

Lecture 2

- 1.5 Classification
- 2.1 Linear Equations
- 2.2 Separable Equations
- 2.3 Differences Between Linear and Nonlinear Equations



Classification

1.5 Classification



ODEs

If only ordinary derivatives appear in a differential equation, then it is called an *ordinary differential equation (ODE)* [adi diferansiyel denklem]. For example

$$\frac{dv}{dt} = 9.8 - \frac{v}{5} \quad (\text{falling object})$$

and

$$\frac{dp}{dt} = \frac{p}{2} - 450 \quad (\text{mice and owls})$$

are ODEs.

PDEs

If the derivatives in a differential equation are partial derivatives, then it is called a *partial differential equation* (**PDE**) [kısımlı türevli diferansiyel denklem]. For example

$$k \frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t} \quad (\text{heat equation})$$

and

$$c^2 \frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2} \quad (\text{wave equation})$$

are PDEs.

Systems

If there is a single function to be found, then one differential equation is enough. However, if there are two or more unknown functions then we need a *system of differential equations*. For example

$$\begin{cases} \frac{dx}{dt} = ax - \alpha xy \\ \frac{dy}{dt} = -cy + \gamma xy \end{cases} \quad (\text{Predator-Prey equations})$$

is a system of differential equations.

Order

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is a **second** order ODE.

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$$\frac{d^2y}{dt^2} + \frac{dy}{dt} + y = 0$$

is a **second** order ODE.

$$y''' + 2e^t y'' + yy' = t^4$$

is a **third** order ODE.

Linear and Non-Linear

The ODE

$$F(t, y, y', \dots, y^{(n)}) = 0$$

is called *linear* iff F is a linear function of $y, y', \dots, y^{(n)}$ (we don't care about t). The *general linear ODE* of order n is

$$a_0(t)y^{(n)} + a_1(t)y^{(n-1)} + \dots + a_n(t)y = g(t). \quad (1)$$

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For example(falling object) and (mice and owls) are linear ODEs. An ODE which is not linear is called *non-linear*. For example

$$y''' + 2e^t y'' + \textcolor{brown}{y} \textcolor{brown}{y}' = t^4$$

is non-linear due to the yy' term.

1.5 Classification

Example

For each ODE below, give the order of the equation and state whether it is linear or non-linear:

■ $\frac{d^3y}{dx^3} + 2\frac{d^5y}{dt^5} + \frac{dy}{dt} - y - e^x \frac{d^2y}{dx^2} = 0$

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- $e^x y^{(7)} - x^3 y^{(99)} + 2x^x y''' - x^2 e^{(\sin x)} = 2021$
ninety-ninth order, linear



First Order Differential Equations

In this chapter, we will consider equations of the form

$$\frac{dy}{dt} = f(t, y). \quad (2)$$

Linear Equations

2.1 Linear Equations



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If the function f in (2) depends linearly on y (we don't care about t), then (2) is a first order *linear* ODE.

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$$\frac{dy}{dt} = -ay + b \quad (3)$$

where the coefficients a and b are constants.

2.1 Linear Equations

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where the coefficients a and b are constants. We will now consider

$$\frac{dy}{dt} + p(t)y = g(t) \quad (4)$$

where the coefficients $p(t)$ and $g(t)$ are functions of t .

2.1 Linear Equations

We have seen how to solve (3):

$$\begin{aligned}
 \frac{dy}{dt} &= -ay + b \\
 \int \frac{dy}{y - \frac{b}{a}} &= \int -a dt \\
 \ln \left| y - \frac{b}{a} \right| &= -at + C \\
 &\vdots \\
 y &= \frac{b}{a} + ce^{-at}.
 \end{aligned}$$

So for example $\frac{dy}{dt} + 2y = 3$ has solution $y = \frac{3}{2} + ce^{-2t}$.

2.1 Linear Equations



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- Find a special function $\mu(t)$ called an integrating factor;
- Multiply the ODE by $\mu(t)$;

2.1 Linear Equations



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- Find a special function $\mu(t)$ called an integrating factor;
- Multiply the ODE by $\mu(t)$;
- Integrate.

2.1 Linear Equations



Example

Use an integrating factor to solve $\frac{dy}{dt} + 2y = 3$.

2.1 Linear Equations

$$\frac{dy}{dt} + 2y = 3$$

First we multiply by an unknown function $\mu(t)$:

$$\mu(t) \frac{dy}{dt} + 2\mu(t)y = 3\mu(t).$$

2.1 Linear Equations

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Notice that

$$\frac{d}{dt} (\mu(t)y) = \mu(t) \frac{dy}{dt} + \frac{d\mu}{dt}(t)y.$$

2.1 Linear Equations

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Notice that

$$\frac{d}{dt}(\mu(t)y) = \mu(t)\frac{dy}{dt} + \frac{d\mu}{dt}(t)y.$$

We want to choose $\mu(t)$ such that

$$\frac{d\mu}{dt} = 2\mu.$$

2.1 Linear Equations



We know how to solve this equation:

$$\int \frac{d\mu}{\mu} = \int 2 dt$$

$$\ln |\mu| = 2t + C$$

⋮

$$\mu(t) = ce^{2t}.$$

2.1 Linear Equations



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Integrating gives

$$e^{2t}y = \frac{3}{2}e^{2t} + c.$$

Therefore

$$y = \frac{3}{2} + ce^{-2t}.$$

2.1 Linear Equations



Remark

For the ODE $\frac{dy}{dt} + 2y = 3$ we use the integrating factor
 $\mu(t) = e^{2t}$.

2.1 Linear Equations



Example

Use an integrating factor to solve $\frac{dy}{dt} + \textcolor{red}{a}y = b$.

2.1 Linear Equations



Example

Use an integrating factor to solve $\frac{dy}{dt} + ay = b$.

If we were to repeat the previous method, we would find that we need the integrating factor $\mu(t) = e^{at}$. (Please check!)

2.1 Linear Equations

Example

Solve $\frac{dy}{dt} + ay = g(t)$.

2.1 Linear Equations

Example

Solve $\frac{dy}{dt} + \textcolor{red}{a}y = g(t)$.

The integrating factor depends only on the coefficient of y . So again we use $\mu(t) = e^{\textcolor{red}{a}t}$.

2.1 Linear Equations

Multiplying the ODE by e^{at} gives

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By integrating, we obtain

$$e^{at}y = \int^t e^{as}g(s) ds + c.$$

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Thus

$$y = e^{-at} \int^t e^{as}g(s) ds + ce^{-at}$$

(5)

2.1 Linear Equations



Example

Solve

$$\begin{cases} \frac{dy}{dt} + \frac{1}{2}y = 2 + t \\ y(0) = 2. \end{cases}$$

2.1 Linear Equations

We multiply the ODE by the integrating factor $e^{\frac{t}{2}}$ to obtain

$$e^{\frac{t}{2}}y' + \frac{1}{2}e^{\frac{t}{2}}y = 2e^{\frac{t}{2}} + te^{\frac{t}{2}}$$

and

$$\frac{d}{dt} \left(e^{\frac{t}{2}}y \right) = 2e^{\frac{t}{2}} + te^{\frac{t}{2}}.$$

Integrating gives us

$$e^{\frac{t}{2}}y = 4e^{\frac{t}{2}} + 2te^{\frac{t}{2}} - 4e^{\frac{t}{2}} + c = 2te^{\frac{t}{2}} + c$$

(where we have used $\int u \frac{dv}{dt} = uv - \int \frac{du}{dt}v$ with $u = t$ and $v = 2e^{\frac{t}{2}}$). Therefore

$$y(t) = 2t + ce^{-\frac{t}{2}}.$$

2.1 Linear Equations



Now

$$2 = y(0) = 0 + c \implies c = 2.$$

Therefore the solution to the IVP is

$$y(t) = 2t + 2e^{-\frac{t}{2}}.$$

2.1 Linear Equations

Example

Solve $\frac{dy}{dt} - 2y = 4 - t$.

Please check that by using $\mu(t) = e^{-2t}$ we obtain
 $y(t) = -\frac{7}{4} + \frac{t}{2} + ce^{2t}$.

2.1 Linear Equations

Now consider

$$\frac{dy}{dt} + p(t)y = g(t).$$

We must find the integrating factor.

2.1 Linear Equations



Now consider

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We must find the integrating factor.

WARNING: The integrating factor is NOT $e^{p(t)}$.

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$$\frac{d}{dt} (\mu y) = \mu \frac{dy}{dt} + \frac{d\mu}{dt} y.$$

So we want

$$\frac{d\mu}{dt} = p(t)\mu.$$

2.1 Linear Equations

We know how to solve this ODE:

$$\begin{aligned} \int \frac{d\mu}{\mu} &= \int p(t) dt \\ \ln |\mu| &= \int p(t) dt + C \\ &\vdots \\ \mu(t) &= c \exp \int p(t) dt. \end{aligned}$$

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As before, we can choose $c = 1$ to obtain

$$\mu(t) = \exp \int p(t) dt = e^{\int p(t) dt}. \quad (6)$$

2.1 Linear Equations



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2.1 Linear Equations

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$$\frac{d}{dt}(\mu y) = \mu g(t)$$

and we calculate that

$$\mu y = \int^t \mu(s)g(s) ds + c$$

and

$$y(t) = \frac{\int^t \mu(s)g(s) ds + c}{\mu(t)}.$$

2.1 Linear Equations



Example

Solve

$$\begin{cases} ty' + 2y = 4t^2 \\ y(1) = 2. \end{cases}$$

2.1 Linear Equations

Example

Solve

$$\begin{cases} ty' + 2y = 4t^2 \\ y(1) = 2. \end{cases}$$

First we must write the equation in the standard form:

$$\frac{dy}{dt} + \frac{2}{t}y = 4t.$$

Here $p(t) = \frac{2}{t}$ and $g(t) = 4t$.

2.1 Linear Equations

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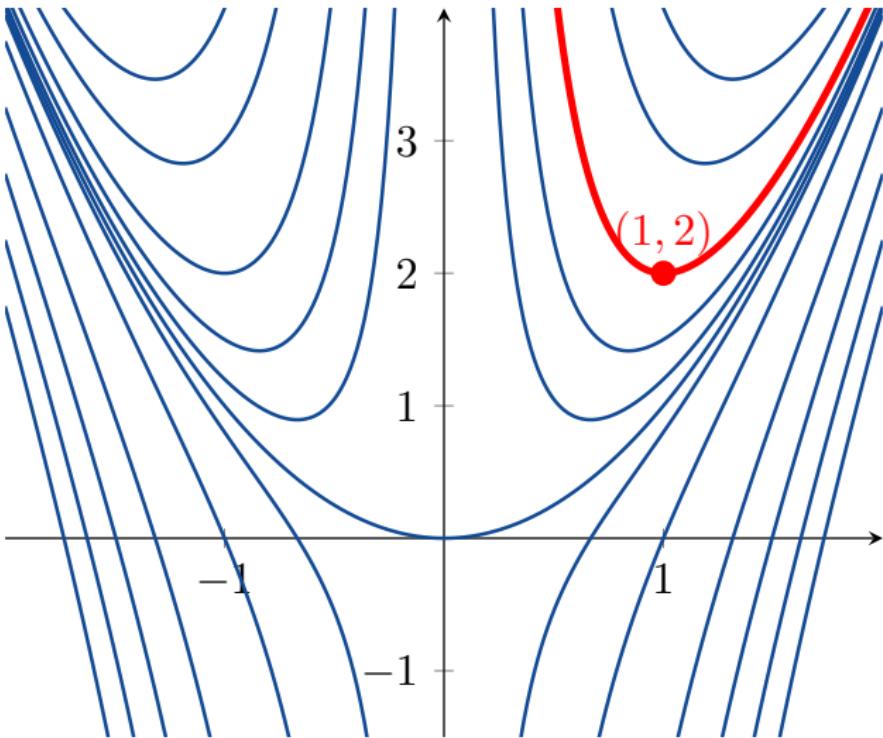
Hence the general solution to the ODE is

$$y(t) = t^2 + \frac{c}{t^2}.$$

To satisfy $y(1) = 2$, we choose $c = 1$. Therefore

$$y(t) = t^2 + \frac{1}{t^2} \quad (t > 0).$$

2.1 Linear Equations



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- 1 the solution satisfying $y(1) = 2$ is a differentiable function
 $y : (0, \infty) \rightarrow \mathbb{R}$.

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- 3 The function $y = t^2 + \frac{1}{t^2}$, $t < 0$ is *not* part of the solution to the IVP. The solution to the IVP only exists for $t \in (0, \infty)$.

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- 3 The function $y = t^2 + \frac{1}{t^2}$, $t < 0$ is *not* part of the solution to the IVP. The solution to the IVP only exists for $t \in (0, \infty)$.
- 4 Solutions for which $c > 0$ (i.e. $y(1) > 1$) are asymptotic to the positive y -axis as $t \searrow 0$. But solutions for which $c < 0$ (i.e. $y(1) < 1$) are asymptotic to the negative y -axis as $t \searrow 0$. So there is an initial value ($y(1) = 0$) where the behaviour changes. This is called a *critical initial value*.



Separable Equations

2.2 Separable Equations



The general first order ODE is

$$\frac{dy}{dx} = f(x, y). \quad (7)$$

2.2 Separable Equations



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$$\frac{dy}{dx} = f(x, y). \quad (7)$$

In the previous section we looked at a special case called “linear equations” – now we will study another special case.

2.2 Separable Equations



$$\frac{dy}{dx} = f(x, y) \quad (7)$$

Equation (7) can *always* be written in the form

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2.2 Separable Equations



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One way would be to write $M = -f$ and $N = 1$, but there may be other ways. If we can do this so that $M(x)$ is a function only of x and $N(y)$ is a function only of y , then (8) becomes

$$M(x) + N(y) \frac{dy}{dx} = 0. \quad (9)$$

2.2 Separable Equations



Definition

A first order ODE is called *separable* if it can be written in the form

$$M(x) + N(y) \frac{dy}{dx} = 0.$$

2.2 Separable Equations

Remark

Note that we can rearrange $M(x) + N(y) \frac{dy}{dx} = 0$ to

$$\underbrace{M(x) dx}_{\text{all } x \text{ terms}} = - \underbrace{N(y) dy}_{\text{all } y \text{ terms}} .$$

In other words, it is possible to “separate” the variables.

2.2 Separable Equations



Example

Consider

$$\frac{dy}{dx} = \frac{x^2}{1 - y^2}.$$

- 1** Show that this ODE is separable.
- 2** Solve this ODE.

2.2 Separable Equations



$$\frac{dy}{dx} = \frac{x^2}{1 - y^2}$$

We can rearrange this ODE to

$$-x^2 + (1 - y^2) \frac{dy}{dx} = 0.$$

This is of the form (9). Therefore this ODE is separable.

2.2 Separable Equations

Note that $\frac{d}{dx} \left(-\frac{1}{3}x^3 \right) = -x^2$ and $\frac{d}{dy} \left(y - \frac{1}{3}y^3 \right) = 1 - y^2$.

2.2 Separable Equations

Note that $\frac{d}{dx} \left(-\frac{1}{3}x^3 \right) = -x^2$ and $\frac{d}{dy} \left(y - \frac{1}{3}y^3 \right) = 1 - y^2$. So our ODE is

$$-x^2 + (1 - y^2) \frac{dy}{dx} = 0$$

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Using the Chain Rule, this is

$$\begin{aligned} \frac{d}{dx} \left(-\frac{1}{3}x^3 \right) + \frac{d}{dx} \left(y - \frac{1}{3}y^3 \right) &= 0 \\ \frac{d}{dx} \left(-\frac{1}{3}x^3 + 1 - \frac{1}{3}y^3 \right) &= 0. \end{aligned}$$

2.2 Separable Equations

$$\frac{d}{dx} \left(-\frac{1}{3}x^3 + 1 - \frac{1}{3}y^3 \right) = 0$$

Therefore

$$-\frac{1}{3}x^3 + 1 - \frac{1}{3}y^3 = C$$

or

$$x^3 - 3y + y^3 = c.$$

2.2 Separable Equations



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$$M(x) + N(y)y' = 0$$

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 $H'_1 = M$ and $H'_2 = N$.

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$$H_1(x) + H_2(y) = c.$$

2.2 Separable Equations



So to recap: To solve $M(x) + N(y)y' = 0$ we must integrate M wrt x and integrate N wrt y .

2.2 Separable Equations

So to recap: To solve $M(x) + N(y)y' = 0$ we must integrate M wrt x and integrate N wrt y . But this is basically what we were doing in Chapter 1, where we did the following:

$$M(x) + N(y)\frac{dy}{dx} = 0$$

$$M(x) = -N(y)\frac{dy}{dx}$$

$$M(x) dx = -N(y) dy$$

$$\int M(x) dx = - \int N(y) dy + c.$$

2.2 Separable Equations



Example

Solve $\begin{cases} \frac{dy}{dx} = \frac{3x^2+4x+2}{2(y-1)} \\ y(0) = -1. \end{cases}$

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Integrating gives

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2.2 Separable Equations

To find c , we use the initial condition $y(0) = 1$ and calculate that

$$1 + 2 = 0 + 0 + 0 + c \quad \Rightarrow \quad c = 3.$$

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This is called an *implicit solution*. Sometimes this is the best that we can do. But in this example, it is possible to solve for y . Since

$$y^2 - 2y - (x^3 + 2x^2 + 2x + 3) = 0$$

is a quadratic equation, we find that

$$y = 1 \pm \sqrt{x^3 + 2x^2 + 2x + 4}.$$

2.2 Separable Equations



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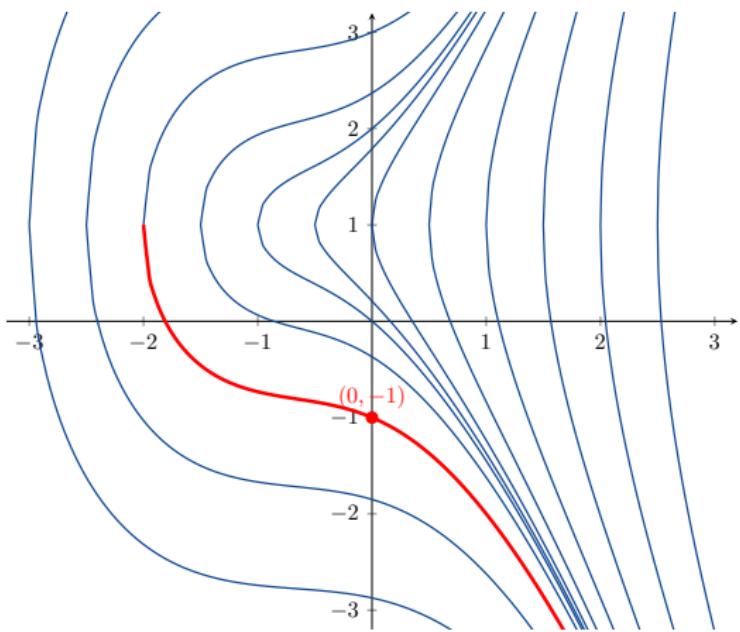
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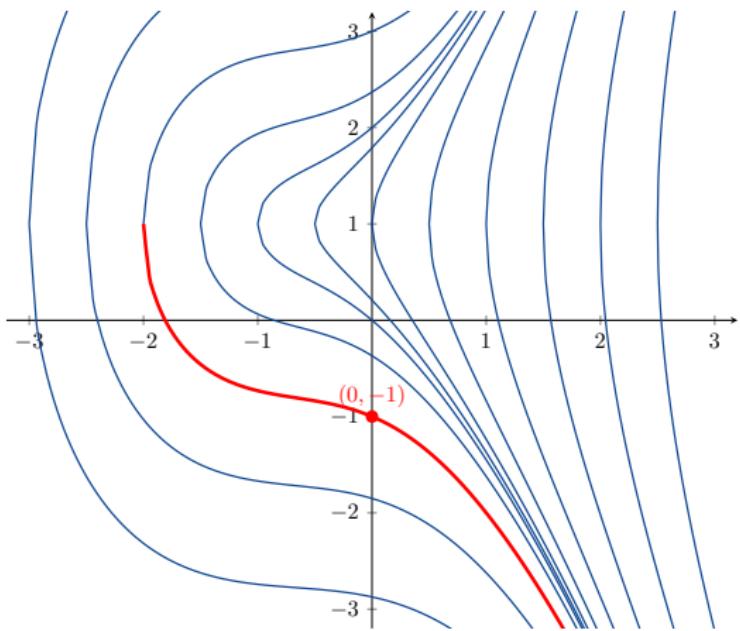
$$y = 1 - \sqrt{x^3 + 2x^2 + 2x + 4}.$$

A solution of the form $y = f(x)$ is called an *explicit solution*.

2.2 Separable Equations



2.2 Separable Equations



Note that the solution satisfying $y(0) = -1$ is a differentiable function $y : (-2, \infty) \rightarrow \mathbb{R}$.

2.2 Separable Equations

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$$\int \frac{1 + 2y^2}{y} dy = \int \cos x dx$$

$$\ln |y| + y^2 = \sin x + c$$

$$y(0) = 1 \quad \Rightarrow \quad \ln 1 + 1^2 = \sin 0 + c \quad \Rightarrow \quad c = 1.$$

$$\boxed{\ln |y| + y^2 = \sin x + 1.}$$

2.2 Separable Equations



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- 1 If $y = 0$, the left-hand side is $-\infty$, but the right-hand side is in $[0, 2]$. This means that $y = 0$ is not possible. Since we know that $y(0) = 1$, we must therefore have $y(x) > 0$ for all x in the domain of the solution.

2.2 Separable Equations



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- 2 The solution exists on $(-\infty, \infty)$ (left for you to prove).



Differences Between Linear and Nonlinear Equations

2.3 Differences Between Linear and Nonlinear Equations

Theorem

Suppose

- p and g are continuous on (α, β) ;
- $t_0 \in (\alpha, \beta)$; and
- $y_0 \in \mathbb{R}$.

Then there exists a unique solution to

$$\begin{cases} y' + p(t)y = g(t) \\ y(t_0) = y_0 \end{cases}$$

on (α, β) .

2.3 Differences Between Linear and Nonlinear Equations



$$\begin{cases} y' + p(t)y = g(t) \\ y(t_0) = y_0 \end{cases}$$

Remark

This theorem says that as long as p and g are continuous, the solution keeps existing. To say this another way: The solution can only stop existing at a discontinuity of either p or g .

2.3 Differences Between Linear and Nonlinear Equations

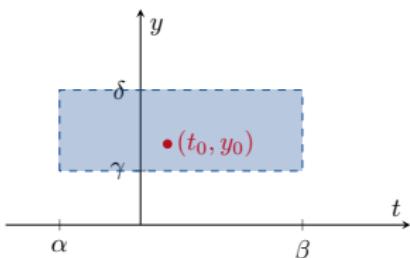


Theorem

Suppose that

- f and $\frac{\partial f}{\partial y}$ are continuous for all $\alpha < t < \beta$ and $\gamma < y < \delta$;
- $t_0 \in (\alpha, \beta)$; and
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2.3 Differences Between Linear and Nonlinear Equations

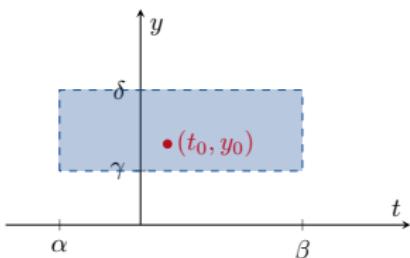


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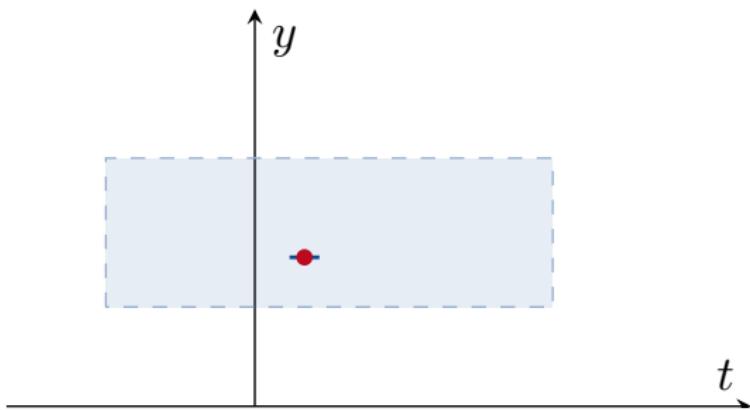
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- $t_0 \in (\alpha, \beta)$; and
- $y_0 \in (\gamma, \delta)$.

Then in some interval $(t_0 - h, t_0 + h) \subseteq (\alpha, \beta)$, there exists a unique solution to

$$\begin{cases} y' = f(t, y) \\ y(t_0) = y_0. \end{cases}$$

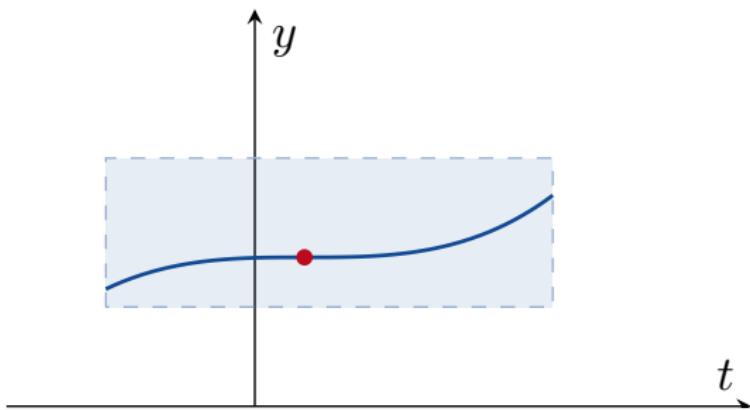
2.3 Differences Between Linear and Nonlinear Equations



Remark

This theorem tells us that “a little bit” of the solution exists. This theorem does not tell us if we only have this little bit of solution or if the solution exists further.

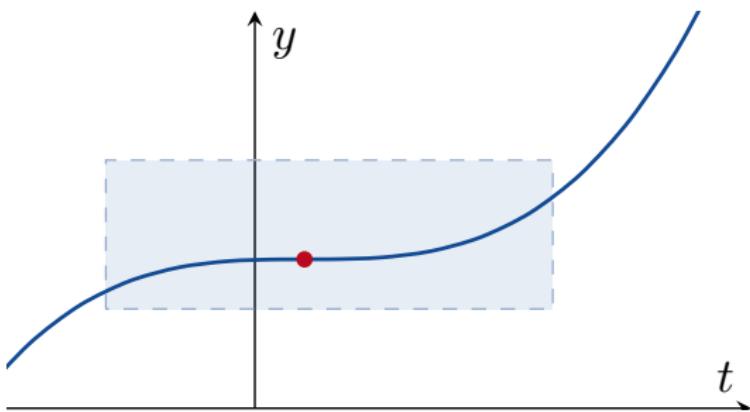
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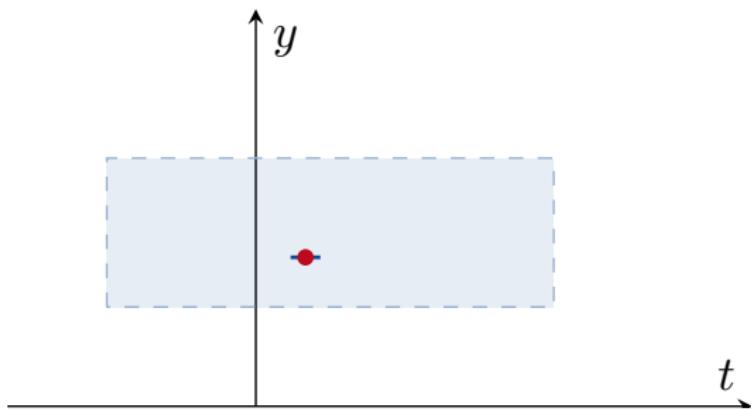
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To understand why: Suppose that two solutions intersect at the point (t_0, y_0) . But then there would be two solutions to

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Solutions to first order ODEs do not intersect !!! (assuming that f and $\frac{\partial f}{\partial y}$ are ...)



Next Time

- 2.4 Autonomous Equations and Population Dynamics
- 2.5 Exact Equations
- 2.6 Substitutions