

# Lecture 12

- 33. The Fundamental Theorem of Calculus
- 34. The Substitution Method
- 35. Area Between Curves



# The Fundamental Theorem of Calculus

## 32. The Fundamental Theorem of Calculus



We don't want to have to use Riemann sums every time we need to calculate a definite integral – we want a better way.

The following theorem is the most important theorem in Calculus. If you can only memorise one theorem for the exams, it should be this one.

## 32. The Fundamental Theorem of Calculus



Theorem (The Fundamental Theorem of Calculus)

*Suppose that  $f : [a, b] \rightarrow \mathbb{R}$  is a continuous function.*

## 32. The Fundamental Theorem of Calculus



- 1 Then the function  $F : [a, b] \rightarrow \mathbb{R}$  defined by

$$F(x) = \int_a^x f(t) \, dt$$

is continuous on  $[a, b]$ ; differentiable on  $(a, b)$ ; and its derivative is

$$F'(x) = \frac{d}{dx} \int_a^x f(t) \, dt = f(x).$$

## 32. The Fundamental Theorem of Calculus



- 2 If  $F$  is any antiderivative of  $f$  on  $[a, b]$ , then

$$\int_a^b f(x) \, dx = F(b) - F(a).$$

## 32. The Fundamental Theorem of Calculus



### Remark

Part (i) of the theorem tells how to differentiate  $\int_a^x f(t) dt$ .

### Example

Find  $\frac{dy}{dx}$  if  $y = \int_a^x (t^3 + 1) dt$ .

*solution:*

$$\frac{dy}{dx} = \frac{d}{dx} \int_a^x (t^3 + 1) dt = x^3 + 1.$$

### Example

Find  $\frac{dy}{dx}$  if  $y = \int_1^x \sin t dt$ .

*solution:*

$$\frac{dy}{dx} = \frac{d}{dx} \int_1^x \sin t dt = \sin x.$$

## 32. The Fundamental Theorem of Calculus



### Example

Find  $\frac{dy}{dx}$  if  $y = \int_0^x \sin \ln \tan e^{t^2} dt$ .

*solution:*

$$\frac{dy}{dx} = \frac{d}{dx} \int_0^x \sin \ln \tan e^{t^2} dt = \sin \ln \tan e^{x^2}.$$

## 32. The Fundamental Theorem of Calculus



### Example

Find  $\frac{dy}{dx}$  if  $y = \int_x^5 3t \sin t \, dt$ .

*solution:*

$$\begin{aligned}\frac{dy}{dx} &= \frac{d}{dx} \int_x^5 3t \sin t \, dt \\ &= \frac{d}{dx} \left( - \int_5^x 3t \sin t \, dt \right) \\ &= -3x \sin x.\end{aligned}$$

## 32. The Fundamental Theorem of Calculus



### Example

Find  $\frac{dy}{dx}$  if  $y = \int_1^{x^2} \cos t \, dt$ .

*solution:* This time we will need to use the Chain rule. Let  $u = x^2$ . Then

$$\begin{aligned}\frac{dy}{dx} &= \frac{dy}{du} \frac{du}{dx} \\&= \left( \frac{d}{du} \int_1^u \cos t \, dt \right) \left( \frac{d}{dx} x^2 \right) \\&= (\cos u) (2x) = 2x \cos x^2.\end{aligned}$$

## 32. The Fundamental Theorem of Calculus



### Remark

Part (ii) of the theorem tells us how to calculate the definite integral of  $f$  over  $[a, b]$ :

- 1 Find an antiderivative  $F$  of  $f$ .
- 2 Calculate  $F(b) - F(a)$ .

### Notation

We will write

$$\left[ F(x) \right]_a^b = F(b) - F(a).$$

## 32. The Fundamental Theorem of Calculus



### Example

$$\int_0^\pi \cos x \, dx = [\sin x]_0^\pi$$

(because  $\frac{d}{dx} \sin x = \cos x$ )

$$= \sin \pi - \sin 0$$
$$= 0 - 0$$
$$= 0$$

## 32. The Fundamental Theorem of Calculus



### Example

$$\begin{aligned}\int_{-\frac{\pi}{4}}^0 \sec x \tan x \, dx &= \left[ \sec x \right]_{-\frac{\pi}{4}}^0 \\&\quad (\text{because } \frac{d}{dx} \sec x = \sec x \tan x) \\&= \sec 0 - \sec -\frac{\pi}{4} \\&= 1 - \sqrt{2}.\end{aligned}$$

## 32. The Fundamental Theorem of Calculus



### Example

$$\int_1^4 \left( \frac{3}{2}\sqrt{x} - \frac{4}{x^2} \right) dx = \left[ x^{\frac{3}{2}} + \frac{4}{x} \right]_1^4$$

(because  $\frac{d}{dx} \left( x^{\frac{3}{2}} + \frac{4}{x} \right) = \frac{3}{2}\sqrt{x} - \frac{4}{x^2}$ )

$$= \left( 4^{\frac{3}{2}} + \frac{4}{4} \right) - \left( 1^{\frac{3}{2}} + \frac{4}{1} \right)$$
$$= (8 + 1) - (1 + 4)$$
$$= 4.$$

## Total Area

### Example

Let  $f(x) = x^2 - 4$ . We have that

$$\begin{aligned}\int_{-2}^2 f(x) \, dx &= \int_{-2}^2 (x^2 - 4) \, dx = \left[ \frac{x^3}{3} - 4x \right]_{-2}^2 \\ &= \left( \frac{8}{3} - 8 \right) - \left( \frac{-8}{3} + 8 \right) = -\frac{32}{3}.\end{aligned}$$

The total area between the graph of  $y = f(x)$  and the  $x$ -axis, over  $[-2, 2]$ , is  $\left| -\frac{32}{3} \right| = \frac{32}{3}$ .

## 32. The Fundamental Theorem of Calculus



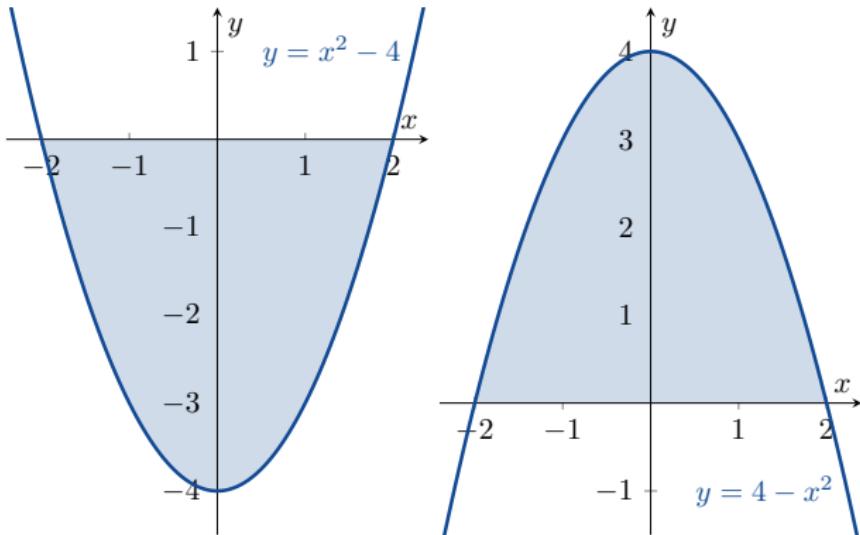
### Example

Let  $g(x) = 4 - x^2$ . We have that

$$\begin{aligned}\int_{-2}^2 g(x) \, dx &= \int_{-2}^2 (4 - x^2) \, dx = \left[ 4x - \frac{x^3}{3} \right]_{-2}^2 \\ &= \left( 8 - \frac{8}{3} \right) - \left( 8 + \frac{-8}{3} \right) = \frac{32}{3}.\end{aligned}$$

The total area between the graph of  $y = g(x)$  and the  $x$ -axis, over  $[-2, 2]$ , is  $\left| \frac{32}{3} \right| = \frac{32}{3}$ .

## 32. The Fundamental Theorem of Calculus



$$\text{integral} = -\frac{32}{3}$$
$$\text{total area} = \frac{32}{3}$$

$$\text{integral} = \frac{32}{3}$$
$$\text{total area} = \frac{32}{3}$$

## 32. The Fundamental Theorem of Calculus

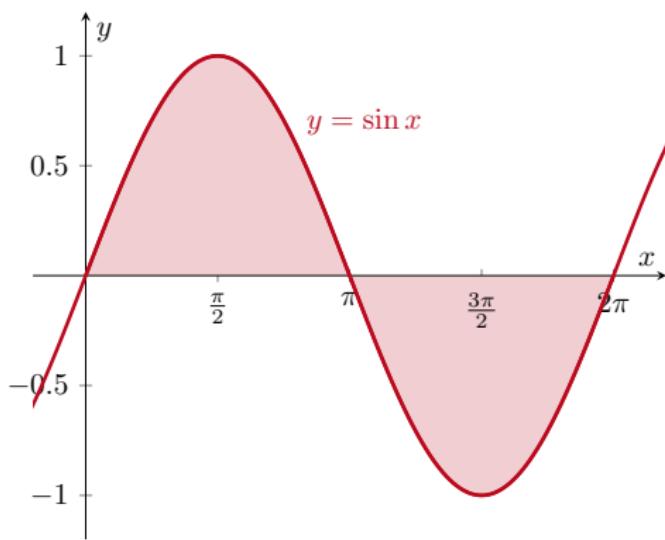


### Example

Let  $f(x) = \sin x$ . Calculate

- 1 the definite integral of  $f$  over  $[0, 2\pi]$ ; and
- 2 the total area between the graph of  $y = f(x)$  and the  $x$ -axis over  $[0, 2\pi]$ .

## 32. The Fundamental Theorem of Calculus



## 32. The Fundamental Theorem of Calculus



*solution:*

1

$$\begin{aligned}\int_0^{2\pi} \sin x \, dx &= \left[ -\cos x \right]_0^{2\pi} = -\cos 2\pi + \cos 0 \\ &= -1 + 1 = 0.\end{aligned}$$

2

$$\begin{aligned}\text{total area} &= \int_0^\pi \sin x \, dx + \left| \int_\pi^{2\pi} \sin x \, dx \right| \\ &= \left[ -\cos x \right]_0^\pi + \left| \left[ -\cos x \right]_\pi^{2\pi} \right| \\ &= -\cos \pi + \cos 0 + |- \cos 2\pi + \cos \pi| \\ &= -(-1) + 1 + |-1 + (-1)| = 4.\end{aligned}$$



### Summary

To find the *total area* between the graph of  $y = f(x)$  and the  $x$ -axis over  $[a, b]$ :

- 1 Divide  $[a, b]$  at the zeroes of  $f$ .
- 2 Integrate  $f$  over each subinterval.
- 3 Add the absolute values of the integrals.

## 32. The Fundamental Theorem of Calculus



### Example

Find the total area between the graph of  $y = x^3 - x^2 - 2x$  and the  $x$ -axis for  $-1 \leq x \leq 2$ .

*solution:*

- 1 Let  $f(x) = x^3 - x^2 - 2x$ .

Since  $0 = f(x) = x^3 - x^2 - 2x = x(x + 1)(x - 2)$  implies that  $x = 0$  or  $x = -1$  or  $x = 2$ , we divide  $[-1, 2]$  into  $[-1, 0]$  and  $[0, 2]$ .

## 32. The Fundamental Theorem of Calculus



2 We calculate that

$$\begin{aligned}\int_{-1}^0 (x^3 - x^2 - 2x) \, dx &= \left[ \frac{x^4}{4} - \frac{x^3}{3} - x^2 \right]_{-1}^0 \\&= (0 - 0 - 0) - \left( \frac{1}{4} + \frac{1}{3} - 1 \right) \\&= \frac{5}{12}\end{aligned}$$

## 32. The Fundamental Theorem of Calculus



and

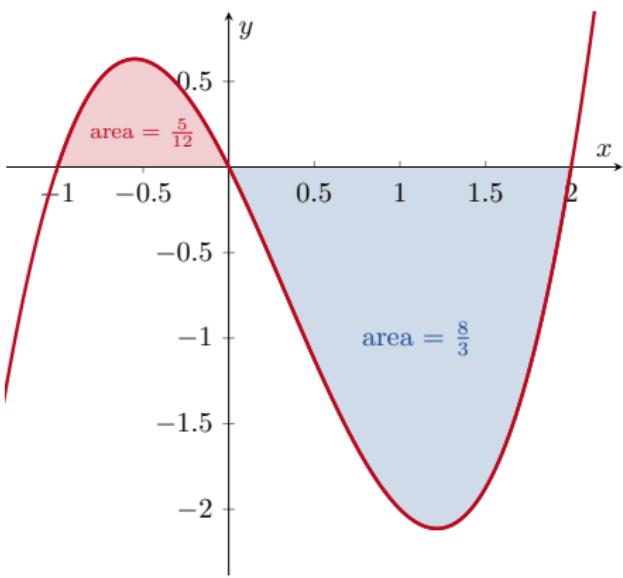
$$\begin{aligned}\int_0^2 (x^3 - x^2 - 2x) \, dx &= \left[ \frac{x^4}{4} - \frac{x^3}{3} - x^2 \right]_0^2 \\&= \left( \frac{16}{4} - \frac{8}{3} - 4 \right) - (0 - 0 - 0) \\&= -\frac{8}{3}.\end{aligned}$$

## 32. The Fundamental Theorem of Calculus



3 Therefore

$$\text{total area} = \left| \frac{5}{12} \right| + \left| -\frac{8}{3} \right| = \frac{37}{12}.$$



### The Average Value of a Continuous Function

The average of  $\{1, 2, 2, 6, 9\}$  is  $\frac{1+2+2+6+9}{5} = \frac{20}{5} = 4$ . We can also calculate the average value of a continuous function.

## 32. The Fundamental Theorem of Calculus



### Definition

If  $f$  is integrable on  $[a, b]$ , then the *average value of  $f$  on  $[a, b]$*  is

$$\text{av}(f) = \frac{1}{b-a} \int_a^b f(x) \, dx.$$

## 32. The Fundamental Theorem of Calculus



### Example

Find the average value of  $f(x) = \sqrt{4 - x^2}$  on  $[-2, 2]$ .

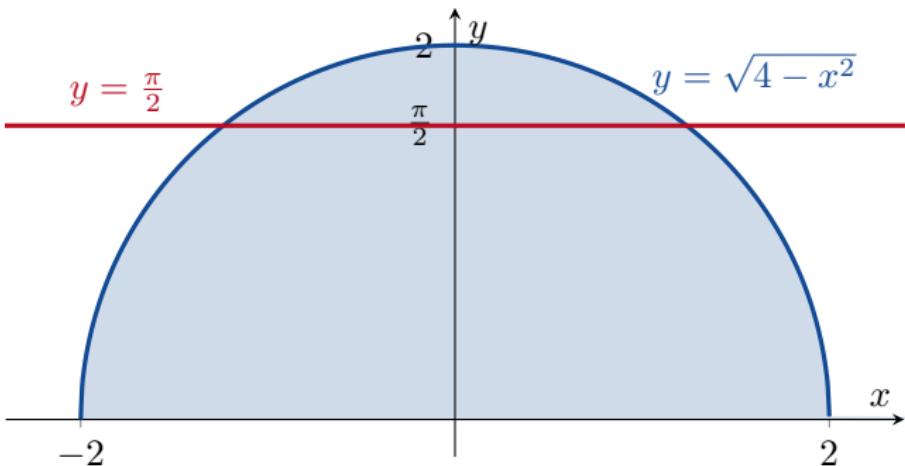
*solution:* Since

$$\begin{aligned}\int_{-2}^2 f(x) \, dx &= \frac{1}{2} \times \text{the area of a circle of radius 2} \\ &= \frac{1}{2}\pi 2^2 = 2\pi,\end{aligned}$$

we have that

$$\text{av}(f) = \frac{1}{2 - (-2)} \int_{-2}^2 f(x) \, dx = \frac{2\pi}{4} = \frac{\pi}{2}.$$

## 32. The Fundamental Theorem of Calculus



## 32. The Fundamental Theorem of Calculus



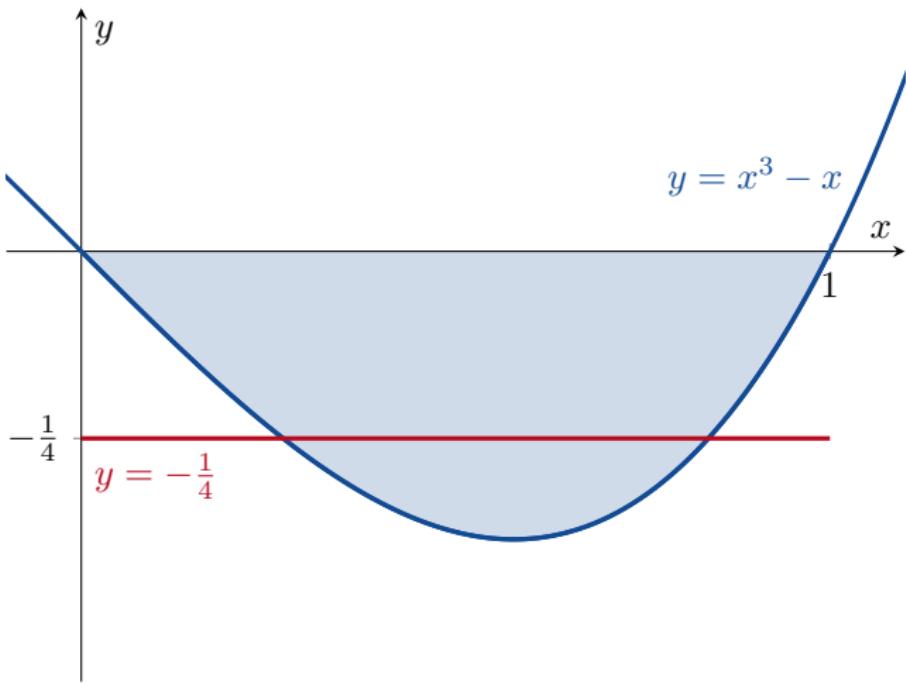
### Example

Find the average value of  $g(x) = x^3 - x$  on  $[0, 1]$ .

*solution:*

$$\begin{aligned}\text{av}(g) &= \frac{1}{1-0} \int_0^1 g(x) \, dx = \int_0^1 (x^3 - x) \, dx \\ &= \left[ \frac{x^4}{4} - \frac{x^2}{2} \right]_0^1 = \frac{1}{4} - \frac{1}{2} = -\frac{1}{4}.\end{aligned}$$

## 32. The Fundamental Theorem of Calculus



## Indefinite Integrals & Definite Integrals

Remember that

$\int f(x) dx$  is a function.

For example

$$\int x \, dx = \frac{x^2}{2} + C$$

and

$$\int \cos x \, dx = \sin x + C.$$

Remember that

$\int_a^b f(x) dx$  is a number.

For example

$$\int_0^1 x \, dx = \frac{1}{2}$$

and

$$\int_0^{\frac{\pi}{2}} \cos x \, dx = 1.$$



# The Substitution Method

### 33. The Substitution Method

By the Chain rule,

$$\frac{d}{dx} \left( \frac{u^{n+1}}{n+1} \right) = u^n \frac{du}{dx}.$$

So

$$\int u^n \frac{du}{dx} dx = \frac{u^{n+1}}{n+1} + C.$$

But we know that

$$\int u^n du = \frac{u^{n+1}}{n+1} + C$$

also. So it looks like

$$du = \frac{du}{dx} dx.$$

### 33. The Substitution Method



#### Example

Find  $\int (x^3 + x)^5(3x^2 + 1) \, dx.$

*solution:* Let  $u = x^3 + x$ . Then  $du = \frac{du}{dx} \, dx = (3x^2 + 1) \, dx$ . By substitution, we have that

$$\begin{aligned}\int (x^3 + x)^5(3x^2 + 1) \, dx &= \int u^5 \, du \\ &= \frac{u^6}{6} + C = \frac{1}{6}(x^3 + x)^6 + C.\end{aligned}$$

### 33. The Substitution Method



#### Example

Find  $\int \sqrt{2x+1} dx$ .

*solution:* Let  $u = 2x + 1$ . Then  $du = \frac{du}{dx} dx = 2dx$ . So  $dx = \frac{1}{2} du$ . Therefore

$$\begin{aligned}\int \sqrt{2x+1} dx &= \int u^{\frac{1}{2}} \left(\frac{1}{2}du\right) = \frac{1}{2} \int u^{\frac{1}{2}} du \\ &= \frac{1}{2} \left( \frac{u^{\frac{3}{2}}}{\frac{3}{2}} \right) + C = \frac{1}{3} (2x+1)^{\frac{3}{2}} + C.\end{aligned}$$

### 33. The Substitution Method



#### Theorem (The Substitution Method)

If

- $u = g(x)$  is differentiable;
- $g : \mathbb{R} \rightarrow I$ ; and
- $f : I \rightarrow \mathbb{R}$  is continuous,

then

$$\int f(g(x))g'(x) \, dx = \int f(u) \, du.$$

### 33. The Substitution Method



#### Example

Find  $\int 5 \sec^2(5t + 1) dt$ .

*solution:* Let  $u = 5t + 1$ . Then  $du = \frac{du}{dt} dt = 5dt$ . So

$$\begin{aligned}\int 5 \sec^2(5t + 1) dt &= \int \sec^2 u du \\&= \tan u + C \\&\quad (\text{because } \frac{d}{du} \tan u = \sec^2 u) \\&= \tan(5t + 1) + C.\end{aligned}$$

### 33. The Substitution Method



#### Example

Find  $\int \cos(7\theta + 3) d\theta$ .

*solution:* Let  $u = 7\theta + 3$ . Then  $du = \frac{du}{d\theta} d\theta = 7d\theta$ . So  $d\theta = \frac{1}{7}du$  and

$$\begin{aligned}\int \cos(7\theta + 3) d\theta &= \frac{1}{7} \int \cos u du \\ &= \frac{1}{7} \sin u + C = \frac{1}{7} \sin(7\theta + 3) + C.\end{aligned}$$

### 33. The Substitution Method



#### Example

Find  $\int x^2 \sin(x^3) dx$ .

*solution:* Let  $u = x^3$ . Then  $du = \frac{du}{dx} dx = 3x^2 dx$ . So  $\frac{1}{3}du = x^2 dx$  and

$$\begin{aligned}\int x^2 \sin(x^3) dx &= \int \frac{1}{3} \sin u du = -\frac{1}{3} \cos u + C \\ &= -\frac{1}{3} \cos(x^3) + C.\end{aligned}$$

### 33. The Substitution Method

#### Example

Find  $\int x\sqrt{2x+1} dx$ .

*solution:* Let  $u = 2x + 1$ . Then  $du = \frac{du}{dx} dx = 2dx$ . So  $dx = \frac{1}{2}du$  and

$$\int x\sqrt{2x+1} dx = \int x\sqrt{u} \frac{1}{2}du.$$

But we still have an  $x$  here. We can't integrate until we change all the  $x$  terms to  $u$  terms. Note that

$$u = 2x + 1 \implies u - 1 = 2x \implies \frac{1}{2}(u - 1) = x.$$

### 33. The Substitution Method



Therefore

$$\begin{aligned}\int x\sqrt{2x+1} \, dx &= \int \frac{1}{2}(u-1)\sqrt{u} \, \frac{1}{2}du \\&= \frac{1}{4} \int u^{\frac{3}{2}} - u^{\frac{1}{2}} \, du \\&= \frac{1}{4} \left( \frac{2}{5}u^{\frac{5}{2}} - \frac{2}{3}u^{\frac{3}{2}} \right) + C \\&= \frac{1}{10}u^{\frac{5}{2}} - \frac{1}{6}u^{\frac{3}{2}} + C \\&= \frac{1}{10}(2x+1)^{\frac{5}{2}} - \frac{1}{6}(2x+1)^{\frac{3}{2}} + C.\end{aligned}$$

### 33. The Substitution Method

#### Example

Find  $\int \frac{2z}{\sqrt[3]{z^2 + 1}} dz$ .

*solution:* Let  $u = z^2 + 1$ . Then  $du = \frac{du}{dx} dx = 2z dz$  and

$$\begin{aligned}\int \frac{2z}{\sqrt[3]{z^2 + 1}} dz &= \int \frac{du}{u^{\frac{1}{3}}} \\&= \int u^{-\frac{1}{3}} du \\&= \frac{u^{\frac{2}{3}}}{\frac{2}{3}} + C \\&= \frac{3}{2}u^{\frac{2}{3}} + C \\&= \frac{3}{2}(z^2 + 1)^{\frac{2}{3}} + C.\end{aligned}$$

### 33. The Substitution Method



#### Example

Find  $\int \sin^2 x \, dx$ .

*solution:* We use the identity

$$\sin^2 x = \frac{1 - \cos 2x}{2}$$

to calculate that

$$\begin{aligned}\int \sin^2 x \, dx &= \int \frac{1 - \cos 2x}{2} \, dx \\&= \frac{1}{2} \int (1 - \cos 2x) \, dx \\&= \frac{1}{2} \left( x - \frac{1}{2} \sin 2x \right) + C \\&= \frac{1}{2}x - \frac{1}{4} \sin 2x + C.\end{aligned}$$

### 33. The Substitution Method



#### Example

Similarly

$$\int \cos^2 x \, dx = \int \frac{1 + \cos 2x}{2} \, dx = \frac{1}{2}x + \frac{1}{4}\sin 2x + C.$$

## The Substitution Method for Definite Integrals

### Theorem (The Substitution Method)

If

- $u = g(x)$  is differentiable on  $[a, b]$ ;
- $g'$  is continuous on  $[a, b]$ ; and
- $f$  is continuous on the range of  $g$ ,

then

$$\int_a^b f(g(x))g'(x) \, dx = \int_{g(a)}^{g(b)} f(u) \, du.$$

### 33. The Substitution Method

#### Example

Calculate  $\int_{-1}^1 3x^2 \sqrt{x^3 + 1} dx.$

*solution 1.* We can use the previous theorem to solve this example. Let  $u = x^3 + 1$ . Then  $du = 3x^2 dx$ . Moreover  $x = -1 \implies u = 0$  and  $x = 1 \implies u = 2$ . So

$$\begin{aligned}\int_{x=-1}^{x=1} 3x^2 \sqrt{x^3 + 1} dx &= \int_{u=0}^{u=2} \sqrt{u} du = \left[ \frac{2}{3} u^{\frac{3}{2}} \right]_0^2 \\ &= \frac{2}{3} \left( 2^{\frac{3}{2}} - 0^{\frac{3}{2}} \right) = \frac{2}{3} 2\sqrt{2} = \frac{4\sqrt{2}}{3}.\end{aligned}$$

### 33. The Substitution Method



*solution 2.* Alternately, we can first find the indefinite integral, then find the required definite integral.

Let  $u = x^3 + 1$ . Then  $du = 3x^2 dx$ . So

$$\int 3x^2 \sqrt{x^3 + 1} dx = \int \sqrt{u} du = \frac{2}{3}u^{\frac{3}{2}} + C = \frac{2}{3}(x^3 + 1)^{\frac{3}{2}} + C.$$

Therefore

$$\begin{aligned}\int_{-1}^1 3x^2 \sqrt{x^3 + 1} dx &= \left[ \frac{2}{3}(x^3 + 1)^{\frac{3}{2}} \right]_{-1}^1 \\&= \left( \frac{2}{3}(1 + 1)^{\frac{3}{2}} \right) - \left( \frac{2}{3}(-1 + 1)^{\frac{3}{2}} \right) \\&= \frac{2}{3} \times 2^{\frac{3}{2}} = \frac{4\sqrt{2}}{3}.\end{aligned}$$

### 33. The Substitution Method



#### Example

Calculate  $\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \cot \theta \cosec^2 \theta \, d\theta$ .

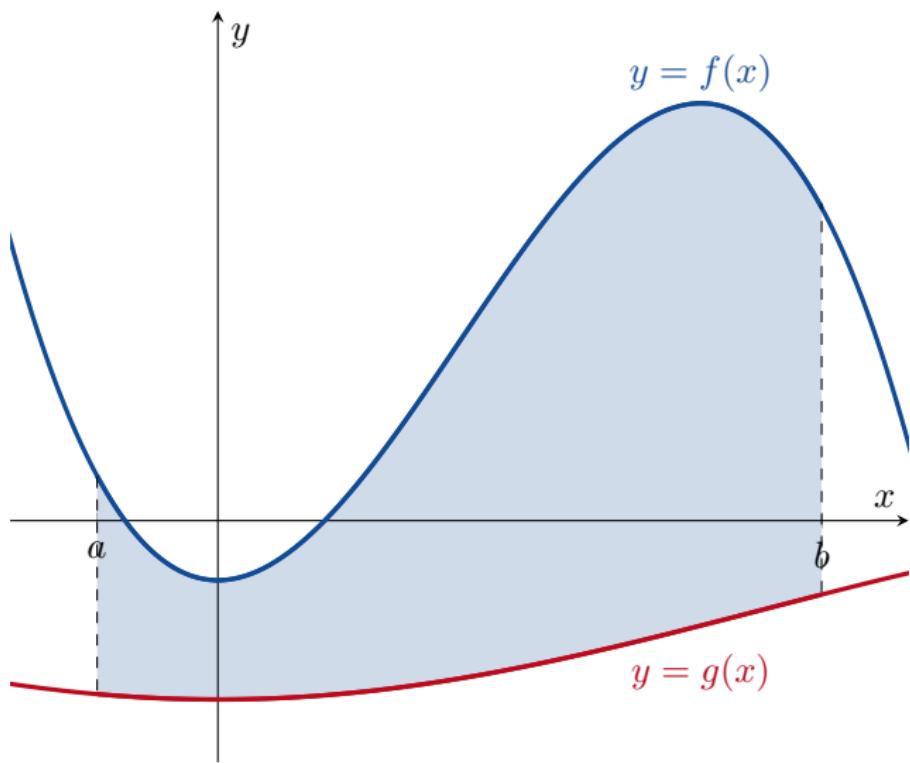
*solution:* Let  $u = \cot \theta$ . Then  $du = \frac{du}{d\theta} d\theta = -\cosec^2 \theta \, d\theta$ . So  $-du = \cosec^2 \theta \, d\theta$ . Moreover  $\theta = \frac{\pi}{4} \implies u = \cot \frac{\pi}{4} = 1$  and  $\theta = \frac{\pi}{2} \implies u = \cot \frac{\pi}{2} = 0$ . Hence

$$\begin{aligned}\int_{\theta=\frac{\pi}{4}}^{\theta=\frac{\pi}{2}} \cot \theta \cosec^2 \theta \, d\theta &= \int_{u=1}^{u=0} u (-du) = - \int_1^0 u \, du \\ &= - \left[ \frac{u^2}{2} \right]_1^0 = - \left( \frac{0^2}{2} - \frac{1^2}{2} \right) = \frac{1}{2}.\end{aligned}$$



# Area Between Curves

## 34. Area Between Curves



## 34. Area Between Curves



### Definition

If

- $f$  is continuous;
- $g$  is continuous; and
- $f(x) \geq g(x)$  on  $[a, b]$ ,

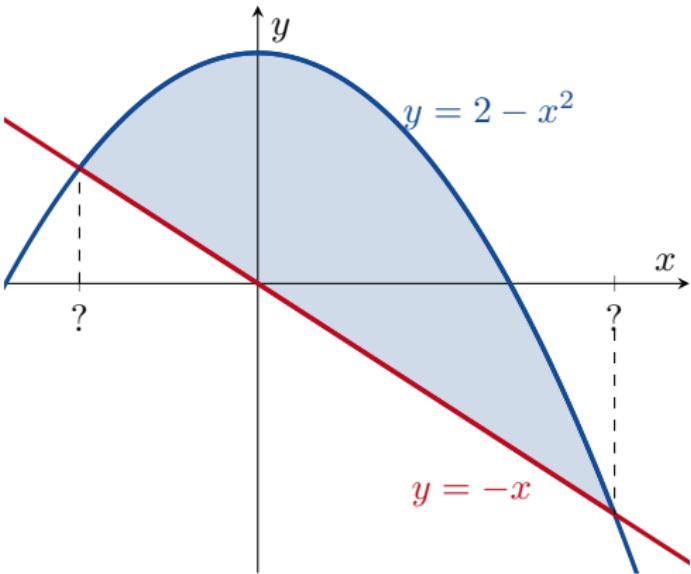
then the *area of the region between the curves  $y = f(x)$  and  $y = g(x)$  for  $a \leq x \leq b$*  is

$$\text{area} = \int_a^b (f(x) - g(x)) \, dx.$$

## 34. Area Between Curves

### Example

Find the area between  $y = 2 - x^2$  and  $y = -x$ .



## 34. Area Between Curves

*solution:* First we need to find the limits of integration:

$$2 - x^2 = -x$$

$$0 = x^2 - x - 2$$

$$0 = (x + 1)(x - 2) \implies x = -1 \text{ or } 2.$$

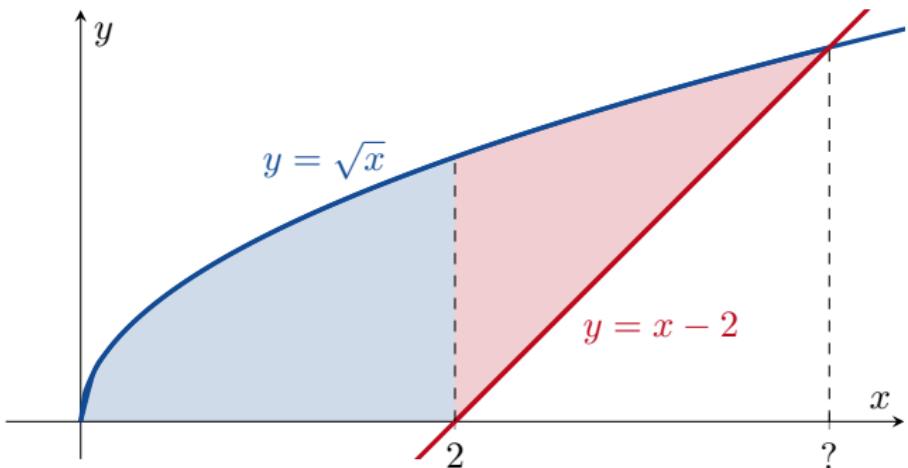
We need to integrate from  $x = -1$  to  $x = 2$ . Therefore

$$\begin{aligned} \text{area} &= \int_{-1}^2 \left( (2 - x^2) - (-x) \right) dx \\ &= \int_{-1}^2 (2 + x - x^2) dx \\ &= \left[ 2x + \frac{1}{2}x^2 - \frac{1}{3}x^3 \right]_{-1}^2 \\ &= \left( 4 + \frac{4}{2} - \frac{8}{3} \right) - \left( -2 + \frac{1}{2} + \frac{1}{3} \right) \\ &= \frac{9}{2}. \end{aligned}$$

## 34. Area Between Curves

### Example

Find the area bounded by  $y = \sqrt{x}$ ,  $y = x - 2$  and the  $x$ -axis, for  $x \geq 0$  and  $y \geq 0$ .



## 34. Area Between Curves



*solution:* First we calculate that

$$\sqrt{x} = x - 2$$

$$x = (x - 2)^2 = x^2 - 4x + 4$$

$$0 = x^2 - 5x + 4 = (x - 1)(x - 4) \implies x = 1 \text{ or } 4.$$

Since  $\sqrt{1} \neq 1 - 2$ , we must have  $x = 4$ .

## 34. Area Between Curves



Therefore

$$\text{area} = \text{blue area} + \text{red area}$$

$$\begin{aligned}&= \int_0^2 \sqrt{x} \, dx + \int_2^4 (\sqrt{x} - (x - 2)) \, dx \\&= \int_0^2 x^{\frac{1}{2}} \, dx + \int_2^4 (x^{\frac{1}{2}} - x + 2) \, dx \\&= \left[ \frac{2}{3}x^{\frac{3}{2}} \right]_0^2 + \left[ \frac{2}{3}x^{\frac{3}{2}} - \frac{1}{2}x^2 + 2x \right]_2^4 \\&= \left( \frac{2}{3}(2)^{\frac{3}{2}} - 0 \right) + \left( \frac{2}{3}(4)^{\frac{3}{2}} - \frac{1}{2}(16) + 2(4) \right) \\&\quad - \left( \frac{2}{3}(2)^{\frac{3}{2}} - \frac{1}{2}(4) + 2(2) \right) \\&= \frac{4\sqrt{2}}{3} + \frac{16}{3} - 8 + 8 - \frac{4\sqrt{2}}{3} + 2 - 4 = \frac{10}{3}.\end{aligned}$$



*The End*

