

Lecture 7

- 4.1 Definition of the Laplace Transform
- 4.2 Solving Initial Value Problems

Recall that $\int_a^\infty f(t) dt$ means $\lim_{R \rightarrow \infty} \int_a^R f(t) dt$.

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Example

Let $c \neq 0$. Then

$$\begin{aligned} \int_0^\infty e^{ct} dt &= \lim_{R \rightarrow \infty} \int_0^R e^{ct} dt = \\ &= \end{aligned}$$

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$$\begin{aligned} \int_0^\infty e^{ct} dt &= \lim_{R \rightarrow \infty} \int_0^R e^{ct} dt = \lim_{R \rightarrow \infty} \left[\frac{1}{c} e^{ct} \right]_0^R \\ &= \end{aligned}$$

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Example

$$\int_1^{\infty} \frac{1}{t} dt =$$

Example

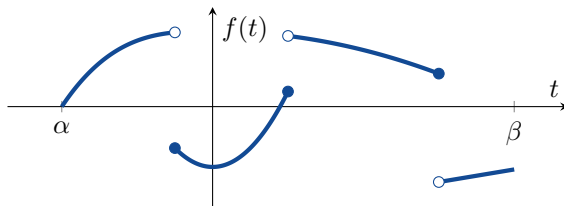
$$\int_1^{\infty} \frac{1}{t} dt = \lim_{R \rightarrow \infty} \int_1^R \frac{1}{t} dt$$

Example

$$\int_1^{\infty} \frac{1}{t} dt = \lim_{R \rightarrow \infty} \int_1^R \frac{1}{t} dt = \lim_{R \rightarrow \infty} [\ln t]_1^R$$

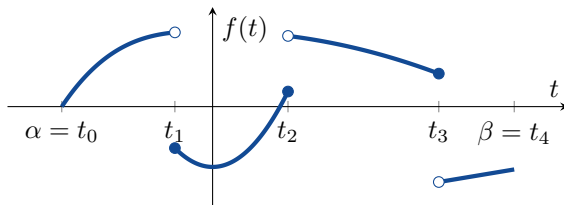
Example

$$\int_1^{\infty} \frac{1}{t} dt = \lim_{R \rightarrow \infty} \int_1^R \frac{1}{t} dt = \lim_{R \rightarrow \infty} [\ln t]_1^R = \lim_{R \rightarrow \infty} (\ln R - 0) = \infty$$



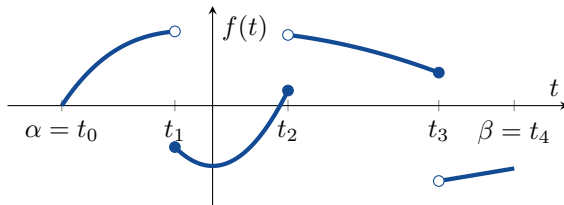
Definition

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A function $f : [\alpha, \beta] \rightarrow \mathbb{R}$ is *piecewise continuous* on $[\alpha, \beta]$ iff there exists a finite partition $\alpha = t_0 < t_1 < t_2 < \dots < t_n = \beta$ such that

- f is continuous on each subinterval (t_{j-1}, t_j) ; and
- every one-sided limit $\lim_{t \searrow t_j} f(t)$ and $\lim_{t \nearrow t_j} f(t)$ is finite.

Definition of the Laplace Transform

4.1 Definition of the Laplace Transform



Pierre-Simon Laplace
FRA, 1749-1827

4.1 Definition of the Laplace Transform



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Definition

Suppose that

- 1 $K > 0, M > 0, a \in \mathbb{R}$;
- 2 f is piecewise continuous on $[0, A]$ for any $A > 0$; and
- 3 $|f(t)| \leq Ke^{at}$ for all $t \geq M$.

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The **Laplace Transform** of $f : [0, \infty) \rightarrow \mathbb{R}$ is a new function defined by

$$F(s) = \mathcal{L}[f](s) = \int_0^{\infty} e^{-st} f(t) dt.$$

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$F(s)$ exists for $s > a$.

4.1 Definition of the Laplace Transform



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Example

$$\mathcal{L}[1](s) =$$

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$$\mathcal{L}[1](s) = \int_0^{\infty} e^{-st} dt = \lim_{R \rightarrow \infty} \left[-\frac{e^{-st}}{s} \right]_0^R$$

4.1 Definition of the Laplace Transform



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4.1 Definition of the Laplace Transform



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4.1 Definition of the Laplace Transform



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$$\mathcal{L}[e^{at}](s) = \int_0^{\infty} e^{-st} e^{at} dt = \int_0^{\infty} e^{-(s-a)t} dt = \frac{1}{s-a} \quad \text{if } s > a.$$

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The Laplace Transform of $e^{at} : [0, \infty) \rightarrow \mathbb{R}$ is $\frac{1}{s-a} : (a, \infty) \rightarrow \mathbb{R}$.

4.1 Definition of the Laplace Transform

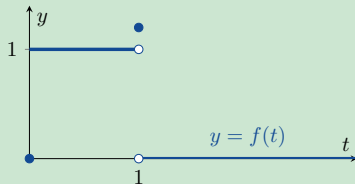


$$F(s) = \mathcal{L}[f](s) = \int_0^{\infty} e^{-st} f(t) dt$$

Example

Let

$$f(t) = \begin{cases} 1 & 0 \leq t < 1 \\ k & t = 1 \\ 0 & t > 1. \end{cases}$$



Then $F(s) = \mathcal{L}[f](s) =$

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4.1 Definition of the Laplace Transform

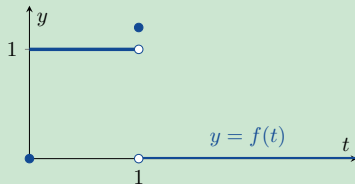


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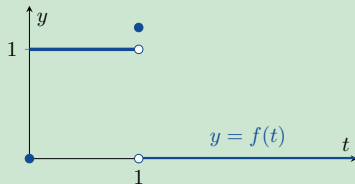


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4.1 Definition of the Laplace Transform

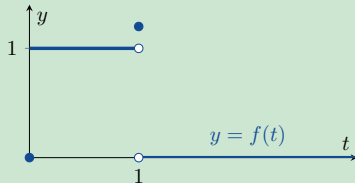


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$$\begin{aligned} \text{Then } F(s) &= \mathcal{L}[f](s) = \int_0^{\infty} e^{-st} f(t) dt = \int_0^1 e^{-st} dt \\ &= \left[-\frac{1}{s} e^{-st} \right]_0^1 = \frac{1 - e^{-s}}{s} \quad \text{if } s > 1. \end{aligned}$$

4.1 Definition of the Laplace Transform



$$F(s) = \mathcal{L}[f](s) = \int_0^{\infty} e^{-st} f(t) dt$$

Example

Find the Laplace Transform of $g(t) = \sin at$ ($t \geq 0$).

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Example

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Using integration by parts ($\int_a^b uv' = [uv]_a^b - \int_a^b u'v$), we have

$$\begin{aligned} G(s) &= \mathcal{L}[g](s) = \int_0^{\infty} e^{-st} \sin at dt = \lim_{R \rightarrow \infty} \int_0^R e^{-st} \sin at dt \\ &= \lim_{R \rightarrow \infty} \left(\right) \\ &= \end{aligned}$$

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4.1 Definition of the Laplace Transform



$$G(s) = \frac{1}{a} - \frac{s}{a} \int_0^{\infty} e^{-st} \cos at \, dt$$

Using integration by parts a second time, we have

$$G(s) = \frac{1}{a} - \frac{s^2}{a^2} \int_0^{\infty} e^{-st} \sin at \, dt = \frac{1}{a} - \frac{s^2}{a^2} G(s).$$

4.1 Definition of the Laplace Transform



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Therefore

$$\mathcal{L}[\sin at](s) = G(s) = \frac{a}{s^2 + a^2} \quad \text{if } s > 0.$$

4.1 Definition of the Laplace Transform



Example

$$\mathcal{L} [\cos at] = \frac{s}{s^2 + a^2} \quad \text{if } s > 0.$$

You prove.

4.1 Definition of the Laplace Transform



Example

$$\mathcal{L} [\sinh at] = \frac{a}{s^2 - a^2} \quad \text{if } s > |a|.$$

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4.1 Definition of the Laplace Transform



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4.1 Definition of the Laplace Transform



Theorem

$$\mathcal{L}[c_1 f_1 + c_2 f_2] = c_1 \mathcal{L}[f_1] + c_2 \mathcal{L}[f_2].$$

You prove.

4.1 Definition of the Laplace Transform



Example

If $h(t) = 5e^{-2t} - 3 \sin 4t$ ($t \geq 0$), then

$$\begin{aligned} H(s) &= \mathcal{L}[h](s) \\ &= \mathcal{L}[5e^{-2t} - 3 \sin 4t](s) \\ &= \\ &= \\ &= \end{aligned}$$

4.1 Definition of the Laplace Transform



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4.1 Definition of the Laplace Transform



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The Inverse Laplace Transform

We also have an *inverse Laplace Transform*:

$$F(s) = \mathcal{L} [f(t)] (s) \quad \Longleftrightarrow \quad f(t) = \mathcal{L}^{-1} [F(s)] (t).$$



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Example

$$\mathcal{L} [1] = \frac{1}{s} \text{ and } \mathcal{L}^{-1} \left[\frac{1}{s} \right] = 1.$$

4.1 Definition of the Laplace Transform



Theorem

$$\mathcal{L}^{-1} [c_1 f_1 + c_2 f_2] = c_1 \mathcal{L}^{-1} [f_1] + c_2 \mathcal{L}^{-1} [f_2] .$$

You prove.

4.1 Definition of the Laplace Transform



Example

Find the inverse Laplace Transform of $\frac{10}{s^2 - 25}$.

4.1 Definition of the Laplace Transform



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4.1 Definition of the Laplace Transform



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We know that $\mathcal{L}[\sinh at] = \frac{a}{s^2 - a^2}$. Therefore

$$\mathcal{L}^{-1}\left[\frac{10}{s^2 - 25}\right] = 2\mathcal{L}^{-1}\left[\frac{5}{s^2 - 25}\right]$$

4.1 Definition of the Laplace Transform



Example

Find the inverse Laplace Transform of $\frac{10}{s^2 - 25}$.

We know that $\mathcal{L}[\sinh at] = \frac{a}{s^2 - a^2}$. Therefore

$$\mathcal{L}^{-1}\left[\frac{10}{s^2 - 25}\right] = 2\mathcal{L}^{-1}\left[\frac{5}{s^2 - 25}\right] = 2 \sinh 5t.$$

4.1 Definition of the Laplace Transform



$$\mathcal{L}[1] = \frac{1}{s} \qquad \mathcal{L}[e^{at}] = \frac{1}{s-a}$$

Example

Find the inverse Laplace Transform of $\frac{1}{s} + \frac{1}{s-2}$.

4.1 Definition of the Laplace Transform



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$$\mathcal{L}^{-1}\left[\frac{1}{s} + \frac{1}{s-2}\right] = \mathcal{L}^{-1}\left[\frac{1}{s}\right] + \mathcal{L}^{-1}\left[\frac{1}{s-2}\right] = 1 + e^{2t}.$$

4.1 Definition of the Laplace Transform



Theorem

$$\mathcal{L} [t^n f(t)] = (-1)^n \frac{d^n F}{ds^n}$$

4.1 Definition of the Laplace Transform



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Proof: First we calculate that

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4.1 Definition of the Laplace Transform



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Therefore the formula holds for $n = 1$.

4.1 Definition of the Laplace Transform



By repeatedly using

$$-\frac{dF}{ds} = \mathcal{L} [tf(t)] ,$$

we can also show that

$$(-1)^2 \frac{d^2 F}{ds^2} = \mathcal{L} [t^2 f(t)]$$



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$$\vdots$$

$$(-1)^n \frac{d^n F}{ds^n} = \mathcal{L} [t^n f(t)] .$$



4.1 Definition of the Laplace Transform



$$\mathcal{L} [t^n f(t)] = (-1)^n \frac{d^n F}{ds^n}$$

$$\mathcal{L} [\cosh at] = \frac{s}{s^2 - a^2}$$

Example

$$\begin{aligned} \mathcal{L} [t^2 \cosh 2t] &= \\ &= \end{aligned}$$

4.1 Definition of the Laplace Transform



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4.1 Definition of the Laplace Transform



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$$\mathcal{L} [\cosh at] = \frac{s}{s^2 - a^2}$$

Example

$$\begin{aligned}\mathcal{L} [t^2 \cosh 2t] &= (-1)^2 \frac{d^2}{ds^2} \mathcal{L} [\cosh 2t] \\ &= \frac{d^2}{ds^2} \left(\frac{s}{s^2 - 2^2} \right) = \dots = \frac{2s(s^2 + 12)}{(s^2 - 4)^3}.\end{aligned}$$

4.1 Definition of the Laplace Transform



$$\mathcal{L}[t^n f(t)] = (-1)^n \frac{d^n F}{ds^n}$$

Example

Find the Laplace Transform of t^n for $n \in \mathbb{N}$.

4.1 Definition of the Laplace Transform



$$\mathcal{L}[t^n f(t)] = (-1)^n \frac{d^n F}{ds^n}$$

Example

Find the Laplace Transform of t^n for $n \in \mathbb{N}$.

$$\mathcal{L}[t^n] = \mathcal{L}[t^n \cdot 1]$$

4.1 Definition of the Laplace Transform



$$\mathcal{L}[t^n f(t)] = (-1)^n \frac{d^n F}{ds^n} \qquad \mathcal{L}[1] = \frac{1}{s}$$

Example

Find the Laplace Transform of t^n for $n \in \mathbb{N}$.

$$\mathcal{L}[t^n] = \mathcal{L}[t^n \cdot 1] = (-1)^n \frac{d^n}{ds^n} \mathcal{L}[1]$$

4.1 Definition of the Laplace Transform



$$\mathcal{L}[t^n f(t)] = (-1)^n \frac{d^n F}{ds^n} \qquad \mathcal{L}[1] = \frac{1}{s}$$

Example

Find the Laplace Transform of t^n for $n \in \mathbb{N}$.

$$\mathcal{L}[t^n] = \mathcal{L}[t^n \cdot 1] = (-1)^n \frac{d^n}{ds^n} \mathcal{L}[1] = (-1)^n \frac{d^n}{ds^n} \left(\frac{1}{s} \right)$$

4.1 Definition of the Laplace Transform



$$\mathcal{L}[t^n f(t)] = (-1)^n \frac{d^n F}{ds^n} \qquad \mathcal{L}[1] = \frac{1}{s}$$

Example

Find the Laplace Transform of t^n for $n \in \mathbb{N}$.

$$\begin{aligned}\mathcal{L}[t^n] &= \mathcal{L}[t^n \cdot 1] = (-1)^n \frac{d^n}{ds^n} \mathcal{L}[1] = (-1)^n \frac{d^n}{ds^n} \left(\frac{1}{s} \right) \\ &= (-1)^n \frac{(-1)^n n!}{s^{n+1}}\end{aligned}$$

4.1 Definition of the Laplace Transform



$$\mathcal{L}[t^n f(t)] = (-1)^n \frac{d^n F}{ds^n} \qquad \mathcal{L}[1] = \frac{1}{s}$$

Example

Find the Laplace Transform of t^n for $n \in \mathbb{N}$.

$$\begin{aligned}\mathcal{L}[t^n] &= \mathcal{L}[t^n \cdot 1] = (-1)^n \frac{d^n}{ds^n} \mathcal{L}[1] = (-1)^n \frac{d^n}{ds^n} \left(\frac{1}{s} \right) \\ &= (-1)^n \frac{(-1)^n n!}{s^{n+1}} = \frac{n!}{s^{n+1}}.\end{aligned}$$

4.1 Definition of the Laplace Transform



$f(t)$	$F(s) = \mathcal{L}[f](s)$	
1	$\frac{1}{s}$	$s > 0$
e^{at}	$\frac{1}{s-a}$	$s > a$
$t^n \quad (n \in \mathbb{N})$	$\frac{n!}{s^{n+1}}$	$s > 0$
$\sin at$	$\frac{a}{s^2+a^2}$	$s > 0$
$\cos at$	$\frac{s}{s^2+a^2}$	$s > 0$
$\sinh at$	$\frac{a}{s^2-a^2}$	$s > a $
$\cosh at$	$\frac{s}{s^2-a^2}$	$s > a $
	\vdots	

4.1 Definition of the Laplace Transform



$f(t)$	$F(s) = \mathcal{L}[f](s)$	
e^{at}	$\frac{1}{s-a}$	$s > a$
$t^n \quad (n \in \mathbb{N})$	$\frac{n!}{s^{n+1}}$	$s > 0$
$\sin at$	$\frac{a}{s^2+a^2}$	$s > 0$
$\cos at$	$\frac{s}{s^2+a^2}$	$s > 0$
$\sinh at$	$\frac{a}{s^2-a^2}$	$s > a $
$\cosh at$	$\frac{s}{s^2-a^2}$	$s > a $
$e^{at} \sin bt$	$\frac{b}{(s-a)^2+b^2}$	$s > a$
	\vdots	

4.1 Definition of the Laplace Transform



$f(t)$	$F(s) = \mathcal{L}[f](s)$	
$t^n \quad (n \in \mathbb{N})$	$\frac{n!}{s^{n+1}}$	$s > 0$
$\sin at$	$\frac{a}{s^2+a^2}$	$s > 0$
$\cos at$	$\frac{s}{s^2+a^2}$	$s > 0$
$\sinh at$	$\frac{a}{s^2-a^2}$	$s > a $
$\cosh at$	$\frac{s}{s^2-a^2}$	$s > a $
$e^{at} \sin bt$	$\frac{b}{(s-a)^2+b^2}$	$s > a$
$e^{at} \cos bt$	$\frac{s-a}{(s-a)^2+b^2}$	$s > a$
	\vdots	

4.1 Definition of the Laplace Transform



$f(t)$	$F(s) = \mathcal{L}[f](s)$	
$\sin at$	$\frac{a}{s^2+a^2}$	$s > 0$
$\cos at$	$\frac{s}{s^2+a^2}$	$s > 0$
$\sinh at$	$\frac{a}{s^2-a^2}$	$s > a $
$\cosh at$	$\frac{s}{s^2-a^2}$	$s > a $
$e^{at} \sin bt$	$\frac{b}{(s-a)^2+b^2}$	$s > a$
$e^{at} \cos bt$	$\frac{s-a}{(s-a)^2+b^2}$	$s > a$
$t^n e^{at} \quad (n \in \mathbb{N})$	$\frac{n!}{(s-a)^{n+1}}$	$s > a$
	\vdots	

4.1 Definition of the Laplace Transform



$f(t)$	$F(s) = \mathcal{L}[f](s)$	
$\cos at$	$\frac{s}{s^2+a^2}$	$s > 0$
$\sinh at$	$\frac{a}{s^2-a^2}$	$s > a $
$\cosh at$	$\frac{s}{s^2-a^2}$	$s > a $
$e^{at} \sin bt$	$\frac{b}{(s-a)^2+b^2}$	$s > a$
$e^{at} \cos bt$	$\frac{s-a}{(s-a)^2+b^2}$	$s > a$
$t^n e^{at} \quad (n \in \mathbb{N})$	$\frac{n!}{(s-a)^{n+1}}$	$s > a$
$u_c(t)$	$\frac{e^{-cs}}{s}$	$s > 0$
	\vdots	

4.1 Definition of the Laplace Transform



$f(t)$	$F(s) = \mathcal{L}[f](s)$	
$\sinh at$	$\frac{a}{s^2 - a^2}$	$s > a $
$\cosh at$	$\frac{s}{s^2 - a^2}$	$s > a $
$e^{at} \sin bt$	$\frac{b}{(s-a)^2 + b^2}$	$s > a$
$e^{at} \cos bt$	$\frac{s-a}{(s-a)^2 + b^2}$	$s > a$
$t^n e^{at} \quad (n \in \mathbb{N})$	$\frac{n!}{(s-a)^{n+1}}$	$s > a$
$u_c(t)$	$\frac{e^{-cs}}{s}$	$s > 0$
$u_c(t)f(t-c)$	$e^{-cs}F(s)$	
	\vdots	

4.1 Definition of the Laplace Transform



$f(t)$	$F(s) = \mathcal{L}[f](s)$	
$\cosh at$	$\frac{s}{s^2 - a^2}$	$s > a $
$e^{at} \sin bt$	$\frac{b}{(s-a)^2 + b^2}$	$s > a$
$e^{at} \cos bt$	$\frac{s-a}{(s-a)^2 + b^2}$	$s > a$
$t^n e^{at} \quad (n \in \mathbb{N})$	$\frac{n!}{(s-a)^{n+1}}$	$s > a$
$u_c(t)$	$\frac{e^{-cs}}{s}$	$s > 0$
$u_c(t)f(t-c)$	$e^{-cs}F(s)$	
$e^{ct}f(t)$	$F(s-c)$	
	\vdots	

4.1 Definition of the Laplace Transform



$f(t)$	$F(s) = \mathcal{L}[f](s)$	
$e^{at} \sin bt$	$\frac{b}{(s-a)^2+b^2}$	$s > a$
$e^{at} \cos bt$	$\frac{s-a}{(s-a)^2+b^2}$	$s > a$
$t^n e^{at} \quad (n \in \mathbb{N})$	$\frac{n!}{(s-a)^{n+1}}$	$s > a$
$u_c(t)$	$\frac{e^{-cs}}{s}$	$s > 0$
$u_c(t)f(t-c)$	$e^{-cs}F(s)$	
$e^{ct}f(t)$	$F(s-c)$	
$f(ct) \quad (c > 0)$	$\frac{1}{c}F\left(\frac{s}{c}\right)$	
	\vdots	

4.1 Definition of the Laplace Transform



$f(t)$	$F(s) = \mathcal{L}[f](s)$	
$e^{at} \cos bt$	$\frac{s-a}{(s-a)^2+b^2}$	$s > a$
$t^n e^{at} \quad (n \in \mathbb{N})$	$\frac{n!}{(s-a)^{n+1}}$	$s > a$
$u_c(t)$	$\frac{e^{-cs}}{s}$	$s > 0$
$u_c(t)f(t-c)$	$e^{-cs}F(s)$	
$e^{ct}f(t)$	$F(s-c)$	
$f(ct) \quad (c > 0)$	$\frac{1}{c}F\left(\frac{s}{c}\right)$	
$\int_0^t f(t-\tau)g(\tau)d\tau$	$F(s)G(s)$	
	\vdots	

4.1 Definition of the Laplace Transform



$f(t)$	$F(s) = \mathcal{L}[f](s)$	
$t^n e^{at} \quad (n \in \mathbb{N})$	$\frac{n!}{(s-a)^{n+1}}$	$s > a$
$u_c(t)$	$\frac{e^{-cs}}{s}$	$s > 0$
$u_c(t)f(t-c)$	$e^{-cs}F(s)$	
$e^{ct}f(t)$	$F(s-c)$	
$f(ct) \quad (c > 0)$	$\frac{1}{c}F\left(\frac{s}{c}\right)$	
$\int_0^t f(t-\tau)g(\tau)d\tau$	$F(s)G(s)$	
$t^n f(t)$	$(-1)^n F^{(n)}(s)$	

4.1 Definition of the Laplace Transform



Example

Find the inverse Laplace Transform of $F(s) = \ln \left(1 + \frac{1}{s^2} \right)$.

4.1 Definition of the Laplace Transform



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Find the inverse Laplace Transform of $F(s) = \ln \left(1 + \frac{1}{s^2}\right)$.

Again we will use the formula

$$\mathcal{L} [t^n f(t)] = (-1)^n \frac{d^n F}{ds^n}.$$

4.1 Definition of the Laplace Transform



Example

Find the inverse Laplace Transform of $F(s) = \ln \left(1 + \frac{1}{s^2}\right)$.

Again we will use the formula

$$\mathcal{L} [t^n f(t)] = (-1)^n \frac{d^n F}{ds^n}.$$

Setting $n = 1$

$$\mathcal{L} [tf(t)] = (-1) \frac{dF}{ds}$$

4.1 Definition of the Laplace Transform



Example

Find the inverse Laplace Transform of $F(s) = \ln \left(1 + \frac{1}{s^2}\right)$.

Again we will use the formula

$$\mathcal{L} [t^n f(t)] = (-1)^n \frac{d^n F}{ds^n}.$$

Setting $n = 1$

$$\mathcal{L} [tf(t)] = (-1) \frac{dF}{ds}$$

and taking \mathcal{L}^{-1} of both sides gives

$$tf(t) = -\mathcal{L}^{-1} \left[\frac{dF}{ds} \right].$$

4.1 Definition of the Laplace Transform



$$F(s) = \ln \left(1 + \frac{1}{s^2} \right)$$

Now

$$\frac{dF}{ds} = \frac{d}{ds} \ln \left(1 + \frac{1}{s^2} \right)$$

4.1 Definition of the Laplace Transform



$$F(s) = \ln \left(1 + \frac{1}{s^2} \right)$$

Now

$$\frac{dF}{ds} = \frac{d}{ds} \ln \left(1 + \frac{1}{s^2} \right) = \frac{\frac{-2}{s^3}}{\left(1 + \frac{1}{s^2} \right)}$$

4.1 Definition of the Laplace Transform



$$F(s) = \ln \left(1 + \frac{1}{s^2} \right)$$

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4.1 Definition of the Laplace Transform



$$F(s) = \ln \left(1 + \frac{1}{s^2} \right)$$

Now

$$\frac{dF}{ds} = \frac{d}{ds} \ln \left(1 + \frac{1}{s^2} \right) = \frac{\frac{-2}{s^3}}{\left(1 + \frac{1}{s^2} \right)} = \frac{-2}{s(s^2 + 1)}.$$

Therefore

$$tf(t) = -\mathcal{L}^{-1} \left[\frac{dF}{ds} \right] = \mathcal{L}^{-1} \left[\frac{2}{s(s^2 + 1)} \right].$$

4.1 Definition of the Laplace Transform



$$F(s) = \ln \left(1 + \frac{1}{s^2} \right)$$

Now

$$\frac{dF}{ds} = \frac{d}{ds} \ln \left(1 + \frac{1}{s^2} \right) = \frac{\frac{-2}{s^3}}{\left(1 + \frac{1}{s^2} \right)} = \frac{-2}{s(s^2 + 1)}.$$

Therefore

$$tf(t) = -\mathcal{L}^{-1} \left[\frac{dF}{ds} \right] = \mathcal{L}^{-1} \left[\frac{2}{s(s^2 + 1)} \right].$$

To proceed, we need to write $\frac{2}{s(s^2+1)}$ in partial fractions.

4.1 Definition of the Laplace Transform



We calculate that

$$\frac{2}{s(s^2 + 1)} = \frac{A}{s} + \frac{Bs + C}{s^2 + 1}$$

4.1 Definition of the Laplace Transform



We calculate that

$$\begin{aligned}\frac{2}{s(s^2 + 1)} &= \frac{A}{s} + \frac{Bs + C}{s^2 + 1} \\ &= \frac{A(s^2 + 1) + Bs^2 + Cs}{s(s^2 + 1)}\end{aligned}$$

4.1 Definition of the Laplace Transform



We calculate that

$$\begin{aligned}\frac{2}{s(s^2 + 1)} &= \frac{A}{s} + \frac{Bs + C}{s^2 + 1} \\ &= \frac{A(s^2 + 1) + Bs^2 + Cs}{s(s^2 + 1)} \\ &= \frac{(A + B)s^2 + Cs + A}{s(s^2 + 1)}\end{aligned}$$

4.1 Definition of the Laplace Transform



We calculate that

$$\begin{aligned}\frac{2}{s(s^2 + 1)} &= \frac{A}{s} + \frac{Bs + C}{s^2 + 1} \\ &= \frac{A(s^2 + 1) + Bs^2 + Cs}{s(s^2 + 1)} \\ &= \frac{(A + B)s^2 + Cs + A}{s(s^2 + 1)}\end{aligned} \quad \Rightarrow \quad \begin{aligned}A &= 2 \\ B &= -2 \\ C &= 0\end{aligned}$$

4.1 Definition of the Laplace Transform



We calculate that

$$\begin{aligned}\frac{2}{s(s^2 + 1)} &= \frac{A}{s} + \frac{Bs + C}{s^2 + 1} \\ &= \frac{A(s^2 + 1) + Bs^2 + Cs}{s(s^2 + 1)} \\ &= \frac{(A + B)s^2 + Cs + A}{s(s^2 + 1)} \\ &= \frac{2}{s} - \frac{2s}{s^2 + 1}.\end{aligned}\quad \Rightarrow \quad \begin{aligned}A &= 2 \\ B &= -2 \\ C &= 0\end{aligned}$$

4.1 Definition of the Laplace Transform



Thus

$$tf(t) = \mathcal{L}^{-1} \left[\frac{2}{s(s^2 + 1)} \right] = \mathcal{L}^{-1} \left[\frac{2}{s} - \frac{2s}{s^2 + 1} \right]$$

4.1 Definition of the Laplace Transform



Thus

$$tf(t) = \mathcal{L}^{-1} \left[\frac{2}{s(s^2 + 1)} \right] = \mathcal{L}^{-1} \left[\frac{2}{s} - \frac{2s}{s^2 + 1} \right]$$

$f(t)$	$F(s) = \mathcal{L}[f](s)$	
1	$\frac{1}{s}$	$s > 0$
e^{at}	$\frac{1}{s-a}$	$s > a$
$t^n \quad (n \in \mathbb{N})$	$\frac{n!}{s^{n+1}}$	$s > 0$
$\sin at$	$\frac{a}{s^2+a^2}$	$s > 0$
$\cos at$	$\frac{s}{s^2+a^2}$	$s > 0$
$\sinh at$	$\frac{a}{s^2-a^2}$	$s > a $

4.1 Definition of the Laplace Transform



Thus

$$tf(t) = \mathcal{L}^{-1} \left[\frac{2}{s(s^2 + 1)} \right] = \mathcal{L}^{-1} \left[\frac{2}{s} - \frac{2s}{s^2 + 1} \right]$$

$f(t)$	$F(s) = \mathcal{L}[f](s)$	
1	$\frac{1}{s}$	$s > 0$
e^{at}	$\frac{1}{s-a}$	$s > a$
$t^n \quad (n \in \mathbb{N})$	$\frac{n!}{s^{n+1}}$	$s > 0$
$\sin at$	$\frac{a}{s^2+a^2}$	$s > 0$
$\cos at$	$\frac{s}{s^2+a^2}$	$s > 0$
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4.1 Definition of the Laplace Transform



Thus

$$tf(t) = \mathcal{L}^{-1} \left[\frac{2}{s(s^2 + 1)} \right] = \mathcal{L}^{-1} \left[\frac{2}{s} - \frac{2s}{s^2 + 1} \right]$$

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1	$\frac{1}{s}$	$s > 0$
e^{at}	$\frac{1}{s-a}$	$s > a$
$t^n \quad (n \in \mathbb{N})$	$\frac{n!}{s^{n+1}}$	$s > 0$
$\sin at$	$\frac{a}{s^2+a^2}$	$s > 0$
$\cos at$	$\frac{s}{s^2+a^2}$	$s > 0$
$\sinh at$	$\frac{a}{s^2-a^2}$	$s > a $

A diagram with two arrows originating from the right side of the table. A green arrow starts from the $\frac{1}{s}$ term in the first row and points to the $\frac{2}{s}$ term in the equation above. An orange arrow starts from the $\frac{s}{s^2+a^2}$ term in the fifth row and points to the $-\frac{2s}{s^2+1}$ term in the equation above.

4.1 Definition of the Laplace Transform



Thus

$$tf(t) = \mathcal{L}^{-1} \left[\frac{2}{s(s^2 + 1)} \right] = \mathcal{L}^{-1} \left[\frac{2}{s} - \frac{2s}{s^2 + 1} \right]$$

$$\mathcal{L} [1] = \frac{1}{s} \qquad \mathcal{L} [\cos at] = \frac{s}{s^2 + a^2}$$

4.1 Definition of the Laplace Transform



Thus

$$\begin{aligned}tf(t) &= \mathcal{L}^{-1} \left[\frac{2}{s(s^2 + 1)} \right] = \mathcal{L}^{-1} \left[\frac{2}{s} - \frac{2s}{s^2 + 1} \right] \\ &= 2\mathcal{L}^{-1} \left[\frac{1}{s} \right] - 2\mathcal{L}^{-1} \left[\frac{s}{s^2 + 1} \right]\end{aligned}$$

$$\mathcal{L} [1] = \frac{1}{s} \qquad \mathcal{L} [\cos at] = \frac{s}{s^2 + a^2}$$

4.1 Definition of the Laplace Transform



Thus

$$\begin{aligned}tf(t) &= \mathcal{L}^{-1} \left[\frac{2}{s(s^2 + 1)} \right] = \mathcal{L}^{-1} \left[\frac{2}{s} - \frac{2s}{s^2 + 1} \right] \\&= 2\mathcal{L}^{-1} \left[\frac{1}{s} \right] - 2\mathcal{L}^{-1} \left[\frac{s}{s^2 + 1} \right] \\&= 2 - 2 \cos t.\end{aligned}$$

$$\mathcal{L} [1] = \frac{1}{s} \qquad \mathcal{L} [\cos at] = \frac{s}{s^2 + a^2}$$

4.1 Definition of the Laplace Transform



Thus

$$\begin{aligned}tf(t) &= \mathcal{L}^{-1} \left[\frac{2}{s(s^2 + 1)} \right] = \mathcal{L}^{-1} \left[\frac{2}{s} - \frac{2s}{s^2 + 1} \right] \\&= 2\mathcal{L}^{-1} \left[\frac{1}{s} \right] - 2\mathcal{L}^{-1} \left[\frac{s}{s^2 + 1} \right] \\&= 2 - 2 \cos t.\end{aligned}$$

Therefore

$$\boxed{f(t) = \frac{2(1 - \cos t)}{t}.$$

$$\mathcal{L} [1] = \frac{1}{s} \qquad \mathcal{L} [\cos at] = \frac{s}{s^2 + a^2}$$

Solving Initial Value Problems

4.2 Solving Initial Value Problems



Theorem

$$1 \quad \mathcal{L}[f'](s) = s\mathcal{L}[f](s) - f(0).$$

4.2 Solving Initial Value Problems



Theorem

- 1 $\mathcal{L}[f'](s) = s\mathcal{L}[f](s) - f(0).$
- 2 $\mathcal{L}[f''](s) = s^2\mathcal{L}[f](s) - sf(0) - f'(0).$

4.2 Solving Initial Value Problems



Theorem

- 1 $\mathcal{L}[f'](s) = s\mathcal{L}[f](s) - f(0).$
- 2 $\mathcal{L}[f''](s) = s^2\mathcal{L}[f](s) - sf(0) - f'(0).$
- 3 $\mathcal{L}[f'''](s) = s^3\mathcal{L}[f](s) - s^2f(0) - sf'(0) - f''(0).$

4.2 Solving Initial Value Problems



Theorem

- 1 $\mathcal{L}[f'](s) = s\mathcal{L}[f](s) - f(0).$
- 2 $\mathcal{L}[f''](s) = s^2\mathcal{L}[f](s) - sf(0) - f'(0).$
- 3 $\mathcal{L}[f'''](s) = s^3\mathcal{L}[f](s) - s^2f(0) - sf'(0) - f''(0).$
- 4 $\mathcal{L}[f^{(n)}](s) = s^n\mathcal{L}[f](s) - s^{n-1}f(0) - s^{n-2}f'(0) - \dots - sf^{(n-2)}(0) - f^{(n-1)}(0).$

4.2 Solving Initial Value Problems



Proof:

- 1 Using integration-by-parts ($\int uv' = uv - \int u'v$) we calculate that

$$\begin{aligned}\mathcal{L}[f'](s) &= \int_0^{\infty} e^{-st} f'(t) dt = [e^{-st} f(t)]_0^{\infty} - \int_0^{\infty} \left(\frac{d}{dt} e^{-st} \right) f(t) dt \\ &= \\ &= \\ &= \end{aligned}$$

4.2 Solving Initial Value Problems



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$$\begin{aligned}\mathcal{L}[f'](s) &= \int_0^{\infty} e^{-st} f'(t) dt = [e^{-st} f(t)]_0^{\infty} - \int_0^{\infty} \left(\frac{d}{dt} e^{-st} \right) f(t) dt \\ &= 0 - f(0) - \int_0^{\infty} -se^{-st} f(t) dt \\ &= \\ &= \end{aligned}$$

4.2 Solving Initial Value Problems



Proof:

- 1 Using integration-by-parts ($\int uv' = uv - \int u'v$) we calculate that

$$\begin{aligned}\mathcal{L}[f'](s) &= \int_0^{\infty} e^{-st} f'(t) dt = [e^{-st} f(t)]_0^{\infty} - \int_0^{\infty} \left(\frac{d}{dt} e^{-st} \right) f(t) dt \\ &= 0 - f(0) - \int_0^{\infty} -s e^{-st} f(t) dt \\ &= -f(0) + s \int_0^{\infty} e^{-st} f(t) dt \\ &= \end{aligned}$$

4.2 Solving Initial Value Problems



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- 1 Using integration-by-parts ($\int uv' = uv - \int u'v$) we calculate that

$$\begin{aligned}\mathcal{L}[f'](s) &= \int_0^{\infty} e^{-st} f'(t) dt = [e^{-st} f(t)]_0^{\infty} - \int_0^{\infty} \left(\frac{d}{dt} e^{-st} \right) f(t) dt \\ &= 0 - f(0) - \int_0^{\infty} -se^{-st} f(t) dt \\ &= -f(0) + s \int_0^{\infty} e^{-st} f(t) dt \\ &= -f(0) + s \mathcal{L}[f](s)\end{aligned}$$

as required.

4.2 Solving Initial Value Problems



$$\mathcal{L}[f'](s) = s\mathcal{L}[f](s) - f(0)$$

- 2 Using part 1, but replacing each f by f' we get

$$\mathcal{L}[f''](s) = s\mathcal{L}[f'](s) - f'(0)$$

$$=$$
$$=$$

4.2 Solving Initial Value Problems



$$\mathcal{L}[f'](s) = s\mathcal{L}[f](s) - f(0)$$

2 Using part 1, but replacing each f by f' we get

$$\begin{aligned}\mathcal{L}[f''](s) &= s\mathcal{L}[f'](s) - f'(0) \\ &= s\left(s\mathcal{L}[f](s) - f(0)\right) - f'(0) \\ &= \end{aligned}$$

4.2 Solving Initial Value Problems



$$\mathcal{L}[f'](s) = s\mathcal{L}[f](s) - f(0)$$

2 Using part 1, but replacing each f by f' we get

$$\begin{aligned}\mathcal{L}[f''](s) &= s\mathcal{L}[f'](s) - f'(0) \\ &= s\left(s\mathcal{L}[f](s) - f(0)\right) - f'(0) \\ &= s^2\mathcal{L}[f](s) - sf(0) - f'(0).\end{aligned}$$

4.2 Solving Initial Value Problems



$$\mathcal{L}[f'](s) = s\mathcal{L}[f](s) - f(0)$$

2 Using part 1, but replacing each f by f' we get

$$\begin{aligned}\mathcal{L}[f''](s) &= s\mathcal{L}[f'](s) - f'(0) \\ &= s\left(s\mathcal{L}[f](s) - f(0)\right) - f'(0) \\ &= s^2\mathcal{L}[f](s) - sf(0) - f'(0).\end{aligned}$$

You prove parts 3 and 4.



4.2 Solving Initial Value Problems



Example

Solve

$$\begin{cases} y'' - y' - 2y = 0 \\ y(0) = 1 \\ y'(0) = 0. \end{cases}$$

4.2 Solving Initial Value Problems



Example

Solve

$$\begin{cases} y'' - y' - 2y = 0 \\ y(0) = 1 \\ y'(0) = 0. \end{cases}$$

solution 1 (method from Chapter 3): The characteristic equation

$$0 = r^2 - r - 2 = (r - 2)(r + 1)$$

has roots $r_1 = -1$ and $r_2 = 2$.

4.2 Solving Initial Value Problems



Example

Solve

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4.2 Solving Initial Value Problems



Example

Solve

$$\begin{cases} y'' - y' - 2y = 0 \\ y(0) = 1 \\ y'(0) = 0. \end{cases}$$

solution 1 (method from Chapter 3): The characteristic equation

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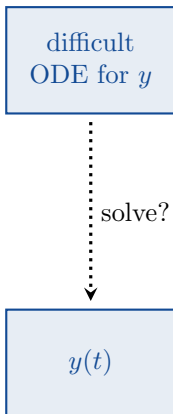
has roots $r_1 = -1$ and $r_2 = 2$. So $y = c_1 e^{-t} + c_2 e^{2t}$. Using the initial conditions we find that $c_1 = \frac{2}{3}$ and $c_2 = \frac{1}{3}$. Therefore

$$y(t) = \frac{2}{3}e^{-t} + \frac{1}{3}e^{2t}.$$

4.2 Solving Initial Value Problems



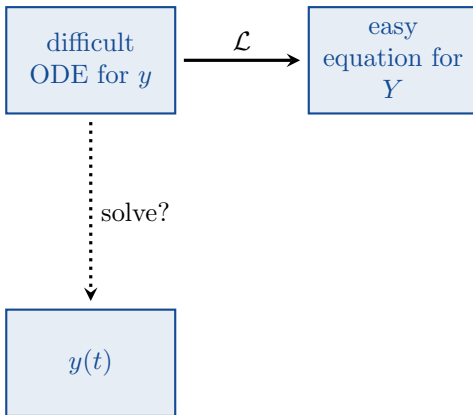
solution 2 (Laplace Transform):



4.2 Solving Initial Value Problems



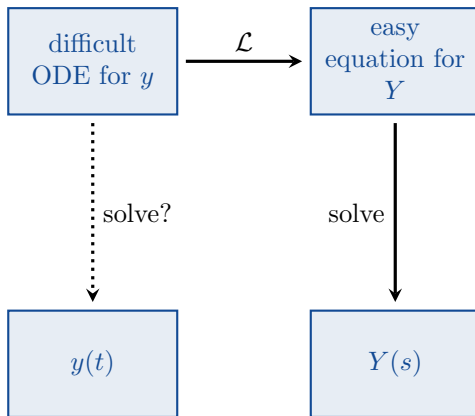
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4.2 Solving Initial Value Problems



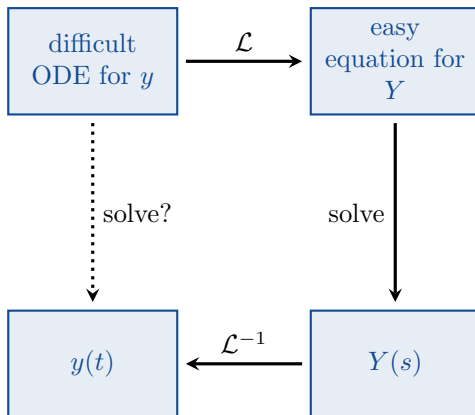
solution 2 (Laplace Transform):



4.2 Solving Initial Value Problems



solution 2 (Laplace Transform):



4.2 Solving Initial Value Problems



$$y'' - y' - 2y = 0$$

First we take the Laplace Transform of the ODE

$$\mathcal{L}[y'' - y' - 2y] = \mathcal{L}[0]$$

4.2 Solving Initial Value Problems



$$y'' - y' - 2y = 0$$

$$\mathcal{L}[y''] = s^2Y - sy(0) - y'(0) \qquad \mathcal{L}[y'] = sY - y(0)$$

First we take the Laplace Transform of the ODE

$$\mathcal{L}[y'' - y' - 2y] = \mathcal{L}[0]$$

$$\mathcal{L}[y''] - \mathcal{L}[y'] - 2\mathcal{L}[y] = 0$$

4.2 Solving Initial Value Problems



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$$\mathcal{L}[y'' - y' - 2y] = \mathcal{L}[0]$$

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It follows that

$$(s^2Y - sy(0) - y'(0)) - (sY - y(0)) - 2Y = 0$$

4.2 Solving Initial Value Problems



$$y'' - y' - 2y = 0$$

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First we take the Laplace Transform of the ODE

$$\mathcal{L}[y'' - y' - 2y] = \mathcal{L}[0]$$

$$\mathcal{L}[y''] - \mathcal{L}[y'] - 2\mathcal{L}[y] = 0$$

It follows that

$$(s^2Y - sy(0) - y'(0)) - (sY - y(0)) - 2Y = 0$$

$$(s^2Y - s - 0) - (sY - 1) - 2Y = 0$$

$$(s^2 - s - 2)Y + (1 - s) = 0.$$

4.2 Solving Initial Value Problems



Thus

$$Y(s) = \frac{s-1}{s^2-s-2} = \frac{s-1}{(s-2)(s+1)}.$$

4.2 Solving Initial Value Problems



Thus

$$Y(s) = \frac{s-1}{s^2-s-2} = \frac{s-1}{(s-2)(s+1)}.$$

Using partial fractions we obtain

$$\begin{aligned} Y(s) &= \frac{s-1}{(s-2)(s+1)} = \frac{A}{s-2} + \frac{B}{s+1} = \frac{As + A + Bs - 2B}{(s-2)(s+1)} \\ &= \frac{1}{3} \left(\frac{1}{s-2} \right) + \frac{2}{3} \left(\frac{1}{s+1} \right). \end{aligned}$$

4.2 Solving Initial Value Problems



Thus

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But recall that $\mathcal{L}[e^{2t}] = \frac{1}{s-2}$ and $\mathcal{L}[e^{-t}] = \frac{1}{s+1}$.

4.2 Solving Initial Value Problems



Thus

$$Y(s) = \frac{s-1}{s^2-s-2} = \frac{s-1}{(s-2)(s+1)}.$$

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But recall that $\mathcal{L}[e^{2t}] = \frac{1}{s-2}$ and $\mathcal{L}[e^{-t}] = \frac{1}{s+1}$. Therefore

$$y(t) = \mathcal{L}^{-1}[Y] = \frac{1}{3}\mathcal{L}^{-1}\left[\frac{1}{s-2}\right] + \frac{2}{3}\mathcal{L}^{-1}\left[\frac{1}{s+1}\right] = \boxed{\frac{1}{3}e^{2t} + \frac{2}{3}e^{-t}}.$$

4.2 Solving Initial Value Problems



Example

Solve

$$\begin{cases} y'' + y = \sin 2t \\ y(0) = 2 \\ y'(0) = 1. \end{cases}$$

4.2 Solving Initial Value Problems



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Solve

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$$y'' + y = \sin 2t$$

4.2 Solving Initial Value Problems



Example

Solve

$$\begin{cases} y'' + y = \sin 2t \\ y(0) = 2 \\ y'(0) = 1. \end{cases}$$

$$\mathcal{L}[y''] + \mathcal{L}[y] = \mathcal{L}[\sin 2t]$$

4.2 Solving Initial Value Problems



Example

Solve

$$\begin{cases} y'' + y = \sin 2t \\ y(0) = 2 \\ y'(0) = 1. \end{cases}$$

$$\mathcal{L}[y''] + \mathcal{L}[y] = \mathcal{L}[\sin 2t]$$

$$(s^2 Y - sy(0) - y'(0)) + Y = \frac{2}{s^2 + 4}$$

4.2 Solving Initial Value Problems



Example

Solve

$$\begin{cases} y'' + y = \sin 2t \\ y(0) = 2 \\ y'(0) = 1. \end{cases}$$

$$\mathcal{L}[y''] + \mathcal{L}[y] = \mathcal{L}[\sin 2t]$$

$$s^2 Y - 2s - 1 + Y = \frac{2}{s^2 + 4}$$

4.2 Solving Initial Value Problems



Example

Solve

$$\begin{cases} y'' + y = \sin 2t \\ y(0) = 2 \\ y'(0) = 1. \end{cases}$$

$$\mathcal{L}[y''] + \mathcal{L}[y] = \mathcal{L}[\sin 2t]$$

$$(s^2 + 1)Y = 2s + 1 + \frac{2}{s^2 + 4}$$

4.2 Solving Initial Value Problems



Example

Solve

$$\begin{cases} y'' + y = \sin 2t \\ y(0) = 2 \\ y'(0) = 1. \end{cases}$$

$$\mathcal{L}[y''] + \mathcal{L}[y] = \mathcal{L}[\sin 2t]$$

$$Y = \frac{2s + 1}{s^2 + 1} + \frac{2}{(s^2 + 1)(s^2 + 4)}$$

4.2 Solving Initial Value Problems



$$Y = \frac{2s + 1}{s^2 + 1} + \frac{2}{(s^2 + 1)(s^2 + 4)} =$$

=

=

=

=

4.2 Solving Initial Value Problems



$$Y = \frac{2s+1}{s^2+1} + \frac{2}{(s^2+1)(s^2+4)} = \frac{2s+1}{s^2+1} + \frac{As+B}{s^2+1} + \frac{Cs+D}{s^2+4}$$

=

=

=

=

4.2 Solving Initial Value Problems



$$\begin{aligned} Y &= \frac{2s+1}{s^2+1} + \frac{2}{(s^2+1)(s^2+4)} = \frac{2s+1}{s^2+1} + \frac{As+B}{s^2+1} + \frac{Cs+D}{s^2+4} \\ &= \frac{2s+1}{s^2+1} + \frac{\frac{2}{3}}{s^2+1} - \frac{\frac{2}{3}}{s^2+4} \\ &= \\ &= \\ &= \end{aligned}$$

4.2 Solving Initial Value Problems



$$\begin{aligned} Y &= \frac{2s+1}{s^2+1} + \frac{2}{(s^2+1)(s^2+4)} = \frac{2s+1}{s^2+1} + \frac{As+B}{s^2+1} + \frac{Cs+D}{s^2+4} \\ &= \frac{2s+1}{s^2+1} + \frac{\frac{2}{3}}{s^2+1} - \frac{\frac{2}{3}}{s^2+4} \\ &= \frac{2s}{s^2+1} + \frac{\frac{5}{3}}{s^2+1} - \frac{\frac{2}{3}}{s^2+4} \\ &= \\ &= \end{aligned}$$

4.2 Solving Initial Value Problems



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4.2 Solving Initial Value Problems



$$\begin{aligned} Y &= \frac{2s+1}{s^2+1} + \frac{2}{(s^2+1)(s^2+4)} = \frac{2s+1}{s^2+1} + \frac{As+B}{s^2+1} + \frac{Cs+D}{s^2+4} \\ &= \frac{2s+1}{s^2+1} + \frac{\frac{2}{3}}{s^2+1} - \frac{\frac{2}{3}}{s^2+4} \\ &= \frac{2s}{s^2+1} + \frac{\frac{5}{3}}{s^2+1} - \frac{\frac{2}{3}}{s^2+4} \\ &= 2 \left(\frac{s}{s^2+1} \right) + \frac{5}{3} \left(\frac{1}{s^2+1} \right) - \frac{1}{3} \left(\frac{2}{s^2+4} \right) \\ &= 2\mathcal{L} [\cos t] + \frac{5}{3}\mathcal{L} [\sin t] - \frac{1}{3}\mathcal{L} [\sin 2t]. \end{aligned}$$

4.2 Solving Initial Value Problems



$$\begin{aligned} Y &= \frac{2s+1}{s^2+1} + \frac{2}{(s^2+1)(s^2+4)} = \frac{2s+1}{s^2+1} + \frac{As+B}{s^2+1} + \frac{Cs+D}{s^2+4} \\ &= \frac{2s+1}{s^2+1} + \frac{\frac{2}{3}}{s^2+1} - \frac{\frac{2}{3}}{s^2+4} \\ &= \frac{2s}{s^2+1} + \frac{\frac{5}{3}}{s^2+1} - \frac{\frac{2}{3}}{s^2+4} \\ &= 2 \left(\frac{s}{s^2+1} \right) + \frac{5}{3} \left(\frac{1}{s^2+1} \right) - \frac{1}{3} \left(\frac{2}{s^2+4} \right) \\ &= 2\mathcal{L}[\cos t] + \frac{5}{3}\mathcal{L}[\sin t] - \frac{1}{3}\mathcal{L}[\sin 2t]. \end{aligned}$$

Therefore

$$y(t) = 2 \cos t + \frac{5}{3} \sin t - \frac{1}{3} \sin 2t.$$

4.2 Solving Initial Value Problems



Example

Solve

$$\begin{cases} y^{(4)} - y = 0 \\ y(0) = 0 \\ y'(0) = 1 \\ y''(0) = 0 \\ y'''(0) = 0. \end{cases}$$

4.2 Solving Initial Value Problems



Example

Solve

$$\begin{cases} y^{(4)} - y = 0 \\ y(0) = 0 \\ y'(0) = 1 \\ y''(0) = 0 \\ y'''(0) = 0. \end{cases}$$

Using the Laplace Transform we calculate that

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4.2 Solving Initial Value Problems



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4.2 Solving Initial Value Problems



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4.2 Solving Initial Value Problems



$$0 = (s^4 - 1)Y - s^2$$

Thus

$$Y(s) = \frac{s^2}{s^4 - 1}$$

4.2 Solving Initial Value Problems



$$0 = (s^4 - 1)Y - s^2$$

Thus

$$Y(s) = \frac{s^2}{s^4 - 1} = \frac{s^2}{(s^2 - 1)(s^2 + 1)}$$

4.2 Solving Initial Value Problems



$$0 = (s^4 - 1)Y - s^2$$

Thus

$$Y(s) = \frac{s^2}{s^4 - 1} = \frac{s^2}{(s^2 - 1)(s^2 + 1)} = \frac{\frac{1}{2}}{s^2 - 1} + \frac{\frac{1}{2}}{s^2 + 1}.$$

4.2 Solving Initial Value Problems



$$0 = (s^4 - 1)Y - s^2$$

Thus

$$Y(s) = \frac{s^2}{s^4 - 1} = \frac{s^2}{(s^2 - 1)(s^2 + 1)} = \frac{\frac{1}{2}}{s^2 - 1} + \frac{\frac{1}{2}}{s^2 + 1}.$$

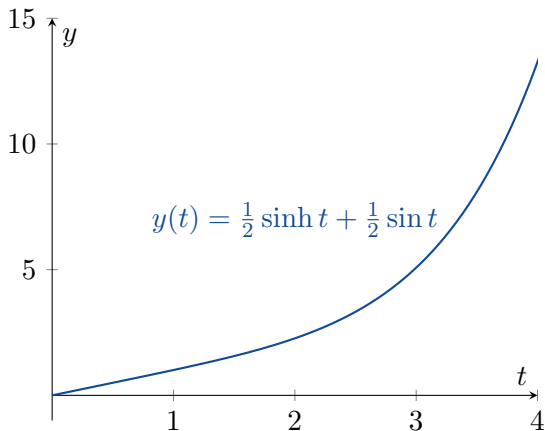
Therefore

$$y = \frac{1}{2}\mathcal{L}^{-1}\left[\frac{1}{s^2 - 1}\right] + \frac{1}{2}\mathcal{L}^{-1}\left[\frac{1}{s^2 + 1}\right] = \boxed{\frac{1}{2}\sinh t + \frac{1}{2}\sin t.}$$

4.2 Solving Initial Value Problems



$$\begin{cases} y^{(4)} - y = 0 \\ y(0) = 0 \\ y'(0) = 1 \\ y''(0) = 0 \\ y'''(0) = 0. \end{cases}$$



Next Time

- 4.3 Solving More Initial Value Problems
- 4.4 Step Functions