Week 14

- 5.5 Fundamental Matrices
- 5.6 Repeated Eigenvalues

Fundamental Matrices

5.5

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where P is an $n \times n$ matrix.

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where P is an $n \times n$ matrix. Suppose that $\mathbf{x}^{(1)}$, $\mathbf{x}^{(2)}$, $\ldots$, $\mathbf{x}^{(n)}$ are linearly independent solutions to this ODE. In other words, suppose that $\mathbf{x}^{(1)}$, $\mathbf{x}^{(2)}$, $\ldots$, $\mathbf{x}^{(n)}$ form a *fundamental set of solutions* to this ODE.

Definition

The matrix

$$\Psi(t) = \begin{bmatrix} \mathbf{x}^{(1)} & \mathbf{x}^{(2)} & \dots & \mathbf{x}^{(n)} \end{bmatrix} = \begin{bmatrix} x_1^{(1)}(t) & x_1^{(2)}(t) & \dots & x_1^{(n)}(t) \\ x_2^{(1)}(t) & x_2^{(2)}(t) & \dots & x_2^{(n)}(t) \\ \vdots & \vdots & & \vdots \\ x_n^{(1)}(t) & x_n^{(2)}(t) & \dots & x_n^{(n)}(t) \end{bmatrix}$$

is called a *fundamental matrix* of $\mathbf{x}' = P(t)\mathbf{x}$.

Example

Find a fundamental matrix for

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Recall that

$$\mathbf{x}^{(1)}(t) = \begin{bmatrix} 1 \\ 2 \end{bmatrix} e^{3t} \qquad \text{and} \qquad \mathbf{x}^{(2)}(t) = \begin{bmatrix} 1 \\ -2 \end{bmatrix} e^{-t}$$

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form a fundamental set of solutions to this ODE. Therefore

$$\Psi(t) = \begin{bmatrix} e^{3t} & e^{-t} \\ 2e^{3t} & -2e^{-t} \end{bmatrix}$$

is a fundamental matrix of this ODE.

Now, the general solution to

$$\mathbf{x}' = A\mathbf{x}$$

is

$$\mathbf{x}(t) = c_1\mathbf{x}^{(1)}(t) + c_2\mathbf{x}^{(2)}(t) + \ldots + c_n\mathbf{x}^{(n)}(t)$$

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is

$$\mathbf{x}(t) = c_1\mathbf{x}^{(1)}(t) + c_2\mathbf{x}^{(2)}(t) + \ldots + c_n\mathbf{x}^{(n)}(t) = \Psi(t)\mathbf{c}$$

where

$$\mathbf{c} = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} \in \mathbb{R}^n.$$

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where

$$\mathbf{c} = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} \in \mathbb{R}^n.$$

If we have an initial condition $\mathbf{x}(t_0) = \mathbf{x}^0$, then

$$\Psi(t_0)\mathbf{c} = \mathbf{x}^0.$$

$$\mathbf{x}(t) = \Psi(t)\mathbf{c} \qquad \Psi(t_0)\mathbf{c} = \mathbf{x}^0$$

But

$\mathbf{x}^{(1)}, \ldots, \mathbf{x}^{(n)}$
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independent

$$\mathbf{x}(t) = \Psi(t)\mathbf{c} \qquad\qquad \Psi(t_0)\mathbf{c} = \mathbf{x}^0$$

But

$\mathbf{x}^{(1)}, \ldots, \mathbf{x}^{(n)}$ are linearly independent $\Longrightarrow$ $\Psi(t)$ is invertible

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$$\implies \mathbf{c} = \Psi^{-1}(t_0)\mathbf{x}^0.$$

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$\mathbf{x}^{(1)}, \ldots, \mathbf{x}^{(n)}$ are linearly independent $\implies$ $\Psi(t)$ is invertible

$\implies$ $\mathbf{c} = \Psi^{-1}(t_0)\mathbf{x}^0.$

Therefore the solution to the IVP

$$\begin{cases} \mathbf{x}' = A\mathbf{x} \\ \mathbf{x}(t_0) = \mathbf{x}^0 \end{cases}$$

is

$$\boxed{\mathbf{x}(t) = \Psi(t)\Psi^{-1}(t_0)\mathbf{x}^0.}$$

Theorem

Suppose that $\Psi(t)$ *is a fundamental matrix for* $\mathbf{x}' = P(t)\mathbf{x}$. *Then* $\Psi(t)$ *solves the differential equation* $\Psi' = P(t)\Psi$.

(You prove)

Remark

It is possible to find a *special fundamental matrix*, $\Phi(t)$, which satisfies

$$\Phi(t_0) = \begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \dots & 1 \end{bmatrix} = I.$$

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We will use Φ for this special fundamental matrix, and Ψ for any fundamental matrix.

Example

Consider

$$\mathbf{x}' = \begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix} \mathbf{x}.$$

Find the special fundamental matrix which satisfies $\Phi(0) = I$.

To find the matrix Φ which satisfies

$$\Phi(0) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

we must solve two IVPs:

$$\begin{cases} \mathbf{x}' = \begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix} \mathbf{x} \\ \mathbf{x}(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \end{cases} \qquad \text{and} \qquad \begin{cases} \mathbf{x}' = \begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix} \mathbf{x} \\ \mathbf{x}(0) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}. \end{cases}$$

The general solution to the ODE is

$$\mathbf{x}(t) = c_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} e^{3t} + c_2 \begin{bmatrix} 1 \\ -2 \end{bmatrix} e^{-t}.$$

We calculate that

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \mathbf{x}(0) = c_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ -2 \end{bmatrix} \quad \Longrightarrow \quad \begin{aligned} c_1 &= \frac{1}{2} \\ c_2 &= \frac{1}{2} \end{aligned}$$

$$\Longrightarrow \quad \mathbf{x}(t) = \begin{bmatrix} \frac{1}{2}e^{3t} + \frac{1}{2}e^{-t} \\ e^{3t} - e^{-t} \end{bmatrix}$$

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$$\implies \mathbf{x}(t) = \begin{bmatrix} \frac{1}{2}e^{3t} + \frac{1}{2}e^{-t} \\ e^{3t} - e^{-t} \end{bmatrix}$$

and

$$\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \mathbf{x}(0) = c_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ -2 \end{bmatrix} \implies \begin{aligned} c_1 &= \frac{1}{4} \\ c_2 &= -\frac{1}{4} \end{aligned}$$

$$\implies \mathbf{x}(t) = \begin{bmatrix} \frac{1}{4}e^{3t} - \frac{1}{4}e^{-t} \\ \frac{1}{2}e^{3t} + \frac{1}{2}e^{-t} \end{bmatrix}.$$

Therefore the special fundamental matrix is

$$\Phi(t) = \begin{bmatrix} \frac{1}{2}e^{3t} + \frac{1}{2}e^{-t} & \frac{1}{4}e^{3t} - \frac{1}{4}e^{-t} \\ e^{3t} - e^{-t} & \frac{1}{2}e^{3t} + \frac{1}{2}e^{-t} \end{bmatrix}.$$

What is e^{At}?

Recall that the solution to

$$\begin{cases} x' = ax \ (a \in \mathbb{R}) \\ x(0) = x_0 \end{cases}$$

is

$$x(t) = x_0 e^{at} = x_0 \exp(at)$$

and recall that

$$\exp(at) = 1 + \sum_{n=1}^{\infty} \frac{a^n t^n}{n!}.$$

Now consider

$$\begin{cases} \mathbf{x}' = A\mathbf{x} \\ \mathbf{x}(0) = \mathbf{x}^0 \end{cases}$$

for $A \in \mathbb{R}^{n \times n}$.

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Definition

$$\exp(At) = I + \sum_{n=1}^{\infty} \frac{A^n t^n}{n!} = I + At + \frac{A^2t^2}{2!} + \frac{A^3t^3}{3!} + \dots.$$

Note that

$$\begin{aligned}\frac{d}{dt}\exp(At) &= \qquad\qquad = \\ &= \qquad\qquad = \\ &= \\ &= \qquad\qquad = \qquad\qquad .\end{aligned}$$

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$$\frac{d}{dt}\exp(At) = \frac{d}{dt}\left(I + \sum_{n=1}^{\infty}\frac{A^n t^n}{n!}\right) =$$

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Note that

$$\frac{d}{dt}\exp(At) = \frac{d}{dt}\left(I + \sum_{n=1}^{\infty}\frac{A^n t^n}{n!}\right) = 0 + \sum_{n=1}^{\infty}\frac{d}{dt}\left(\frac{A^n t^n}{n!}\right)$$

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&= \sum_{n=1}^{\infty}\frac{A^n t^{n-1}}{(n-1)!} = A + \sum_{n=2}^{\infty}\frac{A^n t^{n-1}}{(n-1)!} \\
&= \\
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$$= A + \sum_{k=1}^{\infty}\frac{A^{k+1} t^k}{(k)!} \qquad (k = n-1)$$

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&= \sum_{n=1}^{\infty}\frac{A^n t^{n-1}}{(n-1)!} = A+\sum_{n=2}^{\infty}\frac{A^n t^{n-1}}{(n-1)!} \\
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\end{aligned}$$

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&= A + \sum_{k=1}^{\infty}\frac{A^{k+1} t^k}{(k)!} \qquad\qquad (k = n-1) \\
&= A\left(I + \sum_{k=1}^{\infty}\frac{A^k t^k}{(k)!}\right) = A\exp(At).
\end{aligned}$$

This means that $\exp(At)$ solves

$$\begin{cases} \big(\exp(At)\big)' = A\exp(At) \\ \exp(At)|_{t=0} = I. \end{cases}$$

5.5 Fundamental Matrices

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But remember that Φ solves

$$\begin{cases} \Phi' = A\phi \\ \Phi(0) = I. \end{cases}$$

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Therefore

$$\boxed{\Phi(t) = \exp(At).}$$

Example

Let $A = \begin{bmatrix} 8 & -1 \\ 6 & 1 \end{bmatrix}$. Find $\exp(At)$.

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We have previously found that the general solution to $\mathbf{x}' = A\mathbf{x}$ is

$$\mathbf{x}(t) = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{7t} + c_2 \begin{bmatrix} 1 \\ 6 \end{bmatrix} e^{2t}.$$

To satisfy $\mathbf{x}(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ we require $c_1 = \frac{6}{5}$ and $c_2 = -\frac{1}{5}$. Hence

$$\mathbf{x}(t) = \frac{6}{5}\begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{7t} - \frac{1}{5}\begin{bmatrix} 1 \\ 6 \end{bmatrix} e^{2t} = \begin{bmatrix} \frac{6}{5}e^{7t} - \frac{1}{5}e^{2t} \\ \frac{6}{5}e^{7t} - \frac{6}{5}e^{2t} \end{bmatrix}.$$

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To satisfy $\mathbf{x}(0) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ we require $c_1 = -\frac{1}{5}$ and $c_2 = \frac{1}{5}$. Hence

$$\mathbf{x}(t) = -\frac{1}{5} \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{7t} + \frac{1}{5} \begin{bmatrix} 1 \\ 6 \end{bmatrix} e^{2t} = \begin{bmatrix} -\frac{1}{5}e^{7t} + \frac{1}{5}e^{2t} \\ -\frac{1}{5}e^{7t} + \frac{6}{5}e^{2t} \end{bmatrix}.$$

Therefore the answer is

$$\boxed{\exp(At) = \Phi(t) = \begin{bmatrix} \frac{6}{5}e^{7t} - \frac{1}{5}e^{2t} & -\frac{1}{5}e^{7t} + \frac{1}{5}e^{2t} \\ \frac{6}{5}e^{7t} - \frac{6}{5}e^{2t} & -\frac{1}{5}e^{7t} + \frac{6}{5}e^{2t} \end{bmatrix}.}$$

Diagonalisable Matrices

If

$$D = \begin{bmatrix} r_1 & 0 & 0 & \dots & 0 \\ 0 & r_2 & 0 & \dots & 0 \\ 0 & 0 & r_3 & \dots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \dots & r_n \end{bmatrix}$$

is a diagonal matrix, then it is easy to calculate $\exp(Dt)$. We simply have

$$\exp(Dt) = \begin{bmatrix} e^{r_1 t} & 0 & 0 & \dots & 0 \\ 0 & e^{r_2 t} & 0 & \dots & 0 \\ 0 & 0 & e^{r_3 t} & \dots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \dots & e^{r_n t} \end{bmatrix}.$$

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for $A \in \mathbb{R}^{n\times n}$.Recall how we diagonalise a matrix: If $\boldsymbol{\xi}^{(1)}$, $\boldsymbol{\xi}^{(2)}$, $\ldots$, $\boldsymbol{\xi}^{(n)}$ are the eigenvectors of A, we let

$$T = \begin{bmatrix} \boldsymbol{\xi}^{(1)} & \boldsymbol{\xi}^{(2)} & \ldots & \boldsymbol{\xi}^{(n)} \end{bmatrix}.$$

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Then

$$\det(T) \neq 0 \implies \begin{matrix} T^{-1} \\ \text{exists} \end{matrix}$$

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$$T = \begin{bmatrix} \boldsymbol{\xi}^{(1)} & \boldsymbol{\xi}^{(2)} & \dots & \boldsymbol{\xi}^{(n)} \end{bmatrix}.$$

Then

$\det(T) \neq 0 \implies$ T^{-1} exists $\implies$ $T^{-1}AT$ is diagonal $\implies$ A is diagonalisable.

Example

Diagonalise

$$A = \begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix}.$$

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The eigenvalues are $r_1 = 3$ and $r_2 = -1$. The corresponding eigenvectors are $\boldsymbol{\xi}^{(1)} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $\boldsymbol{\xi}^{(2)} = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$.

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$$T = \begin{bmatrix} 1 & 1 \\ 2 & -2 \end{bmatrix} \qquad \text{and} \qquad T^{-1} = \begin{bmatrix} \frac{1}{2} & \frac{1}{4} \\ \frac{1}{2} & -\frac{1}{4} \end{bmatrix}.$$

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It follows that

$$D = T^{-1}AT = \begin{bmatrix} 3 & 0 \\ 0 & -1 \end{bmatrix}.$$

Now consider

$$\mathbf{x}' = A\mathbf{x}.$$

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Define a new variable $\mathbf{y}$ by

$$\mathbf{x} = T\mathbf{y} \qquad \text{or} \qquad \mathbf{y} = T^{-1}\mathbf{x}.$$

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Then we calculate that

$$\begin{aligned} \mathbf{x}' &= A\mathbf{x} \\ T\mathbf{y}' &= AT\mathbf{y} \end{aligned}$$

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Then we calculate that

$$\begin{aligned} \mathbf{x}' &= A\mathbf{x} \\ T\mathbf{y}' &= AT\mathbf{y} \\ \mathbf{y}' &= T^{-1}AT\mathbf{y} = D\mathbf{y}. \end{aligned}$$

We know that a fundamental matrix for $\mathbf{y}' = D\mathbf{y}$ is

$$\exp(Dt) = \begin{bmatrix} e^{r_1 t} & 0 & \dots & 0 \\ 0 & e^{r_2 t} & \dots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \dots & e^{r_n t} \end{bmatrix}.$$

We know that a fundamental matrix for $\mathbf{y}' = D\mathbf{y}$ is

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Therefore a fundamental matrix for $\mathbf{x}' = A\mathbf{x}$ is

$$\Psi = T \exp(Dt) = \begin{bmatrix} \boldsymbol{\xi}^{(1)} e^{r_1 t} & \boldsymbol{\xi}^{(2)} e^{r_2 t} & \dots & \boldsymbol{\xi}^{(n)} e^{r_n t} \end{bmatrix}.$$

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Recall that $T = \begin{bmatrix} 1 & 1 \\ 2 & -2 \end{bmatrix}$.

Example

Find a fundamental matrix for

$$\mathbf{x}' = \begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix} \mathbf{x}.$$

Recall that $T = \begin{bmatrix} 1 & 1 \\ 2 & -2 \end{bmatrix}$. Letting $\mathbf{y} = T^{-1}\mathbf{x}$, we have

$$\mathbf{y}' = \begin{bmatrix} 3 & 0 \\ 0 & -1 \end{bmatrix} \mathbf{y}.$$

A fundamental matrix for $\mathbf{y}' = \begin{bmatrix} 3 & 0 \\ 0 & -1 \end{bmatrix} \mathbf{y}$ is

$$\exp(Dt) = e^{Dt} = \begin{bmatrix} e^{3t} & 0 \\ 0 & e^{-t} \end{bmatrix}.$$

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Hence a fundamental matrix for $\mathbf{x}' = \begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix} \mathbf{x}$ is

$$\Psi(t) = T \exp(Dt)$$

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A fundamental matrix for $\mathbf{y}' = \begin{bmatrix} 3 & 0 \\ 0 & -1 \end{bmatrix} \mathbf{y}$ is

$$\exp(Dt) = e^{Dt} = \begin{bmatrix} e^{3t} & 0 \\ 0 & e^{-t} \end{bmatrix}.$$

Hence a fundamental matrix for $\mathbf{x}' = \begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix} \mathbf{x}$ is

$$\Psi(t) = T \exp(Dt) = \begin{bmatrix} 1 & 1 \\ 2 & -2 \end{bmatrix} \begin{bmatrix} e^{3t} & 0 \\ 0 & e^{-t} \end{bmatrix} = \begin{bmatrix} e^{3t} & e^{-t} \\ 2e^{3t} & -2e^{-t} \end{bmatrix}.$$

5.6 Repeated Eigenvalues

Example

Find the eigenvalues and eigenvectors of $A = \begin{bmatrix} 1 & -1 \\ 1 & 3 \end{bmatrix}$.

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We calculate that

$$0 = \det(A - rI) = \begin{vmatrix} 1-r & -1 \\ 1 & 3-r \end{vmatrix} = r^2 - 4r + 4 = (r-2)^2.$$

Therefore $r_1 = 2 = r_2$.

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Therefore $r_1 = 2 = r_2$. Moreover

$$\mathbf{0} = (A - rI)\boldsymbol{\xi} = \begin{bmatrix} -1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix} \implies \xi_1 + \xi_2 = 0 \implies \boldsymbol{\xi} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

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Note that A has only one linearly independent eigenvector.

Example

Solve

$$\mathbf{x}' = \begin{bmatrix} 1 & -1 \\ 1 & 3 \end{bmatrix} \mathbf{x}.$$

Example

Solve

$$\mathbf{x}' = \begin{bmatrix} 1 & -1 \\ 1 & 3 \end{bmatrix} \mathbf{x}.$$

We know that

$$\mathbf{x}^{(1)}(t) = \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{2t}$$

is a solution. But we need two solutions.

Guess 1: I guess that

$$\mathbf{x}^{(2)}(t) = \boldsymbol{\xi} t e^{2t}$$

for some $\boldsymbol{\xi} \in \mathbb{R}^2$.

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This guess did not work.

Guess 2: Now I guess that

$$\mathbf{x}^{(2)}(t) = \boldsymbol{\xi} t e^{2t} + \boldsymbol{\eta} e^{2t}$$

for some $\boldsymbol{\xi}, \boldsymbol{\eta} \in \mathbb{R}^2$.

Guess 2: Now I guess that

$$\mathbf{x}^{(2)}(t) = \boldsymbol{\xi} t e^{2t} + \boldsymbol{\eta} e^{2t}$$

for some $\boldsymbol{\xi}, \boldsymbol{\eta} \in \mathbb{R}^2$. Then we have

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$$\boldsymbol{\xi} e^{2t} + 2\boldsymbol{\xi} t e^{2t} + 2\boldsymbol{\eta} e^{2t} = \mathbf{x}^{(2)\prime} = A\mathbf{x}^{(2)} = A\left(\boldsymbol{\xi} t e^{2t} + \boldsymbol{\eta} e^{2t}\right)$$

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and

$$(2\boldsymbol{\xi} - A\boldsymbol{\xi})\, t + (\boldsymbol{\xi} + 2\boldsymbol{\eta} - A\boldsymbol{\eta}) = \mathbf{0}.$$

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$$\mathbf{x}^{(2)}(t) = \boldsymbol{\xi} t e^{2t} + \boldsymbol{\eta} e^{2t}$$

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$$\boldsymbol{\xi} e^{2t} + 2\boldsymbol{\xi} t e^{2t} + 2\boldsymbol{\eta} e^{2t} = \mathbf{x}^{(2)\prime} = A\mathbf{x}^{(2)} = A\left(\boldsymbol{\xi} t e^{2t} + \boldsymbol{\eta} e^{2t}\right)$$

and

$$(2\boldsymbol{\xi} - A\boldsymbol{\xi})\, t + (\boldsymbol{\xi} + 2\boldsymbol{\eta} - A\boldsymbol{\eta}) = \mathbf{0}.$$

Since this must be true $\forall\ t$, we must have

$$(A - 2I)\,\boldsymbol{\xi} = \mathbf{0} \qquad \text{and} \qquad (A - 2I)\,\boldsymbol{\eta} = \boldsymbol{\xi}.$$

$$(A - 2I)\,\boldsymbol{\xi} = \mathbf{0} \qquad (A - 2I)\,\boldsymbol{\eta} = \boldsymbol{\xi}$$

Clearly $\boldsymbol{\xi} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$.

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Clearly $\boldsymbol{\xi} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$. Then we calculate that

$$\begin{bmatrix} -1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \eta_1 \\ \eta_2 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \implies \eta_1 + \eta_2 = -1$$

$$(A-2I)\,\boldsymbol{\xi}=\mathbf{0} \qquad (A-2I)\,\boldsymbol{\eta}=\boldsymbol{\xi}$$

Clearly $\boldsymbol{\xi}=\begin{bmatrix}1\\-1\end{bmatrix}$. Then we calculate that

$$\begin{aligned}\begin{bmatrix}-1 & -1\\ 1 & 1\end{bmatrix}\begin{bmatrix}\eta_1\\ \eta_2\end{bmatrix}=\begin{bmatrix}1\\-1\end{bmatrix} \quad &\Longrightarrow \quad \eta_1+\eta_2=-1\\ &\Longrightarrow \quad \boldsymbol{\eta}=\begin{bmatrix}k\\-1-k\end{bmatrix}=\begin{bmatrix}0\\-1\end{bmatrix}+k\begin{bmatrix}1\\-1\end{bmatrix}\end{aligned}$$

for some k.

$$(A-2I)\boldsymbol{\xi}=\mathbf{0} \qquad (A-2I)\boldsymbol{\eta}=\boldsymbol{\xi}$$

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for some k. So

$$\mathbf{x}^{(2)}(t)=\boldsymbol{\xi}te^{2t}+\boldsymbol{\eta}e^{2t}=\begin{bmatrix}1\\-1\end{bmatrix}te^{2t}+\begin{bmatrix}0\\-1\end{bmatrix}e^{2t}+k\begin{bmatrix}1\\-1\end{bmatrix}e^{2t}$$

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for some k. So

$$\begin{aligned}\mathbf{x}^{(2)}(t)=\boldsymbol{\xi}te^{2t}+\boldsymbol{\eta}e^{2t}&=\begin{bmatrix}1\\-1\end{bmatrix}te^{2t}+\begin{bmatrix}0\\-1\end{bmatrix}e^{2t}+k\begin{bmatrix}1\\-1\end{bmatrix}e^{2t}\\ &=\begin{bmatrix}1\\-1\end{bmatrix}te^{2t}+\begin{bmatrix}0\\-1\end{bmatrix}e^{2t}+k\mathbf{x}^{(1)}(t).\end{aligned}$$

$$\mathbf{x}^{(2)}(t) = \begin{bmatrix} 1 \\ -1 \end{bmatrix} te^{2t} + \begin{bmatrix} 0 \\ -1 \end{bmatrix} e^{2t} + k\mathbf{x}^{(1)}(t)$$

Because we already have $\mathbf{x}^{(1)}(t)$, we can choose $k = 0$. So

$$\mathbf{x}^{(2)}(t) = \boldsymbol{\xi} te^{2t} + \boldsymbol{\eta} e^{2t} = \begin{bmatrix} 1 \\ -1 \end{bmatrix} te^{2t} + \begin{bmatrix} 0 \\ -1 \end{bmatrix} e^{2t}.$$

The general solution of $\mathbf{x}' = \begin{bmatrix} 1 & -1 \\ 1 & 3 \end{bmatrix} \mathbf{x}$ is therefore

$$\boxed{\mathbf{x}(t) = c_1 \mathbf{x}^{(1)} + c_2 \mathbf{x}^{(2)} = c_1 \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{2t} + c_2 \left(\begin{bmatrix} 1 \\ -1 \end{bmatrix} te^{2t} + \begin{bmatrix} 0 \\ -1 \end{bmatrix} e^{2t} \right).}$$

Example

Find a fundamental matrix for

$$\mathbf{x}' = \begin{bmatrix} 1 & -1 \\ 1 & 3 \end{bmatrix} \mathbf{x}.$$

Then find the special fundamental matrix $\Phi(t)$ which satisfies $\Phi(0) = I$.

Since $\mathbf{x}^{(1)}(t) = \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{2t}$ and $\mathbf{x}^{(2)}(t) = \begin{bmatrix} 1 \\ -1 \end{bmatrix} te^{2t} + \begin{bmatrix} 0 \\ -1 \end{bmatrix} e^{2t}$ we have that

$$\Psi(t) = \begin{bmatrix} \mathbf{x}^{(1)} & \mathbf{x}^{(2)} \end{bmatrix} = \begin{bmatrix} e^{2t} & te^{2t} \\ -e^{2t} & -te^{2t} - e^{2t} \end{bmatrix}$$

is a fundamental matrix for this system.

$$\boxed{\Psi(t) = \begin{bmatrix} e^{2t} & te^{2t} \\ -e^{2t} & -te^{2t} - e^{2t} \end{bmatrix}}$$

Now

$$\Psi(0) = \begin{bmatrix} 1 & 0 \\ -1 & -1 \end{bmatrix} \qquad \text{and} \qquad \Psi^{-1}(0) = \begin{bmatrix} 1 & 0 \\ -1 & -1 \end{bmatrix}.$$

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Therefore

$$\exp(At) = \Phi(t) = \Psi(t)\Psi^{-1}(0)$$

$$\boxed{\Psi(t) = \begin{bmatrix} e^{2t} & te^{2t} \\ -e^{2t} & -te^{2t} - e^{2t} \end{bmatrix}}$$

Now

$$\Psi(0) = \begin{bmatrix} 1 & 0 \\ -1 & -1 \end{bmatrix} \qquad \text{and} \qquad \Psi^{-1}(0) = \begin{bmatrix} 1 & 0 \\ -1 & -1 \end{bmatrix}.$$

Therefore

$$\exp(At) = \Phi(t) = \Psi(t)\Psi^{-1}(0) = e^{2t} \begin{bmatrix} 1 & t \\ -1 & -1 - t \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -1 & -1 \end{bmatrix}$$

$$\Psi(t) = \begin{bmatrix} e^{2t} & te^{2t} \\ -e^{2t} & -te^{2t} - e^{2t} \end{bmatrix}$$

Now

$$\Psi(0) = \begin{bmatrix} 1 & 0 \\ -1 & -1 \end{bmatrix} \qquad \text{and} \qquad \Psi^{-1}(0) = \begin{bmatrix} 1 & 0 \\ -1 & -1 \end{bmatrix}.$$

Therefore

$$\begin{aligned} \exp(At) = \Phi(t) = \Psi(t)\Psi^{-1}(0) &= e^{2t} \begin{bmatrix} 1 & t \\ -1 & -1-t \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -1 & -1 \end{bmatrix} \\ &= e^{2t} \begin{bmatrix} 1-t & -t \\ t & 1+t \end{bmatrix}. \end{aligned}$$

Remark

$$\mathbf{x}' = A\mathbf{x}$$

For two repeated eigenvalues (but with only one linearly independent eigenvector), the key equations to remember are

$$\boxed{\mathbf{x}^{(2)}(t) = \boldsymbol{\xi} t e^{rt} + \boldsymbol{\eta} e^{rt}} \qquad \text{and} \qquad \boxed{(A - rI)\boldsymbol{\eta} = \boldsymbol{\xi}}.$$

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Definition

η is called a *generalised eigenvector* of A.

Remark

If you have 2 repeated eigenvalues (but with only one linearly independent eigenvector), the method is:

1. Find the eigenvalues and eigenvectors;
2. The first solution is $\mathbf{x}^{(1)}(t) = \boldsymbol{\xi} e^{rt}$;
3. Use $(A - rI)\boldsymbol{\eta} = \boldsymbol{\xi}$ to find a generalised eigenvector $\boldsymbol{\eta}$;
4. The second solution is $\mathbf{x}^{(2)}(t) = \boldsymbol{\xi} t e^{rt} + \boldsymbol{\eta} e^{rt}$.

Example

Solve

$$\begin{cases} \mathbf{x}' = \begin{bmatrix} -\frac{5}{2} & \frac{3}{2} \\ -\frac{3}{2} & \frac{1}{2} \end{bmatrix} \mathbf{x}, \\ \mathbf{x}(0) = \begin{bmatrix} 3 \\ -1 \end{bmatrix}. \end{cases}$$

The only eigenvalue of the matrix is $r = -1$. The corresponding eigenvector is $\boldsymbol{\xi} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. Therefore one solution of the linear system is

$$\mathbf{x}^{(1)}(t) = \boldsymbol{\xi} e^{rt} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{-t}.$$

We need to find a second, linearly independent solution. We will consider the ansatz

$$\mathbf{x}^{(2)}(t) = \boldsymbol{\xi} t e^{-t} + \boldsymbol{\eta} e^{-t}$$

where $\boldsymbol{\xi} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ as above and $\boldsymbol{\eta}$ is a generalised eigenvector solving $(A - rI)\boldsymbol{\eta} = \boldsymbol{\xi}$.

Solving the latter equation,

$$(A - rI)\boldsymbol{\eta} = \boldsymbol{\xi}$$

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Solving the latter equation,

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$$-\frac{3}{2}\eta_1 + \frac{3}{2}\eta_2 = 1$$

Solving the latter equation,

$$\begin{aligned}(A - rI)\boldsymbol{\eta} &= \boldsymbol{\xi}\\ \begin{bmatrix} -\frac{3}{2} & \frac{3}{2} \\ -\frac{3}{2} & \frac{3}{2} \end{bmatrix} \begin{bmatrix} \eta_1 \\ \eta_2 \end{bmatrix} &= \begin{bmatrix} 1 \\ 1 \end{bmatrix}\\ -\frac{3}{2}\eta_1 + \frac{3}{2}\eta_2 &= 1\\ -\eta_1 + \eta_2 &= \frac{2}{3}\end{aligned}$$

Solving the latter equation,

$$(A - rI)\boldsymbol{\eta} = \boldsymbol{\xi}$$

$$\begin{bmatrix} -\frac{3}{2} & \frac{3}{2} \\ -\frac{3}{2} & \frac{3}{2} \end{bmatrix} \begin{bmatrix} \eta_1 \\ \eta_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$-\frac{3}{2}\eta_1 + \frac{3}{2}\eta_2 = 1$$

$$-\eta_1 + \eta_2 = \frac{2}{3}$$

we can choose $\boldsymbol{\eta} = \begin{bmatrix} 0 \\ \frac{2}{3} \end{bmatrix}$.

Note that we don't need to find *every* generalised eigenvector

$$\boldsymbol{\eta} = \begin{bmatrix} k \\ k + \frac{2}{3} \end{bmatrix} = k \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{2}{3} \end{bmatrix} = k\boldsymbol{\xi} + \begin{bmatrix} 0 \\ \frac{2}{3} \end{bmatrix}$$

because we already have $\mathbf{x}^{(1)}(t) = \boldsymbol{\xi} e^{rt}$.

Note that we don't need to find *every* generalised eigenvector

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Instead we only need to find *one* generalised eigenvector – that means that we can choose any k that we want.

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$$\boldsymbol{\eta} = \begin{bmatrix} k \\ k + \frac{2}{3} \end{bmatrix} = k \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{2}{3} \end{bmatrix} = k\boldsymbol{\xi} + \begin{bmatrix} 0 \\ \frac{2}{3} \end{bmatrix}$$

because we already have $\mathbf{x}^{(1)}(t) = \boldsymbol{\xi} e^{rt}$.

Instead we only need to find *one* generalised eigenvector – that means that we can choose any k that we want.
Hence I have chosen $k = 0$ which gives $\boldsymbol{\eta} = \begin{bmatrix} 0 \\ \frac{2}{3} \end{bmatrix}$.

eigenvector

$$\boldsymbol{\xi} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

generalised eigenvector

$$\boldsymbol{\eta} = \begin{bmatrix} 0 \\ \frac{2}{3} \end{bmatrix}$$

Thus

$$\mathbf{x}^{(1)}(t) = \boldsymbol{\xi} e^{-t} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{-t}$$

and

$$\mathbf{x}^{(2)}(t) = \boldsymbol{\xi} t e^{-t} + \boldsymbol{\eta} e^{-t} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} t e^{-t} + \begin{bmatrix} 0 \\ \frac{2}{3} \end{bmatrix} e^{-t}.$$

eigenvector generalised eigenvector

$$\boldsymbol{\xi} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \qquad \boldsymbol{\eta} = \begin{bmatrix} 0 \\ \frac{2}{3} \end{bmatrix}$$

Thus

$$\mathbf{x}^{(1)}(t) = \boldsymbol{\xi} e^{-t} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{-t}$$

and

$$\mathbf{x}^{(2)}(t) = \boldsymbol{\xi} t e^{-t} + \boldsymbol{\eta} e^{-t} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} t e^{-t} + \begin{bmatrix} 0 \\ \frac{2}{3} \end{bmatrix} e^{-t}.$$

Hence the general solution to the linear system is

$$\mathbf{x}(t) = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{-t} + c_2 \left(\begin{bmatrix} 1 \\ 1 \end{bmatrix} t e^{-t} + \begin{bmatrix} 0 \\ \frac{2}{3} \end{bmatrix} e^{-t} \right).$$

The initial condition gives

$$\begin{bmatrix} 3 \\ -1 \end{bmatrix} = \mathbf{x}(0) = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ \frac{2}{3} \end{bmatrix}$$

which implies that $c_1 = 3$ and $c_2 = -6$.

The initial condition gives

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which implies that $c_1 = 3$ and $c_2 = -6$.

Therefore the solution to the IVP is

$$\boxed{\mathbf{x}(t) = 3 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{-t} - 6 \left(\begin{bmatrix} 1 \\ 1 \end{bmatrix} te^{-t} + \begin{bmatrix} 0 \\ \frac{2}{3} \end{bmatrix} e^{-t} \right) = \begin{bmatrix} 3 \\ -1 \end{bmatrix} e^{-t} - 6 \begin{bmatrix} 1 \\ 1 \end{bmatrix} te^{-t}.}$$

Example

Solve

$$\mathbf{x}' = \begin{bmatrix} 1 & -4 \\ 4 & -7 \end{bmatrix} \mathbf{x}, \qquad \mathbf{x}(0) = \begin{bmatrix} 3 \\ 2 \end{bmatrix}.$$

Example

Solve

$$\mathbf{x}' = \begin{bmatrix} 1 & -4 \\ 4 & -7 \end{bmatrix} \mathbf{x}, \qquad \mathbf{x}(0) = \begin{bmatrix} 3 \\ 2 \end{bmatrix}.$$

The only eigenvalue of the matrix is $r = -3$. The corresponding eigenvector is $\boldsymbol{\xi} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

Example

Solve

$$\mathbf{x}' = \begin{bmatrix} 1 & -4 \\ 4 & -7 \end{bmatrix} \mathbf{x}, \qquad \mathbf{x}(0) = \begin{bmatrix} 3 \\ 2 \end{bmatrix}.$$

The only eigenvalue of the matrix is $r = -3$. The corresponding eigenvector is $\boldsymbol{\xi} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. Therefore one solution of the linear system is

$$\mathbf{x}^{(1)}(t) = \boldsymbol{\xi} e^{rt} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{-3t}.$$

Next we need to find a generalised eigenvector $\boldsymbol{\eta}$.

We calculate that

$$(A - rI)\boldsymbol{\eta} = \boldsymbol{\xi}$$

We calculate that

$$\begin{bmatrix} 4 & -4 \\ 4 & -4 \end{bmatrix} \begin{bmatrix} \eta_1 \\ \eta_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

We calculate that

$$4\eta_1 - 4\eta_2 = 1$$

We calculate that

$$-\eta_1 + \eta_2 = -\frac{1}{4}$$

We calculate that

$$\eta_2 = \eta_1 - \frac{1}{4}.$$

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We can choose any vector $\boldsymbol{\eta}$ that satisfies $\eta_2 = \eta_1 - \frac{1}{4}$.

We calculate that

$$\eta_2 = \eta_1 - \frac{1}{4}.$$

We can choose any vector $\boldsymbol{\eta}$ that satisfies $\eta_2 = \eta_1 - \frac{1}{4}$. Thus we may choose $\boldsymbol{\eta} = \begin{bmatrix} 0 \\ -\frac{1}{4} \end{bmatrix}$.

eigenvector

$$\boldsymbol{\xi} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

generalised eigenvector

$$\boldsymbol{\eta} = \begin{bmatrix} 0 \\ -\frac{1}{4} \end{bmatrix}$$

Therefore

$$\mathbf{x}^{(2)}(t) = \begin{bmatrix} 1 \\ 1 \end{bmatrix} te^{-3t} + \begin{bmatrix} 0 \\ -\frac{1}{4} \end{bmatrix} e^{-3t}.$$

Hence the general solution to the ODE is

$$\mathbf{x}(t) = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{-3t} + c_2 \left(\begin{bmatrix} 1 \\ 1 \end{bmatrix} te^{-3t} + \begin{bmatrix} 0 \\ -\frac{1}{4} \end{bmatrix} e^{-3t} \right).$$

The initial condition gives

$$\begin{bmatrix} 3 \\ 2 \end{bmatrix} = \mathbf{x}(0) = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ -\frac{1}{4} \end{bmatrix}$$

which implies that $c_1 = 3$ and $c_2 = 4$.

Therefore the solution to the IVP is

$$\mathbf{x}(t) = 3\begin{bmatrix}1\\1\end{bmatrix}e^{-3t} + 4\left(\begin{bmatrix}1\\1\end{bmatrix}te^{-3t} + \begin{bmatrix}0\\-\frac{1}{4}\end{bmatrix}e^{-3t}\right)$$

$$=$$

$$= \qquad .$$

Therefore the solution to the IVP is

$$\begin{aligned}\mathbf{x}(t) &= 3\begin{bmatrix}1\\1\end{bmatrix}e^{-3t} + 4\left(\begin{bmatrix}1\\1\end{bmatrix}te^{-3t} + \begin{bmatrix}0\\-\frac{1}{4}\end{bmatrix}e^{-3t}\right)\\ &= \begin{bmatrix}3\\2\end{bmatrix}e^{-3t} + 4\begin{bmatrix}1\\1\end{bmatrix}te^{-3t}\\ &= \qquad\qquad\qquad .\end{aligned}$$

Therefore the solution to the IVP is

$$\begin{aligned}\mathbf{x}(t) &= 3\begin{bmatrix}1\\1\end{bmatrix}e^{-3t} + 4\left(\begin{bmatrix}1\\1\end{bmatrix}te^{-3t} + \begin{bmatrix}0\\-\frac{1}{4}\end{bmatrix}e^{-3t}\right)\\ &= \begin{bmatrix}3\\2\end{bmatrix}e^{-3t} + 4\begin{bmatrix}1\\1\end{bmatrix}te^{-3t}\\ &= \begin{bmatrix}3+4t\\2+4t\end{bmatrix}e^{-3t}.\end{aligned}$$

Next Week

- 5.7 Nonhomogeneous Linear Systems