2014.04.10 **MAT234 Matematik IV – Ara Sınavın Çözümleri** N. Course

Soru 1 (Series).

(a) [10p] Give the definition of " $\sum_{n=1}^{\infty} a_n$ *converges* ".

Define $s_n = \sum_{j=1}^{n} a_j$. We say that $\sum_{n=1}^{\infty} a_n$ converges iff the sequence (s_n) converges.

Now let

$$a_n = \frac{1}{n(n+1)(n+2)} \qquad \text{and} \qquad s_n = \sum_{j=1}^{n} a_j = a_1 + a_2 + a_3 + \ldots + a_n.$$

(b) [10p] Write a_n in partial fractions (i.e. $a_n = \frac{?}{n} + \frac{?}{n+1} + \frac{?}{n+2}$).

$a_n = \frac{1}{n(n+1)(n+2)} = \frac{(\frac{1}{2})}{n} - \frac{1}{n+1} + \frac{(\frac{1}{2})}{n+2} = \frac{1}{2n} - \frac{2}{2(n+1)} + \frac{1}{2(n+2)}$

(c) [20p] Show that

$$s_n = \frac{1}{4} - \frac{1}{2(n+1)} + \frac{1}{2(n+2)}$$

for all $n \in \mathbb{N}$.

METHOD 1:

$$\begin{aligned}
s_n = \sum_{j=1}^{n} a_n &= \sum_{j=1}^{n} \frac{1}{2j} - 2\sum_{j=1}^{n} \frac{1}{2(j+1)} + \sum_{j=1}^{n} \frac{1}{2(j+2)} \\
&= \left(\frac{1}{2} + \frac{1}{4} + \sum_{j=3}^{n} \frac{1}{2j} \right) - \left(\frac{1}{2} + 2\sum_{j=2}^{n-1} \frac{1}{2(j+1)} + \frac{2}{2(n+1)} \right) \\
&\quad + \left(\sum_{j=1}^{n-2} \frac{1}{2(j+2)} + \frac{1}{2(n+1)} + \frac{1}{2(n+2)} \right) \\
&= \frac{1}{4} + \sum_{j=3}^{n} \frac{1}{2j} - 2\sum_{j=3}^{n} \frac{1}{2j} + \sum_{j=3}^{n} \frac{1}{2j} - \frac{1}{2(n+1)} + \frac{1}{2(n+2)} \\
&= \frac{1}{4} - \frac{1}{2(n+1)} + \frac{1}{2(n+2)}
\end{aligned}$$

METHOD 2: We can prove this using Induction. Since $s_1 = a_1 = \frac{1}{1\times2\times3} = \frac{1}{6}$, and since $\frac{1}{4} - \frac{1}{2(n+1)} + \frac{1}{2(n+2)}\Big|_{n=1} = \frac{1}{4} - \frac{1}{4} + \frac{1}{6} = \frac{1}{6}$, the statement is true for $n = 1$.
Now suppose that it is true for $n = k$. Then $s_k = \frac{1}{4} - \frac{1}{2(k+1)} + \frac{1}{2(k+2)}$. It follows that

$$\begin{aligned} s_{k+1} &= s_k + a_{k+1} \\ &= \left(\frac{1}{4} - \frac{1}{2(k+1)} + \frac{1}{2(k+2)}\right) + \left(\frac{1}{2(k+1)} - \frac{2}{2(k+2)} + \frac{1}{2(k+3)}\right) \\ &= \frac{1}{4} - \frac{1}{2(k+2)} + \frac{1}{2(k+3)} \\ &= \frac{1}{4} - \frac{1}{2(n+1)} + \frac{1}{2(n+2)}\Big|_{n=k+1} \end{aligned}$$

as required.
It follows by the principle of mathematical induction that $s_n = \frac{1}{4} - \frac{1}{2(n+1)} + \frac{1}{2(n+2)}$ for all n.

(d) [5p] Show that

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)(n+2)}$$

converges.

It is easy to see that $s_n = \frac{1}{4} - \frac{1}{2(n+1)} + \frac{1}{2(n+2)} \to \frac{1}{4} + 0 + 0 = \frac{1}{4}$ as $n \to \infty$. Therefore the series also converges.

(e) [5p] Calculate

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)(n+2)}.$$

By my answer to (d), $s_n \to \frac{1}{4}$ as $n \to \infty$. Therefore $\sum_{n=1}^{\infty} \frac{1}{n(n+1)(n+2)} = \frac{1}{4}$.

Soru 2 (Cauchy sequences).

(a) [10p] Give the definition of a *Cauchy sequence*.

We say that (a_n) is a *Cauchy sequence* iff, for all $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that

$$n, m > N \implies |a_n - a_m| < \varepsilon.$$

(b) [15p] Let $b_n = 1 + 10^{-n}$ for all $n \in \mathbb{N}$. **Use the definition** that you wrote in part (a) to show that (b_n) is a Cauchy sequence.

Let $\varepsilon > 0$. $\boxed{4}$ Choose $N \geq \log_{10} \frac{1}{\varepsilon}$. $\boxed{4}$ Then

$$\begin{aligned} n > m > N \boxed{2} \implies |b_n - b_m| &= \left|10^{-n} - 10^{-m}\right| \\ &= 10^{-m}(1 - 10^{m-n}) \\ &\leq 10^{-m} \\ &< 10^{-N} \\ &\leq 10^{-\log_{10} \frac{1}{\varepsilon}} \boxed{4} \\ &= 10^{\log_{10} \varepsilon} \\ &= \varepsilon. \end{aligned}$$

Therefore (b_n) is a Cauchy sequence. $\boxed{1}$

(c) [25p] Let (x_n) be a convergent sequence. Show that (x_n) is a Cauchy sequence.

Let $\varepsilon > 0$. $\boxed{4}$ Since $x_n \to x$ as $n \to \infty$, we know that $\exists\ N \in \mathbb{N}$ $\boxed{5}$ such that

$$n > N \implies |x_n - x| < \frac{\varepsilon}{2}. \boxed{4}$$

It follows that

$$\begin{aligned} n, m > N \implies |x_n - x_m| &= |x_n - x + x - x_m| \\ &\leq |x_n - x| + |x - x_m| \boxed{5} \\ &\qquad \text{(by the triangle inequality)} \boxed{2} \\ &= |x_n - x| + |x_m - m| \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\ &= \varepsilon. \boxed{4} \end{aligned}$$

Therefore (x_n) is a Cauchy sequence. $\boxed{1}$

Soru 3 (Convergent sequences and divergent sequences).

(a) [10p] Let (a_n) be a sequence of real numbers and let $l \in \mathbb{R}$. Give the definition of " $a_n \to l$ as $n \to \infty$ ".

We say that (a_n) tends to l ($a_n \to l$ as $n \to \infty$) iff, for all $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that

$$n > N \implies |a_n - l| < \varepsilon.$$

(b) [10p] Let (b_n) be a sequence of real numbers and let $l \in \mathbb{R}$. Give the definition of " $b_n \not\to l$ as $n \to \infty$ ".

We say that $b_n \not\to l$ as $n \to \infty$ iff, there exists $\varepsilon > 0$ such that for all $N \in \mathbb{N}$, there exists $n \in \mathbb{N}$ such that

$$n > N \quad \text{and} \quad |a_n - l| \geq \varepsilon.$$

(c) [15p] Let $c_n = \frac{1}{2} + \frac{(-1)^n}{2}$ for all $n \in \mathbb{N}$. **Use the definition** that you wrote in part (b) to show that $c_n \not\to 1$ as $n \to \infty$.

Choose $\varepsilon = \frac{1}{2}$. Let $N \in \mathbb{N}$. If N is an even number, choose $n = N+1$; if N is an odd number, choose $n = N+2$. Then $n > N$. Moreover

$$|c_n - 1| = |0 - 1| = 1 \geq \frac{1}{2} = \varepsilon.$$

Therefore $c_n \not\to 1$ as $n \to \infty$.

Let

$$d_n = \frac{n + (-1)^n \sqrt{n}}{(n^2+1)^{1/2}}$$

for all $n \in \mathbb{N}$.

(d) [15p]

- Is (d_n) convergent or divergent?
- Does $\lim_{n\to\infty} d_n$ exist?
- If the limit exists, calculate $\lim_{n\to\infty} d_n$.

You must prove your answers. (For this question, you may use any theorem/lemma/corollary/example from the course.)

[HINT: Use the Sandwich Rule.]

Since

$$n^2 \leq n^2 + 1 \leq (n+1)^2,$$

it follows that

$$n \leq (n^2+1)^{\frac{1}{2}} \leq n+1.$$

Hence

$$1 \leftarrow \frac{1 + \frac{(-1)^n}{\sqrt{n}}}{1 + \frac{1}{n}} = \frac{n + (-1)^n \sqrt{n}}{n+1} \leq \frac{n + (-1)^n \sqrt{n}}{(n^2+1)^{1/2}} \leq \frac{n + (-1)^n \sqrt{n}}{n} = 1 + \frac{(-1)^n}{\sqrt{n}} \to 1$$

as $n \to \infty$. It follows by the Sandwich Rule that $d_n \to 1$ as $n \to \infty$. Therefore (d_n) is a convergent sequence and $\lim_{n\to\infty} d_n = 1$.