



Lecture 11

- 5.6 Repeated Eigenvalues
- 5.7 Nonhomogeneous Linear Systems

Repeated Eigenvalues

5.5 Repeated Eigenvalues



Example

Find the eigenvalues and eigenvectors of $A = \begin{bmatrix} 1 & -1 \\ 1 & 3 \end{bmatrix}$.

5.5 Repeated Eigenvalues



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We calculate that

$$0 = \det(A - rI) = \begin{vmatrix} 1-r & -1 \\ 1 & 3-r \end{vmatrix} = r^2 - 4r + 4 = (r-2)^2.$$

Therefore $r_1 = 2 = r_2$.

5.5 Repeated Eigenvalues



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Therefore $r_1 = 2 = r_2$. Moreover

$$\mathbf{0} = (A - rI) \boldsymbol{\xi} = \begin{bmatrix} -1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix} \implies \xi_1 + \xi_2 = 0 \implies \boldsymbol{\xi} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

5.5 Repeated Eigenvalues



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Note that A has only one linearly independent eigenvector.

5.5 Repeated Eigenvalues



Example

Solve

$$\mathbf{x}' = \begin{bmatrix} 1 & -1 \\ 1 & 3 \end{bmatrix} \mathbf{x}.$$

5.5 Repeated Eigenvalues



Example

Solve

$$\mathbf{x}' = \begin{bmatrix} 1 & -1 \\ 1 & 3 \end{bmatrix} \mathbf{x}.$$

We know that

$$\mathbf{x}^{(1)}(t) = \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{2t}$$

is a solution. But we need two solutions.

5.5 Repeated Eigenvalues



Guess 1: I guess that

$$\mathbf{x}^{(2)}(t) = \boldsymbol{\xi} t e^{2t}$$

for some $\boldsymbol{\xi} \in \mathbb{R}^2$.

5.5 Repeated Eigenvalues



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$$\mathbf{x}^{(2)}(t) = \boldsymbol{\xi} t e^{2t}$$

for some $\boldsymbol{\xi} \in \mathbb{R}^2$. Then we have

$$\mathbf{x}^{(2)'} = A\mathbf{x}^{(2)}$$

5.5 Repeated Eigenvalues



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for some $\boldsymbol{\xi} \in \mathbb{R}^2$. Then we have

$$\boldsymbol{\xi} e^{2t} + 2\boldsymbol{\xi} t e^{2t} = \mathbf{x}^{(2)'} = A\mathbf{x}^{(2)}$$

5.5 Repeated Eigenvalues



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5.5 Repeated Eigenvalues



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5.5 Repeated Eigenvalues



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This guess did not work.

5.5 Repeated Eigenvalues



Guess 2: Now I guess that

$$\mathbf{x}^{(2)}(t) = \boldsymbol{\xi}te^{2t} + \boldsymbol{\eta}e^{2t}$$

for some $\boldsymbol{\xi}, \boldsymbol{\eta} \in \mathbb{R}^2$.

5.5 Repeated Eigenvalues



Guess 2: Now I guess that

$$\mathbf{x}^{(2)}(t) = \boldsymbol{\xi}te^{2t} + \boldsymbol{\eta}e^{2t}$$

for some $\boldsymbol{\xi}, \boldsymbol{\eta} \in \mathbb{R}^2$. Then we have

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5.5 Repeated Eigenvalues



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5.5 Repeated Eigenvalues



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$$\mathbf{x}^{(2)}(t) = \boldsymbol{\xi}te^{2t} + \boldsymbol{\eta}e^{2t}$$

for some $\boldsymbol{\xi}, \boldsymbol{\eta} \in \mathbb{R}^2$. Then we have

$$\boldsymbol{\xi}e^{2t} + 2\boldsymbol{\xi}te^{2t} + 2\boldsymbol{\eta}e^{2t} = \mathbf{x}^{(2)'} = A\mathbf{x}^{(2)} = A(\boldsymbol{\xi}te^{2t} + \boldsymbol{\eta}e^{2t})$$

5.5 Repeated Eigenvalues



Guess 2: Now I guess that

$$\mathbf{x}^{(2)}(t) = \xi t e^{2t} + \eta e^{2t}$$

for some $\xi, \eta \in \mathbb{R}^2$. Then we have

$$\xi e^{2t} + 2\xi t e^{2t} + 2\eta e^{2t} = \mathbf{x}^{(2)'} = A\mathbf{x}^{(2)} = A(\xi t e^{2t} + \eta e^{2t})$$

and

$$(2\xi - A\xi)t + (\xi + 2\eta - A\eta) = \mathbf{0}.$$

5.5 Repeated Eigenvalues



Guess 2: Now I guess that

$$\mathbf{x}^{(2)}(t) = \boldsymbol{\xi} t e^{2t} + \boldsymbol{\eta} e^{2t}$$

for some $\boldsymbol{\xi}, \boldsymbol{\eta} \in \mathbb{R}^2$. Then we have

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and

$$(2\boldsymbol{\xi} - A\boldsymbol{\xi}) t + (\boldsymbol{\xi} + 2\boldsymbol{\eta} - A\boldsymbol{\eta}) = \mathbf{0}.$$

Since this must be true $\forall t$, we must have

$$(A - 2I) \boldsymbol{\xi} = \mathbf{0} \quad \text{and} \quad (A - 2I) \boldsymbol{\eta} = \boldsymbol{\xi}.$$

$$(A - 2I) \boldsymbol{\xi} = \mathbf{0} \quad (A - 2I) \boldsymbol{\eta} = \boldsymbol{\xi}$$



Clearly $\boldsymbol{\xi} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$.

$$(A - 2I) \boldsymbol{\xi} = \mathbf{0} \quad (A - 2I) \boldsymbol{\eta} = \boldsymbol{\xi}$$



Clearly $\boldsymbol{\xi} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$. Then we calculate that

$$\begin{bmatrix} -1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \eta_1 \\ \eta_2 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \implies \eta_1 + \eta_2 = -1$$

$$(A - 2I) \boldsymbol{\xi} = \mathbf{0} \quad (A - 2I) \boldsymbol{\eta} = \boldsymbol{\xi}$$



Clearly $\boldsymbol{\xi} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$. Then we calculate that

$$\begin{aligned} \begin{bmatrix} -1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \eta_1 \\ \eta_2 \end{bmatrix} &= \begin{bmatrix} 1 \\ -1 \end{bmatrix} \implies \eta_1 + \eta_2 = -1 \\ \implies \boldsymbol{\eta} &= \begin{bmatrix} k \\ -1 - k \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \end{bmatrix} + k \begin{bmatrix} 1 \\ -1 \end{bmatrix} \end{aligned}$$

for some k .

$$(A - 2I) \boldsymbol{\xi} = \mathbf{0} \quad (A - 2I) \boldsymbol{\eta} = \boldsymbol{\xi}$$



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for some k . So

$$\mathbf{x}^{(2)}(t) = \boldsymbol{\xi} t e^{2t} + \boldsymbol{\eta} e^{2t} = \begin{bmatrix} 1 \\ -1 \end{bmatrix} t e^{2t} + \begin{bmatrix} 0 \\ -1 \end{bmatrix} e^{2t} + k \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{2t}$$

$$(A - 2I) \boldsymbol{\xi} = \mathbf{0} \quad (A - 2I) \boldsymbol{\eta} = \boldsymbol{\xi}$$



Clearly $\boldsymbol{\xi} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$. Then we calculate that

$$\begin{aligned} \begin{bmatrix} -1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \eta_1 \\ \eta_2 \end{bmatrix} &= \begin{bmatrix} 1 \\ -1 \end{bmatrix} \implies \eta_1 + \eta_2 = -1 \\ \implies \boldsymbol{\eta} &= \begin{bmatrix} k \\ -1 - k \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \end{bmatrix} + k \begin{bmatrix} 1 \\ -1 \end{bmatrix} \end{aligned}$$

for some k . So

$$\begin{aligned} \mathbf{x}^{(2)}(t) &= \boldsymbol{\xi} t e^{2t} + \boldsymbol{\eta} e^{2t} = \begin{bmatrix} 1 \\ -1 \end{bmatrix} t e^{2t} + \begin{bmatrix} 0 \\ -1 \end{bmatrix} e^{2t} + k \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{2t} \\ &= \begin{bmatrix} 1 \\ -1 \end{bmatrix} t e^{2t} + \begin{bmatrix} 0 \\ -1 \end{bmatrix} e^{2t} + k \mathbf{x}^{(1)}(t). \end{aligned}$$

5.5 Repeated Eigenvalues



$$\mathbf{x}^{(2)}(t) = \begin{bmatrix} 1 \\ -1 \end{bmatrix} t e^{2t} + \begin{bmatrix} 0 \\ -1 \end{bmatrix} e^{2t} + k \mathbf{x}^{(1)}(t)$$

Because we already have $\mathbf{x}^{(1)}(t)$, we can choose $k = 0$. So

$$\mathbf{x}^{(2)}(t) = \boldsymbol{\xi} t e^{2t} + \boldsymbol{\eta} e^{2t} = \begin{bmatrix} 1 \\ -1 \end{bmatrix} t e^{2t} + \begin{bmatrix} 0 \\ -1 \end{bmatrix} e^{2t}.$$

5.5 Repeated Eigenvalues



The general solution of $\mathbf{x}' = \begin{bmatrix} 1 & -1 \\ 1 & 3 \end{bmatrix} \mathbf{x}$ is therefore

$$\mathbf{x}(t) = c_1 \mathbf{x}^{(1)} + c_2 \mathbf{x}^{(2)} = c_1 \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{2t} + c_2 \left(\begin{bmatrix} 1 \\ -1 \end{bmatrix} t e^{2t} + \begin{bmatrix} 0 \\ -1 \end{bmatrix} e^{2t} \right).$$

5.5 Repeated Eigenvalues



Example

Find a fundamental matrix for

$$\mathbf{x}' = \begin{bmatrix} 1 & -1 \\ 1 & 3 \end{bmatrix} \mathbf{x}.$$

Then find the special fundamental matrix $\Phi(t)$ which satisfies $\Phi(0) = I$.

5.5 Repeated Eigenvalues



Since $\mathbf{x}^{(1)}(t) = \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{2t}$ and $\mathbf{x}^{(2)}(t) = \begin{bmatrix} 1 \\ -1 \end{bmatrix} t e^{2t} + \begin{bmatrix} 0 \\ -1 \end{bmatrix} e^{2t}$ we have that

$$\Psi(t) = \begin{bmatrix} \mathbf{x}^{(1)} & \mathbf{x}^{(2)} \end{bmatrix} = \begin{bmatrix} e^{2t} & t e^{2t} \\ -e^{2t} & -t e^{2t} - e^{2t} \end{bmatrix}$$

is a fundamental matrix for this system.

$$\Psi(t) = \begin{bmatrix} e^{2t} & te^{2t} \\ -e^{2t} & -te^{2t} - e^{2t} \end{bmatrix}$$



Now

$$\Psi(0) = \begin{bmatrix} 1 & 0 \\ -1 & -1 \end{bmatrix} \quad \text{and} \quad \Psi^{-1}(0) = \begin{bmatrix} 1 & 0 \\ -1 & -1 \end{bmatrix}.$$

$$\Psi(t) = \begin{bmatrix} e^{2t} & te^{2t} \\ -e^{2t} & -te^{2t} - e^{2t} \end{bmatrix}$$



Now

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Therefore

$$\exp(At) = \Phi(t) = \Psi(t)\Psi^{-1}(0)$$

$$\Psi(t) = \begin{bmatrix} e^{2t} & te^{2t} \\ -e^{2t} & -te^{2t} - e^{2t} \end{bmatrix}$$



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$$\Psi(0) = \begin{bmatrix} 1 & 0 \\ -1 & -1 \end{bmatrix} \quad \text{and} \quad \Psi^{-1}(0) = \begin{bmatrix} 1 & 0 \\ -1 & -1 \end{bmatrix}.$$

Therefore

$$\exp(At) = \Phi(t) = \Psi(t)\Psi^{-1}(0) = e^{2t} \begin{bmatrix} 1 & t \\ -1 & -1-t \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -1 & -1 \end{bmatrix}$$

$$\Psi(t) = \begin{bmatrix} e^{2t} & te^{2t} \\ -e^{2t} & -te^{2t} - e^{2t} \end{bmatrix}$$



Now

$$\Psi(0) = \begin{bmatrix} 1 & 0 \\ -1 & -1 \end{bmatrix} \quad \text{and} \quad \Psi^{-1}(0) = \begin{bmatrix} 1 & 0 \\ -1 & -1 \end{bmatrix}.$$

Therefore

$$\begin{aligned} \exp(At) = \Phi(t) &= \Psi(t)\Psi^{-1}(0) = e^{2t} \begin{bmatrix} 1 & t \\ -1 & -1-t \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -1 & -1 \end{bmatrix} \\ &= e^{2t} \begin{bmatrix} 1-t & -t \\ t & 1+t \end{bmatrix}. \end{aligned}$$

5.5 Repeated Eigenvalues



Remark

$$\mathbf{x}' = A\mathbf{x}$$

For two repeated eigenvalues (but with only one linearly independent eigenvector), the key equations to remember are

$$\boxed{\mathbf{x}^{(2)}(t) = \boldsymbol{\xi}te^{rt} + \boldsymbol{\eta}e^{rt}}$$

and

$$\boxed{(A - rI)\boldsymbol{\eta} = \boldsymbol{\xi}}.$$

5.5 Repeated Eigenvalues



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and

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Definition

$\boldsymbol{\eta}$ is called a *generalised eigenvector* of A .

5.5 Repeated Eigenvalues



Remark

If you have 2 repeated eigenvalues (but with only one linearly independent eigenvector), the method is:

- 1 Find the eigenvalues and eigenvectors;
- 2 The first solution is $\mathbf{x}^{(1)}(t) = \boldsymbol{\xi}e^{rt}$;
- 3 Use $(A - rI)\boldsymbol{\eta} = \boldsymbol{\xi}$ to find a generalised eigenvector $\boldsymbol{\eta}$;
- 4 The second solution is $\mathbf{x}^{(2)}(t) = \boldsymbol{\xi}te^{rt} + \boldsymbol{\eta}e^{rt}$.

5.5 Repeated Eigenvalues



Example

Solve

$$\begin{cases} \mathbf{x}' = \begin{bmatrix} -\frac{5}{2} & \frac{3}{2} \\ -\frac{3}{2} & \frac{1}{2} \end{bmatrix} \mathbf{x}, \\ \mathbf{x}(0) = \begin{bmatrix} 3 \\ -1 \end{bmatrix}. \end{cases}$$

5.5 Repeated Eigenvalues



The only eigenvalue of the matrix is $r = -1$. The corresponding eigenvector is $\boldsymbol{\xi} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. Therefore one solution of the linear system is

$$\mathbf{x}^{(1)}(t) = \boldsymbol{\xi} e^{rt} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{-t}.$$

5.5 Repeated Eigenvalues



We need to find a second, linearly independent solution. We will consider the ansatz

$$\mathbf{x}^{(2)}(t) = \boldsymbol{\xi} t e^{-t} + \boldsymbol{\eta} e^{-t}$$

where $\boldsymbol{\xi} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ as above and $\boldsymbol{\eta}$ is a generalised eigenvector solving $(A - rI)\boldsymbol{\eta} = \boldsymbol{\xi}$.

5.5 Repeated Eigenvalues



Solving the latter equation,

$$(A - rI)\boldsymbol{\eta} = \boldsymbol{\xi}$$

5.5 Repeated Eigenvalues



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$$(A - rI)\boldsymbol{\eta} = \boldsymbol{\xi}$$
$$\begin{bmatrix} -\frac{3}{2} & \frac{3}{2} \\ -\frac{3}{2} & \frac{3}{2} \end{bmatrix} \begin{bmatrix} \eta_1 \\ \eta_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

5.5 Repeated Eigenvalues



Solving the latter equation,

$$\begin{aligned}(A - rI)\boldsymbol{\eta} &= \boldsymbol{\xi} \\ \begin{bmatrix} -\frac{3}{2} & \frac{3}{2} \\ -\frac{3}{2} & \frac{3}{2} \end{bmatrix} \begin{bmatrix} \eta_1 \\ \eta_2 \end{bmatrix} &= \begin{bmatrix} 1 \\ 1 \end{bmatrix} \\ -\frac{3}{2}\eta_1 + \frac{3}{2}\eta_2 &= 1\end{aligned}$$

5.5 Repeated Eigenvalues



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$$\begin{aligned}(A - rI)\boldsymbol{\eta} &= \boldsymbol{\xi} \\ \begin{bmatrix} -\frac{3}{2} & \frac{3}{2} \\ -\frac{3}{2} & \frac{3}{2} \end{bmatrix} \begin{bmatrix} \eta_1 \\ \eta_2 \end{bmatrix} &= \begin{bmatrix} 1 \\ 1 \end{bmatrix} \\ -\frac{3}{2}\eta_1 + \frac{3}{2}\eta_2 &= 1 \\ -\eta_1 + \eta_2 &= \frac{2}{3}\end{aligned}$$

5.5 Repeated Eigenvalues



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$$\begin{aligned}(A - rI)\boldsymbol{\eta} &= \boldsymbol{\xi} \\ \begin{bmatrix} -\frac{3}{2} & \frac{3}{2} \\ -\frac{3}{2} & \frac{3}{2} \end{bmatrix} \begin{bmatrix} \eta_1 \\ \eta_2 \end{bmatrix} &= \begin{bmatrix} 1 \\ 1 \end{bmatrix} \\ -\frac{3}{2}\eta_1 + \frac{3}{2}\eta_2 &= 1 \\ -\eta_1 + \eta_2 &= \frac{2}{3}\end{aligned}$$

we can choose $\boldsymbol{\eta} = \begin{bmatrix} 0 \\ \frac{2}{3} \end{bmatrix}$.

5.5 Repeated Eigenvalues



Note that we don't need to find *every* generalised eigenvector

$$\boldsymbol{\eta} = \begin{bmatrix} k \\ k + \frac{2}{3} \end{bmatrix} = k \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{2}{3} \end{bmatrix} = k\boldsymbol{\xi} + \begin{bmatrix} 0 \\ \frac{2}{3} \end{bmatrix}$$

because we already have $\mathbf{x}^{(1)}(t) = \boldsymbol{\xi}e^{rt}$.

5.5 Repeated Eigenvalues



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because we already have $\mathbf{x}^{(1)}(t) = \boldsymbol{\xi}e^{rt}$.

Instead we only need to find *one* generalised eigenvector – that means that we can choose any k that we want.

5.5 Repeated Eigenvalues



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$$\boldsymbol{\eta} = \begin{bmatrix} k \\ k + \frac{2}{3} \end{bmatrix} = k \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{2}{3} \end{bmatrix} = k\boldsymbol{\xi} + \begin{bmatrix} 0 \\ \frac{2}{3} \end{bmatrix}$$

because we already have $\mathbf{x}^{(1)}(t) = \boldsymbol{\xi}e^{rt}$.

Instead we only need to find *one* generalised eigenvector – that means that we can choose any k that we want.

Hence I have chosen $k = 0$ which gives $\boldsymbol{\eta} = \begin{bmatrix} 0 \\ \frac{2}{3} \end{bmatrix}$.

5.5 Repeated Eigenvalues



eigenvector

$$\xi = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

generalised eigenvector

$$\eta = \begin{bmatrix} 0 \\ \frac{2}{3} \end{bmatrix}$$

Thus

$$\mathbf{x}^{(1)}(t) = \xi e^{-t} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{-t}$$

and

$$\mathbf{x}^{(2)}(t) = \xi t e^{-t} + \eta e^{-t} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} t e^{-t} + \begin{bmatrix} 0 \\ \frac{2}{3} \end{bmatrix} e^{-t}.$$

5.5 Repeated Eigenvalues



eigenvector

$$\xi = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

generalised eigenvector

$$\eta = \begin{bmatrix} 0 \\ \frac{2}{3} \end{bmatrix}$$

Thus

$$\mathbf{x}^{(1)}(t) = \xi e^{-t} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{-t}$$

and

$$\mathbf{x}^{(2)}(t) = \xi t e^{-t} + \eta e^{-t} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} t e^{-t} + \begin{bmatrix} 0 \\ \frac{2}{3} \end{bmatrix} e^{-t}.$$

Hence the general solution to the linear system is

$$\mathbf{x}(t) = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{-t} + c_2 \left(\begin{bmatrix} 1 \\ 1 \end{bmatrix} t e^{-t} + \begin{bmatrix} 0 \\ \frac{2}{3} \end{bmatrix} e^{-t} \right).$$

5.5 Repeated Eigenvalues



The initial condition gives

$$\begin{bmatrix} 3 \\ -1 \end{bmatrix} = \mathbf{x}(0) = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ \frac{2}{3} \end{bmatrix}$$

which implies that $c_1 = 3$ and $c_2 = -6$.

5.5 Repeated Eigenvalues



The initial condition gives

$$\begin{bmatrix} 3 \\ -1 \end{bmatrix} = \mathbf{x}(0) = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ \frac{2}{3} \end{bmatrix}$$

which implies that $c_1 = 3$ and $c_2 = -6$.

Therefore the solution to the IVP is

$$\mathbf{x}(t) = 3 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{-t} - 6 \left(\begin{bmatrix} 1 \\ 1 \end{bmatrix} t e^{-t} + \begin{bmatrix} 0 \\ \frac{2}{3} \end{bmatrix} e^{-t} \right) = \begin{bmatrix} 3 \\ -1 \end{bmatrix} e^{-t} - 6 \begin{bmatrix} 1 \\ 1 \end{bmatrix} t e^{-t}.$$

5.5 Repeated Eigenvalues



Example

Solve

$$\mathbf{x}' = \begin{bmatrix} 1 & -4 \\ 4 & -7 \end{bmatrix} \mathbf{x}, \quad \mathbf{x}(0) = \begin{bmatrix} 3 \\ 2 \end{bmatrix}.$$

5.5 Repeated Eigenvalues



Example

Solve

$$\mathbf{x}' = \begin{bmatrix} 1 & -4 \\ 4 & -7 \end{bmatrix} \mathbf{x}, \quad \mathbf{x}(0) = \begin{bmatrix} 3 \\ 2 \end{bmatrix}.$$

The only eigenvalue of the matrix is $r = -3$. The corresponding eigenvector is $\boldsymbol{\xi} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

5.5 Repeated Eigenvalues



Example

Solve

$$\mathbf{x}' = \begin{bmatrix} 1 & -4 \\ 4 & -7 \end{bmatrix} \mathbf{x}, \quad \mathbf{x}(0) = \begin{bmatrix} 3 \\ 2 \end{bmatrix}.$$

The only eigenvalue of the matrix is $r = -3$. The corresponding eigenvector is $\boldsymbol{\xi} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. Therefore one solution of the linear system is

$$\mathbf{x}^{(1)}(t) = \boldsymbol{\xi} e^{rt} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{-3t}.$$

5.5 Repeated Eigenvalues



Next we need to find a generalised eigenvector $\boldsymbol{\eta}$.

5.5 Repeated Eigenvalues



We calculate that

$$(A - rI)\boldsymbol{\eta} = \boldsymbol{\xi}$$

5.5 Repeated Eigenvalues



We calculate that

$$\begin{bmatrix} 4 & -4 \\ 4 & -4 \end{bmatrix} \begin{bmatrix} \eta_1 \\ \eta_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

5.5 Repeated Eigenvalues



We calculate that

$$4\eta_1 - 4\eta_2 = 1$$

5.5 Repeated Eigenvalues



We calculate that

$$-\eta_1 + \eta_2 = -\frac{1}{4}$$

5.5 Repeated Eigenvalues



We calculate that

$$\eta_2 = \eta_1 - \frac{1}{4}.$$

5.5 Repeated Eigenvalues



We calculate that

$$\eta_2 = \eta_1 - \frac{1}{4}.$$

We can choose any vector $\boldsymbol{\eta}$ that satisfies $\eta_2 = \eta_1 - \frac{1}{4}$.

5.5 Repeated Eigenvalues



We calculate that

$$\eta_2 = \eta_1 - \frac{1}{4}.$$

We can choose any vector $\boldsymbol{\eta}$ that satisfies $\eta_2 = \eta_1 - \frac{1}{4}$. Thus we may choose $\boldsymbol{\eta} = \begin{bmatrix} 0 \\ -\frac{1}{4} \end{bmatrix}$.

5.5 Repeated Eigenvalues



eigenvector

$$\xi = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

generalised eigenvector

$$\eta = \begin{bmatrix} 0 \\ -\frac{1}{4} \end{bmatrix}$$

Therefore

$$\mathbf{x}^{(2)}(t) = \begin{bmatrix} 1 \\ 1 \end{bmatrix} t e^{-3t} + \begin{bmatrix} 0 \\ -\frac{1}{4} \end{bmatrix} e^{-3t}.$$

5.5 Repeated Eigenvalues



Hence the general solution to the ODE is

$$\mathbf{x}(t) = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{-3t} + c_2 \left(\begin{bmatrix} 1 \\ 1 \end{bmatrix} t e^{-3t} + \begin{bmatrix} 0 \\ -\frac{1}{4} \end{bmatrix} e^{-3t} \right).$$

5.5 Repeated Eigenvalues



The initial condition gives

$$\begin{bmatrix} 3 \\ 2 \end{bmatrix} = \mathbf{x}(0) = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ -\frac{1}{4} \end{bmatrix}$$

which implies that $c_1 = 3$ and $c_2 = 4$.

5.5 Repeated Eigenvalues



Therefore the solution to the IVP is

$$\begin{aligned}\mathbf{x}(t) &= 3 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{-3t} + 4 \left(\begin{bmatrix} 1 \\ 1 \end{bmatrix} t e^{-3t} + \begin{bmatrix} 0 \\ -\frac{1}{4} \end{bmatrix} e^{-3t} \right) \\ &= \\ &= \end{aligned}$$

5.5 Repeated Eigenvalues



Therefore the solution to the IVP is

$$\begin{aligned}\mathbf{x}(t) &= 3 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{-3t} + 4 \left(\begin{bmatrix} 1 \\ 1 \end{bmatrix} t e^{-3t} + \begin{bmatrix} 0 \\ -\frac{1}{4} \end{bmatrix} e^{-3t} \right) \\ &= \begin{bmatrix} 3 \\ 2 \end{bmatrix} e^{-3t} + 4 \begin{bmatrix} 1 \\ 1 \end{bmatrix} t e^{-3t} \\ &= \end{aligned}$$

5.5 Repeated Eigenvalues



Therefore the solution to the IVP is

$$\begin{aligned}\mathbf{x}(t) &= 3 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{-3t} + 4 \left(\begin{bmatrix} 1 \\ 1 \end{bmatrix} t e^{-3t} + \begin{bmatrix} 0 \\ -\frac{1}{4} \end{bmatrix} e^{-3t} \right) \\ &= \begin{bmatrix} 3 \\ 2 \end{bmatrix} e^{-3t} + 4 \begin{bmatrix} 1 \\ 1 \end{bmatrix} t e^{-3t} \\ &= \begin{bmatrix} 3 + 4t \\ 2 + 4t \end{bmatrix} e^{-3t}.\end{aligned}$$

Nonhomogeneous Linear Systems

5.6 Nonhomogeneous Linear Systems



Consider

$$\mathbf{x}' = P(t)\mathbf{x} + \mathbf{g}(t) \quad (1)$$

where $P(t)$ and $\mathbf{g}(t)$ are continuous for $\alpha < t < \beta$.

5.6 Nonhomogeneous Linear Systems



Consider

$$\mathbf{x}' = P(t)\mathbf{x} + \mathbf{g}(t) \quad (1)$$

where $P(t)$ and $\mathbf{g}(t)$ are continuous for $\alpha < t < \beta$. The general solution of (1) can be written as

$$\mathbf{x}(t) = c_1\mathbf{x}^{(1)} + c_2\mathbf{x}^{(2)} + \dots + c_n\mathbf{x}^{(n)} + \mathbf{v}(t)$$

5.6 Nonhomogeneous Linear Systems



Consider

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where

- $c_1\mathbf{x}^{(1)} + c_2\mathbf{x}^{(2)} + \dots + c_n\mathbf{x}^{(n)}$ is the general solution to the homogeneous system $\mathbf{x}' = P(t)\mathbf{x}$; and

5.6 Nonhomogeneous Linear Systems



Consider

$$\mathbf{x}' = P(t)\mathbf{x} + \mathbf{g}(t) \quad (1)$$

where $P(t)$ and $\mathbf{g}(t)$ are continuous for $\alpha < t < \beta$. The general solution of (1) can be written as

$$\mathbf{x}(t) = c_1\mathbf{x}^{(1)} + c_2\mathbf{x}^{(2)} + \dots + c_n\mathbf{x}^{(n)} + \mathbf{v}(t)$$

where

- $c_1\mathbf{x}^{(1)} + c_2\mathbf{x}^{(2)} + \dots + c_n\mathbf{x}^{(n)}$ is the general solution to the homogeneous system $\mathbf{x}' = P(t)\mathbf{x}$; and
- $\mathbf{v}(t)$ is a particular solution to (1).

5.6 Nonhomogeneous Linear Systems



$$\mathbf{x}' = P(t)\mathbf{x} + \mathbf{g}(t) \quad (1)$$

Remark

We will study four methods to solve (1):

5.6 Nonhomogeneous Linear Systems



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Remark

We will study four methods to solve (1):

- 1 Diagonalisation;

5.6 Nonhomogeneous Linear Systems



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We will study four methods to solve (1):

- 1 Diagonalisation;
- 2 Undetermined Coefficients;

5.6 Nonhomogeneous Linear Systems



$$\mathbf{x}' = P(t)\mathbf{x} + \mathbf{g}(t) \quad (1)$$

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We will study four methods to solve (1):

- 1 Diagonalisation;
- 2 Undetermined Coefficients;
- 3 Variation of Parameters;

5.6 Nonhomogeneous Linear Systems



$$\mathbf{x}' = P(t)\mathbf{x} + \mathbf{g}(t) \quad (1)$$

Remark

We will study four methods to solve (1):

- 1 Diagonalisation;
- 2 Undetermined Coefficients;
- 3 Variation of Parameters;
- 4 The Laplace Transform.



Method 1 – Diagonalisation:



Method 1 – Diagonalisation:

Consider

$$\mathbf{x}' = A\mathbf{x} + \mathbf{g}(t).$$

Method 1 – Diagonalisation:

Consider

$$\mathbf{x}' = A\mathbf{x} + \mathbf{g}(t).$$

Suppose that

- $A \in \mathbb{R}^{n \times n}$ is diagonalisable;
- $\mathbf{g} : (\alpha, \beta) \rightarrow \mathbb{R}^n$;
- $\boldsymbol{\xi}^{(1)}, \dots, \boldsymbol{\xi}^{(n)}$ are eigenvectors of A ; and
- $T = \begin{bmatrix} \boldsymbol{\xi}^{(1)} & \dots & \boldsymbol{\xi}^{(n)} \end{bmatrix}.$

5.6 Nonhomogeneous Linear Systems



Then

$$D = T^{-1}AT = \begin{bmatrix} r_1 & 0 & \cdots & 0 \\ 0 & r_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & r_n \end{bmatrix}.$$

5.6 Nonhomogeneous Linear Systems



Let $\mathbf{y} = T^{-1}\mathbf{x}$.

5.6 Nonhomogeneous Linear Systems



Let $\mathbf{y} = T^{-1}\mathbf{x}$. Then $\mathbf{x} = T\mathbf{y}$.

5.6 Nonhomogeneous Linear Systems



Let $\mathbf{y} = T^{-1}\mathbf{x}$. Then $\mathbf{x} = T\mathbf{y}$. It follows that

$$\mathbf{x}' = A\mathbf{x} + \mathbf{g}(t)$$

5.6 Nonhomogeneous Linear Systems



Let $\mathbf{y} = T^{-1}\mathbf{x}$. Then $\mathbf{x} = T\mathbf{y}$. It follows that

$$T\mathbf{y}' = \mathbf{x}' = A\mathbf{x} + \mathbf{g}(t)$$

5.6 Nonhomogeneous Linear Systems



Let $\mathbf{y} = T^{-1}\mathbf{x}$. Then $\mathbf{x} = T\mathbf{y}$. It follows that

$$T\mathbf{y}' = \mathbf{x}' = A\mathbf{x} + \mathbf{g}(t) = AT\mathbf{y} + \mathbf{g}(t)$$

5.6 Nonhomogeneous Linear Systems



Let $\mathbf{y} = T^{-1}\mathbf{x}$. Then $\mathbf{x} = T\mathbf{y}$. It follows that

$$T\mathbf{y}' = \mathbf{x}' = A\mathbf{x} + \mathbf{g}(t) = AT\mathbf{y} + \mathbf{g}(t)$$

and

$$\mathbf{y}' = T^{-1}AT\mathbf{y} + T^{-1}\mathbf{g}(t) \quad (2)$$

5.6 Nonhomogeneous Linear Systems



Let $\mathbf{y} = T^{-1}\mathbf{x}$. Then $\mathbf{x} = T\mathbf{y}$. It follows that

$$T\mathbf{y}' = \mathbf{x}' = A\mathbf{x} + \mathbf{g}(t) = AT\mathbf{y} + \mathbf{g}(t)$$

and

$$\mathbf{y}' = T^{-1}AT\mathbf{y} + T^{-1}\mathbf{g}(t) = D\mathbf{y} + \mathbf{h}(t) \quad (2)$$

where $\mathbf{h} = T^{-1}\mathbf{g}$.

5.6 Nonhomogeneous Linear Systems



But $\mathbf{y}' = D\mathbf{y} + \mathbf{h}(t)$ is just the system

$$\begin{cases} y_1' = r_1 y_1 + h_1(t) \\ y_2' = r_2 y_2 + h_2(t) \\ \vdots \\ y_n' = r_n y_n + h_n(t) \end{cases}$$

5.6 Nonhomogeneous Linear Systems



But $\mathbf{y}' = D\mathbf{y} + \mathbf{h}(t)$ is just the system

$$\begin{cases} y_1' = r_1 y_1 + h_1(t) & \leftarrow \text{only } y_1 \text{ and } t \\ y_2' = r_2 y_2 + h_2(t) \\ \vdots \\ y_n' = r_n y_n + h_n(t) \end{cases}$$

5.6 Nonhomogeneous Linear Systems



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5.6 Nonhomogeneous Linear Systems



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5.6 Nonhomogeneous Linear Systems



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We can solve each of these n first order linear ODEs individually.

5.6 Nonhomogeneous Linear Systems



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We can solve each of these n first order linear ODEs individually. The solution to

$$y_j' - r_j y_j = h_j$$

(see Chapter 2) is

$$y_j(t) = e^{r_j t} \int_{t_0}^t e^{-r_j s} h(s) ds + c_j e^{r_j t}.$$

5.6 Nonhomogeneous Linear Systems



But $\mathbf{y}' = D\mathbf{y} + \mathbf{h}(t)$ is just the system

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We can solve each of these n first order linear ODEs individually. The solution to

$$y_j' - r_j y_j = h_j$$

(see Chapter 2) is

$$y_j(t) = e^{r_j t} \int_{t_0}^t e^{-r_j s} h(s) ds + c_j e^{r_j t}.$$

If we know \mathbf{y} , then we know $\mathbf{x} = T\mathbf{y}$.

5.6 Nonhomogeneous Linear Systems



Example

Solve

$$\mathbf{x}' = \begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 2e^{-t} \\ 3t \end{bmatrix} = A\mathbf{x} + \mathbf{g}(t).$$

5.6 Nonhomogeneous Linear Systems



Example

Solve

$$\mathbf{x}' = \begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 2e^{-t} \\ 3t \end{bmatrix} = A\mathbf{x} + \mathbf{g}(t).$$

The eigenvalues of $A = \begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix}$ are $r_1 = -3$ and $r_2 = -1$.

The eigenvectors are

$$\boldsymbol{\xi}^{(1)} = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \quad \text{and} \quad \boldsymbol{\xi}^{(2)} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

5.6 Nonhomogeneous Linear Systems



Example

Solve

$$\mathbf{x}' = \begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 2e^{-t} \\ 3t \end{bmatrix} = A\mathbf{x} + \mathbf{g}(t).$$

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The eigenvectors are

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So

$$T = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \quad \text{and} \quad T^{-1} = \frac{1}{2} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}.$$

5.6 Nonhomogeneous Linear Systems



Let $\mathbf{y} = T^{-1}\mathbf{x}$.

5.6 Nonhomogeneous Linear Systems



Let $\mathbf{y} = T^{-1}\mathbf{x}$. Then

$$= \mathbf{x}' = A\mathbf{x} + \begin{bmatrix} 2e^{-t} \\ 3t \end{bmatrix} =$$

5.6 Nonhomogeneous Linear Systems



Let $\mathbf{y} = T^{-1}\mathbf{x}$. Then

$$T\mathbf{y}' = \mathbf{x}' = A\mathbf{x} + \begin{bmatrix} 2e^{-t} \\ 3t \end{bmatrix} =$$

5.6 Nonhomogeneous Linear Systems



Let $\mathbf{y} = T^{-1}\mathbf{x}$. Then

$$T\mathbf{y}' = \mathbf{x}' = A\mathbf{x} + \begin{bmatrix} 2e^{-t} \\ 3t \end{bmatrix} = AT\mathbf{y} + \begin{bmatrix} 2e^{-t} \\ 3t \end{bmatrix}$$

5.6 Nonhomogeneous Linear Systems



Let $\mathbf{y} = T^{-1}\mathbf{x}$. Then

$$T\mathbf{y}' = \mathbf{x}' = A\mathbf{x} + \begin{bmatrix} 2e^{-t} \\ 3t \end{bmatrix} = AT\mathbf{y} + \begin{bmatrix} 2e^{-t} \\ 3t \end{bmatrix}$$

$$\mathbf{y}' = T^{-1}AT\mathbf{y} + T^{-1} \begin{bmatrix} 2e^{-t} \\ 3t \end{bmatrix}$$

5.6 Nonhomogeneous Linear Systems



Let $\mathbf{y} = T^{-1}\mathbf{x}$. Then

$$T\mathbf{y}' = \mathbf{x}' = A\mathbf{x} + \begin{bmatrix} 2e^{-t} \\ 3t \end{bmatrix} = AT\mathbf{y} + \begin{bmatrix} 2e^{-t} \\ 3t \end{bmatrix}$$

$$\begin{aligned} \mathbf{y}' &= T^{-1}AT\mathbf{y} + T^{-1} \begin{bmatrix} 2e^{-t} \\ 3t \end{bmatrix} \\ &= D\mathbf{y} + \frac{1}{2} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 2e^{-t} \\ 3t \end{bmatrix} \end{aligned}$$

5.6 Nonhomogeneous Linear Systems



Let $\mathbf{y} = T^{-1}\mathbf{x}$. Then

$$\begin{aligned} T\mathbf{y}' &= \mathbf{x}' = A\mathbf{x} + \begin{bmatrix} 2e^{-t} \\ 3t \end{bmatrix} = AT\mathbf{y} + \begin{bmatrix} 2e^{-t} \\ 3t \end{bmatrix} \\ \mathbf{y}' &= T^{-1}AT\mathbf{y} + T^{-1} \begin{bmatrix} 2e^{-t} \\ 3t \end{bmatrix} \\ &= D\mathbf{y} + \frac{1}{2} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 2e^{-t} \\ 3t \end{bmatrix} \\ &= \begin{bmatrix} -3 & 0 \\ 0 & -1 \end{bmatrix} \mathbf{y} + \frac{1}{2} \begin{bmatrix} 2e^{-t} - 3t \\ 2e^{-t} + 3t \end{bmatrix}. \end{aligned}$$

5.6 Nonhomogeneous Linear Systems



Let $\mathbf{y} = T^{-1}\mathbf{x}$. Then

$$\begin{aligned} T\mathbf{y}' &= \mathbf{x}' = A\mathbf{x} + \begin{bmatrix} 2e^{-t} \\ 3t \end{bmatrix} = AT\mathbf{y} + \begin{bmatrix} 2e^{-t} \\ 3t \end{bmatrix} \\ \mathbf{y}' &= T^{-1}AT\mathbf{y} + T^{-1} \begin{bmatrix} 2e^{-t} \\ 3t \end{bmatrix} \\ &= D\mathbf{y} + \frac{1}{2} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 2e^{-t} \\ 3t \end{bmatrix} \\ &= \begin{bmatrix} -3 & 0 \\ 0 & -1 \end{bmatrix} \mathbf{y} + \frac{1}{2} \begin{bmatrix} 2e^{-t} - 3t \\ 2e^{-t} + 3t \end{bmatrix}. \end{aligned}$$

Therefore

$$\begin{cases} y_1' + 3y_1 = e^{-t} - \frac{3}{2}t \\ y_2' + y_2 = e^{-t} + \frac{3}{2}t. \end{cases}$$

5.6 Nonhomogeneous Linear Systems



$$\begin{cases} y_1' + 3y_1 = e^{-t} - \frac{3}{2}t \\ y_2' + y_2 = e^{-t} + \frac{3}{2}t. \end{cases}$$

You know how to solve first order linear ODEs. The solutions to these two ODEs are

$$\begin{aligned} y_1(t) &= \frac{1}{2}e^{-t} - \frac{t}{2} + \frac{1}{6} + c_1e^{-3t} \\ y_2(t) &= te^{-t} + \frac{3t}{2} - \frac{3}{2} + c_2e^{-t}. \end{aligned}$$

5.6 Nonhomogeneous Linear Systems



Finally we calculate that

$$\mathbf{x} = T\mathbf{y} = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

=

=

5.6 Nonhomogeneous Linear Systems



Finally we calculate that

$$\begin{aligned}\mathbf{x} = T\mathbf{y} &= \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \\ &= \begin{bmatrix} y_1 + y_2 \\ -y_1 + y_2 \end{bmatrix} \\ &= \end{aligned}$$

5.6 Nonhomogeneous Linear Systems



Finally we calculate that

$$\begin{aligned}\mathbf{x} = T\mathbf{y} &= \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \\ &= \begin{bmatrix} y_1 + y_2 \\ -y_1 + y_2 \end{bmatrix} \\ &= c_1 \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{-3t} + c_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{-t} + \frac{1}{2} \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{-t} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} t e^{-t} + \begin{bmatrix} 1 \\ 2 \end{bmatrix} t - \frac{1}{3} \begin{bmatrix} 4 \\ 5 \end{bmatrix}.\end{aligned}$$

5.6 Nonhomogeneous Linear Systems



Example

Solve

$$\mathbf{x}' = \begin{bmatrix} 1 & \sqrt{3} \\ \sqrt{3} & -1 \end{bmatrix} \mathbf{x} + \begin{bmatrix} e^t \\ \sqrt{3}e^{-t} \end{bmatrix}.$$

5.6 Nonhomogeneous Linear Systems



The eigenvalues of $\begin{bmatrix} 1 & \sqrt{3} \\ \sqrt{3} & -1 \end{bmatrix}$ are $r_1 = -2$ and $r_2 = 2$. The corresponding eigenvectors are $\xi^{(1)} = \begin{bmatrix} 1 \\ -\sqrt{3} \end{bmatrix}$ and $\xi^{(2)} = \begin{bmatrix} \sqrt{3} \\ 1 \end{bmatrix}$. Thus

$$T = \begin{bmatrix} 1 & \sqrt{3} \\ -\sqrt{3} & 1 \end{bmatrix},$$

5.6 Nonhomogeneous Linear Systems



The eigenvalues of $\begin{bmatrix} 1 & \sqrt{3} \\ \sqrt{3} & -1 \end{bmatrix}$ are $r_1 = -2$ and $r_2 = 2$. The corresponding eigenvectors are $\xi^{(1)} = \begin{bmatrix} 1 \\ -\sqrt{3} \end{bmatrix}$ and $\xi^{(2)} = \begin{bmatrix} \sqrt{3} \\ 1 \end{bmatrix}$.

Thus

$$T = \begin{bmatrix} 1 & \sqrt{3} \\ -\sqrt{3} & 1 \end{bmatrix},$$

$$T^{-1} = \frac{1}{\det T} \begin{bmatrix} 1 & -\sqrt{3} \\ \sqrt{3} & 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{4} & -\frac{\sqrt{3}}{4} \\ \frac{\sqrt{3}}{4} & \frac{1}{4} \end{bmatrix}$$

and

$$D = T^{-1}AT = \begin{bmatrix} -2 & 0 \\ 0 & 2 \end{bmatrix}.$$

5.6 Nonhomogeneous Linear Systems



Now we must change variables: Let $\mathbf{y} = T^{-1}\mathbf{x}$.

5.6 Nonhomogeneous Linear Systems



Now we must change variables: Let $\mathbf{y} = T^{-1}\mathbf{x}$. Then we have

$$\begin{aligned}\mathbf{y}' &= D\mathbf{y} + T^{-1}\mathbf{g} = \begin{bmatrix} -2 & 0 \\ 0 & 2 \end{bmatrix} \mathbf{y} + \begin{bmatrix} \frac{1}{4} & -\frac{\sqrt{3}}{4} \\ \frac{\sqrt{3}}{4} & \frac{1}{4} \end{bmatrix} \begin{bmatrix} e^t \\ \sqrt{3}e^{-t} \end{bmatrix} \\ &= \begin{bmatrix} -2y_1 \\ 2y_2 \end{bmatrix} + \begin{bmatrix} \frac{1}{4}e^t - \frac{3}{4}e^{-t} \\ \frac{\sqrt{3}}{4}e^t + \frac{\sqrt{3}}{4}e^{-t} \end{bmatrix}.\end{aligned}$$

5.6 Nonhomogeneous Linear Systems



We know how to solve

$$y_1' + 2y_1 = \frac{1}{4}e^t - \frac{3}{4}e^{-t}$$

and

$$y_2' - 2y_2 = \frac{\sqrt{3}}{4}e^t + \frac{\sqrt{3}}{4}e^{-t}.$$

5.6 Nonhomogeneous Linear Systems



We know how to solve

$$y_1' + 2y_1 = \frac{1}{4}e^t - \frac{3}{4}e^{-t}$$

and

$$y_2' - 2y_2 = \frac{\sqrt{3}}{4}e^t + \frac{\sqrt{3}}{4}e^{-t}.$$

The solutions are

$$y_1(t) = \frac{1}{12}e^t - \frac{3}{4}e^{-t} + c_1e^{-2t}$$

and

$$y_2(t) = -\frac{\sqrt{3}}{4}e^t - \frac{\sqrt{3}}{12}e^{-t} + c_2e^{2t}.$$

5.6 Nonhomogeneous Linear Systems



So

$$\mathbf{y} = \begin{bmatrix} \frac{1}{12}e^t - \frac{3}{4}e^{-t} + c_1e^{-2t} \\ -\frac{\sqrt{3}}{4}e^t - \frac{\sqrt{3}}{12}e^{-t} + c_2e^{2t} \end{bmatrix}.$$

5.6 Nonhomogeneous Linear Systems



So

$$\mathbf{y} = \begin{bmatrix} \frac{1}{12}e^t - \frac{3}{4}e^{-t} + c_1e^{-2t} \\ -\frac{\sqrt{3}}{4}e^t - \frac{\sqrt{3}}{12}e^{-t} + c_2e^{2t} \end{bmatrix}.$$

Therefore the general solution to the ODE is

$$\mathbf{x} = T\mathbf{y} = \begin{bmatrix} 1 & \sqrt{3} \\ -\sqrt{3} & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{12}e^t - \frac{3}{4}e^{-t} + c_1e^{-2t} \\ -\frac{\sqrt{3}}{4}e^t - \frac{\sqrt{3}}{12}e^{-t} + c_2e^{2t} \end{bmatrix} = \dots$$



Method 2 – Undetermined Coefficients:



Method 2 – Undetermined Coefficients:

Consider

$$\mathbf{x}' = A\mathbf{x} + \mathbf{g}(t).$$



Method 2 – Undetermined Coefficients:

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(Remember Chapter 3?)



Method 2 – Undetermined Coefficients:

Consider

$$\mathbf{x}' = A\mathbf{x} + \mathbf{g}(t).$$

(Remember Chapter 3?)

The idea is

- 1 Find the general solution to $\mathbf{x}' = A\mathbf{x}$.



Method 2 – Undetermined Coefficients:

Consider

$$\mathbf{x}' = A\mathbf{x} + \mathbf{g}(t).$$

(Remember Chapter 3?)

The idea is

- 1 Find the general solution to $\mathbf{x}' = A\mathbf{x}$.
- 2 Look at $\mathbf{g}(t)$. Make a guess with constants. Find the constants.



Method 2 – Undetermined Coefficients:

Consider

$$\mathbf{x}' = A\mathbf{x} + \mathbf{g}(t).$$

(Remember Chapter 3?)

The idea is

- 1 Find the general solution to $\mathbf{x}' = A\mathbf{x}$.
- 2 Look at $\mathbf{g}(t)$. Make a guess with constants. Find the constants.
- 3 1 + 2.

5.6 Nonhomogeneous Linear Systems



Example

Solve

$$\mathbf{x}' = \begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 2e^{-t} \\ 3t \end{bmatrix} = A\mathbf{x} + \mathbf{g}(t).$$

5.6 Nonhomogeneous Linear Systems



1. The solution of $\mathbf{x}' = \begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix} \mathbf{x}$ is

$$\mathbf{x}(t) = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{-t} + c_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{-3t}.$$

5.6 Nonhomogeneous Linear Systems



2. Since $\mathbf{g}(t) = \begin{bmatrix} 2 \\ 0 \end{bmatrix} e^{-t} + \begin{bmatrix} 0 \\ 3 \end{bmatrix} t$, we try the ansatz

$$\mathbf{x} = \mathbf{v}(t) = \mathbf{a}te^{-t} + \mathbf{b}e^{-t} + \mathbf{c}t + \mathbf{d}.$$

5.6 Nonhomogeneous Linear Systems



2. Since $\mathbf{g}(t) = \begin{bmatrix} 2 \\ 0 \end{bmatrix} e^{-t} + \begin{bmatrix} 0 \\ 3 \end{bmatrix} t$, we try the ansatz

$$\mathbf{x} = \mathbf{v}(t) = \mathbf{a}te^{-t} + \mathbf{b}e^{-t} + \mathbf{c}t + \mathbf{d}.$$

(Note that because $r_1 = -1$ is an eigenvalue of $\begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix}$, we need both te^{-t} and e^{-t} .)

5.6 Nonhomogeneous Linear Systems



2. Since $\mathbf{g}(t) = \begin{bmatrix} 2 \\ 0 \end{bmatrix} e^{-t} + \begin{bmatrix} 0 \\ 3 \end{bmatrix} t$, we try the ansatz

$$\mathbf{x} = \mathbf{v}(t) = \mathbf{a}te^{-t} + \mathbf{b}e^{-t} + \mathbf{c}t + \mathbf{d}.$$

(Note that because $r_1 = -1$ is an eigenvalue of $\begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix}$, we need both te^{-t} and e^{-t} .)

Then we calculate that

$$\mathbf{x}' = \mathbf{A}\mathbf{x} + \mathbf{g}$$

$$\mathbf{a}e^{-t} - \mathbf{a}te^{-t} - \mathbf{b}e^{-t} + \mathbf{c} = \mathbf{A}\mathbf{a}te^{-t} + \mathbf{A}\mathbf{b}e^{-t} + \mathbf{A}\mathbf{c}t + \mathbf{A}\mathbf{d} + \begin{bmatrix} 2 \\ 0 \end{bmatrix} e^{-t} + \begin{bmatrix} 0 \\ 3 \end{bmatrix} t.$$

5.6 Nonhomogeneous Linear Systems



$$\mathbf{a}e^{-t} - \mathbf{a}te^{-t} - \mathbf{b}e^{-t} + \mathbf{c} = \mathbf{A}\mathbf{a}te^{-t} + \mathbf{A}\mathbf{b}e^{-t} + \mathbf{A}\mathbf{c}t + \mathbf{A}\mathbf{d} + \begin{bmatrix} 2 \\ 0 \end{bmatrix} e^{-t} + \begin{bmatrix} 0 \\ 3 \end{bmatrix} t$$

If we look at the te^{-t} terms, we have

$$-\mathbf{a} = \mathbf{A}\mathbf{a}$$

5.6 Nonhomogeneous Linear Systems



$$\mathbf{a}e^{-t} - \mathbf{a}te^{-t} - \mathbf{b}e^{-t} + \mathbf{c} = \mathbf{A}\mathbf{a}te^{-t} + \mathbf{A}\mathbf{b}e^{-t} + \mathbf{A}\mathbf{c}t + \mathbf{A}\mathbf{d} + \begin{bmatrix} 2 \\ 0 \end{bmatrix} e^{-t} + \begin{bmatrix} 0 \\ 3 \end{bmatrix} t$$

If we look at the te^{-t} terms, we have

$$-\mathbf{a} = \mathbf{A}\mathbf{a} \implies \mathbf{a} \text{ is an eigenvector}$$

5.6 Nonhomogeneous Linear Systems



$$\mathbf{a}e^{-t} - \mathbf{a}te^{-t} - \mathbf{b}e^{-t} + \mathbf{c} = A\mathbf{a}te^{-t} + A\mathbf{b}e^{-t} + A\mathbf{c}t + A\mathbf{d} + \begin{bmatrix} 2 \\ 0 \end{bmatrix} e^{-t} + \begin{bmatrix} 0 \\ 3 \end{bmatrix} t$$

If we look at the te^{-t} terms, we have

$$-\mathbf{a} = A\mathbf{a} \implies \mathbf{a} \text{ is an eigenvector} \implies \mathbf{a} = \begin{bmatrix} \alpha \\ \alpha \end{bmatrix}$$

for some $\alpha \in \mathbb{R}$.

5.6 Nonhomogeneous Linear Systems



$$\mathbf{a}e^{-t} - \mathbf{a}te^{-t} - \mathbf{b}e^{-t} + \mathbf{c} = A\mathbf{a}te^{-t} + A\mathbf{b}e^{-t} + A\mathbf{c}t + A\mathbf{d} + \begin{bmatrix} 2 \\ 0 \end{bmatrix} e^{-t} + \begin{bmatrix} 0 \\ 3 \end{bmatrix} t$$

If we look at the e^{-t} terms, we have

$$\mathbf{a} - \mathbf{b} = A\mathbf{b} + \begin{bmatrix} 2 \\ 0 \end{bmatrix}$$

5.6 Nonhomogeneous Linear Systems



$$\mathbf{a}e^{-t} - \mathbf{a}te^{-t} - \mathbf{b}e^{-t} + \mathbf{c} = A\mathbf{a}te^{-t} + A\mathbf{b}e^{-t} + A\mathbf{c}t + A\mathbf{d} + \begin{bmatrix} 2 \\ 0 \end{bmatrix} e^{-t} + \begin{bmatrix} 0 \\ 3 \end{bmatrix} t$$

If we look at the e^{-t} terms, we have

$$\mathbf{a} - \mathbf{b} = A\mathbf{b} + \begin{bmatrix} 2 \\ 0 \end{bmatrix}$$

which becomes

$$\begin{bmatrix} \alpha - 2 \\ \alpha \end{bmatrix} = \mathbf{a} - \begin{bmatrix} 2 \\ 0 \end{bmatrix} = (A + I)\mathbf{b} = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} = \begin{bmatrix} -b_1 + b_2 \\ b_1 - b_2 \end{bmatrix}.$$

5.6 Nonhomogeneous Linear Systems



$$\mathbf{a}e^{-t} - \mathbf{a}te^{-t} - \mathbf{b}e^{-t} + \mathbf{c} = A\mathbf{a}te^{-t} + A\mathbf{b}e^{-t} + A\mathbf{c}t + A\mathbf{d} + \begin{bmatrix} 2 \\ 0 \end{bmatrix} e^{-t} + \begin{bmatrix} 0 \\ 3 \end{bmatrix} t$$

If we look at the e^{-t} terms, we have

$$\mathbf{a} - \mathbf{b} = A\mathbf{b} + \begin{bmatrix} 2 \\ 0 \end{bmatrix}$$

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But this means that

$$\alpha - 2 = -b_1 + b_2 = -(b_1 - b_2) = -\alpha \quad \implies \quad \alpha = 1.$$

5.6 Nonhomogeneous Linear Systems



$$\mathbf{a}e^{-t} - \mathbf{a}te^{-t} - \mathbf{b}e^{-t} + \mathbf{c} = A\mathbf{a}te^{-t} + A\mathbf{b}e^{-t} + A\mathbf{c}t + A\mathbf{d} + \begin{bmatrix} 2 \\ 0 \end{bmatrix} e^{-t} + \begin{bmatrix} 0 \\ 3 \end{bmatrix} t$$

If we look at the e^{-t} terms, we have

$$\mathbf{a} - \mathbf{b} = A\mathbf{b} + \begin{bmatrix} 2 \\ 0 \end{bmatrix}$$

which becomes

$$\begin{bmatrix} \alpha - 2 \\ \alpha \end{bmatrix} = \mathbf{a} - \begin{bmatrix} 2 \\ 0 \end{bmatrix} = (A + I)\mathbf{b} = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} = \begin{bmatrix} -b_1 + b_2 \\ b_1 - b_2 \end{bmatrix}.$$

But this means that

$$\alpha - 2 = -b_1 + b_2 = -(b_1 - b_2) = -\alpha \quad \implies \quad \alpha = 1.$$

$$\text{So } \mathbf{a} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

5.6 Nonhomogeneous Linear Systems



Then we have that

$$b_1 - b_2 = 1 \quad \implies \quad \mathbf{b} = \begin{bmatrix} k \\ k - 1 \end{bmatrix}$$

for any k . If we choose $k = 0$, we get $\mathbf{b} = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$.

5.6 Nonhomogeneous Linear Systems



$$\mathbf{a}e^{-t} - \mathbf{a}te^{-t} - \mathbf{b}e^{-t} + \mathbf{c} = A\mathbf{a}te^{-t} + A\mathbf{b}e^{-t} + \mathbf{A}ct + A\mathbf{d} + \begin{bmatrix} 2 \\ 0 \end{bmatrix} e^{-t} + \begin{bmatrix} 0 \\ 3 \end{bmatrix} t$$

If we look at the t terms, we have

$$0 = A\mathbf{c} + \begin{bmatrix} 0 \\ 3 \end{bmatrix}$$

5.6 Nonhomogeneous Linear Systems



$$\mathbf{a}e^{-t} - \mathbf{a}te^{-t} - \mathbf{b}e^{-t} + \mathbf{c} = A\mathbf{a}te^{-t} + A\mathbf{b}e^{-t} + \mathbf{A}ct + A\mathbf{d} + \begin{bmatrix} 2 \\ 0 \end{bmatrix} e^{-t} + \begin{bmatrix} 0 \\ 3 \end{bmatrix} t$$

If we look at the t terms, we have

$$0 = A\mathbf{c} + \begin{bmatrix} 0 \\ 3 \end{bmatrix} \quad \Rightarrow \quad \mathbf{c} = A^{-1} \begin{bmatrix} 0 \\ -3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}.$$

5.6 Nonhomogeneous Linear Systems



$$\mathbf{a}e^{-t} - \mathbf{a}te^{-t} - \mathbf{b}e^{-t} + \mathbf{c} = A\mathbf{a}te^{-t} + A\mathbf{b}e^{-t} + A\mathbf{c}t + \mathbf{Ad} + \begin{bmatrix} 2 \\ 0 \end{bmatrix} e^{-t} + \begin{bmatrix} 0 \\ 3 \end{bmatrix} t$$

Finally, if we look at the 1 terms, we have

$$\mathbf{c} = A\mathbf{d}$$

5.6 Nonhomogeneous Linear Systems



$$\mathbf{a}e^{-t} - \mathbf{a}te^{-t} - \mathbf{b}e^{-t} + \mathbf{c} = A\mathbf{a}te^{-t} + A\mathbf{b}e^{-t} + A\mathbf{c}t + \mathbf{A}\mathbf{d} + \begin{bmatrix} 2 \\ 0 \end{bmatrix} e^{-t} + \begin{bmatrix} 0 \\ 3 \end{bmatrix} t$$

Finally, if we look at the 1 terms, we have

$$\mathbf{c} = A\mathbf{d} \quad \implies \quad \mathbf{d} = A^{-1}\mathbf{c} = \begin{bmatrix} -4 \\ 3 \\ 5 \\ 3 \end{bmatrix}.$$

5.6 Nonhomogeneous Linear Systems



So

$$\mathbf{v} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} t e^{-t} - \begin{bmatrix} 0 \\ 1 \end{bmatrix} e^{-t} + \begin{bmatrix} 1 \\ 2 \end{bmatrix} t - \frac{1}{3} \begin{bmatrix} 4 \\ 5 \end{bmatrix}.$$

5.6 Nonhomogeneous Linear Systems



3. Therefore the general solution to the ODE is

$$\mathbf{x}(t) = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{-t} + c_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{-3t} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} t e^{-t} - \begin{bmatrix} 0 \\ 1 \end{bmatrix} e^{-t} + \begin{bmatrix} 1 \\ 2 \end{bmatrix} t - \frac{1}{3} \begin{bmatrix} 4 \\ 5 \end{bmatrix}.$$

5.6 Nonhomogeneous Linear Systems



Example

Solve

$$\mathbf{x}' = \begin{bmatrix} 2 & 3 \\ 4 & 1 \end{bmatrix} \mathbf{x} + \begin{bmatrix} e^t \\ -10t - 3 \end{bmatrix}.$$

5.6 Nonhomogeneous Linear Systems



$$\mathbf{x}' = \begin{bmatrix} 2 & 3 \\ 4 & 1 \end{bmatrix} \mathbf{x} + \begin{bmatrix} e^t \\ -10t - 3 \end{bmatrix}$$

We will consider simpler three ODEs:

5.6 Nonhomogeneous Linear Systems



$$\mathbf{x}' = \begin{bmatrix} 2 & 3 \\ 4 & 1 \end{bmatrix} \mathbf{x} + \begin{bmatrix} e^t \\ -10t - 3 \end{bmatrix}$$

We will consider simpler three ODEs:

$$\mathbf{1} \quad \mathbf{x}' = \begin{bmatrix} 2 & 3 \\ 4 & 1 \end{bmatrix} \mathbf{x}$$

5.6 Nonhomogeneous Linear Systems



$$\mathbf{x}' = \begin{bmatrix} 2 & 3 \\ 4 & 1 \end{bmatrix} \mathbf{x} + \begin{bmatrix} e^t \\ -10t - 3 \end{bmatrix}$$

We will consider simpler three ODEs:

1 $\mathbf{x}' = \begin{bmatrix} 2 & 3 \\ 4 & 1 \end{bmatrix} \mathbf{x}$

2 $\mathbf{x}' = \begin{bmatrix} 2 & 3 \\ 4 & 1 \end{bmatrix} \mathbf{x} + \begin{bmatrix} e^t \\ 0 \end{bmatrix}$

5.6 Nonhomogeneous Linear Systems



$$\mathbf{x}' = \begin{bmatrix} 2 & 3 \\ 4 & 1 \end{bmatrix} \mathbf{x} + \begin{bmatrix} e^t \\ -10t - 3 \end{bmatrix}$$

We will consider simpler three ODEs:

$$\text{1 } \mathbf{x}' = \begin{bmatrix} 2 & 3 \\ 4 & 1 \end{bmatrix} \mathbf{x}$$

$$\text{2 } \mathbf{x}' = \begin{bmatrix} 2 & 3 \\ 4 & 1 \end{bmatrix} \mathbf{x} + \begin{bmatrix} e^t \\ 0 \end{bmatrix}$$

$$\text{3 } \mathbf{x}' = \begin{bmatrix} 2 & 3 \\ 4 & 1 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 0 \\ -10t - 3 \end{bmatrix}$$

and then we will add the solutions together.

5.6 Nonhomogeneous Linear Systems



The matrix $\begin{bmatrix} 2 & 3 \\ 4 & 1 \end{bmatrix}$ has eigenvalues $r_1 = 5$ and $r_2 = -2$ and eigenvectors $\boldsymbol{\xi}^{(1)} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\boldsymbol{\xi}^{(2)} = \begin{bmatrix} -3 \\ 4 \end{bmatrix}$. Hence the general solution of $\mathbf{x}' = \begin{bmatrix} 2 & 3 \\ 4 & 1 \end{bmatrix} \mathbf{x}$ is

$$\mathbf{x}(t) = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{5t} + c_2 \begin{bmatrix} -3 \\ 4 \end{bmatrix} e^{-2t}.$$

5.6 Nonhomogeneous Linear Systems



Next we need to find a particular solution to

$$\mathbf{x}' = \begin{bmatrix} 2 & 3 \\ 4 & 1 \end{bmatrix} \mathbf{x} + \begin{bmatrix} e^t \\ 0 \end{bmatrix}.$$

5.6 Nonhomogeneous Linear Systems



Next we need to find a particular solution to

$$\mathbf{x}' = \begin{bmatrix} 2 & 3 \\ 4 & 1 \end{bmatrix} \mathbf{x} + \begin{bmatrix} e^t \\ 0 \end{bmatrix}. \text{ Since } \mathbf{1} \text{ is not an eigenvector of } \begin{bmatrix} 2 & 3 \\ 4 & 1 \end{bmatrix},$$

we try the ansatz $\mathbf{x} = \mathbf{a}e^t$ for some $\mathbf{a} = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} \in \mathbb{R}^2$.

5.6 Nonhomogeneous Linear Systems



Next we need to find a particular solution to

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we try the ansatz $\mathbf{x} = \mathbf{a}e^t$ for some $\mathbf{a} = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} \in \mathbb{R}^2$. Then we calculate that

$$\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} e^t = \mathbf{x}' = \begin{bmatrix} 2 & 3 \\ 4 & 1 \end{bmatrix} \mathbf{x} + \begin{bmatrix} e^t \\ 0 \end{bmatrix} = \begin{bmatrix} 2a_1 + 3a_2 + 1 \\ 4a_1 + a_2 \end{bmatrix} e^t$$

which gives

$$\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} 2a_1 + 3a_2 + 1 \\ 4a_1 + a_2 \end{bmatrix}.$$

5.6 Nonhomogeneous Linear Systems



Next we need to find a particular solution to

$\mathbf{x}' = \begin{bmatrix} 2 & 3 \\ 4 & 1 \end{bmatrix} \mathbf{x} + \begin{bmatrix} e^t \\ 0 \end{bmatrix}$. Since $\mathbf{1}$ is not an eigenvector of $\begin{bmatrix} 2 & 3 \\ 4 & 1 \end{bmatrix}$,

we try the ansatz $\mathbf{x} = \mathbf{a}e^t$ for some $\mathbf{a} = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} \in \mathbb{R}^2$. Then we calculate that

$$\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} e^t = \mathbf{x}' = \begin{bmatrix} 2 & 3 \\ 4 & 1 \end{bmatrix} \mathbf{x} + \begin{bmatrix} e^t \\ 0 \end{bmatrix} = \begin{bmatrix} 2a_1 + 3a_2 + 1 \\ 4a_1 + a_2 \end{bmatrix} e^t$$

which gives

$$\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} 2a_1 + 3a_2 + 1 \\ 4a_1 + a_2 \end{bmatrix}.$$

Hence $a_1 = 0$ and $a_2 = -\frac{1}{3}$. So $\mathbf{x} = \begin{bmatrix} 0 \\ -\frac{1}{3} \end{bmatrix} e^t$.

5.6 Nonhomogeneous Linear Systems



Then we need to find a particular solution to

$$\mathbf{x}' = \begin{bmatrix} 2 & 3 \\ 4 & 1 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 0 \\ -10t - 3 \end{bmatrix}.$$

5.6 Nonhomogeneous Linear Systems



Then we need to find a particular solution to

$$\mathbf{x}' = \begin{bmatrix} 2 & 3 \\ 4 & 1 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 0 \\ -10t - 3 \end{bmatrix}. \text{ We try the ansatz}$$

$$\mathbf{x} = \mathbf{a}t + \mathbf{b} = \begin{bmatrix} a_1t + b_1 \\ a_2t + b_2 \end{bmatrix} \text{ for some } \mathbf{a}, \mathbf{b} \in \mathbb{R}^2$$

5.6 Nonhomogeneous Linear Systems



Then we need to find a particular solution to

$$\mathbf{x}' = \begin{bmatrix} 2 & 3 \\ 4 & 1 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 0 \\ -10t - 3 \end{bmatrix}. \text{ We try the ansatz}$$

$$\mathbf{x} = \mathbf{a}t + \mathbf{b} = \begin{bmatrix} a_1t + b_1 \\ a_2t + b_2 \end{bmatrix} \text{ for some } \mathbf{a}, \mathbf{b} \in \mathbb{R}^2 \text{ and calculate that}$$

$$\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \mathbf{x}' = \begin{bmatrix} 2 & 3 \\ 4 & 1 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 0 \\ t \end{bmatrix} = \begin{bmatrix} 2a_1t + 2b_1 + 3a_2t + 3b_2 \\ 4a_1t + 4b_1 + a_2t + b_2 - 10t - 3 \end{bmatrix}$$

$$\text{which leads to } \begin{cases} 0 = 2a_1 + 3a_2 \\ a_1 = 2b_1 + 3b_2 \\ 0 = 4a_1 + a_2 - 10 \\ a_2 = 4b_1 + b_2 - 3 \end{cases}.$$

5.6 Nonhomogeneous Linear Systems



Then we need to find a particular solution to

$$\mathbf{x}' = \begin{bmatrix} 2 & 3 \\ 4 & 1 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 0 \\ -10t - 3 \end{bmatrix}. \text{ We try the ansatz}$$

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$$\text{which leads to } \begin{cases} 0 = 2a_1 + 3a_2 \\ a_1 = 2b_1 + 3b_2 \\ 0 = 4a_1 + a_2 - 10 \\ a_2 = 4b_1 + b_2 - 3 \end{cases} \text{ .The solution to this linear}$$

$$\text{system is } \mathbf{a} = \begin{bmatrix} 3 \\ -2 \end{bmatrix} \text{ and } \mathbf{b} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}. \text{ Hence } \mathbf{x} = \begin{bmatrix} 3t \\ 1 - 2t \end{bmatrix}.$$

5.6 Nonhomogeneous Linear Systems



Adding all of these together, we find that the general solution to the given ODE is

$$\mathbf{x} = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{5t} + c_2 \begin{bmatrix} -3 \\ 4 \end{bmatrix} e^{-2t} + \begin{bmatrix} 0 \\ -\frac{1}{3} \end{bmatrix} e^t + \begin{bmatrix} 3t \\ 1 - 2t \end{bmatrix}.$$



Method 3 – Variation of Parameters:



Method 3 – Variation of Parameters:

Consider

$$\mathbf{x}' = P(t)\mathbf{x} + \mathbf{g}(t) \quad (1)$$

where

- P and \mathbf{g} are continuous for $\alpha < t < \beta$;
- there exists a fundamental matrix $\Psi(t)$ for the homogeneous system $\mathbf{x}' = P(t)\mathbf{x}$.

5.6 Nonhomogeneous Linear Systems



We know that the general solution to $\mathbf{x}' = P(t)\mathbf{x}$ is $\mathbf{x} = \Psi(t)\mathbf{c}$.

5.6 Nonhomogeneous Linear Systems



We know that the general solution to $\mathbf{x}' = P(t)\mathbf{x}$ is $\mathbf{x} = \Psi(t)\mathbf{c}$.

We guess that

$$\mathbf{x} = \Psi(t)\mathbf{u}(t)$$

is a solution to (1).

5.6 Nonhomogeneous Linear Systems



We know that the general solution to $\mathbf{x}' = P(t)\mathbf{x}$ is $\mathbf{x} = \Psi(t)\mathbf{c}$.

We guess that

$$\mathbf{x} = \Psi(t)\mathbf{u}(t)$$

is a solution to (1). Can we find $\mathbf{u}(t)$?

5.6 Nonhomogeneous Linear Systems



If $\mathbf{x} = \Psi \mathbf{u}$, we can calculate that

$$\mathbf{x}' = P\mathbf{x} + \mathbf{g} \quad (3)$$

5.6 Nonhomogeneous Linear Systems



If $\mathbf{x} = \Psi \mathbf{u}$, we can calculate that

$$\Psi' \mathbf{u} + \Psi \mathbf{u}' = \mathbf{x}' = P\mathbf{x} + \mathbf{g} \quad (3)$$

5.6 Nonhomogeneous Linear Systems



If $\mathbf{x} = \Psi \mathbf{u}$, we can calculate that

$$\Psi' \mathbf{u} + \Psi \mathbf{u}' = \mathbf{x}' = P\mathbf{x} + \mathbf{g} = P\Psi \mathbf{u} + \mathbf{g}. \quad (3)$$

5.6 Nonhomogeneous Linear Systems



If $\mathbf{x} = \Psi \mathbf{u}$, we can calculate that

$$\Psi' \mathbf{u} + \Psi \mathbf{u}' = \mathbf{x}' = P\mathbf{x} + \mathbf{g} = P\Psi \mathbf{u} + \mathbf{g}. \quad (3)$$

But remember that

Ψ is a fundamental matrix for $\mathbf{x}' = P(t)\mathbf{x} \implies \Psi$ solves $\Psi' = P\Psi$.

5.6 Nonhomogeneous Linear Systems



If $\mathbf{x} = \Psi \mathbf{u}$, we can calculate that

$$\cancel{\Psi'} \mathbf{u} + \Psi \mathbf{u}' = \mathbf{x}' = P\mathbf{x} + \mathbf{g} = \cancel{P\Psi} \mathbf{u} + \mathbf{g}. \quad (3)$$

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5.6 Nonhomogeneous Linear Systems



If $\mathbf{x} = \Psi \mathbf{u}$, we can calculate that

$$\cancel{\Psi'} \mathbf{u} + \Psi \mathbf{u}' = \mathbf{x}' = P\mathbf{x} + \mathbf{g} = \cancel{P\Psi} \mathbf{u} + \mathbf{g}. \quad (3)$$

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Hence (3) becomes

$$\Psi \mathbf{u}' = \mathbf{g}.$$

5.6 Nonhomogeneous Linear Systems



If $\mathbf{x} = \Psi \mathbf{u}$, we can calculate that

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But remember that

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Hence (3) becomes

$$\Psi \mathbf{u}' = \mathbf{g}.$$

Therefore

$$\mathbf{u}' = \Psi^{-1} \mathbf{g}$$

and

$$\mathbf{u} = \int \Psi^{-1} \mathbf{g}.$$

5.6 Nonhomogeneous Linear Systems



If $\mathbf{x} = \Psi \mathbf{u}$, we can calculate that

$$\cancel{\Psi'} \mathbf{u} + \Psi \mathbf{u}' = \mathbf{x}' = P\mathbf{x} + \mathbf{g} = \cancel{P\Psi} \mathbf{u} + \mathbf{g}. \quad (3)$$

But remember that

Ψ is a fundamental matrix for $\mathbf{x}' = P(t)\mathbf{x} \implies \Psi$ solves $\Psi' = P\Psi$.

Hence (3) becomes

$$\Psi \mathbf{u}' = \mathbf{g}.$$

Therefore

$$\mathbf{u}' = \Psi^{-1} \mathbf{g}$$

and

$$\mathbf{u} = \int \Psi^{-1} \mathbf{g}.$$

Hence

$$\mathbf{x} = \Psi(t)\mathbf{u}(t) = \Psi(t) \int \Psi^{-1}(s)g(s) ds.$$

5.6 Nonhomogeneous Linear Systems



Remark

To solve $\mathbf{x}' = P(t)\mathbf{x} + \mathbf{g}(t)$, the method is

5.6 Nonhomogeneous Linear Systems



Remark

To solve $\mathbf{x}' = P(t)\mathbf{x} + \mathbf{g}(t)$, the method is

- 1 Find a fundamental matrix Ψ for $\mathbf{x}' = P(t)\mathbf{x}$;

5.6 Nonhomogeneous Linear Systems



Remark

To solve $\mathbf{x}' = P(t)\mathbf{x} + \mathbf{g}(t)$, the method is

- 1 Find a fundamental matrix Ψ for $\mathbf{x}' = P(t)\mathbf{x}$;
- 2 Calculate $\mathbf{x} = \Psi(t) \int \Psi^{-1}(s)g(s) ds$.

5.6 Nonhomogeneous Linear Systems



Example

Solve

$$\mathbf{x}' = \begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 2e^{-t} \\ 3t \end{bmatrix} = A\mathbf{x} + \mathbf{g}(t).$$

5.6 Nonhomogeneous Linear Systems



Example

Solve

$$\mathbf{x}' = \begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 2e^{-t} \\ 3t \end{bmatrix} = A\mathbf{x} + \mathbf{g}(t).$$

The solution of $\mathbf{x}' = \begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix} \mathbf{x}$ is

$$\mathbf{x}(t) = c_1 \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{-3t} + c_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{-t}.$$

5.6 Nonhomogeneous Linear Systems



Example

Solve

$$\mathbf{x}' = \begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 2e^{-t} \\ 3t \end{bmatrix} = A\mathbf{x} + \mathbf{g}(t).$$

The solution of $\mathbf{x}' = \begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix} \mathbf{x}$ is

$$\mathbf{x}(t) = c_1 \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{-3t} + c_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{-t}.$$

So

$$\Psi(t) = \begin{bmatrix} e^{-3t} & e^{-t} \\ -e^{-3t} & e^{-t} \end{bmatrix}$$

is a fundamental matrix.

5.6 Nonhomogeneous Linear Systems



$$\Psi(t) = \begin{bmatrix} e^{-3t} & e^{-t} \\ -e^{-3t} & e^{-t} \end{bmatrix}$$

Then we calculate that

$$\Psi^{-1}(t) = \frac{1}{2e^{-4t}} \begin{bmatrix} e^{-t} & -e^{-t} \\ e^{-3t} & e^{-3t} \end{bmatrix} = \frac{1}{2}e^{4t} \begin{bmatrix} e^{-t} & -e^{-t} \\ e^{-3t} & e^{-3t} \end{bmatrix} = \begin{bmatrix} \frac{1}{2}e^{3t} & -\frac{1}{2}e^{3t} \\ \frac{1}{2}e^t & \frac{1}{2}e^t \end{bmatrix}$$

5.6 Nonhomogeneous Linear Systems



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and

$$\begin{aligned} \int \Psi^{-1}(t)\mathbf{g}(t) dt &= \int \begin{bmatrix} \frac{1}{2}e^{3t} & -\frac{1}{2}e^{3t} \\ \frac{1}{2}e^t & \frac{1}{2}e^t \end{bmatrix} \begin{bmatrix} 2e^{-t} \\ 3t \end{bmatrix} dt \\ &= \int \begin{bmatrix} e^{2t} - \frac{3}{2}te^{3t} \\ 1 + \frac{3}{2}te^t \end{bmatrix} dt = \begin{bmatrix} \frac{1}{2}e^{2t} - \frac{1}{2}te^{3t} + \frac{1}{6}e^{3t} + c_1 \\ t + \frac{3}{2}te^t - \frac{3}{2}e^t + c_2 \end{bmatrix}. \end{aligned}$$

5.6 Nonhomogeneous Linear Systems



Therefore the solution to $\mathbf{x}' = A\mathbf{x} + \mathbf{g}$ is

$$\mathbf{x} = \Psi(t) \int \Psi^{-1}(s) \mathbf{g}(s) ds$$

=

=

5.6 Nonhomogeneous Linear Systems



Therefore the solution to $\mathbf{x}' = A\mathbf{x} + \mathbf{g}$ is

$$\begin{aligned}\mathbf{x} &= \Psi(t) \int \Psi^{-1}(s) \mathbf{g}(s) ds \\ &= \begin{bmatrix} e^{-3t} & e^{-t} \\ -e^{-3t} & e^{-t} \end{bmatrix} \begin{bmatrix} \frac{1}{2}e^{2t} - \frac{1}{2}te^{3t} + \frac{1}{6}e^{3t} + c_1 \\ t + \frac{3}{2}te^t - \frac{3}{2}e^t + c_2 \end{bmatrix} \\ &= \end{aligned}$$

5.6 Nonhomogeneous Linear Systems



Therefore the solution to $\mathbf{x}' = A\mathbf{x} + \mathbf{g}$ is

$$\begin{aligned}\mathbf{x} &= \Psi(t) \int \Psi^{-1}(s) \mathbf{g}(s) ds \\ &= \begin{bmatrix} e^{-3t} & e^{-t} \\ -e^{-3t} & e^{-t} \end{bmatrix} \begin{bmatrix} \frac{1}{2}e^{2t} - \frac{1}{2}te^{3t} + \frac{1}{6}e^{3t} + c_1 \\ t + \frac{3}{2}te^t - \frac{3}{2}e^t + c_2 \end{bmatrix} \\ &= c_1 \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{-3t} + c_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{-t} + \frac{1}{2} \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{-t} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} te^{-t} + \begin{bmatrix} 1 \\ 2 \end{bmatrix} t - \frac{1}{3} \begin{bmatrix} 4 \\ 5 \end{bmatrix}.\end{aligned}$$

5.6 Nonhomogeneous Linear Systems



Example

Solve

$$\mathbf{x}' = \begin{bmatrix} -4 & 2 \\ 2 & -1 \end{bmatrix} \mathbf{x} + \begin{bmatrix} t^{-1} \\ 2t^{-1} + 4 \end{bmatrix}$$

for $t > 0$.

5.6 Nonhomogeneous Linear Systems



Example

Solve

$$\mathbf{x}' = \begin{bmatrix} -4 & 2 \\ 2 & -1 \end{bmatrix} \mathbf{x} + \begin{bmatrix} t^{-1} \\ 2t^{-1} + 4 \end{bmatrix}$$

for $t > 0$.

The eigenvalues of $A = \begin{bmatrix} -4 & 2 \\ 2 & -1 \end{bmatrix}$ are $r_1 = 0$ and $r_2 = -5$; and the eigenvectors are $\xi^{(1)} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $\xi^{(2)} = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$. Thus

$$\Psi(t) = \begin{bmatrix} 1 & -2e^{-5t} \\ 2 & e^{-5t} \end{bmatrix}$$

is a fundamental matrix for $\mathbf{x}' = A\mathbf{x}$.

5.6 Nonhomogeneous Linear Systems



$$\Psi(t) = \begin{bmatrix} 1 & -2e^{-5t} \\ 2 & e^{-5t} \end{bmatrix}$$

Using the formula $\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$ we calculate that

$$\Psi^{-1}(t) = \frac{1}{e^{-5t} + 4e^{-5t}} \begin{bmatrix} e^{-5t} & 2e^{-5t} \\ -2 & 1 \end{bmatrix} = \frac{1}{5} \begin{bmatrix} 1 & 2 \\ -2e^{5t} & e^{5t} \end{bmatrix}.$$

5.6 Nonhomogeneous Linear Systems



Then

$$\begin{aligned}\Psi^{-1}(t)\mathbf{g}(t) &= \frac{1}{5} \begin{bmatrix} 1 & 2 \\ -2e^{5t} & e^{5t} \end{bmatrix} \begin{bmatrix} t^{-1} \\ 2t^{-1} + 4 \end{bmatrix} \\ &= \frac{1}{5} \begin{bmatrix} t^{-1} + 4t^{-1} + 8 \\ -2t^{-1}e^{5t} + 2t^{-1}e^{5t} + 4e^{5t} \end{bmatrix} = \begin{bmatrix} t^{-1} + \frac{8}{5} \\ \frac{4}{5}e^{5t} \end{bmatrix}\end{aligned}$$

5.6 Nonhomogeneous Linear Systems



Then

$$\begin{aligned}\Psi^{-1}(t)\mathbf{g}(t) &= \frac{1}{5} \begin{bmatrix} 1 & 2 \\ -2e^{5t} & e^{5t} \end{bmatrix} \begin{bmatrix} t^{-1} \\ 2t^{-1} + 4 \end{bmatrix} \\ &= \frac{1}{5} \begin{bmatrix} t^{-1} + 4t^{-1} + 8 \\ -2t^{-1}e^{5t} + 2t^{-1}e^{5t} + 4e^{5t} \end{bmatrix} = \begin{bmatrix} t^{-1} + \frac{8}{5} \\ \frac{4}{5}e^{5t} \end{bmatrix}\end{aligned}$$

and

$$\int \Psi^{-1}(t)\mathbf{g}(t) dt = \int \begin{bmatrix} t^{-1} + \frac{8}{5} \\ \frac{4}{5}e^{5t} \end{bmatrix} dt = \begin{bmatrix} \ln t + \frac{8}{5}t + c_1 \\ \frac{4}{25}e^{5t} + c_2 \end{bmatrix}.$$

5.6 Nonhomogeneous Linear Systems



It follows that

$$\mathbf{x}(t) = \Psi(t) \int \Psi^{-1}(s) \mathbf{g}(s) ds$$

5.6 Nonhomogeneous Linear Systems



It follows that

$$\begin{aligned}\mathbf{x}(t) &= \Psi(t) \int \Psi^{-1}(s) \mathbf{g}(s) ds = \begin{bmatrix} 1 & -2e^{-5t} \\ 2 & e^{-5t} \end{bmatrix} \begin{bmatrix} \ln t + \frac{8}{5}t + c_1 \\ \frac{4}{25}e^{5t} + c_2 \end{bmatrix} \\ &= \begin{bmatrix} \ln t + \frac{8}{5}t - \frac{8}{25} + c_1 - 2c_2e^{-5t} \\ 2 \ln t + \frac{16}{5}t + \frac{4}{25} + 2c_1 + c_2e^{-5t} \end{bmatrix} \\ &= c_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + c_2 \begin{bmatrix} -2 \\ 1 \end{bmatrix} e^{-5t} + \begin{bmatrix} 1 \\ 2 \end{bmatrix} \ln t + \frac{8}{5} \begin{bmatrix} 1 \\ 2 \end{bmatrix} t + \frac{4}{25} \begin{bmatrix} -2 \\ 1 \end{bmatrix}.\end{aligned}$$



Method 4 – The Laplace Transform:



Method 4 – The Laplace Transform:

First some notation: If $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$, then $\mathbf{X} = \mathcal{L}[\mathbf{x}] = \begin{bmatrix} \mathcal{L}[x_1] \\ \mathcal{L}[x_2] \\ \vdots \\ \mathcal{L}[x_n] \end{bmatrix}$.

5.6 Nonhomogeneous Linear Systems



Recall from Chapter 6 that $\mathcal{L}[y']$ satisfies

$$\mathcal{L}[y'](s) = sY(s) - y(0).$$

5.6 Nonhomogeneous Linear Systems



Recall from Chapter 6 that $\mathcal{L}[y']$ satisfies

$$\mathcal{L}[y'](s) = sY(s) - y(0).$$

It follows that:

Theorem

$$\mathcal{L}[\mathbf{x}'](s) = s\mathbf{X}(s) - \mathbf{x}(0).$$

5.6 Nonhomogeneous Linear Systems



Example

Solve

$$\begin{cases} \mathbf{x}' = \begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 2e^{-t} \\ 3t \end{bmatrix} = A\mathbf{x} + \mathbf{g}(t), \\ \mathbf{x}(0) = \mathbf{0}. \end{cases}$$

5.6 Nonhomogeneous Linear Systems



Example

Solve

$$\begin{cases} \mathbf{x}' = \begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 2e^{-t} \\ 3t \end{bmatrix} = A\mathbf{x} + \mathbf{g}(t), \\ \mathbf{x}(0) = \mathbf{0}. \end{cases}$$

Taking Laplace Transforms of the ODE gives

$$s\mathbf{X}(s) - \mathbf{x}(0) = A\mathbf{X}(s) + \mathbf{G}(s)$$

$$\text{where } \mathbf{G}(s) = \mathcal{L}[\mathbf{g}](s) = \begin{bmatrix} \frac{2}{s+1} \\ \frac{3}{s^2} \end{bmatrix}.$$

5.6 Nonhomogeneous Linear Systems



$$s\mathbf{X} - \mathbf{x}(0) = A\mathbf{X} + \mathbf{G}$$

Since $\mathbf{x}(0) = \mathbf{0}$ we have that

$$(sI - A)\mathbf{X} = \mathbf{G}$$

5.6 Nonhomogeneous Linear Systems



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Since $\mathbf{x}(0) = \mathbf{0}$ we have that

$$(sI - A)\mathbf{X} = \mathbf{G}$$

and

$$\mathbf{X} = (sI - A)^{-1}\mathbf{G}$$

where

$$(sI - A)^{-1} = \begin{bmatrix} s+2 & -1 \\ -1 & s+2 \end{bmatrix}^{-1} = \frac{1}{(s+1)(s+3)} \begin{bmatrix} s+2 & 1 \\ 1 & s+2 \end{bmatrix}.$$

5.6 Nonhomogeneous Linear Systems



So

$$\begin{aligned}\mathbf{X} &= (sI - A)^{-1} \mathbf{G} \\ &= \frac{1}{(s+1)(s+3)} \begin{bmatrix} s+2 & 1 \\ 1 & s+2 \end{bmatrix} \begin{bmatrix} \frac{2}{s+1} \\ \frac{3}{s^2} \end{bmatrix} \\ &= \begin{bmatrix} \frac{2(s+2)}{(s+1)^2(s+3)} + \frac{3}{s^2(s+1)(s+3)} \\ \frac{2}{(s+1)^2(s+3)} + \frac{3(s+2)}{s^2(s+1)(s+3)} \end{bmatrix}.\end{aligned}$$

5.6 Nonhomogeneous Linear Systems



So

$$\begin{aligned}\mathbf{X} &= (sI - A)^{-1} \mathbf{G} \\ &= \frac{1}{(s+1)(s+3)} \begin{bmatrix} s+2 & 1 \\ 1 & s+2 \end{bmatrix} \begin{bmatrix} \frac{2}{s+1} \\ \frac{3}{s^2} \end{bmatrix} \\ &= \begin{bmatrix} \frac{2(s+2)}{(s+1)^2(s+3)} + \frac{3}{s^2(s+1)(s+3)} \\ \frac{2}{(s+1)^2(s+3)} + \frac{3(s+2)}{s^2(s+1)(s+3)} \end{bmatrix}.\end{aligned}$$

When we take the inverse Laplace Transform of this, we find our solution

$$\mathbf{x} = \mathcal{L}^{-1} [\mathbf{X}] = \begin{bmatrix} 2 \\ 1 \end{bmatrix} e^{-t} - \frac{2}{3} \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{-3t} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} t e^{-t} + \begin{bmatrix} 1 \\ 2 \end{bmatrix} t - \frac{1}{3} \begin{bmatrix} 4 \\ 5 \end{bmatrix}.$$

5.6 Nonhomogeneous Linear Systems



Example

Solve

$$\begin{cases} 2x' + y' - y - t = 0 \\ x' + y' - t^2 = 0 \\ x(0) = 1 \\ y(0) = 0 \end{cases}$$

5.6 Nonhomogeneous Linear Systems



Example

Solve

$$\begin{cases} 2x' + y' - y - t = 0 \\ x' + y' - t^2 = 0 \\ x(0) = 1 \\ y(0) = 0 \end{cases}$$

The ODEs above can be written as

$$\begin{cases} x' = y - t^2 + t \\ y' = -y + 2t^2 - t \end{cases}$$

(please check!).

5.6 Nonhomogeneous Linear Systems



If we write the problem in terms of matrices (using $\mathbf{x} = \begin{bmatrix} x \\ y \end{bmatrix}$) we have

$$\begin{cases} \mathbf{x}' = A\mathbf{x} + \mathbf{g} = \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix} \mathbf{x} + \begin{bmatrix} t - t^2 \\ 2t^2 - t \end{bmatrix} \\ \mathbf{x}(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}. \end{cases}$$

5.6 Nonhomogeneous Linear Systems



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Taking the Laplace transform of the ODE gives

$$(sI - A) \mathbf{X}(s) = \mathbf{x}(0) + \mathbf{G}(s)$$

5.6 Nonhomogeneous Linear Systems



If we write the problem in terms of matrices (using $\mathbf{x} = \begin{bmatrix} x \\ y \end{bmatrix}$) we have

$$\begin{cases} \mathbf{x}' = A\mathbf{x} + \mathbf{g} = \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix} \mathbf{x} + \begin{bmatrix} t - t^2 \\ 2t^2 - t \end{bmatrix} \\ \mathbf{x}(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}. \end{cases}$$

Taking the Laplace transform of the ODE gives

$$\begin{aligned} (sI - A) \mathbf{X}(s) &= \mathbf{x}(0) + \mathbf{G}(s) \\ \begin{bmatrix} s & -1 \\ 0 & s+1 \end{bmatrix} \mathbf{X}(s) &= \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \begin{bmatrix} \frac{1}{s^2} - \frac{2}{s^3} \\ \frac{4}{s^3} - \frac{1}{s^2} \end{bmatrix}. \end{aligned}$$

5.6 Nonhomogeneous Linear Systems



$$\begin{bmatrix} s & -1 \\ 0 & s+1 \end{bmatrix} \mathbf{X}(s) = \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \begin{bmatrix} \frac{1}{s^2} - \frac{2}{s^3} \\ \frac{4}{s^3} - \frac{1}{s^2} \end{bmatrix}$$

Thus

$$\begin{aligned} \mathbf{X}(s) &= \frac{1}{s(s+1)} \begin{bmatrix} s+1 & 1 \\ 0 & s \end{bmatrix} \frac{1}{s^3} \begin{bmatrix} s^3 + s - 2 \\ 4 - s \end{bmatrix} \\ &= \frac{1}{s^4(s+1)} \begin{bmatrix} s^4 + s^3 + s^2 - 2s + 2 \\ 4s - s^2 \end{bmatrix}. \end{aligned}$$

5.6 Nonhomogeneous Linear Systems



Note that

$$\frac{s^4 + s^3 + s^2 - 2s + 2}{s^4(s+1)} = \frac{5}{s+1} - 4\frac{1}{s} + 5\frac{1}{s^2} - 4\frac{1}{s^3} + 2\frac{1}{s^4}$$

and

$$\frac{4s - s^2}{s^4(s+1)} = -5\frac{1}{s+1} + 5\frac{1}{s} - 5\frac{1}{s^2} + 4\frac{1}{s^3}$$

(please check!).

5.6 Nonhomogeneous Linear Systems



It follows that

$$\mathcal{L}^{-1} \left(\frac{s^4 + s^3 + s^2 - 2s + 2}{s^4(s+1)} \right) = 5e^{-t} - 4 + 5t - 2t^2 + \frac{1}{3}t^3$$

and

$$\mathcal{L}^{-1} \left(\frac{4s - s^2}{s^4(s+1)} \right) = -5e^{-t} + 5 - 5t + 2t^2.$$

5.6 Nonhomogeneous Linear Systems



It follows that

$$\mathcal{L}^{-1} \left(\frac{s^4 + s^3 + s^2 - 2s + 2}{s^4 (s + 1)} \right) = 5e^{-t} - 4 + 5t - 2t^2 + \frac{1}{3}t^3$$

and

$$\mathcal{L}^{-1} \left(\frac{4s - s^2}{s^4 (s + 1)} \right) = -5e^{-t} + 5 - 5t + 2t^2.$$

Therefore the solution to the initial value problem is

$$\mathbf{x}(t) = \begin{bmatrix} 5e^{-t} - 4 + 5t - 2t^2 + \frac{1}{3}t^3 \\ -5e^{-t} + 5 - 5t + 2t^2 \end{bmatrix}.$$

The End

A decorative flourish consisting of several overlapping loops and swirls, rendered in a white, elegant script style.