

Lecture 9

- 24. Limits
- 25. Continuity
- 26. Differentiation
- 27. Differentiation Rules



Limits

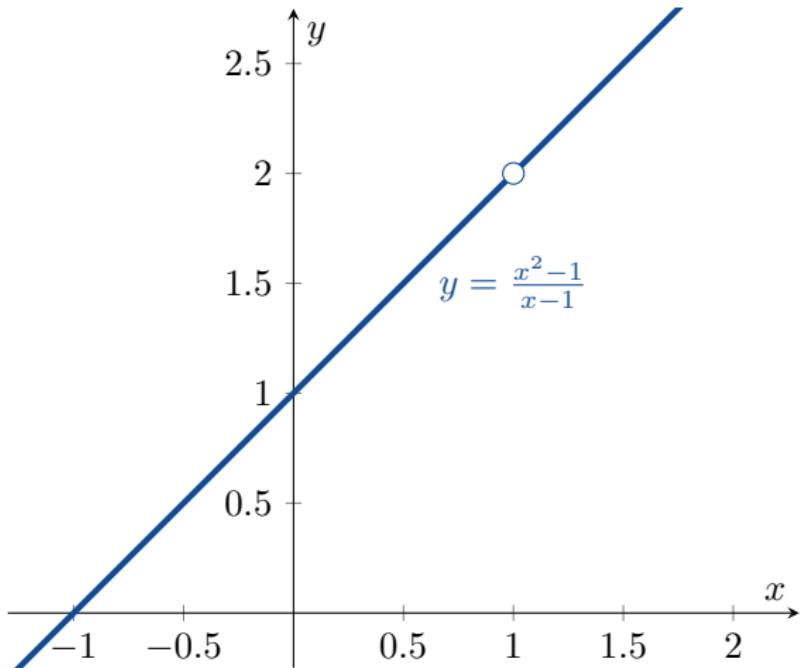
24. Limits



Consider the function $f(x) = \frac{x^2 - 1}{x - 1}$, $f : (-\infty, 1) \cup (1, \infty) \rightarrow \mathbb{R}$.

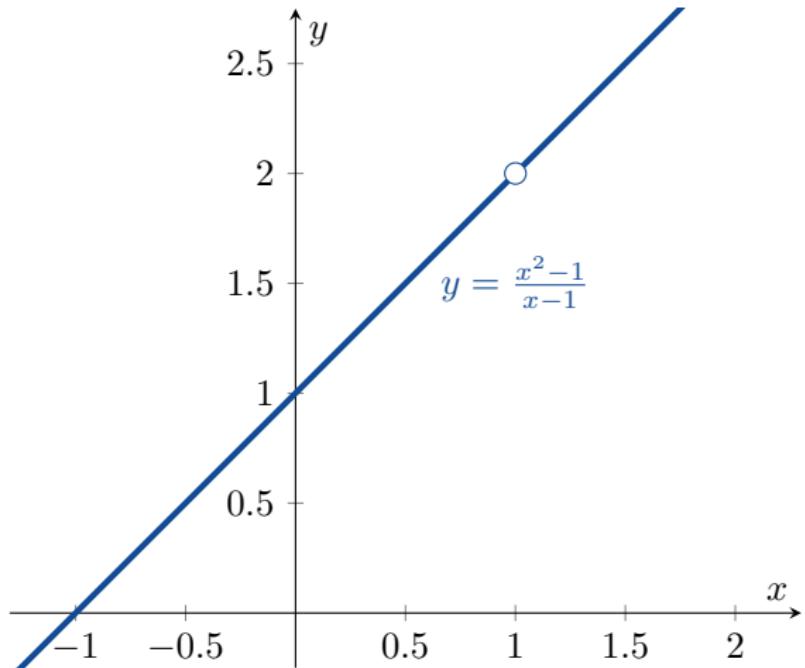
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24. Limits

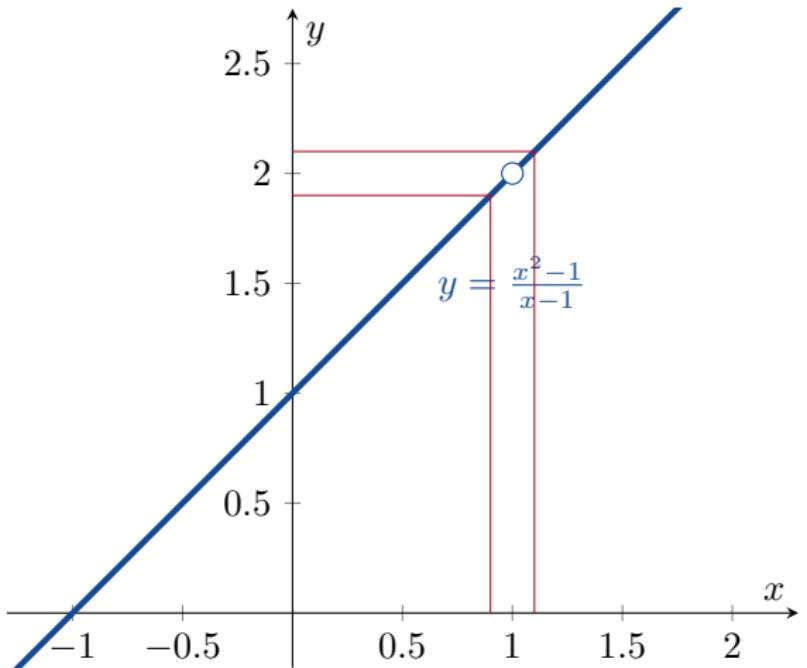
Consider the function $f(x) = \frac{x^2-1}{x-1}$, $f : (-\infty, 1) \cup (1, \infty) \rightarrow \mathbb{R}$.



Question: How does f behave when x is close to 1?

24. Limits

Consider the function $f(x) = \frac{x^2-1}{x-1}$, $f : (-\infty, 1) \cup (1, \infty) \rightarrow \mathbb{R}$.

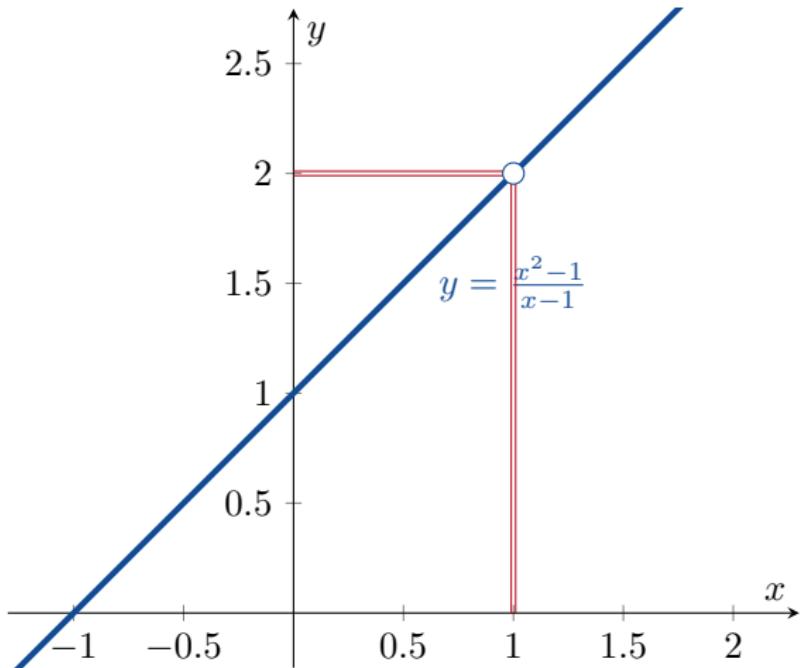


x	$f(x)$
0.9	1.9
1.1	2.1

Question: How does f behave when x is close to 1?

24. Limits

Consider the function $f(x) = \frac{x^2-1}{x-1}$, $f : (-\infty, 1) \cup (1, \infty) \rightarrow \mathbb{R}$.

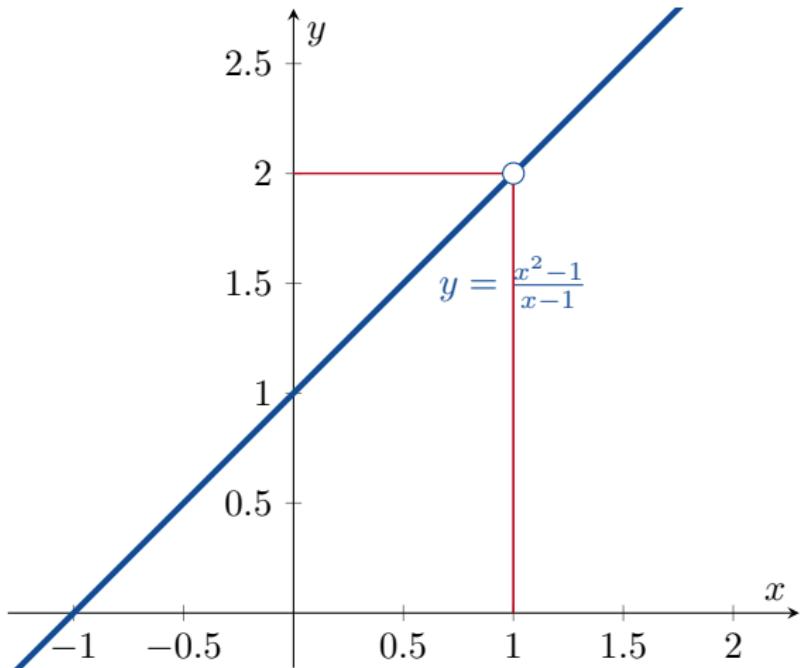


x	$f(x)$
0.9	1.9
1.1	2.1
0.99	1.99
1.01	2.01

Question: How does f behave when x is close to 1?

24. Limits

Consider the function $f(x) = \frac{x^2-1}{x-1}$, $f : (-\infty, 1) \cup (1, \infty) \rightarrow \mathbb{R}$.

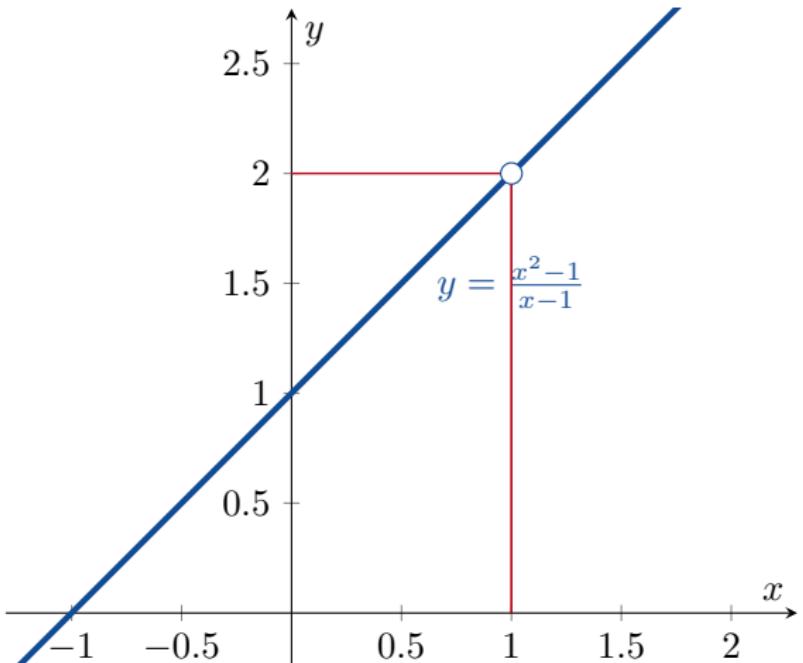


x	$f(x)$
0.9	1.9
1.1	2.1
0.99	1.99
1.01	2.01
0.999	1.999
1.001	2.001

Question: How does f behave when x is close to 1?

24. Limits

Consider the function $f(x) = \frac{x^2-1}{x-1}$, $f : (-\infty, 1) \cup (1, \infty) \rightarrow \mathbb{R}$.



x	$f(x)$
0.9	1.9
1.1	2.1
0.99	1.99
1.01	2.01
0.999	1.999
1.001	2.001

“If x is close to 1, then $f(x)$ is close to 2.”

24. Limits



“If x is close to 1, then $f(x)$ is close to 2.”

Mathematically, we write this as

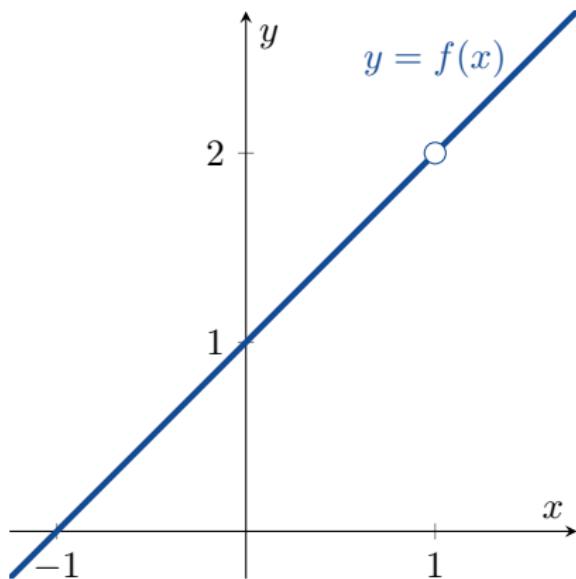
$$\lim_{x \rightarrow 1} f(x) = 2$$

and read it as “the limit, as x tends to 1, of $f(x)$ is equal to 2”.

24. Limits

Example

$$f(x) = \frac{x^2 - 1}{x - 1}$$

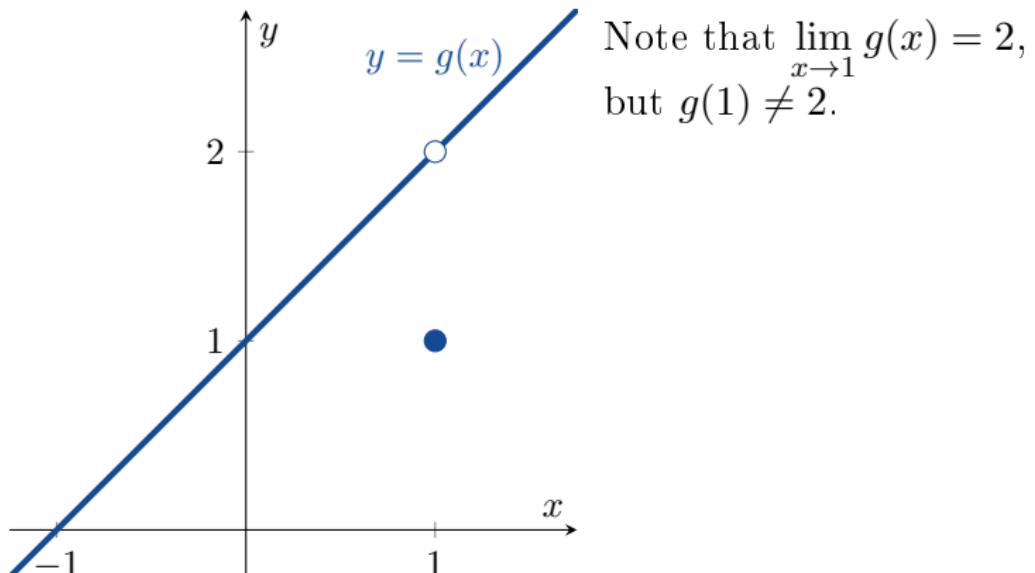


Note that $\lim_{x \rightarrow 1} f(x) = 2$,
but f is not defined at $x = 1$.

24. Limits

Example

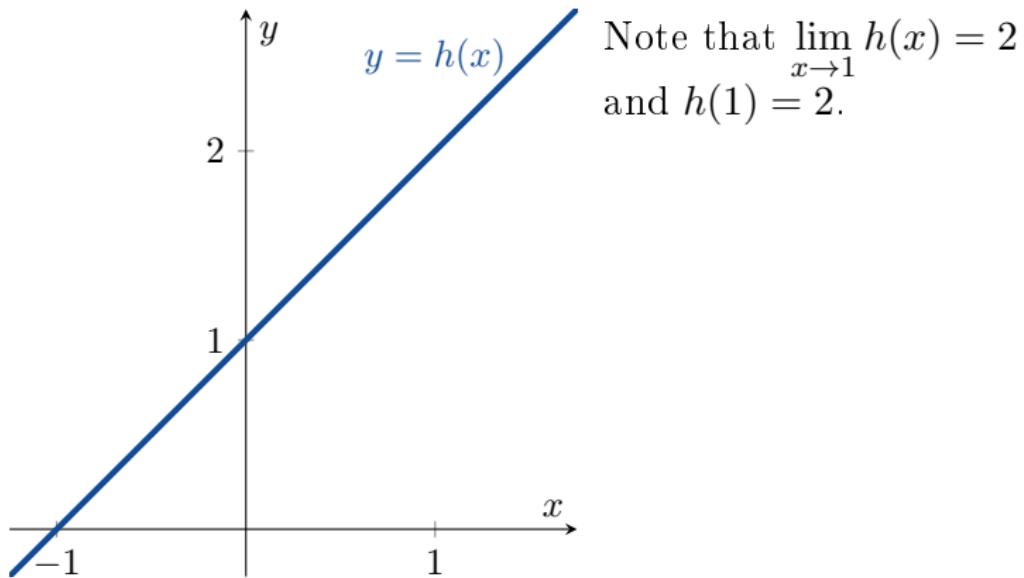
$$g(x) = \begin{cases} \frac{x^2-1}{x-1} & x \neq 1 \\ 1 & x = 1 \end{cases}$$



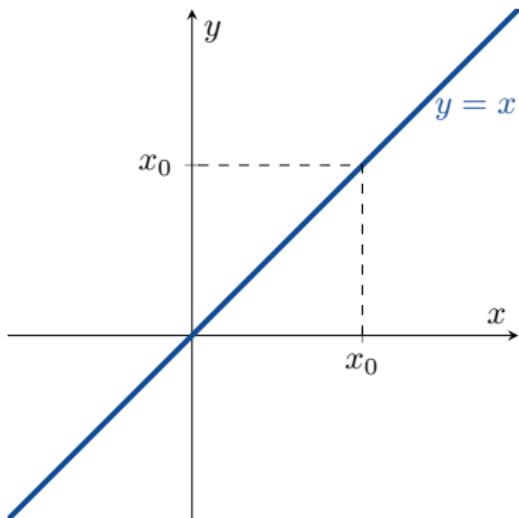
24. Limits

Example

$$h(x) = x + 1$$



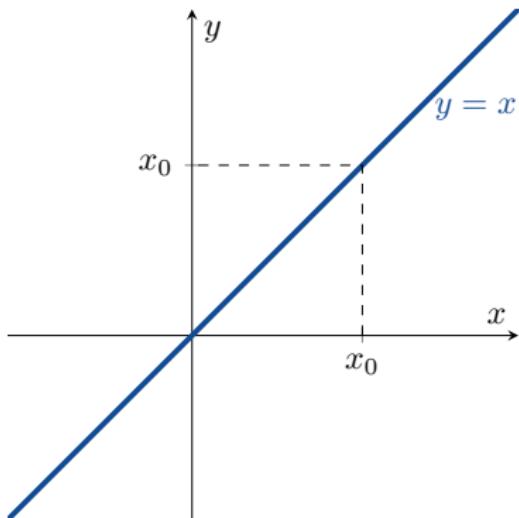
24. Limits



Example (The Identity Function)

$$f(x) = x$$

24. Limits

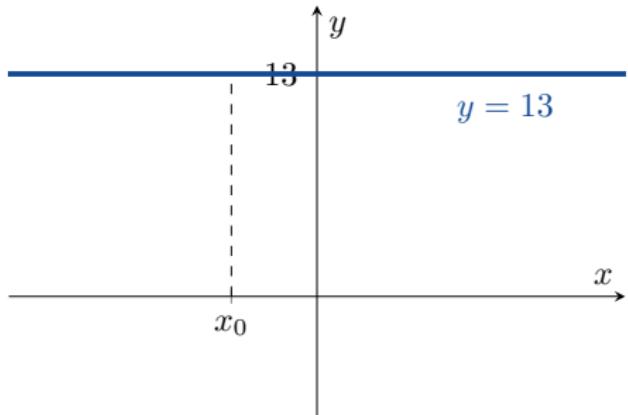


Example (The Identity Function)

$$f(x) = x$$

$$\lim_{x \rightarrow x_0} f(x) = \lim_{x \rightarrow x_0} x = x_0$$

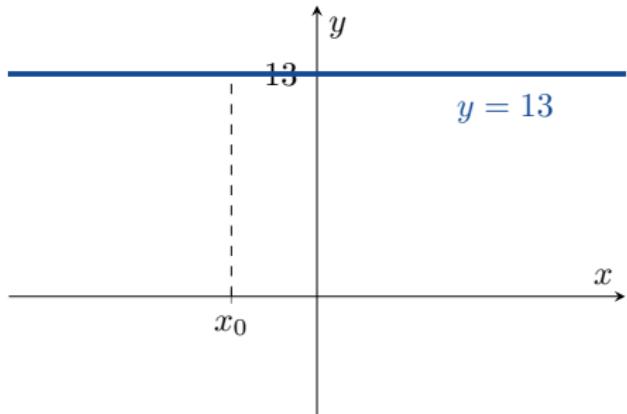
24. Limits



Example (A Constant Function)

$$f(x) = 13$$

24. Limits



Example (A Constant Function)

$$f(x) = 13$$

$$\lim_{x \rightarrow x_0} f(x) = \lim_{x \rightarrow x_0} 13 = 13$$

24. Limits

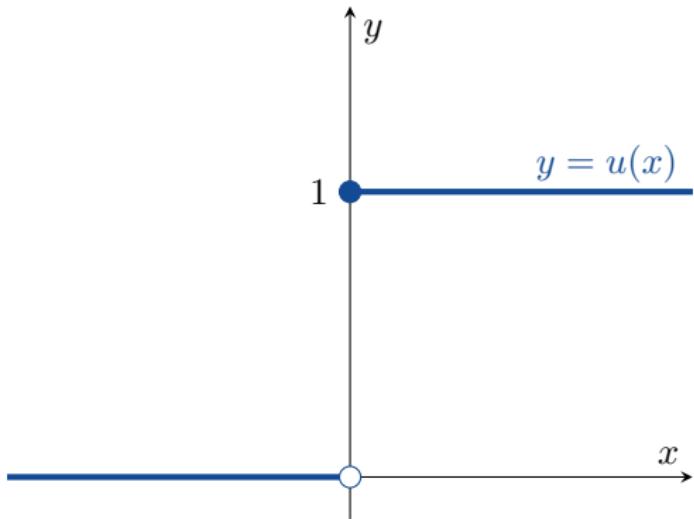


Example (Sometimes Limits Do Not Exist)

Consider the functions

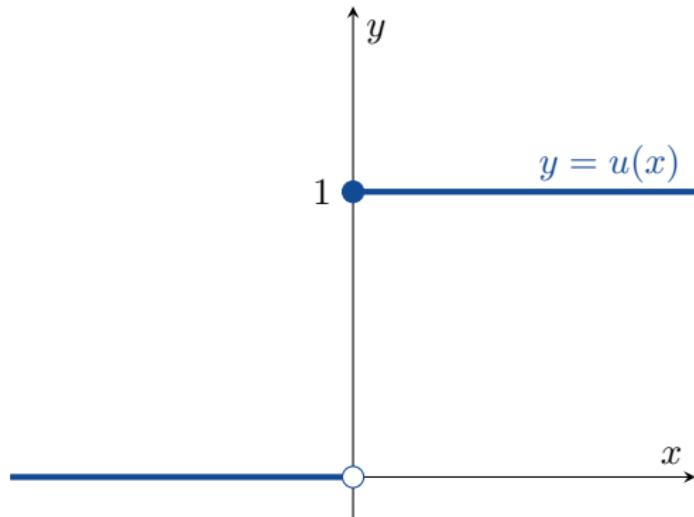
$$u(x) = \begin{cases} 0 & x < 0 \\ 1 & x \geq 0 \end{cases} \quad \text{and} \quad v(x) = \begin{cases} 0 & x \leq 0 \\ \sin \frac{1}{x} & x > 0. \end{cases}$$

24. Limits



Note that $\lim_{x \rightarrow 0} u(x)$ does not exist.

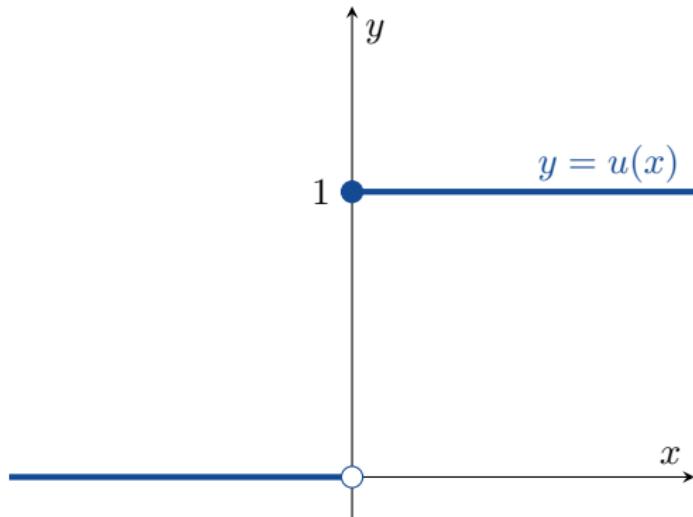
24. Limits



Note that $\lim_{x \rightarrow 0} u(x)$ does not exist. To understand why, we consider x close to 0:

- If x is close to 0 and $x < 0$, then $u(x) = 0$.
- If x is close to 0 and $x > 0$, then $u(x) = 1$.

24. Limits

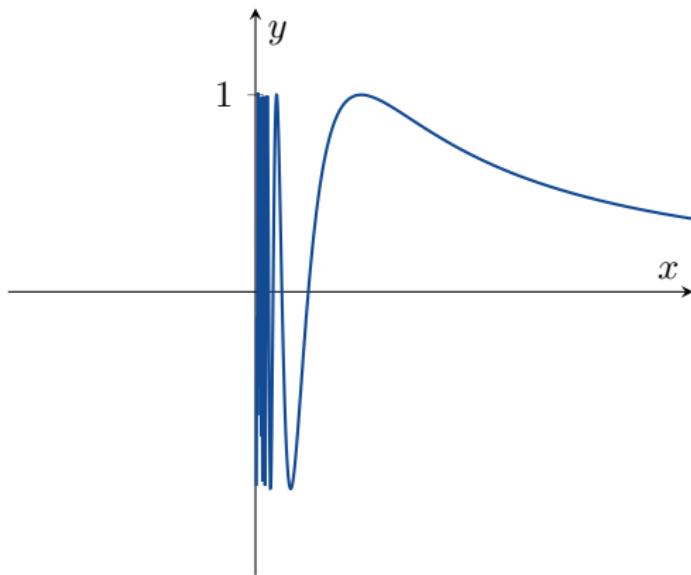


Note that $\lim_{x \rightarrow 0} u(x)$ does not exist. To understand why, we consider x close to 0:

- If x is close to 0 and $x < 0$, then $u(x) = 0$.
- If x is close to 0 and $x > 0$, then $u(x) = 1$.

Because 0 is not close to 1, the limit as $x \rightarrow 0$ can not exist.

24. Limits



Moreover $\lim_{x \rightarrow 0} v(x)$ does not exist because $v(x)$ oscillates up and down too quickly if $x > 0$ and $x \rightarrow 0$.

24. Limits



Theorem (The Limit Laws)

Suppose that

- $L, M, c, k \in \mathbb{R}$;
- f and g are functions;
- $\lim_{x \rightarrow c} f(x) = L$; and
- $\lim_{x \rightarrow c} g(x) = M$.

Then

24. Limits



Theorem (The Limit Laws)

Suppose that

- $L, M, c, k \in \mathbb{R}$;
- f and g are functions;
- $\lim_{x \rightarrow c} f(x) = L$; and
- $\lim_{x \rightarrow c} g(x) = M$.

Then

- 1 Sum Rule:

$$\lim_{x \rightarrow c} (f(x) + g(x)) = L + M;$$

24. Limits

Theorem (The Limit Laws)

Suppose that

- $L, M, c, k \in \mathbb{R}$;
- f and g are functions;
- $\lim_{x \rightarrow c} f(x) = L$; and
- $\lim_{x \rightarrow c} g(x) = M$.

Then

2 Difference Rule:

$$\lim_{x \rightarrow c} (f(x) - g(x)) = L - M;$$

24. Limits

Theorem (The Limit Laws)

Suppose that

- $L, M, c, k \in \mathbb{R}$;
- f and g are functions;
- $\lim_{x \rightarrow c} f(x) = L$; and
- $\lim_{x \rightarrow c} g(x) = M$.

Then

3 Constant Multiple Rule:

$$\lim_{x \rightarrow c} (kf(x)) = kL;$$

24. Limits



Theorem (The Limit Laws)

Suppose that

- $L, M, c, k \in \mathbb{R}$;
- f and g are functions;
- $\lim_{x \rightarrow c} f(x) = L$; and
- $\lim_{x \rightarrow c} g(x) = M$.

Then

- 4 Product Rule:

$$\lim_{x \rightarrow c} (f(x)g(x)) = LM;$$

24. Limits

Theorem (The Limit Laws)

Suppose that

- $L, M, c, k \in \mathbb{R}$;
- f and g are functions;
- $\lim_{x \rightarrow c} f(x) = L$; and
- $\lim_{x \rightarrow c} g(x) = M$.

Then

- 5 Quotient Rule: if $M \neq 0$, then

$$\lim_{x \rightarrow c} \left(\frac{f(x)}{g(x)} \right) = \frac{L}{M};$$

24. Limits

Theorem (The Limit Laws)

Suppose that

- $L, M, c, k \in \mathbb{R}$;
- f and g are functions;
- $\lim_{x \rightarrow c} f(x) = L$; and
- $\lim_{x \rightarrow c} g(x) = M$.

Then

- 6 Power Rule: if $n \in \mathbb{N}$, then

$$\lim_{x \rightarrow c} (f(x))^n = L^n;$$

24. Limits

Theorem (The Limit Laws)

Suppose that

- $L, M, c, k \in \mathbb{R}$;
- f and g are functions;
- $\lim_{x \rightarrow c} f(x) = L$; and
- $\lim_{x \rightarrow c} g(x) = M$.

Then

- 7 Root Rule: if $n \in \mathbb{N}$ and $\sqrt[n]{L}$ exists, then

$$\lim_{x \rightarrow c} \sqrt[n]{f(x)} = \sqrt[n]{L} = L^{\frac{1}{n}}.$$

24. Limits

Example

Find $\lim_{x \rightarrow 2} (x^3 + 4x^2 - 3)$.

solution:

$$\begin{aligned}\lim_{x \rightarrow 2} (x^3 + 4x^2 - 3) &= \left(\lim_{x \rightarrow 2} x^3 \right) + \left(\lim_{x \rightarrow 2} 4x^2 \right) - \left(\lim_{x \rightarrow 2} 3 \right) \\&\quad (\text{sum and difference rules}) \\&= \left(\lim_{x \rightarrow 2} x \right)^3 + 4 \left(\lim_{x \rightarrow 2} x \right)^2 - \left(\lim_{x \rightarrow 2} 3 \right) \\&\quad (\text{power and constant multiple rules}) \\&= 2^3 + 4(2^2) - 3 = 21.\end{aligned}$$

24. Limits

Example

Find $\lim_{x \rightarrow 5} \frac{x^4 + x^2 - 1}{x^2 + 5}$.

solution:

$$\lim_{x \rightarrow 5} \frac{x^4 + x^2 - 1}{x^2 + 5} = \frac{\lim_{x \rightarrow 5}(x^4 + x^2 - 1)}{\lim_{x \rightarrow 5}(x^2 + 5)}$$

(quotient rule)

$$= \frac{\lim_{x \rightarrow 5} x^4 + \lim_{x \rightarrow 5} x^2 - \lim_{x \rightarrow 5} 1}{\lim_{x \rightarrow 5} x^2 + \lim_{x \rightarrow 5} 5}$$

(sum and difference rules)

$$= \frac{5^4 + 5^2 - 1}{5^2 + 5}$$

(power rule)

$$= \frac{649}{30}.$$

24. Limits



Theorem (Limits of Polynomial Functions)

If $P(x) = a_nx^n + a_{n-1}x^{n-1} + \dots + a_1x + a_0$ is a polynomial function, then

$$\lim_{x \rightarrow c} P(x) = P(c).$$

24. Limits



Theorem (Limits of Rational Functions)

If $P(x)$ and $Q(x)$ are polynomial functions and if $Q(c) \neq 0$, then

$$\lim_{x \rightarrow c} \frac{P(x)}{Q(x)} = \frac{P(c)}{Q(c)}.$$

24. Limits



Example

$$\lim_{x \rightarrow -1} \frac{x^3 + 4x^2 - 3}{x^2 + 5} = \frac{(-1)^3 + 4(-1)^2 - 3}{(-1)^2 + 5} = \frac{0}{6} = 0.$$

Eliminating Zero Denominators Algebraically

$$\lim_{x \rightarrow c} \frac{P(x)}{Q(x)}$$

What can we do if $Q(c) = 0$?

24. Limits



Example

$$\lim_{x \rightarrow 1} \frac{x^2 + x - 2}{x^2 - x}$$

If we just put in $x = 1$, we would get “ $\frac{0}{0}$ ” and we never never never want “ $\frac{0}{0}$ ”.

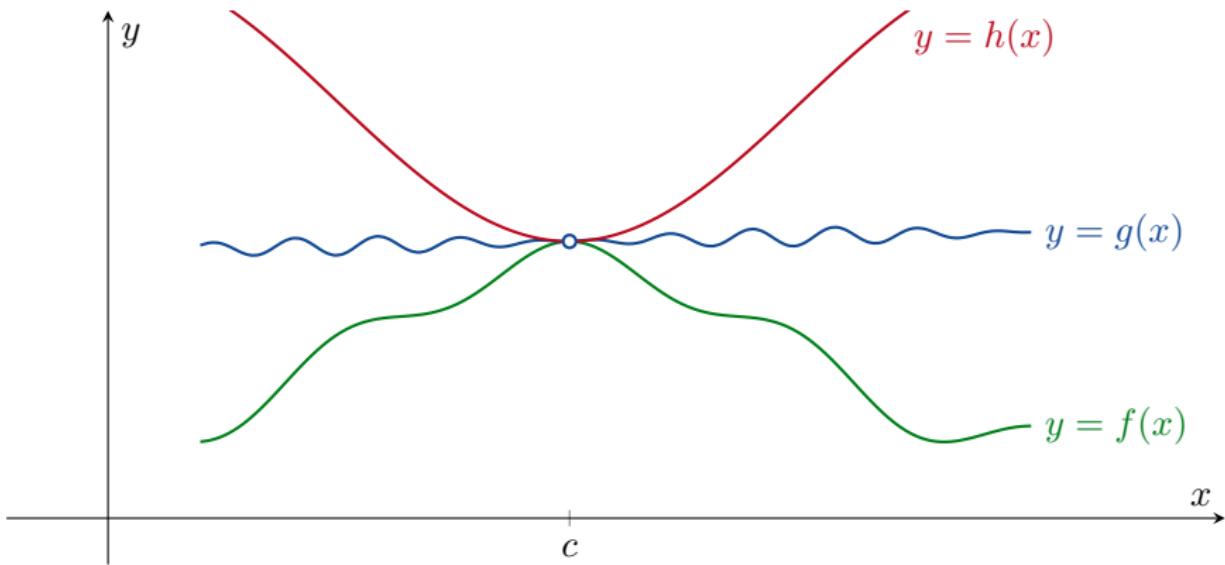
Instead, we try to factor $x^2 + x - 2$ and $x^2 - x$. If $x \neq 1$, we have that

$$\frac{x^2 + x - 2}{x^2 - x} = \frac{(x - 1)(x + 2)}{x(x - 1)} = \frac{x + 2}{x}.$$

So

$$\lim_{x \rightarrow 1} \frac{x^2 + x - 2}{x^2 - x} = \lim_{x \rightarrow 1} \frac{x + 2}{x} = \frac{1 + 2}{1} = 3.$$

The Sandwich Theorem



24. Limits



Theorem (The Sandwich Theorem)

Suppose that

- $f(x) \leq g(x) \leq h(x)$ for all x “close” to c ($x \neq c$); and
- $\lim_{x \rightarrow c} f(x) = \lim_{x \rightarrow c} h(x) = L$.

Then

$$\lim_{x \rightarrow c} g(x) = L$$

also.

24. Limits

Example

The inequality

$$1 - \frac{x^2}{6} < \frac{x \sin x}{2 - 2 \cos x} < 1$$

holds for all x close to 0 ($x \neq 0$). Calculate $\lim_{x \rightarrow 0} \frac{x \sin x}{2 - 2 \cos x}$.

solution: Since $\lim_{x \rightarrow 0} 1 - \frac{x^2}{6} = 1 - \frac{0}{6} = 1$ and $\lim_{x \rightarrow 0} 1 = 1$, it follows by the Sandwich Theorem that $\lim_{x \rightarrow 0} \frac{x \sin x}{2 - 2 \cos x} = 1$.

Theorem

If

- $f(x) \leq g(x)$ for all x close to c ($x \neq c$);
- $\lim_{x \rightarrow c} f(x)$ exists; and
- $\lim_{x \rightarrow c} g(x)$ exists,

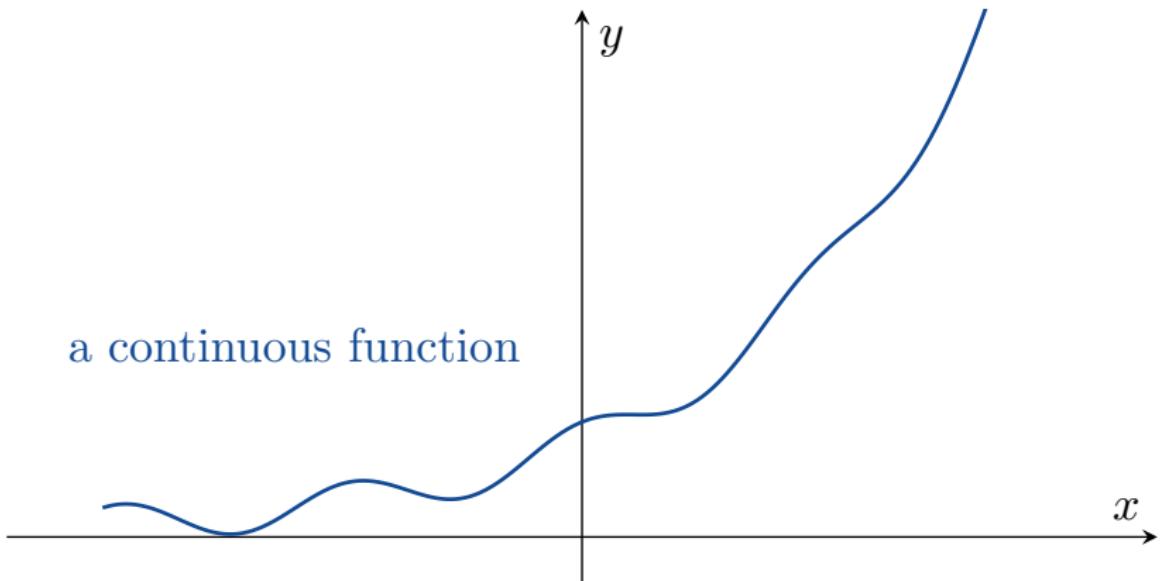
then

$$\lim_{x \rightarrow c} f(x) \leq \lim_{x \rightarrow c} g(x).$$

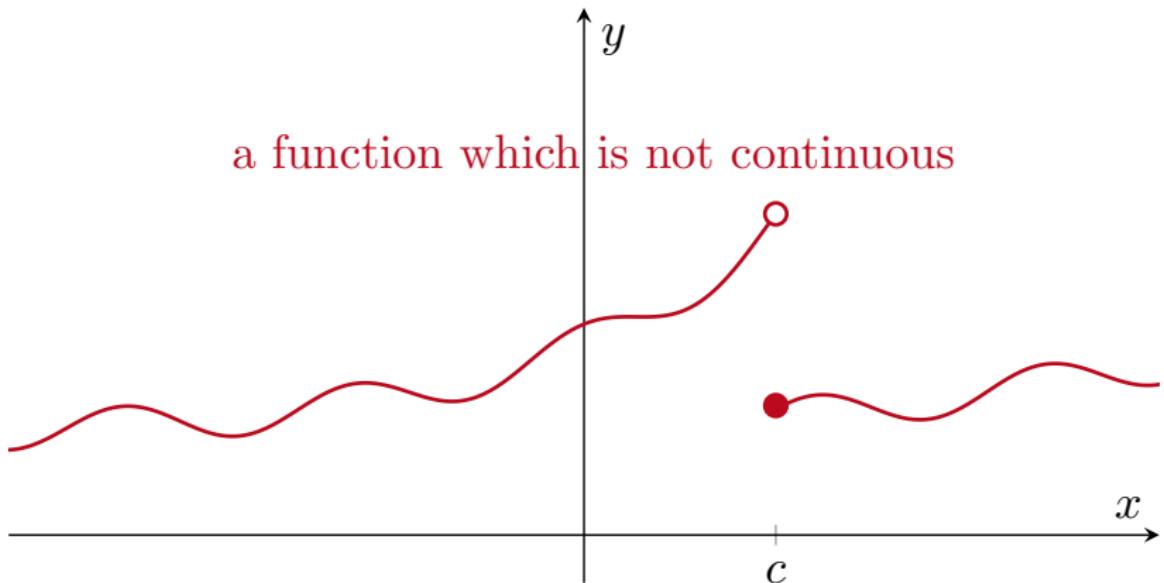


Continuity

25. Continuity



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25. Continuity



Definition

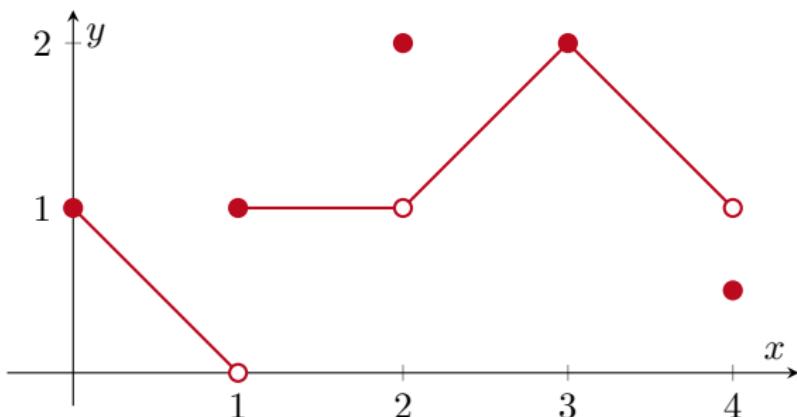
The function $f : D \rightarrow \mathbb{R}$ is *continuous at $c \in D$* if

- $f(c)$ exists;
- $\lim_{x \rightarrow c} f(x)$ exists; and
- $\lim_{x \rightarrow c} f(x) = f(c)$.

Definition

If f is not continuous at c , we say that f is *discontinuous at c* – we say that c is a *point of discontinuity* of f .

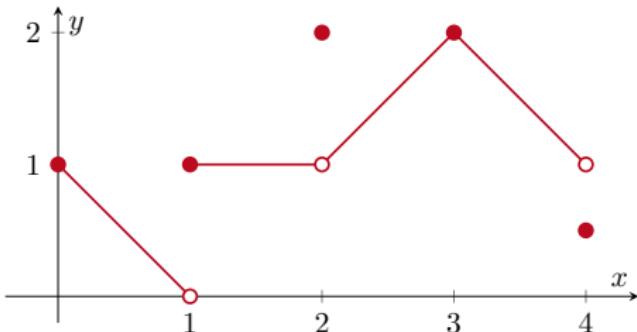
25. Continuity



Example

Consider the function $f : [0, 4] \rightarrow \mathbb{R}$ above. Where is f continuous? Where is f discontinuous?

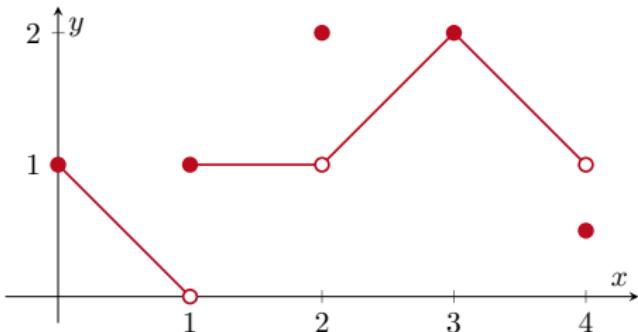
25. Continuity



solution:

c	Is f continuous at c ?	Why?
0	Yes	because $\lim_{x \rightarrow 0} f(x) = 1 = f(0)$
$(0, 1)$	Yes	because $\lim_{x \rightarrow c} f(x) = f(c)$
1	No	because $\lim_{x \rightarrow 1} f(x)$ does not exist

25. Continuity



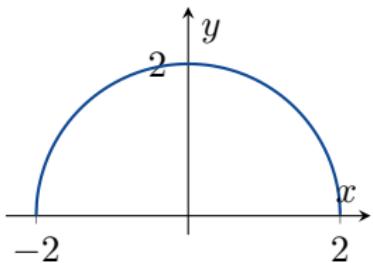
solution:

c	Is f continuous at c ?	Why?
(1, 2)	Yes	because $\lim_{x \rightarrow c} f(x) = f(c)$
2	No	because $\lim_{x \rightarrow 2} f(x) = 1 \neq 2 = f(2)$
(2, 4)	Yes	because $\lim_{x \rightarrow c} f(x) = f(c)$
4	No	because $\lim_{x \rightarrow 4} f(x) = 1 \neq \frac{1}{2} = f(4)$

25. Continuity

Example

$$f : [-2, 2] \rightarrow \mathbb{R}, f(x) = \sqrt{4 - x^2}$$

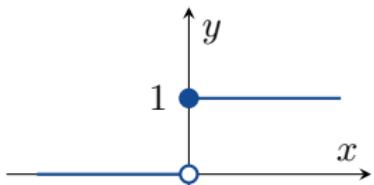


f is continuous at every $c \in [-2, 2]$.

25. Continuity

Example

$$g : \mathbb{R} \rightarrow \mathbb{R}, g(x) = \begin{cases} 0 & x < 0 \\ 1 & x \geq 0 \end{cases}$$

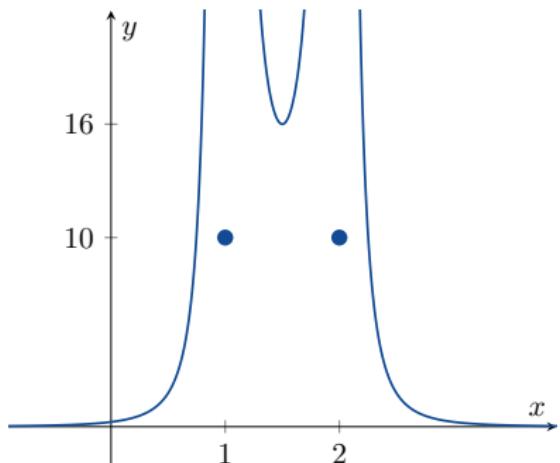


g has a point of discontinuity at $c = 0$. g is continuous at every point $c \neq 0$.

25. Continuity

Example

$$h : \mathbb{R} \rightarrow \mathbb{R}, h(x) = \begin{cases} \frac{1}{(x-1)^2(x-2)^2} & x \neq 1 \text{ or } 2 \\ 10 & x = 1 \text{ or } 2 \end{cases}$$



h is continuous on $(-\infty, 1)$, $(1, 2)$ and $(2, \infty)$. h has a points of discontinuity at $c = 1$ and $c = 2$.

Continuous Functions

Definition

$f : D \rightarrow \mathbb{R}$ is a *continuous function* if it is continuous at every $c \in D$.

25. Continuity



Theorem

If f and g are continuous at c , then $f + g$, $f - g$, kf ($k \in \mathbb{R}$), fg , $\frac{f}{g}$ (if $g(c) \neq 0$) and f^n ($n \in \mathbb{N}$) are all continuous at c . If $\sqrt[n]{f}$ is defined on $(c - \delta, c + \delta)$, then $\sqrt[n]{f}$ is also continuous at c ($n \in \mathbb{N}$).

25. Continuity



Example

Every polynomial

$$P(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_2 x^2 + a_1 x + a_0$$

is continuous.

25. Continuity

Example

If

- P and Q are polynomials; and
- $Q(c) \neq 0$,

then $\frac{P(x)}{Q(x)}$ is continuous at c .

25. Continuity



Example

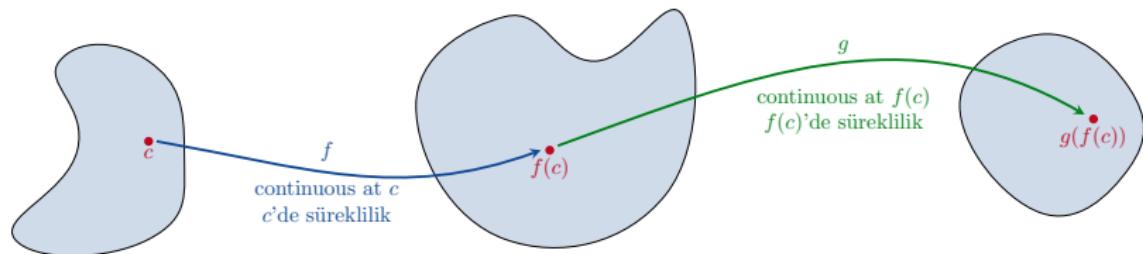
$\sin x$ and $\cos x$ are continuous.

Composites

$$g \circ f(x)$$

$g \circ f(x)$ means $g(f(x))$.

25. Continuity



Theorem

If

- f is continuous at c ; and
- g is continuous at $f(c)$,

then $g \circ f$ is continuous at c .

25. Continuity



Example

Show that $h(x) = \sqrt{x^2 - 2x - 5}$ is continuous on its domain.

solution: The function $g(t) = \sqrt{t}$ is continuous by Theorem ??.
The function $f(x) = x^2 - 2x - 5$ is continuous because all polynomials are continuous. Therefore $h(x) = g \circ f(x)$ is continuous.

25. Continuity



Example

Show that $\frac{x^{\frac{2}{3}}}{1+x^4}$ is continuous.

solution: $x^{\frac{2}{3}}$ and $1 + x^4$ are continuous. Because $1 + x^4 \neq 0$ for all x , we have that $\frac{x^{\frac{2}{3}}}{1+x^4}$ is continuous.

25. Continuity

Theorem

If

- $g(x)$ is continuous at $x = b$; and
- $\lim_{x \rightarrow c} f(x) = b$,

then

$$\lim_{x \rightarrow c} g(f(x)) = g\left(\lim_{x \rightarrow c} f(x)\right).$$

25. Continuity

Example

$$\begin{aligned}\lim_{x \rightarrow \frac{\pi}{2}} \cos \left[2x + \sin \left(\frac{3\pi}{2} + x \right) \right] \\&= \cos \left[\lim_{x \rightarrow \frac{\pi}{2}} \left(2x + \sin \left(\frac{3\pi}{2} + x \right) \right) \right] \\&= \cos \left[\lim_{x \rightarrow \frac{\pi}{2}} (2x) + \lim_{x \rightarrow \frac{\pi}{2}} \left(\sin \left(\frac{3\pi}{2} + x \right) \right) \right] \\&= \cos \left[\pi + \sin \left(\lim_{x \rightarrow \frac{\pi}{2}} \left(\frac{3\pi}{2} + x \right) \right) \right] \\&= \cos [\pi + \sin 2\pi] = \cos [\pi + 0] = -1.\end{aligned}$$

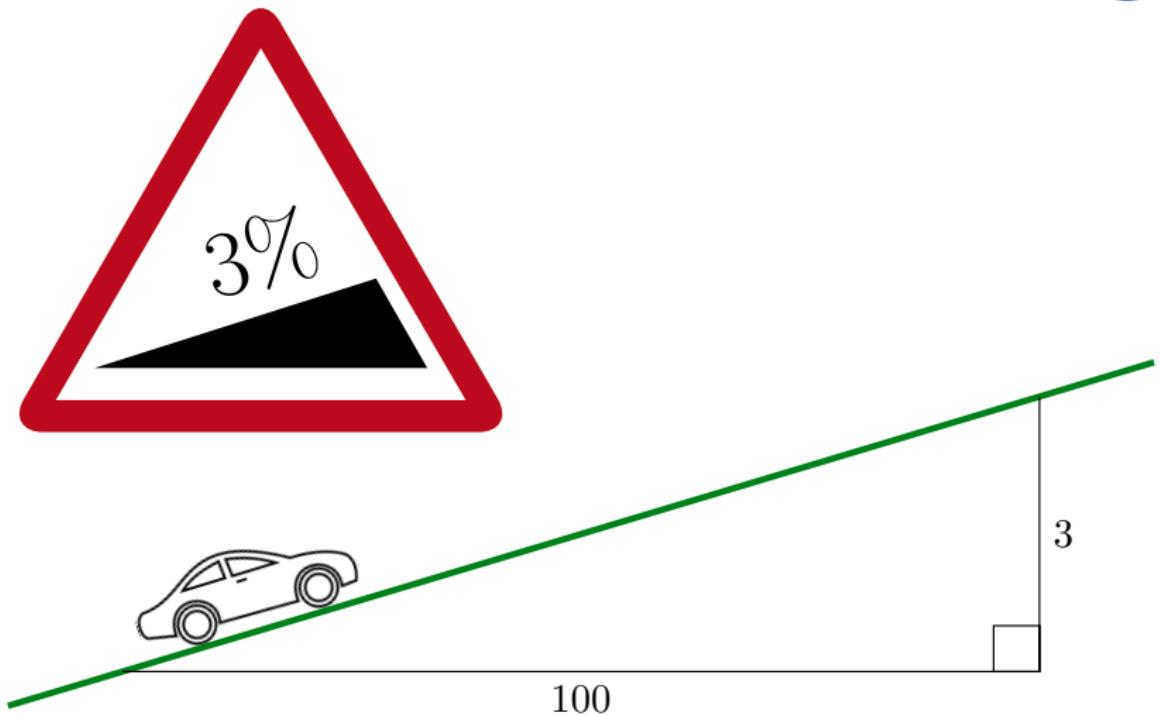


Differentiation

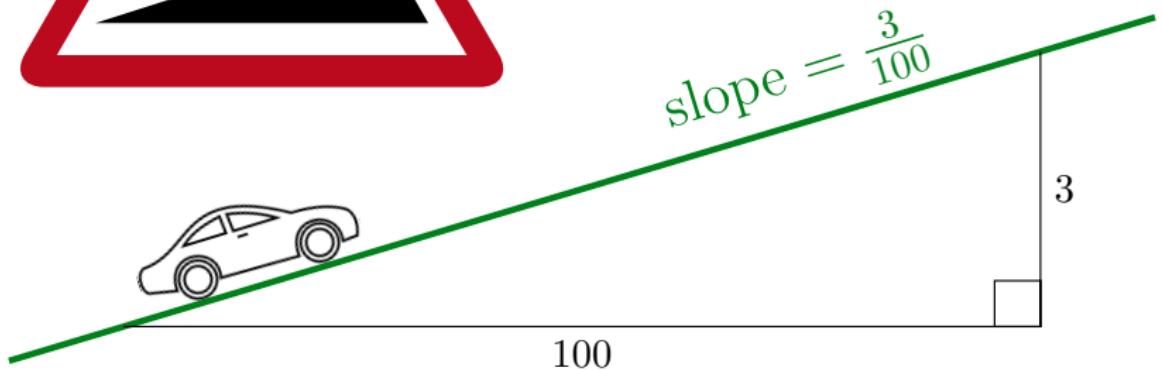
26. Differentiation



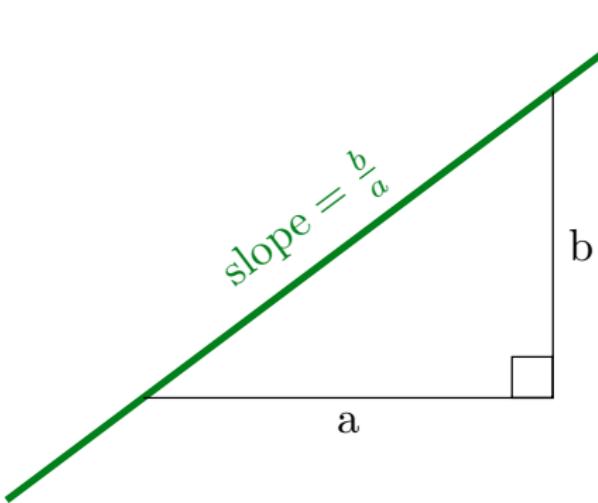
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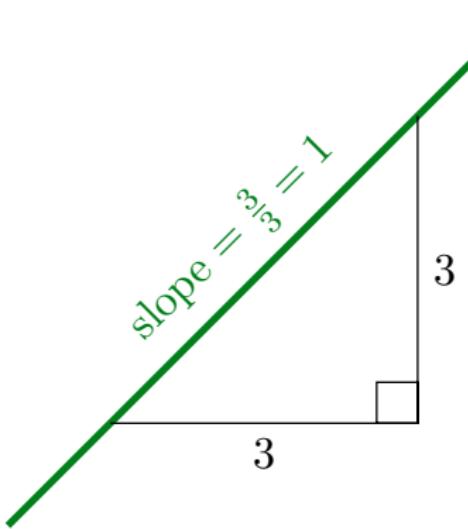
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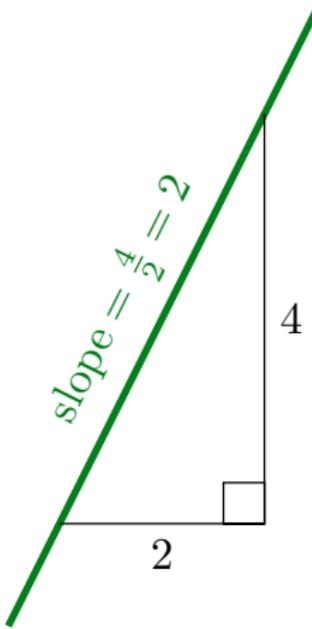
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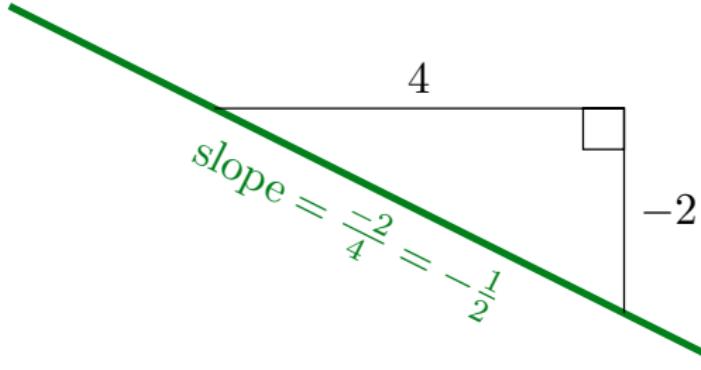
26. Differentiation



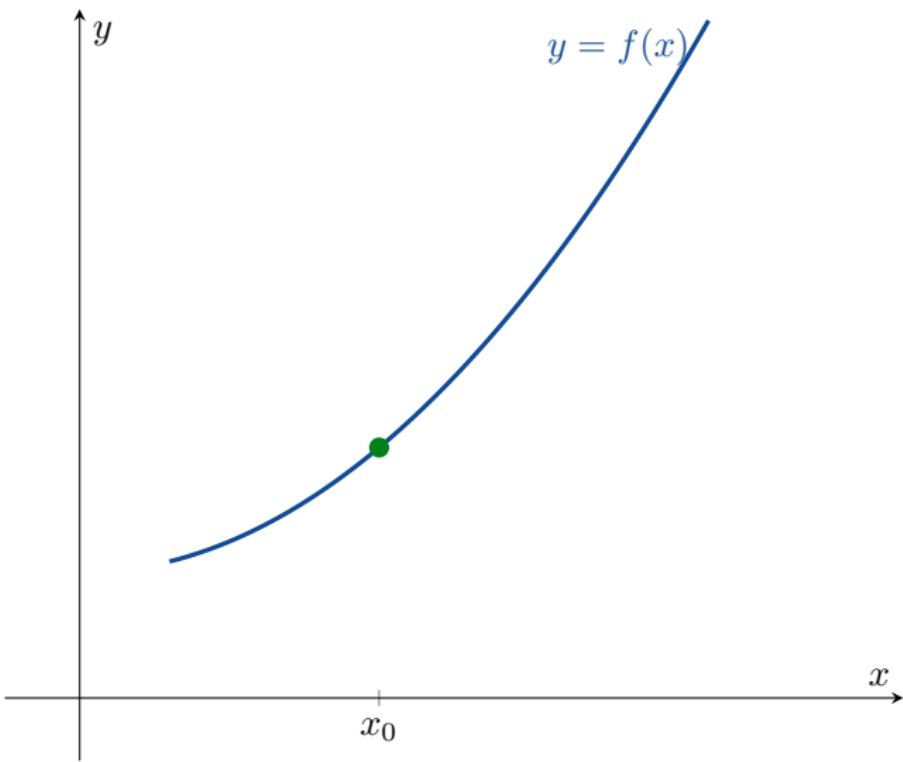
26. Differentiation



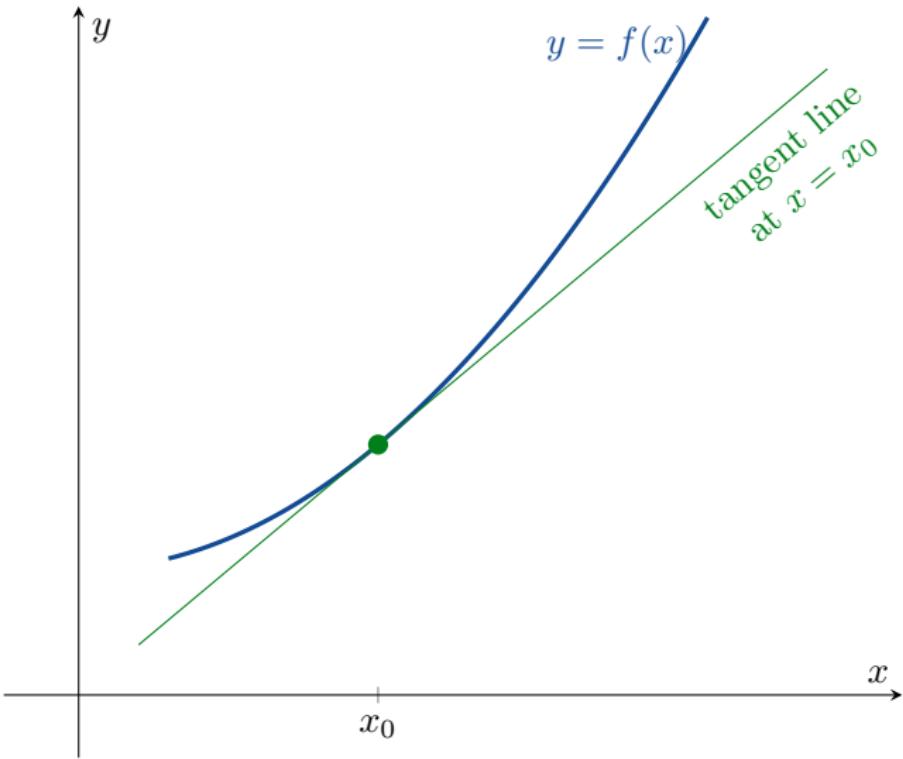
26. Differentiation



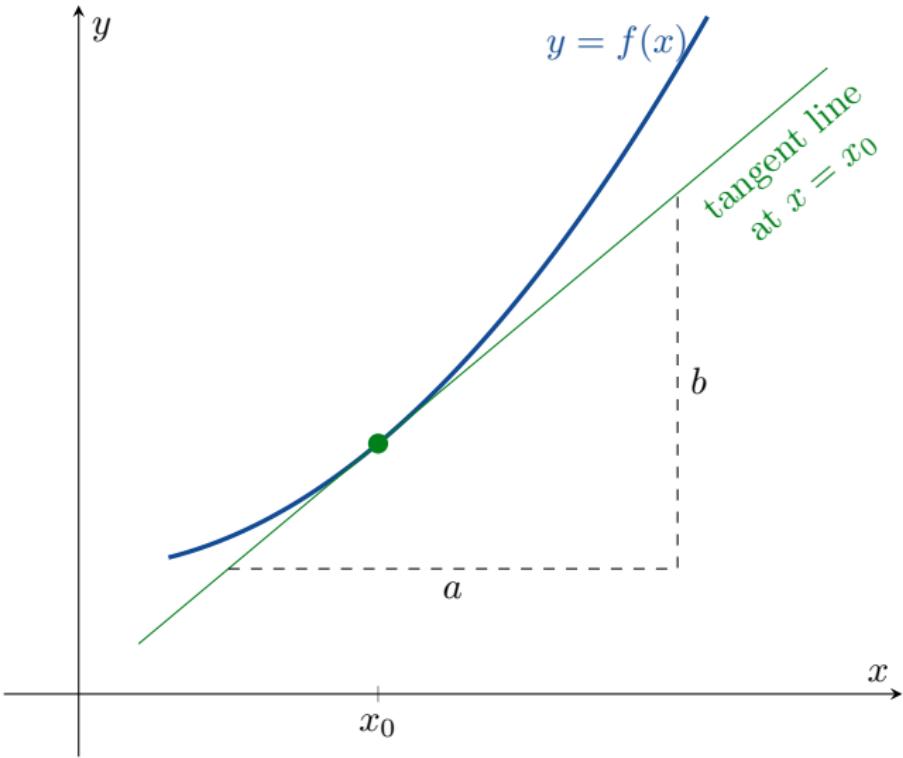
26. Differentiation



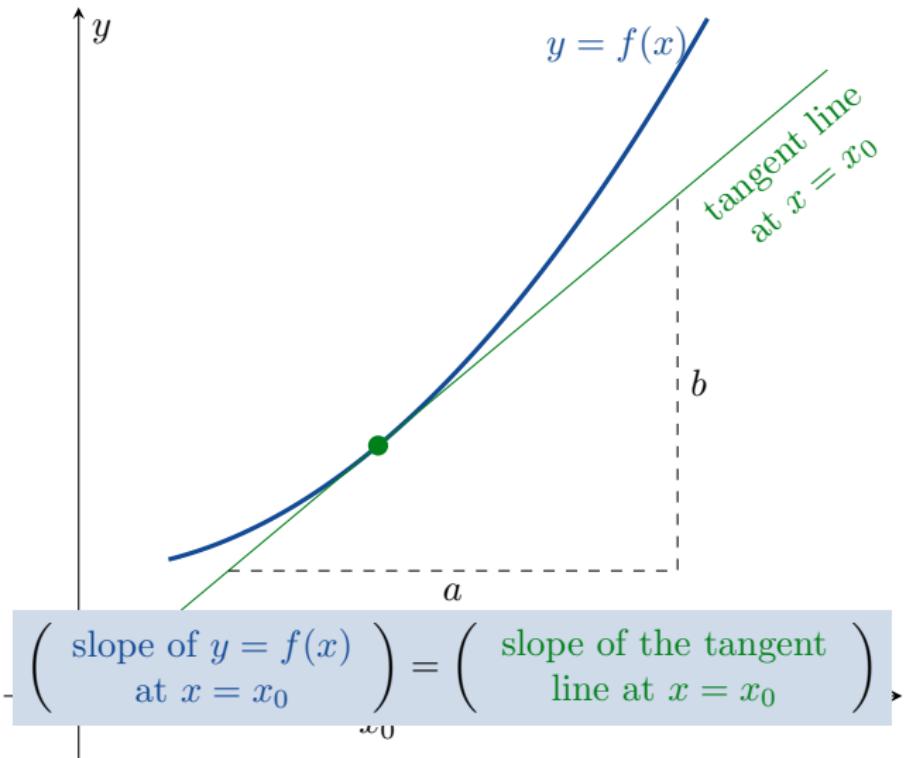
26. Differentiation



26. Differentiation



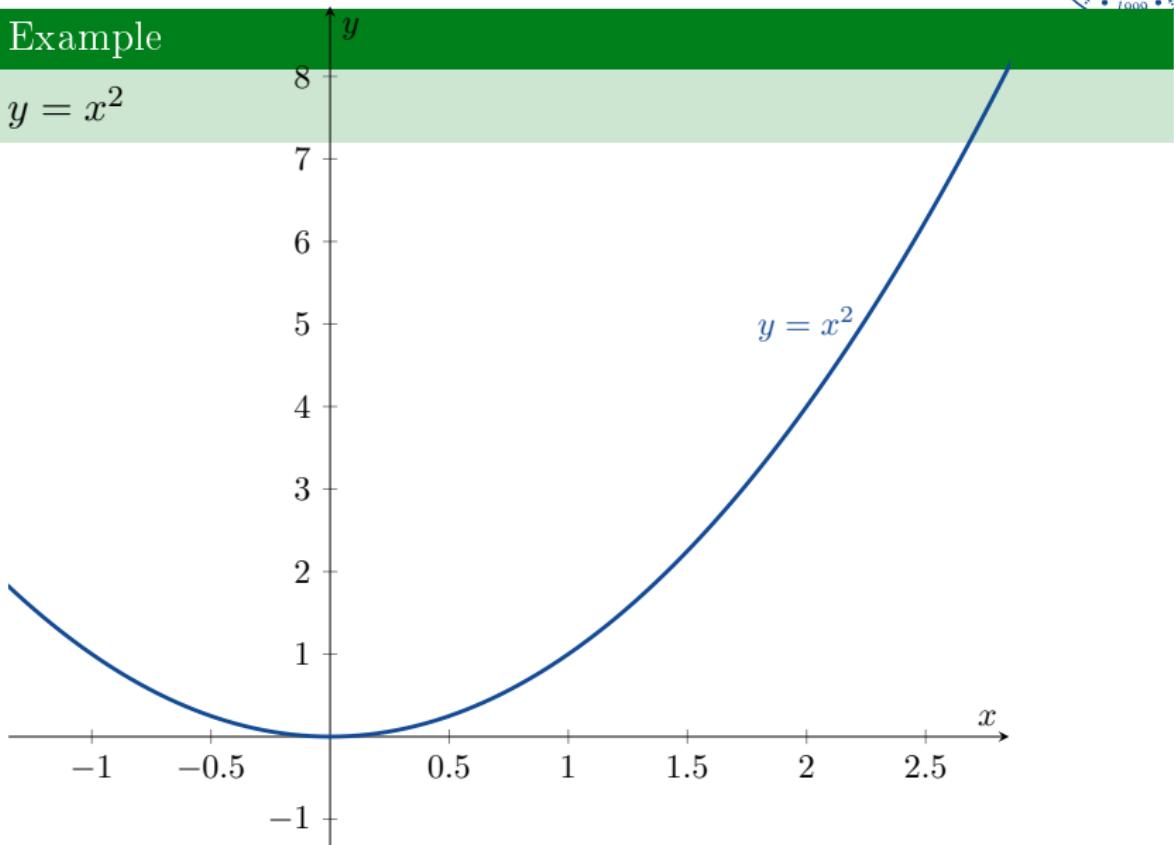
26. Differentiation



26. Differentiation

Example

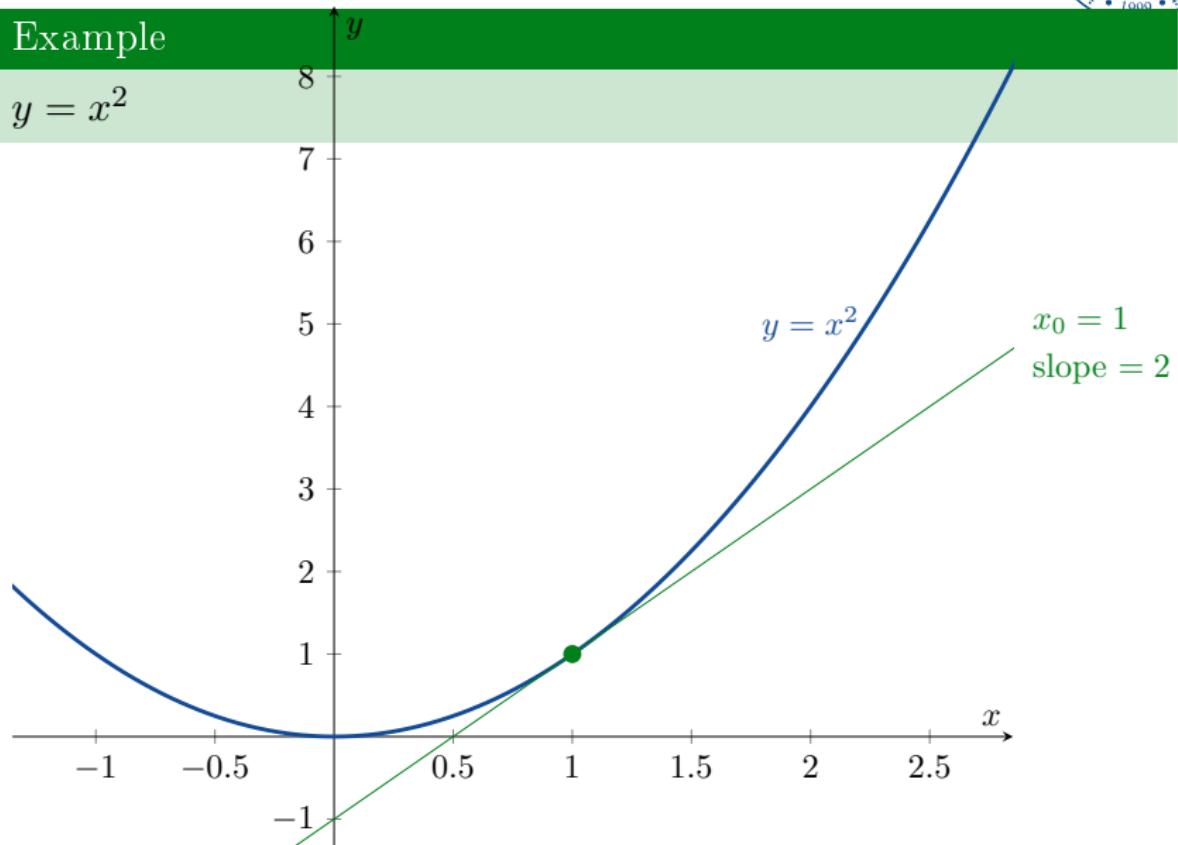
$$y = x^2$$



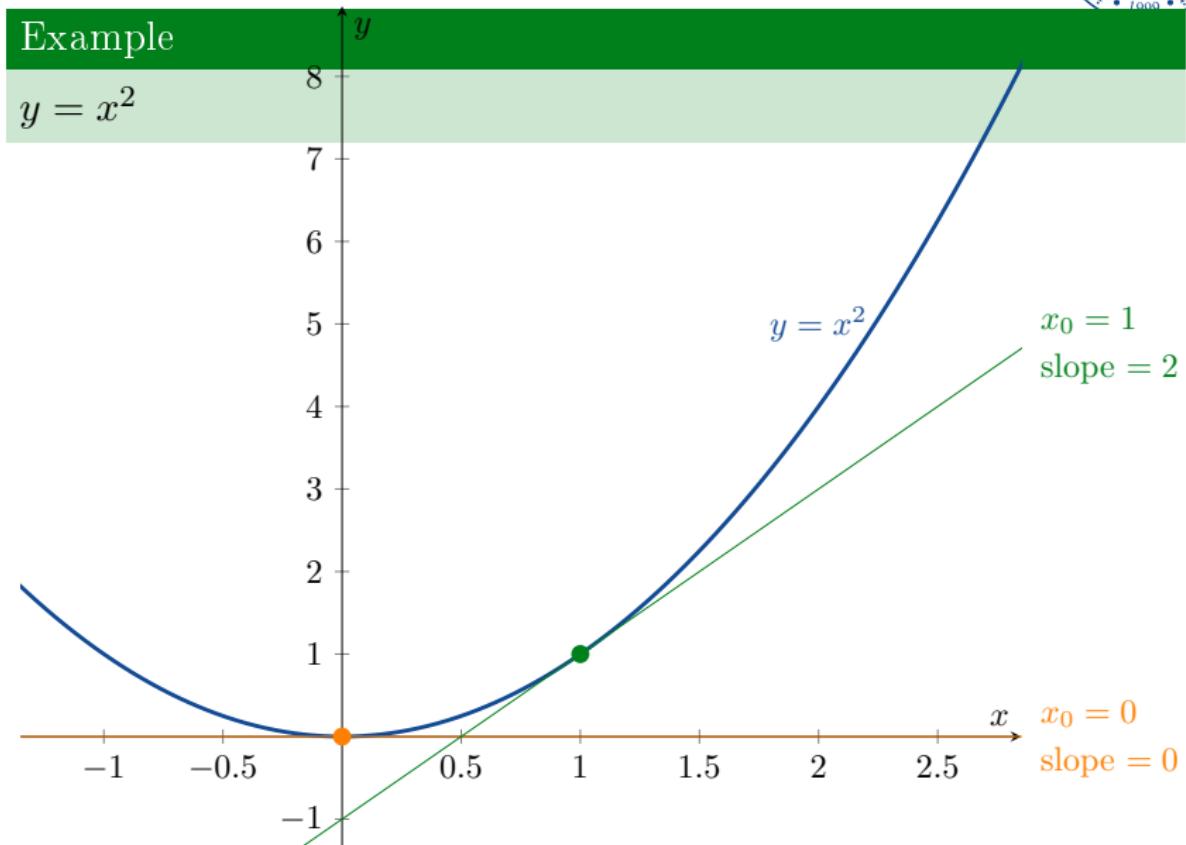
26. Differentiation

Example

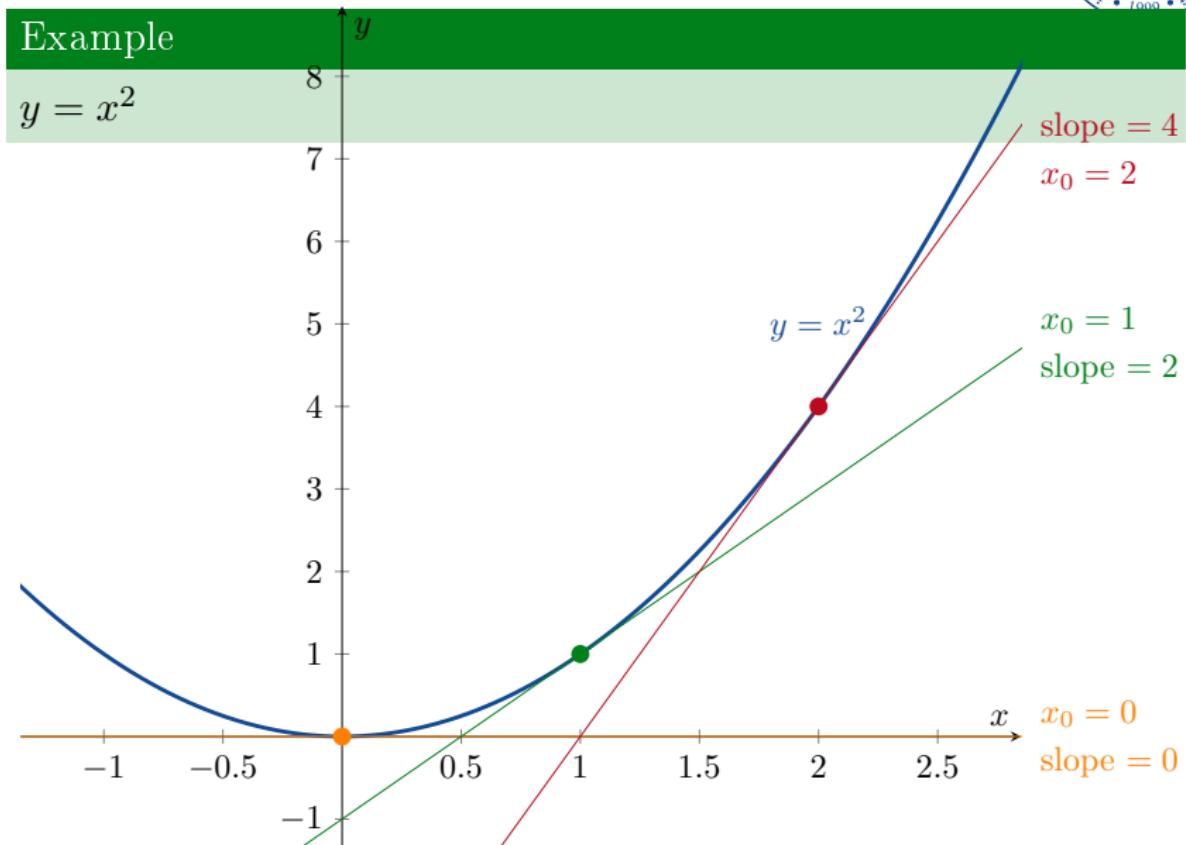
$$y = x^2$$



26. Differentiation



26. Differentiation

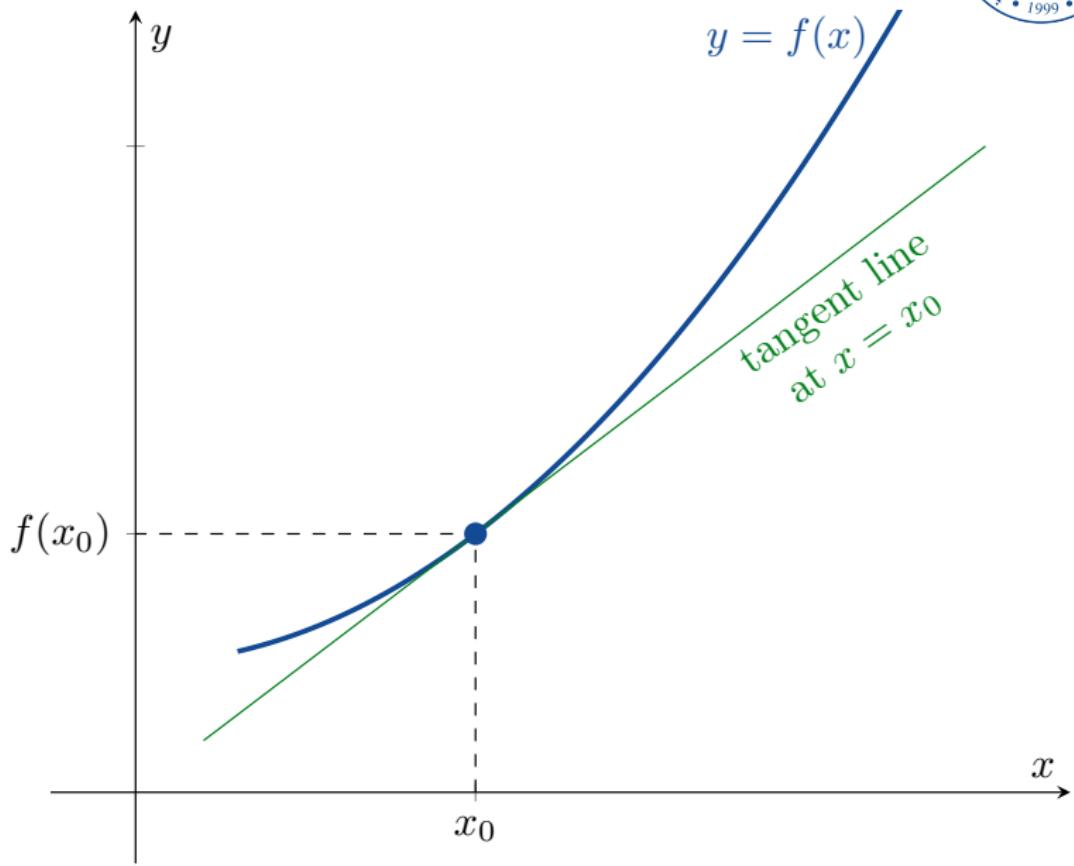


26. Differentiation

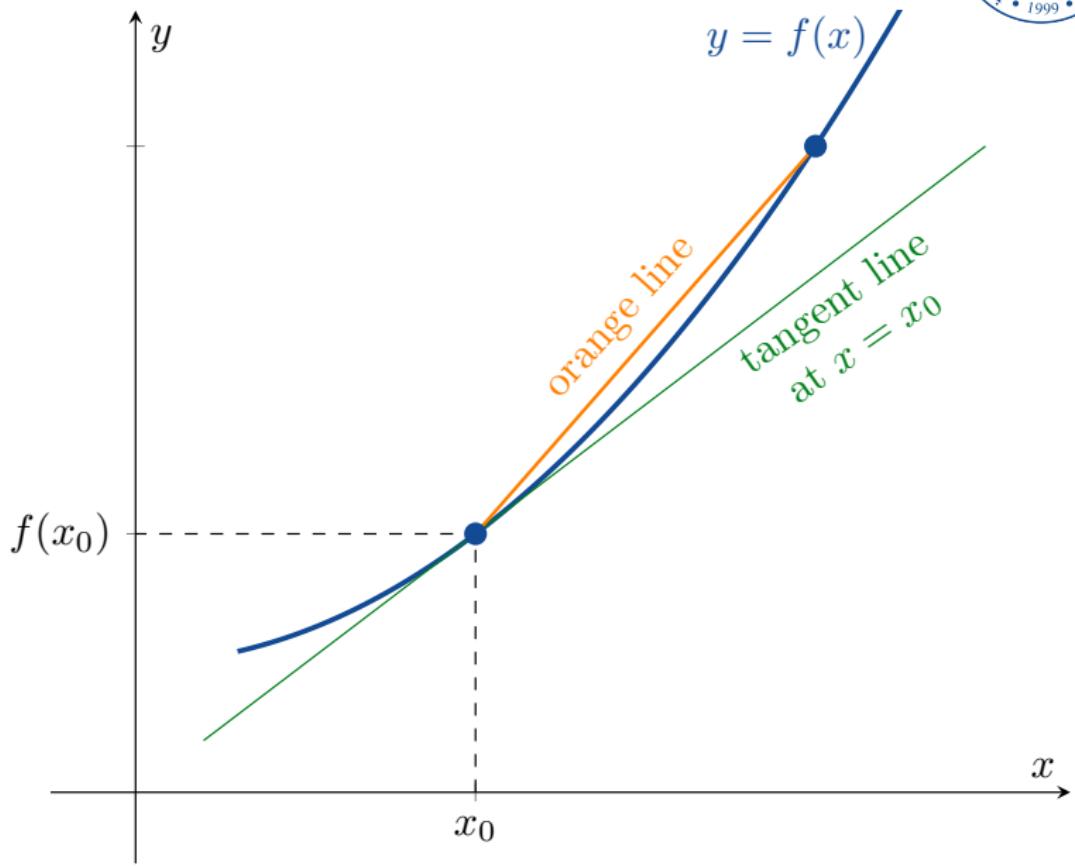


How can we calculate the slope of the tangent line?

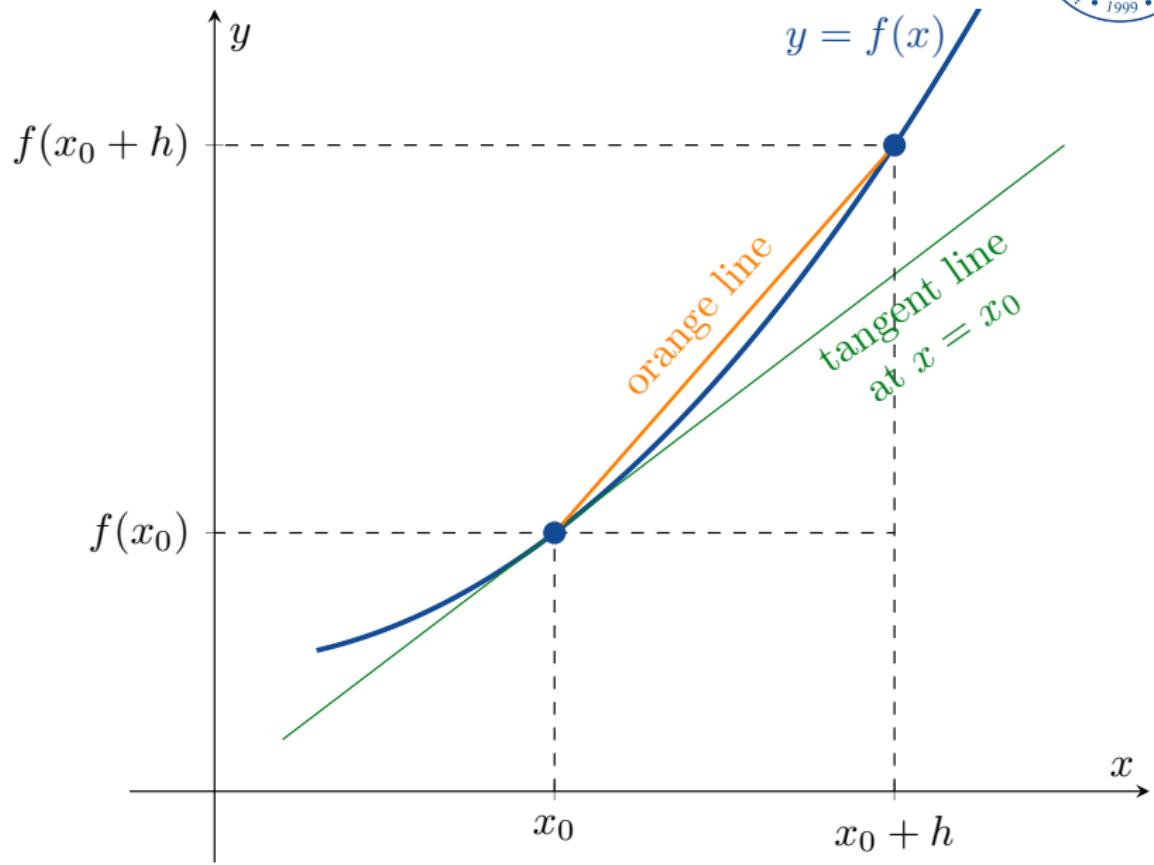
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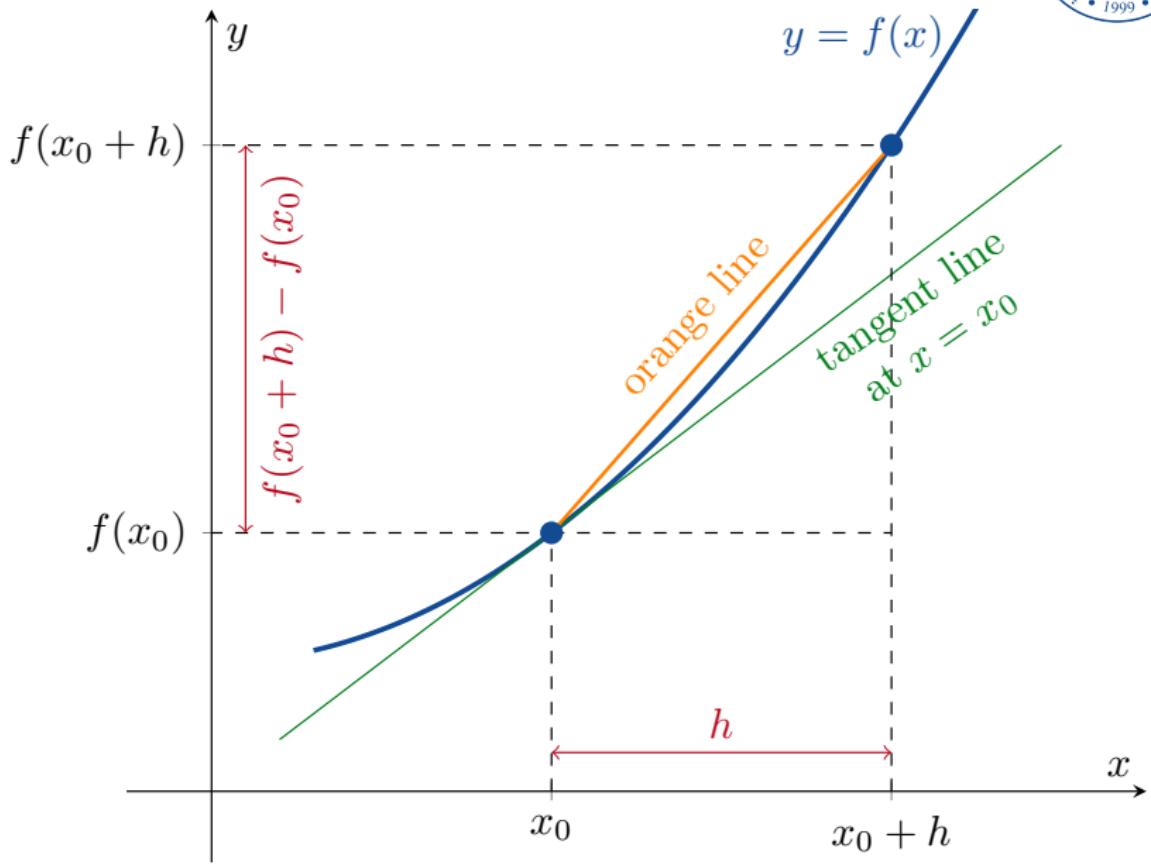
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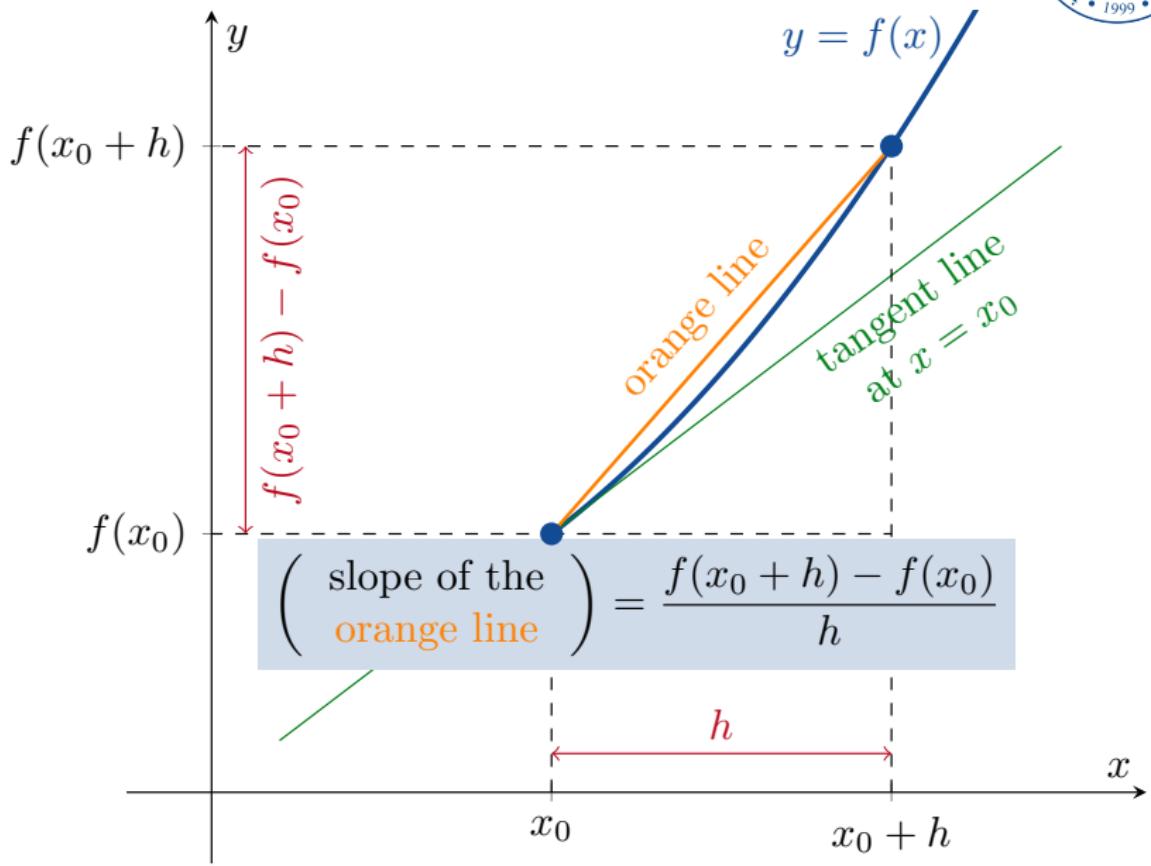
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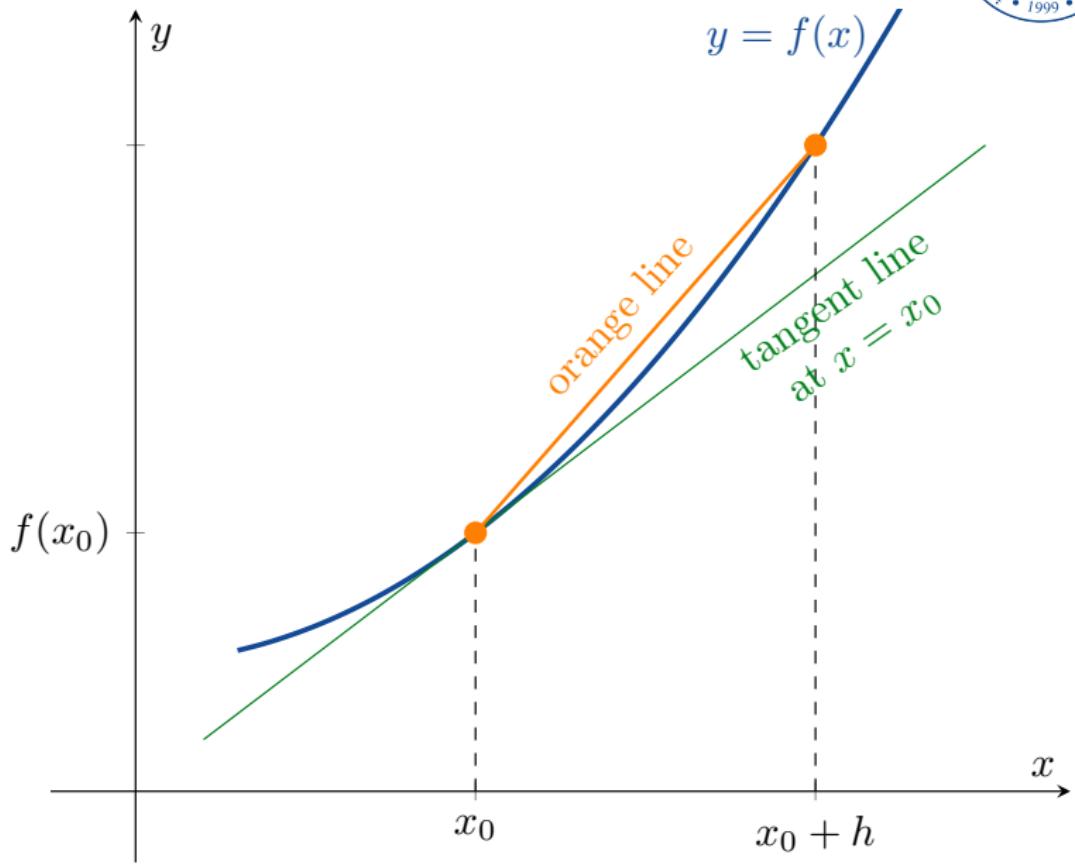
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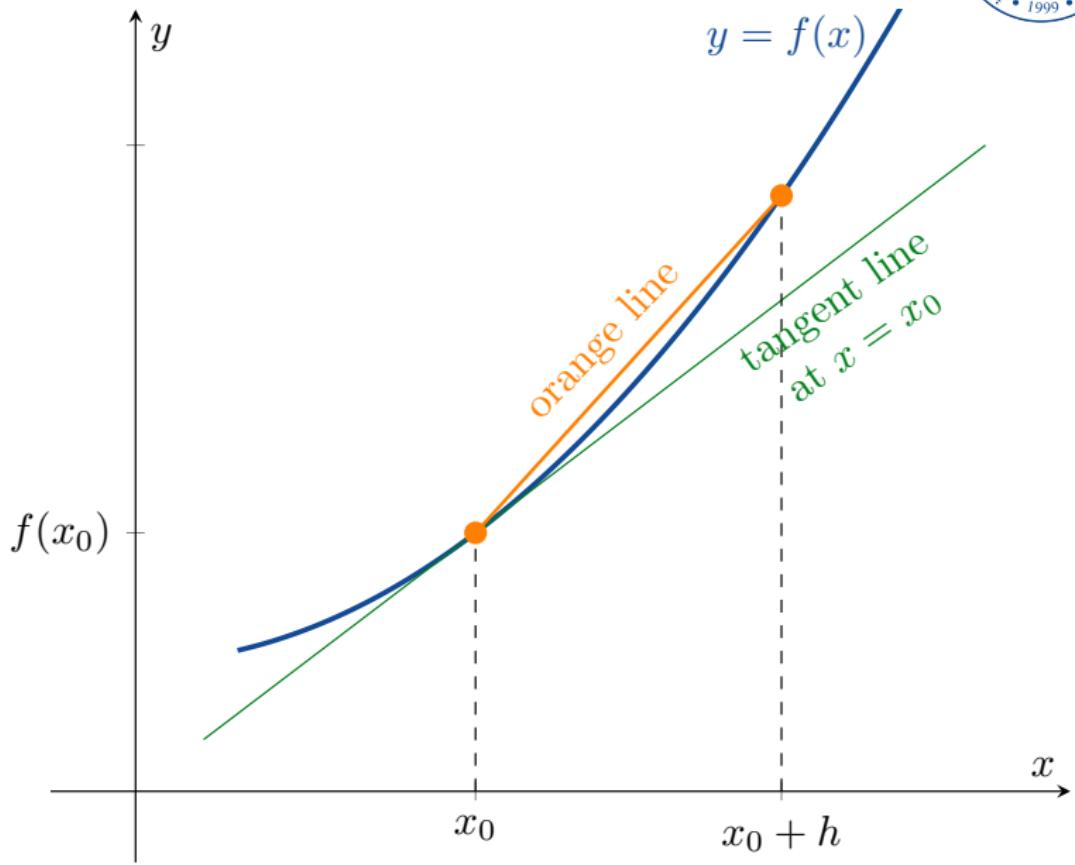
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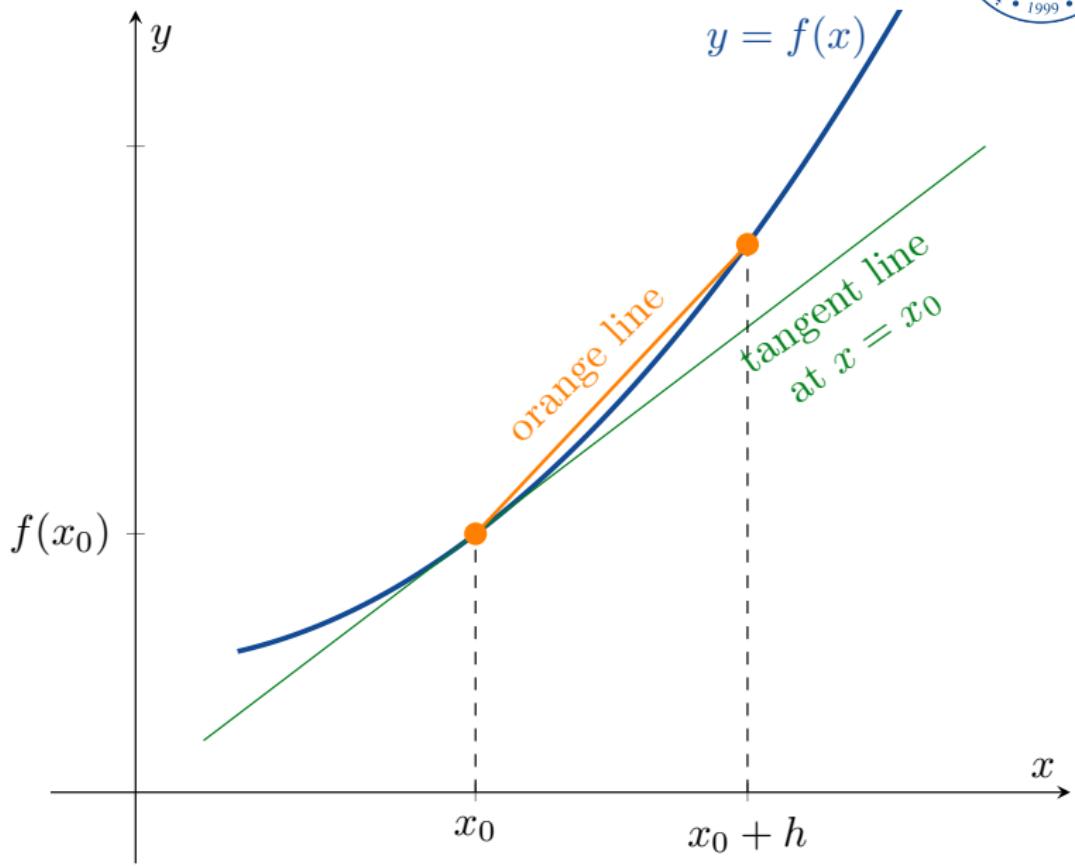
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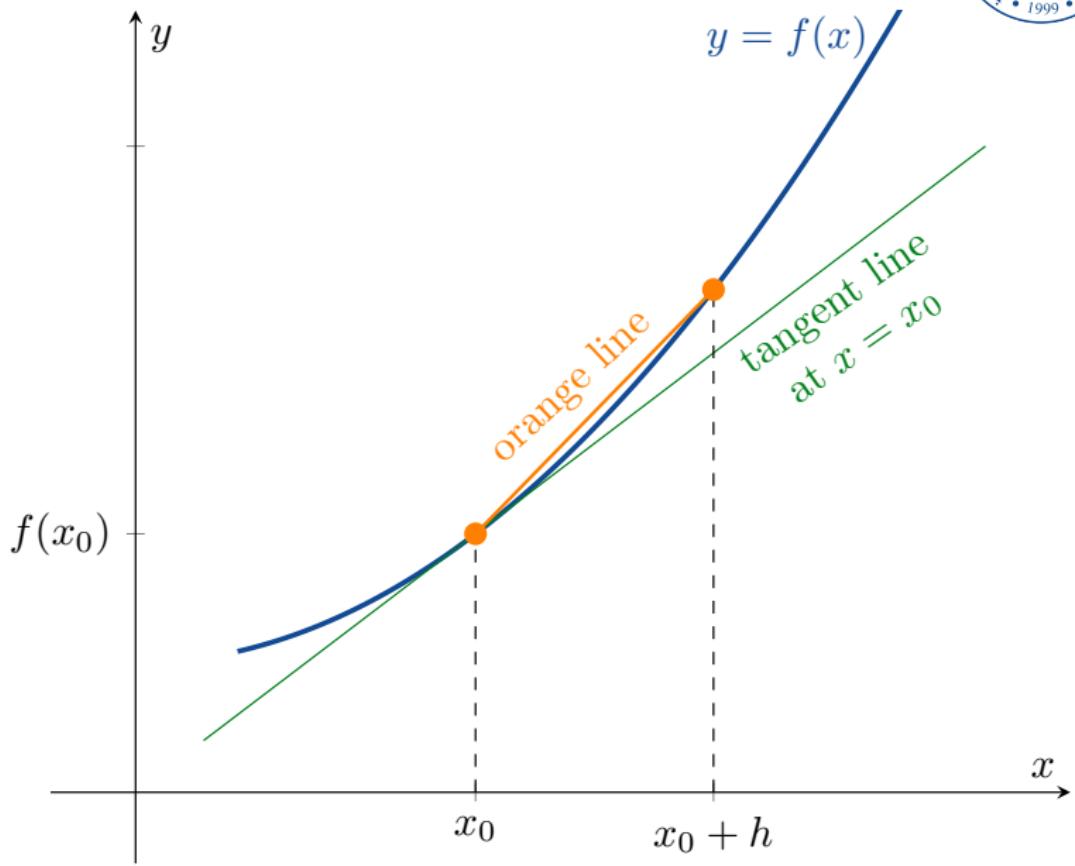
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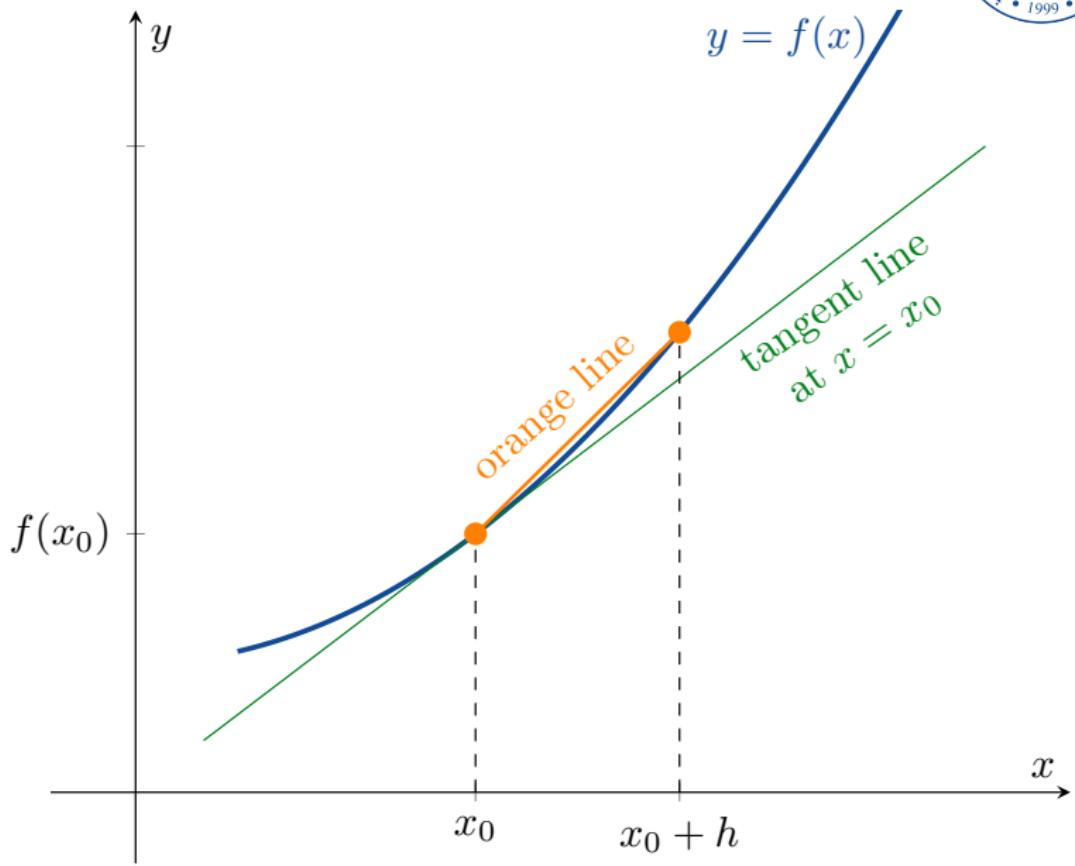
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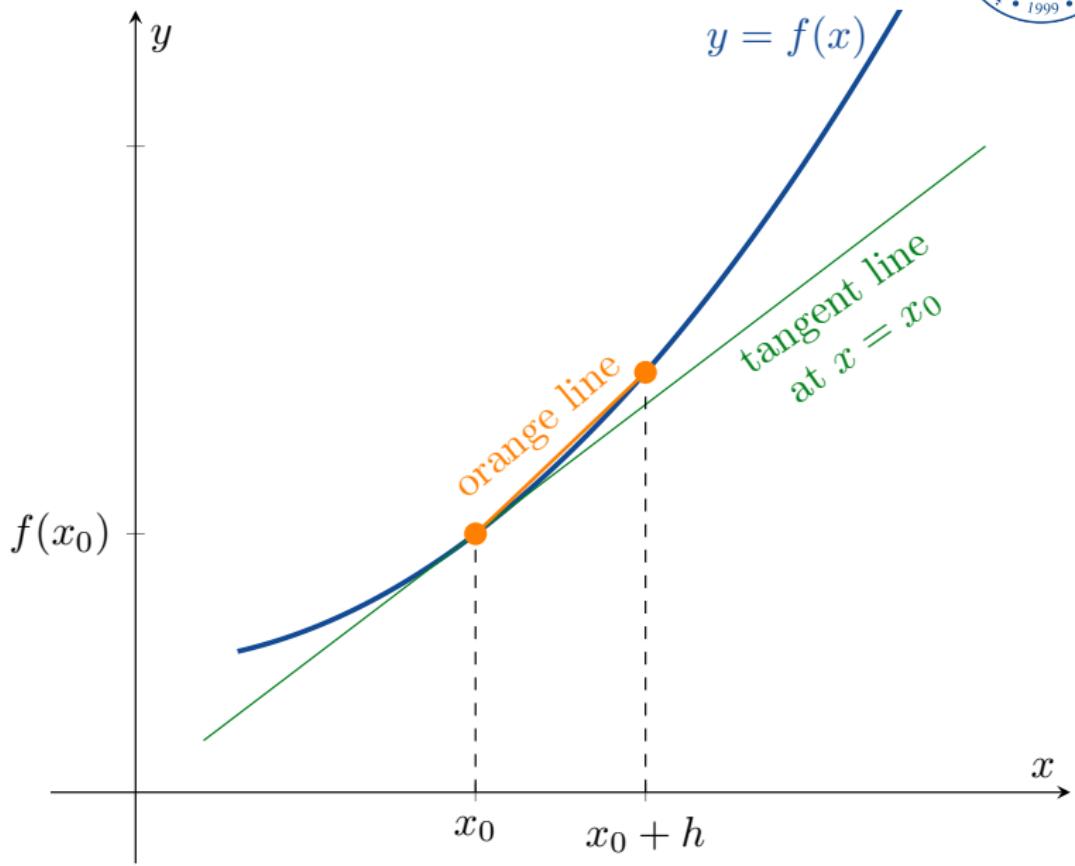
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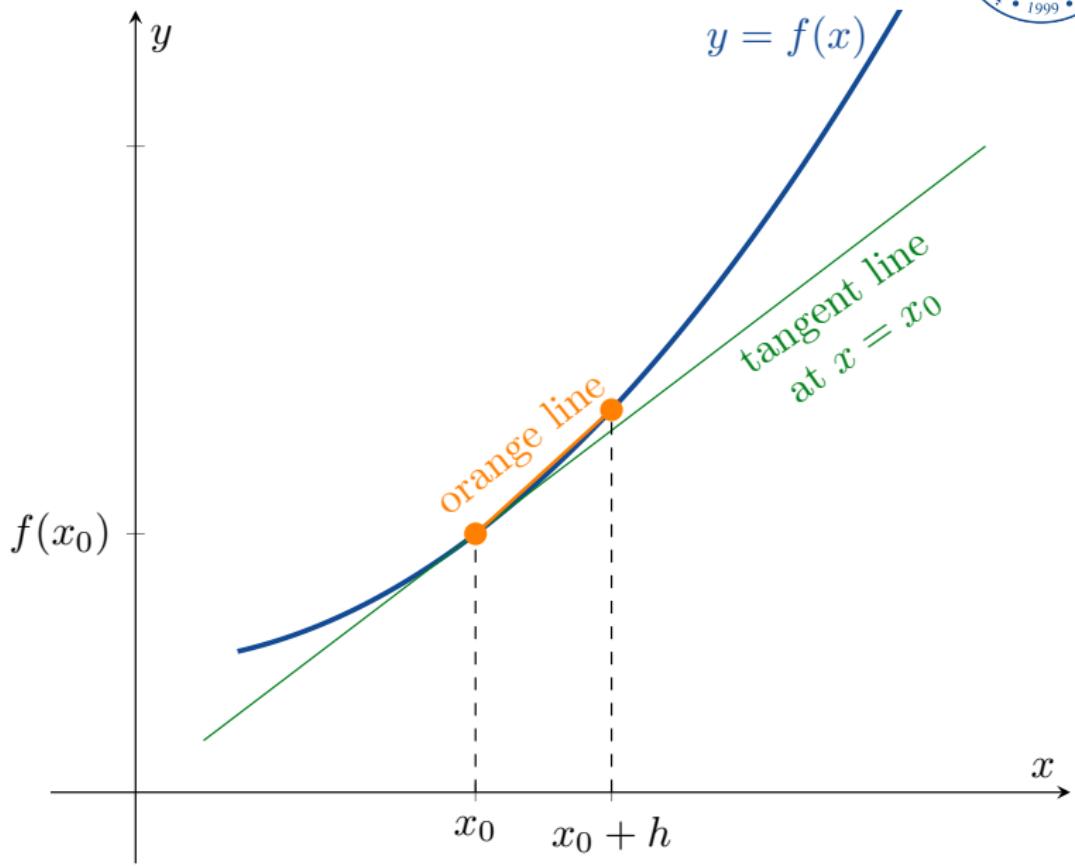
26. Differentiation



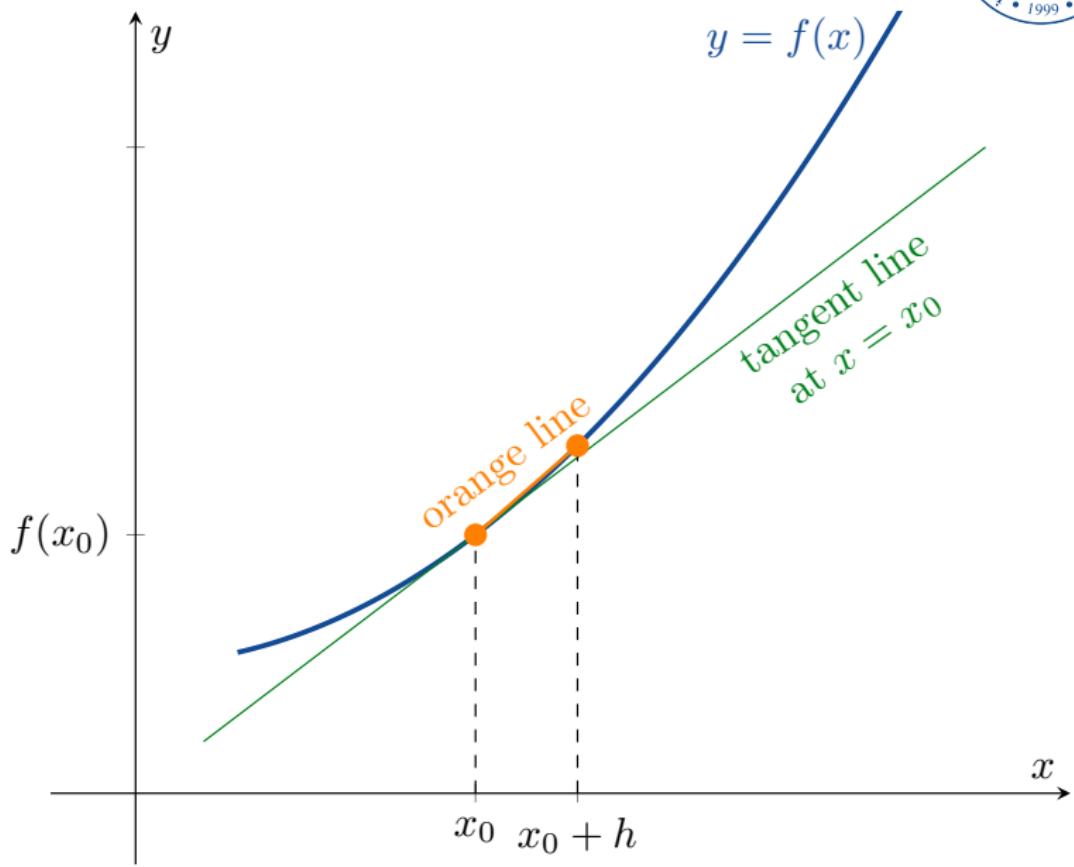
26. Differentiation



26. Differentiation



26. Differentiation



26. Differentiation



If h is very very small, then

$$\left(\begin{array}{l} \text{slope of the} \\ \text{tangent line} \end{array} \right) \approx \left(\begin{array}{l} \text{slope of the} \\ \text{orange line} \end{array} \right) = \frac{f(x_0 + h) - f(x_0)}{h}$$

The Derivative of f

Definition

The *derivative of a function f at a point x_0* is

$$f'(x_0) = \lim_{h \rightarrow 0} \frac{f(x_0 + h) - f(x_0)}{h}$$

if the limit exists.

(f' is pronounced “ f prime”)

26. Differentiation

Example

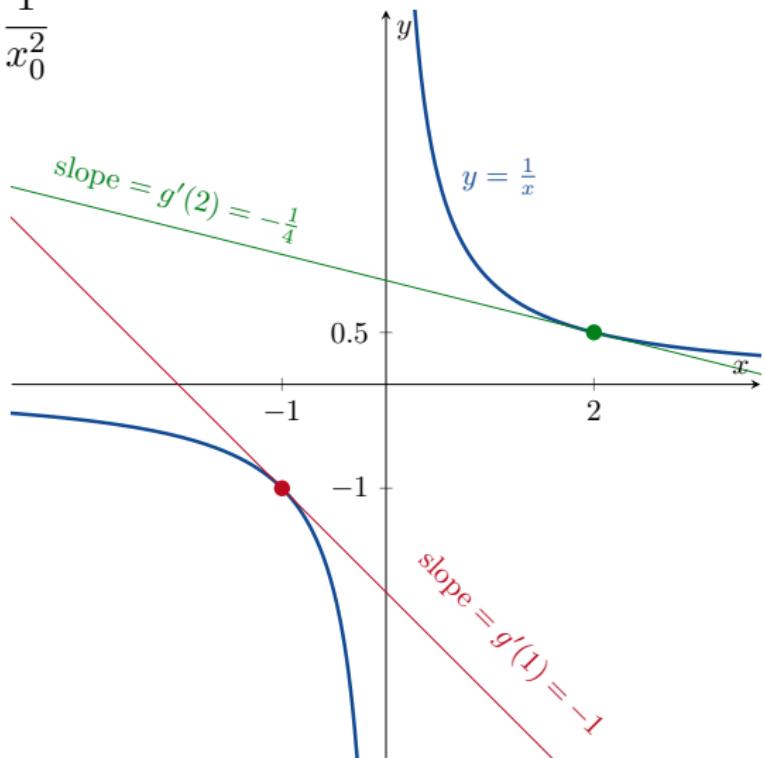
Consider the function $g(x) = \frac{1}{x}$, $x \neq 0$.

If $x_0 \neq 0$, then

$$\begin{aligned} g'(x_0) &= \lim_{h \rightarrow 0} \frac{g(x_0 + h) - g(x_0)}{h} \\ &= \lim_{h \rightarrow 0} \frac{\frac{1}{x_0+h} - \frac{1}{x_0}}{h} \\ &= \lim_{h \rightarrow 0} \frac{\left(\frac{x_0}{x_0(x_0+h)} - \frac{x_0+h}{x_0(x_0+h)} \right)}{h} \\ &= \lim_{h \rightarrow 0} \frac{x_0 - x_0 - h}{hx_0(x_0 + h)} \\ &= \lim_{h \rightarrow 0} \frac{-1}{x_0(x_0 + h)} = -\frac{1}{x_0^2}. \end{aligned}$$

26. Differentiation

$$g'(x_0) = -\frac{1}{x_0^2}$$



26. Differentiation



Definition

If $f'(x_0)$ exists, we say that f is differentiable at x_0 .

26. Differentiation



Definition

Let $f : D \rightarrow \mathbb{R}$ be a function. If f is differentiable at every $x_0 \in D$, we say that f is *differentiable*.

26. Differentiation

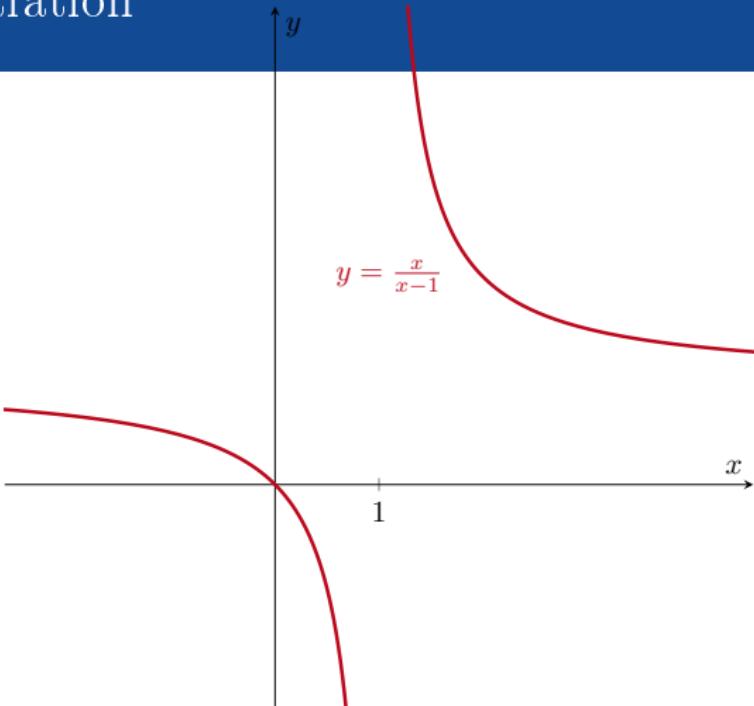


If $f : D \rightarrow \mathbb{R}$ is differentiable, then we have a new function
 $f' : D \rightarrow \mathbb{R}$.

Definition

f' is called the *derivative* of f .

26. Differentiation



Example

Differentiate $f(x) = \frac{x}{x-1}$.

26. Differentiation

solution: First note that $f(x + h) = \frac{x+h}{x+h-1}$. Therefore

$$\begin{aligned}
 f'(x) &= \lim_{h \rightarrow 0} \frac{f(x + h) - f(x)}{h} \\
 &= \lim_{h \rightarrow 0} \frac{\frac{x+h}{x+h-1} - \frac{x}{x-1}}{h} \\
 &= \lim_{h \rightarrow 0} \frac{1}{h} \left(\frac{(x+h)(x-1) - x(x+h-1)}{(x-1)(x+h-1)} \right) \\
 &= \lim_{h \rightarrow 0} \frac{1}{h} \left(\frac{-h}{(x-1)(x+h-1)} \right) \\
 &= \lim_{h \rightarrow 0} \frac{-1}{(x-1)(x+h-1)} \\
 &= \frac{-1}{(x-1)(x+0-1)} \\
 &= \frac{-1}{(x-1)^2}.
 \end{aligned}$$

Notations

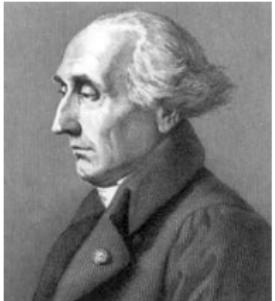
There are many ways to write the derivative of $y = f(x)$.

$$f'(x) = y' = \frac{dy}{dx} = \frac{df}{dx} = \frac{d}{dx}f(x) = \dot{y} = \dot{f}(x)$$



“the derivative of y with respect to x ”

26. Differentiation



Sir Isaac Newton
UK, 1642-1726

Gottfried Leibniz
GER, 1646-1716

Joseph-Louis Lagrange
ITA, 1736-1813

Calculus was started by two men who hated each other: Sir Isaac Newton used f' and \dot{y} . Gottfried Leibniz used $\frac{df}{dx}$ and $\frac{dy}{dx}$.

The f' and y' notation came later from Joseph-Louis Lagrange.

26. Differentiation



If we want the derivative of $y = f(x)$ at the point $x = x_0$, we can write

$$f'(x_0) = \frac{dy}{dx} \Big|_{x=x_0} = \frac{df}{dx} \Big|_{x=x_0} = \frac{d}{dx} f(x) \Big|_{x=x_0}$$



“the derivative of y with respect to x at $x = x_0$ ”

26. Differentiation



For example, if $u(x) = \frac{1}{x}$, then

$$u'(4) = \left. \frac{d}{dx} \left(\frac{1}{x} \right) \right|_{x=4} = \left. \frac{-1}{x^2} \right|_{x=4} = \frac{-1}{4^2} = \frac{-1}{16}.$$

26. Differentiation



Example

Show that $f(x) = |x|$ is differentiable on $(-\infty, 0)$ and on $(0, \infty)$, but is not differentiable at $x = 0$.

26. Differentiation



solution: If $x > 0$ then

$$\frac{df}{dx} = \frac{d}{dx}(|x|) = \frac{d}{dx}(x) = \lim_{h \rightarrow 0} \frac{(x+h)-x}{h} = \lim_{h \rightarrow 0} 1 = 1.$$

26. Differentiation



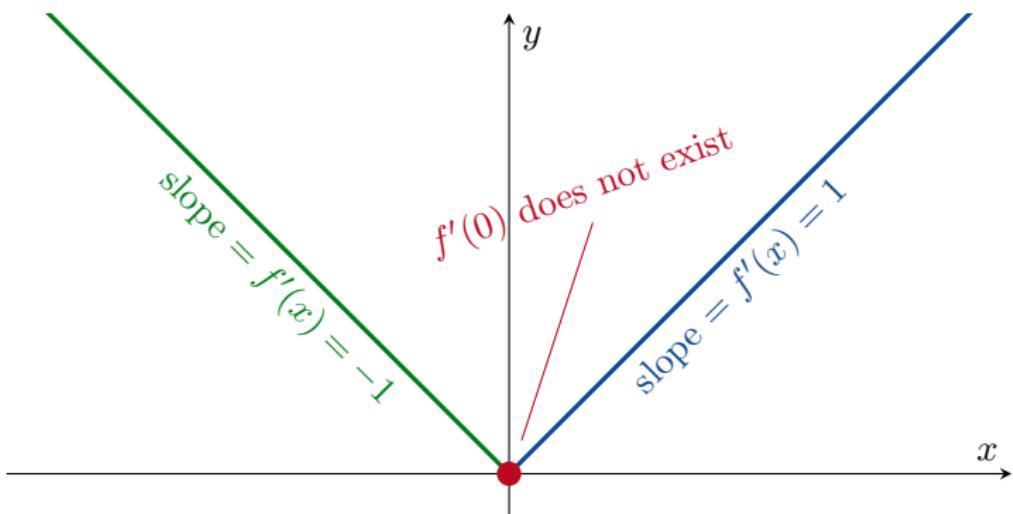
Similarly, if $x < 0$ then

$$\begin{aligned}\frac{df}{dx} &= \frac{d}{dx}(|x|) = \frac{d}{dx}(-x) = \lim_{h \rightarrow 0} \frac{(-x - h) - (-x)}{h} \\ &= \lim_{h \rightarrow 0} -1 = -1.\end{aligned}$$

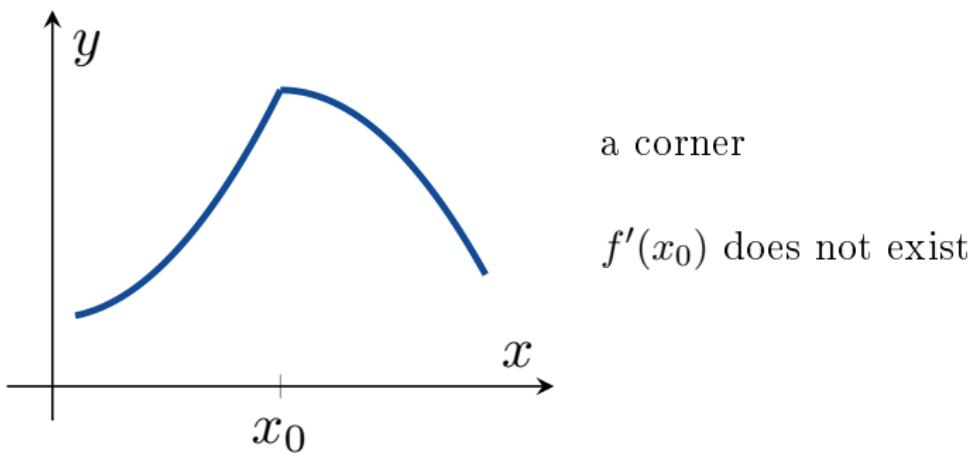
Therefore f is differentiable on $(-\infty, 0)$ and on $(0, \infty)$.

26. Differentiation

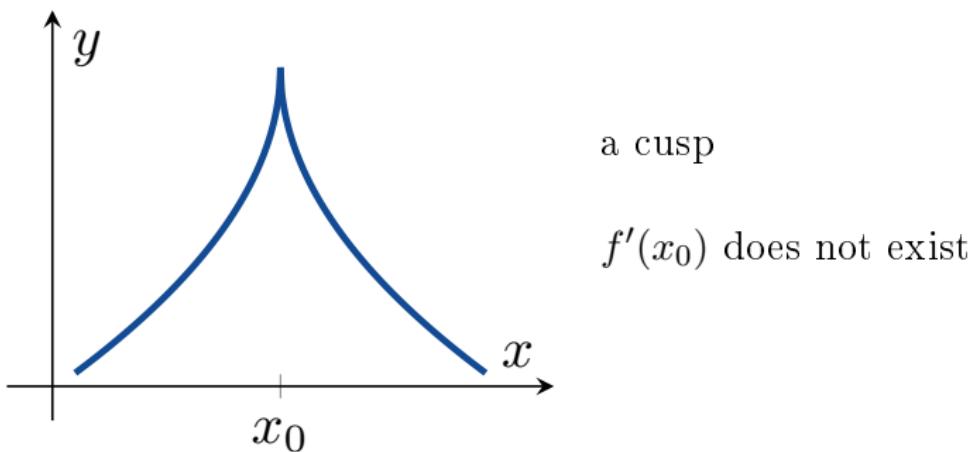
Since $\lim_{h \rightarrow 0} \frac{|0+h| - |0|}{h} = \lim_{h \rightarrow 0} \frac{|h|}{h} = \lim_{h \rightarrow 0} (\pm 1)$ does not exist, f is not differentiable at 0.



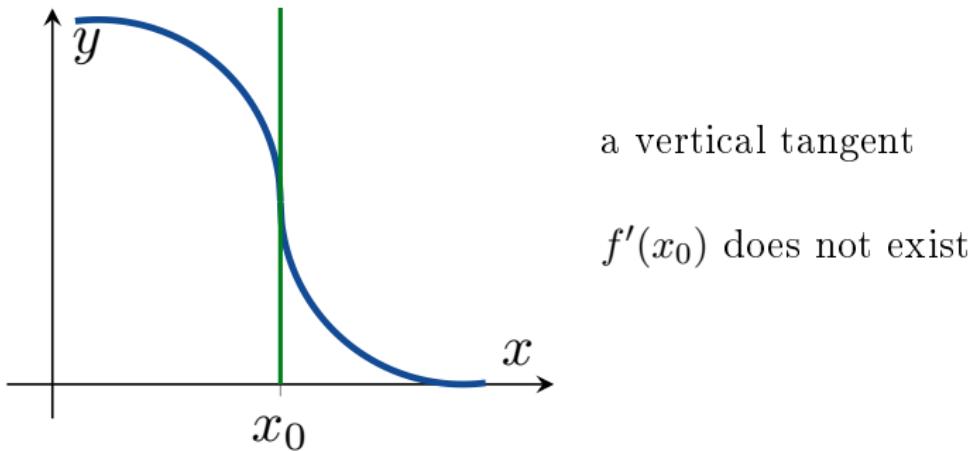
When Does a Function Not Have a Derivative at a Point?



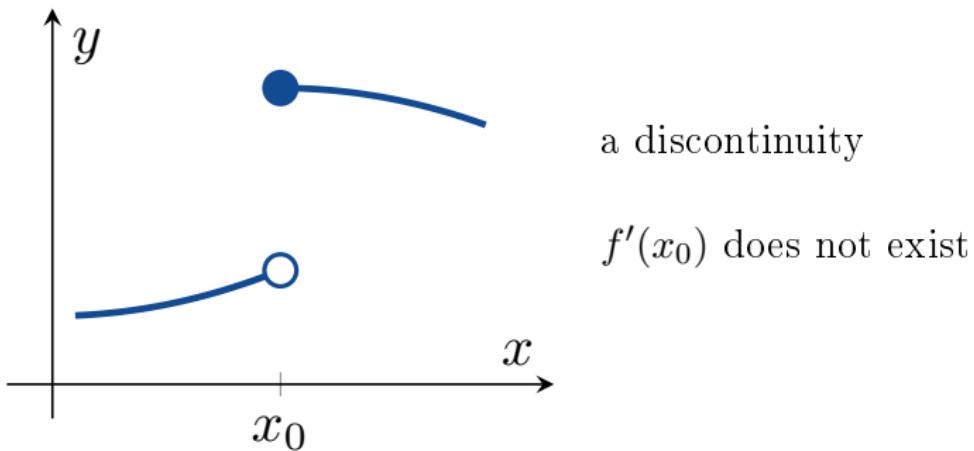
When Does a Function Not Have a Derivative at a Point?



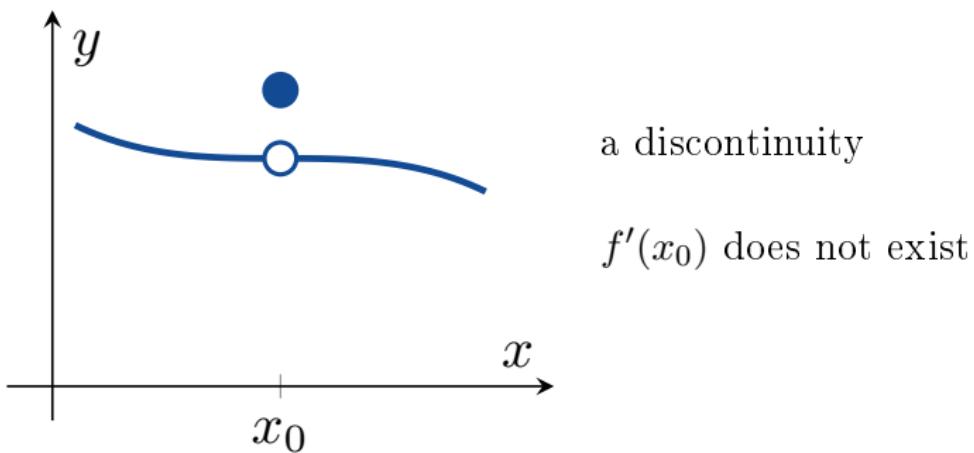
When Does a Function Not Have a Derivative at a Point?



When Does a Function Not Have a Derivative at a Point?



When Does a Function Not Have a Derivative at a Point?



26. Differentiation



Theorem

$$\left(\begin{array}{c} f \text{ has a derivative} \\ \text{at } x = x_0 \end{array} \right) \implies \left(\begin{array}{c} f \text{ is continuous} \\ \text{at } x = x_0 \end{array} \right)$$



Differentiation Rules

27. Differentiation Rules



Constant Function

If $k \in \mathbb{R}$, then

$$\boxed{\frac{d}{dx}(k) = 0.}$$

Power Function

If $n \in \mathbb{R}$, then

$$\frac{d}{dx} (x^n) = nx^{n-1}.$$

Example

$$\frac{d}{dx} (x^3) = 3x^{3-1} = 3x^2$$

27. Differentiation Rules



Example

$$\frac{d}{dx} (\sqrt{x}) = \frac{d}{dx} (x^{\frac{1}{2}}) = \frac{1}{2}x^{\frac{1}{2}-1} = \frac{1}{2}x^{-\frac{1}{2}} = \frac{1}{2\sqrt{x}}$$

Example

$$\frac{d}{dx} \left(\frac{1}{x^4} \right) = \frac{d}{dx} (x^{-4}) = -4x^{-4-1} = -4x^{-5} = -\frac{4}{x^5}$$

The Constant Multiple Rule

If $u(x)$ is differentiable and $k \in \mathbb{R}$, then

$$\boxed{\frac{d}{dx}(ku) = k\frac{du}{dx}}.$$

Proof.

$$\begin{aligned}\frac{d}{dx}(ku) &= \lim_{h \rightarrow 0} \frac{ku(x+h) - ku(x)}{h} \\ &= k \lim_{h \rightarrow 0} \frac{u(x+h) - u(x)}{h} = k\frac{du}{dx}\end{aligned}$$



27. Differentiation Rules



Example

$$\frac{d}{dx} (3x^2) = 3 \frac{d}{dx} (x^2) = 3 \times 2x = 6x$$

Example

$$\frac{d}{dx} (-u) = \frac{d}{dx} (-1 \times u) = -1 \times \frac{du}{dx} = -\frac{du}{dx}$$

The Sum Rule

If $u(x)$ and $v(x)$ are differentiable at x_0 , then $u + v$ is also differentiable at x_0 and

$$\frac{d}{dx} (u + v) = \frac{du}{dx} + \frac{dv}{dx}.$$

27. Differentiation Rules



Example

Differentiate $y = x^3 + \frac{4}{3}x^2 - 5x + 1$.

solution:

$$\begin{aligned}\frac{dy}{dx} &= \frac{d}{dx} \left(x^3 + \frac{4}{3}x^2 - 5x + 1 \right) \\&= \frac{d}{dx} (x^3) + \frac{d}{dx} \left(\frac{4}{3}x^2 \right) - \frac{d}{dx} (5x) + \frac{d}{dx} (1) \\&= 3x^2 + \frac{8}{3}x - 5 + 0\end{aligned}$$

27. Differentiation Rules

Example

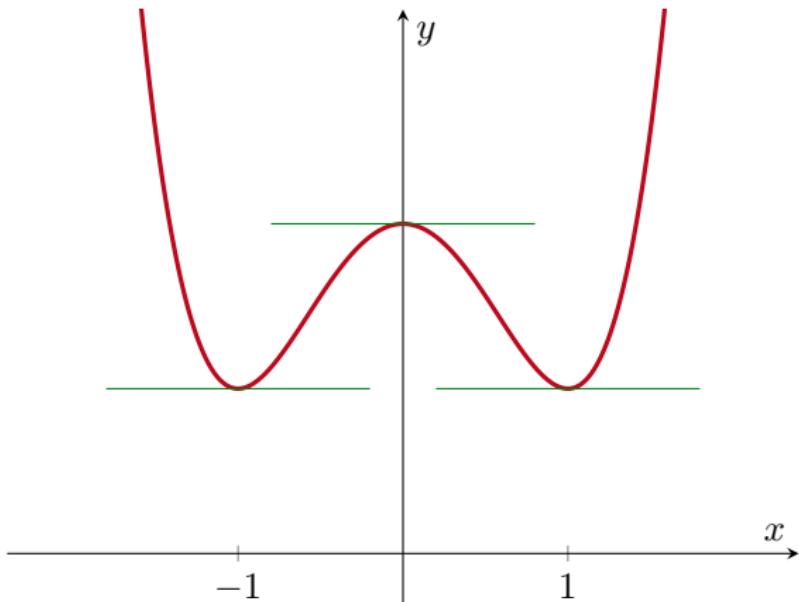
Does the curve $y = x^4 - 2x^2 + 2$ have any points where $\frac{dy}{dx} = 0$?
If so, where?

solution: Since

$$\frac{dy}{dx} = 4x^3 - 4x = 4x(x^2 - 1) = 4x(x - 1)(x + 1),$$

we can see that $\frac{dy}{dx} = 0$ if and only if $x = -1, 0$ or 1 .

27. Differentiation Rules



The Product Rule

If $u(x)$ and $v(x)$ are differentiable at x_0 , then $u(x)v(x)$ is also differentiable at x_0 and

$$\frac{d}{dx} (uv) = u \frac{dv}{dx} + v \frac{du}{dx}.$$

Using prime notation, the product rule is

$$(uv)' = u'v + uv'.$$

27. Differentiation Rules



Example

Differentiate $y = (x^2 + 1)(x^3 + 3)$.

solution 1: We have $y = uv$ with $u = x^2 + 1$ and $v = x^3 + 3$. So

$$\begin{aligned}\frac{dy}{dx} &= (x^2 + 1)'(x^3 + 3) + (x^2 + 1)(x^3 + 3)' \\&= (2x + 0)(x^3 + 3) + (x^2 + 1)(3x^2 + 0) \\&= 2x^4 + 6x + 3x^4 + 3x^2 \\&= 5x^4 + 3x^2 + 6x.\end{aligned}$$

27. Differentiation Rules



solution 2: Since

$$y = (x^2 + 1)(x^3 + 3) = x^5 + x^3 + 3x^2 + 3,$$

we have that

$$\frac{dy}{dx} = 5x^4 + 3x^2 + 6x + 0.$$

The Quotient Rule

If $u(x)$ and $v(x)$ are differentiable at x_0 and if $v(x_0) \neq 0$, then $\frac{u}{v}$ is also differentiable at x_0 and

$$\frac{d}{dx} \left(\frac{u}{v} \right) = \frac{u'v - uv'}{v^2}.$$

27. Differentiation Rules

Example

Differentiate $y = \frac{t^2 - 1}{t^3 + 1}$.

solution: We have $y = \frac{u}{v}$ with $u = t^2 - 1$ and $v = t^3 + 1$.

Therefore

$$\begin{aligned}\frac{dy}{dt} &= \frac{u'v - uv'}{v^2} \\&= \frac{(t^2 - 1)'(t^3 + 1) - (t^2 - 1)(t^3 + 1)'}{(t^3 + 1)^2} \\&= \frac{(2t)(t^3 + 1) - (t^2 - 1)(3t^2)}{(t^3 + 1)^2} \\&= \frac{2t^4 + 2t - 3t^4 + 3t^2}{(t^3 + 1)^2} \\&= \frac{-t^4 + 3t^2 + 2t}{(t^3 + 1)^2}.\end{aligned}$$

27. Differentiation Rules

Example

Differentiate $f(s) = \frac{\sqrt{s}-1}{\sqrt{s}+1}$.

solution: We have $f(s) = \frac{u}{v}$ with $u = \sqrt{s} - 1$ and $v = \sqrt{s} + 1$.

Remember that $\frac{d}{ds}(\sqrt{s}) = \frac{1}{2\sqrt{s}}$. Therefore

$$\begin{aligned}
 \frac{df}{ds} &= \frac{u'v - uv'}{v^2} \\
 &= \frac{(\sqrt{s} - 1)'(\sqrt{s} + 1) - (\sqrt{s} - 1)(\sqrt{s} + 1)'}{(\sqrt{s} + 1)^2} \\
 &= \frac{\left(\frac{1}{2\sqrt{s}}\right)(\sqrt{s} + 1) - (\sqrt{s} - 1)\left(\frac{1}{2\sqrt{s}}\right)}{(\sqrt{s} + 1)^2} \\
 &= \frac{\frac{1}{2} + \frac{1}{2\sqrt{s}} - \frac{1}{2} + \frac{1}{2\sqrt{s}}}{(\sqrt{s} + 1)^2} = \frac{1}{\sqrt{s}(\sqrt{s} + 1)^2}.
 \end{aligned}$$

Second Order Derivatives

If $y = f(x)$ is a differentiable function, then $f'(x)$ is also a function. If $f'(x)$ is also differentiable, then we can differentiate to find a new function called f'' (" f double prime"). f'' is called the *second derivative* of f . We can write

$$f''(x) = \frac{d}{dx} f'(x) = \frac{d}{dx} \left(\frac{dy}{dx} \right) = \frac{d^2y}{dx^2} = \frac{dy'}{dx} = y''$$

↗
"d squared y, dx squared"

27. Differentiation Rules



Example

If $y = x^6$, then $y' = \frac{d}{dx}(x^6) = 6x^5$ and
 $y'' = \frac{d}{dx}(y') = \frac{d}{dx}(6x^5) = 30x^4$. Equivalently, we can write

$$\frac{d^2}{dx^2}(x^6) = \frac{d}{dx}\left(\frac{d}{dx}(x^6)\right) = \frac{d}{dx}(6x^5) = 30x^4.$$

Higher Order Derivatives

If f'' is differentiable, then its derivative $f''' = \frac{d^3 f}{dx^3}$ is the *third derivative* of f .

If f''' is differentiable, then its derivative $f^{(4)} = \frac{d^4 f}{dx^4}$ is the *fourth derivative* of f .

If $f^{(4)}$ is differentiable, then its derivative $f^{(5)} = \frac{d^5 f}{dx^5}$ is the *fifth derivative* of f .

⋮

If $f^{(n-1)}$ is differentiable, then its derivative $f^{(n)} = \frac{d^n f}{dx^n}$ is the *n th derivative* of f .

27. Differentiation Rules



Example

Find the first four derivatives of $y = x^3 - 3x^2 + 2$.

solution:

$$\text{First derivative: } y' = 3x^2 - 6x$$

$$\text{Second derivative: } y'' = 6x - 6$$

$$\text{Third derivative: } y''' = 6$$

$$\text{Fourth derivative: } y^{(4)} = 0.$$

(Note that since $\frac{d}{dx}(0) = 0$, if $n \geq 4$ then $y^{(n)} = 0$ also.)



Next Time

- 28. Derivatives of Trigonometric Functions
- 29. The Chain Rule
- 30. Antiderivatives
- 31. Integration