

Lecture 4

- 3.1 Tangents and the Derivative at a Point
- 3.2 The Derivative as a Function
- 3.3 Differentiation Rules
- 3.4 The Derivative as a Rate of Change

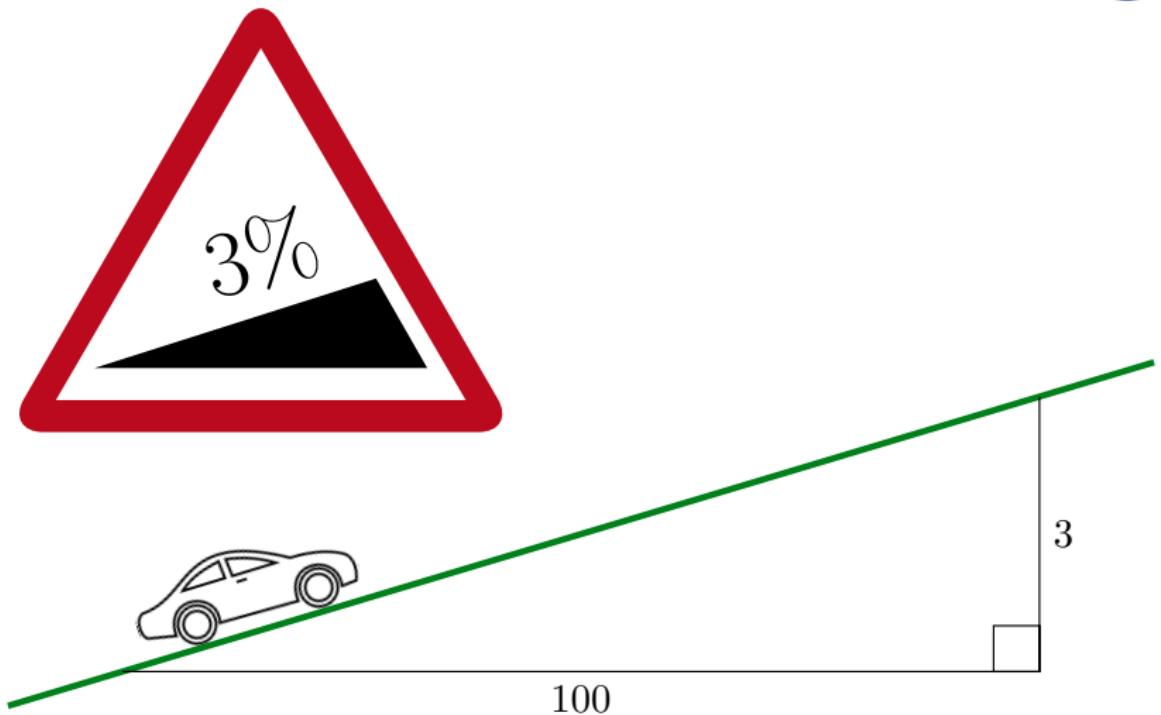


Tangents and the Derivative at a Point

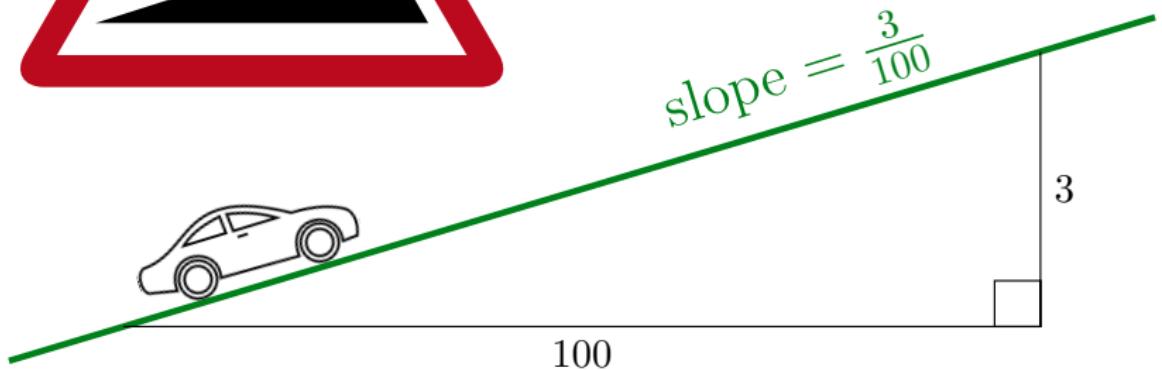
3.2 Tangents and the Derivative at a Point



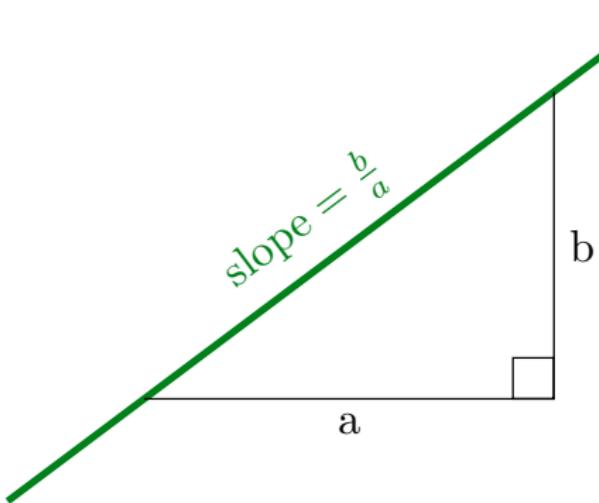
3.2 Tangents and the Derivative at a Point



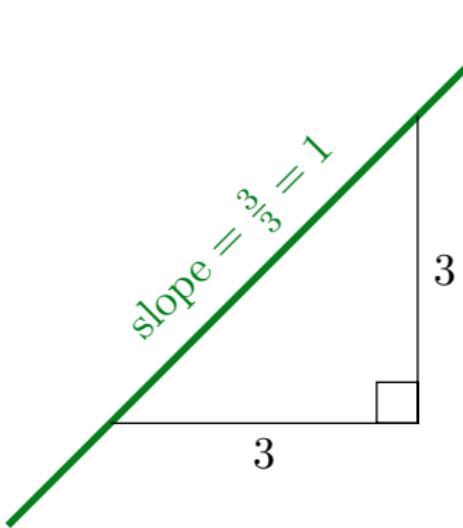
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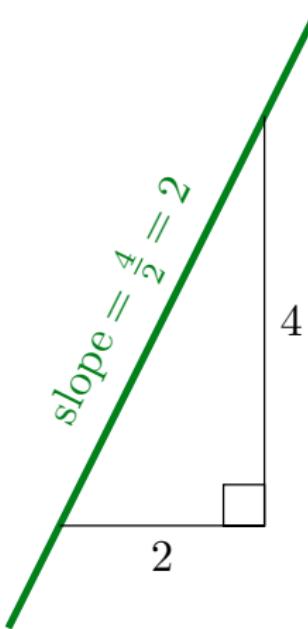
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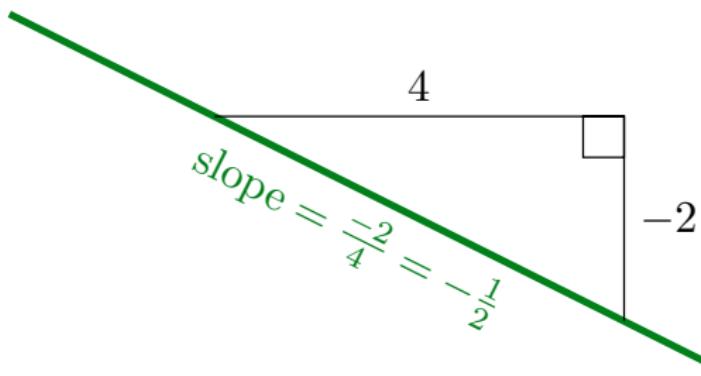
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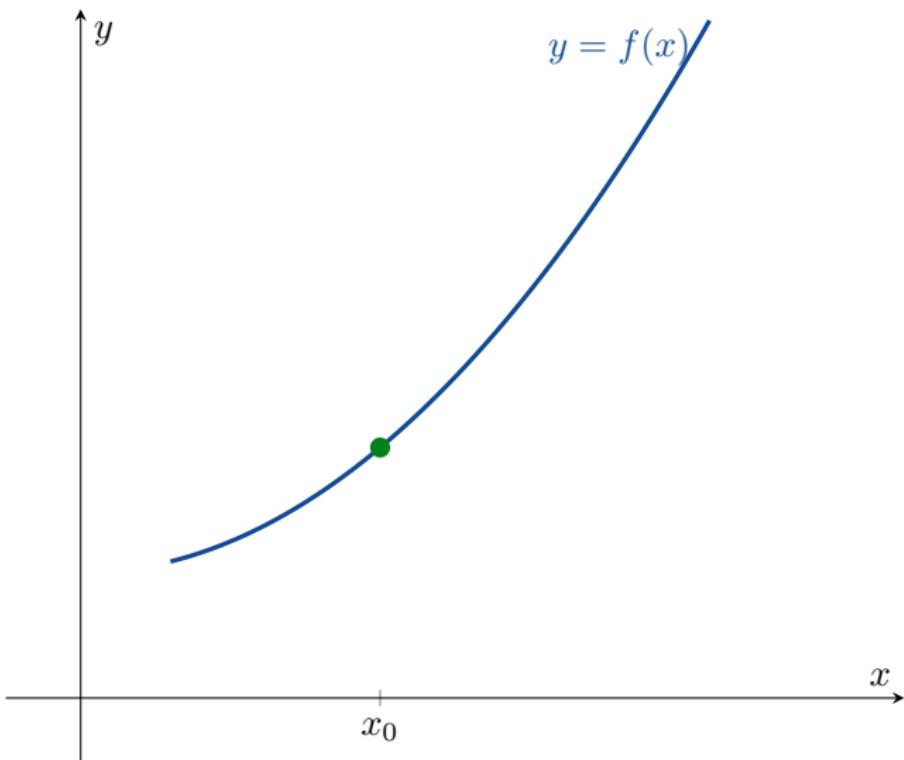
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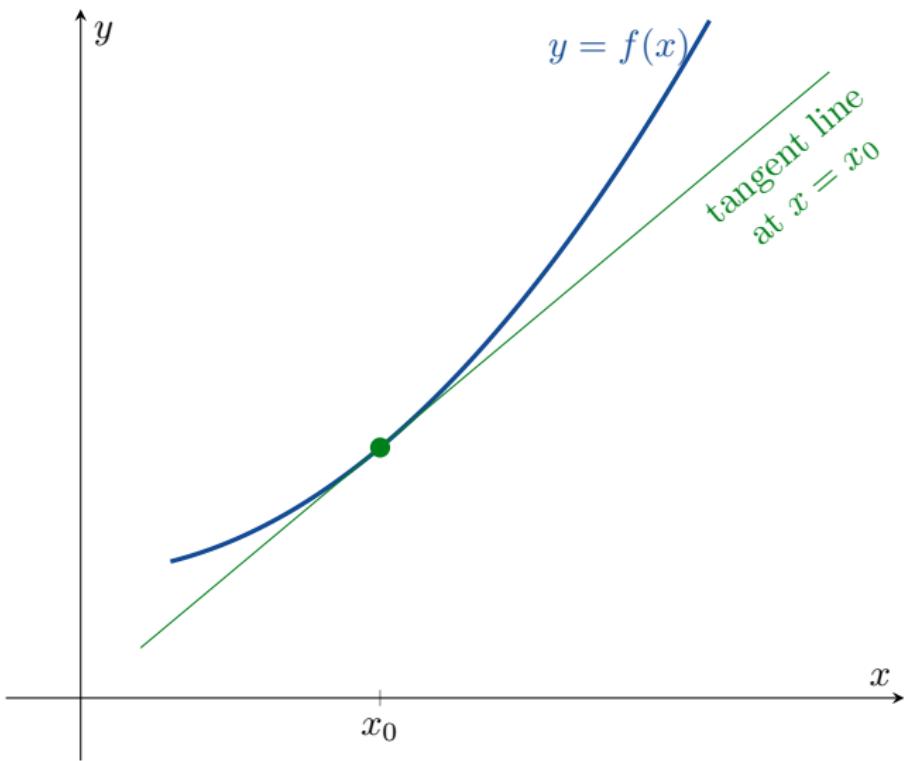
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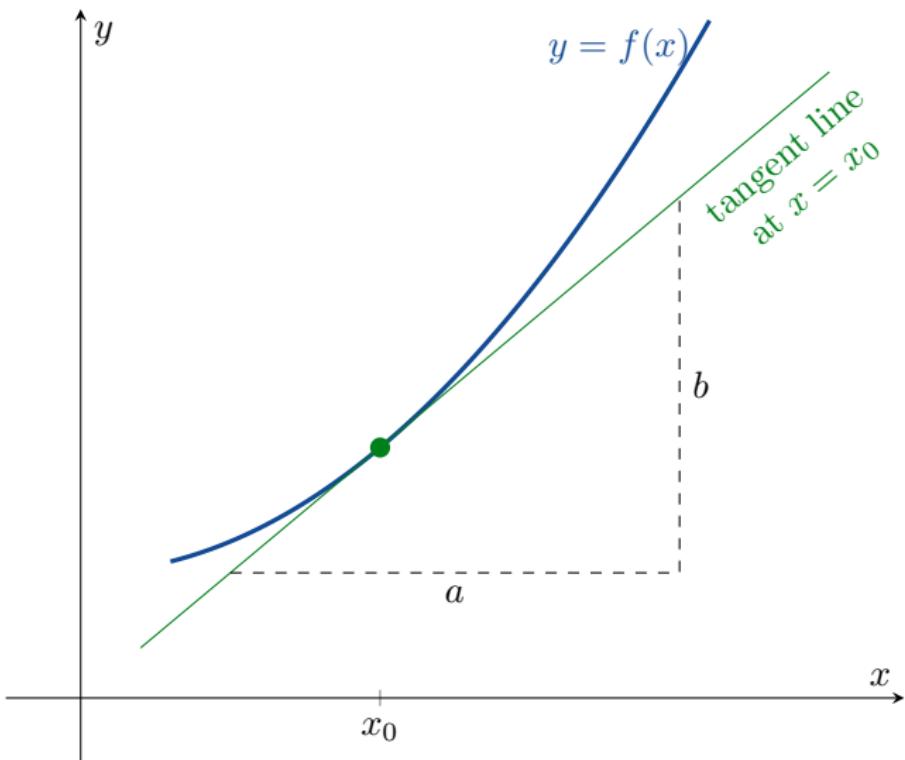
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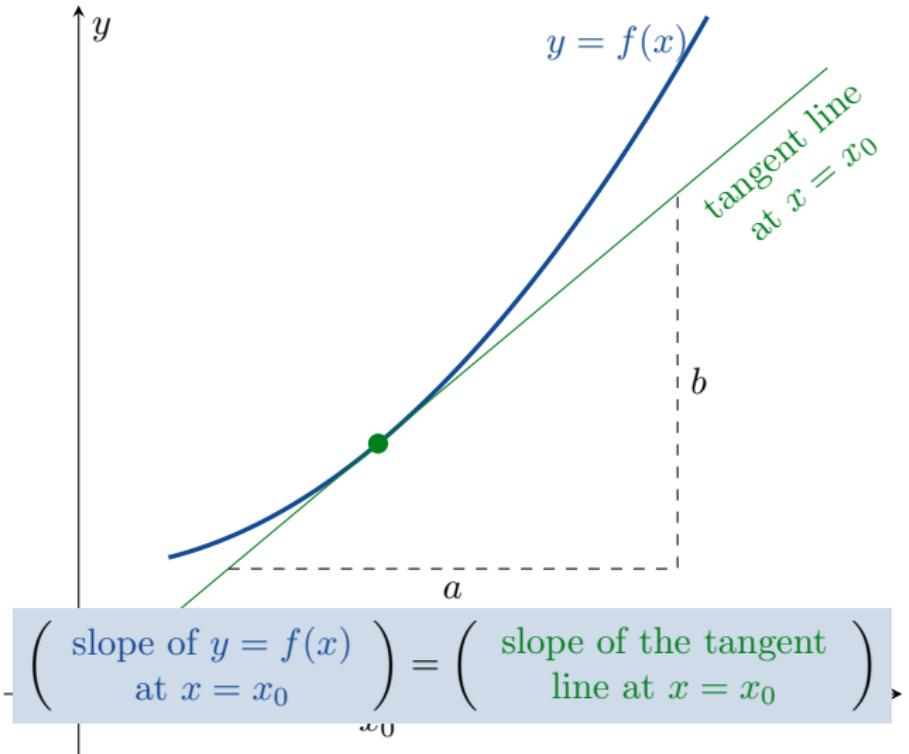
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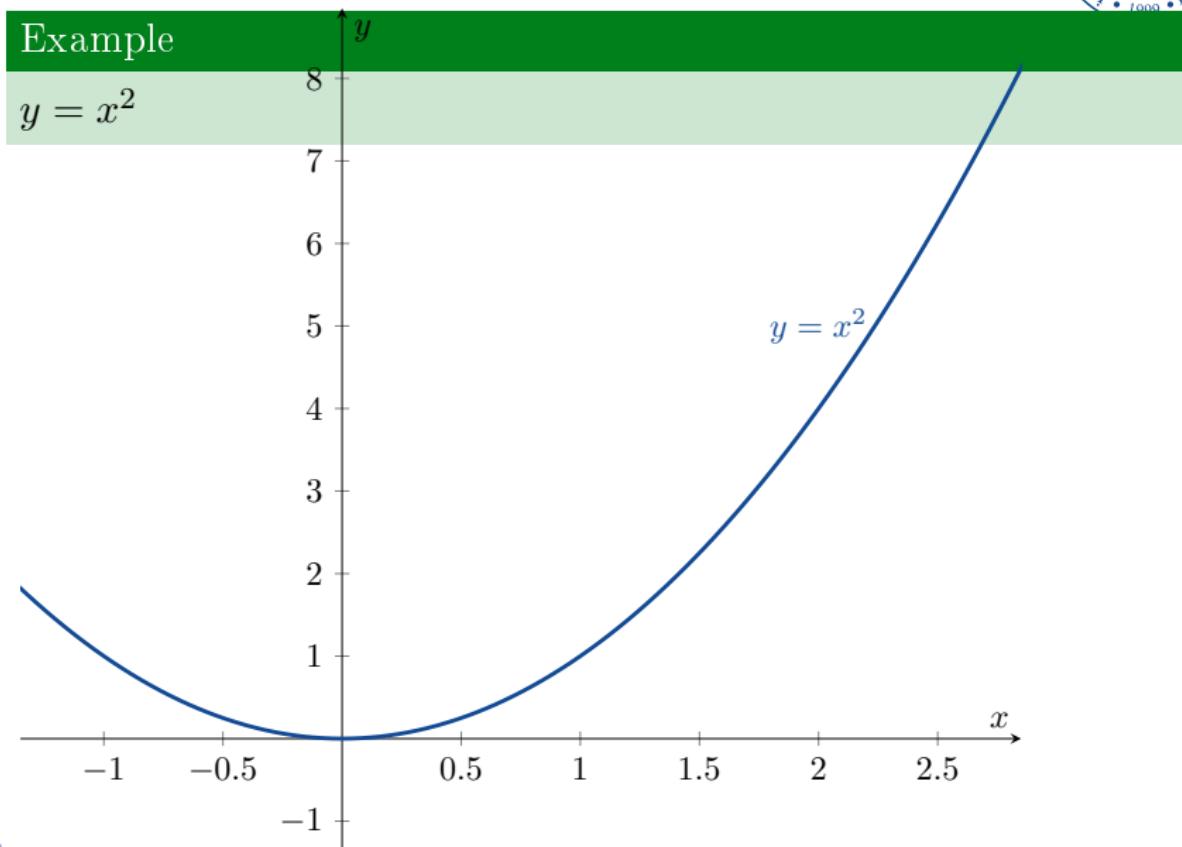
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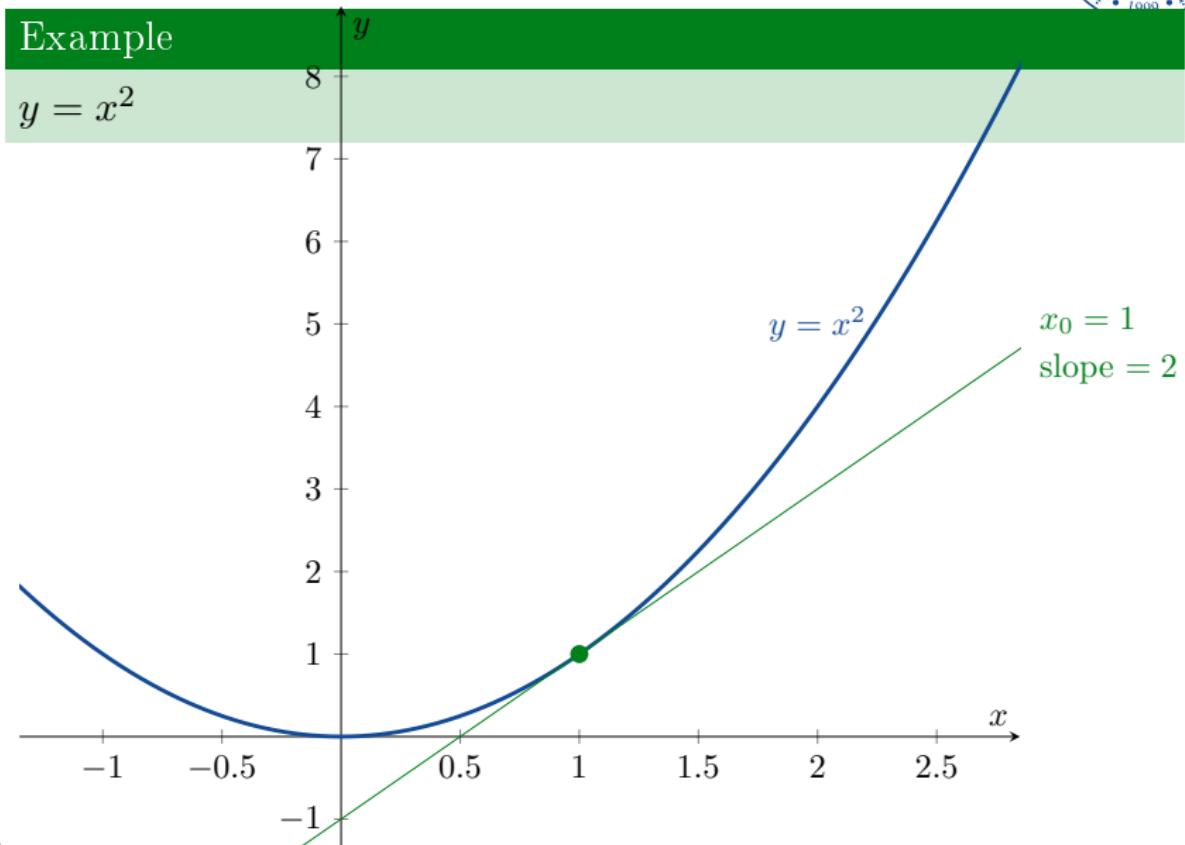
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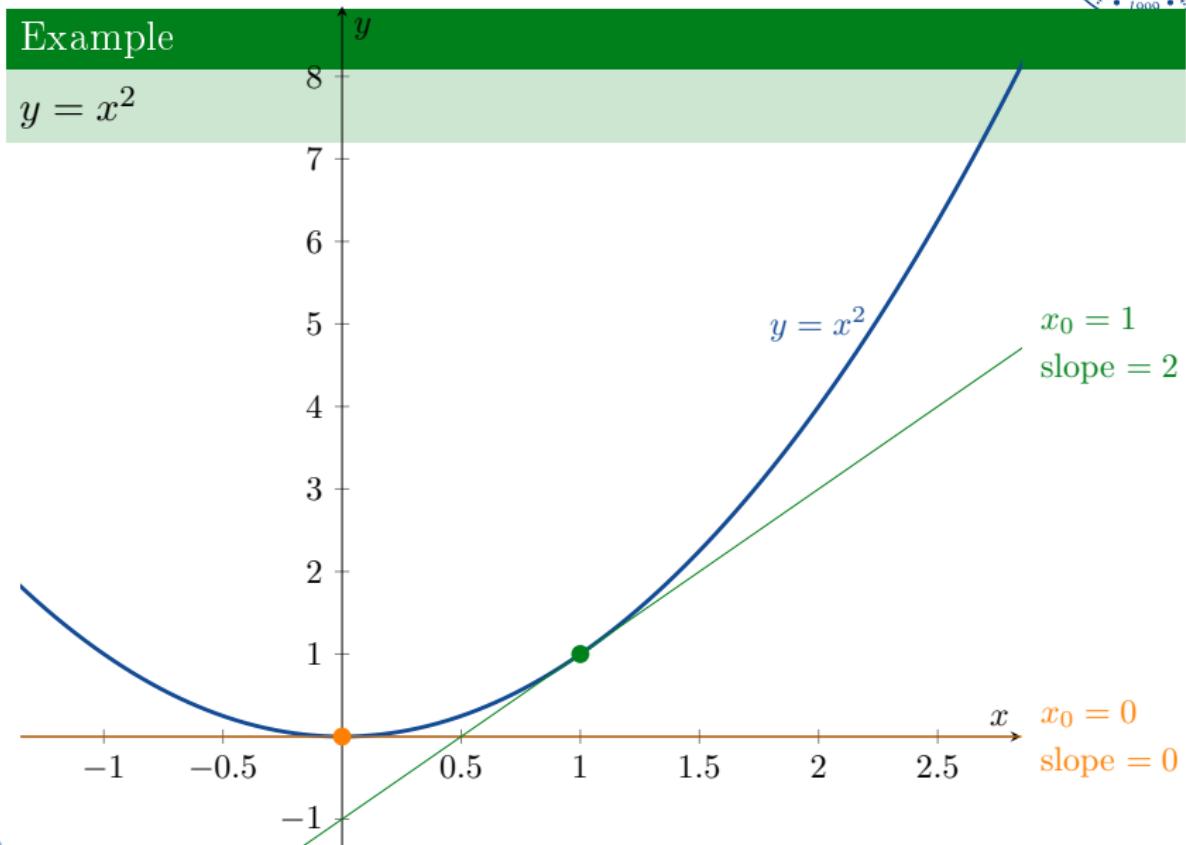
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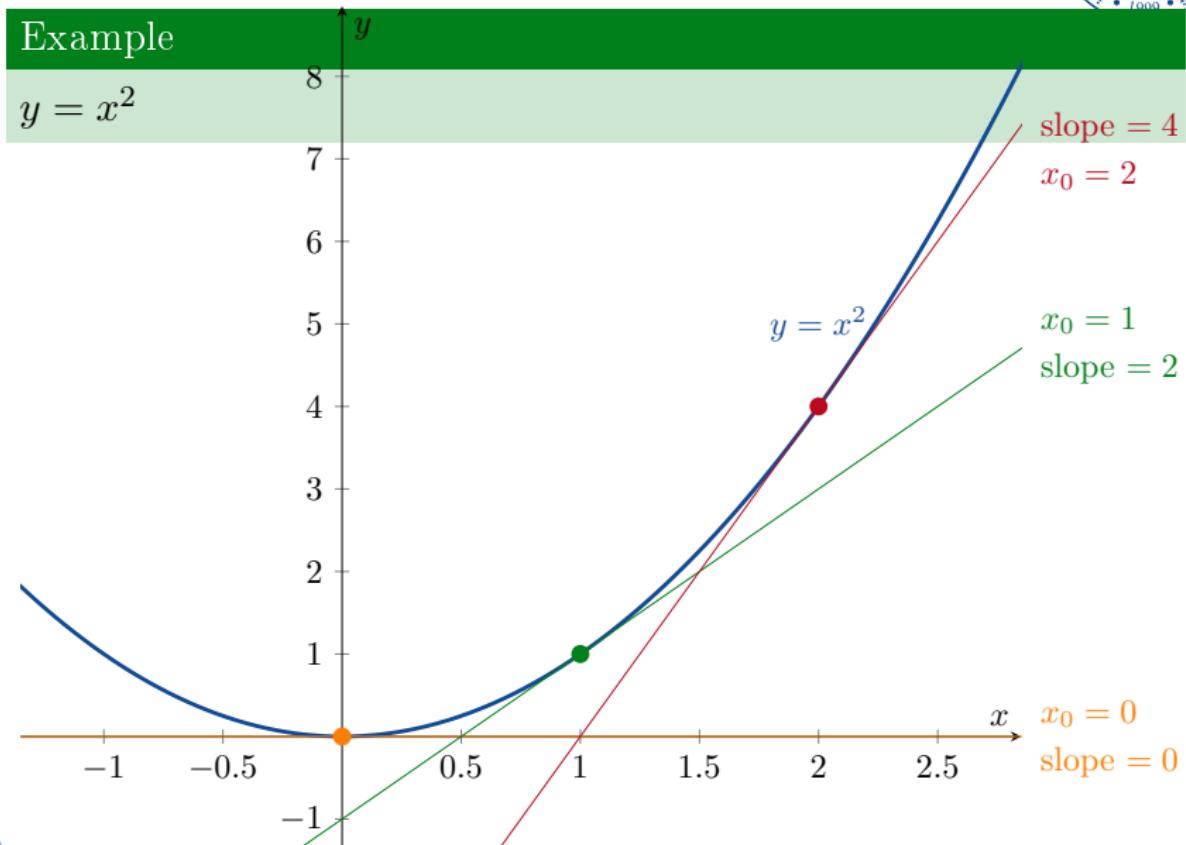
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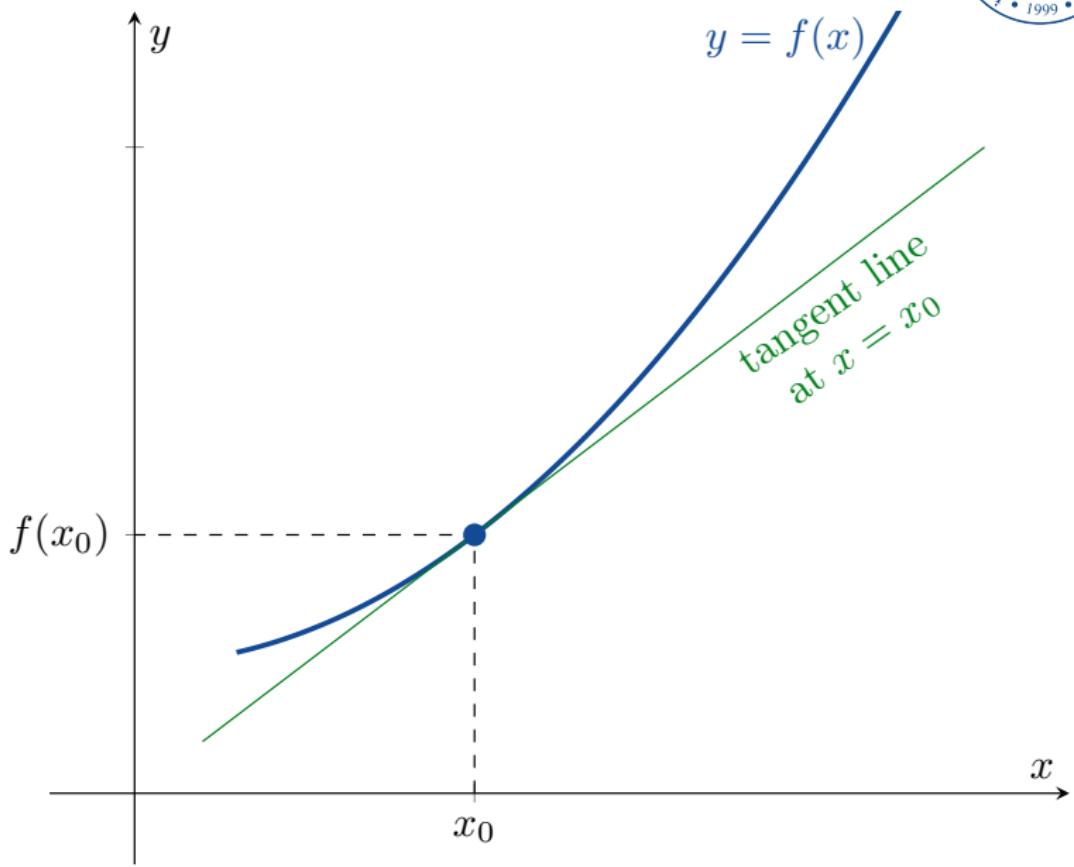


3.2 Tangents and the Derivative at a Point

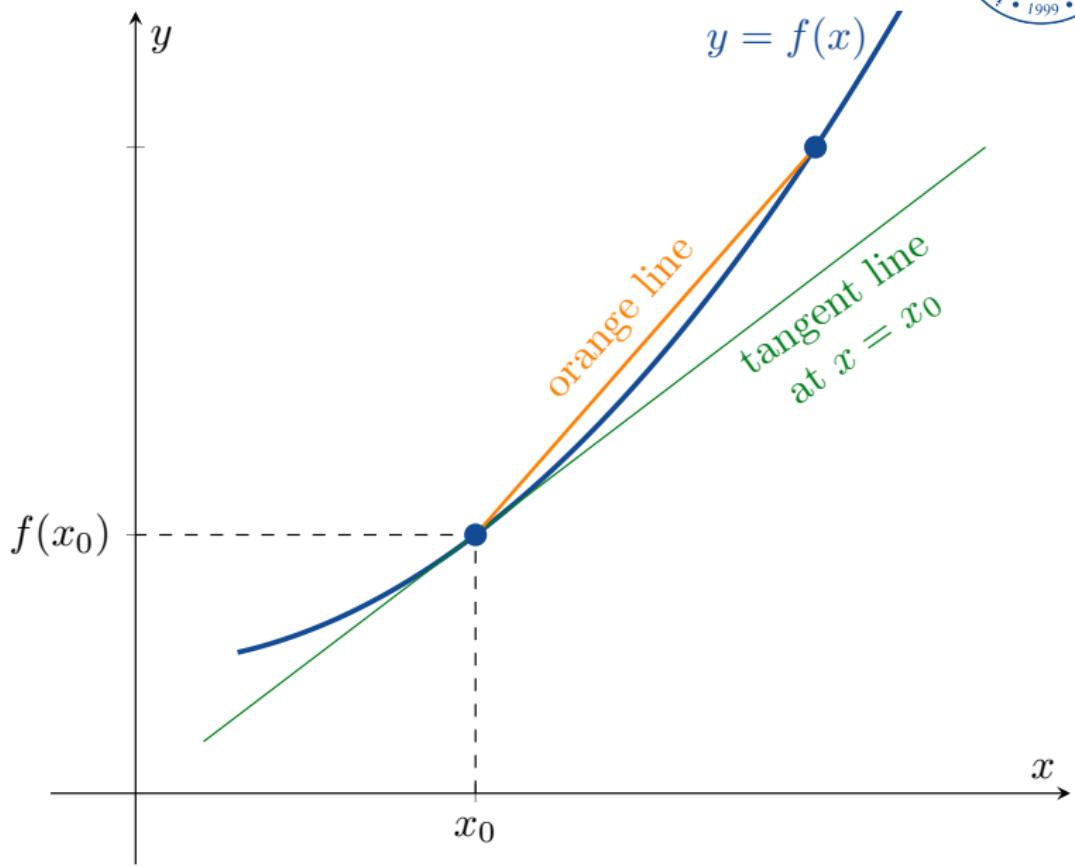


How can we calculate the slope of the tangent line?

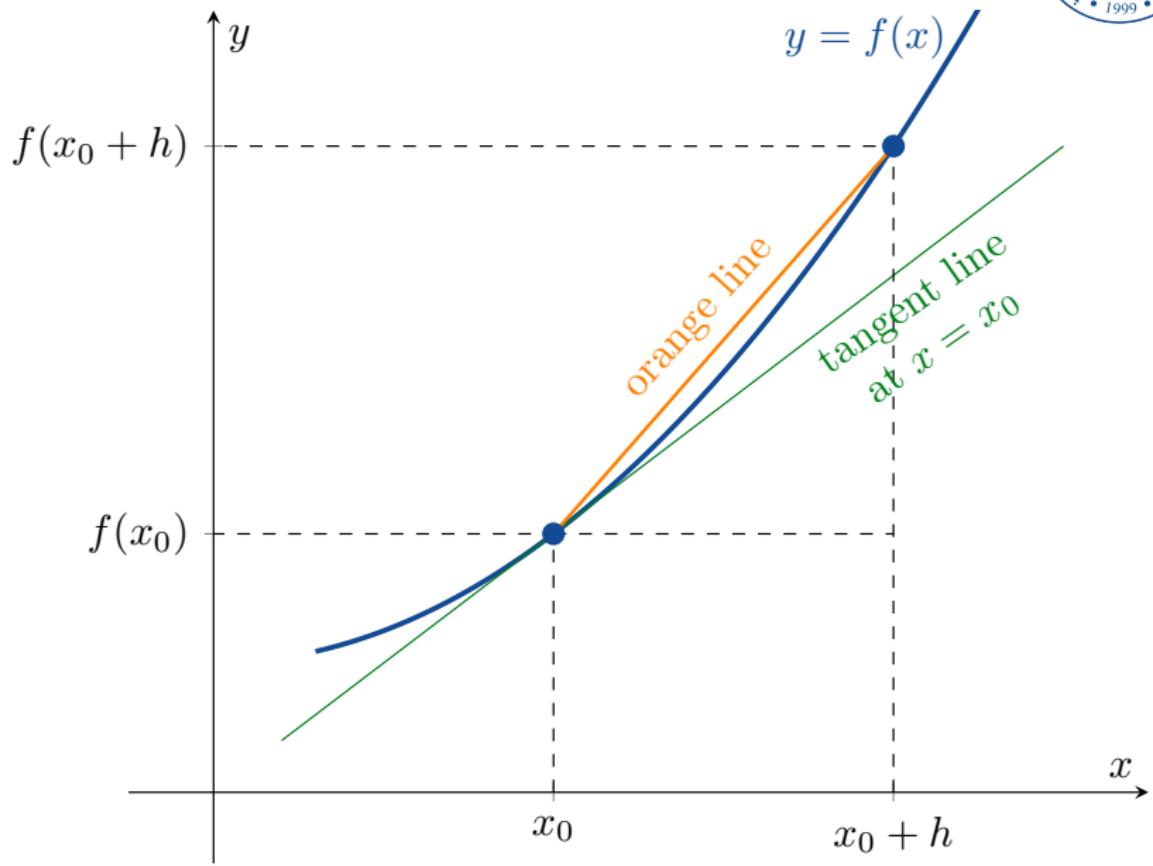
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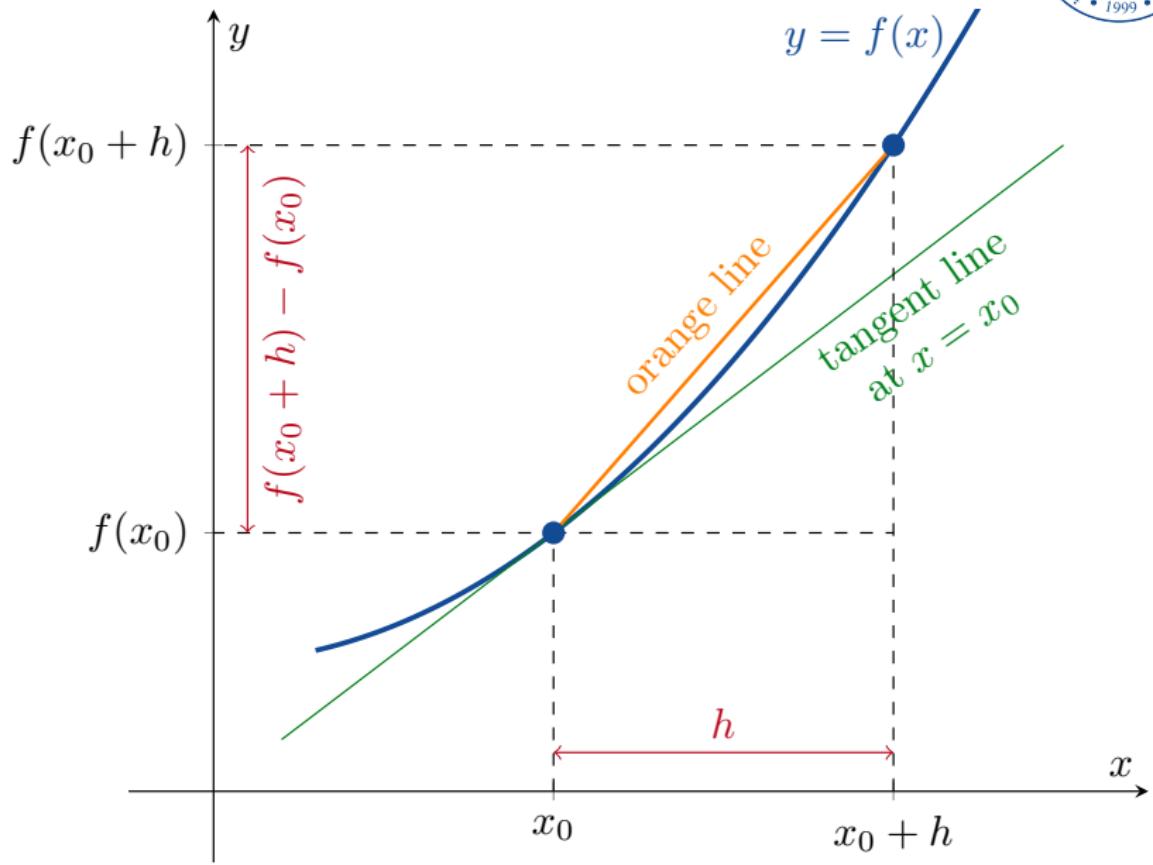
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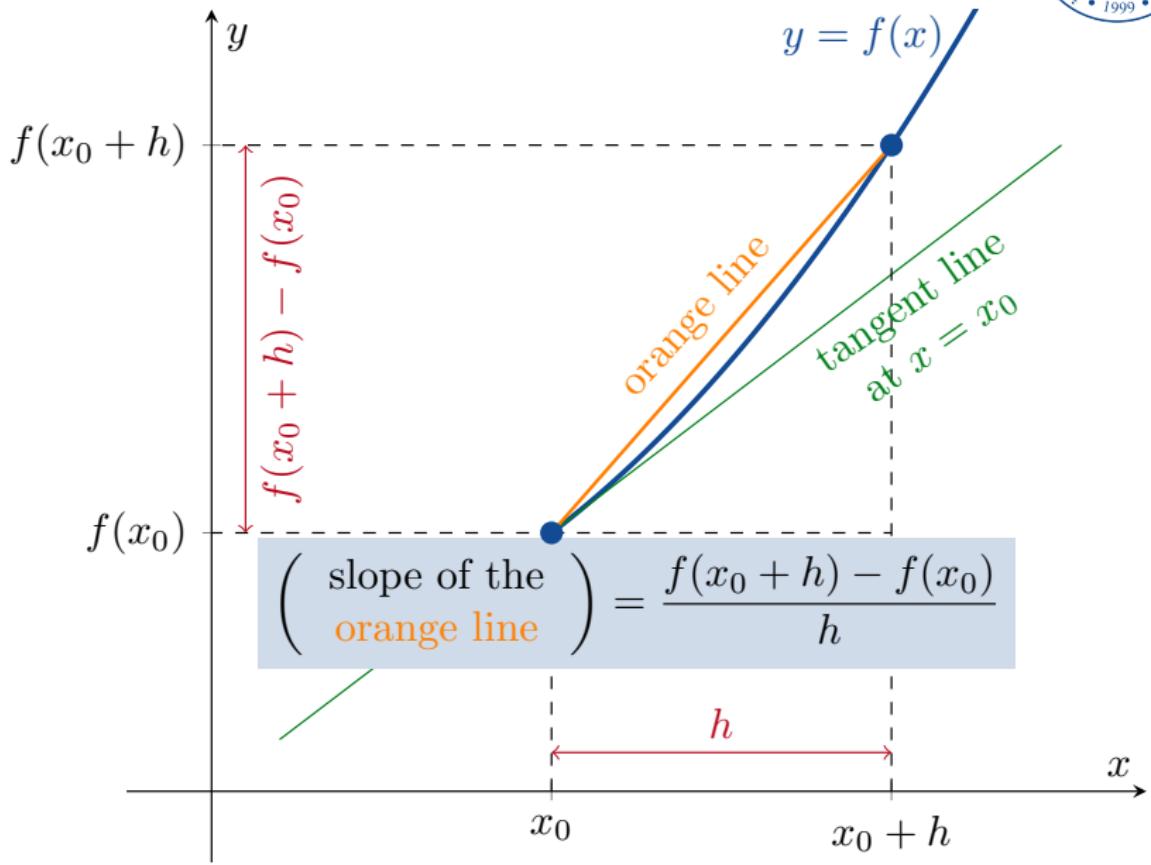
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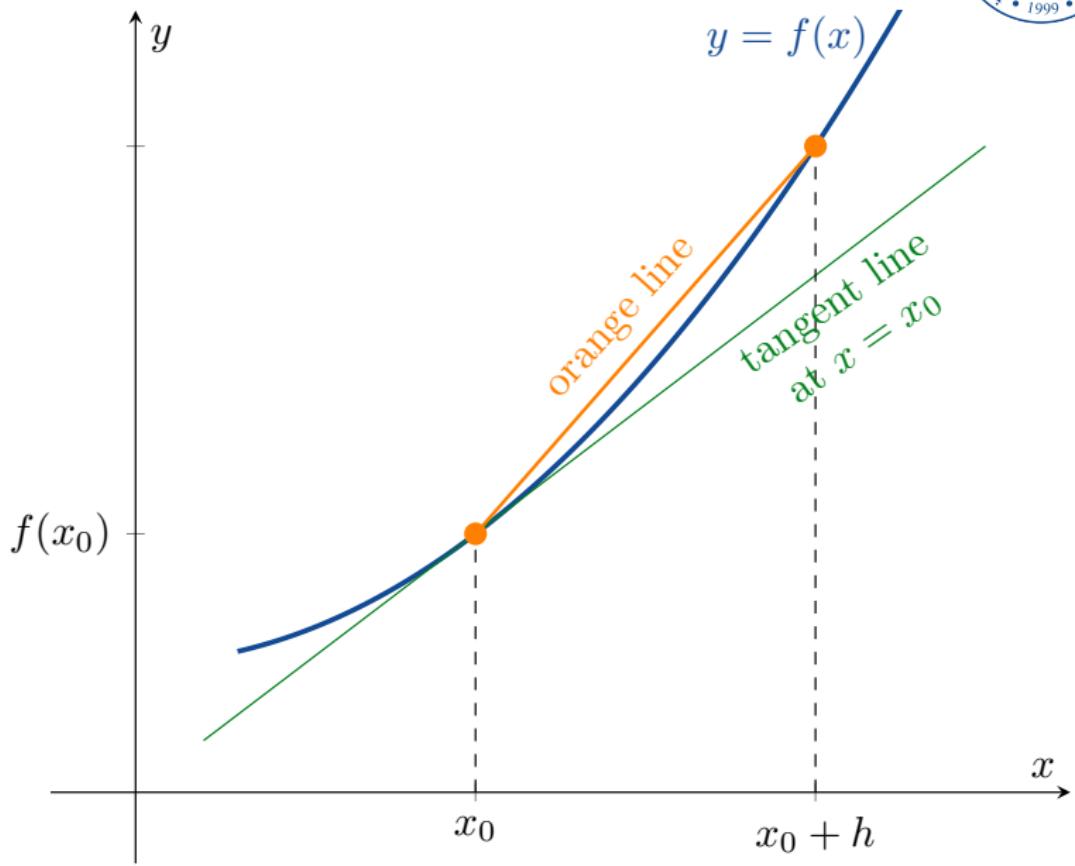
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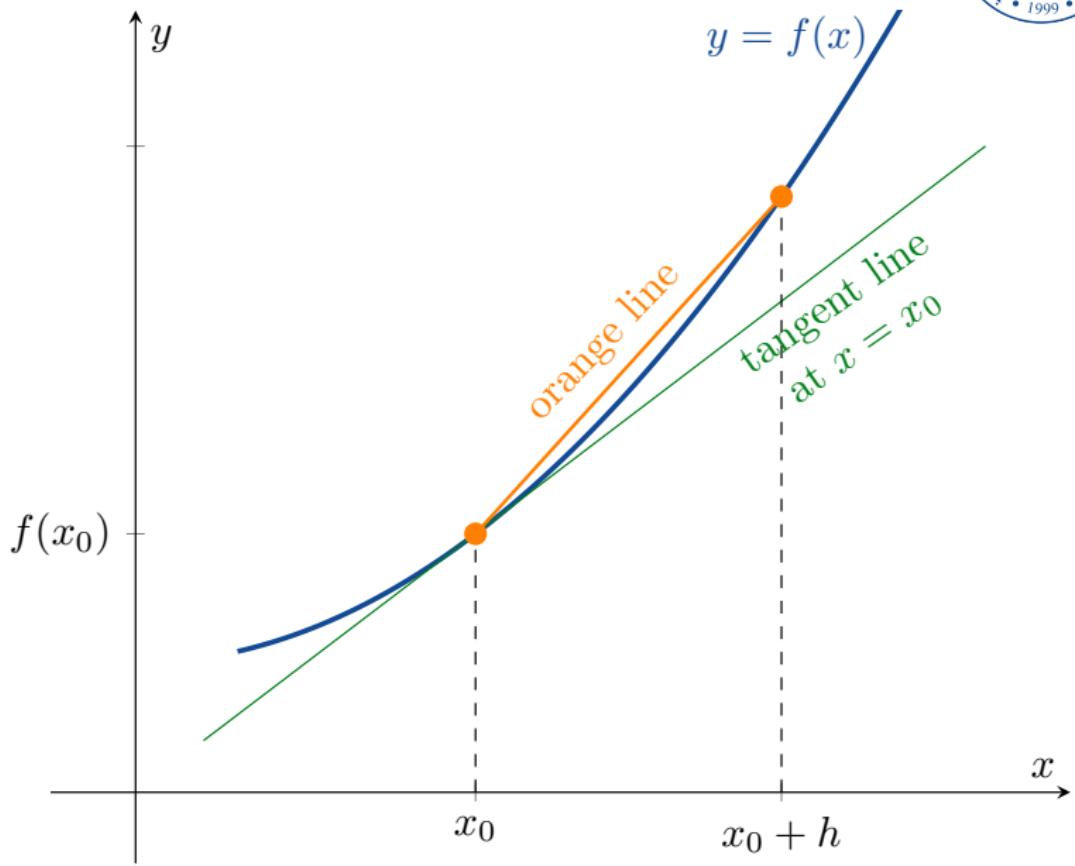
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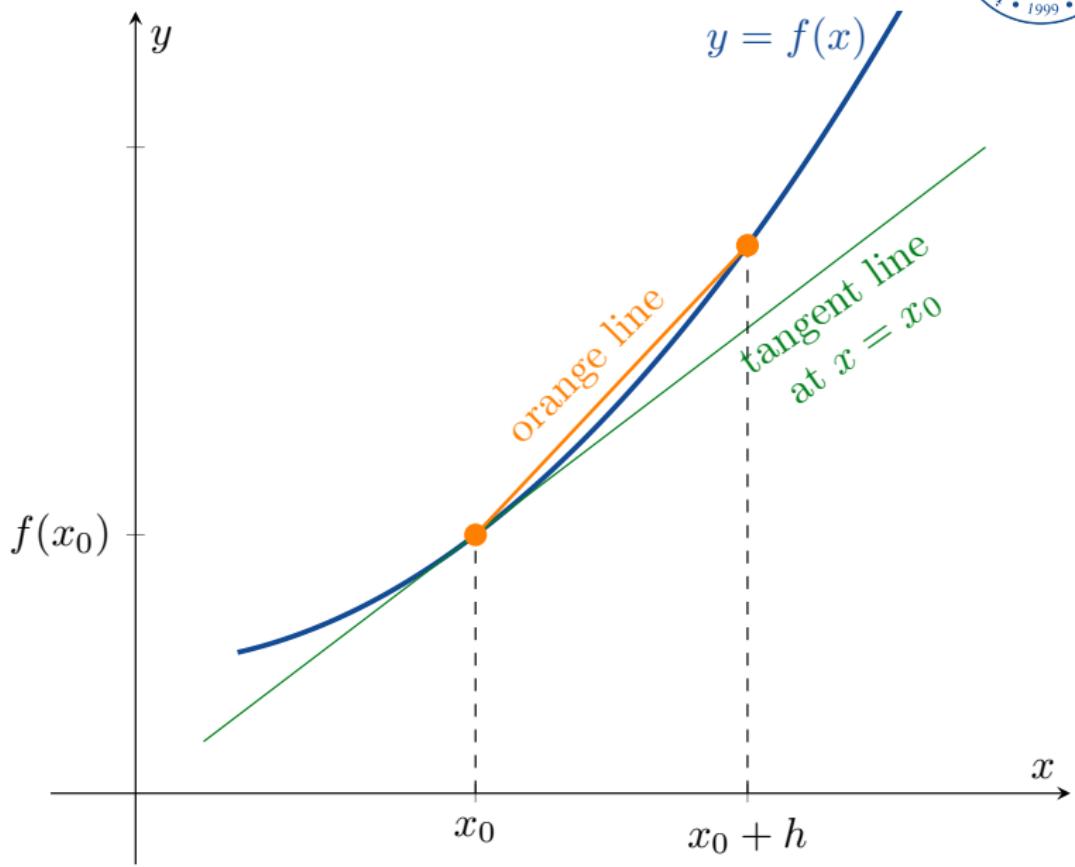
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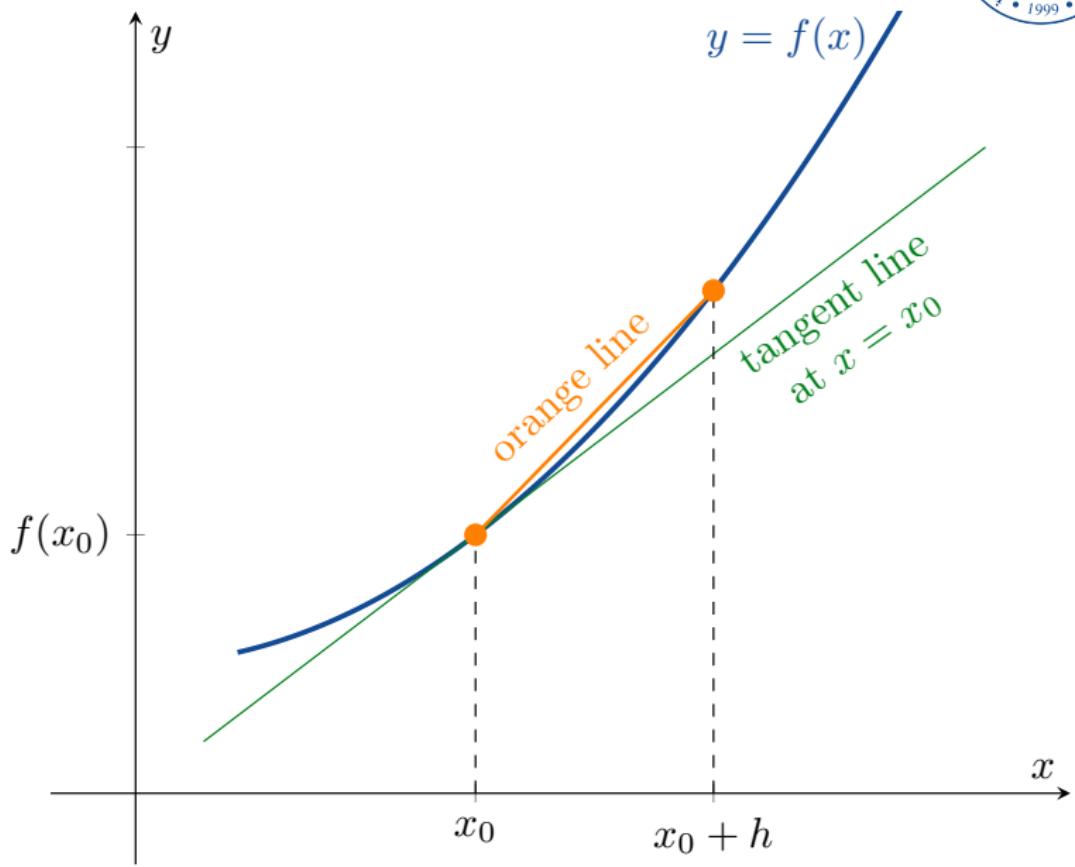
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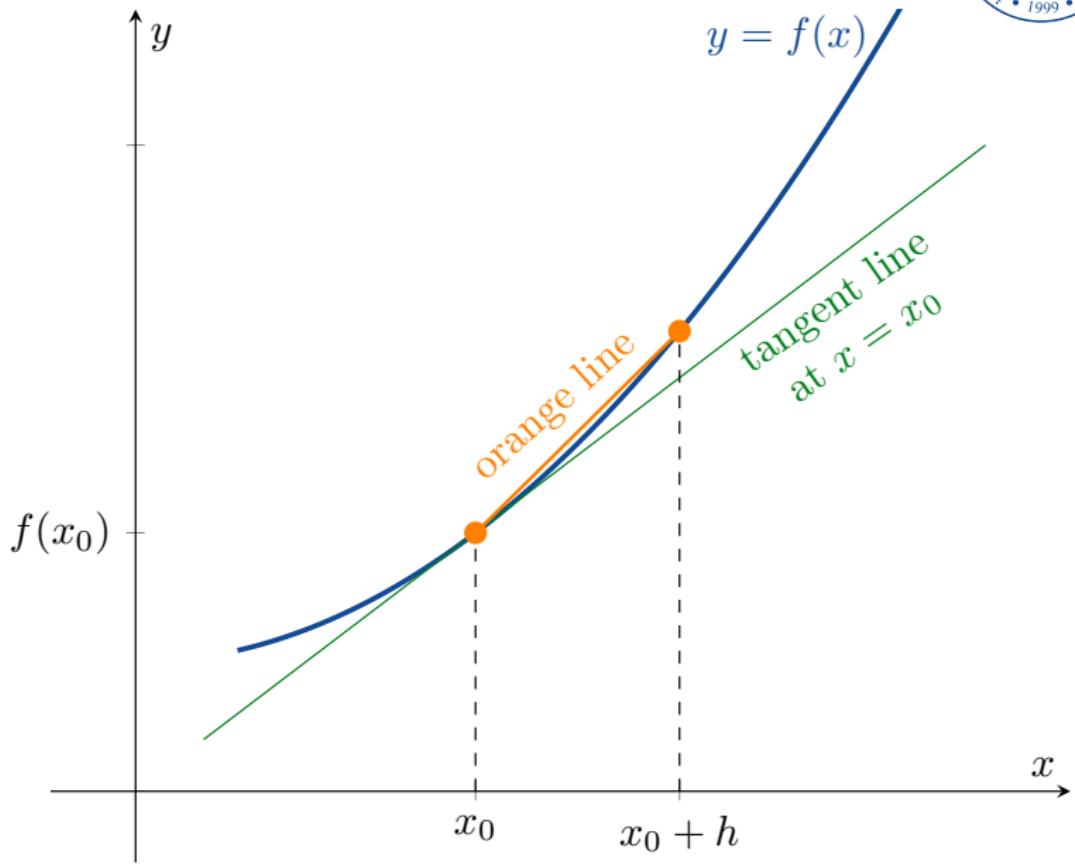
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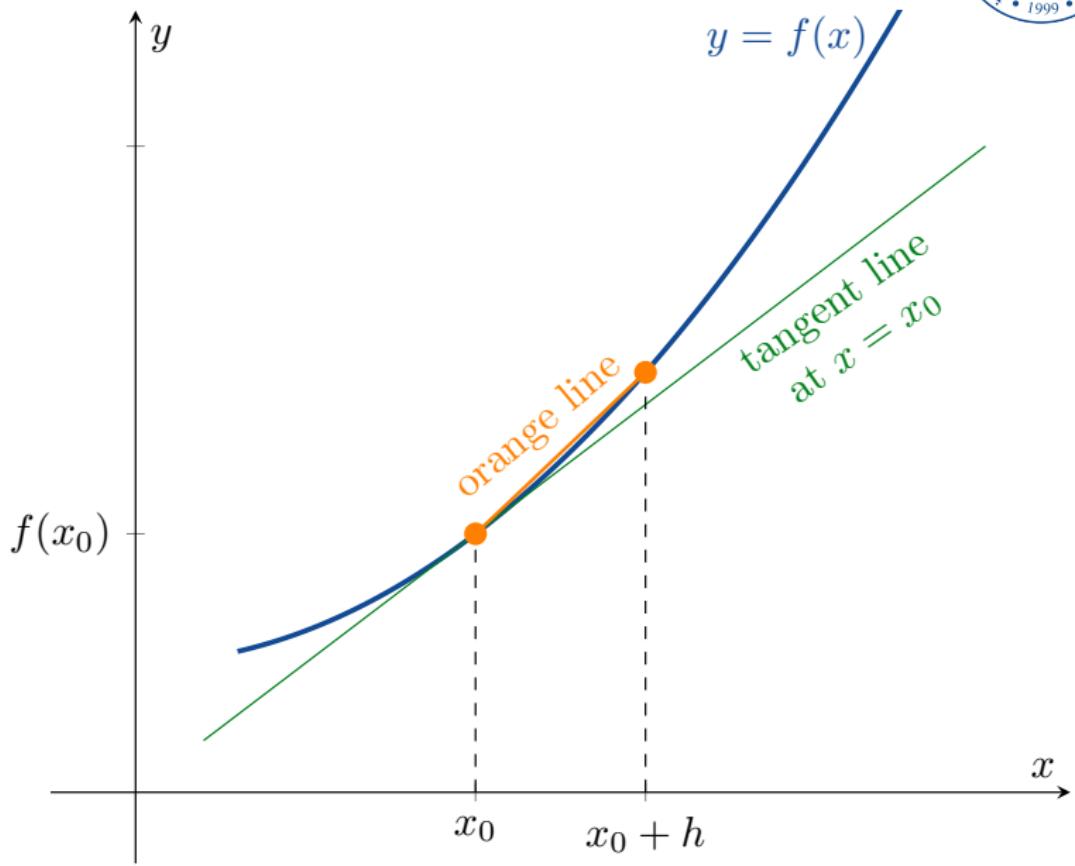
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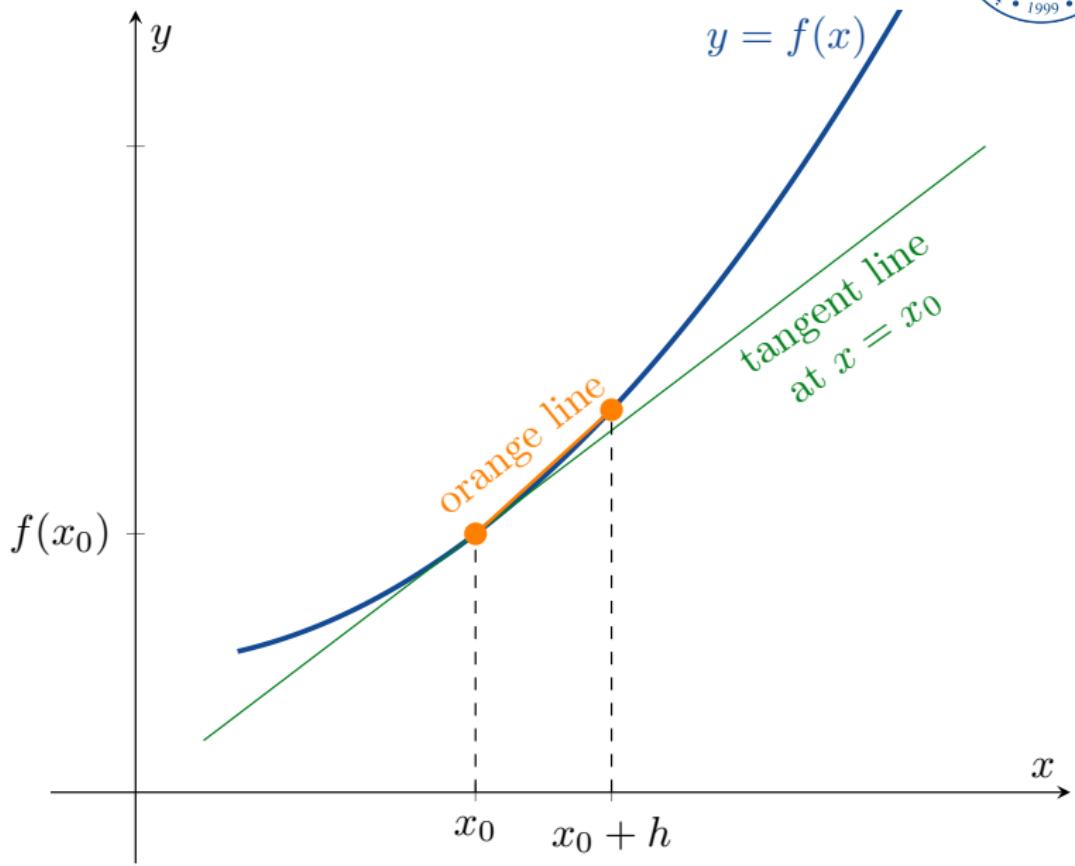
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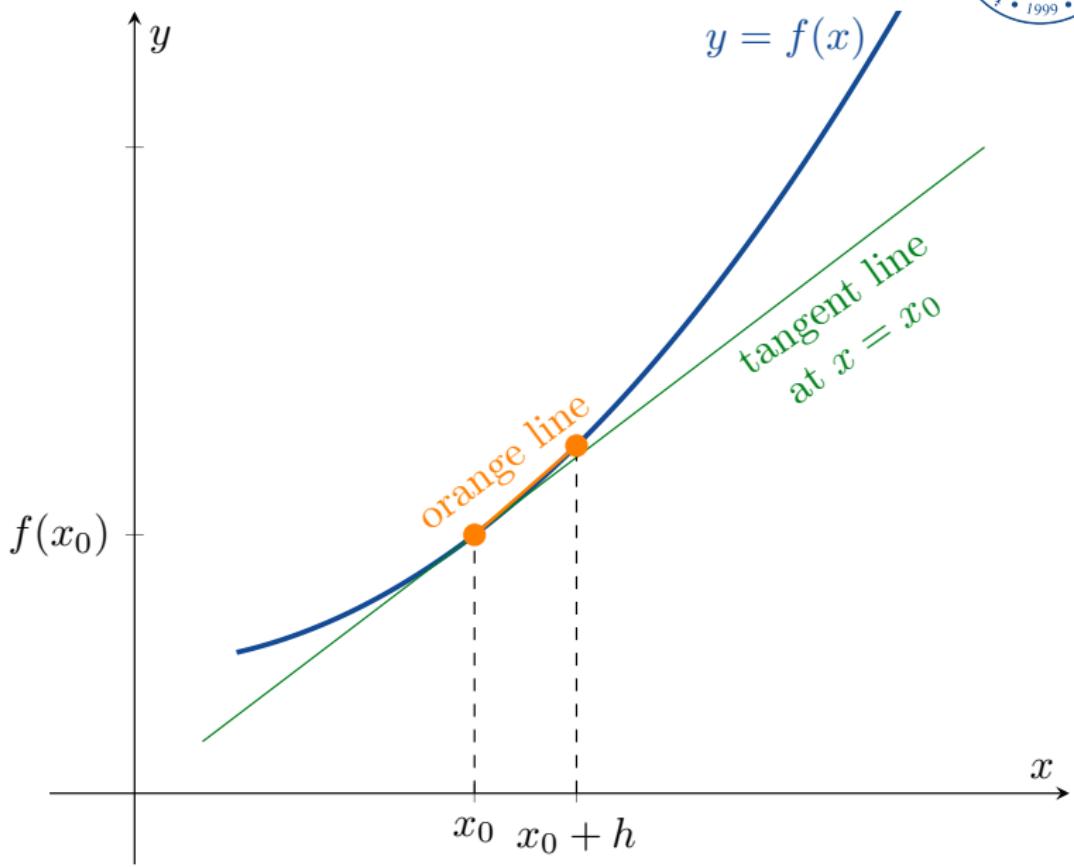
3.2 Tangents and the Derivative at a Point



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3.2 Tangents and the Derivative at a Point



If h is very very small, then

$$\left(\begin{array}{l} \text{slope of the} \\ \text{tangent line} \end{array} \right) \approx \left(\begin{array}{l} \text{slope of the} \\ \text{orange line} \end{array} \right) = \frac{f(x_0 + h) - f(x_0)}{h}$$

3.2 Tangents and the Derivative at a Point



The Derivative of f

Definition

The *derivative of a function f at a point x_0* is

$$f'(x_0) = \lim_{h \rightarrow 0} \frac{f(x_0 + h) - f(x_0)}{h}$$

if the limit exists.

(f' is pronounced “ f prime”)

3.2 Tangents and the Derivative at a Point



Example

Find the derivative of the function $g(x) = \frac{1}{x}$, $x \neq 0$.

3.2 Tangents and the Derivative at a Point



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3.2 Tangents and the Derivative at a Point



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If $x_0 \neq 0$, then

$$\begin{aligned} g'(x_0) &= \lim_{h \rightarrow 0} \frac{g(x_0 + h) - g(x_0)}{h} \\ &= \lim_{h \rightarrow 0} \frac{\frac{1}{x_0+h} - \frac{1}{x_0}}{h} \end{aligned}$$

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3.2 Tangents and the Derivative at a Point



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3.2 Tangents and the Derivative at a Point



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3.2 Tangents and the Derivative at a Point



Example

Find the derivative of the function $g(x) = \frac{1}{x}$, $x \neq 0$.

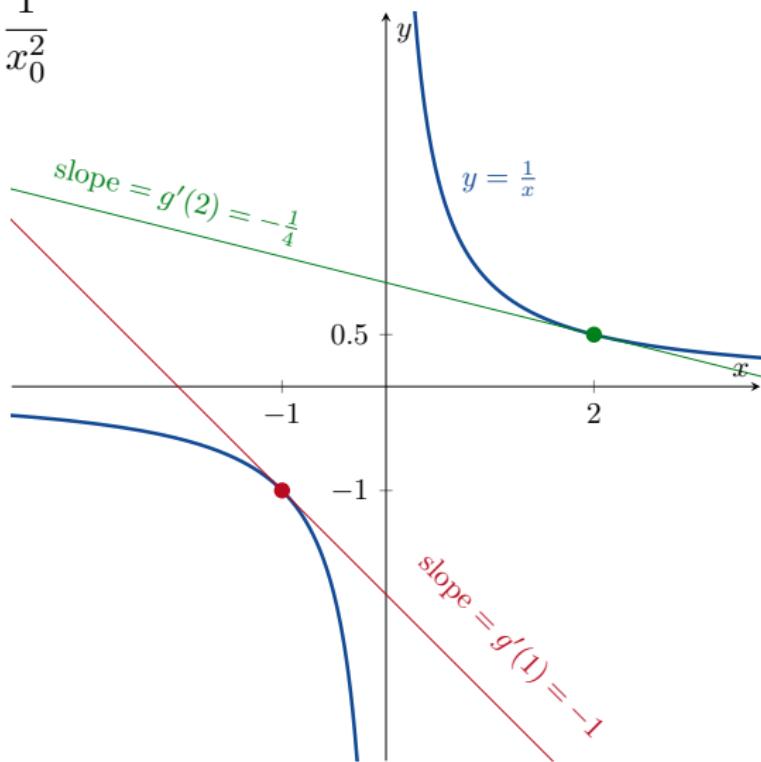
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3.2 Tangents and the Derivative at a Point



$$g'(x_0) = -\frac{1}{x_0^2}$$





The Derivative as a Function

3.3 The Derivative as a Function



Definition

If $f'(x_0)$ exists, we say that f is *differentiable at x_0* .

3.3 The Derivative as a Function



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Let $f : D \rightarrow \mathbb{R}$ be a function. If f is differentiable at every $x_0 \in D$, we say that f is *differentiable*.

3.3 The Derivative as a Function



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Let $f : D \rightarrow \mathbb{R}$ be a function. If f is differentiable at every $x_0 \in D$, we say that f is *differentiable*.

If $f : D \rightarrow \mathbb{R}$ is differentiable, then we have a new function $f' : D \rightarrow \mathbb{R}$.

3.3 The Derivative as a Function



Definition

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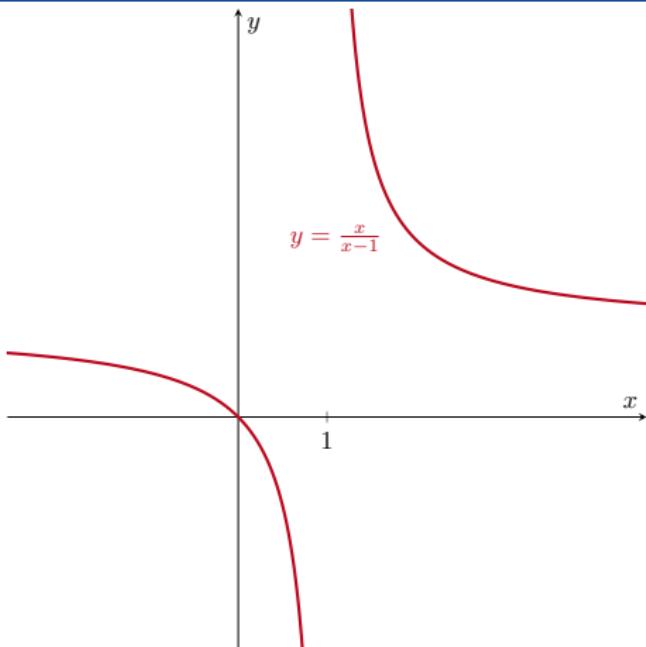
Let $f : D \rightarrow \mathbb{R}$ be a function. If f is differentiable at every $x_0 \in D$, we say that f is *differentiable*.

If $f : D \rightarrow \mathbb{R}$ is differentiable, then we have a new function $f' : D \rightarrow \mathbb{R}$.

Definition

f' is called the *derivative* of f .

3.3 The Derivative as a Function



Example

Differentiate $f(x) = \frac{x}{x-1}$.

3.3 The Derivative as a Function



First note that if $f(x) = \frac{x}{x-1}$ then $f(x+h) = \frac{x+h}{x+h-1}$.

Therefore

$$f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \rightarrow 0} \frac{\frac{x+h}{x+h-1} - \frac{x}{x-1}}{h}$$

=

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3.3 The Derivative as a Function



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3.3 The Derivative as a Function



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3.3 The Derivative as a Function

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 &= \lim_{h \rightarrow 0} \frac{1}{h} \left(\frac{-h}{(x-1)(x+h-1)} \right) = \lim_{h \rightarrow 0} \frac{-1}{(x-1)(x+h-1)} \\
 &=
 \end{aligned}$$

3.3 The Derivative as a Function

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 &= \lim_{h \rightarrow 0} \frac{1}{h} \left(\frac{-h}{(x-1)(x+h-1)} \right) = \lim_{h \rightarrow 0} \frac{-1}{(x-1)(x+h-1)} \\
 &= \frac{-1}{(x-1)(x+0-1)} = \frac{-1}{(x-1)^2}.
 \end{aligned}$$

3.3 The Derivative as a Function



The derivative of $f(x) = \frac{x}{x-1}$ is $f'(x) = \frac{-1}{(x-1)^2}$.

3.3 The Derivative as a Function



Sometimes it is easier to use an alternative formula for the derivative. If we take

$$f'(x) = \lim_{h \rightarrow 0} \frac{f(x + h) - f(x)}{h}$$

and substitute in $z = x + h$, then we get:

3.3 The Derivative as a Function



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Definition

$$f'(x) = \lim_{z \rightarrow x} \frac{f(z) - f(x)}{z - x}.$$

3.3

$$f'(x) = \lim_{z \rightarrow x} \frac{f(z) - f(x)}{z - x}$$



Example

Find the derivative of $f(x) = \sqrt{x}$ for $x > 0$.

3.3

$$f'(x) = \lim_{z \rightarrow x} \frac{f(z) - f(x)}{z - x}$$



Example

Find the derivative of $f(x) = \sqrt{x}$ for $x > 0$.

$$f'(x) = \lim_{z \rightarrow x} \frac{f(z) - f(x)}{z - x} = \lim_{z \rightarrow x} \frac{\sqrt{z} - \sqrt{x}}{z - x}$$

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3.3

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Example

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$$f'(x) = \lim_{z \rightarrow x} \frac{f(z) - f(x)}{z - x} = \lim_{z \rightarrow x} \frac{\sqrt{z} - \sqrt{x}}{z - x}$$

$$= \lim_{z \rightarrow x} \frac{\sqrt{z} - \sqrt{x}}{(\sqrt{z} - \sqrt{x})(\sqrt{z} + \sqrt{x})}$$

=

=

3.3

$$f'(x) = \lim_{z \rightarrow x} \frac{f(z) - f(x)}{z - x}$$



Example

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$$\begin{aligned} f'(x) &= \lim_{z \rightarrow x} \frac{f(z) - f(x)}{z - x} = \lim_{z \rightarrow x} \frac{\sqrt{z} - \sqrt{x}}{z - x} \\ &= \lim_{z \rightarrow x} \frac{\sqrt{z} - \sqrt{x}}{(\sqrt{z} - \sqrt{x})(\sqrt{z} + \sqrt{x})} \\ &= \lim_{z \rightarrow x} \frac{1}{\sqrt{z} + \sqrt{x}} \\ &= \end{aligned}$$

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3.3 The Derivative as a Function



Remark

The line passing through the point (x_0, y_0) with slope k has the equation

$$y = y_0 + k(x - x_0).$$

3.3 The Derivative as a Function



Example

Find the tangent line to the curve $y = \sqrt{x}$ at $x = 4$.

We know that the derivative of $f(x) = \sqrt{x}$ is $f'(x) = \frac{1}{2\sqrt{x}}$.

3.3 The Derivative as a Function



Example

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We know that the derivative of $f(x) = \sqrt{x}$ is $f'(x) = \frac{1}{2\sqrt{x}}$.

Therefore

$$f'(4) = \frac{1}{2\sqrt{4}} = \frac{1}{4}.$$

3.3 The Derivative as a Function



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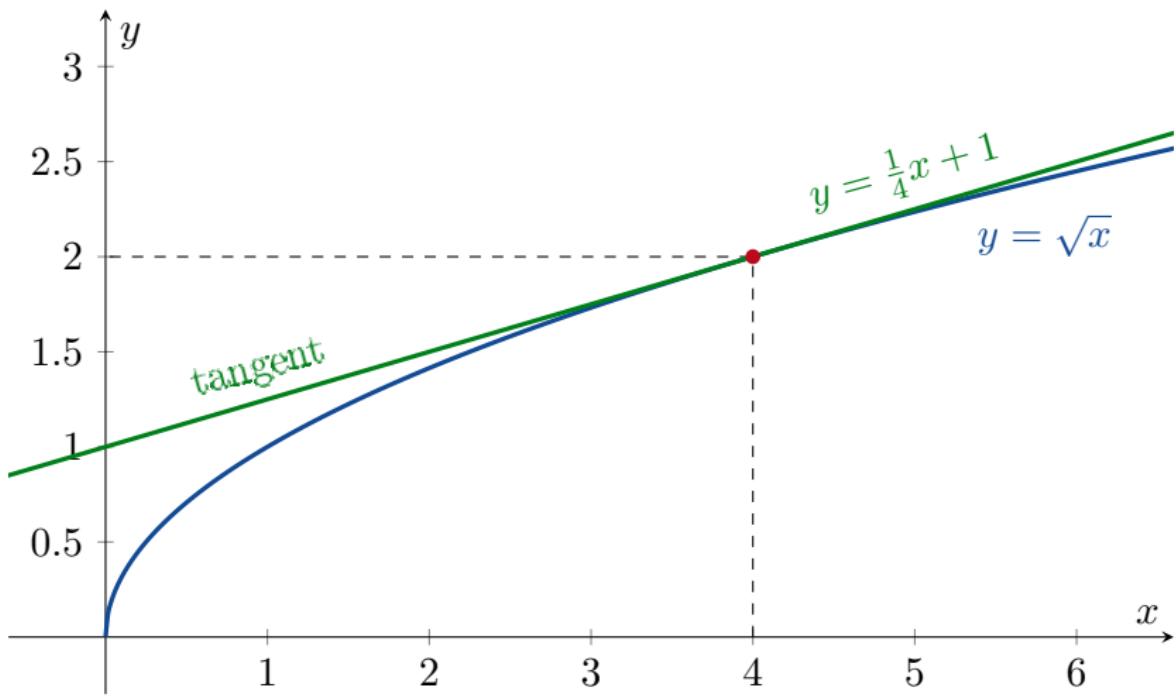
Therefore

$$f'(4) = \frac{1}{2\sqrt{4}} = \frac{1}{4}.$$

The tangent is the line passing through the point $(4, 2)$ with slope $\frac{1}{4}$. So the tangent is

$$y = 2 + \frac{1}{4}(x - 4) = \frac{1}{4}x + 1.$$

3.3 The Derivative as a Function



3.3 The Derivative as a Function



Notation

There are many ways to write the derivative of $y = f(x)$.

$$f'(x) = y' = \frac{dy}{dx} = \frac{df}{dx} = \frac{d}{dx} f(x) = \dot{y} = \dot{f}(x)$$

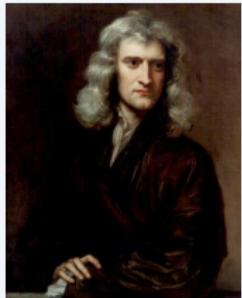
“the derivative of y with respect to x ”

3.3 The Derivative as a Function



Calculus was started by two men who each claimed to have invented it, then grew to hate each other:

3.3 The Derivative as a Function



Sir Isaac Newton

BORN

4 January 1643

DECEASED

31 March 1727 (aged 84)

NATIONALITY

British

Calculus was started by two men who each claimed to have invented it, then grew to hate each other: Sir Isaac Newton used \dot{f} and \dot{y} .

3.3 The Derivative as a Function



Sir Isaac Newton



Gottfried Leibniz

BORN

1 July 1646

DECEASED

14 November 1716 (aged 70)

NATIONALITY

German

Calculus was started by two men who each claimed to have invented it, then grew to hate each other: Sir Isaac Newton used \dot{f} and \dot{y} . Gottfried Leibniz used $\frac{df}{dx}$ and $\frac{dy}{dx}$.

3.3 The Derivative as a Function





Sir Isaac Newton



Gottfried Leibniz



Joseph-Louis Lagrange

BORN
25 January 1736

DECEASED
10 April 1813 (aged 77)

NATIONALITY
Italian

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The f' and y' notation came later from Lagrange.

3.3 The Derivative as a Function



If we want the derivative of $y = f(x)$ at the point $x = x_0$, we can write

$$f'(x_0) = \frac{dy}{dx} \Big|_{x=x_0} = \frac{df}{dx} \Big|_{x=x_0} = \frac{d}{dx} f(x) \Big|_{x=x_0}$$

“the derivative of y with respect to x at $x = x_0$ ”

3.3 The Derivative as a Function



If we want the derivative of $y = f(x)$ at the point $x = x_0$, we can write

$$f'(x_0) = \frac{dy}{dx} \Big|_{x=x_0} = \frac{df}{dx} \Big|_{x=x_0} = \frac{d}{dx} f(x) \Big|_{x=x_0}$$

“the derivative of y with respect to x at $x = x_0$ ”

For example, if $u(x) = \frac{1}{x}$, then

$$u'(4) = \frac{d}{dx} \left(\frac{1}{x} \right) \Big|_{x=4} = \frac{-1}{x^2} \Big|_{x=4} = \frac{-1}{4^2} = \frac{-1}{16}.$$

3.3 The Derivative as a Function



Example

Show that $f(x) = |x|$ is not differentiable at $x = 0$.

3.3 The Derivative as a Function



Example

Show that $f(x) = |x|$ is not differentiable at $x = 0$.

One-Sided Derivatives

Just as we had one-sided limits, we can have one-sided derivatives.

$$\begin{array}{l} \text{Right-hand derivative} \\ \text{of } f \text{ at } a \end{array} = \lim_{h \rightarrow 0^+} \frac{f(a + h) - f(a)}{h}$$

$$\begin{array}{l} \text{Left-hand derivative} \\ \text{of } f \text{ at } b \end{array} = \lim_{h \rightarrow 0^-} \frac{f(b + h) - f(b)}{h}$$

3.3 The Derivative as a Function



Example

Show that $f(x) = |x|$ is not differentiable at $x = 0$.

Right-hand derivative

$$\lim_{h \rightarrow 0^+} \frac{|0 + h| - 0}{h} = \lim_{h \rightarrow 0^+} \frac{h}{h} = \lim_{h \rightarrow 0^+} 1 = 1.$$

Left-hand derivative

$$\lim_{h \rightarrow 0^-} \frac{|0 + h| - 0}{h} = \lim_{h \rightarrow 0^+} \frac{-h}{h} = \lim_{h \rightarrow 0^-} -1 = -1.$$

3.3 The Derivative as a Function



Example

Show that $f(x) = |x|$ is not differentiable at $x = 0$.

Right-hand derivative

$$\lim_{h \rightarrow 0^+} \frac{|0 + h| - 0}{h} = \lim_{h \rightarrow 0^+} \frac{h}{h} = \lim_{h \rightarrow 0^+} 1 = 1.$$

Left-hand derivative

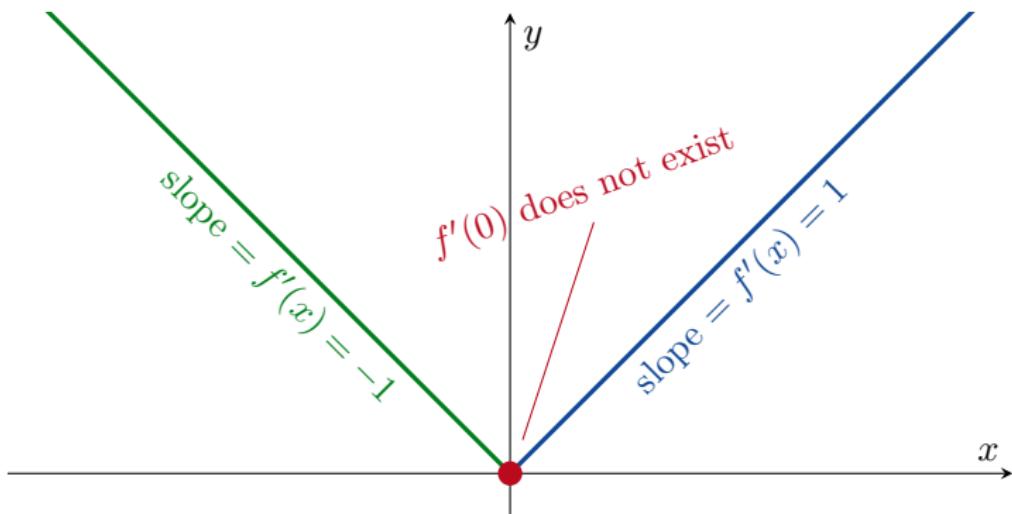
$$\lim_{h \rightarrow 0^-} \frac{|0 + h| - 0}{h} = \lim_{h \rightarrow 0^+} \frac{-h}{h} = \lim_{h \rightarrow 0^-} -1 = -1.$$

Since the right-hand derivative and left-hand derivative at $x = 0$ are not equal, $|x|$ is not differentiable at $x = 0$.

3.3 The Derivative as a Function



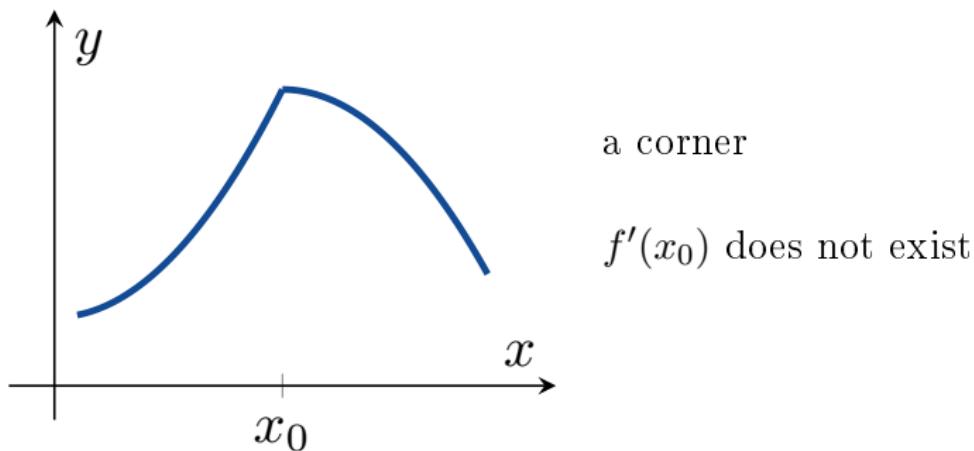
$$f(x) = |x|$$



3.3 The Derivative as a Function



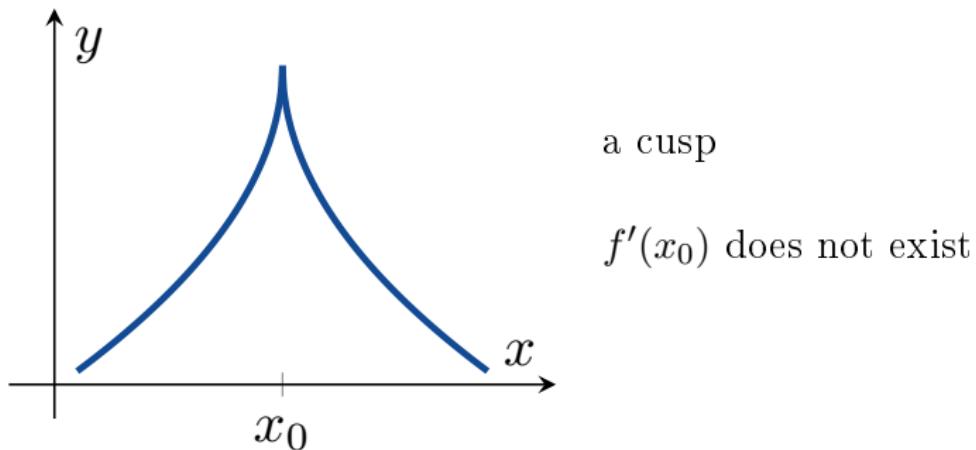
When Does a Function Not Have a Derivative at a Point?



3.3 The Derivative as a Function



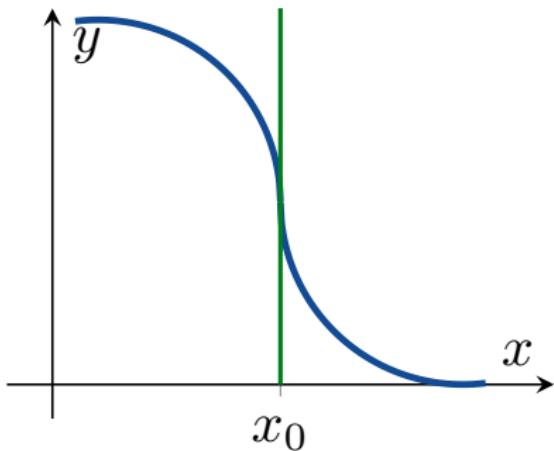
When Does a Function Not Have a Derivative at a Point?



3.3 The Derivative as a Function



When Does a Function Not Have a Derivative at a Point?



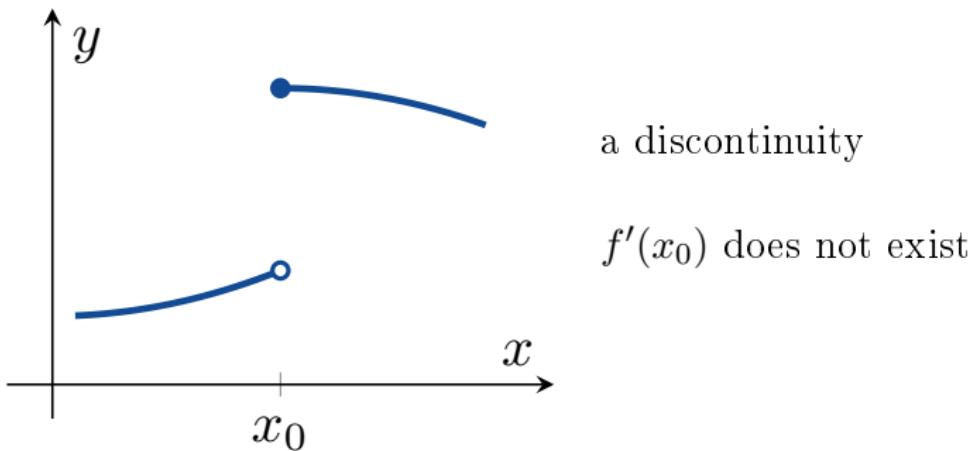
a vertical tangent

$f'(x_0)$ does not exist

3.3 The Derivative as a Function



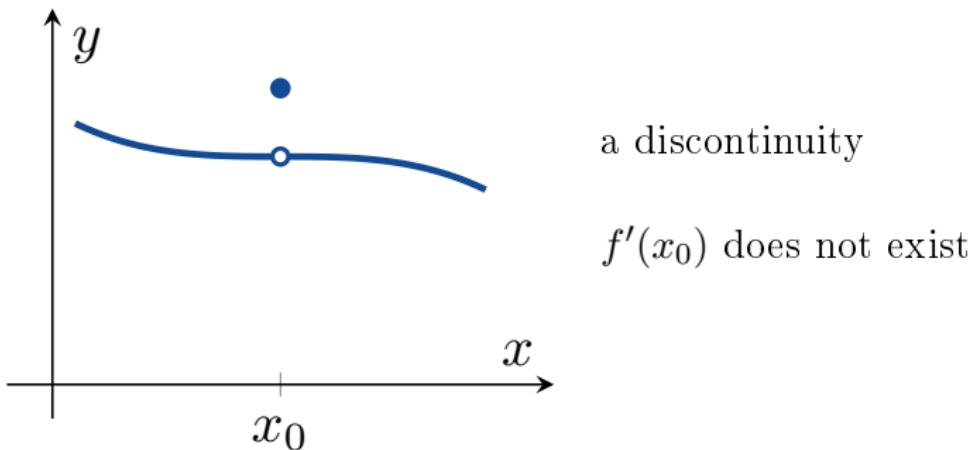
When Does a Function Not Have a Derivative at a Point?



3.3 The Derivative as a Function



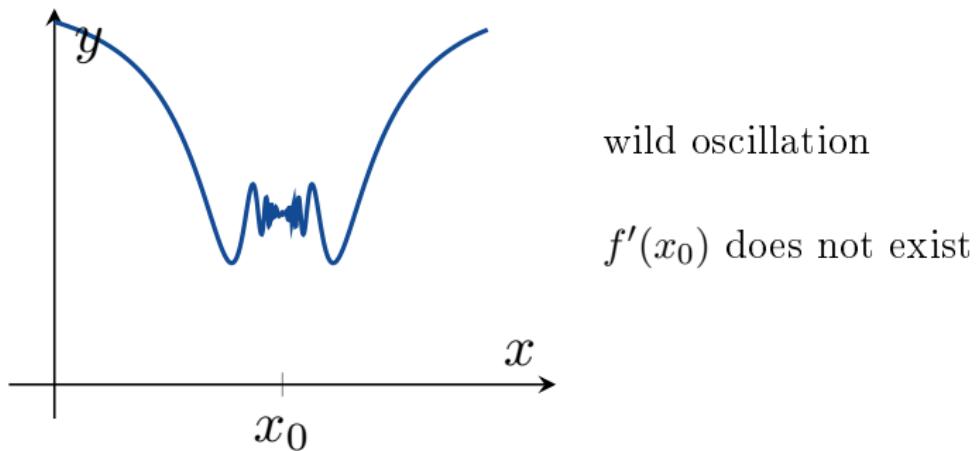
When Does a Function Not Have a Derivative at a Point?



3.3 The Derivative as a Function



When Does a Function Not Have a Derivative at a Point?



3.3 The Derivative as a Function



Theorem

$$\left(\begin{array}{c} f \text{ has a derivative} \\ \text{at } x = c \end{array} \right) \implies \left(\begin{array}{c} f \text{ is continuous} \\ \text{at } x = c \end{array} \right)$$

3.3 The Derivative as a Function



Theorem

$$\left(\begin{array}{c} f \text{ has a derivative} \\ \text{at } x = c \end{array} \right) \implies \left(\begin{array}{c} f \text{ is continuous} \\ \text{at } x = c \end{array} \right)$$

Proof.

We know that $f'(c)$ exists.

3.3 The Derivative as a Function



Theorem

$$\left(\begin{array}{c} f \text{ has a derivative} \\ \text{at } x = c \end{array} \right) \implies \left(\begin{array}{c} f \text{ is continuous} \\ \text{at } x = c \end{array} \right)$$

Proof.

We know that $f'(c)$ exists. Writing $h = c - x$, we have

$$\lim_{x \rightarrow c} f(x) = \lim_{h \rightarrow 0} f(c + h) =$$

=

=

=



3.3 The Derivative as a Function



Theorem

$$\left(\begin{array}{c} f \text{ has a derivative} \\ \text{at } x = c \end{array} \right) \implies \left(\begin{array}{c} f \text{ is continuous} \\ \text{at } x = c \end{array} \right)$$

Proof.

We know that $f'(c)$ exists. Writing $h = c - x$, we have

$$\lim_{x \rightarrow c} f(x) = \lim_{h \rightarrow 0} f(c + h) = \lim_{h \rightarrow 0} (f(c) + f(c + h) - f(c))$$

=

=

=



3.3 The Derivative as a Function



Theorem

$$\left(\begin{array}{c} f \text{ has a derivative} \\ \text{at } x = c \end{array} \right) \implies \left(\begin{array}{c} f \text{ is continuous} \\ \text{at } x = c \end{array} \right)$$

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$$\lim_{x \rightarrow c} f(x) = \lim_{h \rightarrow 0} f(c + h) = \lim_{h \rightarrow 0} (f(c) + f(c + h) - f(c))$$

$$= \lim_{h \rightarrow 0} \left(f(c) + \frac{f(c + h) - f(c)}{h} h \right)$$

=

=



3.3 The Derivative as a Function



Theorem

$$\left(\begin{array}{c} f \text{ has a derivative} \\ \text{at } x = c \end{array} \right) \implies \left(\begin{array}{c} f \text{ is continuous} \\ \text{at } x = c \end{array} \right)$$

Proof.

We know that $f'(c)$ exists. Writing $h = c - x$, we have

$$\begin{aligned}\lim_{x \rightarrow c} f(x) &= \lim_{h \rightarrow 0} f(c + h) = \lim_{h \rightarrow 0} (f(c) + f(c + h) - f(c)) \\&= \lim_{h \rightarrow 0} \left(f(c) + \frac{f(c + h) - f(c)}{h} h \right) \\&= \lim_{h \rightarrow 0} f(c) + \left(\lim_{h \rightarrow 0} \frac{f(c + h) - f(c)}{h} \right) \left(\lim_{h \rightarrow 0} h \right) \\&= \end{aligned}$$



3.3 The Derivative as a Function



Theorem

$$\left(\begin{array}{c} f \text{ has a derivative} \\ \text{at } x = c \end{array} \right) \implies \left(\begin{array}{c} f \text{ is continuous} \\ \text{at } x = c \end{array} \right)$$

Proof.

We know that $f'(c)$ exists. Writing $h = c - x$, we have

$$\begin{aligned}\lim_{x \rightarrow c} f(x) &= \lim_{h \rightarrow 0} f(c + h) = \lim_{h \rightarrow 0} (f(c) + f(c + h) - f(c)) \\&= \lim_{h \rightarrow 0} \left(f(c) + \frac{f(c + h) - f(c)}{h} h \right) \\&= \lim_{h \rightarrow 0} f(c) + \left(\lim_{h \rightarrow 0} \frac{f(c + h) - f(c)}{h} \right) \left(\lim_{h \rightarrow 0} h \right) \\&= f(c) + f'(c) \cdot 0 = f(c).\end{aligned}$$

Therefore f is continuous at c .



3.3 The Derivative as a Function



Remark

$$\left(\begin{array}{l} f \text{ has a derivative} \\ \text{at } x = c \end{array} \right) \implies \left(\begin{array}{l} f \text{ is continuous} \\ \text{at } x = c \end{array} \right)$$

but

$$\left(\begin{array}{l} f \text{ has a derivative} \\ \text{at } x = c \end{array} \right) \not\iff \left(\begin{array}{l} f \text{ is continuous} \\ \text{at } x = c \end{array} \right).$$

For example $f(x) = |x|$ is continuous at $x = 0$, but is not differentiable at $x = 0$.

Break

We will continue at 2pm





Differentiation Rules

3.5 Differentiation Rules



Constant Function

If $k \in \mathbb{R}$, then

$$\frac{d}{dx}(k) = 0.$$

(You can read the proof in your textbook.)

3.5 Differentiation Rules



Power Function

If $n \in \mathbb{R}$, then

$$\frac{d}{dx} (x^n) = nx^{n-1}.$$

(You can read the proof in your textbook.)

3.5 Differentiation Rules



Power Function

If $n \in \mathbb{R}$, then

$$\frac{d}{dx} (x^n) = nx^{n-1}.$$

(You can read the proof in your textbook.)

Example

$$\frac{d}{dx} (x^3) = 3x^{3-1} = 3x^2$$

3.5 Differentiation Rules



Example

$$\frac{d}{dx} (\sqrt{x}) = \frac{d}{dx} (x^{\frac{1}{2}}) = \frac{1}{2}x^{\frac{1}{2}-1} = \frac{1}{2}x^{-\frac{1}{2}} = \frac{1}{2\sqrt{x}}$$

Example

$$\frac{d}{dx} \left(\frac{1}{x^4} \right) = \frac{d}{dx} (x^{-4}) = -4x^{-4-1} = -4x^{-5} = -\frac{4}{x^5}$$

(You can read more examples in the textbook.)

The Constant Multiple Rule

If $u(x)$ is differentiable and $k \in \mathbb{R}$, then

$$\frac{d}{dx} (ku) = k \frac{du}{dx}.$$

3.5 Differentiation Rules



The Constant Multiple Rule

If $u(x)$ is differentiable and $k \in \mathbb{R}$, then

$$\frac{d}{dx} (ku) = k \frac{du}{dx}.$$

Proof.

$$\begin{aligned}\frac{d}{dx} (ku) &= \lim_{h \rightarrow 0} \frac{ku(x+h) - ku(x)}{h} \\&= \lim_{h \rightarrow 0} k \left(\frac{u(x+h) - u(x)}{h} \right)\end{aligned}$$

=

=



The Constant Multiple Rule

If $u(x)$ is differentiable and $k \in \mathbb{R}$, then

$$\boxed{\frac{d}{dx} (ku) = k \frac{du}{dx}.}$$

Proof.

$$\begin{aligned}\frac{d}{dx} (ku) &= \lim_{h \rightarrow 0} \frac{ku(x+h) - ku(x)}{h} \\&= \lim_{h \rightarrow 0} k \left(\frac{u(x+h) - u(x)}{h} \right) \\&= k \lim_{h \rightarrow 0} \frac{u(x+h) - u(x)}{h} =\end{aligned}$$

□

The Constant Multiple Rule

If $u(x)$ is differentiable and $k \in \mathbb{R}$, then

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3.5 Differentiation Rules

Example

$$\frac{d}{dx} (3x^2) = 3 \frac{d}{dx} (x^2) = 3 \times 2x = 6x$$

Example

$$\frac{d}{dx} (-u) = \frac{d}{dx} (-1 \times u) = -1 \times \frac{du}{dx} = -\frac{du}{dx}$$

3.5 Differentiation Rules



The Sum Rule

If $u(x)$ and $v(x)$ are differentiable at x_0 , then $u + v$ is also differentiable at x_0 and

$$\frac{d}{dx} (u + v) = \frac{du}{dx} + \frac{dv}{dx}.$$

(You can read the proof in your textbook.)

3.5 Differentiation Rules



Example

Differentiate $y = x^3 + \frac{4}{3}x^2 - 5x + 1$.

$$\begin{aligned}\frac{dy}{dx} &= \frac{d}{dx} \left(x^3 + \frac{4}{3}x^2 - 5x + 1 \right) \\ &= \frac{d}{dx} (x^3) + \frac{d}{dx} \left(\frac{4}{3}x^2 \right) - \frac{d}{dx} (5x) + \frac{d}{dx} (1) \\ &= 3x^2 + \frac{8}{3}x - 5 + 0.\end{aligned}$$

3.5 Differentiation Rules



Example

Does the curve $y = x^4 - 2x^2 + 2$ have any points where $\frac{dy}{dx} = 0$?
If so, where?

3.5 Differentiation Rules



Example

Does the curve $y = x^4 - 2x^2 + 2$ have any points where $\frac{dy}{dx} = 0$?
If so, where?

Since

$$\frac{dy}{dx} = 4x^3 - 4x = 4x(x^2 - 1) = 4x(x - 1)(x + 1),$$

3.5 Differentiation Rules



Example

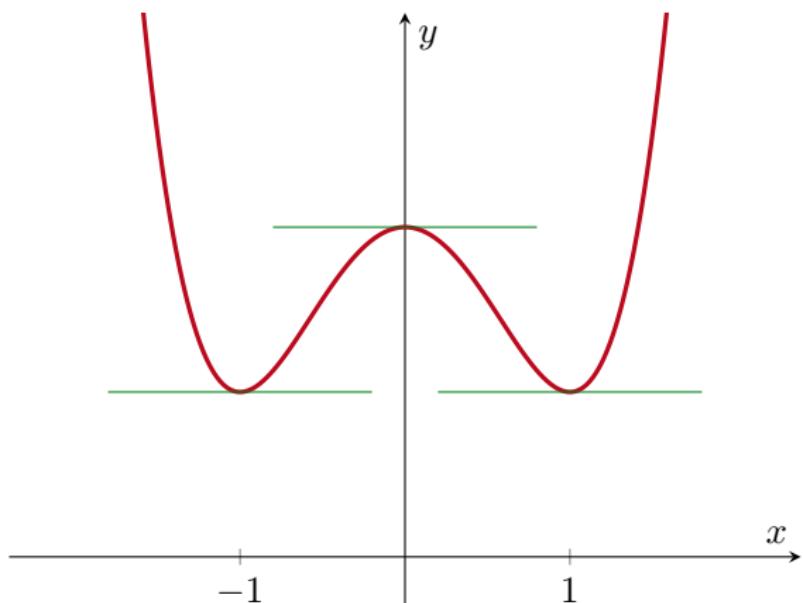
Does the curve $y = x^4 - 2x^2 + 2$ have any points where $\frac{dy}{dx} = 0$?
If so, where?

Since

$$\frac{dy}{dx} = 4x^3 - 4x = 4x(x^2 - 1) = 4x(x - 1)(x + 1),$$

we can see that $\frac{dy}{dx} = 0$ if and only if $x = -1, 0$ or 1 .

3.5 Differentiation Rules



3.5 Differentiation Rules



The Product Rule

If $u(x)$ and $v(x)$ are differentiable at x_0 , then $u(x)v(x)$ is also differentiable at x_0 and

$$\frac{d}{dx}(uv) = \frac{du}{dx}v + u\frac{dv}{dx}.$$

(You can read the proof in your textbook.)

3.5 Differentiation Rules



The Product Rule

If $u(x)$ and $v(x)$ are differentiable at x_0 , then $u(x)v(x)$ is also differentiable at x_0 and

$$\frac{d}{dx}(uv) = \frac{du}{dx}v + u\frac{dv}{dx}.$$

Using prime notation, the product rule is

$$(uv)' = u'v + uv'.$$

(You can read the proof in your textbook.)

$$(uv)' = u'v + uv'$$

Example

Differentiate $y = (x^2 + 1)(x^3 + 3)$.

solution 1: We have $y = uv$ with $u = x^2 + 1$ and $v = x^3 + 3$.

$$(uv)' = u'v + uv'$$

Example

Differentiate $y = (x^2 + 1)(x^3 + 3)$.

solution 1: We have $y = uv$ with $u = x^2 + 1$ and $v = x^3 + 3$. So

$$\frac{dy}{dx} = u'v + uv'$$

$$= (x^2 + 1)'(x^3 + 3) + (x^2 + 1)(x^3 + 3)'$$

=

=

=

$$(uv)' = u'v + uv'$$

Example

Differentiate $y = (x^2 + 1)(x^3 + 3)$.

solution 1: We have $y = uv$ with $u = x^2 + 1$ and $v = x^3 + 3$. So

$$\begin{aligned}\frac{dy}{dx} &= u'v + uv' \\&= (x^2 + 1)'(x^3 + 3) + (x^2 + 1)(x^3 + 3)' \\&= (2x + 0)(x^3 + 3) + (x^2 + 1)(3x^2 + 0) \\&= \\&= \end{aligned}$$

3.5

$$(uv)' = u'v + uv'$$



Example

Differentiate $y = (x^2 + 1)(x^3 + 3)$.

solution 1: We have $y = uv$ with $u = x^2 + 1$ and $v = x^3 + 3$. So

$$\begin{aligned}\frac{dy}{dx} &= u'v + uv' \\&= (x^2 + 1)'(x^3 + 3) + (x^2 + 1)(x^3 + 3)' \\&= (2x + 0)(x^3 + 3) + (x^2 + 1)(3x^2 + 0) \\&= 2x^4 + 6x + 3x^4 + 3x^2 \\&= 5x^4 + 3x^2 + 6x.\end{aligned}$$

3.5 Differentiation Rules



Example

Differentiate $y = (x^2 + 1)(x^3 + 3)$.

solution 2: Since

$$y = (x^2 + 1)(x^3 + 3) = x^5 + x^3 + 3x^2 + 3,$$

3.5 Differentiation Rules



Example

Differentiate $y = (x^2 + 1)(x^3 + 3)$.

solution 2: Since

$$y = (x^2 + 1)(x^3 + 3) = x^5 + x^3 + 3x^2 + 3,$$

we have that

$$\frac{dy}{dx} = 5x^4 + 3x^2 + 6x + 0.$$

3.5 Differentiation Rules



The Quotient Rule

If $u(x)$ and $v(x)$ are differentiable at x_0 and if $v(x_0) \neq 0$, then $\frac{u}{v}$ is also differentiable at x_0 and

$$\boxed{\frac{d}{dx} \left(\frac{u}{v} \right) = \frac{u'v - uv'}{v^2}}.$$

(You can read the proof in your textbook.)

3.5

$$\left(\frac{u}{v}\right)' = \frac{u'v - uv'}{v^2}$$



Example

Differentiate $y = \frac{t^2-1}{t^3+1}$.

We have $y = \frac{u}{v}$ with $u = t^2 - 1$ and $v = t^3 + 1$.

3.5

$$\left(\frac{u}{v}\right)' = \frac{u'v - uv'}{v^2}$$



Example

Differentiate $y = \frac{t^2-1}{t^3+1}$.

We have $y = \frac{u}{v}$ with $u = t^2 - 1$ and $v = t^3 + 1$. Therefore

$$\begin{aligned}\frac{dy}{dt} &= \frac{u'v - uv'}{v^2} \\ &= \frac{(t^2 - 1)'(t^3 + 1) - (t^2 - 1)(t^3 + 1)'}{(t^3 + 1)^2}\end{aligned}$$

=

=

=

3.5

$$\left(\frac{u}{v}\right)' = \frac{u'v - uv'}{v^2}$$



Example

Differentiate $y = \frac{t^2 - 1}{t^3 + 1}$.

We have $y = \frac{u}{v}$ with $u = t^2 - 1$ and $v = t^3 + 1$. Therefore

$$\begin{aligned} \frac{dy}{dt} &= \frac{u'v - uv'}{v^2} \\ &= \frac{(t^2 - 1)'(t^3 + 1) - (t^2 - 1)(t^3 + 1)'}{(t^3 + 1)^2} \\ &= \frac{(2t)(t^3 + 1) - (t^2 - 1)(3t^2)}{(t^3 + 1)^2} \end{aligned}$$

 $=$ $=$

3.5

$$\left(\frac{u}{v}\right)' = \frac{u'v - uv'}{v^2}$$



Example

Differentiate $y = \frac{t^2 - 1}{t^3 + 1}$.

We have $y = \frac{u}{v}$ with $u = t^2 - 1$ and $v = t^3 + 1$. Therefore

$$\begin{aligned} \frac{dy}{dt} &= \frac{u'v - uv'}{v^2} \\ &= \frac{(t^2 - 1)'(t^3 + 1) - (t^2 - 1)(t^3 + 1)'}{(t^3 + 1)^2} \\ &= \frac{(2t)(t^3 + 1) - (t^2 - 1)(3t^2)}{(t^3 + 1)^2} \\ &= \frac{2t^4 + 2t - 3t^4 + 3t^2}{(t^3 + 1)^2} \\ &= \frac{-t^4 + 3t^2 + 2t}{(t^3 + 1)^2}. \end{aligned}$$

3.5

$$\left(\frac{u}{v}\right)' = \frac{u'v - uv'}{v^2}$$



Example

Differentiate $f(s) = \frac{\sqrt{s}-1}{\sqrt{s}+1}$.

3.5

$$\left(\frac{u}{v}\right)' = \frac{u'v - uv'}{v^2}$$



Example

Differentiate $f(s) = \frac{\sqrt{s}-1}{\sqrt{s}+1}$.

We have $f(s) = \frac{u}{v}$ with $u = \sqrt{s} - 1$ and $v = \sqrt{s} + 1$. Remember that $\frac{d}{ds}(\sqrt{s}) = \frac{1}{2\sqrt{s}}$.

3.5

$$\left(\frac{u}{v}\right)' = \frac{u'v - uv'}{v^2}$$



Example

Differentiate $f(s) = \frac{\sqrt{s}-1}{\sqrt{s}+1}$.

We have $f(s) = \frac{u}{v}$ with $u = \sqrt{s} - 1$ and $v = \sqrt{s} + 1$. Remember that $\frac{d}{ds}(\sqrt{s}) = \frac{1}{2\sqrt{s}}$. Therefore

$$\begin{aligned} \frac{df}{ds} &= \frac{u'v - uv'}{v^2} \\ &= \frac{(\sqrt{s}-1)'(\sqrt{s}+1) - (\sqrt{s}-1)(\sqrt{s}+1)'}{(\sqrt{s}+1)^2} \\ &= \frac{\left(\frac{1}{2\sqrt{s}}\right)(\sqrt{s}+1) - (\sqrt{s}-1)\left(\frac{1}{2\sqrt{s}}\right)}{(\sqrt{s}+1)^2} \\ &= \frac{\frac{1}{2} + \frac{1}{2\sqrt{s}} - \frac{1}{2} + \frac{1}{2\sqrt{s}}}{(\sqrt{s}+1)^2} = \frac{1}{\sqrt{s}(\sqrt{s}+1)^2}. \end{aligned}$$

3.5 Differentiation Rules

Example

Differentiate $y = \frac{(x-1)(x^2-2x)}{x^4}$.

It is possible to use the Quotient Rule to answer this question, but there is an easier way.

3.5 Differentiation Rules

Example

Differentiate $y = \frac{(x-1)(x^2-2x)}{x^4}$.

It is possible to use the Quotient Rule to answer this question, but there is an easier way. Note that

$$y = \frac{(x-1)(x^2-2x)}{x^4} = \frac{x^3 - 3x^2 + 2x}{x^4} = x^{-1} - 3x^{-2} + 2x^{-3}.$$

3.5 Differentiation Rules

Example

Differentiate $y = \frac{(x-1)(x^2-2x)}{x^4}$.

It is possible to use the Quotient Rule to answer this question, but there is an easier way. Note that

$$y = \frac{(x-1)(x^2-2x)}{x^4} = \frac{x^3 - 3x^2 + 2x}{x^4} = x^{-1} - 3x^{-2} + 2x^{-3}.$$

Thus

$$y' = -x^{-2} - 3(-2)x^{-3} + 2(-3)x^{-4} = -\frac{1}{x^2} + \frac{6}{x^3} - \frac{6}{x^4}.$$

3.5 Differentiation Rules



Second Order Derivatives

If $y = f(x)$ is a differentiable function, then $f'(x)$ is also a function.

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If $y = f(x)$ is a differentiable function, then $f'(x)$ is also a function. If $f'(x)$ is also differentiable, then we can differentiate to find a new function called f'' (" f double prime").

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f'' is called the *second derivative* of f .

3.5 Differentiation Rules



Second Order Derivatives

If $y = f(x)$ is a differentiable function, then $f'(x)$ is also a function. If $f'(x)$ is also differentiable, then we can differentiate to find a new function called f'' (" f double prime").

f'' is called the *second derivative* of f .

We can write

$$f''(x) = \frac{d}{dx} f'(x) = \frac{d}{dx} \left(\frac{dy}{dx} \right) = \frac{d^2y}{dx^2} = \frac{dy'}{dx} = y''$$

“ d squared y , dx squared”

3.5 Differentiation Rules



Example

If $y = x^6$, then

$$y' = \frac{d}{dx} (x^6) = 6x^5$$

and

$$y'' = \frac{d}{dx} (y') = \frac{d}{dx} (6x^5) = 30x^4.$$

3.5 Differentiation Rules

Example

If $y = x^6$, then

$$y' = \frac{d}{dx} (x^6) = 6x^5$$

and

$$y'' = \frac{d}{dx} (y') = \frac{d}{dx} (6x^5) = 30x^4.$$

Equivalently, we can write

$$\frac{d^2}{dx^2} (x^6) = \frac{d}{dx} \left(\frac{d}{dx} (x^6) \right) = \frac{d}{dx} (6x^5) = 30x^4.$$

Higher Order Derivatives

If f'' is differentiable, then its derivative $f''' = \frac{d^3 f}{dx^3}$ is the *third derivative* of f .

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If f'' is differentiable, then its derivative $f''' = \frac{d^3 f}{dx^3}$ is the *third derivative* of f .

If f''' is differentiable, then its derivative $f^{(4)} = \frac{d^4 f}{dx^4}$ is the *fourth derivative* of f .

Higher Order Derivatives

If f'' is differentiable, then its derivative $f''' = \frac{d^3 f}{dx^3}$ is the *third derivative* of f .

If f''' is differentiable, then its derivative $f^{(4)} = \frac{d^4 f}{dx^4}$ is the *fourth derivative* of f .

If $f^{(4)}$ is differentiable, then its derivative $f^{(5)} = \frac{d^5 f}{dx^5}$ is the *fifth derivative* of f .

3.5 Differentiation Rules



Higher Order Derivatives

If f'' is differentiable, then its derivative $f''' = \frac{d^3 f}{dx^3}$ is the *third derivative* of f .

If f''' is differentiable, then its derivative $f^{(4)} = \frac{d^4 f}{dx^4}$ is the *fourth derivative* of f .

If $f^{(4)}$ is differentiable, then its derivative $f^{(5)} = \frac{d^5 f}{dx^5}$ is the *fifth derivative* of f .

⋮

If $f^{(n-1)}$ is differentiable, then its derivative $f^{(n)} = \frac{d^n f}{dx^n}$ is the *n th derivative* of f .

3.5 Differentiation Rules



Remark

For the 13th derivative of $f(x)$, writing

$$f''''''''''''(x)$$

would just be silly. So we write

$$f^{(13)}(x)$$

instead.

3.5 Differentiation Rules



Example

Find the first four derivatives of $y = x^3 - 3x^2 + 2$.

First derivative: $y' = 3x^2 - 6x$

3.5 Differentiation Rules



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Find the first four derivatives of $y = x^3 - 3x^2 + 2$.

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3.5 Differentiation Rules



Example

Find the first four derivatives of $y = x^3 - 3x^2 + 2$.

First derivative: $y' = 3x^2 - 6x$

Second derivative: $y'' = 6x - 6$

Third derivative: $y''' = 6$

3.5 Differentiation Rules



Example

Find the first four derivatives of $y = x^3 - 3x^2 + 2$.

First derivative: $y' = 3x^2 - 6x$

Second derivative: $y'' = 6x - 6$

Third derivative: $y''' = 6$

Fourth derivative: $y^{(4)} = 0$.

3.5 Differentiation Rules



Example

Find the first four derivatives of $y = x^3 - 3x^2 + 2$.

First derivative: $y' = 3x^2 - 6x$

Second derivative: $y'' = 6x - 6$

Third derivative: $y''' = 6$

Fourth derivative: $y^{(4)} = 0$.

(Note that since $\frac{d}{dx}(0) = 0$, if $n \geq 4$ then $y^{(n)} = 0$ also.)



The Derivative as a Rate of Change

3.6 The Derivative as a Rate of Change



Definition

The *rate of change* of f with respect to x at x_0 is the derivative

$$f'(x_0) = \lim_{h \rightarrow 0} \frac{f(x_0 + h) - f(x_0)}{h}$$

provided the limit exists.

3.6 The Derivative as a Rate of Change

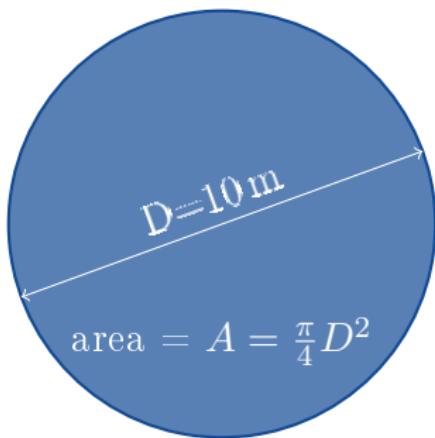


Example

The area A of a circle is related to its diameter by the equation

$$A = \frac{\pi}{4}D^2.$$

How fast does the area change with respect to the diameter when the diameter is 10 m?



3.6 The Derivative as a Rate of Change



Example

The area A of a circle is related to its diameter by the equation

$$A = \frac{\pi}{4}D^2.$$

How fast does the area change with respect to the diameter when the diameter is 10 m?

The rate of change is

$$\frac{dA}{dD} = \frac{\pi}{4} \cdot 2D = \frac{\pi D}{2}.$$

When $D = 10$ m, this is 5π m²/m.

3.6 The Derivative as a Rate of Change



Motion Along a Line; Displacement, Velocity, Speed, Acceleration and Jerk

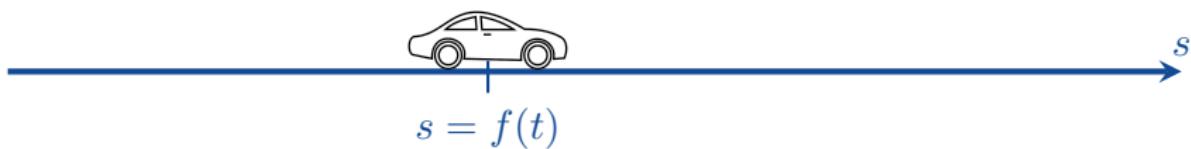


Suppose that an object is moving along a line.

3.6 The Derivative as a Rate of Change



Motion Along a Line; Displacement, Velocity, Speed, Acceleration and Jerk

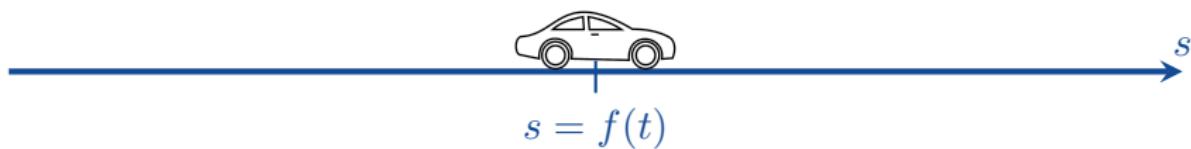


Suppose that an object is moving along a line.

3.6 The Derivative as a Rate of Change



Motion Along a Line; Displacement, Velocity, Speed, Acceleration and Jerk



Suppose that an object is moving along a line. Let

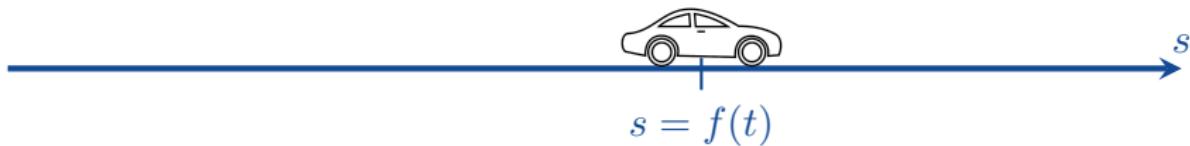
$$s = f(t)$$

be its position at time t .

3.6 The Derivative as a Rate of Change



Motion Along a Line; Displacement, Velocity, Speed, Acceleration and Jerk



Suppose that an object is moving along a line. Let

$$s = f(t)$$

be its position at time t .

3.6 The Derivative as a Rate of Change



Motion Along a Line; Displacement, Velocity, Speed, Acceleration and Jerk

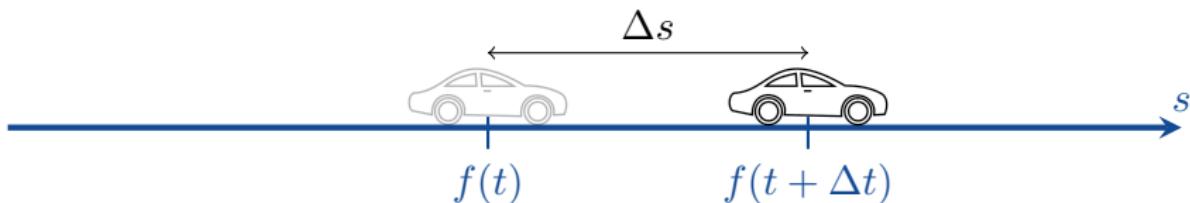


The *displacement* of the object over the time interval from t to $t + \Delta t$ is

3.6 The Derivative as a Rate of Change



Motion Along a Line; Displacement, Velocity, Speed, Acceleration and Jerk



The *displacement* of the object over the time interval from t to $t + \Delta t$ is

$$\Delta s = f(t + \Delta t) - f(t).$$

3.6 The Derivative as a Rate of Change



The *average velocity* of the object between time t and time $t + \Delta t$ is

$$\frac{\text{displacement}}{\text{travel time}} = \frac{\Delta s}{\Delta t} = \frac{f(t + \Delta t) - f(t)}{\Delta t}.$$

Taking the limit as $\Delta t \rightarrow 0$, we obtain:

3.6 The Derivative as a Rate of Change



Definition

$$\text{velocity} = \frac{d}{dt} (\text{position})$$

3.6 The Derivative as a Rate of Change



Definition

velocity = $\frac{d}{dt}$ (position)

$$v(t) = \frac{ds}{dt} = \lim_{\Delta t \rightarrow 0} \frac{f(t + \Delta t) - f(t)}{\Delta t}.$$

3.6 The Derivative as a Rate of Change



Definition

velocity = $\frac{d}{dt}$ (position)

$$v(t) = \frac{ds}{dt} = \lim_{\Delta t \rightarrow 0} \frac{f(t + \Delta t) - f(t)}{\Delta t}.$$

Definition

$$\text{speed} = |\text{velocity}| = \left| \frac{ds}{dt} \right|.$$

3.6 The Derivative as a Rate of Change



$$\text{velocity} = v(t) = \frac{d}{dt} (\text{position}) = \frac{ds}{dt}.$$

Definition

$$\text{acceleration} = a(t) = \frac{d}{dt} (\text{velocity}) = \frac{dv}{dt}.$$

3.6 The Derivative as a Rate of Change



$$\text{velocity} = v(t) = \frac{d}{dt} (\text{position}) = \frac{ds}{dt}.$$

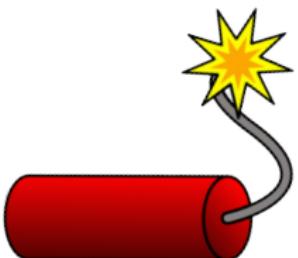
Definition

$$\text{acceleration} = a(t) = \frac{d}{dt} (\text{velocity}) = \frac{dv}{dt}.$$

Definition

$$\text{jerk} = j(t) = \frac{d}{dt} (\text{acceleration}) = \frac{da}{dt}.$$

3.6 The Derivative as a Rate of Change



3.6 The Derivative as a Rate of Change



3.6 The Derivative Rate of Change



3.6 The Derivative as a Rate of Change



Example

A dynamite blast blows a heavy rock straight upwards with a launch velocity of 49 m/s. It reaches a height of

$$s = 49t - 4.9t^2 \text{ m}$$

after t seconds.

3.6 The Derivative as a Rate of Change



Example

A dynamite blast blows a heavy rock straight upwards with a launch velocity of 49 m/s. It reaches a height of

$$s = 49t - 4.9t^2 \text{ m}$$

after t seconds.

- 1 How high does the rock go?
- 2 What are the velocity and speed of the rock when it is 78.4 m above the ground on the way up? On the way down?
- 3 What is the acceleration of the rock at time t during its flight (after the blast)?
- 4 When does the rock hit the ground?

3.6 The Derivative as a Rate of Change

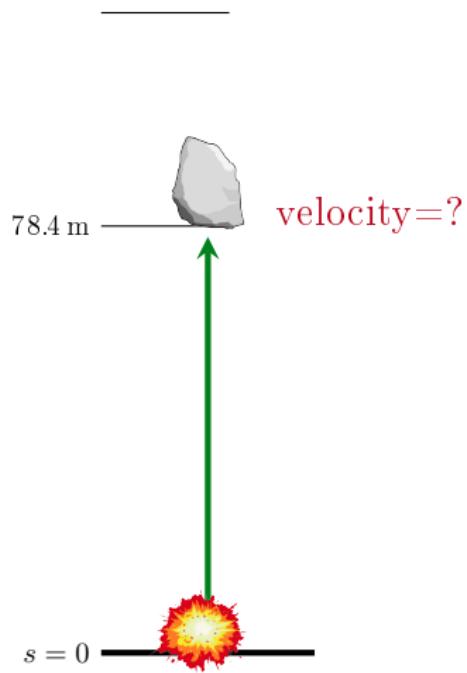


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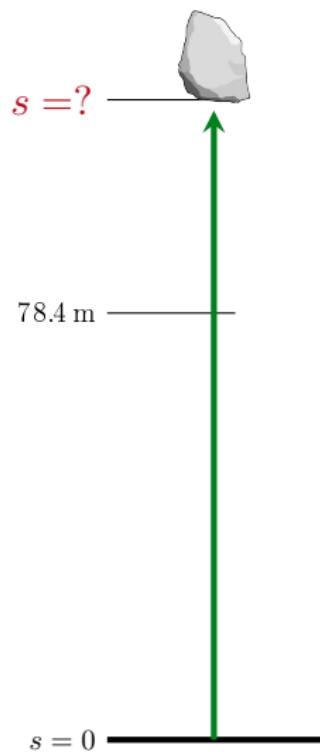
78.4 m —————



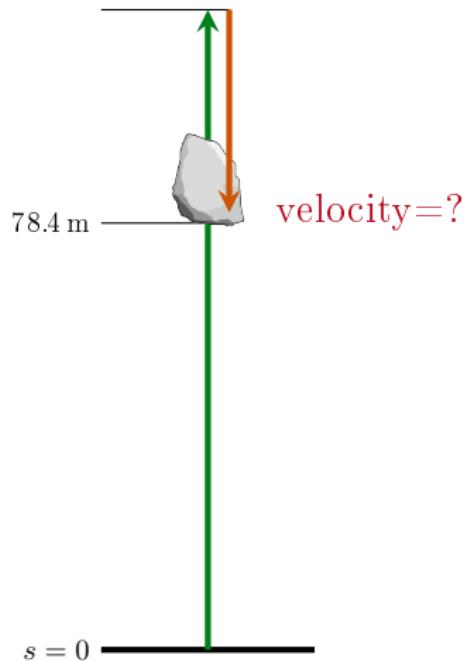
3.6 The Derivative as a Rate of Change



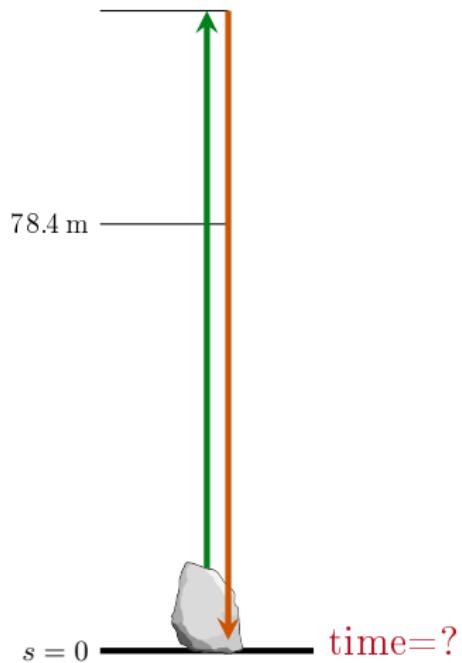
3.6 The Derivative as a Rate of Change



3.6 The Derivative as a Rate of Change



3.6 The Derivative as a Rate of Change

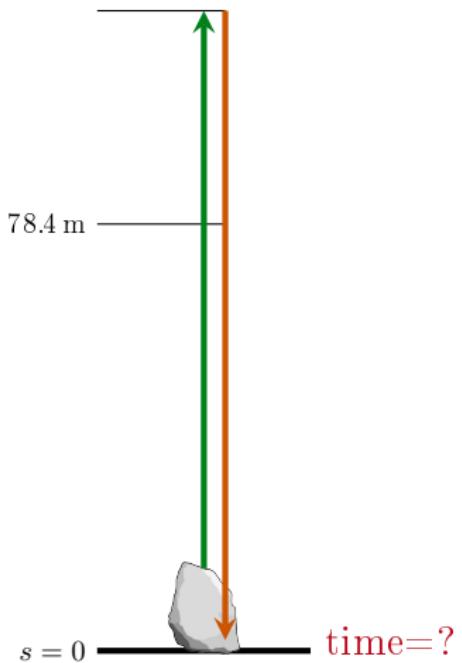


3.6 The Derivative as a Rate of Change

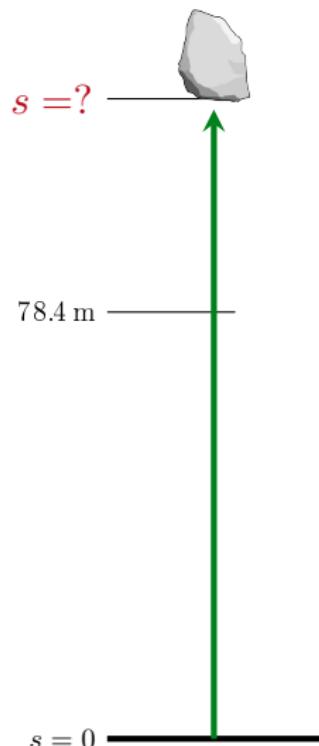


Since $s(t) = 49t - 4.9t^2$ m, we have

$$v(t) = s'(t) = 49 - 9.8t \text{ m/s}.$$



3.6 The Derivative as a Rate of Change



Since $s(t) = 49t - 4.9t^2$ m, we have

$$v(t) = s'(t) = 49 - 9.8t \text{ m/s.}$$

At the top, we have

$$0 = v(t) = 49 - 9.8t \implies t = 5 \text{ s.}$$

So the rock goes

$$s(5) = 49(5) - 4.9(25) = 122.5 \text{ m}$$

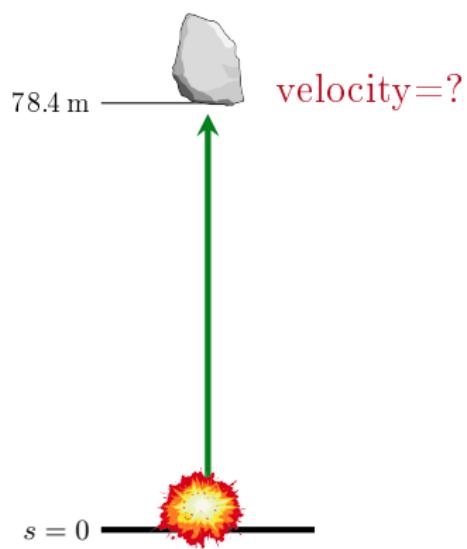
high.

3.6 The Derivative as a Rate of Change



Now

122.5 m —————



$$78.4 = s(t) = 49t - 4.9t^2$$

$$4.9(t^2 - 10t + 16) = 0$$

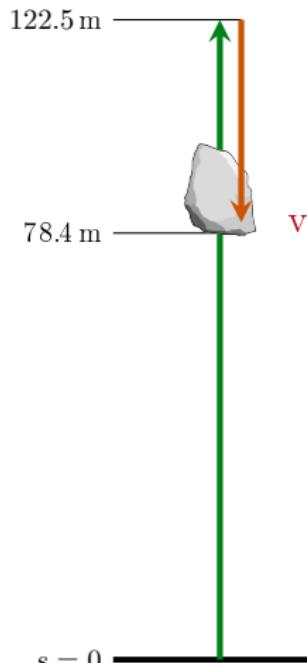
$$4.9(t - 2)(t - 8) = 0$$

implies that $t = 2$ or $t = 8$.

3.6 The Derivative as a Rate of Change



Now



$$78.4 = s(t) = 49t - 4.9t^2$$

$$4.9(t^2 - 10t + 16) = 0$$

$$4.9(t - 2)(t - 8) = 0$$

implies that $t = 2$ or $t = 8$.

We have that

$$v(2) = 49 - 9.8(2) = 29.4 \text{ m/s}$$

$$v(8) = 49 - 9.8(8) = -29.4 \text{ m/s}.$$

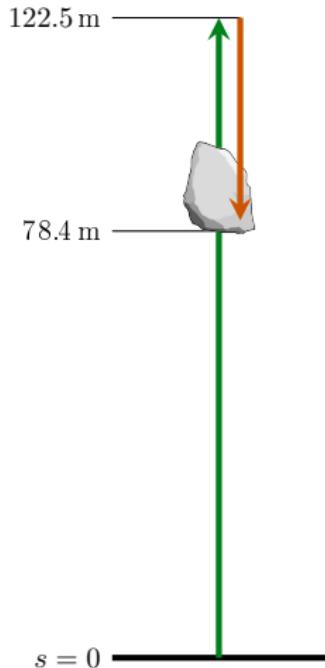
The speed at both points is 29.4 m/s.

3.6 The Derivative as a Rate of Change



Since $v(t) = 49 - 9.8t$ m/s, the acceleration is

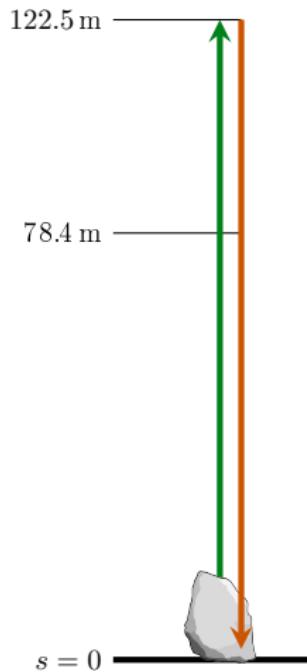
$$a(t) = v'(t) = -9.8 \text{ m/s}^2.$$



3.6 The Derivative as a Rate of Change



Since



$$\begin{aligned}0 &= s(t) = 49t - 4.9t^2 \\&= 4.9(t)(10 - t),\end{aligned}$$

the rock hits the ground after
10 seconds.

Derivatives in Economics

Suppose that

$c(x)$ = the cost to produce x tonnes of steel in one week.

3.6 The Derivative as a Rate of Change



Derivatives in Economics

Suppose that

$c(x)$ = the cost to produce x tonnes of steel in one week.

It costs more to produce $x + h$ tonnes of steel, and the cost difference divided by h is:

$$\text{average cost of producing each addition tonne} = \frac{c(x + h) - c(x)}{h}.$$

Taking the limit as $h \rightarrow 0$, we get the *marginal cost* of producing more steel per week when the current weekly production is x tonnes:

$$\text{marginal cost of production} = \frac{dc}{dx} = \lim_{h \rightarrow 0} \frac{c(x + h) - c(x)}{h}.$$

3.6 The Derivative as a Rate of Change



Example

Suppose that it costs

$$c(x) = x^3 - 6x^2 + 15x$$

lira to produce x radiators per day (when $x \in [8, 30]$) and that you can earn

$$r(x) = x^3 - 3x^2 + 12x$$

lira from selling x radiators. Your factory currently makes 10 radiators per day.

3.6 The Derivative as a Rate of Change



Example

Suppose that it costs

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- 1 How much extra will it cost to produce one more radiator per day?

3.6 The Derivative as a Rate of Change



Example

Suppose that it costs

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lira from selling x radiators. Your factory currently makes 10 radiators per day.

- 1 How much extra will it cost to produce one more radiator per day?
- 2 How much more profit can you make by producing 11 radiators per day instead of 10?

3.6

$$c(x) = x^3 - 6x^2 + 15x \quad r(x) = x^3 - 3x^2 + 12x$$



The marginal cost of production is

$$c'(x) = 3x^2 - 12x + 15$$

$$c'(1) = 3(100) - 12(10) + 15 = 195.$$

3.6

$$c(x) = x^3 - 6x^2 + 15x \quad r(x) = x^3 - 3x^2 + 12x$$



The marginal cost of production is

$$c'(x) = 3x^2 - 12x + 15$$

$$c'(1) = 3(100) - 12(10) + 15 = 195.$$

The marginal revenue is

$$r'(x) = 3x^2 - 6x + 12$$

$$r'(10) = 3(100) - 6(10) + 12 = 252.$$

3.6

$$c(x) = x^3 - 6x^2 + 15x \quad r(x) = x^3 - 3x^2 + 12x$$



The marginal cost of production is

$$c'(x) = 3x^2 - 12x + 15$$

$$c'(1) = 3(100) - 12(10) + 15 = 195.$$

The marginal revenue is

$$r'(x) = 3x^2 - 6x + 12$$

$$r'(10) = 3(100) - 6(10) + 12 = 252.$$

So producing 11 radiators per day instead of 10 will cost approximately 195 liras extra, but will bring in approximately 252 liras more.

We would expect your profit to increase by approximately $252 - 195 = 57$ liras.



Next Time

- 3.5 Derivatives of Trigonometric Functions
- 3.6 The Chain Rule
- 3.7 Implicit Differentiation