

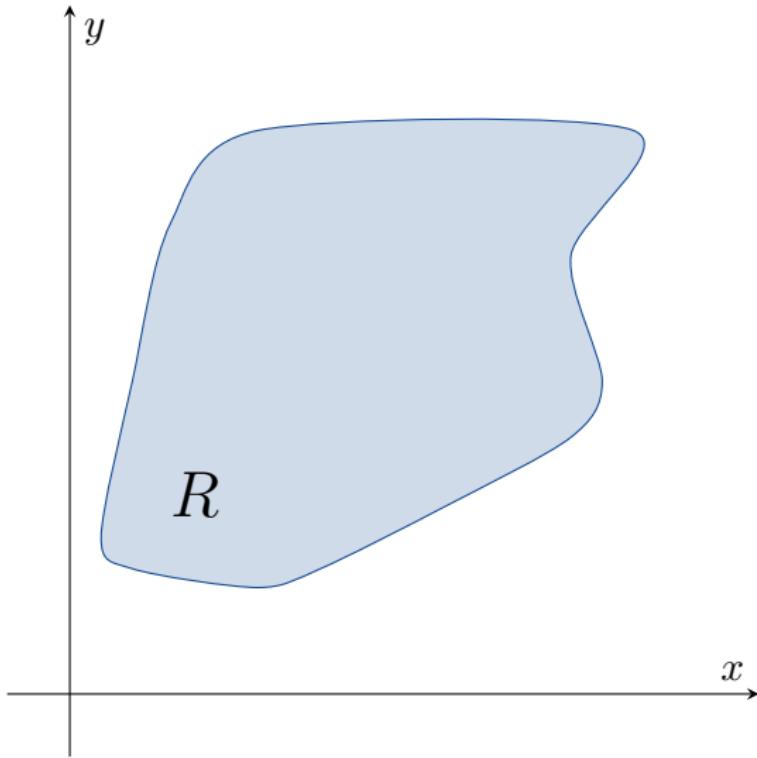
# Lecture 8

- 14.4 Double Integrals in Polar Form
- 14.5 Triple Integrals in Rectangular Coordinates
- 14.7 Triple Integrals in Cylindrical and Spherical Coordinates
- 14.8 Substitutions in Multiple Integrals

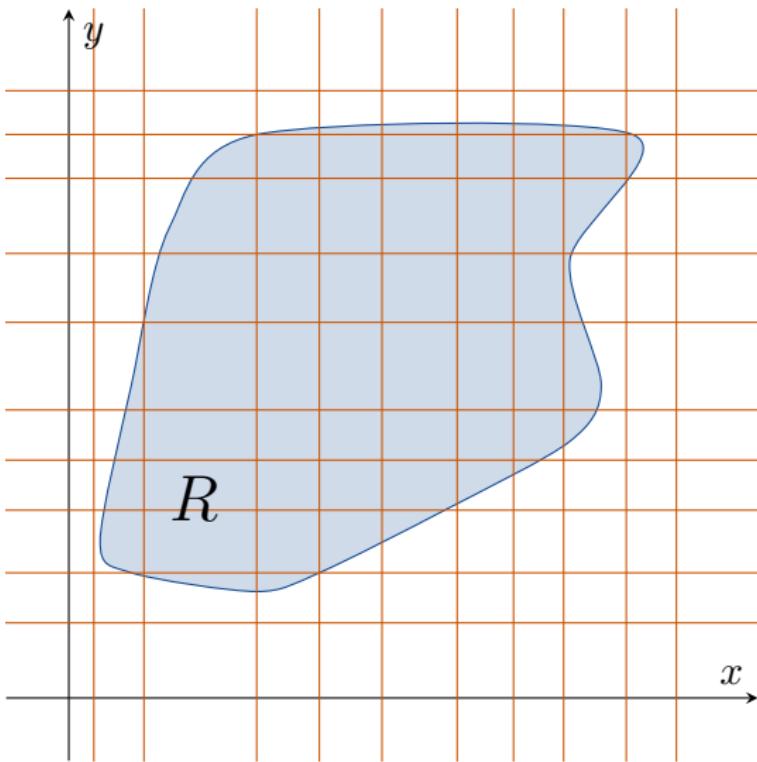


# Double Integrals in Polar Form

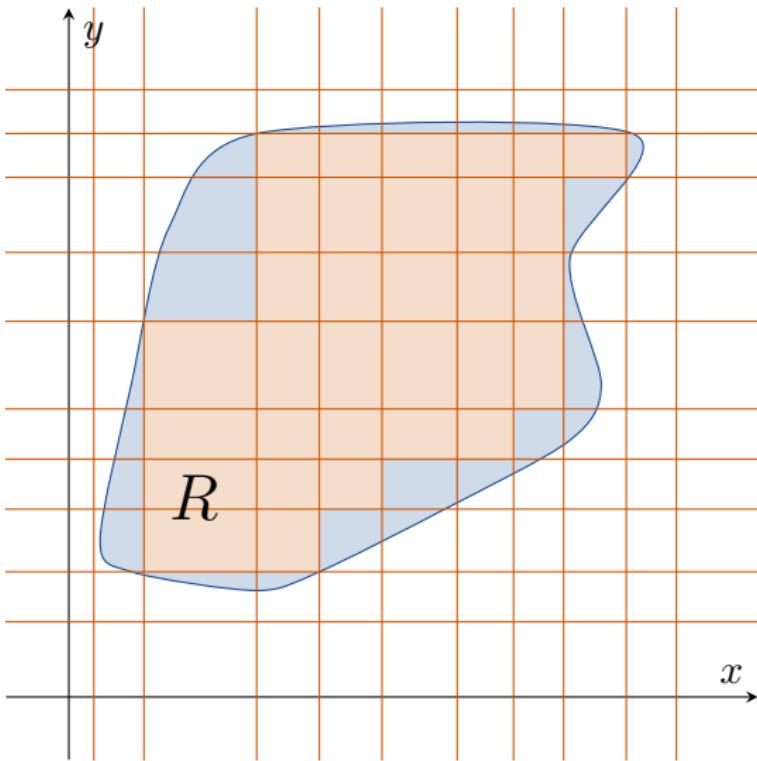
## 14.4 Double Integrals in Polar Form



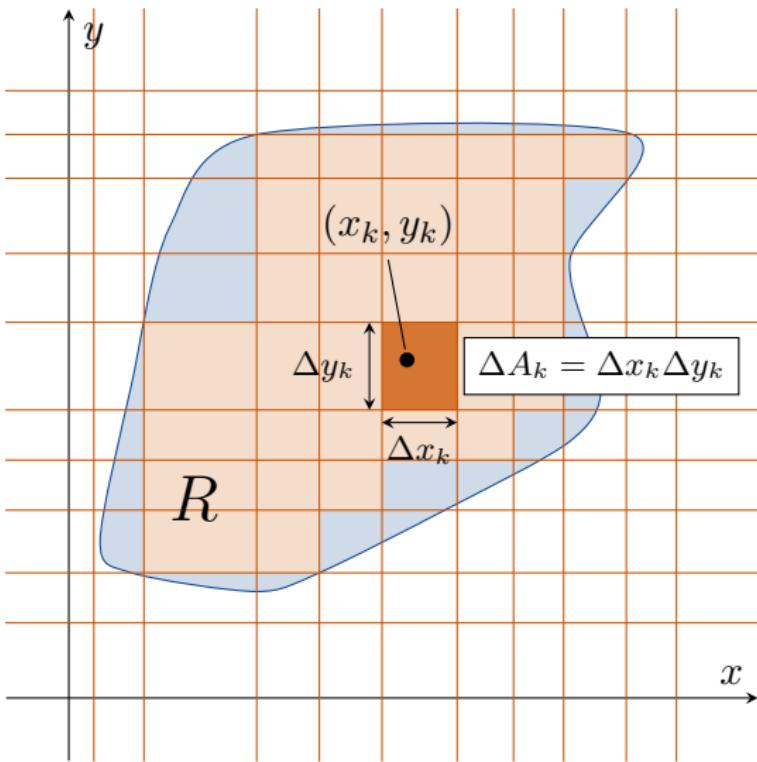
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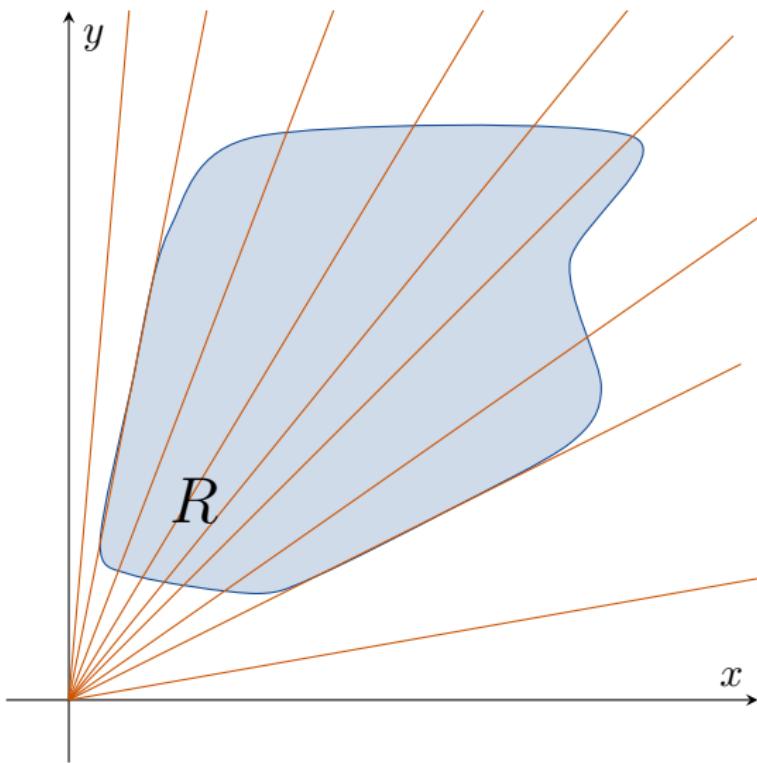
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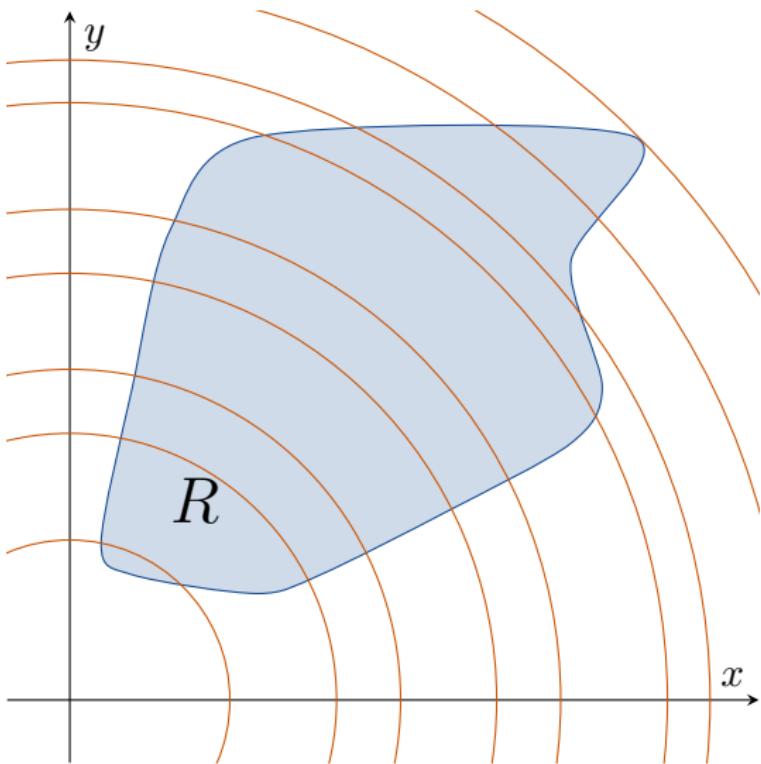
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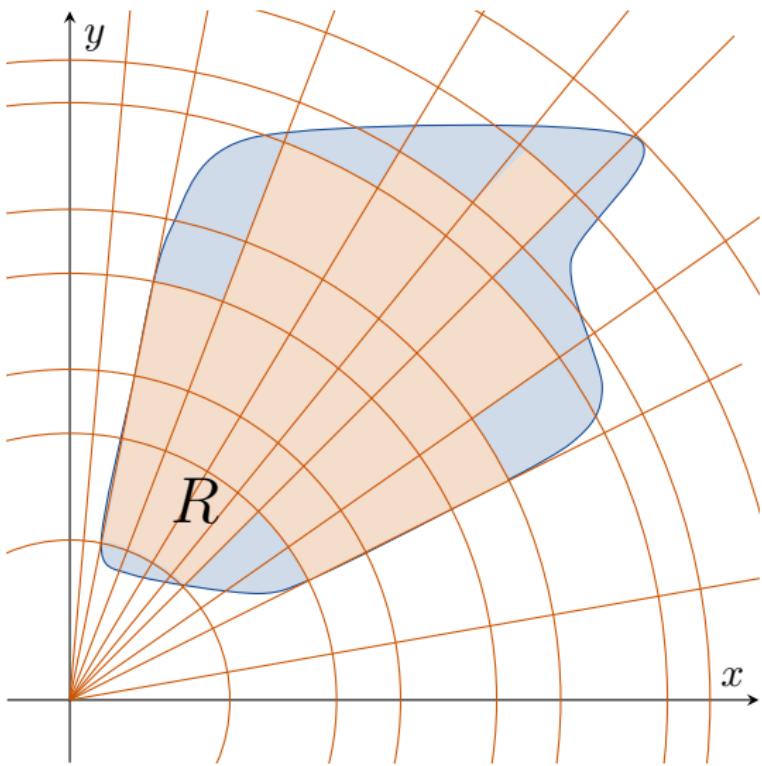
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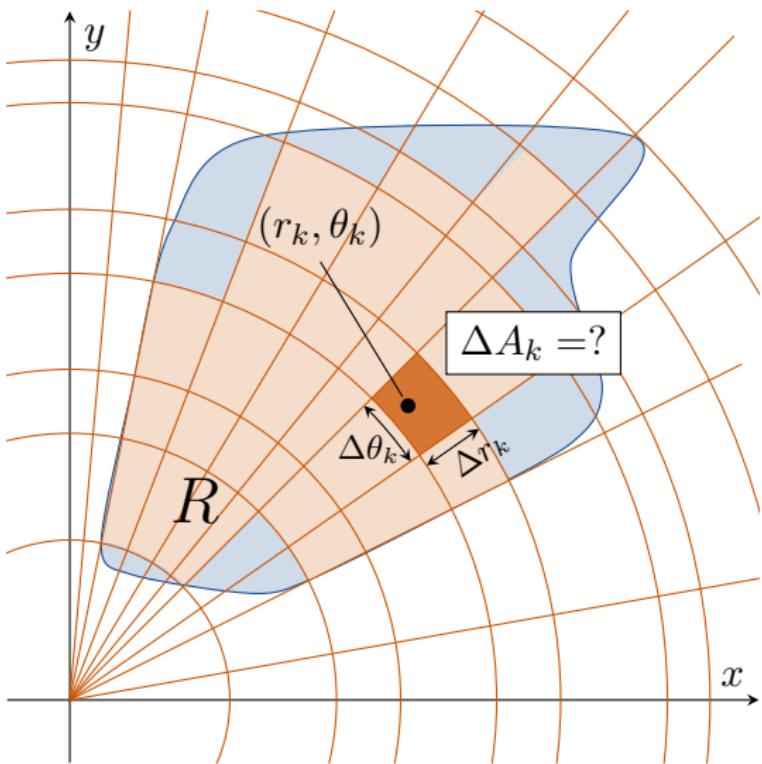
## 14.4 Double Integrals in Polar Form



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## 14.4 Double Integrals in Polar Form



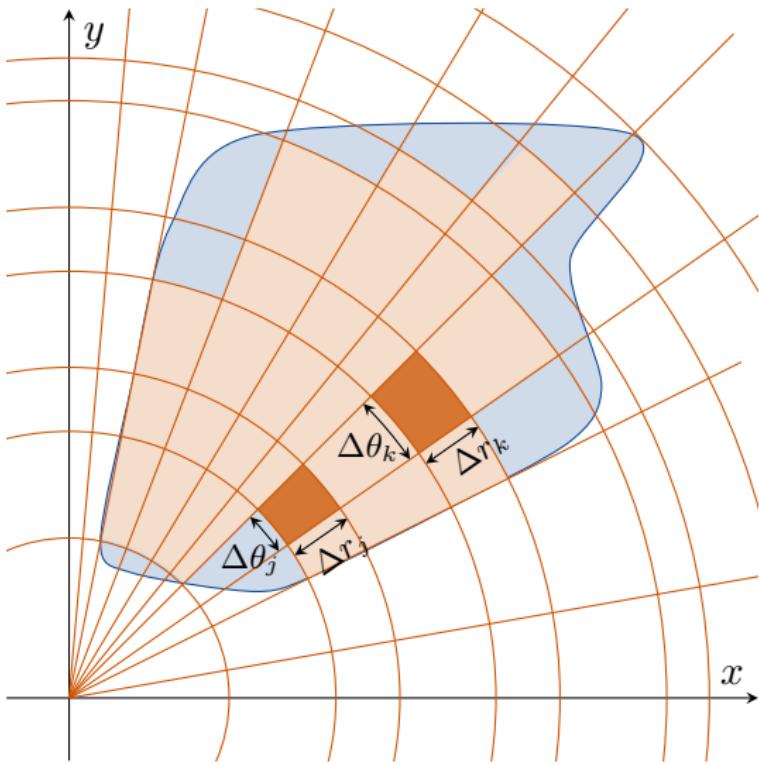
## 14.4 Double Integrals in Polar Form



$$\iint_R f(r, \theta) dA = \lim_{\|P\| \rightarrow 0} \sum_k f(r_k, \theta_k) \Delta A_k$$

But what is  $\Delta A_k$ ?

## 14.4 Double Integrals in Polar Form



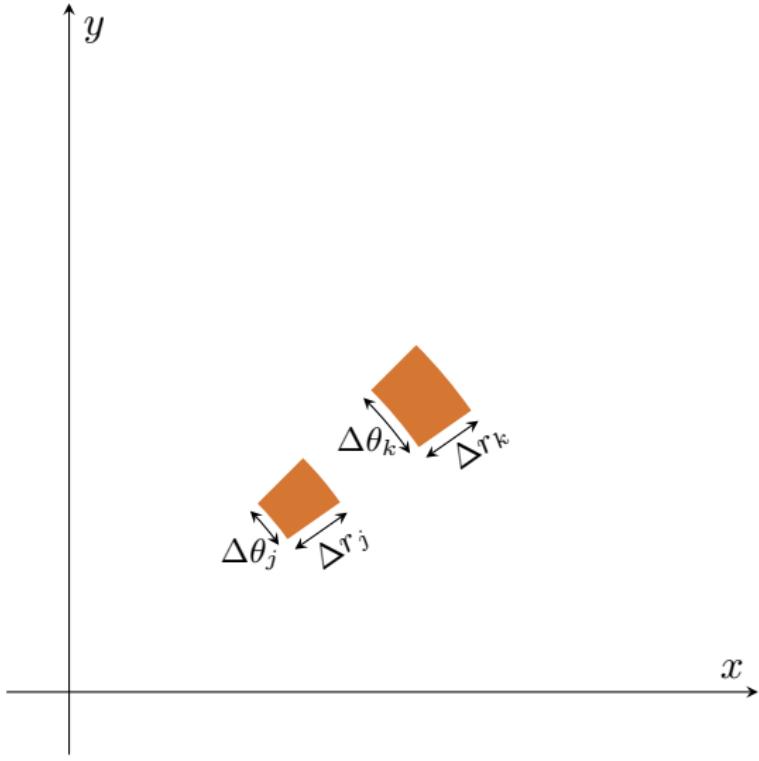
Note that

$$\Delta A_k = \Delta x_k \Delta y_k$$

but

$$\Delta A_k \neq \Delta r_k \Delta \theta_k.$$

## 14.4 Double Integrals in Polar Form



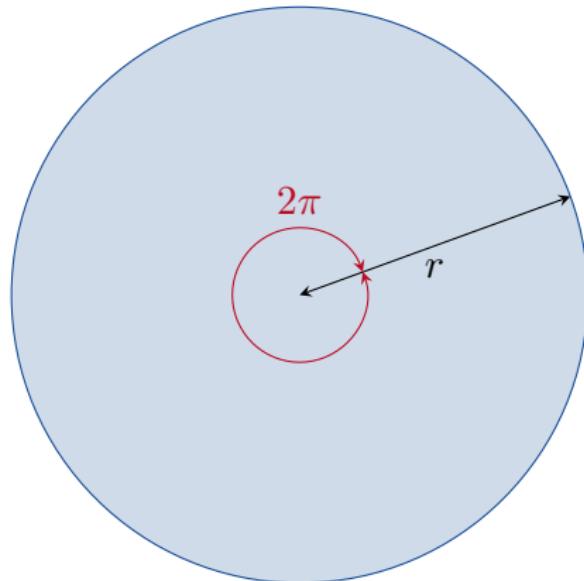
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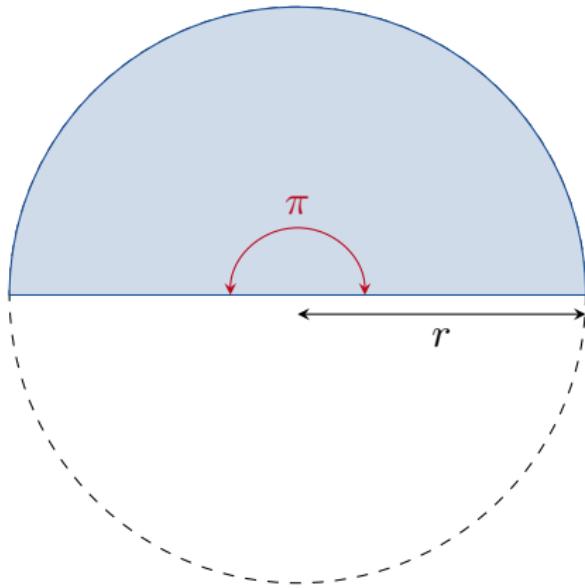
$$\Delta A_k \neq \Delta r_k \Delta \theta_k.$$

## 14.4 Double Integrals in Polar Form



$$\text{area of a circle} = \pi r^2 = \frac{1}{2}(2\pi)r^2$$

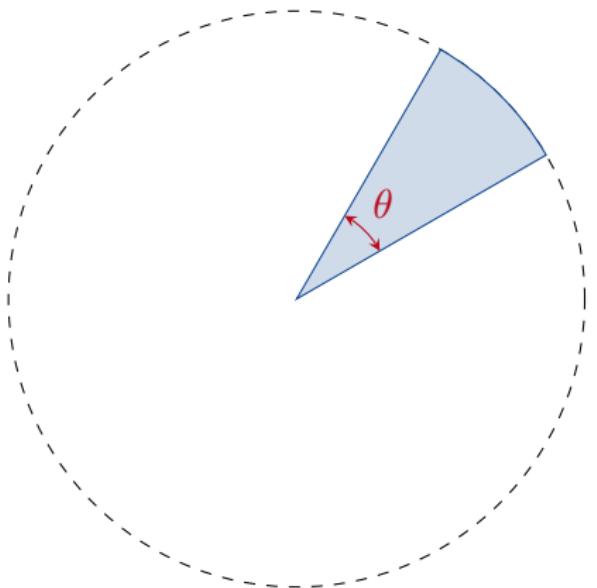
## 14.4 Double Integrals in Polar Form



$$\text{area of a circle} = \pi r^2 = \frac{1}{2}(2\pi)r^2$$

$$\text{area of a semicircle} = \frac{1}{2}\pi r^2$$

## 14.4 Double Integrals in Polar Form

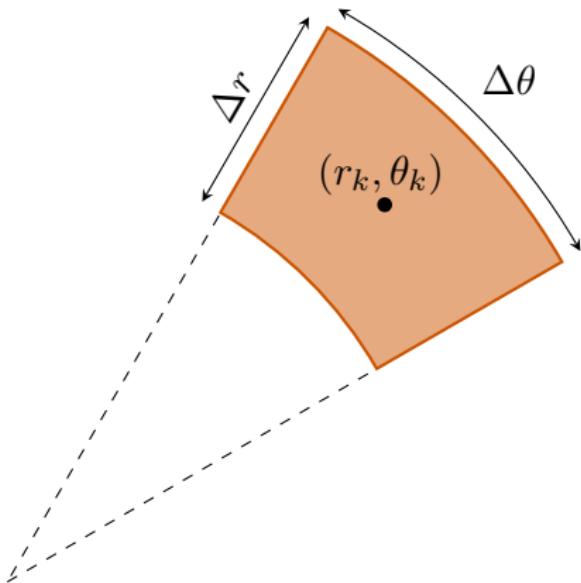


$$\text{area of a circle} = \pi r^2 = \frac{1}{2}(2\pi)r^2$$

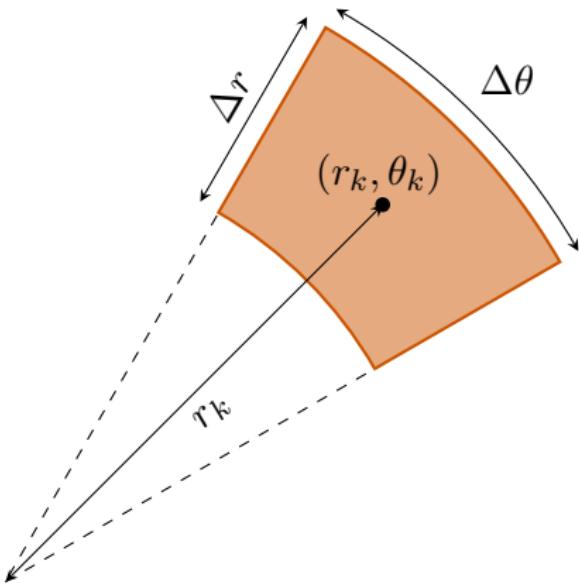
$$\text{area of a semicircle} = \frac{1}{2}\pi r^2$$

$$\text{area of a sector} = \frac{1}{2}\theta r^2$$

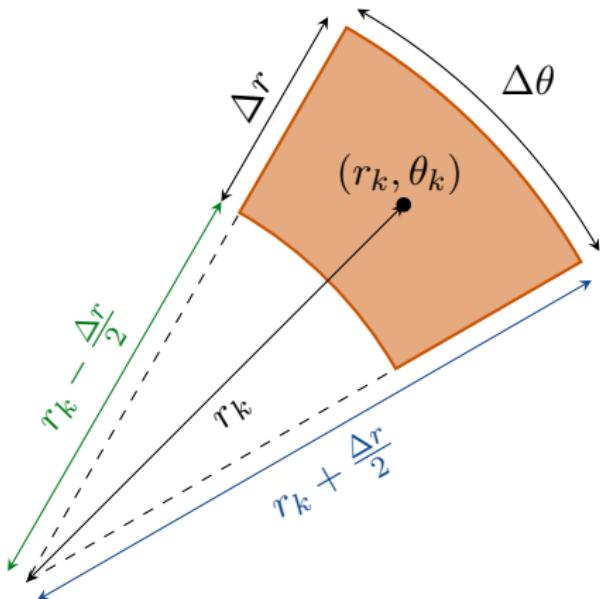
## 14.4 Double Integrals in Polar Form



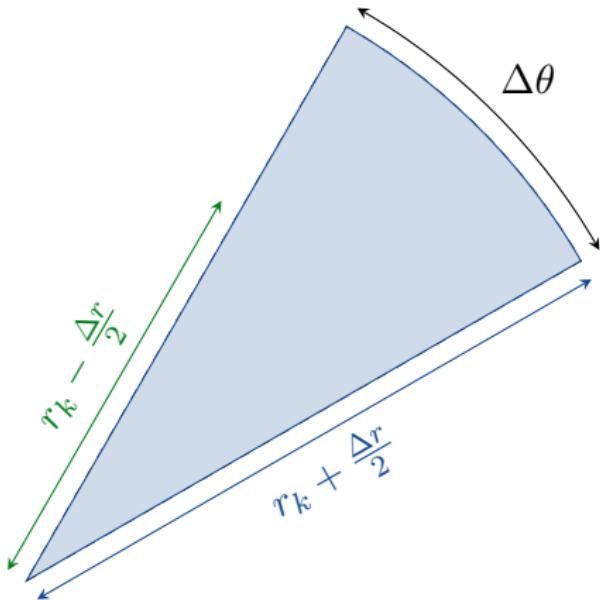
## 14.4 Double Integrals in Polar Form



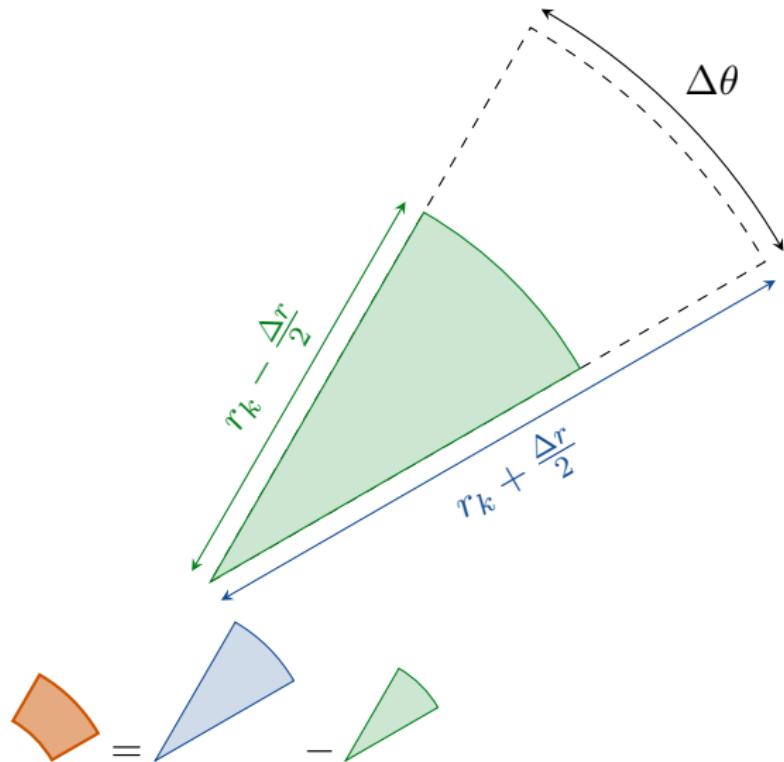
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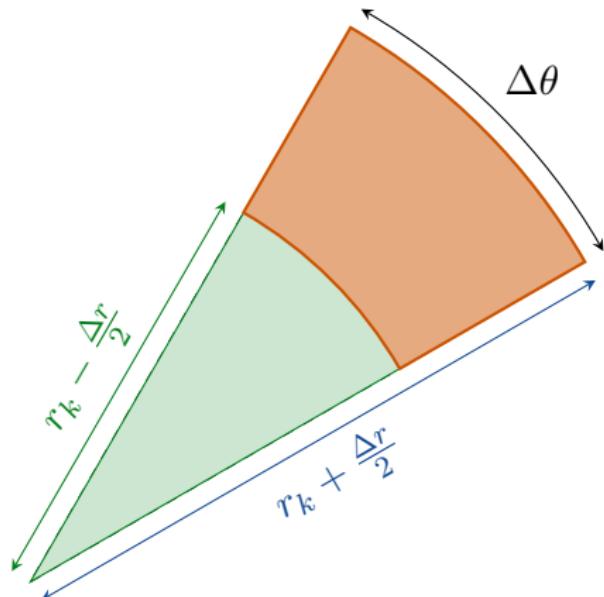
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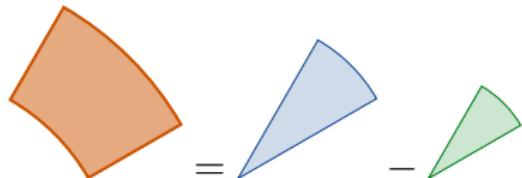


## 14.4 Double Integrals in Polar Form



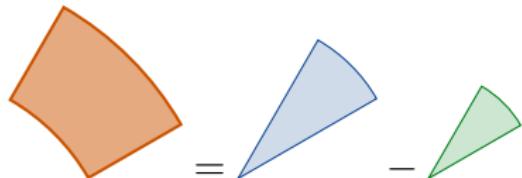
$$\text{orange sector} = \text{blue sector} - \text{green sector} = \frac{1}{2}\Delta\theta \left(r_k + \frac{\Delta r}{2}\right)^2 - \frac{1}{2}\Delta\theta \left(r_k - \frac{\Delta r}{2}\right)^2$$

## 14.4 Double Integrals in Polar Form



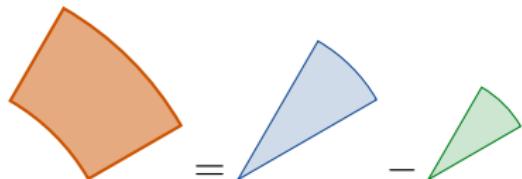
$$= \frac{1}{2} \Delta\theta \left( r_k + \frac{\Delta r}{2} \right)^2 - \frac{1}{2} \Delta\theta \left( r_k - \frac{\Delta r}{2} \right)^2$$

## 14.4 Double Integrals in Polar Form



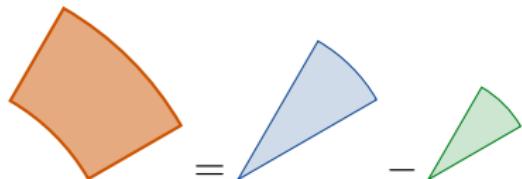
$$\begin{aligned}&= \frac{1}{2} \Delta\theta \left( r_k + \frac{\Delta r}{2} \right)^2 - \frac{1}{2} \Delta\theta \left( r_k - \frac{\Delta r}{2} \right)^2 \\&= \frac{1}{2} \Delta\theta \left( r_k^2 + 2r_k \frac{\Delta r}{2} + \frac{(\Delta r)^2}{4} - r_k^2 + 2r_k \frac{\Delta r}{2} - \frac{(\Delta r)^2}{4} \right)\end{aligned}$$

## 14.4 Double Integrals in Polar Form



$$\begin{aligned}&= \frac{1}{2} \Delta\theta \left( r_k + \frac{\Delta r}{2} \right)^2 - \frac{1}{2} \Delta\theta \left( r_k - \frac{\Delta r}{2} \right)^2 \\&= \frac{1}{2} \Delta\theta \left( r_k^2 + 2r_k \frac{\Delta r}{2} + \frac{(\Delta r)^2}{4} - r_k^2 + 2r_k \frac{\Delta r}{2} - \frac{(\Delta r)^2}{4} \right) \\&= \frac{1}{2} \Delta\theta (2r_k \Delta r)\end{aligned}$$

## 14.4 Double Integrals in Polar Form



$$\begin{aligned}&= \frac{1}{2} \Delta\theta \left( r_k + \frac{\Delta r}{2} \right)^2 - \frac{1}{2} \Delta\theta \left( r_k - \frac{\Delta r}{2} \right)^2 \\&= \frac{1}{2} \Delta\theta \left( r_k^2 + 2r_k \frac{\Delta r}{2} + \frac{(\Delta r)^2}{4} - r_k^2 + 2r_k \frac{\Delta r}{2} - \frac{(\Delta r)^2}{4} \right) \\&= \frac{1}{2} \Delta\theta (2r_k \Delta r) \\&= r_k \Delta r \Delta\theta.\end{aligned}$$

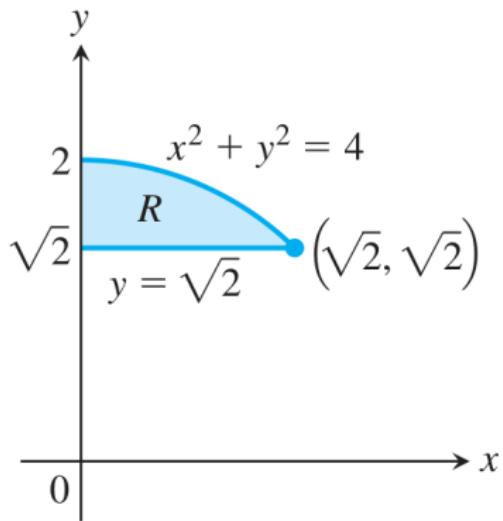
## 14.4 Double Integrals in Polar Form



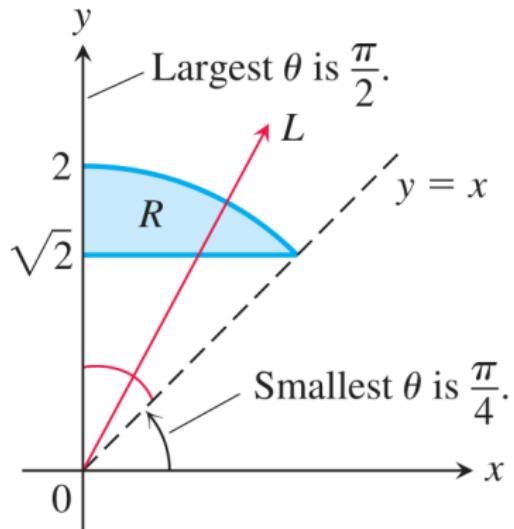
Theorem

$$dA = dx dy = r dr d\theta.$$

## 14.4 Double Integrals in Polar Form



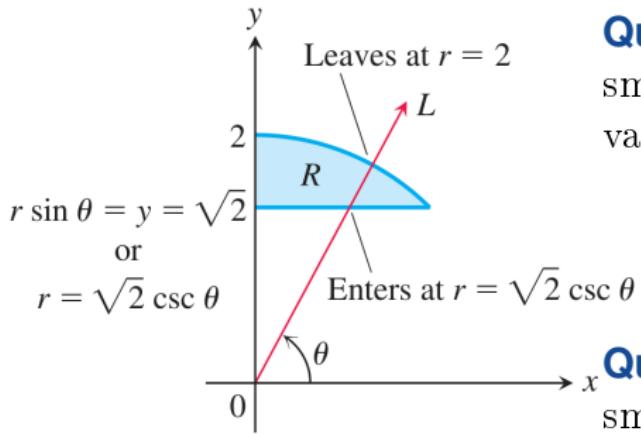
## 14.4 Double Integrals in Polar Form



**Question:** What are the smallest and biggest possible values of  $\theta$  in  $R$ ?

$$\frac{\pi}{4} \leq \theta \leq \frac{\pi}{2}$$

## 14.4 Double Integrals in Polar Form



$$r \sin \theta = y = \sqrt{2}$$

or

$$r = \sqrt{2} \csc \theta$$

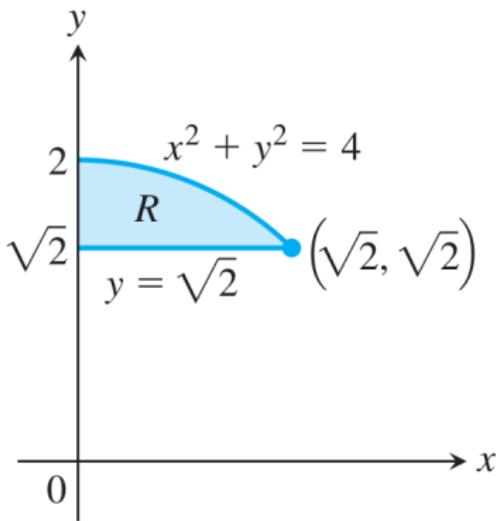
**Question:** What are the smallest and biggest possible values of  $\theta$  in  $R$ ?

$$\frac{\pi}{4} \leq \theta \leq \frac{\pi}{2}$$

**Question:** What are the smallest and biggest possible values of  $r$  in  $R$ ?

$$\sqrt{2} \operatorname{cosec} \theta \leq r \leq 2$$

## 14.4 Double Integrals in Polar Form



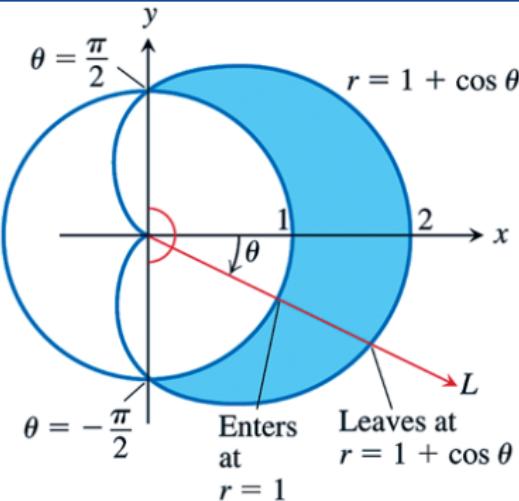
**Question:** What are the smallest and biggest possible values of  $\theta$  in  $R$ ?

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**Question:** What are the smallest and biggest possible values of  $r$  in  $R$ ?

$$\sqrt{2} \operatorname{cosec} \theta \leq r \leq 2$$

$$\iint_R f \, dA = \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \int_{\sqrt{2} \operatorname{cosec} \theta}^2 f(r, \theta) \, r \, dr \, d\theta.$$



**EXAMPLE 1** Find the limits of integration for integrating  $f(r, \theta)$  over the region  $R$  that lies inside the cardioid  $r = 1 + \cos \theta$  and outside the circle  $r = 1$ .

### Solution

1. We first sketch the region and label the bounding curves (Figure 15.25).
2. Next we find the *r-limits of integration*. A typical ray from the origin enters  $R$  where  $r = 1$  and leaves where  $r = 1 + \cos \theta$ .
3. Finally we find the  *$\theta$ -limits of integration*. The rays from the origin that intersect  $R$  run from  $\theta = -\pi/2$  to  $\theta = \pi/2$ . The integral is

$$\int_{-\pi/2}^{\pi/2} \int_1^{1+\cos \theta} f(r, \theta) r dr d\theta.$$

■

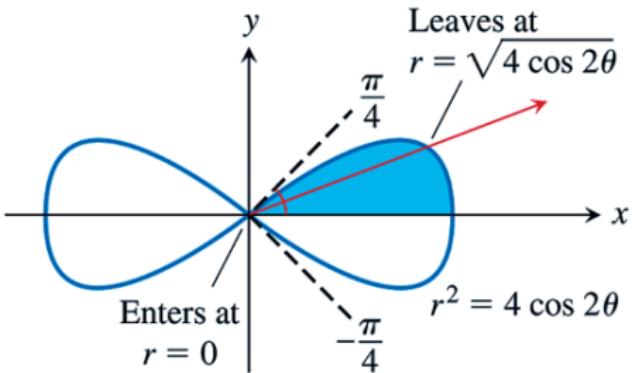
## 14.4 Double Integrals in Polar Form



The area of a closed, bounded region  $R$  is

$$A = \iint_R dA = \iint_R r dr d\theta.$$

## 14.4 Double Integrals in Polar Form



**EXAMPLE 2** Find the area enclosed by the lemniscate  $r^2 = 4 \cos 2\theta$ .

**Solution** We graph the lemniscate to determine the limits of integration (Figure 15.26) and see from the symmetry of the region that the total area is 4 times the first-quadrant portion.

$$\begin{aligned} A &= 4 \int_0^{\pi/4} \int_0^{\sqrt{4 \cos 2\theta}} r \, dr \, d\theta = 4 \int_0^{\pi/4} \left[ \frac{r^2}{2} \right]_{r=0}^{r=\sqrt{4 \cos 2\theta}} d\theta \\ &= 4 \int_0^{\pi/4} 2 \cos 2\theta \, d\theta = 4 \sin 2\theta \Big|_0^{\pi/4} = 4. \end{aligned}$$

■

## 14.4 Double Integrals in Polar Form



Cartesian Integral  $\longrightarrow$  Polar Integral

$$x = r \cos \theta \quad x^2 + y^2 = r^2$$

$$y = r \sin \theta \quad \tan \theta = \frac{y}{x}$$

## 14.4 Double Integrals in Polar Form

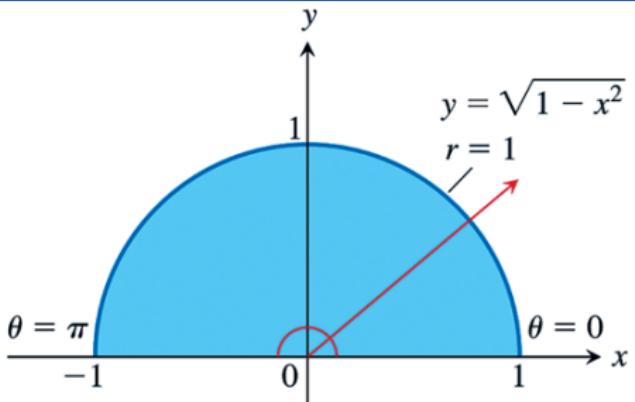


Cartesian Integral  $\longrightarrow$  Polar Integral

$$x = r \cos \theta \quad x^2 + y^2 = r^2$$

$$dxdy = r dr d\theta$$

$$y = r \sin \theta \quad \tan \theta = \frac{y}{x}$$

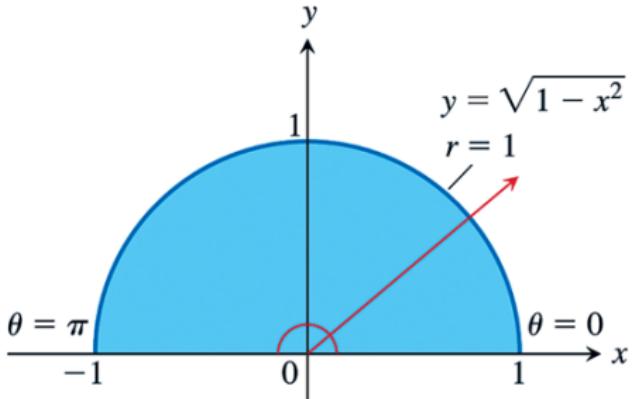


### Example

Calculate

$$\iint_R e^{x^2+y^2} dy dx$$

where  $R$  is the region under  $y = \sqrt{1 - x^2}$ .



## Example

Calculate

$$\iint_R e^{x^2+y^2} dy dx$$

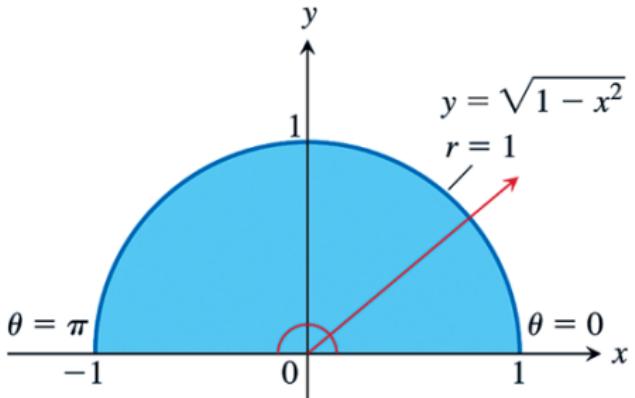
where  $R$  is the region under  $y = \sqrt{1 - x^2}$ .

difficult  
Cartesian  
integral



easy polar  
integral

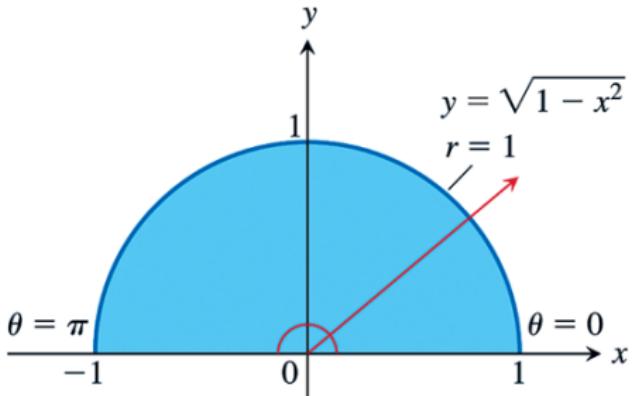
## 14.4 Double Integrals in Polar Form



$$\iint_R e^{x^2+y^2} dy dx = \int \int r dr d\theta$$

=

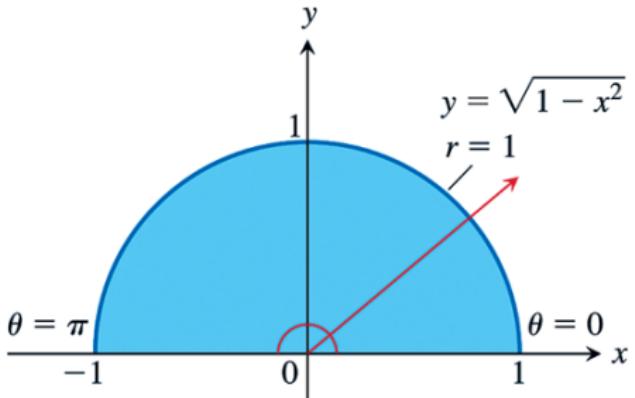
## 14.4 Double Integrals in Polar Form



$$\iint_R e^{x^2+y^2} dy dx = \int \int e^{r^2} r dr d\theta$$

=

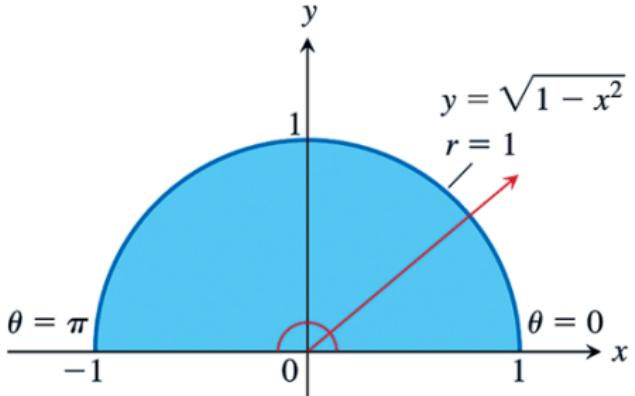
## 14.4 Double Integrals in Polar Form



$$\iint_R e^{x^2+y^2} dy dx = \int_0^\pi \int e^{r^2} r dr d\theta$$

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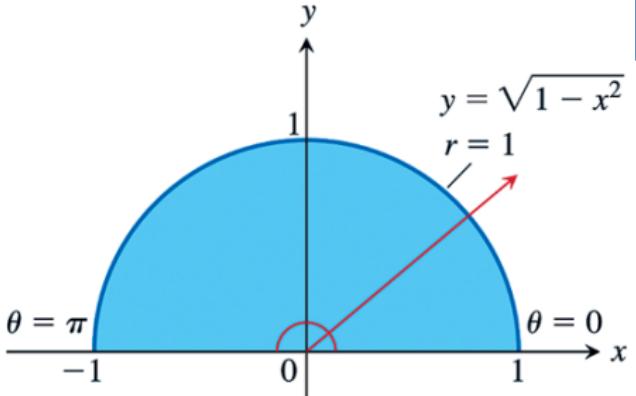
## 14.4 Double Integrals in Polar Form



$$\iint_R e^{x^2+y^2} dy dx = \int_0^\pi \int_0^1 e^{r^2} r dr d\theta$$

=

## 14.4 Double Integrals in Polar Form



$$\begin{aligned}\iint_R e^{x^2+y^2} dy dx &= \int_0^\pi \int_0^1 e^{r^2} r dr d\theta \\&= \int_0^\pi \left[ \frac{1}{2} e^{r^2} \right]_0^1 d\theta = \int_0^\pi \frac{1}{2} (e - 1) d\theta = \frac{\pi}{2} (e - 1).\end{aligned}$$

## 14.4 Double Integrals in Polar Form

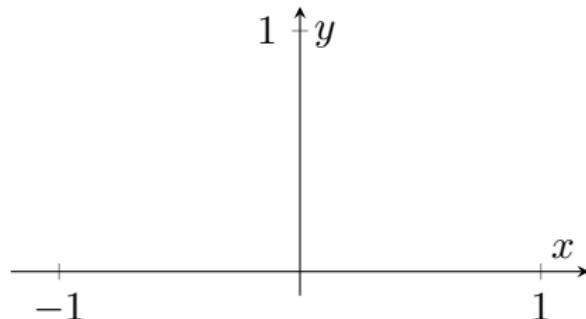
## Example

$$\text{Calculate } \int_0^1 \int_0^{\sqrt{1-x^2}} (x^2 + y^2) dy dx.$$

Note that

$$\int_0^1 \int_0^{\sqrt{1-x^2}} (x^2 + y^2) dy dx = \int_0^1 \left( x^2 \sqrt{1-x^2} + \frac{(1-x^2)^{\frac{3}{2}}}{3} \right) dx,$$

which is not easy to calculate. So let's change to polar coordinates.



## 14.4 Double Integrals in Polar Form

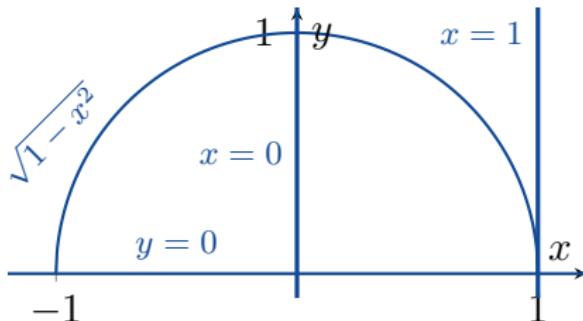
## Example

$$\text{Calculate } \int_0^1 \int_0^{\sqrt{1-x^2}} (x^2 + y^2) dy dx.$$

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which is not easy to calculate. So let's change to polar coordinates.



## 14.4 Double Integrals in Polar Form

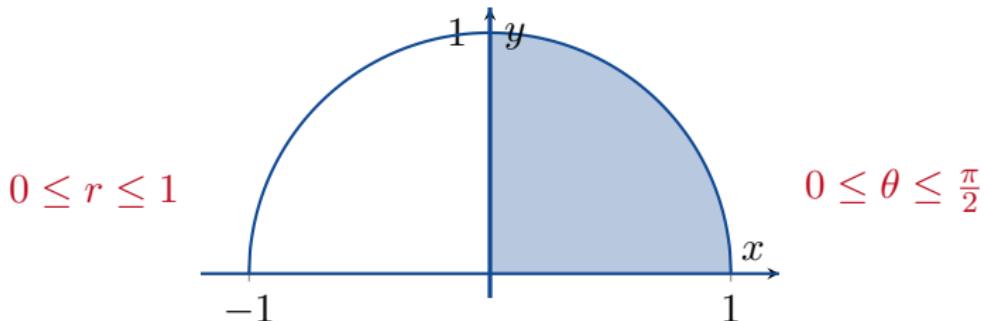
## Example

Calculate  $\int_0^1 \int_0^{\sqrt{1-x^2}} (x^2 + y^2) dy dx.$

Note that

$$\int_0^1 \int_0^{\sqrt{1-x^2}} (x^2 + y^2) dy dx = \int_0^1 \left( x^2 \sqrt{1-x^2} + \frac{(1-x^2)^{\frac{3}{2}}}{3} \right) dx,$$

which is not easy to calculate. So let's change to polar coordinates.



## 14.4 Double Integrals in Polar Form



$$\int_0^1 \int_0^{\sqrt{1-x^2}} (x^2 + y^2) dy dx = \int \int r dr d\theta$$
A diagram illustrating the transformation of a double integral from Cartesian coordinates to polar coordinates. On the left, a Cartesian double integral is shown with limits: the outer integral is from 0 to 1, and the inner integral is from 0 to  $\sqrt{1-x^2}$ . The integrand is  $(x^2 + y^2)$ , with  $dy dx$  written in orange. An orange arrow points from this expression to the right. On the right, a polar double integral is shown with the same limits, but the integrand is  $r dr d\theta$ , also in orange. Another orange arrow points from the original expression to this polar form.

## 14.4 Double Integrals in Polar Form



$$\int_0^1 \int_0^{\sqrt{1-x^2}} (x^2 + y^2) dy dx = \int_0^{\frac{\pi}{2}} \int_0^1 r dr d\theta$$
A diagram showing a quarter circle in the first quadrant of a Cartesian coordinate system. The circle is centered at the origin and passes through the points (1,0), (0,1), and (-1,0). The region is shaded in light gray. Two red arrows point from the origin along the arc of the circle towards the point (1,0).

## 14.4 Double Integrals in Polar Form



$$\int_0^1 \int_0^{\sqrt{1-x^2}} (x^2 + y^2) dy dx = \int_0^{\frac{\pi}{2}} \int_0^1 r^2 r dr d\theta$$
A green curved arrow points from the left-hand double integral to the right-hand polar form integral, indicating the transformation from Cartesian coordinates to polar coordinates.

## 14.4 Double Integrals in Polar Form



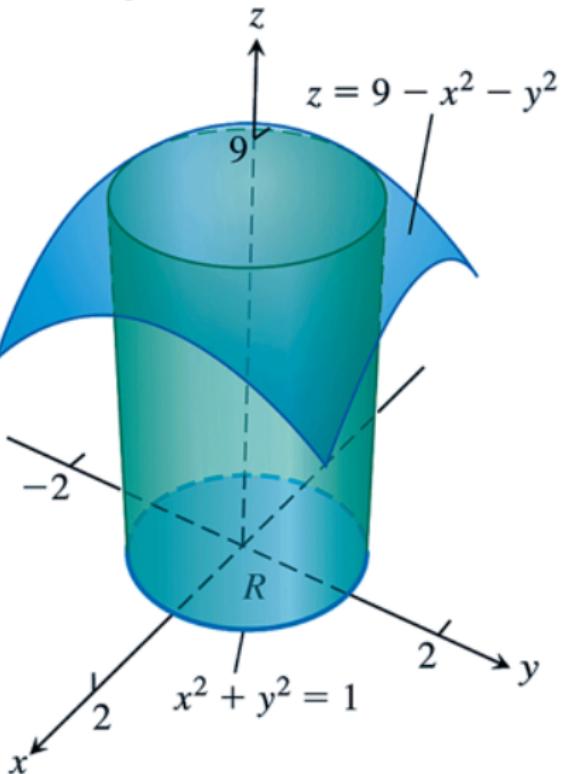
$$\int_0^1 \int_0^{\sqrt{1-x^2}} (x^2 + y^2) dy dx = \int_0^{\frac{\pi}{2}} \int_0^1 r^2 r dr d\theta$$
$$= \dots$$
$$= \frac{\pi}{8}.$$

## 14.4 Double Integrals in Polar Form

**EXAMPLE 5** Find the volume of the solid region bounded above by the paraboloid  $z = 9 - x^2 - y^2$  and below by the unit circle in the  $xy$ -plane.

**Solution** The region of integration  $R$  is bounded by the unit circle  $x^2 + y^2 \leq 1$ . It is described in polar coordinates by  $r = 1$ ,  $0 \leq \theta \leq 2\pi$ . Figure 15.28. The volume is given by the double integral

$$\begin{aligned} \iint_R (9 - x^2 - y^2) dA &= \int_0^{2\pi} \int_0^1 (9 - r^2) r dr d\theta \\ &= \int_0^{2\pi} \int_0^1 (9r - r^3) dr d\theta \\ &= \int_0^{2\pi} \left[ \frac{9}{2}r^2 - \frac{1}{4}r^4 \right]_0^1 d\theta \\ &= \frac{17}{4} \int_0^{2\pi} d\theta = \frac{17\pi}{2} \end{aligned}$$



## 14.4 Double Integrals in Polar Form

**EXAMPLE 5** Find the volume of the solid region bounded above by the paraboloid  $z = 9 - x^2 - y^2$  and below by the unit circle in the  $xy$ -plane.

**Solution** The region of integration  $R$  is bounded by the unit circle  $x^2 + y^2 = 1$ , which is described in polar coordinates by  $r = 1, 0 \leq \theta \leq 2\pi$ . The solid region is shown in Figure 15.28. The volume is given by the double integral

$$\begin{aligned}
 \iint_R (9 - x^2 - y^2) dA &= \int_0^{2\pi} \int_0^1 (9 - r^2) r dr d\theta && r^2 = x^2 + y^2, \quad dA = r dr d\theta. \\
 &= \int_0^{2\pi} \int_0^1 (9r - r^3) dr d\theta \\
 &= \int_0^{2\pi} \left[ \frac{9}{2}r^2 - \frac{1}{4}r^4 \right]_{r=0}^{r=1} d\theta \\
 &= \frac{17}{4} \int_0^{2\pi} d\theta = \frac{17\pi}{2}.
 \end{aligned}$$



## 14.4 Double Integrals in Polar Form



Please read Example 6 in the textbook.



# Triple Integrals in Rectangular Coordinates

## 14.5 Triple Integrals in Rectangular Coordinates



In the last two lectures we have been studying

$$\iint_R f(x, y) dA.$$

## 14.5 Triple Integrals in Rectangular Coordinates



In the last two lectures we have been studying

$$\iint_R f(x, y) dA.$$

Today we will consider

$$\iiint_D f(x, y, z) dV.$$

## 14.5 Triple Integrals in Rectangular Coordinates



### Definition

The *volume* of a closed, bounded region  $D$  in space is

$$V = \iiint_R dV.$$



### Finding Limits of Integration

$$\int \int \int F(x, y, z) dz dy dx.$$



### Finding Limits of Integration

$$\int_{x=a}^{x=b} \int \int F(x, y, z) dz dy d\textcolor{brown}{x}.$$

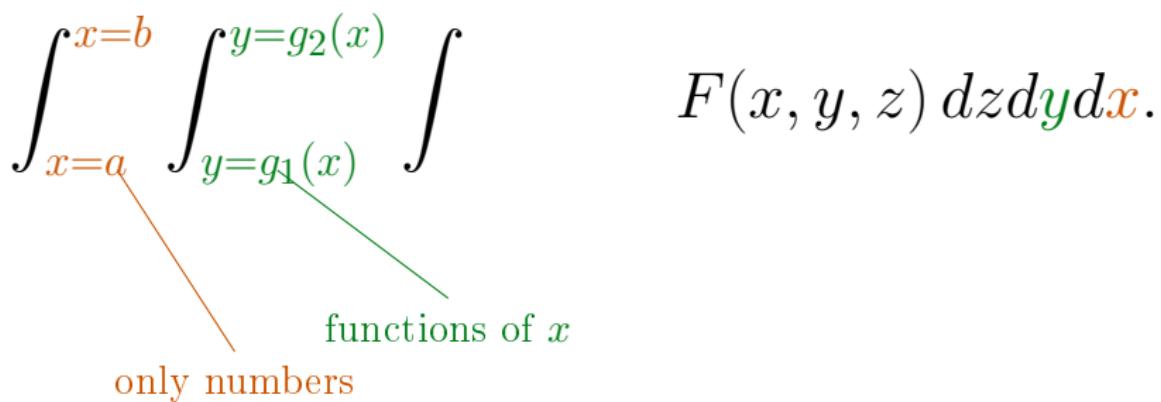
only numbers

## Finding Limits of Integration

$$\int_{x=a}^{x=b} \int_{y=g_1(x)}^{y=g_2(x)} \int F(x, y, z) dz dy dx.$$

functions of  $x$

only numbers



## Finding Limits of Integration

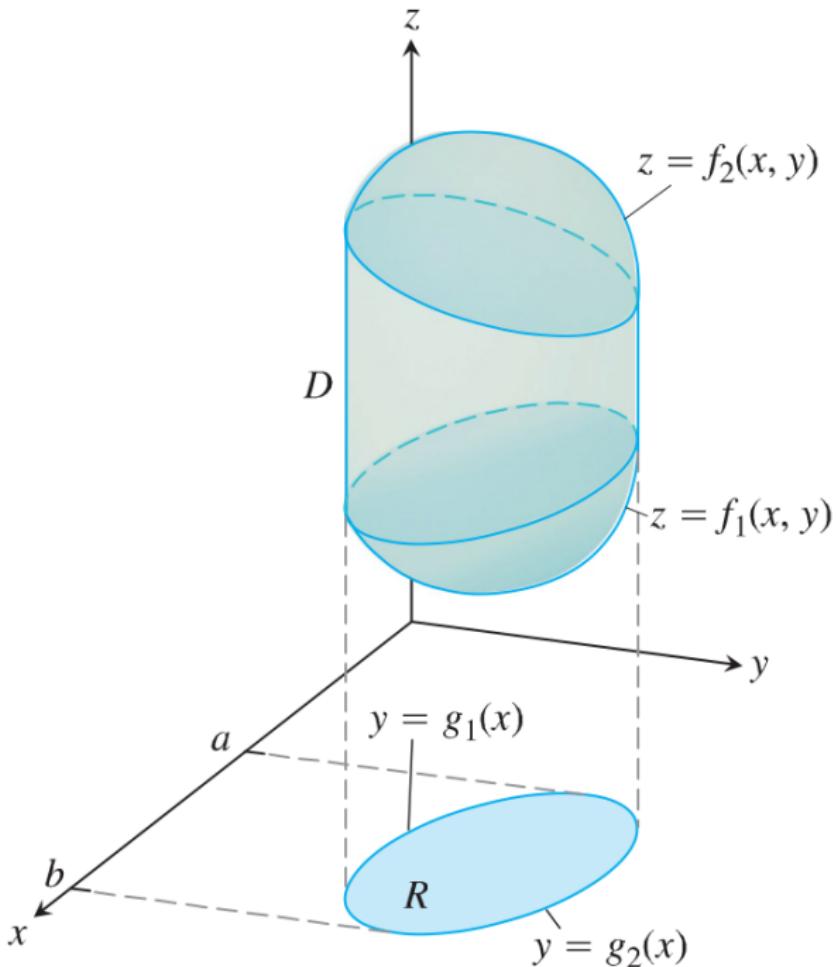
$$\int_{x=a}^{x=b} \int_{y=g_1(x)}^{y=g_2(x)} \int_{z=h_1(x,y)}^{z=h_2(x,y)} F(x, y, z) dz dy dx.$$

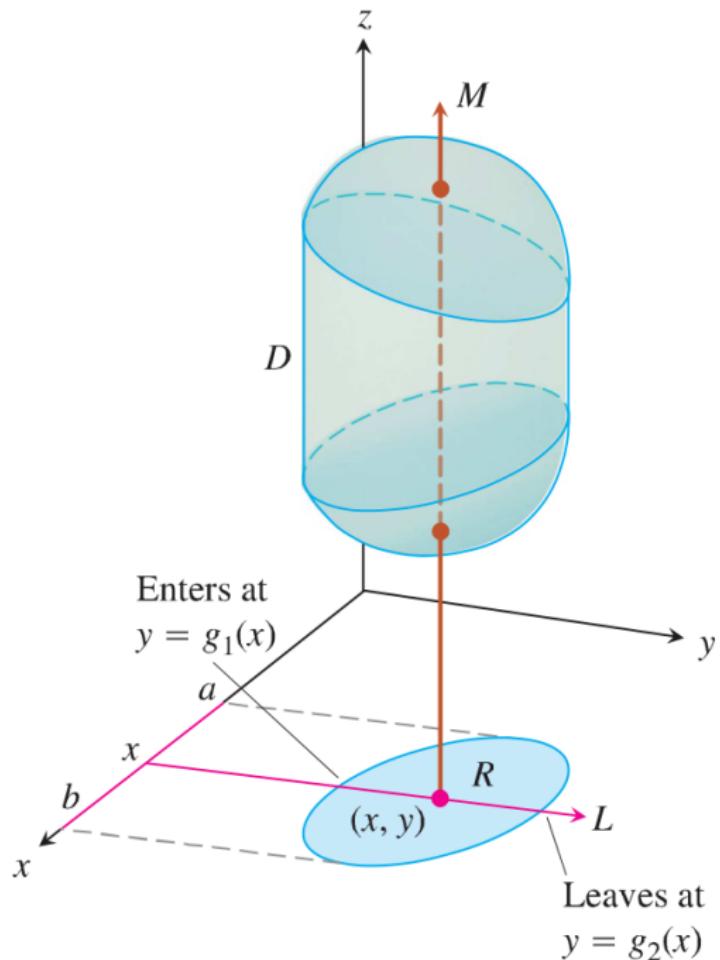
functions of  $x$  and  $y$

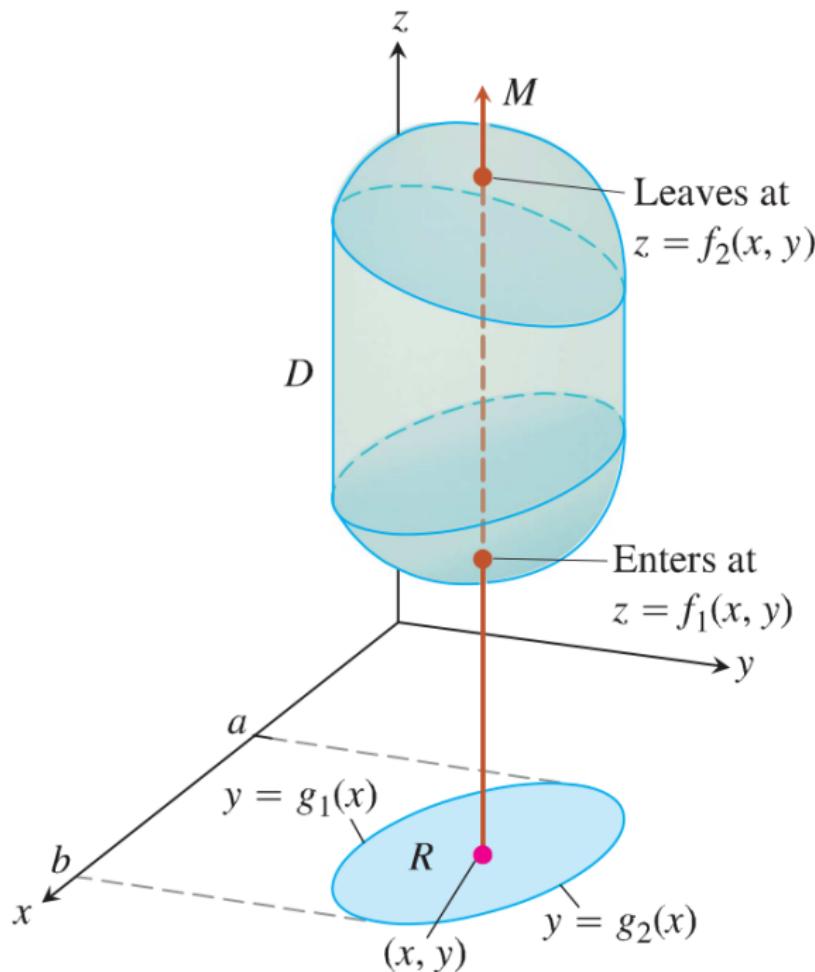
functions of  $x$

only numbers

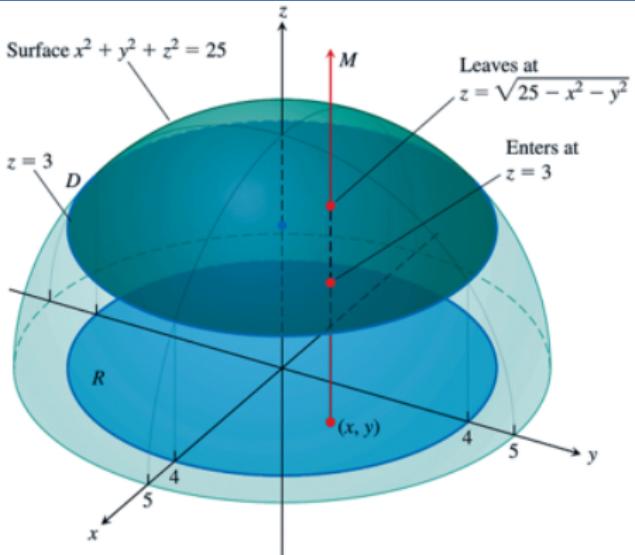
The diagram illustrates the nested integrals for a triple integral. The outermost integral is with respect to  $x$ , ranging from  $a$  to  $b$ . The middle integral is with respect to  $y$ , ranging from  $g_1(x)$  to  $g_2(x)$ . The innermost integral is with respect to  $z$ , ranging from  $h_1(x,y)$  to  $h_2(x,y)$ . Colored lines connect the labels to their corresponding integral components: orange lines point to the limits for  $x$  and the  $z$  function; green lines point to the limits for  $y$  and the  $y$  function; and a red line points to the  $z$  function.







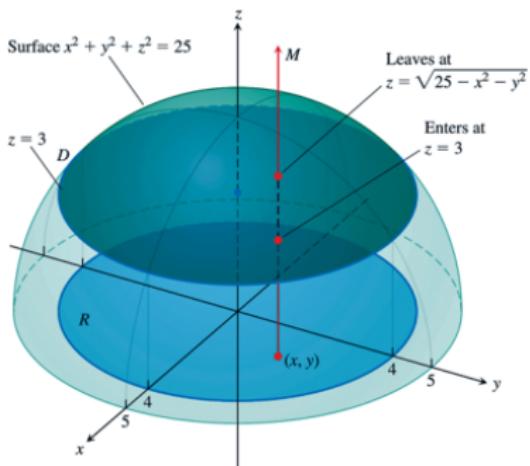
## 14.5 Triple Integrals in Rectangular Coordinates



### Example

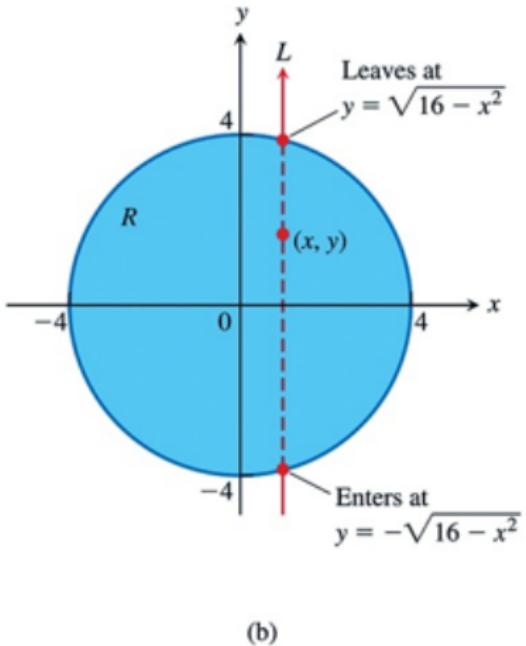
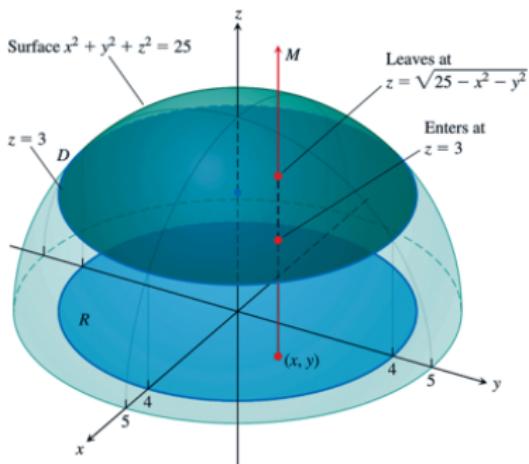
Let  $S$  be the sphere of radius 5 centred at the origin. Let  $D$  be the region under the sphere and above the plane  $z = 3$ . Set up the limits of integration over  $D$ .

## 14.5 Triple Integrals in Rectangular Coordinates



$$z = 3 \implies x^2 + y^2 + 9 = 25 \implies x^2 + y^2 = 16$$

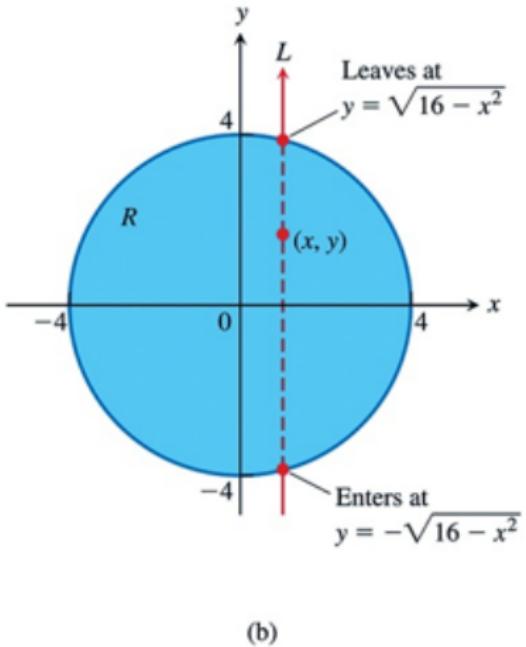
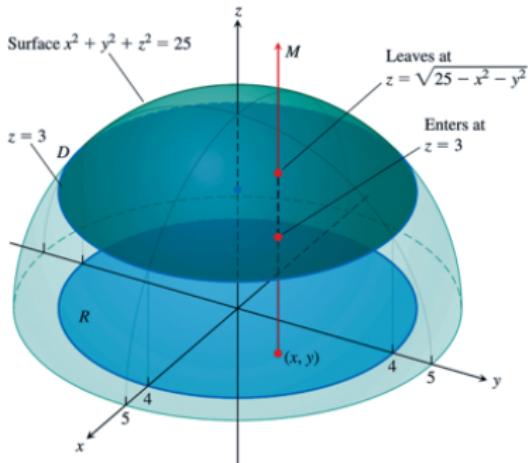
## 14.5 Triple Integrals in Rectangular



$$z = 3 \implies x^2 + y^2 + 9 = 25 \implies x^2 + y^2 = 16$$

$$-4 \leq x \leq 4$$

## 14.5 Triple Integrals in Rectangular

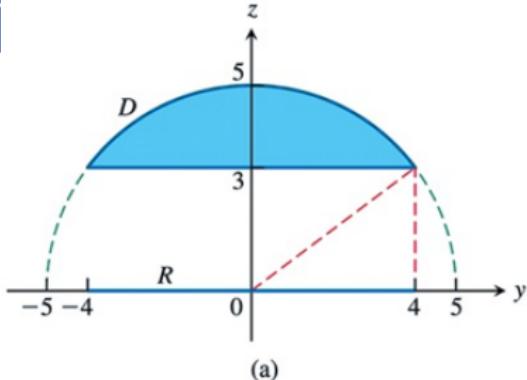
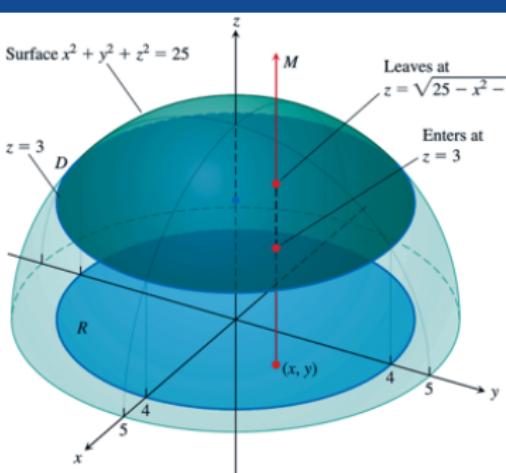


$$z = 3 \implies x^2 + y^2 + 9 = 25 \implies x^2 + y^2 = 16$$

$$-4 \leq x \leq 4 \quad -\sqrt{16 - x^2} \leq y \leq \sqrt{16 - x^2}$$

14.5

## Spherical Coordinates

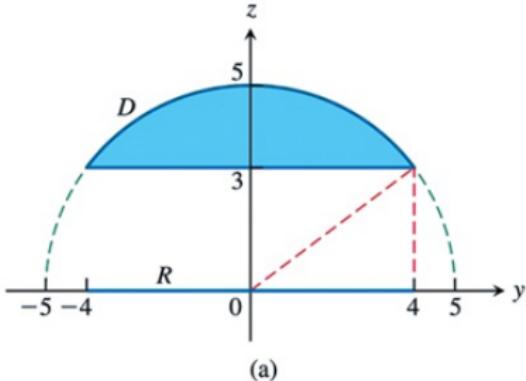
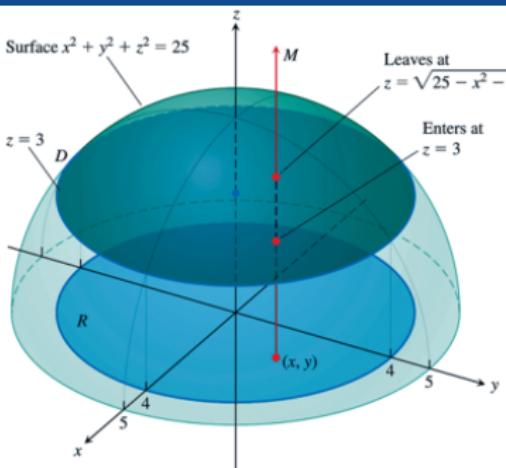


$$-4 \leq x \leq 4$$

$$-\sqrt{16 - x^2} \leq y \leq \sqrt{16 - x^2}$$

14.5

## Spherical Coordinates



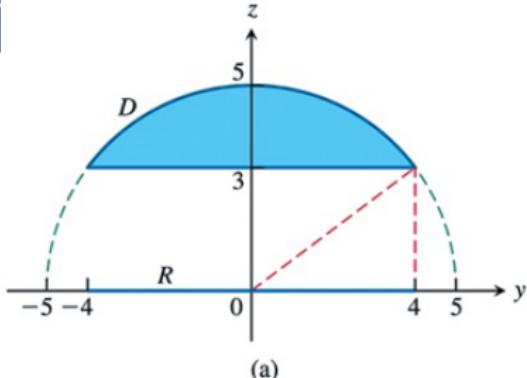
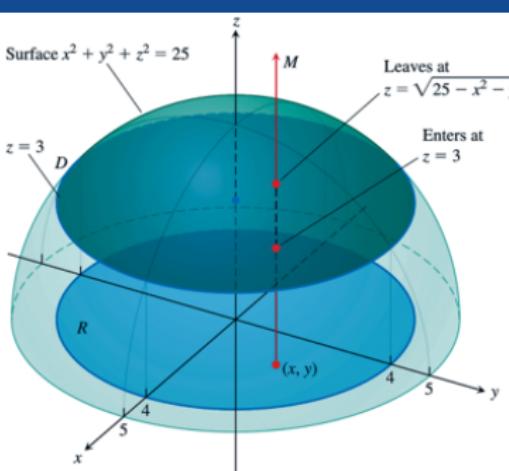
$$-4 \leq x \leq 4$$

$$-\sqrt{16 - x^2} \leq y \leq \sqrt{16 - x^2}$$

$$3 \leq z \leq \sqrt{25 - x^2 - y^2}$$

14.5

## Spherical Coordinates



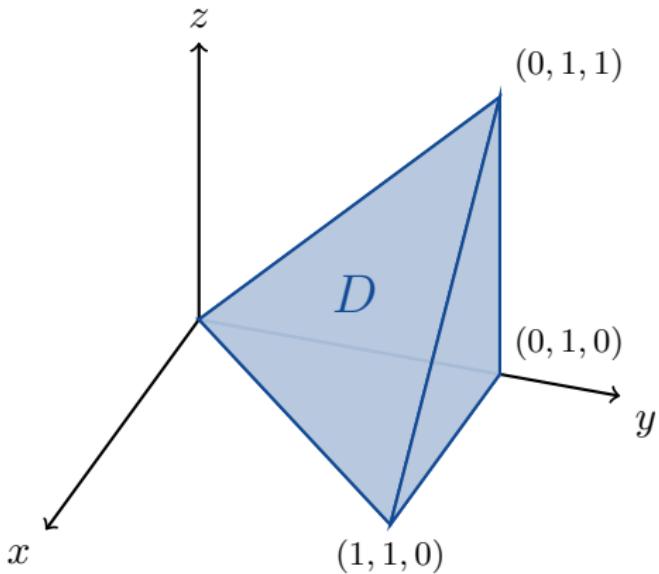
$$-4 \leq x \leq 4$$

$$-\sqrt{16 - x^2} \leq y \leq \sqrt{16 - x^2}$$

$$3 \leq z \leq \sqrt{25 - x^2 - y^2}$$

$$\int_{-4}^4 \int_{-\sqrt{16-x^2}}^{\sqrt{16-x^2}} \int_3^{\sqrt{25-x^2-y^2}} F(x, y, z) dz dy dx.$$

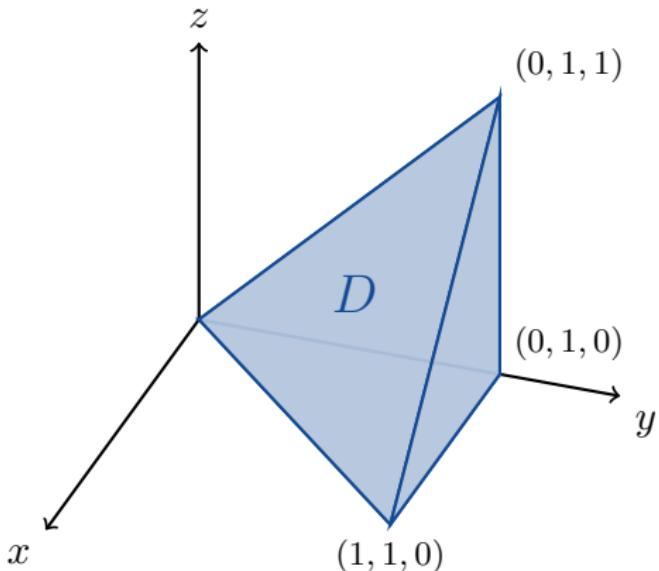
## 14.5 Triple Integrals in Rectangular Coordinates



### Example

Let  $D$  be the tetrahedron whose vertices are  $(0, 0, 0)$ ,  $(1, 1, 0)$ ,  $(0, 1, 0)$  and  $(0, 1, 1)$ .

## 14.5 Triple Integrals in Rectangular Coordinates



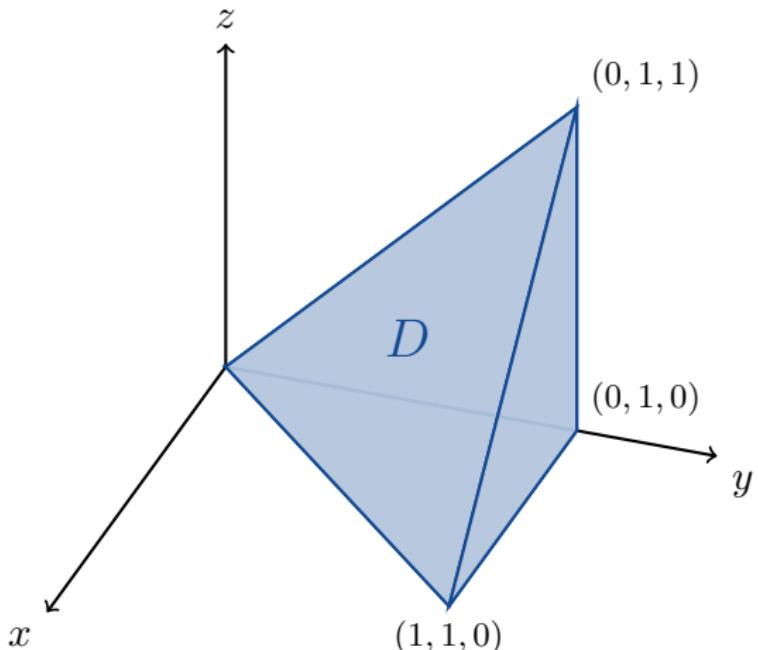
### Example

Let  $D$  be the tetrahedron whose vertices are  $(0, 0, 0)$ ,  $(1, 1, 0)$ ,  $(0, 1, 0)$  and  $(0, 1, 1)$ . Set up the limits of integration over  $D$  using the order  $dxdydz$ .

## 14.5 Triple Integrals in Rectangular Coordinates



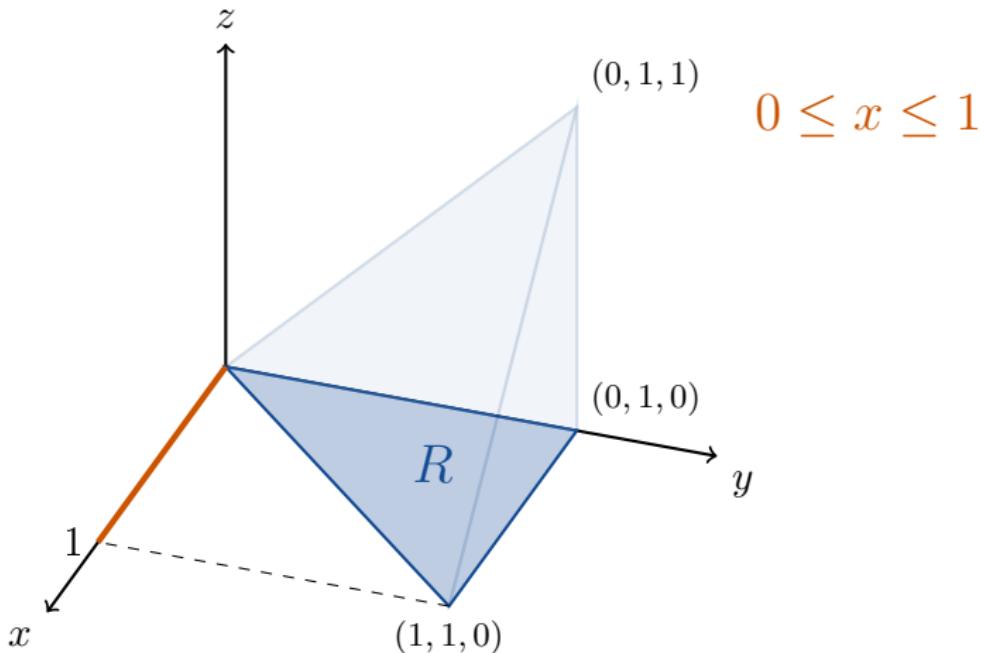
$$\iiint_D F(x, y, z) dz dy dx =$$



## 14.5 Triple Integrals in Rectangular Coordinates



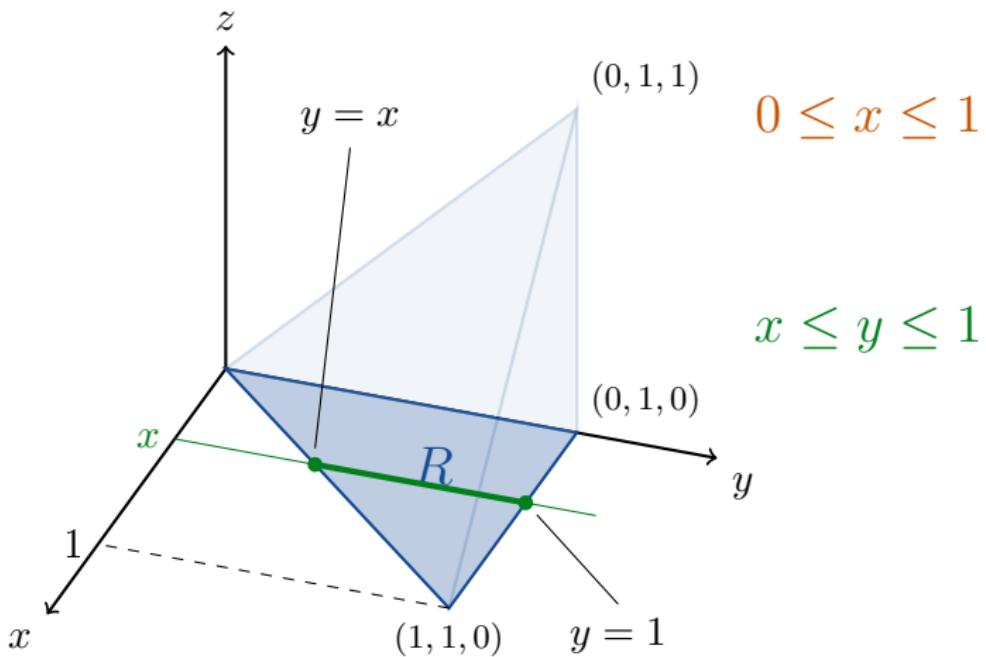
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## 14.5 Triple Integrals in Rectangular Coordinates



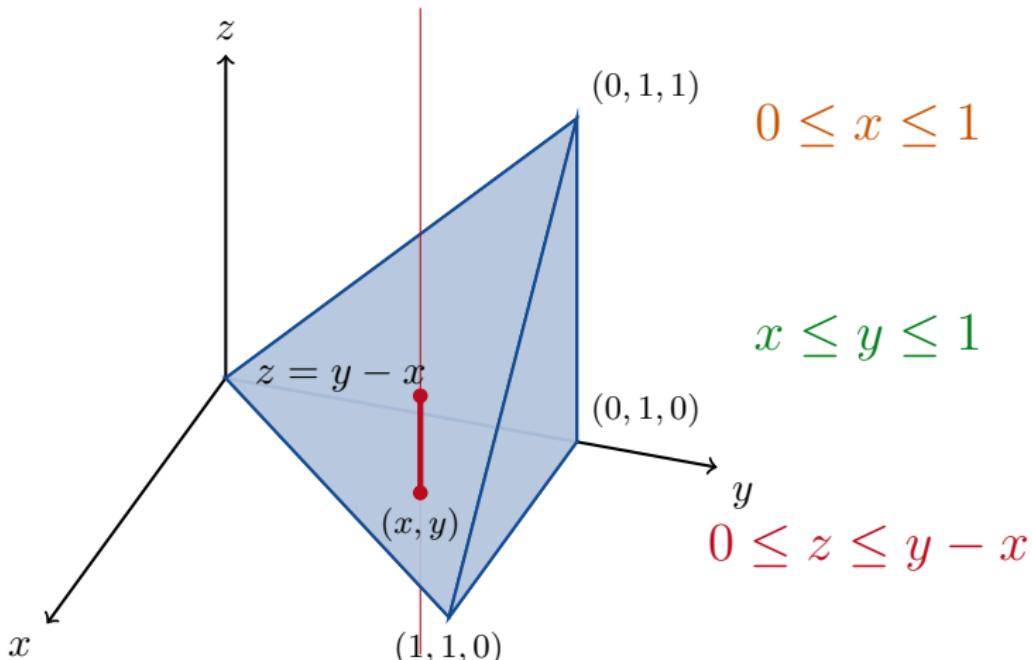
$$\iiint_D F(x, y, z) dz dy dx =$$



## 14.5 Triple Integrals in Rectangular Coordinates



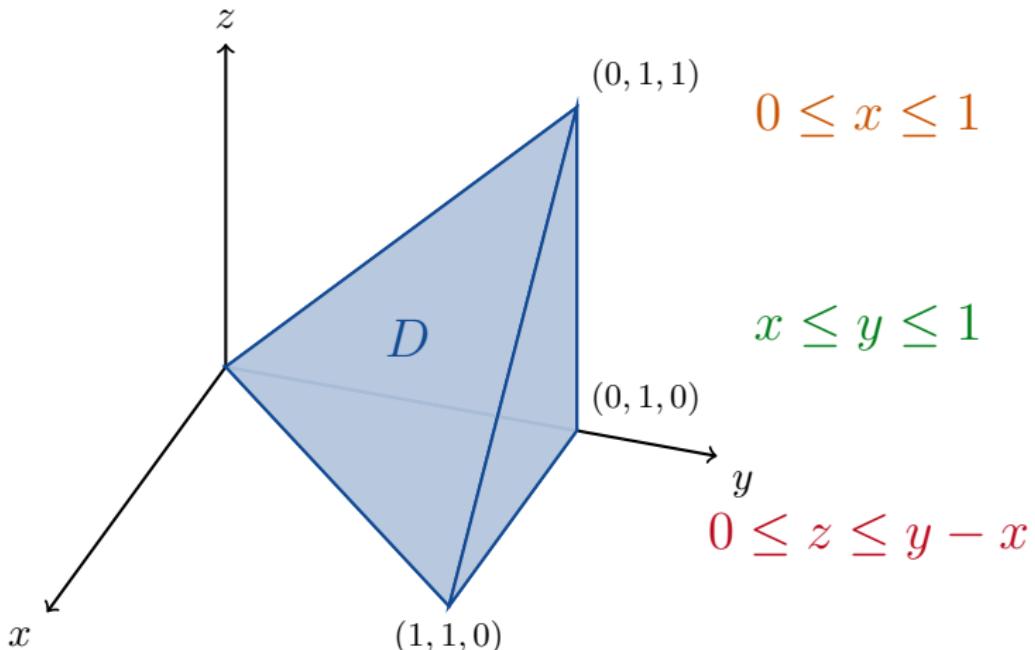
$$\iiint_D F(x, y, z) dz dy dx =$$



## 14.5 Triple Integrals in Rectangular Coordinates



$$\iiint_D F(x, y, z) dz dy dx = \int_0^1 \int_x^1 \int_0^{y-x} F(x, y, z) dz dy dx.$$

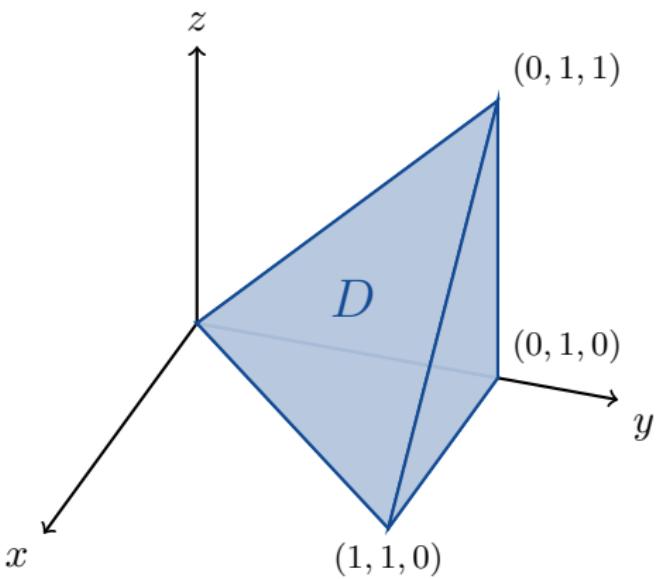


## 14.5 Triple Integrals in Rectangular Coordinates



### Example

Find the volume of this tetrahedron by integrating the function  $F(x, y, z) = 1$  over  $D$  using the order  $dzdydx$ .



## 14.5 Triple Integrals in Rectangular Coordinates



### Example

Find the volume of this tetrahedron by integrating the function  $F(x, y, z) = 1$  over  $D$  using the order  $dzdydx$ .

$$V = \iiint_D dzdydx = \int_0^1 \int_x^1 \int_0^{y-x} dzdydx$$

## 14.5 Triple Integrals in Rectangular Coordinates



### Example

Find the volume of this tetrahedron by integrating the function  $F(x, y, z) = 1$  over  $D$  using the order  $dzdydx$ .

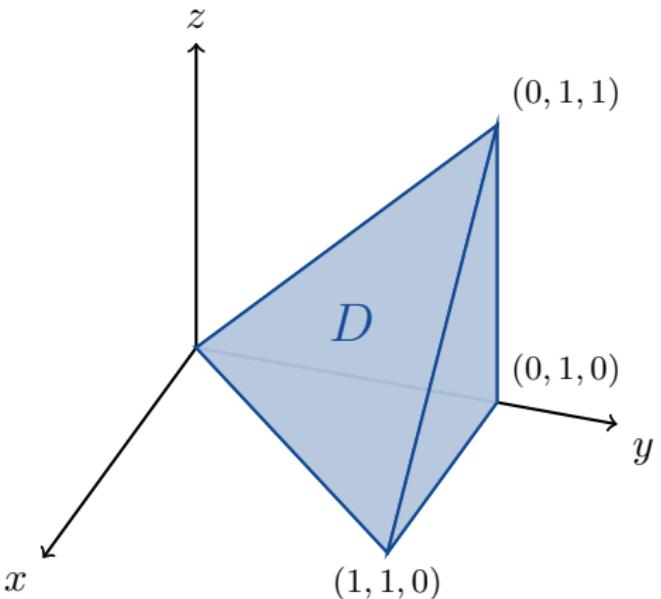
$$\begin{aligned} V &= \iiint_D dzdydx = \int_0^1 \int_x^1 \int_0^{y-x} dzdydx \\ &= \int_0^1 \int_x^1 (y-x) dydx = \int_0^1 \left[ \frac{1}{2}y^2 - xy \right]_{y=x}^{y=1} dx \\ &= \int_0^1 \left( \frac{1}{2} - x + \frac{1}{2}x^2 \right) dx = \left[ \frac{1}{2}x - \frac{1}{2}x^2 + \frac{1}{6}x^3 \right]_0^1 = \frac{1}{6}. \end{aligned}$$

## 14.5 Triple Integrals in Rectangular Coordinates



### Example

Find the volume of this tetrahedron by integrating the function  $F(x, y, z) = 1$  over  $D$  using the order  $dydzdx$ .

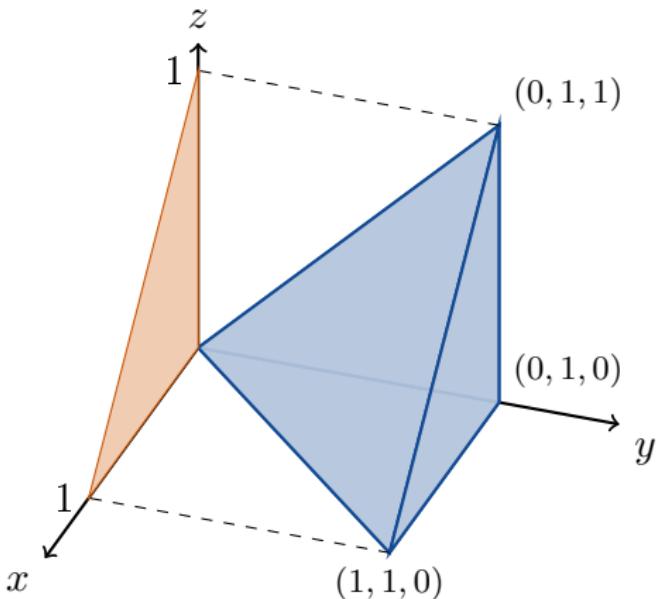


## 14.5 Triple Integrals in Rectangular Coordinates



### Example

Find the volume of this tetrahedron by integrating the function  $F(x, y, z) = 1$  over  $D$  using the order  $dydzdx$ .

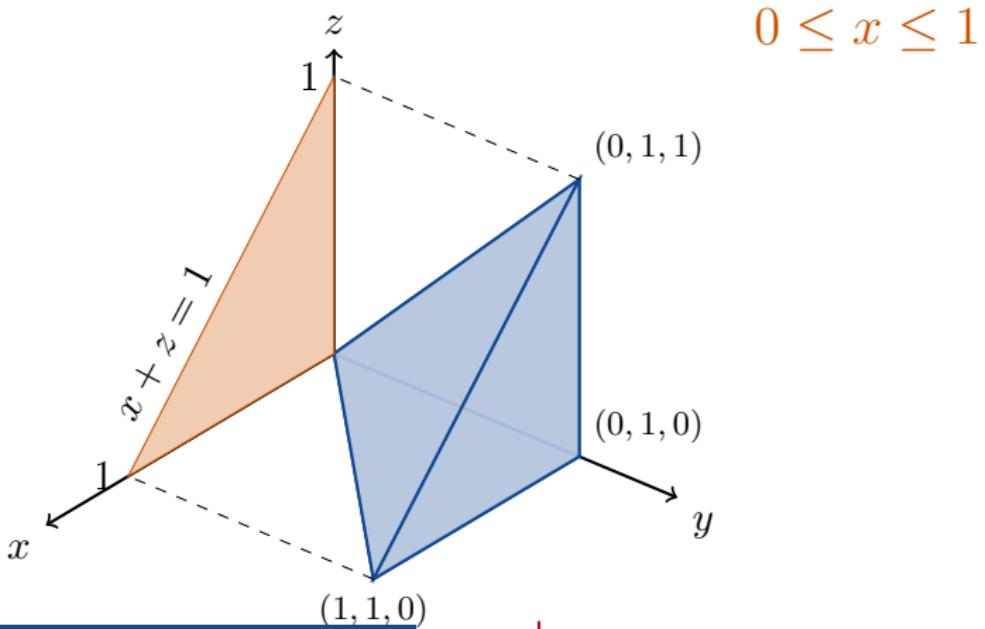


## 14.5 Triple Integrals in Rectangular Coordinates



### Example

Find the volume of this tetrahedron by integrating the function  $F(x, y, z) = 1$  over  $D$  using the order  $\text{dydzdx}$ .

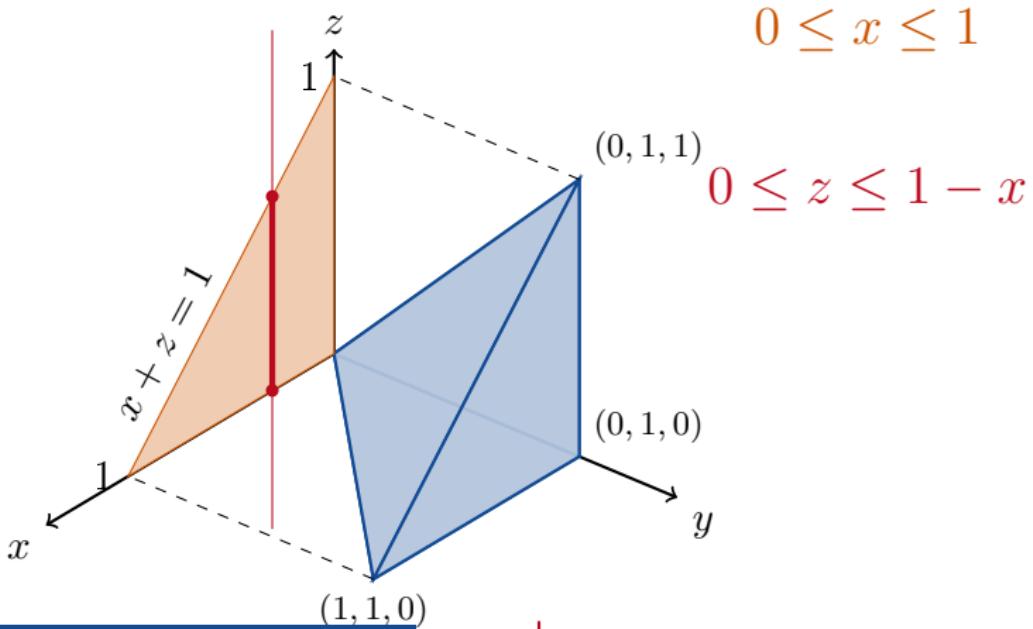


## 14.5 Triple Integrals in Rectangular Coordinates



### Example

Find the volume of this tetrahedron by integrating the function  $F(x, y, z) = 1$  over  $D$  using the order  $\text{dydzdx}$ .

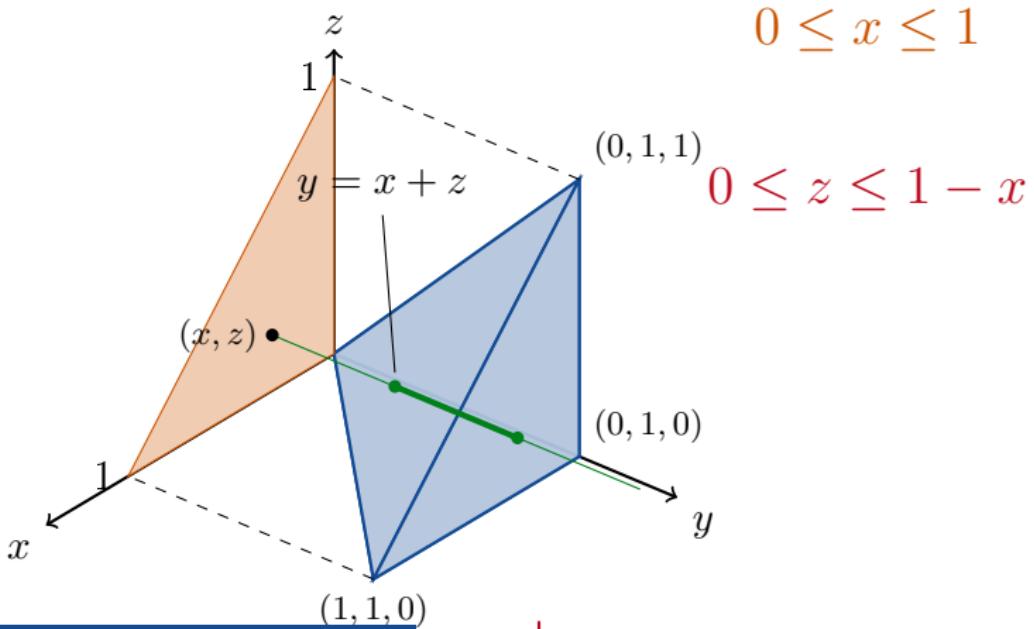


## 14.5 Triple Integrals in Rectangular Coordinates



### Example

Find the volume of this tetrahedron by integrating the function  $F(x, y, z) = 1$  over  $D$  using the order  $\text{dydzdx}$ .

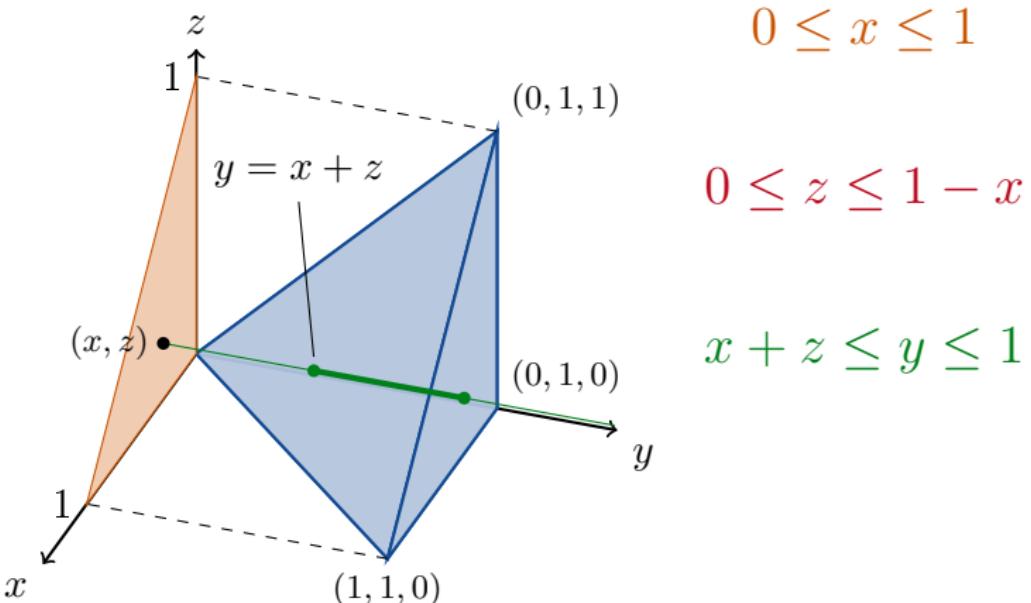


## 14.5 Triple Integrals in Rectangular Coordinates



### Example

Find the volume of this tetrahedron by integrating the function  $F(x, y, z) = 1$  over  $D$  using the order  $\text{dyd}z\text{dx}$ .



## 14.5 Triple Integrals in Rectangular Coordinates



$$0 \leq x \leq 1 \quad 0 \leq z \leq 1 - x \quad x + z \leq y \leq 1$$

$$V = \iiint_D dz dy dx = \int_0^1 \int_0^{1-x} \int_{x+z}^1 dy dz dx$$

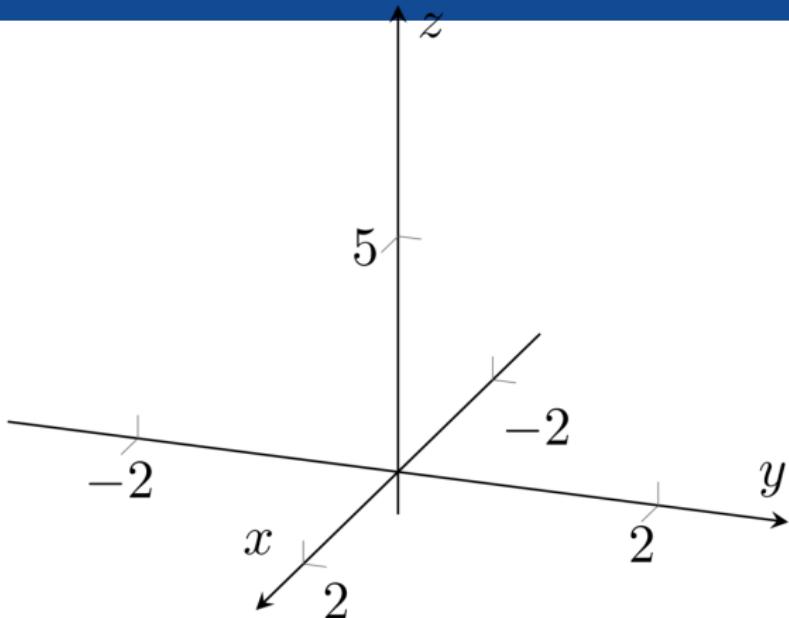
## 14.5 Triple Integrals in Rectangular Coordinates



$$0 \leq x \leq 1 \quad 0 \leq z \leq 1 - x \quad x + z \leq y \leq 1$$

$$\begin{aligned} V &= \iiint_D dz dy dx = \int_0^1 \int_0^{1-x} \int_{x+z}^1 dy dz dx \\ &= \int_0^1 \int_0^{1-x} (1 - x - z) dz dx = \int_0^1 \left[ z - xz - \frac{1}{2}z^2 \right]_{z=0}^{z=1-x} dx \\ &= \int_0^1 \left( 1 - x - x - x^2 - \frac{1}{2}(1-x)^2 \right) dx \\ &= \frac{1}{2} \int_0^1 (1-x)^2 dx = \frac{1}{2} \left[ -\frac{1}{3}(1-x)^3 \right]_0^1 = \frac{1}{6}. \end{aligned}$$

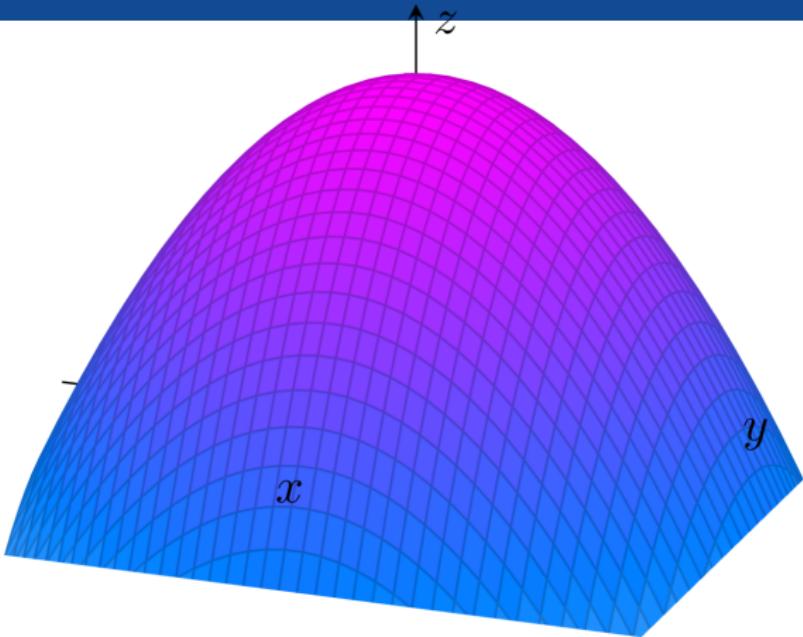
## 14.5 Triple Integrals in Rectangular Coordinates



### Example

Find the volume of the region  $D$  enclosed by the surfaces  $z = x^2 + 3y^2$  and  $z = 8 - x^2 - y^2$ .

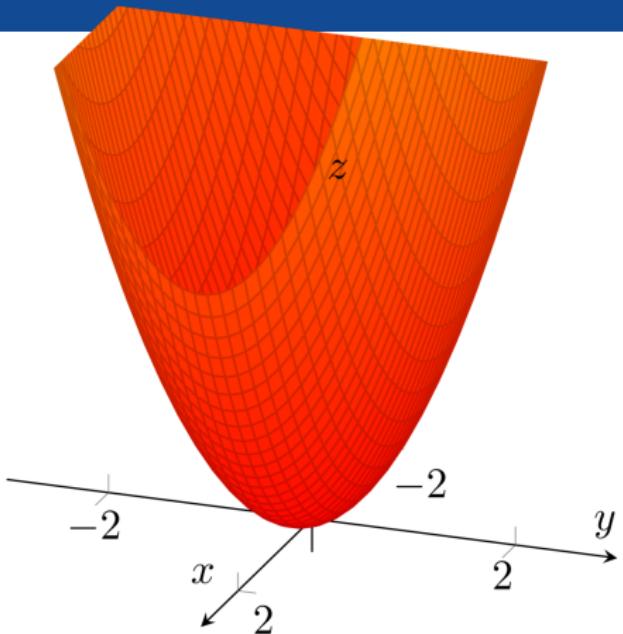
## 14.5 Triple Integrals in Rectangular Coordinates



### Example

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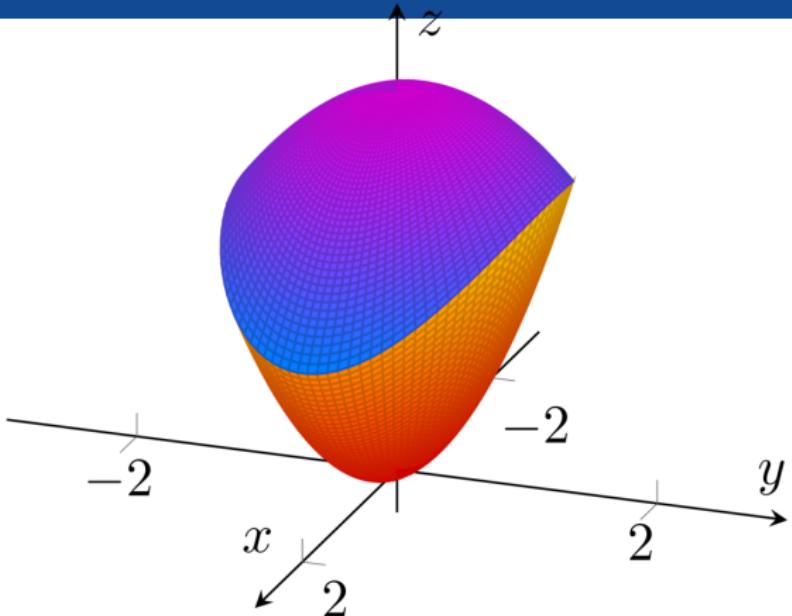
## 14.5 Triple Integrals in Rectangular Coordinates



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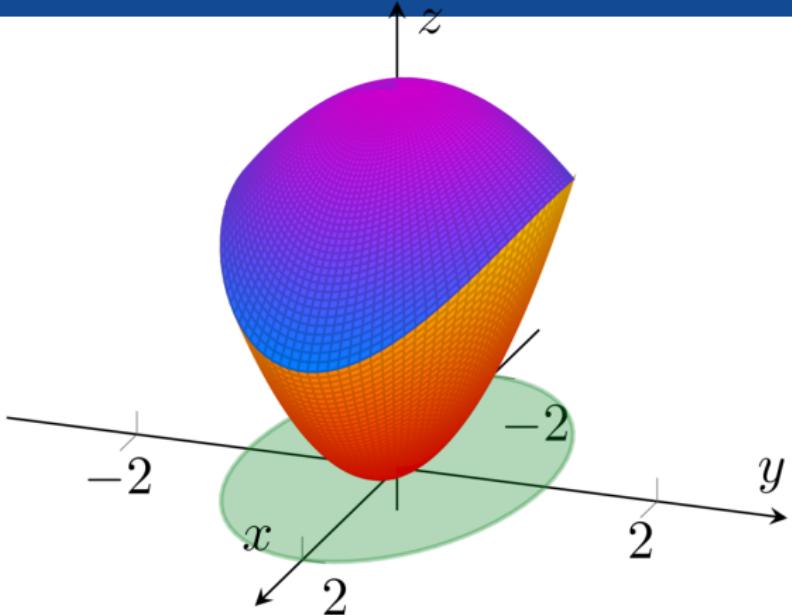
## 14.5 Triple Integrals in Rectangular Coordinates



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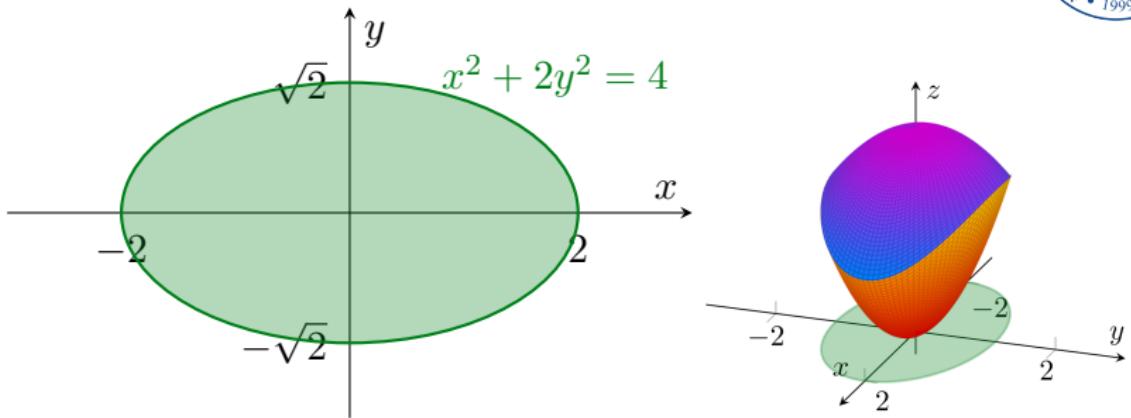
## 14.5 Triple Integrals in Rectangular Coordinates



### Example

Find the volume of the region  $D$  enclosed by the surfaces  $z = x^2 + 3y^2$  and  $z = 8 - x^2 - y^2$ .

## 14.5 Triple Integrals in Rectangular Coordinates

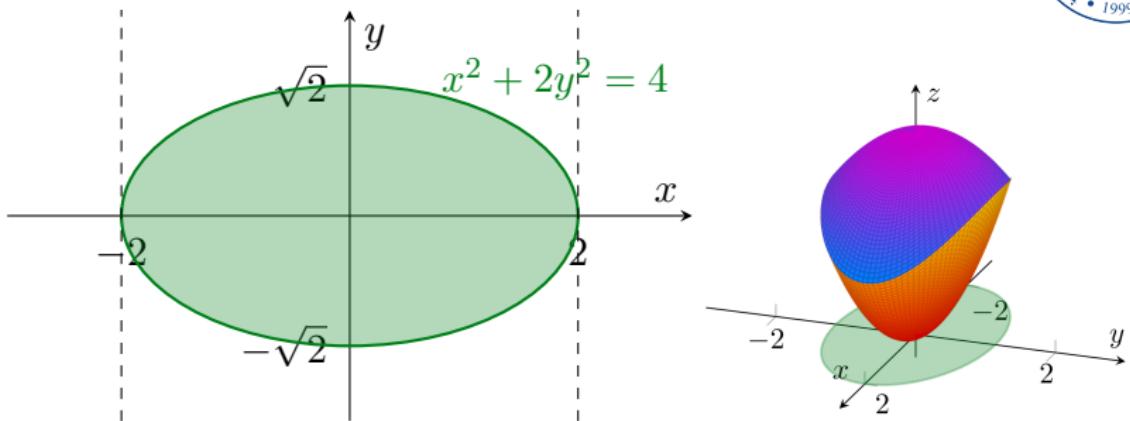


$$x^2 + 3y^2 = z = 8 - x^2 - y^2$$

$$2x^2 + 4y^2 = 8$$

$$x^2 + 2y^2 = 4.$$

## 14.5 Triple Integrals in Rectangular Coordinates



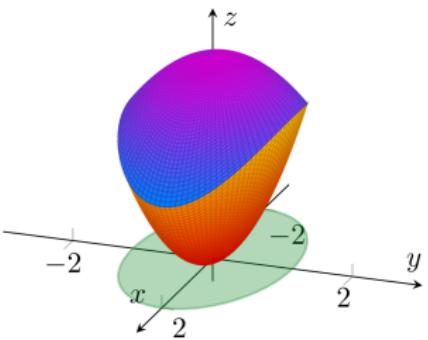
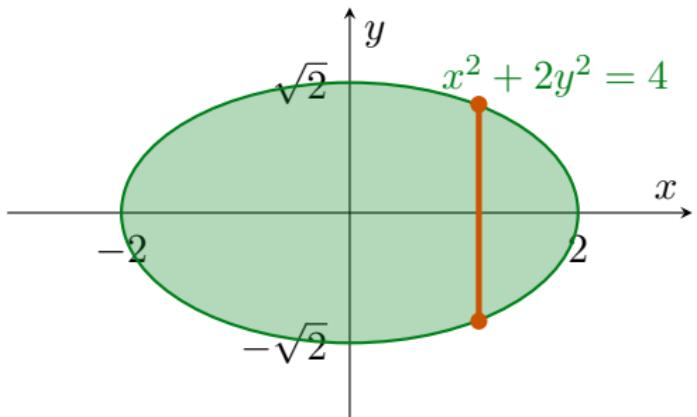
$$-2 \leq x \leq 2$$

$$x^2 + 3y^2 = z = 8 - x^2 - y^2$$

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## 14.5 Triple Integrals in Rectangular Coordinates



$$-2 \leq x \leq 2$$

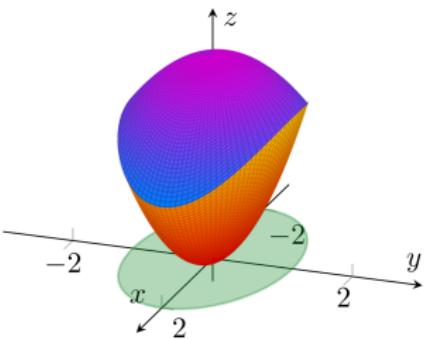
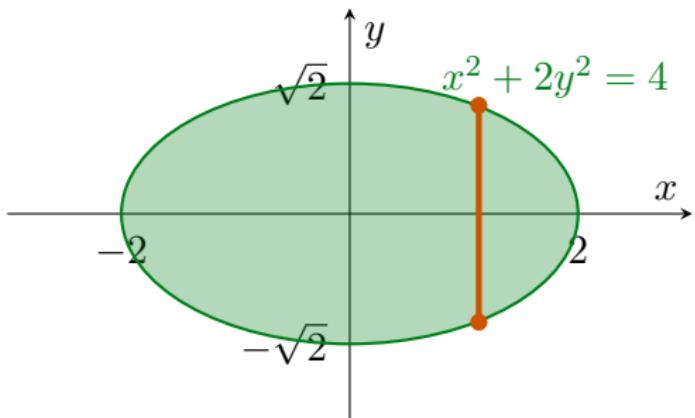
$$x^2 + 3y^2 = z = 8 - x^2 - y^2$$

$$2x^2 + 4y^2 = 8$$

$$x^2 + 2y^2 = 4.$$

$$-\sqrt{\frac{4-x^2}{2}} \leq y \leq \sqrt{\frac{4-x^2}{2}}$$

## 14.5 Triple Integrals in Rectangular Coordinates



$$-2 \leq x \leq 2$$

$$x^2 + 3y^2 = z = 8 - x^2 - y^2$$

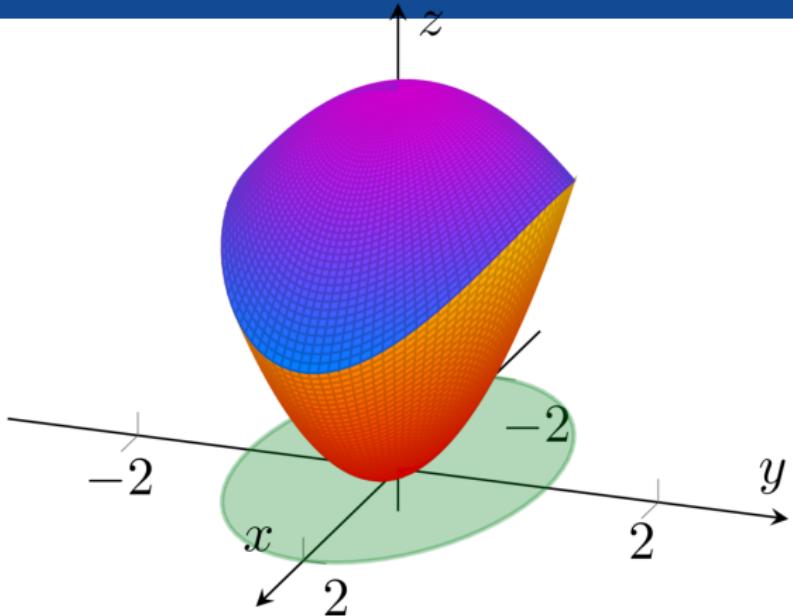
$$2x^2 + 4y^2 = 8$$

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$$-\sqrt{\frac{4-x^2}{2}} \leq y \leq \sqrt{\frac{4-x^2}{2}}$$

$$x^2 + 3y^2 \leq z \leq 8 - x^2 - y^2$$

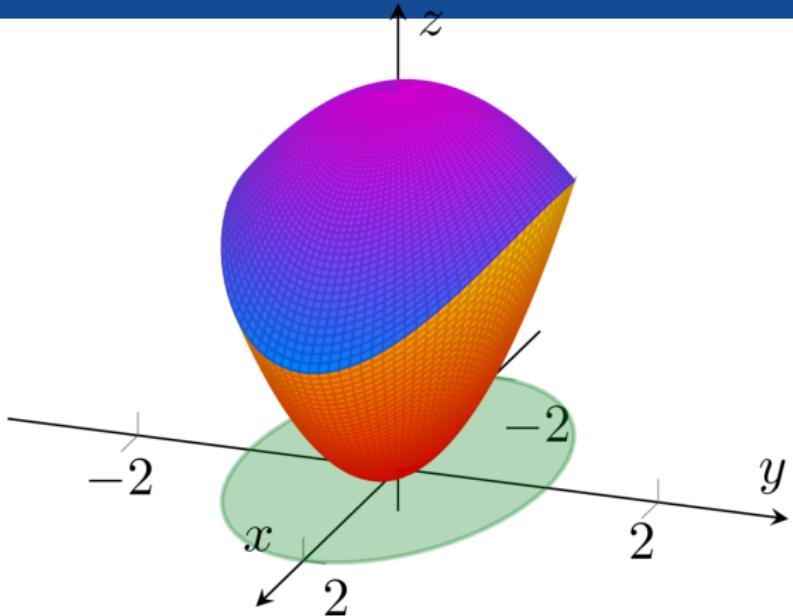
## 14.5 Triple Integrals in Rectangular Coordinates



The volume of  $D$  is

$$V = \int_{-2}^2 \int_{-\sqrt{\frac{4-x^2}{2}}}^{\sqrt{\frac{4-x^2}{2}}} \int_{x^2+3y^2}^{8-x^2-y^2} dz dy dx =$$

## 14.5 Triple Integrals in Rectangular Coordinates



The volume of  $D$  is

$$V = \int_{-2}^2 \int_{-\sqrt{\frac{4-x^2}{2}}}^{\sqrt{\frac{4-x^2}{2}}} \int_{x^2+3y^2}^{8-x^2-y^2} dz dy dx = \dots = 8\pi\sqrt{2}.$$



## Average Value of a Function

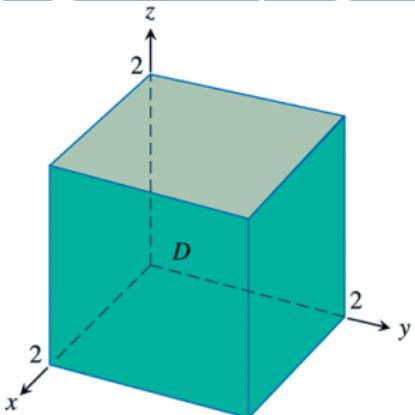
### Definition

The *average value* of a function  $F$  over a region  $D$  is

$$\text{av}(F) = \frac{1}{\text{volume of } D} \iiint_D F \, dV.$$

## 14.5 Triple Integrals

Coordinates

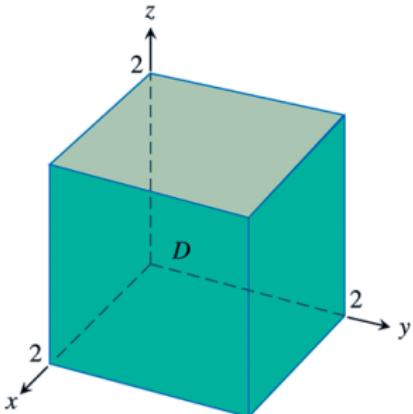


### Example

Find the average value of  $F(x, y, z) = xyz$  over the cube  $[0, 2] \times [0, 2] \times [0, 2]$ .

## 14.5 Triple Integrals

Coordinates



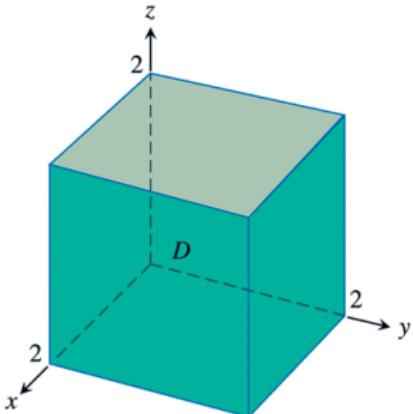
### Example

Find the average value of  $F(x, y, z) = xyz$  over the cube  $[0, 2] \times [0, 2] \times [0, 2]$ .

$$\text{av}(F) = \frac{1}{\text{volume of the cube}} \iiint_{\text{cube}} xyz \, dxdydz$$

## 14.5 Triple Integrals

Coordinates



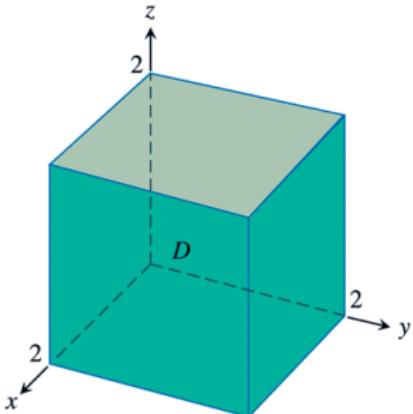
### Example

Find the average value of  $F(x, y, z) = xyz$  over the cube  $[0, 2] \times [0, 2] \times [0, 2]$ .

$$\begin{aligned}\text{av}(F) &= \frac{1}{\text{volume of the cube}} \iiint_{\text{cube}} xyz \, dxdydz \\ &= \frac{1}{8} \int_0^2 \int_0^2 \int_0^2 xyz \, dxdydz\end{aligned}$$

## 14.5 Triple Integrals

Coordinates



### Example

Find the average value of  $F(x, y, z) = xyz$  over the cube  $[0, 2] \times [0, 2] \times [0, 2]$ .

$$\begin{aligned}\text{av}(F) &= \frac{1}{\text{volume of the cube}} \iiint_{\text{cube}} xyz \, dxdydz \\ &= \frac{1}{8} \int_0^2 \int_0^2 \int_0^2 xyz \, dxdydz = \dots = 1.\end{aligned}$$

## 14.5 Triple Integrals in Rectangular Coordinates



### Properties of Triple Integrals



### Properties of Triple Integrals

The same as for double integrals.



# Break

We will continue at 3pm





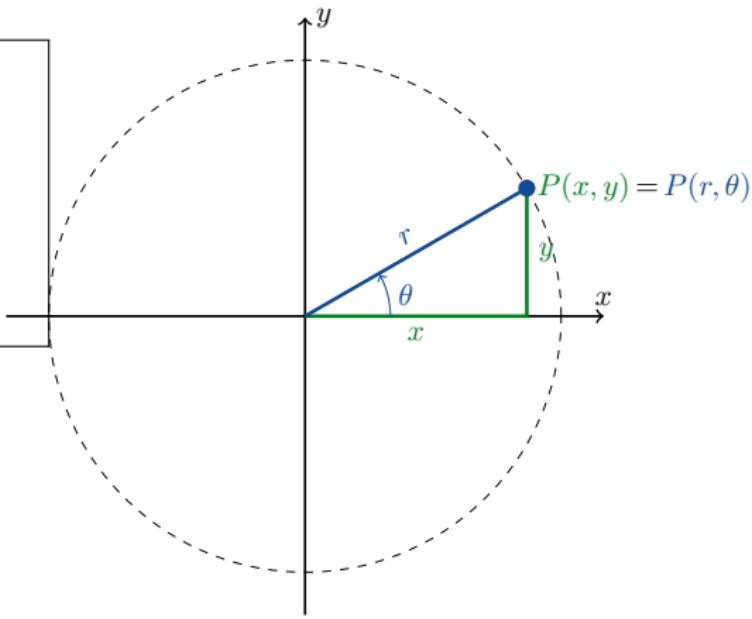
# Triple Integrals in Cylindrical and Spherical Coordinates

## 14.7 Triple Integrals in Cylindrical and Spherical Coordinates



### Polar Coordinates in $\mathbb{R}^2$

$$\begin{array}{ll} x = r \cos \theta & x^2 + y^2 = r^2 \\ y = r \sin \theta & \tan \theta = \frac{y}{x} \end{array}$$



# 14.7 Triple Integrals in Cylindrical and Spherical Coordinates

## Cylindrical Coordinates in $\mathbb{R}^3$

$$x = r \cos \theta$$

$$x^2 + y^2 = r^2$$

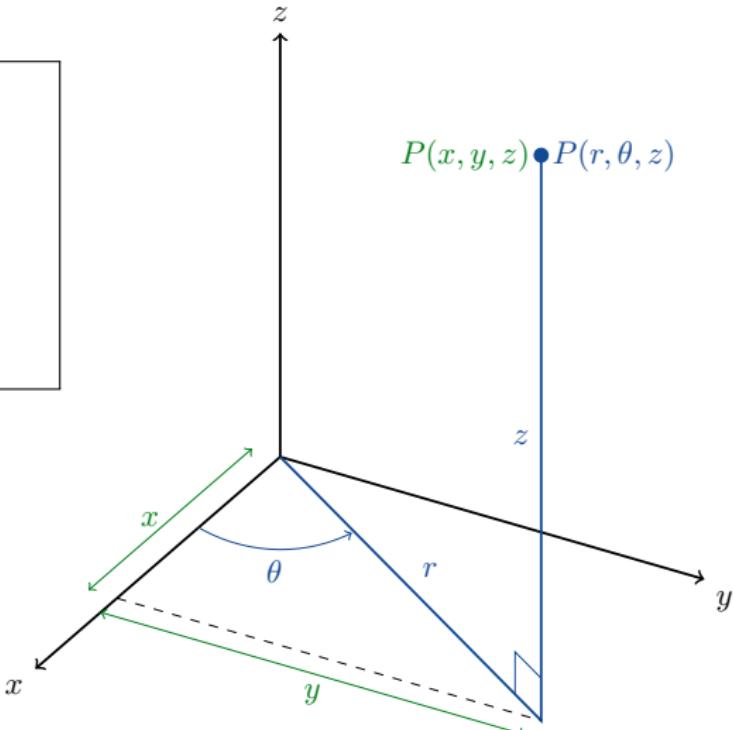
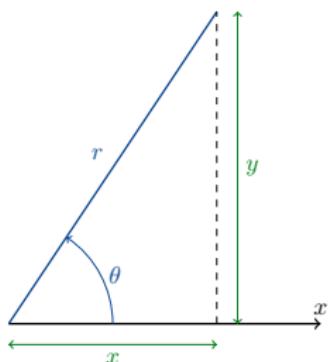
$$y = r \sin \theta$$

$$\tan \theta = \frac{y}{x}$$

$$z = z$$

$$P(x, y, z)$$

$$P(r, \theta, z)$$



## 14.7 Triple Integrals in Cylindrical and Spherical Coordinates



$$x = r \cos \theta \quad y = r \sin \theta \quad z = z$$

### Example

Find cylindrical coordinates for the Cartesian coordinates  $(x, y, z) = (1, 1, 1)$ .

## 14.7 Triple Integrals in Cylindrical and Spherical Coordinates



$$x = r \cos \theta \quad y = r \sin \theta \quad z = z$$

### Example

Find cylindrical coordinates for the Cartesian coordinates  $(x, y, z) = (1, 1, 1)$ .

$$\begin{aligned}(r, \theta, z) &= \left( \sqrt{x^2 + y^2}, \tan^{-1} \frac{y}{x}, z \right) \\&= \left( \sqrt{1^2 + 1^2}, \tan^{-1} 1, 1 \right) = \left( \sqrt{2}, \frac{\pi}{4}, 1 \right).\end{aligned}$$

## 14.7 Triple Integrals in Cylindrical and Spherical Coordinates



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### Example

Convert the cylindrical coordinates  $(r, \theta, z) = \left(2, \frac{\pi}{2}, 2\right)$  to Cartesian coordinates.

## 14.7 Triple Integrals in Cylindrical and Spherical Coordinates



$$x = r \cos \theta \quad y = r \sin \theta \quad z = z$$

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Find cylindrical coordinates for the Cartesian coordinates  $(x, y, z) = (1, 1, 1)$ .

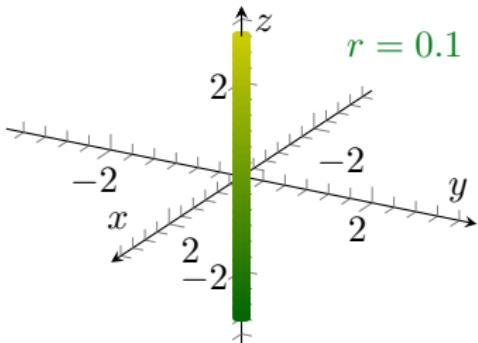
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## 14.7 Triple Integrals in Cylindrical and Spherical Coordinates

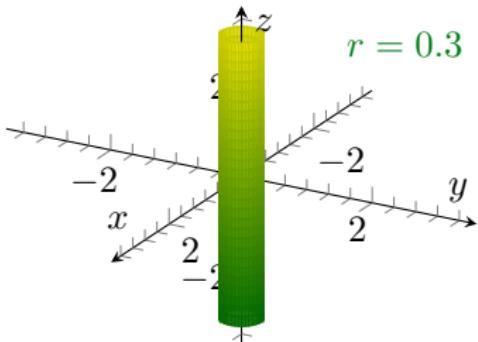


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## 14.7 Triple Integrals in Cylindrical and Spherical Coordinates

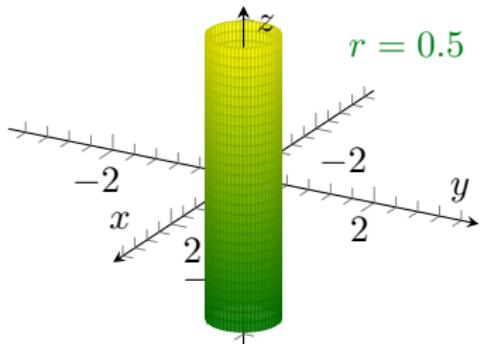


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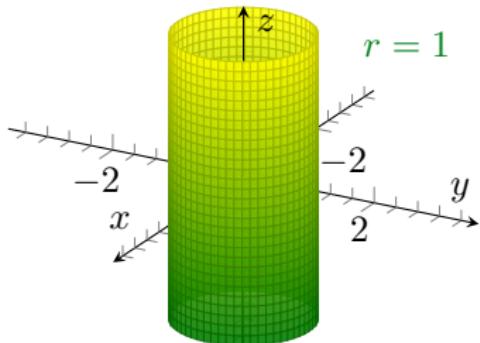


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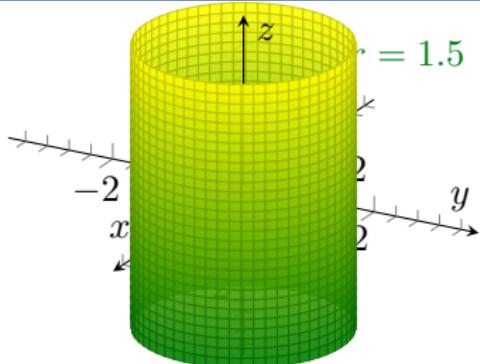


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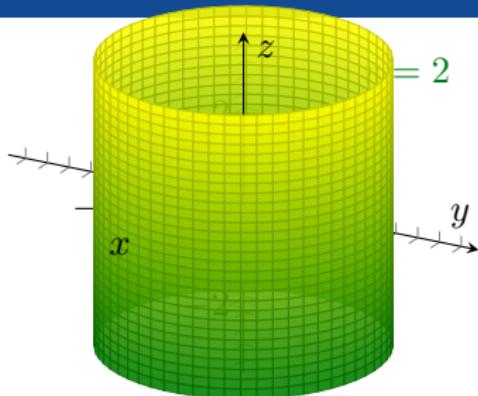


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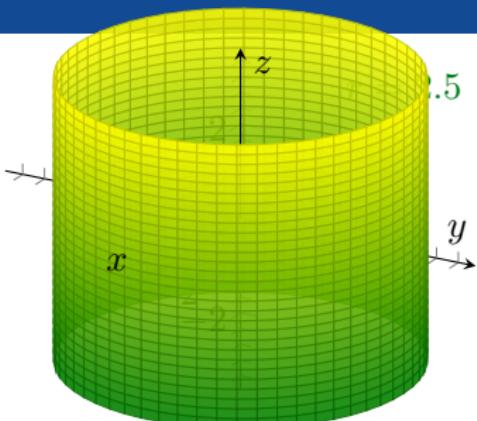


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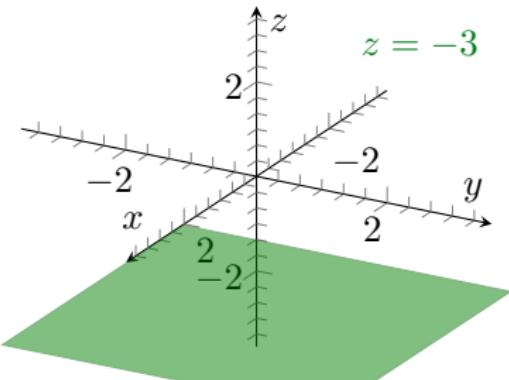


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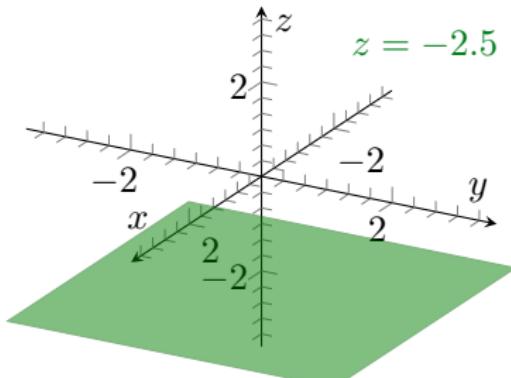


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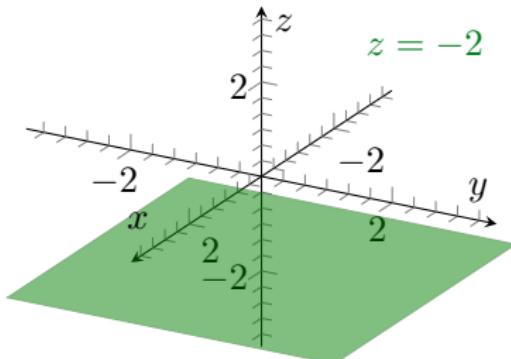


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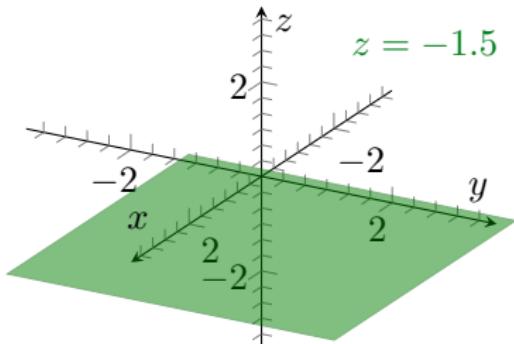


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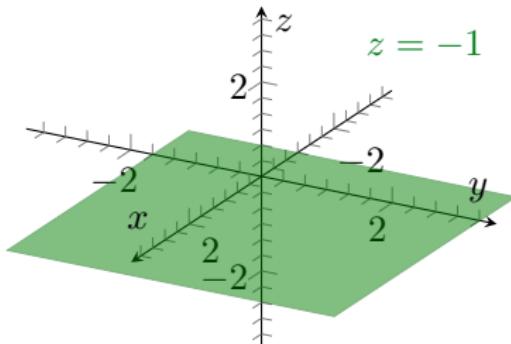


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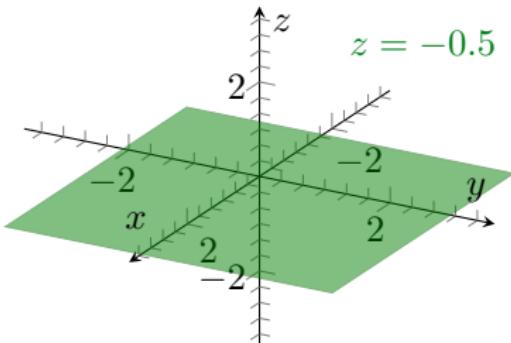


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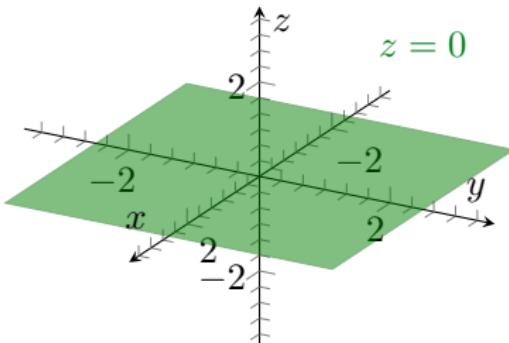


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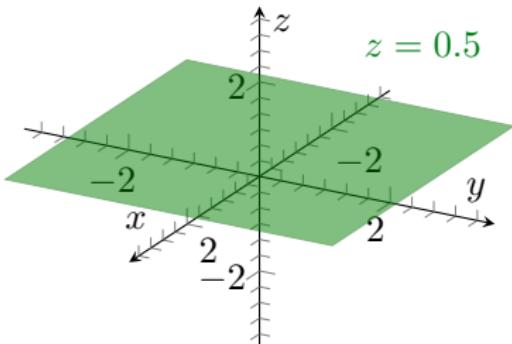


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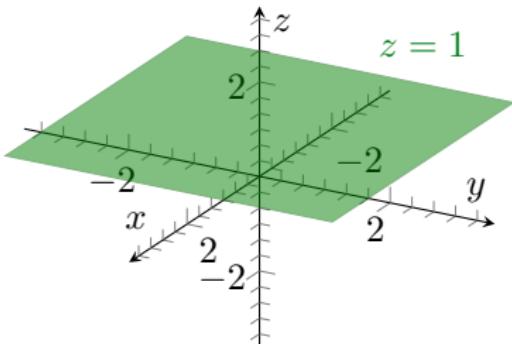


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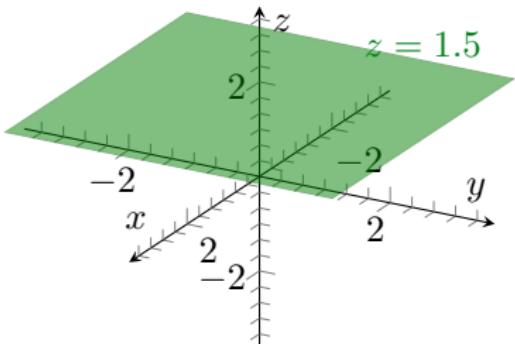


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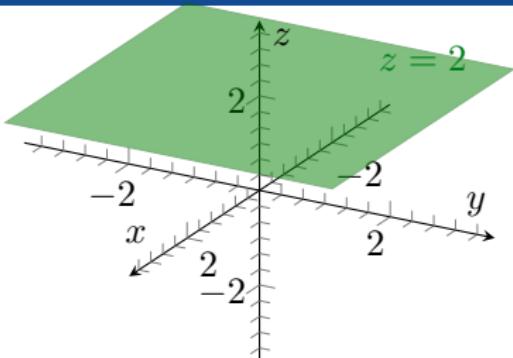


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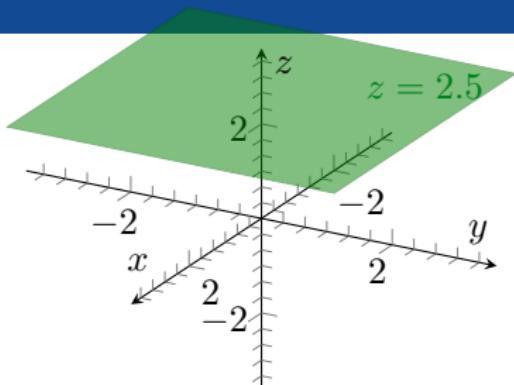


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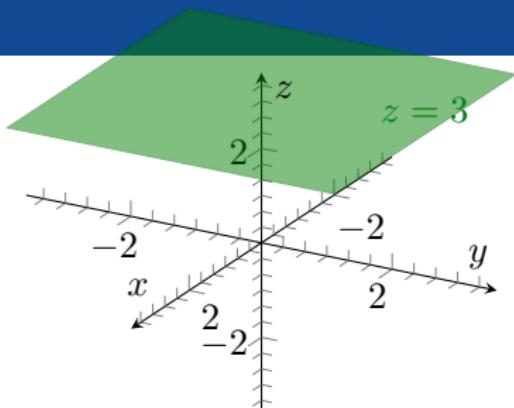


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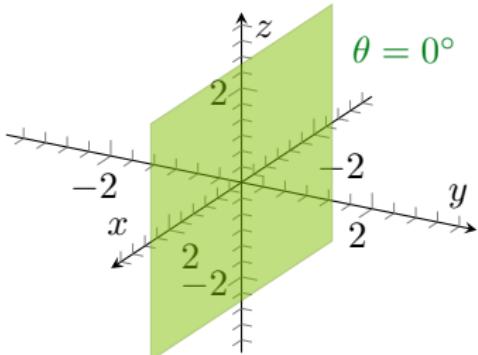


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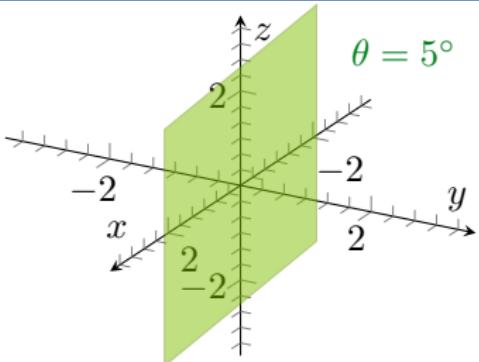


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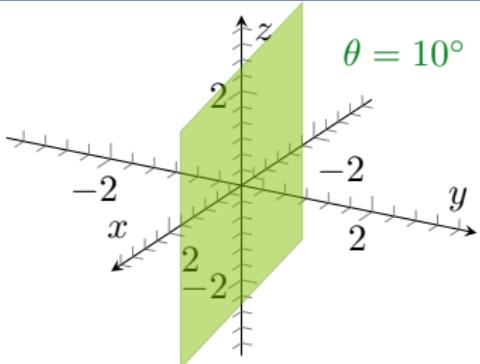


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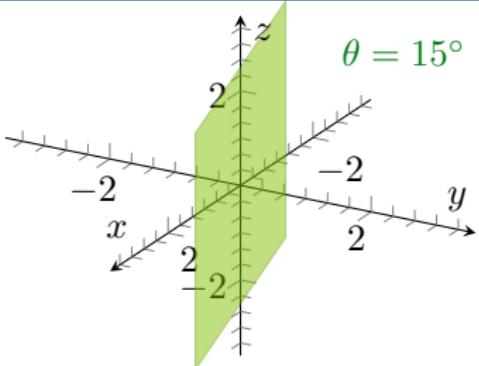


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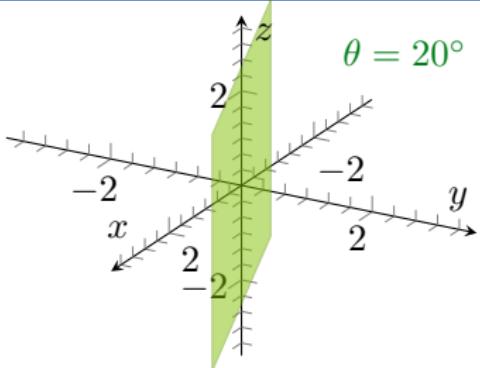


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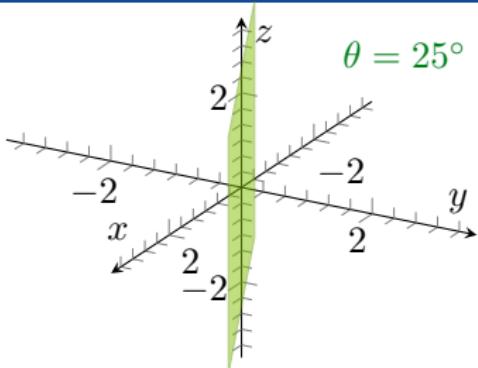


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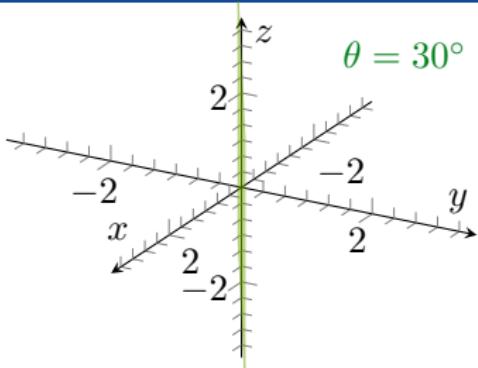


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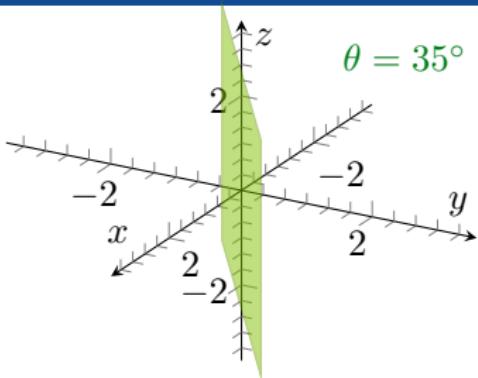


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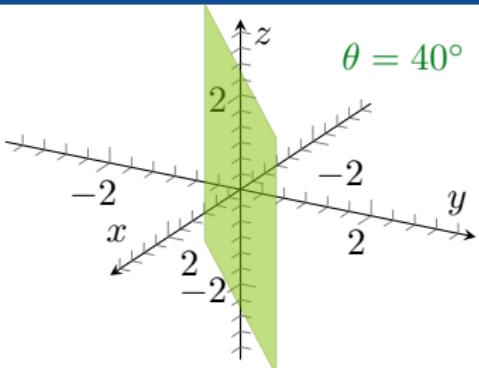


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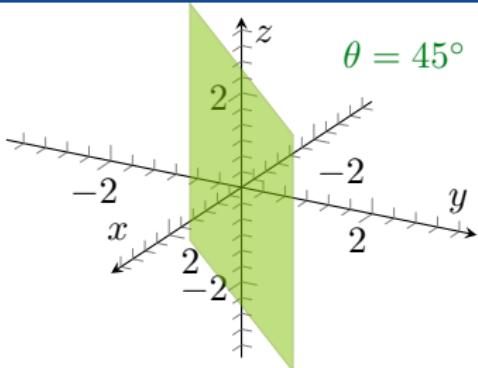


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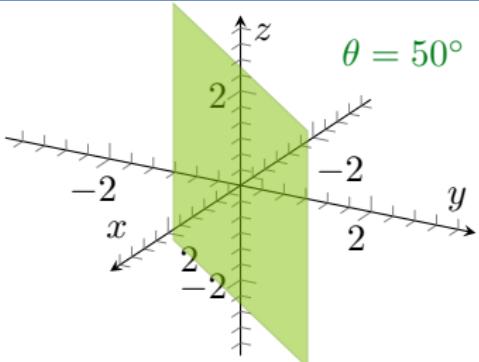


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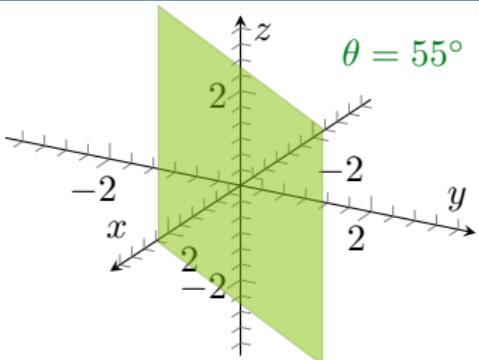


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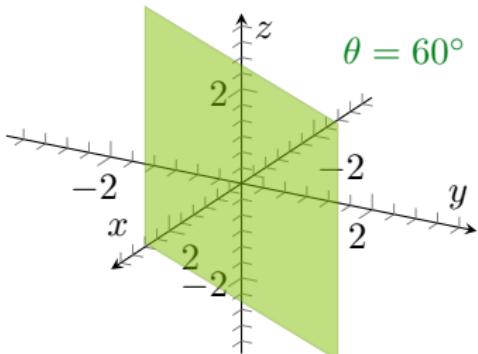


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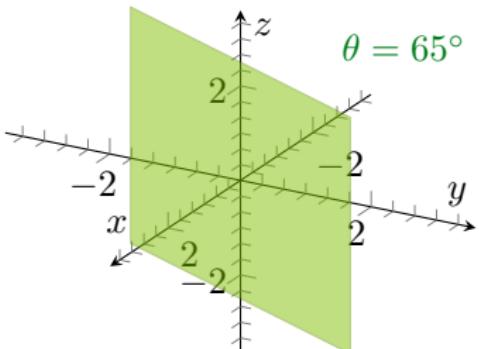


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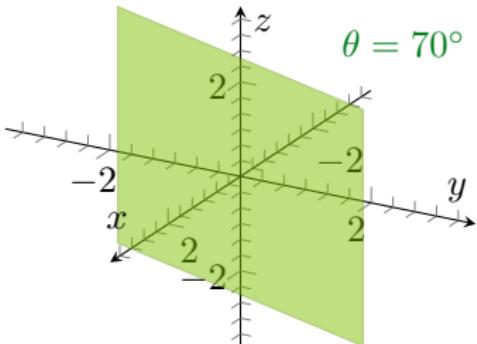


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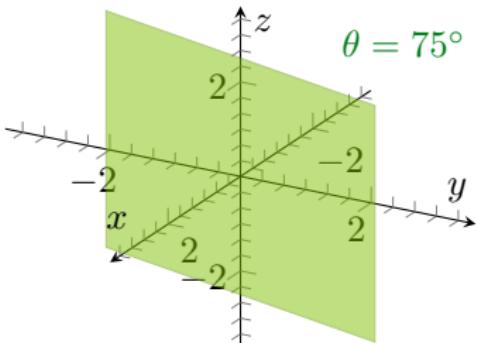


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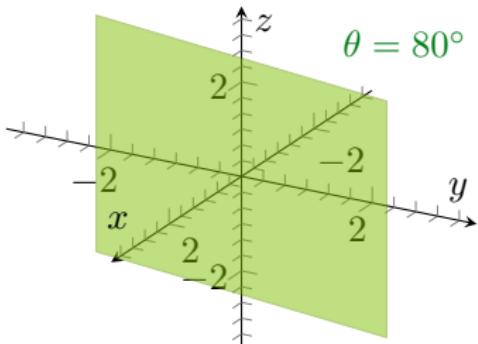


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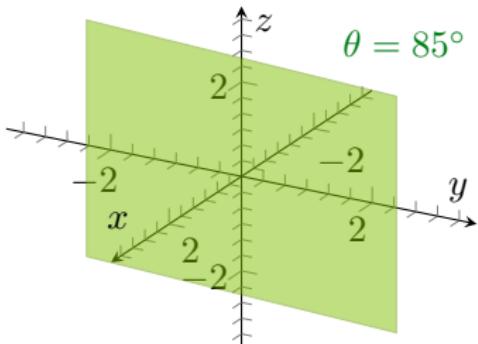


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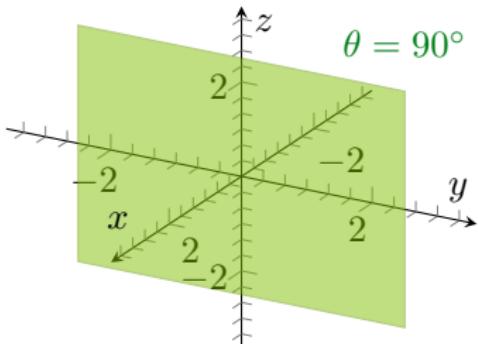


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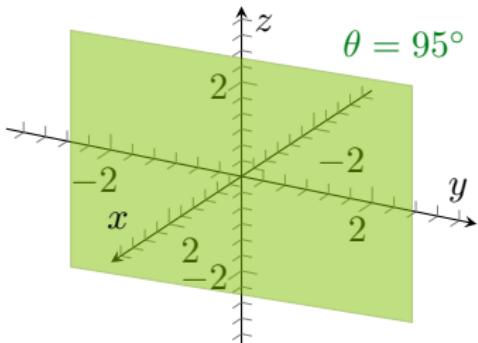


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## 14.7 Triple Integrals in Cylindrical and Spherical Coordinates

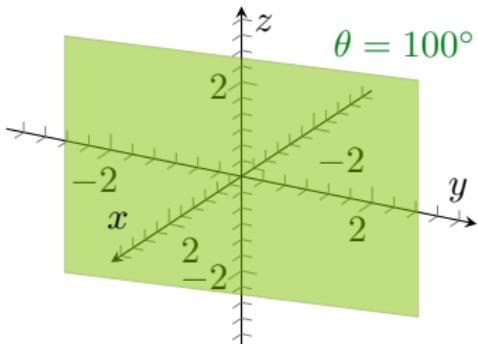


### Remark

Cylindrical coordinates are good for describing:

- vertical cylinders with axis on the  $z$ -axis ( $r = r_0$ );
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## 14.7 Triple Integrals in Cylindrical and Spherical Coordinates

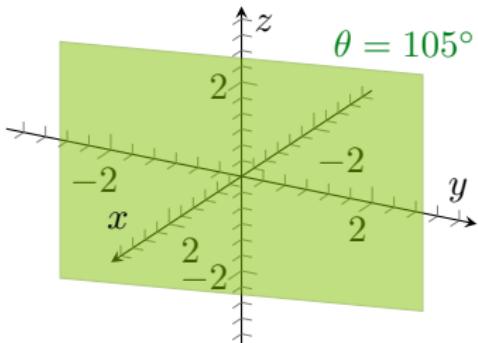


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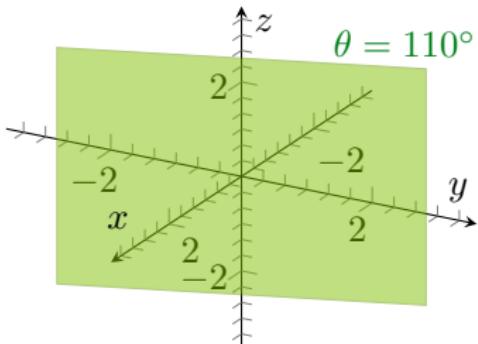


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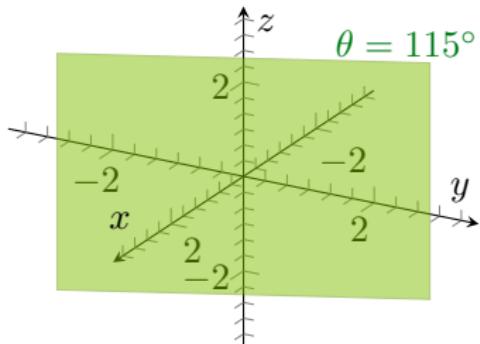


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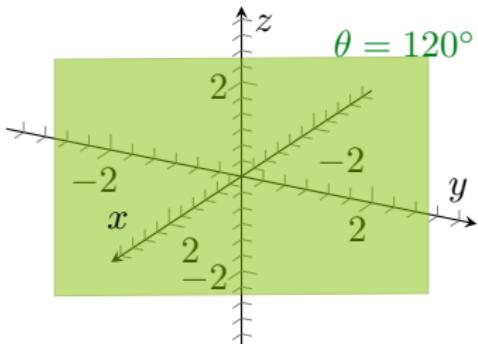


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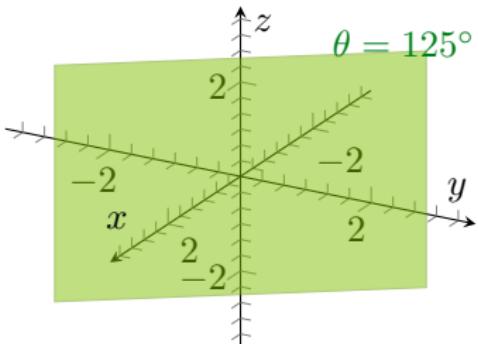


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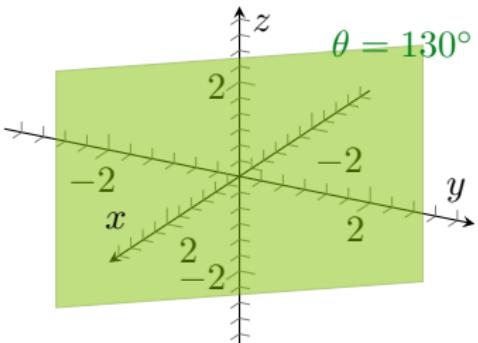


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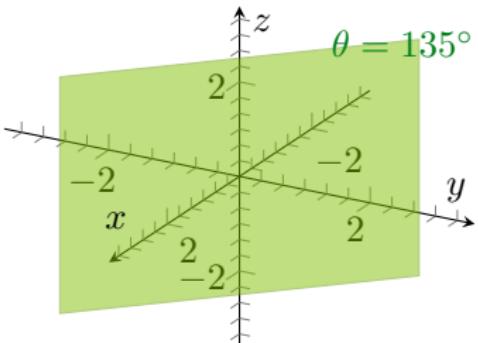


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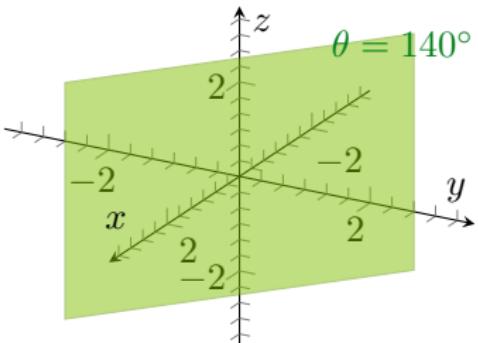


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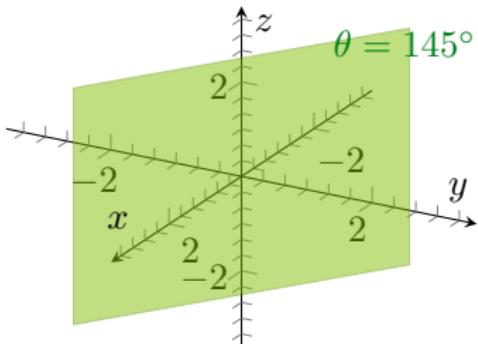


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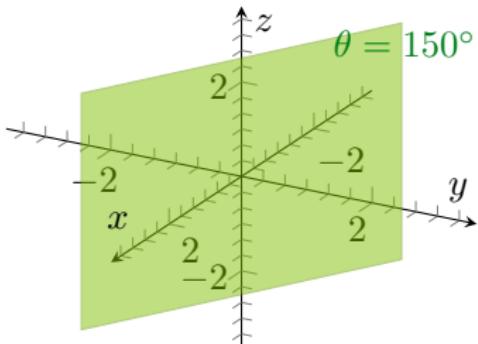


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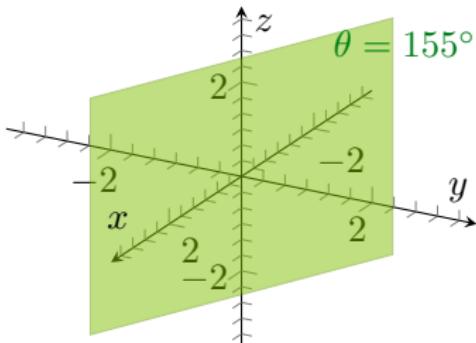


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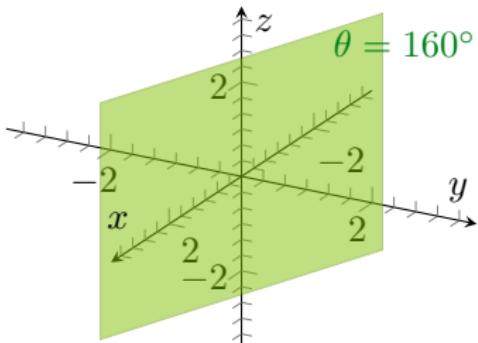


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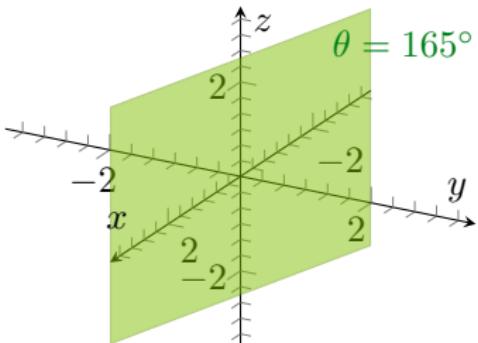


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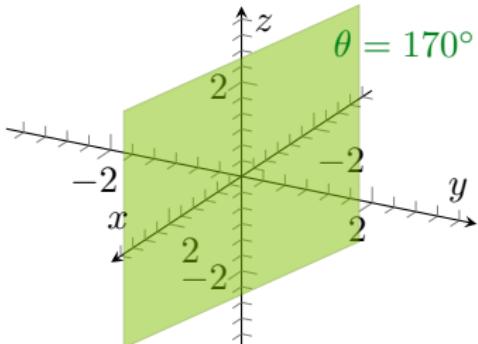


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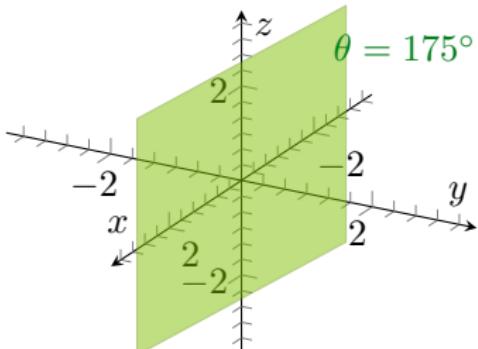


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## 14.7 Triple Integrals in Cylindrical and Spherical Coordinates



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## 14.7 Triple Integrals in Cylindrical and Spherical Coordinates



Recall that

$$dA = dx dy = r dr d\theta.$$

## 14.7 Triple Integrals in Cylindrical and Spherical Coordinates



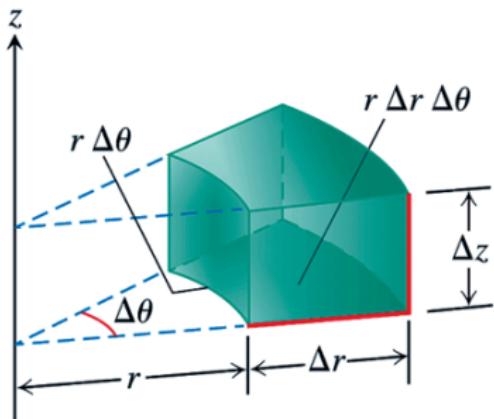
Recall that

$$dA = dx dy = r dr d\theta.$$

Now we have

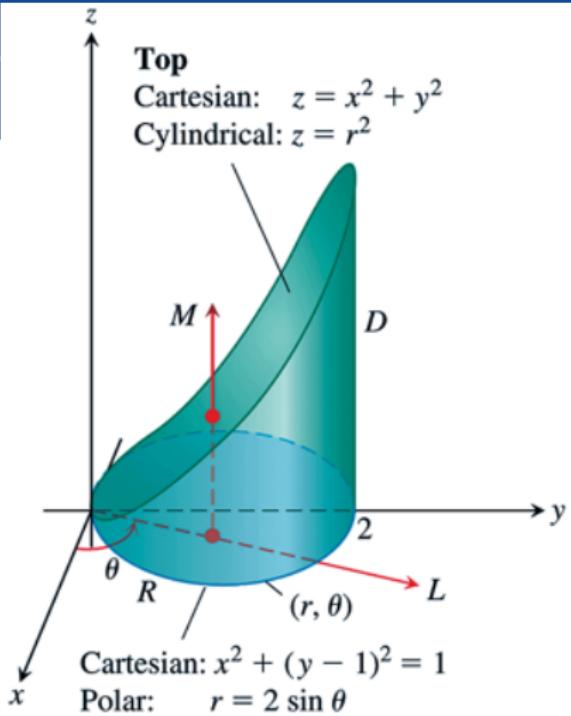
Theorem

$$dV = dx dy dz = r dr d\theta dz.$$



## 14.7 Triple Integrals in Cylindrical Coordinates

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### Example

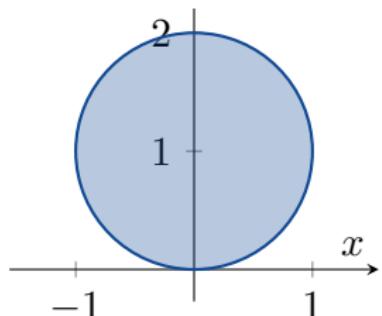
Let  $D$  be the region bounded by  $z = 0$ ,  $x^2 + (y - 1)^2 = 1$  and  $z = x^2 + y^2$ . Find the limits of integration in cylindrical coordinates.

## 14.7 Triple Integrals in Cylindrical and Spherical Coordinates

$$x = r \cos \theta \quad y = r \sin \theta \quad z = z$$



$$x^2 + (y - 1)^2 = 1$$



First note that

$$x^2 + (y - 1)^2 = 1$$

$$x^2 + y^2 - 2y + 1 = 1$$

$$r^2 - 2r \sin \theta = 0$$

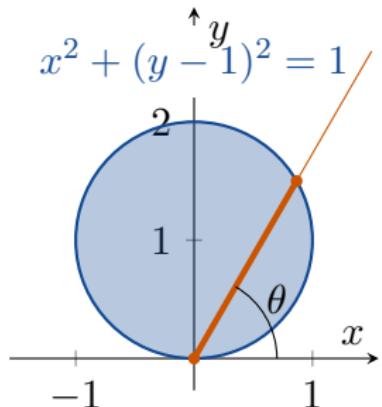
$$r = 2 \sin \theta.$$

## 14.7 Triple Integrals in Cylindrical and Spherical Coordinates



Cylindrical and Spherical

$$x = r \cos \theta \quad y = r \sin \theta \quad z = z$$



So

$$0 \leq \theta \leq \pi$$

$$0 \leq r \leq 2 \sin \theta$$

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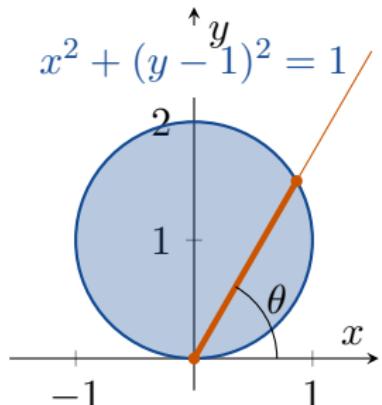
$$r^2 - 2r \sin \theta = 0$$

$$r = 2 \sin \theta.$$

# 14.7 Triple Integrals in Cylindrical and Spherical Coordinates

Cylindrical and Spherical

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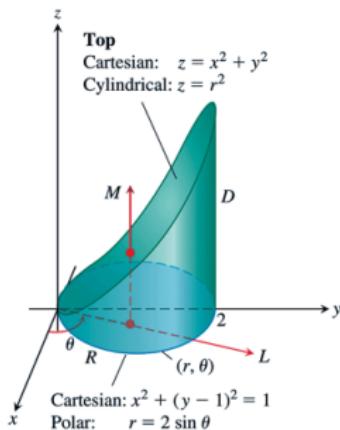
$$r^2 - 2r \sin \theta = 0$$

$$r = 2 \sin \theta.$$

So

$$0 \leq \theta \leq \pi$$

$$0 \leq r \leq 2 \sin \theta$$



$$0 \leq z \leq x^2 + y^2 = r^2$$

## 14.7 Triple Integrals in Cylindrical Coordinates

Calculus and Differential Equations

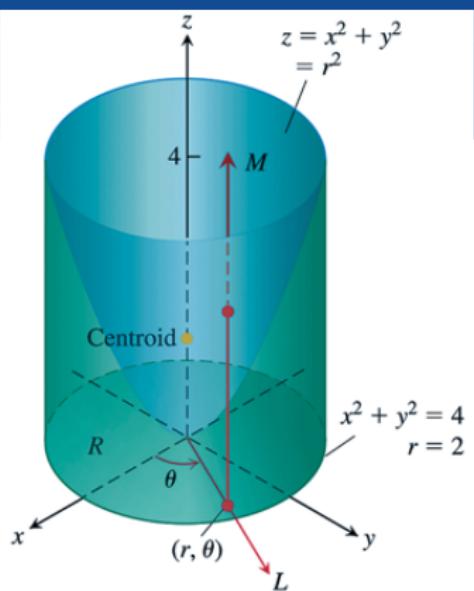
$$dV = dx dy dz = r dr d\theta dz$$



$$0 \leq \theta \leq \pi \quad 0 \leq r \leq 2 \sin \theta \quad 0 \leq z \leq r^2$$

Therefore

$$\iiint_D F(r, \theta, z) dV = \int_0^\pi \int_0^{2 \sin \theta} \int_0^{r^2} F(r, \theta, z) r dz dr d\theta.$$

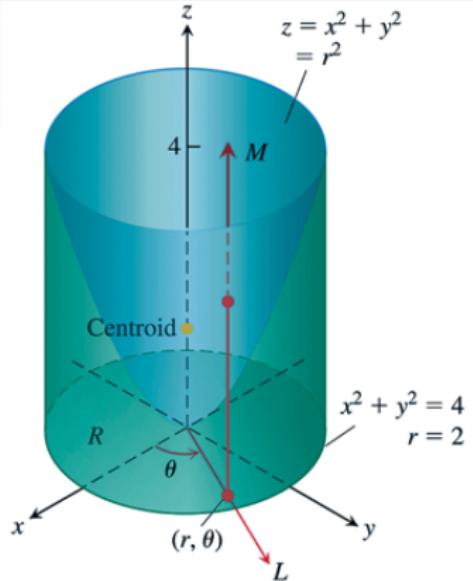


## Example

Calculate

$$\iiint_D z \, dV$$

where  $D$  is the region enclosed by the cylinder  $x^2 + y^2 = 4$ , the  $xy$ -plane and the paraboloid  $z = x^2 + y^2$ .



$$0 \leq \theta \leq 2\pi$$

$$0 \leq r \leq 2$$

$$0 \leq z \leq r^2.$$

### Example

Calculate

$$\iiint_D z \, dV$$

where  $D$  is the region enclosed by the cylinder  $x^2 + y^2 = 4$ , the  $xy$ -plane and the paraboloid  $z = x^2 + y^2$ .

## 14.7 Triple Integrals in Cylindrical Coordinates

## Coordinates

$$dV = dx dy dz = r dr d\theta dz$$



$$0 \leq \theta \leq 2\pi \quad 0 \leq r \leq 2 \quad 0 \leq z \leq r^2$$

Therefore

$$\iiint_D z \, dV = \int_0^{2\pi} \int_0^2 \int_0^{r^2} z \, r \, dz \, r \, d\theta$$

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## 14.7 Triple Integrals in Cylindrical Coordinates

$$dV = dx dy dz = r dr d\theta dz$$



$$0 \leq \theta \leq 2\pi \quad 0 \leq r \leq 2 \quad 0 \leq z \leq r^2$$

Therefore

$$\begin{aligned}\iiint_D z \, dV &= \int_0^{2\pi} \int_0^2 \int_0^{r^2} zr \, dz \, dr \, d\theta \\ &= \int_0^{2\pi} \int_0^2 \left[ \frac{1}{2}z^2 r \right]_{z=0}^{z=r^2} \, dr \, d\theta\end{aligned}$$

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## 14.7 Triple Integrals in Cylindrical Coordinates

$$dV = dx dy dz = r dr d\theta dz$$



$$0 \leq \theta \leq 2\pi \quad 0 \leq r \leq 2 \quad 0 \leq z \leq r^2$$

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$$= \quad = \quad .$$

## 14.7 Triple Integrals in Cylindrical Coordinates

## Coordinates

$$dV = dx dy dz = r dr d\theta dz$$



$$0 \leq \theta \leq 2\pi \quad 0 \leq r \leq 2 \quad 0 \leq z \leq r^2$$

Therefore

$$\begin{aligned}
 \iiint_D z \, dV &= \int_0^{2\pi} \int_0^2 \int_0^{r^2} zr \, dz \, dr \, d\theta \\
 &= \int_0^{2\pi} \int_0^2 \left[ \frac{1}{2}z^2 r \right]_{z=0}^{z=r^2} \, dr \, d\theta \\
 &= \int_0^{2\pi} \int_0^2 \frac{1}{2}r^5 \, dr \, d\theta \\
 &= \int_0^{2\pi} \left[ \frac{1}{12}r^6 \right]_0^2 \, d\theta \\
 &= \dots
 \end{aligned}$$

## 14.7 Triple Integrals in Cylindrical Coordinates

$$dV = dx dy dz = r dr d\theta dz$$



$$0 \leq \theta \leq 2\pi \quad 0 \leq r \leq 2 \quad 0 \leq z \leq r^2$$

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$$\begin{aligned} \iiint_D z \, dV &= \int_0^{2\pi} \int_0^2 \int_0^{r^2} zr \, dz dr d\theta \\ &= \int_0^{2\pi} \int_0^2 \left[ \frac{1}{2}z^2 r \right]_{z=0}^{z=r^2} dr d\theta \\ &= \int_0^{2\pi} \int_0^2 \frac{1}{2}r^5 dr d\theta \\ &= \int_0^{2\pi} \left[ \frac{1}{12}r^6 \right]_0^2 d\theta \\ &= \int_0^{2\pi} \frac{16}{3} d\theta = . \end{aligned}$$

## 14.7 Triple Integrals in Cylindrical Coordinates

$$dV = dx dy dz = r dr d\theta dz$$



$$0 \leq \theta \leq 2\pi \quad 0 \leq r \leq 2 \quad 0 \leq z \leq r^2$$

Therefore

$$\begin{aligned} \iiint_D z \, dV &= \int_0^{2\pi} \int_0^2 \int_0^{r^2} zr \, dz \, dr \, d\theta \\ &= \int_0^{2\pi} \int_0^2 \left[ \frac{1}{2}z^2 r \right]_{z=0}^{z=r^2} \, dr \, d\theta \\ &= \int_0^{2\pi} \int_0^2 \frac{1}{2}r^5 \, dr \, d\theta \\ &= \int_0^{2\pi} \left[ \frac{1}{12}r^6 \right]_0^2 \, d\theta \\ &= \int_0^{2\pi} \frac{16}{3} \, d\theta = \frac{32\pi}{3}. \end{aligned}$$

# 14.7 Triple Integrals in Cylindrical and Spherical Coordinates



## Spherical Coordinates in $\mathbb{R}^3$

$$x = r \cos \theta = \rho \sin \phi \cos \theta$$

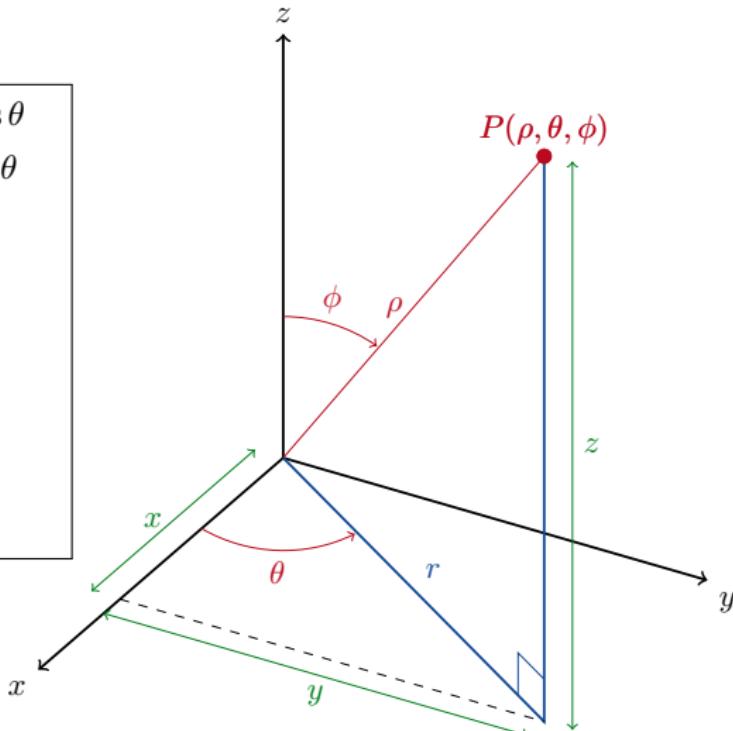
$$y = r \sin \theta = \rho \sin \phi \sin \theta$$

$$z = \rho \cos \phi$$

$$r = \rho \sin \phi$$

$$\tan \theta = \frac{y}{x}$$

$$\begin{aligned}\rho &= \sqrt{x^2 + y^2 + z^2} \\ &= \sqrt{r^2 + z^2}\end{aligned}$$



# 14.7 Triple Integrals in Cylindrical and Spherical Coordinates

## Spherical Coordinates in $\mathbb{R}^3$

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$$y = r \sin \theta = \rho \sin \phi \sin \theta$$

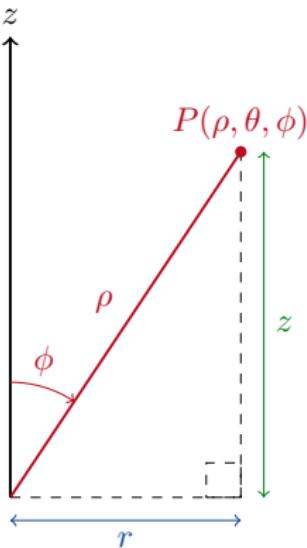
$$z = \rho \cos \phi$$

$$r = \rho \sin \phi$$

$$\tan \theta = \frac{y}{x}$$

$$\rho = \sqrt{x^2 + y^2 + z^2}$$

$$= \sqrt{r^2 + z^2}$$



Typically, we require that  $\rho \geq 0$  and  $0 \leq \phi \leq \pi$ . As before,  $\theta$  can be any number.

## 14.7 Triple Integrals in Cylindrical and Spherical Coordinates

$$\rho = \sqrt{r^2 + z^2} \quad \theta = \theta \quad z = \rho \cos \phi$$



### Example

Convert the point  $P(\sqrt{6}, \frac{\pi}{4}, \sqrt{2})$  from cylindrical to spherical coordinates.

## 14.7 Triple Integrals in Cylindrical and Spherical Coordinates



$$\rho = \sqrt{r^2 + z^2} \quad \theta = \theta \quad z = \rho \cos \phi$$

### Example

Convert the point  $P(\sqrt{6}, \frac{\pi}{4}, \sqrt{2})$  from cylindrical to spherical coordinates.

We have that  $r = \sqrt{6}$ ,  $\theta = \frac{\pi}{4}$  and  $z = \sqrt{2}$ . Therefore

$$\begin{aligned} (\rho, \theta, \phi) &= \left( \sqrt{r^2 + z^2}, \theta, \cos^{-1} \frac{z}{\rho} \right) \\ &= \left( \sqrt{6+2}, \frac{\pi}{4}, \cos^{-1} \frac{\sqrt{2}}{\rho} \right) \\ &= \left( 2\sqrt{2}, \frac{\pi}{4}, \cos^{-1} \frac{\sqrt{2}}{2\sqrt{2}} \right) \\ &= \left( 2\sqrt{2}, \frac{\pi}{4}, \cos^{-1} \frac{1}{2} \right) = \left( 2\sqrt{2}, \frac{\pi}{4}, \frac{\pi}{3} \right). \end{aligned}$$

$$\rho = \sqrt{x^2 + y^2 + z^2} \quad y = \rho \sin \phi \sin \theta \quad z = \rho \cos \phi$$



### Example

Convert the point  $P(-1, 1, -\sqrt{2})$  from Cartesian to spherical polar coordinates.

$$\rho = \sqrt{x^2 + y^2 + z^2} \quad y = \rho \sin \phi \sin \theta \quad z = \rho \cos \phi$$



### Example

Convert the point  $P(-1, 1, -\sqrt{2})$  from Cartesian to spherical polar coordinates.

First we calculate that

$$\rho = \sqrt{x^2 + y^2 + z^2} = \sqrt{(-1)^2 + 1^2 + (-\sqrt{2})^2} = \sqrt{4} = 2.$$

Next we calculate that

$$\phi = \cos^{-1} \frac{z}{\rho} = \cos^{-1} \frac{-\sqrt{2}}{2} = \frac{3\pi}{4}$$

because we want  $\phi \in [0, \pi]$ .

$$\rho = \sqrt{x^2 + y^2 + z^2} \quad y = \rho \sin \phi \sin \theta \quad z = \rho \cos \phi$$

Finally we need a  $\theta$ .

$$\sin \theta = \frac{y}{\rho \sin \phi} = \frac{1}{2 \left( \frac{\sqrt{2}}{2} \right)} = \frac{1}{\sqrt{2}}.$$

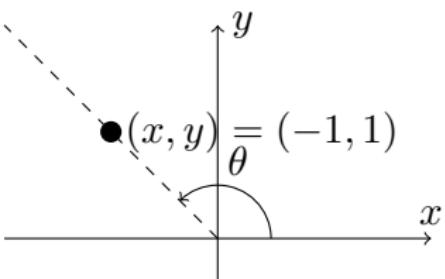
There are infinitely many  $\theta$  that satisfy this equation. Two possible  $\theta$  are  $\theta = \frac{\pi}{4}$  and  $\theta = \frac{3\pi}{4}$ .

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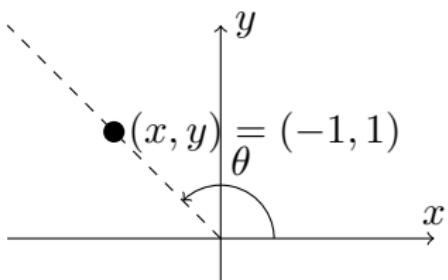
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Finally we need a  $\theta$ .

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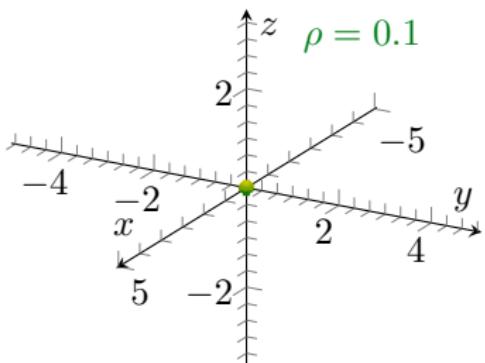
There are infinitely many  $\theta$  that satisfy this equation. Two possible  $\theta$  are  $\theta = \frac{\pi}{4}$  and  $\theta = \frac{3\pi}{4}$ . Only one of these can be correct.



Therefore the answer is

$$(\rho, \theta, \phi) = \left( 2, \frac{3\pi}{4}, \frac{3\pi}{4} \right).$$

## 14.7 Triple Integrals in Cylindrical and Spherical Coordinates

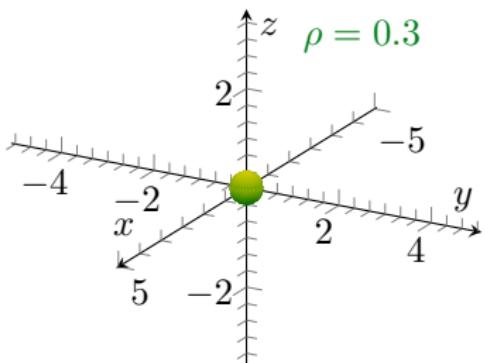


### Remark

Spherical coordinates are good for describing:

- spheres centred at the origin ( $\rho = \rho_0$ );
- cones (with vertex at the origin and axis on the  $z$ -axis) ( $\phi = \phi_0$ ); and
- half planes containing the  $z$ -axis ( $\theta = \theta_0$ ).

## 14.7 Triple Integrals in Cylindrical and Spherical Coordinates

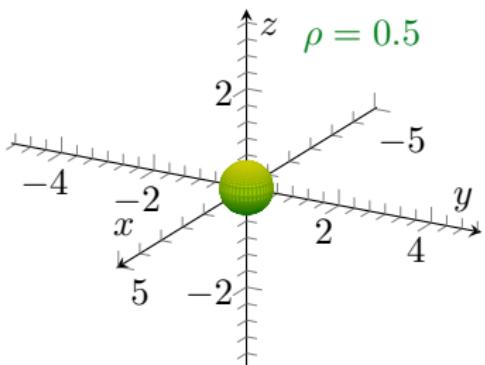


### Remark

Spherical coordinates are good for describing:

- spheres centred at the origin ( $\rho = \rho_0$ );
- cones (with vertex at the origin and axis on the  $z$ -axis) ( $\phi = \phi_0$ ); and
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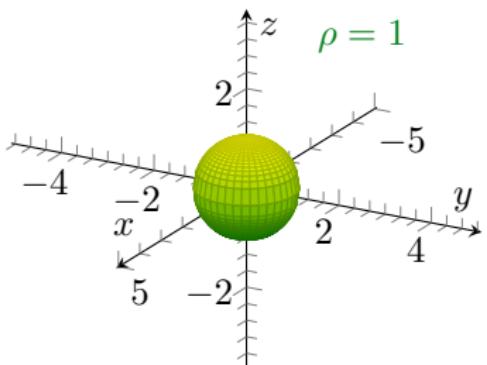


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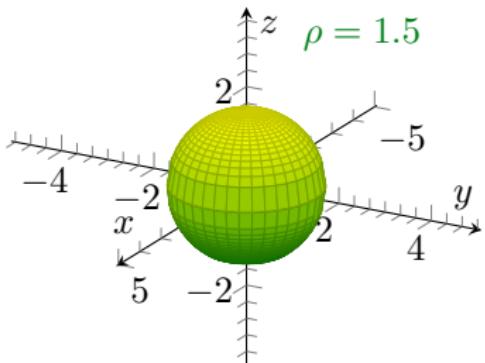


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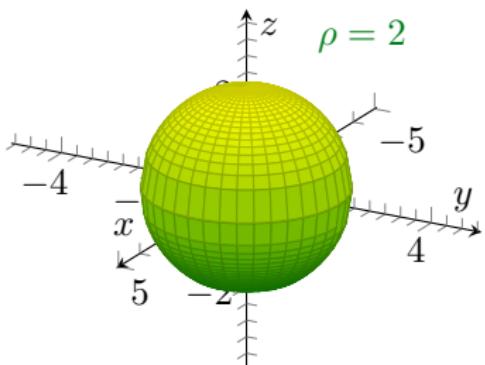


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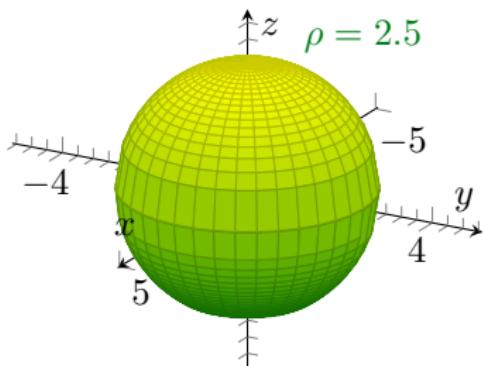


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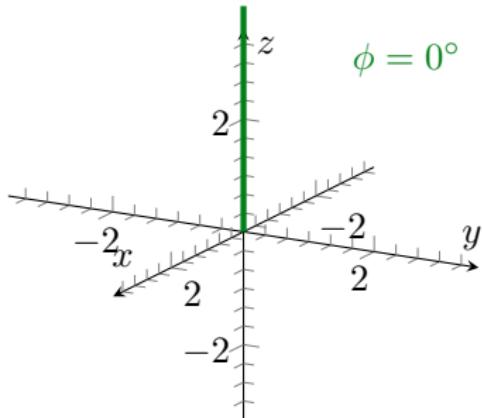


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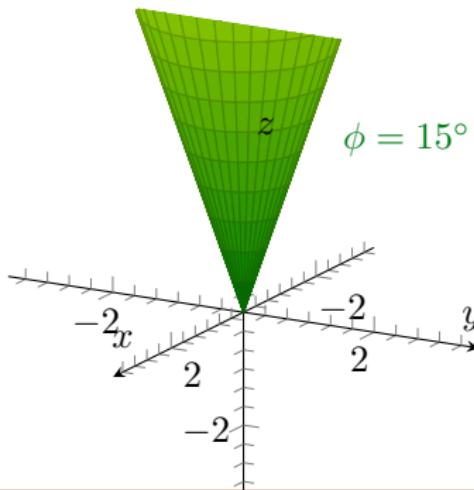


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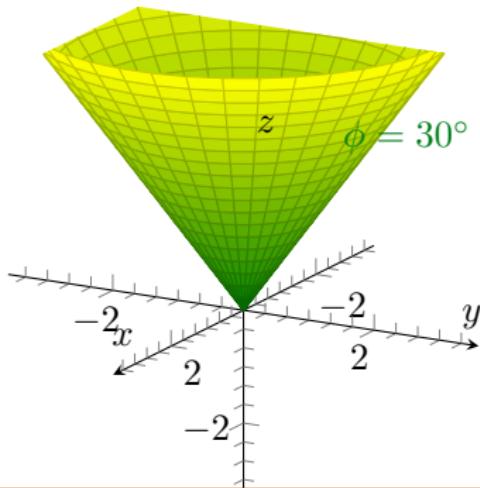


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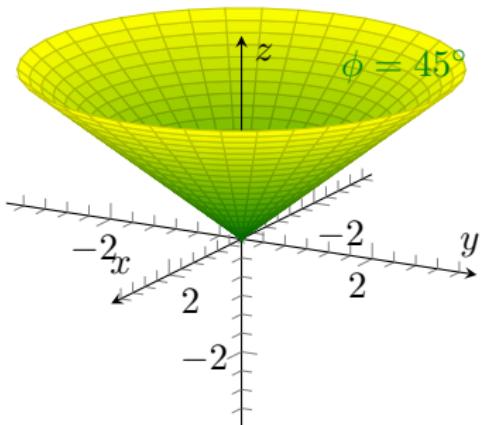


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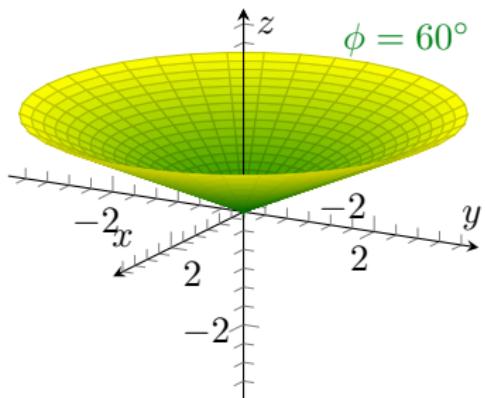


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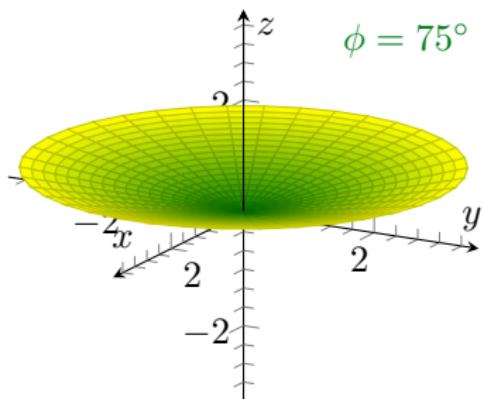


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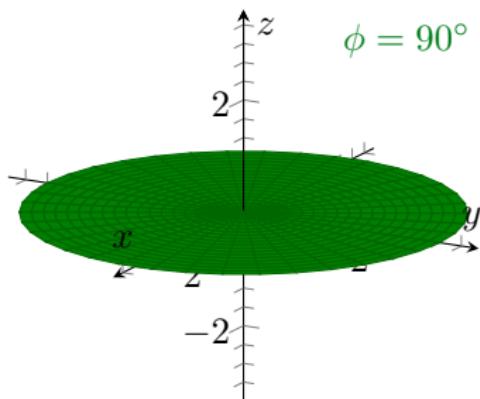


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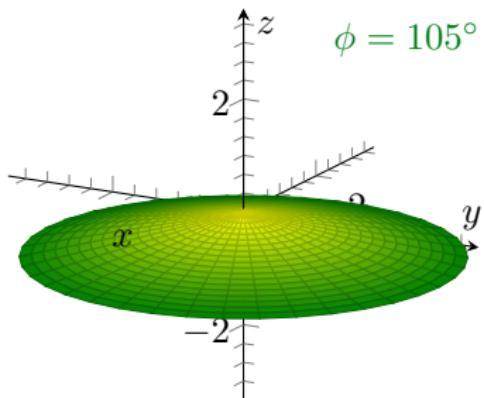


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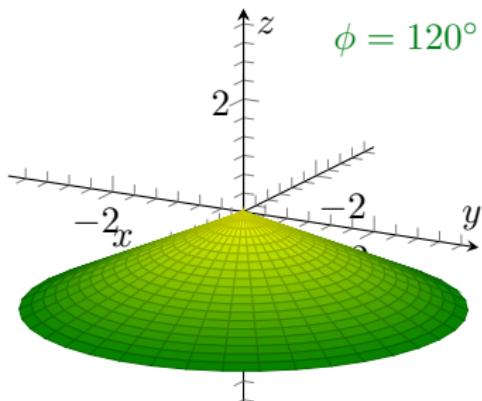


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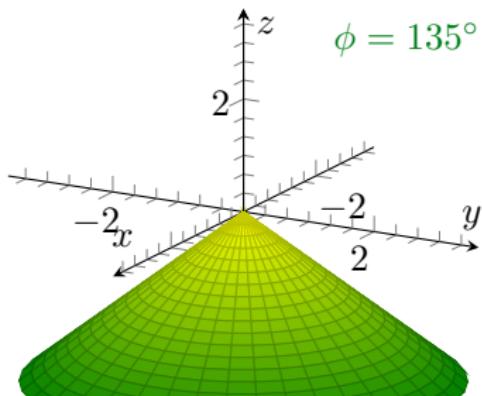


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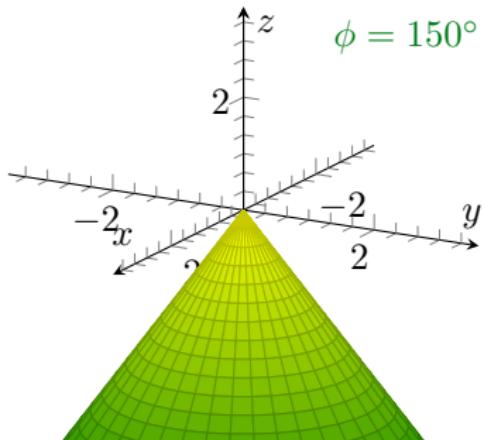


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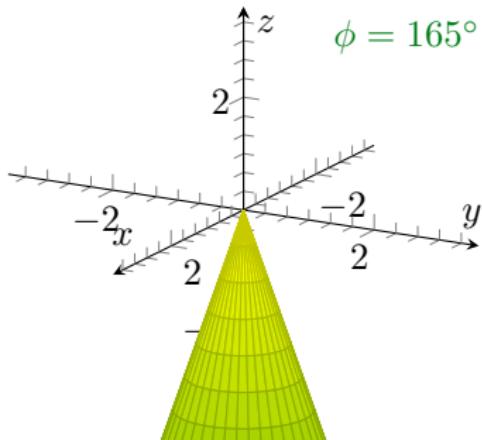


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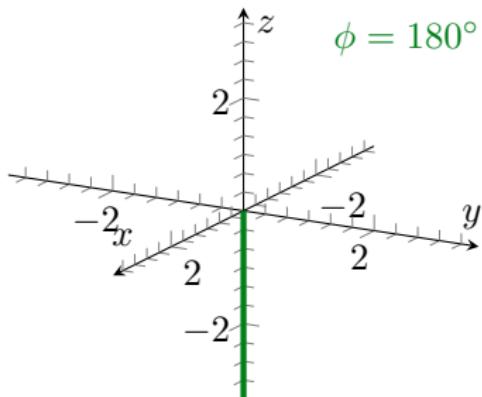


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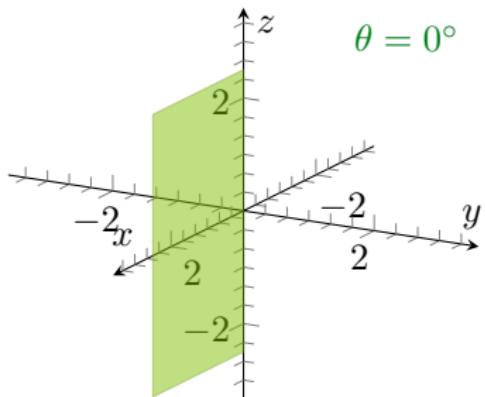


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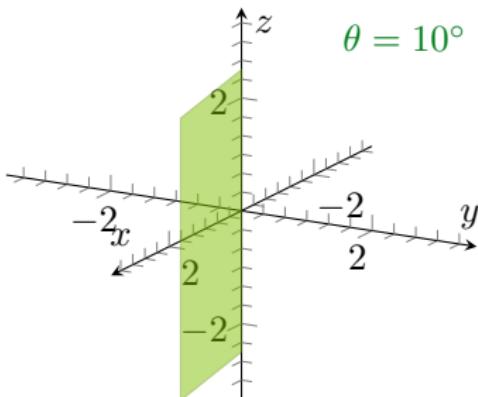


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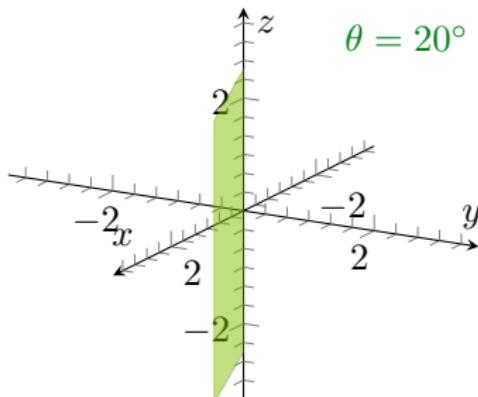


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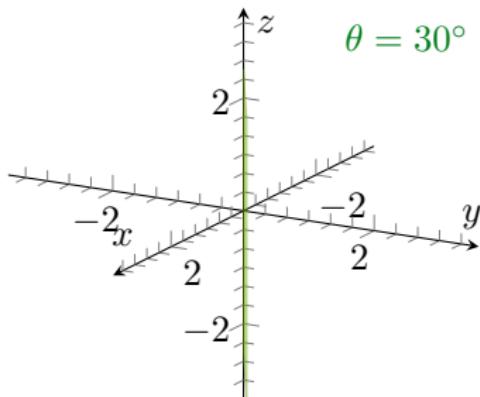


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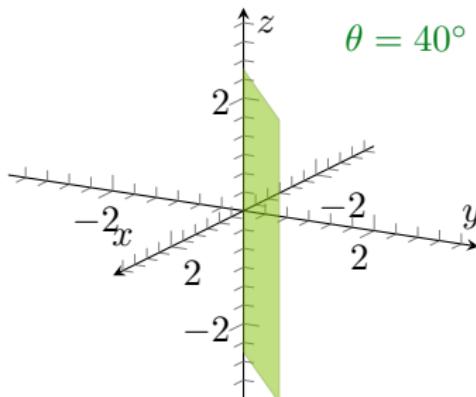


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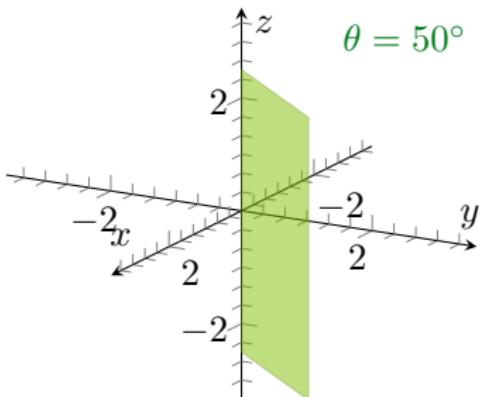


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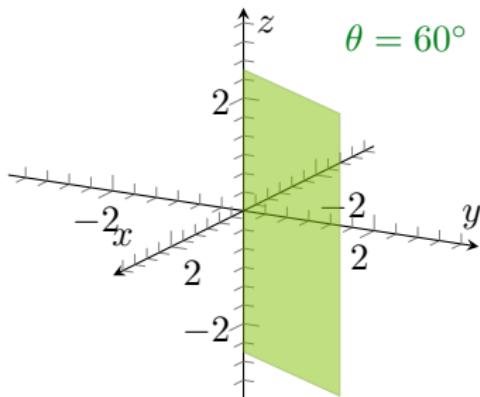


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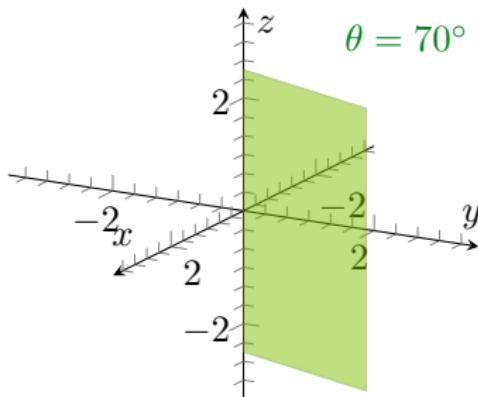


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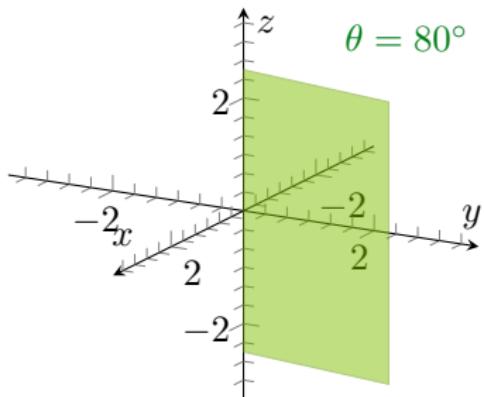


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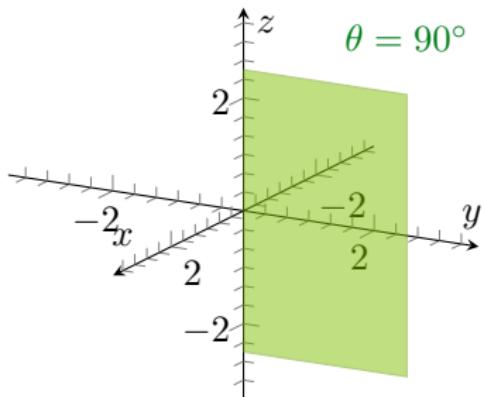


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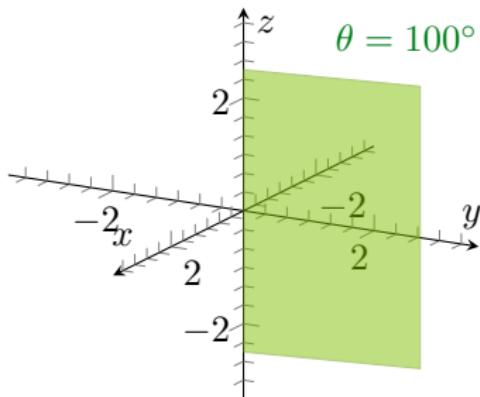


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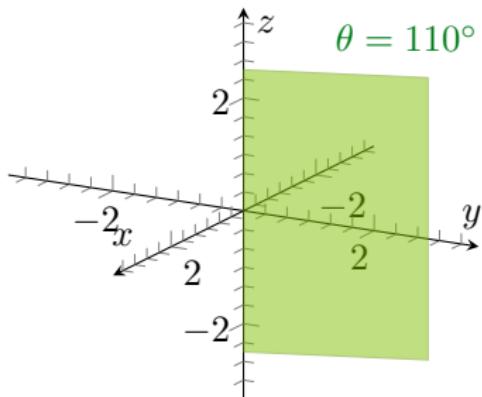


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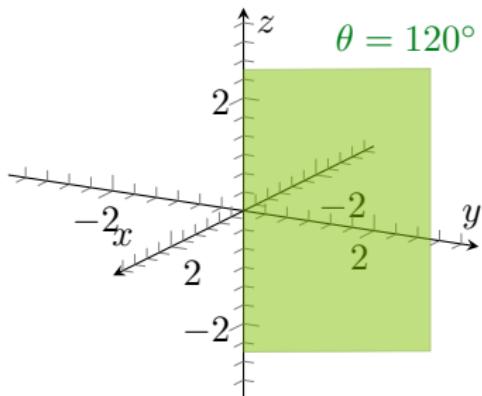


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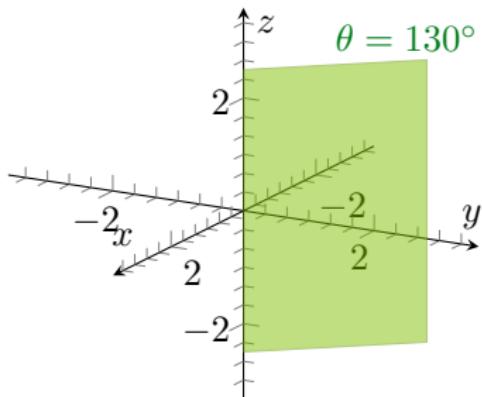


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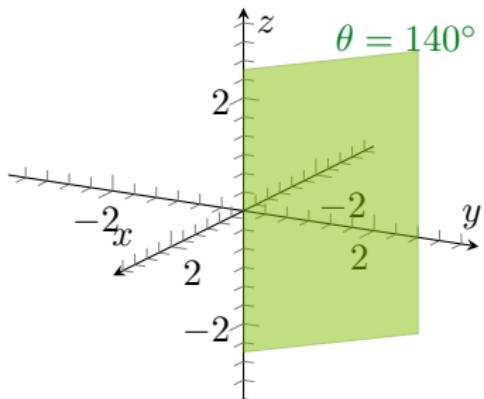


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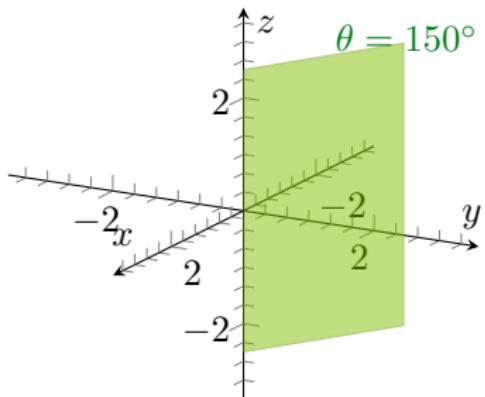


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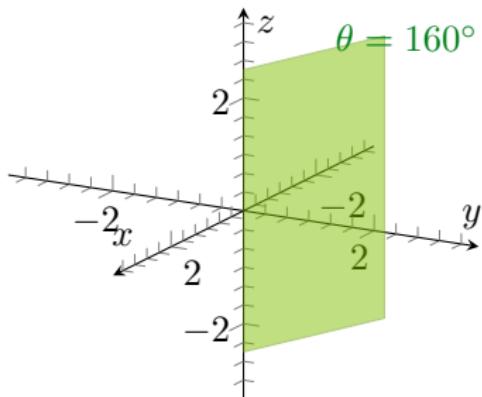


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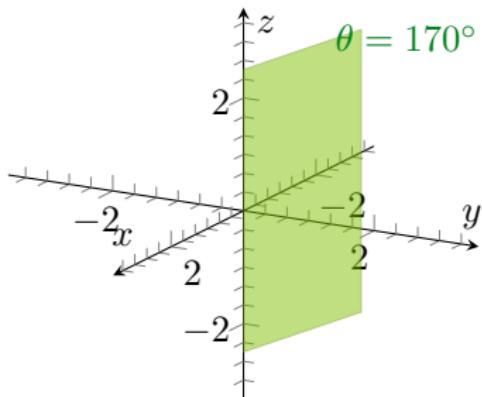


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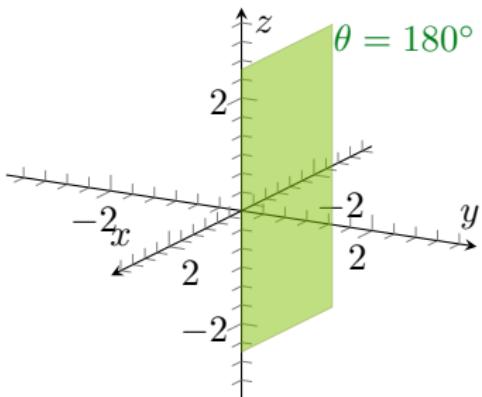


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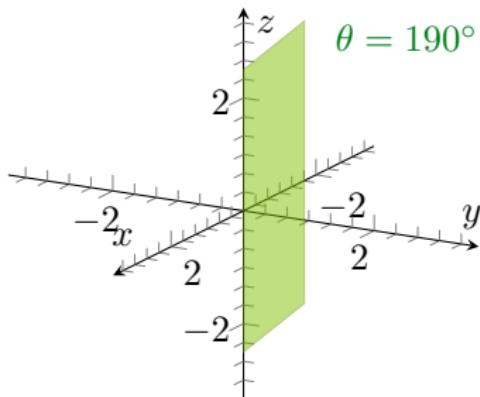


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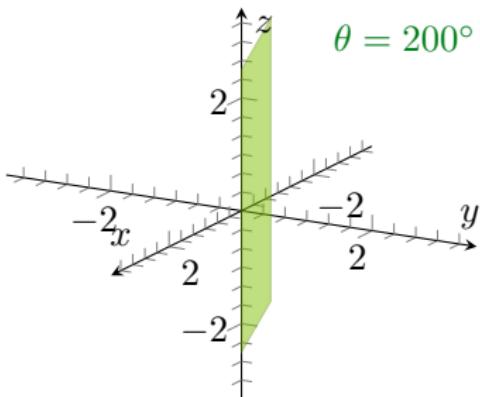


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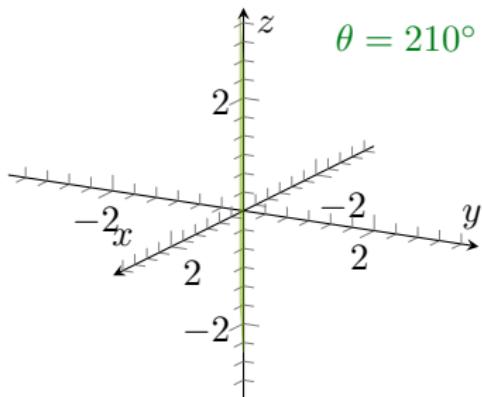


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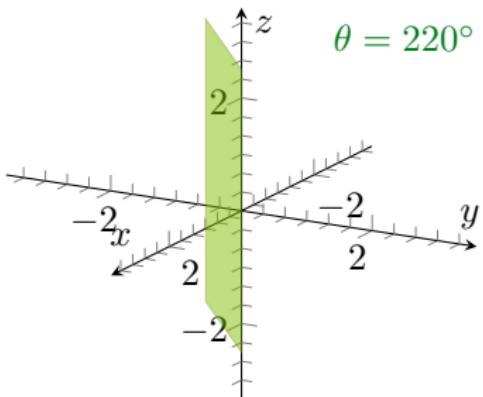


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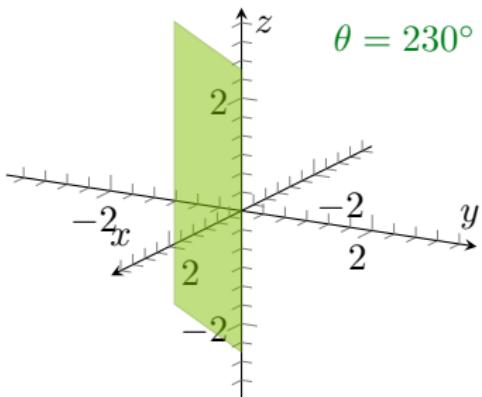


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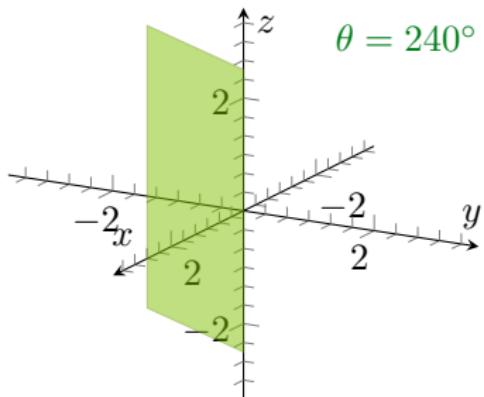


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Spherical coordinates are good for describing:

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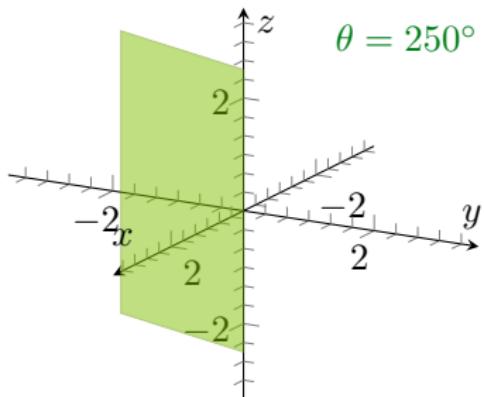


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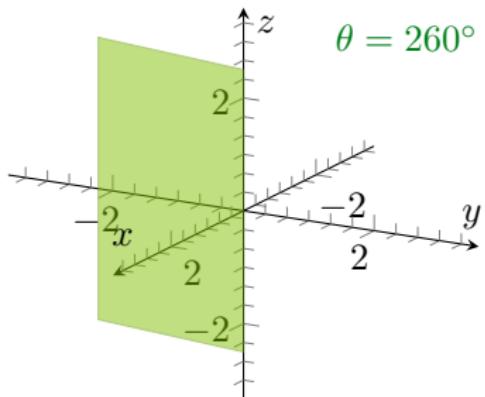


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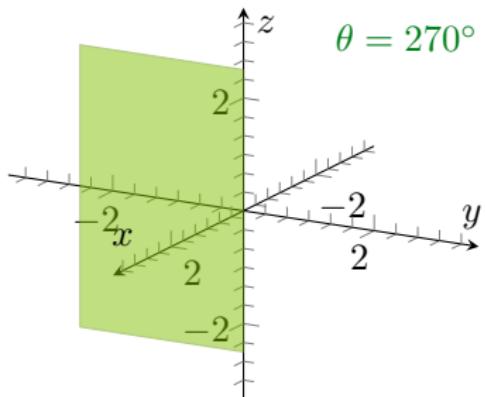


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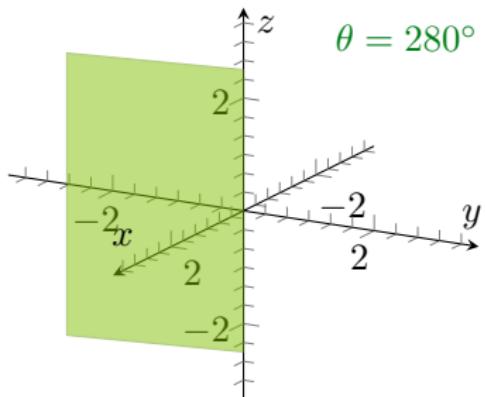


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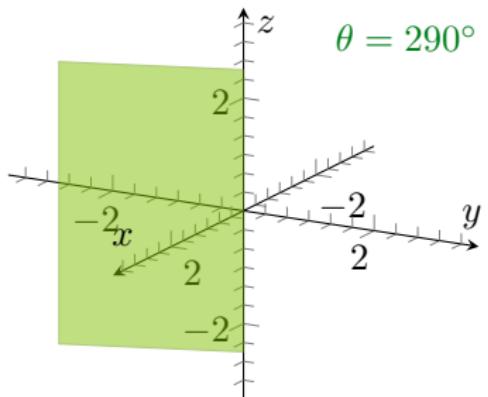


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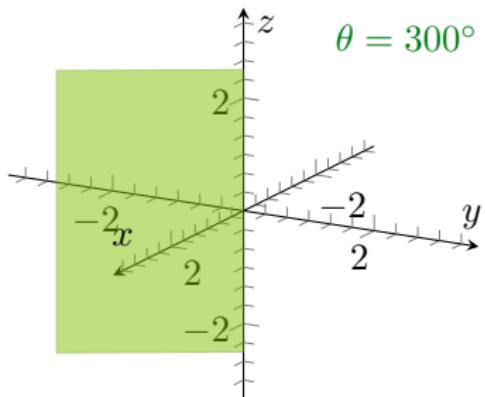


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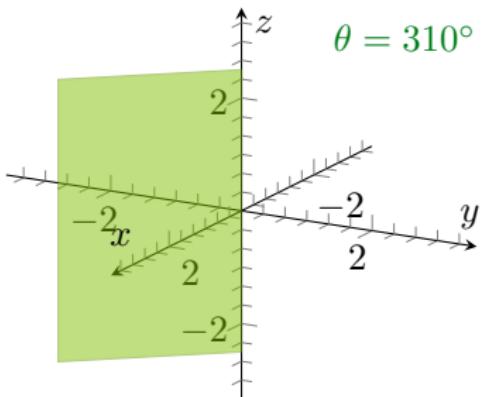


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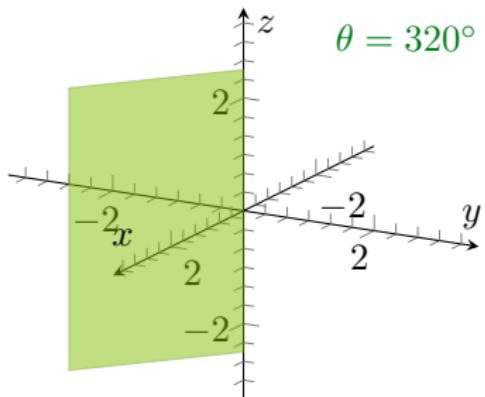


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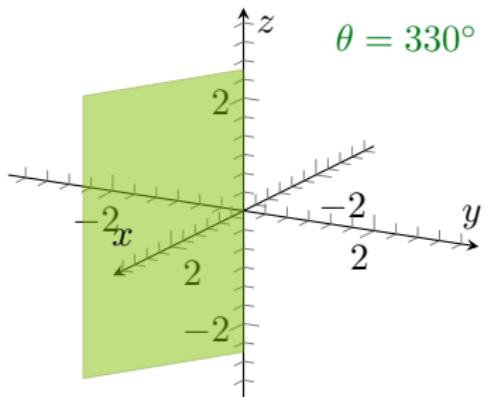


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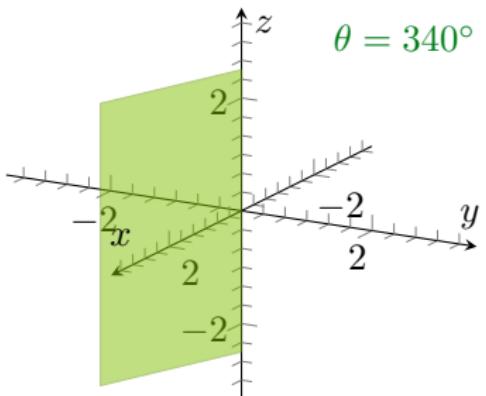


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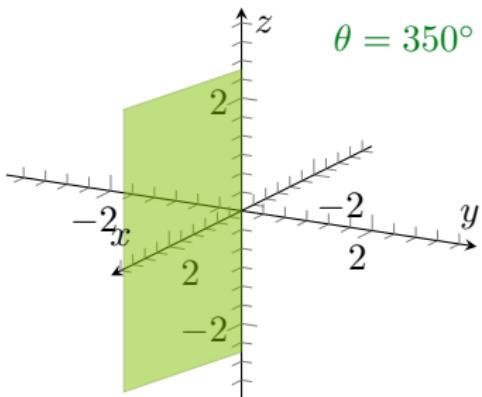


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## 14.7 Triple Integrals in Cylindrical and Spherical Coordinates

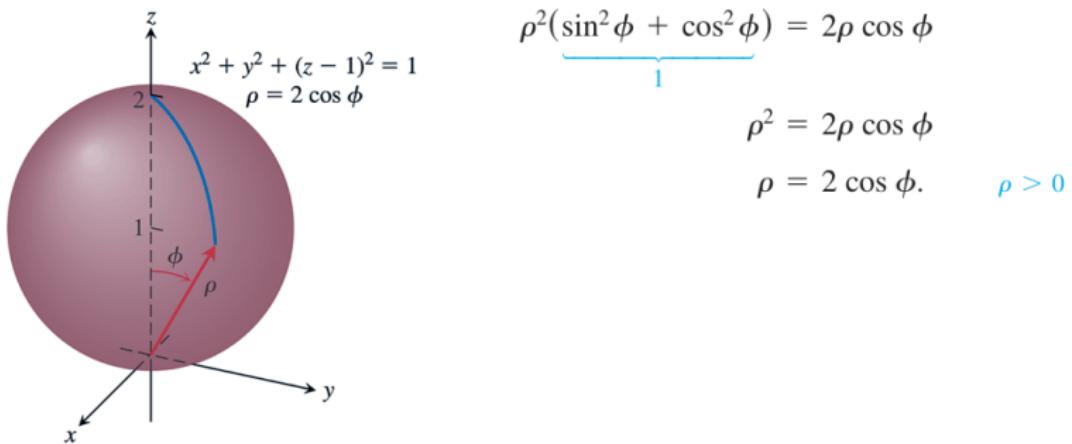


$$x = \rho \sin \phi \cos \theta \quad y = \rho \sin \phi \sin \theta \quad z = \rho \cos \phi$$

**EXAMPLE 3** Find a spherical coordinate equation for the sphere  $x^2 + y^2 + (z - 1)^2 = 1$ .

**Solution** We use Equations (1) to substitute for  $x$ ,  $y$ , and  $z$ :

$$\begin{aligned} x^2 + y^2 + (z - 1)^2 &= 1 \\ \rho^2 \sin^2 \phi \cos^2 \theta + \rho^2 \sin^2 \phi \sin^2 \theta + (\rho \cos \phi - 1)^2 &= 1 && \text{Eqs. (1)} \\ \underbrace{\rho^2 \sin^2 \phi (\cos^2 \theta + \sin^2 \theta) + \rho^2 \cos^2 \phi - 2\rho \cos \phi + 1}_{1} &= 1 \end{aligned}$$



$$\begin{aligned} \rho^2 (\sin^2 \phi + \cos^2 \phi) &= 2\rho \cos \phi \\ \underbrace{\rho^2}_{1} &= 2\rho \cos \phi \\ \rho &= 2 \cos \phi. && \rho > 0 \end{aligned}$$

## 14.7 Triple Integrals in Cylindrical and Spherical Coordinates



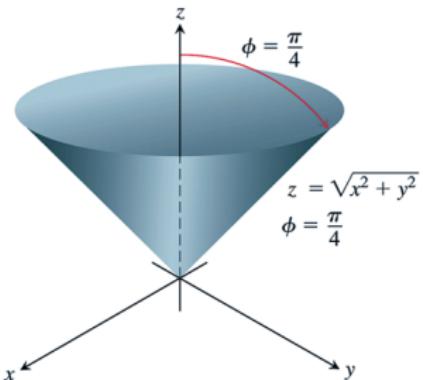
### Coordinate Equations

$$x = \rho \sin \phi \cos \theta \quad y = \rho \sin \phi \sin \theta \quad z = \rho \cos \phi$$

**EXAMPLE 4** Find a spherical coordinate equation for the cone  $z = \sqrt{x^2 + y^2}$ .

**Solution 1** *Use geometry.* The cone is symmetric with respect to the  $z$ -axis and cuts the first quadrant of the  $yz$ -plane along the line  $z = y$ . The angle between the cone and the positive  $z$ -axis is therefore  $\pi/4$  radians. The cone consists of the points whose spherical coordinates have  $\phi$  equal to  $\pi/4$ , so its equation is  $\phi = \pi/4$ . (See Figure 15.54.)

**Solution 2** *Use algebra.* If we use Equations (1) to substitute for  $x$ ,  $y$ , and  $z$  we obtain the same result:



$$z = \sqrt{x^2 + y^2}$$

$$\rho \cos \phi = \sqrt{\rho^2 \sin^2 \phi}$$

$$\rho \cos \phi = \rho \sin \phi$$

$$\cos \phi = \sin \phi$$

$$\phi = \frac{\pi}{4}.$$

Example 3

$\rho > 0, \sin \phi \geq 0$

$0 \leq \phi \leq \pi$



## 14.7 Triple Integrals in Cylindrical and Spherical Coordinates



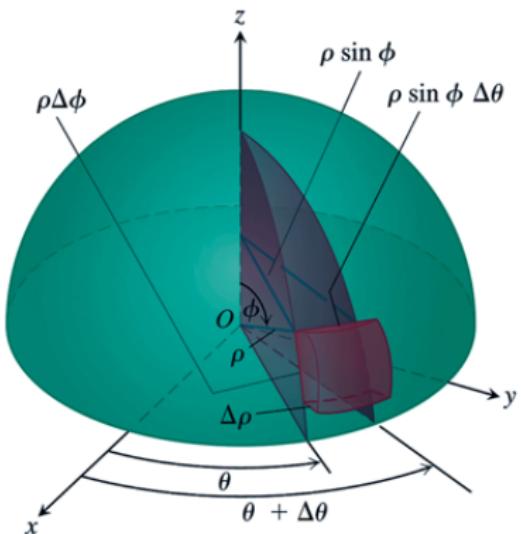
Theorem

$$dV = dx dy dz = r dr d\theta dz = .$$

# 14.7 Triple Integrals in Cylindrical and Spherical Coordinates

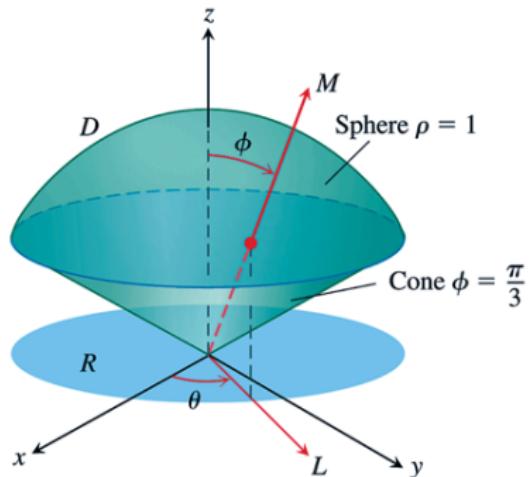
## Theorem

$$dV = dxdydz = r \, dr d\theta dz = \rho^2 \sin \phi \, d\rho d\phi d\theta.$$



## 14.7 Triple Integrals in Cylindrical and Spherical Coordinates

$$dV = \rho^2 \sin \phi \, d\rho d\phi d\theta$$

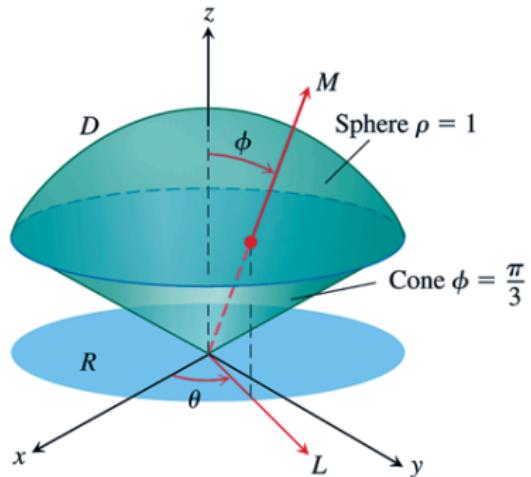


### Example

Calculate the volume of the region enclosed by the sphere  $\rho = 1$  and the cone  $\phi = \frac{\pi}{3}$ .

## 14.7 Triple Integrals in Cylindrical and Spherical Coordinates

$$dV = \rho^2 \sin \phi \, d\rho d\phi d\theta$$



$$0 \leq \theta \leq 2\pi$$

$$0 \leq \phi \leq \frac{\pi}{3}$$

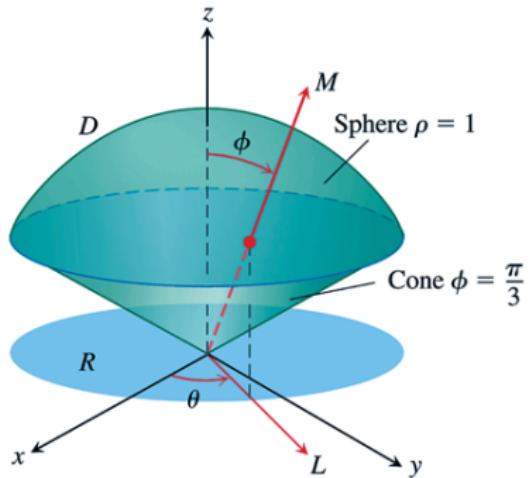
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## 14.7 Triple Integrals in Cylindrical and Spherical Coordinates

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Calculate the volume of the region enclosed by the sphere  $\rho = 1$  and the cone  $\phi = \frac{\pi}{3}$ .

$$\iiint_D dV = \int_0^{2\pi} \int_0^{\frac{\pi}{3}} \int_0^1 \rho^2 \sin \phi \, d\rho d\phi d\theta = \dots = \frac{\pi}{3}.$$

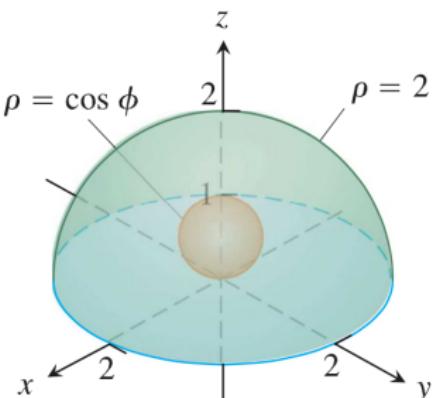
# 14.7 Triple Integrals in Cylindrical and Spherical Coordinates

## Example

Find the limits of integration for the region shown below: The region is bounded by  $\rho = 2$  and the  $xy$ -plane, but the sphere  $\rho = \cos \phi$  has been removed from this large hemisphere.

A     $0 \leq \theta \leq 2\pi$   
 $0 \leq \phi \leq \pi$   
 $0 \leq \rho \leq 2$

B     $0 \leq \theta \leq 2\pi$   
 $0 \leq \phi \leq \frac{\pi}{2}$   
 $0 \leq \rho \leq 2$



C     $0 \leq \theta \leq 2\pi$   
 $0 \leq \phi \leq \frac{\pi}{2}$   
 $\cos \phi \leq \rho \leq 2$

D     $0 \leq \theta \leq \pi$   
 $0 \leq \phi \leq 2$   
 $2 \leq \rho \leq \cos \phi$

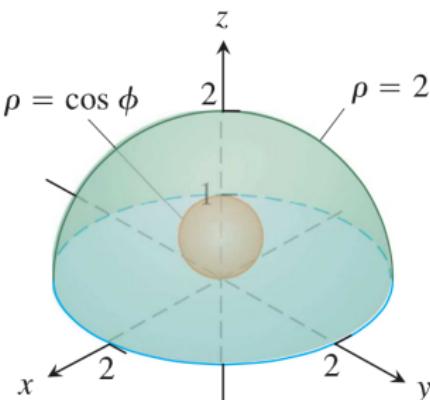
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 $0 \leq \phi \leq \frac{\pi}{2}$   
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C     $0 \leq \theta \leq 2\pi$   
 $0 \leq \phi \leq \frac{\pi}{2}$   
 $\cos \phi \leq \rho \leq 2$

D     $0 \leq \theta \leq \pi$   
 $0 \leq \phi \leq 2$   
 $2 \leq \rho \leq \cos \phi$

## Coordinate Conversion Formulas

### CYLINDRICAL TO RECTANGULAR

$$x = r \cos \theta$$

$$y = r \sin \theta$$

$$z = z$$

### Spherical to Rectangular

$$x = \rho \sin \phi \cos \theta$$

$$y = \rho \sin \phi \sin \theta$$

$$z = \rho \cos \phi$$

### Spherical to Cylindrical

$$r = \rho \sin \phi$$

$$z = \rho \cos \phi$$

$$\theta = \theta$$

Corresponding formulas for  $dV$  in triple integrals:

$$\begin{aligned} dV &= dx \, dy \, dz \\ &= dz \, r \, dr \, d\theta \\ &= \rho^2 \sin \phi \, d\rho \, d\phi \, d\theta \end{aligned}$$



# Substitutions in Multiple Integrals



### Substitutions in Single Integrals

You know that if we write  $u = 2x + 3$  then  $du = 2 dx$  and

$$\int_0^1 2\sqrt{2x+3} dx = \int_3^5 \sqrt{u} du.$$



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$$\int_0^1 2\sqrt{2x+3} dx = \int_3^5 \sqrt{u} du.$$

### Substitutions in Double Integrals

We are going to do the same thing for substitutions in double integrals.

## 14.8 Substitutions in Multiple Integrals



Carl Gustav Jacob Jacobi

BORN

10 December 1804

DECEASED

18 February 1851

NATIONALITY

German

### Definition

The *Jacobian* of the coordinate transformation  $x = g(u, v)$ ,  $y = h(u, v)$  is

$$J(u, v) = \frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \frac{\partial x}{\partial u} \frac{\partial y}{\partial v} - \frac{\partial x}{\partial v} \frac{\partial y}{\partial u}.$$

## 14.8 Substitutions in Multiple

$$\frac{\partial(x, y)}{\partial(u, v)} = \frac{\partial x}{\partial u} \frac{\partial y}{\partial v} - \frac{\partial x}{\partial v} \frac{\partial y}{\partial u}$$



### Example

Find the Jacobian of the polar coordinate transformation  
 $x = r \cos \theta$  and  $y = r \sin \theta$ .

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$$\frac{\partial(x, y)}{\partial(r, \theta)} = \frac{\partial x}{\partial r} \frac{\partial y}{\partial \theta} - \frac{\partial x}{\partial \theta} \frac{\partial y}{\partial r}$$

=

=

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$$\begin{aligned}\frac{\partial(x, y)}{\partial(r, \theta)} &= \frac{\partial x}{\partial r} \frac{\partial y}{\partial \theta} - \frac{\partial x}{\partial \theta} \frac{\partial y}{\partial r} \\&= (\cos \theta) (r \cos \theta) - (-r \sin \theta) (\sin \theta) \\&= r.\end{aligned}$$

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### Remark

Remember that

$$dxdy = r drd\theta.$$

## 14.8 Substitutions in Multiple Integrals



### Theorem

Suppose that  $f(x, y)$  is continuous over the region  $R$ . Let  $G$  be the preimage of  $R$  under the transformation  $x = g(u, v)$ ,  $y = h(u, v)$ , which is assumed to be one-to-one on the interior of  $G$ . If the functions  $g$  and  $h$  have continuous first partial derivatives within the interior of  $G$ , then

$$\iint_R f(x, y) \, dxdy = \iint_G f(g(u, v), h(u, v)) \left| \frac{\partial(x, y)}{\partial(u, v)} \right| \, dudv.$$

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## 14.8 Substitutions in Multiple Integrals



### Example

Calculate

$$\int_0^4 \int_{\frac{y}{2}}^{\frac{y}{2}+1} \frac{2x - y}{2} dx dy$$

by applying the transformation

$$u = \frac{2x - y}{2}, \quad v = \frac{y}{2}$$

and integrating over an appropriate region in the  $uv$ -plane.

## 14.8 Substitutions in Multiple Integrals



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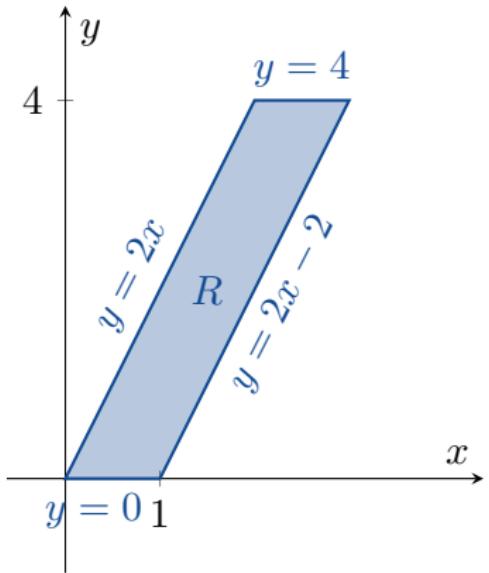
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and integrating over an appropriate region in the  $uv$ -plane.

$$0 \leq y \leq 4 \quad \text{and} \quad \frac{y}{2} \leq x \leq \frac{y}{2} + 1$$

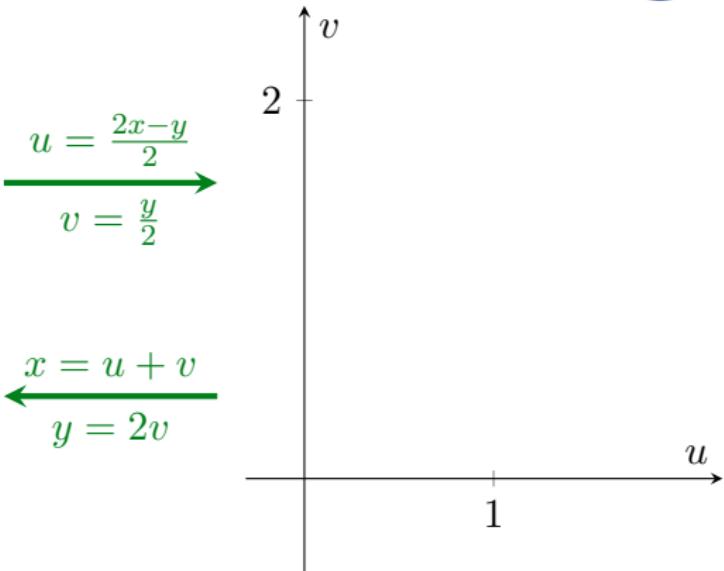
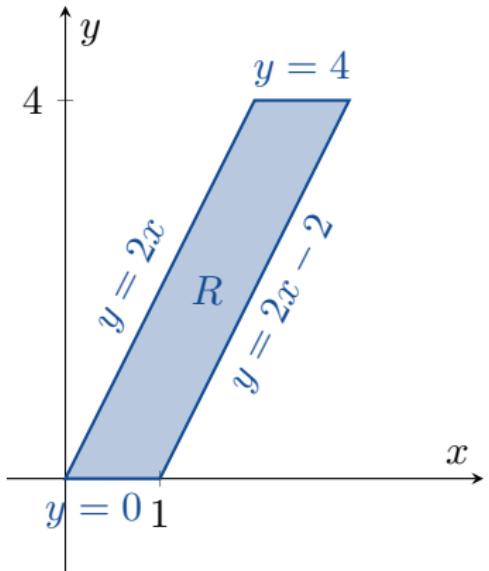
## 14.8 Substitutions in M

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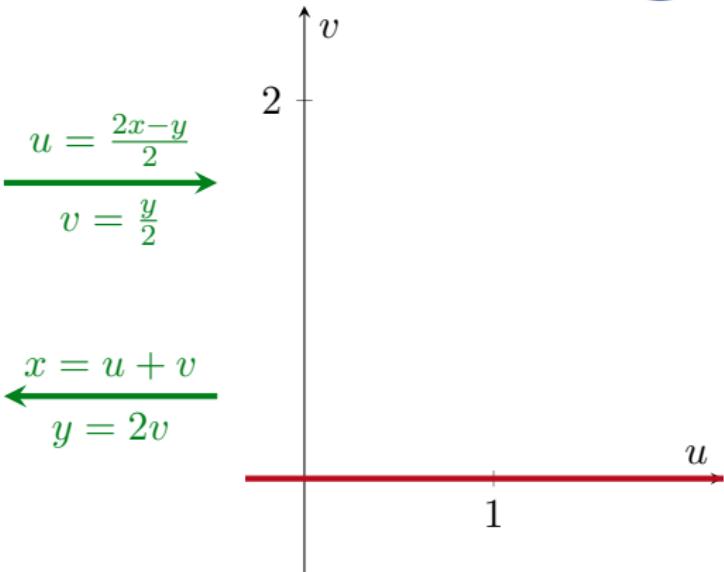
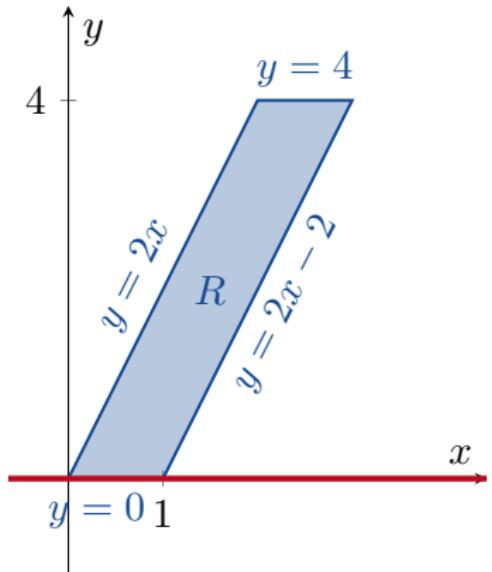
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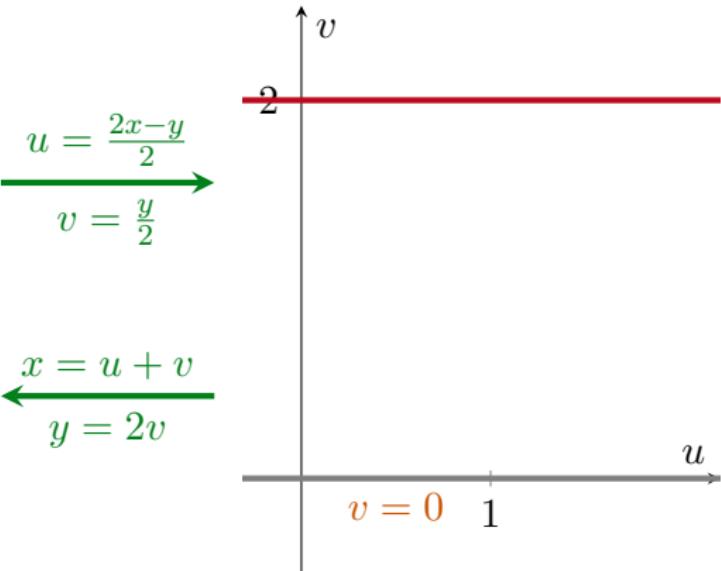
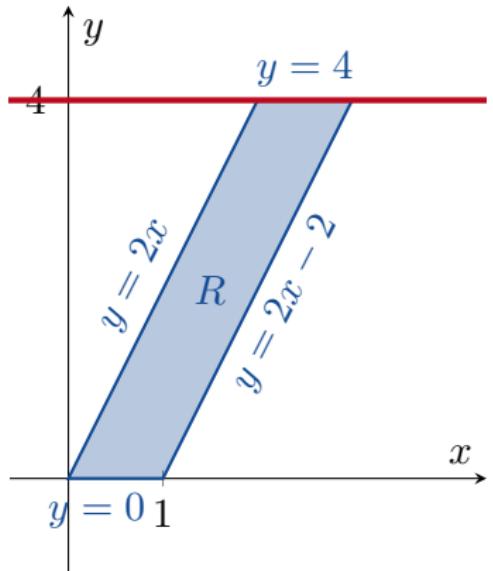
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$$y = 0 \quad \Rightarrow \quad v = \frac{y}{2} = 0$$

## 14.8 Substitutions in M

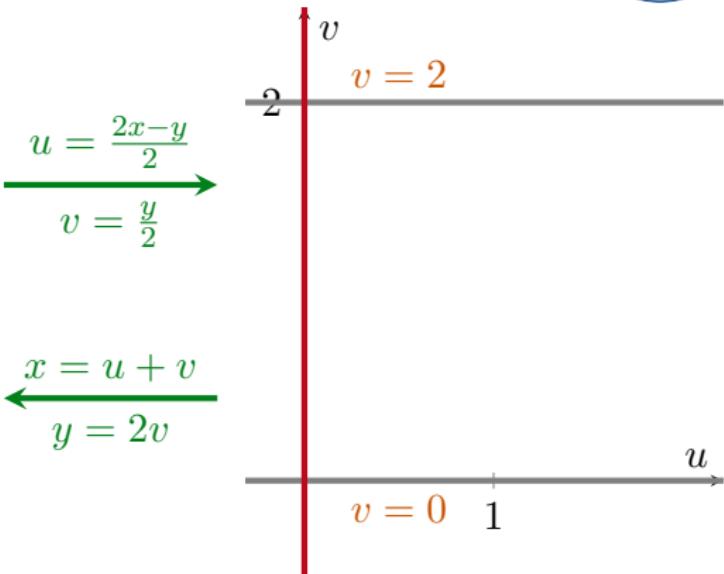
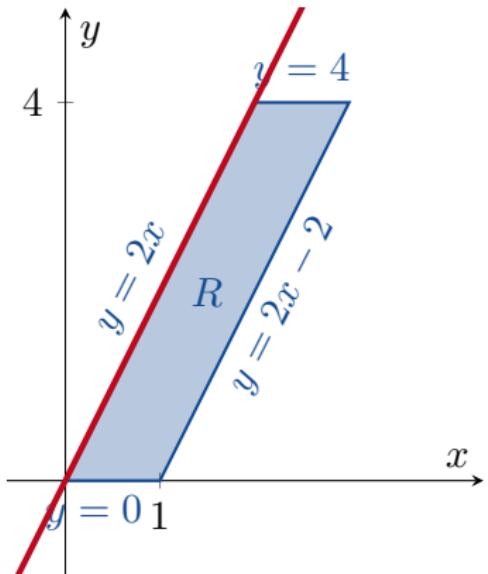
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$$y = 4 \quad \Rightarrow \quad v = \frac{y}{2} = 2$$

## 14.8 Substitutions in M

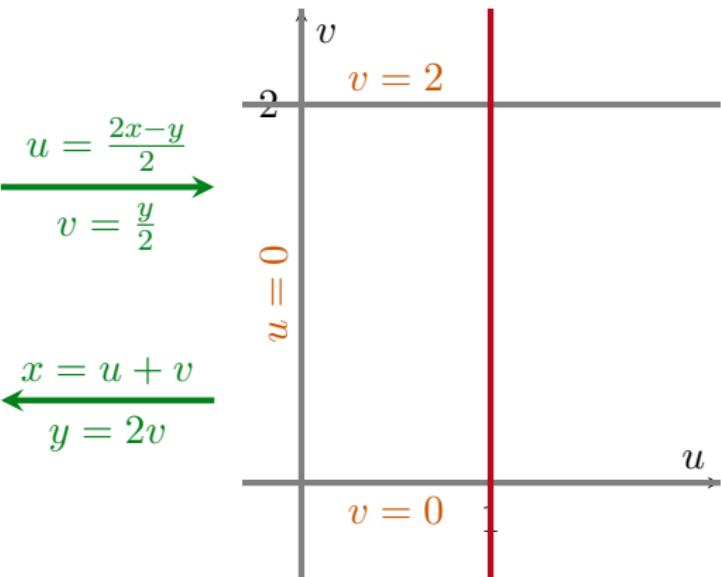
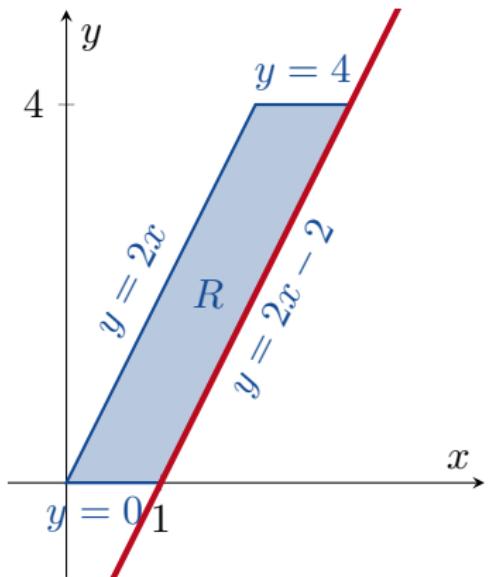
$$0 \leq y \leq 4 \quad \text{and} \quad \frac{y}{2} \leq x \leq \frac{y}{2} + 1$$



$$y = 2x \implies u = \frac{2x - y}{2} = \frac{2x - 2x}{2} = 0$$

## 14.8 Substitutions in M

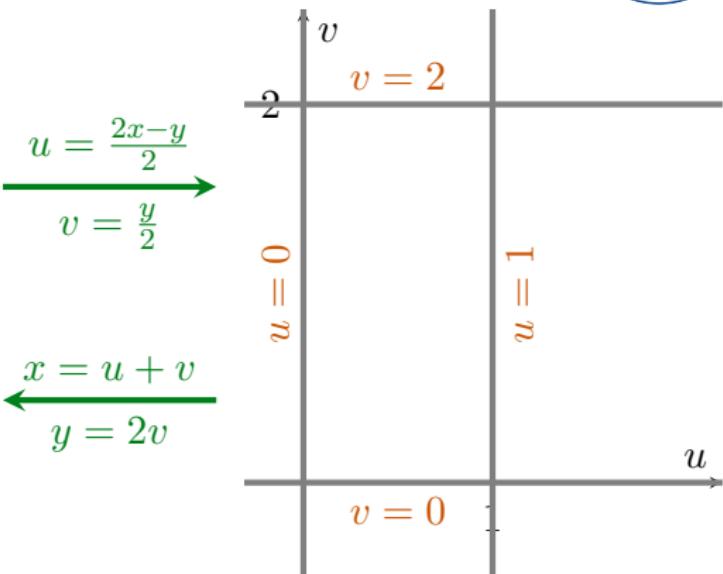
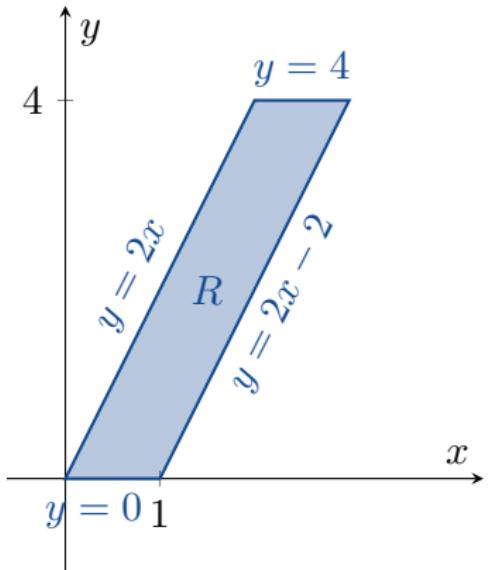
$$0 \leq y \leq 4 \quad \text{and} \quad \frac{y}{2} \leq x \leq \frac{y}{2} + 1$$



$$y = 2x - 2 \quad \Rightarrow \quad u = \frac{2x - y}{2} = \frac{2x - 2x + 2}{2} = 1$$

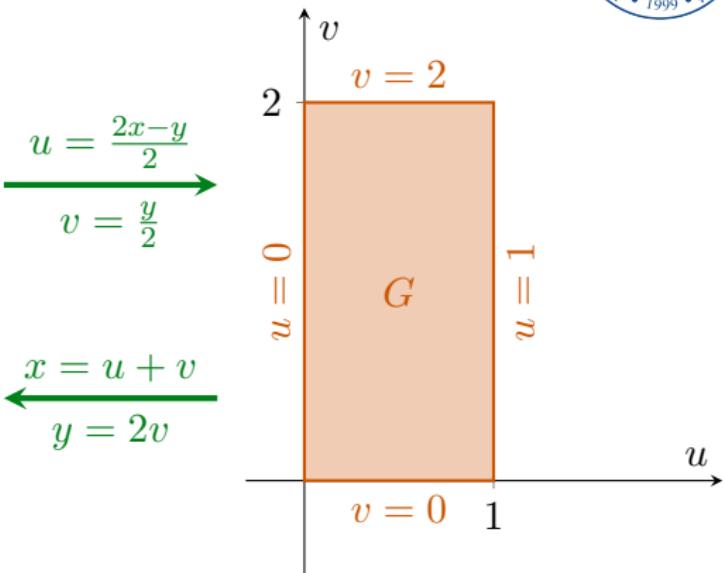
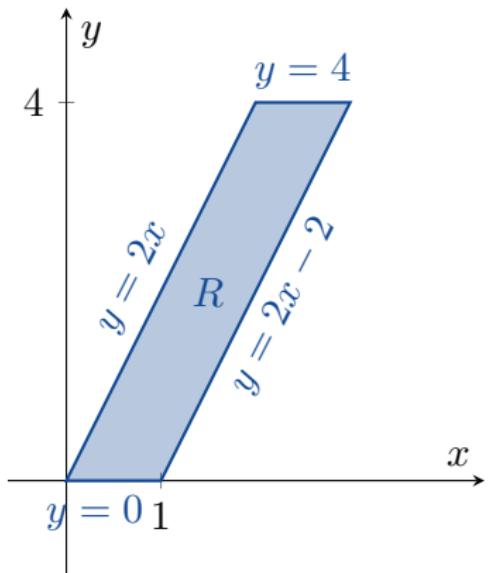
## 14.8 Substitutions in M

$$0 \leq y \leq 4 \quad \text{and} \quad \frac{y}{2} \leq x \leq \frac{y}{2} + 1$$



## 14.8 Substitutions in M

$$0 \leq y \leq 4 \quad \text{and} \quad \frac{y}{2} \leq x \leq \frac{y}{2} + 1$$



$$0 \leq u \leq 1 \quad \text{and} \quad 0 \leq v \leq 2$$

## 14.8 Substitutions in Multiple

$$\frac{\partial(x, y)}{\partial(u, v)} = \frac{\partial x}{\partial u} \frac{\partial y}{\partial v} - \frac{\partial x}{\partial v} \frac{\partial y}{\partial u}$$



$$x = u + v \quad \text{and} \quad y = 2v$$

Next we need the Jacobian of this coordinate transformation:

## 14.8 Substitutions in Multiple

$$\frac{\partial(x, y)}{\partial(u, v)} = \frac{\partial x}{\partial u} \frac{\partial y}{\partial v} - \frac{\partial x}{\partial v} \frac{\partial y}{\partial u}$$



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Next we need the Jacobian of this coordinate transformation:

$$\begin{aligned}\frac{\partial(x, y)}{\partial(u, v)} &= \frac{\partial x}{\partial u} \frac{\partial y}{\partial v} - \frac{\partial x}{\partial v} \frac{\partial y}{\partial u} \\ &= .\end{aligned}$$

## 14.8 Substitutions in Multiple

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$$x = u + v \quad \text{and} \quad y = 2v$$

Next we need the Jacobian of this coordinate transformation:

$$\begin{aligned}\frac{\partial(x, y)}{\partial(u, v)} &= \frac{\partial x}{\partial u} \frac{\partial y}{\partial v} - \frac{\partial x}{\partial v} \frac{\partial y}{\partial u} \\ &= (1)(2) - (1)(0) = 2.\end{aligned}$$

## 14.8 Substitutions in Multiple Integrals



$$u = \frac{2x - y}{2}, \quad v = \frac{y}{2}$$

$$0 \leq u \leq 1 \quad 0 \leq v \leq 2 \quad \frac{\partial(x, y)}{\partial(u, v)} = 2$$

Therefore

$$\int_0^4 \int_{\frac{y}{2}}^{\frac{y}{2}+1} \frac{2x - y}{2} dx dy =$$

## 14.8 Substitutions in Multiple Integrals



$$u = \frac{2x - y}{2}, \quad v = \frac{y}{2}$$

$$0 \leq u \leq 1 \quad 0 \leq v \leq 2 \quad \frac{\partial(x, y)}{\partial(u, v)} = 2$$

Therefore

$$\int_0^4 \int_{\frac{y}{2}}^{\frac{y}{2}+1} \frac{2x - y}{2} dx dy = \int_0^2 \int_0^1 u \left| \frac{\partial(x, y)}{\partial(u, v)} \right| du dv$$

## 14.8 Substitutions in Multiple Integrals



$$u = \frac{2x - y}{2}, \quad v = \frac{y}{2}$$

$$0 \leq u \leq 1 \quad 0 \leq v \leq 2 \quad \frac{\partial(x, y)}{\partial(u, v)} = 2$$

Therefore

$$\begin{aligned} \int_0^4 \int_{\frac{y}{2}}^{\frac{y}{2}+1} \frac{2x - y}{2} dx dy &= \int_0^2 \int_0^1 u \left| \frac{\partial(x, y)}{\partial(u, v)} \right| du dv \\ &= \int_0^2 \int_0^1 2u du dv \end{aligned}$$

## 14.8 Substitutions in Multiple Integrals



$$u = \frac{2x - y}{2}, \quad v = \frac{y}{2}$$

$$0 \leq u \leq 1 \quad 0 \leq v \leq 2 \quad \frac{\partial(x, y)}{\partial(u, v)} = 2$$

Therefore

$$\begin{aligned} \int_0^4 \int_{\frac{y}{2}}^{\frac{y}{2}+1} \frac{2x - y}{2} dx dy &= \int_0^2 \int_0^1 u \left| \frac{\partial(x, y)}{\partial(u, v)} \right| du dv \\ &= \int_0^2 \int_0^1 2u du dv = \dots = 2. \end{aligned}$$

## 14.8 Substitutions in Multiple Integrals



### Remark

To do a substitution, we need to do two things:

- 1 Calculate the Jacobian and write  $dxdy = \left| \frac{\partial(x, y)}{\partial(u, v)} \right| dudv$ ;

and

- 2 change the limits of integration.

## 14.8 Substitutions in Multiple

$$\frac{\partial(x, y)}{\partial(u, v)} = \frac{\partial x}{\partial u} \frac{\partial y}{\partial v} - \frac{\partial x}{\partial v} \frac{\partial y}{\partial u}$$



### Example

Calculate  $\int_0^1 \int_0^{1-x} \sqrt{x+y}(y-2x)^2 dy dx$ .

## 14.8 Substitutions in Multiple

$$\frac{\partial(x, y)}{\partial(u, v)} = \frac{\partial x}{\partial u} \frac{\partial y}{\partial v} - \frac{\partial x}{\partial v} \frac{\partial y}{\partial u}$$



### Example

Calculate  $\int_0^1 \int_0^{1-x} \sqrt{x+y}(y-2x)^2 dy dx$ .

First we need to choose  $u$  and  $v$ .

## 14.8 Substitutions in Multiple

$$\frac{\partial(x, y)}{\partial(u, v)} = \frac{\partial x}{\partial u} \frac{\partial y}{\partial v} - \frac{\partial x}{\partial v} \frac{\partial y}{\partial u}$$



### Example

Calculate  $\int_0^1 \int_0^{1-x} \sqrt{x+y}(y-2x)^2 dy dx.$

First we need to choose  $u$  and  $v$ . I choose

$$u = x + y \quad \text{and} \quad v = y - 2x.$$

## 14.8 Substitutions in Multiple

$$\frac{\partial(x, y)}{\partial(u, v)} = \frac{\partial x}{\partial u} \frac{\partial y}{\partial v} - \frac{\partial x}{\partial v} \frac{\partial y}{\partial u}$$



### Example

Calculate  $\int_0^1 \int_0^{1-x} \sqrt{x+y}(y-2x)^2 dy dx$ .

First we need to choose  $u$  and  $v$ . I choose

$$u = x + y \quad \text{and} \quad v = y - 2x.$$

We can rearrange these to

$$x = \frac{u}{3} - \frac{v}{3} \quad \text{and} \quad y = \frac{2u}{3} + \frac{v}{3}.$$

## 14.8 Substitutions in Multiple Integrals

$$\frac{\partial(x, y)}{\partial(u, v)} = \frac{\partial x}{\partial u} \frac{\partial y}{\partial v} - \frac{\partial x}{\partial v} \frac{\partial y}{\partial u}$$



### Example

Calculate  $\int_0^1 \int_0^{1-x} \sqrt{x+y}(y-2x)^2 dy dx$ .

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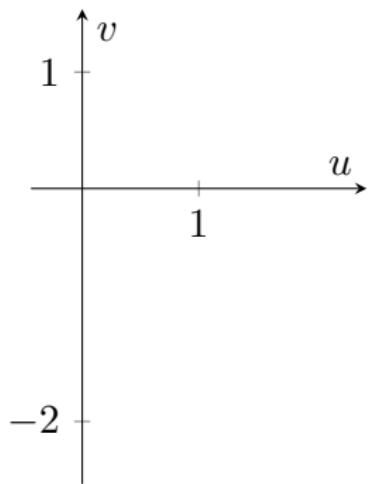
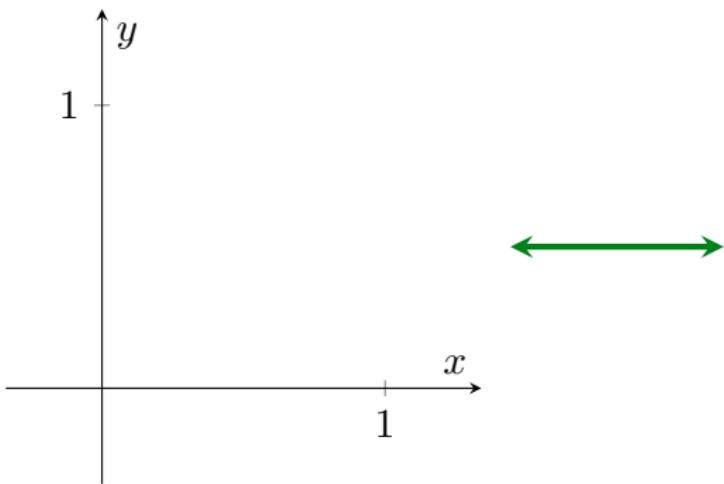
$$x = \frac{u}{3} - \frac{v}{3} \quad \text{and} \quad y = \frac{2u}{3} + \frac{v}{3}.$$

Then the Jacobian is

$$\frac{\partial(x, y)}{\partial(u, v)} = \frac{\partial x}{\partial u} \frac{\partial y}{\partial v} - \frac{\partial x}{\partial v} \frac{\partial y}{\partial u} = \left(\frac{1}{3}\right) \left(\frac{1}{3}\right) - \left(-\frac{1}{3}\right) \left(\frac{2}{3}\right) = \frac{1}{3}.$$

$$+ y \int_0^1 \int_0^{1-x} \sqrt{x+y}(y-2x)^2 dy dx$$

$$\begin{aligned}x &= \frac{u}{3} - \frac{v}{3} \\y &= \frac{2u}{3} + \frac{v}{3}\end{aligned}$$

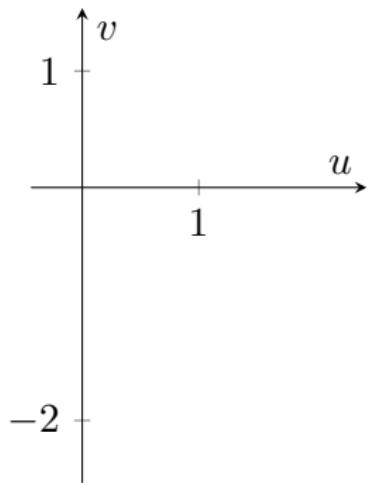
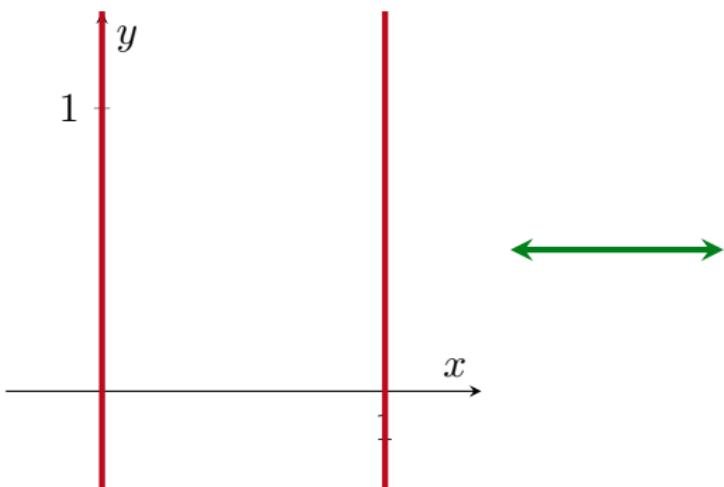


$$+ y \int_0^1 \int_0^{1-x} \sqrt{x+y}(y-2x)^2 dy dx \quad x = \frac{u}{3} - \frac{v}{3}$$

$$- 2x \quad y = \frac{2u}{3} + \frac{v}{3}$$

$$x = 0$$

$$x = 1$$



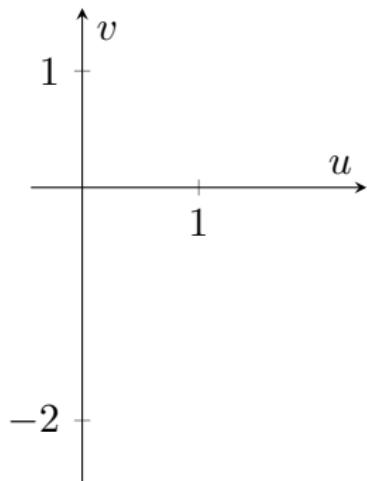
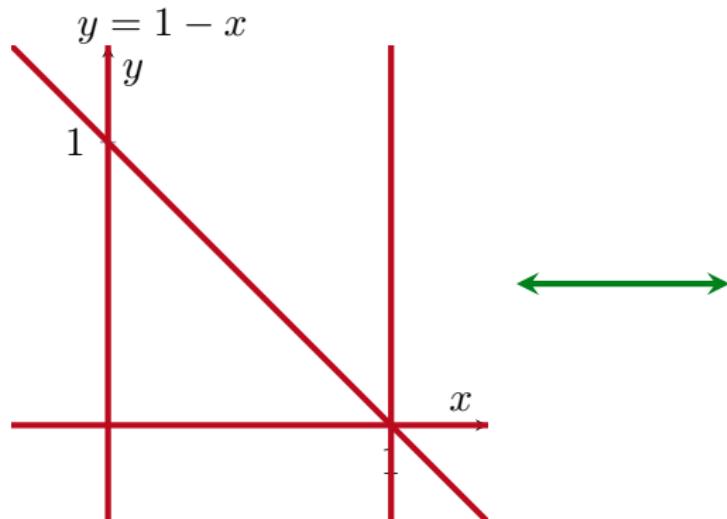
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$$x = 1$$

$$y = 0$$



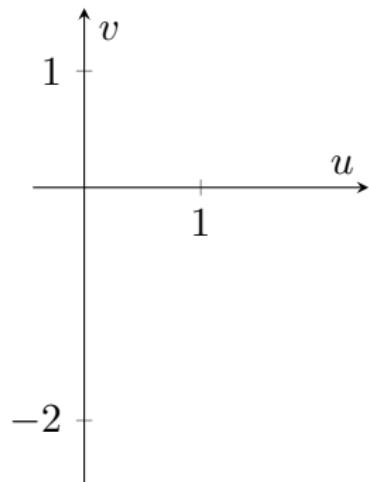
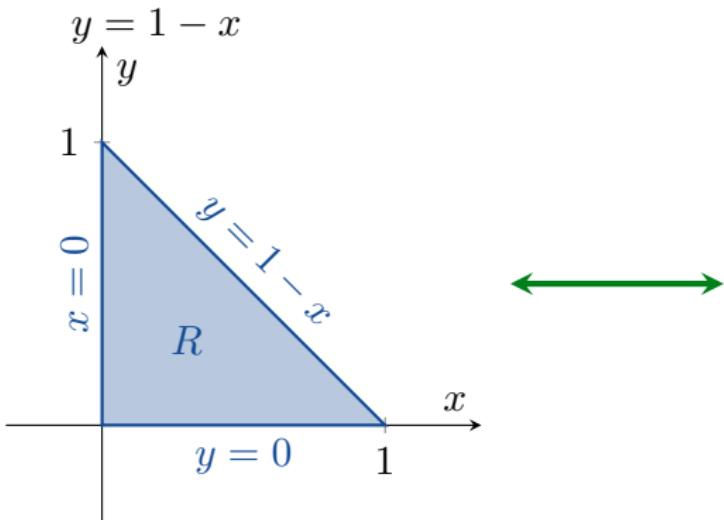
$$+ y \int_0^1 \int_0^{1-x} \sqrt{x+y}(y-2x)^2 dy dx \quad x = \frac{u}{3} - \frac{v}{3}$$

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~~$x = 1$~~

$$y = 0$$



$$+ y \int_0^1 \int_0^{1-x} \sqrt{x+y}(y-2x)^2 dy dx \quad x = \frac{u}{3} - \frac{v}{3}$$

$$- 2x \quad y = \frac{2u}{3} + \frac{v}{3}$$

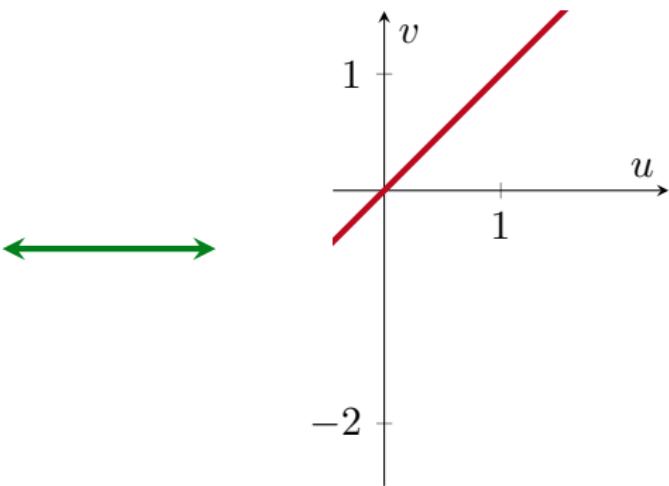
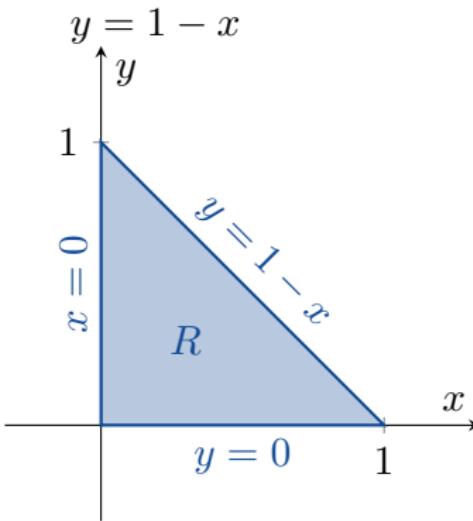
$$x = 0 \quad \Rightarrow$$

$$0 = \frac{u}{3} - \frac{v}{3} \quad \Rightarrow$$

$$u = v$$

~~$x = 1$~~

$$y = 0$$



$$+ y \int_0^1 \int_0^{1-x} \sqrt{x+y}(y-2x)^2 dy dx \quad x = \frac{u}{3} - \frac{v}{3}$$

$$- 2x \quad y = \frac{2u}{3} + \frac{v}{3}$$

$$x = 0 \quad \Rightarrow$$

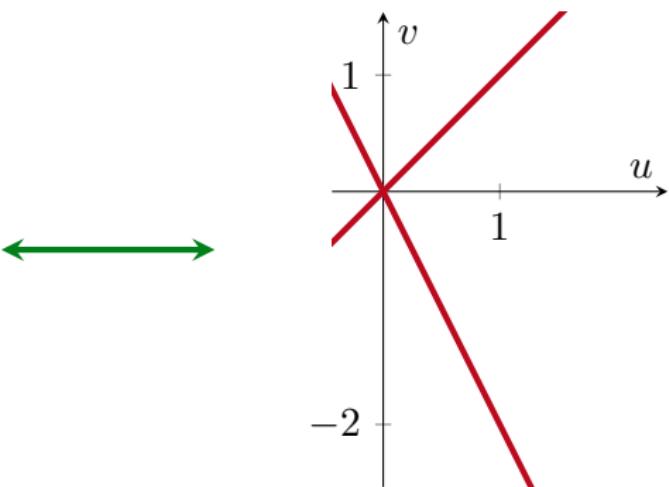
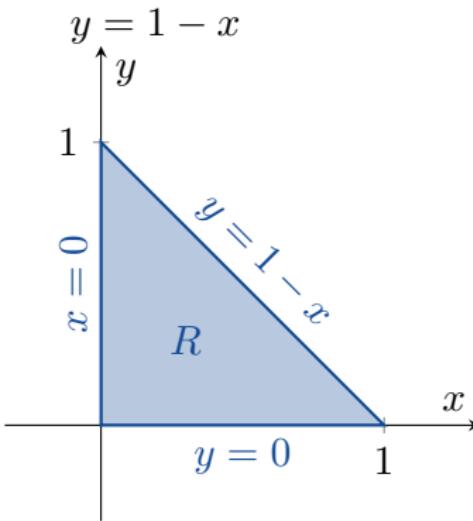
$$0 = \frac{u}{3} - \frac{v}{3} \quad \Rightarrow$$

$$u = v$$

~~$x = 1$~~

$$y = 0 \quad \Rightarrow$$

$$0 = \frac{2u}{3} + \frac{v}{3} \quad \Rightarrow \quad v = -2u$$



$$+ y \int_0^1 \int_0^{1-x} \sqrt{x+y}(y-2x)^2 dy dx \quad x = \frac{u}{3} - \frac{v}{3}$$

$$- 2x \quad y = \frac{2u}{3} + \frac{v}{3}$$

$$x = 0 \implies$$

$$0 = \frac{u}{3} - \frac{v}{3} \implies$$

$$u = v$$

~~$x = 1$~~

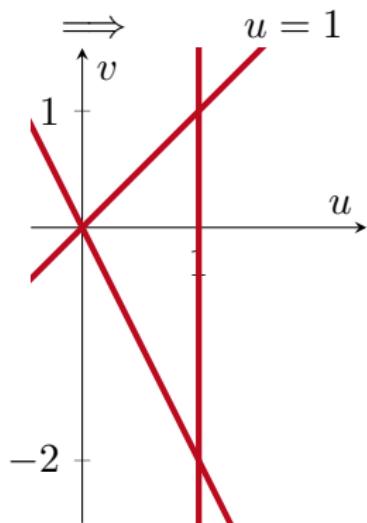
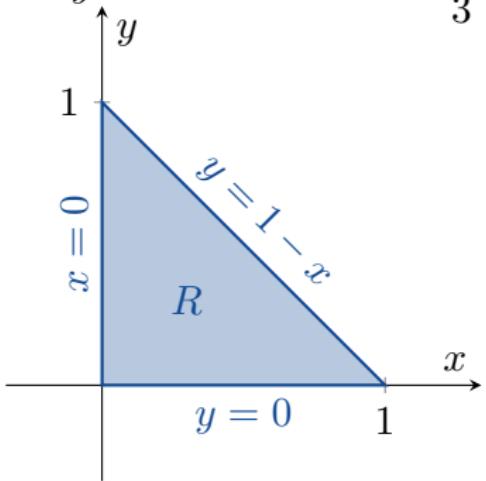
$$y = 0 \implies$$

$$0 = \frac{2u}{3} + \frac{v}{3} \implies v = -2u$$

$$y = 1 - x \implies \frac{2u}{3} + \frac{v}{3} = 1 - \frac{u}{3} + \frac{v}{3}$$

$$\implies$$

$$u = 1$$



$$+ y \int_0^1 \int_0^{1-x} \sqrt{x+y}(y-2x)^2 dy dx \quad x = \frac{u}{3} - \frac{v}{3}$$

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$$x = 0 \implies$$

$$0 = \frac{u}{3} - \frac{v}{3} \implies u = v$$

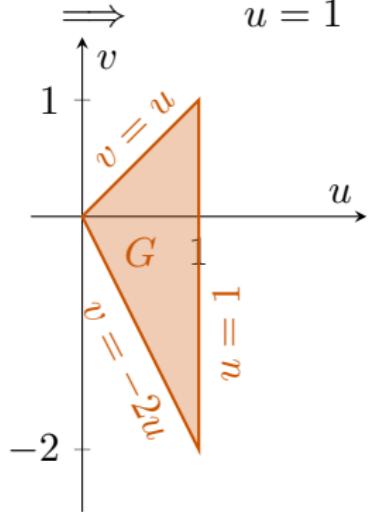
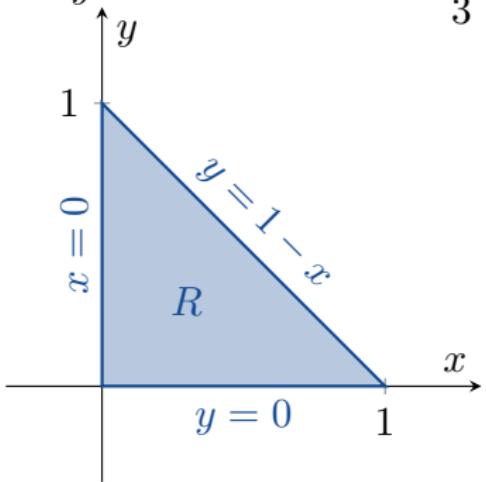
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$$\implies u = 1$$



$$+ y \int_0^1 \int_0^{1-x} \sqrt{x+y}(y-2x)^2 dy dx \quad x = \frac{u}{3} - \frac{v}{3}$$

$$- 2x \quad y = \frac{2u}{3} + \frac{v}{3}$$

$$x = 0 \quad \Rightarrow \quad 0 = \frac{u}{3} - \frac{v}{3} \quad \Rightarrow \quad u = v$$

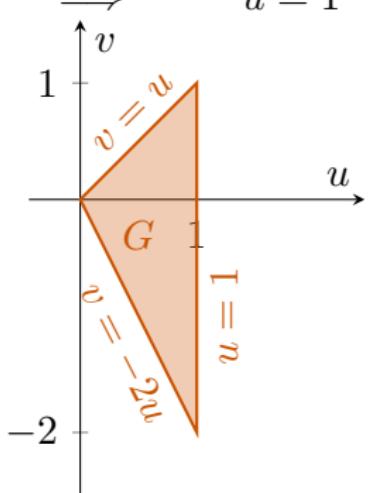
~~x = 1~~

$$y = 0 \quad \Rightarrow \quad 0 = \frac{2u}{3} + \frac{v}{3} \quad \Rightarrow \quad v = -2u$$

$$y = 1 - x \quad \Rightarrow \quad \frac{2u}{3} + \frac{v}{3} = 1 - \frac{u}{3} + \frac{v}{3} \quad \Rightarrow \quad u = 1$$

$$0 \leq u \leq 1$$

$$-2u \leq v \leq u$$



$$= x + y \quad \frac{\partial(x, y)}{\partial(u, v)} = \frac{1}{3} \quad 0 \leq u \leq 1 \quad -2u \leq v \leq u$$

$$= y - 2x$$

Therefore

$$\int_0^1 \int_0^{1-x} \sqrt{x+y}(y-2x)^2 dy dx =$$

=

=

=

=

=

$$= x + y \quad \frac{\partial(x, y)}{\partial(u, v)} = \frac{1}{3} \quad 0 \leq u \leq 1 \quad -2u \leq v \leq u$$

$$= y - 2x$$

Therefore

$$\int_0^1 \int_0^{1-x} \sqrt{x+y}(y-2x)^2 dy dx = \int_0^1 \int_{-2u}^u \sqrt{uv}^2 \frac{1}{3} dv du$$

$$=$$

$$=$$

$$=$$

$$=$$

$$=$$

$$= x + y \quad \frac{\partial(x, y)}{\partial(u, v)} = \frac{1}{3} \quad 0 \leq u \leq 1 \quad -2u \leq v \leq u$$

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Therefore

$$\int_0^1 \int_0^{1-x} \sqrt{x+y}(y-2x)^2 dy dx = \int_0^1 \int_{-2u}^u \sqrt{uv}^2 \frac{1}{3} dv du$$

$$= \dots$$

$$= \dots$$

$$= \dots$$

$$= \dots$$

$$= \frac{2}{9}.$$

## 14.8 Substitutions in Multiple Integrals

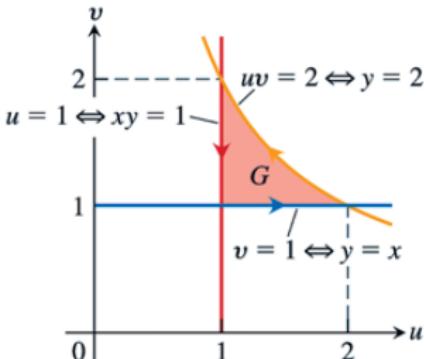
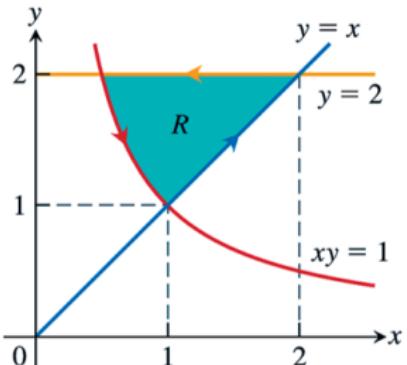
**EXAMPLE 4** Evaluate the integral

$$\int_1^2 \int_{1/y}^y \sqrt{\frac{y}{x}} e^{\sqrt{xy}} dx dy.$$

**Solution** The square root terms in the integrand suggest that we might simplify the integration by substituting  $u = \sqrt{xy}$  and  $v = \sqrt{y/x}$ . Squaring these equations gives  $u^2 = xy$  and  $v^2 = y/x$ , which imply that  $u^2v^2 = y^2$  and  $u^2/v^2 = x^2$ . So we obtain the transformation (in the same ordering of the variables as discussed before)

$$x = \frac{u}{v} \quad \text{and} \quad y = uv,$$

with  $u > 0$  and  $v > 0$ .



Let's first see what happens to the integrand itself under this transformation. The Jacobian of the transformation is not constant:

$$J(u, v) = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} \frac{1}{v} & -\frac{u}{v^2} \\ v & u \end{vmatrix} = \frac{2u}{v}.$$

If  $G$  is the region of integration in the  $uv$ -plane, then by Equation (2) the transformed double integral under the substitution is

$$\iint_R \sqrt{\frac{y}{x}} e^{\sqrt{xy}} dx dy = \iint_G ve^u \frac{2u}{v} du dv = \iint_G 2ue^u du dv.$$

The transformed integrand function is easier to integrate than the original one, so we proceed to determine the limits of integration for the transformed integral.

The region of integration  $R$  of the original integral in the  $xy$ -plane is shown in Figure 15.61. From the substitution equations  $u = \sqrt{xy}$  and  $v = \sqrt{y/x}$ , we see that the image of the left-hand boundary  $xy = 1$  for  $R$  is the vertical line segment  $u = 1$ ,  $2 \geq v \geq 1$ , in  $G$  (see Figure 15.62). Likewise, the right-hand boundary  $y = x$  of  $R$  maps to the horizontal line segment  $v = 1$ ,  $1 \leq u \leq 2$ , in  $G$ . Finally, the horizontal top boundary  $y = 2$  of  $R$

maps to  $uv = 2$ ,  $1 \leq v \leq 2$ , in  $G$ . As we move counterclockwise around the boundary of the region  $R$ , we also move counterclockwise around the boundary of  $G$ , as shown in Figure 15.62. Knowing the region of integration  $G$  in the  $uv$ -plane, we can now write equivalent iterated integrals:

$$\int_1^2 \int_{1/y}^y \sqrt{\frac{y}{x}} e^{\sqrt{xy}} dx dy = \int_1^2 \int_1^{2/u} 2ue^u dv du. \quad \text{Note the order of integration.}$$

We now evaluate the transformed integral on the right-hand side,

$$\begin{aligned} \int_1^2 \int_1^{2/u} 2ue^u dv du &= 2 \int_1^2 \left[ vu e^u \right]_{v=1}^{v=2/u} du \\ &= 2 \int_1^2 (2e^u - ue^u) du \\ &= 2 \int_1^2 (2 - u)e^u du \\ &= 2 \left[ (2 - u)e^u + e^u \right]_{u=1}^{u=2} \\ &= 2(e^2 - (e + e)) = 2e(e - 2). \end{aligned} \quad \text{Integrate by parts.}$$

### Substitutions in Triple Integrals

We use

$$x = g(u, v, w) \quad y = h(u, v, w) \quad z = k(u, v, w)$$

for

$$\iiint_D F \, dxdydz = \iiint_R F \left| \frac{\partial(x, y, z)}{\partial(u, v, w)} \right| \, dudvdw$$

## Substitutions in Triple Integrals

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$$x = g(u, v, w) \quad y = h(u, v, w) \quad z = k(u, v, w)$$

for

$$\iiint_D F \, dxdydz = \iiint_R F \left| \frac{\partial(x, y, z)}{\partial(u, v, w)} \right| \, dudvdw$$

where the *Jacobian* is

$$J(u, v, w) = \frac{\partial(x, y, z)}{\partial(u, v, w)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix}.$$

## 14.8 Substitutions in Multiple Integrals



### Example

Cartesian coordinates  $\rightarrow$  Cylindrical coordinates.

$$x = r \cos \theta, y = r \sin \theta, z = z.$$

$$\frac{\partial(x, y, z)}{\partial(r, \theta, z)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial z} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial z} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial z} \end{vmatrix} = \begin{vmatrix} \cos \theta & -r \sin \theta & 0 \\ \sin \theta & r \cos \theta & 0 \\ 0 & 0 & 1 \end{vmatrix}$$

$$= r \cos^2 \theta + r \sin^2 \theta = r.$$

## 14.8 Substitutions in Multiple Integrals

### Example

Cartesian coordinates  $\rightarrow$  Spherical coordinates.

$$x = \rho \sin \phi \cos \theta, y = \rho \sin \phi \sin \theta, z = \rho \cos \phi.$$

$$\frac{\partial(x, y, z)}{\partial(\rho, \phi, \theta)} = \begin{vmatrix} \frac{\partial x}{\partial \rho} & \frac{\partial x}{\partial \phi} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial \rho} & \frac{\partial y}{\partial \phi} & \frac{\partial y}{\partial \theta} \\ \frac{\partial z}{\partial \rho} & \frac{\partial z}{\partial \phi} & \frac{\partial z}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \sin \phi \cos \theta & \rho \cos \phi \cos \theta & -\rho \sin \phi \sin \theta \\ \sin \phi \sin \theta & \rho \cos \phi \sin \theta & \rho \sin \phi \cos \theta \\ \cos \phi & -\rho \sin \phi & 0 \end{vmatrix}$$

$$= \rho^2 \sin \phi.$$

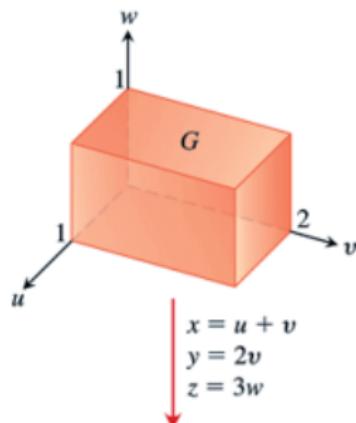
**EXAMPLE 5** Evaluate

$$\int_0^3 \int_0^4 \int_{x=y/2}^{x=(y/2)+1} \left( \frac{2x - y}{2} + \frac{z}{3} \right) dx dy dz$$

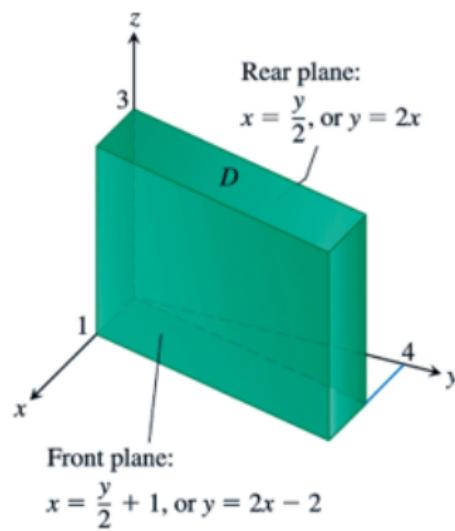
by applying the transformation

$$u = (2x - y)/2, \quad v = y/2, \quad w = z/3 \quad (8)$$

and integrating over an appropriate region in  $uvw$ -space.



$$\begin{aligned}x &= u + v \\y &= 2v \\z &= 3w\end{aligned}$$



**Solution** We sketch the region  $D$  of integration in  $xyz$ -space and identify its boundaries (Figure 15.66). In this case, the bounding surfaces are planes.

To apply Equation (7), we need to find the corresponding  $uvw$ -region  $G$  and the Jacobian of the transformation. To find them, we first solve Equations (8) for  $x$ ,  $y$ , and  $z$  in terms of  $u$ ,  $v$ , and  $w$ . Routine algebra gives

$$x = u + v, \quad y = 2v, \quad z = 3w. \quad (9)$$

We then find the boundaries of  $G$  by substituting these expressions into the equations for the boundaries of  $D$ :

xyz-equations for the boundary of $D$	Corresponding $uvw$ -equations for the boundary of $G$	Simplified $uvw$ -equations
$x = y/2$	$u + v = 2v/2 = v$	$u = 0$
$x = (y/2) + 1$	$u + v = (2v/2) + 1 = v + 1$	$u = 1$
$y = 0$	$2v = 0$	$v = 0$
$y = 4$	$2v = 4$	$v = 2$
$z = 0$	$3w = 0$	$w = 0$
$z = 3$	$3w = 3$	$w = 1$

The Jacobian of the transformation, again from Equations (9), is

$$J(u, v, w) = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix} = \begin{vmatrix} 1 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{vmatrix} = 6.$$

We now have everything we need to apply Equation (7):

$$\begin{aligned} & \int_0^3 \int_0^4 \int_{x=y/2}^{x=(y/2)+1} \left( \frac{2x - y}{2} + \frac{z}{3} \right) dx dy dz \\ &= \int_0^1 \int_0^2 \int_0^1 (u + w) |J(u, v, w)| du dv dw \\ &= \int_0^1 \int_0^2 \int_0^1 (u + w)(6) du dv dw = 6 \int_0^1 \int_0^2 \left[ \frac{u^2}{2} + uw \right]_0^1 dv dw \\ &= 6 \int_0^1 \int_0^2 \left( \frac{1}{2} + w \right) dv dw = 6 \int_0^1 \left[ \frac{v}{2} + vw \right]_0^2 dw = 6 \int_0^1 (1 + 2w) dw \\ &= 6 \left[ w + w^2 \right]_0^1 = 6(2) = 12. \end{aligned}$$



# Next Time

- 9.1 Sequences