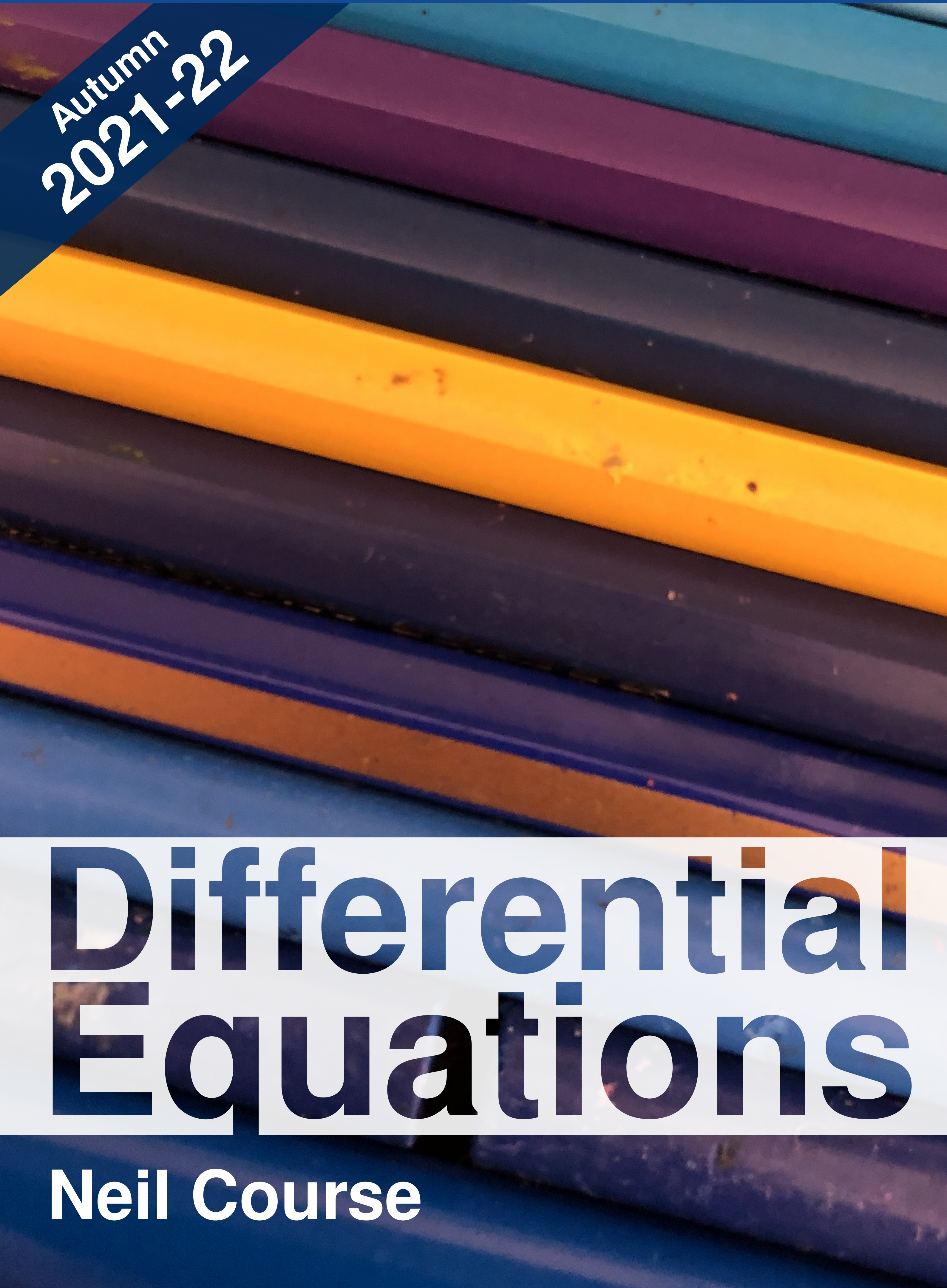
First Order • Second and Higher Order • Laplace Transform • Systems

Autumn 2021-22



Differential Equations

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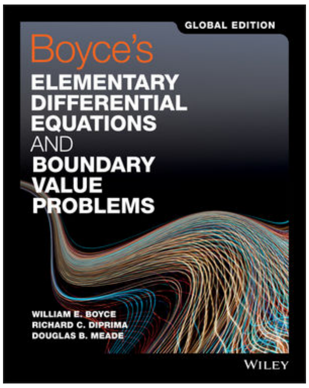
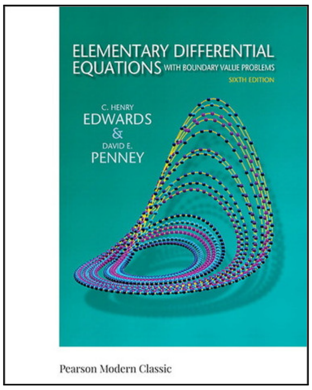
Suggested further reading:



William E. Boyce, Richard C. DiPrima and Douglas B. Meade, ***Boyce's Elementary Differential Equations and Boundary Value Problems***, Wiley.
ISBN: 978-1-119-39074-9

C. Henry Edwards and David E. Penney, ***Elementary Differential Equations with Boundary Value Problems***, Pearson.
ISBN: 978-0-134-99541-0



Thursday 9th December, 2021

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1 Introduction

1.1 Introduction

If I say "Solve $x+3=5$", I am looking for some number x which satisfies this equation. But if I say "Solve $\frac{dy}{dx}=2x$", then I don't want a number; I am looking for a function $y(x)$ which satisfies this equation.

A ***differential equation*** is an equation containing a derivative. For example, the equation $\frac{dy}{dx}=2x$ contains a derivative – so it is a differential equation.

Example 1.1. Solve $\dfrac{dy}{dx}=2x$.

This differential equation is easy to solve:

$$y(x)=\int \frac{dy}{dx}\,dx=\int 2x\,dx=x^2+c.$$

Example 1.2. Solve $\begin{cases}\frac{dy}{dx}=2x\\ y(0)=5.\end{cases}$

This problem has two conditions. The first line says $\frac{dy}{dx}=2x$ which is a differential equation. The second line, $y(0)=5$ is called an ***initial condition***. All together, the problem is called an ***initial value problem*** or IVP.

We know that the solution to the differential equation is $y(x)=x^2+c$. To solve this IVP, we need to choose the correct value of c so that the initial condition is also satisfied.

We calculate that

$$5=y(0)=0^2+c=c \qquad \Longrightarrow \qquad c=5.$$

Therefore the solution to the IVP is

$$\boxed{y(x)=x^2+5.}$$

Example 1.3. Solve $\begin{cases} \frac{dy}{dx} = \sin x \\ y(0) = 3. \end{cases}$

(First we must solve the differential equation. Then we must choose the correct value of c.)

We calculate that

$$y(x) = \int \frac{dy}{dx}\,dx = \int \sin x\,dx = -\cos x + c$$

and

$$3 = y(0) = -\cos 0 + c = -1 + c \qquad \Longrightarrow \qquad c = 4.$$

Therefore the solution to the IVP is

$$\boxed{y(x) = -\cos x + 4.}$$

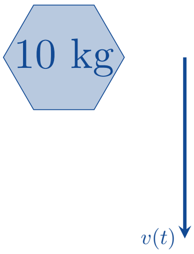
Example 1.4. Solve $\dfrac{dy}{dx} = y.$

This is harder: This time we can't just integrate $\frac{dy}{dx}$ to find $y(x)$. I will show you how to solve this later.

1.2 Some Examples

Many problems in engineering, science and the social sciences can be modelled using differential equations. We start with 3 examples.

Example 1.5 (A Falling Object)**.** Suppose that an object of mass 10 kg is falling.



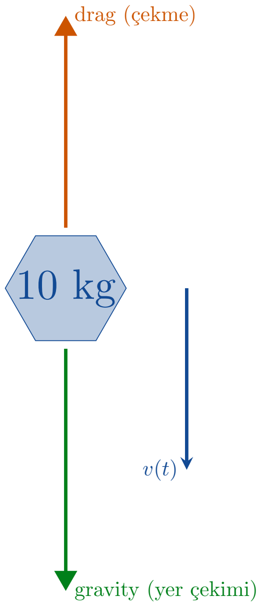
Let

- $v(t)$ denote the velocity (downwards) of the object in ms^{-1}; and
- t denote time in seconds.

Newton's Second Law says

$$\begin{aligned}\text{force} &= \text{mass} \times \text{acceleration}\\ &= 10 \times \frac{dv}{dt}.\end{aligned}$$

Note that $\frac{dv}{dt}$ is measured in $\frac{\text{ms}^{-1}}{\text{s}} = \text{ms}^{-2}$.



Now

$$\text{force} = \text{gravity} - \text{drag}.$$

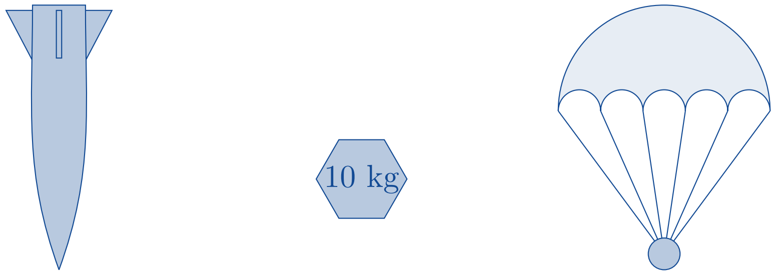
On the Earth, the gravity on an object of mass 10 kg is approximately

$$\text{gravity} = 10g$$

(where $g = 9.8\ \text{ms}^{-2}$). It is reasonable to assume (if the object isn't travelling too quickly) that

$$\begin{aligned}&\text{drag is proportional to velocity}\\ &\text{drag} \propto v\\ &\text{drag} = \gamma v\end{aligned}$$

where $\gamma > 0$ is a constant depending on the shape of the object.



small $\gamma > 0$ $\gamma = 2\ \mathrm{kg\,s^{-1}}$ big $\gamma > 0$

If $\gamma = 2\ \mathrm{kg\,s^{-1}}$, then we have that

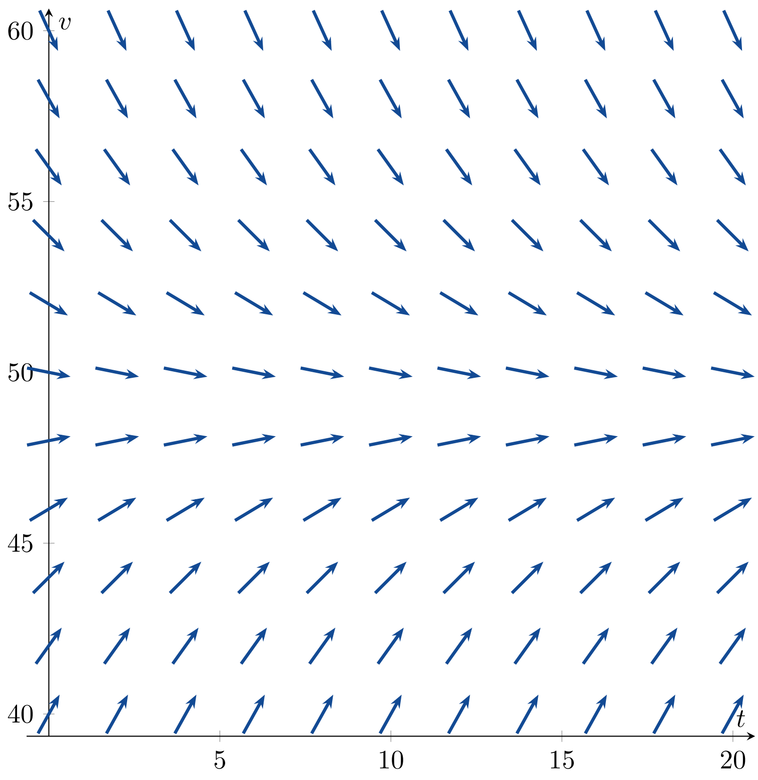
$$10\frac{dv}{dt} = \text{force} = \text{gravity} - \text{drag} = 10g - \gamma v = 98 - 2v.$$

Therefore

$$\boxed{\frac{dv}{dt} = 9.8 - \frac{v}{5}.} \tag{1.1}$$

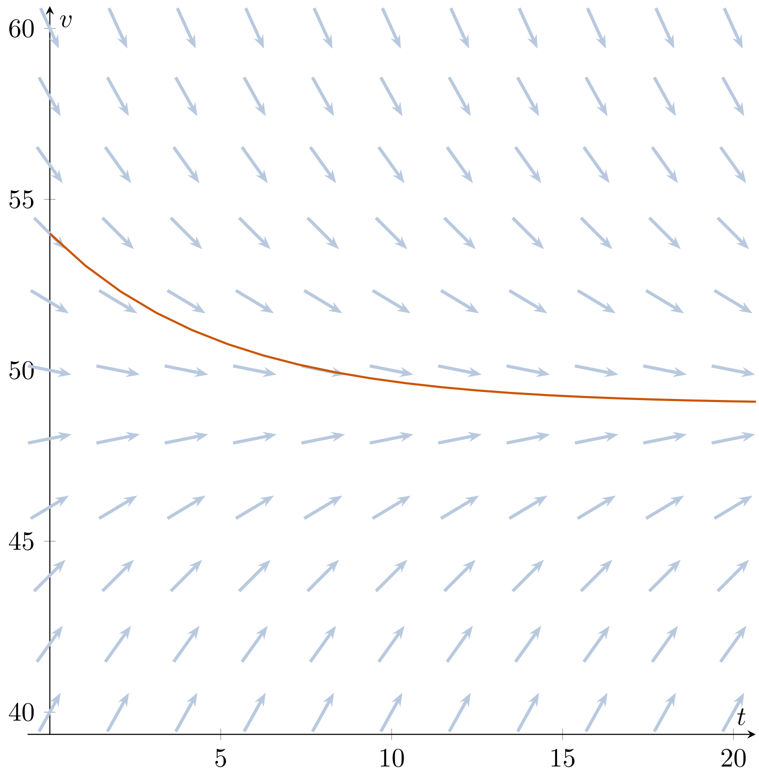
We will solve equation (1.1) later. First we will look at this differential equation's direction field to try to understand it.

A direction field is a grid of arrows in the tv-plane which show the slope of solutions to a differential equation. A direction field for (1.1) looks like this:

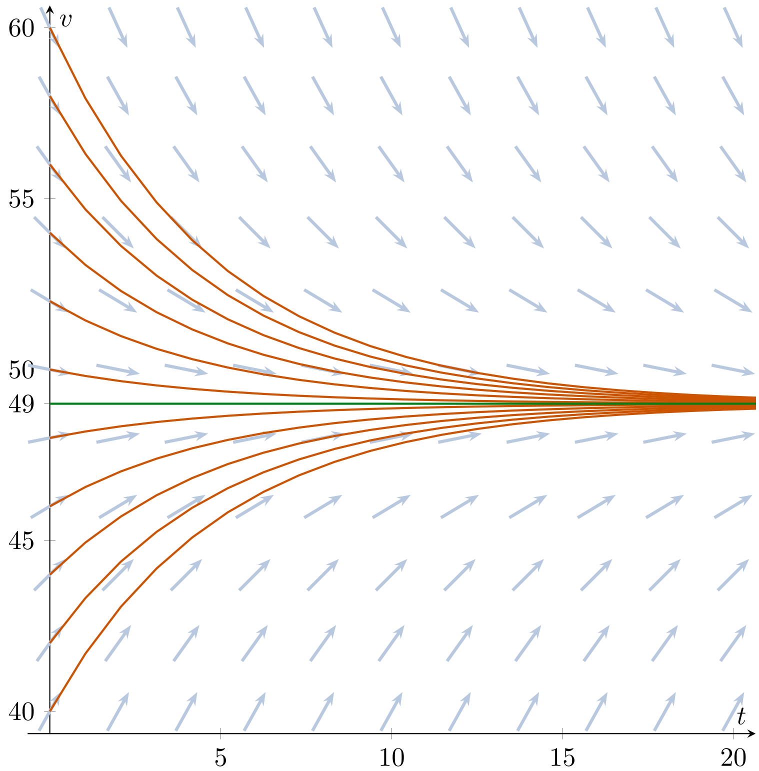


I will show you how to draw direction fields later. For now, I want to see what we can learn about the solutions to (1.1).

If we start at $v(0) = 54$ say, the arrows tell us that the solution is decreasing like this:



We can guess at some more solutions:



Note that if $v = 49$, then we have

$$\frac{dv}{dt} = 9.8 - \frac{49}{5} = 9.8 - 9.8 = 0.$$

Hence $v(t) = 49$ is a constant solution (or ***equilibrium solution***) of (1.1).

Example 1.6 (Mice and Owls)**.** Let $p(t)$ denote the population of mice in an area, where t is measured in months.

We assume that there is plenty of food for the mice to eat so, if nothing eats the mice, $p(t)$ will increase at a rate proportional to $p(t)$.

$$\frac{dp}{dt} \propto p$$
$$\frac{dp}{dt} = rp$$

where $r > 0$ is a constant.

Suppose that $r = 0.5$ per month. Hence

$$\frac{dp}{dt} = \frac{p}{2}.$$

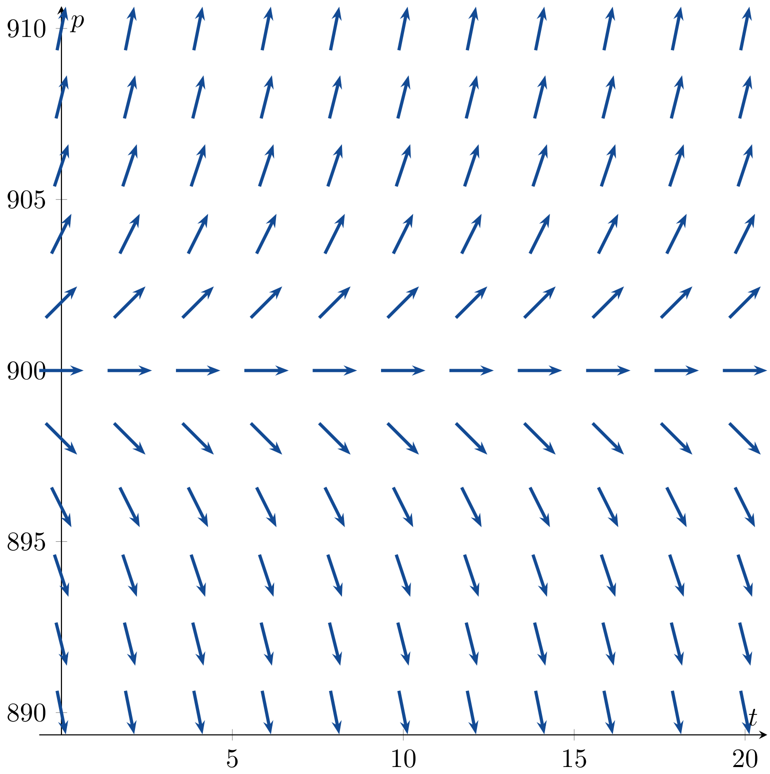
However, suppose that 5 owls also live in this area and suppose that each owl eats 3 mice each day.

$$\begin{aligned} &1 \text{ owl eats } 3 \text{ mice per day} \\ &5 \text{ owls eat } 5 \times 3 = 15 \text{ mice per day} \\ &5 \text{ owls eat } 30 \times 15 = 450 \text{ mice per month.} \end{aligned}$$

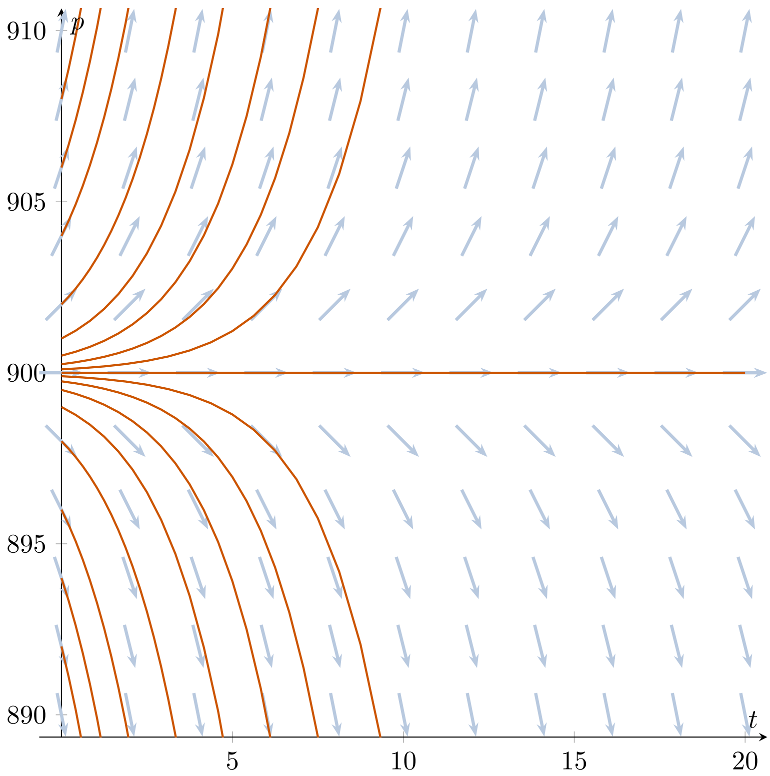
So we change our differential equation to

$$\boxed{\frac{dp}{dt} = \frac{p}{2} - 450.} \tag{1.2}$$

If we look at a direction field for (1.2), then we can guess at some solutions:



then we can guess at some solutions:

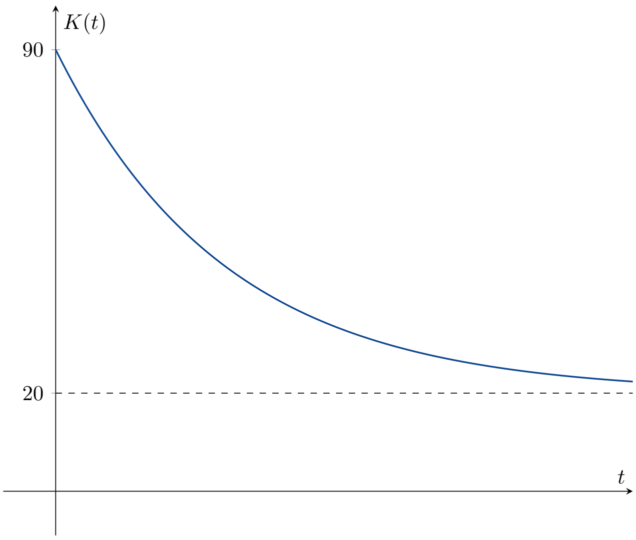


Example 1.7 (A cup of coffee)**.** Newton's law of cooling states that; the temperature of an object changes at a rate proportional to the difference between its temperature and that of its surroundings.

Suppose that the temperature of your cup of coffee obeys Newton's law of cooling; suppose that it has a temperature of 90°C when freshly poured; and suppose that the temperature of your room is 20°C.

Write a differential equation for the temperature of your coffee.

We expect the cup of coffee to cool like this:



When the coffee is hot, it will cool quickly. When it is just above 20°C, it will cool slowly.

Let $K(t)$ denote the temperature of the coffee in °C and let t denote time measured in minutes. Then we know that

$$\frac{dK}{dt} \propto (20 - K).$$

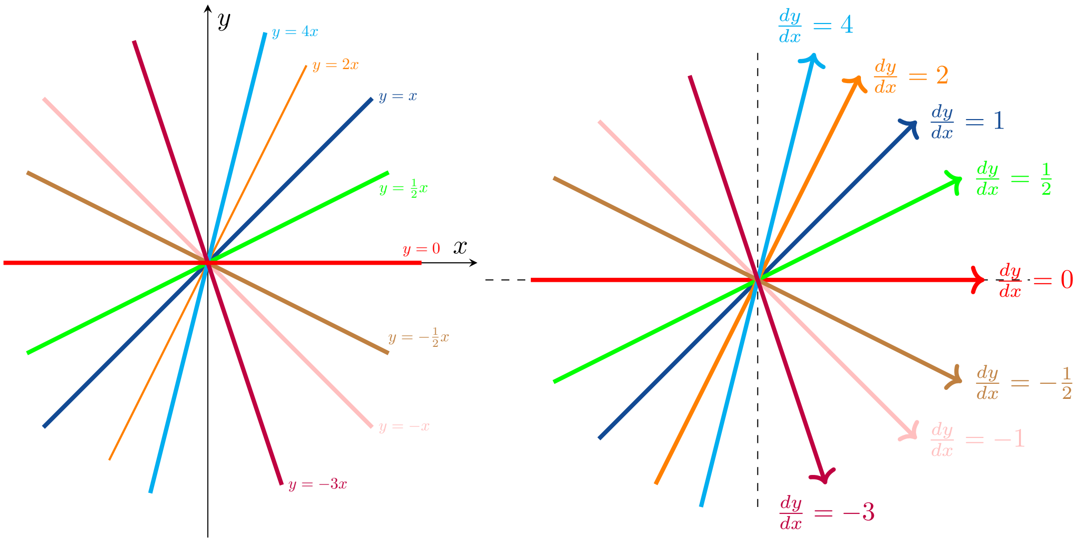
It follows that

$$\boxed{\frac{dK}{dt} = r(20 - K)} \tag{1.3}$$

for some constant r. Since hot coffee cools down (and cold coffee warms up), we must have $r > 0$.

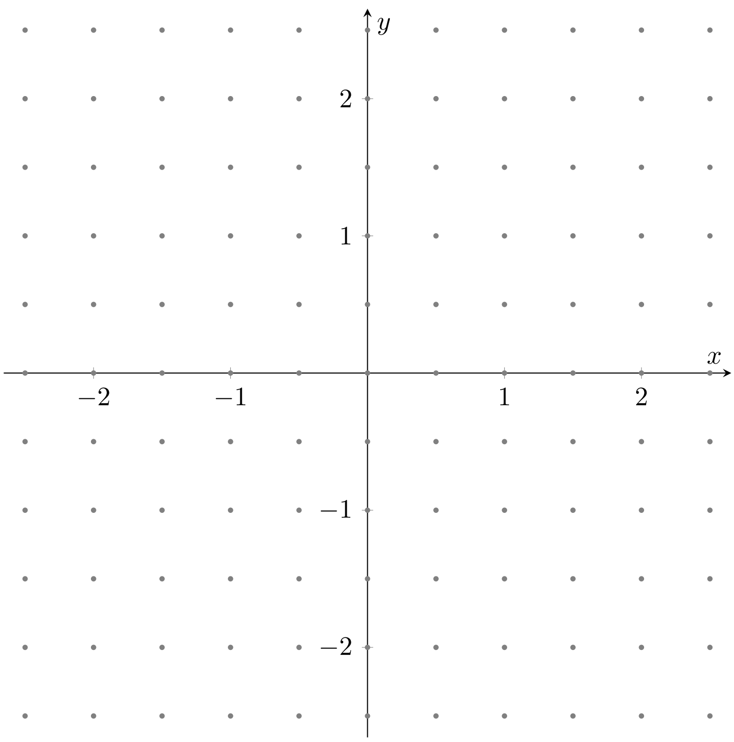
1.3 How to Draw a Direction Field

Consider the graphs of $y = mx$ for different values of $m \in \mathbb{R}$. E.g. $y = 2x$ shopes upwards with slope 2. We will use rightwards arrows to show the slope of solutions of differential equations at various points.



Example 1.8. Draw a direction field for $\frac{dy}{dx} = x + y$.

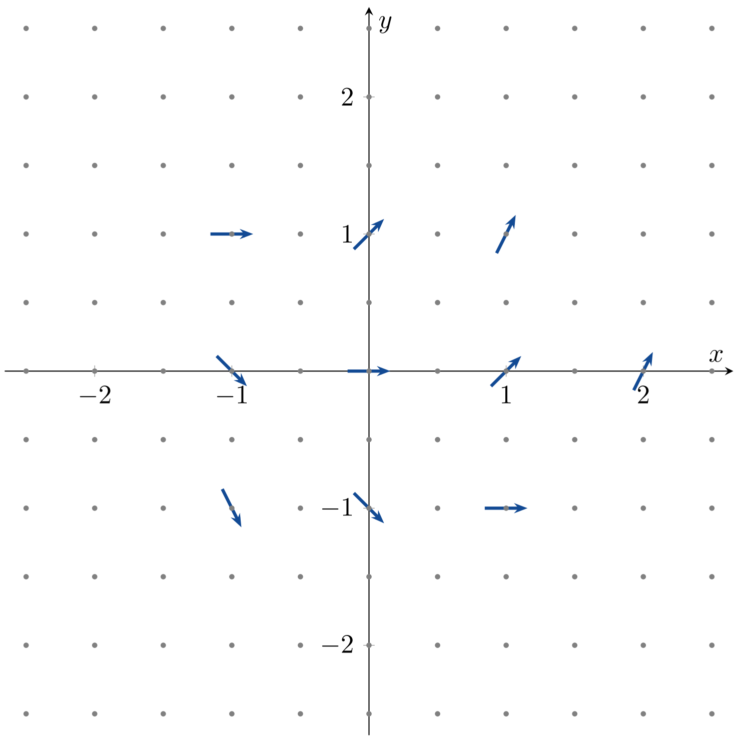
We start with an empty grid. We will need to draw an arrow at each point on this grid. In this example, we will be drawing 121 arrows.



The first step is to calculate $\frac{dy}{dx}$ at a few points. Recall that $\frac{dy}{dx} = 1$ means , etc.

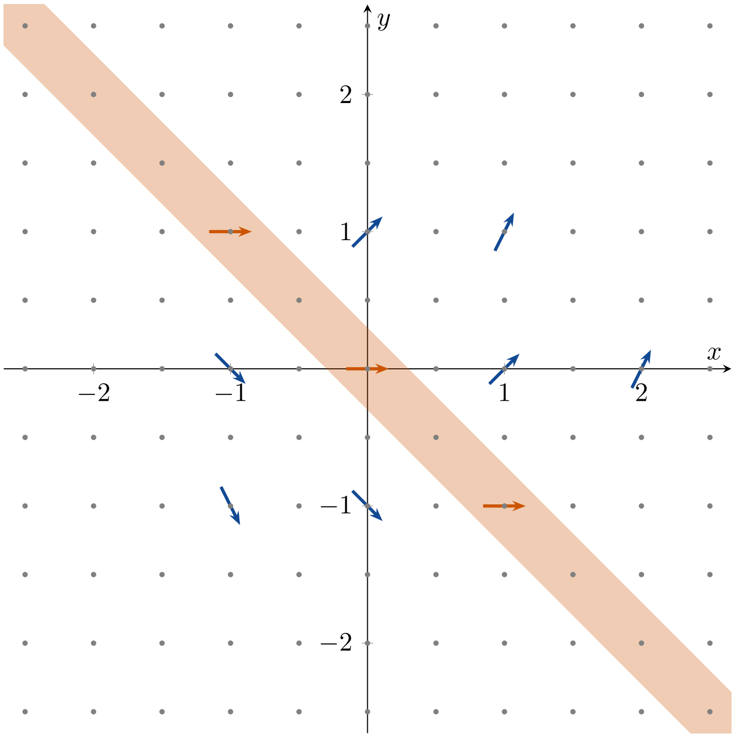
(x, y)	$\frac{dy}{dx} = x + y$	arrow
$(0, 0)$	0	$\rightarrow$
$(1, 0)$	1	$\nearrow$
$(2, 0)$	2	$\nearrow$
$(-1, 0)$	-1	$\searrow$
$(0, 1)$	1	$\nearrow$
$(0, -1)$	-1	$\searrow$
$(1, 1)$	2	$\nearrow$
$(1, -1)$	0	$\rightarrow$
$(-1, 1)$	0	$\rightarrow$
$(-1, -1)$	-2	$\searrow$

Then we draw these arrows on the grid.

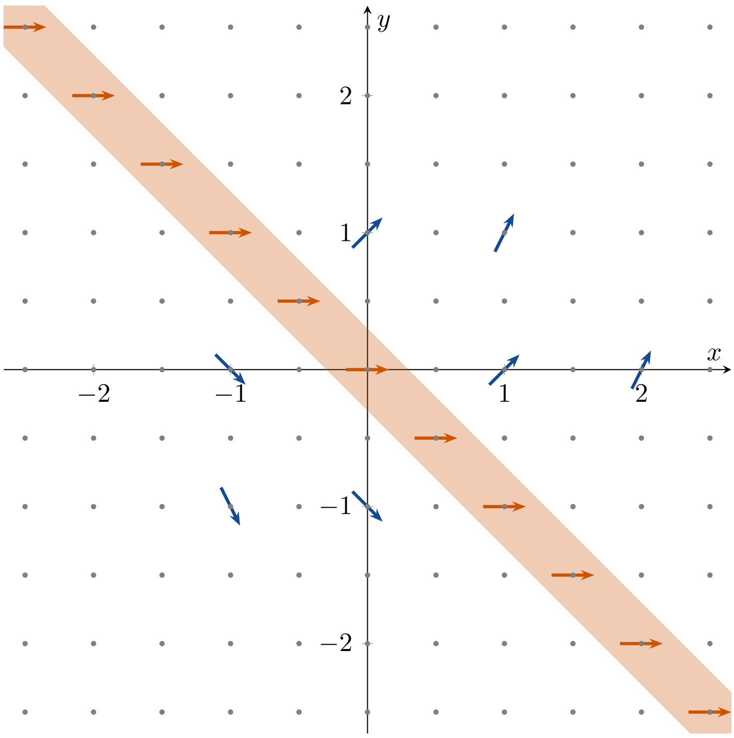


There is a shortcut here. Instead of having to calculate $\frac{dy}{dx}$ at all 121 points, we can look for patterns and guess.

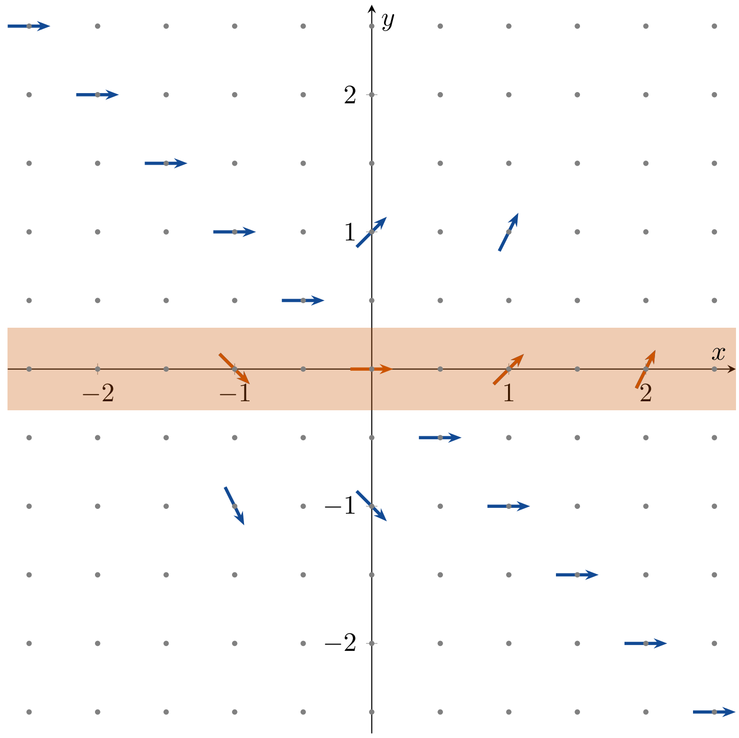
Notice first that along this diagonal line, all the arrows are .



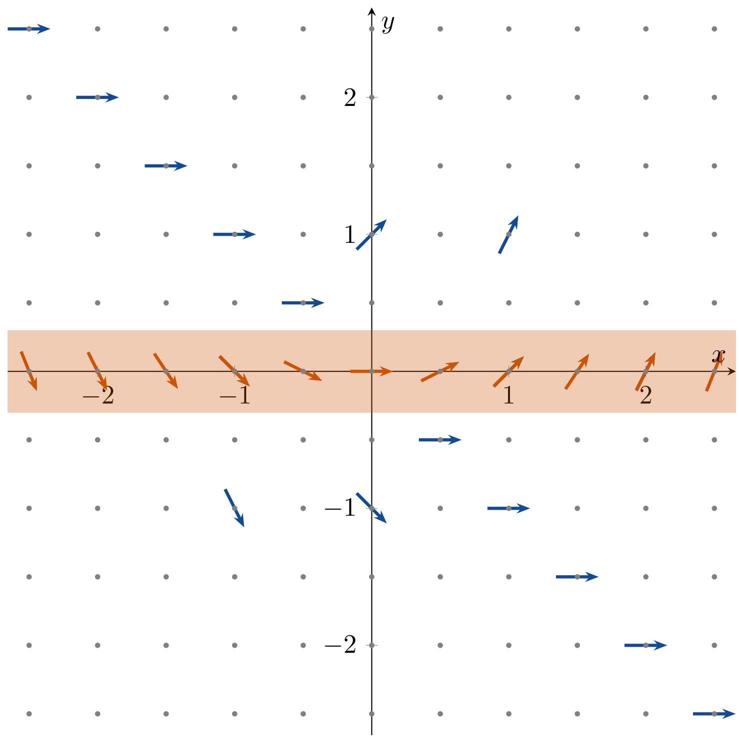
So we guess that every arrow along this line is .



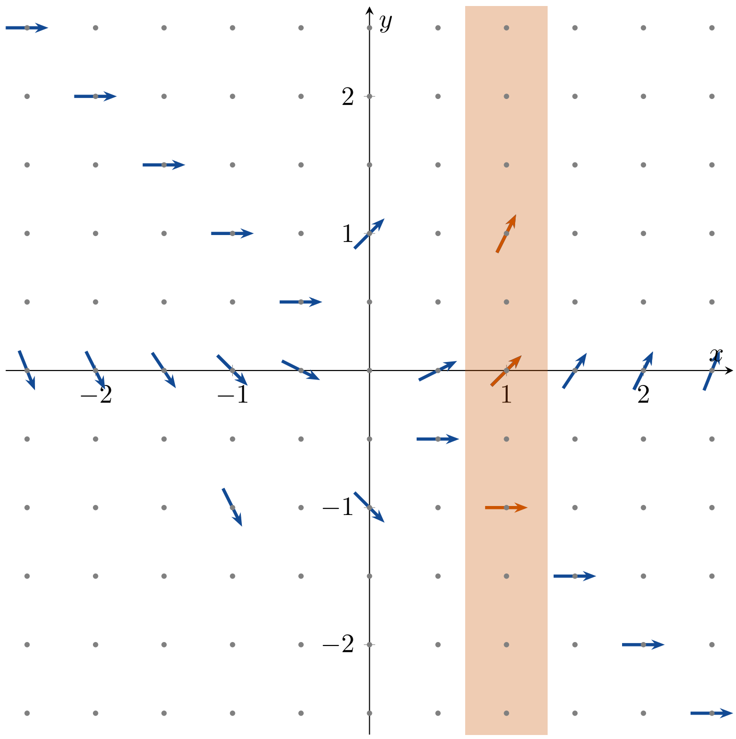
Then we can look at a horizontal line



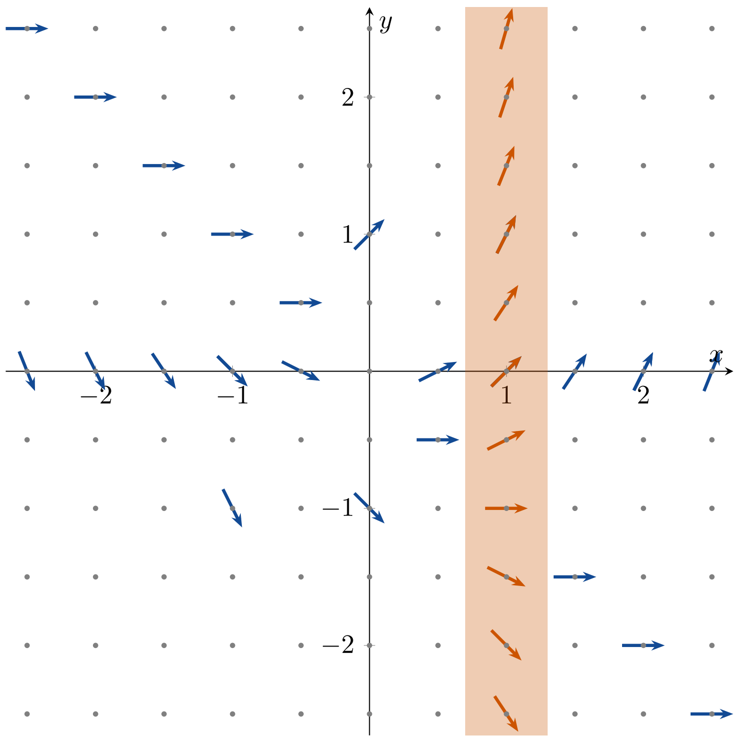
and note that the arrows are turning anticlockwise, as we move from the left to the right. So we guess that all the arrows on this horizontal line look like this:



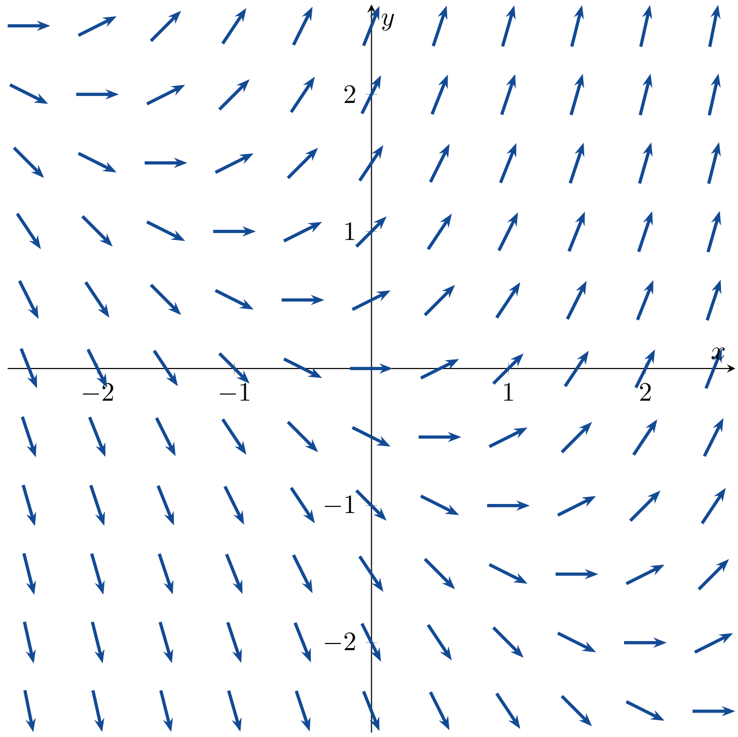
On a vertical line, we see the same thing happening:



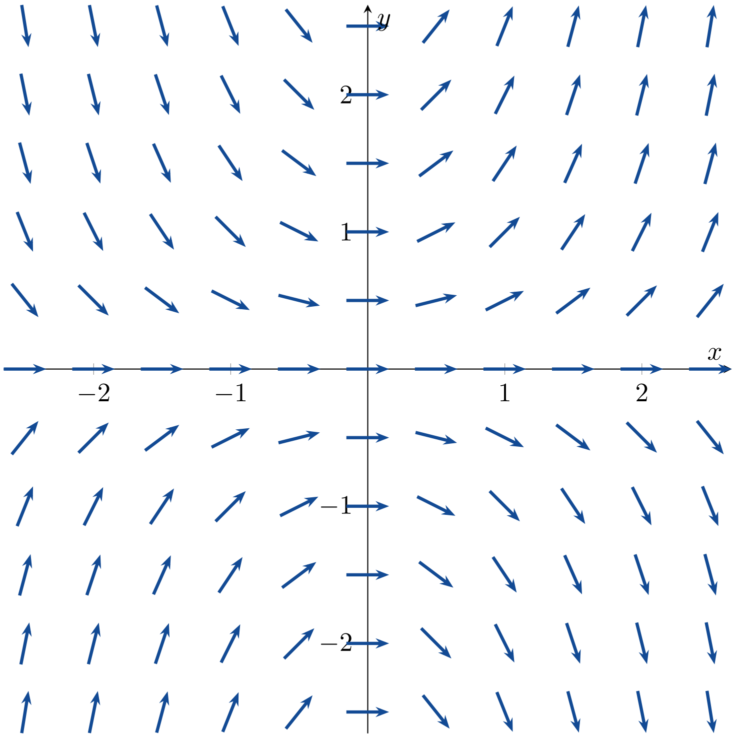
So we guess



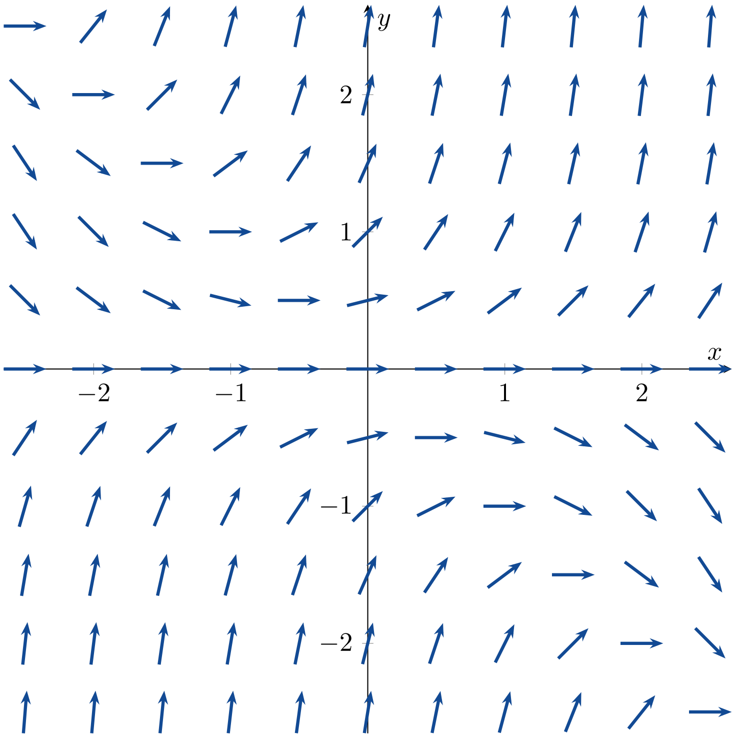
If we keep guessing, then we can quickly finish the direction field:



Example 1.9. Draw a direction field for $\frac{dy}{dx} = xy$.



Example 1.10. Draw a direction field for $\frac{dy}{dx} = y(x + y)$.



1.4 Solving Our First Differential Equations

Both (1.1) and (1.2) are of the form

$$\frac{dy}{dt} = ay - b \tag{1.4}$$

for constants a and b. We will now study how to solve equations like this.

Example 1.11 (Mice and Owls)**.** Recall that we derived the equation

$$\frac{dp}{dt} = \frac{p}{2} - 450 = \frac{p-900}{2}. \tag{1.2}$$

Solve this equation.

If $p \neq 900$, we can rearrange (1.2) to

$$\frac{dp}{p-900} = \frac{1}{2}\,dt.$$

Note that all the terms involving p are on the left, and all the terms involving t are on the right. (Of course $\frac{dp}{dt}$ does not really mean $dp \div dt$, and using this method annoys "Pure Mathematicians", but it works.) **If we can separate the variables like this, then we are allowed to integrate:**

$$\begin{aligned}
\frac{dp}{p-900} &= \frac{1}{2}\,dt \\
\int \frac{dp}{p-900} &= \int \frac{1}{2}\,dt \\
\ln|p-900| &= \frac{t}{2} + K
\end{aligned}$$

where K is a constant. Thus

$$\begin{aligned}
|p-900| &= e^{\frac{t}{2}+K} \\
p-900 &= \pm e^K e^{\frac{t}{2}} \\
p(t) &= 900 \pm e^K e^{\frac{t}{2}}.
\end{aligned}$$

K is a number that we don't know. So e^K is a number that we don't know. So $\pm e^K$ is a number that we don't know. We can give this unknown number a new name: Let $c = \pm e^K$. Then we have

$$\boxed{p(t) = 900 + ce^{\frac{t}{2}}.}$$

Before we go on, let us just check that the function

$$p(t) = 900 + ce^{\frac{t}{2}}$$

really does solve the differential equation

$$\frac{dp}{dt} = \frac{p}{2} - 450.$$

We calculate that

$$\frac{dp}{dt} = \frac{d}{dt}\left(900 + ce^{\frac{t}{2}}\right) = 0 + c\left(\frac{1}{2}\right)e^{\frac{t}{2}} = \frac{c}{2}e^{\frac{t}{2}}$$

and

$$\frac{p}{2} - 450 = \frac{900 + ce^{\frac{t}{2}}}{2} - 450 = 450 + \frac{c}{2}e^{\frac{t}{2}} - 450 = \frac{c}{2}e^{\frac{t}{2}}.$$

Example 1.12 (A Falling Object)**.** Solve

$$\frac{dv}{dt} = 9.8 - \frac{v}{5}. \tag{1.1}$$

We use the same method:

$$\begin{aligned}
\frac{dv}{dt} &= \frac{49 - v}{5} \\
\frac{dv}{v - 49} &= -\frac{1}{5}\,dt \\
\int \frac{dv}{v - 49} &= \int -\frac{1}{5}\,dt \\
\ln|v - 49| &= -\frac{t}{5} + K \\
|v - 49| &= e^{-\frac{t}{5}+K} \\
v - 49 &= \pm e^K e^{-\frac{t}{5}} \\
v(t) &= 49 \pm e^K e^{-\frac{t}{5}} = 49 + ce^{-\frac{t}{5}}.
\end{aligned}$$

Example 1.13. Solve $\dfrac{dy}{dt} = y$.

This is $\dfrac{dy}{dt} = ay - b$ with $a = 1$ and $b = 0$. I leave this for you to solve.

1.5 Classification

ODEs and PDEs

If only ordinary derivatives appear in a differential equation, then it is called an ***ordinary differential equation*** (ODE) [adi diferansiyel denklem]. For example

$$\frac{dv}{dt} = 9.8 - \frac{v}{5} \tag{1.1}$$

and

$$\frac{dp}{dt} = \frac{p}{2} - 450 \tag{1.2}$$

are ODEs. If the derivatives in a differential equation are partial derivatives, then it is called a ***partial differential equation*** (PDE) [kısmi türevli diferansiyel denklem]. For example

$$k\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t} \qquad \text{(heat equation)}$$

and

$$c^2\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2} \qquad \text{(wave equation)}$$

are PDEs.

Systems

If there is a single function to be found, then one differential equation is enough. However, if there are two or more unknown functions then we need a ***system of differential equations***. For example

$$\begin{cases} \frac{dx}{dt} = ax - \alpha xy \\ \frac{dy}{dt} = -cy + \gamma xy \end{cases} \qquad \text{(Predator-Prey equations)}$$

is a system of differential equations.

Order

The ***order*** of a differential equation is the order of the highest derivative. For example (1.1)and (1.2)are first order ODEs.

$$\frac{d^2y}{dt^2} + \frac{dy}{dt} + y = 0$$

is a second order ODE.

$$y''' + 2e^t y'' + yy' = t^4$$

is a third order ODE.

Linear and Non-Linear

The ODE

$$F(t, y, y', \ldots, y^{(n)}) = 0$$

is called ***linear*** iff F is a linear function of y, y', $\ldots$, $y^{(n)}$ (we don't care about t). The ***general linear ODE*** of order n is

$$a_0(t)y^{(n)} + a_1(t)y^{(n-1)} + \ldots + a_n(t)y = g(t). \tag{1.5}$$

For example(1.1)and (1.2)are linear ODEs. An ODE which is not linear is called ***non-linear***. For example

$$y''' + 2e^t y'' + yy' = t^4$$

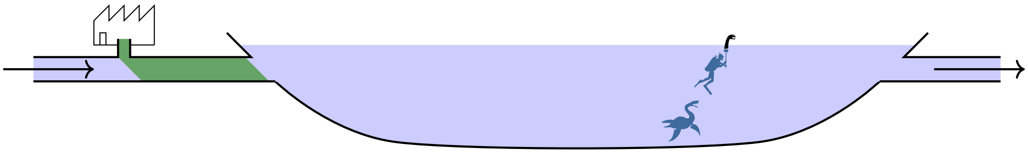
is non-linear due to the yy' term.

Example 1.14. For each ODE below, give the order of the equation and state whether it is linear or non-linear:

- $\dfrac{d^3y}{dx^3} + 2\dfrac{d^5y}{dx^5} + \dfrac{dy}{dx} - y - e^x\dfrac{d^2y}{dx^2} = 0$ fifth order, linear
- $\dfrac{d^3y}{dx^3} + \cos\left(\dfrac{dy}{dx}\right) = \sin x$ third order, non-linear
- $\dfrac{d^3y}{dx^3} + (\cos x)\dfrac{dy}{dx} = \sin x$ third order, linear
- $y'' - y^2 = x^2$ second order, non-linear
- $e^x y^{(7)} - x^3 y^{(99)} + 2x^x y''' - x^2 e^{(\sin x)} = 2021$ ninety-ninth order, linear

Exercises

Exercise 1.1 (Pollution in a Lake).



English

A lake contains 1,000,000 litres of water. At time $t = 0$, the water is pure. Water, containing 0.02 grams/litre of toxic waste, flows into the lake at a rate of 250 litres/hour. Water also flows out of the lake at the same rate, so the amount of water in the lake is always exactly 1,000,000 litres.

Assume that the toxic waste is uniformly distributed throughout the lake. So if there are 2000 grams of toxic waste in the lake, then every litre of water in the lake contains 0.002 grams of toxic waste. Let t be time measured in hours. Let $S(t)$ be the amount (in grams) of toxic waste in the lake at time t.

Türkçe

Bir gölde 1.000.000 litre su vardır. Zaman $t = 0$ olduğunda su saf durumdadır. 1 litrede 0,02 gram zehirli atık yoğunluğu olan akarsu, göle saatte 250 litre hızla akmaktadır. Gölden su aynı hızla da boşalmaktadır, böylece göldeki su her zaman tam olarak 1.000.000 litredir.

Zehirli atıkların göle eşit olarak dağılmış olduğunu varsayın. Yani gölde 2000 gram zehirli atık varsa, göldeki her litre su 0,002 gram zehirli atık içermektedir. t saatle ölçülen zamandır. $S(t)$ ise t zamanında gölde (gram olarak) bulunan zehirli atığın miktarıdır.

(a). How much toxic waste (in grams) enters the lake every hour?

(b). How much toxic waste (as a function of $S(t)$) leaves the lake every hour?

(c). Write an initial value problem for $S(t)$.

[HINT: $\begin{cases} \frac{dS}{dt} = (\text{amount of toxic waste entering the lake per hour}) - (\text{amount of toxic waste leaving the lake per hour}) \\ S(0) = ??? \end{cases}$.]

(d). Solve the initial value problem that you wrote in part (c).

(e). How much toxic waste is in the lake after 1 year?

(f). Sketch a graph of $S(t)$ versus t.

Exercise 1.2 (A cup of coffee).

English

Newton's law of cooling states that; the temperature of an object changes at a rate proportional to the difference between its temperature and that of its surroundings.

Suppose that the temperature of your cup of coffee obeys Newton's law of cooling, and suppose that the temperature of your room is 20°C.

Türkçe

Newton'ın soğuma kanunu der ki, bir nesnenin ısısı, o nesnenin sıcaklığıyla nesnenin içinde bulunduğu çevrenin sıcaklığı arasındaki farkla orantılı olarak değişir.

Varsayın ki kahvenizin ısısı Newton'ın soğuma kanununa uymakta ve varsayın ki odanızın sıcaklığı 20C°.

(a). Write a differential equation for the temperature of your coffee. (You must explain why your differential equation is valid.)

(b). If the coffee has a temperature of 90°C when freshly poured (at time $t = 0$), and 1 minute later has cooled to 80°C, determine how long it takes for the coffee to cool to a temperature of 65°C. (Justify your answer.)

Exercise 1.3 (Direction Fields). Draw direction fields for the following differential equations.

(a). $\dfrac{dy}{dt} = t - y$ (c). $\dfrac{dy}{dt} = y + t$ (e). $\dfrac{dy}{dt} = y^2$ (g). $\dfrac{dy}{dt} = t^2$

(b). $\dfrac{dy}{dt} = 3 - 2y$ (d). $\dfrac{dy}{dt} = -ty$ (f). $\dfrac{dy}{dt} = -y(2-y)$ (h). $\dfrac{dy}{dt} = 0$

Exercise 1.4 (Classification). For each of the following differential equations; give the order of the equation <u>and</u> state whether the equation is linear or non-linear. The first one is done for you.

(ω) $y\frac{d^2y}{dt^2} - t\frac{d^3y}{dt^3} = \frac{dy}{dt}$ (3rd order, non-linear)

(a). $t^2\frac{d^2y}{dt^2} + t\frac{dy}{dt} + 2y = \sin t$

(b). $(1+y^2)\frac{d^2y}{dt^2} + t\frac{dy}{dt} + y = e^t$

(c). $\frac{dy}{dt} + ty^2 = 0$

(d). $\frac{d^4y}{dt^4} + \frac{d^3y}{dt^3} + \frac{d^2y}{dt^2} + \frac{dy}{dt} + y = 1$

(e). $\frac{d^2y}{dt^2} + \sin(t+y) = \sin t$

(f). $t\frac{dy}{dt} + (\cos^2 t)y = t^3 + \frac{d^4y}{dt^4}$

(g). $\frac{dy}{dt} + ty = 0$

(h). $\frac{d^3y}{dt^3} + \sin t = y$

(i). $t^2\frac{d^2y}{dt^2} + t\frac{dy}{dt} + 2y = \sin y$

(j). $\frac{d^3y}{dt^3} - \frac{d^2y}{dt^2} + \frac{dy}{dt} - y = \frac{d^4y}{dt^4} + e^y.$

(k). $y''' + y(y')^3 = 2x.$

(l). $y'' + 2e^{3x}y' + 2y = (x^2+5)^3.$

(m). $\frac{dy}{dx} + 3x^2y = 0.$

(n). $x''(t) - x^2t^2 = 0.$

Exercise 1.5 (Solutions). Show that the following functions are the solutions of the given differential equations. The first one is done for you.

(ω) $y'' + y' = 1;$ $y(t) = t.$
solution: Since $y' = 1$ and $y'' = 0$, we have that $y'' + y' = 0 + 1 = 1$ as required.

(a). $x'' - 2x' + x = 0;$ $x(t) = te^t.$

(b). $y'' + 4y = 0;$ $y(x) = \cos 2x$

2

First Order Differential Equations

In this chapter, we will consider equations of the form

$$\frac{dy}{dt} = f(t, y). \tag{2.1}$$

2.1 Linear Equations

If the function f in (2.1) depends linearly on y (we don't care about t), then (2.1) is a first order ***linear*** ODE. Last week we only talked about equations of the form

$$\frac{dy}{dt} = -ay + b \tag{2.2}$$

where the coefficients a and b are constants. We will now consider

$$\frac{dy}{dt} + p(t)y = g(t) \tag{2.3}$$

where the coefficients $p(t)$ and $g(t)$ are functions of t.

We have seen how to solve (2.2):

$$\begin{aligned}
\frac{dy}{dt} &= -ay + b \\
\int \frac{dy}{y - \frac{b}{a}} &= \int -a\, dt \\
\ln\left|y - \frac{b}{a}\right| &= -at + C \\
&\vdots \\
y &= \frac{b}{a} + ce^{-at}.
\end{aligned}$$

So for example $\frac{dy}{dt} + 2y = 3$ has solution $y = \frac{3}{2} + ce^{-2t}$.

Unfortunately this method can not be used to solve (2.3). So we need a different method – we use a method by Gottfried Leibniz (1646-1716). The idea is

- Find a special function $\mu(t)$ called an integrating factor;
- Multiply the ODE by $\mu(t)$;
- Integrate.

Example 2.1. Use an integrating factor to solve $\dfrac{dy}{dt} + 2y = 3$.

First we multiply by an unknown function $\mu(t)$:

$$\mu(t)\frac{dy}{dt} + 2\mu(t)y = 3\mu(t).$$

How do we find $\mu(t)$ so that the left-hand side is integrable? Notice that

$$\frac{d}{dt}\left(\mu(t)y\right) = \mu(t)\frac{dy}{dt} + \frac{d\mu}{dt}(t)y.$$

We want to choose $\mu(t)$ such that

$$\frac{d\mu}{dt} = 2\mu.$$

We know how to solve this equation:

$$\begin{aligned} \int \frac{d\mu}{\mu} &= \int 2\,dt \\ \ln|\mu| &= 2t + C \\ &\vdots \\ \mu(t) &= ce^{2t}. \end{aligned}$$

We only need to find one $\mu(t)$ which works – so we can choose whichever value of $c \neq 0$ that we wish. I choose $c = 1$. We will use $\mu(t) = e^{2t}$.

Our ODE is then

$$e^{2t}\frac{dy}{dt} + 2e^{2t}y = 3e^{2t}.$$

Because we chose μ carefully, we can use the product rule ($(uv)' = uv' + u'v$) to write this as

$$\frac{d}{dt}\left(e^{2t}y\right) = 3e^{2t}.$$

Integrating gives

$$e^{2t}y = \frac{3}{2}e^{2t} + c.$$

Therefore

$$\boxed{y = \frac{3}{2} + ce^{-2t}.}$$

Remark 2.1. For the ODE $\dfrac{dy}{dt} + 2y = 3$ we use the integrating factor $\mu(t) = e^{2t}$.

Example 2.2. Use an integrating factor to solve $\dfrac{dy}{dt} + ay = b$.

If we were to repeat the previous method, we would find that we need the integrating factor $\mu(t) = e^{at}$. (Please check!)

Example 2.3. Solve $\dfrac{dy}{dt} + ay = g(t)$.

The integrating factor depends only on the coefficient of y. So again we use $\mu(t) = e^{at}$.

Multiplying the ODE by e^{at} gives

$$e^{at}\frac{dy}{dt} + ae^{at}y = e^{at}g(t).$$

So

$$\frac{d}{dt}\left(e^{at}y\right) = e^{at}g(t).$$

By integrating, we obtain

$$e^{at}y = \int^t e^{as}g(s)\,ds + c.$$

Thus

$$\boxed{y = e^{-at}\int^t e^{as}g(s)\,ds + ce^{-at}} \tag{2.4}$$

Example 2.4. Solve

$$\begin{cases} \frac{dy}{dt} + \frac{1}{2}y = 2 + t \\ y(0) = 2. \end{cases}$$

We multiply the ODE by the integrating factor $e^{\frac{t}{2}}$ to obtain

$$e^{\frac{t}{2}}y' + \frac{1}{2}e^{\frac{t}{2}}y = 2e^{\frac{t}{2}} + te^{\frac{t}{2}}$$

and

$$\frac{d}{dt}\left(e^{\frac{t}{2}}y\right) = 2e^{\frac{t}{2}} + te^{\frac{t}{2}}.$$

Integrating gives us

$$e^{\frac{t}{2}}y = 4e^{\frac{t}{2}} + 2te^{\frac{t}{2}} - 4e^{\frac{t}{2}} + c = 2te^{\frac{t}{2}} + c$$

(where we have used $\int u\frac{dv}{dt} = uv - \int \frac{du}{dt}v$ with $u = t$ and $v = 2e^{\frac{t}{2}}$). Therefore

$$y(t) = 2t + ce^{-\frac{t}{2}}.$$

Now

$$2 = y(0) = 0 + c \qquad \Longrightarrow \qquad c = 2.$$

Therefore the solution to the IVP is

$$\boxed{y(t) = 2t + 2e^{-\frac{t}{2}}.}$$

Example 2.5. Solve $\dfrac{dy}{dt} - 2y = 4 - t$.

Please check that by using $\mu(t) = e^{-2t}$ we obtain $y(t) = -\frac{7}{4} + \frac{t}{2} + ce^{2t}$.

Now consider
$$\frac{dy}{dt} + p(t)y = g(t).$$
We must find the integrating factor.

WARNING: The integrating factor is NOT $e^{p(t)}$.

If we multiply by an unknown function $\mu(t)$, we obtain
$$\mu\frac{dy}{dt} + p(t)\mu y = \mu g(t).$$
As before, then left-hand side looks like
$$\frac{d}{dt}(\mu y) = \mu\frac{dy}{dt} + \frac{d\mu}{dt}y.$$
So we want
$$\frac{d\mu}{dt} = p(t)\mu.$$

We know how to solve this ODE:
$$\begin{aligned}
\int \frac{d\mu}{\mu} &= \int p(t)\,dt \\
\ln|\mu| &= \int p(t)\,dt + C \\
&\vdots \\
\mu(t) &= c\exp\int p(t)\,dt.
\end{aligned}$$
As before, we can choose $c = 1$ to obtain
$$\mu(t) = \exp\int p(t)\,dt = e^{\int p(t)\,dt}. \tag{2.5}$$

Then our ODE becomes
$$\frac{d}{dt}(\mu y) = \mu g(t)$$
and we calculate that
$$\mu y = \int^t \mu(s)g(s)\,ds + c$$
and
$$y(t) = \frac{\int^t \mu(s)g(s)\,ds + c}{\mu(t)}.$$

Example 2.6. Solve
$$\begin{cases} ty' + 2y = 4t^2 \\ y(1) = 2. \end{cases}$$

First we must write the equation in the standard form:
$$\frac{dy}{dt} + \frac{2}{t}y = 4t.$$

Here $p(t) = \frac{2}{t}$ and $g(t) = 4t$.

Next we must calculate $\mu(t)$:

$$\mu(t) = \exp \int \frac{2}{t} dt = e^{2\ln|t|} = t^2.$$

Multiplying the ODE by t^2 gives

$$\frac{d}{dt}\left(t^2 y\right) = t^2 \frac{dy}{dt} + 2ty = 4t^3.$$

Integrating gives

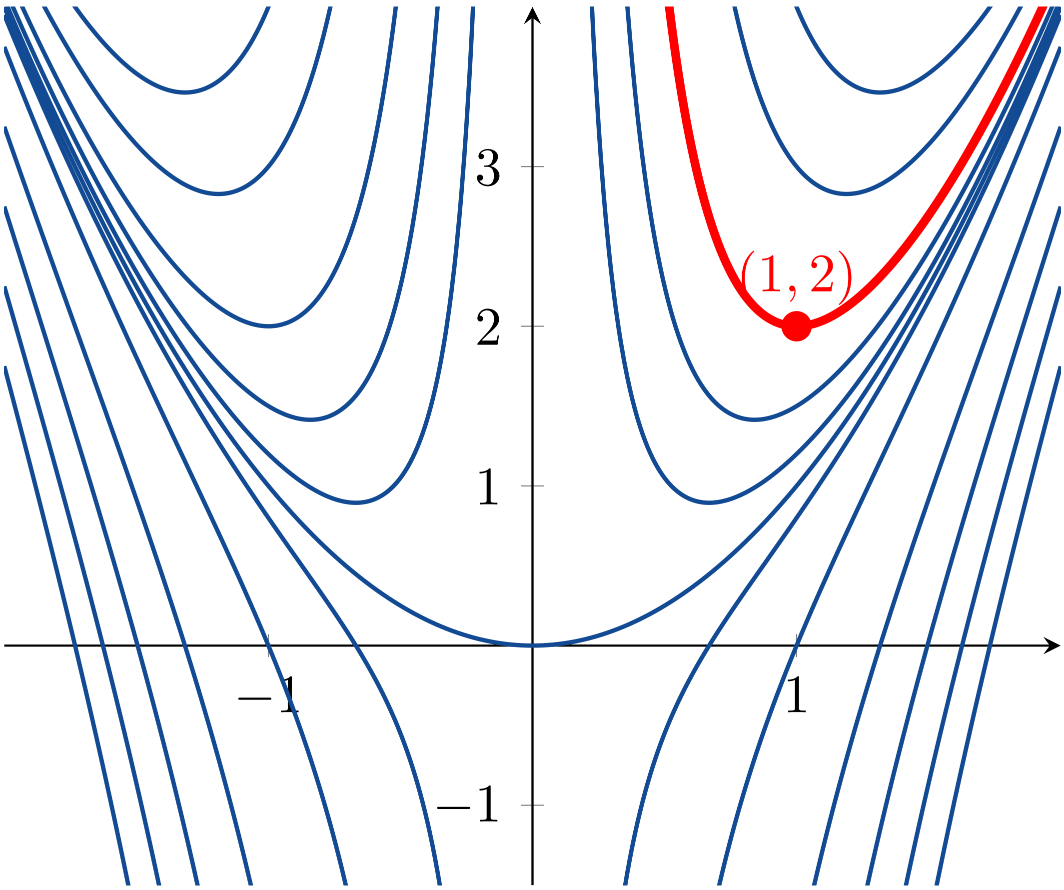
$$t^2 y = t^4 + c.$$

Hence the general solution to the ODE is

$$y(t) = t^2 + \frac{c}{t^2}.$$

To satisfy $y(1) = 2$, we choose $c = 1$. Therefore

$$\boxed{y(t) = t^2 + \frac{1}{t^2} \qquad (t > 0).}$$



Note that

(i). the solution satisfying $y(1) = 2$ is a differentiable function $y : (0, \infty) \to \mathbb{R}$.

(ii). the solution becomes unbounded and asymptotic to the y-axis as $t \searrow 0$. This is because $p(t)$ has a discontinuity at $t = 0$.

(iii). The function $y = t^2 + \frac{1}{t^2}$, $t < 0$ is ***not*** part of the solution to the IVP. The solution to the IVP only exists for $t \in (0, \infty)$.

(iv). Solutions for which $c > 0$ (i.e. $y(1) > 1$) are asymptotic to the positive y-axis as $t \searrow 0$. But solutions for which $c < 0$ (i.e. $y(1) < 1$) are asymptotic to the negative y-axis as $t \searrow 0$. So there is an initial value ($y(1) = 0$) where the behaviour changes. This is called a ***critical initial value***.

2.2 Separable Equations

The general first order ODE is

$$\frac{dy}{dx} = f(x, y). \tag{2.6}$$

In the previous section we looked at a special case called "linear equations" – now we will study another special case.

Equation (2.6) can ***always*** be written in the form

$$M(x, y) + N(x, y)\frac{dy}{dx} = 0. \tag{2.7}$$

One way would be to write $M = -f$ and $N = 1$, but there may be other ways. ***If*** we can do this so that $M(x)$ is a function only of x and $N(y)$ is a function only of y, then (2.7) becomes

$$M(x) + N(y)\frac{dy}{dx} = 0. \tag{2.8}$$

Definition. A first order ODE is called ***separable*** if it can be written in the form (2.8).

Remark 2.2. Note that we can rearrange (2.8) to

$$\underbrace{M(x)\,dx}_{\text{all } x \text{ terms}} = -\underbrace{N(y)\,dy}_{\text{all } y \text{ terms}}.$$

In other words, it is possible to "separate" the variables.

Example 2.7. Consider

$$\frac{dy}{dx} = \frac{x^2}{1 - y^2}.$$

(i). Show that this ODE is separable.

(ii). Solve this ODE.

We can rearrange this ODE to

$$-x^2 + (1 - y^2)\frac{dy}{dx} = 0.$$

This is of the form (2.8). Therefore this ODE is separable.

Note that $\frac{d}{dx}\left(-\frac{1}{3}x^3\right) = -x^2$ and $\frac{d}{dy}\left(y - \frac{1}{3}y^3\right) = 1 - y^2$. So our ODE is

$$-x^2 + (1 - y^2)\frac{dy}{dx} = 0$$

$$\frac{d}{dx}\left(-\frac{1}{3}x^3\right) + \frac{d}{dy}\left(y - \frac{1}{3}y^3\right)\frac{dy}{dx} = 0$$

Using the Chain Rule, this is

$$\begin{aligned} \frac{d}{dx}\left(-\frac{1}{3}x^3\right) + \frac{d}{dx}\left(y - \frac{1}{3}y^3\right) &= 0 \\ \frac{d}{dx}\left(-\frac{1}{3}x^3 + y - \frac{1}{3}y^3\right) &= 0. \end{aligned}$$

Therefore

$$-\frac{1}{3}x^3 + y - \frac{1}{3}y^3 = C$$

or

$$\boxed{x^3 - 3y + y^3 = c.}$$

The same method can be used to solve any separable equation. Consider

$$M(x) + N(y)y' = 0$$

and suppose that $H_1(x)$ and $H_2(y)$ are functions which satisfy $H_1' = M$ and $H_2' = N$. Then our ODE becomes

$$\begin{aligned} M(x) + N(y)\frac{dy}{dx} &= 0 \\ \frac{dH_1}{dx} + \frac{dH_2}{dy}\frac{dy}{dx} &= 0 \\ \frac{dH_1}{dx} + \frac{dH_2}{dx} &= 0 \end{aligned}$$

by the Chain Rule. Then integrating gives the solution

$$H_1(x) + H_2(y) = c.$$

So to recap: To solve $M(x) + N(y)y' = 0$ we must integrate M wrt x and integrate N wrt y. But this is basically what we were doing in Chapter 1, where we did the following:

$$\begin{aligned} M(x) + N(y)\frac{dy}{dx} = 0 & \\ M(x) &= -N(y)\frac{dy}{dx} \\ M(x)\,dx &= -N(y)\,dy \\ \int M(x)\,dx &= -\int N(y)\,dy + c. \end{aligned}$$

Example 2.8. Solve $\begin{cases} \frac{dy}{dx} = \frac{3x^2+4x+2}{2(y-1)} \\ y(0) = -1. \end{cases}$

The ODE can be written as

$$2(y-1)\,dy = (3x^2+4x+2)\,dx.$$

Integrating gives

$$y^2-2y = x^3+2x^2+2x+c.$$

To find c, we use the initial condition $y(0)=1$ and calculate that

$$1+2 = 0+0+0+c \qquad \Longrightarrow \qquad c=3.$$

So the solution to the IVP is given by

$$\boxed{y^2-2y = x^3+2x^2+2x+3.}$$

This is called an ***implicit solution***. Sometimes this is the best that we can do. But in this example, it is possible to solve for y. Since

$$y^2-2y-(x^3+2x^2+2x+3) = 0$$

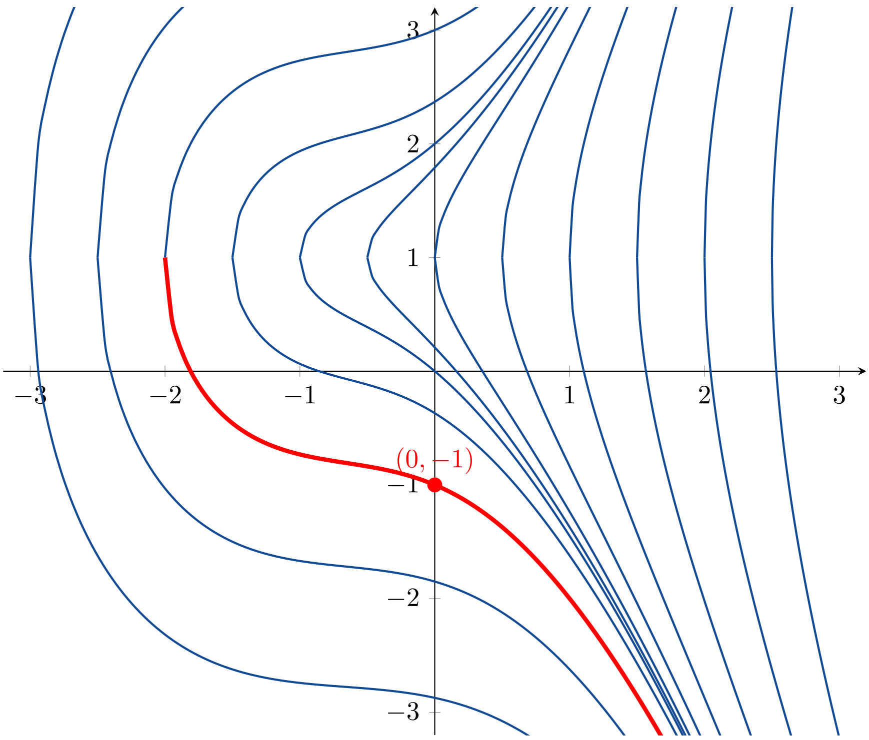
is a quadratic equation, we find that

$$y = 1 \pm \sqrt{x^3+2x^2+2x+4}.$$

There are two solutions here, but only only is correct. Which solution satisfies $y(0)=-1$? The answer is the solution with "$-$". Therefore the solution to the IVP is

$$\boxed{y = 1 - \sqrt{x^3+2x^2+2x+4}.}$$

A solution of the form $y=f(x)$ is called an ***explicit solution***.



Note that the solution satisfying $y(0)=-1$ is a differentiable function $y:(-2,\infty)\to\mathbb{R}$.

Example 2.9. Solve $\begin{cases} \frac{dy}{dx} = \frac{y\cos x}{1+2y^2} \\ y(0) = 1. \end{cases}$

$$\int \frac{1+2y^2}{y} dy = \int \cos x \, dx$$
$$\ln|y| + y^2 = \sin x + c$$
$$y(0) = 1 \qquad \Longrightarrow \qquad \ln 1 + 1^2 = \sin 0 + c \qquad \Longrightarrow \qquad c = 1.$$
$$\boxed{\ln|y| + y^2 = \sin x + 1.}$$

This equation can not be easily solved for y, so we leave it as an implicit solution. What can we say about this solution?

(i). If $y = 0$, the left-hand side is $-\infty$, but the right-hand side is in $[0, 2]$. This means that $y = 0$ is not possible. Since we know that $y(0) = 1$, we must therefore have $y(x) > 0$ for all x in the domain of the solution.

(ii). The solution exists on $(-\infty, \infty)$ (left for you to prove).

2.3 Differences Between Linear and Nonlinear Equations

Theorem 2.1. *Suppose*

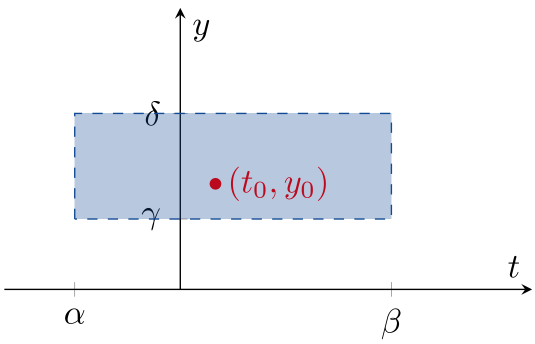
- *p and g are continuous on (α, β);*
- *$t_0 \in (\alpha, \beta)$; and*
- *$y_0 \in \mathbb{R}$.*

Then there exists a unique solution to

$$\begin{cases} y' + p(t)y = g(t) \\ y(t_0) = y_0 \end{cases}$$

on (α, β).

Remark 2.3. This theorem says that as long as p and g are continuous, the solution keeps existing. To say this another way: The solution can only stop existing at a discontinuity of either p or g.

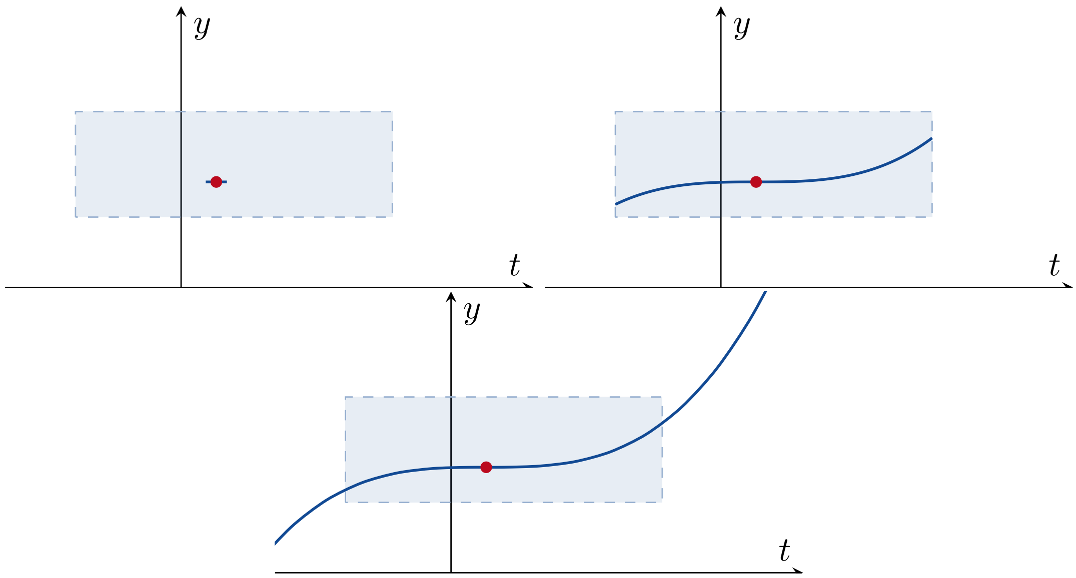


Theorem 2.2. *Suppose that*

- f *and* $\frac{\partial f}{\partial y}$ *are continuous for all* $\alpha < t < \beta$ *and* $\gamma < y < \delta$*;*
- $t_0 \in (\alpha, \beta)$*; and*
- $y_0 \in (\gamma, \delta)$.

Then in some interval $(t_0 - h, t_0 + h) \subseteq (\alpha, \beta)$*, there exists a unique solution to*

$$\begin{cases} y' = f(t, y) \\ y(t_0) = y_0. \end{cases}$$



Remark 2.4. This theorem tells us that "a little bit" of the solution exists. This theorem does not tell us if we only have this little bit of solution or if the solution exists further.

Remark 2.5. This theorem tells us that two solutions to $y' = f(t, y)$ can not intersect.

To understand why: Suppose that two solutions intersect at the point (t_0, y_0). But then there would be two solutions to

$$\begin{cases} y' = f(t, y) \\ y(t_0) = y_0 \end{cases}$$

and the theorem says that this is not possible.
Solutions to first order ODEs do not intersect !!! (assuming that f and $\frac{\partial f}{\partial y}$ are ...)

2.4 Autonomous Equations and Population Dynamics

Equations of the form

$$\frac{dy}{dt} = \underbrace{f(y)}_{\text{only } y} \tag{2.9}$$

are called ***autonomous***.

Example 2.10 (Exponential Growth)**.** Let $y(t)$ denote the number of cats in İstanbul.

The simplest model is to assume that the rate of change of y is proportional to y.

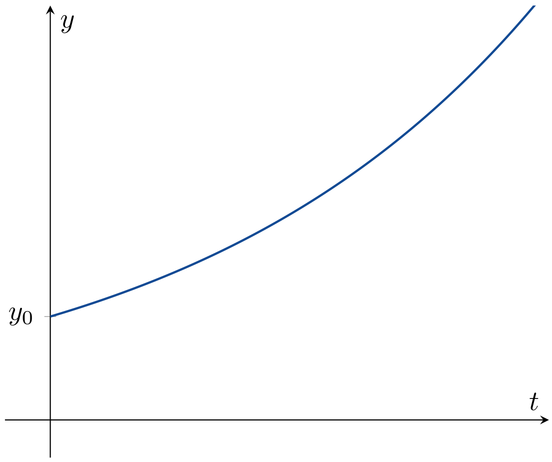
$$\frac{dy}{dt} = ry$$

for some constant r. We will assume that $r > 0$.

The solution to

$$\begin{cases} y' = ry \\ y(0) = y_0 \end{cases}$$

is $y(t) = y_0 e^{rt}$.



This model is good for small y, but it predicts that the number of cats in İstanbul will increase exponentially for all time. This can not be true. At some point

- the food will run out
- there will be no space
- people will get angry

$$\vdots$$

So we need a better model.

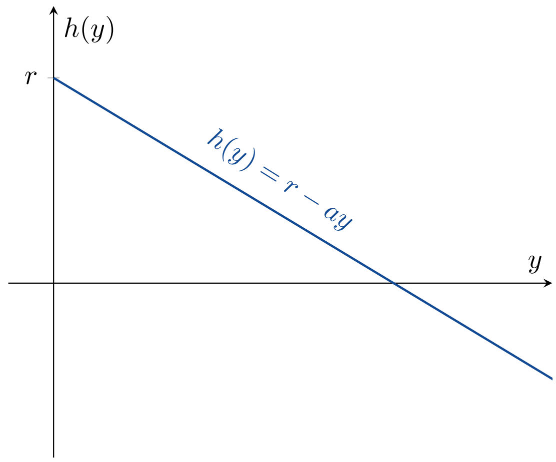
Example 2.11 (Logistic Growth)**.** Now we replace the constant r with a function $h(y)$.

$$\frac{dy}{dt} = h(y)y.$$

We want a function h which satisfies

- $h(y) \approx r$ if y is small;
- $h(y)$ decreases as y grows larger; and
- $h(y) < 0$ for large y.

The simplest such h is $h(y) = r - ay$.



So

$$\frac{dy}{dt} = (r - ay)y$$

which we will write as

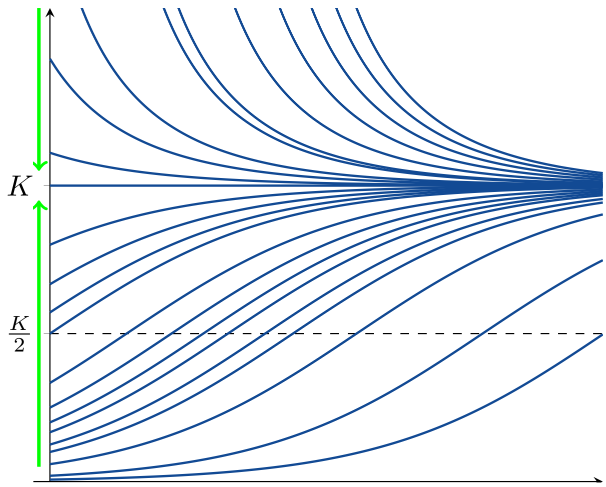
$$\frac{dy}{dt} = r\left(1 - \frac{y}{K}\right)y$$

for $K = \frac{r}{a}$. This is called the ***Logistic Equation***.

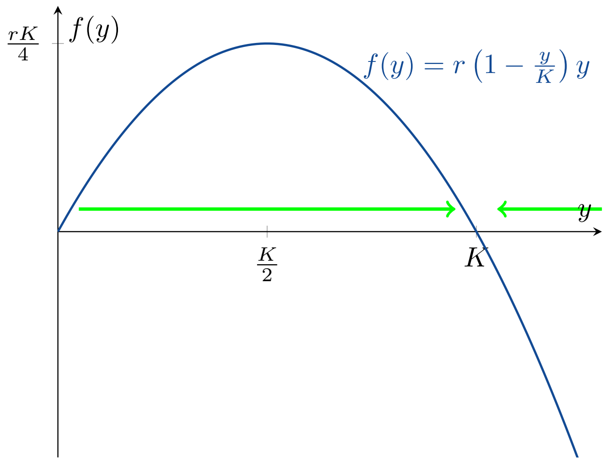
First we look for equilibrium solutions – that is solutions with $\frac{dy}{dt} = 0$ for all t.

$$0 = \frac{dy}{dt} = r\left(1 - \frac{y}{K}\right)y \qquad \Longrightarrow \qquad y = 0 \text{ or } y = K.$$

The equilibrium solutions are important. If we look at some more solutions, we can see that the other solutions converge to $y = K$, but diverge from $y = 0$.



To understand this behaviour, we graph $\frac{dy}{dt}$ against y.



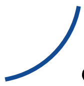
Note that

- $\frac{dy}{dt} > 0 \implies y$ is increasing; and
- $\frac{dy}{dt} < 0 \implies y$ is decreasing; and

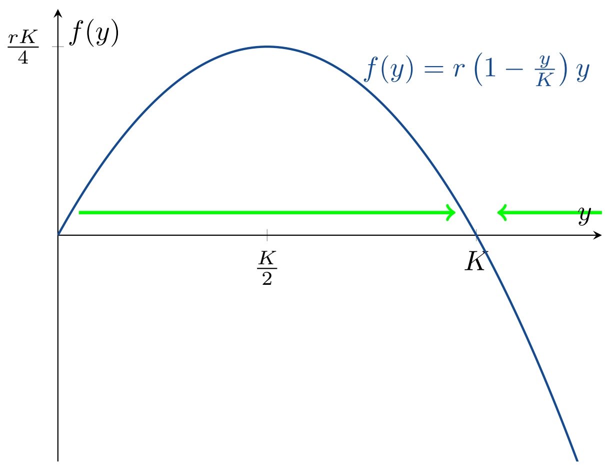
We can show this on the graph by drawing green arrows.

To investigate further, we look at $\frac{d^2y}{dt^2}$: If $\frac{dy}{dt} = f(y)$, then

$$\frac{d^2y}{dt^2} = \frac{d}{dt}\Big(f\big(y(t)\big)\Big) = f'(y)\frac{dy}{dt} = f'(y)f(y).$$

The solution $y(t)$ is concave up ( or ) when $y'' > 0$ (i.e. when both f and f' are both positive or both negative). The solution $y(t)$ is concave down ( or ) when $y'' < 0$ (i.e. when one of f and f' is positive and one is negative).

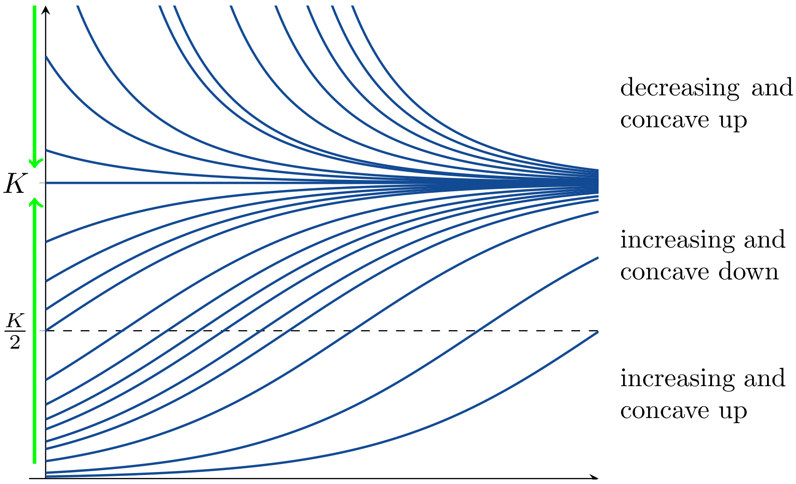
Look again at the graph of $f(y) = r\left(1 - \frac{y}{K}\right)y$ against y.



We can see that

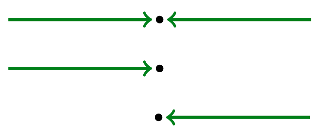
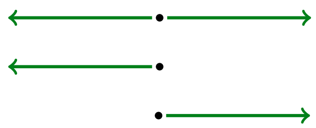
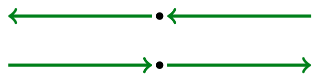
- $y \in (0, \frac{K}{2}) \implies f > 0$ and $f' > 0 \implies y(t)$ is increasing and concave up;
- $y \in (\frac{K}{2}, K) \implies f > 0$ and $f' < 0 \implies y(t)$ is increasing and concave down;
- $y \in (K, \infty) \implies f < 0$ and $f' < 0 \implies y(t)$ is decreasing and concave up;

Moreover, remember that Theorem 2.2 told us that two solutions can not intersect. Hence the solutions look like this:



Because solutions converge to $y = K$, we say that $y = K$ is an ***asymptotically stable equilibrium solution*** or an ***asymptotically stable critical point***. Because solutions diverge from $y = 0$, we say that $y = 0$ is an ***unstable equilibrium solution*** or an ***unstable critical point***.

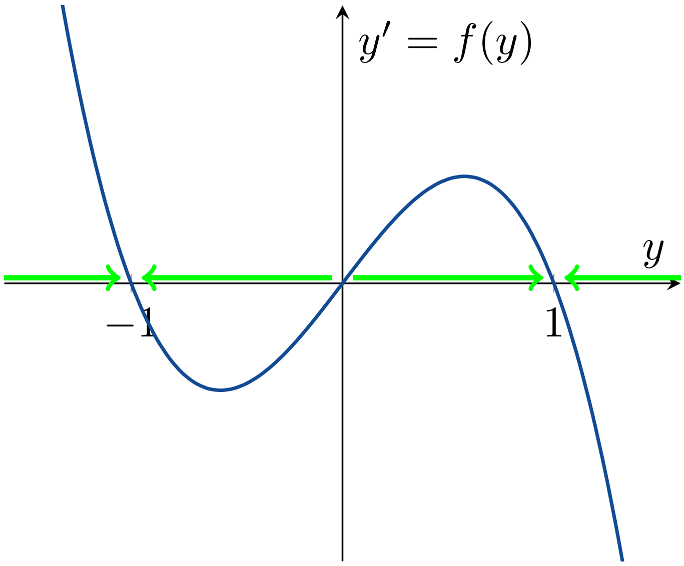
Definition. Equilibrium solutions/critical points can be

	***asymptotically stable;***
	***unstable;***
	***semistable.***

Example 2.12. Find all of the critical points of

$$\frac{dy}{dt} = \underbrace{y(1-y^2)}_{f(y)} \qquad (-\infty < y_0 < \infty)$$

and classify each as asymptotically stable, unstable or semistable.



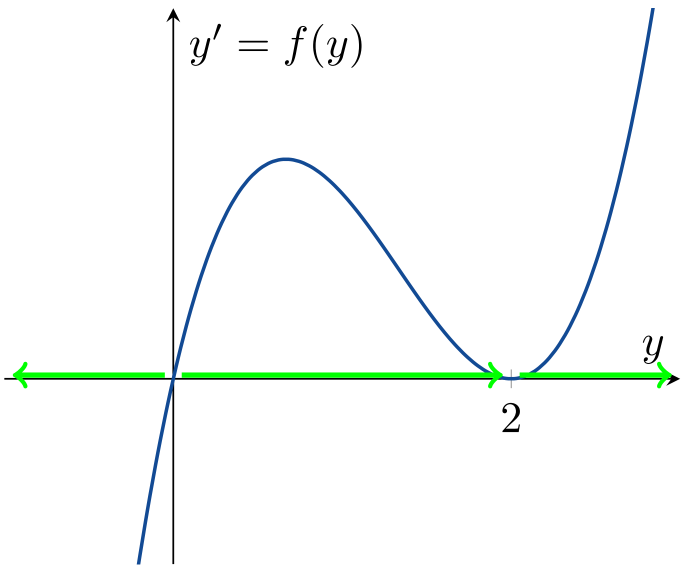
The critical points are $y = -1, 0, 1$.

- $y = -1$ is asymptotically stable;
- $y = 0$ is unstable; and
- $y = 1$ is asymptotically stable.

Example 2.13. Find all of the critical points of

$$\frac{dy}{dt} = \underbrace{y(y-2)^2}_{f(y)} \qquad (-\infty < y_0 < \infty)$$

and classify each as asymptotically stable, unstable or semistable.



The critical points are $y = 0$ and 2.

- $y = 0$ is unstable; and
- $y = 2$ is semistable.

Example 2.14. Consider the autonomous differential equation

$$\frac{dy}{dt} = f(y) = y^4 - 5y^3 + 6y^2. \tag{2.10}$$

(i). Find all of the critical points of (2.10).

(ii). Sketch the graph of $f(y)$ versus y.

(iii). Determine whether each critical point is asymptotically stable, unstable or semistable.

(iv). Sketch 10 (or more) different solutions of (2.10).

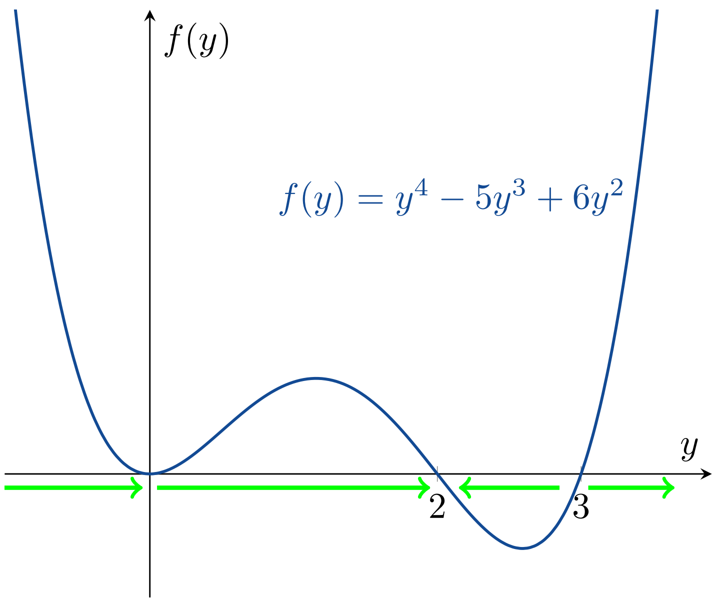
(This is an exam question from 2013: Students had 30 minutes to solve this.)

(i).

$$\frac{dy}{dt} = f(y) = y^4 - 5y^3 + 6y^2 = y^2(y-2)(y-3).$$

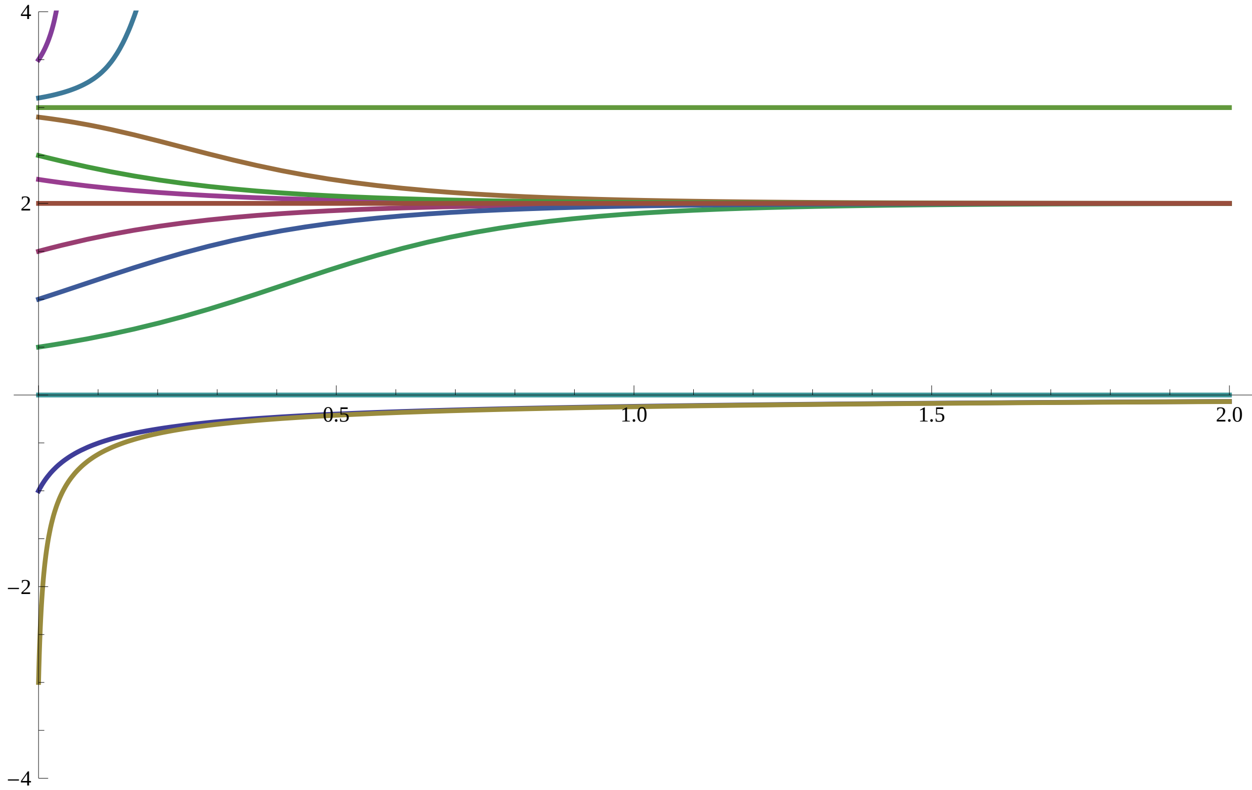
The critical points are $y = 0$, $y = 2$ and $y = 3$.

(ii).



(iii). $y = 0$ is semistable, $y = 2$ is asymptotically stable and $y = 3$ is unstable.

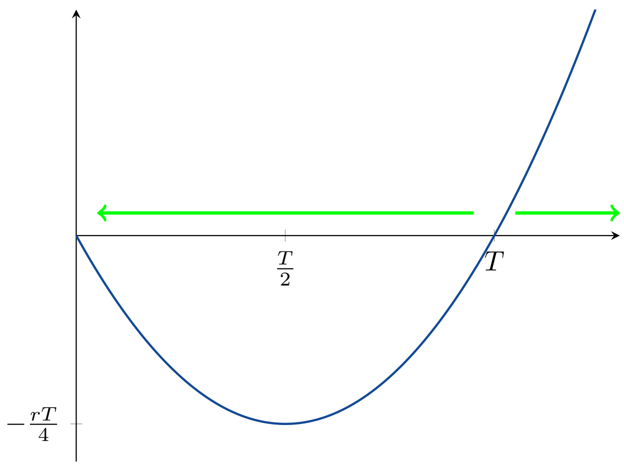
(iv).



Example 2.15 (A Critical Threshold)**.** Now suppose that we can model the number of cats in İstanbul by

$$\frac{dy}{dt} = -r\left(1 - \frac{y}{T}\right)y$$

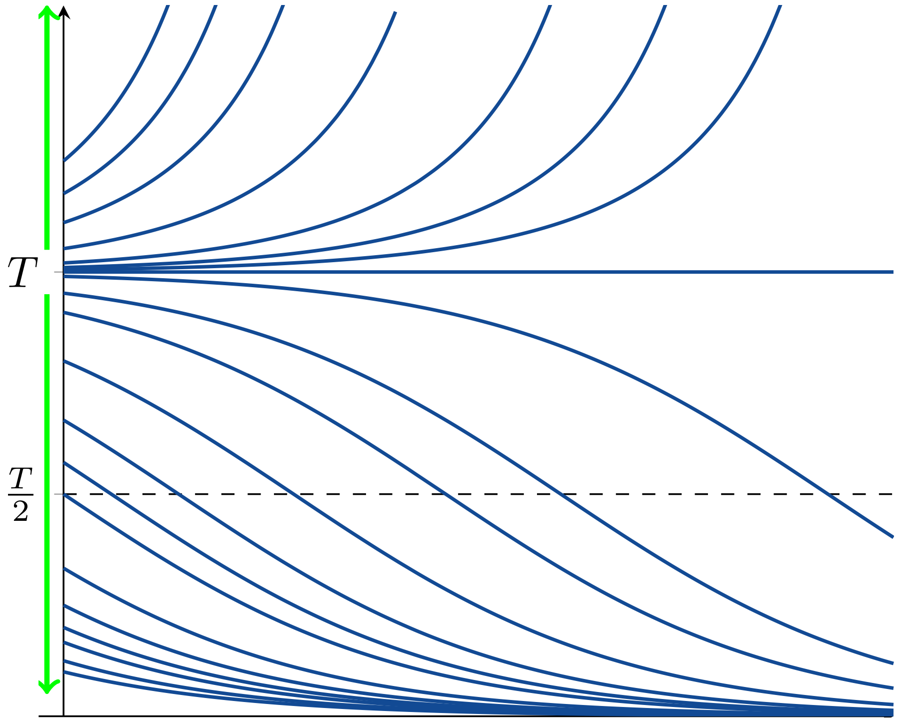
where $T > 0$ and $r > 0$.



The critical points/equilibrium solutions are $y = 0$ and $y = T$.

- $y = 0$ is asymptotically stable; and
- $y = T$ is unstable.

With this information we can sketch some solutions



Depending on y_0 ($y_0 \neq T$), we either have $y \to 0$ or $y \to \infty$. The number T is called a ***threshold level***, below which no growth happens.

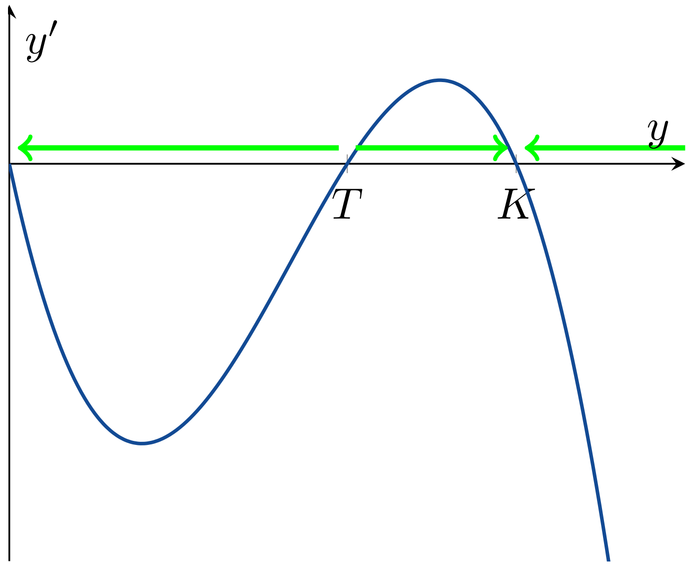
The population of some species have the threshold property: If there are not enough individuals, then the species becomes extinct.

This model predicts that the number of cats in İstanbul will increase to ∞ (if $y_0 > T$), so we need a more advanced model.

Example 2.16 (Logistic Growth with a Threshold)**.** Now consider

$$\frac{dy}{dt} = -r\left(1-\frac{y}{T}\right)\left(1-\frac{y}{K}\right)y$$

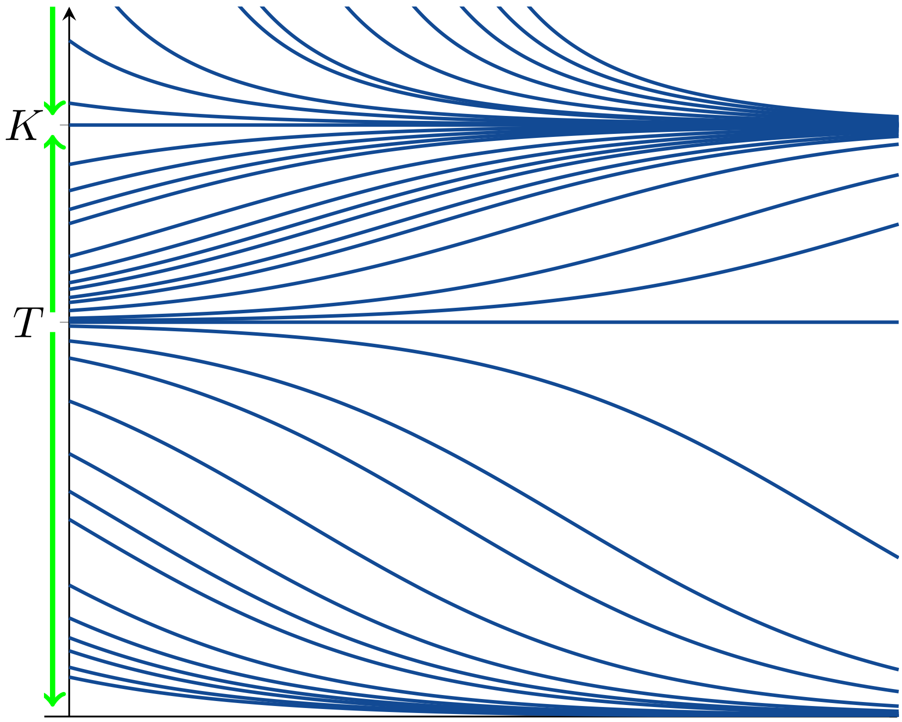
for $0 < T < K$ and $r > 0$.



The critical points/equilibrium solutions are $y = 0$, $y = T$ and $y = K$.

- $y = 0$ is asymptotically stable;
- $y = T$ is unstable; and
- $y = K$ is asymptotically stable.

Solutions look like this:



This is an equation which has been used by biologists to model certain populations of animals.

2.5 Exact Equations

We have looked at linear equations and separable equations. Now we will look at another special type of equation.

Example 2.17. Solve $2x + y^2 + 2xyy' = 0$.

This equation is not linear and is not separable. Note that if $\psi(x, y) = x^2 + xy^2$, then $\frac{\partial\psi}{\partial x} = 2x + y^2$ and $\frac{\partial\psi}{\partial y} = 2xy$. So we can write the ODE as

$$\frac{\partial\psi}{\partial x} + \frac{\partial\psi}{\partial y}\frac{dy}{dx} = 0.$$

Since $y(x)$ is a function of x, we also have that

$$\frac{d}{dx}\Big(\psi\big(x, y(x)\big)\Big) = \frac{\partial\psi}{\partial x} + \frac{\partial\psi}{\partial y}\frac{dy}{dx}$$

by the Chain Rule. So our ODE can be written as

$$\frac{d}{dx}\big(x^2 + xy^2\big) = 0.$$

Therefore

$$x^2 + xy^2 = c.$$

Remark 2.6. The key step was finding $\psi(x, y)$.

Now consider

$$M(x, y) + N(x, y)y' = 0. \tag{2.11}$$

Definition. If we can find a function $\psi(x, y)$ such that

$$\frac{\partial\psi}{\partial x} = M \qquad \text{and} \qquad \frac{\partial\psi}{\partial y} = N,$$

then (2.11) is called an ***exact equation***.

If (2.11) is exact, then

$$0 = M(x, y) + N(x, y)y' = \frac{\partial\psi}{\partial x}(x, y) + \frac{\partial\psi}{\partial y}(x, y)\frac{dy}{dx} = \frac{d}{dx}\Big(\psi\big(x, y(x)\big)\Big)$$

which has solution

$$\psi(x, y) = c.$$

Remark 2.7. To solve an exact equation:

(i). Find $\psi(x, y)$;

(ii). Write $\psi(x, y) = c$.

Notation.

$$y' = \frac{dy}{dx} \qquad f_x = \frac{\partial f}{\partial x} \qquad f_y = \frac{\partial f}{\partial y}$$

Theorem 2.3. *Suppose that M, N, M_y and N_x are continuous on the rectangular region $R = \{(x, y) \ : \ \alpha < x < \beta, \ \gamma < y < \delta\}$. Then*

$$M + Ny' = 0 \text{ is exact} \qquad \Longleftrightarrow \qquad M_y = N_x.$$

Example 2.18. Consider

$$(y \cos x + 2xe^y) + (\sin x + x^2 e^y - 1)y' = 0.$$

Is this ODE exact? If yes, solve it.

$$\begin{aligned} M &= y \cos x + 2xe^y & \qquad M_y &=< 3- > \cos x + 2xe^y \\ N &= \sin x + x^2 e^y - 1 & \qquad N_x &=< 4- > \cos x + 2xe^y \end{aligned}$$

Yes, the ODE is exact. So $\exists \ \psi$ such that

$$\begin{aligned} \psi_x &= M = y \cos x + 2xe^y \\ \psi_y &= N = \sin x + x^2 e^y - 1. \end{aligned}$$

Integrating the first equation (wrt x) gives

$$\psi = \int \psi_x \, dx = y \sin x + x^2 e^y + h(y).$$

Then differentiating (wrt y) gives

$$\psi_y = \sin x + x^2 e^y + h'(y).$$

But we already know that $\psi_y = \sin x + x^2 e^y - 1$. So $h'(y) = -1$ and $h(y) = -y$. So

$$\psi(x, y) = y \sin x + x^2 e^y - y.$$

The solution to the ODE is

$$\boxed{y \sin x + x^2 e^y - y = c.}$$

Example 2.19. Consider
$$ye^{xy} + e^{xy}y' = 0.$$
Is this ODE exact? If yes, solve it.

We have
$$\begin{aligned} M &= ye^{xy} & M_y &= e^{xy} + xye^{xy} \\ N &= e^{xy} & N_x &= ye^{xy}. \end{aligned}$$
Since $M_y \neq N_x$, the ODE is not exact.

Example 2.20. Consider
$$\left(4x^3y^3 + \frac{1}{x}\right) + \left(3x^4y^2 + \frac{1}{y}\right)y' = 0.$$
Is this ODE exact? If yes, solve it.

I leave this one to you to solve. Please check that the solution is
$$\boxed{x^4y^3 + \ln|x| + \ln|y| = c.}$$

Example 2.21. Consider
$$1 + \left(1 + 2y + 3y^2\right)y' = 0.$$
Is this ODE exact? If yes, solve it.

First note that
$$\begin{aligned} M &= 1 & M_y &= 0 \\ N &= 1 + 2y + 3y^2 & N_x &= 0 = M_y \end{aligned}$$
Yes, the ODE is exact. So $\exists\, \psi$ such that
$$\begin{aligned} \psi_x &= 1 \\ \psi_y &= 1 + 2y + 3y^2. \end{aligned}$$
We can start with $\psi_x = 1$ or with $\psi_y = 1 + 2y + 3y^2$.

$$\begin{aligned} \psi_x &= 1 \\ \psi &= \int 1\, dx = x + h(y) \\ \psi_y &= h'(y) \\ h'(y) &= 1 + 2y + 3y^2 \\ h(y) &= y + y^2 + y^3 \\ \psi &= x + y + y^2 + y^3 \end{aligned}$$

$$\begin{aligned} \psi_y &= 1 + 2y + 3y^2 \\ \psi &= \int 1 + 2y + 3y^2\, dy \\ &= y + y^2 + y^3 + h(x) \\ \psi_x &= h'(x) \\ h'(x) &= 1 \\ h(x) &= x \\ \psi &= x + y + y^2 + y^3 \end{aligned}$$

Therefore the solution is $\boxed{x + y + y^2 + y^3 = c.}$

Example 2.22. Consider

$$(3xy + y^2) + (x^2 + xy)y' = 0.$$

Is this ODE exact? If yes, solve it.

First note that

$$\begin{aligned} M &= 3xy + y^2 \qquad & M_y &= 3x + 2y \\ N &= x^2 + xy \qquad & N_x &= 2x + y \neq M_y \end{aligned}$$

Since $M_y \neq N_x$, this ODE is not exact. So our method to solve an exact equation ***will not work***. But we are going to try our method anyway, so that we can see what goes wrong.

Suppose that $\exists\ \psi(x, y)$ such that

$$\begin{aligned} \psi_x &= 3xy + y^2 \\ \psi_y &= x^2 + xy. \end{aligned}$$

Integrating ψ_x with respect to x gives

$$\psi = \frac{3}{2}x^2y + xy^2 + h(y).$$

Thus

$$x^2 + xy = \psi_y = \frac{\partial}{\partial y}\left(\frac{3}{2}x^2y + xy^2 + h(y)\right) = \frac{3}{2}x^2 + 2xy + h'(y).$$

So we need h to satisfy

$$h'(y) = -\frac{1}{2}x^2 - xy.$$

This is not possible!!! $h(y)$ must be a function of y, but $-\frac{1}{2}x^2 - xy$ depends on both x and y. So it is not possible to find h. So it is not possible to find ψ. Our method failed because $M_y \neq N_x$.

Integrating Factors

It is sometimes possible to convert a differential equation which is not exact into an exact equation by multiplying it by an integrating factor. (Do you remember how we solve linear equations?)

Consider

$$M(x, y)\,dx + N(x, y)\,dy = 0. \tag{2.12}$$

Suppose that (2.12) is not exact. If we multiply by $\mu(x, y)$, we obtain

$$\mu(x, y)M(x, y)\,dx + \mu(x, y)N(x, y)\,dy = 0. \tag{2.13}$$

By Theorem 2.3, we know that

$$(2.13) \text{ is exact} \qquad \Longleftrightarrow \qquad (\mu M)_y = (\mu N)_x.$$

Now

$$\begin{aligned} (\mu M)_y &= (\mu N)_x \\ \mu_y M + \mu M_y &= \mu_x N + \mu N_x \end{aligned}$$

$$M\mu_y - N\mu_x + (M_y - N_x)\mu = 0. \tag{2.14}$$

If we can find $\mu(x,y)$ which solves (2.14), then (2.13) is exact and we know how to solve exact equations.

But (2.14) is a first order partial differential equation and PDEs are typically not easy to solve. How can we make this easier? Instead of $\mu(x,y)$, we could look for $\mu(x)$. Then $\mu_y = 0$ and (2.14) becomes

$$0 - N\frac{d\mu}{dx} + (M_y - N_x)\mu = 0$$
$$N\frac{d\mu}{dx} = (M_y - N_x)\mu$$

$$\boxed{\frac{d\mu}{dx} = \left(\frac{M_y - N_x}{N}\right)\mu.} \tag{2.15}$$

If $\frac{M_y - N_x}{N}$ is a function only of x, then there is an integrating factor $\mu(x)$. Please note that (2.15) is both linear and separable.

If instead we looked for $\mu(y)$, we would obtain the ODE

$$\boxed{\frac{d\mu}{dy} = \left(\frac{N_x - M_y}{M}\right)\mu.} \tag{2.16}$$

Remark 2.8. You are expected to remember (2.15) and (2.16).

Example 2.23. Solve

$$(3xy + y^2) + (x^2 + xy)y' = 0.$$

We know that this equation is not exact. So we will try to find an integrating factor: We have that

$$M = 3xy + y^2 \qquad M_y = 3x + 2y$$
$$N = x^2 + xy \qquad N_x = 2x + y \neq M_y$$

So

$$\frac{M_y - N_x}{N} = \frac{(3x+2y) - (2x+y)}{x^2 + xy} = \frac{x+y}{x(x+y)} = \frac{1}{x}$$

and

$$\frac{N_x - M_y}{M} = \frac{(2x+y) - (3x+2y)}{3xy + y^2} = \frac{-x-y}{y(3x+y)}.$$

Note that $\frac{M_y - N_x}{N}$ is a function only of x – so it is possible to find an integrating factor $\mu(x)$. Moreover note that $\frac{N_x - M_y}{M}$ is ***not*** a function only of y – so it is ***not*** possible to find a $\mu(y)$.

We calculate that

$$\frac{d\mu}{dx} = \left(\frac{M_y - N_x}{N}\right)\mu$$
$$\frac{d\mu}{dx} = \frac{\mu}{x}$$
$$\frac{d\mu}{\mu} = \frac{dx}{x}$$
$$\int \frac{d\mu}{\mu} = \int \frac{dx}{x}$$
$$\ln|\mu| = \ln|x| + C$$
$$\mu = cx$$

and we choose $c = 1$ for simplicity. So $\mu(x) = x$.

Multiplying our original ODE by $\mu(x) = x$ gives

$$(3x^2y + xy^2) + (x^3 + x^2y)y' = 0.$$

This ODE is exact ($M_y = 3x^2 + 2xy = N_x$) and we know how to solve exact equations. We must find ψ such that

$$\begin{aligned} \psi_x &= 3x^2y + xy^2 \\ \psi_y &= x^3 + x^2y. \end{aligned}$$

Integrating ψ_x wrt x gives

$$\psi = x^3y + \frac{1}{2}x^2y^2 + h(y).$$

Hence

$$x^3 + x^2y = \psi_y = \frac{\partial}{\partial y}\left(x^3y + \frac{1}{2}x^2y^2 + h(y)\right) = x^3 + x^2y + h'(y)$$

and we see that we may choose $h(y) = 0$. Therefore

$$\psi = x^3y + \frac{1}{2}x^2y^2.$$

So the solution to the ODE is

$$\boxed{x^3y + \frac{1}{2}x^2y^2 = c.}$$

Example 2.24. Solve

$$ye^{xy} + \left(\left(\frac{2}{y} + x\right)e^{xy}\right)y' = 0.$$

This ODE is not exact (you check!).

$$\begin{aligned} \frac{M_y - N_x}{N} &= \frac{e^{xy} + xye^{xy} - e^{xy} - (2 + xy)\,e^{xy}}{\left(\frac{2}{y} + x\right)e^{xy}} = \frac{-2}{\frac{2}{y} + x} \\ \frac{N_x - M_y}{M} &= \frac{2e^{xy}}{ye^{xy}} = \frac{2}{y}. \end{aligned}$$

Since $\frac{N_x - M_y}{M}$ is a function only of y, we look for $\mu(y)$.

•

•

• (you complete this calculation)

•

•

Therefore $\mu(y) = y^2$.

Multiplying our ODE by y^2 gives

$$y^3e^{xy} + \left(\left(2y + xy^2\right)e^{xy}\right)y' = 0.$$

•

•

• (you complete this calculation)

•

•

Hence the solution is

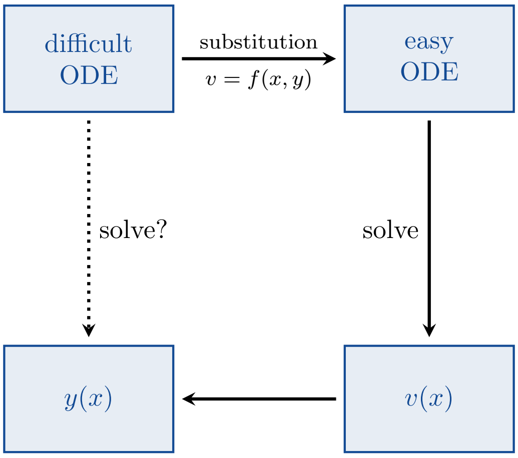
$$\boxed{y^2e^{xy} = c.}$$

2.6 Substitutions

Recall how we calculate an integral such as $\int 3x^2 \sin x^3 \, dx$. We use a substitution, in this case $u = x^3$, to turn a difficult integral into an easy integral:

$$\underbrace{\int 3x^2 \sin x^3 \, dx}_{\text{difficult}} = \underbrace{\int \sin u \, du}_{\text{easy}}.$$

Sometimes we can use the same idea to solve ODEs.



We will use substitutions to solve two types of first order ODE:

- Homogeneous Equations;
- Bernoulli Equations.

Homogeneous Equations

Definition. The first order ODE $\frac{dy}{dx} = f(x, y)$ is called ***homogeneous*** iff we can write it as

$$\frac{dy}{dx} = g\left(\frac{y}{x}\right).$$

For example, the following ODEs are homogeneous:

$$\frac{dy}{dx} = \cos\left(\frac{y}{x}\right) \qquad \frac{dy}{dx} = \left(\frac{y}{x}\right)^3 + \frac{y}{x}$$

$$\frac{dy}{dx} = \cos\left(\frac{x}{y}\right) \qquad \frac{dy}{dx} = \frac{\frac{y}{x} - 4}{1 - \frac{y}{x}}$$

For a homogeneous equation, we use the substitution

$$v(x) = \frac{y}{x}.$$

Note that $y = xv(x)$ and

$$\frac{dy}{dx} = \frac{d}{dx}\big(xv(x)\big) = v + x\frac{dv}{dx}.$$

Example 2.25. Solve $\dfrac{dy}{dx} = \dfrac{y-4x}{x-y}$.

Note first that
$$\frac{dy}{dx} = \frac{y-4x}{x-y} = \frac{\frac{y}{x}-4}{1-\frac{y}{x}}.$$
If we substitute in $v = \frac{y}{x}$ we get
$$\frac{dy}{dx} = \frac{v-4}{1-v}.$$

But remember that $\frac{dy}{dx} = v + x\frac{dv}{dx}$. Hence
$$v + x\frac{dv}{dx} = \frac{v-4}{1-v}$$
and
$$x\frac{dv}{dx} = \frac{v-4}{1-v} - v = \frac{v-4}{1-v} - \frac{v-v^2}{1-v} = \frac{v^2-4}{1-v}$$

Note that
$$x\frac{dv}{dx} = \frac{v^2-4}{1-v}$$
is a separable equation. You know how to solve separable equations – the following should be revision for you. We rearrange to
$$\left(\frac{1-v}{v^2-4}\right) dv = \frac{dx}{x}$$
$$\left(-\frac{3}{4(v+2)} - \frac{1}{4(v-2)}\right) dv = \frac{dx}{x}$$

then integrate to find
$$\begin{aligned}
-\frac{3}{4}\ln|v+2| - \frac{1}{4}\ln|v-2| &= \ln|x| + k \\
\ln|v+2|^3 + \ln|v-2| &= \ln|x|^{-4} - 4k \\
|v+2|^3\,|v-2| &= c\,|x|^{-4} \qquad\qquad (c = \pm e^{-4k}) \\
|x|^4\,|v+2|^3\,|v-2| &= c \\
|vx+2x|^3\,|vx-2x| &= c.
\end{aligned}$$

Now we have an equation for v. The final step is to find an equation for y.

If we substitute $y = vx$ into this equation, we find the solution
$$\boxed{|y+2x|^3\,|y-2x| = c.}$$

Remark 2.9. To solve a homogeneous equation:

STEP 1. Substitute $v = \frac{y}{x}$ (and $\frac{dy}{dx} = v + x\frac{dv}{dx}$);

STEP 2. Solve a separable equation;

STEP 3. Substitute $y = vx$.

Example 2.26. Solve $\dfrac{dy}{dx} = \dfrac{x^2 + 3y^2}{2xy}$.

First we rearrange

$$\frac{dy}{dx} = \frac{1 + 3\frac{y^2}{x^2}}{2\frac{y}{x}}$$

and substitute $v = \frac{y}{x}$ and $\frac{dy}{dx} = v + x\frac{dv}{dx}$ to get

$$v + x\frac{dv}{dx} = \frac{1 + 3v^2}{2v}.$$

Rearranging gives

$$x\frac{dv}{dx} = \frac{1 + 3v^2}{2v} - v = \frac{1 + 3v^2 - 2v^2}{2v} = \frac{1 + v^2}{2v}.$$

This is a separable equation which we can solve:

$$\begin{aligned}
\frac{2v\,dv}{1 + v^2} &= \frac{dx}{x} \\
\int \frac{2v\,dv}{1 + v^2} &= \int \frac{dx}{x} \\
\ln\left|1 + v^2\right| &= \ln|x| + k \\
1 + v^2 &= cx \\
1 + v^2 - cx &= 0.
\end{aligned}$$

Substituting $v = \frac{y}{x}$ then gives

$$1 + \frac{y^2}{x^2} - cx = 0$$

and

$$\boxed{x^2 + y^2 - cx^3 = 0.}$$

Bernoulli Equations

Definition. An equation of the form

$$y' + p(t)y = q(t)y^{n}$$

is called a ***Bernoulli equation***.

For Bernoulli equations, we use the substitution

$$v(x) = y^{1-n}.$$

Example 2.27. Solve $\dfrac{dy}{dx} - \left(\dfrac{3}{2x}\right) y = 2xy^{-1}$.

Note first that this ODE has $n = -1$. Therefore we will use the substitution $v = y^{1-n} = y^{1-(-1)} = y^2$. This means that $y = v^{\frac{1}{2}}$ and

$$\frac{dy}{dx} = \frac{dy}{dv}\frac{dv}{dx} = \frac{1}{2}v^{-\frac{1}{2}}\frac{dv}{dx}.$$

We take our ODE

$$\frac{dy}{dx} - \left(\frac{3}{2x}\right) y = 2xy^{-1}$$

and we substitute in $y = v^{\frac{1}{2}}$ and $\frac{dy}{dx} = \frac{1}{2}v^{-\frac{1}{2}}\frac{dv}{dx}$ to obtain

$$\frac{1}{2}v^{-\frac{1}{2}}\frac{dv}{dx} - \left(\frac{3}{2x}\right) v^{\frac{1}{2}} = 2xv^{-\frac{1}{2}}.$$

Multiplying by $2v^{\frac{1}{2}}$ gives

$$\frac{dv}{dx} - \frac{3}{x}v = 4x$$

which is a linear equation. You know how to solve linear equations, so the following should be revision for you. We multiply by the integrating factor

$$\mu(x) = e^{\int -\frac{3}{x}\,dx} = e^{-3\ln|x|} = \ldots = x^{-3}$$

to get

$$x^{-3}\frac{dv}{dx} - 3x^{-4}v = 4x^{-2}$$

which is

$$\frac{d}{dx}\left(x^{-3}v\right) = 4x^{-2}.$$

Integrating gives

$$\begin{aligned} x^{-3}v &= -4x^{-1} + C \\ v &= -4x^2 + Cx^3. \end{aligned}$$

But $v = y^2$, so the solution is

$$\boxed{y^2 = -4x^2 + Cx^3.}$$

Remark 2.10. To solve a Bernoulli equation:

STEP 1. Substitute $v = y^{1-n}$;

STEP 2. Solve a linear equation;

STEP 3. Substitute $y^{1-n} = v$.

Example 2.28. Solve $x\dfrac{dy}{dx} + 6y = 3xy^{\frac{4}{3}}$.

Note that this time we have $n = \frac{4}{3}$ and $v = y^{1-n} = y^{-\frac{1}{3}}$. Hence $y = v^{-3}$ and

$$\frac{dy}{dx} = \frac{dy}{dv}\frac{dv}{dx} = -3v^{-4}\frac{dv}{dx}.$$

Thus our ODE becomes

$$\begin{aligned} -3xv^{-4}\frac{dv}{dx} + 6v^{-3} &= 3xv^{-4} \\ -x\frac{dv}{dx} + 2v &= x \\ \frac{dv}{dx} - \frac{2}{x}v &= -1. \end{aligned}$$

This is a linear equation which we can solve using the integrating factor $\mu(x) = x^{-2}$. Please check that its solution is

$$v = x + Cx^2.$$

Finally we use $v = y^{-\frac{1}{3}}$ to find that

$$\boxed{y = \frac{1}{(x + Cx^2)^3}.}$$

Exercises

Exercise 2.1 (Linear Equations). Solve the following ODEs:

(a). $y' + y = 5$

(b). $y' + y = te^{-t} + 1$

(c). $y' - 3y = 4e^t$

(d). $ty' - y = t^2 e^{-t}$

(e). $y' + 2ty = 2te^{-t^2}$

(f). $y' + y - 5\sin 2t = 0$

Exercise 2.2 (Initial Value Problems). Solve the following IVPs:

(a). $\begin{cases} \frac{dy}{dt} - y = 2te^{2t} \\ y(0) = 1 \end{cases}$

(b). $\begin{cases} y' + 3y = 12 \\ y(0) = 6 \end{cases}$

(c). $\begin{cases} y' + \left(\frac{2}{t}\right) y = \frac{\cos t}{t^2} \\ y(\pi) = 0 \end{cases}$

Exercise 2.3 (Separable Equations). Solve the following initial value problems:

(a). $\begin{cases} \frac{dy}{dx} = (1-2x)y^2 \\ y(0) = -\frac{1}{6} \end{cases}$

(b). $\begin{cases} x + ye^{-x}\frac{dy}{dx} = 0 \\ y(0) = 1 \end{cases}$

(c). $\begin{cases} \frac{dy}{dx} = \frac{2x}{y+x^2y} \\ y(0) = -2 \end{cases}$

Exercise 2.4 (Stable, Unstable and Semi-Stable Equilibrium Solutions). Each of the following problems involve equations of the form $y' = f(y)$. In each problem, (i) sketch the graph of $f(y)$ versus y; (ii) find the equilibrium solutions (critical points) of the ODE; and (iii) classify each equilibrium solution as asymptotically stable, semi-stable, or unstable.

(a). $\frac{dy}{dt} = ay + by^2$, $a, b > 0$, $y_0 \geq 0$.

(b). $\frac{dy}{dt} = ay + by^2$, $a, b > 0$, $-\infty < y_0 < \infty$.

(c). $\frac{dy}{dt} = y(y-1)(y-2)$, $y_0 \geq 0$.

(d). $\frac{dy}{dt} = y(1-y)^2$, $-\infty < y_0 < \infty$.

(e). $\frac{dy}{dt} = e^y - 1$, $-\infty < y_0 < \infty$.

(f). $\frac{dy}{dt} = e^{-y} - 1$, $-\infty < y_0 < \infty$.

Exercise 2.5 (Sick Students).

Suppose that the students of İstanbul Okan Üniversitesi can be divided into two groups; those who have the flu virus and can infect others, and those who do not have it but are susceptible. Let x be the proportion of susceptible individuals and y the proportion of infectious individuals; then $x+y=1$.

Assume that the disease spreads by contact between sick students and well students, and that the rate of spread $\frac{dy}{dt}$ is proportional to the number of such contacts. So $\frac{dy}{dt} = k_1 \times$ (number of contacts). Further, assume that members of both groups move about freely among each other, so the number of contacts is proportional to the product of x and y. So (number of contacts) $= k_2xy$. Since $x = 1-y$, we obtain the initial value problem

$$\begin{cases} \frac{dy}{dt} = \alpha y(1-y), \\ y(0) = y_0, \end{cases} \tag{2.17}$$

where $\alpha > 0$ is a constant, and $0 \leq y_0 \leq 1$ is the initial proportion of infectious individuals.

İstanbul Okan Üniversitesi öğrencilerinin iki gruba ayrıldıklarını varsayın; grip virüsü taşıyan, diğer öğrencilere bulaştırabilecek olanlar ve virüsü taşımayan ancak hastalığa yakalanabilecek olanlar. Hastalığa yakalanabilecek bireylerin oranı x; hastalığı taşıyan ve bulaştırabilecek olanların oranı y'dir. Bu durumda $x+y=1$.

Hastalığın, hasta öğrencilerle sağlıklı öğrenciler arasında etkileşimle yayıldığını, ve $\frac{dy}{dt}$ olan yayılma hızının etkileşim sayısıyla orantılı olduğunu varsayın. Yani $\frac{dy}{dt} = k_1 \times$ (etkileşim sayısı). Ayrıca, her iki grubun üyelerinin birbirlerinin arasında serbestçe dolaştıklarını varsayın; böylece etkileşim sayısı x ve yenin çarpımları ile orantılıdır. Yani, (etkileş im sayısı) $= k_2xy$. $x = 1-y$ olduğundan, (2.17)'i elde ederiz. $\alpha > 0$ sabit sayıdır, $0 \leq y_0 \leq 1$ hastalık bulaştırabilecek öğrencilerin en baştaki oranıdır.

(a). Find the equilibrium points for the differential equation and determine whether each is asymptotically stable, semi-stable, or unstable.

(b). Draw the graphs of some solutions.

(c). Solve (2.17).

(d). Suppose that $y_0 > 0$. Show that $\lim_{t\to\infty} y(t) = 1$, which means that ultimately all students catch the disease.

Exercise 2.6 (Exact Equations). Determine if each of the following ODEs is an exact equation. If it is exact, find the solution.

(a). $(2x+4y)+(2x-2y)y' = 0$

(b). $(2x+3)+(2y-2)y' = 0$

(c). $(3x^2-2xy+2)+(6y^2-x^2+3)\dfrac{dy}{dx} = 0$

(d). $(2xy^2+2y)+(2x^2y+2x)y' = 0$

(e). $(e^x \sin y - 2y\sin x) + (e^x \cos y + 2\cos x)\dfrac{dy}{dx} = 0$

(f). $(e^x \sin y + 2y)dx + (3x - e^x \sin y)dy = 0$

Exercise 2.7 (Exact Equations). The following equations are not exact. For each one, (i) find an integrating factor ($\mu(x)$ or $\mu(y)$) which changes the equation into an exact equation; and (ii) solve the equation.

(a). $(3x^2y + 2xy + y^3) + (x^2 + y^2)y' = 0$

(b). $y' = e^{2x} + y - 1$

(c). $dx + \left(\dfrac{x}{y} - \sin y\right) dy = 0$

(d). $y + (2xy - e^{-2y})y' = 0$

Exercise 2.8 (Homogeneous Equations). Use the substitution $v(x) = \frac{y}{x}$ (or equivalently $y = v(x)x$ and then $y' = v'(x)x + v(x)$) to solve the following ODEs:

(a). $(x^2 + 3xy + y^2)dx - x^2dy = 0$

(b). $\dfrac{dy}{dx} = \dfrac{x^2 - 3y^2}{2xy}$

(c). $x\dfrac{dy}{dx} = y + \sqrt{x^2 - y^2}$

Exercise 2.9 (Bernoulli Equations). We can use the substitution $v(x) = y^{1-n}$ to solve $y' + p(t)y = q(t)y^n$. Use this technique to solve the following ODEs:

(a). $t^2y' + 2ty - y^3 = 0$

(b). $y' = ry - ky^2$ (where $r > 0$ and $k > 0$ are constants). This is an autonomous equation called the Logistic Equation.

(c). $y' = \varepsilon y - \sigma y^3$ (where $\varepsilon > 0$ and $\sigma > 0$ are constants). This equation occurs in the study of the stability of fluid flow.

In the exams, you will typically not be told if an equation is linear, separable, exact, homogeneous, etc – you should be able to determine this yourself. You can use Exercises 2.10 and 2.11 to practise.

Exercise 2.10 (First Order ODEs). Find the general solutions of the following ODEs:

(a). $9yy' + 4x = 0$.

(b). $y'+(x+1)y^3 = 0$.

(c). $\frac{dx}{dt} = 3t(x+1)$.

(d). $y'+\csc y = 0$.

(e). $x' \sin 2t = x \cos 2t$.

(f). $y' = (y-1)\cot x$.

(g). $\frac{dy}{dx} + \left(\frac{2x+1}{x}\right) y = e^{-2x}$.

(h). $(3x^2 + y^2)dx + 2xydy = 0$.

(i). $y' = \frac{y}{x}+\tan\left(\frac{y}{x}\right)$.

(j). $e^{\frac{x}{y}}(y - x)\frac{dy}{dx} + y(1 + e^{\frac{x}{y}}) = 0$.

(k). $(2x + 3y)dx + (3x + 2y)dy = 0$.

(l). $(x^3 + \frac{y}{x})dx + (y^2 + \ln x)dy = 0$.

(m). $(e^x \sin y + \tan y)dx + (e^x \cos y + x \sec^2 y)dy = 0$.

(n). $ydx + (2x - ye^y)dy = 0$.

(o). $xy' + y = y^{-2}$

(p). $y' = y(xy^3 - 1)$.

(q). $(1 + x^2)y' = 2xy(y^3 - 1)$.

Exercise 2.11 (Initial Value Problems). Solve the following IVPs:

(a). $\begin{cases} y' = x^3 e^{-y} \\ y(2) = 0 \end{cases}$

(b). $\begin{cases} y\frac{dy}{dx} = 4x(y^2+1)^{\frac{1}{2}} \\ y(0) = 1 \end{cases}$

(c). $\begin{cases} y' = y\cot x \\ y(\frac{\pi}{2}) = 2 \end{cases}$

(d). $\begin{cases} y' + 3(y-1) = 2x \\ y(0) = 1 \end{cases}$

(e). $\begin{cases} \frac{dy}{dx} = \frac{10}{(x+y)e^{x+y}} - 1 \\ y(0) = 0 \end{cases}$

(f). $\begin{cases} (4x^2 - 2y^2)y' = 2xy \\ y(3) = -5 \end{cases}$

(g). $\begin{cases} (x-y)dx + (3x+y)dy = 0 \\ y(3) = -2 \end{cases}$

(h). $\begin{cases} \frac{dy}{dx} = \frac{x^3 - xy^2}{x^2 y} \\ y(1) = 1 \end{cases}$

(i). $\begin{cases} (xy+1)ydx + (2y-)dy = 0 \\ y(0) = 3 \end{cases}$.

(j). $\begin{cases} y' - \frac{1}{x}y = y^2 \\ y(1) = 2 \end{cases}$

3

Second and Higher Order Linear ODEs

In this chapter we will consider equations of the form

$$y'' + p(t)y' + q(t)y = g(t)$$

or

$$P(t)y'' + Q(t)y' + R(t)y = G(t).$$

Such equations are ***linear*** second order ODEs.

If $g(t)$ (or $G(t)$) is always zero, then the ODE is called ***homogeneous***. Otherwise it is ***nonhomogeneous***.

3.1 Homogeneous Equations with Constant Coefficients

First we will consider the equation

$$ay'' + by' + cy = 0 \tag{3.1}$$

where $a, b, c \in \mathbb{R}$ are constants.

Example 3.1. Solve $y'' - y = 0$.

We want to find a function $y(t)$ which satisfies

$$\frac{d^2y}{dt^2} = y.$$

- What about e^t? Yes!
- What about e^{-t}? Yes!
- And what about $c_1e^t + c_2e^{-t}$? Yes! In fact, this is the general solution to $y'' - y = 0$.

Example 3.2. Solve

$$\begin{cases} y'' - y = 0 \\ y(0) = 2 \\ y'(0) = -1. \end{cases}$$

First note that this IVP has one 2^{nd} order ODE and two initial conditions.

We know that $y(t) = c_1 e^t + c_2 e^{-t}$. We are looking for the solution which passes through the point $(0, 2)$ and has slope -1 at this point. Using the first initial condition we get that

$$2 = y(0) = c_1 + c_2 \qquad \Longrightarrow \qquad c_1 + c_2 = 2.$$

Next we need to differentiate $y(t)$:

$$y'(t) = \frac{d}{dt}\left(c_1 e^t + c_2 e^{-t}\right) = c_1 e^t - c_2 e^{-t}.$$

Thus

$$-1 = y'(0) = c_1 - c_2 \qquad \Longrightarrow \qquad c_1 - c_2 = -1.$$

To satisfy these two conditions we must have $c_1 = \frac{1}{2}$ and $c_2 = \frac{3}{2}$. Therefore the solution to the IVP is

$$\boxed{y(t) = \frac{1}{2}e^t + \frac{3}{2}e^{-t}.}$$

Now let's go back to

$$ay'' + by' + cy = 0. \tag{3.1}$$

In the previous example, we used exponential functions in our solution. Maybe we always want exponential solutions? We guess that $y(t) = e^{rt}$ might be the solution to (3.1) for some number r that we don't know yet.

Then we calculate that

$$\begin{aligned} y &= e^{rt} \\ y' &= re^{rt} \\ y'' &= r^2 e^{rt} \end{aligned}$$

and

$$0 = ay'' + by' + cy = (ar^2 + br + c)e^{rt}.$$

Since $e^{rt} \neq 0$ for all t, we must have that

$$ar^2 + br + c = 0. \tag{3.2}$$

Definition. (3.2) is called the ***characteristic equation*** of (3.1).

Theorem 3.1.

$$e^{rt} \textit{ solves } (3.1) \qquad \Longleftrightarrow \qquad r \textit{ solves } (3.2).$$

$ar^2 + br + c = 0$ has two roots, r_1 and r_2:

$$r_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

These roots might be

(i). real numbers and different ($r_1, r_2 \in \mathbb{R}$ and $r_1 \neq r_2$);

(ii). complex conjugates ($r_1, r_2 \in \mathbb{C} \setminus \mathbb{R}$, $\overline{r}_1 = r_2$); or

(iii). real numbers but repeated ($r_1, r_2 \in \mathbb{R}$ and $r_1 = r_2$).

We will study these three cases separately. First we study case (i).

Suppose that $r_1, r_2 \in \mathbb{R}$ and $r_1 \neq r_2$. In other words, suppose that $b^2 - 4ac > 0$. Then $y_1(t) = e^{r_1 t}$ and $y_2(t) = e^{r_2 t}$ are both solutions to (3.1). So

$$y(t) = c_1 y_1(t) + c_2 y_2(t) = c_1 e^{r_1 t} + c_2 e^{r_2 t}$$

will also be a solution for any constants $c_1, c_2 \in \mathbb{R}$. This is called the ***general solution*** to (3.1).

Example 3.3. Solve $y'' + 5y' + 6y = 0$.

The first thing that we must do is to write down the characteristic equation. The characteristic equation for this ODE is

$$0 = r^2 + 5r + 6 = (r+2)(r+3).$$

The two roots are $r_1 = -2$ and $r_2 = -3$. Therefore the general solution is

$$\boxed{y(t) = c_1 e^{-2t} + c_2 e^{-3t}.}$$

Example 3.4. Solve

$$\begin{cases} y'' + 5y' + 6y = 0 \\ y(0) = 2 \\ y'(0) = 3. \end{cases}$$

We already found that $y(t) = c_1 e^{-2t} + c_2 e^{-3t}$ is the general solution to the ODE. We just need to find c_1 and c_2. Since $y'(t) = -2c_1 e^{-2t} - 3c_2 e^{-3t}$ we have that

$$2 = y(0) = c_1 + c_2 \qquad \implies \qquad c_1 = 2 - c_2$$

and

$$\begin{aligned} 3 = y'(0) = -2c_1 - 3c_2 = -2(2 - c_2) - 3c_2 = -4 - c_2 \qquad &\implies \qquad c_2 = -7 \\ &\implies \qquad c_1 = 9. \end{aligned}$$

Therefore the solution to the IVP is

$$\boxed{y(t) = 9e^{-2t} - 7e^{-3t}.}$$

Example 3.5. Solve
$$\begin{cases} 4y'' - 8y' + 3y = 0 \\ y(0) = 2 \\ y'(0) = \frac{1}{2}. \end{cases}$$

Since the characteristic equation
$$4r^2 - 8r + 3 = 0$$
has roots,
$$r_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{8 \pm \sqrt{64 - 48}}{8} = 1 \pm \frac{1}{2} = \frac{3}{2} \text{ or } \frac{1}{2},$$
it follows that the general solution to the ODE is
$$y(t) = c_1 e^{\frac{3t}{2}} + c_2 e^{\frac{t}{2}}.$$

Using the initial conditions, we calculate that
$$\begin{aligned} 2 &= y(0) = c_1 + c_2 \\ \frac{1}{2} &= y'(0) = \frac{3}{2}c_1 + \frac{1}{2}c_2 \end{aligned} \qquad \Longrightarrow \qquad c_1 = -\frac{1}{2} \text{ and } c_2 = \frac{5}{2}.$$
Therefore the solution to the IVP is
$$\boxed{y = -\tfrac{1}{2}e^{\frac{3t}{2}} + \tfrac{5}{2}e^{\frac{t}{2}}.}$$

Summary. To solve
$$ay'' + by' + cy = 0$$
we need to find two linearly independent solutions.

(i). If $r_1, r_2 \in \mathbb{R}$ and $r_1 \neq r_2$, then
$$y_1(t) = e^{r_1 t} \qquad \text{and} \qquad y_2(t) = e^{r_2 t};$$

(ii). If the roots are complex numbers, then ?????????????

(iii). If the roots are repeated, then ?????????????

3.2 Fundamental Sets of Solutions

$$y'' + p(t)y' + q(t)y = 0$$

Definition. Let $L = \frac{d^2}{dt^2} + p(t)\frac{d}{dt} + q(t)$.

So

$$L[y] = \frac{d^2 y}{dt^2} + p(t)\frac{dy}{dt} + q(t)y = y'' + p(t)y' + q(t)y$$

and we can write the ODE above as just $L[y] = 0$.

Theorem 3.2. *If y_1 and y_2 are both solutions of $L[y] = 0$, then $c_1y_1 + c_2y_2$ is also a solution to $L[y] = 0$ for all constants c_1, c_2.*

Proof. Since $L[y_1] = 0$ and $L[y_2] = 0$, we have that

$$\begin{aligned} L[y] &= L[c_1y_1 + c_2y_2] \\ &= \frac{d^2}{dt^2}(c_1y_1 + c_2y_2) + p(t)\frac{d}{dt}(c_1y_1 + c_2y_2) + q(t)(c_1y_1 + c_2y_2) \\ &= c_1(y_1'' + p(t)y_1' + q(t)y_1) + c_2(y_2'' + p(t)y_2' + q(t)y_2) \\ &= c_1L[y_1] + c_2L[y_2] \\ &= 0 + 0 = 0. \end{aligned}$$

□

Jósef Maria Hoëné-Wronski POL, 1776-1853

Definition. The ***Wronskian*** of $y_1(t)$ and $y_2(t)$ is

$$W = W(y_1, y_2)(t) = \begin{vmatrix} y_1(t) & y_2(t) \\ y_1'(t) & y_2'(t) \end{vmatrix}$$

Theorem 3.3. *Suppose that*

- *y_1 and y_2 both solve $L[y] = 0$; and*
- *$\exists\ t$ s.t. $W(t) \neq 0$.*

Then $\{c_1y_1 + c_2y_2 \ : \ c_1, c_2 \in \mathbb{R}\}$ contains every solution of $L[y] = 0$.

Definition. Since $y(t) = c_1y_1(t) + c_2y_2(t)$ contains every solution to $L[y] = 0$, $y(t)$ is called the ***general solution*** to $L[y] = 0$.

Definition. In this case, we say that y_1 and y_2 form a ***fundamental set of solutions*** to $L[y] = 0$.

Example 3.6. Show that $y_1(t) = t^{\frac{1}{2}}$ and $y_2(t) = t^{-1}$ form a fundamental set of solutions to

$$2t^2y'' + 3ty' - y = 0$$

for $t > 0$

We must show three things:

(i). that $y_1 = t^{\frac{1}{2}}$ is a solution to the ODE;

(ii). that $y_2 = t^{-1}$ is also a solution to the ODE; and

(iii). that y_1 and y_2 are linearly independent ($W \neq 0$ somewhere).

Since

$$\begin{aligned}
2t^2y_1'' + 3ty_1' - y_1 &= 2t^2\left(t^{\frac{1}{2}}\right)'' + 3t\left(t^{\frac{1}{2}}\right)' - t^{\frac{1}{2}} \\
&= 2t^2\left(-\frac{1}{4}t^{-\frac{3}{2}}\right) + 3t\left(\frac{1}{2}t^{-\frac{1}{2}}\right) - t^{\frac{1}{2}} \\
&= -\frac{1}{2}t^{\frac{1}{2}} + \frac{3}{2}t^{\frac{1}{2}} - t^{\frac{1}{2}} = 0
\end{aligned}$$

and

$$\begin{aligned}
2t^2y_2'' + 3ty_2' - y_2 &= 2t^2\left(t^{-1}\right)'' + 3t\left(t^{-1}\right)' - t^{-1} \\
&= 2t^2\left(2t^{-3}\right) + 3t\left(-t^{-2}\right) - t^{-1} \\
&= 4t^{-1} - 3t^{-1} - t^{-1} = 0,
\end{aligned}$$

y_1 and y_2 both solve the ODE. Moreover since

$$W(y_1, y_2)(t) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} t^{\frac{1}{2}} & t^{-1} \\ \frac{1}{2}t^{-\frac{1}{2}} & -t^{-2} \end{vmatrix} = -t^{-\frac{3}{2}} - \frac{1}{2}t^{-\frac{3}{2}} = -\frac{3}{2}t^{-\frac{3}{2}} \neq 0$$

for all $t > 0$, we have that y_1 and y_2 are linearly independent.

Therefore $y_1 = t^{\frac{1}{2}}$ and $y_2 = t^{-1}$ form a fundamental set of solutions to this ODE.

3.3 Complex Roots of the Characteristic Equation

Now consider

$$ay'' + by' + cy = 0 \tag{3.1}$$

where $b^2 - 4ac < 0$. The two roots of the characteristic equation are complex conjugates. We denote them by

$$r_1 = \lambda + i\mu \qquad \text{and} \qquad r_2 = \lambda - i\mu$$

where $\lambda, \mu \in \mathbb{R}$. The corresponding solutions are

$$y_1(t) = e^{r_1 t} = e^{(\lambda+i\mu)t} \qquad \text{and} \qquad y_2(t) = e^{r_2 t} = e^{(\lambda-i\mu)t}.$$

But what does e to the power of a complex number mean?

Definition.

$$e^{(\lambda+i\mu)t} = e^{\lambda t}\cos\mu t + ie^{\lambda t}\sin\mu t.$$

Remark 3.1. Please note that

$$\begin{aligned}
\frac{d}{dt}\left(e^{r_1 t}\right) &= \frac{d}{dt}\left(e^{\lambda t}\cos\mu t + ie^{\lambda t}\sin\mu t\right) \\
&= \lambda e^{\lambda t}\cos\mu t - \mu e^{\lambda t}\sin\mu t + i\lambda e^{\lambda t}\sin\mu t + i\mu e^{\lambda t}\cos\mu t \\
&= (\lambda + i\mu)\, e^{\lambda t}\cos\mu t + (i\lambda - \mu)\, e^{\lambda t}\sin\mu t \\
&= (\lambda + i\mu)\, e^{\lambda t}\cos\mu t + i\,(\lambda + i\mu)\, e^{\lambda t}\sin\mu t \\
&= (\lambda + i\mu)\left(e^{\lambda t}\cos\mu t + ie^{\lambda t}\sin\mu t\right) \qquad = r_1 e^{r_1 t}.
\end{aligned}$$

Real Valued Solutions

The solutions $y_1(t) = e^{(\lambda+i\mu)t}$ and $y_2(t) = e^{(\lambda-i\mu)t}$ are functions $y_1, y_2 : \mathbb{R} \to \mathbb{C}$. But we want solutions $\mathbb{R} \to \mathbb{R}$. Consider

$$\begin{aligned}
u(t) &= \frac{1}{2}\left(y_1(t) + y_2(t)\right) \\
&= \frac{1}{2}e^{\lambda t}\left(\cos\mu t + i\sin\mu t\right) + \frac{1}{2}e^{\lambda t}\left(\cos\mu t - i\sin\mu t\right) \\
&= e^{\lambda t}\cos\mu t
\end{aligned}$$

and

$$\begin{aligned}
v(t) &= \frac{1}{2i}\left(y_1(t) - y_2(t)\right) \\
&= \frac{1}{2i}e^{\lambda t}\left(\cos\mu t + i\sin\mu t\right) - \frac{1}{2i}e^{\lambda t}\left(\cos\mu t - i\sin\mu t\right) \\
&= \frac{1}{2i}2ie^{\lambda t}\sin\mu t = e^{\lambda t}\sin\mu t.
\end{aligned}$$

Note that $u, v : \mathbb{R} \to \mathbb{R}$ both solve (3.1). But are they linearly independent?

Since

$$\begin{aligned}
W\big(u,v\big)(t) &= \begin{vmatrix} u & v \\ u' & v' \end{vmatrix} \\
&= \begin{vmatrix} e^{\lambda t}\cos\mu t & e^{\lambda t}\sin\mu t \\ \lambda e^{\lambda t}\cos\mu t - \mu e^{\lambda t}\sin\mu t & \lambda e^{\lambda t}\sin\mu t + \mu e^{\lambda t}\cos\mu t \end{vmatrix} \\
&= e^{2\lambda t}\left(\lambda\cos\mu t\sin\mu t + \mu\cos^2\mu t - \lambda\cos\mu t\sin\mu t + \mu\sin^2\mu t\right) = \mu e^{2\lambda t} \neq 0
\end{aligned}$$

(because $\mu \neq 0$), the answer is YES. Therefore $u(t)$ and $v(t)$ form a fundamental set of solutions to (3.1). The general solution to (3.1) is therefore

$$y(t) = c_1 u(t) + c_2 v(t) = c_1 e^{\lambda t} \cos \mu t + c_2 e^{\lambda t} \sin \mu t.$$

Example 3.7. Solve $y'' + y' + y = 0$.

The characteristic equation

$$r^2 + r + 1 = 0$$

has roots

$$r = \frac{-1 \pm \sqrt{1-4}}{2} = \frac{-1 \pm \sqrt{(-1)(3)}}{2} = -\frac{1}{2} \pm i\frac{\sqrt{3}}{2}.$$

So $\lambda = -\frac{1}{2}$ and $\mu = \frac{\sqrt{3}}{2}$.

Therefore the general solution is $\sqrt{-1} = i$

$$\boxed{y(t) = c_1 e^{-\frac{t}{2}} \cos \frac{\sqrt{3}}{2}t + c_2 e^{-\frac{t}{2}} \sin \frac{\sqrt{3}}{2}t.}$$

Example 3.8. Solve $y'' + 9y = 0$.

Since $0 = r^2 + 9 = (r - 3i)(r + 3i)$ we have $r = \pm 3i$ (i.e. $\lambda = 0$ and $\mu = 3$). Therefore the general solution is

$$\boxed{y(t) = c_1 \cos 3t + c_2 \sin 3t.}$$

Example 3.9. Solve

$$\begin{cases} 16y'' - 8y' + 145y = 0 \\ y(0) = -2 \\ y'(0) = 1. \end{cases}$$

The characteristic equation $16r^2 - 8r + 145 = 0$ has roots

$$\begin{aligned} r &= \frac{8 \pm \sqrt{64 - 4(16)(145)}}{32} = \frac{8 \pm \sqrt{(64)(1-145)}}{32} \\ &= \frac{8 \pm \sqrt{(-1)(64)(144)}}{32} = \frac{1}{4} \pm 3i. \end{aligned}$$

Therefore the general solution to the ODE is

$$y(t) = c_1 e^{\frac{t}{4}} \cos 3t + c_2 e^{\frac{t}{4}} \sin 3t.$$

Finally we calculate that

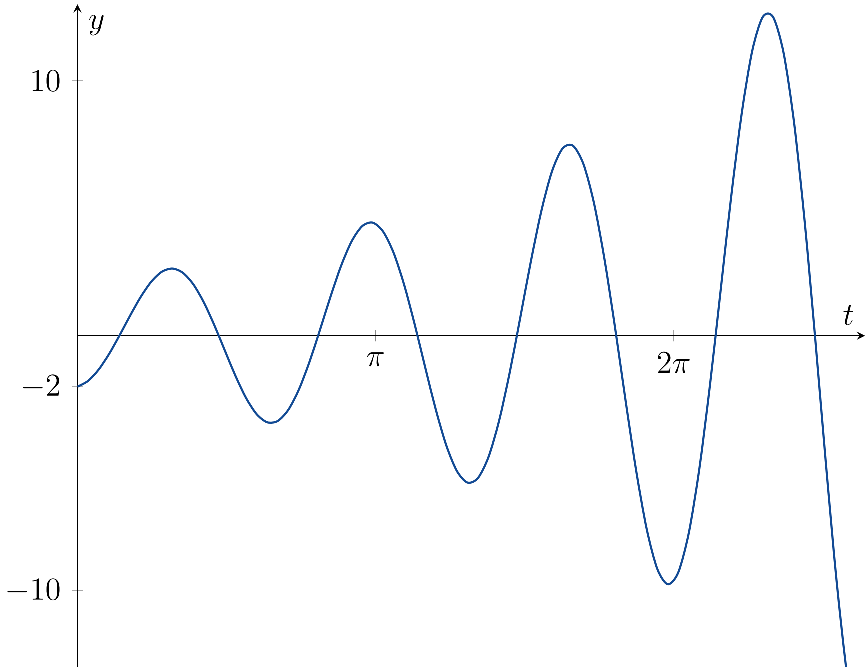
$$y'(t) = \frac{1}{4} c_1 e^{\frac{t}{4}} \cos 3t - 3c_1 e^{\frac{t}{4}} \sin 3t + \frac{1}{4} c_2 e^{\frac{t}{4}} \sin 3t + 3c_2 e^{\frac{t}{4}} \cos 3t$$

and

$$
\begin{aligned}
-2 = y(0) = c_1 + 0 \qquad &\Longrightarrow \qquad c_1 = -2 \\
1 = y'(0) = \frac{1}{4}c_1 + 3c_2 = -\frac{1}{2} + 3c_2 \qquad &\Longrightarrow \qquad c_2 = \frac{1}{2}.
\end{aligned}
$$

Therefore the solution to the IVP is

$$\boxed{y = -2e^{\frac{t}{4}} \cos 3t + \tfrac{1}{2}e^{\frac{t}{4}} \sin 3t.}$$



Summary. To solve

$$ay'' + by' + cy = 0$$

we need to find two linearly independent solutions.

(i). If $r_1, r_2 \in \mathbb{R}$ and $r_1 \neq r_2$, then

$$y_1(t) = e^{r_1 t} \qquad \text{and} \qquad y_2(t) = e^{r_2 t};$$

(ii). If $r_{1,2} = \lambda \pm i\mu$ $(\lambda, \mu \in \mathbb{R})$, then

$$y_1(t) = e^{\lambda t} \cos \mu t \qquad \text{and} \qquad y_2(t) = e^{\lambda t} \sin \mu t;$$

(iii). If the roots are repeated, then ?????????????

3.4 Repeated Roots of the Characteristic Equation

Now consider

$$ay'' + by' + cy = 0 \tag{3.1}$$

where $b^2 - 4ac = 0$. Then the only root of

$$ar^2 + br + c = 0$$

is

$$r = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-b \pm \sqrt{0}}{2a} = -\frac{b}{2a}.$$

We know that $y_1(t) = e^{-\frac{bt}{2a}}$ is a solution of (3.1), but how do we find a linearly independent second solution?

Example 3.10. Solve $y'' + 4y' + 4y = 0$.

The characteristic equation

$$0 = r^2 + 4r + 4 = (r+2)^2$$

has repeated root $r_1 = r_2 = -2$. So one solution is $y_1(t) = e^{-2t}$. To find the general solution, we need to find a second solution.

The idea is:

- We know that $y_1(t)$ is a solution;
- So $cy_1(t)$ is a solution for any $c \in \mathbb{R}$;
- Maybe $v(t)y_1(t)$ is a solution for some non-constant function $v(t)$.

We consider $y_2(t) = v(t)y_1(t)$ for some function $v(t)$ which we don't know yet. Then we calculate that

$$\begin{aligned} y_2 &= ve^{-2t} \\ y_2' &= v'e^{-2t} - 2ve^{-2t} \\ y_2'' &= v''e^{-2t} - 4v'e^{-2t} + 4ve^{-2t} \end{aligned}$$

and that

$$\begin{aligned} 0 &= y_2'' + 4y_2' + 4y_2 \\ &= \left(v''e^{-2t} - 4v'e^{-2t} + 4ve^{-2t}\right) + 4\left(v'e^{-2t} - 2ve^{-2t}\right) + 4\left(ve^{-2t}\right) \\ &= e^{-2t}\left[v'' - 4v' + 4v + 4v' - 8v + 4v\right] \\ &= v''e^{-2t}. \end{aligned}$$

Since $e^{-2t} \neq 0$, we must have $v'' = 0$. We can choose ***any*** non-constant function $v(t)$ which satisfies $v'' = 0$. I like easy functions, so I choose $v(t) = t$. Therefore

$$y_2(t) = te^{-2t}.$$

But are $y_1(t)$ and $y_2(t)$ linearly independent? Since

$$W(y_1, y_2)(t) = \begin{vmatrix} e^{-2t} & te^{-2t} \\ -2e^{-2t} & (1-2t)e^{-2t} \end{vmatrix} = e^{-4t} \neq 0,$$

the answer is YES.

Therefore $y_1(t) = e^{-2t}$ and $y_2(t) = te^{-2t}$ form a fundamental set of solutions and the general solution is

$$y(t) = c_1 e^{-2t} + c_2 t e^{-2t}.$$

For the general equation $ay'' + by' + cy = 0$, we can use the same method: We have $y_1(t) = e^{rt} = e^{-\frac{bt}{2a}}$ and we guess that $y_2(t) = v(t)e^{-\frac{bt}{2a}}$ for some function $v(t)$. Then we calculate (you fill in the details)

$$0 = ay_2'' + by_2' + cy_2 = \ldots = ae^{-\frac{bt}{2a}} v''.$$

So again we want $v'' = 0$ and we can choose $v(t) = t$. Thus $y_2(t) = te^{rt} = te^{-\frac{bt}{2a}}$.

I leave it for you to calculate that $W(e^{rt}, te^{rt}) \neq 0$. Thus e^{rt} and te^{rt} form a fundamental set of solutions to (3.1).

Example 3.11. Solve

$$\begin{cases} y'' - y' + \frac{1}{4}y = 0 \\ y(0) = 2 \\ y'(0) = \frac{1}{3}. \end{cases}$$

The characteristic equation

$$0 = r^2 - r + \frac{1}{4} = \left(r - \frac{1}{2}\right)^2$$

has repeated root $r = \frac{1}{2}$. So we know that the general solution to the ODE is

$$y(t) = c_1 e^{\frac{t}{2}} + c_2 t e^{\frac{t}{2}}.$$

Next we need to look at the initial conditions: Since $y'(t) = \frac{1}{2}c_1 e^{\frac{t}{2}} + c_2 e^{\frac{t}{2}} + \frac{1}{2}c_2 t e^{\frac{t}{2}}$, we have that

$$\begin{aligned} 2 = y(0) &= c_1 + 0 \qquad &\Longrightarrow \qquad & c_1 = 2 \\ \frac{1}{3} = y'(0) &= \frac{1}{2}c_1 + c_2 + 0 \qquad &\Longrightarrow \qquad & c_2 = -\frac{2}{3}. \end{aligned}$$

Therefore the solution to the IVP is

$$y = 2e^{\frac{t}{2}} - \frac{2}{3}te^{\frac{t}{2}}.$$

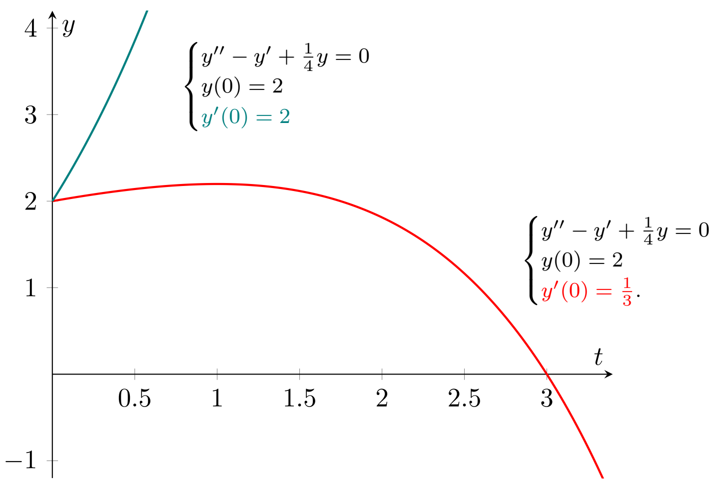
Example 3.12. Now solve

$$\begin{cases} y'' - y' + \frac{1}{4}y = 0 \\ y(0) = 2 \\ y'(0) = 2 \end{cases}$$

You can check that the solution is

$$y = 2e^{\frac{t}{2}} + te^{\frac{t}{2}}.$$

The graph of this solution, and the solution to the previous example, are shown below.



Note that even though these two functions share the same $y(0)$ value, and that their $y'(0)$ value does not differ by much, their behaviour as $t \to \infty$ is very different.

Summary. To solve

$$ay'' + by' + cy = 0$$

we need to find two linearly independent solutions.

(i). If $r_1, r_2 \in \mathbb{R}$ and $r_1 \neq r_2$, then

$$y_1(t) = e^{r_1 t} \qquad \text{and} \qquad y_2(t) = e^{r_2 t};$$

(ii). If $r_{1,2} = \lambda \pm i\mu$ $(\lambda, \mu \in \mathbb{R})$, then

$$y_1(t) = e^{\lambda t}\cos \mu t \qquad \text{and} \qquad y_2(t) = e^{\lambda t}\sin \mu t;$$

(iii). If $r_1, r_2 \in \mathbb{R}$ but $r_1 = r_2$, then

$$y_1(t) = e^{r_1 t} \qquad \text{and} \qquad y_2(t) = te^{r_1 t}.$$

3.5 Reduction of Order

Consider
$$y'' + p(t)y' + q(t)y = 0. \tag{3.6}$$
Suppose that we know that $y_1(t)$ is a solution to (3.6) and suppose that we want to find a second, linearly independent solution.

The main idea in this section is that we guess that
$$y_2(t) = v(t)y_1(t)$$
for some non-constant function $v(t)$. If we can find $v(t)$, then we have our $y_2(t)$. Then we calculate that
$$\begin{aligned} y_2 &= vy_1 \\ y_2' &= v'y_1 + vy_1' \\ y_2'' &= v''y_1 + 2v'y_1' + vy_1'' \end{aligned}$$
and
$$\begin{aligned} 0 &= y_2'' + py_2' + qy_2 \\ &= (v''y_1 + 2v'y_1' + vy_1'') + p(t)(v'y_1 + vy_1') + q(t)(vy_1) \\ &= v''y_1 + v'(2y_1' + py_1) + v\underbrace{(y_1'' + py_1' + qy_1)}_{=0} \\ &= v''y_1 + v'(2y_1' + py_1). \end{aligned}$$

Remark 3.2. Note that since y_1 solves the ODE, we must always get "$0v$" here. We can have v' and v'' terms, but if you do a reduction of order calculation and still have v terms, then you have made a mistake.

Remark 3.3.
$$v''y_1 + v'(2y_1' + py_1) = 0 \tag{3.3}$$
is actually a first order ODE for v'. To see this, we can use the substitution $u = v'$ to convert (3.3) into
$$u'y_1 + u(2y_1' + py_1) = 0. \tag{3.4}$$
If we can find $u(t)$, then we can find $v(t) = \int u(t)\,dt$ and $y_2(t) = v(t)y_1(t)$.

Remark 3.4. Instead of having to solve a second order ODE to find y_2, we only need to solve a first order ODE to find $u(t)$. Hence the name "Reduction of Order".

Remark 3.5. The method is

(i). Guess $y_2 = vy_1$.

(ii). Put this into your ODE and find an equation for v;

(iii). Set $u = v'$;

(iv). Find u;

(v). Integrate to find v;

(vi). Then $y_2(t) = v(t)y_1(t)$.

Example 3.13. Given that $y_1(t) = \frac{1}{t}$ is a solution of

$$2t^2y'' + 3ty' - y = 0, \qquad t > 0$$

find a linearly independent second solution.

Let $y_2(t) = v(t)y_1(t)$. Then we have

$$\begin{aligned} y_2 &= vt^{-1} \\ y_2' &= v't^{-1} - vt^{-2} \\ y_2'' &= v''t^{-1} - 2v't^{-2} + 2vt^{-3} \end{aligned}$$

and

$$\begin{aligned} 0 &= 2t^2y_2'' + 3ty_2' - y_2 \\ &= 2t^2\left(v''t^{-1} - 2v't^{-2} + 2vt^{-3}\right) + 3t\left(v't^{-1} - vt^{-2}\right) - vt^{-1} \\ &= 2tv'' + (-4+3)v' + (4t^{-1} - 3t^{-1} - t^{-1})v \\ &= 2tv'' - v'. \end{aligned}$$

Now let $u = v'$. We need to solve

$$2t\frac{du}{dt} - u = 0.$$

This equation is both linear and separable, so we know 2 ways to solve it.

$$\begin{aligned} 2t\frac{du}{dt} &= u \\ \frac{du}{u} &= \frac{1}{2}\frac{dt}{t} \\ \int \frac{du}{u} &= \int \frac{1}{2}\frac{dt}{t} \\ \ln|u| &= \frac{1}{2}\ln|t| + C \\ e^{\ln|u|} &= e^{\ln|t|^{\frac{1}{2}}}e^C \\ |u| &= |t|^{\frac{1}{2}}e^C \\ u &= \pm e^C t^{\frac{1}{2}} = ct^{\frac{1}{2}}. \end{aligned}$$

Then we have

$$v(t) = \int u(t)\,dt = \int ct^{\frac{1}{2}}\,dt = \frac{2}{3}ct^{\frac{3}{2}} + k$$

and

$$y_2(t) = v(t)t^{-1} = \frac{2}{3}ct^{\frac{1}{2}} + kt^{-1}.$$

Remember that we are trying to find a solution that is linearly independent from $y_1(t) = t^{-1}$. The second term in $y_2(t) = \frac{2}{3}ct^{\frac{1}{2}} + kt^{-1}$ is just a multiple of $y_1(t)$ – we don't need this. So it is ok to choose $k = 0$. Hence

$$y_2(t) = \frac{2}{3}ct^{\frac{1}{2}}$$

Finally, since I like simple functions I choose $c = \frac{3}{2}$ to get

$$\boxed{y_2(t) = t^{\frac{1}{2}}.}$$

I leave it to you to check that $W(t^{-1}, t^{\frac{1}{2}})$ is not always zero.

Example 3.14. Given that $y_1(t) = t$ solves

$$t^2y'' + 2ty' - 2y = 0, \quad t > 0,$$

find a second linearly independent solution $y_2(t)$.

We start with $y_2(t) = v(t)y_1(t) = v(t)t$. Then $y_2' = v't+v$ and $y_2'' = v''t+2v'$. Substituting into the ODE, we calculate that

$$\begin{aligned}
0 &= t^2y_2'' + 2ty_2' - 2y_2 \\
&= t^2(v''t + 2v') + 2t(v't + v) - 2vt \\
&= t^3v'' + v'(2t^2 + 2t^2) + v(2t - 2t) \\
&= t^3v'' + 4t^2v' \\
&= t^2(tv'' + 4v').
\end{aligned}$$

Letting $u = v'$, we obtain the first order ODE

$$t\frac{du}{dt} + 4u = 0.$$

We calculate that

$$\begin{aligned}
t\frac{du}{dt} &= -4u \\
\frac{du}{u} &= -4\frac{dt}{t} \\
\int \frac{du}{u} &= -4\int \frac{dt}{t} \\
\ln|u| &= -4\ln|t| + C \\
u &= \pm e^C t^{-4} = ct^{-4}
\end{aligned}$$

and

$$\begin{aligned}
v &= \int u\ dt = \int ct^{-4}\ dt \\
&= -\frac{1}{3}ct^{-3} + k.
\end{aligned}$$

Thus

$$y_2(t) = v(t)t = -\frac{1}{3}ct^{-2} + kt.$$

Choosing $c = -3$ and $k = 0$, we obtain the solution

$$y_2(t) = t^{-2}.$$

Does $y_2(t) = t^{-2}$ really solve $t^2y'' + 2ty' - 2y = 0$?

Since $y_2' = -2t^{-3}$ and $y_2'' = 6t^{-4}$, we have that

$$\begin{aligned} t^2y_2'' + 2ty_2' - 2y_2 &= t^2(6t^{-4}) + 2t(-2t^{-3}) - 2t^{-2} \\ &= 6t^{-2} - 4t^{-2} - 2t^{-2} \\ &= 0. \end{aligned}$$

The answer is YES!!

Are $y_1(t) = t$ and $y_2(t) = t^{-2}$ linearly independent?

We have that

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} t & t^{-2} \\ 1 & -2t^{-3} \end{vmatrix} = -2t^{-2} - t^{-2} = -3t^{-2} \neq 0$$

since $t > 0$. Therefore y_1 and y_2 are linearly independent.

3.6 Nonhomogeneous Equations

Consider

$$y'' + p(t)y' + q(t)y = g(t). \tag{3.5}$$

The equation

$$y'' + p(t)y' + q(t)y = 0 \tag{3.6}$$

is called the ***homogeneous equation corresponding to*** (3.5).

Theorem 3.4. *The general solution to* (3.5) *can be written in the form*

$$y(t) = c_1 y_1(t) + c_2 y_2(t) + Y(t)$$

where

- y_1 *and* y_2 *form a fundamental set of solutions to the homogeneous equation corresponding to* (3.5)*;*
- c_1 *and* c_2 *are constants; and*
- Y *is a particular solution to* (3.5)*.*

To solve $L[y] = g$

(i). Find the general solution to $L[y] = 0$;

(ii). Find a particular solution to $L[y] = g$;

(iii). $1 + 2$

We will study 2 methods to do step 2.

3.7 The Method of Undetermined Coefficients

$$y'' + p(t)y' + q(t)y = g(t) \tag{3.5}$$

The idea is:

(i). Look at $g(t)$

(ii). Make a guess with constants

(iii). Try to find the constants

Example 3.15. Find a particular solution to $y'' - 3y' - 4y = 3e^{2t}$.

Here we have $g(t) = 3e^{2t}$. We look at this g and we make a guess: g includes e^{2t} so we guess that $Y(t)$ also includes e^{2t}. So we guess that $Y(t) = Ae^{2t}$ for some constant A.

We must try to find A. We calculate that

$$\begin{aligned} Y(t) &= Ae^{2t} \\ Y'(t) &= 2Ae^{2t} \\ Y''(t) &= 4Ae^{2t} \end{aligned}$$

and

$$\begin{aligned} 3e^{2t} &= Y'' - 3Y' - 4Y = 4Ae^{2t} - 3\left(2Ae^{2t}\right) - 4\left(Ae^{2t}\right) \\ &= -6Ae^{2t}. \end{aligned}$$

We must have $A = -\frac{1}{2}$. Therefore a particular solution is

$$\boxed{Y(t) = -\frac{1}{2}e^{2t}.}$$

Example 3.16. Find a particular solution to $y'' - 3y' - 4y = 4t^2 - 1$.

Since $g(t) = 4t^2 - 1$ is a 2nd degree polynomial, we guess that Y is also a second degree polynomial. So we try the ansatz

$$Y(t) = At^2 + Bt + C.$$

I will leave this example for you to finish.

Example 3.17. Find a particular solution to $y'' - 3y' - 4y = 2\sin t$.

First guess: $Y(t) = A\sin t$. Then $Y' = A\cos t$ and $Y'' = -A\sin t$. Hence

$$\begin{aligned} 2\sin t &= Y'' - 3Y' - 4Y \\ &= (-A\sin t) - 3(A\cos t) - 4(A\sin t) = -5A\sin t - 3A\cos t. \end{aligned}$$

We can see that we must have

$$\begin{cases} -5A = 2 \\ -3A = 0. \end{cases}$$

This linear system is inconsistent: It not possible to find a constant A which satisfies both of these equations. Our first guess failed.

Second guess: $Y(t) = A\sin t + B\cos t$. Then we calculate that

$$\begin{aligned} Y' &= A\cos t - B\sin t \\ Y'' &= -A\sin t - B\cos t \end{aligned}$$

and

$$\begin{aligned} 2\sin t &= Y'' - 3Y' - 4Y \\ &= (-A\sin t - B\cos t) - 3(A\cos t - B\sin t) - 4(A\sin t + B\cos t) \\ &= (-5A + 3B)\sin t + (-3A - 5B)\cos t. \end{aligned}$$

So we need A and B to satisfy

$$\begin{cases} -5A + 3B = 2 \\ -3A - 5B = 0. \end{cases}$$

Please check that the solution to this linear system is $A = -\frac{5}{17}$ and $B = \frac{3}{17}$. Therefore a particular solution is

$$\boxed{Y(t) = -\frac{5}{17}\sin t + \frac{3}{17}\cos t.}$$

Remark 3.6. sin and cos are friends! They always go together. If you see either sin or cos in $g(t)$, then your ansatz needs to contain ***both*** sin and cos.

Likewise sinh and cosh always go together.

Example 3.18. Find a particular solution to $y'' - 3y' - 4y = -8e^t\cos 2t$.

We will try the ansatz

$$Y(t) = Ae^t\cos 2t + Be^t\sin 2t.$$

Then

$$\begin{aligned} Y'(t) &= Ae^t\cos 2t - 2Ae^t\sin 2t + Be^t\sin 2t + 2Be^t\cos 2t \\ &= (A + 2B)e^t\cos 2t + (B - 2A)e^t\sin 2t, \\ Y''(t) &= (A + 2B)e^t\cos 2t - 2(A + 2B)e^t\sin 2t + (B - 2A)e^t\sin 2t \\ &\quad + 2(B - 2A)e^t\cos 2t \\ &= (-3A + 4B)e^t\cos 2t + (-4A - 3B)e^t\sin 2t \end{aligned}$$

and

$$\begin{aligned} -8e^t\cos 2t &= Y'' - 3Y' - 4Y \\ &= (-3A + 4B)e^t\cos 2t + (-4A - 3B)e^t\sin 2t \\ &\quad + (-3A - 6B)e^t\cos 2t + (-3B + 6A)e^t\sin 2t \\ &\quad + (-4A)e^t\cos 2t + (-4B)e^t\sin 2t \\ &= (-10A - 2B)e^t\cos 2t + (2A - 10B)e^t\sin 2t. \end{aligned}$$

Thus we must solve

$$\begin{cases} 10A + 2B = 8 \\ 2A - 10B = 0. \end{cases}$$

Please check that the solution to this linear system is $A = \frac{10}{13}$ and $B = \frac{2}{13}$. Therefore a particular solution is

$$\boxed{Y(t) = \frac{10}{13}e^t \cos 2t + \frac{2}{13}e^t \sin 2t.}$$

Theorem 3.5.

$$\left.\begin{array}{l} Y_1 \textit{ solves } ay'' + by' + cy = g_1(t) \\ Y_2 \textit{ solves } ay'' + by' + cy = g_2(t) \end{array}\right\} \implies \begin{array}{c} Y_1 + Y_2 \textit{ solves} \\ ay'' + by' + cy = g_1(t) + g_2(t) \end{array}$$

Example 3.19. Find a particular solution to

$$y'' - 3y' - 4y = 3e^{2t} + 2\sin t - 8e^t \cos 2t. \tag{3.7}$$

We can split this problem up into three easier problems:

$$\begin{aligned} y'' - 3y' - 4y &= 3e^{2t} \\ y'' - 3y' - 4y &= 2\sin t \\ y'' - 3y' - 4y &= -8e^t \cos 2t \end{aligned}$$

We know particular solutions to these three ODEs. Therefore

$$\boxed{Y(t) = -\frac{1}{2}e^{2t} - \frac{5}{17}\sin t + \frac{3}{17}\cos t + \frac{10}{13}e^t \cos 2t + \frac{2}{13}e^t \sin 2t.}$$

is a particular solution to (3.7).

Remark 3.7. To find a particular solution to $ay'' + by' + cy = g(t)$, we have been looking at $g(t)$ and choosing a similar function for $Y(t)$.
This method doesn't always work: There is a difficulty that can occur as we shall see in the next example.

Example 3.20. Find a particular solution to $y'' + 4y = 3\cos 2t$.

First guess: $Y(t) = A\cos 2t + B\sin 2t$.

Then we have that

$$\begin{aligned} Y' &= -2A\sin 2t + 2B\cos 2t \\ Y'' &= -4A\cos 2t - 4B\sin 2t \end{aligned}$$

and

$$\begin{aligned} 3\cos 2t &= Y'' + 4Y \\ &= (-4A\cos 2t - 4B\sin 2t) + 4(A\cos 2t + B\sin 2t) = 0. \end{aligned}$$

This is a FAILURE!!! It not possible to choose A and B such that

$$3\cos 2t = 0$$

for all t.

Why did this happen? Why didn't our usual method work? To understand why, let us solve the homogeneous equation $y'' + 4y = 0$. The characteristic equation is

$$0 = r^2 + 4 = (r + 2i)(r - 2i).$$

So $r = \pm 2i$. It follows that the general solution to the homogeneous equation is

$$y(t) = c_1 \cos 2t + c_2 \sin 2t.$$

Since $\cos 2t$ and $\sin 2t$ appear in the general solution to the homogeneous equation, ***we can not use*** $\cos 2t$ ***and*** $\sin 2t$ ***in a particular solution***.

HINT: If in doubt, multiply by t.

We need two functions which, when differentiated, give us $\cos 2t$ and $\sin 2t$ terms.

We will try $t\cos 2t$ and $t \sin 2t$.

Note that

$$\frac{d}{dt} t\cos 2t = \cos 2t - 2t \sin 2t$$

and

$$\frac{d}{dt} t\sin 2t = \sin 2t + 2t \cos 2t.$$

Second guess: $Y(t) = At\cos 2t + Bt \sin 2t$.

We have that

$$\begin{aligned} Y' &= A\cos 2t - 2At\sin 2t + B\sin 2t + 2Bt\cos 2t \\ &= (A + 2Bt)\cos 2t + (B - 2At)\sin 2t, \\ Y'' &= 2B\cos 2t - 2(A + 2Bt)\sin 2t - 2A\sin 2t + 2(B - 2At)\cos 2t \\ &= (4B - 4At)\cos 2t + (-4A - 4Bt)\sin 2t \end{aligned}$$

and

$$\begin{aligned} 3\cos 2t &= Y'' + 4Y \\ &= (4B - 4At)\cos 2t + (-4A - 4Bt)\sin 2t \\ &\quad + 4At\cos 2t + 4Bt\sin 2t \\ &= 4B\cos 2t - 4A\sin 2t. \end{aligned}$$

Thus

$$\begin{cases} -4A = 0 \\ 4B = 3 \end{cases}$$

which has solution $A = 0$ and $B = \frac{3}{4}$. Therefore a particular solution is

$$\boxed{Y(t) = \frac{3}{4} t \sin 2t.}$$

3.8 Solving Initial Value Problems

Remark 3.8.

$$\begin{cases} ay'' + by' + cy = g(t) \\ y(t_0) = y_0 \\ y'(t_0) = y_1. \end{cases}$$

To solve this IVP, the method is:

(i). Find the general solution to $ay'' + by' + cy = 0$;

(ii). Find a particular solution to $ay'' + by' + cy = g(t)$:

(a) if $g(t)$ **does not** solve the homogeneous equation, then your ansatz should look like $g(t)$;

(b) if $g(t)$ **does** solve the homogeneous equation, then "multiply by t" (repeat as necessary);

(iii). 1+2;

(iv). Find c_1 and c_2.

Remark 3.9. You must do step (iv) last. If you try to find c_1 and c_2 before doing the other steps, you may get the wrong answer.

Example 3.21. Solve

$$\begin{cases} y'' - y = 2e^t \\ y(0) = 1 \\ y'(0) = 2. \end{cases}$$

Correct Solution:

(i). First we consider $y'' - y = 0$. The characteristic equation $r^2 - 1 = 0$ has roots $r_1 = 1$ and $r_2 = -1$. Hence the general solution is $y(t) = c_1e^t + c_2e^{-t}$.

(ii). Next we need to find a particular solution. Since Ae^t solves the homogeneous equation, we must "multiply by t". We try the ansatz $Y(t) = Ate^t$ and we calculate that

$$\begin{aligned} Y' &= Ae^t + Ate^t, \\ Y'' &= 2Ae^t + Ate^t \end{aligned}$$

and

$$\begin{aligned} 2e^t &= Y'' - Y \\ &= 2Ae^t + Ate^t - Ate^t \\ &= 2Ae^t. \end{aligned}$$

We must have $A = 1$. Therefore $Y(t) = te^t$ is a particular solution.

(iii). Thus

$$y(t) = c_1e^t + c_2e^{-t} + te^t$$

is the general solution to the ODE.

(iv). Finally we must satisfy the initial conditions. Since

$$y'(t) = c_1e^t - c_2e^{-t} + e^t + te^t$$

we have

$$\begin{aligned} 1 &= y(0) = c_1 + c_2 + 0 \\ 2 &= y'(0) = c_1 - c_2 + 1 + 0 \end{aligned}$$

which implies that $c_1 = 1$ and $c_2 = 0$. Therefore the solution to the IVP is

$$\boxed{y(t) = e^t + te^t.}$$

Incorrect Solution:

(i). First we consider $y'' - y = 0$. The characteristic equation $r^2 - 1 = 0$ has roots $r_1 = 1$ and $r_2 = -1$. Hence the general solution is $y(t) = c_1e^t + c_2e^{-t}$.

(iv). Next we find c_1 and c_2. Since

$$y'(t) = c_1e^t - c_2e^{-t}$$

we have

$$\begin{aligned} 1 &= y(0) = c_1 + c_2 \\ 2 &= y'(0) = c_1 - c_2 \end{aligned}$$

which implies that $c_1 = \frac{3}{2}$ and $c_2 = -\frac{1}{2}$. Thus

$$y(t) = \frac{3}{2}e^t - \frac{1}{2}e^{-t}.$$

(ii). Next we need to find a particular solution. Since Ae^t solves the homogeneous equation, we must "multiply by t". We try the ansatz $Y(t) = Ate^t$ and we calculate that

$$\begin{aligned} Y' &= Ae^t + Ate^t, \\ Y'' &= 2Ae^t + Ate^t \end{aligned}$$

and

$$\begin{aligned} 2e^t &= Y'' - Y \\ &= 2Ae^t + Ate^t - Ate^t \\ &= 2Ae^t. \end{aligned}$$

We must have $A = 1$. Therefore $Y(t) = te^t$ is a particular solution.

(iii). Finally we add our solutions together to get

$$\boxed{y(t) = \frac{3}{2}e^t - \frac{1}{2}e^{-t} + te^t}$$

which **IS WRONG!!!** This function does not satisfy the initial conditions.

Example 3.22. Solve

$$\begin{cases} -y'' + 6y' - 16y = 1 + 6e^{3t}\sin(2t) \\ y(0) = \frac{15}{16} \\ y'(0) = -1. \end{cases} \tag{3.8}$$

(This is an exam question from 2013: Students had 30 minutes to solve this.)

To solve

$$-y'' + 6y' - 16y = 1 + 6e^{3t}\sin(2t)$$

we will consider 3 ODEs:

- $-y'' + 6y' - 16y = 0$
- $-y'' + 6y' - 16y = 1$
- $-y'' + 6y' - 16y = 6e^{3t}\sin(2t)$.

First consider the homogeneous equation $-y''+6y'-16y = 0$. The characteristic equation is $-r^2 + 6r - 16 = 0$ which has roots $r = 3 \pm i\sqrt{7}$. Therefore the general solution to $-y'' + 6y' - 16y = 0$ is

$$y(t) = c_1e^{3t}\sin(\sqrt{7}t) + c_2e^{3t}\cos(\sqrt{7}t).$$

Next consider $-y'' + 6y' - 16y = 1$. Trying the ansatz $Y(t) = C$, we see that

$$1 = -Y'' + 6Y' - 16Y = -16C.$$

We must choose $C = -\frac{1}{16}$. Hence $Y(t) = -\frac{1}{16}$.

Now consider $-y'' + 6y' - 16y = 6e^{3t}\sin(2t)$. We try the ansatz $Y(t) = Ae^{3t}\cos 2t + Be^{3t}\sin 2t$ and find that

$$\begin{aligned} 6e^{3t}\sin 2t &= -Y'' + 6Y' - 16Y \\ &= -e^{3t}\Big((5A + 12B)\cos 2t + (5B - 12A)\sin 2t\Big) \\ &\quad + 6e^{3t}\big((3A + 2B)\cos 2t + (3B - 2A)\sin 2t\big) \\ &\quad - 16e^{3t}\big(A\cos 2t + B\sin 2t\big) \\ &= e^{3t}\cos 2t\big(-5A - 12B + 16A + 12B - 16A\big) \\ &\quad + e^{3t}\sin 2t\big(-5B + 12A + 18B - 12A - 16B\big) \\ &= e^{3t}\cos 2t\big(-5A\big) + e^{3t}\sin 2t\big(-3B\big). \end{aligned}$$

Thus, we need $A = 0$ and $B = -2$. Hence

$$Y(t) = -2e^{3t}\sin 2t.$$

Next we add these 3 solutions together. Therefore, the general solution to the ODE is

$$y(t) = c_1e^{3t}\sin(\sqrt{7}t) + c_2e^{3t}\cos(\sqrt{7}t) - \frac{1}{16} - 2e^{3t}\sin(2t).$$

The final step is to choose c_1 and c_2 to satisfy the initial conditions.

$$\frac{15}{16} = y(0) = 0 + c_2 - \frac{1}{16} - 0 \qquad \Longrightarrow \qquad c_2 = 1.$$

$$\begin{aligned} -1 &= y'(0) \\ &= 3c_1e^{3t}\sin(\sqrt{7}t) + \sqrt{7}c_1e^{3t}\cos(\sqrt{7}t) + 3e^{3t}\cos(\sqrt{7}t) \\ &\quad - \sqrt{7}e^{3t}\sin(\sqrt{7}t) - 6e^{3t}\sin(2t) - 4e^{3t}\cos(2t)\big|_{t=0} \\ &= 0 + \sqrt{7}c_1 + 3 - 0 - 0 - 4 \qquad \Longrightarrow \qquad c_1 = 0. \end{aligned}$$

Therefore, the solution to the IVP is

$$\boxed{y(t) = e^{3t}\cos(\sqrt{7}t) - \frac{1}{16} - 2e^{3t}\sin(2t).}$$

Remark 3.10.

$$ay'' + by' + cy = g(t)$$

The method of undetermined coefficients works well if $g(t)$ is a nice function: e^{kt}, $\sin kt$, $t^3 + 2t^2 + 3t + 4$, $e^{at}\cosh kt$, ...

However if $g(t)$ is a less nice function, then we may need a different method to find a particular solution.

3.9 The Method of Variation of Parameters

Example 3.23. Find a particular solution to

$$y'' + 4y = 3\operatorname{cosec} t. \tag{3.9}$$

The homogeneous equation $y'' + 4y = 0$ has general solution $y = c_1 \cos 2t + c_2 \sin 2t$. The idea is:

(i). Replace the constants c_1 and c_2 by functions $u_1(t)$ and $u_2(t)$:

$$Y(t) = u_1(t)\cos 2t + u_2(t)\sin 2t.$$

(ii). Try to find u_1 and u_2 so that Y solves (3.9). There will be lots of u_1 and u_2 that we can use, so we will be free to add an extra condition.

So suppose that

$$Y = u_1 \cos 2t + u_2 \sin 2t.$$

Then

$$Y' = u_1' \cos 2t - 2u_1 \sin 2t + u_2' \sin 2t + 2u_2 \cos 2t$$

At this point, it is getting complicated so we will use our chance to add a condition: Suppose that

$$u_1' \cos 2t + u_2' \sin 2t = 0 \tag{3.10}$$

So

$$Y' = -2u_1 \sin 2t + 2u_2 \cos 2t$$

and

$$Y'' = -2u_1' \sin 2t - 4u_1 \cos 2t + 2u_2' \cos 2t - 4u_2 \sin 2t.$$

Then

$$\begin{aligned}
3\operatorname{cosec} t &= Y'' + 4Y \\
&= (-2u_1' \sin 2t - 4u_1 \cos 2t + 2u_2' \cos 2t - 4u_2 \sin 2t) \\
&\quad + 4(u_1 \cos 2t + u_2 \sin 2t) \\
&= -2u_1' \sin 2t + 2u_2' \cos 2t
\end{aligned}$$

We want to find $u_1(t)$ and $u_2(t)$ which satisfy

$$\begin{cases} 3\operatorname{cosec} t = -2u_1' \sin 2t + 2u_2' \cos 2t \\ u_1' \cos 2t + u_2' \sin 2t = 0 \end{cases}$$

From the latter condition, we have $u_2' = -u_1' \frac{\cos 2t}{\sin 2t}$. Putting this into the first condition, we calculate that

$$\begin{aligned}
3\operatorname{cosec} t &= -2u_1' \sin 2t + 2\left(-u_1' \frac{\cos 2t}{\sin 2t}\right)\cos 2t \\
3\operatorname{cosec} t \sin 2t &= -2u_1' \sin^2 2t - 2u_1' \cos^2 2t = -2u_1' \\
u_1' &= \frac{-3\operatorname{cosec} t \sin 2t}{2} = \frac{-3\sin 2t}{2\sin t} = -3\cos t
\end{aligned}$$

and

$$u_2' = \frac{3\cos t \cos 2t}{\sin 2t} = \frac{3\cos t(1-\sin^2 t)}{2\sin t \cos t} = \frac{3}{2}\operatorname{cosec} t - 3\sin t.$$

Integrating gives

$$\begin{aligned}
u_1(t) &= \int u_1'(t)\,dt = \int -3\cos t\,dt = -3\sin t \\
u_2(t) &= \int u_2'(t)\,dt = \int \frac{3}{2}\operatorname{cosec} t - 3\sin t\,dt \\
&= \frac{3}{2}\ln|\operatorname{cosec} t - \cot t| + 3\cos t
\end{aligned}$$

Therefore a particular solution is

$$\begin{aligned}
Y(t) &= u_1(t)\cos 2t + u_2(t)\sin 2t \\
&= -3\sin t\cos 2t + \frac{3}{2}\ln|\operatorname{cosec} t - \cot t|\sin 2t + 3\cos t\sin 2t \\
&= 3\sin t + \frac{3}{2}\ln|\operatorname{cosec} t - \cot t|\sin 2t.
\end{aligned}$$

Summary. Suppose that $c_1y_1 + c_2y_2$ is the general solution of $L[y] = 0$.

(i). Guess $Y = u_1(t)y_1 + u_2(t)y_2$;

(ii). Make the extra condition $u_1'y_1 + u_2'y_2 = 0$;

(iii). Put Y into $L[y] = g(t)$;

(iv). Find u_1' and u_2';

(v). Integrate to get u_1 and u_2;

Then Y is a particular solution to $L[y] = g(t)$.

Example 3.24. Find a particular solution to $y'' - 2y' + y = e^t \ln t$.

The characteristic equation, $0 = r^2 - 2r + 1 = (r-1)^2$ has roots $r_1 = r_2 = 1$. Hence the general solution of the homogeneous equation is $y(t) = c_1 e^t + c_2 t e^t$.

Therefore we guess that $Y = u_1(t)e^t + u_2(t)te^t$.

We make the extra condition that

$$\begin{aligned} u_1' y_1 + u_2' y_2 &= 0 \\ u_1' e^t + u_2' t e^t &= 0 \\ u_1' + u_2' t &= 0. \end{aligned}$$

Then we calculate that

$$\begin{aligned} Y' &= \cancel{u_1' e^t} + u_1 e^t + \cancel{u_2' t e^t} + u_2 e^t + u_2 t e^t \\ &= u_1 e^t + u_2 e^t + u_2 t e^t, \\ Y'' &= \cancel{u_1' e^t} + u_1 e^t + u_2' e^t + u_2 e^t + \cancel{u_2' t e^t} + u_2 e^t + u_2 t e^t \\ &= u_1 e^t + u_2' e^t + 2u_2 e^t + u_2 t e^t \end{aligned}$$

and

$$\begin{aligned} e^t \ln t &= Y'' - 2Y' + Y \\ &= \left(u_1 e^t + u_2' e^t + 2u_2 e^t + u_2 t e^t\right) - 2\left(u_1 e^t + u_2 e^t + u_2 t e^t\right) + \left(u_1 e^t + u_2 t e^t\right) \\ &= u_2' e^t. \end{aligned}$$

It follows that $u_2' = \ln t$ and thus $u_1' = -u_2' t = -t \ln t$.

Next we integrate to find

$$u_1(t) = \int u_1'(t)\,dt = \int -t \ln t\,dt = -\frac{1}{2}t^2 \ln t + \frac{1}{4}t^2$$

and

$$u_2(t) = \int u_2'(t)\,dt = \int \ln t\,dt = t \ln t - t.$$

Therefore a particular solution is

$$\begin{aligned} Y(t) &= u_1(t)e^t + u_2(t)te^t \\ &= \left(-\frac{1}{2}t^2 \ln t + \frac{1}{4}t^2\right)e^t + (t \ln t - t)\,te^t \\ &= \left(\frac{1}{2}\ln t - \frac{3}{4}\right)t^2 e^t. \end{aligned}$$

Isn't there an easier way?

Theorem 3.6. *Suppose that $y_1(t)$ and $y_2(t)$ form a fundamental set of solutions of $y'' + p(t)y' + q(t)y = 0$. Then a particular solution of $y'' + p(t)y' + q(t)y = g(t)$ is given by*

$$Y(t) = -y_1 \int \frac{y_2 g}{W} + y_2 \int \frac{y_1 g}{W} \tag{3.11}$$

where $W = W(y_1, y_2)$ is the Wronskian.

Example 3.25. Find a particular solution to $y'' - 2y' + y = e^t \ln t$.

The characteristic equation $0 = r^2 - 2r + 1 = (r-1)^2$ has roots $r_1 = r_2 = 1$. Hence

$$y_1 = e^t \qquad \text{and} \qquad y_2 = te^t$$

form a fundamental set of solutions to the homogeneous equation $y'' - 2y' + y = 0$.

We calculate that

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} e^t & te^t \\ e^t & e^t + te^t \end{vmatrix} = e^{2t} + te^{2t} - te^{2t} = e^{2t}.$$

It follows that

$$\begin{aligned} Y(t) &= -y_1 \int \frac{y_2 g}{W} + y_2 \int \frac{y_1 g}{W} \\ &= -e^t \int \frac{te^t e^t \ln t}{e^{2t}} dt + te^t \int \frac{e^t e^t \ln t}{e^{2t}} dt \\ &= -e^t \int t \ln t \, dt + te^t \int \ln t \, dt \\ &= -e^t \left(\frac{1}{2} t^2 \ln t - \frac{1}{4} t^2 \right) + te^t (t \ln t - t) \\ &= \left(\frac{1}{2} \ln t - \frac{3}{4} \right) t^2 e^t \end{aligned}$$

is a particular solution to the ODE.

3.10 Higher Order Linear ODEs

We can use the same ideas to solve higher order linear ODEs.

Example 3.26. Solve

$$\begin{cases} y^{(4)} + y''' - 7y'' - y' + 6y = 0 \\ y(0) = 1 \\ y'(0) = 0 \\ y''(0) = -2 \\ y'''(0) = -1. \end{cases}$$

The characteristic equation is

$$r^4 + r^3 - 7r^2 - r + 6 = 0$$

which has roots $r_1 = 1$, $r_2 = -1$, $r_3 = 2$ and $r_4 = -3$. So the general solution to the ODE is

$$y(t) = c_1 e^t + c_2 e^{-t} + c_3 e^{2t} + c_4 e^{-3t}.$$

Then

$$\begin{aligned} 1 = y(0) &= c_1 + c_2 + c_3 + c_4 + 4 \\ 0 = y'(0) &= c_1 - c_2 + 2c_3 - 3c_4 \\ -2 = y''(0) &= c_1 + c_2 + 4c_3 + 9c_4 \\ -1 = y'''(0) &= c_1 - c_2 + 8c_3 - 27c_4 \end{aligned} \qquad \Longrightarrow \qquad \begin{aligned} c_1 &= \frac{11}{8} \\ c_2 &= \frac{5}{12} \\ c_3 &= -\frac{2}{3} \\ c_4 &= -\frac{1}{8} \end{aligned}$$

Therefore the solution to the IVP is

$$\boxed{y = \frac{11}{8}e^t + \frac{5}{12}e^{-t} - \frac{2}{3}e^{2t} - \frac{1}{8}e^{-3t}.}$$

Example 3.27. Solve

$$y''' - 4y'' + 2y' + 3y = 0.$$

The characteristic equation is

$$r^3 - 4r^2 + 2r + 3 = 0.$$

How can we solve this?

One way is to say "our teacher probably gave us an easy one, so atleast one of the roots will be an easy number 0, ±1, ±2, . . ."

Does $r = 0$ work? $0^3 - 4(0^2) + 2(0) + 3 = 3$. No.

Does $r = 1$ work? $1^3 - 4(1^2) + 2(1) + 3 = 2$. No.

Does $r=-1$ work? $(-1)^3-4(-1)^2+2(-1)+3=-4$. No.

Does $r=2$ work? $2^3-4(2^2)+2(2)+3=-1$. No.

Does $r=-2$ work? $(-2)^3-4(-2)^2+2(-2)+3=-25$. No.

Does $r=3$ work? $3^3-4(3^2)+2(3)+3=0$. Yes.
Therefore

$$\begin{aligned}
0 &= r^3-4r^2+2r+3 = (r-3)(r^2+br+c)\\
&= r^3+br^2+cr-3r^2-3br-3c\\
&= r^3+(b-3)r^2+(c-3b)r-3c\\
&\quad b=-1 \qquad c=-1\\
&= (r-3)(r^2-r-1).
\end{aligned}$$

The roots of the characteristic equation are

$$r_1=3, \qquad r_2=\frac{1}{2}+\frac{\sqrt{5}}{2}, \qquad r_3=\frac{1}{2}-\frac{\sqrt{5}}{2}.$$

You can finish this example.

Example 3.28. Solve

$$y^{(4)}-y=e^t$$

The characteristic equation

$$0=r^4-1=(r^2-1)(r^2+1)$$

has roots $r_1=1$, $r_2=-1$, $r_3=i$ and $r_4=-i$. Therefore

$$y=c_1e^t+c_2e^{-t}+c_3\cos t+c_4\sin t$$

is the general solution of the homogenous equation $y^{(4)}-y=0$. Next we need to find a particular solution. Since e^t solves the homogeneous equation, we try the ansatz $Y=Ate^t$. Then

$$\begin{aligned}
Y' &= Ae^t+Ate^t\\
Y'' &= Ae^t+Ae^t+Ate^t=2Ae^t+Ate^t\\
Y''' &= 2Ae^t+Ae^t+Ate^t=3Ae^t+Ate^t\\
Y^{(4)} &= 3Ae^t+Ae^t+Ate^t=4Ae^t+Ate^t
\end{aligned}$$

and

$$e^t=Y^{(4)}-Y=4Ae^t+Ate^t-Ate^t=4Ae^t \qquad \Longrightarrow \qquad A=\tfrac{1}{4}.$$

Therefore $Y(t)=\frac{1}{4}te^t$ is a particular solution to the ODE.
The general solution to the ODE is therefore

$$\boxed{y(t)=c_1e^t+c_2e^{-t}+c_3\cos t+c_4\sin t+\frac{1}{4}te^t.}$$

Remark 3.11. Any time the characteristic equation has a repeated root, just multiply by t. E.g. if the roots are $r_1 = 7$, $r_2 = 7$, $r_3 = 7$, $r_4 = 7$, $r_5 = 7$ and $r_6 = 8$, then the general solution is

$$y(t) = c_1 e^{7t} + c_2 t e^{7t} + c_3 t^2 e^{7t} + c_4 t^3 e^{7t} + c_5 t^4 e^{7t} + c_6 e^{8t}.$$

Example 3.29 (Going backwards)**.** Find a linear, homogeneous ODEs with constant coefficients, which has general solution $y(t) = c_1 e^t + c_2 t e^t + c_3 e^{2t} \sin t + c_4 e^{2t} \cos t + c_5 e^{2t} t \sin t + c_6 e^{2t} t \cos t$.

The first two terms correspond to a double root $r = 1$. The last four terms correspond to a double complex root $r = 2 \pm i$. Consequently, the characteristic equation is

$$\begin{aligned} 0 &= (r-1)^2(r-2-i)^2(r-2+i)^2 \\ &= (r-1)^2(r^2-4r+5)^2 \\ &= r^6 - 10r^5 + 43r^4 - 100r^3 + 131r^2 - 90r + 25. \end{aligned}$$

Then, a differential equation is

$$\frac{d^6y}{dt^6} - 10\frac{d^5y}{dt^5} + 43\frac{d^4y}{dt^4} - 100\frac{d^3y}{dt^3} + 131\frac{d^2y}{dt^2} - 90\frac{dy}{dt} + 25y = 0.$$

Exercises

Exercise 3.1 (Homogeneous Second Order Linear ODEs with constant coefficients)**.** Solve the following IVPs:

(a). $\begin{cases} y'' - 3y' + 2y = 0 \\ y(0) = 1 \\ y'(0) = 1 \end{cases}$ (b). $\begin{cases} y'' + 4y' + 3y = 0 \\ y(0) = 2 \\ y'(0) = -1 \end{cases}$ (c). $\begin{cases} y'' + 3y' = 0 \\ y(0) = -2 \\ y'(0) = 3 \end{cases}$ (d). $\begin{cases} y'' + 5y' + 3y = 0 \\ y(0) = 1 \\ y'(0) = 0 \end{cases}$

Exercise 3.2 (Fundamental Sets of Solutions)**.** In each of the following: Verify that y_1 and y_2 are solutions of the given ODE; calculate the Wronskian of y_1 and y_2; and determine if they form a fundamental set of solutions.

(a). $t^2y'' - 2y = 0; \quad y_1(t) = t^2, \quad y_2(t) = t^{-1}$

(b). $y'' + 4y = 0; \quad y_1(t) = \cos 2t, \quad y_2(t) = \sin 2t$

(c). $y'' - 2y + y = 0; \quad y_1(t) = e^t, \quad y_2(t) = te^t$

(d). $(1 - x\cot x)y'' - xy' + y = 0 \ (0 < x < \pi); \quad y_1(x) = x, \quad y_2(x) = \sin x$

Exercise 3.3 (Homogeneous Second Order Linear ODEs with constant coefficients)**.** Find the general solution of the following ODEs:

(a). $y'' - 2y' + 2y = 0$

(b). $y'' + 2y' + 2y = 0$

(c). $y'' + 2y' - 8y = 0$

(d). $y'' - 2y' + 6y = 0$

(e). $y'' + 6y' + 13y = 0$

(f). $9y'' + 16y = 0$

(g). $y'' - 2y' + y = 0$

(h). $9y'' + 6y' + y = 0$

(i). $4y'' + 12y' + 9y = 0$

(j). $4y'' - 4y' - 3y = 0$

(k). $y'' - 2y' + 10y = 0$

(l). $y'' - 6y' + 9y = 0$

(m). $4y'' + 17y' + 4y = 0$

(n). $4y'' + 20y' + 25y = 0$

(o). $25y'' - 20y' + 4y = 0$

(p). $2y'' + 2y' + y = 0$

Solve the following IVPs:

(q). $\begin{cases} 9y'' + 6y' + 82y = 0 \\ y(0) = -1 \\ y'(0) = 2 \end{cases}$ (r). $\begin{cases} y'' - 6y' + 9y = 0 \\ y(0) = 0 \\ y'(0) = 2 \end{cases}$

Exercise 3.4 (Reduction of Order)**.** In each of the following problems:

(i) Check that y_1 solves the ODE;

(ii) Use the method of reduction of order to find a second, linearly independent solution, y_2
[HINT: Start with $y_2(t) = v(t)y_1(t)$.];

(iii) Check that your y_2 solves the ODE; and

(iv) Calculate the Wronskian of y_1 and y_2.

(a). $t^2y'' + 2ty' - 2y = 0, \quad t > 0; \quad y_1(t) = t$

(b). $t^2y'' - 4ty' + 6y = 0, \quad t > 0; \quad y_1(t) = t^2$

(c). $t^2y'' + 3ty' + y = 0, \quad t > 0; \quad y_1(t) = t^{-1}$

(d). $t^2y'' - t(t+2)y' + (t+2)y = 0, \quad t > 0; \quad y_1(t) = t$

(e). $xy'' - y' + 4x^3y = 0, \quad x > 0; \quad y_1(x) = \sin x^2$

(f). $(x-1)y'' - xy' + y = 0, \quad x > 1; \quad y_1(x) = e^x$

Exercise 3.5 (The Method of Undetermined Coefficients). Find the general solutions of the following ODEs:

(a). $y'' - 2y' - 3y = 3e^{2t}$

(b). $y'' + 2y' + 5y = 3\cos 2t$

(c). $y'' - 2y' - 3y = 2 - 3te^{-t}$

(d). $y'' + 2y' = 3 + 4\sin 2t$

(e). $y'' + 9y = t^2e^{3t} + 6$

(f). $y'' + 2y' + y = 2e^{-t}$

(g). $2y'' + 3y' + y = t^3 + 3\sin t$

(h). $y'' + y = 3\sin 2t + t\cos 2t$

(i). $y'' + y' + 4y = 2\sinh t$

Exercise 3.6 (The Method of Undetermined Coefficients). Solve the following IVPs:

(a). $\begin{cases} y'' + y' - 2y = 2t \\ y(0) = 0 \\ y'(0) = 1 \end{cases}$

(b). $\begin{cases} y'' - 2y' + y = te^t + 4 \\ y(0) = 1 \\ y'(0) = 1 \end{cases}$

(c). $\begin{cases} y'' + 4y = t^2 + 3e^t \\ y(0) = 0 \\ y'(0) = 2 \end{cases}$

(d). $\begin{cases} -y'' + 6y' - 16y = 1 + 6e^{3t}\sin(2t) \\ y(0) = \frac{15}{16} \\ y'(0) = -1 \end{cases}$

Exercise 3.7 (The Method of Variation of Parameters). Find the general solutions of the following ODEs:

(a). $y'' + y = \tan t, \quad 0 < t < \frac{\pi}{2}$

(b). $y'' + 4y = 3\operatorname{cosec} 2t, \quad 0 < t < \frac{\pi}{2}$

(c). $y'' + 4y' + 4y = t^{-2}e^{-2t}, \quad t > 0$

(d). $y'' - 2y' + y = \dfrac{e^t}{1+t^2}$

Exercise 3.8 (Going Backwards). Find linear, homogeneous ODEs with constant coefficients, which have general solutions equal to the functions given below. The first one is done for you.

(ω) $y(t) = c_1e^t + c_2e^{2t} + c_3e^{3t}$.

(a). $y(t) = c_1 + c_2t + c_3e^{3t}\sin t + c_4e^{3t}\cos t + c_5e^{3t}\sin 2t + c_6e^{3t}\cos 2t$

(b). $y(t) = c_1e^t + c_2te^t + c_3e^{2t}\sin t + c_4e^{2t}\cos t + c_5e^{2t}t\sin t + c_6e^{2t}t\cos t$

(c). $y(t) = c_1e^{2t} + c_2te^{2t} + c_3t^2e^{2t} + c_4e^{-t}\sin 3t + c_5e^{-t}\cos 3t$

Exercise 3.9 (Higher Order Linear ODEs)**.**

(a). Given that $\sin t$ is a solution of $y^{(4)} + 2y''' + 6y'' + 2y' + 5y = 0$, find the general solution of this ODE.

(b). Find the general solution of $y^{(4)} + y'' = 3x^2 + 4\sin x - 2\cos x$.

(c). Solve $\begin{cases} \frac{d^3y}{dx^3} - 2\frac{d^2y}{dx^2} + 4\frac{dy}{dx} - 8y = 0 \\ y(0) = 2 \\ y'(0) = 0 \\ y''(0) = 0. \end{cases}$

4

The Laplace Transform

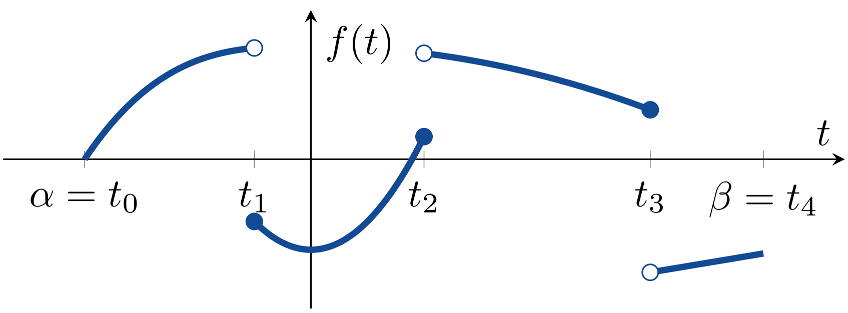
Recall that $\displaystyle\int_a^\infty f(t)\,dt$ means $\displaystyle\lim_{R\to\infty}\int_a^R f(t)\,dt.$

Example 4.1. Let $c \neq 0$. Then

$$\int_0^\infty e^{ct}\,dt = \lim_{R\to\infty}\int_0^R e^{ct}\,dt = \lim_{R\to\infty}\left[\frac{1}{c}e^{ct}\right]_0^R$$
$$= \lim_{R\to\infty}\frac{1}{c}\left(e^{cR}-1\right) = \begin{cases}\infty & c>0\\ -\frac{1}{c} & c<0.\end{cases}$$

Example 4.2.

$$\int_1^\infty \frac{1}{t}\,dt = \lim_{R\to\infty}\int_1^R \frac{1}{t}\,dt = \lim_{R\to\infty}[\ln t]_1^R = \lim_{R\to\infty}(\ln R - 0) = \infty$$




Pierre-Simon Laplace
FRA, 1749-1827

Definition. A function $f : [\alpha,\beta] \to \mathbb{R}$ is ***piecewise continuous*** on $[\alpha,\beta]$ iff <2->there exists a finite partition $\alpha = t_0 < t_1 < t_2 < \ldots < t_n = \beta$ such that<3>

- f is continuous on each subinterval (t_{j-1}, t_j); and
- every one-sided limit $\displaystyle\lim_{t\searrow t_j} f(t)$ and $\displaystyle\lim_{t\nearrow t_j} f(t)$ is finite.

4.1 Definition of the Laplace Transform

$\frac{d}{dt}$ changes a function $f(t)$ into a new function $f'(t)$.

$\mathcal{L}$ changes a function $f(t)$ into a new function $F(s)$.

Definition. Suppose that

(i). $K > 0$, $M > 0$, $a \in \mathbb{R}$;

(ii). f is piecewise continuous on $[0, A]$ for any $A > 0$; and

(iii). $|f(t)| \leq Ke^{at}$ for all $t \geq M$.

The ***Laplace Transform*** of $f : [0, \infty) \to \mathbb{R}$ is a new function defined by

$$F(s) = \mathcal{L}[f](s) = \int_0^\infty e^{-st} f(t)\, dt.$$

$F(s)$ exists for $s > a$.

Example 4.3.

$$\mathcal{L}[0](s) = \int_0^\infty e^{-st} \cdot 0\, dt = \int_0^\infty 0\, dt = 0.$$

Example 4.4.

$$\mathcal{L}[1](s) = \int_0^\infty e^{-st} \cdot 1\, dt = \lim_{R\to\infty} \left[-\frac{e^{-st}}{s}\right]_0^R = \frac{1}{s} \qquad \text{if } s > 0.$$

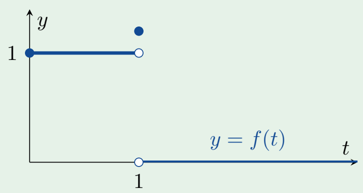
Example 4.5.

$$\mathcal{L}[e^{at}](s) = \int_0^\infty e^{-st} e^{at}\, dt = \int_0^\infty e^{-(s-a)t}\, dt = \frac{1}{s-a} \qquad \text{if } s > a.$$

The Laplace Transform of $e^{at} : [0, \infty) \to \mathbb{R}$ is $\frac{1}{s-a} : (a, \infty) \to \mathbb{R}$.

Example 4.6. Let

$$f(t) = \begin{cases} 1 & 0 \leq t < 1 \\ k & t = 1 \\ 0 & t > 1. \end{cases}$$



Then

$$F(s) = \mathcal{L}[f](s) = \int_0^\infty e^{-st} f(t)\, dt = \int_0^1 e^{-st} \cdot 1\, dt + \int_1^\infty e^{-st} \cdot 0\, dt$$

$$= \left[-\frac{1}{s} e^{-st}\right]_0^1 = \frac{1 - e^{-s}}{s} \qquad \text{if } s > 1.$$

Example 4.7. Find the Laplace Transform of $g(t) = \sin at$ $(t \geq 0)$.

Using integration by parts ($\int_a^b uv' = [uv]_a^b - \int_a^b u'v$), we have

$$\begin{aligned} G(s) = \mathcal{L}\big[g\big](s) &= \int_0^\infty e^{-st} \sin at \, dt = \lim_{R\to\infty} \int_0^R e^{-st} \sin at \, dt \\ &= \lim_{R\to\infty} \left(\left[-\frac{1}{a} e^{-st} \cos at \right]_0^R - \frac{s}{a} \int_0^R e^{-st} \cos at \, dt \right) \\ &= \frac{1}{a} - \frac{s}{a} \int_0^\infty e^{-st} \cos at \, dt. \end{aligned}$$

Using integration by parts a second time, we have

$$G(s) = \frac{1}{a} - \frac{s^2}{a^2} \int_0^\infty e^{-st} \sin at \, dt = \frac{1}{a} - \frac{s^2}{a^2} G(s).$$

Therefore

$$\mathcal{L}\big[\sin at\big](s) = G(s) = \frac{a}{s^2 + a^2} \qquad \text{if } s > 0.$$

Example 4.8.

$$\mathcal{L}\left[\cos at\right](s) = \frac{s}{s^2 + a^2} \qquad \text{if } s > 0.$$

You prove.

Example 4.9.

$$\mathcal{L}\left[\sinh at\right] = \frac{a}{s^2 - a^2} \qquad \text{if } s > |a|.$$

You prove.

Example 4.10.

$$\mathcal{L}\left[\cosh at\right] = \frac{s}{s^2 - a^2} \qquad \text{if } s > |a|.$$

You prove.

Theorem 4.1.

$$\mathcal{L}\big[c_1 f_1 + c_2 f_2\big] = c_1 \mathcal{L}\big[f_1\big] + c_2 \mathcal{L}\big[f_2\big].$$

You prove.

Example 4.11. If $h(t) = 5e^{-2t} - 3\sin 4t$ $(t \geq 0)$, then

$$\begin{aligned} H(s) &= \mathcal{L}\big[h\big](s) \\ &= \mathcal{L}\big[5e^{-2t} - 3\sin 4t\big](s) \\ &= 5\mathcal{L}\big[e^{-2t}\big] - 3\mathcal{L}\big[\sin 4t\big] \\ &= 5\left(\frac{1}{s+2}\right) - 3\left(\frac{4}{s^2+16}\right) \\ &= \frac{5}{s+2} - \frac{12}{s^2+16} \qquad \text{if } s > 0. \end{aligned}$$

The Inverse Laplace Transform

We also have an ***inverse Laplace Transform***:

$$F(s) = \mathcal{L}\left[f(t)\right](s) \qquad \Longleftrightarrow \qquad f(t) = \mathcal{L}^{-1}\left[F(s)\right](t).$$

Example 4.12. $\mathcal{L}\left[1\right] = \frac{1}{s}$ and $\mathcal{L}^{-1}\left[\frac{1}{s}\right] = 1$.

Theorem 4.2.

$$\mathcal{L}^{-1}\left[c_1 f_1 + c_2 f_2\right] = c_1\mathcal{L}^{-1}\left[f_1\right] + c_2\mathcal{L}^{-1}\left[f_2\right].$$

You prove.

Example 4.13. Find the inverse Laplace Transform of $\dfrac{10}{s^2-25}$.

We know that $\mathcal{L}\left[\sinh at\right] = \dfrac{a}{s^2-a^2}$. Therefore

$$\mathcal{L}^{-1}\left[\frac{10}{s^2-25}\right] = 2\mathcal{L}^{-1}\left[\frac{5}{s^2-25}\right] = 2\sinh 5t.$$

Example 4.14. Find the inverse Laplace Transform of $\dfrac{1}{s} + \dfrac{1}{s-2}$.

$$\mathcal{L}^{-1}\left[\frac{1}{s} + \frac{1}{s-2}\right] = \mathcal{L}^{-1}\left[\frac{1}{s}\right] + \mathcal{L}^{-1}\left[\frac{1}{s-2}\right] = 1 + e^{2t}.$$

Theorem 4.3.
$$\mathcal{L}\left[t^n f(t)\right] = (-1)^n \frac{d^n F}{ds^n}$$

Proof: First we calculate that

$$\begin{aligned}
-\frac{dF}{ds} &= -\frac{d}{ds}\int_0^\infty e^{-st} f(t)\ dt = -\int_0^\infty \frac{d}{ds} e^{-st} f(t)\ dt \\
&= -\int_0^\infty -te^{-st} f(t)\ dt \\
&= \int_0^\infty e^{-st} t f(t)\ dt = \mathcal{L}\left[t f(t)\right].
\end{aligned}$$

Therefore the formula holds for $n = 1$.

By repeatedly using
$$-\frac{dF}{ds} = \mathcal{L}\left[t f(t)\right],$$
we can also show that

$$\begin{aligned}
(-1)^2 \frac{d^n F}{ds^2} &= \mathcal{L}\left[t^2 f(t)\right] \\
(-1)^3 \frac{d^n F}{ds^3} &= \mathcal{L}\left[t^3 f(t)\right] \\
(-1)^4 \frac{d^n F}{ds^4} &= \mathcal{L}\left[t^4 f(t)\right] \\
&\vdots \\
(-1)^n \frac{d^n F}{ds^n} &= \mathcal{L}\left[t^n f(t)\right].
\end{aligned}$$

□

Example 4.15.

$$\mathcal{L}\left[t^2 \cosh 2t\right] = (-1)^2 \frac{d^2}{ds^2} \mathcal{L}\left[\cosh 2t\right] = \frac{d^2}{ds^2}\left(\frac{s}{s^2 - 2^2}\right) = \ldots = \frac{2s(s^2+12)}{(s^2-4)^3}.$$

Example 4.16. Find the Laplace Transform of t^n for $n \in \mathbb{N}$.

$$\begin{aligned}
\mathcal{L}\left[t^n\right] &= \mathcal{L}\left[t^n \cdot 1\right] = (-1)^n \frac{d^n}{ds^n} \mathcal{L}\left[1\right] = (-1)^n \frac{d^n}{ds^n}\left(\frac{1}{s}\right) \\
&= (-1)^n \frac{(-1)^n n!}{s^{n+1}} = \frac{n!}{s^{n+1}}.
\end{aligned}$$

$f(t)$	$F(s)=\mathcal{L}\big[f\big](s)$	
1	$\frac{1}{s}$	$s>0$
e^{at}	$\frac{1}{s-a}$	$s>a$
$t^n \qquad (n\in\mathbb{N})$	$\frac{n!}{s^{n+1}}$	$s>0$
$\sin at$	$\frac{a}{s^2+a^2}$	$s>0$
$\cos at$	$\frac{s}{s^2+a^2}$	$s>0$
$\sinh at$	$\frac{a}{s^2-a^2}$	$s>\lvert a\rvert$
$\cosh at$	$\frac{s}{s^2-a^2}$	$s>\lvert a\rvert$
$e^{at}\sin bt$	$\frac{b}{(s-a)^2+b^2}$	$s>a$
$e^{at}\cos bt$	$\frac{s-a}{(s-a)^2+b^2}$	$s>a$
$t^n e^{at} \qquad (n\in\mathbb{N})$	$\frac{n!}{(s-a)^{n+1}}$	$s>a$
$u_c(t)$	$\frac{e^{-cs}}{s}$	$s>0$
$u_c(t)f(t-c)$	$e^{-cs}F(s)$	
$e^{ct}f(t)$	$F(s-c)$	
$f(ct) \qquad (c>0)$	$\frac{1}{c}F\left(\frac{s}{c}\right)$	
$\int_0^t f(t-\tau)g(\tau)d\tau$	$F(s)G(s)$	
$t^n f(t)$	$(-1)^n F^{(n)}(s)$	

$$\mathcal{L}\big[f'\big](s)=s\mathcal{L}\big[f\big](s)-f(0)$$

$$\mathcal{L}\big[f''\big](s)=s^2\mathcal{L}\big[f\big](s)-sf(0)-f'(0)$$

$$\mathcal{L}\big[f^{(n)}\big](s)=s^n\mathcal{L}\big[f\big](s)-s^{n-1}f(0)-\ldots-sf^{(n-2)}(0)-f^{(n-1)}(0)$$

Example 4.17. Find the inverse Laplace Transform of $F(s) = \ln\left(1 + \frac{1}{s^2}\right)$.

Again we will use the formula

$$\mathcal{L}\left[t^n f(t)\right] = (-1)^n \frac{d^n F}{ds^n}.$$

Setting $n = 1$

$$\mathcal{L}\left[t f(t)\right] = (-1)\frac{dF}{ds}$$

and taking $\mathcal{L}^{-1}$ of both sides gives

$$t f(t) = -\mathcal{L}^{-1}\left[\frac{dF}{ds}\right].$$

Now

$$\frac{dF}{ds} = \frac{d}{ds}\ln\left(1 + \frac{1}{s^2}\right) = \frac{\frac{-2}{s^3}}{\left(1 + \frac{1}{s^2}\right)} = \frac{-2}{s\left(s^2+1\right)}.$$

Therefore

$$t f(t) = -\mathcal{L}^{-1}\left[\frac{dF}{ds}\right] = \mathcal{L}^{-1}\left[\frac{2}{s\left(s^2+1\right)}\right].$$

To proceed, we need to write $\frac{2}{s(s^2+1)}$ in partial fractions.
We calculate that

$$\begin{aligned}
\frac{2}{s\left(s^2+1\right)} &= \frac{A}{s} + \frac{Bs + C}{s^2+1} \\
&= \frac{A(s^2+1) + Bs^2 + Cs}{s\left(s^2+1\right)} \\
&= \frac{(A+B)s^2 + Cs + A}{s\left(s^2+1\right)} \\
&= \frac{2}{s} - \frac{2s}{s^2+1}.
\end{aligned}
\qquad \Longrightarrow \qquad
\begin{aligned}
A &= 2 \\
B &= -2 \\
C &= 0
\end{aligned}$$

Thus

$$\begin{aligned}
t f(t) &= \mathcal{L}^{-1}\left[\frac{2}{s\left(s^2+1\right)}\right] = \mathcal{L}^{-1}\left[\frac{2}{s} - \frac{2s}{s^2+1}\right] \\
&= 2\mathcal{L}^{-1}\left[\frac{1}{s}\right] - 2\mathcal{L}^{-1}\left[\frac{s}{s^2+1}\right] \\
&= 2 - 2\cos t.
\end{aligned}$$

Therefore

$$\boxed{f(t) = \frac{2\left(1 - \cos t\right)}{t}.}$$

4.2 Solving Initial Value Problems

Theorem 4.4.

(i). $\mathcal{L}[f'](s) = s\mathcal{L}[f](s) - f(0)$.

(ii). $\mathcal{L}[f''](s) = s^2\mathcal{L}[f](s) - sf(0) - f'(0)$.

(iii). $\mathcal{L}[f'''](s) = s^3\mathcal{L}[f](s) - s^2 f(0) - sf'(0) - f''(0)$.

(iv). $\mathcal{L}[f^{(n)}](s) = s^n\mathcal{L}[f](s) - s^{n-1}f(0) - s^{n-2}f'(0) - \ldots - sf^{(n-2)}(0) - f^{(n-1)}(0)$.

Proof:

(i). Using integration-by-parts ($\int uv' = uv - \int u'v$) we calculate that

$$\begin{aligned}\mathcal{L}[f'](s) &= \int_0^\infty e^{-st} f'(t)\,dt = \left[e^{-st}f(t)\right]_0^\infty - \int_0^\infty \left(\frac{d}{dt}e^{-st}\right) f(t)\,dt \\ &= 0 - f(0) - \int_0^\infty -se^{-st}f(t)\,dt \\ &= -f(0) + s\int_0^\infty e^{-st}f(t)\,dt \\ &= -f(0) + sF(s)\end{aligned}$$

as required.

(ii). Using (i), but replacing each f by f' we get

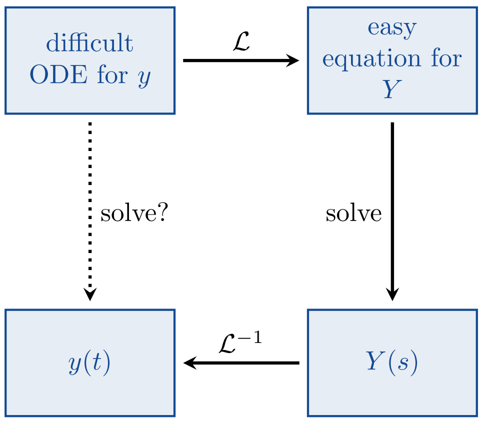
$$\begin{aligned}\mathcal{L}[f''](s) &= s\mathcal{L}[f'](s) - f'(0) \\ &= s\Big(s\mathcal{L}[f](s) - f(0)\Big) - f'(0) \\ &= s^2\mathcal{L}[f](s) - sf(0) - f'(0).\end{aligned}$$

You prove (iii) and (iv). $\square$

Example 4.18. Solve
$$\begin{cases} y'' - y' - 2y = 0 \\ y(0) = 1 \\ y'(0) = 0. \end{cases}$$

solution 1 (method from Chapter 3): The characteristic equation
$$0 = r^2 - r - 2 = (r-2)(r+1)$$
has roots $r_1 = -1$ and $r_2 = 2$. So $y = c_1 e^{-t} + c_2 e^{2t}$. Using the initial conditions we find that $c_1 = \frac{2}{3}$ and $c_2 = \frac{1}{3}$. Therefore
$$\boxed{y(t) = \frac{2}{3}e^{-t} + \frac{1}{3}e^{2t}.}$$

solution 2 (Laplace Transform):



First we take the Laplace Transform of the ODE
$$\mathcal{L}\big[y'' - y' - 2y\big] = \mathcal{L}\big[0\big]$$
$$\mathcal{L}\big[y''\big] - \mathcal{L}\big[y'\big] - 2\mathcal{L}\big[y\big] = 0$$
It follows that
$$\begin{aligned} \big(s^2 Y - sy(0) - y'(0)\big) - (sY - y(0)) - 2Y &= 0 \\ \big(s^2 Y - s - 0\big) - (sY - 1) - 2Y &= 0 \\ (s^2 - s - 2)Y + (1 - s) &= 0. \end{aligned}$$
Thus
$$Y(s) = \frac{s-1}{s^2 - s - 2} = \frac{s-1}{(s-2)(s+1)}.$$
Using partial fractions we obtain
$$\begin{aligned} Y(s) = \frac{s-1}{(s-2)(s+1)} &= \frac{A}{s-2} + \frac{B}{s+1} = \frac{As + A + Bs - 2B}{(s-2)(s+1)} \\ &= \frac{1}{3}\left(\frac{1}{s-2}\right) + \frac{2}{3}\left(\frac{1}{s+1}\right). \end{aligned}$$

$A + B = 1$
$A - 2B = -1$

$A = \frac{1}{3}$
$B = \frac{2}{3}$

But recall that $\mathcal{L}\big[e^{2t}\big] = \frac{1}{s-2}$ and $\mathcal{L}\big[e^{-t}\big] = \frac{1}{s+1}$. Therefore
$$y(t) = \mathcal{L}^{-1}\big[Y\big] = \frac{1}{3}\mathcal{L}^{-1}\left[\frac{1}{s-2}\right] + \frac{2}{3}\mathcal{L}^{-1}\left[\frac{1}{s+1}\right] = \boxed{\frac{1}{3}e^{2t} + \frac{2}{3}e^{-t}.}$$

Example 4.19. Solve

$$\begin{cases} y'' + y = \sin 2t \\ y(0) = 2 \\ y'(0) = 1. \end{cases}$$

$$\begin{aligned} y'' + y &= \sin 2t \\ \mathcal{L}\big[y''\big] + \mathcal{L}\big[y\big] &= \mathcal{L}\big[\sin 2t\big] \\ (s^2Y - sy(0) - y'(0)) + Y &= \frac{2}{s^2+4} \\ s^2Y - 2s - 1 + Y &= \frac{2}{s^2+4} \\ (s^2+1)Y &= 2s + 1 + \frac{2}{s^2+4} \end{aligned}$$

$$\begin{aligned} Y &= \frac{2s+1}{s^2+1} + \frac{2}{(s^2+1)(s^2+4)} = \frac{2s+1}{s^2+1} + \frac{As+B}{s^2+1} + \frac{Cs+D}{s^2+4} \\ &= \frac{2s+1}{s^2+1} + \frac{\frac{2}{3}}{s^2+1} - \frac{\frac{2}{3}}{s^2+4} \\ &= \frac{2s}{s^2+1} + \frac{\frac{5}{3}}{s^2+1} - \frac{\frac{2}{3}}{s^2+4} \\ &= 2\left(\frac{s}{s^2+1}\right) + \frac{5}{3}\left(\frac{1}{s^2+1}\right) - \frac{1}{3}\left(\frac{2}{s^2+4}\right) \\ &= 2\mathcal{L}\left[\cos t\right] + \frac{5}{3}\mathcal{L}\left[\sin t\right] - \frac{1}{3}\mathcal{L}\left[\sin 2t\right]. \end{aligned}$$

Therefore

$$\boxed{y(t) = 2\cos t + \frac{5}{3}\sin t - \frac{1}{3}\sin 2t.}$$

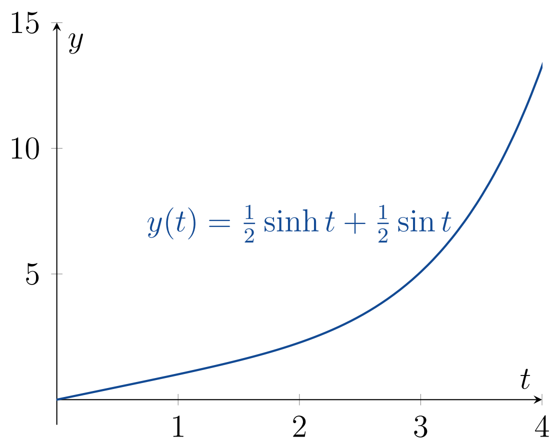
Example 4.20. Solve
$$\begin{cases} y^{(4)} - y = 0 \\ y(0) = 0 \\ y'(0) = 1 \\ y''(0) = 0 \\ y'''(0) = 0. \end{cases}$$

Using the Laplace Transform we calculate that
$$\begin{aligned} 0 &= \mathcal{L}\left[y^{(4)}\right] - \mathcal{L}\left[y\right] \\ &= \big(s^4Y - s^3y(0) - s^2y'(0) - sy''(0) - y'''(0)\big) - Y \\ &= s^4Y - s^2 - Y = (s^4-1)Y - s^2. \end{aligned}$$

Thus
$$Y(s) = \frac{s^2}{s^4-1} = \frac{s^2}{(s^2-1)(s^2+1)} = \frac{\frac{1}{2}}{s^2-1} + \frac{\frac{1}{2}}{s^2+1}.$$

Therefore
$$y = \frac{1}{2}\mathcal{L}^{-1}\left[\frac{1}{s^2-1}\right] + \frac{1}{2}\mathcal{L}^{-1}\left[\frac{1}{s^2+1}\right] = \boxed{\frac{1}{2}\sinh t + \frac{1}{2}\sin t.}$$

$$\begin{cases} y^{(4)} - y = 0 \\ y(0) = 0 \\ y'(0) = 1 \\ y''(0) = 0 \\ y'''(0) = 0. \end{cases}$$



4.3 Solving More Initial Value Problems

Example 4.21. **Use the Laplace Transform** to solve

$$\begin{cases} y'' - 3y' + 2y = \cos t \\ y(0) = 0 \\ y'(0) = 0. \end{cases}$$

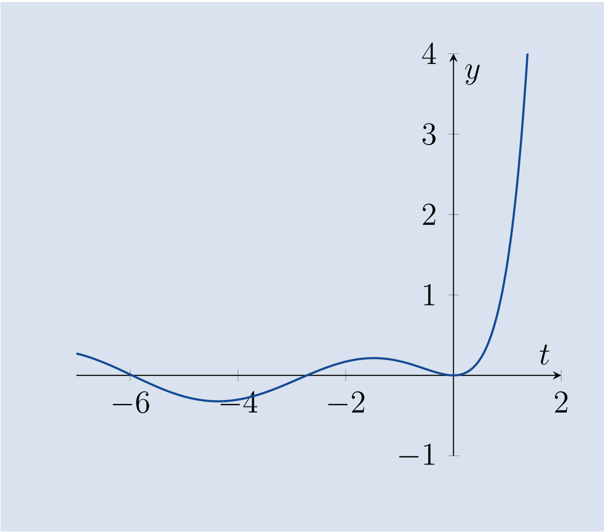
Taking the Laplace Transform of the ODE gives

$$\begin{aligned} \mathcal{L}[y''] - 3\mathcal{L}[y'] + 2\mathcal{L}[y] &= \mathcal{L}[\cos t] \\ \big(s^2Y - sy(0) - y'(0)\big) - 3\big(sY - y(0)\big) + 2Y &= \frac{s}{s^2+1} \\ (s^2 - 3s + 2)Y &= \frac{s}{s^2+1} \end{aligned}$$

$$\begin{aligned} Y(s) &= \frac{s}{(s^2+1)(s^2-3s+2)} = \frac{s}{(s^2+1)(s-2)(s-1)} \\ &= \frac{As+B}{s^2+1} + \frac{C}{s-2} + \frac{D}{s-1} \\ &= \frac{(As+B)(s-2)(s-1) + C(s^2+1)(s-1) + D(s^2+1)(s-2)}{(s^2+1)(s-2)(s-1)} \\ &\quad \left(A = \tfrac{1}{10},\ B = -\tfrac{3}{10},\ C = \tfrac{2}{5},\ D = -\tfrac{1}{2}\right) \\ &= \frac{\frac{1}{10}s - \frac{3}{10}}{s^2+1} + \frac{\frac{2}{5}}{s-2} - \frac{\frac{1}{2}}{s-1} \\ &= \frac{1}{10}\left(\frac{s}{s^2+1}\right) - \frac{3}{10}\left(\frac{1}{s^2+1}\right) + \frac{2}{5}\left(\frac{1}{s-2}\right) - \frac{1}{2}\left(\frac{1}{s-1}\right) \\ &= \frac{1}{10}\mathcal{L}[\cos t] - \frac{3}{10}\mathcal{L}[\sin t] + \frac{2}{5}\mathcal{L}\left[e^{2t}\right] - \frac{1}{2}\mathcal{L}\left[e^{t}\right]. \end{aligned}$$

Therefore the solution to the IVP is

$$\boxed{y(t) = \mathcal{L}^{-1}\left[Y\right](t) = \frac{1}{10}\cos t - \frac{3}{10}\sin t + \frac{2}{5}e^{2t} - \frac{1}{2}e^{t}.}$$



Example 4.22. Use the Laplace Transform to solve

$$\begin{cases} y'' + 2y' + y = 4e^{-t} \\ y(0) = 2 \\ y'(0) = -1. \end{cases}$$

$$\begin{aligned}
y'' \quad + 2 \quad y' \quad + \quad y \quad &= \quad 4e^{-t} \\
\mathcal{L}\left[y''\right] + 2\mathcal{L}\left[y'\right] + \mathcal{L}\left[y\right] &= \mathcal{L}\left[4e^{-t}\right] \\
\mathcal{L}\left[y''\right] + 2\mathcal{L}\left[y'\right] + Y &= \frac{4}{s+1} \\
\mathcal{L}\left[y''\right] + 2(sY - y(0)) + Y &= \frac{4}{s+1} \\
(s^2Y - sy(0) - y'(0)) + 2(sY - y(0)) + Y &= \frac{4}{s+1} \\
(s^2Y - sy(0) - y'(0)) + 2(sY - y(0)) + Y &= \frac{4}{s+1} \\
(s^2Y - 2s + 1) + 2(sY - 2) + Y &= \frac{4}{s+1} \\
(s^2 + 2s + 1)Y - 2s + 1 - 4 &= \frac{4}{s+1} \\
(s^2 + 2s + 1)Y &= \frac{4}{s+1} + 2s + 3 \\
(s+1)^2Y &= \frac{4}{s+1} + 2s + 3 \\
(s+1)^2Y &= \frac{2s^2 + 5s + 7}{s+1} \\
Y &= \frac{2s^2 + 5s + 7}{(s+1)^3} \\
y(t) &= \mathcal{L}^{-1}\left[\frac{2s^2 + 5s + 7}{(s+1)^3}\right]
\end{aligned}$$

I leave it for you to check that if

$$\frac{2s^2 + 5s + 7}{(s+1)^3} = \frac{A}{s+1} + \frac{B}{(s+1)^2} + \frac{C}{(s+1)^3}$$

then $A = 2$, $B = 1$ and $C = 4$.

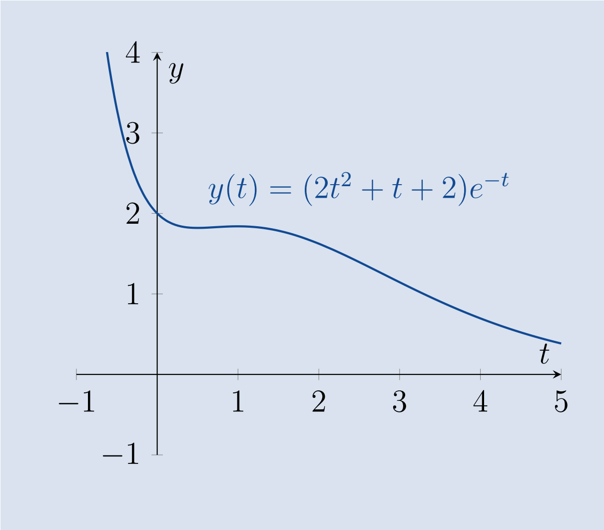
Thus

$$\begin{aligned}
\frac{2s^2 + 5s + 7}{(s+1)^3} &= \frac{2}{s+1} + \frac{1}{(s+1)^2} + \frac{4}{(s+1)^3} \\
&= 2\left(\frac{1}{s+1}\right) + \left(\frac{1}{(s+1)^2}\right) + 2\left(\frac{2}{(s+1)^3}\right).
\end{aligned}$$

In our table of Laplace Transforms, we find that $\mathcal{L}\left[e^{-t}\right] = \frac{1}{s+1}$, $\mathcal{L}\left[te^{-t}\right] = \frac{1}{(s+1)^2}$ and $\mathcal{L}\left[t^2e^{-t}\right] = \frac{2}{(s+1)^3}$.

Therefore the solution to the IVP is

$$\begin{aligned}
y(t) &= \mathcal{L}^{-1}\left[\frac{2s^2+5s+7}{(s+1)^3}\right] \\
&= 2\mathcal{L}^{-1}\left[\frac{1}{s+1}\right] + \mathcal{L}^{-1}\left[\frac{1}{(s+1)^2}\right] + 2\mathcal{L}^{-1}\left[\frac{2}{(s+1)^3}\right] \\
&= 2\left(e^{-t}\right) + \left(te^{-t}\right) + 2\left(t^2e^{-t}\right) \\
&= \boxed{(2t^2+t+2)e^{-t}.}
\end{aligned}$$



Example 4.23. Use the Laplace Transform to solve

$$\begin{cases} y^{(4)} + 2y'' + y = e^{2t} \\ y(0) = 1 \\ y'(0) = 1 \\ y''(0) = 1 \\ y'''(0) = 1. \end{cases}$$

Taking the Laplace Transform of the ODE gives

$$\mathcal{L}\left[y^{(4)}\right] + 2\mathcal{L}\left[y''\right] + \mathcal{L}\left[y\right] = \mathcal{L}\left[e^{2t}\right].$$

Thus

$$\begin{aligned} &\left(s^4 Y(s) - s^3 y(0) - s^2 y'(0) - s y''(0) - y^{(3)}(0)\right) \\ &\qquad + 2\left(s^2 Y(s) - s y(0) - y'(0)\right) + Y(s) = \frac{1}{s-2} \end{aligned}$$

and

$$\left(s^4 Y(s) - s^3 - s^2 - s - 1\right) + 2\left(s^2 Y(s) - s - 1\right) + Y(s) = \frac{1}{s-2}.$$

Thus

$$\left(s^4 + 2s^2 + 1\right) Y(s) - s^3 - s^2 - s - 1 - 2s - 2 = \frac{1}{s-2}.$$

Hence

$$\begin{aligned} \left(s^4 + 2s^2 + 1\right) Y(s) &= \frac{1}{s-2} + s^3 + s^2 + 3s + 3 \\ &= \frac{1}{s-2} + \frac{s^4 - 2s^3}{s-2} + \frac{s^3 - 2s^2}{s-2} \\ &\qquad + \frac{3s^2 - 6s}{s-2} + \frac{3s-6}{s-2} \\ &= \frac{s^4 - s^3 + s^2 - 3s - 5}{s-2}. \end{aligned}$$

$$\begin{aligned} Y(s) &= \frac{s^4 - s^3 + s^2 - 3s - 5}{(s-2)\left(s^4 + 2s^2 + 1\right)} = \frac{s^4 - s^3 + s^2 - 3s - 5}{(s-2)\left(s^2+1\right)^2} \\ &= \frac{\frac{1}{25}}{s-2} + \frac{\frac{24}{25}s + \frac{23}{25}}{s^2+1} + \frac{\frac{9}{5}s + \frac{8}{5}}{\left(s^2+1\right)^2} \\ &= \frac{1}{25}\left(\frac{1}{s-2}\right) + \frac{24}{25}\left(\frac{s}{s^2+1}\right) + \frac{23}{25}\left(\frac{1}{s^2+1}\right) \\ &\qquad + \frac{9}{10}\left(\frac{2s}{\left(s^2+1\right)^2}\right) + \frac{4}{5}\left(\frac{2}{\left(s^2+1\right)^2}\right). \end{aligned}$$

Now $\mathcal{L}\left[e^{2t}\right] = \frac{1}{s-2}$, $\mathcal{L}\left[\cos t\right] = \frac{s}{s^2+1}$ and $\mathcal{L}\left[\sin t\right] = \frac{1}{s^2+1}$. But what do we do with $\frac{2s}{\left(s^2+1\right)^2}$ and $\frac{2}{\left(s^2+1\right)^2}$?

Remember that $\mathcal{L}\left[tf(t)\right] = -F'(s)$. Hence

$$\mathcal{L}\left[t\sin t\right] = -\frac{d}{ds}\mathcal{L}\left[\sin t\right] = -\frac{d}{ds}\left(\frac{1}{s^2+1}\right) = \frac{2s}{\left(s^2+1\right)^2}$$

and

$$\mathcal{L}\left[t\cos t\right] = -\frac{d}{ds}\mathcal{L}\left[\cos t\right] = -\frac{d}{ds}\left(\frac{s}{s^2+1}\right) = \frac{1}{s^2+1} - \frac{2}{(s^2+1)^2}.$$

It follows that

$$\mathcal{L}\left[\sin t - t\cos t\right] = \frac{1}{s^2+1} - \left(\frac{1}{s^2+1} - \frac{2}{(s^2+1)^2}\right) = \frac{2}{(s^2+1)^2}.$$

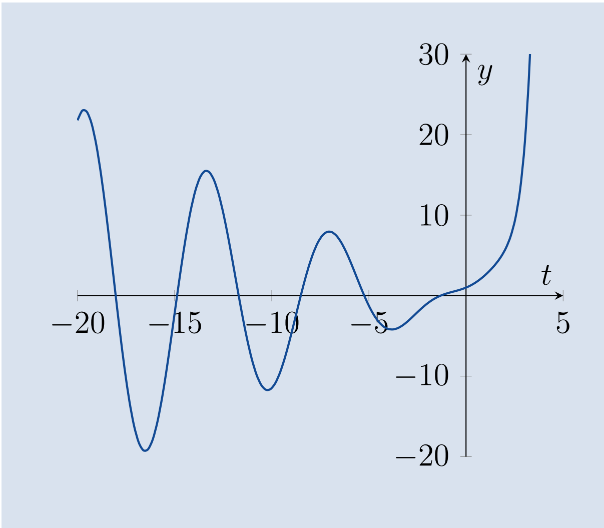
Recall that

$$\begin{aligned} Y(s) = \frac{1}{25}\left(\frac{1}{s-2}\right) + \frac{24}{25}\left(\frac{s}{s^2+1}\right) + \frac{23}{25}\left(\frac{1}{s^2+1}\right) \\ + \frac{9}{10}\left(\frac{2s}{\left(s^2+1\right)^2}\right) + \frac{4}{5}\left(\frac{2}{\left(s^2+1\right)^2}\right). \end{aligned}$$

Hence

$$\boxed{y(t) = \frac{1}{25}\left(e^{2t} + 24\cos t + 23\sin t\right) + \frac{9}{10}t\sin t + \frac{4}{5}(\sin t - t\cos t)}$$

is the solution to the IVP.

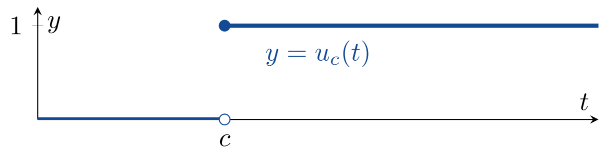


4.4 Step Functions

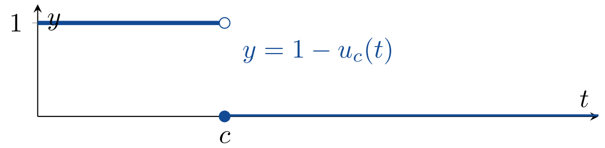
Definition. The ***unit step function*** $u_c : [0, \infty) \to \mathbb{R}$ is defined by

$$u_c(t) = \begin{cases} 0 & t < c \\ 1 & t \geq c \end{cases}$$

for $c \geq 0$.



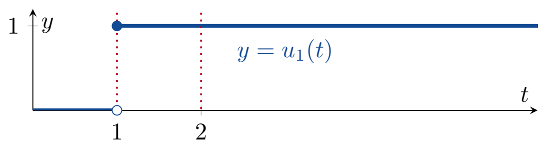
Example 4.24. Draw the graph of $y = 1 - u_c(t)$.

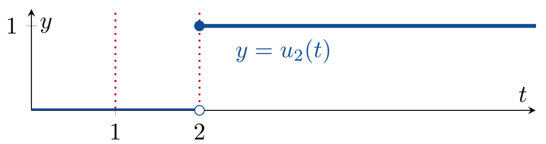


Example 4.25. Draw the graph of $y = u_1(t) - u_2(t)$.

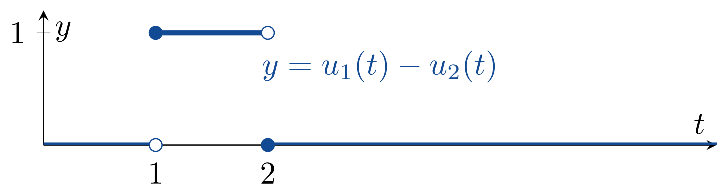
Clearly $t = 1$ and $t = 2$ are important points. So we consider the function on the intervals $[0, 1)$, $[1, 2)$ and $[2, \infty)$.

$$u_1(t) - u_2(t) = \begin{cases} u_1(t) - u_2(t) & 0 \leq t < 1 \\ u_1(t) - u_2(t) & 1 \leq t < 2 \\ u_1(t) - u_2(t) & 2 \leq t \end{cases}$$





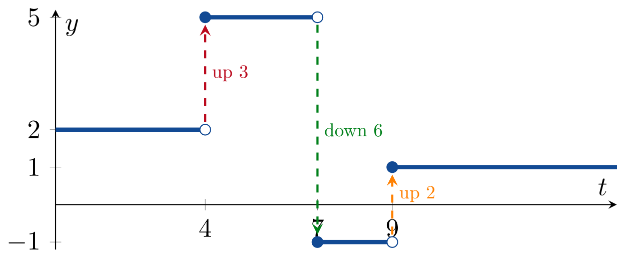
$$\begin{aligned} u_1(t) - u_2(t) &= \begin{cases} u_1(t) - u_2(t) & 0 \leq t < 1 \\ u_1(t) - u_2(t) & 1 \leq t < 2 \\ u_1(t) - u_2(t) & 2 \leq t \end{cases} \\ &= \begin{cases} 0 - 0 & 0 \leq t < 1 \\ 1 - 0 & 1 \leq t < 2 \\ 1 - 1 & 2 \leq t \end{cases} = \begin{cases} 0 & 0 \leq t < 1 \\ 1 & 1 \leq t < 2 \\ 0 & 2 \leq t. \end{cases} \end{aligned}$$



Example 4.26. Write the function

$$f(t) = \begin{cases} 2 & 0 \le t < 4 \\ 5 & 4 \le t < 7 \\ -1 & 7 \le t < 9 \\ 1 & 9 \le t \end{cases}$$

in terms of the unit step function.

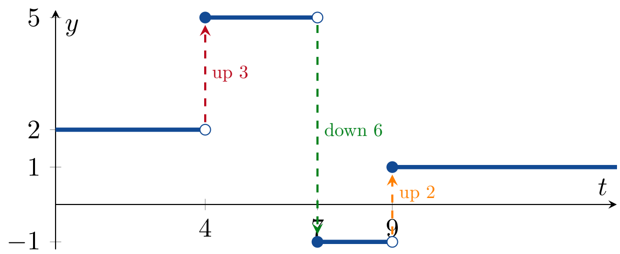


The function starts at $f(0) = 2$. So we will have

$$f(t) = 2 + (\text{something}).$$

At $t = 4$, the function jumps from 2 to 5 (it goes "up 3"). So

$$f(t) = 2 \; +3u_4(t) + (\text{something}).$$



Then it goes "down 6" when $t = 7$. So

$$f(t) = 2 \; +3u_4(t) \; -6u_7(t) + (\text{something}).$$

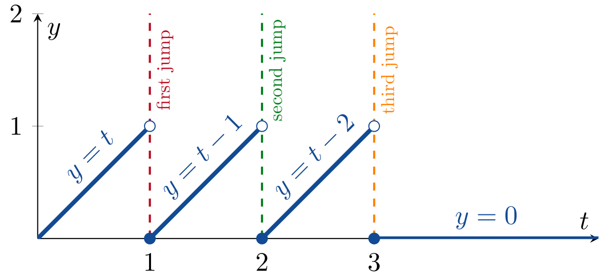
Finally it goes "up 2" when $t = 9$. Therefore

$$f(t) = 2 \; +3u_4(t) \; -6u_7(t) \; +2u_9(t).$$

Example 4.27. Write the function

$$f(t) = \begin{cases} t & 0 \le t < 1 \\ t-1 & 1 \le t < 2 \\ t-2 & 2 \le t < 3 \\ 0 & 3 \le t \end{cases}$$

in terms of the unit step function.



This function starts with $f(t) = t$, then changes when $t = 1$, $t = 2$ and $t = 3$: So we must have

$$f(t) = t + \begin{pmatrix}\text{first}\\ \text{jump}\end{pmatrix} u_1(t) + \begin{pmatrix}\text{second}\\ \text{jump}\end{pmatrix} u_2(t) + \begin{pmatrix}\text{third}\\ \text{jump}\end{pmatrix} u_3(t).$$

At each "jump" we calculate

$$\text{jump} = \begin{pmatrix}\text{function}\\ \text{on right}\end{pmatrix} - \begin{pmatrix}\text{function}\\ \text{on left}\end{pmatrix}.$$

So we have

$$\begin{pmatrix}\text{first}\\ \text{jump}\end{pmatrix} = (t-1) - t = -1$$
$$\begin{pmatrix}\text{second}\\ \text{jump}\end{pmatrix} = (t-2) - (t-1) = -1$$
$$\begin{pmatrix}\text{third}\\ \text{jump}\end{pmatrix} = 0 - (t-2) = 2 - t$$

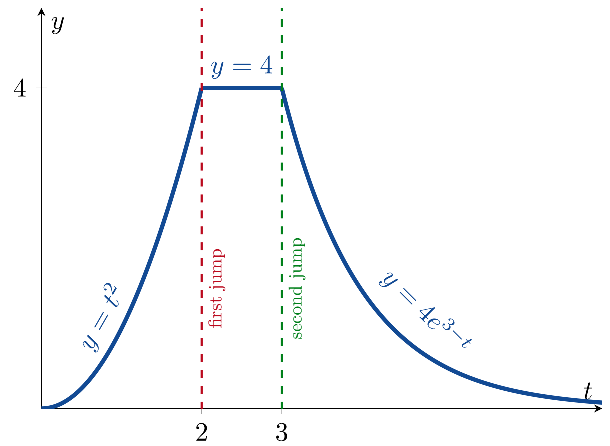
Hence

$$f(t) = t - u_1(t) - u_2(t) + (2-t)u_3(t).$$

Example 4.28. Write the function

$$f(t) = \begin{cases} t^2 & 0 \leq t < 2 \\ 4 & 2 \leq t < 3 \\ 4e^{t-3} & 3 \leq t \end{cases}$$

in terms of the unit step function.



$$f(t) = t^2 + \begin{pmatrix} \text{first} \\ \text{jump} \end{pmatrix} u_2(t) + \begin{pmatrix} \text{second} \\ \text{jump} \end{pmatrix} u_3(t).$$

$$f(t) = t^2 + \left(4 - t^2\right) u_2(t) + \begin{pmatrix} \text{second} \\ \text{jump} \end{pmatrix} u_3(t).$$

$$f(t) = t^2 + \left(4 - t^2\right) u_2(t) + \left(4e^{t-3} - 4\right) u_3(t).$$

What is the Laplace Transform of the unit step function?

We calculate that

$$\mathcal{L}\left[u_c\right](s) = \int_0^\infty e^{-st} u_c(t)\, dt = \int_0^c e^{-st} 0\, dt + \int_c^\infty e^{-st} 1\, dt$$
$$= \int_c^\infty e^{-st}\, dt = \left[-\frac{1}{s}e^{-st}\right]_c^\infty = \frac{e^{-cs}}{s}$$

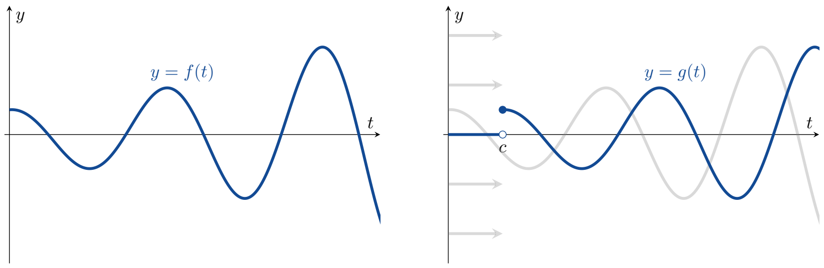
for $s > 0$.

Theorem 4.5.

$$\mathcal{L}\left[u_c\right](s) = \frac{e^{-cs}}{s}$$

Now suppose that we have some function $f : [0, \infty) \to \mathbb{R}$ and we define a new function $g : [0, \infty) \to \mathbb{R}$ by

$$g(t) = \begin{cases} 0 & t < c \\ f(t-c) & t \geq c. \end{cases}$$



We can write $g(t) = u_c(t) f(t-c)$.
What is the Laplace Transform of $g(t) = u_c(t) f(t-c)$?

$$\mathcal{L}\left[g\right] = \mathcal{L}\left[u_c(t) f(t-c)\right] = \int_0^\infty e^{-st} u_c(t) f(t-c)\, dt$$
$$= \int_c^\infty e^{-st} f(t-c)\, dt.$$

Let $u = t - c$. Then $du = dt$ and $t = c \iff u = 0$. Therefore

$$\mathcal{L}\left[g\right] = \int_0^\infty e^{-s(u+c)} f(u)\, du = e^{-cs} \int_0^\infty e^{-su} f(u)\, du = e^{-cs} \mathcal{L}\left[f\right].$$

Theorem 4.6.

$$\mathcal{L}\left[u_c(t) f(t-c)\right](s) = e^{-cs} F(s)$$

Example 4.29. Find the Laplace Transform of

$$f(t) = \begin{cases} t & 0 \le t < 1 \\ t-1 & 1 \le t < 2 \\ t-2 & 2 \le t < 3 \\ 0 & 3 \le t. \end{cases}$$

Since

$$\begin{aligned} f(t) &= t - u_1(t) - u_2(t) + (2-t)u_3(t) \\ &= t - u_1(t) - u_2(t) - u_3(t) - u_3(t)(t-3) \end{aligned}$$

we have that

$$\begin{aligned} F(s) &= \mathcal{L}\left[t\right] - \mathcal{L}\left[u_1\right] - \mathcal{L}\left[u_2\right] - \mathcal{L}\left[u_3\right] - \mathcal{L}\left[u_3(t)(t-3)\right] \\ &= \frac{1}{s^2} - \frac{e^{-s}}{s} - \frac{e^{-2s}}{s} - \frac{e^{-3s}}{s} - \frac{e^{-3s}}{s^2}. \end{aligned}$$

Example 4.30. Find the Laplace Transform of

$$f(t) = \begin{cases} \sin t & 0 \le t \le \frac{\pi}{4} \\ \sin t + \cos(t - \frac{\pi}{4}) & \frac{\pi}{4} \le t. \end{cases}$$

Note that $f(t) = \sin t + g(t)$ where

$$g(t) = \begin{cases} 0 & 0 \le t \le \frac{\pi}{4} \\ \cos(t - \frac{\pi}{4}) & \frac{\pi}{4} \le t \end{cases} = u_{\frac{\pi}{4}}(t)\cos\left(t - \frac{\pi}{4}\right).$$

So

$$\begin{aligned} F(s) &= \mathcal{L}\left[f\right] = \mathcal{L}\left[\sin t\right] + \mathcal{L}\left[u_{\frac{\pi}{4}}(t)\cos\left(t - \frac{\pi}{4}\right)\right] \\ &= \mathcal{L}\left[\sin t\right] + e^{-\frac{\pi s}{4}}\mathcal{L}\left[\cos t\right] = \frac{1}{s^2+1} + e^{-\frac{\pi s}{4}}\frac{s}{s^2+1} \\ &= \frac{1 + se^{-\frac{\pi s}{4}}}{s^2+1}. \end{aligned}$$

Example 4.31. Find the inverse Laplace Transform of $F(s) = \frac{1-e^{-2s}}{s^2}$.

$$\begin{aligned} f(t) &= \mathcal{L}^{-1}\left[F\right] = \mathcal{L}^{-1}\left[\frac{1}{s^2}\right] - \mathcal{L}^{-1}\left[\frac{e^{-2s}}{s^2}\right] = t - u_2(t)(t-2) \\ &= \begin{cases} t & 0 \le t < 2 \\ 2 & t \ge 2. \end{cases} \end{aligned}$$

And what is the Laplace Transform of $e^{ct}f(t)$?

$$\mathcal{L}\left[e^{ct}f(t)\right] = \int_0^\infty e^{-st}e^{ct}f(t)\,dt = \int_0^\infty e^{-(s-c)t}f(t)\,dt = F(s-c).$$

Theorem 4.7.

$$\mathcal{L}\left[e^{ct}f(t)\right] = F(s-c)$$

Example 4.32. Find the inverse Laplace Transform of $G(s) = \dfrac{1}{s^2-4s+5}$.

Note first that

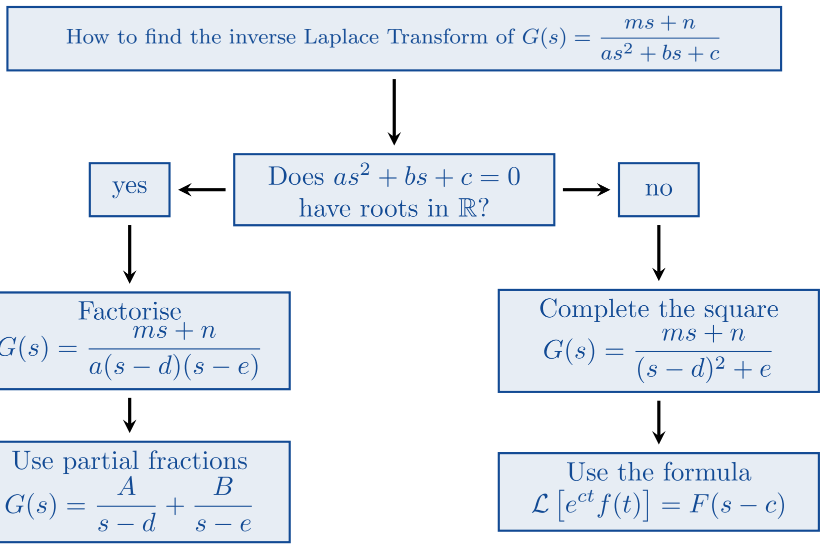
$$G(s) = \frac{1}{s^2-4s+5} = \frac{1}{(s-2)^2+1}.$$

If $F(s) = \frac{1}{s^2+1}$, then we have $G(s) = F(s-2)$. But

$$\mathcal{L}^{-1}\left[F\right] = \mathcal{L}^{-1}\left[\frac{1}{s^2+1}\right] = \sin t.$$

Therefore

$$g(t) = \mathcal{L}^{-1}\left[G\right] = \mathcal{L}^{-1}\left[F(s-2)\right] = e^{2t}\mathcal{L}^{-1}\left[F\right] = e^{2t}\sin t.$$



Example 4.33. Find the inverse Laplace Transform of $G(s) = \dfrac{30s+440}{s^2+32s+240}$.

First note that $s^2+32s+240=0$ has roots $s_1 = -12$ and $s_2 = -20$. In fact

$$G(s) = \frac{30s+440}{s^2+32s+240} = \frac{10}{s+12} + \frac{20}{s+20}.$$

I leave this example for you to finish.

Example 4.34. Find the inverse Laplace Transform of $G(s) = \dfrac{10s+12}{s^2+40s+420}$.

Since the roots of $s^2 + 40s + 420 = 0$ are $s = -20 \pm 2i\sqrt{5}$, we must complete the square. You can check that

$$G(s) = \frac{10s + 12}{s^2 + 40s + 420} = \frac{10s + 12}{(s + 20)^2 + 20}.$$

Now

$$\begin{aligned}
G(s) &= \frac{10s + 12}{(s + 20)^2 + 20} \\
&= 10\left(\frac{s}{(s + 20)^2 + 20}\right) + \frac{12}{\sqrt{20}}\left(\frac{\sqrt{20}}{(s + 20)^2 + 20}\right) \\
&= 10F(s + 20) + \frac{12}{\sqrt{20}}H(s + 20)
\end{aligned}$$

where $F(s) = \frac{s}{s^2+20}$ and $H(s) = \frac{\sqrt{20}}{s^2+20}$.

Note that

$$f(t) = \mathcal{L}^{-1}\left[F\right](t) = \cos\sqrt{20}t$$

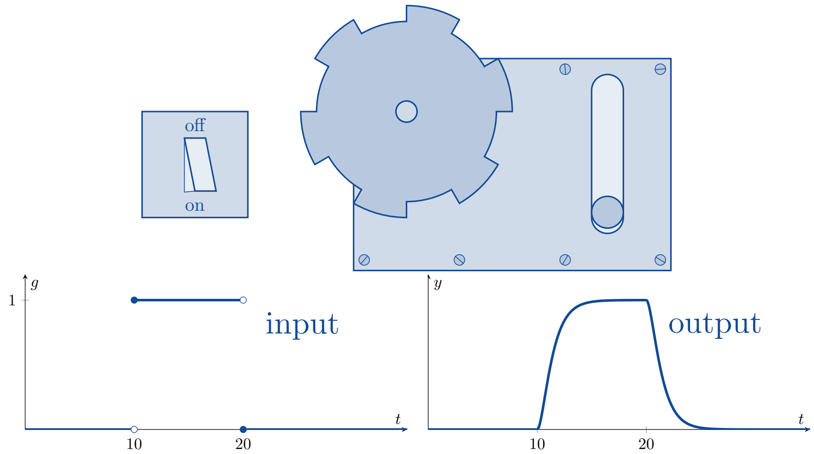
and

$$h(t) = \mathcal{L}^{-1}\left[H\right](t) = \sin\sqrt{20}t.$$

Therefore

$$\begin{aligned}
g(t) &= 10\mathcal{L}^{-1}\left[F(s + 20)\right] + \frac{12}{\sqrt{20}}\mathcal{L}^{-1}\left[H(s + 20)\right] \\
&= 10e^{-20t}\mathcal{L}^{-1}\left[F\right] + \frac{12}{\sqrt{20}}e^{-20t}\mathcal{L}^{-1}\left[H\right] \\
&= 10e^{-20t}\cos\sqrt{20}t + \frac{12}{\sqrt{20}}e^{-20t}\sin\sqrt{20}t.
\end{aligned}$$

4.5 ODEs with Discontinuous Forcing Functions



Example 4.35. Solve

$$\begin{cases} y'' + 4y = f(t) = \begin{cases} 0 & 0 \le t < 5 \\ \frac{1}{5}(t-5) & 5 \le t < 10 \\ 1 & 10 \le t \end{cases} \\ y(0) = 0 \\ y'(0) = 0. \end{cases}$$

Note that

$$\begin{aligned} f(t) &= 0 + \left(\frac{1}{5}(t-5) - 0\right) u_5(t) + \left(1 - \frac{1}{5}(t-5)\right) u_{10}(t) \\ &= \frac{1}{5}\Big(u_5(t)(t-5) - u_{10}(t)(t-10)\Big). \end{aligned}$$

So our IVP is

$$\begin{cases} y'' + 4y = \frac{1}{5}\Big(u_5(t)(t-5) - u_{10}(t)(t-10)\Big) \\ y(0) = 0 \\ y'(0) = 0. \end{cases}$$

Taking the Laplace transform of the ODE gives

$$(s^2+4)Y = \frac{1}{5}\frac{e^{-5s} - e^{-10s}}{s^2}$$

and

$$Y = \frac{1}{5}\frac{e^{-5s} - e^{-10s}}{s^2(s^2+4)}.$$

Let

$$H(s) = \frac{1}{s^2(s^2+4)}.$$

Then

$$Y(s) = \frac{1}{5}e^{-5s}H(s) - \frac{1}{5}e^{-10s}H(s).$$

Since

$$\mathcal{L}\left[u_c(t)h(t-c)\right](s) = e^{-cs}H(s)$$

we have that

$$u_c(t)h(t-c) = \mathcal{L}^{-1}\left[e^{-cs}H(s)\right](t).$$

If we can find $h(t)$, then we can find $y(t)$.

Using partial fractions, we calculate (please check!) that

$$\begin{aligned}
H(s) = \frac{1}{s^2(s^2+4)} &= \frac{As+B}{s^2} + \frac{Cs+D}{s^2+4} \\
&= \frac{As^3+Bs^2+4As+4B+Cs^3+Ds^2}{s^2(s^2+4)} \\
&= \frac{0s+\frac{1}{4}}{s^2} + \frac{0s-\frac{1}{4}}{s^2+4} = \frac{\frac{1}{4}}{s^2} - \frac{\frac{1}{4}}{s^2+4}.
\end{aligned}$$

Hence

$$h(t) = \frac{1}{4}\mathcal{L}^{-1}\left[\frac{1}{s^2}\right] - \frac{1}{8}\mathcal{L}^{-1}\left[\frac{2}{s^2+4}\right] = \frac{t}{4} - \frac{1}{8}\sin 2t.$$

Therefore

$$\begin{aligned}
y(t) &= \mathcal{L}^{-1}\left[\frac{1}{5}e^{-5s}H(s) - \frac{1}{5}e^{-10s}H(s)\right] \\
&= \frac{1}{5}u_5(t)h(t-5) - \frac{1}{5}u_{10}(t)h(t-10) \\
&= u_5(t)\left(\frac{t-5}{20} - \frac{1}{40}\sin(2t-10)\right) \\
&\quad - u_{10}(t)\left(\frac{t-10}{20} - \frac{1}{40}\sin(2t-20)\right).
\end{aligned}$$

Example 4.36. Solve

$$\begin{cases} y'' + 3y' + 2y = f(t) = \begin{cases} 1 & 0 \le t < 10 \\ 0 & 10 \le t \end{cases} \\ y(0) = 1 \\ y'(0) = 0. \end{cases}$$

Since $f(t) = 1 - u_{10}(t)$, the Laplace Transform of the ODE is

$$(s^2 + 3s + 2)Y - (s + 3) = \frac{1 - e^{-10s}}{s}.$$

Thus

$$\begin{aligned} Y(s) &= \frac{1 - e^{-10s}}{s(s^2 + 3s + 2)} + \frac{s + 3}{s^2 + 3s + 2} \\ &= \frac{(s^2 + 3s + 1) - e^{-10s}}{s(s^2 + 3s + 2)}. \end{aligned}$$

Let

$$G(s) = \frac{s^2 + 3s + 1}{s(s^2 + 3s + 2)} \qquad \text{and} \qquad H(s) = \frac{1}{s(s^2 + 3s + 2)}.$$

Then $Y = G(s) - e^{-10s}H(s)$. If we can find $g(t)$ and $h(t)$, then we can find $y(t)$.

Using partial fractions we get

$$G(s) = \frac{A}{s} + \frac{B}{s+1} + \frac{C}{s+2} = \frac{\frac{1}{2}}{s} + \frac{1}{s+1} - \frac{\frac{1}{2}}{s+2}$$

and

$$H(s) = \frac{D}{s} + \frac{E}{s+1} + \frac{F}{s+2} = \frac{\frac{1}{2}}{s} - \frac{1}{s+1} + \frac{\frac{1}{2}}{s+2}$$

(please check!). It follows that

$$g(t) = \frac{1}{2}\left(1 + 2e^{-t} - e^{-2t}\right) \qquad \text{and} \qquad h(t) = \frac{1}{2}\left(1 - 2e^{-t} + e^{-2t}\right).$$

Therefore

$$\begin{aligned} y(t) &= \mathcal{L}^{-1}\left[Y\right] \\ &= \mathcal{L}^{-1}\left[G(s) - e^{-10s}H(s)\right] \\ &= g(t) - u_{10}(t)h(t - 10) \\ &= \frac{1}{2}\left(1 + 2e^{-t} - e^{-2t}\right) - \frac{1}{2}u_{10}(t)\left(1 - 2e^{-(t-10)} + e^{-2(t-10)}\right). \end{aligned}$$

Example 4.37. Solve

$$\begin{cases} y'' + 4y = u_\pi(t) - u_{3\pi}(t) \\ y(0) = 0 \\ y'(0) = 0. \end{cases}$$

Taking the Laplace Transform of the ODE gives

$$(s^2+4)Y(s) = \frac{e^{-\pi s} - e^{-3\pi s}}{s}.$$

Thus

$$Y(s) = \frac{e^{-\pi s} - e^{-3\pi s}}{s(s^2+4)}.$$

Let

$$H(s) = \frac{1}{s(s^2+4)}.$$

Using partial fractions, we calculate that

$$\begin{aligned} H(s) = \frac{1}{s(s^2+4)} &= \frac{A}{s} + \frac{Bs+C}{s^2+4} = \frac{\frac{1}{4}}{s} + \frac{-\frac{1}{4}s+0}{s^2+4} \\ &= \frac{1}{4}\left(\frac{1}{s}\right) - \frac{1}{4}\left(\frac{s}{s^2+4}\right) = \frac{1}{4}\mathcal{L}\left[1\right] - \frac{1}{4}\mathcal{L}\left[\cos 2t\right]. \end{aligned}$$

It follows that

$$h(t) = \frac{1}{4} - \frac{1}{4}\cos 2t$$

and the solution to the IVP is

$$\begin{aligned} y(t) &= \mathcal{L}^{-1}\left[e^{-\pi s}H(s)\right] - \mathcal{L}^{-1}\left[e^{-3\pi s}H(s)\right] \\ &= u_\pi(t)h(t-\pi) - u_{3\pi}(t)h(t-3\pi) \\ &= \frac{1}{4}u_\pi(t)\big(1-\cos(2t-2\pi)\big) - \frac{1}{4}u_{3\pi}(t)\big(1-\cos(2t-6\pi)\big). \end{aligned}$$

4.6 The Convolution Integral

Let $f : [0, \infty) \to \mathbb{R}$ and $g : [0, \infty) \to \mathbb{R}$ be piecewise continuous functions.

Definition. The ***convolution*** of f and g is

$$\big(f * g\big)(t) = \int_0^t f(\tau)g(t-\tau)\,d\tau.$$

Theorem 4.8 (Properties)**.**

- $f * g = g * f$
- $f * (g * h) = (f * g) * h$
- $f * (g + h) = (f * g) + (f * h)$
- $f * 0 = 0 = 0 * f$

Example 4.38.

$$\begin{aligned}
\big(\cos * 1\big)(t) &= \int_0^t \cos\tau \cdot 1\,d\tau = [\sin\tau]_0^t = \sin t - \sin 0 = \sin t \\
\big(1 * \cos\big)(t) &= \int_0^t 1 \cdot \cos(t-\tau)\,d\tau = [-\sin(t-\tau)]_0^t \\
&= -\sin 0 + \sin t = \sin t
\end{aligned}$$

Note that $f * 1 \neq f$ in general.

Example 4.39.

$$\begin{aligned}
\big(\sin * \sin\big)(t) &= \int_0^t \sin\tau\sin(t-\tau)\,d\tau \\
&= \int_0^t \sin\tau\,(\sin t\cos\tau - \cos t\sin\tau)\;d\tau \\
&= \sin t\int_0^t \sin\tau\cos\tau\,d\tau - \cos t\int_0^t \sin^2\tau\,d\tau \\
&= \sin t\left[-\frac{1}{2}\cos^2\tau\right]_0^t - \cos t\left[\frac{1}{2}(\tau - \sin\tau\cos\tau)\right]_0^t \\
&= \frac{1}{2}\sin t\,\big(1 - \cos^2 t\big) - \frac{1}{2}\cos t\,(t - \sin t\cos t) \\
&= \frac{1}{2}\sin t - \frac{t}{2}\cos t.
\end{aligned}$$

Note that $f * f \geq 0$ is ***not*** true in general.

Theorem 4.9.

$$\mathcal{L}\left[f * g\right](s) = F(s)G(s)$$

This means that $\mathcal{L}^{-1}\left[FG\right] = f * g$.

Example 4.40. Find the inverse Laplace Transform of $H(s) = \dfrac{a}{s^2(s^2+a^2)}$.

Note that $H(s) = \left(\dfrac{1}{s^2}\right)\left(\dfrac{a}{s^2+a^2}\right)$. We know that $\mathcal{L}\left[t\right] = \frac{1}{s^2}$ and $\mathcal{L}\left[\sin at\right] = \frac{a}{s^2+a^2}$. So

$$\begin{aligned} h(t) &= \mathcal{L}^{-1}\left[\left(\frac{1}{s^2}\right)\left(\frac{a}{s^2+a^2}\right)\right] = \mathcal{L}^{-1}\left[\frac{1}{s^2}\right] * \mathcal{L}^{-1}\left[\frac{a}{s^2+a^2}\right] \\ &= t * \sin at = \int_0^t \tau \sin a(t-\tau)\, d\tau \\ &= \frac{at - \sin at}{a^2}. \end{aligned}$$

Example 4.41. Solve

$$\begin{cases} y'' + 4y = g(t) \\ y(0) = 3 \\ y'(0) = -1. \end{cases}$$

Taking the Laplace Transform of the ODE gives

$$(s^2 Y - 3s + 1) + 4Y = G(s)$$

which rearranges to

$$\begin{aligned} Y(s) &= \frac{3s-1}{s^2+4} + \frac{G(s)}{s^2+4} \\ &= 3\left(\frac{s}{s^2+4}\right) - \frac{1}{2}\left(\frac{2}{s^2+4}\right) + \frac{1}{2}\left(\frac{2}{s^2+4}\right)G(s). \end{aligned}$$

Hence the solution to the IVP is

$$\begin{aligned} y(t) &= 3\mathcal{L}^{-1}\left[\frac{s}{s^2+4}\right] - \frac{1}{2}\mathcal{L}^{-1}\left[\frac{2}{s^2+4}\right] + \frac{1}{2}\mathcal{L}^{-1}\left[\left(\frac{2}{s^2+4}\right)G(s)\right] \\ &= 3\cos 2t - \frac{1}{2}\sin 2t + \frac{1}{2}\sin 2t * g(t) \\ &= 3\cos 2t - \frac{1}{2}\sin 2t + \frac{1}{2}\int_0^t \sin 2(t-\tau) g(\tau)\, d\tau. \end{aligned}$$

Example 4.42. Find the inverse Laplace Transform of $\dfrac{2}{(s-1)(s^2+4)}$.

$$
\begin{aligned}
\mathcal{L}^{-1}\left[\frac{2}{(s-1)(s^2+4)}\right] &= \mathcal{L}^{-1}\left[\left(\frac{2}{s^2+4}\right)\left(\frac{1}{s-1}\right)\right] = \sin 2t * e^t \\
&= \int_0^t e^{t-\tau}\sin 2\tau \, d\tau = e^t \int_0^t e^{-\tau}\sin 2\tau \, d\tau \\
&= e^t\left[\frac{e^{-\tau}}{5}(-\sin 2\tau - 2\cos 2\tau)\right]_0^t \\
&= \frac{2}{5}e^t - \frac{1}{5}\sin 2t - \frac{2}{5}\cos 2t.
\end{aligned}
$$

Example 4.43. Solve

$$
\begin{cases}
4y'' + y = g(t) \\
y(0) = 3 \\
y'(0) = -7.
\end{cases}
$$

$$4y'' + y = g(t)$$

$$\mathcal{L}\left[4y'' + y\right] = \mathcal{L}\left[g(t)\right]$$

$$
\begin{gathered}
4(s^2Y - sy(0) - y'(0)) + Y = G(s) \\
4(s^2Y - 3s + 7) + Y = G(s) \\
(4s^2+1)Y - 12s + 28 = G(s) \\
(4s^2+1)Y = 12s - 28 + G(s) \\
4\left(s^2 + \frac{1}{4}\right)Y = 12s - 28 + G(s) \\
Y = \frac{12s}{4(s^2+\frac{1}{4})} - \frac{28}{4(s^2+\frac{1}{4})} + \frac{G(s)}{4(s^2+\frac{1}{4})} \\
Y = \frac{3s}{s^2+\frac{1}{4}} - \frac{7}{s^2+\frac{1}{4}} + G(s)\frac{\frac{1}{4}}{s^2+\frac{1}{4}} \\
Y = 3\left(\frac{s}{s^2+\frac{1}{4}}\right) - 14\left(\frac{\frac{1}{2}}{s^2+\frac{1}{4}}\right) + \frac{1}{2}G(s)\left(\frac{\frac{1}{2}}{s^2+\frac{1}{4}}\right) \\
Y = 3\mathcal{L}\left[\cos\frac{t}{2}\right] - 14\mathcal{L}\left[\sin\frac{t}{2}\right] + \frac{1}{2}G(s)\mathcal{L}\left[\sin\frac{t}{2}\right] \\
\boxed{y(t) = 3\cos\frac{t}{2} - 14\sin\frac{t}{2} + \frac{1}{2}g(t) * \sin\frac{t}{2}.}
\end{gathered}
$$

Exercises

Exercise 4.1 (The Laplace Transform). Find the Laplace Transform of the following functions:

(a). $f(t) = e^{-2t}$

(b). $f(t) = 3t^2$

(c). $f(t) = \cos^2 2t$

(d). $f(t) = t\cos t + te^t$

(e). $f(t) = \frac{\sinh t}{t}$

(f). $f(t) = t^2 \cos 2t$

(g). $f(t) = \frac{e^{3t}-1}{t}$

(h). $f(t) = te^{-t}\sin^2 t$

(i). $f(t) = \begin{cases} 2 & 0 < t \leq 3 \\ 0 & t > 3 \end{cases}$

(j). $f(t) = \begin{cases} \sin 2t & \pi \leq t \leq 2\pi \\ 0 & t < \pi \text{ or } t > 2\pi \end{cases}$

Exercise 4.2 (The Inverse Laplace Transform). Find the inverse Laplace Transform of the following functions:

(a). $F(s) = \frac{1}{s-2}$

(b). $F(s) = \frac{1}{s} - \frac{2}{s^{5/2}}$

(c). $F(s) = \frac{3s+1}{s^2+4}$

(d). $F(s) = \frac{2e^{-3s}}{s}$

(e). $F(s) = \frac{1}{s(s-3)}$

(f). $F(s) = \frac{2s+1}{s(s^2+9)}$

(g). $F(s) = \frac{s^3}{(s-4)^4}$

(h). $F(s) = \frac{s^2-2s}{s^4+5s^2+4}$

(i). $F(s) = \frac{2s^3-s^2}{(4s^2-4s+5)^2}$

(j). $F(s) = \arctan\left(\frac{3}{s+2}\right)$

(k). $F(s) = \frac{s}{(s^2+1)^3}$

(l). $F(s) = \frac{e^{-s}}{s+2}$

Hint for (j): Note that since $\mathcal{L}[tf(t)] = (-1)\frac{dF}{ds}$, we have that $-\mathcal{L}^{-1}\left[\frac{dF}{ds}\right] = tf(t)$ and thus $f(t) = -\frac{1}{t}\mathcal{L}^{-1}\left[\frac{dF}{ds}\right]$.

Exercise 4.3 (The Laplace Transform). Use the definition $\mathcal{L}[f](s) = \displaystyle\int_0^\infty e^{-st} f(t)\,dt$ to prove that the following identities are true. The first one is done for you.

(ω). $\mathcal{L}[1](s) = \frac{1}{s}$

$$\mathcal{L}[1](s) = \int_0^\infty e^{-st}(1)\,dt = \lim_{A\to\infty}\int_0^A e^{-st}(1)\,dt = \lim_{A\to\infty}\left[-\frac{e^{-st}}{s}\right]_0^A = \lim_{A\to\infty}\left(-\frac{e^{-sA}}{s} + \frac{e^0}{s}\right) = \frac{1}{s}$$

(a). $\mathcal{L}[t^2](s) = \frac{2}{s^3}$ for $s > 0$

(b). $\mathcal{L}[\cos at](s) = \frac{s}{s^2+a^2}$ for $s > 0$

(c). $\mathcal{L}[\sinh at](s) = \frac{a}{s^2-a^2}$ for $s > a$

(d). $\mathcal{L}[\cosh at](s) = \frac{s}{s^2-a^2}$ for $s > a$

(e). $\mathcal{L}[f(ct)](s) = \frac{1}{c}\mathcal{L}[f]\left(\frac{s}{c}\right)$

(f). $\frac{d}{ds}\mathcal{L}[f](s) = -\mathcal{L}[tf(t)](s)$

Exercise 4.4 (The Laplace Transform)**.** Use the Laplace Transform to solve the following initial value problems:

(a). $\begin{cases} x'' + 4x = 0 \\ x(0) = 5 \\ x'(0) = 0 \end{cases}$

(b). $\begin{cases} x'' - x' - 2x = 0 \\ x(0) = 0 \\ x'(0) = 2 \end{cases}$

(c). $\begin{cases} x'' + 9x = 1 \\ x(0) = 0 \\ x'(0) = 0 \end{cases}$

(d). $\begin{cases} x'' + 6x' + 25x = 0 \\ x(0) = 2 \\ x'(0) = 3 \end{cases}$

(e). $\begin{cases} x'' - 6x' + 8x = 2 \\ x(0) = 0 \\ x'(0) = 0 \end{cases}$

(f). $\begin{cases} x'' - 4x = 3t \\ x(0) = 0 \\ x'(0) = 0 \end{cases}$

(g). $\begin{cases} x'' + 4x' + 8x = e^{-t} \\ x(0) = 0 \\ x'(0) = 0 \end{cases}$

(h). $\begin{cases} x^{(4)} + 8x'' + 16x = 0 \\ x(0) = x'(0) = x''(0) = 0 \\ x^{(3)}(0) = 1 \end{cases}$

(i). $\begin{cases} x^{(3)} + 4x'' + 5x' + 2x = 10\cos t \\ x(0) = x'(0) = 0 \\ x''(0) = 3 \end{cases}$

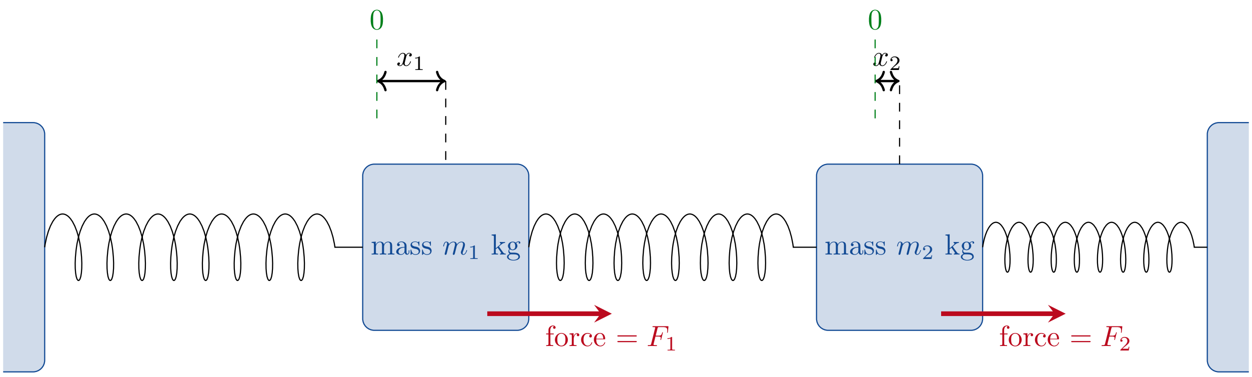
(j). $\begin{cases} x'' + 4x' + 13x = te^{-t} \\ x(0) = 0 \\ x'(0) = 2 \end{cases}$

(k). $\begin{cases} x'' + x = \sin 2t \\ x(\frac{\pi}{2}) = 2 \\ x'(\frac{\pi}{2}) = 0 \end{cases}$

5

Systems of First Order Linear ODEs

5.1 Introduction



Consider the dynamical system shown above. There are two blocks and three springs. Forces F_1 and F_2 act on the blocks as shown.

See https://tinyurl.com/wm2ogdh

We expect that the acceleration of the blocks will depend on

- the displacements x_1 and x_2;
- the forces F_1 and F_2; and
- the masses of the blocks.

So we expect that:

$$\begin{cases} \dfrac{d^2x_1}{dt^2} = f_1(x_1, x_2, F_1, m_1) \\ \dfrac{d^2x_2}{dt^2} = f_2(x_1, x_2, F_2, m_2). \end{cases}$$

This is a system of two ODEs. To find $x_1(t)$ and $x_2(t)$, we would need to solve these equations at the same time.

The most famous system of ODEs is the system of ***Predator-Prey*** equations:

$$\begin{cases} \dfrac{dx}{dt} = \alpha x - \beta xy \\ \dfrac{dy}{dt} = \delta xy - \gamma y \end{cases}$$

where

$$x(t) = \text{number of prey (e.g. mice)}$$
$$y(t) = \text{number of predators (e.g. owls)},$$

which originate circa 1925.

It is possible to convert an nth order linear ODE into a system of n first order linear ODEs. Or vice versa.

$$\begin{array}{c} a_n y^{(n)} + a_{n-1} y^{(n-1)} + \ldots \\ + a_1 y' + a_0 y = g(t) \end{array} \quad \longleftrightarrow \quad \begin{cases} x_1' = b_{11}x_1 + \ldots + b_{1n}x_n + h_1(t) \\ x_2' = b_{21}x_1 + \ldots + b_{2n}x_n + h_2(t) \\ \vdots \\ x_n' = b_{n1}x_1 + \ldots + b_{nn}x_n + h_n(t) \end{cases}$$

Example 5.1. Write

$$u'' + 0.25u' + u = 0$$

as a system of two first order ODEs.

Let $x_1 = u$ and $x_2 = u'$. Then clearly $x_1' = u' = x_2$ and

$$x_2' = u'' = -0.25u' - u = -0.25x_2 - x_1.$$

Therefore

$$\begin{cases} x_1' = x_2 \\ x_2' = -x_1 - 0.25x_2. \end{cases}$$

Remark 5.1. We will need

- matrices,
- eigenvalues,
- eigenvectors,
- the Wronskian,
- linear independence,
- and more

from MATH215 – please either revise your Linear Algebra lecture notes or read your Linear Algebra book or read §7.2-7.3 in the textbook by Boyce and DiPrima.

5.2 Basic Theory of Systems of First Order Linear Equations

$$\begin{cases} x_1' = p_{11}(t)x_1 + p_{12}(t)x_2 + \ldots + p_{1n}(t)x_n + g_1(t) \\ x_2' = p_{21}(t)x_1 + p_{22}(t)x_2 + \ldots + p_{2n}(t)x_n + g_2(t) \\ \quad \vdots \\ x_n' = p_{n1}(t)x_1 + p_{n2}(t)x_2 + \ldots + p_{nn}(t)x_n + g_n(t) \end{cases}$$

is a system of n linear ODEs and n variables: x_1, x_2, $\dots$, x_n.

If we write

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, \ \mathbf{x}' = \begin{bmatrix} x_1' \\ x_2' \\ \vdots \\ x_n' \end{bmatrix}, \ P = \begin{bmatrix} p_{11} & p_{12} & \dots & p_{1n} \\ p_{21} & p_{22} & \dots & p_{2n} \\ \vdots & \vdots & & \vdots \\ p_{n1} & p_{n2} & \dots & p_{nn} \end{bmatrix}, \text{ and } \mathbf{g} = \begin{bmatrix} g_1 \\ g_2 \\ \vdots \\ g_n \end{bmatrix}$$

then we can write this system as

$$\mathbf{x}' = P(t)\mathbf{x} + \mathbf{g}(t).$$

First we will consider the homogeneous system

$$\mathbf{x}' = P(t)\mathbf{x}.$$

In Chapter 3 when we had multiple solutions, we wrote them as $y_1(t)$, $y_2(t)$, $\dots$. But we are already using x_1, x_2, $\dots$ to denote coordinates. So we need a new type of notation.

Notation. We use $\mathbf{x}^{(1)}(t)$, $\mathbf{x}^{(2)}(t)$, $\dots$ to denote different vector solutions.

Recall from Chapter 3 that if $y_1(t)$ and $y_2(t)$ are both solutions to

$$ay'' + by' + cy = 0,$$

then

$$c_1y_1 + c_2y_2$$

is also a solution.

Theorem 5.1. *If $\mathbf{x}^{(1)}(t)$ and $\mathbf{x}^{(2)}(t)$ are solutions to $\mathbf{x}' = P(t)\mathbf{x}$, then*

$$c_1\mathbf{x}^{(1)} + c_2\mathbf{x}^{(2)}$$

is also a solution for any $c_1, c_2 \in \mathbb{R}$.

Example 5.2. $\mathbf{x}^{(1)} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} e^{3t}$ and $\mathbf{x}^{(2)} = \begin{bmatrix} 1 \\ -2 \end{bmatrix} e^{-t}$ are both solutions to $\mathbf{x}' = \begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix} \mathbf{x}$ (we will see this later).Therefore

$$\mathbf{x}(t) = c_1\mathbf{x}^{(1)} + c_2\mathbf{x}^{(2)} = c_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} e^{3t} + c_2 \begin{bmatrix} 1 \\ -2 \end{bmatrix} e^{-t}$$

is also a solution to this system.

(Suppose that $P(t)$ is an $n \times n$ matrix.)

Theorem 5.2. *If $\mathbf{x}^{(1)}(t)$, $\mathbf{x}^{(2)}(t)$, $\ldots$, $\mathbf{x}^{(n)}(t)$ are linearly independent solutions to $\mathbf{x}' = P(t)\mathbf{x}$, then every solution to this system can be written as*

$$\mathbf{x}(t) = c_1\mathbf{x}^{(1)} + c_2\mathbf{x}^{(2)} + \ldots + c_n\mathbf{x}^{(n)}$$

in exactly one way.

Definition. In this case, we say that $\mathbf{x}^{(1)}(t)$, $\mathbf{x}^{(2)}(t)$, $\ldots$, $\mathbf{x}^{(n)}(t)$ form a ***fundamental set of solutions*** to $\mathbf{x}' = P(t)\mathbf{x}$.

Definition. In this case,

$$\mathbf{x}(t) = c_1\mathbf{x}^{(1)} + c_2\mathbf{x}^{(2)} + \ldots + c_n\mathbf{x}^{(n)}$$

is called the ***general solution*** to $\mathbf{x}' = P(t)\mathbf{x}$.

5.3 Homogeneous Linear Systems with Constant Coefficients

Consider

$$\mathbf{x}' = A\mathbf{x}$$

where $A \in \mathbb{R}^{n\times n}$.

If $n = 1$, then we just have

$$\frac{dx}{dt} = ax$$

which has general solution $x(t) = ce^{at}$.

For $n > 1$, we guess that

$$\mathbf{x}(t) = \boldsymbol{\xi} e^{rt}$$

is a solution to $\mathbf{x}' = A\mathbf{x}$, for some number $r \in \mathbb{C}$ and some vector $\boldsymbol{\xi} \in \mathbb{C}^n$.

But if $\mathbf{x}(t) = \boldsymbol{\xi} e^{rt}$, then

$$\begin{aligned} r\boldsymbol{\xi} e^{rt} &= \mathbf{x}' = A\mathbf{x} = A\boldsymbol{\xi} e^{rt} \\ r\boldsymbol{\xi} &= A\boldsymbol{\xi} \\ (A - rI)\boldsymbol{\xi} &= \mathbf{0} \end{aligned}$$

where I is the identity matrix. Hence r must be an eigenvalue of A and $\boldsymbol{\xi}$ must be a corresponding eigenvector of A.

Remark 5.2. So the idea is:

(i). Find the eigenvalues;

(ii). Find the eigenvectors; then

(iii). Write $\mathbf{x}^{(j)}(t) = \boldsymbol{\xi}^{(j)} e^{r_j t}$.

Example 5.3. Solve
$$\mathbf{x}' = \begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix} \mathbf{x}.$$

First we find the eigenvalues. Since

$$\begin{aligned} 0 = \det(A - rI) &= \begin{vmatrix} 1-r & 1 \\ 4 & 1-r \end{vmatrix} = (1-r)^2 - 4 \\ &= r^2 - 2r - 3 = (r+1)(r-3), \end{aligned}$$

the eigenvalues are $r_1 = 3$ and $r_2 = -1$.

Using the first eigenvalue $r_1 = 3$, we calculate that

$$\mathbf{0} = (A - r_1 I)\boldsymbol{\xi} = \begin{bmatrix} -2 & 1 \\ 4 & -2 \end{bmatrix} \begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix} \qquad \Longrightarrow \qquad 0 = -2\xi_1 + \xi_2.$$

Hence we can choose $\boldsymbol{\xi}^{(1)} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$. Then using the second eigenvalue $r_2 = -1$, we calculate that

$$\mathbf{0} = (A - r_2 I)\boldsymbol{\xi} = \begin{bmatrix} 2 & 1 \\ 4 & 2 \end{bmatrix} \begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix} \qquad \Longrightarrow \qquad 0 = 2\xi_1 + \xi_2.$$

Hence we can choose $\boldsymbol{\xi}^{(2)} = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$. This gives us two solutions:

$$\mathbf{x}^{(1)}(t) = \begin{bmatrix} 1 \\ 2 \end{bmatrix} e^{3t} \qquad \text{and} \qquad \mathbf{x}^{(2)}(t) = \begin{bmatrix} 1 \\ -2 \end{bmatrix} e^{-t}.$$

But are these two solutions linearly independent? To find out, we calculate the Wronskian of $\mathbf{x}^{(1)}(t)$ and $\mathbf{x}^{(2)}(t)$:

$$\mathbf{x}^{(1)}(t) = \begin{bmatrix} 1 \\ 2 \end{bmatrix} e^{3t} = \begin{bmatrix} e^{3t} \\ 2e^{3t} \end{bmatrix} \quad \text{and} \quad \mathbf{x}^{(2)}(t) = \begin{bmatrix} 1 \\ -2 \end{bmatrix} e^{-t} = \begin{bmatrix} e^{-t} \\ -2e^{-t} \end{bmatrix}.$$

$$W\left(\mathbf{x}^{(1)}, \mathbf{x}^{(2)}\right)(t) = \begin{vmatrix} e^{3t} & e^{-t} \\ 2e^{3t} & -2e^{-t} \end{vmatrix} = -4e^{2t} \neq 0.$$

Since $W \neq 0$, we have that $\mathbf{x}^{(1)}(t)$ and $\mathbf{x}^{(2)}(t)$ are linearly independent. So $\mathbf{x}^{(1)}(t)$ and $\mathbf{x}^{(2)}(t)$ form a fundamental set of solutions. Therefore the general solution is

$$\boxed{\mathbf{x}(t) = c_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} e^{3t} + c_2 \begin{bmatrix} 1 \\ -2 \end{bmatrix} e^{-t}.}$$

Example 5.4. Solve

$$\begin{cases} \mathbf{x}' = \begin{bmatrix} 8 & -1 \\ 6 & 1 \end{bmatrix} \mathbf{x} \\ \mathbf{x}(0) = \begin{bmatrix} 1 \\ -2 \end{bmatrix}. \end{cases}$$

The eigenvalues are $r_1 = 7$ and $r_2 = 2$. The corresponding eigenvectors are $\boldsymbol{\xi}^{(1)} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\boldsymbol{\xi}^{(2)} = \begin{bmatrix} 1 \\ 6 \end{bmatrix}$. Therefore the general solution to the ODE is

$$\mathbf{x}(t) = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{7t} + c_2 \begin{bmatrix} 1 \\ 6 \end{bmatrix} e^{2t}.$$

Setting $t = 0$, we have

$$\begin{bmatrix} 1 \\ -2 \end{bmatrix} = \mathbf{x}(0) = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 6 \end{bmatrix} = \begin{bmatrix} c_1 + c_2 \\ c_1 + 6c_2 \end{bmatrix} \implies \begin{cases} c_1 + c_2 = 1 \\ c_1 + 6c_2 = -2 \end{cases} \implies \begin{cases} c_1 = \frac{8}{5} \\ c_2 = -\frac{3}{5}. \end{cases}$$

Therefore the solution to the IVP is

$$\boxed{\mathbf{x}(t) = \frac{8}{5} \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{7t} - \frac{3}{5} \begin{bmatrix} 1 \\ 6 \end{bmatrix} e^{2t}.}$$

Example 5.5. Solve

$$\mathbf{x}' = \begin{bmatrix} -3 & \sqrt{2} \\ \sqrt{2} & -2 \end{bmatrix} \mathbf{x}.$$

The eigenvalues are $r_1 = -1$ and $r_2 = -4$. The corresponding eigenvectors are $\boldsymbol{\xi}^{(1)} = \begin{bmatrix} 1 \\ \sqrt{2} \end{bmatrix}$ and $\boldsymbol{\xi}^{(2)} = \begin{bmatrix} -\sqrt{2} \\ 1 \end{bmatrix}$. Hence the general solution is

$$\boxed{\mathbf{x}(t) = c_1 \begin{bmatrix} 1 \\ \sqrt{2} \end{bmatrix} e^{-t} + c_2 \begin{bmatrix} -\sqrt{2} \\ 1 \end{bmatrix} e^{-4t}.}$$

Remark 5.3.

$$\det(A - rI) = 0$$

There are three possibilities for the eigenvalues of A.

(i). All the eigenvalues are real and different;

(ii). Some eigenvalues occur in complex conjugate pairs;

(iii). Some eigenvalues are repeated.

If all the eigenvalues are real and different, then the eigenvectors are linearly independent: So $W\big(\mathbf{x}^{(1)}, \ldots, \mathbf{x}^{(n)}\big)(t) \neq 0$ and $\mathbf{x}^{(1)}(t), \ldots, \mathbf{x}^{(n)}(t)$ form a fundamental set of solutions.

If some eigenvalues are repeated, ***but there are n linearly independent eigenvectors***, then this is also true: $\mathbf{x}^{(1)}(t), \ldots, \mathbf{x}^{(n)}(t)$ form a fundamental set of solutions.

Example 5.6. Solve

$$\mathbf{x}' = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} \mathbf{x}.$$

The eigenvalues and eigenvectors are

$$r_1 = 2 \qquad r_2 = -1 \qquad r_3 = -1$$

$$\boldsymbol{\xi}^{(1)} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \qquad \boldsymbol{\xi}^{(2)} = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \qquad \boldsymbol{\xi}^{(3)} = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$$

which gives us the following three solutions

$$\mathbf{x}^{(1)}(t) = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} e^{2t} \qquad \mathbf{x}^{(2)}(t) = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} e^{-t} \qquad \mathbf{x}^{(3)}(t) = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} e^{-t}.$$

You can check that the Wronskian of $\mathbf{x}^{(1)}$, $\mathbf{x}^{(2)}$ and $\mathbf{x}^{(3)}$ is non-zero. Therefore $\mathbf{x}^{(1)}$, $\mathbf{x}^{(2)}$ and $\mathbf{x}^{(3)}$ form a fundamental set of solutions. The general solution to the ODE is

$$\boxed{\mathbf{x}(t) = c_1 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} e^{2t} + c_2 \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} e^{-t} + c_3 \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} e^{-t}.}$$

Remark 5.4. So if we have repeated eigenvalues with n linearly independent eigenvectors then there is no problem.

In section 5.6 we will study what to do if we do not have enough eigenvectors.

Remark 5.5. Next we will study systems with complex eigenvalues.

5.4 Complex Eigenvalues

Consider
$$\mathbf{x}' = A\mathbf{x}$$
where $A \in \mathbb{R}^{n\times n}$.

Any complex eigenvalues of A must occur in complex conjugate pairs: If $r_1 = \lambda + i\mu$ is an eigenvalue of A, then $r_2 = \overline{r}_1 = \lambda - i\mu$ is also an eigenvalue of A.

Moreover, if $\boldsymbol{\xi}^{(1)}$ is an eigenvector of A corresponding to r_1, then $\boldsymbol{\xi}^{(2)} = \overline{\boldsymbol{\xi}^{(1)}}$ is an eigenvector of A corresponding to $r_2 = \overline{r}_1$.

Two solutions of $\mathbf{x}' = A\mathbf{x}$ are
$$\mathbf{x}^{(1)}(t) = \boldsymbol{\xi}^{(1)}e^{r_1 t} \qquad \text{and} \qquad \mathbf{x}^{(2)}(t) = \overline{\boldsymbol{\xi}^{(1)}}e^{\overline{r}_1 t}.$$
But $\mathbf{x}^{(1)}, \mathbf{x}^{(2)} : \mathbb{R} \to \mathbb{C}^n$ and we want solutions $: \mathbb{R} \to \mathbb{R}^n$.

If $r_1 = \lambda + i\mu$, and $\boldsymbol{\xi}^{(1)} = \mathbf{a} + i\mathbf{b}$ (λ, $\mu \in \mathbb{R}$, $\mathbf{a}$, $\mathbf{b} \in \mathbb{R}^n$), then
$$\begin{aligned}\mathbf{x}^{(1)}(t) &= (\mathbf{a} + i\mathbf{b})e^{(\lambda+i\mu)t} \\ &= (\mathbf{a} + i\mathbf{b})e^{\lambda t}(\cos\mu t + i\sin\mu t) \\ &= e^{\lambda t}(\mathbf{a}\cos\mu t - \mathbf{b}\sin\mu t) + ie^{\lambda t}(\mathbf{a}\sin\mu t + \mathbf{b}\cos\mu t) \\ &= \mathbf{u}(t) + i\mathbf{v}(t).\end{aligned}$$

Remark 5.6.
- The functions $\mathbf{u}(t)$ and $\mathbf{v}(t)$ solve the ODE.
- The functions $\mathbf{u}(t)$ and $\mathbf{v}(t)$ will be linearly independent.
- $\operatorname{span}\{\mathbf{u}(t), \mathbf{v}(t)\} = \operatorname{span}\{\mathbf{x}^{(1)}, \mathbf{x}^{(2)}\}$.

So we can include $\mathbf{u}(t)$ and $\mathbf{v}(t)$ in our fundamental set of solutions instead of $\mathbf{x}^{(1)}(t)$ and $\mathbf{x}^{(2)}(t)$.

Example 5.7. Solve
$$\mathbf{x}' = \begin{bmatrix} -\frac{1}{2} & 1 \\ -1 & -\frac{1}{2}\end{bmatrix}\mathbf{x}.$$

We calculate that
$$0 = \det(A - rI) = \begin{vmatrix} -\frac{1}{2} - r & 1 \\ -1 & -\frac{1}{2} - r\end{vmatrix} = r^2 + r + \frac{5}{4}$$
and
$$r_{1,2} = \frac{-1 \pm \sqrt{1-5}}{2} = \frac{-1 \pm 2i}{2} = -\frac{1}{2} \pm i.$$
So we have $r_1 = -\frac{1}{2} + i$ and $r_2 = -\frac{1}{2} - i$. We will use r_1. We do not need r_2.

Since
$$0 = (A - r_1 I)\boldsymbol{\xi}^{(1)} = \begin{bmatrix} -i & 1 \\ -1 & -i\end{bmatrix}\begin{bmatrix}\xi_1 \\ \xi_2\end{bmatrix} \qquad \Longrightarrow \begin{cases} -i\xi_1 + \xi_2 = 0 \\ -\xi_1 - i\xi_2 = 0\end{cases}$$
we can choose
$$\boldsymbol{\xi}^{(1)} = \begin{bmatrix} 1 \\ i\end{bmatrix}.$$

Note that we also have

$$\boldsymbol{\xi}^{(2)} = \overline{\boldsymbol{\xi}^{(1)}} = \overline{\begin{bmatrix} 1 \\ i \end{bmatrix}} = \begin{bmatrix} 1 \\ -i \end{bmatrix},$$

but we don't need $\boldsymbol{\xi}^{(2)}$.

Next we look at $\mathbf{x}^{(1)}(t)$:

$$\begin{aligned} \mathbf{x}^{(1)}(t) = \boldsymbol{\xi}^{(1)} e^{r_1 t} &= \begin{bmatrix} 1 \\ i \end{bmatrix} e^{-\frac{t}{2}} (\cos t + i \sin t) \\ &= e^{-\frac{t}{2}} \begin{bmatrix} \cos t + i \sin t \\ i \cos t - \sin t \end{bmatrix} \\ &= e^{-\frac{t}{2}} \begin{bmatrix} \cos t \\ -\sin t \end{bmatrix} + i e^{-\frac{t}{2}} \begin{bmatrix} \sin t \\ \cos t \end{bmatrix} \\ &= \mathbf{u}(t) + i\mathbf{v}(t). \end{aligned}$$

Hence we choose

$$\mathbf{u}(t) = e^{-\frac{t}{2}} \begin{bmatrix} \cos t \\ -\sin t \end{bmatrix} \quad \text{and} \quad \mathbf{v}(t) = e^{-\frac{t}{2}} \begin{bmatrix} \sin t \\ \cos t \end{bmatrix}.$$

But are $\mathbf{u}(t)$ and $\mathbf{v}(t)$ linearly independent? Since

$$\begin{aligned} W\big(\mathbf{u}(t), \mathbf{v}(t)\big)(t) = \begin{vmatrix} u_1 & v_1 \\ u_2 & v_2 \end{vmatrix} &= \begin{vmatrix} e^{-\frac{t}{2}} \cos t & e^{-\frac{t}{2}} \sin t \\ -e^{-\frac{t}{2}} \sin t & e^{-\frac{t}{2}} \cos t \end{vmatrix} \\ &= e^{-t} \cos^2 t + e^{-t} \sin^2 t = e^{-t} \\ &\neq 0 \end{aligned}$$

the answer is yes. Therefore $\mathbf{u}(t)$ and $\mathbf{v}(t)$ form a fundamental set of solutions.

Therefore the general solution to $\mathbf{x}' = \begin{bmatrix} -\frac{1}{2} & 1 \\ -1 & -\frac{1}{2} \end{bmatrix} \mathbf{x}$ is

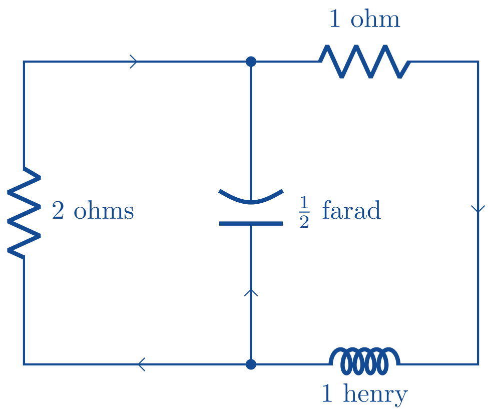
$$\boxed{\mathbf{x}(t) = c_1 e^{-\frac{t}{2}} \begin{bmatrix} \cos t \\ -\sin t \end{bmatrix} + c_2 e^{-\frac{t}{2}} \begin{bmatrix} \sin t \\ \cos t \end{bmatrix}.}$$

Remark 5.7. Our method is

1. Find the eigenvalues;
2. Find the eigenvectors;
3. - If r_j is real, just use the solution $\mathbf{x}^{(j)}(t) = \boldsymbol{\xi}^{(j)} e^{r_j t}$;
   - But if r_j is complex, write

     $$\mathbf{x}^{(j)}(t) = \boldsymbol{\xi}^{(j)} e^{r_j t} = \begin{pmatrix} \text{real valued} \\ \text{function} \end{pmatrix} + i \begin{pmatrix} \text{real valued} \\ \text{function} \end{pmatrix}$$

     and use these two functions.



Example 5.8. The electric circuit shown above is described by

$$\begin{cases} I' = -I - V \\ V' = 2I - V \end{cases}$$

where

$$\begin{aligned} I &= \text{the current through the inductor} \\ V &= \text{the voltage drop across the capacitor.} \end{aligned}$$

(Ask an Electrical Engineer.)

Suppose that at time $t = 0$ the current is 2 amperes and the voltage drop is 2 volts. Find $I(t)$ and $V(t)$.

We must solve the IVP

$$\begin{cases} \dfrac{d}{dt}\begin{bmatrix} I \\ V \end{bmatrix} = \begin{bmatrix} -1 & -1 \\ 2 & -1 \end{bmatrix}\begin{bmatrix} I \\ V \end{bmatrix} \\ \begin{bmatrix} I \\ V \end{bmatrix}(0) = \begin{bmatrix} 2 \\ 2 \end{bmatrix}. \end{cases}$$

The eigenvalues of $\begin{bmatrix} -1 & -1 \\ 2 & -1 \end{bmatrix}$ are $r_1 = -1 + i\sqrt{2}$ and $r_2 = -1 - i\sqrt{2}$ (please check). The corresponding eigenvectors are

$$\boldsymbol{\xi}^{(1)} = \begin{bmatrix} 1 \\ -i\sqrt{2} \end{bmatrix} \qquad \text{and} \qquad \boldsymbol{\xi}^{(2)} = \begin{bmatrix} 1 \\ i\sqrt{2} \end{bmatrix}.$$

Then we calculate that

$$\begin{aligned} \mathbf{x}^{(1)}(t) = \boldsymbol{\xi}^{(1)} e^{r_1 t} &= \begin{bmatrix} 1 \\ -i\sqrt{2} \end{bmatrix} e^{(-1+i\sqrt{2})t} \\ &= \begin{bmatrix} 1 \\ -i\sqrt{2} \end{bmatrix} e^{-t}\big(\cos\sqrt{2}t + i\sin\sqrt{2}t\big) \\ &= e^{-t}\begin{bmatrix} \cos\sqrt{2}t + i\sin\sqrt{2}t \\ -i\sqrt{2}\cos\sqrt{2}t + \sqrt{2}\sin\sqrt{2}t \end{bmatrix} \\ &= e^{-t}\begin{bmatrix} \cos\sqrt{2}t \\ \sqrt{2}\sin\sqrt{2}t \end{bmatrix} + ie^{-t}\begin{bmatrix} \sin\sqrt{2}t \\ -\sqrt{2}\cos\sqrt{2}t \end{bmatrix}. \end{aligned}$$

Hence the general solution to the ODE is

$$\begin{bmatrix} I(t) \\ V(t) \end{bmatrix} = c_1 e^{-t} \begin{bmatrix} \cos\sqrt{2}t \\ \sqrt{2}\sin\sqrt{2}t \end{bmatrix} + c_2 e^{-t} \begin{bmatrix} \sin\sqrt{2}t \\ -\sqrt{2}\cos\sqrt{2}t \end{bmatrix}.$$

Using the initial condition, we calculate that

$$\begin{bmatrix} 2 \\ 2 \end{bmatrix} = \begin{bmatrix} I(0) \\ V(0) \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ -\sqrt{2} \end{bmatrix} \qquad \Longrightarrow \qquad \begin{cases} c_1 = 2 \\ c_2 = -\sqrt{2}. \end{cases}$$

Thus

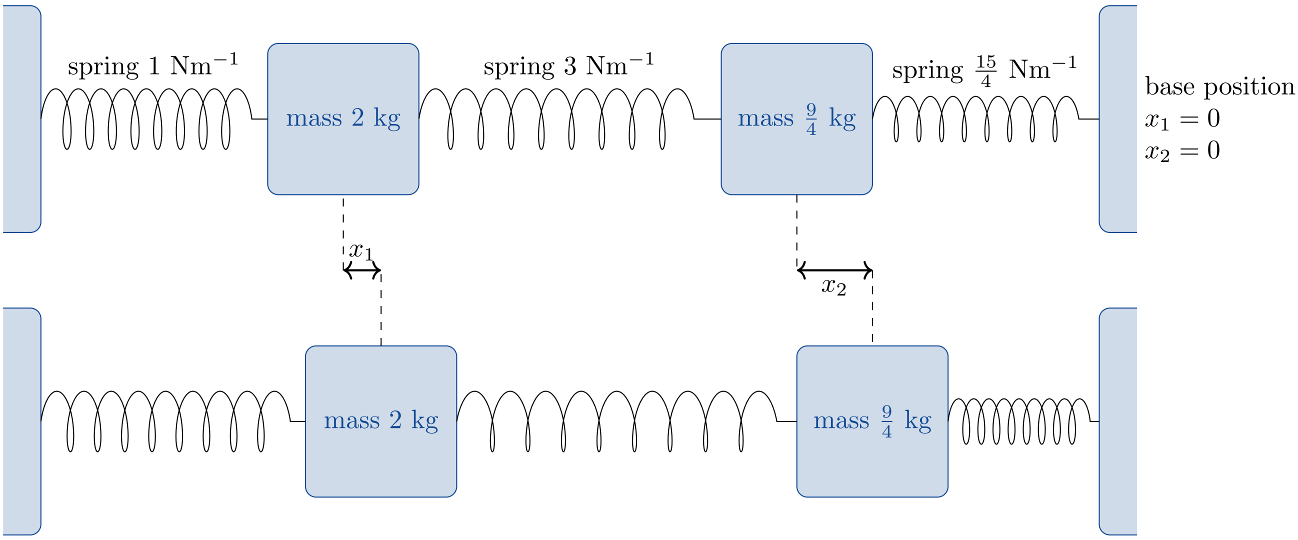
$$\begin{bmatrix} I(t) \\ V(t) \end{bmatrix} = 2\, e^{-t} \begin{bmatrix} \cos\sqrt{2}t \\ \sqrt{2}\sin\sqrt{2}t \end{bmatrix} - \sqrt{2}\, e^{-t} \begin{bmatrix} \sin\sqrt{2}t \\ -\sqrt{2}\cos\sqrt{2}t \end{bmatrix}.$$

So the answers to this problem are

$$\boxed{I(t) = 2e^{-t}\cos\sqrt{2}t - \sqrt{2}e^{-t}\sin\sqrt{2}t}$$

and

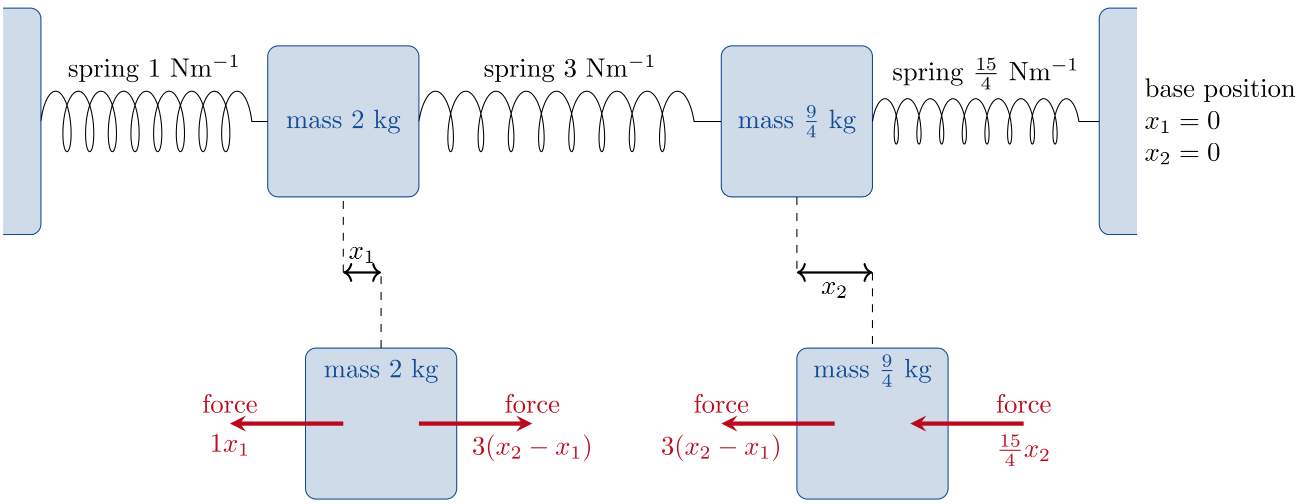
$$\boxed{V(t) = 2\sqrt{2}e^{-t}\sin\sqrt{2}t + 2e^{-t}\cos\sqrt{2}t.}$$



See https://tinyurl.com/wm2ogdh for an animated figure.

Example 5.9. For the dynamical system shown above, find $x_1(t)$ and $x_2(t)$.

As the springs are stretched and compressed, they apply forces on the blocks as shown below (Hooke's Law).



We calculate that

$$2\frac{d^2x_1}{dt^2} = \text{mass} \times \text{acceleration} = \text{force} = -x_1 + 3(x_2 - x_1)$$

$$\frac{9}{4}\frac{d^2x_2}{dt^2} = \text{mass} \times \text{acceleration} = \text{force} = -3(x_2 - x_1) - \frac{15}{4}x_2.$$

This is a system of 2 second order ODEs. We want a system of first order ODEs. Now let $y_1 = x_1$, $y_2 = x_2$, $y_3 = x_1'$ and $y_4 = x_2'$. Then

$$\begin{aligned}
y_1' &= x_1' = y_3 \\
y_2' &= x_2' = y_4 \\
y_3' &= x_1'' = \frac{1}{2}\Big(- x_1 + 3x_2 - 3x_1\Big) = -2y_1 + \frac{3}{2}y_2 \\
y_4' &= x_2'' = \frac{4}{9}\left(-3x_2 + 3x_1 - \frac{15}{4}x_2\right) = \frac{4}{3}y_1 - 3y_2.
\end{aligned}$$

So

$$\mathbf{y}' = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -2 & \frac{3}{2} & 0 & 0 \\ \frac{4}{3} & -3 & 0 & 0 \end{bmatrix} \mathbf{y}.$$

The characteristic polynomial of this matrix is

$$0 = r^4 + 5r^2 + 4 = (r^2+1)(r^2+4).$$

So $r_1 = i$, $r_2 = -i$, $r_3 = 2i$ and $r_4 = -2i$. We will use r_1 and r_3 (we do not need r_2 and r_4). The corresponding eigenvectors (please check) are

$$\boldsymbol{\xi}^{(1)} = \begin{bmatrix} 3 \\ 2 \\ 3i \\ 2i \end{bmatrix} \qquad \text{and} \qquad \boldsymbol{\xi}^{(3)} = \begin{bmatrix} 3 \\ -4 \\ 6i \\ -8i \end{bmatrix}.$$

It follows that

$$\boldsymbol{\xi}^{(1)} e^{r_1 t} = \begin{bmatrix} 3 \\ 2 \\ 3i \\ 2i \end{bmatrix} (\cos t + i \sin t) = \begin{bmatrix} 3\cos t \\ 2\cos t \\ -3\sin t \\ -2\sin t \end{bmatrix} + i \begin{bmatrix} 3\sin t \\ 2\sin t \\ 3\cos t \\ 2\cos t \end{bmatrix} = \mathbf{u}(t) + i\mathbf{v}(t)$$

and

$$\boldsymbol{\xi}^{(3)} e^{r_3 t} = \begin{bmatrix} 3 \\ -4 \\ 6i \\ -8i \end{bmatrix} (\cos 2t + i \sin 2t) = \begin{bmatrix} 3\cos 2t \\ -4\cos 2t \\ -6\sin 2t \\ 8\sin 2t \end{bmatrix} + i \begin{bmatrix} 3\sin 2t \\ -4\sin 2t \\ 6\cos 2t \\ -8\cos 2t \end{bmatrix} = \mathbf{w}(t) + i\mathbf{z}(t)$$

Therefore the general solution is

$$\begin{aligned} \mathbf{y}(t) &= c_1\mathbf{u}(t) + c_2\mathbf{v}(t) + c_3\mathbf{w}(t) + c_4\mathbf{z}(t) \\ &= c_1 \begin{bmatrix} 3\cos t \\ 2\cos t \\ -3\sin t \\ -2\sin t \end{bmatrix} + c_2 \begin{bmatrix} 3\sin t \\ 2\sin t \\ 3\cos t \\ 2\cos t \end{bmatrix} + c_3 \begin{bmatrix} 3\cos 2t \\ -4\cos 2t \\ -6\sin 2t \\ 8\sin 2t \end{bmatrix} + c_4 \begin{bmatrix} 3\sin 2t \\ -4\sin 2t \\ 6\cos 2t \\ -8\cos 2t \end{bmatrix}. \end{aligned}$$

Example 5.10. Suppose that the above system has initial condition

$$\mathbf{y}(0) = \begin{bmatrix} -1 \\ 4 \\ 1 \\ 1 \end{bmatrix}.$$

Sketch graphs of $y_1(t)$ and $y_2(t)$.

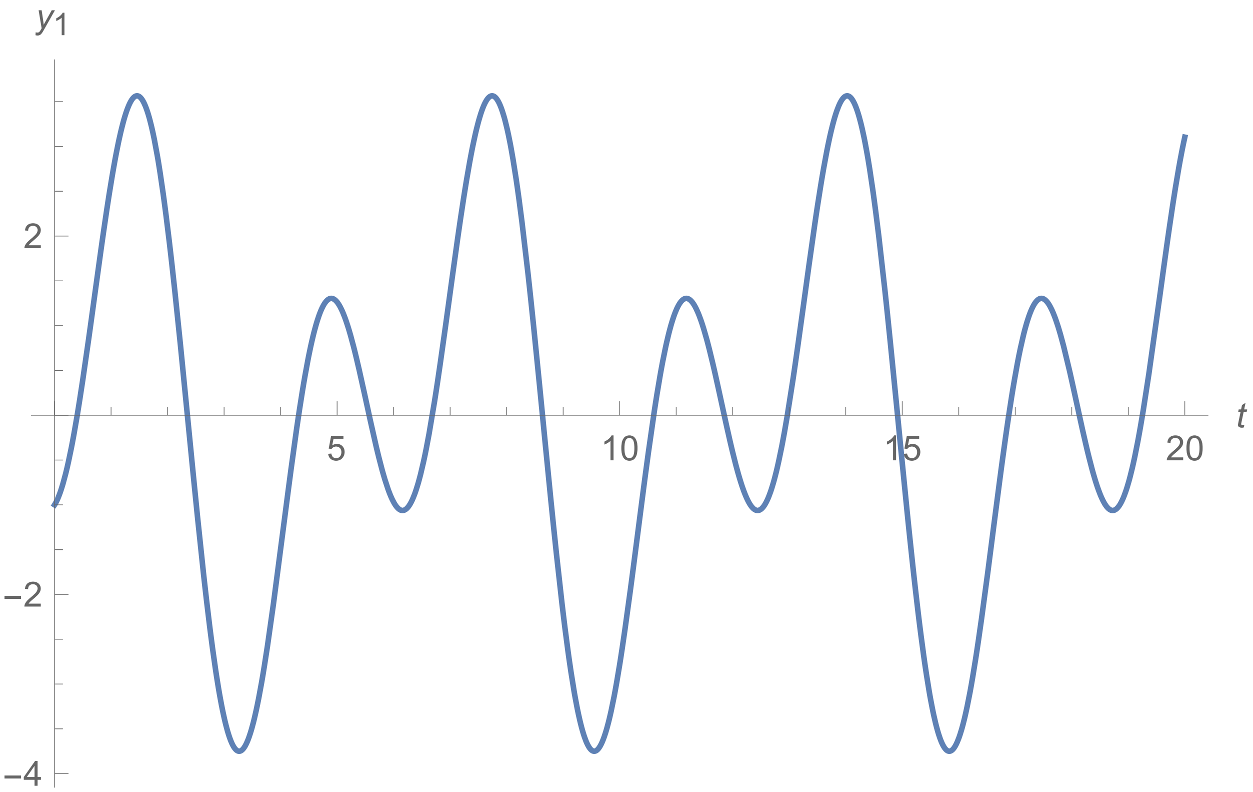
The initial value problem

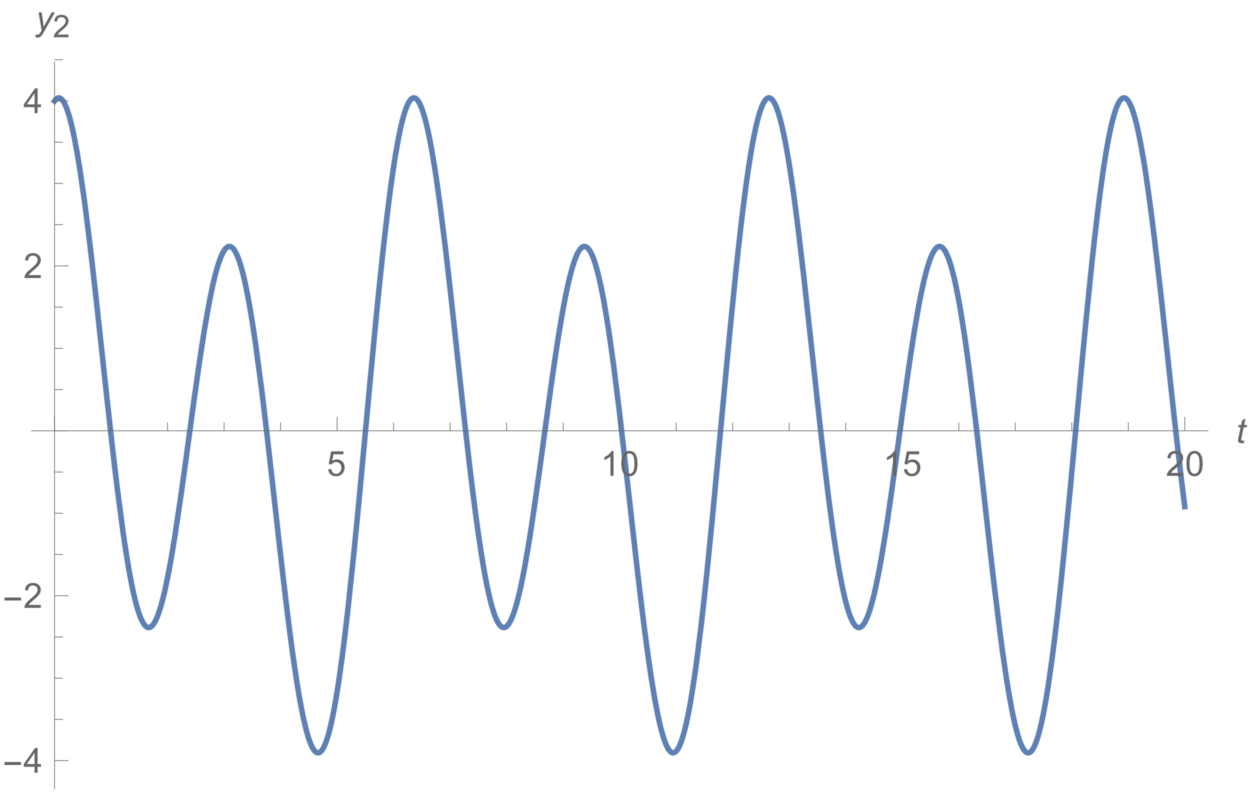
$$\mathbf{y}' = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -2 & \frac{3}{2} & 0 & 0 \\ \frac{4}{3} & -3 & 0 & 0 \end{bmatrix} \mathbf{y}, \qquad \mathbf{y}(0) = \begin{bmatrix} -1 \\ 4 \\ 1 \\ 1 \end{bmatrix}$$

has solution

$$\mathbf{y}(t) = \frac{4}{9}\begin{bmatrix} 3\cos t \\ 2\cos t \\ -3\sin t \\ -2\sin t \end{bmatrix} + \frac{7}{18}\begin{bmatrix} 3\sin t \\ 2\sin t \\ 3\cos t \\ 2\cos t \end{bmatrix} - \frac{7}{9}\begin{bmatrix} 3\cos 2t \\ -4\cos 2t \\ -6\sin 2t \\ 8\sin 2t \end{bmatrix} - \frac{1}{36}\begin{bmatrix} 3\sin 2t \\ -4\sin 2t \\ 6\cos 2t \\ -8\cos 2t \end{bmatrix}.$$

Then we can draw the graphs of y_1 and y_2:





Please see https://tinyurl.com/s7uww7m

5.5 Fundamental Matrices

Now consider

$$\mathbf{x}' = P(t)\mathbf{x}$$

where $P(t)$ is an $n \times n$ matrix function.

Suppose that $\mathbf{x}^{(1)}, \mathbf{x}^{(2)}, \ldots, \mathbf{x}^{(n)}$ are linearly independent solutions to this ODE. In other words, suppose that $\mathbf{x}^{(1)}, \mathbf{x}^{(2)}, \ldots, \mathbf{x}^{(n)}$ form a ***fundamental set of solutions*** to this ODE.

Definition. The matrix

$$\Psi(t) = \begin{bmatrix} \mathbf{x}^{(1)} & \mathbf{x}^{(2)} & \ldots & \mathbf{x}^{(n)} \end{bmatrix} = \begin{bmatrix} x_1^{(1)}(t) & x_1^{(2)}(t) & \ldots & x_1^{(n)}(t) \\ x_2^{(1)}(t) & x_2^{(2)}(t) & \ldots & x_2^{(n)}(t) \\ \vdots & \vdots & & \vdots \\ x_n^{(1)}(t) & x_n^{(2)}(t) & \ldots & x_n^{(n)}(t) \end{bmatrix}$$

is called a ***fundamental matrix*** of $\mathbf{x}' = P(t)\mathbf{x}$.

Example 5.11. Find a fundamental matrix for

$$\mathbf{x}' = \begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix} \mathbf{x}.$$

Recall that

$$\mathbf{x}^{(1)}(t) = \begin{bmatrix} 1 \\ 2 \end{bmatrix} e^{3t} \qquad \text{and} \qquad \mathbf{x}^{(2)}(t) = \begin{bmatrix} 1 \\ -2 \end{bmatrix} e^{-t}$$

form a fundamental set of solutions to this ODE. Therefore

$$\Psi(t) = \begin{bmatrix} e^{3t} & e^{-t} \\ 2e^{3t} & -2e^{-t} \end{bmatrix}$$

is a fundamental matrix of this ODE.

Now, the general solution to

$$\mathbf{x}' = A\mathbf{x}$$

is

$$\mathbf{x}(t) = c_1\mathbf{x}^{(1)}(t) + c_2\mathbf{x}^{(2)}(t) + \ldots + c_n\mathbf{x}^{(n)}(t) = \Psi(t)\mathbf{c}$$

where

$$\mathbf{c} = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} \in \mathbb{R}^n.$$

If we have an initial condition $\mathbf{x}(t_0) = \mathbf{x}^0$, then

$$\Psi(t_0)\mathbf{c} = \mathbf{x}^0.$$

But

$$\begin{array}{c}\mathbf{x}^{(1)}, \ldots, \mathbf{x}^{(n)} \\ \text{are linearly} \\ \text{independent}\end{array} \implies \Psi(t) \text{ is invertible} \implies \mathbf{c} = \Psi^{-1}(t_0)\mathbf{x}^0.$$

Therefore the solution to the IVP

$$\begin{cases} \mathbf{x}' = A\mathbf{x} \\ \mathbf{x}(t_0) = \mathbf{x}^0 \end{cases}$$

is

$$\boxed{\mathbf{x}(t) = \Psi(t)\Psi^{-1}(t_0)\mathbf{x}^0.}$$

Theorem 5.3. *Suppose that $\Psi(t)$ is a fundamental matrix for $\mathbf{x}' = P(t)\mathbf{x}$. Then $\Psi(t)$ solves the differential equation $\Psi' = P(t)\Psi$.*

(You prove)

Remark 5.8. It is possible to find a ***special fundamental matrix***, $\Phi(t)$, which satisfies

$$\Phi(t_0) = \begin{bmatrix} 1 & 0 & \ldots & 0 \\ 0 & 1 & \ldots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \ldots & 1 \end{bmatrix} = I.$$

We will use Φ for this special fundamental matrix, and Ψ for any fundamental matrix.

Example 5.12. Consider

$$\mathbf{x}' = \begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix} \mathbf{x}.$$

Find the special fundamental matrix which satisfies $\Phi(0) = I$.

To find the matrix Φ which satisfies

$$\Phi(0) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

we must solve two IVPs:

$$\begin{cases} \mathbf{x}' = \begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix} \mathbf{x} \\ \mathbf{x}(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \end{cases} \quad \text{and} \quad \begin{cases} \mathbf{x}' = \begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix} \mathbf{x} \\ \mathbf{x}(0) = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \end{cases}.$$

The general solution to the ODE is

$$\mathbf{x}(t) = c_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} e^{3t} + c_2 \begin{bmatrix} 1 \\ -2 \end{bmatrix} e^{-t}.$$

We calculate that

$$\begin{bmatrix}1\\0\end{bmatrix} = \mathbf{x}(0) = c_1\begin{bmatrix}1\\2\end{bmatrix} + c_2\begin{bmatrix}1\\-2\end{bmatrix} \implies \begin{aligned} c_1 &= \frac{1}{2} \\ c_2 &= \frac{1}{2}\end{aligned} \implies \mathbf{x}(t) = \begin{bmatrix}\frac{1}{2}e^{3t} + \frac{1}{2}e^{-t} \\ e^{3t} - e^{-t}\end{bmatrix}$$

and

$$\begin{bmatrix}0\\1\end{bmatrix} = \mathbf{x}(0) = c_1\begin{bmatrix}1\\2\end{bmatrix} + c_2\begin{bmatrix}1\\-2\end{bmatrix} \implies \begin{aligned} c_1 &= \frac{1}{4} \\ c_2 &= -\frac{1}{4}\end{aligned} \implies \mathbf{x}(t) = \begin{bmatrix}\frac{1}{4}e^{3t} - \frac{1}{4}e^{-t} \\ \frac{1}{2}e^{3t} + \frac{1}{2}e^{-t}\end{bmatrix}.$$

Therefore the special fundamental matrix is

$$\Phi(t) = \begin{bmatrix}\frac{1}{2}e^{3t} + \frac{1}{2}e^{-t} & \frac{1}{4}e^{3t} - \frac{1}{4}e^{-t} \\ e^{3t} - e^{-t} & \frac{1}{2}e^{3t} + \frac{1}{2}e^{-t}\end{bmatrix}.$$

What is e^{At}?

Recall that the solution to

$$\begin{cases} x' = ax \ (a \in \mathbb{R}) \\ x(0) = x_0 \end{cases}$$

is

$$x(t) = x_0 e^{at} = x_0 \exp(at)$$

and recall that

$$\exp(at) = 1 + \sum_{n=1}^{\infty} \frac{a^n t^n}{n!}.$$

Now consider

$$\begin{cases} \mathbf{x}' = A\mathbf{x} \\ \mathbf{x}(0) = \mathbf{x}^0 \end{cases}$$

for $A \in \mathbb{R}^{n \times n}$.

Definition.

$$\exp(At) = I + \sum_{n=1}^{\infty} \frac{A^n t^n}{n!} = I + At + \frac{A^2 t^2}{2!} + \frac{A^3 t^3}{3!} + \dots.$$

Note that

$$\begin{aligned} \frac{d}{dt}\exp(At) &= \frac{d}{dt}\left(I + \sum_{n=1}^{\infty} \frac{A^n t^n}{n!}\right) = \mathbf{0} + \sum_{n=1}^{\infty} \frac{d}{dt}\left(\frac{A^n t^n}{n!}\right) \\ &= \sum_{n=1}^{\infty} \frac{A^n t^{n-1}}{(n-1)!} = A + \sum_{n=2}^{\infty} \frac{A^n t^{n-1}}{(n-1)!} \\ &= A + \sum_{k=1}^{\infty} \frac{A^{k+1} t^k}{k!} \qquad\qquad (k = n-1) \\ &= A\left(I + \sum_{k=1}^{\infty} \frac{A^k t^k}{k!}\right) = A\exp(At). \end{aligned}$$

This means that $\exp(At)$ solves

$$\begin{cases} \big(\exp(At)\big)' = A\exp(At) \\ \exp(At)|_{t=0} = I. \end{cases}$$

But remember that Φ solves

$$\begin{cases} \Phi' = A\Phi \\ \Phi(0) = I. \end{cases}$$

Therefore

$$\boxed{\Phi(t) = \exp(At).}$$

Example 5.13. Let $A = \begin{bmatrix} 8 & -1 \\ 6 & 1 \end{bmatrix}$. Find $\exp(At)$.

We have previously found that the general solution to $\mathbf{x}' = A\mathbf{x}$ is

$$\mathbf{x}(t) = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{7t} + c_2 \begin{bmatrix} 1 \\ 6 \end{bmatrix} e^{2t}.$$

To satisfy $\mathbf{x}(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ we require $c_1 = \frac{6}{5}$ and $c_2 = -\frac{1}{5}$. Hence

$$\mathbf{x}(t) = \frac{6}{5} \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{7t} - \frac{1}{5} \begin{bmatrix} 1 \\ 6 \end{bmatrix} e^{2t} = \begin{bmatrix} \frac{6}{5}e^{7t} - \frac{1}{5}e^{2t} \\ \frac{6}{5}e^{7t} - \frac{6}{5}e^{2t} \end{bmatrix}.$$

To satisfy $\mathbf{x}(0) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ we require $c_1 = -\frac{1}{5}$ and $c_2 = \frac{1}{5}$. Hence

$$\mathbf{x}(t) = -\frac{1}{5} \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{7t} + \frac{1}{5} \begin{bmatrix} 1 \\ 6 \end{bmatrix} e^{2t} = \begin{bmatrix} -\frac{1}{5}e^{7t} + \frac{1}{5}e^{2t} \\ -\frac{1}{5}e^{7t} + \frac{6}{5}e^{2t} \end{bmatrix}.$$

Therefore the answer is

$$\boxed{\exp(At) = \Phi(t) = \begin{bmatrix} \frac{6}{5}e^{7t} - \frac{1}{5}e^{2t} & -\frac{1}{5}e^{7t} + \frac{1}{5}e^{2t} \\ \frac{6}{5}e^{7t} - \frac{6}{5}e^{2t} & -\frac{1}{5}e^{7t} + \frac{6}{5}e^{2t} \end{bmatrix}.}$$

Diagonalisable Matrices

If

$$D = \begin{bmatrix} r_1 & 0 & 0 & \dots & 0 \\ 0 & r_2 & 0 & \dots & 0 \\ 0 & 0 & r_3 & \dots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \dots & r_n \end{bmatrix}$$

is a diagonal matrix, then it is easy to calculate $\exp(Dt)$. We simply have

$$\exp(Dt) = \begin{bmatrix} e^{r_1 t} & 0 & 0 & \dots & 0 \\ 0 & e^{r_2 t} & 0 & \dots & 0 \\ 0 & 0 & e^{r_3 t} & \dots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \dots & e^{r_n t} \end{bmatrix}.$$

Now consider

$$\mathbf{x}' = A\mathbf{x}$$

for $A \in \mathbb{R}^{n\times n}$. Recall how we diagonalise a matrix: If $\boldsymbol{\xi}^{(1)}, \boldsymbol{\xi}^{(2)}, \dots, \boldsymbol{\xi}^{(n)}$ are the eigenvectors of A, we let

$$T = \begin{bmatrix} \boldsymbol{\xi}^{(1)} & \boldsymbol{\xi}^{(2)} & \dots & \boldsymbol{\xi}^{(n)} \end{bmatrix}.$$

Then

$$\det(T) \neq 0 \implies T^{-1} \text{ exists} \implies \begin{matrix} T^{-1}AT \\ \text{is diagonal} \end{matrix} \implies \begin{matrix} A \text{ is} \\ \text{diagonalisable.} \end{matrix}$$

Example 5.14. Diagonalise

$$A = \begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix}.$$

The eigenvalues are $r_1 = 3$ and $r_2 = -1$. The corresponding eigenvectors are $\boldsymbol{\xi}^{(1)} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $\boldsymbol{\xi}^{(2)} = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$. Thus

$$T = \begin{bmatrix} 1 & 1 \\ 2 & -2 \end{bmatrix} \qquad \text{and} \qquad T^{-1} = \begin{bmatrix} \frac{1}{2} & \frac{1}{4} \\ \frac{1}{2} & -\frac{1}{4} \end{bmatrix}.$$

It follows that

$$D = T^{-1}AT = \begin{bmatrix} 3 & 0 \\ 0 & -1 \end{bmatrix}.$$

Now consider

$$\mathbf{x}' = A\mathbf{x}.$$

Define a new variable $\mathbf{y}$ by

$$\mathbf{x} = T\mathbf{y} \qquad \text{or} \qquad \mathbf{y} = T^{-1}\mathbf{x}.$$

Then we calculate that

$$\begin{aligned} \mathbf{x}' &= A\mathbf{x} \\ T\mathbf{y}' &= AT\mathbf{y} \\ \mathbf{y}' &= T^{-1}AT\mathbf{y} = D\mathbf{y}. \end{aligned}$$

We know that a fundamental matrix for $\mathbf{y}' = D\mathbf{y}$ is

$$\exp(Dt) = \begin{bmatrix} e^{r_1 t} & 0 & \dots & 0 \\ 0 & e^{r_2 t} & \dots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \dots & e^{r_n t} \end{bmatrix}.$$

Therefore a fundamental matrix for $\mathbf{x}' = A\mathbf{x}$ is

$$\Psi = T\exp(Dt) = \begin{bmatrix} \boldsymbol{\xi}^{(1)} e^{r_1 t} & \boldsymbol{\xi}^{(2)} e^{r_2 t} & \dots & \boldsymbol{\xi}^{(n)} e^{r_n t} \end{bmatrix}.$$

Example 5.15. Find a fundamental matrix for

$$\mathbf{x}' = \begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix}\mathbf{x}.$$

Recall that $T = \begin{bmatrix} 1 & 1 \\ 2 & -2 \end{bmatrix}$. Letting $\mathbf{y} = T^{-1}\mathbf{x}$, we have

$$\mathbf{y}' = \begin{bmatrix} 3 & 0 \\ 0 & -1 \end{bmatrix}\mathbf{y}.$$

A fundamental matrix for $\mathbf{y}' = \begin{bmatrix} 3 & 0 \\ 0 & -1 \end{bmatrix}\mathbf{y}$ is

$$\exp(Dt) = e^{Dt} = \begin{bmatrix} e^{3t} & 0 \\ 0 & e^{-t} \end{bmatrix}.$$

Hence a fundamental matrix for $\mathbf{x}' = \begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix}\mathbf{x}$ is

$$\Psi(t) = T\exp(Dt) = \begin{bmatrix} 1 & 1 \\ 2 & -2 \end{bmatrix}\begin{bmatrix} e^{3t} & 0 \\ 0 & e^{-t} \end{bmatrix} = \begin{bmatrix} e^{3t} & e^{-t} \\ 2e^{3t} & -2e^{-t} \end{bmatrix}.$$

5.6 Repeated Eigenvalues

Example 5.16. Find the eigenvalues and eigenvectors of $A = \begin{bmatrix} 1 & -1 \\ 1 & 3 \end{bmatrix}$.

We calculate that

$$0 = \det(A - rI) = \begin{vmatrix} 1-r & -1 \\ 1 & 3-r \end{vmatrix} = r^2 - 4r + 4 = (r-2)^2.$$

Therefore $r_1 = 2 = r_2$. Moreover

$$\mathbf{0} = (A - rI)\boldsymbol{\xi} = \begin{bmatrix} -1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix} \implies \xi_1 + \xi_2 = 0 \implies \boldsymbol{\xi} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

Note that A has only one linearly independent eigenvector.

Example 5.17. Solve

$$\mathbf{x}' = \begin{bmatrix} 1 & -1 \\ 1 & 3 \end{bmatrix} \mathbf{x}.$$

We know that

$$\mathbf{x}^{(1)}(t) = \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{2t}$$

is a solution. But we need two solutions.

Guess 1: I guess that

$$\mathbf{x}^{(2)}(t) = \boldsymbol{\xi} t e^{2t}$$

for some $\boldsymbol{\xi} \in \mathbb{R}^2$. Then we have

$$\begin{aligned} \boldsymbol{\xi} e^{2t} + 2\boldsymbol{\xi} t e^{2t} = \mathbf{x}^{(2)\prime} = A\mathbf{x}^{(2)} &= A\boldsymbol{\xi} t e^{2t} \\ \boldsymbol{\xi} + (2\boldsymbol{\xi} - A\boldsymbol{\xi})\, t = \mathbf{0} & \qquad \forall t \\ &\implies \boldsymbol{\xi} = \mathbf{0}. \end{aligned}$$

This guess did not work.

Guess 2: Now I guess that

$$\mathbf{x}^{(2)}(t) = \boldsymbol{\xi} t e^{2t} + \boldsymbol{\eta} e^{2t}$$

for some $\boldsymbol{\xi}, \boldsymbol{\eta} \in \mathbb{R}^2$. Then we have

$$\boldsymbol{\xi} e^{2t} + 2\boldsymbol{\xi} t e^{2t} + 2\boldsymbol{\eta} e^{2t} = \mathbf{x}^{(2)\prime} = A\mathbf{x}^{(2)} = A\left(\boldsymbol{\xi} t e^{2t} + \boldsymbol{\eta} e^{2t}\right)$$

and

$$(2\boldsymbol{\xi} - A\boldsymbol{\xi})\, t + (\boldsymbol{\xi} + 2\boldsymbol{\eta} - A\boldsymbol{\eta}) = \mathbf{0}.$$

Since this must be true $\forall\, t$, we must have

$$(A - 2I)\,\boldsymbol{\xi} = \mathbf{0} \qquad \text{and} \qquad (A - 2I)\,\boldsymbol{\eta} = \boldsymbol{\xi}.$$

Clearly $\boldsymbol{\xi} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$. Then we calculate that

$$\begin{aligned} \begin{bmatrix} -1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \eta_1 \\ \eta_2 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \quad &\Longrightarrow \quad \eta_1 + \eta_2 = -1 \\ &\Longrightarrow \quad \boldsymbol{\eta} = \begin{bmatrix} k \\ -1-k \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \end{bmatrix} + k \begin{bmatrix} 1 \\ -1 \end{bmatrix} \end{aligned}$$

for some k. So

$$\begin{aligned} \mathbf{x}^{(2)}(t) = \boldsymbol{\xi} te^{2t} + \boldsymbol{\eta} e^{2t} &= \begin{bmatrix} 1 \\ -1 \end{bmatrix} te^{2t} + \begin{bmatrix} 0 \\ -1 \end{bmatrix} e^{2t} + k \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{2t} \\ &= \begin{bmatrix} 1 \\ -1 \end{bmatrix} te^{2t} + \begin{bmatrix} 0 \\ -1 \end{bmatrix} e^{2t} + k\mathbf{x}^{(1)}(t). \end{aligned}$$

Because we already have $\mathbf{x}^{(1)}(t)$, we can choose $k = 0$. So

$$\mathbf{x}^{(2)}(t) = \boldsymbol{\xi} te^{2t} + \boldsymbol{\eta} e^{2t} = \begin{bmatrix} 1 \\ -1 \end{bmatrix} te^{2t} + \begin{bmatrix} 0 \\ -1 \end{bmatrix} e^{2t}.$$

The general solution of $\mathbf{x}' = \begin{bmatrix} 1 & -1 \\ 1 & 3 \end{bmatrix} \mathbf{x}$ is therefore

$$\boxed{\mathbf{x}(t) = c_1\mathbf{x}^{(1)} + c_2\mathbf{x}^{(2)} = c_1 \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{2t} + c_2 \left(\begin{bmatrix} 1 \\ -1 \end{bmatrix} te^{2t} + \begin{bmatrix} 0 \\ -1 \end{bmatrix} e^{2t} \right).}$$

Example 5.18. Find a fundamental matrix for

$$\mathbf{x}' = \begin{bmatrix} 1 & -1 \\ 1 & 3 \end{bmatrix} \mathbf{x}.$$

Then find the special fundamental matrix $\Phi(t)$ which satisfies $\Phi(0) = I$.

Since $\mathbf{x}^{(1)}(t) = \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{2t}$ and $\mathbf{x}^{(2)}(t) = \begin{bmatrix} 1 \\ -1 \end{bmatrix} te^{2t} + \begin{bmatrix} 0 \\ -1 \end{bmatrix} e^{2t}$ we have that

$$\Psi(t) = \begin{bmatrix} \mathbf{x}^{(1)} & \mathbf{x}^{(2)} \end{bmatrix} = \begin{bmatrix} e^{2t} & te^{2t} \\ -e^{2t} & -te^{2t} - e^{2t} \end{bmatrix}$$

is a fundamental matrix for this system.

Now

$$\Psi(0) = \begin{bmatrix} 1 & 0 \\ -1 & -1 \end{bmatrix} \qquad \text{and} \qquad \Psi^{-1}(0) = \begin{bmatrix} 1 & 0 \\ -1 & -1 \end{bmatrix}.$$

Therefore

$$\begin{aligned} \exp(At) = \Phi(t) = \Psi(t)\Psi^{-1}(0) &= e^{2t} \begin{bmatrix} 1 & t \\ -1 & -1-t \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -1 & -1 \end{bmatrix} \\ &= e^{2t} \begin{bmatrix} 1-t & -t \\ t & 1+t \end{bmatrix}. \end{aligned}$$

Remark 5.9.

$$\mathbf{x}' = A\mathbf{x}$$

For two repeated eigenvalues (but with only one linearly independent eigenvector), the key equations to remember are

$$\boxed{\mathbf{x}^{(2)}(t) = \boldsymbol{\xi} t e^{rt} + \boldsymbol{\eta} e^{rt}} \qquad \text{and} \qquad \boxed{(A - rI)\boldsymbol{\eta} = \boldsymbol{\xi}}.$$

Definition. η is called a ***generalised eigenvector*** of A.

Remark 5.10. If you have 2 repeated eigenvalues (but with only one linearly independent eigenvector), the method is:

(i). Find the eigenvalues and eigenvectors;

(ii). The first solution is $\mathbf{x}^{(1)}(t) = \boldsymbol{\xi} e^{rt}$;

(iii). Use $(A - rI)\boldsymbol{\eta} = \boldsymbol{\xi}$ to find a generalised eigenvector $\boldsymbol{\eta}$;

(iv). The second solution is $\mathbf{x}^{(2)}(t) = \boldsymbol{\xi} t e^{rt} + \boldsymbol{\eta} e^{rt}$.

Example 5.19. Solve

$$\begin{cases} \mathbf{x}' = \begin{bmatrix} -\frac{5}{2} & \frac{3}{2} \\ -\frac{3}{2} & \frac{1}{2} \end{bmatrix} \mathbf{x}, \\ \mathbf{x}(0) = \begin{bmatrix} 3 \\ -1 \end{bmatrix}. \end{cases}$$

The only eigenvalue of the matrix is $r = -1$. The corresponding eigenvector is $\boldsymbol{\xi} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. Therefore one solution of the linear system is

$$\mathbf{x}^{(1)}(t) = \boldsymbol{\xi} e^{rt} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{-t}.$$

We need to find a second, linearly independent solution. We will consider the ansatz

$$\mathbf{x}^{(2)}(t) = \boldsymbol{\xi} t e^{-t} + \boldsymbol{\eta} e^{-t}$$

where $\boldsymbol{\xi} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ as above and $\boldsymbol{\eta}$ is a generalised eigenvector solving $(A - rI)\boldsymbol{\eta} = \boldsymbol{\xi}$.

Solving the latter equation,

$$\begin{gathered} (A - rI)\boldsymbol{\eta} = \boldsymbol{\xi} \\ \begin{bmatrix} -\frac{3}{2} & \frac{3}{2} \\ -\frac{3}{2} & \frac{3}{2} \end{bmatrix} \begin{bmatrix} \eta_1 \\ \eta_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \\ -\frac{3}{2}\eta_1 + \frac{3}{2}\eta_2 = 1 \\ -\eta_1 + \eta_2 = \frac{2}{3} \end{gathered}$$

we can choose $\boldsymbol{\eta} = \begin{bmatrix} 0 \\ \frac{2}{3} \end{bmatrix}$.

Note that we don't need to find ***every*** generalised eigenvector

$$\boldsymbol{\eta} = \begin{bmatrix} k \\ k + \frac{2}{3} \end{bmatrix} = k \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{2}{3} \end{bmatrix} = k\boldsymbol{\xi} + \begin{bmatrix} 0 \\ \frac{2}{3} \end{bmatrix}$$

because we already have $\mathbf{x}^{(1)}(t) = \boldsymbol{\xi} e^{rt}$.

Instead we only need to find ***one*** generalised eigenvector – that means that we can choose any k that we want.

Hence I have chosen $k = 0$ which gives $\boldsymbol{\eta} = \begin{bmatrix} 0 \\ \frac{2}{3} \end{bmatrix}$.

Thus

$$\mathbf{x}^{(2)}(t) = \boldsymbol{\xi} te^{-t} + \boldsymbol{\eta} e^{-t} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} te^{-t} + \begin{bmatrix} 0 \\ \frac{2}{3} \end{bmatrix} e^{-t}.$$

Hence the general solution to the linear system is

$$\mathbf{x}(t) = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{-t} + c_2 \left(\begin{bmatrix} 1 \\ 1 \end{bmatrix} te^{-t} + \begin{bmatrix} 0 \\ \frac{2}{3} \end{bmatrix} e^{-t} \right).$$

The initial condition gives

$$\begin{bmatrix} 3 \\ -1 \end{bmatrix} = \mathbf{x}(0) = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ \frac{2}{3} \end{bmatrix}$$

which implies that $c_1 = 3$ and $c_2 = -6$.

Therefore the solution to the IVP is

$$\boxed{\mathbf{x}(t) = 3 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{-t} - 6 \left(\begin{bmatrix} 1 \\ 1 \end{bmatrix} te^{-t} + \begin{bmatrix} 0 \\ \frac{2}{3} \end{bmatrix} e^{-t} \right) = \begin{bmatrix} 3 \\ -1 \end{bmatrix} e^{-t} - 6 \begin{bmatrix} 1 \\ 1 \end{bmatrix} te^{-t}.}$$

Example 5.20. Solve
$$\mathbf{x}' = \begin{bmatrix} 1 & -4 \\ 4 & -7 \end{bmatrix} \mathbf{x}, \qquad \mathbf{x}(0) = \begin{bmatrix} 3 \\ 2 \end{bmatrix}.$$

The only eigenvalue of the matrix is $r = -3$. The corresponding eigenvector is $\boldsymbol{\xi} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. Therefore one solution of the linear system is

$$\mathbf{x}^{(1)}(t) = \boldsymbol{\xi} e^{rt} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{-3t}.$$

Next we need to find a generalised eigenvector $\boldsymbol{\eta}$.

We calculate that
$$\begin{aligned} (A - rI)\boldsymbol{\eta} &= \boldsymbol{\xi} \\ \begin{bmatrix} 4 & -4 \\ 4 & -4 \end{bmatrix} \begin{bmatrix} \eta_1 \\ \eta_2 \end{bmatrix} &= \begin{bmatrix} 1 \\ 1 \end{bmatrix} \\ 4\eta_1 - 4\eta_2 &= 1 \\ -\eta_1 + \eta_2 &= -\frac{1}{4} \\ \eta_2 &= \eta_1 - \frac{1}{4}. \end{aligned}$$

We can choose any vector $\boldsymbol{\eta}$ that satisfies $\eta_2 = \eta_1 - \frac{1}{4}$. Thus we may choose $\boldsymbol{\eta} = \begin{bmatrix} 0 \\ -\frac{1}{4} \end{bmatrix}$.

Therefore
$$\mathbf{x}^{(2)}(t) = \begin{bmatrix} 1 \\ 1 \end{bmatrix} te^{-3t} + \begin{bmatrix} 0 \\ -\frac{1}{4} \end{bmatrix} e^{-3t}.$$

Hence the general solution to the ODE is

$$\mathbf{x}(t) = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{-3t} + c_2 \left(\begin{bmatrix} 1 \\ 1 \end{bmatrix} te^{-3t} + \begin{bmatrix} 0 \\ -\frac{1}{4} \end{bmatrix} e^{-3t} \right).$$

The initial condition gives

$$\begin{bmatrix} 3 \\ 2 \end{bmatrix} = \mathbf{x}(0) = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ -\frac{1}{4} \end{bmatrix}$$

which implies that $c_1 = 3$ and $c_2 = 4$.

Therefore the solution to the IVP is

$$\begin{aligned} \mathbf{x}(t) &= 3 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{-3t} + 4 \left(\begin{bmatrix} 1 \\ 1 \end{bmatrix} te^{-3t} + \begin{bmatrix} 0 \\ -\frac{1}{4} \end{bmatrix} e^{-3t} \right) \\ &= \begin{bmatrix} 3 \\ 2 \end{bmatrix} e^{-3t} + 4 \begin{bmatrix} 1 \\ 1 \end{bmatrix} te^{-3t} \\ &= \begin{bmatrix} 3 + 4t \\ 2 + 4t \end{bmatrix} e^{-3t}. \end{aligned}$$

5.7 Nonhomogeneous Linear Systems

Consider

$$\mathbf{x}' = P(t)\mathbf{x} + \mathbf{g}(t) \tag{5.1}$$

where $P(t)$ and $\mathbf{g}(t)$ are continuous for $\alpha < t < \beta$. The general solution of (5.1) can be written as

$$\mathbf{x}(t) = c_1\mathbf{x}^{(1)} + c_2\mathbf{x}^{(2)} + \ldots + c_n\mathbf{x}^{(n)} + \mathbf{v}(t)$$

where

- $c_1\mathbf{x}^{(1)} + c_2\mathbf{x}^{(2)} + \ldots + c_n\mathbf{x}^{(n)}$ is the general solution to the homogeneous system $\mathbf{x}' = P(t)\mathbf{x}$; and
- $\mathbf{v}(t)$ is a particular solution to (5.1).

Remark 5.11. We will study four methods to solve (5.1):

(i). Diagonalisation;

(ii). Undetermined Coefficients;

(iii). Variation of Parameters;

(iv). The Laplace Transform.

Method 1 – Diagonalisation:

Consider

$$\mathbf{x}' = A\mathbf{x} + \mathbf{g}(t).$$

Suppose that

- $A \in \mathbb{R}^{n\times n}$ is diagonalisable;
- $\mathbf{g} : (\alpha, \beta) \to \mathbb{R}^n$;
- $\boldsymbol{\xi}^{(1)}, \ldots, \boldsymbol{\xi}^{(n)}$ are eigenvectors of A; and
- $T = \begin{bmatrix} \boldsymbol{\xi}^{(1)} & \cdots & \boldsymbol{\xi}^{(n)} \end{bmatrix}$.

Then

$$D = T^{-1}AT = \begin{bmatrix} r_1 & 0 & \cdots & 0 \\ 0 & r_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & r_n \end{bmatrix}.$$

Let $\mathbf{y} = T^{-1}\mathbf{x}$. Then $\mathbf{x} = T\mathbf{y}$. It follows that

$$T\mathbf{y}' = \mathbf{x}' = A\mathbf{x} + \mathbf{g}(t) = AT\mathbf{y} + \mathbf{g}(t)$$

and

$$\mathbf{y}' = T^{-1}AT\mathbf{y} + T^{-1}\mathbf{g}(t) = D\mathbf{y} + \mathbf{h}(t) \tag{5.2}$$

where $\mathbf{h} = T^{-1}\mathbf{g}$.

But (5.2) is just the system

$$\begin{cases} y_1' = r_1y_1 + h_1(t) & \longleftarrow \text{only } y_1 \text{ and } t \\ y_2' = r_2y_2 + h_2(t) & \longleftarrow \text{only } y_2 \text{ and } t \\ \quad\vdots & \\ y_n' = r_ny_n + h_n(t) & \longleftarrow \text{only } y_n \text{ and } t \end{cases}$$

We can solve each of these n first order linear ODEs individually. The solution to

$$y_j' - r_jy_j = h_j$$

(see Chapter 2) is

$$y_j(t) = e^{r_jt}\int_{t_0}^{t} e^{-r_js}h(s)\,ds + c_je^{r_jt}.$$

If we know $\mathbf{y}$, then we know $\mathbf{x} = T\mathbf{y}$.

Example 5.21. Solve

$$\mathbf{x}' = \begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 2e^{-t} \\ 3t \end{bmatrix} = A\mathbf{x} + \mathbf{g}(t).$$

The eigenvalues of $A = \begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix}$ are $r_1 = -3$ and $r_2 = -1$. The eigenvectors are

$$\boldsymbol{\xi}^{(1)} = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \qquad \text{and} \qquad \boldsymbol{\xi}^{(2)} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

So

$$T = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \qquad \text{and} \qquad T^{-1} = \frac{1}{2} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}.$$

Let $\mathbf{y} = T^{-1}\mathbf{x}$. Then

$$\begin{aligned} T\mathbf{y}' = \mathbf{x}' = A\mathbf{x} + \begin{bmatrix} 2e^{-t} \\ 3t \end{bmatrix} &= AT\mathbf{y} + \begin{bmatrix} 2e^{-t} \\ 3t \end{bmatrix} \\ \mathbf{y}' &= T^{-1}AT\mathbf{y} + T^{-1}\begin{bmatrix} 2e^{-t} \\ 3t \end{bmatrix} \\ &= D\mathbf{y} + \frac{1}{2}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 2e^{-t} \\ 3t \end{bmatrix} \\ &= \begin{bmatrix} -3 & 0 \\ 0 & -1 \end{bmatrix}\mathbf{y} + \frac{1}{2}\begin{bmatrix} 2e^{-t} - 3t \\ 2e^{-t} + 3t \end{bmatrix}. \end{aligned}$$

Therefore

$$\begin{cases} y_1' + 3y_1 = e^{-t} - \frac{3}{2}t \\ y_2' + y_2 = e^{-t} + \frac{3}{2}t. \end{cases}$$

You know how to solve first order linear ODEs. The solutions to these two ODEs are

$$\begin{aligned} y_1(t) &= \frac{1}{2}e^{-t} - \frac{t}{2} + \frac{1}{6} + c_1 e^{-3t} \\ y_2(t) &= te^{-t} + \frac{3t}{2} - \frac{3}{2} + c_2 e^{-t}. \end{aligned}$$

Finally we calculate that

$$\begin{aligned} \mathbf{x} = T\mathbf{y} &= \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \\ &= \begin{bmatrix} y_1 + y_2 \\ -y_1 + y_2 \end{bmatrix} \\ &= c_1 \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{-3t} + c_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{-t} + \frac{1}{2}\begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{-t} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} te^{-t} + \begin{bmatrix} 1 \\ 2 \end{bmatrix} t - \frac{1}{3}\begin{bmatrix} 4 \\ 5 \end{bmatrix}. \end{aligned}$$

Example 5.22. Solve
$$\mathbf{x}' = \begin{bmatrix} 1 & \sqrt{3} \\ \sqrt{3} & -1 \end{bmatrix} \mathbf{x} + \begin{bmatrix} e^t \\ \sqrt{3}e^{-t} \end{bmatrix}.$$

The eigenvalues of $\begin{bmatrix} 1 & \sqrt{3} \\ \sqrt{3} & -1 \end{bmatrix}$ are $r_1 = -2$ and $r_2 = 2$. The corresponding eigenvectors are $\boldsymbol{\xi}^{(1)} = \begin{bmatrix} 1 \\ -\sqrt{3} \end{bmatrix}$ and $\boldsymbol{\xi}^{(2)} = \begin{bmatrix} \sqrt{3} \\ 1 \end{bmatrix}$.

Thus
$$T = \begin{bmatrix} 1 & \sqrt{3} \\ -\sqrt{3} & 1 \end{bmatrix},$$
$$T^{-1} = \frac{1}{\det T} \begin{bmatrix} 1 & -\sqrt{3} \\ \sqrt{3} & 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{4} & -\frac{\sqrt{3}}{4} \\ \frac{\sqrt{3}}{4} & \frac{1}{4} \end{bmatrix}$$
and
$$D = T^{-1}AT = \begin{bmatrix} -2 & 0 \\ 0 & 2 \end{bmatrix}.$$

Now we must change variables: Let $\mathbf{y} = T^{-1}\mathbf{x}$. Then we have

$$\mathbf{y}' = D\mathbf{y} + T^{-1}\mathbf{g} = \begin{bmatrix} -2 & 0 \\ 0 & 2 \end{bmatrix} \mathbf{y} + \begin{bmatrix} \frac{1}{4} & -\frac{\sqrt{3}}{4} \\ \frac{\sqrt{3}}{4} & \frac{1}{4} \end{bmatrix} \begin{bmatrix} e^t \\ \sqrt{3}e^{-t} \end{bmatrix}$$
$$= \begin{bmatrix} -2y_1 \\ 2y_2 \end{bmatrix} + \begin{bmatrix} \frac{1}{4}e^t - \frac{3}{4}e^{-t} \\ \frac{\sqrt{3}}{4}e^t + \frac{\sqrt{3}}{4}e^{-t} \end{bmatrix}.$$

We know how to solve
$$y_1' + 2y_1 = \frac{1}{4}e^t - \frac{3}{4}e^{-t}$$
and
$$y_2' - 2y_2 = \frac{\sqrt{3}}{4}e^t + \frac{\sqrt{3}}{4}e^{-t}.$$
The solutions are
$$y_1(t) = \frac{1}{12}e^t - \frac{3}{4}e^{-t} + c_1e^{-2t}$$
and
$$y_2(t) = -\frac{\sqrt{3}}{4}e^t - \frac{\sqrt{3}}{12}e^{-t} + c_2e^{2t}.$$

So
$$\mathbf{y} = \begin{bmatrix} \frac{1}{12}e^t - \frac{3}{4}e^{-t} + c_1e^{-2t} \\ -\frac{\sqrt{3}}{4}e^t - \frac{\sqrt{3}}{12}e^{-t} + c_2e^{2t} \end{bmatrix}.$$
Therefore the general solution to the ODE is
$$\mathbf{x} = T\mathbf{y} = \begin{bmatrix} 1 & \sqrt{3} \\ -\sqrt{3} & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{12}e^t - \frac{3}{4}e^{-t} + c_1e^{-2t} \\ -\frac{\sqrt{3}}{4}e^t - \frac{\sqrt{3}}{12}e^{-t} + c_2e^{2t} \end{bmatrix} = \dots$$

Method 2 – Undetermined Coefficients:

Consider

$$\mathbf{x}' = A\mathbf{x} + \mathbf{g}(t).$$

(Remember Chapter 3?)

The idea is

(i). Find the general solution to $\mathbf{x}' = A\mathbf{x}$.

(ii). Look at $\mathbf{g}(t)$. Make a guess with constants. Find the constants.

(iii). $1 + 2$.

Example 5.23. Solve

$$\mathbf{x}' = \begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 2e^{-t} \\ 3t \end{bmatrix} = A\mathbf{x} + \mathbf{g}(t).$$

1. The solution of $\mathbf{x}' = \begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix} \mathbf{x}$ is

$$\mathbf{x}(t) = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{-t} + c_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{-3t}.$$

2. Since $\mathbf{g}(t) = \begin{bmatrix} 2 \\ 0 \end{bmatrix} e^{-t} + \begin{bmatrix} 0 \\ 3 \end{bmatrix} t$, we try the ansatz

$$\mathbf{x} = \mathbf{v}(t) = \mathbf{a}te^{-t} + \mathbf{b}e^{-t} + \mathbf{c}t + \mathbf{d}.$$

(Note that because $r_1 = -1$ is an eigenvalue of $\begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix}$, we need both te^{-t} and e^{-t}.)

Then we calculate that

$$\mathbf{x}' = A\mathbf{x} + \mathbf{g}$$

$$\mathbf{a}e^{-t} - \mathbf{a}te^{-t} - \mathbf{b}e^{-t} + \mathbf{c} = A\mathbf{a}te^{-t} + A\mathbf{b}e^{-t} + A\mathbf{c}t + A\mathbf{d} + \begin{bmatrix} 2 \\ 0 \end{bmatrix} e^{-t} + \begin{bmatrix} 0 \\ 3 \end{bmatrix} t.$$

- If we look at the te^{-t} terms, we have

$$-\mathbf{a} = A\mathbf{a} \quad \Longrightarrow \quad \mathbf{a} \text{ is an eigenvector} \quad \Longrightarrow \quad \mathbf{a} = \begin{bmatrix} \alpha \\ \alpha \end{bmatrix} \quad \text{for some } \alpha \in \mathbb{R}.$$

- If we look at the e^{-t} terms, we have

$$\mathbf{a} - \mathbf{b} = A\mathbf{b} + \begin{bmatrix} 2 \\ 0 \end{bmatrix}$$

which becomes

$$\begin{bmatrix} \alpha - 2 \\ \alpha \end{bmatrix} = \mathbf{a} - \begin{bmatrix} 2 \\ 0 \end{bmatrix} = (A + I)\mathbf{b} = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} = \begin{bmatrix} -b_1 + b_2 \\ b_1 - b_2 \end{bmatrix}.$$

But this means that

$$\alpha - 2 = -b_1 + b_2 = -(b_1 - b_2) = -\alpha \quad \Longrightarrow \alpha = 1.$$

So $\mathbf{a} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. Then we have that

$$b_1 - b_2 = 1 \quad \Longrightarrow \quad \mathbf{b} = \begin{bmatrix} k \\ k-1 \end{bmatrix}$$

for any k. If we choose $k = 0$, we get $\mathbf{b} = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$.

- If we look at the t terms, we have

$$0 = A\mathbf{c} + \begin{bmatrix} 0 \\ 3 \end{bmatrix} \quad \Longrightarrow \mathbf{c} = A^{-1} \begin{bmatrix} 0 \\ -3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}.$$

- Finally, if we look at the 1 terms, we have

$$\mathbf{c} = A\mathbf{d} \quad \Longrightarrow \quad \mathbf{d} = A^{-1}\mathbf{c} = \begin{bmatrix} -\frac{4}{3} \\ -\frac{5}{3} \end{bmatrix}.$$

So

$$\mathbf{v} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} te^{-t} - \begin{bmatrix} 0 \\ 1 \end{bmatrix} e^{-t} + \begin{bmatrix} 1 \\ 2 \end{bmatrix} t - \frac{1}{3} \begin{bmatrix} 4 \\ 5 \end{bmatrix}.$$

3. Therefore the general solution to the ODE is

$$\boxed{\mathbf{x}(t) = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{-t} + c_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{-3t} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} te^{-t} - \begin{bmatrix} 0 \\ 1 \end{bmatrix} e^{-t} + \begin{bmatrix} 1 \\ 2 \end{bmatrix} t - \frac{1}{3} \begin{bmatrix} 4 \\ 5 \end{bmatrix}.}$$

Example 5.24. Solve

$$\mathbf{x}' = \begin{bmatrix} 2 & 3 \\ 4 & 1 \end{bmatrix} \mathbf{x} + \begin{bmatrix} e^t \\ -10t - 3 \end{bmatrix}.$$

We will consider three simpler ODEs:

(i). $\mathbf{x}' = \begin{bmatrix} 2 & 3 \\ 4 & 1 \end{bmatrix} \mathbf{x}$

(ii). $\mathbf{x}' = \begin{bmatrix} 2 & 3 \\ 4 & 1 \end{bmatrix} \mathbf{x} + \begin{bmatrix} e^t \\ 0 \end{bmatrix}$

(iii). $\mathbf{x}' = \begin{bmatrix} 2 & 3 \\ 4 & 1 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 0 \\ -10t - 3 \end{bmatrix}$

and then we will add the solutions together.

The matrix $\begin{bmatrix} 2 & 3 \\ 4 & 1 \end{bmatrix}$ has eigenvalues $r_1 = 5$ and $r_2 = -2$ and eigenvectors $\boldsymbol{\xi}^{(1)} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\boldsymbol{\xi}^{(2)} = \begin{bmatrix} -3 \\ 4 \end{bmatrix}$. Hence the general solution of $\mathbf{x}' = \begin{bmatrix} 2 & 3 \\ 4 & 1 \end{bmatrix} \mathbf{x}$ is

$$\mathbf{x}(t) = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{5t} + c_2 \begin{bmatrix} -3 \\ 4 \end{bmatrix} e^{-2t}.$$

Next we need to find a particular solution to $\mathbf{x}' = \begin{bmatrix} 2 & 3 \\ 4 & 1 \end{bmatrix} \mathbf{x} + \begin{bmatrix} e^t \\ 0 \end{bmatrix}$. Since 1 is not an eigenvalue of $\begin{bmatrix} 2 & 3 \\ 4 & 1 \end{bmatrix}$, we try the ansatz $\mathbf{x} = \mathbf{a}e^t$ for some $\mathbf{a} = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} \in \mathbb{R}^2$. Then we calculate that

$$\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} e^t = \mathbf{x}' = \begin{bmatrix} 2 & 3 \\ 4 & 1 \end{bmatrix} \mathbf{x} + \begin{bmatrix} e^t \\ 0 \end{bmatrix} = \begin{bmatrix} 2a_1 + 3a_2 + 1 \\ 4a_1 + a_2 \end{bmatrix} e^t$$

which gives

$$\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} 2a_1 + 3a_2 + 1 \\ 4a_1 + a_2 \end{bmatrix}.$$

Hence $a_1 = 0$ and $a_2 = -\frac{1}{3}$. So $\mathbf{x} = \begin{bmatrix} 0 \\ -\frac{1}{3} \end{bmatrix} e^t$.

Then we need to find a particular solution to $\mathbf{x}' = \begin{bmatrix} 2 & 3 \\ 4 & 1 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 0 \\ -10t - 3 \end{bmatrix}$. We try the ansatz $\mathbf{x} = \mathbf{a}t + \mathbf{b} = \begin{bmatrix} a_1 t + b_1 \\ a_2 t + b_2 \end{bmatrix}$ for some $\mathbf{a}, \mathbf{b} \in \mathbb{R}^2$ and calculate that

$$\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \mathbf{x}' = \begin{bmatrix} 2 & 3 \\ 4 & 1 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 0 \\ t \end{bmatrix} = \begin{bmatrix} 2a_1 t + 2b_1 + 3a_2 t + 3b_2 \\ 4a_1 t + 4b_1 + a_2 t + b_2 - 10t - 3 \end{bmatrix}$$

which leads to $\begin{cases} 0 = 2a_1 + 3a_2 \\ a_1 = 2b_1 + 3b_2 \\ 0 = 4a_1 + a_2 - 10 \\ a_2 = 4b_1 + b_2 - 3 \end{cases}$ .The solution to this linear system is $\mathbf{a} = \begin{bmatrix} 3 \\ -2 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. Hence $\mathbf{x} = \begin{bmatrix} 3t \\ 1 - 2t \end{bmatrix}$.

Adding all of these together, we find that the general solution to the given ODE is

$$\boxed{\mathbf{x} = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{5t} + c_2 \begin{bmatrix} -3 \\ 4 \end{bmatrix} e^{-2t} + \begin{bmatrix} 0 \\ -\frac{1}{3} \end{bmatrix} e^t + \begin{bmatrix} 3t \\ 1 - 2t \end{bmatrix}.}$$

Method 3 – Variation of Parameters:

Consider

$$\mathbf{x}' = P(t)\mathbf{x} + \mathbf{g}(t) \tag{5.1}$$

where

- P and $\mathbf{g}$ are continuous for $\alpha < t < \beta$;
- there exists a fundamental matrix $\Psi(t)$ for the homogeneous system $\mathbf{x}' = P(t)\mathbf{x}$.

We know that the general solution to $\mathbf{x}' = P(t)\mathbf{x}$ is $\mathbf{x} = \Psi(t)\mathbf{c}$.

We guess that

$$\mathbf{x} = \Psi(t)\mathbf{u}(t)$$

is a solution to (5.1). Can we find $\mathbf{u}(t)$?

If $\mathbf{x} = \Psi\mathbf{u}$, we can calculate that

$$\cancel{\Psi'\mathbf{u}} + \Psi\mathbf{u}' = \mathbf{x}' = P\mathbf{x} + \mathbf{g} = \cancel{P\Psi\mathbf{u}} + \mathbf{g}. \tag{5.3}$$

But remember that

$$\Psi \text{ is a fundamental matrix for } \mathbf{x}' = P(t)\mathbf{x} \implies \Psi \text{ solves } \Psi' = P\Psi.$$

Hence (5.3) becomes

$$\Psi\mathbf{u}' = \mathbf{g}.$$

Therefore

$$\mathbf{u}' = \Psi^{-1}\mathbf{g}$$

and

$$\mathbf{u} = \int \Psi^{-1}\mathbf{g}.$$

Hence

$$\mathbf{x} = \Psi(t)\mathbf{u}(t) = \Psi(t)\int \Psi^{-1}(s)g(s)\,ds.$$

Remark 5.12. To solve $\mathbf{x}' = P(t)\mathbf{x} + \mathbf{g}(t)$, the method is

(i). Find a fundamental matrix Ψ for $\mathbf{x}' = P(t)\mathbf{x}$;

(ii). Calculate $\mathbf{x} = \Psi(t)\int \Psi^{-1}(s)g(s)\,ds$.

Example 5.25. Solve

$$\mathbf{x}' = \begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 2e^{-t} \\ 3t \end{bmatrix} = A\mathbf{x} + \mathbf{g}(t).$$

The solution of $\mathbf{x}' = \begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix} \mathbf{x}$ is

$$\mathbf{x}(t) = c_1 \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{-3t} + c_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{-t}.$$

So

$$\Psi(t) = \begin{bmatrix} e^{-3t} & e^{-t} \\ -e^{-3t} & e^{-t} \end{bmatrix}$$

is a fundamental matrix.

Then we calculate that

$$\Psi^{-1}(t) = \frac{1}{2e^{-4t}} \begin{bmatrix} e^{-t} & -e^{-t} \\ e^{-3t} & e^{-3t} \end{bmatrix} = \frac{1}{2} e^{4t} \begin{bmatrix} e^{-t} & -e^{-t} \\ e^{-3t} & e^{-3t} \end{bmatrix} = \begin{bmatrix} \frac{1}{2}e^{3t} & -\frac{1}{2}e^{3t} \\ \frac{1}{2}e^{t} & \frac{1}{2}e^{t} \end{bmatrix}$$

and

$$\begin{aligned} \int \Psi^{-1}(t)\mathbf{g}(t)\,dt &= \int \begin{bmatrix} \frac{1}{2}e^{3t} & -\frac{1}{2}e^{3t} \\ \frac{1}{2}e^{t} & \frac{1}{2}e^{t} \end{bmatrix} \begin{bmatrix} 2e^{-t} \\ 3t \end{bmatrix} dt \\ &= \int \begin{bmatrix} e^{2t} - \frac{3}{2}te^{3t} \\ 1 + \frac{3}{2}te^{t} \end{bmatrix} dt = \begin{bmatrix} \frac{1}{2}e^{2t} - \frac{1}{2}te^{3t} + \frac{1}{6}e^{3t} + c_1 \\ t + \frac{3}{2}te^{t} - \frac{3}{2}e^{t} + c_2 \end{bmatrix}. \end{aligned}$$

Therefore the solution to $\mathbf{x}' = A\mathbf{x} + g$ is

$$\begin{aligned} \mathbf{x} &= \Psi(t) \int \Psi^{-1}(s)\mathbf{g}(s)\,ds \\ &= \begin{bmatrix} e^{-3t} & e^{-t} \\ -e^{-3t} & e^{-t} \end{bmatrix} \begin{bmatrix} \frac{1}{2}e^{2t} - \frac{1}{2}te^{3t} + \frac{1}{6}e^{3t} + c_1 \\ t + \frac{3}{2}te^{t} - \frac{3}{2}e^{t} + c_2 \end{bmatrix} \\ &= c_1 \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{-3t} + c_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{-t} + \frac{1}{2} \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{-t} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} te^{-t} + \begin{bmatrix} 1 \\ 2 \end{bmatrix} t - \frac{1}{3} \begin{bmatrix} 4 \\ 5 \end{bmatrix}. \end{aligned}$$

Example 5.26. Solve
$$\mathbf{x}' = \begin{bmatrix} -4 & 2 \\ 2 & -1 \end{bmatrix} \mathbf{x} + \begin{bmatrix} t^{-1} \\ 2t^{-1} + 4 \end{bmatrix}$$
for $t > 0$.

The eigenvalues of $A = \begin{bmatrix} -4 & 2 \\ 2 & -1 \end{bmatrix}$ are $r_1 = 0$ and $r_2 = -5$; and the eigenvectors are $\xi^{(1)} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $\xi^{(2)} = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$. Thus
$$\Psi(t) = \begin{bmatrix} 1 & -2e^{-5t} \\ 2 & e^{-5t} \end{bmatrix}$$
is a fundamental matrix for $\mathbf{x}' = A\mathbf{x}$.

Using the formula $\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$ we calculate that
$$\Psi^{-1}(t) = \frac{1}{e^{-5t} + 4e^{-5t}} \begin{bmatrix} e^{-5t} & 2e^{-5t} \\ -2 & 1 \end{bmatrix} = \frac{1}{5} \begin{bmatrix} 1 & 2 \\ -2e^{5t} & e^{5t} \end{bmatrix}.$$

Then
$$\begin{aligned} \Psi^{-1}(t)\mathbf{g}(t) &= \frac{1}{5} \begin{bmatrix} 1 & 2 \\ -2e^{5t} & e^{5t} \end{bmatrix} \begin{bmatrix} t^{-1} \\ 2t^{-1} + 4 \end{bmatrix} \\ &= \frac{1}{5} \begin{bmatrix} t^{-1} + 4t^{-1} + 8 \\ -2t^{-1}e^{5t} + 2t^{-1}e^{5t} + 4e^{5t} \end{bmatrix} = \begin{bmatrix} t^{-1} + \frac{8}{5} \\ \frac{4}{5}e^{5t} \end{bmatrix} \end{aligned}$$
and
$$\int \Psi^{-1}(t)\mathbf{g}(t)\, dt = \int \begin{bmatrix} t^{-1} + \frac{8}{5} \\ \frac{4}{5}e^{5t} \end{bmatrix} dt = \begin{bmatrix} \ln t + \frac{8}{5}t + c_1 \\ \frac{4}{25}e^{5t} + c_2 \end{bmatrix}.$$

It follows that
$$\begin{aligned} \mathbf{x}(t) = \Psi(t) \int \Psi^{-1}(s)\mathbf{g}(s)\, ds &= \begin{bmatrix} 1 & -2e^{-5t} \\ 2 & e^{-5t} \end{bmatrix} \begin{bmatrix} \ln t + \frac{8}{5}t + c_1 \\ \frac{4}{25}e^{5t} + c_2 \end{bmatrix} \\ &= \begin{bmatrix} \ln t + \frac{8}{5}t - \frac{8}{25} + c_1 - 2c_2e^{-5t} \\ 2\ln t + \frac{16}{5}t + \frac{4}{25} + 2c_1 + c_2e^{-5t} \end{bmatrix} \\ &= c_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + c_2 \begin{bmatrix} -2 \\ 1 \end{bmatrix} e^{-5t} + \begin{bmatrix} 1 \\ 2 \end{bmatrix} \ln t + \frac{8}{5} \begin{bmatrix} 1 \\ 2 \end{bmatrix} t + \frac{4}{25} \begin{bmatrix} -2 \\ 1 \end{bmatrix}. \end{aligned}$$

Method 4 – The Laplace Transform:

First some notation: If $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$, then $\mathbf{X} = \mathcal{L}\left[\mathbf{x}\right] = \begin{bmatrix} \mathcal{L}\left[x_1\right] \\ \mathcal{L}\left[x_2\right] \\ \vdots \\ \mathcal{L}\left[x_n\right] \end{bmatrix}$.

Recall from Chapter 4 that $\mathcal{L}\left[y'\right]$ satisfies

$$\mathcal{L}\left[y'\right](s) = sY(s) - y(0).$$

It follows that:

Theorem 5.4.

$$\mathcal{L}\left[\mathbf{x}'\right](s) = s\mathbf{X}(s) - \mathbf{x}(0).$$

Example 5.27. Solve

$$\begin{cases} \mathbf{x}' = \begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 2e^{-t} \\ 3t \end{bmatrix} = A\mathbf{x} + \mathbf{g}(t), \\ \mathbf{x}(0) = \mathbf{0}. \end{cases}$$

Taking Laplace Transforms of the ODE gives

$$s\mathbf{X}(s) - \mathbf{x}(0) = A\mathbf{X}(s) + \mathbf{G}(s)$$

where $\mathbf{G}(s) = \mathcal{L}\left[\mathbf{g}\right](s) = \begin{bmatrix} \frac{2}{s+1} \\ \frac{3}{s^2} \end{bmatrix}$.

Since $\mathbf{x}(0) = \mathbf{0}$ we have that

$$(sI - A)\mathbf{X} = \mathbf{G}$$

and

$$\mathbf{X} = (sI - A)^{-1}\mathbf{G}$$

where

$$(sI - A)^{-1} = \begin{bmatrix} s+2 & -1 \\ -1 & s+2 \end{bmatrix}^{-1} = \frac{1}{(s+1)(s+3)} \begin{bmatrix} s+2 & 1 \\ 1 & s+2 \end{bmatrix}.$$

So

$$\begin{aligned} \mathbf{X} &= (sI - A)^{-1}\mathbf{G} \\ &= \frac{1}{(s+1)(s+3)} \begin{bmatrix} s+2 & 1 \\ 1 & s+2 \end{bmatrix} \begin{bmatrix} \frac{2}{s+1} \\ \frac{3}{s^2} \end{bmatrix} \\ &= \begin{bmatrix} \frac{2(s+2)}{(s+1)^2(s+3)} + \frac{3}{s^2(s+1)(s+3)} \\ \frac{2}{(s+1)^2(s+3)} + \frac{3(s+2)}{s^2(s+1)(s+3)} \end{bmatrix}. \end{aligned}$$

When we take the inverse Laplace Transform of this, we find our solution

$$\mathbf{x} = \mathcal{L}^{-1}\left[\mathbf{X}\right] = \begin{bmatrix} 2 \\ 1 \end{bmatrix} e^{-t} - \frac{2}{3} \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{-3t} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} te^{-t} + \begin{bmatrix} 1 \\ 2 \end{bmatrix} t - \frac{1}{3} \begin{bmatrix} 4 \\ 5 \end{bmatrix}.$$

Example 5.28. Solve

$$\begin{cases} 2x' + y' - y - t = 0 \\ x' + y' - t^2 = 0 \\ x(0) = 1 \\ y(0) = 0 \end{cases}$$

The ODEs above can be written as

$$\begin{cases} x' = y - t^2 + t \\ y' = -y + 2t^2 - t \end{cases}$$

(please check!).

If we write the problem in terms of matrices (using $\mathbf{x} = \begin{bmatrix} x \\ y \end{bmatrix}$) we have

$$\begin{cases} \mathbf{x}' = A\mathbf{x} + \mathbf{g} = \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix} \mathbf{x} + \begin{bmatrix} t - t^2 \\ 2t^2 - t \end{bmatrix} \\ \mathbf{x}(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}. \end{cases}$$

Taking the Laplace transform of the ODE gives

$$\begin{aligned} (sI - A)\,\mathbf{X}\,(s) &= \mathbf{x}\,(0) + \mathbf{G}(s) \\ \begin{bmatrix} s & -1 \\ 0 & s+1 \end{bmatrix} \mathbf{X}\,(s) &= \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \begin{bmatrix} \frac{1}{s^2} - \frac{2}{s^3} \\ \frac{4}{s^3} - \frac{1}{s^2} \end{bmatrix}. \end{aligned}$$

Thus

$$\begin{aligned} \mathbf{X}(s) &= \frac{1}{s\,(s+1)} \begin{bmatrix} s+1 & 1 \\ 0 & s \end{bmatrix} \frac{1}{s^3} \begin{bmatrix} s^3 + s - 2 \\ 4 - s \end{bmatrix} \\ &= \frac{1}{s^4\,(s+1)} \begin{bmatrix} s^4 + s^3 + s^2 - 2s + 2 \\ 4s - s^2 \end{bmatrix}. \end{aligned}$$

Note that

$$\frac{s^4 + s^3 + s^2 - 2s + 2}{s^4\,(s+1)} = \frac{5}{s+1} - 4\frac{1}{s} + 5\frac{1}{s^2} - 4\frac{1}{s^3} + 2\frac{1}{s^4}$$

and

$$\frac{4s - s^2}{s^4\,(s+1)} = -5\frac{1}{s+1} + 5\frac{1}{s} - 5\frac{1}{s^2} + 4\frac{1}{s^3}$$

(please check!).

It follows that

$$\mathcal{L}^{-1}\left(\frac{s^4 + s^3 + s^2 - 2s + 2}{s^4\,(s+1)}\right) = 5e^{-t} - 4 + 5t - 2t^2 + \frac{1}{3}t^3$$

and

$$\mathcal{L}^{-1}\left(\frac{4s - s^2}{s^4\,(s+1)}\right) = -5e^{-t} + 5 - 5t + 2t^2.$$

Therefore the solution to the initial value problem is

$$\boxed{\mathbf{x}\,(t) = \begin{bmatrix} 5e^{-t} - 4 + 5t - 2t^2 + \frac{1}{3}t^3 \\ -5e^{-t} + 5 - 5t + 2t^2 \end{bmatrix}}.$$

Exercises

Exercise 5.1 (Systems of Linear Equations)**.** Find the general solutions to the following systems of ODEs:

(a). $\mathbf{x}' = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} \mathbf{x}$

(b). $\mathbf{x}' = \begin{bmatrix} -3 & 2 \\ -3 & 4 \end{bmatrix} \mathbf{x}$

(c). $\mathbf{x}' = \begin{bmatrix} 3 & -1 \\ 5 & -3 \end{bmatrix} \mathbf{x}$.

(d). $\begin{cases} x' = 4x - y \\ y' = x + 2y \end{cases}$

(e). $\begin{cases} x' = 3x - y \\ y' = 4x - y \end{cases}$

(f). $\begin{cases} x' = 5x + 4y \\ y' = -x + y \end{cases}$

(g). $\begin{cases} x' = 3x + 2y \\ y' = -5x + y \end{cases}$

(h). $\begin{cases} x' = x - 4y \\ y' = x + y \end{cases}$

(i). $\begin{cases} x' = x - 3y \\ y' = 3x + y \end{cases}$

(j). $\begin{cases} x' = 4x - 2y \\ y' = 5x + 2y \end{cases}$

(k). $\begin{cases} x' = x + y - z \\ y' = 2x + 3y - 4z \\ z' = 4x + y - 4z \end{cases}$

(l). $\begin{cases} x' = x - y - z \\ y' = x + 3y + z \\ z' = -3x + y - z \end{cases}$

(m). $\begin{cases} x' = 3x + y + z \\ y' = 3y + z \\ z' = 6z \end{cases}$

(n). $\begin{cases} x' = 2x + y - z \\ y' = -4x - 3y - z \\ z' = 4x + 4y + 2z \end{cases}$

Exercise 5.2 (Initial Value Problems)**.** Solve the following IVPs:

(a). $\begin{cases} \mathbf{x}' = \begin{bmatrix} 3 & 4 \\ 3 & 2 \end{bmatrix} \mathbf{x} \\ \mathbf{x}(0) = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \end{cases}$

(b). $\begin{cases} \mathbf{x}' = \begin{bmatrix} 4 & -3 \\ 6 & -7 \end{bmatrix} \mathbf{x} \\ \mathbf{x}(0) = \begin{bmatrix} 8 \\ 0 \end{bmatrix} \end{cases}$

(c). $\begin{cases} x' = 3x + z \\ y' = 9x - y + 2z \\ z' = -9x + 4y - z \\ x(0) = 0 \\ y(0) = 0 \\ z(0) = 17 \end{cases}$

Exercise 5.3 (Systems of First Order Linear Equations)**.** Transform each of the following equations into a system of first order linear ODEs.

(a). $u'' + 0.5u' + 2u = 0$

(b). $u'' + 0.5u' + 2u = 3\sin t$

(c). $t^2u'' + tu' + (t^2 - 0.25)u = 0$

(d). $u^{(4)} - u = 0$

Transform each of the following systems into a single second order ODE for x_1.

(e). $\begin{cases} x_1' = 3x_1 - 2x_2 \\ x_2' = 2x_1 - 2x_2 \end{cases}$

(f). $\begin{cases} x_1' = 1.25x_1 + 0.75x_2 \\ x_2' = 0.75x_1 + 1.25x_2 \end{cases}$

(g). $\begin{cases} x_1' = x_1 - 2x_2 \\ x_2' = 3x_1 - 4x_2 \end{cases}$

(h). $\begin{cases} x_1' = 2x_2 \\ x_2' = -2x_1 \end{cases}$

Exercise 5.4 (Fundamental Matrices)**.**

(a). Suppose that $\Psi(t)$ is a fundamental matrix for $\mathbf{x}' = A\mathbf{x}$, where $A \in \mathbb{R}^{n\times n}$. Show that $\Psi' = A\Psi$.

(b). Let $A = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}$. Find $e^{At} = \exp(At)$.

For each of the following;

(i) Find a fundamental matrix $\Psi(t)$ for the system; and

(ii) Find the special fundamental matrix $\Phi(t)$ which satisfies $\Phi(0) = I$.

The first one is done for you.

(ω) $\mathbf{x}' = \begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix} \mathbf{x}$ (e). $\mathbf{x}' = \begin{bmatrix} 2 & -5 \\ 1 & -2 \end{bmatrix} \mathbf{x}$ (h). $\mathbf{x}' = \begin{bmatrix} -1 & -4 \\ 1 & -1 \end{bmatrix} \mathbf{x}$

(c). $\mathbf{x}' = \begin{bmatrix} 3 & -2 \\ 2 & -2 \end{bmatrix} \mathbf{x}$ (f). $\mathbf{x}' = \begin{bmatrix} 1 & 1 \\ 4 & -2 \end{bmatrix} \mathbf{x}$

(d). $\mathbf{x}' = \begin{bmatrix} -\frac{3}{4} & \frac{1}{2} \\ \frac{1}{8} & -\frac{3}{4} \end{bmatrix} \mathbf{x}$ (g). $\mathbf{x}' = \begin{bmatrix} 2 & -1 \\ 3 & -2 \end{bmatrix} \mathbf{x}$ (i). $\mathbf{x}' = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 1 & -1 \\ -8 & -5 & -3 \end{bmatrix} \mathbf{x}$

Exercise 5.5 (Non-Homogeneous Systems of Equations)**.**
Use the Method of Undetermined Coefficients to solve the following systems of ODEs:

(a). $\mathbf{x}' = \begin{bmatrix} 2 & -1 \\ 3 & -2 \end{bmatrix} \mathbf{x} + \begin{bmatrix} e^t \\ t \end{bmatrix}$ (b). $\mathbf{x}' = \begin{bmatrix} 1 & \sqrt{3} \\ \sqrt{3} & -1 \end{bmatrix} \mathbf{x} + \begin{bmatrix} e^t \\ \sqrt{3}e^{-t} \end{bmatrix}$

Use the Method of Diagonalisation (use the substitution $\mathbf{x} = T\mathbf{y}$) to solve the following systems of ODEs:

(c). $\mathbf{x}' = \begin{bmatrix} 1 & 1 \\ 4 & -2 \end{bmatrix} \mathbf{x} + \begin{bmatrix} e^{-2t} \\ -2e^t \end{bmatrix}$ (d). $\mathbf{x}' = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix} \mathbf{x} + \begin{bmatrix} -\cos t \\ \sin t \end{bmatrix}$

Use the Method of Variation of Parameters ($\mathbf{x}(t) = \Psi(t) \int \Psi^{-1}(s)\mathbf{g}(s)\, ds$) to solve the following systems of ODEs:

(e). $\mathbf{x}' = \begin{bmatrix} -4 & 2 \\ 2 & -1 \end{bmatrix} \mathbf{x} + \begin{bmatrix} t^{-1} \\ 2t^{-1} + 4 \end{bmatrix}, \quad t > 0$ (f). $\mathbf{x}' = \begin{bmatrix} 4 & -2 \\ 8 & -4 \end{bmatrix} \mathbf{x} + \begin{bmatrix} t^{-3} \\ -t^{-2} \end{bmatrix}, \quad t > 0$

Exercise 5.6 (The Laplace Transform)**.** Use the Laplace Transform to solve the following IVPs:

(a). $\begin{cases} x' = x - 2y \\ y' = 5x - y \\ x(0) = -1 \\ y(0) = 2 \end{cases}$ (c). $\begin{cases} 2x' + y' - 2x = 1 \\ x' + y' - 3x - 3y = 2 \\ x(0) = 0 \\ y(0) = 0 \end{cases}$

(b). $\begin{cases} x' = -x + y \\ y' = 2x \\ x(0) = 0 \\ y(0) = 1 \end{cases}$ (d). $\begin{cases} 2x' + y' - y - t = 0 \\ x' + y' - t^2 = 0 \\ x(0) = 1 \\ y(0) = 0 \end{cases}$

[Hint: For (c) and (d), you must first rearrange the ODEs to the form $\begin{cases} x' = f_1(x, y) \\ y' = f_2(x, y) \end{cases}$.]

Solutions to the Exercises

1.1

(a). There are 250 litres/hour $\times$ 0.02 grams/litre = 5 grams/hour of toxic waste flowing into the lake.

(b). The concentration of toxic waste in the lake is $\frac{S(t)}{1000000} = 10^{-6}S(t)$ grams/litre, so 250 litres/hour $\times 10^{-6}S(t)$ grams/litre = $2.5 \times 10^{-4}S(t)$ grams/hour of toxic waste flows out of the lake.

(c). $\begin{cases} S' = 2.5(2 - 10^{-4}S), \\ S(0) = 0. \end{cases}$

(d).

$$\begin{aligned} \frac{dS}{dt} &= 2.5(2 - 10^{-4}S) \\ \frac{dS}{2 - 10^{-4}S} &= 2.5dt \\ \int \frac{dS}{2 - 10^{-4}S} &= \int 2.5dt \\ -10^4 \log\left|2 - 10^{-4}S\right| &= 2.5t + C_1 \\ 2 - 10^{-4}S &= \pm e^{-\frac{2.5t + C_1}{10^4}} = C_2 e^{-2.5t/10^4} \\ S(t) &= 10^4\left(2 - C_2 e^{-2.5t/10^4}\right) \end{aligned}$$

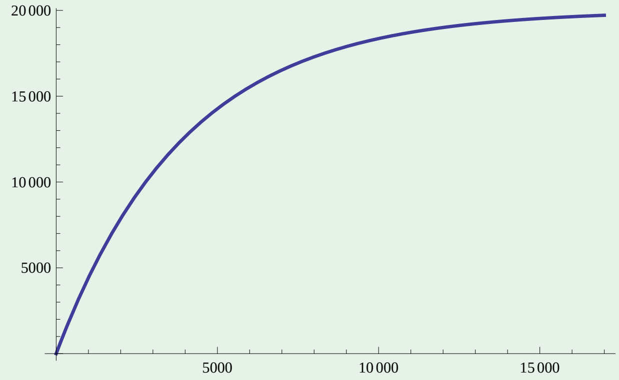
or

$$\begin{aligned} \frac{dS}{dt} &= 2.5(2 - 10^{-4}S) = -2.5 \times 10^{-4}(S - 2 \times 10^4) \\ \frac{dS}{S - 2 \times 10^4} &= -2.5 \times 10^{-4}dt \\ \int \frac{dS}{S - 2 \times 10^4} &= \int -2.5 \times 10^{-4}dt \\ \log\left|S - 2 \times 10^4\right| &= -2.5 \times 10^{-4}t + C_1 \\ S - 2 \times 10^4 &= \pm e^{-\frac{2.5t}{10^4} + C_1} = C_3 e^{-2.5t/10^4} \\ S(t) &= 10^4\left(2 - C_2 e^{-2.5t/10^4}\right) \end{aligned}$$

Finally $S(0) = 0 \implies C_2 = 2$. So $S(t) = 2 \times 10^4\left(1 - e^{-2.5t/10^4}\right)$.

(e). After 1 year, $t = 365 \times 24 = 8760$ hours and $S(8760) \approx 17762g$.

(f).

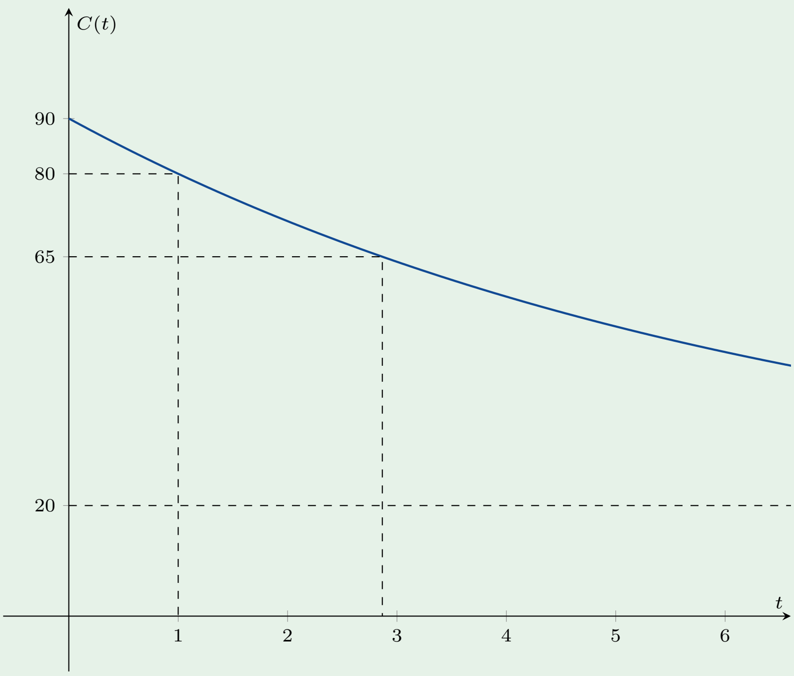


1.2

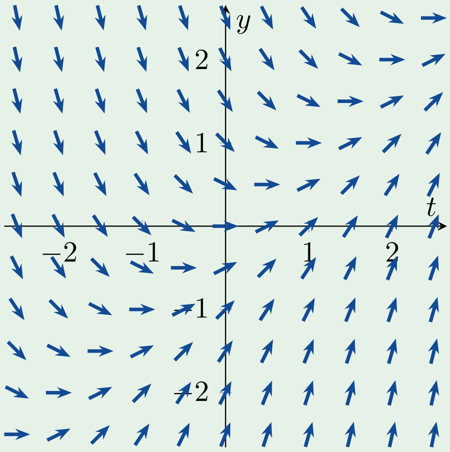
(a). Let $C(t)$ denote the temperature of your cup of coffee in degrees Centigrade. We are told that $\frac{DC}{dt}$ is proportional to $(20 - C)$. Hence $\dfrac{dC}{dt} = r(20 - C)$ for some constant r. Since hot coffee cools and very cold coffee warms up, we must have $r > 0$.

(b). The solution to the ODE is $C(t) = 20 + ce^{-rt}$ for some constant c. Using $C(0) = 90$ we find $c = 70$. Thus $C(t) = 20 + 70e^{-rt}$. We are also told that $80 = C(1) = 20 + 70e^{-r}$. Thus $r = \ln \frac{7}{6} \approx 0.154151$. Then we calculate that

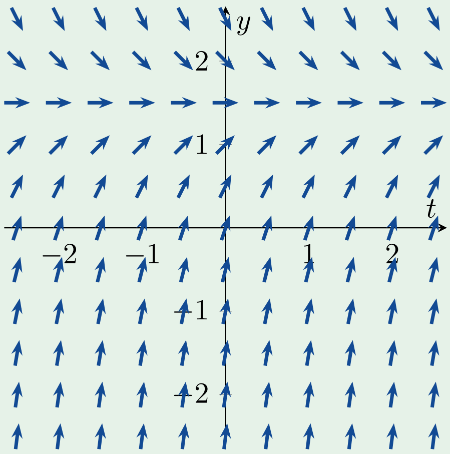
$$\begin{aligned} 65 &= C(T) = 20 + 70e^{-rT} \\ \frac{45}{70} &= e^{-rT} \\ \ln \frac{9}{14} &= (\ln \frac{6}{7})T \\ T &= \frac{\ln \frac{9}{14}}{\ln \frac{6}{7}} \approx 2.86623 \text{ minutes.} \end{aligned}$$



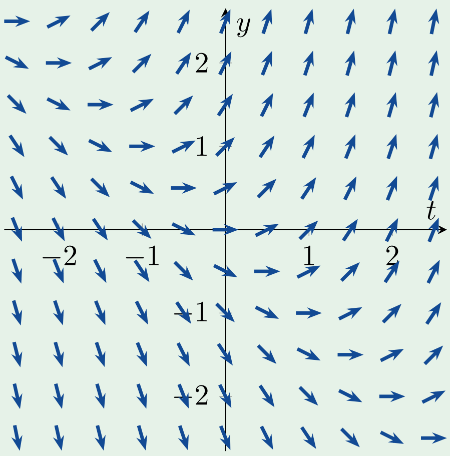
1.3


y
t
2
1
−1
−2
−2
−1
1
2

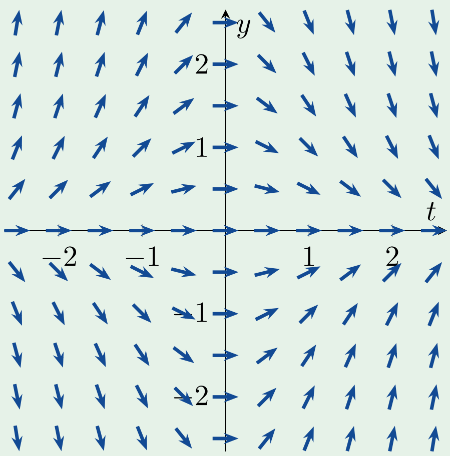
(a).


y
t
2
1
−1
−2
−2
−1
1
2

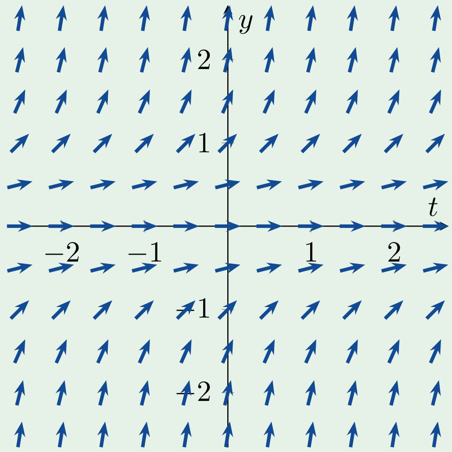
(b).


y
t
2
1
−1
−2
−2
−1
1
2

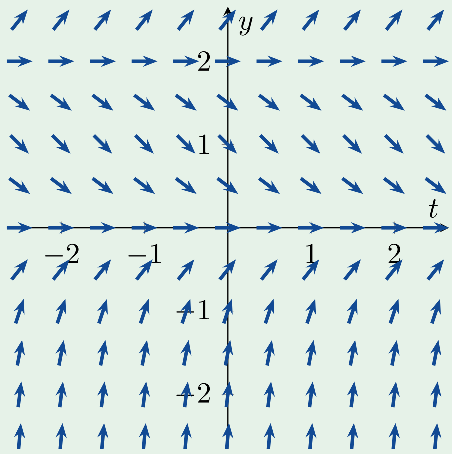
(c).


y
t
2
1
−1
−2
−2
−1
1
2

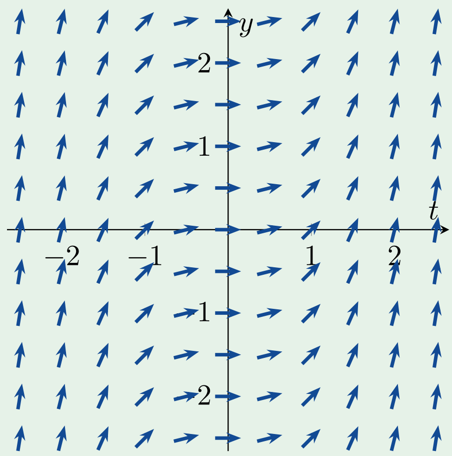
(d).


y
t
2
1
−1
−2
−2
−1
1
2

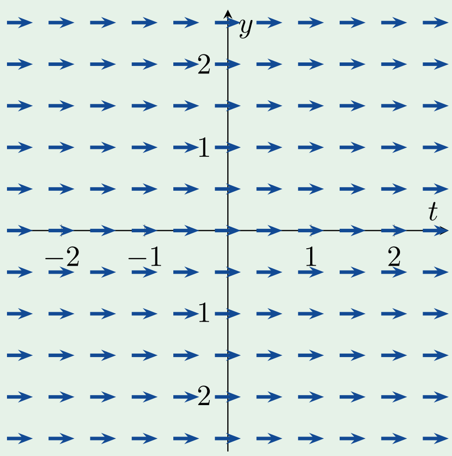
(e).


y
t
2
1
−1
−2
−2
−1
1
2

(f).


y
t
2
1
−1
−2
−2
−1
1
2

(g).


y
t
2
1
−1
−2
−2
−1
1
2

(h).

1.4

(a). first order, linear

(b). second order, non-linear

(c). first order, non-linear

(d). fourth order, linear

(e). second order, non-linear

(f). fourth order, linear

(g). first order, linear

(h). third order, linear

(i). second order, non-linear

(j). fourth order, non-linear

(k). third order, non-linear

(l). second order, linear

(m). first order, linear

(n). second order, non-linear

1.5

(a). Note that

$$\begin{aligned} x' &= (1+t)\,e^t \quad \text{and } x'' = (2+t)\,e^t \Longrightarrow \\ x'' - 2x' + x &= (2+t)\,e^t - 2\,(1+t)\,e^t + te^t = 0. \end{aligned}$$

(b). y'' can be calculated as follows.

$$\begin{aligned} y' &= -2\sin 2x \quad \text{and} \quad y'' = -4\cos 2x \Longrightarrow \\ y'' + 4y &= -4\cos 2x + 4\cos 2x = 0. \end{aligned}$$

2.1

(a). Note that e^x is the integrating factor. Thus, we get

$$\begin{aligned} \frac{d}{dx}\left(ye^x\right) &= 5e^x \Longrightarrow ye^x = \int 5e^x\,dx + C = 5e^x + C \Longrightarrow \\ y &= Ce^{-x} + 5. \end{aligned}$$

(b). $y = ce^{-t} + 1 + \frac{t^2 e^{-t}}{2}$

(c). $y = ce^{3t} - 2e^t$

(d). $y = -te^{-t} + ct$

(e). $y = t^2 e^{-t^2} + ce^{-t^2}$

(f). $y = ce^{-t} + \sin 2t - 2\cos 2t$

2.2

(a). Multiplying the ODE by the integrating factor $\mu(t) = e^{-t}$ gives

$$e^{-t}\frac{dy}{dt} - e^{-t}y = 2te^t.$$

Hence

$$\frac{d}{dt}\left(e^{-t}y\right) = 2te^t.$$

Then

$$e^{-t}y = \int 2te^t\ dt = 2te^t - \int 2e^t\ dt = 2te^t - 2e^t + C$$

so

$$y = Ce^t + 2(t-1)e^{2t}.$$

Finally we use $y(0) = 1$ to find that $C = 3$. Therefore

$$y = 3e^t + 2(t-1)e^{2t}.$$

(b). The integrating factor is e^{3x}. Then, we get

$$\begin{aligned} e^{3x}y' + 3ye^{3x} &= 12e^{3x} \Longrightarrow \frac{d}{dx}\left(e^{3x}y\right) = 12e^{3x} \Longrightarrow e^{3x}y = 4e^{3x} + C \\ y &= 4 + Ce^{-3x}. \end{aligned}$$

Since $y(0) = 6$, we get $6 = y(0) = 4 + Ce^{-3(0)} \Longrightarrow C = 2 \Longrightarrow y = 4 + 2e^{-3x}$.

(c). $y(t) = \dfrac{\sin t}{t^2}$.

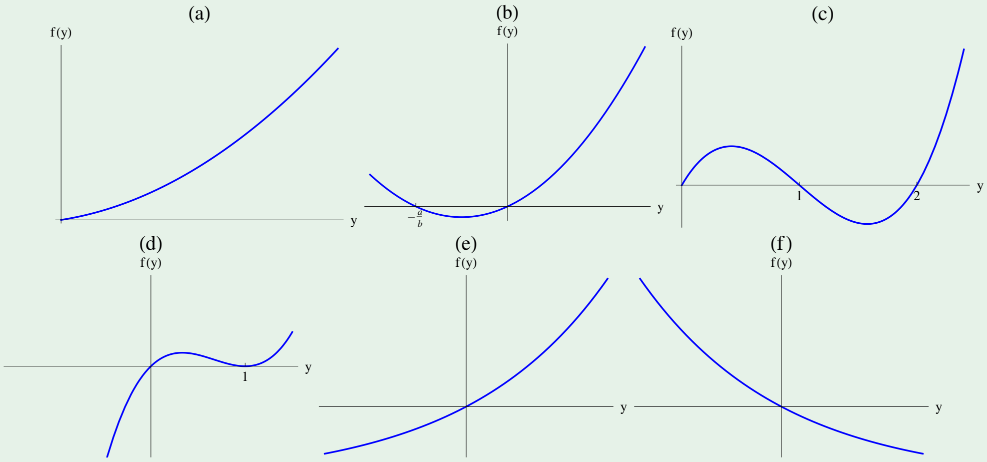
2.3

(a). Rearrange to $\frac{dy}{y^2} = (1-2x)dx$. Then integrate to get $-\frac{1}{y} = x - x^2 + C$, and rearrange to $y = \frac{1}{x^2 - x - C}$. To satisfy $y(0) = -\frac{1}{6}$ we must have $y = \frac{1}{x^2 - x - 6}$. This solution exists for $-2 < x < 3$.

(b). First rearrange to $ydy = -xe^x dx$. Integrating gives $\frac{1}{2}y^2 = (1-x)e^x + C$ which rearranges to $y = \pm\sqrt{2(1-x)e^x + 2C}$. Then we use the initial condition to calculate that $1 = y(0) = +\sqrt{2 + 2C} \Longrightarrow 2C = -1$. Therefore $y = \sqrt{2(1-x)e^x - 1}$.

(c). We can rearrange the ODE to $ydy = \frac{2x}{1+x^2}dx$ and then integrate to obtain $\frac{1}{2}y^2 = \ln(1+x^2) + C$. To satisfy $y(0) = -2$ we must have $C = 2$. Therefore $y = -\sqrt{2\ln(1+x^2) + 4}$.

2.4



(a). $y = 0$ is unstable,

(b). $y = -a/b$ is asymptotically stable, $y = 0$ is unstable,

(c). $y = 1$ is asymptotically stable, $y = 0$ and $y = 2$ are unstable,

(d). $y = 0$ is unstable, $y = 1$ is semi-stable,

(e). $y = 0$ is unstable,

(f). $y = 0$ is asymptotically stable.

2.5

(a). $y = 0$ is unstable, $y = 1$ is asymptotically stable.

(b). omitted.

(c). $y(t) = \dfrac{y_0}{y_0 + (1 - y_0)e^{-\alpha t}}$.

(d). $\lim_{t\to\infty} y(t) = \lim_{t\to\infty} \dfrac{y_0}{y_0 + (1 - y_0)e^{-\alpha t}} = \dfrac{y_0}{y_0 + 0} = 1.$

2.6

(a). The ODE is not exact.

(b). The ODE is exact and has solution $x^2 + 3x + y^2 - 2y = c$.

(c). The ODE is exact and has solution $x^3 - x^2y + 2x + 2y^3 + 3y = c$.

(d). The ODE is exact and has solution $x^2y^2 + 2xy = c$.

(e). The ODE is exact and has solution $e^x \sin y + 2y \cos x = c$.

(f). The ODE is not exact.

2.7

(a). $\mu(x) = e^{3x}$; $\quad (3x^2y + y^3)e^{3x} = c$

(b). $\mu(x) = e^{-x}$; $\quad y = ce^x + 1 + e^{2x}$

(c). $\mu(y) = y$; $\quad xy + y\cos y - \sin y = c$

(d). $\mu(y) = \frac{e^{2y}}{y}$; $\quad xe^{2y} - \ln|y| = c$

2.8

(a). $\frac{x}{x+y}+\ln|x|=c$

(b). Dividing through by x^2 gives

$$\frac{dy}{dx}=\frac{1-3\left(\frac{y}{x}\right)^2}{2\left(\frac{y}{x}\right)}.$$

Now we use the suggested substitutions to obtain

$$v+x\frac{dv}{dx}=\frac{1-3v^2}{2v}$$

which rearranges to

$$x\frac{dv}{dx}=\frac{1-5v^2}{2v}$$

and

$$\frac{2v\,dv}{1-5v^2}=\frac{dx}{x}.$$

Integrating gives

$$-\frac{1}{5}\ln\left|1-5v^2\right|=\ln|x|+C_1.$$

Hence

$$\ln\left|1-5v^2\right|+5\ln|x|=C_2,$$

$$\ln\left|1-5v^2\right|+\ln|x|^5=C_2,$$

$$\left|1-5v^2\right||x|^5=c$$

and

$$\left|x^2-5v^2x^2\right||x|^3=c.$$

Replacing vx by y, we have our implicit solution

$$\left|x^2-5y^2\right||x|^3=c.$$

(c). $y=x\sin(\ln x+C)$

2.9

(a). $y=\pm\sqrt{\frac{5t}{2+5ct^5}}$

(b). $y=\frac{r}{k+cre^{-rt}}$

(c). $y=\pm\sqrt{\frac{\varepsilon}{\sigma+c\varepsilon e^{-2\varepsilon t}}}$

2.10

Thanks to Prof. Eldem for these solutions.

(a). This is a separable equation. Thus, we have

$$\begin{aligned}9y\,dy &= -4x\,dx \Longrightarrow \int 9y\,dy=-\int 4x\,dx+C \Longrightarrow\\ \frac{9}{2}y^2 &= -2x^2+C \Longrightarrow y=\pm\sqrt{\frac{2}{9}C-\frac{4}{9}x^2}=\pm\frac{2}{3}\sqrt{C_1-x^2}. \quad \left(C_1=\frac{C}{2}\right).\end{aligned}$$

(b). This equation can be written as follows.

$$\begin{aligned}\frac{dy}{y^3} &= -(x+1)\,dx \Longrightarrow \int\frac{dy}{y^3}=-\int(x+1)\,dx+C \Longrightarrow\\ \frac{1}{2y^2} &= \frac{x^2}{2}+x+C \Longrightarrow y=\pm\sqrt{\frac{1}{x^2+2x+2C}}.\end{aligned}$$

(c). This separable equation can be written as follows.

$$\begin{aligned}\frac{dx}{(x+1)} &= 3t\,dt \Longrightarrow \int\frac{dx}{(x+1)}=\int 3t\,dt+C \Longrightarrow\\ \ln|(x+1)| &= \frac{3}{2}t^2+C \Longrightarrow x(t)=C_1e^{\frac{3}{2}t^2}-1. \quad \left(C_1=e^C\right).\end{aligned}$$

(d). This separable equation can be solved as follows.

$$\begin{aligned}\frac{dy}{dx} &= -\frac{1}{\sin y} \Longrightarrow -\sin y\,dy=dx \Longrightarrow -\int\sin y\,dy=\int dx+C \Longrightarrow\\ \cos y &= x+C \Longrightarrow y=\arccos(x+C).\end{aligned}$$

(e). This is a separable equation. Therefore, we get

$$\begin{aligned}\frac{dx}{x} &= \cot 2t\,dt \Longrightarrow \ln x=\int\frac{\cos 2t}{\sin 2t}\,dt+C=\frac{1}{2}\ln(\sin 2t)+C \Longrightarrow\\ x &= C_1\sqrt{\sin 2t}. \quad \left(C_1=e^C\right).\end{aligned}$$

(f). Note that this is a separable equation which can be written as follows.

$$\begin{aligned}\frac{dy}{y-1} &= \cot x\,dx \Longrightarrow \int\frac{dy}{y-1}=\int\cot x\,dx+C \Longrightarrow\\ \ln(y-1) &= \ln(\sin x)+C \Longrightarrow y=1+C_1\sin x. \quad \left(C_1=e^C\right).\end{aligned}$$

(g). The integrating factor is

$$e^{\int\left(\frac{2x+1}{x}\right)dx} = xe^{2x}.$$

Consequently, we get

$$\begin{aligned}\frac{d}{dx}\left(yxe^{2x}\right) &= xe^{2x}e^{-2x} = x \Longrightarrow yxe^{2x} = \int x\,dx = \frac{x^2}{2} + C \Longrightarrow \\ y &= \frac{x}{2}e^{-2x} + \frac{C}{x}e^{-2x} = \left(\frac{x}{2} + \frac{C}{x}\right)e^{-2x} = \frac{x^2 + C_1}{2xe^{2x}}.\ (C_1 = 2C).\end{aligned}$$

(h). Let $M(x,y) = 3x^2 + y^2$ and $N(x,y) = 2xy$. Then, we have

$$\frac{\partial M}{\partial y} = 2y = \frac{\partial N}{\partial x},$$

which implies that the equation is exact. Thus, it follows that

$$F(x,y) = \int (3x^2 + y^2)\,dx + g(y) = x^3 + xy^2 + g(y).$$

Taking the derivative with respect to y, we obtain

$$\begin{aligned}\frac{\partial F}{\partial y} &= 2xy + g'(y) = N(x,y) = 2xy \Longrightarrow g'(y) = 0 \Longrightarrow \\ g(y) &= C \Longrightarrow F(x,y) = x^3 + xy^2 = C_1, \quad (C_1 = -C).\end{aligned}$$

(i). This is a homogeneous equation and we let $v = y/x \Longrightarrow y = vx$. Then, we get

$$\begin{aligned}\frac{dy}{dx} &= v + x\frac{dv}{dx} \Longrightarrow v + x\frac{dv}{dx} = v + \tan(v) \Longrightarrow \frac{dv}{dx} = \frac{\tan(v)}{x} \Longrightarrow \int \frac{dv}{\tan(v)} = \int \frac{dx}{x} + C \Longrightarrow \\ \ln(\sin v) &= \ln x + C \Longrightarrow \sin v = C_1 x \Longrightarrow v = \arcsin(C_1 x) \Longrightarrow y = x\arcsin(C_1 x), \quad \left(C_1 = e^C\right).\end{aligned}$$

(j). **Solution 1:** Let $v = x/y \Longrightarrow y = x/v$. This implies that

$$\begin{aligned}\frac{dy}{dx} &= \frac{1}{v} - \frac{x}{v^2}\frac{dv}{dx} \Longrightarrow \frac{1}{v} - \frac{x}{v^2}\frac{dv}{dx} = -\frac{(1+e^v)}{e^v(1-v)} \Longrightarrow \\ \frac{dv}{dx} &= \frac{v^2}{x}\left(\frac{(1+e^v)}{e^v(1-v)} + \frac{1}{v}\right) = \left(\frac{v^2(1+e^v)}{xe^v(1-v)} + \frac{v}{x}\right) \Longrightarrow \\ \frac{e^v(1-v)}{v(v+e^v)}\,dv &= \frac{dx}{x} \Longrightarrow \frac{dv}{v} - \frac{1+e^v}{v+e^v}\,dv = \frac{dx}{x} \Longrightarrow \\ \int \frac{dv}{v} - \int \frac{1+e^v}{v+e^v}\,dv &= \int \frac{dx}{x} + C \Longrightarrow \ln\left(\frac{v}{v+e^v}\right) = \ln x + C \Longrightarrow \\ \frac{v}{v+e^v} &= C_1 x \Longrightarrow \frac{1}{x + ye^{\frac{x}{y}}} = C_1, \quad \left(C_1 = e^C\right) \Longrightarrow \\ x + ye^{\frac{x}{y}} &= C_2.\end{aligned}$$

Solution 2:

$e^{\frac{x}{y}}(y-x)\frac{dy}{dx} + y(1+e^{\frac{x}{y}}) = 0 \Longrightarrow e^{\frac{x}{y}}(y-x) + y(1+e^{\frac{x}{y}})\frac{dx}{dy} = 0$. Then we use the substitution $v = x/y \Longrightarrow x = vy$ and $\frac{dx}{dy} = v + y\frac{dv}{dy}$.Then, we get

$$\begin{aligned}e^v(y - vy) + y(1+e^v)(v + y\frac{dv}{dy}) &= 0 \\ [e^v(1-v) + v(1+e^v)]dy + (1+e^v)ydv &= 0 \\ (e^v + v)dy &= -(1+e^v)ydv \\ \frac{dy}{y} &= -\frac{(1+e^v)}{e^v+v}dv \\ \int \frac{dy}{y} &= -\int \frac{(1+e^v)}{e^v+v}dv + C \\ \ln y &= -\ln(e^v + v) + C \\ y(e^v + v) &= C_1, \left(C_1 = e^C\right) \\ ye^{\frac{x}{y}} + x &= C_1\end{aligned}$$

(k). Let $M(x,y) = 2x + 3y$ and $N(x,y) = 3x + 2y$. Then, we have

$$\frac{\partial M}{\partial y} = 3 = \frac{\partial N}{\partial x},$$

which implies that the equation is exact. Thus, it follows that

$$F(x,y) = \int (2x + 3y)\,dx + g(y) = x^2 + 3xy + g(y).$$

Taking the derivative with respect to y, we obtain

$$\begin{aligned}\frac{\partial F}{\partial y} &= 3x + g'(y) = N(x,y) = 3x + 2y \Longrightarrow g'(y) = 2y \Longrightarrow \\ g(y) &= y^2 + C \Longrightarrow F(x,y) = x^2 + 3xy + y^2 = C_1, \quad (C_1 = -C).\end{aligned}$$

(l). Let $M(x,y) = (x^3 + \frac{y}{x})$ and $N(x,y) = (y^2 + \ln x)$. Then, we have

$$\frac{\partial M}{\partial y} = \frac{1}{x} = \frac{\partial N}{\partial x},$$

which implies that the equation is exact. Thus, it follows that

$$F(x,y) = \int (x^3 + \frac{y}{x})\, dx + g(y) = \frac{x^4}{4} + y \ln x + g(y).$$

Taking the derivative with respect to y, we obtain

$$\begin{aligned} \frac{\partial F}{\partial y} &= \ln x + g'(y) = N(x,y) = y^2 + \ln x \Longrightarrow g'(y) = y^2 \Longrightarrow \\ g(y) &= \frac{y^3}{3} + C \Longrightarrow F(x,y) = \frac{x^4}{4} + y \ln x + \frac{y^3}{3} = C_1, \quad (C_1 = -C). \end{aligned}$$

(m). Let $M(x,y) = (e^x \sin y + \tan y)$ and $N(x,y) = (e^x \cos y + x \sec^2 y)$. Then, we have

$$\frac{\partial M}{\partial y} = e^x \cos y + \sec^2 y = \frac{\partial N}{\partial x},$$

which implies that the equation is exact. Thus, it follows that

$$F(x,y) = \int (e^x \sin y + \tan y)\, dx + g(y) = e^x \sin y + x \tan y + g(y).$$

Taking the derivative with respect to y, we obtain

$$\begin{aligned} \frac{\partial F}{\partial y} &= e^x \cos y + x \sec^2 y + g'(y) = N(x,y) = e^x \cos y + x \sec^2 y \Longrightarrow g'(y) = 0 \Longrightarrow \\ g(y) &= C \Longrightarrow F(x,y) = e^x \sin y + x \tan y = C_1, \quad (C_1 = -C). \end{aligned}$$

(n). Let $M(x,y) = y$ and $N(x,y) = (2x - ye^y)$. Then, we have

$$\frac{\partial M}{\partial y} = 1 \neq \frac{\partial N}{\partial x} = 2$$

Then, we check

$$\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{M} = \frac{-1}{y}.$$

Consequently, y is an integrating factor. Thus, we get

$$M_1(x,y) = y^2 \quad \text{and} \quad N_1(x,y) = (2xy - y^2 e^y)$$

which implies that $M_1(x,y)dx + N_1(x,y)dy = 0$ is exact. Thus, it follows that

$$F(x,y) = \int y^2\, dx + g(y) = y^2 x + g(y).$$

Taking the derivative with respect to y, we obtain

$$\begin{aligned} \frac{\partial F}{\partial y} &= 2xy + g'(y) = N_1(x,y) = (2xy - y^2 e^y) \Longrightarrow g'(y) = -y^2 e^y \Longrightarrow \\ g(y) &= -y^2 e^y + 2ye^y - 2e^y + C \Longrightarrow F(x,y) = y^2 x - e^y (y^2 - 2y + 2) = C_1, \quad (C_1 = -C). \end{aligned}$$

(o). This equation can be written as follows.

$$y' + \frac{1}{x} y = \frac{1}{x} y^{-2}.$$

Hence, we have a Bernoulli equation with $n = -2$. Let $v = y^3 \Longrightarrow v' = 3y^2 y'$. Thus, we have

$$3y^2 y' + 3y^2 \frac{1}{x} y = 3y^2 \frac{1}{x} y^{-2} \Longrightarrow v' + 3\frac{v}{x} = \frac{3}{x}.$$

The integrating factor is x^3 and we get

$$\frac{d}{dx}\left(x^3 v\right) = 3x^2 \Longrightarrow x^3 v = x^3 + C \Longrightarrow v = 1 + \frac{C}{x^3} \Longrightarrow y = \frac{(x^3 + C)^{1/3}}{x}.$$

(p). This equation can be written as follows.

$$y' + y = xy^4.$$

Hence, we have a Bernoulli equation with $n = 4$. Let $v = y^{-3} \Longrightarrow v' = -3y^{-4} y'$. Thus, we have

$$-3y^{-4} y' - 3y^{-4} y = -3x \Longrightarrow v' - 3v = -3x.$$

The integrating factor is e^{-3x} and we get

$$\begin{aligned} \frac{d}{dx}\left(e^{-3x} v\right) &= -3xe^{-3x} \Longrightarrow e^{-3x} v = xe^{-3x} + \frac{1}{3} e^{-3x} + C \Longrightarrow v = \frac{3Ce^{3x} + 3x + 1}{3} \\ \Longrightarrow \quad & y = \left(\frac{3}{3Ce^{3x} + 3x + 1}\right)^{\frac{1}{3}}. \end{aligned}$$

(q). This equation can be written as follows.

$$y' + \frac{2xy}{(1+x^2)} = \frac{2xy^4}{(1+x^2)}.$$

Hence, we have a Bernoulli equation with $n = 4$. Let $v = y^{-3} \Longrightarrow v' = -3y^{-4} y'$. Thus, s solutions. we have

$$-3y^{-4} y' - \frac{6xy^{-3}}{(1+x^2)} = -\frac{6x}{(1+x^2)} \Longrightarrow v' - \frac{6x}{(1+x^2)} v = -\frac{6x}{(1+x^2)}.$$

The integrating factor is $(1+x^2)^{-3}$ and we get

$$\begin{aligned} \frac{d}{dx}\left((1+x^2)^{-3} v\right) &= -6x(1+x^2)^{-4} \Longrightarrow (1+x^2)^{-3} v = (1+x^2)^{-3} + C \Longrightarrow v = 1 + C(1+x^2)^3 \\ \Longrightarrow \quad & y = \left(\frac{1}{1 + C(1+x^2)^3}\right)^{\frac{1}{3}}. \end{aligned}$$

2.11 Thanks to Prof. Eldem for these solutions.

(a). This equation can be written as follows.

$$\frac{dy}{dx} = x^3 e^{-y} \Longrightarrow e^y\, dy = x^3\, dx \Longrightarrow e^y = \frac{x^4}{4} + C \Longrightarrow y = \ln\left(\frac{x^4}{4} + C\right).$$

Since $y(2) = 0$, we get

$$0 = y(2) = \ln\left(\frac{2^4}{4} + C\right) \Longrightarrow C = -3 \Longrightarrow y = \ln\left(\frac{x^4}{4} - 3\right)$$

(b). This equation can be written as follows.

$$\frac{dy}{dx} = \frac{4x(y^2+1)^{\frac{1}{2}}}{y} \Longrightarrow \frac{y}{(y^2+1)^{\frac{1}{2}}}\, dy = 4x\, dx \Longrightarrow (y^2+1)^{\frac{1}{2}} = 2x^2 + C \Longrightarrow y = \sqrt{(2x^2+C)^2 - 1}.$$

Since $y(0) = 1$, we get

$$1 = y(0) = y = \sqrt{\left(2\,(0)^2 + C\right)^2 - 1} \Longrightarrow C = \sqrt{2} \Longrightarrow y = \sqrt{\left(2x^2 + \sqrt{2}\right)^2 - 1}$$

(c). This equation can be expressed as follows.

$$\frac{dy}{dx} = y\cot x \Longrightarrow \frac{dy}{y} = \cot x\, dx \Longrightarrow \ln y = \ln(\sin x) + C \Longrightarrow y = C_1 \sin x, \qquad \left(C_1 = e^C\right).$$

Since $y(\frac{\pi}{2}) = 2$, we get $2 = y(\frac{\pi}{2}) = C_1 \sin(\frac{\pi}{2}) \Longrightarrow C_1 = 2 \Longrightarrow y = 2\sin x$.

(d). This equation can be expressed as follows.

$$\begin{aligned}
\frac{dy}{dx} + 3y &= 2x + 3 \Longrightarrow e^{3x}\frac{dy}{dx} + 3ye^{3x} = (2x+3)\,e^{3x} \Longrightarrow \frac{d}{dx}\left(ye^{3x}\right) = (2x+3)\,e^{3x} \Longrightarrow ye^{3x} = \int (2x+3)\,e^{3x}\, dx \Longrightarrow \\
ye^{3x} &= \frac{2}{3}xe^{3x} - \frac{2}{3}\int e^{3x}\, dx + e^{3x} + C \Longrightarrow ye^{3x} = \frac{2}{3}xe^{3x} + \frac{7}{9}e^{3x} + C \Longrightarrow \\
y &= \frac{1}{9}(6x+7) + Ce^{-3x}.
\end{aligned}$$

Since $y(0) = 1$, we get $1 = y(0) = \frac{1}{9}(6(0)+7) + Ce^{-3(0)} \Longrightarrow C = 2/9 \Longrightarrow y = \frac{1}{9}\left(6x + 2e^{-3x} + 7\right)$.

(e). Let $x + y = v \Longrightarrow y = v - x$. Then, we get

$$\begin{aligned}
\frac{dy}{dx} &= \frac{dv}{dx} - 1 = \frac{10}{ve^v} - 1 \Longrightarrow \frac{dv}{dx} = \frac{10}{ve^v} \Longrightarrow \int ve^v\, dv = \int 10\, dx + C \Longrightarrow \\
ve^v - \int e^v\, dv &= 10x + C \Longrightarrow ve^v - e^v = 10x + C \Longrightarrow (x+y-1)e^{x+y} = 10x + C.
\end{aligned}$$

Since $y(0) = 0 \Longrightarrow C = -1$. Thus, we get

$$(x+y-1)e^{x+y} = 10x - 1.$$

(f). Dividing both sides by x^2, we get $\left(4 - 2\left(\frac{y}{x}\right)^2\right)\frac{dy}{dx} = 2\frac{y}{x}$. Let $v = y/x \Longrightarrow \frac{dy}{dx} = v + x\frac{dv}{dx}$. Then, we have

$$\begin{aligned}
v + x\frac{dv}{dx} &= \frac{2v}{(4 - 2v^2)} \Longrightarrow x\frac{dv}{dx} = \frac{v}{(2-v^2)} - v = \frac{v^3 - v}{(2 - v^2)} \Longrightarrow \\
\frac{dv}{dx} &= \frac{1}{x}\frac{v^3 - v}{(2 - v^2)} \Longrightarrow \int \frac{(2-v^2)}{v^3 - v}\, dv = \int \frac{dx}{x} + C.
\end{aligned}$$

If we use partial fraction expansion for the first integral, we get

$$\frac{(2-v^2)}{v^3 - v} = \frac{A}{v} + \frac{B}{v-1} + \frac{D}{v+1}.$$

where $A = -2$, $B = 1/2$ and $D = 1/2$. This implies that

$$\begin{aligned}
\int \frac{(2-v^2)}{v^3 - v}\, dv &= \int \left(-\frac{2}{v} + \frac{1/2}{v-1} + \frac{1/2}{v+1}\right) = \ln x + C \Longrightarrow \\
\ln\left(\frac{(v^2-1)^{\frac{1}{2}}}{v^2}\right) &= \ln x + C \Longrightarrow \frac{\sqrt{v^2 - 1}}{v^2} = C_1 x \Longrightarrow \\
\frac{\sqrt{y^2 - x^2}}{y^2} &= C_1, \ \left(C_1 = e^C\right).
\end{aligned}$$

Since $y(3) = -5 \Longrightarrow \sqrt{\frac{25-9}{25}} = C_1 \Longrightarrow C_1 = \frac{4}{5}$. Consequently, we get

$$\frac{\sqrt{y^2 - x^2}}{y^2} = \frac{4}{5} \Longrightarrow y^2 - \frac{16}{25}y^4 + x^2 = 0.$$

(g). This equation can be written as follows.

$$\frac{dy}{dx} = -\frac{(x-y)}{(3x+y)} = -\frac{(1-\frac{y}{x})}{(3+\frac{y}{x})}.$$

Let $v = y/x \Longrightarrow \frac{dy}{dx} = v + x\frac{dv}{dx}$. Then, we get

$$\begin{aligned} v + x\frac{dv}{dx} &= -\frac{(1-v)}{(3+v)} \Longrightarrow \frac{dv}{dx} = -\frac{1}{x}\left(\frac{(1-v)}{(3+v)} + v\right) = -\frac{1}{x}\left(\frac{(v^2+2v+1)}{(3+v)}\right) \Longrightarrow \\ \frac{(3+v)\,dv}{(v+1)^2} &= -\frac{dx}{x} \Longrightarrow \int \frac{A\,dv}{(v+1)} + \int \frac{B\,dv}{(v+1)^2} = -\ln x + C, \end{aligned}$$

where $B = 2$ and $A = 1$. Consequently, we have

$$\int \frac{dv}{(v+1)} + \int \frac{2\,dv}{(v+1)^2} = -\ln x + C \Longrightarrow \ln(v+1) - \frac{2}{(v+1)} = -\ln x + C.$$

Substituting $v = y/x$, we get

$$\ln(\frac{y+x}{x}) - \frac{2x}{(y+x)} = -\ln x + C \Longrightarrow \ln(y+x) - \frac{2x}{(y+x)} = C.$$

Since $y(3) = -2$, it follows that

$$\ln(-2+3) - \frac{6}{(-2+3)} = C \Longrightarrow C = -6.$$

Consequently, we get

$$\ln(y+x) - \frac{2x}{(y+x)} + 6 = 0.$$

(h). **Solution 1:** This equation can be rearranged as follows.

$$\frac{dy}{dx} = \frac{x^3 - xy^2}{x^2 y} = \frac{1-(\frac{y}{x})^2}{\frac{y}{x}}.$$

Let $v = y/x \Longrightarrow \frac{dy}{dx} = v + x\frac{dv}{dx}$. Then, we get

$$\begin{aligned} v + x\frac{dv}{dx} &= \frac{1-v^2}{v} \Longrightarrow \frac{dv}{dx} = \frac{1}{x}\left(\frac{1-v^2}{v} - v\right) = \frac{1}{x}\left(\frac{1-2v^2}{v}\right) \Longrightarrow \\ \frac{v\,dv}{(1-2v^2)} &= \frac{dx}{x} \Longrightarrow \int \frac{v\,dv}{(1-2v^2)} = \ln x + C \Longrightarrow -\frac{1}{4}\ln\left|1-2v^2\right| = \ln x + C \Longrightarrow \\ \frac{1}{|(1-2v^2)|^{1/4}} &= e^C x \Longrightarrow \left|\left(1-2v^2\right)\right| = \frac{1}{e^{4C}x^4}. \end{aligned}$$

Since $y(1) = 1$, we get $v(1) = 1$ which implies that $C = 0$. Consequently, we get

$$\left|\left(1 - 2\left(\frac{y}{x}\right)^2\right)\right| = \frac{1}{x^4} \Longrightarrow \left|\left(x^2 - 2y^2\right)\right| = \frac{1}{x^2}.$$

Solution 2 : It is a exact equation also. $\frac{dy}{dx} = \frac{x^3 - xy^2}{x^2 y} \Longrightarrow \left(x^3 - xy^2\right)dx - x^2 y dy = 0.$

Let $M = x^3 - xy^2$ and $N = -x^2 y$.Then

$$\frac{\partial M}{\partial y} = -2xy = \frac{\partial N}{\partial x}$$

Therefore

$$\begin{aligned} F(x,y) &= \int (x^3 - xy^2)dx + g(y) = \frac{x^4}{4} - \frac{x^2y^2}{2} + g(y) \Longrightarrow \\ \frac{\partial F}{\partial y} &= -x^2y + g'(y) = -x^2 y \Longrightarrow g'(y) = 0 \\ g(y) &= C \Longrightarrow F(x,y) = \frac{x^4}{4} - \frac{x^2y^2}{2} + C = 0. \end{aligned}$$

Since $y(1) = 1$, we get $C = -\frac{1}{4} \Longrightarrow x^4 - 2x^2y^2 = 1$.

(i). Let $M = xy^2 + y$ and $N = 2y - x$. Then, we have

$$\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{M} = \frac{2xy + 1 - (-1)}{xy^2 + y} = \frac{2}{y}.$$

This implies that the integrating factor is $p(y) = y^{-2}$. Let $M_1 = x + y^{-1}$ and $N_1 = 2y^{-1} - xy^{-2}$.Then, we have

$$\frac{\partial M_1}{\partial y} = -\frac{1}{y^2} = \frac{\partial N_1}{\partial x}$$

which implies that the equation is exact. Thus, we get

$$\begin{aligned} F(x,y) &= \int \left(x + y^{-1}\right)\,dx + g(y) = \frac{x^2}{2} + \frac{x}{y} + g(y) \Longrightarrow \\ \frac{\partial F}{\partial y} &= -\frac{x}{y^2} + g'(y) = 2y^{-1} - xy^{-2} \Longrightarrow g'(y) = 2y^{-1} \Longrightarrow \\ g(y) &= 2\ln y + C \Longrightarrow F(x,y) = \frac{x^2}{2} + \frac{x}{y} + 2\ln y + C = 0. \end{aligned}$$

Since $y(0) = 3$, we get $C = -2\ln 3$. Therefore, it follows that

$$F(x,y) = \frac{x^2}{2} + \frac{x}{y} + 2\ln y = 2\ln 3.$$

(j). This is a Bernoulli equation with $n = 2$. Let $v = y^{1-2} = y^{-1}$. Then, it follows that

$$\frac{dv}{dx} = -y^{-2}\frac{dy}{dx} \Longrightarrow -y^{-2}y' + \frac{1}{x}y^{-1} = -1 \Longrightarrow \frac{dv}{dx} + \frac{v}{x} = -1.$$

Note that the integrating factor is $e^{\int \frac{dx}{x}} = x$. Thus we get

$$\begin{aligned} x\frac{dv}{dx} + v &= -x \Longrightarrow \frac{d}{dx}(xv) = -x \Longrightarrow xv = -\frac{x^2}{2} + C \Longrightarrow v = \frac{C}{x} - \frac{x}{2} \\ \Longrightarrow & \quad y = \frac{2x}{2C - x^2}. \end{aligned}$$

Since $y(1) = 2$, we get $C = 1$. Consequently, we have

$$y = \frac{2x}{2 - x^2}.$$

3.1

(a). The characteristic equation is $0 = r^2 - 3r + 2 = (r-1)(r-2)$. The roots are $r_1 = 1$ and $r_2 = 2$. Therefore the general solution to the ODE is $y = c_1e^t + c_2e^{2t}$ for constants c_1 and c_2.

The first initial condition gives $1 = y(0) = c_1 + c_2$. Since $y'(x) = c_1e^t + 2c_2e^{2t}$, the second initial condition gives $1 = y'(0) = c_1 + 2c^2$. It follows that $c_1 = 1$ and $c_2 = 0$.

Therefore the solution to the IVP is $y(t) = e^t$.

(b). $y = \frac{5}{2}e^{-t} - \frac{1}{2}e^{-3t}$

(c). $y = -1 - e^{-3t}$

(d). $y = \frac{13 + 5\sqrt{13}}{26}e^{\frac{(-5+\sqrt{13})t}{2}} + \frac{13 - 5\sqrt{13}}{26}e^{\frac{(-5-\sqrt{13})t}{2}}$

3.2

(a). Clearly $t^2y_1'' - 2y_1 = t^2(t^2)'' - 2t^2 = t^2(2) - 2t^2 = 0$ and $t^2y_2'' - 2y_2 = t^2(t^{-1})'' - 2t^{-1} = t^2(2t^{-3} - 2t^{-1} = 0$.

Next we calculate that

$$W(y_1, y_2)(t) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} t^2 & t^{-1} \\ 2t & -t^{-2} \end{vmatrix} = -1 + 2 = 1.$$

Since $W \neq 0$, y_1 and y_2 form a fundamental set of solutions of the ODE.

(b). Yes

(c). Yes

(d). Yes

3.3

(a). The characteristic equation is

$$r^2 - 2r + 2 = 0.$$

Thus

$$r = \frac{2 \pm \sqrt{4-8}}{2} = 1 \pm i.$$

Hence we have complex roots with $\lambda = 1$ and $\mu = 1$. The general solution to the ODE is therefore

$$y = c_1 e^t \cos t + c_2 e^t \sin t.$$

(b). $y = c_1 e^{-t} \cos t + c_2 e^{-t} \sin t$

(c). $y = c_1 e^{2t} + c_2 e^{-4t}$

(d). $y = c_1 e^t \cos \sqrt{5}t + c_2 e^t \sin \sqrt{5}t$

(e). $y = c_1 e^{-3t} \cos 2t + c_2 e^{-3t} \sin 2t$

(f). $y = c_1 \cos \frac{4}{3}t + c_2 \sin \frac{4}{3}t$

(g). $y = c_1 e^t + c_2 t e^t$

(h). $y = c_1 e^{-\frac{t}{3}} + c_2 t e^{-\frac{t}{3}}$

(i). $y = c_1 e^{-\frac{3t}{2}} + c_2 t e^{-\frac{3t}{2}}$

(j). $y = c_1 e^{-\frac{t}{2}} + c_2 e^{\frac{3t}{2}}$

(k). $y = c_1 e^t \cos 3t + c_2 e^t \sin 3t$

(l). $y = c_1 e^{3t} + c_2 t e^{3t}$

(m). $y = c_1 e^{-\frac{t}{4}} + c_2 e^{-4t}$

(n). $y = c_1 e^{-\frac{5t}{2}} + c_2 t e^{-\frac{5t}{2}}$

(o). $y = c_1 e^{\frac{2t}{5}} + c_2 t e^{\frac{2t}{5}}$

(p). $y = c_1 e^{-\frac{t}{2}} \cos \frac{t}{2} + c_2 e^{-\frac{t}{2}} \sin \frac{t}{2}$

(q). $y = -e^{-\frac{t}{3}} \cos 3t + \frac{5}{9} e^{-\frac{t}{3}} \sin 3t$

(r). The characteristic equation is

$$0 = r^2 - 6r + 9 = (r-3)^2$$

which implies that we have the repeated root $r = 3$. Therefore the general solution to the ODE is

$$y = c_1 e^{3t} + c_2 t e^{3t}.$$

Since

$$y' = 3c_1 e^{3t} + c_2 e^{3t} + 3c_2 t e^{3t}$$

we have that

$$\begin{aligned} 0 &= y(0) = c_1 + 0 \\ 2 &= y'(0) = 3c_1 + c_2 + 0, \end{aligned}$$

which imples that $c_1 = 0$ and $c_2 = 2$. Therefore the solution to the IVP is

$$y = 2te^{3t}.$$

3.4

(a). (i) First we calculate that $y_1' = 1$, $y_1'' = 0$ and that

$$t^2 y_1'' + 2ty_1' - 2y_1 = t^2(0) + 2t(1) - 2(t) = 2t - 2t = 0.$$

Hence $y_1(t) = t$ solves the ODE.

(ii) As per the hint, we start with $y_2(t) = v(t)y_1(t) = v(t)t$. Then $y_2' = v't + v$ and $y_2'' = v''t + 2v'$. Substituting into the ODE, we calculate that

$$\begin{aligned} 0 &= t^2 y_2'' + 2ty_2' - 2y_2 \\ &= t^2(v''t + 2v') + 2t(v't + v) - 2vt \\ &= t^3 v'' + v'(2t^2 + 2t^2) + v(2t - 2t) \\ &= t^3 v'' + 4t^2 v' \\ &= t^2(tv'' + 4v'). \end{aligned}$$

Letting $u = v'$, we obtain the first order ODE

$$t\frac{du}{dt} + 4u = 0.$$

We calculate that

$$\begin{aligned} t\frac{du}{dt} &= -4u \\ \frac{du}{u} &= -4\frac{dt}{t} \\ \int \frac{du}{u} &= -4\int \frac{dt}{t} \\ \ln|u| &= -4\ln|t| + C \\ u &= \pm e^C t^{-4} = ct^{-4} \end{aligned}$$

and

$$v = \int u \; dt = \int ct^{-4} \; dt = -\frac{1}{3}ct^{-3} + k.$$

Thus $y_2(t) = v(t)t = -\frac{1}{3}ct^{-2} + kt$. Choosing $c = -3$ and $k = 0$, we obtain the solution

$$y_2(t) = t^{-2}.$$

(iii) Since $y_2' = -2t^{-3}$ and $y_2'' = 6t^{-4}$, we have that

$$\begin{aligned} t^2 y_2'' + 2ty_2' - 2y_2 &= t^2(6t^{-4}) + 2t(-2t^{-3}) - 2t^{-2} \\ &= 6t^{-2} - 4t^{-2} - 2t^{-2} \\ &= 0 \end{aligned}$$

as required.

(iv) We have that

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} t & t^{-2} \\ 1 & -2t^{-3} \end{vmatrix} = -2t^{-2} - t^{-2} = -3t^{-2} \neq 0.$$

Therefore y_1 and y_2 are linearly independent.

(b). $y_2(t) = t^3$

(c). $y_2(t) = t^{-1}\ln t$

(d). $y_2(t) = te^t$

(e). (This is a tricky one. Don't worry if you didn't solve it.) (i), (iii) and (iv) are omitted.

(ii) Let $y_2(x) = v(x)y_1(x)$. Then $y_2' = v'y_1 + vy_1'$, $y_2'' = v''y_1 + 2v'y_1' + vy_1''$ and

$$\begin{aligned} 0 &= xy_2'' - y_2' + 4x^3 y_2 \\ &= xv''y_1 + 2xv'y_1' + xvy_1'' - v'y_1 - vy_1' + 4x^3 vy_1 \\ &= xv''y_1 + (2xy_1' - y_1)v' + (xy_1'' - y_1' + 4x^3 y_1)v \\ &= xy_1 v'' + (2xy_1' - y_1)v' \end{aligned}$$

since y_1 solves the ODE. Let $u = v'$. Then we have the first order ODE

$$u' + \left(2\frac{y_1'}{y_1} - \frac{1}{x}\right)u = 0.$$

Recall that to solve the linear ODE $u' + p(x)u = 0$, we use the integrating factor $\mu(x) = e^{\int p(x)\,dx}$ and calculate that

$$\begin{aligned} u' + pu &= 0 \\ \mu u' + \mu pu &= 0 \\ (\mu u)' &= 0 \\ \mu u &= c \\ u &= \frac{c}{\mu} = ce^{-\int p(x)\,dx}. \end{aligned}$$

It follows that

$$u(x) = ce^{-\int\left(2\frac{y_1'}{y_1} - \frac{1}{x}\right)dx} = ce^{-2\ln y_1 + \ln x} = \frac{cx}{y_1^2} = \frac{cx}{\sin^2 x^2}.$$

Using the substitution $t = x^2$ we calculate that $dt = 2x\,dx$ and

$$\begin{aligned} v(x) &= \int u(x)\,dx = c\int \frac{x}{\sin^2 x^2}\,dx = \frac{c}{2}\int \frac{1}{\sin^2 t}\,dt \\ &= \frac{c}{2}\int \operatorname{cosec}^2 t\,dt = -\frac{c}{2}\cot t + k = -\frac{c}{2}\cot x^2 + k. \end{aligned}$$

Choosing $c = -2$ and $k = 0$ gives $v(x) = \cot x^2$.

Therefore

$$y_2(x) = v(x)y_1(x) = \cot x^2 \sin x^2 = \cos x^2.$$

(f). (ii) Let $y_2(x) = v(x)y_1(x) = ve^x$. Then $y_2' = (v' + v)e^x$, $y_2'' = (v'' + 2v' + v)e^x$ and

$$\begin{aligned} 0 &= (x-1)y'' - xy' + y \\ &= \left[(x-1)(v'' + 2v' + v) - x(v' + v) + v\right]e^x \\ &= \left[(x-1)v'' + (x-2)v'\right]e^x. \end{aligned}$$

Letting $u = v'$ we obtain the first order ODE

$$u' + \left(\frac{x-2}{x-1}\right)u = 0$$

which has solution

$$u(x) = ce^{-x}(x-1).$$

By integrating, we obtain

$$v(x) = \int u(x)\,dx = -cxe^{-x}.$$

Choosing $c = -1$ gives $v(x) = xe^{-x}$. Therefore

$$y_2(x) = v(x)y_1(x) = xe^{-x}e^x = x.$$

3.5

(a). First we must consider the homogeneous equation

$$y'' - 2y' - 3y = 0.$$

The characteristic equation is

$$0 = r^2 - 2r - 3 = (r-3)(r+1)$$

which implies that $r_1 = 3$ and $r_2 = -1$. Hence the general solution of the homogeneous equation is

$$y = c_1 e^{3t} + c_2 e^{-t}.$$

Next we must find a particular solution to our ODE. Since e^{2t} does not solve the homogeneous equation, our ODE does not have resonance. Thus we try the ansatz $Y(t) = Ae^{2t}$ for some constant A. Then we calculate that $Y' = 2Ae^{2t}$, that $Y'' = 4Ae^{2t}$ and that

$$\begin{aligned} 3e^{2t} &= Y'' - 2Y' - 3Y \\ &= 4Ae^{2t} - 2(2Ae^{2t}) - 3(Ae^{2t}) = -3Ae^{2t}. \end{aligned}$$

Thus we must have $A = -1$. Therefore the general solution to our ODE is

$$y = c_1 e^{3t} + c_2 e^{-t} - e^{2t}.$$

(b). $y = c_1 e^{-t}\cos 2t + c_2 e^{-t}\sin 2t + \frac{12}{17}\sin 2t + \frac{3}{17}\cos 2t$

(c). $y = c_1 e^{3t} + c_2 e^{-t} + \frac{1}{192}(72t^2 + 36t + 9 - 128e^t)e^{-t}$

(d). $y = c_1 + c_2 e^{-2t} + \frac{3}{2}t - \frac{1}{2}\sin 2t - \frac{1}{2}\cos 2t$

(e). $y = c_1 \cos 3t + c_2 \sin 3t + \frac{1}{162}(9t^2 - 6t + 1)e^{3t} + \frac{2}{3}$

(f). The homogeneous equation $y'' + 2y' + y = 2e^{-t}$ has characteristic equation

$$0 = r^2 + 2r + r = (r+1)^2$$

and general solution $y = c_1 e^{-t} + c_2 te^{-t}$.

Next we need to find a particular solution to our ODE. Our equation has resonance since both e^{-t} and te^{-t} solve the homogeneous equation. Hence we must multiply by t again and consider the ansatz $Y(t) = At^2e^{-t}$ for some constant A. Then we calculate that $Y' = 2Ate^{-t} - At^2e^{-t}$, that $Y'' = 2Ae^{-t} - 4Ate^{-t} + At^2e^{-t}$ and that

$$\begin{aligned} 2e^{-t} &= Y'' + 2Y' + Y \\ &= e^{-t}\big((2A - 4At + At^2) + 2(2At - At^2) + (At^2)\big) \\ &= 2Ae^{-t}. \end{aligned}$$

Therefore the general solution to our ODE is

$$y = c_1 e^{-t} + c_2 te^{-t} + t^2 e^{-t}.$$

(g). $y = c_1 e^{-t} + c_2 e^{-\frac{t}{2}} + t^3 - 9t^2 + 47t - 90 - \frac{3}{10}\sin t - \frac{9}{10}\cos t$

(h). $y = c_1 \cos t + c_2 \sin t - \frac{1}{3}t\cos 2t - \frac{5}{9}\sin 2t$

(i). First we consider the homogeneous equation

$$y'' + y' + 4y = 0.$$

Its characteristic equation, $r^2 + r + 4 = 0$, has roots

$$r_{1,2} = \frac{-1 \pm \sqrt{1^2 - 16}}{2} = -\frac{1}{2} \pm \frac{\sqrt{15}}{2}i.$$

Hence $\lambda = -\frac{1}{2}$ and $\mu = \frac{\sqrt{15}}{2}$. Therefore this homogeneous ODE has general solution

$$y = c_1 e^{-\frac{t}{2}}\cos\frac{\sqrt{15}t}{2} + c_2 e^{-\frac{t}{2}}\sin\frac{\sqrt{15}t}{2}.$$

Now recall that $\sinh t = \frac{1}{2}(e^t - e^{-t})$. Thus we try the ansatz $Y(t) = Ae^t + Be^{-t}$ for constants A and B. We calculate that $Y' = Ae^t - Be^{-t}$ and $Y'' = Y$. Therefore

$$\begin{aligned} e^t - e^{-t} = 2\sinh t &= Y'' + Y' + 4Y \\ &= (Ae^t + Be^{-t}) + (Ae^t - Be^{-t}) + 4(Ae^t + Be^{-t}) \\ &= 6Ae^t + 4Be^{-t} \end{aligned}$$

which implies that $A = \frac{1}{6}$ and $B = -\frac{1}{4}$. Therefore the general solution to the ODE is

$$y = c_1 e^{-\frac{t}{2}}\cos\frac{\sqrt{15}t}{2} + c_2 e^{-\frac{t}{2}}\sin\frac{\sqrt{15}t}{2} + \frac{1}{6}e^t - \frac{1}{4}e^{-t}.$$

3.6

(a). $y = e^t - \frac{1}{2}e^{-2t} - t - \frac{1}{2}$

(b). $y = 4te^t - 3e^t + \frac{1}{6}t^3e^t + 4$

(c). $y = \frac{7}{10}\sin 2t - \frac{19}{40}\cos 2t + \frac{1}{4}t^2 - \frac{1}{8} + \frac{3}{5}e^t$

(d). First consider the homogeneous equation

$$-y'' + 6y' - 16y = 0.$$

The characteristic equation is

$$-r^2 + 6r - 16 = 0$$

which has roots

$$r = 3 \pm i\sqrt{7}.$$

Therefore the general solution of

$$-y'' + 6y' - 16y = 0$$

is

$$y(t) = c_1e^{3t}\sin(\sqrt{7}t) + c_2e^{3t}\cos(\sqrt{7}t).$$

Next consider

$$-y'' + 6y' - 16y = 1.$$

Trying the ansatz $Y(t) = C$, we see that

$$1 = -Y'' + 6Y' - 16Y = -16C.$$

We must choose $C = -\frac{1}{16}$. Hence $Y(t) = -\frac{1}{16}$.

Now consider

$$-y'' + 6y' - 16y = 6e^{3t}\sin(2t).$$

We try the ansatz

$$Y(t) = Ae^{3t}\cos 2t + Be^{3t}\sin 2t$$

and find that

$$\begin{aligned}
6e^{3t}\sin 2t = -Y'' + 6Y' - 16Y \\
&= -e^{3t}\Big((5A+12B)\cos 2t + (5B-12A)\sin 2t\Big) \\
&\quad + 6e^{2t}\big((3A+2B)\cos 2t + (3B-2A)\sin 2t\big) \\
&\quad - 16e^{3t}(A\cos 2t + B\sin 2t) \\
&= e^{3t}\cos 2t(-5A - 12B + 16A + 12B - 16A) \\
&\quad + e^{3t}\sin 2t(-5B + 12A + 18B - 12A - 16B) \\
&= e^{3t}\cos 2t(-5A) + e^{3t}\sin 2t(-3B).
\end{aligned}$$

Thus, we need $A = 0$ and $B = -2$. Hence

$$Y(t) = -2e^{3t}\sin 2t.$$

Next we add these 3 solutions together. Therefore, the general solution of the ODE is

$$y(t) = c_1e^{3t}\sin(\sqrt{7}t) + c_2e^{3t}\cos(\sqrt{7}t) - 2e^{3t}\sin(2t) - \frac{1}{16}.$$

The final step is to satisfy the initial conditions. We calculate that

$$\frac{15}{16} = y(0) = 0 + c_2 - 0 - \frac{1}{16} \qquad \Longrightarrow \qquad c_2 = 1.$$

and

$$\begin{aligned}
-1 &= y'(0) \\
&= 3c_1e^{3t}\sin(\sqrt{7}t) + \sqrt{7}c_1e^{3t}\cos(\sqrt{7}t) + 3e^{3t}\cos(\sqrt{7}t) \\
&\quad - \sqrt{7}e^{3t}\sin(\sqrt{7}t) - 6e^{3t}\sin(2t) - 4e^{3t}\cos(2t)\Big|_{t=0} \\
&= 0 + \sqrt{7}c_1 + 3 - 0 - 0 - 4 \qquad \Longrightarrow \qquad c_1 = 0.
\end{aligned}$$

Therefore, the solution of the IVP is

$$y(t) = e^{3t}\cos(\sqrt{7}t) - 2e^{3t}\sin(2t) - \frac{1}{16}.$$

3.7

(a). Note first that $y_1(t) = \cos t$ and $y_2(t) = \sin t$ form a fundamental set of solutions of the homogeneous equation $y'' + y = 0$. The Wronskian of y_1 and y_2 is

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{vmatrix} = \cos^2 t + \sin^2 t = 1.$$

Using the theorem from class, we calculate that

$$\begin{aligned}
Y(t) &= -y_1\int \frac{y_2 g}{W} + y_2\int \frac{y_1 g}{W} \\
&= -\cos t\int \sin t\tan t\, dt + \sin t\int \cos t\tan t\, dt \\
&= -\cos t\int \frac{\sin^2 t}{\cos t}dt + \sin t\int \sin t\, dt \\
&= -\cos t\int \frac{1-\cos^2 t}{\cos t}dt + \sin t\int \sin t\, dt \\
&= \cos t\int \cos t - \sec t\, dt + \sin t\int \sin t\, dt \\
&= \cos t(\sin t - \ln(\sec t + \tan t)) + \sin t(-\cos t) \\
&= -(\cos t)\ln(\sec t + \tan t)
\end{aligned}$$

is a particular solution to the non-homogeneous ODE.

Therefore the general solution of the ODE is

$$y(t) = c_1\cos t + c_2\sin t - (\cos t)\ln(\tan t + \sec t).$$

(b). $y = c_1\cos 2t + c_2\sin 2t + \frac{3}{4}(\sin 2t)\ln\sin 2t - \frac{3}{2}t\cos 2t$

(c). $y = c_1e^{-2t} + c_2te^{-2t} - e^{-2t}\ln t$

(d). $y = c_1e^t + c_2te^t - \frac{1}{2}e^t\ln(1+t^2) + te^t\tan^{-1} t$

3.8

(a). $\dfrac{d^6y}{dt^6} - 12\dfrac{d^5y}{dt^5} + 59\dfrac{dy^4}{dt^4} - 138\dfrac{d^3y}{dt^3} + 130\dfrac{d^2y}{dt^2} = 0$

(b). The first two terms correspond to a double root $r = 1$. The last four terms correspond to a double complex root $r = 2 \pm i$. Consequently, the characteristic equation is

$$0 = (r-1)^2(r^2 - 4r + 5)^2 = r^6 - 10r^5 + 43r^4 - 100r^3 + 131r^2 - 90r + 25$$

Then, the differential equation is

$$\frac{d^6y}{dt^6} - 10\frac{d^5y}{dt^5} + 43\frac{d^4y}{dt^4} - 100\frac{d^3y}{dt^3} + 131\frac{d^2y}{dt^2} - 90\frac{dy}{dt} + 25y = 0.$$

(c). $\dfrac{d^5y}{dt} - 6\dfrac{d^4y}{dt^4} + 10\dfrac{d^3y}{dt^3} - 44\dfrac{d^2y}{dt^2} + 104\dfrac{dy}{dt} - 80y = 0.$

3.9

Thank you to Prof. Eldem for the following solutions.

(a). Note that the characteristic equation is $r^4 + 2r^3 + 6r^2 + 2r + 5 = 0$. Since $\sin t$ is a solution, two roots are $\pm i$. Thus the characteristic equation has $(r^2 + 1)$ as a factor. Dividing the characteristic equation by $(r^2 + 1)$, we get

$$r^2 + 2r + 5 = 0.$$

Thus the other roots are $-1 \pm 2i$. Consequently, the general solution is

$$y(t) = c_1 \cos t + c_2 \sin t + c_3 e^{-t} \cos 2t + c_4 e^{-t} \sin 2t.$$

(b). The characteristic equation is $0 = r^4 + r^2 = r^2(r^2 + 1)$ and its roots are 0, 0, $\pm i$. Consequently, the general solution of the homogeneous equation is

$$y(x) = c_1 + c_2 x + c_3 \cos x + c_4 \sin x.$$

There is resonance for all the terms on the right hand side of the equation. For the first term on the right, we try the ansatz $y_{p1} = x^2(a+bx+cx^2)$ because the degree of the zero root is two. For the second term, we try the ansatz $y_{p2} = x(d\cos x+f \sin x)$ because the multiplicity of the imaginary root is one. Thus, we have

$$\begin{aligned} y'_{p1} &= 2ax + 3bx^2 + 4cx^3 \\ y''_{p1} &= 2a + 6bx + 12cx^2 \\ y'''_{p1} &= 6b + 24cx \\ y^{(4)}_{p1} &= 24c. \end{aligned}$$

Using these expressions in the equation, we get

$$24c + 2a + 6bx + 12cx^2 = 3x^2.$$

This implies that $24c + 2a = 0$, $b = 0$ and $c = \frac{1}{4}$. Thus $a = -3$. Consequently, we have $y_{p1} = \frac{1}{4}x^4 - 3x^2$. For the second term, we get

$$\begin{aligned} y'_{p2} &= d(\cos x - x\sin x) + f(\sin x + x\cos x) \\ y''_{p2} &= d(-2\sin x - x\cos x) + f(2\cos x - x\sin x) \\ y'''_{p2} &= d(-3\cos x + x\sin x) + f(-3\sin x - x\cos x) \\ y^{(4)}_{p2} &= d(4\sin x + x\cos x) + f(-4\cos x + x\sin x) \end{aligned}$$

Using these expressions in the equation, we get

$$d(4\sin x + x\cos x) + f(-4\cos x + x\sin x) + d(-2\sin x - x\cos x) + f(2\cos x - x\sin x) = 4\sin x - 2\cos x.$$

This implies that $d = 2$ and $f = 1$. Hence

$$y_{p2} = 2x\cos x + x\sin x.$$

Therefore, the general solution is

$$\begin{aligned} y(x) &= c_1 + c_2 x + c_3 \cos x + c_4 \sin x + y_{p1} + y_{p2} \\ &= c_1 + c_2 x + c_3 \cos x + c_4 \sin x + \frac{1}{4}x^4 - 3x^2 + 2x\cos x + x\sin x \\ &= c_1 + c_2 x - 3x^2 + \frac{1}{4}x^4 + (c_3 + 2x)\cos x + (c_4 + x)\sin x. \end{aligned}$$

(c). The characteristic equation is

$$0 = r^3 - 2r^2 + 4r - 8 = (r^2 + 4)(r - 2)$$

and its roots are 2 and $\pm 2i$. The general solution of the ODE is

$$y(x) = c_1 e^{2x} + c_2 \cos 2x + c_3 \sin 2x.$$

Since $y(0) = 2$, we get $c_1 + c_2 = 2$. Since $y'(x) = 2c_1 e^{2x} - 2c_2 \sin 2x + 2c_3 \cos 2x$ and $y'(0) = 0$, it follows that $2c_1 + 2c_3 = 0$. Furthermore, since $y''(x) = 4c_1 e^{2x} - 4c_2 \cos 2x - 4c_3 \sin 2x$ and $y''(0) = 0$, we also have $4c_1 - 4c_2 = 0$. Thus $c_1 = c_2 = 1$ and $c_3 = -1$. Therefore the solution of the initial value problem is

$$y(x) = e^{2x} + \cos 2x - \sin 2x.$$

4.1

(a). Note that

$$\mathcal{L}\left[e^{-2t}\right](s) = \int_0^\infty e^{-st}e^{-2t}\,dt = \int_0^\infty e^{-t(s+2)}\,dt = \frac{1}{s+2}$$

(b). Note that

$$\mathcal{L}\left[t^2\right] = (-1)^2 \frac{d^2}{ds^2}\left(\mathcal{L}[1]\right) = (-1)^2 \frac{d^2}{ds^2}\left(\frac{1}{s}\right) = (-1)^2 \frac{d}{ds}\left(-\frac{1}{s^2}\right) = \frac{2}{s^3}.$$

Consequently, we get

$$\mathcal{L}\left[3t^2\right] = 3\mathcal{L}\left[t^3\right] = \frac{6}{s^3}.$$

(c). Note that $\cos^2 2t = \frac{1}{2}(1 + \cos 4t)$ which implies that

$$\mathcal{L}\left[\cos^2 2t\right] = \frac{1}{2}\mathcal{L}\left[1 + \cos 4t\right] = \frac{1}{2}\left(\frac{1}{s} + \frac{s}{s^2+16}\right).$$

(d). Since multiplication by t is equivalent to taking derivative with respect to s and multiplying by -1, it follows that

$$\begin{aligned}\mathcal{L}\left[t\cos t + te^t\right] &= (-1)\frac{d}{ds}\mathcal{L}\left[\cos t + e^t\right] = (-1)\frac{d}{ds}\left(\frac{s}{s^2+1}+\frac{1}{s-1}\right)\\ &= (-1)\left(\frac{1-s^2}{s^2+1}-\frac{1}{(s-1)^2}\right) = \frac{s^2-1}{s^2+1}+\frac{1}{(s-1)^2}.\end{aligned}$$

(e). Note first that since

$$(-1)\frac{d}{ds}\mathcal{L}\left[\frac{\sinh t}{t}\right] = \mathcal{L}\left[t\frac{\sinh t}{t}\right] = \mathcal{L}\left[\sinh t\right] = \frac{1}{s^2-1}$$

we have that

$$\mathcal{L}\left[\frac{\sinh t}{t}\right] = -\int\frac{ds}{s^2-1} = -\frac{1}{2}\int\left(\frac{1}{s-1}-\frac{1}{s+1}\right)ds = \frac{1}{2}\ln\left(\frac{s+1}{s-1}\right).$$

(f). We calculate that

$$\begin{aligned}\mathcal{L}\left[t^2\cos 2t\right] &= (-1)^2\frac{d^2}{ds^2}\mathcal{L}\left[\cos 2t\right] = (-1)^2\frac{d^2}{ds^2}\left(\frac{s}{s^2+4}\right) = \frac{d}{ds}\left(\frac{s^2+4-2s^2}{(s^2+4)^2}\right) = \frac{d}{ds}\left(\frac{4-s^2}{(s^2+4)^2}\right)\\ &= \frac{-2s\left(s^2+4\right)^2-\left(4-s^2\right)4s\left(s^2+4\right)}{(s^2+4)^4} = \frac{-2s\left(s^2+4\right)-\left(4-s^2\right)4s}{(s^2+4)^3}\\ &= \frac{2s^3-24s}{(s^2+4)^3}.\end{aligned}$$

(g). Note first that

$$(-1)\frac{d}{ds}\mathcal{L}\left[\frac{e^{3t}-1}{t}\right] = \mathcal{L}\left(t\frac{e^{3t}-1}{t}\right) = \mathcal{L}\left[e^{3t}-1\right] = \frac{1}{s-3}-\frac{1}{s}.$$

It follows that

$$\mathcal{L}\left[\frac{e^{3t}-1}{t}\right] = -\int\left(\frac{1}{s-3}-\frac{1}{s}\right)ds = \ln\left(\frac{s}{s-3}\right).$$

(h). Recall that $\sin^2 t = \frac{1}{2}\left(1-\cos 2t\right)$. This implies that

$$\begin{aligned}\mathcal{L}\left[te^{-t}\sin^2 t\right] &= \mathcal{L}\left[te^{-t}\frac{1}{2}\left(1-\cos 2t\right)\right] = \frac{1}{2}\mathcal{L}\left(te^{-t}-te^{-t}\cos 2t\right) = \frac{1}{2}(-1)\frac{d}{ds}\left(\frac{1}{s+1}-\frac{(s+1)}{4+(s+1)^2}\right)\\ &= \frac{1}{2}(-1)\left(-\frac{1}{(s+1)^2}-\frac{(s+1)^2+4-(s+1)2(s+1)}{\left((s+1)^2+4\right)^2}\right) = \frac{1}{2}\left(\frac{1}{(s+1)^2}+\frac{4-(s+1)^2}{\left((s+1)^2+4\right)^2}\right)\\ &= \frac{1}{2}\left(\frac{(s+1)^4+12(s+1)^2+16-(s+1)^4}{(s+1)^2\left((s+1)^2+4\right)^2}\right) = \left(\frac{6(s+1)^2+8}{(s+1)^2\left((s+1)^2+4\right)^2}\right)\\ &= \frac{\left(6s^2+12s+14\right)}{(s+1)^2\left(s^2+2s+5\right)^2}.\end{aligned}$$

(i). We calculate that

$$\mathcal{L}\left[f(t)\right] = \int_0^\infty e^{-st}f(t)\ dt = \int_0^3 2e^{-st}\ dt = \left[\frac{-2e^{-st}}{s}\right]_0^3 = \frac{2-2e^{-3s}}{s}.$$

(j). Let $\tau = t-\pi$ and $\varphi = t-2\pi$. Then, we get

$$\begin{aligned}\mathcal{L}\left(f(t)\right) &= \int_0^\infty e^{-st}f(t)\,dt\\ &= \int_\pi^{2\pi} e^{-st}f(t)\,dt\\ &= \int_\pi^\infty e^{-st}\sin 2t\ dt - \int_{2\pi}^\infty \sin 2te^{-st}\ dt\\ &= \int_0^\infty e^{-s(\tau+\pi)}\sin\left(2\tau+2\pi\right)\ d\tau - \int_0^\infty \sin\left(2\varphi+4\pi\right)e^{-s(\varphi+2\pi)}\ d\varphi\\ &= \frac{2e^{-\pi s}}{s^2+4}-\frac{2e^{-2\pi s}}{s^2+4}\\ &= \frac{2\left(e^{-\pi s}-e^{-2\pi s}\right)}{s^2+4}.\end{aligned}$$

4.2

(a). $f(t) = e^{2t}$

(b). $f(t) = 1 - \frac{8}{3\sqrt{\pi}} t^{\frac{3}{2}}$

(c). $f(t) = 3\cos 2t + \frac{1}{2}\sin 2t$

(d). $f(t) = 2u_3(t)$

(e). $f(t) = \frac{1}{3}\left(e^{3t} - 1\right)$

(f). $f(t) = \frac{1}{9}\left(6\sin 3t - \cos 3t + 1\right)$

(g). $f(t) = e^{4t}\left(1 + 12t + 24t^2 + \frac{32}{3}t^3\right)$

(h). $f(t) = \frac{1}{3}\left(2\cos 2t + 2\sin 2t - 2\cos t - \sin t\right)$

(i). $f(t) = \frac{1}{64} e^{\frac{t}{2}}\left[(4t+8)\cos t + (4-3t)\sin t\right]$

(j). We use the formula $\mathcal{L}^{-1}\left[\frac{dF}{ds}\right] = (-1)tf(t)$ to calculate that

$$\begin{aligned}
\frac{dF}{ds} &= \frac{\frac{-3}{(s+2)^2}}{1 + \left(\frac{3}{s+2}\right)^2} \\
&= -\frac{3}{s^2+4s+13} = -\frac{3}{(s+2)^2+9} \\
\mathcal{L}^{-1}\left[\frac{dF}{ds}\right] &= -\mathcal{L}^{-1}\left(\frac{3}{(s+2)^2+9}\right) = (-1)tf(t) \\
e^{-2t}\sin 3t &= tf(t) \\
f(t) &= \frac{e^{-2t}\sin 3t}{t}.
\end{aligned}$$

(k). $f(t) = \frac{1}{8}\left(t\sin t - t^2\cos t\right)$

(l).

$$f(t) = \mathcal{L}^{-1}\left(\frac{e^{-s}}{s+2}\right) = \begin{cases} e^{-2(t-1)} & t \geq 1 \\ 0 & t < 1 \end{cases}$$
$$= u_1(t)e^{-2(t-1)}$$

where u is the unit step function.

4.3

(a). Using integration by parts, we calculate that if $s > 0$ then

$$\begin{aligned}
\mathcal{L}[t^2](s) &= \int_0^\infty e^{-st}t^2\,dt \\
&= \left[-t^2\frac{e^{-st}}{s}\right]_0^\infty - \int_0^\infty -\frac{e^{-st}}{s}\,2t\,dt \\
&= \frac{2}{s}\int_0^\infty e^{-st}t\,dt \\
&= \frac{2}{s}\left(\left[-t\frac{e^{-st}}{s}\right]_0^\infty - \int_0^\infty -\frac{e^{-st}}{s}\,1\,dt\right) \\
&= \frac{2}{s^2}\int_0^\infty e^{-st}\,dt \\
&= \frac{2}{s^2}\left[-\frac{e^{-st}}{s}\right]_0^\infty \\
&= \frac{2}{s^3}
\end{aligned}$$

where the notation $[\,\cdot\,]_0^\infty$ means $\lim\limits_{A\to\infty}[\,\cdot\,]_0^A$.

(b). omitted

(c). omitted

(d). For brevity of notation, let $L = \mathcal{L}[\cosh at](s)$. Again using integration by parts, we have that

$$\begin{aligned}
L &= \int_0^\infty e^{-st}\cosh at\,dt \\
&= \left[e^{-st}\frac{1}{a}\sinh at\right]_0^\infty - \int_0^\infty -se^{-st}\frac{1}{a}\sinh at\,dt \\
&= \frac{s}{a}\int_0^\infty e^{-st}\sinh at\,dt \\
&= \frac{s}{a}\left(\left[e^{-st}\frac{1}{a}\cosh at\right]_0^\infty - \int_0^\infty -se^{-st}\frac{1}{a}\cosh at\,dt\right) \\
&= \frac{s}{a}\left(-\frac{1}{a} + \frac{s}{a}\int_0^\infty e^{-st}\cosh at\,dt\right) \\
&= -\frac{s}{a^2} + \frac{s^2}{a^2}L.
\end{aligned}$$

Rearranging this equation gives

$$\begin{aligned}
\frac{s^2}{a^2}L - L &= \frac{s}{a^2} \\
s^2L - a^2L &= s \\
L &= \frac{s}{s^2-a^2}
\end{aligned}$$

as required, if $s > a$.

(e). Let $\xi = ct$. Then $d\xi = c\,dt$ and $dt = \frac{1}{c}\,d\xi$. Thus

$$\begin{aligned}
\mathcal{L}[f(ct)](s) &= \int_0^\infty e^{-st}f(ct)\,dt \\
&= \int_0^\infty e^{-s\frac{\xi}{c}}f(\xi)\,\frac{1}{c}\,d\xi \\
&= \frac{1}{c}\int_0^\infty e^{-\frac{s}{c}\xi}f(\xi)\,d\xi \\
&= \frac{1}{c}\mathcal{L}[f]\left(\frac{s}{c}\right)
\end{aligned}$$

as required.

(f). We calculate that

$$\begin{aligned}
\frac{d}{ds}\mathcal{L}[f](s) &= \frac{d}{ds}\int_0^\infty e^{-st}f(t)\,dt \\
&= \int_0^\infty \frac{d}{ds}e^{-st}f(t)\,dt \\
&= \int_0^\infty -te^{-st}f(t)\,dt \\
&= -\mathcal{L}[tf(t)](s)
\end{aligned}$$

as required.

4.4

(a). We calculate that

$$\begin{aligned}
\mathcal{L}\left[x''+4x\right] &= \mathcal{L}[0] \\
\left[s^2F(s)-sx(0)-x'(0)\right]+4F(s) &= 0 \\
\left(s^2+4\right)F(s)-5s &= 0 \\
F(s) &= \frac{5s}{(s^2+4)} \\
x(t) &= \mathcal{L}^{-1}\left[\frac{5s}{(s^2+4)}\right]=5\cos 2t
\end{aligned}$$

Therefore the solution to the IVP is $x(t)=5\cos 2t$.

(b).

$$\begin{aligned}
\mathcal{L}\left[x''-x'-2x\right] &= \mathcal{L}[0] \\
\left[s^2F(s)-sx(0)-x'(0)\right]-\left[sF(s)-x(0)\right]-2F(s) &= 0 \\
\left(s^2-s-2\right)F(s)-2 &= 0 \\
F(s) &= \frac{2}{(s^2-s-2)} \\
x(t) &= \mathcal{L}^{-1}\left[\frac{2}{3(s-2)}-\frac{2}{3\,(s+1)}\right] \\
x(t) &= \frac{2}{3}e^{2t}-\frac{2}{3}e^{-t}
\end{aligned}$$

(c).

$$\begin{aligned}
\mathcal{L}\left[x''+9x\right] &= \mathcal{L}[1] \\
\left[s^2F(s)-sx(0)-x'(0)\right]+9F(s) &= \frac{1}{s} \\
\left(s^2+9\right)F(s) &= \frac{1}{s} \\
F(s) &= \frac{1}{s\,(s^2+9)} \\
x(t) &= \mathcal{L}^{-1}\left[\frac{1}{9s}-\frac{s}{9\,(s^2+9)}\right] \\
x(t) &= \frac{1}{9}-\frac{1}{9}\cos 3t
\end{aligned}$$

(d).

$$\begin{aligned}
\mathcal{L}\left[x''+6x'+25x\right] &= \mathcal{L}\left[0\right] \\
\left[s^2F(s)-sx(0)-x'(0)\right]+6\left[sF(s)-x(0)\right]+25F(s) &= 0 \\
\left(s^2+6s+25\right)F(s)-2s-3-12 &= 0 \\
F(s) &= \frac{2s+15}{s^2+6s+25} \\
x(t) &= \mathcal{L}^{-1}\left[\frac{2s+15}{(s+3)^2+16}\right]= \\
x(t) &= \mathcal{L}^{-1}\left[\frac{2(s+3)}{(s+3)^2+16}+\frac{9}{(s+3)^2+16}\right] \\
x(t) &= 2e^{-3t}\cos 4t+\frac{9}{4}e^{-3t}\sin 4t
\end{aligned}$$

(e).

$$\begin{aligned}
\mathcal{L}\left[x''-6x'+8x\right] &= \mathcal{L}[2] \\
\left[s^2F(s)-sx(0)-x'(0)\right]-6\left[sF(s)-x(0)\right]+8F(s) &= \frac{2}{s} \\
\left(s^2-6s+8\right)F(s) &= \frac{2}{s} \\
F(s) &= \frac{2}{s\,(s^2-6s+8)} \\
x(t) &= \mathcal{L}^{-1}\left[\frac{1}{4(s-4)}-\frac{1}{2(s-2)}+\frac{1}{4s}\right] \\
x(t) &= \frac{1}{4}e^{4t}-\frac{1}{2}e^{2t}+\frac{1}{4}
\end{aligned}$$

(f).

$$\begin{aligned}
\mathcal{L}\left[x''-4x\right] &= \mathcal{L}\left[3t\right] \\
\left[s^2F(s)-sx(0)-x'(0)\right]-4F(s) &= \frac{3}{s^2} \\
\left(s^2-4\right)F(s) &= \frac{3}{s^2} \\
F(s) &= \frac{3}{s^2\,(s^2-4)} \\
x(t) &= \mathcal{L}^{-1}\left[\frac{3}{16(s-2)}-\frac{3}{4s^2}-\frac{3}{16(s+2)}\right] \\
x(t) &= \frac{3}{16}e^{2t}-\frac{3}{4}t-\frac{3}{16}e^{-2t} \\
x(t) &= \frac{1}{8}\left(-6t+3\sinh 2t\right)
\end{aligned}$$

(g).

$$\begin{aligned}
\mathcal{L}\left[x''+4x'+8x\right] &= \mathcal{L}\left[e^{-t}\right]\\
\left[s^2F(s)-sx(0)-x'(0)\right]+4\left[sF(s)-x(0)\right]+8F(s) &= \frac{1}{s+1}\\
\left(s^2+4s+8\right)F(s) &= \frac{1}{s+1}\\
F(s) &= \frac{1}{(s+1)\left(s^2+4s+8\right)}\\
x(t) &= \mathcal{L}^{-1}\left[\frac{1}{5}\frac{1}{s+1}-\frac{1}{5}\frac{s+3}{(s^2+4s+8)}\right]\\
x(t) &= \mathcal{L}^{-1}\left[\frac{1}{5}\frac{1}{s+1}-\frac{1}{5}\frac{s+2}{(s+2)^2+4}-\frac{1}{10}\frac{2}{(s+2)^2+4}\right]\\
x(t) &= \frac{1}{5}e^{-t}-\frac{1}{5}e^{-2t}\cos 2t-\frac{1}{10}e^{-2t}\sin 2t
\end{aligned}$$

(h).

$$\begin{aligned}
\mathcal{L}\left[x^{(4)}+8x''+16x\right] &= \mathcal{L}\left[0\right]\\
\left[s^4F(s)-s^3x(0)-s^2x'(0)-sx''(0)-x^{(3)}(0)\right]+8\left[s^2F(s)-sx(0)-x'(0)\right]+16F(s) &= 0\\
\left(s^4+8s^2+16\right)F(s)-1 &= 0\\
\left(s^4+8s^2+16\right)F(s) &= 1
\end{aligned}$$

This implies that

$$\begin{aligned}
F(s) &= \frac{1}{s^4+8s^2+16}\\
x(t) &= \mathcal{L}^{-1}\left[\frac{1}{(s^2+4)^2}\right]\\
x(t) &= \frac{1}{8}\left(\sin 2t-t\cos 2t\right).
\end{aligned}$$

(i).

$$\begin{aligned}
\mathcal{L}\left[x^{(3)}+4x''+5x'+2x\right] &= \mathcal{L}\left[10\cos t\right]\\
\left[s^3F(s)-s^2x(0)-sx'(0)-x''(0)\right]+4\left[s^2F(s)-sx(0)-x'(0)\right]+5\left[sF(s)-x(0)\right]+2F(s) &= \frac{10s}{s^2+1}\\
\left(s^3+4s^2+5s+2\right)F(s)-3 &= \frac{10s}{s^2+1}
\end{aligned}$$

$$\begin{aligned}
\left(s^3+4s^2+5s+2\right)F(s) &= \frac{10s}{s^2+1}+3=\frac{3s^2+10s+3}{s^2+1}\\
F(s) &= \frac{3s^2+10s+3}{(s^2+1)(s^3+4s^2+5s+2)}\\
x(t) &= \mathcal{L}^{-1}\left[\frac{3s^2+10s+3}{(s^2+1)(s^3+4s^2+5s+2)}\right]\\
x(t) &= \mathcal{L}^{-1}\left[\frac{2}{s+1}-\frac{2}{(s+1)^2}-\frac{1}{s+2}-\frac{s}{s^2+1}+\frac{2}{s^2+1}\right]\\
x(t) &= 2e^{-t}-2te^{-t}-e^{-2t}-\cos t+2\sin t
\end{aligned}$$

(j).

$$\begin{aligned}
\mathcal{L}\left[x''+4x'+13x\right] &= \mathcal{L}\left[te^{-t}\right]\\
\left[s^2F(s)-sx(0)-x'(0)\right]+4\left[sF(s)-x(0)\right]+13F(s) &= \frac{1}{(s+1)^2}\\
\left(s^2+4s+13\right)F(s)-2 &= \frac{1}{(s+1)^2}\\
\left(s^2+4s+13\right)F(s) &= \frac{1}{(s+1)^2}+2=\frac{4s+2s^2+3}{(s+1)^2}\\
F(s) &= \frac{4s+2s^2+3}{(s+1)^2(s^2+4s+13)}
\end{aligned}$$

and

$$\begin{aligned}
\frac{4s+2s^2+3}{(s+1)^2(s^2+4s+13)} &= \frac{A}{s+1}+\frac{B}{(s+1)^2}+\frac{Cs+D}{s^2+4s+13}\\
&\Rightarrow\ A=\frac{-1}{50},B=\frac{1}{10},C=\frac{1}{50},D=\frac{98}{50}\\
x(t) &= \mathcal{L}^{-1}\left[-\frac{1}{50}\frac{1}{s+1}+\frac{1}{10}\frac{1}{(s+1)^2}+\frac{1}{50}\frac{s+2}{(s+2)^2+9}+\frac{32}{50}\frac{3}{(s+2)^2+9}\right]\\
x(t) &= -\frac{1}{50}e^{-t}+\frac{1}{10}te^{-t}+\frac{1}{50}e^{-2t}\cos 3t+\frac{32}{50}e^{-2t}\sin 3t
\end{aligned}$$

(k). If we use the substitution $k = t - \frac{\pi}{2}$ then we have

$$\begin{aligned}
x'' + x &= \sin(2k + \pi) \\
x'' + x &= -\sin 2k \\
\mathcal{L}[x'' + x] &= -\mathcal{L}[\sin 2k] \\
s^2 F(s) - sx(0) - x'(0) + F(s) &= -\frac{2}{s^2+4} \\
\left(s^2+1\right) F(s) - 2s &= -\frac{2}{s^2+4} \\
\left(s^2+1\right) F(s) &= -\frac{2}{s^2+4} + 2s = \frac{2s^3+8s-2}{s^2+4} \\
F(s) &= \frac{2s^3+8s-2}{(s^2+1)(s^2+4)}
\end{aligned}$$

By using partial fractions we obtain

$$\begin{aligned}
F(s) &= \frac{2s^3+8s-2}{(s^2+1)(s^2+4)} = \frac{As+B}{s^2+1} + \frac{Cs+D}{s^2+4} \\
2s^3+8s-2 &= 4B + D + As^3 + Bs^2 + Cs^3 + s^2 D + 4As + Cs \\
A &= 2, B = -\frac{2}{3}, C = 0, D = \frac{2}{3} \\
F(s) &= \frac{2s^3+8s-2}{(s^2+1)(s^2+4)} = \frac{2s}{s^2+1} - \frac{2}{3}\frac{1}{s^2+1} + \frac{1}{3}\frac{2}{s^2+4} \\
x(k) &= 2\cos k - \frac{2}{3}\sin k + \frac{1}{3}\sin 2k \\
x(t) &= 2\cos\left(t - \frac{\pi}{2}\right) - \frac{2}{3}\sin\left(t - \frac{\pi}{2}\right) + \frac{1}{3}\sin(2t - \pi) \\
x(t) &= -\frac{2}{3}\cos t + 2\sin t - \frac{1}{3}\sin 2t
\end{aligned}$$

5.1

(a). Note that

$$\det(\mathbf{A} - \lambda\mathbf{I}) = \begin{vmatrix} 1-\lambda & 2 \\ 2 & 1-\lambda \end{vmatrix} = \lambda^2 - 2\lambda - 3 = (\lambda - 3)(\lambda + 1).$$

Thus, the eigenvalues of $\mathbf{A}$ are $\{3, -1\}$. Since the eigenvalues of $\mathbf{A}$ are real and distinct, the eigenvectors of $\mathbf{A}$ are linearly independent and can be calculated as follows.

$$\begin{aligned}
\mathbf{0} &= (\mathbf{A} - 3\mathbf{I})\,\mathbf{q}_1 = \begin{bmatrix} -2 & 2 \\ 2 & -2 \end{bmatrix} \mathbf{q}_1 \Longrightarrow \mathbf{q}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \\
\mathbf{0} &= (\mathbf{A} + \mathbf{I})\,\mathbf{q}_2 = \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix} \mathbf{q}_2 \Longrightarrow \mathbf{q}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}.
\end{aligned}$$

Consequently, the general solution is

$$\begin{aligned}
\mathbf{x}(t) &= c_1\mathbf{q}_1 e^{3t} + c_2\mathbf{q}_2 e^{-t} = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{3t} + c_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{-t}, \\
&= \begin{bmatrix} c_1 e^{3t} + c_2 e^{-t} \\ c_1 e^{3t} - c_2 e^{-t} \end{bmatrix}.
\end{aligned}$$

(b). Note that

$$\det(\mathbf{A} - \lambda\mathbf{I}) = \begin{vmatrix} -3-\lambda & 2 \\ -3 & 4-\lambda \end{vmatrix} = \lambda^2 - \lambda - 6 = (\lambda - 3)(\lambda + 2).$$

Thus, the eigenvalues of $\mathbf{A}$ are $\{3, -2\}$. Since the eigenvalues of $\mathbf{A}$ are real and distinct, the eigenvectors of $\mathbf{A}$ are linearly independent and can be calculated as follows.

$$\begin{aligned}
\mathbf{0} &= (\mathbf{A} - 3\mathbf{I})\,\mathbf{q}_1 = \begin{bmatrix} -6 & 2 \\ -3 & 1 \end{bmatrix} \mathbf{q}_1 \Longrightarrow \mathbf{q}_1 = \begin{bmatrix} 1 \\ 3 \end{bmatrix}, \\
\mathbf{0} &= (\mathbf{A} + 2\mathbf{I})\,\mathbf{q}_2 = \begin{bmatrix} -1 & 2 \\ -3 & 6 \end{bmatrix} \mathbf{q}_2 \Longrightarrow \mathbf{q}_2 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}.
\end{aligned}$$

Consequently, the general solution is

$$\begin{aligned}
\mathbf{x}(t) &= c_1\mathbf{q}_1 e^{3t} + c_2\mathbf{q}_2 e^{-2t} = c_1 \begin{bmatrix} 1 \\ 3 \end{bmatrix} e^{3t} + c_2 \begin{bmatrix} 2 \\ 1 \end{bmatrix} e^{-2t}, \\
&= \begin{bmatrix} c_1 e^{3t} + 2c_2 e^{-2t} \\ 3c_1 e^{3t} + c_2 e^{-2t} \end{bmatrix}.
\end{aligned}$$

(c). Note that

$$\det(\mathbf{A} - \lambda\mathbf{I}) = \begin{vmatrix} 3-\lambda & -1 \\ 5 & -3-\lambda \end{vmatrix} = \lambda^2 - 4 = (\lambda - 2)(\lambda + 2).$$

Thus, the eigenvalues of $\mathbf{A}$ are $\{2, -2\}$. Since the eigenvalues of $\mathbf{A}$ are real and distinct, the eigenvectors of $\mathbf{A}$ are linearly independent and can be calculated as follows.

$$\begin{aligned}
\mathbf{0} &= (\mathbf{A} - 2\mathbf{I})\,\mathbf{q}_1 = \begin{bmatrix} 1 & -1 \\ 5 & -5 \end{bmatrix} \mathbf{q}_1 \Longrightarrow \mathbf{q}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \\
\mathbf{0} &= (\mathbf{A} + 2\mathbf{I})\,\mathbf{q}_2 = \begin{bmatrix} 5 & -1 \\ 5 & -1 \end{bmatrix} \mathbf{q}_2 \Longrightarrow \mathbf{q}_2 = \begin{bmatrix} 1 \\ 5 \end{bmatrix}.
\end{aligned}$$

Consequently, the general solution is

$$\begin{aligned}
\mathbf{x}(t) &= c_1\mathbf{q}_1 e^{2t} + c_2\mathbf{q}_2 e^{-2t} = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{2t} + c_2 \begin{bmatrix} 1 \\ 5 \end{bmatrix} e^{-2t}, \\
&= \begin{bmatrix} c_1 e^{2t} + c_2 e^{-2t} \\ c_1 e^{2t} + 5c_2 e^{-2t} \end{bmatrix}.
\end{aligned}$$

(d). The equations above can be written as

$$\mathbf{x}' = \mathbf{A}\mathbf{x} = \begin{bmatrix} 4 & -1 \\ 1 & 2 \end{bmatrix} \mathbf{x}, \text{ where } \mathbf{x} = \begin{bmatrix} x \\ y \end{bmatrix}.$$

Thus, the eigenvalues of $\mathbf{A}$ are

$$\det(\mathbf{A} - \lambda \mathbf{I}) = \begin{vmatrix} 4-\lambda & -1 \\ 1 & 2-\lambda \end{vmatrix} = \lambda^2 - 6\lambda + 9 = (\lambda - 3)^2.$$

In this case, we first calculate the generalized eigenvector $\mathbf{q}_1$ of order one as follows.

$$\mathbf{0} = (\mathbf{A} - 3\mathbf{I})\, \mathbf{q}_1 = \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix} \mathbf{q}_1 \Longrightarrow \mathbf{q}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

Then, we calculate the generalized eigenvector $\mathbf{r}_1$ of order two as follows.

$$(\mathbf{A} - 3\mathbf{I})\, \mathbf{r}_1 = \mathbf{q}_1 \Longrightarrow \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix} \mathbf{r}_1 = \mathbf{q}_1 \Longrightarrow \mathbf{r}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

Consequently, the general solution is

$$\begin{aligned} \mathbf{x}(t) &= c_1 \mathbf{q}_1 e^{3t} + c_2 (t\mathbf{q}_1 + \mathbf{r}_1) e^{3t} = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{3t} + c_2 \left(t \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right) e^{3t}, \\ &= \begin{bmatrix} c_1 e^{3t} + c_2 (t+1) e^{3t} \\ c_1 e^{3t} + c_2 t e^{3t} \end{bmatrix}. \end{aligned}$$

(e). The equations above can be written as

$$\mathbf{x}' = \mathbf{A}\mathbf{x} = \begin{bmatrix} 3 & -1 \\ 4 & -1 \end{bmatrix} \mathbf{x}, \text{ where } \mathbf{x} = \begin{bmatrix} x \\ y \end{bmatrix}.$$

Thus, the eigenvalues of $\mathbf{A}$ are

$$\det(\mathbf{A} - \lambda \mathbf{I}) = \begin{vmatrix} 3-\lambda & -1 \\ 4 & -1-\lambda \end{vmatrix} = \lambda^2 - 2\lambda + 1 = (\lambda - 1)^2.$$

Similar to the previous case, we first calculate the generalized eigenvector $\mathbf{q}_1$ of order one as follows.

$$\mathbf{0} = (\mathbf{A} - \mathbf{I})\, \mathbf{q}_1 = \begin{bmatrix} 2 & -1 \\ 4 & -2 \end{bmatrix} \mathbf{q}_1 \Longrightarrow \mathbf{q}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}.$$

Then, we calculate the generalized eigenvector $\mathbf{r}_1$ of order two as follows.

$$(\mathbf{A} - \mathbf{I})\, \mathbf{r}_1 = \mathbf{q}_1 \Longrightarrow \begin{bmatrix} 2 & -1 \\ 4 & -2 \end{bmatrix} \mathbf{r}_1 = \mathbf{q}_1 \Longrightarrow \mathbf{r}_1 = \begin{bmatrix} 0 \\ -1 \end{bmatrix}.$$

Consequently, the general solution is

$$\begin{aligned} \mathbf{x}(t) &= c_1 \mathbf{q}_1 e^{t} + c_2 (t\mathbf{q}_1 + \mathbf{r}_1) e^{t} = c_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} e^{t} + c_2 \left(t \begin{bmatrix} 1 \\ 2 \end{bmatrix} + \begin{bmatrix} 0 \\ -1 \end{bmatrix} \right) e^{t}, \\ &= \begin{bmatrix} c_1 e^{t} + c_2 t e^{t} \\ 2c_1 e^{t} + c_2 (2t-1) e^{t} \end{bmatrix}. \end{aligned}$$

(f). The equations above can be written as

$$\mathbf{x}' = \mathbf{A}\mathbf{x} = \begin{bmatrix} 5 & 4 \\ -1 & 1 \end{bmatrix} \mathbf{x}, \text{ where } \mathbf{x} = \begin{bmatrix} x \\ y \end{bmatrix}.$$

Thus, the eigenvalues of $\mathbf{A}$ are

$$\det(\mathbf{A} - \lambda \mathbf{I}) = \begin{vmatrix} 5-\lambda & 4 \\ -1 & 1-\lambda \end{vmatrix} = \lambda^2 - 6\lambda + 9 = (\lambda - 3)^2.$$

Similar to the previous case, we first calculate the generalized eigenvector $\mathbf{q}_1$ of order one as follows.

$$\mathbf{0} = (\mathbf{A} - 3\mathbf{I})\, \mathbf{q}_1 = \begin{bmatrix} 2 & 4 \\ -1 & -2 \end{bmatrix} \mathbf{q}_1 \Longrightarrow \mathbf{q}_1 = \begin{bmatrix} 2 \\ -1 \end{bmatrix}.$$

Then, we calculate the generalized eigenvector $\mathbf{r}_1$ of order two as follows.

$$(\mathbf{A} - 3\mathbf{I})\, \mathbf{r}_1 = \mathbf{q}_1 \Longrightarrow \begin{bmatrix} 2 & 4 \\ -1 & -2 \end{bmatrix} \mathbf{r}_1 = \mathbf{q}_1 \Longrightarrow \mathbf{r}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

Consequently, the general solution is

$$\begin{aligned} \mathbf{x}(t) &= c_1 \mathbf{q}_1 e^{3t} + c_2 (t\mathbf{q}_1 + \mathbf{r}_1) e^{3t} = c_1 \begin{bmatrix} 2 \\ -1 \end{bmatrix} e^{3t} + c_2 \left(t \begin{bmatrix} 2 \\ -1 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right) e^{3t}, \\ &= \begin{bmatrix} 2c_1 e^{3t} + c_2 (2t+1) e^{3t} \\ -c_1 e^{3t} - c_2 t e^{3t} \end{bmatrix}. \end{aligned}$$

(g). The equations above can be written as

$$\mathbf{x}' = \mathbf{A}\mathbf{x} = \begin{bmatrix} 3 & 2 \\ -5 & 1 \end{bmatrix} \mathbf{x}, \text{ where } \mathbf{x} = \begin{bmatrix} x \\ y \end{bmatrix}.$$

The characteristic polynomial is

$$\det(\mathbf{A} - \lambda \mathbf{I}) = \begin{vmatrix} 3-\lambda & 2 \\ -5 & 1-\lambda \end{vmatrix} = \lambda^2 - 4\lambda + 13.$$

Note that the characteristic polynomial is of the form $\lambda^2 - 2\sigma\lambda + \sigma^2 + w^2$ where $\sigma = 2$ and $w = 3$. Thus, the eigenvalues of $\mathbf{A}$ are $\{2 \pm 3j\}$. The eigenvectors are $\mathbf{q}_1 \pm j\mathbf{q}_2$ which satisfy the following equation.

$$\begin{aligned} \mathbf{A} \begin{bmatrix} \mathbf{q}_1 & \mathbf{q}_2 \end{bmatrix} &= \begin{bmatrix} \mathbf{q}_1 & \mathbf{q}_2 \end{bmatrix} \begin{bmatrix} 2 & 3 \\ -3 & 2 \end{bmatrix} \Longrightarrow \\ (\mathbf{A} - 2\mathbf{I}) \begin{bmatrix} \mathbf{q}_1 & \mathbf{q}_2 \end{bmatrix} &= \begin{bmatrix} \mathbf{q}_1 & \mathbf{q}_2 \end{bmatrix} \begin{bmatrix} 0 & 3 \\ -3 & 0 \end{bmatrix}. \end{aligned}$$

Since $\mathbf{A}^2 - 2\sigma\mathbf{A} + (\sigma^2 + w^2)\,\mathbf{I} = \mathbf{0}$, $\mathbf{q}_1$ can be chosen as an arbitrary nonzero vector. Let us choose $\mathbf{q}_1 := \begin{bmatrix} 1 & 1 \end{bmatrix}^T$ and calculate $\mathbf{q}_2$ as follows

$$(\mathbf{A} - 2\mathbf{I})\, \mathbf{q}_1 = -3\mathbf{q}_2 \Longrightarrow \begin{bmatrix} 1 & 2 \\ -5 & -1 \end{bmatrix} \mathbf{q}_1 = -3\mathbf{q}_2 \Longrightarrow \mathbf{q}_2 = \begin{bmatrix} -1 \\ 2 \end{bmatrix}.$$

Consequently, the general solution is

$$\begin{aligned} \mathbf{x}(t) &= (c_1 \mathbf{q}_1 + c_2 \mathbf{q}_2) e^{2t} \cos 3t + (c_2 \mathbf{q}_1 - c_1 \mathbf{q}_2) e^{2t} \sin 3t, \\ &= \left(c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} -1 \\ 2 \end{bmatrix} \right) e^{2t} \cos 3t + \left(c_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} - c_1 \begin{bmatrix} -1 \\ 2 \end{bmatrix} \right) e^{2t} \sin 3t \\ &= e^{2t} \begin{bmatrix} (c_1 - c_2) \cos 3t + (c_1 + c_2) \sin 3t \\ (c_1 + 2c_2) \cos 3t - (2c_1 - c_2) \sin 3t \end{bmatrix}. \end{aligned}$$

(h). The equations above can be written as

$$\mathbf{x}' = \mathbf{A}\mathbf{x} = \begin{bmatrix} 1 & -4 \\ 1 & 1 \end{bmatrix} \mathbf{x}, \text{ where } \mathbf{x} = \begin{bmatrix} x \\ y \end{bmatrix}.$$

The characteristic polynomial is

$$\det(\mathbf{A}-\lambda\mathbf{I}) = \begin{vmatrix} 1-\lambda & -4 \\ 1 & 1-\lambda \end{vmatrix} = \lambda^2 - 2\lambda + 5.$$

As in the previous case, the characteristic polynomial is of the form $\lambda^2 - 2\sigma\lambda + \sigma^2 + w^2$ where $\sigma = 1$ and $w = 2$. Thus, the eigenvalues of $\mathbf{A}$ are $\{1 \pm 2j\}$. The eigenvectors are $\mathbf{q}_1 \pm j\mathbf{q}_2$ which satisfy the following equation.

$$\begin{aligned} \mathbf{A}\begin{bmatrix} \mathbf{q}_1 & \mathbf{q}_2 \end{bmatrix} &= \begin{bmatrix} \mathbf{q}_1 & \mathbf{q}_2 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix} \Longrightarrow \\ (\mathbf{A}-\mathbf{I})\begin{bmatrix} \mathbf{q}_1 & \mathbf{q}_2 \end{bmatrix} &= \begin{bmatrix} \mathbf{q}_1 & \mathbf{q}_2 \end{bmatrix} \begin{bmatrix} 0 & 2 \\ -2 & 0 \end{bmatrix}. \end{aligned}$$

Since $\mathbf{A}^2 - 2\sigma\mathbf{A} + (\sigma^2 + w^2)\mathbf{I} = \mathbf{0}$, $\mathbf{q}_1$ can be chosen as an arbitrary nonzero vector. Let us choose $\mathbf{q}_1 := \begin{bmatrix} 0 & 1 \end{bmatrix}^T$ and calculate $\mathbf{q}_2$ as follows

$$(\mathbf{A}-\mathbf{I})\mathbf{q}_1 = -2\mathbf{q}_2 \Longrightarrow \begin{bmatrix} 0 & -4 \\ 1 & 0 \end{bmatrix} \mathbf{q}_1 = -2\mathbf{q}_2 \Longrightarrow \mathbf{q}_2 = \begin{bmatrix} 2 \\ 0 \end{bmatrix}.$$

Consequently, the general solution is

$$\begin{aligned} \mathbf{x}(t) &= (c_1\mathbf{q}_1 + c_2\mathbf{q}_2)\, e^t \cos 2t + (c_2\mathbf{q}_1 - c_1\mathbf{q}_2)\, e^t \sin 2t, \\ &= \left(c_1 \begin{bmatrix} 0 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 2 \\ 0 \end{bmatrix}\right) e^t \cos 2t + \left(c_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix} - c_1 \begin{bmatrix} 2 \\ 0 \end{bmatrix}\right) e^t \sin 2t \\ &= e^t \begin{bmatrix} 2(c_2 \cos 2t - c_1 \sin 2t) \\ c_1 \cos 2t + c_2 \sin 2t \end{bmatrix}. \end{aligned}$$

(i). The equations above can be written as

$$\mathbf{x}' = \mathbf{A}\mathbf{x} = \begin{bmatrix} 1 & -3 \\ 3 & 1 \end{bmatrix} \mathbf{x}, \text{ where } \mathbf{x} = \begin{bmatrix} x \\ y \end{bmatrix}.$$

The characteristic polynomial is

$$\det(\mathbf{A}-\lambda\mathbf{I}) = \begin{vmatrix} 1-\lambda & -3 \\ 3 & 1-\lambda \end{vmatrix} = \lambda^2 - 2\lambda + 10.$$

As in the previous case, the characteristic polynomial is of the form $\lambda^2 - 2\sigma\lambda + \sigma^2 + w^2$ where $\sigma = 1$ and $w = 3$. Thus, the eigenvalues of $\mathbf{A}$ are $\{1 \pm 3j\}$. The eigenvectors are $\mathbf{q}_1 \pm j\mathbf{q}_2$ which satisfy the following equation.

$$\begin{aligned} \mathbf{A}\begin{bmatrix} \mathbf{q}_1 & \mathbf{q}_2 \end{bmatrix} &= \begin{bmatrix} \mathbf{q}_1 & \mathbf{q}_2 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ -3 & 1 \end{bmatrix} \Longrightarrow \\ (\mathbf{A}-\mathbf{I})\begin{bmatrix} \mathbf{q}_1 & \mathbf{q}_2 \end{bmatrix} &= \begin{bmatrix} \mathbf{q}_1 & \mathbf{q}_2 \end{bmatrix} \begin{bmatrix} 0 & 3 \\ -3 & 0 \end{bmatrix}. \end{aligned}$$

Since $\mathbf{A}^2 - 2\sigma\mathbf{A} + (\sigma^2 + w^2)\mathbf{I} = \mathbf{0}$, $\mathbf{q}_1$ can be chosen as an arbitrary nonzero vector. Let us choose $\mathbf{q}_1 := \begin{bmatrix} 1 & 0 \end{bmatrix}^T$ and calculate $\mathbf{q}_2$ as follows

$$(\mathbf{A}-\mathbf{I})\mathbf{q}_1 = -3\mathbf{q}_2 \Longrightarrow \begin{bmatrix} 0 & 3 \\ -3 & 0 \end{bmatrix} \mathbf{q}_1 = -3\mathbf{q}_2 \Longrightarrow \mathbf{q}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

Consequently, the general solution is

$$\begin{aligned} \mathbf{x}(t) &= (c_1\mathbf{q}_1 + c_2\mathbf{q}_2)\, e^t \cos 3t + (c_2\mathbf{q}_1 - c_1\mathbf{q}_2)\, e^t \sin 3t, \\ &= \left(c_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) e^t \cos 3t + \left(c_2 \begin{bmatrix} 1 \\ 0 \end{bmatrix} - c_1 \begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) e^t \sin 3t \\ &= e^t \begin{bmatrix} (c_1 \cos 3t + c_2 \sin 3t) \\ (c_2 \cos 3t - c_1 \sin 3t) \end{bmatrix}. \end{aligned}$$

(j). The equations above can be written as

$$\mathbf{x}' = \mathbf{A}\mathbf{x} = \begin{bmatrix} 4 & -2 \\ 5 & 2 \end{bmatrix} \mathbf{x}, \text{ where } \mathbf{x} = \begin{bmatrix} x \\ y \end{bmatrix}.$$

The characteristic polynomial is

$$\det(\mathbf{A}-\lambda\mathbf{I}) = \begin{vmatrix} 4-\lambda & -2 \\ 5 & 2-\lambda \end{vmatrix} = \lambda^2 - 6\lambda + 18.$$

As in the previous case, the characteristic polynomial is of the form $\lambda^2 - 2\sigma\lambda + \sigma^2 + w^2$ where $\sigma = 3$ and $w = 3$. Thus, the eigenvalues of $\mathbf{A}$ are $\{3 \pm 3j\}$. The eigenvectors are $\mathbf{q}_1 \pm j\mathbf{q}_2$ which satisfy the following equation.

$$\begin{aligned} \mathbf{A}\begin{bmatrix} \mathbf{q}_1 & \mathbf{q}_2 \end{bmatrix} &= \begin{bmatrix} \mathbf{q}_1 & \mathbf{q}_2 \end{bmatrix} \begin{bmatrix} 3 & 3 \\ -3 & 3 \end{bmatrix} \Longrightarrow \\ (\mathbf{A}-3\mathbf{I})\begin{bmatrix} \mathbf{q}_1 & \mathbf{q}_2 \end{bmatrix} &= \begin{bmatrix} \mathbf{q}_1 & \mathbf{q}_2 \end{bmatrix} \begin{bmatrix} 0 & 3 \\ -3 & 0 \end{bmatrix}. \end{aligned}$$

Since $\mathbf{A}^2 - 2\sigma\mathbf{A} + (\sigma^2 + w^2)\mathbf{I} = \mathbf{0}$, $\mathbf{q}_1$ can be chosen as an arbitrary nonzero vector. Let us choose $\mathbf{q}_1 := \begin{bmatrix} 1 & 0 \end{bmatrix}^T$ and calculate $\mathbf{q}_2$ as follows

$$(\mathbf{A}-3\mathbf{I})\mathbf{q}_1 = -3\mathbf{q}_2 \Longrightarrow \begin{bmatrix} 0 & 3 \\ -3 & 0 \end{bmatrix} \mathbf{q}_1 = -3\mathbf{q}_2 \Longrightarrow \mathbf{q}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

Consequently, the general solution is

$$\begin{aligned} \mathbf{x}(t) &= (c_1\mathbf{q}_1 + c_2\mathbf{q}_2)\, e^{3t} \cos 3t + (c_2\mathbf{q}_1 - c_1\mathbf{q}_2)\, e^{3t} \sin 3t, \\ &= \left(c_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) e^{3t} \cos 3t + \left(c_2 \begin{bmatrix} 1 \\ 0 \end{bmatrix} - c_1 \begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) e^{3t} \sin 3t \\ &= e^{3t} \begin{bmatrix} c_1 \cos 3t + c_2 \sin 3t \\ c_2 \cos 3t - c_1 \sin 3t \end{bmatrix}. \end{aligned}$$

(k). The equations above can be written as

$$\mathbf{x}' = \mathbf{A}\mathbf{x} = \begin{bmatrix} 1 & 1 & -1 \\ 2 & 3 & -4 \\ 4 & 1 & -4 \end{bmatrix} \mathbf{x}, \text{ where } \mathbf{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}.$$

The characteristic polynomial is

$$\det(\mathbf{A} - \lambda \mathbf{I}) = \begin{vmatrix} 1-\lambda & 1 & -1 \\ 2 & 3-\lambda & -4 \\ 4 & 1 & -4-\lambda \end{vmatrix} = (\lambda - 1)(\lambda - 2)(\lambda + 3).$$

Thus, the eigenvalues of $\mathbf{A}$ are $\{1, 2, -3\}$. Since the eigenvalues are real and distinct, the eigenvectors of $\mathbf{A}$ are linearly independent and can be calculated as follows.

$$\begin{aligned} \mathbf{0} &= (\mathbf{A} - \mathbf{I})\mathbf{q}_1 = \begin{bmatrix} 0 & 1 & -1 \\ 2 & 2 & -4 \\ 4 & 1 & -5 \end{bmatrix} \mathbf{q}_1 \Longrightarrow \mathbf{q}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \\ \mathbf{0} &= (\mathbf{A} - 2\mathbf{I})\mathbf{q}_2 = \begin{bmatrix} -1 & 1 & -1 \\ 2 & 1 & -4 \\ 4 & 1 & -6 \end{bmatrix} \mathbf{q}_2 \Longrightarrow \mathbf{q}_2 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \\ \mathbf{0} &= (\mathbf{A} + 3\mathbf{I})\mathbf{q}_3 = \begin{bmatrix} 4 & 1 & -1 \\ 2 & 6 & -4 \\ 4 & 1 & -1 \end{bmatrix} \mathbf{q}_3 \Longrightarrow \mathbf{q}_3 = \begin{bmatrix} 1 \\ 7 \\ 11 \end{bmatrix}. \end{aligned}$$

Consequently, the general solution is

$$\begin{aligned} \mathbf{x}(t) &= c_1\mathbf{q}_1 e^{t} + c_2\mathbf{q}_2 e^{2t} + c_3\mathbf{q}_3 e^{-3t} = c_1 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} e^{t} + c_2 \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} e^{2t} + c_3 \begin{bmatrix} 1 \\ 7 \\ 11 \end{bmatrix} e^{-3t}, \\ &= \begin{bmatrix} c_1 e^{t} + c_2 e^{2t} + c_3 e^{-3t} \\ c_1 e^{t} + 2c_2 e^{2t} + 7c_3 e^{-3t} \\ c_1 e^{t} + c_2 e^{2t} + 11c_3 e^{-3t} \end{bmatrix}. \end{aligned}$$

(l). The equations above can be written as

$$\mathbf{x}' = \mathbf{A}\mathbf{x} = \begin{bmatrix} 1 & -1 & -1 \\ 1 & 3 & 1 \\ -3 & 1 & -1 \end{bmatrix} \mathbf{x}, \text{ where } \mathbf{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}.$$

The characteristic polynomial is

$$\det(\mathbf{A} - \lambda \mathbf{I}) = \begin{vmatrix} 1-\lambda & -1 & -1 \\ 1 & 3-\lambda & 1 \\ -3 & 1 & -1-\lambda \end{vmatrix} = (\lambda - 2)(\lambda - 3)(\lambda + 2).$$

Thus, the eigenvalues of $\mathbf{A}$ are $\{2, 3, -2\}$. Since the eigenvalues are real and distinct, the eigenvectors of $\mathbf{A}$ are linearly independent and can be calculated as follows.

$$\begin{aligned} \mathbf{0} &= (\mathbf{A} - 2\mathbf{I})\mathbf{q}_1 = \begin{bmatrix} -1 & -1 & -1 \\ 1 & 1 & 1 \\ -3 & 1 & -3 \end{bmatrix} \mathbf{q}_1 \Longrightarrow \mathbf{q}_1 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \\ \mathbf{0} &= (\mathbf{A} - 3\mathbf{I})\mathbf{q}_2 = \begin{bmatrix} -2 & -1 & -1 \\ 1 & 0 & 1 \\ -3 & 1 & -4 \end{bmatrix} \mathbf{q}_2 \Longrightarrow \mathbf{q}_2 = \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix}, \\ \mathbf{0} &= (\mathbf{A} + 2\mathbf{I})\mathbf{q}_3 = \begin{bmatrix} 3 & -1 & -1 \\ 1 & 5 & 1 \\ -3 & 1 & 1 \end{bmatrix} \mathbf{q}_3 \Longrightarrow \mathbf{q}_3 = \begin{bmatrix} -1 \\ 1 \\ -4 \end{bmatrix}. \end{aligned}$$

Consequently, the general solution is

$$\begin{aligned} \mathbf{x}(t) &= c_1\mathbf{q}_1 e^{2t} + c_2\mathbf{q}_2 e^{3t} + c_3\mathbf{q}_3 e^{-2t} = c_1 \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} e^{2t} + c_2 \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix} e^{3t} + c_3 \begin{bmatrix} -1 \\ 1 \\ -4 \end{bmatrix} e^{-2t}, \\ &= \begin{bmatrix} c_1 e^{2t} + c_2 e^{3t} - c_3 e^{-2t} \\ -c_2 e^{2t} + c_3 e^{-2t} \\ -c_1 e^{t} - c_2 e^{2t} - 4c_3 e^{-2t} \end{bmatrix}. \end{aligned}$$

(m). The equations above can be written as

$$\mathbf{x}' = \mathbf{A}\mathbf{x} = \begin{bmatrix} 3 & 1 & 1 \\ 0 & 3 & 1 \\ 0 & 0 & 6 \end{bmatrix} \mathbf{x}, \text{ where } \mathbf{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}.$$

The characteristic polynomial is

$$\det(\mathbf{A} - \lambda \mathbf{I}) = \begin{vmatrix} 3-\lambda & 1 & 1 \\ 0 & 3-\lambda & 1 \\ 0 & 0 & 6-\lambda \end{vmatrix} = (\lambda - 6)(\lambda - 3)^2.$$

Thus, the eigenvalues of $\mathbf{A}$ are $\{3, 3, 6\}$. The eigenvectors belonging to $\lambda = 3$ are generalized eigenvectors and can be calculated as in Question 6. Therefore, it follows that

$$\begin{aligned} \mathbf{0} &= (\mathbf{A} - 3\mathbf{I})\mathbf{q}_1 = \begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 3 \end{bmatrix} \mathbf{q}_1 \Longrightarrow \mathbf{q}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \\ (\mathbf{A} - 3\mathbf{I})\mathbf{r}_1 &= \mathbf{q}_1 \Longrightarrow \begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 3 \end{bmatrix} \mathbf{r}_1 = \mathbf{q}_1 \Longrightarrow \mathbf{r}_1 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \\ \mathbf{0} &= (\mathbf{A} - 6\mathbf{I})\mathbf{q}_2 = \begin{bmatrix} -3 & 1 & 1 \\ 0 & -3 & 1 \\ 0 & 0 & 0 \end{bmatrix} \mathbf{q}_2 \Longrightarrow \mathbf{q}_2 = \begin{bmatrix} 4 \\ 3 \\ 9 \end{bmatrix}. \end{aligned}$$

Consequently, the general solution is

$$\begin{aligned} \mathbf{x}(t) &= c_1\mathbf{q}_1 e^{3t} + c_2(t\mathbf{q}_1 + \mathbf{r}_1) e^{3t} + c_3\mathbf{q}_2 e^{6t} \\ &= c_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} e^{3t} + c_2 \left(t \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right) e^{3t} + c_3 \begin{bmatrix} 4 \\ 3 \\ 9 \end{bmatrix} e^{6t}, \\ &= \begin{bmatrix} c_1 e^{3t} + c_2 t e^{3t} + 4c_3 e^{6t} \\ c_2 e^{3t} + 3c_3 e^{6t} \\ 9c_3 e^{6t} \end{bmatrix}. \end{aligned}$$

(n). The equations above can be written as

$$\mathbf{x}' = \mathbf{A}\mathbf{x} = \begin{bmatrix} 2 & 1 & -1 \\ -4 & -3 & -1 \\ 4 & 4 & 2 \end{bmatrix} \mathbf{x}, \text{ where } \mathbf{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}.$$

The characteristic polynomial is

$$\det(\mathbf{A}-\lambda\mathbf{I}) = \begin{vmatrix} 2-\lambda & 1 & -1 \\ -4 & -3-\lambda & -1 \\ 4 & 4 & 2-\lambda \end{vmatrix} = (\lambda-1)\left(\lambda^2+4\right).$$

Thus, the eigenvalues of $\mathbf{A}$ are $\{1, \pm 2j\}$. The eigenvectors belonging to $\lambda = \pm 2j$ are $\{\mathbf{q}_1 \pm j\mathbf{q}_2\}$.Note that

$$\mathbf{A}^2 + 4\mathbf{I} = \begin{bmatrix} -4 & -5 & -5 \\ 0 & 1 & 5 \\ 0 & 0 & -4 \end{bmatrix} + 4\mathbf{I} = \begin{bmatrix} 0 & -5 & -5 \\ 0 & 5 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

We can choose $\mathbf{q}_1$ so that $\left(\mathbf{A}^2 + 4\mathbf{I}\right)\mathbf{q}_1 = \mathbf{0}$. Hence, we get

$$\mathbf{0} = \left(\mathbf{A}^2 + 4\mathbf{I}\right)\mathbf{q}_1 = \begin{bmatrix} 0 & -5 & -5 \\ 0 & 5 & 1 \\ 0 & 0 & 0 \end{bmatrix} \mathbf{q}_1 \Longrightarrow \mathbf{q}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}.$$

Then, similar to the problems in $\mathbb{R}^2$, we can choose $\mathbf{q}_2$as follows.

$$(\mathbf{A}-\sigma\mathbf{I})\,\mathbf{q}_1 = -w\mathbf{q}_2 \Longrightarrow \begin{bmatrix} 2 & 1 & -1 \\ -4 & -3 & -1 \\ 4 & 4 & 2 \end{bmatrix} \mathbf{q}_1 = -2\mathbf{q}_2 \Longrightarrow \mathbf{q}_2 = \begin{bmatrix} -1 \\ 2 \\ -2 \end{bmatrix}.$$

Finally, the eigenvector for $\lambda = 1$ can be calculated as follows.

$$\mathbf{0} = (\mathbf{A} - \mathbf{I})\,\mathbf{q}_3 = \begin{bmatrix} 1 & 1 & -1 \\ -4 & -4 & -1 \\ 4 & 4 & 1 \end{bmatrix} \mathbf{q}_3 \Longrightarrow \mathbf{q}_3 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}.$$

Consequently, the general solution is

$$\begin{aligned} \mathbf{x}(t) &= (c_1\mathbf{q}_1 + c_2\mathbf{q}_2)\cos 2t + (c_2\mathbf{q}_1 - c_1\mathbf{q}_2)\sin 2t + c_3\mathbf{q}_3 e^t \\ &= \left(c_1\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + c_2\begin{bmatrix} -1 \\ 2 \\ -2 \end{bmatrix}\right)\cos 2t + \left(c_2\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} - c_1\begin{bmatrix} -1 \\ 2 \\ -2 \end{bmatrix}\right)\sin 2t + c_3\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} e^t, \\ &= \begin{bmatrix} (c_1 - c_2)\cos 2t + (c_1 + c_2)\sin 2t + c_3 e^t \\ 2c_2\cos 2t - 2c_1\sin 2t - c_3 e^t \\ 2c_1\sin 2t - 2c_2\cos 2t \end{bmatrix}. \end{aligned}$$

5.2

(a). Note that

$$\det(\mathbf{A}-\lambda\mathbf{I}) = \begin{vmatrix} 3-\lambda & 4 \\ 3 & 2-\lambda \end{vmatrix} = \lambda^2 - 5\lambda - 6 = (\lambda - 6)(\lambda + 1).$$

Thus, the eigenvalues of $\mathbf{A}$ are $\{6, -1\}$. Since the eigenvalues of $\mathbf{A}$ are real and distinct, the eigenvectors of $\mathbf{A}$ are linearly independent and can be calculated as follows.

$$\begin{aligned} \mathbf{0} &= (\mathbf{A}-6\mathbf{I})\,\mathbf{q}_1 = \begin{bmatrix} -3 & 4 \\ 3 & -4 \end{bmatrix}\mathbf{q}_1 \Longrightarrow \mathbf{q}_1 = \begin{bmatrix} 4 \\ 3 \end{bmatrix}, \\ \mathbf{0} &= (\mathbf{A}+\mathbf{I})\,\mathbf{q}_2 = \begin{bmatrix} 4 & 4 \\ 3 & 3 \end{bmatrix}\mathbf{q}_2 \Longrightarrow \mathbf{q}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}. \end{aligned}$$

Consequently, the general solution is

$$\begin{aligned} \mathbf{x}(t) &= c_1\mathbf{q}_1 e^{6t} + c_2\mathbf{q}_2 e^{-t} = c_1\begin{bmatrix} 4 \\ 3 \end{bmatrix} e^{6t} + c_2\begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{-t}, \\ &= \begin{bmatrix} c_1 e^{6t} + c_2 e^{-t} \\ c_1 e^{6t} - c_2 e^{-t} \end{bmatrix}. \end{aligned}$$

Note that at $t = 0$, we have

$$\begin{aligned} \mathbf{x}(0) &= \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 & 1 \\ 3 & -1 \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \Longrightarrow \\ \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} &= \frac{1}{7}\begin{bmatrix} 1 & 1 \\ 3 & -4 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \frac{1}{7}\begin{bmatrix} 2 \\ -1 \end{bmatrix}. \end{aligned}$$

Thus, the solution of the initial value problem is

$$\mathbf{x}(t) = \frac{1}{7}\begin{bmatrix} 2e^{6t} - e^{-t} \\ 2e^{3t} + e^{-t} \end{bmatrix}.$$

(b). Note that

$$\det(\mathbf{A}-\lambda\mathbf{I}) = \begin{vmatrix} 4-\lambda & -3 \\ 6 & -7-\lambda \end{vmatrix} = \lambda^2 + 3\lambda - 10 = (\lambda - 2)(\lambda + 5).$$

Thus, the eigenvalues of $\mathbf{A}$ are $\{2, -5\}$. Since the eigenvalues of $\mathbf{A}$ are real and distinct, the eigenvectors of $\mathbf{A}$ are linearly independent and can be calculated as follows.

$$\begin{aligned} \mathbf{0} &= (\mathbf{A}-2\mathbf{I})\,\mathbf{q}_1 = \begin{bmatrix} 2 & -3 \\ 6 & -9 \end{bmatrix}\mathbf{q}_1 \Longrightarrow \mathbf{q}_1 = \begin{bmatrix} 3 \\ 2 \end{bmatrix}, \\ \mathbf{0} &= (\mathbf{A}+5\mathbf{I})\,\mathbf{q}_2 = \begin{bmatrix} 9 & -3 \\ 6 & -2 \end{bmatrix}\mathbf{q}_2 \Longrightarrow \mathbf{q}_2 = \begin{bmatrix} 1 \\ 3 \end{bmatrix}. \end{aligned}$$

Consequently, the general solution is

$$\begin{aligned} \mathbf{x}(t) &= c_1\mathbf{q}_1 e^{2t} + c_2\mathbf{q}_2 e^{-5t} = c_1\begin{bmatrix} 3 \\ 2 \end{bmatrix} e^{2t} + c_2\begin{bmatrix} 1 \\ 3 \end{bmatrix} e^{-5t}, \\ &= \begin{bmatrix} 3c_1 e^{2t} + c_2 e^{-5t} \\ 2c_1 e^{2t} + 3c_2 e^{-5t} \end{bmatrix}. \end{aligned}$$

Note that at $t = 0$, we have

$$\begin{aligned} \mathbf{x}(0) &= \begin{bmatrix} 8 \\ 0 \end{bmatrix} = \begin{bmatrix} 3 & 1 \\ 2 & 3 \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \Longrightarrow \\ \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} &= \frac{1}{7}\begin{bmatrix} 3 & -1 \\ -2 & 3 \end{bmatrix}\begin{bmatrix} 8 \\ 0 \end{bmatrix} = \frac{1}{7}\begin{bmatrix} 24 \\ -16 \end{bmatrix}. \end{aligned}$$

Thus, the solution of the initial value problem is

$$\mathbf{x}(t) = \frac{8}{7}\begin{bmatrix} 9e^{2t} - 2e^{-5t} \\ 6e^{2t} - 6e^{-5t} \end{bmatrix}.$$

(c). The equations above can be written as

$$\mathbf{x}' = \mathbf{A}\mathbf{x} = \begin{bmatrix} 3 & 0 & 1 \\ 9 & -1 & 2 \\ -9 & 4 & -1 \end{bmatrix}\mathbf{x}, \text{ where } \mathbf{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}.$$

The characteristic polynomial is

$$\det(\mathbf{A}-\lambda\mathbf{I}) = \begin{vmatrix} 3-\lambda & 0 & 1 \\ 9 & -1-\lambda & 2 \\ -9 & 4 & -1-\lambda \end{vmatrix} = \left(\lambda^2 + 2\lambda + 2\right)(\lambda - 3).$$

Thus, the eigenvalues of $\mathbf{A}$ are $\{3, -1 \pm j\}$. The eigenvectors belonging to $\lambda = -1 \pm j$ are $\{\mathbf{q}_1 \pm j\mathbf{q}_2\}$.Note that

$$\mathbf{A}^2 + 2\mathbf{A} + 2\mathbf{I} = \begin{bmatrix} 8 & 4 & 4 \\ 18 & 9 & 9 \\ 0 & 0 & 0 \end{bmatrix}.$$

We can choose $\mathbf{q}_1$ so that $\left(\mathbf{A}^2 + 2\mathbf{A} + 2\mathbf{I}\right)\mathbf{q}_1 = \mathbf{0}$. Hence, we get

$$\mathbf{0} = \left(\mathbf{A}^2 + 2\mathbf{A} + 2\mathbf{I}\right)\mathbf{q}_1 = \begin{bmatrix} 8 & 4 & 4 \\ 18 & 9 & 9 \\ 0 & 0 & 0 \end{bmatrix}\mathbf{q}_1 \Longrightarrow \mathbf{q}_1 = \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix}.$$

Then, similar to the problems in $\mathbb{R}^2$, we can choose $\mathbf{q}_2$as follows.

$$(\mathbf{A}+\mathbf{I})\,\mathbf{q}_1 = -\mathbf{q}_2 \Longrightarrow \begin{bmatrix} 4 & 0 & 1 \\ 9 & 0 & 2 \\ -9 & 4 & 0 \end{bmatrix}\mathbf{q}_1 = -\mathbf{q}_2 \Longrightarrow \mathbf{q}_2 = \begin{bmatrix} -3 \\ -7 \\ 13 \end{bmatrix}.$$

Finally, the eigenvector for $\lambda = 3$ can be calculated as follows.

$$\mathbf{0} = (\mathbf{A} - 3\mathbf{I})\,\mathbf{q}_3 = \begin{bmatrix} 0 & 0 & 1 \\ 9 & -4 & 2 \\ -9 & 4 & -4 \end{bmatrix} \mathbf{q}_3 \Longrightarrow \mathbf{q}_3 = \begin{bmatrix} 4 \\ 9 \\ 0 \end{bmatrix}.$$

Consequently, the general solution is

$$\begin{aligned} \mathbf{x}(t) &= (c_1\mathbf{q}_1 + c_2\mathbf{q}_2)\, e^{-t}\cos t + (c_2\mathbf{q}_1 - c_1\mathbf{q}_2)\, e^{-t}\sin t + c_3\mathbf{q}_3 e^{3t} \\ &= \left(c_1 \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix} + c_2 \begin{bmatrix} -3 \\ -7 \\ 13 \end{bmatrix} \right) e^{-t}\cos t + \left(c_2 \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix} - c_1 \begin{bmatrix} -3 \\ -7 \\ 13 \end{bmatrix} \right) e^{-t}\sin t + c_3 \begin{bmatrix} 4 \\ 9 \\ 0 \end{bmatrix} e^{3t} \\ &= e^{-t} \begin{bmatrix} (c_1 - 3c_2)\cos t + (3c_1 + c_2)\sin t + 4c_3 e^{4t} \\ -(c_1 + 7c_2)\cos t + (7c_1 - c_2)\sin t + 9c_3 e^{4t} \\ (13c_2 - c_1)\cos t - (13c_1 + c_2)\sin t \end{bmatrix}. \end{aligned}$$

Note that at $t = 0$, we get the solution for the initial value problem.

$$\begin{aligned} \begin{bmatrix} 0 \\ 0 \\ 17 \end{bmatrix} &= \begin{bmatrix} 1 & -3 & 4 \\ -1 & -7 & 9 \\ -1 & 13 & 0 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} \Longrightarrow \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \frac{1}{170} \begin{bmatrix} 117 & -52 & -1 \\ 9 & -4 & 13 \\ 20 & 10 & 10 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 17 \end{bmatrix}, \\ \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} &= \frac{1}{10} \begin{bmatrix} -1 \\ 13 \\ 10 \end{bmatrix} \Longrightarrow \mathbf{x}(t) = e^{-t} \begin{bmatrix} -4\cos t + \sin t + 4e^{4t} \\ -9\cos t - 2\sin t + 9e^{4t} \\ 17\cos t \end{bmatrix}. \end{aligned}$$

Changing back to the notation in the question, we have that

$$\begin{aligned} x(t) &= -4e^{-t}\cos t + e^{-t}\sin t + 4e^{3t} \\ y(t) &= -9e^{-t}\cos t - 2e^{-t}\sin t + 9e^{3t} \\ z(t) &= 17e^{-t}\cos t. \end{aligned}$$

5.3

(a). $\begin{cases} x_1' = x_2 \\ x_2' = -2x_1 - 0.5x_2 \end{cases}$

(b). $\begin{cases} x_1' = x_2 \\ x_2' = -2x_1 - 0.5x_2 + 3\sin t \end{cases}$

(c). $\begin{cases} x_1' = x_2 \\ x_2' = -(1 - 0.25t^{-2})x_1 - t^{-1}x_2 \end{cases}$

(d). $\begin{cases} x_1' = x_2 \\ x_2' = x_3 \\ x_3' = x_4 \\ x_4' = x_1 \end{cases}$

(e). $x_1'' - x_1' - 2x_1 = 0$

(f). $x_1'' - 2.5x_1 + x_1 = 0$

(g). First we solve the first equation for x_2:

$$x_2 = \frac{1}{2}x_1 - \frac{1}{2}x_1'.$$

Then we substitute into the second equation to find

$$\begin{aligned} x_2' &= 3x_1 - 4x_2 \\ \left(\frac{1}{2}x_1 - \frac{1}{2}x_1'\right)' &= 3x_1 - 4\left(\frac{1}{2}x_1 - \frac{1}{2}x_1'\right) \\ \frac{1}{2}x_1' - \frac{1}{2}x_1'' &= 3x_1 - 2x_1 - 2x_1' \\ 0 &= \frac{1}{2}x_1'' - \frac{5}{2}x_1' + x_1 \\ 0 &= x_1'' - 5x_1' + 2x_1 \end{aligned}$$

(h). $x_1'' + 4x_1 = 0$

5.4

(a). We have that

$$A\Psi = A\left[\mathbf{x}^{(1)} \quad \mathbf{x}^{(2)} \quad \ldots \quad \mathbf{x}^{(n)}\right] = \left[A\mathbf{x}^{(1)} \quad A\mathbf{x}^{(2)} \quad \ldots \quad A\mathbf{x}^{(n)}\right] = \left[\mathbf{x}^{(1)\prime} \quad \mathbf{x}^{(2)\prime} \quad \ldots \quad \mathbf{x}^{(n)\prime}\right] = \Psi'$$

since $\mathbf{x}^{(k)}$ solves $A\mathbf{x}^{(k)} = x^{(k)\prime}$ for each k.

(b). Recall first that $e^{At} = \Phi(t)$ where Φ is the special fundamental matrix which satisfies $\Phi(0) = I$. Since the eigenvalues of A are $r_1 = 1$ and $r_2 = 2$; and the eigenvectors are $\xi^{(1)} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\xi^{(2)} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, the general solution of $\mathbf{x}' = A\mathbf{x}$ is $\mathbf{x}(t) = c_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} e^t + c_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{2t}$. Then we calculate that

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \mathbf{x}(0) = c_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} \implies c_1 = 1,\ c_2 = 0 \implies \mathbf{x}^{(1)} = \begin{bmatrix} e^t \\ 0 \end{bmatrix}$$

and

$$\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \mathbf{x}(0) = c_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} \implies c_1 = -1,\ c_2 = 1 \implies \mathbf{x}^{(2)} = \begin{bmatrix} e^{2t} - e^t \\ e^{2t} \end{bmatrix}.$$

Therefore

$$e^{At} = \Phi(t) = \begin{bmatrix} e^{2t} & e^t - e^{2t} \\ 0 & e^t \end{bmatrix}.$$

(c). (i) omitted (ii) $\Phi(t) = \begin{bmatrix} -\frac{1}{3}e^{-t} + \frac{4}{3}e^{2t} & \frac{2}{3}e^{-t} - \frac{2}{3}e^{2t} \\ -\frac{2}{3}e^{-t} + \frac{2}{3}e^{2t} & \frac{4}{3}e^{-t} - \frac{1}{3}e^{2t} \end{bmatrix}$

(d). (i) This matrix has eigenvalues $r_1 = -\frac{1}{2}$ and $r_2 = -1$; and corresponding eigenvectors $\xi^{(1)} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ and $\xi^{(2)} = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$. Therefore the general solution is $\mathbf{x}(t) = c_1 \begin{bmatrix} 2 \\ 1 \end{bmatrix} e^{-\frac{t}{2}} + c_2 \begin{bmatrix} -2 \\ 1 \end{bmatrix} e^{-t}$ and a fundamental matrix is $\Psi(t) = \begin{bmatrix} 2e^{-\frac{t}{2}} & -2e^{-t} \\ e^{-\frac{t}{2}} & e^{-t} \end{bmatrix}$.

(ii) Since

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix} = c_1 \begin{bmatrix} 2 \\ 1 \end{bmatrix} e^{-\frac{0}{2}} + c_2 \begin{bmatrix} -2 \\ 1 \end{bmatrix} e^{-0} \implies c_1 = \frac{1}{4},\ c_2 = -\frac{1}{4} \implies \mathbf{x}^{(1)}(t) = \begin{bmatrix} \frac{1}{2}e^{-\frac{t}{2}} + \frac{1}{2}e^{-t} \\ \frac{1}{4}e^{-\frac{t}{2}} - \frac{1}{4}e^{-t} \end{bmatrix}$$

and

$$\begin{bmatrix} 0 \\ 1 \end{bmatrix} = c_1 \begin{bmatrix} 2 \\ 1 \end{bmatrix} e^{-\frac{0}{2}} + c_2 \begin{bmatrix} -2 \\ 1 \end{bmatrix} e^{-0} \implies c_1 = c_2 = \frac{1}{2} \implies \mathbf{x}^{(1)}(t) = \begin{bmatrix} e^{-\frac{t}{2}} - e^{-t} \\ \frac{1}{2}e^{-\frac{t}{2}} + \frac{1}{2}e^{-t} \end{bmatrix}$$

we have that

$$\Phi(t) = \begin{bmatrix} \frac{1}{2}e^{-\frac{t}{2}} + \frac{1}{2}e^{-t} & e^{-\frac{t}{2}} - e^{-t} \\ \frac{1}{4}e^{-\frac{t}{2}} - \frac{1}{4}e^{-t} & \frac{1}{2}e^{-\frac{t}{2}} + \frac{1}{2}e^{-t} \end{bmatrix}.$$

(e). (i) This matrix has eigenvalues $r_1 = i$ and $r_2 = -i$; and corresponding eigenvectors $\xi^{(1)} = \begin{bmatrix} 2+i \\ 1 \end{bmatrix}$ and $\xi^{(2)} = \begin{bmatrix} 2-i \\ 1 \end{bmatrix}$. Since

$$\begin{aligned} \mathbf{x}^{(1)}(t) = \xi^{(1)} e^{r_1 t} &= \begin{bmatrix} 2+i \\ 1 \end{bmatrix} (\cos t + i \sin t) \\ &= \begin{bmatrix} 2\cos t - \sin t \\ \cos t \end{bmatrix} + i \begin{bmatrix} \cos t + 2\sin t \\ \sin t \end{bmatrix}, \end{aligned}$$

we have general solution

$$\mathbf{x}(t) = c_1 \begin{bmatrix} 2\cos t - \sin t \\ \cos t \end{bmatrix} + c_2 \begin{bmatrix} \cos t + 2\sin t \\ \sin t \end{bmatrix}$$

and fundamental matrix

$$\Psi(t) = \begin{bmatrix} 2\cos t - \sin t & \cos t + 2\sin t \\ \cos t & \sin t \end{bmatrix}.$$

(ii) Then we calculate that

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \mathbf{x}(0) = c_1 \begin{bmatrix} 2 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 0 \end{bmatrix} \implies c_1 = 0,\ c_2 = 1 \implies \mathbf{x}(t) = \begin{bmatrix} \cos t + 2\sin t \\ \sin t \end{bmatrix}$$

and

$$\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \mathbf{x}(0) = c_1 \begin{bmatrix} 2 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 0 \end{bmatrix} \implies c_1 = 1,\ c_2 = -2 \implies \mathbf{x}(t) = \begin{bmatrix} -5\sin t \\ \cos t - 2\sin t \end{bmatrix}.$$

Therefore

$$\Phi(t) = \begin{bmatrix} \cos t + 2\sin t & -5\sin t \\ \sin t & \cos t - 2\sin t \end{bmatrix}.$$

(f). (i) omitted (ii) $\Phi(t) = \begin{bmatrix} \frac{1}{5}e^{-3t} + \frac{4}{5}e^{2t} & -\frac{1}{5}e^{-3t} + \frac{1}{5}e^{2t} \\ -\frac{4}{5}e^{-3t} + \frac{4}{5}e^{2t} & \frac{4}{5}e^{-3t} + \frac{1}{5}e^{2t} \end{bmatrix}$

(g). (i) omitted (ii) $\Phi(t) = \begin{bmatrix} \frac{3}{2}e^{t} - \frac{1}{2}e^{-t} & -\frac{1}{2}e^{t} + \frac{1}{2}e^{-t} \\ \frac{3}{2}e^{t} - \frac{3}{2}e^{-t} & -\frac{1}{2}e^{t} + \frac{3}{2}e^{-t} \end{bmatrix}$

(h). (i) omitted (ii) $\Phi(t) = \begin{bmatrix} e^{-t}\cos 2t & -2e^{-t}\sin 2t \\ \frac{1}{2}e^{-t}\sin 2t & e^{-t}\cos 2t \end{bmatrix}$

(g) (i) omitted (ii) $\Phi(t) = \begin{bmatrix} -2e^{-2t} + 3e^{-t} & -e^{-2t} + e^{-t} & -e^{-2t} + e^{-t} \\ \frac{5}{2}e^{-2t} - 4e^{-t} + \frac{3}{2}e^{2t} & \frac{5}{4}e^{-2t} - \frac{4}{3}e^{-t} + \frac{13}{12}e^{2t} & \frac{5}{4}e^{-2t} - \frac{4}{3}e^{-t} + \frac{1}{12}e^{2t} \\ \frac{7}{2}e^{-2t} - 2e^{-t} - \frac{3}{2}e^{2t} & \frac{7}{4}e^{-2t} - \frac{2}{3}e^{-t} - \frac{13}{12}e^{2t} & \frac{7}{4}e^{-2t} - \frac{2}{3}e^{-t} - \frac{1}{12}e^{2t} \end{bmatrix}$

5.5

(a). Note that our ODE can be written as $\mathbf{x}' = \begin{bmatrix} 2 & -1 \\ 3 & -2 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} e^t + \begin{bmatrix} 0 \\ 1 \end{bmatrix} t.$

The eigenvalues of $A = \begin{bmatrix} 2 & -1 \\ 3 & -2 \end{bmatrix}$ are $r_1 = 1$ and $r_2 = -1$. The corresponding eigenvectors are $\xi^{(1)} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\xi^{(2)} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$. Therefore the general solution of the homogeneous equation $\mathbf{x}' = A\mathbf{x}$ is

$$\mathbf{x}(t) = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^t + c_2 \begin{bmatrix} 1 \\ 3 \end{bmatrix} e^{-t}.$$

Next we need to find a particular solution of $\mathbf{x}' = A\mathbf{x} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} e^t$. Since $r = 1$ is an eigenvalue of A, we try the ansatz $\mathbf{x} = \mathbf{a}te^t + \mathbf{b}e^t$ where $\mathbf{a} = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} \in \mathbb{R}^2$. Then we calculate that

$$\mathbf{a}e^t + \mathbf{a}te^t + \mathbf{b}e^t = \mathbf{x}' = A\mathbf{x} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} e^t = A\mathbf{a}te^t + A\mathbf{b}e^t + \begin{bmatrix} 1 \\ 0 \end{bmatrix} e^t$$

$$\mathbf{a} + \mathbf{a}t + \mathbf{b} = A\mathbf{a}t + A\mathbf{b} + \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

Since this must be true for all t, we must have

$$\begin{cases} \mathbf{a} = A\mathbf{a} \\ A\mathbf{b} - \mathbf{b} = \mathbf{a} - \begin{bmatrix} 1 \\ 0 \end{bmatrix}. \end{cases}$$

The former equation tells us that $\mathbf{a}$ must be an eigenvector of A corresponding to $r = 1$. So $\mathbf{a} = \begin{bmatrix} \alpha \\ \alpha \end{bmatrix}$ for some $\alpha \in \mathbb{R}$ that we need to find. Then the latter equation becomes

$$\begin{bmatrix} b_1 - b_2 \\ 3b_1 - 3b_2 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 3 & -3 \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} = (A - I)\mathbf{b} = A\mathbf{b} - \mathbf{b} = \mathbf{a} - \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \alpha - 1 \\ \alpha \end{bmatrix}.$$

Thus

$$3(\alpha - 1) = 3(b_1 - b_2) = 3b_1 - 3b_2 = \alpha \implies 2\alpha = 3 \implies \alpha = \frac{3}{2}.$$

Then we have $b_1 - b_2 = \frac{1}{2}$ which implies that $\mathbf{b} = \begin{bmatrix} k \\ k - \frac{1}{2} \end{bmatrix}$ for any k. I choose $k = 0$. Therefore

$$\mathbf{x}(t) = \begin{bmatrix} \frac{3}{2} \\ \frac{3}{2} \end{bmatrix} te^t - \begin{bmatrix} 0 \\ \frac{1}{2} \end{bmatrix} e^t.$$

Finally we must find a particular solution of $\mathbf{x}' = A\mathbf{x} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} t$. Here we try the ansatz $\mathbf{x} = \mathbf{c}t + \mathbf{d}$ where $\mathbf{c} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}, \mathbf{d} = \begin{bmatrix} d_1 \\ d_2 \end{bmatrix} \in \mathbb{R}^2$. Then we calculate that

$$\mathbf{c} = \mathbf{x}' = A\mathbf{x} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} t = A\mathbf{c}t + A\mathbf{d} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} t.$$

Since this must be true for all t, we must have $A\mathbf{c} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \mathbf{0}$ and $A\mathbf{d} = \mathbf{c}$. Using the former equation, we calculate that

$$\begin{bmatrix} 0 \\ -1 \end{bmatrix} = A\mathbf{c} = \begin{bmatrix} 2 & -1 \\ 3 & -2 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 2c_1 - c_2 \\ 3c_1 - 2c_2 \end{bmatrix} \implies c_1 = 1,\ c_2 = 2 \implies \mathbf{c} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}.$$

(Or equivalently, we could calculate that $\mathbf{c} = A^{-1} \begin{bmatrix} 0 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$.) Then the latter equation gives us

$$\begin{bmatrix} 1 \\ 2 \end{bmatrix} = A\mathbf{d} = \begin{bmatrix} 2 & -1 \\ 3 & -2 \end{bmatrix} \begin{bmatrix} d_1 \\ d_2 \end{bmatrix} = \begin{bmatrix} 2d_1 - d_2 \\ 3d_1 - 2d_2 \end{bmatrix} \implies d_1 = 0,\ d_2 = -1 \implies \mathbf{d} = \begin{bmatrix} 0 \\ -1 \end{bmatrix}.$$

Hence

$$\mathbf{x} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} t + \begin{bmatrix} 0 \\ -1 \end{bmatrix}.$$

Adding these three solutions together, we obtain the general solution of the problem:

$$\mathbf{x}(t) = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^t + c_2 \begin{bmatrix} 1 \\ 3 \end{bmatrix} e^{-t} + \frac{3}{2} \begin{bmatrix} 1 \\ 1 \end{bmatrix} te^t - \frac{1}{2} \begin{bmatrix} 0 \\ 1 \end{bmatrix} e^t + \begin{bmatrix} 1 \\ 2 \end{bmatrix} t - \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

(b). Using the same method as in (a), we find that $\mathbf{x}(t) = c_1 \begin{bmatrix} \sqrt{3} \\ 1 \end{bmatrix} e^{2t} + c_2 \begin{bmatrix} 1 \\ -\sqrt{3} \end{bmatrix} e^{-2t} - \begin{bmatrix} \frac{2}{3} \\ \frac{1}{\sqrt{3}} \end{bmatrix} e^t + \begin{bmatrix} -1 \\ \frac{2}{\sqrt{3}} \end{bmatrix} e^{-t}$.

(c). The eigenvalues of $A = \begin{bmatrix} 1 & 1 \\ 4 & -2 \end{bmatrix}$ are $r_1 = -3$ and $r_2 = 2$; and the eigenvectors are $\xi^{(1)} = \begin{bmatrix} 1 \\ -4 \end{bmatrix}$ and $\xi^{(2)} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. Let $T = \begin{bmatrix} 1 & 1 \\ -4 & 1 \end{bmatrix}$. Then $T^{-1} = \frac{1}{5} \begin{bmatrix} 1 & -1 \\ 4 & 1 \end{bmatrix}$.

Using the substitution $\mathbf{x} = T\mathbf{y}$ we convert $\mathbf{x}' = A\mathbf{x} + \mathbf{g}$ into two first order linear ODEs as follows:

$$\begin{aligned} \mathbf{x}' &= A\mathbf{x} + \mathbf{g} \\ T\mathbf{y}' &= AT\mathbf{y} + \mathbf{g} \\ \mathbf{y}' &= T^{-1}AT\mathbf{y} + T^{-1}\mathbf{g} \\ \mathbf{y}' &= \begin{bmatrix} -3 & 0 \\ 0 & 2 \end{bmatrix} \mathbf{y} + \frac{1}{5} \begin{bmatrix} 1 & -1 \\ 4 & 1 \end{bmatrix} \begin{bmatrix} e^{-2t} \\ -2e^t \end{bmatrix} = \begin{bmatrix} -3 & 0 \\ 0 & 2 \end{bmatrix} \mathbf{y} + \frac{1}{5} \begin{bmatrix} e^{-2t} + 2e^t \\ 4e^{-2t} - 2e^t \end{bmatrix} \end{aligned}$$

$$\begin{cases} y_1' = -3y_1 + \frac{1}{5}\left(e^{-2t} + 2e^t\right) \\ y_2' = 2y_2 + \frac{1}{5}\left(4e^{-2t} - 2e^t\right) \end{cases}$$

Using the integrating factors $\mu_1(t) = e^{3t}$ and $\mu_2(t) = e^{-2t}$ respectively, we can solve these two first order linear ODEs to obtain

$$\mathbf{y}(t) = \begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} = \begin{bmatrix} \frac{1}{5}e^{-2t} + \frac{1}{10}e^t + c_1 e^{-3t} \\ -\frac{1}{5}e^{-2t} + \frac{2}{5}e^t + c_2 e^{2t} \end{bmatrix}.$$

Finally multiplying by T gives

$$\begin{aligned} \mathbf{x}(t) &= T\mathbf{y} \\ &= \begin{bmatrix} 1 & 1 \\ -4 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{5}e^{-2t} + \frac{1}{10}e^t + c_1 e^{-3t} \\ -\frac{1}{5}e^{-2t} + \frac{2}{5}e^t + c_2 e^{2t} \end{bmatrix} \\ &= c_1 \begin{bmatrix} 1 \\ -4 \end{bmatrix} e^{-3t} + c_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{2t} - \begin{bmatrix} 0 \\ 1 \end{bmatrix} e^{-2t} + \frac{1}{2} \begin{bmatrix} 1 \\ 0 \end{bmatrix} e^t. \end{aligned}$$

(d). $\mathbf{x}(t) = c_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} e^t + c_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{2t} + \frac{1}{5} \begin{bmatrix} 4\cos t - 2\sin t \\ -\cos t - 2\sin t \end{bmatrix}$.

(e). The eigenvalues of $A = \begin{bmatrix} -4 & 2 \\ 2 & -1 \end{bmatrix}$ are $r_1 = 0$ and $r_2 = -5$; and the eigenvectors are $\xi^{(1)} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $\xi^{(2)} = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$. Thus

$$\Psi(t) = \begin{bmatrix} 1 & -2e^{-5t} \\ 2 & e^{-5t} \end{bmatrix}$$

is a fundamental matrix for $\mathbf{x}' = A\mathbf{x}$. Using the formula $\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$ we calculate that

$$\Psi^{-1}(t) = \frac{1}{e^{-5t} + 4e^{-5t}} \begin{bmatrix} e^{-5t} & 2e^{-5t} \\ -2 & 1 \end{bmatrix} = \frac{1}{5} \begin{bmatrix} 1 & 2 \\ -2e^{5t} & e^{5t} \end{bmatrix}.$$

Then

$$\Psi^{-1}(t)\mathbf{g}(t) = \frac{1}{5}\begin{bmatrix} 1 & 2 \\ -2e^{5t} & e^{5t} \end{bmatrix}\begin{bmatrix} t^{-1} \\ 2t^{-1}+4 \end{bmatrix} = \frac{1}{5}\begin{bmatrix} t^{-1}+4t^{-1}+8 \\ -2t^{-1}e^{5t}+2t^{-1}e^{5t}+4e^{5t} \end{bmatrix} = \begin{bmatrix} t^{-1}+\frac{8}{5} \\ \frac{4}{5}e^{5t} \end{bmatrix}$$

and

$$\int \Psi^{-1}(t)\mathbf{g}(t)\,dt = \int \begin{bmatrix} t^{-1}+\frac{8}{5} \\ \frac{4}{5}e^{5t} \end{bmatrix} dt = \begin{bmatrix} \ln t+\frac{8}{5}t+c_1 \\ \frac{4}{25}e^{5t}+c_2 \end{bmatrix}.$$

It follows that

$$\begin{aligned} \mathbf{x}(t) &= \Psi(t)\int \Psi^{-1}(s)\mathbf{g}(s)\,ds = \begin{bmatrix} 1 & -2e^{-5t} \\ 2 & e^{-5t} \end{bmatrix}\begin{bmatrix} \ln t+\frac{8}{5}t+c_1 \\ \frac{4}{25}e^{5t}+c_2 \end{bmatrix} = \begin{bmatrix} \ln t+\frac{8}{5}t-\frac{8}{25}+c_1-2c_2e^{-5t} \\ 2\ln t+\frac{16}{5}t+\frac{4}{25}+2c_1+c_2e^{-5t} \end{bmatrix} \\ &= c_1\begin{bmatrix} 1 \\ 2 \end{bmatrix}+c_2\begin{bmatrix} -2 \\ 1 \end{bmatrix}e^{-5t}+\begin{bmatrix} 1 \\ 2 \end{bmatrix}\ln t+\frac{8}{5}\begin{bmatrix} 1 \\ 2 \end{bmatrix}t+\frac{4}{25}\begin{bmatrix} -2 \\ 1 \end{bmatrix}. \end{aligned}$$

(f). Using the Method of Variation of Parameters, we can find that $\mathbf{x}(t) = c_1\begin{bmatrix} 1 \\ 2 \end{bmatrix}+c_2\left(\begin{bmatrix} 1 \\ 2 \end{bmatrix}t-\frac{1}{2}\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right)-2\begin{bmatrix} 1 \\ 2 \end{bmatrix}\ln t+\begin{bmatrix} 2 \\ 5 \end{bmatrix}t^{-1}-\begin{bmatrix} \frac{1}{2} \\ 0 \end{bmatrix}t^{-2}$.

5.6

(a). The equations above can be written as

$$\mathbf{x}' = \mathbf{A}\mathbf{x} = \begin{bmatrix} 1 & -2 \\ 5 & -1 \end{bmatrix}\mathbf{x}, \text{ where } \mathbf{x} = \begin{bmatrix} x \\ y \end{bmatrix}.$$

If we take the Laplace transform of the both sides of the above equation, we get

$$\begin{aligned} (s\mathbf{I}-\mathbf{A})\,\mathbf{X}(s) &= \mathbf{x}(0) \Longrightarrow \begin{bmatrix} s-1 & 2 \\ -5 & s+1 \end{bmatrix}\mathbf{X}(s) = \begin{bmatrix} -1 \\ 2 \end{bmatrix} \Longrightarrow \\ \mathbf{X}(s) &= \frac{1}{s^2+9}\begin{bmatrix} s+1 & -2 \\ 5 & s-1 \end{bmatrix}\begin{bmatrix} -1 \\ 2 \end{bmatrix} = \frac{1}{s^2+9}\begin{bmatrix} -(s+5) \\ 2s-7 \end{bmatrix}. \end{aligned}$$

Note that

$$\begin{aligned} \frac{-s-5}{s^2+9} &= -\frac{s}{s^2+9}-\frac{5}{3}\frac{3}{s^2+9} \Longrightarrow \mathcal{L}^{-1}\left(\frac{-s-5}{s^2+9}\right) = -\cos 3t-\frac{5}{3}\sin 3t. \\ \frac{2s-7}{s^2+9} &= 2\frac{s}{s^2+9}-\frac{7}{3}\frac{3}{s^2+9} \Longrightarrow \mathcal{L}^{-1}\left(\frac{2s-7}{s^2+9}\right) = 2\cos 3t-\frac{7}{3}\sin 3t. \end{aligned}$$

Then, the solution of the initial value problem is

$$\mathbf{x}(t) = \begin{bmatrix} -\cos 3t-\frac{5}{3}\sin 3t \\ 2\cos 3t-\frac{7}{3}\sin 3t \end{bmatrix}.$$

(b). The equations above can be written as

$$\mathbf{x}' = \mathbf{A}\mathbf{x} = \begin{bmatrix} -1 & 1 \\ 2 & 0 \end{bmatrix}\mathbf{x}, \text{ where } \mathbf{x} = \begin{bmatrix} x \\ y \end{bmatrix}.$$

If we take the Laplace transform of the both sides of the above equation, we get

$$\begin{aligned} (s\mathbf{I}-\mathbf{A})\,\mathbf{X}(s) &= \mathbf{x}(0) \Longrightarrow \begin{bmatrix} s+1 & -1 \\ -2 & s \end{bmatrix}\mathbf{X}(s) = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \Longrightarrow \\ \mathbf{X}(s) &= \frac{1}{(s^2+s-2)}\begin{bmatrix} s & 1 \\ 2 & s+1 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \frac{1}{(s^2+s-2)}\begin{bmatrix} 1 \\ s+1 \end{bmatrix}. \end{aligned}$$

Note that

$$\begin{aligned} \frac{1}{(s^2+s-2)} &= \frac{1}{3}\left(\frac{1}{s-1}-\frac{1}{s+2}\right) \Longrightarrow \mathcal{L}^{-1}\left(\frac{1}{(s^2+s-2)}\right) = \frac{1}{3}\left(e^t-e^{-2t}\right). \\ \frac{s+1}{(s^2+s-2)} &= \frac{2}{3}\frac{1}{s-1}+\frac{1}{3}\frac{1}{s+2} \Longrightarrow \mathcal{L}^{-1}\left(\frac{s+1}{(s^2+s-2)}\right) = \frac{1}{3}\left(2e^t+e^{-2t}\right). \end{aligned}$$

Then, the solution of the initial value problem is

$$\mathbf{x}(t) = \frac{1}{3}\begin{bmatrix} e^t-e^{-2t} \\ 2e^t+e^{-2t} \end{bmatrix}.$$

(c). The equations above can be written as

$$\begin{aligned} x'+x+3y &= -1, \\ y'-4x-6y &= 3. \end{aligned}$$

This implies that

$$\mathbf{x}' = \mathbf{A}\mathbf{x}+\mathbf{h} = \begin{bmatrix} -1 & -3 \\ 4 & 6 \end{bmatrix}\mathbf{x}+\begin{bmatrix} -1 \\ 3 \end{bmatrix}, \text{ where } \mathbf{x} = \begin{bmatrix} x \\ y \end{bmatrix}.$$

If we take the Laplace transfo rm of the both sides of the above equation, we get

$$\begin{aligned} (s\mathbf{I}-\mathbf{A})\,\mathbf{X}(s) &= \mathbf{x}(0)+\mathbf{H}(\mathbf{s}) \Longrightarrow \begin{bmatrix} s+1 & 3 \\ -4 & s-6 \end{bmatrix}\mathbf{X}(s) = \frac{1}{s}\begin{bmatrix} -1 \\ 3 \end{bmatrix} \Longrightarrow \\ \mathbf{X}(s) &= \frac{1}{(s^2-5s+6)}\begin{bmatrix} s-6 & -3 \\ 4 & s+1 \end{bmatrix}\frac{1}{s}\begin{bmatrix} -1 \\ 3 \end{bmatrix} = \frac{1}{s\,(s^2-5s+6)}\begin{bmatrix} -s-3 \\ 3s-1 \end{bmatrix}. \end{aligned}$$

Note that

$$\begin{aligned} \frac{-s-3}{s\,(s^2-5s+6)} &= \frac{-1}{2}\frac{1}{s}+\frac{5}{2}\frac{1}{s-2}-2\frac{1}{s-3} \Longrightarrow \mathcal{L}^{-1}\left(\frac{-s-3}{s\,(s^2-5s+6)}\right) = \frac{-1}{2}+\frac{5}{2}e^{2t}-2e^{3t}. \\ \frac{3s-1}{s\,(s^2-5s+6)} &= \frac{-1}{6}\frac{1}{s}-\frac{5}{2}\frac{1}{s-2}+\frac{8}{3}\frac{1}{s-3} \Longrightarrow \mathcal{L}^{-1}\left(\frac{3s-1}{s\,(s^2-5s+6)}\right) = \frac{-1}{6}-\frac{5}{2}e^{2t}+\frac{8}{3}e^{3t}. \end{aligned}$$

Then, the solution of the initial value problem is

$$\mathbf{x}(t) = \frac{1}{6}\begin{bmatrix} 15e^{2t}-12e^{3t}-3 \\ -15e^{2t}+16e^{3t}-1 \end{bmatrix}.$$

(d). The equations above can be written as

$$\begin{aligned} x' - y + t^2 - t &= 0, \\ y' + y + t - 2t^2 &= 0. \end{aligned}$$

This implies that

$$\mathbf{x}' = \mathbf{A}\mathbf{x} + \mathbf{h} = \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix} \mathbf{x} + \begin{bmatrix} t - t^2 \\ 2t^2 - t \end{bmatrix}, \text{ where } \mathbf{x} = \begin{bmatrix} x \\ y \end{bmatrix}.$$

If we take the Laplace transform of the both sides of the above equation, we get

$$\begin{aligned} (s\mathbf{I} - \mathbf{A})\,\mathbf{X}(s) &= \mathbf{x}(0) + \mathbf{H}(\mathbf{s}) \Longrightarrow \begin{bmatrix} s & -1 \\ 0 & s+1 \end{bmatrix} \mathbf{X}(s) = \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \begin{bmatrix} \frac{1}{s^2} - \frac{2}{s^3} \\ \frac{4}{s^3} - \frac{1}{s^2} \end{bmatrix} \Longrightarrow \\ \mathbf{X}(s) &= \frac{1}{s(s+1)} \begin{bmatrix} s+1 & 1 \\ 0 & s \end{bmatrix} \frac{1}{s^3} \begin{bmatrix} s^3 + s - 2 \\ 4 - s \end{bmatrix} \\ &= \frac{1}{s^4(s+1)} \begin{bmatrix} s^4 + s^3 + s^2 - 2s + 2 \\ 4s - s^2 \end{bmatrix}. \end{aligned}$$

Note that

$$\begin{aligned} \frac{s^4 + s^3 + s^2 - 2s + 2}{s^4(s+1)} &= \frac{5}{s+1} - 4\frac{1}{s} + 5\frac{1}{s^2} - 4\frac{1}{s^3} + 2\frac{1}{s^4} \Longrightarrow \\ \mathcal{L}^{-1}\left(\frac{s^4 + s^3 + s^2 - 2s + 2}{s^4(s+1)}\right) &= 5e^{-t} - 4 + 5t - 2t^2 + \frac{1}{3}t^3. \\ \frac{4s - s^2}{s^4(s+1)} &= -5\frac{1}{s+1} + 5\frac{1}{s} - 5\frac{1}{s^2} + 4\frac{1}{s^3} \Longrightarrow \\ \mathcal{L}^{-1}\left(\frac{4s - s^2}{s^4(s+1)}\right) &= -5e^{-t} + 5 - 5t + 2t^2. \end{aligned}$$

Then, the solution of the initial value problem is

$$\mathbf{x}(t) = \begin{bmatrix} 5e^{-t} - 4 + 5t - 2t^2 + \frac{1}{3}t^3 \\ -5e^{-t} + 5 - 5t + 2t^2 \end{bmatrix}.$$