Your Name

Your Signature

Student ID #

Professor's Name

Your Department

- This exam is closed book.
- Calculators, cell phones are not allowed.
- In order to receive credit, you must **show all of your work**. If you do not indicate the way in which you solved a problem, you may get little or no credit for it, even if your answer is correct. **Show your work in evaluating any limits, derivatives.**
- Place a box around your answer to each question.
- If you need more room, use the backs of the pages and indicate that you have done so.
- Do not ask the invigilator anything.
- Use a **BLUE ball-point pen** to fill the cover sheet. Please make sure that your exam is complete.
- Time limit is 80 min.

Do not write in the table to the right.

| Problem | Points | Score |
|---|---|---|
| 1 | 20 | |
| 2 | 30 | |
| 3 | 20 | |
| 4 | 30 | |
| Total: | 100 | |

1. 20 points Solve the following initial value problem.

$$xy' + 2y = \cos x, \qquad y(2\pi) = 0.$$

**Solution:** it is a linear diff. eq.

$$y' + \frac{2}{x}y = \frac{\cos x}{x}$$

The integrating factor is

$$\lambda = e^{\int \frac{2}{x}dx} = e^{2\ln x} = x^2$$
$$\Rightarrow \lambda y(x) = \int \lambda \frac{\cos x}{x}dx$$
$$x^2 y(x) = \int x\cos x dx$$
$$y(x) = \frac{\sin x}{x} + \frac{\cos x}{x^2} + \frac{C}{x^2}$$

and $C = -1$ so

$$y(x) = \frac{\sin x}{x} + \frac{\cos x}{x^2} - \frac{1}{x^2}$$

2. $\boxed{\text{30 points}}$ Find the solution of the following initial value problem .

$$y''' - 4y'' + 4y' = -3e^{-t}, \qquad y(0) = 0, y'(0) = 0, y''(0) = -1.$$

**Solution: First Way** The characteristic equation of the differential equation above is

$$r^3 - 4r^2 + 4r = 0$$

and its roots are $r_1 = 0$ and $r_2 = r_3 = 2$. Therefore the solution of the homogeneous differential equation $y''' - 4y'' + 4y' = 0$ is

$$y_H(t) = c_1 + c_2 e^{2t} + c_3 t e^{2t}$$

Let us find the $y_P$ by using method of Undetermined Coefficients. Assume that

$$y_P = Ae^{-t} \Rightarrow y_P' = -Ae^{-t} \Rightarrow y_P'' = Ae^{-t} \Rightarrow y_P''' = -Ae^{-t}.$$

When we substitute them, we obtain

$$\begin{aligned} y_P''' - 4y_P'' + 4y_P' &= -3e^{-t} \\ -Ae^{-t} - 4Ae^{-t} - 4Ae^{-t} &= -3e^{-t} \\ -9Ae^{-t} &= -3e^{-t} \end{aligned}$$

The coefficients are caculated as $A = \dfrac{1}{3}$ and the particular solution is $y_P(t) = \dfrac{1}{3}e^{-t}$. Therefore the general solution is

$$y(t) = c_1 + c_2 e^{2t} + c_3 t e^{2t} + \frac{1}{3}e^{-t}$$

Let us find arbitrary constants uniquely by using initial coditions.

$$\begin{aligned} y(0) = 0 \Rightarrow y(0) = c_1 + c_2 + \frac{1}{3} &= 0 \\ y'(0) = 0 \Rightarrow 2c_2 + c_3 - \frac{1}{3} &= 0 \\ y''(0) = 0 \Rightarrow 4c_2 + 4c_3 + \frac{1}{3} &= -1 \end{aligned}$$

Therefore, we obtain $c_1 = -1, c_2 = \dfrac{2}{3}$ and $c_3 = -1$ Then,

$$y(t) = y(t) = -1 + \frac{2}{3}e^{2t} - te^{2t} + \frac{1}{3}e^{-t}$$

**Second Way** Let us calculate the Laplace transform of the differential equation above

$$\begin{aligned} \mathscr{L}\left\{y''' - 4y'' + 4y'\right\} &= \mathscr{L}\left\{-3e^{-t}\right\} \\ \left[s^3\mathscr{L}\{y\} - s^2y(0) - sy'(0) - y''(0)\right] - 4\left[s^2\mathscr{L}\{y\} - sy(0) - y'(0)\right] + 4\left[s\mathscr{L}\{y\} - y(0)\right] &= -\frac{3}{s+1} \\ \left[s^3\mathscr{L}\{y\} + 1\right] - 4s^2\mathscr{L}\{y\} + 4s\mathscr{L}\{y\} &= -\frac{3}{s+1} \\ (s^3 - 4s^2 + 4s)\mathscr{L}\{y\} &= -\frac{3}{s+1} - 1 \\ s(s-2)^2\mathscr{L}\{y\} &= -\frac{s+4}{s+1} \\ \mathscr{L}\{y\} &= -\frac{s+4}{s(s-2)^2(s+1)} \end{aligned}$$

Let us calculate the inverse Laplace Transfonm to determine the $y(t)$ as follows.

$$\begin{aligned} y(t) = \mathscr{L}^{-1}\left\{-\frac{s+4}{s(s-2)^2(s+1)}\right\} &= \mathscr{L}^{-1}\left\{\frac{A}{s} + \frac{B}{s-2} + \frac{C}{(s-2)^2} + \frac{D}{s+1}\right\} \\ y(t) &= \mathscr{L}^{-1}\left\{-\frac{1}{s} + \frac{\frac{2}{3}}{s-2} - \frac{1}{(s-2)^2} + \frac{\frac{1}{3}}{s+1}\right\} \\ y(t) &= -1 + \frac{2}{3}e^{2t} - te^{2t} + \frac{1}{3}e^{-t} \end{aligned}$$

3. $\boxed{\text{20 points}}$ Solve the following system of differential equations.

$$\mathbf{x}' = \begin{bmatrix} 1 & -2 & 0 \\ 2 & 1 & 13 \\ 0 & 0 & 4 \end{bmatrix} \mathbf{x} \qquad \mathbf{x}(0) = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

**Solution:** Let us find the eigenvalues of the matrix above.

$$det(A - \lambda I) = \begin{vmatrix} 1-\lambda & -2 & 0 \\ 2 & 1-\lambda & 13 \\ 0 & 0 & 4-\lambda \end{vmatrix} = (4-\lambda)\left[(1-\lambda)^2 + 4\right] = 0$$

Thus, the eigenvalues are $\lambda_1 = 4$, $\lambda_2 = 1+2i$ and $\lambda_3 = 1-2i$. Let us find the eigenvector corresponding to $\lambda_1 = 4$ as

$$\begin{bmatrix} -3 & -2 & 0 \\ 2 & -3 & 13 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow \mathbf{v} = \begin{bmatrix} -2 \\ 3 \\ 1 \end{bmatrix}$$

and the eigenvector corresponding to $\lambda_2 = 1+2i$ is

$$\begin{bmatrix} -2i & -2 & 0 \\ 2 & -2i & 13 \\ 0 & 0 & 3-2i \end{bmatrix} \Rightarrow \mathbf{w} = \begin{bmatrix} 1 \\ -i \\ 0 \end{bmatrix}.$$

We can determine the third one as $\bar{\mathbf{w}} = \begin{bmatrix} 1 \\ i \\ 0 \end{bmatrix}$. Let us calculate $e^{\lambda t} w$

$$e^{\lambda t} w = e^{(1+2i)t} \begin{bmatrix} 1 \\ -i \\ 0 \end{bmatrix} = e^t (\cos 2t + i \sin 2t) \begin{bmatrix} 1 \\ -i \\ 0 \end{bmatrix} = e^t \begin{bmatrix} \cos 2t + i \sin 2t \\ -i \cos 2t + \sin 2t \\ 0 \end{bmatrix}$$

The general solution is

$$\mathbf{x}(t) = c_1 \begin{bmatrix} \cos 2t \\ \sin 2t \\ 0 \end{bmatrix} e^t + c_2 \begin{bmatrix} \sin 2t \\ -\cos 2t \\ 0 \end{bmatrix} e^t + c_3 \begin{bmatrix} -2 \\ 3 \\ 1 \end{bmatrix} e^{4t}$$

Let us find the particular solution by using initial value.

$$\mathbf{x}(0) = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \Rightarrow \begin{matrix} c_1 - 2c_3 = 1 \\ -c_2 + 3c_3 = 0 \\ c_3 = 1 \end{matrix} \Rightarrow \begin{matrix} c_1 = 3 \\ c_2 = 3 \\ c_3 = 1 \end{matrix}$$

The particular solution is

$$\mathbf{x}(t) = \begin{bmatrix} 3\cos 2t \\ 3\sin 2t \\ 0 \end{bmatrix} e^t + \begin{bmatrix} 3\sin 2t \\ -3\cos 2t \\ 0 \end{bmatrix} e^t + \begin{bmatrix} -2 \\ 3 \\ 1 \end{bmatrix} e^{4t}$$

4. $\boxed{\text{30 points}}$ Solve the following initial value problem.

$$\mathbf{x}' = \begin{bmatrix} 2 & -5 \\ 0 & 1 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 10e^{-3t} \\ 4e^{-3t} - 2 \end{bmatrix}, \qquad \mathbf{x}(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

**Solution: First Way:** Let us find the eigenvalues of the matrix above.

$$det(A - \lambda I) = \begin{vmatrix} 2-\lambda & -5 \\ 0 & 1-\lambda \end{vmatrix} = (2-\lambda)(1-\lambda) = 0$$

Thus, the eigenvalues are $\lambda_1 = 2$ and $\lambda_2 = 1$ and the corresponding eigenvectors are $v = [1 \quad 0]^T$ and $w = [5 \quad 1]^T$ The complementary solution is

$$\mathbf{x_h} = c_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} e^{2t} + c_2 \begin{bmatrix} 5 \\ 1 \end{bmatrix} e^t$$

The particuar solution is can be chooosen as

$$\mathbf{x_p} = \begin{bmatrix} Ae^{-3t} + B \\ Ce^{-3t} + D \end{bmatrix} \Rightarrow \mathbf{x_p}' = \begin{bmatrix} -3Ae^{-3t} \\ -3Ce^{-3t} \end{bmatrix}$$

therefore

$$\mathbf{x}'_\mathbf{p} = \begin{bmatrix} 2 & -5 \\ 0 & 1 \end{bmatrix} \mathbf{x_p} + \begin{bmatrix} 10e^{-3t} \\ 4e^{-3t} - 2 \end{bmatrix}$$

$$\begin{bmatrix} -3Ae^{-3t} \\ -3Ce^{-3t} \end{bmatrix} = \begin{bmatrix} 2 & -5 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} Ae^{-3t} + B \\ Ce^{-3t} + D \end{bmatrix} + \begin{bmatrix} 10e^{-3t} \\ 4e^{-3t} - 2 \end{bmatrix}$$

Therefore $A = -3, B = 5, C = -1$ and $D = 2$. So,

$$\mathbf{x_p} = \begin{bmatrix} -3e^{-3t} + 5 \\ -e^{-3t} + 2 \end{bmatrix}$$

The general solution is

$$\mathbf{x}(t) = c_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} e^{2t} + c_2 \begin{bmatrix} 5 \\ 1 \end{bmatrix} e^t + \begin{bmatrix} -3e^{-3t} + 5 \\ -e^{-3t} + 2 \end{bmatrix}$$

Let us use the initial values to determine $c_1$ and $c_2$.

$$\mathbf{x}(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \Rightarrow \mathbf{x}(0) = c_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 5 \\ 1 \end{bmatrix} + \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

$$\Rightarrow c_1 + 5c_2 + 2 = 1$$

$$c_2 + 1 = 0$$

The arbitrary constants are $c_1 = 4$ and $c_2 = -1$. The solution is

$$\mathbf{x}(t) = 4 \begin{bmatrix} 1 \\ 0 \end{bmatrix} e^{2t} - \begin{bmatrix} 5 \\ 1 \end{bmatrix} e^t + \begin{bmatrix} -3e^{-3t} + 5 \\ -e^{-3t} + 2 \end{bmatrix}$$

$$\mathbf{x}(t) = \begin{bmatrix} 4 \\ 0 \end{bmatrix} e^{2t} - \begin{bmatrix} 5 \\ 1 \end{bmatrix} e^t - \begin{bmatrix} 3 \\ 1 \end{bmatrix} e^{-3t} + \begin{bmatrix} 5 \\ 2 \end{bmatrix}$$

**Second Way:** Let us use the Laplace transform to solve the system above.

$$\mathscr{L}\{\mathbf{x}'\} = \begin{bmatrix} 2 & -5 \\ 0 & 1 \end{bmatrix} \mathscr{L}\{\mathbf{x}\} + \mathscr{L}\left\{ \begin{bmatrix} 10e^{-3t} \\ 4e^{-3t} - 2 \end{bmatrix} \right\}$$

$$s\mathscr{L}\{\mathbf{x}\} - \mathbf{x}(0) = \begin{bmatrix} 2 & -5 \\ 0 & 1 \end{bmatrix} \mathscr{L}\{\mathbf{x}\} + \mathscr{L}\left\{ \begin{bmatrix} 10e^{-3t} \\ 4e^{-3t} - 2 \end{bmatrix} \right\}$$

$$\left( sI - \begin{bmatrix} 2 & -5 \\ 0 & 1 \end{bmatrix} \right) \mathscr{L}\{\mathbf{x}\} = \mathscr{L}\left\{ \begin{bmatrix} 10e^{-3t} \\ 4e^{-3t} - 2 \end{bmatrix} \right\} + \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} s-2 & 5 \\ 0 & s-1 \end{bmatrix} \mathscr{L}\{\mathbf{x}\} = \begin{bmatrix} \dfrac{10}{s+3} \\ \dfrac{4}{s+3} - \dfrac{2}{s} \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\mathscr{L}\{\mathbf{x}\} = \begin{bmatrix} s-2 & 5 \\ 0 & s-1 \end{bmatrix}^{-1} \begin{bmatrix} \dfrac{s+13}{s+3} \\ \dfrac{4}{s+3} - \dfrac{2}{s} \end{bmatrix}$$

$$\mathscr{L}\{\mathbf{x}\} = \frac{1}{(s-2)(s-1)} \begin{bmatrix} s-1 & -5 \\ 0 & s-2 \end{bmatrix} \begin{bmatrix} \dfrac{s+13}{s+3} \\ \dfrac{2s-6}{s(s+3)} \end{bmatrix}$$

$$\mathscr{L}\{\mathbf{x}\} = \begin{bmatrix} \dfrac{s^3+12s^2-23s+30}{s(s-1)(s-2)(s+3)} \\ \dfrac{2s-6}{s(s-1)(s+3)} \end{bmatrix} \Rightarrow \mathbf{x}(t) = \mathscr{L}^{-1}\left\{ \begin{bmatrix} \dfrac{s^3+12s^2-23s+30}{s(s-1)(s-2)(s+3)} \\ \dfrac{2s-6}{(s-1)s(s+3)} \end{bmatrix} \right\}$$

$$\mathbf{x}(t) = \mathscr{L}^{-1}\left\{ \begin{bmatrix} \dfrac{A_1}{s} + \dfrac{B_1}{s-1} + \dfrac{C_1}{s+3} + \dfrac{D_1}{s-2} \\ \dfrac{A_2}{s} + \dfrac{B_2}{s-1} + \dfrac{C_2}{s+3} \end{bmatrix} \right\}$$

$$\mathbf{x}(t) = \mathscr{L}^{-1}\left\{ \begin{bmatrix} \dfrac{5}{s} + \dfrac{-5}{s-1} + \dfrac{-3}{s+3} + \dfrac{4}{s-2} \\ \dfrac{2}{s} + \dfrac{-1}{s-1} + \dfrac{-1}{s+3} \end{bmatrix} \right\}$$

$$\mathbf{x}(t) = \begin{bmatrix} 4 \\ 0 \end{bmatrix} e^{2t} - \begin{bmatrix} 5 \\ 1 \end{bmatrix} e^{t} - \begin{bmatrix} 3 \\ 1 \end{bmatrix} e^{-3t} + \begin{bmatrix} 5 \\ 2 \end{bmatrix}$$

**Third Way:** Let us calculate the characteristic polynomial.

$$\det\left( \begin{bmatrix} 2-\lambda & -5 \\ 0 & 1-\lambda \end{bmatrix} \right) = (\lambda - 2)(\lambda - 1).$$

Thus, the eigenvalues are $\{2, 1\}$. Corresponding eigenvectors can be calculated as follows.

$$\begin{aligned} \mathbf{0} &= \begin{bmatrix} 2-2 & -5 \\ 0 & 1-2 \end{bmatrix} \mathbf{q}_1 = \begin{bmatrix} 0 & -5 \\ 0 & -1 \end{bmatrix} \mathbf{q}_1 \Longrightarrow \mathbf{q}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \\ \mathbf{0} &= \begin{bmatrix} 2-1 & -5 \\ 0 & 1-1 \end{bmatrix} \mathbf{q}_2 = \begin{bmatrix} 1 & -5 \\ 0 & 0 \end{bmatrix} \mathbf{q}_2 \Longrightarrow \mathbf{q}_2 = \begin{bmatrix} 5 \\ 1 \end{bmatrix}. \end{aligned}$$

The general solution of the homogeneous equation can be written as follows.

$$\mathbf{x}_H(t) = \begin{bmatrix} 1 \\ 0 \end{bmatrix} e^{2t} + \begin{bmatrix} 5 \\ 1 \end{bmatrix} e^{t}.$$

Note that a fundamental matrix solution is

$$\mathbf{W}(t) = \begin{bmatrix} 1 & 5 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} e^{2t} & 0 \\ 0 & e^{t} \end{bmatrix} \Longrightarrow \mathbf{W}^{-1}(0) = \begin{bmatrix} 1 & -5 \\ 0 & 1 \end{bmatrix}.$$

Consequently, $e^{\mathbf{A}t}$ can be calculated as follows.

$$\begin{aligned} e^{\mathbf{A}t} &= \mathbf{W}(t)\mathbf{W}^{-1}(0) = \begin{bmatrix} e^{2t} & 5e^{t} \\ 0 & e^{t} \end{bmatrix} \begin{bmatrix} 1 & -5 \\ 0 & 1 \end{bmatrix}, \\ &= \begin{bmatrix} e^{2t} & -5e^{2t} + 5e^{t} \\ 0 & e^{t} \end{bmatrix}. \end{aligned}$$

Then, the solution of the initial value problem is

$$\mathbf{x}(t) = e^{\mathbf{A}t}\left\{\mathbf{x}(0) + \int_0^t e^{-\mathbf{A}\tau}\begin{bmatrix} 10e^{-3\tau} \\ 4e^{-3\tau} - 2 \end{bmatrix} d\tau\right\}.$$

Since the inverse of $e^{\mathbf{A}t}$ is $e^{-\mathbf{A}t}$, we get

$$\mathbf{x}(t) = \begin{bmatrix} 4 \\ 0 \end{bmatrix} e^{2t} - \begin{bmatrix} 5 \\ 1 \end{bmatrix} e^{t} - \begin{bmatrix} 3 \\ 1 \end{bmatrix} e^{-3t} + \begin{bmatrix} 5 \\ 2 \end{bmatrix}$$