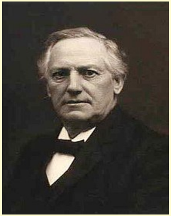
Lecture 11

- The Gram-Schmidt Process
- Orthogonal Matrices
- Orthogonal Diagonalisation

The Gram-Schmidt Process

The Gram-Schmidt Process



Jørgen Pedersen Gram

BORN
27 June 1850

DECEICED
29 April 1916

NATIONALITY
Danish

Erhard Schmidt

BORN
13 January 1876

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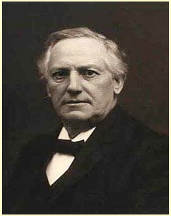
6 December 1959

NATIONALITY
Baltic German

Theorem

Every nonzero finite-dimensional inner product space has an orthonormal basis.

The Gram-Schmidt Process



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Theorem

Every nonzero finite-dimensional inner product space has an orthonormal basis.

But how can we find it?

Start with a basis that we know $\{\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_r\}$ $\longrightarrow$ Construct an orthogonal basis $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_r\}$ $\longrightarrow$ Create an orthonormal basis $\{\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_r\}$

The Gram-Schmidt Process

Just define $\mathbf{u}_i = \dfrac{\mathbf{v}_i}{\|\mathbf{v}_i\|}$ for each i.

Start with a basis that we know $\{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_r\}$ → Construct an orthogonal basis $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_r\}$ → Create an orthonormal basis $\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_r\}$

The Gram-Schmidt Process

Just define $\mathbf{u}_i = \dfrac{\mathbf{v}_i}{\|\mathbf{v}_i\|}$ for each i.

Start with a basis that we know $\{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_r\}$ → Construct an orthogonal basis $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_r\}$ → Create an orthonormal basis $\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_r\}$

Use the Gram-Schmidt process

The Gram-Schmidt Process

To convert a basis $\{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_r\}$ into an orthogonal basis $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_r\}$, perform the following steps:

The Gram-Schmidt Process

To convert a basis $\{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_r\}$ into an orthogonal basis $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_r\}$, perform the following steps:

1. $\mathbf{v}_1 = \mathbf{x}_1$
2. $\mathbf{v}_2 =$
3. $\mathbf{v}_3 =$
4. $\mathbf{v}_4 =$

The Gram-Schmidt Process

To convert a basis $\{\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_r\}$ into an orthogonal basis $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_r\}$, perform the following steps:

1. $\mathbf{v}_1 = \mathbf{x}_1$
2. $\mathbf{v}_2 = \mathbf{x}_2 - \dfrac{\langle \mathbf{x}_2, \mathbf{v}_1 \rangle}{\|\mathbf{v}_1\|^2}\mathbf{v}_1$
3. $\mathbf{v}_3 =$
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The Gram-Schmidt Process

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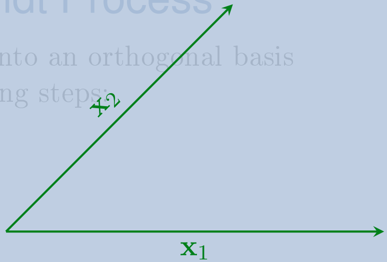

The Gram-Schmidt Process

To convert a basis $\{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_r\}$ into an orthogonal basis $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_r\}$, perform the following steps:

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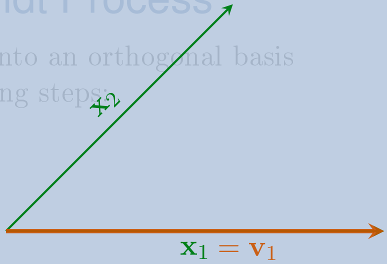

The Gram-Schmidt Process

To convert a basis $\{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_r\}$ into an orthogonal basis $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_r\}$, perform the following steps:

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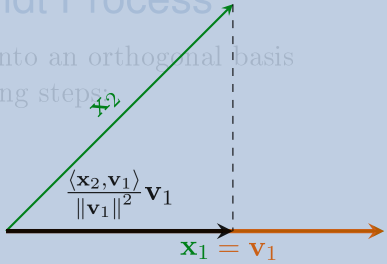

The Gram-Schmidt Process

To convert a basis $\{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_r\}$ into an orthogonal basis $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_r\}$, perform the following steps:

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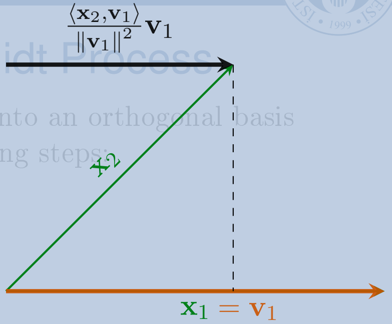

The Gram-Schmidt Process

To convert a basis $\{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_r\}$ into an orthogonal basis $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_r\}$, perform the following steps:

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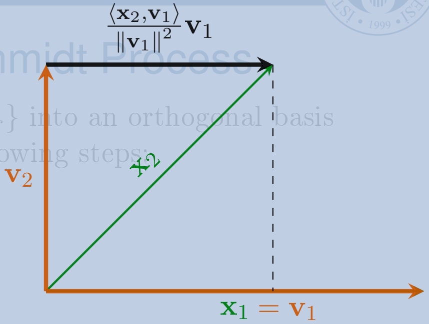

The Gram-Schmidt Process

To convert a basis $\{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_r\}$ into an orthogonal basis $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_r\}$, perform the following steps:

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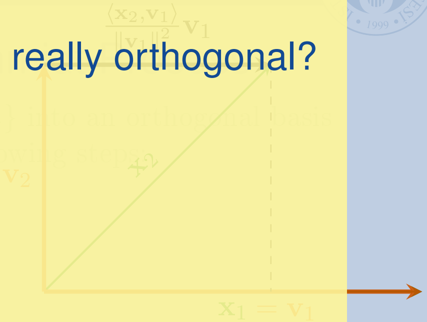

Are $\mathbf{v}_1$ and $\mathbf{v}_2$ really orthogonal?

$$
\begin{aligned}
\langle \mathbf{v}_1, \mathbf{v}_2 \rangle &= \\
&= \\
&= \\
&= \\
&=
\end{aligned}
$$

The Gram-Schmidt Process

To convert a basis $\{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_r\}$ into an orthogonal basis $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_r\}$, perform the following steps:

1. $\mathbf{v}_1 = \mathbf{x}_1$
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3. $\mathbf{v}_3 =$
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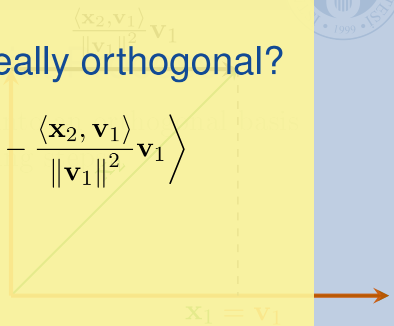

Are $\mathbf{v}_1$ and $\mathbf{v}_2$ really orthogonal?

$$
\begin{aligned}
\langle \mathbf{v}_1, \mathbf{v}_2 \rangle &= \left\langle \mathbf{v}_1, \mathbf{x}_2 - \frac{\langle \mathbf{x}_2, \mathbf{v}_1 \rangle}{\|\mathbf{v}_1\|^2}\mathbf{v}_1 \right\rangle \\
&= \\
&= \\
&= \\
&=
\end{aligned}
$$

The Gram-Schmidt Process

To convert a basis $\{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_r\}$ into an orthogonal basis $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_r\}$, perform the following computations:

1. $\mathbf{v}_1 = \mathbf{x}_1$
2. $\mathbf{v}_2 = \mathbf{x}_2 - \frac{\langle \mathbf{x}_2, \mathbf{v}_1 \rangle}{\|\mathbf{v}_1\|^2}\mathbf{v}_1$
3. $\mathbf{v}_3 =$
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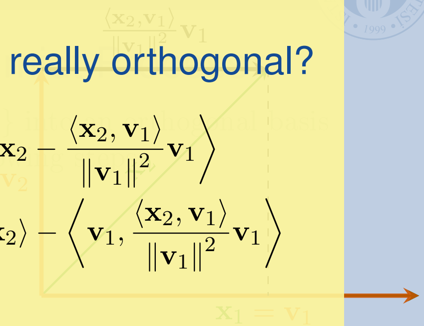

Are $\mathbf{v}_1$ and $\mathbf{v}_2$ really orthogonal?

$$\begin{aligned}
\langle \mathbf{v}_1, \mathbf{v}_2 \rangle &= \left\langle \mathbf{v}_1, \mathbf{x}_2 - \frac{\langle \mathbf{x}_2, \mathbf{v}_1 \rangle}{\|\mathbf{v}_1\|^2}\mathbf{v}_1 \right\rangle \\
&= \langle \mathbf{v}_1, \mathbf{x}_2 \rangle - \left\langle \mathbf{v}_1, \frac{\langle \mathbf{x}_2, \mathbf{v}_1 \rangle}{\|\mathbf{v}_1\|^2}\mathbf{v}_1 \right\rangle \\
&= \\
&= \\
&=
\end{aligned}$$

To convert a basis $\{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_r\}$ into an orthogonal basis $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_r\}$, perform the following computations:

1. $\mathbf{v}_1 = \mathbf{x}_1$
2. $\mathbf{v}_2 = \mathbf{x}_2 - \frac{\langle \mathbf{x}_2, \mathbf{v}_1 \rangle}{\|\mathbf{v}_1\|^2}\mathbf{v}_1$
3. $\mathbf{v}_3 =$
4. $\mathbf{v}_4 =$

Are $\mathbf{v}_1$ and $\mathbf{v}_2$ really orthogonal?

$$
\begin{aligned}
\langle \mathbf{v}_1, \mathbf{v}_2 \rangle &= \left\langle \mathbf{v}_1, \mathbf{x}_2 - \frac{\langle \mathbf{x}_2, \mathbf{v}_1 \rangle}{\|\mathbf{v}_1\|^2}\mathbf{v}_1 \right\rangle \\
&= \langle \mathbf{v}_1, \mathbf{x}_2 \rangle - \left\langle \mathbf{v}_1, \frac{\langle \mathbf{x}_2, \mathbf{v}_1 \rangle}{\|\mathbf{v}_1\|^2}\mathbf{v}_1 \right\rangle \\
&= \langle \mathbf{v}_1, \mathbf{x}_2 \rangle - \frac{\langle \mathbf{x}_2, \mathbf{v}_1 \rangle}{\|\mathbf{v}_1\|^2} \langle \mathbf{v}_1, \mathbf{v}_1 \rangle \\
&= \langle \mathbf{v}_1, \mathbf{x}_2 \rangle - \frac{\langle \mathbf{x}_2, \mathbf{v}_1 \rangle \langle \mathbf{v}_1, \mathbf{v}_1 \rangle}{\|\mathbf{v}_1\|^2} \\
&=
\end{aligned}
$$

To convert a basis $\{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_r\}$ into an orthogonal basis $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_r\}$, perform the following

1. $\mathbf{v}_1 = \mathbf{x}_1$
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Are $\mathbf{v}_1$ and $\mathbf{v}_2$ really orthogonal?

$$
\begin{aligned}
\langle \mathbf{v}_1, \mathbf{v}_2 \rangle &= \left\langle \mathbf{v}_1, \mathbf{x}_2 - \frac{\langle \mathbf{x}_2, \mathbf{v}_1 \rangle}{\|\mathbf{v}_1\|^2}\mathbf{v}_1 \right\rangle \\
&= \langle \mathbf{v}_1, \mathbf{x}_2 \rangle - \left\langle \mathbf{v}_1, \frac{\langle \mathbf{x}_2, \mathbf{v}_1 \rangle}{\|\mathbf{v}_1\|^2}\mathbf{v}_1 \right\rangle \\
&= \langle \mathbf{v}_1, \mathbf{x}_2 \rangle - \frac{\langle \mathbf{x}_2, \mathbf{v}_1 \rangle}{\|\mathbf{v}_1\|^2} \langle \mathbf{v}_1, \mathbf{v}_1 \rangle \\
&= \langle \mathbf{v}_1, \mathbf{x}_2 \rangle - \frac{\langle \mathbf{x}_2, \mathbf{v}_1 \rangle \langle \mathbf{v}_1, \mathbf{v}_1 \rangle}{\|\mathbf{v}_1\|^2} \\
&= \langle \mathbf{v}_1, \mathbf{x}_2 \rangle - \langle \mathbf{x}_2, \mathbf{v}_1 \rangle = 0.
\end{aligned}
$$

YES!

The Gram-Schmidt Process

To convert a basis $\{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_r\}$ into an orthogonal basis $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_r\}$, perform the following steps:

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The Gram-Schmidt Process

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3. $\mathbf{v}_3 = \mathbf{x}_3 - \dfrac{\langle \mathbf{x}_3, \mathbf{v}_1 \rangle}{\|\mathbf{v}_1\|^2}\mathbf{v}_1 - \dfrac{\langle \mathbf{x}_3, \mathbf{v}_2 \rangle}{\|\mathbf{v}_2\|^2}\mathbf{v}_2$
4. $\mathbf{v}_4 =$

The Gram-Schmidt Process

To convert a basis $\{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_r\}$ into an orthogonal basis $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_r\}$, perform the following steps:

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4. $\mathbf{v}_4 =$

Please check that

$$\langle \mathbf{v}_1, \mathbf{v}_3 \rangle = 0$$

and

$$\langle \mathbf{v}_2, \mathbf{v}_3 \rangle = 0.$$

The Gram-Schmidt Process

To convert a basis $\{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_r\}$ into an orthogonal basis $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_r\}$, perform the following steps:

1. $\mathbf{v}_1 = \mathbf{x}_1$
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4. $\mathbf{v}_4 = \mathbf{x}_4 - \dfrac{\langle \mathbf{x}_4, \mathbf{v}_1 \rangle}{\|\mathbf{v}_1\|^2}\mathbf{v}_1 - \dfrac{\langle \mathbf{x}_4, \mathbf{v}_2 \rangle}{\|\mathbf{v}_2\|^2}\mathbf{v}_2 - \dfrac{\langle \mathbf{x}_4, \mathbf{v}_3 \rangle}{\|\mathbf{v}_3\|^2}\mathbf{v}_3$

The Gram-Schmidt Process

To convert a basis $\{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_r\}$ into an orthogonal basis $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_r\}$, perform the following steps:

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4. $\mathbf{v}_4 = \mathbf{x}_4 - \dfrac{\langle \mathbf{x}_4, \mathbf{v}_1 \rangle}{\|\mathbf{v}_1\|^2}\mathbf{v}_1 - \dfrac{\langle \mathbf{x}_4, \mathbf{v}_2 \rangle}{\|\mathbf{v}_2\|^2}\mathbf{v}_2 - \dfrac{\langle \mathbf{x}_4, \mathbf{v}_3 \rangle}{\|\mathbf{v}_3\|^2}\mathbf{v}_3$

$$\vdots$$

(continue until you have $\mathbf{v}_r$)

The Gram-Schmidt Process

Example (Using the Gram-Schmidt Process)

Assume that the vector space $\mathbb{R}^3$ has the Euclidean inner product. Apply the Gram–Schmidt process to transform the basis vectors

$$\mathbf{x}_1 = (1, 1, 1), \qquad \mathbf{x}_2 = (0, 1, 1), \qquad \mathbf{x}_3 = (0, 0, 1)$$

into an orthogonal basis $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$, and then normalise the orthogonal basis vectors to obtain an orthonormal basis $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$.

$$\mathbf{x}_1 = (1,1,1), \quad \mathbf{x}_2 = (0,1,1), \quad \mathbf{x}_3 = (0,0,1)$$

Step 1: $\mathbf{v}_1 = \mathbf{x}_1 = (1,1,1)$

$$\mathbf{x}_1 = (1,1,1), \quad \mathbf{x}_2 = (0,1,1), \quad \mathbf{x}_3 = (0,0,1)$$

Step 1: $\mathbf{v}_1 = \mathbf{x}_1 = (1,1,1)$

Step 2: $\mathbf{v}_2 = \mathbf{x}_2 - \dfrac{\langle \mathbf{x}_2, \mathbf{v}_1 \rangle}{\|\mathbf{v}_1\|^2}\mathbf{v}_1 = (0,1,1) - \dfrac{2}{3}(1,1,1) = \left(-\dfrac{2}{3}, \dfrac{1}{3}, \dfrac{1}{3}\right)$

$$\mathbf{x}_1 = (1,1,1), \quad \mathbf{x}_2 = (0,1,1), \quad \mathbf{x}_3 = (0,0,1)$$

Step 1: $\mathbf{v}_1 = \mathbf{x}_1 = (1,1,1)$

Step 2: $\mathbf{v}_2 = \mathbf{x}_2 - \dfrac{\langle \mathbf{x}_2, \mathbf{v}_1 \rangle}{\|\mathbf{v}_1\|^2}\mathbf{v}_1 = (0,1,1) - \dfrac{2}{3}(1,1,1) = \left(-\dfrac{2}{3}, \dfrac{1}{3}, \dfrac{1}{3}\right)$

$$\begin{aligned}\langle \mathbf{v}_1, \mathbf{v}_2 \rangle &= (1,1,1) \cdot \left(-\frac{2}{3}, \frac{1}{3}, \frac{1}{3}\right) \\ &= (1)\left(-\frac{2}{3}\right) + (1)\left(\frac{1}{3}\right) + (1)\left(\frac{1}{3}\right) = 0 \qquad \checkmark\end{aligned}$$

$$\mathbf{x}_1 = (1,1,1), \quad \mathbf{x}_2 = (0,1,1), \quad \mathbf{x}_3 = (0,0,1)$$

Step 1: $\mathbf{v}_1 = \mathbf{x}_1 = (1,1,1)$

Step 2: $\mathbf{v}_2 = \mathbf{x}_2 - \dfrac{\langle \mathbf{x}_2, \mathbf{v}_1 \rangle}{\|\mathbf{v}_1\|^2}\mathbf{v}_1 = (0,1,1) - \dfrac{2}{3}(1,1,1) = \left(-\dfrac{2}{3}, \dfrac{1}{3}, \dfrac{1}{3}\right)$

$$\begin{aligned}\langle \mathbf{v}_1, \mathbf{v}_2 \rangle &= (1,1,1) \bullet \left(-\frac{2}{3}, \frac{1}{3}, \frac{1}{3}\right) \\ &= (1)\left(-\frac{2}{3}\right) + (1)\left(\frac{1}{3}\right) + (1)\left(\frac{1}{3}\right) = 0 \qquad \checkmark\end{aligned}$$

Step 3: $\mathbf{v}_3 = \mathbf{x}_3 - \dfrac{\langle \mathbf{x}_3, \mathbf{v}_1 \rangle}{\|\mathbf{v}_1\|^2}\mathbf{v}_1 - \dfrac{\langle \mathbf{x}_3, \mathbf{v}_2 \rangle}{\|\mathbf{v}_2\|^2}\mathbf{v}_2$

$$= (0,0,1) - \frac{1}{3}(1,1,1) - \frac{\frac{1}{3}}{\frac{2}{3}}\left(-\frac{2}{3}, \frac{1}{3}, \frac{1}{3}\right) = \left(0, -\frac{1}{2}, \frac{1}{2}\right).$$

$$\mathbf{v}_1 = (1,1,1), \quad \mathbf{v}_2 = \left(-\tfrac{2}{3}, \tfrac{1}{3}, \tfrac{1}{3}\right), \quad \mathbf{v}_3 = \left(0, -\tfrac{1}{2}, \tfrac{1}{2}\right)$$

Let's just finish checking if these are correct:

The Gram-Schmidt Process

$$\mathbf{v}_1 = (1,1,1), \quad \mathbf{v}_2 = \left(-\tfrac{2}{3}, \tfrac{1}{3}, \tfrac{1}{3}\right), \quad \mathbf{v}_3 = \left(0, -\tfrac{1}{2}, \tfrac{1}{2}\right)$$

Let's just finish checking if these are correct:

$$\langle \mathbf{v}_1, \mathbf{v}_3 \rangle = (1,1,1) \bullet \left(0, -\frac{1}{2}, \frac{1}{2}\right) = 0 - \frac{1}{2} + \frac{1}{2} = 0 \qquad \checkmark$$

$$\langle \mathbf{v}_2, \mathbf{v}_3 \rangle = \left(-\frac{2}{3}, \frac{1}{3}, \frac{1}{3}\right) \bullet \left(0, -\frac{1}{2}, \frac{1}{2}\right) = 0 - \frac{1}{6} + \frac{1}{6} = 0 \qquad \checkmark$$

$$\mathbf{v}_1 = (1,1,1), \quad \mathbf{v}_2 = \left(-\tfrac{2}{3}, \tfrac{1}{3}, \tfrac{1}{3}\right), \quad \mathbf{v}_3 = \left(0, -\tfrac{1}{2}, \tfrac{1}{2}\right)$$

Let's just finish checking if these are correct:

$$\langle \mathbf{v}_1, \mathbf{v}_3 \rangle = (1,1,1) \bullet \left(0, -\frac{1}{2}, \frac{1}{2}\right) = 0 - \frac{1}{2} + \frac{1}{2} = 0 \qquad \checkmark$$

$$\langle \mathbf{v}_2, \mathbf{v}_3 \rangle = \left(-\frac{2}{3}, \frac{1}{3}, \frac{1}{3}\right) \bullet \left(0, -\frac{1}{2}, \frac{1}{2}\right) = 0 - \frac{1}{6} + \frac{1}{6} = 0 \qquad \checkmark$$

Therefore $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is an orthogonal basis for $\mathbb{R}^3$.

$$\mathbf{v}_1 = (1,1,1), \quad \mathbf{v}_2 = \left(-\tfrac{2}{3}, \tfrac{1}{3}, \tfrac{1}{3}\right), \quad \mathbf{v}_3 = \left(0, -\tfrac{1}{2}, \tfrac{1}{2}\right)$$

The next step is to normalise these three vectors.

$\mathbf{v}_1 = (1,1,1), \quad \mathbf{v}_2 = \left(-\frac{2}{3}, \frac{1}{3}, \frac{1}{3}\right), \quad \mathbf{v}_3 = \left(0, -\frac{1}{2}, \frac{1}{2}\right)$

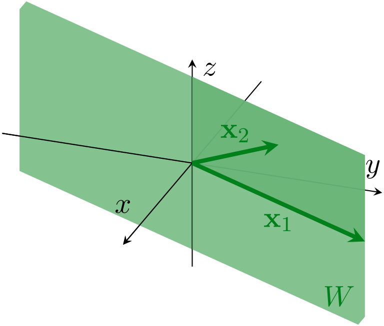
The next step is to normalise these three vectors. Since

$$\|\mathbf{v}_1\| = \sqrt{3}, \qquad \|\mathbf{v}_2\| = \frac{\sqrt{6}}{3}, \qquad \|\mathbf{v}_3\| = \frac{1}{\sqrt{2}},$$

(please check) it follows that

$$\begin{aligned}
\mathbf{u}_1 &= \frac{\mathbf{v}_1}{\|\mathbf{v}_1\|} = \left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right) \\
\mathbf{u}_2 &= \frac{\mathbf{v}_2}{\|\mathbf{v}_2\|} = \left(-\frac{2}{\sqrt{6}}, \frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}\right) \\
\mathbf{u}_3 &= \frac{\mathbf{v}_3}{\|\mathbf{v}_3\|} = \left(0, -\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right).
\end{aligned}$$

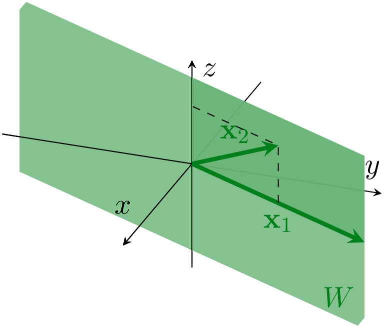
Then $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ is an orthonormal basis for $\mathbb{R}^3$.



Example

Let $V = \mathbb{R}^3$ with the Euclidean inner product and let $W = \text{span}\{\mathbf{x}_1, \mathbf{x}_2\}$ where $\mathbf{x}_1 = \begin{bmatrix} 3 \\ 6 \\ 0 \end{bmatrix}$ and $\mathbf{x}_2 = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$. Construct an orthogonal basis for W.

The Gram-Schmidt Process



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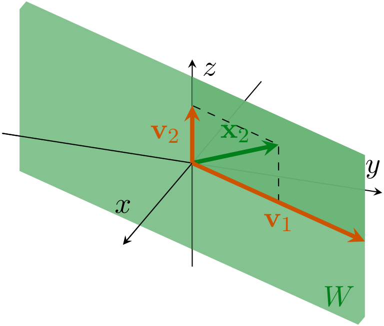

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$$\mathbf{v}_1 = \mathbf{x}_1 = \begin{bmatrix} 3 \\ 6 \\ 0 \end{bmatrix}$$

$$\mathbf{v}_2 =$$

Example

Let $V = \mathbb{R}^3$ with the Euclidean inner product and let $W = \text{span}\{\mathbf{x}_1, \mathbf{x}_2\}$ where $\mathbf{x}_1 = \begin{bmatrix} 3 \\ 6 \\ 0 \end{bmatrix}$ and $\mathbf{x}_2 = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$. Construct an orthogonal basis for W.

$$\mathbf{v}_1 = \mathbf{x}_1 = \begin{bmatrix} 3 \\ 6 \\ 0 \end{bmatrix}$$

$$\mathbf{v}_2 = \mathbf{x}_2 - \frac{\langle \mathbf{x}_2, \mathbf{v}_1 \rangle}{\|\mathbf{v}_1\|^2}\mathbf{v}_1 = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} - \frac{15}{45}\begin{bmatrix} 3 \\ 6 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix}.$$

The Gram-Schmidt Process

Example

Let $V = \mathbb{R}^3$ with the Euclidean inner product and let $W = \operatorname{span}\{\mathbf{x}_1, \mathbf{x}_2\}$ where $\mathbf{x}_1 = \begin{bmatrix} 3 \\ 6 \\ 0 \end{bmatrix}$ and $\mathbf{x}_2 = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$. Construct an orthogonal basis for W.

$$\mathbf{v}_1 = \mathbf{x}_1 = \begin{bmatrix} 3 \\ 6 \\ 0 \end{bmatrix}$$

$$\mathbf{v}_2 = \mathbf{x}_2 - \frac{\langle \mathbf{x}_2, \mathbf{v}_1 \rangle}{\|\mathbf{v}_1\|^2}\mathbf{v}_1 = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} - \frac{15}{45}\begin{bmatrix} 3 \\ 6 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix}.$$

$\{\mathbf{v}_1, \mathbf{v}_2\}$ is an orthogonal basis for W.

Example

Let $\mathbf{x}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$, $\mathbf{x}_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix}$ and $\mathbf{x}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}$ be three vectors in $\mathbb{R}^4$ with the Euclidean inner product. It is easy to see that $\{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3\}$ is linearly independent (Why?).

The Gram-Schmidt Process

Example

Let $\mathbf{x}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$, $\mathbf{x}_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix}$ and $\mathbf{x}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}$ be three vectors in $\mathbb{R}^4$ with the Euclidean inner product. It is easy to see that $\{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3\}$ is linearly independent (Why?).

Construct an orthogonal basis for $W = \mathrm{span}\{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3\}$.

The Gram-Schmidt Process

$$\mathbf{x}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \mathbf{x}_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \mathbf{x}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}$$

$$\mathbf{v}_1 = \mathbf{x}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \quad \mathbf{v}_2 = \mathbf{x}_2 - \frac{\langle \mathbf{x}_2, \mathbf{v}_1 \rangle}{\|\mathbf{v}_1\|^2} \mathbf{v}_1 = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix} - \frac{3}{4} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -\frac{3}{4} \\ \frac{1}{4} \\ \frac{1}{4} \\ \frac{1}{4} \end{bmatrix}.$$

$$\mathbf{x}_1 = \begin{bmatrix}1\\1\\1\\1\end{bmatrix}, \mathbf{x}_2 = \begin{bmatrix}0\\1\\1\\1\end{bmatrix}, \mathbf{x}_3 = \begin{bmatrix}0\\0\\1\\1\end{bmatrix}$$

$$\mathbf{v}_1 = \mathbf{x}_1 = \begin{bmatrix}1\\1\\1\\1\end{bmatrix}, \quad \mathbf{v}_2 = \mathbf{x}_2 - \frac{\langle \mathbf{x}_2, \mathbf{v}_1 \rangle}{\|\mathbf{v}_1\|^2}\mathbf{v}_1 = \begin{bmatrix}0\\1\\1\\1\end{bmatrix} - \frac{3}{4}\begin{bmatrix}1\\1\\1\\1\end{bmatrix} = \begin{bmatrix}-\frac{3}{4}\\\frac{1}{4}\\\frac{1}{4}\\\frac{1}{4}\end{bmatrix}.$$

At this point we can do an optional step to simplify our calculations. Remember that we are trying to find an orthogonal basis. We don't care how big (long) our vectors are, we only care that they are orthogonal.

$$\mathbf{x}_1 = \begin{bmatrix}1\\1\\1\\1\end{bmatrix}, \mathbf{x}_2 = \begin{bmatrix}0\\1\\1\\1\end{bmatrix}, \mathbf{x}_3 = \begin{bmatrix}0\\0\\1\\1\end{bmatrix}$$

$$\mathbf{v}_1 = \mathbf{x}_1 = \begin{bmatrix}1\\1\\1\\1\end{bmatrix}, \quad \mathbf{v}_2 = \mathbf{x}_2 - \frac{\langle \mathbf{x}_2, \mathbf{v}_1 \rangle}{\|\mathbf{v}_1\|^2}\mathbf{v}_1 = \begin{bmatrix}0\\1\\1\\1\end{bmatrix} - \frac{3}{4}\begin{bmatrix}1\\1\\1\\1\end{bmatrix} = \begin{bmatrix}-\frac{3}{4}\\\frac{1}{4}\\\frac{1}{4}\\\frac{1}{4}\end{bmatrix}.$$

At this point we can do an optional step to simplify our calculations. Remember that we are trying to find an orthogonal basis. We don't care how big (long) our vectors are, we only care that they are orthogonal. So we can multiply $\mathbf{v}_2$ by 4 to obtain a new $\mathbf{v}_2$:

$$\mathbf{v}_2 = \begin{bmatrix}-3\\1\\1\\1\end{bmatrix}.$$

The Gram-Schmidt Process

$$\mathbf{x}_1 = \begin{bmatrix}1\\1\\1\\1\end{bmatrix}, \mathbf{x}_2 = \begin{bmatrix}0\\1\\1\\1\end{bmatrix}, \mathbf{x}_3 = \begin{bmatrix}0\\0\\1\\1\end{bmatrix} \qquad \mathbf{v}_1 = \begin{bmatrix}1\\1\\1\\1\end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix}-3\\1\\1\\1\end{bmatrix}$$

Then

$$\mathbf{v}_3 = \mathbf{x}_3 - \frac{\langle \mathbf{x}_3, \mathbf{v}_1 \rangle}{\|\mathbf{v}_1\|^2}\mathbf{v}_1 - \frac{\langle \mathbf{x}_3, \mathbf{v}_2 \rangle}{\|\mathbf{v}_2\|^2}\mathbf{v}_2 = \begin{bmatrix}0\\0\\1\\1\end{bmatrix} - \frac{2}{4}\begin{bmatrix}1\\1\\1\\1\end{bmatrix} - \frac{2}{12}\begin{bmatrix}-3\\1\\1\\1\end{bmatrix} = \begin{bmatrix}0\\-\frac{2}{3}\\\frac{1}{3}\\\frac{1}{3}\end{bmatrix}.$$

$$\mathbf{x}_1 = \begin{bmatrix}1\\1\\1\\1\end{bmatrix}, \mathbf{x}_2 = \begin{bmatrix}0\\1\\1\\1\end{bmatrix}, \mathbf{x}_3 = \begin{bmatrix}0\\0\\1\\1\end{bmatrix} \qquad \mathbf{v}_1 = \begin{bmatrix}1\\1\\1\\1\end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix}-3\\1\\1\\1\end{bmatrix}$$

Then

$$\mathbf{v}_3 = \mathbf{x}_3 - \frac{\langle \mathbf{x}_3, \mathbf{v}_1 \rangle}{\|\mathbf{v}_1\|^2}\mathbf{v}_1 - \frac{\langle \mathbf{x}_3, \mathbf{v}_2 \rangle}{\|\mathbf{v}_2\|^2}\mathbf{v}_2 = \begin{bmatrix}0\\0\\1\\1\end{bmatrix} - \frac{2}{4}\begin{bmatrix}1\\1\\1\\1\end{bmatrix} - \frac{2}{12}\begin{bmatrix}-3\\1\\1\\1\end{bmatrix} = \begin{bmatrix}0\\-\frac{2}{3}\\\frac{1}{3}\\\frac{1}{3}\end{bmatrix}.$$

Again we can multiply $\mathbf{v}_3$ by a scalar if we want to. Let's multiply it by 3 to get a new vector which we will now call $\mathbf{v}_3$:

$$\mathbf{v}_3 = \begin{bmatrix}0\\-2\\1\\1\end{bmatrix}.$$

Hence

$$\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -3 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ -2 \\ 1 \\ 1 \end{bmatrix} \right\}$$

is an orthogonal basis for W.

Adrien-Marie Legendre

BORN
18 September 1752

DECEICED
9 January 1833

NATIONALITY
French

Example (Legendre Polynomials)

Consider the vector space $\mathbb{P}^2$ with the inner product

$$\langle \mathbf{p}, \mathbf{q} \rangle = \int_{-1}^{1} p(x)q(x)\, dx.$$

Apply the Gram-Schmidt process to transform the standard basis $\{1, x, x^2\}$ into an orthogonal basis.

Let $\mathbf{x}_1 = 1$, $\mathbf{x}_2 = x$ and $\mathbf{x}_3 = x^2$.

$\mathbf{x}_1 = 1,\ \mathbf{x}_2 = x,\ \mathbf{x}_3 = x^2$

Step 1: $\mathbf{v}_1 = \mathbf{x}_1 = 1$

$\mathbf{x}_1 = 1,\ \mathbf{x}_2 = x,\ \mathbf{x}_3 = x^2$

Step 1: $\mathbf{v}_1 = \mathbf{x}_1 = 1$

Step 2: $\langle \mathbf{x}_2, \mathbf{v}_1 \rangle = \int_{-1}^{1} x\, dx = 0$

$\mathbf{x}_1 = 1,\ \mathbf{x}_2 = x,\ \mathbf{x}_3 = x^2$

$$\begin{aligned}
\text{Step 1:} \quad \mathbf{v}_1 &= \mathbf{x}_1 = 1 \\
\text{Step 2:} \quad \langle \mathbf{x}_2, \mathbf{v}_1 \rangle &= \int_{-1}^{1} x\,dx = 0 \\
\mathbf{v}_2 &= \mathbf{x}_2 - \frac{\langle \mathbf{x}_2, \mathbf{v}_1 \rangle}{\|\mathbf{v}_1\|^2}\mathbf{v}_1 = x - 0 = x
\end{aligned}$$

$\mathbf{x}_1 = 1,\ \mathbf{x}_2 = x,\ \mathbf{x}_3 = x^2$

Step 1:
$$\mathbf{v}_1 = \mathbf{x}_1 = 1$$

Step 2:
$$\begin{aligned}\langle \mathbf{x}_2, \mathbf{v}_1 \rangle &= \int_{-1}^{1} x\,dx = 0 \\ \mathbf{v}_2 &= \mathbf{x}_2 - \frac{\langle \mathbf{x}_2, \mathbf{v}_1 \rangle}{\|\mathbf{v}_1\|^2}\mathbf{v}_1 = x - 0 = x\end{aligned}$$

Step 3:
$$\begin{aligned}\langle \mathbf{x}_3, \mathbf{v}_1 \rangle &= \int_{-1}^{1} x^2\,dx = \frac{2}{3} \\ \langle \mathbf{x}_3, \mathbf{v}_2 \rangle &= \int_{-1}^{1} x^3\,dx = 0 \\ \|\mathbf{v}_1\|^2 &= \langle \mathbf{v}_1, \mathbf{v}_1 \rangle = \int_{-1}^{1} 1\,dx = 2\end{aligned}$$

$\mathbf{x}_1 = 1,\ \mathbf{x}_2 = x,\ \mathbf{x}_3 = x^2$

Step 1: $\mathbf{v}_1 = \mathbf{x}_1 = 1$

Step 2:
$$\begin{aligned}
\langle \mathbf{x}_2, \mathbf{v}_1 \rangle &= \int_{-1}^{1} x\, dx = 0 \\
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\end{aligned}$$

Step 3:
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\langle \mathbf{x}_3, \mathbf{v}_1 \rangle &= \int_{-1}^{1} x^2\, dx = \frac{2}{3} \\
\langle \mathbf{x}_3, \mathbf{v}_2 \rangle &= \int_{-1}^{1} x^3\, dx = 0 \\
\|\mathbf{v}_1\|^2 &= \langle \mathbf{v}_1, \mathbf{v}_1 \rangle = \int_{-1}^{1} 1\, dx = 2 \\
\mathbf{v}_3 &= \mathbf{x}_3 - \frac{\langle \mathbf{x}_3, \mathbf{v}_1 \rangle}{\|\mathbf{v}_1\|^2}\mathbf{v}_1 - \frac{\langle \mathbf{x}_3, \mathbf{v}_2 \rangle}{\|\mathbf{v}_2\|^2}\mathbf{v}_2 = x^2 - \frac{1}{3} - 0.
\end{aligned}$$

$\mathbf{x}_1 = 1,\ \mathbf{x}_2 = x,\ \mathbf{x}_3 = x^2$

Therefore

$$\left\{1, x, x^2 - \frac{1}{3}\right\}$$

is an orthogonal basis for $\mathbb{P}^2$.

Orthogonal Matrices

Orthogonal Matrices

$\mathbf{x}_1 = 1$, $\mathbf{x}_2 = x$, $\mathbf{x}_3 = x^2$

Definition

An invertible square matrix A is called *orthogonal* iff

$$A^{-1} = A^T.$$

Orthogonal Matrices

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Definition

An invertible square matrix A is called *orthogonal* iff

$$A^{-1} = A^T.$$

This is equivalent to

$$AA^T = A^T A = I.$$

$\mathbf{x}_1 = 1$, $\mathbf{x}_2 = x$, $\mathbf{x}_3 = x^2$

Example

The matrix

$$A = \begin{bmatrix} \frac{3}{7} & \frac{2}{7} & \frac{6}{7} \\ -\frac{6}{7} & \frac{3}{7} & \frac{2}{7} \\ \frac{2}{7} & \frac{6}{7} & -\frac{3}{7} \end{bmatrix}$$

is orthogonal since

$$A^T A = \begin{bmatrix} \frac{3}{7} & -\frac{6}{7} & \frac{2}{7} \\ \frac{2}{7} & \frac{3}{7} & \frac{6}{7} \\ \frac{6}{7} & \frac{2}{7} & -\frac{3}{7} \end{bmatrix} \begin{bmatrix} \frac{3}{7} & \frac{2}{7} & \frac{6}{7} \\ -\frac{6}{7} & \frac{3}{7} & \frac{2}{7} \\ \frac{2}{7} & \frac{6}{7} & -\frac{3}{7} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

(please check).

$\mathbf{x}_1 = 1$, $\mathbf{x}_2 = x$, $\mathbf{x}_3 = x^2$

Example (Rotation and Reflection Matrices are Orthogonal)

Recall from Lecture 8 that the standard matrices for anticlockwise rotation of $\mathbb{R}^2$ by an angle of θ; and for reflection about the x-axis are

$$A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \qquad \text{and} \qquad B = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

respectively.

$\mathbf{x}_1 = 1$, $\mathbf{x}_2 = x$, $\mathbf{x}_3 = x^2$

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respectively. Note that

$$A^T A = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

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and

$$B^T B = B^2 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} -1 \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I.$$

Theorem

Let A be an $n \times n$ matrix. The following are equivalent:

1. *A is orthogonal ($A^{-1} = A^T$).*
2. *The row vectors of A form an orthonormal set in $\mathbb{R}^n$ with the Euclidean inner product.*
3. *The column vectors of A form an orthonormal set in $\mathbb{R}^n$ with the Euclidean inner product.*

$\mathbf{x}_1 = 1,\ \mathbf{x}_2 = x,\ \mathbf{x}_3 = x^2$

Proof.

Let

- $\mathbf{r}_i$ be the ith row vector of A;
- $\mathbf{c}_j$ be the jth column vector of A;
- $\mathbf{r}_i^T$ be the ith row vector of A^T; and
- $\mathbf{c}_j^T$ be the jth column vector of A^T.

Because transposing a matrix swaps rows with columns, and columns with rows, we must have

$$\mathbf{r}_i^T = \mathbf{c}_i \qquad \text{and} \qquad \mathbf{c}_j^T = \mathbf{r}_j.$$

Proof Continued.

Note that

$$AA^T = \begin{bmatrix} \mathbf{r}_1 \\ \mathbf{r}_2 \\ \vdots \\ \mathbf{r}_n \end{bmatrix} \begin{bmatrix} \mathbf{c}_1^T & \mathbf{c}_2^T & \cdots & \mathbf{c}_n^T \end{bmatrix} =$$

$$=$$

Proof Continued.

Note that

$$AA^T = \begin{bmatrix} \mathbf{r}_1 \\ \mathbf{r}_2 \\ \vdots \\ \mathbf{r}_n \end{bmatrix} \begin{bmatrix} \mathbf{c}_1^T & \mathbf{c}_2^T & \cdots & \mathbf{c}_n^T \end{bmatrix} = \begin{bmatrix} \mathbf{r}_1\mathbf{c}_1^T & \mathbf{r}_1\mathbf{c}_2^T & \cdots & \mathbf{r}_1\mathbf{c}_n^T \\ \mathbf{r}_2\mathbf{c}_1^T & \mathbf{r}_2\mathbf{c}_2^T & \cdots & \mathbf{r}_2\mathbf{c}_n^T \\ \vdots & \vdots & & \vdots \\ \mathbf{r}_n\mathbf{c}_1^T & \mathbf{r}_n\mathbf{c}_2^T & \cdots & \mathbf{r}_n\mathbf{c}_n^T \end{bmatrix}$$

$$=$$

Proof Continued.

Note that

$$AA^T = \begin{bmatrix} \mathbf{r}_1 \\ \mathbf{r}_2 \\ \vdots \\ \mathbf{r}_n \end{bmatrix} \begin{bmatrix} \mathbf{c}_1^T & \mathbf{c}_2^T & \cdots & \mathbf{c}_n^T \end{bmatrix} = \begin{bmatrix} \mathbf{r}_1\mathbf{c}_1^T & \mathbf{r}_1\mathbf{c}_2^T & \cdots & \mathbf{r}_1\mathbf{c}_n^T \\ \mathbf{r}_2\mathbf{c}_1^T & \mathbf{r}_2\mathbf{c}_2^T & \cdots & \mathbf{r}_2\mathbf{c}_n^T \\ \vdots & \vdots & & \vdots \\ \mathbf{r}_n\mathbf{c}_1^T & \mathbf{r}_n\mathbf{c}_2^T & \cdots & \mathbf{r}_n\mathbf{c}_n^T \end{bmatrix}$$

$$= \begin{bmatrix} \mathbf{r}_1 \bullet \mathbf{r}_1 & \mathbf{r}_1 \bullet \mathbf{r}_2 & \cdots & \mathbf{r}_1 \bullet \mathbf{r}_n \\ \mathbf{r}_2 \bullet \mathbf{r}_1 & \mathbf{r}_2 \bullet \mathbf{r}_2 & \cdots & \mathbf{r}_2 \bullet \mathbf{r}_n \\ \vdots & \vdots & & \vdots \\ \mathbf{r}_n \bullet \mathbf{r}_1 & \mathbf{r}_n \bullet \mathbf{r}_2 & \cdots & \mathbf{r}_n \bullet \mathbf{r}_n \end{bmatrix}.$$

Proof Continued.

Note that

$$AA^T = \begin{bmatrix} \mathbf{r}_1 \\ \mathbf{r}_2 \\ \vdots \\ \mathbf{r}_n \end{bmatrix} \begin{bmatrix} \mathbf{c}_1^T & \mathbf{c}_2^T & \cdots & \mathbf{c}_n^T \end{bmatrix} = \begin{bmatrix} \mathbf{r}_1\mathbf{c}_1^T & \mathbf{r}_1\mathbf{c}_2^T & \cdots & \mathbf{r}_1\mathbf{c}_n^T \\ \mathbf{r}_2\mathbf{c}_1^T & \mathbf{r}_2\mathbf{c}_2^T & \cdots & \mathbf{r}_2\mathbf{c}_n^T \\ \vdots & \vdots & & \vdots \\ \mathbf{r}_n\mathbf{c}_1^T & \mathbf{r}_n\mathbf{c}_2^T & \cdots & \mathbf{r}_n\mathbf{c}_n^T \end{bmatrix}$$

$$= \begin{bmatrix} \mathbf{r}_1 \bullet \mathbf{r}_1 & \mathbf{r}_1 \bullet \mathbf{r}_2 & \cdots & \mathbf{r}_1 \bullet \mathbf{r}_n \\ \mathbf{r}_2 \bullet \mathbf{r}_1 & \mathbf{r}_2 \bullet \mathbf{r}_2 & \cdots & \mathbf{r}_2 \bullet \mathbf{r}_n \\ \vdots & \vdots & & \vdots \\ \mathbf{r}_n \bullet \mathbf{r}_1 & \mathbf{r}_n \bullet \mathbf{r}_2 & \cdots & \mathbf{r}_n \bullet \mathbf{r}_n \end{bmatrix}.$$

We can see from this formula that $AA^T = I$ if and only if

$$\mathbf{r}_1 \bullet \mathbf{r}_1 = \mathbf{r}_2 \bullet \mathbf{r}_2 = \ldots = \mathbf{r}_n \bullet \mathbf{r}_n = 1$$

and

$$\mathbf{r}_i \bullet \mathbf{r}_k = 0 \qquad \text{for all } i \neq j.$$

$\mathbf{x}_1 = 1$, $\mathbf{x}_2 = x$, $\mathbf{x}_3 = x^2$

Proof Continued.

This proves that

$$AA^T = I \qquad \Longleftrightarrow \qquad \{\mathbf{r}_1, \mathbf{r}_2, \ldots, \mathbf{r}_n\} \text{ is an orthonormal set in } \mathbb{R}^n.$$

$\mathbf{x}_1 = 1,\ \mathbf{x}_2 = x,\ \mathbf{x}_3 = x^2$

Proof Continued.

This proves that

$$AA^T = I \qquad \iff \qquad \{\mathbf{r}_1, \mathbf{r}_2, \ldots, \mathbf{r}_n\} \text{ is an orthonormal set in } \mathbb{R}^n.$$

If we did a similar calculating starting with $A^T A$ instead of AA^T then we could prove that

$$A^T A = I \qquad \iff \qquad \{\mathbf{c}_1, \mathbf{c}_2, \ldots, \mathbf{c}_n\} \text{ is an orthonormal set in } \mathbb{R}^n.$$

(I leave this proof for you to write.) □

$\mathbf{x}_1 = 1,\ \mathbf{x}_2 = x,\ \mathbf{x}_3 = x^2$

Theorem

1. *The transpose of an orthogonal matrix is orthogonal.*
2. *The inverse of an orthogonal matrix is orthogonal.*
3. *A product of orthogonal matrices is orthogonal.*
4. *If A is orthogonal, then* $\det(A) = 1$ *or* $\det(A) = -1$.

I will prove parts 1 and 4. I leave parts 2 and 3 for you to prove.

$\mathbf{x}_1 = 1$, $\mathbf{x}_2 = x$, $\mathbf{x}_3 = x^2$

Proof.

1. Let A be an orthogonal matrix. Then we know that

$$A^T A = I = AA^T.$$

Let $B = A^T$. Then $B^T = (A^T)^T = A$. Hence

$$BB^T = I = B^T B$$

which proves that B is also orthogonal.

4.

$\mathbf{x}_1 = 1,\ \mathbf{x}_2 = x,\ \mathbf{x}_3 = x^2$

Proof.

1. Let A be an orthogonal matrix. Then we know that

$$A^T A = I = AA^T.$$

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4. Suppose that A satisfies $AA^T = I$. Then

$$1 = \det(I) = \det(A)\det(A^T) =$$

$\mathbf{x}_1 = 1$, $\mathbf{x}_2 = x$, $\mathbf{x}_3 = x^2$

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4. Suppose that A satisfies $AA^T = I$. Then

$$1 = \det(I) = \det(A)\det(A^T) = \det(A)\det(A) = \det(A)^2.$$

Hence $\det(A) = \pm 1$.

$\square$

Example

The matrix

$$A = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$$

is orthogonal because its row vectors form an orthonormal set in $\mathbb{R}^2$ (check!).

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is orthogonal because its row vectors form an orthonormal set in $\mathbb{R}^2$ (check!).

I leave it to you to check that the determinant of this matrix is

$$\det(A) = 1.$$

Orthogonal Matrices

$\mathbf{x}_1 = 1,\ \mathbf{x}_2 = x,\ \mathbf{x}_3 = x^2$

Lemma

If $\mathbf{u}$ *and* $\mathbf{v}$ *are vectors in* $\mathbb{R}^n$ *with the Euclidean inner product, then*

$$\mathbf{u}\cdot\mathbf{v} = \frac{1}{4}\left\|\mathbf{u}+\mathbf{v}\right\|^2 - \frac{1}{4}\left\|\mathbf{u}-\mathbf{v}\right\|^2.$$

Proof.

Since

$$\left\|\mathbf{u}+\mathbf{v}\right\|^2 = (\mathbf{u}+\mathbf{v})\cdot(\mathbf{u}+\mathbf{v}) = \left\|\mathbf{u}\right\|^2 + 2\mathbf{u}\cdot\mathbf{v} + \left\|\mathbf{v}\right\|^2$$
$$\left\|\mathbf{u}-\mathbf{v}\right\|^2 = (\mathbf{u}-\mathbf{v})\cdot(\mathbf{u}-\mathbf{v}) = \left\|\mathbf{u}\right\|^2 - 2\mathbf{u}\cdot\mathbf{v} + \left\|\mathbf{v}\right\|^2$$

we have

$$\left\|\mathbf{u}+\mathbf{v}\right\|^2 - \left\|\mathbf{u}-\mathbf{v}\right\|^2 = 4\mathbf{u}\cdot\mathbf{v}.$$

□

Theorem

Let A be an $n \times n$ matrix. The following are equivalent:

1. *A is orthogonal.*
2. *$\|A\mathbf{x}\| = \|\mathbf{x}\|$ for all $\mathbf{x}$ in $\mathbb{R}^n$.*
3. *$A\mathbf{x} \cdot A\mathbf{y} = \mathbf{x} \cdot \mathbf{y}$ for all $\mathbf{x}$ and $\mathbf{y}$ in $\mathbb{R}^n$.*

$\mathbf{x}_1 = 1,\ \mathbf{x}_2 = x,\ \mathbf{x}_3 = x^2$

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Proof.

I will prove that $\boxed{1} \implies \boxed{2} \implies \boxed{3} \implies \boxed{1}$.

1. A is orthogonal.
2. $\|A\mathbf{x}\| = \|\mathbf{x}\|$ for all $\mathbf{x}$ in $\mathbb{R}^n$.
3. $A\mathbf{x} \cdot A\mathbf{y} = \mathbf{x} \cdot \mathbf{y}$ for all $\mathbf{x}$ and $\mathbf{y}$ in $\mathbb{R}^n$.

Proof Continued.

1 $\Longrightarrow$ 2 : Note first that since $\mathbf{x} \cdot \mathbf{y} = \mathbf{y}^T\mathbf{x}$ we have

$$A\mathbf{u} \cdot \mathbf{v} = \mathbf{v}^T(A\mathbf{u})$$

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Proof Continued.

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$$A\mathbf{u} \cdot \mathbf{v} = \mathbf{v}^T(A\mathbf{u}) = (\mathbf{v}^T A)\mathbf{u}$$

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Proof Continued.

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Proof Continued.

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Proof Continued.

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If $A^T A = I$, then

$$\|A\mathbf{x}\|^2 = A\mathbf{x} \cdot A\mathbf{x} = \mathbf{x} \cdot A^T A\mathbf{x}$$

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If $A^T A = I$, then

$$\|A\mathbf{x}\|^2 = A\mathbf{x} \cdot A\mathbf{x} = \mathbf{x} \cdot A^T A\mathbf{x} = \mathbf{x} \cdot I\mathbf{x} = \mathbf{x} \cdot \mathbf{x} = \|\mathbf{x}\|^2.$$

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Proof Continued.

2 $\implies$ 3 : Suppose that $\|A\mathbf{x}\| = \|\mathbf{x}\|$ for all $\mathbf{x} \in \mathbb{R}^n$. By the previous lemma we have

$$\begin{aligned} A\mathbf{x} \cdot A\mathbf{y} &= \\ &= \\ &= \\ &= \mathbf{x} \cdot \mathbf{y}. \end{aligned}$$

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$$\begin{aligned} A\mathbf{x} \cdot A\mathbf{y} &= \frac{1}{4}\|A\mathbf{x} + A\mathbf{y}\|^2 - \frac{1}{4}\|A\mathbf{x} - A\mathbf{y}\|^2 \\ &= \\ &= \\ &= \end{aligned}$$

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Proof Continued.

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Proof Continued.

3 $\implies$ 1 : Suppose that

$$\mathbf{x} \cdot \mathbf{y} = A\mathbf{x} \cdot A\mathbf{y} = \mathbf{x} \cdot A^T A\mathbf{y}$$

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This equation holds for all $\mathbf{x} \in \mathbb{R}^n$. Let $\mathbf{x} = (A^T A - I)\mathbf{y}$.

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Proof Continued.

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This equation holds for all $\mathbf{x} \in \mathbb{R}^n$. Let $\mathbf{x} = (A^T A - I)\mathbf{y}$. Then

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But this equation holds for all $\mathbf{y} \in \mathbb{R}^n$, so we must have $A^T A - I = \mathbf{0}$.

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Proof Continued.

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This equation holds for all $\mathbf{x} \in \mathbb{R}^n$. Let $\mathbf{x} = (A^T A - I)\mathbf{y}$. Then

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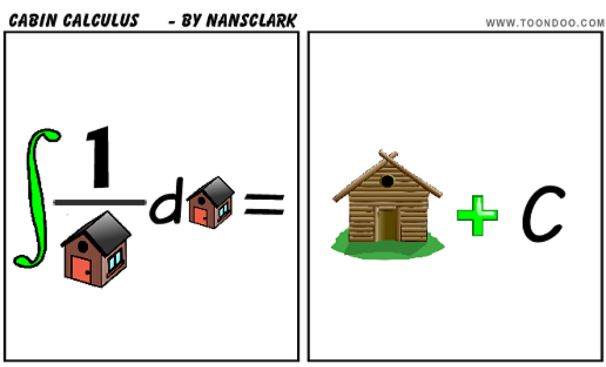
But this equation holds for all $\mathbf{y} \in \mathbb{R}^n$, so we must have $A^T A - I = \mathbf{0}$. Therefore $A^T A = I$ and we are finished.

□

Break

We will continue at 3pm



Orthogonal Diagonalisation

Recall that two square matrices A and B are called *similar* if there exists an invertible matrix P such that

$$P^{-1}AP = B.$$

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If A is similar to a diagonal matrix D, with

$$P^{-1}AP = D$$

then we say that A is *diagonalisable* and that P *diagonalises* A.

(See Lectures 8 and 9.)

Orthogonal Diagonalisation

Sometimes it possible to find an orthogonal matrix P such that

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Definition

If A and B are square matrices, then we say that A and B are *orthogonally similar* if there is an orthogonal matrix P such that

$$P^T AP = B.$$

Definition

If A is orthogonally similar to a diagonal matrix D, with

$$P^T AP = D$$

then we say that A is *orthogonally diagonalisable* and that P *orthogonally diagonalises* A.

Remark

If $P^TAP = D$ then we can multiply on the left by P, and on the right by P^T to get

$$PP^TAPP^T = PDP^T.$$

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But since $P^T P = I = PP^T$, this is just

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Taking the transpose of both sides gives

$$A^T = (PDP^T)^T$$

Remark

If $P^T A P = D$ then we can multiply on the left by P, and on the right by P^T to get

$$P P^T A P P^T = P D P^T.$$

But since $P^T P = I = P P^T$, this is just

$$A = P D P^T.$$

Taking the transpose of both sides gives

$$A^T = (P D P^T)^T = (P^T)^T D^T P^T$$

Remark

If $P^TAP = D$ then we can multiply on the left by P, and on the right by P^T to get

$$PP^TAPP^T = PDP^T.$$

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Taking the transpose of both sides gives

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If $P^TAP = D$ then we can multiply on the left by P, and on the right by P^T to get

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Taking the transpose of both sides gives

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Orthogonal Diagonalisation

Remark

If $P^TAP = D$ then we can multiply on the left by P, and on the right by P^T to get

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But since $P^TP = I = PP^T$, this is just

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Taking the transpose of both sides gives

$$A^T = (PDP^T)^T = (P^T)^T D^T P^T = PDP^T = A.$$

Hence

$$\begin{array}{c} A \text{ is orthogonally} \\ \text{diagonalisable} \end{array} \implies A \text{ is symmetric.}$$

Orthogonal Diagonalisation

Theorem

Let $A \in \mathbb{R}^{n\times n}$. The following are equivalent:

1. *A is orthogonally diagonalizable.*
2. *A has an orthonormal set of n eigenvectors.*
3. *A is symmetric.*

(proof omitted)

Orthogonal Diagonalisation

Theorem

Let $A \in \mathbb{R}^{n\times n}$. The following are equivalent:

1. *A is orthogonally diagonalizable.*
2. *A has an orthonormal set of n eigenvectors.*
3. *A is symmetric.*

(proof omitted)

Theorem

Let $A \in \mathbb{R}^{n\times n}$. If A is symmetric then, the eigenvalues of A are all real numbers.

(proof omitted)

Orthogonal Diagonalisation

Theorem

If $A \in \mathbb{R}^{n \times n}$ is symmetric, then any two eigenvectors from different eigenspaces are orthogonal.

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If $A \in \mathbb{R}^{n \times n}$ is symmetric, then any two eigenvectors from different eigenspaces are orthogonal.

Proof.

Let

- $\mathbf{x}_1$ be an eigenvector corresponding to the eigenvalue λ_1; and
- $\mathbf{x}_2$ be an eigenvector corresponding to the eigenvalue $\lambda_2 \neq \lambda_1$.

We need to prove that $\mathbf{x}_1 \cdot \mathbf{x}_2 = 0$.

Orthogonal Diagonalisation

Proof Continued.

We calculate that

$$\begin{aligned} \lambda_1 \mathbf{x}_1 \bullet \mathbf{x}_2 &= A\mathbf{x}_1 \bullet \mathbf{x}_2 && \text{since } \mathbf{x}_1 \text{ is an eigenvector} \\ &= \\ &= \\ &= \\ &= \end{aligned}$$

Proof Continued.

We calculate that

$$
\begin{aligned}
\lambda_1 \mathbf{x}_1 \cdot \mathbf{x}_2 &= A\mathbf{x}_1 \cdot \mathbf{x}_2 && \text{since } \mathbf{x}_1 \text{ is an eigenvector} \\
&= \mathbf{x}_1 \cdot A^T \mathbf{x}_2 \\
&= \\
&= \\
&=
\end{aligned}
$$

Orthogonal Diagonalisation

Proof Continued.

We calculate that

$$\begin{aligned}
\lambda_1 \mathbf{x}_1 \cdot \mathbf{x}_2 &= A\mathbf{x}_1 \cdot \mathbf{x}_2 && \text{since } \mathbf{x}_1 \text{ is an eigenvector} \\
&= \mathbf{x}_1 \cdot A^T \mathbf{x}_2 \\
&= \mathbf{x}_1 \cdot A\mathbf{x}_2 && \text{since } A \text{ is symmetric} \\
&= \\
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Proof Continued.

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It follows that

$$(\lambda_1 - \lambda_2)\mathbf{x}_1 \bullet \mathbf{x}_2 = 0.$$

But since $\lambda_1 \neq \lambda_2$ we must have $\mathbf{x}_1 \bullet \mathbf{x}_2 = 0$. □

How to Orthogonally Diagonalise an $n \times n$ Symmetric Matrix

Let A be an $n \times n$ symmetric matrix with real entries.

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Then P will orthogonally diagonalise A. The eigenvalues on the main diagonal of $D = P^T AP$ will be in the same order as their corresponding eigenvectors in P.

Example

Find an orthogonal matrix P which diagonalises

$$A = \begin{bmatrix} 4 & 2 & 2 \\ 2 & 4 & 2 \\ 2 & 2 & 4 \end{bmatrix}.$$

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$$A = \begin{bmatrix} 4 & 2 & 2 \\ 2 & 4 & 2 \\ 2 & 2 & 4 \end{bmatrix}.$$

Please check that

$$0 = \det(\lambda I - A) = \begin{vmatrix} \lambda - 4 & -2 & -2 \\ -2 & \lambda - 4 & -2 \\ -2 & -2 & \lambda - 4 \end{vmatrix} = (\lambda - 2)^2(\lambda - 8).$$

Thus the eigenvalues of A are $\lambda = 2$ and $\lambda = 8$.

Please also check that

$$\mathbf{x}_1 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \qquad \text{and} \qquad \mathbf{x}_2 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

form a basis for the eigenspace corresponding to $\lambda = 2$.

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We need to use the Gram-Schmidt process on $\{\mathbf{x}_1, \mathbf{x}_2\}$ to obtain an orthonormal basis.

Orthogonal Diagonalisation

$$\mathbf{x}_1 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \qquad \mathbf{x}_2 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

First we calculate

$$\mathbf{v}_1 = \mathbf{x}_1 =$$

$$\mathbf{v}_2 = \mathbf{x}_2 - \frac{\langle \mathbf{x}_2, \mathbf{v}_1 \rangle}{\|\mathbf{v}_1\|^2}\mathbf{v}_1 =$$

Then we normalise to obtain

$$\mathbf{u}_1 = \frac{\mathbf{v}_1}{\|\mathbf{v}_1\|} = \qquad , \qquad \mathbf{u}_2 = \frac{\mathbf{v}_2}{\|\mathbf{v}_2\|} =$$

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(Please check the calculations on this slide.)

Next please check that the eigenspace corresponding to $\lambda = 8$ has basis

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Now that we know our $\mathbf{u}_1$, $\mathbf{u}_2$ and $\mathbf{u}_3$, we can write

$$P = \begin{bmatrix} \mathbf{u}_1 & \mathbf{u}_2 & \mathbf{u}_3 \end{bmatrix} =$$

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I leave for you to check that

$$P^T A P = \begin{bmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ -\frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{6}} & \frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \end{bmatrix} \begin{bmatrix} 4 & 2 & 2 \\ 2 & 4 & 2 \\ 2 & 2 & 4 \end{bmatrix} \begin{bmatrix} -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ 0 & \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} \end{bmatrix} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 8 \end{bmatrix}.$$

Orthogonal Diagonalisation

Example

Orthogonally diagonalise the matrix $A = \begin{bmatrix} 3 & -6 & 0 \\ -6 & 0 & 6 \\ 0 & 6 & -3 \end{bmatrix}$.

$$A = \begin{bmatrix} 3 & -6 & 0 \\ -6 & 0 & 6 \\ 0 & 6 & -3 \end{bmatrix}$$

The characteristic equation of A is

$$\begin{aligned}
0 = \det(\lambda I - A) &= \begin{vmatrix} \lambda - 3 & 6 & 0 \\ 6 & \lambda & -6 \\ 0 & -6 & \lambda + 3 \end{vmatrix} \\
&= (\lambda - 3) \begin{vmatrix} \lambda & -6 \\ -6 & \lambda + 3 \end{vmatrix} - 6 \begin{vmatrix} 6 & -6 \\ 0 & \lambda + 3 \end{vmatrix} + 0 \\
&= (\lambda - 3)\big(\lambda(\lambda + 3) - 36\big) - 6\big(6\lambda + 18\big) \\
&= (\lambda - 3)\big(\lambda^2 + 3\lambda - 36\big) - 36\lambda - 108 \\
&= \lambda^3 + 3\lambda^2 - 36\lambda - 3\lambda^2 - 9\lambda + 108 - 36\lambda - 108 \\
&= \lambda^3 - 81\lambda = \lambda(\lambda - 9)(\lambda + 9).
\end{aligned}$$

The eigenvalues are $\lambda = 0$, $\lambda = 9$ and $\lambda = -9$.

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Next we need to find bases for the eigenspaces. Since we have three different eigenvalues, each eigenspace will be one-dimensional.

$\lambda = 0,\ \lambda = 9,\ \lambda = -9$ $\qquad A = \begin{bmatrix} 3 & -6 & 0 \\ -6 & 0 & 6 \\ 0 & 6 & -3 \end{bmatrix}$

Since

$$A = \begin{bmatrix} 3 & -6 & 0 \\ -6 & 0 & 6 \\ 0 & 6 & -3 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 & 0 \\ -2 & 0 & 2 \\ 0 & 2 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 & 0 \\ 0 & -4 & 2 \\ 0 & 2 & -1 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & -2 & 0 \\ 0 & 1 & -\frac{1}{2} \\ 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -\frac{1}{2} \\ 0 & 0 & 0 \end{bmatrix},$$

a basis for $\operatorname{Nul} A$ is given by $\mathbf{x}_1 = \begin{bmatrix} 1 \\ \frac{1}{2} \\ 1 \end{bmatrix}$.

(Remember that elementary row operations do not change a null space.)

$\lambda = 0,\ \lambda = 9,\ \lambda = -9$ $\qquad A = \begin{bmatrix} 3 & -6 & 0 \\ -6 & 0 & 6 \\ 0 & 6 & -3 \end{bmatrix}$

The RREF of $A - 9I$ is $\begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{bmatrix}$. Hence a basis for

$\mathrm{Nul}(A - 9I) = \mathrm{Nul}(9I - A)$ is given by $\mathbf{x}_2 = \begin{bmatrix} -2 \\ 2 \\ 1 \end{bmatrix}$.

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$\mathrm{Nul}(A - (-9)I) = \mathrm{Nul}(-9I - A)$ is given by $\mathbf{x}_3 = \begin{bmatrix} -\frac{1}{2} \\ -1 \\ 1 \end{bmatrix}$.

(Please check these!)

$$\mathbf{x}_1 = \begin{bmatrix} 1 \\ \frac{1}{2} \\ 1 \end{bmatrix}, \quad \mathbf{x}_2 = \begin{bmatrix} -2 \\ 2 \\ 1 \end{bmatrix}, \quad \mathbf{x}_3 = \begin{bmatrix} -\frac{1}{2} \\ -1 \\ 1 \end{bmatrix}$$

Since we only have one vector in each basis, we can skip the Gram-Schmidt process and move straight to normalisation.

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$$\|\mathbf{x}_1\|^2 = \mathbf{x}_1 \cdot \mathbf{x}_1 = \frac{9}{4}, \qquad \|\mathbf{x}_2\|^2 = \mathbf{x}_2 \cdot \mathbf{x}_2 = 9, \qquad \|\mathbf{x}_3\|^2 = \mathbf{x}_3 \cdot \mathbf{x}_3 = \frac{9}{4},$$

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we have

$$\mathbf{u}_1 = \frac{\mathbf{x}_1}{\|\mathbf{x}_1\|} = \frac{2}{3}\mathbf{x}_1 = \begin{bmatrix} \frac{2}{3} \\ \frac{1}{3} \\ \frac{2}{3} \end{bmatrix}, \qquad \mathbf{u}_2 = \frac{\mathbf{x}_2}{\|\mathbf{x}_2\|} = \frac{1}{3}\mathbf{x}_2 = \begin{bmatrix} -\frac{2}{3} \\ \frac{2}{3} \\ \frac{1}{3} \end{bmatrix},$$

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$$P = \begin{bmatrix} \mathbf{u}_1 & \mathbf{u}_2 & \mathbf{u}_2 \end{bmatrix} = \begin{bmatrix} \frac{2}{3} & -\frac{2}{3} & -\frac{1}{3} \\ \frac{1}{3} & \frac{2}{3} & -\frac{2}{3} \\ \frac{2}{3} & \frac{1}{3} & \frac{2}{3} \end{bmatrix} = \frac{1}{3} \begin{bmatrix} 2 & -2 & -1 \\ 1 & 2 & -2 \\ 2 & 1 & 2 \end{bmatrix}.$$

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I leave it to you to check that

$$D = P^T A P = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & -9 \end{bmatrix}$$

or equivalently that

$$A = PDP^T.$$

Example (page 396)

Orthogonally Diagonalise the matrix $A = \begin{bmatrix} 3 & -2 & 4 \\ -2 & 6 & 2 \\ 4 & 2 & 3 \end{bmatrix}$, whose characteristic equation is

$$0 = \lambda^3 - 12\lambda^2 + 21\lambda + 98 = (\lambda - 7)^2(\lambda + 2).$$

Orthogonal Diagonalisation

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The usual calculations produce bases for the eigenspaces:

$$\lambda = 7: \ \mathbf{x}_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \ \mathbf{x}_2 = \begin{bmatrix} -\frac{1}{2} \\ 1 \\ 0 \end{bmatrix}; \qquad \lambda = -2: \ \mathbf{x}_3 = \begin{bmatrix} -1 \\ -\frac{1}{2} \\ 1 \end{bmatrix}.$$

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Although $\mathbf{x}_1$ and $\mathbf{x}_2$ are linearly independent, they are not orthogonal. So we must use the Gram-Schmidt process on $\mathbf{x}_1$ and $\mathbf{x}_2$.

$$\lambda = 7: \ \mathbf{x}_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \ \mathbf{x}_2 = \begin{bmatrix} -\frac{1}{2} \\ 1 \\ 0 \end{bmatrix}; \quad \lambda = -2: \ \mathbf{x}_3 = \begin{bmatrix} -1 \\ -\frac{1}{2} \\ 1 \end{bmatrix}$$

We calculate that

$$\mathbf{v}_1 = \mathbf{x}_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

$$\mathbf{v}_2 = \qquad = \qquad .$$

The we normalise these vectors to obtain an orthonormal basis for the eigenspace corresponding to $\lambda = 7$:

$$\mathbf{u}_1 = \frac{\mathbf{v}_1}{\|\mathbf{v}_1\|} = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ \frac{1}{\sqrt{2}} \end{bmatrix}, \qquad \mathbf{u}_2 = \frac{\mathbf{v}_2}{\|\mathbf{v}_2\|} = \begin{bmatrix} -\frac{1}{\sqrt{18}} \\ \frac{4}{\sqrt{18}} \\ \frac{1}{\sqrt{18}} \end{bmatrix}$$

$$\lambda = 7: \ \mathbf{x}_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \ \mathbf{x}_2 = \begin{bmatrix} -\frac{1}{2} \\ 1 \\ 0 \end{bmatrix}; \quad \lambda = -2: \ \mathbf{x}_3 = \begin{bmatrix} -1 \\ -\frac{1}{2} \\ 1 \end{bmatrix}$$

We calculate that

$$\mathbf{v}_1 = \mathbf{x}_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

$$\mathbf{v}_2 = \mathbf{x}_2 - \frac{\langle \mathbf{x}_2, \mathbf{v}_1 \rangle}{\|\mathbf{v}_1\|^2}\mathbf{v}_1 = \qquad .$$

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$$\mathbf{u}_3 = \frac{\mathbf{x}_3}{\|\mathbf{x}_3\|} = \begin{bmatrix} -\frac{2}{3} \\ -\frac{1}{3} \\ \frac{2}{3} \end{bmatrix}.$$

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Let

$$P = \begin{bmatrix} \mathbf{u}_1 & \mathbf{u}_2 & \mathbf{u}_3 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{18}} & -\frac{2}{3} \\ 0 & \frac{4}{\sqrt{18}} & -\frac{1}{3} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{18}} & \frac{2}{3} \end{bmatrix} \quad \text{and} \quad D = \begin{bmatrix} 7 & 0 & 0 \\ 0 & 7 & 0 \\ 0 & 0 & -2 \end{bmatrix}.$$

Then P orthogonally diagonalises A and $A = PDP^T$.

Next Time

- Singular Value Decomposition