CHAPTER 1

1 Introduction and Applications

1.1 Basic Concepts and Definitions

Problems

1. Give the order of each of the following PDEs

a. $u_{xx} + u_{yy} = 0$
b. $u_{xxx} + u_{xy} + a(x)u_y + \log u = f(x,\, y)$
c. $u_{xxx} + u_{xyyy} + a(x)u_{xxy} + u^2 = f(x,\, y)$
d. $u\, u_{xx} + u_{yy}^2 + e^u = 0$
e. $u_x + cu_y = d$

2. Show that
$$u(x,\, t) = \cos(x - ct)$$
is a solution of
$$u_t + cu_x = 0$$

3. Which of the following PDEs is linear? quasilinear? nonlinear? If it is linear, state whether it is homogeneous or not.

a. $u_{xx} + u_{yy} - 2u = x^2$
b. $u_{xy} = u$
c. $u\, u_x + x\, u_y = 0$
d. $u_x^2 + \log u = 2xy$
e. $u_{xx} - 2u_{xy} + u_{yy} = \cos x$
f. $u_x(1 + u_y) = u_{xx}$
g. $(\sin u_x)u_x + u_y = e^x$
h. $2u_{xx} - 4u_{xy} + 2u_{yy} + 3u = 0$
i. $u_x + u_x u_y - u_{xy} = 0$

4. Find the general solution of
$$u_{xy} + u_y = 0$$
(Hint: Let $v = u_y$)

5. Show that
$$u = F(xy) + x\, G(\frac{y}{x})$$
is the general solution of
$$x^2 u_{xx} - y^2 u_{yy} = 0$$

1.
   a. Second order
   b. Third order
   c. Fourth order
   d. Second order
   e. First order

2. $u = \cos(x - ct)$

$$u_t = -c \cdot (-\sin(x - ct)) = c\sin(x - ct)$$
$$u_x = 1 \cdot (-\sin(x - ct)) = -\sin(x - ct)$$
$$\Rightarrow u_t + cu_x = c\sin(x - ct) - c\sin(x - ct) = 0.$$

3.
   a. Linear, inhomogeneous
   b. Linear, homogeneous
   c. Quasilinear, homogeneous
   d. Nonlinear, inhomogeneous
   e. Linear, inhomogeneous
   f. Quasilinear, homogeneous
   g. Nonlinear, inhomogeneous
   h. Linear, homogeneous
   i. Quasilinear, homogeneous

4.

$$u_{xy} + u_y = 0$$

Let $v = u_y$ then the equation becomes

$$v_x + v = 0$$

For fixed y, this is a separable ODE

$$\frac{dv}{v} = -dx$$

$$\ln v = -x + C(y)$$

$$v = K(y)\, e^{-x}$$

In terms of the original variable u we have

$$u_y = K(y)\, e^{-x}$$

$$u = e^{-x}\, q(y) + p(x)$$

You can check your answer by substituting this solution back in the PDE.

5.

$$u = F(xy) + x\,G\left(\frac{y}{x}\right)$$

$$u_x = y\,F'(xy) + G\left(\frac{y}{x}\right) + x\left(-\frac{y}{x^2}\right)G'\left(\frac{y}{x}\right)$$

$$u_{xx} = y^2\,F''(xy) + \left(-\frac{y}{x^2}\right)G'\left(\frac{y}{x}\right) - \frac{y}{x}\left(-\frac{y}{x^2}\right)G''\left(\frac{y}{x}\right) + \left(\frac{y}{x^2}\right)G'\left(\frac{y}{x}\right)$$

$$u_{xx} = y^2\,F''(xy) + \frac{y^2}{x^3}\,G''\left(\frac{y}{x}\right)$$

$$u_y = x\,F'(xy) + x\,\frac{1}{x}G'\left(\frac{y}{x}\right)$$

$$u_{yy} = x^2\,F''(xy) + \frac{1}{x}G''\left(\frac{y}{x}\right)$$

$$x^2u_{xx} - y^2u_{yy} = x^2\left(y^2\,F'' + \frac{y^2}{x^3}\,G''\right) - y^2\left(x^2\,F'' + \frac{1}{x}\,G''\right)$$

Expanding one finds that the first and third terms cancel out and the second and last terms cancel out and thus we get zero.

1.2 Applications

1.3 Conduction of Heat in a Rod

1.4 Boundary Conditions

Problems

1. Suppose the initial temperature of the rod was

$$u(x,\,0) = \begin{cases} 2x & 0 \le x \le 1/2 \\ 2(1-x) & 1/2 \le x \le 1 \end{cases}$$

and the boundary conditions were

$$u(0,\,t) = u(1,\,t) = 0\ ,$$

what would be the behavior of the rod's temperature for later time?

2. Suppose the rod has a constant internal heat source, so that the equation describing the heat conduction is

$$u_t = ku_{xx} + Q, \qquad 0 < x < 1\ .$$

Suppose we fix the temperature at the boundaries

$$\begin{aligned} u(0,\,t) &= 0 \\ u(1,\,t) &= 1\ . \end{aligned}$$

What is the steady state temperature of the rod? (Hint: set $u_t = 0$.)

3. Derive the heat equation for a rod with thermal conductivity $K(x)$.

4. Transform the equation

$$u_t = k(u_{xx} + u_{yy})$$

to polar coordinates and specialize the resulting equation to the case where the function u does NOT depend on θ. (Hint: $r = \sqrt{x^2 + y^2}$, $\tan\theta = y/x$)

5. Determine the steady state temperature for a one-dimensional rod with constant thermal properties and

a. $Q = 0,$ $\quad u(0) = 1,$ $\quad u(L) = 0$
b. $Q = 0,$ $\quad u_x(0) = 0,$ $\quad u(L) = 1$
c. $Q = 0,$ $\quad u(0) = 1,$ $\quad u_x(L) = \varphi$

d. $\frac{Q}{k} = x^2,$ $\quad u(0) = 1,$ $\quad u_x(L) = 0$

e. $Q = 0,$ $\quad u(0) = 1,$ $\quad u_x(L) + u(L) = 0$

1. Since the temperature at both ends is zero (boundary conditions), the temperature of the rod will drop until it is zero everywhere.

2.

$$k\,u_{xx} + Q = 0$$
$$u(0.t) = 0$$
$$u(1,t) = 1$$

$\Rightarrow \qquad u_{xx} = -\frac{Q}{k}$

Integrate with respect to x

$$u_x = -\frac{Q}{k}\,x + A$$

Integrate again

$$u = -\frac{Q}{k}\frac{x^2}{2} + A\,x + B$$

Using the first boundary condition $u(0) = 0$ we get $B = 0$. The other boundary condition will yield

$$-\frac{Q}{k}\frac{1}{2} + A = 1$$

$$\Rightarrow \qquad A = \frac{Q}{2k} + 1$$

$$\Rightarrow \qquad u(x) = \left(1 + \frac{Q}{2k}\right)x - \frac{Q}{2k}\,x^2$$

3. Follow class notes.

4.

$$r = \left(x^2 + y^2\right)^{\frac{1}{2}}$$

$$\theta = \arctan\left(\frac{y}{x}\right)$$

$$r_x = \frac{1}{2}\left(x^2 + y^2\right)^{-\frac{1}{2}} \cdot 2x = \frac{x}{\sqrt{x^2+y^2}}$$

$$\theta_x = \frac{1}{1+\left(\frac{y}{x}\right)^2}\left(-\frac{y}{x^2}\right) = -\frac{y}{x^2+y^2}$$

$$r_y = \frac{1}{2}\left(x^2 + y^2\right)^{-\frac{1}{2}} \cdot 2y = \frac{y}{\sqrt{x^2+y^2}}$$

$$\theta_y = \frac{1}{1+\left(\frac{y}{x}\right)^2}\left(\frac{1}{x}\right) = \frac{x}{x^2+y^2}$$

$$u_x = u_r r_x + u_\theta \theta_x = \frac{x}{\sqrt{x^2+y^2}}\, u_r - \frac{y}{x^2+y^2}\, u_\theta$$

$$u_y = u_r r_y + u_\theta \theta_y = \frac{y}{\sqrt{x^2+y^2}}\, u_r + \frac{x}{x^2+y^2}\, u_\theta$$

$$u_{xx} = \left(\frac{x}{\sqrt{x^2+y^2}}\right)_x u_r + \frac{x}{\sqrt{x^2+y^2}}\,(u_r)_x - \left(\frac{y}{x^2+y^2}\right)_x u_\theta - \frac{y}{x^2+y^2}\,(u_\theta)_x$$

$$u_{xx} = \frac{\sqrt{x^2+y^2} - x\frac{1}{2}\left(x^2+y^2\right)^{-\frac{1}{2}} \cdot 2x}{x^2+y^2}\, u_r + \frac{x}{\sqrt{x^2+y^2}}\left[\frac{x}{\sqrt{x^2+y^2}}\, u_{rr} - \frac{y}{x^2+y^2}\, u_{r\theta}\right]$$
$$- \frac{-2xy}{\left(x^2+y^2\right)^2}\, u_\theta - \frac{y}{x^2+y^2}\left[\frac{x}{\sqrt{x^2+y^2}}\, u_{r\theta} - \frac{y}{x^2+y^2}\, u_{\theta\theta}\right]$$

$$u_{xx} = \frac{x^2}{x^2+y^2}\, u_{rr} - \frac{2xy}{\left(x^2+y^2\right)^{\frac{3}{2}}}\, u_{r\theta} + \frac{y^2}{\left(x^2+y^2\right)^2}\, u_{\theta\theta} + \frac{y^2}{\left(x^2+y^2\right)^{\frac{3}{2}}}\, u_r + \frac{2xy}{\left(x^2+y^2\right)^2}\, u_\theta$$

$$u_{yy} = \left(\frac{y}{\sqrt{x^2+y^2}}\right)_y u_r + \frac{y}{\sqrt{x^2+y^2}}\,(u_r)_y + \left(\frac{x}{x^2+y^2}\right)_y u_\theta + \frac{x}{x^2+y^2}\,(u_\theta)_y$$

$$u_{yy} = \frac{\sqrt{x^2+y^2} - y\frac{1}{2}\left(x^2+y^2\right)^{-\frac{1}{2}} \cdot 2y}{x^2+y^2}\, u_r + \frac{y}{\sqrt{x^2+y^2}}\left[\frac{y}{\sqrt{x^2+y^2}}\, u_{rr} + \frac{x}{x^2+y^2}\, u_{r\theta}\right]$$
$$+ \frac{-2xy}{\left(x^2+y^2\right)^2}\, u_\theta + \frac{x}{x^2+y^2}\left[\frac{y}{\sqrt{x^2+y^2}}\, u_{r\theta} + \frac{x}{x^2+y^2}\, u_{\theta\theta}\right]$$

$$u_{yy} = \frac{y^2}{x^2+y^2}\, u_{rr} + \frac{2xy}{\left(x^2+y^2\right)^{\frac{3}{2}}}\, u_{r\theta} + \frac{x^2}{\left(x^2+y^2\right)^2}\, u_{\theta\theta} + \frac{x^2}{\left(x^2+y^2\right)^{\frac{3}{2}}}\, u_r - \frac{2xy}{\left(x^2+y^2\right)^2}\, u_\theta$$

$$\Rightarrow u_{xx} + u_{yy} = u_{rr} + \frac{1}{r^2} u_{\theta\theta} + \frac{1}{r} u_r$$

$$\boxed{u_t = k\left(u_{rr} + \frac{1}{r} u_r + \frac{1}{r^2} u_{\theta\theta}\right)}$$

In the case u is independent of θ:

$$\boxed{u_t = k\left(u_{rr} + \frac{1}{r} u_r\right)}$$

5. $k\, u_{xx} + Q = 0$

a. $\quad k\, u_{xx} = 0$

Integrate twice with respect to x

$$u(x) = Ax + B$$

Use the boundary conditions

$$u(0) = 1 \qquad \text{implies } B = 1$$

$$u(L) = 0 \qquad \text{implies } AL + B = 0 \qquad \text{that is } A = -\frac{1}{L}$$

Therefore

$$\boxed{u(x) = -\frac{x}{L} + 1}$$

b. $\quad k\, u_{xx} = 0$

Integrate twice with respect to x as in the previous case

$$u(x) = Ax + B$$

Use the boundary conditions

$$u_x(0) = 0 \qquad \text{implies } A = 0$$

$$u(L) = 1 \qquad \text{implies } AL + B = 1 \qquad \text{that is } B = 1$$

Therefore

$$\boxed{u(x) = 1}$$

c. $\quad k\, u_{xx} = 0$

Integrate twice with respect to x as in the previous case

$$u(x) = Ax + B$$

Use the boundary conditions

$$u(0) = 1 \qquad \text{implies } B = 1$$

$$u_x(L) = \varphi \qquad \text{implies } A = \varphi$$

Therefore

$$\boxed{u(x) = \varphi\, x + 1}$$

d. $\quad k\, u_{xx} \,+\, Q \,=\, 0$

$\quad u_{xx} \,=\, -\,\frac{Q}{k} \,=\, -\,x^2$

Integrate with respect to x we get

$$u_x(x) \,=\, -\,\frac{1}{3}\,x^3 \,+\, A$$

Use the boundary condition

$$u_x(L) \,=\, 0 \qquad \text{implies } -\,\frac{1}{3}\,L^3 \,+\, A \,=\, 0 \qquad \text{that is } A \,=\, \frac{1}{3}\,L^3$$

Integrating again with respect to x

$$u \,=\, -\,\frac{x^4}{12} \,+\, \frac{1}{3}\,L^3\,x \,+\, B$$

Use the second boundary condition

$$u(0) \,=\, 1 \qquad \text{implies } B \,=\, 1$$

Therefore

$$\boxed{u(x) \,=\, -\frac{x^4}{12} \,+\, \frac{L^3}{3}\,x \,+\, 1}$$

e. $\quad k\, u_{xx} \,=\, 0$

Integrate twice with respect to x as in the previous case

$$u(x) \,=\, Ax \,+\, B$$

Use the boundary conditions

$$u(0) \,=\, 1 \qquad \text{implies } B \,=\, 1$$

$$u_x(L) \,+\, u(L) = 0 \qquad \text{implies } A \,+\, (AL \,+\, 1) \,=\, 0 \qquad \text{that is } A \,=\, -\,\frac{1}{L+1}$$

Therefore

$$\boxed{u(x) \,=\, -\,\frac{1}{L+1}\,x \,+\, 1}$$

1.5 A Vibrating String

Problems

1. Derive the telegraph equation

$$u_{tt} + au_t + bu = c^2 u_{xx}$$

by considering the vibration of a string under a damping force proportional to the velocity and a restoring force proportional to the displacement.

2. Use Kirchoff's law to show that the current and potential in a wire satisfy

$$\begin{aligned} i_x + C\, v_t + Gv &= 0 \\ v_x + L\, i_t + Ri &= 0 \end{aligned}$$

where i = current, $v = L$ = inductance potential, C = capacitance, G = leakage conductance, R = resistance,

b. Show how to get the one dimensional wave equations for i and v from the above.

1. Follow class notes.

a, b are the proportionality constants for the forces mentioned in the problem.

2. a. Check any physics book on Kirchoff's law.

b. Differentiate the first equation with respect to t and the second with respect to x

$$i_{xt} + C\,v_{tt} + G\,v_t = 0$$
$$v_{xx} + L\,i_{tx} + R\,i_x = 0$$

Solve the first for i_{xt} and substitute in the second

$$i_{xt} = -C\,v_{tt} - G\,v_t$$

$$\Rightarrow v_{xx} - CL\,v_{tt} - GL\,v_t + R\,i_x = 0$$

i_x can be solved for from the original first equation

$$i_x = -C\,v_t - G\,v$$

$$\Rightarrow v_{xx} - CL\,v_{tt} - GL\,v_t - RC\,v_t - RG\,v = 0$$

Or

$$v_{tt} + \left(\frac{G}{C} + \frac{R}{L}\right) v_t + \frac{RG}{CL}\,v = \frac{1}{CL}\,v_{xx}$$

which is the telegraph equation.

In a similar fashion, one can get the equation for i.

1.6 Boundary Conditions

1.7 Diffusion in Three Dimensions

CHAPTER 2

2 Classification and Characteristics

2.1 Physical Classification

2.2 Classification of Linear Second Order PDEs

Problems

1. Classify each of the following as hyperbolic, parabolic or elliptic at every point (x, y) of the domain

a. $x\, u_{xx} + u_{yy} = x^2$
b. $x^2\, u_{xx} - 2xy\, u_{xy} + y^2 u_{yy} = e^x$
c. $e^x u_{xx} + e^y u_{yy} = u$
d. $u_{xx} + u_{xy} - xu_{yy} = 0$ in the left half plane $(x \leq 0)$
e. $x^2 u_{xx} + 2xyu_{xy} + y^2 u_{yy} + xyu_x + y^2 u_y = 0$
f. $u_{xx} + xu_{yy} = 0$ (Tricomi equation)

2. Classify each of the following constant coefficient equations

a. $4u_{xx} + 5u_{xy} + u_{yy} + u_x + u_y = 2$
b. $u_{xx} + u_{xy} + u_{yy} + u_x = 0$
c. $3u_{xx} + 10u_{xy} + 3u_{yy} = 0$
d. $u_{xx} + 2u_{xy} + 3u_{yy} + 4u_x + 5u_y + u = e^x$
e. $2u_{xx} - 4u_{xy} + 2u_{yy} + 3u = 0$
f. $u_{xx} + 5u_{xy} + 4u_{yy} + 7u_y = \sin x$

3. Use any symbolic manipulator (e.g. MACSYMA or MATHEMATICA) to prove (2.1.19). This means that a transformation does NOT change the type of the PDE.

1a. $A = x$ $\quad B = 0$ $\quad C = 1$ $\quad \Delta = -4x$

<u>hyperbolic</u> for $x < 0$

<u>parabolic</u> $x = 0$

<u>elliptic</u> $x > 0$

b. $A = x^2$ $\quad B = 2xy$ $\quad C = y^2$ $\quad \Delta = 0$ <u>parabolic</u>

c. $A = e^x$ $\quad B = 0$ $\quad C = e^y$ $\quad \Delta = -4e^x e^y$ <u>elliptic</u>

d. $A = 1$ $\quad B = 1$ $\quad C = -x$ $\quad \Delta = 1 + 4x$

<u>hyperbolic</u> $0 \geq x > -\frac{1}{4}$

<u>parabolic</u> $x = -\frac{1}{4}$

<u>elliptic</u> $x < -\frac{1}{4}$

e. $A = x^2$ $\quad B = 2xy$ $\quad C = y^2$ $\quad \Delta = 0$ <u>parabolic</u>

f. $A = 1$ $\quad B = 0$ $\quad C = x$ $\quad \Delta = -4x$

<u>hyperbolic</u> $x < 0$

<u>parabolic</u> $x = 0$

<u>elliptic</u> $x > 0$

2.

	A	B	C	Discriminant	
a.	4	5	1	25 - 16 $>$ 0	hyperbolic
b.	1	1	1	1 - 4 $<$ 0	elliptic
c.	3	10	3	100 - 36 $>$ 0	hyperbolic
d.	1	2	3	4 - 12 $<$ 0	elliptic
e.	2	-4	2	16 - 16 $=$ 0	parabolic
f.	1	5	4	25 - 16 $>$ 0	hyperbolic

3. We substitute for A^*, B^*, C^* given by (2.1.12)-(2.1.14) in Δ^*.

$$
\begin{aligned}
\Delta^* &= (B^*)^2 - 4A^*C^* \\
&= \left[2A\xi_x\eta_x + B\left(\xi_x\eta_y + \xi_y\eta_x\right) + 2C\xi_y\eta_y\right]^2 - \\
&\quad 4\left[A\xi_x^2 + B\xi_x\xi_y + C\xi_y^2\right]\left[A\eta_x^2 + B\eta_x\eta_y + C\eta_y^2\right] \\
&= 4A^2\xi_x^2\eta_x^2 + 4A\xi_x\eta_x B\left(\xi_x\eta_y + \xi_y\eta_x\right) + 8A\xi_x\eta_x C\xi_y\eta_y \\
&+ B^2\left(\xi_x\eta_y + \xi_y\eta_x\right)^2 + 4B\left(\xi_x\eta_y + \xi_y\eta_x\right)C\xi_y\eta_y \\
&+ 4C^2\xi_y^2\eta_y^2 - 4A^2\xi_x^2\eta_x^2 - 4A\xi_x^2 B\eta_x\eta_y - 4A\xi_x^2 C\eta_y^2 \\
&- 4B\xi_x\xi_y A\eta_x^2 - 4B^2\xi_x\xi_y\eta_x\eta_y - 4B\xi_x\xi_y C\eta_y^2 \\
&- 4C\xi_y^2 A\eta_x^2 - 4C\xi_y^2 B\eta_x\eta_y - 4C^2\xi_y^2\eta_y^2.
\end{aligned}
$$

Collect terms to find

$$
\begin{aligned}
\Delta^* &= 4AB\xi_x^2\eta_x\eta_y + 4AB\xi_x\xi_y\eta_x^2 + 8AC\xi_x\xi_y\eta_x\eta_y \\
&+ B^2(\xi_x^2\eta_y^2 + 2\xi_x\xi_y\eta_x\eta_y + \xi_y^2\eta_x^2) \\
&+ 4BC\xi_x\xi_y\eta_y^2 + 4BC\eta_x\eta_y\xi_y^2 - 4AB\xi_x^2\eta_x\eta_y \\
&- 4AC\xi_x^2\eta_y^2 - 4AB\xi_x\xi_y\eta_x^2 - 4B^2\xi_x\xi_y\eta_x\eta_y \\
&- 4BC\xi_x\xi_y\eta_y^2 - 4AC\xi_y^2\eta_x^2 - 4BC\xi_y^2\eta_x\eta_y
\end{aligned}
$$

$$
\begin{aligned}
\Delta^* &= -4AC\left(\xi_x^2\eta_y^2 - 2\xi_x\xi_y\eta_x\eta_y + \xi_y^2\eta_x^2\right) \\
&+ B^2\left(\xi_x^2\eta_y^2 - 2\xi_x\xi_y\eta_x\eta_y + \xi_y^2\eta_x^2\right) \\
&= J^2\Delta,
\end{aligned}
$$

since $J = (\xi_x\eta_y - \xi_y\eta_x)$.

2.3 Canonical Forms

Problems

1. Find the characteristic equation, characteristic curves and obtain a canonical form for each

a. $x\,u_{xx} + u_{yy} = x^2$
b. $u_{xx} + u_{xy} - xu_{yy} = 0 \qquad (x \leq 0,\ \text{all } y)$
c. $x^2 u_{xx} + 2xyu_{xy} + y^2 u_{yy} + xyu_x + y^2 u_y = 0$
d. $u_{xx} + xu_{yy} = 0$
e. $u_{xx} + y^2 u_{yy} = y$
f. $\sin^2 x u_{xx} + \sin 2x u_{xy} + \cos^2 x u_{yy} = x$

2. Use Maple to plot the families of characteristic curves for each of the above.

1a. $xu_{xx} + u_{yy} = x^2$

$A = x \qquad B = 0 \qquad C = 1 \qquad \Delta = B^2 - 4AC = -4x$

If $x > 0$ then $\Delta < 0$ <u>elliptic</u>

$\quad = 0 \qquad\qquad = 0$ <u>parabolic</u>

$\quad < 0 \qquad\qquad > 0$ <u>hyperbolic</u>

characteristic equation

$$\frac{dy}{dx} = \frac{\pm\sqrt{-4x}}{2x} = \frac{\pm\sqrt{-x}}{x}$$

<u>Suppose $x < 0$</u> <u>(hyperbolic)</u>

Let $z = -x$ (then $z > 0$). This is done so as not to get confused by the negative sign under the radical.

then $dz = -dx$

and

$\dfrac{dy}{dz} = -\dfrac{dy}{dx} = -\dfrac{\pm\sqrt{z}}{-z} = \pm\dfrac{1}{\sqrt{z}}$

$dy = \pm\dfrac{dz}{z^{1/2}}$

$y = \pm 2\sqrt{z} + c$

$y \mp 2\sqrt{z} = c$

characteristic curves: $y \mp 2\sqrt{z} = c$
2 families as expected.

Transformation: $\xi = y - 2\sqrt{z}$

$\qquad\qquad\qquad \eta = y + 2\sqrt{z}$

We now substitute in the equations for the starred coefficients (see the summary). To this end we list all the necessary derivatives of ξ and η.

$$\begin{array}{ll}
\xi_x = \xi_z\, z_x = -\xi_z & \xi_y = 1 \\
\xi_{xx} = (\xi_x)_x = \left(\frac{1}{\sqrt{z}}\right)_x = \left(\frac{1}{\sqrt{z}}\right)_z z_x = -\left(-\frac{1}{2}\, z^{-3/2}\right) = \frac{1}{2z^{3/2}} & \xi_{xy} = 0 \quad \xi_{yy} = 0 \\
\eta_x = \eta_z\, z_x = -\eta_z & \eta_y = 1 \\
\eta_{xx} = (\eta_x)_x = \left(-\frac{1}{\sqrt{z}}\right)_x = \left(-\frac{1}{\sqrt{z}}\right)_z z_x = -\left(\frac{1}{2}\, z^{-3/2}\right) = \frac{-1}{2z^{3/2}} & \eta_{xy} = 0 \quad \eta_{yy} = 0
\end{array}$$

Note that

$$\xi_z \;=\; -\,2\left(\frac{1}{2}z^{-1/2}\right) \;=\; -\frac{1}{\sqrt{z}}$$

$$\eta_z \;=\; 2\left(\frac{1}{2}z^{-1/2}\right) \;=\; \frac{1}{\sqrt{z}}$$

Therefore

$$\xi_x \;=\; \frac{1}{\sqrt{z}} \qquad \xi_y \;=\; 1 \quad \xi_{xx} \;=\; \frac{1}{2z^{3/2}} \quad \xi_{xy} \;=\; 0 \quad \xi_{yy} \;=\; 0$$

$$\eta_x \;=\; -\frac{1}{\sqrt{z}} \quad \eta_y \;=\; 1 \quad \eta_{xx} \;=\; \frac{-1}{2z^3/2} \quad \eta_{xy} \;=\; 0 \quad \eta_{yy} \;=\; 0$$

Since the problem is Hyperbolic, we know that $A^* \;=\; C^* \;=\; 0$,

$$B^* = 2x(\frac{1}{\sqrt{z}})(-\frac{1}{\sqrt{z}}) + 0 + 2\cdot 1^2 = \frac{2z}{\sqrt{z}^2} + 1 \;=\; 2 + 2 \;=\; 4$$

$$D^* = x\frac{1}{2z^{3/2}} + 0 + 0 + 0 + 0 = -\frac{z}{2z^{3/2}} = -\frac{1}{2\sqrt{z}}$$

$$E^* = x(-\frac{1}{2z^{3/2}}) + 0 + 0 + 0 + 0 = \frac{z}{2z^{3/2}} = \frac{1}{2\sqrt{z}}$$

$$F^* = 0$$

$$G^* = x^2 = (-z)^2 = z^2$$

The equation is then

$$4u_{\xi\eta} - \frac{1}{2\sqrt{z}}u_\xi + \frac{1}{2\sqrt{z}}u_\eta = z^2$$

The last step is to get rid of z

$\xi \,-\, \eta \;=\; -4\sqrt{z}$ (using the transformation)

$$\sqrt{z} \;=\; \frac{\eta \,-\, \xi}{4} \;\Rightarrow\; 2\sqrt{z} \;=\; \frac{\eta \,-\, \xi}{2} \;;\; z \;=\; \left(\frac{\eta \,-\, \xi}{4}\right)^2$$

$$\boxed{4u_{\xi\,\eta} \,-\, \frac{2}{\eta \,-\, \xi}\,u_\xi \,+\, \frac{2}{\eta \,-\, \xi}\,u_\eta \;=\; \left(\frac{\eta \,-\, \xi}{4}\right)^4}$$

For the elliptic case $\underline{x > 0}$

$$\frac{dy}{dx} = \frac{\pm i}{\sqrt{x}}$$

$$dy = \pm i \frac{dx}{\sqrt{x}}$$

$$y = \pm i \, 2\sqrt{x} + c$$

$$\xi = y - 2i\sqrt{x}$$

$$\eta = y + 2i\sqrt{x}$$

$$\alpha = \frac{1}{2}(\xi + \eta) = y$$

$$\beta = \frac{1}{2i}(\xi - \eta) = -2\sqrt{x}$$

Now we find the derivatives of α and β

$$\alpha_x = 0 \; ; \; \alpha_y = 1 \, ; \, \alpha_{xx} = \alpha_{xy} = \alpha_{yy} = 0$$

$$\beta_x = -2 \cdot \frac{1}{2} x^{-1/2} = -x^{-1/2} \, ; \, \beta_y = 0 \, ; \, \beta_{xx} = \frac{1}{2} x^{-3/2} \, ; \, \beta_{xy} = \beta_{yy} = 0$$

For the elliptic case, $B^* = 0$ and $A^* = C^*$, therefore when using the atarred equations (summary) we have

$$A^* = C^* = 0 + 0 + 1 = 1$$

$$D^* = 0 + 0 + 0 + 0 + 0 = 0$$

$$E^* = x(\frac{1}{2} x^{-3/2}) + 0 + 0 + 0 + 0 = \frac{1}{2} x^{-1/2}$$

$$F^* = 0$$

$$G^* = x$$

and the equation

$$u_{\beta\beta} + u_{\alpha\alpha} + \frac{1}{2} x^{-1/2} u_\beta = x^2$$

Now substitute for x

$$\boxed{u_{\beta\beta} + u_{\alpha\alpha} - \frac{1}{\beta} u_\beta = \left(-\frac{\beta}{2}\right)^4}$$

$$\boxed{u_{\alpha\alpha} + u_{\beta\beta} = \frac{1}{\beta} u_\beta + \frac{1}{16} \beta^4}$$

For the parabolic case $\underline{x = 0}$ the equation becomes:

$0 \cdot u_{xx} + u_{yy} = 0$

or $\boxed{u_{yy} = 0}$

which is already in a canonical form

This parabolic case can be solved. Integrate with respect to y holding x fixed (the constant of integration may depend on x)

$u_y = f(x)$

Integrate again:

$\boxed{u(x,\, y) = y\, f(x) + g(x)}$

1b. $u_{xx} + u_{xy} - x\,u_{yy} = 0$

$A = 1$ $\qquad B = 1$ $\qquad C = -x$

$\Delta = 1 + 4x$ $\quad > 0 \quad$ if $x > -\frac{1}{4}$ <u>hyperbolic</u>

$\quad = 0 \quad = -\frac{1}{4}$ <u>parabolic</u>

$\quad < 0 \quad < -\frac{1}{4}$ <u>elliptic</u>

$$\frac{dy}{dx} = \frac{1 \pm \sqrt{1 + 4x}}{2}$$

Consider the hyperbolic case:

$$2dy = (1 \pm \sqrt{1 + 4x})\,dx$$

Integrate to get characteristics

$$2y = x \pm \frac{2}{3} \cdot \frac{1}{4}(1 + 4x)^{3/2} + c$$

$$2y - x \mp \frac{1}{6}(1 + 4x)^{3/2} = c$$

$$\xi = 2y - x - \frac{1}{6}(1 + 4x)^{3/2}$$

$$\eta = 2y - x + \frac{1}{6}(1 + 4x)^{3/2}$$

$$\xi_x = -1 - \frac{1}{6} \cdot \frac{3}{2} \cdot 4\,(1 + 4x)^{1/2} = -1 - \sqrt{1 + 4x}$$

$$\xi_{xx} = -\frac{1}{2}(1 + 4x)^{-1/2} \cdot 4 = -2\,(1 + 4x)^{-1/2}$$

$\xi_y = 2$ $\qquad \xi_{yy} = 0$ $\qquad \xi xy = 0$

$$\eta_x = -1 + \frac{1}{6} \cdot \frac{3}{2} \cdot 4\,(1 + 4x)^{1/2} = -1 + \sqrt{1 + 4x}$$

$$\eta_{xx} = +2\,(1 + 4x)^{-1/2}$$

$\eta_y = 2$ $\qquad \eta_{xy} = 0$ $\qquad \eta_{yy} = 0$

Now we can compute the new coefficients or compute each of the derivative in the equation. We chose the former.

$$A^* \; = \; C^* \; = \; 0$$

$$B^* = 2(1)(-1-\sqrt{1+4x})(-1+\sqrt{1+4x})+1[2(-1-\sqrt{1+4x})+2(-1+\sqrt{1+4x})+2(-x)\cdot 2\cdot 2$$

$$B^* = -8x - 4 - 8x = -4 - 16x$$

$$D^* = -2\,(1\,+\,4x)^{-1/2} + 0 + 0 + 0 + 0 = -2\,(1\,+\,4x)^{-1/2}$$

$$E^* = 2\,(1\,+\,4x)^{-1/2} + 0 + 0 + 0 + 0 = 2\,(1\,+\,4x)^{-1/2}$$

$$F^* = 0$$

$$G^* = 0$$

$$-4\,(1+4x)u_{\xi\,\eta} \;-\; 2(1\,+\,4x)^{-1/2}\;(u_\xi \;-\; u_\eta) \;=\; 0$$

$$u_{\xi\,\eta} \;+\; \frac{1}{2}(1\,+\,4x)^{-3/2}\;(u_\xi \;-\; u_\eta) \;=\; 0$$

Now find $(1\,+\,4x)^{-3/2}$ in terms of ξ , η and substitute

$$\xi \;-\; \eta \;=\; -\frac{1}{3}\,(1\,+\,4x)^{3/2}$$

$$3(\eta \;-\; \xi) \;=\; (1\,+\,4x)^{3/2}$$

$$(1\,+\,4x)^{-3/2} \;=\; [3(\eta \;-\; \xi)]^{-1}$$

$$u_{\xi\,\eta} \;=\; -\frac{1}{2[3(\eta \;-\; \xi)]}\,(u_\xi \;-\; u_\eta)$$

$$u_{\xi\,\eta} \;=\; \frac{1}{6(\eta \;-\; \xi)}\,(u_\eta \;-\; u_\xi)$$

The parabolic case is easier, the only characteristic is

$$y = \frac{1}{2}x + K$$

and so the transformation is

$$\xi = y - \frac{1}{2}x$$

$$\eta = x$$

The last equation is an arbitrary function and one should check the Jacobian. The details are left to the reader. One can easily show that

$$A^* = B^* = 0$$

Also

$$C^* = 1$$

and the rest of the coefficients are zero. Therefore the equation is

$$u_{\eta\eta} = 0$$

In the elliptic case, one can use the transformation $z = -(1+4x)$ so that the characteristic equation becomes

$$\frac{dy}{dx} = \frac{1 \pm \sqrt{z}}{2}$$

or if we eliminate the x dependence

$$\frac{dy}{dz} = \frac{dy}{dx}\frac{dx}{dz} = -\frac{1}{4}\frac{1 \pm \sqrt{z}}{2}$$

Now integrate, and take the real and imaginary part to be the functions ξ and η. The rest is left for the reader.

1c. $x^2\, u_{xx} \,+\, 2xy\, u_{xy} \,+\, y^2\, u_{yy} \,+\, xy\, u_x \,+\, y^2\, u_y \,=\, 0$

$A \,=\, x^2 \qquad\qquad B \,=\, 2xy \qquad\qquad C \,=\, y^2$

$\Delta \,=\, 4x^2\, y^2 \,-\, 4x^2\, y^2 = 0$ <u>parabolic</u>

$$\frac{dy}{dx} \,=\, \frac{2xy}{2x^2} \,=\, \frac{y}{x}$$

$$\frac{dy}{y} \,=\, \frac{dx}{x}$$

$\xi \,=\, \ln\, y - \,\ln\, x \quad\Rightarrow\quad \xi \,=\, \ln\left(\frac{y}{x}\right) \Rightarrow e^{\xi} \,=\, \frac{y}{x}$

$\eta \,=\, x$ arbitrarily chosen since this is parabolic

$\xi_x \,=\, \frac{-1}{x} \qquad \xi_y \,=\, \frac{1}{y} \qquad \xi_{xx} \,=\, \frac{1}{x^2} \qquad \xi_{xy} \,=\, 0 \qquad \xi_{yy} \,=\, -\frac{1}{y^2}$

$\eta_x \,=\, 1 \quad \eta_y \,=\, \eta_{xx} \,=\, \eta_{xy} \,=\, \eta_{yy} \,=\, 0$

$$A^* = B^* = 0 \qquad \text{parabolic}$$

$$C^* = x^2 \cdot 1 + 2xy \cdot 0 + y^2 \cdot 0 = x^2$$

$$D^* = x^2\left(\frac{1}{x^2}\right) + 2xy \cdot 0 + y^2\left(-\frac{1}{y^2}\right) + xy\left(-\frac{1}{x}\right) + y^2\left(\frac{1}{y}\right) = 1 - 1 - y + y = 0$$

$$E^* = 0 + 0 + 0 + xy \cdot 1 + 0 = xy$$

$$F^* = 0$$

$$G^* = 0$$

$x^2\, u_{\eta\,\eta} \,+\, xyu_\eta \,=\, 0$

$\boxed{u_{\eta\,\eta} \,=\, -\,e^{\xi}\, u_\eta}$ $\qquad y \,=\, e^{\xi}\, x$ therefore $y/x \,=\, e^{\xi}$

This equation can be solved.

1d. $u_{xx} + x\,u_{yy} = 0$

$A = 1 \qquad B = 0 \qquad C = x$

$$\Delta = -4x \qquad \begin{array}{lll} > 0 & \text{if } x < 0 & \underline{\text{hyperbolic}} \\ = 0 & x = 0 & \underline{\text{parabolic}} \\ < 0 & x > 0 & \underline{\text{elliptic}} \end{array}$$

<u>Parabolic</u> $x = 0 \quad \Rightarrow \quad u_{xx} = 0$ already in canonical form

<u>Hyperbolic</u> $x < 0$ Let $\zeta = -x$

$$\Delta = 4\zeta > 0$$

$\dfrac{dy}{dx} = \pm\dfrac{2\sqrt{\zeta}}{2} = \pm\sqrt{\zeta}$ Note: $dx = -d\zeta$

$dy = \pm\sqrt{\zeta}\,(-d\zeta)$

$y \pm \dfrac{2}{3}\zeta^{3/2} = c$

$\xi = y + \dfrac{2}{3}\zeta^{3/2}$

$\eta = y - \dfrac{2}{3}\zeta^{3/2}$

Continue as in example in class (See 1a)

1e. $u_{xx} + y^2 u_{yy} = y$

$A = 1 \qquad B = 0 \qquad C = y^2$

$\Delta = -4y^2 < 0$ $\underline{\text{elliptic}}$ if $y \neq 0$

For $y = 0$ the equation is $\underline{\text{parabolic}}$ and it is in canonical form $u_{xx} = 0$

$$\frac{dy}{dx} = \frac{\pm\sqrt{-4y^2}}{2} = \pm iy$$

$$\frac{dy}{y} = \pm i dx$$

$\ln y = \pm ix + c$

$\xi = \ln y + ix$

$\eta = \ln y - ix$

$\alpha = \ln y \qquad \alpha_x = 0 \qquad \alpha_y = \frac{1}{y} \qquad \alpha_{xx} = \alpha_{xy} = 0 \qquad \alpha_{yy} = -\frac{1}{y^2}$

$\beta = x \qquad \beta_x = 1 \qquad \beta_y = 0 \qquad \beta_{xx} = \beta_{xy} = \beta_{yy} = 0$

$$A^* = C^* = 0 + 0 + y^2 \left(\frac{1}{y}\right)^2 = 1$$
$$B^* = 0$$
$$D^* = y^2 \left(-\frac{1}{y^2}\right) = -1$$
$$E^* = 0$$
$$F^* = 0$$
$$G^* = 0$$

$u_{\alpha\alpha} + u_{\beta\beta} - u_\alpha = y$

But $y = e^\alpha$

$\boxed{\Rightarrow \quad u_{\alpha\alpha} + u_{\beta\beta} - u_\alpha = e^\alpha}$

1f. $\sin^2 x\, u_{xx} + \sin 2x\, u_{xy} + \cos^2 x\, u_{yy} = x$

$A = \sin^2 x \qquad B = \sin 2x = 2\sin x \cos x \qquad C = \cos^2 x$

$\Delta = 0$ $\underline{\text{parabolic}}$

$$\frac{dy}{dx} = \frac{2\sin x \cos x}{2\sin^2 x} = \cot x$$

$y = \ln \sin x + c$

$$\xi = y - \ln \sin x \quad \xi_x = -\cot x \quad \xi_y = 1 \quad \xi_{xx} = \frac{1}{\sin^2 x} \quad \xi_{xy} = 0 \quad \xi_{yy} = 0$$

$$\eta = y \quad \eta_x = 0 \quad \eta_y = 1 \quad \eta_{xx} = 0 \quad \eta_{xy} = 0 \quad \eta_{yy} = 0$$

$$A^* = B^* = 0$$
$$C^* = 0 + 0 + \cos^2 x \cdot 1 = \cos^2 x$$
$$D^* = \sin^2 x \left(\frac{1}{\sin^2 x}\right) + 0 + 0 + 0 + 0 = 1$$
$$E^* = 0 + 0 + 0 + 0 + 0 = 0$$
$$F^* = 0$$
$$G^* = x$$

Therefore the equation becomes:

$\cos^2 x\, u_{\eta\eta} + u_\xi = x$

$\ln \sin x = y - \xi = \eta - \xi$

$\sin x = e^{\eta - \xi} \quad \Rightarrow \quad \cos^2 x = 1 - \sin^2 x = 1 - e^{2(\eta - \xi)}$

$x = \arcsin e^{\eta - \xi}$

$$\boxed{[1 - e^{2(\eta - \xi)}]\, u_{\eta\eta} + u_\xi = \arcsin e^{\eta - \xi}}$$

2a. $y \pm 2\sqrt{z} = c \qquad z > 0$

eq: $y + 2 * sqrt(z) = c$; $\quad\longleftarrow\quad$ maple command to give the equation

char:=solve (eq, y); $\quad\longleftarrow\quad$ maple command to solve for y

chars:=seq (char, c= -5..5); $\quad\longleftarrow\quad$ maple command to create several characteristic curves for a variety of c's.

plot ($\{$chars$\}$, $z = 0..10, y = -5..5$); $\quad\longleftarrow\quad$ maple command to plot all those curves

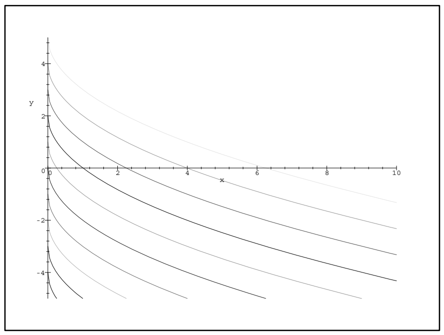

Figure 1: Maple plot of characteristics for 2.2 2a

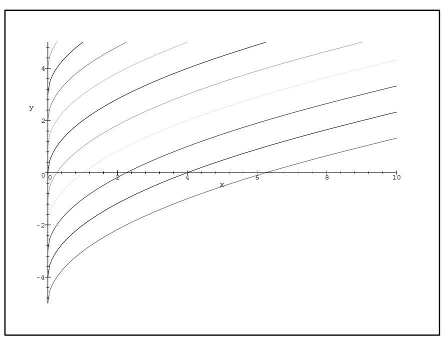

Figure 2: Maple plot of characteristics for 2.2 2a

2b. $y = \dfrac{1}{2}x \pm \dfrac{1}{12}(1 + 4x)^{3/2} + c$

$$1 + 4\,x \geq 0$$

$$4\,x \geq -1$$

$$x \geq -.25$$

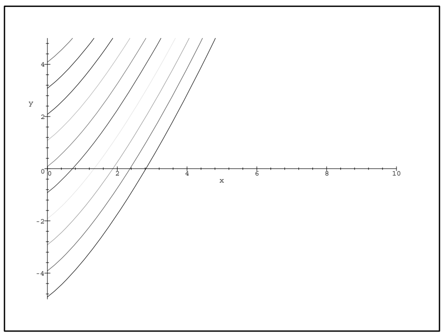

Figure 3: Maple plot of characteristics for 2.2 2b

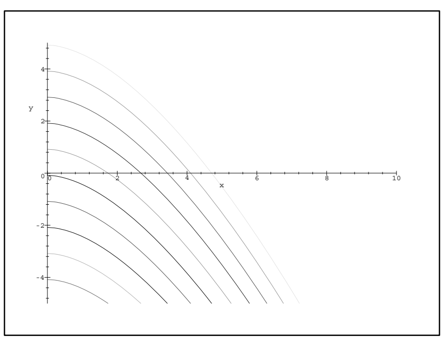

Figure 4: Maple plot of characteristics for 2.2 2b

2c. $\quad \ln \dfrac{y}{x} = c \quad$ parabolic

$\ln y = xe^c = kx$

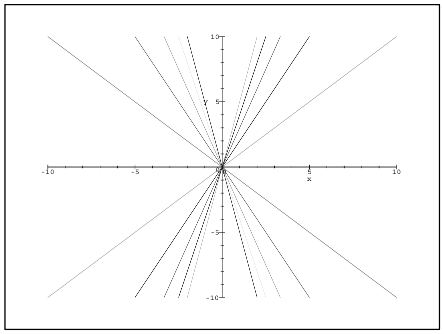

Figure 5: Maple plot of characteristics for 2.2 2c

2d. $\quad y \pm \dfrac{2}{3} z^{3/2} = c$

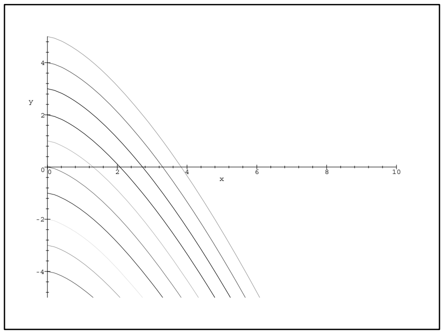

Figure 6: Maple plot of characteristics for 2.2 2d

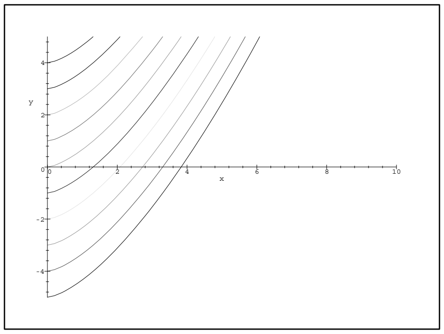

Figure 7: Maple plot of characteristics for 2.2 2d

2e. elliptic. no real characteristic

2f. $y = \ln \sin x + c$

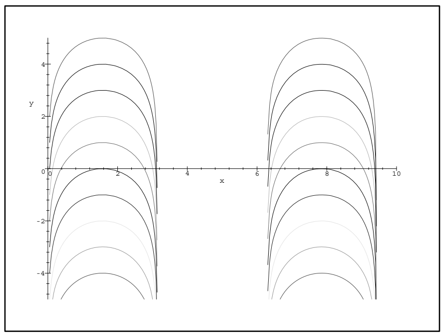

Figure 8: Maple plot of characteristics for 2.2 2f

2.4 Equations with Constant Coefficients

Problems

1. Find the characteristic equation, characteristic curves and obtain a canonical form for

   a. $4u_{xx} + 5u_{xy} + u_{yy} + u_x + u_y = 2$
   b. $u_{xx} + u_{xy} + u_{yy} + u_x = 0$
   c. $3u_{xx} + 10u_{xy} + 3u_{yy} = x + 1$
   d. $u_{xx} + 2u_{xy} + 3u_{yy} + 4u_x + 5u_y + u = e^x$
   e. $2u_{xx} - 4u_{xy} + 2u_{yy} + 3u = 0$
   f. $u_{xx} + 5u_{xy} + 4u_{yy} + 7u_y = \sin x$

2. Use Maple to plot the families of characteristic curves for each of the above.

1a. $4u_{xx} + 5u_{xy} + u_{yy} + u_x + u_y = 2$

$$A = 4 \qquad B = 5 \qquad C = 1$$

$\Delta = 5^2 - 4 \cdot 4 \cdot 1 = 25 - 16 = 9 > 0$ <u>hyperbolic</u>

$$\frac{dy}{dx} = \frac{5 \pm \sqrt{9}}{2 \cdot 4} = \frac{5 \pm 3}{8} \begin{matrix} \nearrow 1/4 \\ \searrow 1 \end{matrix}$$

$$dy = dx \qquad dy = \frac{1}{4}\,dx$$

$$y = x + c \qquad y = \frac{1}{4}\,x + c$$

$$\xi = y - x \qquad \eta = y - \frac{1}{4}x$$

$$\begin{array}{lllll} \xi_x = -1 & \xi_y = 1 & \xi_{xx} = 0 & \xi_{xy} = 0 & \xi_{yy} = 0 \\ \eta_x = -\frac{1}{4} & \eta_y = 1 & \eta_{xx} = 0 & \eta_{xy} = 0 & \eta_{yy} = 0 \end{array}$$

$$A^* = C^* = 0$$
$$B^* = 2 \cdot 4(-1)(-\frac{1}{4}) + 5(-1 \cdot 1 + 1(-\frac{1}{4})) + 2 \cdot 1 \cdot 1 \cdot 1 = -\frac{9}{4}$$
$$D^* = 0 + 0 + 0 + 1(-1) + 1 \cdot 1 = 0$$
$$E^* = 0 + 0 + 0 + 1(-\frac{1}{4}) + 1 \cdot 1 = \frac{3}{4}$$
$$F^* = 0$$
$$G^* = 2$$

$$-\frac{9}{4}\,u_{\xi\eta} + \frac{3}{4}\,u_\eta = 2$$

$$\boxed{u_{\xi\eta} = \tfrac{1}{3}\,u_\eta - \tfrac{8}{9}}$$

This equation can be solved as follows:

Let $\nu = u_\eta$ then $u_{\xi\eta} = \nu_\xi$

$\nu_\xi = \frac{1}{3}\,\nu - \frac{8}{9}$

This is Linear 1^{st} order ODE

$$\nu' - \frac{1}{3}\nu = -\frac{8}{9}$$

Integrating factor is $e^{-\frac{1}{3}\xi}$

$(\nu e^{-\frac{1}{3}\xi})' = -\frac{8}{9}\, e^{-\frac{1}{3}\xi}$

$\nu e^{-\frac{1}{3}\xi} = -\frac{8}{9}\int e^{-\frac{1}{3}\xi}\, d_\xi = \frac{8}{3}\, e^{-\frac{1}{3}\xi} + C\,(\eta)$

$\boxed{\nu = \frac{8}{3} + C\,(\eta)\, e^{\frac{1}{3}\xi}}$

To find u we integrate with respect to η

$$u_\eta = \frac{8}{3} + C(\eta)\, e^{\frac{1}{3}\xi}$$

$$u = \frac{8}{3}\eta + e^{\frac{1}{3}\xi} \qquad \underbrace{c_1(\eta)}_{\text{integral of } C(\eta)} \qquad + K(\xi)$$

To check the solution, we differentiate it and substitute in the canonical form:

$$u_\xi = 0 + \frac{1}{3}\, e^{\frac{1}{3}\xi}\, c_1\,(\eta) + K'\,(\xi)$$

$$u_{\xi\,\eta} = \frac{1}{3}\, e^{\frac{1}{3}\xi}\, c_1'\,(\eta)$$

$$u_\eta = \frac{8}{3} + e^{\frac{1}{3}\xi}\, c_1'\,(\eta)$$

$$\Rightarrow \quad \frac{1}{3}\, u_\eta = \frac{8}{9} + \frac{1}{3}\, e^{\frac{1}{3}\xi}\, c_1'\,(\eta)$$

Substitute in the PDE in canonical form

$$\frac{1}{3}\, e^{\frac{1}{3}\xi}\, c_1'\,(\xi) = \frac{8}{9} + \frac{1}{3}\, e^{\frac{1}{3}\xi}\, c_1'\,(\eta) - \frac{8}{9}$$

Identity

$\boxed{\text{In terms of original variables} \qquad u(x,\, y) = \frac{8}{3}\,(y - \frac{1}{4}x) + e^{\frac{1}{3}\,(y-x)}\, c_1\,(y - \frac{1}{4}x) + K\,(y - x)}$

1b. $u_{xx} + u_{xy} + u_{yy} + u_x = 0$

$A = 1 \qquad B = 1 \qquad C = 1 \qquad \Delta = 1 - 4 = -3 < 0$ $\qquad$ <u>elliptic</u>

$$\frac{dy}{dx} = \frac{1 \pm \sqrt{-3}}{2}$$

$$2dy = (1 \pm \sqrt{3}i)\, dx$$

$$\xi = 2y - (1 + \sqrt{3}i)x \qquad\qquad \eta = 2y - (1 - \sqrt{3}i)x$$

$$\alpha = \frac{1}{2}(\xi + \eta) = 2y - x$$

$$\beta = \frac{1}{2i}(\xi - \eta) = -\sqrt{3}x$$

$$\alpha_x = -1 \qquad \alpha_y = 2 \qquad \alpha_{xx} = 0 \qquad \alpha_{xy} = 0 \qquad \alpha_{yy} = 0$$

$$\beta_x = -\sqrt{3} \qquad \beta_y = 0 \qquad \beta_{xx} = 0 \qquad \beta_{xy} = 0 \qquad \beta_{yy} = 0$$

$$A^* = C^* = 1(-1)^2 + 1(-1)2 + 1(2)^2 = 1 - 2 + 4 = 3$$
$$B^* = 0$$
$$D^* = 0 + 0 + 0 + 1(-1) + 0 = -1$$
$$E^* = 0 + 0 + 0 + 1(-\sqrt{3}) + 0 = -\sqrt{3}$$
$$F^* = 0$$
$$G^* = 0$$

$$3u_{\alpha\alpha} + 3u_{\beta\beta} - u_\alpha - \sqrt{3}\, u_\beta = 0$$

$$\boxed{u_{\alpha\alpha} + u_{\beta\beta} = \frac{1}{3}u_\alpha + \frac{\sqrt{3}}{3}\, u_\beta}$$

1c. $3u_{xx} + 10u_{xu} + 3u_{yy} = x + 1$

$$A = C = 3 \qquad B = 10 \qquad \Delta = 100 - 36 = 64 > 0 \qquad \underline{\text{hyperbolic}}$$

$$\frac{dy}{dx} = \frac{10 \pm 8}{6} \begin{array}{l} \nearrow 3 \\ \searrow 1/3 \end{array}$$

$$\xi = y - 3x \qquad \eta = y - \frac{1}{3}x$$

$$\xi_x = -3 \qquad \xi_y = 1 \qquad \xi_{xx} = 0 \qquad \xi_{xy} = 0 \qquad \xi_{yy} = 0$$

$$\eta_x = -\frac{1}{3} \qquad \eta_y = 1 \qquad \eta_{xx} = 0 \qquad \eta_{xy} = 0 \qquad \eta_{yy} = 0$$

$$A^* = C^* = 0$$
$$B^* = 2 \cdot 3(-3)(-\frac{1}{3}) + 10(-3 - \frac{1}{3}) + 2 \cdot 3(1)(1) = 6 - \frac{100}{3} + 6 = -\frac{64}{3}$$
$$D^* = 0 + 0 + 0 + 0 + 0 = 0$$
$$E^* = 0 + 0 + 0 + 0 + 0 = 0$$
$$F^* = 0$$
$$G^* = x + 1$$

$$-\tfrac{64}{3}u_{\xi\eta} = x + 1$$

$$\left.\begin{array}{l} \xi = y - 3x \\[1em] \underline{\eta = y - \frac{1}{3}x} \end{array}\right\} -$$

$$\xi - \eta = -\frac{8}{3}x$$

$$x = \frac{3}{8}(\eta - \xi)$$

$$-\frac{64}{3}u_{\xi\eta} = \frac{3}{8}(\eta - \xi) + 1$$

$$\boxed{u_{\xi\eta} = -\tfrac{9}{512}(\eta - \xi) - \tfrac{3}{64}}$$

To Find the general solution !

$$u_{\xi\,\eta} \;=\; -\,\frac{9}{512}\,(\eta \,-\, \xi) \,-\, \frac{3}{64}$$

$$u_{\xi} \;=\; -\,\frac{9}{512}\,(\frac{1}{2}\eta^2 \,-\, \eta\,\xi) \,-\, \frac{3}{64}\eta \,+\, f(\xi)$$

$$u \;=\; -\,\frac{9}{512}\,(\frac{1}{2}\eta^2\,\xi \,-\, \frac{1}{2}\,\xi^2\,\eta) \,-\, \frac{3}{64}\eta\,\xi \,+\, F\,(\xi) \,+\, G\,(\eta)$$

$$=\; \frac{9}{512}\,\cdot\,\frac{1}{2}\eta\,\xi(\xi \,-\, \eta) \,-\, \frac{3}{64}\,\eta\,\xi \,+\, F\,(\xi) \,+\, G\,(\eta)$$

$$u(x,\,y) \;=\; \frac{9}{1024}\,\left(y \,-\, \frac{1}{3}x\right)\,(y \,-\, 3x)\,\underbrace{\left(\frac{1}{3}x \,-\, 3x\right)}_{-\frac{8}{3}x} - \frac{3}{64}\,\left(y \,-\, \frac{1}{3}x\right)\,(y \,-\, 3x)$$

$$+\,F\,(y \,-\, 3x) \,+\, G\,(y - \tfrac{1}{3}x)$$

$$=\; \frac{9}{1024}\,\cdot\,\frac{-8}{3}x\,\left(y \,-\, \frac{1}{3}x\right)\,(y \,-\, 3x) \,-\, \frac{3}{64}\,\left(y \,-\, \frac{1}{3}x\right)\,(y \,-\, 3x) \,+\, F\,(y \,-\, 3x)$$
$$+\,G\,(y - \tfrac{1}{3}x)$$

$$\boxed{u(x\,,\,y) \;=\; \left(-\frac{3}{128}x \,-\, \frac{3}{64}\right)(y \,-\, \frac{1}{3}x)\,(y \,-\, 3x) + \,F\,(y \,-\, 3x) \,+\, G\,\left(y - \frac{1}{3}x\right)}$$

check !

$$u_x \;=\; -\frac{3}{128}\,(y \,-\, \frac{1}{3}x)\,(y \,-\, 3x) + \,(-\frac{3}{128}x \,-\, \frac{3}{64})\,(-\frac{1}{3})\,(y \,-\, 3x)$$

$$+\,\left(-\frac{3}{128}x \,-\, \frac{3}{64}\right)\left(y \,-\, \frac{1}{3}x\right)\,(-3) \,-\, 3F'\,(y \,-\, 3x) \,-\frac{1}{3}\,G'\,\left(y - \frac{1}{3}x\right)$$

$$u_y \;=\; \left(-\frac{3}{128}x \,-\, \frac{3}{64}\right)\,(y - 3x) + \left(-\frac{3}{128}x \,-\, \frac{3}{64}\right)\,\left(y - \frac{1}{3}x\right) + F'\,(y - 3x) + G'\,\left(y - \frac{1}{3}x\right)$$

$$u_{xx} \;=\; -\frac{3}{128}\left(-\frac{1}{3}\right)\,(y \,-\, 3x) \,+\, \frac{9}{128}\,(y - \frac{1}{3}x) \,+\, (-\frac{3}{128}x \,-\, \frac{3}{64}) \,-\, \frac{1}{3}\,(-\frac{3}{128})\,(y \,-\, 3x)$$

$$-3\,\left(-\frac{3}{128}\right)\,\left(y - \frac{1}{3}x\right) \,-3\,(-\frac{1}{3})\,(-\frac{3}{128}x \,-\, \frac{3}{64}) \,+\, 9F'' \,+\, \frac{1}{9}G''$$

$$u_{xx} \;=\; \frac{1}{64}\,(y \,-\, 3x) + \,\frac{9}{64}\,(y - \frac{1}{3}x) \,+2\,(-\frac{3}{128}x \,-\, \frac{3}{64}) \,+9F''\,(y \,-\, 3x) \,+\, \frac{1}{9}G''(y - \frac{1}{3}x)$$

$$u_{xy} = -\frac{3}{128}(y - 3x) - 3\left(-\frac{3}{128}x - \frac{3}{64}\right) - \frac{3}{128}(y - \frac{1}{3}x) - \frac{1}{3}\left(-\frac{3}{128}x - \frac{3}{64}\right)$$

$$-3F''(y - 3x) - \frac{1}{3}G''(y - \frac{1}{3}x)$$

$$u_{yy} = -\frac{3}{128}x - \frac{3}{64} - \frac{3}{128}x - \frac{3}{64} + F''(y - 3x) + G''\left(y - \frac{1}{3}x\right)$$

$$3u_{xx} + 10u_{xy} + 3u_{yy} = \frac{3}{64}(y - 3x) + \frac{27}{64}(y - \frac{1}{3}x) + 6\left(-\frac{3}{128}x - \frac{3}{64}\right) + 27F'' + \frac{1}{3}G''$$

$$-\frac{30}{128}(y - 3x) - \frac{15}{64}(y - \frac{1}{3}x) - \frac{100}{3}\left(-\frac{3}{128}x - \frac{3}{64}\right) - 30F'' - \frac{10}{3}G''$$

$$+6\left(-\frac{3}{128}x - \frac{3}{64}\right) + 3F'' + 3G''$$

$$= -\frac{12}{64}(y - 3x) + \frac{12}{64}(y - \frac{1}{3}x) - \frac{64}{3}\left(-\frac{3}{128}x - \frac{3}{64}\right)$$

$$= \frac{9}{16}x - \frac{1}{16}x + \frac{1}{2}x + 1 \qquad = x + 1$$

checks

1d. $u_{xx} + 2u_{xy} + 3u_{yy} + 4u_x + 5u_y + u = e^x$

$$A = 1 \qquad B = 2 \qquad C = 3 \qquad \Delta = 4 - 12 = -8 < 0 \qquad \underline{\text{elliptic}}$$

$$\frac{dy}{dx} = \frac{2 \pm \sqrt{-8}}{2} = 1 \pm i\sqrt{2}$$

$$y = (1 \pm i\sqrt{2})x + C$$

$$\xi = y - (1 + i\sqrt{2})x$$

$$\eta = y - (1 - i\sqrt{2})x$$

$$\alpha = y - x$$

$$\beta = -\sqrt{2}x \qquad \Rightarrow x = -\frac{\beta}{\sqrt{2}}$$

$$\alpha_x = -1 \qquad \alpha_y = 1 \qquad \alpha_{xx} = \alpha_{xy} = \alpha_{yy} = 0$$

$$\beta_x = -\sqrt{2} \qquad \beta_y = 0 \qquad \beta_{xx} = \beta_{xy} = \beta_{yy} = 0$$

$$A^* = C^* = 1(-1)^2 + 2(-1)1 + 3(1)^2 = 1 - 2 + 3 = 2$$
$$B^* = 0$$
$$D^* = 0 + 0 + 0 + 4(-1) + 5(1) = 1$$
$$E^* = 0 + 0 + 0 + 4(-\sqrt{2}) + 0 = -4\sqrt{2}$$
$$F^* = 1$$
$$G^* = e^x$$

$$2u_{\alpha\alpha} + 2u_{\beta\beta} + u_\alpha - 4\sqrt{2}\,u_\beta + u = e^x$$

$$\boxed{u_{\alpha\alpha} + u_{\beta\beta} = -\tfrac{1}{2}u_\alpha + 2\sqrt{2}\,u_\beta - \tfrac{1}{2}u + \tfrac{1}{2}e^{-\beta/\sqrt{2}}}$$

1e. $2u_{xx} - 4u_{xy} + 2u_{yy} + 3u = 0$

$$A = C = 2 \qquad B = -4 \qquad \Delta = 16 - 16 = 0 \qquad \underline{\text{parabolic}}$$

$$\frac{dy}{dx} = \frac{-4 \pm 0}{4} = -1$$

$$dy = -dx$$

$$\begin{cases} \xi = y + x & \xi_x = 1 \quad \xi_y = 1 \quad \xi_{xx} = \xi_{xy} = \xi_{yy} = 0 \\ \eta = x & \eta_x = 1 \quad \eta_y = 0 \quad \eta_{xx} = \eta_{xy} = \eta_{yy} = 0 \end{cases}$$

$$A^* = 0$$
$$B^* = 0$$
$$C^* = 2 - 4 \cdot 0 + 2 \cdot 0 = 2$$
$$D^* = 0 + 0 + 0 + 0 + 0 = 0$$
$$E^* = 0 + 0 + 0 + 0 + 0 = 0$$
$$F^* = 3$$
$$G^* = 0$$

$$2u_{\eta\eta} + 3u = 0$$

$$\boxed{u_{\eta\eta} = -\tfrac{3}{2}u}$$

1f. $u_{xx} + 5u_{xy} + 4u_{yy} + 7u_y = \sin x$

$$A = 1 \qquad B = 5 \qquad C = 4 \qquad \Delta = 25 - 16 = 9 > 0 \qquad \underline{\text{hyperbolic}}$$

$$\frac{dy}{dx} = \frac{5 \pm 3}{2} \begin{matrix} \nearrow 4 \\ \searrow 1 \end{matrix}$$

$$\begin{cases} \xi = y - 4x & \xi_x = -4 \quad \xi_y = 1 \quad \xi_{xx} = \xi_{xy} = \xi_{yy} = 0 \\ \eta = y - x & \eta_x = -1 \quad \eta_y = 1 \quad \eta_{xx} = \eta_{xy} = \eta_{yy} = 0 \end{cases}$$

$$A^* = C^* = 0$$

$$B^* = 2(1)(-4)(-1) + 5((-4)(1) + 1(-1)) + 2(4)(1)(1) = 8 - 25 + 8 = -9$$

$$D^* = 0 + 0 + 0 + 0 + 7(1) = 7$$

$$E^* = 0 + 0 + 0 + 0 + 7(1) = 7$$

$$F^* = 0$$

$$G^* = \sin x$$

$$-9\, u_{\xi\eta} + 7(u_\xi + u_\eta) = \sin x$$

$$u_{\xi\eta} = \frac{7}{9}(u_\xi + u_\eta) - \frac{1}{9}\sin x$$

$$\xi - \eta = -3x$$

$$x = \frac{\eta - \xi}{3}$$

$$\boxed{u_{\xi\eta} = \tfrac{7}{9}(u_\xi + u_\eta) - \tfrac{1}{9}\sin\left(\tfrac{\eta-\xi}{3}\right)}$$

2a. $y = x + c$

$y = \frac{1}{4}x + c$

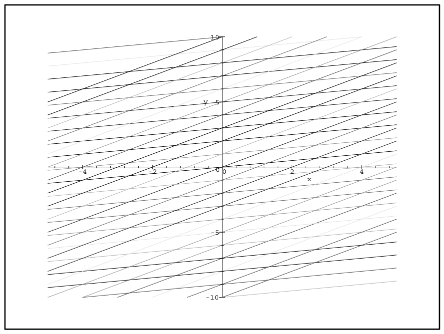

Figure 9: Maple plot of characteristics for 2.3 2a

2b. elliptic . no real characteristics

2c. $y = 3x + c$

$y = \frac{1}{3}x + c$

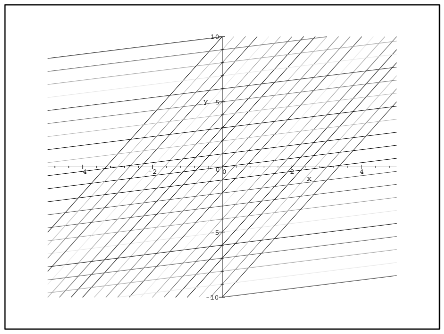

Figure 10: Maple plot of characteristics for 2.3 2c

2d. elliptic . no real characteristics

2e. $y = x + c$ see 2a

2f. $y = 4x + c$

$y = x + c \quad \rightarrow \quad$ (see 2a)

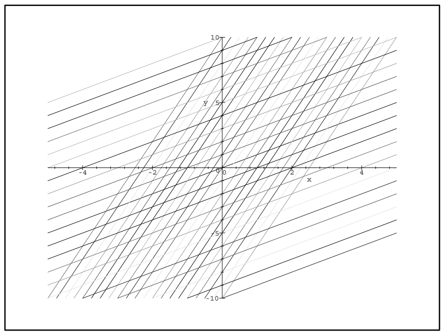

Figure 11: Maple plot of characteristics for 2.3 2f

2.5 Linear Systems

2.6 General Solution

Problems

1. Determine the general solution of

a. $u_{xx} - \frac{1}{c^2} u_{yy} = 0 \qquad c = \text{constant}$
b. $u_{xx} - 3u_{xy} + 2u_{yy} = 0$
c. $u_{xx} + u_{xy} = 0$
d. $u_{xx} + 10u_{xy} + 9u_{yy} = y$

2. Transform the following equations to

$$U_{\xi\eta} = cU$$

by introducing the new variables

$$U = ue^{-(\alpha\xi + \beta\eta)}$$

where α, β to be determined

a. $u_{xx} - u_{yy} + 3u_x - 2u_y + u = 0$
b. $3u_{xx} + 7u_{xy} + 2u_{yy} + u_y + u = 0$

(Hint: First obtain a canonical form)

3. Show that

$$u_{xx} = au_t + bu_x - \frac{b^2}{4}u + d$$

is parabolic for a, b, d constants. Show that the substitution

$$u(x,t) = v(x,t)e^{\frac{b}{2}x}$$

transforms the equation to

$$v_{xx} = av_t + de^{-\frac{b}{2}x}$$

1a. $u_{xx} - \frac{1}{c^2} u_{yy} = 0$

$A = 1 \qquad B = 0 \qquad C = -\frac{1}{c^2} \qquad \Delta = \frac{4}{c^2} > 0 \qquad \underline{\text{hyperbolic}}$

$$\frac{dy}{dx} = \frac{\pm \frac{2}{c}}{2} = \pm \frac{1}{c}$$

$$y = \pm \frac{1}{c} x + K$$

$$\xi = y + \frac{1}{c} x$$

$$\eta = y - \frac{1}{c} x$$

Canonical form:

$$u_{\xi \eta} = 0$$

The solution is:

$$u = f(\xi) + g(\eta)$$

Substitute for ξ and η to get the solution in the original domain:

$$u(x, y) = f(y + \frac{1}{c} x) + g(y - \frac{1}{c} x)$$

1b. $u_{xx} - 3u_{xy} + 2u_{yy} = 0$

$A = 1 \qquad B = -3 \qquad C = 2 \qquad \Delta = 9 - 8 = 1 \qquad$ <u>hyperbolic</u>

$$\frac{dy}{dx} = \frac{-3 \pm 1}{2} \begin{matrix} \nearrow -2 \\ \searrow -1 \end{matrix}$$

$y = -2x + K_1$

$y = -x + K_2$

$\xi = y + 2x \qquad \xi_x = 2 \qquad \xi_y = 1$

$\eta = y + x \qquad \eta_x = 1 \qquad \eta_y = 1$

$u_x = 2u_\xi + u_\eta$

$u_y = u_\xi + u_\eta$

$u_{xx} = 2\,(2u_{\xi\xi} + u_{\xi\eta}) + 2u_{\xi\eta} + u_{\eta\eta}$

$\Rightarrow \quad u_{xx} = 4u_{\xi\xi} + 4u_{\xi\eta} + u_{\eta\eta}$

$u_{xy} = 2\,(u_{\xi\xi} + u_{\xi\eta}) + u_{\xi\eta} + u_{\eta\eta} = 2u_{\xi\xi} + 3u_{\xi\eta} + u_{\eta\eta}$

$u_{yy} = u_{\xi\xi} + 2u_{\xi\eta} + u_{\eta\eta}$

$$\begin{aligned} u_{xx} - 3u_{xy} + 2u_{yy} &= 4u_{\xi\xi} + 4u_{\xi\eta} + u_{\eta\eta} - 3\,(2u_{\xi\xi} + 3u_{\xi\eta} + u_{\eta\eta}) + 2\,(u_{\xi\xi} + 2u_{\xi\eta} + u_{\eta\eta}) \\ &= -u_{\xi\eta} \end{aligned}$$

$\Rightarrow u_{\xi\eta} = 0$

The solution in the original domain is then:

$u(x, y) = f\,(y + 2x) + g\,(y + x)$

1c. $u_{xx} + u_{xy} = 0$

$A = 1 \qquad B = 1 \qquad C = 0 \qquad \Delta = 1 \qquad \underline{\text{hyperbolic}}$

$$\frac{dy}{dx} = \frac{+1 \pm 1}{2} \begin{array}{l} \nearrow +1 \\ \searrow 0 \end{array}$$

$y = +x + K_1$

$y = K_2$

$$\begin{cases} \xi = y - x & \xi_x = -1 \quad \xi_y = 1 \\ \eta = y & \eta_x = 0 \quad \eta_y = 1 \end{cases}$$

$u_x = -u_\xi + u_\eta \underbrace{\eta_x}_{=0} = -u_\xi$

$u_y = u_\xi + u_\eta$

$$u_{xx} = u_{\xi\xi}$$

$$u_{xy} = -u_{\xi\xi} - u_{\xi\eta}$$

$u_{xx} + u_{xy} = -u_{\xi\eta} = 0$

The solution in the original domain is then:
$u = f(y - x) + g(y)$

1d. $u_{xx} + 10u_{xy} + 9u_{yy} = y$

$A = 1 \qquad B = 10 \qquad C = 9 \qquad \Delta = 100 - 36 = 64 \qquad \underline{\text{hyperbolic}}$

$$\frac{dy}{dx} = \frac{10 \pm 8}{2} \begin{matrix} \nearrow 9 \\ \searrow 1 \end{matrix}$$

$\xi = y - 9x \quad \xi_x = -9 \quad \xi_y = 1$

$\eta = y - x \quad \eta_x = -1 \quad \eta_y = 1$

$u_x = -9u_\xi - u_\eta$

$u_y = u_\xi + u_\eta$

$$\begin{aligned} u_{xx} &= -9\left(-9u_{\xi\xi} - u_{\xi\eta}\right) - \left(-9u_{\xi\eta} - u_{\eta\eta}\right) \\ &= 81u_{\xi\xi} + 18u_{\xi\eta} + u_{\eta\eta} \end{aligned}$$

$$\begin{aligned} u_{xy} &= -9\left(u_{\xi\xi} + u_{\xi\eta}\right) - \left(u_{\xi\eta} + u_{yy}\right) \\ &= -9u_{\xi\xi} - 10u_{\xi\eta} - u_{\eta\eta} \end{aligned}$$

$u_{yy} = u_{\xi\xi} + 2u_{\xi\eta} + u_{\eta\eta}$

$$u_{xx} + 10u_{xy} + 9u_{yy} = \underbrace{(81 - 90 + 9)}_{=0} u_{\xi\xi} + (18 - 100 + 18)u_{\xi\eta} + \underbrace{(1 - 10 + 9)}_{=0} u_{\eta\eta} = y$$

$$-64u_{\xi\eta} = y$$

Substitute for y by using the transformation

$$\left.\begin{aligned} \xi &= y - 9x \\ 9\eta &= 9y - 9x \end{aligned}\right\} -$$

$$\overline{\xi - 9\eta = -8y}$$

$$y = \frac{9\eta - \xi}{8}$$

$$u_{\xi\eta} = \frac{\frac{9\eta - \xi}{8}}{-64} = \frac{\xi}{512} - \frac{9\eta}{512}$$

$$u_{\xi\eta} = \frac{\xi}{512} - \frac{9\eta}{512}$$

To solve this PDE let ξ be fixed and integrate with respect to η

$$\Rightarrow\ u_\xi = \frac{\xi}{512}\,\eta - \frac{9}{512}\,\frac{1}{2}\eta^2 + f\,(\xi)$$

$$u = \frac{1}{2}\,\frac{\xi^2\eta}{512} - \frac{9}{2}\,\frac{1}{512}\,\xi\,\eta^2 + F(\xi) + g\,(\eta)$$

The solution in xy domain is:

$$u(x,\,y) = \frac{(y - 9x)^2(y - x)}{1024} - \frac{9}{1024}\,(y - 9x)(y - x)^2 + F(y - 9x) + g(y - x)$$

2a. $u_{xx} - u_{yy} + 3u_x - 2u_y + u = 0$

$$U = u\,e^{-(\alpha\xi + \beta\eta)}$$

$A = 1 \qquad B = 0 \qquad C = -1 \qquad \Delta = 4 \qquad$ <u>hyperbolic</u>

$$\frac{dy}{dx} = \frac{\pm 2}{2} = \pm 1$$

$$\xi = y - x$$

$$\eta = y + x$$

$$u_x = -u_\xi + u_\eta$$

$$u_y = u_\xi + u_\eta$$

$$u_{xx} = -(-u_{\xi\xi} + u_{\xi\eta}) + (-u_{\xi\eta} + u_{\eta\eta}) = u_{\xi\xi} - 2u_{\xi\eta} + u_{\eta\eta}$$

$$u_{yy} = u_{\xi\xi} + 2u_{\xi\eta} + u_{\eta\eta}$$

$$-4u_{\xi\eta} - 3u_\xi + 3u_\eta - 2u_\xi - 2u_\eta + u = 0$$

$$-4u_{\xi\eta} - 5u_\xi + u_\eta + u = 0$$

$$U = u\,e^{-(\alpha\xi + \beta\eta)} \Rightarrow u = U\,e^{(\alpha\xi + \beta\eta)}$$

$$u_\xi = U_\xi\,e^{(\alpha\xi + \beta\eta)} + \alpha\,U\,e^{(\alpha\xi + \beta\eta)}$$

$$u_\eta = U_\eta\,e^{(\alpha\xi + \beta\eta)} + \beta\,U\,e^{(\alpha\xi + \beta\eta)}$$

$$u_{\xi\eta} = U_{\xi\eta}\,e^{(\alpha\xi + \beta\eta)} + \beta\,U_\xi\,e^{(\alpha\xi + \beta\eta)} + \alpha\,U_\eta\,e^{(\alpha\xi + \beta\eta)} + \alpha\,\beta\,U\,e^{(\alpha\xi + \beta\eta)}$$

$$-4U_{\xi\eta} - 4\beta\,U_\xi - 4\alpha\,U_\eta - 4\alpha\,\beta\,U - 5U_\xi - 5\alpha\,U + U_\eta + \beta\,U + U = 0$$

$$-4U_{\xi\eta} + (-4\beta - 5)U_\xi + (-4\alpha + 1)U_\eta + (-4\alpha\,\beta - 5\alpha + \beta + 1)\,U = 0$$

$$\Downarrow \qquad\qquad \Downarrow \qquad\qquad \Downarrow$$

$$\beta = -5/4 \qquad \alpha = 1/4 \qquad -4(1/4)(-5/4) - 5(1/4) + (-5/4) + 1 = -1/4$$

$$-4U_{\xi\eta} - \frac{1}{4}U = 0$$

$\boxed{U_{\xi\eta} = -\frac{1}{16}U}$ required form

2b. $3u_{xx} + 7u_{xy} + 2u_{yy} + u_y + u = 0$

$A = 3 \qquad B = 7 \qquad C = 2 \qquad \Delta = 49 - 24 = 25 \qquad \underline{\text{hyperbolic}}$

$$\frac{dy}{dx} = \frac{7 \pm 5}{6} \begin{array}{l} \nearrow 2 \\ \searrow \frac{1}{3} \end{array}$$

$\xi = y - 2x \qquad \xi_x = -2 \qquad \xi_y = 1$

$\eta = y - \frac{1}{3}x \qquad \eta_x = -\frac{1}{3} \qquad \eta_y = 1$

$u_x = -2u_\xi - \frac{1}{3}u_\eta$

$u_y = u_\xi + u_\eta$

$u_{xx} = -2\left(-2u_{\xi\xi} - \frac{1}{3}u_{\xi\eta}\right) - \frac{1}{3}\left(-2u_{\xi\eta} - \frac{1}{3}u_{\eta\eta}\right)$

$u_{xx} = 4u_{\xi\xi} + \frac{4}{3}u_{\xi\eta} + \frac{1}{9}u_{\eta\eta}$

$u_{xy} = -2\left(u_{\xi\xi} + u_{\xi\eta}\right) - \frac{1}{3}\left(u_{\xi\eta} + u_{\eta\eta}\right)$

$u_{xy} = -2u_{\xi\xi} - \frac{7}{3}u_{\xi\eta} - \frac{1}{3}u_{\eta\eta}$

$u_{yy} = u_{\xi\xi} + 2u_{\xi\eta} + u_{\eta\eta}$

$4u_{\xi\eta} - \frac{49}{3}u_{\xi\eta} + 4u_{\xi\eta} + u_\xi + u_\eta + u = 0$

$\boxed{-\frac{25}{3}u_{\xi\eta} + u_\xi + u_\eta + u = 0}$

Use last page:

$$\frac{-25}{3}\left(U_{\xi\eta} + \beta U_\xi + \alpha U_\eta + \alpha\beta U\right) + U_\xi + \alpha U + U_\eta + \beta U + U = 0$$

$$\frac{-25}{3}U_{\xi\eta} + \underbrace{\left(\frac{-25}{3}\beta + 1\right)}_{\Downarrow}U_\xi + \underbrace{\left(\frac{-25}{3}\alpha + 1\right)}_{\Downarrow}U_\eta + \underbrace{\left(\frac{-25}{3}\alpha\beta + \alpha + \beta + 1\right)}_{\Downarrow}U = 0$$

$$\beta = 3/25 \qquad\qquad \alpha = 3/25 \qquad\qquad -\frac{3}{25} + \frac{3}{25} + \frac{3}{25} + 1 = \frac{28}{25}$$

$\frac{-25}{3}u_{\xi\eta} + \frac{28}{25}U = 0 \qquad \Rightarrow \qquad \boxed{U_{\xi\eta} = \frac{3}{25}\,\frac{28}{25}\,U}$

3.

$$u_{xx} = au_t + bu_x - \frac{b^2}{4}u + d$$

$A = 1 \qquad B = C = 0 \qquad \Rightarrow \qquad \Delta = 0$ <u>parabolic</u>

$\dfrac{dx}{dt} = 0$ already in canonical form since u_{xx} is the only 2^{nd} order term

$$u = \upsilon e^{\frac{b}{2}x}$$

$$u_x = \upsilon_x e^{\frac{b}{2}x} + \frac{b}{2}\upsilon e^{\frac{b}{2}x}$$

$$u_{xx} = \upsilon_{xx} e^{\frac{b}{2}x} + b\,\upsilon_x e^{\frac{b}{2}x} + \frac{b^2}{4}\upsilon e^{\frac{b}{2}x}$$

$$u_t = \upsilon_t e^{\frac{b}{2}x}$$

$$\Rightarrow \quad \upsilon_{xx} + b\upsilon_x + \frac{b^2}{4}\upsilon = a\upsilon_t + b\left(\upsilon_x + \frac{b}{2}\upsilon\right) - \frac{b^2}{4}\upsilon + de^{-\frac{b}{2}x}$$

Since υ_x and υ terms cancel out we have:

$$\upsilon_{xx} = a\upsilon_t + de^{-\frac{b}{2}x}$$

CHAPTER 3

3 Method of Characteristics

3.1 Advection Equation (first order wave equation)

Problems

1. Solve

$$\frac{\partial w}{\partial t} - 3\frac{\partial w}{\partial x} = 0$$

subject to

$$w(x,0) = \sin x$$

2. Solve using the method of characteristics

   a. $\dfrac{\partial u}{\partial t} + c\dfrac{\partial u}{\partial x} = e^{2x}$ subject to $u(x,0) = f(x)$

   b. $\dfrac{\partial u}{\partial t} + x\dfrac{\partial u}{\partial x} = 1$ subject to $u(x,0) = f(x)$

   c. $\dfrac{\partial u}{\partial t} + 3t\dfrac{\partial u}{\partial x} = u$ subject to $u(x,0) = f(x)$

   d. $\dfrac{\partial u}{\partial t} - 2\dfrac{\partial u}{\partial x} = e^{2x}$ subject to $u(x,0) = \cos x$

   e. $\dfrac{\partial u}{\partial t} - t^2\dfrac{\partial u}{\partial x} = -u$ subject to $u(x,0) = 3e^{x}$

3. Show that the characteristics of

$$\frac{\partial u}{\partial t} + 2u\frac{\partial u}{\partial x} = 0$$
$$u(x,0) = f(x)$$

are straight lines.

4. Consider the problem

$$\frac{\partial u}{\partial t} + 2u\frac{\partial u}{\partial x} = 0$$

$$u(x,0) = f(x) = \begin{cases} 1 & x < 0 \\ 1 + \frac{x}{L} & 0 < x < L \\ 2 & L < x \end{cases}$$

   a. Determine equations for the characteristics
   b. Determine the solution $u(x,t)$
   c. Sketch the characteristic curves.
   d. Sketch the solution $u(x,t)$ for fixed t.

1. The PDE can be rewrriten as a system of two ODEs

$$\frac{dx}{dt} = -3$$

$$\frac{dw}{dt} = 0$$

The solution of the first gives the characteristic curve

$$x + 3t = x_0$$

and the second gives

$$w(x(t), t) = w(x(0), 0) = \sin x_0 = \sin(x + 3t)$$

$\boxed{w(x, t) = \sin(x + 3t)}$

2. a. The ODEs in this case are

$$\frac{dx}{dt} = c$$

$$\frac{du}{dt} = e^{2x}$$

Solve the characteristic equation

$$x = ct + x_0$$

Now solve the second ODE. To do that we have to plug in for x

$$\frac{du}{dt} = e^{2(x_0 + ct)} = e^{2x_0} e^{2ct}$$

$$u(x, t) = e^{2x_0} \frac{1}{2c} e^{2ct} + K$$

The constant of integration can be found from the initial condition

$$f(x_0) = u(x_0, 0) = \frac{1}{2c} e^{2x_0} + K$$

Therefore

$$K = f(x_0) - \frac{1}{2c} e^{2x_0}$$

Plug this K in the solution

$$u(x, t) = \frac{1}{2c} e^{2x_0 + 2ct} + f(x_0) - \frac{1}{2c} e^{2x_0}$$

Now substitute for x_0 from the characteristic curve $\boxed{u(x, t) = \frac{1}{2c} e^{2x} + f(x - ct) - \frac{1}{2c} e^{2(x - ct)}}$

2. b. The ODEs in this case are

$$\frac{dx}{dt} = x$$
$$\frac{du}{dt} = 1$$

Solve the characteristic equation

$$\ln x = t + \ln x_0 \qquad \text{or} \qquad x = x_0 \, e^t$$

The solution of the second ODE is

$$u = t + K \qquad \text{and the constant is} \qquad f(x_0)$$

$$u(x,t) = t + f(x_0)$$

Substitute x_0 from the characteristic curve $\boxed{u(x,t) = t + f\left(x\, e^{-t}\right)}$

2. c. The ODEs in this case are

$$\frac{dx}{dt} = 3t$$
$$\frac{du}{dt} = u$$

Solve the characteristic equation

$$x = \frac{3}{2} t^2 + x_0$$

The second ODE can be written as

$$\frac{du}{u} = dt$$

Thus the solution of the second ODE is

$$\ln u = t + \ln K \qquad \text{and the constant is} \qquad f(x_0)$$

$$u(x,t) = f(x_0)\, e^t$$

Substitute x_0 from the characteristic curve $\boxed{u(x,t) = f\left(x - \frac{3}{2} t^2\right) e^t}$

2. d. The ODEs in this case are

$$\frac{dx}{dt} = -2$$

$$\frac{du}{dt} = e^{2x}$$

Solve the characteristic equation

$$x = -2t + x_0$$

Now solve the second ODE. To do that we have to plug in for x

$$\frac{du}{dt} = e^{2(x_0 - 2t)} = e^{2x_0}\, e^{-4t}$$

$$u(x,t) = e^{2x_0}\left(-\frac{1}{4}\, e^{-4t}\right) + K$$

The constant of integration can be found from the initial condition

$$\cos(x_0) = u(x_0, 0) = -\frac{1}{4}\, e^{2x_0} + K$$

Therefore

$$K = \cos(x_0) + \frac{1}{4}\, e^{2x_0}$$

Plug this K in the solution and substitute for x_0 from the characteristic curve

$$u(x,t) = -\frac{1}{4}\, e^{2(x+2t)}\, e^{-4t} + \cos(x + 2t) + \frac{1}{4}\, e^{2(x+2t)}$$

$$\boxed{u(x,t) = \frac{1}{4}\, e^{2x}\left(e^{4t} - 1\right) + \cos(x + 2t)}$$

To check the answer, we differentiate

$$u_x = \frac{1}{2}\, e^{2x}\left(e^{4t} - 1\right) - \sin(x + 2t)$$

$$u_t = \frac{1}{4}\, e^{2x}\left(4\, e^{4t}\right) - 2\sin(x + 2t)$$

Substitute in the PDE

$$u_t - 2u_x = e^{2x}\, e^{4t} - 2\sin(x + 2t) - 2\left\{\frac{1}{2}\, e^{2x}\left(e^{4t} - 1\right) - \sin(x + 2t)\right\}$$

$$= e^{2x}\, e^{4t} - 2\sin(x + 2t) - e^{2x}\, e^{4t} + e^{2x} + 2\sin(x + 2t)$$

$$= e^{2x} \qquad \text{which is the right hand side of the PDE}$$

2. e. The ODEs in this case are

$$\frac{dx}{dt} = -t^2$$

$$\frac{du}{dt} = -u$$

Solve the characteristic equation

$$x = -\frac{t^3}{3} + x_0$$

Now solve the second ODE. To do that we rewrite it as

$$\frac{du}{u} = -dt$$

Therefore the solution as in 2c

$$\ln u = -t + \ln K \qquad \text{and the constant is} \qquad 3\,e^{x_0}$$

Plug this K in the solution and substitute for x_0 from the characteristic curve

$$\ln u(x,t) = \ln\left[3\,e^{x + \frac{1}{3}t^3}\right] - t$$

$$\boxed{u(x,t) = 3\,e^{x + \frac{1}{3}t^3}\,e^{-t}}$$

To check the answer, we differentiate

$$u_t = 3\,e^x\left(t^2 - 1\right)e^{\frac{1}{3}t^3 - t}$$

$$u_x = 3\,e^x\,e^{\frac{1}{3}t^3 - t}$$

Substitute in the PDE

$$u_t - t^2 u_x = 3\,e^x\,e^{\frac{1}{3}t^3 - t} - t^2\left\{3\,e^x\left(t^2 - 1\right)e^{\frac{1}{3}t^3 - t}\right\}$$

$$= 3\,e^x\,e^{\frac{1}{3}t^3 - t}\left[\left(t^2 - 1\right) - t^2\right] = -\,3\,e^{x + \frac{1}{3}t^3 - t} = -u$$

3. The ODEs in this case are

$$\frac{dx}{dt} = 2u$$
$$\frac{du}{dt} = 0$$

Since the first ODE contains x, t and u, we solve the second ODE first

$$u(x,t) = u(x(0),0) = f(x(0))$$

Plug this u in the first ODE, we get

$$\frac{dx}{dt} = 2f(x(0))$$

The solution is

$$x = x_0 + 2tf(x_0)$$

These are characteristics lines all with slope

$$\frac{1}{2f(x_0)}$$

Note that the characteristic through $x_1(0)$ will have a different slope than the one through $x_2(0)$. That is the straight line are NOT parallel.

4. The ODEs in this case are

$$\frac{dx}{dt} = 2u$$

$$\frac{du}{dt} = 0$$

with

$$u(x,0) = f(x) = \begin{cases} 1 & x < 0 \\ 1 + \frac{x}{L} & 0 < x < L \\ 2 & L < x \end{cases}$$

a. Since the first ODE contains x, t and u, we solve the second ODE first

$$u(x,t) = u(x(0),0) = f(x(0))$$

Plug this u in the first ODE, we get

$$\frac{dx}{dt} = 2f(x(0))$$

The solution is

$$x = x_0 + 2tf(x_0)$$

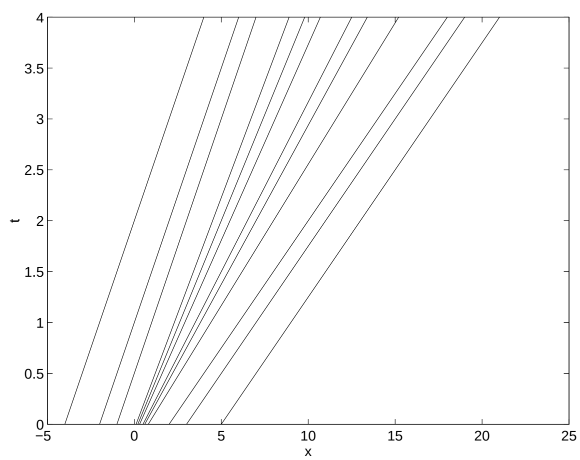

Figure 12: Characteristics for problem 4

b. For $x_0 < 0$ then $f(x_0) = 1$ and the solution is

$$u(x,t) = 1 \qquad \text{on } x = x_0 + 2t$$

or

$$u(x,t) = 1 \qquad \text{on } x < 2t$$

For $x_0 > L$ then $f(x_0) = 2$ and the solution is

$$u(x,t) = 2 \qquad \text{on } x > 4t + L$$

For $0 < x_0 < L$ then $f(x_0) = 1 + x_0/L$ and the solution is

$$u(x,t) = 1 + \frac{x_0}{L} \qquad \text{on } x = 2t\left(1 + \frac{x_0}{L}\right) + x_0$$

That is

$$x_0 = \frac{x - 2t}{2t + L} L$$

Thus the solution on this interval is

$$u(x,t) = 1 + \frac{x - 2t}{2t + L} = \frac{2t + L + x - 2t}{2t + L} = \frac{x + L}{2t + L}$$

Notice that u is continuous.

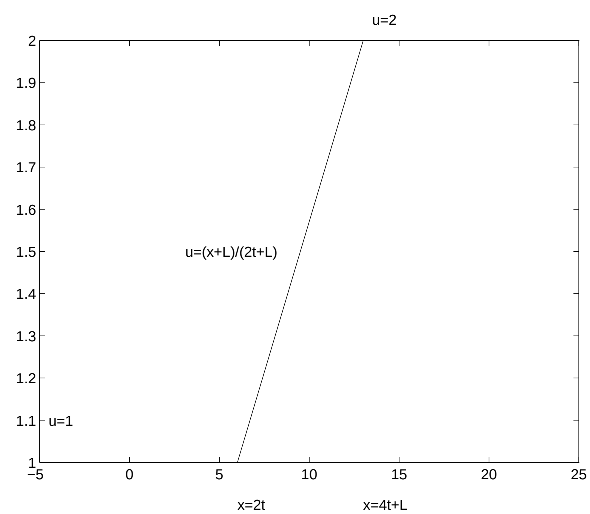

Figure 13: Solution for problem 4