

Lecture 5

- 3.4 Repeated Roots of the Characteristic Equation
- 3.5 Reduction of Order
- 3.6 Nonhomogeneous Equations
- 3.7 The Method of Undetermined Coefficients

Summary

To solve

$$ay'' + by' + cy = 0$$

we need to find two linearly independent solutions, $y_1(t)$ and $y_2(t)$. Then the general solution to the ODE is

$$y(t) = c_1 y_1(t) + c_2 y_2(t).$$

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$$y(t) = c_1 y_1(t) + c_2 y_2(t).$$

First we solve the characteristic equation

$$ar^2 + br + c = 0$$

and find the roots r_1 and r_2 .

Summary

- 1 If $r_1, r_2 \in \mathbb{R}$ and $r_1 \neq r_2$, then

$$y_1(t) = e^{r_1 t} \quad \text{and} \quad y_2(t) = e^{r_2 t};$$

- 2 If $r_{1,2} = \lambda \pm i\mu$ ($\lambda, \mu \in \mathbb{R}$), then

$$y_1(t) = e^{\lambda t} \cos \mu t \quad \text{and} \quad y_2(t) = e^{\lambda t} \sin \mu t;$$

- 3 If the roots are repeated, then ?????????????

Repeated Roots of the Characteristic Equation

3.4 Repeated Roots of the Characteristic Equation



Now consider

$$ay'' + by' + cy = 0 \quad (1)$$

where $b^2 - 4ac = 0$. Then the only root of

$$ar^2 + br + c = 0$$

is

$$r = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-b \pm \sqrt{0}}{2a} = -\frac{b}{2a}.$$

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We know that $y_1(t) = e^{-\frac{bt}{2a}}$ is a solution of (1), but how do we find a linearly independent second solution?

3.4 Repeated Roots of the Characteristic Equation



Example

Solve $y'' + 4y' + 4y = 0$.

3.4 Repeated Roots of the Characteristic Equation



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The characteristic equation

$$0 = r^2 + 4r + 4 = (r + 2)^2$$

has repeated root $r_1 = r_2 = -2$.

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- We know that $y_1(t)$ is a solution;

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- We know that $y_1(t)$ is a solution;
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The idea is:

- We know that $y_1(t)$ is a solution;
- So $cy_1(t)$ is a solution for any $c \in \mathbb{R}$;
- Maybe $v(t)y_1(t)$ is a solution for some non-constant function $v(t)$.

3.4 Repeated Roots of the Characteristic Equation



We consider $y_2(t) = v(t)y_1(t)$ for some function $v(t)$ which we don't know yet.

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We consider $y_2(t) = v(t)y_1(t)$ for some function $v(t)$ which we don't know yet. Then we calculate that

$$y_2 = ve^{-2t}$$

$$y_2' = v'e^{-2t} - 2ve^{-2t}$$

$$y_2'' = v''e^{-2t} - 4ve^{-2t} + 4ve^{-2t}$$

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and that

$$\begin{aligned} 0 &= y_2'' + 4y_2' + 4y_2 \\ &= (v''e^{-2t} - 4ve^{-2t} + 4ve^{-2t}) + 4(v'e^{-2t} - 2ve^{-2t}) + 4(ve^{-2t}) \\ &= e^{-2t} [v'' - 4v' + 4v + 4v' - 8v + 4v] \\ &= v''e^{-2t}. \end{aligned}$$

3.4 Repeated Roots of the Characteristic Equation



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Since $e^{-2t} \neq 0$, we must have $v'' = 0$. We can choose *any* non-constant function $v(t)$ which satisfies $v'' = 0$. I like easy functions, so I choose $v(t) = t$. Therefore

$$y_2(t) = te^{-2t}.$$

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$$y_2(t) = te^{-2t}.$$

But are $y_1(t)$ and $y_2(t)$ linearly independent? Since

$$W(y_1, y_2)(t) = \begin{vmatrix} e^{-2t} & te^{-2t} \\ -2e^{-2t} & (1-2t)e^{-2t} \end{vmatrix} = e^{-4t} \neq 0,$$

the answer is YES.

3.4 Repeated Roots of the Characteristic Equation



Therefore $y_1(t) = e^{-2t}$ and $y_2(t) = te^{-2t}$ form a fundamental set of solutions and the general solution is

$$y(t) = c_1e^{-2t} + c_2te^{-2t}.$$

3.4 Repeated Roots of the Characteristic Equation



For the general equation $ay'' + by' + cy = 0$, we can use the same method:

3.4 Repeated Roots of the Characteristic Equation



For the general equation $ay'' + by' + cy = 0$, we can use the same method: We have $y_1(t) = e^{rt} = e^{-\frac{bt}{2a}}$ and we guess that $y_2(t) = v(t)e^{-\frac{bt}{2a}}$ for some function $v(t)$.

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$$0 = ay_2'' + by_2' + cy_2 = \dots = ae^{-\frac{bt}{2a}}v''.$$

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$$0 = ay_2'' + by_2' + cy_2 = \dots = ae^{-\frac{bt}{2a}}v''.$$

So again we want $v'' = 0$ and we can choose $v(t) = t$. Thus $y_2(t) = te^{rt} = te^{-\frac{bt}{2a}}$.

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For the general equation $ay'' + by' + cy = 0$, we can use the same method: We have $y_1(t) = e^{rt} = e^{-\frac{bt}{2a}}$ and we guess that $y_2(t) = v(t)e^{-\frac{bt}{2a}}$ for some function $v(t)$. Then we calculate (you fill in the details)

$$0 = ay_2'' + by_2' + cy_2 = \dots = ae^{-\frac{bt}{2a}}v''.$$

So again we want $v'' = 0$ and we can choose $v(t) = t$. Thus $y_2(t) = te^{rt} = te^{-\frac{bt}{2a}}$.

I leave it for you to calculate that $W(e^{rt}, te^{rt}) \neq 0$. Thus e^{rt} and te^{rt} form a fundamental set of solutions to (1).

3.4 Repeated Roots of the Characteristic Equation



Example

Solve

$$\begin{cases} y'' - y' + \frac{1}{4}y = 0 \\ y(0) = 2 \\ y'(0) = \frac{1}{3}. \end{cases}$$

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$$\begin{cases} y'' - y' + \frac{1}{4}y = 0 \\ y(0) = 2 \\ y'(0) = \frac{1}{3}. \end{cases}$$

The characteristic equation

$$0 = r^2 - r + \frac{1}{4} = \left(r - \frac{1}{2}\right)^2$$

has repeated root $r = \frac{1}{2}$.

3.4 Repeated Roots of the Characteristic Equation



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The characteristic equation

$$0 = r^2 - r + \frac{1}{4} = \left(r - \frac{1}{2}\right)^2$$

has repeated root $r = \frac{1}{2}$. So we know that the general solution to the ODE is

$$y(t) = c_1 e^{\frac{t}{2}} + c_2 t e^{\frac{t}{2}}.$$

3.4 Repeated Roots of the Characteristic Equation



Next we need to look at the initial conditions: Since $y'(t) = \frac{1}{2}c_1e^{\frac{t}{2}} + c_2e^{\frac{t}{2}} + \frac{1}{2}c_2te^{\frac{t}{2}}$, we have that

$$\begin{aligned} 2 = y(0) &= c_1 + 0 &\implies c_1 &= 2 \\ \frac{1}{3} = y'(0) &= \frac{1}{2}c_1 + c_2 + 0 &\implies c_2 &= -\frac{2}{3}. \end{aligned}$$

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Therefore the solution to the IVP is

$$y = 2e^{\frac{t}{2}} - \frac{2}{3}te^{\frac{t}{2}}.$$

3.4 Repeated Roots of the Characteristic Equation



Example

Now solve

$$\begin{cases} y'' - y' + \frac{1}{4}y = 0 \\ y(0) = 2 \\ y'(0) = 2 \end{cases}$$

3.4 Repeated Roots of the Characteristic Equation



Example

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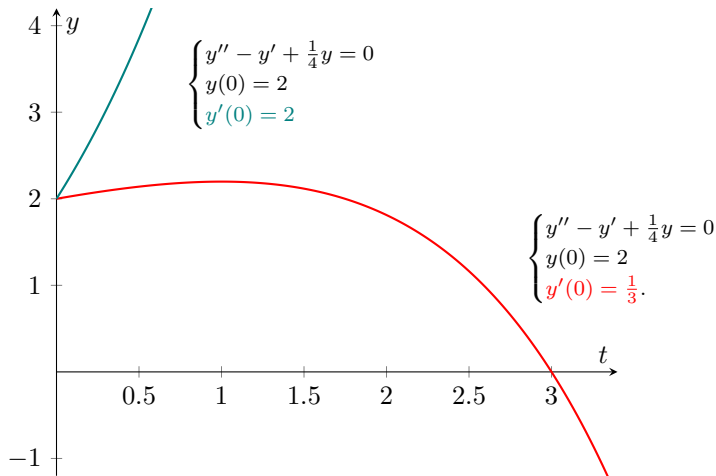
$$\begin{cases} y'' - y' + \frac{1}{4}y = 0 \\ y(0) = 2 \\ y'(0) = 2 \end{cases}$$

You can check that the solution is

$$y = 2e^{\frac{t}{2}} + te^{\frac{t}{2}}.$$

The graph of this solution, and the solution to the previous example, are shown on the next slide.

3.4 Repeated Roots of the Characteristic Equation



Note that even though these two functions share the same $y(0)$ value, and that their $y'(0)$ value does not differ by much, their behaviour as $t \rightarrow \infty$ is very different.

3.4 Repeated Roots of the Characteristic Equation



Summary

To solve

$$ay'' + by' + cy = 0$$

we need to find two linearly independent solutions.

- 1** If $r_1, r_2 \in \mathbb{R}$ and $r_1 \neq r_2$, then

$$y_1(t) = e^{r_1 t} \quad \text{and} \quad y_2(t) = e^{r_2 t};$$

- 2** If $r_{1,2} = \lambda \pm i\mu$ ($\lambda, \mu \in \mathbb{R}$), then

$$y_1(t) = e^{\lambda t} \cos \mu t \quad \text{and} \quad y_2(t) = e^{\lambda t} \sin \mu t;$$

- 3** If $r_1, r_2 \in \mathbb{R}$ but $r_1 = r_2$, then

$$y_1(t) = e^{r_1 t} \quad \text{and} \quad y_2(t) = te^{r_1 t}.$$

Reduction of Order

3.5 Reduction of Order



Consider

$$y'' + p(t)y' + q(t)y = 0. \quad (2)$$

3.5 Reduction of Order



Consider

$$y'' + p(t)y' + q(t)y = 0. \quad (2)$$

Suppose that we know that $y_1(t)$ is a solution to (2) and suppose that we want to find a second, linearly independent solution.

3.5 Reduction of Order



Consider

$$y'' + p(t)y' + q(t)y = 0. \quad (2)$$

Suppose that we know that $y_1(t)$ is a solution to (2) and suppose that we want to find a second, linearly independent solution.

The main idea in this section is that we guess that

$$y_2(t) = v(t)y_1(t)$$

for some non-constant function $v(t)$. If we can find $v(t)$, then we have our $y_2(t)$.

3.5 Reduction of Order



Then we calculate that

$$y_2 = v y_1$$

$$y_2' = v' y_1 + v y_1'$$

$$y_2'' = v'' y_1 + 2v' y_1' + v y_1''$$

and

$$0 = y_2'' + p y_2' + q y_2$$

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and

$$\begin{aligned} 0 &= y_2'' + p y_2' + q y_2 \\ &= (v'' y_1 + 2v' y_1' + v y_1'') + p(t) (v' y_1 + v y_1') + q(t) (v y_1) \\ &= \\ &= \end{aligned}$$

3.5 Reduction of Order



Then we calculate that

$$y_2 = vy_1$$

$$y_2' = v'y_1 + vy_1'$$

$$y_2'' = v''y_1 + 2v'y_1' + vy_1''$$

and

$$\begin{aligned} 0 &= y_2'' + py_2' + qy_2 \\ &= (v''y_1 + 2v'y_1' + vy_1'') + p(t)(v'y_1 + vy_1') + q(t)(vy_1) \\ &= v''y_1 + v'(2y_1' + py_1) + \underbrace{v(y_1'' + py_1' + qy_1)}_{=0} \\ &= \end{aligned}$$

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Then we calculate that

$$y_2 = v y_1$$

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and

$$\begin{aligned} 0 &= y_2'' + p y_2' + q y_2 \\ &= (v'' y_1 + 2v' y_1' + v y_1'') + p(t) (v' y_1 + v y_1') + q(t) (v y_1) \\ &= v'' y_1 + v' (2y_1' + p y_1) + \underbrace{v(y_1'' + p y_1' + q y_1)}_{=0} \\ &= v'' y_1 + v' (2y_1' + p y_1). \end{aligned}$$

3.5 Reduction of Order



$$0 = v''y_1 + v' (2y_1' + py_1) + 0v$$

Remark

Note that since y_1 solves the ODE, we must always get “ $0v$ ” here. We can have v' and v'' terms, but if you do a reduction of order calculation and still have v terms, then you have made a mistake.

Remark

$$v''y_1 + v'(2y_1' + py_1) = 0 \quad (3)$$

is actually a first order ODE for v' .

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$$u'y_1 + u(2y_1' + py_1) = 0. \quad (4)$$

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If we can find $u(t)$, then we can find $v(t) = \int u(t) dt$ and $y_2(t) = v(t)y_1(t)$.

3.5 Reduction of Order



Remark

Instead of having to solve a second order ODE to find y_2 , we only need to solve a first order ODE to find $u(t)$. Hence the name “Reduction of Order”.

3.5 Reduction of Order



Remark

The method is

- 1 Guess $y_2 = vy_1$.
- 2 Put this into your ODE and find an equation for v ;
- 3 Set $u = v'$;
- 4 Find u ;
- 5 Integrate to find v ;
- 6 Then $y_2(t) = v(t)y_1(t)$.

3.5 Reduction of Order



Example

Given that $y_1(t) = \frac{1}{t}$ is a solution of

$$2t^2y'' + 3ty' - y = 0, \quad t > 0$$

find a linearly independent second solution.

3.5 Reduction of Order



$$y_1(t) = \frac{1}{t}$$

Let $y_2(t) = v(t)y_1(t)$. Then we have

$$y_2 = vt^{-1}$$

$$y_2' = v't^{-1} - vt^{-2}$$

$$y_2'' = v''t^{-1} - 2v't^{-2} + 2vt^{-3}$$

and

$$0 = 2t^2 y_2'' + 3t y_2' - y_2$$

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3.5 Reduction of Order



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Let $y_2(t) = v(t)y_1(t)$. Then we have

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3.5 Reduction of Order



$$2tv'' - v' = 0$$

Now let $u = v'$.

3.5 Reduction of Order



$$2tv'' - v' = 0$$

Now let $u = v'$. We need to solve

$$2t \frac{du}{dt} - u = 0.$$

3.5 Reduction of Order



$$2tv'' - v' = 0$$

Now let $u = v'$. We need to solve

$$2t \frac{du}{dt} - u = 0.$$

This equation is both linear and separable, so we know 2 ways to solve it.

3.5 Reduction of Order



$$2t \frac{du}{dt} = u$$

$$\frac{du}{u} = \frac{1}{2} \frac{dt}{t}$$

$$\int \frac{du}{u} = \int \frac{1}{2} \frac{dt}{t}$$

$$\ln |u| = \frac{1}{2} \ln |t| + C$$

$$e^{\ln |u|} = e^{\ln |t|^{\frac{1}{2}}} e^C$$

$$|u| = |t|^{\frac{1}{2}} e^C$$

$$u = \pm e^C t^{\frac{1}{2}} = c t^{\frac{1}{2}}.$$

3.5 Reduction of Order



$$u(t) = ct^{\frac{1}{2}}$$

Then we have

$$v(t) = \int u(t) dt = \int ct^{\frac{1}{2}} dt = \frac{2}{3}ct^{\frac{3}{2}} + k$$

3.5 Reduction of Order



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and

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and

$$y_2(t) = v(t)t^{-1} = \frac{2}{3}ct^{\frac{1}{2}} + kt^{-1}.$$

Remember that we are trying to find a solution that is linearly independent from $y_1(t) = t^{-1}$. The second term in $y_2(t) = \frac{2}{3}ct^{\frac{1}{2}} + kt^{-1}$ is just a multiple of $y_1(t)$ – we don't need this. So it is ok to choose $k = 0$.

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Remember that we are trying to find a solution that is linearly independent from $y_1(t) = t^{-1}$. The second term in $y_2(t) = \frac{2}{3}ct^{\frac{1}{2}} + kt^{-1}$ is just a multiple of $y_1(t)$ – we don't need this. So it is ok to choose $k = 0$. Hence

$$y_2(t) = \frac{2}{3}ct^{\frac{1}{2}}$$

3.5 Reduction of Order



$$y_2(t) = \frac{2}{3}ct^{\frac{1}{2}}$$

Finally, since I like simple functions I choose $c = \frac{3}{2}$ to get

$$y_2(t) = t^{\frac{1}{2}}.$$

I leave it to you to check that $W(t^{-1}, t^{\frac{1}{2}})$ is not always zero.

3.5 Reduction of Order



Example

Given that $y_1(t) = t$ solves

$$t^2 y'' + 2ty' - 2y = 0, \quad t > 0,$$

find a second linearly independent solution $y_2(t)$.

3.5 Reduction of Order



Example

Given that $y_1(t) = t$ solves

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find a second linearly independent solution $y_2(t)$.

We start with $y_2(t) = v(t)y_1(t) = v(t)t$.

3.5 Reduction of Order



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Given that $y_1(t) = t$ solves

$$t^2 y'' + 2ty' - 2y = 0, \quad t > 0,$$

find a second linearly independent solution $y_2(t)$.

We start with $y_2(t) = v(t)y_1(t) = v(t)t$. Then $y_2' = v't + v$ and $y_2'' = v''t + 2v'$.

3.5 Reduction of Order



Example

Given that $y_1(t) = t$ solves

$$t^2 y'' + 2ty' - 2y = 0, \quad t > 0,$$

find a second linearly independent solution $y_2(t)$.

We start with $y_2(t) = v(t)y_1(t) = v(t)t$. Then $y_2' = v't + v$ and $y_2'' = v''t + 2v'$. Substituting into the ODE, we calculate that

$$\begin{aligned} 0 &= t^2 y_2'' + 2ty_2' - 2y_2 \\ &= t^2(v''t + 2v') + 2t(v't + v) - 2vt \\ &= t^3 v'' + v'(2t^2 + 2t^2) + v(2t - 2t) \\ &= t^3 v'' + 4t^2 v' \\ &= t^2(tv'' + 4v'). \end{aligned}$$

3.5 Reduction of Order



$$t^2(tv'' + 4v') = 0$$

Letting $u = v'$, we obtain the first order ODE

$$t \frac{du}{dt} + 4u = 0.$$

3.5 Reduction of Order



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Letting $u = v'$, we obtain the first order ODE

$$t \frac{du}{dt} + 4u = 0.$$

We calculate that

$$\begin{aligned} t \frac{du}{dt} &= -4u \\ \frac{du}{u} &= -4 \frac{dt}{t} \\ \int \frac{du}{u} &= -4 \int \frac{dt}{t} \\ \ln |u| &= -4 \ln |t| + C \\ u &= \pm e^C t^{-4} = ct^{-4} \end{aligned}$$

3.5 Reduction of Order



$$y_1(t) = t \qquad v' = u \qquad u = ct^{-4}$$

and

$$\begin{aligned} v &= \int u \, dt = \int ct^{-4} \, dt \\ &= -\frac{1}{3}ct^{-3} + k. \end{aligned}$$

3.5 Reduction of Order



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Thus

$$y_2(t) = v(t)t = -\frac{1}{3}ct^{-2} + kt.$$

3.5 Reduction of Order



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Thus

$$y_2(t) = v(t)t = -\frac{1}{3}ct^{-2} + kt.$$

Choosing $c = -3$ and $k = 0$, we obtain the solution

$$y_2(t) = t^{-2}.$$

3.5 Reduction of Order



Does $y_2(t) = t^{-2}$ really solve $t^2 y'' + 2ty' - 2y = 0$?

3.5 Reduction of Order



Does $y_2(t) = t^{-2}$ really solve $t^2 y'' + 2ty' - 2y = 0$?

Since $y_2' = -2t^{-3}$ and $y_2'' = 6t^{-4}$, we have that

$$\begin{aligned} t^2 y_2'' + 2ty_2' - 2y_2 &= t^2(6t^{-4}) + 2t(-2t^{-3}) - 2t^{-2} \\ &= 6t^{-2} - 4t^{-2} - 2t^{-2} \\ &= 0. \end{aligned}$$

The answer is YES!!

3.5 Reduction of Order



Are $y_1(t) = t$ and $y_2(t) = t^{-2}$ linearly independent?

Are $y_1(t) = t$ and $y_2(t) = t^{-2}$ linearly independent?

We have that

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} t & t^{-2} \\ 1 & -2t^{-3} \end{vmatrix} = -2t^{-2} - t^{-2} = -3t^{-2} \neq 0$$

since $t > 0$. Therefore y_1 and y_2 are linearly independent.

Nonhomogeneous Equations

3.6 Nonhomogeneous Equations



Consider

$$y'' + p(t)y' + q(t)y = g(t). \quad (5)$$

3.6 Nonhomogeneous Equations



Consider

$$y'' + p(t)y' + q(t)y = g(t). \quad (5)$$

The equation

$$y'' + p(t)y' + q(t)y = 0 \quad (6)$$

is called the *homogeneous equation corresponding to (5)*.

3.6 Nonhomogeneous Equations



$$y'' + p(t)y' + q(t)y = g(t) \quad (5)$$

Theorem

The general solution to (5) can be written in the form

$$y(t) = c_1 y_1(t) + c_2 y_2(t) + Y(t)$$

where

3.6 Nonhomogeneous Equations



$$y'' + p(t)y' + q(t)y = g(t) \quad (5)$$

Theorem

The general solution to (5) can be written in the form

$$y(t) = c_1 y_1(t) + c_2 y_2(t) + Y(t)$$

where

- *y_1 and y_2 form a fundamental set of solutions to the homogeneous equation corresponding to (5);*
- *c_1 and c_2 are constants; and*
- *Y is a particular solution to (5).*

3.6 Nonhomogeneous Equations



To solve $L[y] = g$

- 1 Find the general solution to $L[y] = 0$;
- 2 Find a particular solution to $L[y] = g$;
- 3 1 + 2

3.6 Nonhomogeneous Equations



To solve $L[y] = g$

- 1 Find the general solution to $L[y] = 0$;
- 2 Find a particular solution to $L[y] = g$;
- 3 1 + 2

We will study 2 methods to do step 2. One method this week and one method next week.

The Method of Undetermined Coefficients

3.7 The Method of Undetermined Coefficients



$$y'' + p(t)y' + q(t)y = g(t) \quad (5)$$

The idea is:

- 1 Look at $g(t)$
- 2 Make a guess with constants
- 3 Try to find the constants

3.7 The Method of Undetermined Coefficients



Example

Find a particular solution to $y'' - 3y' - 4y = 3e^{2t}$.

3.7 The Method of Undetermined Coefficients



Example

Find a particular solution to $y'' - 3y' - 4y = 3e^{2t}$.

Here we have $g(t) = 3e^{2t}$. We look at this g and we make a guess:

3.7 The Method of Undetermined Coefficients



Example

Find a particular solution to $y'' - 3y' - 4y = 3e^{2t}$.

Here we have $g(t) = 3e^{2t}$. We look at this g and we make a guess: g includes e^{2t} so we guess that $Y(t)$ also includes e^{2t} . So we guess that $Y(t) = Ae^{2t}$ for some constant A .

3.7 The Method of Undetermined Coefficients



Example

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We must try to find A . We calculate that

$$Y(t) = Ae^{2t} \quad Y'(t) = 2Ae^{2t} \quad Y''(t) = 4Ae^{2t}$$

3.7 The Method of Undetermined Coefficients



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and

$$\begin{aligned} 3e^{2t} &= Y'' - 3Y' - 4Y = 4Ae^{2t} - 3(2Ae^{2t}) - 4(Ae^{2t}) \\ &= -6Ae^{2t}. \end{aligned}$$

3.7 The Method of Undetermined Coefficients



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We must have $A = -\frac{1}{2}$.

3.7 The Method of Undetermined Coefficients



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We must have $A = -\frac{1}{2}$. Therefore a particular solution is

$$Y(t) = -\frac{1}{2}e^{2t}.$$

3.7 The Method of Undetermined Coefficients



Example

Find a particular solution to $y'' - 3y' - 4y = 4t^2 - 1$.

3.7 The Method of Undetermined Coefficients



Example

Find a particular solution to $y'' - 3y' - 4y = 4t^2 - 1$.

Since $g(t) = 4t^2 - 1$ is a 2nd degree polynomial, we guess that Y is also a second degree polynomial. So we try the ansatz

$$Y(t) = At^2 + Bt + C.$$

I will leave this example for you to finish.

3.7 The Method of Undetermined Coefficients



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Find a particular solution to $y'' - 3y' - 4y = 4t^2 - 1$.

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I will leave this example for you to finish.

(ansatz = a mathematical guess)

3.7 The Method of Undetermined Coefficients



Example

Find a particular solution to $y'' - 3y' - 4y = 2 \sin t$.

3.7 The Method of Undetermined Coefficients



Example

Find a particular solution to $y'' - 3y' - 4y = 2 \sin t$.

First guess: $Y(t) = A \sin t$.

3.7 The Method of Undetermined Coefficients



Example

Find a particular solution to $y'' - 3y' - 4y = 2 \sin t$.

First guess: $Y(t) = A \sin t$. Then $Y' = A \cos t$ and $Y'' = -A \sin t$.

3.7 The Method of Undetermined Coefficients



Example

Find a particular solution to $y'' - 3y' - 4y = 2 \sin t$.

First guess: $Y(t) = A \sin t$. Then $Y' = A \cos t$ and $Y'' = -A \sin t$. Hence

$$\begin{aligned} 2 \sin t &= Y'' - 3Y' - 4Y \\ &= (-A \sin t) - 3(A \cos t) - 4(A \sin t) = -5A \sin t - 3A \cos t. \end{aligned}$$

3.7 The Method of Undetermined Coefficients



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We can see that we must have

$$\begin{cases} -5A = 2 \\ -3A = 0. \end{cases}$$

3.7 The Method of Undetermined Coefficients



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We can see that we must have

$$\begin{cases} -5A = 2 \\ -3A = 0. \end{cases}$$

This linear system is inconsistent: It not possible to find a constant A which satisfies both of these equations. Our first guess failed.

3.7 The Method of Undetermined Coefficients



Second guess: $Y(t) = A \sin t + B \cos t$.

3.7 The Method of Undetermined Coefficients



Second guess: $Y(t) = A \sin t + B \cos t$. Then we calculate that

$$Y' = A \cos t - B \sin t, \quad Y'' = -A \sin t - B \cos t$$

and

$$\begin{aligned} 2 \sin t &= Y'' - 3Y' - 4Y \\ &= (-A \sin t - B \cos t) - 3(A \cos t - B \sin t) - 4(A \sin t + B \cos t) \\ &= (-5A + 3B) \sin t + (-3A - 5B) \cos t. \end{aligned}$$

3.7 The Method of Undetermined Coefficients



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So we need A and B to satisfy

$$\begin{cases} -5A + 3B = 2 \\ -3A - 5B = 0. \end{cases}$$

3.7 The Method of Undetermined Coefficients



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Please check that the solution to this linear system is $A = -\frac{5}{17}$ and $B = \frac{3}{17}$.

3.7 The Method of Undetermined Coefficients



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Please check that the solution to this linear system is $A = -\frac{5}{17}$ and $B = \frac{3}{17}$. Therefore a particular solution is

$$Y(t) = -\frac{5}{17} \sin t + \frac{3}{17} \cos t.$$

3.7 The Method of Undetermined Coefficients



Remark

\sin and \cos are friends! They always go together. If you see either \sin or \cos in $g(t)$, then your ansatz needs to contain *both* \sin and \cos .

Likewise \sinh and \cosh always go together.

3.7 The Method of Undetermined Coefficients



Example

Find a particular solution to $y'' - 3y' - 4y = -8e^t \cos 2t$.

3.7 The Method of Undetermined Coefficients



Example

Find a particular solution to $y'' - 3y' - 4y = -8e^t \cos 2t$.

We will try the ansatz

$$Y(t) = Ae^t \cos 2t + Be^t \sin 2t.$$

Then

3.7 The Method of Undetermined Coefficients



Example

Find a particular solution to $y'' - 3y' - 4y = -8e^t \cos 2t$.

We will try the ansatz

$$Y(t) = Ae^t \cos 2t + Be^t \sin 2t.$$

Then

$$\begin{aligned} Y'(t) &= Ae^t \cos 2t - 2Ae^t \sin 2t + Be^t \sin 2t + 2Be^t \cos 2t \\ &= (A + 2B)e^t \cos 2t + (B - 2A)e^t \sin 2t, \\ Y''(t) &= (A + 2B)e^t \cos 2t - 2(A + 2B)e^t \sin 2t + (B - 2A)e^t \sin 2t \\ &\quad + 2(B - 2A)e^t \cos 2t \\ &= (-3A + 4B)e^t \cos 2t + (-4A - 3B)e^t \sin 2t \end{aligned}$$

3.7 The Method of Undetermined Coefficients



and

$$\begin{aligned}-8e^t \cos 2t &= Y'' - 3Y' - 4Y \\ &= (-3A + 4B)e^t \cos 2t + (-4A - 3B)e^t \sin 2t \\ &\quad + (-3A - 6B)e^t \cos 2t + (-3B + 6A)e^t \sin 2t \\ &\quad + (-4A)e^t \cos 2t + (-4B)e^t \sin 2t \\ &= (-10A - 2B)e^t \cos 2t + (2A - 10B)e^t \sin 2t.\end{aligned}$$

3.7 The Method of Undetermined Coefficients



and

$$\begin{aligned}-8e^t \cos 2t &= Y'' - 3Y' - 4Y \\&= (-3A + 4B)e^t \cos 2t + (-4A - 3B)e^t \sin 2t \\&\quad + (-3A - 6B)e^t \cos 2t + (-3B + 6A)e^t \sin 2t \\&\quad + (-4A)e^t \cos 2t + (-4B)e^t \sin 2t \\&= (-10A - 2B)e^t \cos 2t + (2A - 10B)e^t \sin 2t.\end{aligned}$$

Thus we must solve

$$\begin{cases} 10A + 2B = 8 \\ 2A - 10B = 0. \end{cases}$$

3.7 The Method of Undetermined Coefficients



and

$$\begin{aligned}-8e^t \cos 2t &= Y'' - 3Y' - 4Y \\ &= (-3A + 4B)e^t \cos 2t + (-4A - 3B)e^t \sin 2t \\ &\quad + (-3A - 6B)e^t \cos 2t + (-3B + 6A)e^t \sin 2t \\ &\quad + (-4A)e^t \cos 2t + (-4B)e^t \sin 2t \\ &= (-10A - 2B)e^t \cos 2t + (2A - 10B)e^t \sin 2t.\end{aligned}$$

Thus we must solve

$$\begin{cases} 10A + 2B = 8 \\ 2A - 10B = 0. \end{cases}$$

Please check that the solution to this linear system is $A = \frac{10}{13}$ and $B = \frac{2}{13}$.

3.7 The Method of Undetermined Coefficients



and

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Thus we must solve

$$\begin{cases} 10A + 2B = 8 \\ 2A - 10B = 0. \end{cases}$$

Please check that the solution to this linear system is $A = \frac{10}{13}$ and $B = \frac{2}{13}$. Therefore a particular solution is

$$Y(t) = \frac{10}{13}e^t \cos 2t + \frac{2}{13}e^t \sin 2t.$$

3.7 The Method of Undetermined Coefficients



Theorem

$$\left. \begin{array}{l} Y_1 \text{ solves} \\ ay'' + by' + cy = g_1(t) \\ \\ Y_2 \text{ solves} \\ ay'' + by' + cy = g_2(t) \end{array} \right\} \Rightarrow Y_1 + Y_2 \text{ solves} \\ ay'' + by' + cy = g_1(t) + g_2(t)$$

3.7 The Method of Undetermined Coefficients



Example

Find a particular solution to

$$y'' - 3y' - 4y = 3e^{2t} + 2 \sin t - 8e^t \cos 2t. \quad (7)$$

3.7 The Method of Undetermined Coefficients



Example

Find a particular solution to

$$y'' - 3y' - 4y = 3e^{2t} + 2\sin t - 8e^t \cos 2t. \quad (7)$$

We can split this problem up into three easier problems:

$$y'' - 3y' - 4y = 3e^{2t}$$

$$y'' - 3y' - 4y = 2\sin t$$

$$y'' - 3y' - 4y = -8e^t \cos 2t$$

We know particular solutions to these three ODEs.

3.7 The Method of Undetermined Coefficients



Example

Find a particular solution to

$$y'' - 3y' - 4y = 3e^{2t} + 2\sin t - 8e^t \cos 2t. \quad (7)$$

We can split this problem up into three easier problems:

$$y'' - 3y' - 4y = 3e^{2t}$$

$$y'' - 3y' - 4y = 2\sin t$$

$$y'' - 3y' - 4y = -8e^t \cos 2t$$

We know particular solutions to these three ODEs. Therefore

$$Y(t) = -\frac{1}{2}e^{2t} - \frac{5}{17}\sin t + \frac{3}{17}\cos t + \frac{10}{13}e^t \cos 2t + \frac{2}{13}e^t \sin 2t.$$

is a particular solution to (7).

3.7 The Method of Undetermined Coefficients



Remark

To find a particular solution to $ay'' + by' + cy = g(t)$, we have been looking at $g(t)$ and choosing a similar function for $Y(t)$.

3.7 The Method of Undetermined Coefficients



Remark

To find a particular solution to $ay'' + by' + cy = g(t)$, we have been looking at $g(t)$ and choosing a similar function for $Y(t)$.

This method doesn't always work: There is a difficulty that can occur as we shall see in the next example.

3.7 The Method of Undetermined Coefficients



Example

Find a particular solution to $y'' + 4y = 3 \cos 2t$.

First guess: $Y(t) = A \cos 2t + B \sin 2t$.

3.7 The Method of Undetermined Coefficients



Example

Find a particular solution to $y'' + 4y = 3 \cos 2t$.

First guess: $Y(t) = A \cos 2t + B \sin 2t$.

Then we have that

$$Y' = -2A \sin 2t + 2B \cos 2t$$

$$Y'' = -4A \cos 2t - 4B \sin 2t$$

and

$$\begin{aligned} 3 \cos 2t &= Y'' + 4Y \\ &= (-4A \cos 2t - 4B \sin 2t) + 4(A \cos 2t + B \sin 2t) = 0. \end{aligned}$$

3.7 The Method of Undetermined Coefficients



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$$\begin{aligned} 3 \cos 2t &= Y'' + 4Y \\ &= (-4A \cos 2t - 4B \sin 2t) + 4(A \cos 2t + B \sin 2t) = 0. \end{aligned}$$

This is a FAILURE!!! It not possible to choose A and B such that

$$3 \cos 2t = 0$$

for all t .

3.7 The Method of Undetermined Coefficients



Why did this happen? Why didn't our usual method work?

3.7 The Method of Undetermined Coefficients



Why did this happen? Why didn't our usual method work? To understand why, let us solve the homogeneous equation $y'' + 4y = 0$.

3.7 The Method of Undetermined Coefficients



Why did this happen? Why didn't our usual method work? To understand why, let us solve the homogeneous equation $y'' + 4y = 0$. The characteristic equation is

$$0 = r^2 + 4 = (r + 2i)(r - 2i).$$

So $r = \pm 2i$.

3.7 The Method of Undetermined Coefficients



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So $r = \pm 2i$. It follows that the general solution to the homogeneous equation is

$$y(t) = c_1 \cos 2t + c_2 \sin 2t.$$

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Since $\cos 2t$ and $\sin 2t$ appear in the general solution to the homogeneous equation, we can not use $\cos 2t$ and $\sin 2t$ in a *particular solution*.

3.7 The Method of Undetermined Coefficients



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RULE: If in doubt, multiply by t .

3.7 The Method of Undetermined Coefficients



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Since $\cos 2t$ and $\sin 2t$ appear in the general solution to the homogeneous equation, we can not use $\cos 2t$ and $\sin 2t$ in a particular solution.

RULE: If in doubt, multiply by t .

We need two functions which, when differentiated, give us $\cos 2t$ and $\sin 2t$. We will try $t \cos 2t$ and $t \sin 2t$ because

$$\frac{d}{dt} t \cos 2t = \cos 2t - 2t \sin 2t \text{ and } \frac{d}{dt} t \sin 2t = \sin 2t + 2t \cos 2t.$$

3.7 The Method of Undetermined Coefficients



Second guess: $Y(t) = At \cos 2t + Bt \sin 2t$.

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We have that

$$\begin{aligned} Y' &= A \cos 2t - 2At \sin 2t + B \sin 2t + 2Bt \cos 2t \\ &= (A + 2Bt) \cos 2t + (B - 2At) \sin 2t, \\ Y'' &= 2B \cos 2t - 2(A + 2Bt) \sin 2t - 2A \sin 2t + 2(B - 2At) \cos 2t \\ &= (4B - 4At) \cos 2t + (-4A - 4Bt) \sin 2t \end{aligned}$$

and

$$\begin{aligned} 3 \cos 2t &= Y'' + 4Y \\ &= (4B - 4At) \cos 2t + (-4A - 4Bt) \sin 2t \\ &\quad + 4At \cos 2t + 4Bt \sin 2t \\ &= 4B \cos 2t - 4A \sin 2t. \end{aligned}$$

3.7 The Method of Undetermined Coefficients



$$3 \cos 2t = 4B \cos 2t - 4A \sin 2t$$

Thus

$$\begin{cases} -4A = 0 \\ 4B = 3 \end{cases}$$

which has solution $A = 0$ and $B = \frac{3}{4}$.

3.7 The Method of Undetermined Coefficients



$$3 \cos 2t = 4B \cos 2t - 4A \sin 2t$$

Thus

$$\begin{cases} -4A = 0 \\ 4B = 3 \end{cases}$$

which has solution $A = 0$ and $B = \frac{3}{4}$. Therefore a particular solution is

$$Y(t) = \frac{3}{4}t \sin 2t.$$

Next Time

- 3.8 Solving Initial Value Problems
- 3.9 The Method of Variation of Parameters
- 3.10 Higher Order Linear ODEs