

Lecture 6

- 3.8 Related Rates
- 3.9 Linearisation and Differentials
- 4.1 Extreme Values of Functions on Closed Intervals
- 4.2 The Mean Value Theorem



Related Rates

3.8 Related Rates



$$A \leftrightarrow B$$

Sometimes the rate of change of one quantity affects the rate of change of a different quantity.

3.8 Related Rates



$$V = \frac{4}{3}\pi r^3$$

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$$\frac{dV}{dt} = \frac{dV}{dr} \frac{dr}{dt} = 4\pi r^2 \frac{dr}{dt}.$$

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3.8 Related Rates

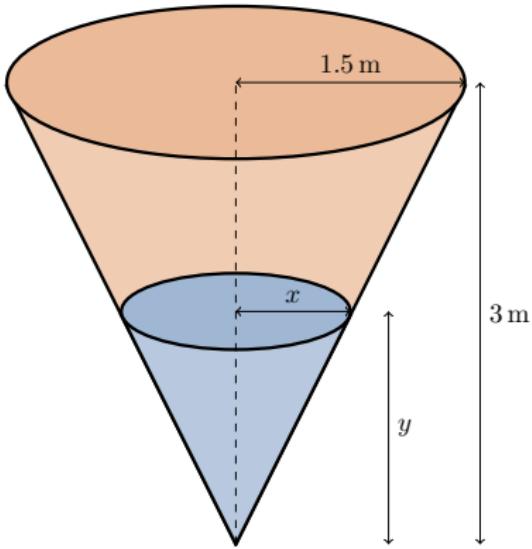


$$\frac{dV}{dt} = \frac{dV}{dr} \frac{dr}{dt} = 4\pi r^2 \frac{dr}{dt}.$$

If we know r and $\frac{dV}{dt}$, then we can find $\frac{dr}{dt}$.

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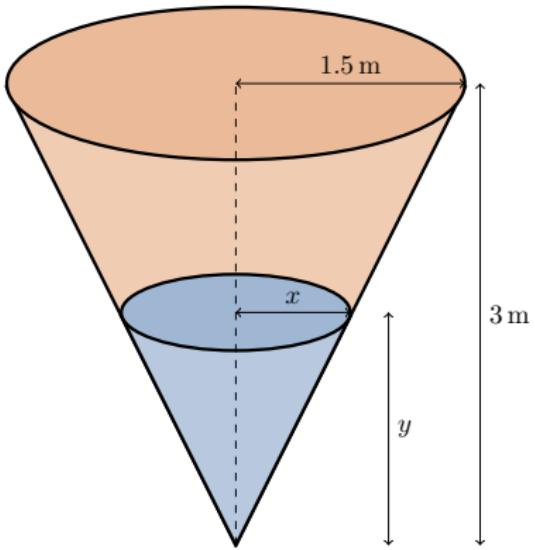
3.8 Related Rates



Example

Water runs into a conical tank at the rate of $0.25 \text{ m}^3/\text{min}$. The tank stands point down and has a height of 3 m and a base radius of 1.5 m. How fast is the water level rising when the water is 1.8 m deep?

3.8 Related Rates



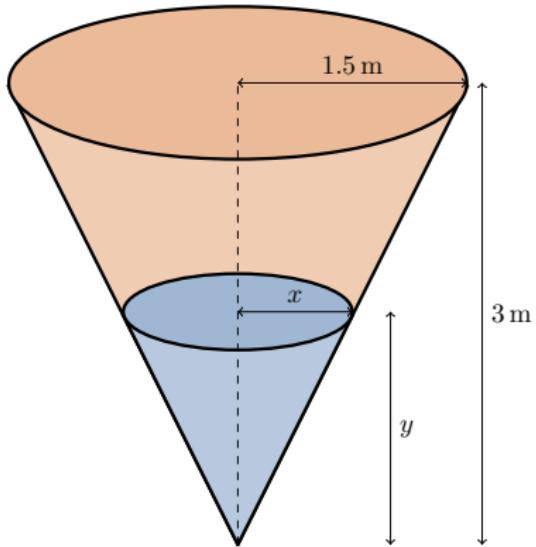
t = time (min)

V = volume (m^3) of the water in the tank at time

x = radius (m) of the water at time t

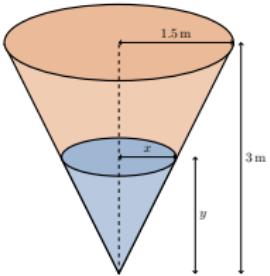
y = height (m) of the water at time t .

3.8 Related Rates



We know $y = 1.8 \text{ m}$ and $\frac{dV}{dt} = 0.25 \text{ m}^3/\text{min}$. We are asked to find $\frac{dy}{dt}$.

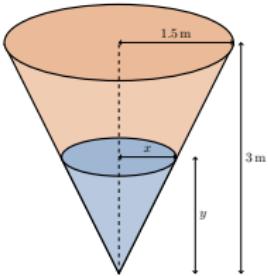
3.8 Related Rates



Since $x = \frac{y}{2}$, the volume of the water is

$$V = \frac{1}{3}\pi x^2 y = \frac{1}{3}\pi \left(\frac{y}{2}\right)^2 y = \frac{\pi}{12}y^3.$$

3.8 Related Rates



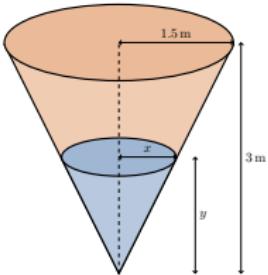
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$$\frac{dV}{dt} = \frac{\pi}{4}y^2 \frac{dy}{dt}$$

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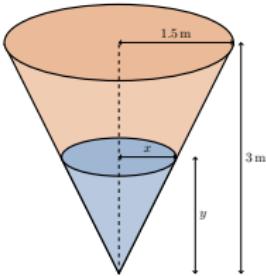
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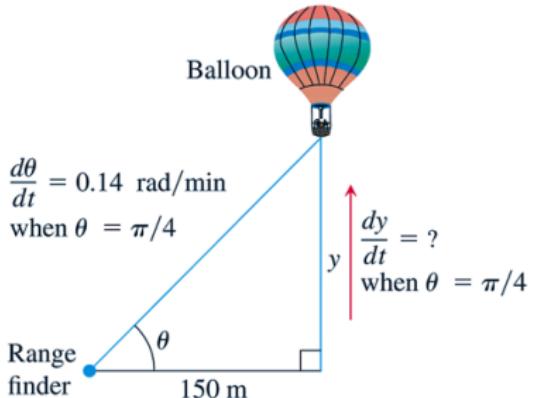
It follows that

$$\frac{dy}{dt} = \frac{1}{3.24\pi} \approx 0.098 \text{ m/min.}$$

Related Rates Problem Strategy

1. *Draw a picture and name the variables and constants.* Use t for time. Assume that all variables are differentiable functions of t .
2. *Write down the numerical information* (in terms of the symbols you have chosen).
3. *Write down what you are asked to find* (usually a rate, expressed as a derivative).
4. *Write an equation that relates the variables.* You may have to combine two or more equations to get a single equation that relates the variable whose rate you want to the variables whose rates you know.
5. *Differentiate with respect to t .* Then express the rate you want in terms of the rates and variables whose values you know.
6. *Evaluate.* Use known values to find the unknown rate.

3.8 Related Rates



EXAMPLE 2 A hot air balloon rising straight up from a level field is tracked by a range finder 150 m from the liftoff point. At the moment the range finder's elevation angle is $\pi/4$, the angle is increasing at the rate of 0.14 rad/min . How fast is the balloon rising at that moment?

Solution We answer the question in the six strategy steps.

1. Draw a picture and name the variables and constants (Figure 3.34). The variables in the picture are

θ = the angle in radians the range finder makes with the ground.

y = the height in meters of the balloon above the ground.

We let t represent time in minutes and assume that θ and y are differentiable functions of t .

The one constant in the picture is the distance from the range finder to the liftoff point (150 m). There is no need to give it a special symbol.

- 2.** Write down the additional numerical information.

$$\frac{d\theta}{dt} = 0.14 \text{ rad/min} \quad \text{when} \quad \theta = \frac{\pi}{4}$$

- 3.** Write down what we are to find. We want dy/dt when $\theta = \pi/4$.

- 4.** Write an equation that relates the variables y and θ .

$$\frac{y}{150} = \tan \theta \quad \text{or} \quad y = 150 \tan \theta$$

- 5.** Differentiate with respect to t using the Chain Rule. The result tells how dy/dt (which we want) is related to $d\theta/dt$ (which we know).

$$\frac{dy}{dt} = 150 (\sec^2 \theta) \frac{d\theta}{dt}$$

- 6.** Evaluate with $\theta = \pi/4$ and $d\theta/dt = 0.14$ to find dy/dt .

$$\frac{dy}{dt} = 150(\sqrt{2})^2(0.14) = 42 \quad \sec \frac{\pi}{4} = \sqrt{2}$$

At the moment in question, the balloon is rising at the rate of 42 m/min. ■

3.8 Related Rates

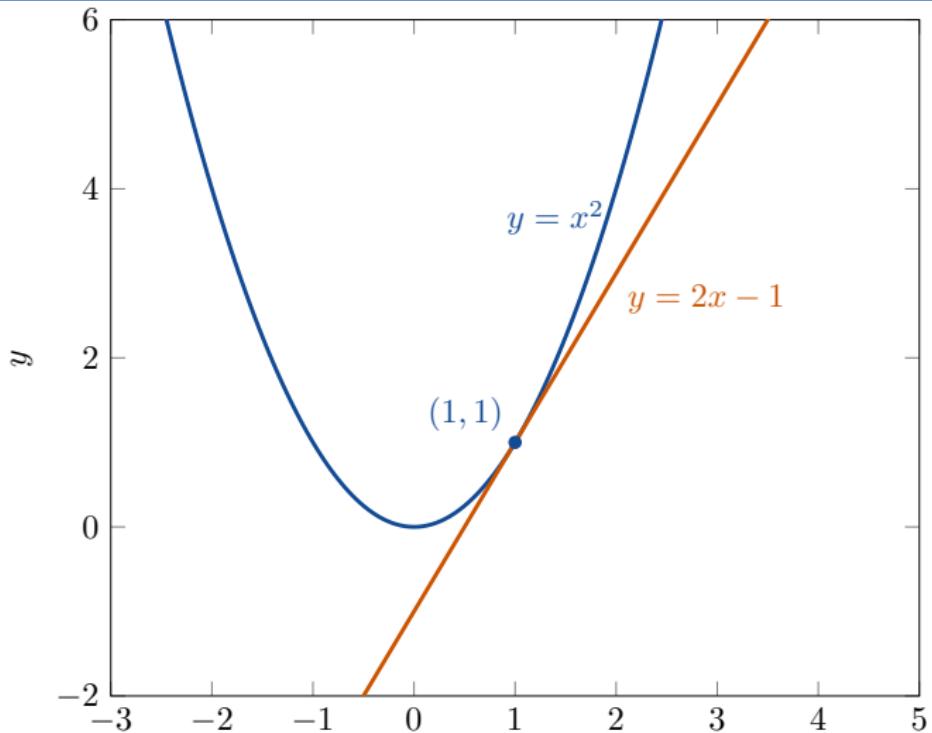


Please read examples 3-6 in your textbook.



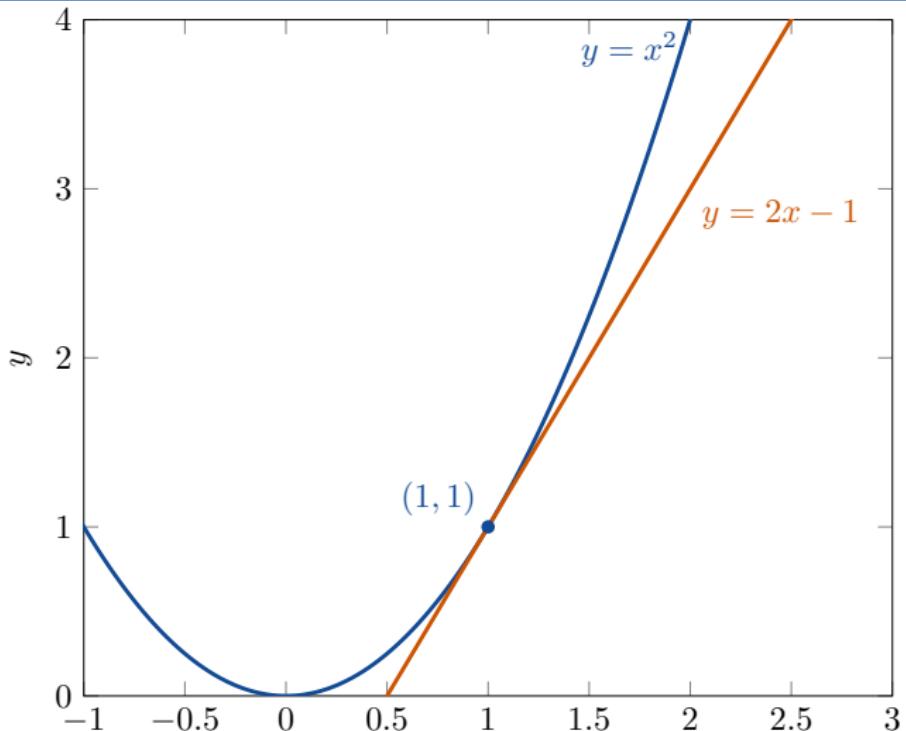
Linearisation and Differentials

3.9 Linearisation and Differentials



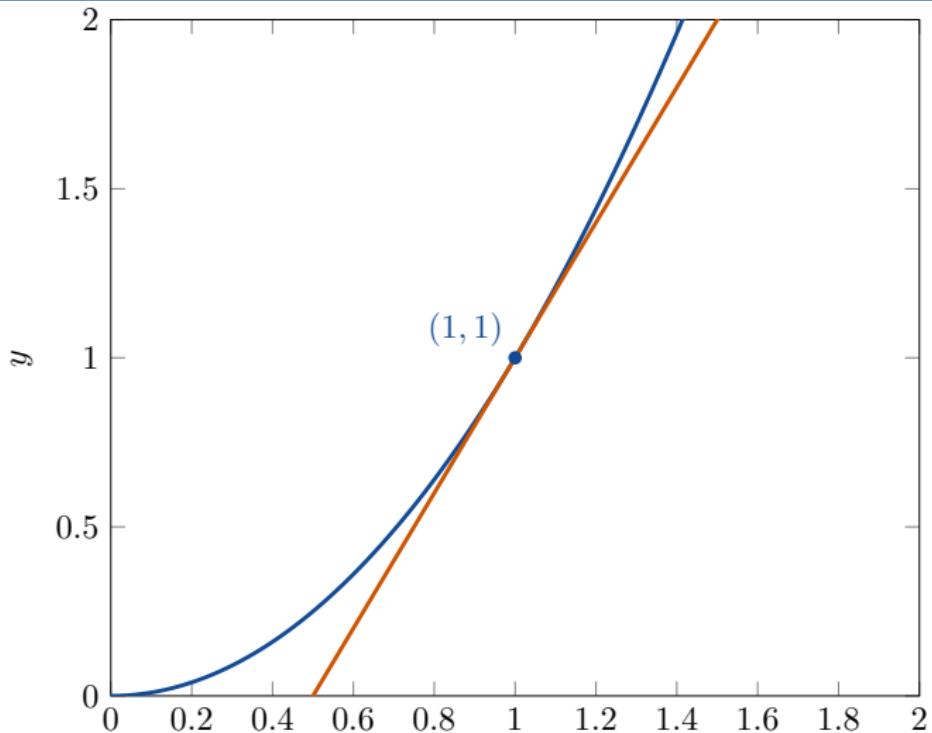
Consider the tangent to $y = x^2$ at $\overset{x}{(1, 1)}$. Let's zoom in.

3.9 Linearisation and Differentials



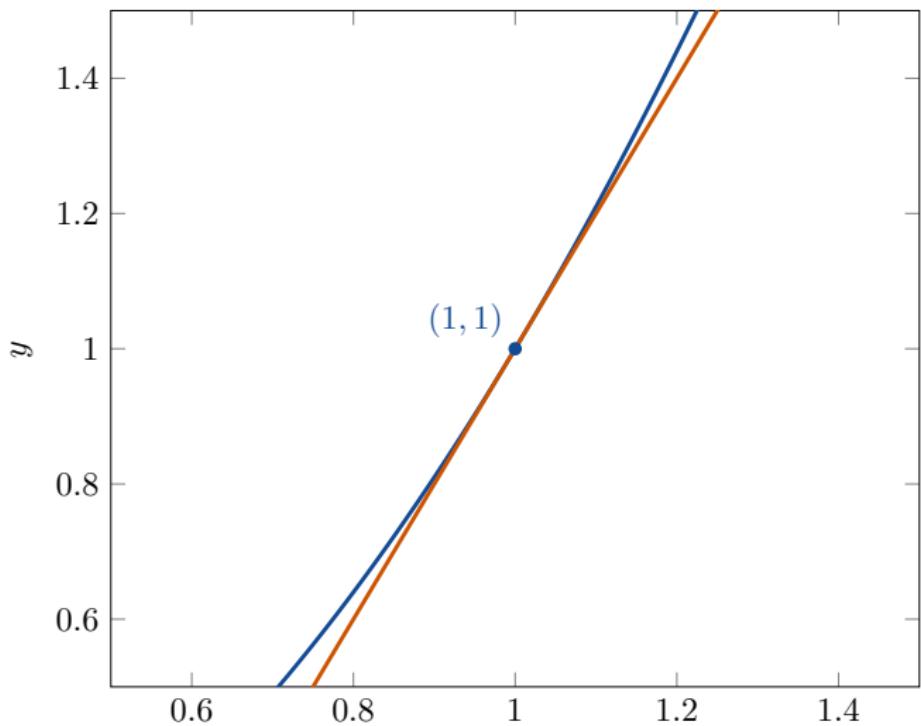
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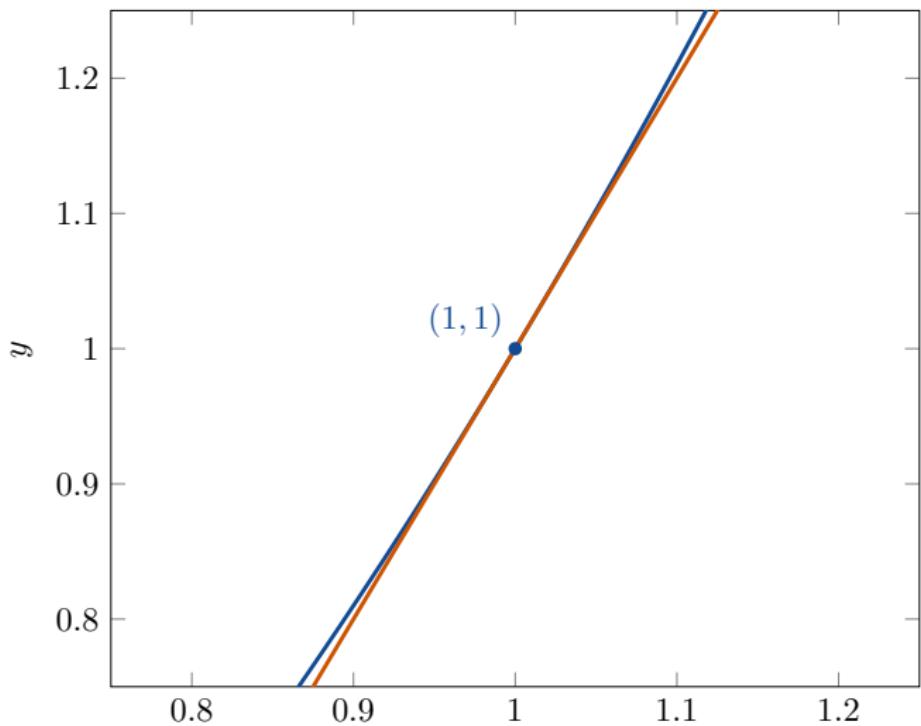
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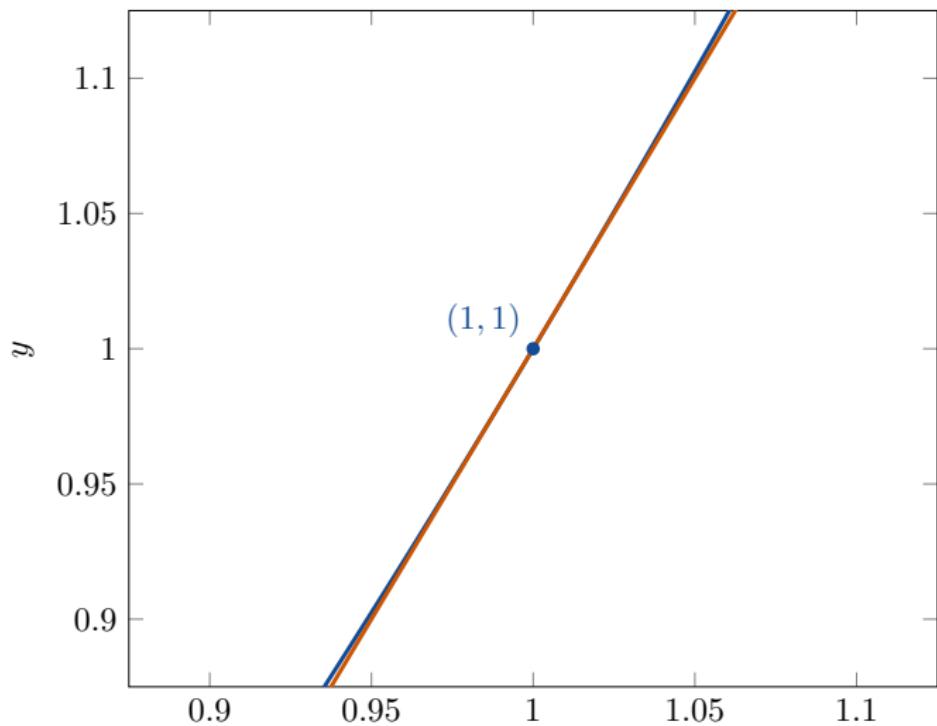
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3.9 Linearisation and Differentials



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3.9 Linearisation and Differentials



Consider the tangent to $y = x^2$ at $(1, 1)$. Let's zoom in. If x is close to 1, then $x^2 \approx 2x - 1$.

3.9 Linearisation and Differentials



Definition

If f is differentiable at $x = a$, then the approximating function

$$L(x) = f(a) + f'(a)(x - a)$$

is the *linearisation* of f at a .

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The approximation

$$f(x) \approx L(x)$$

of f by L is called the *standard linear approximation* of f at a .
The point $x = a$ is the *centre* of the approximation.

EXAMPLE 1 Find the linearization of $f(x) = \sqrt{1 + x}$ at $x = 0$ (Figure 3.41).

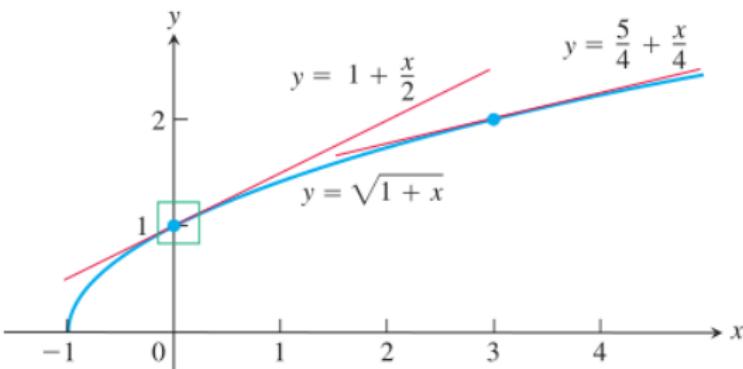


FIGURE 3.41 The graph of $y = \sqrt{1 + x}$ and its linearizations at $x = 0$ and $x = 3$. Figure 3.42 shows a magnified view of the small window about 1 on the y -axis.

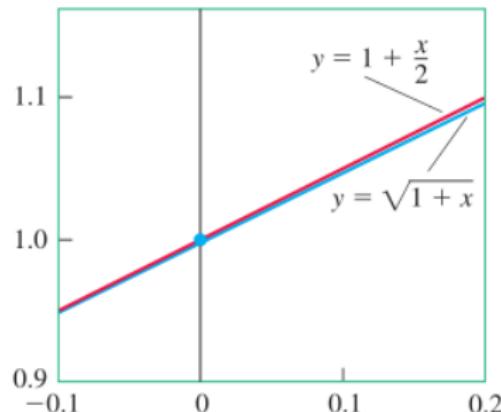


FIGURE 3.42 Magnified view of the window in Figure 3.41.

Solution Since

$$f'(x) = \frac{1}{2}(1 + x)^{-1/2},$$

we have $f(0) = 1$ and $f'(0) = 1/2$, giving the linearization

$$L(x) = f(a) + f'(a)(x - a) = 1 + \frac{1}{2}(x - 0) = 1 + \frac{x}{2}.$$

See Figure 3.42. ■

The following table shows how accurate the approximation $\sqrt{1+x} \approx 1 + (x/2)$ from Example 1 is for some values of x near 0. As we move away from zero, we lose accuracy. For example, for $x = 2$, the linearization gives 2 as the approximation for $\sqrt{3}$, which is not even accurate to one decimal place.

Approximation	True value	True value – approximation
$\sqrt{1.005} \approx 1 + \frac{0.005}{2} = 1.00250$	1.002497	$0.000003 < 10^{-5}$
$\sqrt{1.05} \approx 1 + \frac{0.05}{2} = 1.025$	1.024695	$0.000305 < 10^{-3}$
$\sqrt{1.2} \approx 1 + \frac{0.2}{2} = 1.10$	1.095445	$0.004555 < 10^{-2}$

EXAMPLE 2 Find the linearization of $f(x) = \sqrt{1 + x}$ at $x = 3$. (See Figure 3.41.)

Solution We evaluate the equation defining $L(x)$ at $a = 3$. With

$$f(3) = 2, \quad f'(3) = \frac{1}{2}(1 + x)^{-1/2} \Big|_{x=3} = \frac{1}{4},$$

we have

$$L(x) = 2 + \frac{1}{4}(x - 3) = \frac{5}{4} + \frac{x}{4}.$$

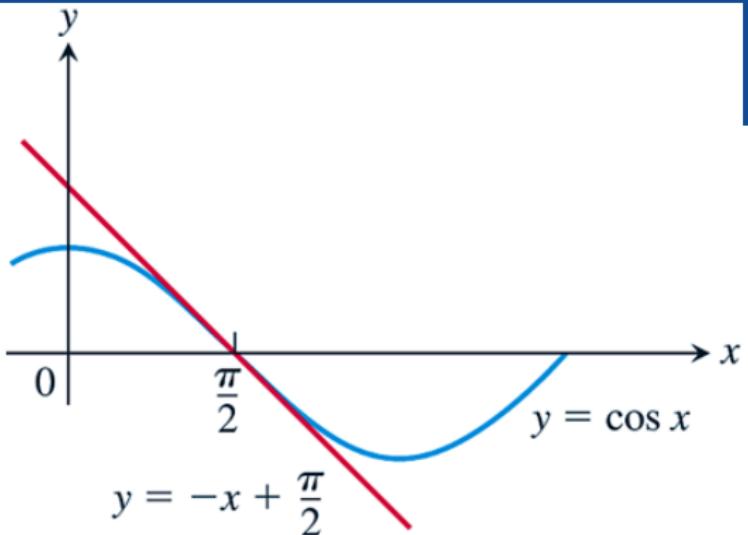
At $x = 3.2$, the linearization in Example 2 gives

$$\sqrt{1 + x} = \sqrt{1 + 3.2} \approx \frac{5}{4} + \frac{3.2}{4} = 1.250 + 0.800 = 2.050,$$

which differs from the true value $\sqrt{4.2} \approx 2.04939$ by less than one one-thousandth. The linearization in Example 1 gives

$$\sqrt{1 + x} = \sqrt{1 + 3.2} \approx 1 + \frac{3.2}{2} = 1 + 1.6 = 2.6,$$

a result that is off by more than 25%. ■



EXAMPLE 3 Find the linearization of $f(x) = \cos x$ at $x = \pi/2$ (Figure 3.43).

Solution Since $f(\pi/2) = \cos(\pi/2) = 0$, $f'(x) = -\sin x$, and $f'(\pi/2) = -\sin(\pi/2) = -1$, we find the linearization at $a = \pi/2$ to be

$$\begin{aligned}L(x) &= f(a) + f'(a)(x - a) \\&= 0 + (-1)\left(x - \frac{\pi}{2}\right) \\&= -x + \frac{\pi}{2}.\end{aligned}$$



3.9 Linearisation and Differ

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Now consider

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Now consider

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close to $x = 0$ for any number k .

Since

$$\begin{aligned}f'(x) &= k(1 + x)^{k-1} \\f'(0) &= k(1 + 0)^{k-1} = k,\end{aligned}$$

we have that

$$L(x) = f(0) + f'(0)(x - 0) = 1 + kx.$$

3.9 Linearisation and Differentials

$$(1 + x)^k \approx 1 + kx$$



So for example,

$$\sqrt{1 + x} = (1 + x)^{\frac{1}{2}} \approx 1 + \frac{1}{2}x$$

3.9 Linearisation and Differentials

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3.9 Linearisation and Differentials

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$$\sqrt[3]{1+5x^4} = (1+5x^4)^{\frac{1}{3}} \approx 1 + \frac{1}{3}(5x^4) = 1 + \frac{5}{3}x^4$$

3.9 Linearisation and Differentials

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$$\frac{1}{\sqrt{1-x^2}} = (1-x^2)^{-\frac{1}{2}} \approx 1 + \left(-\frac{1}{2}\right)(-x^2) = 1 + \frac{1}{2}x^2$$

Differentials

Recall that

$$\frac{dy}{dx}$$

means the derivative of y . It does not mean dy divided by dx .

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Unless we create two new variables dy and dx such that dy divided by dx is equal to the derivative of y .

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Then

$$dy \div dx = \frac{f'(x) dx}{dx} = f'(x) = \frac{dy}{dx}.$$

EXAMPLE 4

- (a) Find dy if $y = x^5 + 37x$.
- (b) Find the value of dy when $x = 1$ and $dx = 0.2$.

Solution

- (a) $dy = (5x^4 + 37) dx$
- (b) Substituting $x = 1$ and $dx = 0.2$ in the expression for dy , we have

$$dy = (5 \cdot 1^4 + 37) 0.2 = 8.4.$$



3.9 Linearisation and Differentials



If $f(x) = 3x^2 - 6$, then the *differential of f* is

$$df = d(3x^2 - 6) = 6x \, dx.$$

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If $f(x) = 3x^2 - 6$, then the *differential of f* is

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Every differential formula like

$$\frac{d}{dx}(u + v) = \frac{du}{dx} + \frac{dv}{dx} \quad \text{or} \quad \frac{d}{dx}(\sin u) = \cos u \frac{du}{dx}$$

has a corresponding differential form like

$$d(u + v) = du + dv \quad \text{or} \quad d(\sin u) = \cos u \, du.$$

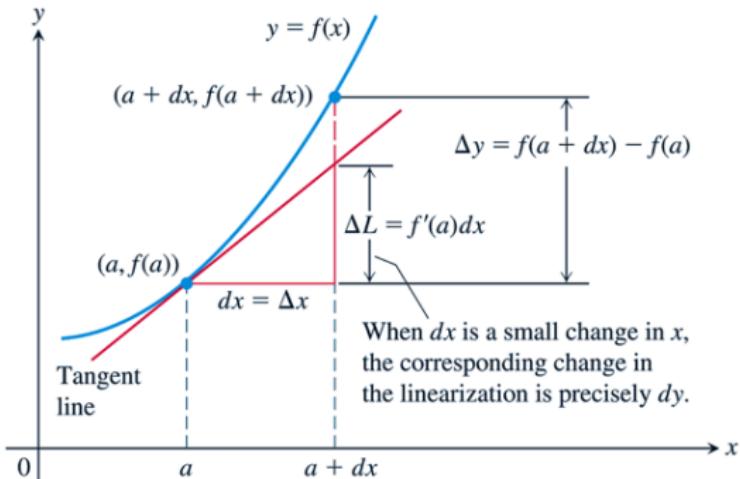
EXAMPLE 5 We can use the Chain Rule and other differentiation rules to find differentials of functions.

(a) $d(\tan 2x) = \sec^2(2x) d(2x) = 2 \sec^2 2x dx$

(b) $d\left(\frac{x}{x+1}\right) = \frac{(x+1)dx - x d(x+1)}{(x+1)^2} = \frac{xdx + dx - xdx}{(x+1)^2} = \frac{dx}{(x+1)^2}$ ■

$$dy = f'(x) dx$$

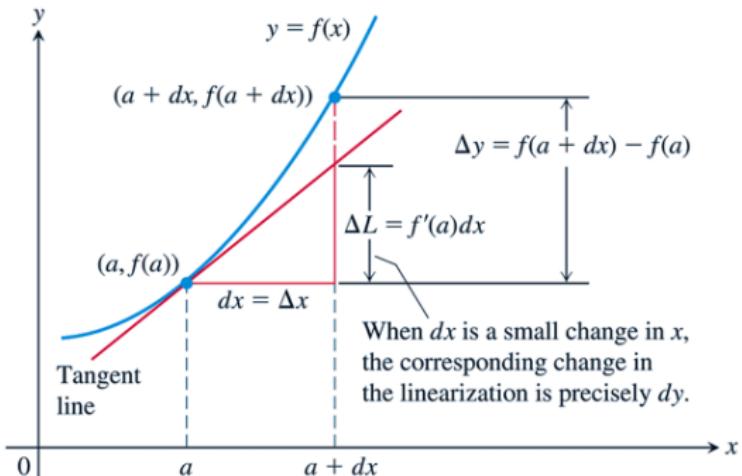
Estimating with Differentials



Suppose that we know the value of $f(x)$ at $x = a$ and we want to estimate how much this value will change if we move to $x = a + dx$.

$$dy = f'(x) dx$$

Estimating with Differentials



If dx is small, then

$$f(a + dx) \approx f(a) + dy.$$

3.9 Linearisatio

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Example

Use differentials to estimate $7.97^{\frac{1}{3}}$.

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Use differentials to estimate $7.97^{\frac{1}{3}}$.

Note first that if $y = f(x) = x^{\frac{1}{3}}$, then

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We are close to $a = 8$ ($7.97 = a - .03$). Using $dx = -.03$,

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We are close to $a = 8$ ($7.97 = a - .03$). Using $dx = -.03$, we have

$$\begin{aligned} f(7.97) &= f(a + dx) \approx f(a) + dy \\ &= 8^{\frac{1}{3}} + \frac{1}{3(8)^{\frac{2}{3}}}(-0.03) \\ &= 2 + \frac{1}{12}(-0.03) = 1.9975. \end{aligned}$$

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The true value is $7.97^{\frac{1}{3}} = 1.99749686847326240\dots$

EXAMPLE 6 The radius r of a circle increases from $a = 10$ m to 10.1 m (Figure 3.45). Use dA to estimate the increase in the circle's area A . Estimate the area of the enlarged circle and compare your estimate to the true area found by direct calculation.

Solution Since $A = \pi r^2$, the estimated increase is

$$dA = A'(a) dr = 2\pi a dr = 2\pi(10)(0.1) = 2\pi \text{ m}^2.$$

Thus, since $A(r + \Delta r) \approx A(r) + dA$, we have

$$\begin{aligned} A(10 + 0.1) &\approx A(10) + 2\pi \\ &= \pi(10)^2 + 2\pi = 102\pi. \end{aligned}$$

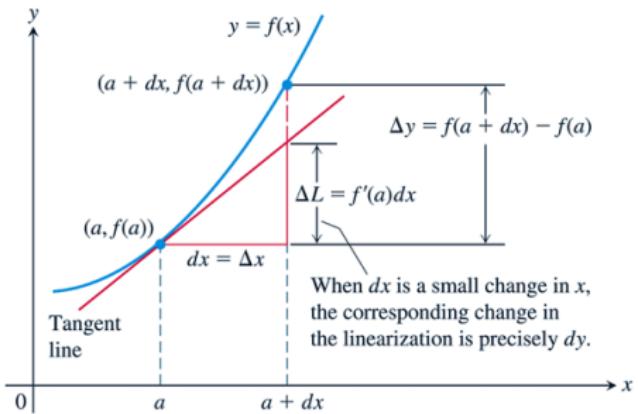
The area of a circle of radius 10.1 m is approximately $102\pi \text{ m}^2$.

The true area is

$$\begin{aligned} A(10.1) &= \pi(10.1)^2 \\ &= 102.01\pi \text{ m}^2. \end{aligned}$$

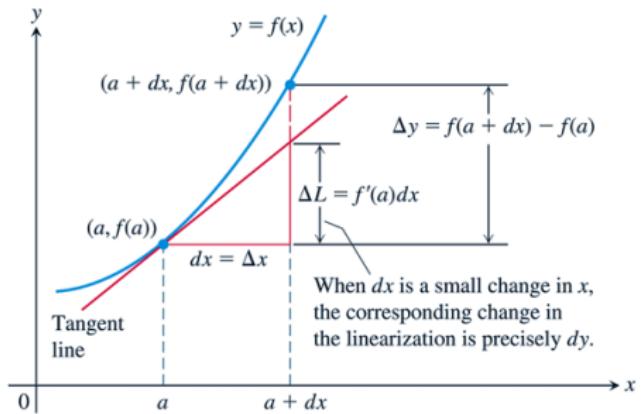
The error in our estimate is $0.01\pi \text{ m}^2$, which is the difference $\Delta A - dA$. ■

Error in Differential Approximation



How well does $f(a) + dy$ approximate $f(a + dx)$?

Error in Differential Approximation



How well does $f(a) + dy$ approximate $f(a + dx)$?

$$\text{true change: } \Delta y = f(a + dx) - f(a)$$

$$\text{approximation change: } dy = f'(a) dx$$

Error in Differential Approximation

How well does $f(a) + dy$ approximate $f(a + dx)$?

true change: $\Delta y = f(a + dx) - f(a)$

approximation change: $dy = f'(a) dx$

$$\text{error} = \Delta y - dy$$

$$= f(a + dx) - f(a) - f'(a) dx$$

=

=

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$$= f(a + dx) - f(a) - f'(a) dx$$

$$= \left(\frac{f(a + dx) - f(a)}{dx} - f'(a) \right) dx$$

$$= \varepsilon dx.$$

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$$\text{error} = \Delta y - dy$$

$$= f(a + dx) - f(a) - f'(a) dx$$

$$= \left(\frac{f(a + dx) - f(a)}{dx} - f'(a) \right) dx$$

$$= \varepsilon dx.$$

Therefore

$$\boxed{\Delta y = f'(a) dx + \varepsilon dx.}$$

In Example 6 we found that

$$\Delta A = \pi(10.1)^2 - \pi(10)^2 = (102.01 - 100)\pi = (\underbrace{2\pi}_{dA} + \underbrace{0.01\pi}_{\text{error}}) \text{ m}^2$$

so the approximation error is $\Delta A - dA = \varepsilon \Delta r = 0.01\pi$ and $\varepsilon = 0.01\pi/\Delta r = 0.01\pi/0.1 = 0.1\pi$ m.



Break

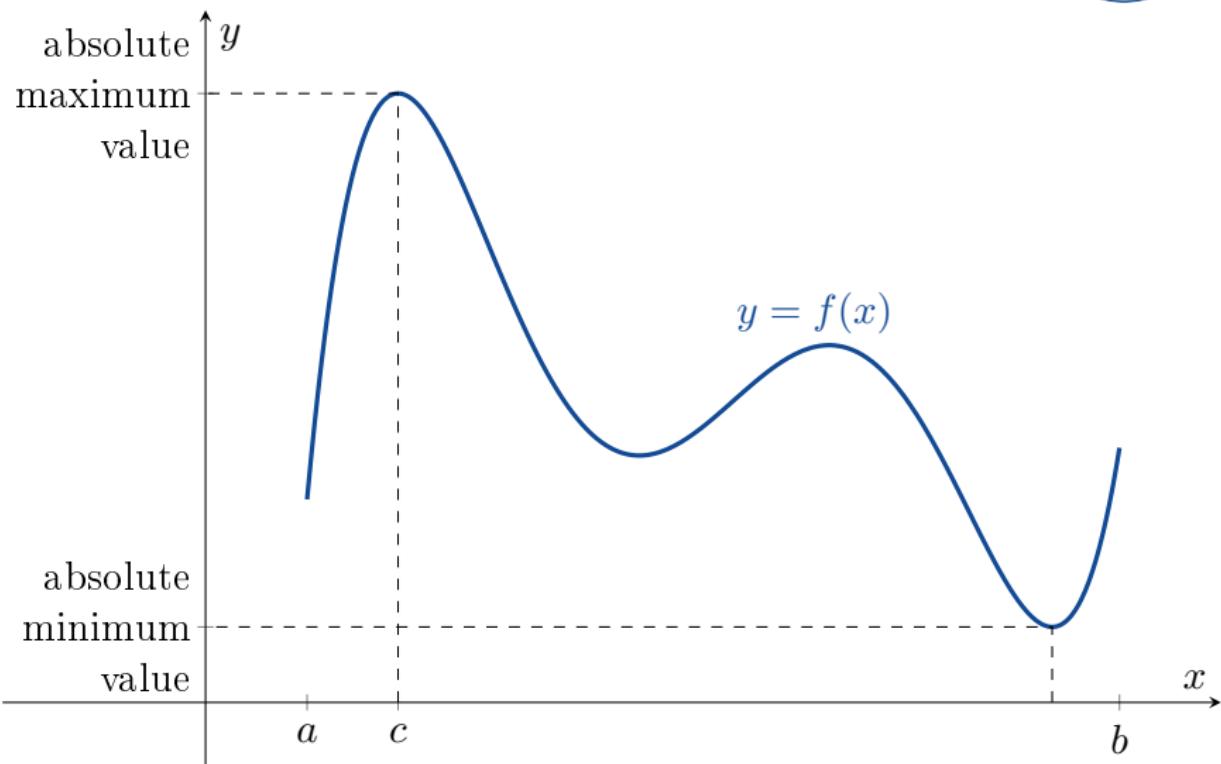
We will continue at 3pm



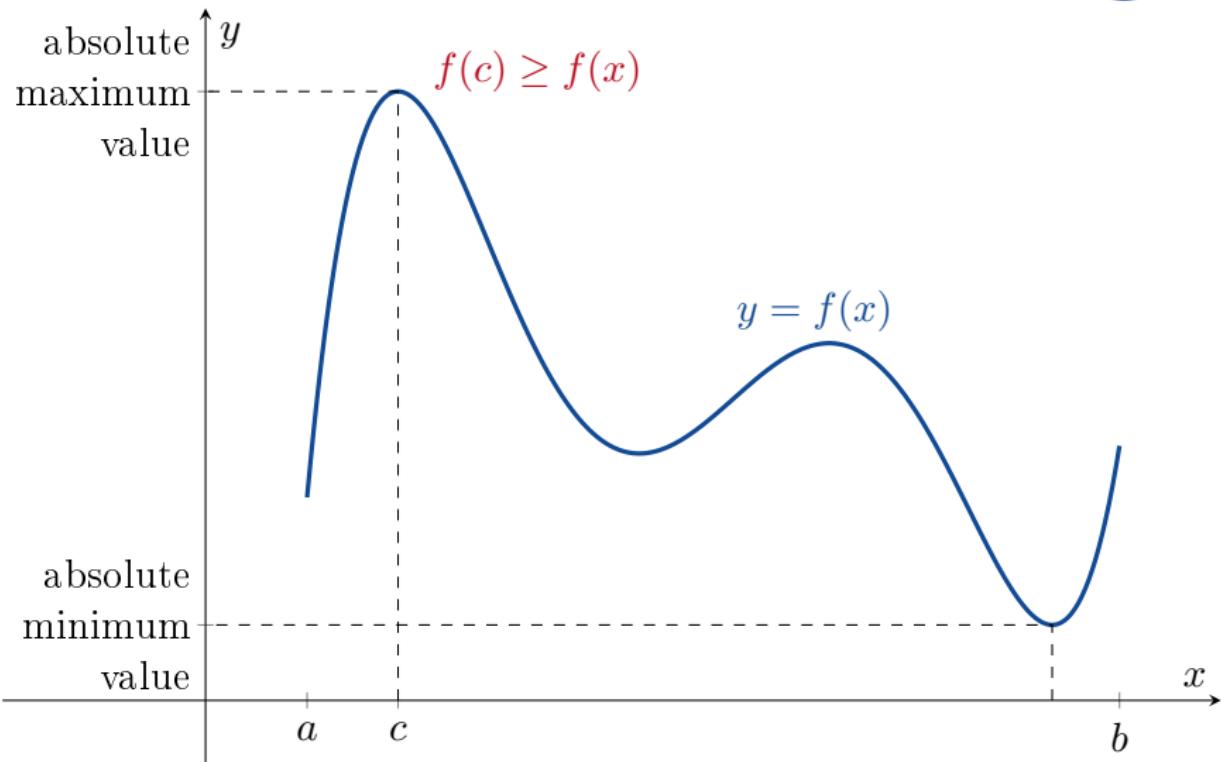


Extreme Values of Functions on Closed Intervals

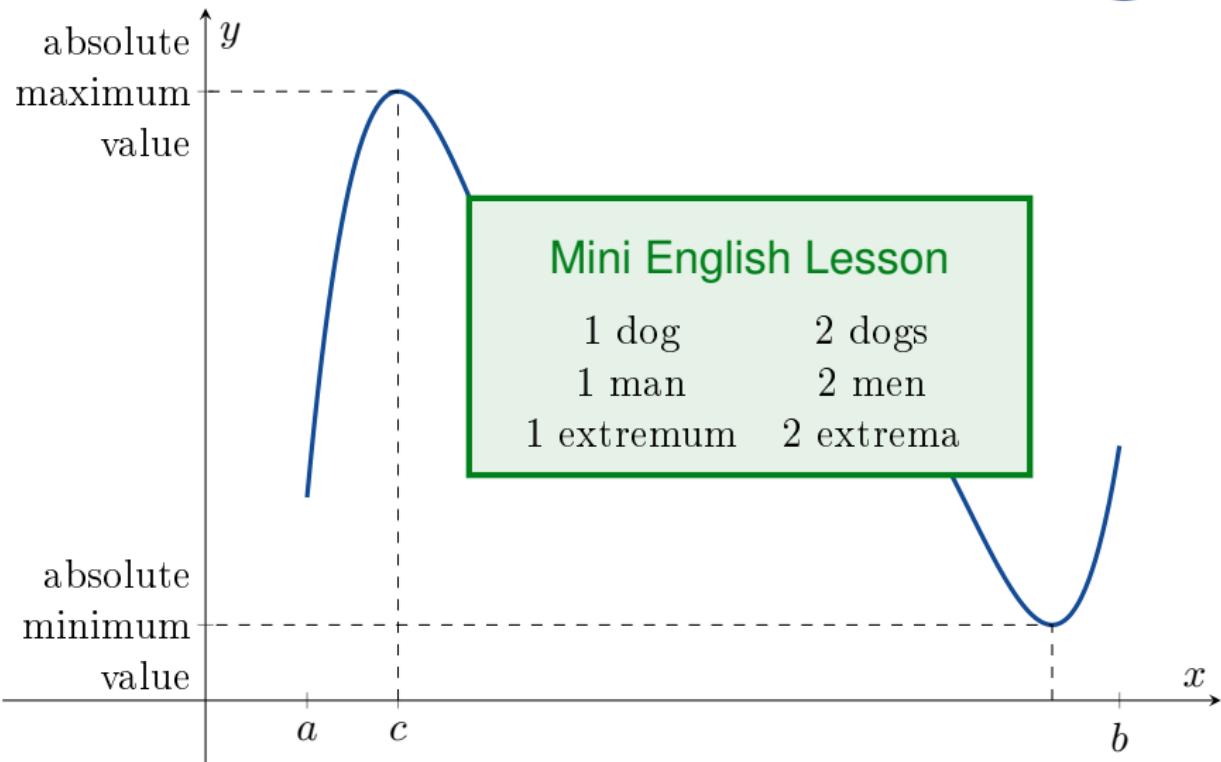
4.1 Extreme Values of Functions on Closed Intervals



4.1 Extreme Values of Functions on Closed Intervals



4.1 Extreme Values of Functions on Closed Intervals



4.1 Extreme Values of Functions on Closed Intervals



Definition

Let $f : D \rightarrow \mathbb{R}$ be a function.

- f has an *absolute maximum value* on D at a point c if

$$f(x) \leq f(c)$$

for all $x \in D$.

- f has an *absolute minimum value* on D at a point c if

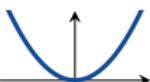
$$f(x) \geq f(c)$$

for all $x \in D$.

Maximum and minimum values are called *extrema/extreme values*.

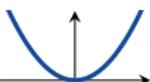
4.1 Extreme Values of Functions on Closed Intervals

Example

function	domain, D	graph	absolute extrema on D
$y = x^2$	$(-\infty, \infty)$		No absolute maximum. Absolute minimum of 0 at $x = 0$.
$y = x^2$	$[0, 2]$		
$y = x^2$	$(0, 2]$		
$y = x^2$	$(0, 2)$		

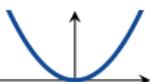
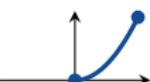
4.1 Extreme Values of Functions on Closed Intervals

Example

function	domain, D	graph	absolute extrema on D
$y = x^2$	$(-\infty, \infty)$	 A graph of the function $y = x^2$ showing a parabola opening upwards with its vertex at the origin (0,0). The graph extends infinitely in both directions along the positive and negative x-axis.	No absolute maximum. Absolute minimum of 0 at $x = 0$.
$y = x^2$	$[0, 2]$	 A graph of the function $y = x^2$ on the closed interval $[0, 2]$. The curve starts at the origin (0,0) and ends at the point (2, 4). The endpoint at (0,0) is marked with a solid blue dot, while the endpoint at (2, 4) is marked with an open blue circle.	Absolute maximum of 4 at $x = 2$. Absolute minimum of 0 at $x = 0$.
$y = x^2$	$(0, 2]$		
$y = x^2$	$(0, 2)$		

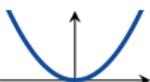
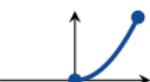
4.1 Extreme Values of Functions on Closed Intervals

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function	domain, D	graph	absolute extrema on D
$y = x^2$	$(-\infty, \infty)$		No absolute maximum. Absolute minimum of 0 at $x = 0$.
$y = x^2$	$[0, 2]$		Absolute maximum of 4 at $x = 2$. Absolute minimum of 0 at $x = 0$.
$y = x^2$	$(0, 2]$		Absolute maximum of 4 at $x = 2$. No absolute minimum.
$y = x^2$	$(0, 2)$		

4.1 Extreme Values of Functions on Closed Intervals

Example

function	domain, D	graph	absolute extrema on D
$y = x^2$	$(-\infty, \infty)$		No absolute maximum. Absolute minimum of 0 at $x = 0$.
$y = x^2$	$[0, 2]$		Absolute maximum of 4 at $x = 2$. Absolute minimum of 0 at $x = 0$.
$y = x^2$	$(0, 2]$		Absolute maximum of 4 at $x = 2$. No absolute minimum.
$y = x^2$	$(0, 2)$		No absolute extrema.

4.1 Extreme Values of Functions on Closed Intervals



Theorem (The Extreme Value Theorem)

Suppose that

- $f : D \rightarrow \mathbb{R}$ is continuous; and
- $D = [a, b]$ is a closed interval.

Then f attains both an absolute maximum value M and an absolute minimum value m in $[a, b]$.

4.1 Extreme Values of Functions on Closed Intervals



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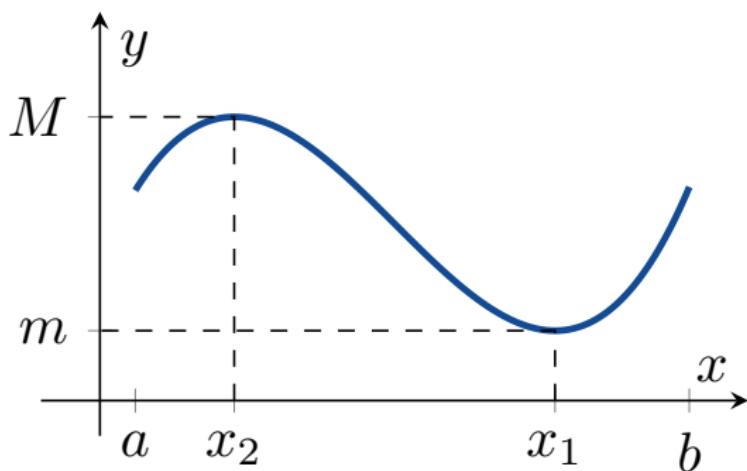
Then f attains both an absolute maximum value M and an absolute minimum value m in $[a, b]$.

Remark

The Extreme Value Theorem says that there are numbers $x_1, x_2 \in [a, b]$ such that

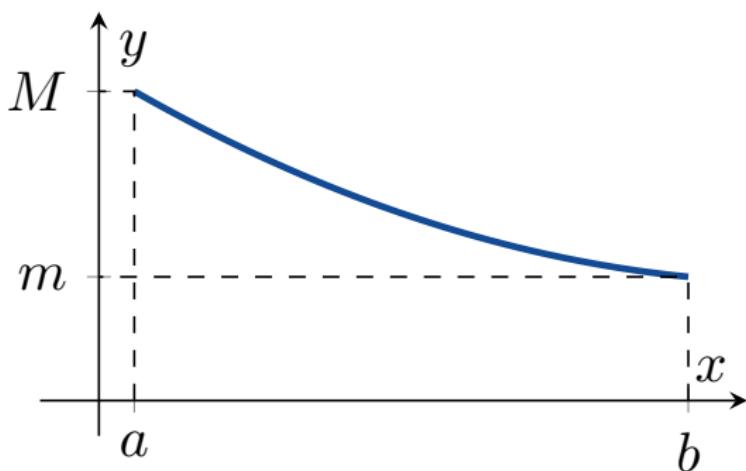
- $f(x_1) = m$;
- $f(x_2) = M$; and
- $m \leq f(x) \leq M$ for all $x \in [a, b]$.

4.1 Extreme Values of Functions on Closed Intervals



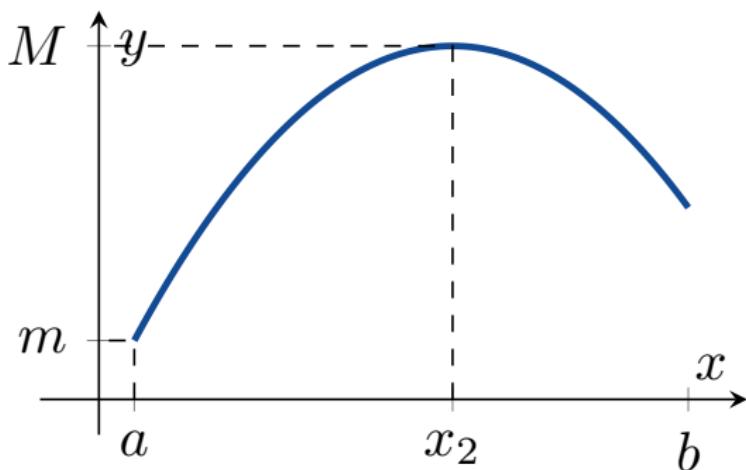
The absolute maximum and absolute minimum are at interior points.

4.1 Extreme Values of Functions on Closed Intervals



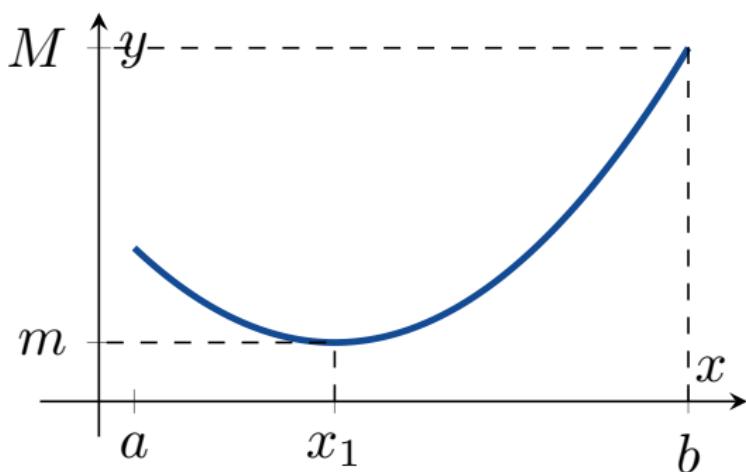
The absolute maximum and absolute minimum are at endpoints.

4.1 Extreme Values of Functions on Closed Intervals



The absolute maximum is at an interior point. The absolute minimum is at an endpoint.

4.1 Extreme Values of Functions on Closed Intervals

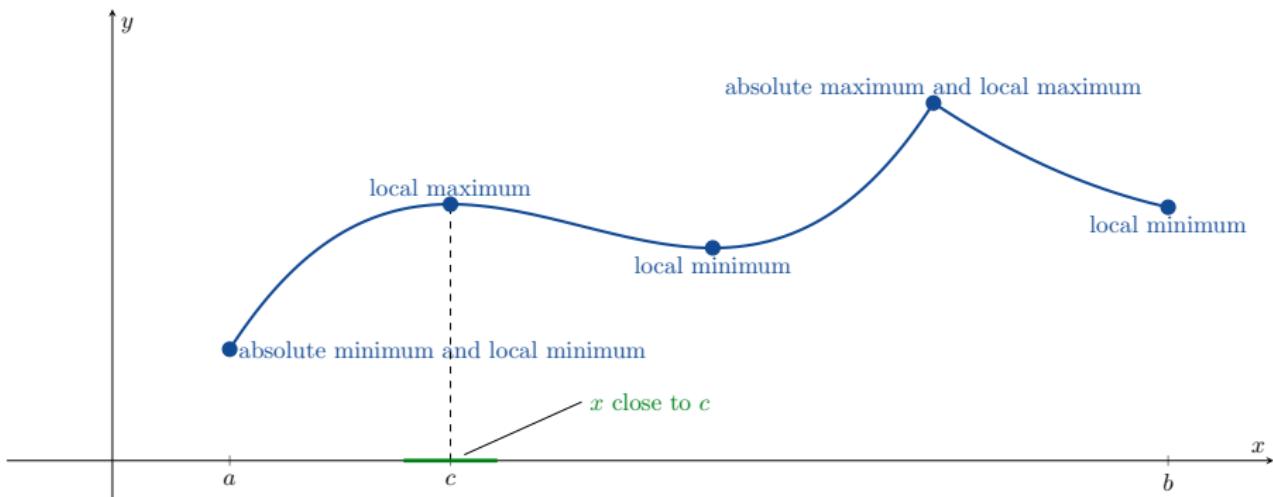


The absolute maximum is at an endpoint. The absolute minimum is at an interior point.

4.1 Extreme Values of Functions on Closed Intervals



Local Extreme Values



4.1 Extreme Values of Functions on Closed Intervals



Definition

Let $f : D \rightarrow \mathbb{R}$ be a function.

- f has a *local maximum value* at a point $c \in D$ if

$$f(x) \leq f(c)$$

for all x close to c .

- f has a *local minimum value* at a point $c \in D$ if

$$f(x) \geq f(c)$$

for all x close to c .

An absolute maximum is always a local maximum too. An absolute minimum is always a local minimum too.

4.1 Extreme Values of Functions on Closed Intervals

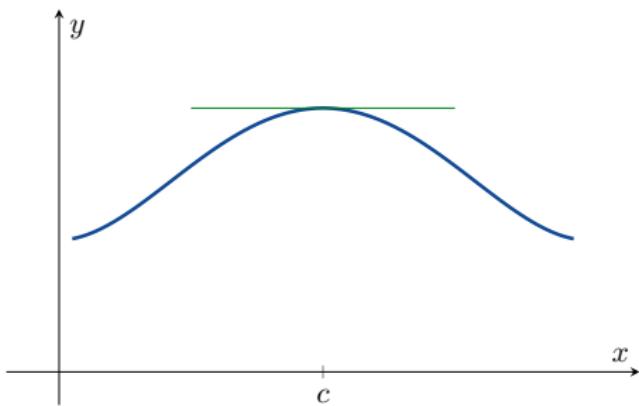


Theorem (The First Derivative Test for Local Extrema)

Suppose that

- f has a local maximum/minimum value at an interior point $c \in D$; and
- $f'(c)$ exists.

4.1 Extreme Values of Functions on Closed Intervals



Theorem (The First Derivative Test for Local Extrema)

Suppose that

- f has a local maximum/minimum value at an interior point $c \in D$; and
- $f'(c)$ exists.

Then $f'(c) = 0$.

4.1 Extreme Values of Functions on Closed Intervals



Remark

The First Derivative Test tells us that the only places where $f : D \rightarrow \mathbb{R}$ can have an extreme value are

- interior points where $f'(c) = 0$;
- interior points where $f'(c)$ does not exist; and
- endpoints of D .

4.1 Extreme Values of Functions on Closed Intervals



Remark

The First Derivative Test tells us that the only places where $f : D \rightarrow \mathbb{R}$ can have an extreme value are

- interior points where $f'(c) = 0$;
- interior points where $f'(c)$ does not exist; and
- endpoints of D .

Definition

An interior point of D where either

- $f' = 0$; or
- f' does not exist,

is called a *critical point* of f .

4.1 Extreme Values of Functions on Closed Intervals



How to find the absolute extrema of a continuous function $f : [a, b] \rightarrow \mathbb{R}$

- 1 Find the critical points of f .

4.1 Extreme Values of Functions on Closed Intervals



How to find the absolute extrema of a continuous function $f : [a, b] \rightarrow \mathbb{R}$

- 1 Find the critical points of f .
- 2 Calculate $f(x)$ at all of the critical points.

4.1 Extreme Values of Functions on Closed Intervals



How to find the absolute extrema of a continuous function $f : [a, b] \rightarrow \mathbb{R}$

- 1 Find the critical points of f .
- 2 Calculate $f(x)$ at all of the critical points.
- 3 Calculate $f(a)$ and $f(b)$.

4.1 Extreme Values of Functions on Closed Intervals



How to find the absolute extrema of a continuous function $f : [a, b] \rightarrow \mathbb{R}$

- 1 Find the critical points of f .
- 2 Calculate $f(x)$ at all of the critical points.
- 3 Calculate $f(a)$ and $f(b)$.
- 4 Take the largest and smallest values.

4.1 Extreme Values of Functions on Closed Intervals



Example

Find the absolute maximum and absolute minimum values of $f(x) = x^2$ on $[-2, 1]$.

4.1 Extreme Values of Functions on Closed Intervals



Example

Find the absolute maximum and absolute minimum values of $f(x) = x^2$ on $[-2, 1]$.

- 1 We know that $f(x) = x^2$ is differentiable on $[-2, 1]$. So $f'(x)$ exists for all interior points $x \in (-2, 1)$. The only critical point is

$$0 = f'(x) = 2x \implies x = 0.$$

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4.1 Extreme Values of Functions on Closed Intervals



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- 2 $f(0) = 0$.
- 3 $f(-2) = 4$ and $f(1) = 1$.
- 4 The largest and smallest numbers in $\{0, 1, 4\}$ are 4 and 0. Therefore the absolute maximum value of $f(x) = x^2$ on $[-2, 1]$ is 4 and the absolute minimum value of f on $[-2, 1]$ is 0.

4.1 Extreme Values of Functions on Closed Intervals



Example

Find the absolute maximum and absolute minimum values of $g(t) = 8t - t^4$ on $[-2, 1]$.

4.1 Extreme Values of Functions on Closed Intervals



Example

Find the absolute maximum and absolute minimum values of $g(t) = 8t - t^4$ on $[-2, 1]$.

① $g'(t)$ exists for all $t \in (-2, 1)$. Since

$$0 = g'(t) = 8 - 4t^3 \implies t^3 = \frac{8}{4} = 2 \implies t = \sqrt[3]{2} > 1,$$

g does not have any critical points in $[-2, 1]$.

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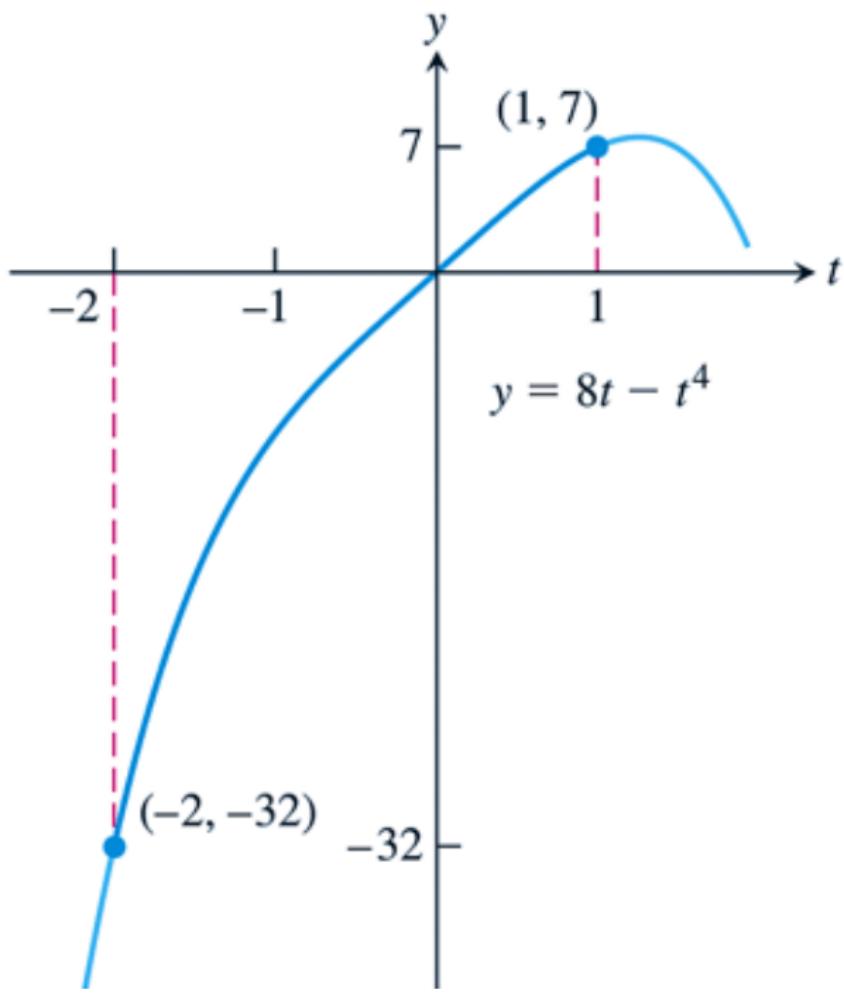
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- 3 $g(-2) = -32$ and $g(1) = 7$.

- 4 Therefore

$$\max_{t \in [-2, 1]} g(t) = 7 \quad \text{and} \quad \min_{t \in [-2, 1]} g(t) = -32.$$



4.1 Extreme Values of Functions on Closed Intervals



Example

Find the absolute maximum and absolute minimum values of $h(x) = x^{\frac{2}{3}}$ on $[-2, 3]$.

1 We calculate that

$$h'(x) = \frac{d}{dx} \left(x^{\frac{2}{3}} \right) = \frac{2}{3} x^{-\frac{1}{3}} = \frac{2}{3\sqrt[3]{x}}.$$

Hence h' does not exist if $x = 0$. We can also see that $h'(x) \neq 0$ if $x \in [-2, 0)$ or $x \in (0, 3]$. The only critical point is $x = 0$

4.1 Extreme Values of Functions on Closed Intervals



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4.1 Extreme Values of Functions on Closed Intervals



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4.1 Extreme Values of Functions on Closed Intervals



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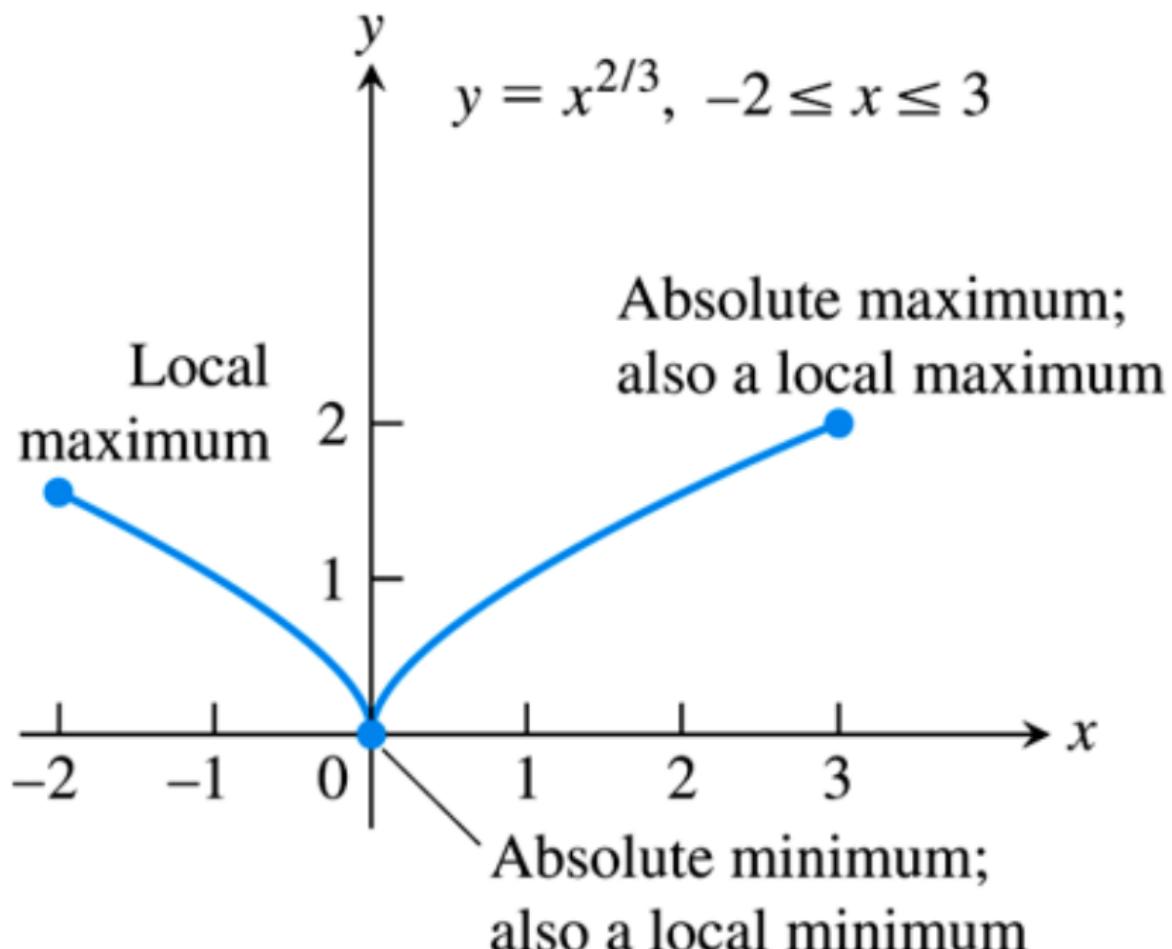
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3 $h(-2) = (-2)^{\frac{2}{3}} = (4)^{\frac{1}{3}} = \sqrt[3]{4}$ and $h(3) = (3)^{\frac{2}{3}} = (9)^{\frac{1}{3}} = \sqrt[3]{9}$.

4 Therefore

$$\max_{x \in [-2, 3]} h(x) = \sqrt[3]{9} \approx 2.08 \quad \text{and} \quad \min_{x \in [-2, 3]} h(t) = 0.$$





The Mean Value Theorem

4.2 The Mean Value Theorem



Michel Rolle

BORN

21 April 1652

DECEASED

8 November 1719

NATIONALITY

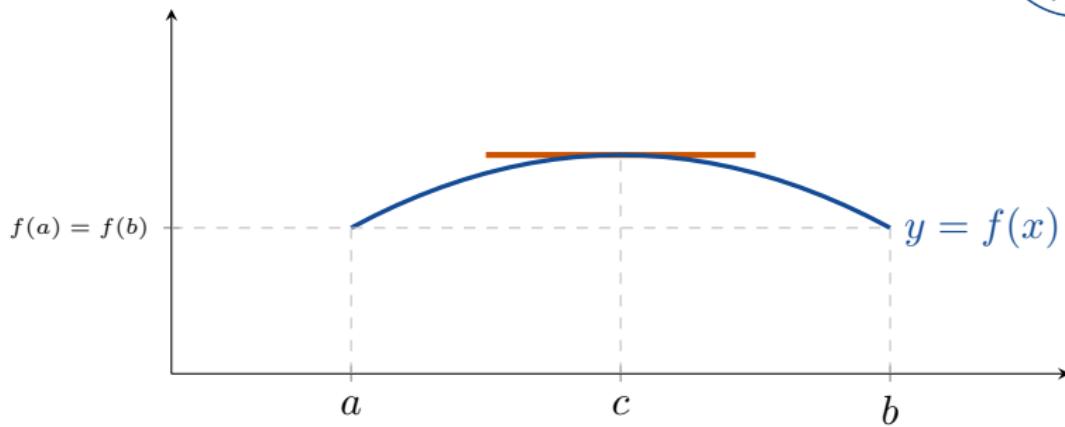
French

Theorem (Rolle's Theorem)

Suppose that

- 1 $f : [a, b] \rightarrow \mathbb{R}$ is continuous;
- 2 f is differentiable on (a, b) ; and
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4.2 The Mean Value Theorem

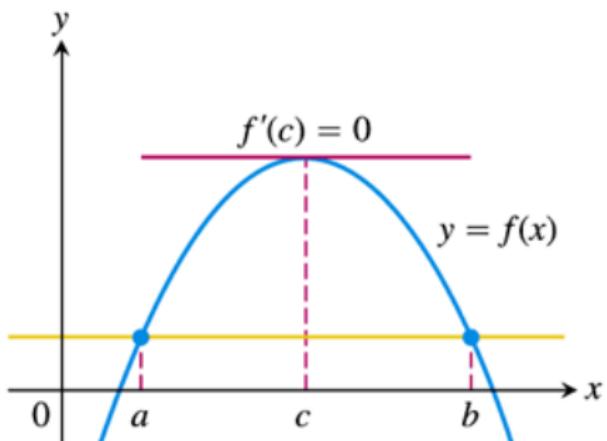


Theorem (Rolle's Theorem)

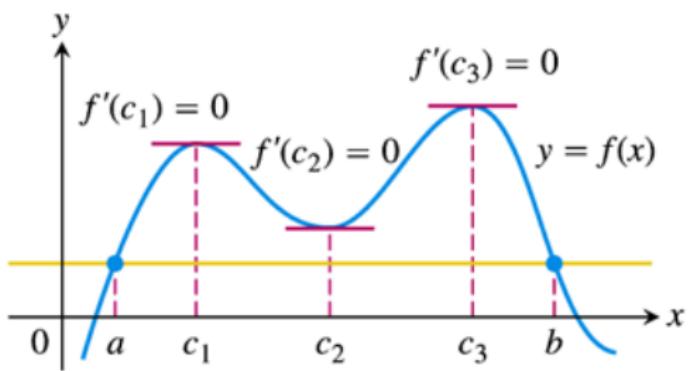
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- 1 $f : [a, b] \rightarrow \mathbb{R}$ is continuous;
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- 3 $f(a) = f(b)$.

Then $\exists c \in (a, b)$ such that $f'(c) = 0$.



(a)



(b)



Augustin-Louis Cauchy

BORN

21 August 1789

DECEASED

23 May 1857

NATIONALITY

French

Theorem (The Mean Value Theorem)

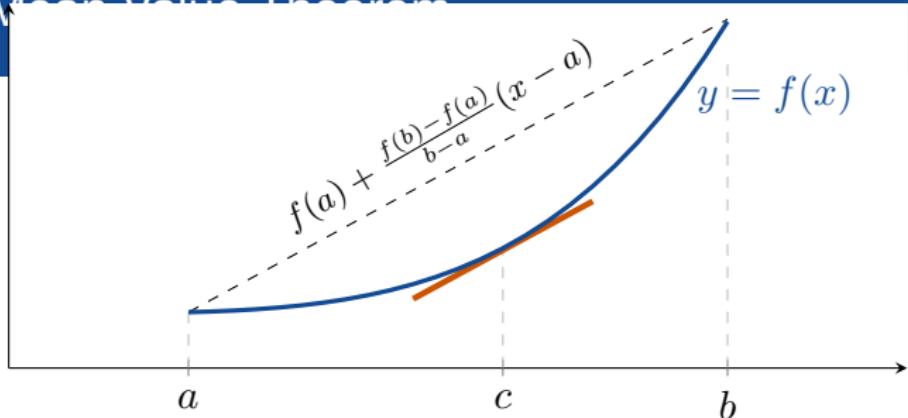
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Then $\exists c \in (a, b)$ such that

$$\frac{f(b) - f(a)}{b - a} = f'(c).$$

4.2 The Mean Value Theorem



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Use the Mean Value Theorem to show that the function $f(x) = x^2$ has a point where $f'(c) = 2$.

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Use the Mean Value Theorem to show that the function $f(x) = x^2$ has a point where $f'(c) = 2$.

Note that $f(x) = x^2$ is

- continuous on $[0, 2]$;
- differentiable on $(0, 2)$;
- $f(0) = 0$ and $f(2) = 4$.

4.2 The Mean Value Theorem



Example

Use the Mean Value Theorem to show that the function $f(x) = x^2$ has a point where $f'(c) = 2$.

Note that $f(x) = x^2$ is

- continuous on $[0, 2]$;
- differentiable on $(0, 2)$;
- $f(0) = 0$ and $f(2) = 4$.

By the Mean Value Theorem, there exists $c \in (0, 2)$ at which

$$f'(c) = \frac{f(2) - f(0)}{2 - 0} = \frac{4 - 0}{2 - 0} = 2.$$

4.2 The Mean Value Theorem



Mathematical Consequences

Corollary

If $f'(x) = 0$ at each point $x \in (a, b)$, then $f(x) = C$ for all $x \in (a, b)$, where C is a constant.

Corollary

If $f'(x) = g'(x)$ at each point $x \in (a, b)$, then $f(x) = g(x) + C$ for all $x \in (a, b)$, where C is a constant.

Next Time

- 4.3 Monotonic Functions and the First Derivative Test
- 4.4 Concavity and Curve Sketching
- 4.5 Applied Optimisation
- 4.7 Antiderivatives