

Lecture 9

- 14.5 Triple Integrals in Rectangular Coordinates
- 14.7 Triple Integrals in Cylindrical and Spherical Coordinates
- 14.8 Substitutions in Multiple Integrals



Triple Integrals in Rectangular Coordinates

14.5 Triple Integrals in Rectangular Coordinates



In the last two lectures we have been studying

$$\iint_R f(x, y) dA.$$

14.5 Triple Integrals in Rectangular Coordinates



In the last two lectures we have been studying

$$\iint_R f(x, y) \, dA.$$

Today we will consider

$$\iiint_D f(x, y, z) \, dV.$$

14.5 Triple Integrals in Rectangular Coordinates



Definition

The *volume* of a closed, bounded region D in space is

$$V = \iiint_R dV.$$



Finding Limits of Integration

$$\int \int \int F(x, y, z) dz dy dx.$$

Finding Limits of Integration

$$\int_{x=a}^{x=b} \int \int F(x, y, z) dz dy d\textcolor{brown}{x}.$$

only numbers



Finding Limits of Integration

$$\int_{x=a}^{x=b} \int_{y=g_1(x)}^{y=g_2(x)} \int F(x, y, z) dz dy dx.$$

functions of x
only numbers



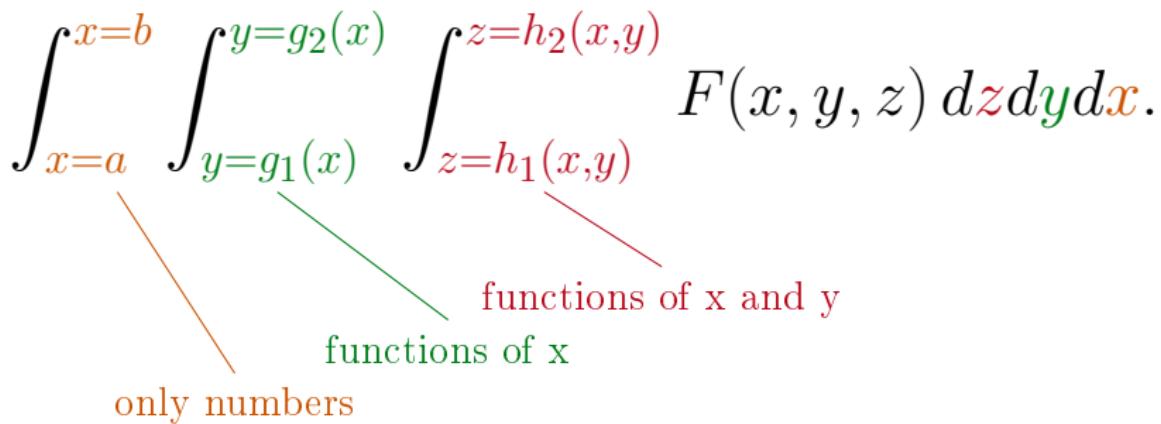
Finding Limits of Integration

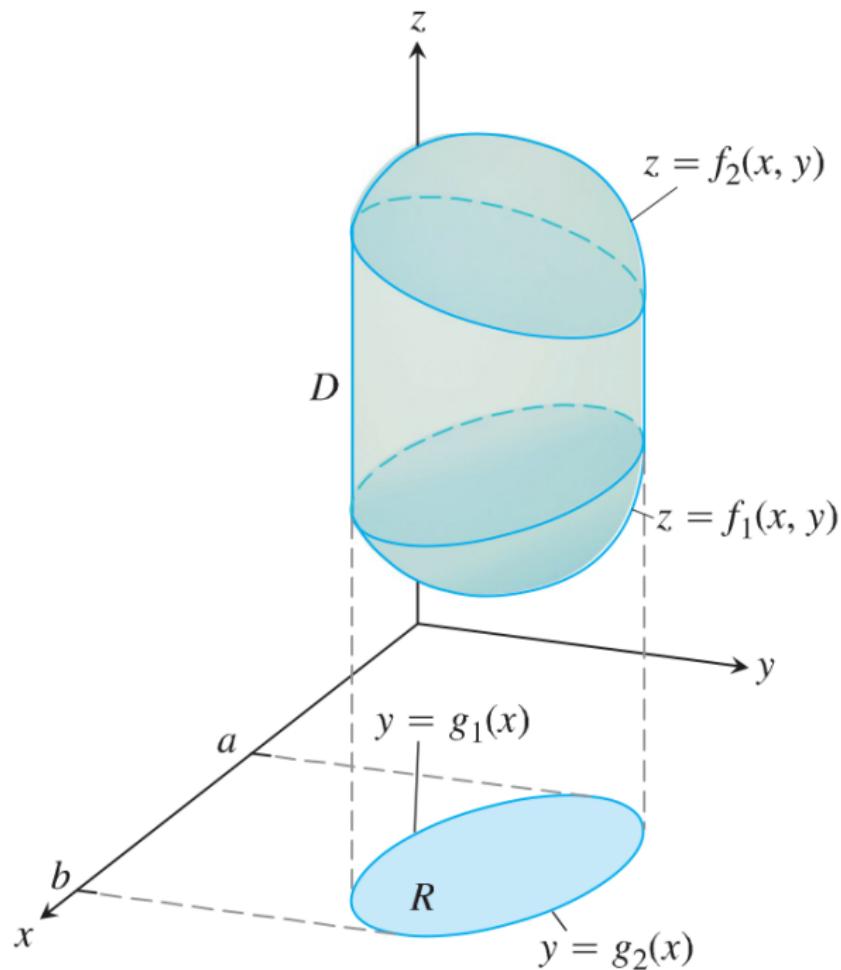
$$\int_{x=a}^{x=b} \int_{y=g_1(x)}^{y=g_2(x)} \int_{z=h_1(x,y)}^{z=h_2(x,y)} F(x, y, z) dz dy dx.$$

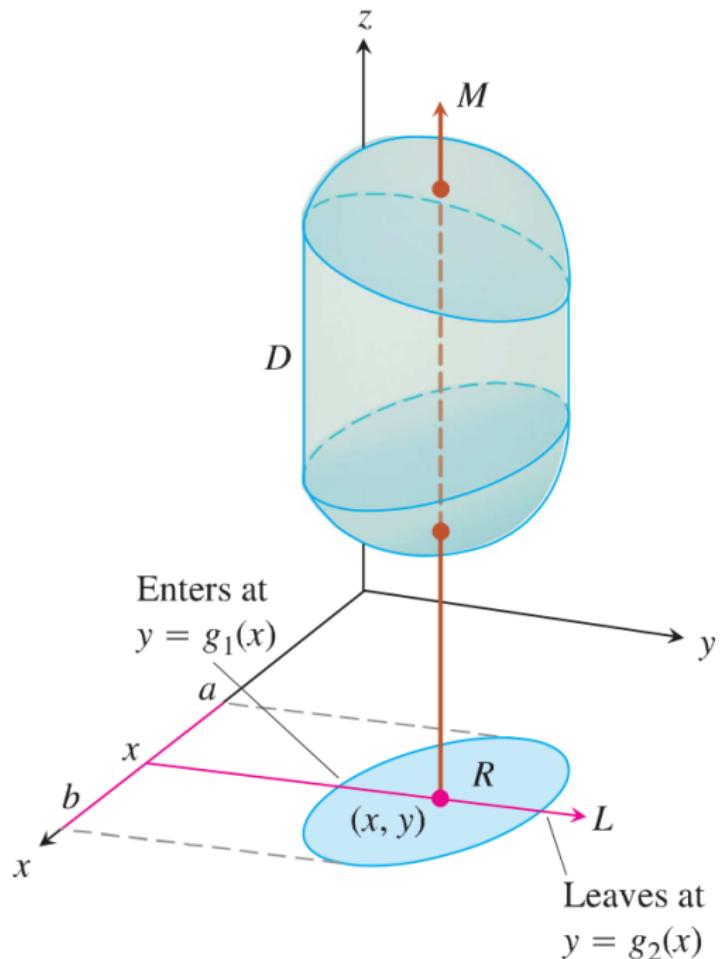
functions of x and y

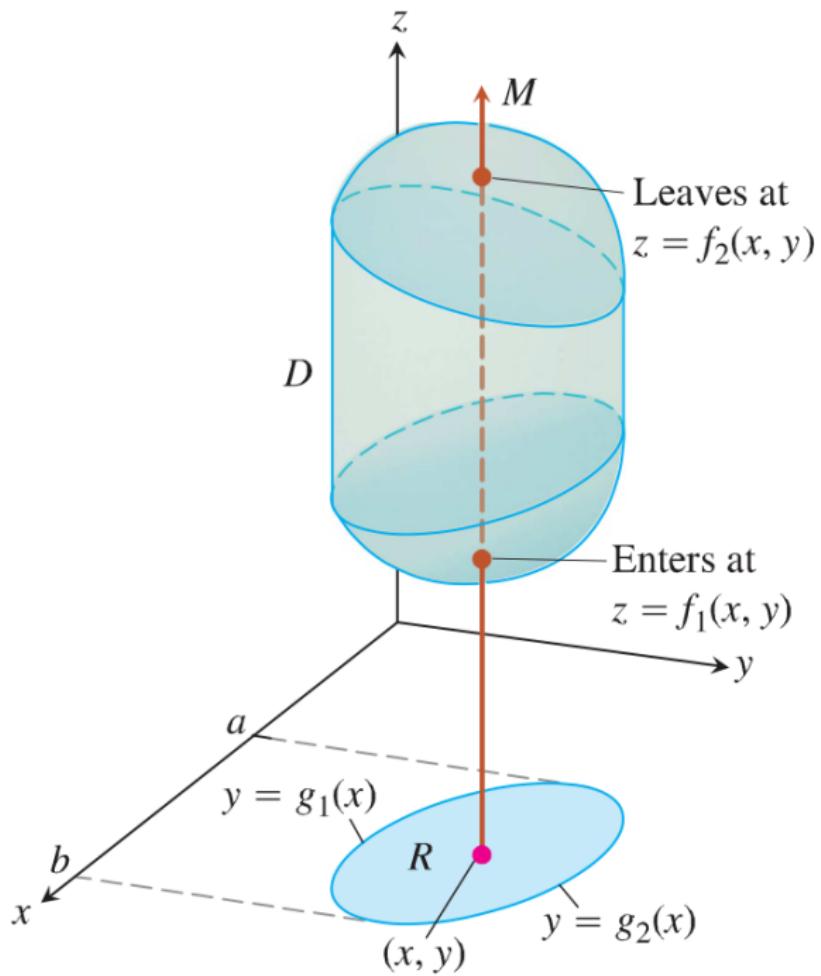
functions of x

only numbers

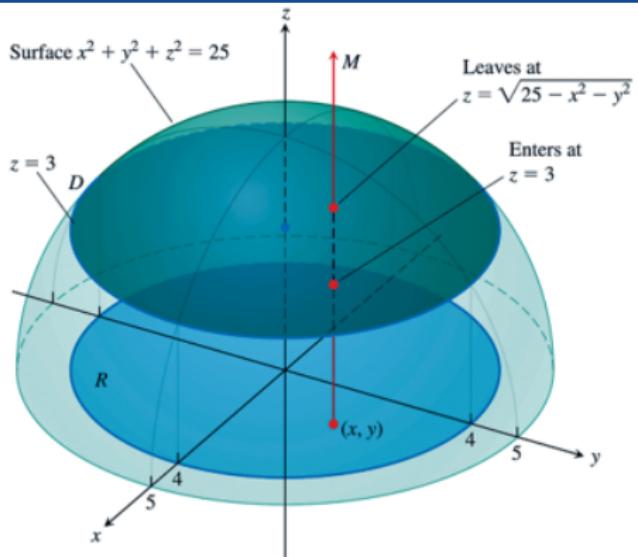








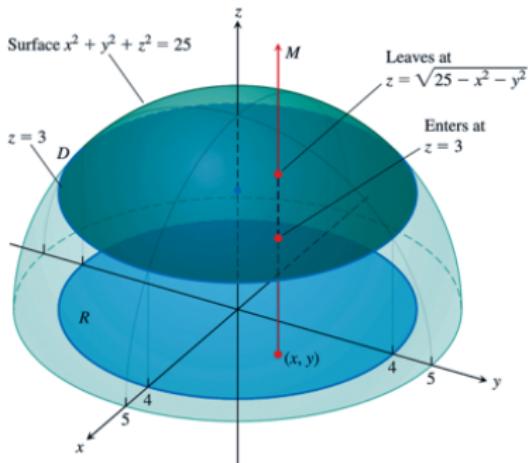
14.5 Triple Integrals in Rectangular Coordinates



Example

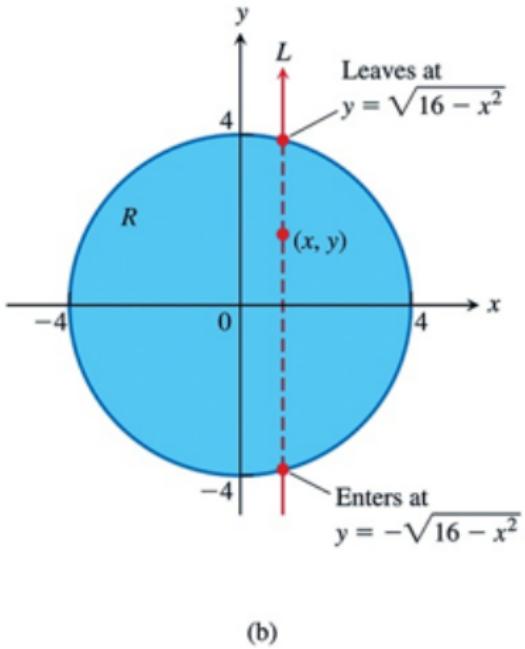
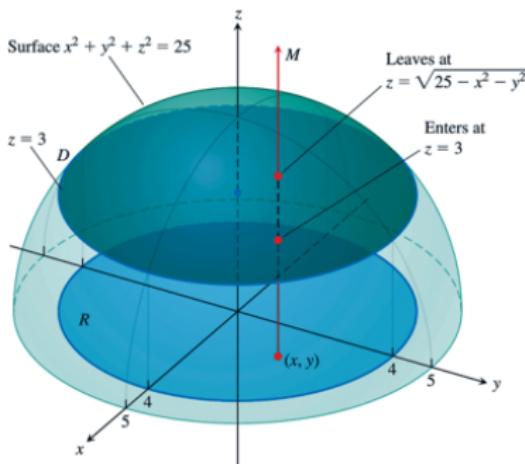
Let S be the sphere of radius 5 centred at the origin. Let D be the region under the sphere and above the plane $z = 3$. Set up the limits of integration over D .

14.5 Triple Integrals in Rectangular Coordinates



$$z = 3 \implies x^2 + y^2 + 9 = 25 \implies x^2 + y^2 = 16$$

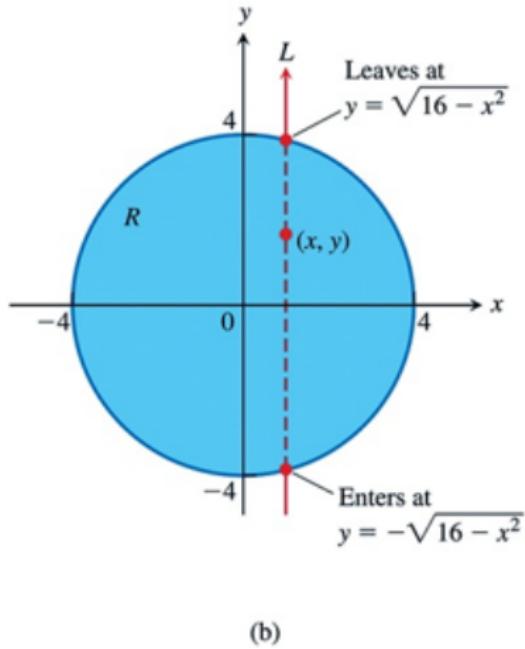
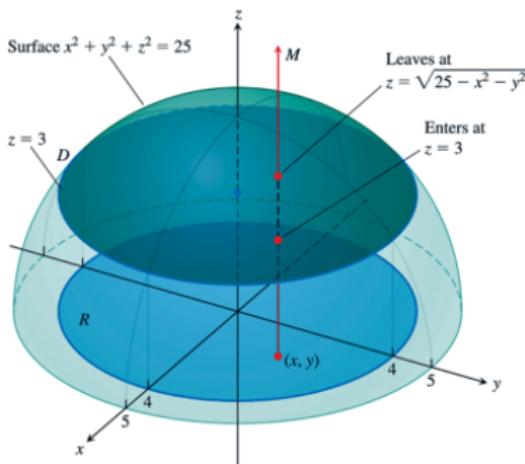
14.5 Triple Integrals in Rectangular



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$$-4 \leq x \leq 4$$

14.5 Triple Integrals in Rectangular

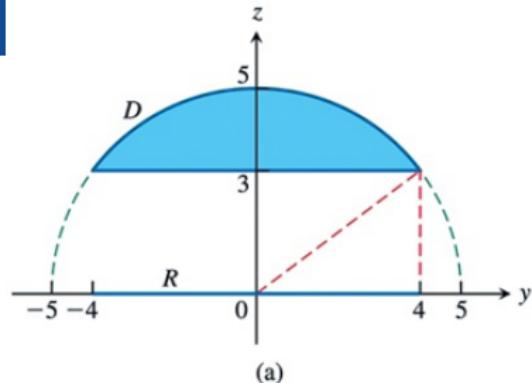
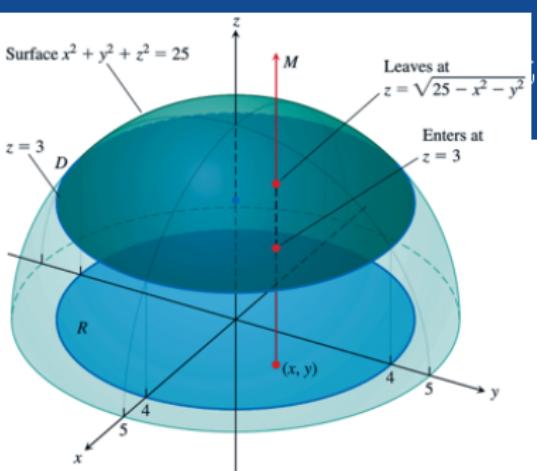


$$z = 3 \implies x^2 + y^2 + 9 = 25 \implies x^2 + y^2 = 16$$

$$-4 \leq x \leq 4 \quad -\sqrt{16 - x^2} \leq y \leq \sqrt{16 - x^2}$$

14.5

Spherical Coordinates

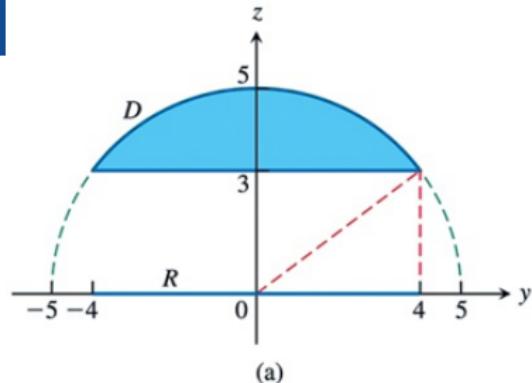
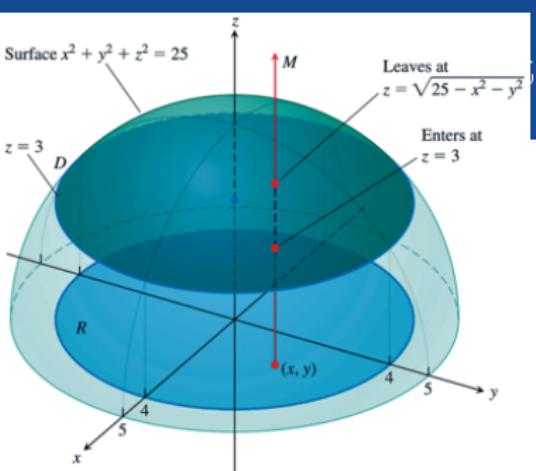


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14.5

Spherical Coordinates



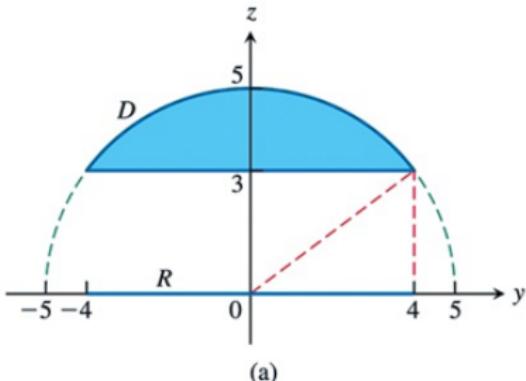
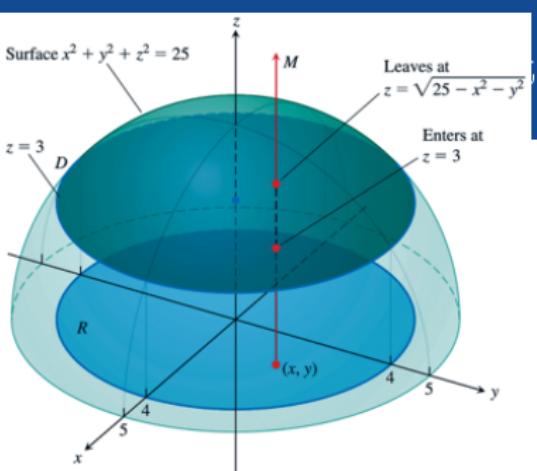
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$$3 \leq z \leq \sqrt{25 - x^2 - y^2}$$

14.5

Spherical Coordinates



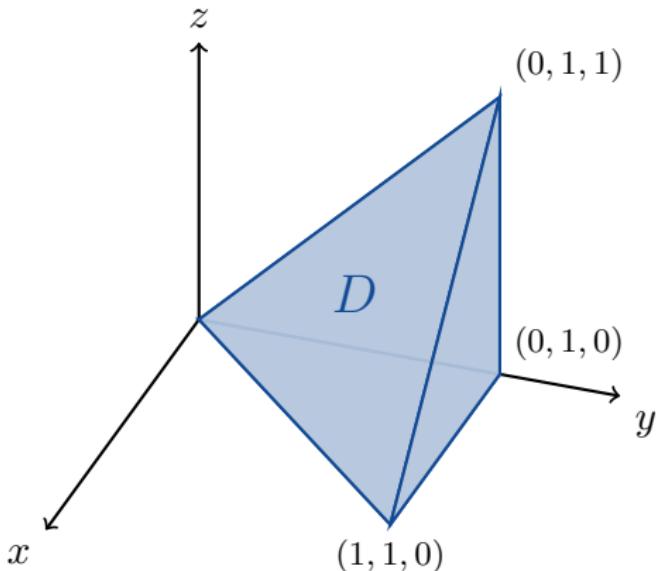
$$-4 \leq x \leq 4$$

$$-\sqrt{16 - x^2} \leq y \leq \sqrt{16 - x^2}$$

$$3 \leq z \leq \sqrt{25 - x^2 - y^2}$$

$$\int_{-4}^4 \int_{-\sqrt{16-x^2}}^{\sqrt{16-x^2}} \int_3^{\sqrt{25-x^2-y^2}} F(x, y, z) dz dy dx.$$

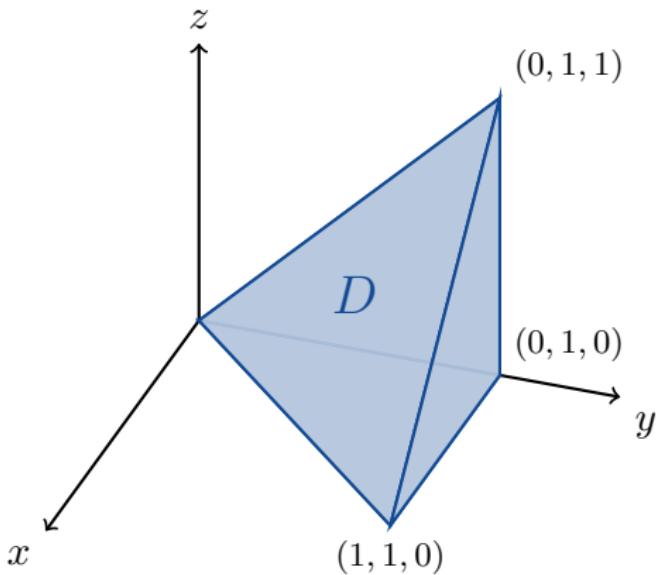
14.5 Triple Integrals in Rectangular Coordinates



Example

Let D be the tetrahedron whose vertices are $(0, 0, 0)$, $(1, 1, 0)$, $(0, 1, 0)$ and $(0, 1, 1)$.

14.5 Triple Integrals in Rectangular Coordinates

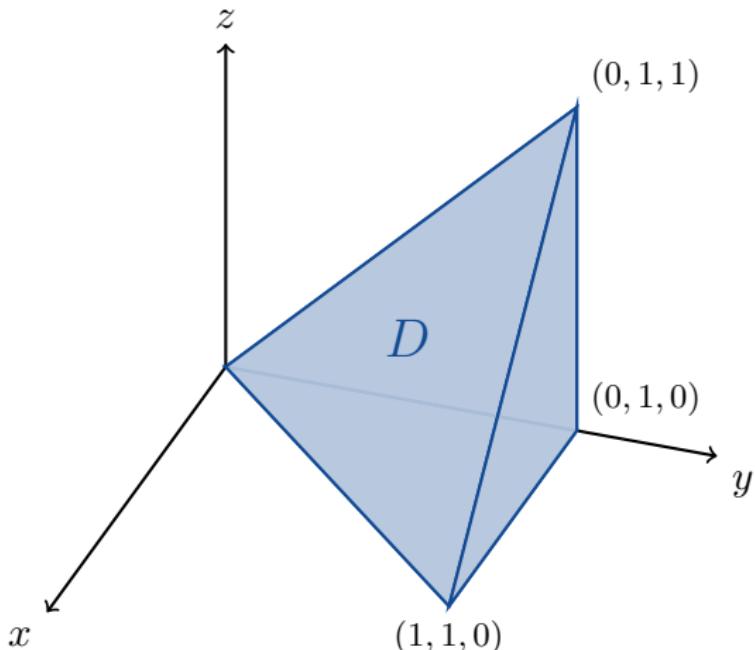


Example

Let D be the tetrahedron whose vertices are $(0,0,0)$, $(1,1,0)$, $(0,1,0)$ and $(0,1,1)$. Set up the limits of integration over D using the order $dxdydz$.

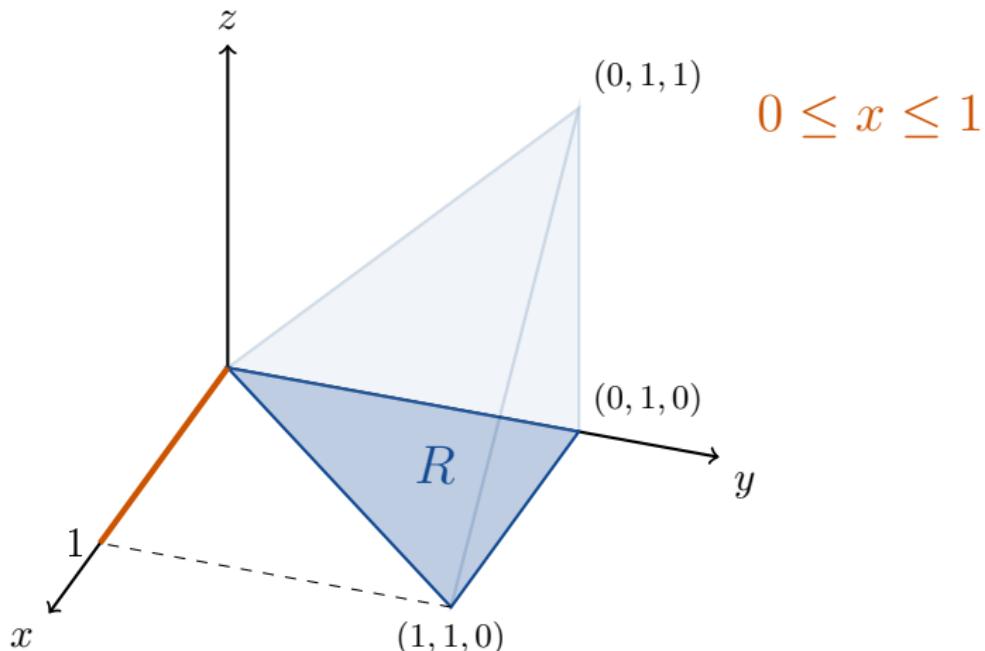
14.5 Triple Integrals in Rectangular Coordinates

$$\iiint_D F(x, y, z) dz dy dx =$$



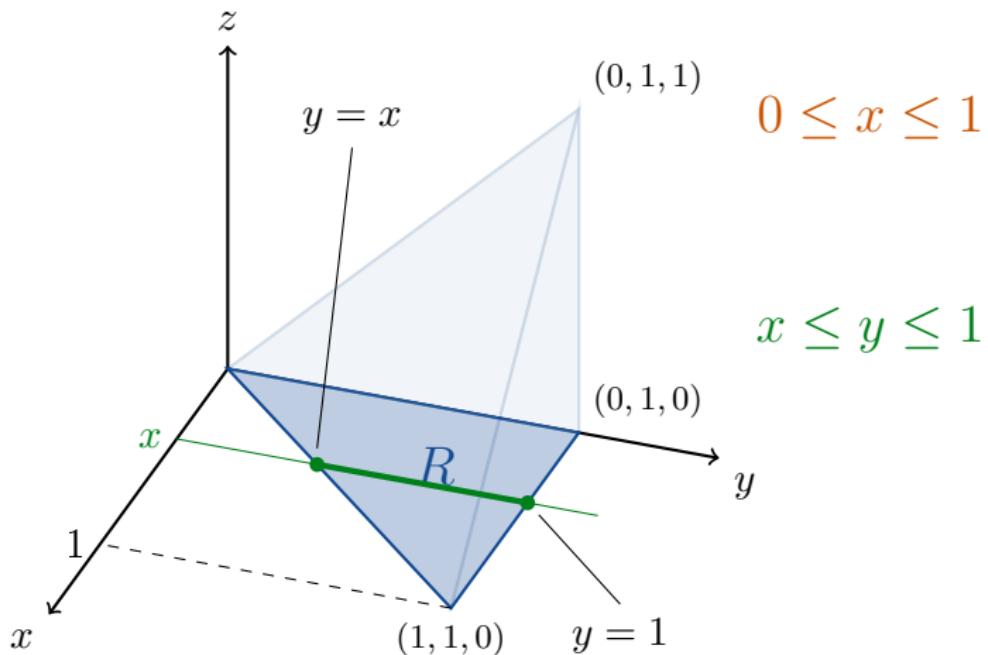
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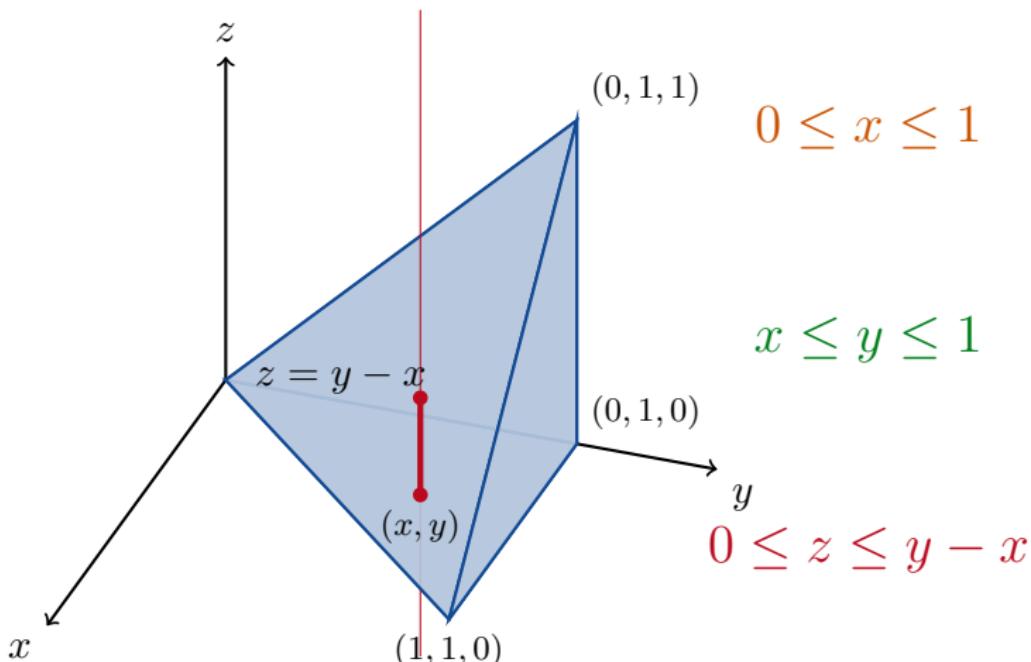
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14.5 Triple Integrals in Rectangular Coordinates



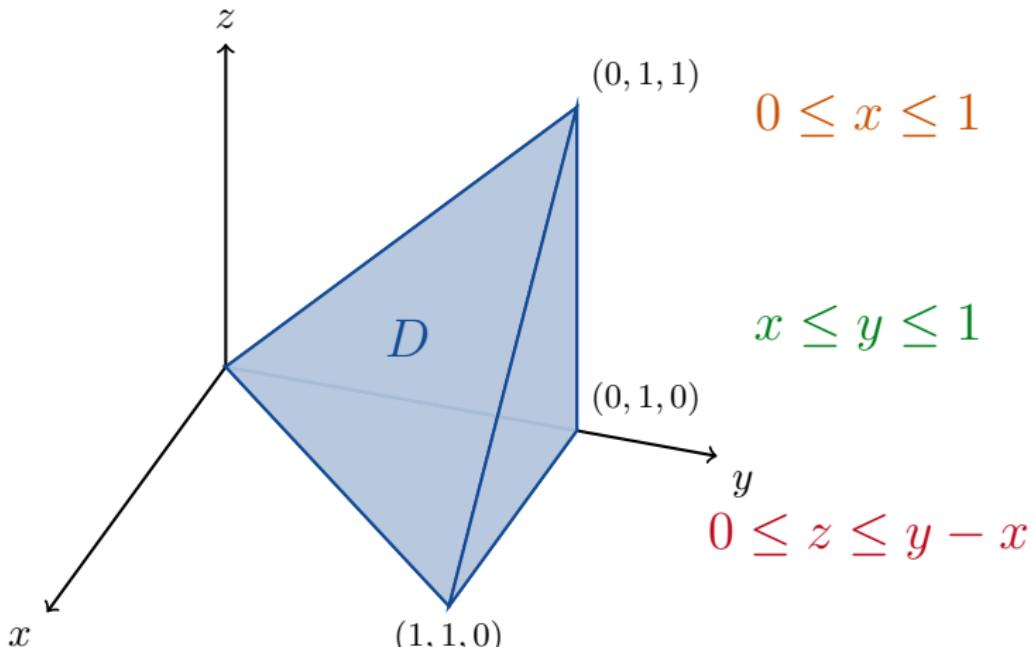
$$\iiint_D F(x, y, z) dz dy dx =$$



14.5 Triple Integrals in Rectangular Coordinates



$$\iiint_D F(x, y, z) dz dy dx = \int_0^1 \int_x^1 \int_0^{y-x} F(x, y, z) dz dy dx.$$

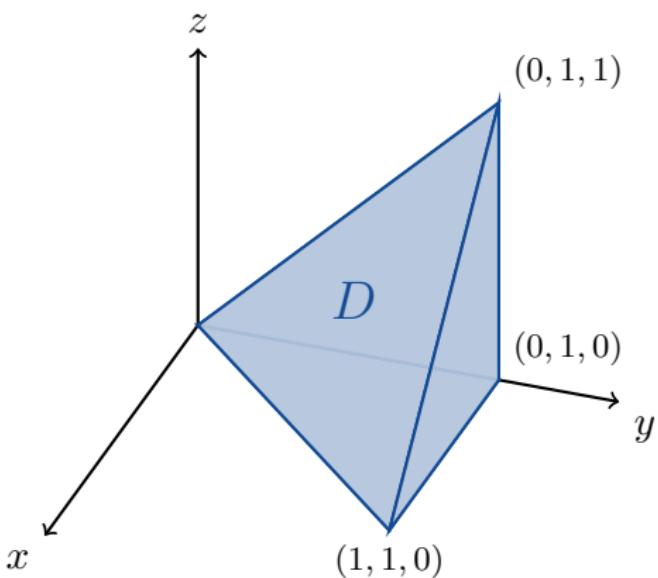


14.5 Triple Integrals in Rectangular Coordinates



Example

Find the volume of this tetrahedron by integrating the function $F(x, y, z) = 1$ over D using the order $dzdydx$.



14.5 Triple Integrals in Rectangular Coordinates



Example

Find the volume of this tetrahedron by integrating the function $F(x, y, z) = 1$ over D using the order $dzdydx$.

$$V = \iiint_D dzdydx = \int_0^1 \int_x^1 \int_0^{y-x} dzdydx$$

14.5 Triple Integrals in Rectangular Coordinates

Example

Find the volume of this tetrahedron by integrating the function $F(x, y, z) = 1$ over D using the order $dzdydx$.

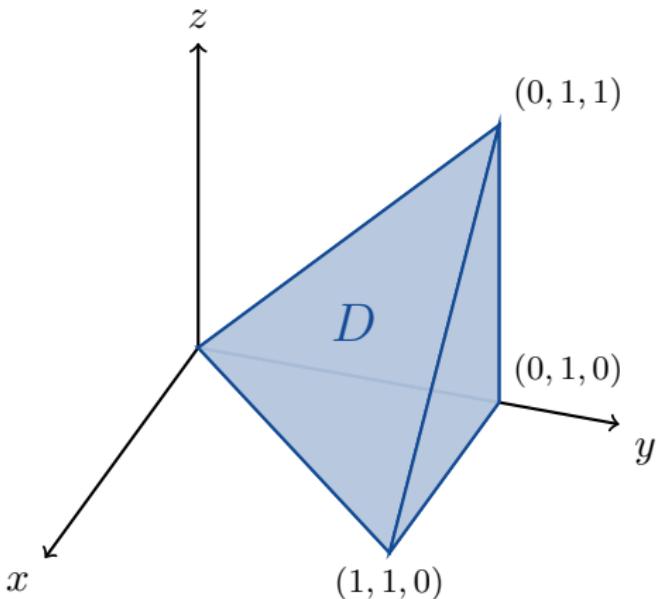
$$\begin{aligned} V &= \iiint_D dz dy dx = \int_0^1 \int_x^1 \int_0^{y-x} dz dy dx \\ &= \int_0^1 \int_x^1 (y - x) dy dx = \int_0^1 \left[\frac{1}{2}y^2 - xy \right]_{y=x}^{y=1} dx \\ &= \int_0^1 \left(\frac{1}{2} - x + \frac{1}{2}x^2 \right) dx = \left[\frac{1}{2}x - \frac{1}{2}x^2 + \frac{1}{6}x^3 \right]_0^1 = \frac{1}{6}. \end{aligned}$$

14.5 Triple Integrals in Rectangular Coordinates



Example

Find the volume of this tetrahedron by integrating the function $F(x, y, z) = 1$ over D using the order $\textcolor{red}{dydzdx}$.

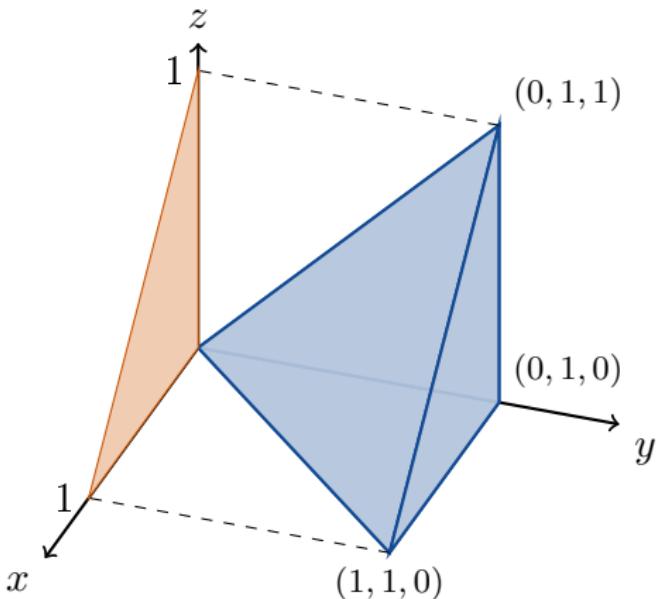


14.5 Triple Integrals in Rectangular Coordinates



Example

Find the volume of this tetrahedron by integrating the function $F(x, y, z) = 1$ over D using the order dydzdx .

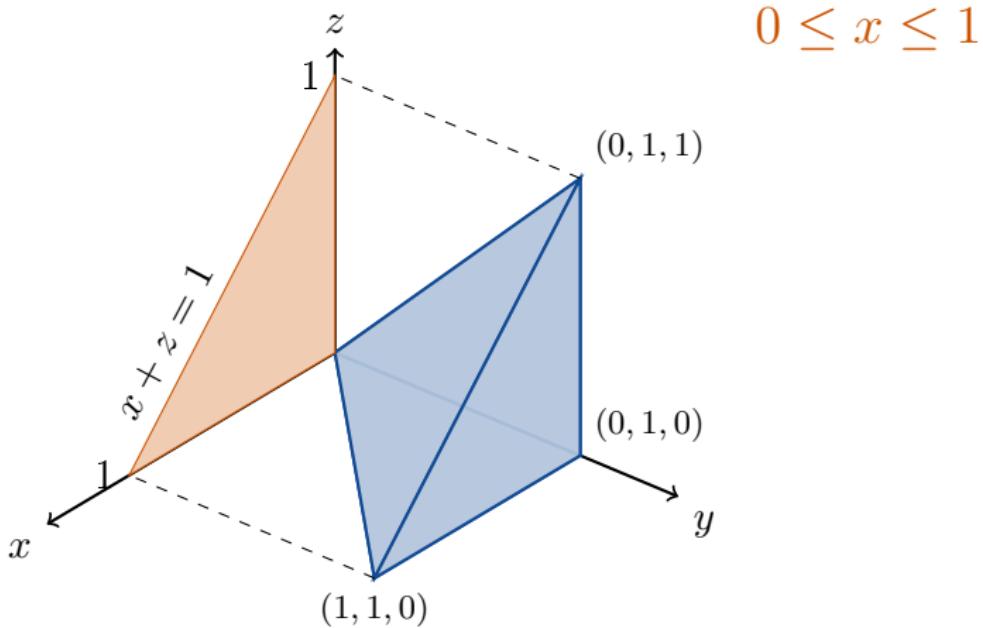


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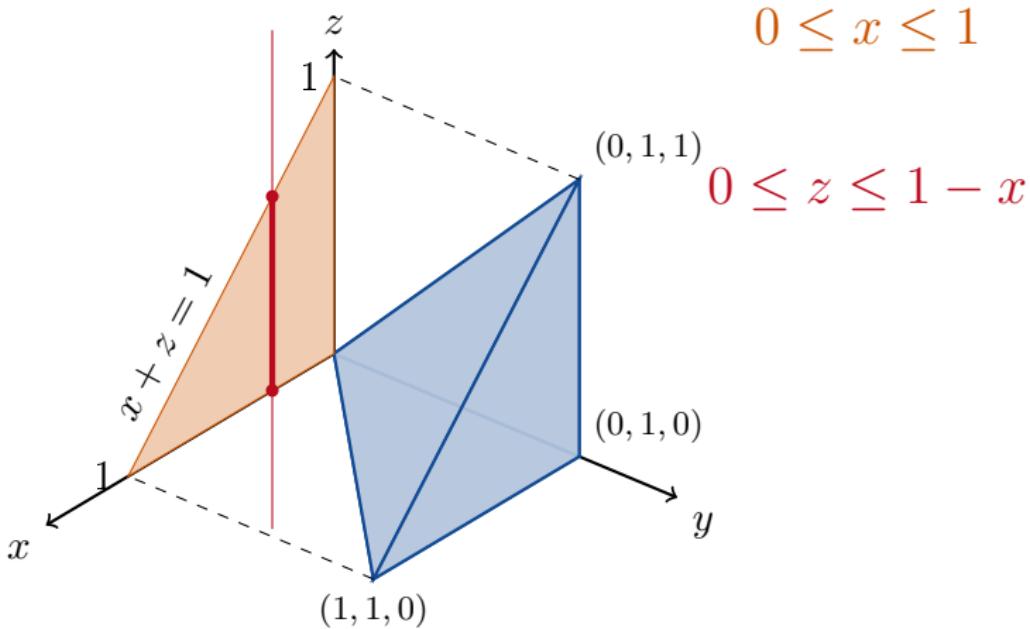


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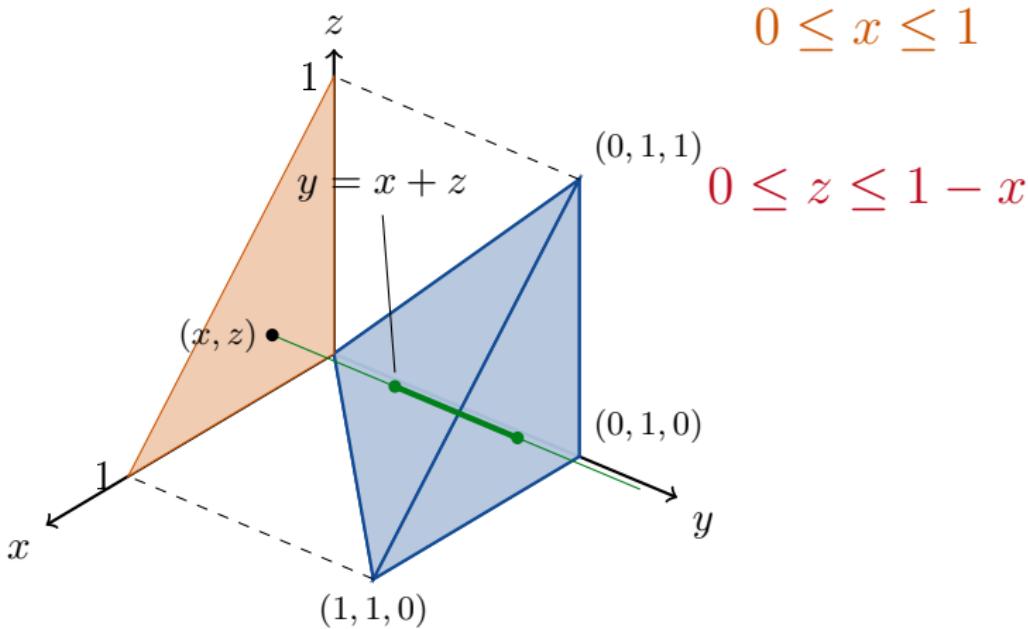


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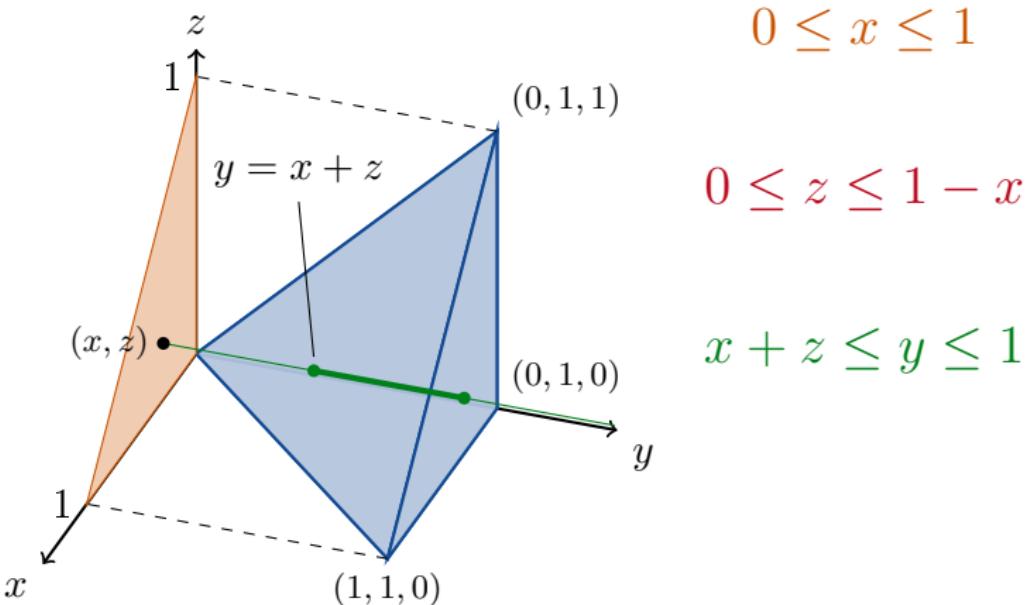


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Find the volume of this tetrahedron by integrating the function $F(x, y, z) = 1$ over D using the order dydzdx .



14.5 Triple Integrals in Rectangular Coordinates



$$0 \leq x \leq 1 \quad 0 \leq z \leq 1 - x \quad x + z \leq y \leq 1$$

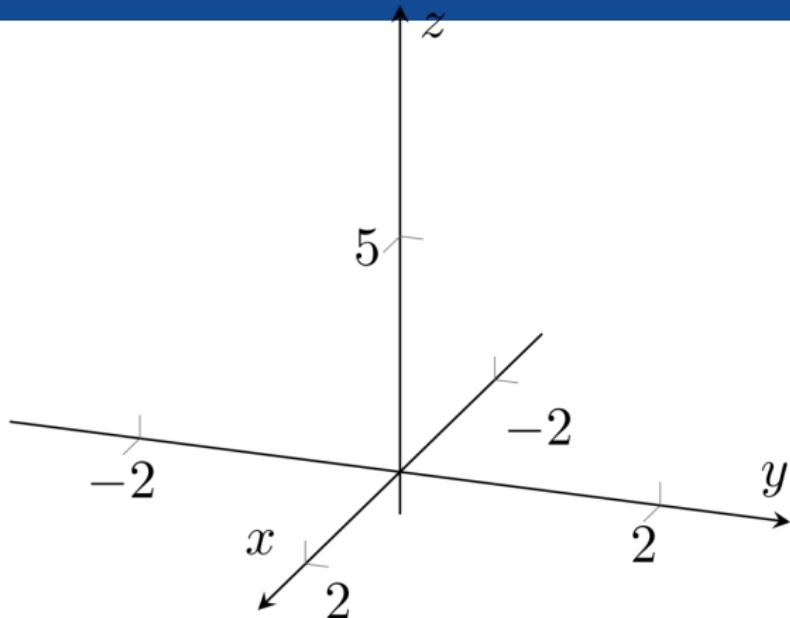
$$V = \iiint_D dz dy dx = \int_0^1 \int_0^{1-x} \int_{x+z}^1 dy dz dx$$

14.5 Triple Integrals in Rectangular Coordinates

$$0 \leq x \leq 1 \quad 0 \leq z \leq 1 - x \quad x + z \leq y \leq 1$$

$$\begin{aligned}
 V &= \iiint_D dz dy dx = \int_0^1 \int_0^{1-x} \int_{x+z}^1 dy dz dx \\
 &= \int_0^1 \int_0^{1-x} (1 - x - z) dz dx = \int_0^1 \left[z - xz - \frac{1}{2}z^2 \right]_{z=0}^{z=1-x} dx \\
 &= \int_0^1 \left(1 - x - x - x^2 - \frac{1}{2}(1-x)^2 \right) dx \\
 &= \frac{1}{2} \int_0^1 (1-x)^2 dx = \frac{1}{2} \left[-\frac{1}{3}(1-x)^3 \right]_0^1 = \frac{1}{6}.
 \end{aligned}$$

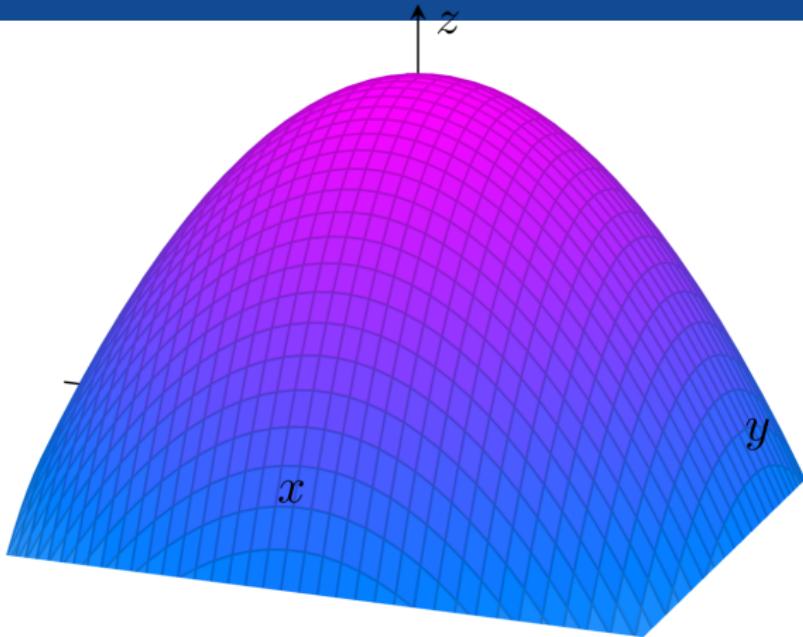
14.5 Triple Integrals in Rectangular Coordinates



Example

Find the volume of the region D enclosed by the surfaces $z = x^2 + 3y^2$ and $z = 8 - x^2 - y^2$.

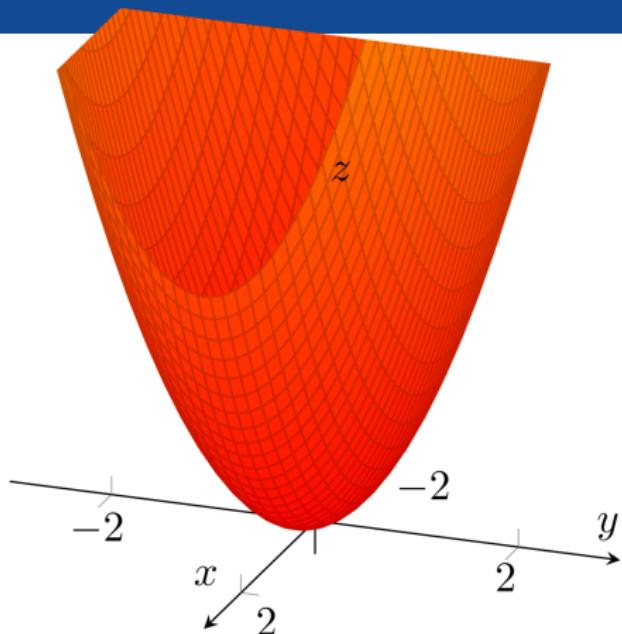
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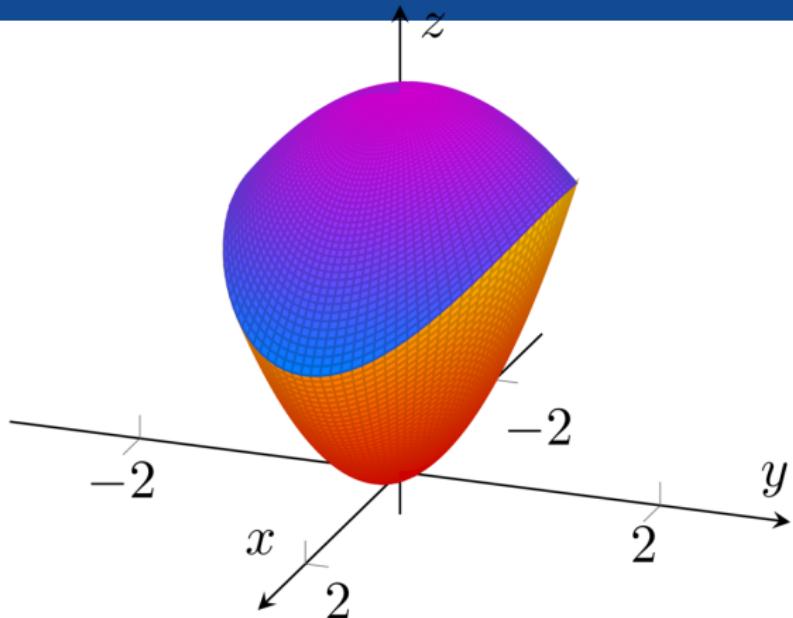
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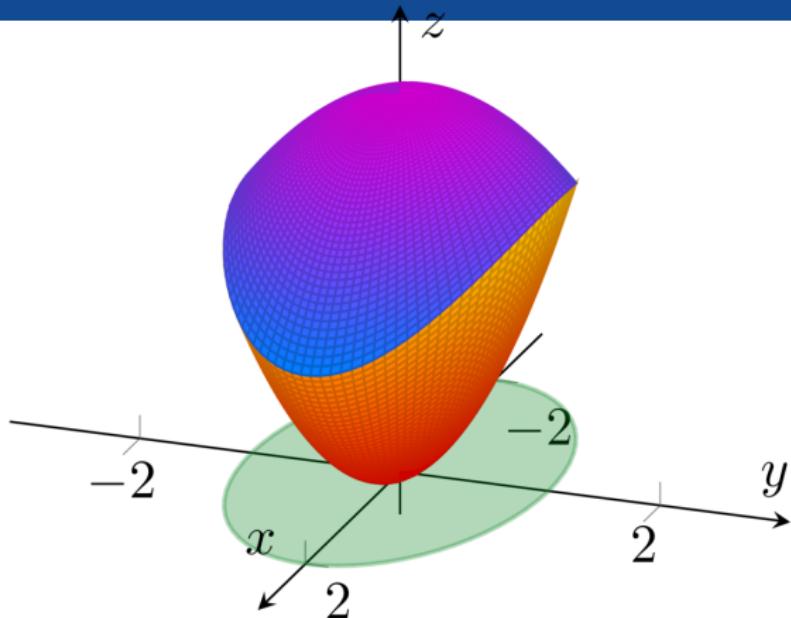
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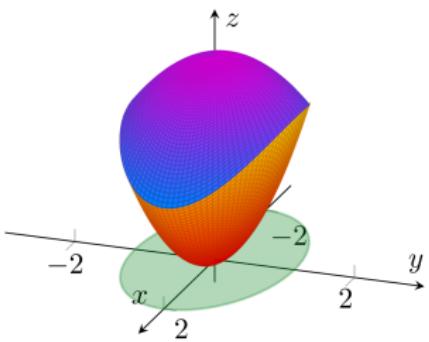
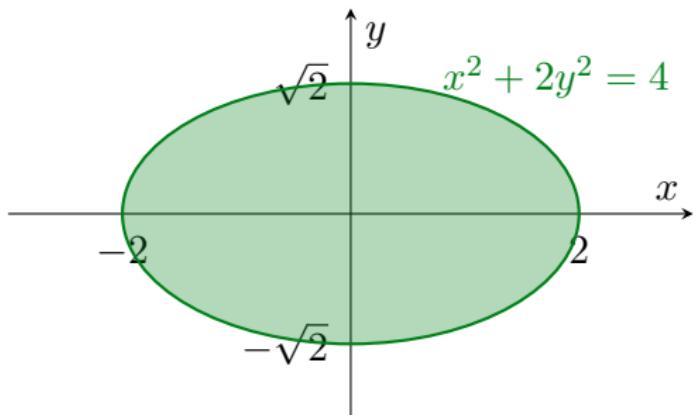
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14.5 Triple Integrals in Rectangular Coordinates

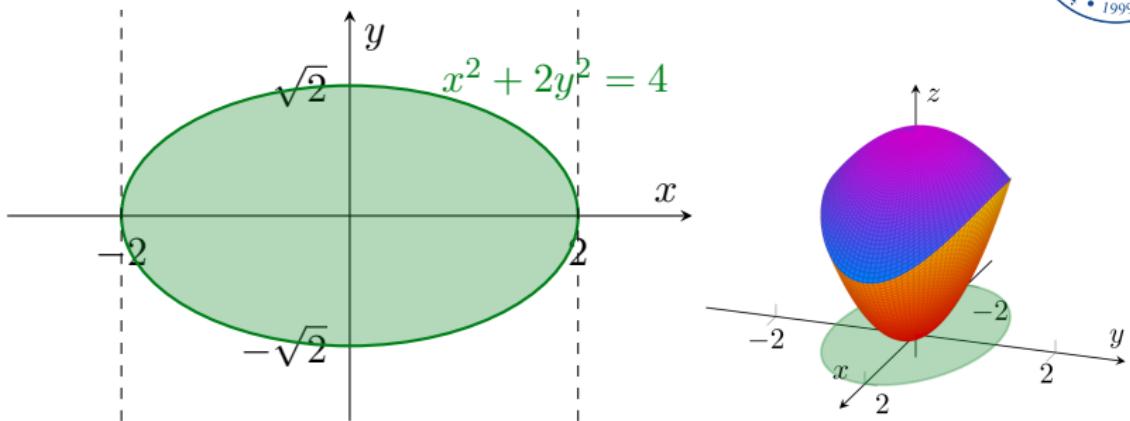


$$x^2 + 3y^2 = z = 8 - x^2 - y^2$$

$$2x^2 + 4y^2 = 8$$

$$x^2 + 2y^2 = 4.$$

14.5 Triple Integrals in Rectangular Coordinates



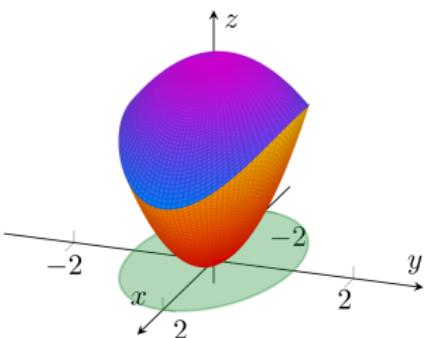
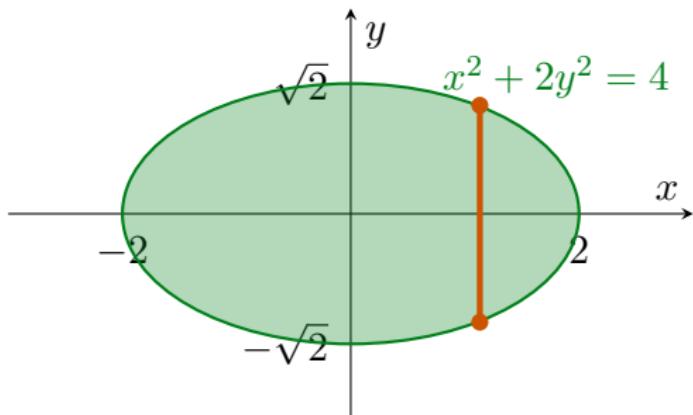
$$-2 \leq x \leq 2$$

$$x^2 + 3y^2 = z = 8 - x^2 - y^2$$

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14.5 Triple Integrals in Rectangular Coordinates



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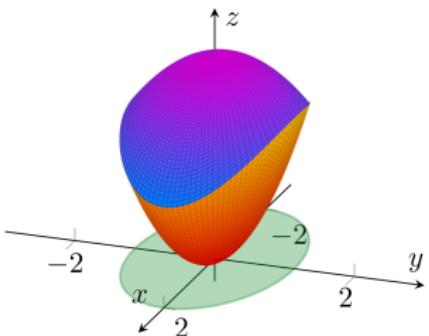
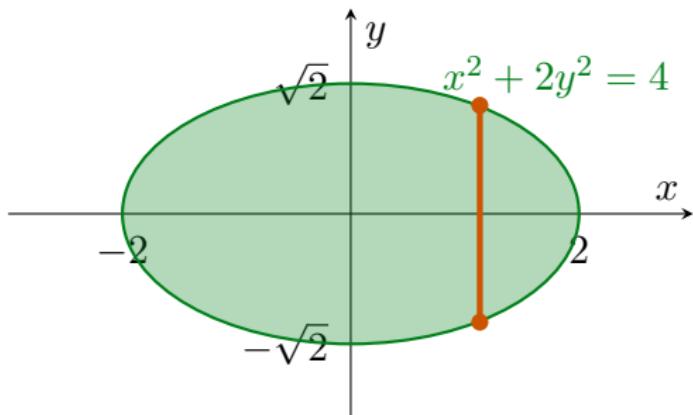
$$x^2 + 3y^2 = z = 8 - x^2 - y^2$$

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$$-\sqrt{\frac{4-x^2}{2}} \leq y \leq \sqrt{\frac{4-x^2}{2}}$$

14.5 Triple Integrals in Rectangular Coordinates



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$$x^2 + 3y^2 = z = 8 - x^2 - y^2$$

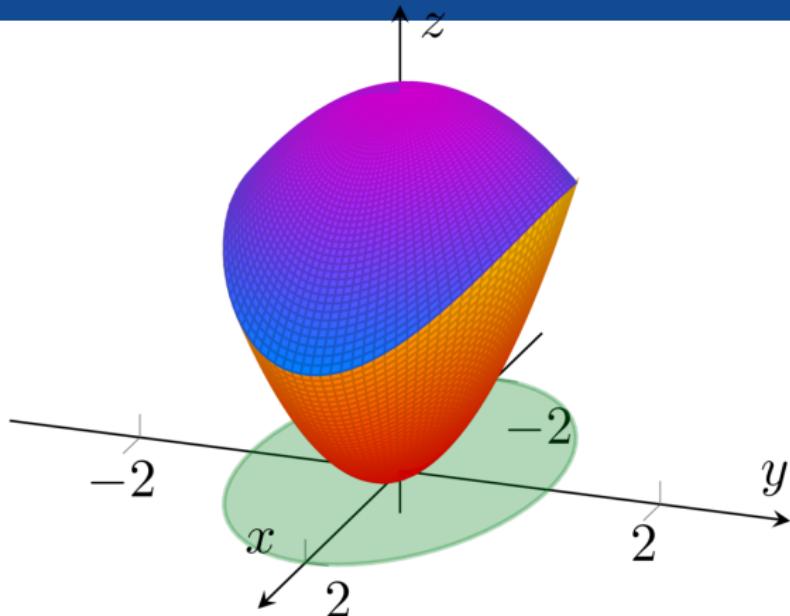
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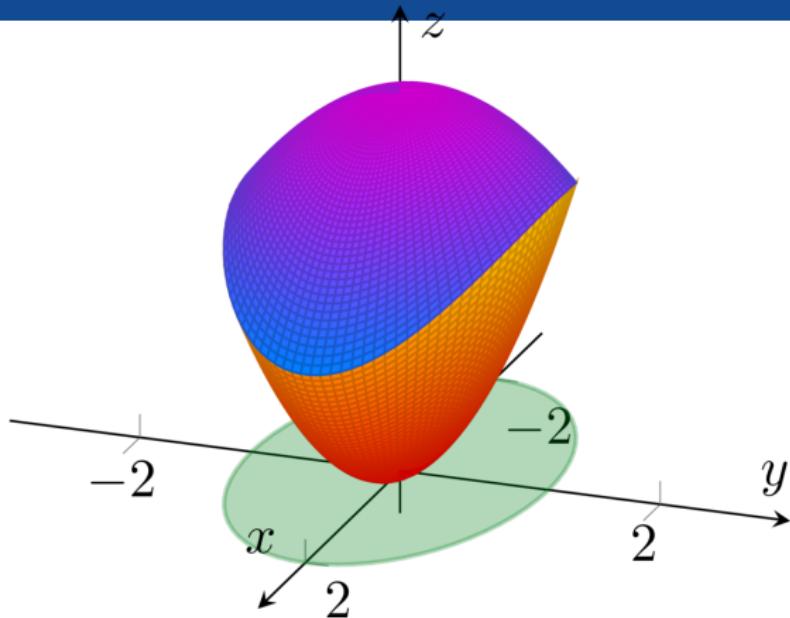
14.5 Triple Integrals in Rectangular Coordinates



The volume of D is

$$V = \int_{-2}^2 \int_{-\sqrt{\frac{4-x^2}{2}}}^{\sqrt{\frac{4-x^2}{2}}} \int_{x^2+3y^2}^{8-x^2-y^2} dz dy dx =$$

14.5 Triple Integrals in Rectangular Coordinates



The volume of D is

$$V = \int_{-2}^2 \int_{-\sqrt{\frac{4-x^2}{2}}}^{\sqrt{\frac{4-x^2}{2}}} \int_{x^2+3y^2}^{8-x^2-y^2} dz dy dx = \dots = 8\pi\sqrt{2}.$$

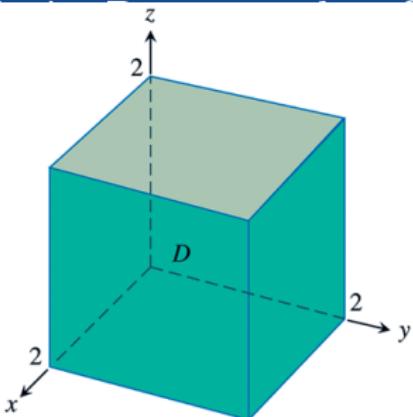


Average Value of a Function

Definition

The *average value* of a function F over a region D is

$$\text{av}(F) = \frac{1}{\text{volume of } D} \iiint_D F \, dV.$$

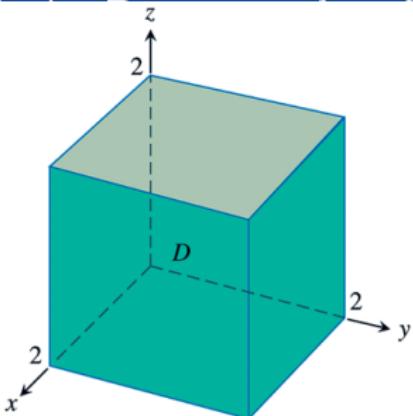


Example

Find the average value of $F(x, y, z) = xyz$ over the cube $[0, 2] \times [0, 2] \times [0, 2]$.

14.5 Triple Integrals

Coordinates



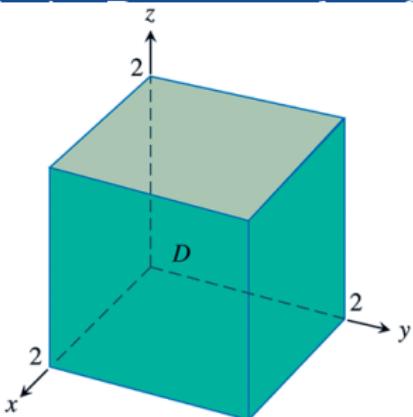
Example

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$$\text{av}(F) = \frac{1}{\text{volume of the cube}} \iiint_{\text{cube}} xyz \, dxdydz$$

14.5 Triple Integrals

Coordinates



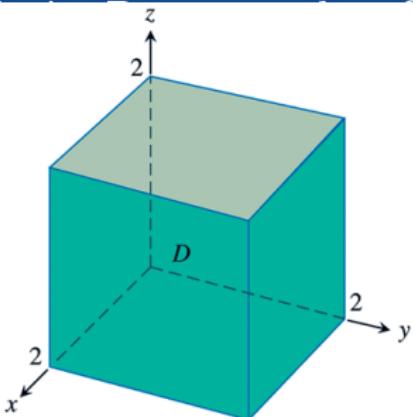
Example

Find the average value of $F(x, y, z) = xyz$ over the cube $[0, 2] \times [0, 2] \times [0, 2]$.

$$\begin{aligned}\text{av}(F) &= \frac{1}{\text{volume of the cube}} \iiint_{\text{cube}} xyz \, dxdydz \\ &= \frac{1}{8} \int_0^2 \int_0^2 \int_0^2 xyz \, dxdydz\end{aligned}$$

14.5 Triple Integrals

Coordinates



Example

Find the average value of $F(x, y, z) = xyz$ over the cube $[0, 2] \times [0, 2] \times [0, 2]$.

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Properties of Triple Integrals



Properties of Triple Integrals

The same as for double integrals.



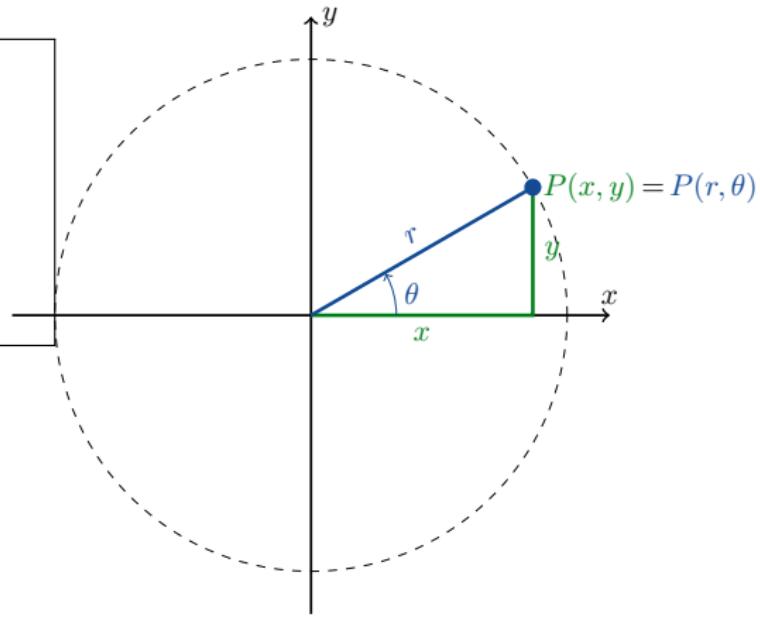
Triple Integrals in Cylindrical and Spherical Coordinates

14.7 Triple Integrals in Cylindrical and Spherical Coordinates



Polar Coordinates in \mathbb{R}^2

$$\begin{array}{ll} x = r \cos \theta & x^2 + y^2 = r^2 \\ y = r \sin \theta & \tan \theta = \frac{y}{x} \end{array}$$



14.7 Triple Integrals in Cylindrical and Spherical Coordinates



Cylindrical Coordinates in \mathbb{R}^3

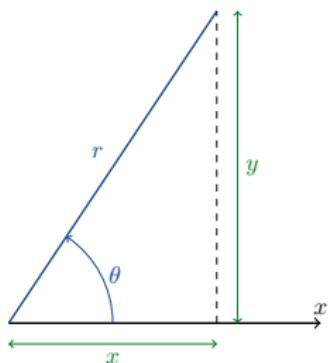
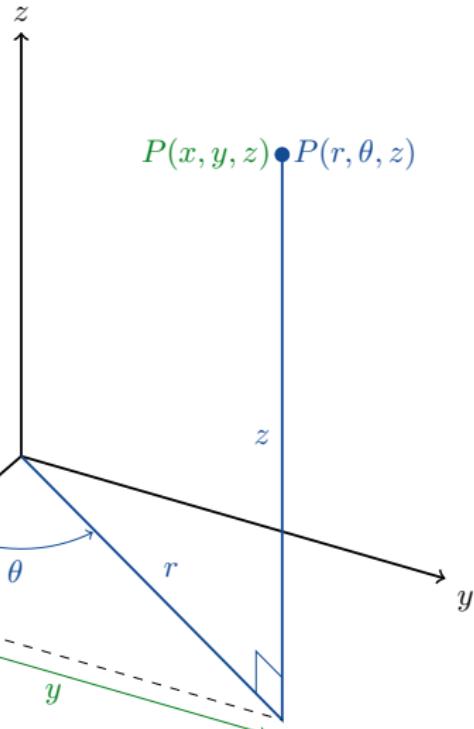
$$x = r \cos \theta$$

$$x^2 + y^2 = r^2$$

$$y = r \sin \theta$$

$$\tan \theta = \frac{y}{x}$$

$$z = z$$



$$x = r \cos \theta \quad y = r \sin \theta \quad z = z$$

Example

Find cylindrical coordinates for the Cartesian coordinates $(x, y, z) = (1, 1, 1)$.

$$x = r \cos \theta \quad y = r \sin \theta \quad z = z$$

Example

Find cylindrical coordinates for the Cartesian coordinates $(x, y, z) = (1, 1, 1)$.

$$\begin{aligned}(r, \theta, z) &= \left(\sqrt{x^2 + y^2}, \tan^{-1} \frac{y}{x}, z \right) \\ &= \left(\sqrt{1^2 + 1^2}, \tan^{-1} 1, 1 \right) = \left(\sqrt{2}, \frac{\pi}{4}, 1 \right).\end{aligned}$$

$$x = r \cos \theta \quad y = r \sin \theta \quad z = z$$

Example

Find cylindrical coordinates for the Cartesian coordinates $(x, y, z) = (1, 1, 1)$.

$$\begin{aligned}(r, \theta, z) &= \left(\sqrt{x^2 + y^2}, \tan^{-1} \frac{y}{x}, z \right) \\ &= \left(\sqrt{1^2 + 1^2}, \tan^{-1} 1, 1 \right) = \left(\sqrt{2}, \frac{\pi}{4}, 1 \right).\end{aligned}$$

Example

Convert the cylindrical coordinates $(r, \theta, z) = \left(2, \frac{\pi}{2}, 2\right)$ to Cartesian coordinates.

$$x = r \cos \theta \quad y = r \sin \theta \quad z = z$$

Example

Find cylindrical coordinates for the Cartesian coordinates $(x, y, z) = (1, 1, 1)$.

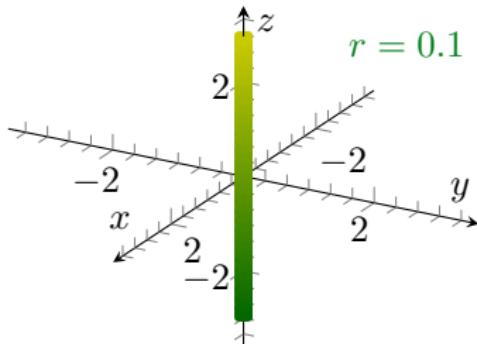
$$\begin{aligned} (r, \theta, z) &= \left(\sqrt{x^2 + y^2}, \tan^{-1} \frac{y}{x}, z \right) \\ &= \left(\sqrt{1^2 + 1^2}, \tan^{-1} 1, 1 \right) = \left(\sqrt{2}, \frac{\pi}{4}, 1 \right). \end{aligned}$$

Example

Convert the cylindrical coordinates $(r, \theta, z) = \left(2, \frac{\pi}{2}, 2\right)$ to Cartesian coordinates.

$$\begin{aligned} (x, y, z) &= (x \cos \theta, y \sin \theta, z) \\ &= \left(2 \cos \frac{\pi}{2}, 2 \sin \frac{\pi}{2}, 2 \right) = (0, 2, 2). \end{aligned}$$

14.7 Triple Integrals in Cylindrical and Spherical Coordinates

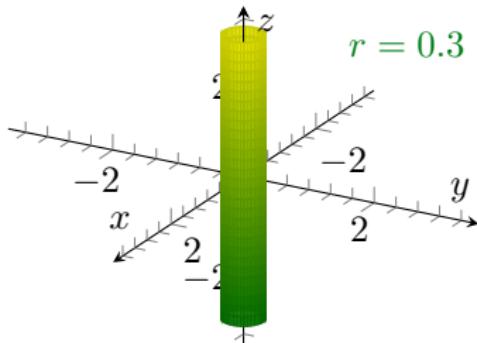


Remark

Cylindrical coordinates are good for describing:

- vertical cylinders with axis on the z -axis ($r = r_0$);
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14.7 Triple Integrals in Cylindrical and Spherical Coordinates

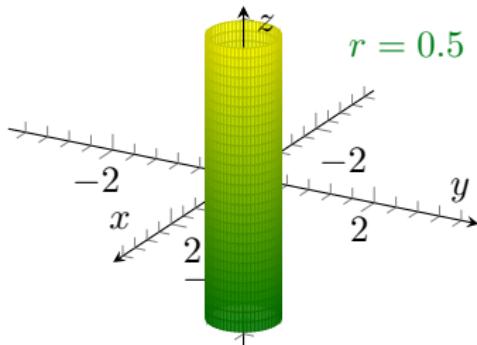


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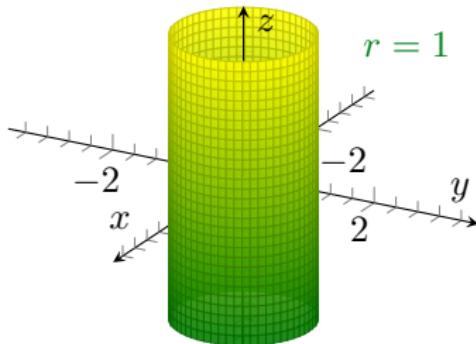


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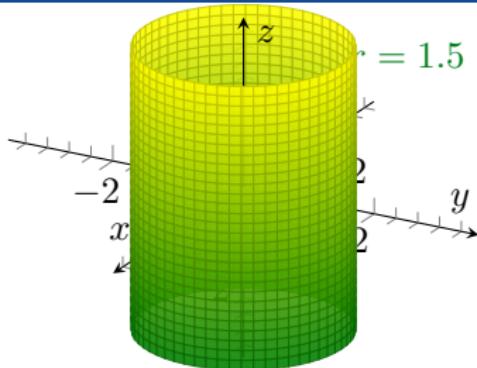


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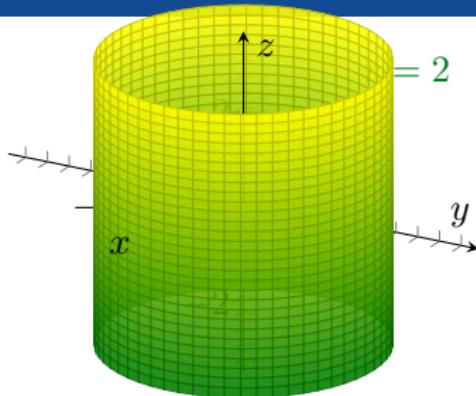


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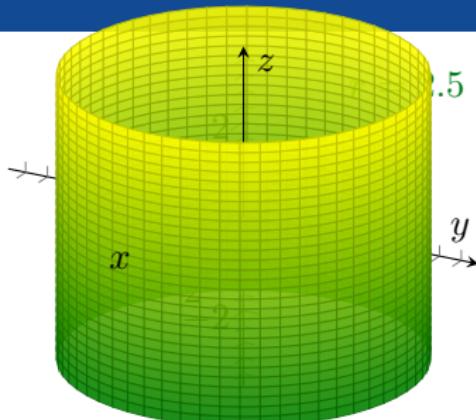


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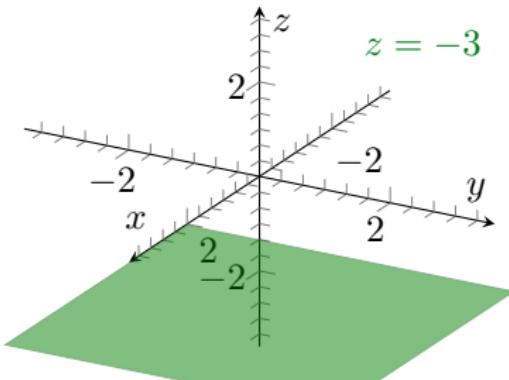


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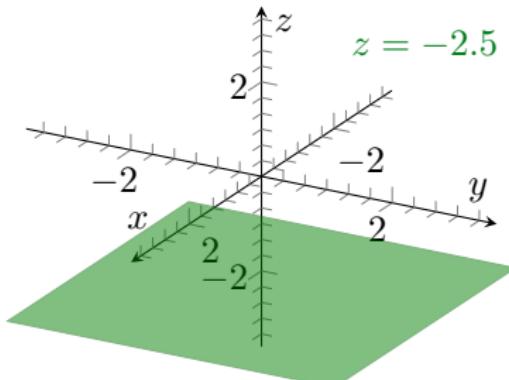


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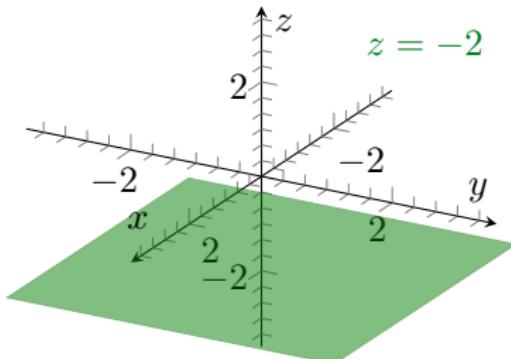


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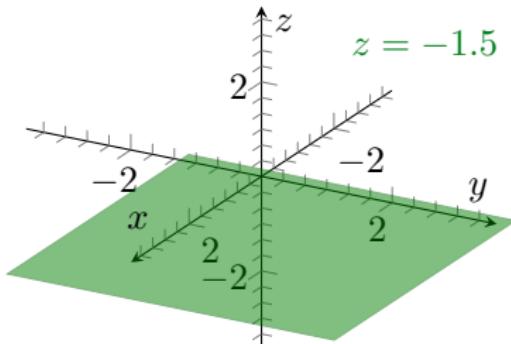


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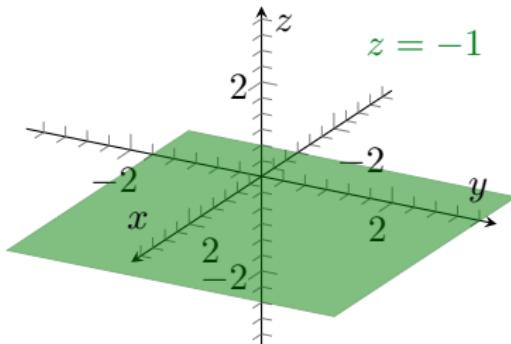


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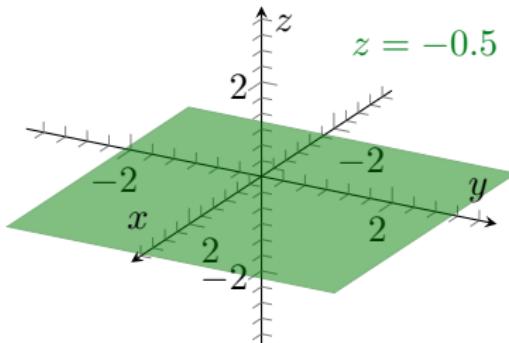


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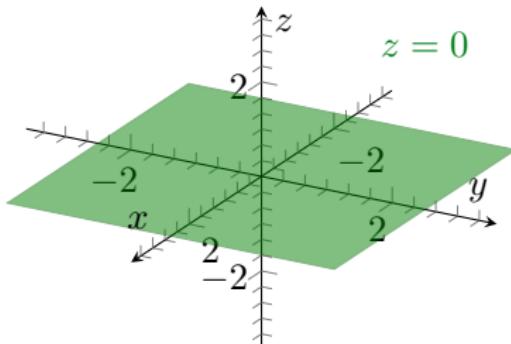


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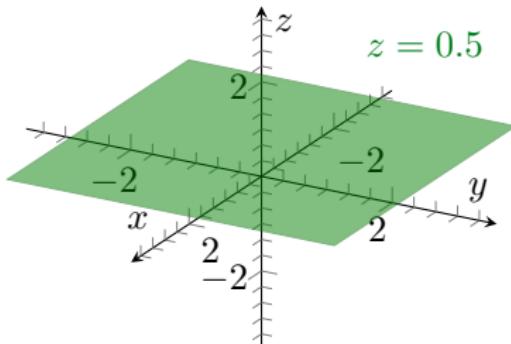


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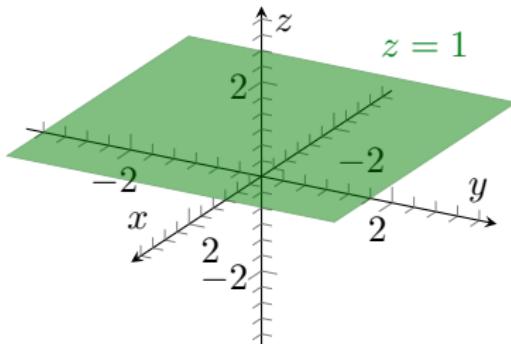


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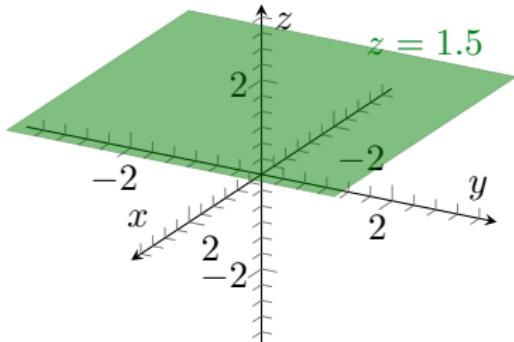


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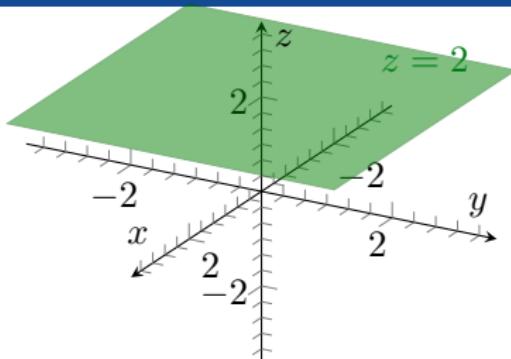


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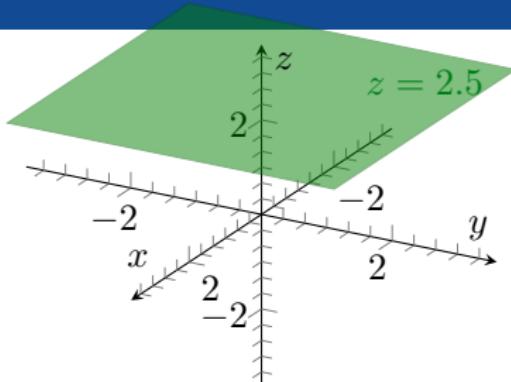


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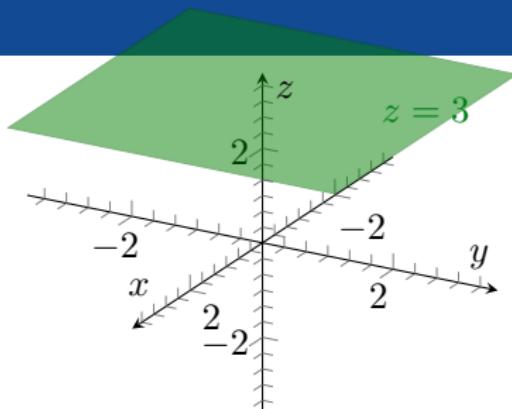


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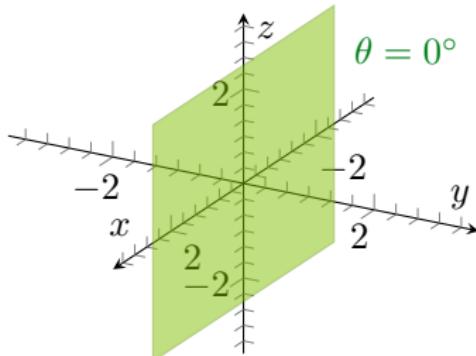


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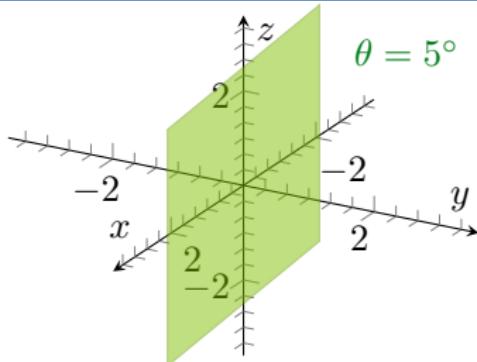


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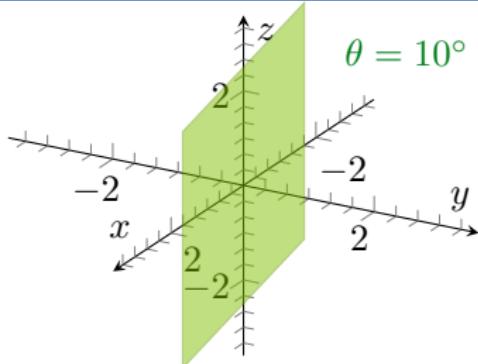


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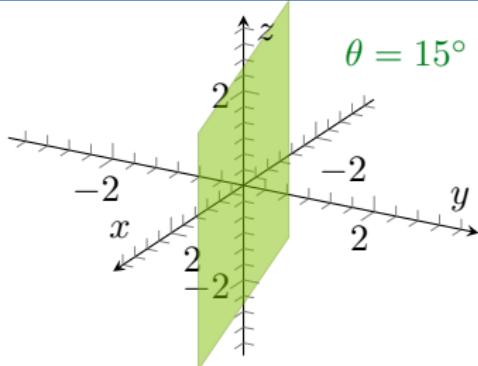


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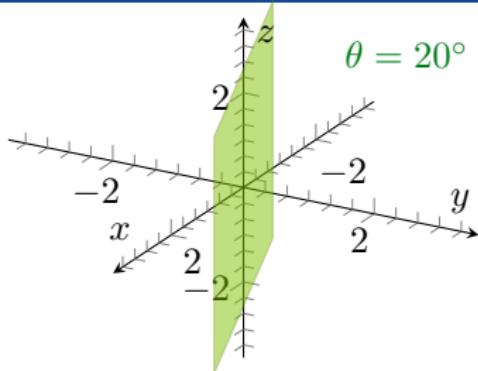


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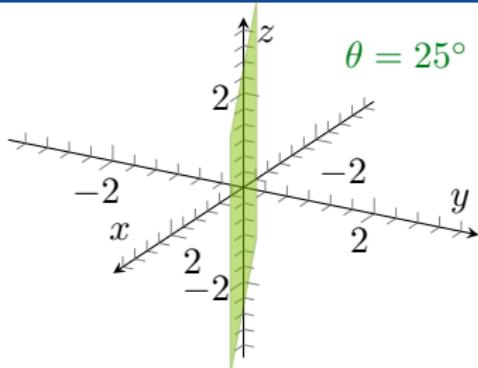


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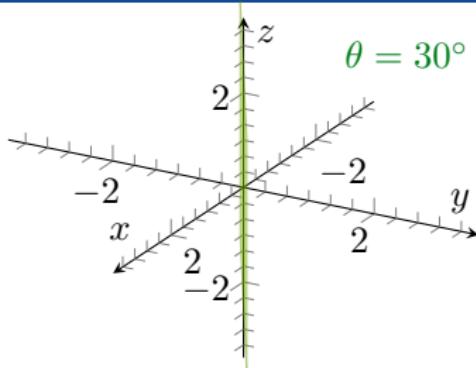


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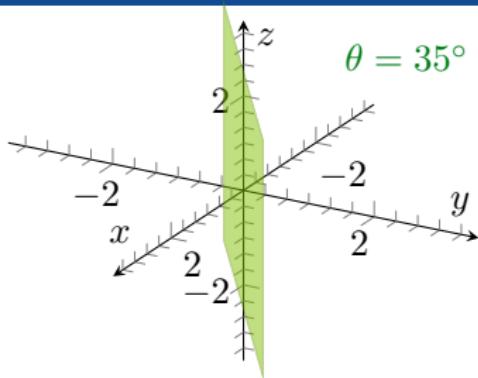


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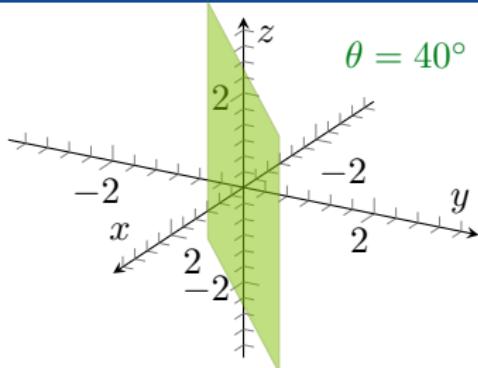


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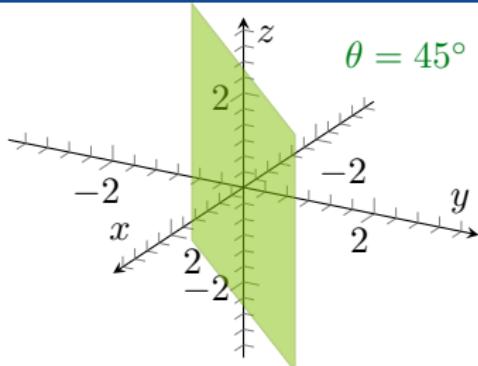


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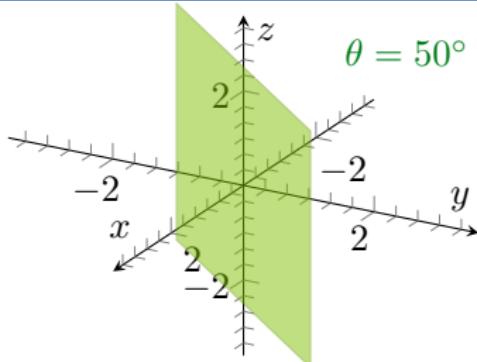


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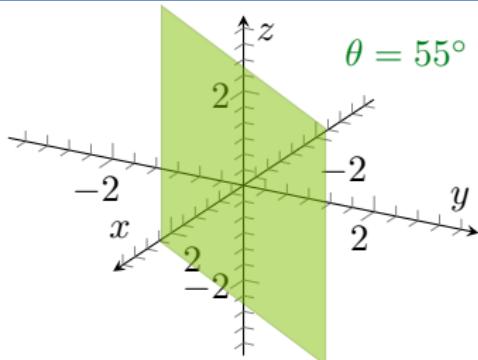


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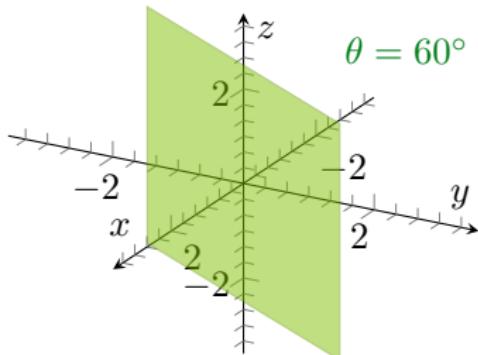


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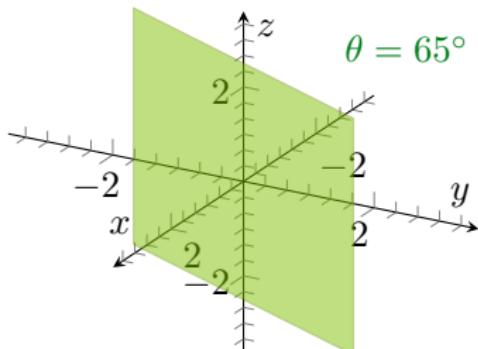


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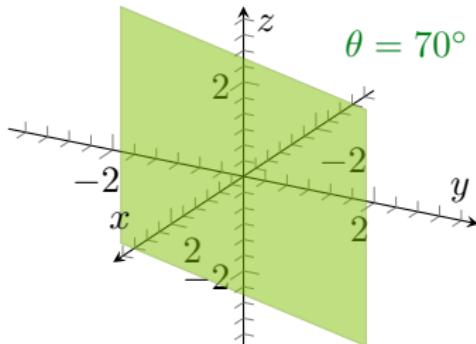


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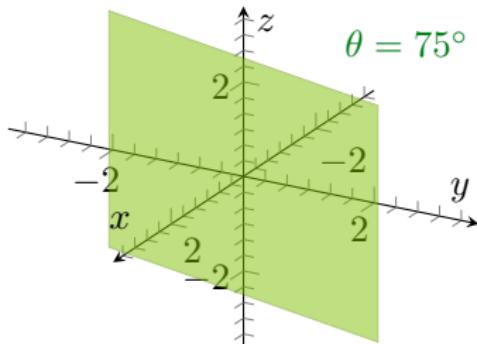


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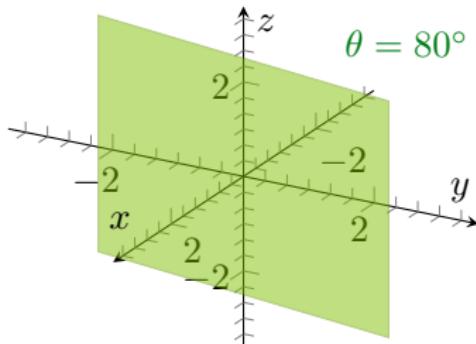


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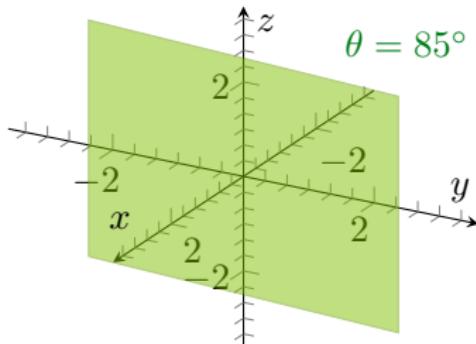


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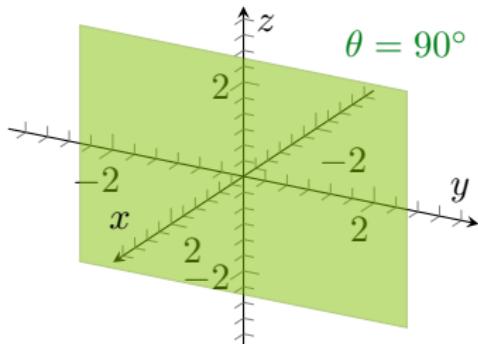


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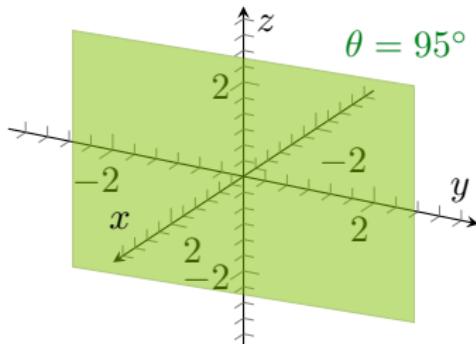


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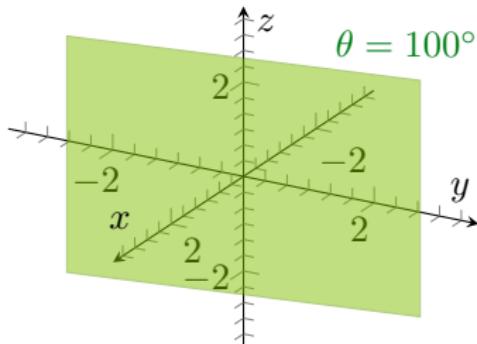


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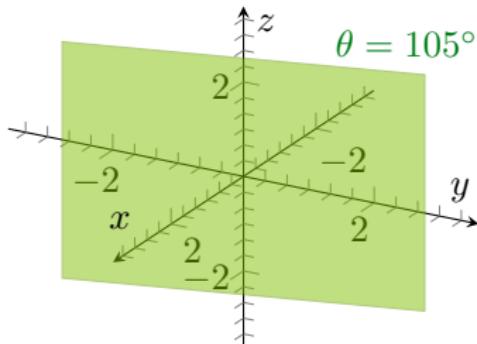


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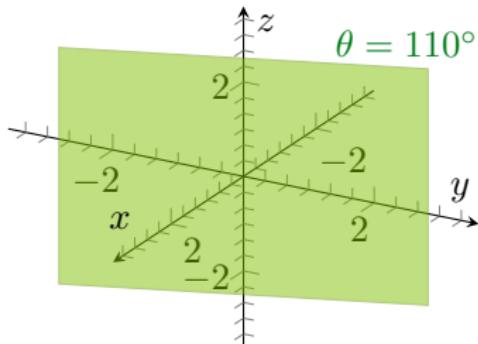


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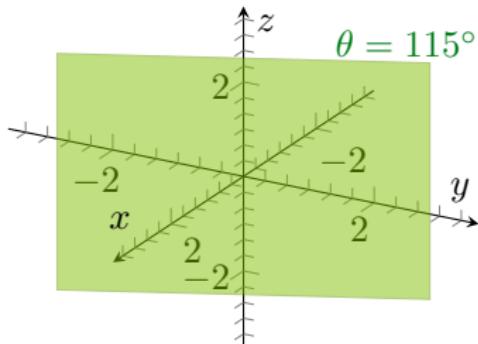


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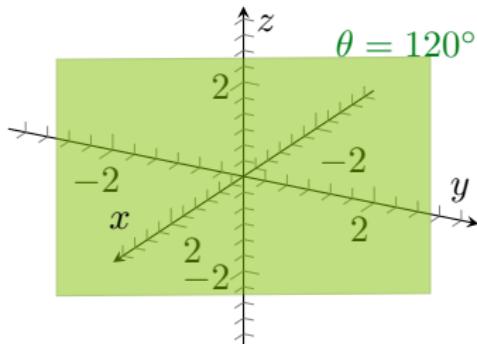


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- planes containing the z -axis ($\theta = \theta_0$).

14.7 Triple Integrals in Cylindrical and Spherical Coordinates

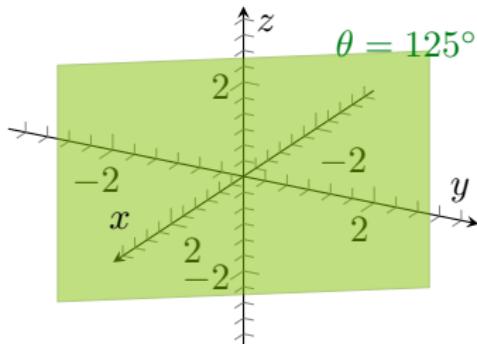


Remark

Cylindrical coordinates are good for describing:

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14.7 Triple Integrals in Cylindrical and Spherical Coordinates

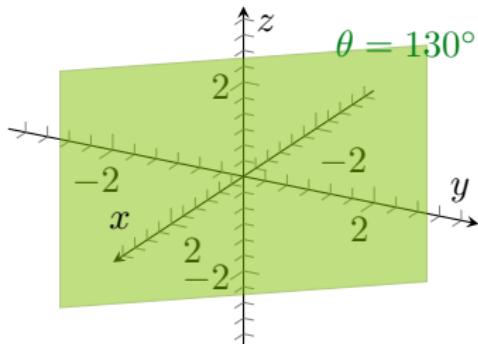


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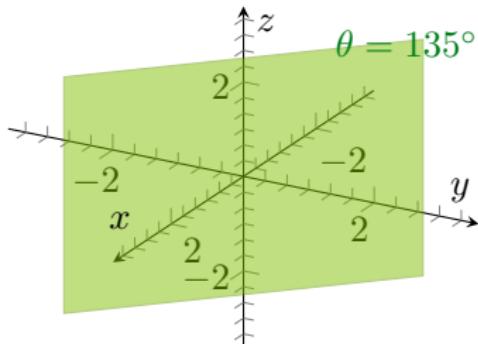


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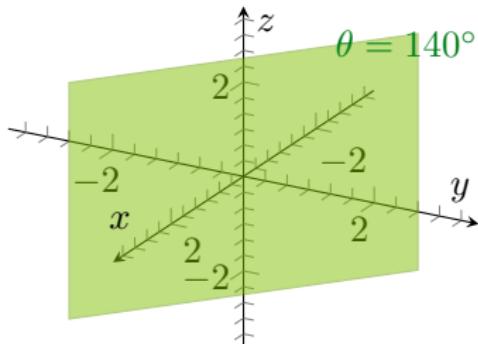


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14.7 Triple Integrals in Cylindrical and Spherical Coordinates

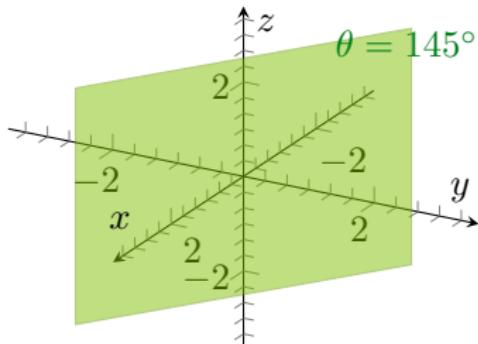


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14.7 Triple Integrals in Cylindrical and Spherical Coordinates

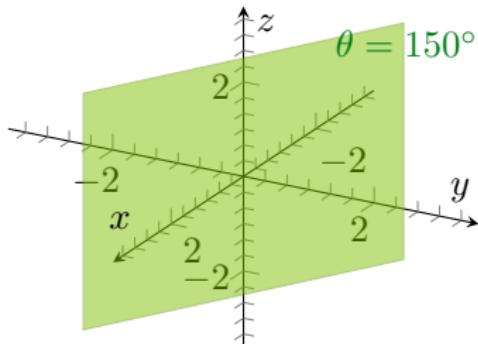


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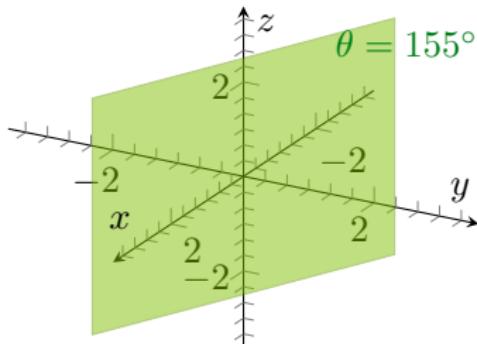


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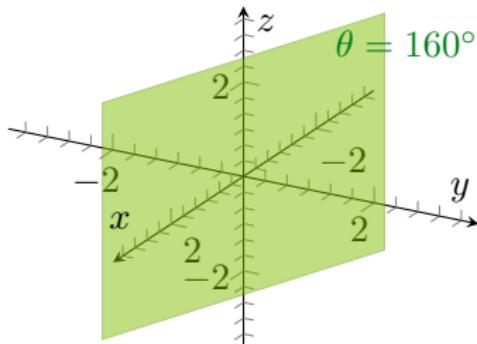


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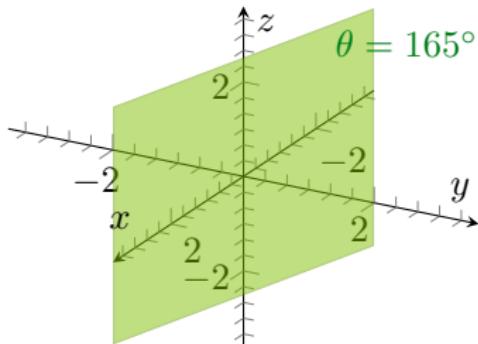


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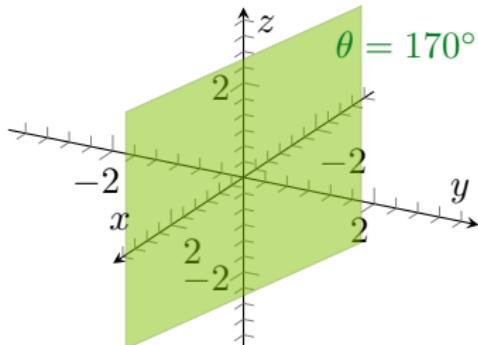


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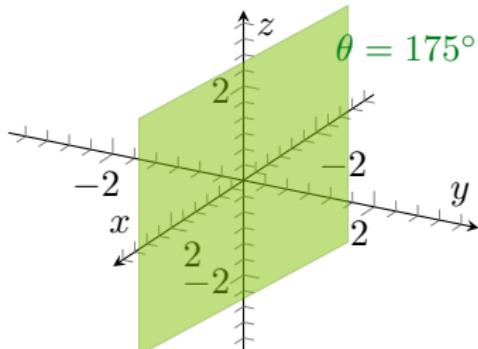


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14.7 Triple Integrals in Cylindrical and Spherical Coordinates



Recall that

$$dA = dx dy = r dr d\theta.$$

14.7 Triple Integrals in Cylindrical and Spherical Coordinates



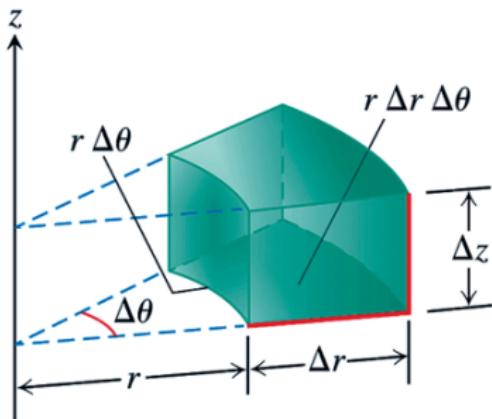
Recall that

$$dA = dx dy = r dr d\theta.$$

Now we have

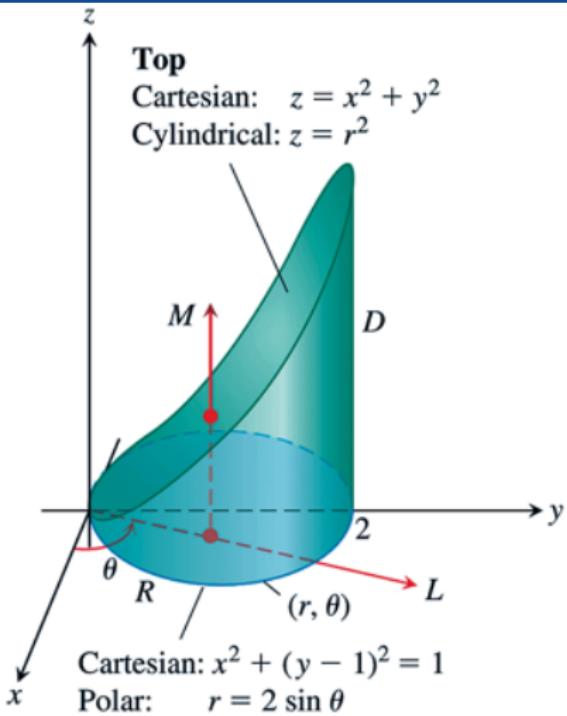
Theorem

$$dV = dx dy dz = r dr d\theta dz.$$



14.7 Triple Integrals in Cylindrical Coordinates

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Example

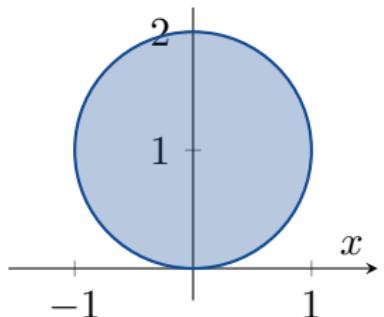
Let D be the region bounded by $z = 0$, $x^2 + (y - 1)^2 = 1$ and $z = x^2 + y^2$. Find the limits of integration in cylindrical coordinates.

14.7

$$x = r \cos \theta \quad y = r \sin \theta \quad z = z$$



$$x^2 + (y - 1)^2 = 1$$



First note that

$$x^2 + (y - 1)^2 = 1$$

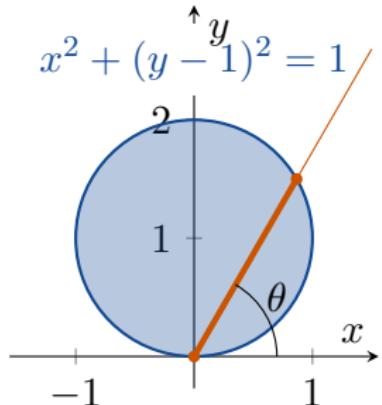
$$x^2 + y^2 - 2y + 1 = 1$$

$$r^2 - 2r \sin \theta = 0$$

$$r = 2 \sin \theta.$$

14.7

$$x = r \cos \theta \quad y = r \sin \theta \quad z = z$$



So

$$0 \leq \theta \leq \pi$$

$$0 \leq r \leq 2 \sin \theta$$

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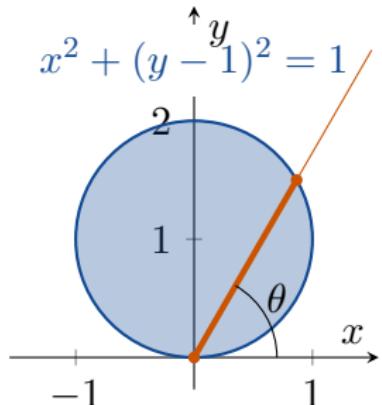
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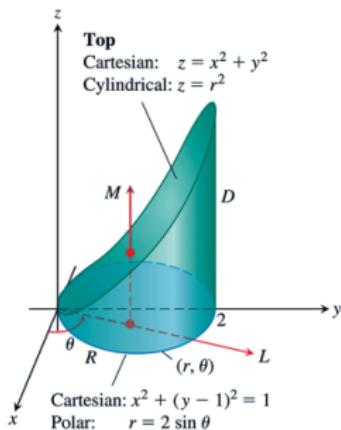
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$$x^2 + y^2 - 2y + 1 = 1$$

$$r^2 - 2r \sin \theta = 0$$

$$r = 2 \sin \theta.$$



$$0 \leq z \leq x^2 + y^2 = r^2$$

14.7

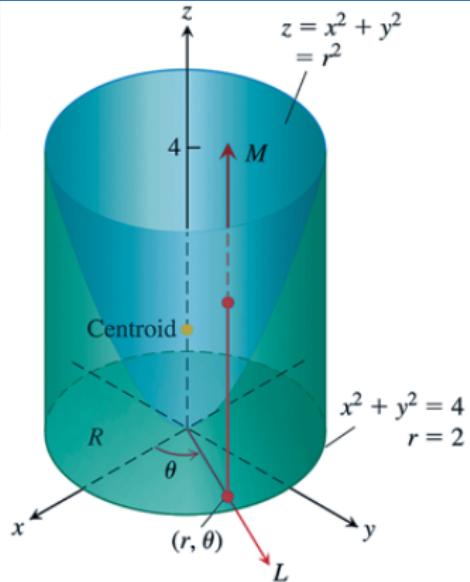
$$dV = dx dy dz = r dr d\theta dz$$



$$0 \leq \theta \leq \pi \quad 0 \leq r \leq 2 \sin \theta \quad 0 \leq z \leq r^2$$

Therefore

$$\iiint_D F(r, \theta, z) dV = \int_0^\pi \int_0^{2 \sin \theta} \int_0^{r^2} F(r, \theta, z) \textcolor{red}{r} dz dr d\theta.$$

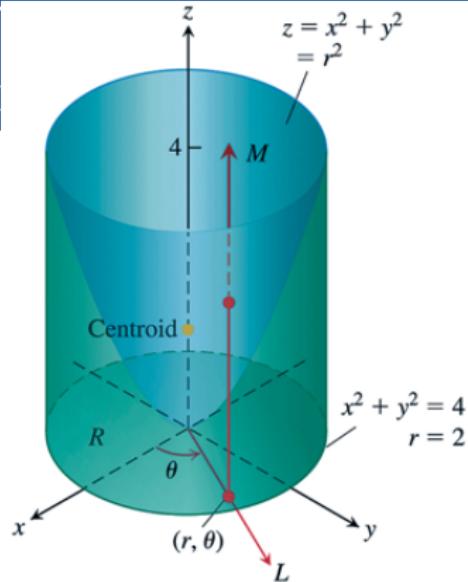


Example

Calculate

$$\iiint_D z \, dV$$

where D is the region enclosed by the cylinder $x^2 + y^2 = 4$, the xy -plane and the paraboloid $z = x^2 + y^2$.



$$0 \leq \theta \leq 2\pi$$

$$0 \leq r \leq 2$$

$$0 \leq z \leq r^2.$$

Example

Calculate

$$\iiint_D z \, dV$$

where D is the region enclosed by the cylinder $x^2 + y^2 = 4$, the xy -plane and the paraboloid $z = x^2 + y^2$.

14.7

$$dV = dx dy dz = r dr d\theta dz$$



$$0 \leq \theta \leq 2\pi \quad 0 \leq r \leq 2 \quad 0 \leq z \leq r^2$$

Therefore

$$\iiint_D z \, dV = \int_0^{2\pi} \int_0^2 \int_0^{r^2} z \, r \, dz \, dr \, d\theta$$

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14.7

$$dV = dx dy dz = r dr d\theta dz$$



$$0 \leq \theta \leq 2\pi \quad 0 \leq r \leq 2 \quad 0 \leq z \leq r^2$$

Therefore

$$\begin{aligned} \iiint_D z \, dV &= \int_0^{2\pi} \int_0^2 \int_0^{r^2} zr \, dz dr d\theta \\ &= \int_0^{2\pi} \int_0^2 \left[\frac{1}{2}z^2 r \right]_{z=0}^{z=r^2} dr d\theta \\ &= \end{aligned}$$

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$$dV = dx dy dz = r dr d\theta dz$$

$$0 \leq \theta \leq 2\pi \quad 0 \leq r \leq 2 \quad 0 \leq z \leq r^2$$

Therefore

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14.7

$$dV = dx dy dz = r dr d\theta dz$$



$$0 \leq \theta \leq 2\pi \quad 0 \leq r \leq 2 \quad 0 \leq z \leq r^2$$

Therefore

$$\begin{aligned}
 \iiint_D z \, dV &= \int_0^{2\pi} \int_0^2 \int_0^{r^2} zr \, dz dr d\theta \\
 &= \int_0^{2\pi} \int_0^2 \left[\frac{1}{2}z^2 r \right]_{z=0}^{z=r^2} dr d\theta \\
 &= \int_0^{2\pi} \int_0^2 \frac{1}{2}r^5 dr d\theta \\
 &= \int_0^{2\pi} \left[\frac{1}{12}r^6 \right]_0^2 d\theta \\
 &= \dots
 \end{aligned}$$

14.7

$$dV = dx dy dz = r dr d\theta dz$$



$$0 \leq \theta \leq 2\pi \quad 0 \leq r \leq 2 \quad 0 \leq z \leq r^2$$

Therefore

$$\begin{aligned}
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 &= \int_0^{2\pi} \int_0^2 \left[\frac{1}{2}z^2 r \right]_{z=0}^{z=r^2} dr d\theta \\
 &= \int_0^{2\pi} \int_0^2 \frac{1}{2}r^5 dr d\theta \\
 &= \int_0^{2\pi} \left[\frac{1}{12}r^6 \right]_0^2 d\theta \\
 &= \int_0^{2\pi} \frac{16}{3} d\theta = .
 \end{aligned}$$

14.7

$$dV = dx dy dz = r dr d\theta dz$$



$$0 \leq \theta \leq 2\pi \quad 0 \leq r \leq 2 \quad 0 \leq z \leq r^2$$

Therefore

$$\begin{aligned} \iiint_D z \, dV &= \int_0^{2\pi} \int_0^2 \int_0^{r^2} zr \, dz dr d\theta \\ &= \int_0^{2\pi} \int_0^2 \left[\frac{1}{2}z^2 r \right]_{z=0}^{z=r^2} dr d\theta \\ &= \int_0^{2\pi} \int_0^2 \frac{1}{2}r^5 dr d\theta \\ &= \int_0^{2\pi} \left[\frac{1}{12}r^6 \right]_0^2 d\theta \\ &= \int_0^{2\pi} \frac{16}{3} d\theta = \frac{32\pi}{3}. \end{aligned}$$

14.7 Triple Integrals in Cylindrical and Spherical Coordinates



Spherical Coordinates in \mathbb{R}^3

$$x = r \cos \theta = \rho \sin \phi \cos \theta$$

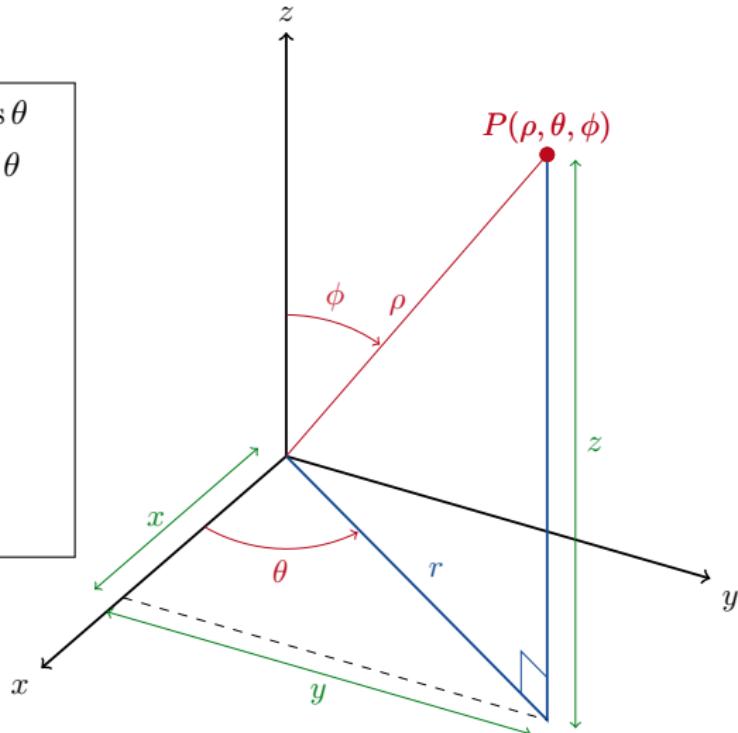
$$y = r \sin \theta = \rho \sin \phi \sin \theta$$

$$z = \rho \cos \phi$$

$$r = \rho \sin \phi$$

$$\tan \theta = \frac{y}{x}$$

$$\begin{aligned}\rho &= \sqrt{x^2 + y^2 + z^2} \\ &= \sqrt{r^2 + z^2}\end{aligned}$$



14.7 Triple Integrals in Cylindrical and Spherical Coordinates



Spherical Coordinates in \mathbb{R}^3

$$x = r \cos \theta = \rho \sin \phi \cos \theta$$

$$y = r \sin \theta = \rho \sin \phi \sin \theta$$

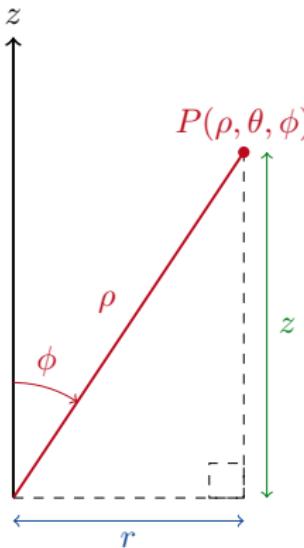
$$z = \rho \cos \phi$$

$$r = \rho \sin \phi$$

$$\tan \theta = \frac{y}{x}$$

$$\rho = \sqrt{x^2 + y^2 + z^2}$$

$$= \sqrt{r^2 + z^2}$$



Typically, we require that $\rho \geq 0$ and $0 \leq \phi \leq \pi$. As before, θ can be any number.

14.7

$$\rho = \sqrt{r^2 + z^2} \quad \theta = \theta \quad z = \rho \cos \phi$$



Example

Convert the point $P(\sqrt{6}, \frac{\pi}{4}, \sqrt{2})$ from cylindrical to spherical coordinates.

14.7

$$\rho = \sqrt{r^2 + z^2} \quad \theta = \theta \quad z = \rho \cos \phi$$



Example

Convert the point $P(\sqrt{6}, \frac{\pi}{4}, \sqrt{2})$ from cylindrical to spherical coordinates.

We have that $r = \sqrt{6}$, $\theta = \frac{\pi}{4}$ and $z = \sqrt{2}$. Therefore

$$\begin{aligned} (\rho, \theta, \phi) &= \left(\sqrt{r^2 + z^2}, \theta, \cos^{-1} \frac{z}{\rho} \right) \\ &= \left(\sqrt{6+2}, \frac{\pi}{4}, \cos^{-1} \frac{\sqrt{2}}{\rho} \right) \\ &= \left(2\sqrt{2}, \frac{\pi}{4}, \cos^{-1} \frac{\sqrt{2}}{2\sqrt{2}} \right) \\ &= \left(2\sqrt{2}, \frac{\pi}{4}, \cos^{-1} \frac{1}{2} \right) = \left(2\sqrt{2}, \frac{\pi}{4}, \frac{\pi}{3} \right). \end{aligned}$$

14.7

$$\rho = \sqrt{x^2 + y^2 + z^2} \quad y = \rho \sin \phi \sin \theta \quad z = \rho \cos \phi$$



Example

Convert the point $P(-1, 1, -\sqrt{2})$ from Cartesian to spherical polar coordinates.

$$\rho = \sqrt{x^2 + y^2 + z^2} \quad y = \rho \sin \phi \sin \theta \quad z = \rho \cos \phi$$

Example

Convert the point $P(-1, 1, -\sqrt{2})$ from Cartesian to spherical polar coordinates.

First we calculate that

$$\rho = \sqrt{x^2 + y^2 + z^2} = \sqrt{(-1)^2 + 1^2 + (-\sqrt{2})^2} = \sqrt{4} = 2.$$

Next we calculate that

$$\phi = \cos^{-1} \frac{z}{\rho} = \cos^{-1} \frac{-\sqrt{2}}{2} = \frac{3\pi}{4}$$

because we want $\phi \in [0, \pi]$.

14.7

$$\rho = \sqrt{x^2 + y^2 + z^2} \quad y = \rho \sin \phi \sin \theta \quad z = \rho \cos \phi$$



Finally we need a θ .

$$\sin \theta = \frac{y}{\rho \sin \phi} = \frac{1}{2 \left(\frac{\sqrt{2}}{2} \right)} = \frac{1}{\sqrt{2}}.$$

There are infinitely many θ that satisfy this equation. Two possible θ are $\theta = \frac{\pi}{4}$ and $\theta = \frac{3\pi}{4}$.

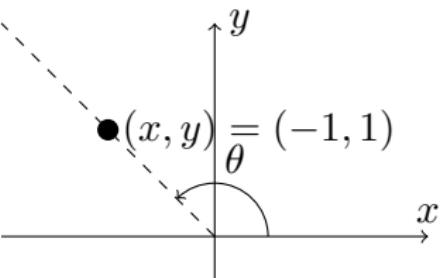
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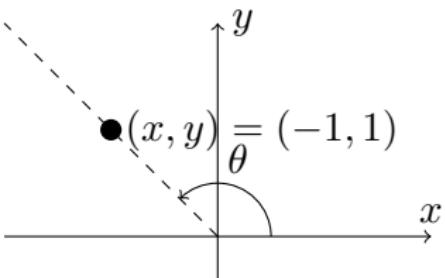
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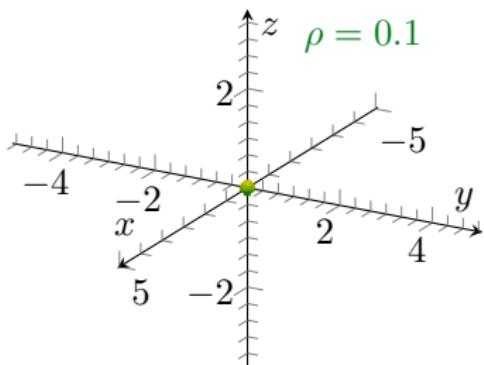
There are infinitely many θ that satisfy this equation. Two possible θ are $\theta = \frac{\pi}{4}$ and $\theta = \frac{3\pi}{4}$. Only one of these can be correct.



Therefore the answer is

$$(\rho, \theta, \phi) = \left(2, \frac{3\pi}{4}, \frac{3\pi}{4} \right).$$

14.7 Triple Integrals in Cylindrical and Spherical Coordinates

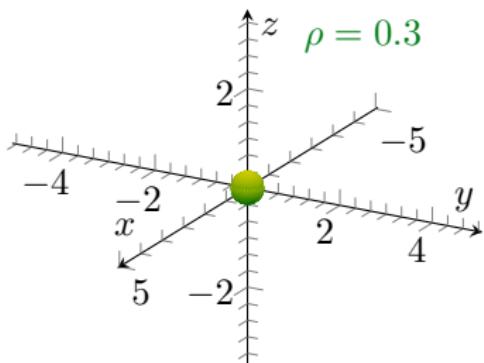


Remark

Spherical coordinates are good for describing:

- spheres centred at the origin ($\rho = \rho_0$);
- cones (with vertex at the origin and axis on the z -axis) ($\phi = \phi_0$); and
- half planes containing the z -axis ($\theta = \theta_0$).

14.7 Triple Integrals in Cylindrical and Spherical Coordinates

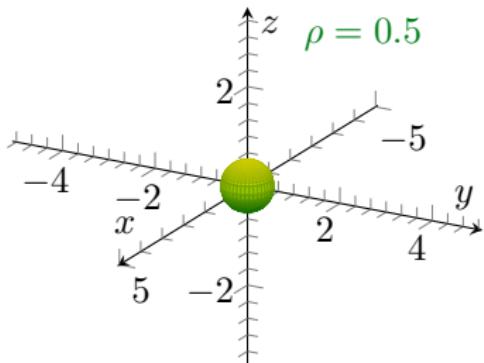


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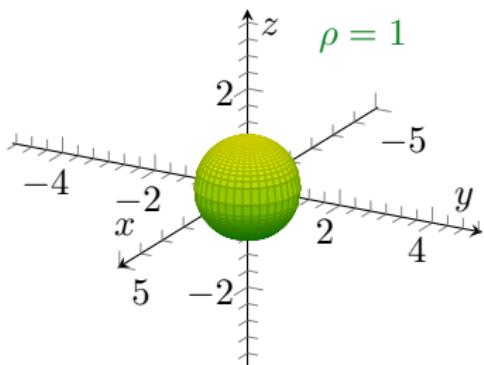


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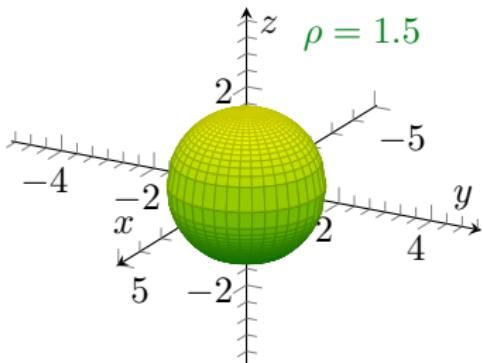


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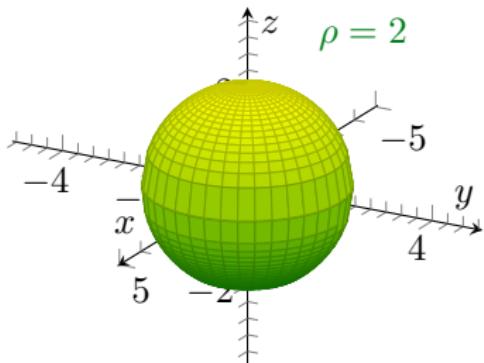


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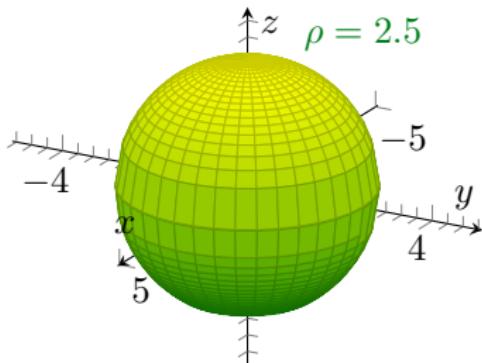


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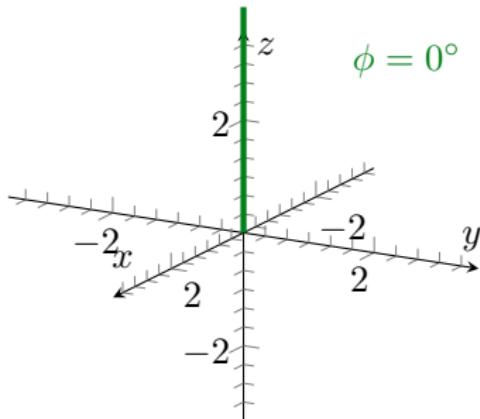


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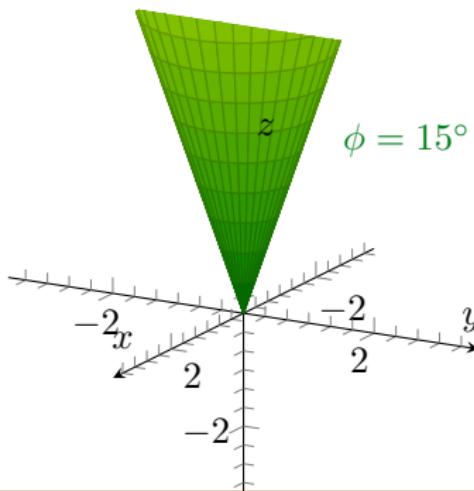


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14.7 Triple Integrals in Cylindrical and Spherical Coordinates

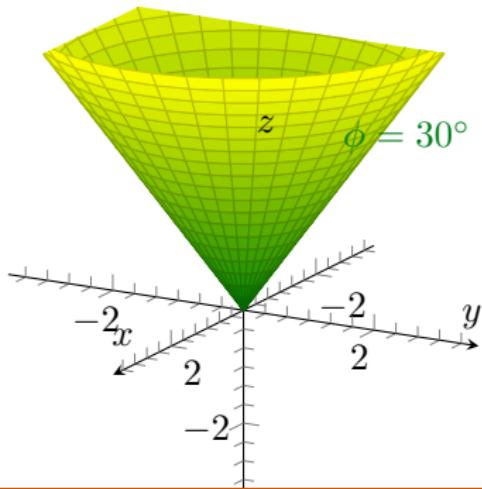


Remark

Spherical coordinates are good for describing:

- spheres centred at the origin ($\rho = \rho_0$);
- cones (with vertex at the origin and axis on the z -axis) ($\phi = \phi_0$); and
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14.7 Triple Integrals in Cylindrical and Spherical Coordinates

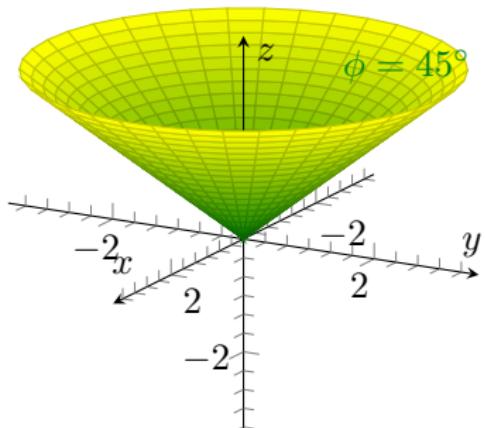


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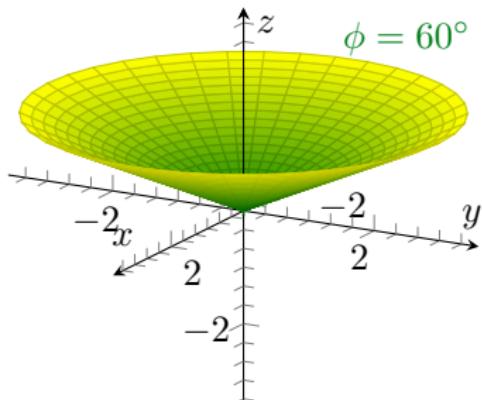


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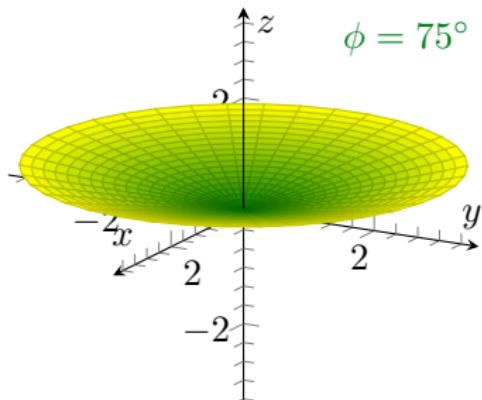


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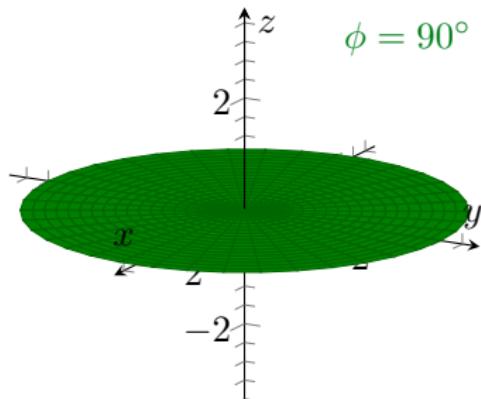


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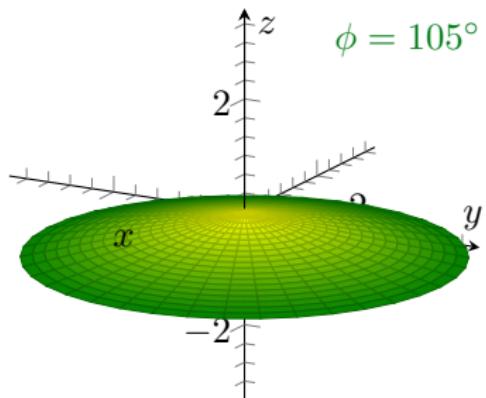


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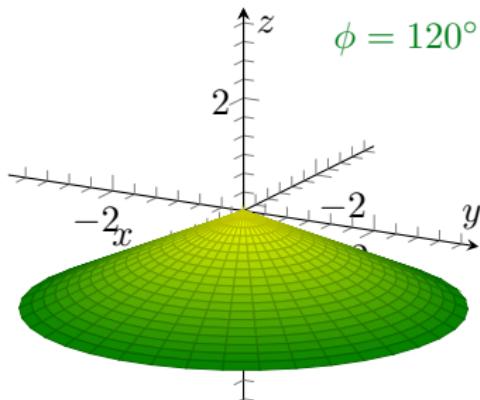


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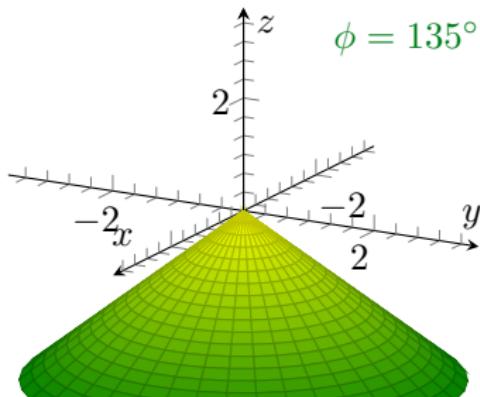


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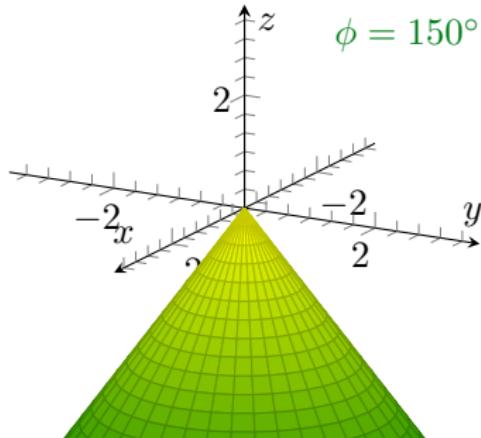


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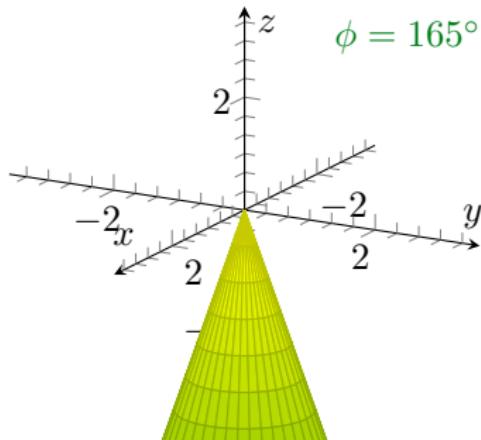


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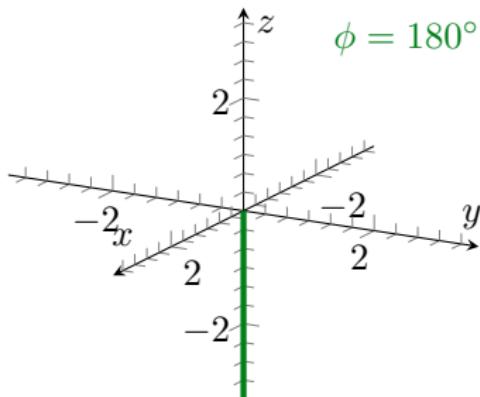


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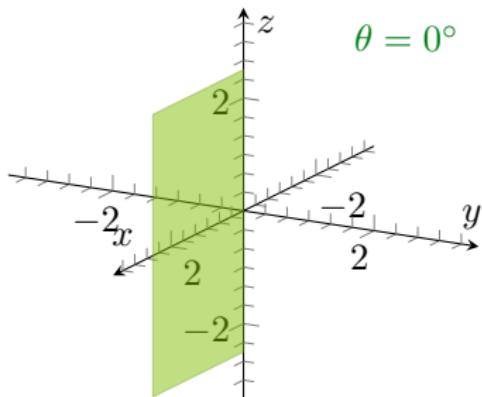


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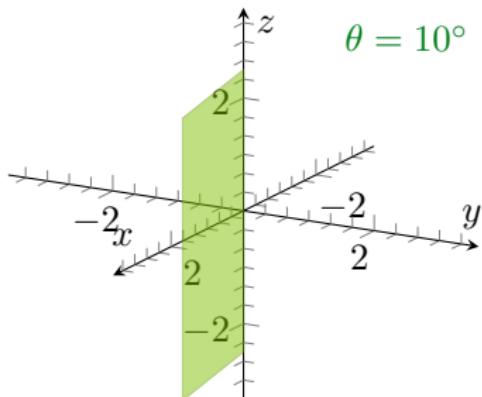


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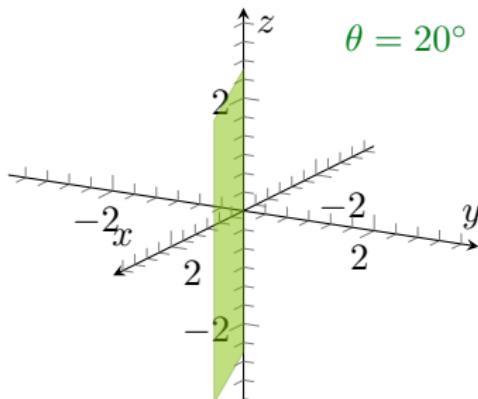


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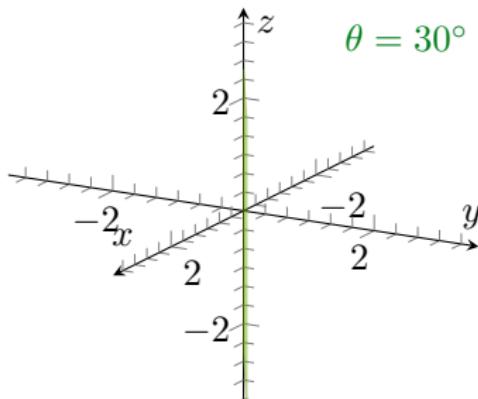


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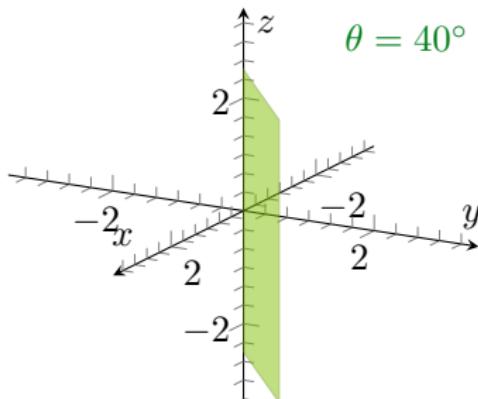


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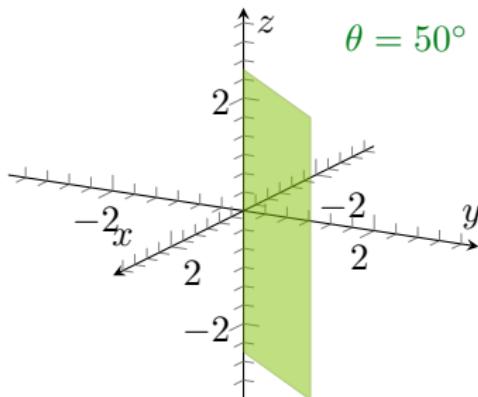


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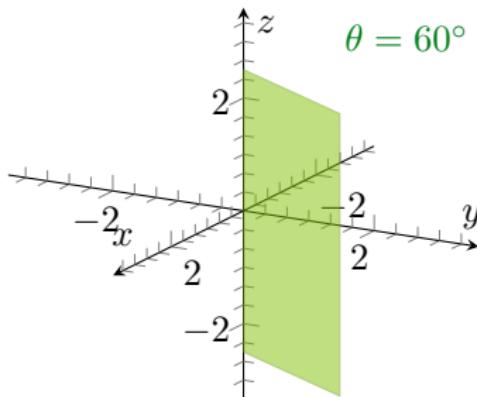


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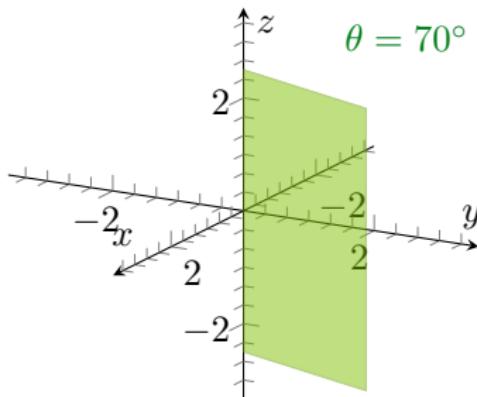


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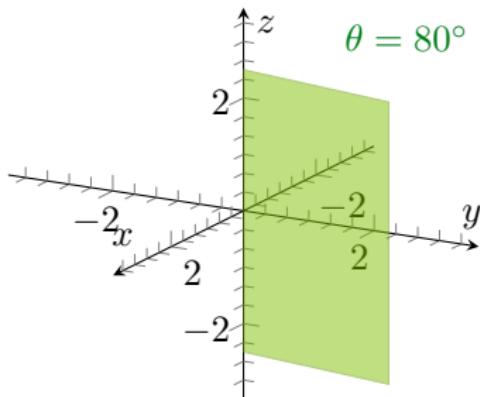


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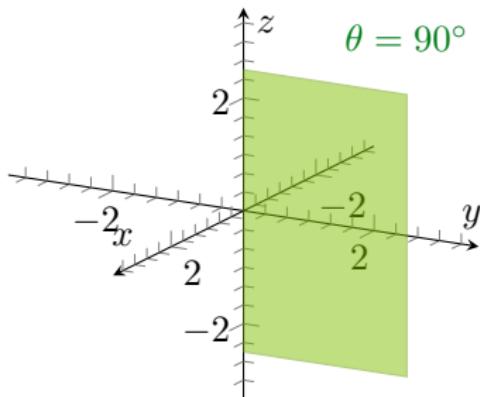


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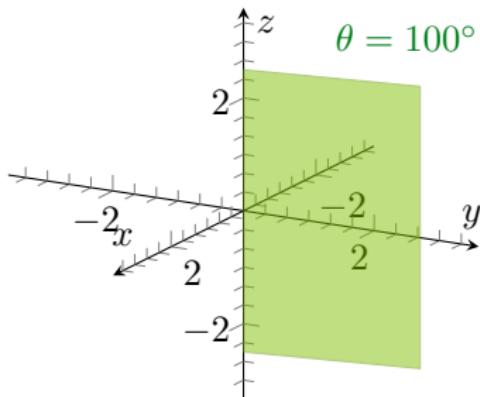


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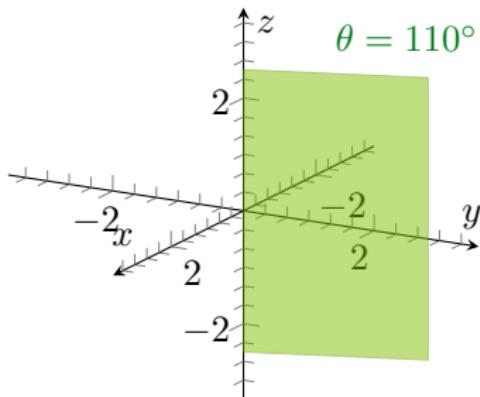


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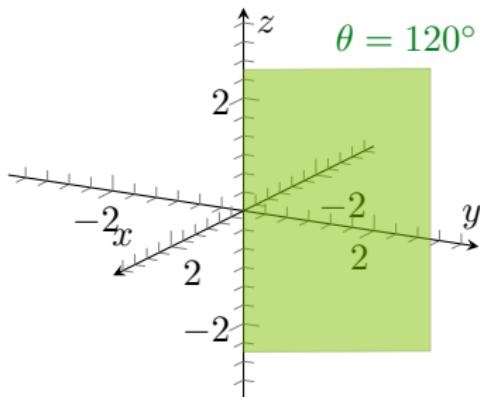


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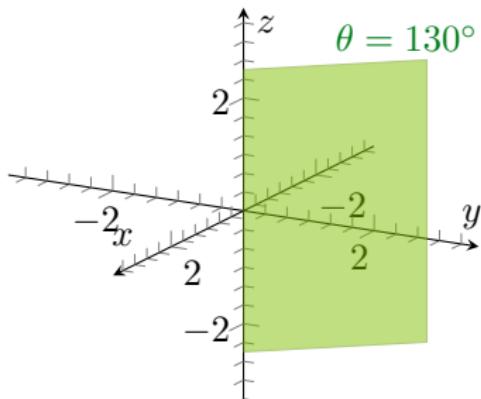


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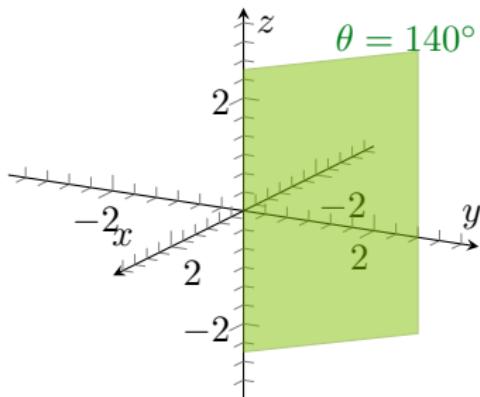


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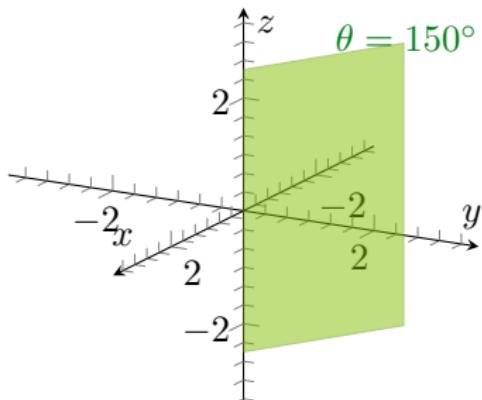


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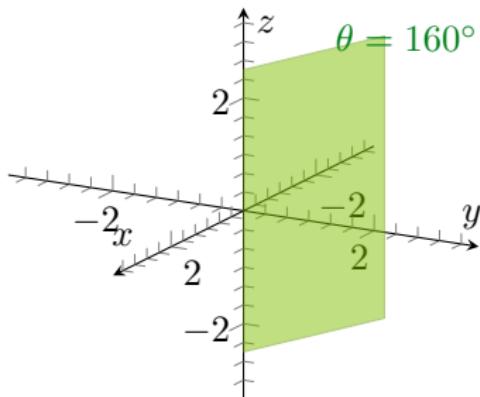


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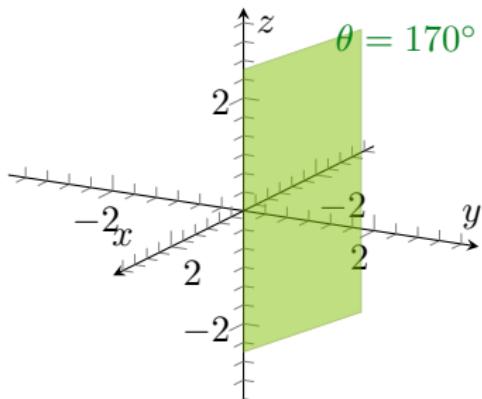


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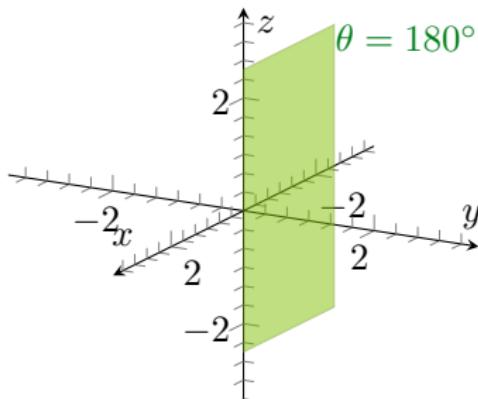


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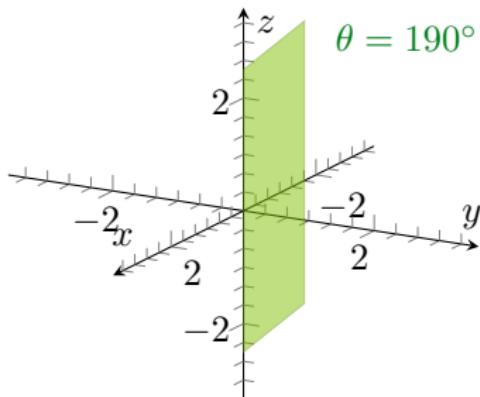


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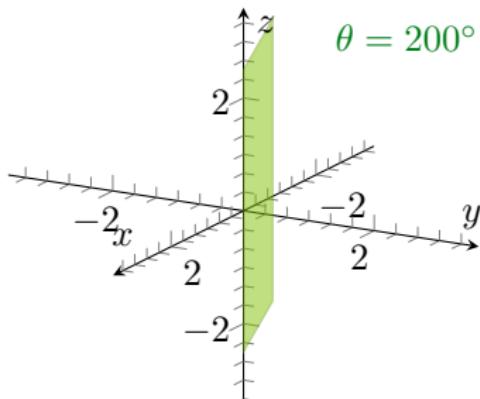


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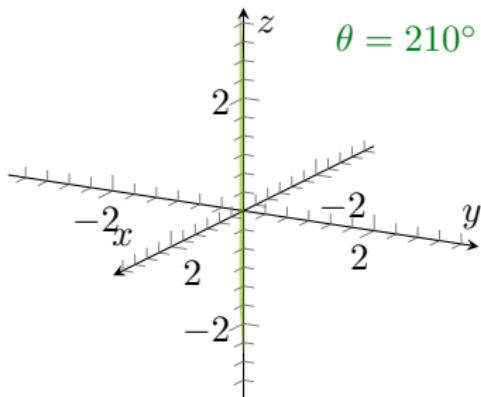


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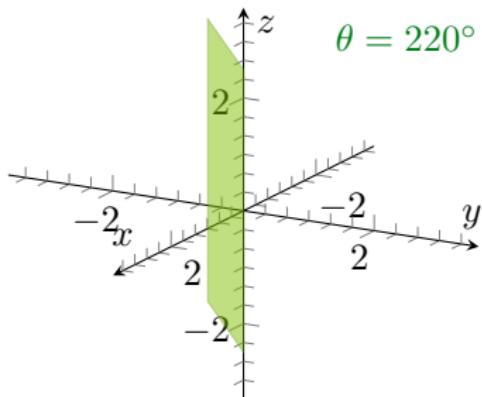


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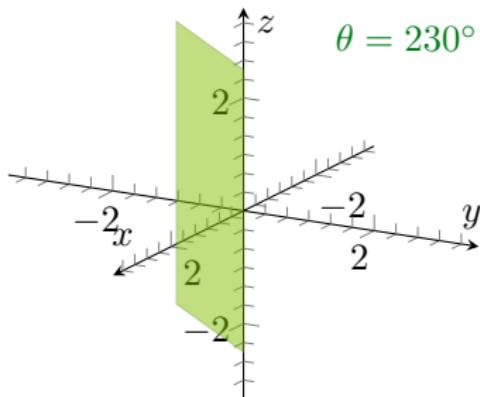


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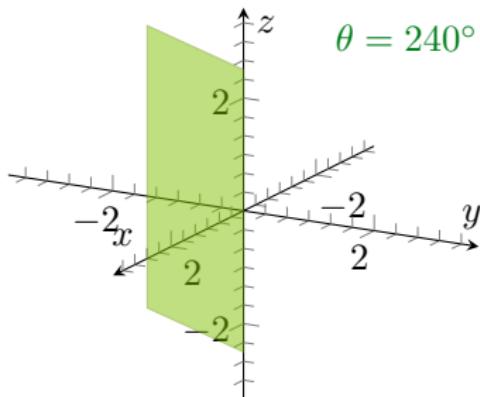


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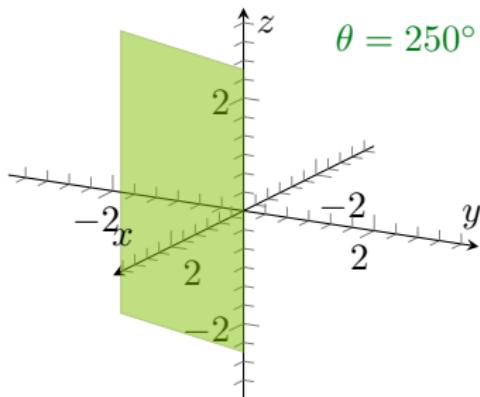


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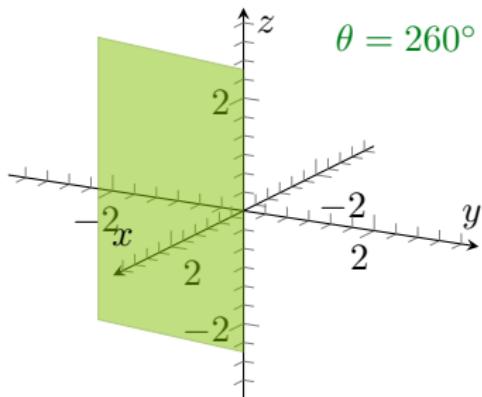


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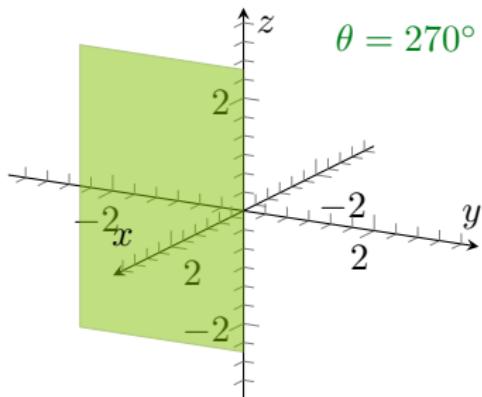


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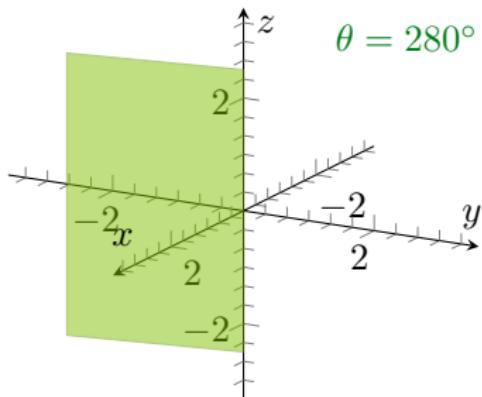


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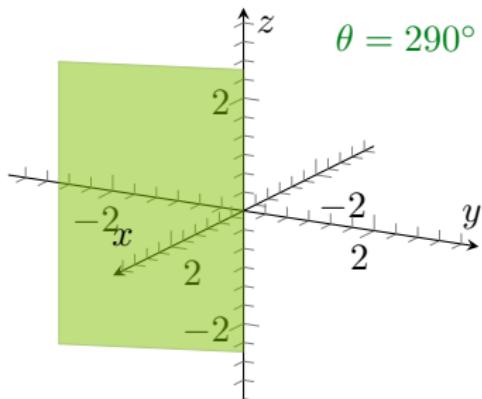


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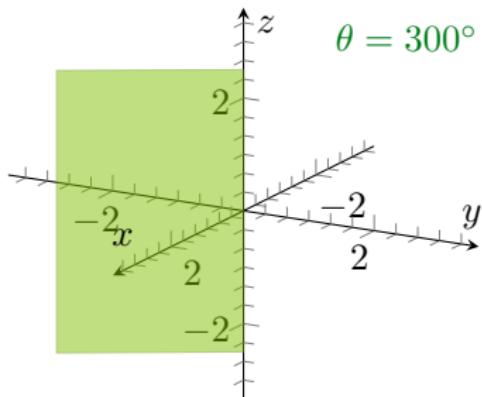


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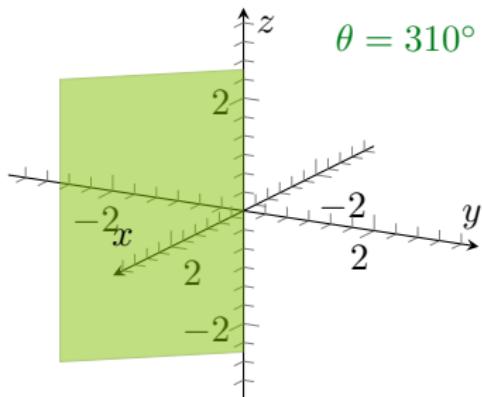


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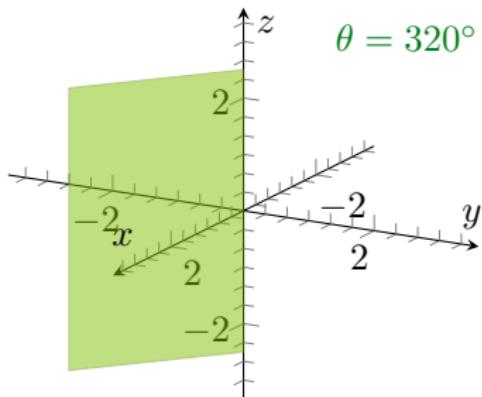


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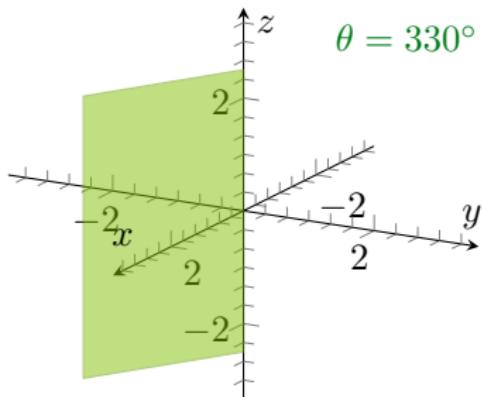


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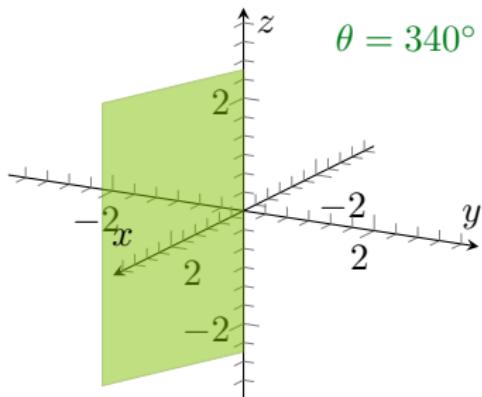


Remark

Spherical coordinates are good for describing:

- spheres centred at the origin ($\rho = \rho_0$);
- cones (with vertex at the origin and axis on the z -axis) ($\phi = \phi_0$); and
- half planes containing the z -axis ($\theta = \theta_0$).

14.7 Triple Integrals in Cylindrical and Spherical Coordinates

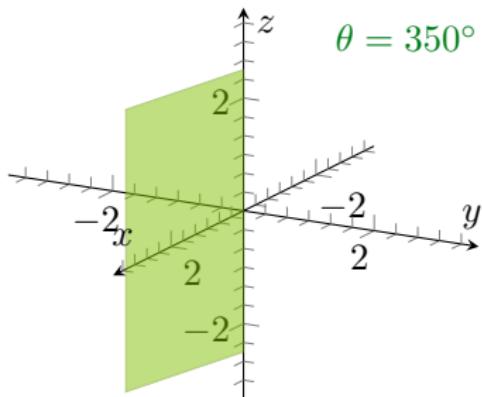


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14.7 Triple Integrals in Cylindrical and Spherical Coordinates



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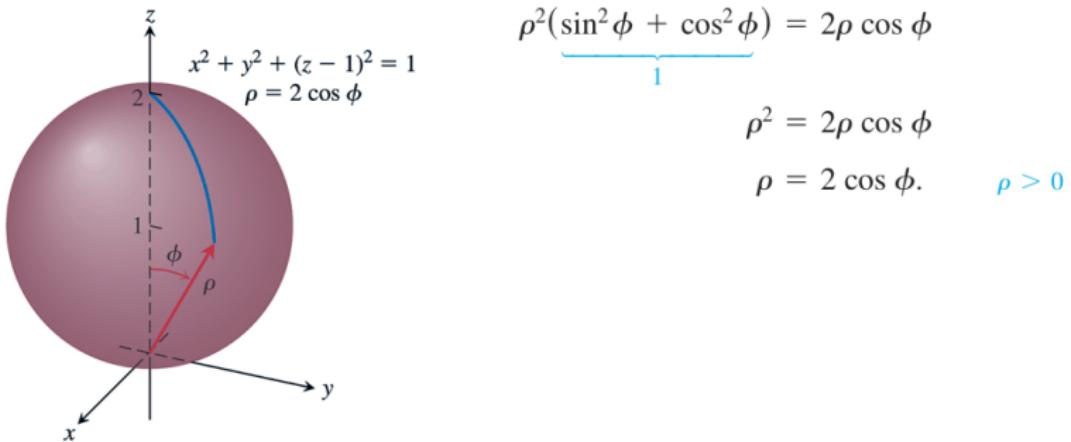
- spheres centred at the origin ($\rho = \rho_0$);
- cones (with vertex at the origin and axis on the z -axis) ($\phi = \phi_0$); and
- half planes containing the z -axis ($\theta = \theta_0$).

$$x = \rho \sin \phi \cos \theta \quad y = \rho \sin \phi \sin \theta \quad z = \rho \cos \phi$$

EXAMPLE 3 Find a spherical coordinate equation for the sphere $x^2 + y^2 + (z - 1)^2 = 1$.

Solution We use Equations (1) to substitute for x , y , and z :

$$\begin{aligned} x^2 + y^2 + (z - 1)^2 &= 1 \\ \rho^2 \sin^2 \phi \cos^2 \theta + \rho^2 \sin^2 \phi \sin^2 \theta + (\rho \cos \phi - 1)^2 &= 1 && \text{Eqs. (1)} \\ \underbrace{\rho^2 \sin^2 \phi (\cos^2 \theta + \sin^2 \theta) + \rho^2 \cos^2 \phi - 2\rho \cos \phi + 1}_{1} &= 1 \end{aligned}$$



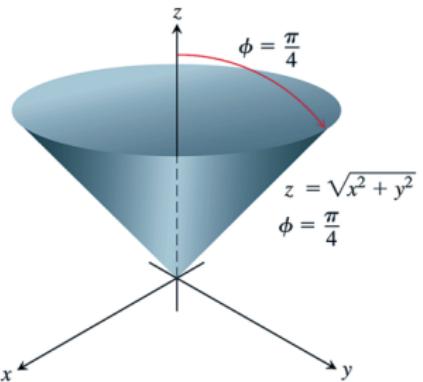
$$\begin{aligned} \rho^2 (\sin^2 \phi + \cos^2 \phi) &= 2\rho \cos \phi && \text{1} \\ \rho^2 &= 2\rho \cos \phi \\ \rho &= 2 \cos \phi. && \rho > 0 \end{aligned}$$

$$x = \rho \sin \phi \cos \theta \quad y = \rho \sin \phi \sin \theta \quad z = \rho \cos \phi$$

EXAMPLE 4 Find a spherical coordinate equation for the cone $z = \sqrt{x^2 + y^2}$.

Solution 1 *Use geometry.* The cone is symmetric with respect to the z -axis and cuts the first quadrant of the yz -plane along the line $z = y$. The angle between the cone and the positive z -axis is therefore $\pi/4$ radians. The cone consists of the points whose spherical coordinates have ϕ equal to $\pi/4$, so its equation is $\phi = \pi/4$. (See Figure 15.54.)

Solution 2 *Use algebra.* If we use Equations (1) to substitute for x , y , and z we obtain the same result:



$$z = \sqrt{x^2 + y^2}$$

$$\rho \cos \phi = \sqrt{\rho^2 \sin^2 \phi}$$

$$\rho \cos \phi = \rho \sin \phi$$

$$\cos \phi = \sin \phi$$

Example 3

$\rho > 0, \sin \phi \geq 0$

$$\phi = \frac{\pi}{4}.$$

$0 \leq \phi \leq \pi$



14.7 Triple Integrals in Cylindrical and Spherical Coordinates



Theorem

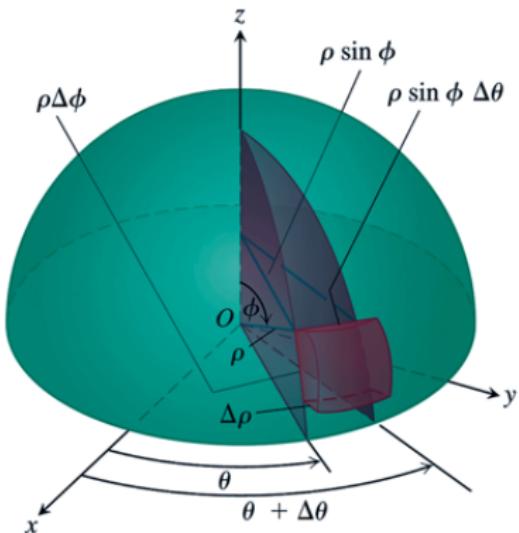
$$dV = dx dy dz = r dr d\theta dz = .$$

14.7 Triple Integrals in Cylindrical and Spherical Coordinates



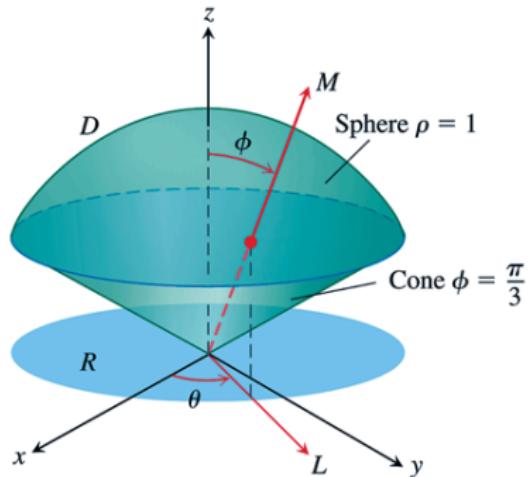
Theorem

$$dV = dxdydz = r \, dr d\theta dz = \rho^2 \sin \phi \, d\rho d\phi d\theta.$$



14.7

$$dV = \rho^2 \sin \phi \, d\rho d\phi d\theta$$

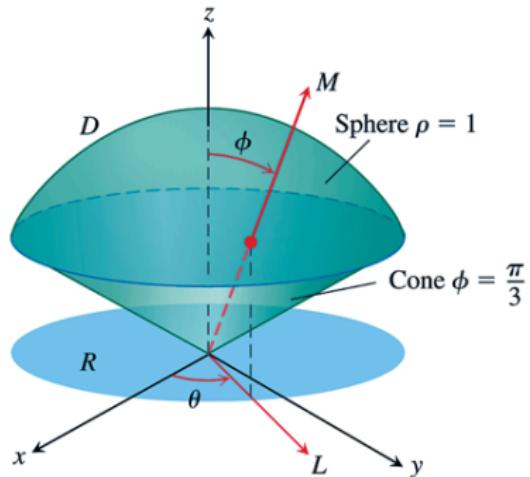


Example

Calculate the volume of the region enclosed by the sphere $\phi = 1$ and the cone $\phi = \frac{\pi}{3}$.

14.7

$$dV = \rho^2 \sin \phi \, d\rho d\phi d\theta$$



$$0 \leq \theta \leq 2\pi$$

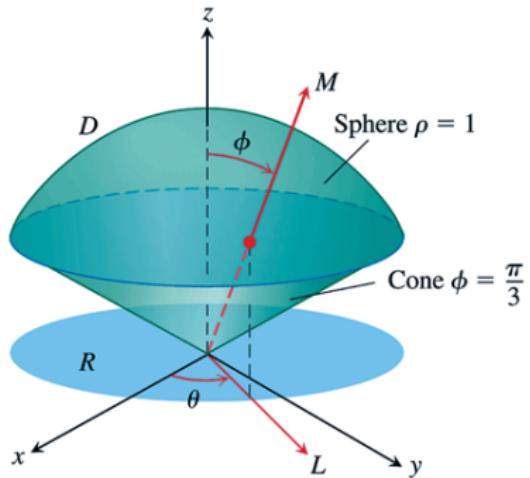
$$0 \leq \phi \leq \frac{\pi}{3}$$

$$0 \leq \rho \leq 1.$$

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Calculate the volume of the region enclosed by the sphere $\phi = 1$ and the cone $\phi = \frac{\pi}{3}$.

$$dV = \rho^2 \sin \phi \, d\rho d\phi d\theta$$



$$0 \leq \theta \leq 2\pi$$

$$0 \leq \phi \leq \frac{\pi}{3}$$

$$0 \leq \rho \leq 1.$$

Example

Calculate the volume of the region enclosed by the sphere $\phi = 1$ and the cone $\phi = \frac{\pi}{3}$.

$$\iiint_D dV = \int_0^{2\pi} \int_0^{\frac{\pi}{3}} \int_0^1 \rho^2 \sin \phi \, d\rho d\phi d\theta = \dots = \frac{\pi}{3}.$$

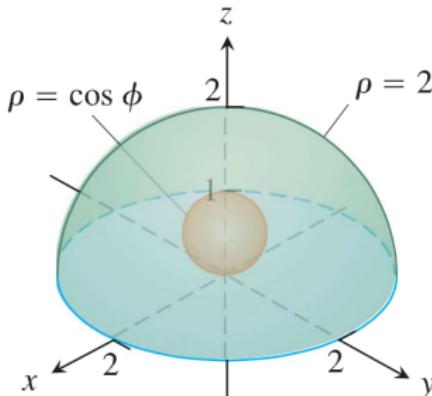
14.7 Triple Integrals in Cylindrical and Spherical Coordinates

Example

Find the limits of integration for the region shown below: The region is bounded by $\rho = 2$ and the xy -plane, but the sphere $\rho = \cos \phi$ has been removed from this large hemisphere.

A $0 \leq \theta \leq 2\pi$
 $0 \leq \phi \leq \pi$
 $0 \leq \rho \leq 2$

B $0 \leq \theta \leq 2\pi$
 $0 \leq \phi \leq \frac{\pi}{2}$
 $0 \leq \rho \leq 2$



C $0 \leq \theta \leq 2\pi$
 $0 \leq \phi \leq \frac{\pi}{2}$
 $\cos \phi \leq \rho \leq 2$

D $0 \leq \theta \leq \pi$
 $0 \leq \phi \leq 2$
 $2 \leq \rho \leq \cos \phi$

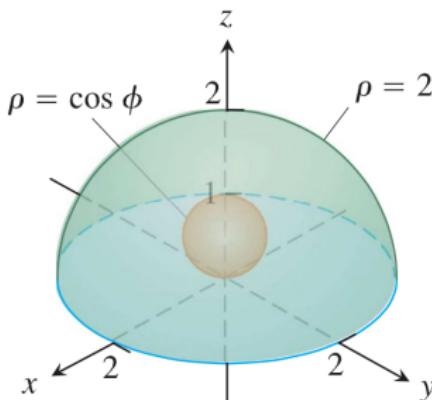
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 $0 \leq \phi \leq \frac{\pi}{2}$
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 $\cos \phi \leq \rho \leq 2$

D $0 \leq \theta \leq \pi$
 $0 \leq \phi \leq 2$
 $2 \leq \rho \leq \cos \phi$

Coordinate Conversion Formulas

CYLINDRICAL TO RECTANGULAR

$$x = r \cos \theta$$

$$y = r \sin \theta$$

$$z = z$$

SPHERICAL TO RECTANGULAR

$$x = \rho \sin \phi \cos \theta$$

$$y = \rho \sin \phi \sin \theta$$

$$z = \rho \cos \phi$$

SPHERICAL TO CYLINDRICAL

$$r = \rho \sin \phi$$

$$z = \rho \cos \phi$$

$$\theta = \theta$$

Corresponding formulas for dV in triple integrals:

$$\begin{aligned} dV &= dx \, dy \, dz \\ &= dz \, r \, dr \, d\theta \\ &= \rho^2 \sin \phi \, d\rho \, d\phi \, d\theta \end{aligned}$$



Break

We will continue at 2pm





Substitutions in Multiple Integrals



Substitutions in Single Integrals

You know that if we write $u = 2x + 3$ then $du = 2 dx$ and

$$\int_0^1 2\sqrt{2x+3} dx = \int_3^5 \sqrt{u} du.$$

14.8 Substitutions in Multiple Integrals



Substitutions in Single Integrals

You know that if we write $u = 2x + 3$ then $du = 2 dx$ and

$$\int_0^1 2\sqrt{2x+3} dx = \int_3^5 \sqrt{u} du.$$

Substitutions in Double Integrals

We are going to do the same thing for substitutions in double integrals.

14.8 Substitutions in Multiple Integrals



Carl Gustav Jacob Jacobi

BORN

10 December 1804

DECEASED

18 February 1851

NATIONALITY

German

Definition

The *Jacobian* of the coordinate transformation $x = g(u, v)$, $y = h(u, v)$ is

$$J(u, v) = \frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \frac{\partial x}{\partial u} \frac{\partial y}{\partial v} - \frac{\partial x}{\partial v} \frac{\partial y}{\partial u}.$$

14.8

$$\frac{\partial(x, y)}{\partial(u, v)} = \frac{\partial x}{\partial u} \frac{\partial y}{\partial v} - \frac{\partial x}{\partial v} \frac{\partial y}{\partial u}$$



Example

Find the Jacobian of the polar coordinate transformation
 $x = r \cos \theta$ and $y = r \sin \theta$.

14.8

$$\frac{\partial(x, y)}{\partial(u, v)} = \frac{\partial x}{\partial u} \frac{\partial y}{\partial v} - \frac{\partial x}{\partial v} \frac{\partial y}{\partial u}$$



Example

Find the Jacobian of the polar coordinate transformation
 $x = r \cos \theta$ and $y = r \sin \theta$.

$$\frac{\partial(x, y)}{\partial(r, \theta)} = \frac{\partial x}{\partial r} \frac{\partial y}{\partial \theta} - \frac{\partial x}{\partial \theta} \frac{\partial y}{\partial r}$$

=

=

14.8

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Example

Find the Jacobian of the polar coordinate transformation
 $x = r \cos \theta$ and $y = r \sin \theta$.

$$\begin{aligned}\frac{\partial(x, y)}{\partial(r, \theta)} &= \frac{\partial x}{\partial r} \frac{\partial y}{\partial \theta} - \frac{\partial x}{\partial \theta} \frac{\partial y}{\partial r} \\ &= (\cos \theta) (r \cos \theta) - (-r \sin \theta) (\sin \theta) \\ &= r.\end{aligned}$$

14.8

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Remark

Remember that

$$dxdy = r dr d\theta.$$

14.8 Substitutions in Multiple Integrals



Theorem

Suppose that $f(x, y)$ is continuous over the region R . Let G be the preimage of R under the transformation $x = g(u, v)$, $y = h(u, v)$, which is assumed to be one-to-one on the interior of G . If the functions g and h have continuous first partial derivatives within the interior of G , then

$$\iint_R f(x, y) \, dxdy = \iint_G f(g(u, v), h(u, v)) \left| \frac{\partial(x, y)}{\partial(u, v)} \right| \, dudv.$$

14.8 Substitutions in Multiple Integrals



Theorem

Suppose that $f(x, y)$ is continuous over the region R . Let G be the preimage of R under the transformation $x = g(u, v)$, $y = h(u, v)$, which is assumed to be one-to-one on the interior of G . If the functions g and h have continuous first partial derivatives within the interior of G , then

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14.8 Substitutions in Multiple Integrals



Example

Calculate

$$\int_0^4 \int_{\frac{y}{2}}^{\frac{y}{2}+1} \frac{2x - y}{2} dx dy$$

by applying the transformation

$$u = \frac{2x - y}{2}, \quad v = \frac{y}{2}$$

and integrating over an appropriate region in the uv -plane.

14.8 Substitutions in Multiple Integrals



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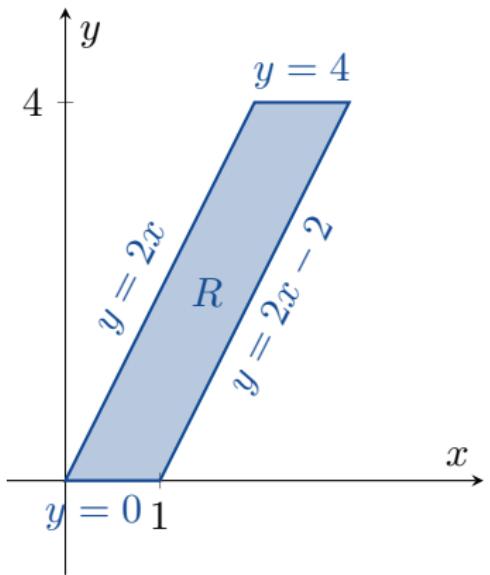
$$u = \frac{2x - y}{2}, \quad v = \frac{y}{2}$$

and integrating over an appropriate region in the uv -plane.

$$0 \leq y \leq 4 \quad \text{and} \quad \frac{y}{2} \leq x \leq \frac{y}{2} + 1$$

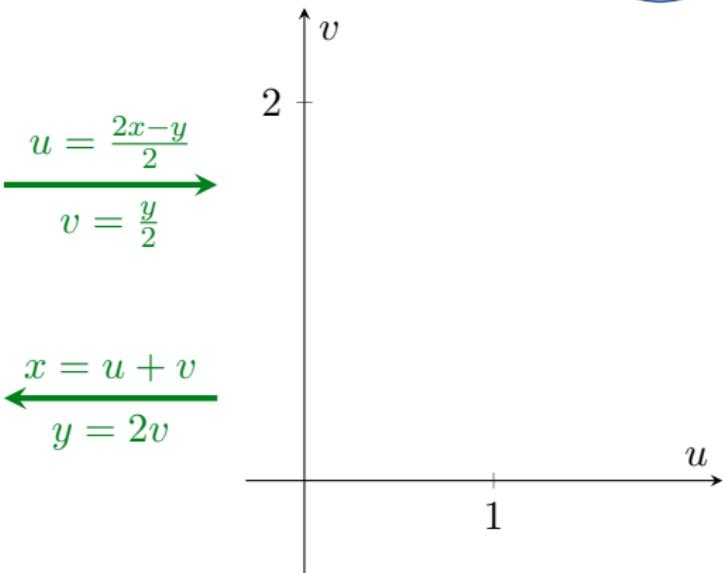
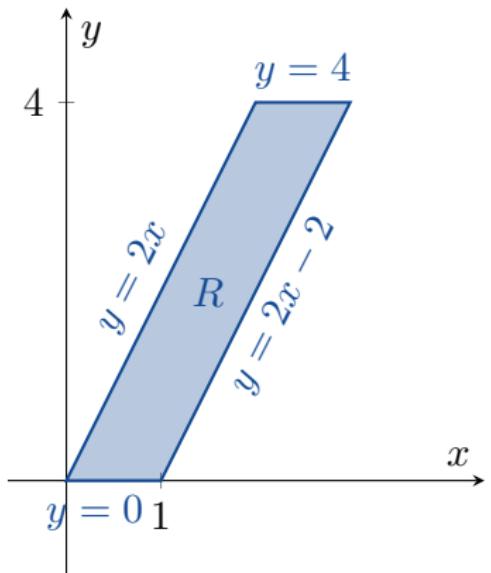
14.8

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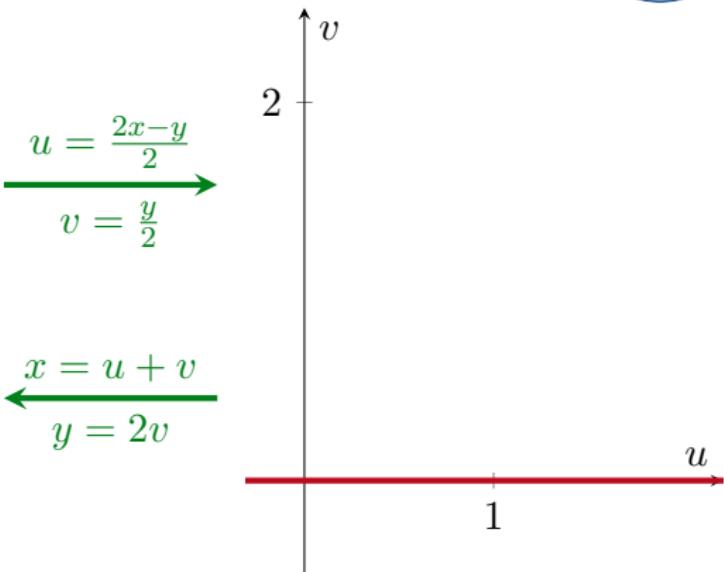
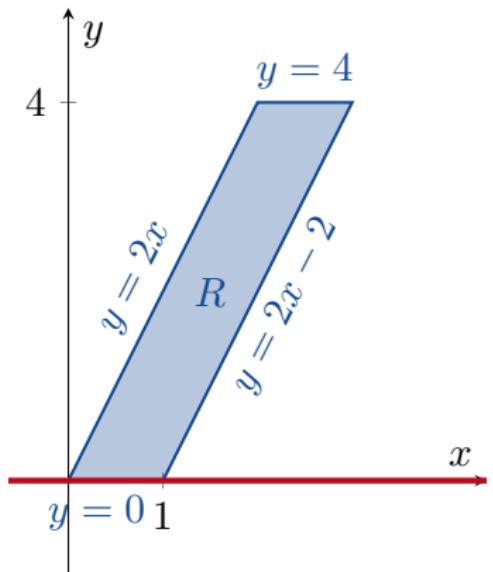
14.8

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14.8

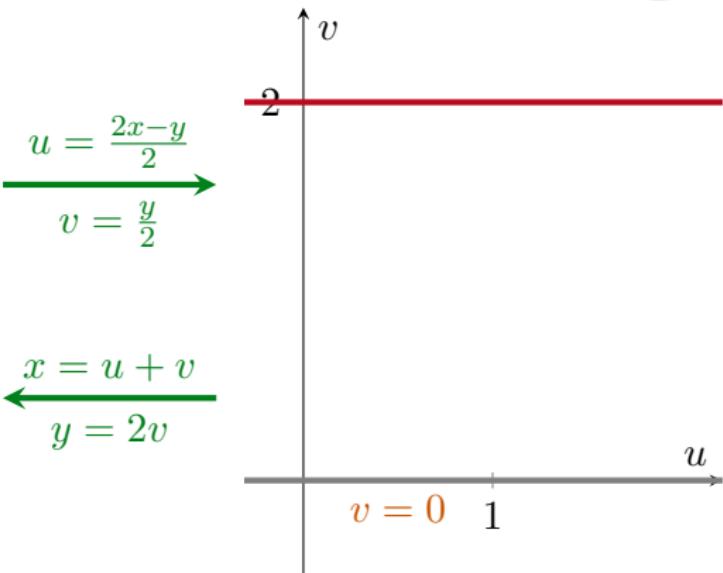
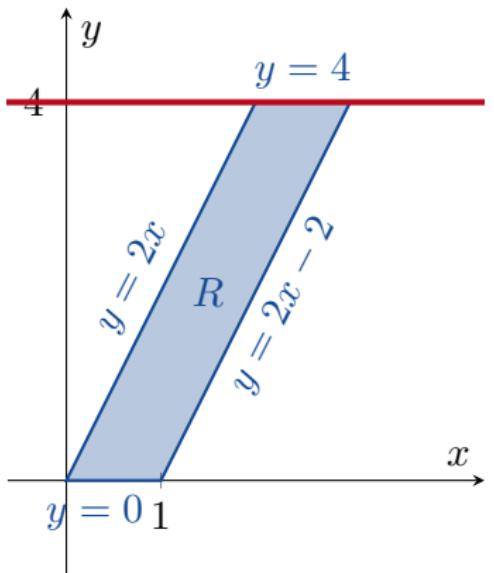
$$0 \leq y \leq 4 \quad \text{and} \quad \frac{y}{2} \leq x \leq \frac{y}{2} + 1$$



$$y = 0 \quad \Rightarrow \quad v = \frac{y}{2} = 0$$

14.8

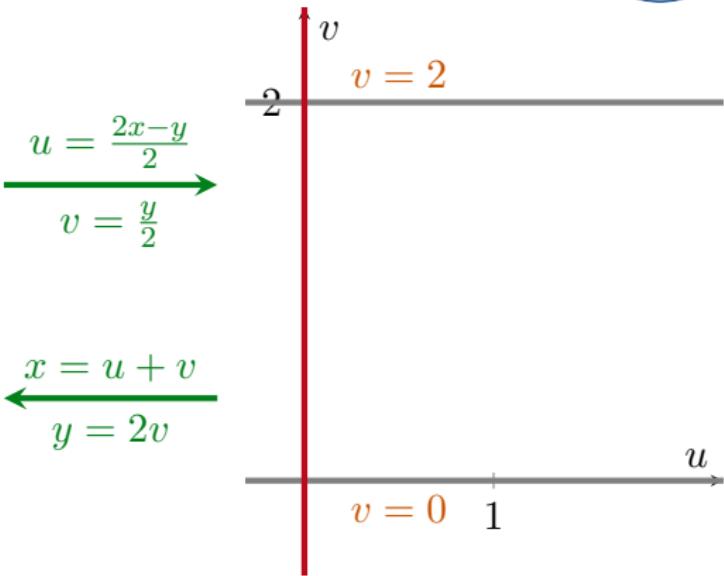
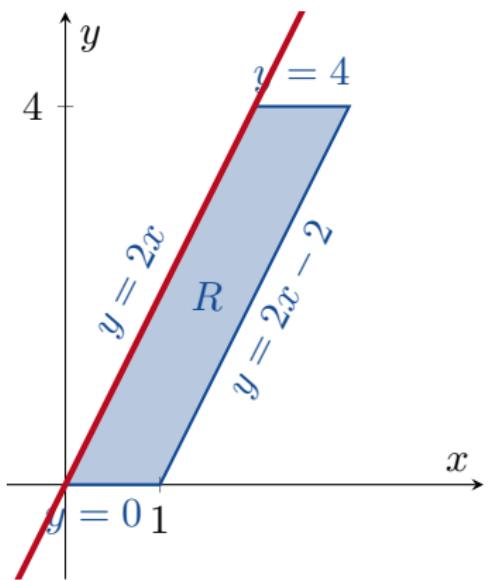
$$0 \leq y \leq 4 \quad \text{and} \quad \frac{y}{2} \leq x \leq \frac{y}{2} + 1$$



$$y = 4 \quad \Rightarrow \quad v = \frac{y}{2} = 2$$

14.8

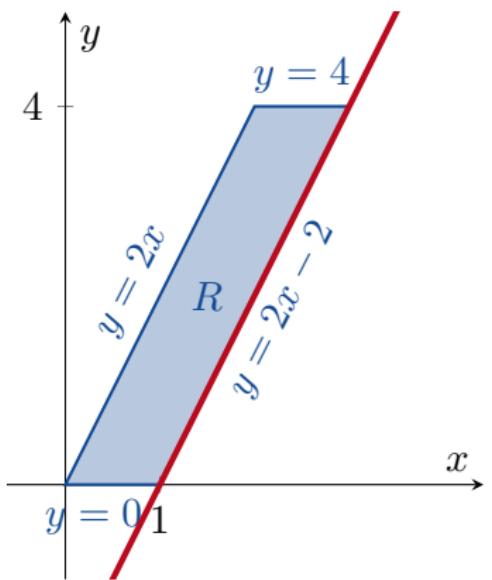
$$0 \leq y \leq 4 \quad \text{and} \quad \frac{y}{2} \leq x \leq \frac{y}{2} + 1$$



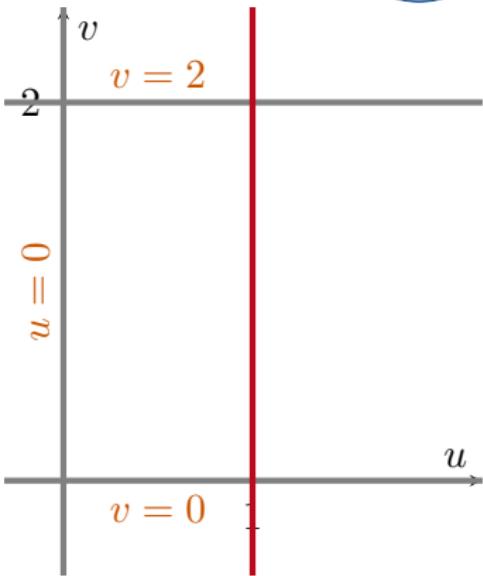
$$y = 2x \implies u = \frac{2x - y}{2} = \frac{2x - 2x}{2} = 0$$

14.8

$$0 \leq y \leq 4 \quad \text{and} \quad \frac{y}{2} \leq x \leq \frac{y}{2} + 1$$



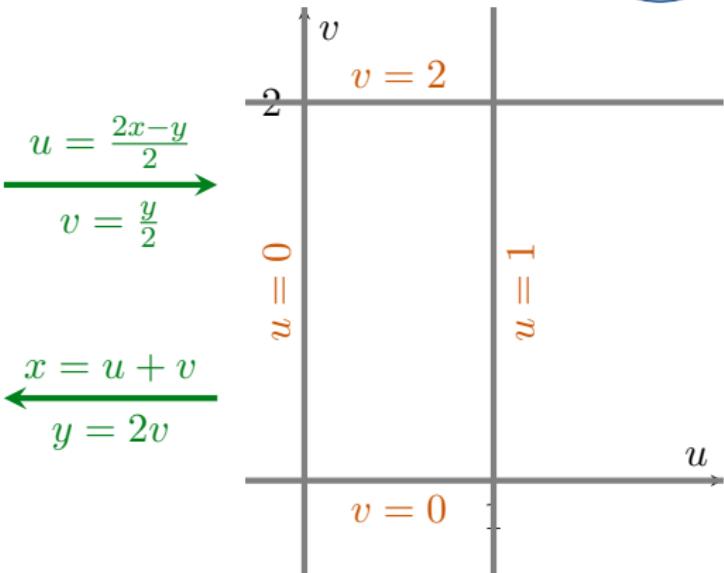
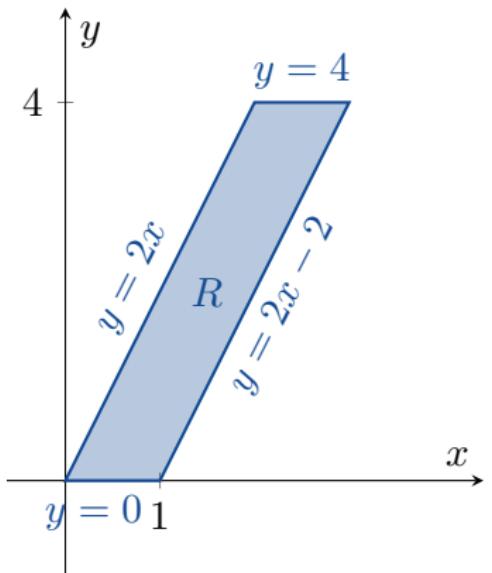
$$\begin{aligned} u &= \frac{2x-y}{2} \\ v &= \frac{y}{2} \\ x &= u+v \\ y &= 2v \end{aligned}$$



$$y = 2x - 2 \implies u = \frac{2x - y}{2} = \frac{2x - 2x + 2}{2} = 1$$

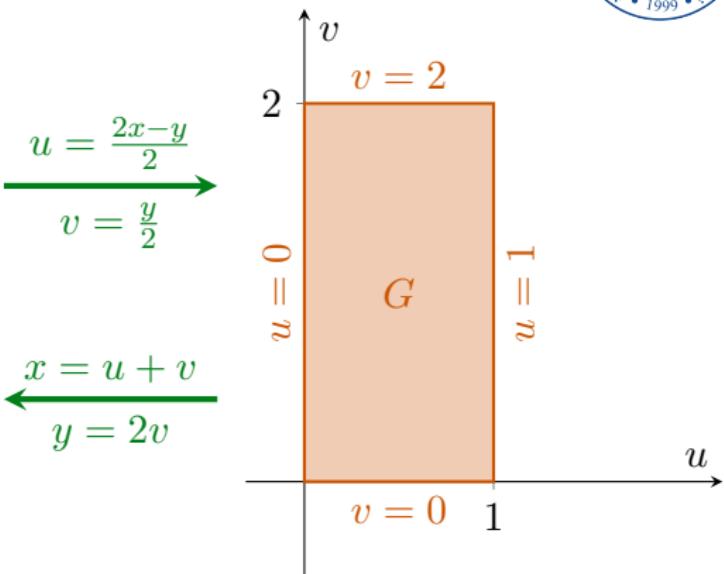
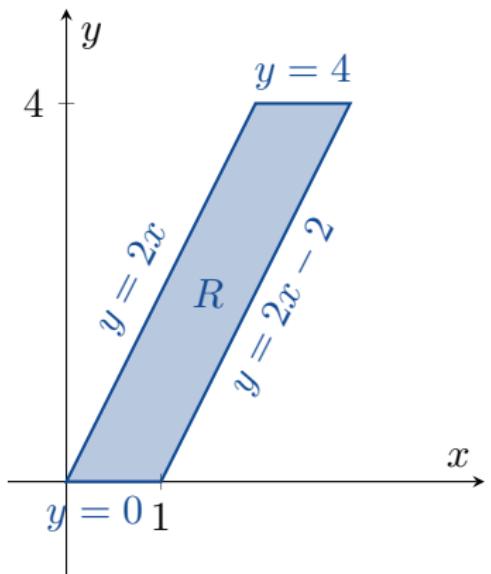
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14.8

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$$0 \leq u \leq 1 \quad \text{and} \quad 0 \leq v \leq 2$$

14.8

$$\frac{\partial(x, y)}{\partial(u, v)} = \frac{\partial x}{\partial u} \frac{\partial y}{\partial v} - \frac{\partial x}{\partial v} \frac{\partial y}{\partial u}$$



$$x = u + v \quad \text{and} \quad y = 2v$$

Next we need the Jacobian of this coordinate transformation:

14.8

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$$\begin{aligned}\frac{\partial(x, y)}{\partial(u, v)} &= \frac{\partial x}{\partial u} \frac{\partial y}{\partial v} - \frac{\partial x}{\partial v} \frac{\partial y}{\partial u} \\ &= (1)(2) - (1)(0) = 2.\end{aligned}$$

14.8 Substitutions in Multiple Integrals



$$u = \frac{2x - y}{2}, \quad v = \frac{y}{2}$$

$$0 \leq u \leq 1 \quad 0 \leq v \leq 2 \quad \frac{\partial(x, y)}{\partial(u, v)} = 2$$

Therefore

$$\int_0^4 \int_{\frac{y}{2}}^{\frac{y}{2}+1} \frac{2x - y}{2} dx dy =$$

14.8 Substitutions in Multiple Integrals



$$u = \frac{2x - y}{2}, \quad v = \frac{y}{2}$$

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Therefore

$$\int_0^4 \int_{\frac{y}{2}}^{\frac{y}{2}+1} \frac{2x - y}{2} dx dy = \int_0^2 \int_0^1 u \left| \frac{\partial(x, y)}{\partial(u, v)} \right| du dv$$

14.8 Substitutions in Multiple Integrals



$$u = \frac{2x - y}{2}, \quad v = \frac{y}{2}$$

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$$\begin{aligned} \int_0^4 \int_{\frac{y}{2}}^{\frac{y}{2}+1} \frac{2x - y}{2} dx dy &= \int_0^2 \int_0^1 u \left| \frac{\partial(x, y)}{\partial(u, v)} \right| du dv \\ &= \int_0^2 \int_0^1 2u du dv \end{aligned}$$

14.8 Substitutions in Multiple Integrals



$$u = \frac{2x - y}{2}, \quad v = \frac{y}{2}$$

$$0 \leq u \leq 1 \quad 0 \leq v \leq 2 \quad \frac{\partial(x, y)}{\partial(u, v)} = 2$$

Therefore

$$\begin{aligned} \int_0^4 \int_{\frac{y}{2}}^{\frac{y}{2}+1} \frac{2x - y}{2} dx dy &= \int_0^2 \int_0^1 u \left| \frac{\partial(x, y)}{\partial(u, v)} \right| du dv \\ &= \int_0^2 \int_0^1 2u du dv = \dots = 2. \end{aligned}$$

14.8 Substitutions in Multiple Integrals



Remark

To do a substitution, we need to do two things:

- 1 Calculate the Jacobian and write $dxdy = \left| \frac{\partial(x, y)}{\partial(u, v)} \right| dudv$;

and

- 2 change the limits of integration.

14.8

$$\frac{\partial(x, y)}{\partial(u, v)} = \frac{\partial x}{\partial u} \frac{\partial y}{\partial v} - \frac{\partial x}{\partial v} \frac{\partial y}{\partial u}$$



Example

Calculate $\int_0^1 \int_0^{1-x} \sqrt{x+y}(y-2x)^2 dy dx$.

14.8

$$\frac{\partial(x, y)}{\partial(u, v)} = \frac{\partial x}{\partial u} \frac{\partial y}{\partial v} - \frac{\partial x}{\partial v} \frac{\partial y}{\partial u}$$



Example

Calculate $\int_0^1 \int_0^{1-x} \sqrt{x+y}(y-2x)^2 dy dx$.

First we need to choose u and v .

14.8

$$\frac{\partial(x, y)}{\partial(u, v)} = \frac{\partial x}{\partial u} \frac{\partial y}{\partial v} - \frac{\partial x}{\partial v} \frac{\partial y}{\partial u}$$



Example

Calculate $\int_0^1 \int_0^{1-x} \sqrt{x+y}(y-2x)^2 dydx.$

First we need to choose u and v . I choose

$$u = x + y \quad \text{and} \quad v = y - 2x.$$

14.8

$$\frac{\partial(x, y)}{\partial(u, v)} = \frac{\partial x}{\partial u} \frac{\partial y}{\partial v} - \frac{\partial x}{\partial v} \frac{\partial y}{\partial u}$$



Example

Calculate $\int_0^1 \int_0^{1-x} \sqrt{x+y}(y-2x)^2 dydx.$

First we need to choose u and v . I choose

$$u = x + y \quad \text{and} \quad v = y - 2x.$$

We can rearrange these to

$$x = \frac{u}{3} - \frac{v}{3} \quad \text{and} \quad y = \frac{2u}{3} + \frac{v}{3}.$$

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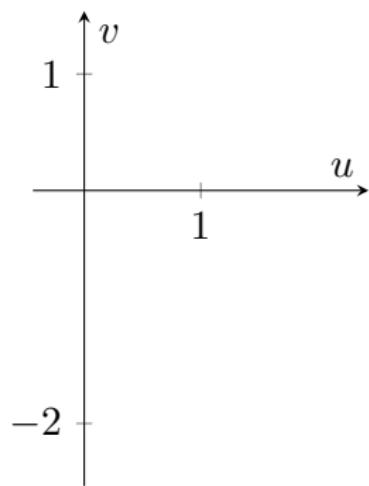
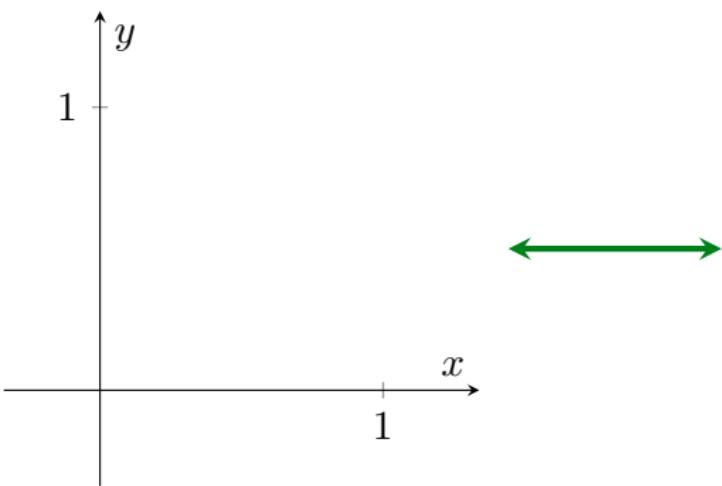
We can rearrange these to

$$x = \frac{u}{3} - \frac{v}{3} \quad \text{and} \quad y = \frac{2u}{3} + \frac{v}{3}.$$

Then the Jacobian is

$$\frac{\partial(x, y)}{\partial(u, v)} = \frac{\partial x}{\partial u} \frac{\partial y}{\partial v} - \frac{\partial x}{\partial v} \frac{\partial y}{\partial u} = \left(\frac{1}{3}\right) \left(\frac{1}{3}\right) - \left(-\frac{1}{3}\right) \left(\frac{2}{3}\right) = \frac{1}{3}.$$

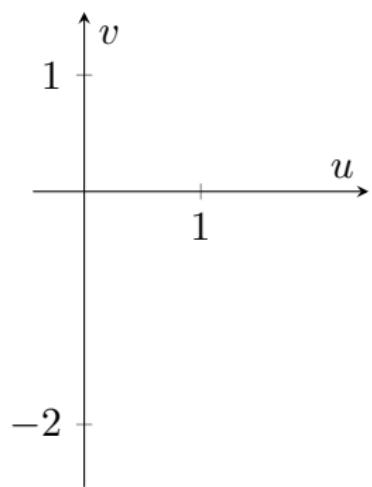
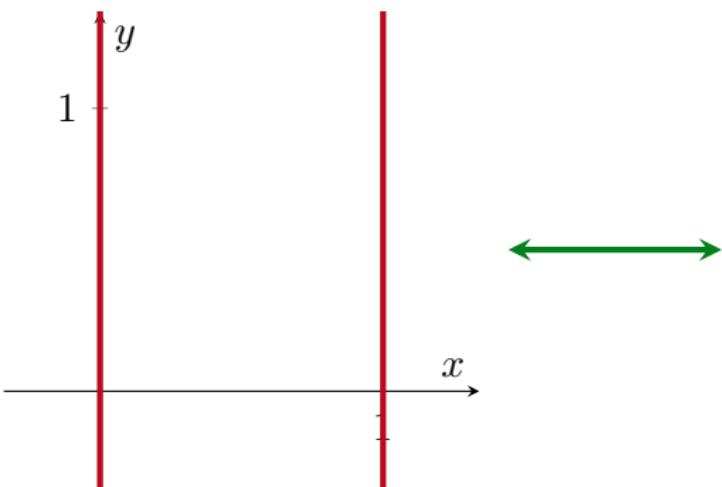
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$$x = 0$$

$$x = 1$$



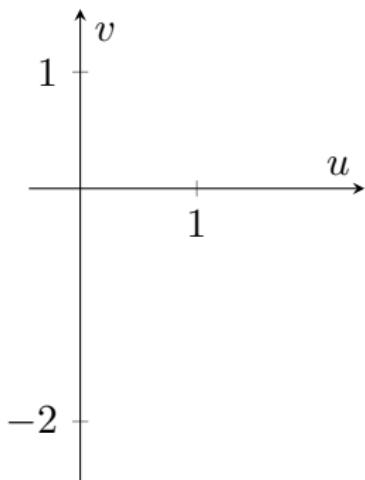
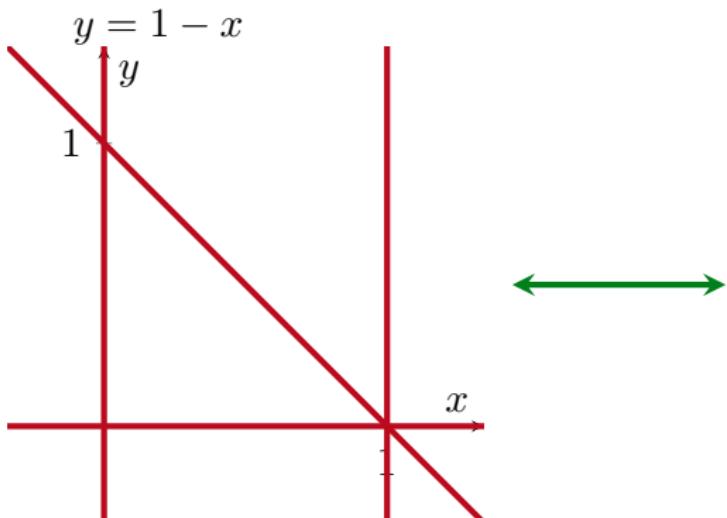
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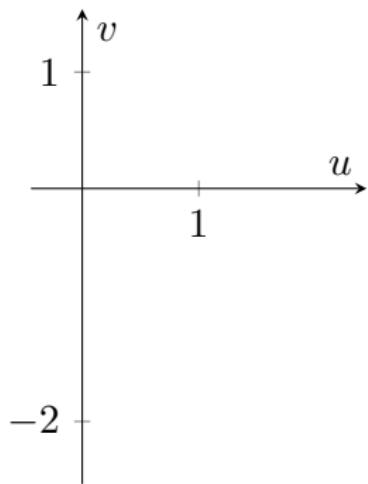
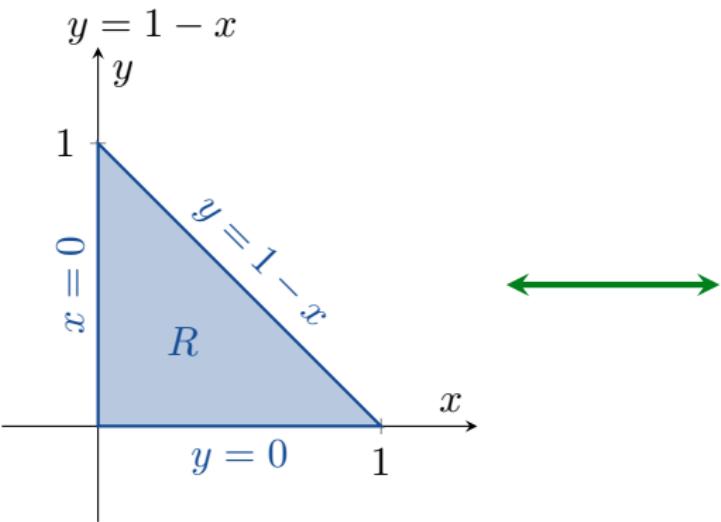
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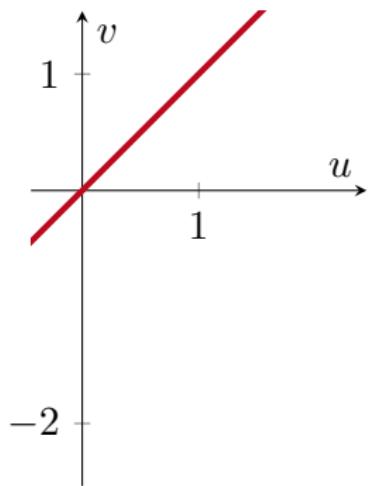
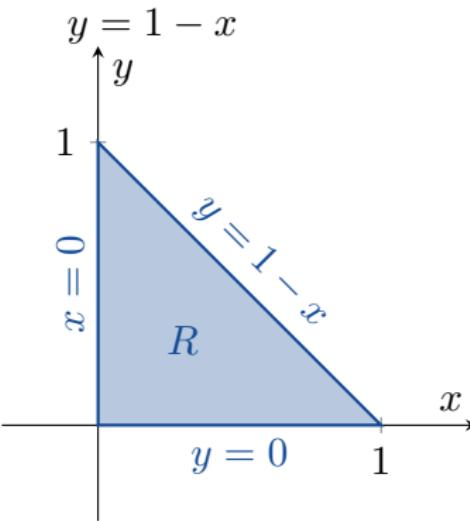
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$$x = 0 \implies$$

$$0 = \frac{u}{3} - \frac{v}{3} \implies u = v$$

$$\cancel{x=1}$$

$$y = 0$$

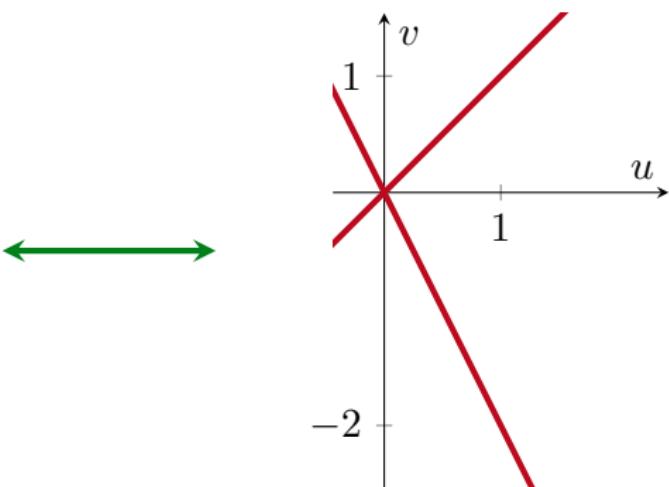
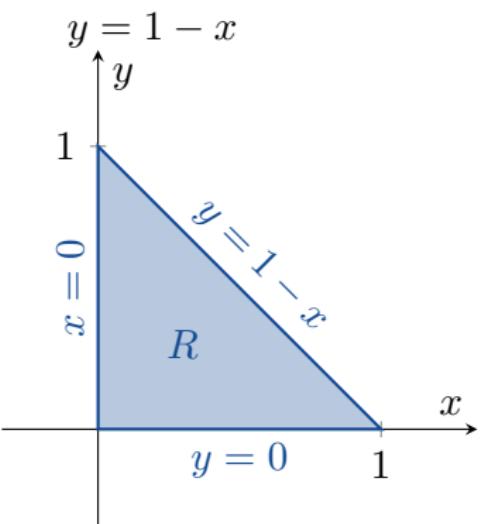


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$$y = 0 \implies 0 = \frac{2u}{3} + \frac{v}{3} \implies v = -2u$$



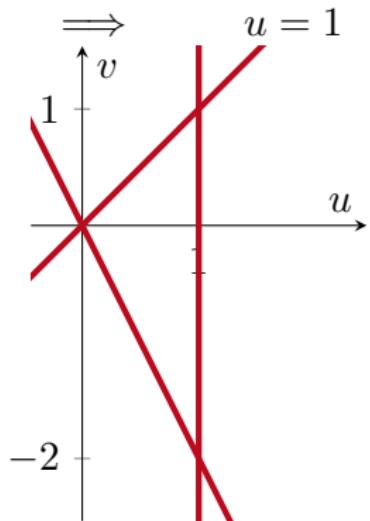
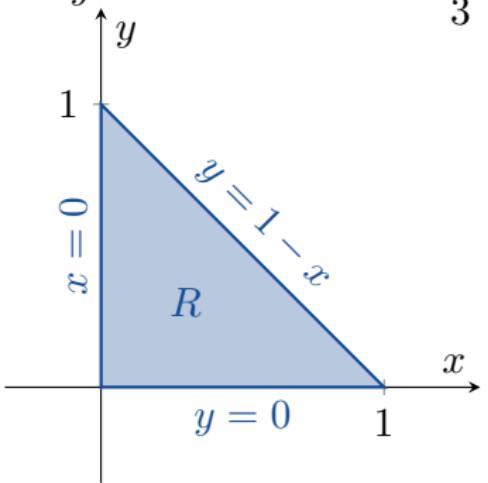
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$$x = 0 \implies 0 = \frac{u}{3} - \frac{v}{3} \implies u = v$$

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$$y = 0 \implies 0 = \frac{2u}{3} + \frac{v}{3} \implies v = -2u$$

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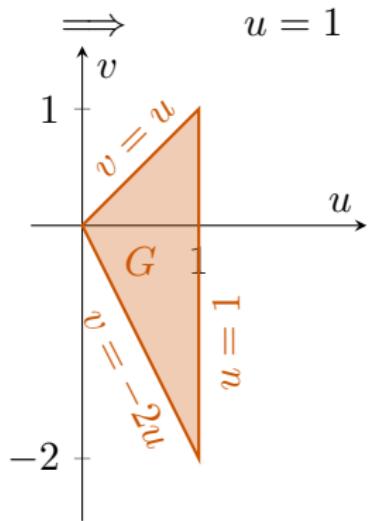
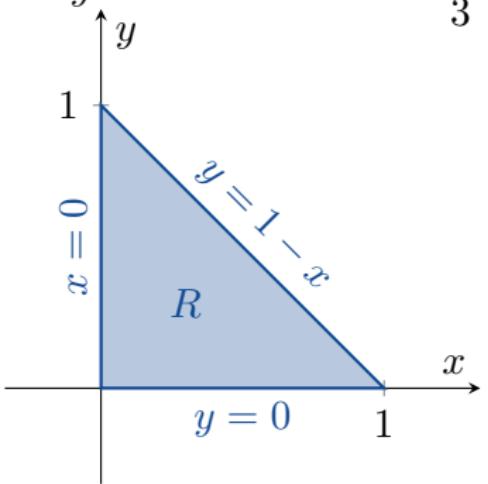
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$$x = 0 \implies 0 = \frac{u}{3} - \frac{v}{3} \implies u = v$$

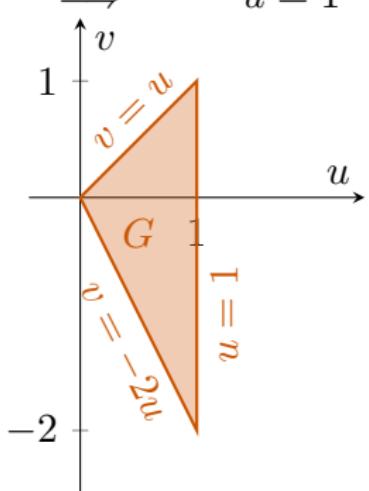
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$$0 \leq u \leq 1$$

$$-2u \leq v \leq u$$



14

$$\begin{aligned} u &= x + y & \frac{\partial(x, y)}{\partial(u, v)} &= \frac{1}{3} & 0 \leq u \leq 1 & \quad -2u \leq v \leq u \\ v &= y - 2x \end{aligned}$$



Therefore

$$\int_0^1 \int_0^{1-x} \sqrt{x+y}(y-2x)^2 dy dx =$$

=

=

=

=

=

14

$$\begin{aligned} u &= x + y & \frac{\partial(x, y)}{\partial(u, v)} &= \frac{1}{3} & 0 \leq u \leq 1 & -2u \leq v \leq u \\ v &= y - 2x \end{aligned}$$



Therefore

$$\begin{aligned}
 \int_0^1 \int_0^{1-x} \sqrt{x+y}(y-2x)^2 dy dx &= \int_0^1 \int_{-2u}^u \sqrt{uv}^2 \frac{1}{3} dv du \\
 &= \\
 &= \\
 &= \\
 &= \\
 &=
 \end{aligned}$$

14

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Therefore

$$\begin{aligned} \int_0^1 \int_0^{1-x} \sqrt{x+y}(y-2x)^2 dy dx &= \int_0^1 \int_{-2u}^u \sqrt{uv}^2 \frac{1}{3} dv du \\ &= \dots \\ &= \dots \\ &= \dots \\ &= \dots \\ &= \frac{2}{9}. \end{aligned}$$

14.8 Substitutions in Multiple Integrals

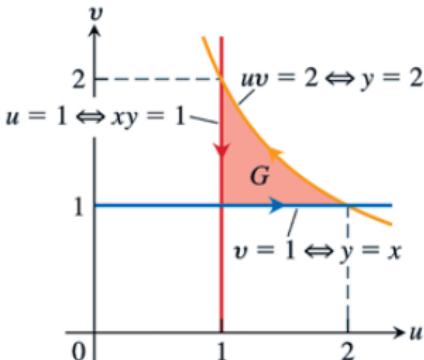
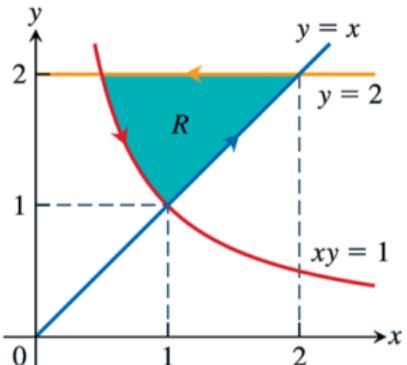
EXAMPLE 4 Evaluate the integral

$$\int_1^2 \int_{1/y}^y \sqrt{\frac{y}{x}} e^{\sqrt{xy}} dx dy.$$

Solution The square root terms in the integrand suggest that we might simplify the integration by substituting $u = \sqrt{xy}$ and $v = \sqrt{y/x}$. Squaring these equations gives $u^2 = xy$ and $v^2 = y/x$, which imply that $u^2v^2 = y^2$ and $u^2/v^2 = x^2$. So we obtain the transformation (in the same ordering of the variables as discussed before)

$$x = \frac{u}{v} \quad \text{and} \quad y = uv,$$

with $u > 0$ and $v > 0$.



Let's first see what happens to the integrand itself under this transformation. The Jacobian of the transformation is not constant:

$$J(u, v) = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} \frac{1}{v} & -\frac{u}{v^2} \\ v & u \end{vmatrix} = \frac{2u}{v}.$$

If G is the region of integration in the uv -plane, then by Equation (2) the transformed double integral under the substitution is

$$\iint_R \sqrt{\frac{y}{x}} e^{\sqrt{xy}} dx dy = \iint_G ve^u \frac{2u}{v} du dv = \iint_G 2ue^u du dv.$$

The transformed integrand function is easier to integrate than the original one, so we proceed to determine the limits of integration for the transformed integral.

The region of integration R of the original integral in the xy -plane is shown in Figure 15.61. From the substitution equations $u = \sqrt{xy}$ and $v = \sqrt{y/x}$, we see that the image of the left-hand boundary $xy = 1$ for R is the vertical line segment $u = 1, 2 \geq v \geq 1$, in G (see Figure 15.62). Likewise, the right-hand boundary $y = x$ of R maps to the horizontal line segment $v = 1, 1 \leq u \leq 2$, in G . Finally, the horizontal top boundary $y = 2$ of R

maps to $uv = 2$, $1 \leq v \leq 2$, in G . As we move counterclockwise around the boundary of the region R , we also move counterclockwise around the boundary of G , as shown in Figure 15.62. Knowing the region of integration G in the uv -plane, we can now write equivalent iterated integrals:

$$\int_1^2 \int_{1/y}^y \sqrt{\frac{y}{x}} e^{\sqrt{xy}} dx dy = \int_1^2 \int_1^{2/u} 2ue^u dv du. \quad \text{Note the order of integration.}$$

We now evaluate the transformed integral on the right-hand side,

$$\begin{aligned} \int_1^2 \int_1^{2/u} 2ue^u dv du &= 2 \int_1^2 \left[vu e^u \right]_{v=1}^{v=2/u} du \\ &= 2 \int_1^2 (2e^u - ue^u) du \\ &= 2 \int_1^2 (2 - u)e^u du \\ &= 2 \left[(2 - u)e^u + e^u \right]_{u=1}^{u=2} \quad \text{Integrate by parts.} \\ &= 2(e^2 - (e + e)) = 2e(e - 2). \end{aligned}$$



Substitutions in Triple Integrals

We use

$$x = g(u, v, w) \quad y = h(u, v, w) \quad z = k(u, v, w)$$

for

$$\iiint_D F \, dxdydz = \iiint_R F \left| \frac{\partial(x, y, z)}{\partial(u, v, w)} \right| \, dudvdw$$

Substitutions in Triple Integrals

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where the *Jacobian* is

$$J(u, v, w) = \frac{\partial(x, y, z)}{\partial(u, v, w)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix}.$$

14.8 Substitutions in Multiple Integrals



Example

Cartesian coordinates \rightarrow Cylindrical coordinates.

$$x = r \cos \theta, y = r \sin \theta, z = z.$$

$$\frac{\partial(x, y, z)}{\partial(r, \theta, z)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial z} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial z} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial z} \end{vmatrix} = \begin{vmatrix} \cos \theta & -r \sin \theta & 0 \\ \sin \theta & r \cos \theta & 0 \\ 0 & 0 & 1 \end{vmatrix}$$

$$= r \cos^2 \theta + r \sin^2 \theta = r.$$

14.8 Substitutions in Multiple Integrals

Example

Cartesian coordinates \rightarrow Spherical coordinates.

$$x = \rho \sin \phi \cos \theta, y = \rho \sin \phi \sin \theta, z = \rho \cos \phi.$$

$$\frac{\partial(x, y, z)}{\partial(\rho, \phi, \theta)} = \begin{vmatrix} \frac{\partial x}{\partial \rho} & \frac{\partial x}{\partial \phi} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial \rho} & \frac{\partial y}{\partial \phi} & \frac{\partial y}{\partial \theta} \\ \frac{\partial z}{\partial \rho} & \frac{\partial z}{\partial \phi} & \frac{\partial z}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \sin \phi \cos \theta & \rho \cos \phi \cos \theta & -\rho \sin \phi \sin \theta \\ \sin \phi \sin \theta & \rho \cos \phi \sin \theta & \rho \sin \phi \cos \theta \\ \cos \phi & -\rho \sin \phi & 0 \end{vmatrix}$$

$$= \rho^2 \sin \phi.$$

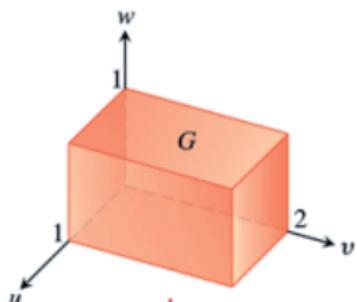
EXAMPLE 5 Evaluate

$$\int_0^3 \int_0^4 \int_{x=y/2}^{x=(y/2)+1} \left(\frac{2x - y}{2} + \frac{z}{3} \right) dx dy dz$$

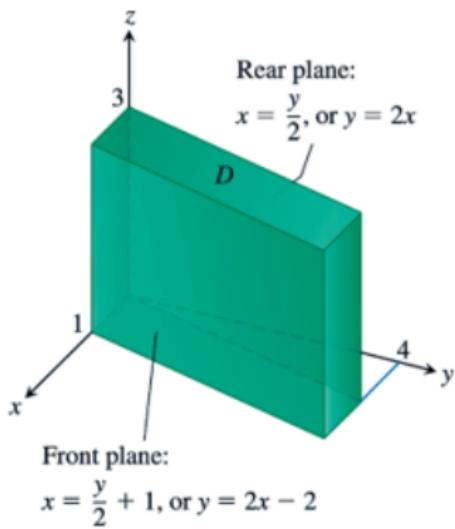
by applying the transformation

$$u = (2x - y)/2, \quad v = y/2, \quad w = z/3 \tag{8}$$

and integrating over an appropriate region in uvw -space.



$$\begin{aligned}x &= u + v \\y &= 2v \\z &= 3w\end{aligned}$$



Rear plane:
 $x = \frac{y}{2}, \text{ or } y = 2x$

Solution We sketch the region D of integration in xyz -space and identify its boundaries (Figure 15.66). In this case, the bounding surfaces are planes.

To apply Equation (7), we need to find the corresponding uvw -region G and the Jacobian of the transformation. To find them, we first solve Equations (8) for x , y , and z in terms of u , v , and w . Routine algebra gives

$$x = u + v, \quad y = 2v, \quad z = 3w. \quad (9)$$

We then find the boundaries of G by substituting these expressions into the equations for the boundaries of D :

xyz-equations for the boundary of D	Corresponding uvw -equations for the boundary of G	Simplified uvw -equations
$x = y/2$	$u + v = 2v/2 = v$	$u = 0$
$x = (y/2) + 1$	$u + v = (2v/2) + 1 = v + 1$	$u = 1$
$y = 0$	$2v = 0$	$v = 0$
$y = 4$	$2v = 4$	$v = 2$
$z = 0$	$3w = 0$	$w = 0$
$z = 3$	$3w = 3$	$w = 1$

The Jacobian of the transformation, again from Equations (9), is

$$J(u, v, w) = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix} = \begin{vmatrix} 1 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{vmatrix} = 6.$$

We now have everything we need to apply Equation (7):

$$\begin{aligned} & \int_0^3 \int_0^4 \int_{x=y/2}^{x=(y/2)+1} \left(\frac{2x - y}{2} + \frac{z}{3} \right) dx dy dz \\ &= \int_0^1 \int_0^2 \int_0^1 (u + w) |J(u, v, w)| du dv dw \\ &= \int_0^1 \int_0^2 \int_0^1 (u + w)(6) du dv dw = 6 \int_0^1 \int_0^2 \left[\frac{u^2}{2} + uw \right]_0^1 dv dw \\ &= 6 \int_0^1 \int_0^2 \left(\frac{1}{2} + w \right) dv dw = 6 \int_0^1 \left[\frac{v}{2} + vw \right]_0^2 dw = 6 \int_0^1 (1 + 2w) dw \\ &= 6 \left[w + w^2 \right]_0^1 = 6(2) = 12. \end{aligned}$$



Next Time

- 9.1 Sequences