An Introduction to Analysis

Neil Course

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Preface

These are the lecture notes for the course ***MAT234 Matematik IV*** aka ***Advanced Calculus***.

Because this is version 1 of these notes, it is likely that it includes some misprints/typos/mistakes/errors. Please keep a list of all the mistakes that you find, and then inform me periodically. If you can find a mistake, then it is good for you because it means that you understand the material well enough to find the errors.

These notes are based on ***Guide to Analysis*** by Mary Hart and ***Thomas' Calculus***. Students are advised to read Mary Hart's book for further detail.

Schedule[1]

Week #	Dates			Independent study expected
1	1 Feb	–	5 Feb	read Chapters 1-2
2	8 Feb	–	12 Feb	read Chapters 2-3
3	15 Feb	–	19 Feb	read Chapter 3 complete Homework 1
4	22 Feb	–	26 Feb	read Chapters 3-4 complete Homework 2
5	29 Feb	–	4 Mar	read Chapters 5-6
6	7 Mar	–	11 Mar	read Chapters 6-7 complete Homework 3
7	14 Mar	–	18 Mar	read Chapters 8-9 complete Homework 4
8	21 Mar	–	25 Mar	read Chapters 10-11
9	28 Mar	–	1 Apr	*midterm exam*
10	4 Apr	–	8 Apr	read Chapters 12-14 complete Homework 5
11	11 Apr	–	15 Apr	read Chapters 15-17 complete Homework 6
12	18 Apr	–	22 Apr	read Chapters 18-19
13	25 Apr	–	29 Apr	read Chapter 20 complete Homework 7
14	2 May	–	6 May	read Chapter 21 complete Homework 8

[1] subject to change

Chapter 1

Symbolic Logic (Sembolik Mantık)

1.1 Introduction

Definition. A ***proposition*** is a statement which is either *true* or *false* (but not both).

Example 1.1.

- "Grass is green" (true)
- "2+5=5" (false)
- "Benim adım Neil" (true)

are propositions, but

- "Close the door"
- "Bugün soğuk mu?"
- "1"

are not propositions.

Notation. The symbol for ***or*** (veya) is $\vee$.

Example 1.2. If P is the proposition "It is snowing today" and Q is the proposition "It is raining today", then $P \vee Q$ is the proposition "It is snowing or raining today".

Example 1.3. If $M = (x \in A)$ and $N = (x \in B)$, then $M \vee N = (x \in A \cup B)$

Truth Table. (T = true, F = false, D = doğru, Y = yanlış)

P	Q	$P \vee Q$
T	T	T
T	F	T
F	T	T
F	F	F

P	Q	$P \vee Q$
D	D	D
D	Y	D
Y	D	D
Y	Y	Y

Notation. The symbol for ***and*** (ve) is $\wedge$.

Example 1.4. If P = "Ben açım" and Q = "Uykum var", then $P \wedge Q$ = "Ben açım ve uykum var".

Example 1.5. If $M = (x \in A)$ and $N = (x \in B)$, then $M \wedge N = (x \in A \cap B)$

Truth Table.

P	Q	$P \wedge Q$
T	T	T
T	F	F
F	T	F
F	F	F

P	Q	$P \wedge Q$
D	D	D
D	Y	Y
Y	D	Y
Y	Y	Y

Notation. The symbol for ***not*** (değil) is $\neg$.

Example 1.6. If P = "Sizin hocanız kahve seviyor", then $\neg P$ = "Sizin hocanız kahve sevmiyor".

Example 1.7. If $M = (x \geq 7)$, then $\neg M = (x < 7)$

Truth Table.

P	$\neg P$
T	F
F	T

P	$\neg P$
D	Y
Y	D

Notation. The symbol for ***if and only if*** (iff/ancak ve ancak) is $\iff$.

Truth Table.

P	Q	$P \iff Q$
T	T	T
T	F	F
F	T	F
F	F	T

P	Q	$P \iff Q$
D	D	D
D	Y	Y
Y	D	Y
Y	Y	D

Notation. The symbol for ***implies*** (ise) is $\implies$.

Truth Table.

P	Q	$P \implies Q$
T	T	T
T	F	F
F	T	T
F	F	T

P	Q	$P \implies Q$
D	D	D
D	Y	Y
Y	D	D
Y	Y	D

Remark. We must only write "$P \implies Q$" if both P and Q are propositions. I don't want to see nonsense like

$$\int_0^1 3x^2 \, dx = \left[x^3\right]_0^1 \implies 1$$

in your work. Yes, "$\int_0^1 3x^2 \, dx = \left[x^3\right]_0^1$" is a proposition. In fact, it is a *true* proposition. But "1" is not a proposition.

Be warned: From now on, I am subtracting points every time I see the misuse of "$\implies$".

Remark. If P and Q are propositions, then $(P \vee Q)$, $(P \wedge Q)$, $(\neg P)$, $(P \implies Q)$ and $(P \iff Q)$ are also propositions.

Definition. The ***converse*** (zıt) of $(P \implies Q)$ is $(Q \implies P)$.

Definition. The ***contapositive*** (devrik) of $(P \implies Q)$ is $(\neg Q \implies \neg P)$.

Example 1.8.
$P =$ "It is raining"
$Q =$ "I get wet"
$(P \implies Q) =$ "If it is raining, then I get wet"
converse: $(Q \implies P) =$ "If I get wet, then it is raining"
contrapositive: $(\neg Q \implies \neg P) =$ "If I do not get wet, then it is not raining"

1.2 The 22 Identities.

1. $(P \vee P) = P$
2. $(P \wedge P) = P$
3. $(P \vee Q) = (Q \vee P)$
4. $(P \wedge Q) = (Q \wedge P)$
5. $((P \vee Q) \vee R) = (P \vee (Q \vee R))$
6. $((P \wedge Q) \wedge R) = (P \wedge (Q \wedge R))$
7. $\neg(P \vee Q) = (\neg P \wedge \neg Q)$
8. $\neg(P \wedge Q) = (\neg P \vee \neg Q)$
9. $(P \wedge (Q \vee R)) = ((P \wedge Q) \vee (P \wedge R))$
10. $(P \vee (Q \wedge R)) = ((P \vee Q) \wedge (P \vee R))$
11. $(P \vee \text{true}) = \text{true}$
12. $(P \wedge \text{false}) = \text{false}$
13. $(P \vee \text{false}) = P$
14. $(P \wedge \text{true}) = P$
15. $(P \vee \neg P) = \text{true}$
16. $(P \wedge \neg P) = \text{false}$
17. $\neg(\neg P) = P$
18. $(P \implies Q) = (\neg P \vee Q)$
19. $(P \iff Q) = ((P \implies Q) \wedge (Q \implies P))$
20. $((P \wedge Q) \implies R) = (P \implies (Q \implies R))$
21. $((P \implies Q) \wedge (P \implies \neg Q)) = \neg P$
22. $(P \implies Q) = (\neg Q \implies \neg P)$

Proof of Identity 18.

P	Q	$P \implies Q$	$\neg P$	Q	$\neg P \vee Q$
T	T	T	F	T	T
T	F	F	F	F	F
F	T	T	T	T	T
F	F	T	T	F	T

Note that the 3rd and 6th columns are the same : T,F,T,T. Therefore $(P \implies Q) = (\neg P \vee Q)$. □

Proof of Identity 22.

P	Q	$P \implies Q$	$\neg Q$	$\neg P$	$\neg Q \implies \neg P$
T	T	T	F	F	T
T	F	F	T	F	F
F	T	T	F	T	T
F	F	T	T	T	T

Therefore $(P \implies Q) = (\neg Q \implies \neg P)$. □

Example 1.9. Prove that $\neg(P \implies Q) = (P \wedge \neg Q)$

solution:

$$\neg(P \implies Q) \overset{18}{=} \neg(\neg P \vee Q) \overset{7}{=} (\neg\neg P \wedge \neg Q) \overset{17}{=} (P \wedge \neg Q).$$

Example 1.10. Prove that $(P \vee (P \wedge Q)) = P$

solution:

$$\begin{aligned}(P \vee (P \wedge Q)) &\overset{14}{=} ((P \wedge \text{true}) \vee (P \wedge Q)) \overset{9}{=} (P \wedge (\text{true} \vee Q)) \\ &\overset{3}{=} (P \wedge (Q \vee \text{true})) \overset{11}{=} (P \wedge \text{true}) \overset{14}{=} P.\end{aligned}$$

Notation. The symbol for ***for all*** (her) is $\forall$.

Notation. The symbol for ***there exists*** (vardır) is $\exists$.

Example 1.11. Let $f : \mathbb{R} \to \mathbb{R}$ be a function and let $x_0, L \in \mathbb{R}$. The definition of $f(x) \to L$ as $x \to x_0$ (i.e. $\lim_{x \to x_0} f(x) = L$) is "for all $\varepsilon > 0$, there exists $\delta > 0$ such that

$$|x - x_0| < \delta \implies |f(x) - L| < \varepsilon."$$

In symbolic logic, this is

$$(\forall \varepsilon > 0)(\exists \delta > 0)(\forall x \in \mathbb{R})\big((|x - x_0| < \delta) \implies (|f(x) - L| < \varepsilon)\big)$$

Remark. In the English version of the definition, we didn't say what x was. We assume that everyone knows that x is a real number. But when we use symbolic logic we must be precise. We must say that x is any real number BEFORE we write $|x - x_0| < \delta$.

1.3 Negating a proposition.

To find the opposite of a proposition like this, we must do the following:

1. change all the $\forall$ to $\exists$;
2. change all the $\exists$ to $\forall$;
3. put a $\neg$ on the final parentheses.

Example 1.12. The definition of "$f(x) \to L$ as $x \to x_0$" is

$$(\forall \varepsilon > 0)(\exists \delta > 0)(\forall x \in \mathbb{R})\big((|x - x_0| < \delta) \implies (|f(x) - L| < \varepsilon)\big)$$

Therefore, the definition of "$f(x) \not\to L$ as $x \to x_0$" is

$$(\exists \varepsilon > 0)(\forall \delta > 0)(\exists x \in \mathbb{R})\neg\big((|x - x_0| < \delta) \implies (|f(x) - L| < \varepsilon)\big)$$

But remember that $\neg(P \implies Q) = (P \wedge \neg Q)$. So the definition of "$f(x) \not\to L$ as $x \to x_0$" is

$$(\exists \varepsilon > 0)(\forall \delta > 0)(\exists x \in \mathbb{R})\big((|x - x_0| < \delta) \wedge (|f(x) - L| \geq \varepsilon)\big)$$

Then we can change it back to English (or Turkish):

"there exists $\varepsilon > 0$ such that for all $\delta > 0$, there exists $x \in \mathbb{R}$ such that
$$|x - x_0| < \delta \text{ and } |f(x) - L| \geq \varepsilon."$$

Remark. It is important to understand how to negate definitions.

Example 1.13. If $P = (\exists x \in [0,1])(x \geq \frac{1}{2})$, find $\neg P$.

solution:

$$\neg P = (\forall x \in [0,1])\neg(x \geq \tfrac{1}{2}) = (\forall x \in [0,1])(x < \tfrac{1}{2})$$

Note that P is true and $\neg P$ is false.

Example 1.14. Negate

$$(\forall a \in \mathbb{R})(\forall b \in (a, \infty))(\exists M \in \mathbb{R})(\forall x \in (a,b))(x \leq M).$$

solution:

$$(\exists a \in \mathbb{R})(\exists b \in (a, \infty))(\forall M \in \mathbb{R})(\exists x \in (a,b))(x > M).$$

Chapter 2

Four Types of Proof (Dört Tür Kanıt)

This chapter is about ***proving*** results. Some of the lemmata in this chapter are not particularly interesting in themselves – the important thing is ***how*** we prove them and how to write proofs ***clearly*** and ***logically***.

2.1 Direct Proof (Dolaysız Kanıt)

The simpliest, most common type of proof is called a Direct Proof. If we want to prove that

$$P \implies Q,$$

we start with what we know (P) and we end with what we want (Q). Precisely, a direct proof should look something like this:

Suppose that P is true.

(some mathematics)

Therefore Q is true.

Notice how this is structured logically. We want to prove that $P \implies Q$; in other words, we want to prove that "if P is true, then Q is true". So we start with "Suppose that P is true" and we end with "Therefore Q is true". Consider the following easy lemma.

Definition. The ***odd numbers*** are the numbers $1, 3, 5, 7, \ldots$ and the ***even numbers*** are the numbers $2, 4, 6, 8, \ldots$.

Lemma 2.1. *Suppose $x \in \mathbb{N}$. Then*

$$x \textit{ is even} \implies x^2 \textit{ is even.}$$

Proof. Suppose x is even. Then we can write $x = 2k$ for some $k \in \mathbb{N}$. It follows that

$$x^2 = (2k)^2 = 4k^2 = 2(2k^2).$$

Since $2k^2 \in \mathbb{N}$, we have that $x^2 = 2 \times$ (a natural number). Therefore x^2 is even. □

2.2 Proof by Induction (Tümevarımla Kanıt)

Suppose that we want to prove that a proposition P_n is true for all $n \in \mathbb{N}$. If we prove P_1, then prove P_2, then prove P_3, then prove P_4, then $\ldots$, we would spend our whole life trying to prove the result, but never finish.

The idea behind Induction is that we must do 2 things:

(i). Prove that P_1 is true;

(ii). Prove that $P_k \implies P_{k+1}$.

If we have done these 2 things, then

$$(P_1 \text{ is true}) \implies (P_2 \text{ is true}) \implies (P_3 \text{ is true}) \implies \dots$$

and we know that P_n is true for every n.

Lemma 2.2.

$$1^2 + 2^2 + 3^3 + 4^2 + \ldots + n^2 = \frac{1}{6}n(n+1)(2n+1)$$

for all $n \in \mathbb{N}$.

Remark. Our proposition is

$$P_n = \text{“}1^2 + 2^2 + 3^3 + 4^2 + \ldots + n^2 = \frac{1}{6}n(n+1)(2n+1)\text{”}.$$

We must prove that P_n is true for all $n \in \mathbb{N}$. First we will prove that P_1 is true. Then we will prove that $P_k \implies P_{k+1}$. We will finish our proof by writing “By the principle of mathematical induction...”.

Proof of Lemma 2.2. Step 1: First, $1^2 = 1 = \frac{1}{6} \times 1 \times (1+1)(2+1)$. So P_1 is true.

Step 2: Now suppose that P_k is true. Then we know that

$$1^2 + 2^2 + 3^2 + \ldots + k^2 = \frac{1}{6}k(k+1)(2k+1).$$

So

$$\begin{aligned}
&1^2 + 2^2 + 3^2 + \ldots + k^2 + (k+1)^2 \\
&\qquad = \left(1^2 + 2^2 + 3^2 + \ldots + k^2\right) + (k+1)^2 \\
&\qquad = \frac{1}{6}k(k+1)(2k+1) + (k+1)^2 \\
&\qquad = \frac{1}{6}(k+1)\left[k(2k+1) + 6(k+1)\right] \\
&\qquad = \frac{1}{6}(k+1)\left[2k^2 + 7k + 6\right] \\
&\qquad = \frac{1}{6}(k+1)(k+2)(2k+3) \\
&\qquad = \frac{1}{6}(k+1)(k+2)(2(k+1)+1).
\end{aligned}$$

So P_{k+1} is true. *(Notice what we have done here. We have proved that if P_k is true, then P_{k+1} is also true. In other words, we have proved that $P_k \implies P_{k+1}$.)*

Step 3: By the principle of mathematical induction, P_n is true $\forall n \in \mathbb{N}$. □

Lemma 2.3.

$$1 + 3 + 5 + 7 + \ldots + (2n - 1) = n^2$$

for all $n \in \mathbb{N}$.

Remark. This lemma includes the phrase "for all $n \in \mathbb{N}$". Every time we see this, we should think "maybe we should use proof by induction".

Proof of Lemma 2.3. Let

$$P_n := \text{“}1 + 3 + 5 + 7 + \ldots + (2n - 1) = n^2\text{”}$$

Step 1: First; $1 = 1^2$, so P_1 is true.

Step 2: Now suppose that P_k is true. Then we have that

$$1 + 3 + 5 + 7 + \ldots + (2k - 1) = k^2.$$

It follows that

$$\begin{aligned}
&1 + 3 + 5 + 7 + \ldots + (2(k + 1) - 1) \\
&\qquad = (1 + 3 + 5 + 7 + \ldots + (2k - 1)) + (2(k + 1) - 1) \\
&\qquad = k^2 + (2k + 1) \\
&\qquad = (k + 1)^2.
\end{aligned}$$

So P_{k+1} is true. I.e. $P_k \implies P_{k+1}$.

Step 3: By the principle of mathematical induction, P_n is true $\forall n \in \mathbb{N}$. □

Remark. If you wish, you can write your induction proof without talking about P_n. You can just write "the proposition" instead.

Lemma 2.4. *Suppose that* $t_k = t_{k-1} + h$ *and* $y_k = (1+h)y_{k-1} + h - ht_{k-1}$ *for all* $k = 1, 2, 3, \ldots$. *Then*

$$y_n = (1+h)^n(y_0 - t_0) + t_n$$

for all $n \in \mathbb{N}$.

Proof. First

$$\begin{aligned} y_1 &= (1+h)y_0 + h - ht_0 \\ &= (1+h)y_0 - (1+h)t_0 + t_0 + h \\ &= (1+h)(y_0 - t_0) + t_1. \end{aligned}$$

So the proposition is true for $n = 1$.

Now suppose that the proposition is true for $n = k$. Then

$$y_k = (1+h)^k(y_0 - t_0) + t_k.$$

So

$$\begin{aligned} y_{k+1} &= (1+h)y_k + h - ht_k \\ &= (1+h)\left[(1+h)^k(y_0 - t_0) + t_k\right] + h - ht_k \\ &= (1+h)^{k+1}(y_0 - t_0) + (1+h)t_k + h - ht_k \\ &= (1+h)^{k+1}(y_0 - t_0) + t_k + h \\ &= (1+h)^{k+1}(y_0 - t_0) + t_{k+1}. \end{aligned}$$

So the proposition is also true for $n = k + 1$.

By the principle of mathematical induction, the proposition is true $\forall n \in \mathbb{N}$. □

Remark. DO NOT forget to prove that P_1 is true. To use Proof by Induction, we must prove that $P_k \implies P_{k+1}$ AND we must prove that P_1 is true.

2.3 Proof by Contrapositive

Recall from Chapter 1 that

$$(P \implies Q) = (\neg Q \implies \neg P)$$

If we can prove that $(\neg Q \implies \neg P)$ is true, then we also have that $(P \implies Q)$ is true.

"Prove one, get one free"

Sometimes it is easier to prove $(\neg Q \implies \neg P)$, than it to prove $(P \implies Q)$.

Lemma 2.5. *Suppose $x \in \mathbb{N}$. Then*

$$x^2 \textit{ is even} \implies x \textit{ is even.}$$

Remark. Here

$$\begin{aligned} P &= \text{“}x^2 \text{ is even”} \\ Q &= \text{“}x \text{ is even”} \\ \neg P &= \text{“}x^2 \text{ is odd”} \\ \neg Q &= \text{“}x \text{ is odd”} \end{aligned}$$

We want to prove $P \implies Q$. We **will** prove $\neg Q \implies \neg P$. So we will start by supposing that $\neg Q$ is true – we will start by supposing that x is odd – and then we will prove that $\neg P$ is true.

Proof of Lemma 2.5. Suppose that x is an odd number. Then we can write

$$x = 2k - 1$$

for some $k \in \mathbb{N}$. It follows that

$$x^2 = (2k-1)^2 = 4k^2 - 4k + 1 = 2(2k^2 - 2k) + 1$$

which is an odd number. Therefore $\neg Q \implies \neg P$. Therefore $P \implies Q$. □

Lemma 2.6. *Let $x, y \in \mathbb{N}$. Then*

$$xy \textit{ is even} \implies \textit{atleast one of } x \textit{ and } y \textit{ must be even}$$

Proof. Let

$$\begin{aligned} P &= \text{“}xy \text{ is even”} \\ Q &= \text{“}x \text{ is even or } y \text{ is even”} \\ \neg P &= \text{“}xy \text{ is odd”} \\ \neg Q &= \text{“both } x \text{ and } y \text{ are odd”} \end{aligned}$$

We will prove that $\neg Q \implies \neg P$.

Suppose that both x and y are odd. Then we can write $x = 2k-1$ and $y = 2l-1$ for some $k, l \in \mathbb{N}$. So

$$xy = (2k-1)(2l-1) = 4kl - 2k - 2l + 1 = 2(2kl - k - l) + 1$$

which is odd. Therefore $\neg Q \implies \neg P$. Therefore $P \implies Q$. $\square$

2.4 Proof by Contradiction (Olmayana Ergi)

A ***contradiction*** (çelişki) is a logical incompatibility between two or more propositions. The symbol for a contradiction is $\lightning$.

For example, suppose that for a number x, we have that $x > 7$ and $x < 3$. We know that this doesn't make sense; we know that we can not have a number which is both greater than 7 *and* less than 3. So this is a contradiction.

Now suppose that we want to prove that some proposition P is true. Proving that P is true is the same as proving that $\neg P$ is false. The idea is: If

$$\neg P \implies \lightning$$

then $\neg P$ must be false. So P must be true.

A proof by contradiction will look like:

Suppose that $\neg P$ is true.

$\vdots$ (some mathematics)

Contradiction
Therefore P is true.

Lemma 2.7. $\sqrt{2} \notin \mathbb{Q}$.

Remark. We want to prove that $\sqrt{2}$ is not a rational number. To use "proof by contradiction", we start with the opposite – we start by supposing that $\sqrt{2}$ *is* a rational number. Then we find a contradiction. Because

$$\sqrt{2} \in \mathbb{Q} \quad \Longrightarrow \quad \text{contradiction},$$

we know that $(\sqrt{2} \in \mathbb{Q})$ must be false. In other words, we know that $\sqrt{2} \notin \mathbb{Q}$ must be true.

Proof of Lemma 2.7. Suppose that $\sqrt{2} \in \mathbb{Q}$. Then there exists $a, b \in \mathbb{Z}$ $(b \neq 0)$ such that

$$\sqrt{2} = \frac{a}{b}.$$

(Note: There are many ways to write a fraction – for example,

$$\frac{\mathbf{2}}{\mathbf{3}} = \frac{4}{6} = \frac{6}{9} = \frac{126}{189} = \dots$$

We can assume that we have chosen the simplest one. So...) Suppose that a and b do not have any common factors. In particular, this means that <u>a and b are not both even numbers</u>. *(This is important.)*

Now, since $\sqrt{2} = \frac{a}{b}$, we have that $2 = \frac{a^2}{b^2}$ and hence

$$a^2 = 2b^2.$$

Since $b \in \mathbb{Z}$, we can see that a^2 must be an even number. By lemma 2.5, we have that <u>a is also an even number</u>. Define $k = \frac{1}{2}a \in \mathbb{Z}$.

It follows that $2b^2 = a^2 = (2k)^2 = 4k^2$. Therefore

$$b^2 = 2k^2.$$

Because $k^2 \in \mathbb{Z}$, we have that b^2 is an even number. So <u>b is an even number</u>.

However, we said earlier that a and b are not both even numbers. This is a contradiction. Therefore $\sqrt{2} \notin \mathbb{Q}$. □

Remark. Through out this course, we will be using Direct Proof, Proof by Induction, Proof by Contrapositive and Proof by Contradiction many times.

Chapter 3

Sequences (Sonsuz Diziler)

A ***sequence*** is a(n infinitely long) list of real numbers.

$$a_1, a_2, a_3, a_4, \ldots$$

in a particular order. Each of the a_j represents a number. These are the ***terms*** or the sequence. For example, the sequence

$$2, 4, 6, 8, 10, 12, 14, 16, \ldots, 2n, \ldots$$

has first term $a_1 = 2$, second term $a_2 = 4$ and n^{th} term $a_n = 2n$.

We write $(a_n)_{n=1}^{\infty}$ – or sometimes just (a_n) – to denote to the sequence

$$a_1, a_2, a_3, a_4, \ldots$$

If we remove the first four terms, we would get the sequence

$$a_5, a_6, a_7, a_8, \ldots$$

which we denote by $(a_n)_{n=5}^{\infty}$.

Example 3.1. Let $b_n = \frac{1}{n}$ for all $n \in \mathbb{N}$. Then $(b_n)_{n=1}^{\infty}$ is the sequence

$$1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \frac{1}{6}, \ldots$$

Example 3.2. The sequence $\left((-1)^n \frac{1}{n}\right)_{n=1}^{\infty}$ is

$$-1, \frac{1}{2}, -\frac{1}{3}, \frac{1}{4}, -\frac{1}{5}, \frac{1}{6}, \ldots$$

Example 3.3. Let $x_n = \cos n\pi$ for all $n \in \mathbb{N}$. Then $(x_n)_{n=1}^{\infty}$ is the sequence

$$-1, 1, -1, 1, -1, 1, -1, 1, -1, 1, \ldots$$

Example 3.4. The sequence $\left(\frac{1}{n^2}\right)_{n=5}^{\infty}$ is

$$\frac{1}{25}. \frac{1}{36}, \frac{1}{49}, \frac{1}{64}, \frac{1}{81}, \ldots$$

Let (a_n) be a sequence. Note that for every number $n \in \mathbb{N}$, we have a number $a_n \in \mathbb{R}$. So we have a function $\mathbb{N} \to \mathbb{R}$. We use this idea to formally define a sequence:

Definition. A ***sequence*** of real numbers is a function $a : \mathbb{N} \to \mathbb{R}$. We write $a_n := a(n)$.

Definition. $\mathbb{R}^{\mathbb{N}} := \{f : \mathbb{N} \to \mathbb{R}\} = \{\text{all sequences of real numbers}\}$

Remark. One notation for the sequence $(\frac{1}{n})_{n=1}^{\infty}$ is

$$\left(1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \frac{1}{6}, \frac{1}{7}, \frac{1}{8}, \frac{1}{9}, \ldots\right).$$

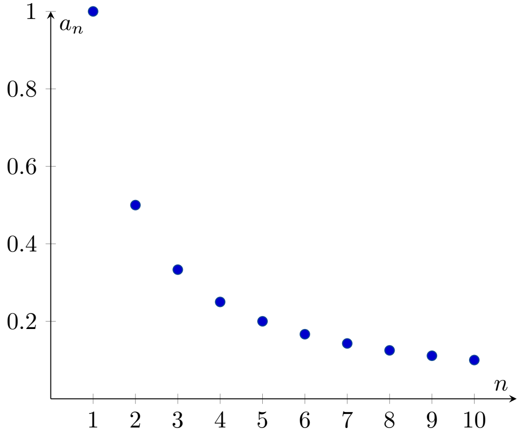
Just as $\mathbb{R}^3$ is the set of all vectors (x, y, z), we might expect the set of all sequences to be denoted $\mathbb{R}^{\infty}$ – but what is "∞"? ∞ has many different types. The notation $\mathbb{R}^{\mathbb{N}}$ is more precise.

More generally, B^A denotes the set of all maps from A to B, but we won't need this in this course.

Remark. Even though sequences are really functions, we will usually think of them as lists of numbers.

For sequences, the important things are: the order in which the numbers appear, and the behaviour of the terms as $n \to \infty$.

Example 3.5.

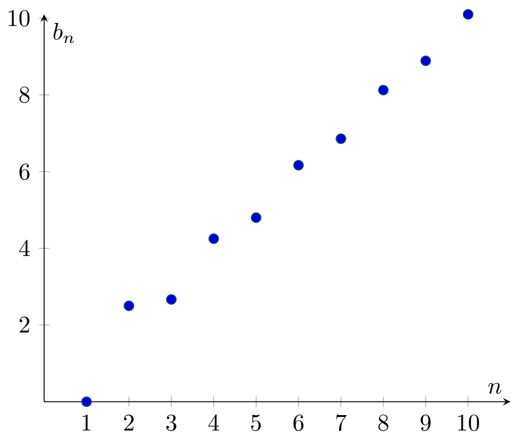


Let $a_n = \frac{1}{n}$ for all $n \in \mathbb{N}$. Then $(a_n)_{n=1}^{\infty}$ is the sequence

$$1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \frac{1}{6}, \frac{1}{7}, \frac{1}{8}, \frac{1}{9}, \frac{1}{10}, \ldots$$

It looks like a_n "goes to" 0 as $n \to \infty$.

Example 3.6.

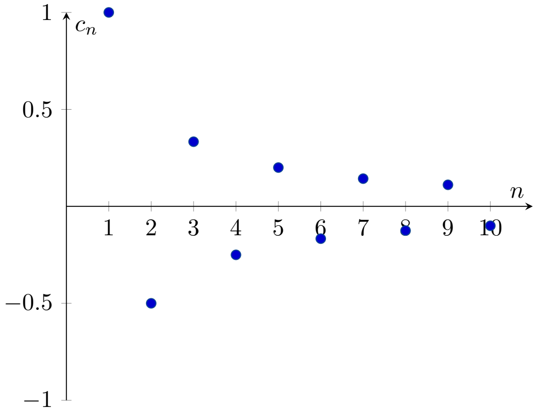


Let $b_n = n + (-1)^n \frac{1}{n}$ for all $n \in \mathbb{N}$. Then $(b_n)_{n=1}^{\infty}$ is the sequence

$$0, 2\tfrac{1}{2}, 2\tfrac{2}{3}, 4\tfrac{1}{4}, 4\tfrac{4}{5}, 6\tfrac{1}{6}, \ldots$$

It looks like b_n "goes to" ∞ as $n \to \infty$.

Example 3.7.

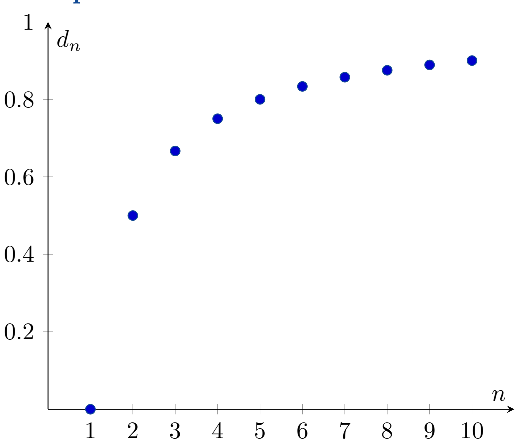


Let $c_n = \frac{(-1)^{n+1}}{n}$ for all $n \in \mathbb{N}$. Then $(c_n)_{n=1}^{\infty}$ is the sequence

$$1, -\frac{1}{2}, \frac{1}{3}, -\frac{1}{4}, \frac{1}{5}, -\frac{1}{6}, \ldots$$

It looks like c_n "goes to" 0 as $n \to \infty$.

Example 3.8.



Let $d_n = 1 - \frac{1}{n}$ for all $n \in \mathbb{N}$. Then $(d_n)_{n=1}^{\infty}$ is the sequence

$$0, \tfrac{1}{2}, \tfrac{2}{3}, \tfrac{3}{4}, \tfrac{4}{5}, \tfrac{5}{6}, \ldots$$

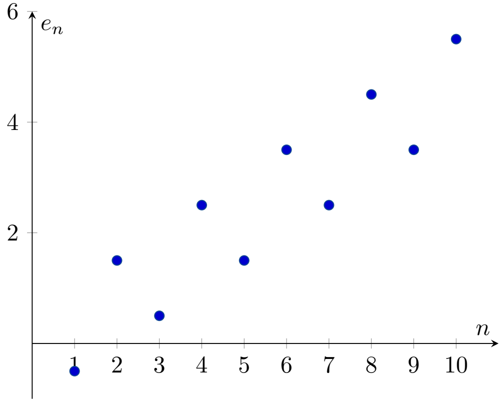
It looks like d_n "goes to" 1 as $n \to \infty$.

Definition. The ***floor function***, $\lfloor \cdot \rfloor : \mathbb{R} \to \mathbb{Z}$, is defined by

$$\lfloor x \rfloor = \max\{p \in \mathbb{Z} : p \leq x\}.$$

For example $\lfloor 3.79 \rfloor = 3$ and $\lfloor 4 \rfloor = 4$.

Example 3.9.



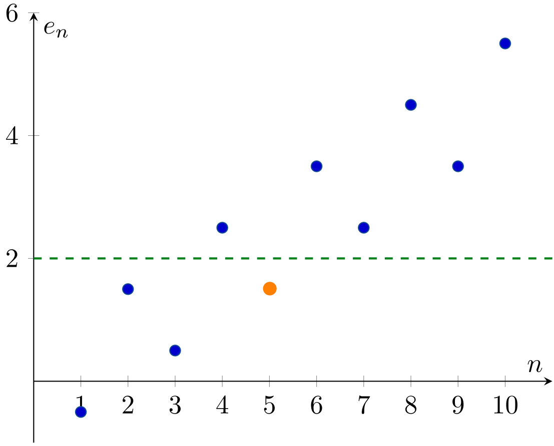
Let $e_n = \left\lfloor \frac{n}{2} \right\rfloor + \frac{(-1)^n}{2}$ for all $n \in \mathbb{N}$. Then $(e_n)_{n=1}^{\infty}$ is the sequence

$$-\tfrac{1}{2}, \tfrac{3}{2}, \tfrac{1}{2}, \tfrac{5}{2}, \tfrac{3}{2}, \tfrac{7}{2}, \tfrac{5}{2}, \tfrac{9}{2}, \ldots$$

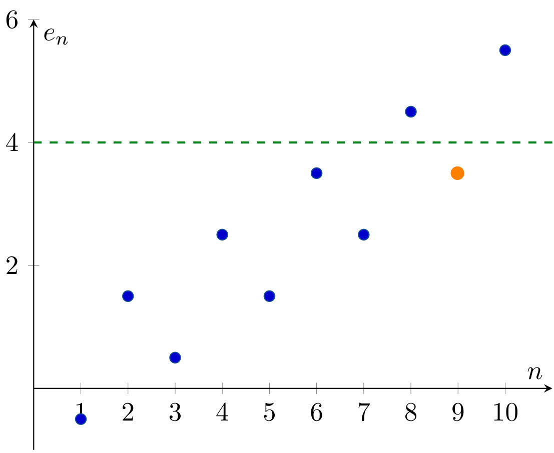
Note that $e_{2000} = 1000 + \frac{1}{2}$ and $e_{2000000} = 1000000 + \frac{1}{2}$. It looks like e_n "goes to" ∞ as $n \to \infty$.

Remark. In these last five examples, we have said "looks like" and "goes to" a lot. But what does this mean mathematically? We need to be more precise.

What does "goes to ∞" really mean? It doesn't mean "gets bigger" because d_n gets bigger, but we think that d_n "goes to" 1. Furthermore, we think that e_n "goes to" ∞, but e_n gets bigger, smaller, bigger, smaller, bigger, smaller, bigger, smaller, $\ldots$.



Notice that if we draw a green line at height 2, then 4 points at the start of the sequence are underneath the line and all of the rest of the sequence is above the line.



If we draw a green line at height 4, then a finite number of points at the start of the sequence are underneath the line and all of the rest of the sequence is above the line.

Now we are getting somewhere.

In general, if I choose any number $A \in \mathbb{R}$ and draw a green line at height A, then there will be a finite number of points underneath the line and an infinite number of points above the line. One of the points under the green line must be the last one. Call this point e_N. This means that

$$e_{N+1}, e_{N+2}, e_{N+3}, e_{N+4}, e_{N+5}, \ldots$$

are all above the green line. In other words, $\exists\, N \in \mathbb{N}$ such that $e_n > A$ for all $n > N$.

Obviously the number N will depend on A. We will write $N = N(A)$ so that we remember this.

If we choose $A = 10$, then note that

$$\begin{aligned} n > 21 \quad \Longrightarrow \quad e_n &= \left\lfloor \tfrac{n}{2} \right\rfloor + \tfrac{(-1)^n}{2} \geq \left\lfloor \tfrac{22}{2} \right\rfloor + \tfrac{(-1)^n}{2} \\ &= 11 + \tfrac{(-1)^n}{2} > 10 = A \end{aligned}$$

which means that we can choose $N(10) = 21$. In fact, we don't have to choose the "best" N – any N which works is good enough. So if we wanted to, we could choose $N(10) = 1000000$ and the calculation above still works. If $n > 1000000$, then $e_n > 10$ (check it!!!).

If we choose $A = 100$, then $e_n > 100 = A$ for all $n > 201$ (you check!), so we can choose $N(100) = 201$.

In general, for any given $A > 0$, we can always find an $N = N(A)$ for the sequence (e_n). If we choose

$$N = \text{"the smallest integer such that } N > 2A + 3\text{"},$$

then

$$\begin{aligned} n > N \quad \Longrightarrow \quad n \geq 2A + 4 \quad \Longrightarrow \quad e_n &= \left\lfloor \tfrac{n}{2} \right\rfloor + \tfrac{(-1)^n}{2} \\ &\geq \lfloor A + 2 \rfloor + \tfrac{(-1)^n}{2} > A. \end{aligned}$$

Definition. A sequence of real numbers (a_n) ***tends to infinity*** iff for all $A > 0$, there exists $N = N(A) \in \mathbb{N}$ such that

$$n > N \quad \Longrightarrow \quad a_n > A.$$

We write "$a_n \to \infty$ as $n \to \infty$" in this case.

Remark. In symbolic logic, $a_n \to \infty$ as $n \to \infty$ iff

$$(\forall\ A > 0)(\exists\ N \in \mathbb{N})(\forall\ n \in \mathbb{N})(n > N \implies a_n > A).$$

Example 3.10. Let $a_n = \sqrt{n}$ for all $n \in \mathbb{N}$. Show that $a_n \to \infty$ as $n \to \infty$.

solution: Let $A > 0$. Choose $N \in \mathbb{N}$ such that $N \geq A^2$. Then for all $n \in \mathbb{N}$,

$$n > N \implies a_n = \sqrt{n} > \sqrt{N} \geq A.$$

Therefore $a_n \to \infty$ as $n \to \infty$.

Example 3.11. Let $b_n = \log n$ for all $n \in \mathbb{N}$. Show that $b_n \to \infty$ as $n \to \infty$.

solution: Let $A > 0$. Choose $N \in \mathbb{N}$ such that $N \geq e^A$. Then for all $n \in \mathbb{N}$,

$$n > N \implies b_n = \log n > \log N \geq \log e^A = A.$$

Therefore $b_n \to \infty$ as $n \to \infty$.

Example 3.12. Let

$$c_n = \begin{cases} \frac{n^2\sqrt{n}+n^2+1}{n^2-43} & n \geq 7 \\ 0 & 1 \leq n \leq 6 \end{cases}$$

for all $n \in \mathbb{N}$. Show that $c_n \to \infty$ as $n \to \infty$.

solution: First note that if $n \geq 7$ then

$$c_n = \frac{n^2\sqrt{n}+n^2+1}{n^2-43} > \frac{n^2\sqrt{n}}{n^2-43} > \frac{n^2\sqrt{n}}{n^2} = \sqrt{n}.$$

Let $A > 0$. Choose $N \in \mathbb{N}$ such that $N \geq \max\{A^2, 7\}$. Then for all $n \in \mathbb{N}$,

$$n > N \implies c_n > \sqrt{n} > \sqrt{N} \geq A.$$

Therefore $c_n \to \infty$ as $n \to \infty$.

Remark. Remember that we don't need to find the "best" or smallest N. We only need to find one which works.

Remark. In examples 3.10 and 3.11 it was easy to find an N. In example 3.12, we used an inequality first so that finding an N was easier.

Theorem 3.1. *Let (a_n) and (b_n) be two sequences of real numbers such that*

$$a_n \geq b_n$$

for all $n > N_0 \in \mathbb{N}$. If $b_n \to \infty$ as $n \to \infty$, then $a_n \to \infty$ as $n \to \infty$.

Proof. Let $A > 0$. Since $b_n \to \infty$ as $n \to \infty$, $\exists\ N_1 \in \mathbb{N}$ such that

$$n > N_1 \implies b_n > A.$$

Choose $N = \max\{N_0, N_1\}$. Then

$$n > N \implies a_n \geq b_n > A.$$

Therefore $a_n \to \infty$ as $n \to \infty$. $\square$

Example 3.13. Let $a_n = n^2 + n\cos n\pi$ and $b_n = \frac{1}{2}n^2$. Let $A > 0$. Choose $N \geq \sqrt{2A}$. Then

$$n > N \implies b_n = \tfrac{1}{2}n^2 > \tfrac{1}{2}N^2 \geq A.$$

Hence $b_n \to \infty$ as $n \to \infty$.

Moreover, if $n \geq 2$ then

$$a_n = n^2 + n\cos n\pi = n^2 + n(-1)^n \geq n^2 - n \geq n^2 - \tfrac{1}{2}n^2 = \tfrac{1}{2}n^2 = b_n.$$

Therefore $a_n \to \infty$ as $n \to \infty$ by Theorem 3.1.

Example 3.14. Let $a_n := \frac{n^2+\sqrt{n}}{n+\cos n}$. Show that $a_n \to \infty$ as $n \to \infty$.

solution: If $n \geq 2$, then

$$a_n < \frac{n^2}{n+\cos n} \geq \frac{n^2}{n+1} > \frac{n^2}{n+n} = \tfrac{1}{2}n.$$

Now let $b_n = \frac{1}{2}n$. Since $b_n \to \infty$ as $n \to \infty$ (you check!!!) and since $a_n > b_n$ for all $n \geq 2$, if follows by Theorem 3.1 that $a_n \to \infty$ as $n \to \infty$.

Exercise 3.15. Let $c_n = n!$ for all $n \in \mathbb{N}$. Show that $c_n \to \infty$ as $n \to \infty$.

Definition. A sequence of real numbers a_n ***tends to minus infinity*** ($a_n \to -\infty$ as $n \to \infty$) iff for all $A > 0$, there exists $N = N(A) \in \mathbb{N}$ such that

$$n > N \implies a_n < -A.$$

Remark. In symbolic logic, $a_n \to -\infty$ as $n \to \infty$ iff

$$(\forall\ A > 0)(\exists\ N \in \mathbb{N})(\forall\ n \in \mathbb{N})(n > N \implies a_n < -A).$$

Exercise 3.16. Prove that

$$a_n \to -\infty \text{ as } n \to \infty \iff -a_n \to \infty \text{ as } n \to \infty.$$

Exercise 3.17. Let $d_n = (-1)^n n - n^2$ for all $n \in \mathbb{N}$. Show that $d_n \to -\infty$ as $n \to \infty$.

Theorem 3.2. *Let (a_n) and (b_n) be two sequences of real numbers such that*

$$a_n \leq b_n$$

for all $n > N_0 \in \mathbb{N}$. If $b_n \to -\infty$ as $n \to \infty$, then $a_n \to -\infty$ as $n \to \infty$.

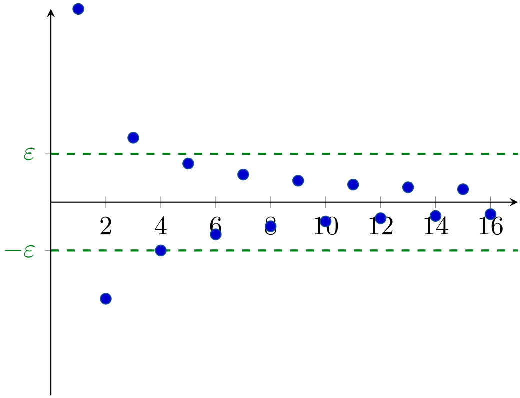
Proof. Since $a_n \leq b_n\ \forall\, n \geq N_0$, it follows that $-a_n \geq -b_n\ \forall\, n \geq N_0$. Thus

$$\begin{aligned} b_n \to -\infty \text{ as } n \to \infty &\implies -b_n \to \infty \text{ as } n \to \infty \\ &\implies -a_n \to \infty \text{ as } n \to \infty \\ &\implies a_n \to -\infty \text{ as } n \to \infty \end{aligned}$$

by Theorem 3.1 and Exercise 3.16. □

Example 3.18. Now let $y_n = \frac{(-1)^{n+1}}{n}$ for all $n \in \mathbb{N}$. Then $(y_n)_{n=1}^{\infty}$ is the sequence

$$1, -\tfrac{1}{2}, \tfrac{1}{3}, -\tfrac{1}{4}, \tfrac{1}{5}, -\tfrac{1}{6}, \tfrac{1}{7}, -\tfrac{1}{8}, \tfrac{1}{9}, \ldots$$



It "looks like" y_n "goes to" 0 as $n \to \infty$. But what does this mean? How can we be more precise?

If we draw green lines at heights ε and $-\varepsilon$, then (apart from a finite number of points) the sequence will be between the two green lines.

E.g. Let $\varepsilon = \frac{1}{100}$. Then $-\frac{1}{100} < y_n < \frac{1}{100}$ for all $n > 100$.

In general: If we choose $N \geq \frac{1}{\varepsilon}$, then

$$n > N \implies -\varepsilon < y_n < \varepsilon.$$

In other words, if $N \geq \frac{1}{\varepsilon}$ then

$$n > N \implies |y_n| < \varepsilon.$$

Definition. A sequence of real numbers a_n ***tends to zero*** ($a_n \to 0$ as $n \to \infty$) iff for all $\varepsilon > 0$, there exists $N = N(\varepsilon) \in \mathbb{N}$ such that

$$n > N \implies |a_n| < \varepsilon.$$

Remark. In symbolic logic, $a_n \to 0$ as $n \to \infty$ iff

$$(\forall\, \varepsilon > 0)(\exists\, N \in \mathbb{N})(\forall\, n \in \mathbb{N})(n > N \implies |a_n| < \varepsilon).$$

Remark. In Mathematics; we usually use ε for arbitrarily small numbers and a capital letter (e.g. A) for arbitrarily large numbers.

Definition. A sequence of real numbers a_n is called a ***null sequence*** iff $a_n \to 0$ as $n \to \infty$.

Example 3.19. Let $a_n = n^{-7}$ for all $n \in \mathbb{N}$. Show that (a_n) is a null sequence.

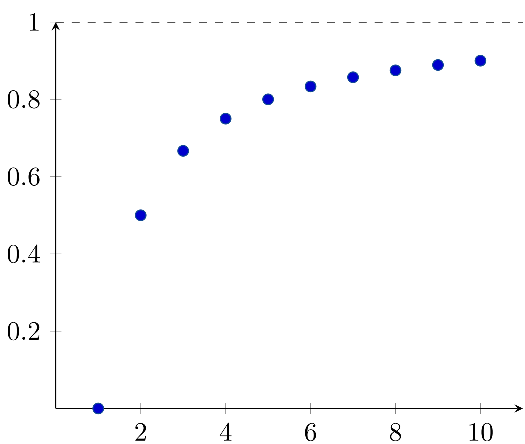
solution: We have to show that $a_n \to 0$ as $n \to \infty$. Let $\varepsilon > 0$. Choose $N \geq \varepsilon^{-\frac{1}{7}}$. Then for all $n \in \mathbb{N}$,

$$n > N \implies |a_n| = \left|\frac{1}{n^7}\right| = \frac{1}{n^7} < \frac{1}{N^7} \leq \varepsilon.$$

Therefore (a_n) is a null sequence.

Example 3.20. Let $z_n = 1 - \frac{1}{n}$ for all $n \in \mathbb{N}$. Then (z_n) is the sequence

$$0, \tfrac{1}{2}, \tfrac{2}{3}, \tfrac{3}{4}, \tfrac{4}{5}, \tfrac{5}{6}, \ldots$$



It "looks like" z_n "goes to" 1 as $n \to \infty$. But this is Mathematics, so we need to be precise.

Definition. A sequence of real numbers a_n ***tends to*** l ($a_n \to l$ as $n \to \infty$) iff for all $\varepsilon > 0$, there exists $N = N(\varepsilon) \in \mathbb{N}$ such that

$$n > N \implies |a_n - l| < \varepsilon.$$

Remark. In symbolic logic, $a_n \to l$ as $n \to \infty$ iff

$$(\forall\, \varepsilon > 0)(\exists\, N \in \mathbb{N})(\forall\, n \in \mathbb{N})(n > N \implies |a_n - l| < \varepsilon).$$

Remark. We can also write $\lim_{n\to\infty} a_n = l$ if a_n tends to l.

Exercise 3.21. Prove that

$$a_n \to l \text{ as } n \to \infty \iff (a_n - l) \text{ is a null sequence.}$$

Example 3.22. Let $u_n = \begin{cases} 7 & n \geq 7 \\ n & 1 \leq n \leq 6. \end{cases}$ Show that $u_n \to 7$ as $n \to \infty$.

solution: Let $\varepsilon > 0$. Choose $N = 6$. Then

$$n > N \implies n \geq 7 \implies u_n = 7 \implies |u_n - 7| = 0 < \varepsilon.$$

Therefore $u_n \to 7$ as $n \to \infty$.

Example 3.23. Let $v_n = \frac{n^2+n+1}{2n^2+1}$ for all $n \in \mathbb{N}$. Show that $v_n \to \frac{1}{2}$ as $n \to \infty$.

solution: Let $\varepsilon > 0$. First note that

$$\begin{aligned} v_n - \frac{1}{2} &= \left(\frac{n^2+n+1}{2n^2+1}\right) - \frac{1}{2} = \left(\frac{n^2+n+1}{2n^2+1}\right) - \left(\frac{n^2+\frac{1}{2}}{2n^2+1}\right) \\ &= \frac{2n+1}{2(2n^2+1)}. \end{aligned}$$

So

$$\left|v_n - \tfrac{1}{2}\right| < \frac{2n+1}{4n^2} \leq \frac{2n+n}{4n^2} = \frac{3}{4n}.$$

Now choose $N > \frac{3}{4\varepsilon}$. Then for all $n \in \mathbb{N}$,

$$n > N \implies \left|v_n - \tfrac{1}{2}\right| < \frac{3}{4n} < \frac{3}{4N} < \varepsilon.$$

Therefore $v_n \to \frac{1}{2}$ as $n \to \infty$.

Example 3.24. Recall that π = 3.14159265358979323846264
33832795028841971693993751058209749445923078164062862089
98628034825342117067982148086513282306647093844609550582
23172535940812848111745028410270193852110555964462294895
49303819644288109756659334461284756482337867831652712019
09145648566923460348610454326648213393607260249141273724
58700660631558817488152092096282925409171536436789259036
00113305305488204665213841469519415116094330572703657595
91953092186117381932611793105118548074462379962749567351
88575272489122793818301194912983367336244065664308602139
49463952247371907021798609437027705392171762931767523846
74818467669405132000568127145263560827785771342757789609
17363717872146844090122495343014654958537105079227968925
89235420199561121290219608640344181598136297747713099605
18707211349999998372978049951059731732816096318595024459
45534690830264252230825334468503526193118817101000313783
87528865875332083814206171776691473035982534904287554687
31159562863882353787593751957781857780532171226806613001
92787661119590921642019893809525720106548586327886593615
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75283479131515574857242454150695950829533116861727855889
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05224894077267194782684826014769909026401363944374553050
68203496252451749399651431429809190659250937221696461515
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68038447734549202605414665925201497442850732518666002132
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35732654893823911932597463667305836041428138830320382490
37589852437441702913276561809377344403070746921120191302
03303801976211011004492932151608424448596376698389522868
47831235526582131449576857262433441893039686426243410773
22697802807318915441101044682325271620105265227211166040 ...

Define a sequence (p_n) by

$$\begin{aligned} p_1 &= 3 \\ p_2 &= 3.1 \\ p_3 &= 3.14 \\ p_4 &= 3.141 \\ p_5 &= 3.1415 \\ &\vdots \\ p_n &= \text{the first } n \text{ digits of } \pi \\ &\vdots \end{aligned}$$

Show that $p_n \to \pi$ as $n \to \infty$.

solution: First note that

$$\begin{aligned} |p_1 - \pi| &= 0.141592\ldots < 1 = 10^0 \\ |p_2 - \pi| &= 0.041592\ldots < 0.1 = 10^{-1} \\ |p_3 - \pi| &= 0.001592\ldots < 0.01 = 10^{-2} \\ &\vdots \\ |p_n - \pi| &< 10^{1-n} \\ &\vdots \end{aligned}$$

Let $\varepsilon > 0$. Choose $N > 1 - \log_{10} \varepsilon$. Then for all $n \in \mathbb{N}$,

$$n > N \implies |p_n - \pi| < 10^{1-n} < 10^{1-N} < 10^{1-(1-\log_{10} \varepsilon)} = \varepsilon.$$

Therefore $p_n \to \pi$ as $n \to \infty$.

Theorem 3.3. *A sequence of real numbers cannot have more than one limit.*

Proof. Let (a_n) be a sequence.

CASE 1: Suppose first that $a_n \to l \in \mathbb{R}$ and $a_n \to m \in \mathbb{R}$ as $n \to \infty$. We will use proof by contradiction to prove that $l = m$: Assume that $l \neq m$. Then $l - m \neq 0$ and $|l - m| > 0$. Let $\varepsilon = \frac{1}{2}|l - m| > 0$.

Since $a_n \to l$ as $n \to \infty$, $\exists N_0 \in \mathbb{N}$ such that

$$n > N_0 \implies |a_n - l| < \varepsilon.$$

Similarly, since $a_n \to m$ as $n \to \infty$, $\exists N_2 \in \mathbb{N}$ such that

$$n > N_0 \implies |a_n - m| < \varepsilon.$$

Let $N = \max\{N_0, N_1\}$. Then $\forall\ n > N$ we have that

$$\begin{aligned} |l - m| &= |l - a_n + a_n - m| \leq |l - a_n| + |a_n - m| \\ &= |a_n - l| + |a_n - m| < \varepsilon + \varepsilon = |l - m| \end{aligned}$$

by the triangle inequality. But $|l - m| < |l - m|$ is a contradiction $\lightning$. Since $l \neq m$ leads to a contradiction, we must have $l = m$. This means that a sequence cannot have two different finite limits.

CASE 2: Moreover, if $a_n \to l$ as $n \to \infty$, then $\exists N \in \mathbb{N}$ such that

$$n > N \implies |a_n - l| < 1 \implies l - 1 < a_n < l + 1.$$

Hence $a_n \not\to \infty$ and $a_n \not\to -\infty$ as $n \to \infty$. Therefore a sequence cannot have both a finite limit and an infinite limit.

CASE 3: Finally, a_n cannot tend to both ∞ and $-\infty$ by Exercise 3.27.

Therefore a sequence cannot have two different limits. $\square$

Remark. Recall that

$$\begin{aligned} &(a_n \to l \text{ as } n \to \infty) \\ &\qquad \iff (\forall\ \varepsilon > 0)(\exists\ N \in \mathbb{N})(\forall\ n \in \mathbb{N})(n > N \implies |a_n - l| < \end{aligned}$$

So

$$\begin{aligned} &(a_n \not\to l \text{ as } n \to \infty) \\ &\qquad \iff \neg(a_n \to l \text{ as } n \to \infty) \\ &\qquad \iff (\exists\ \varepsilon > 0)(\forall\ N \in \mathbb{N})(\exists\ n \in \mathbb{N})(n > N\ \wedge\ |a_n - l| \geq \varepsilon \end{aligned}$$

Changing back into English, we get: A sequence of real numbers a_n ***does not tend to*** l ($a_n \not\to l$ as $n \to \infty$) iff there exists $\varepsilon > 0$ such that for all $N \in \mathbb{N}$, there exists $n \in \mathbb{N}$ such that

$$n > N \text{ and } |a_n - l| \geq \varepsilon.$$

Example 3.25. Let (z_n) be the sequence

$$1, 0, 1, 0, 1, 0, 1, 0, 1, 0, 1, 0, \ldots$$

Show that $z_n \not\to 0$ as $n \to \infty$.

solution: Choose $\varepsilon = \frac{1}{2}$. Let N be any natural number. If N is odd, choose $n = N + 2$. If N is even, choose $n = N + 1$. Then clearly $n > N$. Since n is odd, we have that

$$|z_n| = 1 \geq \frac{1}{2} = \varepsilon.$$

Therefore $z_n \not\to 0$ as $n \to \infty$.

Exercise 3.26. Write a definition for $a_n \not\to \infty$ as $n \to \infty$. Write a definition for $a_n \not\to -\infty$ as $n \to \infty$.

Exercise 3.27. Let (a_n) be a sequence of real numbers. Prove that if $a_n \to \infty$ as $n \to \infty$, then $a_n \not\to -\infty$ as $n \to \infty$.

Definition. A sequence of real numbers (a_n) is called a ***convergent sequence*** iff $\exists\ l \in \mathbb{R}$ such that $a_n \to l$ as $n \to \infty$.

Remark. We know from Theorem 3.3 that a convergent sequence has only one limit.

Definition. A sequence which is not convergent is called a ***divergent sequence***.

Definition. If $a_n \to \infty$ as $n \to \infty$, we say that (a_n) ***diverges to infinity*** (sonsuza ıraksar).

Definition. If $a_n \to -\infty$ as $n \to \infty$, we say that (a_n) ***diverges to minus infinity***.

Example 3.28. Let $a_n = (-1)^n$, $b_n = (-1)^n n$ and $c_n = n^2$. Then (a_n), (b_n) and (c_n) are divergent sequences. (a_n) and (b_n) do not have a finite limit or an infinite limit. (c_n) diverges to infinity.

Definition. A sequence of real numbers (a_n) is called a ***bounded sequence*** (sınırlı dizi) iff $\exists\ M > 0$ such that $|a_n| \leq M$ for all $n \in \mathbb{N}$.

Theorem 3.4. *Every convergent sequence is bounded.*

Proof. Let (a_n) be a convergent sequence. Then $\exists\ a \in \mathbb{R}$ such that $a_n \to a$ as $n \to \infty$. So $\exists\ N \in \mathbb{N}$ such that

$$n > N \implies |a_n - a| < 1 \implies |a_n| < |a| + 1.$$

Now let $M := \max\{|a_1|, |a_2|, |a_3|, \ldots, |a_N|, |a| + 1\}$. Then $|a_n| \leq M$ for all $n \in \mathbb{N}$. Therefore (a_n) is a bounded sequence. □

Lemma 3.5. *Suppose that $a_n \to a \in \mathbb{R}$ and $b_n \to b \in \mathbb{R}$ as $n \to \infty$. Then $a_n + b_n \to a + b$ as $n \to \infty$.*

Proof. Let $\varepsilon > 0$. Since $a_n \to a \in \mathbb{R}$ and $b_n \to b \in \mathbb{R}$ as $n \to \infty$, $\exists N \in \mathbb{N}$ such that

$$n > N \implies a - \tfrac{\varepsilon}{2} < a_n < a + \tfrac{\varepsilon}{2} \qquad \text{and} \qquad b - \tfrac{\varepsilon}{2} < b_n < b + \tfrac{\varepsilon}{2}.$$

Adding these inequalities together, we see that

$$n > N \implies a + b - \varepsilon < a_n + b_n < a + b + \varepsilon.$$

Therefore $a_n + b_n \to a + b$ as $n \to \infty$. □

Lemma 3.6. *Suppose that $a_n \to a \in \mathbb{R}$ and $b_n \to b \in \mathbb{R}$ as $n \to \infty$. Then $a_n b_n \to ab$ as $n \to \infty$.*

Proof. Let $\varepsilon > 0$. First

$$b_n \to b \text{ as } n \to \infty \implies (b_n) \text{ is convergent} \implies (b_n) \text{ is bounded}$$

by Theorem 3.4. So $\exists M > 0$ such that $|b_n| \leq M\ \forall n \in \mathbb{N}$. Note that $\frac{\varepsilon}{M+|a|} > 0$.

Since $a_n \to a$ and $b_n \to b$ as $n \to \infty$, $\exists N \in \mathbb{N}$ such that

$$n > N \implies |a_n - a| < \frac{\varepsilon}{M + |a|} \qquad \text{and} \qquad |b_n - b| < \frac{\varepsilon}{M + |a|}.$$

But then

$$\begin{aligned} n > N \implies |a_n b_n - ab| &= |a_n b_n - a b_n + a b_n - ab| \\ &\leq |a_n b_n - a b_n| + |a b_n - ab| \\ &= |a_n - a|\,|b_n| + |a|\,|b_n - b| \\ &< \frac{\varepsilon}{M + |a|} M + |a| \frac{\varepsilon}{M + |a|} = \varepsilon. \end{aligned}$$

Therefore $a_n b_n \to ab$ as $n \to \infty$. □

Lemma 3.7. *Suppose that $a_n \to a \in \mathbb{R}$ and $b_n \to b \in \mathbb{R}$ as $n \to \infty$. Suppose that $b \neq 0$. Then $\frac{a_n}{b_n} \to \frac{a}{b}$ as $n \to \infty$.*

Proof. We only need to prove that $\frac{1}{b_n} \to \frac{1}{b}$ as $n \to \infty$. Then we can use Lemma 3.6 to finish.

Since $b_n \to b \neq 0$ as $n \to \infty$, $\exists N_0$ such that

$$n > N_0 \implies |b_n - b| < \tfrac{1}{2}|b|.$$

But then

$$n > N_0 \implies |b_n| = |b_n - b + b| \geq |b| - |b_n - b| > |b| - \tfrac{1}{2}|b| = \tfrac{1}{2}|b|.$$

Let $\varepsilon > 0$. Since $b_n \to b$ as $n \to \infty$, $\exists N_1 \in \mathbb{N}$ such that

$$n > N_1 \implies |b_n - b| < \frac{\varepsilon |b|^2}{2}.$$

Let $N = \max\{N_0, N_1\}$. Then

$$n > N \implies \left|\frac{1}{b_n} - \frac{1}{b}\right| = \left|\frac{b - b_n}{bb_n}\right| = \frac{|b - b_n|}{|b|\,|b_n|} < \frac{\varepsilon |b|^2}{2}\frac{1}{|b|}\frac{2}{|b|} = \varepsilon.$$

Therefore $\frac{1}{b_n} \to \frac{1}{b}$ as $n \to \infty$. By Lemma 3.6, we have that $\frac{a_n}{b_n} = \frac{1}{b_n}a_n \to \frac{1}{b}a = \frac{a}{b}$ as $n \to \infty$. □

Example 3.29. Let $a_n = \frac{n^5+7n^3+5n^2+8}{5n^5+3n^4+27}$. Then

$$a_n = \frac{n^5 + 7n^3 + 5n^2 + 8}{5n^5 + 3n^4 + 27} = \frac{1 + 7n^{-2} + 5n^{-3} + 8n^{-5}}{5 + 3n^{-1} + 27n^{-5}} \to \frac{1 + 0 + 0 +}{5 + 0 + 0}$$

as $n \to \infty$ by Lemmata 3.5-3.7.

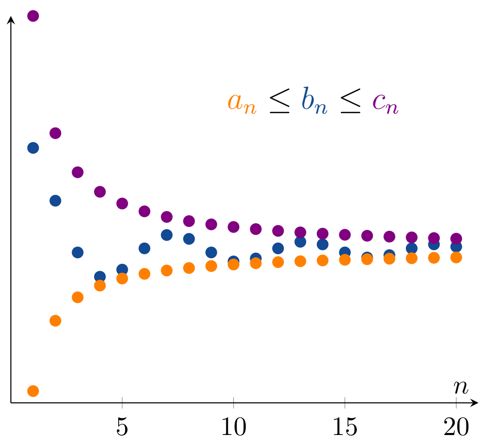
Chapter 4

The Sandwich Rule (Sandviç Kuralı)

Theorem 4.1 (The Sandwich Rule). *Let (a_n), (b_n) and (c_n) be three sequences of real numbers such that*

$$a_n \leq b_n \leq c_n$$

for all $n > N_0 \in \mathbb{N}$. If $a_n \to l$ and $c_n \to l$ as $n \to \infty$, then $b_n \to l$ as $n \to \infty$ also.



Proof. Let $\varepsilon > 0$. Since $a_n \to l$ as $n \to \infty$, $\exists N_1 \in \mathbb{N}$ such that

$$n > N_1 \implies l - \varepsilon < a_n < l + \varepsilon.$$

Since $c_n \to l$ as $n \to \infty$, $\exists N_2 \in \mathbb{N}$ such that

$$n > N_2 \implies l - \varepsilon < c_n < l + \varepsilon.$$

Let $N = \max\{N_0, N_1, N_2\}$. Then

$$n > N \implies l - \varepsilon < a_n \leq b_n \leq c_n < l + \varepsilon.$$

Therefore $b_n \to l$ as $n \to \infty$. □

Theorem 4.2. *Let (c_n) be a sequence of real numbers such that*

$$c_n \geq 0$$

for all $n > N_0 \in \mathbb{N}$. Suppose that $c_n \to c$ as $n \to \infty$. Then $c \geq 0$.

Proof. We will use proof by contradiction: Suppose that $c < 0$.

Let $\varepsilon = \frac{1}{2}|c| > 0$. Since $c_n \to c$ as $n \to \infty$, $\exists N_1 \in \mathbb{N}$ such that

$$n > N_1 \implies |c_n - c| < \varepsilon = -\frac{c}{2}.$$

Let $N = \max\{N_0, N_1\}$. Since $c_n \geq 0$ $\forall n \in \mathbb{N}$ and $c < 0$, we have that

$$n > N \implies c_n - c = |c_n - c| < -\frac{c}{2}.$$

So $c_n < \frac{c}{2} < 0$ for all $n > N$. This is a contradiction $\lightning$. So we must have that $c \geq 0$. □

Corollary 4.2.1. *Let (a_n) and (b_n) be sequences of real numbers such that*

$$a_n \leq b_n$$

for all $n > N_0 \in \mathbb{N}$. Suppose that $a_n \to a$ and $b_n \to b$ as $n \to \infty$. Then $a \leq b$.

Proof. Let $c_n = b_n - a_n$. By Theorem 4.2 it follows that $b - a \geq 0$. □

Remark. So

$$a_n \le b_n \implies \lim_{n\to\infty} a_n \le \lim_{n\to\infty} b_n$$

if the limits exist. But is "$a_n < b_n \implies \lim_{n\to\infty} a_n < \lim_{n\to\infty} b_n$." true? The answer is NO!!!!

Example 4.1. Let $a_n = \frac{1}{n}$ and $b_n = \frac{1}{n^2}$. Then $a_n < b_n$ for all $n > 1$, but $\lim_{n\to\infty} a_n = 0 = \lim_{n\to\infty} b_n$.

Remark. Be careful when taking limits of inequalities!

Remark.

$$a_n < b_n \implies \lim_{n\to\infty} a_n \le \lim_{n\to\infty} b_n$$

if the limits exist.

Theorem 4.3. *Let (a_n) be a sequence. If $a_n \to \infty$ as $n \to \infty$, then $\frac{1}{a_n} \to 0$ as $n \to \infty$.*

Proof. Let $\varepsilon > 0$. Then let $A = \frac{1}{\varepsilon} > 0$. Since $a_n \to \infty$ as $n \to \infty$, $\exists N \in \mathbb{N}$ such that

$$n > N \implies a_n > A.$$

So

$$n > N \implies 0 < \frac{1}{a_n} < \frac{1}{A} = \varepsilon.$$

Therefore $\frac{1}{a_n} \to 0$ as $n \to \infty$. $\square$

Example 4.2. Since $0 \leftarrow -\frac{1}{n} \le \frac{\cos n}{n} \le \frac{1}{n} \to 0$ as $n \to \infty$, it follows by the Sandwich Rule that $\frac{\cos n}{n} \to 0$ as $n \to \infty$.

Example 4.3.

$$\frac{n^5 + n^4 \cos n + 6}{4n^5 + n^3 + \cos n} = \frac{1 + \frac{\cos n}{n} + 6n^{-5}}{4 + \frac{1}{n} + \frac{\cos n}{n^5}} \to \frac{1+0+0}{4+0+0} = \frac{1}{4}$$

as $n \to \infty$.

Example 4.4. Let (a_n) be a sequence. Suppose that

- $a_n \geq 0$ for all $n \in \mathbb{N}$;
- $a \geq 0$; and
- $a_n^2 \to a^2$ as $n \to \infty$.

Prove that $a_n \to a$ as $n \to \infty$.

solution:
CASE 1: $a = 0$
Suppose that $a_n^2 \to 0$ as $n \to \infty$. Let $\varepsilon > 0$. Then $\varepsilon^2 > 0$. So $\exists$ N such that

$$n > N \implies \left|a_n^2\right| < \varepsilon^2.$$

So $|a_n| < \varepsilon$ for all $n > N$. Hence $a_n \to 0$ as $n \to \infty$.

CASE 2: $a > 0$
Suppose that $a_n^2 \to a^2$ as $n \to \infty$. Since $a_n \geq 0$ and $a > 0$,

$$\begin{aligned} \left|a_n^2 - a^2\right| &= |(a_n - a)(a_n + a)| = |a_n - a|\,|a_n + a| \\ &= |a_n - a|\,(a_n + a) \geq |a_n - a|\,a. \end{aligned}$$

Let $\varepsilon > 0$. Then $a\varepsilon > 0$. So $\exists$ N such that

$$n > N \implies \left|a_n^2 - a^2\right| < a\varepsilon.$$

But then

$$n > N \implies |a_n - a| \leq \frac{1}{a}\left|a_n^2 - a^2\right| < \frac{1}{a}a\varepsilon = \varepsilon.$$

Therefore $a_n \to a$ as $n \to \infty$.

Chapter 5

Standard Limits of Sequences

5.1 n^α

Lemma 5.1. *Let $\alpha \in \mathbb{R}$. Then*

$$n^\alpha \to \begin{cases} \infty & \text{if } \alpha > 0 \\ 1 & \text{if } \alpha = 0 \\ 0 & \text{if } \alpha < 0 \end{cases}$$

as $n \to \infty$.

Proof.
CASE 1 ($\alpha > 0$): Let $A > 0$. Choose N such that $\alpha \log N \geq \log A$. Then

$$n > N \implies n^\alpha > N^\alpha = e^{\log N^\alpha} = e^{\alpha \log N} \geq e^{\log A} = A.$$

So $n^\alpha \to \infty$ as $n \to \infty$.

CASE 2 ($\alpha = 0$): Clearly $n^\alpha = n^0 = 1\ \forall n \in \mathbb{N}$. So (n^α) is the sequence

$$1, 1, 1, 1, 1, 1, \ldots$$

which must converge to 1.

CASE 3 ($\alpha < 0$): Let $\beta = -\alpha > 0$. Then $n^\beta \to \infty$ as $n \to \infty$ by Case 1. Therefore

$$n^\alpha = \frac{1}{n^\beta} \to 0$$

as $n \to \infty$, by Theorem 4.3.

$\square$

5.2 a^n

Lemma 5.2. *Let $a \in \mathbb{R}$. Then*

$$a^n \to \begin{cases} \infty & \text{if } a > 1 \\ 1 & \text{if } a = 1 \\ 0 & \text{if } |a| < 1 \end{cases}$$

as $n \to \infty$, and a^n does not have a limit if $a \leq -1$.

Proof.
CASE 1 ($a > 1$): Let $h = a - 1 > 0$. Then

$$\begin{aligned} a^n &= (1+h)^n = 1 + nh + \tfrac{n(n-1)}{2!}h^2 + \tfrac{n(n-1)(n-2)}{3!}h^3 + \ldots + h^n \\ &\geq 1 + nh \to \infty \end{aligned}$$

as $n \to \infty$. It follows that $a^n \to \infty$ as $n \to \infty$, by Theorem 3.1.

CASE 2 ($a = 1$): Since $a^n = 1 \ \forall n$, we must have that $a^n \to 1$ as $n \to \infty$.

CASE 3 ($0 < a < 1$): Let $b = \frac{1}{a} > 1$. Then $b^n \to \infty$ as $n \to \infty$, by Case 1. Therefore $a^n = \left(\frac{1}{b}\right)^n = \frac{1}{b^n} \to 0$ as $n \to \infty$, by Theorem 4.3.

CASE 4 ($a = 0$): Another easy case. Since $a^n = 0 \ \forall n$, we have that $a^n \to 0$ as $n \to \infty$.

CASE 5 ($-1 < a < 0$): Since $0 < |a| < 1$, we have that

$$0 \leftarrow -|a|^n = -|a^n| \leq a^n \leq |a^n| = |a|^n \to 0$$

as $n \to \infty$. By the Sandwich Rule, $a^n \to 0$ as $n \to \infty$.

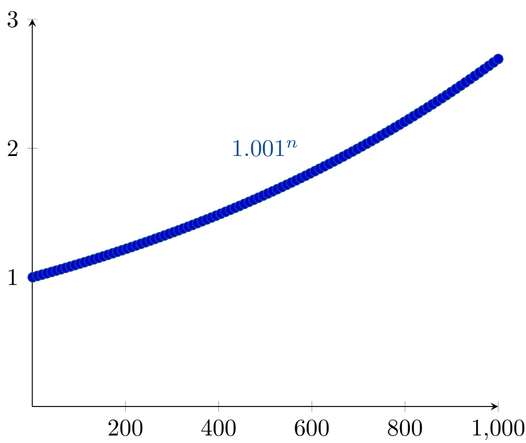
CASE 6 ($a \leq -1$): Now we have $a^n = (-1)^n |a|^n$. Since $|a|^n \geq 1$, $a^n \leq -1$ if n is odd and $a^n \geq 1$ if n is even. Therefore a^n cannot tend to any finite or infinite limit as $n \to \infty$. $\square$

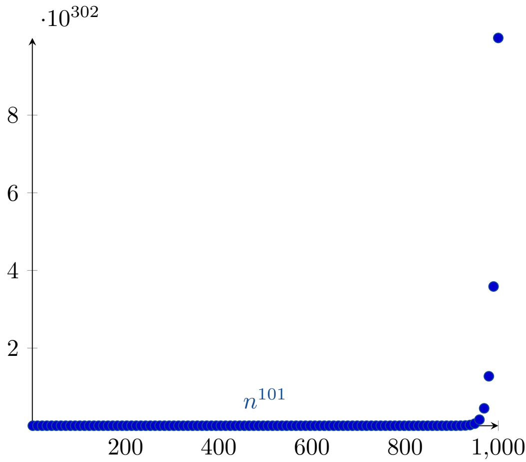
5.3 $\frac{a^n}{n^\alpha}$

Now suppose that $a > 1$ and $\alpha > 0$. We know that $a^n \to \infty$ and $n^\alpha \to \infty$ as $n \to \infty$.

QUESTION: $\dfrac{a^n}{n^\alpha} \to ?$ as $n \to \infty$

Example 5.1. Let $a = 1.001$ and $\alpha = 101$.





Note that $1.001^{1000} \approx 2.7$, but $1000^{101} = 10^{303}$. So what will happen to $\frac{a^n}{n^\alpha}$ as $n \to \infty$?

It might surprise you to learn that

$$\frac{1.001^n}{n^{101}} \to \infty$$

as $n \to \infty$.

Lemma 5.3. *Let $a > 1$ and $\alpha > 0$. Then*

$$\frac{a^n}{n^\alpha} \to \infty$$

as $n \to \infty$.

Proof. Let $p \in \mathbb{N}$ and $p \geq \alpha$. Then

$$\frac{a^n}{n^\alpha} \geq \frac{a^n}{n^p}$$

for all $n \in \mathbb{N}$.

We will prove that $\frac{a^n}{n^p} \to \infty$ as $n \to \infty$. For general p, the notation in the proof is complicated – so we will only prove it for $p = 2$.

Since $a > 1$, we have that $h := a - 1 > 0$. So

$$\begin{aligned} n \geq 4 \quad \Longrightarrow \quad \frac{a^n}{n^2} &= \frac{(1+h)^n}{n^2} \\ &= \frac{1}{n^2}\Big(1 + nh + \frac{n(n-1)}{2!}h^2 + \frac{n(n-1)(n-2)}{3!}h^3 \\ &\qquad + \ldots + h^n\Big) \\ &> \frac{n(n-1)(n-2)}{3!n^2}h^3 \\ &= \frac{(n-1)(n-2)}{6n}h^3 \\ &> \frac{(\frac{1}{2}n)(\frac{1}{2}n)}{6n}h^3 \\ &= \frac{nh^3}{24}. \end{aligned}$$

Since $\frac{nh^3}{24} \to \infty$ as $n \to \infty$, it follows that $\frac{a^n}{n^2} \to \infty$ as $n \to \infty$.

A similar argument shows that $\frac{a^n}{n^p} \to \infty$ as $n \to \infty$ for all $p \in \mathbb{N}$. Then Theorem 3.1 tells us that $\frac{a^n}{n^\alpha} \to \infty$ as $n \to \infty$. □

Corollary 5.3.1. *Let $a > 1$ and $\alpha > 0$. Then $\frac{n^\alpha}{a^n} \to 0$ as $n \to \infty$.*

Corollary 5.3.2. *Let $\alpha > 0$ and $|b| < 1$. Then $n^\alpha b^n \to 0$ as $n \to \infty$.*

Sketch Proof.
CASE 1 ($0 < b < 1$): Let $a = \frac{1}{b} > 1$. Use Corollary 5.3.1.
CASE 2 ($b = 0$): $n^\alpha b^n = 0 \ \forall \ n$.
CASE 3 ($-1 > b > 0$): Use the Sandwich Rule. □

Exercise 5.2. Write a full proof to Corollary 5.3.2.

5.4 $a^{\frac{1}{n}}$

Lemma 5.4. *Let $a > 0$. Then $a^{\frac{1}{n}} \to 1$ as $n \to \infty$.*

Proof.
CASE 1 ($a > 1$): Let $h_n = a^{\frac{1}{n}} - 1 > 0$. Then

$$a = (1 + h_n)^n = 1 + nh_n + \frac{n(n-1)}{2!}h_n^2 + \ldots + h_n^n > nh_n.$$

So

$$0 < h_n < \frac{a}{n} \to 0$$

as $n \to \infty$. It follows by the Sandwich Rule that $h_n \to 0$ as $n \to \infty$. Therefore $a^{\frac{1}{n}} = 1 + h_n \to 1$ as $n \to \infty$.

CASE 2 ($a = 1$): Clearly $a^{\frac{1}{n}} = 1\ \forall n$. Hence $a^{\frac{1}{n}} \to 1$ as $n \to \infty$.

CASE 3 ($0 < a < 1$): Let $b = \frac{1}{a} > 1$. Then

$$a^{\frac{1}{n}} = \left(\frac{1}{b}\right)^{\frac{1}{n}} \to \frac{1}{1} = 1$$

as $n \to \infty$.

Therefore $a^{\frac{1}{n}} \to 1$ as $n \to \infty\ \forall a > 0$. □

5.5 $n^{\frac{1}{n}}$

Lemma 5.5. *$n^{\frac{1}{n}} \to 1$ as $n \to \infty$.*

Proof. Let $k_n := n^{\frac{1}{n}} - 1$. If $n > 1$, then $n^{\frac{1}{n}} > 1$ and $k_n > 0$. So

$$\begin{aligned} n \geq 2 \implies \quad n = (1 + k_n)^n &= 1 + nk_n + \frac{n(n-1)}{2!}k_n^2 + \ldots + k_n^n \\ &> \frac{n(n-1)}{2!}k_n^2. \end{aligned}$$

Thus

$$0 < k_n < \sqrt{\frac{2}{n-1}}$$

for all $n \geq 2$. By the Sandwich Rule, $k_n \to 0$ as $n \to \infty$. Therefore $n^{\frac{1}{n}} \to 1$ as $n \to \infty$. □

5.6 $\frac{a^n}{n!}$

Lemma 5.6. *Let $a \in \mathbb{R}$. Then $\frac{a^n}{n!} \to 0$ as $n \to \infty$.*

Proof. Let N be the smallest number in $\mathbb{N}$ such that $N \geq 2\,|a|$. Then

$$p \in \mathbb{N},\ p \geq N \implies \frac{|a|}{p} \leq \frac{|a|}{N} \leq \frac{1}{2}.$$

So

$$\begin{aligned} n > N \implies 0 \leq \left|\frac{a^n}{n!}\right| &= \left(\tfrac{|a|}{n}\right)\left(\tfrac{|a|}{n-1}\right)\left(\tfrac{|a|}{n-2}\right)\cdots\left(\tfrac{|a|}{N+1}\right)\left(\tfrac{|a|^N}{N!}\right) \\ &\leq \left(\frac{1}{2}\right)\left(\frac{1}{2}\right)\left(\frac{1}{2}\right)\cdots\left(\frac{1}{2}\right)\left(\frac{|a|^N}{N!}\right) \\ &= \left(\frac{1}{2}\right)^{n-N}\frac{|a|^N}{N!} \to 0 \end{aligned}$$

as $n \to \infty$. Therefore $\frac{a^n}{n!} \to 0$ as $n \to \infty$, by the Sandwich Rule. □

5.7 $\frac{n!}{n^n}$

Lemma 5.7. $\frac{n!}{n^n} \to 0$ *as $n \to \infty$.*

Proof. Since

$$\begin{aligned} 0 < \frac{n!}{n^n} &= \frac{n(n-1)(n-2)\cdots(2)(1)}{nnn\cdots nn} \\ &= 1(1-\tfrac{1}{n})(1-\tfrac{2}{n})\cdots\tfrac{2}{n}\tfrac{1}{n} \leq \tfrac{1}{n} \to 0 \end{aligned}$$

as $n \to \infty$, the result follows by the Sandwich Rule. □

5.8 Summary

As $n \to \infty$,

- $\alpha > 0 \implies n^\alpha \to \infty$
- $\alpha = 0 \implies n^\alpha \to 1$
- $\alpha < 0 \implies n^\alpha \to 0$

- $a > 1 \implies a^n \to \infty$
- $a = 1 \implies a^n \to 1$
- $|a| < 1 \implies a^n \to 0$
- $a < -1 \implies a^n$ does not have a limit

- $a > 1, \alpha > 0 \implies \frac{a^n}{n^\alpha} \to \infty$
- $a > 0 \implies a^{\frac{1}{n}} \to 1$
- $n^{\frac{1}{n}} \to 1$
- $\frac{a^n}{n!} \to 0$
- $\frac{n!}{n^n} \to 0$

Example 5.3.

$$\frac{n^7 7^n + n^5 5^n}{3^n + 8^n} = \frac{n^7 \left(\frac{7}{8}\right)^n + n^5 \left(\frac{5}{8}\right)^n}{\left(\frac{3}{8}\right)^n + 1} \to \frac{0+0}{0+1} = 0$$

as $n \to \infty$.

Example 5.4.

$$\frac{n! + 8^n}{7^n + n!} = \frac{1 + \frac{8^n}{n!}}{\frac{7^n}{n!} + 1} \to \frac{1+0}{0+1} = 1$$

as $n \to \infty$.

Example 5.5. Let $r > 0$ and $a_n = \left(4^{10} + r^n\right)^{\frac{1}{n}}$. Calculate $\lim_{n\to\infty} a_n$.

solution:
CASE 1 ($0 < r \leq 1$): Since

$$1 \leftarrow (4^{10})^{\frac{1}{n}} \leq a_n \leq (4^{10} + 1)^{\frac{1}{n}} \to 1$$

as $n \to \infty$, it follows by the Sandwich Rule that if $0 < r \leq 1$, we have that $\lim_{n\to\infty} a_n = 1$.

CASE 2 ($r > 1$): In this case $r^n \to \infty$ as $n \to \infty$. So $\exists\ N \in \mathbb{N}$ such that $r^n > 4^{10}$ for all $n > N$. So

$$n > N \implies r = (r^n)^{\frac{1}{n}} < (4^{10} + r^n)^{\frac{1}{n}} < (r^n + r^n)^{\frac{1}{n}} = 2^{\frac{1}{n}} r \to r$$

as $n \to \infty$. By the Sandwich Rule, $\lim_{n\to\infty} a_n = r$ if $r > 1$.

Chapter 6

Monotonic Sequences (Monoton Diziler)

Definition. A sequence (a_n) is called an ***increasing sequence*** (artan dizi) iff

$$a_n \leq a_{n+1}$$

for all $n \in \mathbb{N}$.

Definition. A sequence (a_n) is called a ***strictly increasing sequence*** iff

$$a_n < a_{n+1}$$

for all $n \in \mathbb{N}$.

Definition. A sequence (a_n) is called a ***decreasing sequence*** (azalan dizi) iff

$$a_n \geq a_{n+1}$$

for all $n \in \mathbb{N}$.

Definition. A sequence (a_n) is called a ***strictly decreasing sequence*** iff

$$a_n > a_{n+1}$$

for all $n \in \mathbb{N}$.

Definition. A sequence (a_n) is called a ***monotonic sequence*** (monoton dizi) iff it is either an increasing sequence or a decreasing sequence.

Definition. A sequence (a_n) is said to be ***bounded above*** (üstten sınırlı) iff $\exists\ M \in \mathbb{R}$ such that

$$a_n \leq M$$

for all $n \in \mathbb{N}$. The number M is called an ***upper bound*** (üst sınırdır) for (a_n).

Definition. A sequence (a_n) is said to be ***bounded below*** (alttan sınırlı) iff $\exists\ m \in \mathbb{R}$ such that

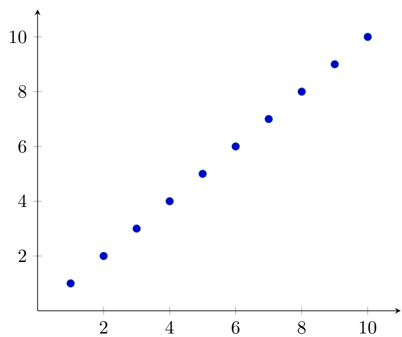
$$a_n \geq m$$

for all $n \in \mathbb{N}$. The number m is called a ***lower bound*** (alt sınırdır) for (a_n).

Example 6.1. Let $b_n = n$ for all $n \in \mathbb{N}$.

Then (b_n) is

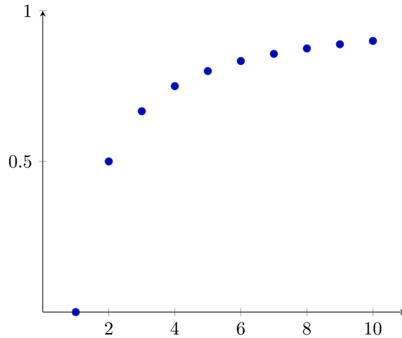
- increasing;
- strictly increasing;
- monotonic;
- bounded below ($b_n \geq 0\ \forall n$);
- not bounded above.



Example 6.2. Let $c_n = 1 - \frac{1}{n}$ for all $n \in \mathbb{N}$.

Then (c_n) is

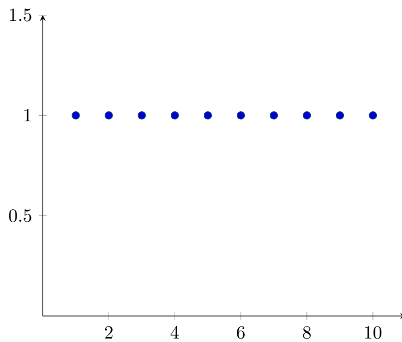
- increasing;
- strictly increasing;
- monotonic;
- bounded above ($c_n \leq 1\ \forall n$);
- bounded below ($c_n \geq 0\ \forall n$).



Example 6.3. Let $d_n = 1$ for all $n \in \mathbb{N}$.

Then (d_n) is

- increasing;
- not strictly increasing;
- decreasing;
- not strictly decreasing;
- monotonic;
- bounded below ($b_n \geq 0\ \forall n$);
- bounded above ($b_n \leq 567\ \forall n$).



Theorem 6.1. *Let (a_n) be an increasing sequence.*

(i). *If (a_n) is bounded above, then (a_n) converges.*

(ii). *If (a_n) is not bounded above, then $a_n \to \infty$ as $n \to \infty$.*

We need some more theory before we can prove this theorem.

Definition. Let $S \subseteq \mathbb{R}$ be a set. We say that S is ***bounded above*** iff $\exists\ M \in \mathbb{R}$ such that

$$x \leq M$$

for all $x \in S$. M is called an ***upper bound*** for S.

Definition. Let $S \subseteq \mathbb{R}$ be a set. We say that S is ***bounded below*** iff $\exists\ m \in \mathbb{R}$ such that

$$x \geq m$$

for all $x \in S$. m is called a ***lower bound*** for S.

Example 6.4. Let $S = \{1, 2, 3, 4\}$. Then $x \leq 7$ for all $x \in S$. So 7 is an upper bound for S.

Note that 5 is also an upper bound for S. So is 4. In fact, 4 is the least upper bound for S.

Definition. Let $S \subseteq \mathbb{R}$. The ***supremum*** of S, $\sup S$, is the least upper bound (en küçük üst sınır) for S.

If S is empty, we define $\sup S = -\infty$.
If S is not bounded above, we define $\sup S = \infty$.

Example 6.5.

$$\sup\{1, 2, 3\} = 3$$
$$\sup[0, 1] = 1$$
$$\sup(0, 1) = 1$$
$$\sup \mathbb{Z} = \infty$$
$$\sup \varnothing = -\infty$$

Definition. Let $S \subseteq \mathbb{R}$. The ***infinum*** of S, $\inf S$, is the greatest lower bound (en büyük alt sınır) for S.

If S is empty, we define $\sup S = \infty$.
If S is not bounded above, we define $\sup S = -\infty$.

Example 6.6.

$$\inf\{-1, 0, 7, 11\} = -1$$
$$\inf(0, 1] = 0$$
$$\inf \mathbb{Z} = -\infty$$
$$\inf \mathbb{N} = 1$$

Lemma 6.2. *Let $S \subseteq \mathbb{R}$ and $S \neq \varnothing$. Then*

$$\sup S = \alpha \iff \begin{array}{l} \text{(i) } x \leq \alpha \quad \forall x \in S; \textit{ and} \\ \text{(ii) } \forall \varepsilon > 0 \quad \exists x_0 \in S \textit{ such that } \alpha - \varepsilon < x_0 \leq \alpha. \end{array}$$

Proof.
"$\Longleftarrow$"
(i) $\implies$ α is an upper bound for S.
(ii) $\implies$ $\alpha - \varepsilon$ is not an upper bound for S $\forall\, \varepsilon > 0$.
Therefore α is the least upper bound.

"$\implies$"
$\sup S = \alpha \implies \alpha$ is the least upper bound $\implies$ (i) and (ii) are true. □

Completeness Axiom. *Every non-empty set of real numbers, which is bounded above, has a supremum.*

Proof of Theorem 6.1. (i) Let $S = \{a_n : n \in \mathbb{N}\}$. If (a_n) is bounded above, then S is bounded above. By the completeness axiom, S has a supremum.

Let $\alpha = \sup S < \infty$. Let $\varepsilon > 0$. Then $\alpha - \varepsilon$ is not an upper bound of S. So $\exists\, a_N \in S$ such that $\alpha - \varepsilon < a_N < \alpha$. Since (a_n) is increasing,

$$n > N \implies \alpha - \varepsilon < a_n < \alpha \implies |a_n - \alpha| < \varepsilon.$$

Therefore $a_n \to \alpha$ as $n \to \infty$.

(ii) Let $A > 0$. If (a_n) is not bounded above, then A is not an upper bound for (a_n). Hence $\exists\, a_N$ such that $a_N > A$. Since (a_n) is increasing,

$$n > N \implies a_n \geq a_N > A.$$

Therefore $a_n \to \infty$ as $n \to \infty$.

$\square$

Corollary 6.1.1. *Let (a_n) be an increasing sequence. Then*

$$(a_n) \text{ is convergent } \iff (a_n) \text{ is bounded above.}$$

Theorem 6.3. *Let (a_n) be an decreasing sequence.*

(i). *If (a_n) is bounded below, then (a_n) converges.*

(ii). *If (a_n) is not bounded below, then $a_n \to -\infty$ as $n \to \infty$.*

Exercise 6.7. Use Theorem 6.1 to prove Theorem 6.3.

Corollary 6.3.1. *Let (a_n) be an decreasing sequence. Then*

$$(a_n) \text{ is convergent } \iff (a_n) \text{ is bounded below.}$$

We can rewrite Theorems 6.1 and 6.3 as:

Theorem 6.4. ⋆⋆⋆⋆
Every bounded monotonic sequence converges.

Example 6.8. Consider $t_n = (1 + \frac{1}{n})^n$.

Note that $1 + \frac{1}{n} \to 1$ as $n \to \infty$. A common mistake is to think that this implies that t_n tends to 1 too. This is false.

For each $n \in \mathbb{N}$,

$$\begin{aligned} t_n = (1 + \tfrac{1}{n})^n &= 1 + n(\tfrac{1}{n}) + \frac{n(n-1)}{2!}(\tfrac{1}{n})^2 + \ldots + (\tfrac{1}{n})^n \\ &= 1 + 1 + \frac{n-1}{2n} + \ldots + (\tfrac{1}{n})^n \\ &\geq 2. \end{aligned}$$

So *if* $\lim_{n\to\infty} t_n$ exists, then $\lim_{n\to\infty} t_n \geq 2$. We must show that t_n is convergent.

First, (t_n) is the sequence

$$2, \frac{9}{4}, \frac{64}{27}, \ldots$$

It looks like (t_n) is an increasing sequence – we need to prove this. We need to show that $t_n \leq t_{n+1}$ for all n. Compare

$$\begin{aligned} t_n &= 1 + n(\tfrac{1}{n}) + \tfrac{n(n-1)}{2!}(\tfrac{1}{n})^2 + \ldots + (\tfrac{1}{n})^n \\ &= 1 + 1 + \tfrac{1}{2!}(1 - \tfrac{1}{n}) + \ldots \\ &\qquad + \tfrac{1}{r!}(1 - \tfrac{1}{n})(1 - \tfrac{2}{n}) \cdots (1 - \tfrac{r-1}{n}) + \ldots \\ &\qquad + \tfrac{1}{n!}(1 - \tfrac{1}{n})(1 - \tfrac{2}{n}) \cdots (1 - \tfrac{n-1}{n}) \end{aligned}$$

with

$$\begin{aligned} t_{n+1} &= 1 + 1 + \tfrac{1}{2!}(1 - \tfrac{1}{n+1}) + \ldots \\ &\qquad + \tfrac{1}{r!}(1 - \tfrac{1}{n+1})(1 - \tfrac{2}{n+1}) \cdots (1 - \tfrac{r-1}{n+1}) + \ldots \\ &\qquad + \tfrac{1}{n!}(1 - \tfrac{1}{n+1})(1 - \tfrac{2}{n+1}) \cdots (1 - \tfrac{n-1}{n+1}) \\ &\qquad + \tfrac{1}{(n+1)!}(1 - \tfrac{1}{n+1})(1 - \tfrac{2}{n+1}) \cdots (1 - \tfrac{n}{n+1}). \end{aligned}$$

The first two terms of t_n and t_{n+1} are the same. The r^{th} term of t_{n+1} is greater than the r^{th} term of t_n. Finally t_{n+1} has an extra positive term on the end. Therefore $t_n \leq t_{n+1}$ for all n. Hence (t_n) is an increasing sequence.

Moreover

$$\begin{aligned} t_n &= 1 + 1 + \tfrac{1}{2!}(1 - \tfrac{1}{n}) + \tfrac{1}{3!}(1 - \tfrac{1}{n})(1 - \tfrac{2}{n}) + \ldots \\ &\qquad + \tfrac{1}{n!}(1 - \tfrac{1}{n})(1 - \tfrac{2}{n}) \ldots (1 - \tfrac{n-1}{n}) \\ &\leq 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \ldots + \frac{1}{n!} \\ &\leq 1 + 1 + \frac{1}{2} + \frac{1}{2^2} + \ldots + \frac{1}{2^{n-1}} \end{aligned}$$

since

$$\begin{aligned} \frac{1}{r!} &= \frac{1}{r(r-1)(r-2)\cdots(3)(2)(1)} \\ &\leq \frac{1}{2 \times 2 \times 2 \times \cdots \times 2 \times 2 \times 1} = \frac{1}{2^{r-1}} \end{aligned}$$

for all $r \geq 2$.

It follows[1] that

$$t_n \leq 1 + \frac{1 - (\frac{1}{2})^n}{1 - \frac{1}{2}} < 1 + \frac{1}{1 - \frac{1}{2}} = 1 + 2 = 3.$$

[1] see Example 9.2.

So (t_n) is bounded above. By Theorem 6.1, (t_n) is convergent. So $\lim_{n\to\infty} t_n$ exists. We haven't found this limit, but we have proved that it exists and $2 \leq \lim_{n\to\infty} t_n \leq 3$. In fact $\lim_{n\to\infty} t_n = e$.

Example 6.9. Define a sequence (a_n) by

$$a_1 = \frac{5}{2} \qquad \text{and} \qquad 5a_{n+1} = a_n^2 + 6 \qquad \forall n \in \mathbb{N}.$$

(E.g. $5a_2 = a_1^2 + 6 = \frac{25}{4} + \frac{24}{4} = \frac{49}{4} \implies a_2 = \frac{49}{20}$, etc.)

(i). Show that $2 < a_n < 3$ $\forall n \in \mathbb{N}$.

(ii). Show that (a_n) is decreasing.

(iii). Show that (a_n) is convergent.

(iv). Find $\lim_{n\to\infty} a_n$, if it exists.

solution:
(i) We will use Proof by Induction (see section 2.2). The statement is true for $n = 1$ because $2 < a_1 = \frac{5}{2} < 3$.

Assume that $2 < a_k < 3$. Then

$$5a_{k+1} = a_k^2 + 6 < 3^2 + 6 = 15 \implies a_{k+1} < 3$$

and

$$5a_{k+1} = a_k^2 + 6 > 2^2 + 6 = 10 \implies a_{k+1} > 2.$$

So

$$2 < a_k < 3 \implies 2 < a_{k+1} < 3.$$

By the principle of mathematical induction, $2 < a_n < 3$ $\forall n$.

(ii) We have that

$$\begin{aligned} a_{n+1} - a_n &= \frac{1}{5}\left(a_n^2 + 6\right) - a_n = \frac{1}{5}\left(a_n^2 - 5a_n + 6\right) \\ &= \frac{1}{5}(a_n - 3)(a_n - 2) < 0 \end{aligned}$$

since $2 < a_n < 3$. So $a_{n+1} < a_n$ $\forall n$. Therefore (a_n) is decreasing.

(iii) Since (a_n) is decreasing and bounded below (by 2), it follows by Theorem 6.3 that (a_n) is convergent.

(iv) Let $a := \lim_{n\to\infty} a_n$. Then

$$5a \leftarrow 5a_{n+1} = a_n^2 + 6 \to a^2 + 6$$

as $n \to \infty$. Since limits are unique, we must have that $5a = a^2 + 6$ which rearranges to $(a-3)(a-2) = 0$. So $a = 2$ or $a = 3$. Which is correct?

Since $a_1 = \frac{5}{2}$ and (a_n) is decreasing, we must have that $a = 2$.

Exercise 6.10. Define a sequence (b_n) by

$$b_1 = 1 \qquad \text{and} \qquad 7b_{n+1} = b_n^2 + 12 \qquad \forall n \in \mathbb{N}.$$

(i). Show that $0 \leq b_n \leq 3$ $\forall n \in \mathbb{N}$.

(ii). Show that (b_n) is increasing.

(iii). Show that (b_n) is convergent.

(iv). Find $\lim_{n\to\infty} b_n$.

Exercise 6.11. Define a sequence (c_n) by

$$c_1 = 2 \qquad \text{and} \qquad 6c_{n+1} = c_n^2 + 5 \qquad \forall n \in \mathbb{N}.$$

(i). Show that $1 \leq c_n \leq 5$ $\forall n \in \mathbb{N}$.

(ii). Show that (c_n) is decreasing.

(iii). Show that (c_n) is convergent.

(iv). Find $\lim_{n\to\infty} c_n$.

Chapter 7

Subsequences (Altdiziler)

Take a sequence $(a_n)_{n=1}^{\infty}$ and delete some of the terms:

$$\begin{array}{cccccccccccccc} a_1, & a_2, & a_3, & a_4, & a_5, & a_6, & a_7, & a_8, & a_9, & a_{10}, & a_{11}, & a_{12}, & \dots \\ & & a_3, & & & a_6, & & & a_9, & & & a_{12}, & \dots \end{array}$$

If we write $n_1 = 3$, $n_2 = 6$, $n_3 = 9$, $\dots$, $n_k = 3k$, $\dots$, then the second sequence above is the sequence

$$a_{n_1}, a_{n_2}, a_{n_3}, a_{n_4}, a_{n_5}, a_{n_6}, \dots$$

which we donote by $(a_{n_k})_{k=1}^{\infty}$. We can write $(a_{n_k})_{k=1}^{\infty} \subseteq (a_n)_{n=1}^{\infty}$.

Remark. Because we want the order to stay the same, we must always have

$$n_1 < n_2 < n_3 < n_4 < n_5 < n_6 < \dots$$

Definition. Let $(n_k)_{k=1}^{\infty}$ be a strictly increasing sequence of natural numbers. Then $(a_{n_k})_{k=1}^{\infty}$ is called a ***subsequence*** of $(a_n)_{n=1}^{\infty}$.

Example 7.1. Let $b_n = \frac{1}{n}$ for all $n \in \mathbb{N}$. Then (b_n) is the sequence

$$1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \frac{1}{6}, \frac{1}{7}, \frac{1}{8}, \frac{1}{9}, \frac{1}{10}, \frac{1}{11}, \frac{1}{12}, \frac{1}{13}, \frac{1}{14}, \frac{1}{15}, \frac{1}{16}, \frac{1}{17}, \frac{1}{18}, \dots$$

and

- $1, \frac{1}{3}, \frac{1}{5}, \frac{1}{7}, \frac{1}{9}, \frac{1}{11}, \frac{1}{13}, \frac{1}{15}, \frac{1}{17}, \frac{1}{19}, \ldots$ is a subsequence of (b_n).
- $\frac{1}{2}, \frac{1}{4}, \frac{1}{6}, \frac{1}{8}, \frac{1}{10}, \frac{1}{12}, \frac{1}{14}, \frac{1}{16}, \frac{1}{18}, \frac{1}{20}, \ldots$ is a subsequence of (b_n).
- $\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \frac{1}{6}, \frac{1}{7}, \frac{1}{8}, \frac{1}{9}, \frac{1}{10}, \frac{1}{11}, \frac{1}{12}, \ldots$ is a subsequence of (b_n).
- $\frac{1}{2}, \frac{1}{3}, \frac{1}{5}, \frac{1}{7}, \frac{1}{11}, \frac{1}{13}, \frac{1}{17}, \frac{1}{19}, \frac{1}{23}, \frac{1}{29}, \ldots$ is a subsequence of (b_n).

Lemma 7.1. *Let (a_n) be a convergent sequence. Suppose that $a_n \to a$ as $n \to \infty$. Then every subsequence of (a_n) also converges to a.*

Proof. Let $(a_{n_k})_{k=1}^{\infty}$ be any subsequence of (a_n). Then $n_1 < n_2 < n_3 < n_4 < n_5 < n_6 < \ldots$ and $n_k \geq k$ for all k.

Let $\varepsilon > 0$. Since $a_n \to a$ as $n \to \infty$, $\exists N$ such that

$$n > N \implies |a_n - a| < \varepsilon.$$

But then

$$k > N \implies n_k > N \implies |a_{n_k} - a| < \varepsilon.$$

Hence $a_{n_k} \to a$ as $k \to \infty$. □

Corollary 7.1.1. *If (a_n) has two subsequences which converge to two different limits, then (a_n) is not convergent.*

Example 7.2. Let $b_n = (-1)^n$ for all $n \in \mathbb{N}$. Let $n_k = 2k$ and $m_k = 2k - 1$. (b_{n_k}) is the sequence $1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, \ldots$ Clearly $b_{n_k} \to 1$ as $k \to \infty$. (b_{m_k}) is the sequence $-1, -1, -1, -1, -1, \ldots$ Clearly $b_{m_k} \to -1$ as $k \to \infty$. Therefore (b_n) is not convergent.

Lemma 7.2. *Let (a_n) be a sequence. Suppose that $a_n \to \infty$ as $n \to \infty$. Then every subsequence of (a_n) also diverges to infinity.*

Lemma 7.3. *Let (a_n) be a sequence. Suppose that $a_n \to -\infty$ as $n \to \infty$. Then every subsequence of (a_n) also diverges to minus infinity.*

Exercise 7.3. Prove Lemmata 7.2 and 7.3.

Example 7.4. Let $\lambda \in \mathbb{R}$, $\lambda \leq -1$ and $b_n = \lambda^n$ for all $n \in \mathbb{N}$. Then $b_{2n} = \lambda^{2n} = |\lambda|^{2n}$ and $b_{2n-1} = \lambda^{2n-1} = -|\lambda|^{2n-1}$.

If $\lambda = -1$, then $b_{2n} \to 1$ and $b_{2n-1} \to -1$ as $n \to \infty$ and so (b_n) does not have a limit.

If $\lambda < -1$, then $b_{2n} \to \infty$ and $b_{2n-1} \to -\infty$ as $n \to \infty$ and so (b_n) does not have a limit.

Example 7.5. Let $t_n = (1 + \frac{1}{n})^n$. Then $\left((1 + \frac{1}{2n})^{2n}\right)_{n=1}^{\infty}$ is a subsequence of (t_n).

Recall that in Example 6.8, we said that $t_n \to e$ as $n \to \infty$. Every subsequence also converges to e. Thus

$$\lim_{n\to\infty} (1 + \tfrac{1}{2n})^{2n} = e.$$

It follows that

$$\lim_{n\to\infty} (1 + \tfrac{1}{2n})^n = \lim_{n\to\infty} \sqrt{(1 + \tfrac{1}{2n})^{2n}} = \sqrt{\lim_{n\to\infty} (1 + \tfrac{1}{2n})^{2n}} = \sqrt{e}.$$

Example 7.6. Let (z_n) be a sequence. Suppose that we know that the subsequences (z_{2n}) and (z_{2n-1}) both converge to $l \in \mathbb{R}$.

Is it true that $z_n \to l$ as $n \to \infty$? Answer: Yes

Let $\varepsilon > 0$. Since $z_{2n} \to l$ as $n \to \infty$, $\exists N_1$ such that

$$n > N_1 \implies |z_{2n} - l| < \varepsilon.$$

Since $z_{2n-1} \to l$ as $n \to \infty$, $\exists N_2$ such that

$$n > N_2 \implies |z_{2n-1} - l| < \varepsilon.$$

Now let $N = \max\{2N_1 + 1, 2N_2 + 2\}$. Then

$$n > N \implies |z_n - l| < \varepsilon.$$

Therefore $z_n \to l$ as $n \to \infty$.

Lemma 7.4. *Every sequence of real numbers has a monotonic subsequence.*

Definition. Let (a_n) be a sequence of real numbers. We call a_p a ***terrace point*** iff

$$a_p \geq a_n \qquad \forall n \geq p.$$

Proof of Lemma 7.4. Let (a_n) be a sequence of real numbers. Either

(i). (a_n) has an infinite number of terrace points; or

(ii). (a_n) does not have an infinite number of terrace points.

CASE 1: Suppose that (a_n) does has an infinite number of terrace points. Label these terrace points

$$a_{n_1}, a_{n_2}, a_{n_3}, a_{n_4}, a_{n_5}, a_{n_6}, \ldots$$

where $n_1 < n_2 < n_3 < n_4 < \ldots$

Now,

$$\begin{aligned} a_{n_k} \text{ is a terrace point} &\implies a_{n_k} \geq a_n \ \forall n \geq n_k \\ &\implies a_{n_k} \geq a_{n_{k+1}}. \end{aligned}$$

Hence $(a_{n_k})_{k=1}^{\infty}$ is a decreasing sequence.

CASE 2: Now suppose that (a_n) does not have an infinite number of terrace points. Then it must have a finite number (might be zero) of terrace points.

Since $\{p \in \mathbb{N} : a_p \text{ is a terrace point}\}$ is a finite set, $\max\big(\{0\} \cup \{p \in \mathbb{N} : a_p \text{ is a terrace point}\}\big)$ exists. Choosing $n_1 \in \mathbb{N}$ greater than this maximum, we see that all the terrace points a_p have $p < n_1$.

Then a_{n_1} is not a terrace point. So $\exists\ n_2 > n_1$ such that $a_{n_1} < a_{n_2}$. But a_{n_2} is not a terrace point. So $\exists\ n_3 > n_2 > n_1$ such that $a_{n_1} < a_{n_2} < a_{n_3}$.

Repeating, we get a strictly increasing subsequence $(a_{n_k})_{k=1}^{\infty}$. $\square$

Theorem 7.5 (The Bolzano-Weierstraß Theorem).
★★★★★
Every bounded sequence has a convergent subsequence.

Proof. Every sequence (a_n) has a monotonic subsequence $(a_{n_k})_{k=1}^{\infty}$ by Lemma 7.4. Since (a_n) is bounded, the subsequence $(a_{n_k})_{k=1}^{\infty}$ must also be bounded. Therefore $(a_{n_k})_{k=1}^{\infty}$ is convergent by Theorem 6.4. □

Example 7.7. The sequence

$$-1, 1, -1, 1, -1, 1, -1, 1, -1, 1, -1, 1, -1, 1, -1, 1, \ldots$$

is bounded but is not convergent. The subsequence

$$1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, \ldots$$

is convergent.

Chapter 8

Cauchy Sequences (Cauchy Dizileri)

Definition. A sequence (a_n) is called a ***Cauchy sequence*** iff $\forall \varepsilon > 0\ \exists N = N(\varepsilon) \in \mathbb{N}$ such that

$$n, m > N \implies |a_n - a_m| < \varepsilon.$$

Remark. In symbolic logic, (a_n) is Cauchy iff

$$(\forall \varepsilon > 0)(\exists N \in \mathbb{N})(\forall n, m \in \mathbb{N})(n, m > N \implies |a_n - a_m| < \varepsilon)$$

Remark. Since $|a_n - a_m| = |a_m - a_n|$, we can always assume that $n > m$.

Example 8.1. Let $b_n = \frac{1}{n}$ for all $n \in \mathbb{N}$. Show that (b_n) is a Cauchy sequence.

solution: Let $\varepsilon > 0$. Choose $N \geq \frac{1}{\varepsilon}$. Then

$$\begin{aligned} n > m > N \implies |b_n - b_m| &= \left| \frac{1}{n} - \frac{1}{m} \right| = \left| \frac{m-n}{nm} \right| < \frac{n}{nm} \\ &= \frac{1}{m} < \frac{1}{N} \leq \varepsilon. \end{aligned}$$

Therefore (b_n) is a Cauchy sequence.

Example 8.2. Let $c_n = 1 + 10^{-n}$ for all $n \in \mathbb{N}$. Show that (c_n) is a Cauchy sequence.

solution: Let $\varepsilon > 0$. Choose $N \geq \log_{10} \frac{1}{\varepsilon}$. Then

$$\begin{aligned} n > m > N \implies |c_n - c_m| &= \left|10^{-n} - 10^{-m}\right| \\ &= 10^{-m}(1 - 10^{m-n}) \\ &\leq 10^{-m} < 10^{-N} \leq 10^{-\log_{10}\frac{1}{\varepsilon}} \\ &= 10^{\log_{10}\varepsilon} = \varepsilon. \end{aligned}$$

Therefore (c_n) is a Cauchy sequence.

Exercise 8.3. Let $d_n = 7 - \frac{1}{n}$ for all $n \in \mathbb{N}$. Show that (d_n) is a Cauchy sequence.

Example 8.4. Let $e_n = n$ for all $n \in \mathbb{N}$. Show that (e_n) is *not* a Cauchy sequence.

solution: First we need to negate the definition of a Cauchy sequence to understand what it means for a sequence to not be a Cauchy sequence.

Since

$$\begin{array}{c} (a_n) \text{ is a} \\ \text{Cauchy sequence} \end{array} \iff \begin{array}{c} (\forall \varepsilon > 0)(\exists N \in \mathbb{N})(\forall n, m \in \mathbb{N}) \\ (n, m > N \implies |a_n - a_m| < \varepsilon) \end{array}$$

we have that

$$\begin{array}{c} (a_n) \text{ is not a} \\ \text{Cauchy sequence} \end{array} \iff \begin{array}{c} (\exists \varepsilon > 0)(\forall N \in \mathbb{N})(\exists n, m \in \mathbb{N}) \\ (n, m > N \wedge |a_n - a_m| \geq \varepsilon). \end{array}$$

Now we just need to satisfy this statement.

Choose $\varepsilon = \frac{1}{2}$. Let $N \in \mathbb{N}$. Choose $n = N + 2$ and $m = N + 1$. Then $n, m > N$ and

$$|e_n - e_m| = |n - m| = |(N + 2) - (N + 1)| = 1 \geq \frac{1}{2} = \varepsilon.$$

Therefore (e_n) is not a Cauchy sequence.

Lemma 8.1. *Every Cauchy sequence is bounded.*

Proof. Suppose that (a_n) is a Cauchy sequence. Then $\exists N$ such that

$$\begin{aligned} n > N &\implies |a_n - a_{N+1}| < 1 \\ &\implies |a_n| = |a_n - a_{N+1} + a_{N+1}| \\ &\qquad\quad \leq |a_n - a_{N+1}| + |a_{N+1}| < 1 + |a_{N+1}| . \end{aligned}$$

Now let

$$K := \max \left\{ |a_1|, |a_2|, |a_3|, \ldots, |a_N|, 1 + |a_{N+1}| \right\}.$$

Then $|a_n| \leq K$ for all $n \in \mathbb{N}$. So (a_n) is bounded. □

Theorem 8.2. ★★★★★
Let (a_n) be a sequence of real numbers. Then

$$(a_n) \text{ is convergent} \iff (a_n) \text{ is Cauchy.}$$

Remark. This is true in $\mathbb{R}$. This is not true in all spaces, as you will learn in your Topology course. For example, it is not true in $\mathbb{Q}$: Some Cauchy sequences in $\mathbb{Q}$ are not convergent.

But in this course, we are talking about sequences of real numbers.

Proof of Theorem 8.2.
" $\implies$ "
Suppose $a_n \to a$ as $n \to \infty$. Let $\varepsilon > 0$. Then $\exists N$ such that

$$n > N \implies |a_n - a| < \tfrac{\varepsilon}{2}.$$

So

$$\begin{aligned} n, m > N \implies |a_n - a_m| &= |a_n - a + a - a_m| \\ &\leq |a_n - a| + |a - a_m| < \tfrac{\varepsilon}{2} + \tfrac{\varepsilon}{2} = \varepsilon. \end{aligned}$$

So (a_n) is a Cauchy sequence.

" $\impliedby$ "
Suppose that (a_n) is a Cauchy sequence. By Lemma 8.1, (a_n) is

bounded. By Theorem 7.5, (a_n) has a convergent subsequence $(a_{n_k})_{k=1}^{\infty}$. Suppose that $a_{n_k} \to a$ as $k \to \infty$.

Let $\varepsilon > 0$. Then $\exists N_0, K_0 \in \mathbb{N}$ such that

$$n, m > N_0 \implies |a_n - a_m| < \tfrac{\varepsilon}{2}$$

and

$$k > K_0 \implies |a_{n_k} - a| < \tfrac{\varepsilon}{2}.$$

Let $N := \max\{N_0, K_0\}$. Since $n_k \geq k \ \forall k$, we have that

$$\begin{aligned} n_k \geq k > N \implies |a_n - a| &= |a_k - a_{n_k} + a_{n_k} - a| \\ &\leq |a_k - a_{n_k}| + |a_{n_k} - a| \\ &< \tfrac{\varepsilon}{2} + \tfrac{\varepsilon}{2} = \varepsilon \end{aligned}$$

Therefore $a_k \to a$ as $k \to \infty$. Hence (a_n) is a convergent sequence.

$\square$

Chapter 9

Series (Seriler)

Let $(a_n)_{n=1}^{\infty}$ be a sequence:

$$a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8, a_9, a_{10}, a_{11}, a_{12}, a_{13}, a_{14}, a_{15}, a_{16}, \dots$$

Then

$$\sum_{k=1}^{\infty} a_k = a_1 + a_2 + a_3 + a_4 + a_5 + a_6 + a_7 + a_8 + a_9 + a_{10} + a_{11} + \dots$$

is a ***series***.

Definition. Let (a_n) be a sequence of real numbers. Let

$$s_n := \sum_{k=1}^{n} a_k = a_1 + a_2 + a_3 + a_4 + a_5 + a_6 + a_7 + \dots + a_n.$$

Then s_n is called a ***partial sum*** of the series $\sum_{k=1}^{\infty} a_k$.

Definition. We say that $\sum_{k=1}^{\infty} a_k$ ***converges*** iff (s_n) converges.

Definition. If $s_n \to s$ as $n \to \infty$, we say that s is the ***sum of the series*** and we write

$$\sum_{k=1}^{\infty} a_k = s.$$

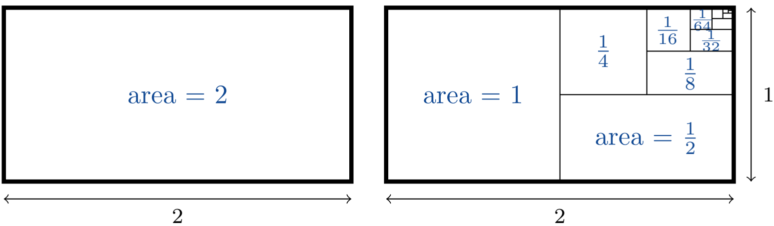
Definition. If $\sum_{k=1}^{\infty} a_k$ does not converge, then we say that $\sum_{k=1}^{\infty} a_k$ ***diverges***.

Remark. Sometimes the notation "$\sum_{k=1}^{\infty} a_k$" means $a_1+a_2+a_3+a_4+a_5+a_6+a_7+\ldots$ which might converge or might diverge. Sometimes the notation "$\sum_{k=1}^{\infty} a_k$" means the sum of the series, i.e.

$$\sum_{k=1}^{\infty} a_k = \lim_{n\to\infty} \sum_{k=1}^{n} a_k = \lim_{n\to\infty} s_n = s.$$

You need to be able to understand what I mean every time I write "$\sum_{k=1}^{\infty} a_k$".

Example 9.1.



Looking at the areas of the rectangles above, we can see that

$$2 = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \frac{1}{64} + \frac{1}{128} + \frac{1}{256} + \ldots$$

Example 9.2. Consider the series

$$1 + x + x^2 + x^3 + x^4 + x^5 + x^6 + \ldots$$

Let $s_n = 1 + x + x^2 + x^3 + \ldots + x^{n-1}$. Then we can see that $xs_n = x + x^2 + x^3 + x^4 + \ldots + x^n$. So

$$\begin{aligned}(1-x)s_n &= s_n - xs_n \\ &= \left(1 + x + x^2 + x^3 + \ldots + x^{n-1}\right) \\ &\qquad - \left(x + x^2 + x^3 + x^4 + \ldots + x^n\right) \\ &= 1 - x^n.\end{aligned}$$

If $x \neq 1$, then

$$s_n = \frac{1-x^n}{1-x}.$$

Now

- If $x = 1$, then $s_n = 1 + 1 + 1 + 1 + \ldots + 1 = n \to \infty$ as $n \to \infty$. So

$$x = 1 \quad \Longrightarrow \quad \sum_{k=0}^{\infty} x^k \text{ diverges.}$$

- If $|x| < 1$, then $s_n = \frac{1-x^n}{1-x} \to \frac{1}{1-x}$ as $n \to \infty$. So

$$|x| < 1 \quad \Longrightarrow \quad \sum_{k=0}^{\infty} x^k \text{ converges and } \sum_{k=0}^{\infty} x^k = \frac{1}{1-x}.$$

- If $x = -1$, then $s_n = \frac{1-(-1)^n}{2}$ and (s_n) does not have a limit as $n \to \infty$. So

$$x = -1 \quad \Longrightarrow \quad \sum_{k=0}^{\infty} x^k \text{ diverges.}$$

- If $|x| > 1$, then $|s_n| = \frac{|x^n-1|}{|x-1|} \geq \frac{|x|^n-1}{|x|+1} \to \infty$ as $n \to \infty$. So

$$|x| > 1 \quad \Longrightarrow \quad \sum_{k=0}^{\infty} x^k \text{ diverges.}$$

Therefore $\sum_{k=0}^{\infty} x^k \begin{cases} \text{converges if } |x| < 1 \\ \text{diverges if } |x| \geq 1. \end{cases}$ Moreover, if $|x| < 1$ then

$$\sum_{k=0}^{\infty} x^k = \frac{1}{1-x}.$$

Example 9.3.

$$\begin{aligned} 7 + \frac{7}{3} + \frac{7}{27} + \frac{7}{81} + \frac{7}{243} + \ldots &= 7\left(1 + \frac{1}{3} + \frac{1}{27} + \frac{1}{81} + \frac{1}{243} + \ldots\right) \\ &= 7\left(\frac{1}{1-\frac{1}{3}}\right) \\ &= \frac{21}{2} = 10.5. \end{aligned}$$

Example 9.4. Let $a_n = \frac{1}{n(n+1)}$ for all $n \in \mathbb{N}$. If we write a_n in partial fractions, we get $a_n = \frac{1}{n} - \frac{1}{n+1}$. So

$$\begin{aligned}
a_1 &= 1 - \tfrac{1}{2} \\
a_2 &= \tfrac{1}{2} - \tfrac{1}{3} \\
a_3 &= \tfrac{1}{3} - \tfrac{1}{4} \\
&\vdots \\
a_{n-1} &= \tfrac{1}{n-1} - \tfrac{1}{n} \\
a_n &= \tfrac{1}{n} - \tfrac{1}{n+1}.
\end{aligned}$$

Thus

$$\begin{aligned}
s_n &= a_1 + a_2 + a_3 + a_4 + \ldots + a_{n-1} + a_n \\
&= \left(1 - \tfrac{1}{2}\right) + \left(\tfrac{1}{2} - \tfrac{1}{3}\right) + \left(\tfrac{1}{3} - \tfrac{1}{4}\right) + \left(\tfrac{1}{4} - \tfrac{1}{5}\right) + \ldots \\
&\qquad + \left(\tfrac{1}{n-1} - \tfrac{1}{n}\right) + \left(\tfrac{1}{n} - \tfrac{1}{n+1}\right) \\
&= 1 - \tfrac{1}{2} + \tfrac{1}{2} - \tfrac{1}{3} + \tfrac{1}{3} - \tfrac{1}{4} + \tfrac{1}{4} - \tfrac{1}{5} + \ldots + \tfrac{1}{n-1} - \tfrac{1}{n} + \tfrac{1}{n} - \tfrac{1}{n+1} \\
&= 1 - \tfrac{1}{n+1} \\
&\to 1
\end{aligned}$$

as $n \to \infty$. Therefore $\sum_{k=1}^{\infty} a_k$ converges and $\sum_{k=1}^{\infty} \frac{1}{k(k+1)} = 1$.

Remark. Let $\sum_{k=1}^{\infty} a_k$ and $\sum_{k=1}^{\infty} b_k$ be convergent series. Let $s_n = a_1 + a_2 + a_3 + \ldots a_n$ and $t_n = b_1 + b_2 + b_3 + \ldots + b_n$ be the partial sums of $\sum_{k=1}^{\infty} a_k$ and $\sum_{k=1}^{\infty} b_k$ respectively.

Suppose that $s = \sum_{k=1}^{\infty} a_k$ and $t = \sum_{k=1}^{\infty} b_k$. Then $s_n \to s$ and $t_n \to t$ as $n \to \infty$. So

$$\begin{aligned}
\sum_{k=1}^{n}(a_k + b_k) &= (a_1 + b_1) + (a_2 + b_2) + (a_3 + b_3) + \ldots + (a_n + b_n) \\
&= s_n + t_n \to s + t
\end{aligned}$$

as $n \to \infty$. Therefore $\sum_{k=1}^{\infty}(a_k + b_k)$ converges and

$$\sum_{k=1}^{\infty}(a_k + b_k) = \sum_{k=1}^{\infty} a_k + \sum_{k=1}^{\infty} b_k.$$

So if two series are convergent, we can add them.

Chapter 10

Tests for Convergence

Theorem 10.1 (The Divergence Test / Iraksaklık Testi). *If* $a_n \not\to 0$ *as* $n \to \infty$, *then* $\sum_{n=1}^{\infty} a_n$ *diverges.*

Proof. Let $s_n = a_1 + a_2 + a_3 + \ldots + a_n$. We will use proof by contrapositive: Suppose that $\sum_{k=1}^{\infty} a_k$ converges. Then $s_n \to s$ as $n \to \infty$. But then $s_{n-1} \to s$ as $n \to \infty$ also. Hence

$$a_n = s_n - s_{n-1} \to s - s = 0$$

as $n \to \infty$. So

$$\sum_{k=1}^{\infty} a_k \text{ converges } \implies a_n \to 0 \text{ as } n \to \infty.$$

Therefore

$$a_n \not\to 0 \text{ as } n \to \infty \implies \sum_{k=1}^{\infty} a_k \text{ diverges.}$$

$\square$

Corollary 10.1.1. *If* $\sum_{k=1}^{\infty} a_k$ *converges, then* $a_n \to 0$ *as* $n \to \infty$.

Remark.

$$\sum_{k=1}^{\infty} a_k \text{ converges } \implies a_n \to 0 \text{ as } n \to \infty.$$

But

$$a_n \to 0 \text{ as } n \to \infty \not\Longrightarrow \sum_{k=1}^{\infty} a_k \text{ converges.}$$

Be careful!!!

Example 10.1. Let $a_n = \frac{1}{n}$ for all $n \in \mathbb{N}$. Then $a_n \to 0$ as $n \to \infty$.

Let $s_n = a_1 + a_2 + a_3 + \ldots + a_n = 1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n}$. Clearly (s_n) is an increasing sequence. Since

$$\begin{aligned} s_{2^n} &= 1 + \tfrac{1}{2} + \left(\tfrac{1}{3} + \tfrac{1}{4}\right) + \left(\tfrac{1}{5} + \tfrac{1}{6} + \tfrac{1}{7} + \tfrac{1}{8}\right) + \ldots \\ &\qquad + \left(\tfrac{1}{2^{n-1}+1} + \ldots + \tfrac{1}{2^n}\right) \\ &\geq 1 + \tfrac{1}{2} + \left(\tfrac{1}{4} + \tfrac{1}{4}\right) + \left(\tfrac{1}{8} + \tfrac{1}{8} + \tfrac{1}{8} + \tfrac{1}{8}\right) + \ldots + \left(\tfrac{1}{2^n} + \ldots + \tfrac{1}{2^n}\right) \\ &= 1 + \tfrac{1}{2} + \tfrac{1}{2} + \tfrac{1}{2} + \ldots + \tfrac{1}{2} = 1 + n\left(\tfrac{1}{2}\right) = \tfrac{n+2}{2} \to \infty \end{aligned}$$

as $n \to \infty$, we have that (s_n) is not bounded above. Hence $s_n \to \infty$ as $n \to \infty$ by Theorem 6.1. Therefore $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges. This is an important series, so we should write it as a lemma. . .

Lemma 10.2. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$ *diverges.*

Example 10.2. Since $b_n = \left(\frac{3n+1}{5n+1}\right)^4 \to \left(\frac{3}{5}\right)^4 \neq 0$ as $n \to \infty$, it follows that $\sum_{n=1}^{\infty} \left(\frac{3n+1}{5n+1}\right)^4$ diverges by the Divergence Test

Example 10.3. Let $a_n = \frac{1}{\sqrt{n}}$ and $b_n = \frac{1}{n}$. Then

$$a_n = \frac{1}{\sqrt{n}} \geq \frac{1}{n} = b_n$$

for all $n \in \mathbb{N}$. Let $s_n = a_1 + a_2 + a_3 + \ldots + a_n$ and $t_n = b_1 + b_2 + b_3 + \ldots + b_n$. Then

$$s_n \geq t_n$$

for all $n \in \mathbb{N}$. Since $t_n \to \infty$ as $n \to \infty$ (by Lemma 10.2), we have that $s_n \to \infty$ as $n \to \infty$ also. Therefore $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ diverges.

Theorem 10.3 (The Comparison Test/Karşılaştırma Testi). *Let $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ be two series of non-negative real numbers (i.e. $a_n \geq 0$ and $b_n \geq 0$ for all n). Suppose that*

(i). $0 \leq a_n \leq K b_n$ for all $n \in \mathbb{N}$ and for some $K > 0$; and

(ii). $\sum_{n=1}^{\infty} b_n$ converges.

Then $\sum_{n=1}^{\infty} a_n$ converges.

Proof. Let $s_n = a_1 + a_2 + a_3 + \ldots + a_n$ and $t_n = b_1 + b_2 + b_3 + \ldots + b_n$. Since $a_k \geq 0$ and $b_k \geq 0$ for all $k \in \mathbb{N}$, (s_n) and (t_n) are increasing sequences.

Since $\sum_{n=1}^{\infty} b_n$ converges, $\exists t \in \mathbb{R}$ such that $t_n \to t$ as $n \to \infty$. So

$$\begin{aligned} s_n &= a_1 + a_2 + a_3 + \ldots + a_n \\ &\leq K b_1 + K b_2 + K b_3 + \ldots + K b_n = K t_n \leq K t \end{aligned}$$

for all $n \in \mathbb{N}$. So (s_n) is an increasing sequence which is bounded above. Therefore (s_n) is convergent by Theorem 6.1. □

Corollary 10.3.1. *Let $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} c_n$ be two series of non-negative real numbers. Suppose that*

(i). $a_n \geq k c_n \geq 0$ for all $n \in \mathbb{N}$ and for some $k > 0$; and

(ii). $\sum_{n=1}^{\infty} c_n$ diverges.

Then $\sum_{n=1}^{\infty} a_n$ diverges.

Proof. Let $K = \frac{1}{k}$. Then $c_n \leq K a_n$ for all $n \in \mathbb{N}$. By the Comparison Test we have that

$$\sum_{n=1}^{\infty} a_n \text{ converges} \implies \sum_{n=1}^{\infty} c_n \text{ converges.}$$

By proof by contrapositive, we have that

$$\sum_{n=1}^{\infty} c_n \text{ diverges} \implies \sum_{n=1}^{\infty} a_n \text{ diverges.}$$

□

Corollary 10.3.2. *Let $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ be two series of non-negative real numbers. Suppose that*

(i). $0 \leq a_n \leq K b_n$ *for all* $n \geq N_0$ *for some* $N_0 \in \mathbb{N}$ *and* $K > 0$*; and*

(ii). $\sum_{n=1}^{\infty} b_n$ *converges.*

Then $\sum_{n=1}^{\infty} a_n$ *also converges.*

Theorem 10.4 (The Limit Comparison Test). (Limit Karşılaştırma Testi). *Let $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ be two series of strictly positive real numbers (i.e. $a_n > 0$ and $b_n > 0$ for all n). Suppose that*

(i). $\frac{a_n}{b_n} \to l$ *as* $n \to \infty$*;*

(ii). $l \in \mathbb{R}$*; and*

(iii). $l \neq 0$.

Then either

- $\sum_{n=1}^{\infty} a_n$ *and* $\sum_{n=1}^{\infty} b_n$ *both converge; or*
- $\sum_{n=1}^{\infty} a_n$ *and* $\sum_{n=1}^{\infty} b_n$ *both diverge.*

Proof. Since $a_n > 0$, $b_n > 0$, $l \neq 0$ and $\frac{a_n}{b_n} \to l$ as $n \to \infty$, we must have that $l > 0$.

So $\exists N \in \mathbb{N}$ such that

$$\begin{aligned} n > N \implies& \left| \frac{a_n}{b_n} - l \right| < \frac{l}{2} \\ \implies& \frac{l}{2} < \frac{a_n}{b_n} < \frac{3l}{2} \\ \implies& \frac{l}{2} b_n < a_n < \frac{3l}{2} b_n. \end{aligned}$$

Now

- $\sum_{n=1}^{\infty} b_n$ converges $\implies \sum_{n=1}^{\infty} a_n$ converges, by Corollary 10.3.2, since $0 < a_n < \left(\frac{3l}{2}\right) b_n$ for all $n > N$; and
- $\sum_{n=1}^{\infty} a_n$ converges $\implies \sum_{n=1}^{\infty} b_n$ converges, by Corollary 10.3.2, since $0 < b_n < \left(\frac{2}{l}\right) a_n$ for all $n > N$.

So the two series both converge, or both diverge. □

Lemma 10.5. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2}$ *converges.*

Proof. Let $a_n = \frac{1}{n^2}$ and $b_n = \frac{1}{n(n+1)}$. In chapter 9 we showed that $\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \frac{1}{n(n+1)}$ converges. Note that $\forall n \in \mathbb{N}$, $a_n > 0$ and $b_n > 0$. Moreover $\frac{a_n}{b_n} = \frac{n(n+1)}{n^2} = 1 + \frac{1}{n} \to 1$ as $n \to \infty$. It follows by the Limit Comparison Test that $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{1}{n^2}$ also converges. □

Lemma 10.6. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^\alpha}$ *converges for all* $\alpha \geq 2$.

Proof. Let $a_n = \frac{1}{n^\alpha}$ where $\alpha \geq 2$ and $b_n = \frac{1}{n^2}$. Then $\forall n \in \mathbb{N}$, $a_n > 0$, $b_n > 0$ and $0 < a_n = \frac{1}{n^\alpha} \leq \frac{1}{n^2} = b_n$. Since $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{1}{n^2}$ converges, it follows by the Comparison Test that $\sum_{n=1}^{\infty} a_n$ also converges. □

Lemma 10.7. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^\alpha}$ *diverges for all* $\alpha \leq 1$.

Proof. Let $a_n = \frac{1}{n}$ and $b_n = \frac{1}{n^\alpha}$ where $\alpha \leq 1$. Then $\forall n \in \mathbb{N}$, $a_n > 0$, $b_n > 0$ and $0 < a_n = \frac{1}{n} \leq \frac{1}{n^\alpha} = b_n$. By Corollary 10.3.1, it follows that $\sum_{n=1}^{\infty} b_n$ also diverges. □

Lemma 10.8. $\displaystyle\sum_{n=1}^{\infty} \sin\frac{1}{n}$ *diverges.*

Proof. Let $a_n = \sin\frac{1}{n}$ and $b_n = \frac{1}{n}$. Then $\forall n \in \mathbb{N}$, $0 < \frac{1}{n} \leq 1 < \frac{\pi}{2}$. So $\sin\frac{1}{n} > 0$ for all $n \in \mathbb{N}$. Hence $\forall n \in \mathbb{N}$, $a_n > 0$, $b_n > 0$ and

$$\frac{a_n}{b_n} = \frac{\sin\frac{1}{n}}{\frac{1}{n}} \to 1$$

as $n \to \infty$. Since $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges, it follows by the Limit Comparison Test that $\sum_{n=1}^{\infty} a_n$ also diverges. □

Theorem 10.9 (The Ratio Test / Oran Testi). *Let $\sum_{n=1}^{\infty} a_n$ be a series of strictly positive real numbers (i.e. $a_n > 0$ for all n). Suppose that $\frac{a_{n+1}}{a_n} \to l \in \mathbb{R}$ as $n \to \infty$.*

(i). If $l < 1$, then $\sum_{n=1}^{\infty} a_n$ converges.

(ii). If $l > 1$, then $\sum_{n=1}^{\infty} a_n$ diverges.

Proof.
CASE 1 ($l < 1$): Let $k \in (l, 1)$. Then $k - l > 0$. Since $\frac{a_{n+1}}{a_n} \to l$ as $n \to \infty$, $\exists N \in \mathbb{N}$ such that

$$n > N \implies \left| \frac{a_{n+1}}{a_n} - l \right| < k - l \implies \frac{a_{n+1}}{a_n} < k \implies a_{n+1} < k a_n.$$

Thus

$$\begin{aligned} n > N + 1 \implies 0 < a_n < k a_{n-1} < k^2 a_{n-2} < \ldots \\ < k^{n-N-1} a_{N+1} = k^n \left(\frac{a_{N+1}}{k^{N+1}} \right). \end{aligned}$$

Now $\frac{a_{N+1}}{k^{N+1}}$ is a constant, so $0 < a_n < k^n C$ for all $n > N + 1$. We know that $\sum_{k=1}^{\infty} k^n$ converges, since $0 < k < 1$. By the Comparison Test, $\sum_{n=1}^{\infty} a_n$ also converges.

CASE 2 ($l > 1$): Since $\frac{a_{n+1}}{a_n} \to l$ as $n \to \infty$, $\exists M$ such that

$$n > M \implies \frac{a_{n+1}}{a_n} > 1 \implies a_{n+1} > a_n.$$

So

$$n > M + 1 \implies a_n > a_{n-1} > a_{n-2} > \ldots > a_{M+1}.$$

So $a_n \not\to 0$ as $n \to \infty$. Therefore $\sum_{n=1}^{\infty} a_n$ diverges. □

Corollary 10.9.1. *Let $\sum_{n=1}^{\infty} a_n$ be a series of positive real numbers. Suppose that $\frac{a_{n+1}}{a_n} \to \infty$ as $n \to \infty$. Then $\sum_{n=1}^{\infty} a_n$ diverges.*

Remark. If $l = 1$, the Ratio Test tells us nothing.

For example, let $a_n = \frac{1}{n}$ and $b_n = \frac{1}{n^2}$. We know that $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges and $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges. But

$$\frac{a_{n+1}}{a_n} = \frac{\frac{1}{n+1}}{\frac{1}{n}} = \frac{n}{n+1} = \frac{1}{1 + \frac{1}{n}} \to 1$$

as $n \to \infty$ and

$$\frac{b_{n+1}}{b_n} = \frac{\frac{1}{(n+1)^2}}{\frac{1}{n^2}} = \frac{n^2}{(n+1)^2} = \frac{1}{(1+\frac{1}{n})^2} \to 1$$

as $n \to \infty$.

If we get $\frac{a_{n+1}}{a_n} \to 1$ as $n \to \infty$, then we cannot use the Ratio Test – we have to use a different test to see if $\sum_{n=1}^{\infty} a_n$ converges or diverges.

Remark. Moreover, note that if $a_n = \frac{1}{n}$, then $\frac{a_{n+1}}{a_n} < 1$ for all $n \in \mathbb{N}$. Please remember that when we use the Ratio Test, we look at $\lim_{n\to\infty} \frac{a_{n+1}}{a_n}$, not at $\frac{a_{n+1}}{a_n}$.

Example 10.4. Let $z_n = \frac{(2n)!}{7^n(n!)^2}$. Then $z_n > 0$ for all $n \in \mathbb{N}$ and

$$\begin{aligned}\frac{z_{n+1}}{z_n} &= \frac{(2n+2)!}{7^{n+1}((n+1)!)^2} \cdot \frac{7^n(n!)^2}{(2n)!} = \frac{(2n+2)(2n+1)}{7(n+1)^2} \\ &= \frac{(2+\frac{2}{n})(2+\frac{1}{n})}{7(1+\frac{1}{n})(1+\frac{1}{n})} \to \frac{4}{7} < 1\end{aligned}$$

as $n \to \infty$. By the Ratio Test, $\sum_{n=1}^{\infty} \frac{(2n)!}{7^n(n!)^2}$ converges.

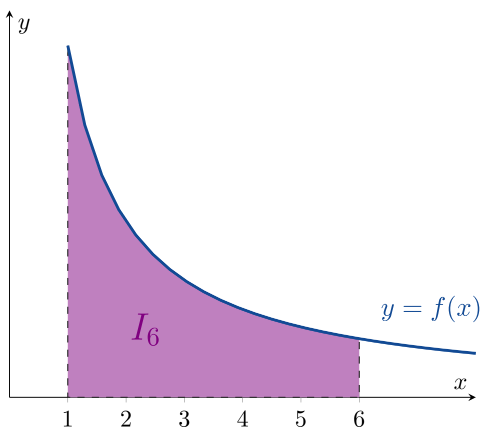
Example 10.5. Let $y_n = n^2 e^{-n(n+1)}$. Then $y_n > 0$ for all $n \in \mathbb{N}$ and

$$\frac{y_{n+1}}{y_n} = \frac{(n+1)^2 e^{-(n+1)(n+2)}}{n^2 e^{-n(n+1)}} = \left(1+\frac{1}{n}\right)^2 e^{-2(n+1)} \to 0$$

as $n \to \infty$. By the Ratio Test, $\sum_{n=1}^{\infty} n^2 e^{-n(n+1)}$ converges.

Chapter 11

The Integral Test



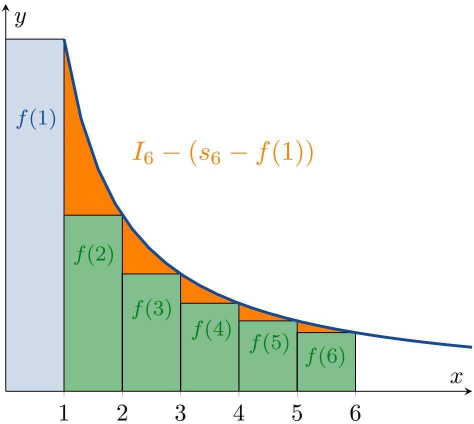
Let $f : [1, \infty) \to [0, \infty)$ be a continuous function. Suppose that f is decreasing ($x_1 < x_2 \implies f(x_1) \geq f(x_2)$) and positive ($f(x) \geq 0 \ \forall x \in [1, \infty)$). Define

$$I_n := \int_1^n f(x)\ dx$$

and

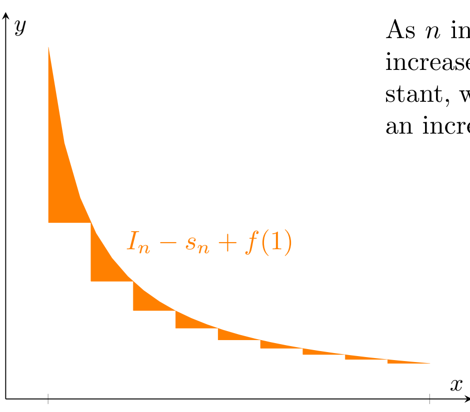
$$s_n := f(1) + f(2) + f(3) + \ldots + f(n) = \sum_{k=1}^{n} f(k)$$

for all $n \in \mathbb{N}$.

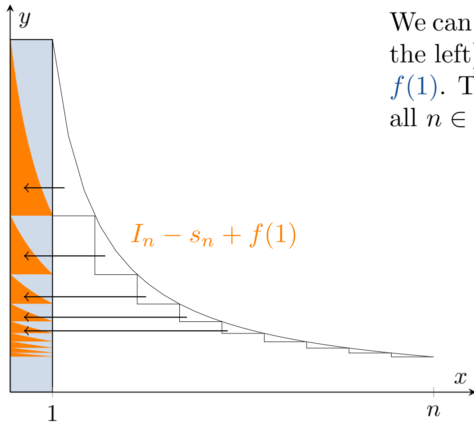


Notice that $f(2) + f(3) + f(3) + \ldots + f(n) \leq I_n$. The difference is

$$I_n - f(2) - f(3) - f(3) - \ldots - f(n) = I_n - (s_n - f(1)) = I_n - s_n + f(1).$$



As n increases, $I_n - s_n + f(1)$ increases. Since $f(1)$ is a constant, we have that $(I_n - s_n)$ is an increasing sequence.



We can see from the picture (to the left) that $I_n - s_n + f(1) \leq f(1)$. Therefore $I_n - s_n \leq 0$ for all $n \in \mathbb{N}$.

So $(I_n - s_n)$ is an increasing sequence which is bounded above. Therefore $(I_n - s_n)$ is convergent by Theorem 6.1. This gives us the following lemma:

Lemma 11.1. *Let $f : [1, \infty) \to [0, \infty)$ be a positive, decreasing, continuous function. Let $s_n = f(1) + f(2) + f(3) + \ldots + f(n)$ and $I_n := \int_1^n f(x)\, dx$ for all $n \in \mathbb{N}$. Then $(I_n - s_n)$ is convergent.*

Theorem 11.2 (The Integral Test / İntegral Testi). *Let $f : [1, \infty) \to [0, \infty)$ be a positive, decreasing, continuous function.*

(i). *If $\int_1^\infty f(x)\, dx < \infty$, then $\sum_{n=1}^\infty f(n)$ converges.*

(ii). *If $\int_1^\infty f(x)\, dx = \infty$, then $\sum_{n=1}^\infty f(n)$ diverges.*

Proof. Let s_n and I_n be as defined above. Let $c_n = s_n - I_n$. By Lemma 11.1, we know that $c_n \to c$ as $n \to \infty$.

Since $f(x) > 0$ for all $x \in [1, \infty)$, (s_n) and (I_n) are both increasing sequences. Either

(i). (I_n) is bounded above; or

(ii). (I_n) is not bounded above.

CASE 1: If (I_n) is increasing and bounded above, then (I_n) must converge, $I_n \to I$ as $n \to \infty$. But then $s_n = c_n + I_n \to c + I$ as $n \to \infty$. So $\sum_{n=1}^\infty f(n)$ converges.

CASE 2: If (I_n) is increasing and not bounded above, then $I_n \to \infty$ as $n \to \infty$ and we have that $s_n = c_n + I_n \to \infty$ as $n \to \infty$ also. So $\sum_{n=1}^{\infty} f(n)$ diverges. □

Example 11.1. Let $f(x) = \frac{1}{x^\alpha}$ for some $\alpha > 0$. Then f is continuous, decreasing and positive $\forall x \geq 1$. So

$$I_n = \int_1^n f(x)\, dx = \int_1^n \frac{1}{x^\alpha} dx = \begin{cases} \frac{1}{\alpha-1}\left[-\frac{1}{x^{\alpha-1}}\right]_1^n & \text{if } \alpha \neq 1 \\ [\log x]_1^n & \text{if } \alpha = 1 \end{cases}$$

- Suppose that $\alpha > 1$. Then $I_n = \frac{1}{\alpha-1}\left(1 - \frac{1}{n^{\alpha-1}}\right) \to \frac{1}{\alpha-1} < \infty$ as $n \to \infty$. So $\sum_{n=1}^{\infty} \frac{1}{n^\alpha}$ converges by the Integral Test.
- Suppose that $\alpha = 1$. Then $I_n = \log n - \log 1 = \log n \to \infty$ as $n \to \infty$. So $\sum_{n=1}^{\infty} \frac{1}{n^\alpha}$ diverges by the Integral Test.
- Suppose that $0 < \alpha < 1$. Then $I_n = \frac{1}{1-\alpha}\left(n^{1-\alpha} - 1\right) \to \infty$ as $n \to \infty$. So $\sum_{n=1}^{\infty} \frac{1}{n^\alpha}$ diverges by the Integral Test.

Example 11.2. Consider $\sum_{n=2}^{\infty} \frac{1}{n \log n}$. (Q: Why am I starting at $n = 2$?) Use the Integral Test to decide if this series converges or diverges.

solution: Let $f(x) = \frac{1}{x \log x}$ for $x \geq 2$. Then $f : [2, \infty) \to \mathbb{R}$ is continuous, decreasing and positive. Moreover, for $n \geq 2$,

$$\begin{aligned} I_n &= \int_2^n f(x)\, dx = \int_2^n \frac{1}{x \log x} dx = [\log(\log x)]_2^n \\ &= \log\log n - \log\log 2 \to \infty \end{aligned}$$

as $n \to \infty$. Therefore $\sum_{n=2}^{\infty} \frac{1}{n \log n}$ diverges.

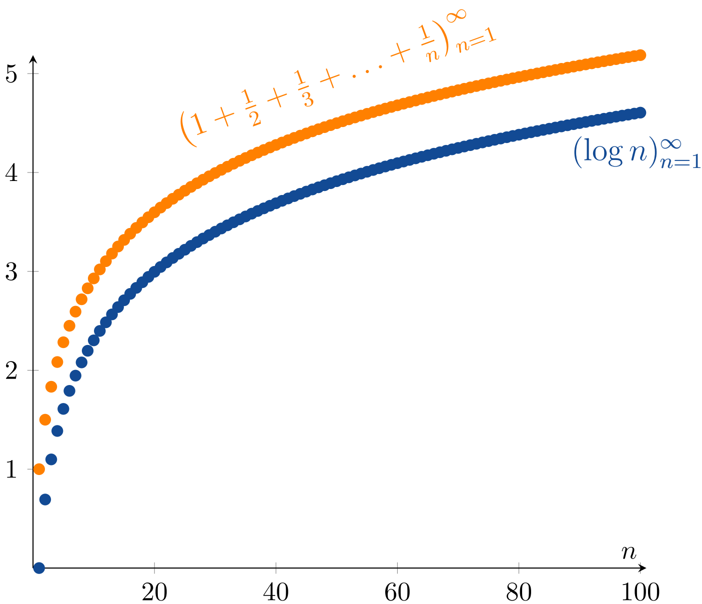
Example 11.3 (The Euler-Mascheroni constant). Let $f(x) = \frac{1}{x}$ for $x \geq 1$. Then $f : [1, \infty) \to \mathbb{R}$ is continuous, decreasing and positive. We have that

$$I_n = \int_1^n \frac{1}{x} dx = [\log x]_1^n = \log n$$

and

$$s_n = \sum_{k=1}^{n} \frac{1}{k} = 1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n}.$$

Let $\gamma_n := s_n - I_n$. Lemma 11.1 tells us that (γ_n) converges. Let $\gamma = \lim_{n\to\infty} \gamma_n = \lim_{n\to\infty}(s_n - I_n)$. γ is called the ***Euler-Mascheroni constant*** and $\gamma = 0.577215\,664901\,532860\,606512\,090082\,402431\,042159\,335939\,92\ldots$



Now $s_n = I_n + \gamma_n = \log n + \gamma_n$ and $\gamma_n \to \gamma \approx 0.6$ as $n \to \infty$. So for large n, we have that

$$1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n} \approx \log n.$$

Example 11.4. Consider

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \ldots = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n}.$$

Let $t_n = \sum_{k=1}^{n} (-1)^{k+1} \frac{1}{k}$. Then we can calculate that

$$\begin{aligned}
t_{2n} &= 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \ldots + \frac{1}{2n-1} - \frac{1}{2n} \\
&= \left(1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ldots + \frac{1}{2n-1} + \frac{1}{2n}\right) \\
&\qquad - 2\left(\frac{1}{2} + \frac{1}{4} + \frac{1}{6} + \ldots + \frac{1}{2n}\right) \\
&= \left(1 + \frac{1}{2} + \frac{1}{3} + + \ldots + \frac{1}{2n}\right) - \left(1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n}\right) \\
&= (\log 2n + \gamma_{2n}) - (\log n + \gamma_n) = \log 2n - \log n + \gamma_{2n} - \gamma_n \\
&= \log \frac{2n}{n} + \gamma_{2n} - \gamma_n = \log 2 + \gamma_{2n} - \gamma_n \\
&\to \log 2 + \gamma - \gamma = \log 2
\end{aligned}$$

as $n \to \infty$. Moreover

$$\begin{aligned}
t_{2n+1} &= 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \ldots + \frac{1}{2n-1} - \frac{1}{2n} + \frac{1}{2n+1} \\
&= t_{2n} + \frac{1}{2n+1} \to \log 2 + 0 = \log 2
\end{aligned}$$

as $n \to \infty$. Since $t_{2n} \to \log 2$ and $t_{2n+1} \to \log 2$ as $n \to \infty$, it follows that $t_n \to \log 2$ as $n \to \infty$. Therefore

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \ldots = \log 2$$

Example 11.5. Take the above series and rearrange it to

$$\underbrace{1 + \tfrac{1}{3} + \tfrac{1}{5} + \tfrac{1}{7}}_{\text{4 positive terms}} \quad \underbrace{-\tfrac{1}{2}}_{\substack{\text{1}\\\text{negative}\\\text{term}}} \quad \underbrace{+\tfrac{1}{9} + \tfrac{1}{11} + \tfrac{1}{13} + \tfrac{1}{15}}_{\text{4 positive terms}} \quad \underbrace{-\tfrac{1}{4}}_{\substack{\text{1}\\\text{negative}\\\text{term}}} \quad + \tfrac{1}{17} + \tfrac{1}{19} + \ldots$$

It is possible to show that this series converges and has sum $\log 4$. Read page 99 of Mary Hart's book for more details.

These last two examples show that we cannot reorder a conditionally convergent series without affecting its sum.

Chapter 12

Alternating Series

In this chapter, we will consider sequences of the form

$$\sum_{n=1}^{\infty}(-1)^{n+1}a_n = a_1 - a_2 + a_3 - a_4 + a_5 - a_6 + a_7 - a_8 + a_9 - a_{10} + \dots$$

where $a_n > 0$ $\forall n$.

Theorem 12.1 (The Alternating Series Test). (Alterne Seri Testi). *Let (a_n) be a sequence such that*

(i) $a_n > 0$ for all n;

(ii) (a_n) is decreasing (i.e. $a_n \geq a_{n+1}$ for all n); and

(iii) $a_n \to 0$ as $n \to \infty$.

Then $\sum_{n=1}^{\infty}(-1)^{n+1}a_n$ converges.

Proof. Let $s_n = a_1 - a_2 + a_3 - a_4 + a_5 - a_6 + \dots + (-1)^{n+1}a_n$. Then $s_{2n} = a_1 - a_2 + a_3 - a_4 + a_5 - a_6 + \dots - a_{2n}$. So

$$s_{2n+2} - s_{2n} = a_{2n+1} - a_{2n+2} \geq 0.$$

Therefore the sequence (s_{2n}) is increasing.

Moreover, since (a_n) is positive and decreasing, we have that

$$\begin{aligned} s_{2n} &= a_1 - a_2 + a_3 - a_4 + a_5 - a_6 + \ldots - a_{2n-2} + a_{2n-1} - a_{2n} \\ &= a_1 - (a_2 - a_3) - (a_4 - a_5) - \ldots - (a_{2n-2} - a_{2n-1}) - a_{2n} \\ &\leq a_1 - 0 - 0 - 0 - \ldots - 0 - 0 = a_1. \end{aligned}$$

So (s_{2n}) is bounded above. Therefore (s_{2n}) is convergent. Let $s = \lim_{n\to\infty} s_{2n}$. Then $s_{2n} \to s$ as $n \to \infty$. Furthermore

$$\begin{aligned} s_{2n+1} &= a_1 - a_2 + a_3 - a_4 + a_5 - a_6 + \ldots - a_{2n} + a_{2n+1} \\ &= s_{2n} + a_{2n+1} \to s + 0 = s \end{aligned}$$

as $n \to \infty$. It follows by Example 7.6 that $s_n \to s$ as $n \to \infty$ also. Therefore $\sum_{n=1}^{\infty}(-1)^{n+1}a_n$ is convergent. □

Remark. If $a_n \not\to 0$ as $n \to \infty$, then $(-1)^{n+1}a_n \not\to 0$ as $n \to \infty$ and $\sum_{n=1}^{\infty}(-1)^{n+1}a_n$ diverges by the Divergence Test.

Example 12.1. Let $a_n = \sin\frac{1}{n}$ for all $n \in \mathbb{N}$. Note that $0 < \frac{1}{n+1} < \frac{1}{n} \leq 1 < \frac{\pi}{2}$ for all $n \in \mathbb{N}$. Thus $a_n = \sin\frac{1}{n} > 0$ and

$$a_{n+1} = \sin\frac{1}{n+1} < \sin\frac{1}{n} = a_n.$$

So (a_n) is a decreasing sequence of positive numbers. Moreover, $a_n = \sin\frac{1}{n} \to 0$ as $n \to \infty$. Therefore

$$\sum_{n=1}^{\infty}(-1)^{n+1}\sin\frac{1}{n}$$

converges by the Alternating Series Test.

Example 12.2. Since $a_n = \cos\frac{1}{n} \to 1 \neq 0$ as $n \to \infty$, it follows that

$$\sum_{n=1}^{\infty}(-1)^{n+1}\cos\frac{1}{n}$$

diverges by the Divergence Test.

Example 12.3. Now consider

$$\sum_{n=1}^{\infty}(-1)^{n+1}\sin^2\frac{1}{n}.$$

Let $a_n = \sin^2\frac{1}{n}$. Then $a_n > a_{n+1} > 0\ \forall n$ and $a_n \to 0$ as $n \to \infty$. Therefore $\sum_{n=1}^{\infty}(-1)^{n+1}\sin^2\frac{1}{n}$ converges by the Alternating Series Test.

Chapter 13

Absolute Convergence and Conditional Convergence

Definition. Let $\sum_{n=1}^{\infty} a_n$ be a series of real numbers. If $\sum_{n=1}^{\infty} |a_n|$ is convergent, then we say that $\sum_{n=1}^{\infty} a_n$ is ***absolutely convergent***. (We can also say that $\sum_{n=1}^{\infty} a_n$ ***converges absolutely*** in this case.)

Theorem 13.1. *Every absolutely convergent series is convergent.*

Remark. The theorem says that

$$\sum_{n=1}^{\infty} |a_n| \text{ converges} \implies \sum_{n=1}^{\infty} a_n \text{ converges.}$$

Proof. Let $\sum_{n=1}^{\infty} a_n$ be absolutely convergent. Then $\sum_{n=1}^{\infty} |a_n|$ converges.

Some of the a_n might be ≥ 0 and some might be < 0. We want to separate these two types of a_n. Define

$$b_n := \begin{cases} a_n & \text{if } a_n \geq 0 \\ 0 & \text{if } a_n < 0 \end{cases} \qquad \text{and} \qquad c_n := \begin{cases} 0 & \text{if } a_n \geq 0 \\ -a_n & \text{if } a_n < 0. \end{cases}$$

Note that $b_n \geq 0\ \forall n$, $c_n \geq 0\ \forall n$ and $a_n = b_n - c_n$.

Let

$$\begin{aligned} s_n &= |a_1| + |a_2| + |a_3| + \ldots + |a_n|, \\ t_n &= a_1 + a_2 + a_3 + \ldots + a_n, \\ r_n &= b_1 + b_2 + b_3 + \ldots + b_n \\ \text{and } u_n &= c_1 + c_2 + c_3 + \ldots + c_n. \end{aligned}$$

Now

$$\sum_{n=1}^{\infty} |a_n| \text{ converges} \implies s_n \to s \text{ as } n \to \infty.$$

Since $|a_k| \geq 0$ for all $k \in \mathbb{N}$, (s_n) is increasing. So $s_n \leq s$ for all $n \in \mathbb{N}$. Hence

$$r_n = b_1 + b_2 + b_3 + \ldots + b_n \leq |a_1| + |a_2| + |a_3| + \ldots + |a_n| = s_n \leq s$$

for all $n \in \mathbb{N}$. Since $b_k \geq 0$ for all $k \in \mathbb{N}$, (b_k) is an increasing sequence which is bounded above. So $r_n \to r$ as $n \to \infty$. Similarly $u_n \to u$ as $n \to \infty$. Therefore $t_n = r_n - u_n \to r - u$ as $n \to \infty$. So $\sum_{n=1}^{\infty} a_n$ converges. □

Remark.

$$\sum_{n=1}^{\infty} a_n \text{ is absolutely convergent} \implies \sum_{n=1}^{\infty} a_n \text{ is convergent.}$$

But

$$\sum_{n=1}^{\infty} a_n \text{ is convergent} \;\not\Longrightarrow\; \sum_{n=1}^{\infty} a_n \text{ is absolutely convergent.}$$

For example, consider $a_n = \frac{(-1)^{n+1}}{n}$. The series $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} - \frac{1}{8} + \ldots$ is convergent, but the series $\sum_{n=1}^{\infty} \left| \frac{(-1)^{n+1}}{n} \right| = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \ldots$ is divergent.

Corollary 13.1.1 (The Triangle Inequality). *If $\sum_{n=1}^{\infty} a_n$ is absolutely convergent, then*

$$\left|\sum_{n=1}^{\infty} a_n\right| \leq \sum_{n=1}^{\infty} |a_n| .$$

Exercise 13.1. Prove the Triangle Inequality.

Definition. If a series $\sum_{n=1}^{\infty} a_n$ is convergent, but is not absolutely convergent, then we say that it is ***conditionally convergent***. (Equivalently, we can say that the series ***converges conditionally***.)

Example 13.2. $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$ is conditionally convergent.

Example 13.3. $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2}$ is absolutely convergent.

Example 13.4. In the previous chapter, we used the Alternating Series Test to prove that $\sum_{n=1}^{\infty}(-1)^{n+1} \sin^2 \frac{1}{n}$ is convergent. Is it absolutely convergent?

solution: First note that $\left|(-1)^{n+1} \sin^2 \frac{1}{n}\right| = \sin^2 \frac{1}{n}$. Let $a_n = \sin^2 \frac{1}{n}$ and $b_n = \frac{1}{n^2}$ for all $n \in \mathbb{N}$. Then $a_n > 0$ and $b_n > 0$ for all n, and

$$\frac{a_n}{b_n} = \left(\frac{\sin \frac{1}{n}}{\frac{1}{n}}\right)\left(\frac{\sin \frac{1}{n}}{\frac{1}{n}}\right) \to 1 \cdot 1 = 1$$

as $n \to \infty$. We know that $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges. By the Limit Comparison Test, $\sum_{n=1}^{\infty} \sin^2 \frac{1}{n}$ also converges. Hence

$$\sum_{n=1}^{\infty} \left|(-1)^{n+1} \sin^2 \frac{1}{n}\right|$$

converges and therefore $\sum_{n=1}^{\infty}(-1)^{n+1} \sin^2 \frac{1}{n}$ converges absolutely.

Theorem 13.2 (The Ratio Test v2). *Let $\sum_{n=1}^{\infty} a_n$ be a series of non-zero real numbers (i.e. $a_n \neq 0\ \forall n$). Suppose that*

$$\left|\frac{a_{n+1}}{a_n}\right| \to l \in \mathbb{R}$$

as $n \to \infty$.

(i). If $l < 1$, then $\sum_{n=1}^{\infty} a_n$ is absolutely convergent.

(ii). If $l > 1$, then $\sum_{n=1}^{\infty} a_n$ is divergent.

Proof. Let $b_n = |a_n|$. Then $b_n > 0\ \forall n$ and

$$\frac{b_{n+1}}{b_n} = \frac{|a_{n+1}|}{|a_n|} = \left|\frac{a_{n+1}}{a_n}\right| \to l$$

as $n \to \infty$. Then we can use the Ratio Test to see that

$$l < 1 \implies \sum_{n=1}^{\infty} b_n \text{ converges} \implies \sum_{n=1}^{\infty} a_n \text{ converges absolutely.}$$

If $l > 1$, then $\exists N$ such that $\frac{b_{n+1}}{b_n} > 1\ \forall n > N$. Hence $b_n > b_{N+1}\ \forall n > N+1$. Therefore $b_n \not\to 0$ as $n \to \infty$. So $a_n \not\to 0$ as $n \to \infty$ and thus $\sum_{n=1}^{\infty} a_n$ diverges by the Divergence Test. $\square$

Corollary 13.2.1. *If $a_n \neq 0\ \forall n$ and $\left|\frac{a_{n+1}}{a_n}\right| \to \infty$ as $n \to \infty$, then $\sum_{n=1}^{\infty} a_n$ diverges.*

Remark. If $l = 1$, then the Ratio Test v2 tells us nothing.

Chapter 14

Power Series (Kuvvet Serileri)

Let $(a_n)_{n=0}^{\infty}$ be a sequence. Then

$$\sum_{n=0}^{\infty} a_n (x-c)^n$$

is a ***power series*** (kuvvet serisi).

Example 14.1. The following are power series:

- $1+2x+3x^2+4x^3+5x^4+\ldots = \sum_{n=1}^{\infty} nx^{n-1} = \sum_{n=0}^{\infty}(n+1)x^n;$
- $1+x+x^2+x^3+x^4+\ldots = \sum_{n=0}^{\infty} x^n;$
- $1+x+2x^2+6x^3+24x^4+\ldots = \sum_{n=0}^{\infty} n!x^n;$
- $1+x^2+x^4+x^6+x^8+\ldots = \sum_{n=0}^{\infty} \left(\frac{1+(-1)^2}{2}\right) x^n;$

- $1+(x-2)+(x-2)^2+(x-2)^3+(x-2)^4+\ldots=\sum_{n=0}^{\infty}(x-2)^n$;

Definition. The constant c is called the ***centre of expansion*** of the power series $\sum_{n=0}^{\infty} a_n(x-c)^n$.

Remark. To make things easier, we start by looking at power series with $c=0$. We will be considering

$$\sum_{n=0}^{\infty} a_n x^n$$

in this chapter. We will discuss power series with $c \neq 0$ in Chapter 18.

Remark. We wish to answer the following three questions about power series:

- How does a power series behave?
- Does this depend on x?
- Is it possible for a power series to converge for some x, but diverge for other x?

Example 14.2. Recall that

$$\sum_{n=0}^{\infty} x^n \begin{cases} \text{converges if } |x|<1 \\ \text{diverges if } |x| \geq 1. \end{cases}$$

Example 14.3. Consider $\displaystyle\sum_{n=0}^{\infty} \frac{x^n}{n!}$.

- If $x=0$, then $\sum_{n=0}^{\infty} \frac{x^n}{n!}=1+0+0+0+0+0+\ldots$ is convergent.
- Suppose that $x \neq 0$. Let $b_n := \frac{x^n}{n!}$. Then

  $$\left|\frac{b_{n+1}}{b_n}\right| = \left|\frac{x^{n+1}}{(n+1)!}\frac{n!}{x^n}\right| = \frac{|x|}{n+1} \to 0$$

  as $n \to \infty$. Hence $\sum_{n=0}^{\infty} b_n$ is absolutely convergent by the Ratio Test v2.

Therefore $\displaystyle\sum_{n=0}^{\infty} \frac{x^n}{n!}$ converges $\forall x \in \mathbb{R}$.

Example 14.4. Consider $\sum_{n=0}^{\infty} n!x^n$.

- If $x = 0$, then $\sum_{n=0}^{\infty} n!x^n = 1 + 0 + 0 + 0 + 0 + 0 + \ldots$ is convergent.
- Suppose that $x \neq 0$. Let $b_n := n!x^n$ and $t = \frac{1}{x}$. Recall that $\frac{t^n}{n!} \to 0$ as $n \to \infty$ $\forall t \in \mathbb{R}$. So $|b_n| = |n!x^n| = \left|\frac{n!}{t^n}\right| \to \infty$ as $n \to \infty$. So $|b_n| \not\to 0$ as $n \to \infty$. By the Divergence Test, $\sum_{n=0}^{\infty} b_n$ diverges.

Therefore $\displaystyle\sum_{n=0}^{\infty} \frac{x^n}{n!} \begin{cases} \text{converges if } x = 0 \\ \text{diverges if } x \neq 0. \end{cases}$

Example 14.5. Consider $\sum_{n=1}^{\infty} nx^{n-1} = 1 + 2x + 3x^2 + 4x^3 + 5x^4 + \ldots$

- If $x = 0$, then $\sum_{n=1}^{\infty} nx^{n-1} = 1 + 0 + 0 + 0 + 0 + 0 + \ldots$ converges.
- Suppose that $x \neq 0$. Let $b_n := nx^{n-1}$. Then

$$\left|\frac{b_{n+1}}{b_n}\right| = \frac{(n+1)|x|^n}{n|x|^{n-1}} = (1 + \tfrac{1}{n})|x| \to |x|$$

  as $n \to \infty$.

  By the Ratio Test v2, $\sum_{n=1}^{\infty} nx^{n-1} \begin{cases} \text{converges if } 0 < |x| < 1 \\ \text{diverges if } |x| > 1. \end{cases}$

- Suppose that $|x| = 1$. Then $\left|nx^{n-1}\right| = n$ which means that $nx^{n-1} \not\to 0$ as $n \to \infty$. So $\sum_{n=1}^{\infty} b_n$ diverges if $|x| = 1$.

Therefore $\displaystyle\sum_{n=1}^{\infty} nx^{n-1} \begin{cases} \text{converges if } |x| < 1 \\ \text{diverges if } |x| \geq 1. \end{cases}$

Remark. $\sum_{n=0}^{\infty} x^n$ converges for $|x| < 1$ and diverges for $|x| \geq 1$. If we differentiate each term (are we allowed to do this?), we get

$$0 + 1 + 2x + 3x^2 + 4x^3 + 5x^4 + \ldots = \sum_{n=1}^{\infty} nx^{n-1}$$

which also converges for $|x| < 1$ and diverges for $|x| \geq 1$. Interesting!

Theorem 14.1. *Let $x_0 \neq 0$. Suppose that $\sum_{n=0}^{\infty} a_n x_0^n$ converges. Then $\sum_{n=0}^{\infty} a_n x^n$ converges absolutely for all $|x| < |x_0|$.*

Example 14.6. Let $a_n = \left(\frac{1}{4}\right)^n$ and $x_0 = -2$. Then $\sum_{n=0}^{\infty} a_n x_0^n = \sum_{n=0}^{\infty} \left(\frac{-1}{2}\right)^n$ which converges. By Theorem 14.1, we know that $\sum_{n=0}^{\infty} a_n x^n$ converges absolutely for all $x \in (-2, 2)$.

Proof of Theorem 14.1. Suppose that $\sum_{n=0}^{\infty} a_n x_0^n$ converges. The Divergence Test tells us that $a_n x_0^n \to 0$ as $n \to \infty$, and Theorem 3.4 says that "every convergent sequence is bounded". So $\exists K > 0$ such that $|a_n x_0^n| < K$ for all n.

If $|x| < |x_0|$, then

$$|a_n x^n| = |a_n x_0^n| \frac{|x|^n}{|x_0|^n} \leq K \left(\frac{|x|}{|x_0|}\right)^n$$

and $\sum_{n=0}^{n} K \left(\frac{|x|}{|x_0|}\right)^n$ converges, because $\frac{|x|}{|x_0|} < 1$.

By the Comparison Test, $\sum_{n=0}^{n} a_n x^n$ converges. Therefore $\sum_{n=0}^{\infty} a_n x^n$ is absolutely convergent. □

Theorem 14.2. *A power series $\sum_{n=0}^{\infty} a_n x^n$ satisfies one and only one of the following:*

(*i*). *It converges absolutely $\forall x$;*

(*ii*). *It converges for $x = 0$ and diverges $\forall x \neq 0$; or*

(*iii*). *$\exists R > 0$ such that $\sum_{n=0}^{\infty} a_n x^n \begin{cases} \text{converges if } |x| < R \\ \text{diverges if } |x| > R. \end{cases}$*

Proof. Let

$$S := \{x \in \mathbb{R} \ : \ x \geq 0 \text{ and } \sum_{n=0}^{\infty} a_n x^n \text{ converges}\}.$$

Then $0 \in S$ since $\sum_{n=0}^{\infty} a_n 0^n = a_0 + 0 + 0 + 0 + 0 + 0 + \ldots$ always converges. So $S \neq \varnothing$. Either

- S is not bounded above; or
- S is bounded above.

CASE 1: Suppose that S is not bounded above. Let $x \in \mathbb{R}$. Then $|x|$ is not an upper bound of S. So $\exists\, x_0 \in S$ such that $|x| \le x_0$. But

$$\begin{aligned} x_0 \in S &\implies \sum_{n=0}^{\infty} a_n x_0^n \text{ converges} \\ &\implies \sum_{n=0}^{\infty} a_n x^n \text{ converges absolutely} \end{aligned}$$

by Theorem 14.1. Therefore $\sum_{n=0}^{\infty} a_n x^n$ is absolutely convergent $\forall x$. This is (i).

CASE 2: Now suppose that S is bounded above. Since $S \neq \varnothing$, we have that $\sup S$ exists in $\mathbb{R}$. Let $R := \sup S$. We know that $R \ge 0$ since $0 \in S$.

Claim 1. *$\sum_{n=0}^{\infty} a_n x^n$ diverges $\forall\, |x| > R$.*

> *Proof of Claim 1.* We will use proof by contradiction: Suppose that $\exists\, x_0 \in \mathbb{R}$ such that $|x_0| > R$ and $\sum_{n=0}^{\infty} a_n x_0^n$ converges.
>
> Let $x_1 = \frac{1}{2}(|x_0| + R)$. Then $0 \le R < x_1 < |x_0|$. So $\sum_{n=0}^{\infty} a_n x_1^n$ converges by Theorem 14.1. So $x_1 \in S$.
>
> This is impossible because $x_1 > R$ and R is an upper bound of S. So we have our contradiction.
>
> So $\nexists\, x_0 \in \mathbb{R}$ such that $|x_0| > R$ and $\sum_{n=0}^{\infty} a_n x_0^n$ converges. Hence $\sum_{n=0}^{\infty} a_n x^n$ diverges $\forall\, |x| > R$. $\square$

Claim 2. *If $R > 0$, then $\sum_{n=0}^{\infty} a_n x^n$ converges absolutely $\forall\, |x| < R$.*

> *Proof of Claim 2.* Choose $x \in \mathbb{R}$ such that $|x| < R$. Then $|x|$ is not an upper bound of S. So $\exists\, x_0 \in S$

such that $|x| < x_0 < R$. Then

$$\begin{aligned} x_0 \in S &\implies \sum_{n=0}^{\infty} a_n x_0^n \text{ converges} \\ &\implies \sum_{n=0}^{\infty} a_n x^n \text{ converges absolutely} \end{aligned}$$

by Theorem 14.1. Therefore $\sum_{n=0}^{\infty} a_n x^n$ is absolutely convergent $\forall\, |x| < R$. $\square$

Finally, if $R = 0$ then $S = \{0\}$, which is (ii). $\square$

Chapter 15

Radius of Convergence

Definition. Let $R \in [0, \infty) \cup \{\infty\}$. If $\sum_{n=0}^{\infty} a_n x^n$ converges $\forall\, |x| < R$ and diverges $\forall\, |x| > R$, then R is called the ***radius of convergence*** (yakınsaklık yarıçapı) of the power series $\sum_{n=0}^{\infty} a_n x^n$.

Definition. If $R = \infty$, then we say that $\sum_{n=0}^{\infty} a_n x^n$ has ***infinite radius of convergence***. (This means that $\sum_{n=0}^{\infty} a_n x^n$ converges $\forall x$.)

Definition. If $R = 0$, then we say that $\sum_{n=0}^{\infty} a_n x^n$ has ***zero radius of convergence***. (This means that $\sum_{n=0}^{\infty} a_n x^n$ converges if $x = 0$ and diverges $\forall x \neq 0$.)

Definition. If $R > 0$ or $R = \infty$, then the open interval $(-R, R)$ is called the ***open interval of convergence*** of $\sum_{n=0}^{\infty} a_n x^n$.

Theorem 15.1. *Suppose that*

$$\left| \frac{a_n}{a_{n+1}} \right| \to R \in \mathbb{R}$$

as $n \to \infty$. Then $\sum_{n=0}^{\infty} a_n x^n$ has radius of convergence R.

Remark. By Theorem 14.2, we know that a power series *always* has a radius of convergence, even if $\lim_{n\to\infty} \left| \frac{a_n}{a_{n+1}} \right|$ doesn't exist. Theorem 15.1 just gives us an easy way to find R, if this limit does exist. If the limit does not exist, then we need to use a different method to find R.

Remark. Never, never, never forget to use $|\cdot|$ when you use Theorem 15.1.

Remark. The Ratio Test v2 uses $\left|\frac{a_{n+1}}{a_n}\right|$, but Theorem 15.1 uses $\left|\frac{a_n}{a_{n+1}}\right|$. Don't get these mixed up.

Proof of Theorem 15.1. Suppose that $\left|\frac{a_n}{a_{n+1}}\right| \to R$ as $n \to \infty$. Then $R \geq 0$.

Claim 1. *If $R > 0$, then $\sum_{n=0}^{\infty} a_n x^n$ converges absolutely $\forall\, |x| < R$.*

> *Proof of Claim 1.* Clearly if $x = 0$, then $\sum_{n=0}^{\infty} a_n x^n = a_0 + 0 + 0 + 0 + 0 + 0 + 0 + \ldots$ converges.
>
> Let $0 < |x| < R$. Then
>
> $$\left|\frac{a_{n+1}x^{n+1}}{a_n x^n}\right| = \left|\frac{a_{n+1}}{a_n}\right| |x| \to \frac{|x|}{R} < 1$$
>
> as $n \to \infty$. It follows by the Ratio Test v2 that $\sum_{n=0}^{\infty} a_n x^n$ converges absolutely. □

Claim 2. *If $R \geq 0$, then $\sum_{n=0}^{\infty} a_n x^n$ diverges absolutely $\forall\, |x| > R$.*

> *Proof of Claim 2.* Suppose that $|x| > R$. Then
>
> $$\left|\frac{a_{n+1}x^{n+1}}{a_n x^n}\right| = \left|\frac{a_{n+1}}{a_n}\right| |x| \to \begin{cases} \frac{|x|}{R} > 1 & \text{if } R \neq 0 \\ \infty & \text{if } R = 0. \end{cases}$$
>
> It follows by the Ratio Test v2 that $\sum_{n=0}^{\infty} a_n x^n$ diverges. □

Therefore R is the radius of convergence. □

Corollary 15.1.1. *If*

$$\left|\frac{a_n}{a_{n+1}}\right| \to \infty$$

as $n \to \infty$, then the power series $\sum_{n=0}^{\infty} a_n x^n$ has an infinite radius of convergence.

Proof. Clearly $\sum_{n=0}^{\infty} a_n x^n$ converges if $x = 0$. If $x \neq 0$, then

$$\left|\frac{a_{n+1}x^{n+1}}{a_n x^n}\right| = \left|\frac{a_{n+1}}{a_n}\right| |x| \to 0$$

as $n \to \infty$. Therefore $\sum_{n=0}^{\infty} a_n x^n$ is absolutely convergent $\forall x \in \mathbb{R}$. □

We have seen that $\exists\ R$ such that

$$\sum_{n=0}^{\infty} a_n x^n \begin{cases} \text{converges if } |x| < R \\ \text{diverges if } |x| > R. \end{cases}$$

Suppose that $0 < R < \infty$.

Question. What happens when $|x| = R$?

Example 15.1. Consider $\sum\limits_{n=0}^{\infty} x^n$. This is a power series with $a_n = 1\ \forall n$. Since

$$\left|\frac{a_n}{a_{n+1}}\right| = \frac{1}{1} = 1,$$

the radius of convergence is $R = 1$. This means that

$$\sum_{n=0}^{\infty} a_n x^n \begin{cases} \text{converges if } |x| < 1 \\ \text{diverges if } |x| > 1. \end{cases}$$

In Chapter 9, we saw that $\sum_{n=0}^{\infty} x^n$ also diverges for $|x| = 1$.

For this power series, we have divergence when $x = \pm R$.

Example 15.2. Now consider $\sum\limits_{n=0}^{\infty} \dfrac{x^n}{n+1}$. For this power series, $a_n = \frac{1}{n+1}\ \forall n \in \mathbb{N} \cup \{0\}$ and

$$\left|\frac{a_n}{a_{n+1}}\right| = \frac{n+2}{n+1} = \frac{1+\frac{2}{n}}{1+\frac{1}{n}} \to 1$$

as $n \to \infty$. Thus, the radius of convergence is $R = 1$ again. This means that $\sum_{n=0}^{\infty} a_n x^n \begin{cases} \text{converges if } |x| < 1 \\ \text{diverges if } |x| > 1. \end{cases}$

When $x = 1$, the series is $\sum_{n=0}^{\infty} \frac{1^n}{n+1} = 1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\frac{1}{6}+\ldots = \sum_{n=1}^{\infty} \frac{1}{n}$ which we know diverges. When $x = -1$, the series is $\sum_{n=0}^{\infty} \frac{(-1)^n}{n+1} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \ldots$ which we know converges.

For this power series, we have convergence when $x = -R$ and divergence when $x = R$.

Example 15.3. Consider $\sum_{n=0}^{\infty} \frac{x^n}{(n+1)^2}$. For this power series, $a_n = \frac{1}{(n+1)^2}$ $\forall n \in \mathbb{N} \cup \{0\}$ and

$$\left| \frac{a_n}{a_{n+1}} \right| = \frac{(n+2)^2}{(n+1)^2} \to 1$$

as $n \to \infty$. Thus, the radius of convergence is $R = 1$ again. This means that $\sum_{n=0}^{\infty} a_n x^n \begin{cases} \text{converges if } |x| < 1 \\ \text{diverges if } |x| > 1. \end{cases}$

When $|x| = R = 1$, $\sum_{n=0}^{\infty} \frac{|x|^n}{(n+1)^2} = \sum_{n=0}^{\infty} \frac{1}{(n+1)^2} = \sum_{n=1}^{\infty} \frac{1}{n^2}$ which converges.

For this power series, we have convergence when $x = \pm R$.

Remark. The previous three examples show that when $|x| = R \in (0, \infty)$, we can have divergence, conditional convergence or absolute convergence.

Example 15.4. Consider $\sum_{n=0}^{\infty} (-2)^n x^n$. Since

$$\left| \frac{a_n}{a_{n+1}} \right| = \left| \frac{(-2)^n}{(-2)^{n+1}} \right| = \frac{1}{2},$$

this power series has radius of convergence $R = \frac{1}{2}$. The open interval of convergence is $(-\frac{1}{2}, \frac{1}{2})$.

Example 15.5. Consider $\sum_{n=0}^{\infty}(n+1)x^n$. Since

$$\left|\frac{a_n}{a_{n+1}}\right| = \frac{n+1}{n+2} \to 1$$

as $n \to \infty$, this power series has radius of convergence $R = 1$. The open interval of convergence is $(-1, 1)$.

Example 15.6. Consider $\sum_{n=0}^{\infty}(\cosh n)x^n$. Since

$$\left|\frac{a_n}{a_{n+1}}\right| = \left|\frac{\cosh n}{\cosh(n+1)}\right| = \frac{e^n + e^{-n}}{e^{n+1} + e^{-n-1}} = \frac{1 + e^{-2n}}{e + e^{-2n-1}} \to \frac{1+0}{e+1} =$$

as $n \to \infty$, this power series has radius of convergence $R = \frac{1}{e}$. The open interval of convergence is $(-\frac{1}{e}, \frac{1}{e})$.

Example 15.7. For the power series $\sum_{n=0}^{\infty} \frac{x^n}{n!}$, we have that

$$R = \lim_{n\to\infty} \left|\frac{a_n}{a_{n+1}}\right| = \infty$$

as $n \to \infty$. The radius of convergence $R = \infty$. The open interval of convergence is $(-\infty, \infty)$.

Example 15.8. Consider

$$\sum_{n=0}^{\infty} \frac{x^{2n}}{\cosh n} = \frac{1}{\cosh 0} + 0 + \frac{x^2}{\cosh 1} + 0 + \frac{x^4}{\cosh 2} + 0 + \frac{x^6}{\cosh 3} + \dots$$

We have a problem here: If we write this power series as $\sum_{n=0}^{\infty} a_n x^n$, then $0 = a_1 = a_3 = a_5 = a_7 = \dots$ and $\frac{a_n}{a_{n+1}}$ is undefined if n is even.

Instead, let $t = x^2$. Then we have

$$\sum_{n=0}^{\infty} b_n t^n = \sum_{n=0}^{\infty} \frac{t^n}{\cosh n} = \sum_{n=0}^{\infty} \frac{x^{2n}}{\cosh n}$$

and

$$\left|\frac{b_n}{b_{n+1}}\right| = \frac{\cosh n+1}{\cosh n} = \frac{e^{n+1} + e^{-n-1}}{e^n + e^{-n}} = \frac{e + e^{-2n-1}}{1 + e^{-2n}} \to e$$

as $n \to \infty$. So $\sum_{n=0}^{\infty} \frac{t^n}{\cosh n}$ has radius of convergence $R = e$. This means that

$$\sum_{n=0}^{\infty} \frac{t^n}{\cosh n} \begin{cases} \text{converges if } |t| < e \\ \text{diverges if } |t| > e. \end{cases}$$

But then

$$\sum_{n=0}^{\infty} \frac{x^{2n}}{\cosh n} \begin{cases} \text{converges if } \left|x^2\right| < e \\ \text{diverges if } \left|x^2\right| > e. \end{cases}$$

So

$$\sum_{n=0}^{\infty} \frac{x^{2n}}{\cosh n} \begin{cases} \text{converges if } |x| < \sqrt{e} \\ \text{diverges if } |x| > \sqrt{e}. \end{cases}$$

Hence $\sum_{n=0}^{\infty} \frac{x^{2n}}{\cosh n}$ has radius of convergence $R = \sqrt{e}$ and open internal of convergence $(-\sqrt{e}, \sqrt{e})$.

Chapter 16

Taylor's Theorem

Recall Rolle's Theorem and the Mean Value Theorem from MAT113 Matematik I (see Thomas' Calculus Chapter 4):

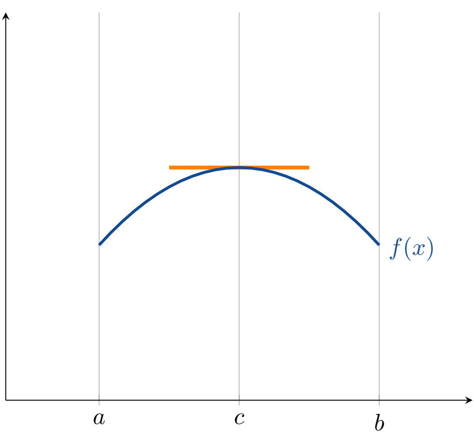
Theorem 16.1 (Rolle's Theorem). *Suppose that*

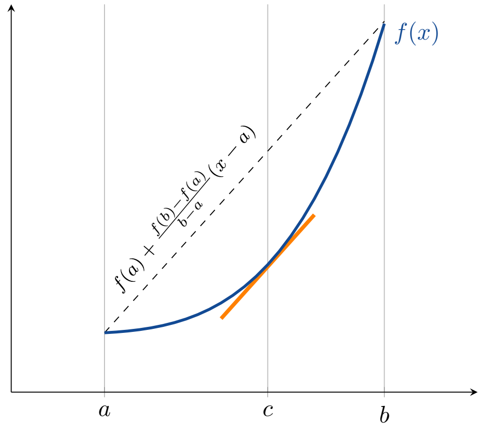
(i). $f : [a, b] \to \mathbb{R}$ *is continuous;*

(ii). f *is differentiable on* (a, b)*; and*

(iii). $f(a) = f(b)$.

Then $\exists\ c \in (a, b)$ *such that* $f'(c) = 0$.





Theorem 16.2 (The Mean Value Theorem). *Suppose that*

(i). $f:[a,b]\to\mathbb{R}$ *is continuous; and*

(ii). f *is differentiable on* (a,b).

Then $\exists\ c\in(a,b)$ *such that*

$$\frac{f(b)-f(a)}{b-a}=f'(c).$$

Remark. In other words, $\exists\ c$ such that $a<c<b$ and

$$f(b)=f(a)+f'(c)(b-a).$$

Theorem 16.3 (Taylor's Theorem). ⋆⋆⋆⋆⋆
Let $f:[a,b]\to\mathbb{R}$ *be a function. Suppose that*

(i). $f,f',f'',f''',\ldots,f^{(n-1)}$ *exist and are continuous on* $[a,b]$; *and*

(ii). $f^{(n)}$ *exists and is continuous on* (a,b).

Then $\exists\ c\in(a,b)$ *such that*

$$f(b)=f(a)+f'(a)(b-a)+\frac{f''(a)}{2!}(b-a)^2+\ldots$$
$$+\frac{f^{(n-1)}(a)}{(n-1)!}(b-a)^{n-1}+\frac{f^{(n)}(c)}{n!}(b-a)^n.$$

Proof. Let

$$k := \frac{f(b) - f(a) - \sum_{j=1}^{n-1} \frac{f^{(j)}(a)}{j!}(b-a)^j}{(b-a)^n} \in \mathbb{R} \tag{16.1}$$

and

$$\begin{aligned} g(x) := f(x) - f(a) - f'(a)(x-a) - \frac{f''(a)}{2!}(x-a)^2 - \ldots \\ - \frac{f^{(n-1)}(a)}{(n-1)!}(x-a)^{n-1} - k(x-a)^n. \end{aligned} \tag{16.2}$$

Then $g(a) = 0 = g(b)$, g is continuous on $[a, b]$ and differentiable on (a, b). By Rolle's Theorem, it follows that $\exists\ c_1 \in (a, b)$ such that $g'(c_1) = 0$.

Differentiating (16.2) gives

$$\begin{aligned} g'(x) := f'(x) - 0 - f'(a) - f''(a)(x-a) - \ldots \\ - \frac{f^{(n-1)}(a)}{(n-2)!}(x-a)^{n-2} - kn(x-a)^{n-1} \end{aligned} \tag{16.3}$$

and we can see that $g'(a) = 0 = g'(c_1)$.

If $n \geq 2$, then g' is continuous on $[a, b]$ and differentiable on (a, b), so we can use Rolle's Theorem again to see that $\exists\ a < c_2 < c_1 < b$ such that $g''(c_2) = 0$. Repeating, we get $a < c_n < c_{n-1} < \ldots < c_2 < c_1 < b$ and $g^{(n)}(c_n) = 0$. But differentiating (16.2) n times gives $g^{(n)}(x) = f^{(n)}(x) - kn!$ for all $a < x < b$. Therefore $f^{(n)}(c_n) = kn!$. Finally we use (16.1) to finish the proof. □

Chapter 17

Differentiation and Integration of Power Series

Let $\sum_{n=0}^{\infty} a_n x^n$ be a power series with radius of convergence $R \neq 0$. Then we can define a function $f : (-R, R) \to \mathbb{R}$ by

$$f(x) = \sum_{n=0}^{\infty} a_n x^n.$$

Question. What properties does this function have?

Question. Is f continuous?

Question. Is f differentiable?

Question. If so, what is $f'(x)$?

Question. Is f integrable?

Lemma 17.1. *Let $\sum_{n=0}^{\infty} a_n x^n$ be a power series with radius of convergence R. Then the power series $\sum_{n=1}^{\infty} n a_n x^{n-1}$ has the same radius of convergence.*

Proof. Suppose that $\sum_{n=1}^{\infty} n a_n x^{n-1}$ has radius of convergence S.

Claim 1. $S \le R$

Proof of Claim 1. First note that

$$
\begin{aligned}
|x| < S \implies & \sum_{n=1}^{\infty} n a_n x^{n-1} \text{ is absolutely convergent} \\
\implies & \sum_{n=1}^{\infty} n a_n x^{n} \text{ is absolutely convergent}
\end{aligned}
$$

But $0 \le |a_n x^n| \le |n a_n x^n|$ for all $n \ge 1$. So, by the Comparison Test, $\sum_{n=1}^{\infty} a_n x^n$ converges absolutely for all $|x| < S$. Therefore $S \le R$. □

Claim 2. $S \ge R$

Proof of Claim 2. Suppose that $|x| < R$. Choose $x_0 \in \mathbb{R}$ such that $|x| < x_0 < R$. Then

$$
\begin{aligned}
\sum_{n=0}^{\infty} a_n x_0^n \text{ is abs. convergent} & \\
\implies & a_n x_0^n \to 0 \text{ as } n \to \infty \\
\implies & (a_n x_0^n) \text{ is a bounded sequence (Thm 3.4)} \\
\implies & \exists K \text{ such that } |a_n x_0^n| < K \ \forall n.
\end{aligned}
$$

So

$$
\begin{aligned}
\left| n a_n x^{n-1} \right| &= n \left| a_n \right| \frac{\left| x_0 \right|^{n-1}}{\left| x_0 \right|^{n-1}} \left| x \right|^{n-1} \\
&= \left| a_n x_0^{n-1} \right| n \frac{\left| x \right|^{n-1}}{\left| x_0 \right|^{n-1}} \\
&< \frac{K}{x_0} n \left| \frac{x}{x_0} \right|^{n-1}.
\end{aligned}
$$

Since $0 < \left| \frac{x}{x_0} \right|^{n-1} < 1$, it follows that $\sum_{n=1}^{\infty} n \left| \frac{x}{x_0} \right|^{n-1}$ converges. Then the Comparison Test tells us that $\sum_{n=1}^{\infty} n a_n x^{n-1}$ converges absolutely $\forall \, |x| < R$. Therefore $R \le S$. □

Therefore $S = R$. □

Corollary 17.1.1. *If the power series $\sum_{n=0}^{\infty} a_n x^n$ has radius of convergence R, then the power series $\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$ has the same radius of convergence.*

Lemma 17.2. *Suppose $x, x_0 \in \mathbb{R}$ and $x \neq x_0$. Let $X = \max\{x, x_0\}$. Then for all $n \in \mathbb{N}$, $n \geq 2$,*

$$\left| \frac{x^n - x_0^n}{x - x_0} - n x_0^{n-1} \right| \leq \frac{n(n-1)}{2} X^{n-2} \left| x - x_0 \right|.$$

Proof. Let $n \in \mathbb{N}$ and $n \geq 2$. Define $g : [\min\{x_0, x\}, \max\{x_0, x\}] \to \mathbb{R}$ by $g(t) = t^n$. Then g is continuous and differentiable infinitely many times. By Taylor's Theorem (with $n = 2$),

$$g(x) = g(x_0) + (x - x_0) g'(x_0) + \frac{(x - x_0)^2}{2} g''(c)$$

for some c between x_0 and x (either $x_0 < c < x$ or $x < c < x_0$). So

$$x^n = x_0^n + (x - x_0) n x_0^{n-1} + \frac{(x - x_0)^2}{2} n(n-1) c^{n-2}.$$

Thus

$$\frac{x^n - x_0^n}{x - x_0} = n x_0^{n-1} + \frac{(x - x_0)}{2} n(n-1) c^{n-2}.$$

Since $|x| < X$, if follows that

$$\begin{aligned} \left| \frac{x^n - x_0^n}{x - x_0} - n x_0^{n-1} \right| &= \left| x - x_0 \right| \frac{n(n-1)}{2} \left| c \right|^{n-2} \\ &\leq \frac{n(n-1)}{2} X^{n-2} \left| x - x_0 \right|. \end{aligned}$$

$\square$

Theorem 17.3. ★★★★
Let $\sum_{n=0}^{\infty} a_n x^n$ be a power series with radius of convergence $R > 0$. Define $f, g : (-R, R) \to \mathbb{R}$ by $f(x) = \sum_{n=0}^{\infty} a_n x^n$ and $g(x) = \sum_{n=1}^{\infty} n a_n x^{n-1}$. Then f is differentiable and $f'(x) = g(x)$ for all $x \in (-R, R)$.

Proof. Let $x, x_0 \in \mathbb{R}$, $x \neq x_0$, $|x| < R$ and $|x_0| < R$. Let $X = \max\{x, x_0\} < R$. Then

$$
\begin{aligned}
\frac{f(x)-f(x_0)}{x-x_0} - g(x_0) &= \frac{1}{x-x_0}\left(\sum_{n=0}^{\infty} a_n x^n - \sum_{n=0}^{\infty} a_n x_0^n\right) \\
&\quad - \sum_{n=1}^{\infty} n a_n x_0^{n-1} \\
&= \frac{1}{x-x_0}\Bigg(\left(a_0 + a_1 x + \sum_{n=2}^{\infty} a_n x^n\right) \\
&\quad - \left(a_0 + a_1 x_0 + \sum_{n=2}^{\infty} a_n x_0^n\right)\Bigg) \\
&\quad - a_1 - \sum_{n=2}^{\infty} n a_n x_0^{n-1} \\
&= \sum_{n=2}^{\infty} a_n \left(\frac{x^n - x_0^n}{x-x_0} - n x_0^{n-1}\right).
\end{aligned}
$$

(Note that all these series are absolutely convergent for $|x| < R$ and $|x_0| < R$, so we are allowed to add and subtract them.)

It follows that

$$
\begin{aligned}
\left|\frac{f(x)-f(x_0)}{x-x_0} - g(x_0)\right| &\le \sum_{n=2}^{\infty} |a_n| \left|\frac{x^n - x_0^n}{x-x_0} - n x_0^{n-1}\right| \\
&\le \frac{|x-x_0|}{2} \sum_{n=2}^{\infty} |a_n|\, n(n-1) X^{n-2}
\end{aligned}
$$

by the Triangle Inequality and Lemma 17.2. The power series $\sum_{n=2}^{\infty} |a_n|\, n(n-1) X^{n-2}$ converges by Corollary 17.1.1, because $0 < X < R$. Thus

$$
\left|\frac{f(x)-f(x_0)}{x-x_0} - g(x_0)\right| \to 0
$$

as $x \to x_0$. Therefore f is differentiable at x_0 and $f'(x_0) = g(x_0)$. This is true $\forall x_0 \in (-R, R)$. □

Corollary 17.3.1. *Let $\sum_{n=0}^{\infty} a_n x^n$ be a power series with radius of convergence $R > 0$. Define $f : (-R, R) \to \mathbb{R}$ by $f(x) = \sum_{n=0}^{\infty} a_n x^n$. Then f is differentiable infinitely many times and*

$$
f^{(k)}(x) = \sum_{n=k}^{\infty} \frac{n!}{(n-k)!} a_n x^{n-k}.
$$

We can also integrate power series.

Theorem 17.4. *Let $\sum_{n=0}^{\infty} a_n x^n$ be a power series with radius of convergence $R > 0$. Define $f : (-R, R) \to \mathbb{R}$ by $f(x) = \sum_{n=0}^{\infty} a_n x^n$. Then $\forall\, |x| < R$,*

$$\int f(x)\ dx = \left(\sum_{n=0}^{\infty} a_n \frac{x^{n+1}}{n+1} \right) + c$$

where c is a constant.

Proof. By Lemma 17.1, $\sum_{n=0}^{\infty} a_n x^n$ and $\sum_{n=0}^{\infty} a_n \frac{x^{n+1}}{n+1}$ have the same radius of convergence. Define $F : (-R, R) \to \mathbb{R}$ by $F(x) = \sum_{n=0}^{\infty} a_n \frac{x^{n+1}}{n+1}$. Then $F'(x) = f(x)$ for all $x \in (-R, R)$, by Theorem 17.3. It follows that $\int f(x)\ dx = \int F'(x)\ dx = F(x) + c$. $\square$

Summary. If $|x| < R$, then

$$\frac{d}{dx} \left(\sum_{n=0}^{\infty} a_n x^n \right) = \sum_{n=0}^{\infty} \left(\frac{d}{dx} a_n x^n \right)$$

and

$$\int \left(\sum_{n=0}^{\infty} a_n x^n \right) dx = \sum_{n=0}^{\infty} \left(\int a_n x^n\ dx \right).$$

Chapter 18

Power Series with Centre of Expansion c

The results that we have proved for the power series $\sum_{n=0}^{\infty} a_n x^n$ are also true for the power series $\sum_{n=0}^{\infty} a_n (x-c)^n$.

Example 18.1. Recall that $\sum_{n=0}^{\infty} x^n$ has radius of convergence $R = 1$. Therefore $\sum_{n=0}^{\infty}(x-c)^n$ also has radius of convergence $R = 1$. Since

$$\sum_{n=0}^{\infty} x^n \begin{cases} \text{converges absolutely } \forall\, |x| < 1 \\ \text{diverges } \forall\, |x| > 1 \end{cases}$$

we have that

$$\sum_{n=0}^{\infty} (x-c)^n \begin{cases} \text{converges absolutely } \forall\, |x-c| < 1 \\ \text{diverges } \forall\, |x-c| > 1. \end{cases}$$

The open interval of convergence for $\sum_{n=0}^{\infty}(x-c)^n$ is $(c-1, c+1)$.

Example 18.2. Since $\sum_{n=0}^{\infty} \frac{x^n}{n!}$ has radius of convergence $R = \infty$, it follows that $\sum_{n=0}^{\infty} \frac{(x-c)^n}{n!}$ converges absolutely $\forall x$. The radius of convergence of $\sum_{n=0}^{\infty} \frac{(x-c)^n}{n!}$ is $R = \infty$ and the open interval of convergence is $(-\infty, \infty)$.

Example 18.3. Recall that $\sum_{n=0}^{\infty} \frac{x^n}{(n+1)^2}$ has radius of convergence $R = 1$. So

$$\sum_{n=0}^{\infty} \frac{(x-c)^n}{(n+1)^2} \begin{cases} \text{converges absolutely } \forall\, |x-c| < 1 \\ \text{diverges } \forall\, |x-c| > 1. \end{cases}$$

The open interval of convergence of $\sum_{n=0}^{\infty} \frac{(x-c)^n}{(n+1)^2}$ is $(c-1, c+1)$.

If $x \in (c-1, c+1)$, then $\frac{d}{dx}\left(\sum_{n=0}^{\infty} \frac{(x-c)^n}{(n+1)^2}\right) = \sum_{n=0}^{\infty} \frac{d}{dx}\left(\frac{(x-c)^n}{(n+1)^2}\right)$ and $\int \left(\sum_{n=0}^{\infty} \frac{(x-c)^n}{(n+1)^2}\right) dx = \sum_{n=0}^{\infty} \left(\int \frac{(x-c)^n}{(n+1)^2}\, dx\right)$

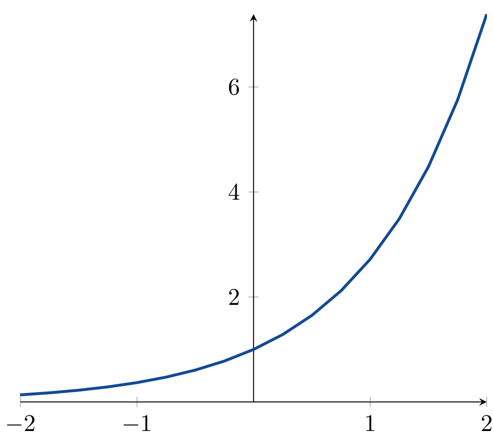
Chapter 19

The Exponential Function

Recall that

$$\sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \frac{x^6}{6!} + \frac{x^7}{7!} + \frac{x^8}{8!} + \frac{x^9}{9!} + \dots$$

has radius of convergence $R = \infty$. So $\sum_{n=0}^{\infty} \frac{x^n}{n!}$ is a function $\mathbb{R} \to \mathbb{R}$.



Definition. The ***exponential function*** $e^{\cdot} : \mathbb{R} \to \mathbb{R}$ is defined by

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}.$$

Lemma 19.1. *$e^x \cdot e^y = e^{x+y}$ for all $x, y \in \mathbb{R}$.*

Proof. Let $x, y \in \mathbb{R}$. Then $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots$ and $e^y = 1 + y + \frac{y^2}{2!} + \frac{y^3}{3!} + \ldots$. So

$$\begin{aligned} e^x e^y &= \left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots\right)\left(1 + y + \frac{y^2}{2!} + \frac{y^3}{3!} + \ldots\right) \\ &= 1 + (x+y) + \frac{1}{2!}\left(x^2 + xy + y^2\right) + \frac{1}{3!}\left(x^3 + 3x^2y + 3xy^2 + y^3 \right. \\ &= 1 + (x+y) + \frac{1}{2!}(x+y)^2 + \frac{1}{3!}(x+y)^3 + \ldots \\ &= e^{x+y}. \end{aligned}$$

Note: We can only do this because $\sum_{n=0}^{\infty} \frac{x^n}{n!}$ is absolutely convergence (see Mary Hart's book for details). □

Lemma 19.2. *$e^x > 0$ for all $x \in \mathbb{R}$.*

Proof. First, $e^0 = 1 + 0 + \frac{0}{2!} + \frac{0}{3!} + \ldots = 1$, and $e^x e^{-x} = e^0 = 1$. If $x \geq 0$, then $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots \geq 1$. If $x \leq 0$, then $-x \geq 0$ and $e^{-x} > 0$ which means that $e^x = \frac{1}{e^{-x}} > 0$. Therefore $e^x > 0$ for all $x \in \mathbb{R}$. □

Lemma 19.3. *$e^x \to \infty$ as $x \to \infty$.*

Proof. If $x \geq 0$, then $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots \geq 1 + x \to \infty$ as $x \to \infty$. □

Lemma 19.4. *$e^x \to 0$ as $x \to -\infty$.*

Proof. If $x < 0$, then $e^x = \frac{1}{e^{-x}} \to 0$ as $x \to -\infty$. □

Lemma 19.5. *$\frac{d}{dx}(e^x) = e^x$ for all $x \in \mathbb{R}$.*

Proof. By Theorem 17.3,

$$\begin{aligned}
\frac{d}{dx}e^x = \frac{d}{dx}\sum_{n=0}^{\infty}\frac{x^n}{n!} &= \frac{d}{dx}\left(1+\sum_{n=1}^{\infty}\frac{x^n}{n!}\right)\\
&= \sum_{n=1}^{\infty}\frac{d}{dx}\left(\frac{x^n}{n!}\right) = \sum_{n=1}^{\infty}\frac{x^{n-1}}{(n-1)!}\\
&= \sum_{m=0}^{\infty}\frac{x^m}{m!} = e^x
\end{aligned}$$

where $m = n-1$. □

Definition. A function f is called ***strictly increasing*** iff

$$x < y \implies f(x) < f(y).$$

Lemma 19.6. *The exponential function is strictly increasing.*

Proof. Since $\frac{d}{dx}e^x = e^x > 0\ \forall x \in \mathbb{R}$, e^x is strictly increasing. □

Lemma 19.7. $e \notin \mathbb{Q}$.

Proof. We will use proof by contradiction: Suppose that $e \in \mathbb{Q}$. Then we can write

$$e = \frac{p}{q}$$

for some $p, q \in \mathbb{N}$. Now

$$e = e^1 = 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \frac{1}{5!} + \frac{1}{6!} + \dots$$

Let $s_n = \sum_{k=0}^{n}\frac{1}{k!}$. Recall that if $|a| < 1$, then $\sum_{k=0}^{\infty} a^k = \frac{1}{1-a}$. Thus

$$\begin{aligned}
0 < e - s_n &= \tfrac{1}{(n+1)!} + \tfrac{1}{(n+2)!} + \tfrac{1}{(n+3)!} + \tfrac{1}{(n+4)!} + \tfrac{1}{(n+5)!} + \dots\\
&< \tfrac{1}{(n+1)!}\left(1 + \tfrac{1}{n+1} + \tfrac{1}{(n+1)^2} + \tfrac{1}{(n+1)^3} + \tfrac{1}{(n+1)^4} + \dots\right)\\
&= \tfrac{1}{(n+1)!}\left(\frac{1}{1-\left(\frac{1}{n+1}\right)}\right) = \tfrac{1}{(n+1)!}\left(\frac{n+1}{(n+1)-1}\right) = \frac{1}{n!n}
\end{aligned}$$

for all $n \in \mathbb{N}$. But $q \in \mathbb{N}$, so we have that

$$0 < q!(e - s_q) < \frac{1}{q} \leq 1.$$

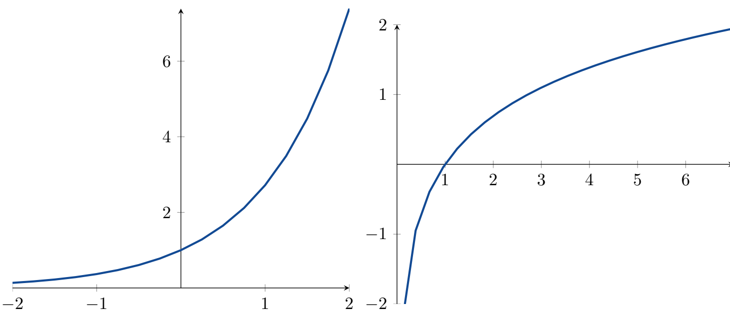
Now

$$e = \frac{p}{q} \implies qe = p \in \mathbb{N} \implies q!e \in \mathbb{N}$$

and

$$q!s_q = q!\left(1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \ldots + \frac{1}{q!}\right) \in \mathbb{N}.$$

So $q!(e - s_q) \in \mathbb{N}$ and $0 < q!(e - s_q) < 1$. This is a contradiction. Hence $e \notin \mathbb{Q}$. □



Definition. By Lemmata 19.3, 19.4 and 19.6, $e^{\cdot} : \mathbb{R} \to (0, \infty)$ is a bijection. So there exists an inverse function $\log : (0, \infty) \to \mathbb{R}$ called the ***natural logarithm***:

$$y = e^x \iff x = \log y.$$

Lemma 19.8. *$\log ab = \log a + \log b$ for all $a, b \in (0, \infty)$.*

Proof. Let $a, b > 0$. Let $x = \log a$ and $y = \log b$. Then $a = e^x$ and $b = e^y$. By Lemma 19.1, $ab = e^x e^y = e^{x+y}$. So $\log ab = x + y = \log a + \log b$. □

Lemma 19.9. *$\log 1 = 0$*

Proof. Since $e^0 = 1 + 0 + \frac{0}{2!} + \frac{0}{3!} + \ldots = 1$, we have that $\log 1 = 0$. □

Lemma 19.10. $\frac{d}{dx}(\log x) = \frac{1}{x}\ \forall x > 0$

Proof. Let $x > 0$ and let $y = \log x$. Then $x = e^y$ and $\frac{dx}{dy} = \frac{d}{dy}e^y = e^y = x$. Hence $\frac{dy}{dx} = \frac{1}{\left(\frac{dx}{dy}\right)} = \frac{1}{x}$. □

Lemma 19.11. *The natural logarithm is strictly increasing.*

Proof. Since $\frac{d}{dx}\log x = \frac{1}{x} > 0$ for all $x > 0$, $\log x$ is strictly increasing. □

Lemma 19.12. $\log x = \int_1^x \frac{1}{t}\ dt\ \forall x > 0.$

Proof. Let $x, t > 0$ and let $y = \log t$. Then

$$\int_1^x \frac{dt}{t} = \int_1^x \frac{dy}{dt}dt = [y(t)]_1^x = [\log t]_1^x = \log x - \log 1 = \log x.$$

□

Lemma 19.13. *Let $\alpha > 0$. Then $\frac{e^x}{x^\alpha} \to \infty$ as $x \to \infty$.*

Proof. Choose $p \in \mathbb{N}$ such that $p > \alpha$. Then $\forall x > 0$, $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots + \frac{x^p}{p!} + \ldots > \frac{x^p}{p!}$. So $\forall x > 0$,

$$\frac{e^x}{x^\alpha} > \frac{x^p}{p!x^\alpha} = \frac{1}{p!}x^{p-\alpha} \to \infty$$

as $x \to \infty$. □

Lemma 19.14. *Let $\alpha > 0$. Then $\frac{x^\alpha}{e^x} \to 0$ as $x \to \infty$.*

Proof. Since $\frac{e^x}{x^\alpha} > 0\ \forall x > 0$,

$$\frac{x^\alpha}{e^x} = \frac{1}{\left(\frac{e^x}{x^\alpha}\right)} \to 0$$

as $x \to \infty$. □

Lemma 19.15. *Let $\alpha > 0$. Then $\frac{\log x}{x^\alpha} \to 0$ as $x \to \infty$.*

Proof. By Lemma 19.14 (with $\alpha = 1$), we know that $\frac{x}{e^x} \to 0$ as $x \to \infty$. Let $\alpha > 0$ and $x = \alpha t$. Then $\frac{\alpha t}{e^{\alpha t}} \to 0$ as $t \to \infty$. Thus $\frac{t}{e^{\alpha t}} \to 0$ as $t \to \infty$.

Now write $y = e^t$. Then $t = \log y$. Since $t \to \infty \iff y \to \infty$, we have that

$$\frac{\log y}{y^\alpha} = \frac{\log y}{e^{\alpha \log y}} = \frac{t}{e^\alpha t} \to 0$$

as $y \to \infty$. □

Lemma 19.16. *Let $\alpha > 0$. Then $x^\alpha \log x \to 0$ as $x \searrow 0$.*

Proof. Let $x = \frac{1}{y}$. Then $y \to \infty \iff x \searrow 0$. Since $\log \frac{1}{x} = -\log x$,

$$x^\alpha \log x = -\frac{\log \frac{1}{x}}{\left(\frac{1}{x}\right)^\alpha} = -\frac{\log y}{y^\alpha} \to 0$$

as $u \searrow 0$. □

Chapter 20

Taylor Series

Let $a, x \in \mathbb{R}$ and $a \neq x$. Let $I := [\min\{a, x\}, \max\{a, x\}]$ be the closed interval with end points a and x. In other words, $I = [a, x]$ or $I = [x, a]$ depending on if $a < x$ or if $x < a$. Suppose that $f : I \to \mathbb{R}$ is differentiable infinitely many times. In other words, suppose that $\frac{d^n f}{dx^n}(y)$ exists and is continuous $\forall y \in I$ and $n \in \mathbb{N}$.

By Taylor's Theorem, $\forall n \in \mathbb{N}\ \exists c\ (a < c < x$ or $x < c < a)$ such that

$$\begin{aligned} f(x) &= f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \dots \\ &\quad + \frac{f^{(n-1)}(a)}{(n-1)!}(x-a)^{n-1} + \frac{f^{(n)}(c)}{n!}(x-a)^n \\ &= \sum_{k=0}^{n-1} \frac{f^{(k)}(a)}{k!}(x-a)^k + R_n(c) \end{aligned}$$

where

$$\begin{aligned} R_n(c) &= \frac{f^{(n)}(c)}{n!}(x-a)^n \\ &= f(x) - \sum_{k=0}^{n-1} \frac{f^{(k)}(a)}{k!}(x-a)^k \end{aligned}$$

is called the ***remainder term***. If we can show that the remainder term tends to zero, then $\sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!}(x-a)^k$ converges and

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!}(x-a)^k = f(x).$$

This power series is called the ***Taylor Series of $f(x)$ with centre a.***

Example 20.1. Let $f(x) = e^x$. Then $\frac{d^k f}{dx^k}$ exists and is continuous $\forall x$ and $\forall k$. Let $a = 0$ and $x \neq 0$. By Taylor's Theorem,

$$\begin{aligned} f(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \dots \\ + \frac{f^{(n-1)}(0)}{(n-1)!}x^{n-1} + \frac{f^{(n)}(c)}{n!}x^n \end{aligned}$$

for some c between 0 and x $(0 < c < x$ or $x < c < 0)$.

Because $\frac{d}{dx}e^x = e^x$, it is easy to see that $f^{(k)}(0) = 1\ \forall k$. So

$$\begin{aligned} e^x &= f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \dots \\ &\quad + \frac{f^{(n-1)}(0)}{(n-1)!}x^{n-1} + \frac{f^{(n)}(c)}{n!}x^n \\ &= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots + \frac{x^{n-1}}{(n-1)!} + \frac{e^c}{n!}x^n. \end{aligned}$$

Since $0 < |c| < |x|$,

$$0 \leq \left| \frac{e^c}{n!}x^n \right| \leq \frac{e^{|x|}\,|x|^n}{n!} \to 0$$

as $n \to \infty$. Hence the remainder term $R_c(x) = \frac{e^c}{n!}x^n$ tends to zero. Therefore

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

is the Taylor Series of e^x with centre 0. (We have already seen this as the definition of e^x).

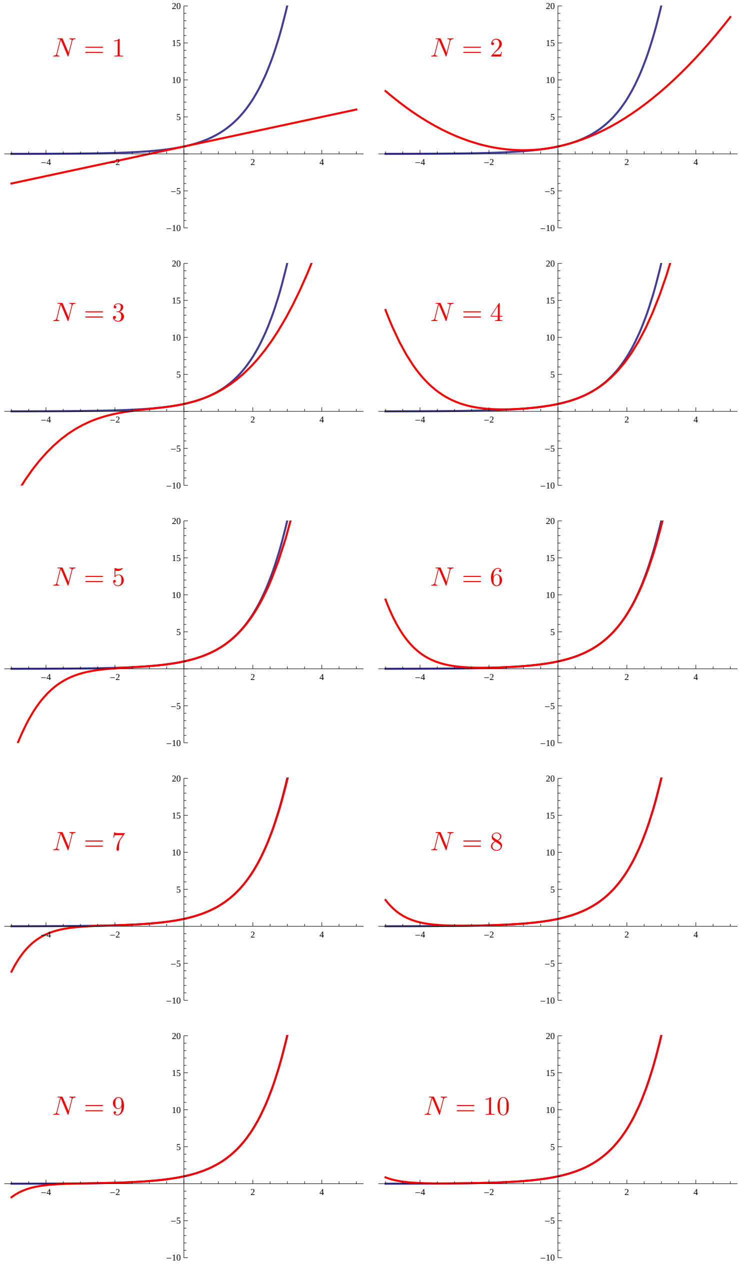


Figure 20.1: Graphs of $y = e^x$ and $y = \sum_{n=0}^{N} \frac{x^n}{n!}$ for $N \in \{1, 2, 3, \ldots, 10\}$.

Example 20.2. Let $f(x) = \sin x$. Then $\frac{d^k f}{dx^k}$ exists and is continuous $\forall x$ and $\forall k$. Let $a = 0$ and $x \neq 0$. By Taylor's Theorem,

$$\begin{aligned} f(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \dots \\ + \frac{f^{(n-1)}(0)}{(n-1)!}x^{n-1} + \frac{f^{(n)}(c)}{n!}x^n \end{aligned}$$

for some c between 0 and x ($0 < c < x$ or $x < c < 0$).

Because $\frac{d^k}{dx^k}\sin x = \cos x$ or $-\sin x$ or $-\cos x$ or $\sin x$, it is easy to see that

$$f^{(k)}(0) = \begin{cases} 1 & \text{if } k = 1, 5, 9, 13, \dots \\ 0 & \text{if } k = 0, 2, 4, 6, 8, \dots \\ -1 & \text{if } k = 3, 7, 11, 15, \dots \end{cases}$$

So

$$\begin{aligned} \sin x &= f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \dots \\ &\quad + \frac{f^{(n-1)}(0)}{(n-1)!}x^{n-1} + \frac{f^{(n)}(c)}{n!}x^n \\ &= 0 + x + 0 - \frac{x^3}{3!} + 0 + \frac{x^5}{5!} + 0 - \frac{x^7}{7!} + \dots \\ &\quad + \frac{f^{(n-1)}(0)}{(n-1)!}x^{n-1} + \frac{e^c}{n!}x^n. \end{aligned}$$

Since $-1 \leq \sin c \leq 1$ and $-1 \leq \cos c \leq 1$ $\forall z$,

$$0 \leq \left| \frac{f^{(n)}(c)}{n!}x^n \right| \leq \frac{|x|^n}{n!} \to 0$$

as $n \to \infty$, the remainder term $R_c(x) = \frac{f^{(n)}(c)}{n!}x^n$ tends to zero. Therefore

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots$$

is the Taylor Series of $\sin x$ with centre 0.

Exercise 20.3. Let $f : (-1, \infty) \to \mathbb{R}$ be defined by $f(x) = \log(1 + x)$. Use proof by induction to prove that

$$\frac{d^k f}{dx^k}(x) = (-1)^{k-1}\frac{(k-1)!}{(1+x)^k}.$$

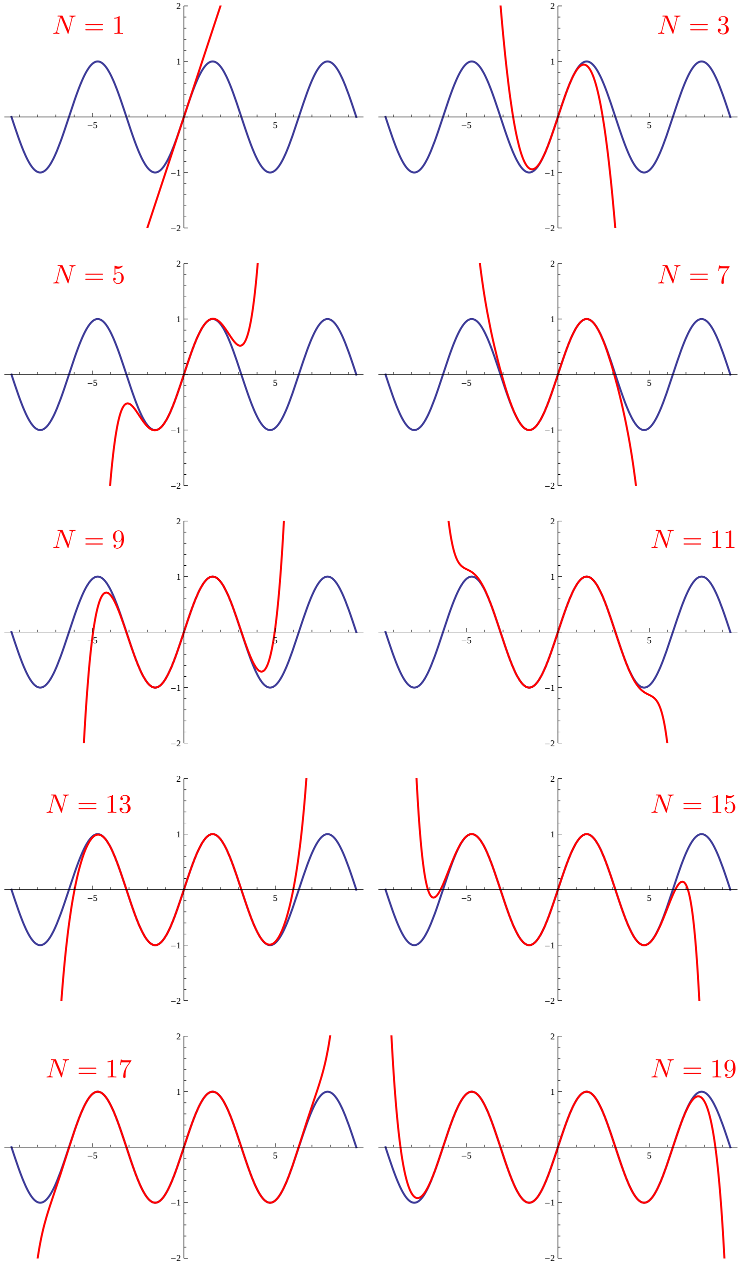


Figure 20.2: Graphs of $y = \sin x$ and $y = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots + (-1)^{\frac{N-1}{2}} \frac{x^N}{N!}$ for $N = 1, 3, 5, 7, 9, 11, 13, 15, 17, 19$.

Example 20.4. Let $f : (-1, \infty) \to \mathbb{R}$ be defined by $f(x) = \log(1+x)$. Let $a = 0$ and let $0 < x \leq 1$. Then f and its derivatives exist and are continuous on the closed interval $[0, x]$.

By Taylor's Theorem,

$$f(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \dots + \frac{f^{(n-1)}(0)}{(n-1)!}x^{n-1} + \frac{f^{(n)}(c)}{n!}x^n$$

for some c satisfying $0 < c < x$. Using Exercise 20.3, we can calculate that $f(0) = \log 1 = 0$ and $f^{(k)}(0) = (-1)^{k-1}(k-1)!$. So

$$\log(1+x) = f(x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots + \frac{(-1)^{n-1}x^{n-1}}{n-1} + \frac{f^{(n)}(c)x^n}{n!}$$

Since $0 < c < x \leq 1$, it follows that

$$|R_n(c)| = \left|\frac{f^{(n)}(c)x^n}{n!}\right| = \left|\frac{(n-1)!x^n}{(1+c)^n n!}\right| = \frac{x^n}{(1+c)^n n} \leq \frac{x^n}{n} \to 0$$

as $n \to \infty$. Therefore, if $0 < x \leq 1$, then

$$\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots = \sum_{n=1}^{\infty}(-1)^{n-1}\frac{x^n}{n}$$

is the Taylor Series of $\log(1+x)$ centred at 0, on the interval $[0, 1]$.

If can be proved (more difficult) that this series also converges to $\log(1+x)$ $\forall x \in (-1, 0)$. If $x > 1$, then the series diverges. This is illustrated in Figure 20.3 on page 124.

Example 20.5. Let $y = x + 1$. Then

$$\log y = (y-1) - \frac{1}{2}(y-1)^2 + \frac{1}{3}(y-1)^3 - \frac{1}{4}(y-1)^4 + \frac{1}{5}(y-1)^5 - \frac{1}{6}(y-1)^6 + \dots$$

is the Taylor Series of $\log y$ with center $a = 1$. If converges for all $y \in (0, 2]$.

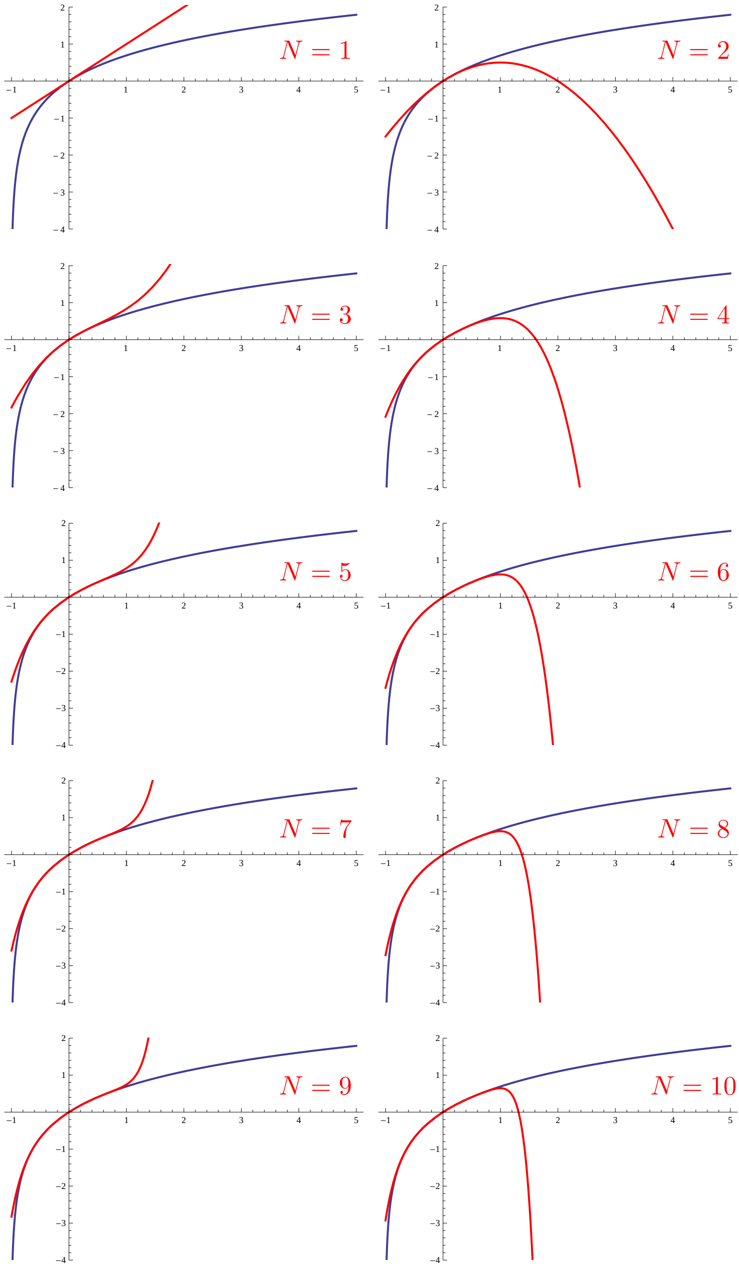
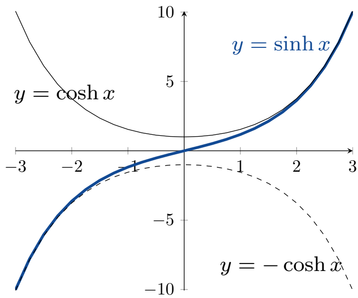


Figure 20.3: Graphs of $y = \log(1+x)$ and $y = \sum_{n=1}^{N} (-1)^{n-1} \frac{x^n}{n}$ for $N \in \{1, 2, 3, \ldots, 10\}$.

Definition. A Taylor Series with centre 0 is also called a ***Maclaurin Series***.

Example 20.6. Calculate the Maclaurin Series for $f(x) = \sinh x$.

solution:



Since $\frac{d}{dx} \sinh x = \cosh x$ and $\frac{d}{dx} \cosh x = \sinh x$, we know that $f^{(n)}(c) = \sinh x$ or $\cosh x \ \forall n \in \mathbb{N}$.

Let $x \neq 0$ and let c be between 0 and x. So $0 < c < x$ or $x < c < 0$. Then

$$\left|f^{(n)}(c)\right| < \left|f^{(n)}(x)\right| \leq \cosh x.$$

So

$$0 \leq \left|\frac{f^{(n)}(c)x^n}{n!}\right| \leq \cosh x \frac{|x|^n}{n!} \to 0$$

as $n \to \infty$. By the Sandwich Rule, it follows that $R_c(x) = \frac{f^{(n)}(c)x^n}{n!} \to 0$ as $n \to \infty$.

Now, since

$$f^{(n)}(x) = \begin{cases} \sinh x & \text{if } n = 0, 2, 4, 6, 8, \ldots \\ \cosh x & \text{if } n = 1, 3, 5, 7, 9, \ldots \end{cases}$$

we have that

$$f^{(n)}(0) = \begin{cases} 0 & \text{if } n = 0, 2, 4, 6, 8, \ldots \\ 1 & \text{if } n = 1, 3, 5, 7, 9, \ldots \end{cases}$$

Therefore

$$\begin{aligned}
\sinh x &= f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \frac{f^{(4)}(0)}{4!}x^4 \dots \\
&= 0 + 1x + \frac{0}{2!}x^2 + \frac{1}{3!}x^3 + \frac{0}{4!}x^4 \dots \\
&= x + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} + \frac{x^9}{9!} + \dots \\
&= \sum_{k=0}^{\infty} \frac{x^{2k+1}}{(2k+1)!}.
\end{aligned}$$

Example 20.7. Calculate the Taylor Series for $f(x) = \frac{1}{x}$ with centre $a = 2$. For which $x \in \mathbb{R}$ does the series converge?

solution: Since

$$\begin{aligned}
f(x) &= x^{-1} & f(2) &= \tfrac{1}{2} \\
f'(x) &= -x^{-2} & f'(2) &= -\tfrac{1}{4} \\
f''(x) &= 2x^{-3} & \tfrac{f''(2)}{2!} &= 2\tfrac{1}{8} \\
f'''(x) &= -6x^{-4} & \tfrac{f'''(2)}{3!} &= -\tfrac{1}{16} \\
&\vdots & &\vdots \\
f^{(n)}(x) &= (-1)^n n! x^{-n-1} & \tfrac{f^{(n)}(x)}{n!} &= \tfrac{(-1)^n}{2^{n+1}} \\
&\vdots & &\vdots
\end{aligned}$$

the Taylor Series is

$$\begin{aligned}
\frac{1}{x} &= f(2) + f'(2)(x-2) + \frac{f''(2)}{2!}(x-2)^2 + \frac{f'''(2)}{3!}(x-2)^3 + \dots \\
&= \frac{1}{2} - \frac{x-2}{4} + \frac{(x-2)^2}{8} - \frac{(x-2)^3}{16} + \frac{(x-2)^4}{32} - \dots \\
&= \frac{1}{2}\left(1 + r + r^2 + r^3 + r^4 + \dots\right)
\end{aligned}$$

where $r = -\frac{x-2}{2}$.

This series converges absolutely for $|r| < 1$ and diverges for $|r| \geq 1$. Therefore, the Taylor Series converges for $0 < x < 4$.

Chapter 21

Applications of Power Series

Example 21.1. Calculate $\int \sin(x^2)\ dx$.

solution: The Taylor Series for sin with centre 0 is

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots$$

So

$$\sin x^2 = x^2 - \frac{x^6}{3!} + \frac{x^{10}}{5!} - \frac{x^{14}}{7!} + \dots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{4n+2}}{(2n+1)!}$$

Therefore

$$\begin{aligned}
\int \sin x^2\ dx &= \int \left(\sum_{n=0}^{\infty} \frac{(-1)^n x^{4n+2}}{(2n+1)!} \right) dx \\
&= \sum_{n=0}^{\infty} \left(\int \frac{(-1)^n x^{4n+2}}{(2n+1)!}\ dx \right) \\
&= \sum_{n=0}^{\infty} \frac{(-1)^n x^{4n+3}}{(4n+3)(2n+1)!}\ dx + c \\
&= c + \frac{x^3}{3} - \frac{x^7}{7 \cdot 3!} + \frac{x^{11}}{11 \cdot 5!} - \frac{x^{15}}{15 \cdot 7!} + \frac{x^{19}}{19 \cdot 9!} - \dots
\end{aligned}$$

Example 21.2. Estimate $\int_0^1 \sin(x^2)\,dx$ with an error < 0.001.

solution:

$$\int \sin x^2 \ dx = \frac{1}{3} - \frac{1}{7 \cdot 3!} + \frac{1}{11 \cdot 5!} - \frac{1}{15 \cdot 7!} + \frac{1}{19 \cdot 9!} - \dots$$

Because the series is alternating, we have that

$$\left| \int_0^1 \sin(x^2) \ dx - \left(\frac{1}{3} - \frac{1}{7 \cdot 3!} \right) \right| \le \frac{1}{11 \cdot 5!} \approx 0.00076 < 0.001$$

Therefore

$$\int_0^1 \sin(x^2) \ dx \approx \frac{1}{3} - \frac{1}{7 \cdot 3!} \approx 0.310.$$

Since $\frac{1}{23 \cdot 11!} < 10^{-6}$, we could also calculate

$$\int_0^1 \sin(x^2) \ dx \approx \frac{1}{3} - \frac{1}{7 \cdot 3!} + \frac{1}{11 \cdot 5!} - \frac{1}{15 \cdot 7!} + \frac{1}{19 \cdot 9!} \approx 0.310268.$$

Example 21.3. Calculate the Taylor Series (with centre 0) for $\tan^{-1} x$.

solution: We know that $\frac{d}{dx} \tan^{-1} x = \frac{1}{1+x^2} = 1 - x^2 + x^4 - x^6 + x^8 - \dots$. So

$$\begin{aligned} \tan^{-1} x &= \int \left(1 - x^2 + x^4 - x^6 + x^8 - \dots\right) \ dx \\ &= c + x - \frac{1}{3}x^3 + \frac{1}{5}x^5 - \frac{1}{7}x^7 + \frac{1}{9}x^9 - \dots \end{aligned}$$

Since $\tan 0 = 0 \implies \tan^{-1} 0 = 0 \implies c = 0$, we have

$$\tan^{-1} x = x - \frac{1}{3}x^3 + \frac{1}{5}x^5 - \frac{1}{7}x^7 + \frac{1}{9}x^9 - \dots$$

Now since $\tan \frac{\pi}{4} = 1$, we can calculate

$$\pi = 4 \tan^{-1} 1 = 4 \left(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} + \dots \right).$$

Example 21.4. Calculate $\lim\limits_{n\to 1}\left(\dfrac{\log x}{x-1}\right)$.

solution: Recall that

$$\begin{aligned}\log x = (x-1) - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3 - \frac{1}{4}(x-1)^4 + \frac{1}{5}(x-1)^5 \\ - \frac{1}{6}(x-1)^6 + \dots\end{aligned}$$

So

$$\begin{aligned}\lim_{n\to 1}\left(\frac{\log x}{x-1}\right) &= \lim_{x\to 1}\left(1 - \frac{(x-1)}{2} + \frac{(x-1)^2}{3} - \frac{(x-1)^3}{4} + \dots\right) \\ &= 1.\end{aligned}$$

Example 21.5. Calculate $\lim\limits_{n\to 0}\left(\dfrac{\sin x - \tan x}{x^3}\right)$.

solution: Since

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots$$

and

$$\tan x = x + \frac{x^3}{3} + \frac{2x^5}{15} + \frac{17x^7}{315} + \dots$$

we have that

$$\sin x - \tan x = -\frac{x^3}{2} - \frac{x^5}{8} - \dots = x^3\left(-\frac{1}{2} - \frac{x^2}{8} - \dots\right)$$

and

$$\lim_{n\to 0}\left(\frac{\sin x - \tan x}{x^3}\right) = \lim_{n\to 0}\left(-\frac{1}{2} - \frac{x^2}{8} - \dots\right) = -\frac{1}{2}.$$

Example 21.6. Calculate $\lim\limits_{n\to 0}\left(\dfrac{1}{\sin x} - \dfrac{1}{x}\right)$.

solution: Since

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots$$

we have that

$$\begin{aligned}
\frac{1}{\sin x} - \frac{1}{x} &= \frac{x - \sin x}{x \sin x} \\
&= \frac{x - \left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots\right)}{x\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots\right)} \\
&= \frac{x^3\left(\frac{1}{3!} - \frac{x^2}{5!} + \frac{x^4}{7!} - \ldots\right)}{x^2\left(1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \frac{x^6}{7!} + \ldots\right)} \\
&= \frac{x\left(\frac{1}{3!} - \frac{x^2}{5!} + \frac{x^4}{7!} - \ldots\right)}{1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \frac{x^6}{7!} + \ldots} \\
&\to 0
\end{aligned}$$

as $x \to 0$.

Moreover if $|x|$ is small,

$$\frac{1}{\sin x} - \frac{1}{x} = \frac{x\left(\frac{1}{3!} - \frac{x^2}{5!} + \frac{x^4}{7!} - \ldots\right)}{1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \frac{x^6}{7!} + \ldots} \approx \frac{x}{3!} = \frac{x}{6}.$$

It follows that

$$\operatorname{cosec} x = \frac{1}{\sin x} \approx \frac{1}{x} + \frac{x}{6}$$

for small $|x|$.

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[3] George B. Thomas, Jr., Maurice D. Weir, and Joel Hass. *Thomas' Calculus*. Addison-Wesley, Boston, twelfth edition, 2010.

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