

Lecture 4

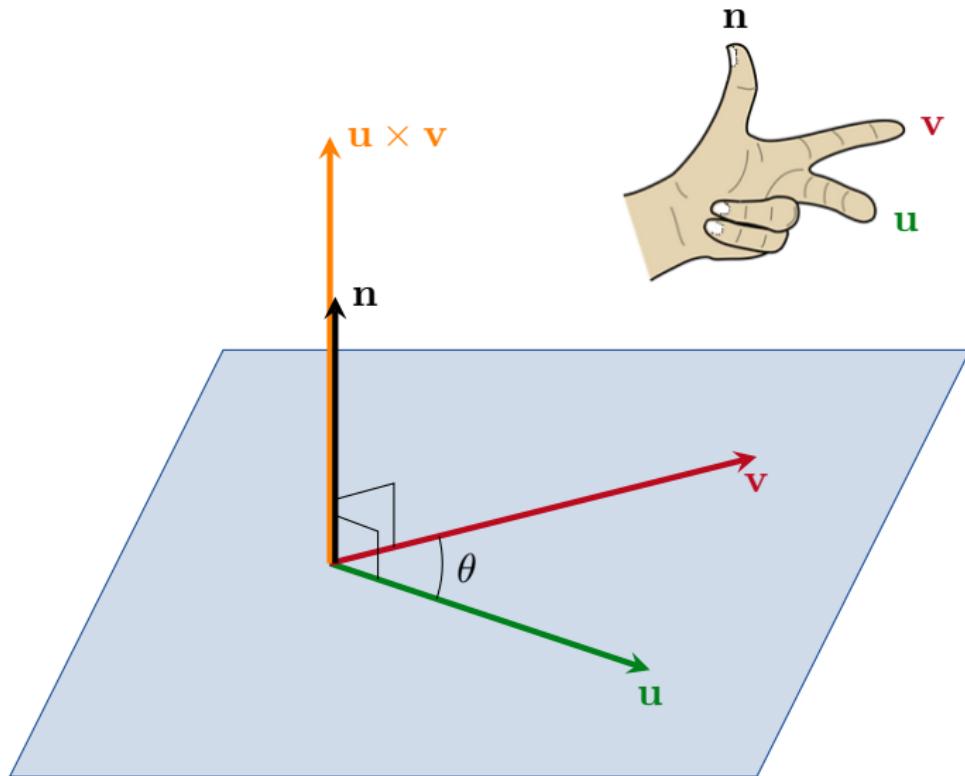
- 11.4 The Cross Product
- 11.5 Lines and Planes in Space



Happy Guy Fawkes Night.

The Cross Product

11.4 The Cross Product

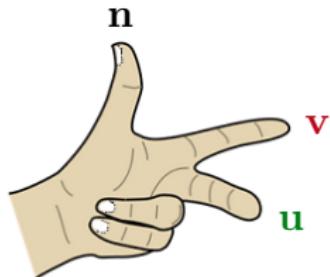


11.4 The Cross Product



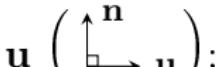
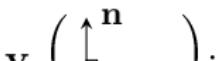
Let \mathbf{n} be a unit vector which satisfies

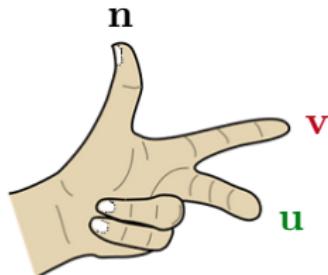
- 1 \mathbf{n} is orthogonal to \mathbf{u} ($\begin{smallmatrix} \mathbf{n} \\ \perp \\ \mathbf{u} \end{smallmatrix}$);
- 2 \mathbf{n} is orthogonal to \mathbf{v} ($\begin{smallmatrix} \mathbf{n} \\ \perp \\ \mathbf{v} \end{smallmatrix}$); and
- 3 the direction of \mathbf{n} is chosen using the left-hand rule.



11.4 The Cross Product

Let \mathbf{n} be a unit vector which satisfies

- 1 \mathbf{n} is orthogonal to \mathbf{u} () ;
- 2 \mathbf{n} is orthogonal to \mathbf{v} () ; and
- 3 the direction of \mathbf{n} is chosen using the left-hand rule.



Definition

The *cross product* of \mathbf{u} and \mathbf{v} is

$$\mathbf{u} \times \mathbf{v} = \|\mathbf{u}\| \|\mathbf{v}\| (\sin \theta) \mathbf{n}.$$

11.4 The Cross Product



Remark

- $\mathbf{u} \cdot \mathbf{v}$ is a number.
- $\mathbf{u} \times \mathbf{v}$ is a vector.

$$\mathbf{u} \times \mathbf{v} = \|\mathbf{u}\| \|\mathbf{v}\| (\sin \theta) \mathbf{n}$$

Remark

$$\begin{pmatrix} \mathbf{u} \text{ and } \mathbf{v} \\ \text{are} \\ \text{parallel} \end{pmatrix} \iff \theta = 0^\circ \text{ or } 180^\circ$$
$$\implies \sin \theta = 0 \implies \mathbf{u} \times \mathbf{v} = \mathbf{0}.$$

11.4 The Cross Product



Properties of the Cross Product

Let \mathbf{u} , \mathbf{v} and \mathbf{w} be vectors. Let r and s be numbers. Then

1 $(r\mathbf{u}) \times (s\mathbf{v}) = (rs)(\mathbf{u} \times \mathbf{v});$

11.4 The Cross Product



Properties of the Cross Product

Let \mathbf{u} , \mathbf{v} and \mathbf{w} be vectors. Let r and s be numbers. Then

- 1 $(r\mathbf{u}) \times (s\mathbf{v}) = (rs)(\mathbf{u} \times \mathbf{v});$
- 2 $\mathbf{u} \times (\mathbf{v} + \mathbf{w}) = (\mathbf{u} \times \mathbf{v}) + (\mathbf{u} \times \mathbf{w});$

11.4 The Cross Product



Properties of the Cross Product

Let \mathbf{u} , \mathbf{v} and \mathbf{w} be vectors. Let r and s be numbers. Then

- 1 $(r\mathbf{u}) \times (s\mathbf{v}) = (rs)(\mathbf{u} \times \mathbf{v});$
- 2 $\mathbf{u} \times (\mathbf{v} + \mathbf{w}) = (\mathbf{u} \times \mathbf{v}) + (\mathbf{u} \times \mathbf{w});$
- 3 $\mathbf{v} \times \mathbf{u} = -\mathbf{u} \times \mathbf{v};$

11.4 The Cross Product



Properties of the Cross Product

Let \mathbf{u} , \mathbf{v} and \mathbf{w} be vectors. Let r and s be numbers. Then

- 1 $(r\mathbf{u}) \times (s\mathbf{v}) = (rs)(\mathbf{u} \times \mathbf{v});$
- 2 $\mathbf{u} \times (\mathbf{v} + \mathbf{w}) = (\mathbf{u} \times \mathbf{v}) + (\mathbf{u} \times \mathbf{w});$
- 3 $\mathbf{v} \times \mathbf{u} = -\mathbf{u} \times \mathbf{v};$
- 4 $(\mathbf{v} + \mathbf{w}) \times \mathbf{u} = (\mathbf{v} \times \mathbf{u}) + (\mathbf{w} \times \mathbf{u});$

11.4 The Cross Product



Properties of the Cross Product

Let \mathbf{u} , \mathbf{v} and \mathbf{w} be vectors. Let r and s be numbers. Then

- 1 $(r\mathbf{u}) \times (s\mathbf{v}) = (rs)(\mathbf{u} \times \mathbf{v});$
- 2 $\mathbf{u} \times (\mathbf{v} + \mathbf{w}) = (\mathbf{u} \times \mathbf{v}) + (\mathbf{u} \times \mathbf{w});$
- 3 $\mathbf{v} \times \mathbf{u} = -\mathbf{u} \times \mathbf{v};$
- 4 $(\mathbf{v} + \mathbf{w}) \times \mathbf{u} = (\mathbf{v} \times \mathbf{u}) + (\mathbf{w} \times \mathbf{u});$
- 5 $\mathbf{0} \times \mathbf{u} = \mathbf{0};$ and

11.4 The Cross Product



Properties of the Cross Product

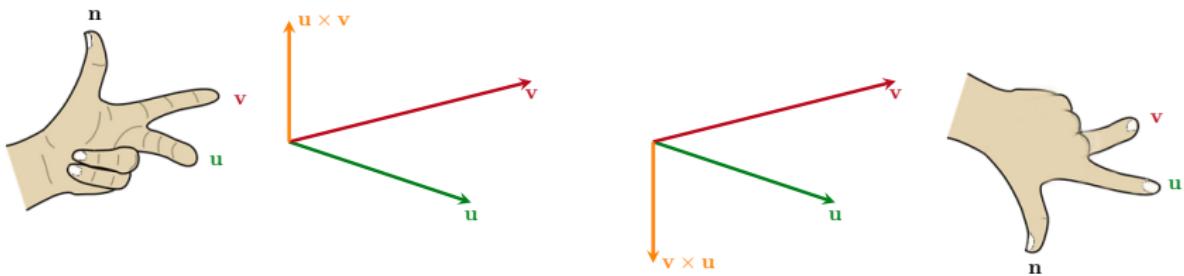
Let \mathbf{u} , \mathbf{v} and \mathbf{w} be vectors. Let r and s be numbers. Then

- 1 $(r\mathbf{u}) \times (s\mathbf{v}) = (rs)(\mathbf{u} \times \mathbf{v});$
- 2 $\mathbf{u} \times (\mathbf{v} + \mathbf{w}) = (\mathbf{u} \times \mathbf{v}) + (\mathbf{u} \times \mathbf{w});$
- 3 $\mathbf{v} \times \mathbf{u} = -\mathbf{u} \times \mathbf{v};$
- 4 $(\mathbf{v} + \mathbf{w}) \times \mathbf{u} = (\mathbf{v} \times \mathbf{u}) + (\mathbf{w} \times \mathbf{u});$
- 5 $\mathbf{0} \times \mathbf{u} = \mathbf{0};$ and
- 6 $\mathbf{u} \times (\mathbf{v} \times \mathbf{w}) = (\mathbf{u} \cdot \mathbf{w})\mathbf{v} - (\mathbf{u} \cdot \mathbf{v})\mathbf{w}.$

11.4 The Cross Product



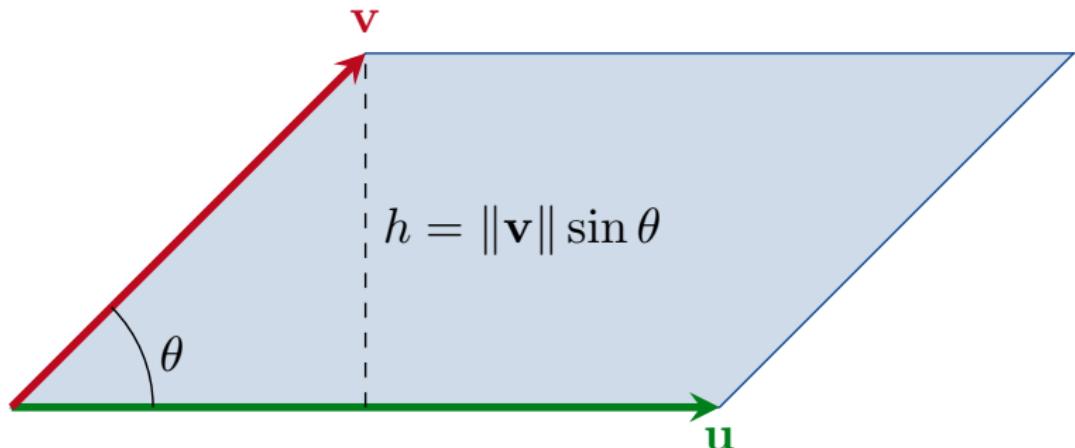
Property (iii)



$$\mathbf{v} \times \mathbf{u} = -\mathbf{u} \times \mathbf{v}$$

$$\mathbf{u} \times \mathbf{v} = \|\mathbf{u}\| \|\mathbf{v}\| (\sin \theta) \mathbf{n}$$

Area of a Parallelogram

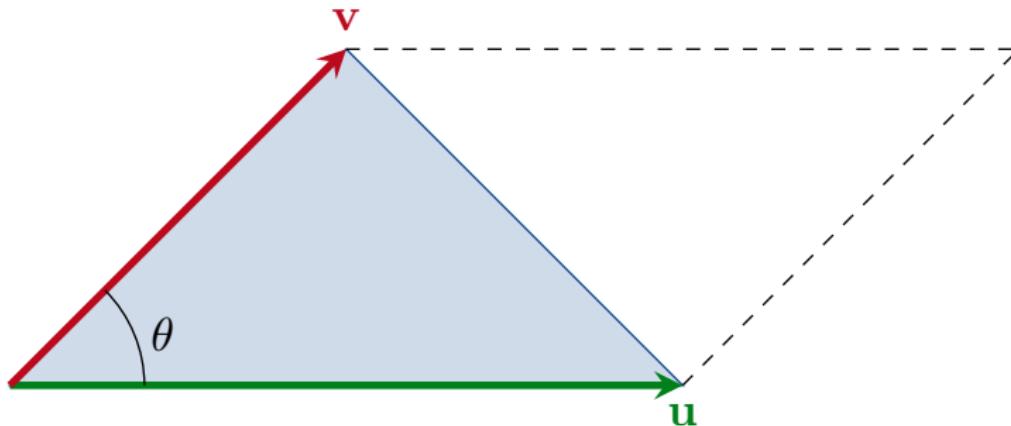


$$\text{area} = (\text{base}) (\text{height}) = \|\mathbf{u}\| \|\mathbf{v}\| \sin \theta = \|\mathbf{u} \times \mathbf{v}\| .$$

11.4 The Cross Product



Area of a Triangle

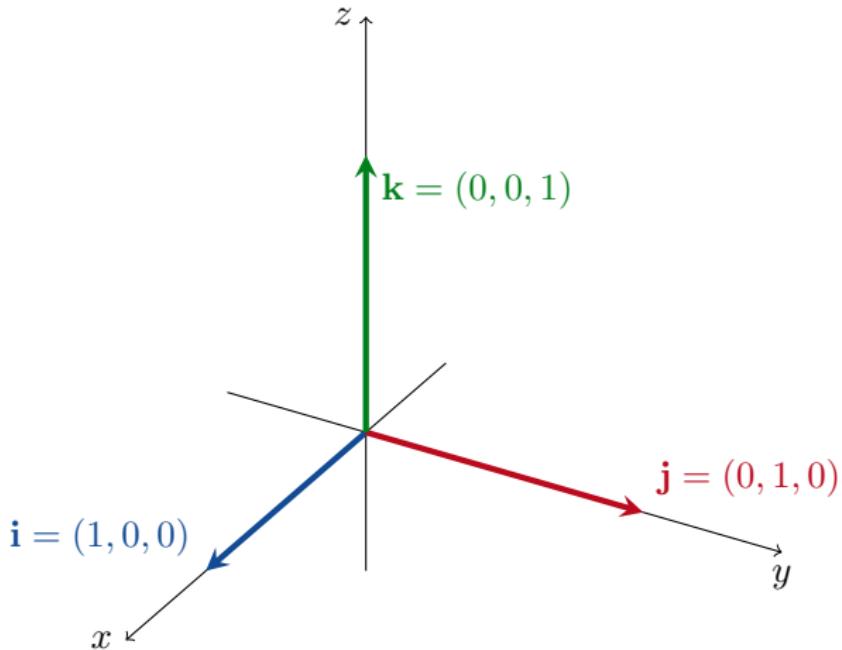


$$\begin{aligned}\text{area of triangle} &= \frac{1}{2} (\text{area of parallelogram}) \\ &= \frac{1}{2} \|\mathbf{u} \times \mathbf{v}\|.\end{aligned}$$

11.4 The Cross Product



A Formula for $\mathbf{u} \times \mathbf{v}$



11.4

$$\mathbf{u} \times \mathbf{v} = \|\mathbf{u}\| \|\mathbf{v}\| (\sin \theta) \mathbf{n}$$



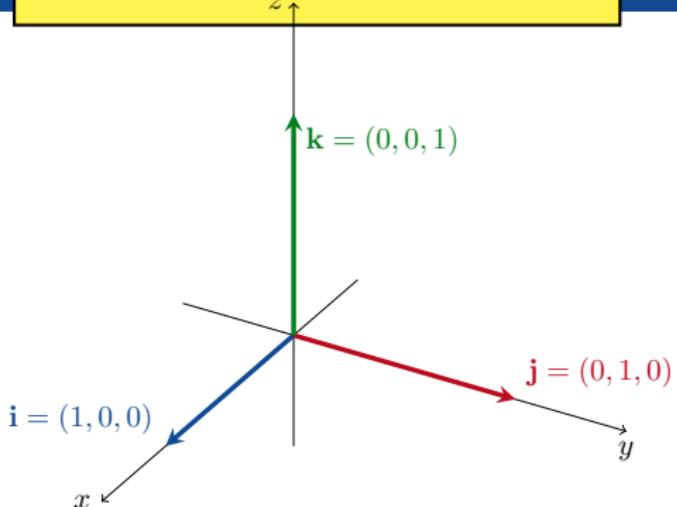
Note first that

$$\mathbf{i} \times \mathbf{i} = \|\mathbf{i}\| \|\mathbf{i}\| \sin 0^\circ \mathbf{n} = \mathbf{0}.$$

Similarly $\mathbf{j} \times \mathbf{j} = \mathbf{0}$ and $\mathbf{k} \times \mathbf{k} = \mathbf{0}$ also.

11.4

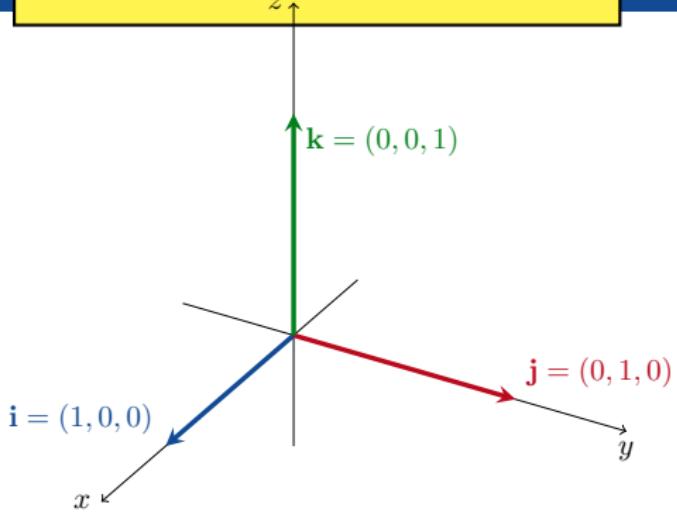
$$\mathbf{u} \times \mathbf{v} = \|\mathbf{u}\| \|\mathbf{v}\| (\sin \theta) \mathbf{n}$$



Next note that $\mathbf{i} \times \mathbf{j}$ must point in the same direction as \mathbf{k} by the left-hand rule.

11.4

$$\mathbf{u} \times \mathbf{v} = \|\mathbf{u}\| \|\mathbf{v}\| (\sin \theta) \mathbf{n}$$

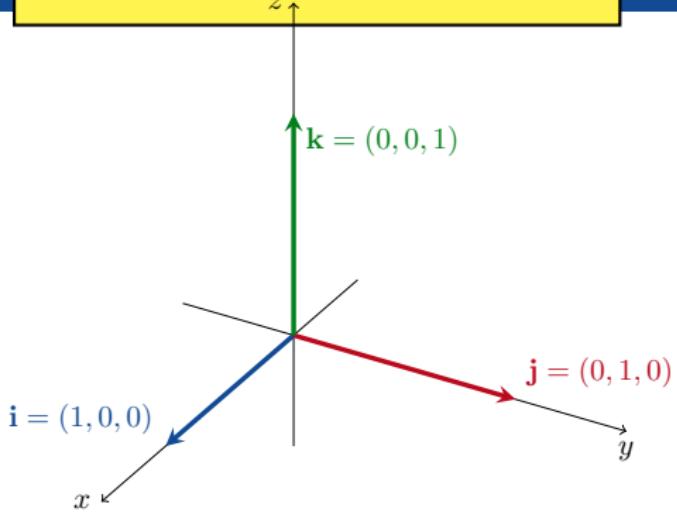


Next note that $\mathbf{i} \times \mathbf{j}$ must point in the same direction as \mathbf{k} by the left-hand rule. Thus

$$\mathbf{i} \times \mathbf{j} = \|\mathbf{i}\| \|\mathbf{j}\| \sin 90^\circ \mathbf{k} = \mathbf{k}.$$

11.4

$$\mathbf{u} \times \mathbf{v} = \|\mathbf{u}\| \|\mathbf{v}\| (\sin \theta) \mathbf{n}$$



Next note that $\mathbf{i} \times \mathbf{j}$ must point in the same direction as \mathbf{k} by the left-hand rule. Thus

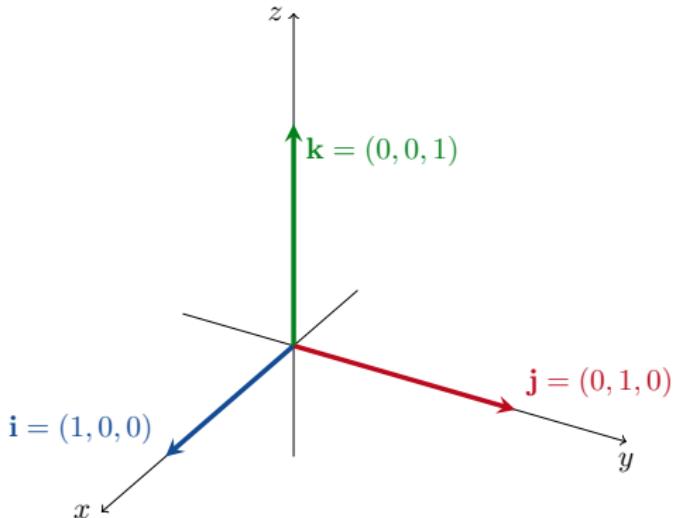
$$\mathbf{i} \times \mathbf{j} = \|\mathbf{i}\| \|\mathbf{j}\| \sin 90^\circ \mathbf{k} = \mathbf{k}.$$

We then immediately also have

$$\mathbf{j} \times \mathbf{i} = -(\mathbf{i} \times \mathbf{j}) = -\mathbf{k}.$$

11.4

$$\mathbf{u} \times \mathbf{v} = \|\mathbf{u}\| \|\mathbf{v}\| (\sin \theta) \mathbf{n}$$



I leave it for you to check that

$$\mathbf{j} \times \mathbf{k} = \mathbf{i}, \quad \mathbf{k} \times \mathbf{j} = -\mathbf{i}, \quad \mathbf{i} \times \mathbf{k} = -\mathbf{j} \quad \text{and} \quad \mathbf{k} \times \mathbf{i} = \mathbf{j}.$$

11.4 The Cross Product



Now suppose that $\mathbf{u} = u_1\mathbf{i} + u_2\mathbf{j} + u_3\mathbf{k}$
and $\mathbf{v} = v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}$.

11.4 The Cross Product



Now suppose that $\mathbf{u} = u_1\mathbf{i} + u_2\mathbf{j} + u_3\mathbf{k}$
and $\mathbf{v} = v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}$. Then we can calculate
that

$$\mathbf{u} \times \mathbf{v} = (u_1\mathbf{i} + u_2\mathbf{j} + u_3\mathbf{k}) \times (v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k})$$

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11.4 The Cross Product

Now suppose that $\mathbf{u} = u_1\mathbf{i} + u_2\mathbf{j} + u_3\mathbf{k}$
and $\mathbf{v} = v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}$. Then we can calculate
that

$$\begin{aligned}\mathbf{u} \times \mathbf{v} &= (u_1\mathbf{i} + u_2\mathbf{j} + u_3\mathbf{k}) \times (v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}) \\&= u_1v_1\mathbf{i} \times \mathbf{i} + u_1v_2\mathbf{i} \times \mathbf{j} + u_1v_3\mathbf{i} \times \mathbf{k} + u_2v_1\mathbf{j} \times \mathbf{i} + u_2v_2\mathbf{j} \times \mathbf{j} \\&\quad + u_2v_3\mathbf{j} \times \mathbf{k} + u_3v_1\mathbf{k} \times \mathbf{i} + u_3v_2\mathbf{k} \times \mathbf{j} + u_3v_3\mathbf{k} \times \mathbf{k} \\&= \\&= \end{aligned}$$

11.4 The Cross Product

$$\begin{aligned}\mathbf{i} \times \mathbf{j} &= \mathbf{k} \\ \mathbf{j} \times \mathbf{i} &= -\mathbf{k} \\ \mathbf{j} \times \mathbf{k} &= \mathbf{i} \\ \mathbf{k} \times \mathbf{j} &= -\mathbf{i} \\ \mathbf{i} \times \mathbf{k} &= -\mathbf{j} \\ \mathbf{k} \times \mathbf{i} &= \mathbf{j}\end{aligned}$$

Now suppose that $\mathbf{u} = u_1\mathbf{i} + u_2\mathbf{j} + u_3\mathbf{k}$
and $\mathbf{v} = v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}$. Then we can calculate
that

$$\begin{aligned}\mathbf{u} \times \mathbf{v} &= (u_1\mathbf{i} + u_2\mathbf{j} + u_3\mathbf{k}) \times (v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}) \\ &= u_1v_1\mathbf{i} \times \mathbf{i} + u_1v_2\mathbf{i} \times \mathbf{j} + u_1v_3\mathbf{i} \times \mathbf{k} + u_2v_1\mathbf{j} \times \mathbf{i} + u_2v_2\mathbf{j} \times \mathbf{j} \\ &\quad + u_2v_3\mathbf{j} \times \mathbf{k} + u_3v_1\mathbf{k} \times \mathbf{i} + u_3v_2\mathbf{k} \times \mathbf{j} + u_3v_3\mathbf{k} \times \mathbf{k} \\ &= \\ &= \end{aligned}$$

11.4 The Cross Product

Now suppose that $\mathbf{u} = u_1\mathbf{i} + u_2\mathbf{j} + u_3\mathbf{k}$
and $\mathbf{v} = v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}$. Then we can calculate
that

$$\begin{aligned}\mathbf{i} \times \mathbf{j} &= \mathbf{k} \\ \mathbf{j} \times \mathbf{i} &= -\mathbf{k} \\ \mathbf{j} \times \mathbf{k} &= \mathbf{i} \\ \mathbf{k} \times \mathbf{j} &= -\mathbf{i} \\ \mathbf{i} \times \mathbf{k} &= -\mathbf{j} \\ \mathbf{k} \times \mathbf{i} &= \mathbf{j}\end{aligned}$$

$$\begin{aligned}\mathbf{u} \times \mathbf{v} &= (u_1\mathbf{i} + u_2\mathbf{j} + u_3\mathbf{k}) \times (v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}) \\ &= u_1v_1\mathbf{i} \times \mathbf{i} + u_1v_2\mathbf{i} \times \mathbf{j} + u_1v_3\mathbf{i} \times \mathbf{k} + u_2v_1\mathbf{j} \times \mathbf{i} + u_2v_2\mathbf{j} \times \mathbf{j} \\ &\quad + u_2v_3\mathbf{j} \times \mathbf{k} + u_3v_1\mathbf{k} \times \mathbf{i} + u_3v_2\mathbf{k} \times \mathbf{j} + u_3v_3\mathbf{k} \times \mathbf{k} \\ &= \mathbf{0} + u_1v_2\mathbf{k} - u_1v_3\mathbf{j} - u_2v_1\mathbf{k} + \mathbf{0} + u_2v_3\mathbf{i} + u_3v_1\mathbf{j} - u_3v_2\mathbf{i} + \mathbf{0} \\ &= \end{aligned}$$

11.4 The Cross Product

$$\begin{aligned}\mathbf{i} \times \mathbf{j} &= \mathbf{k} \\ \mathbf{j} \times \mathbf{i} &= -\mathbf{k} \\ \mathbf{j} \times \mathbf{k} &= \mathbf{i} \\ \mathbf{k} \times \mathbf{j} &= -\mathbf{i} \\ \mathbf{i} \times \mathbf{k} &= -\mathbf{j} \\ \mathbf{k} \times \mathbf{i} &= \mathbf{j}\end{aligned}$$

Now suppose that $\mathbf{u} = u_1\mathbf{i} + u_2\mathbf{j} + u_3\mathbf{k}$
and $\mathbf{v} = v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}$. Then we can calculate
that

$$\begin{aligned}\mathbf{u} \times \mathbf{v} &= (u_1\mathbf{i} + u_2\mathbf{j} + u_3\mathbf{k}) \times (v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}) \\ &= u_1v_1\mathbf{i} \times \mathbf{i} + u_1v_2\mathbf{i} \times \mathbf{j} + u_1v_3\mathbf{i} \times \mathbf{k} + u_2v_1\mathbf{j} \times \mathbf{i} + u_2v_2\mathbf{j} \times \mathbf{j} \\ &\quad + u_2v_3\mathbf{j} \times \mathbf{k} + u_3v_1\mathbf{k} \times \mathbf{i} + u_3v_2\mathbf{k} \times \mathbf{j} + u_3v_3\mathbf{k} \times \mathbf{k} \\ &= \mathbf{0} + u_1v_2\mathbf{k} - u_1v_3\mathbf{j} - u_2v_1\mathbf{k} + \mathbf{0} + u_2v_3\mathbf{i} + u_3v_1\mathbf{j} - u_3v_2\mathbf{i} + \mathbf{0} \\ &= (u_2v_3 - u_3v_2)\mathbf{i} - (u_1v_3 - u_3v_1)\mathbf{j} + (u_1v_2 - u_2v_1)\mathbf{k}.\end{aligned}$$

11.4 The Cross Product



Theorem

If $\mathbf{u} = u_1\mathbf{i} + u_2\mathbf{j} + u_3\mathbf{k}$ and $\mathbf{v} = v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}$, then

$$\mathbf{u} \times \mathbf{v} = (u_2v_3 - u_3v_2)\mathbf{i} - (u_1v_3 - u_3v_1)\mathbf{j} + (u_1v_2 - u_2v_1)\mathbf{k}.$$

11.4 The Cross Product



If you studied matrices and determinants at high school, then you may prefer to use the following symbolic determinant formula instead.

$$\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}.$$

$$\mathbf{u} \times \mathbf{v} = (u_2 v_3 - u_3 v_2) \mathbf{i} - (u_1 v_3 - u_3 v_1) \mathbf{j} + (u_1 v_2 - u_2 v_1) \mathbf{k}$$



Example

Find $\mathbf{u} \times \mathbf{v}$ and $\mathbf{v} \times \mathbf{u}$ if $\mathbf{u} = 2\mathbf{i} + \mathbf{j} + \mathbf{k}$ and $\mathbf{v} = -4\mathbf{i} + 3\mathbf{j} + \mathbf{k}$.

$$\mathbf{u} \times \mathbf{v} = (u_2 v_3 - u_3 v_2) \mathbf{i} - (u_1 v_3 - u_3 v_1) \mathbf{j} + (u_1 v_2 - u_2 v_1) \mathbf{k}$$



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$$\mathbf{u} \times \mathbf{v} = (1 - 3)\mathbf{i} - (2 - -4)\mathbf{j} + (6 - -4)\mathbf{k} = -2\mathbf{i} - 6\mathbf{j} + 10\mathbf{k}$$

$$\mathbf{u} \times \mathbf{v} = (u_2 v_3 - u_3 v_2) \mathbf{i} - (u_1 v_3 - u_3 v_1) \mathbf{j} + (u_1 v_2 - u_2 v_1) \mathbf{k}$$



Example

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$$\mathbf{u} \times \mathbf{v} = (1 - 3)\mathbf{i} - (2 - -4)\mathbf{j} + (6 - -4)\mathbf{k} = -2\mathbf{i} - 6\mathbf{j} + 10\mathbf{k}$$

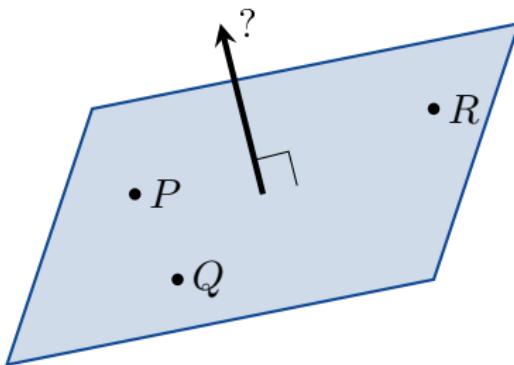
and

$$\mathbf{v} \times \mathbf{u} = -\mathbf{u} \times \mathbf{v} = 2\mathbf{i} + 6\mathbf{j} - 10\mathbf{k}.$$

$$\mathbf{u} \times \mathbf{v} = (u_2 v_3 - u_3 v_2) \mathbf{i} - (u_1 v_3 - u_3 v_1) \mathbf{j} + (u_1 v_2 - u_2 v_1) \mathbf{k}$$

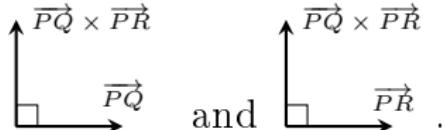
Example

Find a vector perpendicular to the plane containing the three points $P(1, -1, 0)$, $Q(2, 1, -1)$ and $R(-1, 1, 2)$.

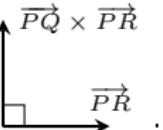


$$\mathbf{u} \times \mathbf{v} = (u_2 v_3 - u_3 v_2) \mathbf{i} - (u_1 v_3 - u_3 v_1) \mathbf{j} + (u_1 v_2 - u_2 v_1) \mathbf{k}$$

The vector $\overrightarrow{PQ} \times \overrightarrow{PR}$ is perpendicular to the plane because



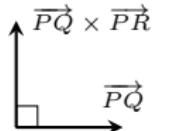
and

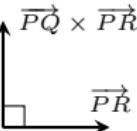


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$$\mathbf{u} \times \mathbf{v} = (u_2 v_3 - u_3 v_2) \mathbf{i} - (u_1 v_3 - u_3 v_1) \mathbf{j} + (u_1 v_2 - u_2 v_1) \mathbf{k}$$

The vector $\overrightarrow{PQ} \times \overrightarrow{PR}$ is perpendicular to the plane because



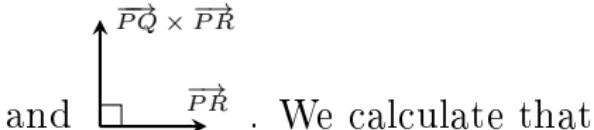
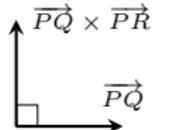
and  . We calculate that

$$\begin{aligned}\overrightarrow{PQ} &= Q - P = (2, 1, -1) - (1, -1, 0) \\ &= (2 - 1, 1 + 1, -1 - 0) = \mathbf{i} + 2\mathbf{j} - \mathbf{k}\end{aligned}$$

$$\begin{aligned}\overrightarrow{PR} &= R - P = (-1, 1, 2) - (1, -1, 0) \\ &= (-1 - 1, 1 + 1, 2 - 0) = -2\mathbf{i} + 2\mathbf{j} + 2\mathbf{k}\end{aligned}$$

$$\mathbf{u} \times \mathbf{v} = (u_2 v_3 - u_3 v_2) \mathbf{i} - (u_1 v_3 - u_3 v_1) \mathbf{j} + (u_1 v_2 - u_2 v_1) \mathbf{k}$$

The vector $\overrightarrow{PQ} \times \overrightarrow{PR}$ is perpendicular to the plane because



and . We calculate that

$$\begin{aligned}\overrightarrow{PQ} &= Q - P = (2, 1, -1) - (1, -1, 0) \\ &= (2 - 1, 1 + 1, -1 - 0) = \mathbf{i} + 2\mathbf{j} - \mathbf{k}\end{aligned}$$

$$\begin{aligned}\overrightarrow{PR} &= R - P = (-1, 1, 2) - (1, -1, 0) \\ &= (-1 - 1, 1 + 1, 2 - 0) = -2\mathbf{i} + 2\mathbf{j} + 2\mathbf{k}\end{aligned}$$

and

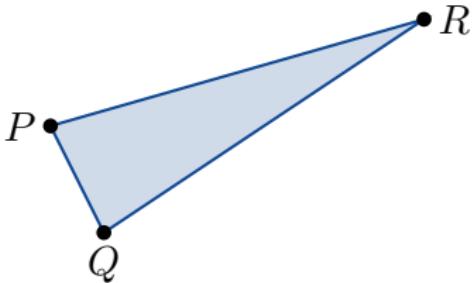
$$\overrightarrow{PQ} \times \overrightarrow{PR} = (4 + 2)\mathbf{i} - (2 - 2)\mathbf{j} + (2 + 4)\mathbf{k} = 6\mathbf{i} + 6\mathbf{k}.$$

11.4 The Cross Product

Example

Find the area of triangle PQR .

$P(1, -1, 0)$, $Q(2, 1, -1)$ and $R(-1, 1, 2)$

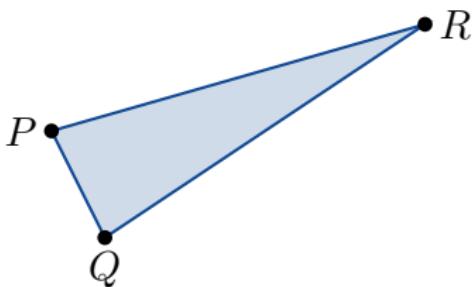


11.4 The Cross Product

Example

Find the area of triangle PQR .

$P(1, -1, 0)$, $Q(2, 1, -1)$ and $R(-1, 1, 2)$



The area of the triangle is

$$\begin{aligned}\text{area} &= \frac{1}{2} \left\| \overrightarrow{PQ} \times \overrightarrow{PR} \right\| = \frac{1}{2} \|6\mathbf{i} + 6\mathbf{k}\| \\ &= \frac{1}{2} \sqrt{6^2 + 0^2 + 6^2} = 3\sqrt{2}.\end{aligned}$$

11.4 The Cross Product



Example

Find a unit vector perpendicular to the plane containing P , Q and R .

$P(1, -1, 0)$, $Q(2, 1, -1)$ and $R(-1, 1, 2)$

11.4 The Cross Product

Example

Find a unit vector perpendicular to the plane containing P , Q and R .

$P(1, -1, 0)$, $Q(2, 1, -1)$ and $R(-1, 1, 2)$

We know that $\overrightarrow{PQ} \times \overrightarrow{PR}$ is perpendicular to the plane. We just need to normalise this vector to find a unit vector.

11.4 The Cross Product



Example

Find a unit vector perpendicular to the plane containing P , Q and R .

$P(1, -1, 0)$, $Q(2, 1, -1)$ and $R(-1, 1, 2)$

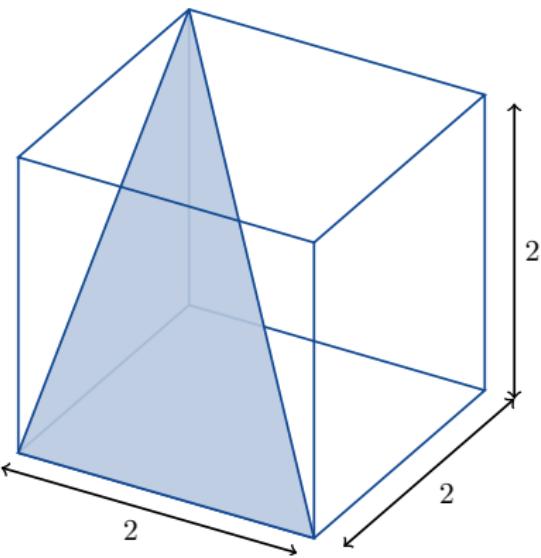
We know that $\overrightarrow{PQ} \times \overrightarrow{PR}$ is perpendicular to the plane. We just need to normalise this vector to find a unit vector.

$$\mathbf{n} = \frac{\overrightarrow{PQ} \times \overrightarrow{PR}}{\|\overrightarrow{PQ} \times \overrightarrow{PR}\|} = \frac{6\mathbf{i} + 6\mathbf{k}}{6\sqrt{2}} = \frac{1}{\sqrt{2}}\mathbf{i} + \frac{1}{\sqrt{2}}\mathbf{k}.$$

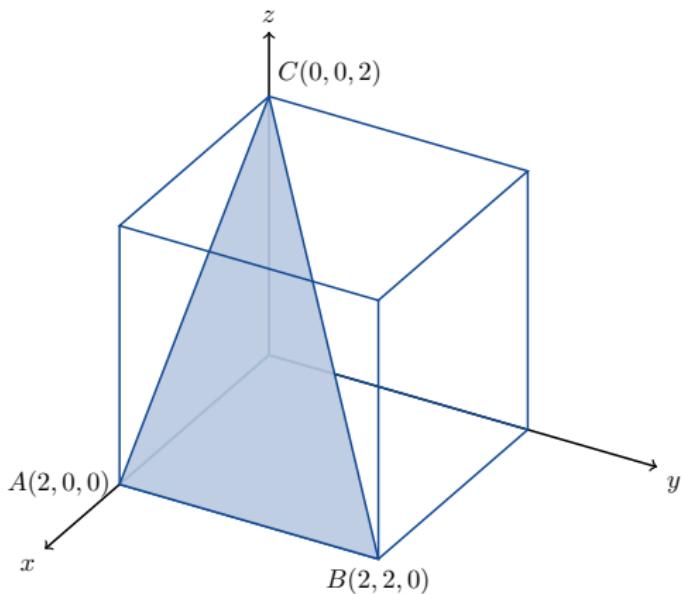
11.4 The Cross Product

Example

A triangle is inscribed inside a cube of side 2 as shown below.
Use the cross product to find the area of the triangle.



11.4 The Cross Product



First we draw coordinate axes and assign coordinates to the vertices of the triangle.

11.4 The Cross Product

Then we can calculate

$$\overrightarrow{AB} = B - A = (2, 2, 0) - (2, 0, 0) = (0, 2, 0) = 2\mathbf{j}$$

and

$$\overrightarrow{AC} = C - A = (0, 0, 2) - (2, 0, 0) = (-2, 0, 2) = -2\mathbf{i} + 2\mathbf{k}.$$

11.4 The Cross Product

Then we can calculate

$$\overrightarrow{AB} = B - A = (2, 2, 0) - (2, 0, 0) = (0, 2, 0) = 2\mathbf{j}$$

and

$$\overrightarrow{AC} = C - A = (0, 0, 2) - (2, 0, 0) = (-2, 0, 2) = -2\mathbf{i} + 2\mathbf{k}.$$

It follows that

$$\begin{aligned}\overrightarrow{AB} \times \overrightarrow{AC} &= (2\mathbf{j}) \times (-2\mathbf{i} + 2\mathbf{k}) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 2 & 0 \\ -2 & 0 & 2 \end{vmatrix} \\ &= \mathbf{i}(4 - 0) - \mathbf{j}(0 - 0) + \mathbf{k}(0 - -4) = 4\mathbf{i} + 4\mathbf{k}.\end{aligned}$$

11.4 The Cross Product

Then we can calculate

$$\overrightarrow{AB} = B - A = (2, 2, 0) - (2, 0, 0) = (0, 2, 0) = 2\mathbf{j}$$

and

$$\overrightarrow{AC} = C - A = (0, 0, 2) - (2, 0, 0) = (-2, 0, 2) = -2\mathbf{i} + 2\mathbf{k}.$$

It follows that

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Therefore

$$\begin{aligned}\text{area of triangle} &= \frac{1}{2} \left\| \overrightarrow{AB} \times \overrightarrow{AC} \right\| = \frac{1}{2} \sqrt{4^2 + 0^2 + 4^2} \\ &= \frac{1}{2} \sqrt{32} = \frac{1}{2} \sqrt{4} \sqrt{8} = \sqrt{8} = 2\sqrt{2}.\end{aligned}$$

11.4 The Cross Product



The Triple Scalar Product

Definition

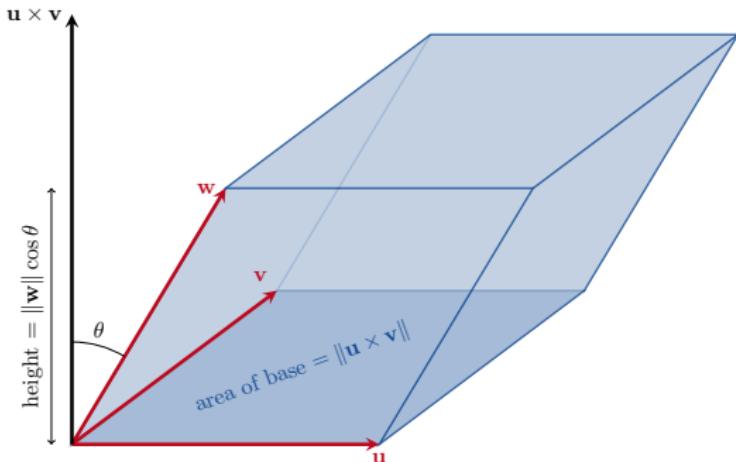
The *triple scalar product* of \mathbf{u} , \mathbf{v} and \mathbf{w} is

$$(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w}.$$

11.4 The Cross Product



The Volume of a Parallelepiped



$$\text{volume} = (\text{area of base})(\text{height}) = \|\mathbf{u} \times \mathbf{v}\| \|\mathbf{w}\| \cos \theta = |(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w}|$$

11.4 The Cross Product



One Final Comment

We can do the dot product in both \mathbb{R}^2 and \mathbb{R}^3 . But we can only do the cross product in \mathbb{R}^3 . There is no cross product in \mathbb{R}^2 .



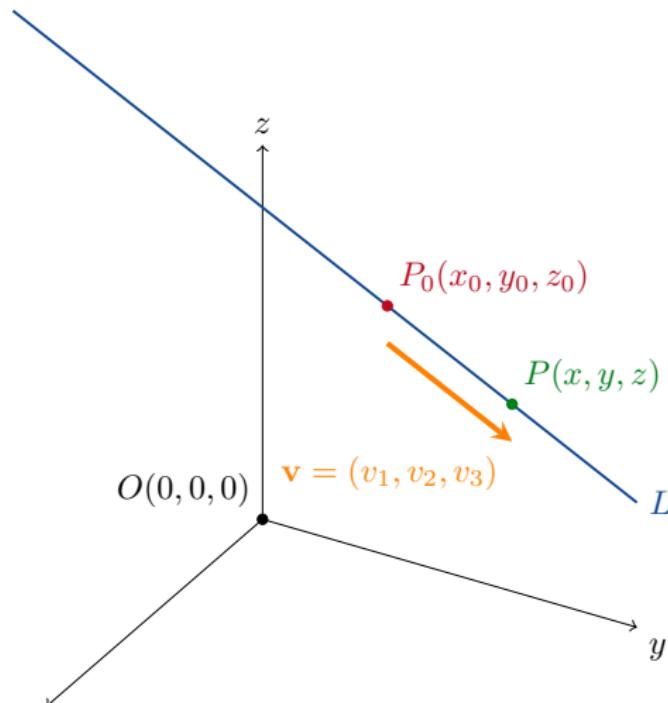
1 Lines and Planes in Space 5

11.5 Lines and Planes in Space



To describe a line in \mathbb{R}^3 , we need

- a point $P_0(x_0, y_0, z_0)$ which the line passes through; and
- a vector \mathbf{v} which gives the direction of the line.



11.5 Lines and Planes in Space



Let $\mathbf{r}_0 = \overrightarrow{OP_0}$ and $\mathbf{r} = \overrightarrow{OP}$.

Definition

The *line L passing through $P_0(x_0, y_0, z_0)$ parallel to $\mathbf{v} = (v_1, v_2, v_3)$* has the vector equation

$$\mathbf{r} = \mathbf{r}_0 + t\mathbf{v}, \quad -\infty < t < \infty.$$

11.5 Lines and Planes in Space



This equation is equivalent to

$$(x, y, z) = (x_0, y_0, z_0) + t(v_1, v_2, v_3)$$

or to the set of three equations

$$x = x_0 + tv_1, \quad y = y_0 + tv_2, \quad z = z_0 + tv_3.$$

11.5 Lines and Planes in Space



Definition

The *parametric equations* for the line L passing through $P_0(x_0, y_0, z_0)$ parallel to $\mathbf{v} = (v_1, v_2, v_3)$ are

$$x = x_0 + tv_1, \quad y = y_0 + tv_2, \quad z = z_0 + tv_3.$$

11.5 Lines and Planes in Space



Example

Find parametric equations for the line passing through $P_0(-2, 0, 4)$ parallel to $\mathbf{v} = 2\mathbf{i} + 4\mathbf{j} - 2\mathbf{k}$.

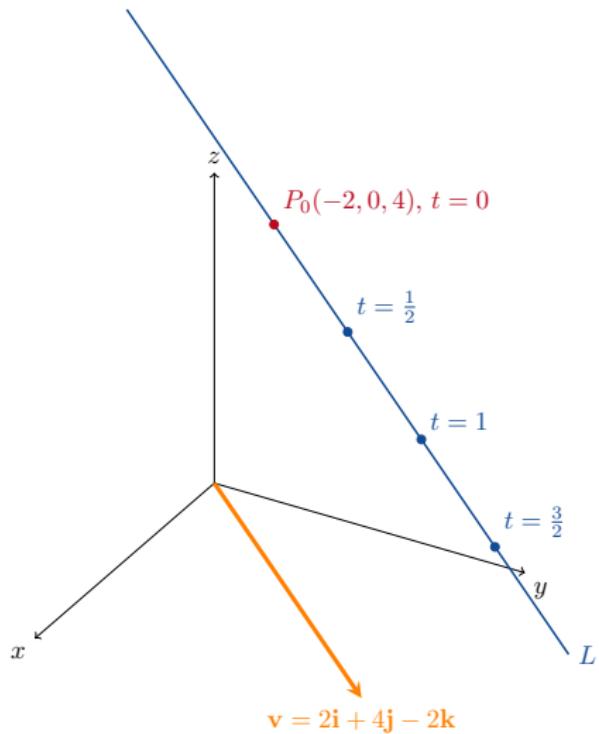
We can write

$$x = -2 + 2t, \quad y = 4t, \quad z = 4 - 2t.$$

11.5 Lines and Planes in Space



$$\begin{aligned}x &= -2 + 2t \\y &= 4t \\z &= 4 - 2t\end{aligned}$$



11.5 Lines and Planes in Space



Example

Find parametric equations for the line passing through $P(-3, 2, -3)$ and $Q(1, -1, 4)$.

Choose $P_0 = P$ and $\mathbf{v} = \overrightarrow{PQ} = (4, -3, 7) = 4\mathbf{i} - 3\mathbf{j} + 7\mathbf{k}$. Then we can write

$$x = -3 + 4t, \quad y = 2 - 3t, \quad z = -3 + 7t.$$

11.5 Lines and Planes in Space



Definition

The vector equation

$$\mathbf{r} = \mathbf{r}_0 + t\mathbf{v}, \quad a \leq t \leq b$$

denotes a *line segment*.

11.5 Lines and Planes in Space



Example

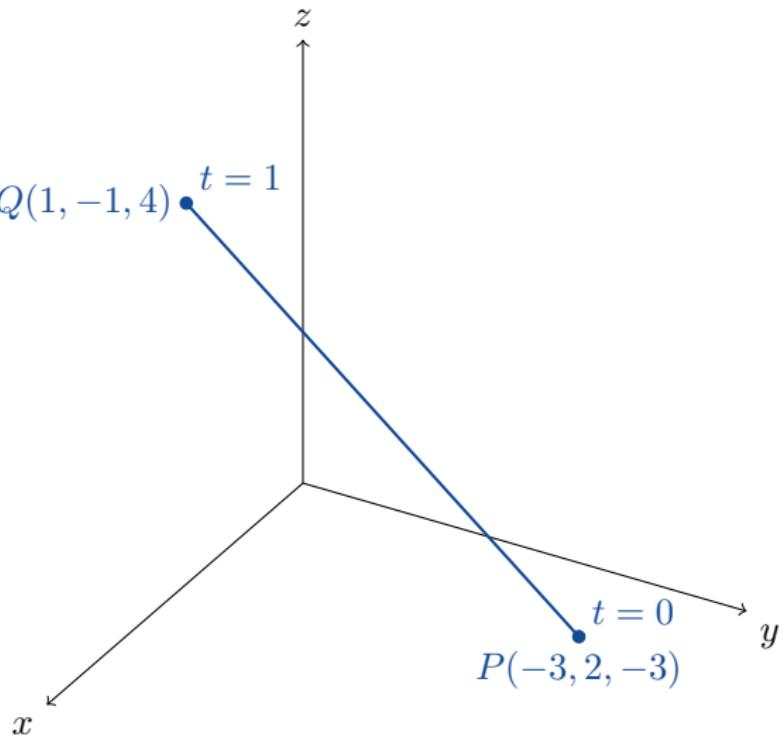
Parametrise the line segment joining $P(-3, 2, -3)$ and $Q(1, -1, 4)$.

We know that $x = -3 + 4t$, $y = 2 - 3t$ and $z = -3 + 7t$. The line passes through P then $t = 0$ and passed through Q when $t = 1$. Therefore

$$x = -3 + 4t, \quad y = 2 - 3t, \quad z = -3 + 7t, \quad 0 \leq t \leq 1$$

denotes the line segment from P to Q .

11.5 Lines and Planes in Space



EXAMPLE 4 A helicopter is to fly directly from a helipad at the origin in the direction of the point $(1, 1, 1)$ at a speed of 60 m/sec. What is the position of the helicopter after 10 sec?

Solution We place the origin at the starting position (helipad) of the helicopter. Then the unit vector

$$\mathbf{u} = \frac{1}{\sqrt{3}}\mathbf{i} + \frac{1}{\sqrt{3}}\mathbf{j} + \frac{1}{\sqrt{3}}\mathbf{k}$$

gives the flight direction of the helicopter. From Equation (4), the position of the helicopter at any time t is

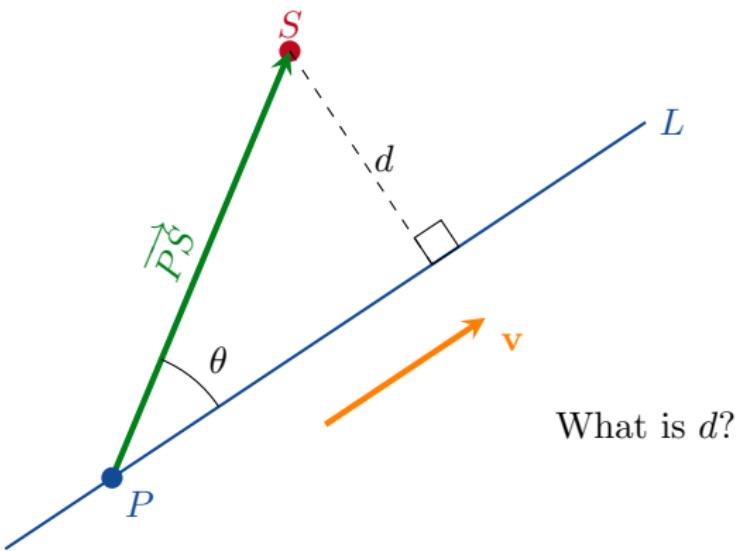
$$\begin{aligned}\mathbf{r}(t) &= \mathbf{r}_0 + t(\text{speed})\mathbf{u} \\ &= \mathbf{0} + t(60)\left(\frac{1}{\sqrt{3}}\mathbf{i} + \frac{1}{\sqrt{3}}\mathbf{j} + \frac{1}{\sqrt{3}}\mathbf{k}\right) \\ &= 20\sqrt{3}t(\mathbf{i} + \mathbf{j} + \mathbf{k}).\end{aligned}$$

When $t = 10$ sec,

$$\begin{aligned}\mathbf{r}(10) &= 200\sqrt{3}(\mathbf{i} + \mathbf{j} + \mathbf{k}) \\ &= \langle 200\sqrt{3}, 200\sqrt{3}, 200\sqrt{3} \rangle.\end{aligned}$$

After 10 sec of flight from the origin toward $(1, 1, 1)$, the helicopter is located at the point $(200\sqrt{3}, 200\sqrt{3}, 200\sqrt{3})$ in space. It has traveled a distance of $(60 \text{ m/sec})(10 \text{ sec}) = 600 \text{ m}$, which is the length of the vector $\mathbf{r}(10)$. ■

The Distance from a Point to a Line

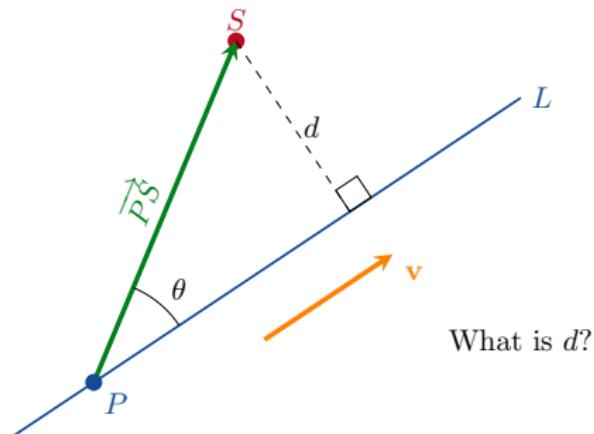


11.5 Lines and Planes in Space



Let d be the shortest distance from the point S to the line L .
We can see from this diagram that

$$d = \|\overrightarrow{PS}\| \sin \theta.$$



11.5 Lines and Planes in Space



Let d be the shortest distance from the point S to the line L . We can see from this diagram that

$$d = \|\overrightarrow{PS}\| \sin \theta.$$

But remember that $\overrightarrow{PS} \times \mathbf{v} = \|\overrightarrow{PS}\| \|\mathbf{v}\| \sin \theta \mathbf{n}$. Therefore

$$d = \frac{\|\overrightarrow{PS} \times \mathbf{v}\|}{\|\mathbf{v}\|}.$$

$$\mathbf{u} \times \mathbf{v} = (u_2 v_3 - u_3 v_2) \mathbf{i} - (u_1 v_3 - u_3 v_1) \mathbf{j} + (u_1 v_2 - u_2 v_1) \mathbf{k}$$



Example

Find the distance from the point $S(1, 1, 5)$ to the line

$$x = 1 + t, \quad y = 3 - t, \quad z = 2t.$$

$$\mathbf{u} \times \mathbf{v} = (u_2 v_3 - u_3 v_2) \mathbf{i} - (u_1 v_3 - u_3 v_1) \mathbf{j} + (u_1 v_2 - u_2 v_1) \mathbf{k}$$



Example

Find the distance from the point $S(1, 1, 5)$ to the line

$$x = 1 + t, \quad y = 3 - t, \quad z = 2t.$$

The line passes through the point $P(1, 3, 0)$ in the direction $\mathbf{v} = \mathbf{i} - \mathbf{j} + 2\mathbf{k}$.

$$\mathbf{u} \times \mathbf{v} = (u_2 v_3 - u_3 v_2) \mathbf{i} - (u_1 v_3 - u_3 v_1) \mathbf{j} + (u_1 v_2 - u_2 v_1) \mathbf{k}$$



Example

Find the distance from the point $S(1, 1, 5)$ to the line

$$x = 1 + t, \quad y = 3 - t, \quad z = 2t.$$

The line passes through the point $P(1, 3, 0)$ in the direction $\mathbf{v} = \mathbf{i} - \mathbf{j} + 2\mathbf{k}$. Thus

$$\overrightarrow{PS} = S - P = (1, 1, 5) - (1, 3, 0) = (0, -2, 5) = -2\mathbf{j} + 5\mathbf{k}$$

and

$$\overrightarrow{PS} \times \mathbf{v} = (-4 + 5)\mathbf{i} - (0 - 5)\mathbf{j} + (0 + 2)\mathbf{k} = \mathbf{i} + 5\mathbf{j} + 2\mathbf{k}.$$

$$\mathbf{u} \times \mathbf{v} = (u_2 v_3 - u_3 v_2) \mathbf{i} - (u_1 v_3 - u_3 v_1) \mathbf{j} + (u_1 v_2 - u_2 v_1) \mathbf{k}$$

Example

Find the distance from the point $S(1, 1, 5)$ to the line

$$x = 1 + t, \quad y = 3 - t, \quad z = 2t.$$

The line passes through the point $P(1, 3, 0)$ in the direction $\mathbf{v} = \mathbf{i} - \mathbf{j} + 2\mathbf{k}$. Thus

$$\overrightarrow{PS} = S - P = (1, 1, 5) - (1, 3, 0) = (0, -2, 5) = -2\mathbf{j} + 5\mathbf{k}$$

and

$$\overrightarrow{PS} \times \mathbf{v} = (-4 + 5)\mathbf{i} - (0 - 5)\mathbf{j} + (0 + 2)\mathbf{k} = \mathbf{i} + 5\mathbf{j} + 2\mathbf{k}.$$

Therefore

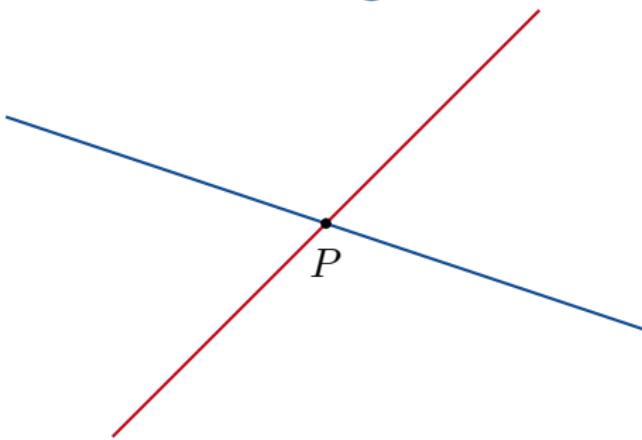
$$d = \frac{\|\overrightarrow{PS} \times \mathbf{v}\|}{\|\mathbf{v}\|} = \frac{\sqrt{1^2 + 5^2 + 2^2}}{\sqrt{1^2 + 1^2 + 2^2}} = \frac{\sqrt{30}}{\sqrt{6}} = \sqrt{5}.$$

Break

We will continue at 2pm

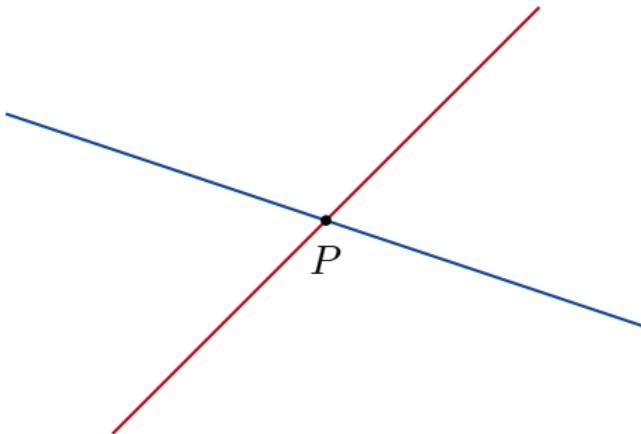


Intersecting Lines¹



¹not in book

Intersecting Lines¹



Definition

Two lines intersect at a point P if and only if P lies on both lines.

¹not in book

11.5 Lines and Planes in Space

Example

Do the following two lines intersect? If yes, where?

- 1 $x = 7 - t, y = 3 + 3t, z = 2t.$
- 2 $x = -1 + 2s, y = 3s, z = 1 + s.$

11.5 Lines and Planes in Space

Example

Do the following two lines intersect? If yes, where?

- 1 $x = 7 - t, y = 3 + 3t, z = 2t.$
- 2 $x = -1 + 2s, y = 3s, z = 1 + s.$

The two lines intersect if and only if there exist $s, t \in \mathbb{R}$ such that

$$7 - t = x = -1 + 2s$$

$$3 + 3t = y = 3s$$

$$2t = z = 1 + s$$

11.5 Lines and Planes in Space



Example

Do the following two lines intersect? If yes, where?

- 1 $x = 7 - t, y = 3 + 3t, z = 2t.$
- 2 $x = -1 + 2s, y = 3s, z = 1 + s.$

The two lines intersect if and only if there exist $s, t \in \mathbb{R}$ such that

$$7 - t = x = -1 + 2s \qquad \Rightarrow \qquad t = 8 - 2s$$

$$3 + 3t = y = 3s$$

$$2t = z = 1 + s$$

11.5 Lines and Planes in Space



Example

Do the following two lines intersect? If yes, where?

- 1 $x = 7 - t, y = 3 + 3t, z = 2t.$
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The two lines intersect if and only if there exist $s, t \in \mathbb{R}$ such that

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$$3 + 3t = y = 3s \qquad \Rightarrow \qquad s = t + 1$$

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11.5 Lines and Planes in Space

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$$3 + 3t = y = 3s \qquad \Rightarrow \qquad s = t + 1$$

$$2t = z = 1 + s$$

The first equation tells us that $t = 8 - 2s$.

11.5 Lines and Planes in Space

Example

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- 2 $x = -1 + 2s, y = 3s, z = 1 + s.$

The two lines intersect if and only if there exist $s, t \in \mathbb{R}$ such that

$$\begin{aligned} 7 - t &= x = -1 + 2s && \implies t = 8 - 2s \\ 3 + 3t &= y = 3s && \implies s = t + 1 \\ 2t &= z = 1 + s \end{aligned}$$

The first equation tells us that $t = 8 - 2s$. Putting this into the second equation gives $s = t + 1 = (8 - 2s) + 1 = 9 - 2s$ which implies that $s = 3$ and $t = 2$.

11.5 Lines and Planes in Space

Example

Do the following two lines intersect? If yes, where?

- 1 $x = 7 - t, y = 3 + 3t, z = 2t.$
- 2 $x = -1 + 2s, y = 3s, z = 1 + s.$

The two lines intersect if and only if there exist $s, t \in \mathbb{R}$ such that

$$\begin{aligned} 7 - t &= x = -1 + 2s && \implies t = 8 - 2s \\ 3 + 3t &= y = 3s && \implies s = t + 1 \\ 2t &= z = 1 + s \end{aligned}$$

The first equation tells us that $t = 8 - 2s$. Putting this into the second equation gives $s = t + 1 = (8 - 2s) + 1 = 9 - 2s$ which implies that $s = 3$ and $t = 2$. We must check the third equation:

11.5 Lines and Planes in Space

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Do the following two lines intersect? If yes, where?

- 1 $x = 7 - t, y = 3 + 3t, z = 2t.$
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The two lines intersect if and only if there exist $s, t \in \mathbb{R}$ such that

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 $2t = 2 \times 2 = 4 = 1 + 3 = 1 + s.$

11.5 Lines and Planes in Space

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The two lines intersect if and only if there exist $s, t \in \mathbb{R}$ such that

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The first equation tells us that $t = 8 - 2s$. Putting this into the second equation gives $s = t + 1 = (8 - 2s) + 1 = 9 - 2s$ which implies that $s = 3$ and $t = 2$. We must check the third equation: $2t = 2 \times 2 = 4 = 1 + 3 = 1 + s$. Because the third equation is also true, we know that the two lines intersect at $P(5, 9, 4)$.

11.5 Lines and Planes in Space

Example

Do the following two lines intersect? If yes, where?

- 1 $x = 1 + t, y = 3t, z = 3 + 3t.$
- 2 $x = -1 + 2s, y = 3s, z = 1 + s.$

11.5 Lines and Planes in Space



Example

Do the following two lines intersect? If yes, where?

- 1 $x = 1 + t, y = 3t, z = 3 + 3t.$
- 2 $x = -1 + 2s, y = 3s, z = 1 + s.$

Can we find $s, t \in \mathbb{R}$ such that

$$1 + t = x = -1 + 2s$$

$$3t = y = 3s$$

$$3 + 3t = z = 1 + s$$

are all true?

11.5 Lines and Planes in Space

Example

Do the following two lines intersect? If yes, where?

- 1 $x = 1 + t, y = 3t, z = 3 + 3t.$
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Can we find $s, t \in \mathbb{R}$ such that

$$1 + t = x = -1 + 2s$$

$$3t = y = 3s \qquad \qquad \qquad \Rightarrow \quad s = t$$

$$3 + 3t = z = 1 + s$$

are all true?

11.5 Lines and Planes in Space

Example

Do the following two lines intersect? If yes, where?

- 1 $x = 1 + t, y = 3t, z = 3 + 3t.$
- 2 $x = -1 + 2s, y = 3s, z = 1 + s.$

Can we find $s, t \in \mathbb{R}$ such that

$$1 + t = x = -1 + 2s \quad \Rightarrow \quad 2 + t = 2s \quad \Rightarrow \quad t = 2$$

$$3t = y = 3s \quad \Rightarrow \quad s = t$$

$$3 + 3t = z = 1 + s$$

are all true?

11.5 Lines and Planes in Space



Example

Do the following two lines intersect? If yes, where?

- 1 $x = 1 + t, y = 3t, z = 3 + 3t.$
- 2 $x = -1 + 2s, y = 3s, z = 1 + s.$

Can we find $s, t \in \mathbb{R}$ such that

$$1 + t = x = -1 + 2s \implies 2 + t = 2s \implies t = 2$$

$$3t = y = 3s \implies s = t$$

$$3 + 3t = z = 1 + s \implies 2 + 3t = 1 + t \implies 2 + 2t = 0 \implies t = -2 \neq 2$$

are all true?

11.5 Lines and Planes in Space



Example

Do the following two lines intersect? If yes, where?

- 1 $x = 1 + t, y = 3t, z = 3 + 3t.$
- 2 $x = -1 + 2s, y = 3s, z = 1 + s.$

Can we find $s, t \in \mathbb{R}$ such that

$$1 + t = x = -1 + 2s \implies 2 + t = 2s \implies t = 2$$

$$3t = y = 3s \implies s = t$$

$$3 + 3t = z = 1 + s \implies 2 + 3t = 1 + t \implies 2 + 2t = 0 \implies t = -2 \neq 2$$

are all true?

Therefore it is not possible to find an s and a t . Hence the lines do not intersect.

11.5 Lines and Planes in Space



The Distance Between Two Lines²

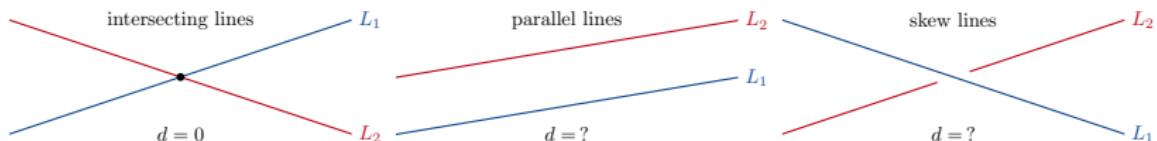
There are three cases to consider:

²not in book

The Distance Between Two Lines²

There are three cases to consider:

- the lines intersect;

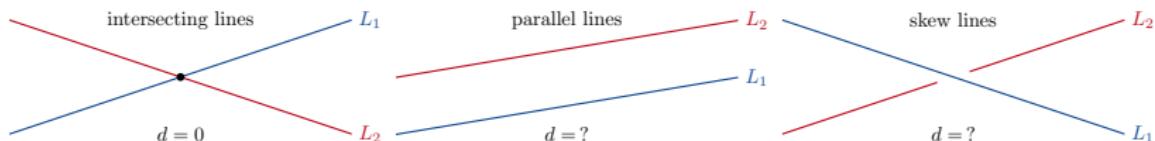


²not in book

The Distance Between Two Lines²

There are three cases to consider:

- the lines intersect;
- the lines do not intersect and are parallel ($\mathbf{v}_1 = k\mathbf{v}_2$ for some $k \in \mathbb{R}$); or

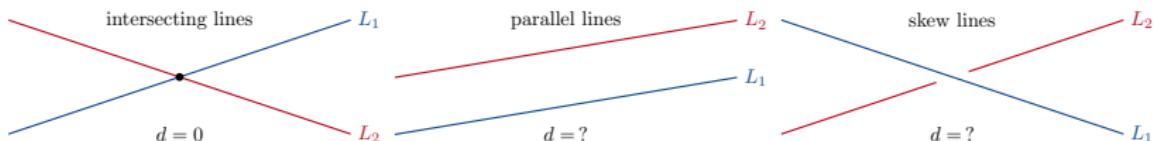


²not in book

The Distance Between Two Lines²

There are three cases to consider:

- the lines intersect;
- the lines do not intersect and are parallel ($\mathbf{v}_1 = k\mathbf{v}_2$ for some $k \in \mathbb{R}$); or
- the lines do not intersect and are skew ($\mathbf{v}_1 \neq k\mathbf{v}_2$ for all $k \in \mathbb{R}$).



²not in book

11.5 Lines and Planes in Space



Intersecting Lines

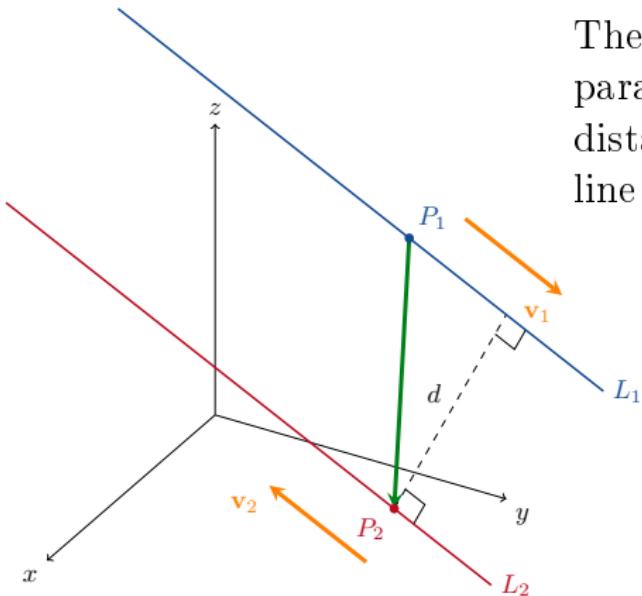
Clearly the distance between intersecting lines is zero. Hence

$$d = 0.$$

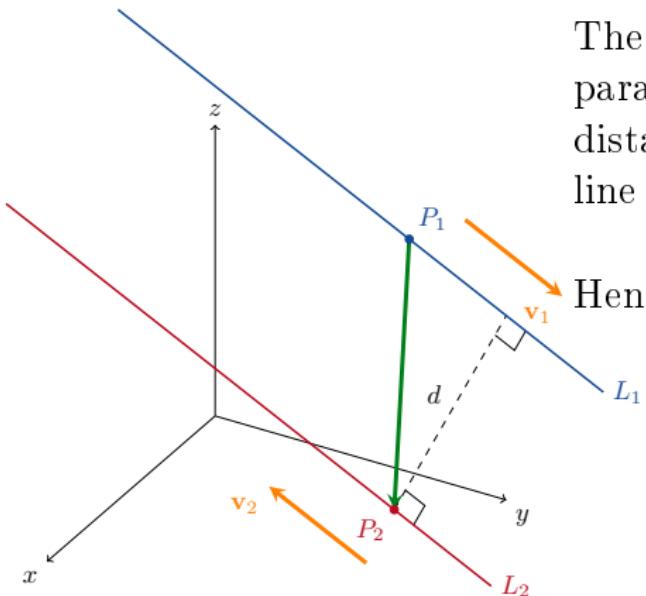
11.5 Lines and Planes in Space



Parallel Lines ($\mathbf{v}_1 \times \mathbf{v}_2 = 0$)



The distance between the two parallel lines is the same as the distance between P_2 and the line L_1 .

Parallel Lines ($\mathbf{v}_1 \times \mathbf{v}_2 = \mathbf{0}$)

The distance between the two parallel lines is the same as the distance between P_2 and the line L_1 .

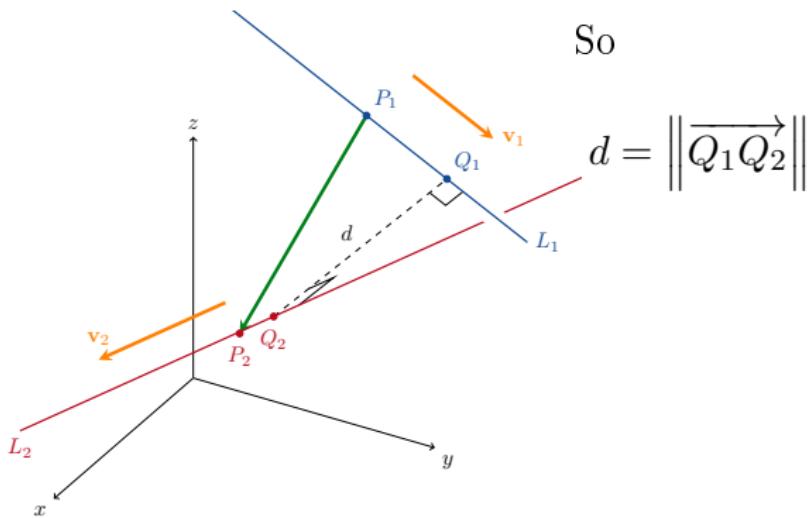
Hence

$$d = \frac{\|\overrightarrow{P_1P_2} \times \mathbf{v}_1\|}{\|\mathbf{v}_1\|}.$$

11.5 Lines and Planes in Space



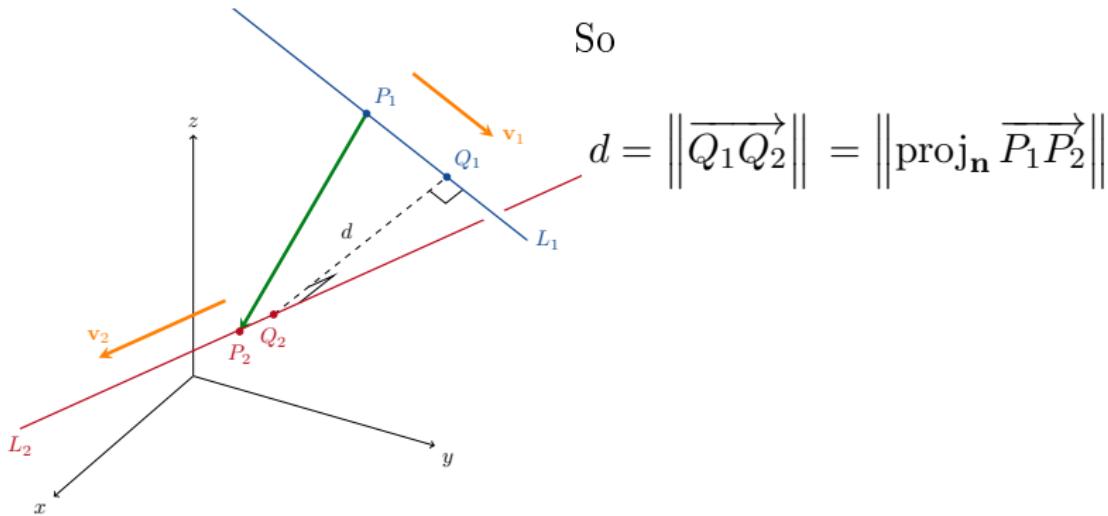
Skew Lines ($\mathbf{v}_1 \times \mathbf{v}_2 \neq \mathbf{0}$)



Let $\mathbf{n} = \mathbf{v}_1 \times \mathbf{v}_2$. Then \mathbf{n} is orthogonal to both \mathbf{v}_1 and \mathbf{v}_2 .

11.5 Lines and Planes in Space

Skew Lines ($\mathbf{v}_1 \times \mathbf{v}_2 \neq \mathbf{0}$)

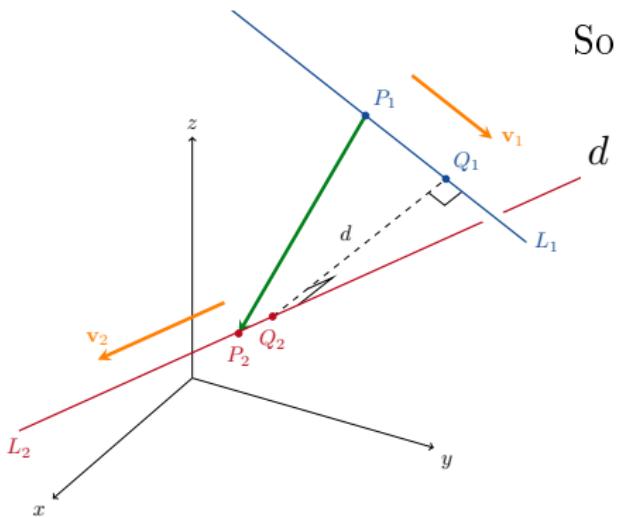


Let $\mathbf{n} = \mathbf{v}_1 \times \mathbf{v}_2$. Then \mathbf{n} is orthogonal to both \mathbf{v}_1 and \mathbf{v}_2 .

11.5 Lines and Planes in Space



Skew Lines ($\mathbf{v}_1 \times \mathbf{v}_2 \neq \mathbf{0}$)



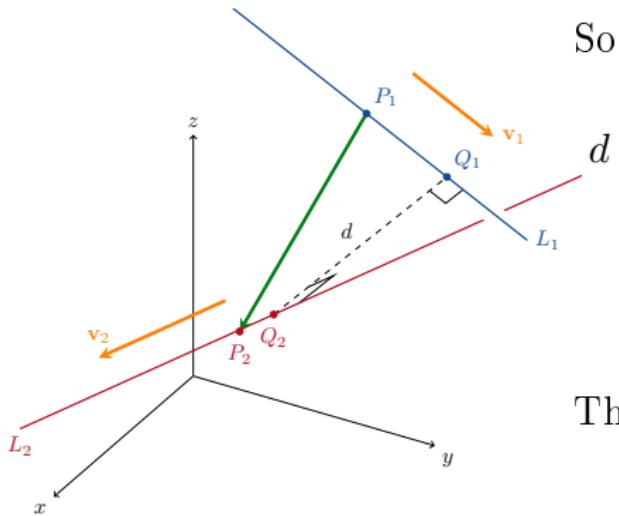
So

$$\begin{aligned} d &= \left\| \overrightarrow{Q_1 Q_2} \right\| = \left\| \text{proj}_{\mathbf{n}} \overrightarrow{P_1 P_2} \right\| \\ &= \frac{\left| \overrightarrow{P_1 P_2} \cdot \mathbf{n} \right|}{\|\mathbf{n}\|}. \end{aligned}$$

Let $\mathbf{n} = \mathbf{v}_1 \times \mathbf{v}_2$. Then \mathbf{n} is orthogonal to both \mathbf{v}_1 and \mathbf{v}_2 .

11.5 Lines and Planes in Space

Skew Lines ($\mathbf{v}_1 \times \mathbf{v}_2 \neq 0$)



So

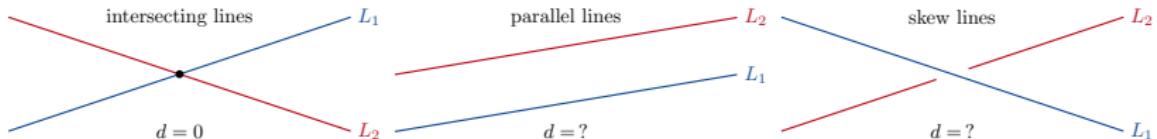
$$\begin{aligned} d &= \left\| \overrightarrow{Q_1 Q_2} \right\| = \left\| \text{proj}_{\mathbf{n}} \overrightarrow{P_1 P_2} \right\| \\ &= \frac{\left| \overrightarrow{P_1 P_2} \cdot \mathbf{n} \right|}{\|\mathbf{n}\|}. \end{aligned}$$

Thus

$$d = \frac{\left| \overrightarrow{P_1 P_2} \cdot (\mathbf{v}_1 \times \mathbf{v}_2) \right|}{\|\mathbf{v}_1 \times \mathbf{v}_2\|}.$$

Let $\mathbf{n} = \mathbf{v}_1 \times \mathbf{v}_2$. Then \mathbf{n} is orthogonal to both \mathbf{v}_1 and \mathbf{v}_2 .

11.5 Lines and Planes in Space



- Intersecting Lines: $d = 0$.

- Parallel Lines ($\mathbf{v}_1 \times \mathbf{v}_2 = \mathbf{0}$): $d = \frac{\left\| \overrightarrow{P_1 P_2} \times \mathbf{v}_1 \right\|}{\|\mathbf{v}_1\|}$.

- Skew Lines ($\mathbf{v}_1 \times \mathbf{v}_2 \neq \mathbf{0}$): $d = \frac{\left| \overrightarrow{P_1 P_2} \cdot (\mathbf{v}_1 \times \mathbf{v}_2) \right|}{\|\mathbf{v}_1 \times \mathbf{v}_2\|}$.

11.5 Lines and Planes in Space

Example

Find the distance between the following two lines.

$$\text{line 1: } x = 0, y = -t, z = t,$$

$$\text{line 2: } x = 1 + 2s, y = s, z = -3s.$$

11.5 Lines and Planes in Space

Example

Find the distance between the following two lines.

$$\text{line 1: } x = 0, y = -t, z = t,$$

$$\text{line 2: } x = 1 + 2s, y = s, z = -3s.$$

We have that $P_1(0, 0, 0)$, $\mathbf{v}_1 = -\mathbf{j} + \mathbf{k}$, $P_2(1, 0, 0)$ and $\mathbf{v}_2 = 2\mathbf{i} + \mathbf{j} - 3\mathbf{k}$. Since

$$\mathbf{v}_1 \times \mathbf{v}_2 = 2\mathbf{i} + 2\mathbf{j} + 2\mathbf{k} \neq \mathbf{0},$$

the lines are skew. (Recall that we have $\mathbf{v}_1 \times \mathbf{v}_2 = \mathbf{0}$ for parallel vectors.)

11.5 Lines and Planes in Space

Example

Find the distance between the following two lines.

line 1: $x = 0, y = -t, z = t,$

line 2: $x = 1 + 2s, y = s, z = -3s.$

We have that $P_1(0, 0, 0)$, $\mathbf{v}_1 = -\mathbf{j} + \mathbf{k}$, $P_2(1, 0, 0)$ and $\mathbf{v}_2 = 2\mathbf{i} + \mathbf{j} - 3\mathbf{k}$. Since

$$\mathbf{v}_1 \times \mathbf{v}_2 = 2\mathbf{i} + 2\mathbf{j} + 2\mathbf{k} \neq \mathbf{0},$$

the lines are skew. (Recall that we have $\mathbf{v}_1 \times \mathbf{v}_2 = \mathbf{0}$ for parallel vectors.) Moreover note that $\overrightarrow{P_1 P_2} = \mathbf{i}$. Then we calculate that

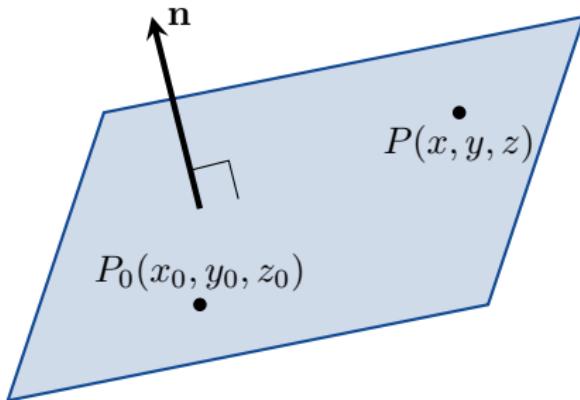
$$\begin{aligned} d &= \frac{\left| \overrightarrow{P_1 P_2} \cdot (\mathbf{v}_1 \times \mathbf{v}_2) \right|}{\|\mathbf{v}_1 \times \mathbf{v}_2\|} = \frac{|(\mathbf{i}) \cdot (2\mathbf{i} + 2\mathbf{j} + 2\mathbf{k})|}{\|2\mathbf{i} + 2\mathbf{j} + 2\mathbf{k}\|} \\ &= \frac{|2 + 0 + 0|}{\sqrt{2^2 + 2^2 + 2^2}} = \frac{1}{\sqrt{3}}. \end{aligned}$$

An Equation for a Plane in Space

To describe a plane, we need

- a point $P_0(x_0, y_0, z_0)$ which the plane passes through; and
- a vector $\mathbf{n} = A\mathbf{i} + B\mathbf{j} + C\mathbf{k}$ which is perpendicular to the plane.

The vector \mathbf{n} is said to be *normal* to the plane.



11.5 Lines and Planes in Space



Definition

The plane passing through the point $P_0(x_0, y_0, z_0)$ with normal vector $\mathbf{n} = A\mathbf{i} + B\mathbf{j} + C\mathbf{k}$ has the vector equation

$$\mathbf{n} \cdot \overrightarrow{P_0P} = 0.$$

11.5 Lines and Planes in Space

Definition

The plane passing through the point $P_0(x_0, y_0, z_0)$ with normal vector $\mathbf{n} = A\mathbf{i} + B\mathbf{j} + C\mathbf{k}$ has the vector equation

$$\mathbf{n} \cdot \overrightarrow{P_0P} = 0.$$

Writing this equation in coordinates, we have

$$A(x - x_0) + B(y - y_0) + C(z - z_0) = 0$$

or

$$Ax + By + Cz = D$$

where $D = Ax_0 + By_0 + Cz_0$ is a constant.

11.5 Lines and Planes in Space

Example

Find an equation for the plane passing through $P_0(-3, 0, 7)$ normal to $\mathbf{n} = 5\mathbf{i} + 2\mathbf{j} - \mathbf{k}$.

11.5 Lines and Planes in Space



Example

Find an equation for the plane passing through $P_0(-3, 0, 7)$ normal to $\mathbf{n} = 5\mathbf{i} + 2\mathbf{j} - \mathbf{k}$.

$$A(x - x_0) + B(y - y_0) + C(z - z_0) = 0$$

$$5(x - (-3)) + 2(y - 0) + (-1)(z - 7) = 0$$

$$5x - 15 + 2y - z + 7 = 0$$

$$5x + 2y - z = -22.$$

11.5 Lines and Planes in Space



Remark

The vector $\mathbf{n} = A\mathbf{i} + B\mathbf{j} + C\mathbf{k}$ is normal to the plane
 $Ax + By + Cz = D$.

11.5 Lines and Planes in Space

Remark

The vector $\mathbf{n} = A\mathbf{i} + B\mathbf{j} + C\mathbf{k}$ is normal to the plane $Ax + By + Cz = D$.

Example

Find a vector normal to the plane $x + 2y + 3z = 4$.

11.5 Lines and Planes in Space



Remark

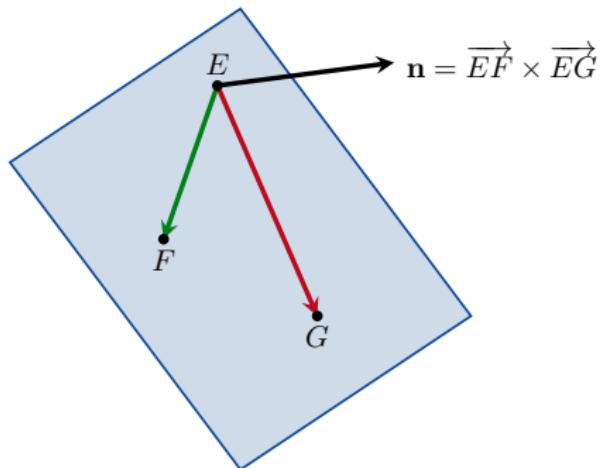
The vector $\mathbf{n} = A\mathbf{i} + B\mathbf{j} + C\mathbf{k}$ is normal to the plane $Ax + By + Cz = D$.

Example

Find a vector normal to the plane $x + 2y + 3z = 4$.

We can immediately write down $\mathbf{n} = \mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$.

11.5 Lines and Planes in Space



Example

Find an equation for the plane containing the points $E(0, 0, 1)$, $F(2, 0, 0)$ and $G(0, 3, 0)$.

11.5 Lines and Planes in Space



First we need to find a vector normal to the plane. Since $\overrightarrow{EF} = 2\mathbf{i} - \mathbf{k}$ and $\overrightarrow{EG} = 3\mathbf{j} - \mathbf{k}$, we have that

$$\begin{aligned}\mathbf{n} &= \overrightarrow{EF} \times \overrightarrow{EG} = (0 - -3)\mathbf{i} - (-2 - 0)\mathbf{j} + (6 - 0)\mathbf{k} \\ &= 3\mathbf{i} + 2\mathbf{j} + 6\mathbf{k}\end{aligned}$$

is normal to the plane.

11.5 Lines and Planes in Space



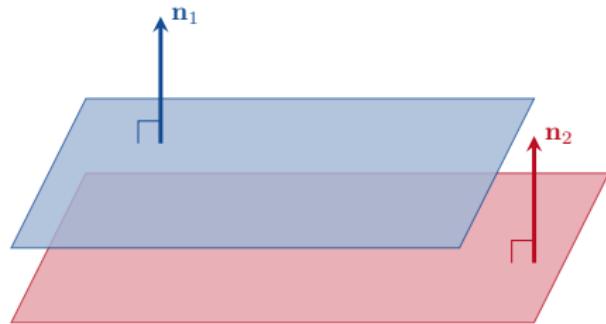
First we need to find a vector normal to the plane. Since $\overrightarrow{EF} = 2\mathbf{i} - \mathbf{k}$ and $\overrightarrow{EG} = 3\mathbf{j} - \mathbf{k}$, we have that

$$\begin{aligned}\mathbf{n} &= \overrightarrow{EF} \times \overrightarrow{EG} = (0 - -3)\mathbf{i} - (-2 - 0)\mathbf{j} + (6 - 0)\mathbf{k} \\ &= 3\mathbf{i} + 2\mathbf{j} + 6\mathbf{k}\end{aligned}$$

is normal to the plane. Using $P_0 = E(0, 0, 1)$, the equation for the plane is

$$\begin{aligned}3(x - 0) + 2(y - 0) + 6(z - 1) &= 0 \\ 3x + 2y + 6z &= 6.\end{aligned}$$

Lines of Intersection

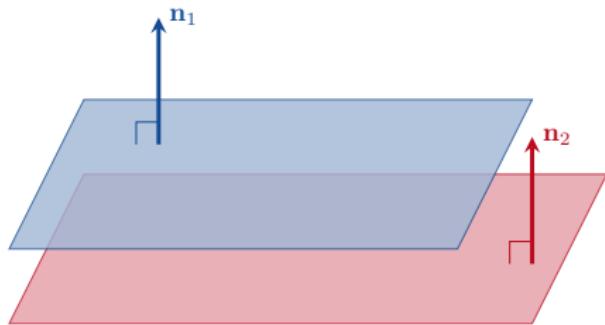


Two planes are parallel \iff
 $\mathbf{n}_1 = k\mathbf{n}_2$ for some $k \in \mathbb{R}$.

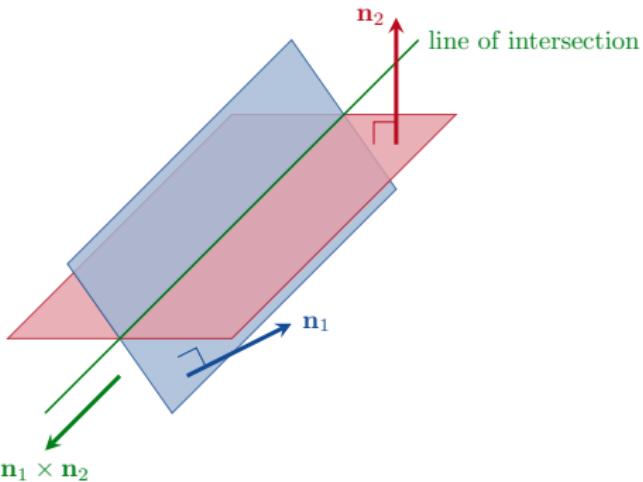
11.5 Lines and Planes in Space



Lines of Intersection



Two planes are parallel \iff
 $\mathbf{n}_1 = k\mathbf{n}_2$ for some $k \in \mathbb{R}$.



Two planes intersect in a line
 $\iff \mathbf{n}_1 \neq k\mathbf{n}_2$ for all $k \in \mathbb{R}$.

11.5 Lines and Planes in Space



Example

Find a vector parallel of the line of intersection of the planes
 $3x - 6y - 2z = 15$ and $2x + y - 2z = 5$.

11.5 Lines and Planes in Space



Example

Find a vector parallel of the line of intersection of the planes
 $3x - 6y - 2z = 15$ and $2x + y - 2z = 5$.

We can immediately write down $\mathbf{n}_1 = 3\mathbf{i} - 6\mathbf{j} - 2\mathbf{k}$ and
 $\mathbf{n}_2 = 2\mathbf{i} + \mathbf{j} - 2\mathbf{k}$.

11.5 Lines and Planes in Space



Example

Find a vector parallel of the line of intersection of the planes
 $3x - 6y - 2z = 15$ and $2x + y - 2z = 5$.

We can immediately write down $\mathbf{n}_1 = 3\mathbf{i} - 6\mathbf{j} - 2\mathbf{k}$ and
 $\mathbf{n}_2 = 2\mathbf{i} + \mathbf{j} - 2\mathbf{k}$. A vector parallel to the line of intersection is

$$\mathbf{n}_1 \times \mathbf{n}_2 = (12 + 2)\mathbf{i} - (-6 + 4)\mathbf{j} + (3 + 12)\mathbf{k} = 14\mathbf{i} + 2\mathbf{j} + 15\mathbf{k}.$$

11.5 Lines and Planes in Space

Example

Find the point where the line $x = \frac{8}{3} + 2t$, $y = -2t$, $z = 1 + t$ intersects the plane $3x + 2y + 6z = 6$.

11.5 Lines and Planes in Space

Example

Find the point where the line $x = \frac{8}{3} + 2t$, $y = -2t$, $z = 1 + t$ intersects the plane $3x + 2y + 6z = 6$.

We calculate that

$$3x + 2y + 6z = 6$$

11.5 Lines and Planes in Space

Example

Find the point where the line $x = \frac{8}{3} + 2t$, $y = -2t$, $z = 1 + t$ intersects the plane $3x + 2y + 6z = 6$.

We calculate that

$$3x + 2y + 6z = 6$$

$$3\left(\frac{8}{3} + 2t\right) + 2(-2t) + 6(1 + t) = 6$$

11.5 Lines and Planes in Space

Example

Find the point where the line $x = \frac{8}{3} + 2t$, $y = -2t$, $z = 1 + t$ intersects the plane $3x + 2y + 6z = 6$.

We calculate that

$$\begin{aligned}3x + 2y + 6z &= 6 \\3\left(\frac{8}{3} + 2t\right) + 2(-2t) + 6(1 + t) &= 6 \\8 + 6t - 4t + 6 + 6t &= 6\end{aligned}$$

11.5 Lines and Planes in Space

Example

Find the point where the line $x = \frac{8}{3} + 2t$, $y = -2t$, $z = 1 + t$ intersects the plane $3x + 2y + 6z = 6$.

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$$8t = -8$$

$$t = -1.$$

11.5 Lines and Planes in Space



Example

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The point of intersection is

$$P(x, y, z)|_{t=-1} =$$

11.5 Lines and Planes in Space

Example

Find the point where the line $x = \frac{8}{3} + 2t$, $y = -2t$, $z = 1 + t$ intersects the plane $3x + 2y + 6z = 6$.

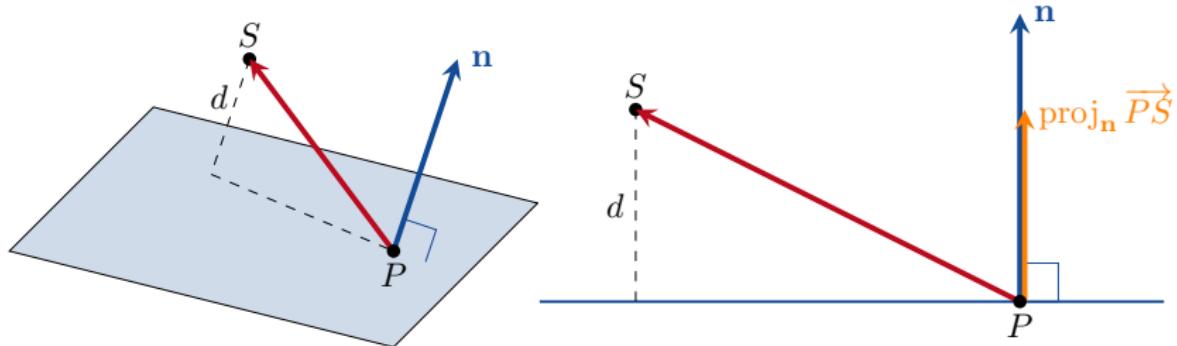
We calculate that

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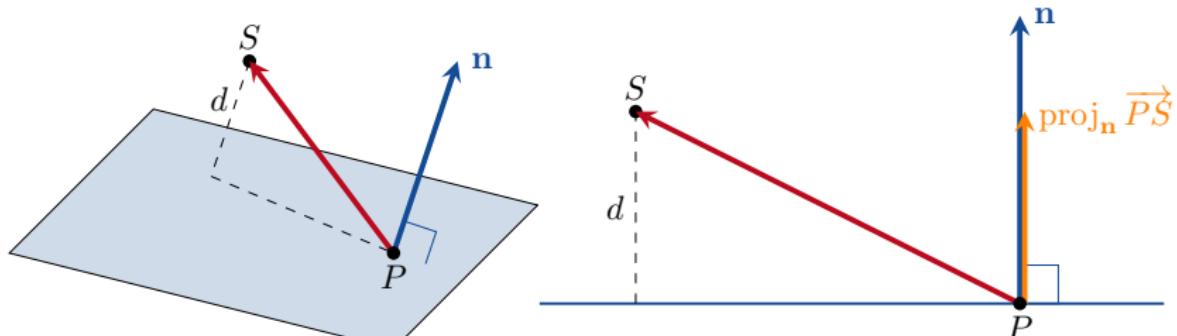
The point of intersection is

$$P(x, y, z)|_{t=-1} = P\left(\frac{8}{3} + 2t, -2t, 1 + t\right)\Big|_{t=-1} = P\left(\frac{2}{3}, 2, 0\right).$$

The Distance from a Point to a Plane

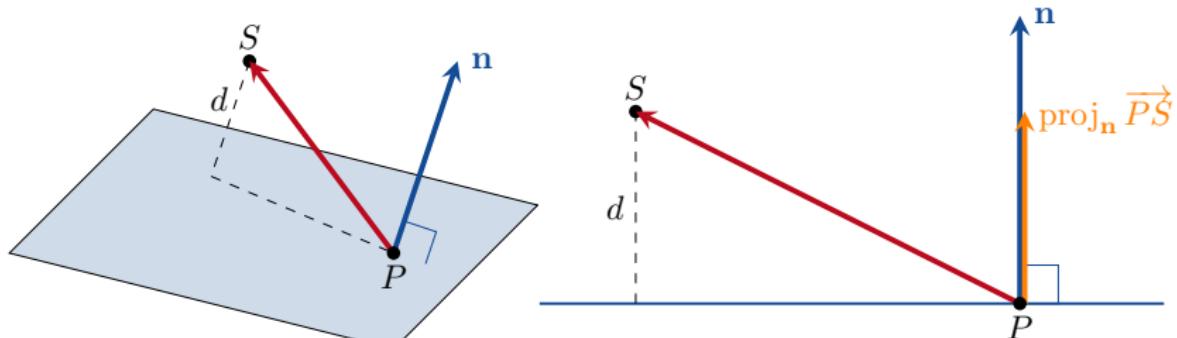


The Distance from a Point to a Plane



We can see that $d = \|\text{proj}_{\mathbf{n}} \vec{PS}\|$.

The Distance from a Point to a Plane



We can see that $d = \|\text{proj}_n \overrightarrow{PS}\|$. Therefore the distance from a point S to a plane with normal \mathbf{n} containing the point P is

$$d = \frac{|\overrightarrow{PS} \cdot \mathbf{n}|}{\|\mathbf{n}\|}.$$

$$d = \frac{|\overrightarrow{PS} \cdot \mathbf{n}|}{\|\mathbf{n}\|}$$

Example

Find the distance from the point $S(1, 2, 3)$ to the plane
 $x + 2y + 3z = 4$.

First we need a point in the plane.

$$d = \frac{|\overrightarrow{PS} \cdot \mathbf{n}|}{\|\mathbf{n}\|}$$



Example

Find the distance from the point $S(1, 2, 3)$ to the plane
 $x + 2y + 3z = 4$.

First we need a point in the plane. Setting $y = 0$ and $z = 0$ we must have $x = 4 - 2y - 3z = 4$. Therefore $P(4, 0, 0)$ is in the plane.

$$d = \frac{|\overrightarrow{PS} \cdot \mathbf{n}|}{\|\mathbf{n}\|}$$



Example

Find the distance from the point $S(1, 2, 3)$ to the plane
 $x + 2y + 3z = 4$.

First we need a point in the plane. Setting $y = 0$ and $z = 0$ we must have $x = 4 - 2y - 3z = 4$. Therefore $P(4, 0, 0)$ is in the plane. Clearly $\mathbf{n} = \mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$.

$$d = \frac{|\overrightarrow{PS} \cdot \mathbf{n}|}{\|\mathbf{n}\|}$$

Example

Find the distance from the point $S(1, 2, 3)$ to the plane $x + 2y + 3z = 4$.

First we need a point in the plane. Setting $y = 0$ and $z = 0$ we must have $x = 4 - 2y - 3z = 4$. Therefore $P(4, 0, 0)$ is in the plane. Clearly $\mathbf{n} = \mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$.

Therefore the required distance is

$$\begin{aligned} d &= \frac{|\overrightarrow{PS} \cdot \mathbf{n}|}{\|\mathbf{n}\|} = \frac{|(-3\mathbf{i} + 2\mathbf{j} + 3\mathbf{k}) \cdot (\mathbf{i} + 2\mathbf{j} + 3\mathbf{k})|}{\|\mathbf{i} + 2\mathbf{j} + 3\mathbf{k}\|} \\ &= \frac{|-3 + 4 + 9|}{\sqrt{1^2 + 2^2 + 3^2}} = \frac{10}{\sqrt{14}}. \end{aligned}$$

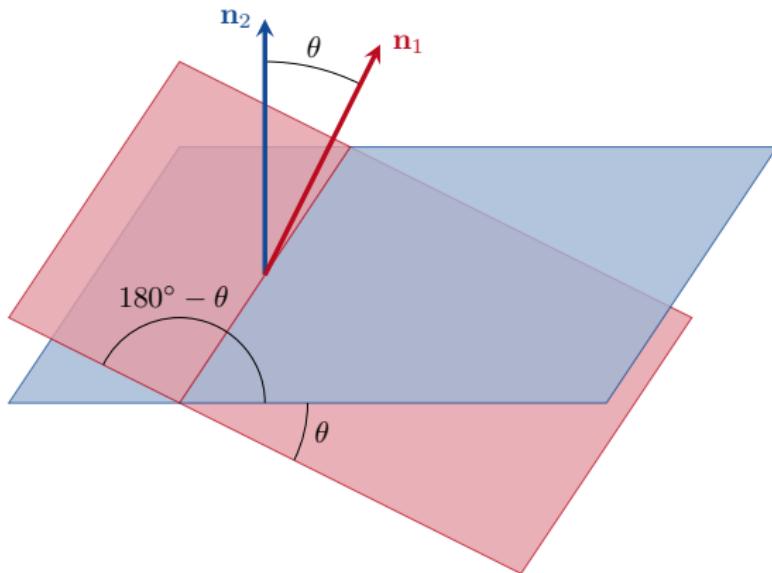
11.5 Lines and Planes in Space



Please read Example 11 in the textbook.

Angles Between Planes

There are two possible angles that can be measured between planes. We are interested in the smaller angle.



11.5 Lines and Planes in Space



Definition

The angle between two planes is defined to be equal to whichever of the following angles is smaller

- the angle between \mathbf{n}_1 and \mathbf{n}_2 ;
- 180° minus the angle between \mathbf{n}_1 and \mathbf{n}_2 .

The angle between two planes will always be between 0° and 90° .

11.5 Lines and Planes in Space



Example

Find the angle between the planes $3x - 6y - 2z = 15$ and $-2x - y + 2z = 5$.

11.5 Lines and Planes in Space

Example

Find the angle between the planes $3x - 6y - 2z = 15$ and $-2x - y + 2z = 5$.

We have normal vectors $\mathbf{n}_1 = 3\mathbf{i} - 6\mathbf{j} - 2\mathbf{k}$ and $\mathbf{n}_2 = -2\mathbf{i} - \mathbf{j} + 2\mathbf{k}$. The angle between \mathbf{n}_1 and \mathbf{n}_2 is

$$\theta = \cos^{-1} \left(\frac{\mathbf{n}_1 \cdot \mathbf{n}_2}{\|\mathbf{n}_1\| \|\mathbf{n}_2\|} \right) = \cos^{-1} \left(\frac{-4}{21} \right) \approx 101^\circ.$$

11.5 Lines and Planes in Space

Example

Find the angle between the planes $3x - 6y - 2z = 15$ and $-2x - y + 2z = 5$.

We have normal vectors $\mathbf{n}_1 = 3\mathbf{i} - 6\mathbf{j} - 2\mathbf{k}$ and $\mathbf{n}_2 = -2\mathbf{i} - \mathbf{j} + 2\mathbf{k}$. The angle between \mathbf{n}_1 and \mathbf{n}_2 is

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Because $101^\circ > 90^\circ$, the angle between the two planes is approximately $180 - 101^\circ = 79^\circ$.



Next Time

- 13.1 Functions of Several Variables
- 13.2 Limits and Continuity in Higher Dimensions
- 13.3 Partial Derivatives
- 13.4 The Chain Rule