

Lecture 8

- 14.3 Area by Double Integration
- 10.3 Polar Coordinates
- 14.4 Double Integrals in Polar Form

11 Area by Double Integration 3

14.3 Area by Double Integration



Definition

The *area* of a closed, bounded region R is

$$A = \iint_R dA.$$

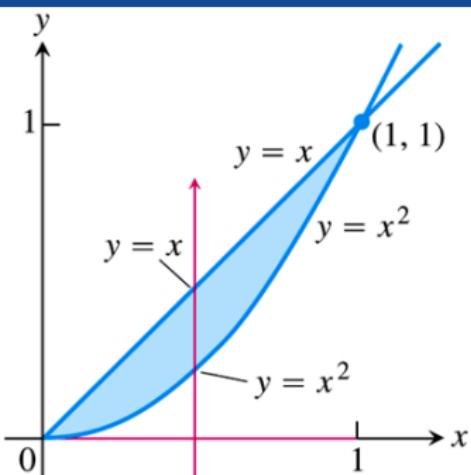
14.3 Area by Double Integration



Example

Find the area of the region R bounded by $y = x$ and $y = x^2$ in the first quadrant.

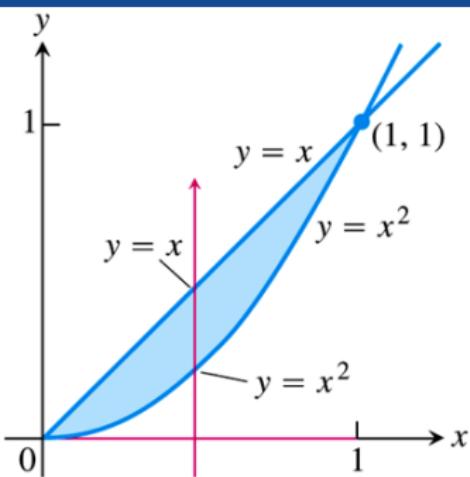
14.3 Area by Dou



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14.3 Area by Dou

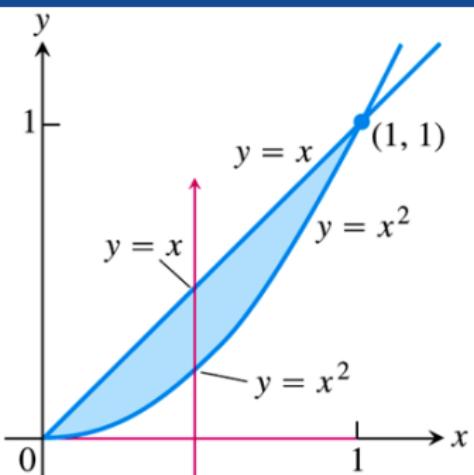


Example

Find the area of the region R bounded by $y = x$ and $y = x^2$ in the first quadrant.

$$A = \int_0^1 \int_{x^2}^x dy dx$$

14.3 Area by Double Integration



Example

Find the area of the region R bounded by $y = x$ and $y = x^2$ in the first quadrant.

$$\begin{aligned}A &= \int_0^1 \int_{x^2}^x dy dx = \int_0^1 [y]_{x^2}^x dx = \int_0^1 (x - x^2) dx \\&= \left[\frac{x^2}{2} - \frac{x^3}{3} \right]_0^1 = \frac{1}{6}.\end{aligned}$$

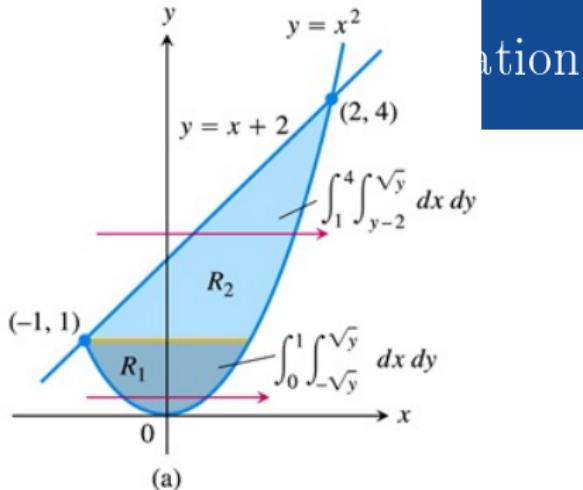
14.3 Area by Double Integration



Example

Find the area of the region R bounded by $y = x$ and $y = x^2$ in the first quadrant.

14.3 A

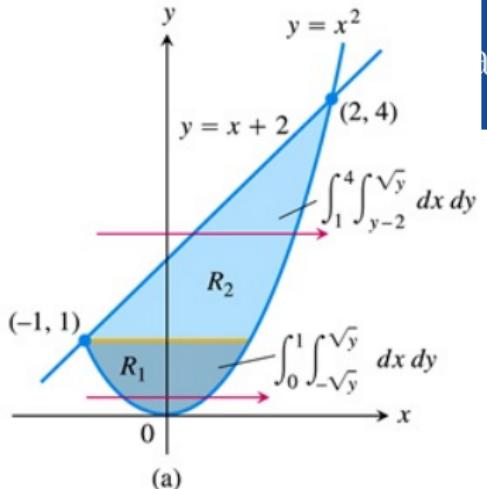


Example

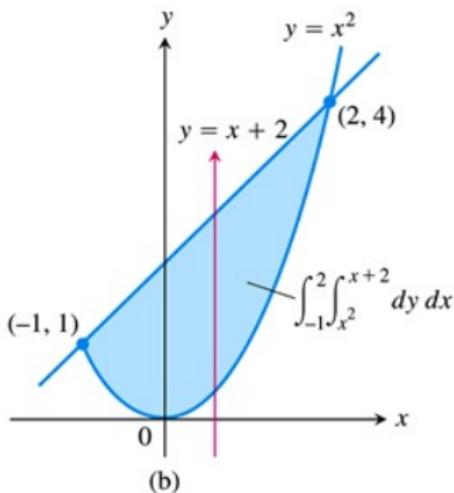
Find the area of the region R bounded by $y = x$ and $y = x^2$ in the first quadrant.

$$A = \iint_{R_1} dA + \iint_{R_2} dA = \int_0^1 \int_{-\sqrt{y}}^{\sqrt{y}} dx dy + \int_1^4 \int_{y-2}^{\sqrt{y}} dx dy = \dots$$

14.3 A



(a)



(b)

Example

Find the area of the region R bounded by $y = x$ and $y = x^2$ in the first quadrant.

$$A = \iint_{R_1} dA + \iint_{R_2} dA = \int_0^1 \int_{-\sqrt{y}}^{\sqrt{y}} dx dy + \int_1^4 \int_{y-2}^{\sqrt{y}} dx dy = \dots$$

$$A = \int_{-1}^2 \int_{x^2}^{x+2} dy dx = \int_{-1}^2 [y]_{x^2}^{x+2} = \int_{-1}^2 (x+2-x^2) dx = \dots = \frac{9}{2}.$$

14.3 Area by Double Integration

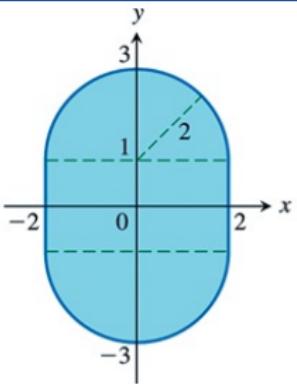


Example

Find the area of the region R described by $-2 \leq x \leq 2$ and $-1 - \sqrt{4 - x^2} \leq y \leq 1 + \sqrt{4 - x^2}$.

14.3 Area by

ration



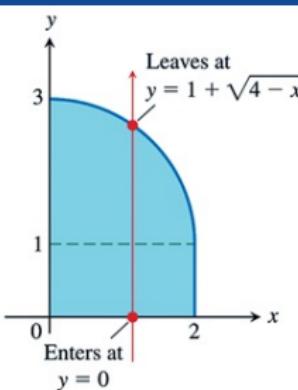
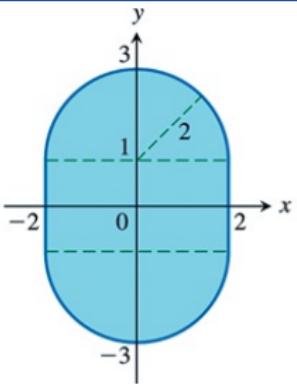
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$$A = \iint_R dA$$

14.3 Area by

ratio



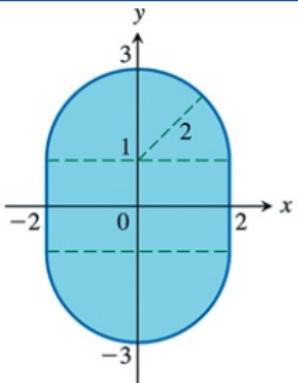
Example

Find the area of the region R described by $-2 \leq x \leq 2$ and $-1 - \sqrt{4 - x^2} \leq y \leq 1 + \sqrt{4 - x^2}$.

$$A = \iint_R dA = 4 \int_0^2 \int_{0}^{1+\sqrt{4-x^2}} dy dx = \dots = 8 + 4\pi.$$

14.3 Area by

ration



Example

Find the area of the region R described by $-2 \leq x \leq 2$ and $-1 - \sqrt{4 - x^2} \leq y \leq 1 + \sqrt{4 - x^2}$.

$$A = \iint_R dA = 4 \int_0^2 \int_{0}^{1+\sqrt{4-x^2}} dy dx = \dots = 8 + 4\pi.$$

or

$$A = \left(\begin{array}{l} \text{area of a} \\ \text{circle of} \\ \text{radius 2} \end{array} \right) + \left(\begin{array}{l} \text{area of a } 4 \times 2 \\ \text{rectangle} \end{array} \right) = 4\pi + 8.$$

14.3 Area by Double Integration



Average Value of a Function

Definition

The *average value* of f over R is

$$\text{av}(f) = \frac{1}{\text{area of } R} \iint_R f \, dA$$

14.3 Area by Double Integration



Average Value of a Function

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$$\text{av}(f) = \frac{1}{\text{area of } R} \iint_R f \, dA = \frac{\iint_R f \, dA}{\iint_R dA}.$$

14.3 Area by Double Integration



Example

Find the average value of $f(x, y) = x \cos xy$ over the rectangle $R = [0, \pi] \times [0, 1]$.

$$\text{av}(f) = \frac{1}{\text{area of } R} \iint_R f \, dA$$

14.3 Area by Double Integration

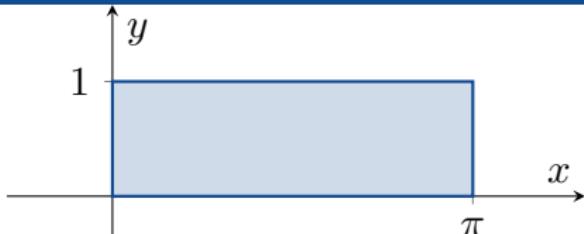


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14.3 Area by Double Integration

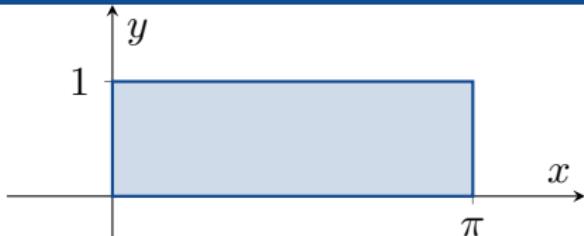


Example

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$$\begin{aligned}\text{av}(f) &= \frac{1}{\text{area of } R} \iint_R f \, dA = \frac{1}{\pi} \int_0^\pi \int_0^1 x \cos xy \, dy \, dx \\ &= \frac{1}{\pi} \int_0^\pi \left[\sin xy \right]_{y=0}^{y=1} dx = \frac{1}{\pi} \int_0^\pi (\sin x - 0) \, dx\end{aligned}$$

14.3 Area by Double Integration



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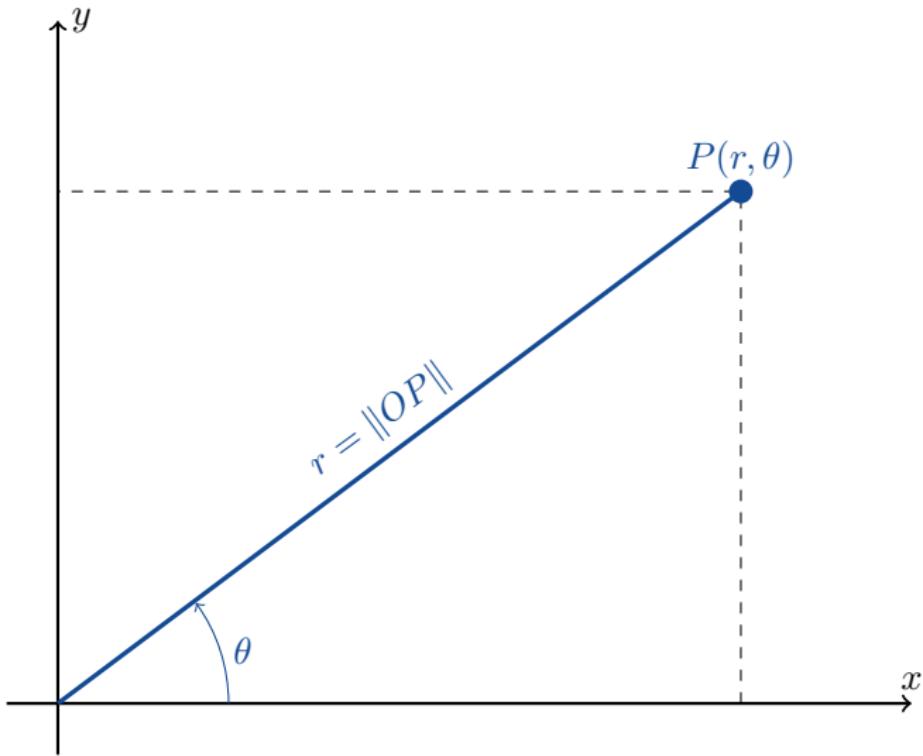
$$\begin{aligned}
 \text{av}(f) &= \frac{1}{\text{area of } R} \iint_R f \, dA = \frac{1}{\pi} \int_0^\pi \int_0^1 x \cos xy \, dy \, dx \\
 &= \frac{1}{\pi} \int_0^\pi \left[\sin xy \right]_{y=0}^{y=1} \, dx = \frac{1}{\pi} \int_0^\pi (\sin x - 0) \, dx \\
 &= \frac{1}{\pi} \left[-\cos x \right]_0^\pi = \frac{1}{\pi} (1 + 1) = \frac{2}{\pi}.
 \end{aligned}$$



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Polar Coordinates

10.3 Polar Coordinates



10.3 Polar Coordinates

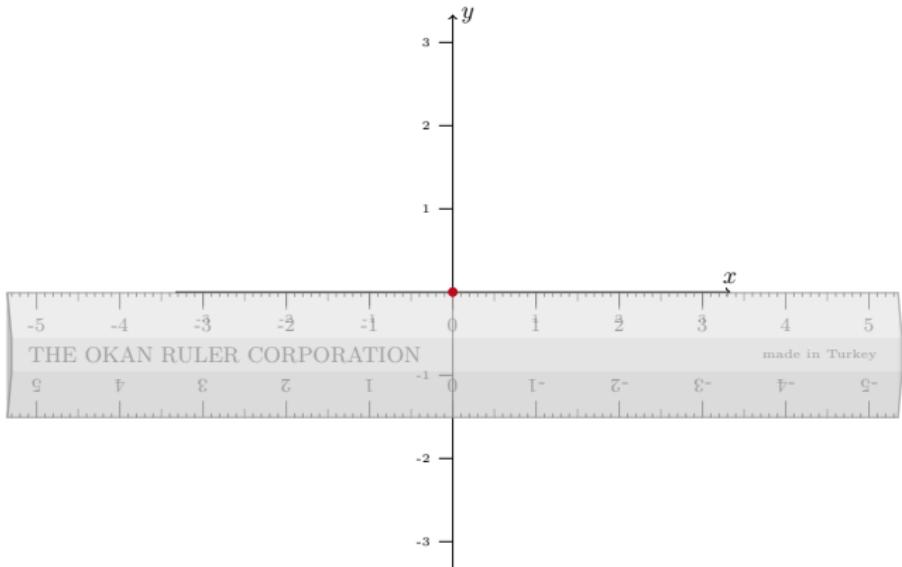


anticlockwise = positive angle
saat yönünün tersi = pozitif açı

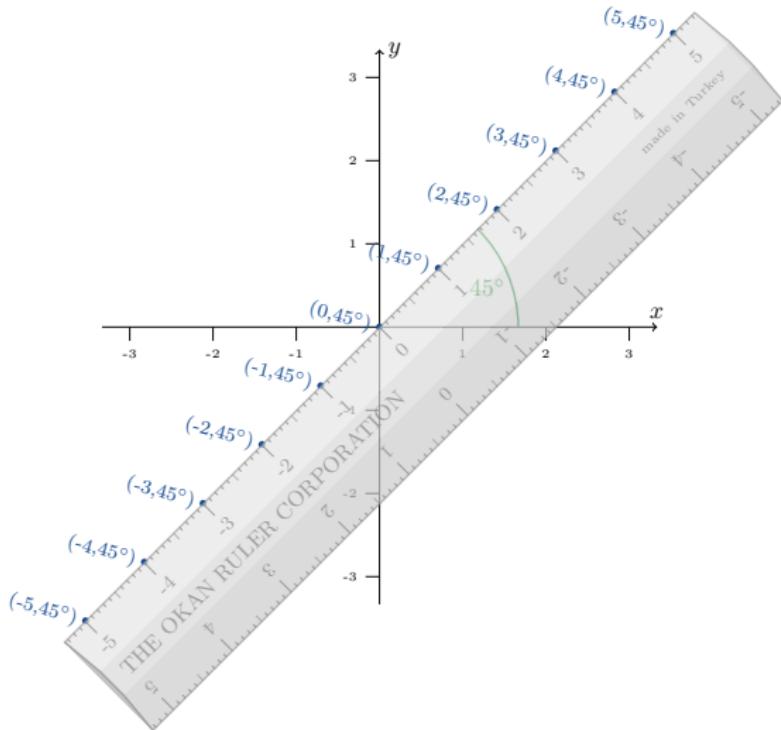


clockwise = negative angle
saat yönünde = negatif açı

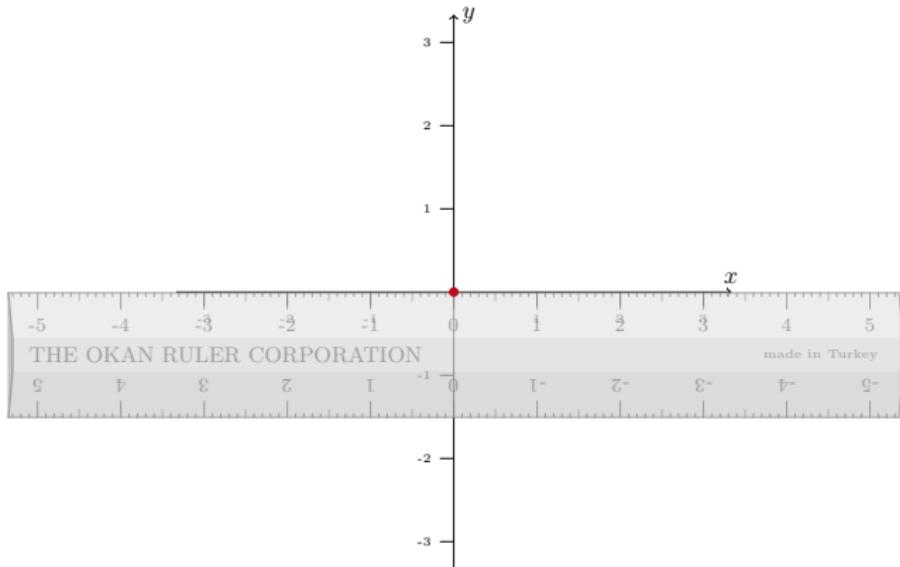
10.3 Polar Coordinates



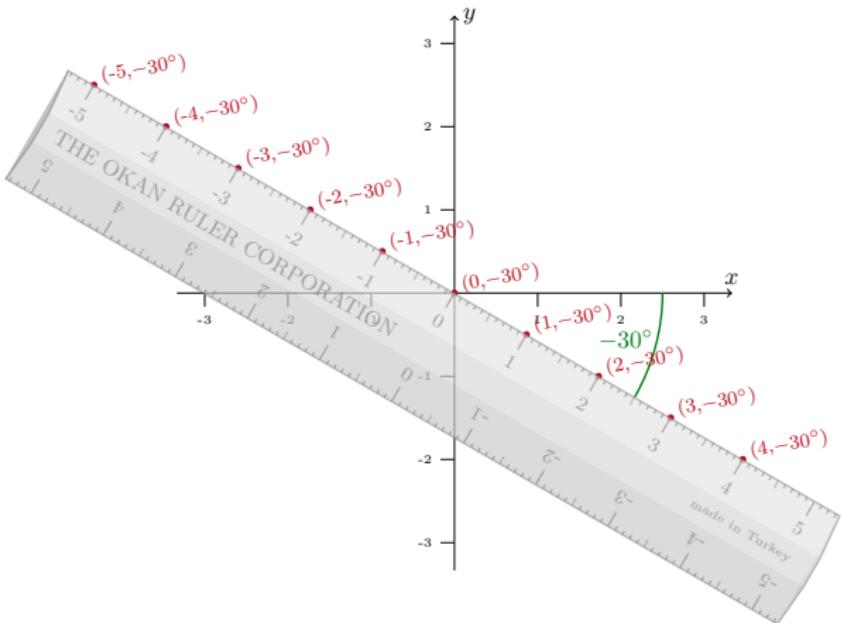
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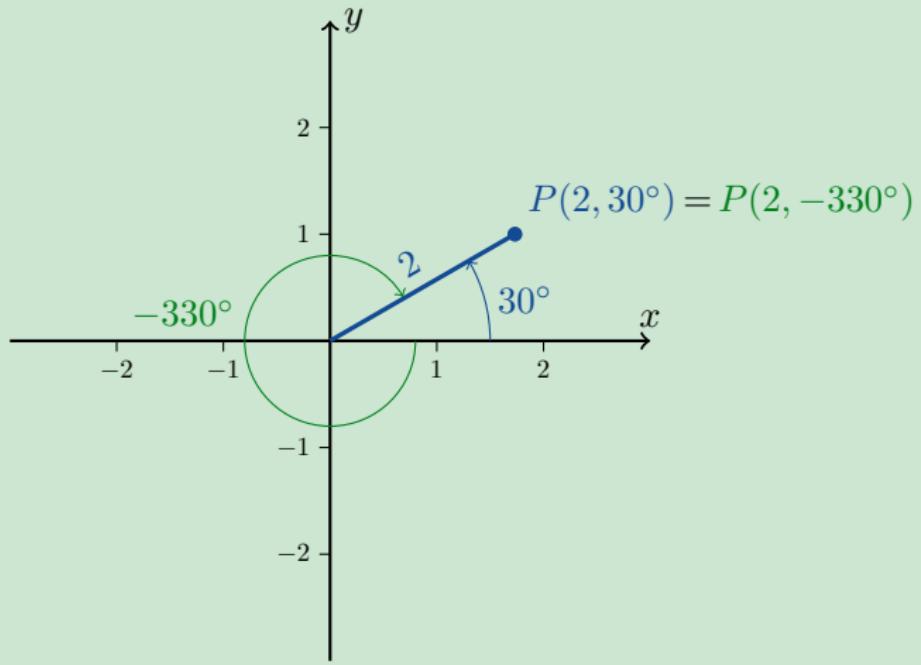


10.3 Polar Coordinates



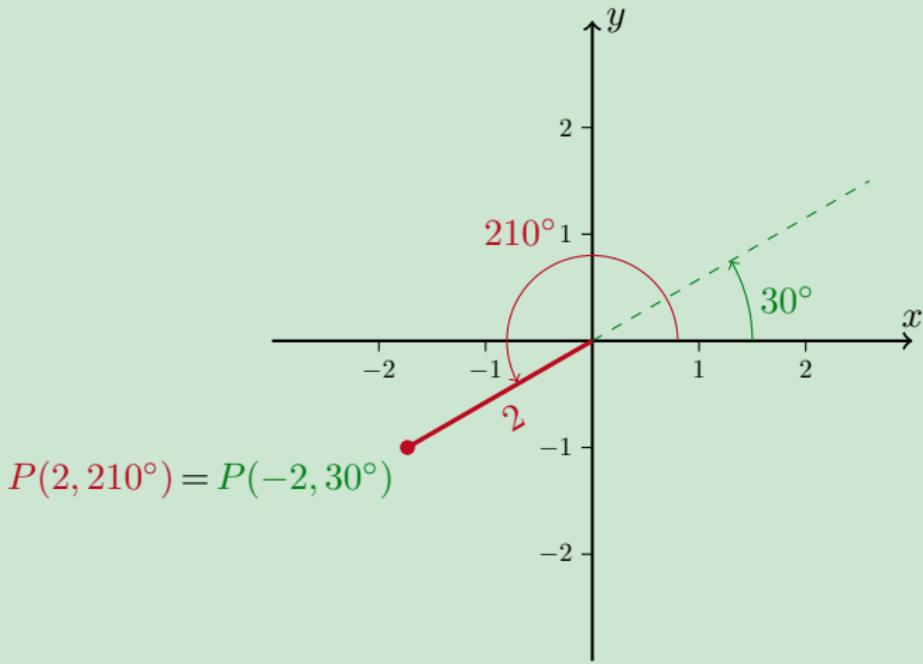
10.3 Polar Coordinates

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Example

Find all the polar coordinates of $P(2, 30^\circ)$.

We can have either $r = 2$ or $r = -2$.

10.3 Polar Coordinates



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We can have either $r = 2$ or $r = -2$. For $r = 2$, we can have

$$\theta = 30^\circ, 30^\circ \pm 360^\circ, 30^\circ \pm 720^\circ, 30^\circ \pm 1080^\circ, \dots$$

10.3 Polar Coordinates



Example

Find all the polar coordinates of $P(2, 30^\circ)$.

We can have either $r = 2$ or $r = -2$. For $r = 2$, we can have

$$\theta = 30^\circ, 30^\circ \pm 360^\circ, 30^\circ \pm 720^\circ, 30^\circ \pm 1080^\circ, \dots$$

For $r = -2$, we can have

$$\theta = 210^\circ, 210^\circ \pm 360^\circ, 210^\circ \pm 720^\circ, 210^\circ \pm 1080^\circ, \dots$$

10.3 Polar Coordinates



Example

Find all the polar coordinates of $P(2, 30^\circ)$.

We can have either $r = 2$ or $r = -2$. For $r = 2$, we can have

$$\theta = 30^\circ, 30^\circ \pm 360^\circ, 30^\circ \pm 720^\circ, 30^\circ \pm 1080^\circ, \dots$$

For $r = -2$, we can have

$$\theta = 210^\circ, 210^\circ \pm 360^\circ, 210^\circ \pm 720^\circ, 210^\circ \pm 1080^\circ, \dots$$

Therefore

$$P(2, 30^\circ) = P\left(2, (30 + 360n)^\circ\right) = P\left(-2, (210 + 360m)^\circ\right)$$

for all $m, n \in \mathbb{Z}$.

10.3 Polar Coordinates



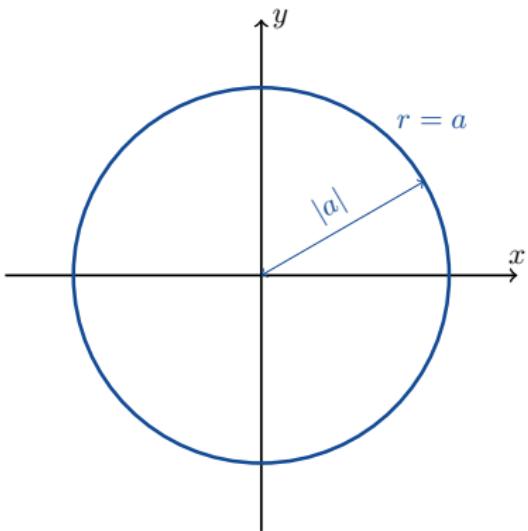
Example

Draw the graph of $r = a$.

10.3 Polar Coordinates

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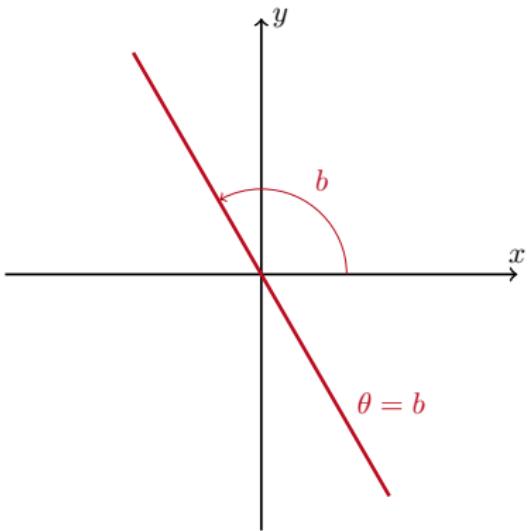
Example

Draw the graph of $\theta = b$.

10.3 Polar Coordinates

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Draw the graph of $\theta = b$.



10.3 Polar Coordinates



Remark

$r = 1$ and $r = -1$ are both equations for a circle of radius 1 centred at the origin.

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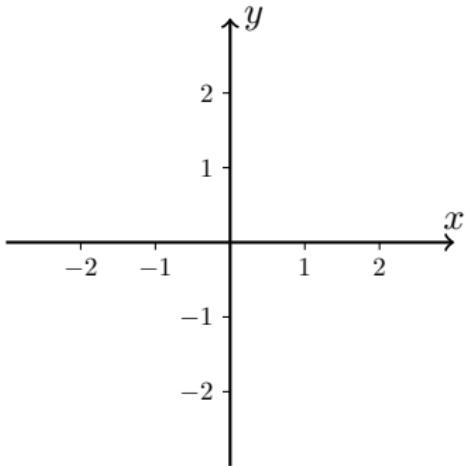
Remark

$\theta = 30^\circ$, $\theta = 210^\circ$ and $\theta = -150^\circ$ are all equations for the same line.

10.3 Polar Coordinates

Example

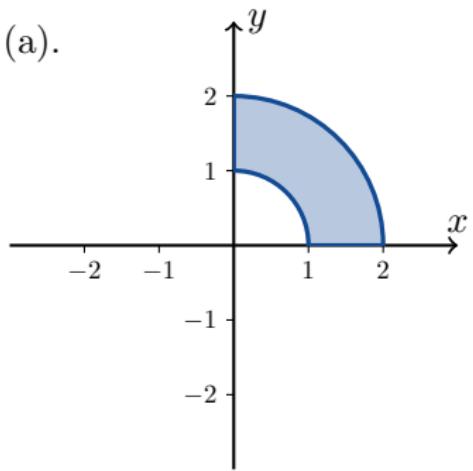
Draw the sets of points whose polar coordinates satisfy the following: $1 \leq r \leq 2$ and $0 \leq \theta \leq 90^\circ$.



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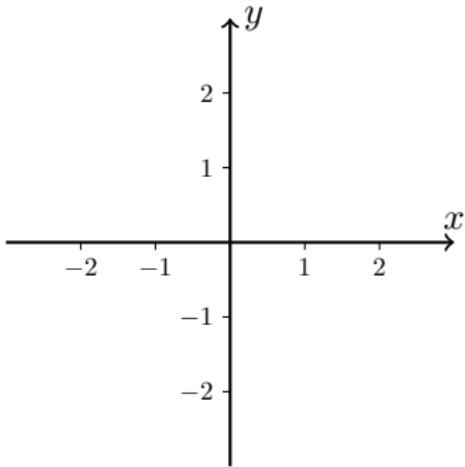


10.3 Polar Coordinates



Example

Draw the sets of points whose polar coordinates satisfy the following: $-3 \leq r \leq 2$ and $\theta = 45^\circ$.

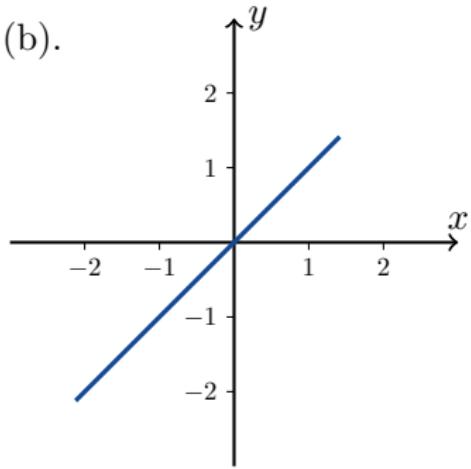


10.3 Polar Coordinates



Example

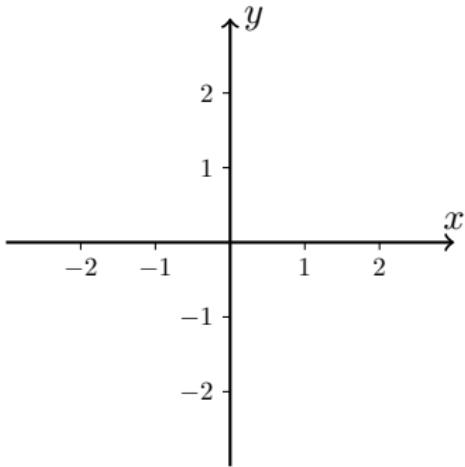
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10.3 Polar Coordinates

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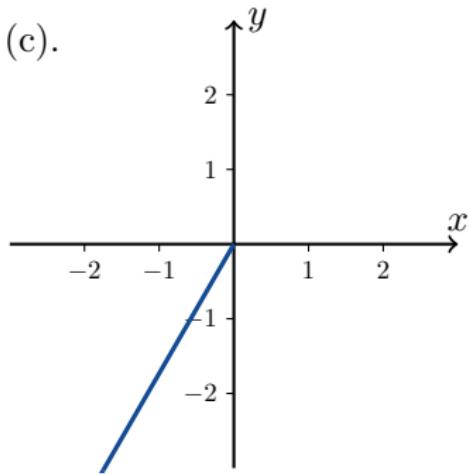
Draw the sets of points whose polar coordinates satisfy the following: $r \leq 0$ and $\theta = 60^\circ$.



10.3 Polar Coordinates

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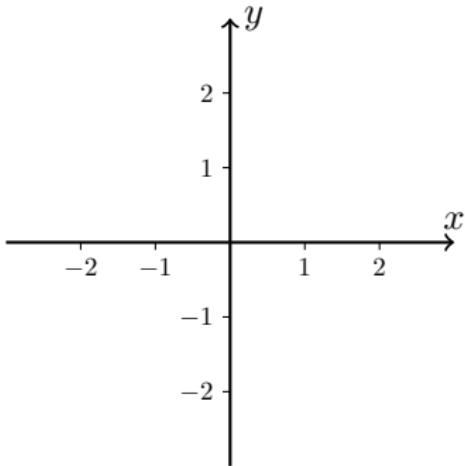


10.3 Polar Coordinates



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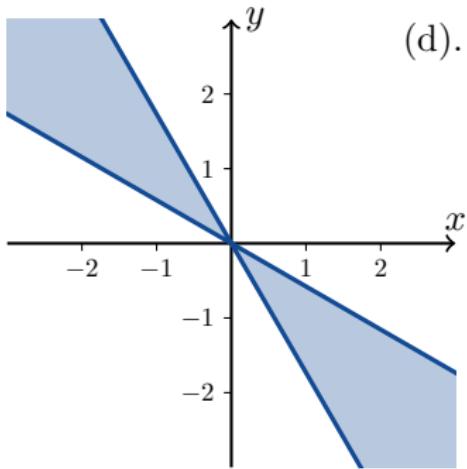
Draw the sets of points whose polar coordinates satisfy the following: $120^\circ \leq \theta \leq 150^\circ$.



10.3 Polar Coordinates

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Draw the sets of points whose polar coordinates satisfy the following: $120^\circ \leq \theta \leq 150^\circ$.



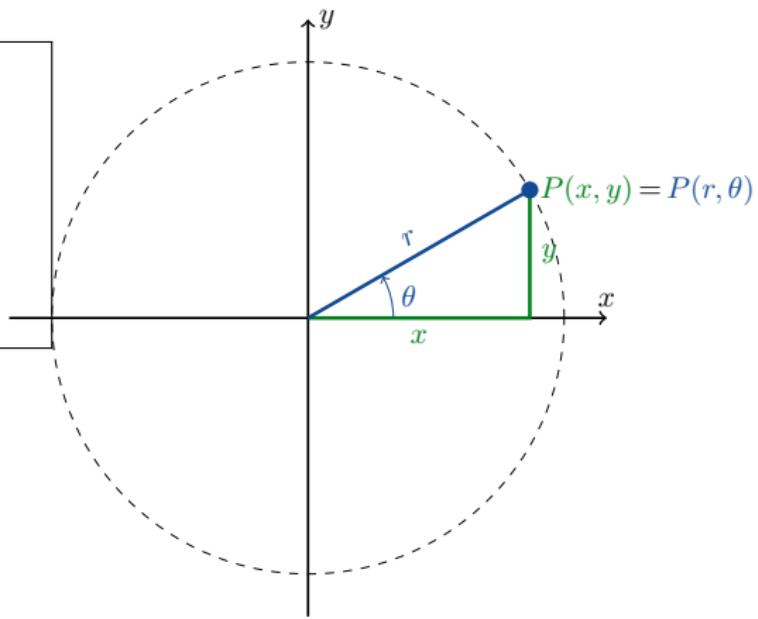
Relating Polar and Cartesian Coordinates

$$x = r \cos \theta$$

$$x^2 + y^2 = r^2$$

$$y = r \sin \theta$$

$$\tan \theta = \frac{y}{x}$$



10.3

$$x = r \cos \theta \quad y = r \sin \theta \quad x^2 + y^2 = r^2 \quad \tan \theta = \frac{y}{x}$$



Example

Convert the polar coordinates $(r, \theta) = (-3, 90^\circ)$ into Cartesian coordinates.

10.3

$$x = r \cos \theta \quad y = r \sin \theta \quad x^2 + y^2 = r^2 \quad \tan \theta = \frac{y}{x}$$



Example

Convert the polar coordinates $(r, \theta) = (-3, 90^\circ)$ into Cartesian coordinates.

$$(x, y) = (r \cos \theta, r \sin \theta) = (-3 \cos 90^\circ, -3 \sin 90^\circ) = (0, -3).$$

10.3

$$x = r \cos \theta \quad y = r \sin \theta \quad x^2 + y^2 = r^2 \quad \tan \theta = \frac{y}{x}$$



Example

Find polar coordinates for the Cartesian coordinates
 $(x, y) = (5, -12)$.

$$x = r \cos \theta \quad y = r \sin \theta \quad x^2 + y^2 = r^2 \quad \tan \theta = \frac{y}{x}$$

Example

Find polar coordinates for the Cartesian coordinates $(x, y) = (5, -12)$.

Choosing $r > 0$, we calculate that

$$r = \sqrt{x^2 + y^2} = \sqrt{25 + 144} = \sqrt{169} = 13.$$

To find θ we use the equation $y = r \sin \theta$ to calculate that

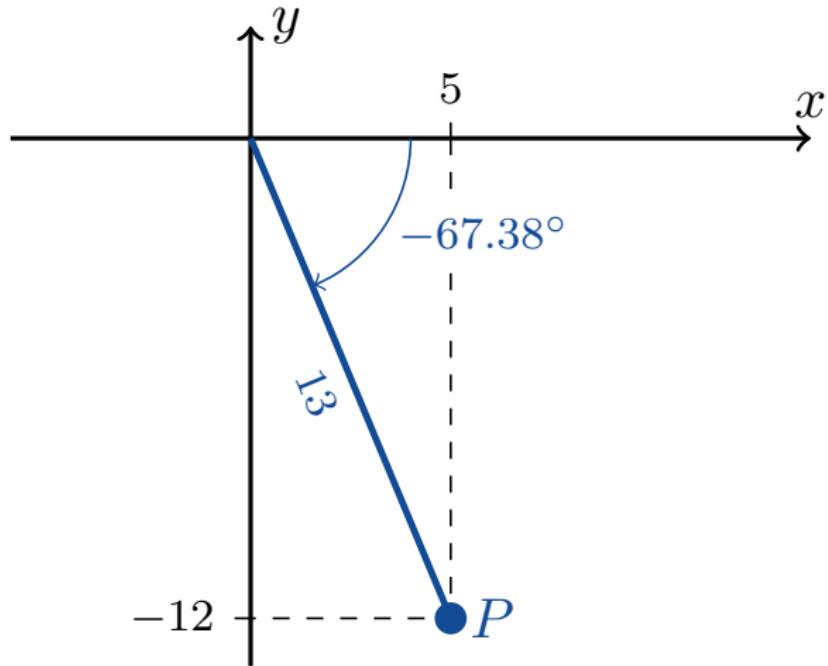
$$\theta = \sin^{-1} \frac{y}{r} = \sin^{-1} \frac{-12}{13} \approx -67.38^\circ.$$

Therefore

$$(r, \theta) = (13, -67.38^\circ).$$

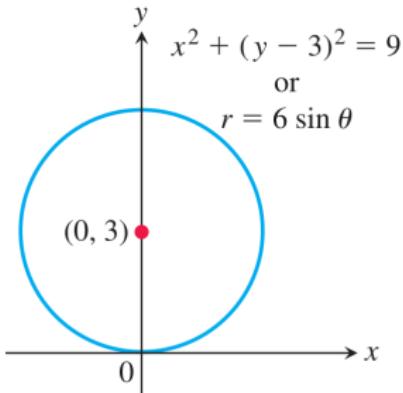
10.3

$$x = r \cos \theta \quad y = r \sin \theta \quad x^2 + y^2 = r^2 \quad \tan \theta = \frac{y}{x}$$



10.3

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EXAMPLE 5 Find a polar equation for the circle $x^2 + (y - 3)^2 = 9$

Solution We apply the equations relating polar and Cartesian coordinates:

$$x^2 + (y - 3)^2 = 9$$

$$x^2 + y^2 - 6y + 9 = 9 \qquad \text{Expand } (y - 3)^2.$$

$$x^2 + y^2 - 6y = 0 \qquad \text{Cancelation}$$

$$r^2 - 6r \sin \theta = 0 \qquad x^2 + y^2 = r^2, y = r \sin \theta$$

$$r = 0 \quad \text{or} \quad r - 6 \sin \theta = 0$$

$$r = 6 \sin \theta \qquad \text{Includes both possibilities}$$





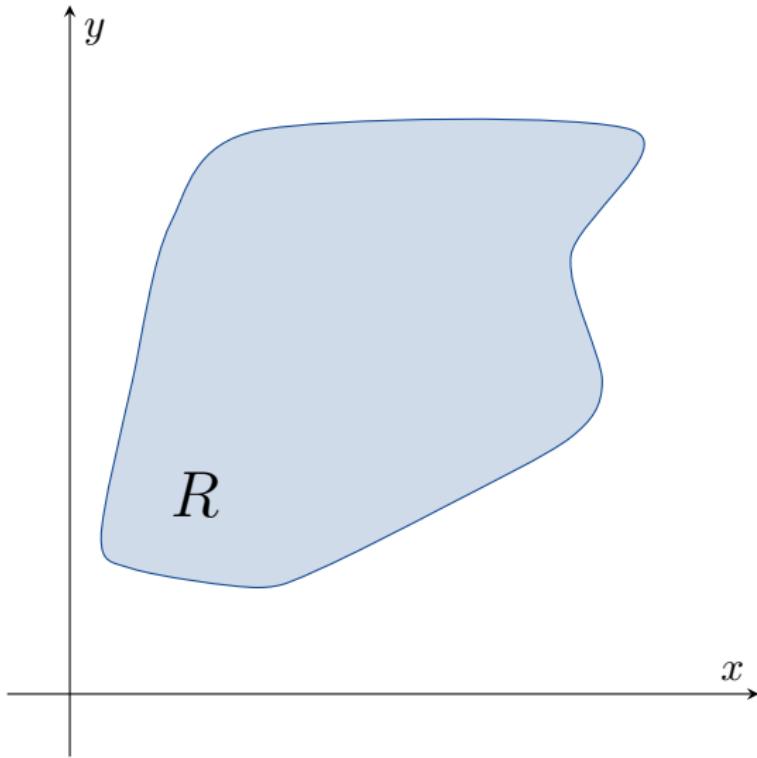
Break

We will continue at 2pm

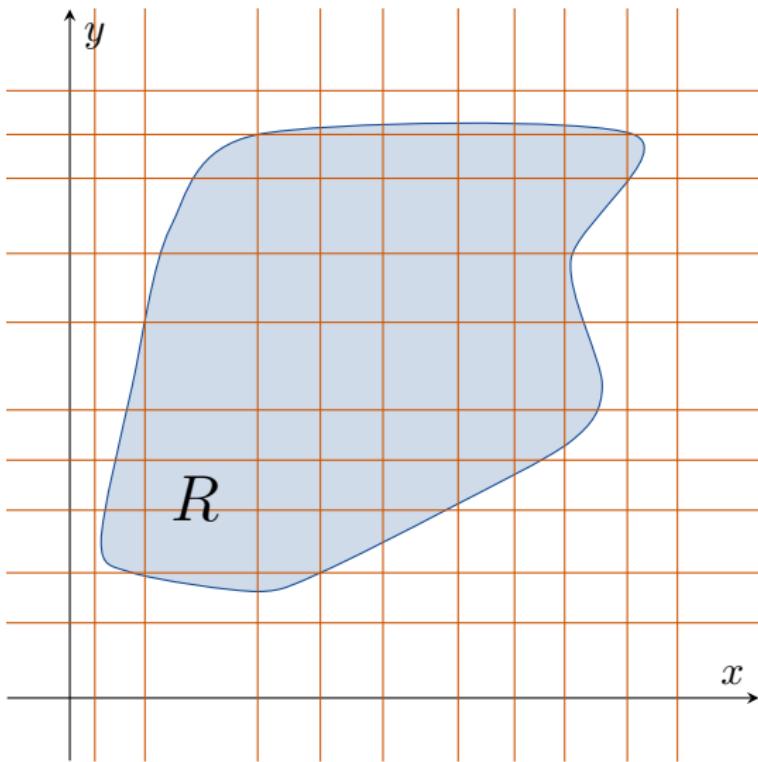


11 Double Integrals in Polar Form

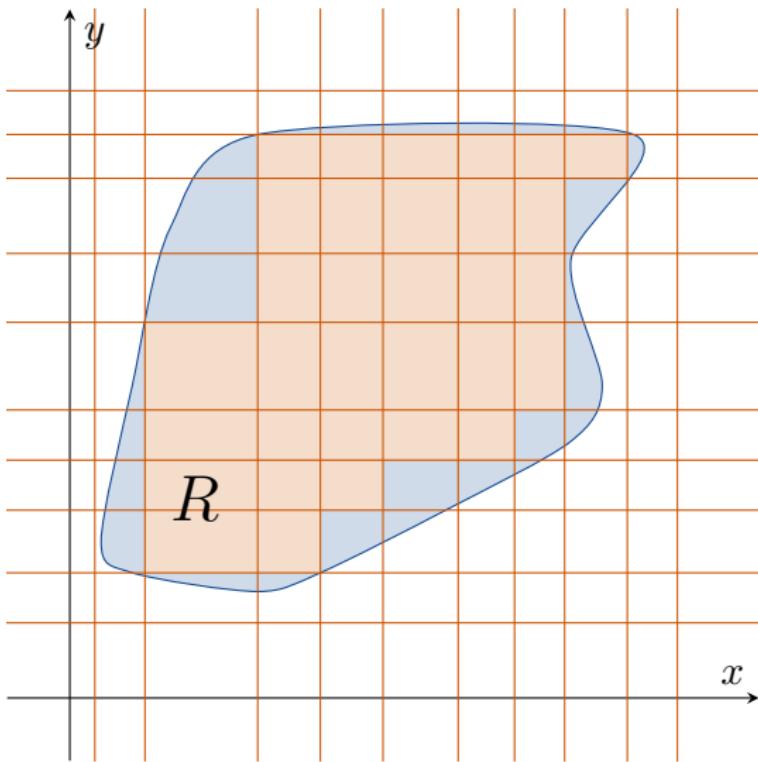
14.4 Double Integrals in Polar Form



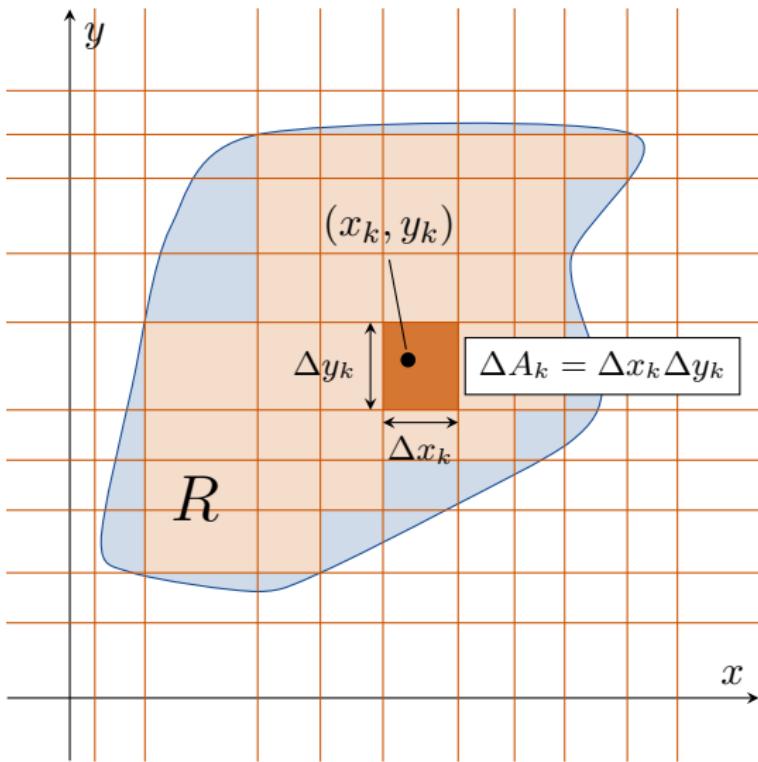
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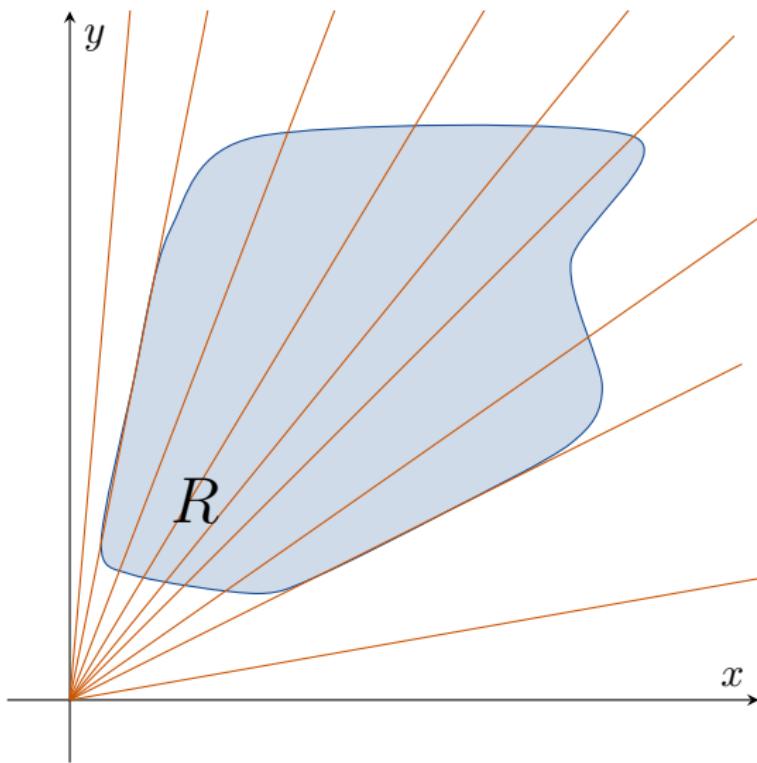
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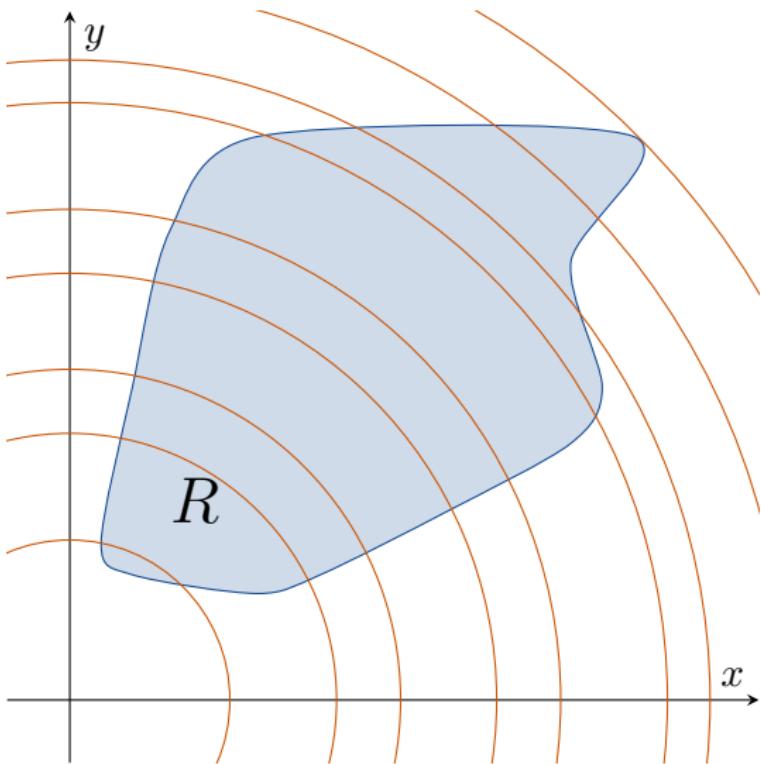
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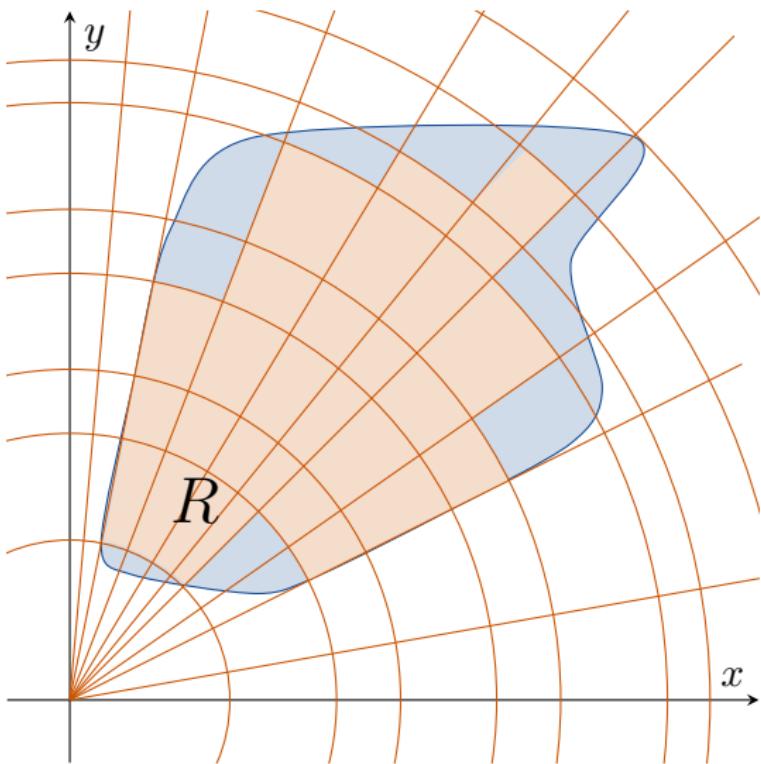
14.4 Double Integrals in Polar Form



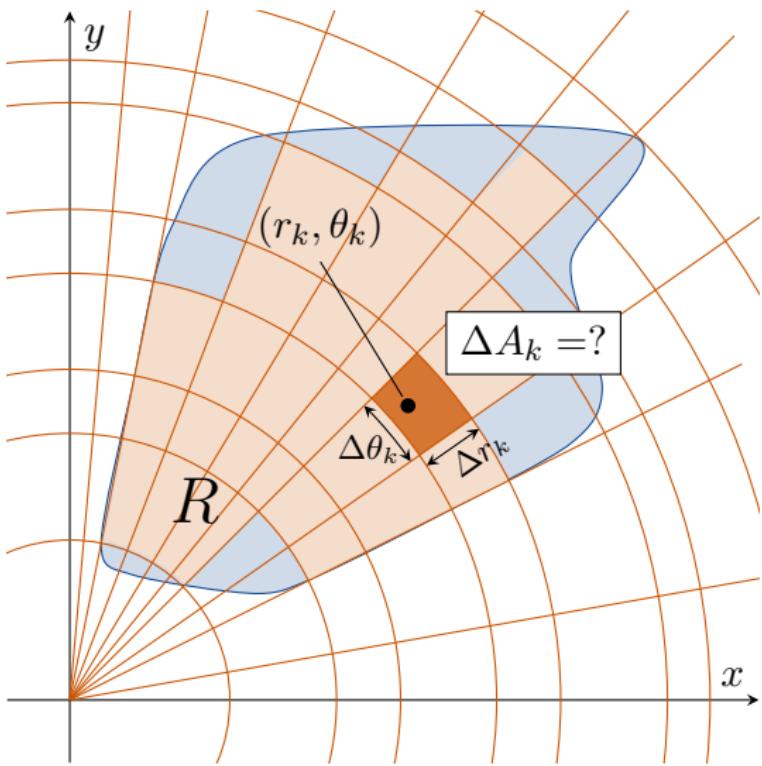
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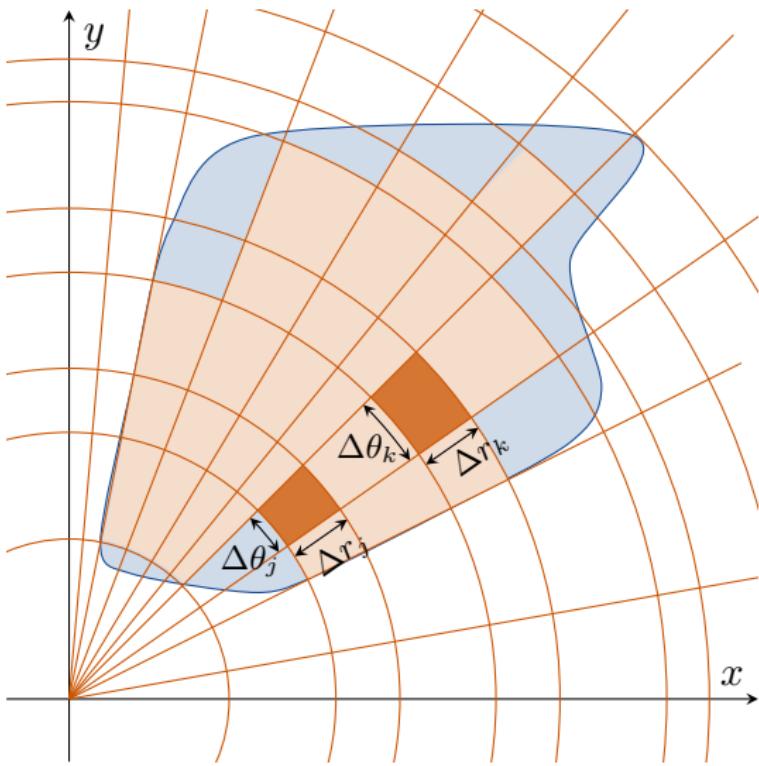
14.4 Double Integrals in Polar Form



$$\iint_R f(r, \theta) dA = \lim_{\|P\| \rightarrow 0} \sum_k f(r_k, \theta_k) \Delta A_k$$

But what is ΔA_k ?

14.4 Double Integrals in Polar Form



Note that

$$\Delta A_k = \Delta x_k \Delta y_k$$

but

$$\Delta A_k \neq \Delta r_k \Delta \theta_k.$$

14.4 Double Integrals in Polar Form

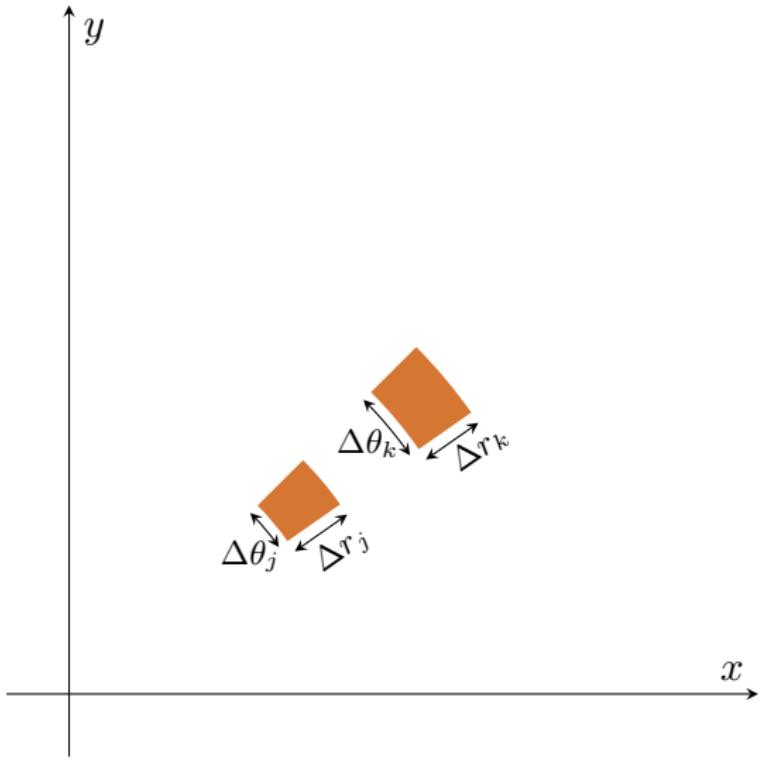


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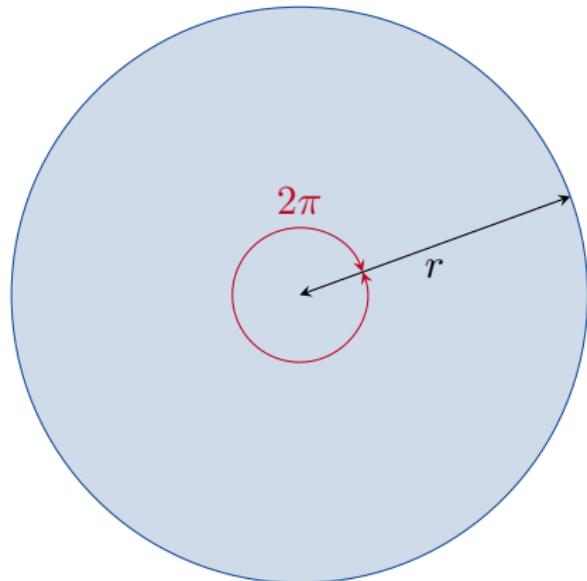
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but

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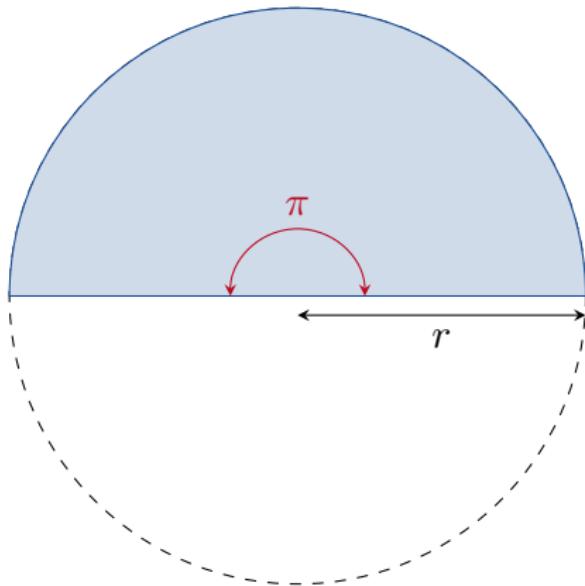


14.4 Double Integrals in Polar Form



$$\text{area of a circle} = \pi r^2 = \frac{1}{2}(2\pi)r^2$$

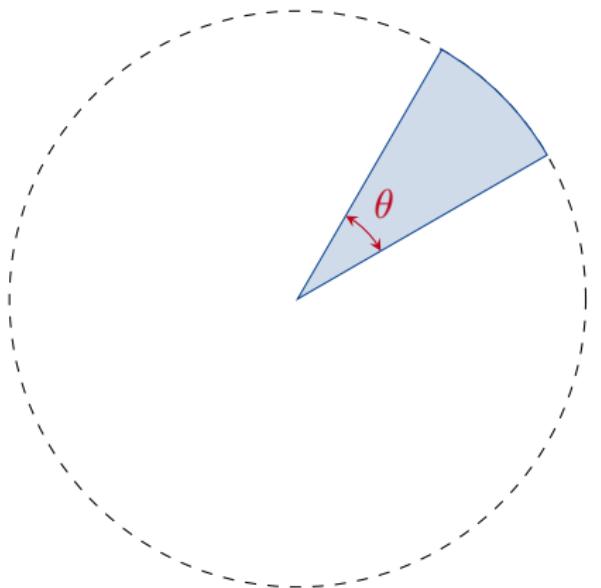
14.4 Double Integrals in Polar Form



$$\text{area of a circle} = \pi r^2 = \frac{1}{2}(2\pi)r^2$$

$$\text{area of a semicircle} = \frac{1}{2}\pi r^2$$

14.4 Double Integrals in Polar Form

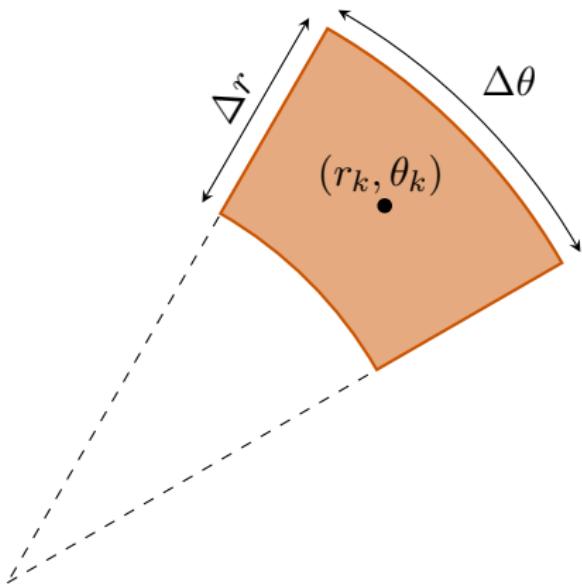


$$\begin{array}{l} \text{area of a} \\ \text{circle} \end{array} = \pi r^2 = \frac{1}{2}(2\pi)r^2$$

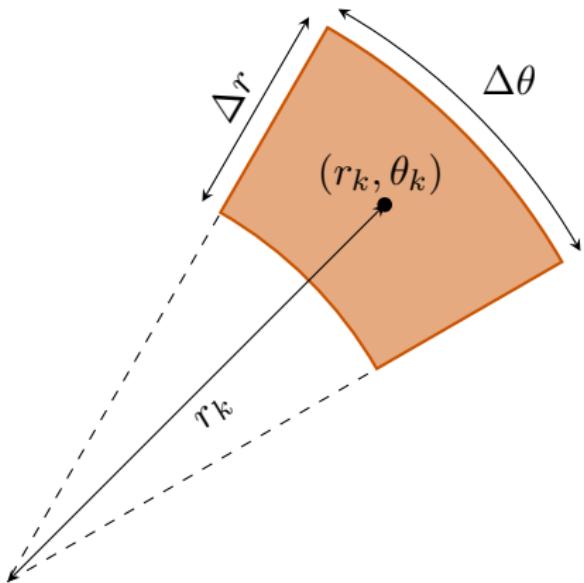
$$\begin{array}{l} \text{area of a} \\ \text{semicircle} \end{array} = \frac{1}{2}\pi r^2$$

$$\begin{array}{l} \text{area of a} \\ \text{sector} \end{array} = \frac{1}{2}\theta r^2$$

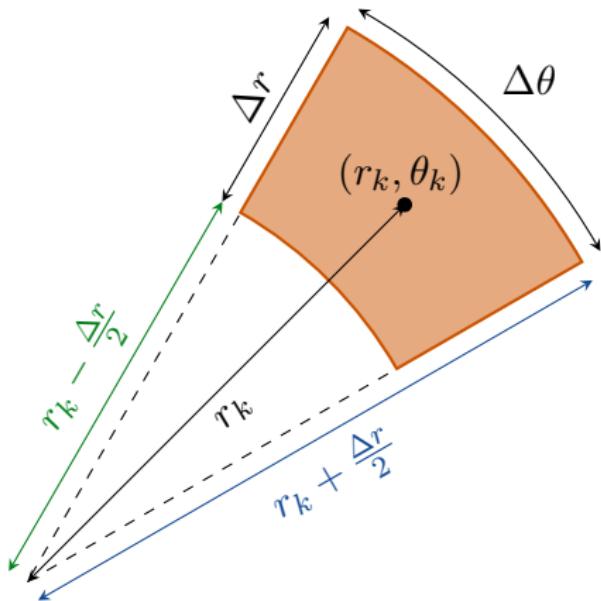
14.4 Double Integrals in Polar Form



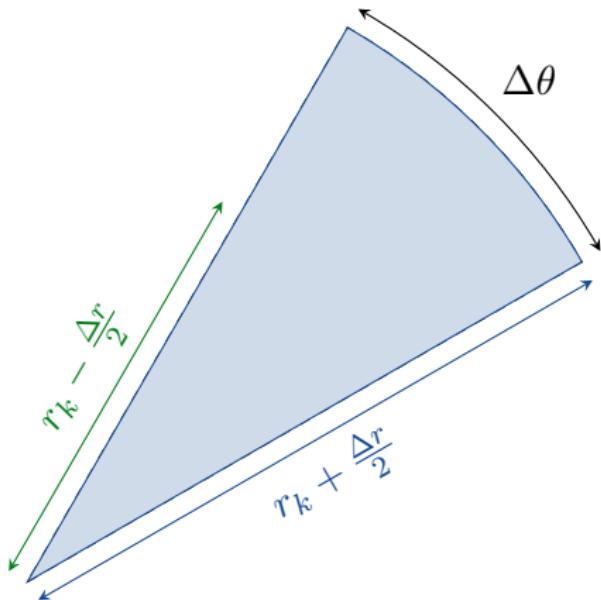
14.4 Double Integrals in Polar Form



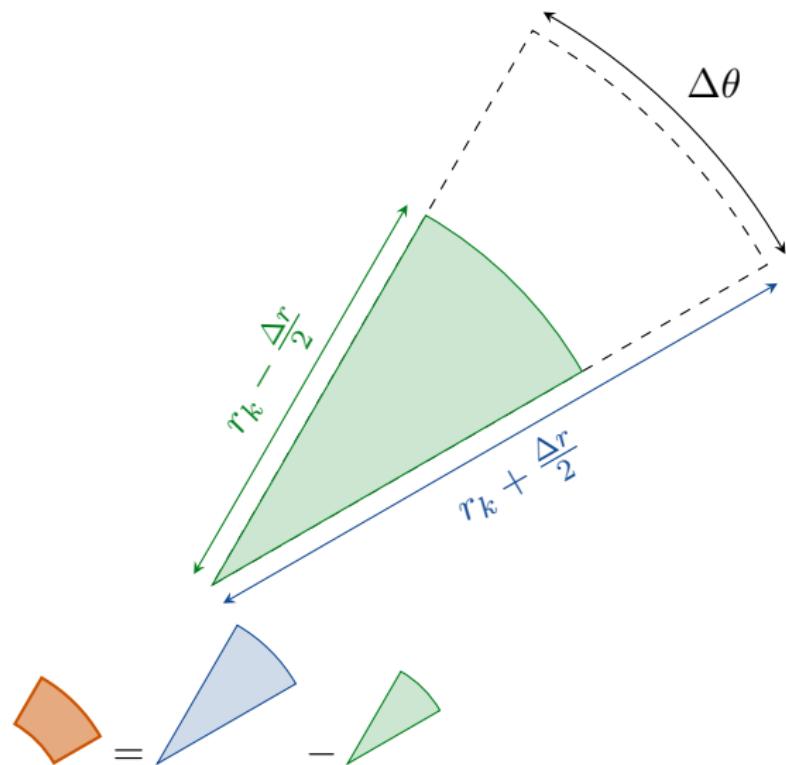
14.4 Double Integrals in Polar Form



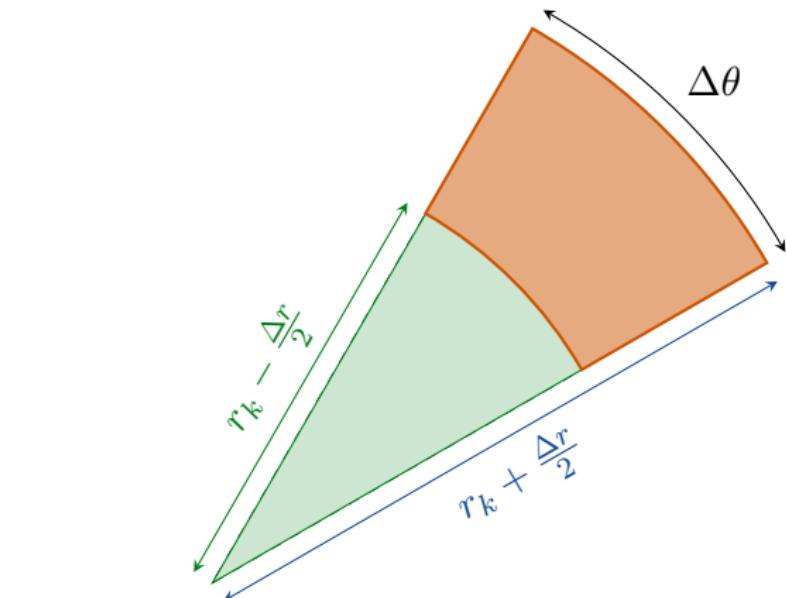
14.4 Double Integrals in Polar Form



14.4 Double Integrals in Polar Form



14.4 Double Integrals in Polar Form



$$\text{orange sector} = \text{blue sector} - \text{green triangle} = \frac{1}{2}\Delta\theta \left(r_k + \frac{\Delta r}{2}\right)^2 - \frac{1}{2}\Delta\theta \left(r_k - \frac{\Delta r}{2}\right)^2$$

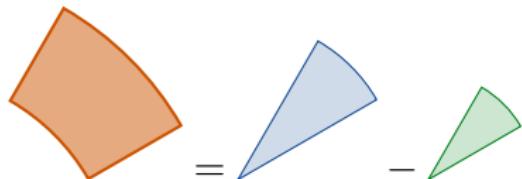
14.4 Double Integrals in Polar Form



A diagram illustrating the subtraction of two sectors from a larger orange sector. The orange sector is divided into two smaller sectors by a radius line. A blue sector is subtracted from the right side, and a green sector is subtracted from the left side. This visualizes the formula for the area of a sector in polar coordinates.

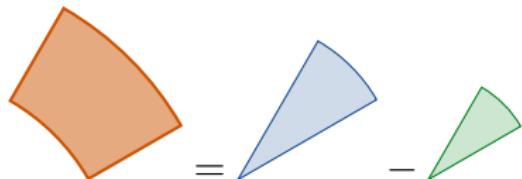
$$= \frac{1}{2} \Delta\theta \left(r_k + \frac{\Delta r}{2} \right)^2 - \frac{1}{2} \Delta\theta \left(r_k - \frac{\Delta r}{2} \right)^2$$

14.4 Double Integrals in Polar Form



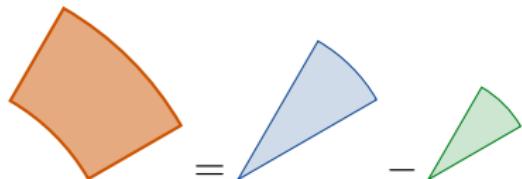
$$\begin{aligned}&= \frac{1}{2} \Delta\theta \left(r_k + \frac{\Delta r}{2} \right)^2 - \frac{1}{2} \Delta\theta \left(r_k - \frac{\Delta r}{2} \right)^2 \\&= \frac{1}{2} \Delta\theta \left(r_k^2 + 2r_k \frac{\Delta r}{2} + \frac{(\Delta r)^2}{4} - r_k^2 + 2r_k \frac{\Delta r}{2} - \frac{(\Delta r)^2}{4} \right)\end{aligned}$$

14.4 Double Integrals in Polar Form



$$\begin{aligned}&= \frac{1}{2} \Delta\theta \left(r_k + \frac{\Delta r}{2} \right)^2 - \frac{1}{2} \Delta\theta \left(r_k - \frac{\Delta r}{2} \right)^2 \\&= \frac{1}{2} \Delta\theta \left(r_k^2 + 2r_k \frac{\Delta r}{2} + \frac{(\Delta r)^2}{4} - r_k^2 + 2r_k \frac{\Delta r}{2} - \frac{(\Delta r)^2}{4} \right) \\&= \frac{1}{2} \Delta\theta (2r_k \Delta r)\end{aligned}$$

14.4 Double Integrals in Polar Form



$$\begin{aligned}&= \frac{1}{2} \Delta\theta \left(r_k + \frac{\Delta r}{2} \right)^2 - \frac{1}{2} \Delta\theta \left(r_k - \frac{\Delta r}{2} \right)^2 \\&= \frac{1}{2} \Delta\theta \left(r_k^2 + 2r_k \frac{\Delta r}{2} + \frac{(\Delta r)^2}{4} - r_k^2 + 2r_k \frac{\Delta r}{2} - \frac{(\Delta r)^2}{4} \right) \\&= \frac{1}{2} \Delta\theta (2r_k \Delta r) \\&= r_k \Delta r \Delta\theta.\end{aligned}$$

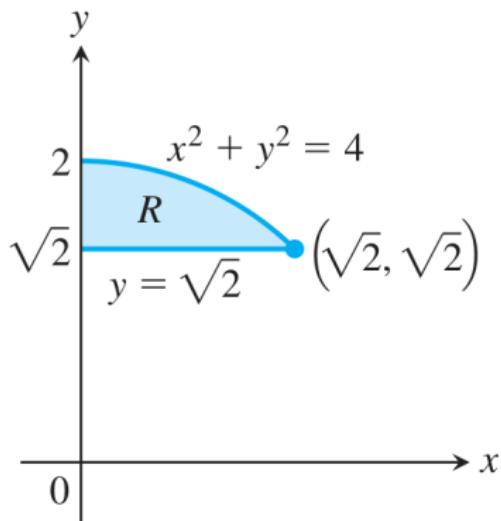
14.4 Double Integrals in Polar Form



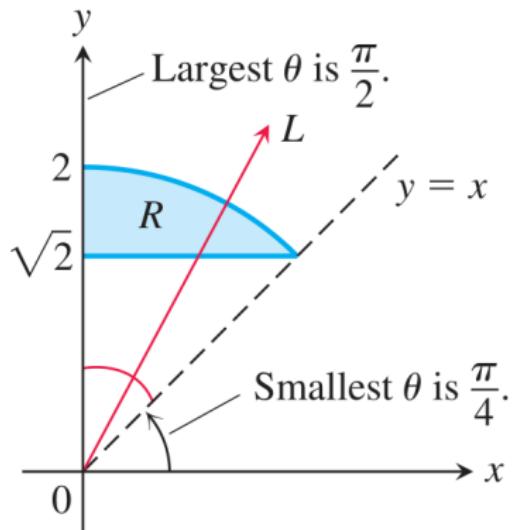
Theorem

$$dA = dx dy = r dr d\theta.$$

14.4 Double Integrals in Polar Form



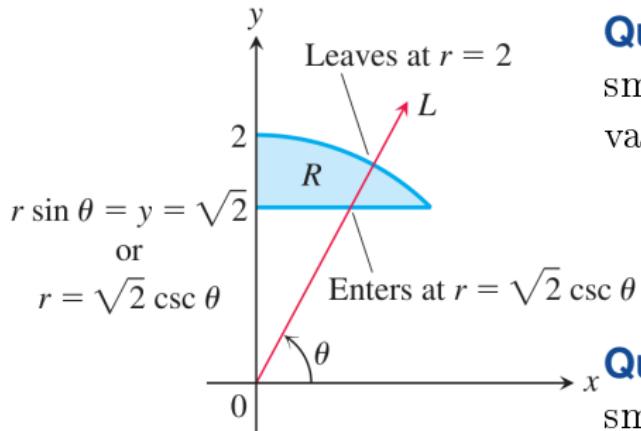
14.4 Double Integrals in Polar Form



Question: What are the smallest and biggest possible values of θ in R ?

$$\frac{\pi}{4} \leq \theta \leq \frac{\pi}{2}$$

14.4 Double Integrals in Polar Form



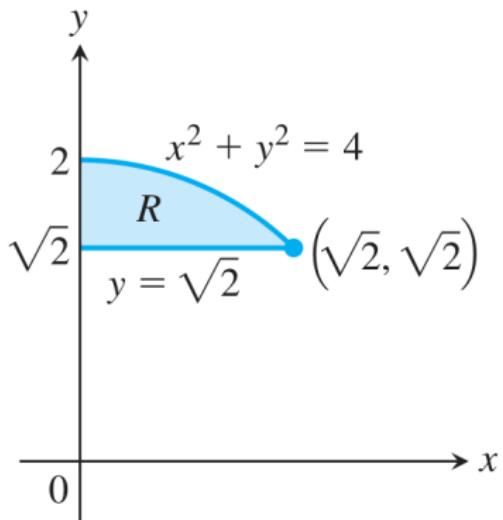
Question: What are the smallest and biggest possible values of θ in R ?

$$\frac{\pi}{4} \leq \theta \leq \frac{\pi}{2}$$

Question: What are the smallest and biggest possible values of r in R ?

$$\sqrt{2} \operatorname{cosec} \theta \leq r \leq 2$$

14.4 Double Integrals in Polar Form



Question: What are the smallest and biggest possible values of θ in R ?

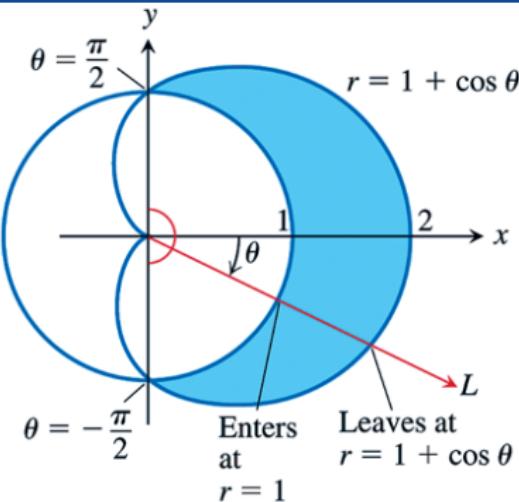
$$\frac{\pi}{4} \leq \theta \leq \frac{\pi}{2}$$

Question: What are the smallest and biggest possible values of r in R ?

$$\sqrt{2} \operatorname{cosec} \theta \leq r \leq 2$$

$$\iint_R f \, dA = \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \int_{\sqrt{2} \operatorname{cosec} \theta}^2 f(r, \theta) \, r \, dr \, d\theta.$$

14.4 Double Integrals



EXAMPLE 1 Find the limits of integration for integrating $f(r, \theta)$ over the region R that lies inside the cardioid $r = 1 + \cos \theta$ and outside the circle $r = 1$.

Solution

1. We first sketch the region and label the bounding curves (Figure 15.25).
2. Next we find the *r-limits of integration*. A typical ray from the origin enters R where $r = 1$ and leaves where $r = 1 + \cos \theta$.
3. Finally we find the *θ -limits of integration*. The rays from the origin that intersect R run from $\theta = -\pi/2$ to $\theta = \pi/2$. The integral is

$$\int_{-\pi/2}^{\pi/2} \int_1^{1+\cos \theta} f(r, \theta) r dr d\theta.$$

■

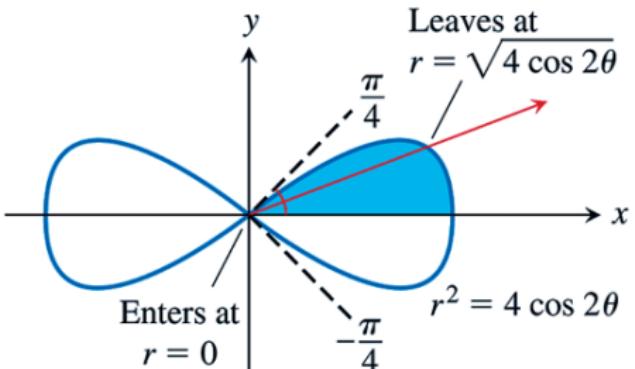
14.4 Double Integrals in Polar Form



The area of a closed, bounded region R is

$$A = \iint_R dA = \iint_R r dr d\theta.$$

14.4 Double Integrals in Polar Form



EXAMPLE 2 Find the area enclosed by the lemniscate $r^2 = 4 \cos 2\theta$.

Solution We graph the lemniscate to determine the limits of integration (Figure 15.26) and see from the symmetry of the region that the total area is 4 times the first-quadrant portion.

$$\begin{aligned} A &= 4 \int_0^{\pi/4} \int_0^{\sqrt{4 \cos 2\theta}} r dr d\theta = 4 \int_0^{\pi/4} \left[\frac{r^2}{2} \right]_{r=0}^{r=\sqrt{4 \cos 2\theta}} d\theta \\ &= 4 \int_0^{\pi/4} 2 \cos 2\theta d\theta = 4 \sin 2\theta \Big|_0^{\pi/4} = 4. \end{aligned}$$

■

14.4 Double Integrals in Polar Form



Cartesian Integral \longrightarrow Polar Integral

$$x = r \cos \theta \quad x^2 + y^2 = r^2$$

$$y = r \sin \theta \quad \tan \theta = \frac{y}{x}$$

14.4 Double Integrals in Polar Form



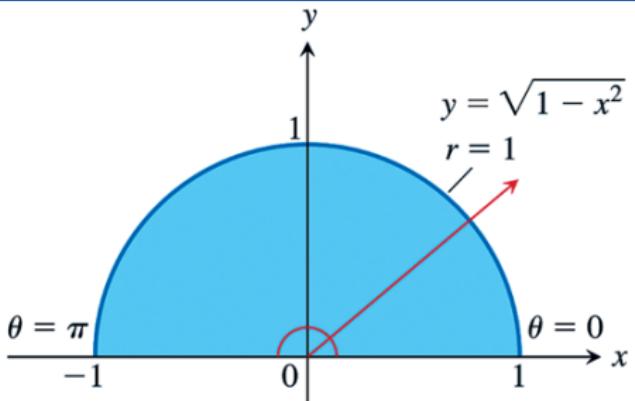
Cartesian Integral \longrightarrow Polar Integral

$$x = r \cos \theta \quad x^2 + y^2 = r^2$$

$$dxdy = r dr d\theta$$

$$y = r \sin \theta \quad \tan \theta = \frac{y}{x}$$

14.4 Double Integrals



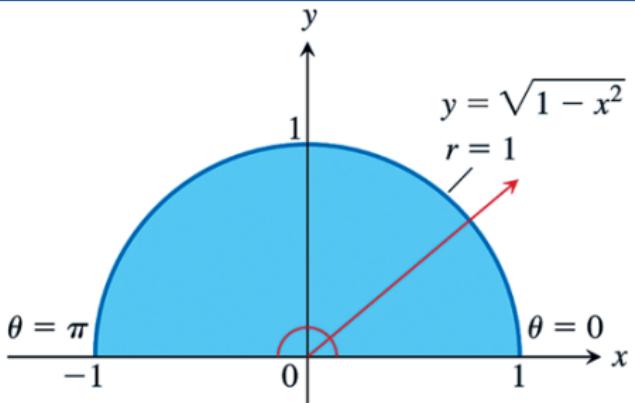
Example

Calculate

$$\iint_R e^{x^2+y^2} dy dx$$

where R is the region under $y = \sqrt{1 - x^2}$.

14.4 Double Integrals



Example

Calculate

$$\iint_R e^{x^2+y^2} dy dx$$

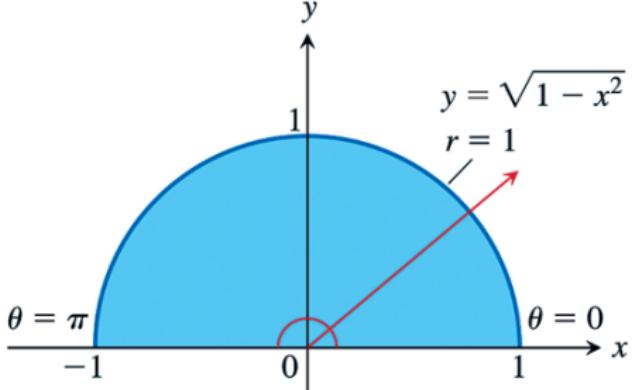
where R is the region under $y = \sqrt{1 - x^2}$.

difficult
Cartesian
integral



easy polar
integral

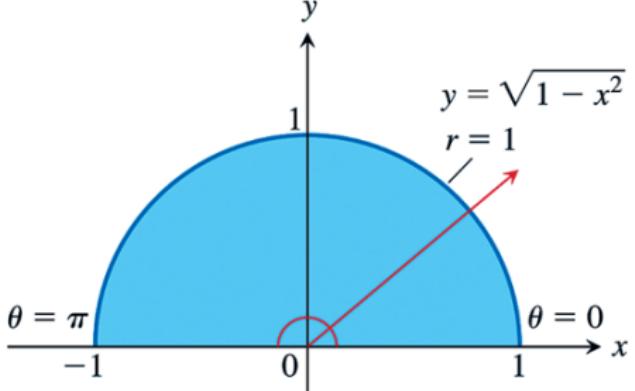
14.4 Double Integrals in Polar Form



$$\iint_R e^{x^2+y^2} dy dx = \int \int r dr d\theta$$

=

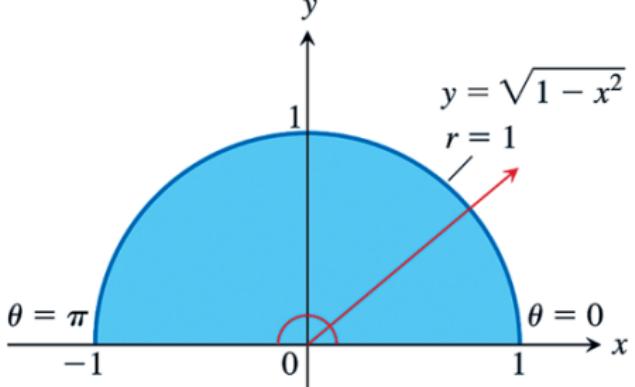
14.4 Double Integrals in Polar Form



$$\iint_R e^{x^2+y^2} dy dx = \int \int e^{r^2} r dr d\theta$$

=

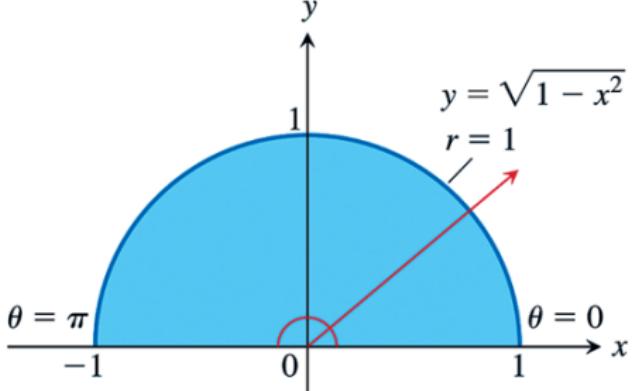
14.4 Double Integrals in Polar Form



$$\iint_R e^{x^2+y^2} dy dx = \int_0^\pi \int e^{r^2} r dr d\theta$$

=

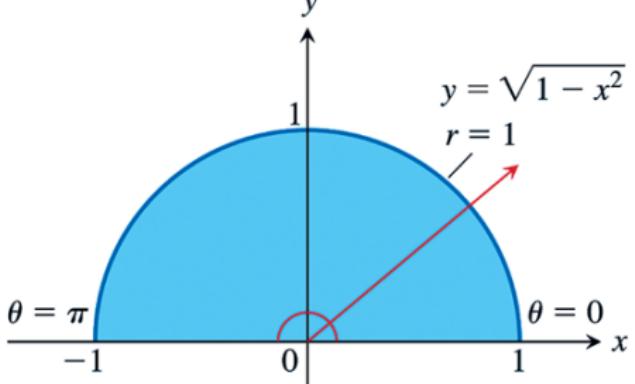
14.4 Double Integrals in Polar Form



$$\iint_R e^{x^2+y^2} dy dx = \int_0^\pi \int_0^1 e^{r^2} r dr d\theta$$

=

14.4 Double Integrals in Polar Form



$$\begin{aligned}\iint_R e^{x^2+y^2} dy dx &= \int_0^\pi \int_0^1 e^{r^2} r dr d\theta \\ &= \int_0^\pi \left[\frac{1}{2} e^{r^2} \right]_0^1 d\theta = \int_0^\pi \frac{1}{2} (e - 1) d\theta = \frac{\pi}{2} (e - 1).\end{aligned}$$

EXAMPLE 4

Evaluate the integral

$$\int_0^1 \int_0^{\sqrt{1-x^2}} (x^2 + y^2) dy dx.$$

Solution Integration with respect to y gives

$$\int_0^1 \left(x^2 \sqrt{1 - x^2} + \frac{(1 - x^2)^{3/2}}{3} \right) dx,$$

which is difficult to evaluate without tables. Things go better if we change the original integral to polar coordinates. The region of integration in Cartesian coordinates is given by the inequalities $0 \leq y \leq \sqrt{1 - x^2}$ and $0 \leq x \leq 1$, which correspond to the interior of the unit quarter circle $x^2 + y^2 = 1$ in the first quadrant. (See Figure 15.27, first quadrant.) Substituting the polar coordinates $x = r \cos \theta$, $y = r \sin \theta$, $0 \leq \theta \leq \pi/2$, and $0 \leq r \leq 1$, and replacing $dy dx$ by $r dr d\theta$ in the double integral, we get

$$\begin{aligned} \int_0^1 \int_0^{\sqrt{1-x^2}} (x^2 + y^2) dy dx &= \int_0^{\pi/2} \int_0^1 (r^2) r dr d\theta \\ &= \int_0^{\pi/2} \left[\frac{r^4}{4} \right]_{r=0}^{r=1} d\theta = \int_0^{\pi/2} \frac{1}{4} d\theta = \frac{\pi}{8}. \end{aligned}$$

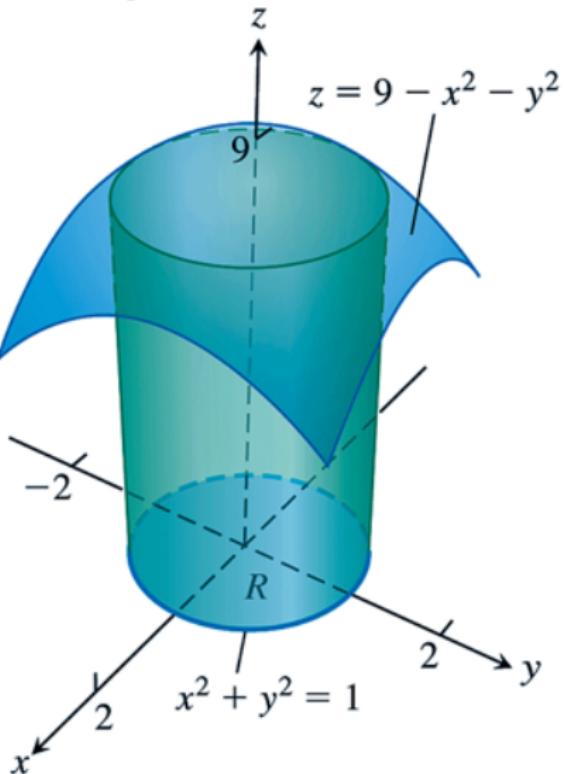
The polar coordinate transformation is effective here because $x^2 + y^2$ simplifies to r^2 and the limits of integration become constants. ■

14.4 Double Integrals in Polar Form

EXAMPLE 5 Find the volume of the solid region bounded above by the paraboloid $z = 9 - x^2 - y^2$ and below by the unit circle in the xy -plane.

Solution The region of integration R is bounded by the unit circle $x^2 + y^2 \leq 1$. It is described in polar coordinates by $r = 1$, $0 \leq \theta \leq 2\pi$. Figure 15.28. The volume is given by the double integral

$$\begin{aligned} \iint_R (9 - x^2 - y^2) dA &= \int_0^{2\pi} \int_0^1 (9 - r^2) r dr d\theta \\ &= \int_0^{2\pi} \int_0^1 (9r - r^3) dr d\theta \\ &= \int_0^{2\pi} \left[\frac{9}{2}r^2 - \frac{1}{4}r^4 \right]_0^1 d\theta \\ &= \frac{17}{4} \int_0^{2\pi} d\theta = \frac{17\pi}{2} \end{aligned}$$



14.4 Double Integrals in Polar Form

EXAMPLE 5 Find the volume of the solid region bounded above by the paraboloid $z = 9 - x^2 - y^2$ and below by the unit circle in the xy -plane.

Solution The region of integration R is bounded by the unit circle $x^2 + y^2 = 1$, which is described in polar coordinates by $r = 1, 0 \leq \theta \leq 2\pi$. The solid region is shown in Figure 15.28. The volume is given by the double integral

$$\begin{aligned}
 \iint_R (9 - x^2 - y^2) dA &= \int_0^{2\pi} \int_0^1 (9 - r^2) r dr d\theta && r^2 = x^2 + y^2, \quad dA = r dr d\theta. \\
 &= \int_0^{2\pi} \int_0^1 (9r - r^3) dr d\theta \\
 &= \int_0^{2\pi} \left[\frac{9}{2}r^2 - \frac{1}{4}r^4 \right]_{r=0}^{r=1} d\theta \\
 &= \frac{17}{4} \int_0^{2\pi} d\theta = \frac{17\pi}{2}.
 \end{aligned}$$



14.4 Double Integrals in Polar Form



Please read Example 6 in the textbook.



Next Time

- 14.5 Triple Integrals in Rectangular Coordinates
- 14.7 Triple Integrals in Cylindrical and Spherical Coordinates
- 14.8 Substitutions in Multiple Integrals