Lecture 7

- 14.1 Double and Iterated Integrals over Rectangles
- 14.2 Double Integrals over General Regions
- 14.3 Area by Double Integration
- 10.3 Polar Coordinates

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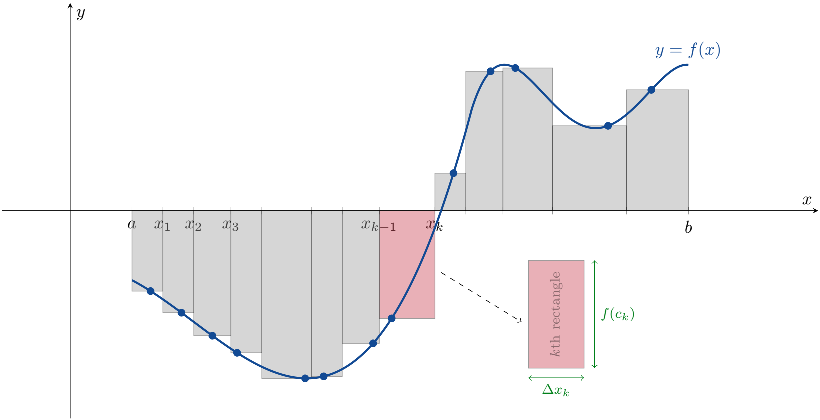
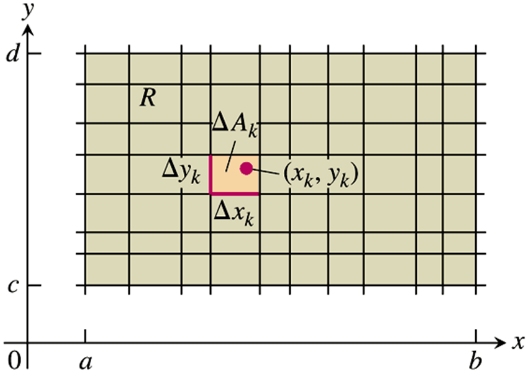
Double and Iterated Integrals over Rectangles

Riemann Sums



$$\int_a^b f(x)\,dx = \lim_{\|P\|\to 0} \sum_{k=1}^{n} f(c_k)\Delta x_k.$$

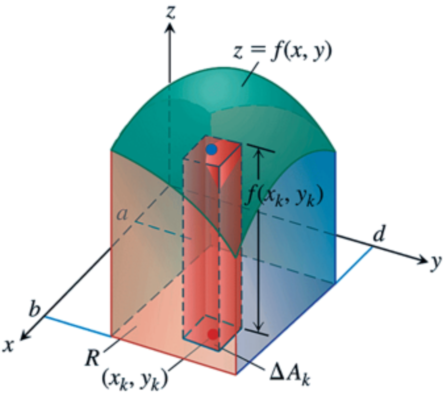
Riemann Sums for Double Integrals



Partition a rectangle R into small rectangles of area

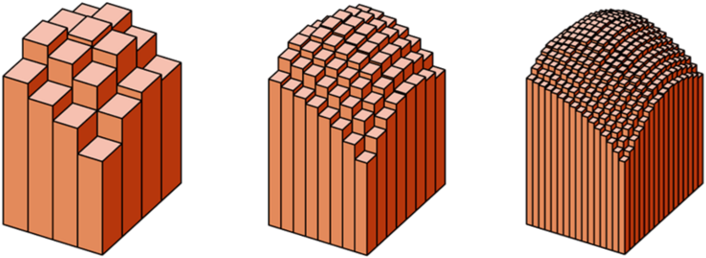
$$\Delta A_k = \Delta x_k \Delta y_k.$$

Riemann Sums for Double Integrals



Then approximate the graph of $z = f(x, y)$ by drawing a cuboid on each small rectangle.

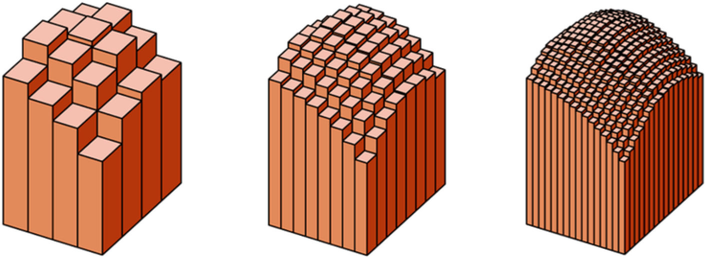
Riemann Sums for Double Integrals



(a) $n = 16$ (b) $n = 64$ (c) $n = 256$

$$\sum_k f(x_k, y_k)\Delta A_k.$$

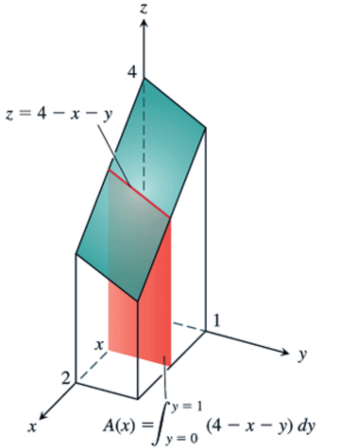
Riemann Sums for Double Integrals



(a) $n = 16$ (b) $n = 64$ (c) $n = 256$

Then take the limit,

$$\iint\limits_R f\, dA = \lim_{\|P\| \to 0} \sum_k f(x_k, y_k)\Delta A_k.$$



Example

Calculate the volume under the plane $z = 4 - x - y$ over the rectangle $R = \{(x, y)\,|\,0 \leq x \leq 2,\ 0 \leq y \leq 1\}$.

Since $R = \{(x,y) \,|\, 0 \le x \le 2,\ 0 \le y \le 1\}$, we calculate that

$$\iint\limits_R (4 - x - y)\, dA = \int_{x=0}^{x=2} \int_{y=0}^{y=1} (4 - x - y)\, dydx$$

$$=$$

$$=$$

$$=$$

$$=$$

Since $R = \{(x,y)\,|\, 0 \leq x \leq 2,\ 0 \leq y \leq 1\}$, we calculate that

$$\begin{aligned}
\iint\limits_R (4-x-y)\,dA &= \int_{x=0}^{x=2}\int_{y=0}^{y=1} (4-x-y)\,dydx \\
&= \int_{x=0}^{x=2} \left(\int_{y=0}^{y=1} (4-x-y)\,dy \right) dx \\
&= \\
&= \\
&=
\end{aligned}$$

Since $R = \{(x,y) \,|\, 0 \leq x \leq 2,\ 0 \leq y \leq 1\}$, we calculate that

$$\begin{aligned}
\iint\limits_R (4-x-y)\,dA &= \int_{x=0}^{x=2} \int_{y=0}^{y=1} (4-x-y)\,dydx \\
&= \int_{x=0}^{x=2} \left(\int_{y=0}^{y=1} (4-x-y)\,dy \right) dx \\
&= \int_{x=0}^{x=2} \left[4y - xy - \frac{y^2}{2} \right]_{y=0}^{y=1} dx \\
&= \\
&=
\end{aligned}$$

Since $R = \{(x,y)\,|\,0 \leq x \leq 2,\ 0 \leq y \leq 1\}$, we calculate that

$$\begin{aligned}
\iint_R (4-x-y)\,dA &= \int_{x=0}^{x=2}\int_{y=0}^{y=1}(4-x-y)\,dydx \\
&= \int_{x=0}^{x=2}\left(\int_{y=0}^{y=1}(4-x-y)\,dy\right)dx \\
&= \int_{x=0}^{x=2}\left[4y-xy-\frac{y^2}{2}\right]_{y=0}^{y=1}dx \\
&= \int_{x=0}^{x=2}\left(\frac{7}{2}-x\right)dx \\
&=
\end{aligned}$$

Since $R = \{(x,y) \,|\, 0 \le x \le 2,\ 0 \le y \le 1\}$, we calculate that

$$\begin{aligned}
\iint\limits_R (4 - x - y)\, dA &= \int_{x=0}^{x=2} \int_{y=0}^{y=1} (4 - x - y)\, dydx \\
&= \int_{x=0}^{x=2} \left(\int_{y=0}^{y=1} (4 - x - y)\, dy \right) dx \\
&= \int_{x=0}^{x=2} \left[4y - xy - \frac{y^2}{2} \right]_{y=0}^{y=1} dx \\
&= \int_{x=0}^{x=2} \left(\frac{7}{2} - x \right) dx \\
&= \left[\frac{7}{2}x - \frac{x^2}{2} \right]_{x=0}^{x=2} = 5.
\end{aligned}$$

Remark

You don't need to write "$x =$" and "$y =$" if understand which integral is which:

$$\int_{x=0}^{x=2}\int_{y=0}^{y=1}(4-x-y)\,dy\,dx = \int_0^2\int_0^1 (4-x-y)\,dy\,dx.$$

Remark

You don't need to write "$x =$" and "$y =$" if understand which integral is which:

$$\int_{x=0}^{x=2}\int_{y=0}^{y=1}(4-x-y)\,dy\,dx = \int_0^2\int_0^1(4-x-y)\,dy\,dx.$$

Remark

You don't need to write "$x =$" and "$y =$" if understand which integral is which:

$$\int_{x=0}^{x=2} \int_{y=0}^{y=1} (4 - x - y)\, dy dx = \int_0^2 \int_0^1 (4 - x - y)\, dy dx.$$

Previously we did

$$\int_0^2 \int_0^1 (4 - x - y)\, dy dx = 5.$$

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$$\int_0^2 \int_0^1 (4 - x - y)\, dy dx = 5.$$

But note that

$$\begin{aligned}\int_0^1 \int_0^2 (4 - x - y)\, dx dy &= \int_0^1 \left(\int_0^2 (4 - x - y)\, dx \right) dy \\ &= \int_0^1 \left[4x - \frac{x^2}{2} - xy \right]_0^2 dy \\ &= \int_0^1 (6 - 2y)\, dy \\ &= \left[6y - y^2 \right]_0^1 = 5\end{aligned}$$

also.

Guido Fubini

BORN
19 January 1879

DECEICED
6 June 1943

NATIONALITY
Italian

Theorem (Fubini's Theorem)

If $f(x, y)$ is continuous on the rectangle

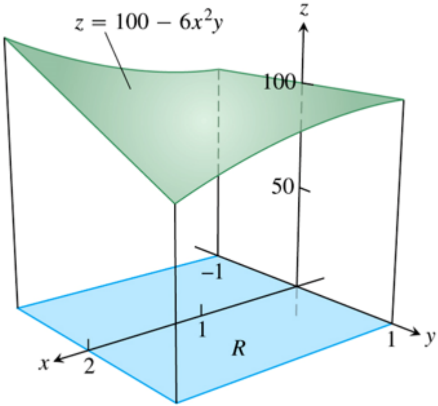
$$R = \{(x, y) \,|\, a \le x \le b,\ c \le y \le d\} = [a, b] \times [c, d],$$

then

$$\iint\limits_R f(x, y)\, dA = \int_a^b \int_c^d f(x, y)\, dy dx = \int_c^d \int_a^b f(x, y)\, dx dy.$$

EXAMPLE 1 Calculate $\iint_R f(x, y)\, dA$ for

$$f(x, y) = 100 - 6x^2y \quad \text{and} \quad R: \ 0 \le x \le 2, \ -1 \le y \le 1.$$



Solution Figure 15.6 displays the volume beneath the surface. By Fubini's Theorem,

$$\iint_R f(x, y)\, dA = \int_{-1}^{1}\int_{0}^{2} (100 - 6x^2y)\, dx\, dy = \int_{-1}^{1} \Big[100x - 2x^3y\Big]_{x=0}^{x=2} dy$$

$$= \int_{-1}^{1} (200 - 16y)\, dy = \Big[200y - 8y^2\Big]_{-1}^{1} = 400.$$

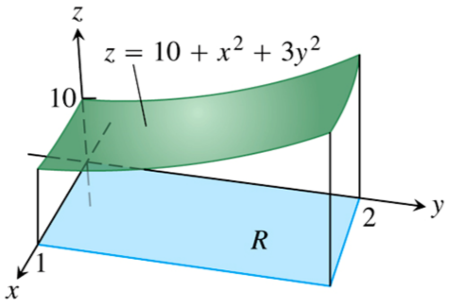
Reversing the order of integration gives the same answer:

$$\int_{0}^{2}\int_{-1}^{1} (100 - 6x^2y)\, dy\, dx = \int_{0}^{2} \Big[100y - 3x^2y^2\Big]_{y=-1}^{y=1} dx$$

$$= \int_{0}^{2} \big[(100 - 3x^2) - (-100 - 3x^2)\big]\, dx$$

$$= \int_{0}^{2} 200\, dx = 400.$$

■



EXAMPLE 2 Find the volume of the region bounded above by the elliptical paraboloid $z = 10 + x^2 + 3y^2$ and below by the rectangle R: $0 \le x \le 1$, $0 \le y \le 2$.

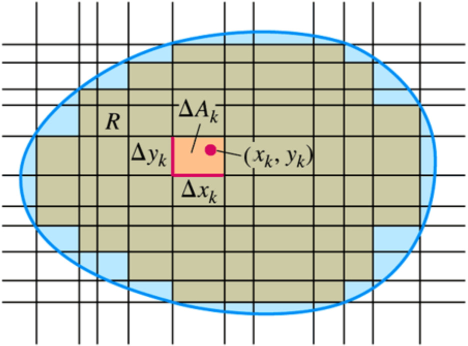
Solution The surface and volume are shown in Figure 15.7. The volume is given by the double integral

$$V = \iint_R (10 + x^2 + 3y^2)\, dA = \int_0^1 \int_0^2 (10 + x^2 + 3y^2)\, dy\, dx$$

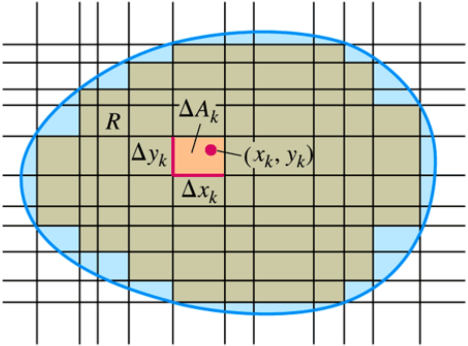
$$= \int_0^1 \left[10y + x^2 y + y^3 \right]_{y=0}^{y=2} dx$$

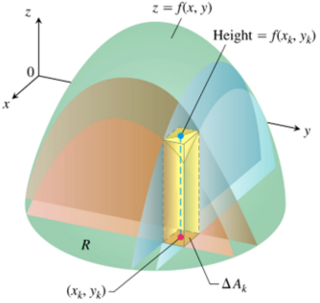
$$= \int_0^1 (20 + 2x^2 + 8)\, dx = \left[20x + \frac{2}{3}x^3 + 8x \right]_0^1 = \frac{86}{3}.$$

14.2 Double Integrals over General Regions



We only use rectangles that are completely inside R.





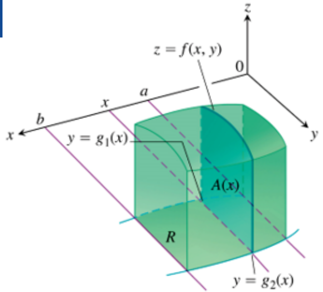
Volume $= \lim \sum f(x_k, y_k)\,\Delta A_k = \iint_R f(x, y)\,dA$

We only use rectangles that are completely inside R.

$$\iint_R f\,dA = \lim_{\|P\|\to 0} \sum_k f(x_k, y_k)\Delta A_k.$$

Theorem (Fubini's Theorem)

Let $f(x, y)$ be continuous on R. Let g_1, g_2, h_1 and h_2 be continuous.



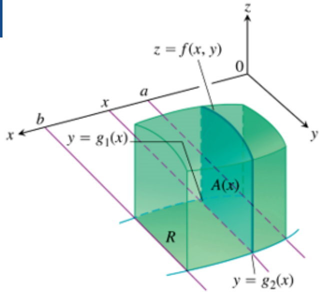
Theorem (Fubini's Theorem)

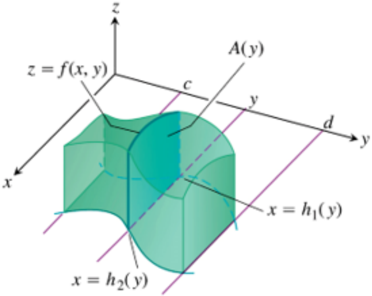
1. *If R is defined by $a \leq x \leq b$ and $g_1(x) \leq y \leq g_2(x)$, then*

$$\iint_R f\, dA = \int_{x=a}^{x=b} \int_{y=g_1(x)}^{y=g_2(x)} f(x,y)\, dy dx$$

only numbers

you can have functions here





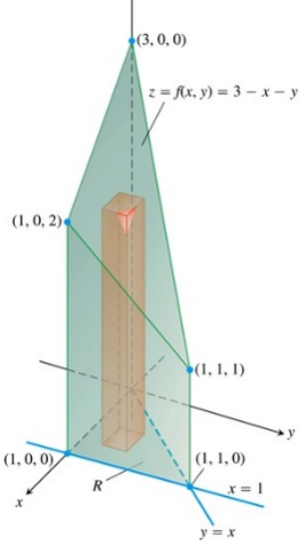
Theorem (Fubini's Theorem)

2 *If R is defined by $c \le y \le d$ and $h_1(x) \le x \le h_2(x)$, then*

$$\iint_R f\,dA = \int_{y=c}^{y=d} \int_{x=h_1(x)}^{x=h_2(x)} f(x,y)\,dx dy$$

only numbers

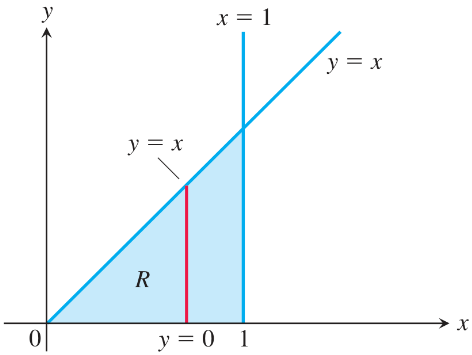
you can have functions here



(a)

EXAMPLE 1 Find the volume of the prism whose base is the triangle in the xy-plane bounded by the x-axis and the lines $y = x$ and $x = 1$ and whose top lies in the plane

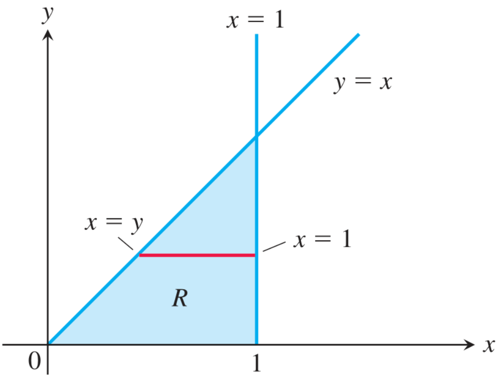
$$z = f(x, y) = 3 - x - y.$$



Solution See Figure 15.12. For any x between 0 and 1, y may vary from $y = 0$ to $y = x$ (Figure 15.12b). Hence,

$$V = \int_0^1 \int_0^x (3 - x - y)\, dy\, dx = \int_0^1 \left[3y - xy - \frac{y^2}{2} \right]_{y=0}^{y=x} dx$$

$$= \int_0^1 \left(3x - \frac{3x^2}{2} \right) dx = \left[\frac{3x^2}{2} - \frac{x^3}{2} \right]_{x=0}^{x=1} = 1.$$



When the order of integration is reversed (Figure 15.12c), the integral for the volume is

$$
\begin{aligned}
V &= \int_0^1 \int_y^1 (3 - x - y)\, dx\, dy = \int_0^1 \left[3x - \frac{x^2}{2} - xy \right]_{x=y}^{x=1} dy \\
&= \int_0^1 \left(3 - \frac{1}{2} - y - 3y + \frac{y^2}{2} + y^2 \right) dy \\
&= \int_0^1 \left(\frac{5}{2} - 4y + \frac{3}{2} y^2 \right) dy = \left[\frac{5}{2} y - 2y^2 + \frac{y^3}{2} \right]_{y=0}^{y=1} = 1.
\end{aligned}
$$

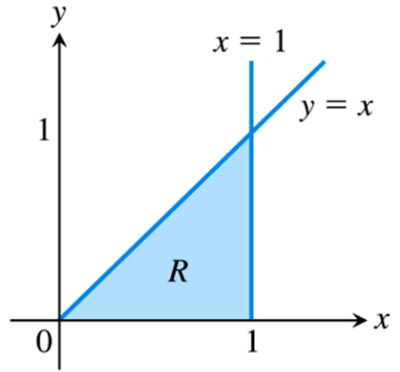
The two integrals are equal, as they should be. ■

Remark

Fubini's Theorem tells us that

$$\iint f\, dxdy = \iint f\, dydx.$$

But sometimes one is easier to calculate than the other.



EXAMPLE 2 Calculate

$$\iint_R \frac{\sin x}{x}\, dA,$$

where R is the triangle in the xy-plane bounded by the x-axis, the line $y = x$, and the line $x = 1$.

Solution The region of integration is shown in Figure 15.13. If we integrate first with respect to y and next with respect to x, then because x is held fixed in the first integration, we find

$$\int_0^1 \left(\int_0^x \frac{\sin x}{x}\, dy \right) dx = \int_0^1 \left[y \frac{\sin x}{x} \right]_{y=0}^{y=x} dx = \int_0^1 \sin x\, dx = -\cos(1) + 1 \approx 0.46.$$

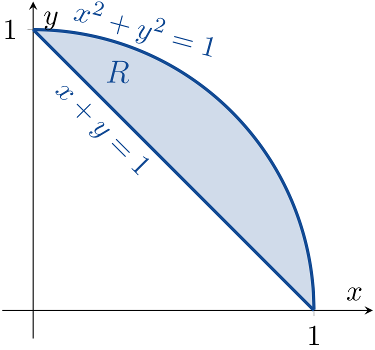
If we reverse the order of integration and attempt to calculate

$$\int_0^1 \int_y^1 \frac{\sin x}{x}\, dx\, dy,$$

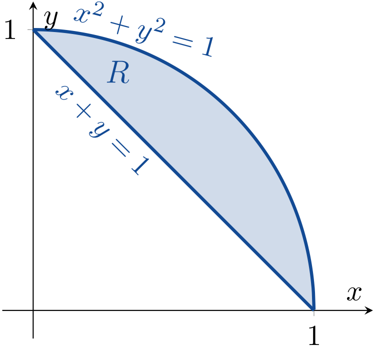
we run into a problem because $\int ((\sin x)/x)\, dx$ cannot be expressed in terms of elementary functions (there is no simple antiderivative).

There is no general rule for predicting which order of integration will be the good one in circumstances like these. If the order you first choose doesn't work, try the other. Sometimes neither order will work, and then we may need to use numerical approximations. ■

Finding Limits of Integration



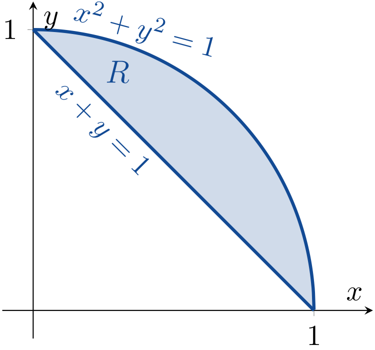
Finding Limits of Integration



Question 1: Shall we do $dxdy$ or $dydx$?

$$\boxed{\int \int \qquad f(x, y)}$$

Finding Limits of Integration

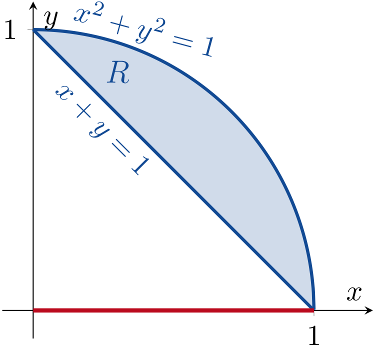


Question 1: Shall we do $dxdy$ or $dydx$?

$dydx$

$$\boxed{\int \int \qquad f(x,y)\, dydx}$$

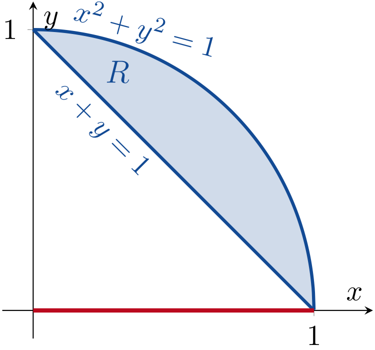
Finding Limits of Integration



Question 2: What are the smallest and biggest possible values of x in R?

$$\boxed{\int \int \qquad f(x,y)\,dydx}$$

Finding Limits of Integration

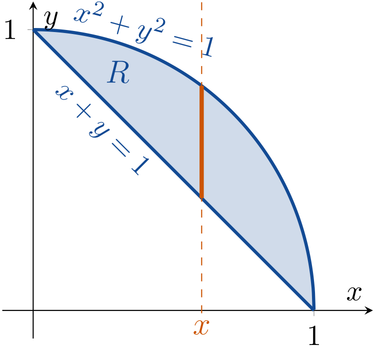


Question 2: What are the smallest and biggest possible values of x in R?

$$0 \leq x \leq 1$$

$$\boxed{\int_0^1 \int \qquad f(x,y)\,dydx}$$

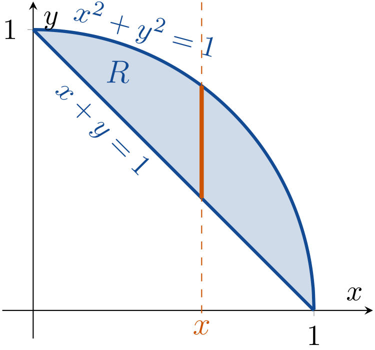
Finding Limits of Integration



Now choose an x between 0 and 1 and draw the vertical line though x.

$$\int_0^1 \int \qquad f(x,y)\,dy\,dx$$

Finding Limits of Integration



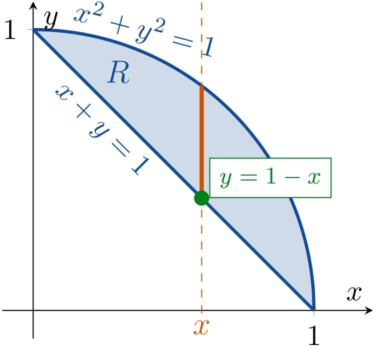
Now choose an x between 0 and 1 and draw the vertical line though x.

Question 3: What are the smallest and biggest possible values of y (on this line) in R?

$$\le y \le$$

$$\boxed{\int_0^1 \int \qquad f(x,y)\,dydx}$$

Finding Limits of Integration



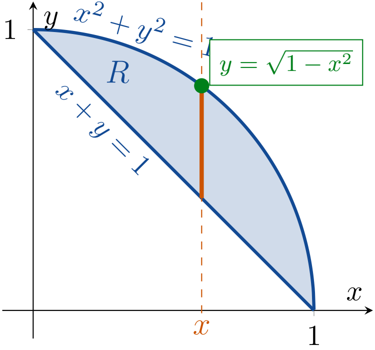
Now choose an x between 0 and 1 and draw the vertical line though x.

Question 3: What are the smallest and biggest possible values of y (on this line) in R?

$$1-x \leq y \leq$$

$$\boxed{\int_0^1 \int_{1-x} \qquad f(x,y)\,dydx}$$

Finding Limits of Integration



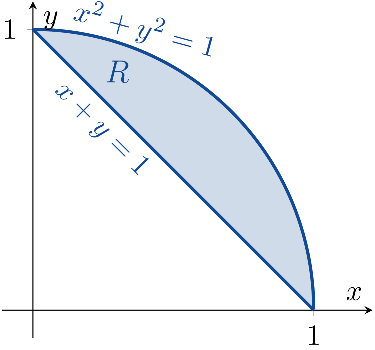
Now choose an x between 0 and 1 and draw the vertical line though x.

Question 3: What are the smallest and biggest possible values of y (on this line) in R?

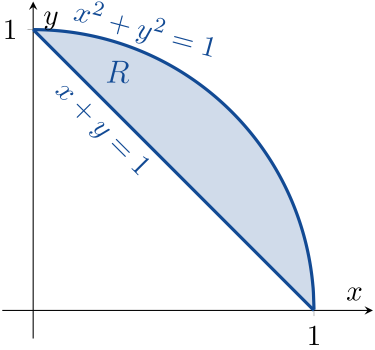
$$1 - x \leq y \leq \sqrt{1-x^2}$$

$$\boxed{\int_0^1 \int_{1-x}^{\sqrt{1-x^2}} f(x,y)\, dy dx}$$

Finding Limits of Integration (again)



Finding Limits of Integration (again)

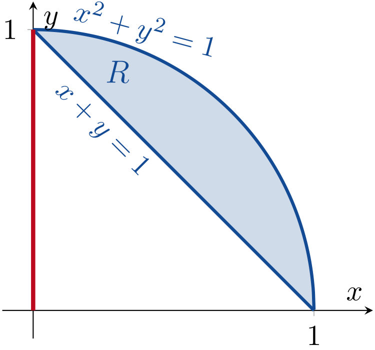


Question 1: Shall we do $dxdy$ or $dydx$?

$dxdy$

$$\boxed{\int \int \qquad f(x,y)\, dxdy}$$

Finding Limits of Integration (again)

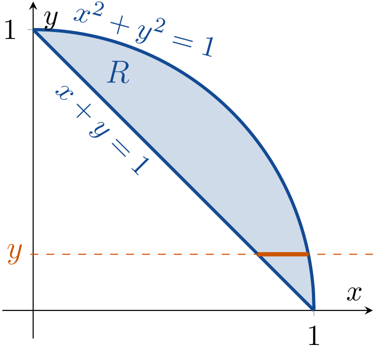


Question 2: What are the smallest and biggest possible values of y in R?

$$0 \leq y \leq 1$$

$$\boxed{\int_0^1 \int \qquad f(x, y)\, dx dy}$$

Finding Limits of Integration (again)

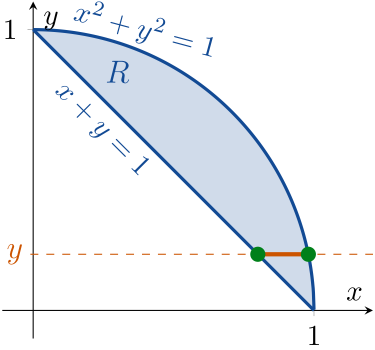


Now choose an y between 0 and 1 and draw the horizontal line though y.

Question 3: What are the smallest and biggest possible values of x (on this line) in R?

$$\int_0^1 \int \qquad f(x,y)\,dxdy$$

Finding Limits of Integration (again)



Now choose an y between 0 and 1 and draw the horizontal line though y.

Question 3: What are the smallest and biggest possible values of x (on this line) in R?

$$1 - y \leq x \leq \sqrt{1 - y^2}$$

$$\boxed{\int_0^1 \int_{1-y}^{\sqrt{1-y^2}} f(x, y)\, dx dy}$$

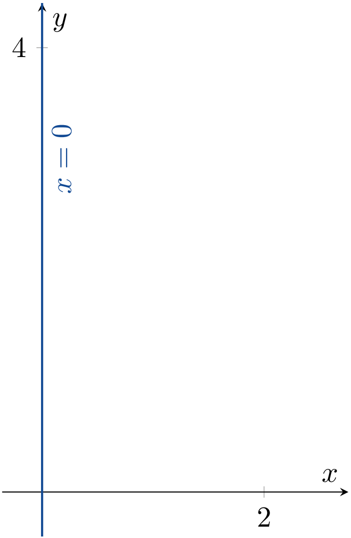
Example

Sketch the region of integration for the integral

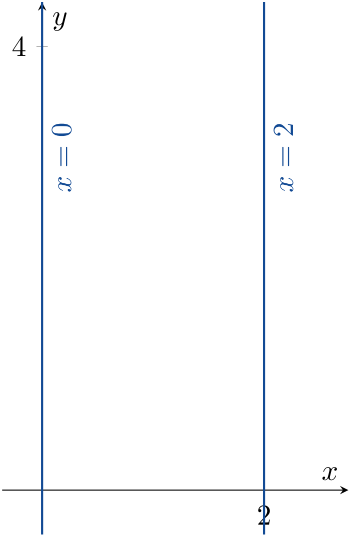
$$\int_0^2 \int_{x^2}^{2x} (4x+2)\, dydx.$$

Then write an equivalent integral with $dxdy$.

$$\int_0^2 \int_{x^2}^{2x} (4x + 2)\, dydx$$

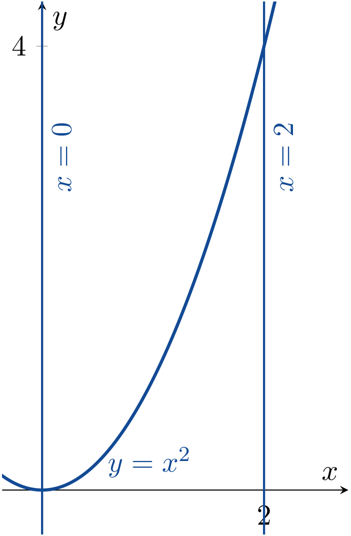


$$\int_0^2 \int_{x^2}^{2x} (4x+2)\, dy dx$$

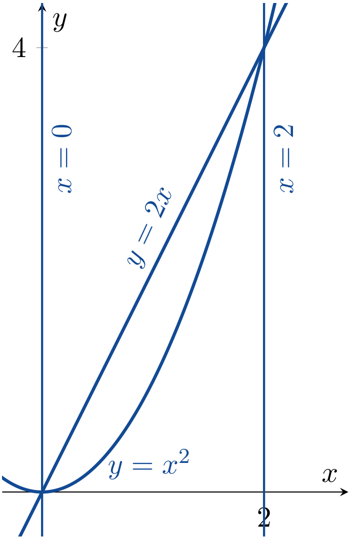


$$\int_0^{2}\int_{x^2}^{2x}(4x+2)\,dydx$$

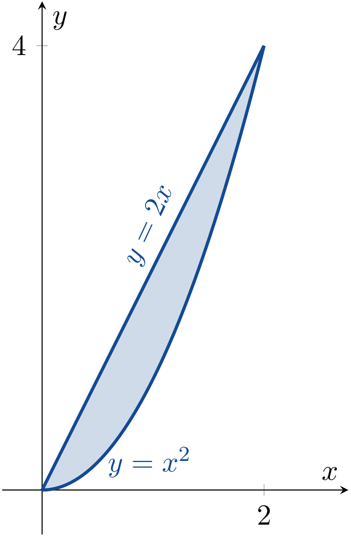
14.2 Double Integrals over General Regions



$$\int_0^2 \int_{x^2}^{2x} (4x+2)\, dy dx$$

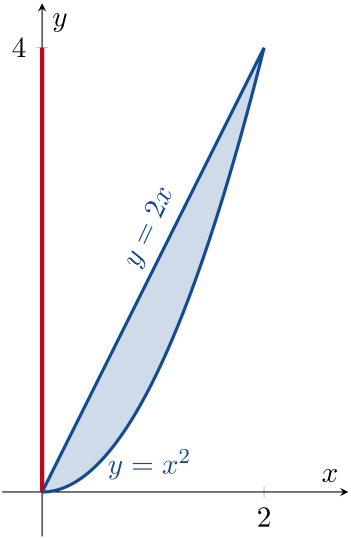


$$\int_0^2 \int_{x^2}^{2x} (4x+2)\, dy dx$$



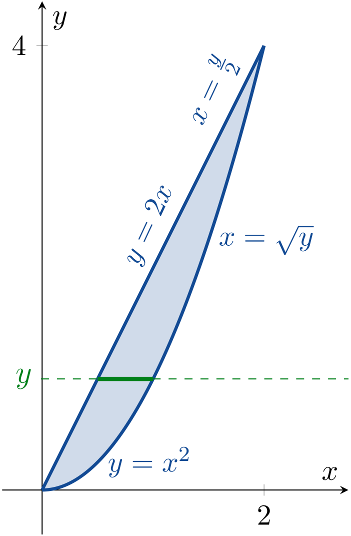
$$\int_0^2 \int_{x^2}^{2x} (4x+2)\,dydx$$

$$= \int \int \quad (4x+2)\,dxdy.$$



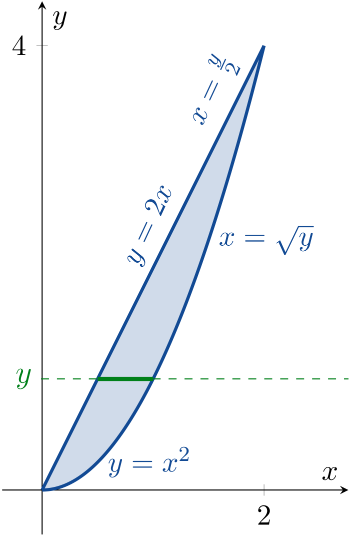
$$\int_0^2 \int_{x^2}^{2x} (4x+2)\,dydx$$

$$= \int_0^4 \int \quad (4x+2)\,dxdy.$$



$$\int_0^2 \int_{x^2}^{2x} (4x+2)\,dydx$$

$$= \int_0^4 \int \quad (4x+2)\,dxdy.$$



$$\int_0^2 \int_{x^2}^{2x} (4x+2)\,dydx$$

$$= \int_0^4 \int_{\frac{y}{2}}^{\sqrt{y}} (4x+2)\,dxdy.$$

Properties of Double Integrals

Suppose that $f(x, y)$ and $g(x, y)$ are continuous on the bounded region R.

Theorem

Properties of Double Integrals

Suppose that $f(x, y)$ and $g(x, y)$ are continuous on the bounded region R.

Theorem

$$\iint\limits_R cf\, dA = c \iint\limits_R f\, dA$$

for any number c.

Properties of Double Integrals

Suppose that $f(x, y)$ and $g(x, y)$ are continuous on the bounded region R.

Theorem

$$\iint\limits_R (f+g)\,dA = \iint\limits_R f\,dA + \iint\limits_R g\,dA.$$

Properties of Double Integrals

Suppose that $f(x, y)$ and $g(x, y)$ are continuous on the bounded region R.

Theorem

$$f \geq 0 \ \ on \ R \qquad \Longrightarrow \qquad \iint\limits_R f \, dA \geq 0.$$

Properties of Double Integrals

Suppose that $f(x, y)$ and $g(x, y)$ are continuous on the bounded region R.

Theorem

$$f \geq g \text{ on } R \quad \Longrightarrow \quad \iint_R f\,dA \geq \iint_R g\,dA.$$

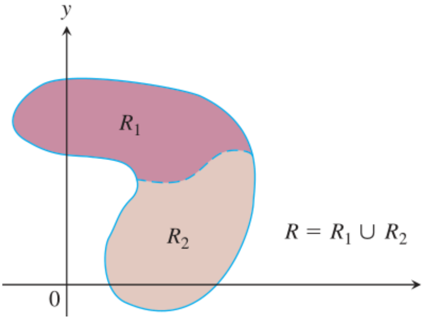
Properties of Double Integrals

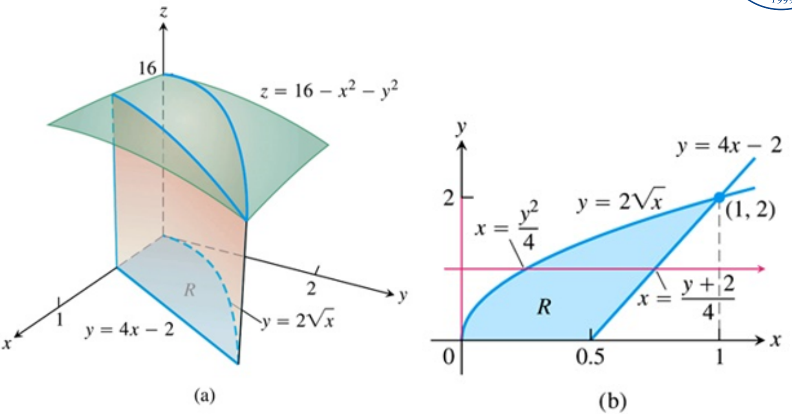
Suppose that $f(x, y)$ and $g(x, y)$ are continuous on the bounded region R.

Theorem

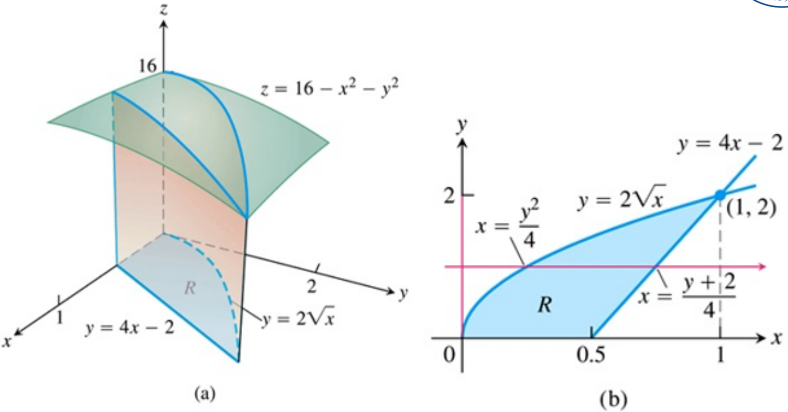
If $R = R_1 \cup R_2$ where R_1 and R_2 don't overlay, then

$$\iint_R f\,dA = \iint_{R_1} f\,dA + \iint_{R_2} f\,dA.$$

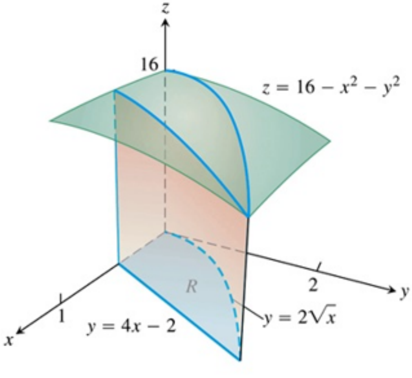




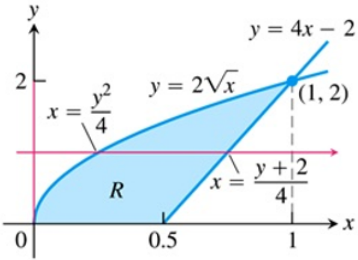
EXAMPLE 4 Find the volume of the wedgelike solid that lies beneath the surface $z = 16 - x^2 - y^2$ and above the region R bounded by the curve $y = 2\sqrt{x}$, the line $y = 4x - 2$, and the x-axis.



$$0 \leq y \leq 2$$



(a)



(b)

$$0 \leq y \leq 2$$

$$\frac{y^2}{4} \leq x \leq \frac{y+2}{4}$$

Solution Figure 15.18a shows the surface and the "wedgelike" solid whose volume we want to calculate. Figure 15.18b shows the region of integration in the xy-plane. If we integrate in the order $dy\,dx$ (first with respect to y and then with respect to x), two integrations will be required because y varies from $y = 0$ to $y = 2\sqrt{x}$ for $0 \le x \le 0.5$, and then varies from $y = 4x - 2$ to $y = 2\sqrt{x}$ for $0.5 \le x \le 1$. So we choose to integrate in the order $dx\,dy$, which requires only one double integral whose limits of integration are indicated in Figure 15.18b. The volume is then calculated as the iterated integral:

$$\iint_R (16 - x^2 - y^2)\, dA$$

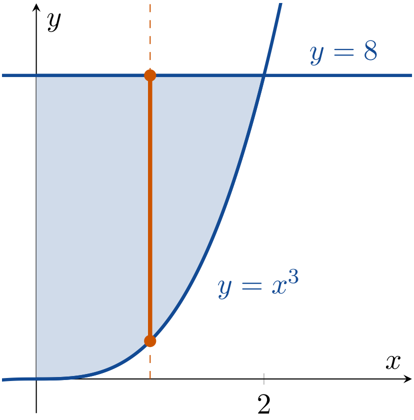
$$= \int_0^2 \int_{y^2/4}^{(y+2)/4} (16 - x^2 - y^2)\, dx\, dy$$

$$= \int_0^2 \left[16x - \frac{x^3}{3} - xy^2 \right]_{x=y^2/4}^{x=(y+2)/4} dx$$

$$= \int_0^2 \left[4(y+2) - \frac{(y+2)^3}{3 \cdot 64} - \frac{(y+2)y^2}{4} - 4y^2 + \frac{y^6}{3 \cdot 64} + \frac{y^4}{4} \right] dy$$

$$= \left[\frac{191y}{24} + \frac{63y^2}{32} - \frac{145y^3}{96} - \frac{49y^4}{768} + \frac{y^5}{20} + \frac{y^7}{1344} \right]_0^2 = \frac{20803}{1680} \approx 12.4. \quad \blacksquare$$

Now some problems for you.



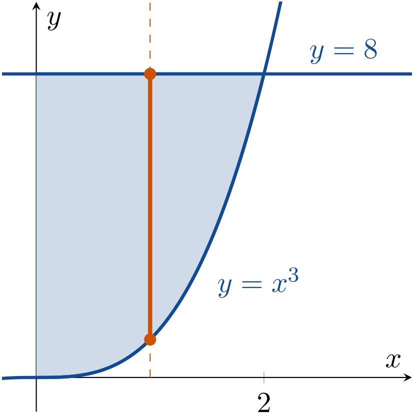
Which is correct?

A $\displaystyle\int_0^2 \int_0^8 f(x,y)\,dydx$

B $\displaystyle\int_0^2 \int_{x^3}^8 f(x,y)\,dydx$

C $\displaystyle\int_{x^3}^8 \int_0^2 f(x,y)\,dydx$

D $\displaystyle\int_0^1 \int_0^{x^3} f(x,y)\,dydx$



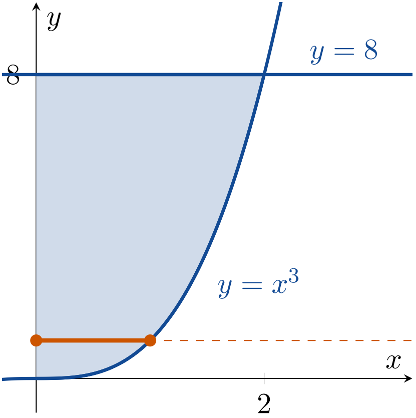
Which is correct?

A $\displaystyle\int_0^2 \int_0^8 f(x,y)\,dydx$

B $\displaystyle\int_0^2 \int_{x^3}^8 f(x,y)\,dydx$

C $\displaystyle\int_{x^3}^8 \int_0^2 f(x,y)\,dydx$

D $\displaystyle\int_0^1 \int_0^{x^3} f(x,y)\,dydx$



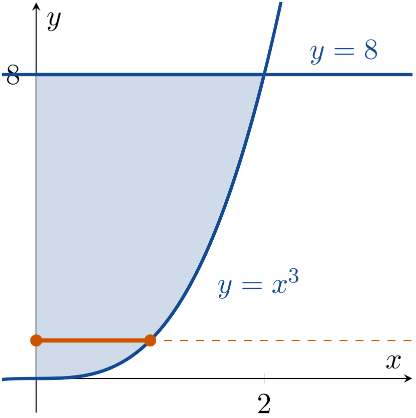
Which is correct?

A $\displaystyle\int_0^8 \int_0^{y^{\frac{1}{3}}} f(x,y)\,dxdy$

B $\displaystyle\int_0^8 \int_0^{y^{\frac{1}{3}}} f(x,y)\,dydx$

C $\displaystyle\int_0^8 \int_0^{x^3} f(x,y)\,dxdy$

D $\displaystyle\int_0^8 \int_0^{3} f(x,y)\,dydx$



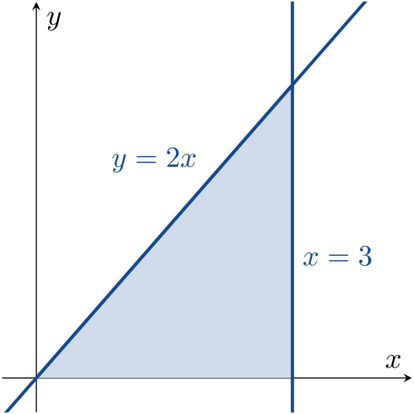
Which is correct?

A $\displaystyle\int_0^8 \int_0^{y^{\frac{1}{3}}} f(x,y)\,dxdy$

B $\displaystyle\int_0^8 \int_0^{y^{\frac{1}{3}}} f(x,y)\,dydx$

C $\displaystyle\int_0^8 \int_0^{x^3} f(x,y)\,dxdy$

D $\displaystyle\int_0^8 \int_0^{3} f(x,y)\,dydx$



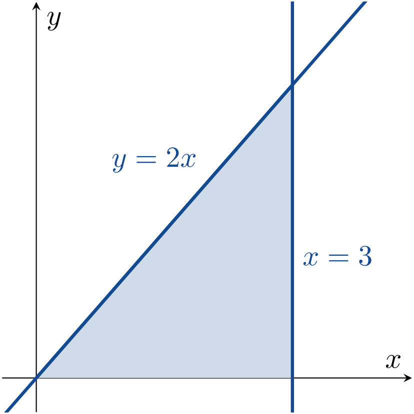
Which is correct?

A $\displaystyle\int_0^6 \int_0^{\frac{y}{2}} f(x,y)\,dxdy$

B $\displaystyle\int_0^3 \int_0^{2x} f(x,y)\,dxdy$

C $\displaystyle\int_0^3 \int_0^{2x} f(x,y)\,dydx$

D $\displaystyle\int_0^6 \int_{\frac{y}{2}}^{3} f(x,y)\,dxdy$



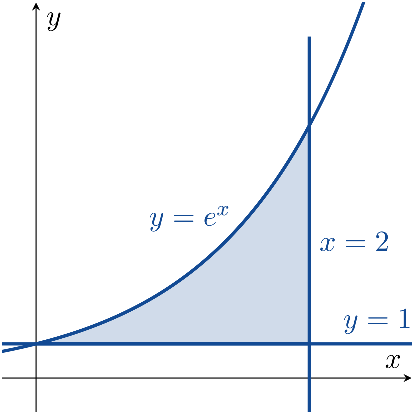
Which is correct?

A $\int_0^6 \int_0^{\frac{y}{2}} f(x,y)\,dxdy$

B $\int_0^3 \int_0^{2x} f(x,y)\,dxdy$

C $\int_0^3 \int_0^{2x} f(x,y)\,dydx$

D $\int_0^6 \int_{\frac{y}{2}}^{3} f(x,y)\,dxdy$



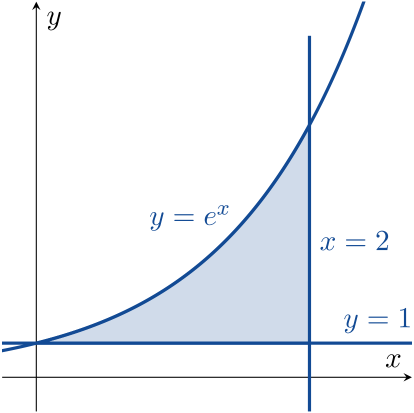
Which is correct?

A $\displaystyle\int_1^{e^x}\int_0^2 f(x,y)\,dxdy$

B $\displaystyle\int_0^2\int_0^{e^x} f(x,y)\,dxdy$

C $\displaystyle\int_0^2\int_0^{e^x} f(x,y)\,dydx$

D $\displaystyle\int_0^2\int_1^{e^x} f(x,y)\,dydx$



Which is correct?

A $\displaystyle\int_1^{e^x}\int_0^2 f(x,y)\,dxdy$

B $\displaystyle\int_0^2\int_0^{e^x} f(x,y)\,dxdy$

C $\displaystyle\int_0^2\int_0^{e^x} f(x,y)\,dydx$

D $\displaystyle\int_0^2\int_1^{e^x} f(x,y)\,dydx$

Break

We will continue at 3pm

"The next part of this recipe will involve some calculus."

Area by Double Integration

14.3 Area by Double Integration

Definition

The *area* of a closed, bounded region R is

$$A = \iint\limits_R \, dA.$$

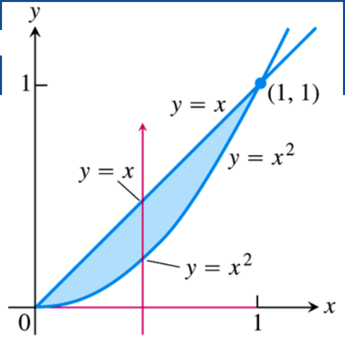
Definition

The *area* of a closed, bounded region R is

$$A = \iint\limits_R 1 \, dA.$$

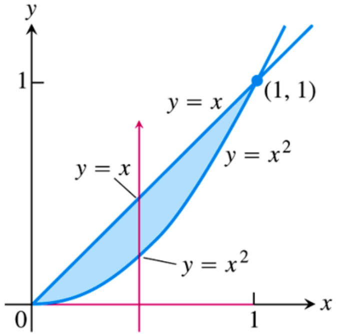
Example

Find the area of the region R bounded by $y = x$ and $y = x^2$ in the first quadrant.



Example

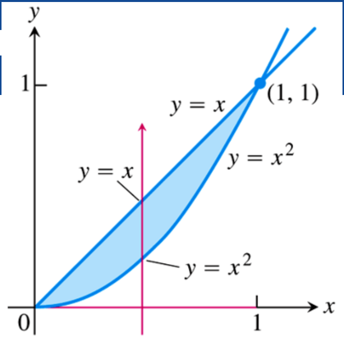
Find the area of the region R bounded by $y = x$ and $y = x^2$ in the first quadrant.



Example

Find the area of the region R bounded by $y = x$ and $y = x^2$ in the first quadrant.

$$A = \int_0^1 \int_{x^2}^{x} dydx$$



Example

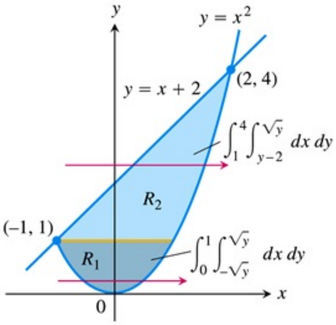
Find the area of the region R bounded by $y = x$ and $y = x^2$ in the first quadrant.

$$A = \int_0^1 \int_{x^2}^{x} dydx = \int_0^1 \Big[y\Big]_{x^2}^{x} dx = \int_0^1 (x - x^2)\, dx$$

$$= \left[\frac{x^2}{2} - \frac{x^3}{3}\right]_0^1 = \frac{1}{6}.$$

Example

Find the area of the region R bounded by $y = x + 2$ and $y = x^2$ in the first quadrant.

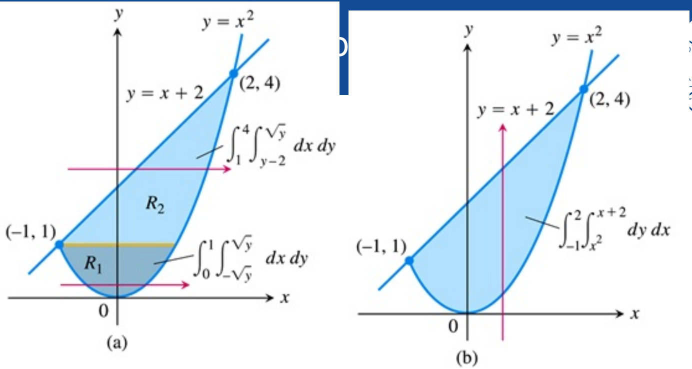


(a)

Example

Find the area of the region R bounded by $y = x + 2$ and $y = x^2$ in the first quadrant.

$$A = \iint_{R_1} dA + \iint_{R_2} dA = \int_0^1 \int_{-\sqrt{y}}^{\sqrt{y}} dxdy + \int_1^4 \int_{y-2}^{\sqrt{y}} dxdy = \dots$$



Example

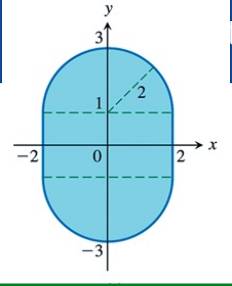
Find the area of the region R bounded by $y = x + 2$ and $y = x^2$ in the first quadrant.

$$A = \iint_{R_1} dA + \iint_{R_2} dA = \int_0^1 \int_{-\sqrt{y}}^{\sqrt{y}} dxdy + \int_1^4 \int_{y-2}^{\sqrt{y}} dxdy = \ldots$$

$$A = \int_{-1}^2 \int_{x^2}^{x+2} dydx = \int_{-1}^2 \Big[y\Big]_{x^2}^{x+2} = \int_{-1}^2 (x+2-x^2)\, dx = \ldots = \frac{9}{2}.$$

Example

Find the area of the region R described by $-2 \leq x \leq 2$ and $-1 - \sqrt{4 - x^2} \leq y \leq 1 + \sqrt{4 - x^2}$.

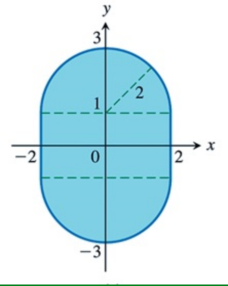


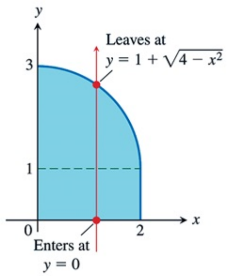
Example

Find the area of the region R described by $-2 \leq x \leq 2$ and $-1 - \sqrt{4 - x^2} \leq y \leq 1 + \sqrt{4 - x^2}$.

$$A = \iint\limits_R dA$$

14.3 Area by … tion

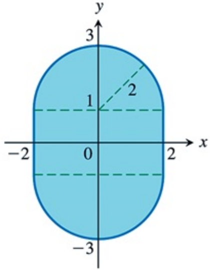




Example

Find the area of the region R described by $-2 \leq x \leq 2$ and $-1-\sqrt{4-x^2} \leq y \leq 1+\sqrt{4-x^2}$.

$$A = \iint\limits_R dA = 4\int_0^2 \int_0^{1+\sqrt{4-x^2}} dydx = \ldots = 8 + 4\pi.$$



Example

Find the area of the region R described by $-2 \le x \le 2$ and $-1-\sqrt{4-x^2} \le y \le 1+\sqrt{4-x^2}$.

$$A = \iint\limits_R dA = 4\int_0^2 \int_0^{1+\sqrt{4-x^2}} dydx = \ldots = 8+4\pi.$$

or

$$A = \begin{pmatrix} \text{area of a} \\ \text{circle of} \\ \text{radius } 2 \end{pmatrix} + \begin{pmatrix} \text{area of a } 4\times 2 \\ \text{rectangle} \end{pmatrix} = 4\pi + 8.$$

Average Value of a Function

Definition

The *average value* of f over R is

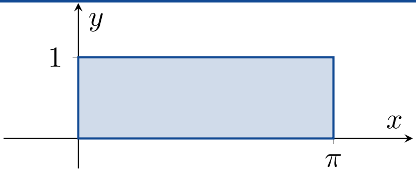
$$\operatorname{av}(f) = \frac{1}{\text{area of } R} \iint\limits_R f \, dA$$

Average Value of a Function

Definition

The *average value* of f over R is

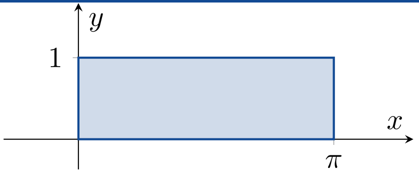
$$\operatorname{av}(f) = \frac{1}{\text{area of } R} \iint_R f\,dA = \frac{\iint_R f\,dA}{\iint_R dA}.$$



Example

Find the average value of $f(x,y) = x\cos xy$ over the rectangle $R = [0,\pi] \times [0,1]$.

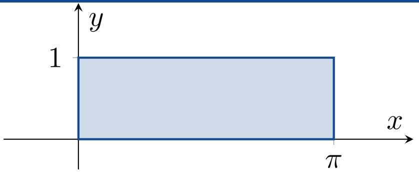
$$\operatorname{av}(f) = \frac{1}{\text{area of } R} \iint_R f \, dA$$



Example

Find the average value of $f(x, y) = x \cos xy$ over the rectangle $R = [0, \pi] \times [0, 1]$.

$$\text{av}(f) = \frac{1}{\text{area of } R} \iint\limits_R f\, dA = \frac{1}{\pi} \int_0^{\pi} \int_0^1 x \cos xy \, dy dx$$

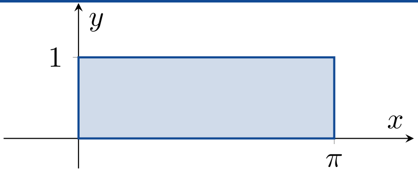


Example

Find the average value of $f(x, y) = x\cos xy$ over the rectangle $R = [0, \pi] \times [0, 1]$.

$$\operatorname{av}(f) = \frac{1}{\text{area of } R} \iint_R f\,dA = \frac{1}{\pi}\int_0^\pi \int_0^1 x\cos xy\,dydx$$

$$= \frac{1}{\pi}\int_0^\pi \Big[\sin xy\Big]_{y=0}^{y=1} dx = \frac{1}{\pi}\int_0^\pi (\sin x - 0)\,dx$$



Example

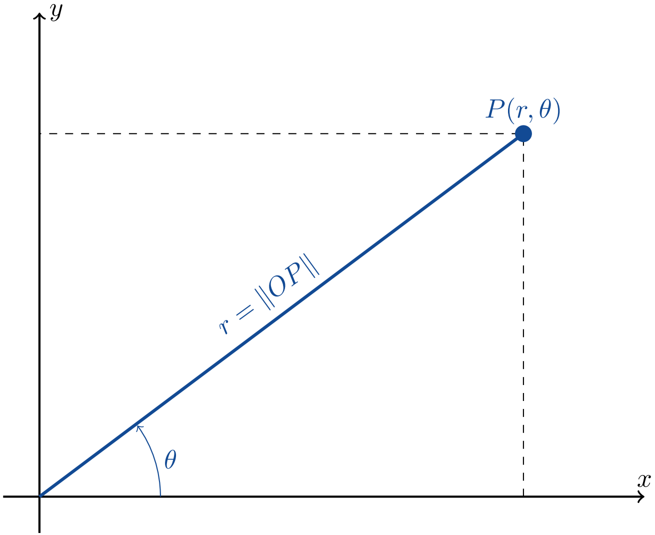
Find the average value of $f(x, y) = x \cos xy$ over the rectangle $R = [0, \pi] \times [0, 1]$.

$$\mathrm{av}(f) = \frac{1}{\text{area of } R} \iint\limits_R f \, dA = \frac{1}{\pi} \int_0^{\pi} \int_0^1 x \cos xy \, dy dx$$

$$= \frac{1}{\pi} \int_0^{\pi} \Big[\sin xy \Big]_{y=0}^{y=1} dx = \frac{1}{\pi} \int_0^{\pi} (\sin x - 0) \, dx$$

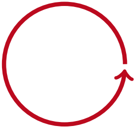
$$= \frac{1}{\pi} \Big[-\cos x \Big]_0^{\pi} = \frac{1}{\pi}(1 + 1) = \frac{2}{\pi}.$$

10.3 Polar Coordinates

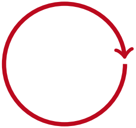


y
$P(r, \theta)$
$r = \|OP\|$
θ
x

10.3 Polar Coordinates

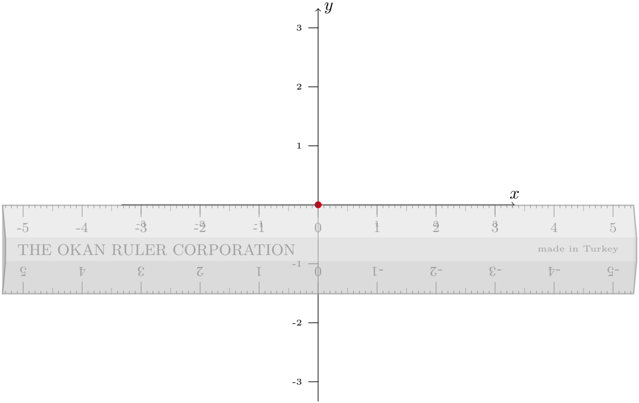


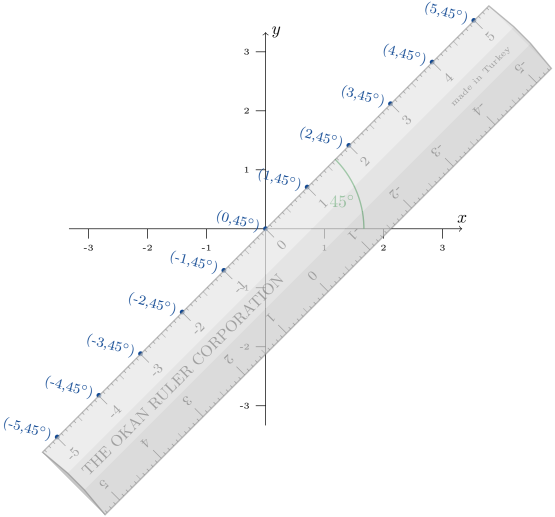
anticlockwise = positive angle
saat yönünün tersi = pozitif açı



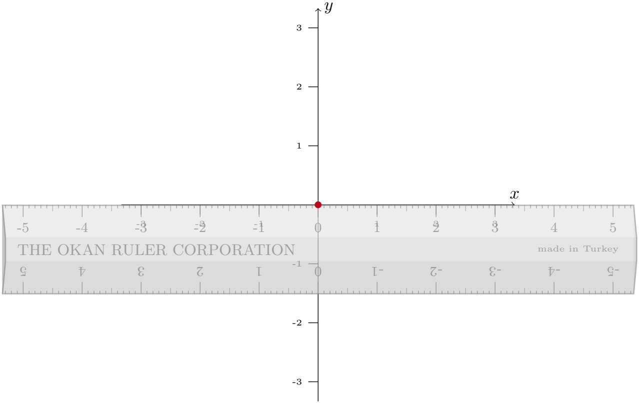
clockwise = negative angle
saat yönünde = negatif açı

10.3 Polar Coordinates

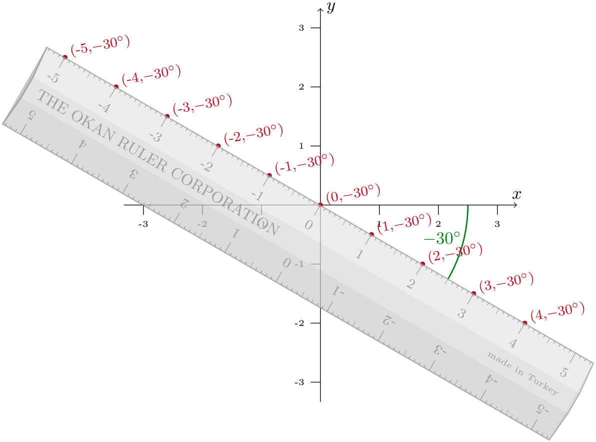




(5,45°)
(4,45°)
(3,45°)
(2,45°)
(1,45°)
(0,45°)
(-1,45°)
(-2,45°)
(-3,45°)
(-4,45°)
(-5,45°)
45°
x
y
THE OKAN RULER CORPORATION
made in Turkey

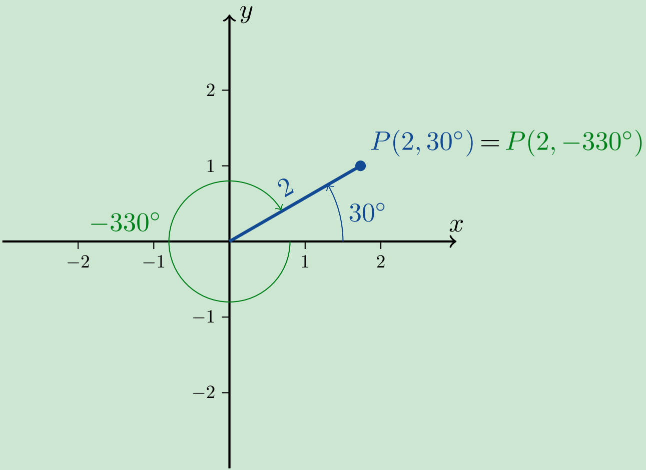


y
x
3
2
1
-1
-2
-3
-5 -4 -3 -2 -1 0 1 2 3 4 5
THE OKAN RULER CORPORATION
made in Turkey

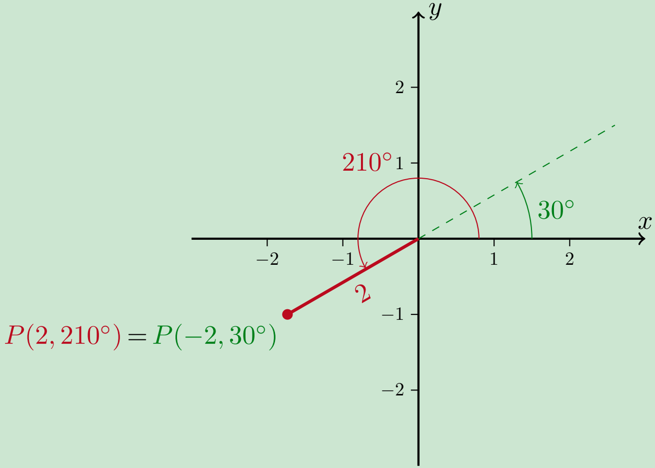


y
x
(-5,−30°)
(-4,−30°)
(-3,−30°)
(-2,−30°)
(-1,−30°)
(0,−30°)
(1,−30°)
(2,−30°)
(3,−30°)
(4,−30°)
−30°
THE OKAN RULER CORPORATION
made in Turkey

Example



Example



Example

Find all the polar coordinates of $P(2, 30°)$.

We can have either $r = 2$ or $r = -2$.

Example

Find all the polar coordinates of $P(2, 30°)$.

We can have either $r = 2$ or $r = -2$. For $r = 2$, we can have

$$\theta = 30°, 30° \pm 360°, 30° \pm 720°, 30° \pm 1080°, \ldots$$

Example

Find all the polar coordinates of $P(2, 30^\circ)$.

We can have either $r = 2$ or $r = -2$. For $r = 2$, we can have

$$\theta = 30^\circ, 30^\circ \pm 360^\circ, 30^\circ \pm 720^\circ, 30^\circ \pm 1080^\circ, \ldots$$

For $r = -2$, we can have

$$\theta = 210^\circ, 210^\circ \pm 360^\circ, 210^\circ \pm 720^\circ, 210^\circ \pm 1080^\circ, \ldots$$

Example

Find all the polar coordinates of $P(2, 30^\circ)$.

We can have either $r = 2$ or $r = -2$. For $r = 2$, we can have

$$\theta = 30^\circ, 30^\circ \pm 360^\circ, 30^\circ \pm 720^\circ, 30^\circ \pm 1080^\circ, \ldots$$

For $r = -2$, we can have

$$\theta = 210^\circ, 210^\circ \pm 360^\circ, 210^\circ \pm 720^\circ, 210^\circ \pm 1080^\circ, \ldots$$

Therefore

$$P(2, 30^\circ) = P\big(2, (30 + 360n)^\circ\big) = P\big(-2, (210 + 360m)^\circ\big)$$

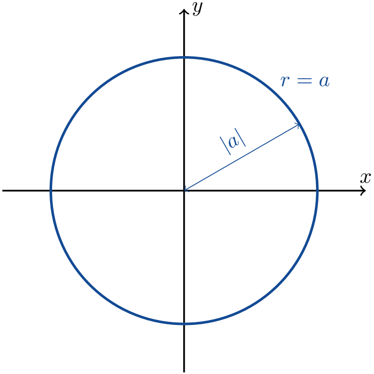
for all $m, n \in \mathbb{Z}$.

Example

Draw the graph of $r = a$.

Example

Draw the graph of $r = a$.

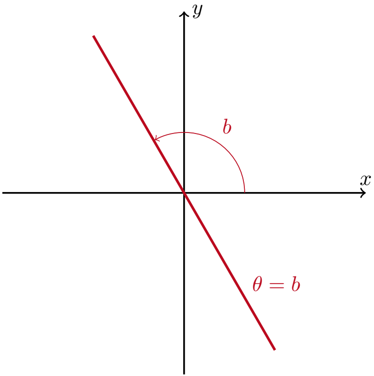


Example

Draw the graph of $\theta = b$.

Example

Draw the graph of $\theta = b$.



Remark

$r = 1$ and $r = -1$ are both equations for a circle of radius 1 centred at the origin.

Remark

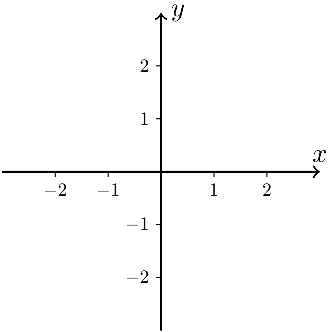
$r = 1$ and $r = -1$ are both equations for a circle of radius 1 centred at the origin.

Remark

$\theta = 30°$, $\theta = 210°$ and $\theta = -150°$ are all equations for the same line.

Example

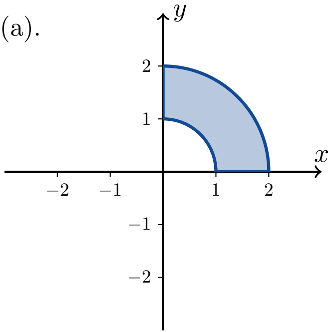
Draw the sets of points whose polar coordinates satisfy the following: $1 \leq r \leq 2$ and $0 \leq \theta \leq 90°$.



Example

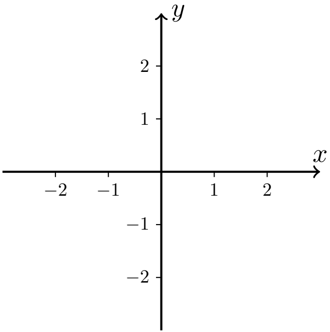
Draw the sets of points whose polar coordinates satisfy the following: $1 \leq r \leq 2$ and $0 \leq \theta \leq 90°$.

(a).



Example

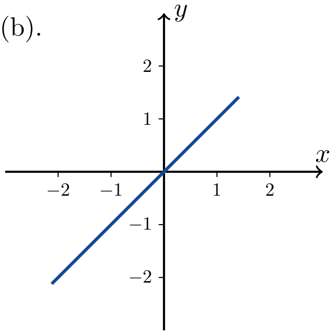
Draw the sets of points whose polar coordinates satisfy the following: $-3 \leq r \leq 2$ and $\theta = 45°$.



Example

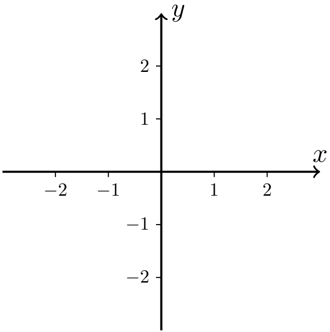
Draw the sets of points whose polar coordinates satisfy the following: $-3 \leq r \leq 2$ and $\theta = 45°$.

(b).



Example

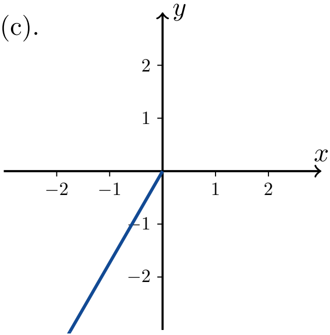
Draw the sets of points whose polar coordinates satisfy the following: $r \leq 0$ and $\theta = 60°$.



Example

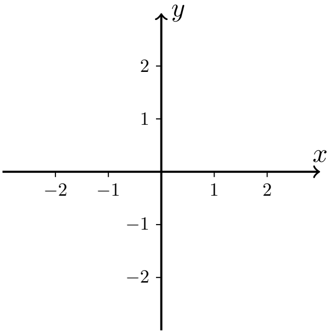
Draw the sets of points whose polar coordinates satisfy the following: $r \leq 0$ and $\theta = 60°$.

(c).



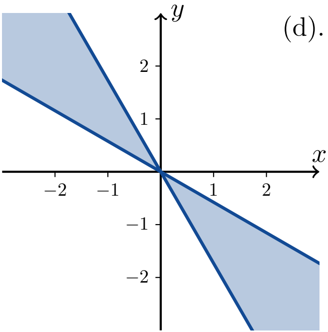
Example

Draw the sets of points whose polar coordinates satisfy the following: $120° \le \theta \le 150°$.



Example

Draw the sets of points whose polar coordinates satisfy the following: $120^\circ \le \theta \le 150^\circ$.

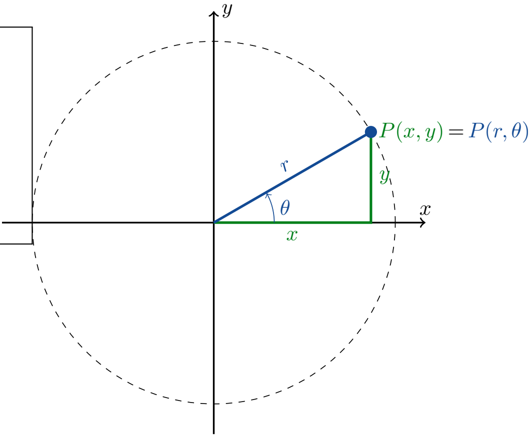


(d).

Relating Polar and Cartesian Coordinates

$$x = r\cos\theta \qquad x^2 + y^2 = r^2$$

$$y = r\sin\theta \qquad \tan\theta = \frac{y}{x}$$



$x = r\cos\theta \quad y = r\sin\theta \quad x^2 + y^2 = r^2 \quad \tan\theta = \frac{y}{x}$

Example

Convert the polar coordinates $(r, \theta) = (-3, 90^\circ)$ into Cartesian coordinates.

$x = r\cos\theta \quad y = r\sin\theta \quad x^2 + y^2 = r^2 \quad \tan\theta = \frac{y}{x}$

Example

Convert the polar coordinates $(r, \theta) = (-3, 90^\circ)$ into Cartesian coordinates.

$$(x, y) = (r\cos\theta, r\sin\theta) = (-3\cos 90^\circ, -3\sin 90^\circ) = (0, -3).$$

$x = r\cos\theta \quad y = r\sin\theta \quad x^2 + y^2 = r^2 \quad \tan\theta = \frac{y}{x}$

Example

Find polar coordinates for the Cartesian coordinates $(x, y) = (5, -12)$.

$x = r\cos\theta \quad y = r\sin\theta \quad x^2 + y^2 = r^2 \quad \tan\theta = \frac{y}{x}$

Example

Find polar coordinates for the Cartesian coordinates $(x, y) = (5, -12)$.

Choosing $r > 0$, we calculate that

$$r = \sqrt{x^2 + y^2} = \sqrt{25 + 144} = \sqrt{169} = 13.$$

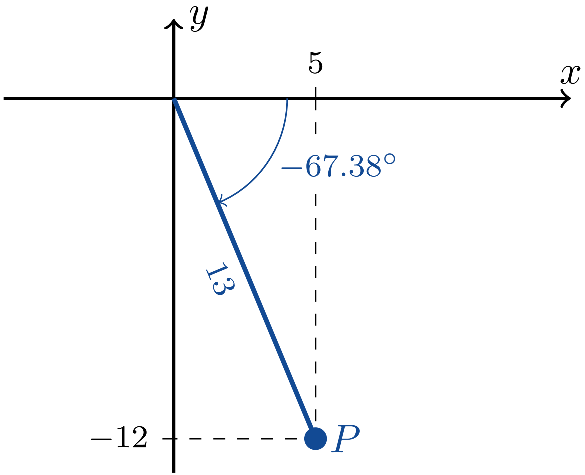
To find θ we use the equation $y = r\sin\theta$ to calculate that

$$\theta = \sin^{-1}\frac{y}{r} = \sin^{-1}\frac{-12}{13} \approx -67.38^\circ.$$

Therefore

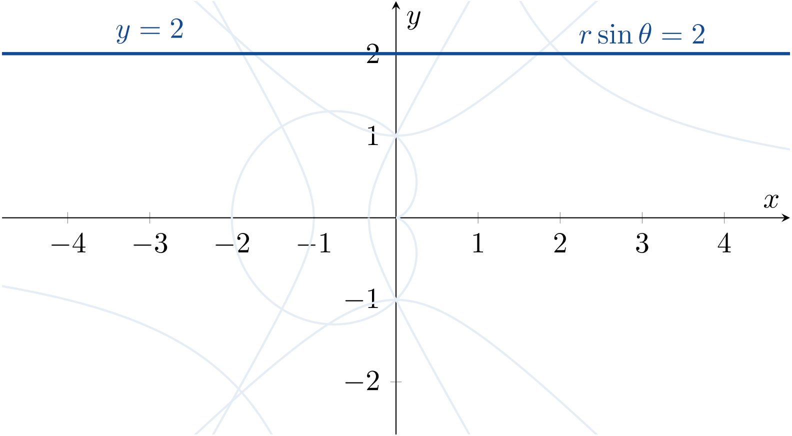
$$(r, \theta) = (13, -67.38^\circ).$$

$$x = r\cos\theta \quad y = r\sin\theta \quad x^2 + y^2 = r^2 \quad \tan\theta = \frac{y}{x}$$



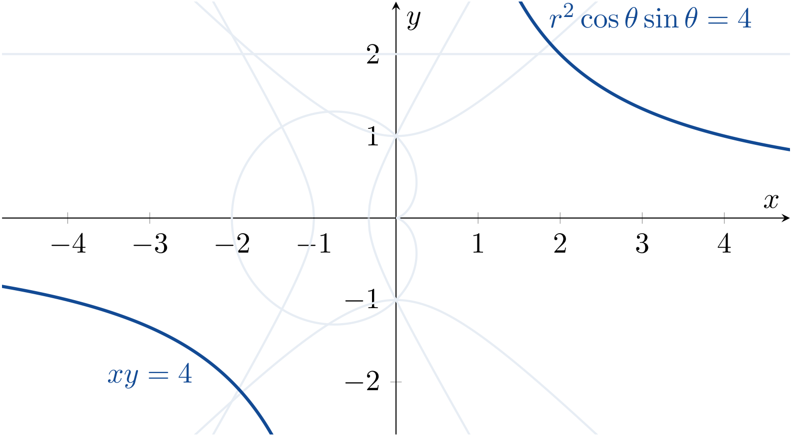
$x = r\cos\theta \quad y = r\sin\theta \quad x^2 + y^2 = r^2 \quad \tan\theta = \frac{y}{x}$

Cartesian Equation $\longleftrightarrow$ Polar Equation



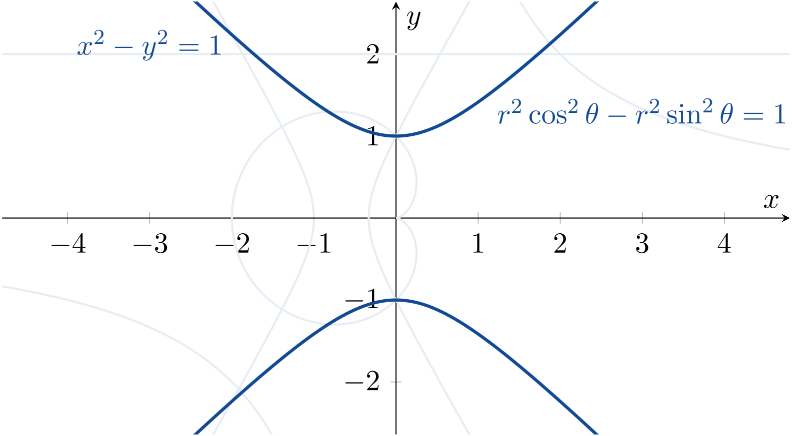
$x = r\cos\theta \quad y = r\sin\theta \quad x^2 + y^2 = r^2 \quad \tan\theta = \frac{y}{x}$

Cartesian Equation $\longleftrightarrow$ Polar Equation



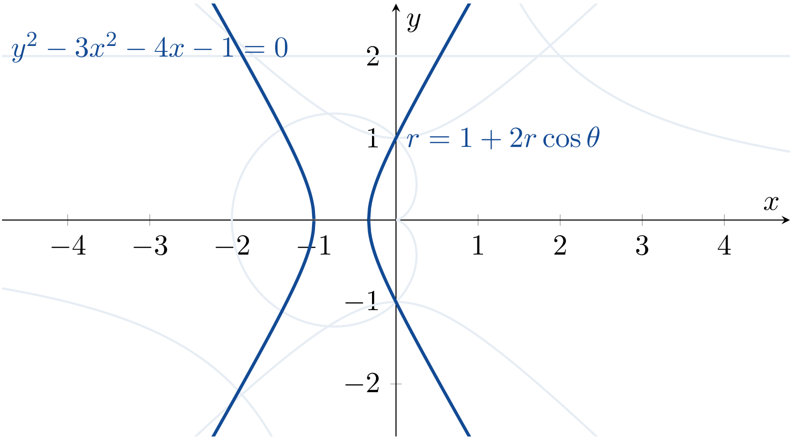
$x = r\cos\theta \quad y = r\sin\theta \quad x^2 + y^2 = r^2 \quad \tan\theta = \frac{y}{x}$

Cartesian Equation $\longleftrightarrow$ Polar Equation



$x = r\cos\theta \quad y = r\sin\theta \quad x^2 + y^2 = r^2 \quad \tan\theta = \frac{y}{x}$

Cartesian Equation $\longleftrightarrow$ Polar Equation



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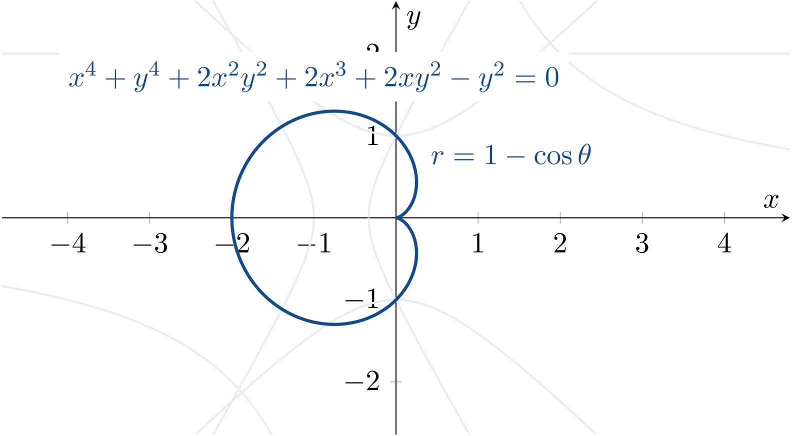
Cartesian Equation $\longleftrightarrow$ Polar Equation



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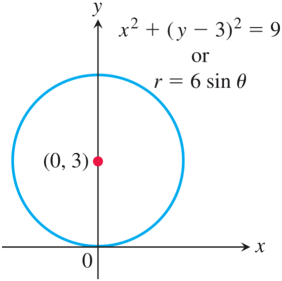

EXAMPLE 5 Find a polar equation for the circle $x^2 + (y-3)^2 = 9$

Solution We apply the equations relating polar and Cartesian coordinates:

$$x^2 + (y-3)^2 = 9$$

$$x^2 + y^2 - 6y + 9 = 9 \qquad \text{Expand } (y-3)^2.$$

$$x^2 + y^2 - 6y = 0 \qquad \text{Cancelation}$$

$$r^2 - 6r\sin\theta = 0 \qquad x^2 + y^2 = r^2,\ y = r\sin\theta$$

$$r = 0 \quad \text{or} \quad r - 6\sin\theta = 0$$

$$r = 6\sin\theta \qquad \text{Includes both possibilities}$$

Please quickly look through **10.4 Graphing in Polar Coordinates** (only 3 pages) in your textbook.

We will not be testing you on section 10.4, but it can help you to understand section 14.4 which we will study after the break.

Next Time

- 14.4 Double Integrals in Polar Form
- 14.5 Triple Integrals in Rectangular Coordinates
- 14.7 Triple Integrals in Cylindrical and Spherical Coordinates
- 14.8 Substitutions in Multiple Integrals