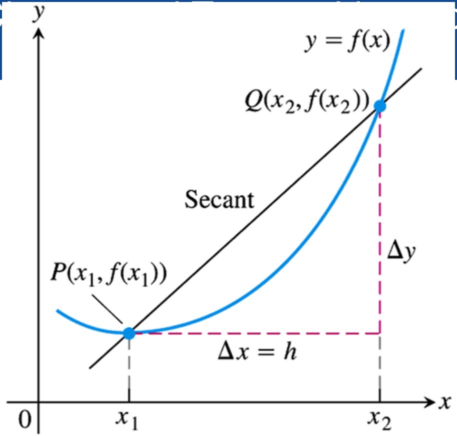
Lecture 2

- 2.1 Rates of Change and Tangents to Curves
- 2.2 Limit of a Function and Limit Laws
- 2.3 The Precise Definition of a Limit

2.1 Rates of Change and Tangent Lines to Curves

2.1 Rates of C… Curves

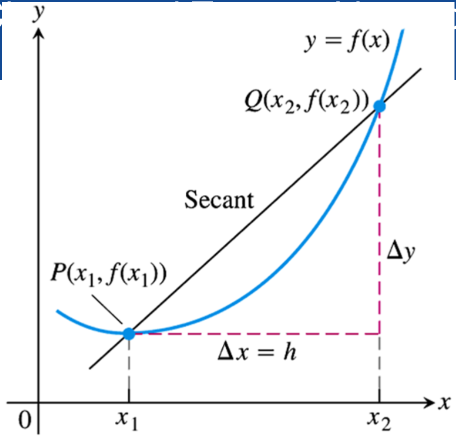


Definition

The *average rate of change* of $y = f(x)$ with respect to x over the interval $[x_1, x_2]$ is

$$\frac{\Delta y}{\Delta x}$$

2.1 Rates of C
Curves



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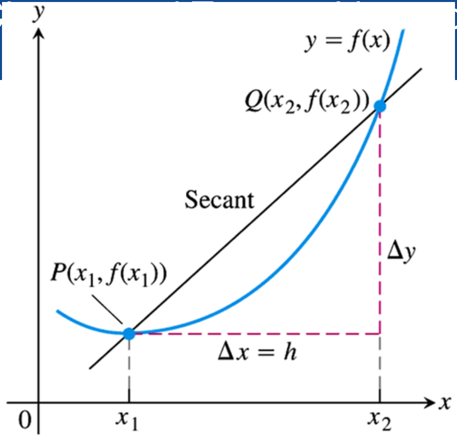
The *average rate of change* of $y = f(x)$ with respect to x over the interval $[x_1, x_2]$ is

$$\frac{\Delta y}{\Delta x} = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$



Definition

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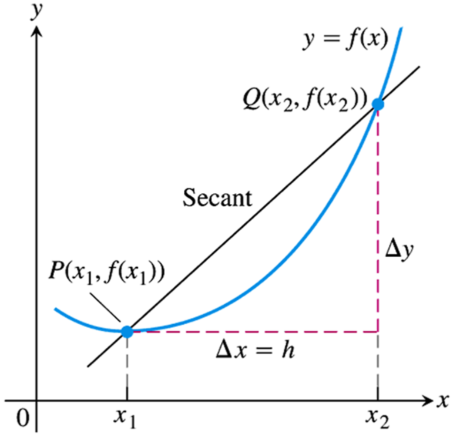
$$\frac{\Delta y}{\Delta x} = \frac{f(x_2) - f(x_1)}{x_2 - x_1} = \frac{f(x_1 + h) - f(x_1)}{h}.$$

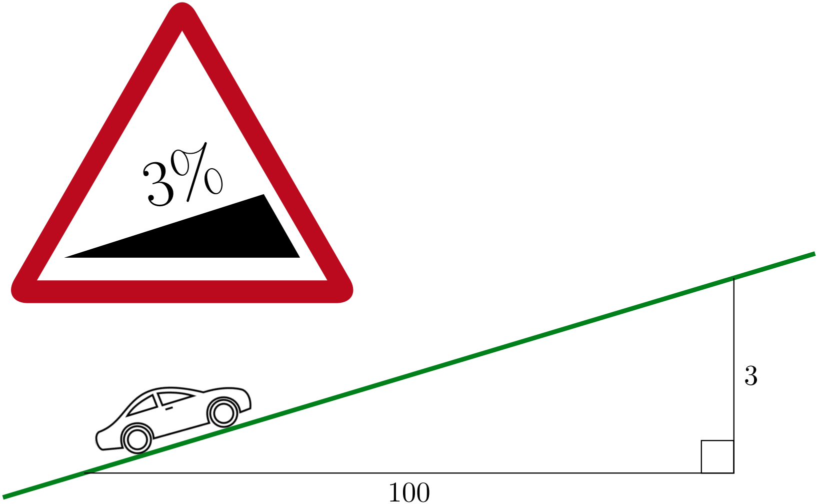


Definition

A line joining 2 points on a curve is called a *secant line.*

Slopes of Lines

Slopes of Lines



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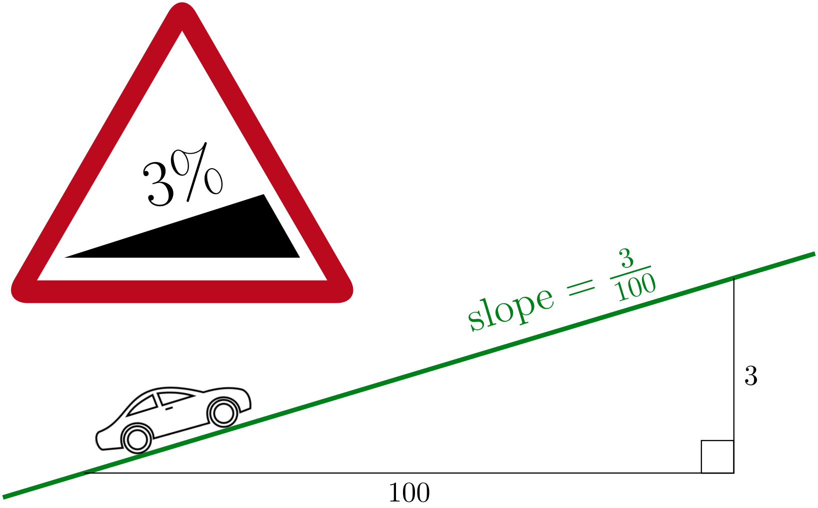
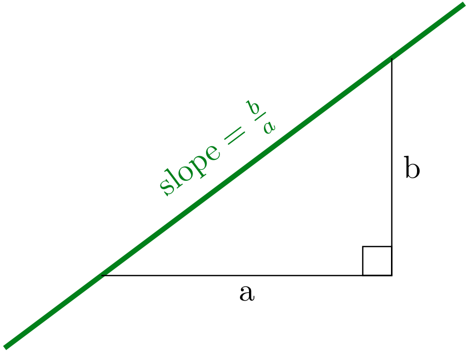


2.1 Rates of Change and Tangent Lines to Curves



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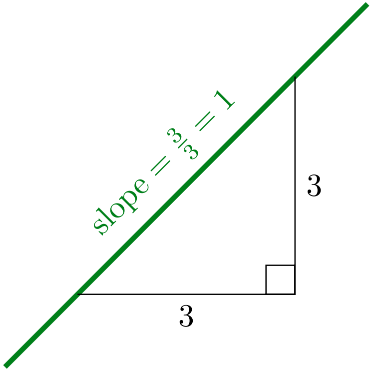
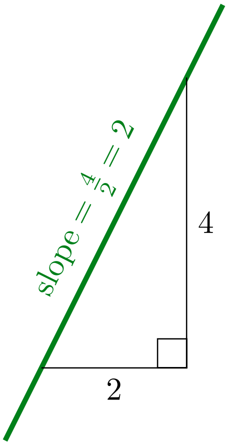
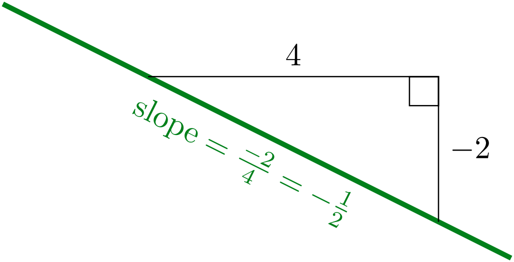



slope $= \frac{4}{2} = 2$
4
2

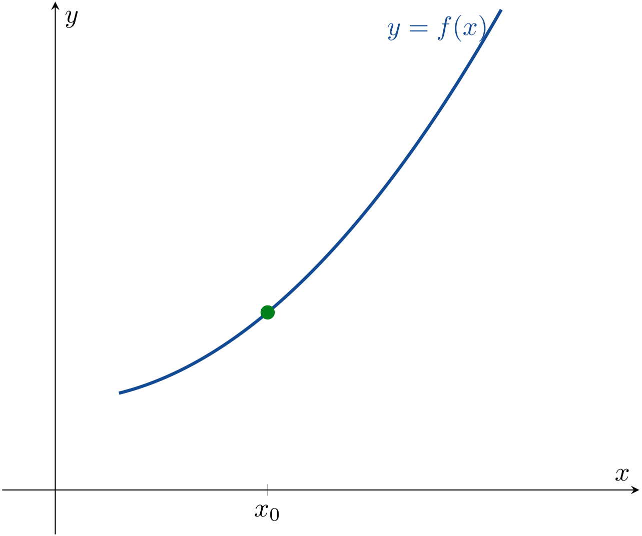


4
−2
slope = −2/4 = −1/2

Slopes of Curves

But how can we find the slope of a curve $y = f(x)$ at a point x_0?

2.1 Rates of Change and Tangent Lines to Curves



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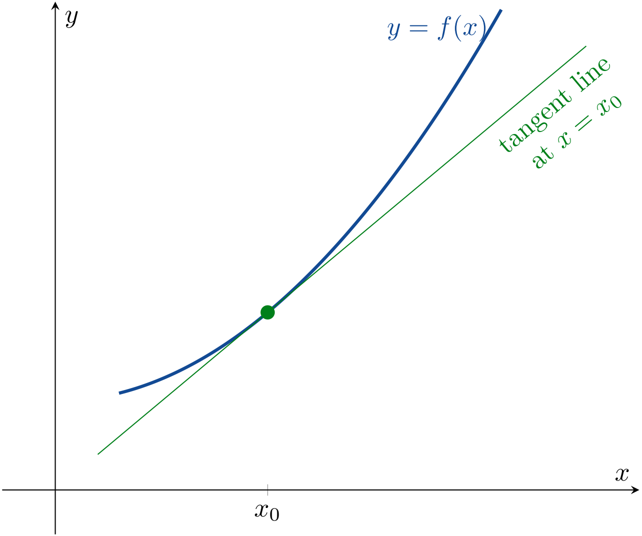

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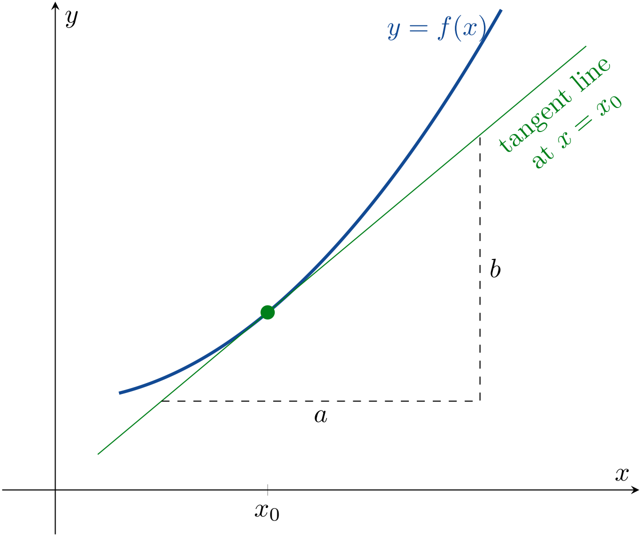

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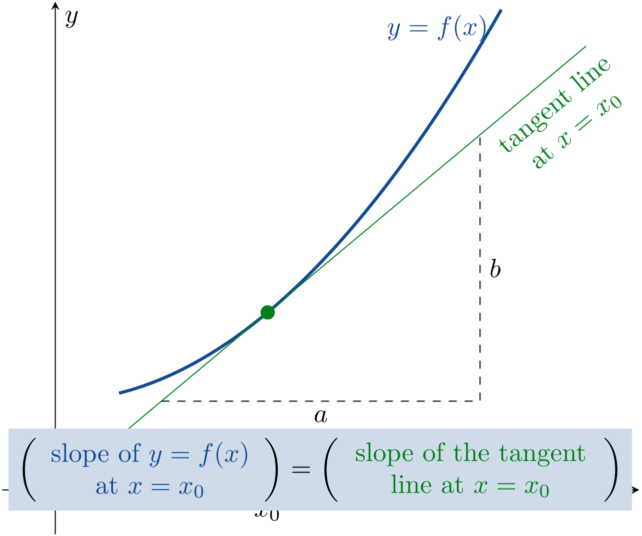

$$\left(\begin{array}{c} \text{slope of } y = f(x) \\ \text{at } x = x_0 \end{array} \right) = \left(\begin{array}{c} \text{slope of the tangent} \\ \text{line at } x = x_0 \end{array} \right)$$

We will talk more about the slopes of curves in Lecture 4.

2.2 Limit of a Function and Limit Laws

Consider the function $f(x) = \frac{x^2-1}{x-1}$, $f : (-\infty, 1) \cup (1, \infty) \to \mathbb{R}$.

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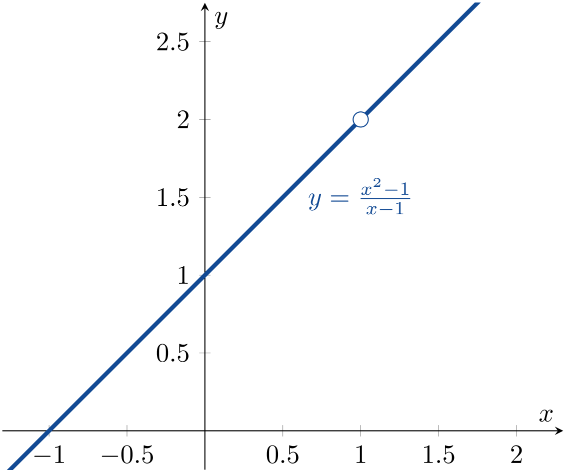

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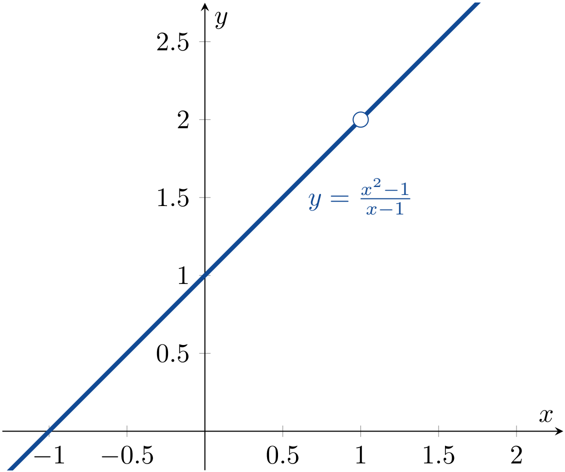
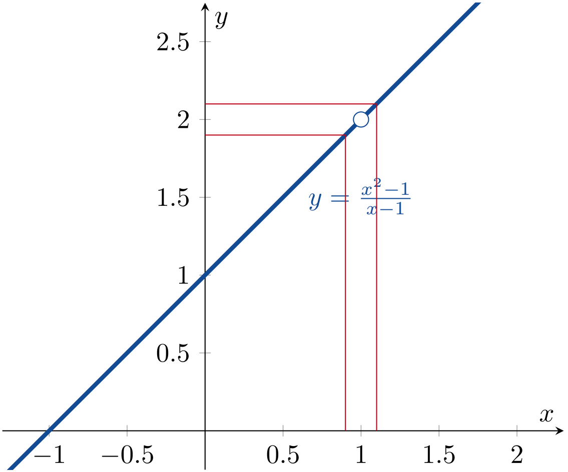


Question: How does f behave when x is close to 1?

Consider the function $f(x) = \frac{x^2-1}{x-1}$, $f : (-\infty, 1) \cup (1, \infty) \to \mathbb{R}$.



x	$f(x)$
0.9	1.9
1.1	2.1

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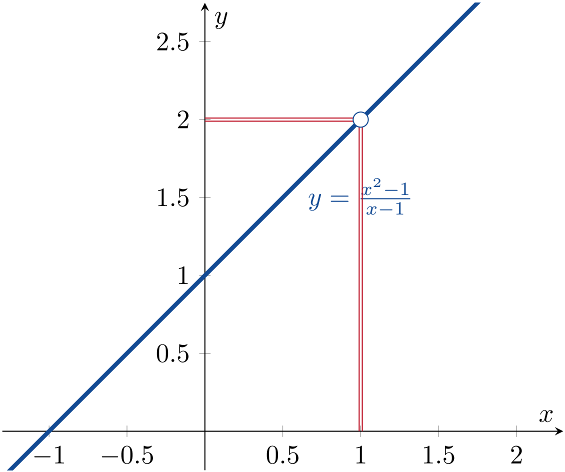

x	$f(x)$
0.9	1.9
1.1	2.1
0.99	1.99
1.01	2.01

Question: How does f behave when x is close to 1?

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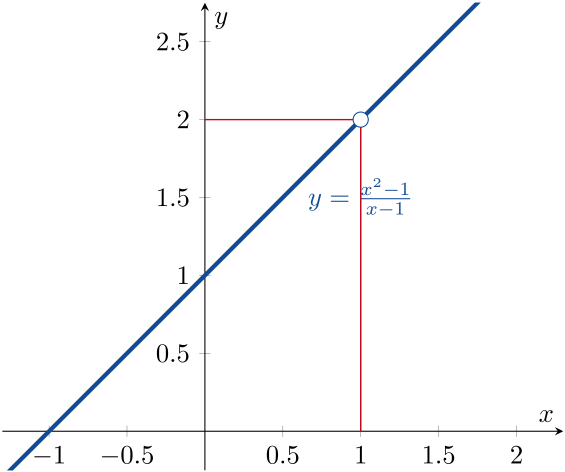

x	$f(x)$
0.9	1.9
1.1	2.1
0.99	1.99
1.01	2.01
0.999	1.999
1.001	2.001

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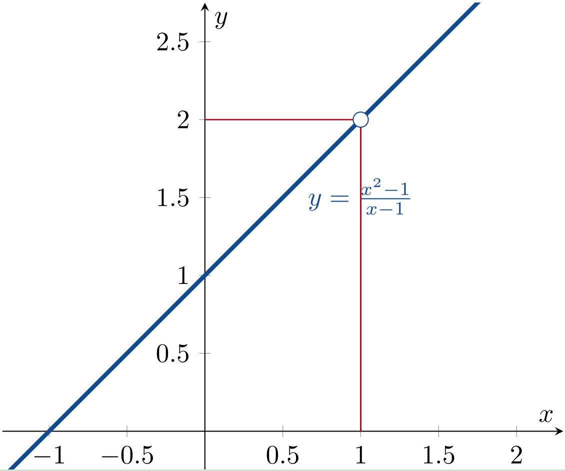

x	$f(x)$
0.9	1.9
1.1	2.1
0.99	1.99
1.01	2.01
0.999	1.999
1.001	2.001

"If x is close to 1, then $f(x)$ is close to 2."

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Mathematically, we write this as

$$\lim_{x \to 1} f(x) = 2$$

and read it as "the limit, as x tends to 1, of $f(x)$ is equal to 2".

Example

$$f(x) = \frac{x^2 - 1}{x - 1}$$

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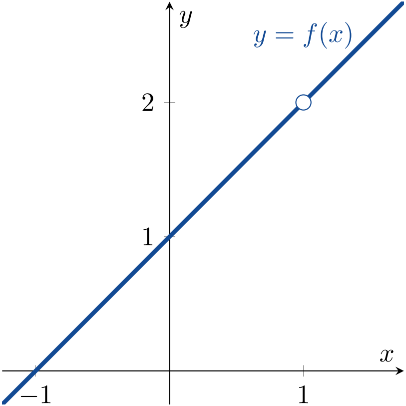

Note that

$$\lim_{x \to 1} f(x) = 2,$$

but f is not defined at $x = 1$.

Example

$$g(x) = \begin{cases} \frac{x^2-1}{x-1} & x \neq 1 \\ 1 & x = 1 \end{cases}$$

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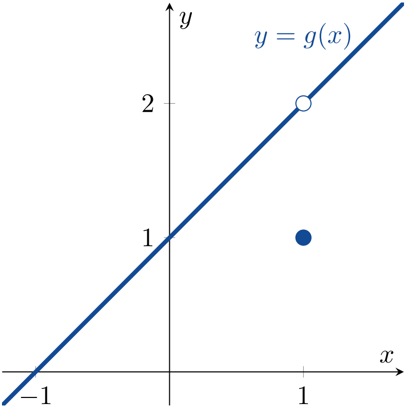

Note that

$$\lim_{x \to 1} g(x) = 2,$$

but

$$g(1) \neq 2.$$

Example

$h(x) = x + 1$

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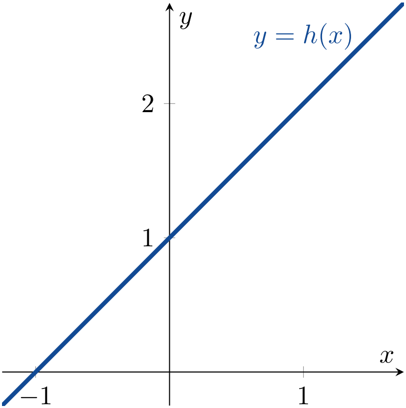
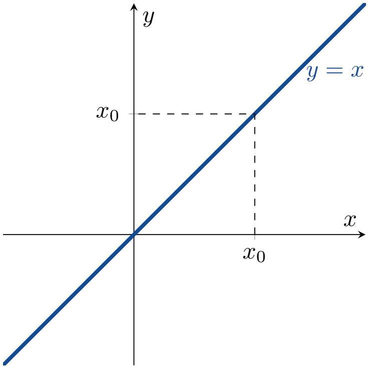


Note that

$$\lim_{x \to 1} h(x) = 2$$

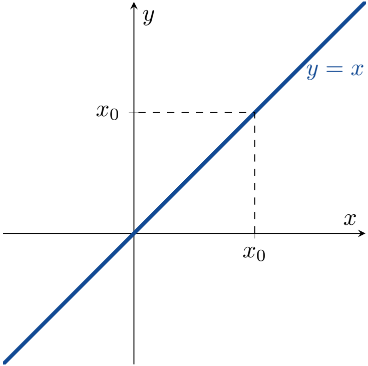
and

$$h(1) = 2.$$



Example (The Identity Function)

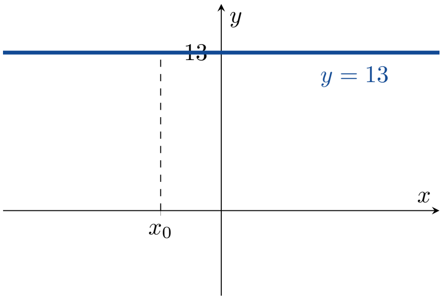
$f(x) = x$



Example (The Identity Function)

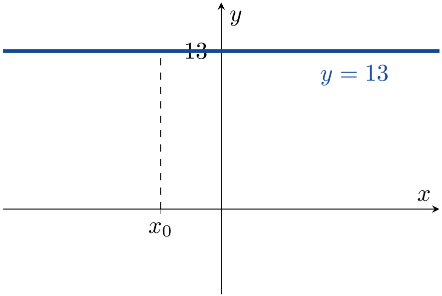
$f(x) = x$

$$\lim_{x \to x_0} f(x) = \lim_{x \to x_0} x = x_0$$



Example (A Constant Function)

$f(x) = 13$



Example (A Constant Function)

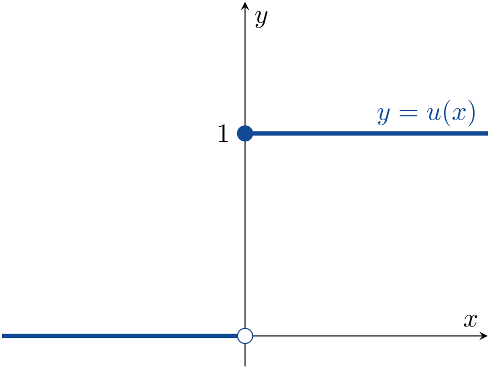
$f(x) = 13$

$$\lim_{x \to x_0} f(x) = \lim_{x \to x_0} 13 = 13$$

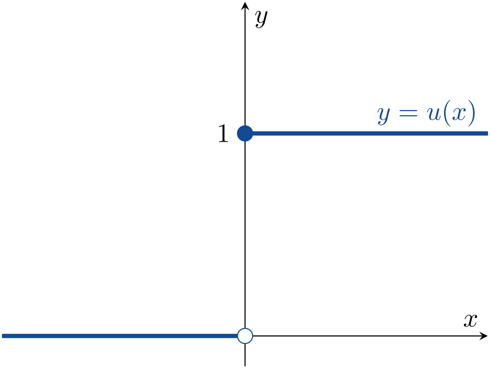
Example (Sometimes Limits Do Not Exist)

Consider the functions

$$u(x) = \begin{cases} 0 & x < 0 \\ 1 & x \geq 0 \end{cases} \qquad \text{and} \qquad v(x) = \begin{cases} 0 & x \leq 0 \\ \sin \frac{1}{x} & x > 0. \end{cases}$$

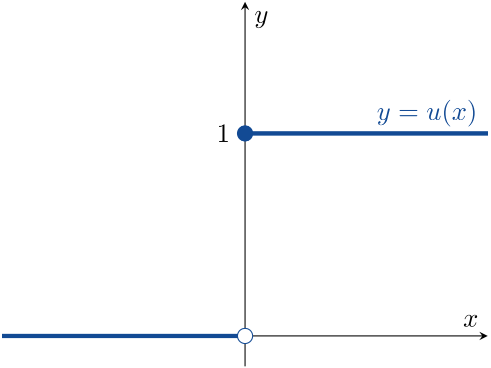


Note that $\lim_{x \to 0} u(x)$ does not exist.



Note that $\lim_{x\to 0} u(x)$ does not exist. To understand why, we consider x close to 0:

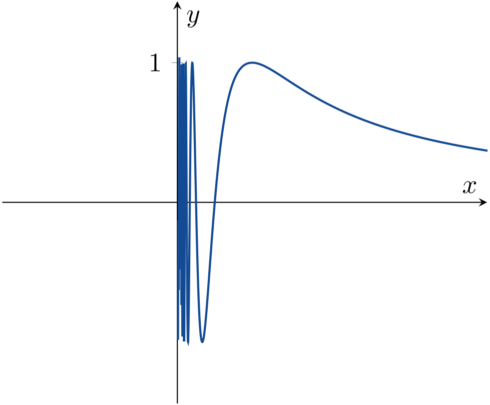
- If x is close to 0 and $x < 0$, then $u(x) = 0$.
- If x is close to 0 and $x > 0$, then $u(x) = 1$.



Note that $\lim_{x \to 0} u(x)$ does not exist. To understand why, we consider x close to 0:

- If x is close to 0 and $x < 0$, then $u(x) = 0$.
- If x is close to 0 and $x > 0$, then $u(x) = 1$.

Because 0 is not close to 1, the limit as $x \to 0$ can not exist.



Moreover $\lim_{x \to 0} v(x)$ does not exist because $v(x)$ oscillates up and down too quickly if $x > 0$ and $x \to 0$.

Theorem (The Limit Laws)

Suppose that

- $L, M, c, k \in \mathbb{R}$;
- f *and* g *are functions;*
- $\lim\limits_{x \to c} f(x) = L$; *and*
- $\lim\limits_{x \to c} g(x) = M$.

Then

Theorem (The Limit Laws)

Suppose that

- $L, M, c, k \in \mathbb{R}$;
- f *and* g *are functions;*
- $\lim\limits_{x \to c} f(x) = L$; *and*
- $\lim\limits_{x \to c} g(x) = M$.

Then

1. *Sum Rule:*
$$\lim_{x \to c} \big(f(x) + g(x)\big) = L + M;$$

2.2 Limit of a Function and Limit Laws

Theorem (The Limit Laws)

Suppose that

- $L, M, c, k \in \mathbb{R}$;
- f *and* g *are functions;*
- $\lim_{x \to c} f(x) = L$; *and*
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Then

2. *Difference Rule:*

$$\lim_{x \to c} \big(f(x) - g(x)\big) = L - M;$$

2.2 Limit of a Function and Limit Laws

Theorem (The Limit Laws)

Suppose that

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- $\lim_{x \to c} f(x) = L$; *and*
- $\lim_{x \to c} g(x) = M$.

Then

3. *Constant Multiple Rule:*

$$\lim_{x \to c} \big(k f(x) \big) = kL;$$

Theorem (The Limit Laws)

Suppose that

- $L, M, c, k \in \mathbb{R}$;
- f *and* g *are functions;*
- $\lim\limits_{x\to c} f(x) = L$; *and*
- $\lim\limits_{x\to c} g(x) = M$.

Then

4. *Product Rule:*

$$\lim_{x\to c}\big(f(x)g(x)\big) = LM;$$

Theorem (The Limit Laws)

Suppose that

- $L, M, c, k \in \mathbb{R}$;
- f *and* g *are functions;*
- $\lim_{x \to c} f(x) = L$; *and*
- $\lim_{x \to c} g(x) = M$.

Then

5. *Quotient Rule: if* $M \neq 0$, *then*

$$\lim_{x \to c} \left(\frac{f(x)}{g(x)} \right) = \frac{L}{M};$$

Theorem (The Limit Laws)

Suppose that

- $L, M, c, k \in \mathbb{R}$;
- f *and* g *are functions;*
- $\lim_{x \to c} f(x) = L$; *and*
- $\lim_{x \to c} g(x) = M$.

Then

6. *Power Rule: if* $n \in \mathbb{N}$, *then*

$$\lim_{x \to c} \big(f(x)\big)^n = L^n;$$

Theorem (The Limit Laws)

Suppose that

- $L, M, c, k \in \mathbb{R}$;
- f *and* g *are functions;*
- $\lim_{x \to c} f(x) = L$; *and*
- $\lim_{x \to c} g(x) = M$.

Then

7. *Root Rule: if* $n \in \mathbb{N}$ *and* $\sqrt[n]{L}$ *exists, then*

$$\lim_{x \to c} \sqrt[n]{f(x)} = \sqrt[n]{L} = L^{\frac{1}{n}}.$$

Example

Find $\lim_{x\to 2}(x^3 + 4x^2 - 3)$.

Example

Find $\lim_{x\to 2}(x^3 + 4x^2 - 3)$.

$$\begin{aligned}
\lim_{x\to 2}(x^3 + 4x^2 - 3) &= \Big(\lim_{x\to 2} x^3\Big) + \Big(\lim_{x\to 2} 4x^2\Big) - \Big(\lim_{x\to 2} 3\Big) \\
&\quad \text{(sum and difference rules)} \\
&= \Big(\lim_{x\to 2} x\Big)^3 + 4\Big(\lim_{x\to 2} x\Big)^2 - \Big(\lim_{x\to 2} 3\Big) \\
&\quad \text{(power and constant multiple rules)} \\
&= 2^3 + 4(2^2) - 3 = 21.
\end{aligned}$$

Example

Find $\lim_{x\to 6} 8(x-5)(x-7)$.

$$\begin{aligned}
\lim_{x\to 6} 8(x-5)(x-7) &= 8\lim_{x\to 6}(x-5)(x-7) \\
&\quad \text{(constant multiple rule)} \\
&= 8\Big(\lim_{x\to 6}(x-5)\Big)\Big(\lim_{x\to 6}(x-7)\Big) \\
&\quad \text{(product rule)} \\
&= 8(1)(-1) = -8.
\end{aligned}$$

Example

Find $\lim\limits_{x \to 5} \dfrac{x^4 + x^2 - 1}{x^2 + 5}$.

$$\lim_{x \to 5} \frac{x^4 + x^2 - 1}{x^2 + 5} = \frac{\lim_{x \to 5}(x^4 + x^2 - 1)}{\lim_{x \to 5}(x^2 + 5)}$$

(quotient rule)

$$= \frac{\lim_{x \to 5} x^4 + \lim_{x \to 5} x^2 - \lim_{x \to 5} 1}{\lim_{x \to 5} x^2 + \lim_{x \to 5} 5}$$

(sum and difference rules)

$$= \frac{5^4 + 5^2 - 1}{5^2 + 5} = \frac{649}{30}.$$

(power rule)

Example

Find $\displaystyle\lim_{x\to-5}\frac{x^2+3x-11}{x+6}$.

$$\lim_{x\to-5}\frac{x^2+3x-11}{x+6}=\frac{\lim_{x\to-5}(x^2+3x-11)}{\lim_{x\to-5}(x+6)}$$

(quotient rule)

$$=\frac{\lim_{x\to-5}x^2+\lim_{x\to-5}3x-\lim_{x\to-5}11}{\lim_{x\to-5}x+\lim_{x\to-5}6}$$

(sum and difference rules)

$$=\frac{(-5)^2-15-11}{-5+6}=\frac{-1}{1}=-1.$$

(power rule)

Is there an easier way?

Theorem (Limits of Polynomial Functions)

If $P(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_0$ *is a polynomial function, then*

$$\lim_{x \to c} P(x) = P(c).$$

Theorem (Limits of Polynomial Functions)

If $P(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_0$ is a polynomial function, then

$$\lim_{x \to c} P(x) = P(c).$$

Theorem (Limits of Rational Functions)

If $P(x)$ and $Q(x)$ are polynomial functions and if $Q(c) \neq 0$, then

$$\lim_{x \to c} \frac{P(x)}{Q(x)} = \frac{P(c)}{Q(c)}.$$

Example

$$\lim_{x\to -1} \frac{x^3+4x^2-3}{x^2+5} = \frac{(-1)^3+4(-1)^2-3}{(-1)^2+5} = \frac{0}{6} = 0.$$

Example

$$\lim_{x\to 2} \frac{x^3+4x^2-3}{x^2+5} = \frac{(2)^3+4(2)^2-3}{(2)^2+5} = \frac{8+16-3}{4+5} = \frac{21}{9} = \frac{7}{3}.$$

Eliminating Zero Denominators Algebraically

$$\lim_{x \to c} \frac{P(x)}{Q(x)}$$

What can we do if $Q(c) = 0$?

Example

Find
$$\lim_{x \to 1} \frac{x^2 + x - 2}{x^2 - x}.$$

Example

Find
$$\lim_{x \to 1} \frac{x^2 + x - 2}{x^2 - x}.$$

If we just put in $x = 1$, we would get "$\frac{0}{0}$" and we never never never want "$\frac{0}{0}$". So what can we do?

Example

Find
$$\lim_{x\to 1} \frac{x^2+x-2}{x^2-x}.$$

If we just put in $x = 1$, we would get "$\frac{0}{0}$" and we never never never want "$\frac{0}{0}$". So what can we do?
Instead, we try to factor $x^2 + x - 2$ and $x^2 - x$.

Example

Find
$$\lim_{x\to 1} \frac{x^2+x-2}{x^2-x}.$$

If we just put in $x = 1$, we would get "$\frac{0}{0}$" and we never never never want "$\frac{0}{0}$". So what can we do?
Instead, we try to factor $x^2 + x - 2$ and $x^2 - x$. If $x \neq 1$, we have that

$$\frac{x^2+x-2}{x^2-x} = \frac{(x-1)(x+2)}{x(x-1)} = \frac{x+2}{x}.$$

Example

Find
$$\lim_{x\to 1} \frac{x^2+x-2}{x^2-x}.$$

If we just put in $x = 1$, we would get "$\frac{0}{0}$" and we never never never want "$\frac{0}{0}$". So what can we do?
Instead, we try to factor $x^2 + x - 2$ and $x^2 - x$. If $x \neq 1$, we have that

$$\frac{x^2+x-2}{x^2-x} = \frac{(x-1)(x+2)}{x(x-1)} = \frac{x+2}{x}.$$

So
$$\lim_{x\to 1} \frac{x^2+x-2}{x^2-x} = \lim_{x\to 1} \frac{x+2}{x} = \frac{1+2}{1} = 3.$$

Example

Find

$$\lim_{x \to -5} \frac{x^2 + 3x - 10}{x^2 + 5x}.$$

Example

Find
$$\lim_{x \to -5} \frac{x^2 + 3x - 10}{x^2 + 5x}.$$

We must try to factor $x^2 + 3x - 10$ and $x^2 + 5x$. If $x \neq -5$, we have that

$$\frac{x^2 + 3x - 10}{x^2 + 5x} = \frac{(x+5)(x-2)}{x(x+5)} = \frac{x-2}{x}.$$

So
$$\lim_{x \to -5} \frac{x^2 + 3x - 10}{x^2 + 5x} = \lim_{x \to -5} \frac{x-2}{x} = \frac{-5-2}{-5} = \frac{7}{5}.$$

Example

Find

$$\lim_{x\to 0} \frac{\sqrt{x^2+100}-10}{x^2}.$$

This is another "$\frac{0}{0}$" limit.

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There is a trick we can use if we have $A - B$ in a limit: We multiply by $(A + B)$ because $(A - B)(A + B) = A^2 - B^2$.

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So we multiply top and bottom by $(\sqrt{x^2 + 100} + 10)$.

$$\lim_{x\to 0} \frac{\sqrt{x^2+100}-10}{x^2} = \lim_{x\to 0} \frac{(\sqrt{x^2+100}-10)(\sqrt{x^2+100}+10)}{x^2(\sqrt{x^2+100}+10)}$$

$$=$$

$$=$$

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$$=$$

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$$=$$

$$=$$

$$\begin{aligned}
\lim_{x\to 0}\frac{\sqrt{x^2+100}-10}{x^2} &= \lim_{x\to 0}\frac{(\sqrt{x^2+100}-10)(\sqrt{x^2+100}+10)}{x^2(\sqrt{x^2+100}+10)} \\
&= \lim_{x\to 0}\frac{x^2+100-100}{x^2(\sqrt{x^2+100}+10)} \\
&= \lim_{x\to 0}\frac{x^2}{x^2(\sqrt{x^2+100}+10)} \\
&= \\
&=
\end{aligned}$$

$$
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&= \lim_{x\to 0} \frac{1}{\sqrt{x^2+100}+10} \\
&=
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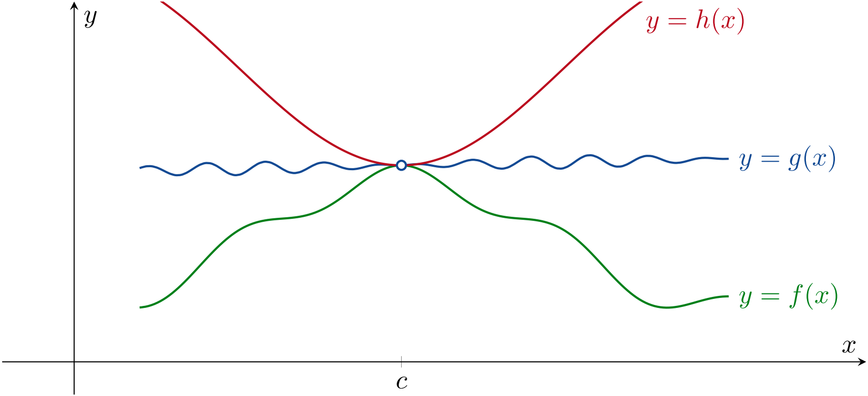
&= \lim_{x\to 0}\frac{1}{\sqrt{x^2+100}+10} \\
&= \frac{1}{\sqrt{0^2+100}+10} = \frac{1}{20}.
\end{aligned}$$

The Sandwich Theorem



Theorem (The Sandwich Theorem)

Suppose that

- $f(x) \leq g(x) \leq h(x)$ *for all* x *"close" to* c $(x \neq c)$*; and*
- $\lim\limits_{x \to c} f(x) = \lim\limits_{x \to c} h(x) = L.$

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Then

$$\lim_{x \to c} g(x) = L$$

also.

Example

The inequality

$$1 - \frac{x^2}{6} < \frac{x \sin x}{2 - 2\cos x} < 1$$

holds for all x close to 0 ($x \neq 0$). Calculate $\displaystyle\lim_{x \to 0} \frac{x \sin x}{2 - 2\cos x}$.

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Since $\lim_{x \to 0} 1 - \frac{x^2}{6} = 1 - \frac{0}{6} = 1$

Example

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Since $\lim\limits_{x \to 0} 1 - \dfrac{x^2}{6} = 1 - \dfrac{0}{6} = 1$ and $\lim\limits_{x \to 0} 1 = 1$,

Example

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Since $\lim\limits_{x \to 0} 1 - \dfrac{x^2}{6} = 1 - \dfrac{0}{6} = 1$ and $\lim\limits_{x \to 0} 1 = 1$, it follows by the Sandwich Theorem that

$$\lim_{x \to 0} \frac{x \sin x}{2 - 2\cos x} = 1$$

also.

2 important limits

Remember that last lecture we found that

$$-|\theta| \leq \sin\theta \leq |\theta| \qquad \text{and} \qquad -|\theta| \leq 1 - \cos\theta \leq |\theta| .$$

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So it follows by the Sandwich Theorem that

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and hence that

$$\lim_{\theta\to 0} \cos\theta = 1.$$

Theorem

$$\lim_{\theta \to 0} \sin\theta = 0 \qquad \textit{and} \qquad \lim_{\theta \to 0} \cos\theta = 1.$$

We will need these later in the course.

Theorem

If

- $f(x) \leq g(x)$ *for all* x *close to* c $(x \neq c)$;
- $\lim_{x\to c} f(x)$ *exists; and*
- $\lim_{x\to c} g(x)$ *exists,*

then

$$\lim_{x\to c} f(x) \leq \lim_{x\to c} g(x).$$

Remark

Warning:

$$f(x) < g(x) \qquad \not\Longrightarrow \qquad \lim_{x\to c} f(x) < \lim_{x\to c} g(x).$$

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$$f(x) < g(x) \qquad \not\Longrightarrow \qquad \lim_{x\to c} f(x) < \lim_{x\to c} g(x).$$

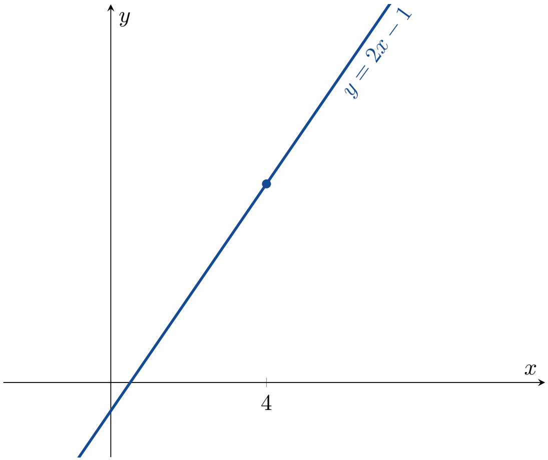
The actually result is

$$f(x) < g(x) \qquad \Longrightarrow \qquad \lim_{x\to c} f(x) \leq \lim_{x\to c} g(x).$$

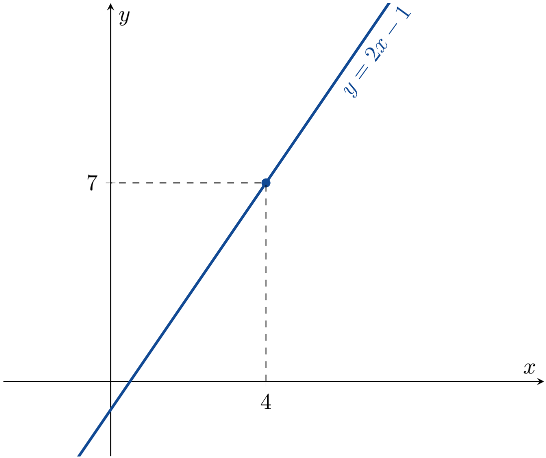
Break

We will continue at 2pm

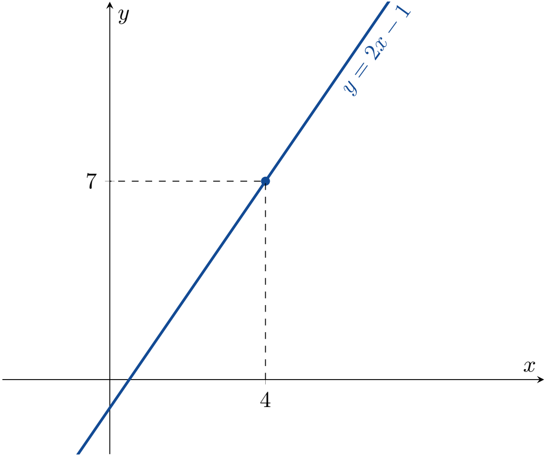
The Precise Definition of a Limit



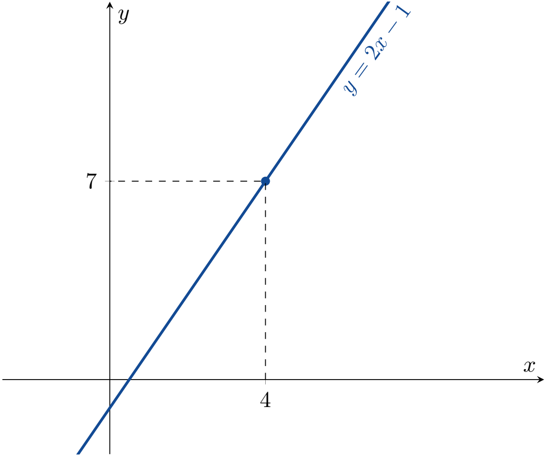
Consider the function $y = 2x - 1$ close to $x = 4$.



We think that if x is "close to 4" (but $x \neq 4$), then y is "close to 7".

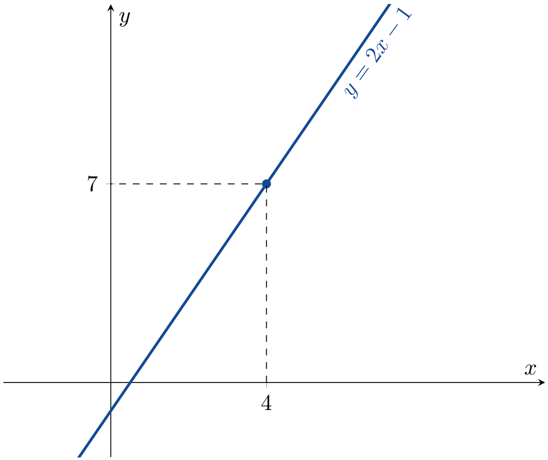


We think that if x is "close to 4" (but $x \neq 4$), then y is "close to 7". So we think that $\lim_{x \to 4}(2x - 1) = 7$.

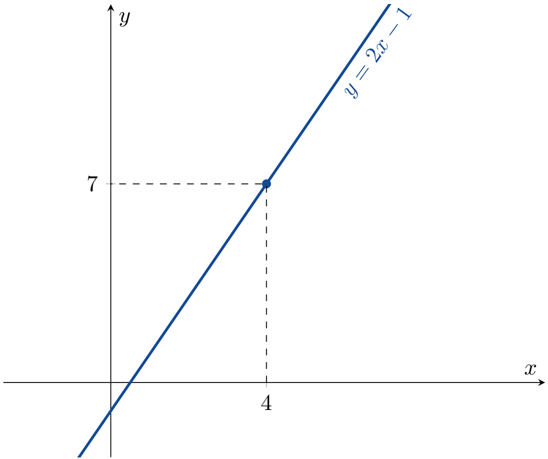


But what does this really mean? How can we make “close to” precise?

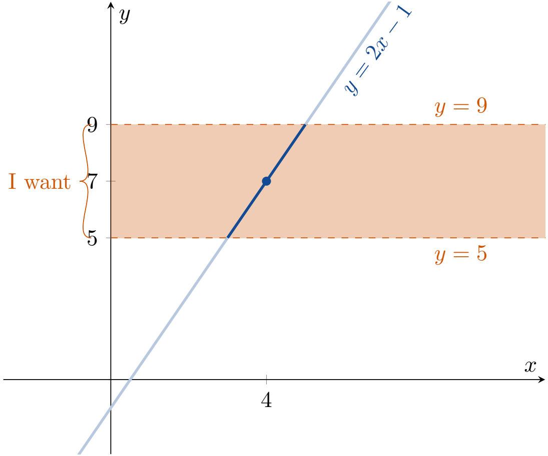
2.3 The Precise Definition of a Limit



How close to 4 does x need to be to make y close to 7?



How close to 4 does x need to be to make $|y - 7| < 2$?



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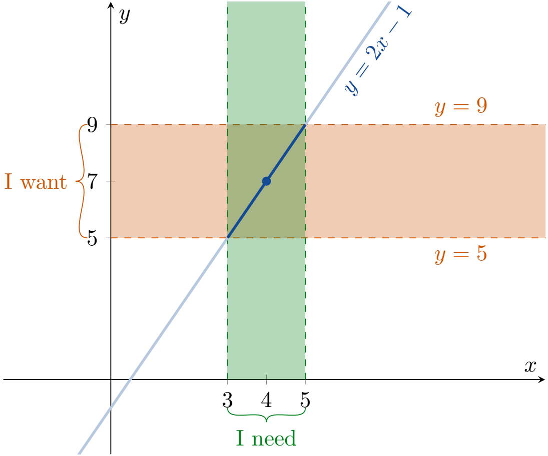

If I want $|y - 7| < 2$, then I need to have $3 < x < 5$.

$y = 2x - 1$ close to $x = 4$

I want $\quad |y - 7| < 2$

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$$\text{I want} \qquad |y - 7| < 2$$

$$-2 < y - 7 < 2$$

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$$5 < 2x - 1 < 9$$

$$6 < 2x < 10$$

$$3 < x < 5$$

$$3 - 4 < x - 4 < 5 - 4$$

$$-1 < x - 4 < 1$$

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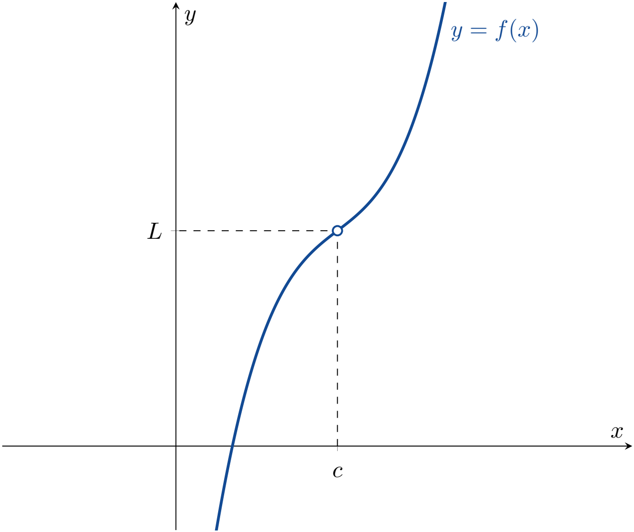
$$-1 < x - 4 < 1$$
$$\text{I need} \qquad |x - 4| < 1$$

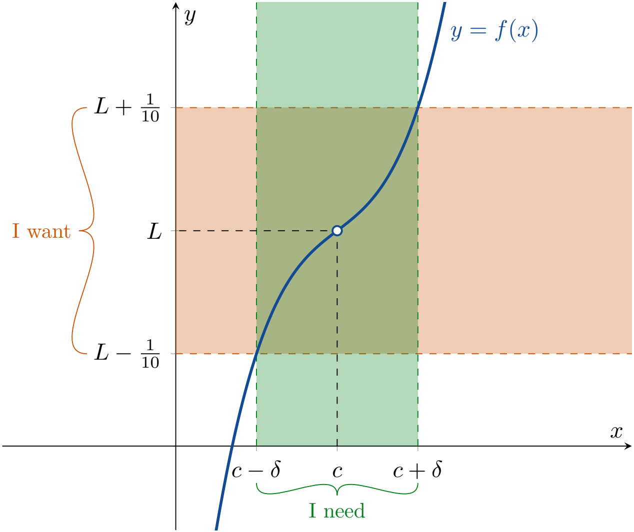
We can write this as

$$|x - 4| < 1 \qquad \Longrightarrow \qquad |y - 7| < 2.$$

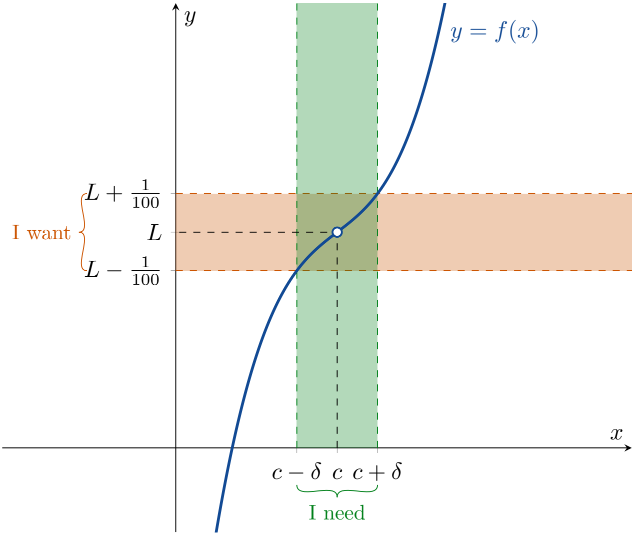
This is the idea that we will use to precisely define what a limit is.

2.3 The Precise Definition of a Limit





y
$y = f(x)$
$L + \frac{1}{10}$
I want
L
$L - \frac{1}{10}$
x
$c - \delta$
c
$c + \delta$
I need



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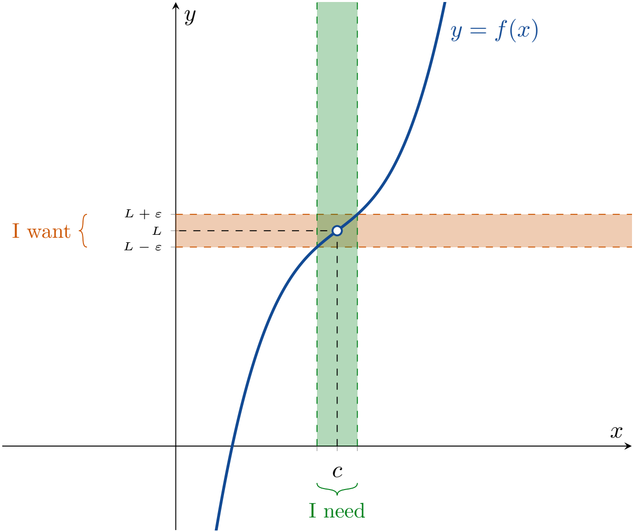
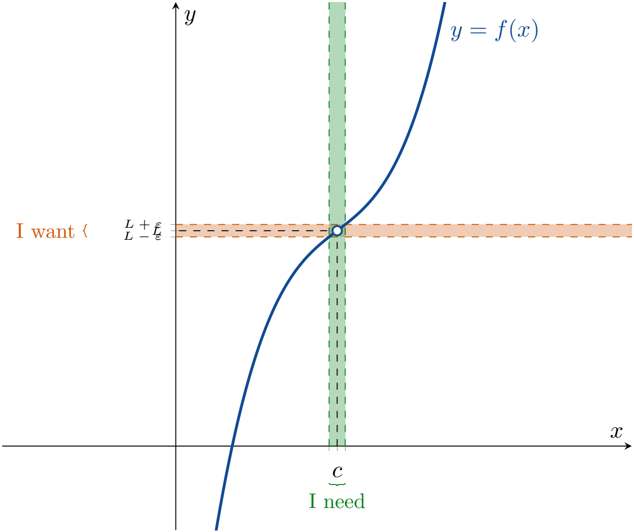
x
$c - \delta$ c $c + \delta$
I need



y
$y = f(x)$
I want
$L + \varepsilon$
L
$L - \varepsilon$
x
c
I need



y
$y = f(x)$
I want
$L + \varepsilon$
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I need

Any time you see δ (delta) or ε (epsilon), think "small number".

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We want

$$x \text{ is close to } c \text{ (but } x \neq c) \qquad \Longrightarrow \qquad f(x) \text{ is close to } L.$$

"if x is close to c (but $x \neq c$) then $f(x)$ is close to L"

Any time you see δ (delta) or ε (epsilon), think "small number".

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"if x is close to c (but $x \neq c$) then $f(x)$ is close to L"

So we want

$$0 < |x - c| < \delta \qquad \Longrightarrow \qquad |f(x) - L| < \varepsilon$$

and we want this to **always be true**, no matter how small an $\varepsilon > 0$ we have.

2.3 The Precise Definition of a Limit

Definition

We write $\lim_{x \to c} f(x) = L$ iff for all $\varepsilon > 0$, there exists $\delta > 0$ such that

$$0 < |x - c| < \delta \qquad \Longrightarrow \qquad |f(x) - L| < \varepsilon.$$

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Example

Show that $\lim_{x \to 1} (5x - 3) = 2$.

(Here we have $f(x) = 5x - 3$, $c = 1$ and $L = 2$.)

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Let $\varepsilon > 0$. Choose $\delta =$ ______. Then

$0 < |x - 1| < \delta \implies |(5x - 3) - 2|$ ____________________ $< \varepsilon$.

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scrap paper

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Therefore $\lim_{x\to 1}(5x - 3) = 2$

— — — — — scrap paper — — — — —

We want

$$|(5x - 3) - 2| < \varepsilon$$
$$|5x - 5| < \varepsilon$$
$$5\,|x - 1| < \varepsilon$$
$$|x - 1| < \frac{\varepsilon}{5}$$

Definition

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scrap paper

We want

$$|(5x - 3) - 2| < \varepsilon$$
$$|5x - 5| < \varepsilon$$
$$5|x - 1| < \varepsilon$$
$$|x - 1| < \frac{\varepsilon}{5}$$

We can choose $\delta = \frac{\varepsilon}{5}$

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Let $\varepsilon > 0$. Choose $\delta = \underline{\;\frac{\varepsilon}{5}\;}$. Then

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Example

Show that $\lim_{x \to 1} (5x - 3) = 2$.

(Here we have $f(x) = 5x - 3$, $c = 1$ and $L = 2$.)

Let $\varepsilon > 0$. Choose $\delta = \frac{\varepsilon}{5}$. Then

$$0 < |x - 1| < \delta \implies |(5x - 3) - 2| = |5(x - 1)| = 5\,|x - 1| < \varepsilon.$$

Therefore $\lim_{x \to 1} (5x - 3) = 2$.

Example

For the limit $\lim_{x \to 5} \sqrt{x-1} = 2$, find a $\delta > 0$ which works for $\varepsilon = 1$.

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In other words: Find a $\delta > 0$ such that

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First we calculate that

$$\left|\sqrt{x-1} - 2\right| < 1$$

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First we calculate that

$$\begin{aligned} \left|\sqrt{x-1} - 2\right| &< 1 \\ -1 < \sqrt{x-1} - 2 &< 1 \\ 1 < \sqrt{x-1} &< 3 \end{aligned}$$

Example

For the limit $\lim_{x\to 5}\sqrt{x-1}=2$, find a $\delta>0$ which works for $\varepsilon=1$.

In other words: Find a $\delta>0$ such that

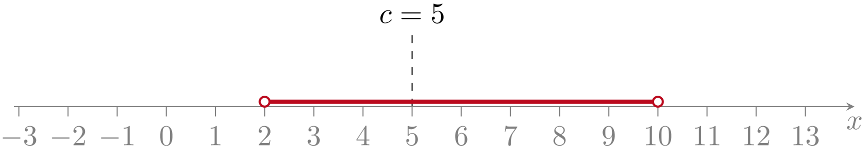
$$0<|x-5|<\delta \qquad \Longrightarrow \qquad \left|\sqrt{x-1}-2\right|<1.$$

First we calculate that

$$\begin{aligned}\left|\sqrt{x-1}-2\right|&<1\\ -1<\sqrt{x-1}-2&<1\\ 1<\sqrt{x-1}&<3\\ 1<x-1&<9\\ 2<x&<10.\end{aligned}$$

So we need to have

$$2 < x < 10.$$



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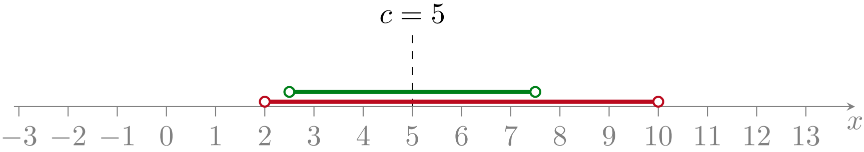

So we need $(5 - \delta, 5 + \delta) \subseteq (2, 10)$.

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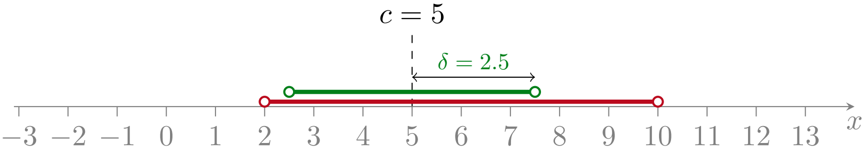

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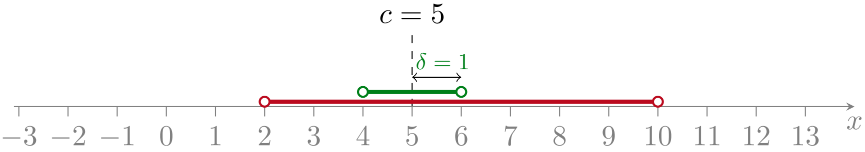

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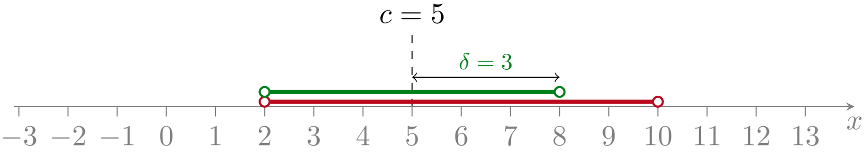

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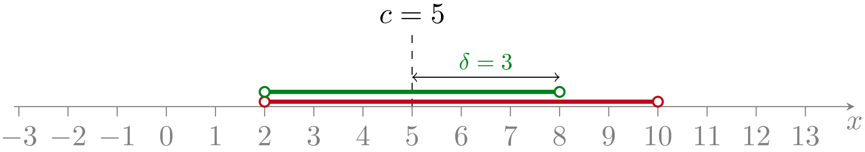

So we need $(5-\delta, 5+\delta) \subseteq (2, 10)$.

To answer this question, we don't need to find the 'best' δ or 'biggest' δ. We only need to find a $\delta > 0$ which works.

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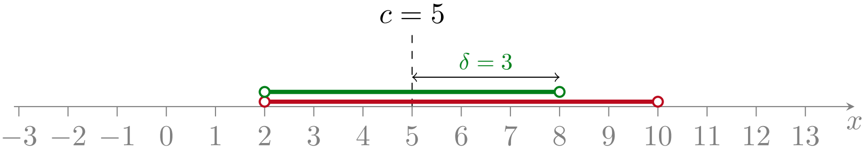

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From the picture, we can see that we can choose any δ in $(0, 3]$.

So we need to have

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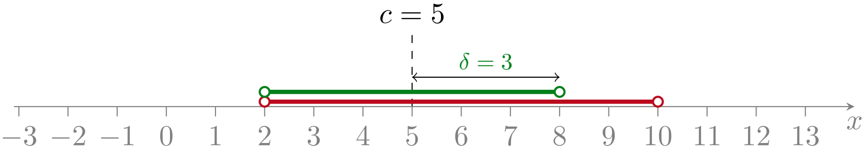

So we need $(5 - \delta, 5 + \delta) \subseteq (2, 10)$.

To answer this question, we don't need to find the 'best' δ or 'biggest' δ. We only need to find a $\delta > 0$ which works.

From the picture, we can see that we can choose any δ in $(0, 3]$. I choose $\delta = 3$.

Then

$$\begin{aligned} 0 < |x-5| < 3 \quad &\implies \quad -3 < x-5 < 3 \\ &\implies \quad 2 < x < 8 \\ &\implies \quad 2 < x < 10 \\ &\implies \quad 1 < x-1 < 9 \\ &\implies \quad 1 < \sqrt{x-1} < 3 \\ &\implies \quad -1 < \sqrt{x-1}-2 < 1 \\ &\implies \quad \left|\sqrt{x-1}-2\right| < 1. \end{aligned}$$

Then

$$
\begin{aligned}
0 < |x-5| < 3 \qquad &\implies \qquad -3 < x-5 < 3 \\
&\implies \qquad 2 < x < 8 \\
&\implies \qquad 2 < x < 10 \\
&\implies \qquad 1 < x-1 < 9 \\
&\implies \qquad 1 < \sqrt{x-1} < 3 \\
&\implies \qquad -1 < \sqrt{x-1} - 2 < 1 \\
&\implies \qquad \left|\sqrt{x-1} - 2\right| < 1.
\end{aligned}
$$

Note: $\delta = 2.5$, $\delta = 2$, $\delta = 1$, etc. are also correct answers to this problem. $\delta = 3.0000001$ is not a correct answer.

$\forall \varepsilon > 0$, $\exists \delta > 0$ such that $0 < |x - c| < \delta \implies |f(x) - L| < \varepsilon$.

$$\forall = \text{ "for all"} \qquad\qquad \exists = \text{ "there exists"}$$

Theorem

$$\lim_{x \to c} x = c$$

$\forall \varepsilon > 0$, $\exists \delta > 0$ such that $0 < |x - c| < \delta \implies |f(x) - L| < \varepsilon$.

$\forall =$ "for all" $\qquad \exists =$ "there exists"

Theorem

$$\lim_{x \to c} x = c$$

Proof.

Let $\varepsilon > 0$.

$\forall \varepsilon > 0$, $\exists \delta > 0$ such that $0 < |x - c| < \delta \implies |f(x) - L| < \varepsilon$.

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Proof.

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$\square$

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Let $\varepsilon > 0$. Choose $\delta = \varepsilon$. Then

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$\square$

Theorem

$$\lim_{x \to c} k = k.$$

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Proof.

Let $\varepsilon > 0$.

Theorem

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Proof.

Let $\varepsilon > 0$. Choose $\delta = 123456789$.

Theorem

$$\lim_{x\to c} k = k.$$

Proof.

Let $\varepsilon > 0$. Choose $\delta = 123456789$. Then

$$0 < |x-c| < \delta \qquad \implies \qquad |k-k| = 0 < \varepsilon.$$

$\square$

Example

Prove that $\lim_{x \to 2} f(x) = 4$ if

$$f(x) = \begin{cases} x^2 & x \neq 2 \\ 1 & x = 2. \end{cases}$$

$\forall \varepsilon > 0$, $\exists \delta > 0$ such that $0 < |x - c| < \delta \implies |f(x) - L| < \varepsilon$.

Example

Prove that $\lim_{x \to 2} f(x) = 4$ if

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Example

Prove that $\lim_{x \to 2} f(x) = 4$

$f($

Assume $x \neq 2$.

$$|f(x) - 4| < \varepsilon$$

$$\left|x^2 - 4\right| < \varepsilon$$

Let $\varepsilon > 0$. Choose $\delta =$

$0 < |x - 2| < \delta \quad \implies$

$\forall \varepsilon > 0$, $\exists \delta > 0$ such that $0 < |x - c| < \delta \implies |f(x) - L| < \varepsilon$.

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Let $\varepsilon > 0$. Choose $\delta =$

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Assume $x \neq 2$.

$$|f(x) - 4| < \varepsilon$$
$$|x^2 - 4| < \varepsilon$$
$$-\varepsilon < x^2 - 4 < \varepsilon$$
$$4 - \varepsilon < x^2 < 4 + \varepsilon$$
$$\sqrt{4-\varepsilon} < x < \sqrt{4+\varepsilon}$$
$$\sqrt{4-\varepsilon} - 2 < x - 2 < \sqrt{4+\varepsilon} - 2$$

$\forall \varepsilon > 0$, $\exists \delta > 0$ such that $0 < |x - c| < \delta \implies |f(x) - L| < \varepsilon$.

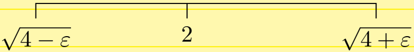
Example

Prove that $\lim_{x \to 2} f(x) = 4$ if

$f(x) =$

$$\sqrt{4-\varepsilon} < x < \sqrt{4+\varepsilon}$$

$$\sqrt{4-\varepsilon} - 2 < x - 2 < \sqrt{4+\varepsilon} - 2$$



Let $\varepsilon > 0$. Choose $\delta =$

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$\forall \varepsilon > 0$, $\exists \delta > 0$ such that $0 < |x - c| < \delta \implies |f(x) - L| < \varepsilon$.

Example

Prove that $\lim\limits_{x \to 2} f(x) = 4$ if

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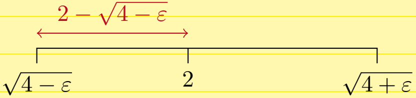

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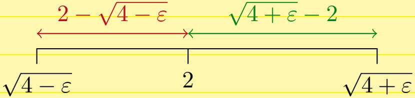

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$$0 < |x - 2| < \delta \quad \implies$$

$\forall \varepsilon > 0$, $\exists \delta > 0$ such that $0 < |x - c| < \delta \implies |f(x) - L| < \varepsilon$.

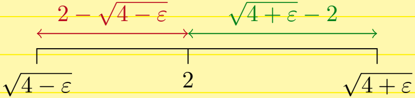
Example

Prove that $\lim_{x \to 2} f(x) = 4$ if

$f(x) =$

$$\sqrt{4 - \varepsilon} < x < \sqrt{4 + \varepsilon}$$

$$\sqrt{4 - \varepsilon} - 2 < x - 2 < \sqrt{4 + \varepsilon} - 2$$



Which is smaller?

$2 - \sqrt{4 - \varepsilon}$ or $\sqrt{4 + \varepsilon} - 2$?

Let $\varepsilon > 0$. Choose $\delta =$

$$0 < |x - 2| < \delta \implies$$

$\forall \varepsilon > 0$, $\exists \delta > 0$ such that $0 < |x - c| < \delta \implies |f(x) - L| < \varepsilon$.

Example

Prove that $\lim_{x \to 2} f(x) = 4$ if

$$f(x) = \begin{cases} x^2 & x \neq 2 \\ 1 & x = 2. \end{cases}$$

Let $\varepsilon > 0$. Choose $\delta = \min\{2 - \sqrt{4 - \varepsilon}, \sqrt{4 + \varepsilon} - 2\}$. Then

$$0 < |x - 2| < \delta \quad \implies$$

$\forall \varepsilon > 0$, $\exists \delta > 0$ such that $0 < |x - c| < \delta \implies |f(x) - L| < \varepsilon$.

Example

Prove that $\lim_{x \to 2} f(x) = 4$ if

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Let $\varepsilon > 0$. Choose $\delta = \min\{2 - \sqrt{4 - \varepsilon}, \sqrt{4 + \varepsilon} - 2\}$. Then

$$\begin{aligned} 0 < |x - 2| < \delta &\implies \sqrt{4 - \varepsilon} - 2 < x - 2 < \sqrt{4 + \varepsilon} - 2 \\ &\implies \sqrt{4 - \varepsilon} < x < \sqrt{4 + \varepsilon} \\ &\implies 4 - \varepsilon < x^2 < 4 + \varepsilon \\ &\implies -\varepsilon < x^2 - 4 < \varepsilon \\ &\implies \left| x^2 - 4 \right| < \varepsilon. \end{aligned}$$

$\forall \varepsilon > 0$, $\exists \delta > 0$ such that $0 < |x - c| < \delta \implies |f(x) - L| < \varepsilon$.

Example (page 81, exercise 43.)

Prove that $\displaystyle\lim_{x \to 1} \frac{1}{x} = 1$.

Let $\varepsilon > 0$. Choose $\delta =$. Then

$$0 < |x - 1| < \delta \quad \implies$$

$\forall \varepsilon > 0$, $\exists \delta > 0$ such that $0 < |x - c| < \delta \implies |f(x) - L| < \varepsilon$.

Example (page 81, exerc

Prove that $\lim_{x \to 1} \frac{1}{x} = 1$.

Let $\varepsilon > 0$. Choose $\delta =$

$$0 < |x - 1| < \delta \quad =$$

$$\left|\tfrac{1}{x} - 1\right| < \varepsilon$$

$$-\varepsilon < \tfrac{1}{x} - 1 < \varepsilon$$

$$1 - \varepsilon < \tfrac{1}{x} < 1 + \varepsilon$$

$$\tfrac{1}{1-\varepsilon} > x > \tfrac{1}{1+\varepsilon}$$

$$\tfrac{1}{1+\varepsilon} < x < \tfrac{1}{1-\varepsilon}$$

$$\tfrac{1}{1+\varepsilon} - 1 < x - 1 < \tfrac{1}{1-\varepsilon} - 1$$

$$-\tfrac{\varepsilon}{1+\varepsilon} < x - 1 < \tfrac{\varepsilon}{1-\varepsilon}$$

Since $\frac{\varepsilon}{1+\varepsilon} < \frac{\varepsilon}{1-\varepsilon}$,
we will choose $\delta = \frac{\varepsilon}{1+\varepsilon}$.

Example (page 81, exercise 43.)

Prove that $\lim\limits_{x\to 1} \dfrac{1}{x} = 1$.

Let $\varepsilon > 0$. Choose $\delta = \frac{\varepsilon}{1+\varepsilon}$. Then

$$0 < |x - 1| < \delta \quad \implies$$

$\forall \varepsilon > 0$, $\exists \delta > 0$ such that $0 < |x - c| < \delta \implies |f(x) - L| < \varepsilon$.

Example (page 81, exercise 43.)

Prove that $\lim\limits_{x \to 1} \dfrac{1}{x} = 1$.

Let $\varepsilon > 0$. Choose $\delta = \frac{\varepsilon}{1+\varepsilon}$. Then

$$
\begin{aligned}
0 < |x - 1| < \delta \quad &\implies \quad -\tfrac{\varepsilon}{1+\varepsilon} < x - 1 < \tfrac{\varepsilon}{1+\varepsilon} < \tfrac{\varepsilon}{1-\varepsilon} \\
&\implies \quad 1 - \tfrac{\varepsilon}{1+\varepsilon} < x < 1 + \tfrac{\varepsilon}{1-\varepsilon} \\
&\implies \quad \tfrac{1}{1+\varepsilon} < x < \tfrac{1}{1-\varepsilon} \\
&\implies \quad 1 + \varepsilon > \tfrac{1}{x} > 1 - \varepsilon \\
&\implies \quad 1 - \varepsilon < \tfrac{1}{x} < 1 + \varepsilon \\
&\implies \quad -\varepsilon < \tfrac{1}{x} - 1 < \varepsilon \\
&\implies \quad \left| \tfrac{1}{x} - 1 \right| < \varepsilon.
\end{aligned}
$$

$\forall\varepsilon > 0$, $\exists\delta > 0$ such that $0 < |x - c| < \delta \implies |f(x) - L| < \varepsilon$.

Example (page 81, exercise 45.)

Prove that $\displaystyle\lim_{x\to -3} \frac{x^2 - 9}{x + 3} = -6.$

Let $\varepsilon > 0$. Choose $\delta =$. Then

$$0 < |x - (-3)| < \delta \quad \implies$$

$\forall \varepsilon > 0$, $\exists \delta > 0$ such that $0 < |x - c| < \delta \implies |f(x) - L| < \varepsilon$.

Example (page 81, exerc

Prove that $\lim\limits_{x \to -3} \dfrac{x^2 - 9}{x + 3}$ =

Let $\varepsilon > 0$. Choose $\delta =$.

$$0 < |x - (-3)| < \delta$$

$$\left| \tfrac{x^2-9}{x+3} - (-6) \right| < \varepsilon$$

$$-\varepsilon < \tfrac{x^2-9}{x+3} + 6 < \varepsilon$$

$$-\varepsilon < \tfrac{(x+3)(x-3)}{x+3} + 6 < \varepsilon$$

$$-\varepsilon < (x-3) + 6 < \varepsilon$$

$$-\varepsilon < x + 3 < \varepsilon$$

$$-\varepsilon < x - (-3) < \varepsilon$$

$$|x - (-3)| < \varepsilon$$

Choose $\delta = \varepsilon$.

$\forall \varepsilon > 0$, $\exists \delta > 0$ such that $0 < |x - c| < \delta \implies |f(x) - L| < \varepsilon$.

Example (page 81, exercise 45.)

Prove that $\lim_{x \to -3} \dfrac{x^2 - 9}{x + 3} = -6$.

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Example (page 81, exercise 45.)

Prove that $\lim_{x \to -3} \dfrac{x^2 - 9}{x + 3} = -6$.

Let $\varepsilon > 0$. Choose $\delta = \varepsilon$. Then

$$
\begin{aligned}
0 < |x - (-3)| < \delta \quad &\implies \quad -\varepsilon = -\delta < x + 3 < \delta = \varepsilon \\
&\implies \quad -\varepsilon < (x - 3) + 6 < \varepsilon \\
&\implies \quad -\varepsilon < \tfrac{(x-3)(x+3)}{x+3} + 6 < \varepsilon \\
&\implies \quad -\varepsilon < \tfrac{x^2-9}{x+3} - (-6) < \varepsilon \\
&\implies \quad \left| \tfrac{x^2-9}{x+3} - (-6) \right| < \varepsilon.
\end{aligned}
$$

Theorem (Sum Rule for Limits)

Suppose that $\lim_{x \to c} f(x) = L$ *and* $\lim_{x \to c} g(x) = M$. *Then*

$$\lim_{x \to c} \big(f(x) + g(x) \big) = L + M.$$

Proof.

Let $\varepsilon > 0$.

Proof.

Let $\varepsilon > 0$.

Since $\lim_{x \to c} f(x) = L$, we know that there exists a number $\delta_1 > 0$ such that

$$0 < |x - c| < \delta_1 \qquad \Longrightarrow \qquad |f(x) - L| < \frac{\varepsilon}{2}.$$

Proof.

Let $\varepsilon > 0$.

Since $\lim_{x \to c} f(x) = L$, we know that there exists a number $\delta_1 > 0$ such that

$$0 < |x - c| < \delta_1 \qquad \Longrightarrow \qquad |f(x) - L| < \frac{\varepsilon}{2}.$$

Since $\lim_{x \to c} g(x) = M$, we know that there exists a number $\delta_2 > 0$ such that

$$0 < |x - c| < \delta_2 \qquad \Longrightarrow \qquad |g(x) - M| < \frac{\varepsilon}{2}.$$

Proof.

Let $\varepsilon > 0$.

Since $\lim_{x \to c} f(x) = L$, we know that there exists a number $\delta_1 > 0$ such that

$$0 < |x - c| < \delta_1 \qquad \Longrightarrow \qquad |f(x) - L| < \frac{\varepsilon}{2}.$$

Since $\lim_{x \to c} g(x) = M$, we know that there exists a number $\delta_2 > 0$ such that

$$0 < |x - c| < \delta_2 \qquad \Longrightarrow \qquad |g(x) - M| < \frac{\varepsilon}{2}.$$

Choose $\delta = \min\{\delta_1, \delta_2\}$.

$$|a+b| \leq |a| + |b|$$

Proof continued.

Then

$$\begin{aligned} 0 < |x-c| < \delta \quad \implies \quad & \big|\big(f(x)+g(x)\big) - (L+M)\big| \\ & \quad = \big|\big(f(x)-L\big) + \big(g(x)-M\big)\big| \\ & \quad \leq |f(x)-L| + |g(x)-M| \end{aligned}$$

$$|a+b| \le |a| + |b|$$

Proof continued.

Then

$$\begin{aligned} 0 < |x-c| < \delta \qquad \Longrightarrow \qquad & \big|\big(f(x)+g(x)\big) - (L+M)\big| \\ & \qquad = \big|\big(f(x)-L\big) + \big(g(x)-M\big)\big| \\ & \qquad \le |f(x)-L| + |g(x)-M| \\ & \qquad < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \end{aligned}$$

$$|a+b| \le |a| + |b|$$

Proof continued.

Then

$$\begin{aligned} 0 < |x-c| < \delta \quad \implies \quad & \big|\big(f(x)+g(x)\big) - (L+M)\big| \\ &= \big|\big(f(x)-L\big) + \big(g(x)-M\big)\big| \\ &\le |f(x)-L| + |g(x)-M| \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \end{aligned}$$

Therefore $\lim_{x\to c} \big(f(x)+g(x)\big) = L+M.$ $\square$

Next Time

- 2.4 One-Sided Limits
- 2.5 Continuity
- 2.6 Limits Involving Infinity; Asymptotes of Graphs