

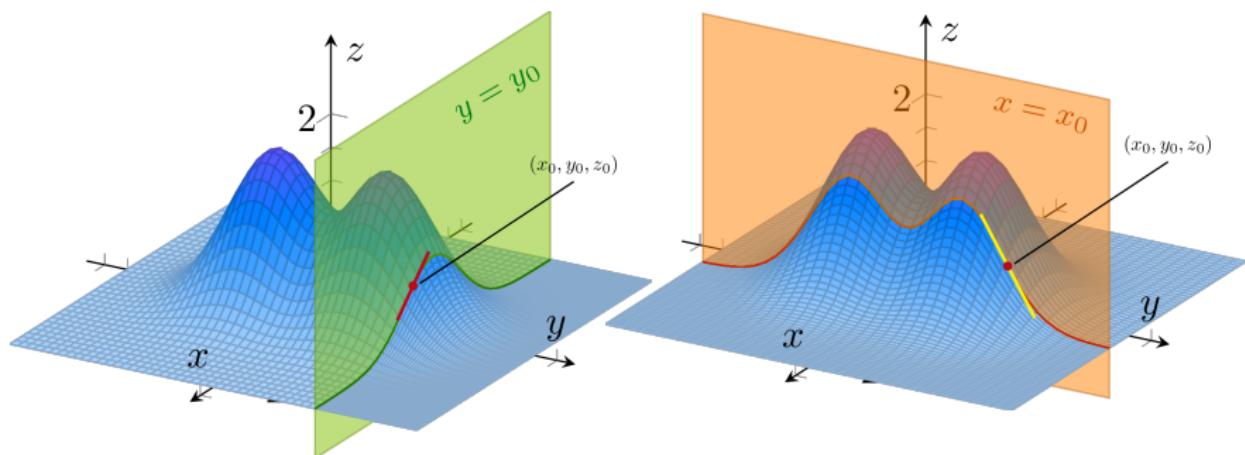
# Lecture 6

- 13.5 Directional Derivatives and Gradient Vectors
- 13.6 Tangent Planes and Differentials
- 13.7 Extreme Values and Saddle Points
- 13.8 Lagrange Multipliers

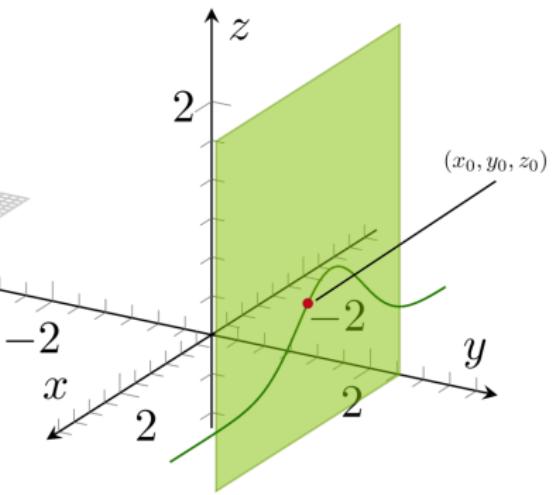
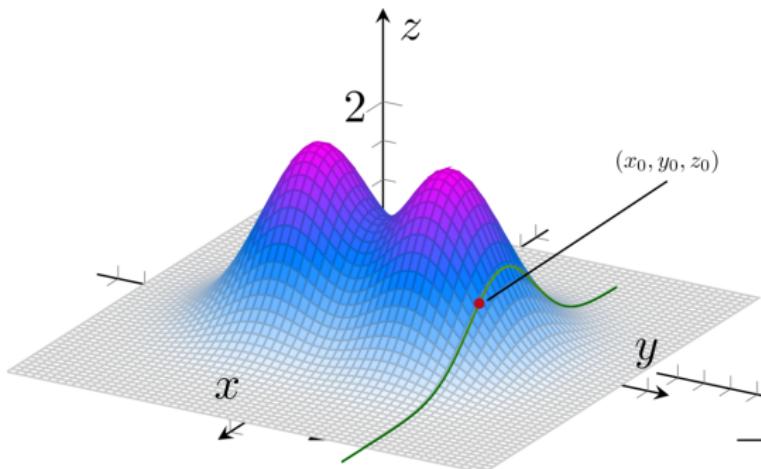


# Directional Derivatives and Gradient Vectors

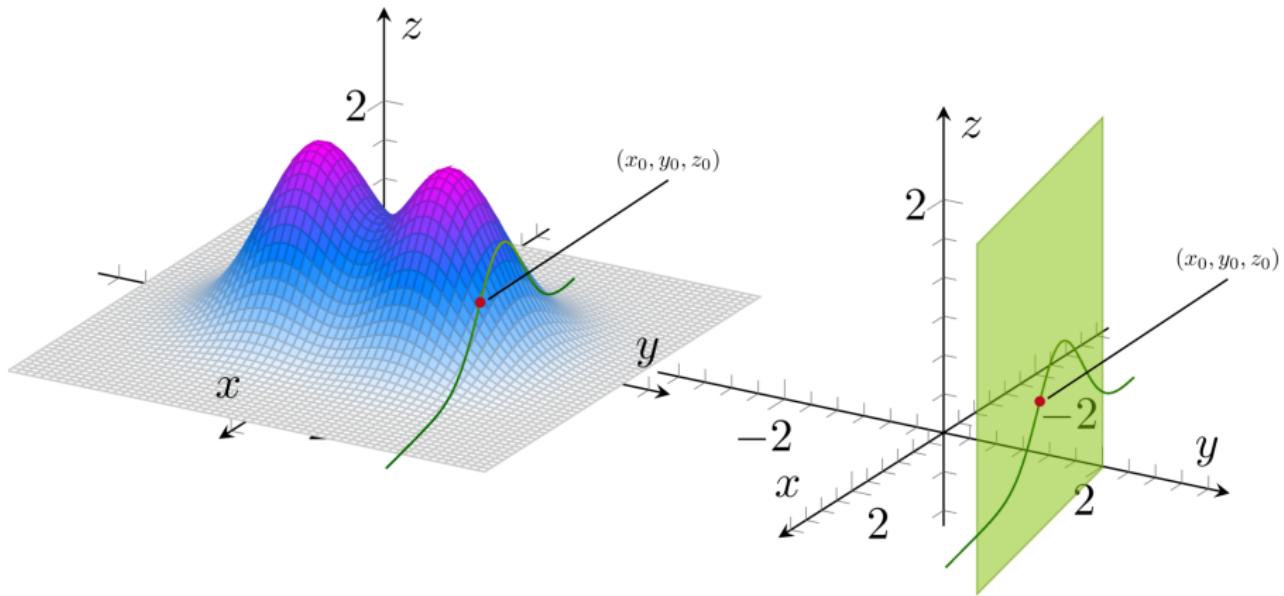
## Partial Derivatives (revision)



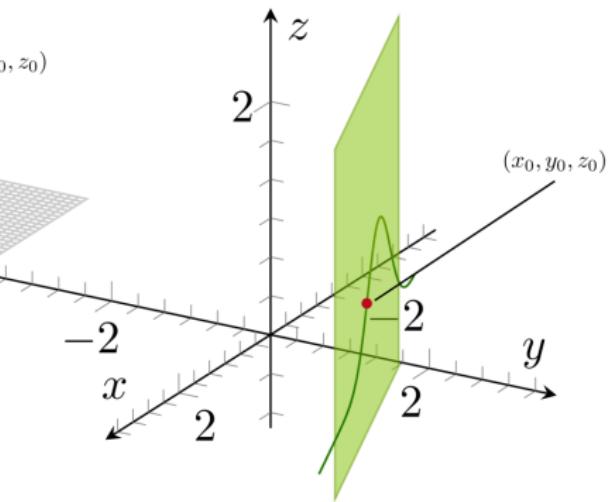
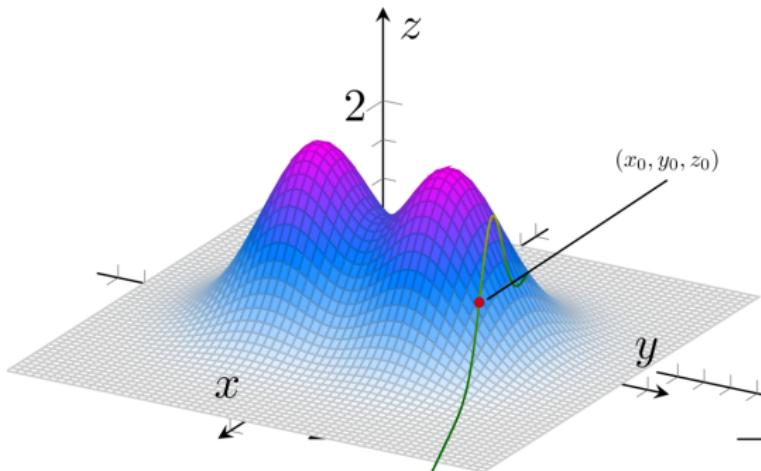
## Directional Derivatives



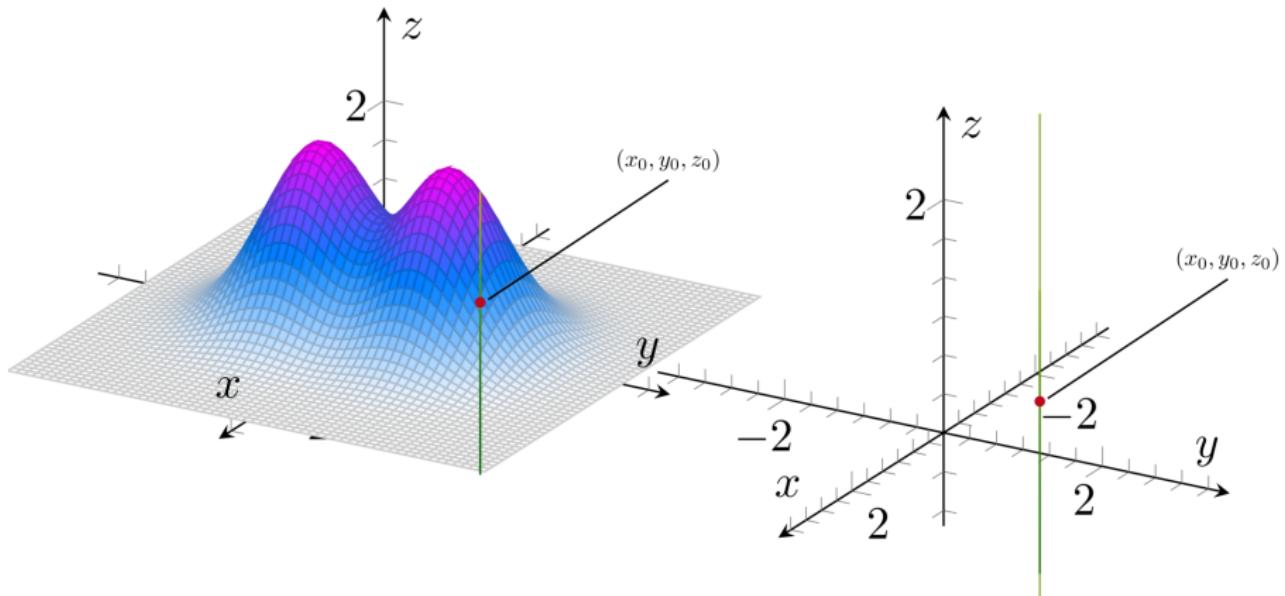
## Directional Derivatives



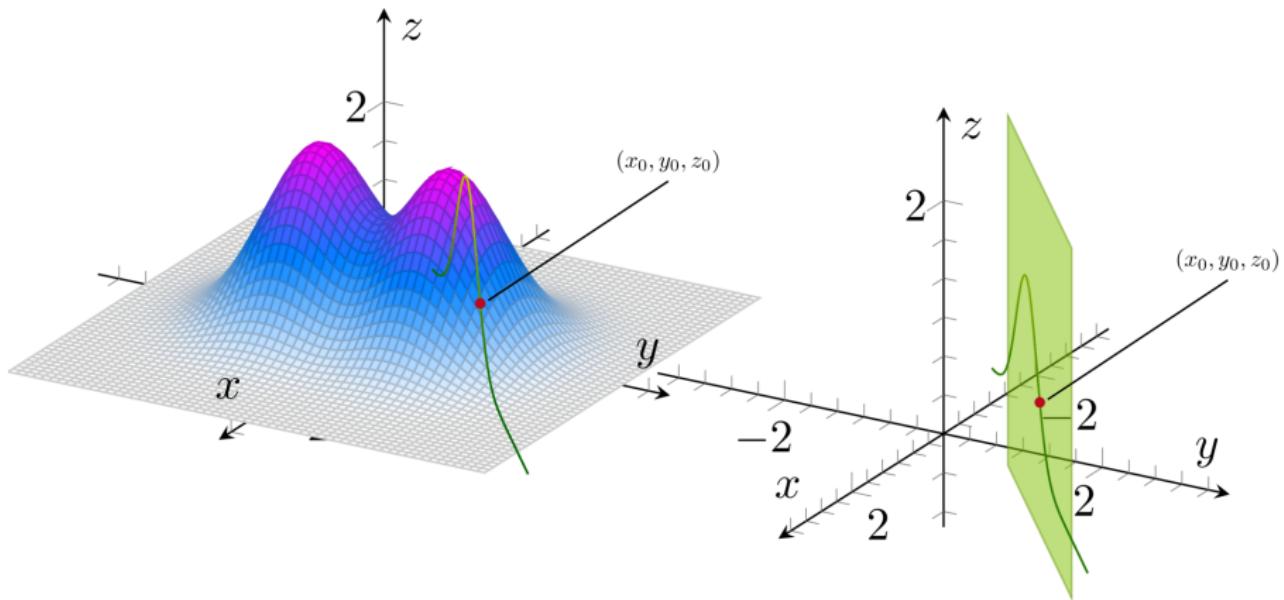
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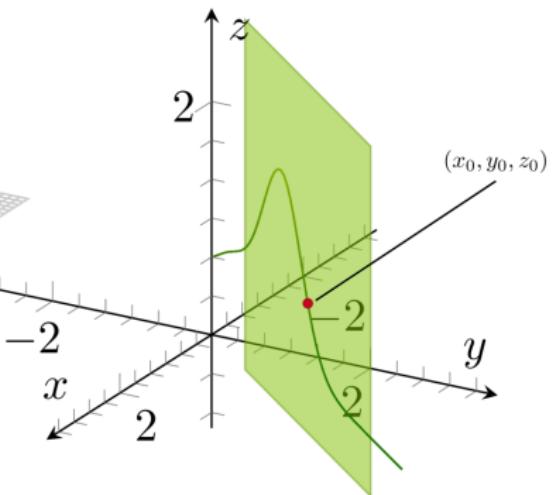
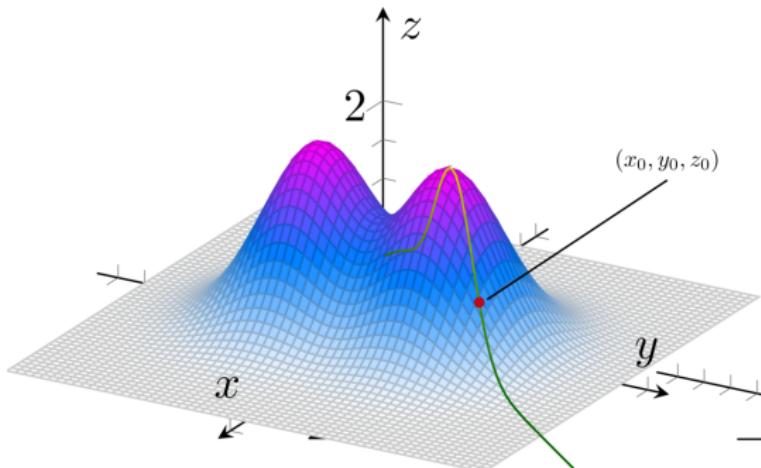
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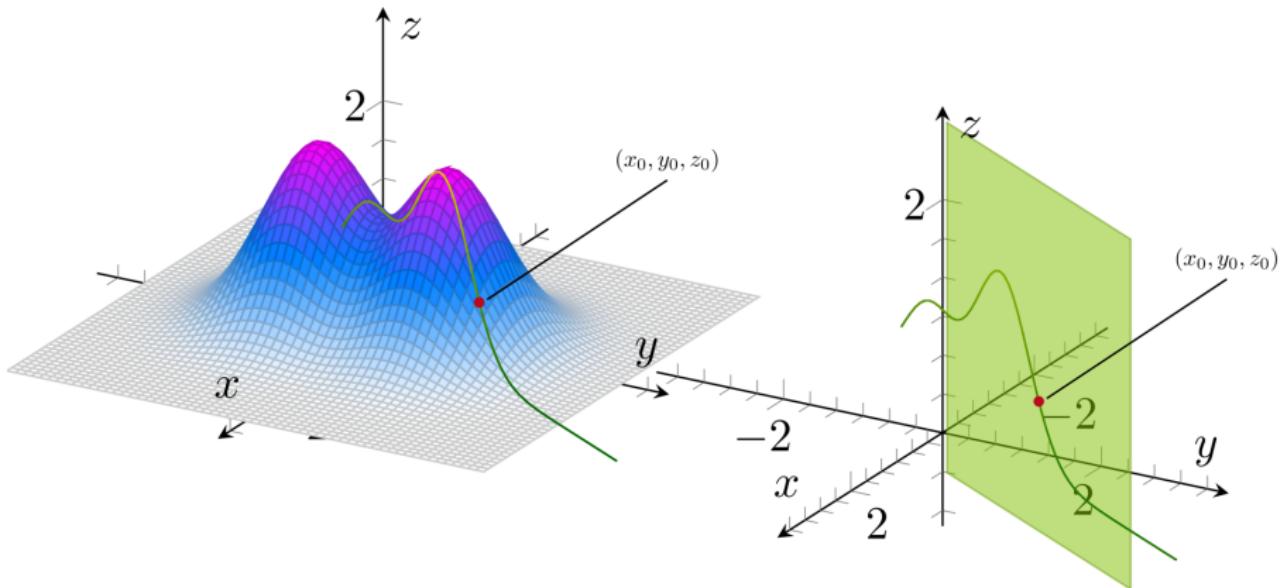
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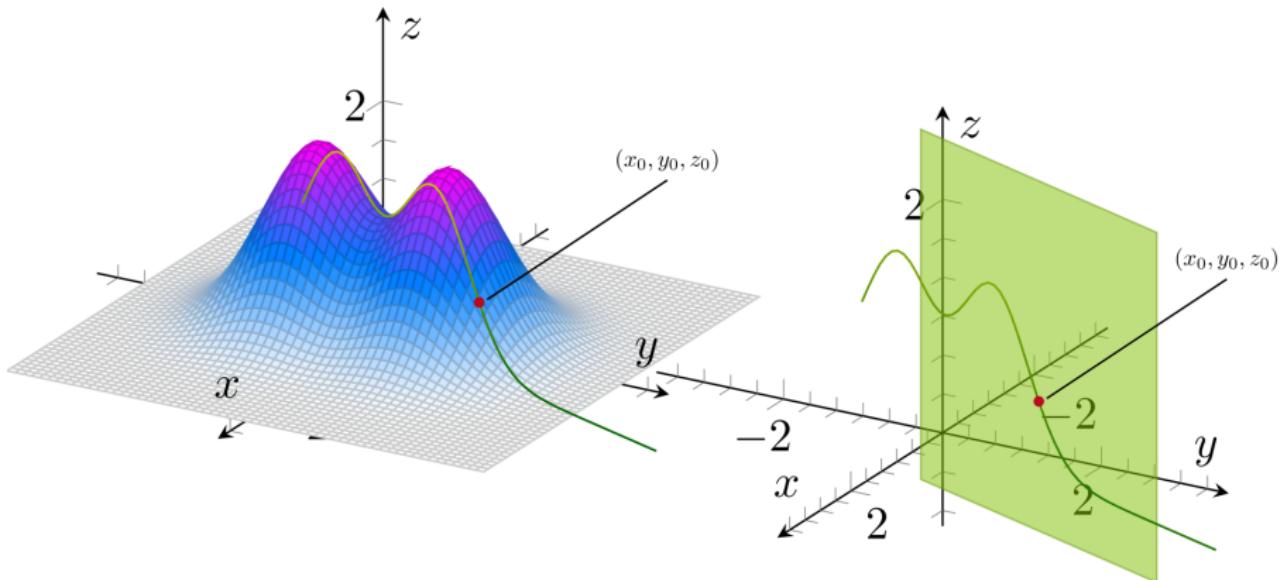
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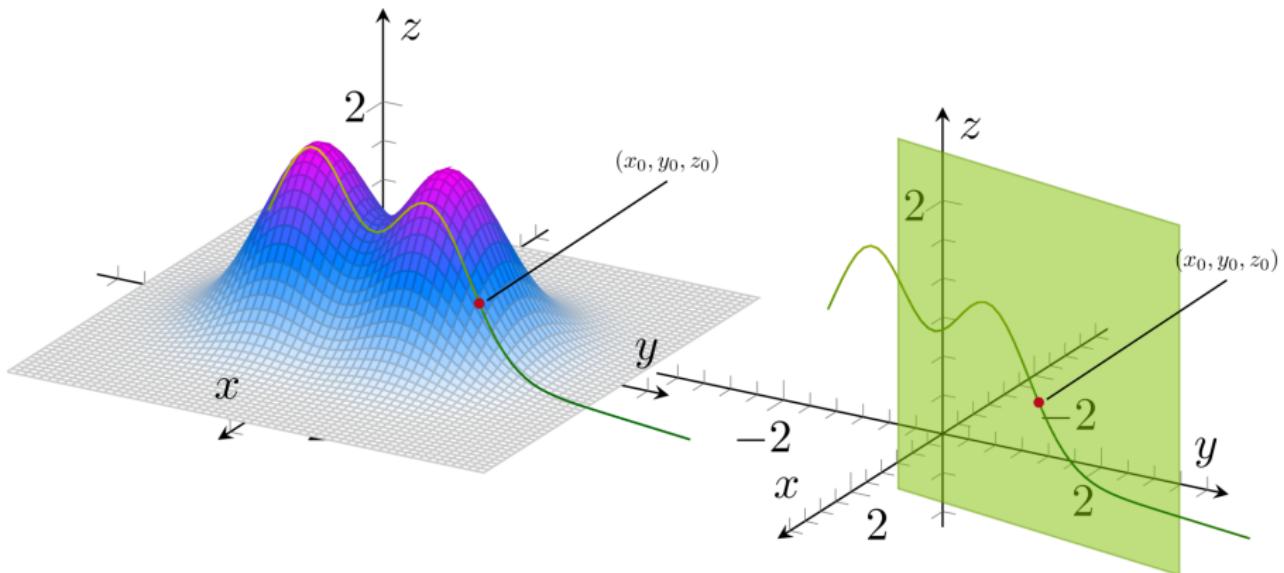
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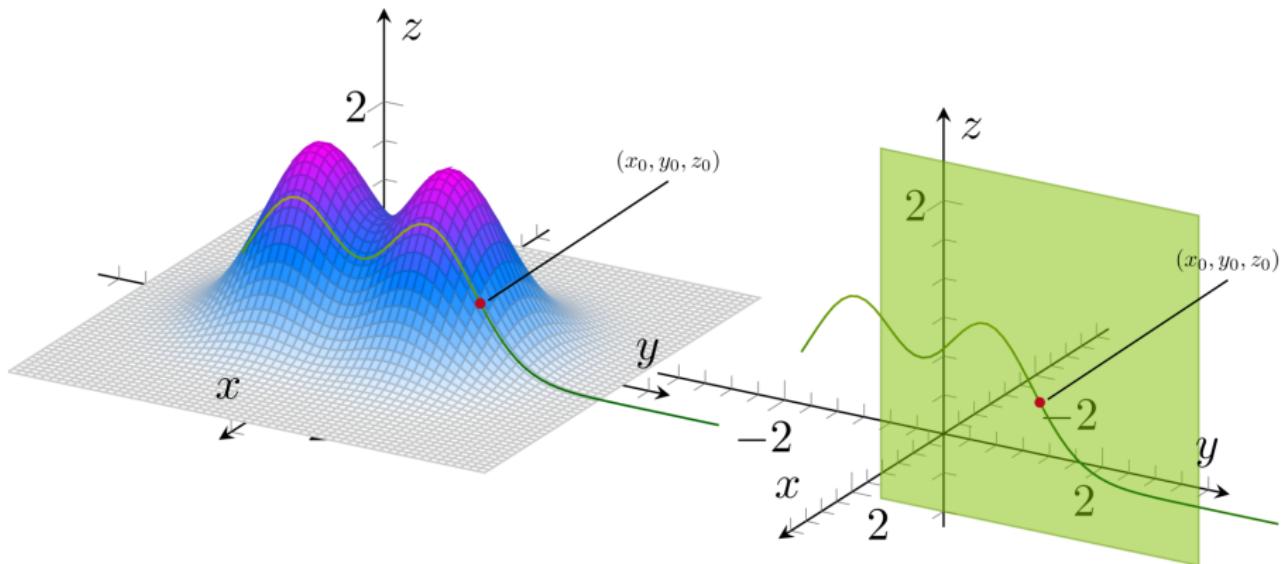
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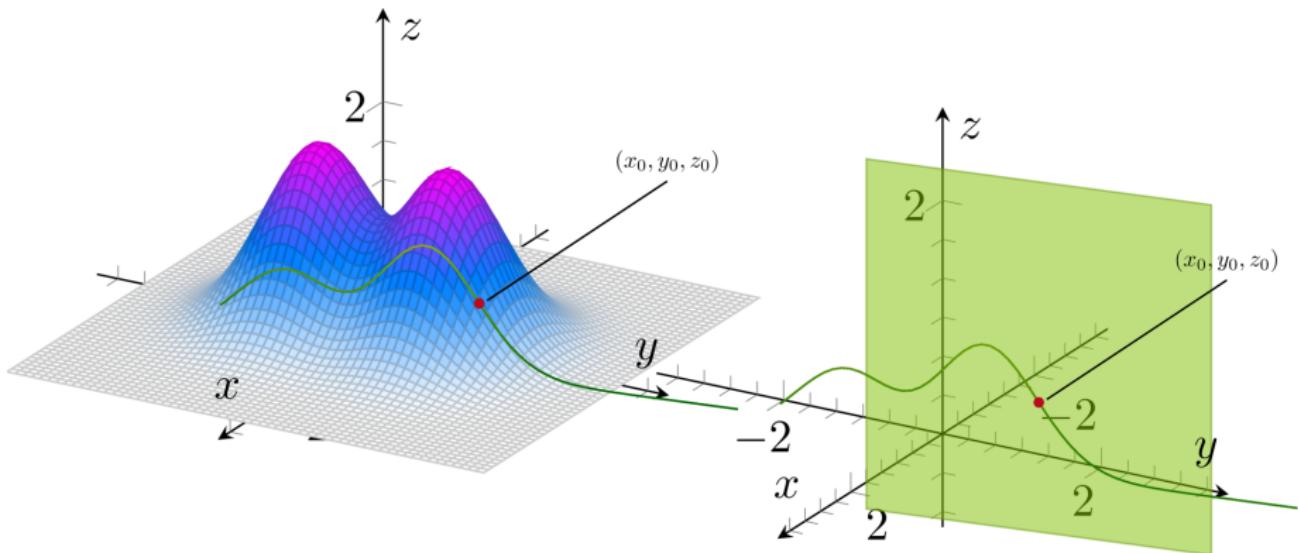
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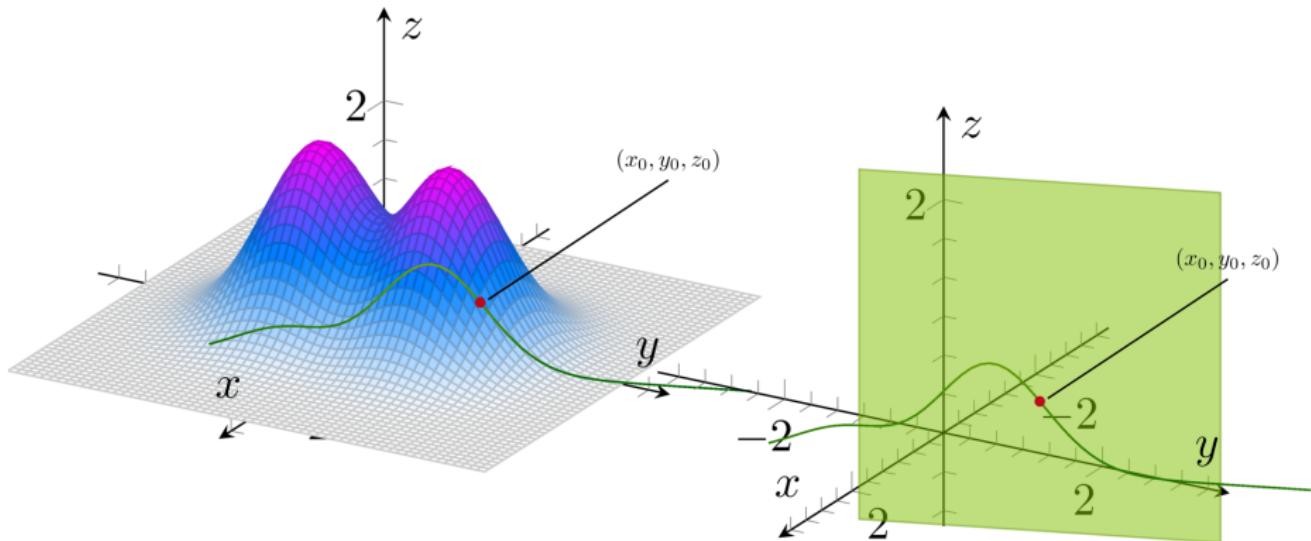
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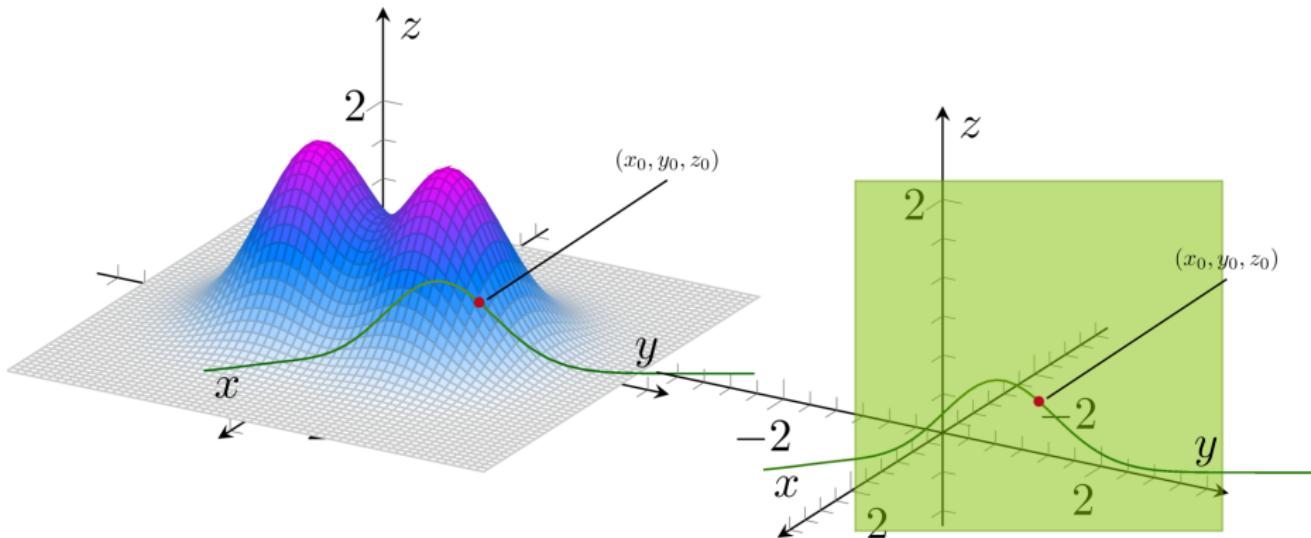
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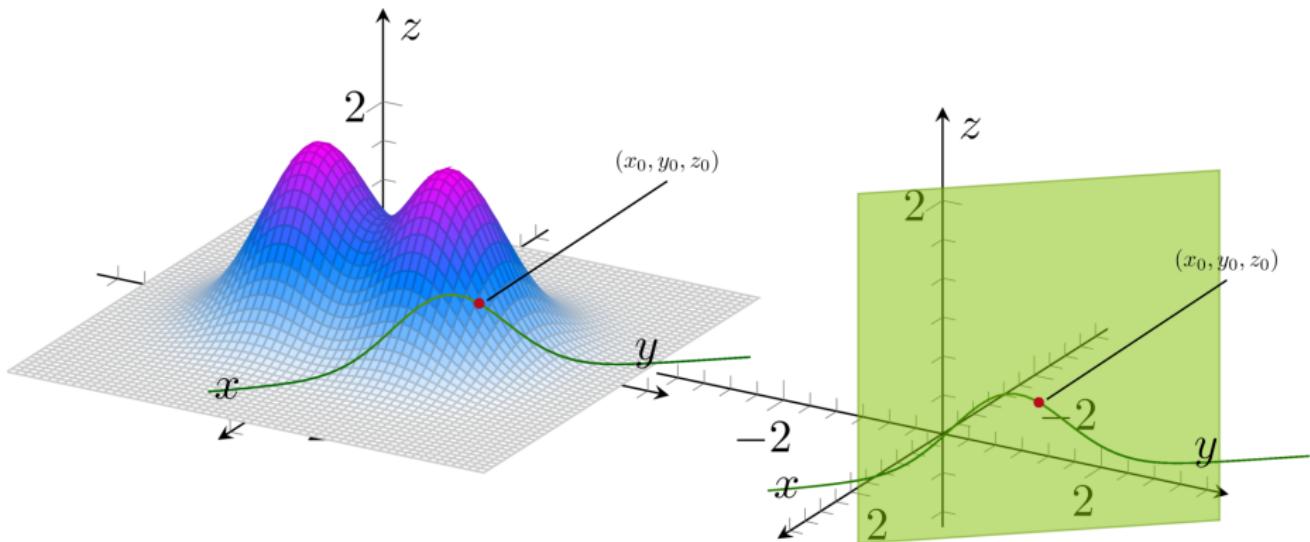
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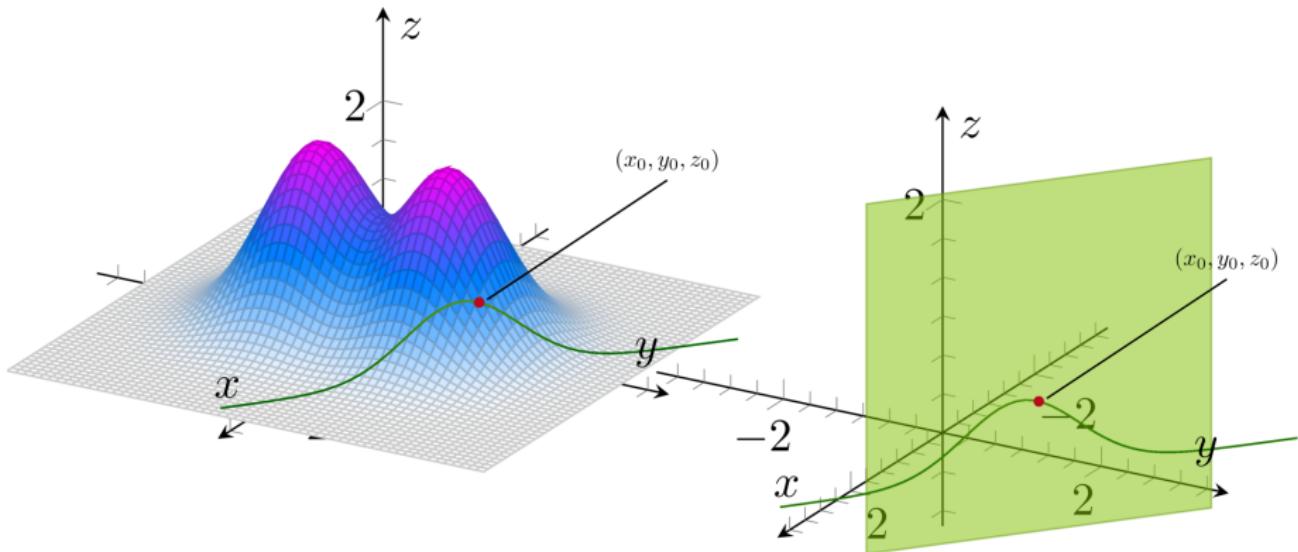
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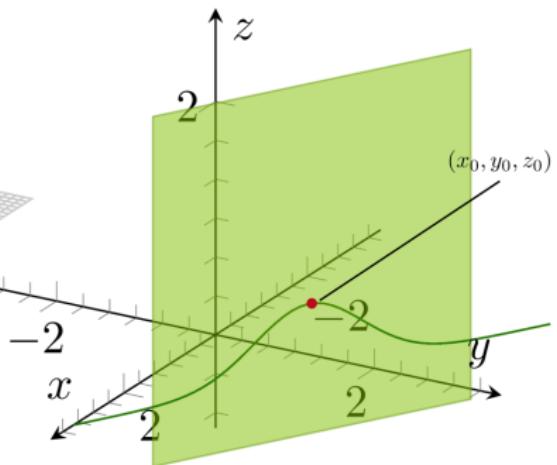
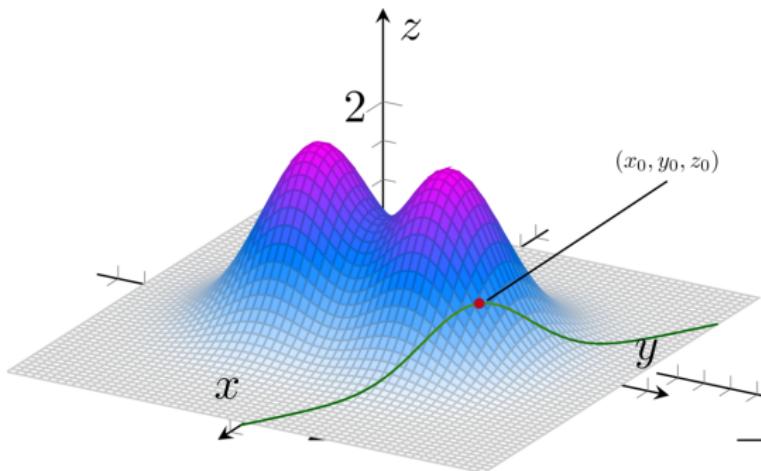
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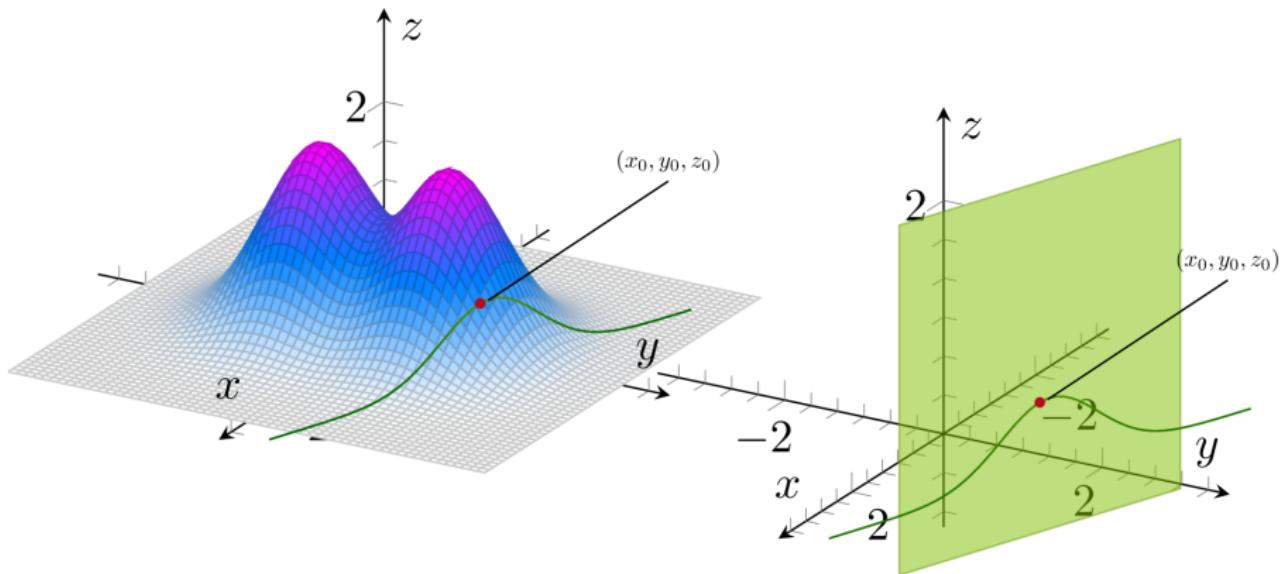
## Directional Derivatives



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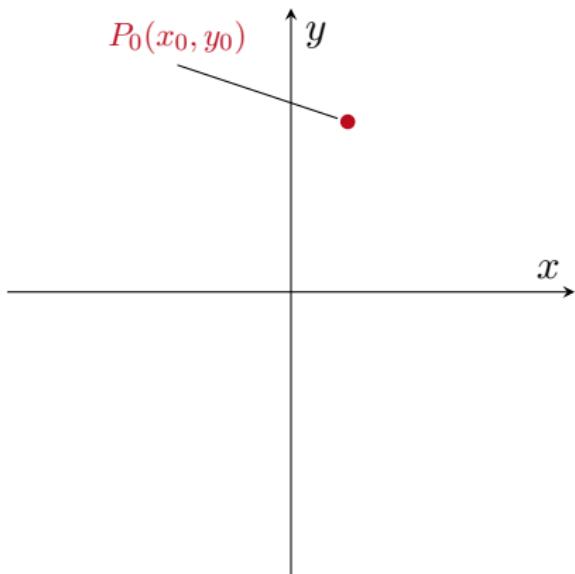
## Directional Derivatives



## 13.5 Directional Derivatives and Gradient Vector



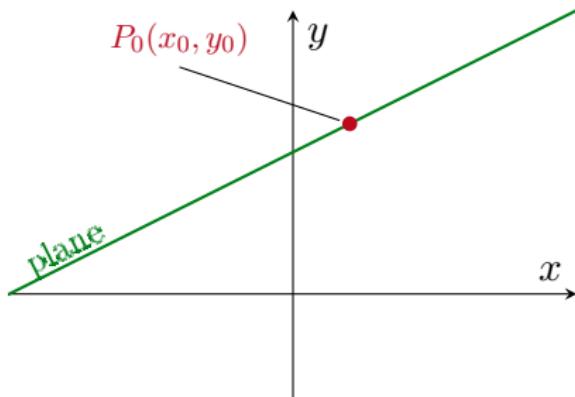
top view



## 13.5 Directional Derivatives and Gradient Vector



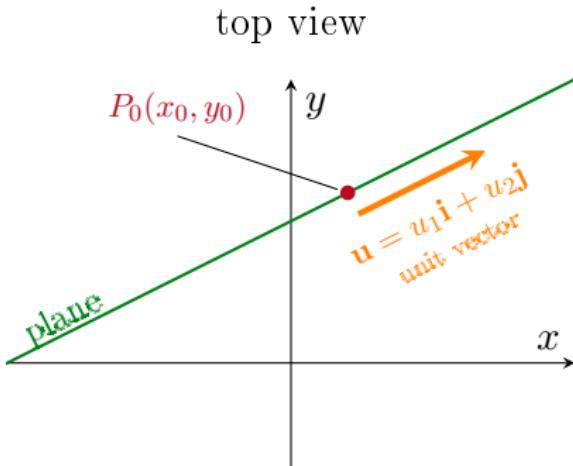
top view



Definition

The *derivative of  $f$  at  $P_0(x_0, y_0)$*

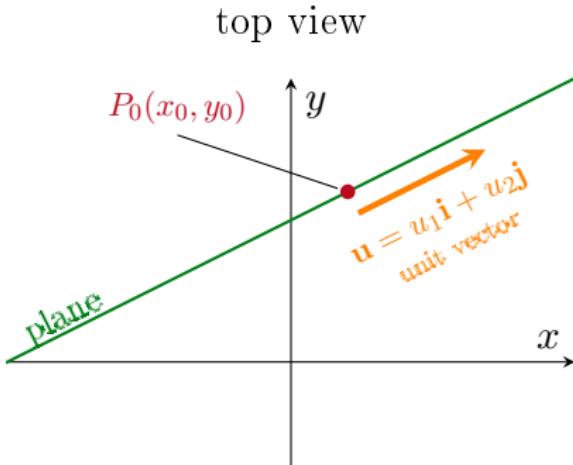
## 13.5 Directional Derivatives and Gradient Vector



### Definition

The derivative of  $f$  at  $P_0(x_0, y_0)$  in the direction of the unit vector  $\mathbf{u} = u_1\mathbf{i} + u_2\mathbf{j}$

## 13.5 Directional Derivatives and Gradient Vector



### Definition

The derivative of  $f$  at  $P_0(x_0, y_0)$  in the direction of the unit vector  $\mathbf{u} = u_1 \mathbf{i} + u_2 \mathbf{j}$  is

$$D_{\mathbf{u}} f(P_0) = \lim_{s \rightarrow 0} \frac{f(x_0 + su_1, y_0 + su_2) - f(x_0, y_0)}{s}$$

## 13.5 Directional Derivatives and Gradient Vector



$$D_{\mathbf{u}} f(P_0) = \left( \frac{df}{ds} \right)_{\mathbf{u}, P_0}$$

**EXAMPLE 1** Using the definition, find the derivative of

$$f(x, y) = x^2 + xy$$

at  $P_0(1, 2)$  in the direction of the unit vector  $\mathbf{u} = (1/\sqrt{2})\mathbf{i} + (1/\sqrt{2})\mathbf{j}$ .

**Solution** Applying the definition in Equation (1), we obtain

$$\begin{aligned}\left(\frac{df}{ds}\right)_{\mathbf{u}, P_0} &= \lim_{s \rightarrow 0} \frac{f(x_0 + su_1, y_0 + su_2) - f(x_0, y_0)}{s} \quad \text{Eq. (1)} \\ &= \lim_{s \rightarrow 0} \frac{f\left(1 + s \cdot \frac{1}{\sqrt{2}}, 2 + s \cdot \frac{1}{\sqrt{2}}\right) - f(1, 2)}{s} \\ &= \lim_{s \rightarrow 0} \frac{\left(1 + \frac{s}{\sqrt{2}}\right)^2 + \left(1 + \frac{s}{\sqrt{2}}\right)\left(2 + \frac{s}{\sqrt{2}}\right) - (1^2 + 1 \cdot 2)}{s} \\ &= \lim_{s \rightarrow 0} \frac{\left(1 + \frac{2s}{\sqrt{2}} + \frac{s^2}{2}\right) + \left(2 + \frac{3s}{\sqrt{2}} + \frac{s^2}{2}\right) - 3}{s} \\ &= \lim_{s \rightarrow 0} \frac{\frac{5s}{\sqrt{2}} + s^2}{s} = \lim_{s \rightarrow 0} \left(\frac{5}{\sqrt{2}} + s\right) = \frac{5}{\sqrt{2}}.\end{aligned}$$

The rate of change of  $f(x, y) = x^2 + xy$  at  $P_0(1, 2)$  in the direction  $\mathbf{u}$  is  $5/\sqrt{2}$ .

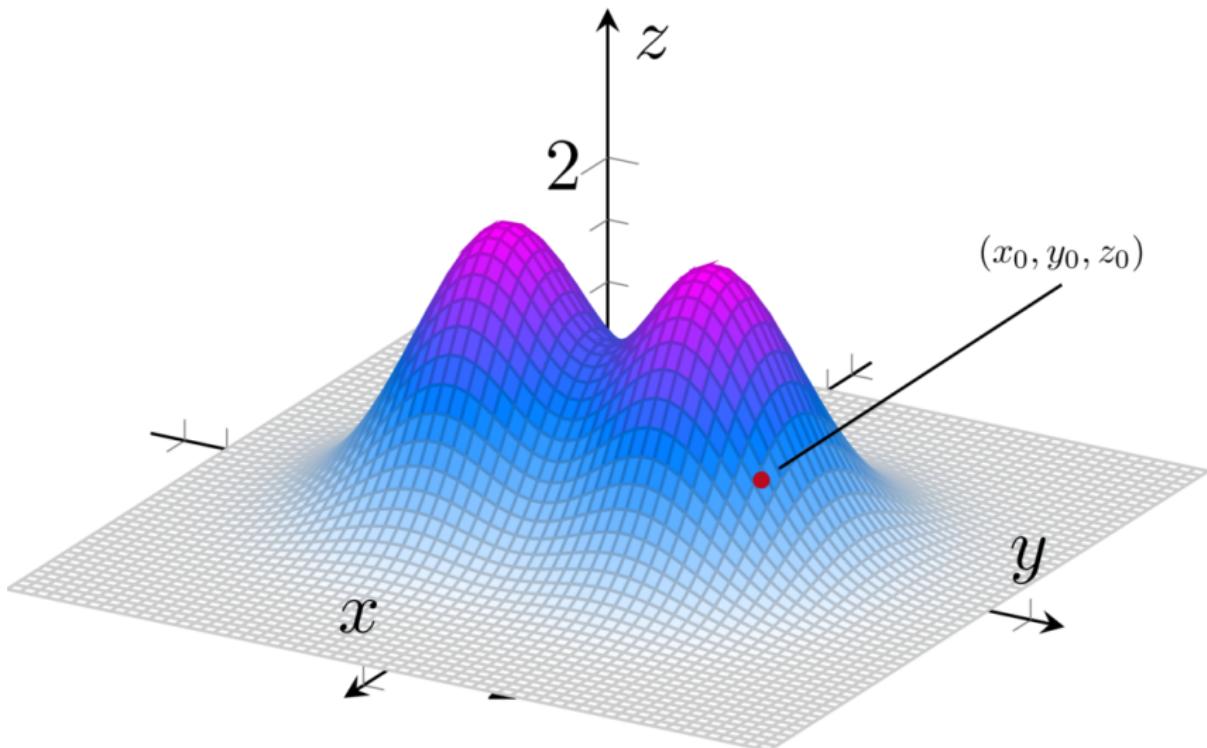
## 13.5 Directional Derivatives and Gradient Vector



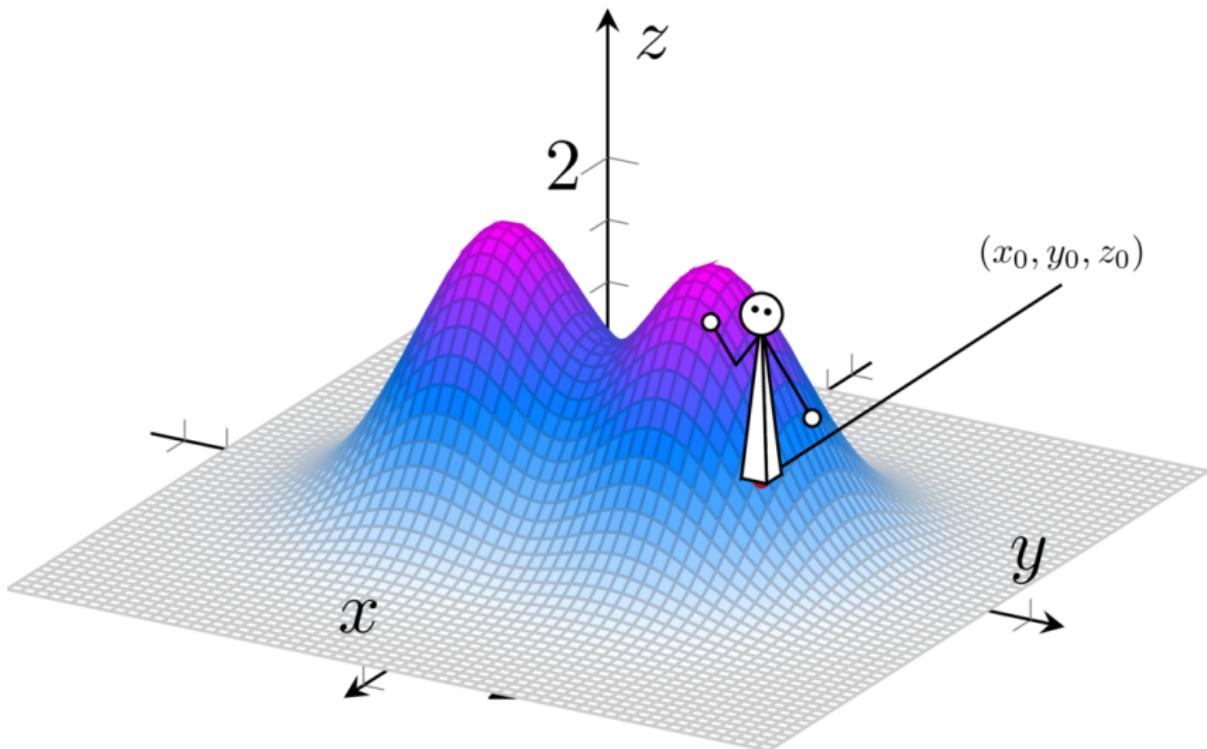
### Remark

But it is easier to calculate directional derivatives if we use gradient vectors.

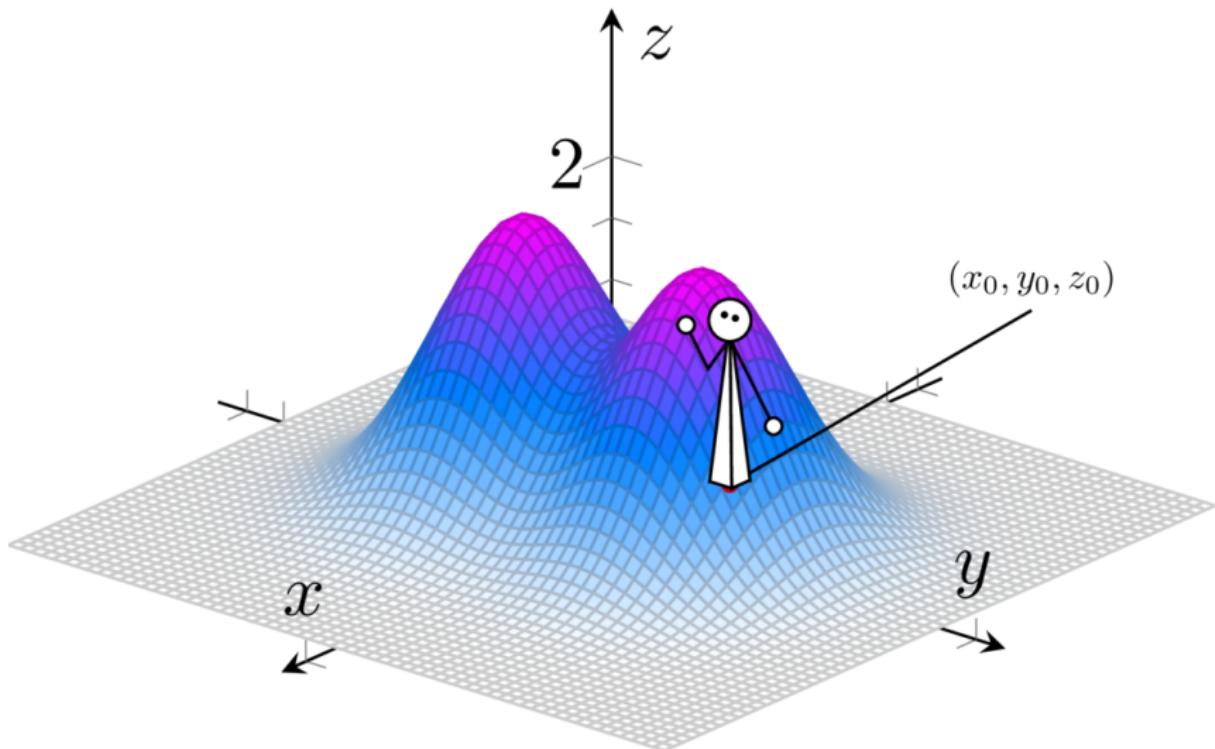
## What is a Gradient Vector?



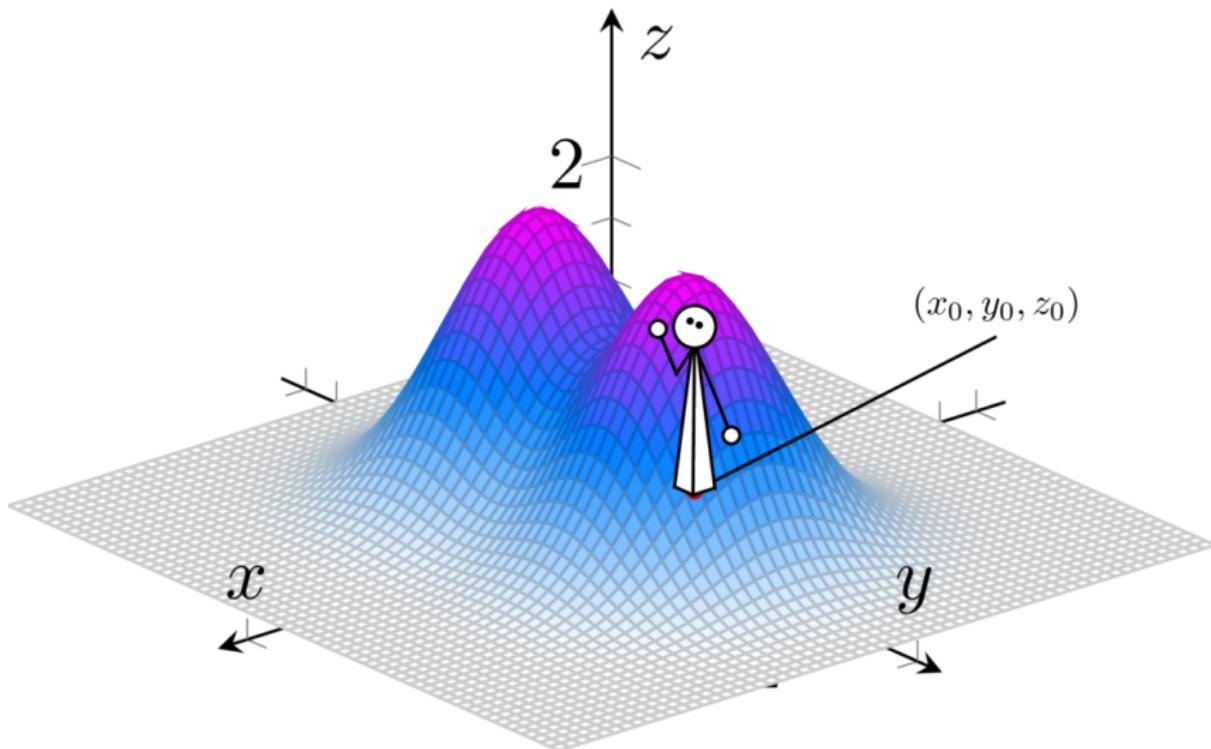
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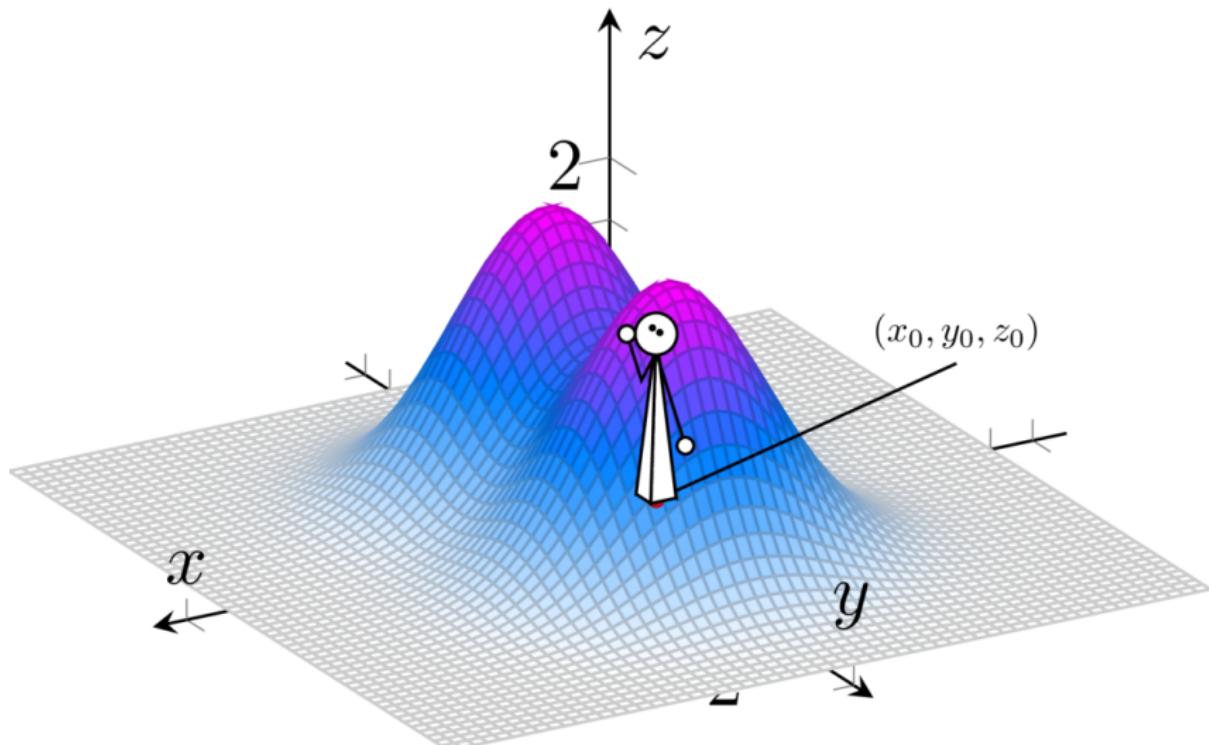
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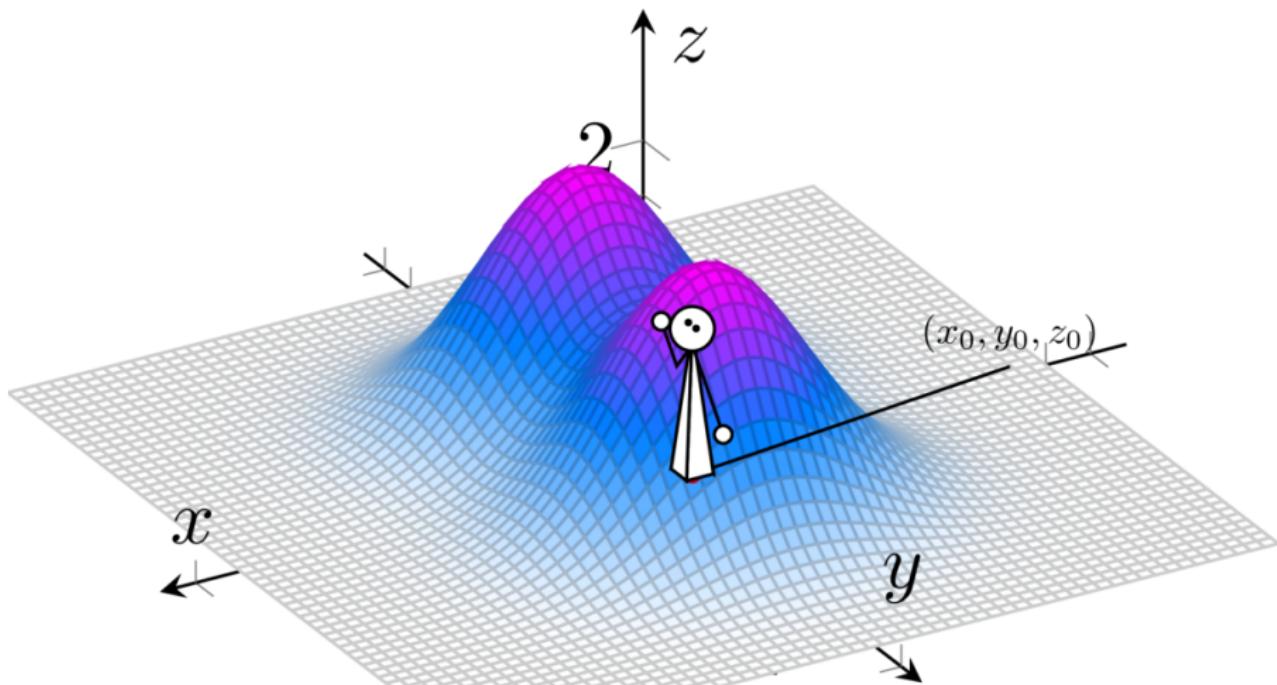
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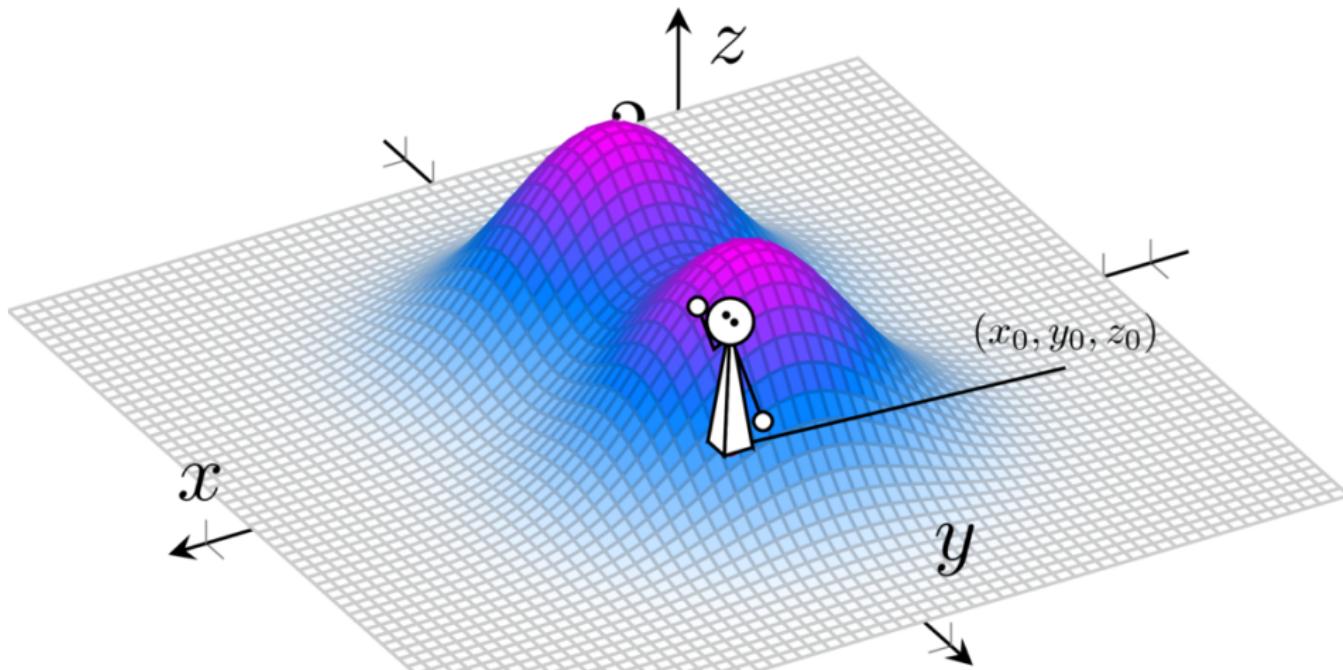
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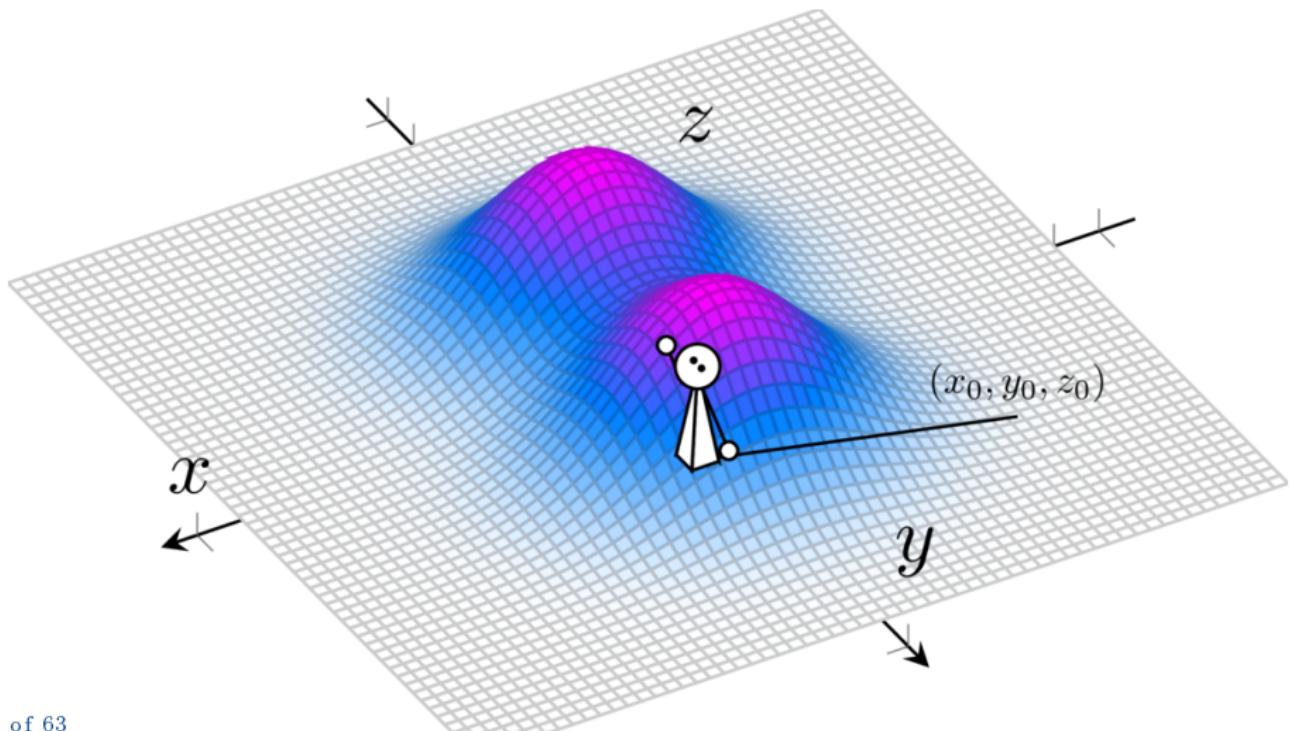
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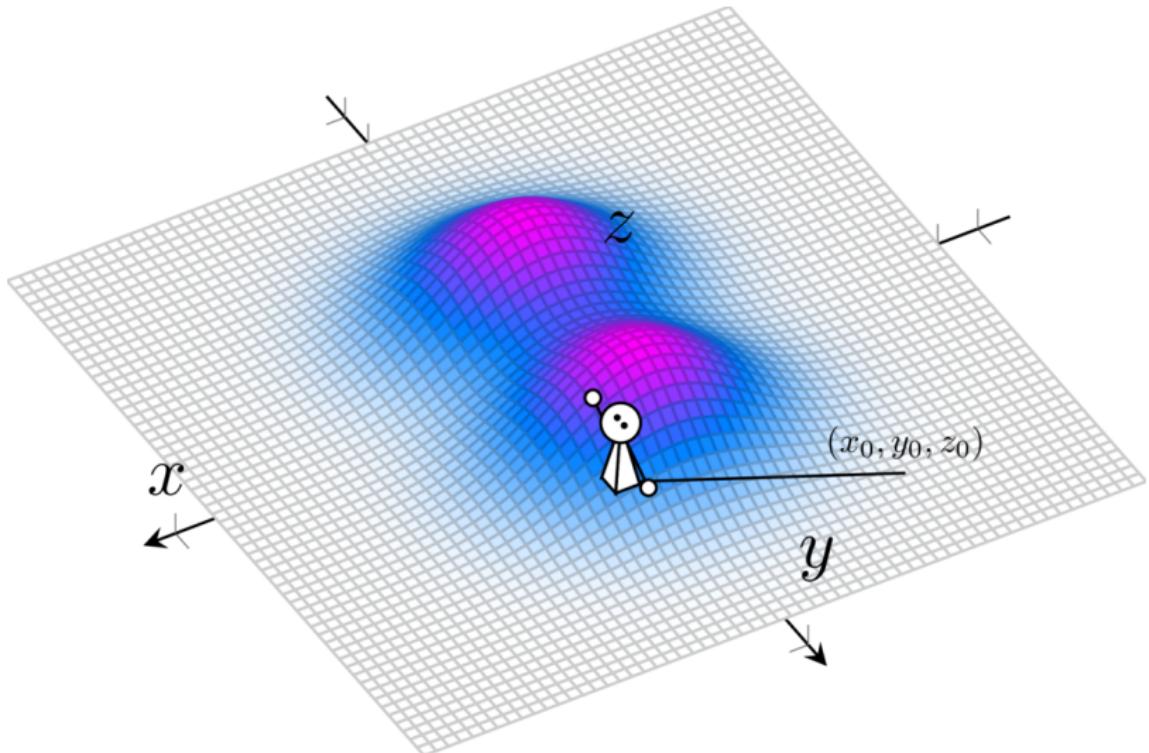
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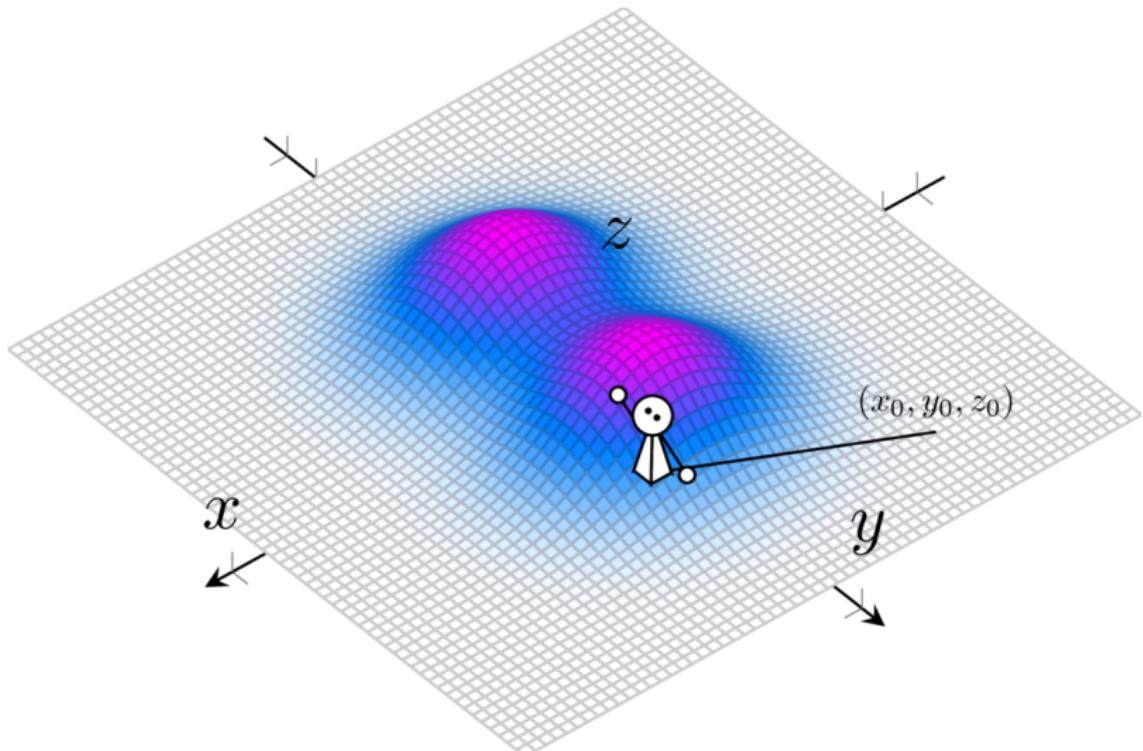
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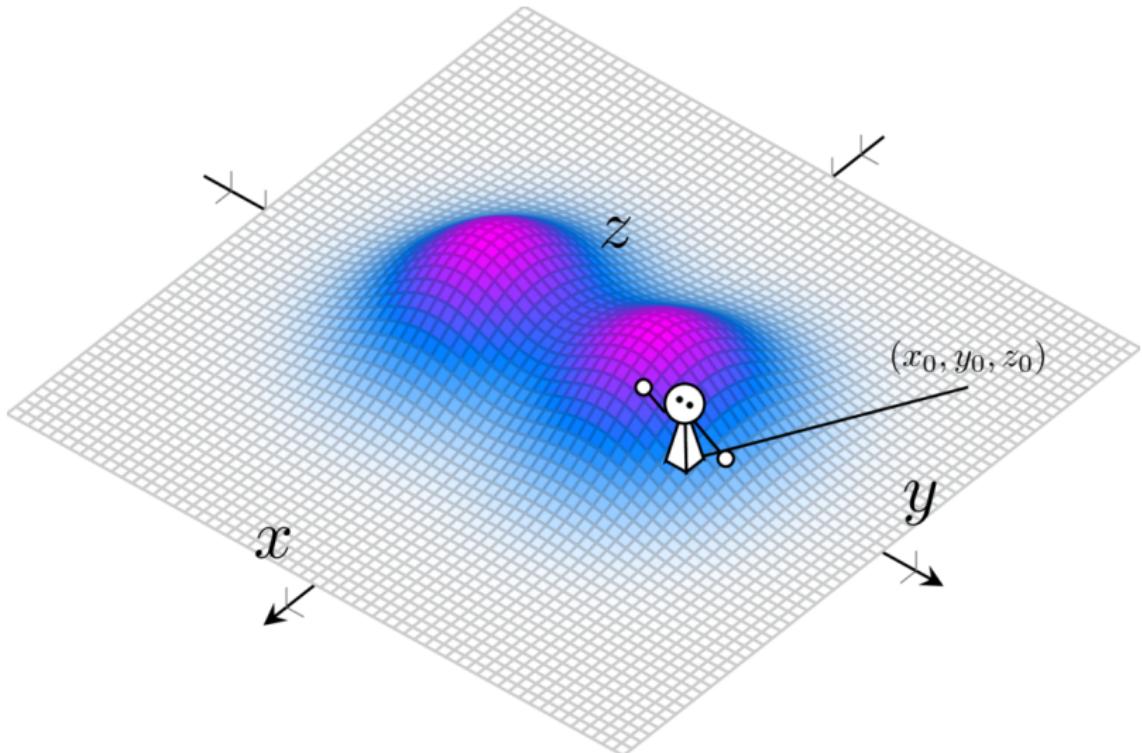
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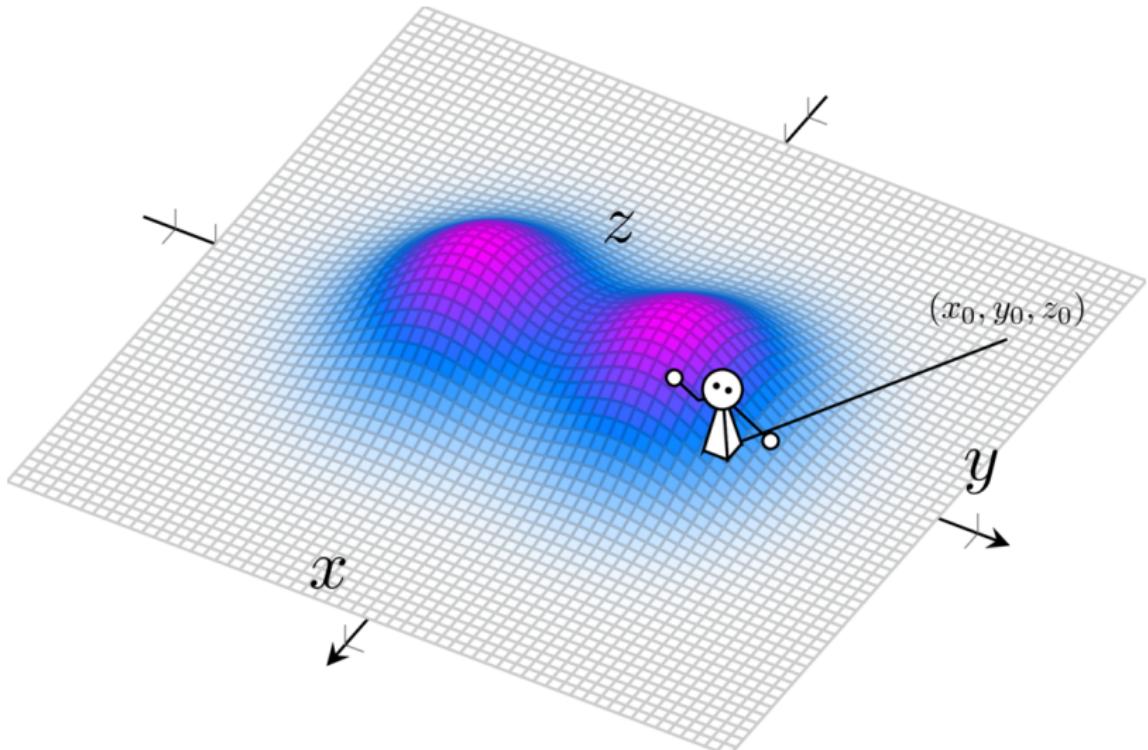
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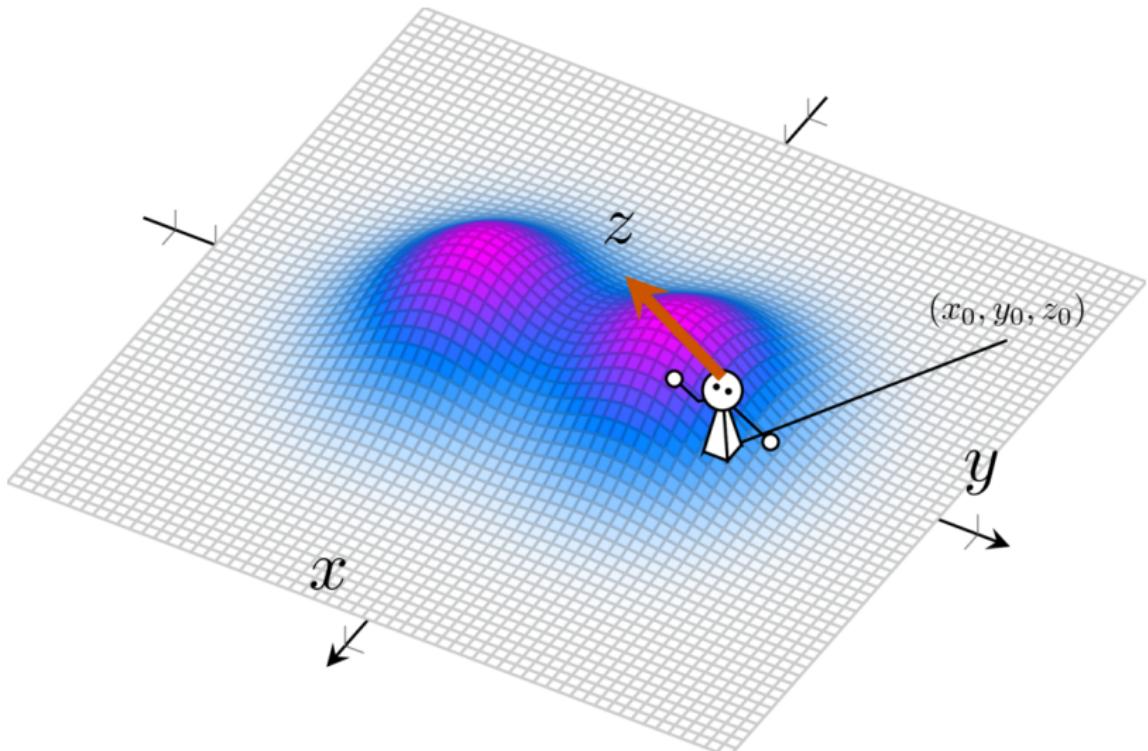
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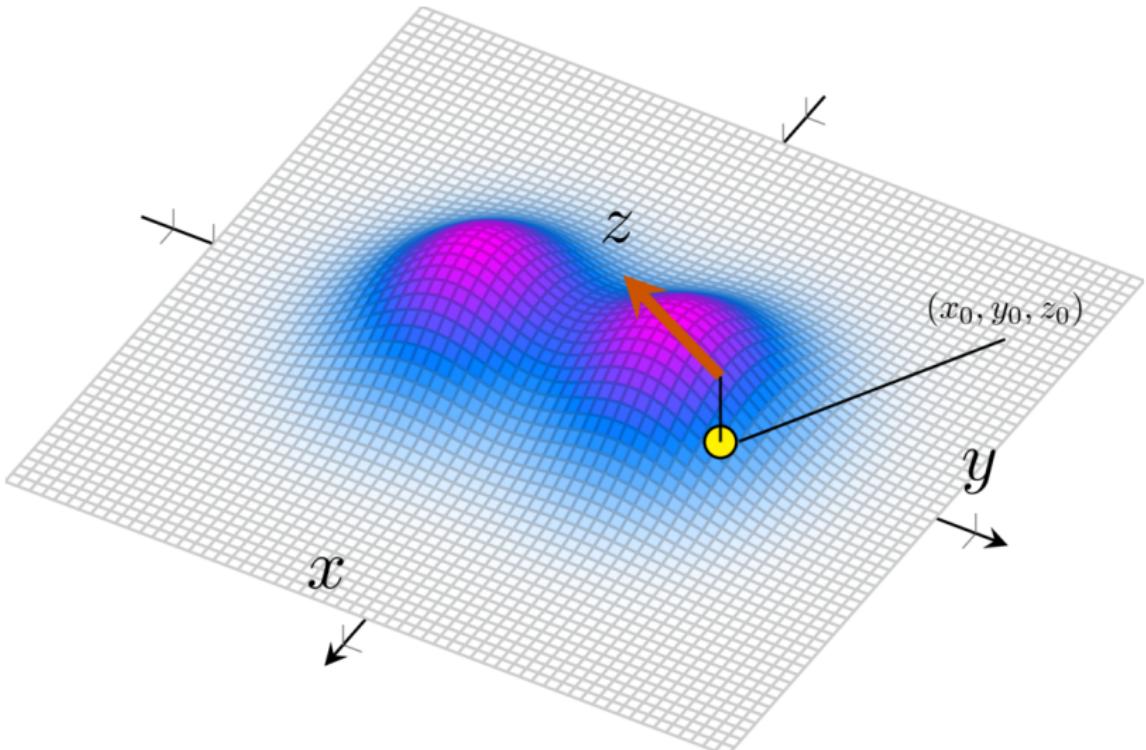
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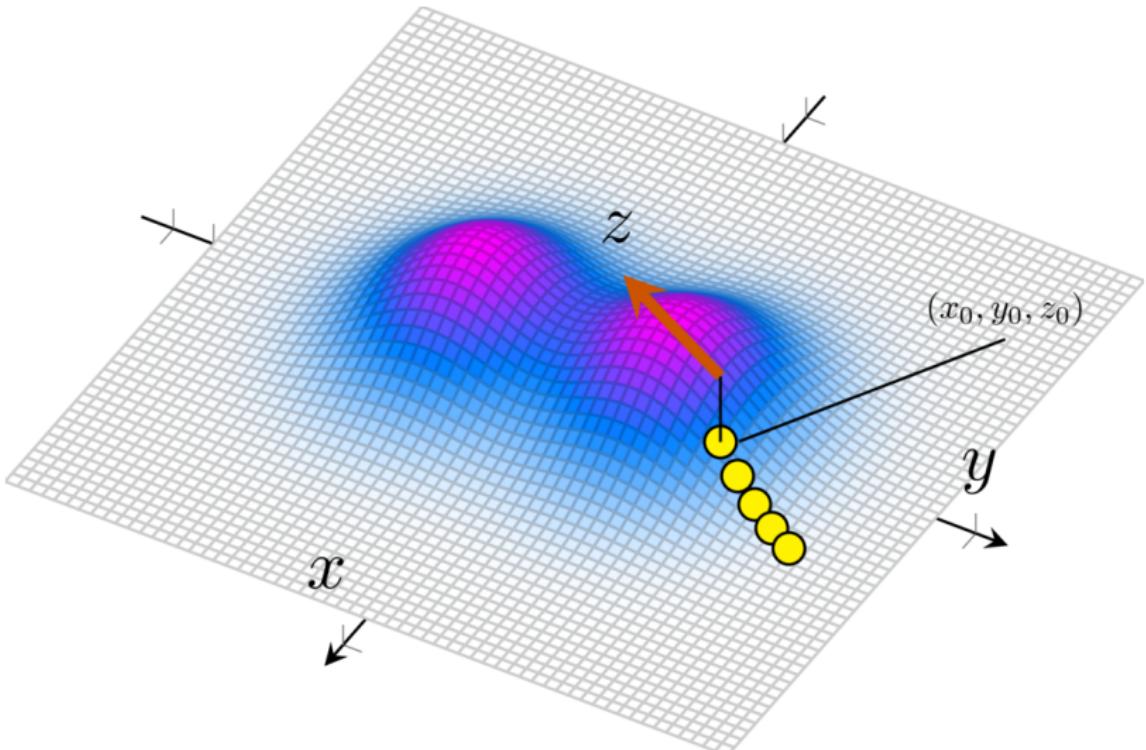
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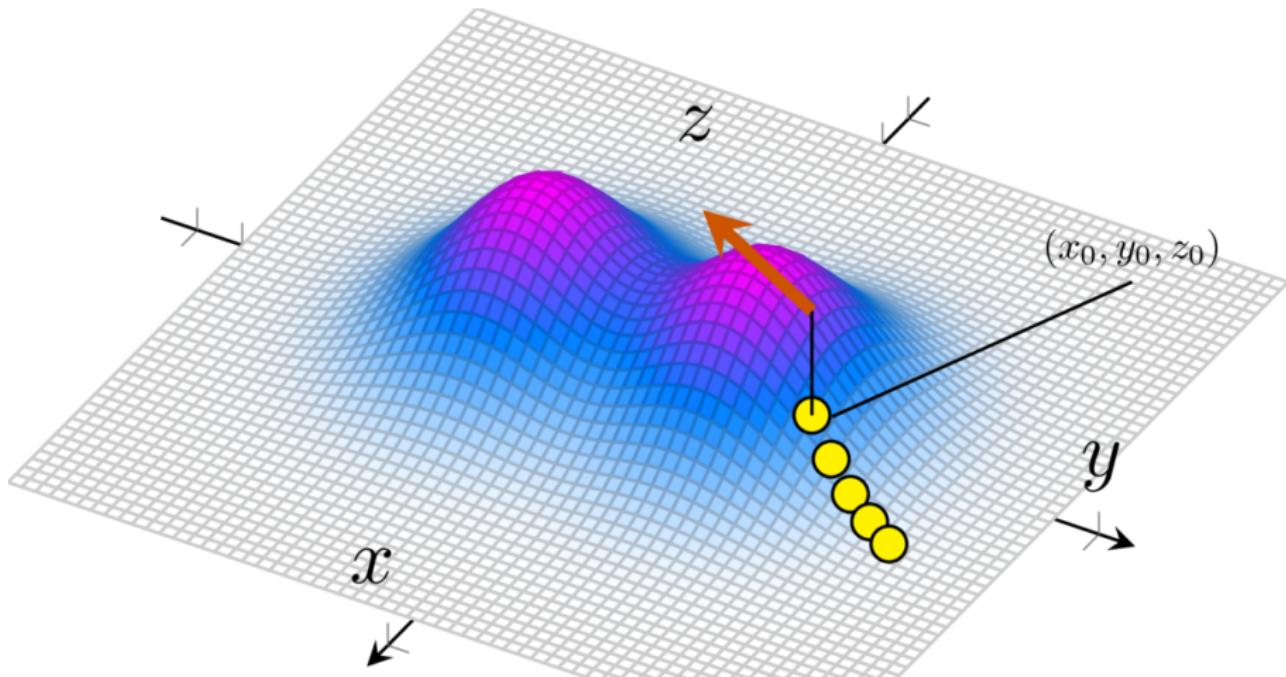
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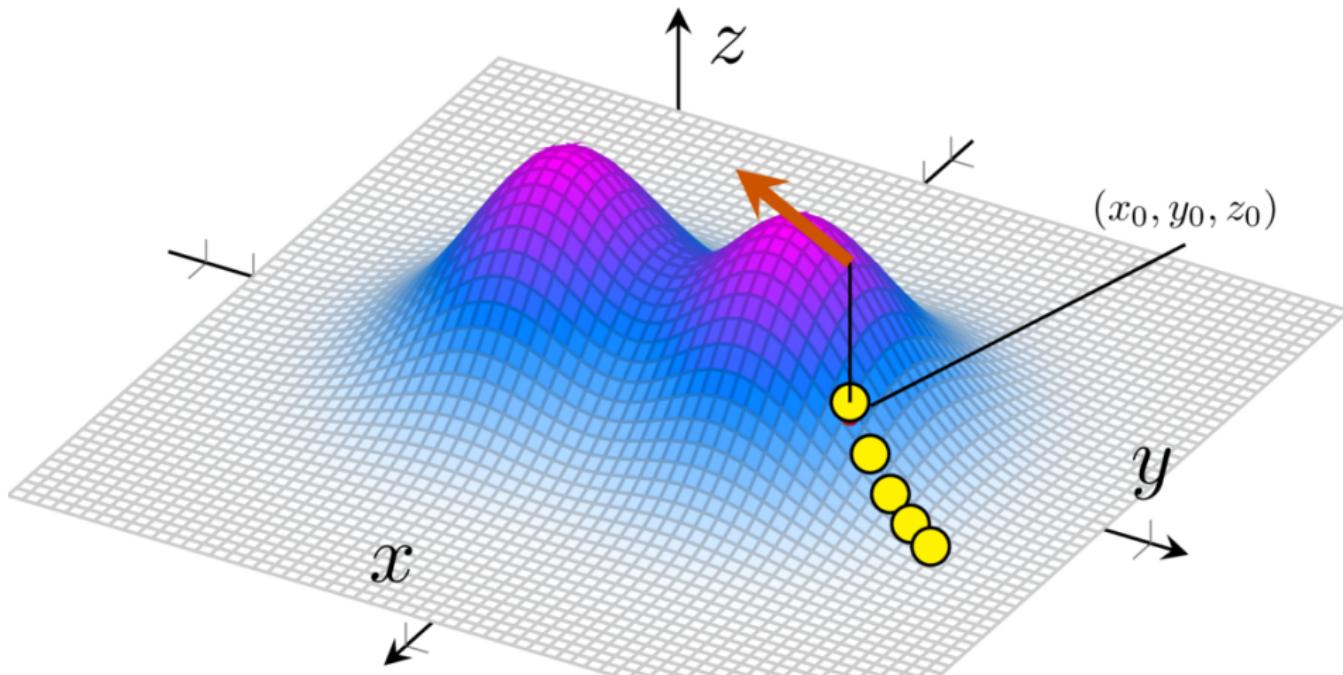
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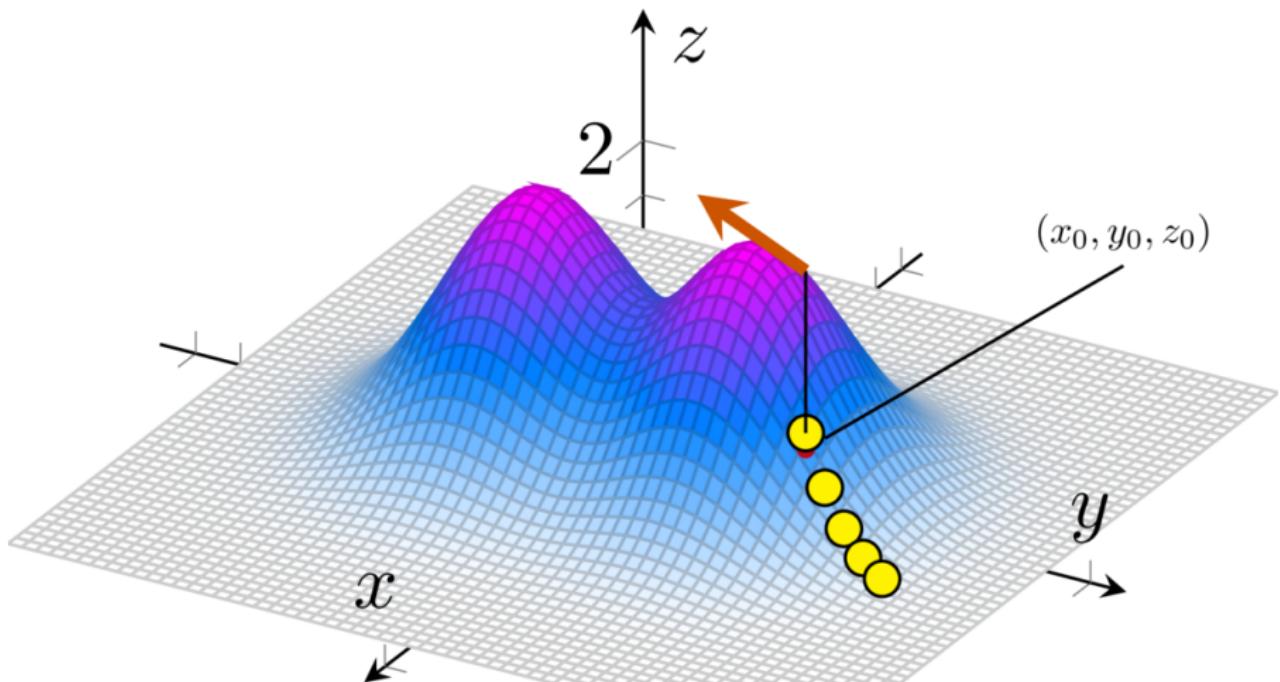
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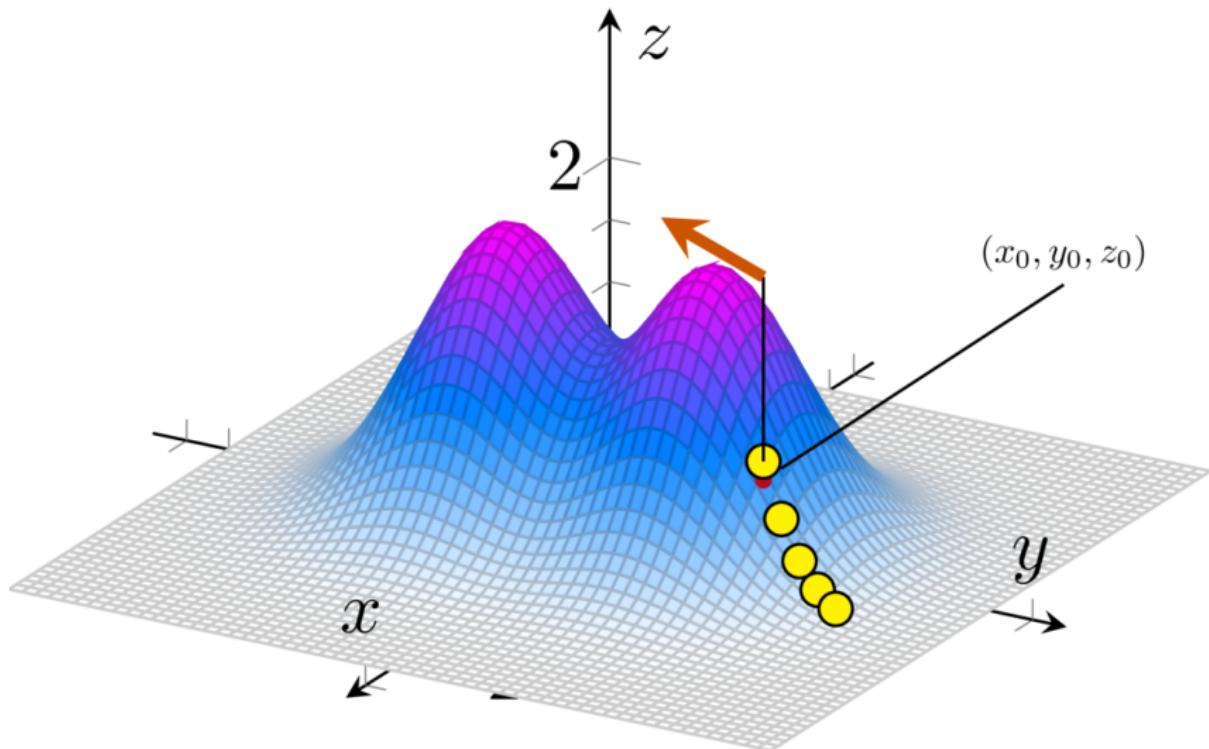
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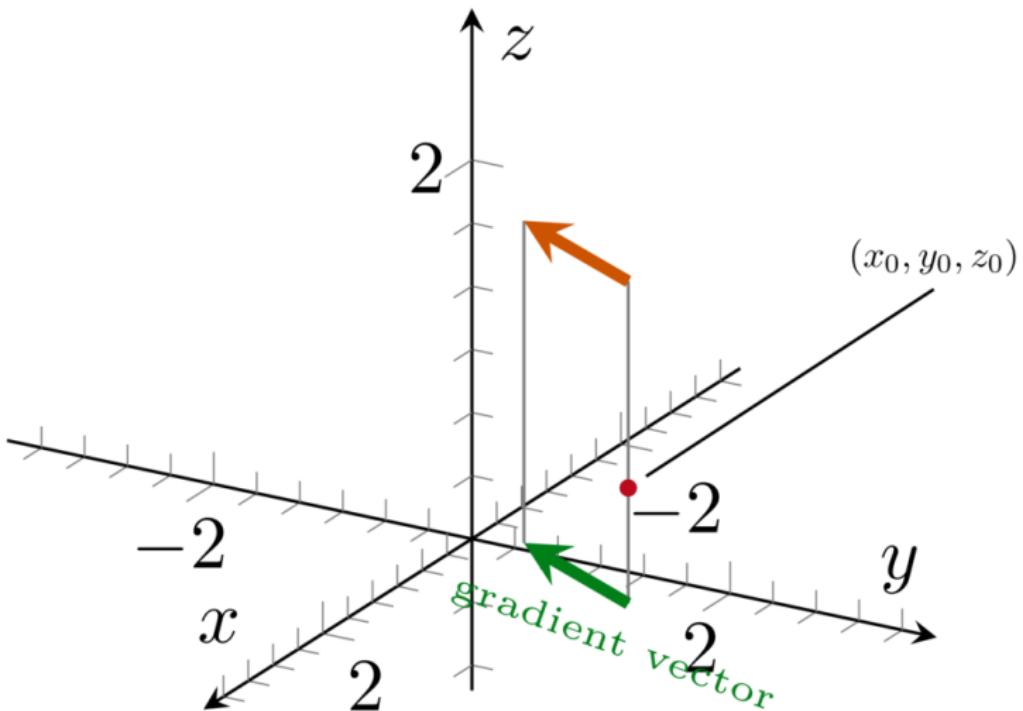
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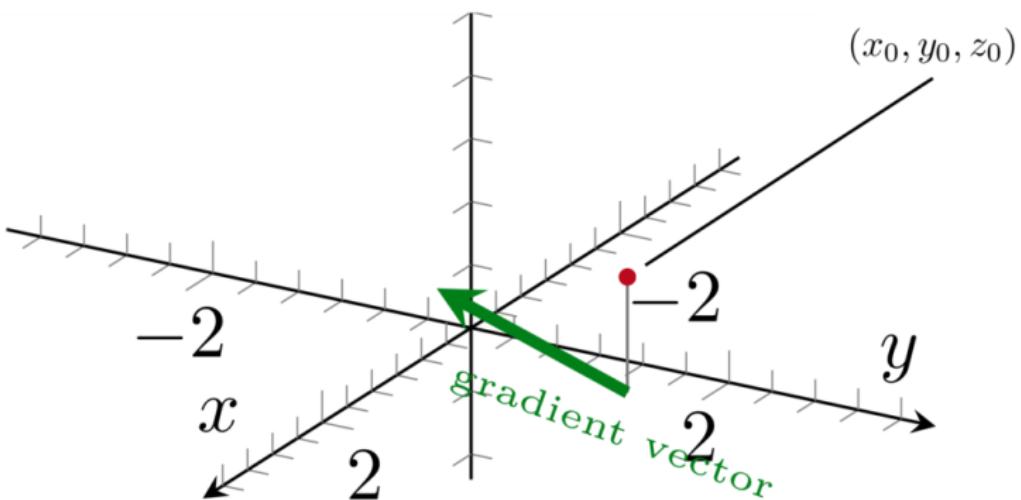
## What is a Gradient Vector?



## What is a Gradient Vector?

↑  
 $z$

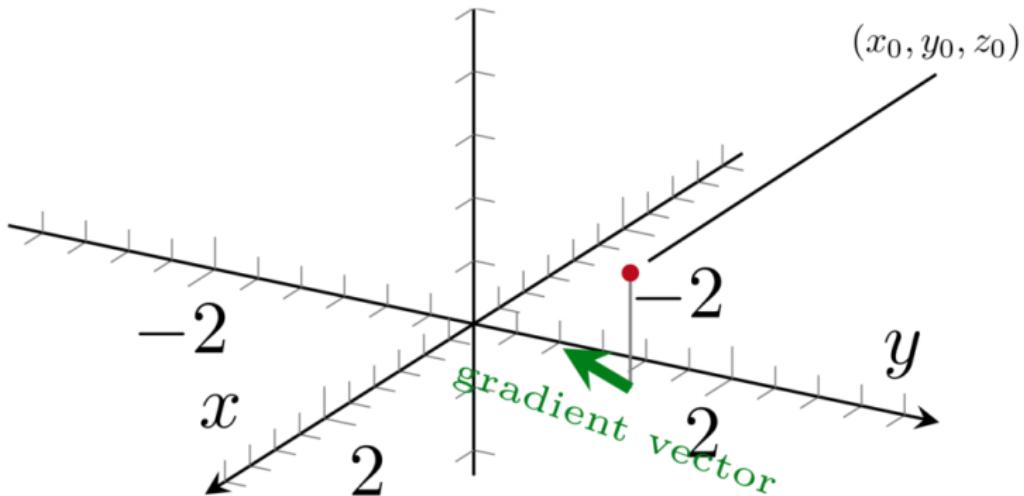
steep slope=long arrow



## What is a Gradient Vector?



shallow slope=short arrow



## 13.5 Directional Derivatives and Gradient Vector



### Definition

The *gradient vector* of  $f(x, y)$  is the vector

$$\nabla f = \frac{\partial f}{\partial x} \mathbf{i} + \frac{\partial f}{\partial y} \mathbf{j}.$$

$\nabla$  is pronounced “nabla” or “del”.



Harps, p. 984.

$$\nabla f = \frac{\partial f}{\partial x} \mathbf{i} + \frac{\partial f}{\partial y} \mathbf{j}$$

## Example

Find the gradient vector of  $f(x, y) = xe^y + \cos(xy)$  at the point  $(2, 0)$ .

$$\nabla f = \frac{\partial f}{\partial x} \mathbf{i} + \frac{\partial f}{\partial y} \mathbf{j}$$

## Example

Find the gradient vector of  $f(x, y) = xe^y + \cos(xy)$  at the point  $(2, 0)$ .

We calculate that

$$f_x(2, 0) =$$

$$f_y(2, 0) =$$

and

$$\nabla f \Big|_{(2,0)} = f_x(2, 0) \mathbf{i} + f_y(2, 0) \mathbf{j} = \quad .$$

$$\nabla f = \frac{\partial f}{\partial x} \mathbf{i} + \frac{\partial f}{\partial y} \mathbf{j}$$

## Example

Find the gradient vector of  $f(x, y) = xe^y + \cos(xy)$  at the point  $(2, 0)$ .

We calculate that

$$f_x(2, 0) = e^y - y \sin(xy) \Big|_{(2,0)} = e^0 - 0 = 1,$$

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and

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and

$$\nabla f \Big|_{(2,0)} = f_x(2, 0) \mathbf{i} + f_y(2, 0) \mathbf{j} = \mathbf{i} + 2\mathbf{j}.$$

## 13.5 Directional Derivatives and Gradient Vector



So how can we use gradient vectors to find directional derivatives?

## 13.5 Directional Derivatives and Gradient Vector



So how can we use gradient vectors to find directional derivatives?

Theorem

$$D_{\mathbf{u}} f = \nabla f \cdot \mathbf{u}.$$

$$D_{\mathbf{u}} f = \nabla f \cdot \mathbf{u} \quad \nabla f = \frac{\partial f}{\partial x} \mathbf{i} + \frac{\partial f}{\partial y} \mathbf{j}$$

## Example

Find the derivative of  $f(x, y) = xe^y + \cos(xy)$  at the point  $(2, 0)$  in the direction of  $\mathbf{v} = 3\mathbf{i} - 4\mathbf{j}$ .

13.5

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### Example

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Recall that  $\nabla f|_{(2,0)} = \mathbf{i} + 2\mathbf{j}$ .

$$D_{\mathbf{u}}f = \nabla f \cdot \mathbf{u} \quad \nabla f = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j}$$

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Recall that  $\nabla f|_{(2,0)} = \mathbf{i} + 2\mathbf{j}$ . We need to find a unit vector  $\mathbf{u}$  which points in the same direction as  $\mathbf{v}$ ,

$$D_{\mathbf{u}}f = \nabla f \cdot \mathbf{u} \quad \nabla f = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j}$$

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Recall that  $\nabla f|_{(2,0)} = \mathbf{i} + 2\mathbf{j}$ . We need to find a unit vector  $\mathbf{u}$  which points in the same direction as  $\mathbf{v}$ , so we calculate that

$$\mathbf{u} = \frac{\mathbf{v}}{\|\mathbf{v}\|} = \frac{3\mathbf{i} - 4\mathbf{j}}{\sqrt{3^2 + (-4)^2}} = \frac{3}{5}\mathbf{i} - \frac{4}{5}\mathbf{j}.$$

$$D_{\mathbf{u}}f = \nabla f \cdot \mathbf{u} \quad \nabla f = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j}$$

## Example

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Therefore

$$D_{\mathbf{u}}f(2, 0) = \nabla f|_{(2,0)} \cdot \mathbf{u} =$$

$$D_{\mathbf{u}}f = \nabla f \cdot \mathbf{u} \quad \nabla f = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j}$$

## Example

Find the derivative of  $f(x, y) = xe^y + \cos(xy)$  at the point  $(2, 0)$  in the direction of  $\mathbf{v} = 3\mathbf{i} - 4\mathbf{j}$ .

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Therefore

$$D_{\mathbf{u}}f(2, 0) = \nabla f|_{(2,0)} \cdot \mathbf{u} = (\mathbf{i} + 2\mathbf{j}) \cdot \left(\frac{3}{5}\mathbf{i} - \frac{4}{5}\mathbf{j}\right) = \frac{3}{5} - \frac{8}{5} = -1.$$

$$\mathbf{a} \cdot \mathbf{b} = \|\mathbf{a}\| \|\mathbf{b}\| \cos \theta$$

Note that

$$D_{\mathbf{u}} f = \nabla f \cdot \mathbf{u} = \|\nabla f\| \|\mathbf{u}\| \cos \theta = \|\nabla f\| \cos \theta$$

since  $\|\mathbf{u}\| = 1$ .

$$\mathbf{a} \cdot \mathbf{b} = \|\mathbf{a}\| \|\mathbf{b}\| \cos \theta$$

Note that

$$D_{\mathbf{u}} f = \nabla f \cdot \mathbf{u} = \|\nabla f\| \|\mathbf{u}\| \cos \theta = \|\nabla f\| \cos \theta$$

since  $\|\mathbf{u}\| = 1$ .

So we must always have

$$-\|\nabla f\| \leq D_{\mathbf{u}} f \leq \|\nabla f\|.$$

$$D_{\mathbf{u}} f = \|\nabla f\| \cos \theta$$

### Remark

$f$  increases  
mostly rapidly

$$\implies \cos \theta = 1 \implies \theta = 0$$

$\mathbf{u}$  points in the  
same direction  
as  $\nabla f$

$\nabla f$  points ‘uphill’

13.5

$$D_{\mathbf{u}} f = \|\nabla f\| \cos \theta$$



## Remark

$f$  increases  
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$\nabla f$  points ‘uphill’

## Remark

$f$  decreases  
mostly rapidly  $\implies \cos \theta = -1 \implies \theta = 180^\circ \implies \mathbf{u}$  points in  
the opposite  
direction from  
 $\nabla f$

a ball on a hill rolls in the direction  $-\nabla f$

$$D_{\mathbf{u}} f = \|\nabla f\| \cos \theta$$

### Remark

$f$  increases mostly rapidly  $\implies \cos \theta = 1 \implies \theta = 0 \implies \mathbf{u}$  points in the same direction as  $\nabla f$

$\nabla f$  points ‘uphill’

### Remark

$f$  decreases mostly rapidly  $\implies \cos \theta = -1 \implies \theta = 180^\circ \implies \mathbf{u}$  points in the opposite direction from  $\nabla f$

a ball on a hill rolls in the direction  $-\nabla f$

### Remark

$$\theta = 90^\circ \implies D_{\mathbf{u}} f = 0.$$

**EXAMPLE 3** Find the directions in which  $f(x, y) = (x^2/2) + (y^2/2)$

- (a) increases most rapidly at the point  $(1, 1)$ .
- (b) decreases most rapidly at  $(1, 1)$ .
- (c) What are the directions of zero change in  $f$  at  $(1, 1)$ ?

**Solution**

- (a) The function increases most rapidly in the direction of  $\nabla f$  at  $(1, 1)$ . The gradient there is

$$(\nabla f)_{(1,1)} = (x\mathbf{i} + y\mathbf{j})_{(1,1)} = \mathbf{i} + \mathbf{j}.$$

Its direction is

$$\mathbf{u} = \frac{\mathbf{i} + \mathbf{j}}{|\mathbf{i} + \mathbf{j}|} = \frac{\mathbf{i} + \mathbf{j}}{\sqrt{(1)^2 + (1)^2}} = \frac{1}{\sqrt{2}}\mathbf{i} + \frac{1}{\sqrt{2}}\mathbf{j}.$$

**(b)** The function decreases most rapidly in the direction of  $-\nabla f$  at  $(1, 1)$ , which is

$$-\mathbf{u} = -\frac{1}{\sqrt{2}}\mathbf{i} - \frac{1}{\sqrt{2}}\mathbf{j}.$$

**(c)** The directions of zero change at  $(1, 1)$  are the directions orthogonal to  $\nabla f$ :

$$\mathbf{n} = -\frac{1}{\sqrt{2}}\mathbf{i} + \frac{1}{\sqrt{2}}\mathbf{j} \quad \text{and} \quad -\mathbf{n} = \frac{1}{\sqrt{2}}\mathbf{i} - \frac{1}{\sqrt{2}}\mathbf{j}.$$

$$D_{\mathbf{u}} f = \|\nabla f\| \cos \theta$$

## Algebra Rules for $\nabla$

### Theorem

- 1 *Sum Rule:*  $\nabla(f + g) = \nabla f + \nabla g$
- 2 *Difference Rule:*  $\nabla(f - g) = \nabla f - \nabla g$
- 3 *Constant Multiple Rule:*  $\nabla(kf) = k\nabla f$       (for  $k \in \mathbb{R}$ )
- 4 *Product Rule:*  $\nabla(fg) = g\nabla f + f\nabla g$
- 5 *Quotient Rule:*  $\nabla\left(\frac{f}{g}\right) = \frac{g\nabla f - f\nabla g}{g^2}.$

**EXAMPLE 5**

We illustrate two of the rules with

$$\begin{aligned}f(x, y) &= x - y & g(x, y) &= 3y \\ \nabla f &= \mathbf{i} - \mathbf{j} & \nabla g &= 3\mathbf{j}.\end{aligned}$$

We have

1.  $\nabla(f - g) = \nabla(x - 4y) = \mathbf{i} - 4\mathbf{j} = \nabla f - \nabla g$  Rule 2
2.  $\nabla(fg) = \nabla(3xy - 3y^2) = 3y\mathbf{i} + (3x - 6y)\mathbf{j}$

and

$$\begin{aligned}f\nabla g + g\nabla f &= (x - y)3\mathbf{j} + 3y(\mathbf{i} - \mathbf{j}) && \text{Substitute.} \\ &= 3y\mathbf{i} + (3x - 6y)\mathbf{j}. && \text{Simplify.}\end{aligned}$$

We have therefore verified for this example that  $\nabla(fg) = f\nabla g + g\nabla f$ .

$$D_{\mathbf{u}} f = \|\nabla f\| \cos \theta$$



## Functions of Three Variables

$$\nabla f = \frac{\partial f}{\partial x} \mathbf{i} + \frac{\partial f}{\partial y} \mathbf{j} + \frac{\partial f}{\partial z} \mathbf{k}$$

$$D_{\mathbf{u}} f = \nabla f \cdot \mathbf{u}.$$

## EXAMPLE 6

- (a) Find the derivative of  $f(x, y, z) = x^3 - xy^2 - z$  at  $P_0(1, 1, 0)$  in the direction of  $\mathbf{v} = 2\mathbf{i} - 3\mathbf{j} + 6\mathbf{k}$ .
- (b) In what directions does  $f$  change most rapidly at  $P_0$ , and what are the rates of change in these directions?

### Solution

- (a) The direction of  $\mathbf{v}$  is obtained by dividing  $\mathbf{v}$  by its length:

$$|\mathbf{v}| = \sqrt{(2)^2 + (-3)^2 + (6)^2} = \sqrt{49} = 7$$

$$\mathbf{u} = \frac{\mathbf{v}}{|\mathbf{v}|} = \frac{2}{7}\mathbf{i} - \frac{3}{7}\mathbf{j} + \frac{6}{7}\mathbf{k}.$$

The partial derivatives of  $f$  at  $P_0$  are

$$f_x = (3x^2 - y^2) \Big|_{(1, 1, 0)} = 2, \quad f_y = -2xy \Big|_{(1, 1, 0)} = -2, \quad f_z = -1 \Big|_{(1, 1, 0)} = -1.$$

The gradient of  $f$  at  $P_0$  is

$$\nabla f \Big|_{(1, 1, 0)} = 2\mathbf{i} - 2\mathbf{j} - \mathbf{k}.$$

The derivative of  $f$  at  $P_0$  in the direction of  $\mathbf{v}$  is therefore

$$\begin{aligned}D_{\mathbf{u}}f|_{(1,1,0)} &= \nabla f|_{(1,1,0)} \cdot \mathbf{u} = (2\mathbf{i} - 2\mathbf{j} - \mathbf{k}) \cdot \left(\frac{2}{7}\mathbf{i} - \frac{3}{7}\mathbf{j} + \frac{6}{7}\mathbf{k}\right) \\&= \frac{4}{7} + \frac{6}{7} - \frac{6}{7} = \frac{4}{7}.\end{aligned}$$

- (b) The function increases most rapidly in the direction of  $\nabla f = 2\mathbf{i} - 2\mathbf{j} - \mathbf{k}$  and decreases most rapidly in the direction of  $-\nabla f$ . The rates of change in the directions are, respectively,

$$|\nabla f| = \sqrt{(2)^2 + (-2)^2 + (-1)^2} = \sqrt{9} = 3 \quad \text{and} \quad -|\nabla f| = -3.$$

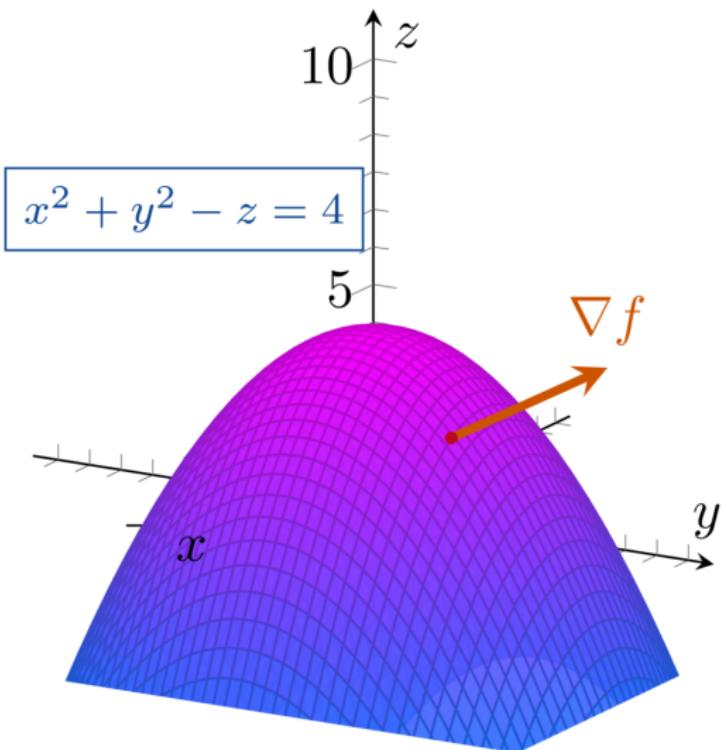




# 11 Tangent Planes and Differentials

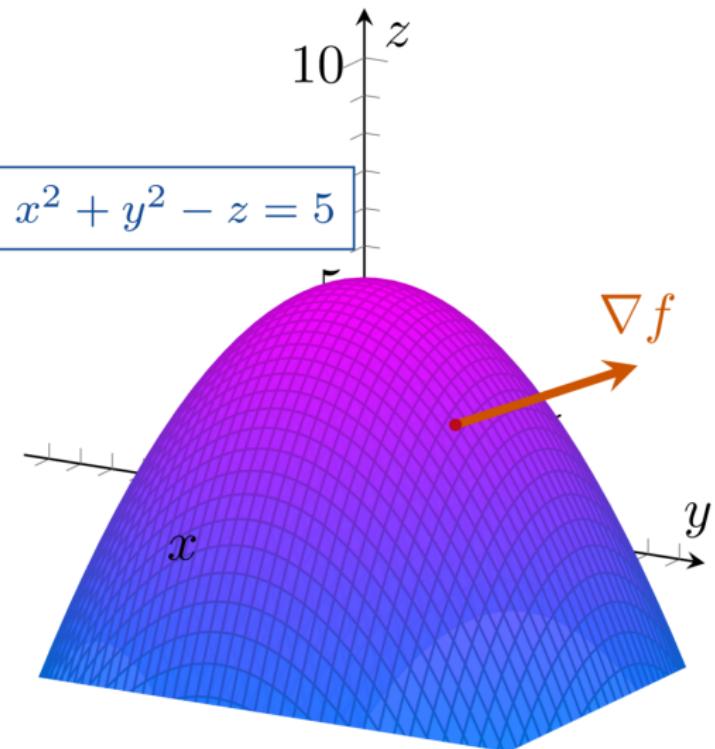
$$D_{\mathbf{u}} f = \|\nabla f\| \cos \theta$$

## Tangent Planes and Normal Lines



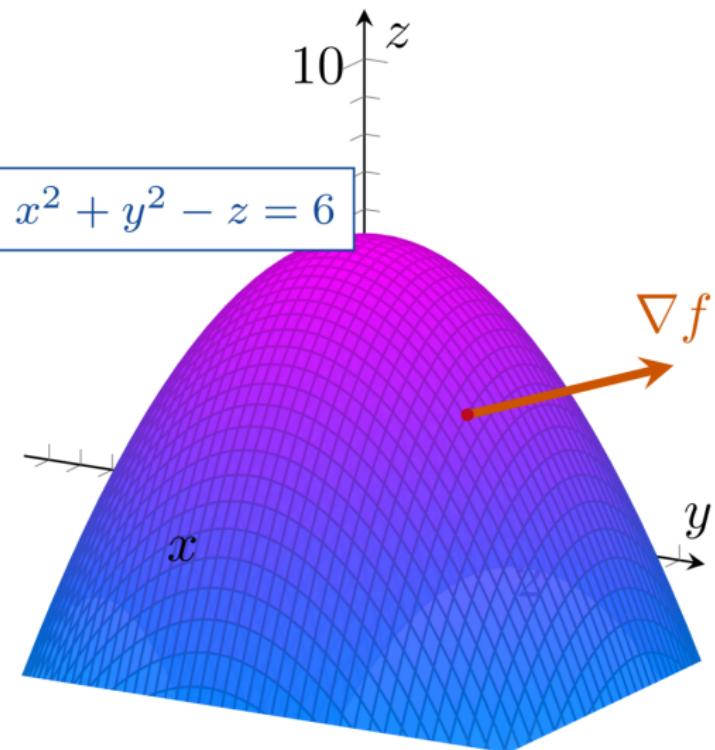
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## Tangent Planes and Normal Lines



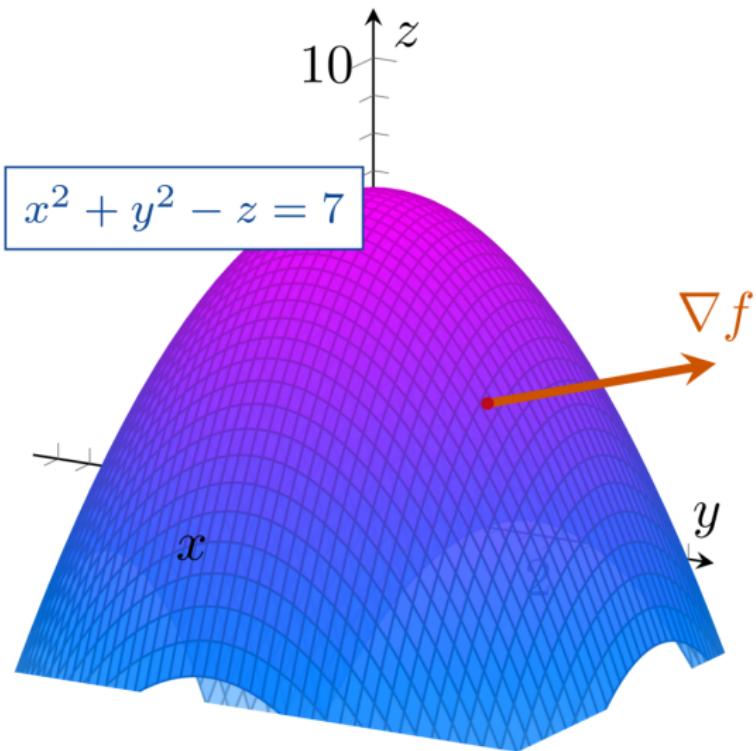
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## Tangent Planes and Normal Lines



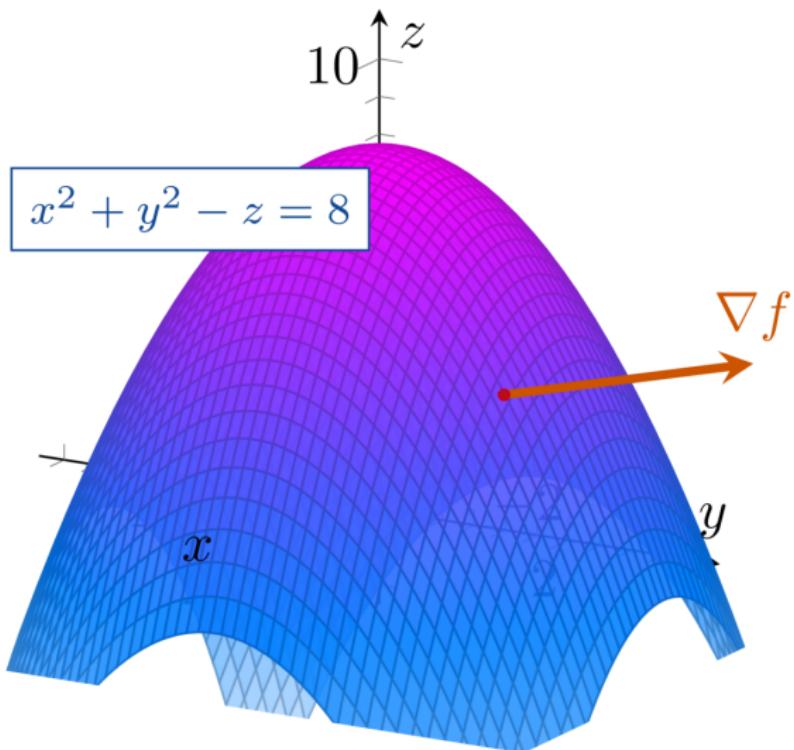
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## Tangent Planes and Normal Lines



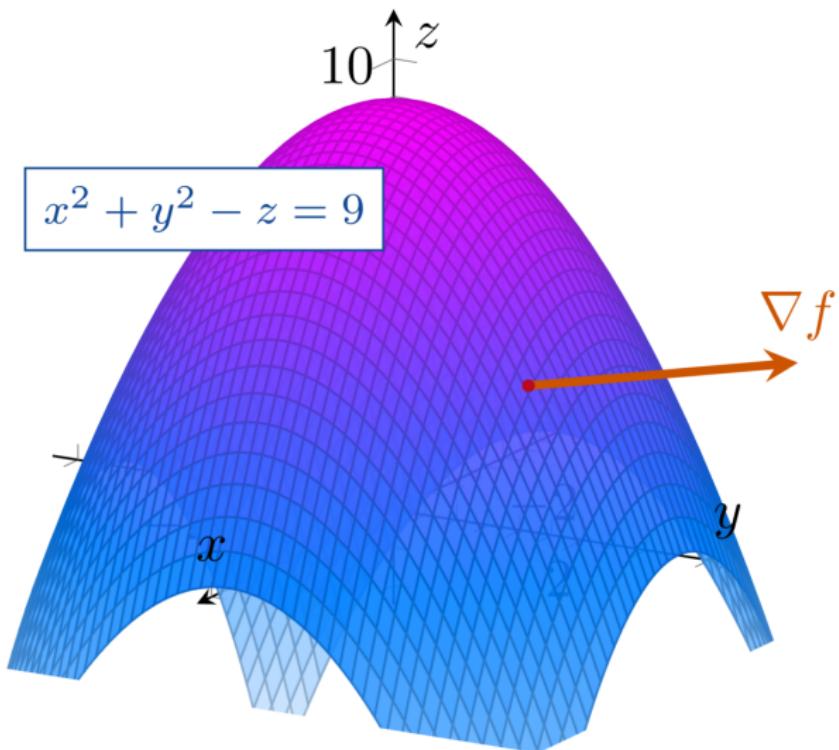
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## Tangent Planes and Normal Lines



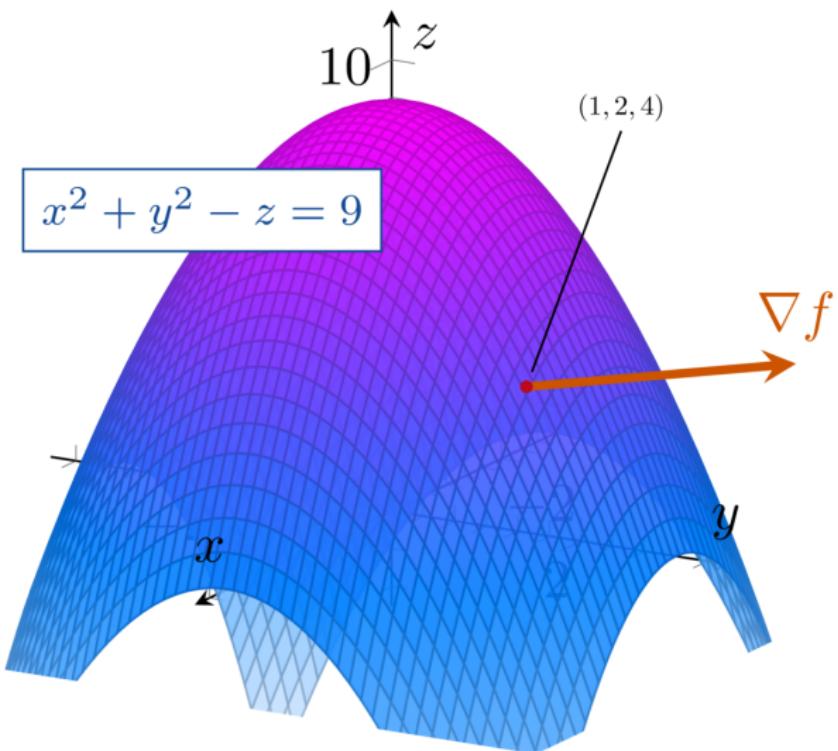
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## Tangent Planes and Normal Lines



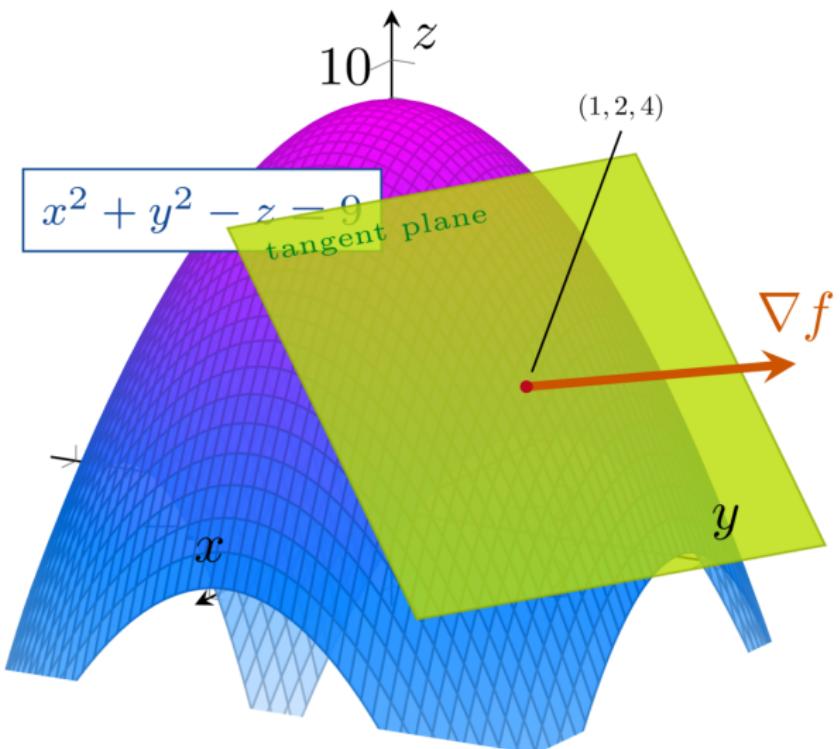
$$D_{\mathbf{u}} f = \|\nabla f\| \cos \theta$$

## Tangent Planes and Normal Lines



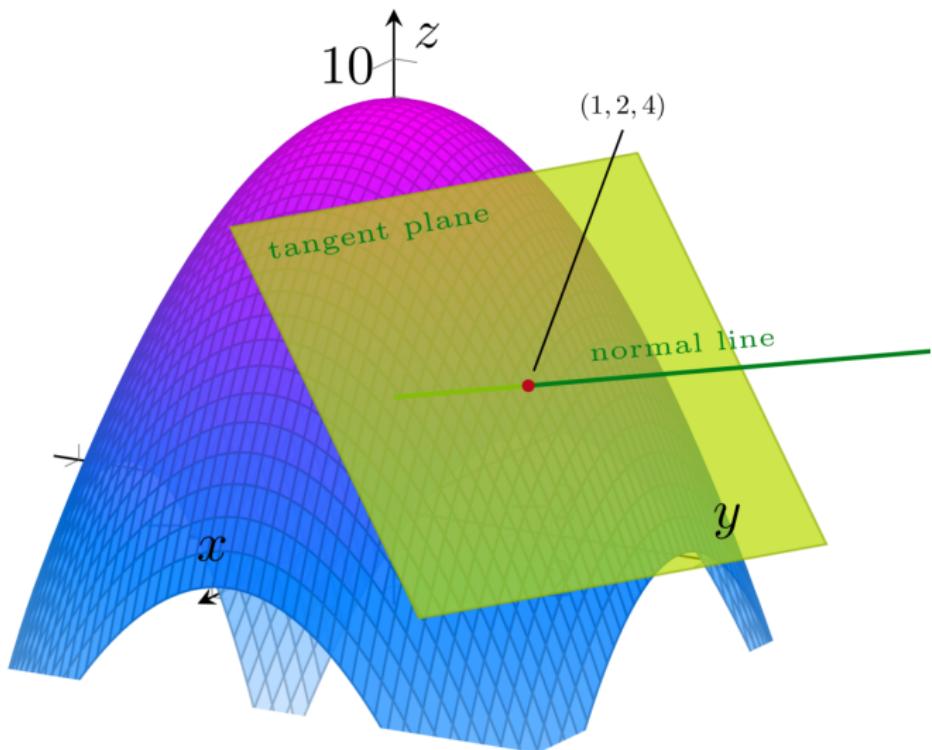
$$D_{\mathbf{u}} f = \|\nabla f\| \cos \theta$$

## Tangent Planes and Normal Lines



$$D_{\mathbf{u}} f = \|\nabla f\| \cos \theta$$

## Tangent Planes and Normal Lines



$$D_{\mathbf{u}} f = \|\nabla f\| \cos \theta$$

## Definition

The *tangent plane* to the surface  $f(x, y, z) = c$  at the point  $P(x_0, y_0, z_0)$  (where the gradient is not zero) is the plane through  $P$  with normal vector  $\nabla f|_P$ .

$$D_{\mathbf{u}} f = \|\nabla f\| \cos \theta$$

## Definition

The *tangent plane* to the surface  $f(x, y, z) = c$  at the point  $P(x_0, y_0, z_0)$  (where the gradient is not zero) is the plane through  $P$  with normal vector  $\nabla f|_P$ .

$$f_x(P)(x - x_0) + f_y(P)(y - y_0) + f_z(P)(z - z_0) = 0.$$

$$D_{\mathbf{u}} f = \|\nabla f\| \cos \theta$$

## Definition

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$$f_x(P)(x - x_0) + f_y(P)(y - y_0) + f_z(P)(z - z_0) = 0.$$

## Definition

The *normal line* to the surface  $f(x, y, z) = c$  at the point  $P$  is the line through  $P$  parallel to  $\nabla f|_P$ .

$$D_{\mathbf{u}} f = \|\nabla f\| \cos \theta$$

## Definition

The *tangent plane* to the surface  $f(x, y, z) = c$  at the point  $P(x_0, y_0, z_0)$  (where the gradient is not zero) is the plane through  $P$  with normal vector  $\nabla f|_P$ .

$$f_x(P)(x - x_0) + f_y(P)(y - y_0) + f_z(P)(z - z_0) = 0.$$

## Definition

The *normal line* to the surface  $f(x, y, z) = c$  at the point  $P$  is the line through  $P$  parallel to  $\nabla f|_P$ .

$$x = x_0 + f_x(P)t \quad y = y_0 + f_y(P)t \quad z = z_0 + f_z(P)t.$$

**EXAMPLE 1** Find the tangent plane and normal line of the level surface

$$f(x, y, z) = x^2 + y^2 + z - 9 = 0 \quad \text{A circular paraboloid}$$

at the point  $P_0(1, 2, 4)$ .

**Solution** The surface is shown in Figure 14.34.

The tangent plane is the plane through  $P_0$  perpendicular to the gradient of  $f$  at  $P_0$ . The gradient is

$$\nabla f|_{P_0} = (2x\mathbf{i} + 2y\mathbf{j} + \mathbf{k}) \Big|_{(1, 2, 4)} = 2\mathbf{i} + 4\mathbf{j} + \mathbf{k}.$$

The tangent plane is therefore the plane

$$2(x - 1) + 4(y - 2) + (z - 4) = 0, \quad \text{or} \quad 2x + 4y + z = 14.$$

The line normal to the surface at  $P_0$  is

$$x = 1 + 2t, \quad y = 2 + 4t, \quad z = 4 + t. \quad \blacksquare$$

13.6

$$D_{\mathbf{u}} f = \|\nabla f\| \cos \theta$$



Now consider

$$z = f(x, y).$$

13.6

$$D_{\mathbf{u}} f = \|\nabla f\| \cos \theta$$



Now consider

$$z = f(x, y).$$

This is equivalent to

$$F(x, y, z) = f(x, y) - z = 0.$$

$$D_{\mathbf{u}} f = \|\nabla f\| \cos \theta$$

Now consider

$$z = f(x, y).$$

This is equivalent to

$$F(x, y, z) = f(x, y) - z = 0.$$

## Definition

The *tangent plane* to the surface  $z = f(x, y)$  at the point  $P(x_0, y_0, z_0) = (x_0, y_0, f(x_0, y_0))$  is

$$f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0) - (z - z_0) = 0.$$

**EXAMPLE 2** Find the plane tangent to the surface  $z = x \cos y - ye^x$  at  $(0, 0, 0)$ .

**Solution** We calculate the partial derivatives of  $f(x, y) = x \cos y - ye^x$  and use Equation (3):

$$f_x(0, 0) = (\cos y - ye^y) \Big|_{(0, 0)} = 1 - 0 \cdot 1 = 1$$

$$f_y(0, 0) = (-x \sin y - e^y) \Big|_{(0, 0)} = 0 - 1 = -1.$$

The tangent plane is therefore

$$1 \cdot (x - 0) - 1 \cdot (y - 0) - (z - 0) = 0, \quad \text{Eq. (3)}$$

or

$$x - y - z = 0.$$



$$D_{\mathbf{u}} f = \|\nabla f\| \cos \theta$$

**EXAMPLE 3**

The surfaces

$$f(x, y, z) = x^2 + y^2 - 2 = 0 \quad \text{A cylinder}$$

and

$$g(x, y, z) = x + z - 4 = 0 \quad \text{A plane}$$

meet in an ellipse  $E$  (Figure 14.35). Find parametric equations for the line tangent to  $E$  at the point  $P_0(1, 1, 3)$ .

$$D_{\mathbf{u}} f = \|\nabla f\| \cos \theta$$

### EXAMPLE 3

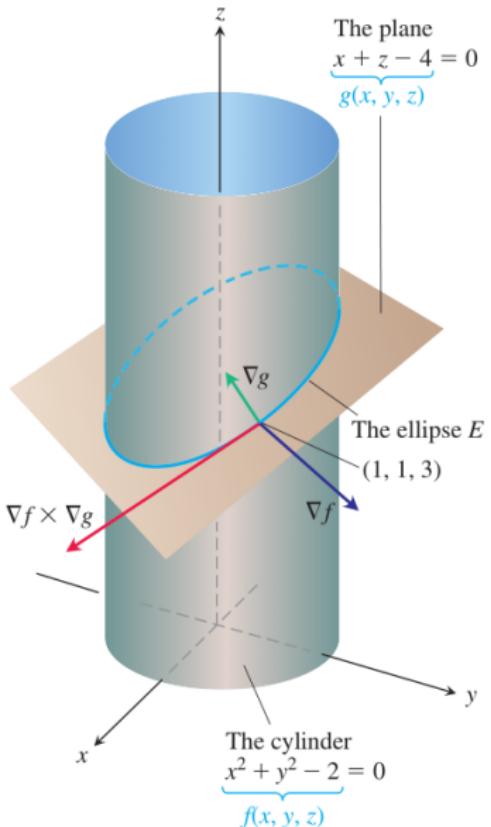
The surfaces

$$f(x, y, z) = x^2 + y^2 - 2$$

and

$$g(x, y, z) = x + z - 4$$

meet in an ellipse  $E$  (Figure 14.35). Find parameters of the point  $P_0(1, 1, 3)$ .



$$D_{\mathbf{u}} f = \|\nabla f\| \cos \theta$$

**EXAMPLE 3**

The surfaces

$$f(x, y, z) = x^2 + y^2 - 2 = 0 \quad \text{A cylinder}$$

and

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meet in an ellipse  $E$  (Figure 14.35). Find parametric equations for the line tangent to  $E$  at the point  $P_0(1, 1, 3)$ .

**Solution** The tangent line is orthogonal to both  $\nabla f$  and  $\nabla g$  at  $P_0$ , and therefore parallel to  $\mathbf{v} = \nabla f \times \nabla g$ . The components of  $\mathbf{v}$  and the coordinates of  $P_0$  give us equations for the line. We have

$$\nabla f|_{(1, 1, 3)} = (2x\mathbf{i} + 2y\mathbf{j}) \Big|_{(1, 1, 3)} = 2\mathbf{i} + 2\mathbf{j}$$

$$\nabla g|_{(1, 1, 3)} = (\mathbf{i} + \mathbf{k}) \Big|_{(1, 1, 3)} = \mathbf{i} + \mathbf{k}$$

$$\mathbf{v} = (2\mathbf{i} + 2\mathbf{j}) \times (\mathbf{i} + \mathbf{k}) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 2 & 0 \\ 1 & 0 & 1 \end{vmatrix} = 2\mathbf{i} - 2\mathbf{j} - 2\mathbf{k}.$$

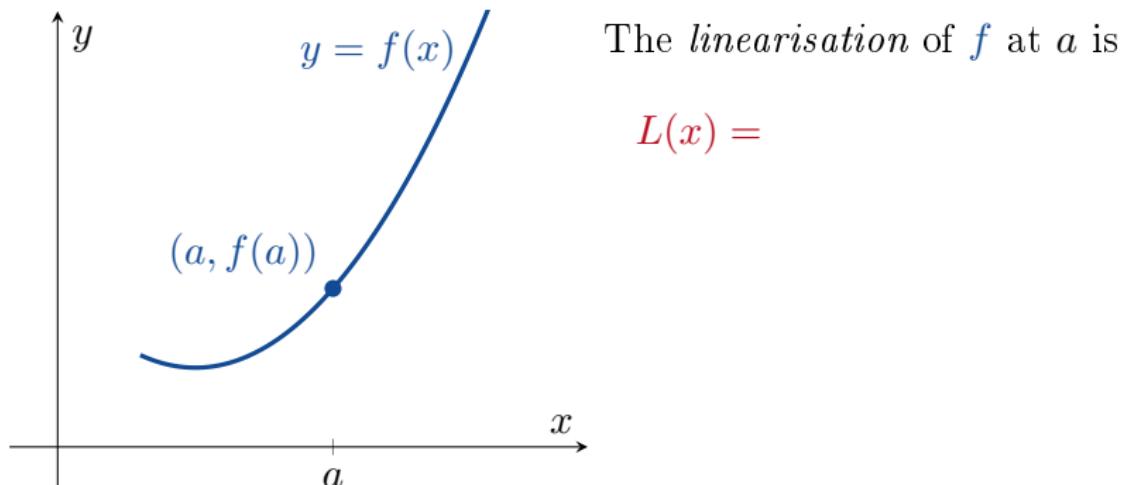
The tangent line to the ellipse of intersection is

$$x = 1 + 2t, \quad y = 1 - 2t, \quad z = 3 - 2t.$$



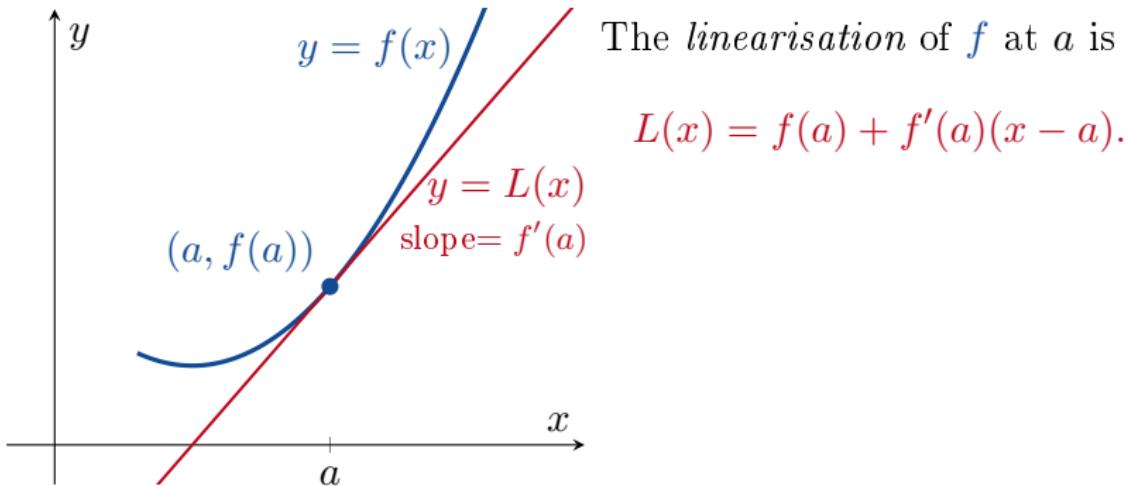
$$D_{\mathbf{u}} f = \|\nabla f\| \cos \theta$$

## Linearisation of a Function of One Variable



$$D_{\mathbf{u}} f = \|\nabla f\| \cos \theta$$

## Linearisation of a Function of One Variable



$$L(x) = f(a) + f'(a)(x - a)$$

## Linearisation of a Function of Two Variable

### Definition

The *linearisation* of a function  $f(x, y)$  at a point  $(x_0, y_0)$  is

$$L(x, y) = f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0).$$

**EXAMPLE 5**

Find the linearization of

$$f(x, y) = x^2 - xy + \frac{1}{2}y^2 + 3$$

at the point  $(3, 2)$ .**Solution** We first evaluate  $f$ ,  $f_x$ , and  $f_y$  at the point  $(x_0, y_0) = (3, 2)$ :

$$f(3, 2) = \left. \left( x^2 - xy + \frac{1}{2}y^2 + 3 \right) \right|_{(3, 2)} = 8$$

$$f_x(3, 2) = \left. \frac{\partial}{\partial x} \left( x^2 - xy + \frac{1}{2}y^2 + 3 \right) \right|_{(3, 2)} = \left. (2x - y) \right|_{(3, 2)} = 4$$

$$f_y(3, 2) = \left. \frac{\partial}{\partial y} \left( x^2 - xy + \frac{1}{2}y^2 + 3 \right) \right|_{(3, 2)} = \left. (-x + y) \right|_{(3, 2)} = -1,$$

giving

$$\begin{aligned} L(x, y) &= f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0) \\ &= 8 + (4)(x - 3) + (-1)(y - 2) = 4x - y - 2. \end{aligned}$$

The linearization of  $f$  at  $(3, 2)$  is  $L(x, y) = 4x - y - 2$



# Break

We will continue at 2pm



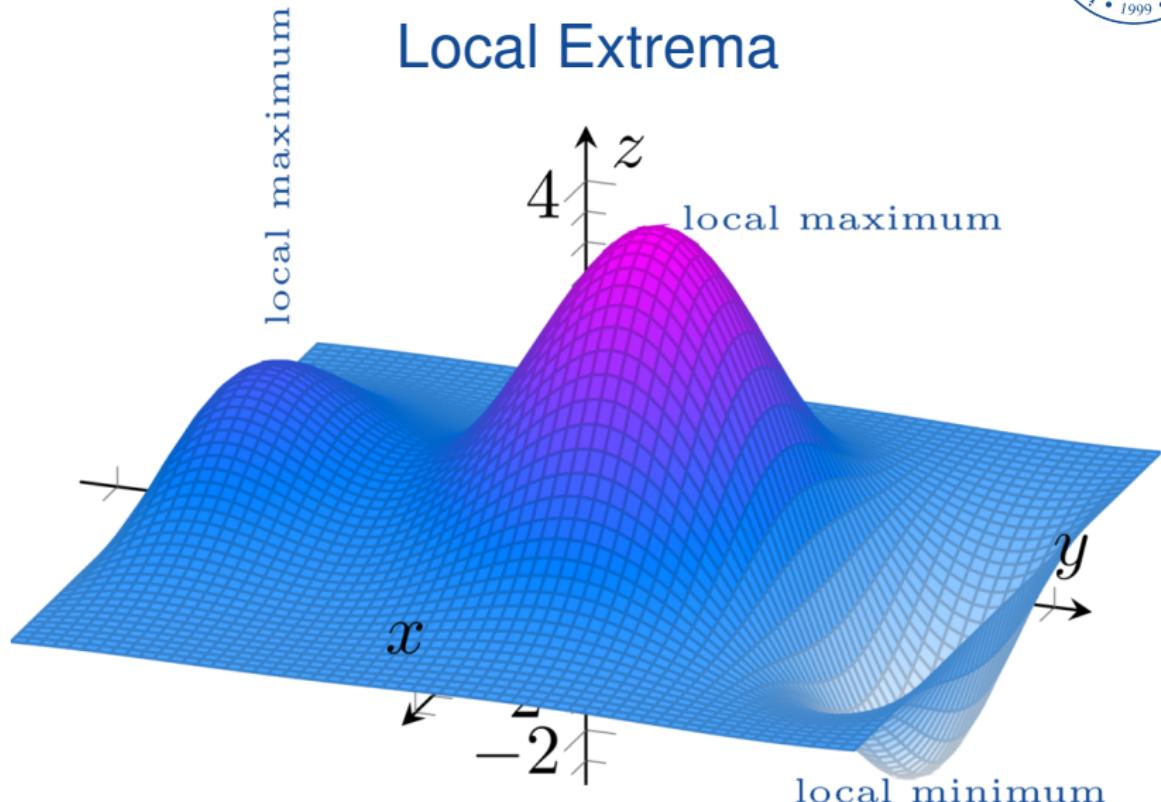


# Extreme Values and Saddle Points

## 13.7 Extreme Values and Saddle Points



### Local Extrema



## 13.7 Extreme Values and Saddle Points



### Definition

- 1  $f(a, b)$  is a local maximum value of  $f(x, y)$  iff

$$f(a, b) \geq f(x, y)$$

for all  $(x, y)$  close to  $(a, b)$ .

## 13.7 Extreme Values and Saddle Points



### Definition

- 1  $f(a, b)$  is a local maximum value of  $f(x, y)$  iff

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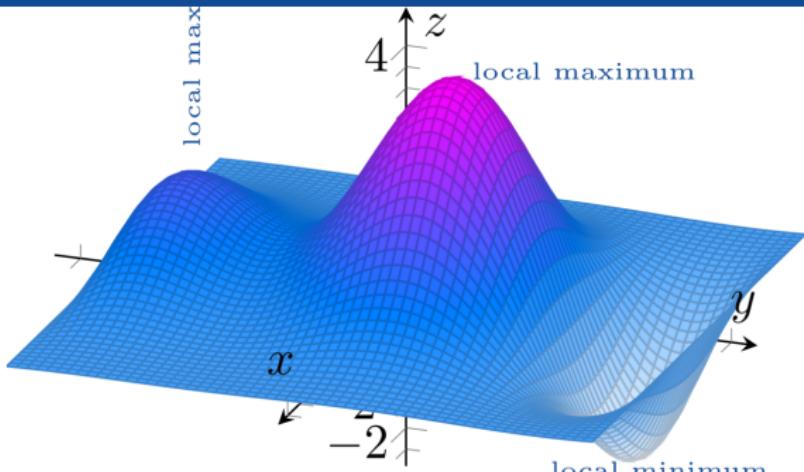
for all  $(x, y)$  close to  $(a, b)$ .

- 2  $f(a, b)$  is a local **minimum** value of  $f(x, y)$  iff

$$f(a, b) \leq f(x, y)$$

for all  $(x, y)$  close to  $(a, b)$ .

## 13.7 Extreme Values and Saddle Points



Theorem (First Derivative Test)

$$\left( \begin{array}{l} f(x,y) \text{ has a local} \\ \text{extrema at an interior} \\ \text{point } (a,b) \text{ of its} \\ \text{domain} \end{array} \right) \implies \begin{array}{l} f_x(a,b) = 0 \\ \text{and} \\ f_y(a,b) = 0 \end{array}$$

if  $f_x(a,b)$  and  $f_y(a,b)$  both exist.

## 13.7 Extreme Values and Saddle Points



### Definition

An interior point of the domain of  $f(x, y)$  where either

- 1  $f_x = f_y = 0$ ;
- 2  $f_x$  does not exist; or
- 3  $f_y$  does not exist

is called a *critical point* of  $f$ .

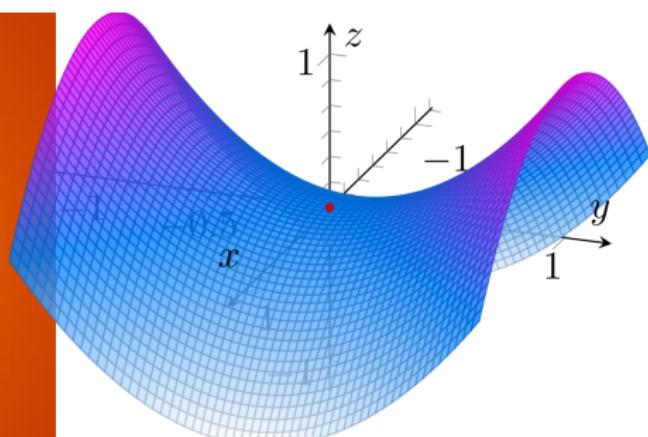
## 13.7 Extreme Values and Saddle Points



### Saddle Points



### Saddle Points



The point  $(0, 0)$  is a *saddle point* of  $z = y^2 - x^2$ .

## 13.7 Extreme Values and Saddle Points



### Example

Find the local extrema of  $f(x, y) = y^2 - 4y + x^2 + 9$ .

domain:

partial derivatives:

$$\begin{aligned} 0 &= f_x = \\ 0 &= f_y = \end{aligned} \qquad \implies \qquad (x, y) =$$

## 13.7 Extreme Values and Saddle Points



### Example

Find the local extrema of  $f(x, y) = y^2 - 4y + x^2 + 9$ .

domain:  $\mathbb{R}^2$

partial derivatives:

$$\begin{aligned} 0 &= f_x = 2x \\ 0 &= f_y = 2y - 4 \end{aligned} \qquad \implies \qquad (x, y) = (0, 2)$$

## 13.7 Extreme Values and Saddle Points



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Therefore the only possible place for an extrema is  $(0, 2)$ , where  $f(0, 2) = 5$ . Is this a local minimum or a local maximum?

## 13.7 Extreme Values and Saddle Points



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Therefore the only possible place for an extrema is  $(0, 2)$ , where  $f(0, 2) = 5$ . Is this a local minimum or a local maximum?

Since

$$(y - 2)^2 + x^2 + 5 \geq 5$$

for all  $(x, y)$ , this must be a local minimum.

## 13.7 Extreme Values and Saddle Points

### Example

Find the local extrema of  $f(x, y) = y^2 - 4y - x^2 + 9$ .

domain:

partial derivatives:

$$\begin{aligned} 0 &= f_x = \\ 0 &= f_y = \end{aligned} \implies (x, y) =$$

## 13.7 Extreme Values and Saddle Points



### Example

Find the local extrema of  $f(x, y) = y^2 - 4y - x^2 + 9$ .

domain:  $\mathbb{R}^2$

partial derivatives:

$$\begin{aligned} 0 &= f_x = -2x \\ 0 &= f_y = 2y - 4 \end{aligned} \implies (x, y) = (0, 2)$$

## 13.7 Extreme Values and Saddle Points

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domain:  $\mathbb{R}^2$

partial derivatives:

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Therefore the only possible place for an extrema is  $(0, 2)$ , where  $f(0, 2) = 5$ . Is this a local minimum or a local maximum?

## 13.7 Extreme Values and Saddle Points



### Example

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domain:  $\mathbb{R}^2$

partial derivatives:

$$\begin{aligned} 0 &= f_x = -2x \\ 0 &= f_y = 2y - 4 \end{aligned} \implies (x, y) = (0, 2)$$

Therefore the only possible place for an extrema is  $(0, 2)$ , where  $f(0, 2) = 5$ . Is this a local minimum or a local maximum?

No. Fixing  $x = 0$  we have  $f(0, y) = (y - 2)^2 + 5$  which curves upwards. But fixing  $y = 2$  we have  $f(x, 2) = 5 - x^2$  which curves downwards.

So  $(0, 2)$  must be a saddle point.

## 13.7 Extreme Values and Saddle Points

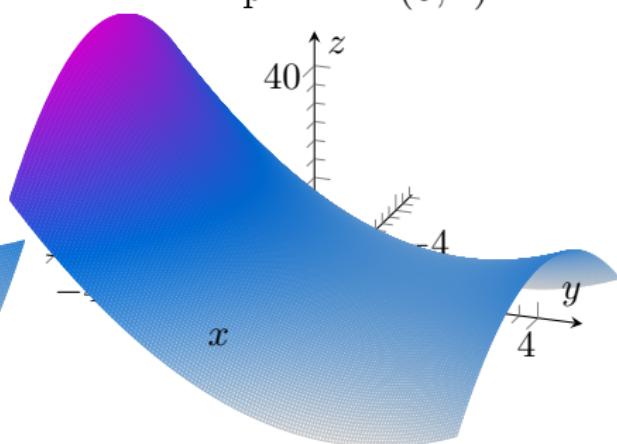
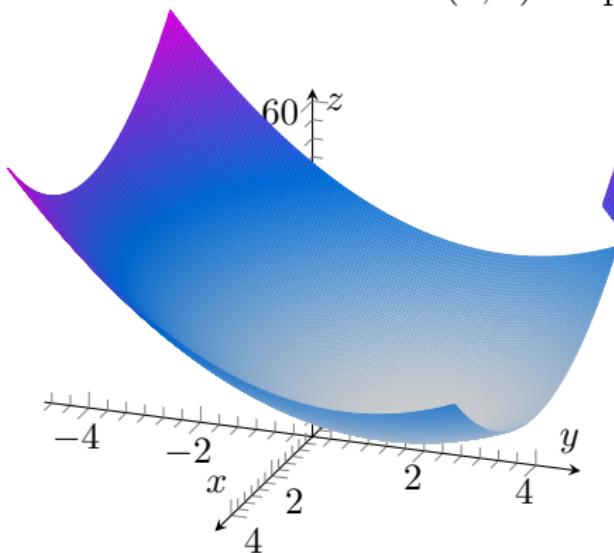


$$z = y^2 - 4y + x^2 + 9$$

has a local minimum at  $(0, 2)$ .

$$z = y^2 - 4y - x^2 + 9$$

has a saddle point at  $(0, 2)$ .



## 13.7 Extreme Values and Saddle Points



### Theorem (Second Derivative Test)

Suppose that

- $f(x, y), f_x, f_y, f_{xx}, f_{yy}$  and  $f_{xy}$  are all continuous on an open disk centred at  $(a, b)$ ; and
- $f_x(a, b) = 0 = f_y(a, b)$ .

<i>If at <math>(a, b)</math> we have</i>	<i>then</i>

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	$f_{xx}f_{yy} - f_{xy}^2 = 0$	we don't know

## 13.7 Extreme Values and Saddle Points



Otto Hesse

BORN

22 April 1811

DECEASED

4 August 1874

NATIONALITY

German

### Definition

$f_{xx}f_{yy} - f_{xy}^2$  is called the *Hessian* (or *discriminant*) of  $f$ .

**EXAMPLE 3** Find the local extreme values of the function

$$f(x, y) = xy - x^2 - y^2 - 2x - 2y + 4.$$

**Solution** The function is defined and differentiable for all  $x$  and  $y$ , and its domain has no boundary points. The function therefore has extreme values only at the points where  $f_x$  and  $f_y$  are simultaneously zero. This leads to

$$f_x = y - 2x - 2 = 0, \quad f_y = x - 2y - 2 = 0,$$

or

$$x = y = -2.$$

Therefore, the point  $(-2, -2)$  is the only point where  $f$  may take on an extreme value. To see if it does so, we calculate

$$f_{xx} = -2, \quad f_{yy} = -2, \quad f_{xy} = 1.$$

The discriminant of  $f$  at  $(a, b) = (-2, -2)$  is

$$f_{xx}f_{yy} - f_{xy}^2 = (-2)(-2) - (1)^2 = 4 - 1 = 3.$$

The combination

$$f_{xx} < 0 \quad \text{and} \quad f_{xx}f_{yy} - f_{xy}^2 > 0$$

tells us that  $f$  has a local maximum at  $(-2, -2)$ . The value of  $f$  at this point is  $f(-2, -2) = 8$ . ■

**EXAMPLE 4** Find the local extreme values of  $f(x, y) = 3y^2 - 2y^3 - 3x^2 + 6xy$ .

**Solution** Since  $f$  is differentiable everywhere, it can assume extreme values only where

$$f_x = 6y - 6x = 0 \quad \text{and} \quad f_y = 6y - 6y^2 + 6x = 0.$$

From the first of these equations we find  $x = y$ , and substitution for  $y$  into the second equation then gives

$$6x - 6x^2 + 6x = 0 \quad \text{or} \quad 6x(2 - x) = 0.$$

The two critical points are therefore  $(0, 0)$  and  $(2, 2)$ .

To classify the critical points, we calculate the second derivatives:

$$f_{xx} = -6, \quad f_{yy} = 6 - 12y, \quad f_{xy} = 6.$$

The discriminant is given by

$$f_{xx}f_{yy} - f_{xy}^2 = (-36 + 72y) - 36 = 72(y - 1).$$

At the critical point  $(0, 0)$  we see that the value of the discriminant is the negative number  $-72$ , so the function has a saddle point at the origin. At the critical point  $(2, 2)$  we see that the discriminant has the positive value  $72$ . Combining this result with the negative value of the second partial  $f_{xx} = -6$ , Theorem 11 says that the critical point  $(2, 2)$  gives a local maximum value of  $f(2, 2) = 12 - 16 - 12 + 24 = 8$ . A graph of the surface is shown in Figure 14.48. ■

**EXAMPLE 5** Find the critical points of the function  $f(x, y) = 10xye^{-(x^2+y^2)}$  and use the Second Derivative Test to classify each point as one where a saddle, local minimum, or local maximum occurs.

**Solution** First we find the partial derivatives  $f_x$  and  $f_y$  and set them simultaneously to zero in seeking the critical points:

$$\begin{aligned}f_x &= 10ye^{-(x^2+y^2)} - 20x^2ye^{-(x^2+y^2)} = 10y(1 - 2x^2)e^{-(x^2+y^2)} = 0 \Rightarrow y = 0 \text{ or } 1 - 2x^2 = 0, \\f_y &= 10xe^{-(x^2+y^2)} - 20xy^2e^{-(x^2+y^2)} = 10x(1 - 2y^2)e^{-(x^2+y^2)} = 0 \Rightarrow x = 0 \text{ or } 1 - 2y^2 = 0.\end{aligned}$$

Since both partial derivatives are continuous everywhere, the only critical points are

$$(0, 0), \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right), \left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right), \left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right), \text{ and } \left(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right).$$

Next we calculate the second partial derivatives in order to evaluate the discriminant at each critical point:

$$\begin{aligned}f_{xx} &= -20xy(1 - 2x^2)e^{-(x^2+y^2)} - 40xye^{-(x^2+y^2)} = -20xy(3 - 2x^2)e^{-(x^2+y^2)}, \\f_{xy} = f_{yx} &= 10(1 - 2x^2)e^{-(x^2+y^2)} - 20y^2(1 - 2x^2)e^{-(x^2+y^2)} = 10(1 - 2x^2)(1 - 2y^2)e^{-(x^2+y^2)}, \\f_{yy} &= -20xy(1 - 2y^2)e^{-(x^2+y^2)} - 40xye^{-(x^2+y^2)} = -20xy(3 - 2y^2)e^{-(x^2+y^2)}.\end{aligned}$$

The following table summarizes the values needed by the Second Derivative Test.

Critical Point	$f_{xx}$	$f_{xy}$	$f_{yy}$	Discriminant $D$
(0, 0)	0	10	0	-100
$\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$	$-\frac{20}{e}$	0	$-\frac{20}{e}$	$\frac{400}{e^2}$
$\left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$	$\frac{20}{e}$	0	$\frac{20}{e}$	$\frac{400}{e^2}$
$\left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)$	$\frac{20}{e}$	0	$\frac{20}{e}$	$\frac{400}{e^2}$
$\left(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)$	$-\frac{20}{e}$	0	$-\frac{20}{e}$	$\frac{400}{e^2}$

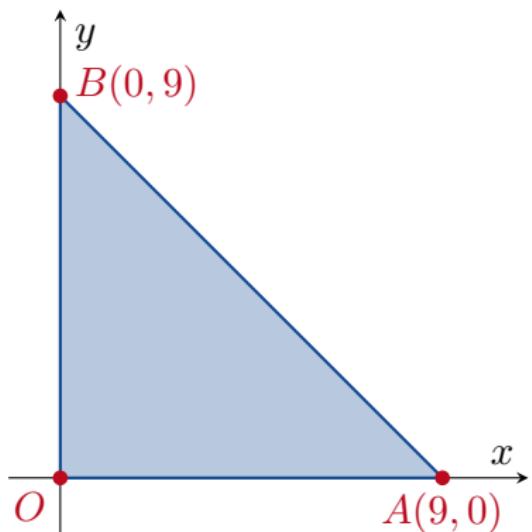
From the table we find that  $D < 0$  at the critical point (0, 0), giving a saddle;  $D > 0$  and  $f_{xx} < 0$  at the critical points  $(1/\sqrt{2}, 1/\sqrt{2})$  and  $(-1/\sqrt{2}, -1/\sqrt{2})$ , giving local maximum values there; and  $D > 0$  and  $f_{xx} > 0$  at the critical points  $(-1/\sqrt{2}, 1/\sqrt{2})$  and  $(1/\sqrt{2}, -1/\sqrt{2})$ , each giving local minimum values. A graph of the surface is shown in Figure 14.49. ■

## Example

Find the absolute maximum and minimum values of

$$f(x, y) = 2 + 2x + 4y - x^2 - y^2$$

on the triangular region in the first quadrant bounded by the lines  $x = 0$ ,  $y = 0$  and  $y = 9 - x$ .

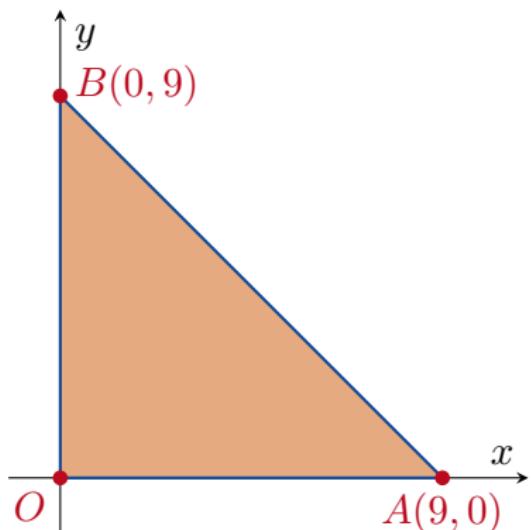


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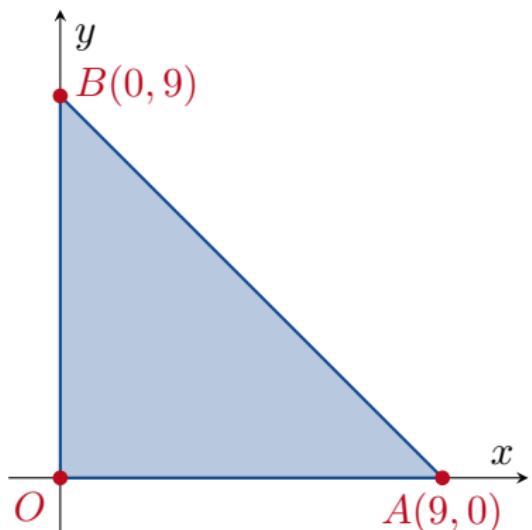
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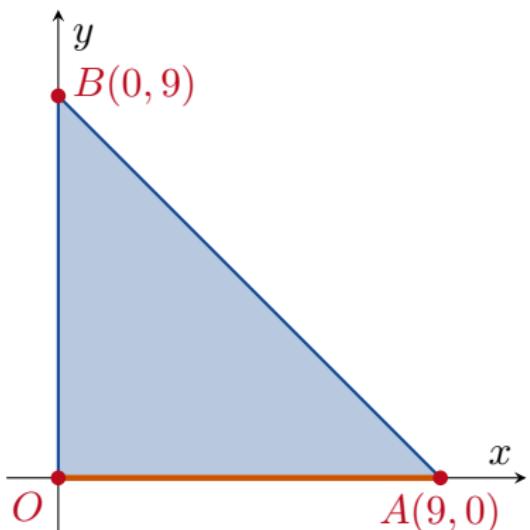
- 1 at the interior of the region;
- 2 at the corners  $O$ ,  $A$  and  $B$ ;

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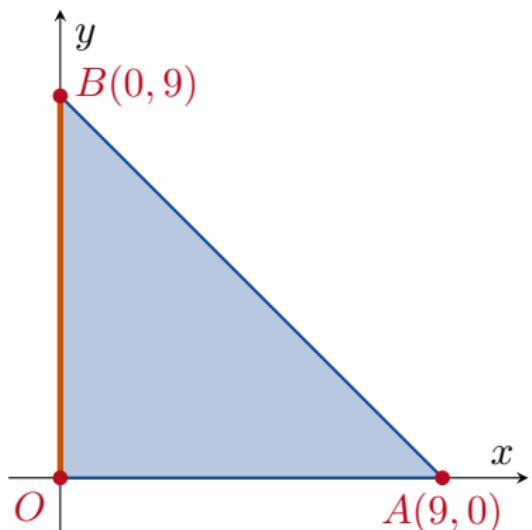
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- 3 at the line  $OA$ ;

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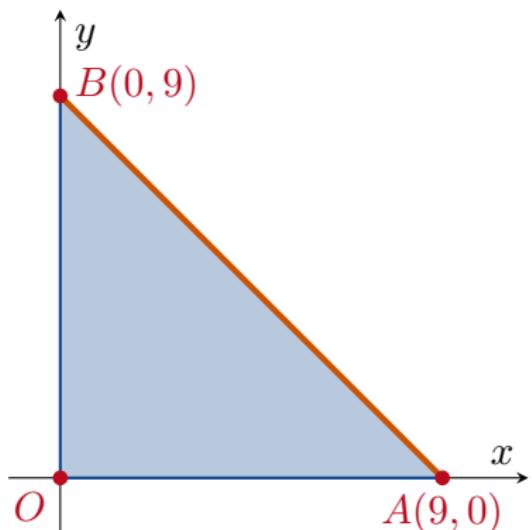
- 1 at the interior of the region;
- 2 at the corners  $O$ ,  $A$  and  $B$ ;
- 3 at the line  $OA$ ;
- 4 at the line  $OB$ ; and

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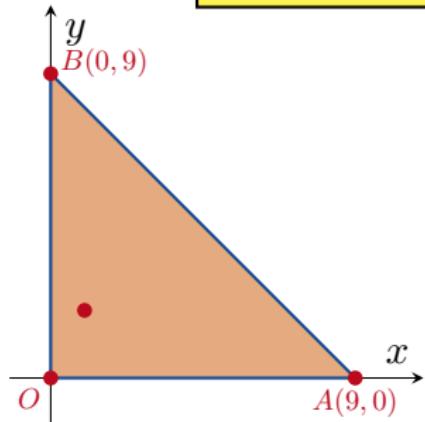


We will look

- 1 at the interior of the region;
- 2 at the corners  $O$ ,  $A$  and  $B$ ;
- 3 at the line  $OA$ ;
- 4 at the line  $OB$ ; and
- 5 at the line  $AB$ .

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$$f(x, y) = 2 + 2x + 4y - x^2 - y^2$$



- 1 Consider the interior of the region.  
We need to look for the critical points

$$\begin{aligned} 0 &= f_x = 2 - 2x \\ 0 &= f_y = 4 - 2y \end{aligned} \implies (x, y) = (1, 2).$$

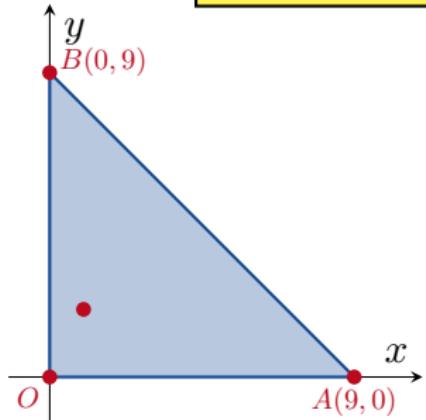
Then we calculate that

$$f(1, 2) = 2 + 2 + 8 - 1 - 4 = 7.$$

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$$f(x, y) = 2 + 2x + 4y - x^2 - y^2$$

$$f(1, 2) = 7$$



2 Consider the corners of the region.

We calculate that

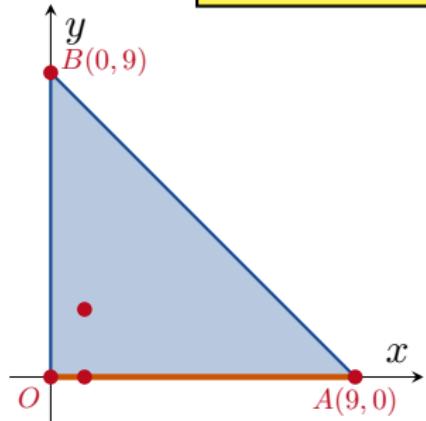
$$f(0, 0) = 2 + 0 + 0 - 0 - 0 = 2$$

$$f(9, 0) = 2 + 18 + 0 - 81 - 0 = -61$$

$$f(0, 9) = 2 + 0 + 36 - 0 - 81 = -43.$$

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$$f(x, y) = 2 + 2x + 4y - x^2 - y^2$$



$f(1, 2) = 7$
$f(0, 0) = 2$
$f(9, 0) = -61$
$f(0, 9) = -43$

3 Consider the line  $OA$ .

If we set  $y = 0$ , then we have a new function

$$g(x) = f(x, 0) = 2 + 2x - x^2.$$

Since

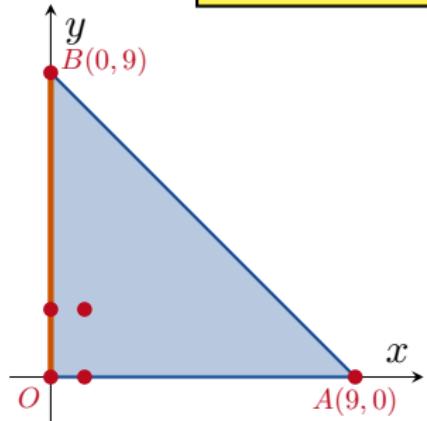
$$0 = g'(x) = 2 - 2x \implies x = 1$$

we calculate

$$g(1) = f(1, 0) = 2 + 2 - 1 = 3.$$

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$$f(x, y) = 2 + 2x + 4y - x^2 - y^2$$



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$f(0, 9) = -43$
$f(1, 0) = 3$

4 Consider the line  $OB$ .

If we set  $x = 0$ , then we have a new function

$$h(y) = f(0, y) = 2 + 4y - y^2.$$

Since

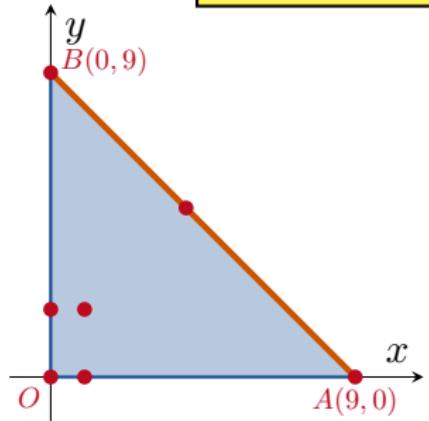
$$0 = h'(x) = 4 - 2y \implies y = 2$$

we calculate

$$h(2) = f(0, 2) = 2 + 8 - 4 = 6.$$

13.7

$$f(x, y) = 2 + 2x + 4y - x^2 - y^2$$



$f(1, 2) = 7$
$f(0, 0) = 2$
$f(9, 0) = -61$
$f(0, 9) = -43$
$f(1, 0) = 3$
$f(0, 2) = 6$

5 Finally consider the line  $AB$ .

If we set  $y = 9 - x$ , then we have a new function

$$k(x) = f(x, 9-x) = 2+2x+4(9-x)-x^2-(9-x)^2 = -43+16x-2x^2.$$

Since

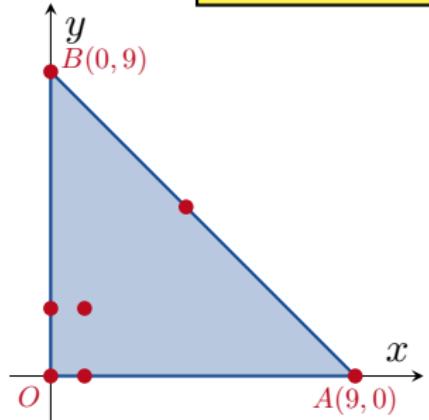
$$0 = k'(x) = 16 - 4x \implies x = 4$$

we calculate

$$k(4) = f(4, 5) = -43 + 64 - 32 = -11.$$

13.7

$$f(x, y) = 2 + 2x + 4y - x^2 - y^2$$



$f(1, 2) = 7$
$f(0, 0) = 2$
$f(9, 0) = -61$
$f(0, 9) = -43$
$f(1, 0) = 3$
$f(0, 2) = 6$
$f(4, 5) = -11$

We have found the values

$$7, 2, -61, -43, 3, 6, -11.$$

The biggest of these numbers is 7 and the least is -61.

Therefore the absolute maximum value of  $f$  on this region is 7 and the absolute minimum value of  $f$  on this region is -61.

## 13.7 Extreme Values and Saddle Points



Please read Example 7 in the textbook.



# 123 Lagrange Multipliers

## 13.8 Lagrange Multipliers

### Example

Find the point  $P(x, y, z)$  on the plane  $2x + y - z - 5 = 0$  that is closest to the origin.

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We need to find the minimum of

$$\|\overrightarrow{OP}\| = \sqrt{x^2 + y^2 + z^2}$$

subject to the constraint that

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## 13.8 Lagrange Multipliers

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Let  $f(x, y) = x^2 + y^2 + z^2$ . We will study  $f$  instead of  $\|\overrightarrow{OP}\|$ .

## 13.8 Lagrange Multipliers



We will let  $x$  and  $y$  be the independent variables and write

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$$\begin{aligned} 0 &= h_x = 2x + 2(2x + y - 5)(2) \implies 10x + 4y = 20 \\ 0 &= h_y = 2y + 2(2x + y - 5) \implies 4x + 4y = 10 \end{aligned}$$

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Then we have

$$z = 2x + y - 5 = \frac{10}{3} + \frac{5}{6} - 5 = -\frac{5}{6}.$$

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Then we have

$$z = 2x + y - 5 = \frac{10}{3} + \frac{5}{6} - 5 = -\frac{5}{6}.$$

The point on this plane which is closest to the origin is

$$P\left(\frac{5}{3}, \frac{5}{6}, -\frac{5}{6}\right).$$

# The Method of Lagrange Multipliers

Suppose that we want to find the maximum/minimum of

$$f(x, y, z)$$

subject to the constraint that

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Theorem (The Method of Lagrange Multipliers)

*We only need to find  $x, y, z$  and  $\lambda$  which satisfy*

$$\nabla f = \lambda \nabla g \quad \text{and} \quad g(x, y, z) = 0.$$

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Example (repeat)

Find the point  $P(x, y, z)$  on the plane  $2x + y - z - 5 = 0$  that is closest to the origin.

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$$\nabla f = \lambda \nabla g \quad g(x, y, z) = 0.$$



Example (repeat)

Find the point  $P(x, y, z)$  on the plane  $2x + y - z - 5 = 0$  that is closest to the origin.

Let  $f(x, y, z) = x^2 + y^2 + z^2$  and  $g(x, y, z) = 2x + y - z - 5$ .

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$$\nabla f = \lambda \nabla g \quad g(x, y, z) = 0.$$



### Example (repeat)

Find the point  $P(x, y, z)$  on the plane  $2x + y - z - 5 = 0$  that is closest to the origin.

Let  $f(x, y, z) = x^2 + y^2 + z^2$  and  $g(x, y, z) = 2x + y - z - 5$ . Then

$$= \nabla f = \lambda \nabla g =$$

13.8

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### Example (repeat)

Find the point  $P(x, y, z)$  on the plane  $2x + y - z - 5 = 0$  that is closest to the origin.

Let  $f(x, y, z) = x^2 + y^2 + z^2$  and  $g(x, y, z) = 2x + y - z - 5$ . Then

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Hence

$$0 = 2x + y - z - 5 = 2\lambda + \frac{\lambda}{2} - \left(-\frac{\lambda}{2}\right) - 5 = 3\lambda - 5$$

13.8

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$$0 = 2x + y - z - 5 = 2\lambda + \frac{\lambda}{2} - \left(-\frac{\lambda}{2}\right) - 5 = 3\lambda - 5 \implies \lambda = \frac{5}{3}.$$

$$\nabla f = \lambda \nabla g \quad g(x, y, z) = 0.$$

### Example (repeat)

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Let  $f(x, y, z) = x^2 + y^2 + z^2$  and  $g(x, y, z) = 2x + y - z - 5$ . Then

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Hence

$$0 = 2x + y - z - 5 = 2\lambda + \frac{\lambda}{2} - \left(-\frac{\lambda}{2}\right) - 5 = 3\lambda - 5 \implies \lambda = \frac{5}{3}.$$

Therefore

$$P(x, y, z) = \left(\lambda, \frac{\lambda}{2}, -\frac{\lambda}{2}\right) = \left(\frac{5}{3}, \frac{5}{6}, -\frac{5}{6}\right).$$

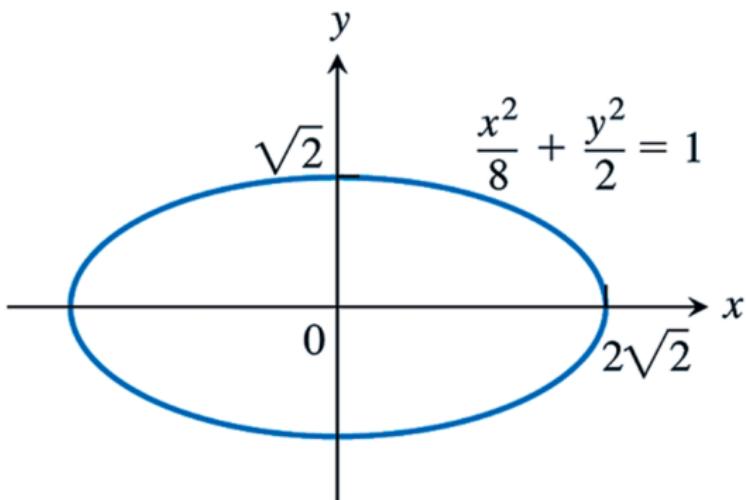
## 13.8 Lagrange Multipliers

**EXAMPLE 3** Find the greatest and smallest values that the function

$$f(x, y) = xy$$

takes on the ellipse (Figure 14.55)

$$\frac{x^2}{8} + \frac{y^2}{2} = 1.$$



**Solution** We want to find the extreme values of  $f(x, y) = xy$  subject to the constraint

$$g(x, y) = \frac{x^2}{8} + \frac{y^2}{2} - 1 = 0.$$

To do so, we first find the values of  $x$ ,  $y$ , and  $\lambda$  for which

$$\nabla f = \lambda \nabla g \quad \text{and} \quad g(x, y) = 0.$$

The gradient equation in Equations (1) gives

$$y\mathbf{i} + x\mathbf{j} = \frac{\lambda}{4}x\mathbf{i} + \lambda y\mathbf{j},$$

from which we find

$$y = \frac{\lambda}{4}x, \quad x = \lambda y, \quad \text{and} \quad y = \frac{\lambda}{4}(\lambda y) = \frac{\lambda^2}{4}y,$$

so that  $y = 0$  or  $\lambda = \pm 2$ . We now consider these two cases.

## 13.8 Lagrange Multipliers

**Case 1:** If  $y = 0$ , then  $x = y = 0$ . But  $(0, 0)$  is not on the ellipse. Hence,  $y \neq 0$ .

**Case 2:** If  $y \neq 0$ , then  $\lambda = \pm 2$  and  $x = \pm 2y$ . Substituting this in the equation  $g(x, y) = 0$  gives

$$\frac{(\pm 2y)^2}{8} + \frac{y^2}{2} = 1, \quad 4y^2 + 4y^2 = 8 \quad \text{and} \quad y = \pm 1.$$

The function  $f(x, y) = xy$  therefore takes on its extreme values on the ellipse at the four points  $(\pm 2, 1), (\pm 2, -1)$ . The extreme values are  $xy = 2$  and  $xy = -2$ .

**EXAMPLE 4** Find the maximum and minimum values of the function  $f(x, y) = 3x + 4y$  on the circle  $x^2 + y^2 = 1$ .

**Solution** We model this as a Lagrange multiplier problem with

$$f(x, y) = 3x + 4y, \quad g(x, y) = x^2 + y^2 - 1$$

and look for the values of  $x$ ,  $y$ , and  $\lambda$  that satisfy the equations

$$\begin{aligned}\nabla f = \lambda \nabla g: \quad & 3\mathbf{i} + 4\mathbf{j} = 2x\lambda\mathbf{i} + 2y\lambda\mathbf{j} \\ g(x, y) = 0: \quad & x^2 + y^2 - 1 = 0.\end{aligned}$$

The gradient equation in Equations (1) implies that  $\lambda \neq 0$  and gives

$$x = \frac{3}{2\lambda}, \quad y = \frac{2}{\lambda}.$$

These equations tell us, among other things, that  $x$  and  $y$  have the same sign. With these values for  $x$  and  $y$ , the equation  $g(x, y) = 0$  gives

$$\left(\frac{3}{2\lambda}\right)^2 + \left(\frac{2}{\lambda}\right)^2 - 1 = 0,$$

so

$$\frac{9}{4\lambda^2} + \frac{4}{\lambda^2} = 1, \quad 9 + 16 = 4\lambda^2, \quad 4\lambda^2 = 25, \quad \text{and} \quad \lambda = \pm \frac{5}{2}.$$

Thus,

$$x = \frac{3}{2\lambda} = \pm \frac{3}{5}, \quad y = \frac{2}{\lambda} = \pm \frac{4}{5},$$

and  $f(x, y) = 3x + 4y$  has extreme values at  $(x, y) = \pm(3/5, 4/5)$ .

By calculating the value of  $3x + 4y$  at the points  $\pm(3/5, 4/5)$ , we see that its maximum and minimum values on the circle  $x^2 + y^2 = 1$  are

$$3\left(\frac{3}{5}\right) + 4\left(\frac{4}{5}\right) = \frac{25}{5} = 5 \quad \text{and} \quad 3\left(-\frac{3}{5}\right) + 4\left(-\frac{4}{5}\right) = -\frac{25}{5} = -5.$$



# Next Time

- 14.1 Double and Iterated Integrals over Rectangles
- 14.2 Double Integrals over General Regions
- 14.3 Area by Double Integration
- 10.3 Polar Coordinates