Week 4

- 2.5 Exact Equations
- 2.6 Substitutions

Exact Equations

Previously we have looked at linear equations and separable equations. Now we will look at another special type of equation.

Example

Solve $2x + y^2 + 2xyy' = 0$.

This equation is not linear and is not separable.

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Note that if $\psi(x,y) = x^2 + xy^2$, then $\frac{\partial\psi}{\partial x} = 2x + y^2$ and $\frac{\partial\psi}{\partial y} = 2xy$. So we can write the ODE as

$$\frac{\partial\psi}{\partial x} + \frac{\partial\psi}{\partial y}\frac{dy}{dx} = 0.$$

Since $y(x)$ is a function of x, we also have that

$$\frac{\partial}{\partial x}\Big(\psi\big(x, y(x)\big)\Big) = \frac{\partial \psi}{\partial x} + \frac{\partial \psi}{\partial y}\frac{dy}{dx}$$

by the Chain Rule.

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$$\frac{\partial}{\partial x}\big(x^2 + xy^2\big) = 0.$$

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by the Chain Rule. So our ODE can be written as

$$\frac{\partial}{\partial x}\big(x^2 + xy^2\big) = 0.$$

Therefore

$$x^2 + xy^2 = c.$$

Remark

The key step was finding $\psi(x, y)$.

Now consider

$$M(x, y) + N(x, y)y' = 0. \tag{1}$$

Definition

If we can find a function $\psi(x, y)$ such that

$$\frac{\partial \psi}{\partial x} = M \qquad \text{and} \qquad \frac{\partial \psi}{\partial y} = N,$$

then (1) is called an *exact equation*.

2.4 Exact Equations

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then (1) is called an *exact equation*.

If (1) is exact, then

$$0 = M(x,y) + N(x,y)y' = \frac{\partial\psi}{\partial x}(x,y) + \frac{\partial\psi}{\partial y}(x,y)\frac{dy}{dx} = \frac{d}{dx}\Big(\psi\big(x,y(x)\big)\Big)$$

which has solution

$$\psi(x,y) = c.$$

Remark

To solve an exact equation:

1. Find $\psi(x, y)$;
2. Write $\psi(x, y) = c$.

Notation

$$y' = \frac{dy}{dx} \qquad f_x = \frac{\partial f}{\partial x} \qquad f_y = \frac{\partial f}{\partial y}$$

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Suppose that M, N, M_y and N_x are continuous on the rectangular region $R = \{(x,y) \ : \ \alpha < x < \beta,\ \gamma < y < \delta\}$.

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Theorem

Suppose that M, N, M_y and N_x are continuous on the rectangular region $R = \{(x,y) \ : \ \alpha < x < \beta,\ \gamma < y < \delta\}$. Then

$$M + Ny' = 0 \textit{ is exact} \qquad \Longleftrightarrow \qquad M_y = N_x.$$

Example

Consider

$$(y\cos x + 2xe^y) + (\sin x + x^2e^y - 1)y' = 0.$$

Is this ODE exact? If yes, solve it.

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Yes, the ODE is exact.

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$$M = y\cos x + 2xe^y \qquad M_y = \cos x + 2xe^y$$
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Yes, the ODE is exact. So $\exists\ \psi$ such that

$$\psi_x = M = y\cos x + 2xe^y$$
$$\psi_y = N = \sin x + x^2e^y - 1.$$

2.4 Exact Equations

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Integrating the first equation (wrt x) gives

$$\psi = \int \psi_x \, dx = y\sin x + x^2e^y + h(y).$$

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Then differentiating (wrt y) gives

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$$\psi(x, y) = y\sin x + x^2e^y - y.$$

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$$\psi(x,y) = y\sin x + x^2e^y - y.$$

The solution to the ODE is

$$\boxed{y\sin x + x^2e^y - y = c.}$$

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$$ye^{xy} + e^{xy}y' = 0.$$

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We have

$$\begin{aligned} M &= ye^{xy} & \qquad M_y &= e^{xy} + xye^{xy} \\ N &= e^{xy} & \qquad N_x &= ye^{xy}. \end{aligned}$$

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We have

$$\begin{aligned} M &= ye^{xy} & M_y &= e^{xy} + xye^{xy} \\ N &= e^{xy} & N_x &= ye^{xy}. \end{aligned}$$

Since $M_y \neq N_x$, the ODE is not exact.

Example

Consider

$$\left(4x^3y^3 + \frac{1}{x}\right) + \left(3x^4y^2 + \frac{1}{y}\right) y' = 0.$$

Is this ODE exact? If yes, solve it.

Example

Consider
$$\left(4x^3y^3 + \frac{1}{x}\right) + \left(3x^4y^2 + \frac{1}{y}\right) y' = 0.$$
Is this ODE exact? If yes, solve it.

I leave this one to you to solve. Please check that the solution is

$$\boxed{x^4y^3 + \ln|x| + \ln|y| = c.}$$

Example

Consider

$$1 + \left(1 + 2y + 3y^2\right) y' = 0.$$

Is this ODE exact? If yes, solve it.

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Consider
$$1+\left(1+2y+3y^2\right)y'=0.$$
Is this ODE exact? If yes, solve it.

First note that

$$\begin{aligned} M &= 1 & M_y &= 0 \\ N &= 1+2y+3y^2 & N_x &= 0 = M_y \end{aligned}$$

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Yes, the ODE is exact. So $\exists\ \psi$ such that

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First note that

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Yes, the ODE is exact. So $\exists\ \psi$ such that

$$\begin{aligned} \psi_x &= 1 \\ \psi_y &= 1 + 2y + 3y^2. \end{aligned}$$

We can start with $\psi_x = 1$ or with $\psi_y = 1 + 2y + 3y^2$.

$$\psi_x = 1$$
$$\psi_y = 1 + 2y + 3y^2$$

$$\begin{aligned}
\psi_x &= 1 \\
\psi &= \int 1\,dx = x + h(y) \\
\psi_y &= h'(y) \\
h'(y) &= 1 + 2y + 3y^2 \\
h(y) &= y + y^2 + y^3 \\
\psi &= x + y + y^2 + y^3
\end{aligned}$$

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\psi_y &= 1 + 2y + 3y^2 \\
\psi &= \int 1 + 2y + 3y^2\,dy \\
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\psi_x &= h'(x) \\
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\end{aligned}$$

Therefore the solution is $\boxed{x + y + y^2 + y^3 = c.}$

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Since $M_y \neq N_x$, this ODE is not exact. So our method to solve an exact equation *will not work*.

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Since $M_y \neq N_x$, this ODE is not exact. So our method to solve an exact equation *will not work*. But we are going to try our method anyway, so that we can see what goes wrong.

Suppose that $\exists\ \psi(x, y)$ such that

$$\psi_x = 3xy + y^2$$
$$\psi_y = x^2 + xy.$$

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Integrating ψ_x with respect to x gives

$$\psi = \frac{3}{2}x^2y + xy^2 + h(y).$$

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Thus

$$x^2 + xy = \psi_y = \frac{\partial}{\partial y}\left(\frac{3}{2}x^2y + xy^2 + h(y)\right) = \frac{3}{2}x^2 + 2xy + h'(y).$$

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So we need h to satisfy

$$h'(y) = -\frac{1}{2}x^2 - xy.$$

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This is not possible!!! $h(y)$ must be a function of y, but $-\frac{1}{2}x^2 - xy$ depends on both x and y. So it is not possible to find h. So it is not possible to find ψ. Our method failed because $M_y \neq N_x$.

Integrating Factors

It is sometimes possible to convert a differential equation which is not exact into an exact equation by multiplying it by an integrating factor. (Do you remember how we solve linear equations?)

Consider

$$M(x,y)\,dx + N(x,y)\,dy = 0. \tag{2}$$

Suppose that (2) is not exact.

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Suppose that (2) is not exact. If we multiply by $\mu(x,y)$, we obtain

$$\mu(x,y)M(x,y)\,dx + \mu(x,y)N(x,y)\,dy = 0. \tag{3}$$

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By 3, we know that

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By 3, we know that

$$\text{(3) is exact} \qquad \Longleftrightarrow \qquad (\mu M)_y = (\mu N)_x.$$

Now

$$\begin{aligned} (\mu M)_y &= (\mu N)_x \\ \mu_y M + \mu M_y &= \mu_x N + \mu N_x \\ M\mu_y - N\mu_x + (M_y - N_x)\mu &= 0. \end{aligned} \tag{4}$$

If we can find $\mu(x,y)$ which solves (4), then (3) is exact and we know how to solve exact equations.

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$$\boxed{\frac{d\mu}{dx} = \left(\frac{M_y - N_x}{N}\right)\mu.} \tag{5}$$

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$$\boxed{\frac{d\mu}{dx} = \left(\frac{M_y - N_x}{N}\right)\mu.} \tag{5}$$

If $\frac{M_y - N_x}{N}$ is a function only of x, then there is an integrating factor $\mu(x)$. Please note that (5) is both linear and separable.

If instead we looked for $\mu(y)$, we would obtain the ODE

$$\boxed{\frac{d\mu}{dy} = \left(\frac{N_x - M_y}{M}\right)\mu.} \tag{6}$$

Remark

If we were having classroom exams, you would be expected to remember (5) and (6).

Example

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$$M = 3xy + y^2 \qquad M_y = 3x + 2y$$
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So

$$\frac{M_y - N_x}{N} =$$

and

$$\frac{N_x - M_y}{M} =$$

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So

$$\frac{M_y - N_x}{N} = \frac{(3x + 2y) - (2x + y)}{x^2 + xy} = \frac{x + y}{x(x + y)} = \frac{1}{x}$$

and

$$\frac{N_x - M_y}{M} =$$

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So

$$\frac{M_y - N_x}{N} = \frac{(3x + 2y) - (2x + y)}{x^2 + xy} = \frac{x + y}{x(x + y)} = \frac{1}{x}$$

and

$$\frac{N_x - M_y}{M} = \frac{(2x + y) - (3x + 2y)}{3xy + y^2} = \frac{-x - y}{y(3x + y)}.$$

Note that $\frac{M_y - N_x}{N}$ is a function only of x – so it is possible to find an integrating factor $\mu(x)$. Moreover note that $\frac{N_x - M_y}{M}$ is *not* a function only of y – so it is *not* possible to find a $\mu(y)$.

We calculate that

$$\begin{aligned}
\frac{d\mu}{dx} &= \left(\frac{M_y - N_x}{N}\right)\mu \\
\frac{d\mu}{dx} &= \frac{\mu}{x} \\
\frac{d\mu}{\mu} &= \frac{dx}{x} \\
\int \frac{d\mu}{\mu} &= \int \frac{dx}{x} \\
\ln|\mu| &= \ln|x| + C \\
\mu &= cx
\end{aligned}$$

and we choose $c = 1$ for simplicity. So $\mu(x) = x$.

$$(3xy + y^2) + (x^2 + xy)y' = 0$$

Multiplying our original ODE by $\mu(x) = x$ gives

$$(3x^2y + xy^2) + (x^3 + x^2y)y' = 0.$$

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This ODE is exact ($M_y = 3x^2 + 2xy = N_x$) and we know how to solve exact equations.

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$$(3x^2y + xy^2) + (x^3 + x^2y)y' = 0.$$

This ODE is exact ($M_y = 3x^2 + 2xy = N_x$) and we know how to solve exact equations. We must find ψ such that

$$\psi_x = 3x^2y + xy^2$$
$$\psi_y = x^3 + x^2y.$$

2.4 Exact Equations

$$\psi_x = 3x^2y + xy^2$$
$$\psi_y = x^3 + x^2y$$

Integrating ψ_x wrt x gives

$$\psi = x^3y + \frac{1}{2}x^2y^2 + h(y).$$

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Hence

$$x^3 + x^2y = \psi_y = \frac{\partial}{\partial y}\left(x^3y + \frac{1}{2}x^2y^2 + h(y)\right) = x^3 + x^2y + h'(y)$$

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$$\psi = x^3y + \frac{1}{2}x^2y^2.$$

So the solution to the ODE is

$$\boxed{x^3y + \frac{1}{2}x^2y^2 = c.}$$

Example

Solve
$$ye^{xy} + \left(\left(\frac{2}{y} + x\right) e^{xy}\right) y' = 0.$$

This ODE is not exact (you check!).

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This ODE is not exact (you check!).

$$\frac{M_y - N_x}{N} = \frac{e^{xy} + xye^{xy} - e^{xy} - (2 + xy)\, e^{xy}}{\left(\frac{2}{y} + x\right) e^{xy}} = \frac{-2}{\frac{2}{y} + x}$$

$$\frac{N_x - M_y}{M} = \frac{2e^{xy}}{ye^{xy}} = \frac{2}{y}.$$

Since $\frac{N_x - M_y}{M}$ is a function only of y, we look for $\mu(y)$.

$$\frac{d\mu}{dy} = \frac{N_x - M_y}{M}\mu = \frac{2e^{xy}}{ye^{xy}}\mu = \frac{2}{y}\mu$$

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-
-
- (you complete this calculation)
-
-

Therefore $\mu(y) = y^2$.

Multiplying our ODE by y^2 gives

$$y^3 e^{xy} + \left(\left(2y + xy^2 \right) e^{xy} \right) y' = 0.$$

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Hence the solution is

$$\boxed{y^2 e^{xy} = c.}$$

2.5 Substitutions

Recall how we calculate an integral such as $\int 3x^2 \sin x^3 \; dx$.

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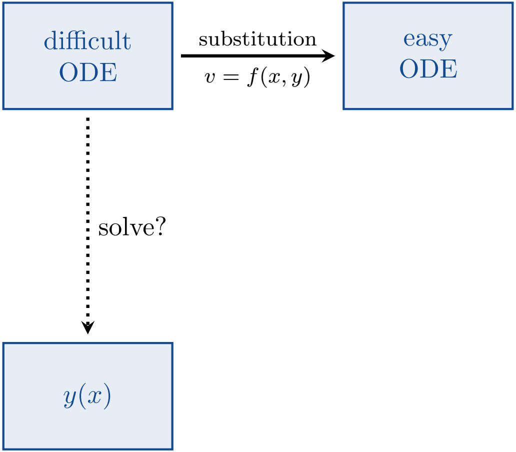
$$\underbrace{\int 3x^2 \sin x^3 \, dx}_{\text{difficult}} = \underbrace{\int \sin u \, du}_{\text{easy}}.$$

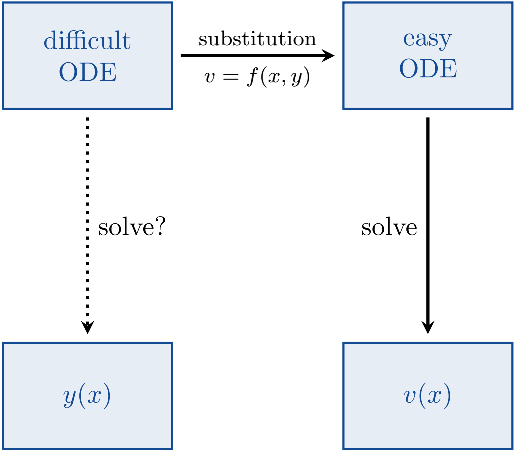
Sometimes we can use the same idea to solve ODEs.

2.5 Substitutions

$$
\begin{tikzcd}
\boxed{\text{difficult ODE}} \arrow[d, dotted, "\text{solve?}"] \\
\boxed{y(x)}
\end{tikzcd}
$$

2.5 Substitutions




difficult ODE
substitution
$v = f(x, y)$
easy ODE
solve?
solve
$y(x)$
$v(x)$

2.5 Substitutions

$$
\begin{tikzcd}[column sep=huge, row sep=huge]
\text{difficult ODE} \arrow[r, "\text{substitution}", "v = f(x,y)"'] \arrow[d, dotted, "\text{solve?}"] & \text{easy ODE} \arrow[d, "\text{solve}"'] \\
y(x) & v(x) \arrow[l]
\end{tikzcd}
$$

2.5 Substitutions

We will use substitutions to solve two types of first order ODE:

- Homogeneous Equations;
- Bernoulli Equations.

Homogeneous Equations

Definition

The first order ODE $\frac{dy}{dx} = f(x, y)$ is called *homogeneous* iff we can write it as

$$\frac{dy}{dx} = g\left(\frac{y}{x}\right).$$

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For example, the following ODEs are homogeneous:

$$\frac{dy}{dx} = \cos\left(\frac{y}{x}\right) \qquad \frac{dy}{dx} = \left(\frac{y}{x}\right)^3 + \frac{y}{x}$$

$$\frac{dy}{dx} = \cos\left(\frac{x}{y}\right) \qquad \frac{dy}{dx} = \frac{\frac{y}{x} - 4}{1 - \frac{y}{x}}$$

For a homogeneous equation, we use the substitution

$$v(x) = \frac{y}{x}.$$

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Note that $y = xv(x)$ and

$$\frac{dy}{dx} = \frac{d}{dx}\big(xv(x)\big) = v + x\frac{dv}{dx}.$$

Example

Solve $\dfrac{dy}{dx} = \dfrac{y-4x}{x-y}$.

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Note first that
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$$\frac{dy}{dx} = \frac{y-4x}{x-y} = \frac{\frac{y}{x}-4}{1-\frac{y}{x}}.$$

If we substitute in $v = \frac{y}{x}$ we get

$$\frac{dy}{dx} = \frac{v-4}{1-v}.$$

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But remember that $\frac{dy}{dx} = v + x\frac{dv}{dx}$.

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$$v + x\frac{dv}{dx} = \frac{v-4}{1-v}$$

and

$$x\frac{dv}{dx} = \frac{v-4}{1-v} - v$$

$$\frac{dy}{dx} = \frac{v-4}{1-v}$$

But remember that $\frac{dy}{dx} = v + x\frac{dv}{dx}$. Hence

$$v + x\frac{dv}{dx} = \frac{v-4}{1-v}$$

and

$$x\frac{dv}{dx} = \frac{v-4}{1-v} - v = \frac{v-4}{1-v} - \frac{v-v^2}{1-v} = \frac{v^2-4}{1-v}$$

Note that

$$x\frac{dv}{dx} = \frac{v^2 - 4}{1 - v}$$

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$$x\frac{dv}{dx} = \frac{v^2 - 4}{1 - v}$$

is a separable equation. You know how to solve separable equations – the following should be revision for you. We rearrange to

$$\left(\frac{1 - v}{v^2 - 4}\right) dv = \frac{dx}{x}$$

$$\left(-\frac{3}{4(v + 2)} - \frac{1}{4(v - 2)}\right) dv = \frac{dx}{x}$$

$$\left(-\frac{3}{4(v+2)} - \frac{1}{4(v-2)}\right) dv = \frac{dx}{x}$$

then integrate to find

$$\begin{aligned}
-\frac{3}{4}\ln|v+2| - \frac{1}{4}\ln|v-2| &= \ln|x| + k \\
\ln|v+2|^3 + \ln|v-2| &= \ln|x|^{-4} - 4k \\
|v+2|^3\,|v-2| &= c\,|x|^{-4} \qquad\qquad (c = \pm e^{-4k}) \\
|x|^4\,|v+2|^3\,|v-2| &= c \\
|vx+2x|^3\,|vx-2x| &= c.
\end{aligned}$$

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$$\begin{aligned}
-\frac{3}{4}\ln|v+2|-\frac{1}{4}\ln|v-2| &= \ln|x|+k \\
\ln|v+2|^3+\ln|v-2| &= \ln|x|^{-4}-4k \\
|v+2|^3\,|v-2| &= c\,|x|^{-4} \qquad\qquad (c=\pm e^{-4k}) \\
|x|^4\,|v+2|^3\,|v-2| &= c \\
|vx+2x|^3\,|vx-2x| &= c.
\end{aligned}$$

Now we have an equation for v. The final step is to find an equation for y.

$$|vx + 2x|^3 \, |vx - 2x| = c.$$

If we substitute $y = vx$ into this equation, we find the solution

$$\boxed{|y + 2x|^3 \, |y - 2x| = c.}$$

Remark

To solve a homogeneous equation:

1. Substitute $v = \frac{y}{x}$ (and $\frac{dy}{dx} = v + x\frac{dv}{dx}$);
2. Solve a separable equation;
3. Substitute $y = vx$.

Example

Solve $\dfrac{dy}{dx} = \dfrac{x^2 + 3y^2}{2xy}$.

Example

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First we rearrange

$$\frac{dy}{dx} = \frac{1 + 3\frac{y^2}{x^2}}{2\frac{y}{x}}$$

and substitute $v = \frac{y}{x}$ and $\frac{dy}{dx} = v + x\frac{dv}{dx}$ to get

$$v + x\frac{dv}{dx} = \frac{1 + 3v^2}{2v}.$$

Rearranging gives

$$x\frac{dv}{dx} = \frac{1+3v^2}{2v} - v = \frac{1+3v^2-2v^2}{2v} = \frac{1+v^2}{2v}.$$

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$$x\frac{dv}{dx} = \frac{1+3v^2}{2v} - v = \frac{1+3v^2-2v^2}{2v} = \frac{1+v^2}{2v}.$$

This is a separable equation which we can solve:

$$\begin{aligned}
\frac{2v\,dv}{1+v^2} &= \frac{dx}{x} \\
\int \frac{2v\,dv}{1+v^2} &= \int \frac{dx}{x} \\
\ln\left|1+v^2\right| &= \ln|x| + k \\
1+v^2 &= cx \\
1+v^2-cx &= 0.
\end{aligned}$$

Substituting $v = \frac{y}{x}$ then gives

$$1 + \frac{y^2}{x^2} - cx = 0$$

and

$$\boxed{x^2 + y^2 - cx^3 = 0.}$$

Bernoulli Equations

Definition

An equation of the form

$$y' + p(t)y = q(t)y^{n}$$

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Solve $\dfrac{dy}{dx} - \left(\dfrac{3}{2x}\right) y = 2xy^{-1}$.

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Note first that this ODE has $n = -1$. Therefore we will use the substitution $v = y^{1-n} = y^{1-(-1)} = y^2$. This means that $y = v^{\frac{1}{2}}$ and

$$\frac{dy}{dx} = \frac{dy}{dv}\frac{dv}{dx} = \frac{1}{2}v^{-\frac{1}{2}}\frac{dv}{dx}.$$

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We take our ODE

$$\frac{dy}{dx} - \left(\frac{3}{2x}\right) y = 2xy^{-1}$$

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and we substitute in $y = v^{\frac{1}{2}}$ and $\frac{dy}{dx} = \frac{1}{2}v^{-\frac{1}{2}}\frac{dv}{dx}$ to obtain

$$\frac{1}{2}v^{-\frac{1}{2}}\frac{dv}{dx} - \left(\frac{3}{2x}\right) v^{\frac{1}{2}} = 2xv^{-\frac{1}{2}}.$$

Multiplying by $2v^{\frac{1}{2}}$ gives

$$\frac{dv}{dx} - \frac{3}{x}v = 4x$$

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which is a linear equation. You know how to solve linear equations, so the following should be revision for you. We multiply by the integrating factor

$$\mu(x) = e^{\int -\frac{3}{x}\,dx} = e^{-3\ln|x|} = \ldots = x^{-3}$$

to get

$$x^{-3}\frac{dv}{dx} - 3x^{-4}v = 4x^{-2}$$

which is

$$\frac{d}{dx}\left(x^{-3}v\right) = 4x^{-2}.$$

Integrating gives

$$\begin{aligned} x^{-3}v &= -4x^{-1} + C \\ v &= -4x^2 + Cx^3. \end{aligned}$$

Integrating gives

$$x^{-3}v = -4x^{-1} + C$$
$$v = -4x^2 + Cx^3.$$

But $v = y^2$, so the solution is

$$\boxed{y^2 = -4x^2 + Cx^3.}$$

Remark

To solve a Bernoulli equation:

1. Substitute $v = y^{1-n}$;
2. Solve a linear equation;
3. Substitute $y^{1-n} = v$.

Example

Solve $x\dfrac{dy}{dx} + 6y = 3xy^{\frac{4}{3}}$.

Example

Solve $x\dfrac{dy}{dx} + 6y = 3xy^{\frac{4}{3}}$.

Note that this time we have $n = \frac{4}{3}$ and $v = y^{1-n} = y^{-\frac{1}{3}}$. Hence $y = v^{-3}$ and

$$\frac{dy}{dx} = \frac{dy}{dv}\frac{dv}{dx} = -3v^{-4}\frac{dv}{dx}.$$

Thus our ODE becomes

$$
\begin{aligned}
-3xv^{-4}\frac{dv}{dx} + 6v^{-3} &= 3xv^{-4} \\
-x\frac{dv}{dx} + 2v &= x \\
\frac{dv}{dx} - \frac{2}{x}v &= -1.
\end{aligned}
$$

Thus our ODE becomes

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This is a linear equation which we can solve using the integrating factor $\mu(x) = x^{-2}$. Please check that its solution is

$$v = x + Cx^2.$$

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$$v = x + Cx^2.$$

Finally we use $v = y^{-\frac{1}{3}}$ to find that

$$\boxed{y = \frac{1}{(x + Cx^2)^3}}.$$

Next Week

- 3.1 Homogeneous Equations with Constant Coefficients
- 3.2 Fundamental Sets of Solutions
- 3.3 Complex Roots of the Characteristic Equation