 1 & 3 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 3 & 5 \\ 1 & 2 \end{bmatrix}.$$

Then

$$AB = \begin{bmatrix} 2 & -5 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} 3 & 5 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

and

$$BA = \begin{bmatrix} 3 & 5 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 2 & -5 \\ 1 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

Therefore A and B are invertible and each is the inverse of the other.

$AB = I = B$

A square matrix with a row or column or zeros is always singular.

$AB = I = B$

A square matrix with a row or column or zeros is always singular.

Example

Consider $A = \begin{bmatrix} 1 & 4 & 0 \\ 2 & 5 & 0 \\ 3 & 6 & 0 \end{bmatrix}$. I want to show that such a B does not exist.

$AB = I = B$

A square matrix with a row or column or zeros is always singular.

Example

Consider $A = \begin{bmatrix} 1 & 4 & 0 \\ 2 & 5 & 0 \\ 3 & 6 & 0 \end{bmatrix}$. I want to show that such a B does not exist. We calculate that

$$BA = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \begin{bmatrix} 1 & 4 & 0 \\ 2 & 5 & 0 \\ 3 & 6 & 0 \end{bmatrix}$$

$AB = I = B$

A square matrix with a row or column or zeros is always singular.

Example

Consider $A = \begin{bmatrix} 1 & 4 & 0 \\ 2 & 5 & 0 \\ 3 & 6 & 0 \end{bmatrix}$. I want to show that such a B does not exist. We calculate that

$$BA = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \begin{bmatrix} 1 & 4 & 0 \\ 2 & 5 & 0 \\ 3 & 6 & 0 \end{bmatrix} = \begin{bmatrix} a+2b+3c & 4a+5b+6c & 0 \\ d+2e+3f & 4d+5e+6f & 0 \\ g+2h+3i & 4g+5h+6i & 0 \end{bmatrix}.$$

The 3rd column will always be full of zeros. It is not possible to choose $a, \ldots, i$ to make $(BA)_{33} = 1$.

A square matrix with a row or column or zeros is always singular.

Example

Consider $A = \begin{bmatrix} 1 & 4 & 0 \\ 2 & 5 & 0 \\ 3 & 6 & 0 \end{bmatrix}$. I want to show that such a B does not exist. We calculate that

$$BA = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \begin{bmatrix} 1 & 4 & 0 \\ 2 & 5 & 0 \\ 3 & 6 & 0 \end{bmatrix} = \begin{bmatrix} a+2b+3c & 4a+5b+6c & 0 \\ d+2e+3f & 4d+5e+6f & 0 \\ g+2h+3i & 4g+5h+6i & 0 \end{bmatrix}.$$

The 3rd column will always be full of zeros. It is not possible to choose $a, \ldots, i$ to make $(BA)_{33} = 1$. So A does not have an inverse. Therefore A is singular.

Can an invertible matrix have more than one inverse?

$AB = I = B$

Can an invertible matrix have more than one inverse? No.

Theorem

If B and C are both inverses of A, then $B = C$.

$AB = I = B$

Can an invertible matrix have more than one inverse? No.

Theorem

If B and C are both inverses of A, then $B = C$.

Proof.

Suppose that

$$AB = I = BA \qquad \text{and} \qquad AC = I = CA.$$

$AB = I = B$

Can an invertible matrix have more than one inverse? No.

Theorem

If B and C are both inverses of A, then $B = C$.

Proof.

Suppose that

$$AB = I = BA \qquad \text{and} \qquad AC = I = CA.$$

Then we have that

$$B = BI$$

$AB = I = B$

Can an invertible matrix have more than one inverse? No.

Theorem

If B and C are both inverses of A, then $B = C$.

Proof.

Suppose that

$$AB = I = BA \qquad \text{and} \qquad AC = I = CA.$$

Then we have that

$$B = BI = B(AC)$$

$AB = I = B$

Can an invertible matrix have more than one inverse? No.

Theorem

If B and C are both inverses of A, then $B = C$.

Proof.

Suppose that

$$AB = I = BA \qquad \text{and} \qquad AC = I = CA.$$

Then we have that

$$B = BI = B(AC) = (BA)C$$

$AB = I = B$

Can an invertible matrix have more than one inverse? No.

Theorem

If B and C are both inverses of A, then $B = C$.

Proof.

Suppose that

$$AB = I = BA \qquad \text{and} \qquad AC = I = CA.$$

Then we have that

$$B = BI = B(AC) = (BA)C = IC$$

$AB = I = BA$

Can an invertible matrix have more than one inverse? No.

Theorem

If B and C are both inverses of A, then $B = C$.

Proof.

Suppose that

$$AB = I = BA \qquad \text{and} \qquad AC = I = CA.$$

Then we have that

$$B = BI = B(AC) = (BA)C = IC = C.$$

□

Definition

~~An~~ The inverse of A is denoted by A^{-1}. Thus

$$\boxed{AA^{-1} = I = A^{-1}A.}$$

Theorem

The matrix

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

is invertible if and only if $ad - bc \neq 0$, *in which case its inverse is given by*

$$A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

We will generalise this to bigger square matrices later in the course.

$$ad - bc \neq 0 \qquad A^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

Definition

The number $ad - bc$ is called the *determinant* of $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and is denoted by

$$\det(A) = ad - bc$$

or by

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc.$$

$$\det(A) = ad - bc \neq 0 \qquad A^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

Example

Is $B = \begin{bmatrix} 6 & 1 \\ 5 & 2 \end{bmatrix}$ invertible? If so, find its inverse.

$\det(A) = ad - bc \neq 0 \qquad A^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

Example

Is $B = \begin{bmatrix} 6 & 1 \\ 5 & 2 \end{bmatrix}$ invertible? If so, find its inverse.

The determinant of B is

$$\det(B) = 6 \cdot 2 - 1 \cdot 5 = 7 \neq 0.$$

Therefore B is invertible.

$$\det(A) = ad - bc \neq 0 \qquad A^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

Example

Is $B = \begin{bmatrix} 6 & 1 \\ 5 & 2 \end{bmatrix}$ invertible? If so, find its inverse.

The determinant of B is

$$\det(B) = 6 \cdot 2 - 1 \cdot 5 = 7 \neq 0.$$

Therefore B is invertible. Its inverse is

$$B^{-1} = \frac{1}{7}\begin{bmatrix} 2 & -1 \\ -5 & 6 \end{bmatrix} = \begin{bmatrix} \frac{2}{7} & -\frac{1}{7} \\ -\frac{5}{7} & \frac{6}{7} \end{bmatrix}.$$

$$\det(A) = ad - bc \neq 0 \qquad A^{-1} = \frac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

Example

Is $C = \begin{bmatrix} -1 & 2 \\ 3 & -6 \end{bmatrix}$ invertible? If so, find its inverse.

$$\det(A) = ad - bc \neq 0 \qquad A^{-1} = \frac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

Example

Is $C = \begin{bmatrix} -1 & 2 \\ 3 & -6 \end{bmatrix}$ invertible? If so, find its inverse.

The determinant of C is

$$\det(C) = (-1) \cdot (-6) - 2 \cdot 3 = 0.$$

Therefore C is not invertible. C is singular.

$$B = \begin{bmatrix} 6 & 1 \\ 5 & 2 \end{bmatrix} \qquad B^{-1} = \begin{bmatrix} \frac{2}{7} & -\frac{1}{7} \\ -\frac{5}{7} & \frac{6}{7} \end{bmatrix}$$

Inverse matrices can be used to solve linear systems.

Example

Solve $\begin{cases} 6x_1 + x_2 = 2 \\ 5x_1 + 2x_2 = 3. \end{cases}$

$$B = \begin{bmatrix} 6 & 1 \\ 5 & 2 \end{bmatrix} \qquad B^{-1} = \begin{bmatrix} \frac{2}{7} & -\frac{1}{7} \\ -\frac{5}{7} & \frac{6}{7} \end{bmatrix}$$

Inverse matrices can be used to solve linear systems.

Example

Solve $\begin{cases} 6x_1 + x_2 = 2 \\ 5x_1 + 2x_2 = 3. \end{cases}$

We can write this as

$$\begin{bmatrix} 6x_1 + x_2 \\ 5x_1 + 2x_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

$$B = \begin{bmatrix} 6 & 1 \\ 5 & 2 \end{bmatrix} \qquad B^{-1} = \begin{bmatrix} \frac{2}{7} & -\frac{1}{7} \\ -\frac{5}{7} & \frac{6}{7} \end{bmatrix}$$

Inverse matrices can be used to solve linear systems.

Example

Solve $\begin{cases} 6x_1 + x_2 = 2 \\ 5x_1 + 2x_2 = 3. \end{cases}$

We can write this as

$$\begin{bmatrix} 6x_1 + x_2 \\ 5x_1 + 2x_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

or as

$$\begin{bmatrix} 6 & 1 \\ 5 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}.$$

$$B = \begin{bmatrix} 6 & 1 \\ 5 & 2 \end{bmatrix} \qquad B^{-1} = \begin{bmatrix} \frac{2}{7} & -\frac{1}{7} \\ -\frac{5}{7} & \frac{6}{7} \end{bmatrix}$$

Inverse matrices can be used to solve linear systems.

Example

Solve $\begin{cases} 6x_1 + x_2 = 2 \\ 5x_1 + 2x_2 = 3. \end{cases}$

We can write this as

$$\begin{bmatrix} 6x_1 + x_2 \\ 5x_1 + 2x_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

or as

$$\begin{bmatrix} 6 & 1 \\ 5 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}.$$

Multiplying both sides on the left by B^{-1} gives

$$\begin{bmatrix} \frac{2}{7} & -\frac{1}{7} \\ -\frac{5}{7} & \frac{6}{7} \end{bmatrix} \begin{bmatrix} 6 & 1 \\ 5 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \frac{2}{7} & -\frac{1}{7} \\ -\frac{5}{7} & \frac{6}{7} \end{bmatrix} \begin{bmatrix} 2 \\ 3 \end{bmatrix} =$$

$$B = \begin{bmatrix} 6 & 1 \\ 5 & 2 \end{bmatrix} \qquad B^{-1} = \begin{bmatrix} \frac{2}{7} & -\frac{1}{7} \\ -\frac{5}{7} & \frac{6}{7} \end{bmatrix}$$

Inverse matrices can be used to solve linear systems.

Example

Solve $\begin{cases} 6x_1 + \ \ x_2 = 2 \\ 5x_1 + 2x_2 = 3. \end{cases}$

We can write this as

$$\begin{bmatrix} 6x_1 + x_2 \\ 5x_1 + 2x_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

or as

$$\begin{bmatrix} 6 & 1 \\ 5 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}.$$

Multiplying both sides on the left by B^{-1} gives

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \frac{2}{7} & -\frac{1}{7} \\ -\frac{5}{7} & \frac{6}{7} \end{bmatrix} \begin{bmatrix} 6 & 1 \\ 5 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \frac{2}{7} & -\frac{1}{7} \\ -\frac{5}{7} & \frac{6}{7} \end{bmatrix} \begin{bmatrix} 2 \\ 3 \end{bmatrix} =$$

$$B = \begin{bmatrix} 6 & 1 \\ 5 & 2 \end{bmatrix} \qquad B^{-1} = \begin{bmatrix} \frac{2}{7} & -\frac{1}{7} \\ -\frac{5}{7} & \frac{6}{7} \end{bmatrix}$$

Inverse matrices can be used to solve linear systems.

Example

Solve $\begin{cases} 6x_1 + x_2 = 2 \\ 5x_1 + 2x_2 = 3. \end{cases}$

We can write this as

$$\begin{bmatrix} 6x_1 + x_2 \\ 5x_1 + 2x_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

or as

$$\begin{bmatrix} 6 & 1 \\ 5 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}.$$

Multiplying both sides on the left by B^{-1} gives

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \frac{2}{7} & -\frac{1}{7} \\ -\frac{5}{7} & \frac{6}{7} \end{bmatrix} \begin{bmatrix} 6 & 1 \\ 5 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \frac{2}{7} & -\frac{1}{7} \\ -\frac{5}{7} & \frac{6}{7} \end{bmatrix} \begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} \frac{1}{7} \\ \frac{8}{7} \end{bmatrix}.$$

Remark

In general, the linear system

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots a_{mn}x_n = b_m. \end{cases}$$

can be written as the matrix equation

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

Remark

In general, the linear system

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots a_{mn}x_n = b_m. \end{cases}$$

can be written as the matrix equation

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

or just

$$A\mathbf{x} = \mathbf{b}.$$

Remark

Clearly, if A is a square matrix and is invertible, then the solution to

$$A\mathbf{x} = \mathbf{b}$$

is

$$\mathbf{x} = A^{-1}\mathbf{b}.$$

Theorem

If A and B are invertible matrices of the same size, then AB is invertible and

$$(AB)^{-1} = B^{-1}A^{-1}.$$

Theorem

If A and B are invertible matrices of the same size, then AB is invertible and

$$(AB)^{-1} = B^{-1}A^{-1}.$$

Proof.

We calculate that

$$(AB)(B^{-1}A^{-1}) =$$

Theorem

If A and B are invertible matrices of the same size, then AB is invertible and

$$(AB)^{-1} = B^{-1}A^{-1}.$$

Proof.

We calculate that

$$(AB)(B^{-1}A^{-1}) = A(BB^{-1})A^{-1}$$

2.2 The Inverse of a Matrix

Theorem

If A and B are invertible matrices of the same size, then AB is invertible and

$$(AB)^{-1} = B^{-1}A^{-1}.$$

Proof.

We calculate that

$$(AB)(B^{-1}A^{-1}) = A(BB^{-1})A^{-1} = AIA^{-1}$$

Theorem

If A and B are invertible matrices of the same size, then AB is invertible and

$$(AB)^{-1} = B^{-1}A^{-1}.$$

Proof.

We calculate that

$$(AB)(B^{-1}A^{-1}) = A(BB^{-1})A^{-1} = AIA^{-1} = AA^{-1}$$

2.2 The Inverse of a Matrix

Theorem

If A and B are invertible matrices of the same size, then AB is invertible and

$$(AB)^{-1} = B^{-1}A^{-1}.$$

Proof.

We calculate that

$$(AB)(B^{-1}A^{-1}) = A(BB^{-1})A^{-1} = AIA^{-1} = AA^{-1} = I.$$

Theorem

If A and B are invertible matrices of the same size, then AB is invertible and

$$(AB)^{-1} = B^{-1}A^{-1}.$$

Proof.

We calculate that

$$(AB)(B^{-1}A^{-1}) = A(BB^{-1})A^{-1} = AIA^{-1} = AA^{-1} = I.$$

Similarly

$$(B^{-1}A^{-1})(AB) = B^{-1}(A^{-1}A)B = B^{-1}IB = B^{-1}B = I.$$

Therefore AB is invertible and $B^{-1}A^{-1}$ is the inverse of AB. $\square$

Remark

This is also true if we have more than 2 matrices. For example

$$(ABC)^{-1} = C^{-1}B^{-1}A^{-1},$$

$$(ABCD)^{-1} = D^{-1}C^{-1}B^{-1}A^{-1},$$

$$\vdots$$

Example

Let

$$A = \begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix} \qquad \text{and} \qquad B = \begin{bmatrix} 3 & 2 \\ 2 & 2 \end{bmatrix}.$$

I leave it for you to check the following:

$$AB = \begin{bmatrix} 7 & 6 \\ 9 & 8 \end{bmatrix} \qquad A^{-1} = \begin{bmatrix} 3 & -2 \\ -1 & 1 \end{bmatrix}$$

$$(AB)^{-1} = \begin{bmatrix} 4 & -3 \\ -\frac{9}{2} & \frac{7}{2} \end{bmatrix} \qquad B^{-1} = \begin{bmatrix} 1 & -1 \\ -1 & \frac{3}{2} \end{bmatrix}$$

$$B^{-1}A^{-1} = \begin{bmatrix} 1 & -1 \\ -1 & \frac{3}{2} \end{bmatrix} \begin{bmatrix} 3 & -2 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 4 & -3 \\ -\frac{9}{2} & \frac{7}{2} \end{bmatrix}$$

Powers of a Matrix

Definition

Let A is a square matrix and let $k \in \mathbb{N}$. We define

$$A^0 = I \qquad \text{and} \qquad A^k = \underbrace{A \cdot A \cdot \ldots \cdot A}_{k}.$$

Powers of a Matrix

Definition

Let A is a square matrix and let $k \in \mathbb{N}$. We define

$$A^0 = I \qquad \text{and} \qquad A^k = \underbrace{A \cdot A \cdot \ldots \cdot A}_{k}.$$

If A is invertible, then we define

$$A^{-k} = (A^{-1})^k = \underbrace{A^{-1} \cdot A^{-1} \cdot \ldots \cdot A^{-1}}_{k}.$$

Powers of a Matrix

Definition

Let A is a square matrix and let $k \in \mathbb{N}$. We define

$$A^0 = I \qquad \text{and} \qquad A^k = \underbrace{A \cdot A \cdot \ldots \cdot A}_{k}.$$

If A is invertible, then we define

$$A^{-k} = (A^{-1})^k = \underbrace{A^{-1} \cdot A^{-1} \cdot \ldots \cdot A^{-1}}_{k}.$$

Note that

$$A^r A^s = A^{r+s} \qquad \text{and} \qquad (A^r)^s = A^{rs}$$

if $r, s \geq 0$ as expected.

2.2 The Inverse of a Matrix

Theorem

Let A be an invertible matrix. Let $n \in \mathbb{N} \cup \{0\}$ and $k \neq 0$. Then:

1. A^{-1} *is invertible and* $(A^{-1})^{-1} = A$.

Theorem

Let A be an invertible matrix. Let $n \in \mathbb{N} \cup \{0\}$ and $k \neq 0$. Then:

1. A^{-1} *is invertible and* $(A^{-1})^{-1} = A$.
2. A^n *is invertible and* $(A^n)^{-1} = A^{-n} = (A^{-1})^n$.

Theorem

Let A be an invertible matrix. Let $n \in \mathbb{N} \cup \{0\}$ and $k \neq 0$. Then:

1. A^{-1} *is invertible and* $(A^{-1})^{-1} = A$.
2. A^n *is invertible and* $(A^n)^{-1} = A^{-n} = (A^{-1})^n$.
3. kA *is invertible and* $(kA)^{-1} = k^{-1}A^{-1}$.

I will prove part 3. I leave parts 1 and 2 for you to prove.

Theorem

Let A be an invertible matrix. Let $n \in \mathbb{N} \cup \{0\}$ and $k \neq 0$. Then:

1. *A^{-1} is invertible and $(A^{-1})^{-1} = A$.*
2. *A^n is invertible and $(A^n)^{-1} = A^{-n} = (A^{-1})^n$.*
3. *kA is invertible and $(kA)^{-1} = k^{-1}A^{-1}$.*

I will prove part 3. I leave parts 1 and 2 for you to prove.

Proof of Part 3.

We have that

$$(kA)(k^{-1}A^{-1}) = k^{-1}(kA)A^{-1} = (k^1 k)(AA^{-1}) = (1)(I) = I.$$

Similarly $(k^{-1}A^{-1})(kA) = I$. $\square$

Example

Let $A = \begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix}$ and $A^{-1} = \begin{bmatrix} 3 & -2 \\ -1 & 1 \end{bmatrix}$. Then

$$\begin{aligned} A^{-3} = A^{-1}A^{-1}A^{-1} &= \begin{bmatrix} 3 & -2 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 3 & -2 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 3 & -2 \\ -1 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 41 & -30 \\ -15 & 11 \end{bmatrix}. \end{aligned}$$

Example

Let $A = \begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix}$ and $A^{-1} = \begin{bmatrix} 3 & -2 \\ -1 & 1 \end{bmatrix}$. Then

$$\begin{aligned} A^{-3} = A^{-1}A^{-1}A^{-1} &= \begin{bmatrix} 3 & -2 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 3 & -2 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 3 & -2 \\ -1 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 41 & -30 \\ -15 & 11 \end{bmatrix}. \end{aligned}$$

Moreover

$$A^3 = AAA = \begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix} = \begin{bmatrix} 11 & 30 \\ 15 & 41 \end{bmatrix}$$

Example

Let $A = \begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix}$ and $A^{-1} = \begin{bmatrix} 3 & -2 \\ -1 & 1 \end{bmatrix}$. Then

$$\begin{aligned} A^{-3} = A^{-1}A^{-1}A^{-1} &= \begin{bmatrix} 3 & -2 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 3 & -2 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 3 & -2 \\ -1 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 41 & -30 \\ -15 & 11 \end{bmatrix}. \end{aligned}$$

Moreover

$$A^3 = AAA = \begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix} = \begin{bmatrix} 11 & 30 \\ 15 & 41 \end{bmatrix}$$

and

$$(A^3)^{-1} = \frac{1}{(11 \cdot 41 - 30 \cdot 15)} \begin{bmatrix} 41 & -30 \\ -15 & 11 \end{bmatrix} = \begin{bmatrix} 41 & -30 \\ -15 & 11 \end{bmatrix} = (A^{-1})^3$$

as expected.

Remark

If a and b are numbers then $ab = ba$, so we can calculate

$$(a+b)^2 = a^2 + ab + ba + b^2 = a^2 + ab + ab + b^2 = a^2 + 2ab + b^2.$$

Remark

If a and b are numbers then $ab = ba$, so we can calculate

$$(a+b)^2 = a^2 + ab + ba + b^2 = a^2 + ab + ab + b^2 = a^2 + 2ab + b^2.$$

But remember that $AB \neq BA$ in general for matrices. So the best that we can do is

$$(A+B)^2 = A^2 + AB + BA + B^2.$$

Properties of the Transpose

Theorem

(If the sizes are correct, then)

1. $(A^T)^T = A$
2. $(A+B)^T = A^T + B^T$
3. $(kA)^T = kA^T$

Properties of the Transpose

Theorem

(If the sizes are correct, then)

1. $(A^T)^T = A$
2. $(A+B)^T = A^T + B^T$
3. $(kA)^T = kA^T$
4. $(AB)^T = B^T A^T$

Properties of the Transpose

Theorem

(If the sizes are correct, then)

1. $(A^T)^T = A$
2. $(A+B)^T = A^T + B^T$
3. $(kA)^T = kA^T$
4. $(AB)^T = B^T A^T$

Theorem

If A is an invertible matrix, then A^T is also invertible and

$$(A^T)^{-1} = (A^{-1})^T.$$

Elementary Matrices

Definition

An *elementary matrix* is one that is obtained by performing a single elementary row operation on I.

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

identity matrix

elementary matrix

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -4 & 0 & 1 \end{bmatrix} \xleftarrow{-3R_1 + R_3} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

identity matrix

elementary matrix　　　　　　　　　　elementary matrix

$$\begin{tikzcd}
\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -4 & 0 & 1 \end{bmatrix} & & \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \\
& \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \arrow[ul, "-3R_1 + R_3"] \arrow[ur, "R_1 \leftrightarrow R_2"'] &
\end{tikzcd}$$

identity matrix

2.2 The Inverse of a Matrix

elementary matrix

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{-3R_1 + R_3} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -4 & 0 & 1 \end{bmatrix}$$

elementary matrix

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{R_1 \leftrightarrow R_2} \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

identity matrix

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{5R_3} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 5 \end{bmatrix}$$

elementary matrix

$$\underset{\text{identity matrix}}{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}} \xrightarrow{-3R_1 + R_3} \underset{\text{elementary matrix}}{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -4 & 0 & 1 \end{bmatrix}}$$

1 elementary row
operation away
from I

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{-3R_1 + R_3} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -4 & 0 & 1 \end{bmatrix} \xrightarrow{R_1 \leftrightarrow R_2} \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ -4 & 0 & 1 \end{bmatrix}$$

identity matrix

elementary matrix

not an elementary matrix

1 elementary row operation away from I

2 elementary row operations away from I

Example

Let

$$E_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -4 & 0 & 1 \end{bmatrix}, \quad E_2 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad E_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 5 \end{bmatrix},$$

$$A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}.$$

Calculate E_1A, E_2A and E_3A.

I leave it for you to check that

$$E_1A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -4 & 0 & 1 \end{bmatrix} \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} = \begin{bmatrix} a & b & c \\ d & e & f \\ g-4a & h-4b & i-4c \end{bmatrix}$$

$$E_2A = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} = \begin{bmatrix} d & e & f \\ a & b & c \\ g & h & i \end{bmatrix}$$

$$E_3A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 5 \end{bmatrix} \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} = \begin{bmatrix} a & b & c \\ d & e & f \\ 5g & 5h & 5i \end{bmatrix}.$$

$$E_1 A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -4 & 0 & 1 \end{bmatrix} \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} = \begin{bmatrix} a & b & c \\ d & e & f \\ g-4a & h-4b & i-4c \end{bmatrix}$$

But note that

$$I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{-4R_1+R_3} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -4 & 0 & 1 \end{bmatrix} = E_1$$

and

$$A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \xrightarrow{-4R_1+R_3} \begin{bmatrix} a & b & c \\ d & e & f \\ g-4a & h-4b & i-4c \end{bmatrix} = E_1 A.$$

Similarly (please check)

$$I \xrightarrow{R_1 \leftrightarrow R_2} E_2 \qquad \text{and} \qquad A \xrightarrow{R_1 \leftrightarrow R_2} E_2 A$$

$$I \xrightarrow{5R_3} E_3 \qquad \text{and} \qquad A \xrightarrow{5R_3} E_3 A$$

Similarly (please check)

$$I \xrightarrow{R_1 \leftrightarrow R_2} E_2 \qquad \text{and} \qquad A \xrightarrow{R_1 \leftrightarrow R_2} E_2 A$$

$$I \xrightarrow{5R_3} E_3 \qquad \text{and} \qquad A \xrightarrow{5R_3} E_3 A$$

Remark

Multiplying (on the left) by an elementary matrix is the same as doing the equivalent elementary row operation.

Similarly (please check)

$$I \xrightarrow{R_1 \leftrightarrow R_2} E_2 \qquad \text{and} \qquad A \xrightarrow{R_1 \leftrightarrow R_2} E_2 A$$

$$I \xrightarrow{5R_3} E_3 \qquad \text{and} \qquad A \xrightarrow{5R_3} E_3 A$$

Remark

Multiplying (on the left) by an elementary matrix is the same as doing the equivalent elementary row operation.

Remark

Since every elementary row operation is reversible, every elementary matrix is invertible.

Theorem

An $n \times n$ matrix A is invertible if and only if A is row equivalent to I_n.

Theorem

An $n \times n$ matrix A is invertible if and only if A is row equivalent to I_n.

If A is invertible, then any sequence of elementary row operations that reduces A to I_n also transforms I_n into A^{-1}.

(proof in book)

An algorithm for finding A^{-1}

1. Glue A and I together side-by-side to form $[A\ I]$.

An algorithm for finding A^{-1}

1. Glue A and I together side-by-side to form $[A\ I]$.
2. Use Gauss-Jordan Elimination to reduce this augmented matrix to RREF.

An algorithm for finding A^{-1}

1. Glue A and I together side-by-side to form $[A\ I]$.
2. Use Gauss-Jordan Elimination to reduce this augmented matrix to RREF.
3. If A is invertible, then you will obtain $[I\ A^{-1}]$. If you don't get this, then you know that A is singular.

Example

Find the inverse of the matrix $A = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 0 & 3 \\ 4 & -3 & 8 \end{bmatrix}$, if it exists.

Example

Find the inverse of the matrix $A = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 0 & 3 \\ 4 & -3 & 8 \end{bmatrix}$, if it exists.

We start with

$$[A \ I] = \begin{bmatrix} 0 & 1 & 2 & 1 & 0 & 0 \\ 1 & 0 & 3 & 0 & 1 & 0 \\ 4 & -3 & 8 & 0 & 0 & 1 \end{bmatrix}$$

and we need to row reduce this to RREF.

$$[A\ \ I] = \begin{bmatrix} 0 & 1 & 2 & 1 & 0 & 0 \\ 1 & 0 & 3 & 0 & 1 & 0 \\ 4 & -3 & 8 & 0 & 0 & 1 \end{bmatrix} \xrightarrow{R_1 \leftrightarrow R_2} \begin{bmatrix} 1 & 0 & 3 & 0 & 1 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 \\ 4 & -3 & 8 & 0 & 0 & 1 \end{bmatrix}$$

$$\xrightarrow{-4R_1+R_3} \begin{bmatrix} 1 & 0 & 3 & 0 & 1 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 \\ 0 & -3 & -4 & 0 & -4 & 1 \end{bmatrix} \xrightarrow{3R_2+R_3} \begin{bmatrix} 1 & 0 & 3 & 0 & 1 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 \\ 0 & 0 & 2 & 3 & -4 & 1 \end{bmatrix}$$

$$\xrightarrow{-2R_1} \begin{bmatrix} -2 & 0 & -6 & 0 & -2 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 \\ 0 & 0 & 2 & 3 & -4 & 1 \end{bmatrix} \xrightarrow[-R_3+R_2]{3R_3+R_1} \begin{bmatrix} -2 & 0 & 0 & 9 & -14 \\ 0 & 1 & 0 & -2 & 4 \\ 0 & 0 & 2 & 3 & -4 \end{bmatrix}$$

$$\xrightarrow[\frac{1}{2}R_3]{-\frac{1}{2}R_1} \begin{bmatrix} 1 & 0 & 0 & -\frac{9}{2} & 7 & -\frac{3}{2} \\ 0 & 1 & 0 & -2 & 4 & -1 \\ 0 & 0 & 1 & \frac{3}{2} & -2 & \frac{1}{2} \end{bmatrix} = [I\ \ A^{-1}].$$

Since

$$\begin{bmatrix} 0 & 1 & 2 & 1 & 0 & 0 \\ 1 & 0 & 3 & 0 & 1 & 0 \\ 4 & -3 & 8 & 0 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & -\frac{9}{2} & 7 & -\frac{3}{2} \\ 0 & 1 & 0 & -2 & 4 & -1 \\ 0 & 0 & 1 & \frac{3}{2} & -2 & \frac{1}{2} \end{bmatrix}$$

we have that

$$A^{-1} = \begin{bmatrix} -\frac{9}{2} & 7 & -\frac{3}{2} \\ -2 & 4 & -1 \\ \frac{3}{2} & -2 & \frac{1}{2} \end{bmatrix}.$$

Let's just check our answer to make sure that we didn't make a mistake in our calculation:

$$AA^{-1} = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 0 & 3 \\ 4 & -3 & 8 \end{bmatrix} \begin{bmatrix} -\frac{9}{2} & 7 & -\frac{3}{2} \\ -2 & 4 & -1 \\ \frac{3}{2} & -2 & \frac{1}{2} \end{bmatrix}$$

Let's just check our answer to make sure that we didn't make a mistake in our calculation:

$$AA^{-1} = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 0 & 3 \\ 4 & -3 & 8 \end{bmatrix} \begin{bmatrix} -\frac{9}{2} & 7 & -\frac{3}{2} \\ -2 & 4 & -1 \\ \frac{3}{2} & -2 & \frac{1}{2} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}. \qquad \checkmark$$

Example

Does $A = \begin{bmatrix} 1 & -2 & 3 \\ 0 & 1 & 4 \\ -2 & 5 & -2 \end{bmatrix}$ have an inverse?

Example

Does $A = \begin{bmatrix} 1 & -2 & 3 \\ 0 & 1 & 4 \\ -2 & 5 & -2 \end{bmatrix}$ have an inverse?

$$\begin{bmatrix} 1 & -2 & 3 & 1 & 0 & 0 \\ 0 & 1 & 4 & 0 & 1 & 0 \\ -2 & 5 & -2 & 0 & 0 & 1 \end{bmatrix} \xrightarrow{2R_1+R_3} \begin{bmatrix} 1 & -2 & 3 & 1 & 0 & 0 \\ 0 & 1 & 4 & 0 & 1 & 0 \\ 0 & 1 & 4 & 2 & 0 & 1 \end{bmatrix}$$

$$\xrightarrow{-R_2+R_3} \begin{bmatrix} 1 & -2 & 3 & 1 & 0 & 0 \\ 0 & 1 & 4 & 0 & 1 & 0 \\ 0 & 0 & 0 & 2 & -1 & 1 \end{bmatrix}$$

Example

Does $A = \begin{bmatrix} 1 & -2 & 3 \\ 0 & 1 & 4 \\ -2 & 5 & -2 \end{bmatrix}$ have an inverse?

$$\left[\begin{array}{cccccc} 1 & -2 & 3 & 1 & 0 & 0 \\ 0 & 1 & 4 & 0 & 1 & 0 \\ -2 & 5 & -2 & 0 & 0 & 1 \end{array}\right] \xrightarrow{2R_1+R_3} \left[\begin{array}{cccccc} 1 & -2 & 3 & 1 & 0 & 0 \\ 0 & 1 & 4 & 0 & 1 & 0 \\ 0 & 1 & 4 & 2 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{-R_2+R_3} \left[\begin{array}{cccccc} 1 & -2 & 3 & 1 & 0 & 0 \\ 0 & 1 & 4 & 0 & 1 & 0 \\ 0 & 0 & 0 & 2 & -1 & 1 \end{array}\right]$$

Since the first three entries in R_3 are zeros, we cannot row reduce A to I. This means that A does not have an inverse.

Next Time

- 2.3 Characterisations of Invertible Matrices
- 0.0 Diagonal, triangular, and symmetric matrices
- 1.6 Applications of linear systems.
- 3.1 Introduction to Determinants