Lecture 6

- Coordinates and Basis
- Dimension
- Change of Basis
- Row Space, Column Space, and Null Space

Coordinates and Basis

Definition

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If $S = \{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\}$ is a set of vectors in a finite-dimensional vector space V , then S is called a *basis* for V if:

1. S spans V;[1] and
2. S is linearly independent.

(The plural of basis is bases.)

[1] i.e. if $\operatorname{span} S = V$.

$$\mathbf{i} = (1,0,0) \qquad \mathbf{j} = (0,1,0) \qquad \mathbf{k} = (0,0,1)$$

Example

Recall from last week that the standard unit vectors

$$\mathbf{e}_1 = (1,0,0,\ldots,0),\ \mathbf{e}_2 = (0,1,0,\ldots,0),\ \ldots,\ \mathbf{e}_n = (0,0,0,\ldots,1).$$

span $\mathbb{R}^n$. These vectors are linearly independent (I only proved it for $n = 3$.)

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Example

$S = \{\mathbf{i}, \mathbf{j}, \mathbf{k}\}$ is called the *standard basis for* $\mathbb{R}^3$.

Example (The standard basis for $\mathbb{P}^n$)

Show that $S = \{1, x, x^2, \ldots, x^n\}$ is a basis for the vector space $\mathbf{P}^n$ of polynomials of degree n or less.

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Last week we showed that $\operatorname{span} S = \mathbb{P}^n$ and that S is linearly independent. Therefore S is a basis for $\mathbb{P}^n$.

Example (Another Basis for R3)

Show that the vectors $\mathbf{v}_1 = (1, 2, 1)$, $\mathbf{v}_2 = (2, 9, 0)$, and $\mathbf{v}_3 = (3, 3, 4)$ form a basis for $\mathbb{R}^3$.

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$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3 = \mathbf{0}$$

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has only the trivial solution.For 2., we must show that

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3 = \mathbf{b}$$

is consistent for every $\mathbf{b} \in \mathbb{R}^3$.

Coordinates and Basis

So we have two linear systems to consider:

$$\begin{cases} c_1 + 2c_2 + 3c_3 = 0 \\ 2c_1 + 9c_2 + 3c_3 = 0 \\ c_1 \qquad\quad + 4c_3 = 0 \end{cases} \qquad \text{and} \qquad \begin{cases} c_1 + 2c_2 + 3c_3 = b_1 \\ 2c_1 + 9c_2 + 3c_3 = b_2 \\ c_1 \qquad\quad + 4c_3 = b_3. \end{cases}$$

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These two linear systems have the same coefficient matrix

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 9 & 3 \\ 1 & 0 & 4 \end{bmatrix}.$$

If we can show that $\det(A) \neq 0$, then we can prove both things at the same time.

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If $\det(A) \neq 0$, then

$$A\mathbf{c} = \mathbf{0} \implies \mathbf{c} = A^{-1}\mathbf{0} = \mathbf{0}$$

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I leave it for you to prove that $\det(A) = -1$. Hence $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$ form a basis for $\mathbb{R}^3$.

Example (The Standard Basis for $\mathbb{R}^{m\times n} = M_{mn}$)

Show that the matrices

$$M_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix},\ M_2 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix},\ M_3 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix},\ M_4 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

form a basis for the vector space $\mathbb{R}^{2\times 2}$ of 2×2 matrices.

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Hence

$$c_1 = c_2 = c_3 = c_4 = 0$$

which proves that these four matrices are linearly independent.

Coordinates and Basis

Let $B = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ be any matrix in $\mathbb{R}^{2\times 2}$.

$$c_1M_1 + c_2M_2 + c_3M_3 + c_4M_4 = B$$
$$c_1\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + c_2\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + c_3\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + c_4\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

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Hence

$$\begin{cases} c_1 = a \\ c_2 = b \\ c_3 = c \\ c_4 = d. \end{cases}$$

This proves that these four matrices span $\mathbb{R}^{2\times 2}$.

Therefore M_1, M_2, M_3 and M_4 for a basis for $\mathbb{R}^{2\times 2}$ called the *standard basis for* $\mathbb{R}^{2\times 2} = M_{22}$.

Example

Show that the vector space $\mathbb{P}$ is infinite dimensional[2].

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Thus, there would be no way to express the polynomial x^{n+1} as a linear combination of the polynomials in S, contradicting the fact that the vectors in S span $\mathbb{P}$.

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Example

Some other infinite dimensional vector spaces are

$$\mathbb{R}^{\mathbb{N}} = \{\text{all sequences of real numbers}\}$$

$$F(-\infty,\infty) = \{\text{all functions } f:\mathbb{R}\to\mathbb{R}\}$$

$$C(-\infty,\infty) = \{\text{all continuous functions } f:\mathbb{R}\to\mathbb{R}\}$$

$$C^k(-\infty,\infty) = \left\{\begin{matrix}\text{all functions } f:\mathbb{R}\to\mathbb{R} \text{ which are} \\ \text{continuously differentiable } k \text{ times}\end{matrix}\right\}$$

$$C^\infty(-\infty,\infty) = \left\{\begin{matrix}\text{all functions } f:\mathbb{R}\to\mathbb{R} \text{ which} \\ \text{can be differentiated } \infty \text{ times}\end{matrix}\right\}$$

Coordinates Relative to a Basis

Theorem

If $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is a basis for a vector space V, then every vector $\mathbf{v}$ in V can be expressed in the form

$$\mathbf{v} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \ldots + c_n\mathbf{v}_n$$

in exactly one way.

Now we can start talking about coordinates in a general vector space V.

Coordinates and Basis

Definition

If $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is a basis for a vector space V, and

$$\mathbf{v} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \ldots + c_n\mathbf{v}_n$$

is the expression for a vector $\mathbf{v}$ in terms of the basis S, then the scalars c_1, c_2, …, c_n are called the *coordinates* of $\mathbf{v}$ relative to the basis S.

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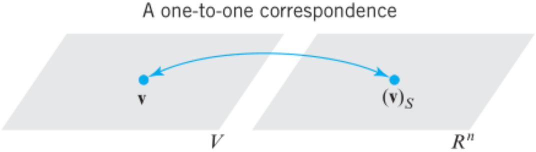
is the expression for a vector $\mathbf{v}$ in terms of the basis S, then the scalars c_1, c_2, …, c_n are called the *coordinates* of $\mathbf{v}$ relative to the basis S. The vector $(c_1, c_2, \ldots, c_n)$ in $\mathbb{R}^n$ constructed from these coordinates is called the *coordinate vector of* $\mathbf{v}$ *relative to* S; it is denoted by

$$(\mathbf{v})_S = (c_1, c_2, \ldots, c_n).$$

Remark

When we do coordinated, the order of the vectors in S is important. Some books use the term *ordered basis* which means a basis in which the order of the vectors is fixed.

Observe that $(\mathbf{v})_S$ is a vector in R^n, so that once an ordered basis S is given for a vector space V, Theorem 4.4.1 establishes a one-to-one correspondence between vectors in V and vectors in R^n (Figure 4.4.6).



$$\mathbf{i} = (1, 0, 0) \qquad \mathbf{j} = (0, 1, 0) \qquad \mathbf{k} = (0, 0, 1)$$

Example

$V = \mathbb{R}^3$. Standard basis $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$.

If $\mathbf{v} = (a, b, c)$ is any vector in $\mathbb{R}^3$, then

$$\mathbf{v} = a\mathbf{i} + b\mathbf{j} + c\mathbf{k} \qquad \Longrightarrow \qquad (\mathbf{v})_S = (a, b, c).$$

Example

Find the coordinate vector for the polynomial

$$\mathbf{p}(x) = c_0 + c_1 x + c_2 x^2 + \ldots + c_n x^n$$

relative to the standard basis for the vector space $\mathbb{P}^n$.

The standard basis for $\mathbb{P}^n$ is $S = \{1, x, x^2, \ldots, x^n\}$. So clearly

$$(\mathbf{p})_S = (c_0, c_1, c_2, \ldots, c_n).$$

Example

Find the coordinate vector of

$$B = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

relative to the standard basis for $\mathbb{R}^{2\times 2}$.

Since

$$B = \begin{bmatrix} a & b \\ c & d \end{bmatrix} = a\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + b\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + c\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + d\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

we have

$$(B)_S = (a, b, c, d).$$

Example (Coordinates in $\mathbb{R}^3$)

We showed in a previous example that the vectors

$$\mathbf{v}_1 = (1,2,1), \qquad \mathbf{v}_2 = (2,9,0), \qquad \mathbf{v}_3 = (3,3,4)$$

form a basis for $\mathbb{R}^3$.

1. Find the coordinate vector of $\mathbf{v} = (5,-1,9)$ relative to the basis $S = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$.
2. Find the vector $\mathbf{u}$ in $\mathbb{R}^3$ whose coordinate vector relative to S is $(\mathbf{u})_S = (-1,3,2)$.

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1. Find the coordinate vector of $\mathbf{v} = (5, -1, 9)$ relative to the basis $S = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$.

We need to find scalars c_1, c_2, c_3 such that

$$\mathbf{v} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3.$$

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I.e. such that

$$(5,-1,9) = c_1(1,2,1) + c_2(2,9,0) + c_3(3,3,4).$$

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So we need to solve the linear system

$$\begin{cases} c_1 + 2c_2 + 3c_3 = 5 \\ 2c_1 + 9c_2 + 3c_3 = -1 \\ c_1 \qquad\quad + 4c_3 = 9. \end{cases}$$

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I leave it to you to check that the solution is $c_1 = 1$, $c_2 = -1$ and $c_3 = 2$. Therefore

$$(\mathbf{v})_S = (1,-1,2).$$

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2. Find the vector $\mathbf{u}$ in $\mathbb{R}^3$ whose coordinate vector relative to S is $(\mathbf{u})_S = (-1,3,2)$.

This one is easier.

$$\begin{aligned}\mathbf{u} &= (-1)\mathbf{v}_1 + 3\mathbf{v}_2 + 2\mathbf{v}_3\\ &= \\ &= \end{aligned}$$

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Dimension

Theorem

All bases for a finite-dimensional vector space have the same number of vectors.

Definition

The dimension of a finite-dimensional vector space V, denoted by $\dim(V)$, is defined to be the number of vectors in a basis for V.

The dimension of the zero vector space $\{\mathbf{0}\}$ is defined to be 0.

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Engineers often use the term ***degrees of freedom*** as a synonym for dimension.

Dimension

Theorem

Let V be an n-dimensional vector space, and let $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ be any basis.

1. *If a set in V has more than n vectors, then it is linearly dependent.*
2. *If a set in V has fewer than n vectors, then it does not span V.*

Example

- $\dim(\mathbb{R}^n) = n$ because the standard basis has n vectors in it.

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- $\dim(\mathbb{P}^n) = n + 1$. Recall that the standard basis for $\mathbb{P}^n$ is $\{1, x, x^2, \ldots, x^n\}$.

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- $\dim(\mathbb{R}^n) = n$ because the standard basis has n vectors in it.
- $\dim(\mathbb{P}^n) = n+1$. Recall that the standard basis for $\mathbb{P}^n$ is $\{1, x, x^2, \ldots, x^n\}$.
- $\dim\left(\mathbb{R}^{2\times 2}\right) = 4$ because the standard basis contains the following four vectors

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}.$$

Dimension

Example

- $\dim(\mathbb{R}^n) = n$ because the standard basis has n vectors in it.

- $\dim(\mathbb{P}^n) = n + 1$. Recall that the standard basis for $\mathbb{P}^n$ is $\{1, x, x^2, \ldots, x^n\}$.

- $\dim\left(\mathbb{R}^{2\times 2}\right) = 4$ because the standard basis contains the following four vectors

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}.$$

- Similarly $\dim(\mathbb{R}^{m\times n}) = mn$.

▶ EXAMPLE 2 Dimension of Span(S)

If $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_r\}$ then every vector in span(S) is expressible as a linear combination of the vectors in S. Thus, if the vectors in S are *linearly independent*, they automatically form a basis for span(S), from which we can conclude that

$$\dim[\operatorname{span}\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_r\}] = r$$

In words, the dimension of the space spanned by a linearly independent set of vectors is equal to the number of vectors in that set.

Example (Dimension of a Solution Space)

Find a basis for and the dimension of the solution space of the homogeneous linear system

$$\begin{cases} x_1 + 3x_2 - 2x_3 + 2x_5 = 0 \\ 2x_1 + 6x_2 - 5x_3 - 2x_4 + 4x_5 - 3x_6 = 0 \\ 5x_3 + 10x_4 + 15x_6 = 0 \\ 2x_1 + 6x_2 + 8x_4 + 4x_5 + 18x_6 = 0. \end{cases}$$

I leave it to you to check that the solution is

$$\begin{cases} x_1 = -3x_2 - 4x_4 - 2x_5 \\ x_2 \text{ is free} \\ x_3 = -2x_4 \\ x_4 \text{ is free} \\ x_5 \text{ is free} \\ x_6 = 0. \end{cases}$$

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We can write this as

$$\begin{aligned} (x_1, x_2, x_3, x_4, x_5, x_6) &= (-3x_2 - 4x_4 - 2x_5, x_2, -2x_4, x_4, x_5, 0) \\ &= x_2(-3, 1, 0, 0, 0, 0) + x_4(-4, 0, -2, 1, 0, 0) \\ &\qquad + x_5(-2, 0, 0, 0, 1, 0). \end{aligned}$$

This shows that the vectors

$$\mathbf{v}_1 = (-3, 1, 0, 0, 0, 0),\ \mathbf{v}_2 = (-4, 0, -2, 1, 0, 0),\ \mathbf{v}_3 = (-2, 0, 0, 0, 1, 0)$$

span the solution space.

This shows that the vectors

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span the solution space. I leave it to you to check that these vectors are linearly independent.

Therefore the solution space has dimension 3.

Some More Theorems

Theorem (The Basis Theorem[3])

Let V be an n-dimensional vector space, and let S be a set in V with exactly n vectors. Then the following are equivalent:

1. *S is a basis for V;*
2. *S spans V;*
3. *S is linearly independent.*

[3] page 245 in your textbook

▶ EXAMPLE 5 Bases by Inspection

(a) Explain why the vectors $\mathbf{v}_1 = (-3, 7)$ and $\mathbf{v}_2 = (5, 5)$ form a basis for R^2.

(b) Explain why the vectors $\mathbf{v}_1 = (2, 0, -1)$, $\mathbf{v}_2 = (4, 0, 7)$, and $\mathbf{v}_3 = (-1, 1, 4)$ form a basis for R^3.

Solution (a) Since neither vector is a scalar multiple of the other, the two vectors form a linearly independent set in the two-dimensional space R^2, and hence they form a basis by Theorem 4.5.4.

Solution (b) The vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ form a linearly independent set in the xz-plane (why?). The vector $\mathbf{v}_3$ is outside of the xz-plane, so the set $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is also linearly independent. Since R^3 is three-dimensional, Theorem 4.5.4 implies that $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is a basis for the vector space R^3. ◀

Theorem

Let S be a finite set of vectors in a finite-dimensional vector space V.

1. *If S spans V but is not a basis for V, then S can be reduced to a basis for V by removing appropriate vectors from S.*
2. *If S is a linearly independent set that is not already a basis for V, then S can be enlarged to a basis for V by inserting appropriate vectors into S.*

Theorem

If W is a subspace of a finite-dimensional vector space V, then:

1. *W is finite-dimensional;*
2. $\dim(W) \leq \dim(V)$; *and*
3. *$W = V$ if and only if* $\dim(W) = \dim(V)$.

Change of Basis

Consider the vector space $V = \mathbb{R}^2$. Suppose that $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2\}$ and $\mathcal{C} = \{\mathbf{c}_1, \mathbf{c}_2\}$ are two bases for V.

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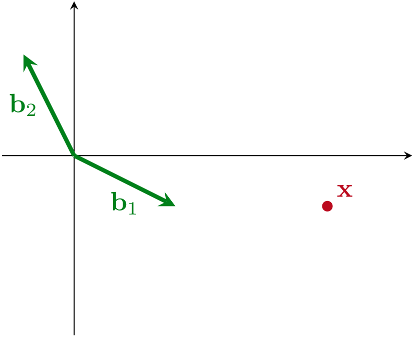

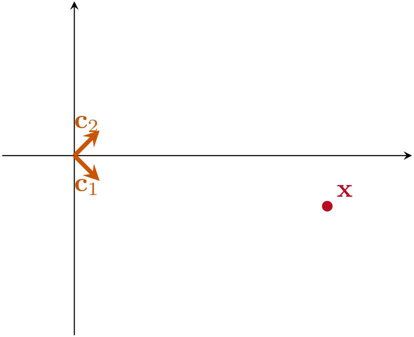

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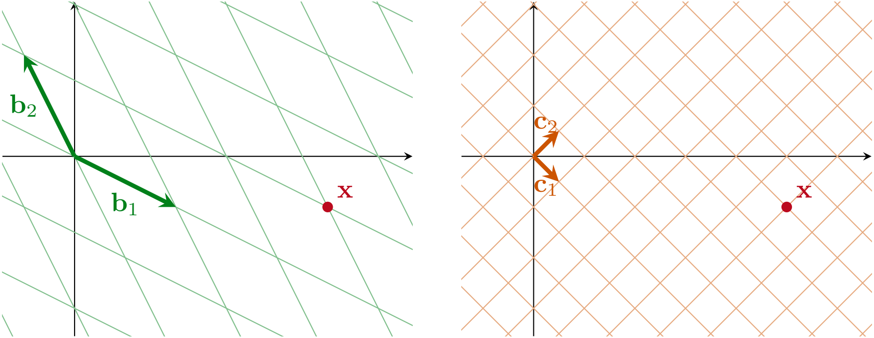

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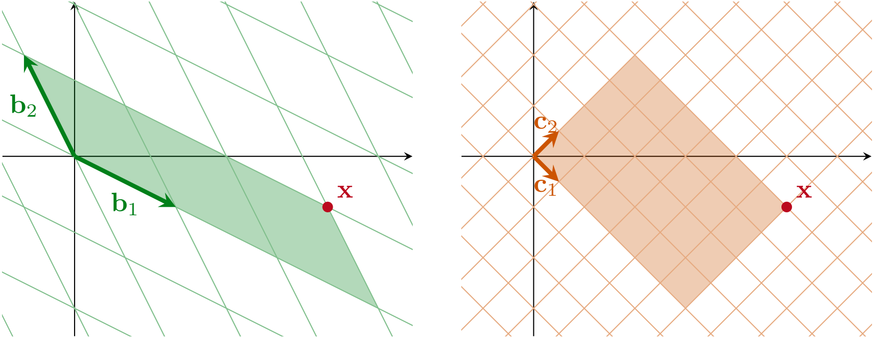

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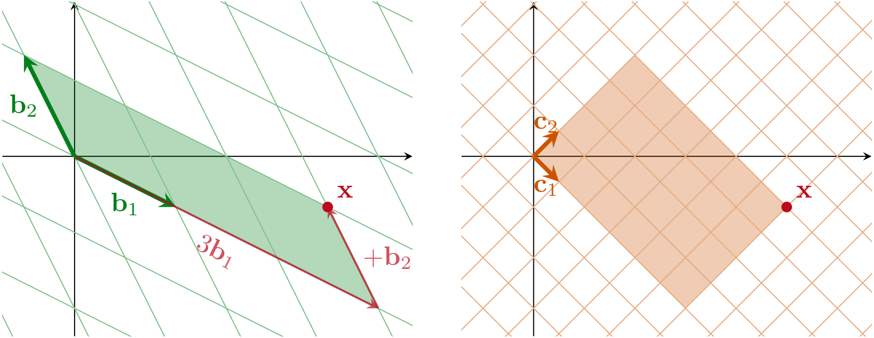
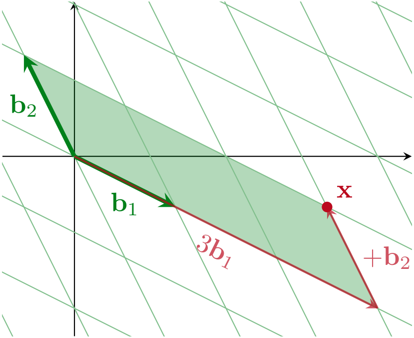
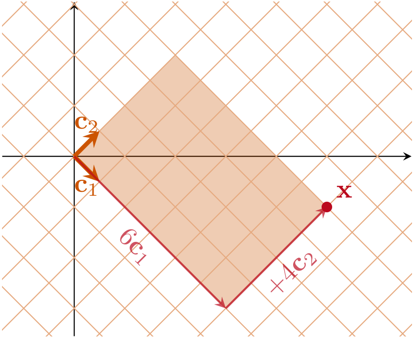


$$\mathbf{x} = 3\mathbf{b}_1 + \mathbf{b}_2$$

$$(\mathbf{x})_{\mathcal{B}} = (3, 1) \qquad [\mathbf{x}]_{\mathcal{B}} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

Consider the vector space $V = \mathbb{R}^2$. Suppose that $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2\}$ and $\mathcal{C} = \{\mathbf{c}_1, \mathbf{c}_2\}$ are two bases for V.





$$\mathbf{x} = 3\mathbf{b}_1 + \mathbf{b}_2$$

$$(\mathbf{x})_{\mathcal{B}} = (3, 1) \qquad [\mathbf{x}]_{\mathcal{B}} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

$$\mathbf{x} = 6\mathbf{c}_1 + 4\mathbf{c}_2$$

$$(\mathbf{x})_{\mathcal{C}} = (6, 4) \qquad [\mathbf{x}]_{\mathcal{C}} = \begin{bmatrix} 6 \\ 4 \end{bmatrix}$$

$$[\mathbf{x}]_{\mathcal{B}} = \begin{bmatrix} 3 \\ 1 \end{bmatrix} \xrightarrow{\text{HOW?}} [\mathbf{x}]_{\mathcal{C}} = \begin{bmatrix} 6 \\ 4 \end{bmatrix}$$

$$[\mathbf{x}]_{\mathcal{B}} = \begin{bmatrix} 3 \\ 1 \end{bmatrix} \xrightarrow{\text{HOW?}} [\mathbf{x}]_{\mathcal{C}} = \begin{bmatrix} 6 \\ 4 \end{bmatrix}$$

Example

Suppose that $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2\}$ and $\mathcal{C} = \{\mathbf{c}_1, \mathbf{c}_2\}$ are two bases for a vector space V. Suppose you know that

$$\mathbf{b}_1 = 4\mathbf{c}_1 + \mathbf{c}_2, \qquad \mathbf{b}_2 = -6\mathbf{c}_1 + \mathbf{c}_2$$

and that the vector $\mathbf{x}$ has coordinates $[\mathbf{x}]_{\mathcal{B}} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$. Find $[\mathbf{x}]_{\mathcal{C}}$.

Note that

$$[\mathbf{x}]_{\mathcal{B}} = \begin{bmatrix} 3 \\ 1 \end{bmatrix} \qquad \Longrightarrow \qquad \mathbf{x} = 3\mathbf{b}_1 + \mathbf{b}_2.$$

Hence

$$[\mathbf{x}]_{\mathcal{C}} = [3\mathbf{b}_1 + \mathbf{b}_2]_{\mathcal{C}} = 3\,[\mathbf{b}_1]_{\mathcal{C}} + [\mathbf{b}_2]_{\mathcal{C}}$$

$$=$$

Note that

$$[\mathbf{x}]_{\mathcal{B}} = \begin{bmatrix} 3 \\ 1 \end{bmatrix} \qquad \Longrightarrow \qquad \mathbf{x} = 3\mathbf{b}_1 + \mathbf{b}_2.$$

Hence

$$\begin{aligned} [\mathbf{x}]_{\mathcal{C}} &= [3\mathbf{b}_1 + \mathbf{b}_2]_{\mathcal{C}} = 3\,[\mathbf{b}_1]_{\mathcal{C}} + [\mathbf{b}_2]_{\mathcal{C}} \\ &= \begin{bmatrix} [\mathbf{b}_1]_{\mathcal{C}} & [\mathbf{b}_2]_{\mathcal{C}} \end{bmatrix} \begin{bmatrix} 3 \\ 1 \end{bmatrix}. \end{aligned}$$

Note that

$$[\mathbf{x}]_{\mathcal{B}} = \begin{bmatrix} 3 \\ 1 \end{bmatrix} \qquad \Longrightarrow \qquad \mathbf{x} = 3\mathbf{b}_1 + \mathbf{b}_2.$$

Hence

$$\begin{aligned} [\mathbf{x}]_{\mathcal{C}} &= [3\mathbf{b}_1 + \mathbf{b}_2]_{\mathcal{C}} = 3\,[\mathbf{b}_1]_{\mathcal{C}} + [\mathbf{b}_2]_{\mathcal{C}} \\ &= \begin{bmatrix} [\mathbf{b}_1]_{\mathcal{C}} & [\mathbf{b}_2]_{\mathcal{C}} \end{bmatrix} \begin{bmatrix} 3 \\ 1 \end{bmatrix}. \end{aligned}$$

So what are $[\mathbf{b}_1]_{\mathcal{C}}$ and $[\mathbf{b}_2]_{\mathcal{C}}$?

The question told us $\mathbf{b}_1 = 4\mathbf{c}_1 + \mathbf{c}_2$ and $\mathbf{b}_2 = -6\mathbf{c}_1 + \mathbf{c}_2$. Hence

$$[\mathbf{b}_1]_{\mathcal{C}} = \begin{bmatrix} 4 \\ 1 \end{bmatrix} \qquad \text{and} \qquad [\mathbf{b}_2]_{\mathcal{C}} = \begin{bmatrix} -6 \\ 1 \end{bmatrix}.$$

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Therefore

$$\begin{aligned} [\mathbf{x}]_{\mathcal{C}} &= [3\mathbf{b}_1 + \mathbf{b}_2]_{\mathcal{C}} = 3\,[\mathbf{b}_1]_{\mathcal{C}} + [\mathbf{b}_2]_{\mathcal{C}} \\ &= \begin{bmatrix} [\mathbf{b}_1]_{\mathcal{C}} & [\mathbf{b}_2]_{\mathcal{C}} \end{bmatrix} \begin{bmatrix} 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 & -6 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 6 \\ 4 \end{bmatrix}. \end{aligned}$$

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Remark

We did

$$[\mathbf{x}]_{\mathcal{C}} = \begin{bmatrix} 4 & -6 \\ 1 & 1 \end{bmatrix} [\mathbf{x}]_{\mathcal{B}}.$$

Theorem

Let $\mathcal{B} = \{\mathbf{b}_1, \ldots, \mathbf{b}_n\}$ and $\mathcal{C} = \{\mathbf{c}_1, \ldots, \mathbf{c}_n\}$ be bases of a vector space V. Then there is a unique $n \times n$ matrix $\underset{\mathcal{C}\leftarrow\mathcal{B}}{P}$ such that

$$[\mathbf{x}]_{\mathcal{C}} = \underset{\mathcal{C}\leftarrow\mathcal{B}}{P}\,[\mathbf{x}]_{\mathcal{B}}\,.$$

The columns of $\underset{\mathcal{C}\leftarrow\mathcal{B}}{P}$ are the $\mathcal{C}$-coordinate vectors of the vectors in the basis $\mathcal{B}$. That is,

$$\underset{\mathcal{C}\leftarrow\mathcal{B}}{P} = \begin{bmatrix} [\mathbf{b}_1]_{\mathcal{C}} & [\mathbf{b}_2]_{\mathcal{C}} & \cdots & [\mathbf{b}_n]_{\mathcal{C}} \end{bmatrix}.$$

Remark

Some books write $\underset{\mathcal{C}\leftarrow\mathcal{B}}{P}$, other books write $\underset{\mathcal{B}\rightarrow\mathcal{C}}{P}$.

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Definition

The matrix $\underset{\mathcal{C}\leftarrow\mathcal{B}}{P}$ is called the *change-of-coordinates matrix from* $\mathcal{B}$ *to* $\mathcal{C}$ or the *transition matrix from* $\mathcal{B}$ *to* $\mathcal{C}$.

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Theorem

The change-of-coordinates matrix $\underset{\mathcal{C}\leftarrow\mathcal{B}}{P}$ *is invertible and*

$$\left(\underset{\mathcal{C}\leftarrow\mathcal{B}}{P}\right)^{-1} = \underset{\mathcal{B}\leftarrow\mathcal{C}}{P}.$$

Break

We will continue at 3pm

Example

Consider $V = \mathbb{R}^2$ with the bases $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2\}$ and $\mathcal{C} = \{\mathbf{c}_1, \mathbf{c}_2\}$, where

$$\mathbf{b}_1 = \begin{bmatrix} -9 \\ 1 \end{bmatrix}, \quad \mathbf{b}_2 = \begin{bmatrix} -5 \\ -1 \end{bmatrix}, \quad \mathbf{c}_1 = \begin{bmatrix} 1 \\ -4 \end{bmatrix} \text{ and } \mathbf{c}_2 = \begin{bmatrix} 3 \\ -5 \end{bmatrix}.$$

Find the change-of-coordinates matrix from $\mathcal{B}$ to $\mathcal{C}$.

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Find the change-of-coordinates matrix from $\mathcal{B}$ to $\mathcal{C}$.

Theorem

$$\begin{bmatrix} \mathbf{c}_1 & \mathbf{c}_2 & \mathbf{b}_1 & \mathbf{b}_2 \end{bmatrix} \sim \begin{bmatrix} I & \underset{\mathcal{C}\leftarrow\mathcal{B}}{P} \end{bmatrix}.$$

Change of Basis

$$\begin{bmatrix} \mathbf{c}_1 & \mathbf{c}_2 & \mathbf{b}_1 & \mathbf{b}_2 \end{bmatrix} = \begin{bmatrix} 1 & 3 & -9 & -5 \\ -4 & -5 & 1 & -1 \end{bmatrix}$$

$$\begin{bmatrix} \mathbf{c}_1 & \mathbf{c}_2 & \mathbf{b}_1 & \mathbf{b}_2 \end{bmatrix} = \begin{bmatrix} 1 & 3 & -9 & -5 \\ -4 & -5 & 1 & -1 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 3 & -9 & -5 \\ 0 & 7 & -35 & -21 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 & -9 & -5 \\ 0 & 1 & -5 & -3 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 6 & 4 \\ 0 & 1 & -5 & -3 \end{bmatrix} = \begin{bmatrix} I & \underset{\mathcal{C}\leftarrow\mathcal{B}}{P} \end{bmatrix}.$$

$$\begin{bmatrix} \mathbf{c}_1 & \mathbf{c}_2 & \mathbf{b}_1 & \mathbf{b}_2 \end{bmatrix} = \begin{bmatrix} 1 & 3 & -9 & -5 \\ -4 & -5 & 1 & -1 \end{bmatrix}$$

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$$\sim \begin{bmatrix} 1 & 0 & 6 & 4 \\ 0 & 1 & -5 & -3 \end{bmatrix} = \begin{bmatrix} I & \underset{\mathcal{C}\leftarrow\mathcal{B}}{P} \end{bmatrix}.$$

Therefore

$$\underset{\mathcal{C}\leftarrow\mathcal{B}}{P} = \begin{bmatrix} 6 & 4 \\ -5 & -3 \end{bmatrix}.$$

Example

Let $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2\}$ and $\mathcal{C} = \{\mathbf{c}_1, \mathbf{c}_2\}$ be two bases for $V = \mathbb{R}^2$, where

$$\mathbf{b}_1 = \begin{bmatrix} 1 \\ 3 \end{bmatrix}, \quad \mathbf{b}_2 = \begin{bmatrix} -2 \\ 4 \end{bmatrix}, \quad \mathbf{c}_1 = \begin{bmatrix} -7 \\ 9 \end{bmatrix} \text{ and } \mathbf{c}_2 = \begin{bmatrix} -5 \\ 7 \end{bmatrix}.$$

1. Find the change-of-coordinates matrix from $\mathcal{C}$ to $\mathcal{B}$.
2. Find the change-of-coordinates matrix from $\mathcal{B}$ to $\mathcal{C}$.

$$\mathbf{b}_1 = \begin{bmatrix} 1 \\ 3 \end{bmatrix}, \quad \mathbf{b}_2 = \begin{bmatrix} -2 \\ 4 \end{bmatrix}, \quad \mathbf{c}_1 = \begin{bmatrix} -7 \\ 9 \end{bmatrix}, \quad \mathbf{c}_2 = \begin{bmatrix} -5 \\ 7 \end{bmatrix}.$$

1. Note that we need to find $\underset{\mathcal{B}\leftarrow\mathcal{C}}{P}$ this time.

$$\mathbf{b}_1 = \begin{bmatrix} 1 \\ 3 \end{bmatrix}, \quad \mathbf{b}_2 = \begin{bmatrix} -2 \\ 4 \end{bmatrix}, \quad \mathbf{c}_1 = \begin{bmatrix} -7 \\ 9 \end{bmatrix}, \quad \mathbf{c}_2 = \begin{bmatrix} -5 \\ 7 \end{bmatrix}.$$

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we have that

$$\underset{\mathcal{B}\leftarrow\mathcal{C}}{P} = \begin{bmatrix} 5 & 3 \\ 6 & 4 \end{bmatrix}.$$

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2. We calculate that

$$\underset{\mathcal{C}\leftarrow\mathcal{B}}{P} = \left(\underset{\mathcal{B}\leftarrow\mathcal{C}}{P}\right)^{-1} =$$

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$$\underset{\mathcal{B}\leftarrow\mathcal{C}}{P} = \begin{bmatrix} 5 & 3 \\ 6 & 4 \end{bmatrix}.$$

2. We calculate that

$$\underset{\mathcal{C}\leftarrow\mathcal{B}}{P} = \left(\underset{\mathcal{B}\leftarrow\mathcal{C}}{P}\right)^{-1} = \begin{bmatrix} 5 & 3 \\ 6 & 4 \end{bmatrix}^{-1} = \frac{1}{2}\begin{bmatrix} 4 & -3 \\ -6 & 5 \end{bmatrix} = \begin{bmatrix} 2 & -\frac{3}{2} \\ -3 & \frac{5}{2} \end{bmatrix}.$$

3. $\boxed{\text{25 points}}$ Let $\mathcal{A} = \{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3\}$ and $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3\}$ be bases for the vector space V, and suppose $\mathbf{a}_1 = 4\mathbf{b}_1 - \mathbf{b}_2$, $\mathbf{a}_2 = -\mathbf{b}_1 + \mathbf{b}_2 + \mathbf{b}_3$, and $\mathbf{a}_3 = \mathbf{b}_2 - 2\mathbf{b}_3$.

(a) Find the change of coordinates matrix from $\mathcal{A}$ to $\mathcal{B}$.

Solution:

$$[\mathbf{a}_1]_{\mathcal{B}} = \begin{bmatrix} 4 \\ -1 \\ 0 \end{bmatrix}, \quad [\mathbf{a}_2]_{\mathcal{B}} = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}, \quad [\mathbf{a}_3]_{\mathcal{B}} = \begin{bmatrix} 0 \\ 1 \\ -2 \end{bmatrix} \Rightarrow P = \begin{bmatrix} 4 & -1 & 0 \\ -1 & 1 & 1 \\ 0 & 1 & -2 \end{bmatrix}$$

(b) Find $[\mathbf{x}]_{\mathcal{B}}$ for $\mathbf{x} = 3\mathbf{a}_1 + 4\mathbf{a}_2 + \mathbf{a}_3$.

Solution:

$$[\mathbf{x}]_{\mathcal{A}} = \begin{bmatrix} 3 \\ 4 \\ 1 \end{bmatrix} \Rightarrow [\mathbf{x}]_{\mathcal{B}} = P[\mathbf{x}]_{\mathcal{A}} = \begin{bmatrix} 4 & -1 & 0 \\ -1 & 1 & 1 \\ 0 & 1 & -2 \end{bmatrix} \begin{bmatrix} 3 \\ 4 \\ 1 \end{bmatrix} = \begin{bmatrix} 8 \\ 2 \\ 2 \end{bmatrix}$$

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5 December 2018 [16:00-17:10] MATH215, Second Exam

3. Let $V = \left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ -3 \\ -3 \end{bmatrix}, \begin{bmatrix} 1 \\ -3 \\ -5 \end{bmatrix} \right\}$ and $W = \left\{ \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ 6 \end{bmatrix} \right\}$ be two bases for $\mathbb{R}^3$.

   (a) $\boxed{\text{10 points}}$ Find the coordinates of $\mathbf{v} = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$ relative to the basis V.

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3. Let $V = \left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ -3 \\ -3 \end{bmatrix}, \begin{bmatrix} 1 \\ -3 \\ -5 \end{bmatrix} \right\}$ and $W = \left\{ \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ 6 \end{bmatrix} \right\}$ be two bases for $\mathbb{R}^3$.

(a) $\boxed{\text{10 points}}$ Find the coordinates of $\mathbf{v} = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$ relative to the basis V.

$$\mathbf{v} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3$$

$$\begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix} = c_1 \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 2 \\ -3 \\ -3 \end{bmatrix} + c_3 \begin{bmatrix} 1 \\ -3 \\ -5 \end{bmatrix} \Rightarrow \begin{bmatrix} -1 & 2 & 1 \\ 1 & -3 & -3 \\ 0 & -3 & -5 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$$

$$\left[\begin{array}{ccc|c} -1 & 2 & 1 & 2 \\ 1 & -3 & -3 & -1 \\ 0 & -3 & -5 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & -2 & -1 & -2 \\ 0 & 1 & 2 & -1 \\ 0 & -3 & -5 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 3 & -4 \\ 0 & 1 & 2 & -1 \\ 0 & 0 & 1 & -2 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & -2 \end{array}\right]$$

$$[\mathbf{v}]_V = \begin{bmatrix} 2 \\ 3 \\ -2 \end{bmatrix}$$

(b) $\boxed{\text{10 points}}$ Find the change of coordinates matrix $\underset{W \leftarrow V}{P}$ from V to W.

(b) $\boxed{\text{10 points}}$ Find the change of coordinates matrix $\underset{W \leftarrow V}{P}$ from V to W.

Solution: We can use $\underset{W \leftarrow V}{P} = W^{-1}V$, $[W|V] \sim \left[I|\underset{W \leftarrow V}{P}\right]$ or $\underset{W \leftarrow V}{P} = \left[[\mathbf{v}_1]_W \quad [\mathbf{v}_2]_W \quad [\mathbf{v}_3]_W\right]$

$$[W|V] \sim \left[I|\underset{W \leftarrow V}{P}\right] \Rightarrow \left[\begin{array}{ccc|ccc} 1 & 1 & 2 & -1 & 2 & 1 \\ 2 & 2 & 3 & 1 & -3 & -3 \\ 4 & 3 & 6 & 0 & -3 & -5 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 1 & 2 & -1 & 2 & 1 \\ 0 & 0 & -1 & 3 & -7 & -5 \\ 0 & -1 & -2 & 4 & -11 & -9 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 1 & 2 & -1 & 2 & 1 \\ 0 & 1 & 2 & -4 & 11 & 9 \\ 0 & 0 & 1 & -3 & 7 & 5 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 3 & -9 & -8 \\ 0 & 1 & 2 & -4 & 11 & 9 \\ 0 & 0 & 1 & -3 & 7 & 5 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 3 & -9 & -8 \\ 0 & 1 & 0 & 2 & -3 & -1 \\ 0 & 0 & 1 & -3 & 7 & 5 \end{array}\right]$$

$$\underset{W \leftarrow V}{P} = \begin{bmatrix} 3 & -9 & -8 \\ 2 & -3 & -1 \\ -3 & 7 & 5 \end{bmatrix}$$

(c) $\boxed{\text{5 points}}$ Find the coordinates of **v** relative to W by using $\underset{W \leftarrow V}{P}$.

Solution:

$$[\mathbf{v}]_W = \underset{W \leftarrow V}{P}[\mathbf{v}]_V = \begin{bmatrix} 3 & -9 & -8 \\ 2 & -3 & -1 \\ -3 & 7 & 5 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \\ -2 \end{bmatrix} = \begin{bmatrix} -5 \\ -3 \\ 5 \end{bmatrix}$$

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3. Let $V = \left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ -3 \\ -3 \end{bmatrix}, \begin{bmatrix} 1 \\ -3 \\ -5 \end{bmatrix} \right\}$ and $W = \left\{ \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ 6 \end{bmatrix} \right\}$ be two bases for $\mathbb{R}^3$.

   (a) $\boxed{\text{10 points}}$ Find the coordinates of $\mathbf{v} = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$ relative to the basis V.

Remark

I think that the 2019 question (below) is easier than the 2018 question (above), but I remember that the 2019 marks were lower than the 2018 marks.

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3. $\boxed{\text{25 points}}$ Let $\mathcal{A} = \{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3\}$ and $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3\}$ be bases for the vector space V, and suppose $\mathbf{a}_1 = 4\mathbf{b}_1 - \mathbf{b}_2$, $\mathbf{a}_2 = -\mathbf{b}_1 + \mathbf{b}_2 + \mathbf{b}_3$, and $\mathbf{a}_3 = \mathbf{b}_2 - 2\mathbf{b}_3$.

   (a) Find the change of coordinates matrix from $\mathcal{A}$ to $\mathcal{B}$.

Row Space, Column Space, and Null Space

Consider an $m \times n$ matrix

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}.$$

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Definition

The *row vectors of* A are the vectors

$$\mathbf{r}_1 = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \end{bmatrix}$$
$$\mathbf{r}_2 = \begin{bmatrix} a_{21} & a_{22} & \cdots & a_{2n} \end{bmatrix}$$
$$\vdots$$
$$\mathbf{r}_m = \begin{bmatrix} a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

Note that $\mathbf{r}_k \in \mathbb{R}^n$ for each $1 \leq k \leq m$.

Row Space, Column Space, and Null Space

Consider an $m \times n$ matrix

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}.$$

Definition

The *column vectors of* A are the vectors

$$\mathbf{c}_1 = \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix}, \quad \mathbf{c}_2 = \begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{bmatrix}, \quad \ldots, \quad \mathbf{c}_m = \begin{bmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{bmatrix}$$

Note that $\mathbf{c}_k \in \mathbb{R}^m$ for each $1 \leq k \leq n$.

Example

Find the row and column vectors of

$$A = \begin{bmatrix} 2 & 1 & 0 \\ 3 & -1 & 4 \end{bmatrix}.$$

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The row vectors of A are

$$\mathbf{r}_1 = \begin{bmatrix} 2 & 1 & 1 \end{bmatrix} \quad \text{and} \quad \mathbf{r}_2 = \begin{bmatrix} 3 & -1 & 4 \end{bmatrix}.$$

The column vectors of A are

$$\mathbf{c}_1 = \begin{bmatrix} 2 \\ 3 \end{bmatrix}, \qquad \mathbf{c}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \text{ and } C_3 = \begin{bmatrix} 0 \\ 4 \end{bmatrix}.$$

Let A be an $m \times n$ matrix.

Definition

The subspace of $\mathbb{R}^n$ spanned by the row vectors of A is called the *row space* of A. This is denoted by Row A.

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Definition

Row Space, Column Space, and Null Space

Let A be an $m \times n$ matrix.

Definition

The subspace of $\mathbb{R}^n$ spanned by the row vectors of A is called the *row space* of A. This is denoted by $\operatorname{Row} A$.

Definition

The subspace of $\mathbb{R}^m$ spanned by the column vectors of A is called the *column space* of A. This is denoted by $\operatorname{Col} A$.

Definition

The *null space* of A is the subspace

$$\operatorname{Nul} A = \{\mathbf{x} \in \mathbb{R}^n \mid A\mathbf{x} = \mathbf{0}\} \subseteq \mathbb{R}^n.$$

Consider $A\mathbf{x} = \mathbf{b}$ where

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \quad \text{and} \quad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}.$$

Note that

$$\mathbf{b} = A\mathbf{x} = x_1\mathbf{c}_1 + x_2\mathbf{c}_2 + \ldots + x_n\mathbf{c}_n$$

where $\mathbf{c}_1$, $\mathbf{c}_2$, …, $\mathbf{c}_n$ are the column vectors of A.

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Note that

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where $\mathbf{c}_1$, $\mathbf{c}_2$, ..., $\mathbf{c}_n$ are the column vectors of A. So $A\mathbf{x} = \mathbf{b}$ is consistent iff $\mathbf{b}$ is a linear combination of the column vectors of A.

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where $\mathbf{c}_1$, $\mathbf{c}_2$, ..., $\mathbf{c}_n$ are the column vectors of A. So $A\mathbf{x} = \mathbf{b}$ is consistent iff $\mathbf{b}$ is a linear combination of the column vectors of A.

Theorem

$$A\mathbf{x} = \mathbf{b} \textit{ is consistent} \qquad \Longleftrightarrow \qquad \mathbf{b} \in \mathrm{Col}\, A$$

EXAMPLE 2 A Vector b in the Column Space of A

Let $A\mathbf{x} = \mathbf{b}$ be the linear system

$$\begin{bmatrix} -1 & 3 & 2 \\ 1 & 2 & -3 \\ 2 & 1 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ -9 \\ -3 \end{bmatrix}$$

Show that $\mathbf{b}$ is in the column space of A by expressing it as a linear combination of the column vectors of A.

Solution Solving the system by Gaussian elimination yields (verify)

$$x_1 = 2, \quad x_2 = -1, \quad x_3 = 3$$

It follows from this and Formula (2) that

$$2\begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix} - \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} + 3\begin{bmatrix} 2 \\ -3 \\ -2 \end{bmatrix} = \begin{bmatrix} 1 \\ -9 \\ -3 \end{bmatrix}$$ ◀

Example

Let

$$A = \begin{bmatrix} 2 & 4 & -2 & 1 \\ -2 & -5 & 7 & 3 \\ 3 & 7 & -8 & 6 \end{bmatrix}, \quad \mathbf{u} = \begin{bmatrix} 3 \\ -2 \\ -1 \\ 0 \end{bmatrix}, \quad \text{and} \quad \mathbf{v} = \begin{bmatrix} 3 \\ -1 \\ 3 \end{bmatrix}.$$

1. Is $\mathbf{u}$ in $\operatorname{Nul} A$?
2. Is $\mathbf{u}$ in $\operatorname{Col} A$?
3. Is $\mathbf{v}$ in $\operatorname{Nul} A$?
4. Is $\mathbf{v}$ in $\operatorname{Col} A$?

$$A = \begin{bmatrix} 2 & 4 & -2 & 1 \\ -2 & -5 & 7 & 3 \\ 3 & 7 & -8 & 6 \end{bmatrix}, \quad \mathbf{u} = \begin{bmatrix} 3 \\ -2 \\ -1 \\ 0 \end{bmatrix}, \quad \text{and} \quad \mathbf{v} = \begin{bmatrix} 3 \\ -1 \\ 3 \end{bmatrix}.$$

1. Is $\mathbf{u}$ in $\operatorname{Nul} A$?

Recall that

$$\operatorname{Nul} A = \{\mathbf{x} \in \mathbb{R}^4 \mid A\mathbf{x} = \mathbf{0}\} \subseteq \mathbb{R}^4$$

since A is a 3×4 matrix. So $\mathbf{u} \in \operatorname{Nul} A$ if and only if $A\mathbf{u} = \mathbf{0}$.

$$A = \begin{bmatrix} 2 & 4 & -2 & 1 \\ -2 & -5 & 7 & 3 \\ 3 & 7 & -8 & 6 \end{bmatrix}, \quad \mathbf{u} = \begin{bmatrix} 3 \\ -2 \\ -1 \\ 0 \end{bmatrix}, \quad \text{and} \quad \mathbf{v} = \begin{bmatrix} 3 \\ -1 \\ 3 \end{bmatrix}.$$

1. Is $\mathbf{u}$ in $\operatorname{Nul} A$?

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since A is a 3×4 matrix. So $\mathbf{u} \in \operatorname{Nul} A$ if and only if $A\mathbf{u} = \mathbf{0}$.

We calculate that

$$A\mathbf{u} = \begin{bmatrix} 2 & 4 & -2 & 1 \\ -2 & -5 & 7 & 3 \\ 3 & 7 & -8 & 6 \end{bmatrix} \begin{bmatrix} 3 \\ -2 \\ -1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

Therefore the answer is YES.

$$A = \begin{bmatrix} 2 & 4 & -2 & 1 \\ -2 & -5 & 7 & 3 \\ 3 & 7 & -8 & 6 \end{bmatrix}, \quad \mathbf{u} = \begin{bmatrix} 3 \\ -2 \\ -1 \\ 0 \end{bmatrix}, \quad \text{and} \quad \mathbf{v} = \begin{bmatrix} 3 \\ -1 \\ 3 \end{bmatrix}.$$

2. Is $\mathbf{u}$ in $\operatorname{Col} A$?

Since A is a 3×4 matrix, $\operatorname{Col} A$ is a subspace of $\mathbb{R}^3$.

$$A = \begin{bmatrix} 2 & 4 & -2 & 1 \\ -2 & -5 & 7 & 3 \\ 3 & 7 & -8 & 6 \end{bmatrix}, \quad \mathbf{u} = \begin{bmatrix} 3 \\ -2 \\ -1 \\ 0 \end{bmatrix}, \quad \text{and} \quad \mathbf{v} = \begin{bmatrix} 3 \\ -1 \\ 3 \end{bmatrix}.$$

2 Is $\mathbf{u}$ in $\mathrm{Col}\,A$?

Since A is a 3×4 matrix, $\mathrm{Col}\,A$ is a subspace of $\mathbb{R}^3$. But $\mathbf{u} \in \mathbb{R}^4$. Therefore the answer is NO.

$$A = \begin{bmatrix} 2 & 4 & -2 & 1 \\ -2 & -5 & 7 & 3 \\ 3 & 7 & -8 & 6 \end{bmatrix}, \quad \mathbf{u} = \begin{bmatrix} 3 \\ -2 \\ -1 \\ 0 \end{bmatrix}, \quad \text{and} \quad \mathbf{v} = \begin{bmatrix} 3 \\ -1 \\ 3 \end{bmatrix}.$$

3. Is $\mathbf{v}$ in $\operatorname{Nul} A$?

NO, because $\mathbf{v} \in \mathbb{R}^3$, but $\operatorname{Nul} A \subseteq \mathbb{R}^4$.

$$A = \begin{bmatrix} 2 & 4 & -2 & 1 \\ -2 & -5 & 7 & 3 \\ 3 & 7 & -8 & 6 \end{bmatrix}, \quad \mathbf{u} = \begin{bmatrix} 3 \\ -2 \\ -1 \\ 0 \end{bmatrix}, \quad \text{and} \quad \mathbf{v} = \begin{bmatrix} 3 \\ -1 \\ 3 \end{bmatrix}.$$

4. Is $\mathbf{v}$ in $\operatorname{Col} A$?

To answer this, we need to reduce $\begin{bmatrix} A & \mathbf{v} \end{bmatrix}$ to REF.

$$A = \begin{bmatrix} 2 & 4 & -2 & 1 \\ -2 & -5 & 7 & 3 \\ 3 & 7 & -8 & 6 \end{bmatrix}, \quad \mathbf{u} = \begin{bmatrix} 3 \\ -2 \\ -1 \\ 0 \end{bmatrix}, \quad \text{and} \quad \mathbf{v} = \begin{bmatrix} 3 \\ -1 \\ 3 \end{bmatrix}.$$

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$$A = \begin{bmatrix} 2 & 4 & -2 & 1 \\ -2 & -5 & 7 & 3 \\ 3 & 7 & -8 & 6 \end{bmatrix}, \quad \mathbf{u} = \begin{bmatrix} 3 \\ -2 \\ -1 \\ 0 \end{bmatrix}, \quad \text{and} \quad \mathbf{v} = \begin{bmatrix} 3 \\ -1 \\ 3 \end{bmatrix}.$$

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$$\begin{bmatrix} A & \mathbf{v} \end{bmatrix} = \begin{bmatrix} 2 & 4 & -2 & 1 & 3 \\ -2 & -5 & 7 & 3 & -1 \\ 3 & 7 & -8 & 6 & 3 \end{bmatrix} \sim \begin{bmatrix} 2 & 4 & -2 & 1 & 3 \\ 0 & 1 & -5 & -4 & -2 \\ 0 & 0 & 0 & 17 & 1 \end{bmatrix}$$

$$A = \begin{bmatrix} 2 & 4 & -2 & 1 \\ -2 & -5 & 7 & 3 \\ 3 & 7 & -8 & 6 \end{bmatrix}, \quad \mathbf{u} = \begin{bmatrix} 3 \\ -2 \\ -1 \\ 0 \end{bmatrix}, \quad \text{and} \quad \mathbf{v} = \begin{bmatrix} 3 \\ -1 \\ 3 \end{bmatrix}.$$

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$$\begin{bmatrix} A & \mathbf{v} \end{bmatrix} = \begin{bmatrix} 2 & 4 & -2 & 1 & 3 \\ -2 & -5 & 7 & 3 & -1 \\ 3 & 7 & -8 & 6 & 3 \end{bmatrix} \sim \begin{bmatrix} 2 & 4 & -2 & 1 & 3 \\ 0 & 1 & -5 & -4 & -2 \\ 0 & 0 & 0 & 17 & 1 \end{bmatrix}$$

We can now see that the linear system $A\mathbf{x} = \mathbf{v}$ is consistent. Therefore the answer to this question is YES.

Theorem

Suppose that

- $A\mathbf{x} = \mathbf{b}$ *is consistent;*
- $\mathbf{x}_0$ *is any solution of* $A\mathbf{x} = \mathbf{b}$*;*
- $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$ *is a basis for* $\operatorname{Nul} A$.

Theorem

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- $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$ *is a basis for* $\operatorname{Nul} A$*.*

Then every solution of $A\mathbf{x} = \mathbf{b}$ *can be written in the form*

$$\mathbf{x} = \mathbf{x}_0 + c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \ldots + c_k\mathbf{v}_k.$$

Theorem

Suppose that

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Then every solution of $A\mathbf{x} = \mathbf{b}$ *can be written in the form*

$$\mathbf{x} = \mathbf{x}_0 + c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \ldots + c_k\mathbf{v}_k.$$

Conversely, the vector $\mathbf{x}$ *in this formula is a solution of* $A\mathbf{x} = \mathbf{b}$ *for any choice of scalars* $c_j \in \mathbb{R}$*.*

$$\mathbf{x} = \mathbf{x}_0 + c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \ldots + c_k\mathbf{v}_k$$

$$\mathbf{x} = \underbrace{\mathbf{x}_0}_{\substack{\text{a particular} \\ \text{solution of} \\ A\mathbf{x} = \mathbf{b}}} + c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \ldots + c_k\mathbf{v}_k$$

$$\mathbf{x} = \underbrace{\mathbf{x}_0}_{\substack{\text{a particular} \\ \text{solution of} \\ A\mathbf{x}=\mathbf{b}}} + \underbrace{c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \ldots + c_k\mathbf{v}_k}_{\text{the general solution of } A\mathbf{x}=\mathbf{0}}$$

$$\mathbf{x} = \overbrace{\underbrace{\mathbf{x}_0}_{\substack{\text{a particular}\\ \text{solution of}\\ A\mathbf{x}=\mathbf{b}}} + \underbrace{c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \ldots + c_k\mathbf{v}_k}_{\text{the general solution of } A\mathbf{x}=\mathbf{0}}}^{\text{the general solution of } A\mathbf{x}=\mathbf{b}}$$

Bases for Row Spaces, Column Spaces, and Null Spaces

Theorem

Elementary row operations do not change the null space of a matrix.

Theorem

Elementary row operations do not change the row space of a matrix.

Remark

Elementary row operations <u>can</u> change the column space of a matrix.

Remark

Elementary row operations can change the column space of a matrix.

For example, consider

$$A = \begin{bmatrix} 1 & 3 \\ 2 & 6 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 & 3 \\ 0 & 0 \end{bmatrix}.$$

Note that A and B are row equivalent,

Remark

Elementary row operations can change the column space of a matrix.

For example, consider

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Note that A and B are row equivalent, but

$$\operatorname{Col} A = \operatorname{span}\left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right\}$$

Remark

Elementary row operations can change the column space of a matrix.

For example, consider

$$A = \begin{bmatrix} 1 & 3 \\ 2 & 6 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 & 3 \\ 0 & 0 \end{bmatrix}.$$

Note that A and B are row equivalent, but

$$\operatorname{Col} A = \operatorname{span}\left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right\}$$

and

$$\operatorname{Col} B = \operatorname{span}\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\} \neq \operatorname{Col} A.$$

Row Space, Column Space, and Null Space

Example

Find a basis the null space of the matrix

$$A = \begin{bmatrix} 1 & 3 & -2 & 0 & 2 & 0 \\ 2 & 6 & -5 & -2 & 4 & -3 \\ 0 & 0 & 5 & 10 & 0 & 15 \\ 2 & 6 & 0 & 8 & 4 & 18 \end{bmatrix}.$$

Example

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We need to find the solution space of $A\mathbf{x} = \mathbf{0}$. I leave it to you to check that

$$\left[\begin{array}{cccccc|c} 1 & 3 & -2 & 0 & 2 & 0 & 0 \\ 2 & 6 & -5 & -2 & 4 & -3 & 0 \\ 0 & 0 & 5 & 10 & 0 & 15 & 0 \\ 2 & 6 & 0 & 8 & 4 & 18 & 0 \end{array}\right] \sim \left[\begin{array}{cccccc|c} 1 & 3 & 0 & 4 & 2 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right].$$

Therefore the solution of $A\mathbf{x} = \mathbf{0}$ is

$$\begin{cases} x_1 = -3x_2 - 4x_4 - 2x_5 \\ x_2 \text{ is free} \\ x_3 = -2x_4 \\ x_4 \text{ is free} \\ x_5 \text{ is free} \\ x_6 = 0 \end{cases}$$

Row Space, Column Space, and Null Space

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or in vectors,

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} = \begin{bmatrix} -3x_2 - 4x_4 - 2x_5 \\ x_2 \\ -2x_4 \\ x_4 \\ x_5 \\ 0 \end{bmatrix} = x_2 \begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -4 \\ 0 \\ -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} -2 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}.$$

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Therefore

$$\left\{ \mathbf{v}_1 = \begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} -4 \\ 0 \\ -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \mathbf{v}_3 = \begin{bmatrix} -2 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \right\}$$

is a basis for $\operatorname{Nul} A$.

Remark

In the previous example we found that that

$$\begin{cases} x_1 = -3x_2 - 4x_4 - 2x_5 \\ x_2 \text{ is free} \\ x_3 = -2x_4 \\ x_4 \text{ is free} \\ x_5 \text{ is free} \\ x_6 = 0 \end{cases}$$

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Note that if we set $x_2 = 1$ and $x_4 = x_5 = 0$ then we get

$$\mathbf{v}_1 = \begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

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Setting $x_4 = 1$ and then $x_5 = 1$ (and the other two to 0) gives us the other two vectors in $\operatorname{Nul} A$.

Row Space, Column Space, and Null Space

Theorem

Suppose that a matrix R is in row echelon form. Then

- *the row vectors with the pivots (i.e. the nonzero row vectors) form a basis for the row space of R; and*
- *the column vectors with the pivots (i.e. the pivot columns) form a basis for the column space of R.*

Example

Find bases for the row and column spaces of the matrix

$$R = \begin{bmatrix} 1 & -2 & 5 & 0 & 3 \\ 0 & 1 & 3 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

This matrix is in REF, so we can use the previous theorem.

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Find bases for the row and column spaces of the matrix

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- $\{\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3\}$ is a basis for Row R since the pivots are in rows 1, 2 and 3.

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- $\{\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3\}$ is a basis for Row R since the pivots are in rows 1, 2 and 3.
- $\{\mathbf{c}_1, \mathbf{c}_2, \mathbf{c}_4\}$ is a basis for Col R since the pivots are in columns 1, 2 and 4.

Example

Find a basis for the row space of the matrix

$$A = \begin{bmatrix} 1 & -3 & 4 & -2 & 5 & 4 \\ 2 & -6 & 9 & -1 & 8 & 2 \\ 2 & -6 & 9 & -1 & 9 & 7 \\ -1 & 3 & -4 & 2 & -5 & -4 \end{bmatrix}.$$

Recall that elementary row operations do not change the row space of a matrix.

Example

Find a basis for the row space of the matrix

$$A = \begin{bmatrix} 1 & -3 & 4 & -2 & 5 & 4 \\ 2 & -6 & 9 & -1 & 8 & 2 \\ 2 & -6 & 9 & -1 & 9 & 7 \\ -1 & 3 & -4 & 2 & -5 & -4 \end{bmatrix}.$$

Recall that elementary row operations do not change the row space of a matrix. So our method is:

1. Reduce A to REF. Call this new matrix R;
2. Take the row vectors of R which contain a pivot.

As always, I leave it for you to check that

$$A = \begin{bmatrix} 1 & -3 & 4 & -2 & 5 & 4 \\ 2 & -6 & 9 & -1 & 8 & 2 \\ 2 & -6 & 9 & -1 & 9 & 7 \\ -1 & 3 & -4 & 2 & -5 & -4 \end{bmatrix} \sim R = \begin{bmatrix} 1 & -3 & 4 & -2 & 5 & 4 \\ 0 & 0 & 1 & 3 & -2 & -6 \\ 0 & 0 & 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Therefore

$$\begin{aligned} \mathbf{r}_1 &= \begin{bmatrix} 1 & -3 & 4 & -2 & 5 & 4 \end{bmatrix} \\ \mathbf{r}_2 &= \begin{bmatrix} 0 & 0 & 1 & 3 & -2 & -6 \end{bmatrix} \\ \mathbf{r}_3 &= \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 5 \end{bmatrix} \end{aligned}$$

form a basis for Row A = Row R.

Finding a Basis for $\operatorname{Col} A$

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The pivot columns of A form a basis of $\operatorname{Col} A$.

Finding a Basis for $\mathrm{Col}\,A$

Elementary row operations can change the column space of a matrix A, but they do not change *which columns* of A we want in our basis.

Theorem

The pivot columns of A form a basis of $\mathrm{Col}\,A$.

So our method is:

1. Reduce A to REF. Call this new matrix R;
2. Find the pivot columns of R;
3. Go back to looking at the original matrix A;
4. Take the pivot columns <u>of A</u>.

Example

Find a basis for the column space of the matrix

$$A = \begin{bmatrix} 1 & -3 & 4 & -2 & 5 & 4 \\ 2 & -6 & 9 & -1 & 8 & 2 \\ 2 & -6 & 9 & -1 & 9 & 7 \\ -1 & 3 & -4 & 2 & -5 & -4 \end{bmatrix}.$$

Recall that

$$A = \begin{bmatrix} 1 & -3 & 4 & -2 & 5 & 4 \\ 2 & -6 & 9 & -1 & 8 & 2 \\ 2 & -6 & 9 & -1 & 9 & 7 \\ -1 & 3 & -4 & 2 & -5 & -4 \end{bmatrix} \sim R = \begin{bmatrix} 1 & -3 & 4 & -2 & 5 & 4 \\ 0 & 0 & 1 & 3 & -2 & -6 \\ 0 & 0 & 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Look at R. The pivot columns of R are the first, third and fifth columns.

Row Space, Column Space, and Null Space

Recall that

$$A = \begin{bmatrix} 1 & -3 & 4 & -2 & 5 & 4 \\ 2 & -6 & 9 & -1 & 8 & 2 \\ 2 & -6 & 9 & -1 & 9 & 7 \\ -1 & 3 & -4 & 2 & -5 & -4 \end{bmatrix} \sim R = \begin{bmatrix} 1 & -3 & 4 & -2 & 5 & 4 \\ 0 & 0 & 1 & 3 & -2 & -6 \\ 0 & 0 & 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Look at R. The pivot columns of R are the first, third and fifth columns. Therefore the pivot columns of A are also the first, third and fifth columns.

Row Space, Column Space, and Null Space

Recall that

$$A = \begin{bmatrix} 1 & -3 & 4 & -2 & 5 & 4 \\ 2 & -6 & 9 & -1 & 8 & 2 \\ 2 & -6 & 9 & -1 & 9 & 7 \\ -1 & 3 & -4 & 2 & -5 & -4 \end{bmatrix} \sim R = \begin{bmatrix} 1 & -3 & 4 & -2 & 5 & 4 \\ 0 & 0 & 1 & 3 & -2 & -6 \\ 0 & 0 & 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Look at R. The pivot columns of R are the first, third and fifth columns. Therefore the pivot columns of A are also the first, third and fifth columns. Hence

$$\left\{ \mathbf{c}_1 = \begin{bmatrix} 1 \\ 2 \\ 2 \\ -1 \end{bmatrix}, \ \mathbf{c}_3 = \begin{bmatrix} 4 \\ 9 \\ 9 \\ -4 \end{bmatrix}, \ \mathbf{c}_5 = \begin{bmatrix} 5 \\ 8 \\ 9 \\ -5 \end{bmatrix} \right\}$$

is a basis for $\operatorname{Col} A$.

Example

Find a subset of the vectors

$$\mathbf{v}_1 = (1, -2, 0, 3), \quad \mathbf{v}_2 = (2, -5, -3, 6), \quad \mathbf{v}_3 = (0, 1, 3, 0),$$

$$\mathbf{v}_4 = (2, -1, 4, -7), \quad \mathbf{v}_5 = (5, -8, 1, 2)$$

that forms a basis for span$\{\mathbf{v}_1, \ldots, \mathbf{v}_5\}$.

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Find a subset of the vectors

$$\mathbf{v}_1 = (1, -2, 0, 3), \quad \mathbf{v}_2 = (2, -5, -3, 6), \quad \mathbf{v}_3 = (0, 1, 3, 0),$$

$$\mathbf{v}_4 = (2, -1, 4, -7), \quad \mathbf{v}_5 = (5, -8, 1, 2)$$

that forms a basis for $\operatorname{span}\{\mathbf{v}_1, \dots, \mathbf{v}_5\}$.

If we write these vectors as columns of a matrix

$$A = \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3 & \mathbf{v}_4 & \mathbf{v}_4 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 0 & 2 & 5 \\ -2 & -5 & 1 & -1 & -8 \\ 0 & -3 & 3 & 4 & 1 \\ 3 & 6 & 0 & -7 & 2 \end{bmatrix}$$

then we just need to find a basis for $\operatorname{Col} A$.

Since

$$A \sim \begin{bmatrix} 1 & 0 & 2 & 0 & 1 \\ 0 & 1 & -1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

(please check) we can see that the pivot columns are the first, second and fourth columns.

Since

$$A \sim \begin{bmatrix} 1 & 0 & 2 & 0 & 1 \\ 0 & 1 & -1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

(please check) we can see that the pivot columns are the first, second and fourth columns.

Therefore $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_4\}$ is a basis for $\text{span}\{\mathbf{v}_1, \dots, \mathbf{v}_5\}$.

Next Time

- Rank and Nullity
- The Fundamental Matrix Spaces
- Linear Transformations
- Composition and Inverse Transformations
- Isomorphisms