

Mathematics IV

Dr. Neil Course

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2 First Order ODEs

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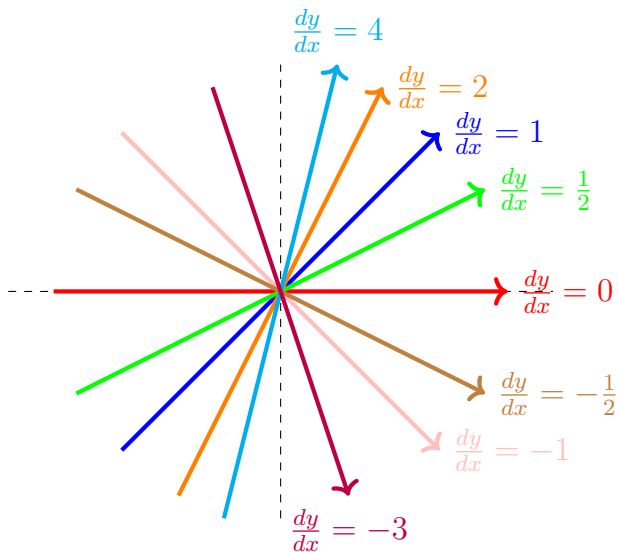
7 Systems of First Order Linear ODEs

- Homogeneous Linear Systems with Constant Coefficients
- Complex Eigenvalues
- Repeated Eigenvectors



1 Introduction

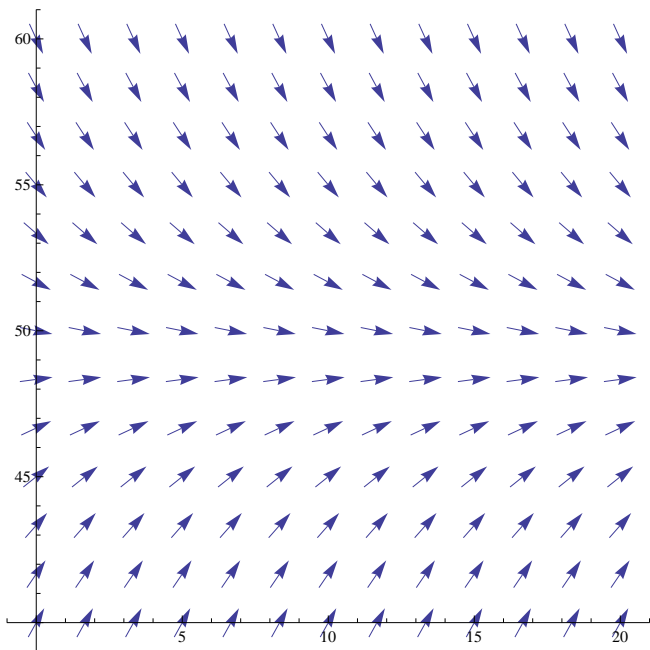
Direction Fields



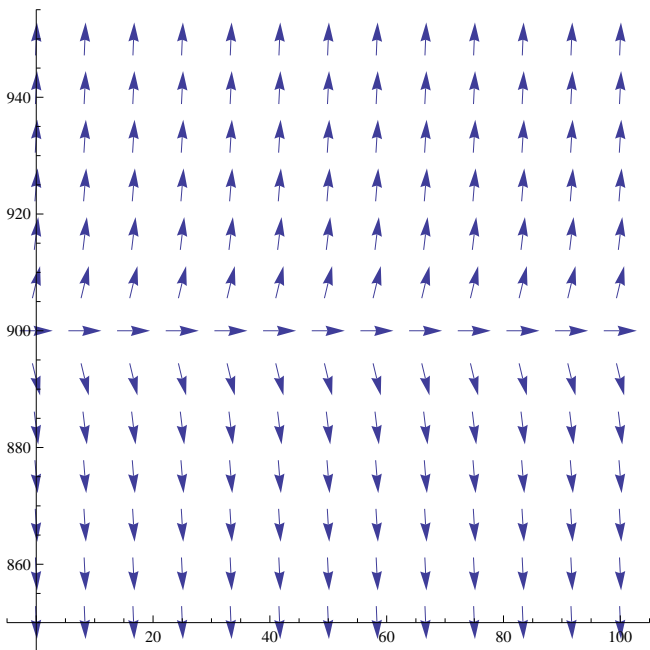
A Falling Object



$$\frac{dv}{dt} = 9.8 - \frac{v}{5}$$



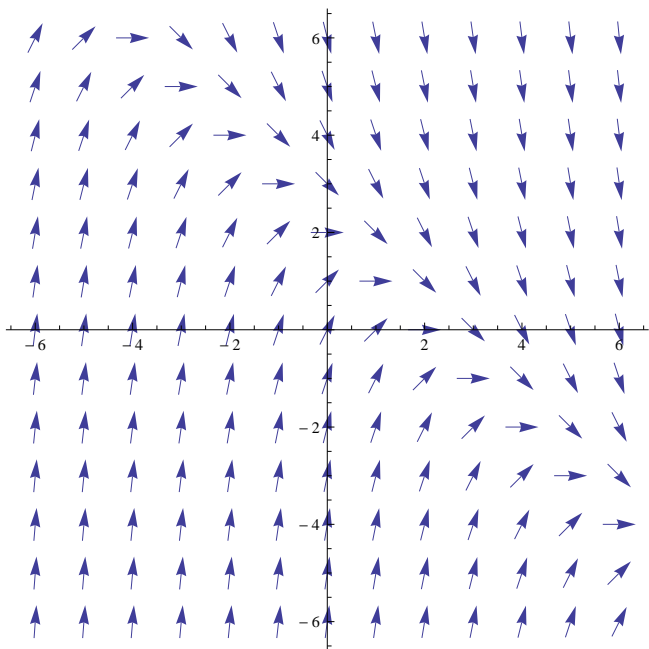
$$\frac{dp}{dt} = \frac{p}{2} - 450$$



Example



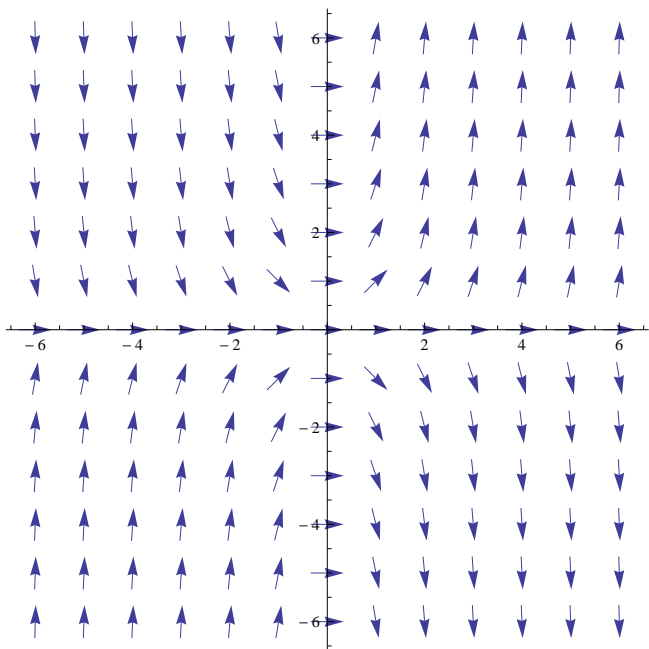
$$\frac{dy}{dx} = 2 - x - y$$



Example



$$\frac{dy}{dx} = xy$$



- ODE = ordinary differential equation
(adi diferansiyel denklemler)
- PDE = partial differential equation
(kısmi türevli diferansiyel denklemler)
- IVP = initial value problem
- BVP = boundary value problem

alpha	A	α
beta	B	β
gamma	Γ	γ
delta	Δ	δ
epsilon	E	$\epsilon \varepsilon$
zeta	Z	ζ
eta	N	η
theta	Θ	θ
iota	I	ι
kappa	K	κ
lambda	Λ	λ
mu	M	μ
nu	N	ν
xi	Ξ	ξ
omicron	O	o
pi	Π	π
rho	P	ρ
sigma	Σ	σ
tau	T	τ
upsilon	Y	υ
phi	Φ	$\phi \varphi$
chi	X	χ
psi	Ψ	ψ
omega	Ω	ω



First Order ODEs

A Linear Equation

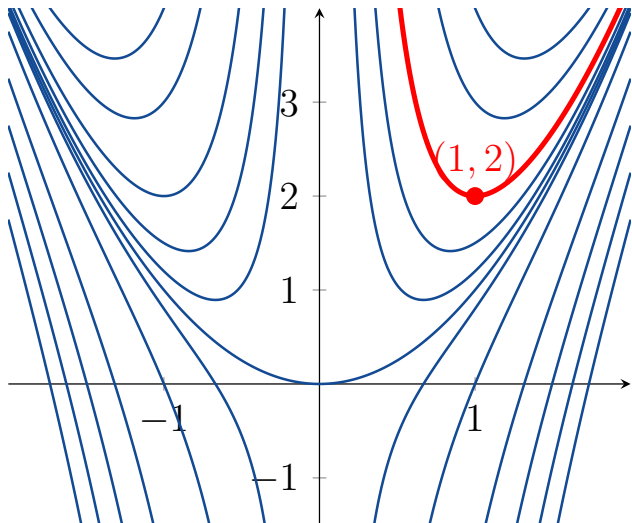


The solution to

$$ty' + 2y = 4t$$

is

$$y(t) = t^2 + \frac{c}{t^2}.$$



Note that the solution satisfying $y(1) = 2$ is a differentiable function $y : (0, \infty) \rightarrow \mathbb{R}$.

A Separable Equation

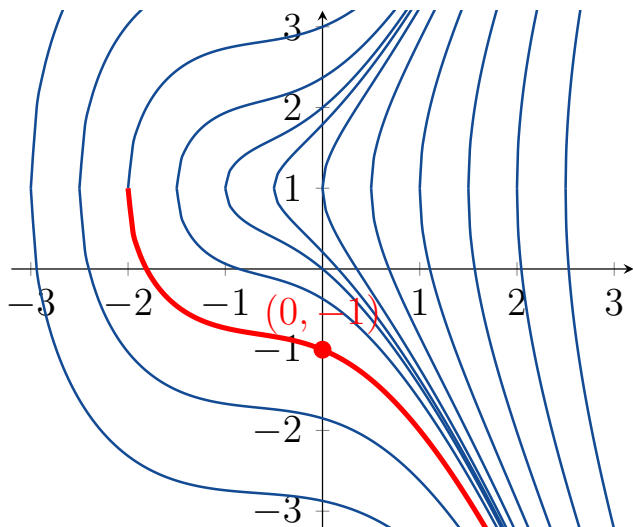


The solution to

$$\frac{dy}{dx} = \frac{3x^2 + 4x + 2}{2(y - 1)}$$

is

$$y(x) = 1 \pm \sqrt{x^3 + 2x^2 + 2x + c + 1}.$$

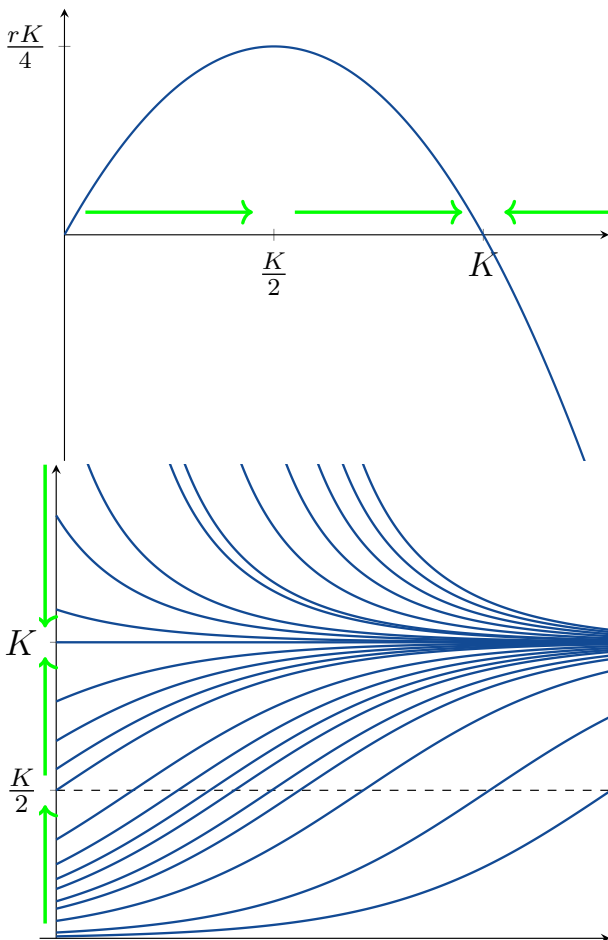


Note that the solution satisfying $y(0) = -1$ is a differentiable function $y : (-2, \infty) \rightarrow \mathbb{R}$.

Logistic Growth



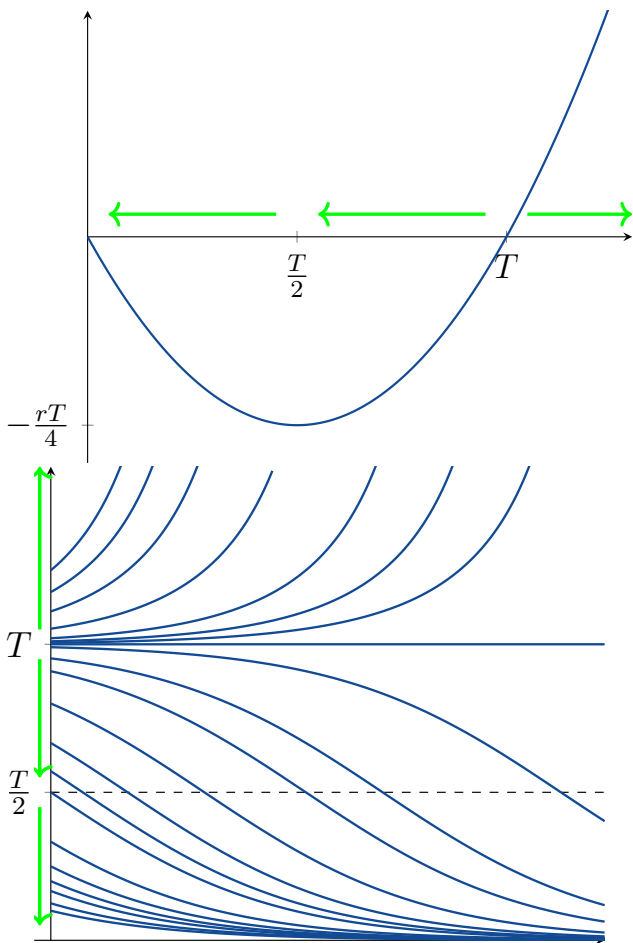
$$\frac{dy}{dt} = r \left(1 - \frac{y}{K} \right) y$$



A Critical Threshold



$$\frac{dy}{dt} = -r \left(1 - \frac{y}{T}\right) y$$

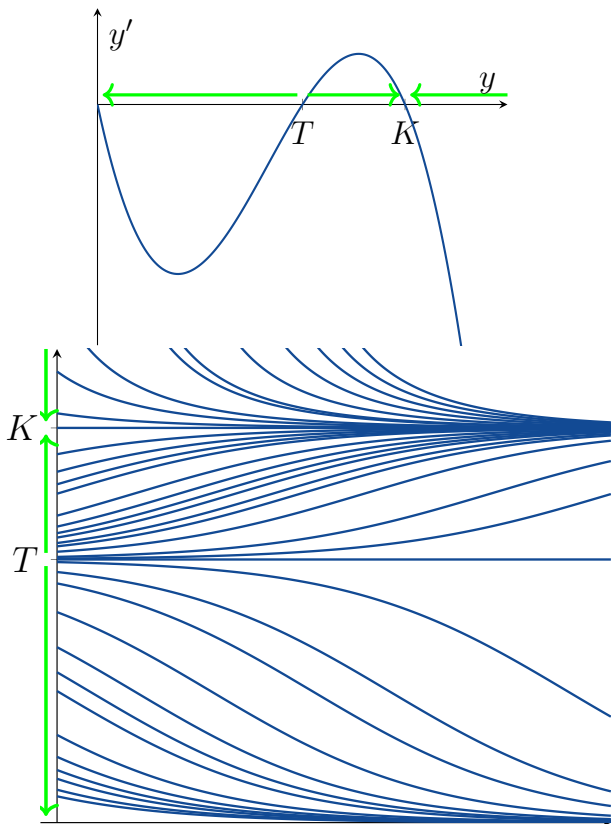


Logistic Growth with a Threshold



$$0 < T < K, r > 0$$

$$\frac{dy}{dt} = -r \left(1 - \frac{y}{T}\right) \left(1 - \frac{y}{K}\right) y$$





Second Order Linear ODEs

Example

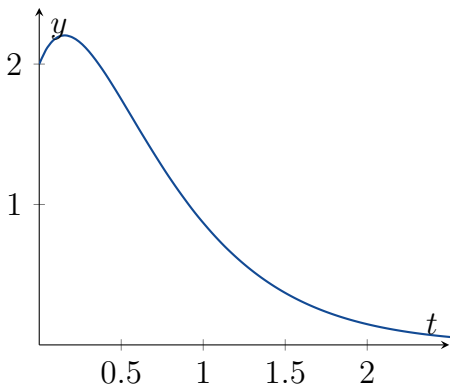


The initial value problem

$$\begin{cases} y'' + 5y' + 6y = 0 \\ y(0) = 2 \\ y'(0) = 3 \end{cases}$$

has solution

$$y = 9e^{-2t} - 7e^{-3t}.$$



Example

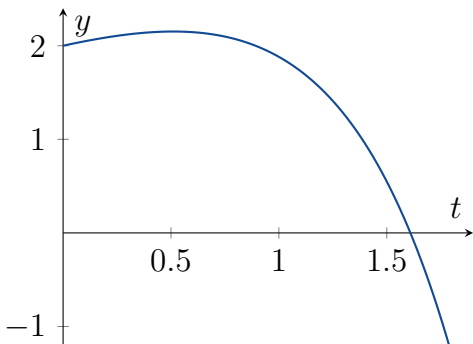


The initial value problem

$$\begin{cases} 4y'' - 8y' + 3y = 0 \\ y(0) = 2 \\ y'(0) = \frac{1}{2} \end{cases}$$

has solution

$$y = -\frac{1}{2}e^{\frac{3t}{2}} + \frac{5}{2}e^{\frac{t}{2}}.$$



Example

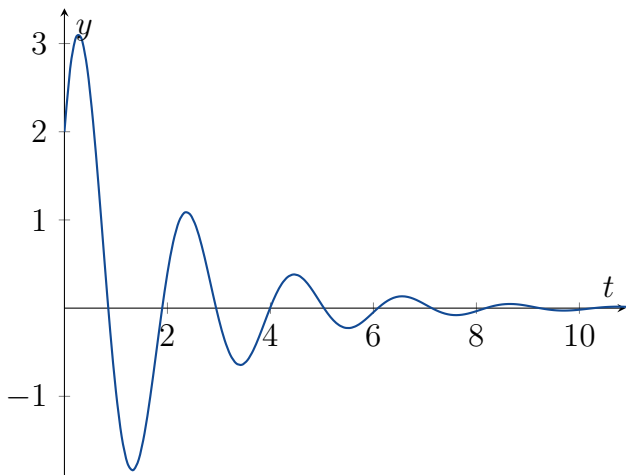


The initial value problem

$$\begin{cases} y'' + y' + 9.25y = 0 \\ y(0) = 2 \\ y'(0) = 8 \end{cases}$$

has solution

$$y = e^{-\frac{t}{2}} (2 \cos 3t + 3 \sin 3t) .$$



Example

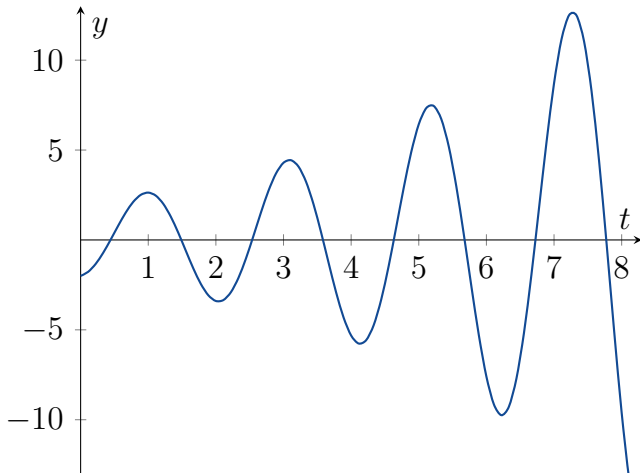


The initial value problem

$$\begin{cases} 16y'' - 8y' + 145y = 0 \\ y(0) = -2 \\ y'(0) = 1 \end{cases}$$

has solution

$$y = -2e^{\frac{t}{4}} \cos 3t + \frac{1}{2}e^{\frac{t}{4}} \sin 3t.$$



Example

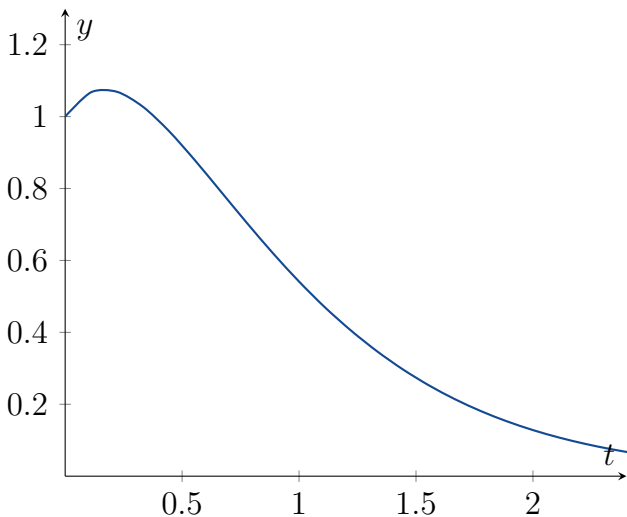


The initial value problem

$$\begin{cases} y'' + 4y' + 4y = 0 \\ y(0) = 1 \\ y'(0) = 1 \end{cases}$$

has solution

$$y = e^{-2t} + 3te^{-2t}.$$



Example



The initial value problems

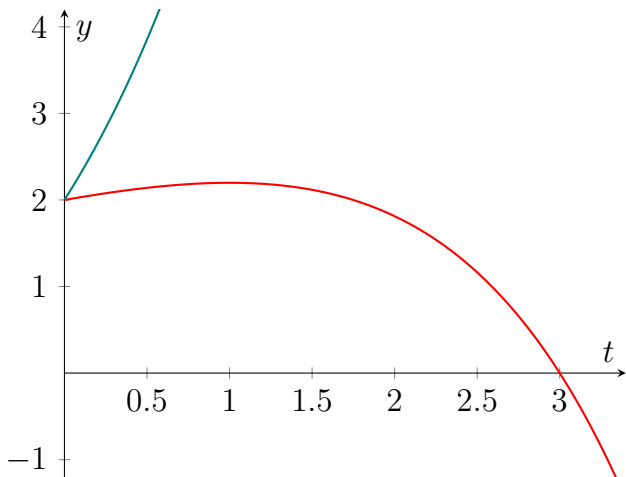
$$\begin{cases} y'' - y' + \frac{1}{4}y = 0 \\ y(0) = 2 \\ y'(0) = \frac{1}{3} \end{cases} \quad \text{and} \quad \begin{cases} y'' - y' + \frac{1}{4}y = 0 \\ y(0) = 2 \\ y'(0) = 2 \end{cases}$$

have solutions

$$y = 2e^{\frac{t}{2}} - \frac{2}{3}te^{\frac{t}{2}}$$

and

$$y = 2e^{\frac{t}{2}} + te^{\frac{t}{2}}.$$





Higher Order Linear ODEs

Example

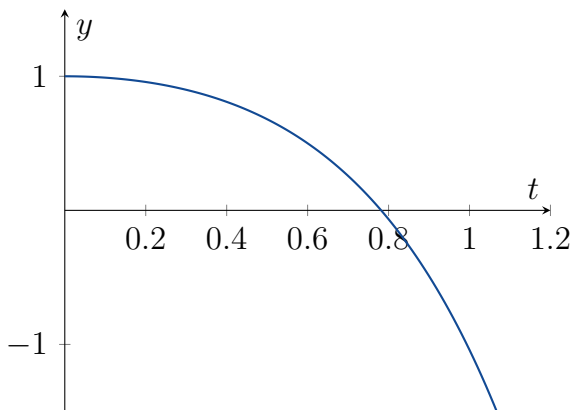


The initial value problem

$$\begin{cases} y^{(4)} + y''' - 7y'' - y' + 6y = 0 \\ y(0) = 1 \\ y'(0) = 0 \\ y''(0) = -2 \\ y'''(0) = -1 \end{cases}$$

has solution

$$y = \frac{11}{8}e^t + \frac{5}{12}e^{-t} - \frac{2}{3}e^{2t} - \frac{1}{8}e^{-3t}.$$



Example



The initial value problems

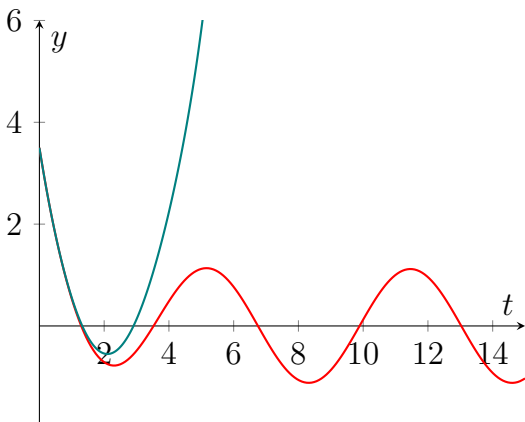
$$\begin{cases} y^{(4)} - y = 0 \\ y(0) = \frac{7}{2} \\ y'(0) = -4 \\ y''(0) = \frac{5}{2} \\ y'''(0) = -2 \end{cases} \quad \text{and} \quad \begin{cases} y^{(4)} - y = 0 \\ y(0) = \frac{7}{2} \\ y'(0) = -4 \\ y''(0) = \frac{5}{2} \\ y'''(0) = -\frac{15}{8} \end{cases}$$

have solutions

$$y = 3e^{-t} + \frac{1}{2} \cos t - \sin t$$

and

$$y = \frac{1}{32}e^t + \frac{95}{32}e^{-t} + \frac{1}{2} \cos t - \frac{17}{16} \sin t.$$





The Laplace Transform

Definition: Suppose that

- (i) $A > 0$, $K > 0$, $M > 0$, $a \in \mathbb{R}$;
- (ii) f is piecewise continuous on $[0, A]$; and
- (iii) $|f(t)| \leq Ke^{at}$ for all $t \geq M$.

The **Laplace Transform** of f is

$$F(s) = \mathcal{L}[f](s) = \int_0^{\infty} e^{-st} f(t) dt.$$

Note that

$$\mathcal{L}[f'](s) = sF(s) - f(0)$$

$$\mathcal{L}[f''](s) = s^2F(s) - sf(0) - f'(0)$$

$$\begin{aligned} \mathcal{L}[f^{(n)}](s) &= s^n F(s) - s^{n-1}f(0) - s^{(n-2)}f'(0) \\ &\quad - \dots - sf^{(n-2)}(0) - f^{(n-1)}(0). \end{aligned}$$

Elementary Laplace Transforms



$f(t)$	$F(s) = \mathcal{L}[f](s)$	
1	$\frac{1}{s}$	$s > 0$
e^{at}	$\frac{1}{s-a}$	$s > a$
$t^n \quad (n \in \mathbb{N})$	$\frac{n!}{s^{n+1}}$	$s > 0$
$\sin at$	$\frac{a}{s^2+a^2}$	$s > 0$
$\cos at$	$\frac{s}{s^2+a^2}$	$s > 0$
$\sinh at$	$\frac{a}{s^2-a^2}$	$s > a $
$\cosh at$	$\frac{s}{s^2-a^2}$	$s > a $
$e^{at} \sin bt$	$\frac{b}{(s-a)^2+b^2}$	$s > a$
$e^{at} \cos bt$	$\frac{s-a}{(s-a)^2+b^2}$	$s > a$
$t^n e^{at} \quad (n \in \mathbb{N})$	$\frac{n!}{(s-a)^{n+1}}$	$s > a$
$u_c(t)$	$\frac{e^{-cs}}{s}$	$s > 0$
$u_c(t)f(t-c)$	$e^{-cs}F(s)$	
$e^{ct}f(t)$	$F(s-c)$	
$f(ct) \quad (c > 0)$	$\frac{1}{c}F\left(\frac{s}{c}\right)$	
$\int_0^t f(t-\tau)g(\tau)d\tau$	$F(s)G(s)$	
$t^n f(t)$	$(-1)^n F^{(n)}(s)$	

Example

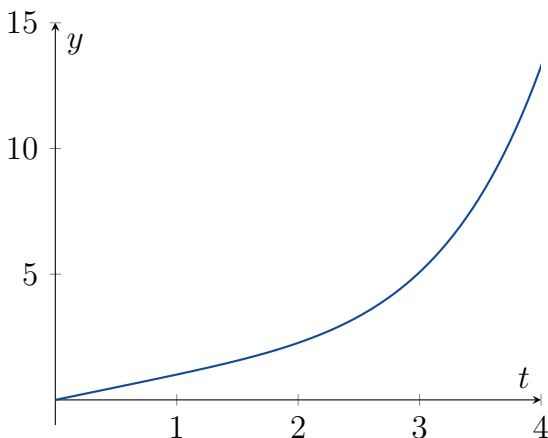


The initial value problem

$$\begin{cases} y^{(4)} - y = 0 \\ y(0) = 0 \\ y'(0) = 1 \\ y''(0) = 0 \\ y'''(0) = 0 \end{cases}$$

has solution

$$y = \frac{1}{2} \sinh t + \frac{1}{2} \sin t.$$





Systems of First Order Linear ODEs

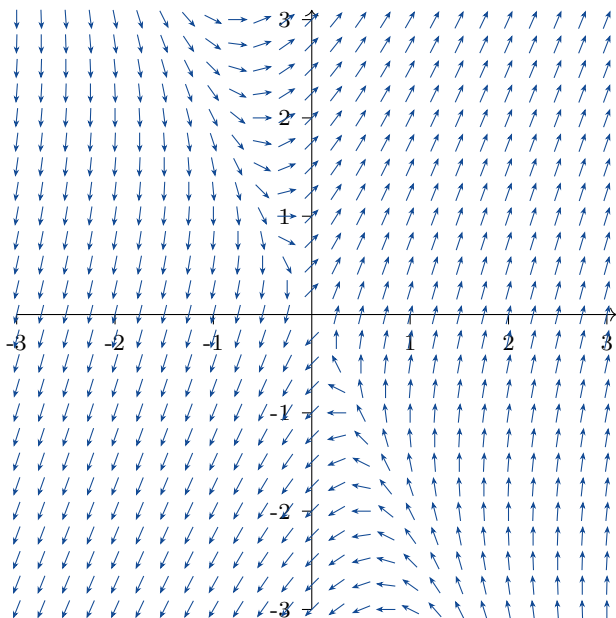
Example



A direction field for the linear system

$$\mathbf{x}' = \begin{pmatrix} 1 & 1 \\ 4 & 1 \end{pmatrix} \mathbf{x}$$

is



Example

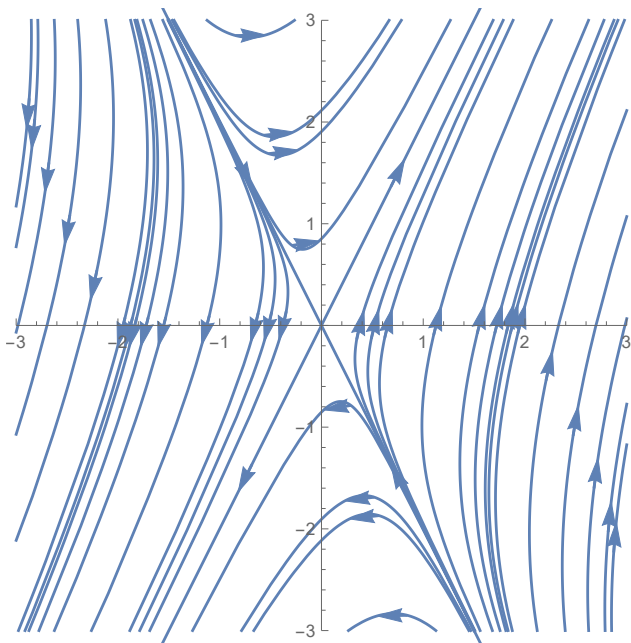


The linear system

$$\mathbf{x}' = \begin{pmatrix} 1 & 1 \\ 4 & 1 \end{pmatrix} \mathbf{x}$$

has general solution

$$\mathbf{x}(t) = c_1 \begin{pmatrix} 1 \\ 2 \end{pmatrix} e^{3t} + c_2 \begin{pmatrix} 1 \\ -2 \end{pmatrix} e^{-t}.$$



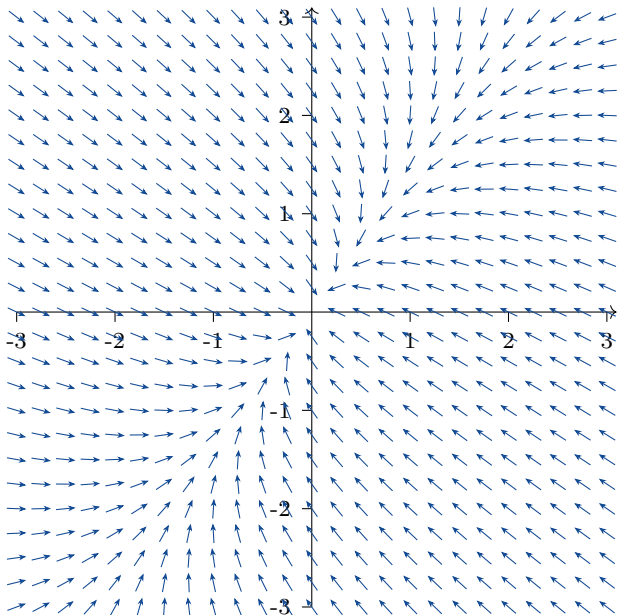
Example



A direction field for the linear system

$$\mathbf{x}' = \begin{pmatrix} -3 & \sqrt{2} \\ \sqrt{2} & -2 \end{pmatrix} \mathbf{x}$$

is



Example

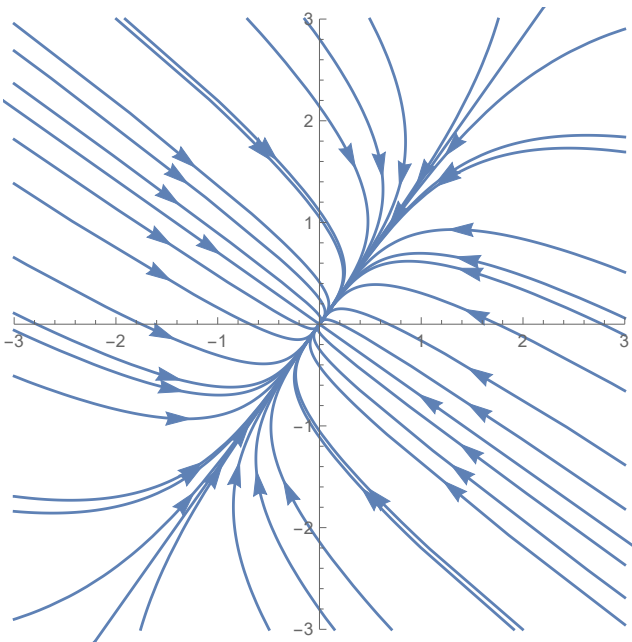


The linear system

$$\mathbf{x}' = \begin{pmatrix} -3 & \sqrt{2} \\ \sqrt{2} & -2 \end{pmatrix} \mathbf{x}$$

has general solution

$$\mathbf{x}(t) = c_1 \begin{pmatrix} 1 \\ \sqrt{2} \end{pmatrix} e^{-t} + c_2 \begin{pmatrix} -\sqrt{2} \\ 1 \end{pmatrix} e^{-4t}.$$



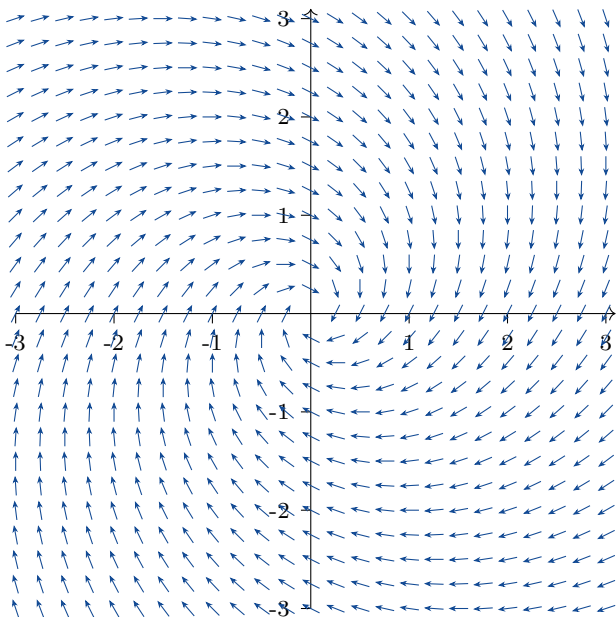
Example



A direction field for the linear system

$$\mathbf{x}' = \begin{pmatrix} -\frac{1}{2} & 1 \\ -1 & -\frac{1}{2} \end{pmatrix} \mathbf{x}$$

is



Example

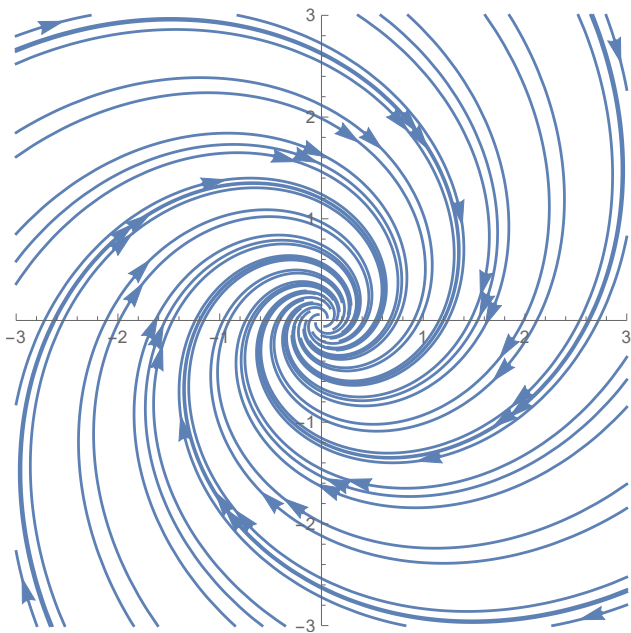


The linear system

$$\mathbf{x}' = \begin{pmatrix} -\frac{1}{2} & 1 \\ -1 & -\frac{1}{2} \end{pmatrix} \mathbf{x}$$

has general solution

$$\mathbf{x}(t) = c_1 \begin{pmatrix} \cos t \\ -\sin t \end{pmatrix} e^{-\frac{t}{2}} + c_2 \begin{pmatrix} -\sin t \\ \cos t \end{pmatrix} e^{-\frac{t}{2}}.$$



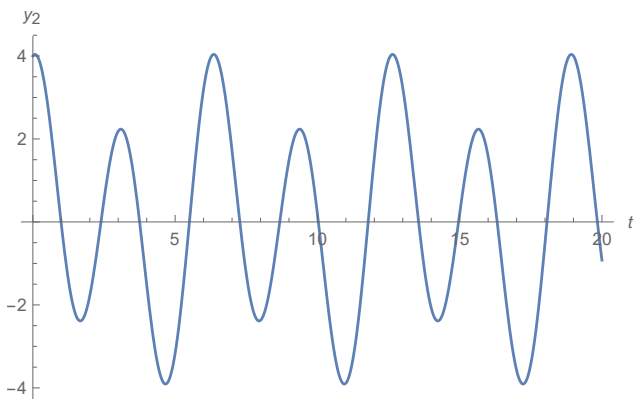
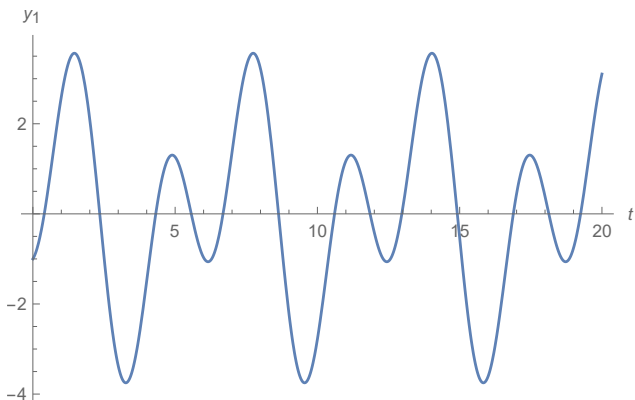
The initial value problem

$$\mathbf{y}' = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -2 & \frac{3}{2} & 0 & 0 \\ \frac{4}{3} & -3 & 0 & 0 \end{pmatrix} \mathbf{y}, \quad \mathbf{y}(0) = \begin{pmatrix} -1 \\ 4 \\ 1 \\ 1 \end{pmatrix}$$

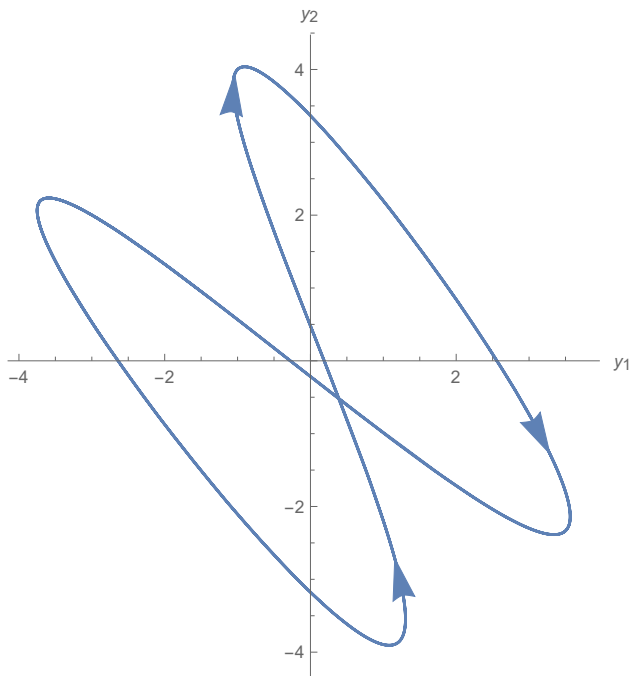
has solution

$$\begin{aligned} y(t) = & \frac{4}{9} \begin{pmatrix} 3 \cos t \\ 2 \cos t \\ -3 \sin t \\ -2 \sin t \end{pmatrix} + \frac{7}{18} \begin{pmatrix} 3 \sin t \\ 2 \sin t \\ 3 \cos t \\ 2 \cos t \end{pmatrix} \\ & - \frac{7}{9} \begin{pmatrix} 3 \cos 2t \\ -4 \cos 2t \\ -6 \sin 2t \\ 8 \sin 2t \end{pmatrix} - \frac{1}{36} \begin{pmatrix} 3 \sin 2t \\ -4 \sin 2t \\ 6 \cos 2t \\ -8 \cos 2t \end{pmatrix} \end{aligned}$$

Example (continued)



Example (continued)



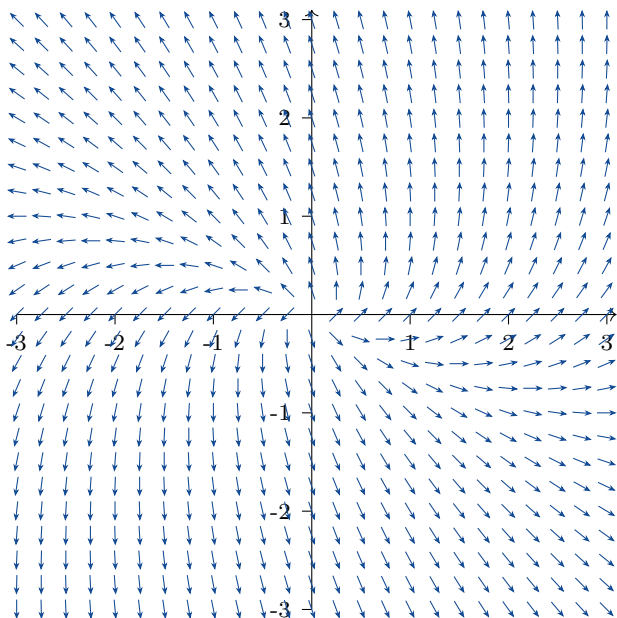
Example



A direction field for the linear system

$$\mathbf{x}' = \begin{pmatrix} 1 & -1 \\ 1 & 3 \end{pmatrix} \mathbf{x}$$

is



Example



The linear system

$$\mathbf{x}' = \begin{pmatrix} 1 & -1 \\ 1 & 3 \end{pmatrix} \mathbf{x}$$

has general solution

$$\mathbf{x}(t) = c_1 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{2t} + c_2 \left[\begin{pmatrix} 1 \\ -1 \end{pmatrix} t + \begin{pmatrix} 0 \\ -1 \end{pmatrix} \right] e^{2t}.$$

