Lecture 12

- 9.6 Alternating Series and Conditional Convergence
- 9.7 Power Series
- 9.8 Taylor and Maclaurin Series

9.6 Alternating Series and Conditional Convergence

Alternating Series

Now let's talk about sequences of the form

$$\sum_{n=1}^{\infty}(-1)^{n+1}a_n = a_1 - a_2 + a_3 - a_4 + a_5 - a_6 + a_7 - a_8 + a_9 - a_{10} + \ldots$$

where $a_n > 0\ \forall n$.

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \dots + \frac{(-1)^{n+1}}{n} + \dots$$

$$1 - 2 + 4 - 8 + 16 - 32 + \dots$$

$$4 - 2 + 1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{8} + \frac{1}{16} - \dots$$

Theorem (The Alternating Series Test / Alterne Seri Testi)

Let (a_n) *be a sequence such that*

1. $a_n > 0$ *for all* n*;*
2. (a_n) *is decreasing (i.e.* $a_n \geq a_{n+1}$ *for all* n*); and*
3. $a_n \to 0$ *as* $n \to \infty$*.*

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Then $\displaystyle\sum_{n=1}^{\infty} (-1)^{n+1} a_n$ *converges.*

Of course we can write condition 2 as "(a_n) **is decreasing eventually (i.e.** $a_n \geq a_{n+1}$ **for all** $n > N$ **for some** $N \in \mathbb{N}$**)**" since we don't care what happens at the start of a sequence/series.

Proof.

Let

$$s_n = a_1 - a_2 + a_3 - a_4 + a_5 - a_6 + \ldots + (-1)^{n+1} a_n.$$

Then

$$s_{2n} = a_1 - a_2 + a_3 - a_4 + a_5 - a_6 + \ldots - a_{2n}.$$

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So

$$s_{2n+2} - s_{2n} = a_{2n+1} - a_{2n+2} \geq 0.$$

Therefore the sequence (s_{2n}) is increasing.

Proof continued.

Moreover, since (a_n) is positive and decreasing, we have that

$$s_{2n} = a_1 - a_2 + a_3 - a_4 + a_5 - a_6 + \ldots - a_{2n-2} + a_{2n-1} - a_{2n}$$

Proof continued.

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$$
\begin{aligned}
s_{2n} &= a_1 - a_2 + a_3 - a_4 + a_5 - a_6 + \ldots - a_{2n-2} + a_{2n-1} - a_{2n} \\
&= a_1 - (a_2 - a_3) - (a_4 - a_5) - \ldots - (a_{2n-2} - a_{2n-1}) - a_{2n}
\end{aligned}
$$

Proof continued.

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$$\begin{aligned} s_{2n} &= a_1 - a_2 + a_3 - a_4 + a_5 - a_6 + \ldots - a_{2n-2} + a_{2n-1} - a_{2n} \\ &= a_1 - (a_2 - a_3) - (a_4 - a_5) - \ldots - (a_{2n-2} - a_{2n-1}) - a_{2n} \\ &\leq a_1 - 0 - 0 - \ldots - 0 - 0 = a_1. \end{aligned}$$

$$\begin{aligned} a_2 &\geq a_3 \\ a_2 - a_3 &\geq 0 \\ -(a_2 - a_3) &\leq -0 \end{aligned}$$

Proof continued.

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So (s_{2n}) is bounded above.

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So (s_{2n}) is bounded above.

$$\begin{cases} (s_{2n}) \text{ is increasing} \\ (s_{2n}) \text{ is bounded above} \end{cases} \implies (s_{2n}) \text{ is convergent.}$$

Proof continued.

Let $s = \lim_{n\to\infty} s_{2n}$. Then $s_{2n} \to s$ as $n \to \infty$. Furthermore

$$\begin{aligned} s_{2n+1} &= a_1 - a_2 + a_3 - a_4 + a_5 - a_6 + \ldots - a_{2n} + a_{2n+1} \\ &= s_{2n} + a_{2n+1} \to s + 0 = s \end{aligned}$$

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It follows (you prove) that $s_n \to s$ as $n \to \infty$ also.

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It follows (you prove) that $s_n \to s$ as $n \to \infty$ also.

Therefore $\sum_{n=1}^{\infty} (-1)^{n+1} a_n$ is convergent. □

Remark

If $a_n \not\to 0$ as $n \to \infty$, then $(-1)^{n+1}a_n \not\to 0$ as $n \to \infty$ and

$$\sum_{n=1}^{\infty}(-1)^{n+1}a_n$$

diverges by the Divergence Test.

Example

Let $a_n = \sin \frac{1}{n}$ for all $n \in \mathbb{N}$. Note that $0 < \frac{1}{n+1} < \frac{1}{n} \leq 1 < \frac{\pi}{2}$ for all $n \in \mathbb{N}$. Thus

$$a_n = \sin \frac{1}{n} > 0$$

and

$$a_{n+1} = \sin \frac{1}{n+1} < \sin \frac{1}{n} = a_n.$$

So (a_n) is a decreasing sequence of positive numbers. Moreover, $a_n = \sin \frac{1}{n} \to 0$ as $n \to \infty$.

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$$\sum_{n=1}^{\infty}(-1)^{n+1}\sin\frac{1}{n}$$

converges by the Alternating Series Test.

Example

Since $a_n = \cos\frac{1}{n} \to 1 \neq 0$ as $n \to \infty$, it follows that

$$\sum_{n=1}^{\infty} (-1)^{n+1} \cos\frac{1}{n}$$

diverges by the Divergence Test.

Example

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Let $a_n = \sin^2\dfrac{1}{n}$. Then $a_n > a_{n+1} > 0\ \forall n$ and $a_n \to 0$ as $n \to \infty$.

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Let $a_n = \frac{10n}{n^2+16}$ and $f(x) = \frac{10x}{x^2+16}$. Note that $a_n > 0$ for all n and $a_n \to 0$ as $n \to \infty$.

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$$f'(x) = \frac{10(16-x^2)}{(x^2+16)^2} \leq 0.$$

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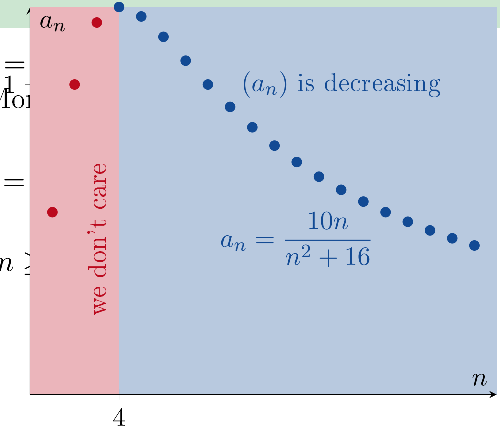

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So (a_n) is decreasing for $n \geq 4$.

Therefore $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1} 10n}{n^2+16}$ converges.

Definition

If a series $\displaystyle\sum_{k=1}^{\infty} a_k$ is convergent, but is not absolutely convergent, then we say that it is *conditionally convergent*.

(Equivalently, we can say that the series *converges conditionally*.)

Example

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \text{ is conditionally convergent.}$$

(because $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$ diverges)

Example

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2} \text{ is absolutely convergent.}$$

Remark

We can rearrange an absolutely convergent series without changing its sum.

This is **not true** for conditionally convergent series.

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For example, it is possible (see page 99 of Mary Hart's book) to show that

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \ldots = \ln 2$$

and

$$\underbrace{1 + \tfrac{1}{3} + \tfrac{1}{5} + \tfrac{1}{7}}_{\text{4 positive terms}} \; \underbrace{-\tfrac{1}{2}}_{\substack{\text{1}\\\text{negative}\\\text{term}}} \; \underbrace{+\tfrac{1}{9} + \tfrac{1}{11} + \tfrac{1}{13} + \tfrac{1}{15}}_{\text{4 positive terms}} \; \underbrace{-\tfrac{1}{4}}_{\substack{\text{1}\\\text{negative}\\\text{term}}} \; + \tfrac{1}{17} + \tfrac{1}{19} + \ldots = \ln 4.$$

9.7 Power Series

Let $(a_n)_{n=0}^{\infty}$ be a sequence. Then

$$\sum_{n=0}^{\infty} a_n(x-c)^n$$

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"infinite polynomials"

Example

The following are power series:

- $1 + 2x + 3x^2 + 4x^3 + 5x^4 + \ldots = \sum_{n=1}^{\infty} nx^{n-1} = \sum_{n=0}^{\infty}(n+1)x^n;$
- $1 + x + x^2 + x^3 + x^4 + \ldots = \sum_{n=0}^{\infty} x^n;$
- $1 + x + 2x^2 + 6x^3 + 24x^4 + \ldots = \sum_{n=0}^{\infty} n!x^n;$
- $1 + x^2 + x^4 + x^6 + x^8 + \ldots = \sum_{n=0}^{\infty} \left(\frac{1+(-1)^2}{2}\right) x^n;$
- $1 + (x-2) + (x-2)^2 + (x-2)^3 + (x-2)^4 + \ldots = \sum_{n=0}^{\infty}(x-2)^n.$

Definition

The constant c is called the *centre of expansion* of the power series $\displaystyle\sum_{n=0}^{\infty} a_n(x-c)^n$.

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Remark

To make things easier, we start by looking at power series with $c = 0$. So first we will consider

$$\sum_{n=0}^{\infty} a_n x^n,$$

then we will discuss power series with $c \neq 0$ later.

Remark

We wish to answer the following three questions about power series:

- How does a power series behave?
- Does this depend on x?
- Is it possible for a power series to converge for some x, but diverge for other x?

Example

Recall that

$$\sum_{n=0}^{\infty} x^n \begin{cases} \text{converges if } |x| < 1 \\ \text{diverges if } |x| \geq 1. \end{cases}$$

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If $-1 < x < 1$, then

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + x^4 + x^5 + x^6 + \dots.$$

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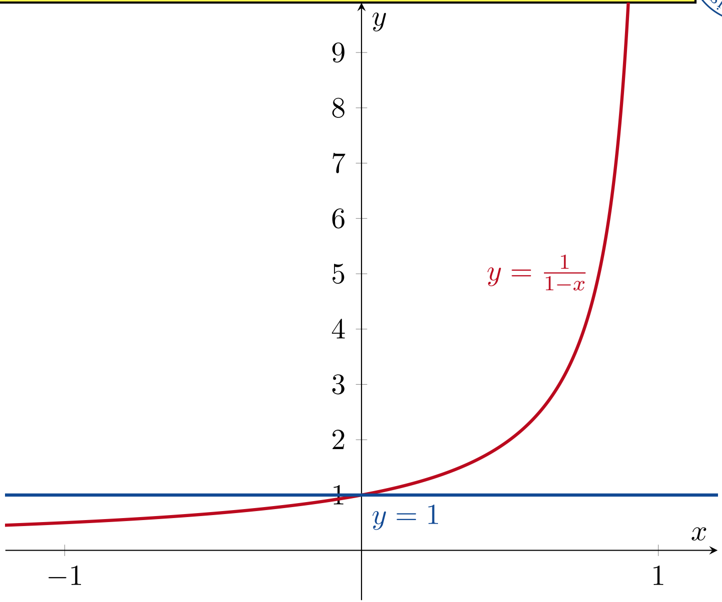

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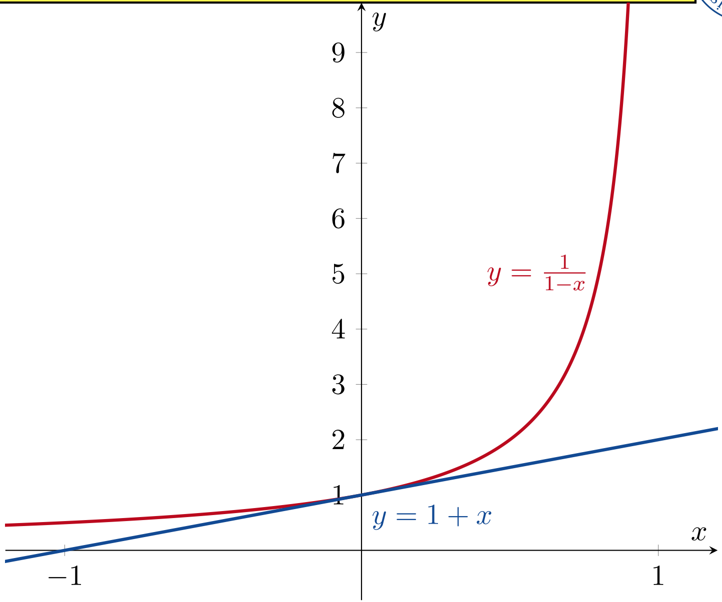

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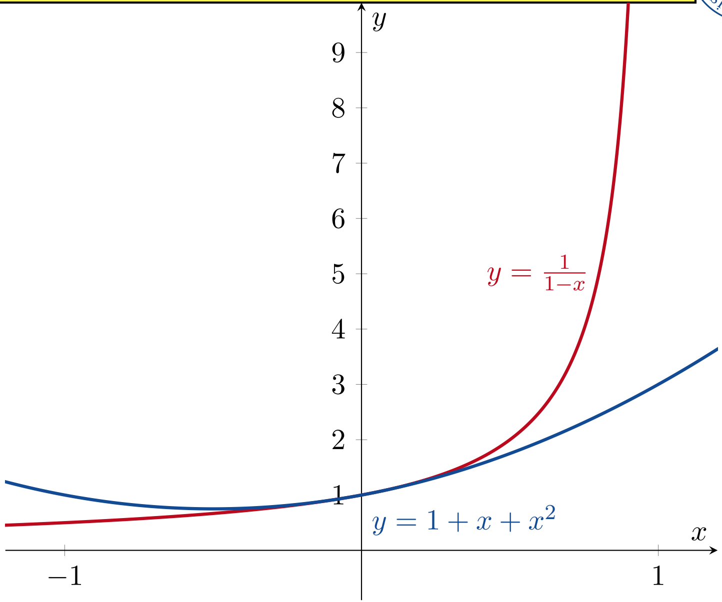

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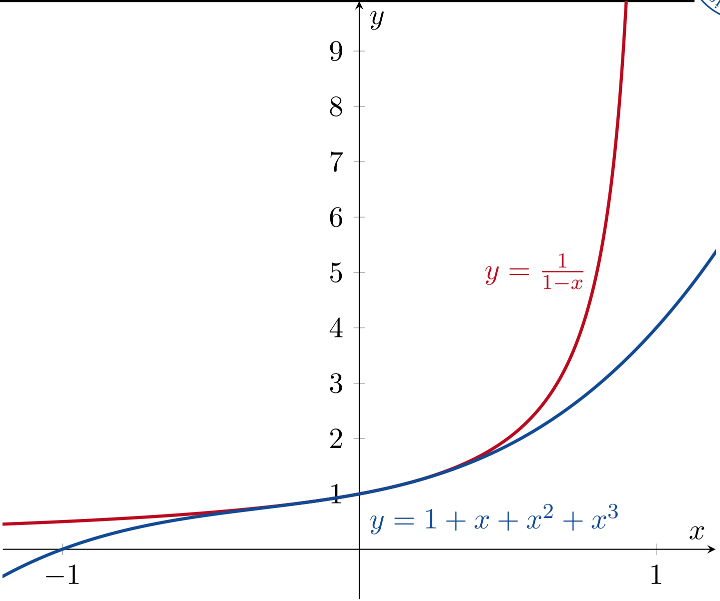

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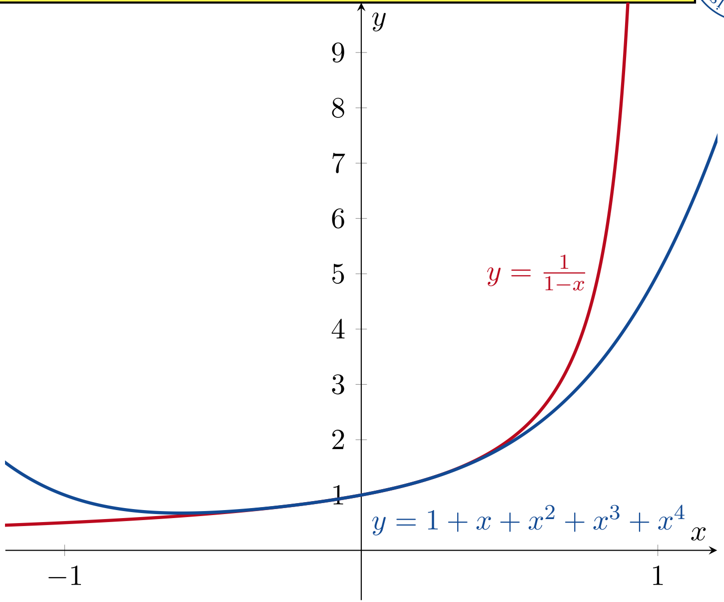

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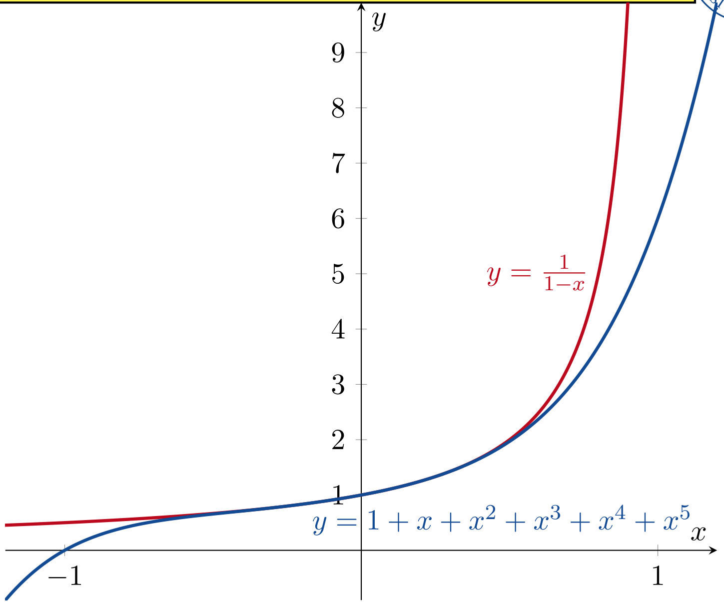

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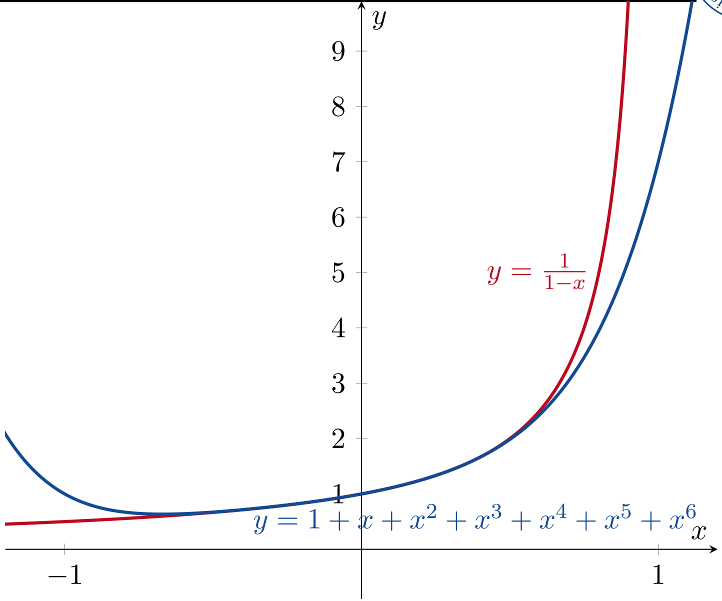

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- If $x = 0$, then $\sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + 0 + 0 + 0 + 0 + 0 + \dots$ is convergent.
- Suppose that $x \neq 0$. Let $b_n := \frac{x^n}{n!}$. Then

$$\left|\frac{b_{n+1}}{b_n}\right| = \left|\frac{x^{n+1}}{(n+1)!}\frac{n!}{x^n}\right| = \frac{|x|}{n+1} \to 0$$

as $n \to \infty$. Hence $\sum_{n=0}^{\infty} b_n$ is absolutely convergent by the Ratio Test v2.

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Therefore $\displaystyle\sum_{n=0}^{\infty} \frac{x^n}{n!}$ converges $\forall x \in \mathbb{R}$.

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Therefore $\displaystyle\sum_{n=0}^{\infty} \frac{x^n}{n!} \begin{cases} \text{converges if } x = 0 \\ \text{diverges if } x \neq 0. \end{cases}$

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  Then
  $$\left|\frac{b_{n+1}}{b_n}\right| = \frac{(n+1)\left|x\right|^n}{n\left|x\right|^{n-1}} = \left(1 + \tfrac{1}{n}\right)\left|x\right| \to \left|x\right|$$
  as $n \to \infty$.

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  By the Ratio Test v2,
  $$\sum_{n=1}^{\infty} nx^{n-1} \begin{cases} \text{converges if } 0 < |x| < 1 \\ \text{diverges if } |x| > 1. \end{cases}$$

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Consider $\sum_{n=1}^{\infty} nx^{n-1} = 1 + 2x + 3x^2 + 4x^3 + 5x^4 + \dots$

- Suppose that $|x| = 1$.

Example

Consider $\sum_{n=1}^{\infty} nx^{n-1} = 1 + 2x + 3x^2 + 4x^3 + 5x^4 + \dots$

- Suppose that $|x| = 1$. Then $\left|nx^{n-1}\right| = n$ which means that $nx^{n-1} \not\to 0$ as $n \to \infty$.

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  So $\sum_{n=1}^{\infty} b_n$ diverges if $|x| = 1$.

Therefore $\sum_{n=1}^{\infty} nx^{n-1} \begin{cases} \text{converges if } |x| < 1 \\ \text{diverges if } |x| \geq 1. \end{cases}$

You can read more examples in the textbook.

Remark

$\sum_{n=0}^{\infty} x^n$ converges for $|x| < 1$ and diverges for $|x| \geq 1$. If we differentiate each term (are we allowed to do this?), we get

$$0 + 1 + 2x + 3x^2 + 4x^3 + 5x^4 + \ldots = \sum_{n=1}^{\infty} nx^{n-1}$$

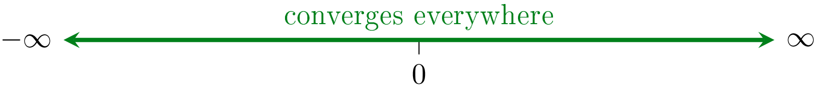
which also converges for $|x| < 1$ and diverges for $|x| \geq 1$. Interesting!

Theorem

A power series $\displaystyle\sum_{n=0}^{\infty} a_n x^n$ *satisfies one and only one of the following:*

1. *It converges absolutely* $\forall x$*;*
2. *It converges for* $x = 0$ *and diverges* $\forall x \neq 0$*; or*
3. $\exists R > 0$ *such that* $\sum_{n=0}^{\infty} a_n x^n \begin{cases} \text{converges if } |x| < R \\ \text{diverges if } |x| > R. \end{cases}$

1



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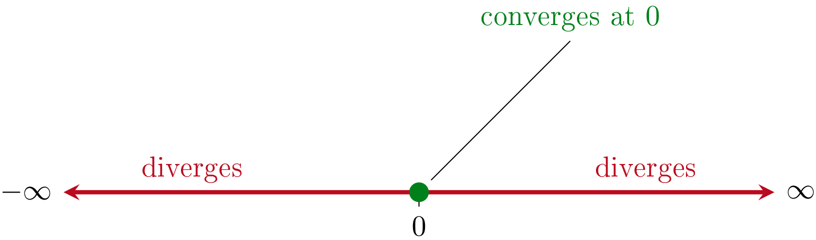
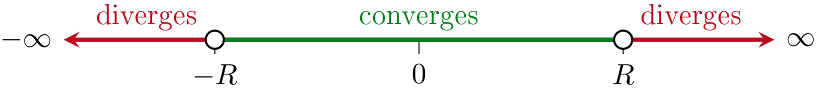


9.7 Power Series

3



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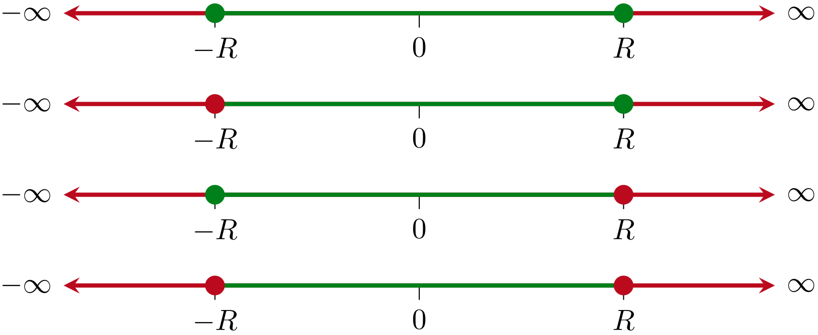
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$-R$
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Radius of Convergence

Definition

Let $R \in [0, \infty) \cup \{\infty\}$.

If $\displaystyle\sum_{n=0}^{\infty} a_n x^n$ converges $\forall\, |x| < R$ and diverges $\forall\, |x| > R$, then R is called the *radius of convergence* (yakınsaklık yarıçapı) of the power series $\displaystyle\sum_{n=0}^{\infty} a_n x^n$.

Definition

If $R = \infty$, then we say that $\sum_{n=0}^{\infty} a_n x^n$ has *infinite radius of convergence.* (This means that $\sum_{n=0}^{\infty} a_n x^n$ converges $\forall x$.)

Definition

If $R = 0$, then we say that $\sum_{n=0}^{\infty} a_n x^n$ has *zero radius of convergence.* (This means that $\sum_{n=0}^{\infty} a_n x^n$ converges if $x = 0$ and diverges $\forall x \neq 0$.)

Definition

If $R > 0$ or $R = \infty$, then the open interval $(-R, R)$ is called the *open interval of convergence* of $\sum_{n=0}^{\infty} a_n x^n$.

Is there an easy way to find R?

Is there an easy way to find R?

Theorem

Suppose that

$$\left|\frac{a_n}{a_{n+1}}\right| \to R \in \mathbb{R} \cup \{\infty\}$$

as $n \to \infty$. *Then* $\sum_{n=0}^{\infty} a_n x^n$ *has radius of convergence* R.

Remark

A power series *always* has a radius of convergence, even if $\lim_{n\to\infty}\left|\frac{a_n}{a_{n+1}}\right|$ doesn't exist.

This theorem just gives us an easy way to find R, if this limit does exist.

If the limit does not exist, then we need to use a different method to find R.

Remark

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This theorem just gives us an easy way to find R, if this limit does exist.

If the limit does not exist, then we need to use a different method to find R.

Remark

Never, never, never forget to use $|\cdot|$ when you use this theorem.

Remark

The Ratio Test v2 uses $\left|\dfrac{a_{n+1}}{a_n}\right|$, but this theorem uses $\left|\dfrac{a_n}{a_{n+1}}\right|$.

Don't get these mixed up.

We have seen that $\exists\ R$ such that

$$\sum_{n=0}^{\infty} a_n x^n \begin{cases} \text{converges if } |x| < R \\ \text{diverges if } |x| > R. \end{cases}$$

Suppose that $0 < R < \infty$.

We have seen that $\exists\ R$ such that

$$\sum_{n=0}^{\infty} a_n x^n \begin{cases} \text{converges if } |x| < R \\ \text{diverges if } |x| > R. \end{cases}$$

Suppose that $0 < R < \infty$.

What happens when $|x| = R$?

Example

Consider $\displaystyle\sum_{n=0}^{\infty} x^n$.

Example

Consider $\sum_{n=0}^{\infty} x^n$.

This is a power series with $a_n = 1 \ \forall n$. Since

$$\left| \frac{a_n}{a_{n+1}} \right| = \frac{1}{1} = 1,$$

the radius of convergence is $R = 1$.

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This is a power series with $a_n = 1 \ \forall n$. Since

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$$\sum_{n=0}^{\infty} a_n x^n \begin{cases} \text{converges if } |x| < 1 \\ \text{diverges if } |x| > 1. \end{cases}$$

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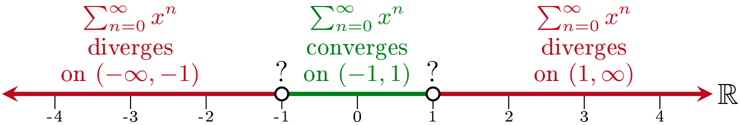
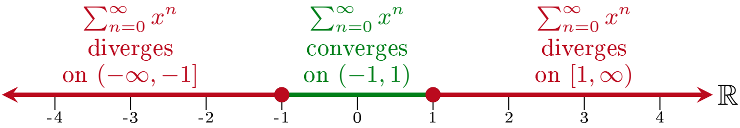


9.7 Power Series

Previously we saw that $\sum_{n=0}^{\infty} x^n$ also diverges for $|x| = 1$.



For this power series, we have divergence when $x = \pm R$.

Example

Now consider $\displaystyle\sum_{n=0}^{\infty} \frac{x^n}{n+1}$.

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For this power series, $a_n = \frac{1}{n+1}$ $\forall n \in \mathbb{N} \cup \{0\}$ and

$$\left|\frac{a_n}{a_{n+1}}\right| = \frac{n+2}{n+1} = \frac{1+\frac{2}{n}}{1+\frac{1}{n}} \to 1$$

as $n \to \infty$. Thus, the radius of convergence is $R = 1$ again.

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as $n \to \infty$. Thus, the radius of convergence is $R = 1$ again.

This means that $\displaystyle\sum_{n=0}^{\infty} \frac{x^n}{n+1} \begin{cases} \text{converges if } |x| < 1 \\ \text{diverges if } |x| > 1. \end{cases}$

When $x = 1$, the series is

$$\sum_{n=0}^{\infty} \frac{1^n}{n+1} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \ldots = \sum_{n=1}^{\infty} \frac{1}{n}$$

which we know diverges.

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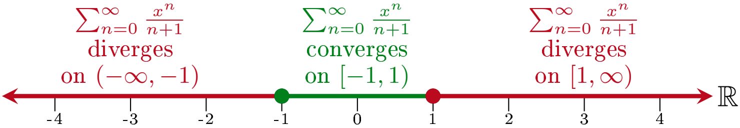
which we know diverges.

When $x = -1$, the series is

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{n+1} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \ldots$$

which we know converges.



For this power series, we have convergence when $x = -R$ and divergence when $x = R$.

Example

Consider $\displaystyle\sum_{n=0}^{\infty} \frac{x^n}{(n+1)^2}$.

Example

Consider $\displaystyle\sum_{n=0}^{\infty} \frac{x^n}{(n+1)^2}$.

For this power series, $a_n = \frac{1}{(n+1)^2}$ $\forall n \in \mathbb{N} \cup \{0\}$ and

$$\left|\frac{a_n}{a_{n+1}}\right| = \frac{(n+2)^2}{(n+1)^2} \to 1$$

as $n \to \infty$. Thus, the radius of convergence is $R = 1$ again.

When $|x| = R = 1$,

$$\sum_{n=0}^{\infty} \frac{|x|^n}{(n+1)^2} = \sum_{n=0}^{\infty} \frac{1}{(n+1)^2} = \sum_{n=1}^{\infty} \frac{1}{n^2}$$

which converges.

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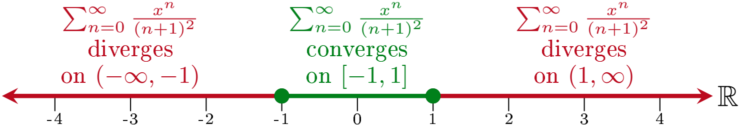
which converges.



For this power series, we have convergence when $x = \pm R$.

Remark

The previous three examples show that when $|x| = R \in (0, \infty)$, we can have divergence, conditional convergence or absolute convergence.

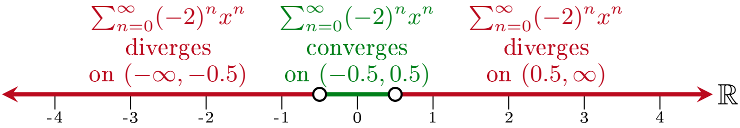
Example

Consider $\displaystyle\sum_{n=0}^{\infty}(-2)^n x^n$.

Since

$$\left|\frac{a_n}{a_{n+1}}\right| = \left|\frac{(-2)^n}{(-2)^{n+1}}\right| = \frac{1}{2},$$

this power series has radius of convergence $R = \frac{1}{2}$. The open interval of convergence is $(-\frac{1}{2}, \frac{1}{2})$.



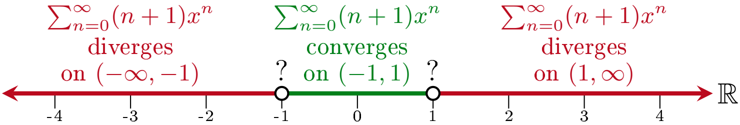
Example

Consider $\displaystyle\sum_{n=0}^{\infty}(n+1)x^n$.

Since

$$\left|\frac{a_n}{a_{n+1}}\right| = \frac{n+1}{n+2} \to 1$$

as $n \to \infty$, this power series has radius of convergence $R = 1$. The open interval of convergence is $(-1, 1)$.



$$\boxed{\cosh x = \frac{e^x + e^{-x}}{2}}$$

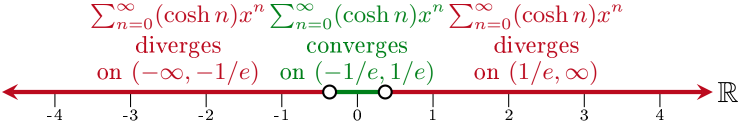
Example

Consider $\displaystyle\sum_{n=0}^{\infty} (\cosh n) x^n$.

Since

$$\left|\frac{a_n}{a_{n+1}}\right| = \left|\frac{\cosh n}{\cosh(n+1)}\right| = \frac{e^n + e^{-n}}{e^{n+1} + e^{-n-1}} = \frac{1 + e^{-2n}}{e + e^{-2n-1}} \to \frac{1+0}{e+1} = \frac{1}{e}$$

as $n \to \infty$, this power series has radius of convergence $R = \frac{1}{e}$. The open interval of convergence is $(-\frac{1}{e}, \frac{1}{e})$.

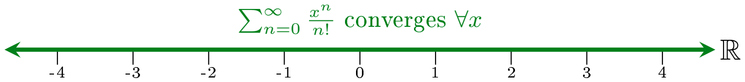


Example

For the power series $\sum_{n=0}^{\infty} \frac{x^n}{n!}$, we have that

$$R = \lim_{n\to\infty} \left| \frac{a_n}{a_{n+1}} \right| = \infty$$

as $n \to \infty$. The radius of convergence $R = \infty$. The open interval of convergence is $(-\infty, \infty)$.



Term-by-Term Differentiation and Integration

A power series is a function. So can we differentiate it? Can we integrate it?

Theorem

Let $\displaystyle\sum_{n=0}^{\infty} a_n x^n$ *be a power series with radius of convergence* $R > 0$.

If $|x| < R$, *then*

$$\frac{d}{dx}\left(\sum_{n=0}^{\infty} a_n x^n\right) = \sum_{n=0}^{\infty}\left(\frac{d}{dx} a_n x^n\right)$$

and

$$\int \left(\sum_{n=0}^{\infty} a_n x^n\right) dx = \sum_{n=0}^{\infty}\left(\int a_n x^n \, dx\right).$$

EXAMPLE 4 Find series for $f'(x)$ and $f''(x)$ if

$$f(x) = \frac{1}{1-x} = 1 + x + x^2 + x^3 + x^4 + \cdots + x^n + \cdots$$

$$= \sum_{n=0}^{\infty} x^n, \qquad -1 < x < 1.$$

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$$= \sum_{n=0}^{\infty} x^n, \qquad -1 < x < 1.$$

Solution We differentiate the power series on the right term by term:

$$f'(x) = \frac{1}{(1-x)^2} = 1 + 2x + 3x^2 + 4x^3 + \cdots + nx^{n-1} + \cdots$$

$$= \sum_{n=1}^{\infty} nx^{n-1}, \qquad -1 < x < 1;$$

$$f''(x) = \frac{2}{(1-x)^3} = 2 + 6x + 12x^2 + \cdots + n(n-1)x^{n-2} + \cdots$$

$$= \sum_{n=2}^{\infty} n(n-1)x^{n-2}, \qquad -1 < x < 1. \qquad \blacksquare$$

EXAMPLE 6 The series

$$\frac{1}{1+t} = 1 - t + t^2 - t^3 + \cdots$$

converges on the open interval $-1 < t < 1$. Therefore,

$$\begin{aligned}\ln(1+x) &= \int_0^x \frac{1}{1+t}\,dt = t - \frac{t^2}{2} + \frac{t^3}{3} - \frac{t^4}{4} + \cdots \Big]_0^x \\ &= x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots\end{aligned}$$

or

$$\ln(1+x) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}\,x^n}{n}, \qquad -1 < x < 1.$$

Power Series with Centre of Expansion c

The results that we have proved for the power series $\displaystyle\sum_{n=0}^{\infty} a_n x^n$ are also true for the power series $\displaystyle\sum_{n=0}^{\infty} a_n (x - c)^n$.

Example

Recall that $\sum_{n=0}^{\infty} x^n$ has radius of convergence $R = 1$. Therefore $\sum_{n=0}^{\infty} (x-c)^n$ also has radius of convergence $R = 1$.

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$$\sum_{n=0}^{\infty} x^n \begin{cases} \text{converges absolutely } \forall\,|x| < 1 \\ \text{diverges } \forall\,|x| > 1 \end{cases}$$

we have that

$$\sum_{n=0}^{\infty}(x-c)^n \begin{cases} \text{converges absolutely } \forall\,|x-c| < 1 \\ \text{diverges } \forall\,|x-c| > 1. \end{cases}$$

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Recall that $\sum_{n=0}^{\infty} x^n$ has radius of convergence $R = 1$. Therefore $\sum_{n=0}^{\infty} (x-c)^n$ also has radius of convergence $R = 1$. Since

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$$\sum_{n=0}^{\infty} (x-c)^n \begin{cases} \text{converges absolutely } \forall\, |x-c| < 1 \\ \text{diverges } \forall\, |x-c| > 1. \end{cases}$$

The open interval of convergence for $\displaystyle\sum_{n=0}^{\infty} (x-c)^n$ is $(c-1, c+1)$.

Example

Since $\sum_{n=0}^{\infty} \frac{x^n}{n!}$ has radius of convergence $R = \infty$, it follows that $\sum_{n=0}^{\infty} \frac{(x-c)^n}{n!}$ converges absolutely $\forall x$.

The radius of convergence of $\sum_{n=0}^{\infty} \frac{(x-c)^n}{n!}$ is $R = \infty$ and the open interval of convergence is $(-\infty, \infty)$.

Example

Recall that $\sum_{n=0}^{\infty} \frac{x^n}{(n+1)^2}$ has radius of convergence $R = 1$. So

$$\sum_{n=0}^{\infty} \frac{(x-c)^n}{(n+1)^2} \begin{cases} \text{converges absolutely } \forall \, |x-c| < 1 \\ \text{diverges } \forall \, |x-c| > 1. \end{cases}$$

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The open interval of convergence of $\sum_{n=0}^{\infty} \frac{(x-c)^n}{(n+1)^2}$ is $(c-1, c+1)$.

If $x \in (c-1, c+1)$, then

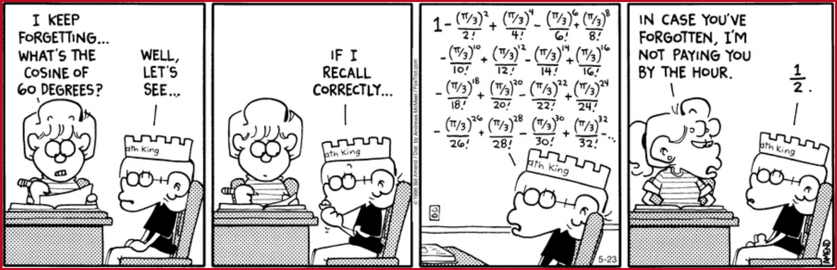
$$\frac{d}{dx}\left(\sum_{n=0}^{\infty} \frac{(x-c)^n}{(n+1)^2}\right) = \sum_{n=0}^{\infty} \frac{d}{dx}\left(\frac{(x-c)^n}{(n+1)^2}\right)$$

and

$$\int \left(\sum_{n=0}^{\infty} \frac{(x-c)^n}{(n+1)^2}\right) dx = \sum_{n=0}^{\infty} \left(\int \frac{(x-c)^n}{(n+1)^2}\, dx\right).$$

Break

We will continue at 2pm



Taylor and Maclaurin Series

Recall Rolle's Theorem and the Mean Value Theorem from MATH113 Mathematics I (see chapter 4 of Thomas' Calculus):

Michel Rolle

BORN
21 April 1652

DECEICED
8 November 1719

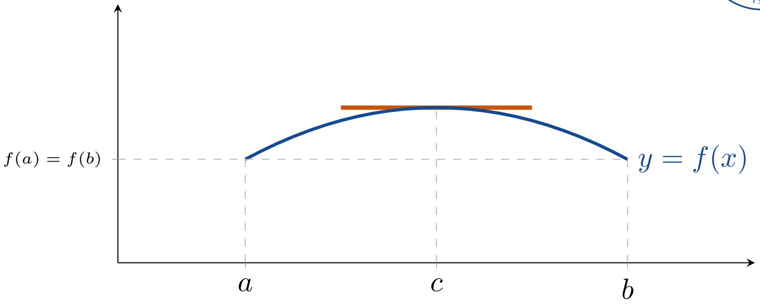
NATIONALITY
French

Theorem (Rolle's Theorem)

Suppose that

1. $f : [a, b] \to \mathbb{R}$ *is continuous;*
2. f *is differentiable on* (a, b)*; and*
3. $f(a) = f(b)$.

Then $\exists\ c \in (a, b)$ *such that* $f'(c) = 0$.



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Augustin-Louis Cauchy

BORN
21 August 1789

DECEICED
23 May 1857

NATIONALITY
French

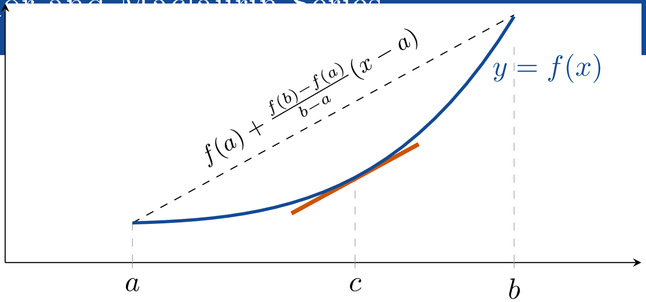
Theorem (The Mean Value Theorem)

Suppose that

1. $f : [a, b] \to \mathbb{R}$ *is continuous; and*
2. f *is differentiable on* (a, b).

Then $\exists\ c \in (a, b)$ *such that*

$$\frac{f(b) - f(a)}{b - a} = f'(c).$$



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Suppose that

1. $f : [a, b] \to \mathbb{R}$ *is continuous; and*
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Then $\exists\ c \in (a, b)$ *such that*

$$\frac{f(b) - f(a)}{b - a} = f'(c).$$

Remark

In other words, $\exists\ c$ such that $a < c < b$ and

$$f(b) = f(a) + f'(c)(b - a).$$

Remark

In other words, $\exists\ c$ such that $a < c < b$ and

$$f(b) = f(a) + f'(c)(b - a).$$

Taylor's Theorem takes this formula and expands it to more terms.

Brook Taylor

BORN
18 August 1685

DECEICED
29 December 1731

NATIONALITY
British

Theorem (Taylor's Theorem)

Let $f : [a, b] \to \mathbb{R}$ *be a function. Suppose that*

1. $f, f', f'', f''', \ldots, f^{(n-1)}$ *exist and are continuous on* $[a, b]$*; and*
2. $f^{(n)}$ *exists and is continuous on* (a, b)*.*

Theorem (Taylor's Theorem)

Let $f : [a, b] \to \mathbb{R}$ be a function. Suppose that

1. $f, f', f'', f''', \dots, f^{(n-1)}$ *exist and are continuous on* $[a, b]$*; and*
2. $f^{(n)}$ *exists and is continuous on* (a, b).

Then $\exists\ c \in (a, b)$ *such that*

$$f(b) = f(a) + f'(a)(b-a) + \frac{f''(a)}{2!}(b-a)^2 + \dots + \frac{f^{(n-1)}(a)}{(n-1)!}(b-a)^{n-1} + \frac{f^{(n)}(c)}{n!}(b-a)^n.$$

Replacing b by x we have

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \dots + \frac{f^{(n-1)}(a)}{(n-1)!}(x-a)^{n-1} + \frac{f^{(n)}(c)}{n!}(x-a)^n.$$

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This is called the *Taylor Series of $f(x)$ with centre a.*

Definition

The *Taylor Series of* $f(x)$ *with centre* a is the power series

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n.$$

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$$\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n.$$

$$R_n(c) = \frac{f^{(n)}(c)}{n!}(x-a)^n = f(x) - \sum_{k=0}^{n-1} \frac{f^{(k)}(a)}{k!}(x-a)^k$$

is called the *remainder term*.

Remark

The Taylor Series converges to $f(x)$ $\iff$ $R_n(c) \to 0$ as $n \to \infty$.

Example

Find the Taylor Series for e^x centred at 0.

Let $f(x) = e^x$. Then $\dfrac{d^k f}{dx^k}$ exists and is continuous $\forall x$ and $\forall k$.

Let $a = 0$ and $x \neq 0$. By Taylor's Theorem,

$$\begin{aligned} f(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \ldots \\ + \frac{f^{(n-1)}(0)}{(n-1)!}x^{n-1} + \frac{f^{(n)}(c)}{n!}x^n \end{aligned}$$

for some c between 0 and x $(0 < c < x$ or $x < c < 0)$.

$$f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \frac{f^{(4)}(a)}{4!}(x-a)^4 + \ldots$$

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$$
\begin{aligned}
e^x &= f(0)+f'(0)x+\frac{f''(0)}{2!}x^2+\frac{f'''(0)}{3!}x^3+\dots \\
&\quad +\frac{f^{(n-1)}(0)}{(n-1)!}x^{n-1}+\frac{f^{(n)}(c)}{n!}x^n \\
&= 1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\dots+\frac{x^{n-1}}{(n-1)!}+\frac{e^c}{n!}x^n.
\end{aligned}
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Because $\frac{d}{dx}e^x = e^x$, it is easy to see that $f^{(k)}(0) = 1 \; \forall k$. So

$$\begin{aligned} e^x &= f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \ldots \\ &\quad + \frac{f^{(n-1)}(0)}{(n-1)!}x^{n-1} + \frac{f^{(n)}(c)}{n!}x^n \\ &= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots + \frac{x^{n-1}}{(n-1)!} + \frac{e^c}{n!}x^n. \end{aligned}$$

Since $0 < |c| < |x|$,

$$0 \le \left| \frac{e^c}{n!}x^n \right| \le \frac{e^{|x|}\,|x|^n}{n!} \to 0$$

as $n \to \infty$. Hence the remainder term $R_c(x) = \frac{e^c}{n!}x^n$ tends to zero.

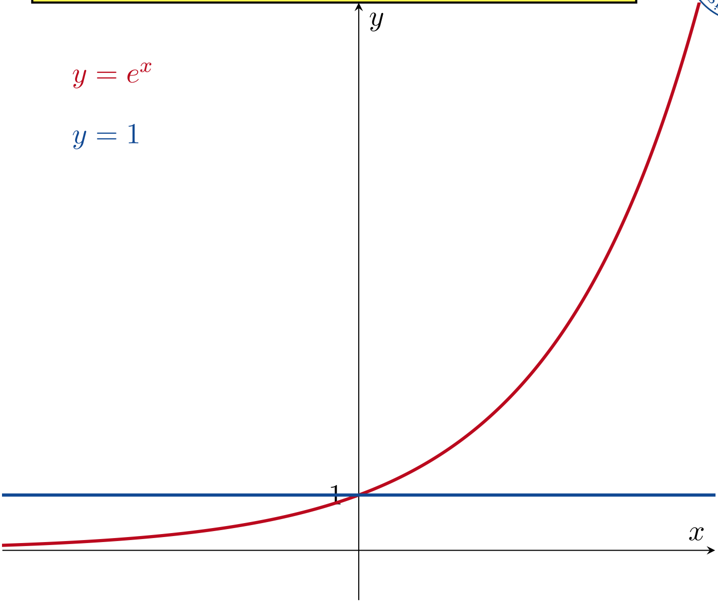
Therefore

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

is the Taylor Series of e^x with centre 0.

(Some people use this as the definition of e^x, then define $\ln x$ as the inverse of this.)

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \dots$$



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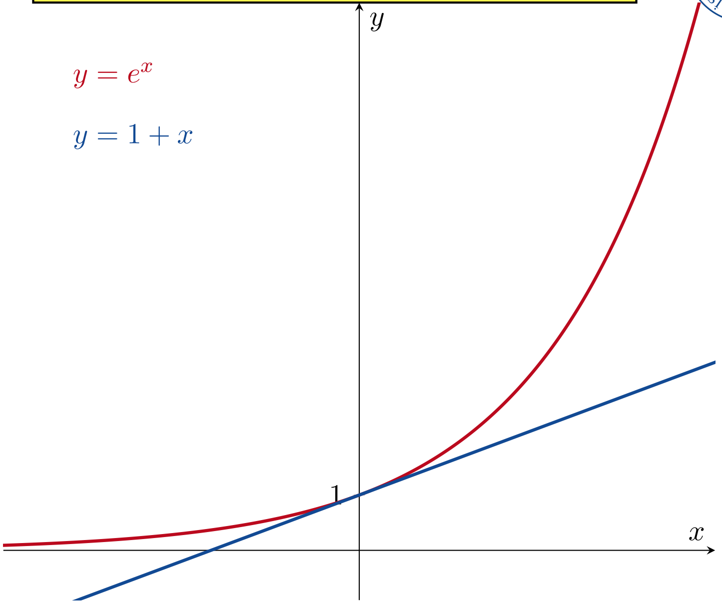

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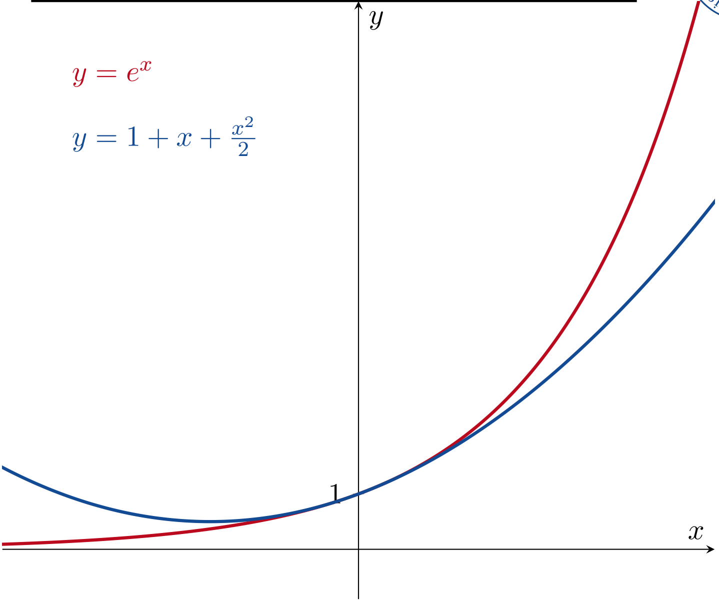

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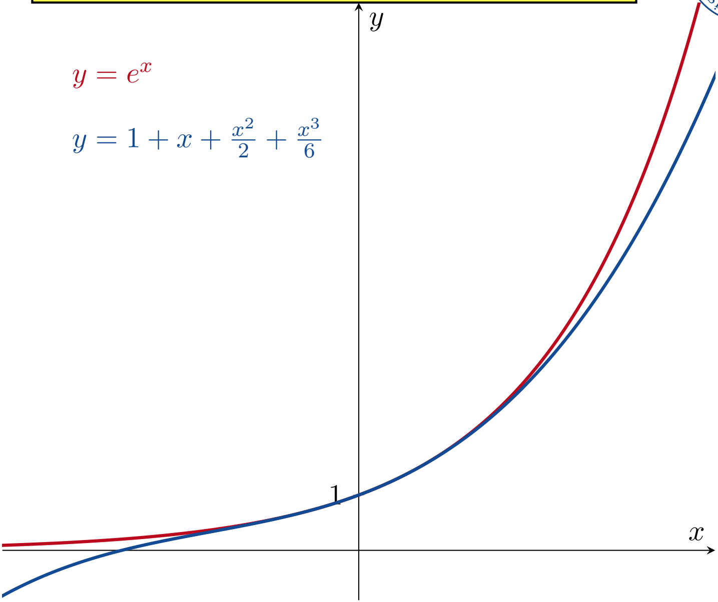

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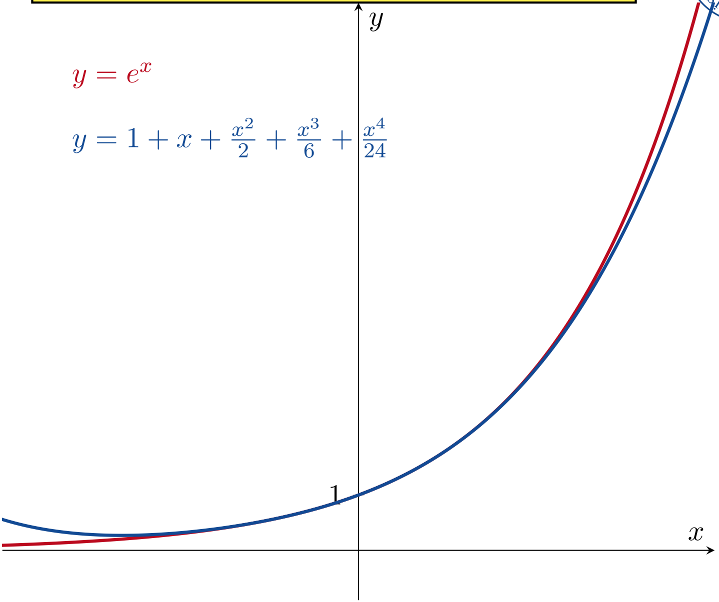

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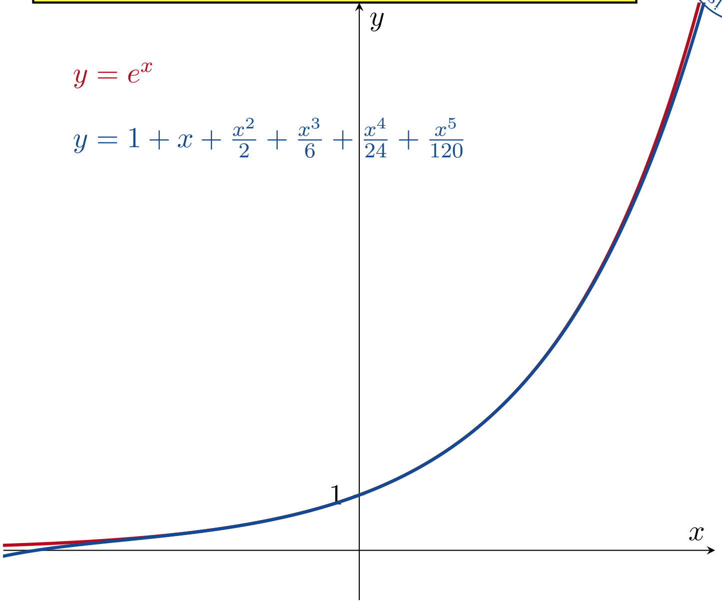

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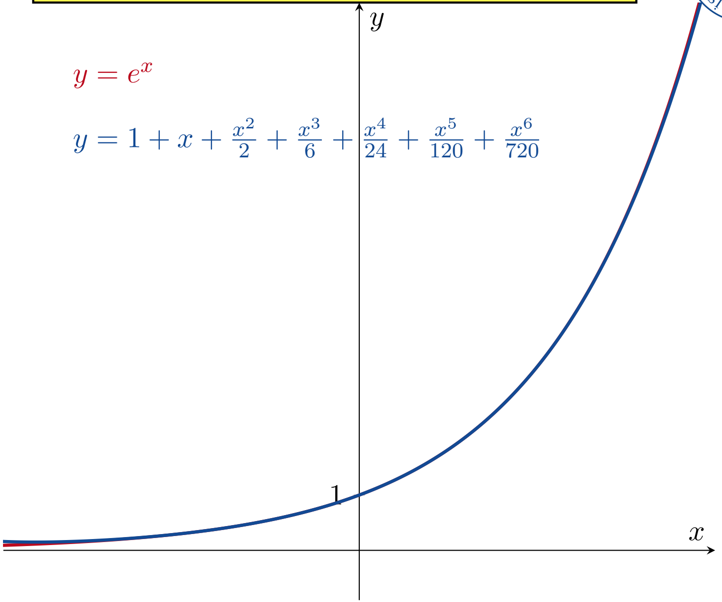

Example

Find the Taylor Series for $\sin x$ centred at 0.

Let $f(x) = \sin x$. Then $\frac{d^k f}{dx^k}$ exists and is continuous $\forall x$ and $\forall k$.
Let $a = 0$ and $x \neq 0$.

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Let $f(x) = \sin x$. Then $\frac{d^k f}{dx^k}$ exists and is continuous $\forall x$ and $\forall k$. Let $a = 0$ and $x \neq 0$.

We need

- to find $\frac{d^k f}{dx^k}$ for all k;
- to show that the remainder term tends to zero; and
- to calculate $\frac{d^k f}{dx^k}(0)$ for all k.

First note that

$$\frac{d^k}{dx^k} \sin x = \cos x \quad \text{or} \quad -\sin x \quad \text{or} \quad -\cos x \quad \text{or} \quad \sin x.$$

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Since $-1 \leq \sin c \leq 1$ and $-1 \leq \cos c \leq 1$ $\forall c$, we have that

$$0 \leq \left| \frac{f^{(n)}(c)}{n!} x^n \right| \leq \frac{|x|^n}{n!} \to 0$$

as $n \to \infty$.

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I leave it for you to check that

$$f^{(k)}(0) = \begin{cases} 1 & \text{if } k = 1, 5, 9, 13, \ldots \\ 0 & \text{if } k = 0, 2, 4, 6, 8, \ldots \\ -1 & \text{if } k = 3, 7, 11, 15, \ldots \end{cases}$$

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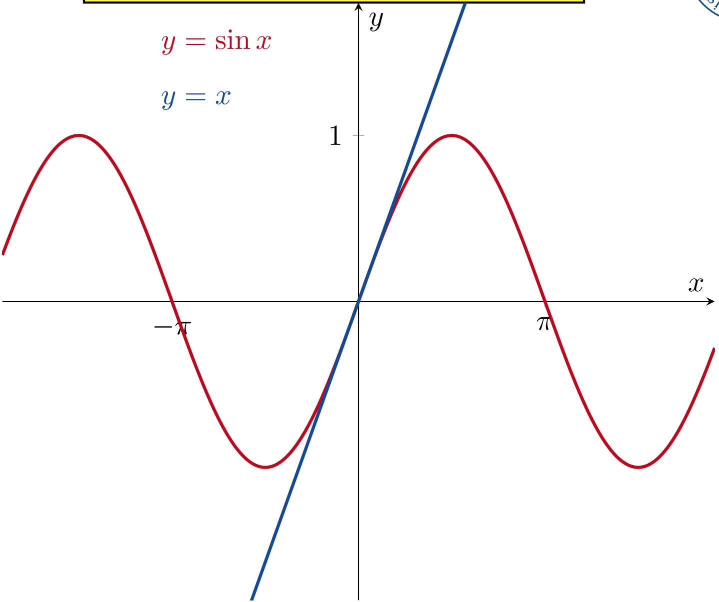
Therefore

$$\begin{aligned} \sin x &= f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \frac{f^{(4)}(0)}{4!}x^4 + \frac{f^{(5)}(0)}{5!}x^5 + \ldots \\ &= 0 + x + 0 - \frac{x^3}{3!} + 0 + \frac{x^5}{5!} + \ldots \\ &= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots \end{aligned}$$

is the Taylor Series of $\sin x$ with centre 0.

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots$$



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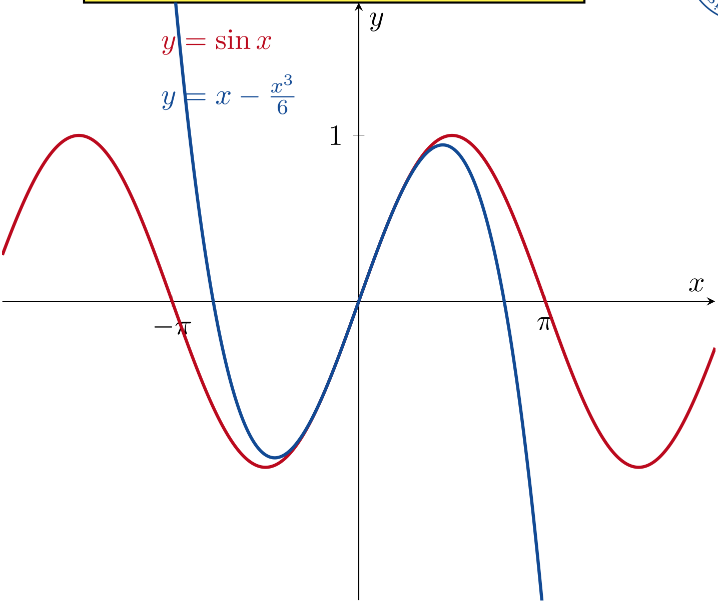

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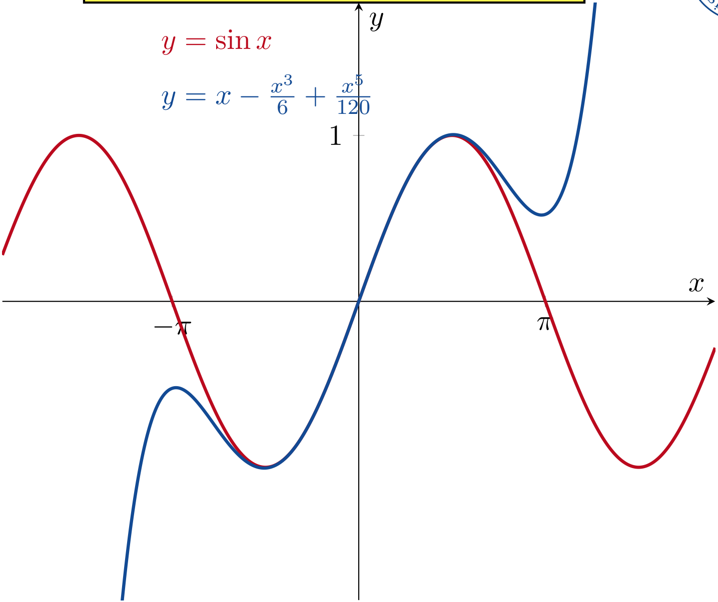

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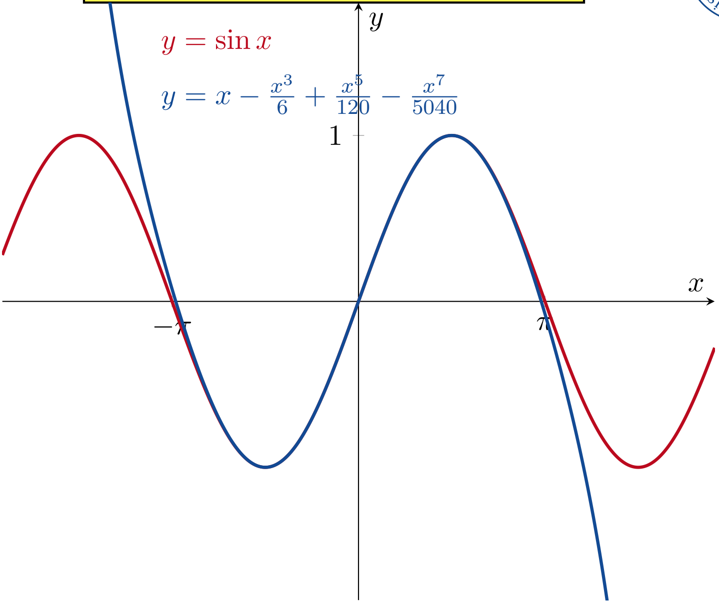

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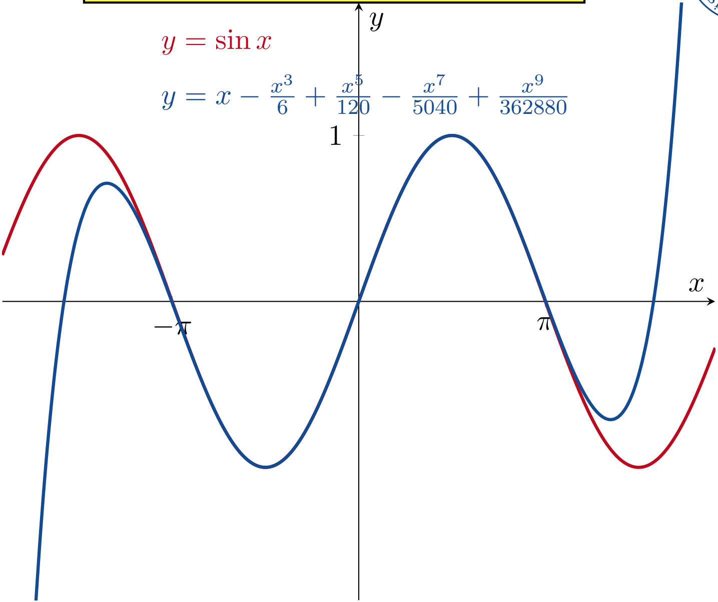

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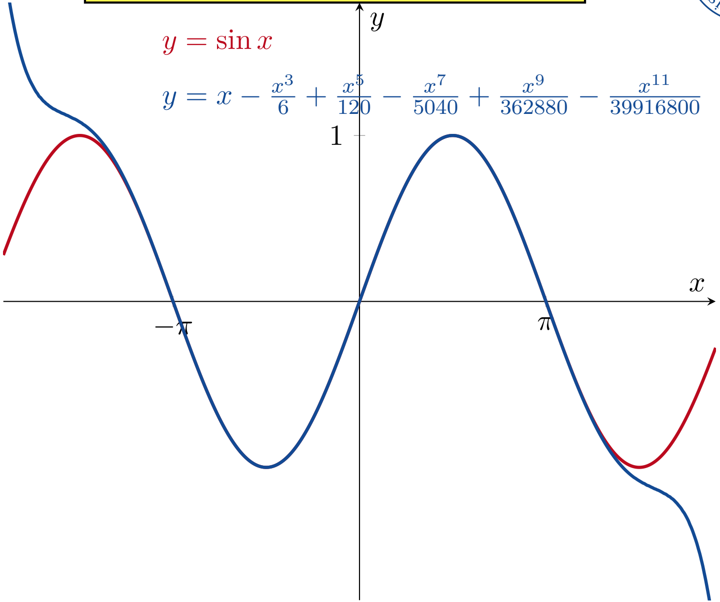

Example

Find the Taylor Series for $\ln(1+x)$ centred at 0.

Let $f : (-1, \infty) \to \mathbb{R}$ be defined by $f(x) = \ln(1+x)$. Let $a = 0$ and let $0 < x \leq 1$. Then f and its derivatives exist and are continuous on the closed interval $[0, x]$.

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I leave it for you to check that

$$f(0)=\ln 1=0$$
$$f^{(k)}(x)=\frac{(-1)^{k-1}(k-1)!}{(1+x)^n}$$
$$f^{(k)}(0)=(-1)^{k-1}(k-1)!$$

for $k\in\mathbb{N}$.

Since $0 < c < x \leq 1$, it follows that

$$|R_n(c)| = \left|\frac{f^{(n)}(c)x^n}{n!}\right| = \left|\frac{(n-1)!x^n}{(1+c)^n n!}\right| = \frac{x^n}{(1+c)^n n} \leq \frac{x^n}{n} \to 0$$

as $n \to \infty$.

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as $n \to \infty$.

Therefore, if $0 < x \leq 1$, then

$$\begin{aligned} \ln(1+x) &= f(0)+f'(0)x+\frac{f''(0)}{2!}x^2+\frac{f'''(0)}{3!}x^3+\frac{f^{(4)}(0)}{4!}x^4+\frac{f^{(5)}(0)}{5!}x^5+\dots \\ &= x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+\dots = \sum_{n=1}^{\infty}(-1)^{n-1}\frac{x^n}{n} \end{aligned}$$

is the Taylor Series of $\ln(1+x)$ centred at 0, on the interval $[0,1]$.

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as $n \to \infty$.

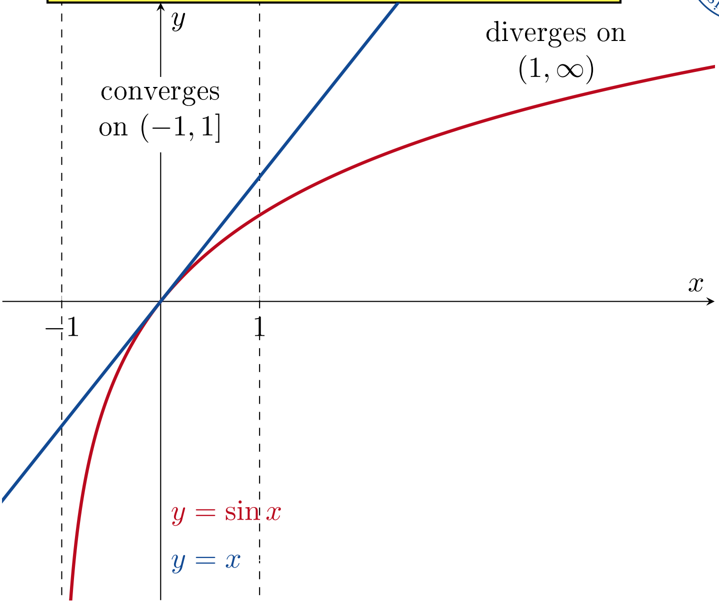
Therefore, if $0 < x \leq 1$, then

$$\begin{aligned} \ln(1+x) &= f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \frac{f^{(4)}(0)}{4!}x^4 + \frac{f^{(5)}(0)}{5!}x^5 + \ldots \\ &= x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{n} \end{aligned}$$

is the Taylor Series of $\ln(1+x)$ centred at 0, on the interval $[0, 1]$.

If can be proved (more difficult) that this series also converges to $\ln(1+x)$ $\forall x \in (-1, 0)$. If $x > 1$, then the series diverges.

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots$$



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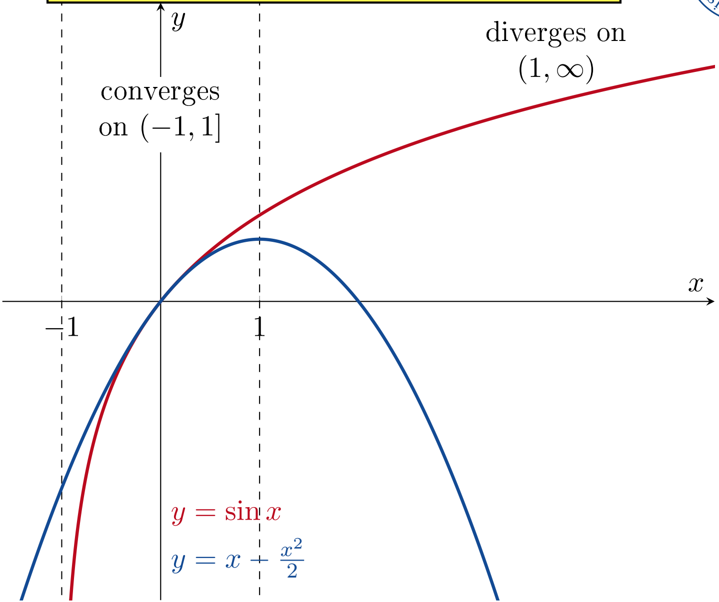

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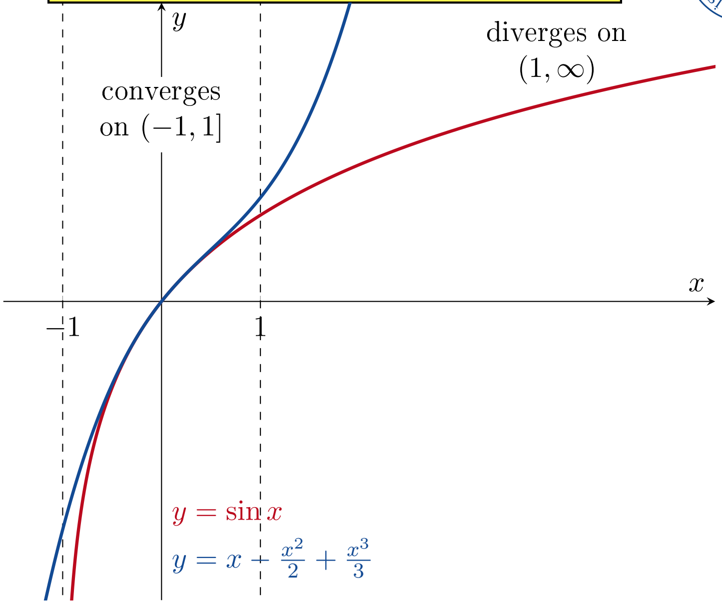

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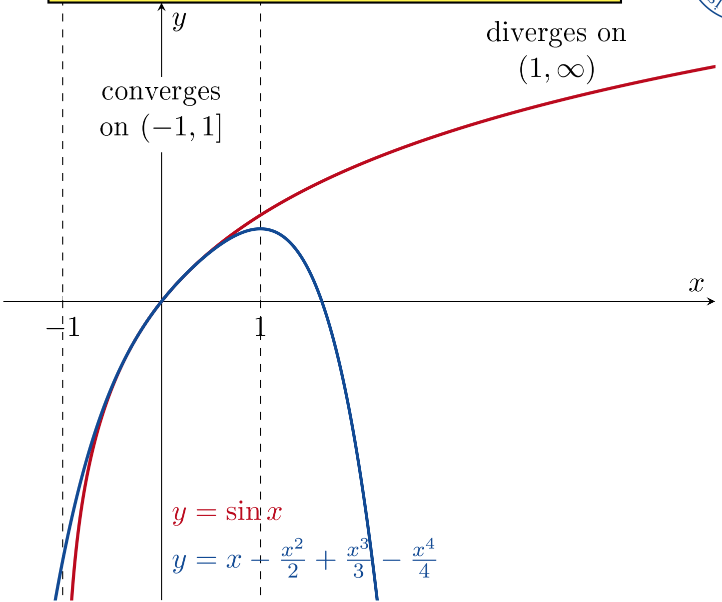

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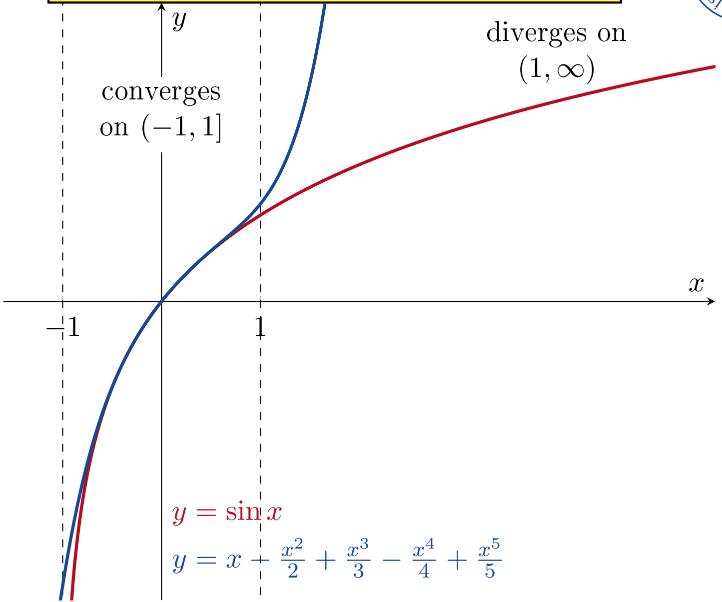

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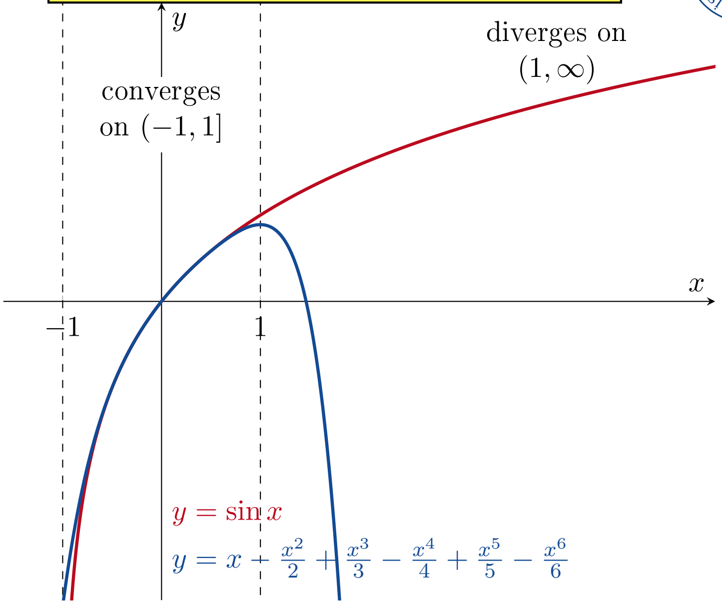

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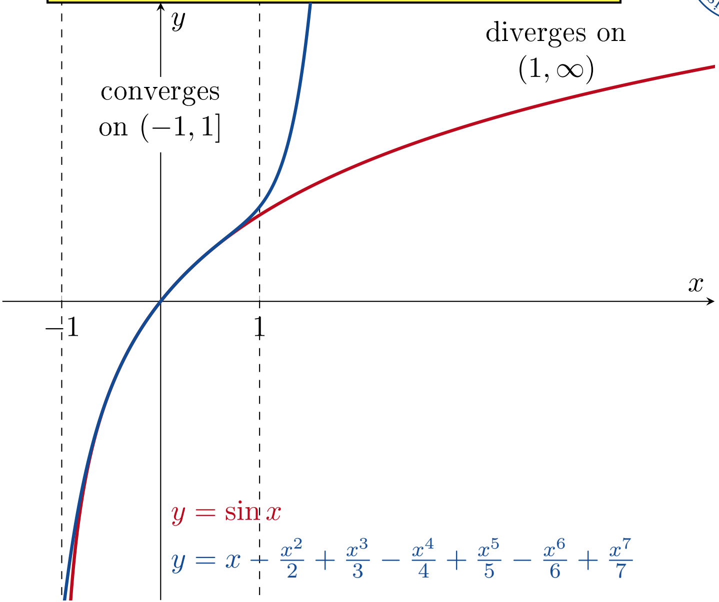


$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots$$

Example

Let $y = x + 1$. Then

$$\ln y = (y-1) - \frac{1}{2}(y-1)^2 + \frac{1}{3}(y-1)^3 - \frac{1}{4}(y-1)^4 + \frac{1}{5}(y-1)^5 - \frac{1}{6}(y-1)^6 + \dots$$

is the Taylor Series of $\ln y$ with centre $a = 1$. It converges for all $y \in (0, 2]$.

Colin Maclaurin

BORN
February 1698

DECEICED
14 June 1746

NATIONALITY
British

Definition

A Taylor Series with centre 0 is also called a *Maclaurin Series*.

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n$$

$$f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \frac{f^{(4)}(0)}{4!}x^4 + \dots$$

Example

Calculate the Maclaurin Series for $f(x) = \sinh x$.

$$f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \frac{f^{(4)}(0)}{4!}x^4 + \dots$$

Example

Calculate the Maclaurin Series for $f(x) = \sinh x$.

This is the same as:

Example

Calculate the Taylor Series for $f(x) = \sinh x$ centred at 0.

$$f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \frac{f^{(4)}(0)}{4!}x^4 + \dots$$

Example

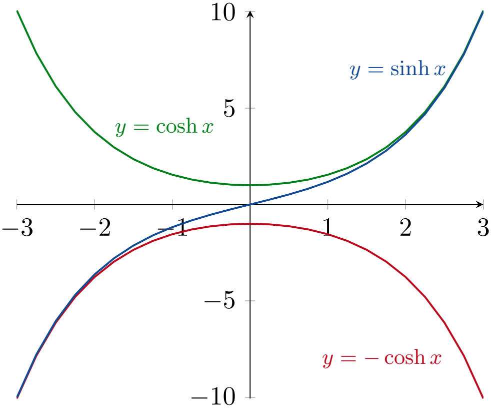
Calculate the Maclaurin Series for $f(x) = \sinh x$.

Since $\frac{d}{dx}\sinh x = \cosh x$ and $\frac{d}{dx}\cosh x = \sinh x$, we know that

$$f^{(n)}(x) = \sinh x \quad \text{or} \quad \cosh x$$

for all $n \in \mathbb{N}$.

$$f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \frac{f^{(4)}(0)}{4!}x^4 + \dots$$



Note that

$$-\cosh x \leq \sinh x \leq \cosh x$$

for all $x \in \mathbb{R}$.

$$f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \frac{f^{(4)}(0)}{4!}x^4 + \dots$$

Let $x \neq 0$ and let c be between 0 and x. (So $0 < c < x$ or $x < c < 0$.) Then

$$\left|f^{(n)}(c)\right| < \left|f^{(n)}(x)\right| \leq \cosh x.$$

So

$$0 \leq \left|\frac{f^{(n)}(c)x^n}{n!}\right| \leq \cosh x \frac{|x|^n}{n!} \to 0$$

as $n \to \infty$. By the Sandwich Rule, it follows that

$$R_c(x) = \frac{f^{(n)}(c)x^n}{n!} \to 0$$

as $n \to \infty$.

$$f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \frac{f^{(4)}(0)}{4!}x^4 + \dots$$

Now, since

$$f^{(n)}(x) = \begin{cases} \sinh x & \text{if } n = 0, 2, 4, 6, 8, \dots \\ \cosh x & \text{if } n = 1, 3, 5, 7, 9, \dots \end{cases}$$

we have that

$$f^{(n)}(0) = \begin{cases} 0 & \text{if } n = 0, 2, 4, 6, 8, \dots \\ 1 & \text{if } n = 1, 3, 5, 7, 9, \dots \end{cases}$$

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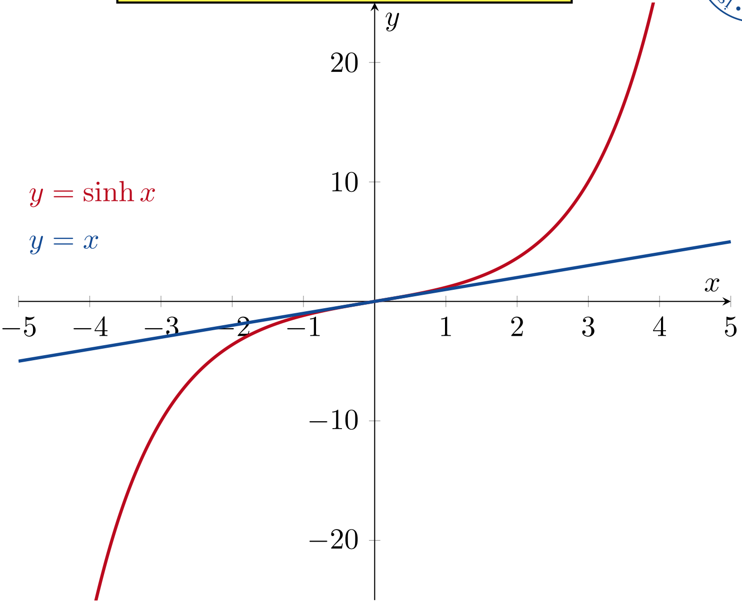
$$f^{(n)}(0) = \begin{cases} 0 & \text{if } n = 0, 2, 4, 6, 8, \ldots \\ 1 & \text{if } n = 1, 3, 5, 7, 9, \ldots \end{cases}$$

Therefore

$$\begin{aligned} \sinh x &= f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \frac{f^{(4)}(0)}{4!}x^4 \ldots \\ &= 0 + 1x + \frac{0}{2!}x^2 + \frac{1}{3!}x^3 + \frac{0}{4!}x^4 \ldots \\ &= x + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} + \frac{x^9}{9!} + \ldots = \sum_{k=0}^{\infty} \frac{x^{2k+1}}{(2k+1)!}. \end{aligned}$$

$$x + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} + \frac{x^9}{9!} + \dots$$



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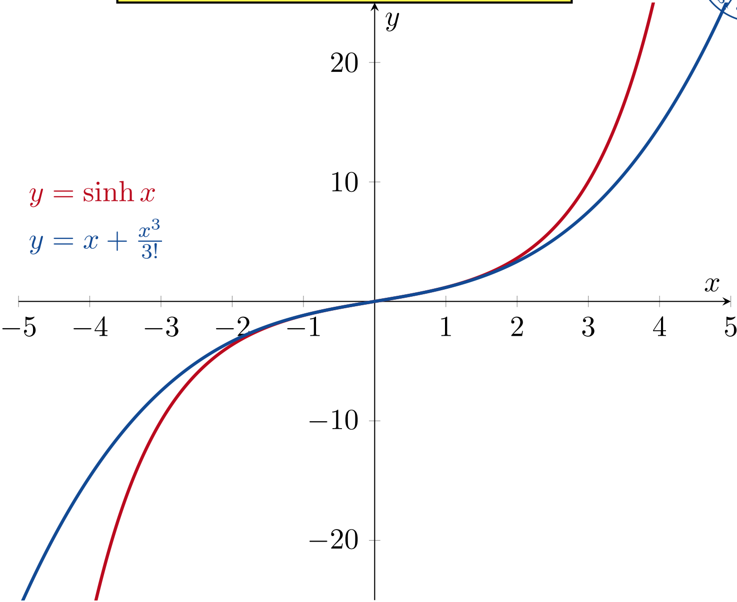

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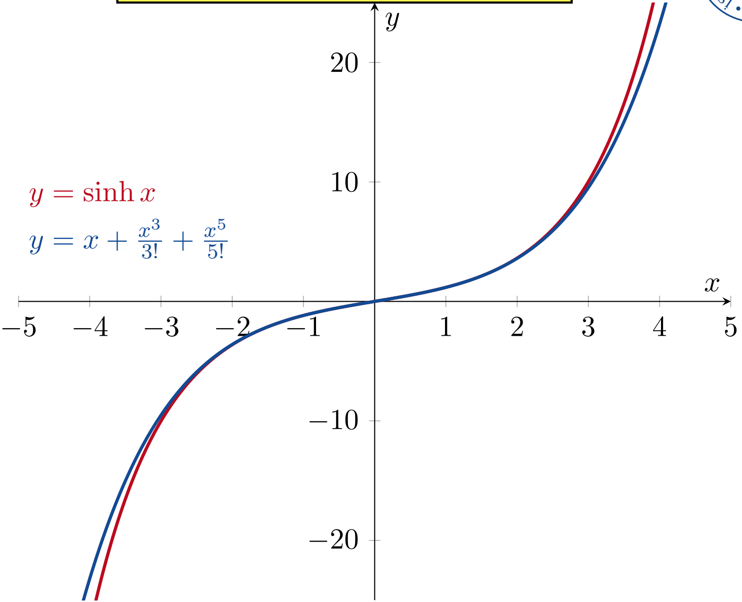

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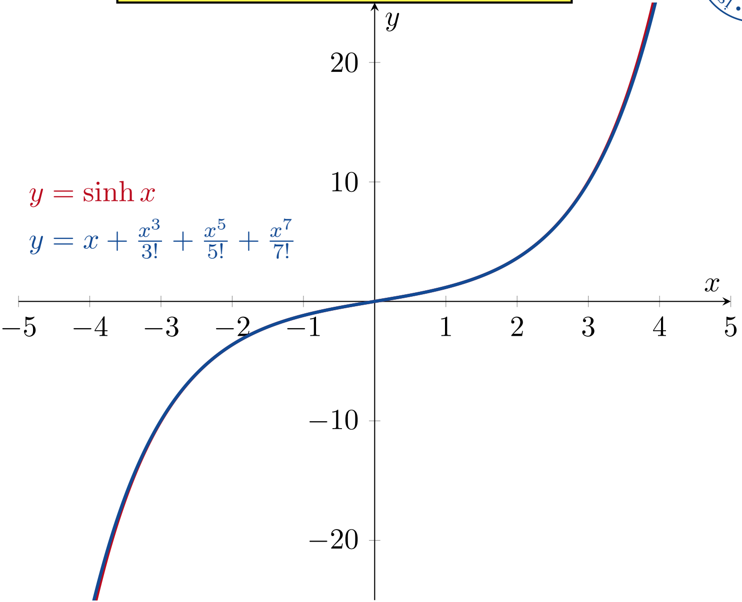

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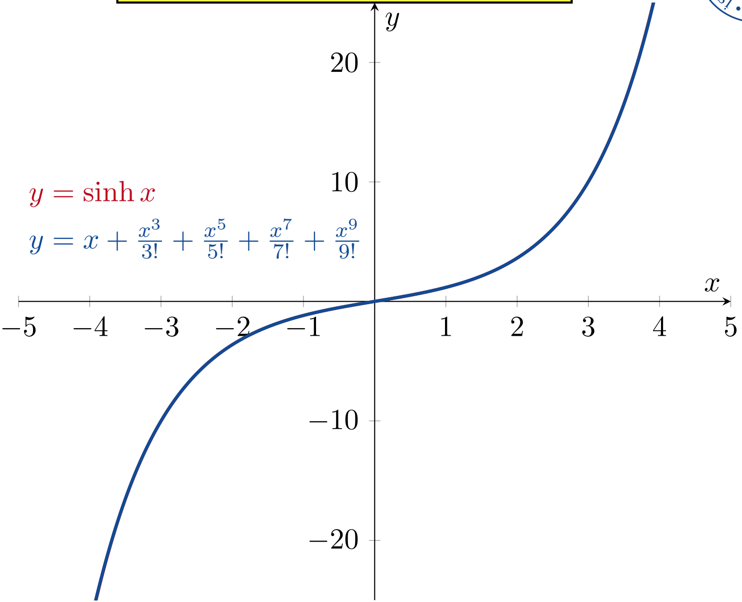

Example

Calculate the Taylor Series for $f(x) = \frac{1}{x}$ with centre $a = 2$. For which $x \in \mathbb{R}$ does the series converge?

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Since

$$\begin{aligned} f(x) &= x^{-1} & f(2) &= \tfrac{1}{2} \\ f'(x) &= -x^{-2} & f'(2) &= -\tfrac{1}{4} \\ f''(x) &= 2x^{-3} & \tfrac{f''(2)}{2!} &= \tfrac{1}{8} \\ f'''(x) &= -6x^{-4} & \tfrac{f'''(2)}{3!} &= -\tfrac{1}{16} \\ &\vdots & &\vdots \\ f^{(n)}(x) &= (-1)^n n! x^{-n-1} & \tfrac{f^{(n)}(2)}{n!} &= \tfrac{(-1)^n}{2^{n+1}} \\ &\vdots & &\vdots \end{aligned}$$

the Taylor Series is

$$\begin{aligned}\frac{1}{x} &= f(2) + f'(2)(x-2) + \frac{f''(2)}{2!}(x-2)^2 + \frac{f'''(2)}{3!}(x-2)^3 + \frac{f^{(4)}(2)}{4!}(x-2)^4 + \ldots \\ &= \frac{1}{2} - \frac{x-2}{4} + \frac{(x-2)^2}{8} - \frac{(x-2)^3}{16} + \frac{(x-2)^4}{32} - \ldots \\ &= \frac{1}{2}\left(1 + r + r^2 + r^3 + r^4 + \ldots\right)\end{aligned}$$

where $r = -\frac{x-2}{2}$.

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$$
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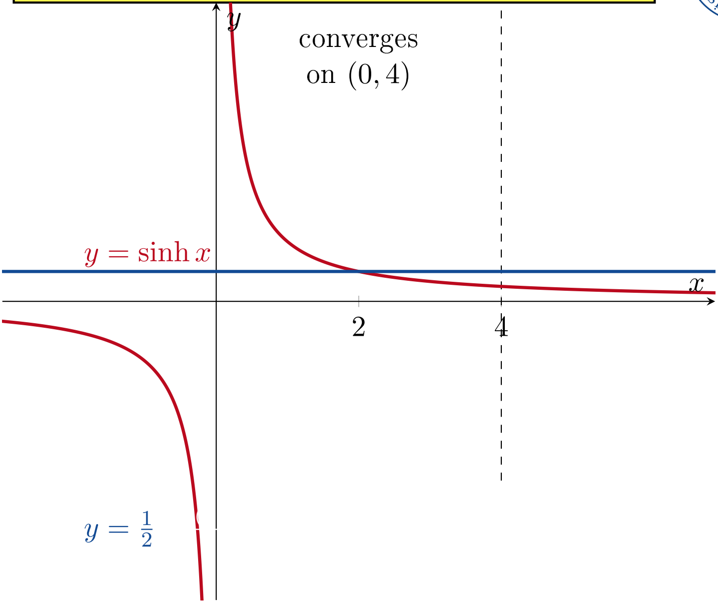
\end{aligned}
$$

where $r = -\frac{x-2}{2}$.

This series converges absolutely for $|r| < 1$ and diverges for $|r| \geq 1$. Therefore, the Taylor Series converges for $0 < x < 4$.

$$\frac{1}{2} - \frac{x-2}{4} + \frac{(x-2)^2}{8} - \frac{(x-2)^3}{16} + \frac{(x-2)^4}{32} - \dots$$



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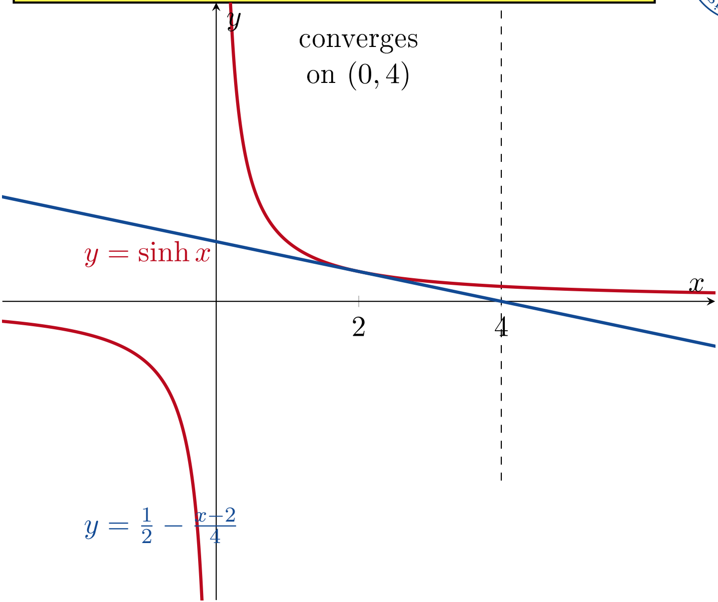

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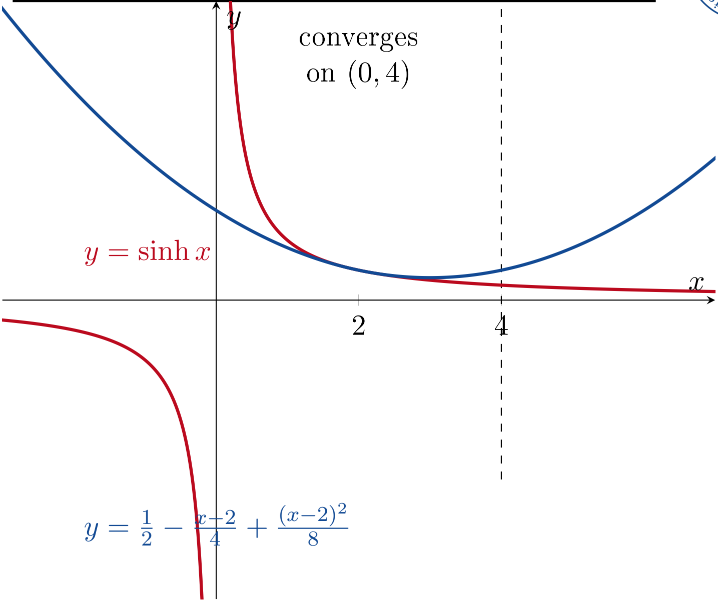

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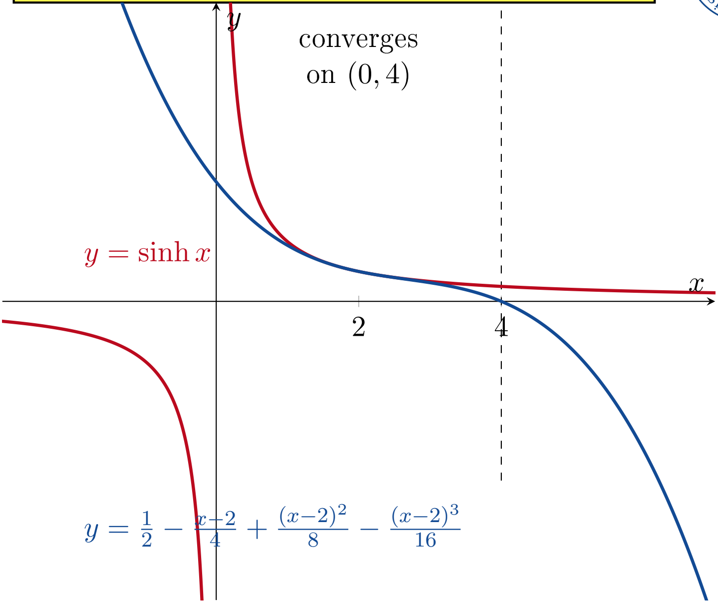

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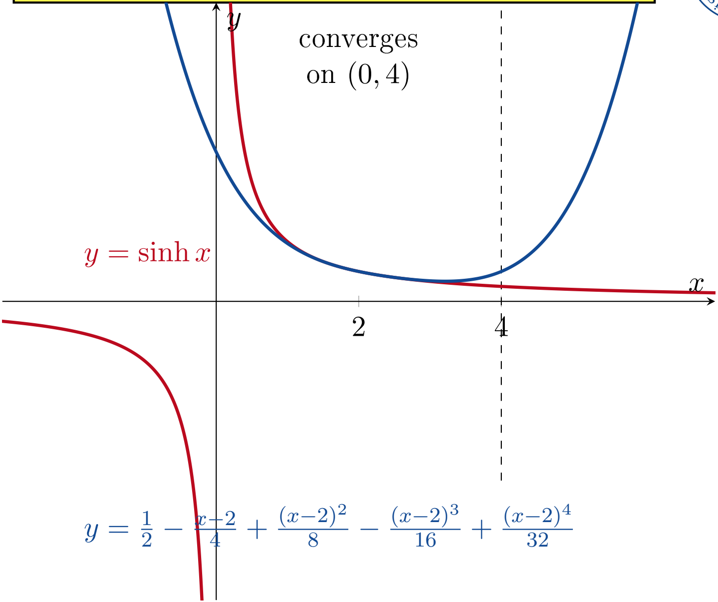

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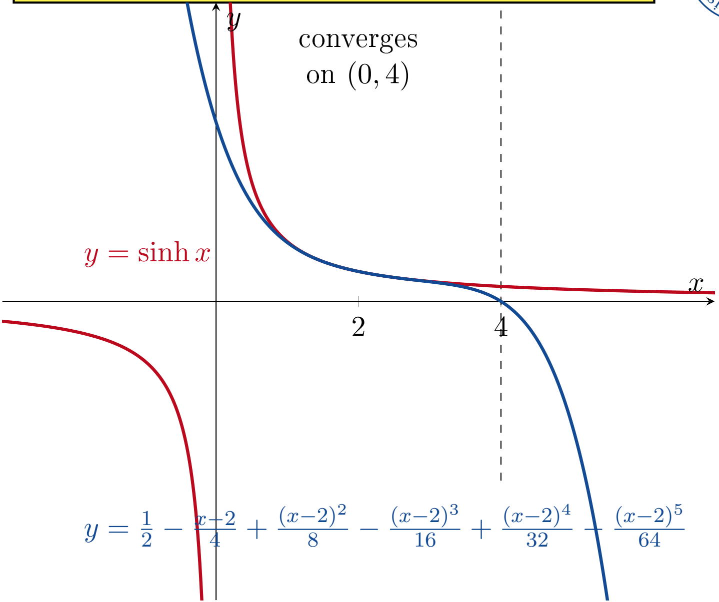

The End