

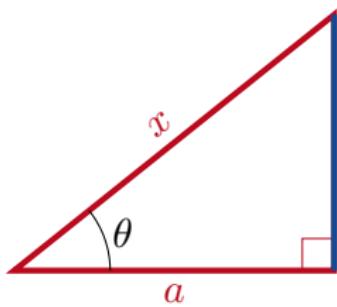
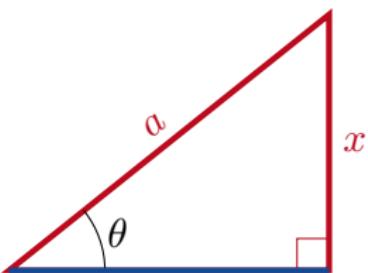
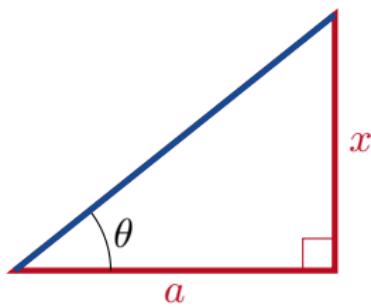
Lecture 2

- 8.4 Trigonometric Substitutions
- 8.5 Integration of Rational Functions by Partial Fractions
- 8.8 Improper Integrals



Trigonometric Substitutions

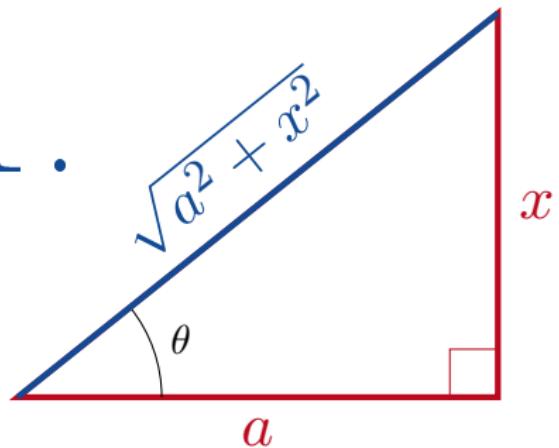
8.4 Trigonometric Substitutions



8.4 Trigonometric Substitutions



1.



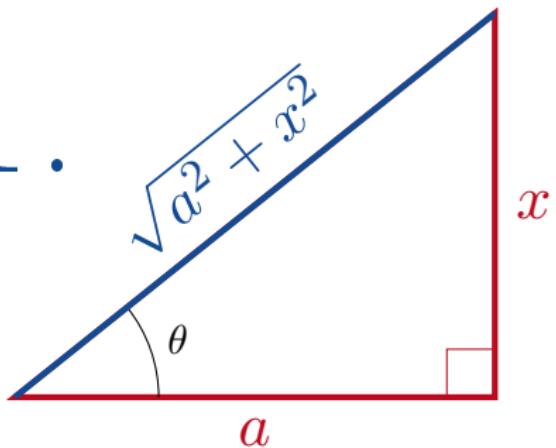
$$x = a \tan \theta$$

$$a^2 + x^2 = \quad = \quad = .$$

8.4 Trigonometric Substitutions



1.



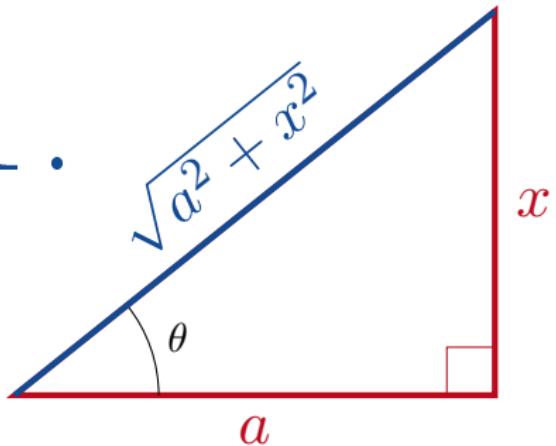
$$x = a \tan \theta$$

$$a^2 + x^2 = a^2 + a^2 \tan^2 \theta = \quad = \quad .$$

8.4 Trigonometric Substitutions



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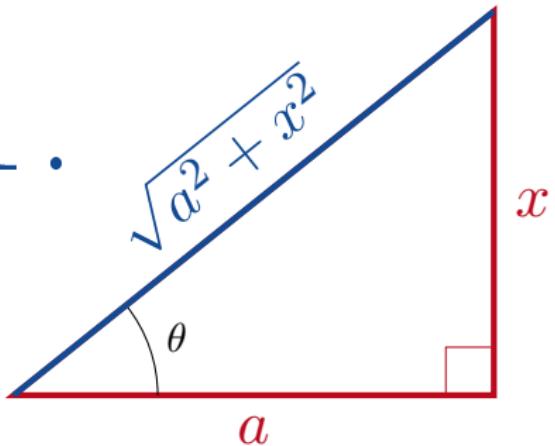
$$x = a \tan \theta$$

$$a^2 + x^2 = a^2 + a^2 \tan^2 \theta = a^2(1 + \tan^2 \theta) = .$$

8.4 Trigonometric Substitutions



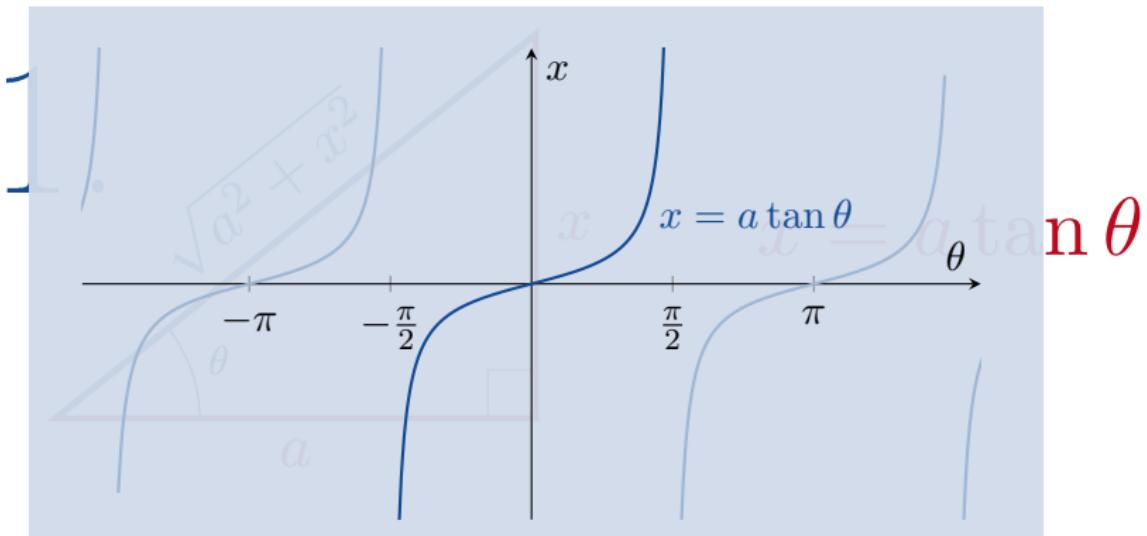
1.



$$x = a \tan \theta$$

$$a^2 + x^2 = a^2 + a^2 \tan^2 \theta = a^2(1 + \tan^2 \theta) = a^2 \sec^2 \theta.$$

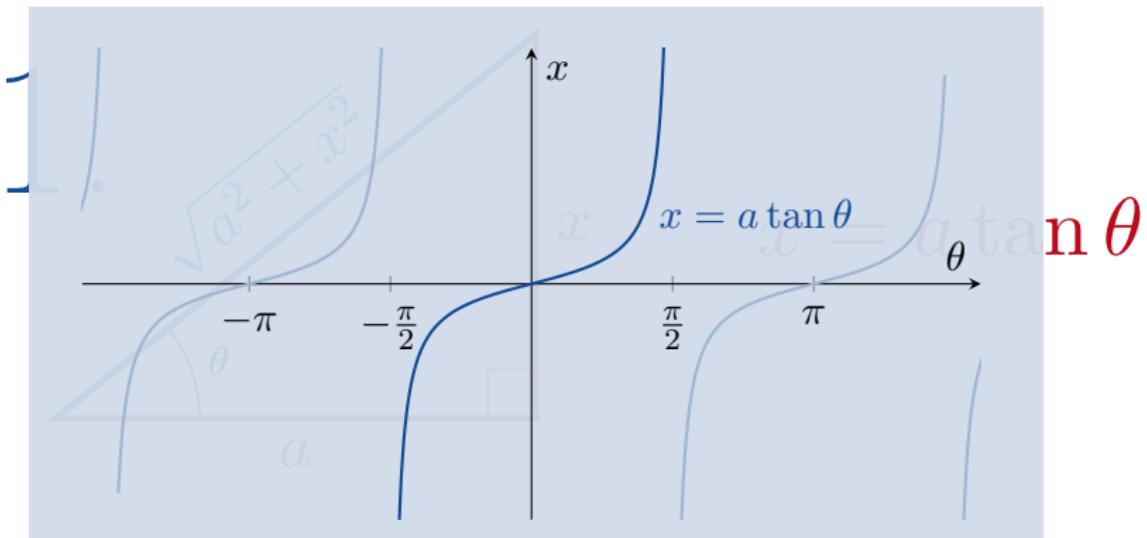
8.4 Trigonometric Substitutions



$$a^2 + x^2 = a^2 + a^2 \tan^2 \theta = a^2(1 + \tan^2 \theta) = a^2 \sec^2 \theta.$$



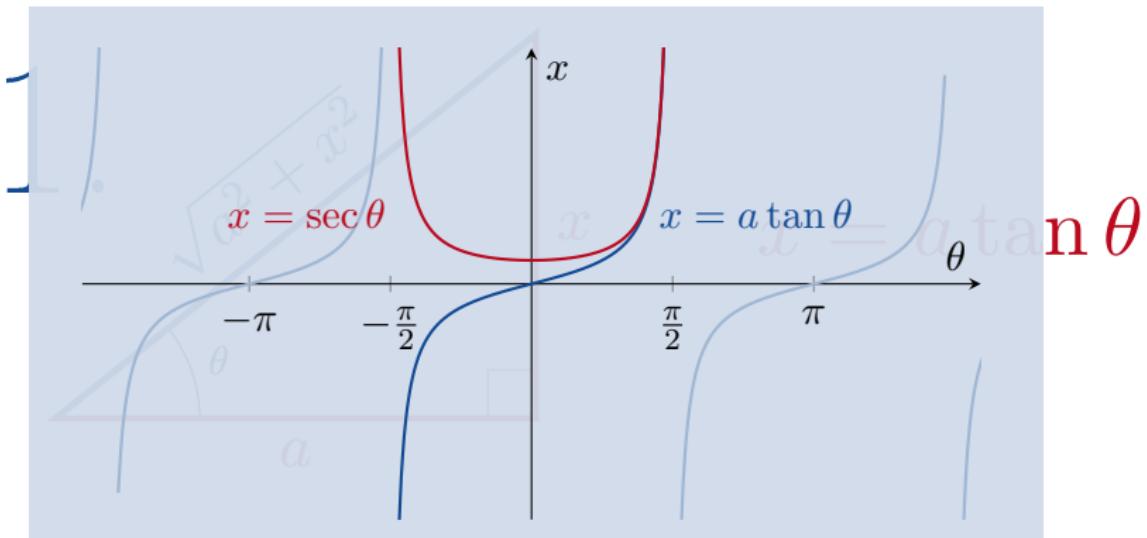
8.4 Trigonometric Substitutions



$$a^2 + x^2 = a^2 + a^2 \tan^2 \theta = a^2(1 + \tan^2 \theta) = a^2 \sec^2 \theta.$$

$$-\frac{\pi}{2} < \theta < \frac{\pi}{2}.$$

8.4 Trigonometric Substitutions



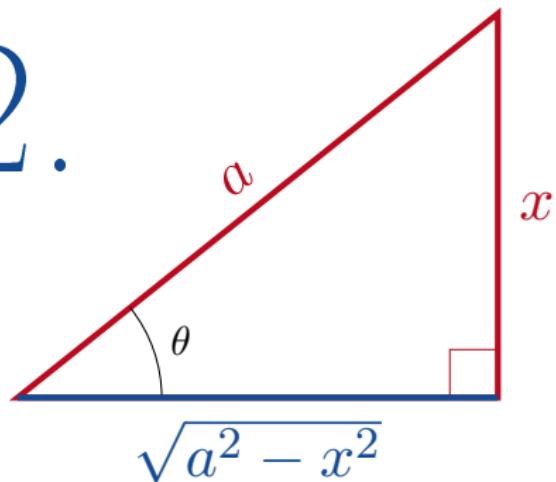
$$a^2 + x^2 = a^2 + a^2 \tan^2 \theta = a^2(1 + \tan^2 \theta) = a^2 \sec^2 \theta.$$

$$\boxed{\sqrt{a^2 + x^2} = a \sec \theta \quad -\frac{\pi}{2} < \theta < \frac{\pi}{2}.}$$

8.4 Trigonometric Substitutions



2.



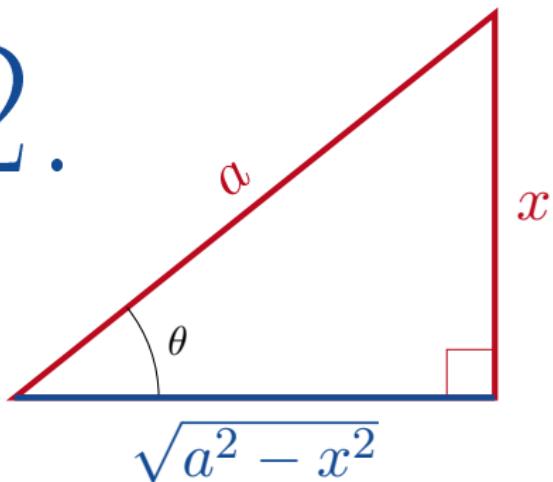
$$x = a \sin \theta$$

$$a^2 - x^2 = \quad = \quad .$$

8.4 Trigonometric Substitutions



2.



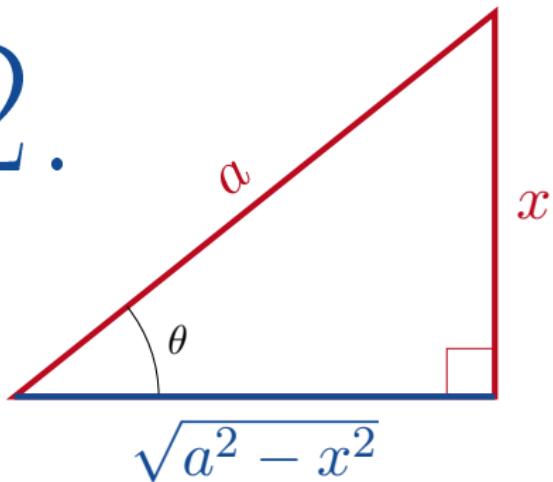
$$x = a \sin \theta$$

$$a^2 - x^2 = a^2 - a^2 \sin^2 \theta = \quad = \quad .$$

8.4 Trigonometric Substitutions



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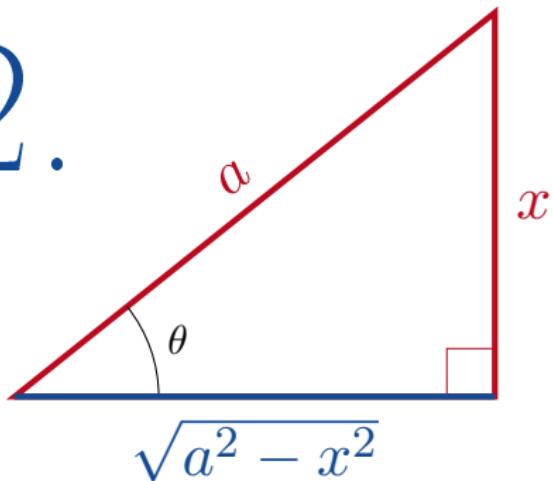
$$x = a \sin \theta$$

$$a^2 - x^2 = a^2 - a^2 \sin^2 \theta = a^2(1 - \sin^2 \theta) = .$$

8.4 Trigonometric Substitutions



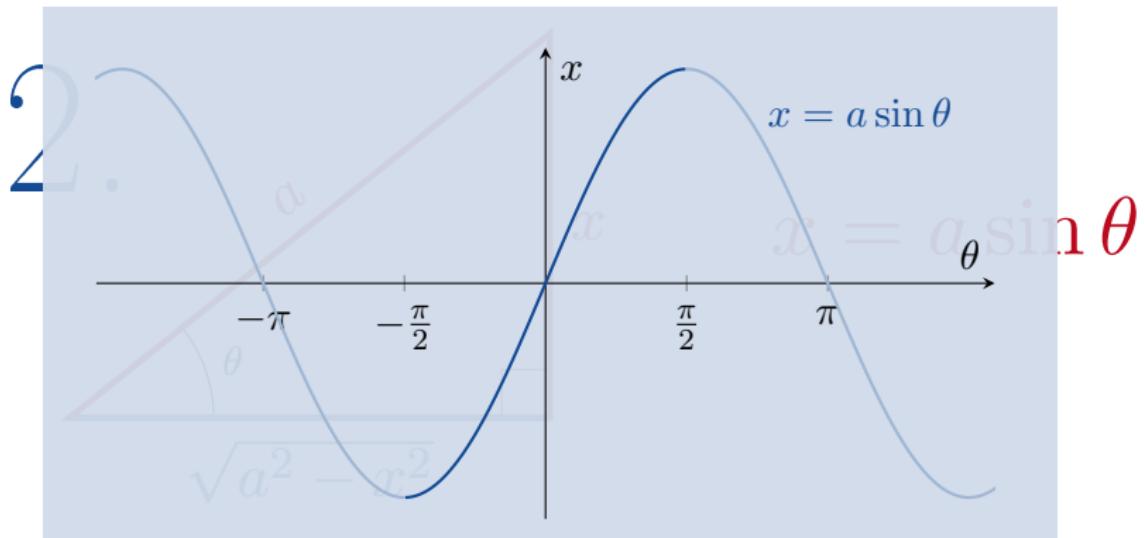
2.



$$x = a \sin \theta$$

$$a^2 - x^2 = a^2 - a^2 \sin^2 \theta = a^2(1 - \sin^2 \theta) = a^2 \cos^2 \theta.$$

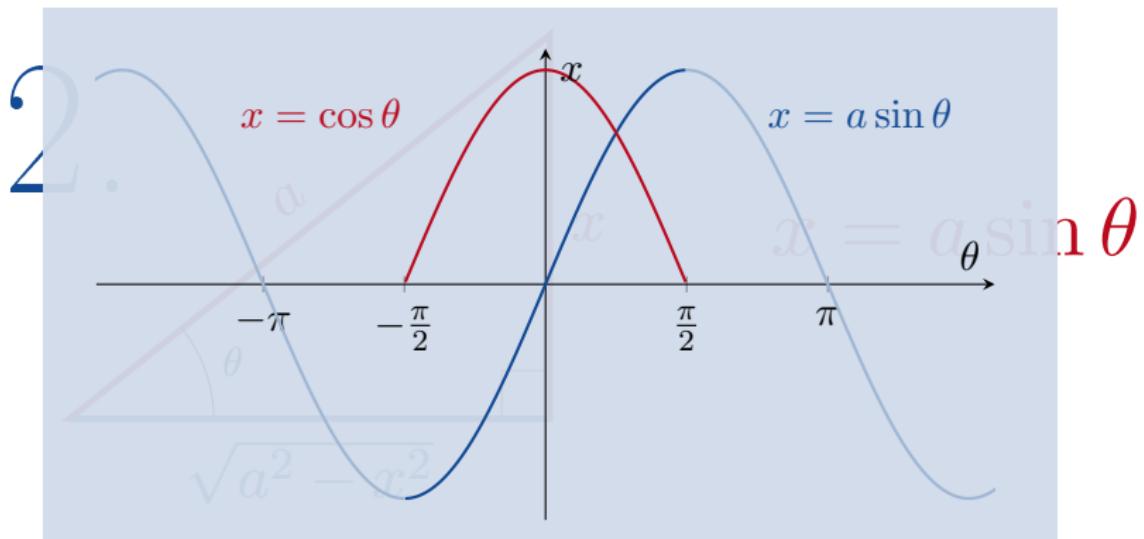
8.4 Trigonometric Substitutions



$$a^2 - x^2 = a^2 - a^2 \sin^2 \theta = a^2(1 - \sin^2 \theta) = a^2 \cos^2 \theta.$$

$$-\frac{\pi}{2} < \theta < \frac{\pi}{2}.$$

8.4 Trigonometric Substitutions



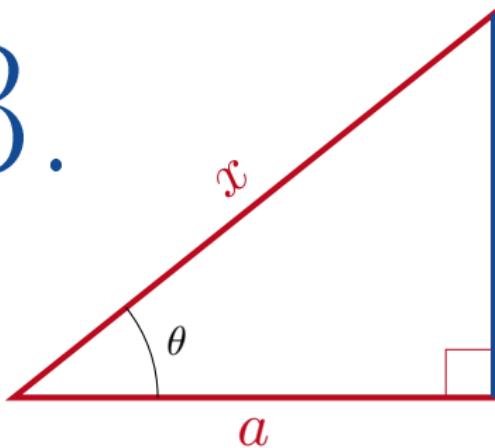
$$a^2 - x^2 = a^2 - a^2 \sin^2 \theta = a^2(1 - \sin^2 \theta) = a^2 \cos^2 \theta.$$

$$\boxed{\sqrt{a^2 - x^2} = a \cos \theta \quad -\frac{\pi}{2} < \theta < \frac{\pi}{2}.}$$

8.4 Trigonometric Substitutions



3.



$$\sqrt{x^2 - a^2}$$

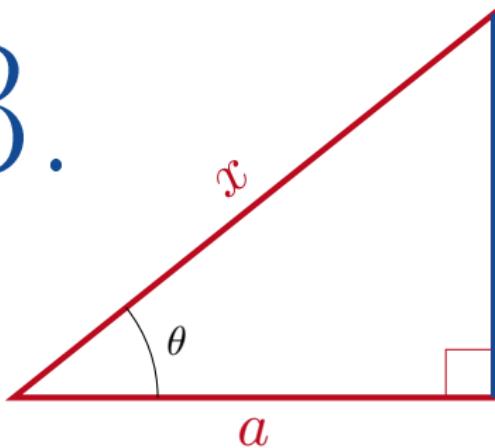
$$x = a \sec \theta$$

$$x^2 - a^2 = \quad = \quad .$$

8.4 Trigonometric Substitutions



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$$\sqrt{x^2 - a^2}$$

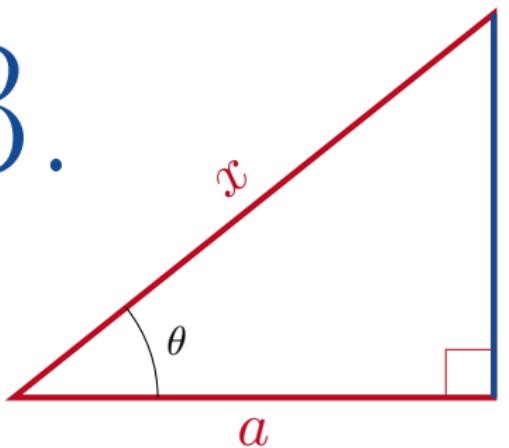
$$x = a \sec \theta$$

$$x^2 - a^2 = a^2 \sec^2 \theta - a^2 = \quad = \quad .$$

8.4 Trigonometric Substitutions



3.



$$\sqrt{x^2 - a^2}$$

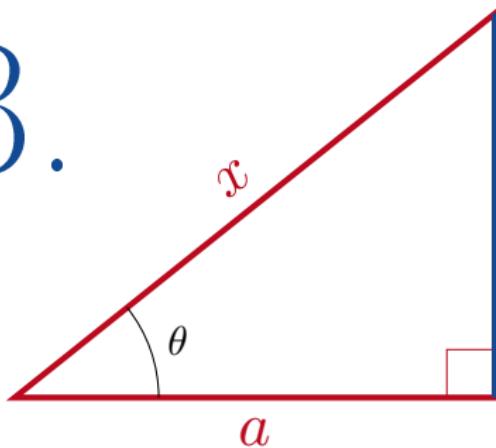
$$x = a \sec \theta$$

$$x^2 - a^2 = a^2 \sec^2 \theta - a^2 = a^2(\sec^2 \theta - 1) = .$$

8.4 Trigonometric Substitutions



3.

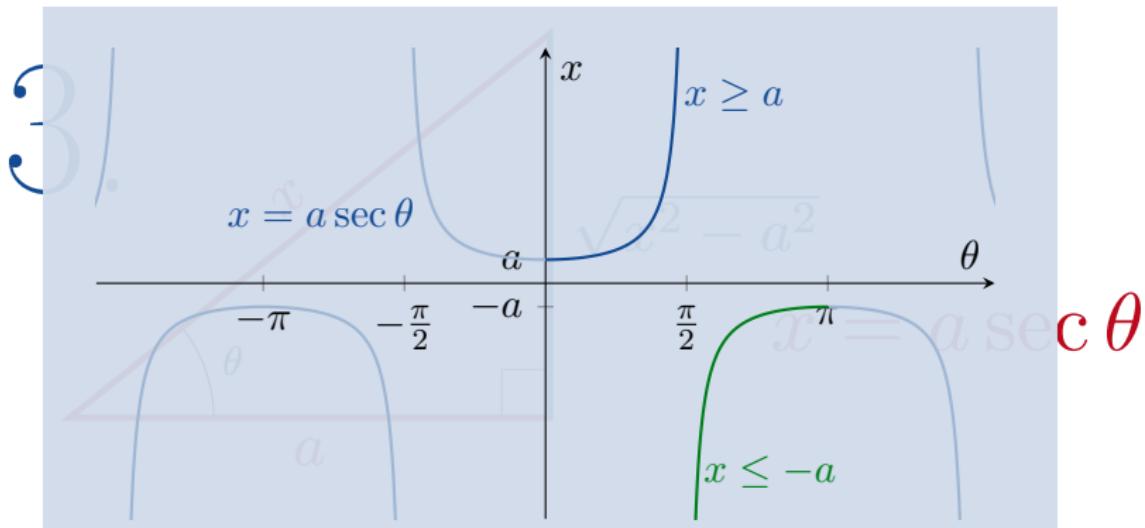


$$\sqrt{x^2 - a^2}$$

$$x = a \sec \theta$$

$$x^2 - a^2 = a^2 \sec^2 \theta - a^2 = a^2(\sec^2 \theta - 1) = a^2 \tan^2 \theta.$$

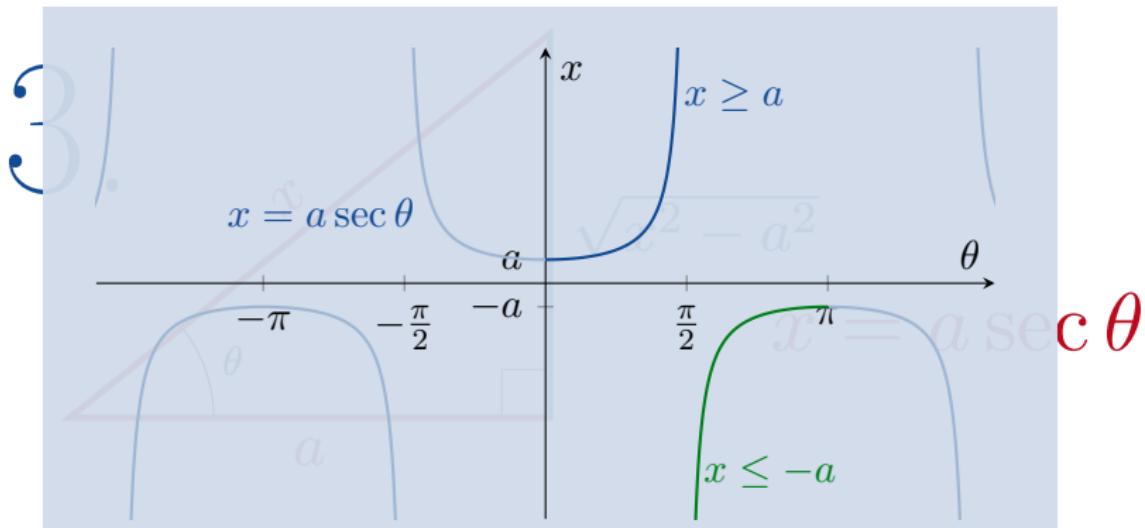
8.4 Trigonometric Substitutions



$$x^2 - a^2 = a^2 \sec^2 \theta - a^2 = a^2 (\sec^2 \theta - 1) = a^2 \tan^2 \theta.$$



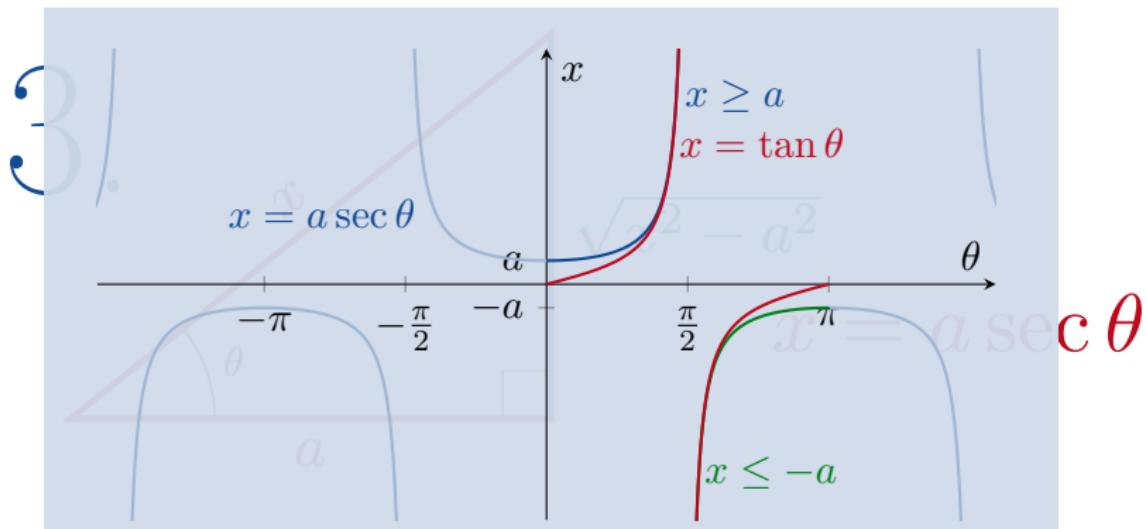
8.4 Trigonometric Substitutions



$$x^2 - a^2 = a^2 \sec^2 \theta - a^2 = a^2 (\sec^2 \theta - 1) = a^2 \tan^2 \theta.$$

$$\begin{cases} 0 \leq \theta < \frac{\pi}{2} & x \geq a \\ \frac{\pi}{2} < \theta \leq \pi & x \leq -a \end{cases}.$$

8.4 Trigonometric Substitutions



$$\sqrt{x^2 - a^2} = a |\tan x| \quad \begin{cases} 0 \leq \theta < \frac{\pi}{2} & x \geq a \\ \frac{\pi}{2} < \theta \leq \pi & x \leq -a \end{cases}.$$

$$\begin{array}{lll} x = a \tan \theta & x = a \sin \theta & x = a \sec \theta \\ \sqrt{a^2 + x^2} = a \sec \theta & \sqrt{a^2 - x^2} = a \cos \theta & \sqrt{x^2 - a^2} = a |\tan \theta| \\ -\frac{\pi}{2} < \theta < \frac{\pi}{2} & -\frac{\pi}{2} < \theta < \frac{\pi}{2} & \begin{cases} 0 \leq \theta < \frac{\pi}{2} & x \geq a \\ \frac{\pi}{2} < \theta \leq \pi & x \leq -a \end{cases} \end{array}$$



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Example

Find $\int \frac{dx}{\sqrt{4+x^2}}$.

$$\begin{array}{lll} x = a \tan \theta & x = a \sin \theta & x = a \sec \theta \\ \sqrt{a^2 + x^2} = a \sec \theta & \sqrt{a^2 - x^2} = a \cos \theta & \sqrt{x^2 - a^2} = a |\tan \theta| \\ -\frac{\pi}{2} < \theta < \frac{\pi}{2} & -\frac{\pi}{2} < \theta < \frac{\pi}{2} & \begin{cases} 0 \leq \theta < \frac{\pi}{2} & x \geq a \\ \frac{\pi}{2} < \theta \leq \pi & x \leq -a \end{cases} \end{array}$$



Example

Find $\int \frac{dx}{\sqrt{4+x^2}}$.

Let $x = 2 \tan \theta$.

$$\begin{array}{lll} x = a \tan \theta & x = a \sin \theta & x = a \sec \theta \\ \sqrt{a^2 + x^2} = a \sec \theta & \sqrt{a^2 - x^2} = a \cos \theta & \sqrt{x^2 - a^2} = a |\tan \theta| \\ -\frac{\pi}{2} < \theta < \frac{\pi}{2} & -\frac{\pi}{2} < \theta < \frac{\pi}{2} & \begin{cases} 0 \leq \theta < \frac{\pi}{2} & x \geq a \\ \frac{\pi}{2} < \theta \leq \pi & x \leq -a \end{cases} \end{array}$$



Example

Find $\int \frac{dx}{\sqrt{4+x^2}}$.

Let $x = 2 \tan \theta$. Then $dx = 2 \sec^2 \theta d\theta$ and $\sqrt{4+x^2} = 2 \sec \theta$.

$x = a \tan \theta$	$x = a \sin \theta$	$x = a \sec \theta$
$\sqrt{a^2 + x^2} = a \sec \theta$	$\sqrt{a^2 - x^2} = a \cos \theta$	$\sqrt{x^2 - a^2} = a \tan \theta $
$-\frac{\pi}{2} < \theta < \frac{\pi}{2}$	$-\frac{\pi}{2} < \theta < \frac{\pi}{2}$	$\begin{cases} 0 \leq \theta < \frac{\pi}{2} & x \geq a \\ \frac{\pi}{2} < \theta \leq \pi & x \leq -a \end{cases}$



Example

Find $\int \frac{dx}{\sqrt{4+x^2}}$.

Let $x = 2 \tan \theta$. Then $dx = 2 \sec^2 \theta d\theta$ and $\sqrt{4+x^2} = 2 \sec \theta$.
Therefore

$$\int \frac{dx}{\sqrt{4+x^2}} = \int \frac{2 \sec^2 \theta d\theta}{2 \sec \theta}$$

=

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$x = a \tan \theta$	$x = a \sin \theta$	$x = a \sec \theta$
$\sqrt{a^2 + x^2} = a \sec \theta$	$\sqrt{a^2 - x^2} = a \cos \theta$	$\sqrt{x^2 - a^2} = a \tan \theta $
$-\frac{\pi}{2} < \theta < \frac{\pi}{2}$	$-\frac{\pi}{2} < \theta < \frac{\pi}{2}$	$\begin{cases} 0 \leq \theta < \frac{\pi}{2} & x \geq a \\ \frac{\pi}{2} < \theta \leq \pi & x \leq -a \end{cases}$



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Find $\int \frac{dx}{\sqrt{4+x^2}}$.

Let $x = 2 \tan \theta$. Then $dx = 2 \sec^2 \theta d\theta$ and $\sqrt{4+x^2} = 2 \sec \theta$.
Therefore

$$\begin{aligned}
 \int \frac{dx}{\sqrt{4+x^2}} &= \int \frac{2 \sec^2 \theta d\theta}{2 \sec \theta} \\
 &= \int \sec \theta d\theta \\
 &= \\
 &=
 \end{aligned}$$

$x = a \tan \theta$	$x = a \sin \theta$	$x = a \sec \theta$
$\sqrt{a^2 + x^2} = a \sec \theta$	$\sqrt{a^2 - x^2} = a \cos \theta$	$\sqrt{x^2 - a^2} = a \tan \theta $
$-\frac{\pi}{2} < \theta < \frac{\pi}{2}$	$-\frac{\pi}{2} < \theta < \frac{\pi}{2}$	$\begin{cases} 0 \leq \theta < \frac{\pi}{2} & x \geq a \\ \frac{\pi}{2} < \theta \leq \pi & x \leq -a \end{cases}$



Example

Find $\int \frac{dx}{\sqrt{4+x^2}}$.

Let $x = 2 \tan \theta$. Then $dx = 2 \sec^2 \theta d\theta$ and $\sqrt{4+x^2} = 2 \sec \theta$.
Therefore

$$\begin{aligned} \int \frac{dx}{\sqrt{4+x^2}} &= \int \frac{2 \sec^2 \theta d\theta}{2 \sec \theta} \\ &= \int \sec \theta d\theta \\ &= \ln |\sec \theta + \tan \theta| + C \end{aligned}$$

=

.

$x = a \tan \theta$ $\sqrt{a^2 + x^2} = a \sec \theta$ $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$	$x = a \sin \theta$ $\sqrt{a^2 - x^2} = a \cos \theta$ $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$	$x = a \sec \theta$ $\sqrt{x^2 - a^2} = a \tan \theta $ $\begin{cases} 0 \leq \theta < \frac{\pi}{2} & x \geq a \\ \frac{\pi}{2} < \theta \leq \pi & x \leq -a \end{cases}$
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Find $\int \frac{dx}{\sqrt{4+x^2}}$.

Let $x = 2 \tan \theta$. Then $dx = 2 \sec^2 \theta d\theta$ and $\sqrt{4+x^2} = 2 \sec \theta$.
Therefore

$$\begin{aligned}
 \int \frac{dx}{\sqrt{4+x^2}} &= \int \frac{2 \sec^2 \theta d\theta}{2 \sec \theta} \\
 &= \int \sec \theta d\theta \\
 &= \ln |\sec \theta + \tan \theta| + C \\
 &= \ln \left| \frac{\sqrt{4+x^2}}{2} + \frac{x}{2} \right| + C.
 \end{aligned}$$

$$\begin{array}{lll} x = a \tan \theta & x = a \sin \theta & x = a \sec \theta \\ \sqrt{a^2 + x^2} = a \sec \theta & \sqrt{a^2 - x^2} = a \cos \theta & \sqrt{x^2 - a^2} = a |\tan \theta| \\ -\frac{\pi}{2} < \theta < \frac{\pi}{2} & -\frac{\pi}{2} < \theta < \frac{\pi}{2} & \begin{cases} 0 \leq \theta < \frac{\pi}{2} & x \geq a \\ \frac{\pi}{2} < \theta \leq \pi & x \leq -a \end{cases} \end{array}$$



Example

Calculate $\int_0^{\frac{3}{2}} \frac{dx}{\sqrt{9 - x^2}}$.

$$\begin{array}{lll} x = a \tan \theta & x = a \sin \theta & x = a \sec \theta \\ \sqrt{a^2 + x^2} = a \sec \theta & \sqrt{a^2 - x^2} = a \cos \theta & \sqrt{x^2 - a^2} = a |\tan \theta| \\ -\frac{\pi}{2} < \theta < \frac{\pi}{2} & -\frac{\pi}{2} < \theta < \frac{\pi}{2} & \begin{cases} 0 \leq \theta < \frac{\pi}{2} & x \geq a \\ \frac{\pi}{2} < \theta \leq \pi & x \leq -a \end{cases} \end{array}$$



Example

Calculate $\int_0^{\frac{3}{2}} \frac{dx}{\sqrt{9 - x^2}}$.

Let $x = 3 \sin \theta$.

$x = a \tan \theta$	$x = a \sin \theta$	$x = a \sec \theta$
$\sqrt{a^2 + x^2} = a \sec \theta$	$\sqrt{a^2 - x^2} = a \cos \theta$	$\sqrt{x^2 - a^2} = a \tan \theta $
$-\frac{\pi}{2} < \theta < \frac{\pi}{2}$	$-\frac{\pi}{2} < \theta < \frac{\pi}{2}$	$\begin{cases} 0 \leq \theta < \frac{\pi}{2} & x \geq a \\ \frac{\pi}{2} < \theta \leq \pi & x \leq -a \end{cases}$

Example

Calculate $\int_0^{\frac{3}{2}} \frac{dx}{\sqrt{9 - x^2}}$.

Let $x = 3 \sin \theta$. $dx = 3 \cos \theta d\theta$ and $\sqrt{9 - x^2} = 3 \cos \theta$.

$x = a \tan \theta$	$x = a \sin \theta$	$x = a \sec \theta$
$\sqrt{a^2 + x^2} = a \sec \theta$	$\sqrt{a^2 - x^2} = a \cos \theta$	$\sqrt{x^2 - a^2} = a \tan \theta $
$-\frac{\pi}{2} < \theta < \frac{\pi}{2}$	$-\frac{\pi}{2} < \theta < \frac{\pi}{2}$	$\begin{cases} 0 \leq \theta < \frac{\pi}{2} & x \geq a \\ \frac{\pi}{2} < \theta \leq \pi & x \leq -a \end{cases}$



Example

Calculate $\int_0^{\frac{3}{2}} \frac{dx}{\sqrt{9 - x^2}}$.

Let $x = 3 \sin \theta$. $dx = 3 \cos \theta d\theta$ and $\sqrt{9 - x^2} = 3 \cos \theta$.

Moreover $x = 0 \implies \theta = \sin^{-1} \frac{x}{3} = \sin^{-1} 0 = 0$
 and $x = \frac{3}{2} \implies \theta = \sin^{-1} \frac{x}{3} = \sin^{-1} \frac{1}{2} = \frac{\pi}{6}$.

$x = a \tan \theta$ $\sqrt{a^2 + x^2} = a \sec \theta$ $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$	$x = a \sin \theta$ $\sqrt{a^2 - x^2} = a \cos \theta$ $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$	$x = a \sec \theta$ $\sqrt{x^2 - a^2} = a \tan \theta $ $\begin{cases} 0 \leq \theta < \frac{\pi}{2} & x \geq a \\ \frac{\pi}{2} < \theta \leq \pi & x \leq -a \end{cases}$
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Example

Calculate $\int_0^{\frac{3}{2}} \frac{dx}{\sqrt{9 - x^2}}$.

Let $x = 3 \sin \theta$. $dx = 3 \cos \theta d\theta$ and $\sqrt{9 - x^2} = 3 \cos \theta$.

Moreover $x = 0 \implies \theta = \sin^{-1} \frac{x}{3} = \sin^{-1} 0 = 0$

and $x = \frac{3}{2} \implies \theta = \sin^{-1} \frac{x}{3} = \sin^{-1} \frac{1}{2} = \frac{\pi}{6}$. Therefore

$$\int_0^{\frac{3}{2}} \frac{dx}{\sqrt{9 - x^2}} = \int_0^{\frac{\pi}{6}} \frac{3 \cos \theta d\theta}{3 \cos \theta} = \int_0^{\frac{\pi}{6}} d\theta = \frac{\pi}{6}.$$

$x = a \tan \theta$	$x = a \sin \theta$	$x = a \sec \theta$
$\sqrt{a^2 + x^2} = a \sec \theta$	$\sqrt{a^2 - x^2} = a \cos \theta$	$\sqrt{x^2 - a^2} = a \tan \theta $
$-\frac{\pi}{2} < \theta < \frac{\pi}{2}$	$-\frac{\pi}{2} < \theta < \frac{\pi}{2}$	$\begin{cases} 0 \leq \theta < \frac{\pi}{2} & x \geq a \\ \frac{\pi}{2} < \theta \leq \pi & x \leq -a \end{cases}$

Example

Calculate $\int_0^{\frac{3}{2}} \frac{dx}{\sqrt{9 - x^2}}$.

Let $x = 3 \sin \theta$. $dx = 3 \cos \theta d\theta$ and $\sqrt{9 - x^2} = 3 \cos \theta$.

Moreover $x = 0 \implies \theta = \sin^{-1} \frac{x}{3} = \sin^{-1} 0 = 0$

and $x = \frac{3}{2} \implies \theta = \sin^{-1} \frac{x}{3} = \sin^{-1} \frac{1}{2} = \frac{\pi}{6}$. Therefore

$$\int_0^{\frac{3}{2}} \frac{dx}{\sqrt{9 - x^2}} = \int_0^{\frac{\pi}{6}} \frac{3 \cos \theta d\theta}{3 \cos \theta} = \int_0^{\frac{\pi}{6}} d\theta = \frac{\pi}{6}.$$

Or

$$\int_0^{\frac{3}{2}} \frac{dx}{\sqrt{9 - x^2}} = \left[\sin^{-1} \frac{x}{3} \right]_0^{\frac{3}{2}} = \frac{\pi}{6} - 0 = \frac{\pi}{6}.$$

EXAMPLE 2 Here we find an expression for the inverse hyperbolic sine function in terms of the natural logarithm. Following the same procedure as in Example 1, we find that

$$\begin{aligned}\int \frac{dx}{\sqrt{a^2 + x^2}} &= \int \sec \theta d\theta & x = a \tan \theta, dx = a \sec^2 \theta d\theta \\ &= \ln |\sec \theta + \tan \theta| + C \\ &= \ln \left| \frac{\sqrt{a^2 + x^2}}{a} + \frac{x}{a} \right| + C & \text{Fig. 8.2}\end{aligned}$$

From Table 7.11, $\sinh^{-1}(x/a)$ is also an antiderivative of $1/\sqrt{a^2 + x^2}$, so the two antiderivatives differ by a constant, giving

$$\sinh^{-1} \frac{x}{a} = \ln \left| \frac{\sqrt{a^2 + x^2}}{a} + \frac{x}{a} \right| + C.$$

Setting $x = 0$ in this last equation, we find $0 = \ln |1| + C$, so $C = 0$. Since $\sqrt{a^2 + x^2} > |x|$, we conclude that

$$\sinh^{-1} \frac{x}{a} = \ln \left(\frac{\sqrt{a^2 + x^2}}{a} + \frac{x}{a} \right)$$

EXAMPLE 3 Evaluate

$$\int \frac{x^2 dx}{\sqrt{9 - x^2}}.$$

Solution We set

$$x = 3 \sin \theta, \quad dx = 3 \cos \theta d\theta, \quad -\frac{\pi}{2} < \theta < \frac{\pi}{2}$$
$$9 - x^2 = 9 - 9 \sin^2 \theta = 9(1 - \sin^2 \theta) = 9 \cos^2 \theta.$$

Then

$$\begin{aligned}\int \frac{x^2 dx}{\sqrt{9 - x^2}} &= \int \frac{9 \sin^2 \theta \cdot 3 \cos \theta d\theta}{|3 \cos \theta|} \\&= 9 \int \sin^2 \theta d\theta && \cos \theta > 0 \text{ for } -\frac{\pi}{2} < \theta < \frac{\pi}{2} \\&= 9 \int \frac{1 - \cos 2\theta}{2} d\theta \\&= \frac{9}{2} \left(\theta - \frac{\sin 2\theta}{2} \right) + C \\&= \frac{9}{2} (\theta - \sin \theta \cos \theta) + C && \sin 2\theta = 2 \sin \theta \cos \theta \\&= \frac{9}{2} \left(\sin^{-1} \frac{x}{3} - \frac{x}{3} \cdot \frac{\sqrt{9 - x^2}}{3} \right) + C && \text{From Fig. 8.5} \\&= \frac{9}{2} \sin^{-1} \frac{x}{3} - \frac{x}{2} \sqrt{9 - x^2} + C.\end{aligned}$$

EXAMPLE 4

Evaluate

$$\int \frac{dx}{\sqrt{25x^2 - 4}}, \quad x > \frac{2}{5}.$$

Solution We first rewrite the radical as

$$\begin{aligned}\sqrt{25x^2 - 4} &= \sqrt{25\left(x^2 - \frac{4}{25}\right)} \\ &= 5\sqrt{x^2 - \left(\frac{2}{5}\right)^2} \quad \text{with } a = \frac{2}{5}\end{aligned}$$

to put the radicand in the form $x^2 - a^2$. We then substitute

$$x = \frac{2}{5} \sec \theta, \quad dx = \frac{2}{5} \sec \theta \tan \theta d\theta, \quad 0 < \theta < \frac{\pi}{2}.$$

We then get

$$x^2 - \left(\frac{2}{5}\right)^2 = \frac{4}{25} \sec^2 \theta - \frac{4}{25} = \frac{4}{25} (\sec^2 \theta - 1) = \frac{4}{25} \tan^2 \theta$$

and

$$\sqrt{x^2 - \left(\frac{2}{5}\right)^2} = \frac{2}{5} |\tan \theta| = \frac{2}{5} \tan \theta. \quad \begin{matrix} \tan \theta > 0 \text{ for} \\ 0 < \theta < \pi/2 \end{matrix}$$

With these substitutions, we have

$$\begin{aligned} \int \frac{dx}{\sqrt{25x^2 - 4}} &= \int \frac{dx}{5\sqrt{x^2 - (4/25)}} = \int \frac{(2/5) \sec \theta \tan \theta d\theta}{5 \cdot (2/5) \tan \theta} \\ &= \frac{1}{5} \int \sec \theta d\theta = \frac{1}{5} \ln |\sec \theta + \tan \theta| + C \\ &= \frac{1}{5} \ln \left| \frac{5x}{2} + \frac{\sqrt{25x^2 - 4}}{2} \right| + C. \end{aligned} \quad \text{From Fig. 8.6}$$



Integration of Rational Functions by Partial Fractions

8.5 Integration of Rational Functions by Partial Fractions



$$\frac{2}{x+1} + \frac{3}{x-3} = \frac{\text{something?}}{\text{something?}}$$

8.5 Integration of Rational Functions by Partial Fractions



$$\frac{2}{x+1} + \frac{3}{x-3} = \frac{2}{(x+1)} + \frac{3}{(x-3)}$$

8.5 Integration of Rational Functions by Partial Fractions



$$\frac{2}{x+1} + \frac{3}{x-3} = \frac{2(x-3)}{(x+1)(x-3)} + \frac{3}{(x-3)}$$

8.5 Integration of Rational Functions by Partial Fractions



$$\frac{2}{x+1} + \frac{3}{x-3} = \frac{2(x-3)}{(x+1)(x-3)} + \frac{3(x+1)}{(x-3)(x+1)}$$

8.5 Integration of Rational Functions by Partial Fractions



$$\begin{aligned}\frac{2}{x+1} + \frac{3}{x-3} &= \frac{2(x-3)}{(x+1)(x-3)} + \frac{3(x+1)}{(x-3)(x+1)} \\ &= \frac{2x-6}{x^2-2x-3} + \frac{3x+3}{x^2-2x-3}\end{aligned}$$

8.5 Integration of Rational Functions by Partial Fractions



$$\begin{aligned}\frac{2}{x+1} + \frac{3}{x-3} &= \frac{2(x-3)}{(x+1)(x-3)} + \frac{3(x+1)}{(x-3)(x+1)} \\&= \frac{2x-6}{x^2-2x-3} + \frac{3x+3}{x^2-2x-3} \\&= \frac{5x-3}{x^2-2x-3}.\end{aligned}$$

8.5 Integration of Rational Functions by Partial Fractions



$$\begin{aligned}\frac{2}{x+1} + \frac{3}{x-3} &= \frac{2(x-3)}{(x+1)(x-3)} + \frac{3(x+1)}{(x-3)(x+1)} \\&= \frac{2x-6}{x^2-2x-3} + \frac{3x+3}{x^2-2x-3} \\&= \frac{5x-3}{x^2-2x-3}.\end{aligned}$$

But how do we do the opposite?

$$\frac{13x+1}{x^2-9} = \frac{\text{something?}}{x-3} + \frac{\text{something?}}{x+3}.$$

8.5 Integration of Rational Functions by Partial Fractions



$$\frac{5x - 3}{x^2 - 2x - 3} = \frac{A}{x + 1} + \frac{B}{x - 3}.$$

Pretend for a moment that we don't know that $A = 2$ and $B = 3$. How can we find A and B ?

8.5 Integration of Rational Functions by Partial Fractions



$$\frac{5x - 3}{x^2 - 2x - 3} = \frac{A}{x + 1} + \frac{B}{x - 3}.$$

Pretend for a moment that we don't know that $A = 2$ and $B = 3$. How can we find A and B ?

$$\frac{5x - 3}{x^2 - 2x - 3} = \frac{A}{(x + 1)} + \frac{B}{(x - 3)}$$

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8.5 Integration of Rational Functions by Partial Fractions



$$\frac{5x - 3}{x^2 - 2x - 3} = \frac{A}{x + 1} + \frac{B}{x - 3}.$$

Pretend for a moment that we don't know that $A = 2$ and $B = 3$. How can we find A and B ?

$$\frac{5x - 3}{x^2 - 2x - 3} = \frac{A(x - 3)}{(x + 1)(x - 3)} + \frac{B(x + 1)}{(x - 3)(x + 1)}$$

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8.5 Integration of Rational Functions by Partial Fractions



$$\frac{5x - 3}{x^2 - 2x - 3} = \frac{A}{x + 1} + \frac{B}{x - 3}.$$

Pretend for a moment that we don't know that $A = 2$ and $B = 3$. How can we find A and B ?

$$\begin{aligned}\frac{5x - 3}{x^2 - 2x - 3} &= \frac{A(x - 3)}{(x + 1)(x - 3)} + \frac{B(x + 1)}{(x - 3)(x + 1)} \\&= \frac{Ax - 3A}{x^2 - 2x - 3} + \frac{Bx + B}{x^2 - 2x - 3} \\&= \end{aligned}$$

8.5 Integration of Rational Functions by Partial Fractions



$$\frac{5x - 3}{x^2 - 2x - 3} = \frac{A}{x + 1} + \frac{B}{x - 3}.$$

Pretend for a moment that we don't know that $A = 2$ and $B = 3$. How can we find A and B ?

$$\begin{aligned}\frac{5x - 3}{x^2 - 2x - 3} &= \frac{A(x - 3)}{(x + 1)(x - 3)} + \frac{B(x + 1)}{(x - 3)(x + 1)} \\&= \frac{Ax - 3A}{x^2 - 2x - 3} + \frac{Bx + B}{x^2 - 2x - 3} \\&= \frac{(A + B)x + (B - 3A)}{x^2 - 2x - 3}.\end{aligned}$$

8.5 Integration of Rational Functions by Partial Fractions



$$\frac{5x - 3}{x^2 - 2x - 3} = \frac{A}{x + 1} + \frac{B}{x - 3}.$$

Pretend for a moment that we don't know that $A = 2$ and $B = 3$. How can we find A and B ?

$$\begin{aligned}\frac{5x - 3}{x^2 - 2x - 3} &= \frac{A(x - 3)}{(x + 1)(x - 3)} + \frac{B(x + 1)}{(x - 3)(x + 1)} \\&= \frac{Ax - 3A}{x^2 - 2x - 3} + \frac{Bx + B}{x^2 - 2x - 3} \\&= \frac{(A + B)x + (B - 3A)}{x^2 - 2x - 3}.\end{aligned}$$

8.5 Integration of Rational Functions by Partial Fractions



$$\frac{5x - 3}{x^2 - 2x - 3} = \frac{A}{x + 1} + \frac{B}{x - 3}.$$

Pretend for a moment that we don't know that $A = 2$ and $B = 3$. How can we find A and B ?

$$\begin{aligned}\frac{5x - 3}{x^2 - 2x - 3} &= \frac{A(x - 3)}{(x + 1)(x - 3)} + \frac{B(x + 1)}{(x - 3)(x + 1)} \\&= \frac{Ax - 3A}{x^2 - 2x - 3} + \frac{Bx + B}{x^2 - 2x - 3} \\&= \frac{(A + B)x + (B - 3A)}{x^2 - 2x - 3}.\end{aligned}$$

Hence

$$\begin{cases} A + B = 5 \\ B - 3A = -3 \end{cases}$$

8.5 Integration of Rational Functions by Partial Fractions



$$\frac{5x - 3}{x^2 - 2x - 3} = \frac{A}{x + 1} + \frac{B}{x - 3}.$$

Pretend for a moment that we don't know that $A = 2$ and $B = 3$. How can we find A and B ?

$$\begin{aligned}\frac{5x - 3}{x^2 - 2x - 3} &= \frac{A(x - 3)}{(x + 1)(x - 3)} + \frac{B(x + 1)}{(x - 3)(x + 1)} \\&= \frac{Ax - 3A}{x^2 - 2x - 3} + \frac{Bx + B}{x^2 - 2x - 3} \\&= \frac{(A + B)x + (B - 3A)}{x^2 - 2x - 3}.\end{aligned}$$

Hence

$$\begin{cases} A + B = 5 \\ B - 3A = -3 \end{cases} \implies \begin{cases} A = 2 \\ B = 3. \end{cases}$$

8.5 Integration of Rational Functions by Partial Fractions



We can use *partial fractions* on $\frac{f(x)}{g(x)}$

8.5 Integration of Rational Functions by Partial Fractions



We can use *partial fractions* on $\frac{f(x)}{g(x)}$ if

- $\left(\begin{array}{c} \text{the degree} \\ \text{of } f(x) \end{array} \right) < \left(\begin{array}{c} \text{the degree} \\ \text{of } g(x) \end{array} \right);$

8.5 Integration of Rational Functions by Partial Fractions



We can use *partial fractions* on $\frac{f(x)}{g(x)}$ if

- $\left(\begin{array}{c} \text{the degree} \\ \text{of } f(x) \end{array} \right) < \left(\begin{array}{c} \text{the degree} \\ \text{of } g(x) \end{array} \right)$; and
- we can factorise $g(x)$.

8.5 Integration of Rational Functions by Partial Fractions



Example

Write $\frac{x^2 + x + 2}{(x + 1)(x^2 + 1)}$ in partial fractions.

8.5 Integration of Rational Functions by Partial Fractions



Example

Write $\frac{x^2 + x + 2}{(x + 1)(x^2 + 1)}$ in partial fractions.

$$\frac{x^2 + x + 2}{(x + 1)(x^2 + 1)} = \frac{A}{x + 1} + \frac{Bx + C}{x^2 + 1}$$

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8.5 Integration of Rational Functions by Partial Fractions



Example

Write $\frac{x^2 + x + 2}{(x + 1)(x^2 + 1)}$ in partial fractions.

$$\begin{aligned}\frac{x^2 + x + 2}{(x + 1)(x^2 + 1)} &= \frac{A}{x + 1} + \frac{Bx + C}{x^2 + 1} \\&= \frac{A(x^2 + 1)}{(x + 1)(x^2 + 1)} + \frac{(Bx + C)(x + 1)}{(x^2 + 1)(x + 1)} \\&= \\&= \\&= .\end{aligned}$$

8.5 Integration of Rational Functions by Partial Fractions



Example

Write $\frac{x^2 + x + 2}{(x + 1)(x^2 + 1)}$ in partial fractions.

$$\begin{aligned}\frac{x^2 + x + 2}{(x + 1)(x^2 + 1)} &= \frac{A}{x + 1} + \frac{Bx + C}{x^2 + 1} \\&= \frac{A(x^2 + 1)}{(x + 1)(x^2 + 1)} + \frac{(Bx + C)(x + 1)}{(x^2 + 1)(x + 1)} \\&= \frac{Ax^2 + A + Bx^2 + Bx + Cx + C}{(x + 1)(x^2 + 1)}\end{aligned}$$

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8.5 Integration of Rational Functions by Partial Fractions



Example

Write $\frac{x^2 + x + 2}{(x + 1)(x^2 + 1)}$ in partial fractions.

$$\begin{aligned}\frac{x^2 + x + 2}{(x + 1)(x^2 + 1)} &= \frac{A}{x + 1} + \frac{Bx + C}{x^2 + 1} \\&= \frac{A(x^2 + 1)}{(x + 1)(x^2 + 1)} + \frac{(Bx + C)(x + 1)}{(x^2 + 1)(x + 1)} \\&= \frac{Ax^2 + A + Bx^2 + Bx + Cx + C}{(x + 1)(x^2 + 1)} \\&= \frac{(A + B)x^2 + (B + C)x + (A + C)}{(x + 1)(x^2 + 1)} \\&= \end{aligned}$$

8.5 Integration of Rational Functions by Partial Fractions



Example

Write $\frac{x^2 + x + 2}{(x + 1)(x^2 + 1)}$ in partial fractions.

$$\begin{aligned}\frac{x^2 + x + 2}{(x + 1)(x^2 + 1)} &= \frac{A}{x + 1} + \frac{Bx + C}{x^2 + 1} \\&= \frac{A}{(x + 1)} + \frac{(Bx + C)}{(x^2 + 1)} \\&= \frac{Ax^2 + A + Bx^2 + Bx + Cx + C}{(x + 1)(x^2 + 1)} \\&= \frac{(A + B)x^2 + (B + C)x + (A + C)}{(x + 1)(x^2 + 1)} \\&= \end{aligned}$$

$$\begin{aligned}A + B &= 1 \\B + C &= 1 \\A + C &= 2\end{aligned}$$

$$\frac{(x + C)(x + 1)}{(x^2 + 1)(x + 1)}$$

8.5 Integration of Rational Functions by Partial Fractions



Example

Write $\frac{x^2 + x + 2}{(x + 1)(x^2 + 1)}$ in partial fractions.

$$\begin{aligned}\frac{x^2 + x + 2}{(x + 1)(x^2 + 1)} &= \frac{A}{x + 1} + \frac{Bx + C}{x^2 + 1} \\&= \frac{A}{(x + 1)} + \frac{Bx + C}{x^2 + 1} \\&= \frac{Ax^2 + A + Bx^2 + Bx + C}{(x + 1)(x^2 + 1)} \\&= \frac{(A + B)x^2 + (B + C)x + (A + C)}{(x + 1)(x^2 + 1)} \\&= \end{aligned}$$

A + B = 1
B + C = 1
A + C = 2

A = 1
B = 0
C = 1

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8.5 Integration of Rational Functions by Partial Fractions



Example

Write $\frac{x^2 + x + 2}{(x + 1)(x^2 + 1)}$ in partial fractions.

$$\begin{aligned}\frac{x^2 + x + 2}{(x + 1)(x^2 + 1)} &= \frac{A}{x + 1} + \frac{Bx + C}{x^2 + 1} \\&= \frac{A(x^2 + 1)}{(x + 1)(x^2 + 1)} + \left| \begin{array}{l} A = 1 \\ B = 0 \\ C = 1 \end{array} \right| \\&= \frac{Ax^2 + A + Bx^2 + C}{(x + 1)(x^2 + 1)} \\&= \frac{(A + B)x^2 + (B + C)x + (A + C)}{(x + 1)(x^2 + 1)} \\&= \frac{1}{x + 1} + \frac{1}{x^2 + 1}.\end{aligned}$$

8.5 Integration of Rational Functions by Partial Fractions



Example

Write $\frac{3x^3 + 2x^2 + x}{(x + 3)^4}$ in partial fractions.

8.5 Integration of Rational Functions by Partial Fractions



Example

Write $\frac{3x^3 + 2x^2 + x}{(x + 3)^4}$ in partial fractions.

$$\frac{3x^3 + 2x^2 + x}{(x + 3)^4} = \frac{A}{x + 3} + \frac{B}{(x + 3)^2} + \frac{C}{(x + 3)^3} + \frac{D}{(x + 3)^4} = \dots$$

8.5 Integration of Rational Functions by Partial Fractions



Example

Write $\frac{3x^3 + 2x^2 + x}{(x + 3)^4}$ in partial fractions.

$$\frac{3x^3 + 2x^2 + x}{(x + 3)^4} = \frac{A}{x + 3} + \frac{B}{(x + 3)^2} + \frac{C}{(x + 3)^3} + \frac{D}{(x + 3)^4} = \dots$$

Example

Write $\frac{71}{(x + 3)(x^2 + 2x + 3)^2}$ in partial fractions.

8.5 Integration of Rational Functions by Partial Fractions



Example

Write $\frac{3x^3 + 2x^2 + x}{(x + 3)^4}$ in partial fractions.

$$\frac{3x^3 + 2x^2 + x}{(x + 3)^4} = \frac{A}{x + 3} + \frac{B}{(x + 3)^2} + \frac{C}{(x + 3)^3} + \frac{D}{(x + 3)^4} = \dots$$

Example

Write $\frac{71}{(x + 3)(x^2 + 2x + 3)^2}$ in partial fractions.

$$\begin{aligned}\frac{71}{(x + 3)(x^2 + 2x + 3)^2} &= \frac{A}{x + 3} + \frac{Bx + C}{(x^2 + 2x + 3)} + \frac{Dx + E}{(x^2 + 2x + 3)^2} \\ &= \dots\end{aligned}$$

Method of Partial Fractions When $f(x)/g(x)$ Is Proper

- Let $x - r$ be a linear factor of $g(x)$. Suppose that $(x - r)^m$ is the highest power of $x - r$ that divides $g(x)$. Then, to this factor, assign the sum of the m partial fractions:

$$\frac{A_1}{(x - r)} + \frac{A_2}{(x - r)^2} + \cdots + \frac{A_m}{(x - r)^m}.$$

Do this for each distinct linear factor of $g(x)$.

- Let $x^2 + px + q$ be an irreducible quadratic factor of $g(x)$ so that $x^2 + px + q$ has no real roots. Suppose that $(x^2 + px + q)^n$ is the highest power of this factor that divides $g(x)$. Then, to this factor, assign the sum of the n partial fractions:

$$\frac{B_1x + C_1}{(x^2 + px + q)} + \frac{B_2x + C_2}{(x^2 + px + q)^2} + \cdots + \frac{B_nx + C_n}{(x^2 + px + q)^n}.$$

Do this for each distinct quadratic factor of $g(x)$.

- Set the original fraction $f(x)/g(x)$ equal to the sum of all these partial fractions. Clear the resulting equation of fractions and arrange the terms in decreasing powers of x .
- Equate the coefficients of corresponding powers of x and solve the resulting equations for the undetermined coefficients.

8.5 Integration of Rational Functions by Partial Fractions



Example

Use partial fractions to find $\int \frac{x^2 + 4x + 1}{(x - 1)(x + 1)(x + 3)} dx$.

8.5 Integration of Rational Functions by Partial Fractions



Since

$$\frac{x^2 + 4x + 1}{(x - 1)(x + 1)(x + 3)} = \frac{A}{x - 1} + \frac{B}{x + 1} + \frac{C}{x + 3}$$

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8.5 Integration of Rational Functions by Partial Fractions



Since

$$\begin{aligned}& \frac{x^2 + 4x + 1}{(x - 1)(x + 1)(x + 3)} \\&= \frac{A}{x - 1} + \frac{B}{x + 1} + \frac{C}{x + 3} \\&= \frac{A(x + 1)(x + 3) + B(x - 1)(x + 3) + C(x - 1)(x + 1)}{(x - 1)(x + 1)(x + 3)}\end{aligned}$$

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8.5 Integration of Rational Functions by Partial Fractions



Since

$$\begin{aligned}& \frac{x^2 + 4x + 1}{(x - 1)(x + 1)(x + 3)} \\&= \frac{A}{x - 1} + \frac{B}{x + 1} + \frac{C}{x + 3} \\&= \frac{A(x + 1)(x + 3) + B(x - 1)(x + 3) + C(x - 1)(x + 1)}{(x - 1)(x + 1)(x + 3)} \\&= \frac{A(x^2 + 4x + 3) + B(x^2 + 2x - 3) + C(x^2 - 1)}{(x - 1)(x + 1)(x + 3)}\end{aligned}$$

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8.5 Integration of Rational Functions by Partial Fractions



Since

$$\begin{aligned} & \frac{x^2 + 4x + 1}{(x - 1)(x + 1)(x + 3)} \\ &= \frac{A}{x - 1} + \frac{B}{x + 1} + \frac{C}{x + 3} \\ &= \frac{A(x + 1)(x + 3) + B(x - 1)(x + 3) + C(x - 1)(x + 1)}{(x - 1)(x + 1)(x + 3)} \\ &= \frac{A(x^2 + 4x + 3) + B(x^2 + 2x - 3) + C(x^2 - 1)}{(x - 1)(x + 1)(x + 3)} \\ &= \frac{(A + B + C)x^2 + (4A + 2B)x + (3A - 3B - C)}{(x - 1)(x + 1)(x + 3)} \\ &= \end{aligned}$$

8.5 Integration of Rational Functions by Partial Fractions



Since

$$\begin{aligned} & \frac{x^2 + 4x + 1}{(x - 1)(x + 1)(x + 3)} \\ &= \frac{A}{x - 1} + \frac{B}{x + 1} + \boxed{\begin{array}{l} A + B + C = 1 \\ 4A + 2B = 4 \\ 3A - 3B - C = 1 \end{array}} \frac{(x - 1)(x + 1)}{x^2 - 1} \\ &= \frac{A(x^2 + 4x + 3)}{(x - 1)(x + 1)(x + 3)} \\ &= \frac{(A + B + C)x^2 + (4A + 2B)x + (3A - 3B - C)}{(x - 1)(x + 1)(x + 3)} \end{aligned}$$

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8.5 Integration of Rational Functions by Partial Fractions



Since

$$\begin{aligned} & \frac{x^2 + 4x + 1}{(x - 1)(x + 1)(x + 3)} \\ &= \frac{A}{x - 1} + \frac{B}{x + 1} + \boxed{\begin{array}{l} A + B + C = 1 \\ 4A + 2B = 4 \\ 3A - 3B - C = 1 \end{array}} \\ &= \frac{A(x + 1)(x + 3)}{(x - 1)(x + 1)(x + 3)} \\ &= \frac{A(x^2 + 4x + 3)}{(x - 1)(x + 1)(x + 3)} \\ &= \frac{(A + B + C)x^2 + (4A + 2B)x + (3A - 3B - C)}{(x - 1)(x + 1)(x + 3)} \\ &= \end{aligned}$$

$A = \frac{3}{4}$
 $B = \frac{1}{2}$
 $C = -\frac{1}{4}$

1)

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8.5 Integration of Rational Functions by Partial Fractions



Since

$$\begin{aligned} & \frac{x^2 + 4x + 1}{(x - 1)(x + 1)(x + 3)} \\ &= \frac{A}{x - 1} + \frac{B}{x + 1} + \frac{C}{x + 3} \\ &= \frac{A(x + 1)(x + 3) + B(x - 1)(x + 3) + C(x - 1)}{(x - 1)(x + 1)(x + 3)} \\ &= \frac{A(x^2 + 4x + 3) + B(x^2 + 2x - 3) + Cx - C}{(x - 1)(x + 1)(x + 3)} \\ &= \frac{(A + B + C)x^2 + (4A + 2B)x + (3A - 3B - C)}{(x - 1)(x + 1)(x + 3)} \\ &= \frac{\frac{3}{4}}{x - 1} + \frac{\frac{1}{2}}{x + 1} + \frac{-\frac{1}{4}}{x + 3} \end{aligned}$$

$A = \frac{3}{4}$
 $B = \frac{1}{2}$
 $C = -\frac{1}{4}$

1)

8.5

$$\frac{x^2+4x+1}{(x-1)(x+1)(x+3)} = \frac{\frac{3}{4}}{x-1} + \frac{\frac{1}{2}}{x+1} + \frac{-\frac{1}{4}}{x+3}$$



We have that

$$\begin{aligned}\int \frac{x^2+4x+1}{(x-1)(x+1)(x+3)} dx \\ &= \int \frac{\frac{3}{4}}{x-1} + \frac{\frac{1}{2}}{x+1} + \frac{-\frac{1}{4}}{x+3} dx\end{aligned}$$

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8.5

$$\frac{x^2+4x+1}{(x-1)(x+1)(x+3)} = \frac{\frac{3}{4}}{x-1} + \frac{\frac{1}{2}}{x+1} + \frac{-\frac{1}{4}}{x+3}$$



We have that

$$\begin{aligned} & \int \frac{x^2 + 4x + 1}{(x-1)(x+1)(x+3)} dx \\ &= \int \frac{\frac{3}{4}}{x-1} + \frac{\frac{1}{2}}{x+1} + \frac{-\frac{1}{4}}{x+3} dx \\ &= \frac{3}{4} \int \frac{dx}{x-1} + \frac{1}{2} \int \frac{dx}{x+1} - \frac{1}{4} \int \frac{dx}{x+2} \\ &= \end{aligned}$$

8.5

$$\frac{x^2+4x+1}{(x-1)(x+1)(x+3)} = \frac{\frac{3}{4}}{x-1} + \frac{\frac{1}{2}}{x+1} + \frac{-\frac{1}{4}}{x+3}$$



We have that

$$\begin{aligned} & \int \frac{x^2+4x+1}{(x-1)(x+1)(x+3)} dx \\ &= \int \frac{\frac{3}{4}}{x-1} + \frac{\frac{1}{2}}{x+1} + \frac{-\frac{1}{4}}{x+3} dx \\ &= \frac{3}{4} \int \frac{dx}{x-1} + \frac{1}{2} \int \frac{dx}{x+1} - \frac{1}{4} \int \frac{dx}{x+2} \\ &= \frac{3}{4} \ln|x-1| + \frac{1}{2} \ln|x+1| - \frac{1}{4} \ln|x+2| + K. \end{aligned}$$

(I already used C)

EXAMPLE 2 Use partial fractions to evaluate

$$\int \frac{6x + 7}{(x + 2)^2} dx.$$

Solution First we express the integrand as a sum of partial fractions with undetermined coefficients.

$$\frac{6x + 7}{(x + 2)^2} = \frac{A}{x + 2} + \frac{B}{(x + 2)^2}$$

Two terms because $(x + 2)$ is squared

$$\begin{aligned} 6x + 7 &= A(x + 2) + B \\ &= Ax + (2A + B) \end{aligned}$$

Multiply both sides by $(x + 2)^2$.

Equating coefficients of corresponding powers of x gives

$$A = 6 \quad \text{and} \quad 2A + B = 12 + B = 7, \quad \text{or} \quad A = 6 \quad \text{and} \quad B = -5.$$

Therefore,

$$\begin{aligned} \int \frac{6x + 7}{(x + 2)^2} dx &= \int \left(\frac{6}{x + 2} - \frac{5}{(x + 2)^2} \right) dx \\ &= 6 \int \frac{dx}{x + 2} - 5 \int (x + 2)^{-2} dx \\ &= 6 \ln |x + 2| + 5(x + 2)^{-1} + C. \end{aligned}$$



EXAMPLE 3

Use partial fractions to evaluate

$$\int \frac{2x^3 - 4x^2 - x - 3}{x^2 - 2x - 3} dx.$$

Solution First we divide the denominator into the numerator to get a polynomial plus a proper fraction.

$$\begin{array}{r} 2x \\ x^2 - 2x - 3 \overline{)2x^3 - 4x^2 - x - 3} \\ 2x^3 - 4x^2 - 6x - 3 \\ \hline 5x - 3 \end{array}$$

Then we write the improper fraction as a polynomial plus a proper fraction.

$$\frac{2x^3 - 4x^2 - x - 3}{x^2 - 2x - 3} = 2x + \frac{5x - 3}{x^2 - 2x - 3}$$

We found the partial fraction decomposition of the fraction on the right in the opening example, so

$$\begin{aligned} \int \frac{2x^3 - 4x^2 - x - 3}{x^2 - 2x - 3} dx &= \int 2x dx + \int \frac{5x - 3}{x^2 - 2x - 3} dx \\ &= \int 2x dx + \int \frac{2}{x + 1} dx + \int \frac{3}{x - 3} dx \\ &= x^2 + 2 \ln |x + 1| + 3 \ln |x - 3| + C. \end{aligned}$$



EXAMPLE 4 Use partial fractions to evaluate

$$\int \frac{-2x + 4}{(x^2 + 1)(x - 1)^2} dx.$$

Solution The denominator has an irreducible quadratic factor $x^2 + 1$ as well as a repeated linear factor $(x - 1)^2$, so we write

$$\frac{-2x + 4}{(x^2 + 1)(x - 1)^2} = \frac{Ax + B}{x^2 + 1} + \frac{C}{x - 1} + \frac{D}{(x - 1)^2}. \quad (2)$$

Clearing the equation of fractions gives

$$\begin{aligned}-2x + 4 &= (Ax + B)(x - 1)^2 + C(x - 1)(x^2 + 1) + D(x^2 + 1) \\&= (A + C)x^3 + (-2A + B - C + D)x^2 \\&\quad + (A - 2B + C)x + (B - C + D).\end{aligned}$$

Equating coefficients of like terms gives

$$\text{Coefficients of } x^3: \quad 0 = A + C$$

$$\text{Coefficients of } x^2: \quad 0 = -2A + B - C + D$$

$$\text{Coefficients of } x^1: \quad -2 = A - 2B + C$$

$$\text{Coefficients of } x^0: \quad 4 = B - C + D$$

We solve these equations simultaneously to find the values of A , B , C , and D :

$$-4 = -2A, \quad A = 2 \quad \text{Subtract fourth equation from second.}$$

$$C = -A = -2 \quad \text{From the first equation}$$

$$B = (A + C + 2)/2 = 1 \quad \text{From the third equation and } C = -A$$

$$D = 4 - B + C = 1. \quad \text{From the fourth equation}$$

We substitute these values into Equation (2), obtaining

$$\frac{-2x + 4}{(x^2 + 1)(x - 1)^2} = \frac{2x + 1}{x^2 + 1} - \frac{2}{x - 1} + \frac{1}{(x - 1)^2}.$$

Finally, using the expansion above we can integrate:

$$\begin{aligned}\int \frac{-2x + 4}{(x^2 + 1)(x - 1)^2} dx &= \int \left(\frac{2x + 1}{x^2 + 1} - \frac{2}{x - 1} + \frac{1}{(x - 1)^2} \right) dx \\&= \int \left(\frac{2x}{x^2 + 1} + \frac{1}{x^2 + 1} - \frac{2}{x - 1} + \frac{1}{(x - 1)^2} \right) dx \\&= \ln(x^2 + 1) + \tan^{-1} x - 2 \ln|x - 1| - \frac{1}{x - 1} + C.\end{aligned}$$

■

EXAMPLE 5 Use partial fractions to evaluate

$$\int \frac{dx}{x(x^2 + 1)^2}.$$

Solution The form of the partial fraction decomposition is

$$\frac{1}{x(x^2 + 1)^2} = \frac{A}{x} + \frac{Bx + C}{x^2 + 1} + \frac{Dx + E}{(x^2 + 1)^2}.$$

Multiplying by $x(x^2 + 1)^2$, we have

$$\begin{aligned} 1 &= A(x^2 + 1)^2 + (Bx + C)x(x^2 + 1) + (Dx + E)x \\ &= A(x^4 + 2x^2 + 1) + B(x^4 + x^2) + C(x^3 + x) + Dx^2 + Ex \\ &= (A + B)x^4 + Cx^3 + (2A + B + D)x^2 + (C + E)x + A. \end{aligned}$$

If we equate coefficients, we get the system

$$A + B = 0, \quad C = 0, \quad 2A + B + D = 0, \quad C + E = 0, \quad A = 1.$$

Solving this system gives $A = 1$, $B = -1$, $C = 0$, $D = -1$, and $E = 0$. Thus,

Solving this system gives $A = 1$, $B = -1$, $C = 0$, $D = -1$, and $E = 0$. Thus,

$$\begin{aligned}\int \frac{dx}{x(x^2 + 1)^2} &= \int \left[\frac{1}{x} + \frac{-x}{x^2 + 1} + \frac{-x}{(x^2 + 1)^2} \right] dx \\&= \int \frac{dx}{x} - \int \frac{x \, dx}{x^2 + 1} - \int \frac{x \, dx}{(x^2 + 1)^2} \\&= \int \frac{dx}{x} - \frac{1}{2} \int \frac{du}{u} - \frac{1}{2} \int \frac{du}{u^2} && u = x^2 + 1, \\&= \ln |x| - \frac{1}{2} \ln |u| + \frac{1}{2u} + K \\&= \ln |x| - \frac{1}{2} \ln (x^2 + 1) + \frac{1}{2(x^2 + 1)} + K \\&= \ln \frac{|x|}{\sqrt{x^2 + 1}} + \frac{1}{2(x^2 + 1)} + K.\end{aligned}$$



8.5 Integration of Rational Functions by Partial Fractions



Remark

When we have

$$\frac{f(x)}{(x - r_1)(x - r_2) \cdots (x - r_n)},$$

where r_1, r_2, \dots, r_n are all different, there is a quicker way to find partial fractions.

EXAMPLE 6 Find A , B , and C in the partial fraction expansion

$$\frac{x^2 + 1}{(x - 1)(x - 2)(x - 3)} = \frac{A}{x - 1} + \frac{B}{x - 2} + \frac{C}{x - 3}. \quad (3)$$

Solution If we multiply both sides of Equation (3) by $(x - 1)$ to get

$$\frac{x^2 + 1}{(x - 2)(x - 3)} = A + \frac{B(x - 1)}{x - 2} + \frac{C(x - 1)}{x - 3}$$

and set $x = 1$, the resulting equation gives the value of A :

$$\begin{aligned}\frac{(1)^2 + 1}{(1 - 2)(1 - 3)} &= A + 0 + 0, \\ A &= 1.\end{aligned}$$

In exactly the same way, we can multiply both sides by $(x - 2)$ and then substitute in $x = 2$. This gives

$$\frac{(2)^2 + 1}{(2 - 1)(2 - 3)} = B.$$

So $B = -5$. Finally, we multiply both sides by $(x - 3)$ and then substitute in $x = 3$, which yields

$$\frac{(3)^2 + 1}{(3 - 1)(3 - 2)} = C,$$

and $C = 5$.



8.5 Integration of Rational Functions by Partial Fractions



Example

$$\text{Find } \int \frac{x+4}{x^3 + 3x^2 - 10x} dx.$$

8.5 Integration of Rational Functions by Partial Fractions



Example

Find $\int \frac{x+4}{x^3 + 3x^2 - 10x} dx.$

First we have

$$\frac{x+4}{x^3 + 3x^2 - 10x} = \frac{x+4}{x(x-2)(x+5)}$$

=

8.5 Integration of Rational Functions by Partial Fractions



Example

Find $\int \frac{x+4}{x^3 + 3x^2 - 10x} dx$.

First we have

$$\begin{aligned}\frac{x+4}{x^3 + 3x^2 - 10x} &= \frac{x+4}{x(x-2)(x+5)} \\ &= \frac{A}{x} + \frac{B}{x-2} + \frac{C}{x+5}.\end{aligned}$$

8.5 Integration of Rational Functions by Partial Fractions



Example

Find $\int \frac{x+4}{x^3 + 3x^2 - 10x} dx.$

First we have

$$\begin{aligned}\frac{x+4}{x^3 + 3x^2 - 10x} &= \frac{x+4}{x(x-2)(x+5)} \\ &= \frac{A}{x} + \frac{B}{x-2} + \frac{C}{x+5}.\end{aligned}$$

- 1 multiply by x , then set $x = 0$;
- 2 multiply by $(x - 2)$, then set $x = 2$;
- 3 multiply by $(x + 5)$, then set $x = -5$.

8.5

$$\frac{x+4}{x(x-2)(x+5)} = \frac{A}{x} + \frac{B}{x-2} + \frac{C}{x+5}$$



- 1 multiply by x , then set $x = 0$;

$$\frac{x+4}{(x-2)(x+5)} = A + \frac{Bx}{x-2} + \frac{Cx}{x+5}$$

$$\frac{4}{(-2)(5)} = A + 0 + 0$$

$$-\frac{2}{5} = A$$

8.5

$$\frac{x+4}{x(x-2)(x+5)} = \frac{A}{x} + \frac{B}{x-2} + \frac{C}{x+5}$$



- 2 multiply by $(x - 2)$, then set $x = 2$;

$$\frac{x+4}{x(x+5)} = \frac{A(x-2)}{x} + B + \frac{C(x-2)}{x+5}$$

$$\frac{2+4}{(2)(7)} = 0 + B + 0$$

$$\frac{3}{7} = B$$

8.5

$$\frac{x+4}{x(x-2)(x+5)} = \frac{A}{x} + \frac{B}{x-2} + \frac{C}{x+5}$$



- 3 multiply by $(x + 5)$, then set $x = -5$.

$$\begin{aligned}\frac{x+4}{x(x-2)} &= \frac{A(x+5)}{x} + \frac{B(x+5)}{x-2} + C \\ \frac{-5+4}{(-5)(-7)} &= 0 + 0 + C \\ -\frac{1}{35} &= C\end{aligned}$$

8.5 Integration of Rational Functions by Partial Fractions



Therefore

$$\frac{x+4}{x(x-2)(x+5)} = \frac{-\frac{2}{5}}{x} + \frac{\frac{3}{7}}{x-2} + \frac{-\frac{1}{35}}{x+5}$$

8.5 Integration of Rational Functions by Partial Fractions



Therefore

$$\frac{x+4}{x(x-2)(x+5)} = \frac{-\frac{2}{5}}{x} + \frac{\frac{3}{7}}{x-2} + \frac{-\frac{1}{35}}{x+5}$$

and thus

$$\int \frac{x+4}{x(x-2)(x+5)} dx = -\frac{2}{5} \ln|x| + \frac{3}{7} \ln|x-2| - \frac{1}{35} \ln|x+5| + C.$$

8.5 Integration of Rational Functions by Partial Fractions



Remark

We can also use differentiation to find partial fractions.

EXAMPLE 7 Find A , B , and C in the equation

$$\frac{x - 1}{(x + 1)^3} = \frac{A}{x + 1} + \frac{B}{(x + 1)^2} + \frac{C}{(x + 1)^3}$$

by clearing fractions, differentiating the result, and substituting $x = -1$.

Solution We first clear fractions:

$$x - 1 = A(x + 1)^2 + B(x + 1) + C.$$

Substituting $x = -1$ shows $C = -2$. We then differentiate both sides with respect to x , obtaining

$$1 = 2A(x + 1) + B.$$

Substituting $x = -1$ shows $B = 1$. We differentiate again to get $0 = 2A$, which shows $A = 0$. Hence,

$$\frac{x - 1}{(x + 1)^3} = \frac{1}{(x + 1)^2} - \frac{2}{(x + 1)^3}.$$



8.5 Integration of Rational Functions by Partial Fractions



Remark

Sometimes we can just try putting in small numbers $x = 0$, $x = \pm 1$, $x = \pm 2$, etc. to find the coefficients A, B, C, \dots

EXAMPLE 8 Find A , B , and C in the expression

$$\frac{x^2 + 1}{(x - 1)(x - 2)(x - 3)} = \frac{A}{x - 1} + \frac{B}{x - 2} + \frac{C}{x - 3}$$

by assigning numerical values to x .

Solution Clear fractions to get

$$x^2 + 1 = A(x - 2)(x - 3) + B(x - 1)(x - 3) + C(x - 1)(x - 2).$$

Then let $x = 1, 2, 3$ successively to find A , B , and C :

$$x = 1: \quad (1)^2 + 1 = A(-1)(-2) + B(0) + C(0)$$

$$2 = 2A$$

$$A = 1$$

$$x = 2: \quad (2)^2 + 1 = A(0) + B(1)(-1) + C(0)$$

$$5 = -B$$

$$B = -5$$

$$x = 3: \quad (3)^2 + 1 = A(0) + B(0) + C(2)(1)$$

$$10 = 2C$$

$$C = 5.$$

Conclusion:

$$\frac{x^2 + 1}{(x - 1)(x - 2)(x - 3)} = \frac{1}{x - 1} - \frac{5}{x - 2} + \frac{5}{x - 3}.$$

Break

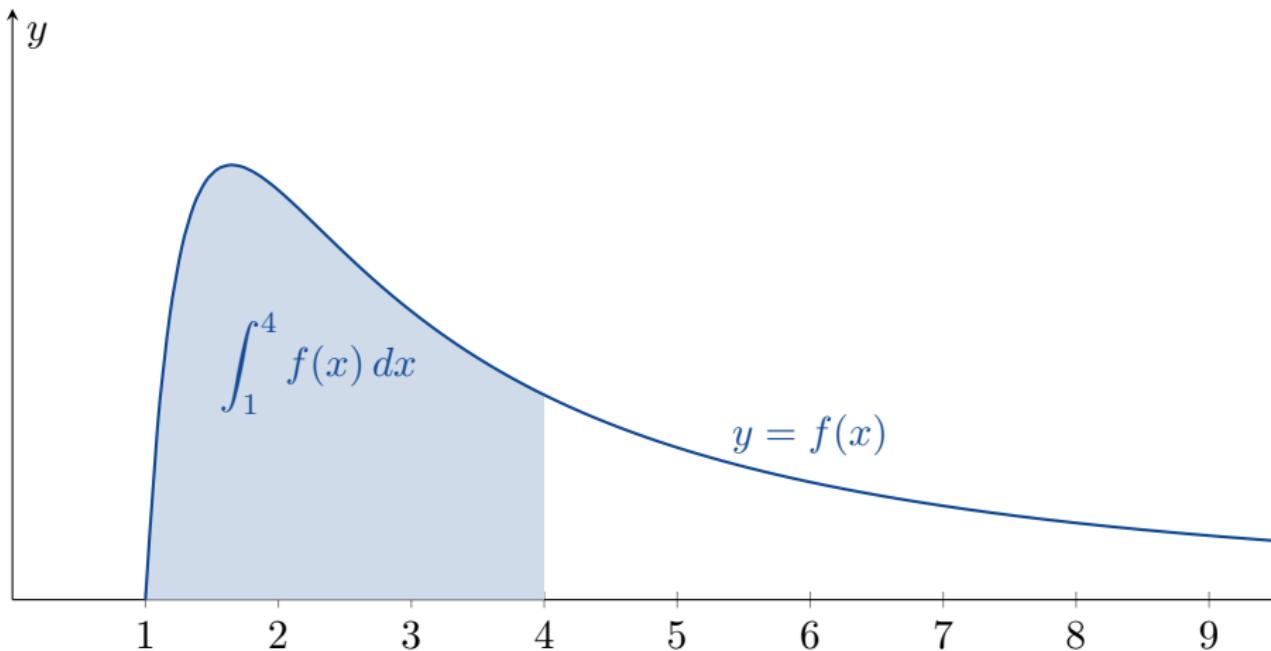
We will continue at 2pm



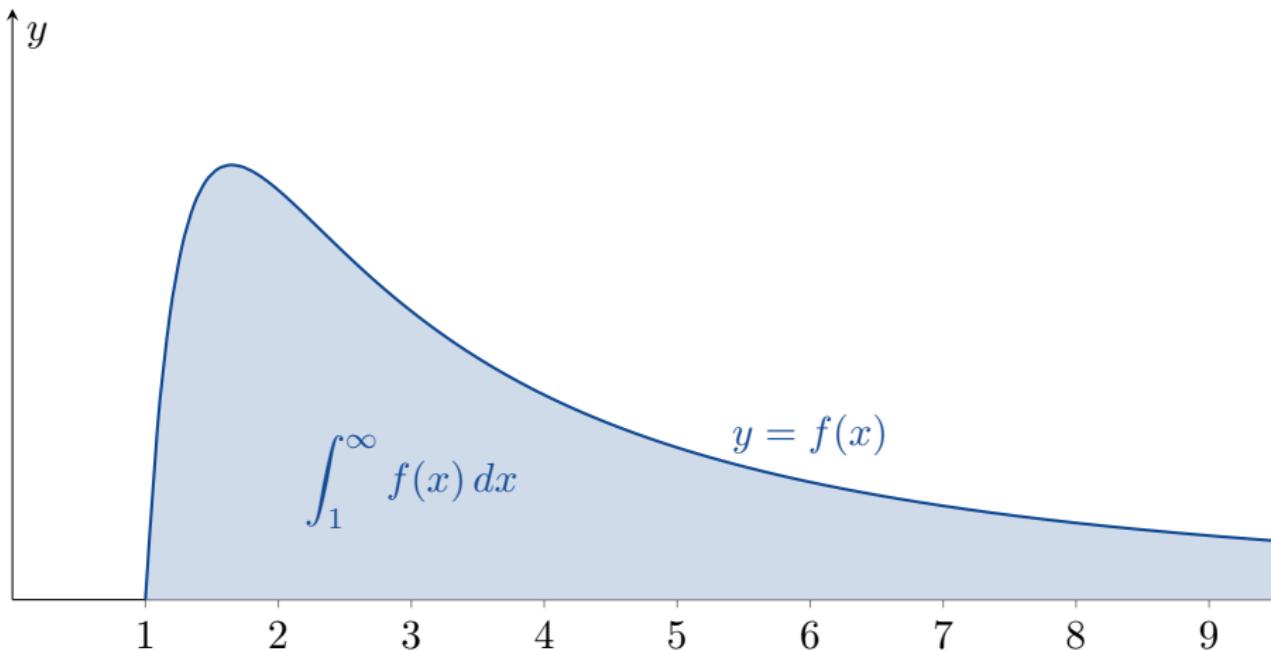


Improper Integrals

8.8 Improper Integrals



8.8 Improper Integrals



8.8 Improper Integrals



We need to use limits.

8.8 Improper Integrals

Example

Calculate $\int_0^\infty e^{-\frac{x}{2}} dx.$

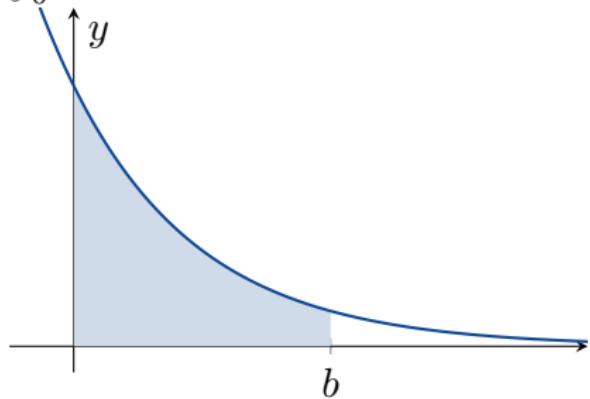
8.8 Improper Integrals

Example

Calculate $\int_0^\infty e^{-\frac{x}{2}} dx.$

Step 1:

$$\int_0^b e^{-\frac{x}{2}} dx = ?$$



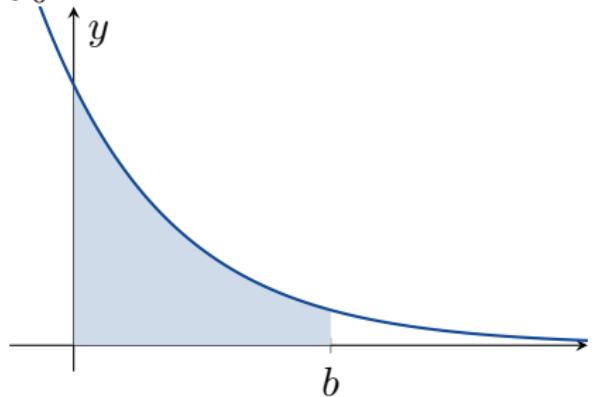
8.8 Improper Integrals

Example

Calculate $\int_0^\infty e^{-\frac{x}{2}} dx$.

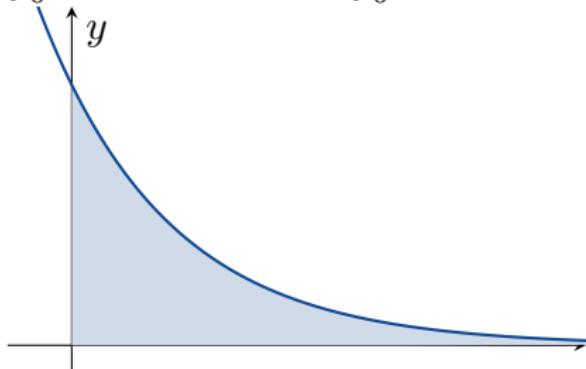
Step 1:

$$\int_0^b e^{-\frac{x}{2}} dx = ?$$



Step 2:

$$\int_0^\infty e^{-\frac{x}{2}} dx = \lim_{b \rightarrow \infty} \int_0^b e^{-\frac{x}{2}} dx$$



8.8 Improper Integrals

Since

$$\int_0^b e^{-\frac{x}{2}} dx = \left[-2e^{-\frac{x}{2}} \right]_0^b = -2e^{-\frac{b}{2}} + 2,$$

8.8 Improper Integrals

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we have that

$$\int_0^\infty e^{-\frac{x}{2}} dx = \lim_{b \rightarrow \infty} \int_0^b e^{-\frac{x}{2}} dx = \lim_{b \rightarrow \infty} \left(-2e^{-\frac{b}{2}} + 2 \right) = 2.$$

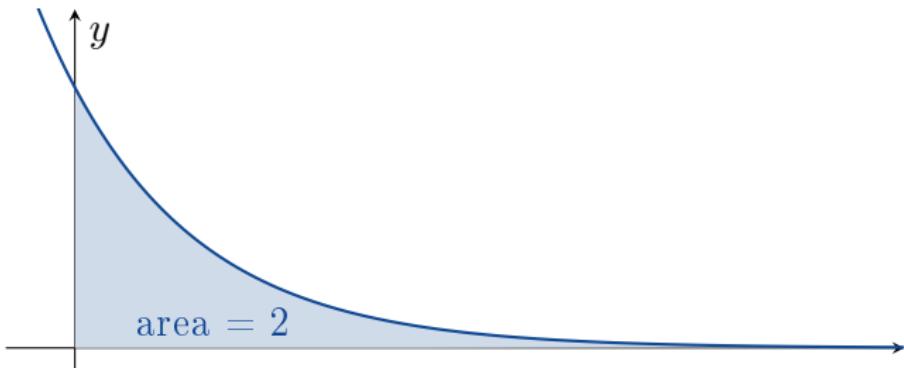
8.8 Improper Integrals

Since

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DEFINITION Integrals with infinite limits of integration are **improper integrals of Type I**.

1. If $f(x)$ is continuous on $[a, \infty)$, then

$$\int_a^{\infty} f(x) dx = \lim_{b \rightarrow \infty} \int_a^b f(x) dx.$$

2. If $f(x)$ is continuous on $(-\infty, b]$, then

$$\int_{-\infty}^b f(x) dx = \lim_{a \rightarrow -\infty} \int_a^b f(x) dx.$$

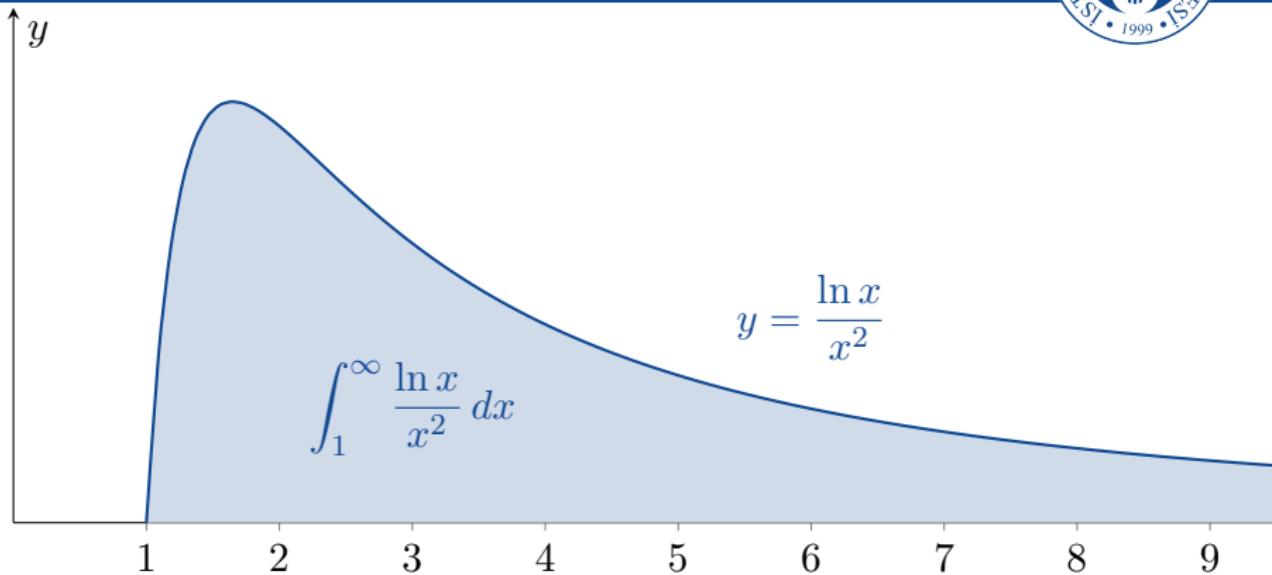
3. If $f(x)$ is continuous on $(-\infty, \infty)$, then

$$\int_{-\infty}^{\infty} f(x) dx = \int_{-\infty}^c f(x) dx + \int_c^{\infty} f(x) dx,$$

where c is any real number.

In each case, if the limit exists and is finite, we say that the improper integral **converges** and that the limit is the **value** of the improper integral. If the limit fails to exist, the improper integral **diverges**.

8.8 Improper Integrals



Example

Is the area under the curve $y = \frac{\ln x}{x^2}$, from $x = 1$ to $x = \infty$, finite? Is so, what is its value?

8.8 Improper Integrals

Since

$$\begin{aligned}\int_1^b \frac{\ln x}{x^2} dx &= \left[(\ln x) \left(-\frac{1}{x} \right) \right]_1^b - \int_1^b \left(-\frac{1}{x} \right) \left(\frac{1}{x} \right) dx \\ &= -\frac{\ln b}{b} - \left[\frac{1}{x} \right]_1^b = -\frac{\ln b}{b} - \frac{1}{b} + 1,\end{aligned}$$

8.8 Improper Integrals

Since

$$\begin{aligned}\int_1^b \frac{\ln x}{x^2} dx &= \left[(\ln x) \left(-\frac{1}{x} \right) \right]_1^b - \int_1^b \left(-\frac{1}{x} \right) \left(\frac{1}{x} \right) dx \\ &= -\frac{\ln b}{b} - \left[\frac{1}{x} \right]_1^b = -\frac{\ln b}{b} - \frac{1}{b} + 1,\end{aligned}$$

we have that

$$\int_1^\infty \frac{\ln x}{x^2} dx = \lim_{b \rightarrow \infty} \int_1^b \frac{\ln x}{x^2} dx = \lim_{b \rightarrow \infty} \left(-\frac{\ln b}{b} - \frac{1}{b} + 1 \right)$$

=

=

.

8.8 Improper Integrals

Since

$$\begin{aligned}\int_1^b \frac{\ln x}{x^2} dx &= \left[(\ln x) \left(-\frac{1}{x} \right) \right]_1^b - \int_1^b \left(-\frac{1}{x} \right) \left(\frac{1}{x} \right) dx \\ &= -\frac{\ln b}{b} - \left[\frac{1}{x} \right]_1^b = -\frac{\ln b}{b} - \frac{1}{b} + 1,\end{aligned}$$

we have that

$$\begin{aligned}\int_1^\infty \frac{\ln x}{x^2} dx &= \lim_{b \rightarrow \infty} \int_1^b \frac{\ln x}{x^2} dx = \lim_{b \rightarrow \infty} \left(-\frac{\ln b}{b} - \frac{1}{b} + 1 \right) \\ &= \lim_{b \rightarrow \infty} \left(-\frac{\ln b}{b} \right) - 0 + 1 \\ &= \end{aligned}$$

.

8.8 Improper Integrals

Since

$$\begin{aligned}\int_1^b \frac{\ln x}{x^2} dx &= \left[(\ln x) \left(-\frac{1}{x} \right) \right]_1^b - \int_1^b \left(-\frac{1}{x} \right) \left(\frac{1}{x} \right) dx \\ &= -\frac{\ln b}{b} - \left[\frac{1}{x} \right]_1^b = -\frac{\ln b}{b} - \frac{1}{b} + 1,\end{aligned}$$

we have that

$$\begin{aligned}\int_1^\infty \frac{\ln x}{x^2} dx &= \lim_{b \rightarrow \infty} \int_1^b \frac{\ln x}{x^2} dx = \lim_{b \rightarrow \infty} \left(-\frac{\ln b}{b} - \frac{1}{b} + 1 \right) \\ &= \lim_{b \rightarrow \infty} \left(-\frac{\ln b}{b} \right) - 0 + 1 \\ &= \lim_{b \rightarrow \infty} \left(-\frac{\frac{1}{b}}{1} \right) + 1 \quad (\text{l'Hôpital's Rule})\end{aligned}$$

.

8.8 Improper Integrals

Since

$$\begin{aligned}\int_1^b \frac{\ln x}{x^2} dx &= \left[(\ln x) \left(-\frac{1}{x} \right) \right]_1^b - \int_1^b \left(-\frac{1}{x} \right) \left(\frac{1}{x} \right) dx \\ &= -\frac{\ln b}{b} - \left[\frac{1}{x} \right]_1^b = -\frac{\ln b}{b} - \frac{1}{b} + 1,\end{aligned}$$

we have that

$$\begin{aligned}\int_1^\infty \frac{\ln x}{x^2} dx &= \lim_{b \rightarrow \infty} \int_1^b \frac{\ln x}{x^2} dx = \lim_{b \rightarrow \infty} \left(-\frac{\ln b}{b} - \frac{1}{b} + 1 \right) \\ &= \lim_{b \rightarrow \infty} \left(-\frac{\ln b}{b} \right) - 0 + 1 \\ &= \lim_{b \rightarrow \infty} \left(-\frac{\frac{1}{b}}{1} \right) + 1 \quad (\text{l'Hôpital's Rule}) \\ &= 0 + 1 = 1.\end{aligned}$$

Therefore the integral converges and the area has finite value 1.

EXAMPLE 2 Evaluate

$$\int_{-\infty}^{\infty} \frac{dx}{1 + x^2}.$$

Solution According to the definition (Part 3), we can choose $c = 0$ and write

$$\int_{-\infty}^{\infty} \frac{dx}{1 + x^2} = \int_{-\infty}^0 \frac{dx}{1 + x^2} + \int_0^{\infty} \frac{dx}{1 + x^2}.$$

Next we evaluate each improper integral on the right side of the equation above.

$$\begin{aligned}\int_{-\infty}^0 \frac{dx}{1 + x^2} &= \lim_{a \rightarrow -\infty} \int_a^0 \frac{dx}{1 + x^2} \\&= \lim_{a \rightarrow -\infty} \left[\tan^{-1} x \right]_a^0 \\&= \lim_{a \rightarrow -\infty} (\tan^{-1} 0 - \tan^{-1} a) = 0 - \left(-\frac{\pi}{2} \right) = \frac{\pi}{2}\end{aligned}$$

$$\begin{aligned}
 \int_0^\infty \frac{dx}{1+x^2} &= \lim_{b \rightarrow \infty} \int_0^b \frac{dx}{1+x^2} \\
 &= \lim_{b \rightarrow \infty} \left[\tan^{-1} x \right]_0^b \\
 &= \lim_{b \rightarrow \infty} (\tan^{-1} b - \tan^{-1} 0) = \frac{\pi}{2} - 0 = \frac{\pi}{2}
 \end{aligned}$$

Thus,

$$\int_{-\infty}^{\infty} \frac{dx}{1+x^2} = \frac{\pi}{2} + \frac{\pi}{2} = \pi.$$

Since $1/(1+x^2) > 0$, the improper integral can be interpreted as the (finite) area beneath the curve and above the x -axis (Figure 8.15). ■

8.8 Improper Integrals

Remark

$$\int_{-\infty}^{\infty} f(x) dx = \int_{-\infty}^c f(x) dx + \int_c^{\infty} f(x) dx.$$

8.8 Improper Integrals

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8.8 Improper Integrals

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For example, $\int_0^{\infty} \frac{2x}{x^2 + 1} dx$ diverges

8.8 Improper Integrals

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For example, $\int_0^{\infty} \frac{2x}{x^2 + 1} dx$ diverges and hence $\int_{-\infty}^{\infty} \frac{2x}{x^2 + 1}$ diverges.

8.8 Improper Integrals

Remark

$$\int_{-\infty}^{\infty} f(x) dx = \int_{-\infty}^c f(x) dx + \int_c^{\infty} f(x) dx.$$

This is not the same as $\lim_{b \rightarrow \infty} \int_{-b}^b f(x) dx !!!$

For example, $\int_0^{\infty} \frac{2x}{x^2 + 1} dx$ diverges and hence $\int_{-\infty}^{\infty} \frac{2x}{x^2 + 1}$ diverges. However

$$\lim_{b \rightarrow \infty} \int_{-b}^b \frac{2x}{x^2 + 1} dx = 0.$$

(Left for you to prove.)

EXAMPLE 3 For what values of p does the integral $\int_1^\infty dx/x^p$ converge? When the integral does converge, what is its value?

Solution If $p \neq 1$,

$$\int_1^b \frac{dx}{x^p} = \left[\frac{x^{-p+1}}{-p+1} \right]_1^b = \frac{1}{1-p} (b^{-p+1} - 1) = \frac{1}{1-p} \left(\frac{1}{b^{p-1}} - 1 \right).$$

Thus,

$$\begin{aligned}\int_1^\infty \frac{dx}{x^p} &= \lim_{b \rightarrow \infty} \int_1^b \frac{dx}{x^p} \\ &= \lim_{b \rightarrow \infty} \left[\frac{1}{1-p} \left(\frac{1}{b^{p-1}} - 1 \right) \right] = \begin{cases} \frac{1}{p-1}, & p > 1 \\ \infty, & p < 1 \end{cases}\end{aligned}$$

because

$$\lim_{b \rightarrow \infty} \frac{1}{b^{p-1}} = \begin{cases} 0, & p > 1 \\ \infty, & p < 1. \end{cases}$$

Therefore, the integral converges to the value $1/(p-1)$ if $p > 1$ and it diverges if $p < 1$.

8.8 Improper Integrals

If $p = 1$, the integral also diverges:

$$\begin{aligned}
 \int_1^\infty \frac{dx}{x^p} &= \int_1^\infty \frac{dx}{x} \\
 &= \lim_{b \rightarrow \infty} \int_1^b \frac{dx}{x} \\
 &= \lim_{b \rightarrow \infty} \left[\ln x \right]_1^b \\
 &= \lim_{b \rightarrow \infty} (\ln b - \ln 1) = \infty.
 \end{aligned}$$

8.8 Improper Integrals

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$$\begin{aligned} \int_1^\infty \frac{dx}{x^p} &= \int_1^\infty \frac{dx}{x} \\ &= \lim_{b \rightarrow \infty} \int_1^b \frac{dx}{x} \\ &= \lim_{b \rightarrow \infty} \left[\ln x \right]_1^b \\ &= \lim_{b \rightarrow \infty} (\ln b - \ln 1) = \infty. \end{aligned}$$

Theorem

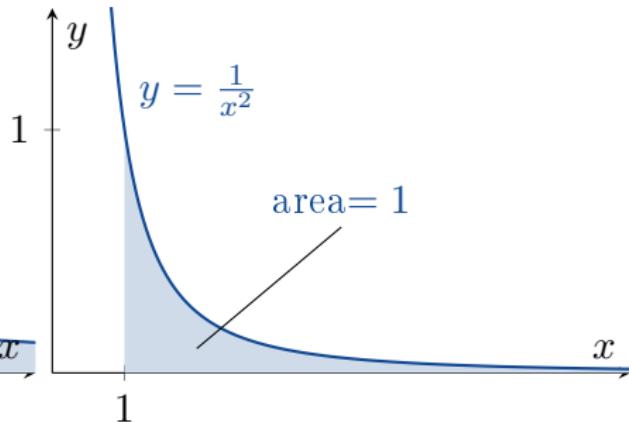
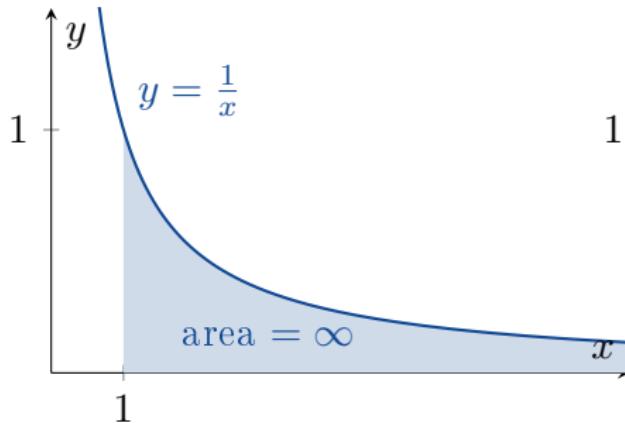
$$\int_1^\infty \frac{dx}{x^p} \quad \begin{cases} \text{converges if } p > 1, \\ \text{diverges if } p \leq 1. \end{cases}$$

8.8 Improper Integrals

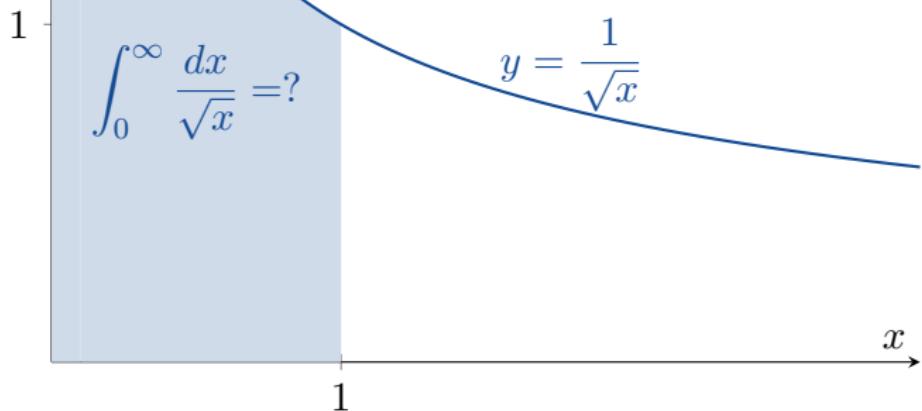
Remark

In particular, please remember that

$$\int_1^{\infty} \frac{dx}{x} \quad \text{diverges} \quad \text{and} \quad \int_1^{\infty} \frac{dx}{x^2} \quad \text{converges.}$$



Integrands with Vertical Asymptotes

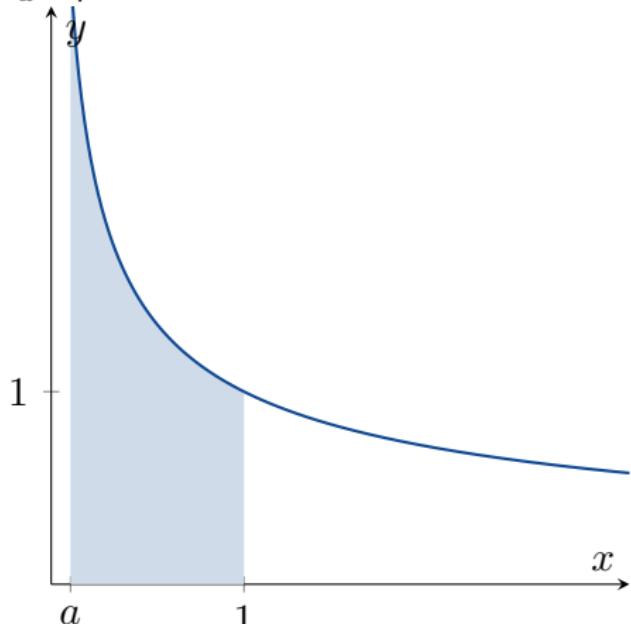


8.8 Improper Integrals



Step 1:

$$\int_a^1 \frac{dx}{\sqrt{x}} = ?$$

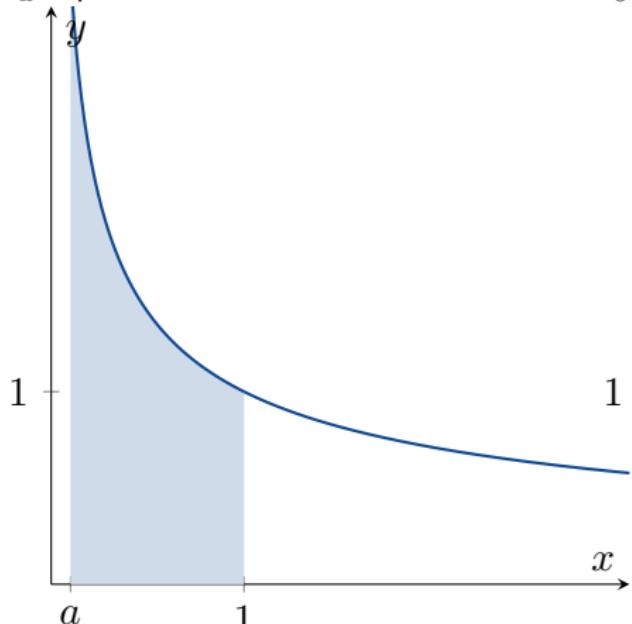


8.8 Improper Integrals



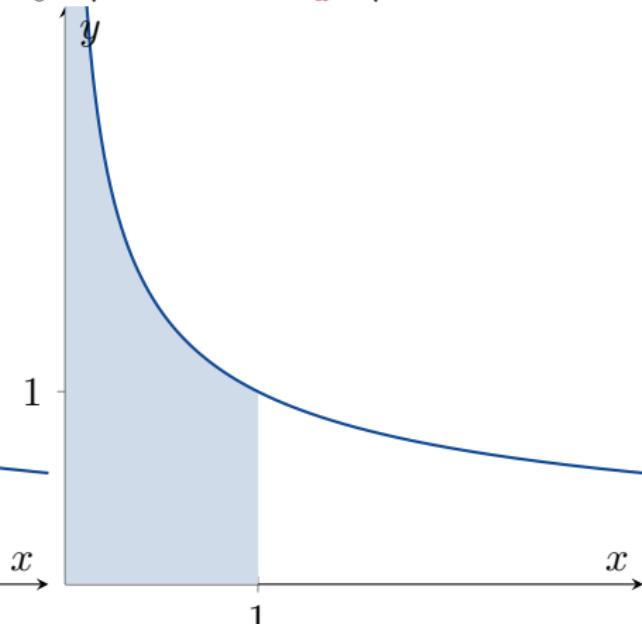
Step 1:

$$\int_a^1 \frac{dx}{\sqrt{x}} = ?$$



Step 2:

$$\int_0^1 \frac{dx}{\sqrt{x}} = \lim_{a \rightarrow 0^+} \int_a^1 \frac{dx}{\sqrt{x}}$$



8.8 Improper Integrals



Since

$$\int_a^1 \frac{dx}{\sqrt{x}} = \left[2\sqrt{x} \right]_a^1 = 2 - 2\sqrt{a},$$

8.8 Improper Integrals



Since

$$\int_a^1 \frac{dx}{\sqrt{x}} = \left[2\sqrt{x} \right]_a^1 = 2 - 2\sqrt{a},$$

we have that

$$\int_0^1 \frac{dx}{\sqrt{x}} = \lim_{a \rightarrow 0^+} \int_a^1 \frac{dx}{\sqrt{x}} = \lim_{a \rightarrow 0^+} (2 - 2\sqrt{a}) = 2.$$

DEFINITION Integrals of functions that become infinite at a point within the interval of integration are **improper integrals of Type II**.

1. If $f(x)$ is continuous on $(a, b]$ and discontinuous at a , then

$$\int_a^b f(x) dx = \lim_{c \rightarrow a^+} \int_c^b f(x) dx.$$

2. If $f(x)$ is continuous on $[a, b)$ and discontinuous at b , then

$$\int_a^b f(x) dx = \lim_{c \rightarrow b^-} \int_a^c f(x) dx.$$

3. If $f(x)$ is discontinuous at c , where $a < c < b$, and continuous on $[a, c) \cup (c, b]$, then

$$\int_a^b f(x) dx = \int_a^c f(x) dx + \int_c^b f(x) dx.$$

In each case, if the limit exists and is finite, we say the improper integral **converges** and that the limit is the **value** of the improper integral. If the limit does not exist, the integral **diverges**.

EXAMPLE 4 Investigate the convergence of

$$\int_0^1 \frac{1}{1-x} dx.$$

Solution The integrand $f(x) = 1/(1 - x)$ is continuous on $[0, 1)$ but is discontinuous at $x = 1$ and becomes infinite as $x \rightarrow 1^-$ (Figure 8.17). We evaluate the integral as

$$\begin{aligned}\lim_{b \rightarrow 1^-} \int_0^b \frac{1}{1-x} dx &= \lim_{b \rightarrow 1^-} [-\ln|1-x|]_0^b \\ &= \lim_{b \rightarrow 1^-} [-\ln(1-b) + 0] = \infty.\end{aligned}$$

The limit is infinite, so the integral diverges. ■

EXAMPLE 5 Evaluate

$$\int_0^3 \frac{dx}{(x - 1)^{2/3}}.$$

Solution The integrand has a vertical asymptote at $x = 1$ and is continuous on $[0, 1)$ and $(1, 3]$ (Figure 8.18). Thus, by Part 3 of the definition above,

$$\int_0^3 \frac{dx}{(x - 1)^{2/3}} = \int_0^1 \frac{dx}{(x - 1)^{2/3}} + \int_1^3 \frac{dx}{(x - 1)^{2/3}}.$$

Next, we evaluate each improper integral on the right-hand side of this equation.

$$\begin{aligned}\int_0^1 \frac{dx}{(x - 1)^{2/3}} &= \lim_{b \rightarrow 1^-} \int_0^b \frac{dx}{(x - 1)^{2/3}} \\&= \lim_{b \rightarrow 1^-} \left[3(x - 1)^{1/3} \right]_0^b \\&= \lim_{b \rightarrow 1^-} [3(b - 1)^{1/3} + 3] = 3\end{aligned}$$

$$\begin{aligned}
 \int_1^3 \frac{dx}{(x-1)^{2/3}} &= \lim_{c \rightarrow 1^+} \int_c^3 \frac{dx}{(x-1)^{2/3}} \\
 &= \lim_{c \rightarrow 1^+} \left[3(x-1)^{1/3} \right]_c^3 \\
 &= \lim_{c \rightarrow 1^+} [3(3-1)^{1/3} - 3(c-1)^{1/3}] = 3\sqrt[3]{2}
 \end{aligned}$$

We conclude that

$$\int_0^3 \frac{dx}{(x-1)^{2/3}} = 3 + 3\sqrt[3]{2}.$$



8.8 Improper Integrals



Remark

Sometimes we cannot evaluate an improper integral, but we can still determine whether it converges or diverges.

8.8 Improper Integrals

Remark

Sometimes we cannot evaluate an improper integral, but we can still determine whether it converges or diverges.

Example

Does $\int_1^\infty e^{-x^2} dx$ converge or diverge?

We can not calculate $\int_1^b e^{-x^2} dx$ because it is nonelementary.
But we can answer this example another way.

8.8 Improper Integrals



Since $e^{-x^2} > 0$, we know that $I(b) = \int_1^b e^{-x^2} dx$ is an increasing function of b .

8.8 Improper Integrals

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So either

- $\lim_{b \rightarrow \infty} \int_1^b e^{-x^2} dx = \infty$; or
- $\lim_{b \rightarrow \infty} \int_1^b e^{-x^2} dx$ is a finite number.

8.8 Improper Integrals



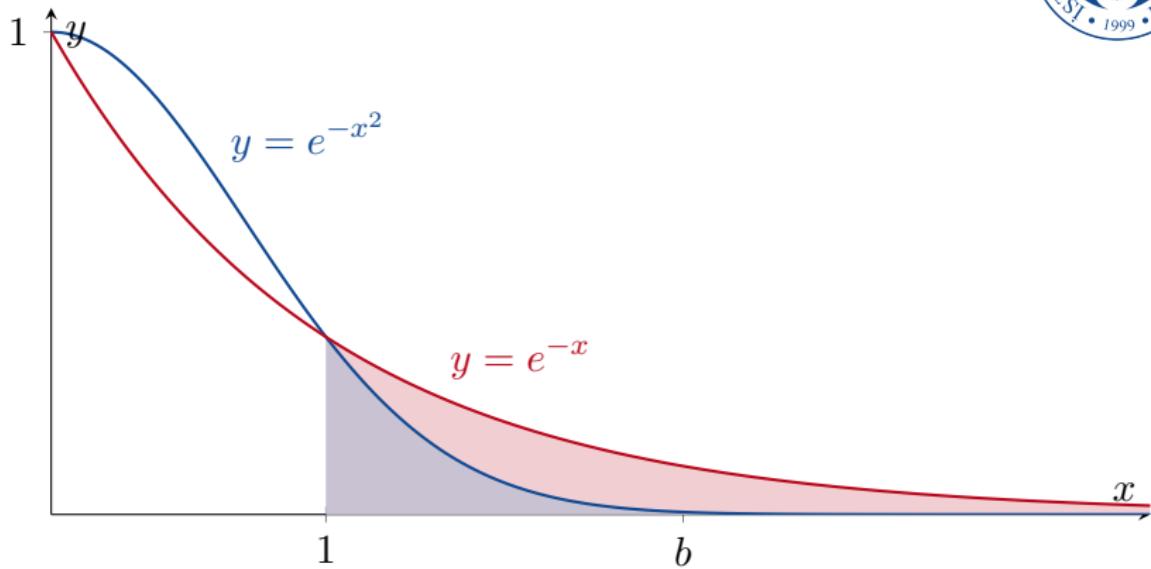
Since $e^{-x^2} > 0$, we know that $I(b) = \int_1^b e^{-x^2} dx$ is an increasing function of b .

So either

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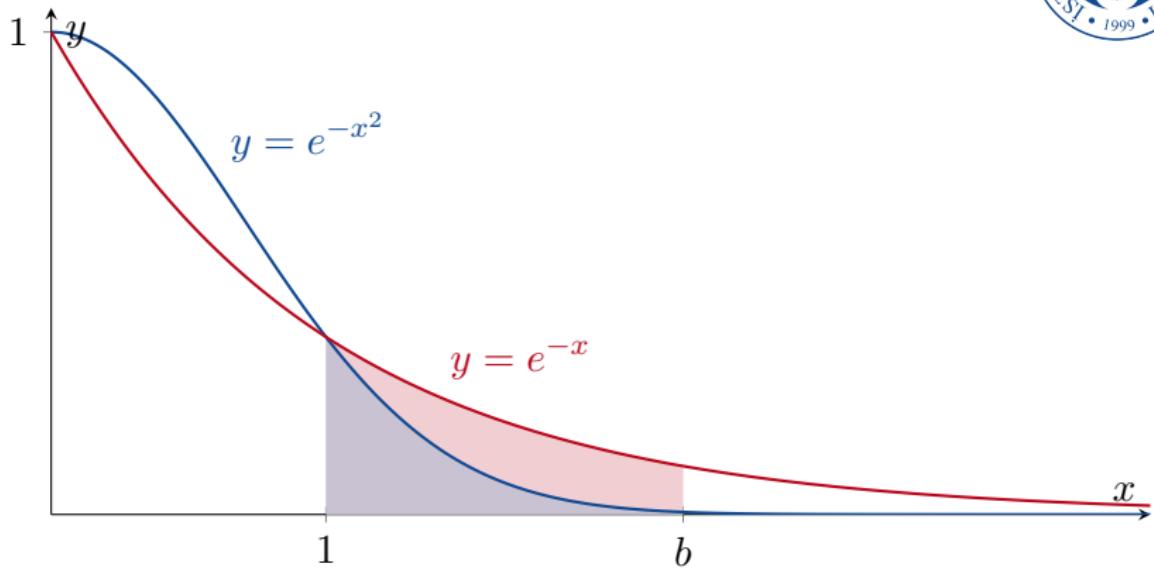
I am going to prove to you that $\lim_{b \rightarrow \infty} \int_1^b e^{-x^2} dx$ is finite.

8.8 Improper Integrals



Note that $e^{-x^2} \leq e^{-x}$ for all $x \geq 1$.

8.8 Improper Integrals

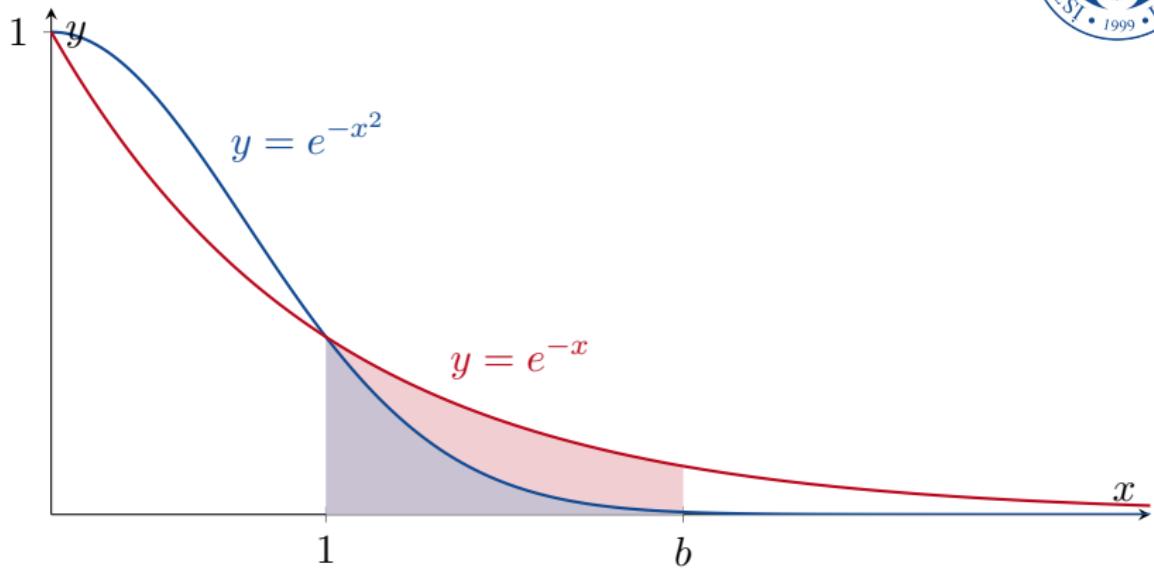


Note that $e^{-x^2} \leq e^{-x}$ for all $x \geq 1$. So

$$\int_1^b e^{-x^2} dx \leq \int_1^b e^{-x} dx$$

for any $b > 1$.

8.8 Improper Integrals



Note that $e^{-x^2} \leq e^{-x}$ for all $x \geq 1$. So

$$\int_1^b e^{-x^2} dx \leq \int_1^b e^{-x} dx = -e^{-b} + e^{-1} < e^{-1} \approx 0.36788$$

for any $b > 1$.

8.8 Improper Integrals



Therefore

$$\int_1^{\infty} e^{-x^2} dx = \lim_{b \rightarrow \infty} \int_1^b e^{-x^2} dx$$

converges to a finite value.



8.8 Improper Integrals

Theorem (Direct Comparison Test)

Let $f : [a, \infty) \rightarrow \mathbb{R}$ and $g : [a, \infty) \rightarrow \mathbb{R}$ be continuous functions.
Suppose that

$$0 \leq f(x) \leq g(x)$$

for all $x \in [a, \infty)$.

8.8 Improper Integrals



Theorem (Direct Comparison Test)

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for all $x \in [a, \infty)$. Then

1 $\int_a^\infty g(x) dx$ converges $\implies \int_a^\infty f(x) dx$ converges;

8.8 Improper Integrals



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- 1 $\int_a^\infty g(x) dx$ converges $\implies \int_a^\infty f(x) dx$ converges;
- 2 $\int_a^\infty f(x) dx$ diverges $\implies \int_a^\infty g(x) dx$ diverges.

8.8 Improper Integrals



Theorem (Direct Comparison Test)

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Suppose that

$$0 \leq f(x) \leq g(x)$$

for all $x \in [a, \infty)$. Then

- 1 $\int_a^\infty g(x) dx$ converges $\implies \int_a^\infty f(x) dx$ converges;
- 2 $\int_a^\infty f(x) dx$ diverges $\implies \int_a^\infty g(x) dx$ diverges.

(you can read the proof in the book)

EXAMPLE 7

These examples illustrate how we use Theorem 2.

(a) $\int_1^\infty \frac{\sin^2 x}{x^2} dx$ converges because

$$0 \leq \frac{\sin^2 x}{x^2} \leq \frac{1}{x^2} \quad \text{on } [1, \infty) \quad \text{and} \quad \int_1^\infty \frac{1}{x^2} dx \quad \text{converges.}$$

Example 3

(b) $\int_1^\infty \frac{1}{\sqrt{x^2 - 0.1}} dx$ diverges because

$$\frac{1}{\sqrt{x^2 - 0.1}} \geq \frac{1}{x} \quad \text{on } [1, \infty) \quad \text{and} \quad \int_1^\infty \frac{1}{x} dx \quad \text{diverges.}$$

Example 3

(c) $\int_0^{\pi/2} \frac{\cos x}{\sqrt{x}} dx$ converges because

$$0 \leq \frac{\cos x}{\sqrt{x}} \leq \frac{1}{\sqrt{x}} \quad \text{on} \quad \left[0, \frac{\pi}{2}\right], \quad 0 \leq \cos x \leq 1 \text{ on } \left[0, \frac{\pi}{2}\right]$$

and

$$\begin{aligned} \int_0^{\pi/2} \frac{dx}{\sqrt{x}} &= \lim_{a \rightarrow 0^+} \int_a^{\pi/2} \frac{dx}{\sqrt{x}} \\ &= \lim_{a \rightarrow 0^+} \left. \sqrt{4x} \right|_a^{\pi/2} \quad 2\sqrt{x} = \sqrt{4x} \\ &= \lim_{a \rightarrow 0^+} (\sqrt{2\pi} - \sqrt{4a}) = \sqrt{2\pi} \quad \text{converges.} \end{aligned}$$



8.8 Improper Integrals

Theorem (Limit Comparison Test)

Suppose that

- $f : [a, \infty) \rightarrow \mathbb{R}$ and $g : [a, \infty) \rightarrow \mathbb{R}$ are continuous;
- $f > 0$ and $g > 0$;
- $\lim_{x \rightarrow \infty} \frac{f(x)}{g(x)}$ exists and $0 < \lim_{x \rightarrow \infty} \frac{f(x)}{g(x)} < \infty$.

8.8 Improper Integrals

Theorem (Limit Comparison Test)

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8.8 Improper Integrals

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- $f > 0$ and $g > 0$;
- $\lim_{x \rightarrow \infty} \frac{f(x)}{g(x)}$ exists and $0 < \lim_{x \rightarrow \infty} \frac{f(x)}{g(x)} < \infty$.

Then

$$\int_a^{\infty} f(x) dx \quad \text{and} \quad \int_a^{\infty} g(x) dx$$

both converge or both diverge.

8.8 Improper Integrals

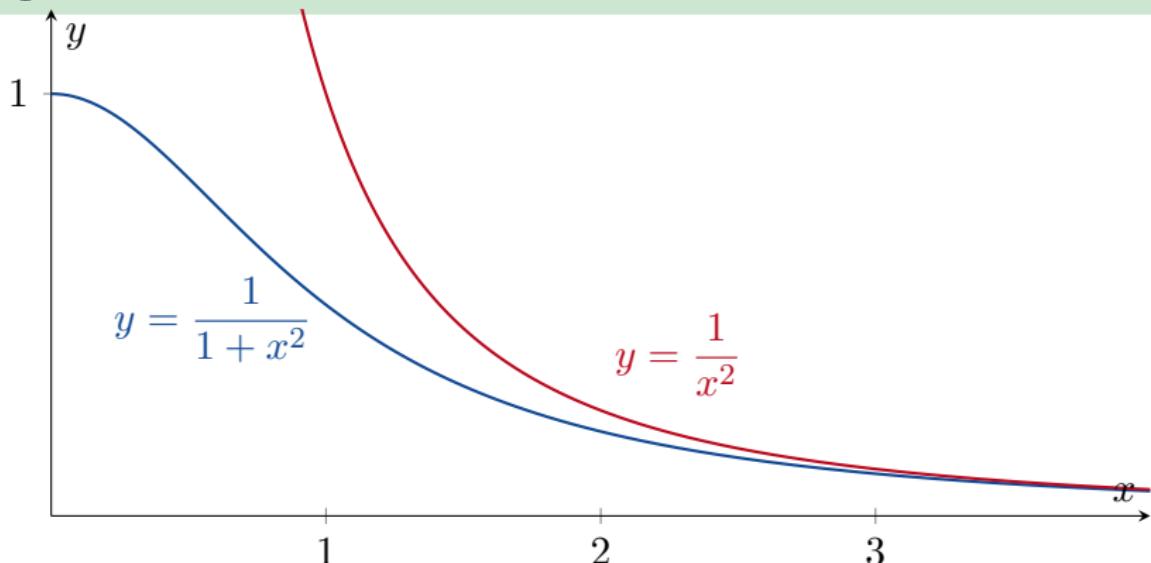
Example

Show that $\int_1^\infty \frac{dx}{1+x^2}$ diverges, by comparing it with $\int_1^\infty \frac{1}{x^2} dx$.

8.8 Improper Integrals

Example

Show that $\int_1^\infty \frac{dx}{1+x^2}$ diverges, by comparing it with $\int_1^\infty \frac{1}{x^2} dx$.



Solution The functions $f(x) = 1/x^2$ and $g(x) = 1/(1 + x^2)$ are positive and continuous on $[1, \infty)$. Also,

$$\begin{aligned}\lim_{x \rightarrow \infty} \frac{f(x)}{g(x)} &= \lim_{x \rightarrow \infty} \frac{1/x^2}{1/(1+x^2)} = \lim_{x \rightarrow \infty} \frac{1+x^2}{x^2} \\&= \lim_{x \rightarrow \infty} \left(\frac{1}{x^2} + 1 \right) = 0 + 1 = 1,\end{aligned}$$

which is a positive finite limit (Figure 8.20). Therefore, $\int_1^\infty \frac{dx}{1+x^2}$ converges because $\int_1^\infty \frac{dx}{x^2}$ converges.

The integrals converge to different values, however:

$$\int_1^\infty \frac{dx}{x^2} = \frac{1}{2-1} = 1 \quad \text{Example 3}$$

and

$$\int_1^\infty \frac{dx}{1+x^2} = \lim_{b \rightarrow \infty} \int_1^b \frac{dx}{1+x^2} = \lim_{b \rightarrow \infty} [\tan^{-1} b - \tan^{-1} 1] = \frac{\pi}{2} - \frac{\pi}{4} = \frac{\pi}{4}. \quad \blacksquare$$

EXAMPLE 9 Investigate the convergence of $\int_1^\infty \frac{1 - e^{-x}}{x} dx$.

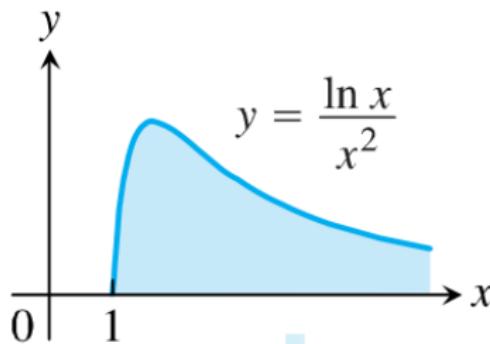
Solution The integrand suggests a comparison of $f(x) = (1 - e^{-x})/x$ with $g(x) = 1/x$. However, we cannot use the Direct Comparison Test because $f(x) \leq g(x)$ and the integral of $g(x)$ diverges. On the other hand, using the Limit Comparison Test we find that

$$\lim_{x \rightarrow \infty} \frac{f(x)}{g(x)} = \lim_{x \rightarrow \infty} \left(\frac{1 - e^{-x}}{x} \right) \left(\frac{x}{1} \right) = \lim_{x \rightarrow \infty} (1 - e^{-x}) = 1,$$

which is a positive finite limit. Therefore, $\int_1^\infty \frac{1 - e^{-x}}{x} dx$ diverges because $\int_1^\infty \frac{dx}{x}$ diverges. Approximations to the improper integral are given in Table 8.5. Note that the values do not appear to approach any fixed limiting value as $b \rightarrow \infty$. ■

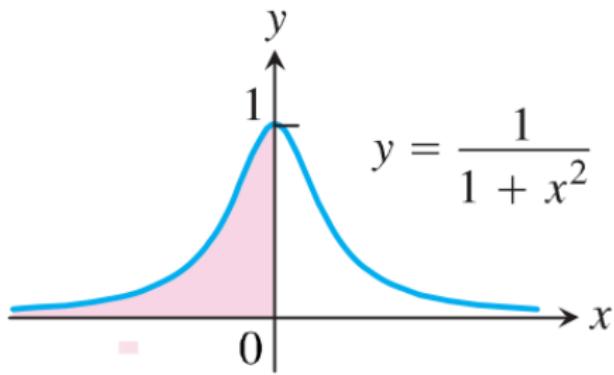
Upper limit

$$\int_1^{\infty} \frac{\ln x}{x^2} dx = \lim_{b \rightarrow \infty} \int_1^b \frac{\ln x}{x^2} dx$$



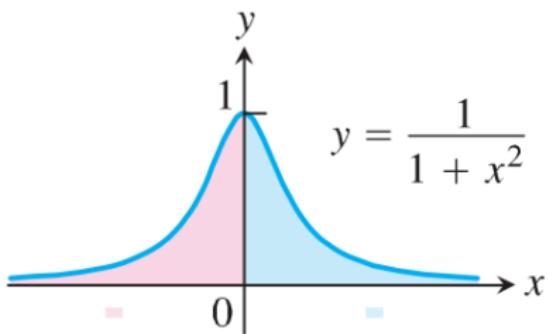
Lower limit

$$\int_{-\infty}^0 \frac{dx}{1+x^2} = \lim_{a \rightarrow -\infty} \int_a^0 \frac{dx}{1+x^2}$$



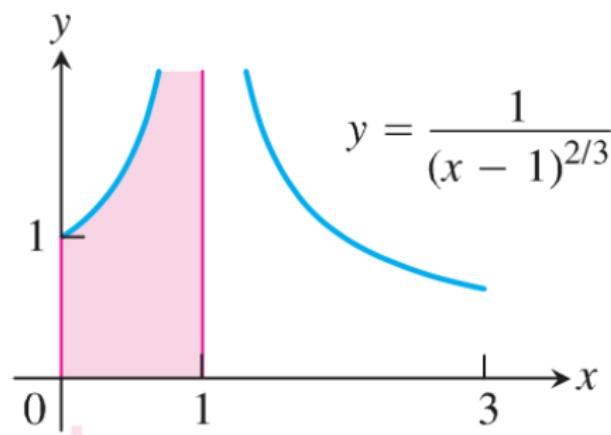
Both limits

$$\int_{-\infty}^{\infty} \frac{dx}{1+x^2} = \lim_{b \rightarrow -\infty} \int_b^0 \frac{dx}{1+x^2} + \lim_{c \rightarrow \infty} \int_0^c \frac{dx}{1+x^2}$$



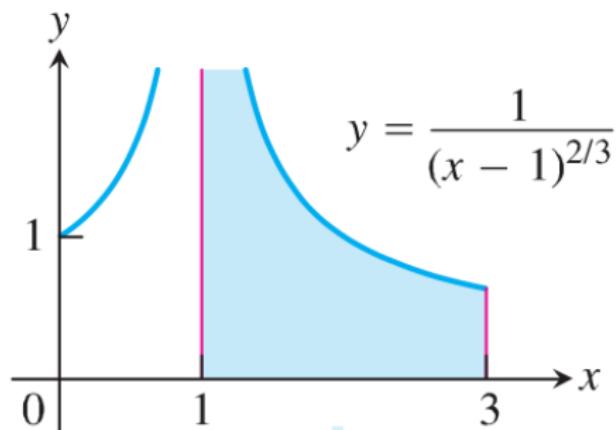
Upper endpoint

$$\int_0^1 \frac{dx}{(x - 1)^{2/3}} = \lim_{b \rightarrow 1^-} \int_0^b \frac{dx}{(x - 1)^{2/3}}$$



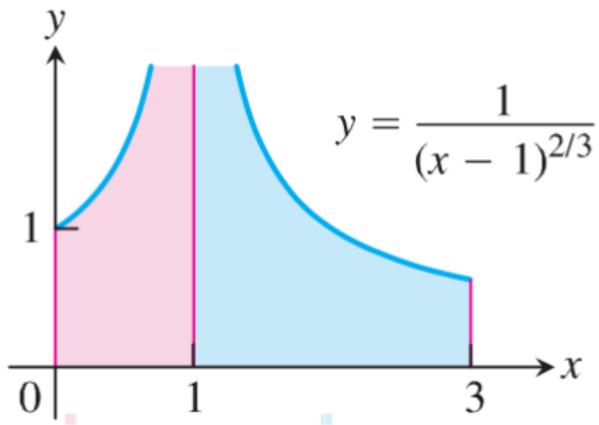
Lower endpoint

$$\int_1^3 \frac{dx}{(x - 1)^{2/3}} = \lim_{d \rightarrow 1^+} \int_d^3 \frac{dx}{(x - 1)^{2/3}}$$



Interior point

$$\int_0^3 \frac{dx}{(x - 1)^{2/3}} = \int_0^1 \frac{dx}{(x - 1)^{2/3}} + \int_1^3 \frac{dx}{(x - 1)^{2/3}}$$





Next Time

- 11.1 Three-Dimensional Coordinate Systems
- 11.2 Vectors
- 11.3 The Dot Product