

Lecture 7

- Rank and Nullity
- The Fundamental Matrix Spaces
- Linear Transformations
- Composition and Inverse Transformations
- Isomorphisms



Rank and Nullity

Rank and Nullity

Recall, to find a basis for the row space of a matrix A , we reduce it to REF, then take the rows with a pivot.

$$A = \begin{bmatrix} 1 & -3 & 4 & -2 & 5 & 4 \\ 2 & -6 & 9 & -1 & 8 & 2 \\ 2 & -6 & 9 & -1 & 9 & 7 \\ -1 & 3 & -4 & 2 & -5 & -4 \end{bmatrix} \sim R = \begin{bmatrix} 1 & -3 & 4 & -2 & 5 & 4 \\ 0 & 0 & 1 & 3 & -2 & -6 \\ 0 & 0 & 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

For example

$$\mathbf{r}_1 = [1 \quad -3 \quad 4 \quad -2 \quad 5 \quad 4]$$

$$\mathbf{r}_2 = [0 \quad 0 \quad 1 \quad 3 \quad -2 \quad -6]$$

$$\mathbf{r}_3 = [0 \quad 0 \quad 0 \quad 0 \quad 1 \quad 5]$$

form a basis for Row A = Row R . Since R has three pivots, the dimension of Row A is 3.

Rank and Nullity



And to find a basis for a column space, we take the pivot columns.

$$A = \begin{bmatrix} 1 & -3 & 4 & -2 & 5 & 4 \\ 2 & -6 & 9 & -1 & 8 & 2 \\ 2 & -6 & 9 & -1 & 9 & 7 \\ -1 & 3 & -4 & 2 & -5 & -4 \end{bmatrix} \sim R = \begin{bmatrix} 1 & -3 & 4 & -2 & 5 & 4 \\ 0 & 0 & 1 & 3 & -2 & -6 \\ 0 & 0 & 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

The pivot columns of A are also the **first**, **third** and **fifth** columns.
Hence

$$\left\{ \mathbf{c}_1 = \begin{bmatrix} 1 \\ 2 \\ 2 \\ -1 \end{bmatrix}, \mathbf{c}_3 = \begin{bmatrix} 4 \\ 9 \\ 9 \\ -4 \end{bmatrix}, \mathbf{c}_5 = \begin{bmatrix} 5 \\ 8 \\ 9 \\ -5 \end{bmatrix} \right\}$$

is a basis for $\text{Col } A$. Since A has three pivot columns, the dimension of $\text{Col } A$ is 3.

Rank and Nullity

Theorem

The row space and the column space of a matrix A have the same dimension.

Rank and Nullity

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Rank and Nullity

Definition

The *rank* of a matrix A is

$$\text{rank } A = \dim (\text{Col } A) = \dim (\text{Row } A).$$

Theorem

The row space and the column space of a matrix A have the same dimension.

Rank and Nullity

Definition

The *rank* of a matrix A is

$$\text{rank } A = \dim (\text{Col } A) = \dim (\text{Row } A).$$

Definition

The *nullity* of A is

$$\text{nullity } A = \dim (\text{Nul } A).$$

Rank and Nullity

Example

Find the rank and nullity of the matrix

$$A = \begin{bmatrix} -1 & 2 & 0 & 4 & 5 & -3 \\ 3 & -7 & 2 & 0 & 1 & 4 \\ 2 & -5 & 2 & 4 & 6 & 1 \\ 4 & -9 & 2 & -4 & -4 & 7 \end{bmatrix}.$$

Rank and Nullity

Example

Find the rank and nullity of the matrix

$$A = \begin{bmatrix} -1 & 2 & 0 & 4 & 5 & -3 \\ 3 & -7 & 2 & 0 & 1 & 4 \\ 2 & -5 & 2 & 4 & 6 & 1 \\ 4 & -9 & 2 & -4 & -4 & 7 \end{bmatrix}.$$

Please check that A is row equivalent to the RREF matrix

$$R = \begin{bmatrix} 1 & 0 & -4 & -28 & -37 & 13 \\ 0 & 1 & -2 & -12 & -16 & 5 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Rank and Nullity

The matrix

$$R = \begin{bmatrix} 1 & 0 & -4 & -28 & -37 & 13 \\ 0 & 1 & -2 & -12 & -16 & 5 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

has two pivots. So both Row A and Col A are two dimensional.
Hence $\text{rank } A = 2$.

Rank and Nullity

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To find the nullity of A , we need to find the dimension of the solution space of $A\mathbf{x} = \mathbf{0}$.

Rank and Nullity

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has two pivots. So both Row A and Col A are two dimensional.
Hence $\text{rank } A = 2$.

To find the nullity of A , we need to find the dimension of the solution space of $A\mathbf{x} = \mathbf{0}$. We can see from R that the general solution of $A\mathbf{x} = \mathbf{0}$ is

$$\begin{cases} x_1 = 4x_3 + 28x_4 + 37x_5 - 13x_6 \\ x_2 = 2x_3 + 12x_4 + 16x_5 - 5x_6 \\ x_3, x_4, x_5, x_6 \text{ are free.} \end{cases}$$

Rank and Nullity



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In vector form, the general solution is

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} = r \begin{bmatrix} 4 \\ 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + s \begin{bmatrix} 28 \\ 12 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} 37 \\ 16 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} + u \begin{bmatrix} -13 \\ -5 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

for any $r, s, t, u \in \mathbb{R}$.

Rank and Nullity



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for any $r, s, t, u \in \mathbb{R}$. These four vectors (on the right hand side) form a basis for $\text{Nul } A$. Hence nullity $A = 4$.

Rank and Nullity



Note that

$$\text{rank } A = \text{number of pivot positions}$$

and

$$\text{nullity } A = \text{number of free variables.}$$

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Theorem

If A is an $m \times n$ matrix (i.e. with n columns), then

$$\text{rank } A + \text{nullity } A = n.$$



Example

The matrix

$$A = \begin{bmatrix} -1 & 2 & 0 & 4 & 5 & -3 \\ 3 & -7 & 2 & 0 & 1 & 4 \\ 2 & -5 & 2 & 4 & 6 & 1 \\ 4 & -9 & 2 & -4 & -4 & 7 \end{bmatrix}$$

has 6 columns. Therefore

$$\text{rank } A + \text{nullity } A = 6.$$



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has 6 columns. Therefore

$$\text{rank } A + \text{nullity } A = 6.$$

In the previous example we showed that $\text{rank } A = 2$ and $\text{nullity } A = 4$.

$$\text{rank } A + \text{nullity } A = \text{number of columns}$$



Ask the audience 1/2

Suppose that A is a 5×7 matrix of rank 3. Find the nullity of A .

1 nullity $A = 1$

3 nullity $A = 3$

2 nullity $A = 2$

4 nullity $A = 4$

$$\text{rank } A + \text{nullity } A = \text{number of columns}$$



Ask the audience 1/2

Suppose that A is a 5×7 matrix of rank 3. Find the nullity of A .

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Ask the audience 2/2

Suppose that the solution space of $B\mathbf{x} = \mathbf{0}$ is two dimensional.
If B is a 5×7 , find the rank of B .

1 rank $B = 4$

3 rank $B = 6$

2 rank $B = 5$

4 rank $B = 7$



Ask the audience 2/2

Suppose that the solution space of $B\mathbf{x} = \mathbf{0}$ is two dimensional.
If B is a 5×7 , find the rank of B .

1 $\text{rank } B = 4$

3 $\text{rank } B = 6$

2 $\text{rank } B = 5$

4 $\text{rank } B = 7$

Theorem

If $A\mathbf{x} = \mathbf{b}$ is a consistent linear system of m equations in n unknowns, and if A has rank r , then the general solution of the system contains $n - r$ free variables (parameters).



The Fundamental Matrix Spaces

The Fundamental Matrix Spaces



Consider the following six vector spaces:

- the row space of A
- the column space of A
- the null space of A
- the row space of A^T
- the column space of A^T
- the null space of A^T

The Fundamental Matrix Spaces



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When we take the transpose of a matrix, rows become columns and columns become rows. That means that

$$\text{Row } A = \text{Col } A^T$$

and

$$\text{Col } A = \text{Row } A^T.$$

The Fundamental Matrix Spaces



Definition

The *fundamental spaces* of a matrix A are

- 1 the row space of A
- 2 the column space of A
- 3 the null space of A
- 4 the null space of A^T .

We will talk about how these four vector spaces are related.

The Fundamental Matrix Spaces



Theorem

If A is any matrix, then

$$\text{rank } A = \text{rank } A^T.$$

The Fundamental Matrix Spaces

Theorem

If A is any matrix, then

$$\text{rank } A = \text{rank } A^T.$$

Proof.

$$\begin{aligned}\text{rank } A &= \dim(\text{row space of } A) \\ &= \dim(\text{column space of } A^T) = \text{rank } A^T.\end{aligned}$$



The Fundamental Matrix Spaces



If A is an $m \times n$ matrix, then A^T is an $n \times m$ matrix.

The Fundamental Matrix Spaces



If A is an $m \times n$ matrix, then A^T is an $n \times m$ matrix. So

$$\text{rank } A^T + \text{nullity } A^T = \textcolor{red}{m}.$$

The Fundamental Matrix Spaces



If A is an $m \times n$ matrix, then A^T is an $n \times m$ matrix. So

$$\text{rank } A^T + \text{nullity } A^T = \textcolor{red}{m}.$$

If $\text{rank } A = r$, then

$$\dim (\text{Row } A) = r$$

$$\dim (\text{Col } A) = r$$

$$\dim (\text{Nul } A) = n - r$$

$$\dim (\text{Nul } A^T) = m - r.$$

The Fundamental Matrix Spaces

Now suppose that \mathbf{x} is a solution of

$$\begin{bmatrix} 1 & 3 & -2 & 0 & 2 & 0 \\ 2 & 6 & -5 & -2 & 4 & -3 \\ 0 & 0 & 5 & 10 & 0 & 15 \\ 2 & 6 & 0 & 8 & 4 & 18 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

The Fundamental Matrix Spaces

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Note that

$$\mathbf{r}_1 \cdot \mathbf{x} = (1, 3, -2, 0, 2, 0) \cdot (x_1, x_2, x_3, x_4, x_5, x_6)$$

=

=

The Fundamental Matrix Spaces

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The Fundamental Matrix Spaces

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$$= [1 \quad 3 \quad -2 \quad 0 \quad 2 \quad 0] \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} = 0.$$

So

\mathbf{x} solves $A\mathbf{x} = \mathbf{0} \implies \mathbf{x}$ is orthogonal to \mathbf{r}_1 .

The Fundamental Matrix Spaces



$$\begin{bmatrix} 1 & 3 & -2 & 0 & 2 & 0 \\ 2 & 6 & -5 & -2 & 4 & -3 \\ 0 & 0 & 5 & 10 & 0 & 15 \\ 2 & 6 & 0 & 8 & 4 & 18 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

Similarly $\mathbf{r}_2 \cdot \mathbf{x} = 0$, $\mathbf{r}_3 \cdot \mathbf{x} = 0$ and $\mathbf{r}_4 \cdot \mathbf{x} = 0$.

The Fundamental Matrix Spaces



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Similarly $\mathbf{r}_2 \cdot \mathbf{x} = 0$, $\mathbf{r}_3 \cdot \mathbf{x} = 0$ and $\mathbf{r}_4 \cdot \mathbf{x} = 0$.

Theorem

If A is an $m \times n$ matrix, then the null space of A (i.e. the solution space of $A\mathbf{x} = \mathbf{0}$) consists of all vectors in \mathbb{R}^n that are orthogonal to every row vector of A .

The Fundamental Matrix Spaces

Definition

If W is a subspace of \mathbb{R}^n , then the set of all vectors in \mathbb{R}^n that are orthogonal to every vector in W is called the orthogonal complement of W and is denoted by the symbol W^\perp .

In other words,

$$W^\perp = \{\mathbf{v} \in V \mid \mathbf{v} \cdot \mathbf{w} = 0 \text{ for all } \mathbf{w} \in W\}.$$

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Theorem

If W is a subspace of \mathbb{R}^n , then:

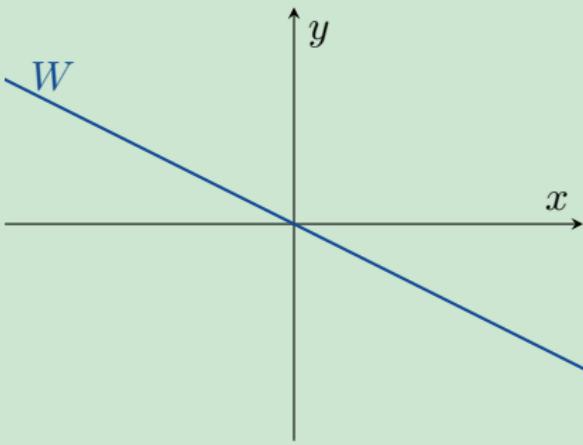
- 1 W^\perp is also a subspace of \mathbb{R}^n .
- 2 $W \cap W^\perp = \{\mathbf{0}\}$.
- 3 $(W^\perp)^\perp = W$.

The Fundamental Matrix Spaces



Example

In \mathbb{R}^2 :

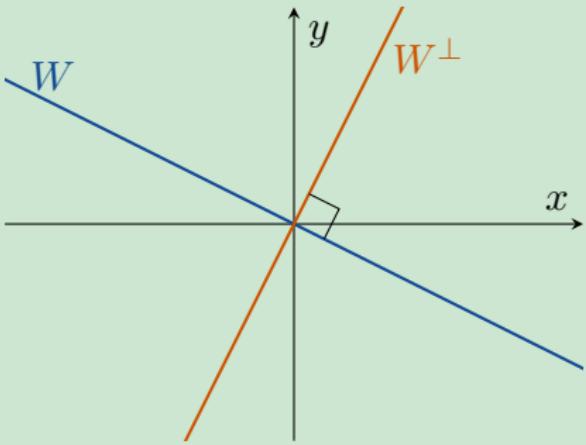


The Fundamental Matrix Spaces



Example

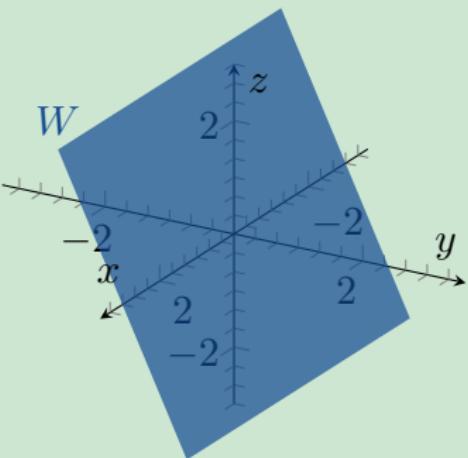
In \mathbb{R}^2 :



The Fundamental Matrix Spaces

Example

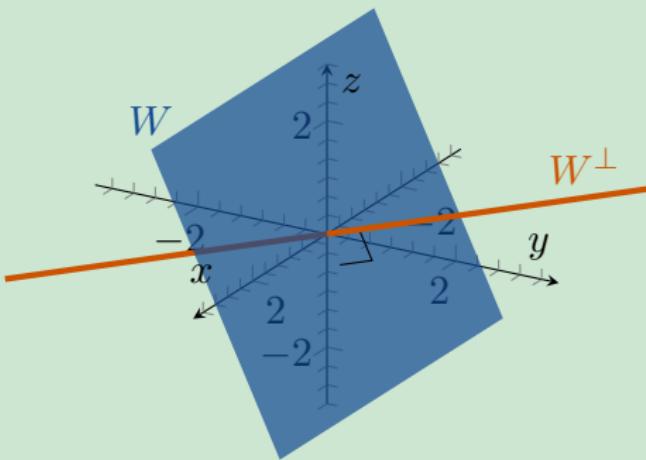
In \mathbb{R}^3 :



The Fundamental Matrix Spaces

Example

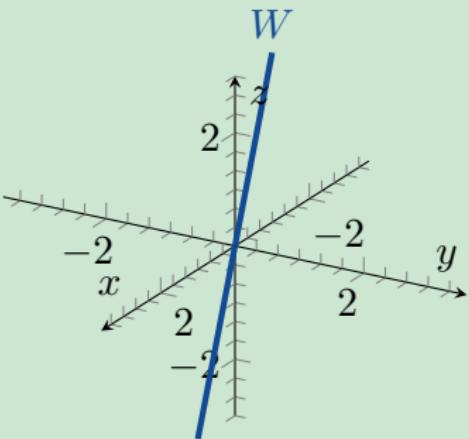
In \mathbb{R}^3 :



The Fundamental Matrix Spaces

Example

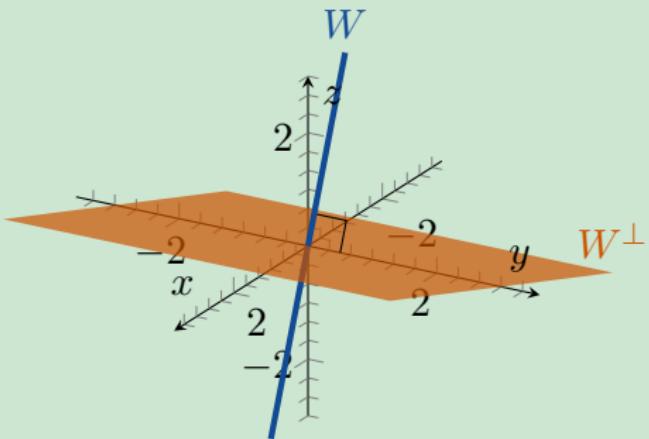
In \mathbb{R}^3 :



The Fundamental Matrix Spaces

Example

In \mathbb{R}^3 :



The Fundamental Matrix Spaces



Remark

Note that

$$V^\perp = \{\mathbf{0}\}$$

because only the zero vector is orthogonal to every vector in V ;
and hence

$$\{\mathbf{0}\}^\perp = V.$$

The Fundamental Matrix Spaces



Theorem

Let A be an $m \times n$ matrix. Then

- 1 The null space of A is the orthogonal complement of the row space of A , in \mathbb{R}^n .

$$\text{Nul } A = (\text{Row } A)^\perp \quad \text{and} \quad \text{Row } A = (\text{Nul } A)^\perp.$$

The Fundamental Matrix Spaces

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- 2 The null space of A^T is the orthogonal complement of the column space of A , in \mathbb{R}^m .

$$\text{Nul } A^T = (\text{Col } A)^\perp \quad \text{and} \quad \text{Col } A = (\text{Nul } A^T)^\perp.$$

The Fundamental Matrix Spaces

Theorem

Let A be an $n \times n$ matrix. The following statements are equivalent:

- 1 A is invertible. 9
- 2 $A\mathbf{x} = \mathbf{0}$ has only the trivial solution. 10
- 3 The reduced row echelon form of A is I_n . 11
- 4 A is expressible as a product of elementary matrices. 12
- 5 $A\mathbf{x} = \mathbf{b}$ is consistent for every $n \times 1$ matrix \mathbf{b} . 13
- 6 $A\mathbf{x} = \mathbf{b}$ has exactly one solution for every $n \times 1$ matrix \mathbf{b} . 14
- 7 $\det(A) \neq 0$. 15
- 8 16

The Fundamental Matrix Spaces

Theorem

Let A be an $n \times n$ matrix. The following statements are equivalent:

- | | |
|--|--|
| 1 A is invertible. | 9 The row vectors of A are linearly independent. |
| 2 $A\mathbf{x} = \mathbf{0}$ has only the trivial solution. | 10 The column vectors of A span \mathbb{R}^n . |
| 3 The reduced row echelon form of A is I_n . | 11 The row vectors of A span \mathbb{R}^n . |
| 4 A is expressible as a product of elementary matrices. | 12 The column vectors of A form a basis for \mathbb{R}^n . |
| 5 $A\mathbf{x} = \mathbf{b}$ is consistent for every $n \times 1$ matrix \mathbf{b} . | 13 The row vectors of A form a basis for \mathbb{R}^n . |
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The Fundamental Matrix Spaces

Theorem

Let A be an $n \times n$ matrix. The following statements are equivalent:

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- 2** $A\mathbf{x} = \mathbf{0}$ has only the trivial solution.
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- 4** A is expressible as a product of elementary matrices.
- 5** $A\mathbf{x} = \mathbf{b}$ is consistent for every $n \times 1$ matrix \mathbf{b} .
- 6** $A\mathbf{x} = \mathbf{b}$ has exactly one solution for every $n \times 1$ matrix \mathbf{b} .
- 7** $\det(A) \neq 0$.
- 8** The column vectors of A are linearly independent.
- 9** The row vectors of A are linearly independent.
- 10** The column vectors of A span \mathbb{R}^n .
- 11** The row vectors of A span \mathbb{R}^n .
- 12** The column vectors of A form a basis for \mathbb{R}^n .
- 13** The row vectors of A form a basis for \mathbb{R}^n .
- 14** $\text{rank } A = n$.
- 15** $\text{nullity } A = 0$.
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The Fundamental Matrix Spaces

Theorem

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- 11** The row vectors of A span \mathbb{R}^n .
- 12** The column vectors of A form a basis for \mathbb{R}^n .
- 13** The row vectors of A form a basis for \mathbb{R}^n .
- 14** $\text{rank } A = n$.
- 15** $\text{nullity } A = 0$.
- 16** $(\text{Nul } A)^\perp = \mathbb{R}^n$.
- 17** $(\text{Row } A)^\perp = \{\mathbf{0}\}$.



Linear Transformations

Definition

Let V and W be vector spaces. A function $T : V \rightarrow W$ is called a *linear transformation* iff

- 1 $T(\mathbf{u} + \mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v})$
- 2 $T(k\mathbf{u}) = kT(\mathbf{u})$; and

for all vectors $\mathbf{u}, \mathbf{v} \in V$ and all scalars k .

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for all vectors $\mathbf{u}, \mathbf{v} \in V$ and all scalars k .

Definition

If $V = W$, then the linear transformation $T : V \rightarrow V$ is called a *linear operator* on V .

$$T(\mathbf{u} + \mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v}) \text{ and } T(k\mathbf{u}) = kT(\mathbf{u})$$



Remark

I have tried to avoid any mention of linear transformations up to now. Your textbook introduces them in section 1.8.

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Remark

“Transformation” is just another word for “function”. The important word in “linear transformation” is “linear”.

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Remark

We can combine rules 1 and 2 to show that

$$T(k_1\mathbf{v}_1 + k_2\mathbf{v}_2) = k_1T(\mathbf{v}_1) + k_2T(\mathbf{v}_2)$$

Linear Trans

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$$T(k_1\mathbf{v}_1 + k_2\mathbf{v}_2) = k_1T(\mathbf{v}_1) + k_2T(\mathbf{v}_2)$$

or more generally that

$$T(k_1\mathbf{v}_1 + k_2\mathbf{v}_2 + \dots + k_r\mathbf{v}_r) = k_1T(\mathbf{v}_1) + k_2T(\mathbf{v}_2) + \dots + k_rT(\mathbf{v}_r).$$

$$T(\mathbf{u} + \mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v}) \text{ and } T(k\mathbf{u}) = kT(\mathbf{u})$$



Theorem

If $T : V \rightarrow W$ is a linear transformation, then

$$T(\mathbf{0}) = \mathbf{0}.$$

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Theorem

If $T : V \rightarrow W$ is a linear transformation, then

$$T(\mathbf{0}) = \mathbf{0}.$$

Proof.

Let $\mathbf{u} \in V$ be any vector. Since $0\mathbf{u} = \mathbf{0}$, it follows that

$$T(\mathbf{0}) = T(0\mathbf{u}) = 0T(\mathbf{u}) = \mathbf{0}.$$



$$T(\mathbf{u} + \mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v}) \text{ and } T(k\mathbf{u}) = kT(\mathbf{u})$$



Example (Matrix Transformations)

Let A be an $m \times n$ matrix. We can define a function
 $T_A : \mathbb{R}^n \rightarrow \mathbb{R}^m$ by

$$T_A(\mathbf{x}) = A\mathbf{x}.$$

$$T(\mathbf{u} + \mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v}) \text{ and } T(k\mathbf{u}) = kT(\mathbf{u})$$



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Note that if $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$ are any vectors, and k is any number, then

$$T_A(\mathbf{u} + \mathbf{v}) = A(\mathbf{u} + \mathbf{v}) = A\mathbf{u} + A\mathbf{v} = T_A(\mathbf{u}) + T_A(\mathbf{v})$$

and

$$T_A(k\mathbf{u}) = A(k\mathbf{u}) = k(A\mathbf{u}) = kT_A(\mathbf{u}).$$

Therefore T_A is a linear transformation.

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Therefore T_A is a linear transformation.

Functions of the form $T_A : \mathbb{R}^n \rightarrow \mathbb{R}^m$ are called *matrix transformations*.

$$T(\mathbf{u} + \mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v}) \text{ and } T(k\mathbf{u}) = kT(\mathbf{u})$$



Example (The Zero Transformation)

Let V and W be any two vector spaces. Define a function $T : V \rightarrow W$ by $T(\mathbf{v}) = \mathbf{0}$ for all $\mathbf{v} \in V$.

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and

$$T(k\mathbf{u}) = \mathbf{0} = k\mathbf{0} = kT(\mathbf{u})$$

for all \mathbf{u} , \mathbf{v} and k . Hence T is linear.

$$T(\mathbf{u} + \mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v}) \text{ and } T(k\mathbf{u}) = kT(\mathbf{u})$$



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and

$$T(k\mathbf{u}) = \mathbf{0} = k\mathbf{0} = kT(\mathbf{u})$$

for all \mathbf{u}, \mathbf{v} and k . Hence T is linear.

Example (The Identity Operator)

Let V be any vector space. The function $I : V \rightarrow V$ defined by $I(\mathbf{v}) = \mathbf{v}$ is called the *identity operator* on V . I leave it to you to show that I is linear.

$$T(\mathbf{u} + \mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v}) \text{ and } T(k\mathbf{u}) = kT(\mathbf{u})$$



Example (A Linear Transformation from \mathbb{P}^n to \mathbb{P}^{n+1})

Let $\mathbf{p} = a_0 + a_1x + \dots + a_nx^n$ be a polynomial in \mathbb{P}^n . Define $T : \mathbb{P}^n \rightarrow \mathbb{P}^{n+1}$ by $T(\mathbf{p}) = x\mathbf{p}$. That is

$$T(\mathbf{p}) = a_0x + a_1x^2 + \dots + a_nx^{n+1}.$$

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$$T(\mathbf{p}) = a_0x + a_1x^2 + \dots + a_nx^{n+1}.$$

This function is a linear transformation because

$$T(\mathbf{p}_1 + \mathbf{p}_2) = x(\mathbf{p}_1 + \mathbf{p}_2) = x\mathbf{p}_1 + x\mathbf{p}_2 = T(\mathbf{p}_1) + T(\mathbf{p}_2)$$

and

$$T(k\mathbf{p}) = x(k\mathbf{p}) = k(x\mathbf{p}) = kT(\mathbf{p}).$$

$$T(\mathbf{u} + \mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v}) \text{ and } T(k\mathbf{u}) = kT(\mathbf{u})$$



Example

Let $\mathbb{R}^{n \times n} = M_{nn}$ be the vector space of $n \times n$ matrices.

- 1 Is $T_1 : \mathbb{R}^{n \times n} \rightarrow \mathbb{R}^{n \times n}$, $T_1(A) = A^T$ linear?
- 2 Is $T_2 : \mathbb{R}^{n \times n} \rightarrow \mathbb{R}$, $T_2(A) = \det(A)$ linear?

$$T(\mathbf{u} + \mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v}) \text{ and } T(k\mathbf{u}) = kT(\mathbf{u})$$



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- 2 Is $T_2 : \mathbb{R}^{n \times n} \rightarrow \mathbb{R}$, $T_2(A) = \det(A)$ linear?

- 1 Since

$$T_1(A + B) = (A + B)^T = A^T + B^T = T_1(A) + T_1(B)$$

and

$$T_2(kA) = (kA)^T = kA^T = kT_2(A)$$

the answer is YES.

Example

Let $\mathbb{R}^{n \times n} = M_{nn}$ be the vector space of $n \times n$ matrices.

- 1 Is $T_1 : \mathbb{R}^{n \times n} \rightarrow \mathbb{R}^{n \times n}$, $T_1(A) = A^T$ linear?
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- 1 Since

$$T_1(A + B) = (A + B)^T = A^T + B^T = T_1(A) + T_1(B)$$

and

$$T_2(kA) = (kA)^T = kA^T = kT_2(A)$$

the answer is YES.

- 2 Recall from Lecture 4 that

$$T_2(kA) = \det(kA) = k^n \det(A) = k^n T_2(A).$$

If $n \geq 2$, then we have $T_2(kA) \neq kT_2(A)$. So the answer is NO (if $n \geq 2$).

$$T(\mathbf{u} + \mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v}) \text{ and } T(k\mathbf{u}) = kT(\mathbf{u})$$



Recall that Linear Transformations always have $T(\mathbf{0}) = \mathbf{0}$.

Example (A Nonlinear Transformation)

Let $\mathbf{b} \in \mathbb{R}^2$ such that $\mathbf{b} \neq \mathbf{0}$. Define $T : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ by $T(\mathbf{x}) = \mathbf{x} + \mathbf{b}$. Note that T is not linear because

$$T(\mathbf{0}) = \mathbf{b} \neq \mathbf{0}.$$

$$T(\mathbf{u} + \mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v}) \text{ and } T(k\mathbf{u}) = kT(\mathbf{u})$$



Example

$\frac{d}{dx} : C^1(-\infty, \infty) \rightarrow F(-\infty, \infty)$ is a linear operator because

$$\frac{d}{dx}(f + g) = \frac{d}{dx}(f) + \frac{d}{dx}(g)$$

and

$$\frac{d}{dx}(kf) = k \frac{d}{dx}(f).$$

Finding Linear Transformations from Images of Basis Vectors

Theorem

Let $T : V \rightarrow W$ be a linear transformation, where V is finite-dimensional. If $S = \{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\}$ is a basis for V , then the image of any vector $\mathbf{v} \in V$ can be expressed as

$$T(\mathbf{v}) = c_1 T(\mathbf{v}_1) + c_2 T(\mathbf{v}_2) + \dots + c_n T(\mathbf{v}_n)$$

where c_1, c_2, \dots, c_n are the coefficients required to express \mathbf{v} as a linear combination of the vectors in the basis S .

► EXAMPLE 10 Computing with Images of Basis Vectors

Consider the basis $S = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ for \mathbb{R}^3 , where

$$\mathbf{v}_1 = (1, 1, 1), \quad \mathbf{v}_2 = (1, 1, 0), \quad \mathbf{v}_3 = (1, 0, 0)$$

Let $T : \mathbb{R}^3 \rightarrow \mathbb{R}^2$ be the linear transformation for which

$$T(\mathbf{v}_1) = (1, 0), \quad T(\mathbf{v}_2) = (2, -1), \quad T(\mathbf{v}_3) = (4, 3)$$

Find a formula for $T(x_1, x_2, x_3)$, and then use that formula to compute $T(2, -3, 5)$.

Solution We first need to express $\mathbf{x} = (x_1, x_2, x_3)$ as a linear combination of $\mathbf{v}_1, \mathbf{v}_2$, and \mathbf{v}_3 . If we write

$$(x_1, x_2, x_3) = c_1(1, 1, 1) + c_2(1, 1, 0) + c_3(1, 0, 0)$$

then on equating corresponding components, we obtain

$$c_1 + c_2 + c_3 = x_1$$

$$c_1 + c_2 = x_2$$

$$c_1 = x_3$$

which yields $c_1 = x_3, c_2 = x_2 - x_3, c_3 = x_1 - x_2$, so

$$\begin{aligned}(x_1, x_2, x_3) &= x_3(1, 1, 1) + (x_2 - x_3)(1, 1, 0) + (x_1 - x_2)(1, 0, 0) \\&= x_3\mathbf{v}_1 + (x_2 - x_3)\mathbf{v}_2 + (x_1 - x_2)\mathbf{v}_3\end{aligned}$$

Thus

$$\begin{aligned}T(x_1, x_2, x_3) &= x_3T(\mathbf{v}_1) + (x_2 - x_3)T(\mathbf{v}_2) + (x_1 - x_2)T(\mathbf{v}_3) \\&= x_3(1, 0) + (x_2 - x_3)(2, -1) + (x_1 - x_2)(4, 3) \\&= (4x_1 - 2x_2 - x_3, 3x_1 - 4x_2 + x_3)\end{aligned}$$

From this formula we obtain

$$T(2, -3, 5) = (9, 23)$$

Kernal and Range

Definition

If $T : V \rightarrow W$ is a linear transformation, then the set of vectors in V that T maps into 0 is called the kernel of T and is denoted by $\ker(T)$.

$$\ker(T) = \{\mathbf{v} \in V \mid T(\mathbf{v}) = \mathbf{0}\}$$

Kernal and Range

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If $T : V \rightarrow W$ is a linear transformation, then the set of vectors in V that T maps into 0 is called the kernel of T and is denoted by $\ker(T)$.

$$\ker(T) = \{\mathbf{v} \in V \mid T(\mathbf{v}) = \mathbf{0}\}$$

Definition

The set of all vectors in W that are images under T of at least one vector in V is called the range of T and is denoted by $R(T)$.

$$R(T) = \{\mathbf{w} \in W \mid \exists \mathbf{v} \in V \text{ such that } T(\mathbf{v}) = \mathbf{w}\}$$

Linear Transformations



Example (Matrix Transformations)

If $T_A : \mathbb{R}^n \rightarrow \mathbb{R}^m$ is a matrix transformation (i.e. $T_A(\mathbf{x}) = A\mathbf{x}$), then

$$\ker(T_A) = \text{Nul } A \quad \text{and} \quad R(T_A) = \text{Col}(A).$$

Example (Matrix Transformations)

If $T_A : \mathbb{R}^n \rightarrow \mathbb{R}^m$ is a matrix transformation (i.e. $T_A(\mathbf{x}) = A\mathbf{x}$), then

$$\ker(T_A) = \text{Nul } A \quad \text{and} \quad R(T_A) = \text{Col}(A).$$

Example (Zero Transformation)

Let $T : V \rightarrow W$ is the zero transformation. Since T maps every vector in V to $\mathbf{0}$, it follows that

$$\ker(T) = V \quad \text{and} \quad R(T) = \{\mathbf{0}\}.$$

► EXAMPLE 15 Kernel and Range of the Identity Operator

Let $I: V \rightarrow V$ be the identity operator. Since $I(\mathbf{v}) = \mathbf{v}$ for all vectors in V , *every* vector in V is the image of some vector (namely, itself); thus $R(I) = V$. Since the *only* vector that I maps into $\mathbf{0}$ is $\mathbf{0}$, it follows that $\ker(I) = \{\mathbf{0}\}$.

Linear Transformations



Example

Let $T : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ be the orthogonal projection onto the xy -plane (i.e. $T(x, y, z) = (x, y, 0)$).

Linear Transformations



Example

Let $T : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ be the orthogonal projection onto the xy -plane (i.e. $T(x, y, z) = (x, y, 0)$). Then

$$\begin{aligned}(x, y, z) \in \ker(T) &\implies (0, 0, 0) = T(x, y, z) = (x, y, 0) \\ &\implies x = y = 0.\end{aligned}$$

Linear Transformations



Example

Let $T : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ be the orthogonal projection onto the xy -plane (i.e. $T(x, y, z) = (x, y, 0)$). Then

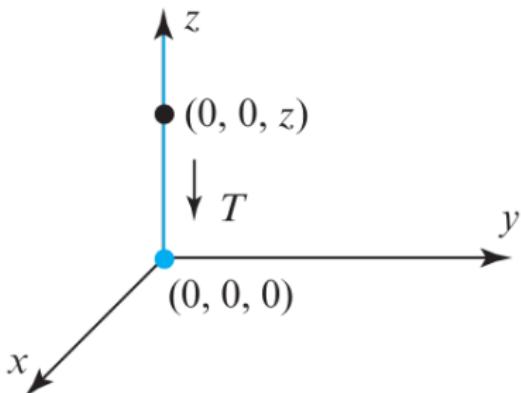
$$\begin{aligned}(x, y, z) \in \ker(T) &\implies (0, 0, 0) = T(x, y, z) = (x, y, 0) \\ &\implies x = y = 0.\end{aligned}$$

Moreover $R(T)$ is the set of points of the form $(x, y, 0)$.

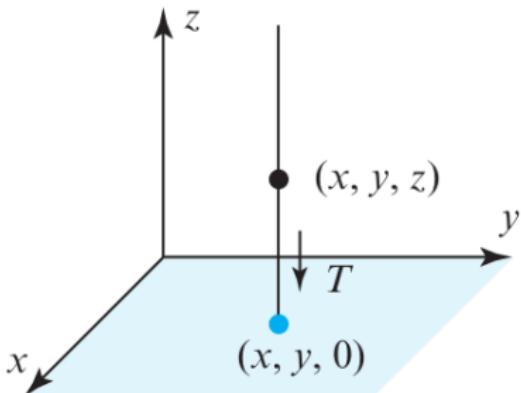
Linear Transformations

Example

Let $T : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ be the orthogonal projection onto the xy -plane (i.e. $T(x, y, z) = (x, y, 0)$). Then



(a) $\ker(T)$ is the z -axis.



(b) $R(T)$ is the entire xy -plane

► EXAMPLE 17 Kernel and Range of a Rotation

Let $T: R^2 \rightarrow R^2$ be the linear operator that rotates each vector in the xy -plane through the angle θ (Figure 8.1.3). Since *every* vector in the xy -plane can be obtained by rotating some vector through the angle θ , it follows that $R(T) = R^2$. Moreover, the only vector that rotates into $\mathbf{0}$ is $\mathbf{0}$, so $\ker(T) = \{\mathbf{0}\}$.

Properties of Kernel and Range

Theorem

If $T : V \rightarrow W$ is a linear transformation, then:

- 1 The kernel of T is a subspace of V .
- 2 The range of T is a subspace of W .

Rank and Nullity

Let $T : V \rightarrow W$ be a linear transformation.

Definition

If the range of T is finite-dimensional, then its dimension is called the *rank of T* .

Definition

If the kernel of T is finite-dimensional, then its dimension is called the nullity of T .

Theorem

If $T : V \rightarrow W$ is a linear transformation from a finite-dimensional vector space V to a vector space W , then the range of T is finite-dimensional, and

$$\text{rank } T + \text{nullity } T = \dim(V).$$

Break

We will continue at 3pm



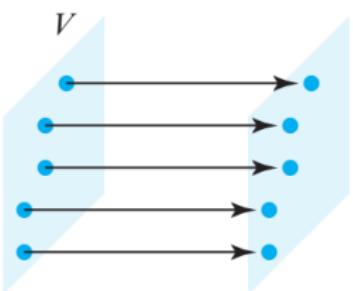


Composition and Inverse Transforma- tions

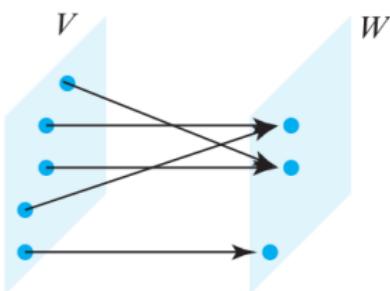
One-to-One and Onto

Definition

If $T : V \rightarrow W$ is a linear transformation from a vector space V to a vector space W , then T is said to be *one-to-one* if T maps distinct vectors in V into distinct vectors in W .



One-to-one. Distinct vectors in V have distinct images in W .

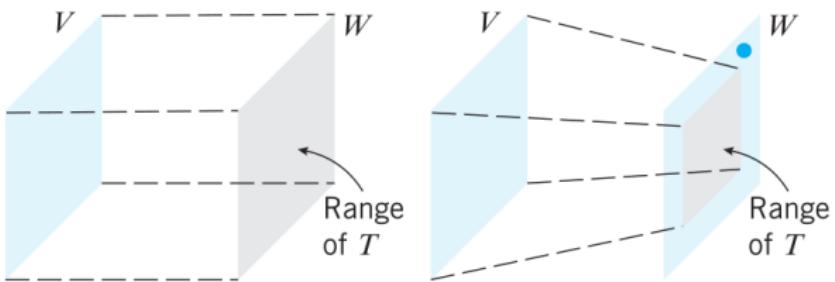


Not one-to-one. There exist distinct vectors in V with the same image.

Composition and Inverse Transformations

Definition

If $T : V \rightarrow W$ is a linear transformation from a vector space V to a vector space W , then T is said to be *onto* if every vector in W is the image of at least one vector in V .



Onto W . Every vector in W is the image of some vector in V .

Not onto W . Not every vector in W is the image of some vector in V .

Theorem

If $T : V \rightarrow W$ is a linear transformation, then the following statements are equivalent:

- 1 T is one-to-one.
- 2 $\ker(T) = \{\mathbf{0}\}$.

Composition and Inverse Transformations



Theorem

If $T : V \rightarrow W$ is a linear transformation, then the following statements are equivalent:

- 1 T is one-to-one.
- 2 $\ker(T) = \{\mathbf{0}\}$.

Proof.

1 \implies 2: We know that $T(\mathbf{0}) = \mathbf{0}$ because T is linear. But if T is one-to-one, then $\mathbf{0}$ is the only vector that maps to $\mathbf{0}$. Hence $\ker(T) = \{\mathbf{0}\}$.

Composition and Inverse Transformations



Theorem

If $T : V \rightarrow W$ is a linear transformation, then the following statements are equivalent:

- 1 T is one-to-one.
- 2 $\ker(T) = \{\mathbf{0}\}$.

Proof.

1 \implies 2: We know that $T(\mathbf{0}) = \mathbf{0}$ because T is linear. But if T is one-to-one, then $\mathbf{0}$ is the only vector that maps to $\mathbf{0}$. Hence $\ker(T) = \{\mathbf{0}\}$.

1 \iff 2: Suppose that $\ker(T) = \{\mathbf{0}\}$. Let \mathbf{u}, \mathbf{v} be two different vectors in V . So $\mathbf{u} - \mathbf{v} \neq \mathbf{0}$ which implies that $T(\mathbf{u} - \mathbf{v}) \neq \mathbf{0}$.

Composition and Inverse Transformations



Theorem

If $T : V \rightarrow W$ is a linear transformation, then the following statements are equivalent:

- 1 T is one-to-one.
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Proof.

1 \implies 2: We know that $T(\mathbf{0}) = \mathbf{0}$ because T is linear. But if T is one-to-one, then $\mathbf{0}$ is the only vector that maps to $\mathbf{0}$. Hence $\ker(T) = \{\mathbf{0}\}$.

1 \iff 2: Suppose that $\ker(T) = \{\mathbf{0}\}$. Let \mathbf{u}, \mathbf{v} be two different vectors in V . So $\mathbf{u} - \mathbf{v} \neq \mathbf{0}$ which implies that $T(\mathbf{u} - \mathbf{v}) \neq \mathbf{0}$. But then

$$T(\mathbf{u}) - T(\mathbf{v}) = T(\mathbf{u} - \mathbf{v}) \neq \mathbf{0}$$

Composition and Inverse Transformations



Theorem

If $T : V \rightarrow W$ is a linear transformation, then the following statements are equivalent:

- 1 T is one-to-one.
- 2 $\ker(T) = \{\mathbf{0}\}$.

Proof.

1 \implies 2: We know that $T(\mathbf{0}) = \mathbf{0}$ because T is linear. But if T is one-to-one, then $\mathbf{0}$ is the only vector that maps to $\mathbf{0}$. Hence $\ker(T) = \{\mathbf{0}\}$.

1 \iff 2: Suppose that $\ker(T) = \{\mathbf{0}\}$. Let \mathbf{u}, \mathbf{v} be two different vectors in V . So $\mathbf{u} - \mathbf{v} \neq \mathbf{0}$ which implies that $T(\mathbf{u} - \mathbf{v}) \neq \mathbf{0}$. But then

$$T(\mathbf{u}) - T(\mathbf{v}) = T(\mathbf{u} - \mathbf{v}) \neq \mathbf{0} \implies T(\mathbf{u}) \neq T(\mathbf{v}).$$



Theorem

If V and W are finite-dimensional vector spaces with the same dimension, and if $T : V \rightarrow W$ is a linear transformation, then the following statements are equivalent:

- 1 T is one-to-one.
- 2 $\ker(T) = \{\mathbf{0}\}$.
- 3 T is onto [i.e., $R(T) = W$].

Composition and Inverse Transformations



Example (Matrix Transformations)

Consider $T_A : \mathbb{R}^n \rightarrow \mathbb{R}^m$, $T_A(\mathbf{x}) = A\mathbf{x}$ where A is an $m \times n$ matrix.

Composition and Inverse Transformations



Example (Matrix Transformations)

Consider $T_A : \mathbb{R}^n \rightarrow \mathbb{R}^m$, $T_A(\mathbf{x}) = A\mathbf{x}$ where A is an $m \times n$ matrix.

- If $m < n$, then T_A is not one-to-one.

Composition and Inverse Transformations



Example (Matrix Transformations)

Consider $T_A : \mathbb{R}^n \rightarrow \mathbb{R}^m$, $T_A(\mathbf{x}) = A\mathbf{x}$ where A is an $m \times n$ matrix.

- If $m < n$, then T_A is not one-to-one.
- If $m > n$, then T_A is not onto.

Composition and Inverse Transformations



Example (Matrix Transformations)

Consider $T_A : \mathbb{R}^n \rightarrow \mathbb{R}^m$, $T_A(\mathbf{x}) = A\mathbf{x}$ where A is an $m \times n$ matrix.

- If $m < n$, then T_A is not one-to-one.
- If $m > n$, then T_A is not onto.
- If $m = n$, then

T_A is both one-to-one and onto $\iff A$ is invertible

Composition and Inverse Transformations



Example

The linear transformation $T_1 : \mathbb{P}^3 \rightarrow \mathbb{R}^4$,

$$T_1(a + bx + cx^2 + dx^3) = (a, b, c, d)$$

is both one-to-one and onto because $\ker(T_1) = \{\mathbf{0}\}$ (you prove).

Composition and Inverse Transformations



Example

The linear transformation $T_1 : \mathbb{P}^3 \rightarrow \mathbb{R}^4$,

$$T_1(a + bx + cx^2 + dx^3) = (a, b, c, d)$$

is both one-to-one and onto because $\ker(T_1) = \{\mathbf{0}\}$ (you prove).

Example

The linear transformation $T_2 : \mathbb{R}^{2 \times 2} \rightarrow \mathbb{R}^4$,

$$T_2 \left(\begin{bmatrix} a & b \\ c & d \end{bmatrix} \right) = (a, b, c, d)$$

is both one-to-one and onto because $\ker(T_2) = \{\mathbf{0}\}$ (you prove).

Composition and Inverse Transformations



Example

Let $T : \mathbb{P}^n \rightarrow \mathbb{P}^{n+1}$ be the linear transformation

$$T(\mathbf{p}) = x\mathbf{p}.$$

If $\mathbf{p} \neq \mathbf{q}$, then $x\mathbf{p} \neq x\mathbf{q}$. Therefore T is one-to-one.

Composition and Inverse Transformations



Example

Let $T : \mathbb{P}^n \rightarrow \mathbb{P}^{n+1}$ be the linear transformation

$$T(\mathbf{p}) = x\mathbf{p}.$$

If $\mathbf{p} \neq \mathbf{q}$, then $x\mathbf{p} \neq x\mathbf{q}$. Therefore T is one-to-one.

However, T is not onto because all images under T have a zero constant term. Thus, for example, there does not exist a polynomial \mathbf{p} in \mathbb{P}^n such that $T(\mathbf{p}) = 1$.

Composition and Inverse Transformations



Example

Let $T : \mathbb{P}^n \rightarrow \mathbb{P}^{n+1}$ be the linear transformation

$$T(\mathbf{p}) = x\mathbf{p}.$$

If $\mathbf{p} \neq \mathbf{q}$, then $x\mathbf{p} \neq x\mathbf{q}$. Therefore T is one-to-one.

However, T is not onto because all images under T have a zero constant term. Thus, for example, there does not exist a polynomial \mathbf{p} in \mathbb{P}^n such that $T(\mathbf{p}) = 1$.

Example

$\frac{d}{dx} : C^1(-\infty, \infty) \rightarrow F(-\infty, \infty)$ is not one-to-one because, for example,

$$\frac{d}{dx}(x^2) = \frac{d}{dx}(x^2 + 1).$$

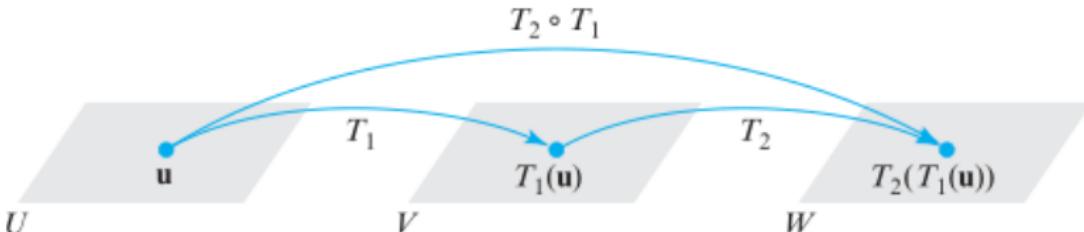
Compositions of Linear Transformations

Definition

If $T_1 : U \rightarrow V$ and $T_2 : V \rightarrow W$ are linear transformations, then the composition of T_2 with T_1 , denoted by $T_2 \circ T_1$, is the function defined by the formula

$$(T_2 \circ T_1)(\mathbf{u}) = T_2(T_1(\mathbf{u}))$$

where \mathbf{u} is a vector in U .



Theorem

If $T_1 : U \rightarrow V$ and $T_2 : V \rightarrow W$ are linear transformations, then $(T_2 \circ T_1) : U \rightarrow W$ is also a linear transformation.

► EXAMPLE 6 Composition of Linear Transformations

Let $T_1: P_1 \rightarrow P_2$ and $T_2: P_2 \rightarrow P_2$ be the linear transformations given by the formulas

$$T_1(p(x)) = xp(x) \quad \text{and} \quad T_2(p(x)) = p(2x + 4)$$

Then the composition $(T_2 \circ T_1): P_1 \rightarrow P_2$ is given by the formula

$$(T_2 \circ T_1)(p(x)) = T_2(T_1(p(x))) = T_2(xp(x)) = (2x + 4)p(2x + 4)$$

In particular, if $p(x) = c_0 + c_1x$, then

$$\begin{aligned}(T_2 \circ T_1)(p(x)) &= (T_2 \circ T_1)(c_0 + c_1x) = (2x + 4)(c_0 + c_1(2x + 4)) \\ &= c_0(2x + 4) + c_1(2x + 4)^2\end{aligned}$$

► EXAMPLE 7 Composition with the Identity Operator

If $T: V \rightarrow V$ is any linear operator, and if $I: V \rightarrow V$ is the identity operator (Example 3 of Section 8.1), then for all vectors \mathbf{v} in V , we have

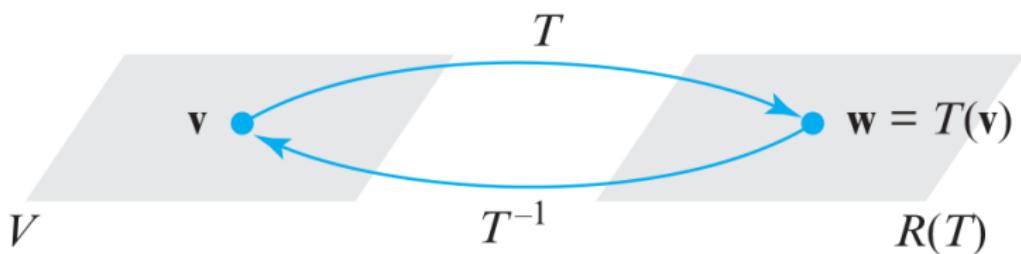
$$(T \circ I)(\mathbf{v}) = T(I(\mathbf{v})) = T(\mathbf{v})$$

$$(I \circ T)(\mathbf{v}) = I(T(\mathbf{v})) = T(\mathbf{v})$$

It follows that $T \circ I$ and $I \circ T$ are the same as T ; that is,

$$T \circ I = T \quad \text{and} \quad I \circ T = T \quad \blacktriangleleft \quad (2)$$

Inverse Linear Transformations



If $T : V \rightarrow W$ is a linear transformation, then its *inverse* $T^{-1} : R(T) \rightarrow V$ is also a linear transformation.

$$T(\mathbf{v}) = \mathbf{w} \iff \mathbf{v} = T^{-1}(\mathbf{w}).$$

► EXAMPLE 8 An Inverse Transformation

We showed in Example 3 of this section that the linear transformation $T: P_n \rightarrow P_{n+1}$ given by

$$T(\mathbf{p}) = T(p(x)) = xp(x)$$

is one-to-one but not onto. The fact that it is not onto can be seen explicitly from the formula

$$T(c_0 + c_1x + \cdots + c_nx^n) = c_0x + c_1x^2 + \cdots + c_nx^{n+1} \quad (6)$$

which makes it clear that the range of T consists of all polynomials in P_{n+1} that have zero constant term. Since T is one-to-one it has an inverse, and from (6) this inverse is given by the formula

$$T^{-1}(c_0x + c_1x^2 + \cdots + c_nx^{n+1}) = c_0 + c_1x + \cdots + c_nx^n$$

For example, in the case where $n \geq 3$,

$$T^{-1}(2x - x^2 + 5x^3 + 3x^4) = 2 - x + 5x^2 + 3x^3$$

Composition and Inverse Transformations



Theorem

If $T_1 : U \rightarrow V$ and $T_2 : V \rightarrow W$ are one-to-one linear transformations, then:

- 1 $T_2 \circ T_1$ is one-to-one.
- 2 $(T_2 \circ T_1)^{-1} = T_1^{-1} \circ T_2^{-1}$.



Isomorphisms

Isomorphisms

Definition

A linear transformation $T : V \rightarrow W$ that is both one-to-one and onto is called an *isomorphism*.

Isomorphisms

Definition

A linear transformation $T : V \rightarrow W$ that is both one-to-one and onto is called an *isomorphism*.

Definition

If there exists an isomorphism $V \rightarrow W$, then we say that V and W are *isomorphic*.

Remark

$T : V \rightarrow W$
is an isomorphism $\implies T^{-1} : W \rightarrow V$
is also an isomorphism

(you prove)

What does isomorphic really mean?

Consider the following isomorphism $T : \mathbb{P}^2 \rightarrow \mathbb{R}^3$

$$a_0 + a_1x + a_2x^2 \quad \begin{array}{c} \xrightarrow{T} \\ \xleftarrow{T^{-1}} \end{array} \quad (a_0, a_1, a_2)$$

What does isomorphic really mean?

Consider the following isomorphism $T : \mathbb{P}^2 \rightarrow \mathbb{R}^3$

$$a_0 + a_1x + a_2x^2 \quad \begin{array}{c} \xrightarrow{T} \\ \xleftarrow{T^{-1}} \end{array} \quad (a_0, a_1, a_2)$$

On the left we have a polynomial function. On the right we have a vector. But the numbers a_0 , a_1 and a_2 are the same.

Isomorphisms

The behaviour under vector addition and scalar multiplication are also the same:

$$\begin{array}{ccc} (a_0 + a_1x + a_2x^2) + (b_0 + b_1x + b_2x^2) & \xrightleftharpoons[T]{T^{-1}} & (a_0, a_1, a_2) + (b_0, b_1, b_2) \\ (a_0 + b_0) + (a_1 + b_1)x + (a_2 + b_2)x^2 & \xrightleftharpoons[T]{T^{-1}} & (a_0 + b_0, a_1 + b_1, a_2 + b_2) \end{array}$$

Isomorphisms

The behaviour under vector addition and scalar multiplication are also the same:

$$\begin{array}{ccc}
 (a_0 + a_1x + a_2x^2) + (b_0 + b_1x + b_2x^2) & \xrightarrow{T} & (a_0, a_1, a_2) + (b_0, b_1, b_2) \\
 \\
 (a_0 + b_0) + (a_1 + b_1)x + (a_2 + b_2)x^2 & \xleftrightarrow[T]{T^{-1}} & (a_0 + b_0, a_1 + b_1, a_2 + b_2)
 \end{array}$$

and

$$\begin{array}{ccc}
 k(a_0 + a_1x + a_2x^2) & \xrightarrow{T} & k(a_0, a_1, a_2) \\
 \\
 ka_0 + ka_1x + ka_2x^2 & \xleftrightarrow[T]{T^{-1}} & (ka_0, ka_1, ka_2)
 \end{array}$$

Isomorphisms

Remark

In the Greek language,

- ‘morphē’ means ‘form’; and
- ‘iso’ means ‘identical’.

Isomorphisms

Remark

In the Greek language,

- ‘morphe’ means ‘form’; and
- ‘iso’ means ‘identical’.

An Important Theorem

Theorem (★ ★ ★ ★ ★)

Every real n -dimensional vector space is isomorphic to \mathbb{R}^n .

Isomorphisms

Theorem

If $S = \{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\}$ is a basis for a vector space V , then the coordinate map

$$T : \mathbf{u} \mapsto (\mathbf{u})_S$$

is an isomorphism between V and \mathbb{R}^n .

Isomorphisms

Theorem

If $S = \{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\}$ is a basis for a vector space V , then the coordinate map

$$T : \mathbf{u} \mapsto (\mathbf{u})_S$$

is an isomorphism between V and \mathbb{R}^n .

Example (The Natural Isomorphism Between \mathbb{P}^{n-1} and \mathbb{R}^n)

The linear transformation $T : \mathbb{P}^{n-1} \rightarrow \mathbb{R}^n$,

$$T : a_0 + a_1x + a_2X^2 + \dots + a_{n-1}x^{n-1} \mapsto (a_0, a_1, a_2, \dots, a_{n-1})$$

is an isomorphism between \mathbb{P}^{n-1} and \mathbb{R}^n . This is called the *natural isomorphism* between these vector spaces.

Isomorphisms

Example (The Natural Isomorphism Between $\mathbb{R}^{2 \times 2}$ and \mathbb{R}^4)

The coordinate map

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \mapsto (a, b, c, d)$$

defines an isomorphism between $\mathbb{R}^{2 \times 2} = M_{22}$ and \mathbb{R}^4 . We call this the *natural isomorphism between $\mathbb{R}^{2 \times 2}$ and \mathbb{R}^4* .

Working with Isomorphisms

Example

Are the following polynomials linearly independent?

$$p_1 = 1 + 2x - 3x^2 + 4x^3 + x^5$$

$$p_2 = 1 + 3x - 4x^2 + 6x^3 + 5x^4 + 4x^5$$

$$p_3 = 3 + 8x - 11x^2 - 16x^3 + 10x^4 + 9x^5$$

Isomorphisms



We will use the fact that \mathbb{P}^5 is isomorphic to \mathbb{R}^6 .

Isomorphisms



We will use the fact that \mathbb{P}^5 is isomorphic to \mathbb{R}^6 .

The following two questions are equivalent: Are the following polynomials linearly independent?

$$\begin{aligned}\mathbf{p}_1 &= 1 + 2x - 3x^2 + 4x^3 + x^5 \\ \mathbf{p}_2 &= 1 + 3x - 4x^2 + 6x^3 + 5x^4 + 4x^5 \\ \mathbf{p}_3 &= 3 + 8x - 11x^2 - 16x^3 + 10x^4 + 9x^5\end{aligned}$$

Are the following vectors linearly independent?

$$\begin{pmatrix} 1, & 2, & -3, & 4, & 0 & 1 \\ 1, & 3, & -4, & 6, & 5, & 4 \\ 3, & 8, & -11, & -16, & 10, & 9 \end{pmatrix}$$

Isomorphisms



Another equivalent question is: Is the row space of the following matrix 3 dimensional?

$$A = \begin{bmatrix} 1 & 2 & -3 & 4 & 0 & 1 \\ 1 & 3 & -4 & 6 & 5 & 4 \\ 3 & 8 & -11 & 16 & 10 & 9 \end{bmatrix}$$

Isomorphisms

Another equivalent question is: Is the row space of the following matrix 3 dimensional?

$$A = \begin{bmatrix} 1 & 2 & -3 & 4 & 0 & 1 \\ 1 & 3 & -4 & 6 & 5 & 4 \\ 3 & 8 & -11 & 16 & 10 & 9 \end{bmatrix}$$

I leave it to you to check that A is row equivalent to

$$R = \begin{bmatrix} 1 & 2 & -3 & 4 & 0 & 1 \\ 0 & 1 & -1 & 2 & 5 & 3 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

We can see that the answer to all three questions is **NO**.

Isomorphisms



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5 December 2018 [16:00-17:10]

MATH215, Second Exam

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4. (a) 10 points Let $L : \mathbb{R}^3 \rightarrow \mathbb{R}^4$ be a linear transformation defined by $L\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2y + 3z \\ x + y - 2z \\ 4x + y \\ 3x - y - z \end{pmatrix}$. Find the matrix representation of L .

- (b) 15 points Define $T : \mathbb{P}_2 \rightarrow \mathbb{R}^2$ by $T(\mathbf{p}) = \begin{bmatrix} \mathbf{p}(0) \\ \mathbf{p}(1) \end{bmatrix}$. Find a polynomial $\mathbf{p} \in \mathbb{P}_2$ which is a basis for kernel of T .

Solution:

$$\ker T = \{\mathbf{p} : \mathbf{p} \in \mathbb{P}_2 \text{ and } T(\mathbf{p}) = \mathbf{0}\}$$

$$\mathbf{p}(t) = a + bt + ct^2 \Rightarrow T(\mathbf{p}(t)) = \begin{bmatrix} \mathbf{p}(0) \\ \mathbf{p}(1) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} \mathbf{p}(0) \\ \mathbf{p}(1) \end{bmatrix} = \begin{bmatrix} a \\ a + b + c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{array}{l} a = 0 \\ b + c = 0 \end{array} \Rightarrow \mathbf{p}(t) = -ct + ct^2$$

$$\ker T = \{\mathbf{p} : \mathbf{p}(t) = (-t + t^2)c, c \in \mathbb{R}\} = \text{Span}\{-t + t^2\}$$

$$\mathbf{p}(t) = -t + t^2$$



Next Time

- Matrices for Linear Transformations
- Similarity
- Complex Numbers