Lecture 4

- Introduction to Determinants
- Evaluating Determinants by Row Reduction
- Properties of Determinants
- Cramer's Rule

Introduction to Determinants

Recall that a 2×2 matrix

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

is invertible if and only if the special number $\det(A) = ad - bc$ is nonzero. This special number is called the *determinant* of the matrix.

The determinant of a A is written as $\det(A)$ or as $\begin{vmatrix} a & b \\ c & d \end{vmatrix}$.

Now it is time to generalise this idea to all square matrices.

Remark

In today's lecture, all the matrices will be square matrices unless I specifically say otherwise.

(Remember that only square matrices have determinants.)

1×1 matrices are easy, we just define

$$\det \left[a_{11} \right] = a_{11}.$$

Introduction to Determinants

Definition

If A is a square matrix, then the *minor of entry* a_{ij} is denoted by M_{ij} and is defined to be the determinant of the submatrix that remains after the ith row and jth column are deleted from A.

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Definition

The number

$$C_{ij} = (-1)^{i+j} M_{ij}$$

is called *the cofactor of entry* a_{ij}.

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc \qquad C_{ij} = (-1)^{i+j} M_{ij}$$

Example

Let

$$A = \begin{bmatrix} 3 & 1 & -4 \\ 2 & 5 & 6 \\ 1 & 4 & 8 \end{bmatrix}.$$

Find

1. the minor of entry a_{11};
2. the cofactor of entry a_{11};
3. the minor of entry a_{32}; and
4. the cofactor of entry a_{32}.

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc \qquad C_{ij} = (-1)^{i+j} M_{ij}$$

$$A = \begin{bmatrix} 3 & 1 & -4 \\ 2 & 5 & 6 \\ 1 & 4 & 8 \end{bmatrix}$$

1. For the minor of entry a_{11}, we need to cross out the first row and the first column.

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc \qquad C_{ij} = (-1)^{i+j} M_{ij}$$

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$$M_{11} = \begin{vmatrix} 5 & 6 \\ 4 & 8 \end{vmatrix} = 5 \cdot 8 - 6 \cdot 4 = 16.$$

Introduction to Determinants

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2. The cofactor of entry a_{11} is then

$$C_{11} = (-1)^{1+1} M_{11} = (-1)^2 \cdot 16 = 16.$$

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc \qquad C_{ij} = (-1)^{i+j} M_{ij}$$

3. The minor of entry a_{32} is

$$M_{32} = \begin{vmatrix} 3 & 1 & -4 \\ 2 & 5 & 6 \\ 1 & 4 & 8 \end{vmatrix}$$

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc \qquad C_{ij} = (-1)^{i+j} M_{ij}$$

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$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc \qquad C_{ij} = (-1)^{i+j} M_{ij}$$

3. The minor of entry a_{32} is

$$M_{32} = \begin{vmatrix} 3 & 1 & -4 \\ 2 & 5 & 6 \\ 1 & 4 & 8 \end{vmatrix} = \begin{vmatrix} 3 & -4 \\ 2 & 6 \end{vmatrix} = 3 \cdot 6 - (-4) \cdot 2 = 26$$

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc \qquad C_{ij} = (-1)^{i+j} M_{ij}$$

3 The minor of entry a_{32} is

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4 and the cofactor of entry a_{32} is then

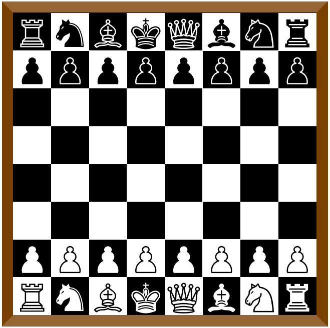
$$C_{32} = (-1)^{3+2} M_{32} = (-1)^3 \cdot 26 = -26.$$

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc \qquad C_{ij} = (-1)^{i+j} M_{ij}$$

Remark

We always have $C_{ij} = +M_{ij}$ or $C_{ij} = -M_{ij}$.

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc \qquad C_{ij} = (-1)^{i+j} M_{ij}$$



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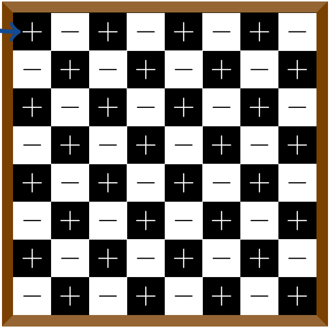
We always have $C_{ij} = +M_{ij}$ or $C_{ij} = -M_{ij}$.

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc \qquad C_{ij} = (-1)^{i+j} M_{ij}$$

top left is always $+$



Remark

We always have $C_{ij} = +M_{ij}$ or $C_{ij} = -M_{ij}$.

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc \qquad C_{ij} = (-1)^{i+j} M_{ij}$$

$C_{51} = +M_{51}$

+	−	+	−	+	−	+	−
−	+	−	+	−	+	−	+
+	−	+	−	+	−	+	−
−	+	−	+	−	+	−	+
+	−	+	−	+	−	+	−
−	+	−	+	−	+	−	+
+	−	+	−	+	−	+	−
−	+	−	+	−	+	−	+

Remark

We always have $C_{ij} = +M_{ij}$ or $C_{ij} = -M_{ij}$.

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc \qquad C_{ij} = (-1)^{i+j} M_{ij}$$

+	−	+	−	+	−	+	−
−	+	−	+	−	+	−	+
+	−	+	−	+	−	+	−
−	+	−	+	−	+	−	+
+	−	+	−	+	−	+	−
−	+	−	+	−	+	−	+
+	−	+	−	+	−	+	−
−	+	−	+	−	+	−	+

$C_{34} = -M_{34}$

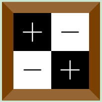
$C_{51} = +M_{51}$

Remark

We always have $C_{ij} = +M_{ij}$ or $C_{ij} = -M_{ij}$.

Example

The checkerboard pattern for a 2×2 matrix $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ is



This means that

$$\begin{aligned} C_{11} &= M_{11} = a_{22} & C_{12} &= -M_{12} = -a_{21} \\ C_{21} &= -M_{21} = -a_{12} & C_{22} &= M_{22} = a_{11}. \end{aligned}$$

$$C_{11} = M_{11} = a_{22} \qquad C_{12} = -M_{12} = -a_{21}$$
$$C_{21} = -M_{21} = -a_{12} \qquad C_{22} = M_{22} = a_{11}.$$

Remark

Note that

$$\begin{aligned} \det(A) = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} &= a_{11}a_{22} - a_{12}a_{21} \\ &= \\ &= \\ &= \\ &= \end{aligned}$$

$$C_{11} = M_{11} = a_{22} \qquad C_{12} = -M_{12} = -a_{21}$$
$$C_{21} = -M_{21} = -a_{12} \qquad C_{22} = M_{22} = a_{11}.$$

Remark

Note that

$$\begin{aligned} \det(A) = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} &= a_{11}a_{22} - a_{12}a_{21} \\ &= a_{11}C_{11} + a_{12}C_{12} \\ &= \\ &= \\ &= \end{aligned}$$

$$C_{11} = M_{11} = a_{22} \qquad C_{12} = -M_{12} = -a_{21}$$
$$C_{21} = -M_{21} = -a_{12} \qquad C_{22} = M_{22} = a_{11}.$$

Remark

Note that

$$\begin{aligned}\det(A) = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} &= a_{11}a_{22} - a_{12}a_{21} \\ &= a_{11}C_{11} + a_{12}C_{12} \\ &= a_{21}C_{21} + a_{22}C_{22} \\ &= a_{11}C_{11} + a_{21}C_{21} \\ &= a_{12}C_{12} + a_{22}C_{22}.\end{aligned}$$

These last four lines are called *cofactor expansions* of $\det(A)$.

The Definition of the Determinant

Theorem

If A is an $n \times n$ matrix, then regardless of which row or column of A is chosen, the number obtained by multiplying the entries in that row or column by the corresponding cofactors and adding the resulting products is always the same.

Definition

If A is an $n \times n$ matrix, then the number obtained by multiplying the entries in any row or column of A by the corresponding cofactors and adding the resulting products is called the *determinant* of A, and the sums themselves are called *cofactor expansions of A*. That is,

$$\det(A) = \underbrace{a_{1j}C_{1j} + a_{2j}C_{2j} + \ldots + a_{nj}C_{nj}}_{\text{cofactor expansion along the } j\text{th column}}$$

and

$$\det(A) = \underbrace{a_{i1}C_{i1} + a_{i2}C_{i2} + \ldots + a_{in}C_{in}}_{\text{cofactor expansion along the } i\text{th row}}.$$

Remark

$$\begin{bmatrix} 3 & 1 & 0 \\ -2 & -4 & 3 \\ 5 & 4 & -2 \end{bmatrix} \qquad \text{and} \qquad \begin{pmatrix} 3 & 1 & 0 \\ -2 & -4 & 3 \\ 5 & 4 & -2 \end{pmatrix}$$

are matrices, but

$$\begin{vmatrix} 3 & 1 & 0 \\ -2 & -4 & 3 \\ 5 & 4 & -2 \end{vmatrix}$$

is a number.

Example

Find the determinant of

$$A = \begin{bmatrix} 3 & 1 & 0 \\ -2 & -4 & 3 \\ 5 & 4 & -2 \end{bmatrix}$$

by cofactor expansion along the first row.

Introduction to Determinants

Example

Find the determinant of

$$A = \begin{bmatrix} 3 & 1 & 0 \\ -2 & -4 & 3 \\ 5 & 4 & -2 \end{bmatrix}$$

$$\begin{matrix} + & - & + \\ - & + & - \\ + & - & + \end{matrix}$$

by cofactor expansion along the first row.

$$\det(A) = \begin{vmatrix} 3 & 1 & 0 \\ -2 & -4 & 3 \\ 5 & 4 & -2 \end{vmatrix} = + \qquad\qquad - \qquad\qquad +$$

$$=$$

Introduction to Determinants

Example

Find the determinant of

$$A = \begin{bmatrix} 3 & 1 & 0 \\ -2 & -4 & 3 \\ 5 & 4 & -2 \end{bmatrix}$$

$$\begin{matrix} + & - & + \\ - & + & - \\ + & - & + \end{matrix}$$

by cofactor expansion along the first row.

$$\det(A) = \begin{vmatrix} 3 & 1 & 0 \\ -2 & -4 & 3 \\ 5 & 4 & -2 \end{vmatrix} = +3 \qquad\qquad - 1 \qquad\qquad + 0$$

$$=$$

Introduction to Determinants

Example

Find the determinant of

$$A = \begin{bmatrix} 3 & 1 & 0 \\ -2 & -4 & 3 \\ 5 & 4 & -2 \end{bmatrix}$$

$$\begin{matrix} + & - & + \\ - & + & - \\ + & - & + \end{matrix}$$

by cofactor expansion along the first row.

$$\det(A) = \begin{vmatrix} 3 & 1 & 0 \\ -2 & -4 & 3 \\ 5 & 4 & -2 \end{vmatrix} = +3 \begin{vmatrix} -4 & 3 \\ 4 & -2 \end{vmatrix} - 1 \qquad\qquad + 0$$

$$=$$

Introduction to Determinants

Example

Find the determinant of

$$A = \begin{bmatrix} 3 & 1 & 0 \\ -2 & -4 & 3 \\ 5 & 4 & -2 \end{bmatrix}$$

$$\begin{array}{|c|c|c|} \hline + & - & + \\ \hline - & + & - \\ \hline + & - & + \\ \hline \end{array}$$

by cofactor expansion along the first row.

$$\det(A) = \begin{vmatrix} 3 & 1 & 0 \\ -2 & -4 & 3 \\ 5 & 4 & -2 \end{vmatrix} = +3 \begin{vmatrix} -4 & 3 \\ 4 & -2 \end{vmatrix} - 1 \begin{vmatrix} -2 & 3 \\ 5 & -2 \end{vmatrix} + 0$$

$$=$$

Introduction to Determinants

Example

Find the determinant of

$$A = \begin{bmatrix} 3 & 1 & 0 \\ -2 & -4 & 3 \\ 5 & 4 & -2 \end{bmatrix}$$

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$$=$$

Introduction to Determinants

Example

Find the determinant of

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$$\det(A) = \begin{vmatrix} 3 & 1 & 0 \\ -2 & -4 & 3 \\ 5 & 4 & -2 \end{vmatrix} = +3 \begin{vmatrix} -4 & 3 \\ 4 & -2 \end{vmatrix} - 1 \begin{vmatrix} -2 & 3 \\ 5 & -2 \end{vmatrix} + 0 \begin{vmatrix} -2 & -4 \\ 5 & 4 \end{vmatrix}$$

$$= 3(-4) - 1(-11) + 0(12) = -1.$$

Introduction to Determinants

Example

Find the determinant of the same matrix by cofactor expansion along the first column.

$$\begin{matrix} + & - & + \\ - & + & - \\ + & - & + \end{matrix}$$

$$\det(A) = \begin{vmatrix} 3 & 1 & 0 \\ -2 & -4 & 3 \\ 5 & 4 & -2 \end{vmatrix} =$$

$$=$$

Introduction to Determinants

Example

Find the determinant of the same matrix by cofactor expansion along the first column.

$$\begin{matrix} + & - & + \\ - & + & - \\ + & - & + \end{matrix}$$

$$\det(A) = \begin{vmatrix} 3 & 1 & 0 \\ -2 & -4 & 3 \\ 5 & 4 & -2 \end{vmatrix} = +3 \qquad\qquad -(-2) \qquad\qquad +5$$

$$=$$

Introduction to Determinants

Example

Find the determinant of the same matrix by cofactor expansion along the first column.

$$\begin{bmatrix} + & - & + \\ - & + & - \\ + & - & + \end{bmatrix}$$

$$\det(A) = \begin{vmatrix} 3 & 1 & 0 \\ -2 & -4 & 3 \\ 5 & 4 & -2 \end{vmatrix} = +3 \begin{vmatrix} -4 & 3 \\ 4 & -2 \end{vmatrix} - (-2) \begin{vmatrix} 1 & 0 \\ 4 & -2 \end{vmatrix} + 5 \begin{vmatrix} 1 & 0 \\ -4 & 3 \end{vmatrix}$$

$$=$$

Introduction to Determinants

Example

Find the determinant of the same matrix by cofactor expansion along the first column.

$$\begin{matrix} + & - & + \\ - & + & - \\ + & - & + \end{matrix}$$

$$\det(A) = \begin{vmatrix} 3 & 1 & 0 \\ -2 & -4 & 3 \\ 5 & 4 & -2 \end{vmatrix} = +3 \begin{vmatrix} -4 & 3 \\ 4 & -2 \end{vmatrix} - (-2) \begin{vmatrix} 1 & 0 \\ 4 & -2 \end{vmatrix} + 5 \begin{vmatrix} 1 & 0 \\ -4 & 3 \end{vmatrix}$$

$$= 3(-4) + 2(-2) + 5(3) = -1$$

again.

Sometimes it is easier to use cofactor expansion along some rows or columns than others.

Example (A Smart Choice of Row or Column)

Calculate the determinant of

$$A = \begin{bmatrix} 1 & 0 & 0 & -1 \\ 3 & 1 & 2 & 2 \\ 1 & 0 & -2 & 1 \\ 2 & 0 & 0 & 1 \end{bmatrix}.$$

Sometimes it is easier to use cofactor expansion along some rows or columns than others.

Example (A Smart Choice of Row or Column)

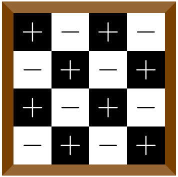
Calculate the determinant of

$$A = \begin{bmatrix} 1 & 0 & 0 & -1 \\ 3 & 1 & 2 & 2 \\ 1 & 0 & -2 & 1 \\ 2 & 0 & 0 & 1 \end{bmatrix}.$$

Look at the 2nd column. It has the most zeros. It will be easiest for us if we choose this column.

Introduction to Determinants

$$A = \begin{bmatrix} 1 & 0 & 0 & -1 \\ 3 & 1 & 2 & 2 \\ 1 & 0 & -2 & 1 \\ 2 & 0 & 0 & 1 \end{bmatrix}$$

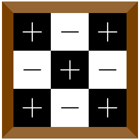


We calculate that

$$\det(A) = -0 + 1 \cdot \begin{vmatrix} 1 & 0 & -1 \\ 1 & -2 & 1 \\ 2 & 0 & 1 \end{vmatrix} - 0 + 0 = \begin{vmatrix} 1 & 0 & -1 \\ 1 & -2 & 1 \\ 2 & 0 & 1 \end{vmatrix}.$$

Now which row or column should I choose in this 3×3 determinant?

Introduction to Determinants



$$\det(A) = \begin{vmatrix} 1 & 0 & -1 \\ 1 & -2 & 1 \\ 2 & 0 & 1 \end{vmatrix} =$$

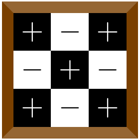
$$=$$

+	−	+
−	+	−
+	−	+

$$\det(A) = \begin{vmatrix} 1 & 0 & -1 \\ 1 & -2 & 1 \\ 2 & 0 & 1 \end{vmatrix} = -0 + (-2)\begin{vmatrix} 1 & -1 \\ 2 & 1 \end{vmatrix} - 0$$

$$=$$

Introduction to Determinants



$$\det(A) = \begin{vmatrix} 1 & 0 & -1 \\ 1 & -2 & 1 \\ 2 & 0 & 1 \end{vmatrix} = -0 + (-2)\begin{vmatrix} 1 & -1 \\ 2 & 1 \end{vmatrix} - 0$$

$$= (-2)(3) = -6.$$

Example (Determinant of a Lower Triangular Matrix)

Calculate

$$\begin{vmatrix} a_{11} & 0 & 0 & 0 \\ a_{21} & a_{22} & 0 & 0 \\ a_{31} & a_{32} & a_{33} & 0 \\ a_{41} & a_{42} & a_{43} & a_{44} \end{vmatrix}.$$

Example (Determinant of a Lower Triangular Matrix)

Calculate

$$\begin{vmatrix} a_{11} & 0 & 0 & 0 \\ a_{21} & a_{22} & 0 & 0 \\ a_{31} & a_{32} & a_{33} & 0 \\ a_{41} & a_{42} & a_{43} & a_{44} \end{vmatrix}.$$

I am going to be using cofactor expansion along the first rows:

$$\begin{vmatrix} a_{11} & 0 & 0 & 0 \\ a_{21} & a_{22} & 0 & 0 \\ a_{31} & a_{32} & a_{33} & 0 \\ a_{41} & a_{42} & a_{43} & a_{44} \end{vmatrix} = a_{11} \begin{vmatrix} a_{22} & 0 & 0 \\ a_{32} & a_{33} & 0 \\ a_{42} & a_{43} & a_{44} \end{vmatrix} =$$

$$=$$

Example (Determinant of a Lower Triangular Matrix)

Calculate

$$\begin{vmatrix} a_{11} & 0 & 0 & 0 \\ a_{21} & a_{22} & 0 & 0 \\ a_{31} & a_{32} & a_{33} & 0 \\ a_{41} & a_{42} & a_{43} & a_{44} \end{vmatrix}.$$

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$$\begin{vmatrix} a_{11} & 0 & 0 & 0 \\ a_{21} & a_{22} & 0 & 0 \\ a_{31} & a_{32} & a_{33} & 0 \\ a_{41} & a_{42} & a_{43} & a_{44} \end{vmatrix} = a_{11} \begin{vmatrix} a_{22} & 0 & 0 \\ a_{32} & a_{33} & 0 \\ a_{42} & a_{43} & a_{44} \end{vmatrix} = a_{11}a_{22} \begin{vmatrix} a_{33} & 0 \\ a_{43} & a_{44} \end{vmatrix}$$

$$=$$

Example (Determinant of a Lower Triangular Matrix)

Calculate

$$\begin{vmatrix} a_{11} & 0 & 0 & 0 \\ a_{21} & a_{22} & 0 & 0 \\ a_{31} & a_{32} & a_{33} & 0 \\ a_{41} & a_{42} & a_{43} & a_{44} \end{vmatrix}.$$

I am going to be using cofactor expansion along the first rows:

$$\begin{vmatrix} a_{11} & 0 & 0 & 0 \\ a_{21} & a_{22} & 0 & 0 \\ a_{31} & a_{32} & a_{33} & 0 \\ a_{41} & a_{42} & a_{43} & a_{44} \end{vmatrix} = a_{11} \begin{vmatrix} a_{22} & 0 & 0 \\ a_{32} & a_{33} & 0 \\ a_{42} & a_{43} & a_{44} \end{vmatrix} = a_{11}a_{22} \begin{vmatrix} a_{33} & 0 \\ a_{43} & a_{44} \end{vmatrix}$$

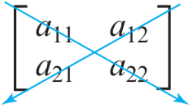
$$= a_{11}a_{22}a_{33}a_{44}.$$

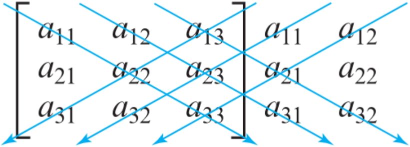
Theorem

If A is an $n \times n$ triangular matrix (upper triangular, lower triangular, or diagonal), then

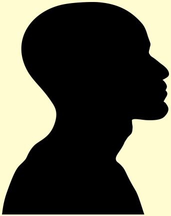
$$\det(A) = a_{11}a_{22}\dots a_{nn}.$$

Rule of Sarrus





Determinants of 2×2 and 3×3 matrices can be evaluated very efficiently using the pattern shown above.



Pierre Frédéric Sarrus

BORN
10 March 1798

DECEICED
20 November 1861

NATIONALITY
French

Rule of Sarrus

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \qquad \left[\begin{array}{ccc} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{array}\right] \begin{array}{cc} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{array}$$

Determinants of 2×2 and 3×3 matrices can be evaluated very efficiently using the pattern shown above.

We sum the products of the rightwards arrows, then subtract the products on the leftward arrows.

Remark

The rule of Sarrus only works for 2×2 and 3×3 matrices.

Don't try to use it for bigger matrices.

EXAMPLE 7 A Technique for Evaluating 2 × 2 and 3 × 3 Determinants

$$\begin{vmatrix} 3 & 1 \\ 4 & -2 \end{vmatrix} = \begin{vmatrix} 3 & 1 \\ 4 & -2 \end{vmatrix} = (3)(-2) - (1)(4) = -10$$

$$\begin{vmatrix} 1 & 2 & 3 \\ -4 & 5 & 6 \\ 7 & -8 & 9 \end{vmatrix} = \left|\begin{matrix} 1 & 2 & 3 \\ -4 & 5 & 6 \\ 7 & -8 & 9 \end{matrix}\right| \begin{matrix} 1 & 2 \\ -4 & 5 \\ 7 & -8 \end{matrix}$$

$$= [45 + 84 + 96] - [105 - 48 - 72] = 240$$

Evaluating Determinants by Row Reduction

Evaluating Determinants by Row Reduction

$$\begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 7 & 8 & 9 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 3 & 4 & 5 & 6 & 7 & 8 \\ 9 & 0 & 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 & 9 & 0 \end{bmatrix} \qquad \begin{bmatrix} 1 & 2 & 3 & 0 & 5 \\ 7 & 8 & 9 & 0 & 1 \\ 3 & 4 & 5 & 0 & 7 \\ 9 & 0 & 1 & 0 & 3 \\ 5 & 6 & 7 & 0 & 9 \end{bmatrix}$$

Theorem

If A has a row of zeros, or a column or zeros, then $\det(A) = 0$.

$$\begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 7 & 8 & 9 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 3 & 4 & 5 & 6 & 7 & 8 \\ 9 & 0 & 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 & 9 & 0 \end{bmatrix} \qquad \begin{bmatrix} 1 & 2 & 3 & 0 & 5 \\ 7 & 8 & 9 & 0 & 1 \\ 3 & 4 & 5 & 0 & 7 \\ 9 & 0 & 1 & 0 & 3 \\ 5 & 6 & 7 & 0 & 9 \end{bmatrix}$$

Theorem

If A has a row of zeros, or a column or zeros, then $\det(A) = 0$.

Proof.

Let's just do this for the left matrix above.

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$$\begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 7 & 8 & 9 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 3 & 4 & 5 & 6 & 7 & 8 \\ 9 & 0 & 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 & 9 & 0 \end{bmatrix} \qquad \begin{bmatrix} 1 & 2 & 3 & 0 & 5 \\ 7 & 8 & 9 & 0 & 1 \\ 3 & 4 & 5 & 0 & 7 \\ 9 & 0 & 1 & 0 & 3 \\ 5 & 6 & 7 & 0 & 9 \end{bmatrix}$$

Theorem

If A has a row of zeros, or a column or zeros, then $\det(A) = 0$.

Proof.

Let's just do this for the left matrix above. If we use cofactor expansion along the 3rd row, then we get

$$\det(A) = 0C_{31} + 0C_{32} + 0C_{33} + 0C_{34} + 0C_{35} + 0C_{36} = 0.$$

□

Evaluating Determinants by Row Reduction

Theorem

$$\det(A) = \det(A^T).$$

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Theorem

$$\det(A) = \det(A^T).$$

Proof.

Recall that transposing a matrix swaps it columns to rows and rows to columns.

So cofactor expansion along the ith row of A is exactly the same as cofactor expansion along the ith column of A^T. Hence, both A and A^T must have the same determinant. $\square$

Next I want to explain how the elementary row operations affect the determinant.

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1. What happens if we multiply a row by a number k?

$$\begin{vmatrix} ka_{11} & ka_{12} & ka_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} =$$

$$=$$

$$=$$

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$$\begin{vmatrix} ka_{11} & ka_{12} & ka_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = ka_{11}C_{11} + ka_{12}C_{12} + ka_{13}C_{13}$$

$$=$$

$$=$$

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$$= k\big(a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13}\big)$$

$$=$$

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$$= k\big(a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13}\big)$$

$$= k\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}.$$

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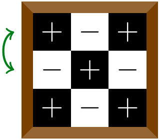
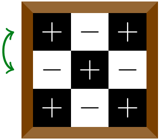
$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$



2. What happens if we swap two rows?

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$



2. What happens if we swap two rows?

$$\begin{vmatrix} a_{21} & a_{22} & a_{23} \\ a_{11} & a_{12} & a_{13} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = - \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

Evaluating Determinants by Row Reduction

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \longrightarrow \begin{bmatrix} a_{31} & a_{32} & a_{33} \\ a_{21} & a_{22} & a_{23} \\ a_{11} & a_{12} & a_{13} \end{bmatrix}$$

Evaluating Determinants by Row Reduction

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \longrightarrow \begin{bmatrix} a_{31} & a_{32} & a_{33} \\ a_{21} & a_{22} & a_{23} \\ a_{11} & a_{12} & a_{13} \end{bmatrix}$$

$$\begin{array}{ccc} + & - & + \\ - & + & - \\ + & - & + \end{array} \qquad\qquad \begin{array}{ccc} + & - & + \\ - & + & - \\ + & - & + \end{array}$$

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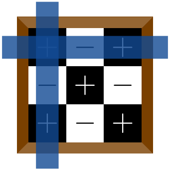
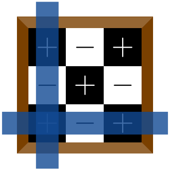
$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \longrightarrow \begin{bmatrix} a_{31} & a_{32} & a_{33} \\ a_{21} & a_{22} & a_{23} \\ a_{11} & a_{12} & a_{13} \end{bmatrix}$$

$$\begin{matrix} + & - & + \\ - & + & - \\ + & - & + \end{matrix} \qquad\qquad \begin{matrix} + & - & + \\ - & + & - \\ + & - & + \end{matrix}$$

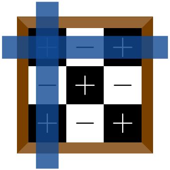
$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \longrightarrow \begin{bmatrix} a_{31} & a_{32} & a_{33} \\ a_{21} & a_{22} & a_{23} \\ a_{11} & a_{12} & a_{13} \end{bmatrix}$$

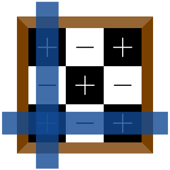




Evaluating Determinants by Row Reduction

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \longrightarrow \begin{bmatrix} a_{31} & a_{32} & a_{33} \\ a_{21} & a_{22} & a_{23} \\ a_{11} & a_{12} & a_{13} \end{bmatrix}$$





We still get

$$\begin{vmatrix} a_{31} & a_{32} & a_{33} \\ a_{21} & a_{22} & a_{23} \\ a_{11} & a_{12} & a_{13} \end{vmatrix} = - \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

Evaluating Determinants by Row Reduction

3. And what happens if add a multiple of one row to another row?

$$\begin{vmatrix} a_{11} + ka_{21} & a_{12} + ka_{22} & a_{13} + ka_{23} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

$=$

$=$

$=$

Evaluating Determinants by Row Reduction

3. And what happens if add a multiple of one row to another row?

$$\begin{vmatrix} a_{11} + ka_{21} & a_{12} + ka_{22} & a_{13} + ka_{23} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$
$$= (a_{11} + ka_{21})C_{11} + (a_{12} + ka_{22})C_{12} + (a_{13} + ka_{23})C_{13}$$
$$=$$

$$=$$

3. And what happens if add a multiple of one row to another row?

$$\begin{vmatrix} a_{11}+ka_{21} & a_{12}+ka_{22} & a_{13}+ka_{23} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

$$= (a_{11}+ka_{21})C_{11} + (a_{12}+ka_{22})C_{12} + (a_{13}+ka_{23})C_{13}$$

$$= a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13} + k\big(a_{21}C_{11} + a_{22}C_{12} + a_{23}C_{13}\big)$$

$$=$$

3. And what happens if add a multiple of one row to another row?

$$
\begin{aligned}
&\begin{vmatrix} a_{11}+ka_{21} & a_{12}+ka_{22} & a_{13}+ka_{23} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} \\
&= (a_{11}+ka_{21})C_{11} + (a_{12}+ka_{22})C_{12} + (a_{13}+ka_{23})C_{13} \\
&= a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13} + k\big(a_{21}C_{11} + a_{22}C_{12} + a_{23}C_{13}\big) \\
&= \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} + k\big(a_{21}C_{11} + a_{22}C_{12} + a_{23}C_{13}\big).
\end{aligned}
$$

But what is $a_{21}C_{11} + a_{22}C_{12} + a_{23}C_{13}$?

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

I leave it for you check that

$$\begin{aligned} a_{21}C_{11} + a_{22}C_{12} + a_{23}C_{13} &= a_{21}\big(a_{22}a_{33} - a_{23}a_{32}\big) \\ &\quad - a_{22}\big(a_{21}a_{33} - a_{23}a_{31}\big) \\ &\quad + a_{23}\big(a_{21}a_{32} - a_{22}a_{31}\big) \\ &= \end{aligned}$$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

I leave it for you check that

$$\begin{aligned} a_{21}C_{11} + a_{22}C_{12} + a_{23}C_{13} &= a_{21}\big(a_{22}a_{33} - a_{23}a_{32}\big) \\ &\quad - a_{22}\big(a_{21}a_{33} - a_{23}a_{31}\big) \\ &\quad + a_{23}\big(a_{21}a_{32} - a_{22}a_{31}\big) \\ &= 0. \end{aligned}$$

Therefore

$$\begin{vmatrix} a_{11} + ka_{21} & a_{12} + ka_{22} & a_{13} + ka_{23} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}.$$

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Theorem

1. *If one row of A is multiplied by a number k to produce a matrix B, then*
$$\det(B) = k\det(A).$$

(This is also true for columns.)

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2. *If two rows of A are swapped to produce B, then*
$$\det(B) = -\det(A).$$
3. *If a multiple of one row of A is added to another row to produce B, then*
$$\det(B) = \det(A).$$

(This is also true for columns.)

Evaluating Determinants by Row Reduction

Example (Determinants of Elementary Matrices)

$$I_4 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \xrightarrow{3R_2} B = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\det(I_4) = 1 \qquad \det(B) = 3$$

Evaluating Determinants by Row Reduction

Example (Determinants of Elementary Matrices)

$$I_4 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \xrightarrow{R_1 \leftrightarrow R_3} C = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\det(I_4) = 1 \qquad \det(C) = -1$$

Evaluating Determinants by Row Reduction

Example (Determinants of Elementary Matrices)

$$I_4 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \xrightarrow{7R_4+R_1} D = \begin{bmatrix} 1 & 0 & 0 & 7 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\det(I_4) = 1 \qquad \det(D) = 1$$

Theorem

If A is a square matrix with two proportional rows or two proportional columns, then $\det(A) = 0$.

Evaluating Determinants by Row Reduction

Theorem

If A is a square matrix with two proportional rows or two proportional columns, then $\det(A) = 0$.

Example

$$\begin{vmatrix} -1 & 4 \\ -2 & 8 \end{vmatrix} = 0, \quad \begin{vmatrix} 1 & -2 & 7 \\ -4 & 8 & 5 \\ 2 & -4 & 3 \end{vmatrix} = 0, \quad \begin{vmatrix} 3 & -1 & 5 & -5 \\ 6 & -2 & 5 & 2 \\ 5 & 8 & 1 & 4 \\ -9 & 3 & -15 & 15 \end{vmatrix} = 0.$$

Why?

Evaluating Determinants by Row Reduction

Why? Because the elementary row operation $3R_1 + R_4 \to R_4$ doesn't change the determinant and a matrix with a row of zeros always has zero determinant.

$$\begin{vmatrix} 3 & -1 & 5 & -5 \\ 6 & -2 & 5 & 2 \\ 5 & 8 & 1 & 4 \\ -9 & 3 & -15 & 15 \end{vmatrix} \xlongequal{3R_1+R_4} \begin{vmatrix} 3 & -1 & 5 & -5 \\ 6 & -2 & 5 & 2 \\ 5 & 8 & 1 & 4 \\ 0 & 0 & 0 & 0 \end{vmatrix} = 0.$$

Evaluating Determinants by Row Reduction

Example (Using Row Reduction to Calculate $\det(A)$)

Calculate the determinant of

$$A = \begin{bmatrix} 0 & 1 & 5 \\ 3 & -6 & 9 \\ 2 & 6 & 1 \end{bmatrix}.$$

We will reduce A to REF (which will be upper triangular), then calculate its determinant.

Evaluating Determinants by Row Reduction

$$\det(A) = \begin{vmatrix} 0 & 1 & 5 \\ 3 & -6 & 9 \\ 2 & 6 & 1 \end{vmatrix} \overset{R_1 \leftrightarrow R_2}{=\!=\!=\!=} - \begin{vmatrix} 3 & -6 & 9 \\ 0 & 1 & 5 \\ 2 & 6 & 1 \end{vmatrix} \qquad \text{(multiply by } -1\text{)}$$

$(\qquad)$

$(\qquad)$

$(\qquad)$

$$\det(A) = \begin{vmatrix} 0 & 1 & 5 \\ 3 & -6 & 9 \\ 2 & 6 & 1 \end{vmatrix} \xlongequal{R_1 \leftrightarrow R_2} - \begin{vmatrix} 3 & -6 & 9 \\ 0 & 1 & 5 \\ 2 & 6 & 1 \end{vmatrix} \qquad \big(\text{multiply by } -1\big)$$

$$\xlongequal{\frac{1}{3}R_1} -3 \begin{vmatrix} 1 & -2 & 3 \\ 0 & 1 & 5 \\ 2 & 6 & 1 \end{vmatrix} \qquad \big(\text{multiply by } 3\big)$$

$$\big(\qquad \big)$$

$$\big(\qquad \big)$$

Evaluating Determinants by Row Reduction

$$\det(A) = \begin{vmatrix} 0 & 1 & 5 \\ 3 & -6 & 9 \\ 2 & 6 & 1 \end{vmatrix} \xlongequal{R_1 \leftrightarrow R_2} - \begin{vmatrix} 3 & -6 & 9 \\ 0 & 1 & 5 \\ 2 & 6 & 1 \end{vmatrix} \qquad \text{(multiply by } -1\text{)}$$

$$\xlongequal{\frac{1}{3}R_1} -3 \begin{vmatrix} 1 & -2 & 3 \\ 0 & 1 & 5 \\ 2 & 6 & 1 \end{vmatrix} \qquad \text{(multiply by } 3\text{)}$$

$$\xlongequal{-2R_1 + R_3} -3 \begin{vmatrix} 1 & -2 & 3 \\ 0 & 1 & 5 \\ 0 & 10 & -5 \end{vmatrix} \qquad \text{(no change)}$$

$$(\qquad\qquad)$$

Evaluating Determinants by Row Reduction

$$\det(A) = \begin{vmatrix} 0 & 1 & 5 \\ 3 & -6 & 9 \\ 2 & 6 & 1 \end{vmatrix} \xlongequal{R_1 \leftrightarrow R_2} - \begin{vmatrix} 3 & -6 & 9 \\ 0 & 1 & 5 \\ 2 & 6 & 1 \end{vmatrix} \qquad (\text{multiply by } -1)$$

$$\xlongequal{\frac{1}{3}R_1} -3 \begin{vmatrix} 1 & -2 & 3 \\ 0 & 1 & 5 \\ 2 & 6 & 1 \end{vmatrix} \qquad (\text{multiply by } 3)$$

$$\xlongequal{-2R_1 + R_3} -3 \begin{vmatrix} 1 & -2 & 3 \\ 0 & 1 & 5 \\ 0 & 10 & -5 \end{vmatrix} \qquad (\text{no change})$$

$$\xlongequal{-10R_2 + R_3} -3 \begin{vmatrix} 1 & -2 & 3 \\ 0 & 1 & 5 \\ 0 & 0 & -55 \end{vmatrix} \qquad (\text{no change})$$

Evaluating Determinants by Row Reduction

$$\det(A) = \begin{vmatrix} 0 & 1 & 5 \\ 3 & -6 & 9 \\ 2 & 6 & 1 \end{vmatrix} \xlongequal{R_1 \leftrightarrow R_2} - \begin{vmatrix} 3 & -6 & 9 \\ 0 & 1 & 5 \\ 2 & 6 & 1 \end{vmatrix} \qquad \text{(multiply by } -1\text{)}$$

$$\xlongequal{\frac{1}{3}R_1} -3 \begin{vmatrix} 1 & -2 & 3 \\ 0 & 1 & 5 \\ 2 & 6 & 1 \end{vmatrix} \qquad \text{(multiply by } 3\text{)}$$

$$\xlongequal{-2R_1 + R_3} -3 \begin{vmatrix} 1 & -2 & 3 \\ 0 & 1 & 5 \\ 0 & 10 & -5 \end{vmatrix} \qquad \text{(no change)}$$

$$\xlongequal{-10R_2 + R_3} -3 \begin{vmatrix} 1 & -2 & 3 \\ 0 & 1 & 5 \\ 0 & 0 & -55 \end{vmatrix} \qquad \text{(no change)}$$

$$-3(1 \cdot 1 \cdot -55) = 165.$$

Example

Calculate the determinant of

$$A = \begin{bmatrix} 2 & -8 & 6 & 8 \\ 3 & -9 & 5 & 10 \\ -3 & 0 & 1 & -2 \\ 1 & -4 & 0 & 6 \end{bmatrix}.$$

Evaluating Determinants by Row Reduction

$$\begin{vmatrix} 2 & -8 & 6 & 8 \\ 3 & -9 & 5 & 10 \\ -3 & 0 & 1 & -2 \\ 1 & -4 & 0 & 6 \end{vmatrix} = 2\begin{vmatrix} 1 & -4 & 3 & 4 \\ 3 & -9 & 5 & 10 \\ -3 & 0 & 1 & -2 \\ 1 & -4 & 0 & 6 \end{vmatrix} = 2\begin{vmatrix} 1 & -4 & 3 & 4 \\ 0 & 3 & -4 & -2 \\ 0 & -12 & 10 & 10 \\ 0 & 0 & -3 & 2 \end{vmatrix}$$

$$= 2\begin{vmatrix} 1 & -4 & 3 & 4 \\ 0 & 3 & -4 & -2 \\ 0 & 0 & -6 & 2 \\ 0 & 0 & -3 & 2 \end{vmatrix} = 2\begin{vmatrix} 1 & -4 & 3 & 4 \\ 0 & 3 & -4 & -2 \\ 0 & 0 & -6 & 2 \\ 0 & 0 & 0 & 1 \end{vmatrix}$$

$$= 2(1 \cdot 3 \cdot (-6) \cdot 1) = -36.$$

Example (Row Reduction AND Cofactor Expansion)

Evaluate $\det(A)$ where

$$A = \begin{bmatrix} 3 & 5 & -2 & 6 \\ 1 & 2 & -1 & 1 \\ 2 & 4 & 1 & 5 \\ 3 & 7 & 5 & 3 \end{bmatrix}.$$

Evaluating Determinants by Row Reduction

$$\det(A) = \begin{vmatrix} 3 & 5 & -2 & 6 \\ 1 & 2 & -1 & 1 \\ 2 & 4 & 1 & 5 \\ 3 & 7 & 5 & 3 \end{vmatrix} \xlongequal[-3R_2+R_4]{\substack{-3R_2+R_1 \\ -2R_2+R_3}} \begin{vmatrix} 0 & -1 & 1 & 3 \\ 1 & 2 & -1 & 1 \\ 0 & 0 & 3 & 3 \\ 0 & 1 & 8 & 0 \end{vmatrix}$$

$$=$$

$$\xlongequal{\qquad\qquad}$$

$$=$$

$$\det(A) = \begin{vmatrix} 3 & 5 & -2 & 6 \\ 1 & 2 & -1 & 1 \\ 2 & 4 & 1 & 5 \\ 3 & 7 & 5 & 3 \end{vmatrix} \overset{\substack{-3R_2+R_1 \\ -2R_2+R_3}}{\underset{-3R_2+R_4}{=\!=\!=\!=\!=}} \begin{vmatrix} 0 & -1 & 1 & 3 \\ 1 & 2 & -1 & 1 \\ 0 & 0 & 3 & 3 \\ 0 & 1 & 8 & 0 \end{vmatrix}$$

$$\begin{matrix} + & - & + & - \\ - & + & - & + \\ + & - & + & - \\ - & + & - & + \end{matrix}$$

$$=$$

$$=\!=\!=\!=\!=$$

$$=$$

Evaluating Determinants by Row Reduction

$$\det(A) = \begin{vmatrix} 3 & 5 & -2 & 6 \\ 1 & 2 & -1 & 1 \\ 2 & 4 & 1 & 5 \\ 3 & 7 & 5 & 3 \end{vmatrix} \underset{-3R_2+R_4}{\overset{\substack{-3R_2+R_1 \\ -2R_2+R_3}}{=\!=\!=\!=\!=}} \begin{vmatrix} 0 & -1 & 1 & 3 \\ 1 & 2 & -1 & 1 \\ 0 & 0 & 3 & 3 \\ 0 & 1 & 8 & 0 \end{vmatrix} \qquad \begin{matrix} + & - & + & - \\ - & + & - & + \\ + & - & + & - \\ - & + & - & + \end{matrix}$$

$$= -1C_{21} = -\begin{vmatrix} -1 & 1 & 3 \\ 0 & 3 & 3 \\ 1 & 8 & 0 \end{vmatrix}$$

$$=\!=\!=\!=\!=$$

$$=$$

Evaluating Determinants by Row Reduction

$$\det(A) = \begin{vmatrix} 3 & 5 & -2 & 6 \\ 1 & 2 & -1 & 1 \\ 2 & 4 & 1 & 5 \\ 3 & 7 & 5 & 3 \end{vmatrix} \underset{-3R_2+R_4}{\overset{\substack{-3R_2+R_1 \\ -2R_2+R_3}}{=\!=\!=\!=}} \begin{vmatrix} 0 & -1 & 1 & 3 \\ 1 & 2 & -1 & 1 \\ 0 & 0 & 3 & 3 \\ 0 & 1 & 8 & 0 \end{vmatrix}$$

+	−	+	−
−	+	−	+
+	−	+	−
−	+	−	+

$$= -1C_{21} = -\begin{vmatrix} -1 & 1 & 3 \\ 0 & 3 & 3 \\ 1 & 8 & 0 \end{vmatrix}$$

$$\overset{R_1+R_3}{=\!=\!=\!=} -\begin{vmatrix} -1 & 1 & 3 \\ 0 & 3 & 3 \\ 0 & 9 & 3 \end{vmatrix}$$

$$=$$

Evaluating Determinants by Row Reduction

$$\det(A) = \begin{vmatrix} 3 & 5 & -2 & 6 \\ 1 & 2 & -1 & 1 \\ 2 & 4 & 1 & 5 \\ 3 & 7 & 5 & 3 \end{vmatrix} \xlongequal[-3R_2+R_4]{\substack{-3R_2+R_1 \\ -2R_2+R_3}} \begin{vmatrix} 0 & -1 & 1 & 3 \\ 1 & 2 & -1 & 1 \\ 0 & 0 & 3 & 3 \\ 0 & 1 & 8 & 0 \end{vmatrix} \qquad \begin{matrix} + & - & + & - \\ - & + & - & + \\ + & - & + & - \\ - & + & - & + \end{matrix}$$

$$= -1C_{21} = -\begin{vmatrix} -1 & 1 & 3 \\ 0 & 3 & 3 \\ 1 & 8 & 0 \end{vmatrix}$$

$$\xlongequal{R_1+R_3} \; -\begin{vmatrix} -1 & 1 & 3 \\ 0 & 3 & 3 \\ 0 & 9 & 3 \end{vmatrix} \qquad \begin{matrix} + & - & + \\ - & + & - \\ + & - & + \end{matrix}$$

$$=$$

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$$= -(-1)\begin{vmatrix} 3 & 3 \\ 9 & 3 \end{vmatrix} = -18.$$

Example (Exam question from 2017)

Suppose that $\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = 7.$

Find $\begin{vmatrix} d & e & f \\ a & b & c \\ g & h & i \end{vmatrix}$ and $\begin{vmatrix} a & b & c \\ 2d+a & 2e+b & 2f+c \\ g & h & i \end{vmatrix}$.

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Suppose that $\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = 7.$

Find $\begin{vmatrix} d & e & f \\ a & b & c \\ g & h & i \end{vmatrix}$ and $\begin{vmatrix} a & b & c \\ 2d+a & 2e+b & 2f+c \\ g & h & i \end{vmatrix}$.

First

$$\begin{vmatrix} d & e & f \\ a & b & c \\ g & h & i \end{vmatrix} = -7$$

because swapping two rows multiplies the determinant by -1.

Multiplying a row by a constant k, also multiplies the determinant by k. So

$$\begin{vmatrix} a & b & c \\ 2d & 2e & 2f \\ g & h & i \end{vmatrix} = 2 \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = 14.$$

Since adding one row to another does not change the determinant, we have

$$\begin{vmatrix} a & b & c \\ 2d+a & 2e+b & 2f+c \\ g & h & i \end{vmatrix} = \begin{vmatrix} a & b & c \\ 2d & 2e & 2f \\ g & h & i \end{vmatrix} = 14$$

also.

Break

We will continue at 3pm

Properties of Determinants

Properties of Determinants

- What is $\det(kA)$?
- What is $\det(A + B)$?
- What is $\det(AB)$?

First note that

$$\begin{vmatrix} ka_{11} & ka_{12} & ka_{13} \\ ka_{21} & ka_{22} & ka_{23} \\ ka_{31} & ka_{32} & ka_{33} \end{vmatrix} = k \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ ka_{21} & ka_{22} & ka_{23} \\ ka_{31} & ka_{32} & ka_{33} \end{vmatrix} =$$

$$=$$

Properties of Determinants

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$$= kkk \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}.$$

Theorem

If A is an $n \times n$ matrix and if k is a number, then

$$\det(kA) = k^n \det(A).$$

Example ($\det(A+B) \neq \det(A) + \det(B)$)

Consider

$$A = \begin{bmatrix} 1 & 2 \\ 2 & 5 \end{bmatrix}, \qquad B = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix} \quad \text{and} \quad A + B = \begin{bmatrix} 4 & 3 \\ 3 & 8 \end{bmatrix}.$$

Note that $\det(A) = 1$, $\det(B) = 8$ and $\det(A+B) = 23$.
Therefore

$$\det(A+B) \neq \det(A) + \det(B).$$

Properties of Determinants

In general $\det(A + B) \neq \det(A) + \det(B)$.

However there is one special case to consider: When two matrices are the same except for one row.

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Note that these matrices have the same first row and only the second row differs.

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However there is one special case to consider: When two matrices are the same except for one row. Let

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Note that these matrices have the same first row and only the second row differs. In this special case we have

$$\begin{aligned} \det(A) + \det(B) &= (a_{11}a_{22} - a_{12}a_{21}) + (a_{11}b_{22} - a_{12}b_{21}) \\ &= a_{11}(a_{22} + b_{22}) - a_{12}(a_{21} + b_{21}) \\ &= \begin{vmatrix} a_{11} & a_{12} \\ a_{21} + b_{21} & a_{22} + b_{22} \end{vmatrix}. \end{aligned}$$

Properties of Determinants

Theorem

Let A, B, and C be $n \times n$ matrices that differ only in a single row, say the rth row, and assume that the rth row of C can be obtained by adding the rth rows of A and B. Then

$$\det(C) = \det(A) + \det(B).$$

(The same result holds for columns.)

Properties of Determinants

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$$\det(C) = \det(A) + \det(B).$$

(The same result holds for columns.)

Example

Please check that

$$\begin{vmatrix} 1 & 7 & 5 \\ 2 & 0 & 3 \\ 1+0 & 4+1 & 7+(-1) \end{vmatrix} = \begin{vmatrix} 1 & 7 & 5 \\ 2 & 0 & 3 \\ 1 & 4 & 7 \end{vmatrix} + \begin{vmatrix} 1 & 7 & 5 \\ 2 & 0 & 3 \\ 0 & 1 & -1 \end{vmatrix}.$$

Properties of Determinants

Theorem (Most important theorem of today's lecture)

A square matrix A is invertible if and only if $\det(A) \neq 0$.

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Example

Is $A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 0 & 1 \\ 2 & 4 & 6 \end{bmatrix}$ invertible?

Theorem (Most important theorem of today's lecture)

A square matrix A is invertible if and only if $\det(A) \neq 0$.

Example

Is $A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 0 & 1 \\ 2 & 4 & 6 \end{bmatrix}$ invertible?

Since row 1 and row 3 are proportional ($R_3 = 2R_1$), we have $\det(A) = 0$. Therefore A is not invertible. A is singular.

Properties of Determinants

Theorem

$$\det(AB) = \det(A)\det(B).$$

Properties of Determinants

Theorem

$$\det(AB) = \det(A)\det(B).$$

Example

Let

$$A = \begin{bmatrix} 6 & 1 \\ 3 & 2 \end{bmatrix}, \qquad B = \begin{bmatrix} 4 & 3 \\ 1 & 2 \end{bmatrix}, \quad \text{and} \quad AB = \begin{bmatrix} 25 & 20 \\ 14 & 13 \end{bmatrix}.$$

Note that

$$\det(A)\det(B) = 9 \cdot 5 = 45 = \det(AB).$$

Properties of Determinants

$$\boxed{\det(AB) = \det(A)\det(B)}$$

Recall that

$$A^{-1}A = I.$$

So

$$\det(A^{-1})\underbrace{\det(A)}_{\neq 0} = \det(I) = 1.$$

$$\det(AB) = \det(A)\det(B)$$

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$$A^{-1}A = I.$$

So

$$\det(A^{-1})\underbrace{\det(A)}_{\neq 0} = \det(I) = 1.$$

Theorem

If A is invertible, then

$$\det(A^{-1}) = \frac{1}{\det(A)}.$$

Properties of Determinants

$$\det(AB) = \det(A)\det(B)$$

Because

$$\begin{aligned} \det(AA) &= \det(A)\det(A) \\ \det(AAA) &= \det(A)\det(A)\det(A) \\ \det(AAAA) &= \det(A)\det(A)\det(A)\det(A) \\ &\vdots \end{aligned}$$

we have

$\det(AB) = \det(A)\det(B)$

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we have

Theorem

$$\det(A^n) = \big(\det(A)\big)^n$$

for any $n \in \mathbb{N}$.

Example (Exam question from 2018)

Compute $\det(2A^3)$ where $A = \begin{bmatrix} 1 & 0 & 2 \\ 1 & 1 & 4 \\ 1 & 3 & 3 \end{bmatrix}$.

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we have that

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we have that

$$\det(2A^3) = 2^3 \det(A^3) = 8\big(\det(A)\big)^3$$

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Since

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we have that

$$\det(2A^3) = 2^3 \det(A^3) = 8\big(\det(A)\big)^3 = 8(-5)^3 = -1000.$$

Example

Let A, B and C be 4×4 matrices with $\det(A) = 1$, $\det(B) = 2$ and $\det(C) = 3$. Find $\det\left(3A^{-1}B^TC^2\right)$.

$$\begin{aligned}\det\left(3A^{-1}B^TC^2\right) &= 3^4 \det\left(A^{-1}\right) \det\left(B^T\right) \det\left(C^2\right) \\ &= \\ &= \end{aligned}$$

Properties of Determinants

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$$\begin{aligned}
\det\left(3A^{-1}B^TC^2\right) &= 3^4 \det\left(A^{-1}\right) \det\left(B^T\right) \det\left(C^2\right) \\
&= 3^4 \frac{1}{\det(A)} \det(B) \left(\det(C)\right)^2 \\
&= \ldots
\end{aligned}$$

Adjoint of a Matrix

Let A be an $n \times n$ square matrix and let C_{ij} denote the cofactors of A.

Definition

The matrix

$$\begin{bmatrix} C_{11} & C_{12} & \dots & C_{1n} \\ C_{21} & C_{22} & \dots & C_{2n} \\ \vdots & \vdots & & \vdots \\ C_{n1} & C_{n2} & \dots & C_{nn} \end{bmatrix}$$

is called the *matrix of cofactors from* A.

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is called the *matrix of cofactors from* A.

Definition

The transpose of the matrix of cofactors from A is called the *adjoint of* A and is denoted by $\operatorname{adj}(A)$.

Example

Find the adjoint of $A = \begin{bmatrix} 3 & 2 & -1 \\ 1 & 6 & 3 \\ 2 & -4 & 0 \end{bmatrix}$.

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1. Find the cofactors of A.

2. Write the matrix of cofactors.

3. Take the transpose.

Example

Find the adjoint of $A = \begin{bmatrix} 3 & 2 & -1 \\ 1 & 6 & 3 \\ 2 & -4 & 0 \end{bmatrix}$.

1. Find the cofactors of A.

$$\begin{array}{lll} C_{11} = 12 & C_{12} = 6 & C_{13} = -16 \\ C_{21} = 4 & C_{22} = 2 & C_{23} = 16 \\ C_{31} = 12 & C_{32} = -10 & C_{33} = 16 \end{array}$$

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2. Write the matrix of cofactors.

$$\begin{bmatrix} 12 & 6 & -16 \\ 4 & 2 & 16 \\ 12 & -10 & 16 \end{bmatrix}$$

3. Take the transpose.

$$\operatorname{adj}(A) = \begin{bmatrix} 12 & 4 & 12 \\ 6 & 2 & -10 \\ -16 & 16 & 16 \end{bmatrix}$$

Theorem

If A is an invertible matrix, then

$$A^{-1} = \frac{1}{\det(A)}\,\mathrm{adj}(A).$$

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Example

Use this theorem to find the inverse of $A = \begin{bmatrix} 3 & 2 & -1 \\ 1 & 6 & 3 \\ 2 & -4 & 0 \end{bmatrix}$.

In the last example we found that $\mathrm{adj}(A) = \begin{bmatrix} 12 & 4 & 12 \\ 6 & 2 & -10 \\ -16 & 16 & 16 \end{bmatrix}$.

So we just need to find $\det(A)$ then use the theorem.

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So we just need to find $\det(A)$ then use the theorem.

Using cofactor expansion along the first row, we calculate that

$$\det(A) = 3C_{11} + 2C_{12} + (-1)C_{13} = 36 + 12 + 16 = 64.$$

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Hence

$$A^{-1} = \frac{1}{\det(A)} \operatorname{adj}(A) = \frac{1}{64} \begin{bmatrix} 12 & 4 & 12 \\ 6 & 2 & -10 \\ -16 & 16 & 16 \end{bmatrix}$$

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Hence

$$A^{-1} = \frac{1}{\det(A)}\,\mathrm{adj}(A) = \frac{1}{64}\begin{bmatrix} 12 & 4 & 12 \\ 6 & 2 & -10 \\ -16 & 16 & 16 \end{bmatrix} = \begin{bmatrix} \frac{12}{64} & \frac{4}{64} & \frac{12}{64} \\ \frac{6}{64} & \frac{2}{64} & -\frac{10}{64} \\ -\frac{16}{64} & \frac{16}{64} & \frac{16}{64} \end{bmatrix}.$$

EXAMPLE 3 Find the inverse of the matrix $A = \begin{bmatrix} 2 & 1 & 3 \\ 1 & -1 & 1 \\ 1 & 4 & -2 \end{bmatrix}$.

SOLUTION The nine cofactors are

$$C_{11} = +\begin{vmatrix} -1 & 1 \\ 4 & -2 \end{vmatrix} = -2, \quad C_{12} = -\begin{vmatrix} 1 & 1 \\ 1 & -2 \end{vmatrix} = 3, \quad C_{13} = +\begin{vmatrix} 1 & -1 \\ 1 & 4 \end{vmatrix} = 5$$

$$C_{21} = -\begin{vmatrix} 1 & 3 \\ 4 & -2 \end{vmatrix} = 14, \quad C_{22} = +\begin{vmatrix} 2 & 3 \\ 1 & -2 \end{vmatrix} = -7, \quad C_{23} = -\begin{vmatrix} 2 & 1 \\ 1 & 4 \end{vmatrix} = -7$$

$$C_{31} = +\begin{vmatrix} 1 & 3 \\ -1 & 1 \end{vmatrix} = 4, \quad C_{32} = -\begin{vmatrix} 2 & 3 \\ 1 & 1 \end{vmatrix} = 1, \quad C_{33} = +\begin{vmatrix} 2 & 1 \\ 1 & -1 \end{vmatrix} = -3$$

The adjugate matrix is the *transpose* of the matrix of cofactors. [For instance, C_{12} goes in the $(2, 1)$ position.] Thus

$$\operatorname{adj} A = \begin{bmatrix} -2 & 14 & 4 \\ 3 & -7 & 1 \\ 5 & -7 & -3 \end{bmatrix}$$

We could compute det A directly, but the following computation provides a check on the calculations above *and* produces det A:

$$(\text{adj } A) \cdot A = \begin{bmatrix} -2 & 14 & 4 \\ 3 & -7 & 1 \\ 5 & -7 & -3 \end{bmatrix} \begin{bmatrix} 2 & 1 & 3 \\ 1 & -1 & 1 \\ 1 & 4 & -2 \end{bmatrix} = \begin{bmatrix} 14 & 0 & 0 \\ 0 & 14 & 0 \\ 0 & 0 & 14 \end{bmatrix} = 14I$$

Since $(\text{adj } A)A = 14I$, Theorem 8 shows that det $A = 14$ and

$$A^{-1} = \frac{1}{14} \begin{bmatrix} -2 & 14 & 4 \\ 3 & -7 & 1 \\ 5 & -7 & -3 \end{bmatrix} = \begin{bmatrix} -1/7 & 1 & 2/7 \\ 3/14 & -1/2 & 1/14 \\ 5/14 & -1/2 & -3/14 \end{bmatrix} \quad \blacksquare$$

Cramer's Rule

Cramer's Rule

Suppose we have an $n \times n$ matrix and an $n \times 1$ vector.

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}, \qquad \mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}.$$

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I want to replace the first column of A by $\mathbf{b}$ and call this new matrix A_1.

$$A_1 = \begin{bmatrix} b_1 & a_{12} & \cdots & a_{1n} \\ b_2 & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ b_n & a_{n2} & \cdots & a_{nn} \end{bmatrix}.$$

Similarly I want to replace the second column of A by $\mathbf{b}$ and call this new matrix A_2.

$$A_2 = \begin{bmatrix} a_{11} & b_1 & \cdots & a_{1n} \\ a_{21} & b_2 & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & b_n & \cdots & a_{nn} \end{bmatrix}.$$

Similarly I want to replace the second column of A by $\mathbf{b}$ and call this new matrix A_2.

$$A_2 = \begin{bmatrix} a_{11} & b_1 & \cdots & a_{1n} \\ a_{21} & b_2 & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & b_n & \cdots & a_{nn} \end{bmatrix}.$$

$$\vdots$$

And I want to replace the nth column of A by $\mathbf{b}$ and call this new matrix A_n.

$$A_n = \begin{bmatrix} a_{11} & a_{12} & \cdots & b_1 \\ a_{21} & a_{22} & \cdots & b_2 \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & b_n \end{bmatrix}.$$

Cramer's Rule

Theorem (Cramer's Rule)

If

$$A\mathbf{x} = \mathbf{b}$$

is a system of n *linear equations in* n *unknowns and if* $\det(A) \neq 0$*, then this linear system has a unique solution which is given by*

$$x_1 = \frac{\det(A_1)}{\det(A)}, \quad x_2 = \frac{\det(A_2)}{\det(A)}, \quad \dots, \quad x_n = \frac{\det(A_n)}{\det(A)}.$$

Cramer's Rule

$$x_1 = \frac{\det(A_1)}{\det(A)},\ x_2 = \frac{\det(A_2)}{\det(A)},\ \dots,\ x_n = \frac{\det(A_n)}{\det(A)}.$$

Example

Use Cramer's Rule to solve $\begin{cases} 3x_1 - 2x_2 = 6 \\ -5x_1 + 4x_2 = 8 \end{cases}.$

$$x_1 = \frac{\det(A_1)}{\det(A)},\ x_2 = \frac{\det(A_2)}{\det(A)},\ \ldots,\ x_n = \frac{\det(A_n)}{\det(A)}.$$

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Use Cramer's Rule to solve $\begin{cases} 3x_1 - 2x_2 = 6 \\ -5x_1 + 4x_2 = 8 \end{cases}.$

We have $A\mathbf{x} = \mathbf{b}$ where

$$A = \begin{bmatrix} 3 & -2 \\ -5 & 4 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 6 \\ 8 \end{bmatrix}, \quad A_1 = \begin{bmatrix} 6 & -2 \\ 8 & 4 \end{bmatrix} \text{ and } A_2 = \begin{bmatrix} 3 & 6 \\ -5 & 8 \end{bmatrix}.$$

$$x_1 = \frac{\det(A_1)}{\det(A)},\ x_2 = \frac{\det(A_2)}{\det(A)},\ \ldots,\ x_n = \frac{\det(A_n)}{\det(A)}.$$

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Since $\det(A) = 2 \neq 0$, the linear system has a unique solution.

$$x_1 = \frac{\det(A_1)}{\det(A)},\ x_2 = \frac{\det(A_2)}{\det(A)},\ \ldots,\ x_n = \frac{\det(A_n)}{\det(A)}.$$

Example

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$$A = \begin{bmatrix} 3 & -2 \\ -5 & 4 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 6 \\ 8 \end{bmatrix}, \quad A_1 = \begin{bmatrix} 6 & -2 \\ 8 & 4 \end{bmatrix} \text{ and } A_2 = \begin{bmatrix} 3 & 6 \\ -5 & 8 \end{bmatrix}.$$

Since $\det(A) = 2 \neq 0$, the linear system has a unique solution. By Cramer's Rule, this solution is

$$x_1 = \frac{\det(A_1)}{\det(A)} = \frac{24+16}{2} = 20$$

$$x_2 = \frac{\det(A_2)}{\det(A)} = \frac{24+30}{2} = 27.$$

Cramer's Rule

$$x_1 = \frac{\det(A_1)}{\det(A)}, \, x_2 = \frac{\det(A_2)}{\det(A)}, \, \ldots, \, x_n = \frac{\det(A_n)}{\det(A)}.$$

Cramer's Rule is useful if you only want to find some variables and you don't care about the rest.

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$$x_1 = \frac{\det(A_1)}{\det(A)},\ x_2 = \frac{\det(A_2)}{\det(A)},\ \ldots,\ x_n = \frac{\det(A_n)}{\det(A)}.$$

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Example

Find x_3 if $\begin{cases} x_1 \phantom{{}+4x_2} + 2x_3 = 6 \\ -3x_1 + 4x_2 + 6x_3 = 30 \\ -x_1 - 2x_2 + 3x_3 = 8. \end{cases}$

$$x_1 = \frac{\det(A_1)}{\det(A)},\ x_2 = \frac{\det(A_2)}{\det(A)},\ \dots,\ x_n = \frac{\det(A_n)}{\det(A)}.$$

Cramer's Rule is useful if you only want to find some variables and you don't care about the rest.

Example

Find x_3 if $\begin{cases} x_1 \qquad\quad + 2x_3 = 6 \\ -3x_1 + 4x_2 + 6x_3 = 30 \\ -x_1 - 2x_2 + 3x_3 = 8. \end{cases}$

Here we have $A\mathbf{x} = \mathbf{b}$ with

$$A = \begin{bmatrix} 1 & 0 & 2 \\ -3 & 4 & 6 \\ -1 & -2 & 3 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 6 \\ 30 \\ 8 \end{bmatrix} \quad \text{and} \quad A_3 = \begin{bmatrix} 1 & 0 & 6 \\ -3 & 4 & 30 \\ -1 & -2 & 8 \end{bmatrix}.$$

I leave it for you to check that $\det(A) = 44$ and $\det(A_3) = 152$.

$$x_1 = \frac{\det(A_1)}{\det(A)}, x_2 = \frac{\det(A_2)}{\det(A)}, \ldots, x_n = \frac{\det(A_n)}{\det(A)}.$$

Cramer's Rule is useful if you only want to find some variables and you don't care about the rest.

Example

Find x_3 if $\begin{cases} x_1 \qquad\qquad + 2x_3 = 6 \\ -3x_1 + 4x_2 + 6x_3 = 30 \\ -x_1 - 2x_2 + 3x_3 = 8. \end{cases}$

Here we have $A\mathbf{x} = \mathbf{b}$ with

$$A = \begin{bmatrix} 1 & 0 & 2 \\ -3 & 4 & 6 \\ -1 & -2 & 3 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 6 \\ 30 \\ 8 \end{bmatrix} \quad \text{and} \quad A_3 = \begin{bmatrix} 1 & 0 & 6 \\ -3 & 4 & 30 \\ -1 & -2 & 8 \end{bmatrix}.$$

I leave it for you to check that $\det(A) = 44$ and $\det(A_3) = 152$. Since $\det(A) \neq 0$, the linear system has a unique solution. By Cramer's Rule, we calculate that $x_3 = \frac{\det(A_3)}{\det(A)} = \frac{152}{44} = \frac{38}{11}$.

$$x_1 = \frac{\det(A_1)}{\det(A)},\ x_2 = \frac{\det(A_2)}{\det(A)},\ \ldots,\ x_n = \frac{\det(A_n)}{\det(A)}.$$

Example

Consider the following system in which s is an unspecified parameter. Determine the values of s for which the system has a unique solution, and use Cramer's rule to find the solution.

$$\begin{cases} 3sx_1 - 2x_2 = 4 \\ -6x_1 + sx_2 = 1. \end{cases}$$

$$x_1 = \frac{\det(A_1)}{\det(A)},\ x_2 = \frac{\det(A_2)}{\det(A)},\ \dots,\ x_n = \frac{\det(A_n)}{\det(A)}.$$

Example

Consider the following system in which s is an unspecified parameter. Determine the values of s for which the system has a unique solution, and use Cramer's rule to find the solution.

$$\begin{cases} 3sx_1 - 2x_2 = 4 \\ -6x_1 + sx_2 = 1. \end{cases}$$

We have $A\mathbf{x} = \mathbf{b}$ with

$$A = \begin{bmatrix} 3s & -2 \\ -6 & s \end{bmatrix}, \quad A_1 = \begin{bmatrix} 4 & -2 \\ 1 & s \end{bmatrix} \text{ and } A_2 = \begin{bmatrix} 3s & 4 \\ -6 & 1 \end{bmatrix}.$$

Since

$$\det(A) = 3s^2 - 12 = 3(s+2)(s-2),$$

the linear system has a unique solution when $s \neq \pm 2$.

$$x_1 = \frac{\det(A_1)}{\det(A)},\ x_2 = \frac{\det(A_2)}{\det(A)},\ \ldots,\ x_n = \frac{\det(A_n)}{\det(A)}.$$

$$A = \begin{bmatrix} 3s & -2 \\ -6 & s \end{bmatrix}, \quad A_1 = \begin{bmatrix} 4 & -2 \\ 1 & s \end{bmatrix} \text{ and } A_2 = \begin{bmatrix} 3s & 4 \\ -6 & 1 \end{bmatrix}$$

By Cramer's Rule, this solution is

$$x_1 = \frac{\det(A_1)}{\det(A)} = \frac{4s+2}{3(s+2)(s-2)}$$

$$x_2 = \frac{\det(A_2)}{\det(A)} = \frac{3s+24}{3(s+2)(s-2)} = \frac{s+8}{(s+2)(s-2)}.$$

for $s \neq \pm 2$.

FORENAME:

SURNAME:

STUDENT NO:

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SIGNATURE:

Question	Points	Score
1	25	
2	25	
3	25	
4	25	
Total:	100	

- The time limit is 75 minutes.
- Any attempts at cheating or plagiarizing and assisting of such actions in any form would result in getting an automatic zero (0) from the exam. Disciplinary action will also be taken in accordance with the regulations of the Council of Higher Education.
- Give your answers in exact form (for example $\frac{\pi}{3}$ or $5\sqrt{3}$), except as noted in particular problems.
- Calculators, mobile phones, smart watches, etc. are not allowed.
- In order to receive credit, you must **show all of your work**. If you do not indicate the way in which you solved a problem, you may get little or no credit for it, even if your answer is correct.
- Place $\boxed{\text{a box around your answer}}$ to each question.
- Use a **BLUE ball-point pen** to fill the cover sheet. Please make sure that your exam is complete.
- Do not write in the table above.

1. $\boxed{\text{25 points}}$ For which values of k does the system $\begin{aligned} x+2y+6z&=2 \\ y+2kz&=0 \\ kx+2z&=1 \end{aligned}$ have

   (a) no solution.

   (b) infinitely many solutions.

   (c) a unique solution.

1. $\boxed{\text{25 points}}$ For which values of k does the system $\begin{aligned} x + 2y + 6z &= 2 \\ y + 2kz &= 0 \\ kx + 2z &= 1 \end{aligned}$ have

(a) no solution.

(b) infinitely many solutions.

(c) a unique solution.

Solution: Let us transform the augmented matrix to row echelon form.

$$\begin{bmatrix} 1 & 2 & 6 & 2 \\ 0 & 1 & 2k & 0 \\ k & 0 & 2 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 6 & 2 \\ 0 & 1 & 2k & 0 \\ 0 & -2k & 2-6k & 1-2k \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 6 & 2 \\ 0 & 1 & 2k & 0 \\ 0 & 0 & 4k^2-6k+2 & 1-2k \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 6 & 2 \\ 0 & 1 & 2k & 0 \\ 0 & 0 & 2(2k-1)(k-1) & 1-2k \end{bmatrix}$$

If $k = 1$, then we obtain $\begin{bmatrix} 1 & 2 & 6 & 2 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$. Therefore, the system has no solution.

If $k = \frac{1}{2}$, then we obtain $\begin{bmatrix} 1 & 2 & 6 & 2 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$. Therefore, the system has infinitely many solutions.

If $k \neq 1$ and $k \neq \frac{1}{2}$, then we obtain $\begin{bmatrix} 1 & 2 & 6 & 2 \\ 0 & 1 & 2k & 0 \\ 0 & 0 & 1 & -\dfrac{1}{2(k-1)} \end{bmatrix}$ Therefore, the system have a unique solution.

2. (a) $\boxed{\text{15 points}}$ Calculate the determinant of the matrix $A = \begin{bmatrix} 1 & 2 & 3 & -1 \\ 0 & 0 & 4 & -2 \\ 3 & -2 & 1 & 1 \\ 2 & 0 & 1 & 1 \end{bmatrix}$.

Solution:

$$\begin{vmatrix} 1 & 2 & 3 & -1 \\ 0 & 0 & 4 & -2 \\ 3 & -2 & 1 & 1 \\ 2 & 0 & 1 & 1 \end{vmatrix} = \begin{vmatrix} 1 & 2 & 3 & -1 \\ 0 & 0 & 4 & -2 \\ 4 & 0 & 4 & 0 \\ 2 & 0 & 1 & 1 \end{vmatrix} = 2(-1)^{1+2} \begin{vmatrix} 0 & 4 & -2 \\ 4 & 4 & 0 \\ 2 & 1 & 1 \end{vmatrix} + 0 + 0 + 0$$

$$= (-2) \begin{vmatrix} 0 & 4 & -2 \\ 0 & 2 & -2 \\ 2 & 1 & 1 \end{vmatrix} = (-2)\left[(2)(-1)^{3+1} \begin{vmatrix} 4 & -2 \\ 2 & -2 \end{vmatrix}\right] = (-4)(-8+4) = 16$$

2. (a) $\boxed{\text{15 points}}$ Calculate the determinant of the matrix $A = \begin{bmatrix} 1 & 2 & 3 & -1 \\ 0 & 0 & 4 & -2 \\ 3 & -2 & 1 & 1 \\ 2 & 0 & 1 & 1 \end{bmatrix}$.

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$$= (-2) \begin{vmatrix} 0 & 4 & -2 \\ 0 & 2 & -2 \\ 2 & 1 & 1 \end{vmatrix} = (-2)\left[(2)(-1)^{3+1} \begin{vmatrix} 4 & -2 \\ 2 & -2 \end{vmatrix}\right] = (-4)(-8+4) = 16$$

(b) $\boxed{\text{10 points}}$ Suppose that $A\mathbf{x} = \mathbf{b}$ where $A = \begin{bmatrix} 1 & 1 & 3 & -1 \\ 0 & 2 & 4 & -2 \\ 3 & 0 & 1 & 1 \\ 2 & 4 & 1 & 1 \end{bmatrix}$, $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} 2 \\ 0 \\ -2 \\ 0 \end{bmatrix}$ and $\det A = (-2)$. Use Cramer's rule to find x_2.

Solution:

$$x_2 = \frac{|A_2|}{|A|} = \frac{\begin{vmatrix} 1 & 2 & 3 & -1 \\ 0 & 0 & 4 & -2 \\ 3 & -2 & 1 & 1 \\ 2 & 0 & 1 & 1 \end{vmatrix}}{-2} = \frac{16}{-2} = -8$$

3. (a) $\boxed{\text{10 points}}$ Find the adjoint matrix $(Adj A)$ of $A = \begin{bmatrix} 1 & 2 & 6 \\ 0 & 1 & 6 \\ 0 & 0 & 2 \end{bmatrix}$.

Solution:

$$C_{11} = (-1)^{1+1}\begin{vmatrix} 1 & 6 \\ 0 & 2 \end{vmatrix} = 2 \quad C_{21} = (-1)^{2+1}\begin{vmatrix} 2 & 6 \\ 0 & 2 \end{vmatrix} = -4 \quad C_{31} = (-1)^{3+1}\begin{vmatrix} 2 & 6 \\ 1 & 6 \end{vmatrix} = 6$$

$$C_{12} = (-1)^{1+2}\begin{vmatrix} 0 & 6 \\ 0 & 2 \end{vmatrix} = 0 \quad C_{22} = (-1)^{2+2}\begin{vmatrix} 1 & 6 \\ 0 & 2 \end{vmatrix} = 2 \quad C_{32} = (-1)^{3+2}\begin{vmatrix} 1 & 6 \\ 0 & 6 \end{vmatrix} = -6$$

$$C_{13} = (-1)^{1+3}\begin{vmatrix} 0 & 1 \\ 0 & 0 \end{vmatrix} = 0 \quad C_{23} = (-1)^{2+3}\begin{vmatrix} 1 & 2 \\ 0 & 0 \end{vmatrix} = 0 \quad C_{33} = (-1)^{3+3}\begin{vmatrix} 1 & 2 \\ 0 & 1 \end{vmatrix} = 1$$

$$Adj A = \begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix}^T = \begin{bmatrix} 2 & 0 & 0 \\ -4 & 2 & 0 \\ 6 & -6 & 1 \end{bmatrix}^T = \begin{bmatrix} 2 & -4 & 6 \\ 0 & 2 & -6 \\ 0 & 0 & 1 \end{bmatrix}$$

(b) $\boxed{\text{15 points}}$ Find the inverse of the matrix $A = \begin{bmatrix} 1 & 2 & 6 \\ 0 & 1 & 6 \\ 0 & 0 & 2 \end{bmatrix}$.

Solution: First Way:

$$A^{-1} = \frac{1}{\det A} Adj A = \frac{1}{2} \begin{bmatrix} 2 & -4 & 6 \\ 0 & 2 & -6 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -2 & 3 \\ 0 & 1 & -3 \\ 0 & 0 & 0.5 \end{bmatrix}$$

Second Way:

$$[A|I] \sim [I|A^{-1}]$$

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 6 & 1 & 0 & 0 \\ 0 & 1 & 6 & 0 & 1 & 0 \\ 0 & 0 & 2 & 0 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 0 & -6 & 1 & -2 & 0 \\ 0 & 1 & 6 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0.5 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -2 & 3 \\ 0 & 1 & 0 & 0 & 1 & -3 \\ 0 & 0 & 1 & 0 & 0 & 0.5 \end{array}\right]$$

4. (a) $\boxed{\text{10 points}}$ Suppose an $n \times n$ matrix A satisfies the equation $A^2 - 2A + I = 0$. Show that $A^3 = 3A - 2I$.

Solution:

$$A^2 = 2A - I \Rightarrow A^3 = 2A^2 - A = 2(2A - I) - A = 3A - 2I$$

(b) $\boxed{\text{15 points}}$ Let A, B and C be 3×3 matrices with $\det A = -3$, $\det B = 4$ and $\det C = 2$. Compute $\det(2A^2B^{-2}C^T)$.

Solution:

$$\det(2A^2B^{-2}C^T) = 2^3(\det A)^2 \frac{1}{(\det B)^2}(\det C) = 8(-3)^2 \frac{1}{4^2} 2 = 9$$

Next Time

- Vector Spaces
- Subspaces
- Linear Independence