

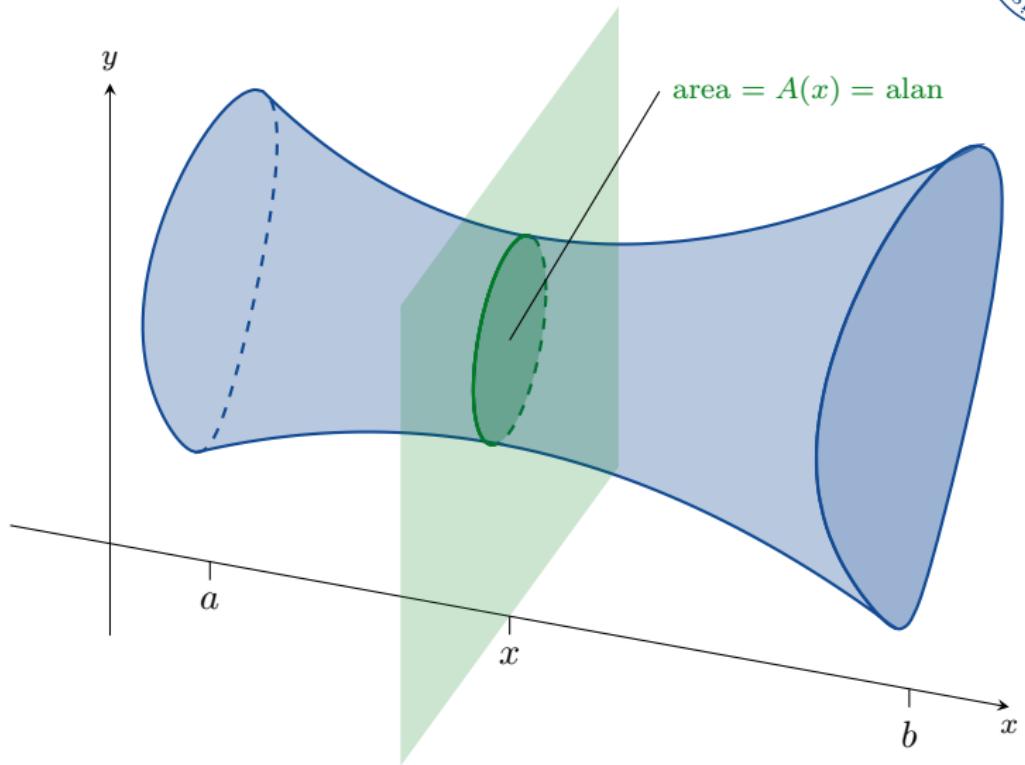
Lecture 10

- 6.1 Volumes Using Cross-Sections
- 6.2 Volumes Using Cylindrical Shells
- 6.3 Arc Length
- 6.4 Areas of Surfaces of Revolution



Volumes Using Cross- Sections

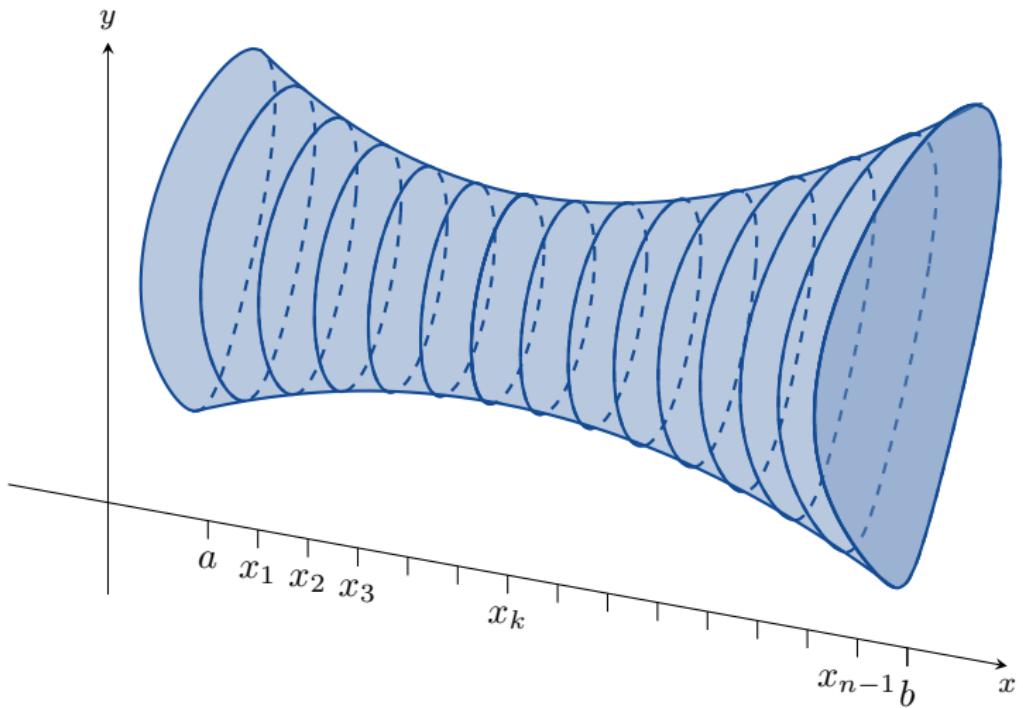
6.1 Volumes Using Cross-Sections



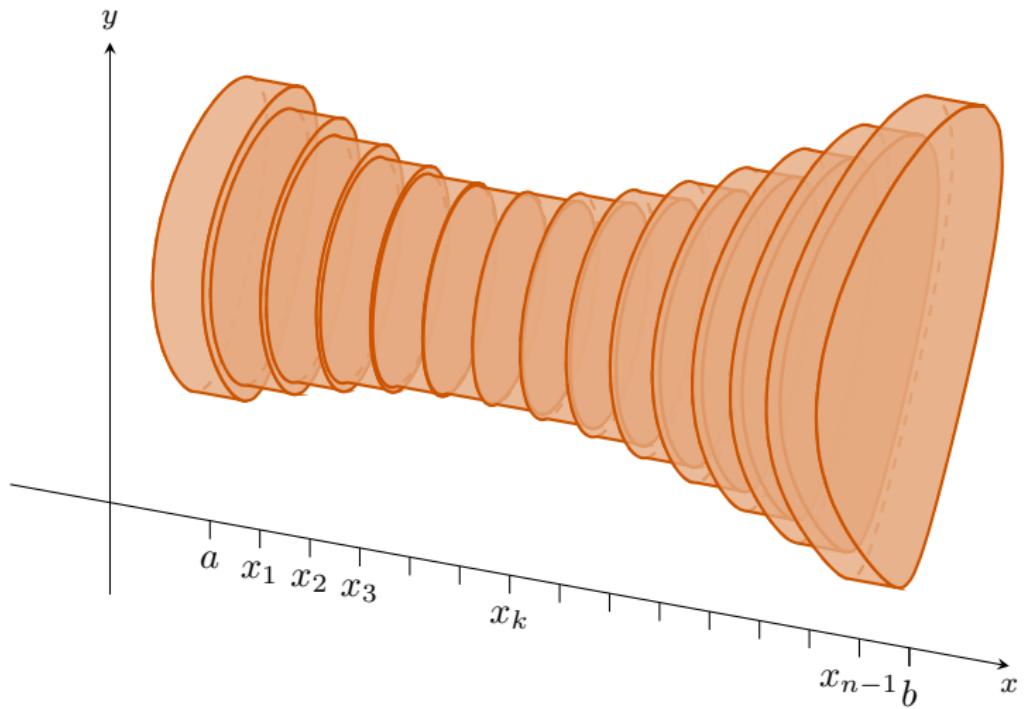
6.1 Volumes Using Cross-Sections



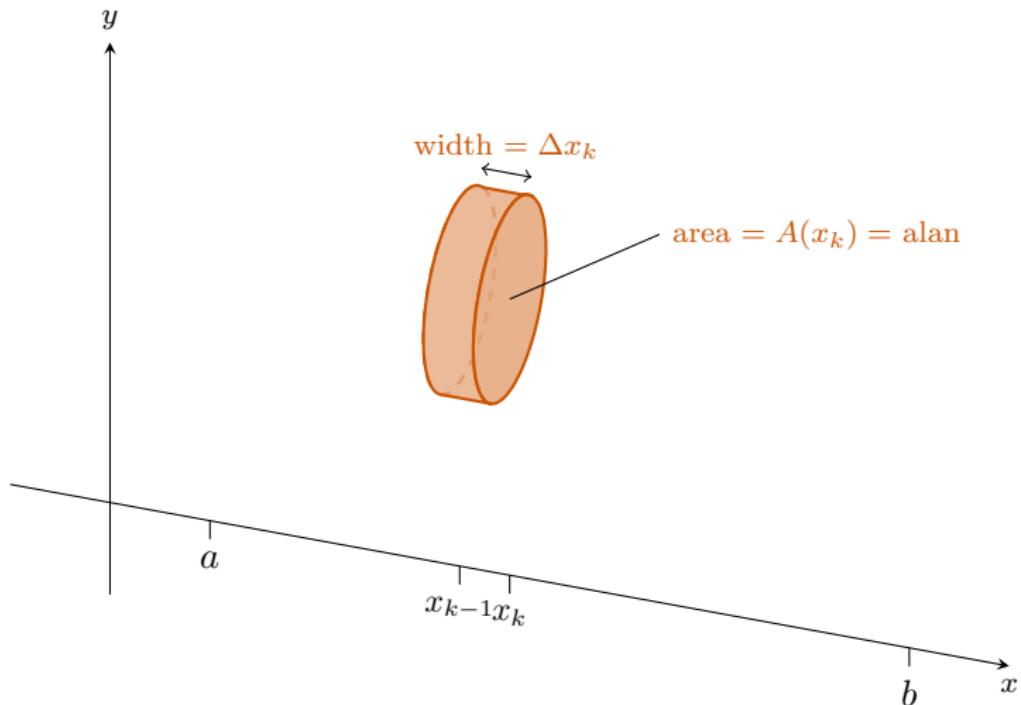
6.1 Volumes Using Cross-Sections



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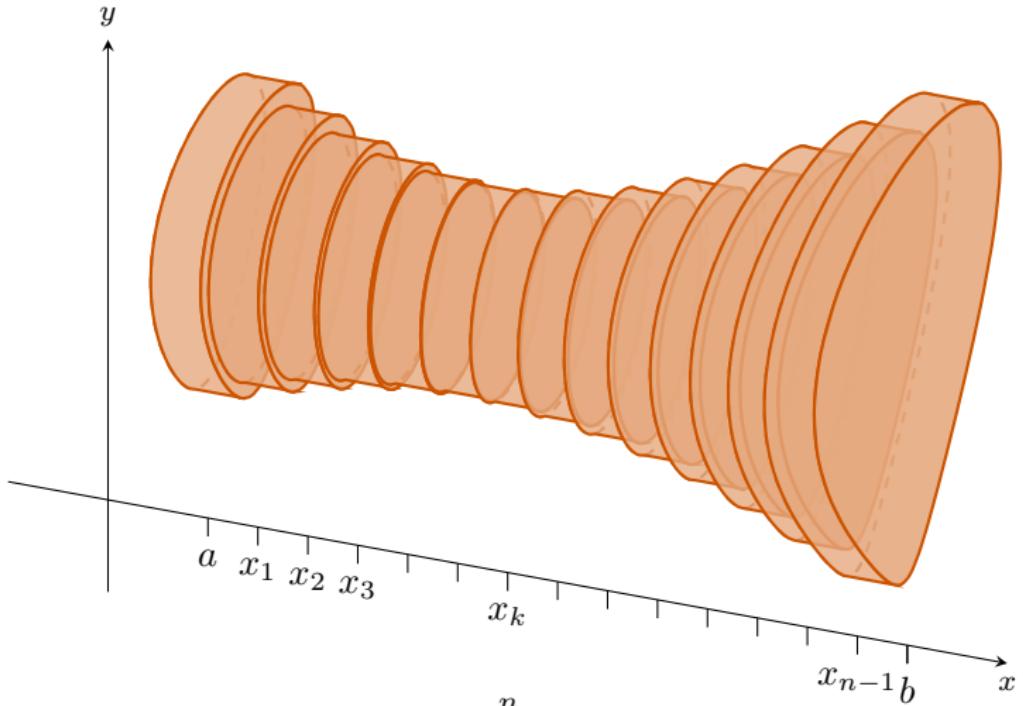


6.1 Volumes Using Cross-Sections



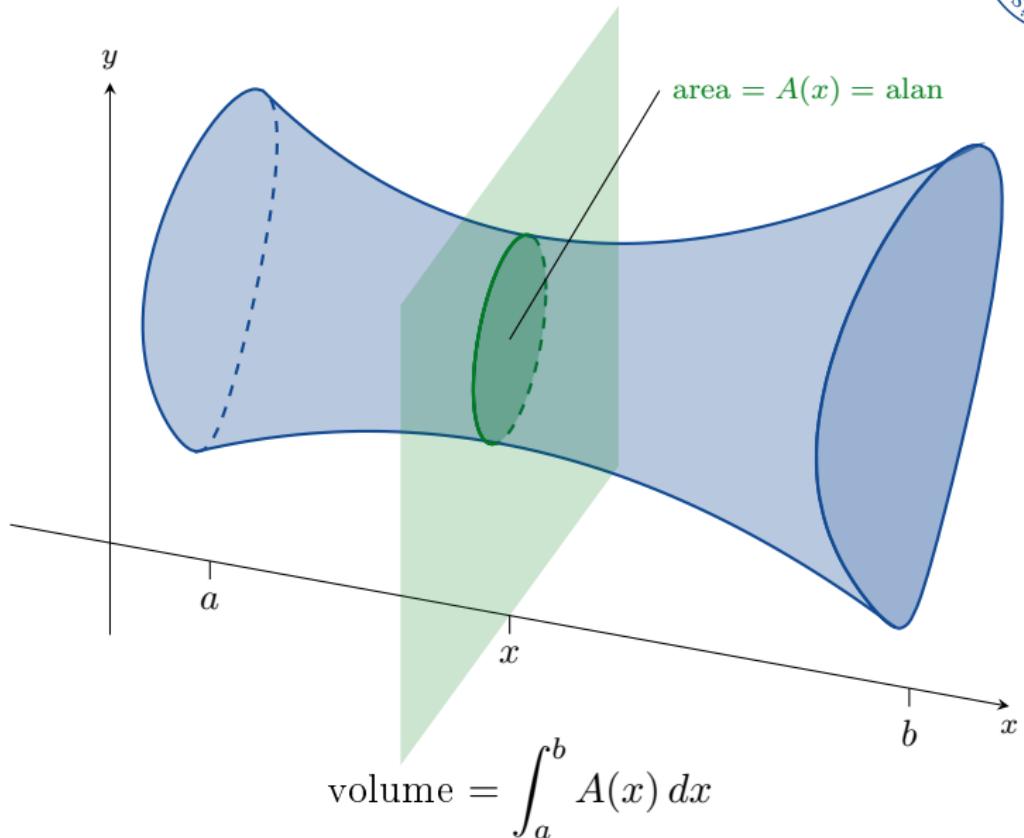
$$\text{volume} = (\text{area})(\text{width}) = A(x_k)\Delta x_k$$

6.1 Volumes Using Cross-Sections



$$\text{volume} = \sum_{k=1}^n A(x_k) \Delta x_k$$

6.1 Volumes Using Cross-Sections



Calculating the Volume of a Solid

1. Sketch the solid and a typical cross-section.
2. Find a formula for $A(x)$, the area of a typical cross-section.
3. Find the limits of integration.
4. Integrate $A(x)$ to find the volume.

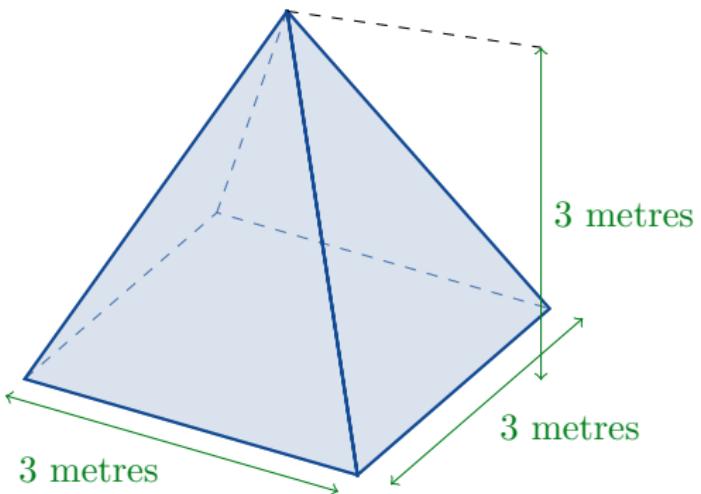
6.1 Volumes Using Cross-Sections

$$\text{volume} = \int_a^b A(x) dx$$



Example

A 3 metres tall pyramid has a square $3 \text{ metres} \times 3 \text{ metres}$ base, as shown below. The cross-section x metres from the vertex is an $x \text{ m} \times x \text{ m}$ square. Find the volume of the pyramid.

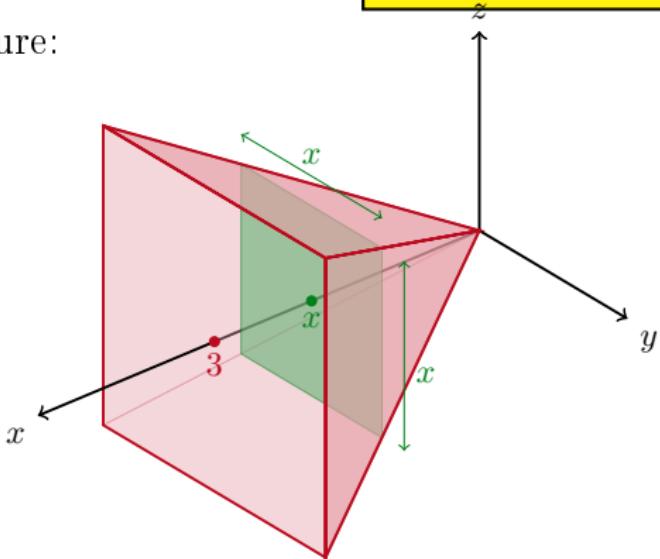


6.1 Volumes Using Cross-Sections

$$\text{volume} = \int_a^b A(x) dx$$



- 1 Draw a picture:

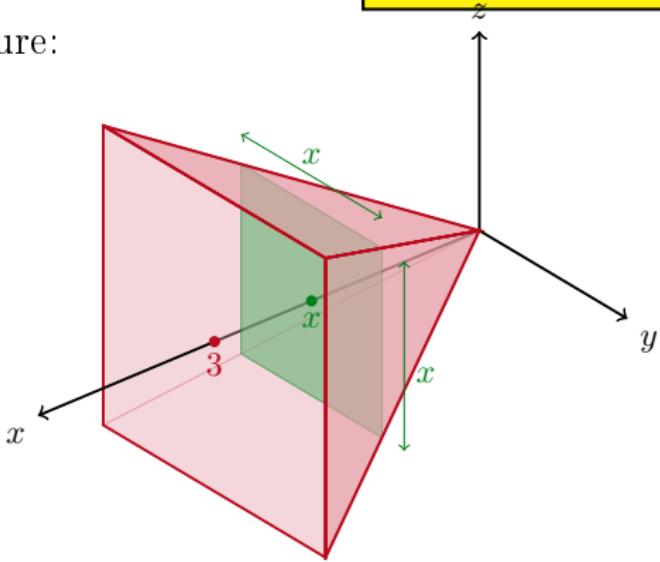


6.1 Volumes Using Cross-Sections

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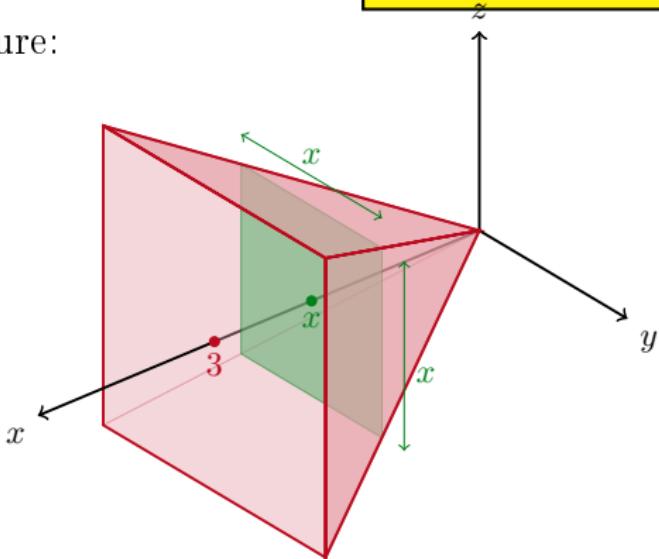
- 2 Find a formula for $A(x)$: $A(x) = x^2$.

6.1 Volumes Using Cross-Sections

$$\text{volume} = \int_a^b A(x) dx$$



- 1 Draw a picture:



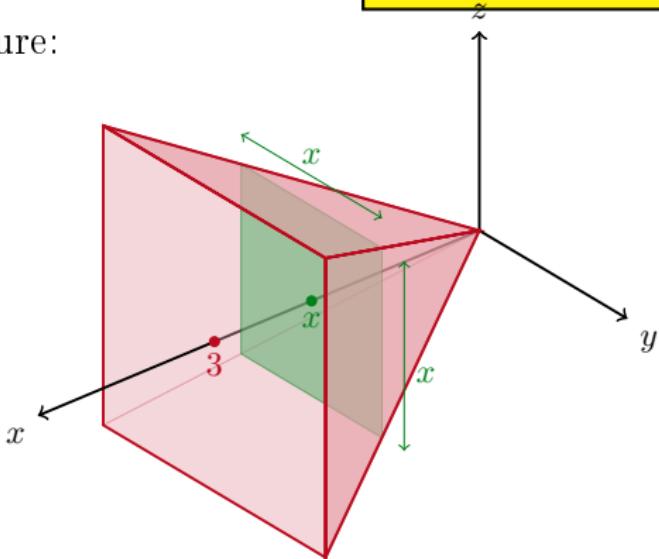
- 2 Find a formula for $A(x)$: $A(x) = x^2$.
- 3 Find the limits of integration: $0 \leq x \leq 3$.

6.1 Volumes Using Cross-Sections

$$\text{volume} = \int_a^b A(x) dx$$



- 1 Draw a picture:



- 2 Find a formula for $A(x)$: $A(x) = x^2$.
- 3 Find the limits of integration: $0 \leq x \leq 3$.
- 4 Integrate:

$$\text{volume} = \int_a^b A(x) dx = \int_0^3 x^2 dx = \left[\frac{1}{3}x^3 \right]_0^3 = 9 \text{ m}^3.$$

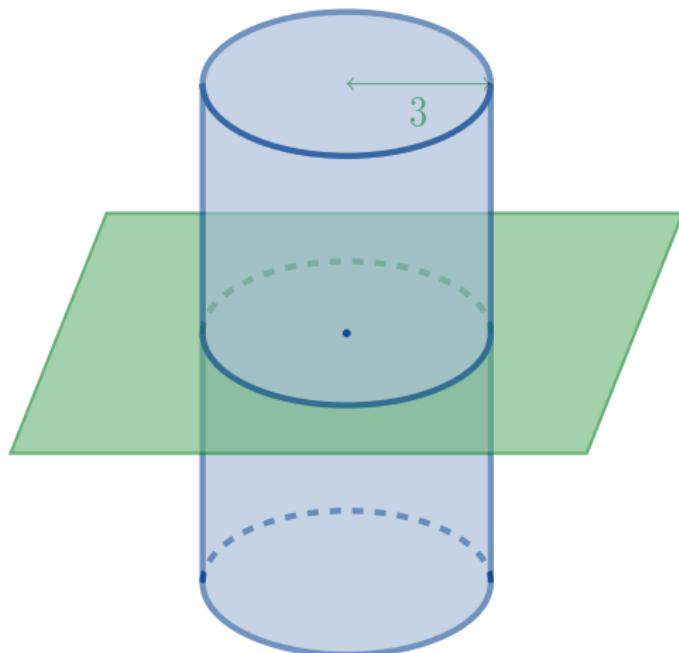
6.1 Example

A curved wedge is cut from a cylinder of radius 3 by two planes.



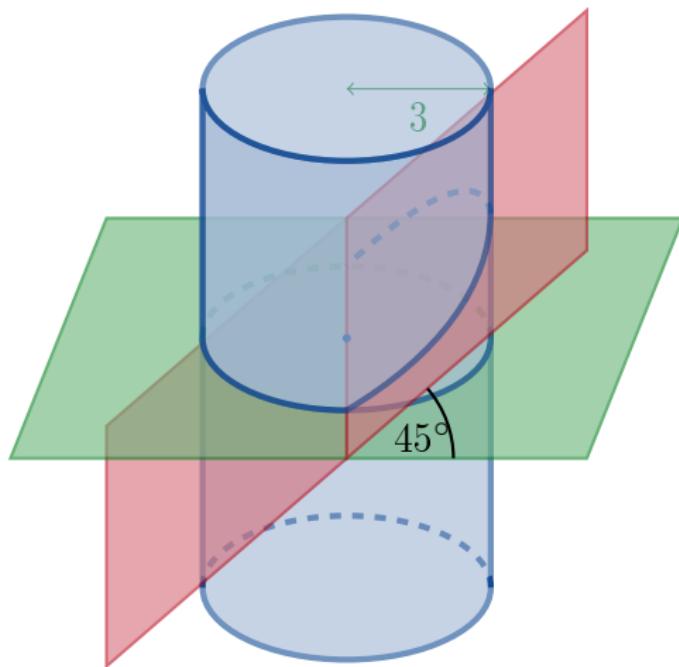
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A curved wedge is cut from a cylinder of radius 3 by two planes. The **first plane** is perpendicular to the axis of the cylinder.



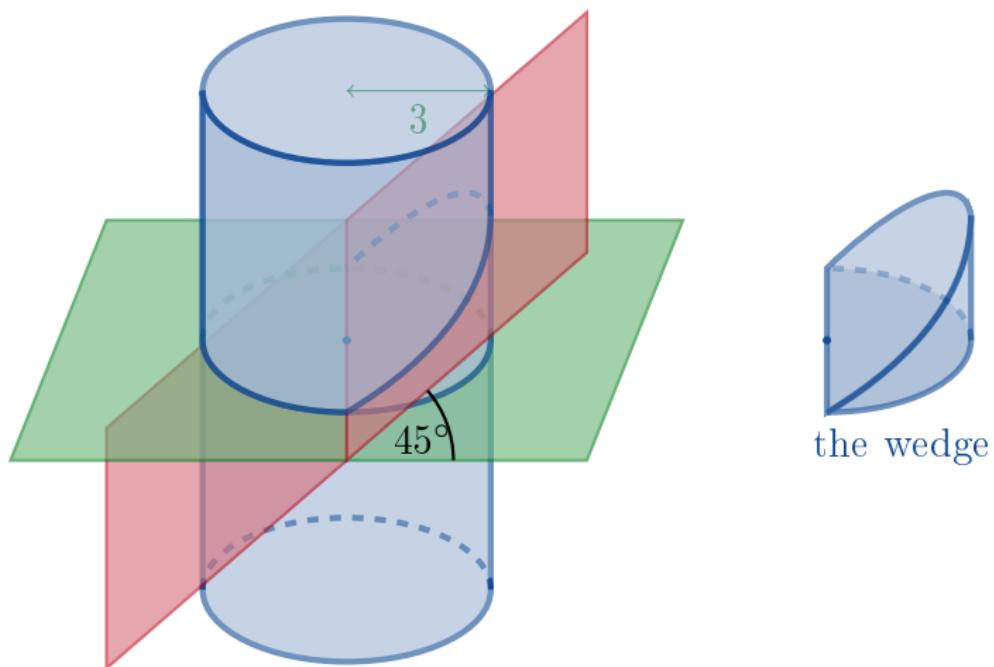
6.1 Example

A curved wedge is cut from a cylinder of radius 3 by two planes. The **first plane** is perpendicular to the axis of the cylinder. The **second plane** crosses the first plane with an angle of $45^\circ = \frac{\pi}{4}$ at the centre of the cylinder.



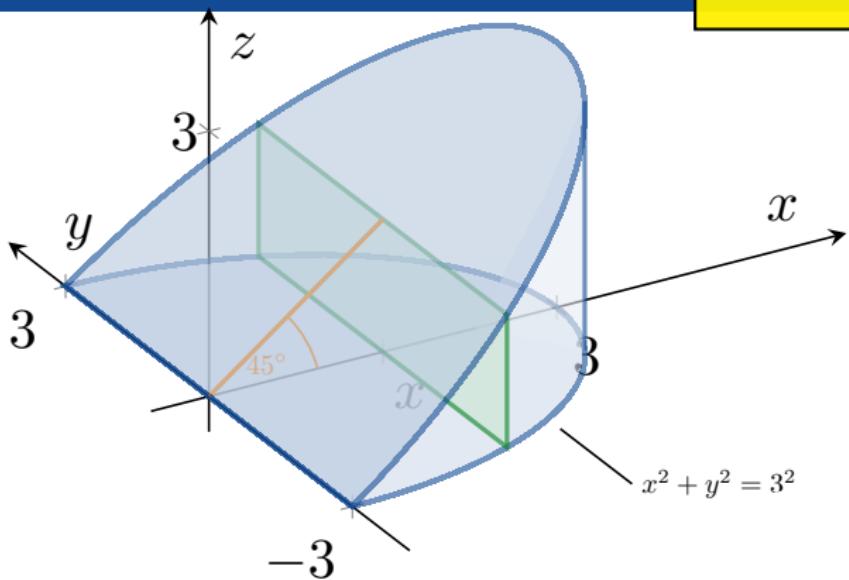
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A curved wedge is cut from a cylinder of radius 3 by two planes. The **first plane** is perpendicular to the axis of the cylinder. The **second plane** crosses the first plane with an angle of $45^\circ = \frac{\pi}{4}$ at the centre of the cylinder. Find the volume of the wedge.



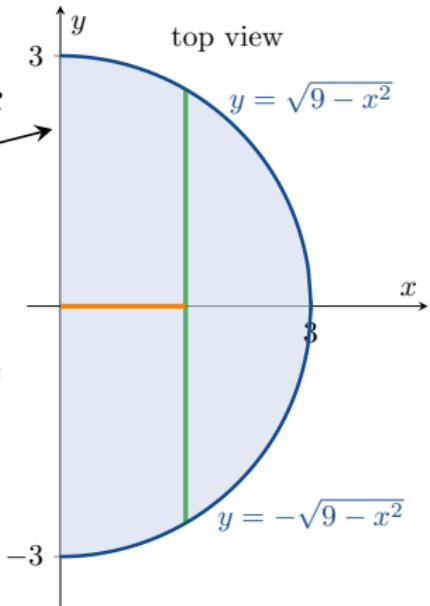
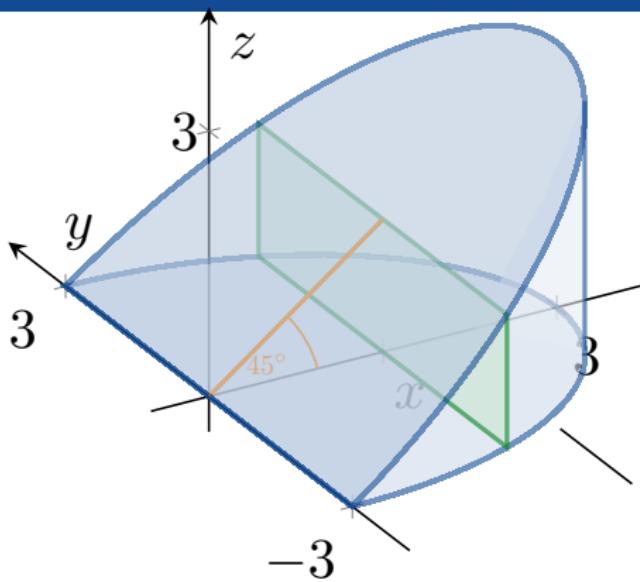
6.1 Volumes Using Cross-Sections

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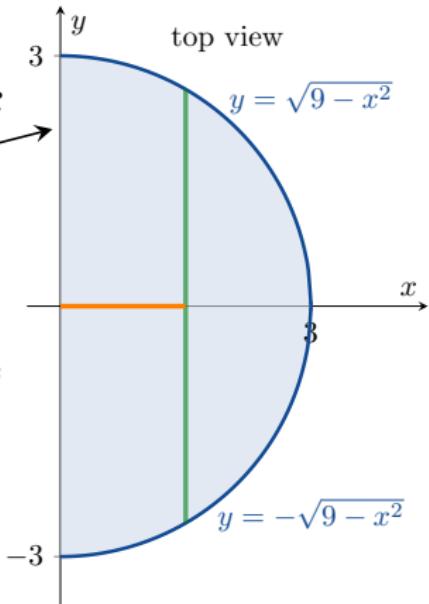
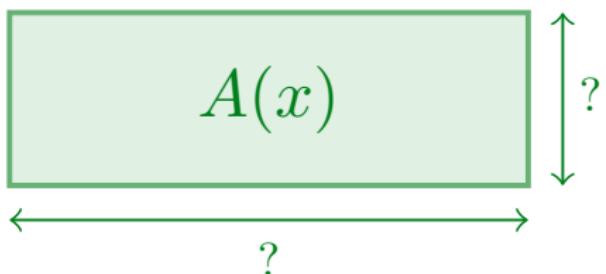
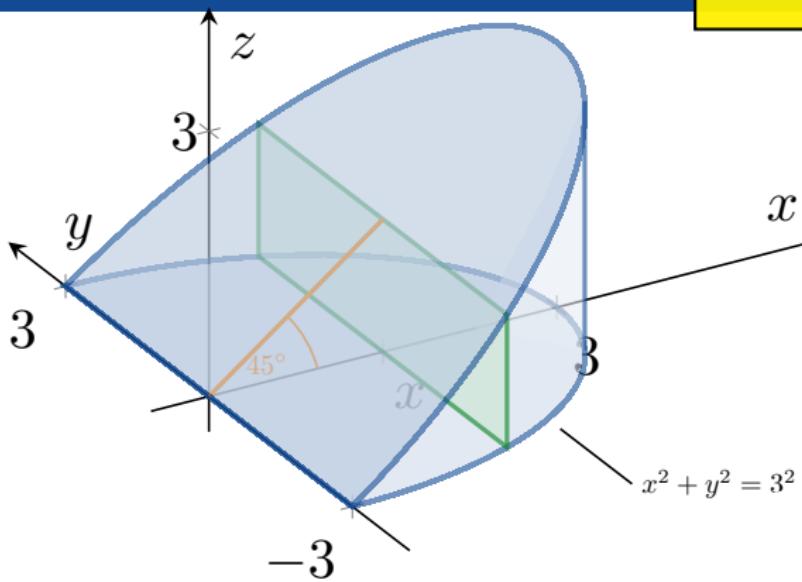
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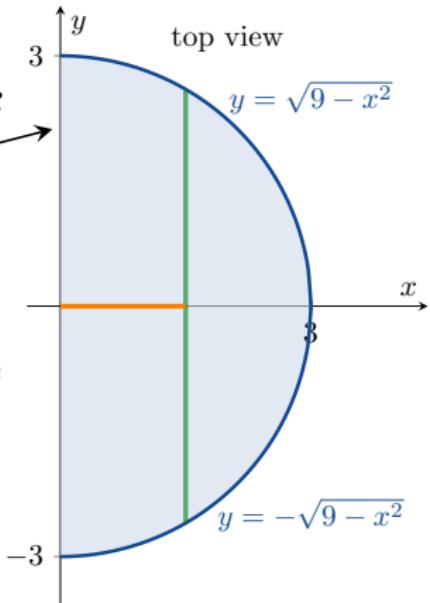
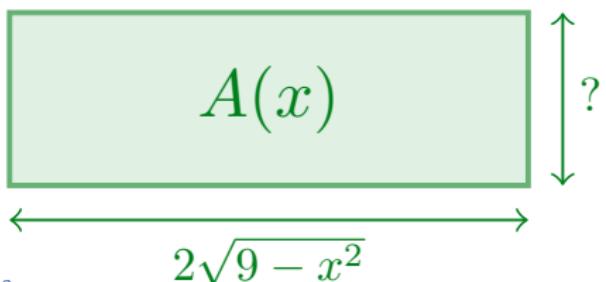
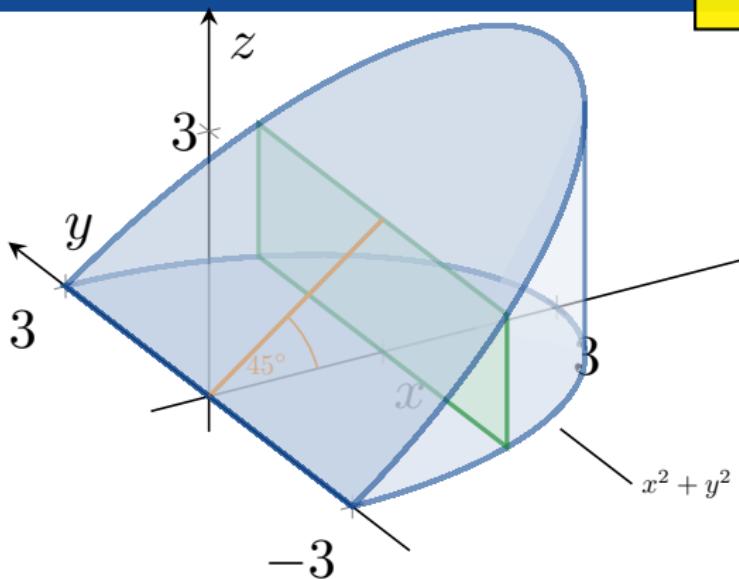
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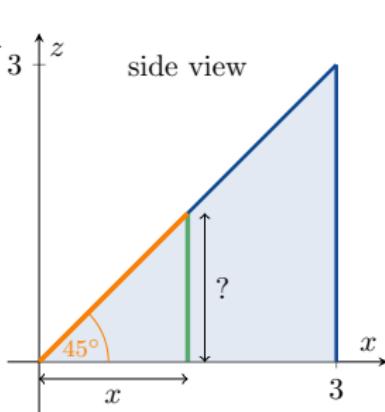
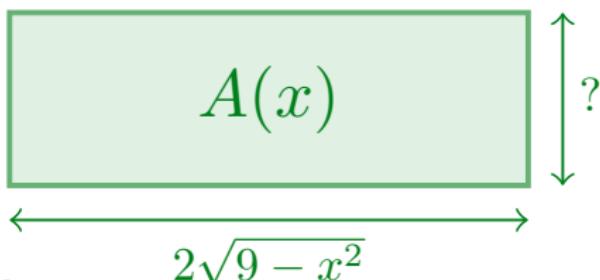
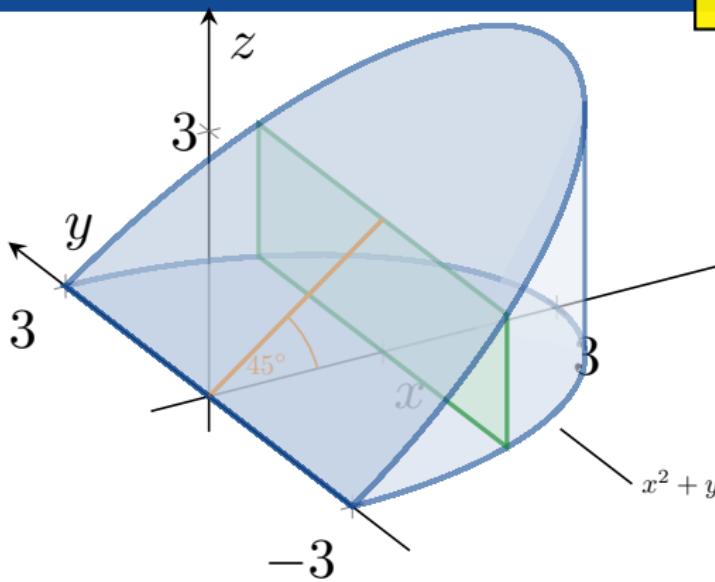
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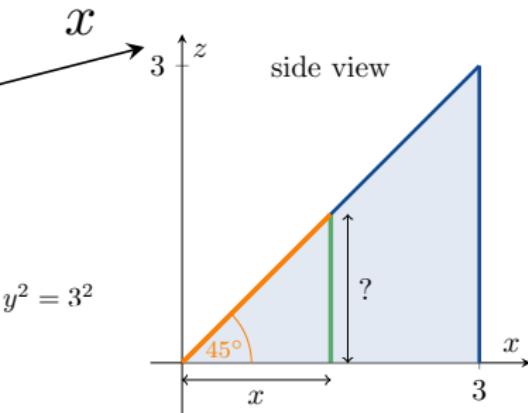
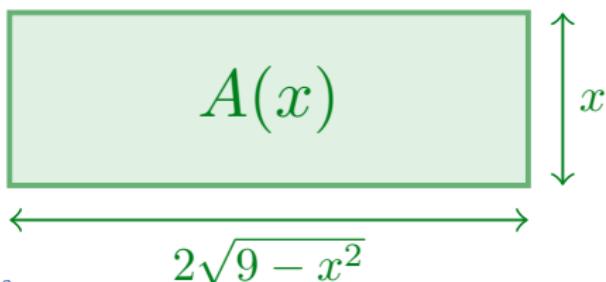
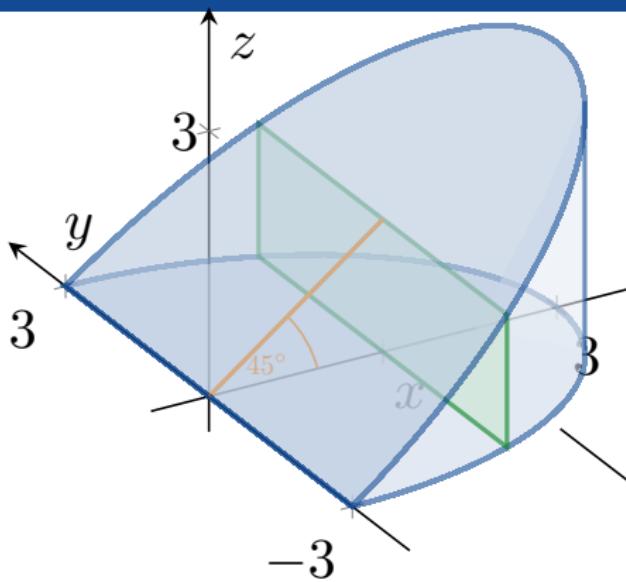
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$$A(x) = 2x\sqrt{9 - x^2}$$

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The cross-sectional area is

$$A(x) = 2x\sqrt{9 - x^2}$$

for $0 \leq x \leq 3$. Therefore

$$\text{volume} = \int_0^3 2x\sqrt{9 - x^2} dx$$

6.1 Volumes Using Cross-Sections

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$$\text{volume} = \int_0^3 2x\sqrt{9 - x^2} dx$$

We need to make a substitution. Let $u = 9 - x^2$. Then
 $du = -2x dx$ and

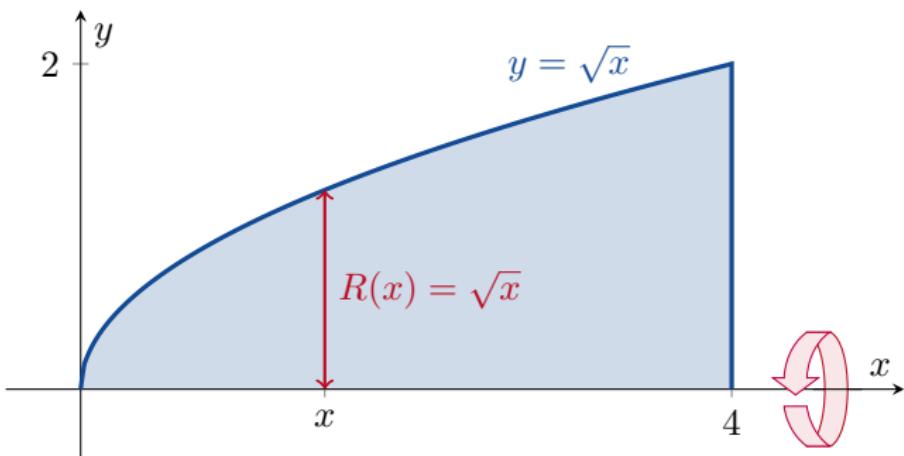
$$\begin{aligned}\text{volume} &= \int_{x=0}^{x=3} -u^{\frac{1}{2}} du = \left[-\frac{2}{3}u^{\frac{3}{2}} \right]_{x=0}^{x=3} \\ &= \left[-\frac{2}{3}(9 - x^2)^{\frac{3}{2}} \right]_{x=0}^{x=3} = 0 - \frac{2}{3}(9)^{\frac{3}{2}} \\ &= 18.\end{aligned}$$

6.1 Volumes Using Cross-Sections

$$\text{volume} = \int_a^b A(x) dx$$

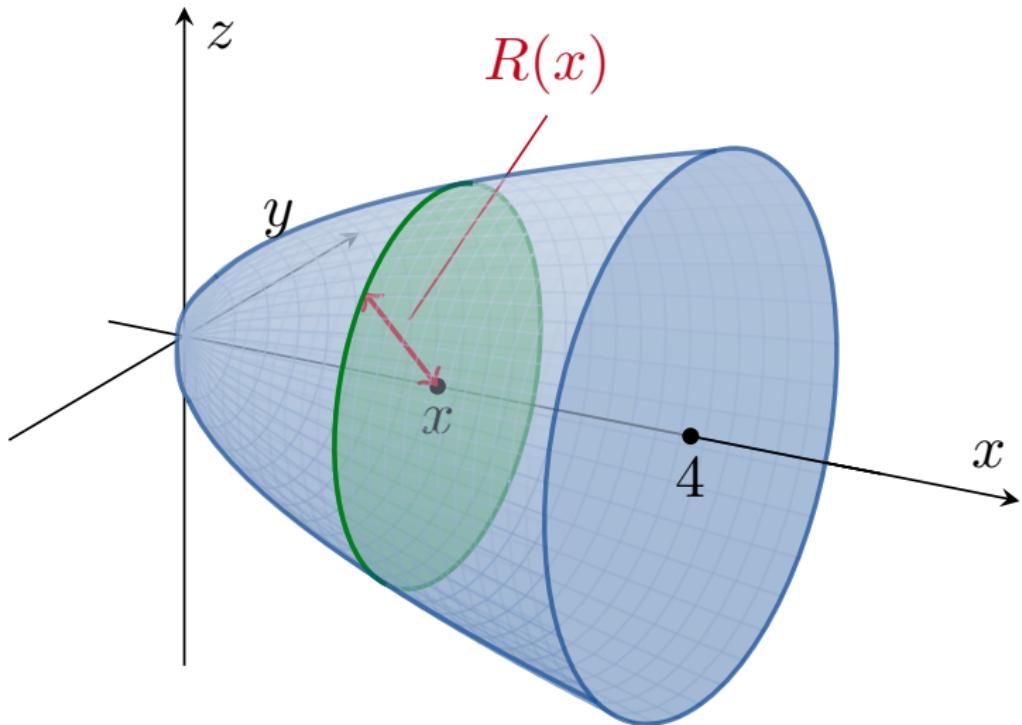


Solids of Revolution: The Disk Method



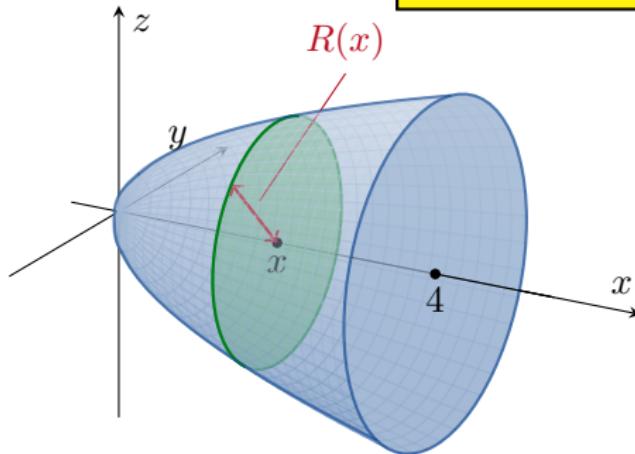
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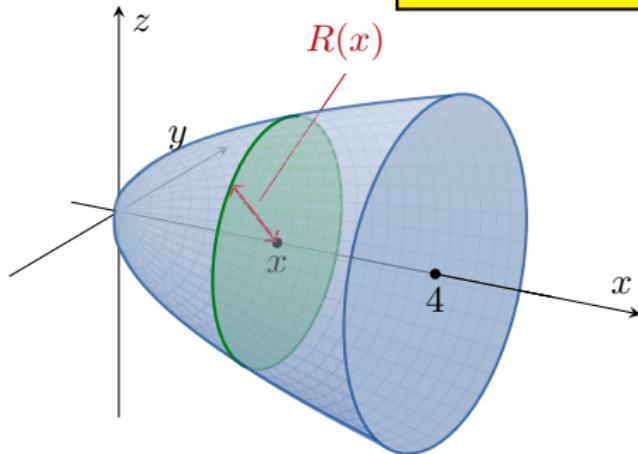
Definition

The solid generated by rotating a plane region about a line in the plane is called a *solid of revolution*.

$$\text{volume} = \int_a^b A(x) dx$$

6.1 Volumes Using Cross-Sections

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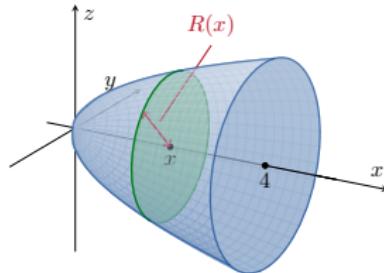
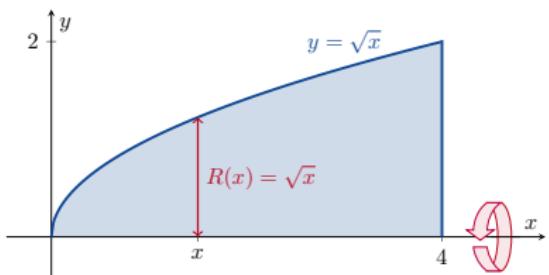
Definition

The solid generated by rotating a plane region about a line in the plane is called a *solid of revolution*.

$$\text{volume} = \int_a^b A(x) dx = \int_a^b \pi(R(x))^2 dx$$

6.1 Volumes Using Cross-Sections

$$\text{volume} = \int_a^b \pi(R(x))^2 dx$$



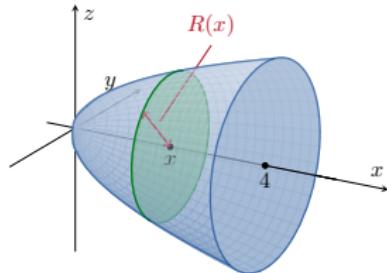
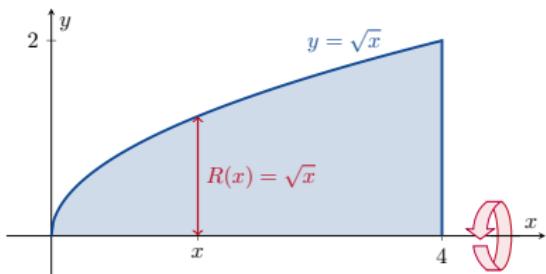
Example

The region between the curve $y = \sqrt{x}$ and the x -axis, for $0 \leq x \leq 4$, is rotated about the x -axis to generate a solid. Find its volume.

$$\text{volume} = \int_a^b \pi(R(x))^2 dx =$$

6.1 Volumes Using Cross-Sections

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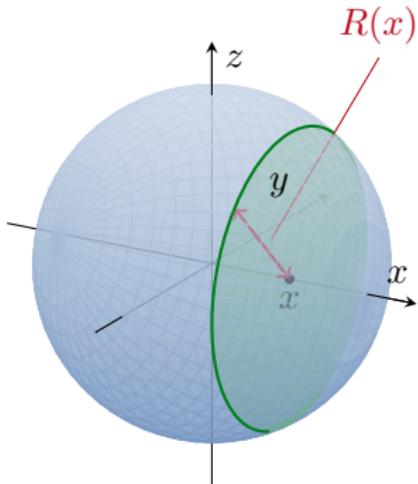
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$$\begin{aligned}\text{volume} &= \int_a^b \pi(R(x))^2 dx = \int_0^4 \pi(\sqrt{x})^2 dx \\ &= \pi \int_0^4 x dx = \pi \left[\frac{1}{2}x^2 \right]_0^4 = 8\pi.\end{aligned}$$

6.1 Volumes Using Cross-Sections

$$\text{volume} = \int_a^b \pi(R(x))^2 dx$$

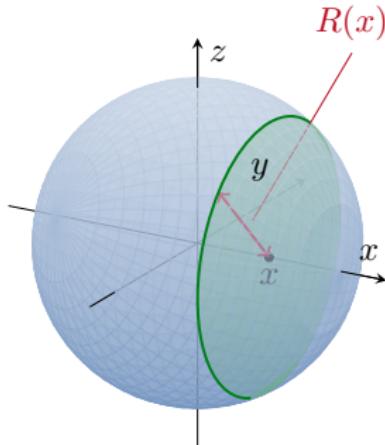
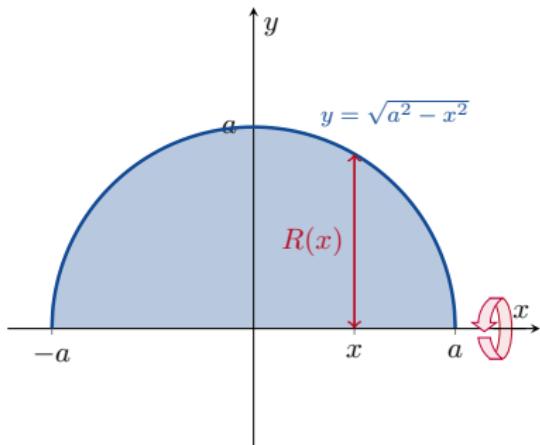


Example

Find the volume of a sphere of radius a .

6.1 Volumes Using Cross-Sections

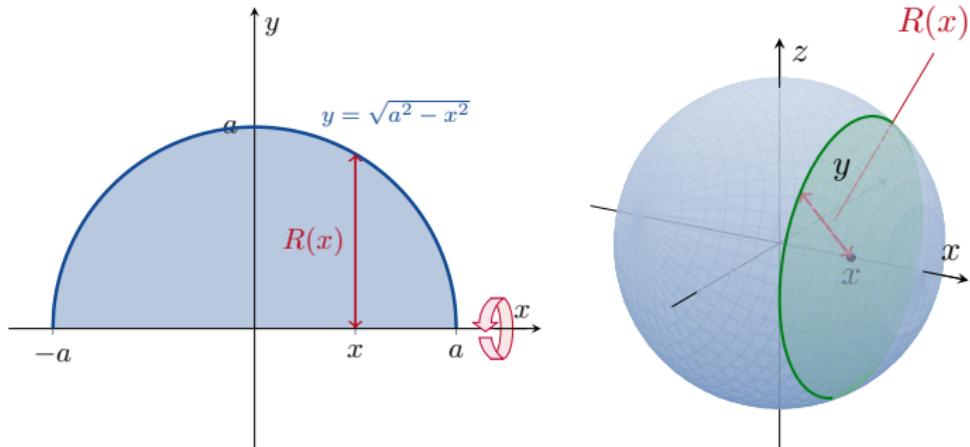
$$\text{volume} = \int_a^b \pi(R(x))^2 dx$$



To generate a sphere, we rotate the area between $y = \sqrt{a^2 - x^2}$ and the x -axis about the x -axis.

6.1 Volumes Using Cross-Sections

$$\text{volume} = \int_a^b \pi(R(x))^2 dx$$

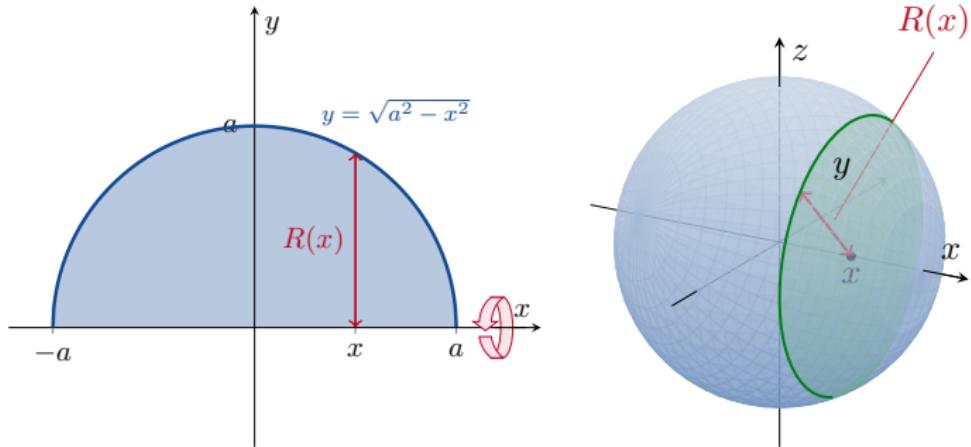


To generate a sphere, we rotate the area between $y = \sqrt{a^2 - x^2}$ and the x -axis about the x -axis. Its volume is

$$\text{volume} = \int_{-a}^a \pi(R(x))^2 dx = \int_{-a}^a \pi(\sqrt{a^2 - x^2})^2 dx$$

6.1 Volumes Using Cross-Sections

$$\text{volume} = \int_a^b \pi(R(x))^2 dx$$

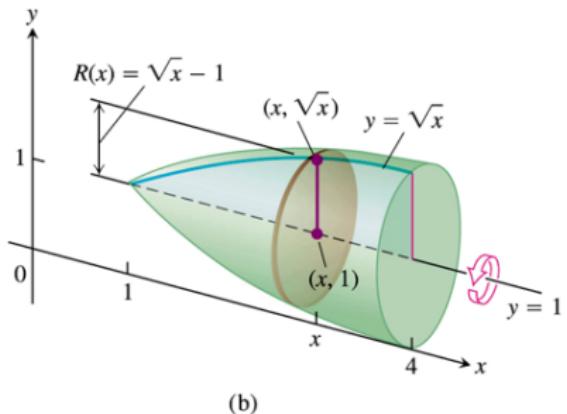
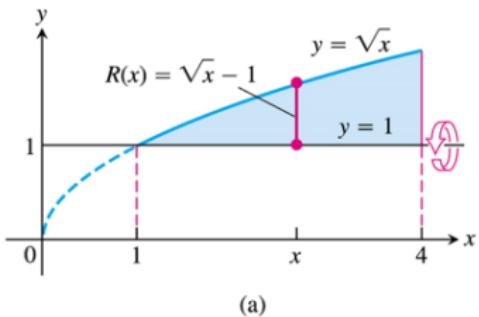


To generate a sphere, we rotate the area between $y = \sqrt{a^2 - x^2}$ and the x -axis about the x -axis. Its volume is

$$\begin{aligned}\text{volume} &= \int_{-a}^a \pi(R(x))^2 dx = \int_{-a}^a \pi(\sqrt{a^2 - x^2})^2 dx \\ &= \pi \int_{-a}^a (a^2 - x^2) dx = \pi \left[a^2x - \frac{x^3}{3} \right]_{-a}^a = \frac{4}{3}\pi a^3.\end{aligned}$$

6.1 Volumes Using Cross-Sections

$$\text{volume} = \int_a^b \pi(R(x))^2 dx$$

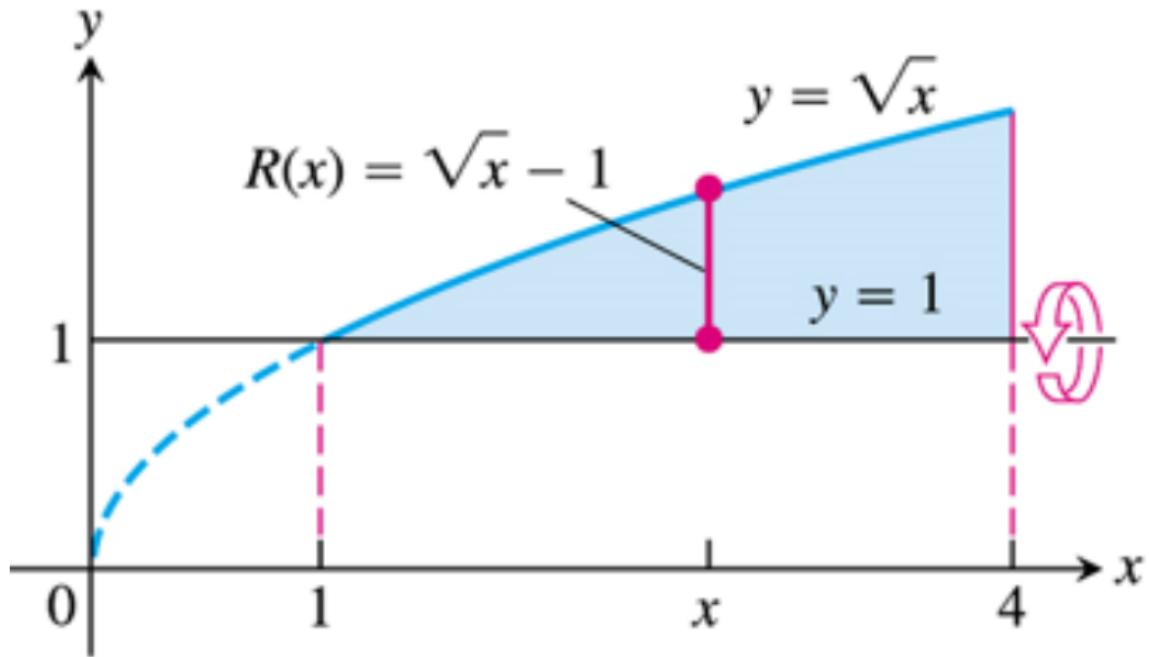


Example

Find the volume of the solid generated by revolving the region bounded by $y = \sqrt{x}$ and the lines $y = 1$ and $x = 4$, about the line $y = 1$.

6.1 Volumes Using Cross-Sections

$$\text{volume} = \int_a^b \pi(R(x))^2 dx$$



EXAMPLE 6 Find the volume of the solid generated by revolving the region bounded by $y = \sqrt{x}$ and the lines $y = 1, x = 4$ about the line $y = 1$.

Solution We draw figures showing the region, a typical radius, and the generated solid (Figure 6.10). The volume is

$$\begin{aligned} V &= \int_1^4 \pi [R(x)]^2 dx \\ &= \int_1^4 \pi [\sqrt{x} - 1]^2 dx && \text{Radius } R(x) = \sqrt{x} - 1 \text{ for} \\ &&& \text{rotation around } y = 1. \\ &= \pi \int_1^4 [x - 2\sqrt{x} + 1] dx && \text{Expand integrand.} \\ &= \pi \left[\frac{x^2}{2} - 2 \cdot \frac{2}{3}x^{3/2} + x \right]_1^4 = \frac{7\pi}{6}. && \text{Integrate.} \end{aligned}$$

6.1 Volumes Using Cross-Sections

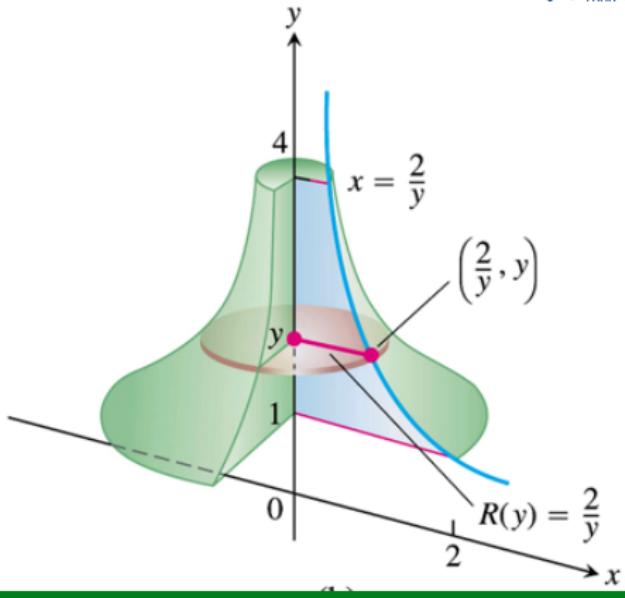
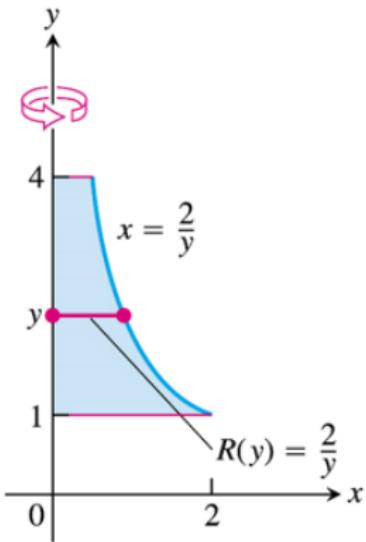


Volume by Disks for Rotation About the y -Axis

$$\text{volume} = \int_c^d A(y) dy = \int_c^d \pi(R(y))^2 dy$$

6.1 Volumes Using Cross-Sections

$$\text{volume} = \int_c^d \pi(R(y))^2 dy$$

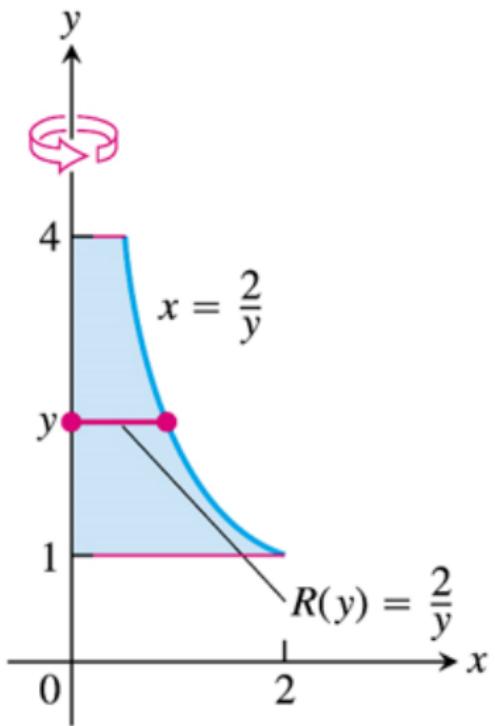


Example

Find the volume of the solid generated by revolving the region between the y -axis and the curve $x = \frac{2}{y}$, $1 \leq y \leq 4$ about the y -axis.

6.1 Volumes Using Cross-Sections

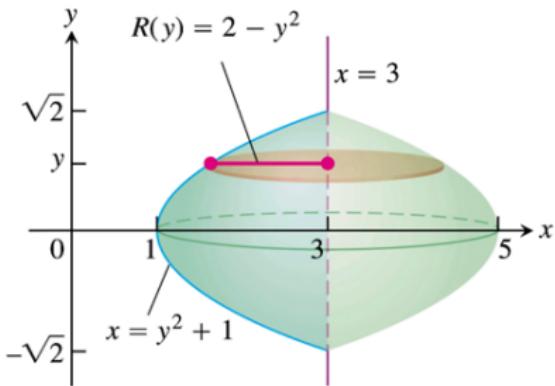
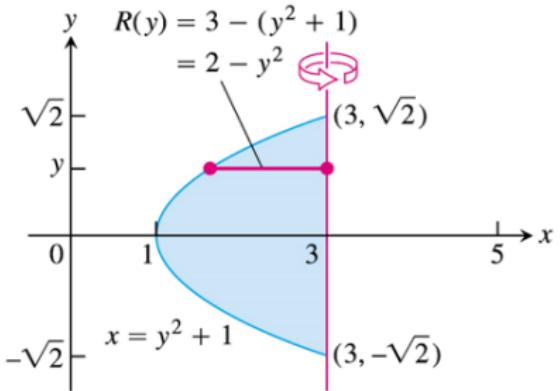
$$\text{volume} = \int_c^d \pi (R(y))^2 dy$$



$$\begin{aligned}\text{volume} &= \int_1^4 \pi \left(\frac{2}{y}\right)^2 dy \\ &= \dots\end{aligned}$$

6.1 Volumes Using Cross-Sections

$$\text{volume} = \int_c^d \pi(R(y))^2 dy$$

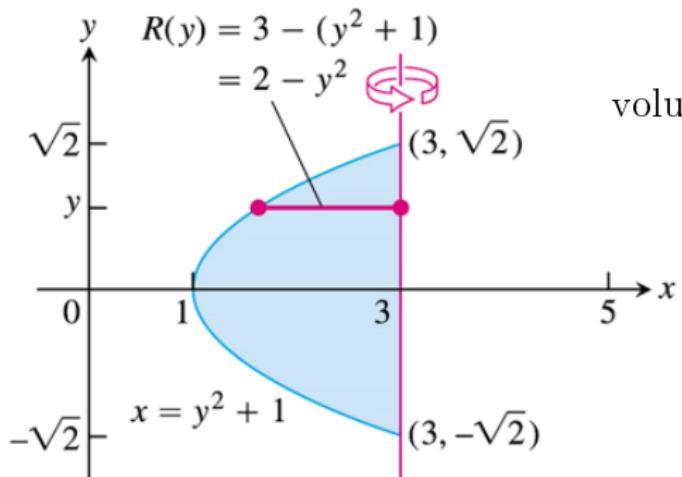


Example

Find the volume of the solid generated by revolving the region between the parabola $x = y^2 + 1$ and the line $x = 3$ about the line $x = 3$.

6.1 Volumes Using Cross-Sections

$$\text{volume} = \int_c^d \pi (R(y))^2 dy$$

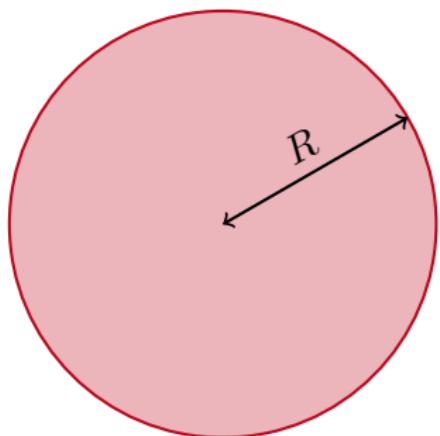


$$\begin{aligned}\text{volume} &= \int_{-\sqrt{2}}^{\sqrt{2}} \pi (2 - y^2)^2 dy \\ &= \dots\end{aligned}$$

6.1 Volumes Using Cross-Sections

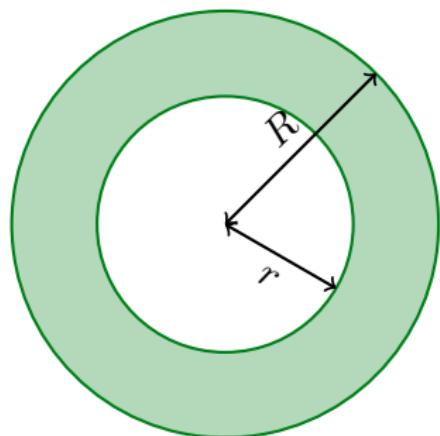


Solids of Revolution: The Washer Method



a disk

$$\text{area} = \pi R^2$$

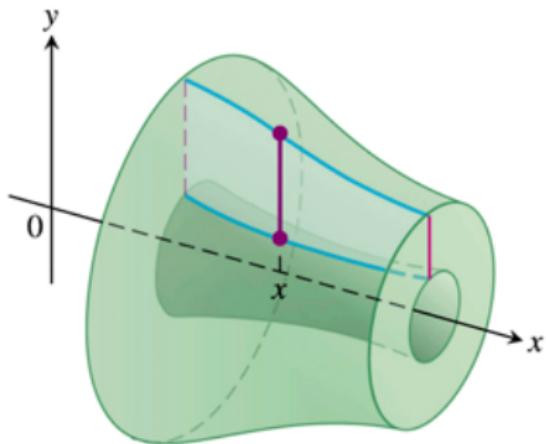
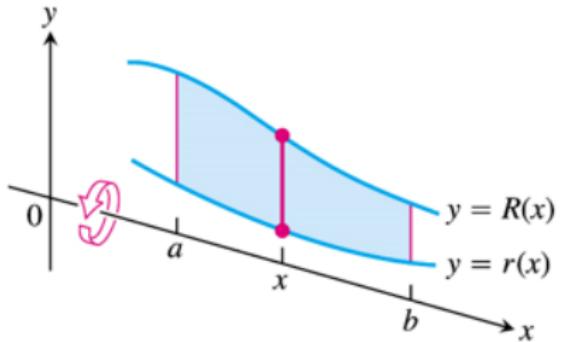


a washer

$$\text{area} = \pi R^2 - \pi r^2$$

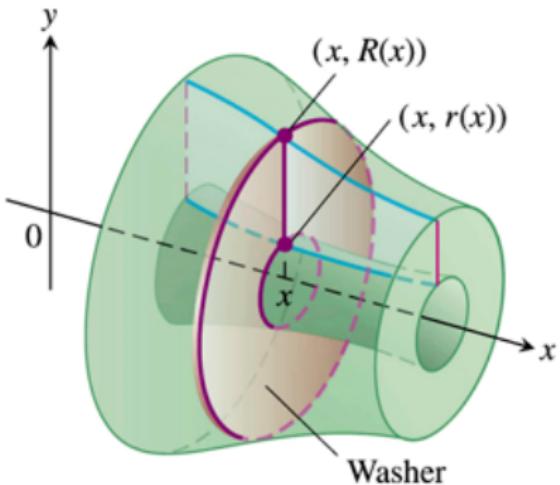
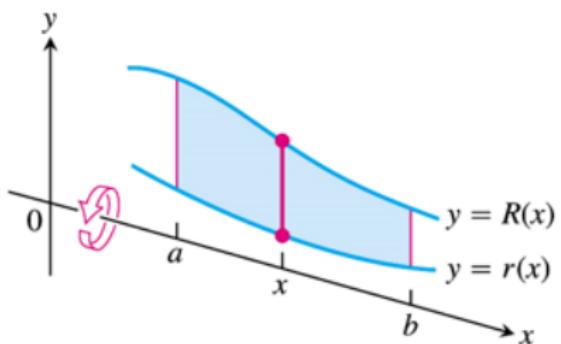
6.1 Volumes Using Cross-Sections

Volume by Washers for Rotation About the x -Axis



6.1 Volumes Using Cross-Sections

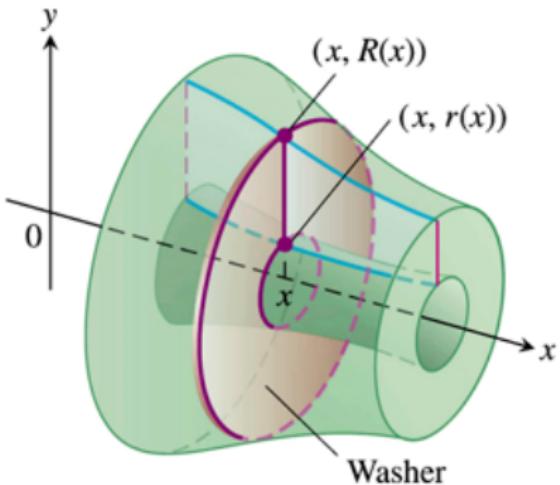
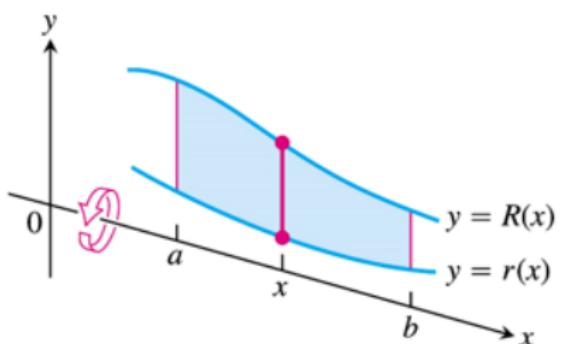
Volume by Washers for Rotation About the x -Axis



6.1 Volumes Using Cross-Sections



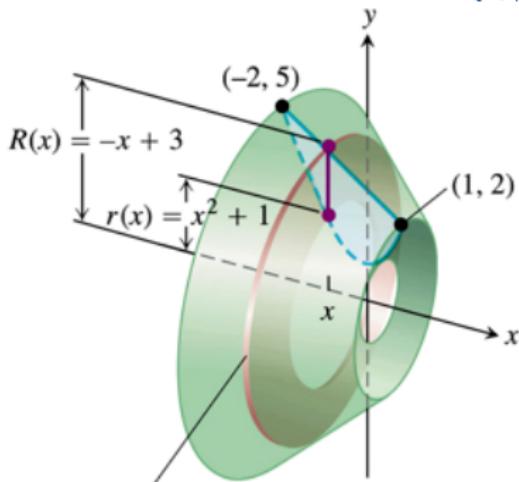
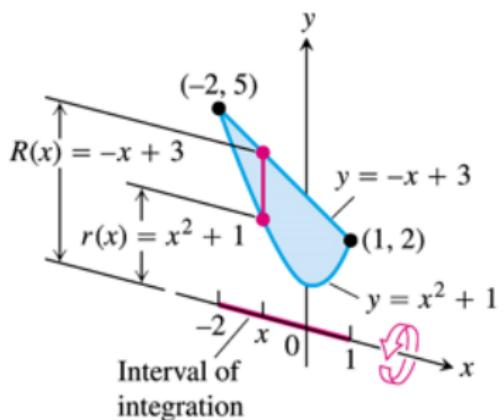
Volume by Washers for Rotation About the x -Axis



$$\text{volume} = \int_a^b A(x) dx = \int_a^b \pi \left((R(x))^2 - (r(x))^2 \right) dx.$$

6.1 Volumes Using

$$\text{volume} = \int_a^b \pi \left((R(x))^2 - (r(x))^2 \right) dx$$



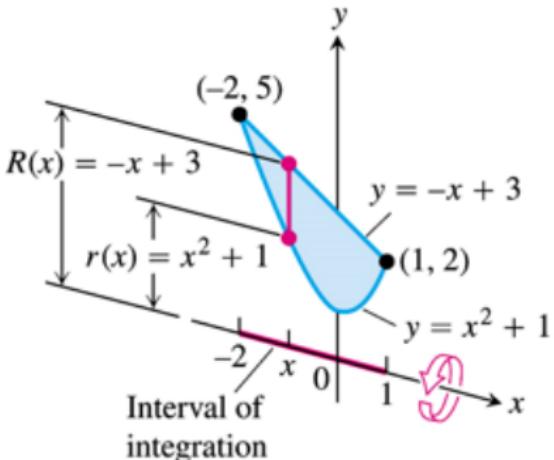
Outer radius: $R(x) = -x + 3$
 Inner radius: $r(x) = x^2 + 1$

Example

The region bounded by the curve $y = x^2 + 1$ and the line $y = -x + 3$ is revolved about the x -axis to generate a solid. Find the volume of the solid.

6.1 Volumes Using

$$\text{volume} = \int_a^b \pi \left((R(x))^2 - (r(x))^2 \right) dx$$



$$\begin{aligned}\text{volume} &= \int_{-2}^1 \pi \left((-x + 3)^2 - (x^2 + 1)^2 \right) dx \\ &= \pi \int_{-2}^1 8 - 6x - x^2 - x^4 dx \\ &= \pi \left[8x - 3x^2 - \frac{x^3}{3} - \frac{x^5}{5} \right]_{-2}^1 = \frac{117\pi}{5}.\end{aligned}$$

6.1 Volumes Using Cross-Sections

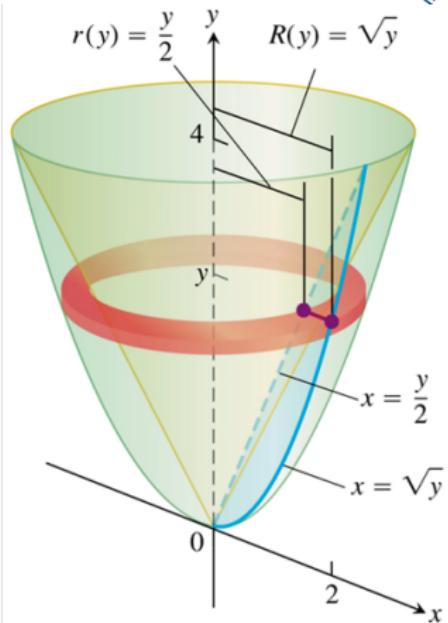
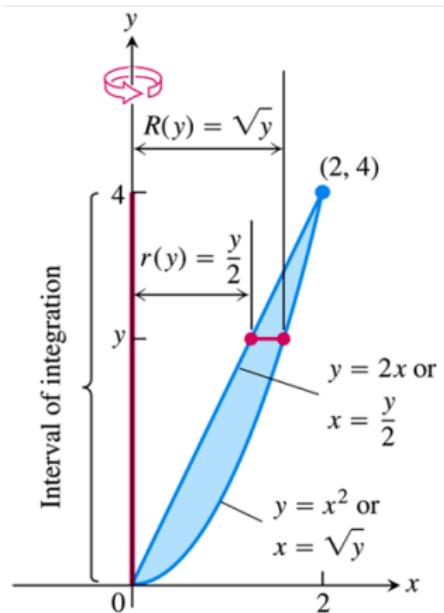


Volume by Washers for Rotation About the y -Axis

$$\text{volume} = \int_c^d A(y) dy = \int_c^d \pi \left((R(y))^2 - (r(y))^2 \right) dy.$$

6.1 Volumes Using

$$\text{volume} = \int_c^d \pi \left((R(y))^2 - (r(y))^2 \right) dy$$

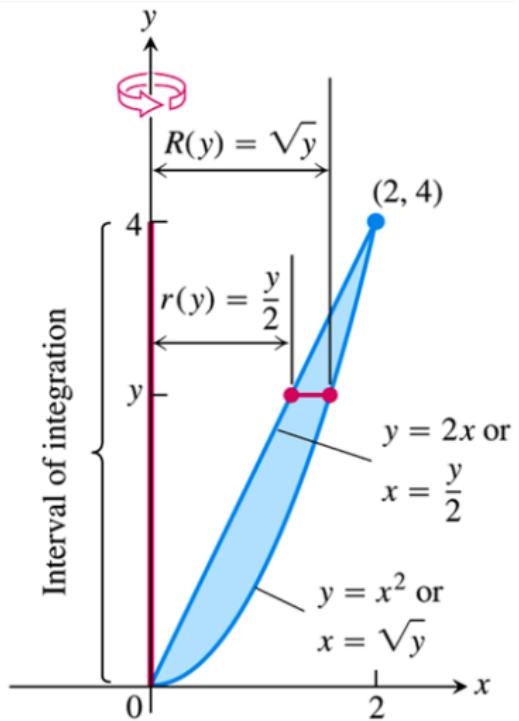


Example

The region bounded by the parabola $y = x^2$ and the line $y = 2x$ in the first quadrant is revolved about the y -axis to generate a solid. Find the volume of the solid.

6.1 Volumes Using

$$\text{volume} = \int_c^d \pi \left((R(y))^2 - (r(y))^2 \right) dy$$



$$\begin{aligned} \text{volume} &= \int_0^4 \pi \left((\sqrt{y})^2 - \left(\frac{y}{2}\right)^2 \right) dy \\ &= \dots \end{aligned}$$



Volumes Using Cylindrical Shells

6.2 Volumes Using Cylindrical Shells

$$\text{volume} = \int_c^d \pi \left((R(y))^2 - (r(y))^2 \right) dy$$



The Disk and Washer
methods.

6.2 Volumes Using Cylindrical Shells



$$\text{volume} = \int_c^d \pi \left((R(y))^2 - (r(y))^2 \right) dy$$



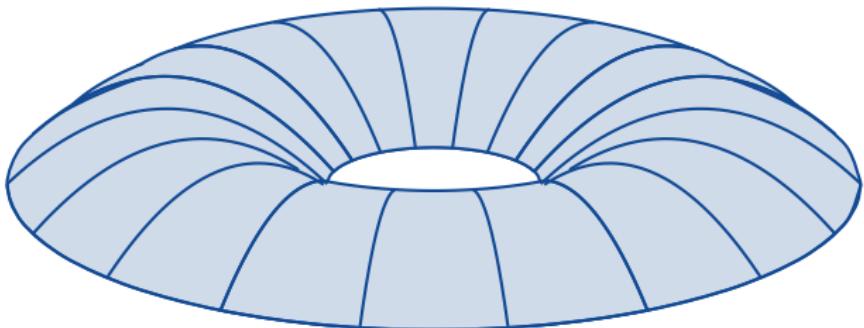
The Disk and Washer
methods.



The Shell method.

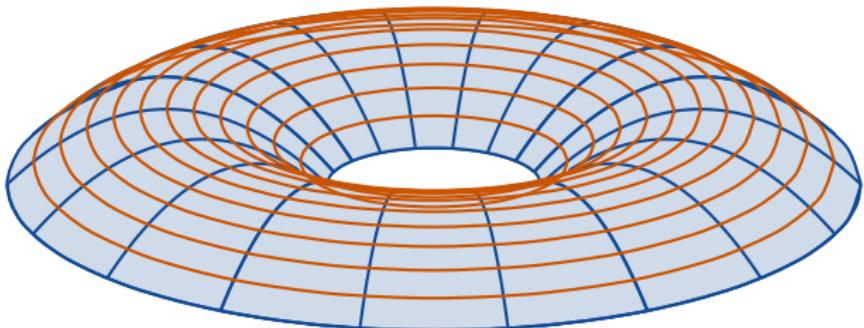
6.2 Volumes Using Cylindrical Shells

$$\text{volume} = \int_c^d \pi \left((R(y))^2 - (r(y))^2 \right) dy$$



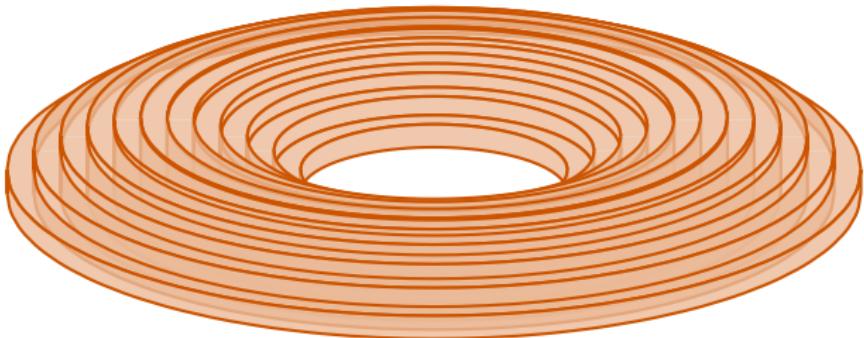
6.2 Volumes Using Cylindrical Shells

$$\text{volume} = \int_c^d \pi \left((R(y))^2 - (r(y))^2 \right) dy$$



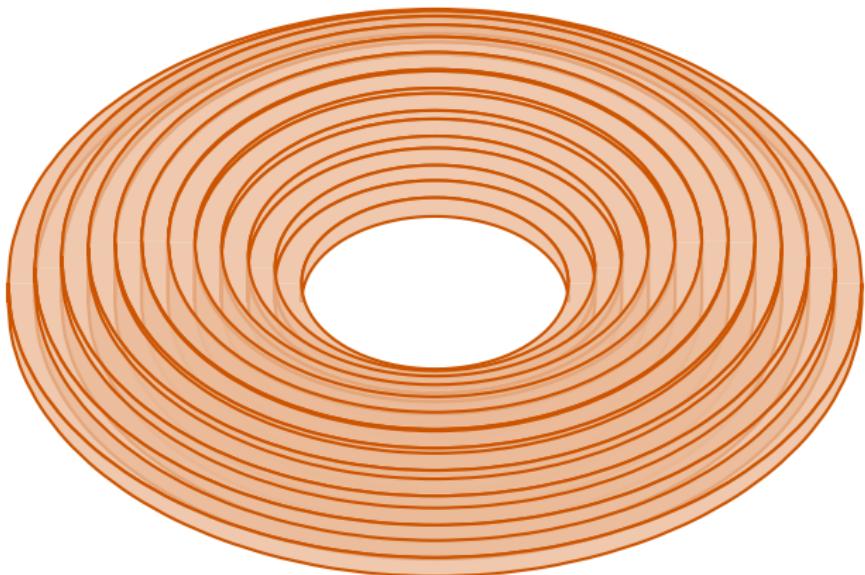
6.2 Volumes Using Cylindrical Shells

$$\text{volume} = \int_c^d \pi \left((R(y))^2 - (r(y))^2 \right) dy$$



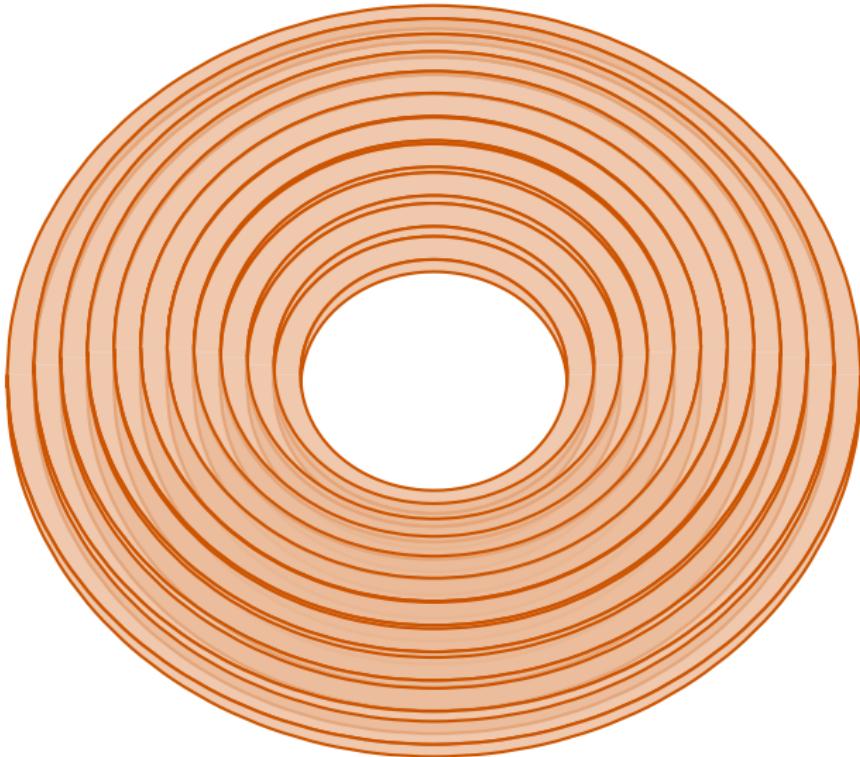
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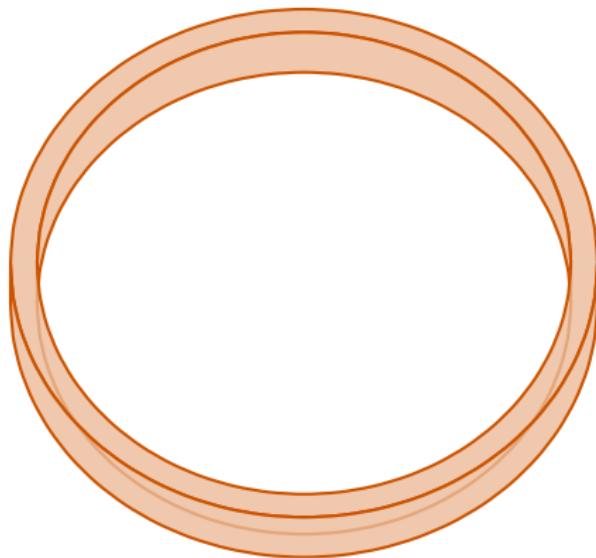
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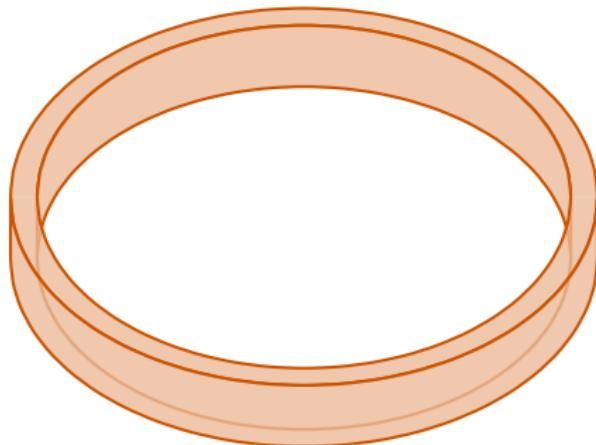
6.2 Volumes Using Cylindrical Shells

$$\text{volume} = \int_c^d \pi \left((R(y))^2 - (r(y))^2 \right) dy$$



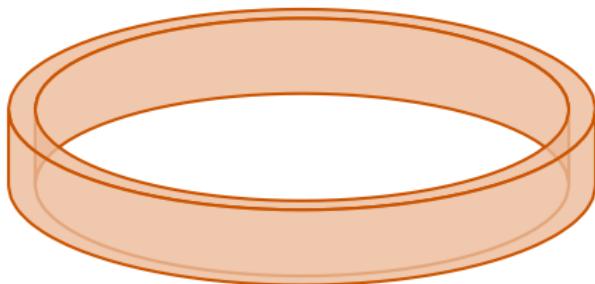
6.2 Volumes Using Cylindrical Shells

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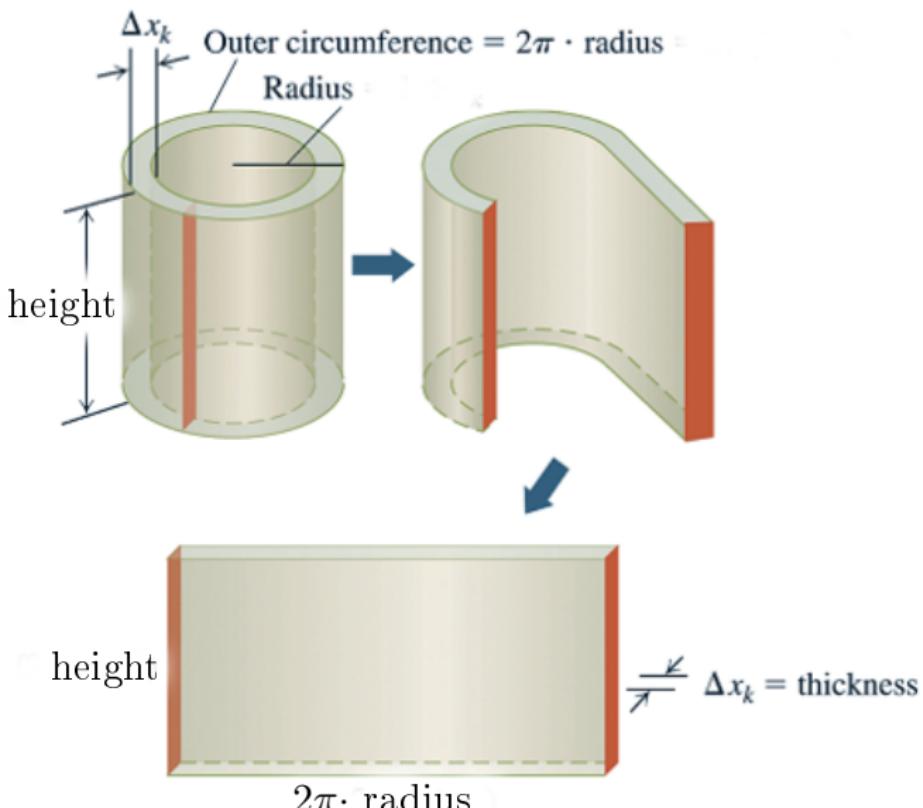
6.2 Volumes Using Cylindrical Shells

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6.2 Volumes Using Cylindrical Shells

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6.2 Volumes Using Cylindrical Shells

$$\text{volume} = \int_c^d \pi \left((R(y))^2 - (r(y))^2 \right) dy$$



The volume of one of these shells is approximately

$$2\pi \left(\begin{matrix} \text{shell} \\ \text{radius} \end{matrix} \right) \left(\begin{matrix} \text{shell} \\ \text{height} \end{matrix} \right) \Delta x_k$$

6.2 Volumes Using Cylindrical Shells

$$\text{volume} = \int_c^d \pi \left((R(y))^2 - (r(y))^2 \right) dy$$



The volume of one of these shells is approximately

$$2\pi \left(\frac{\text{shell}}{\text{radius}} \right) \left(\frac{\text{shell}}{\text{height}} \right) \Delta x_k$$

So the volume of all of the shells is the Riemann sum

$$\sum_{k=1}^n 2\pi \left(\frac{\text{shell}}{\text{radius}} \right) \left(\frac{\text{shell}}{\text{height}} \right) \Delta x_k$$

6.2 Volumes Using Cylindrical Shells

$$\text{volume} = \int_c^d \pi \left((R(y))^2 - (r(y))^2 \right) dy$$



The volume of one of these shells is approximately

$$2\pi \left(\frac{\text{shell}}{\text{radius}} \right) \left(\frac{\text{shell}}{\text{height}} \right) \Delta x_k$$

So the volume of all of the shells is the Riemann sum

$$\sum_{k=1}^n 2\pi \left(\frac{\text{shell}}{\text{radius}} \right) \left(\frac{\text{shell}}{\text{height}} \right) \Delta x_k$$

So the volume of the solid object is

$$\text{volume} = \int_a^b 2\pi \left(\frac{\text{shell}}{\text{radius}} \right) \left(\frac{\text{shell}}{\text{height}} \right) dx.$$

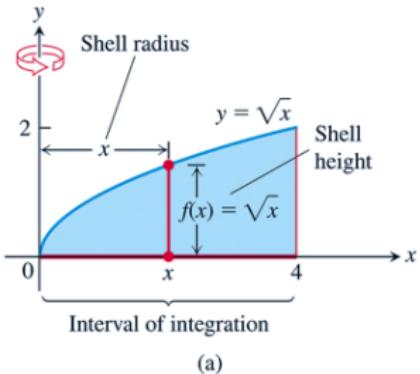
Shell Formula for Revolution About a Vertical Line

The volume of the solid generated by revolving the region between the x -axis and the graph of a continuous function $y = f(x) \geq 0, L \leq a \leq x \leq b$, about a vertical line $x = L$ is

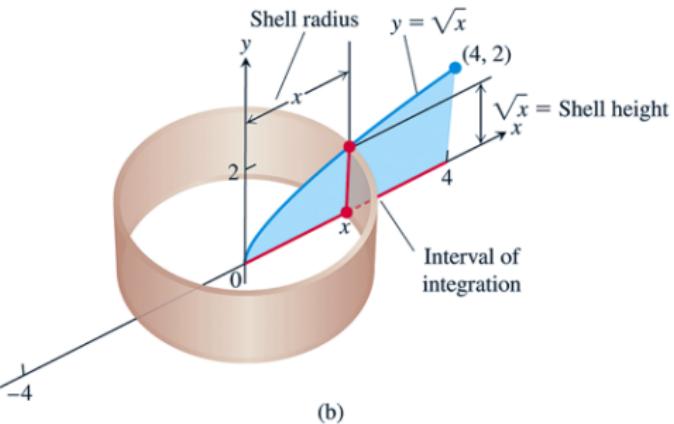
$$V = \int_a^b 2\pi \left(\begin{matrix} \text{shell} \\ \text{radius} \end{matrix} \right) \left(\begin{matrix} \text{shell} \\ \text{height} \end{matrix} \right) dx.$$

6.2 Volumes Using Cylindrical Shells

$$\text{volume} = \int_a^b 2\pi(\text{radius})(\text{height}) dx$$



(a)



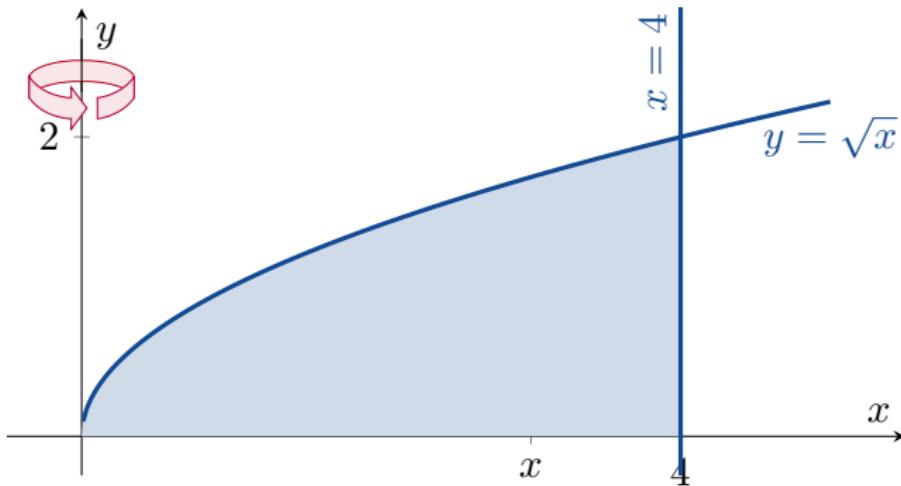
(b)

Example

The region bounded by the curve $y = \sqrt{x}$, the x -axis and the line $x = 4$ is revolved about the y -axis to generate a solid. Find the volume of the solid.

6.2 Volumes Using Cylindrical Shells

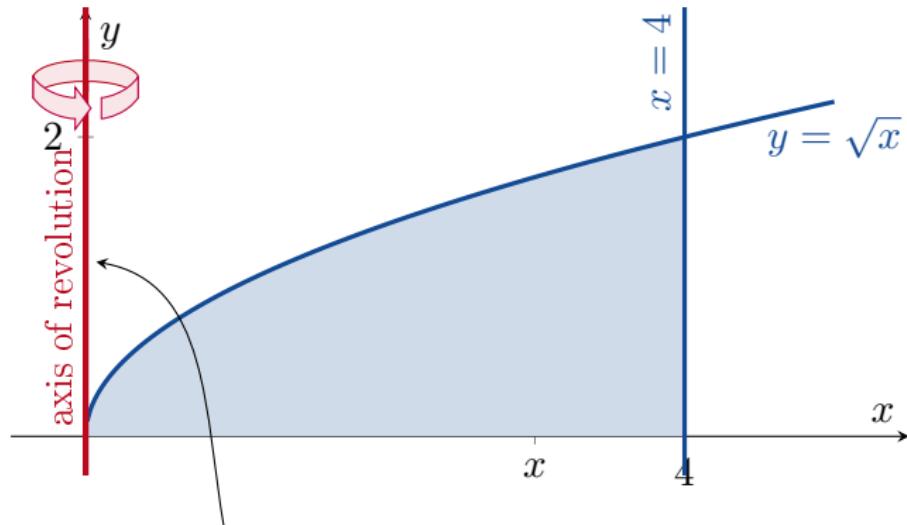
$$\text{volume} = \int_a^b 2\pi(\text{shell radius})(\text{shell height}) dx$$



the s | ell method

6.2 Volumes Using Cylindrical Shells

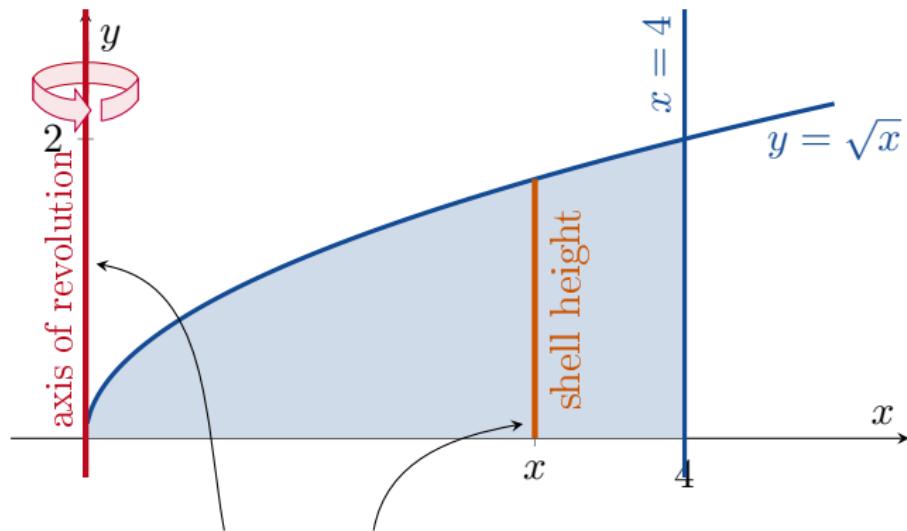
$$\text{volume} = \int_a^b 2\pi(\text{shell radius})(\text{shell height}) dx$$



the s ell method

6.2 Volumes Using Cylindrical Shells

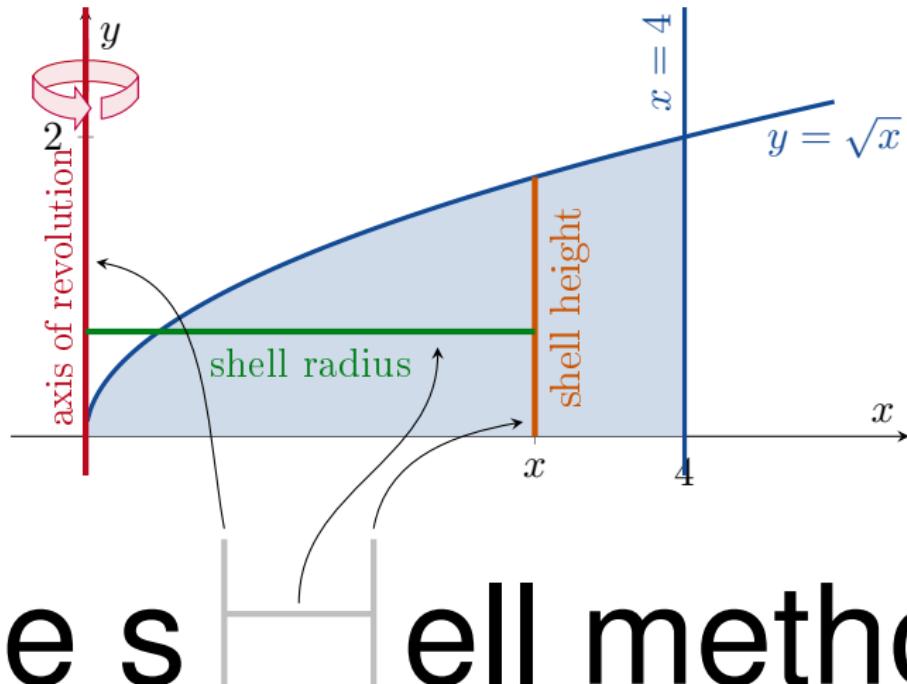
$$\text{volume} = \int_a^b 2\pi(\text{radius})(\text{height}) dx$$



the s — ell method

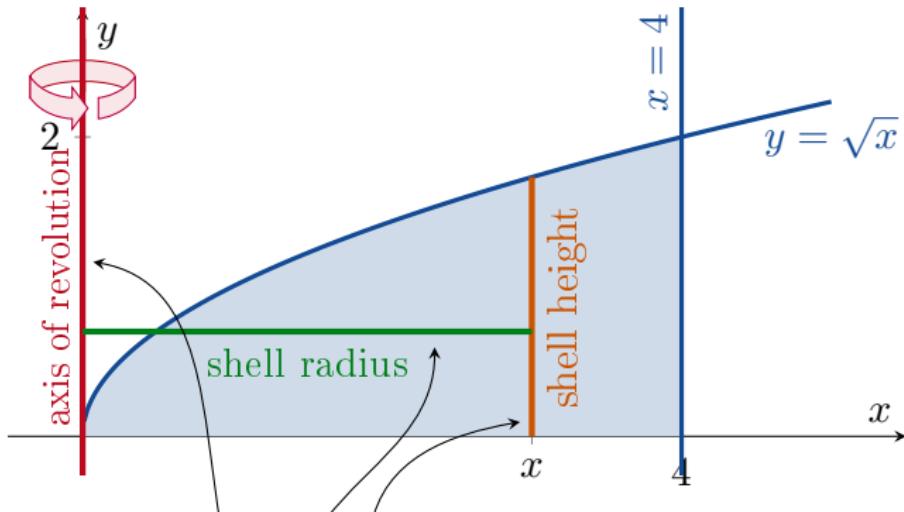
6.2 Volumes Using Cylindrical Shells

$$\text{volume} = \int_a^b 2\pi(\text{radius})(\text{height}) dx$$



6.2 Volumes Using Cylindrical Shells

$$\text{volume} = \int_a^b 2\pi(\text{radius})(\text{height}) dx$$



$$0 \leq x \leq 4$$

$$\text{shell height} = \sqrt{x}$$

$$\text{shell radius} = x$$

the shell method

6.2 Volumes Using Cylindrical Shells

$$\text{volume} = \int_a^b 2\pi(\text{shell radius})(\text{height}) dx$$



$$0 \leq x \leq 4$$

$$\text{shell height} = \sqrt{x}$$

$$\text{shell radius} = x$$

Therefore

$$\text{volume} = \int_a^b 2\pi(\text{shell radius})(\text{height}) dx$$

=

=

=

=

.

6.2 Volumes Using Cylindrical Shells

$$\text{volume} = \int_a^b 2\pi(\text{shell radius})(\text{height}) dx$$



$$0 \leq x \leq 4$$

$$\text{shell height} = \sqrt{x}$$

$$\text{shell radius} = x$$

Therefore

$$\begin{aligned}\text{volume} &= \int_a^b 2\pi(\text{shell radius})(\text{height}) dx \\ &= \int_0^4 2\pi(x) (\sqrt{x}) dx\end{aligned}$$

$$= \quad = \quad = \quad .$$

6.2 Volumes Using Cylindrical Shells

$$\text{volume} = \int_a^b 2\pi(\text{shell radius})(\text{height}) dx$$



$$0 \leq x \leq 4$$

$$\text{shell height} = \sqrt{x}$$

$$\text{shell radius} = x$$

Therefore

$$\begin{aligned}\text{volume} &= \int_a^b 2\pi(\text{shell radius})(\text{height}) dx \\ &= \int_0^4 2\pi(x)(\sqrt{x}) dx \\ &= 2\pi \int_0^4 x^{\frac{3}{2}} dx = 2\pi \left[\frac{2}{5}x^{\frac{5}{2}} \right]_0^4 = \frac{128\pi}{5}.\end{aligned}$$

6.2 Volumes Using Cylindrical Shells

$$\text{volume} = \int_a^b 2\pi (\text{shell radius})(\text{shell height}) \, dx$$



Remark

When you do the shell method, it doesn't really matter if you get the **shell radius** and the **shell height** mixed up because

- you are just going to multiply them together;
- so you still get the correct answer; and
- you will only lose a few points in an exam.

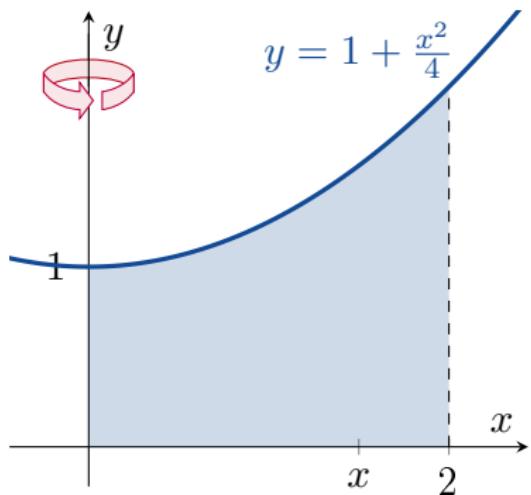
6.2 Volumes Using Cylindrical Shells

$$\text{volume} = \int_a^b 2\pi (\text{shell radius})(\text{shell height}) dx$$



Example (Page 348, Exercise 1)

The region bounded by the curve $y = 1 + \frac{x^2}{4}$ and the x -axis, for $0 \leq x \leq 2$, is revolved about the y -axis to generate a solid. Find the volume of the solid.



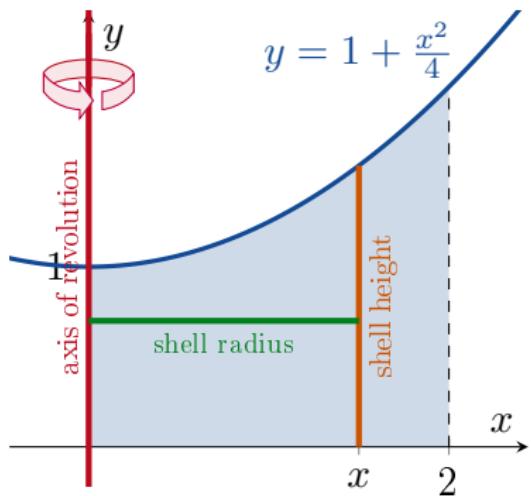
6.2 Volumes Using Cylindrical Shells

$$\text{volume} = \int_a^b 2\pi (\text{radius})(\text{height}) dx$$



Example (Page 348, Exercise 1)

The region bounded by the curve $y = 1 + \frac{x^2}{4}$ and the x -axis, for $0 \leq x \leq 2$, is revolved about the y -axis to generate a solid. Find the volume of the solid.



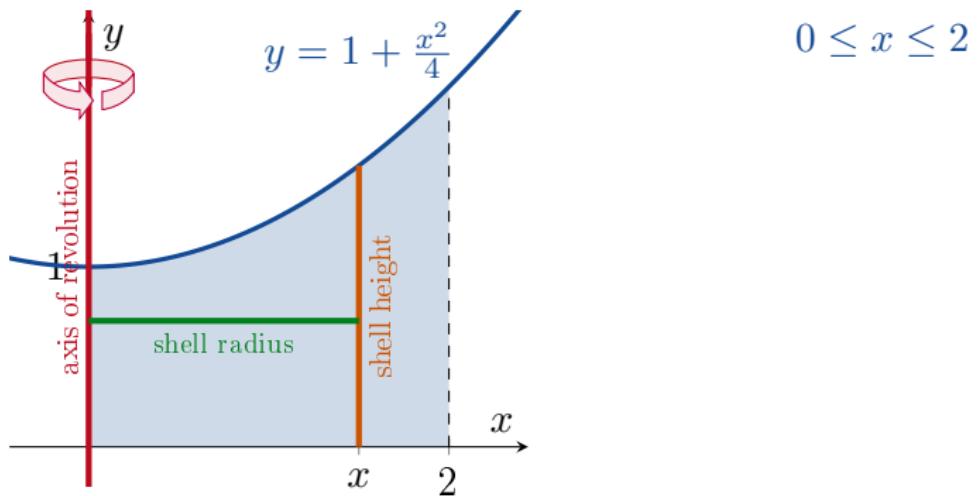
6.2 Volumes Using Cylindrical Shells

$$\text{volume} = \int_a^b 2\pi (\text{radius})(\text{height}) dx$$



Example (Page 348, Exercise 1)

The region bounded by the curve $y = 1 + \frac{x^2}{4}$ and the x -axis, for $0 \leq x \leq 2$, is revolved about the y -axis to generate a solid. Find the volume of the solid.



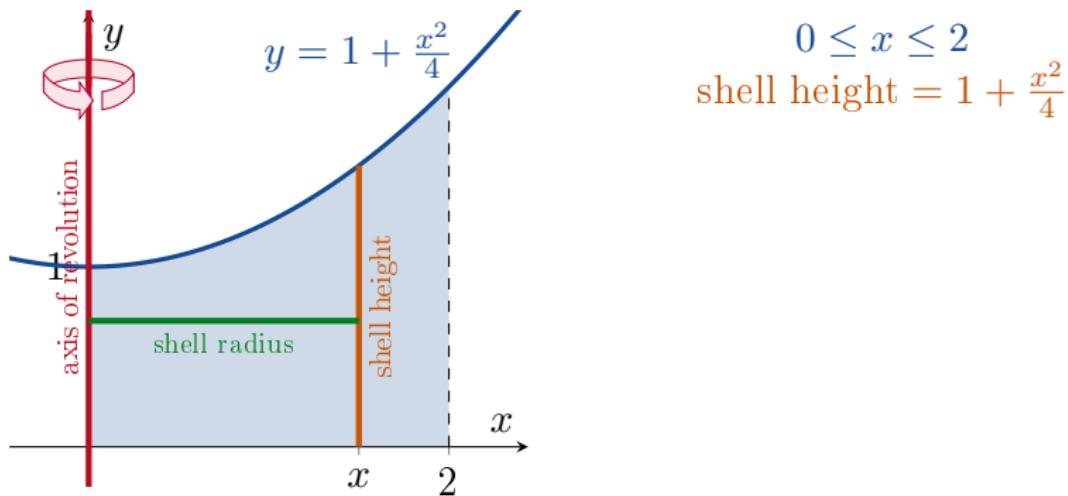
6.2 Volumes Using Cylindrical Shells

$$\text{volume} = \int_a^b 2\pi (\text{radius})(\text{height}) dx$$



Example (Page 348, Exercise 1)

The region bounded by the curve $y = 1 + \frac{x^2}{4}$ and the x -axis, for $0 \leq x \leq 2$, is revolved about the y -axis to generate a solid. Find the volume of the solid.



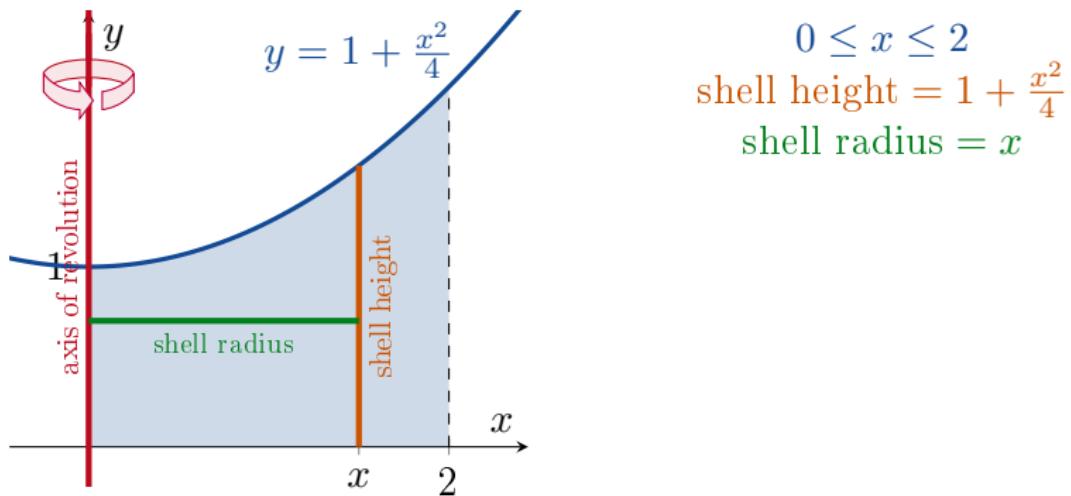
6.2 Volumes Using Cylindrical Shells

$$\text{volume} = \int_a^b 2\pi (\text{radius})(\text{height}) dx$$



Example (Page 348, Exercise 1)

The region bounded by the curve $y = 1 + \frac{x^2}{4}$ and the x -axis, for $0 \leq x \leq 2$, is revolved about the y -axis to generate a solid. Find the volume of the solid.



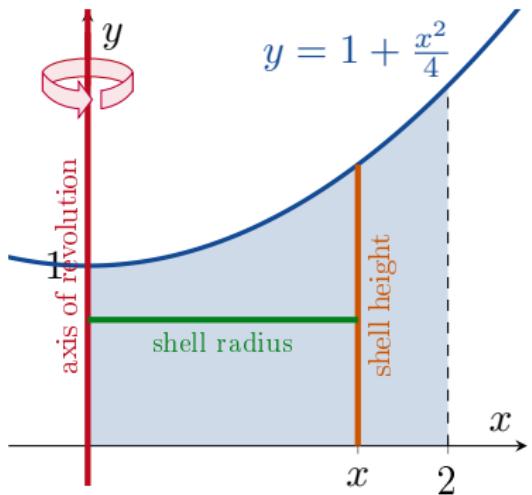
6.2 Volumes Using Cylindrical Shells

$$\text{volume} = \int_a^b 2\pi (\text{radius}) (\text{height}) dx$$



Example (Page 348, Exercise 1)

The region bounded by the curve $y = 1 + \frac{x^2}{4}$ and the x -axis, for $0 \leq x \leq 2$, is revolved about the y -axis to generate a solid. Find the volume of the solid.



$$\begin{aligned}0 &\leq x \leq 2 \\ \text{shell height} &= 1 + \frac{x^2}{4} \\ \text{shell radius} &= x\end{aligned}$$

$$\begin{aligned}\text{volume} &= \int_0^2 2\pi (x) \left(1 + \frac{x^2}{4}\right) dx \\ &= \dots\end{aligned}$$

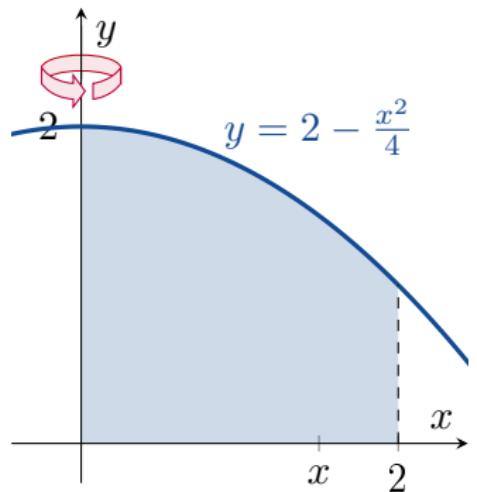
6.2 Volumes Using Cylindrical Shells

$$\text{volume} = \int_a^b 2\pi (\text{radius}) (\text{height}) dx$$



Example (Page 348, Exercise 2)

The region shown below is revolved about the y -axis to generate a solid. Find the volume of the solid.



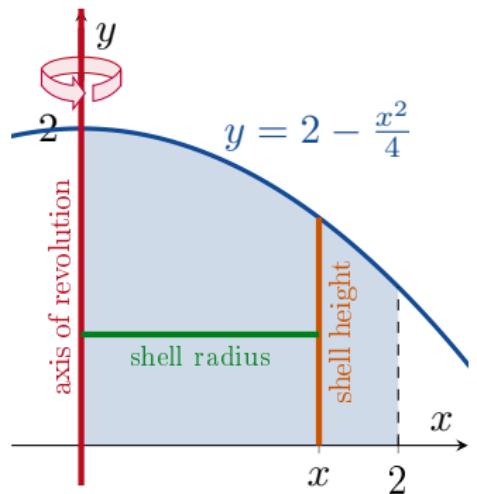
6.2 Volumes Using Cylindrical Shells

$$\text{volume} = \int_a^b 2\pi (\text{radius})(\text{height}) dx$$



Example (Page 348, Exercise 2)

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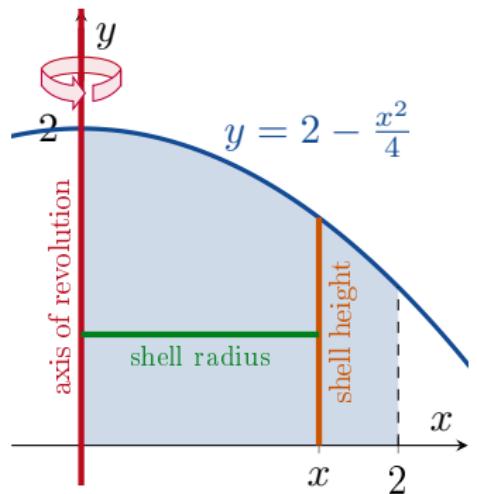
6.2 Volumes Using Cylindrical Shells

$$\text{volume} = \int_a^b 2\pi (\text{radius})(\text{height}) dx$$



Example (Page 348, Exercise 2)

The region shown below is revolved about the y -axis to generate a solid. Find the volume of the solid.



$$0 \leq x \leq 2$$

$$\text{shell height} = 2 - \frac{x^2}{4}$$

$$\text{shell radius} = x$$

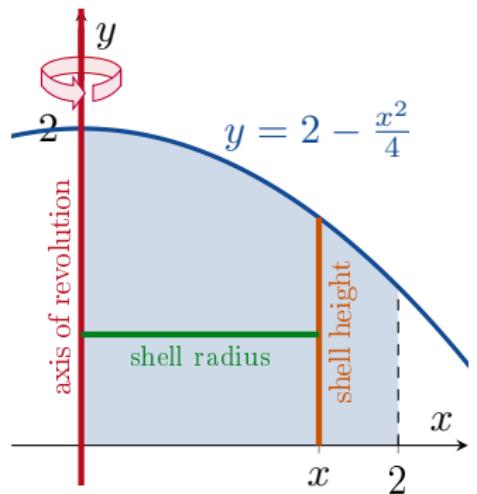
6.2 Volumes Using Cylindrical Shells

$$\text{volume} = \int_a^b 2\pi(\text{radius})(\text{height}) dx$$



Example (Page 348, Exercise 2)

The region shown below is revolved about the y -axis to generate a solid. Find the volume of the solid.



$$0 \leq x \leq 2$$

$$\text{shell height} = 2 - \frac{x^2}{4}$$

$$\text{shell radius} = x$$

$$\begin{aligned}\text{volume} &= \int_0^2 2\pi(x) \left(2 - \frac{x^2}{4}\right) dx \\ &= \dots\end{aligned}$$

6.2 Volumes Using Cylindrical Shells

$$\text{volume} = \int_a^b 2\pi(\text{shell radius})(\text{shell height}) dx$$



We can use the

the shell method

when we rotate about a
vertical line.

$$V = \int_a^b 2\pi(\text{shell radius})(\text{shell height}) dx.$$

6.2 Volumes Using Cylindrical Shells

$$\text{volume} = \int_a^b 2\pi(\text{shell radius})(\text{shell height}) dx$$



We can use the
the shell method

when we rotate about a
vertical line.

$$V = \int_a^b 2\pi(\text{shell radius})(\text{shell height}) dx.$$

And we can use the
the shell method

when we rotate about a
horizontal line.

$$V = \int_a^b 2\pi(\text{shell radius})(\text{shell height}) dy.$$

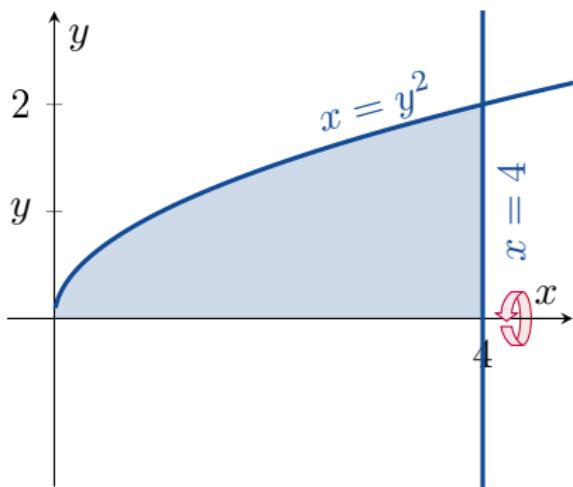
6.2 Volumes Using Cylindrical Shells

$$\text{volume} = \int_a^b 2\pi (\text{shell radius})(\text{shell height}) dx$$



Example

The region bounded by the curve $y = \sqrt{x}$, the x -axis and the line $x = 4$ is revolved about the **x -axis** to generate a solid. Find the volume of the solid.



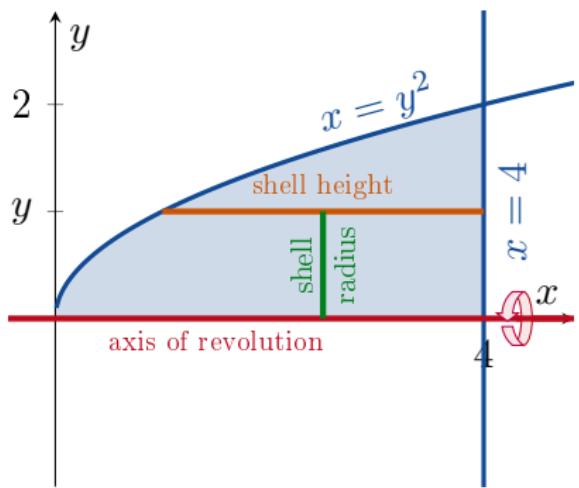
6.2 Volumes Using Cylindrical Shells

$$\text{volume} = \int_a^b 2\pi (\text{radius})(\text{height}) dx$$



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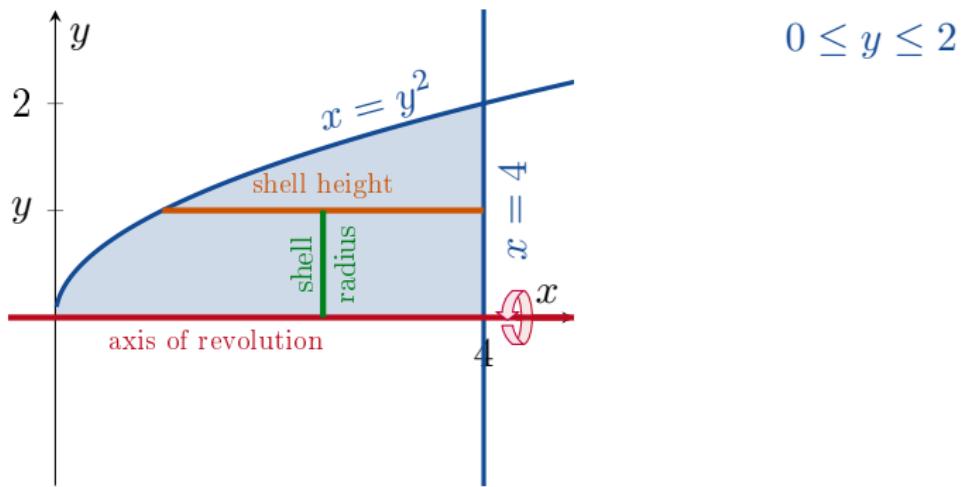
6.2 Volumes Using Cylindrical Shells

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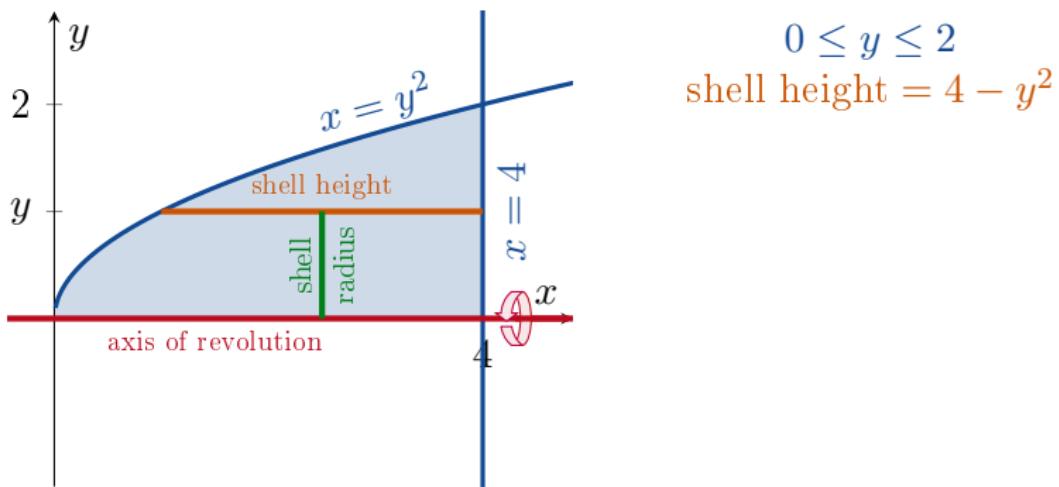
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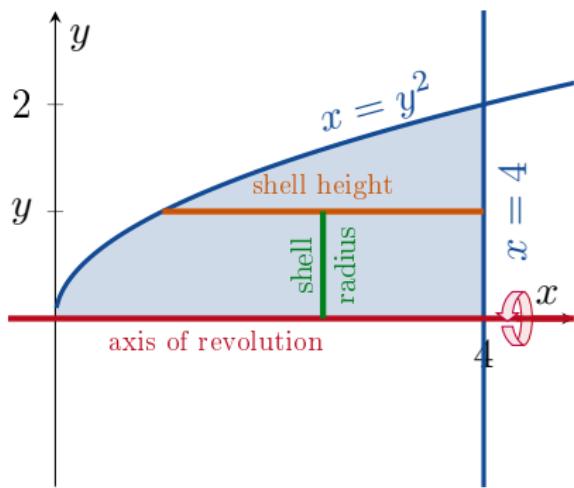
6.2 Volumes Using Cylindrical Shells

$$\text{volume} = \int_a^b 2\pi (\text{radius})(\text{height}) dx$$



Example

The region bounded by the curve $y = \sqrt{x}$, the x -axis and the line $x = 4$ is revolved about the **x -axis** to generate a solid. Find the volume of the solid.



$$0 \leq y \leq 2$$

$$\text{shell height} = 4 - y^2$$

$$\text{shell radius} = y$$

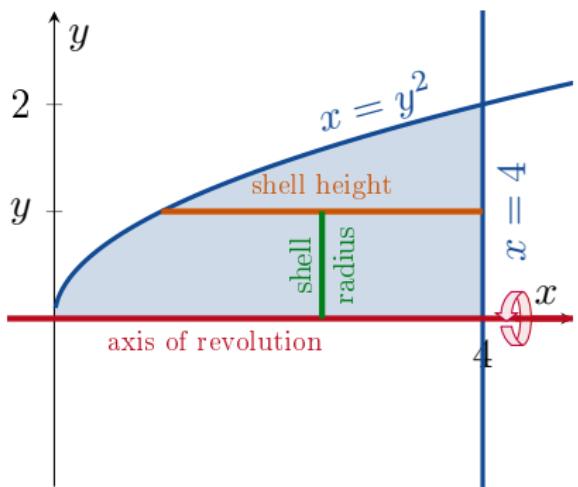
6.2 Volumes Using Cylindrical Shells

$$\text{volume} = \int_a^b 2\pi (\text{radius})(\text{height}) dy$$



Example

The region bounded by the curve $y = \sqrt{x}$, the x -axis and the line $x = 4$ is revolved about the **x -axis** to generate a solid. Find the volume of the solid.



$$0 \leq y \leq 2$$
$$\text{shell height} = 4 - y^2$$
$$\text{shell radius} = y$$

$$\begin{aligned}\text{volume} &= \int_0^2 2\pi (y) (4 - y^2) dy \\ &= \dots\end{aligned}$$

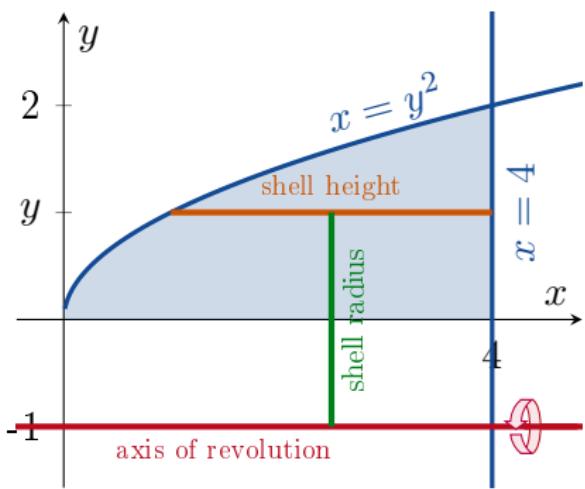
6.2 Volumes Using Cylindrical Shells

$$\text{volume} = \int_a^b 2\pi (\text{radius})(\text{height}) dx$$



Example

The region bounded by the curve $y = \sqrt{x}$, the x -axis and the line $x = 4$ is revolved about the line $y = -1$ to generate a solid. Find the volume of the solid.



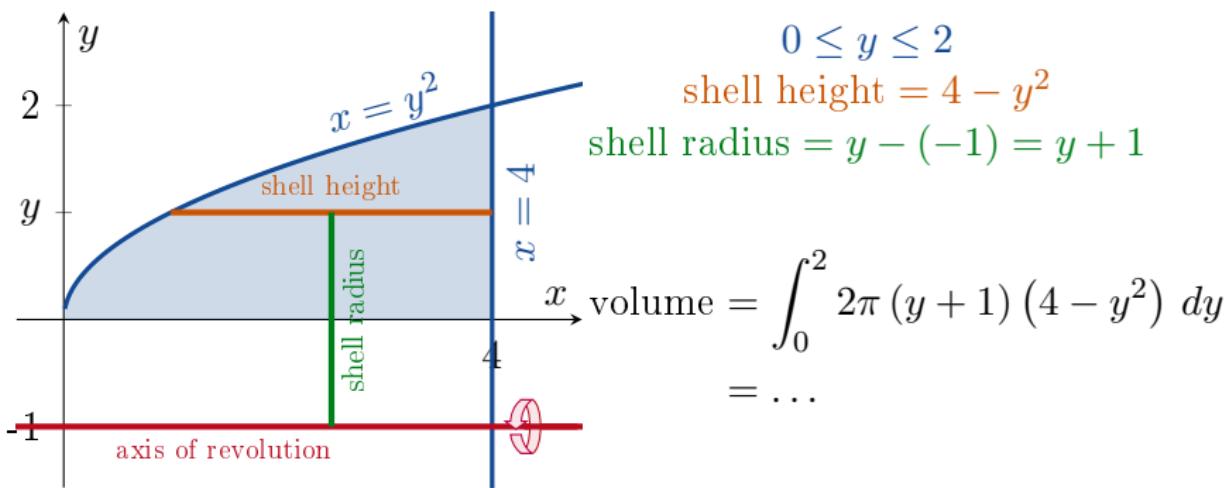
6.2 Volumes Using Cylindrical Shells

$$\text{volume} = \int_a^b 2\pi (\text{radius})(\text{height}) \, dy$$

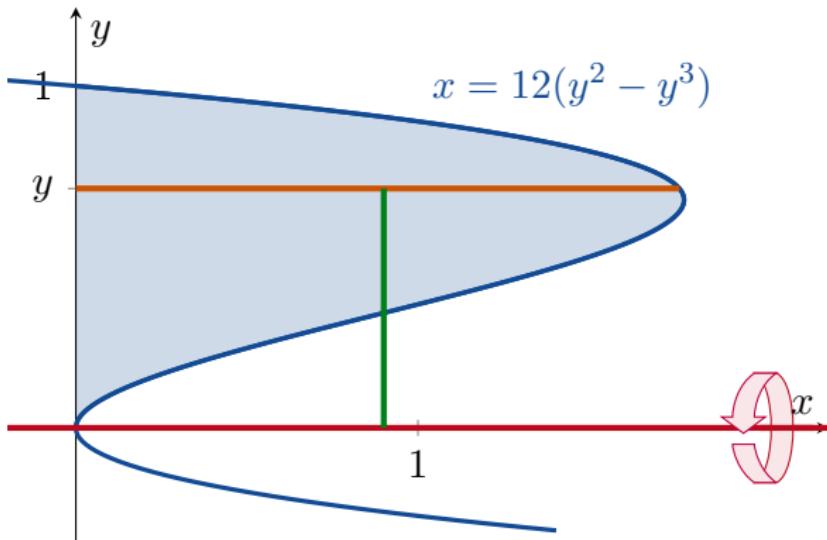


Example

The region bounded by the curve $y = \sqrt{x}$, the x -axis and the line $x = 4$ is revolved about the line $y = -1$ to generate a solid. Find the volume of the solid.



6.2 Volumes Using Cylindrical Shells



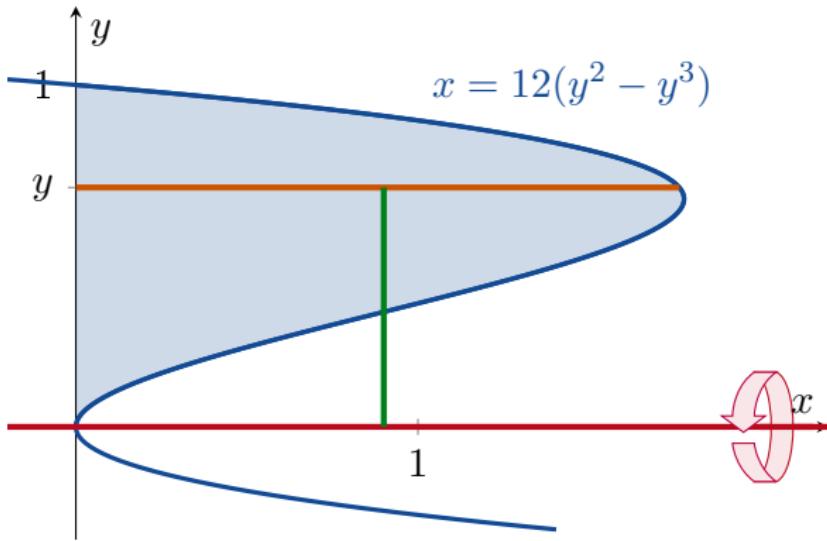
1 shell height = $12y^3$
shell radius = $12y^2$

3 shell height = $12(y^2 - y^3)$
shell radius = y

2 shell height = y
shell radius = $12(y^2 - y^3)$

4 shell height = $12y^2$
shell radius = $12y^3$

6.2 Volumes Using Cylindrical Shells



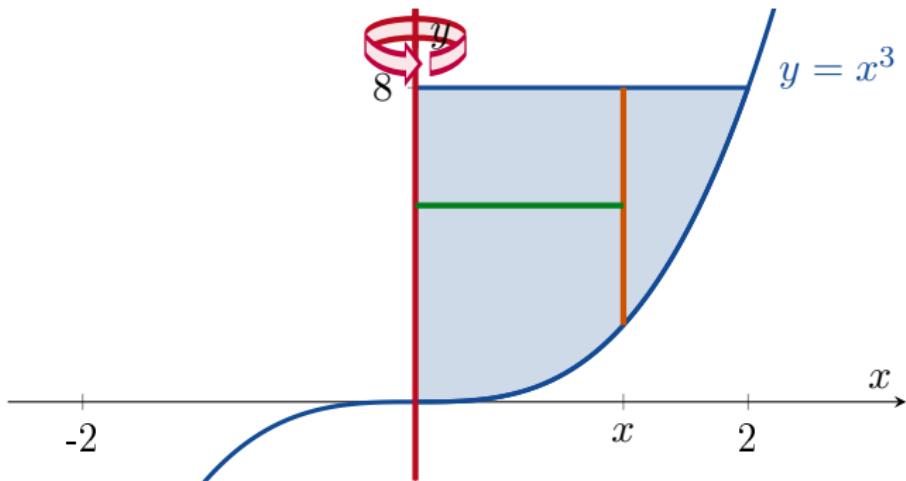
1 shell height = $12y^3$
shell radius = $12y^2$

2 shell height = y
shell radius = $12(y^2 - y^3)$

3 shell height = $12(y^2 - y^3)$
shell radius = y

4 shell height = $12y^2$
shell radius = $12y^3$

6.2 Volumes Using Cylindrical Shells



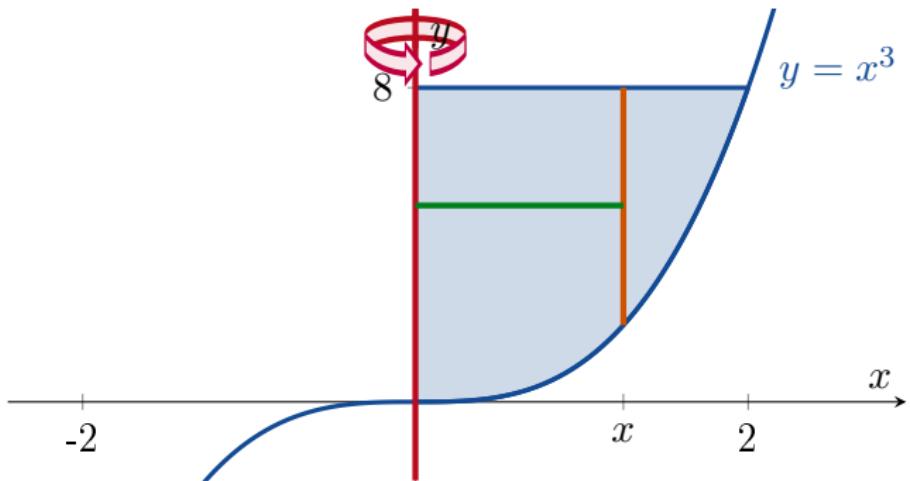
1 shell height = $8 - x^2$
shell radius = x

3 shell height = $8 - x^2$
shell radius = $x + 2$

2 shell height = $y^{\frac{1}{3}}$
shell radius = $8 - y$

4 shell height = $y^{\frac{1}{3}}$
shell radius = y

6.2 Volumes Using Cylindrical Shells



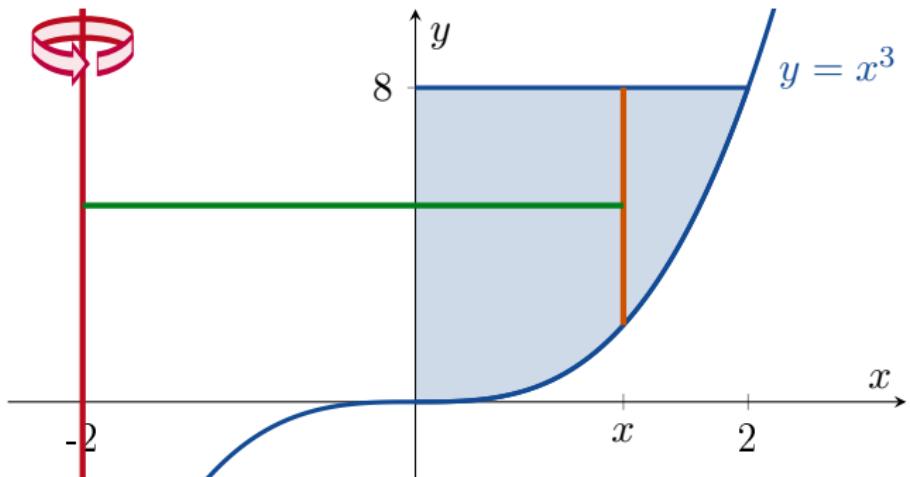
1 shell height = $8 - x^2$
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shell radius = $x + 2$

2 shell height = $y^{\frac{1}{3}}$
shell radius = $8 - y$

4 shell height = $y^{\frac{1}{3}}$
shell radius = y

6.2 Volumes Using Cylindrical Shells



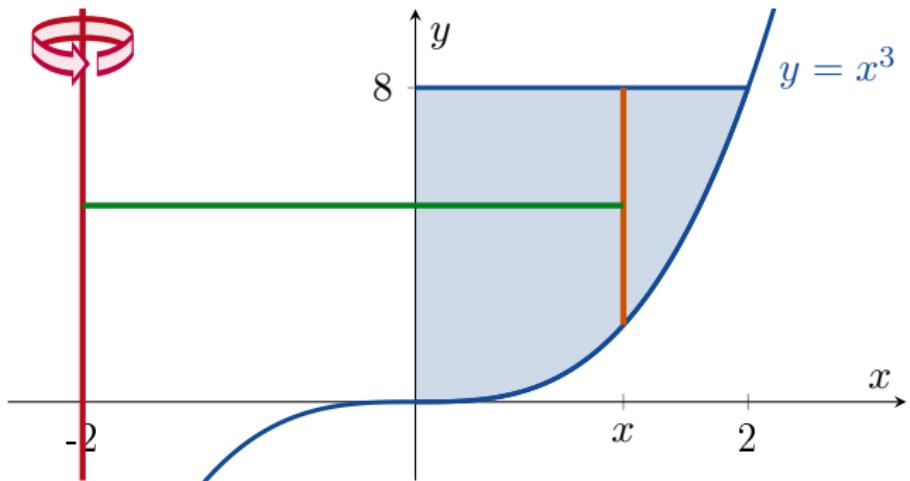
1 shell height = $8 - x^2$
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shell radius = $x + 2$

4 shell height = $y^{\frac{1}{3}}$
shell radius = y

6.2 Volumes Using Cylindrical Shells



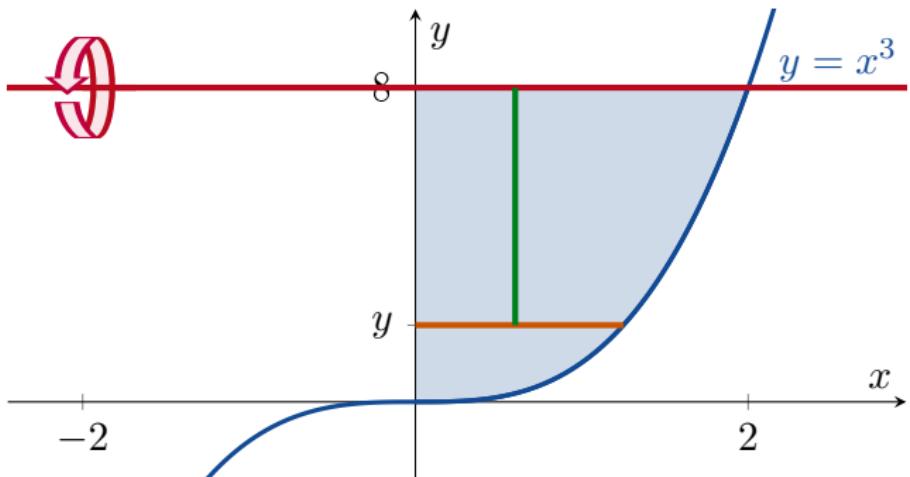
1 shell height = $8 - x^2$
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2 shell height = $y^{\frac{1}{3}}$
shell radius = $8 - y$

3 shell height = $8 - x^2$
shell radius = $x + 2$

4 shell height = $y^{\frac{1}{3}}$
shell radius = y

6.2 Volumes Using Cylindrical Shells



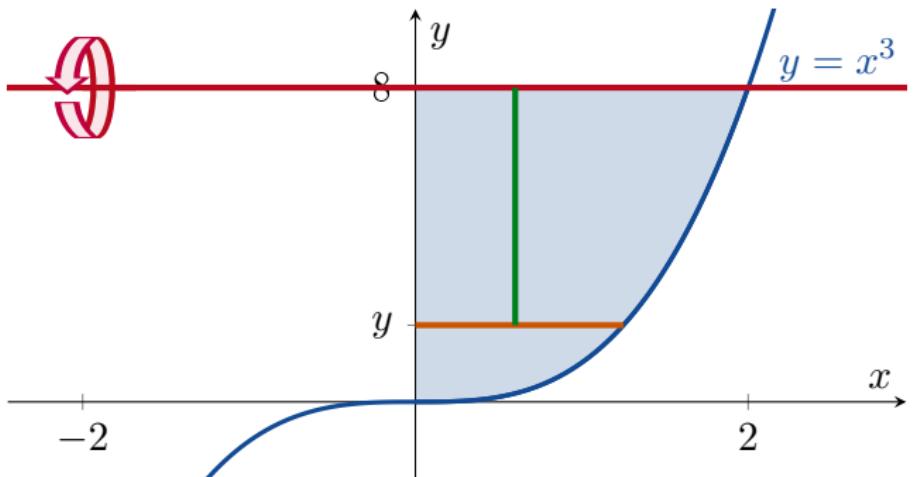
1 shell height = $8 - x^2$
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shell radius = $x + 2$

2 shell height = $y^{\frac{1}{3}}$
shell radius = $8 - y$

4 shell height = $y^{\frac{1}{3}}$
shell radius = y

6.2 Volumes Using Cylindrical Shells



1 shell height $= 8 - x^2$
shell radius $= x$

3 shell height $= 8 - x^2$
shell radius $= x + 2$

2 shell height $= y^{\frac{1}{3}}$
shell radius $= 8 - y$

4 shell height $= y^{\frac{1}{3}}$
shell radius $= y$



Break

We will continue at 2pm



KEEP
CALM
AND
PASS
CALCULUS



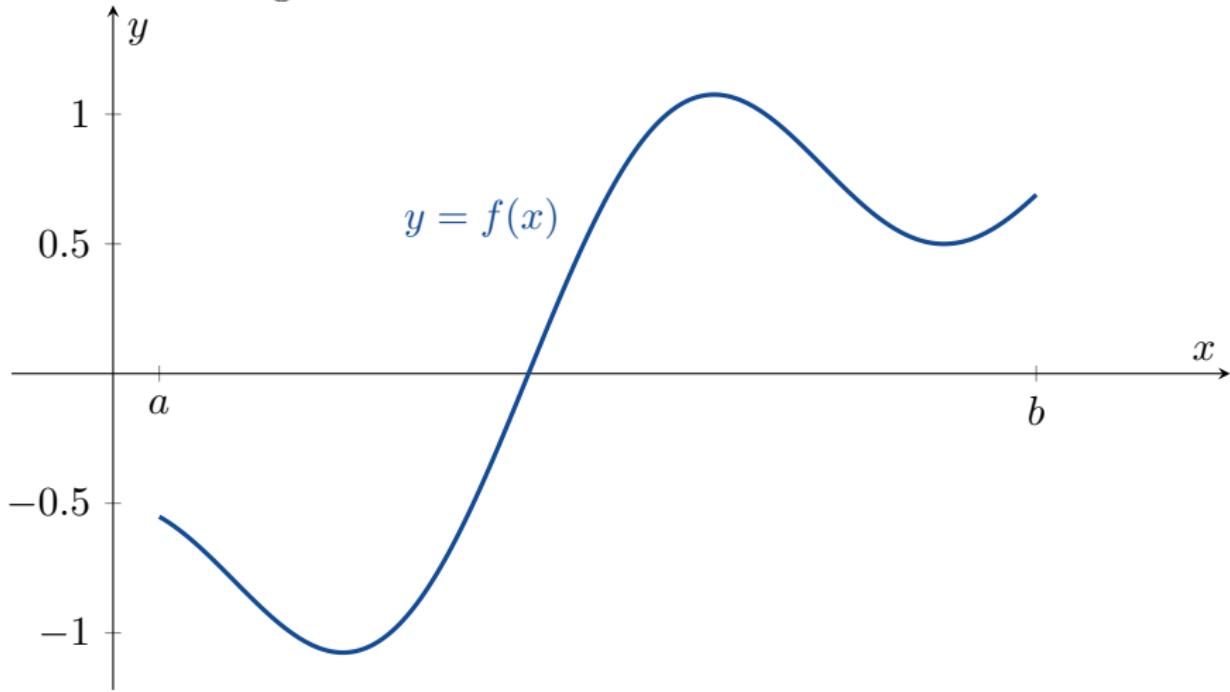
Arc Length



6.3 Arc Length



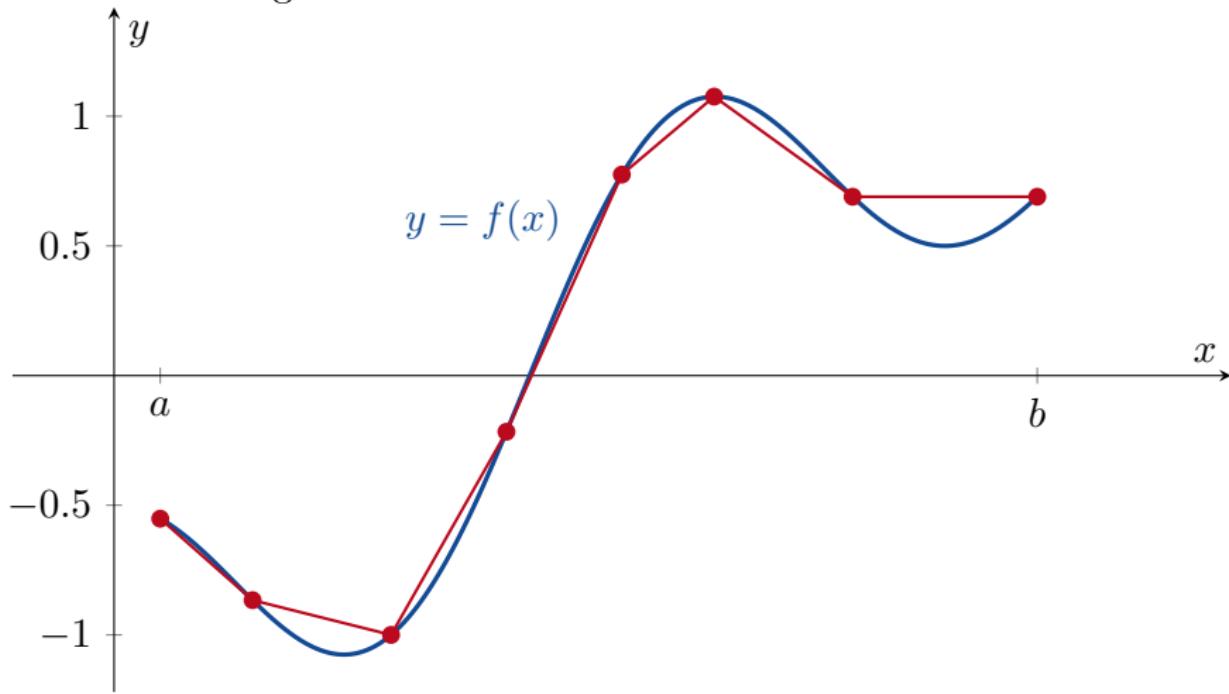
What is the length of this curve?



6.3 Arc Length

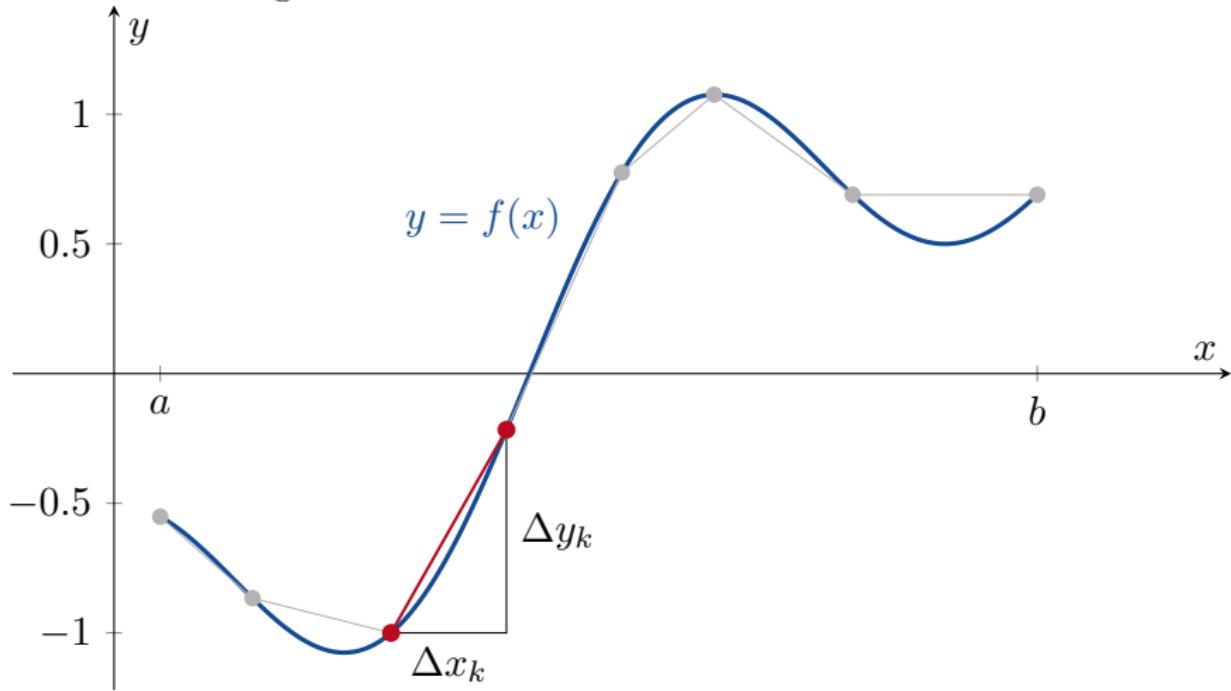


What is the length of this curve?



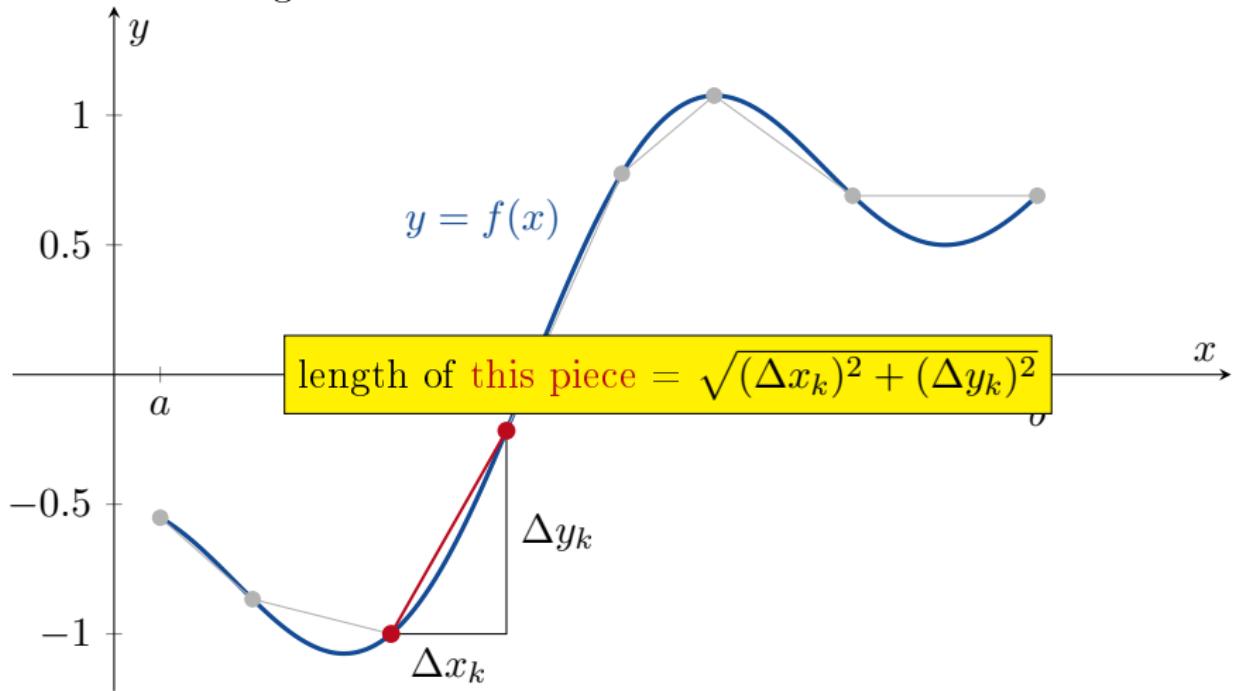
6.3 Arc Length

What is the length of this curve?



6.3 Arc Length

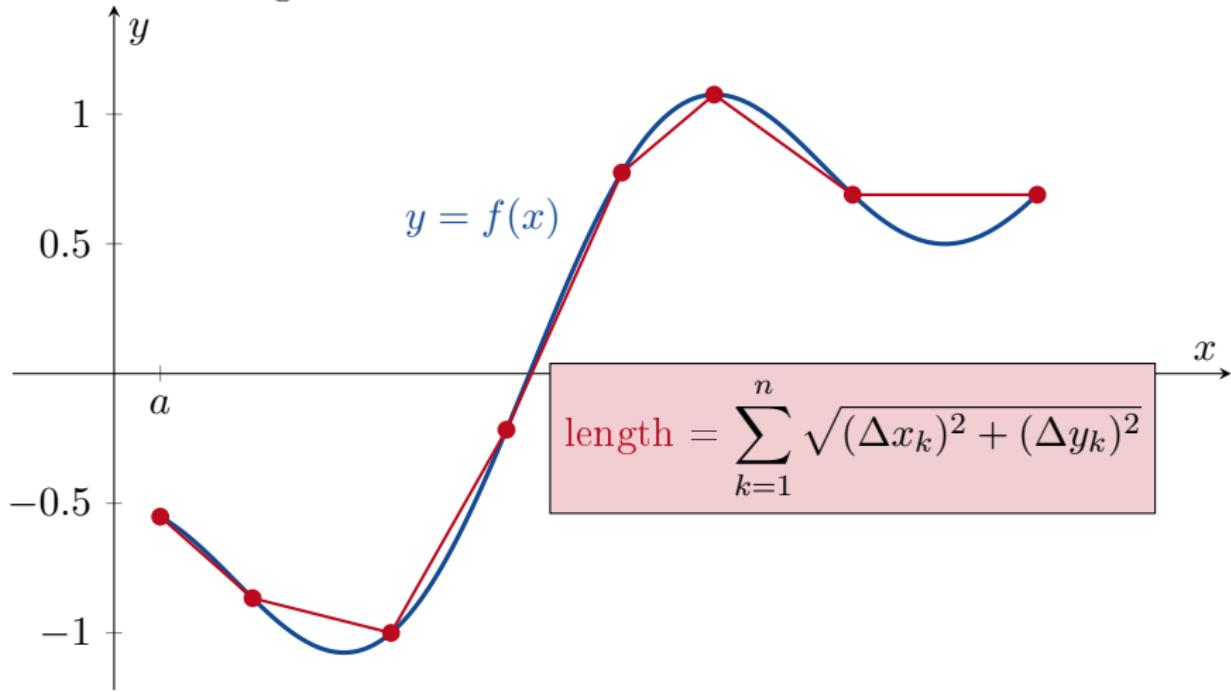
What is the length of this curve?



6.3 Arc Length



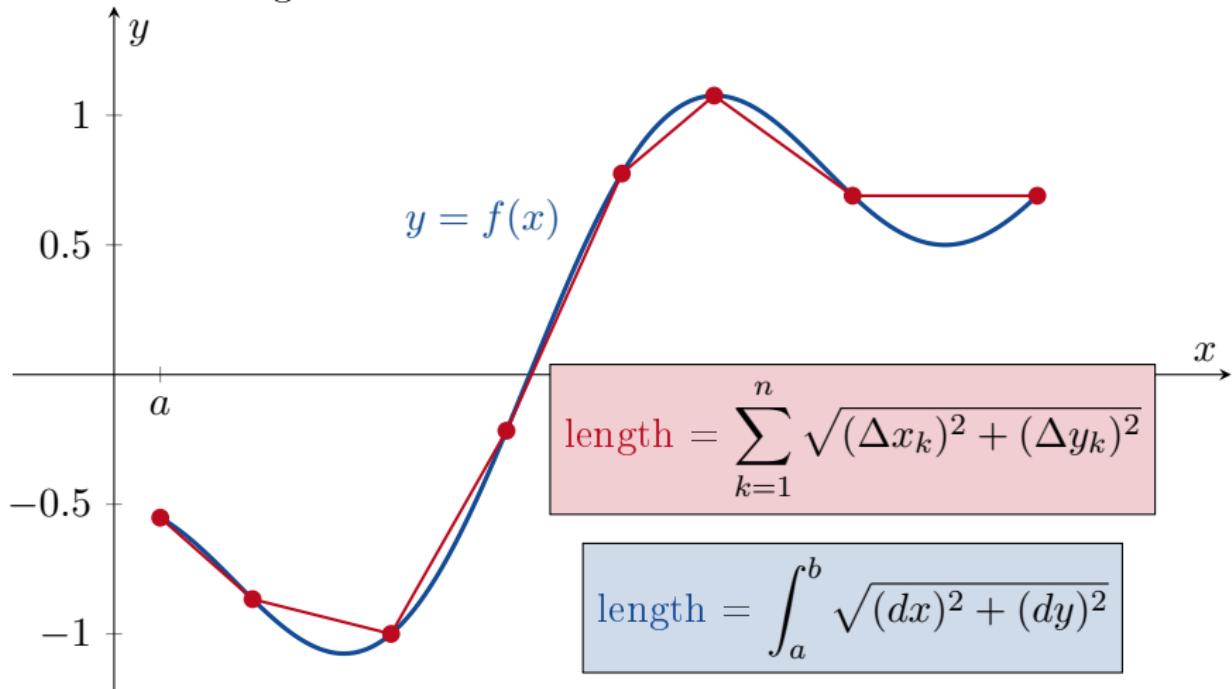
What is the length of this curve?



6.3 Arc Length



What is the length of this curve?



6.3 Arc Length

OK, so

$$\text{length} = \int_a^b \sqrt{(dx)^2 + (dy)^2}$$

but what is $\sqrt{(dx)^2 + (dy)^2}$?

6.3 Arc Length

OK, so

$$\text{length} = \int_a^b \sqrt{(dx)^2 + (dy)^2}$$

but what is $\sqrt{(dx)^2 + (dy)^2}$?

$$\sqrt{(dx)^2 + (dy)^2} = \sqrt{(dx)^2 \left(1 + \frac{(dy)^2}{(dx)^2}\right)}$$

=

=

6.3 Arc Length

OK, so

$$\text{length} = \int_a^b \sqrt{(dx)^2 + (dy)^2}$$

but what is $\sqrt{(dx)^2 + (dy)^2}$?

$$\begin{aligned}\sqrt{(dx)^2 + (dy)^2} &= \sqrt{(dx)^2 \left(1 + \frac{(dy)^2}{(dx)^2}\right)} \\ &= \sqrt{(dx)^2 \left(1 + \left(\frac{dy}{dx}\right)^2\right)}\end{aligned}$$

=

6.3 Arc Length

OK, so

$$\text{length} = \int_a^b \sqrt{(dx)^2 + (dy)^2}$$

but what is $\sqrt{(dx)^2 + (dy)^2}$?

$$\begin{aligned}\sqrt{(dx)^2 + (dy)^2} &= \sqrt{(dx)^2 \left(1 + \frac{(dy)^2}{(dx)^2}\right)} \\ &= \sqrt{(dx)^2 \left(1 + \left(\frac{dy}{dx}\right)^2\right)} \\ &= \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx\end{aligned}$$

6.3 Arc Length



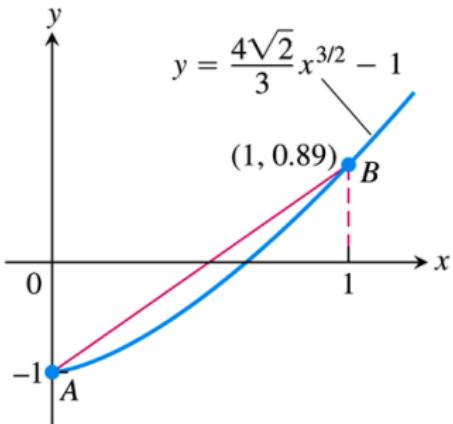
Definition

If f' is continuous on $[a, b]$, then the *length* of the curve $y = f(x)$ from $x = a$ to $x = b$ is

$$\text{length} = \int_a^b \sqrt{1 + (f'(x))^2} dx = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx.$$

6.3 Arc Length

$$\text{length} = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx$$



Example

Find the length of the graph of $y = \frac{4\sqrt{2}}{3}x^{\frac{3}{2}} - 1$ for $0 \leq x \leq 1$.

6.3 Arc Length

$$\text{length} = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx$$



We calculate that

$$y = \frac{4\sqrt{2}}{3}x^{\frac{3}{2}} - 1$$

$$\frac{dy}{dx} =$$

$$\left(\frac{dy}{dx}\right)^2 =$$

$$\text{length} = \int_0^1 \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx =$$

6.3 Arc Length

$$\text{length} = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx$$



We calculate that

$$y = \frac{4\sqrt{2}}{3}x^{\frac{3}{2}} - 1$$

$$\frac{dy}{dx} = \frac{4\sqrt{2}}{3} \cdot \frac{3}{2}x^{\frac{1}{2}} = 2\sqrt{2}x^{\frac{1}{2}}$$

$$\left(\frac{dy}{dx}\right)^2 =$$

$$\text{length} = \int_0^1 \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx =$$

6.3 Arc Length

$$\text{length} = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx$$



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$$y = \frac{4\sqrt{2}}{3}x^{\frac{3}{2}} - 1$$

$$\frac{dy}{dx} = \frac{4\sqrt{2}}{3} \cdot \frac{3}{2}x^{\frac{1}{2}} = 2\sqrt{2}x^{\frac{1}{2}}$$

$$\left(\frac{dy}{dx}\right)^2 = \left(2\sqrt{2}x^{\frac{1}{2}}\right)^2 = 8x$$

$$\text{length} = \int_0^1 \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx =$$

6.3 Arc Length

$$\text{length} = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx$$



We calculate that

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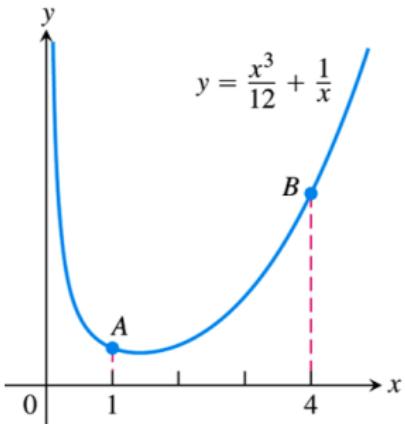
$$\frac{dy}{dx} = \frac{4\sqrt{2}}{3} \cdot \frac{3}{2}x^{\frac{1}{2}} = 2\sqrt{2}x^{\frac{1}{2}}$$

$$\left(\frac{dy}{dx}\right)^2 = \left(2\sqrt{2}x^{\frac{1}{2}}\right)^2 = 8x$$

$$\text{length} = \int_0^1 \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx = \int_0^1 \sqrt{1 + 8x} dx = \dots$$

6.3 Arc Length

$$\text{length} = \int_a^b \sqrt{1 + (f'(x))^2} dx$$



Example

Find the length of the graph of $f(x) = \frac{x^3}{12} + \frac{1}{x}$ for $1 \leq x \leq 4$.

6.3 Arc Length

$$\text{length} = \int_a^b \sqrt{1 + (f'(x))^2} dx$$



We calculate that

$$f(x) = \frac{x^3}{12} + \frac{1}{x}$$

$$f'(x) =$$

$$1 + (f'(x))^2 =$$

$$\text{length} = \int_1^4 \sqrt{1 + (f'(x))^2} dx =$$

6.3 Arc Length

$$\text{length} = \int_a^b \sqrt{1 + (f'(x))^2} dx$$



We calculate that

$$f(x) = \frac{x^3}{12} + \frac{1}{x}$$

$$f'(x) = \frac{x^2}{4} - \frac{1}{x^2}$$

$$1 + (f'(x))^2 =$$

$$\text{length} = \int_1^4 \sqrt{1 + (f'(x))^2} dx =$$

6.3 Arc Length

$$\text{length} = \int_a^b \sqrt{1 + (f'(x))^2} dx$$



We calculate that

$$f(x) = \frac{x^3}{12} + \frac{1}{x}$$

$$f'(x) = \frac{x^2}{4} - \frac{1}{x^2}$$

$$1 + (f'(x))^2 = 1 + \left(\frac{x^2}{4} - \frac{1}{x^2} \right)^2 = 1 + \left(\frac{x^4}{16} - \frac{1}{2} + \frac{1}{x^4} \right)$$

$$= \frac{x^4}{16} + \frac{1}{2} + \frac{1}{x^4} = \left(\frac{x^2}{4} + \frac{1}{x^2} \right)^2$$

$$\text{length} = \int_1^4 \sqrt{1 + (f'(x))^2} dx =$$

6.3 Arc Length

$$\text{length} = \int_a^b \sqrt{1 + (f'(x))^2} dx$$



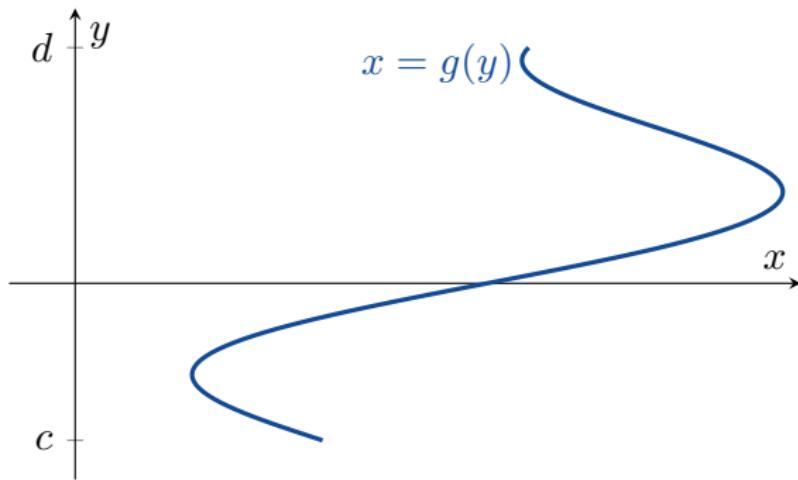
We calculate that

$$f(x) = \frac{x^3}{12} + \frac{1}{x}$$

$$f'(x) = \frac{x^2}{4} - \frac{1}{x^2}$$

$$\begin{aligned}1 + (f'(x))^2 &= 1 + \left(\frac{x^2}{4} - \frac{1}{x^2} \right)^2 = 1 + \left(\frac{x^4}{16} - \frac{1}{2} + \frac{1}{x^4} \right) \\&= \frac{x^4}{16} + \frac{1}{2} + \frac{1}{x^4} = \left(\frac{x^2}{4} + \frac{1}{x^2} \right)^2 \\ \text{length} &= \int_1^4 \sqrt{1 + (f'(x))^2} dx = \int_1^4 \left(\frac{x^2}{4} + \frac{1}{x^2} \right) dx = \dots\end{aligned}$$

6.3 Arc Length

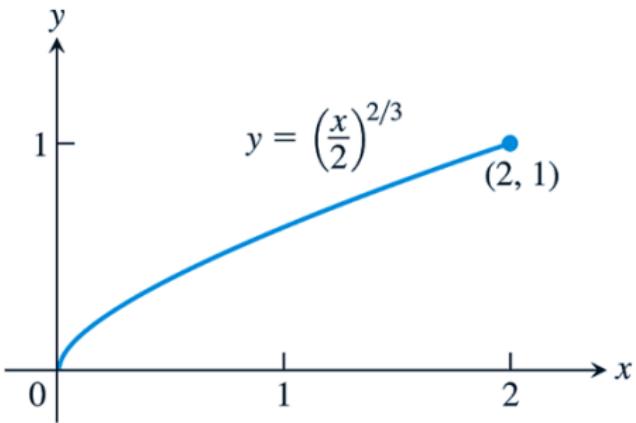


Definition

If g' is continuous on $[c, d]$, then the *length* of the curve $x = g(y)$ from $y = c$ to $y = d$ is

$$\text{length} = \int_c^d \sqrt{1 + (g'(y))^2} dy = \int_c^d \sqrt{1 + \left(\frac{dx}{dy}\right)^2} dy.$$

6.3 Arc Length



Example

Find the length of the graph of $y = \left(\frac{x}{2}\right)^{\frac{2}{3}}$ for $0 \leq x \leq 2$.

6.3 Arc Length

Note first that if $y = \left(\frac{x}{2}\right)^{\frac{2}{3}}$ then

$$\frac{dy}{dx} = \frac{2}{3} \left(\frac{x}{2}\right)^{-\frac{1}{3}} \left(\frac{1}{2}\right) = \frac{1}{3} \left(\frac{2}{x}\right)^{\frac{1}{3}}$$

is not defined at $x = 0$.

6.3 Arc Length

Note first that if $y = \left(\frac{x}{2}\right)^{\frac{2}{3}}$ then

$$\frac{dy}{dx} = \frac{2}{3} \left(\frac{x}{2}\right)^{-\frac{1}{3}} \left(\frac{1}{2}\right) = \frac{1}{3} \left(\frac{2}{x}\right)^{\frac{1}{3}}$$

is not defined at $x = 0$. This means that we can not use the formula

$$\text{length} = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx.$$

6.3 Arc Length

Note first that if $y = \left(\frac{x}{2}\right)^{\frac{2}{3}}$ then

$$\frac{dy}{dx} = \frac{2}{3} \left(\frac{x}{2}\right)^{-\frac{1}{3}} \left(\frac{1}{2}\right) = \frac{1}{3} \left(\frac{2}{x}\right)^{\frac{1}{3}}$$

is not defined at $x = 0$. This means that we can not use the formula

$$\text{length} = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx.$$

Instead, we need to use

$$\text{length} = \int_c^d \sqrt{1 + \left(\frac{dx}{dy}\right)^2} dy.$$

6.3 Arc Length

$$\text{length} = \int_c^d \sqrt{1 + \left(\frac{dx}{dy}\right)^2} dy$$



First we calculate that

$$y = \left(\frac{x}{2}\right)^{\frac{2}{3}} \implies y^{\frac{3}{2}} = \frac{x}{2} \implies x = 2y^{\frac{3}{2}}.$$

6.3 Arc Length

$$\text{length} = \int_c^d \sqrt{1 + \left(\frac{dx}{dy}\right)^2} dy$$



First we calculate that

$$y = \left(\frac{x}{2}\right)^{\frac{2}{3}} \implies y^{\frac{3}{2}} = \frac{x}{2} \implies x = 2y^{\frac{3}{2}}.$$

Then we differentiate

$$\frac{dx}{dy} = 2 \left(\frac{3}{2}\right) y^{\frac{1}{2}} = 3y^{\frac{1}{2}}.$$

6.3 Arc Length

$$\text{length} = \int_c^d \sqrt{1 + \left(\frac{dx}{dy}\right)^2} dy$$



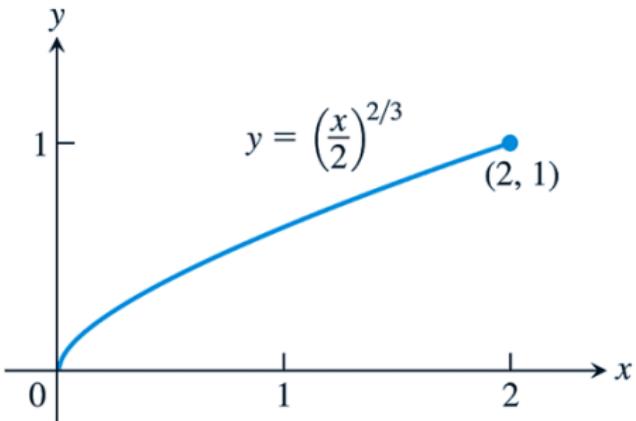
First we calculate that

$$y = \left(\frac{x}{2}\right)^{\frac{2}{3}} \implies y^{\frac{3}{2}} = \frac{x}{2} \implies x = 2y^{\frac{3}{2}}.$$

Then we differentiate

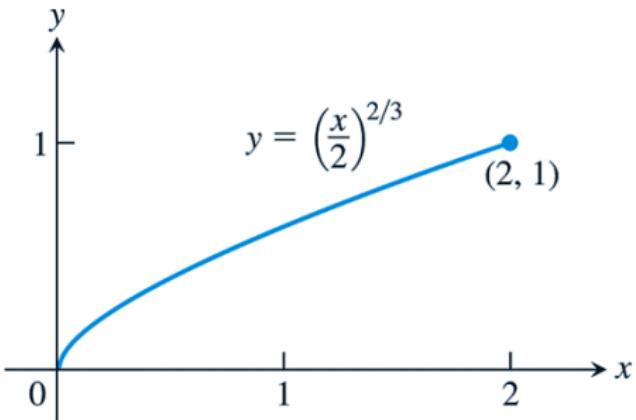
$$\frac{dx}{dy} = 2 \left(\frac{3}{2}\right) y^{\frac{1}{2}} = 3y^{\frac{1}{2}}.$$

This function is continuous on $[0, 1]$.



6.3 Arc Length

$$\text{length} = \int_c^d \sqrt{1 + \left(\frac{dx}{dy}\right)^2} dy$$



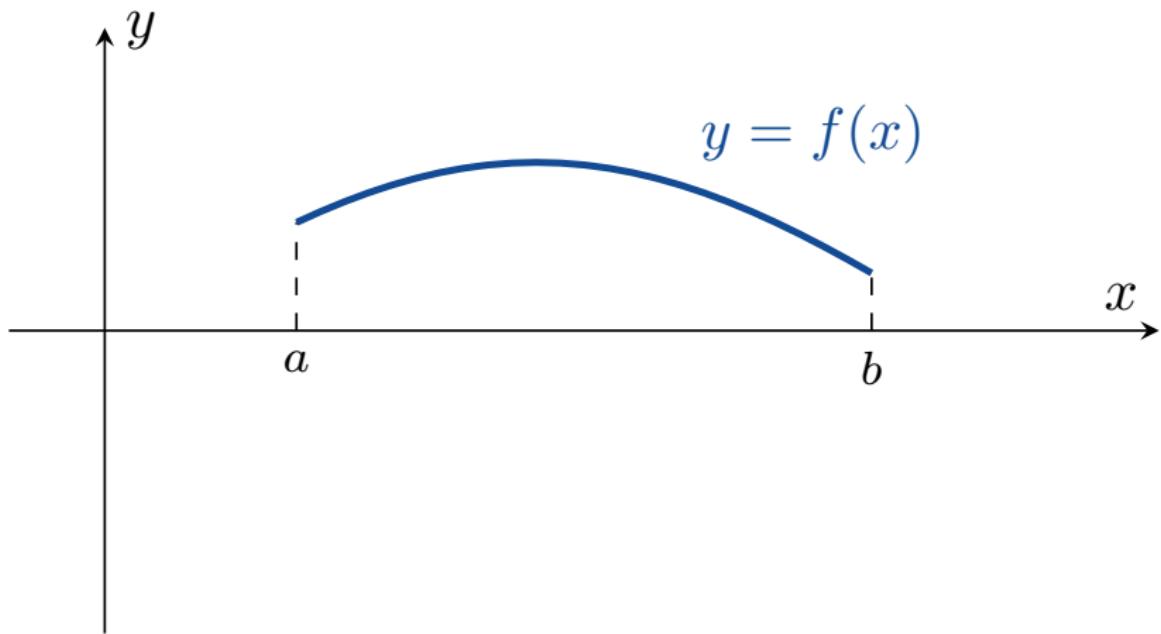
Therefore

$$\text{length} = \int_c^d \sqrt{1 + \left(\frac{dx}{dy}\right)^2} dy = \int_0^1 \sqrt{1 + 9y} dy = \dots$$

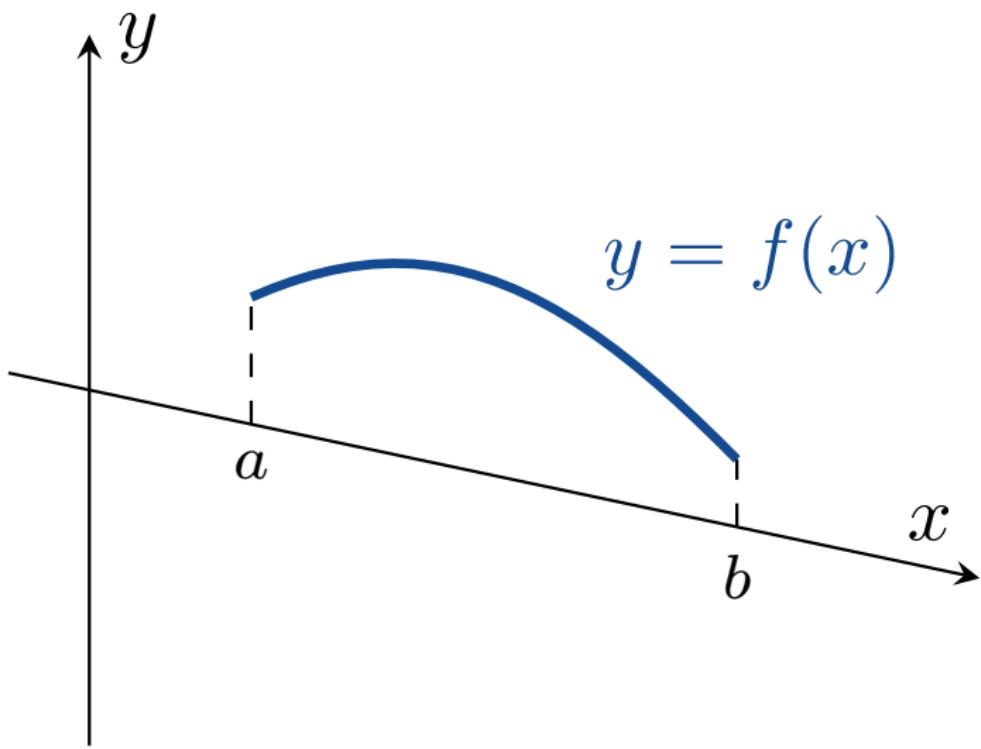


Areas of Surfaces of Revolution

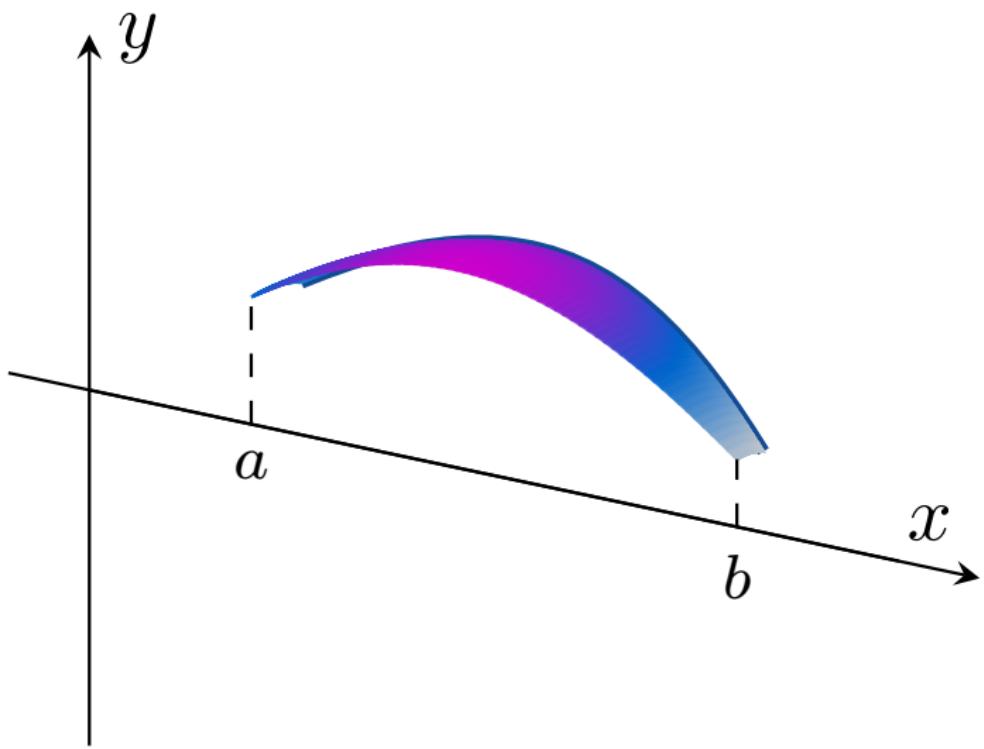
6.4 Areas of Surfaces of Revolution



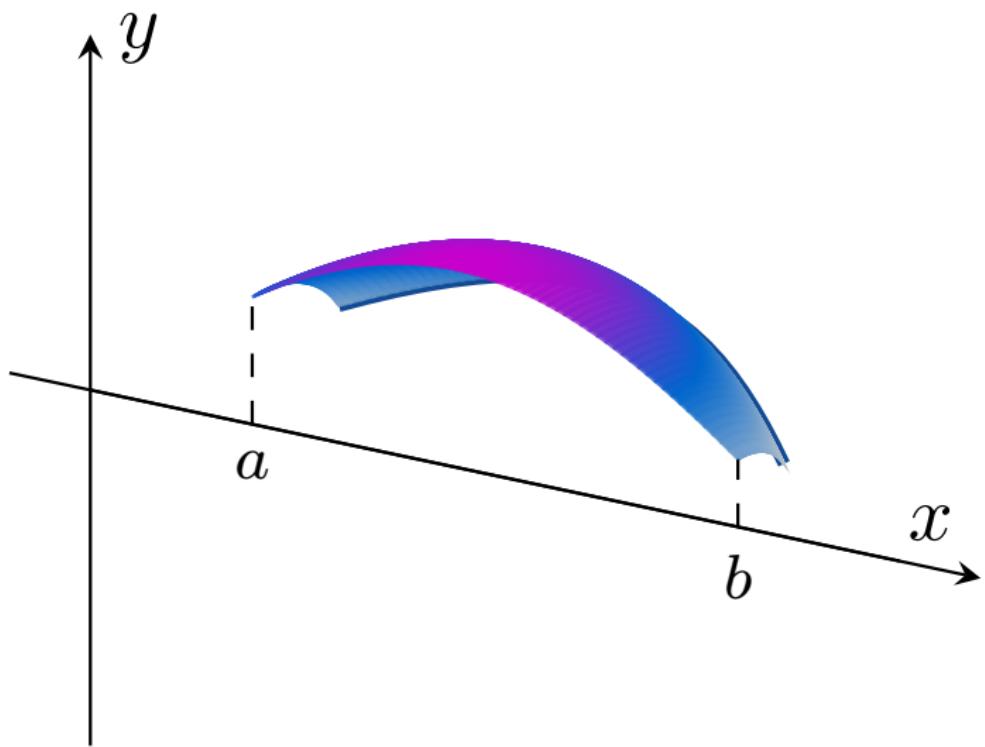
6.4 Areas of Surfaces of Revolution



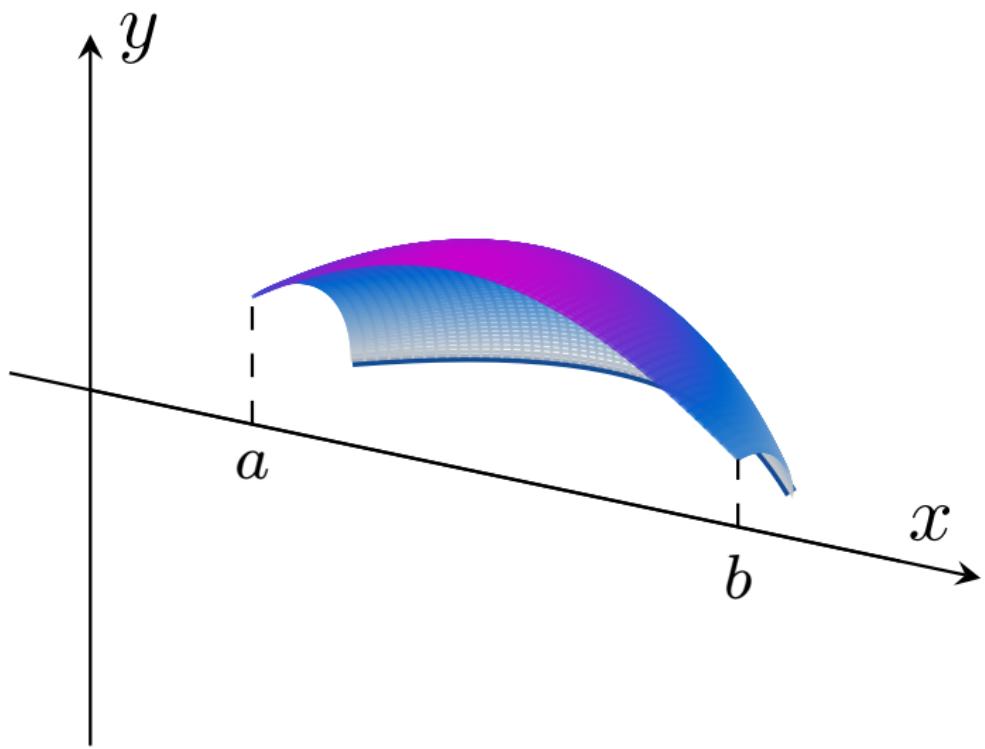
6.4 Areas of Surfaces of Revolution



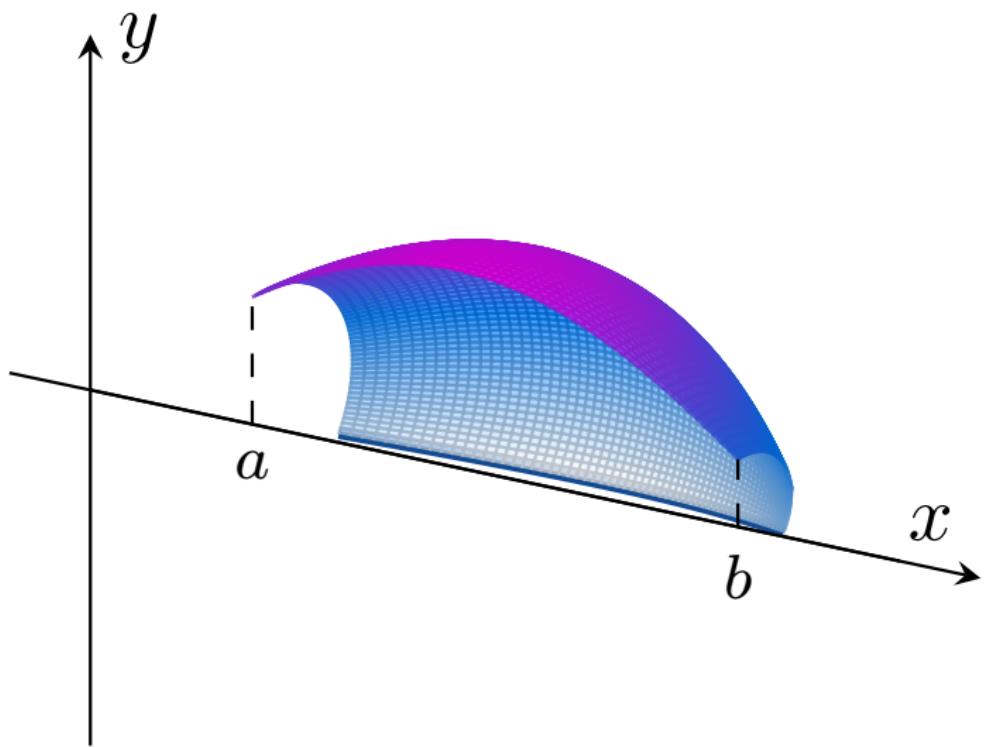
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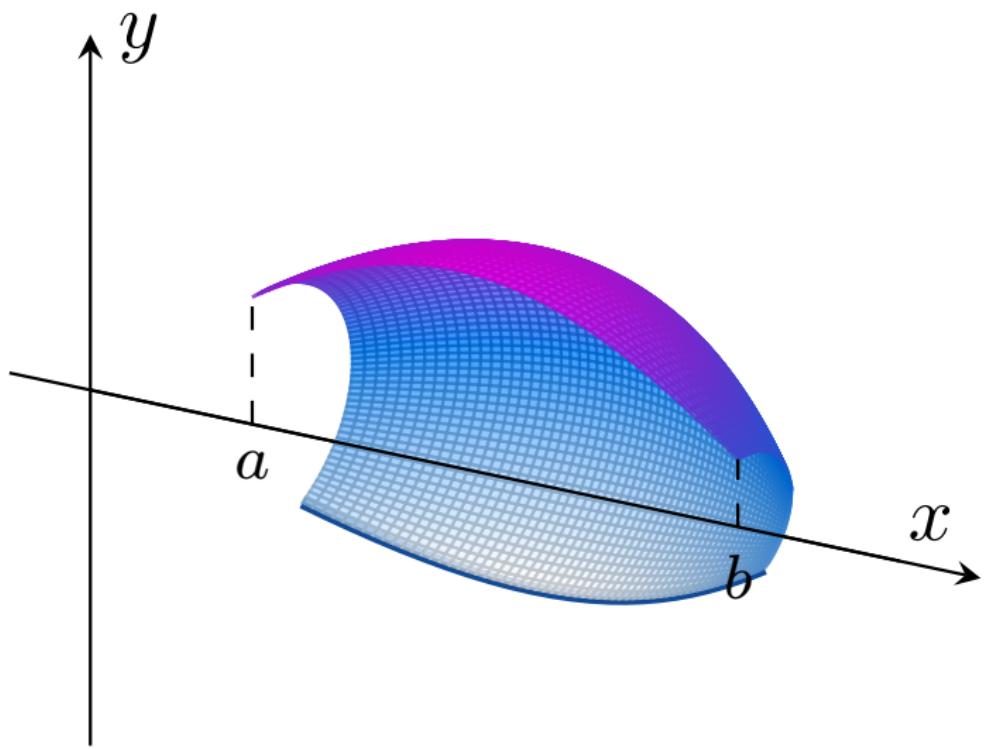
6.4 Areas of Surfaces of Revolution



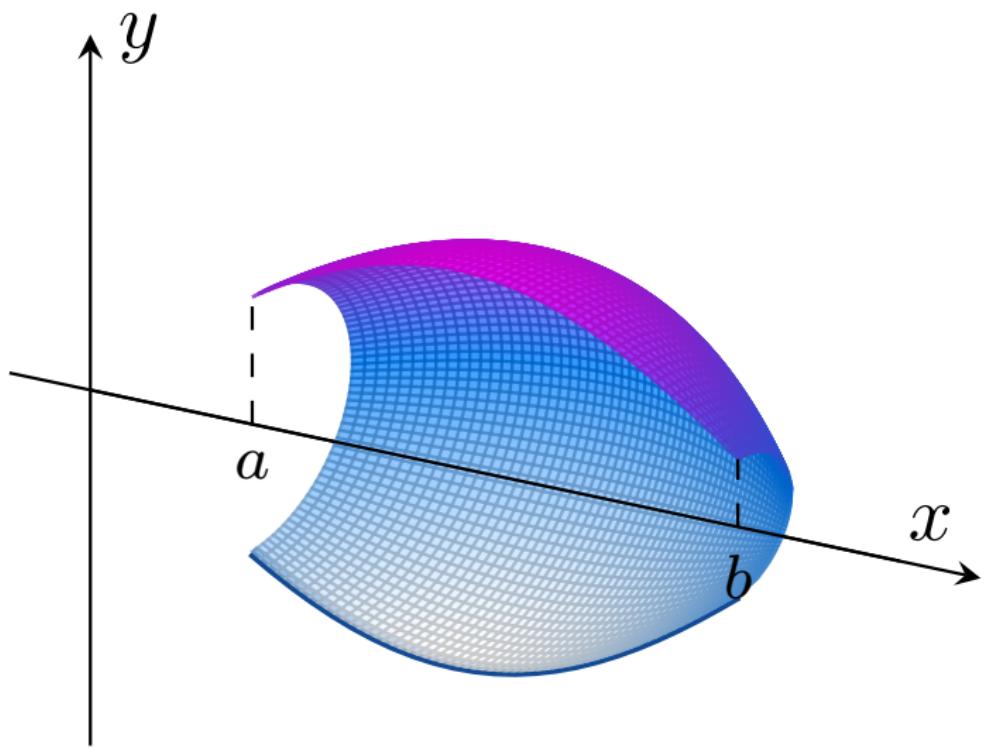
6.4 Areas of Surfaces of Revolution



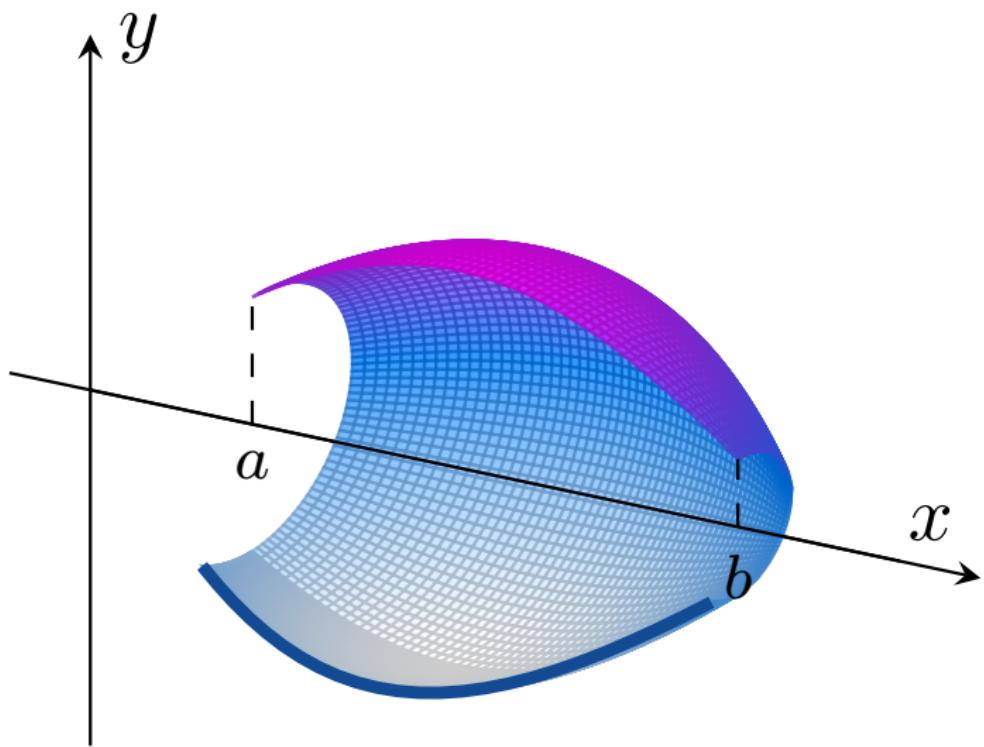
6.4 Areas of Surfaces of Revolution



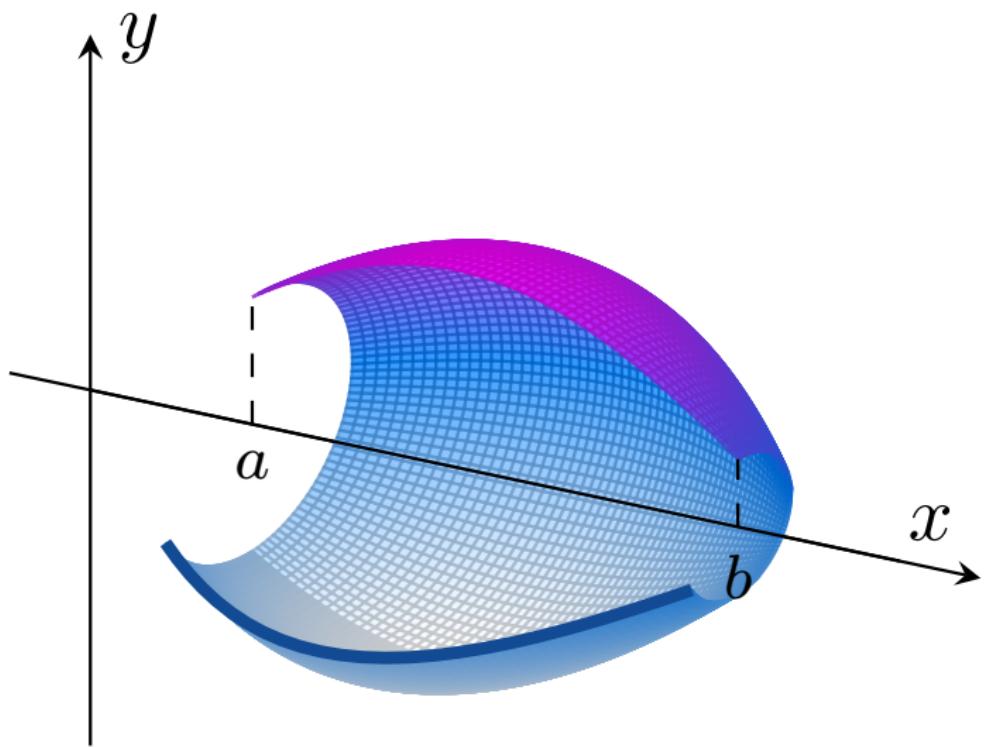
6.4 Areas of Surfaces of Revolution



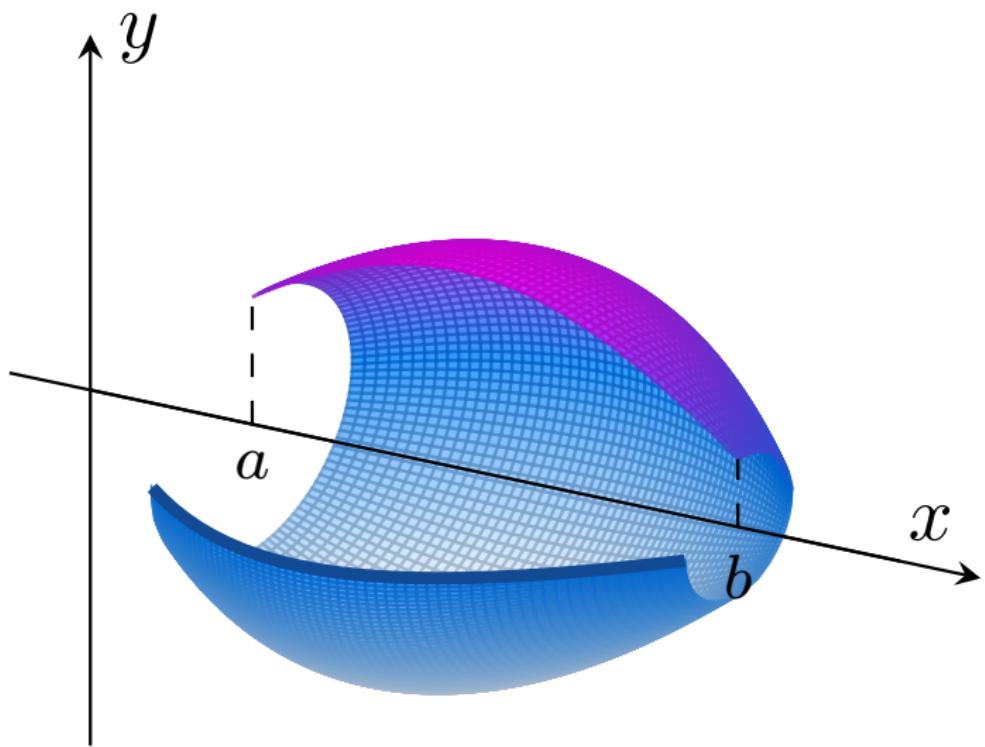
6.4 Areas of Surfaces of Revolution



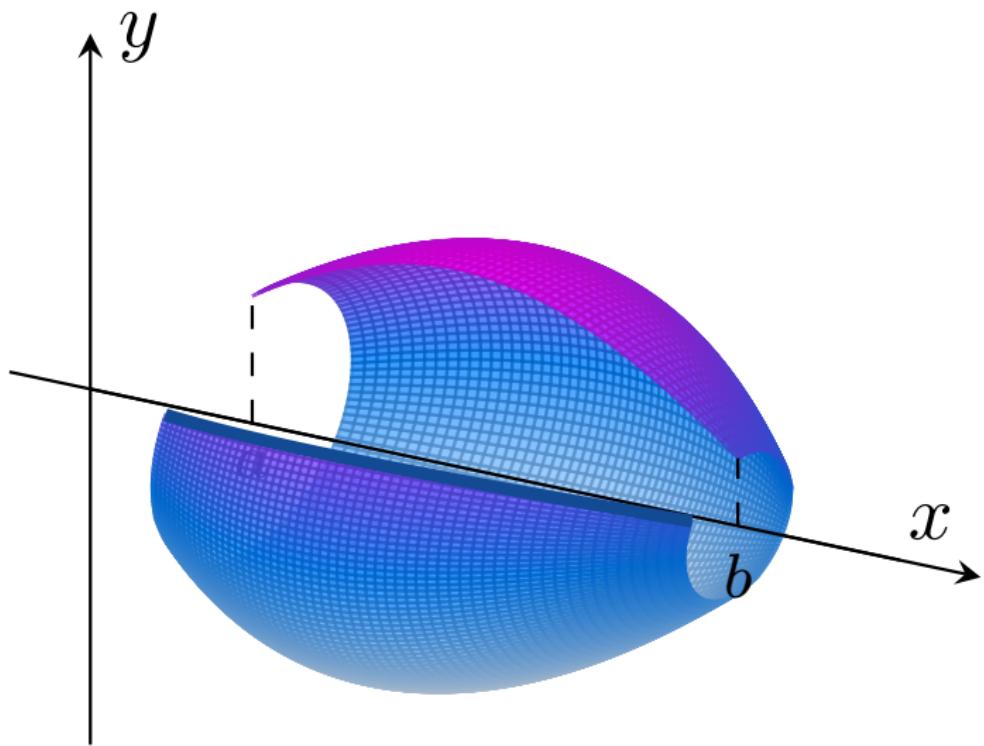
6.4 Areas of Surfaces of Revolution



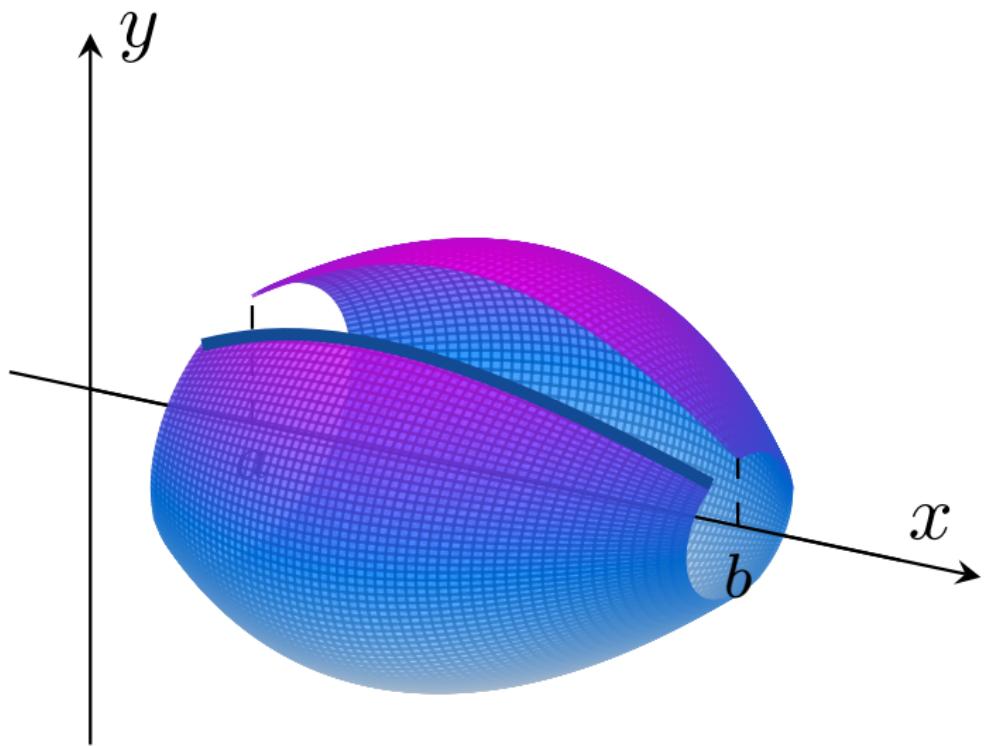
6.4 Areas of Surfaces of Revolution



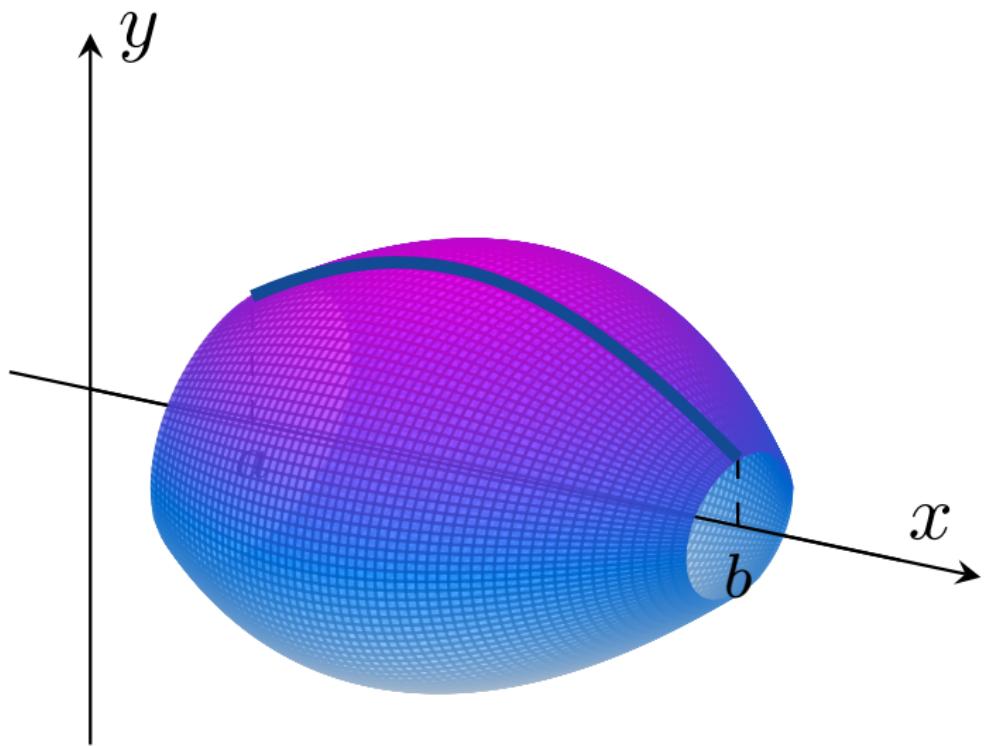
6.4 Areas of Surfaces of Revolution



6.4 Areas of Surfaces of Revolution



6.4 Areas of Surfaces of Revolution



$$\text{length} = \int_c^d \sqrt{1 + \left(\frac{dx}{dy}\right)^2} dy$$



Revolution About the x -Axis

This time I will just tell you the formula without showing how to derive it.

6.4 Areas of Surfaces of Revolution

$$\text{length} = \int_c^d \sqrt{1 + \left(\frac{dx}{dy}\right)^2} dy$$



Revolution About the x -Axis

This time I will just tell you the formula without showing how to derive it.

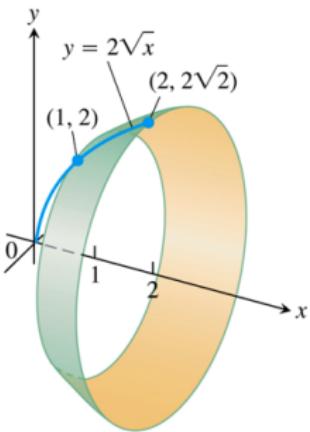
Definition

If the function $f(x) \geq 0$ is continuously differentiable on $[a, b]$, the *area of the surface* generated by revolving the graph $y = f(x)$ about the x -axis is

$$\text{surface area} = \int_a^b 2\pi y \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx.$$

6.4 Areas of Surfaces

$$\text{surface area} = \int_a^b 2\pi y \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx$$

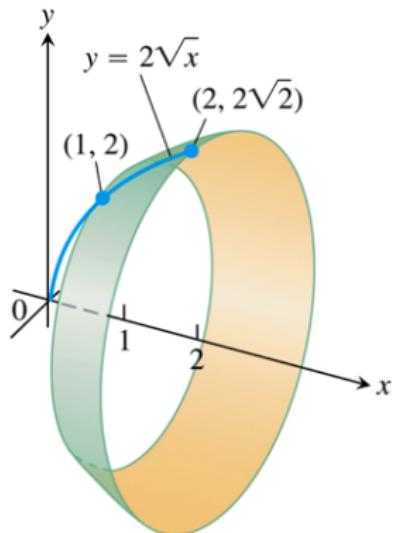


Example

Find the area of the surface generated by revolving the curve $y = 2\sqrt{x}$, $1 \leq x \leq 2$, about the x -axis.

6.4 Areas of Surfaces

surface area = $\int_a^b 2\pi y \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx$



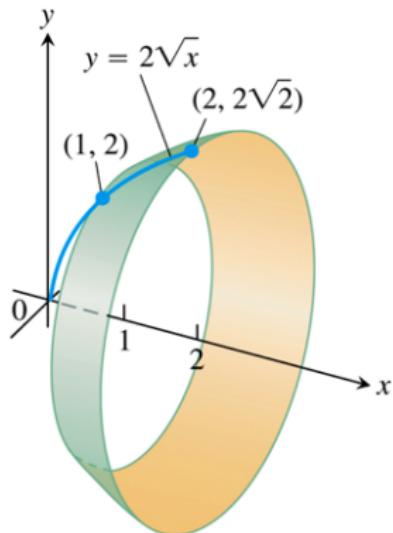
$$y = 2\sqrt{x}$$

$$\frac{dy}{dx} = \frac{1}{\sqrt{x}}$$

$$\begin{aligned}1 + \left(\frac{dy}{dx}\right)^2 &= 1 + \left(\frac{1}{\sqrt{x}}\right)^2 \\&= 1 + \frac{1}{x} = \frac{x+1}{x}\end{aligned}$$

6.4 Areas of Surfaces

$$\text{surface area} = \int_a^b 2\pi y \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx$$



$$y = 2\sqrt{x}$$

$$\frac{dy}{dx} = \frac{1}{\sqrt{x}}$$

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$$\text{surface area} = \int_a^b 2\pi y \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx = \int_1^2 2\pi \cdot 2\sqrt{x} \cdot \frac{\sqrt{x+1}}{\sqrt{x}} dx = \dots$$

6.4 Areas of Surfaces

$$\text{surface area} = \int_a^b 2\pi y \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx$$



Revolution About the y -Axis

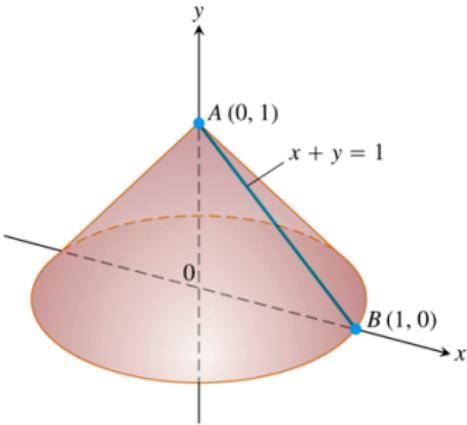
Definition

If the function $x = g(y) \geq 0$ is continuously differentiable on $[c, d]$, the *area of the surface* generated by revolving the graph $x = g(y)$ about the y -axis is

$$\text{surface area} = \int_c^d 2\pi x \sqrt{1 + \left(\frac{dx}{dy}\right)^2} dy.$$

6.4 Areas of Surfaces of Revolution

$$\text{surface area} = \int_c^d 2\pi x \sqrt{1 + \left(\frac{dx}{dy}\right)^2} dy$$

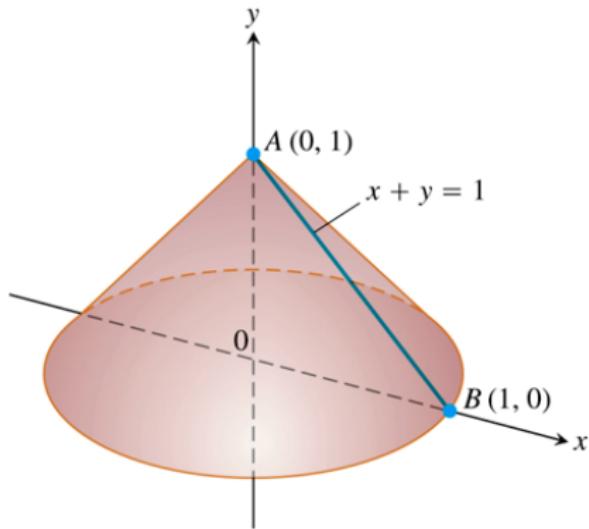


Example

The line segment $x = 1 - y$, $0 \leq y \leq 1$, is revolved about the y -axis to generate the cone shown above. Find its surface area.

6.4 Areas of Surfaces of Revolution

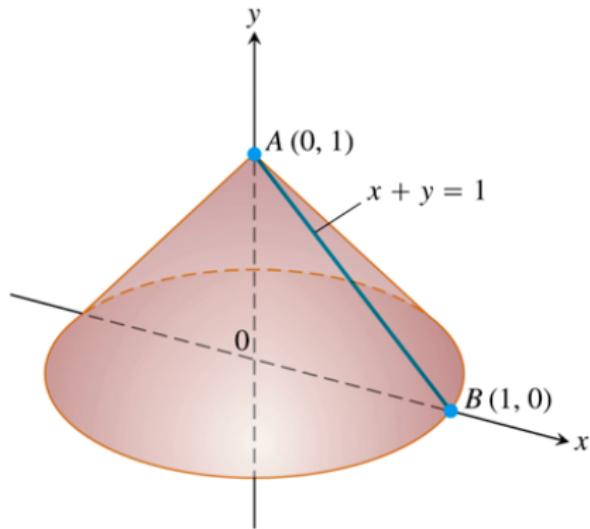
$$\text{surface area} = \int_c^d 2\pi x \sqrt{1 + \left(\frac{dx}{dy}\right)^2} dy$$



$$\begin{aligned}x &= 1 - y \\ \frac{dx}{dy} &= -1 \\ 1 + \left(\frac{dx}{dy}\right)^2 &= 1 + (-1)^2 = 2\end{aligned}$$

6.4 Areas of Surfaces of Revolution

$$\text{surface area} = \int_c^d 2\pi x \sqrt{1 + \left(\frac{dx}{dy}\right)^2} dy$$



$$x = 1 - y$$

$$\frac{dx}{dy} = -1$$

$$1 + \left(\frac{dx}{dy}\right)^2 = 1 + (-1)^2 = 2$$

$$\text{surface area} = \int_c^d 2\pi x \sqrt{1 + \left(\frac{dx}{dy}\right)^2} dy = \int_0^1 2\pi(1-y)\sqrt{2} dy = \dots$$

6.4 Areas of Surfaces of Revolution



volumes by cross sections

$$V = \int_a^b A(x) dx$$

the shell method

$$V = \int_a^b 2\pi \left(\begin{matrix} \text{shell} \\ \text{radius} \end{matrix} \right) \left(\begin{matrix} \text{shell} \\ \text{height} \end{matrix} \right) dx$$

the disk method

$$V = \int_a^b \pi \left(R(x) \right)^2 dx$$

arc length

$$L = \int_a^b \sqrt{1 + \left(\frac{dy}{dx} \right)^2} dx$$

the washer method

$$V = \int_a^b \pi \left(\left(R(x) \right)^2 - \left(r(x) \right)^2 \right) dx$$

surfaces of revolution

$$S = \int_a^b 2\pi y \sqrt{1 + \left(\frac{dy}{dx} \right)^2} dx$$



Next Time

- 7.1 Inverse Functions and Their Derivatives
- 7.2 Natural Logarithms
- 7.3 Exponential Functions