Lecture 3

- 2.3 Characterisations of Invertible Matrices
- 0.0 Diagonal, triangular, and symmetric matrices
- 1.6 Applications of linear systems.
- 3.1 Introduction to Determinants

2.3 Characterisation of Invertible Matrices

Recall that a square matrix A is called *invertible* if there exists a matrix A^{-1} of the same size such that

$$AA^{-1} = I \qquad \text{and} \qquad A^{-1}A = I.$$

A matrix which is not invertible is called *singular*.

Theorem (The Invertible Matrix Theorem)

Let A be a square $n \times n$ matrix. The following statements are equivalent (i.e. for a given A, they are either all true, or all false):

Theorem (The Invertible Matrix Theorem)

Let A be a square $n \times n$ matrix. The following statements are equivalent (i.e. for a given A, they are either all true, or all false):

1. *A is invertible;*

Theorem (The Invertible Matrix Theorem)

Let A be a square $n \times n$ matrix. The following statements are equivalent (i.e. for a given A, they are either all true, or all false):

1. *A is invertible;*
2. *A is row equivalent to I;*

2.3 Characterisations of Invertible Matrices

Theorem (The Invertible Matrix Theorem)

Let A be a square $n \times n$ matrix. The following statements are equivalent (i.e. for a given A, they are either all true, or all false):

1. *A is invertible;*
2. *A is row equivalent to I;*
3. *A is expressible as a product of elementary matrices;*

Theorem (The Invertible Matrix Theorem)

Let A be a square $n \times n$ matrix. The following statements are equivalent (i.e. for a given A, they are either all true, or all false):

1. *A is invertible;*
2. *A is row equivalent to I;*
3. *A is expressible as a product of elementary matrices;*
4. *A has n pivot positions;*

Theorem (The Invertible Matrix Theorem)

Let A be a square $n \times n$ matrix. The following statements are equivalent (i.e. for a given A, they are either all true, or all false):

1. *A is invertible;*
2. *A is row equivalent to I;*
3. *A is expressible as a product of elementary matrices;*
4. *A has n pivot positions;*
5. *$A\mathbf{x} = \mathbf{0}$ has only the trivial solution;*

Theorem (The Invertible Matrix Theorem)

Let A be a square $n \times n$ matrix. The following statements are equivalent (i.e. for a given A, they are either all true, or all false):

1. *A is invertible;*
2. *A is row equivalent to I;*
3. *A is expressible as a product of elementary matrices;*
4. *A has n pivot positions;*
5. *$A\mathbf{x} = \mathbf{0}$ has only the trivial solution;*
6. *$A\mathbf{x} = \mathbf{b}$ is consistent;*

Theorem (The Invertible Matrix Theorem)

Let A be a square $n \times n$ matrix. The following statements are equivalent (i.e. for a given A, they are either all true, or all false):

1. *A is invertible;*
2. *A is row equivalent to I;*
3. *A is expressible as a product of elementary matrices;*
4. *A has n pivot positions;*
5. *$A\mathbf{x} = \mathbf{0}$ has only the trivial solution;*
6. *$A\mathbf{x} = \mathbf{b}$ is consistent;*
7. *$A\mathbf{x} = \mathbf{b}$ has exactly one solution;*

Theorem (The Invertible Matrix Theorem)

Let A be a square $n \times n$ matrix. The following statements are equivalent (i.e. for a given A, they are either all true, or all false):

1. *A is invertible;*
2. *A is row equivalent to I;*
3. *A is expressible as a product of elementary matrices;*
4. *A has n pivot positions;*
5. *$A\mathbf{x} = \mathbf{0}$ has only the trivial solution;*
6. *$A\mathbf{x} = \mathbf{b}$ is consistent;*
7. *$A\mathbf{x} = \mathbf{b}$ has exactly one solution;*
8. *The columns of A form a linearly independent set;*

Theorem (The Invertible Matrix Theorem)

Let A be a square $n \times n$ matrix. The following statements are equivalent (i.e. for a given A, they are either all true, or all false):

1. *A is invertible;*
2. *A is row equivalent to I;*
3. *A is expressible as a product of elementary matrices;*
4. *A has n pivot positions;*
5. *$A\mathbf{x} = \mathbf{0}$ has only the trivial solution;*
6. *$A\mathbf{x} = \mathbf{b}$ is consistent;*
7. *$A\mathbf{x} = \mathbf{b}$ has exactly one solution;*
8. *The columns of A form a linearly independent set;*
9. *The linear transformation $\mathbf{x} \mapsto A\mathbf{x}$ is one-to-one;*

Theorem (The Invertible Matrix Theorem)

Let A be a square $n \times n$ matrix. The following statements are equivalent (i.e. for a given A, they are either all true, or all false):

1. *A is invertible;*
2. *A is row equivalent to I;*
3. *A is expressible as a product of elementary matrices;*
4. *A has n pivot positions;*
5. *$A\mathbf{x} = \mathbf{0}$ has only the trivial solution;*
6. *$A\mathbf{x} = \mathbf{b}$ is consistent;*
7. *$A\mathbf{x} = \mathbf{b}$ has exactly one solution;*
8. *The columns of A form a linearly independent set;*
9. *The linear transformation $\mathbf{x} \mapsto A\mathbf{x}$ is one-to-one;*
10. *The columns of A span $\mathbb{R}^n$;*

Theorem (The Invertible Matrix Theorem)

Let A be a square $n \times n$ matrix. The following statements are equivalent (i.e. for a given A, they are either all true, or all false):

1. *A is invertible;*
2. *A is row equivalent to I;*
3. *A is expressible as a product of elementary matrices;*
4. *A has n pivot positions;*
5. *$A\mathbf{x} = \mathbf{0}$ has only the trivial solution;*
6. *$A\mathbf{x} = \mathbf{b}$ is consistent;*
7. *$A\mathbf{x} = \mathbf{b}$ has exactly one solution;*
8. *The columns of A form a linearly independent set;*
9. *The linear transformation $\mathbf{x} \mapsto A\mathbf{x}$ is one-to-one;*
10. *The columns of A span $\mathbb{R}^n$;*
11. *The linear transformation $\mathbf{x} \mapsto A\mathbf{x}$ maps $\mathbb{R}^n$ onto $\mathbb{R}^n$;*

Theorem (The Invertible Matrix Theorem)

Let A be a square $n \times n$ matrix. The following statements are equivalent (i.e. for a given A, they are either all true, or all false):

1. *A is invertible;*
2. *A is row equivalent to I;*
3. *A is expressible as a product of elementary matrices;*
4. *A has n pivot positions;*
5. *$A\mathbf{x} = \mathbf{0}$ has only the trivial solution;*
6. *$A\mathbf{x} = \mathbf{b}$ is consistent;*
7. *$A\mathbf{x} = \mathbf{b}$ has exactly one solution;*
8. *The columns of A form a linearly independent set;*
9. *The linear transformation $\mathbf{x} \mapsto A\mathbf{x}$ is one-to-one;*
10. *The columns of A span $\mathbb{R}^n$;*
11. *The linear transformation $\mathbf{x} \mapsto A\mathbf{x}$ maps $\mathbb{R}^n$ onto $\mathbb{R}^n$;*
12. *There is an $n \times n$ matrix C such that $CA = I$;*

Theorem (The Invertible Matrix Theorem)

Let A be a square $n \times n$ matrix. The following statements are equivalent (i.e. for a given A, they are either all true, or all false):

1. *A is invertible;*
2. *A is row equivalent to I;*
3. *A is expressible as a product of elementary matrices;*
4. *A has n pivot positions;*
5. *$A\mathbf{x} = \mathbf{0}$ has only the trivial solution;*
6. *$A\mathbf{x} = \mathbf{b}$ is consistent;*
7. *$A\mathbf{x} = \mathbf{b}$ has exactly one solution;*
8. *The columns of A form a linearly independent set;*
9. *The linear transformation $\mathbf{x} \mapsto A\mathbf{x}$ is one-to-one;*
10. *The columns of A span $\mathbb{R}^n$;*
11. *The linear transformation $\mathbf{x} \mapsto A\mathbf{x}$ maps $\mathbb{R}^n$ onto $\mathbb{R}^n$;*
12. *There is an $n \times n$ matrix C such that $CA = I$;*
13. *There is an $n \times n$ matrix D such that $AD = I$;*

Theorem (The Invertible Matrix Theorem)

Let A be a square $n \times n$ matrix. The following statements are equivalent (i.e. for a given A, they are either all true, or all false):

1. *A is invertible;*
2. *A is row equivalent to I;*
3. *A is expressible as a product of elementary matrices;*
4. *A has n pivot positions;*
5. *$A\mathbf{x} = \mathbf{0}$ has only the trivial solution;*
6. *$A\mathbf{x} = \mathbf{b}$ is consistent;*
7. *$A\mathbf{x} = \mathbf{b}$ has exactly one solution;*
8. *The columns of A form a linearly independent set;*
9. *The linear transformation $\mathbf{x} \mapsto A\mathbf{x}$ is one-to-one;*
10. *The columns of A span $\mathbb{R}^n$;*
11. *The linear transformation $\mathbf{x} \mapsto A\mathbf{x}$ maps $\mathbb{R}^n$ onto $\mathbb{R}^n$;*
12. *There is an $n \times n$ matrix C such that $CA = I$;*
13. *There is an $n \times n$ matrix D such that $AD = I$;*
14. *A^T is an invertible matrix.*

Remark

If A is invertible, then statements 2-14 are all true.

If A is singluar, then statements 2-14 are all false.

Remark

If A is invertible, then statements 2-14 are all true.

If A is singluar, then statements 2-14 are all false.

Remark

Properties 1, 12 and 13 were

1. A is invertible;
12. There is an $n \times n$ matrix C such that $CA = I$;
13. There is an $n \times n$ matrix D such that $AD = I$.

Remark

If A is invertible, then statements 2-14 are all true.

If A is singluar, then statements 2-14 are all false.

Remark

Properties 1, 12 and 13 were

1. A is invertible;
12. There is an $n \times n$ matrix C such that $CA = I$;
13. There is an $n \times n$ matrix D such that $AD = I$.

This means that we don't need to prove both

$$AA^{-1} = I \qquad \text{and} \qquad A^{-1}A = I,$$

we only need to satisfy one of these.

Example

Use the Invertible Matrix Theorem to decide if

$$A = \begin{bmatrix} 1 & 0 & -2 \\ 3 & 1 & -2 \\ -5 & -1 & 9 \end{bmatrix}$$ is invertible.

Example

Use the Invertible Matrix Theorem to decide if
$A = \begin{bmatrix} 1 & 0 & -2 \\ 3 & 1 & -2 \\ -5 & -1 & 9 \end{bmatrix}$ is invertible.

Since

$$A = \begin{bmatrix} 1 & 0 & -2 \\ 3 & 1 & -2 \\ -5 & -1 & 9 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 4 \\ -5 & -1 & 9 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 4 \\ 0 & -1 & -1 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 4 \\ 0 & 0 & 3 \end{bmatrix},$$

we can see that A has 3 pivot positions.

Example

Use the Invertible Matrix Theorem to decide if
$A = \begin{bmatrix} 1 & 0 & -2 \\ 3 & 1 & -2 \\ -5 & -1 & 9 \end{bmatrix}$ is invertible.

Since

$$A = \begin{bmatrix} 1 & 0 & -2 \\ 3 & 1 & -2 \\ -5 & -1 & 9 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 4 \\ -5 & -1 & 9 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 4 \\ 0 & -1 & -1 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 4 \\ 0 & 0 & 3 \end{bmatrix},$$

we can see that A has 3 pivot positions. Hence A is invertible by the Invertible Matrix Theorem.

Example

Does the linear system

$$\begin{cases} x_1 \phantom{{}+x_2} - 2x_3 = 0 \\ 3x_1 + x_2 - 2x_3 = 0 \\ -5x_1 - x_2 + 9x_3 = 0 \end{cases}$$

have any nontrivial solutions?

Recall that $A = \begin{bmatrix} 1 & 0 & -2 \\ 3 & 1 & -2 \\ -5 & -1 & 9 \end{bmatrix}$ is invertible.

Example

Does the linear system

$$\begin{cases} x_1 \qquad\quad - 2x_3 = 0 \\ 3x_1 + x_2 - 2x_3 = 0 \\ -5x_1 - x_2 + 9x_3 = 0 \end{cases}$$

have any nontrivial solutions?

Recall that $A = \begin{bmatrix} 1 & 0 & -2 \\ 3 & 1 & -2 \\ -5 & -1 & 9 \end{bmatrix}$ is invertible.

By the theorem, the linear system $A\mathbf{x} = 0$ has only the trivial solution. So the answer is "no".

Number of Solutions of a Linear System

In Lecture 1 I said that

Theorem

A linear system has either

1. *zero solutions; or*
2. *exactly one solution; or*
3. *infinitely many solutions.*

There are no other possibilities.

Now it is time to prove this.

Proof.

Consider the linear system $A\mathbf{x} = \mathbf{b}$. Exactly one of the following must be true:

a. $A\mathbf{x} = \mathbf{b}$ has no solutions;
b. $A\mathbf{x} = \mathbf{b}$ has exactly one solution; or
c. $A\mathbf{x} = \mathbf{b}$ has more than one solution.

Proof.

Consider the linear system $A\mathbf{x} = \mathbf{b}$. Exactly one of the following must be true:

a. $A\mathbf{x} = \mathbf{b}$ has no solutions;
b. $A\mathbf{x} = \mathbf{b}$ has exactly one solution; or
c. $A\mathbf{x} = \mathbf{b}$ has more than one solution.

We only need to prove that $A\mathbf{x} = \mathbf{b}$ has infinitely many solutions in case c.

Proof.

Consider the linear system $A\mathbf{x} = \mathbf{b}$. Exactly one of the following must be true:

- a $A\mathbf{x} = \mathbf{b}$ has no solutions;
- b $A\mathbf{x} = \mathbf{b}$ has exactly one solution; or
- c $A\mathbf{x} = \mathbf{b}$ has more than one solution.

We only need to prove that $A\mathbf{x} = \mathbf{b}$ has infinitely many solutions in case c.

In other words, we are going to prove that

$$\begin{array}{c}\text{there are 2}\\ \text{different solutions}\end{array} \implies \begin{array}{c}\text{there are } \infty\\ \text{solutions.}\end{array}$$

Proof continued.

Suppose that $\mathbf{x}_1$ and $\mathbf{x}_2$ be two <u>different</u> solutions.

Proof continued.

Suppose that $\mathbf{x}_1$ and $\mathbf{x}_2$ be two different solutions.So we are assuming that

- $A\mathbf{x}_1 = \mathbf{b}$,
- $A\mathbf{x}_2 = \mathbf{b}$ and
- $\mathbf{x}_1 \neq \mathbf{x}_2$

Proof continued.

Suppose that $\mathbf{x}_1$ and $\mathbf{x}_2$ be two different solutions.So we are assuming that

- $A\mathbf{x}_1 = \mathbf{b}$,
- $A\mathbf{x}_2 = \mathbf{b}$ and
- $\mathbf{x}_1 \neq \mathbf{x}_2$

Let $\mathbf{x}_0 = \mathbf{x}_1 - \mathbf{x}_2 \neq \mathbf{0}$.

Proof continued.

Suppose that $\mathbf{x}_1$ and $\mathbf{x}_2$ be two different solutions.So we are assuming that

- $A\mathbf{x}_1 = \mathbf{b}$,
- $A\mathbf{x}_2 = \mathbf{b}$ and
- $\mathbf{x}_1 \neq \mathbf{x}_2$

Let $\mathbf{x}_0 = \mathbf{x}_1 - \mathbf{x}_2 \neq \mathbf{0}$. Then

$$A\mathbf{x}_0 = A(\mathbf{x}_1 - \mathbf{x}_2) = A\mathbf{x}_1 - A\mathbf{x}_2 = \mathbf{b} - \mathbf{b} = 0.$$

Proof continued.

Suppose that $\mathbf{x}_1$ and $\mathbf{x}_2$ be two <u>different</u> solutions.So we are assuming that

- $A\mathbf{x}_1 = \mathbf{b}$,
- $A\mathbf{x}_2 = \mathbf{b}$ and
- $\mathbf{x}_1 \neq \mathbf{x}_2$

Let $\mathbf{x}_0 = \mathbf{x}_1 - \mathbf{x}_2 \neq \mathbf{0}$. Then

$$A\mathbf{x}_0 = A(\mathbf{x}_1 - \mathbf{x}_2) = A\mathbf{x}_1 - A\mathbf{x}_2 = \mathbf{b} - \mathbf{b} = 0.$$

Now let $k \in \mathbb{R}$ be any number. Then

$$A(\mathbf{x}_1 + k\mathbf{x}_0) =$$

Proof continued.

Suppose that $\mathbf{x}_1$ and $\mathbf{x}_2$ be two <u>different</u> solutions.So we are assuming that

- $A\mathbf{x}_1 = \mathbf{b}$,
- $A\mathbf{x}_2 = \mathbf{b}$ and
- $\mathbf{x}_1 \neq \mathbf{x}_2$

Let $\mathbf{x}_0 = \mathbf{x}_1 - \mathbf{x}_2 \neq \mathbf{0}$. Then

$$A\mathbf{x}_0 = A(\mathbf{x}_1 - \mathbf{x}_2) = A\mathbf{x}_1 - A\mathbf{x}_2 = \mathbf{b} - \mathbf{b} = 0.$$

Now let $k \in \mathbb{R}$ be any number. Then

$$A(\mathbf{x}_1 + k\mathbf{x}_0) = A\mathbf{x}_1 + kA\mathbf{x}_0 = \mathbf{b} + \mathbf{0} = \mathbf{b}.$$

Proof continued.

Suppose that $\mathbf{x}_1$ and $\mathbf{x}_2$ be two <u>different</u> solutions.So we are assuming that

- $A\mathbf{x}_1 = \mathbf{b}$,
- $A\mathbf{x}_2 = \mathbf{b}$ and
- $\mathbf{x}_1 \neq \mathbf{x}_2$

Let $\mathbf{x}_0 = \mathbf{x}_1 - \mathbf{x}_2 \neq \mathbf{0}$. Then

$$A\mathbf{x}_0 = A(\mathbf{x}_1 - \mathbf{x}_2) = A\mathbf{x}_1 - A\mathbf{x}_2 = \mathbf{b} - \mathbf{b} = 0.$$

Now let $k \in \mathbb{R}$ be any number. Then

$$A(\mathbf{x}_1 + k\mathbf{x}_0) = A\mathbf{x}_1 + kA\mathbf{x}_0 = \mathbf{b} + \mathbf{0} = \mathbf{b}.$$

So $(\mathbf{x}_1 + k\mathbf{x}_0)$ is a solution for any k.

Proof continued.

Suppose that $\mathbf{x}_1$ and $\mathbf{x}_2$ be two <u>different</u> solutions.So we are assuming that

- $A\mathbf{x}_1 = \mathbf{b}$,
- $A\mathbf{x}_2 = \mathbf{b}$ and
- $\mathbf{x}_1 \neq \mathbf{x}_2$

Let $\mathbf{x}_0 = \mathbf{x}_1 - \mathbf{x}_2 \neq \mathbf{0}$. Then

$$A\mathbf{x}_0 = A(\mathbf{x}_1 - \mathbf{x}_2) = A\mathbf{x}_1 - A\mathbf{x}_2 = \mathbf{b} - \mathbf{b} = 0.$$

Now let $k \in \mathbb{R}$ be any number. Then

$$A(\mathbf{x}_1 + k\mathbf{x}_0) = A\mathbf{x}_1 + kA\mathbf{x}_0 = \mathbf{b} + \mathbf{0} = \mathbf{b}.$$

So $(\mathbf{x}_1 + k\mathbf{x}_0)$ is a solution for any k. So $A\mathbf{x} = \mathbf{b}$ has infinitely many solutions. $\square$

Solving Linear Systems by Matrix Inversion

Theorem

If A is an invertible $n \times n$ matrix, then for each $n \times 1$ matrix $\mathbf{b}$, the linear system $A\mathbf{x} = \mathbf{b}$ has exactly one solution,

Solving Linear Systems by Matrix Inversion

Theorem

If A is an invertible $n \times n$ matrix, then for each $n \times 1$ matrix $\mathbf{b}$, the linear system $A\mathbf{x} = \mathbf{b}$ has exactly one solution, namely $\mathbf{x} = A^{-1}\mathbf{b}$.

Solving Linear Systems by Matrix Inversion

Theorem

If A is an invertible $n \times n$ matrix, then for each $n \times 1$ matrix $\mathbf{b}$, the linear system $A\mathbf{x} = \mathbf{b}$ has exactly one solution, namely $\mathbf{x} = A^{-1}\mathbf{b}$.

Proof.

Since $A(A^{-1}\mathbf{b}) = \mathbf{b}$, it follows that $A^{-1}\mathbf{b}$ is a solution of $A\mathbf{x} = \mathbf{b}$. We need to show that $A^{-1}\mathbf{b}$ is the only solution.

Solving Linear Systems by Matrix Inversion

Theorem

If A is an invertible $n \times n$ matrix, then for each $n \times 1$ matrix $\mathbf{b}$, the linear system $A\mathbf{x} = \mathbf{b}$ has exactly one solution, namely $\mathbf{x} = A^{-1}\mathbf{b}$.

Proof.

Since $A(A^{-1}\mathbf{b}) = \mathbf{b}$, it follows that $A^{-1}\mathbf{b}$ is a solution of $A\mathbf{x} = \mathbf{b}$. We need to show that $A^{-1}\mathbf{b}$ is the only solution.

Let $\mathbf{x}$ be any solution of $A\mathbf{x} = \mathbf{b}$. We need to show that $\mathbf{x} = A^{-1}\mathbf{b}$. So we calculate that

$$A\mathbf{x} = \mathbf{b}$$

Solving Linear Systems by Matrix Inversion

Theorem

If A is an invertible $n \times n$ matrix, then for each $n \times 1$ matrix $\mathbf{b}$, the linear system $A\mathbf{x} = \mathbf{b}$ has exactly one solution, namely $\mathbf{x} = A^{-1}\mathbf{b}$.

Proof.

Since $A(A^{-1}\mathbf{b}) = \mathbf{b}$, it follows that $A^{-1}\mathbf{b}$ is a solution of $A\mathbf{x} = \mathbf{b}$. We need to show that $A^{-1}\mathbf{b}$ is the only solution.

Let $\mathbf{x}$ be any solution of $A\mathbf{x} = \mathbf{b}$. We need to show that $\mathbf{x} = A^{-1}\mathbf{b}$. So we calculate that

$$A\mathbf{x} = \mathbf{b}$$

$$A^{-1}A\mathbf{x} = A^{-1}\mathbf{b}.$$

Solving Linear Systems by Matrix Inversion

Theorem

If A is an invertible $n \times n$ matrix, then for each $n \times 1$ matrix $\mathbf{b}$, the linear system $A\mathbf{x} = \mathbf{b}$ has exactly one solution, namely $\mathbf{x} = A^{-1}\mathbf{b}$.

Proof.

Since $A(A^{-1}\mathbf{b}) = \mathbf{b}$, it follows that $A^{-1}\mathbf{b}$ is a solution of $A\mathbf{x} = \mathbf{b}$. We need to show that $A^{-1}\mathbf{b}$ is the only solution.

Let $\mathbf{x}$ be any solution of $A\mathbf{x} = \mathbf{b}$. We need to show that $\mathbf{x} = A^{-1}\mathbf{b}$. So we calculate that

$$A\mathbf{x} = \mathbf{b}$$

$$\mathbf{x} = I\mathbf{x} = A^{-1}A\mathbf{x} = A^{-1}\mathbf{b}.$$

$\square$

Example

Solve $\begin{cases} \quad\quad\;\; x_2 + 2x_3 = 1 \\ x_1 \quad\quad\;\; + 3x_3 = 2 \\ 4x_1 - 3x_2 + 8x_3 = 3 \end{cases}.$

We can write this as

$$A\mathbf{x} = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 0 & 3 \\ 4 & -3 & 8 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \mathbf{b}.$$

Example

Solve $\begin{cases} x_2 + 2x_3 = 1 \\ x_1 + 3x_3 = 2 \\ 4x_1 - 3x_2 + 8x_3 = 3 \end{cases}$.

We can write this as

$$A\mathbf{x} = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 0 & 3 \\ 4 & -3 & 8 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \mathbf{b}.$$

Last week we found that the inverse of A is

$$A^{-1} = \begin{bmatrix} -\frac{9}{2} & 7 & -\frac{3}{2} \\ -2 & 4 & -1 \\ \frac{3}{2} & -2 & \frac{1}{2} \end{bmatrix}.$$

Example

Solve $\begin{cases} x_2 + 2x_3 = 1 \\ x_1 + 3x_3 = 2 \\ 4x_1 - 3x_2 + 8x_3 = 3 \end{cases}.$

We can write this as

$$A\mathbf{x} = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 0 & 3 \\ 4 & -3 & 8 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \mathbf{b}.$$

Last week we found that the inverse of A is

$$A^{-1} = \begin{bmatrix} -\frac{9}{2} & 7 & -\frac{3}{2} \\ -2 & 4 & -1 \\ \frac{3}{2} & -2 & \frac{1}{2} \end{bmatrix}.$$

Therefore the solution is

$$\mathbf{x} = A^{-1}\mathbf{b} = \begin{bmatrix} -\frac{9}{2} & 7 & -\frac{3}{2} \\ -2 & 4 & -1 \\ \frac{3}{2} & -2 & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 5 \\ 3 \\ -1 \end{bmatrix}.$$

Example (Solving 2 Linear Systems at Once)

Solve $\begin{cases} x_1 + 2x_2 + 3x_3 = 4 \\ 2x_1 + 5x_2 + 3x_3 = 5 \\ x_1 \qquad\quad + 8x_3 = 9 \end{cases}$ and $\begin{cases} x_1 + 2x_2 + 3x_3 = 1 \\ 2x_1 + 5x_2 + 3x_3 = 6 \\ x_1 \qquad\quad + 8x_3 = -6 \end{cases}$.

Example (Solving 2 Linear Systems at Once)

Solve $\begin{cases} x_1 + 2x_2 + 3x_3 = 4 \\ 2x_1 + 5x_2 + 3x_3 = 5 \\ x_1 \qquad\quad + 8x_3 = 9 \end{cases}$ and $\begin{cases} x_1 + 2x_2 + 3x_3 = 1 \\ 2x_1 + 5x_2 + 3x_3 = 6 \\ x_1 \qquad\quad + 8x_3 = -6 \end{cases}$.

Since the two systems have the same coefficient matrix, we can write one augmented matrix which includes both systems:

$$\begin{bmatrix} 1 & 2 & 3 & 4 & 1 \\ 2 & 5 & 3 & 5 & 6 \\ 1 & 0 & 8 & 9 & -6 \end{bmatrix}.$$

Example (Solving 2 Linear Systems at Once)

Solve $\begin{cases} x_1 + 2x_2 + 3x_3 = 4 \\ 2x_1 + 5x_2 + 3x_3 = 5 \\ x_1 + 8x_3 = 9 \end{cases}$ and $\begin{cases} x_1 + 2x_2 + 3x_3 = 1 \\ 2x_1 + 5x_2 + 3x_3 = 6 \\ x_1 + 8x_3 = -6 \end{cases}$.

Since the two systems have the same coefficient matrix, we can write one augmented matrix which includes both systems:

$$\begin{bmatrix} 1 & 2 & 3 & 4 & 1 \\ 2 & 5 & 3 & 5 & 6 \\ 1 & 0 & 8 & 9 & -6 \end{bmatrix}.$$

After using Gauss-Jordan Elimination (please check), we obtain:

$$\begin{bmatrix} 1 & 0 & 0 & 1 & 2 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & -1 \end{bmatrix}.$$

Example (Solving 2 Linear Systems at Once)

Solve $\begin{cases} x_1 + 2x_2 + 3x_3 = 4 \\ 2x_1 + 5x_2 + 3x_3 = 5 \\ x_1 \qquad\quad + 8x_3 = 9 \end{cases}$ and $\begin{cases} x_1 + 2x_2 + 3x_3 = 1 \\ 2x_1 + 5x_2 + 3x_3 = 6 \\ x_1 \qquad\quad + 8x_3 = -6 \end{cases}$.

After using Gauss-Jordan Elimination (please check), we obtain:

$$\begin{bmatrix} 1 & 0 & 0 & 1 & 2 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & -1 \end{bmatrix}.$$

So the solutions are $\begin{cases} x_1 = 1 \\ x_2 = 0 \\ x_3 = 1 \end{cases}$ and $\begin{cases} x_1 = 2 \\ x_2 = 1 \\ x_3 = -1 \end{cases}$.

Theorem

Let A and B be square matrices of the same size. If AB is invertible, then A and B must also be invertible.

Theorem

Let A and B be square matrices of the same size. If AB is invertible, then A and B must also be invertible.

Proof.

Suppose that AB is invertible.

The theorem at the start of today's lecture tells us that

B is invertible	$\iff$	$B\mathbf{x} = \mathbf{0}$ has only the trivial solution.

First we will use this to prove that B is invertible. Then we will prove that A is also invertible.

Proof continued.

Suppose that $\mathbf{x}$ is a solution to $B\mathbf{x} = \mathbf{0}$. Then

$$(AB)\mathbf{x} = A(B\mathbf{x}) = A\mathbf{0} = \mathbf{0}.$$

Proof continued.

Suppose that $\mathbf{x}$ is a solution to $B\mathbf{x} = \mathbf{0}$. Then

$$(AB)\mathbf{x} = A(B\mathbf{x}) = A\mathbf{0} = \mathbf{0}.$$

Since AB is invertible, this implies that $\mathbf{x} = \mathbf{0}$. Hence the trivial solution is the only solution to $B\mathbf{x} = \mathbf{0}$. Therefore B must be invertible.

Proof continued.

Suppose that $\mathbf{x}$ is a solution to $B\mathbf{x} = \mathbf{0}$. Then

$$(AB)\mathbf{x} = A(B\mathbf{x}) = A\mathbf{0} = \mathbf{0}.$$

Since AB is invertible, this implies that $\mathbf{x} = \mathbf{0}$. Hence the trivial solution is the only solution to $B\mathbf{x} = \mathbf{0}$. Therefore B must be invertible.

Now since both AB and B^{-1} are invertible matrices, it follows that

$$A = AI = A(BB^{-1}) = (AB)B^{-1}$$

is the product of two invertible matrices and hence is also invertible. □

Diagonal, Triangular, and Symmetric Matrices

Remark

Your textbook doesn't have a section on this. Instead these ideas are spread through various sections and exercises.

I think that it makes sense to introduce these concepts now so that you are familar with them when we need them later in the course.

Diagonal Matrices

Definition

A square matrix in which all the entries off the main diagonal are zero is called a *diagonal matrix*.

Example

$$\begin{bmatrix} 2 & 0 \\ 0 & -5 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} 6 & 0 & 0 & 0 \\ 0 & -4 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 8 \end{bmatrix}, \quad \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

are diagonal matrices.

A general $n \times n$ diagonal matrix D can be written as

$$D = \begin{bmatrix} d_1 & 0 & \cdots & 0 \\ 0 & d_2 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & d_n \end{bmatrix}.$$

A general $n \times n$ diagonal matrix D can be written as

$$D = \begin{bmatrix} d_1 & 0 & \cdots & 0 \\ 0 & d_2 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & d_n \end{bmatrix}.$$

D is invertible if and only if $d_k \neq 0$ for all k;

A general $n \times n$ diagonal matrix D can be written as

$$D = \begin{bmatrix} d_1 & 0 & \cdots & 0 \\ 0 & d_2 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & d_n \end{bmatrix}.$$

D is invertible if and only if $d_k \neq 0$ for all k; in this case its inverse is

$$D^{-1} = \begin{bmatrix} \frac{1}{d_1} & 0 & \cdots & 0 \\ 0 & \frac{1}{d_2} & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & \frac{1}{d_n} \end{bmatrix}.$$

(Check what you get if you multiply D and D^{-1} together.)

Powers of diagonal matrices are easy to calculate. I leave it for you to check that if

$$D = \begin{bmatrix} d_1 & 0 & \cdots & 0 \\ 0 & d_2 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & d_n \end{bmatrix}$$

and if $k \in \mathbb{N}$, then

$$D^k = \begin{bmatrix} d_1^k & 0 & \cdots & 0 \\ 0 & d_2^k & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & d_n^k \end{bmatrix}.$$

Example

If

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & 2 \end{bmatrix},$$

then

$$A^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -\frac{1}{3} & 0 \\ 0 & 0 & \frac{1}{2} \end{bmatrix},$$

$$A^5 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -243 & 0 \\ 0 & 0 & 32 \end{bmatrix} \quad \text{and} \quad A^{-5} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -\frac{1}{243} & 0 \\ 0 & 0 & \frac{1}{32} \end{bmatrix}.$$

It is easy to calculate the product of two matrices if one is a diagonal matrix.

It is easy to calculate the product of two matrices if one is a diagonal matrix.

$$\begin{bmatrix} d_1 & 0 & 0 \\ 0 & d_2 & 0 \\ 0 & 0 & d_3 \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \end{bmatrix} =$$

It is easy to calculate the product of two matrices if one is a diagonal matrix.

$$\begin{bmatrix} d_1 & 0 & 0 \\ 0 & d_2 & 0 \\ 0 & 0 & d_3 \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \end{bmatrix} = \begin{bmatrix} d_1 a_{11} & d_1 a_{12} & d_1 a_{13} & d_1 a_{14} \\ d_2 a_{21} & d_2 a_{22} & d_2 a_{23} & d_2 a_{24} \\ d_3 a_{31} & d_3 a_{32} & d_3 a_{33} & d_3 a_{34} \end{bmatrix}$$

It is easy to calculate the product of two matrices if one is a diagonal matrix.

$$\begin{bmatrix} d_1 & 0 & 0 \\ 0 & d_2 & 0 \\ 0 & 0 & d_3 \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \end{bmatrix} = \begin{bmatrix} d_1a_{11} & d_1a_{12} & d_1a_{13} & d_1a_{14} \\ d_2a_{21} & d_2a_{22} & d_2a_{23} & d_2a_{24} \\ d_3a_{31} & d_3a_{32} & d_3a_{33} & d_3a_{34} \end{bmatrix}$$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \end{bmatrix} \begin{bmatrix} d_1 & 0 & 0 & 0 \\ 0 & d_2 & 0 & 0 \\ 0 & 0 & d_3 & 0 \\ 0 & 0 & 0 & d_4 \end{bmatrix} = \begin{bmatrix} d_1a_{11} & d_2a_{12} & d_3a_{13} & d_4a_{14} \\ d_1a_{21} & d_2a_{22} & d_3a_{23} & d_4a_{24} \\ d_1a_{31} & d_2a_{32} & d_3a_{33} & d_4a_{34} \end{bmatrix}$$

Triangular Matrices

$$\underbrace{\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ 0 & a_{22} & a_{23} & a_{24} \\ 0 & 0 & a_{33} & a_{34} \\ 0 & 0 & 0 & a_{44} \end{bmatrix}}_{\text{upper triangular } 4\times 4}$$

Definition

A square matrix in which all the entries below the main diagonal are zero is called *upper triangular*.

Triangular Matrices

$$\underbrace{\begin{bmatrix} a_{11} & 0 & 0 & 0 \\ a_{21} & a_{22} & 0 & 0 \\ a_{31} & a_{32} & a_{33} & 0 \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix}}_{\text{lower triangular } 4\times 4}$$

Definition

A square matrix in which all the entries above the main diagonal are zero is called *lower triangular*.

Triangular Matrices

$$\underbrace{\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ 0 & a_{22} & a_{23} & a_{24} \\ 0 & 0 & a_{33} & a_{34} \\ 0 & 0 & 0 & a_{44} \end{bmatrix}}_{\text{upper triangular } 4\times 4} \quad \text{or} \quad \underbrace{\begin{bmatrix} a_{11} & 0 & 0 & 0 \\ a_{21} & a_{22} & 0 & 0 \\ a_{31} & a_{32} & a_{33} & 0 \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix}}_{\text{lower triangular } 4\times 4}$$

Definition

A matrix that is either upper triangular or lower triangular is called *triangular*.

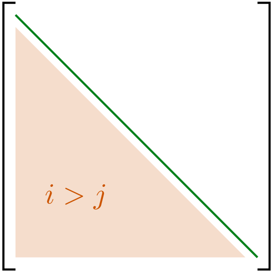
Remark

Note that diagonal matrices are both upper triangular and lower triangular.

Remark

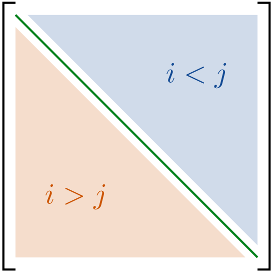
A square matrix in row echelon form is upper triangular since it has zeros below the main diagonal.

0.0 Diagonal, Triangular, and Symmetric Matrices



A square matrix $A = [a_{ij}]$ is

- *upper triangular* $\iff$ $a_{ij} = 0$ for all $i > j$;



A square matrix $A = [a_{ij}]$ is

- *upper triangular* $\iff$ $a_{ij} = 0$ for all $i > j$;
- *lower triangular* $\iff$ $a_{ij} = 0$ for all $i < j$.

Let

$$L = \text{a lower triangular matrix}$$
$$U = \text{an upper triangular matrix}$$

Theorem

1. $L^T = U$
2. $U^T = L$

Let

$$L = \text{a lower triangular matrix}$$
$$U = \text{an upper triangular matrix}$$

Theorem

1. $L^T = U$
2. $U^T = L$
3. $L_1 L_2 = L$
4. $U_1 U_2 = U$.

Let

$$L = \text{a lower triangular matrix}$$
$$U = \text{an upper triangular matrix}$$

Theorem

1. $L^T = U$
2. $U^T = L$
3. $L_1 L_2 = L$
4. $U_1 U_2 = U$.
5. *A triangular matrix is invertible iff its diagonal entries are all nonzero.*

Let

$$L = \text{a lower triangular matrix}$$
$$U = \text{an upper triangular matrix}$$

Theorem

1. $L^T = U$
2. $U^T = L$
3. $L_1 L_2 = L$
4. $U_1 U_2 = U$.
5. *A triangular matrix is invertible iff its diagonal entries are all nonzero.*
6. L^{-1} *(if it exists) is lower triangular.*
7. U^{-1} *(if it exists) is upper triangular.*

Example

Consider the upper triangular matrices

$$A = \begin{bmatrix} 1 & 3 & -1 \\ 0 & 2 & 4 \\ 0 & 0 & 5 \end{bmatrix} \qquad \text{and} \qquad B = \begin{bmatrix} 3 & -2 & 2 \\ 0 & 0 & -1 \\ 0 & 0 & 1 \end{bmatrix}.$$

Since all the entries on the main diagonal of A are nonzero, A must be invertible. Since B has a 0 on its main diagonal, B is singular.

Example

Consider the upper triangular matrices

$$A = \begin{bmatrix} 1 & 3 & -1 \\ 0 & 2 & 4 \\ 0 & 0 & 5 \end{bmatrix} \qquad \text{and} \qquad B = \begin{bmatrix} 3 & -2 & 2 \\ 0 & 0 & -1 \\ 0 & 0 & 1 \end{bmatrix}.$$

Since all the entries on the main diagonal of A are nonzero, A must be invertible. Since B has a 0 on its main diagonal, B is singular.

The theorem tells us that A^{-1}, AB and BA will also be upper triangular.

Example

Consider the upper triangular matrices

$$A = \begin{bmatrix} 1 & 3 & -1 \\ 0 & 2 & 4 \\ 0 & 0 & 5 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 3 & -2 & 2 \\ 0 & 0 & -1 \\ 0 & 0 & 1 \end{bmatrix}.$$

Since all the entries on the main diagonal of A are nonzero, A must be invertible. Since B has a 0 on its main diagonal, B is singular.

The theorem tells us that A^{-1}, AB and BA will also be upper triangular. I leave it for you to check that

$$A^{-1} = \begin{bmatrix} 1 & -\frac{3}{2} & \frac{7}{5} \\ 0 & \frac{1}{2} & -\frac{2}{5} \\ 0 & 0 & \frac{1}{5} \end{bmatrix}, \; AB = \begin{bmatrix} 3 & -2 & -2 \\ 0 & 0 & 2 \\ 0 & 0 & 5 \end{bmatrix}, \; BA = \begin{bmatrix} 3 & 5 & -1 \\ 0 & 0 & -5 \\ 0 & 0 & 5 \end{bmatrix}.$$

Symmetric Matrices

Definition

A square matrix A is called *symmetric* if $A = A^T$.

Example

$$\begin{bmatrix} 7 & -3 \\ -3 & 5 \end{bmatrix}, \qquad \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 4 & 5 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} d_1 & 0 & 0 & 0 \\ 0 & d_2 & 0 & 0 \\ 0 & 0 & d_3 & 0 \\ 0 & 0 & 0 & d_4 \end{bmatrix}$$

are symmetric matrices.

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 4 & 5 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 4 & 5 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 4 & 5 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 4 & 5 \end{bmatrix}$$

Remark

The matrix $A = [a_{ij}]$ is symmetrical iff

$$a_{ij} = a_{ja}$$

for all i and j.

Theorem

Let A and B be symmetric matrices with the same size, and let k be a number. Then

1. A^T *is symmetric;*
2. $A + B$ *and* $A - B$ *are symmetric;*
3. kA *is symmetric.*

Remark

It is not true, in general, that the product of two symmetric matrices is symmetric.

Since

$$(AB)^T = B^T A^T = BA$$

(if A and B are symmetric) we have $(AB)^T = AB$ if and only if $AB = BA$. Thus...

Remark

It is not true, in general, that the product of two symmetric matrices is symmetric.

Since

$$(AB)^T = B^T A^T = BA$$

(if A and B are symmetric) we have $(AB)^T = AB$ if and only if $AB = BA$. Thus...

Theorem

The product of two symmetric matrices A and B is symmetric if and only if A and B commute (i.e. if $AB = BA$).

Example

Note that

$$\begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} -4 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} -2 & 1 \\ -5 & 2 \end{bmatrix}$$

and

$$\begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} -4 & 3 \\ 3 & -1 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix}.$$

Example

Note that

$$\begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} -4 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} -2 & 1 \\ -5 & 2 \end{bmatrix}$$

and

$$\begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} -4 & 3 \\ 3 & -1 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix}.$$

So the latter pair of symmetric commute, but the first pair do not.

Invertibility of Symmetric Matrices

Note that the matrix

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix}$$

is symmetric but not invertible (because it has a zero on its main diagonal).

Invertibility of Symmetric Matrices

Note that the matrix

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix}$$

is symmetric but not invertible (because it has a zero on its main diagonal).

Theorem

If A is an invertible symmetric matrix, then A^{-1} is symmetric.

Proof.

$$A = A^T \implies (A^{-1})^T = (A^T)^{-1} = A^{-1} \implies A^{-1} \text{ is symmetric.}$$

$\square$

AA^T and A^TA

Note that if A is an $m \times n$ matrix, then A^T is an $n \times m$ matrix, so the products AA^T and A^TA are both square matrices.

AA^T and A^TA

Note that if A is an $m \times n$ matrix, then A^T is an $n \times m$ matrix, so the products AA^T and A^TA are both square matrices. Moreover, since $(AB)^T = B^TA^T$, we have

$$\left(AA^T\right)^T = (A^T)^T A^T$$

AA^T and A^TA

Note that if A is an $m \times n$ matrix, then A^T is an $n \times m$ matrix, so the products AA^T and A^TA are both square matrices. Moreover, since $(AB)^T = B^TA^T$, we have

$$\left(AA^T\right)^T = (A^T)^T A^T = AA^T$$

AA^T and $A^T A$

Note that if A is an $m \times n$ matrix, then A^T is an $n \times m$ matrix, so the products AA^T and $A^T A$ are both square matrices. Moreover, since $(AB)^T = B^T A^T$, we have

$$\left(AA^T\right)^T = (A^T)^T A^T = AA^T$$

and

$$\left(A^T A\right)^T = A^T (A^T)^T = A^T A$$

which shows that both AA^T and $A^T A$ are symmetric.

Example

Let

$$A = \begin{bmatrix} 1 & -2 & 4 \\ 3 & 0 & -5 \end{bmatrix}.$$

Please check that

$$A^T A = \begin{bmatrix} 1 & 3 \\ -2 & 0 \\ 4 & -5 \end{bmatrix} \begin{bmatrix} 1 & -2 & 4 \\ 3 & 0 & -5 \end{bmatrix} = \begin{bmatrix} 10 & -2 & -11 \\ -2 & 4 & -8 \\ -11 & -8 & 41 \end{bmatrix}$$

and

$$AA^T = \begin{bmatrix} 1 & -2 & 4 \\ 3 & 0 & -5 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ -2 & 0 \\ 4 & -5 \end{bmatrix} = \begin{bmatrix} 21 & -17 \\ -17 & 34 \end{bmatrix}.$$

Theorem

If A is an invertible matrix, then AA^T and A^TA are also invertible.

Proof.

A is invertible $\implies$ A^T is invertible. Recall that the product of two invertible matrices is invertible. $\square$

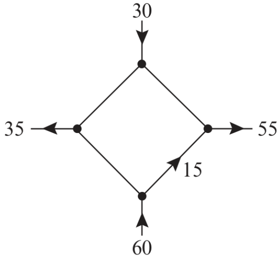
That's enough about AA^T and A^TA for now. We will come back to them later in the course.

Break

We will continue at 3pm

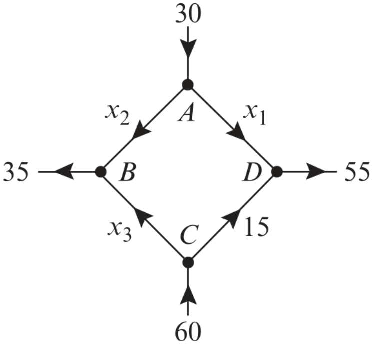
Applications of linear systems.

Network Analysis

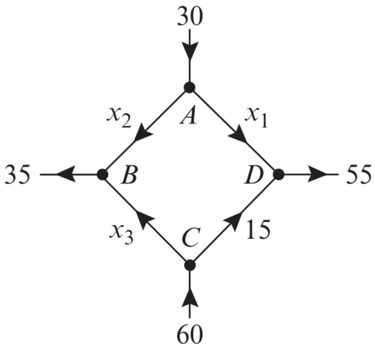


Example

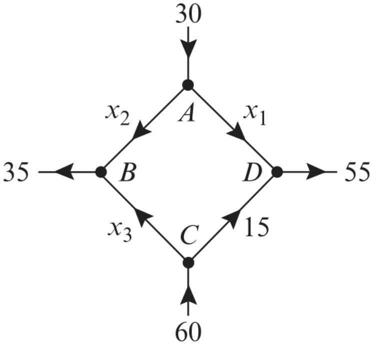
Consider a network with four nodes in which the flow rate and direction of flow in certain branches are known. Find the flow rates and directions of flow in the remaining branches.



- At node A we have $x_1 + x_2 = 30$;

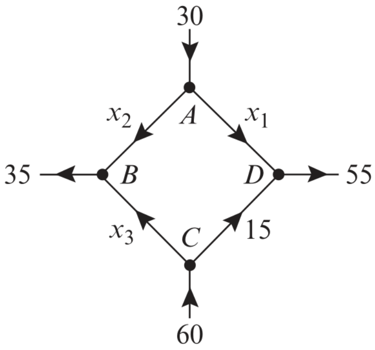


- At node A we have $x_1 + x_2 = 30$;
- At node B we have $x_2 + x_3 = 35$;

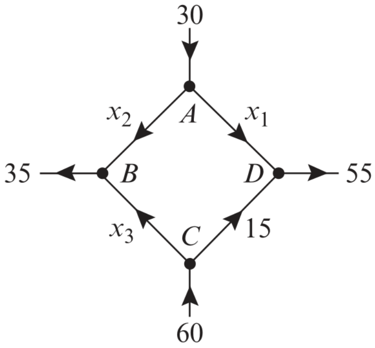


- At node A we have $x_1 + x_2 = 30$;
- At node B we have $x_2 + x_3 = 35$;
- At node C we have $x_3 + 15 = 60$;

1.6 Applications of linear systems.

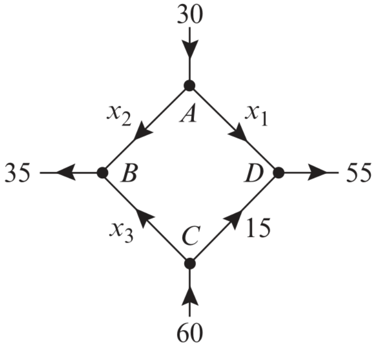


- At node A we have $x_1 + x_2 = 30$;
- At node B we have $x_2 + x_3 = 35$;
- At node C we have $x_3 + 15 = 60$; and
- At node D we have $x_1 + 15 = 55$.



So we have a linear system

$$\begin{cases} x_1 + x_2 & = 30 \\ \qquad x_2 + x_3 & = 35 \\ \qquad\qquad x_3 + 15 & = 60 \\ x_1 \qquad\qquad + 15 & = 55. \end{cases}$$



So we have a linear system

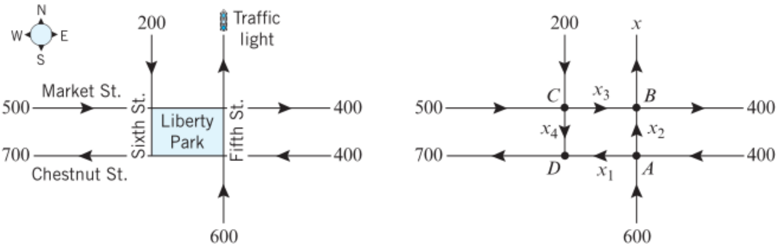
$$\begin{cases} x_1 + x_2 \qquad\qquad = 30 \\ \qquad x_2 + x_3 \qquad = 35 \\ \qquad\qquad x_3 + 15 = 60 \\ x_1 \qquad\qquad + 15 = 55. \end{cases} \quad \Longrightarrow \quad \begin{cases} x_1 = \ \ 40 \\ x_2 = -10 \\ x_3 = \ \ 45. \end{cases}$$

▶ EXAMPLE 2 Design of Traffic Patterns

The network in Figure 1.9.3 shows a proposed plan for the traffic flow around a new park that will house the Liberty Bell in Philadelphia, Pennsylvania. The plan calls for a computerized traffic light at the north exit on Fifth Street, and the diagram indicates the average number of vehicles per hour that are expected to flow in and out of the streets that border the complex. All streets are one-way.

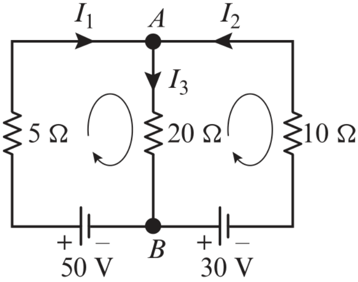
(a) How many vehicles per hour should the traffic light let through to ensure that the average number of vehicles per hour flowing into the complex is the same as the average number of vehicles flowing out?

(b) Assuming that the traffic light has been set to balance the total flow in and out of the complex, what can you say about the average number of vehicles per hour that will flow along the streets that border the complex?



(a) (b)

Electric Circuits



▲ Figure 1.9.9

▶ EXAMPLE 4 **A Circuit with Three Closed Loops**

Determine the currents I_1, I_2, and I_3 in the circuit shown in Figure 1.9.9.

Balancing Chemical Equations

Example

Balance the chemical equation

$$\underset{\text{hydrochloric acid}}{HCl} + \underset{\text{sodium phosphate}}{Na_3PO_4} \longrightarrow \underset{\text{phosphoric acid}}{H_3PO_4} + \underset{\text{sodium chloride}}{NaCl}$$

Balancing Chemical Equations

Example

Balance the chemical equation

$$\mathrm{HCl} + \mathrm{Na_3PO_4} \longrightarrow \mathrm{H_3PO_4} + \mathrm{NaCl}$$

hydrochloric acid

sodium phosphate

phosphoric acid

sodium chloride

We need to find natural numbers x_1, x_2, x_3, x_4 such that

$$x_1\,\mathrm{HCl} + x_2\,\mathrm{Na_3PO_4} \longrightarrow x_3\,\mathrm{H_3PO_4} + x_4\,\mathrm{NaCl}$$

is balanced (same number of each atom on each side).

1.6 Applications of linear systems.

$$x_1\,\mathrm{HCl} + x_2\,\mathrm{Na_3PO_4} \longrightarrow x_3\,\mathrm{H_3PO_4} + x_4\,\mathrm{NaCl}$$

$$1x_1 = 3x_3 \qquad \text{(Hydrogen H)}$$

(Chlorine Cl)

(Sodium Na)

(Phosphorus P)

(Oxygen O)

1.6 Applications of linear systems.

$$x_1\,\mathrm{HCl} + x_2\,\mathrm{Na_3PO_4} \longrightarrow x_3\,\mathrm{H_3PO_4} + x_4\,\mathrm{NaCl}$$

$$\begin{aligned}
1x_1 &= 3x_3 && \text{(Hydrogen H)}\\
1x_1 &= 1x_4 && \text{(Chlorine Cl)}\\
& && \text{(Sodium Na)}\\
& && \text{(Phosphorus P)}\\
& && \text{(Oxygen O)}
\end{aligned}$$

1.6 Applications of linear systems.

$$x_1\,\mathrm{HCl} + x_2\,\mathrm{Na_3PO_4} \longrightarrow x_3\,\mathrm{H_3PO_4} + x_4\,\mathrm{NaCl}$$

$$1x_1 = 3x_3 \qquad \text{(Hydrogen H)}$$
$$1x_1 = 1x_4 \qquad \text{(Chlorine Cl)}$$
$$3x_2 = 1x_4 \qquad \text{(Sodium Na)}$$
$$\text{(Phosphorus P)}$$
$$\text{(Oxygen O)}$$

$$x_1\,\mathrm{HCl} + x_2\,\mathrm{Na_3PO_4} \longrightarrow x_3\,\mathrm{H_3PO_4} + x_4\,\mathrm{NaCl}$$

$$\begin{aligned}
1x_1 &= 3x_3 && \text{(Hydrogen H)}\\
1x_1 &= 1x_4 && \text{(Chlorine Cl)}\\
3x_2 &= 1x_4 && \text{(Sodium Na)}\\
1x_2 &= 1x_3 && \text{(Phosphorus P)}\\
& && \text{(Oxygen O)}
\end{aligned}$$

$$x_1\,\mathrm{HCl} + x_2\,\mathrm{Na_3PO_4} \longrightarrow x_3\,\mathrm{H_3PO_4} + x_4\,\mathrm{NaCl}$$

$$\begin{aligned}
1x_1 &= 3x_3 && \text{(Hydrogen H)}\\
1x_1 &= 1x_4 && \text{(Chlorine Cl)}\\
3x_2 &= 1x_4 && \text{(Sodium Na)}\\
1x_2 &= 1x_3 && \text{(Phosphorus P)}\\
4x_2 &= 4x_3 && \text{(Oxygen O)}
\end{aligned}$$

So we have a linear system

$$\begin{cases} x_1 \qquad\quad - 3x_3 \qquad\;\; = 0 \\ x_1 \qquad\qquad\qquad - x_4 = 0 \\ \qquad 3x_2 \qquad\quad - x_4 = 0 \\ \qquad\; x_2 - \;\; x_3 \qquad\;\; = 0 \\ \qquad 4x_2 - 4x_3 \qquad\;\; = 0 \end{cases}$$

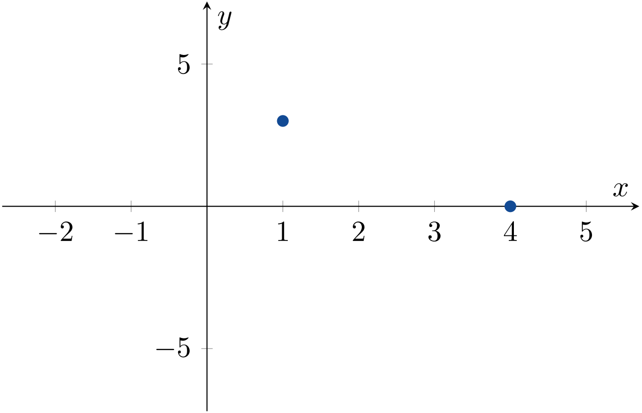
which has solution

$$\begin{cases} x_1 = x_4 \\ x_2 = \frac{1}{3}x_4 \\ x_3 = \frac{1}{3}x_4 \\ x_4 \text{ is free.} \end{cases}$$

Since we want natural numbers, we choose $x_4 = 3$. Then we have $x_1 = 3$, $x_2 = 1$ and $x_3 = 1$. The balanced equation is

$$3\,\mathrm{HCl} + \mathrm{Na_3PO_4} \longrightarrow \mathrm{H_3PO_4} + 3\,\mathrm{NaCl}.$$

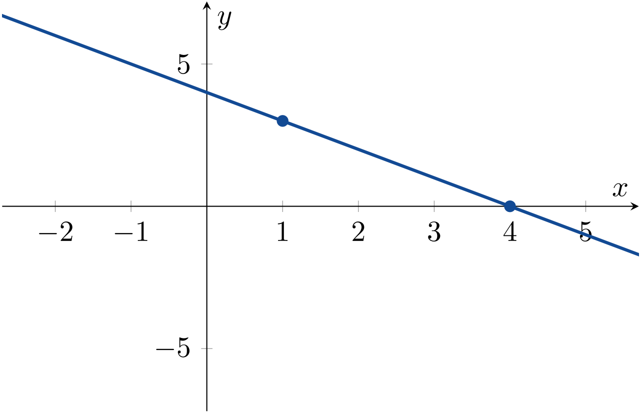
Polynomial Interpolation



If I have 2 points, I can find a unique line through them.

$y = a_0 + a_1 x$

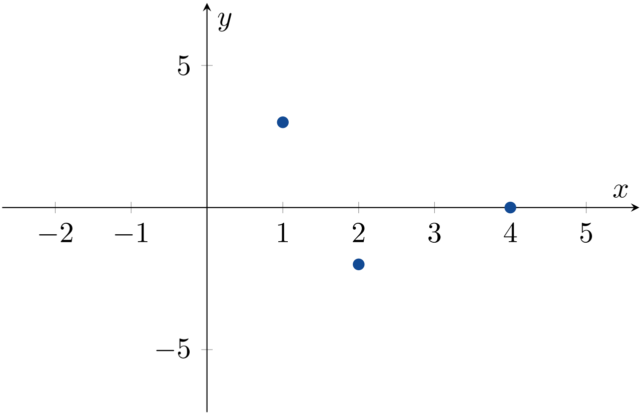
Polynomial Interpolation



If I have 2 points, I can find a unique line through them.

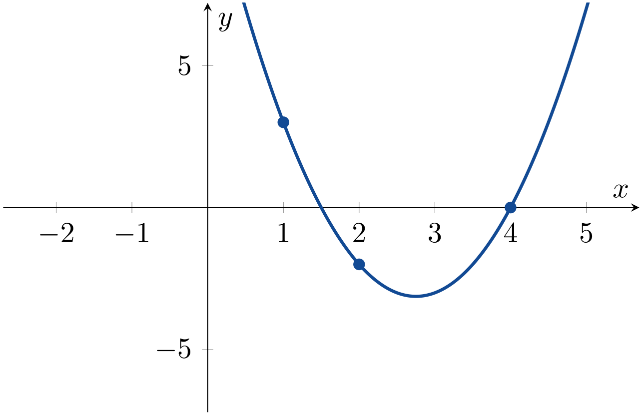
$y = a_0 + a_1 x$

Polynomial Interpolation



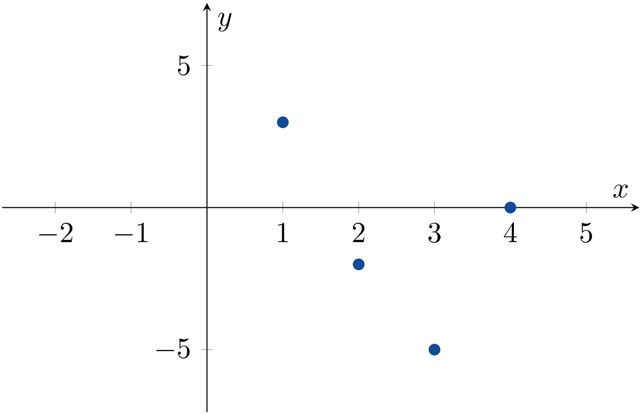
If I have 3 points, I can find a unique quadratic polynomial through them. $y = a_0 + a_1 x + a_2 x^2$

Polynomial Interpolation



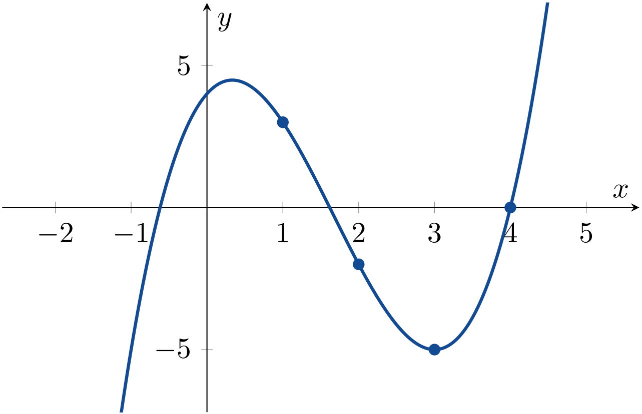
If I have 3 points, I can find a unique quadratic polynomial through them. $y = a_0 + a_1 x + a_2 x^2$

Polynomial Interpolation



If I have 4 points, I can find a unique cubic polynomial through them. $y = a_0 + a_1 x + a_2 x^2 + a_3 x^3$

Polynomial Interpolation



If I have 4 points, I can find a unique cubic polynomial through them. $y = a_0 + a_1 x + a_2 x^2 + a_3 x^3$

Example

Find a cubic polynomial whose graph passes through the points

$$(1,3), \qquad (2,-2), \qquad (3,-5), \qquad (4,0).$$

We are looking for a function

$$y = a_0 + a_1 x + a_2 x^2 + a_3 x^3$$

which passes through these four points.

Example

Find a cubic polynomial whose graph passes through the points

$$(1,3), \qquad (2,-2), \qquad (3,-5), \qquad (4,0).$$

We are looking for a function

$$y = a_0 + a_1 x + a_2 x^2 + a_3 x^3$$

which passes through these four points.

At the point $(x, y) = (1, 3)$ we have

$$a_0 + a_1 + a_2 + a_3 = 3 \qquad (1,3)$$

$$(2,-2)$$

$$(3,-5)$$

$$(4,0)$$

Example

Find a cubic polynomial whose graph passes through the points

$$(1, 3), \qquad (2, -2), \qquad (3, -5), \qquad (4, 0).$$

We are looking for a function

$$y = a_0 + a_1 x + a_2 x^2 + a_3 x^3$$

which passes through these four points.

At the point $(x, y) = (1, 3)$ we have

$$a_0 + a_1 + a_2 + a_3 = 3 \tag{1,3}$$

Similarly

$$\begin{aligned} a_0 + a_1 2 + a_2 4 + a_3 8 &= -2 && (2,-2) \\ a_0 + a_1 3 + a_2 9 + a_3 27 &= -5 && (3,-5) \\ a_0 + a_1 4 + a_2 16 + a_3 64 &= 0. && (4,0) \end{aligned}$$

So we have a linear system with augmented matrix

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 3 \\ 1 & 2 & 4 & 8 & -2 \\ 1 & 3 & 9 & 27 & -5 \\ 1 & 4 & 16 & 64 & 0 \end{bmatrix}$$

So we have a linear system with augmented matrix

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 3 \\ 1 & 2 & 4 & 8 & -2 \\ 1 & 3 & 9 & 27 & -5 \\ 1 & 4 & 16 & 64 & 0 \end{bmatrix}$$

which we can row reduce to

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 4 \\ 0 & 1 & 0 & 0 & 3 \\ 0 & 0 & 1 & 0 & -5 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix}$$

So we have a linear system with augmented matrix

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 3 \\ 1 & 2 & 4 & 8 & -2 \\ 1 & 3 & 9 & 27 & -5 \\ 1 & 4 & 16 & 64 & 0 \end{bmatrix}$$

which we can row reduce to

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 4 \\ 0 & 1 & 0 & 0 & 3 \\ 0 & 0 & 1 & 0 & -5 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix} \qquad \Longrightarrow \qquad \begin{cases} a_0 = 4 \\ a_1 = 3 \\ a_2 = -5 \\ a_3 = 1 \end{cases}$$

So we have a linear system with augmented matrix

$$\left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 3 \\ 1 & 2 & 4 & 8 & -2 \\ 1 & 3 & 9 & 27 & -5 \\ 1 & 4 & 16 & 64 & 0 \end{array}\right]$$

which we can row reduce to

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 4 \\ 0 & 1 & 0 & 0 & 3 \\ 0 & 0 & 1 & 0 & -5 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix} \implies \begin{cases} a_0 = 4 \\ a_1 = 3 \\ a_2 = -5 \\ a_3 = 1 \end{cases}$$

Hence our function is

$$y = 4 + 3x - 5x^2 + x^3.$$

There are more examples in your textbook. See sections 1.6 and 1.10.

Google

Search engines, such as Google, rely on linear algebra.

After you finish this course, I encourage you to read section 10.2 in your textbook to understand how Google's PageRank works.

3.1 Introduction to Determinants

Next Time

-