

Lecture 2

- Solutions of Linear Systems
- Introduction to Matrices
- The Inverse of a Matrix



Solutions of Linear Systems

Solutions of Linear Systems



Definition

Variables which correspond to pivot columns are called *basic variables*.

Other variables are called *free variables*.

Solutions of Linear Systems

Example

Consider

$$\begin{bmatrix} 1 & 0 & -5 & 1 \\ 0 & 1 & 1 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \left\{ \begin{array}{l} x_1 - 5x_3 = 1 \\ x_2 + x_3 = 4 \\ 0 = 0. \end{array} \right.$$

Solutions of Linear Systems



Example

Consider

$$\left[\begin{array}{cccc} 1 & 0 & -5 & 1 \\ 0 & 1 & 1 & 4 \\ 0 & 0 & 0 & 0 \end{array} \right] \quad \left\{ \begin{array}{lcl} x_1 & - 5x_3 & = 1 \\ x_2 + x_3 & = 4 \\ 0 & & 0 = 0. \end{array} \right.$$

Columns 1 and 2 are the pivot columns.

Therefore x_1 and x_2 are basic variables. x_3 is a free variable.

Solutions of Linear Systems



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$$\left[\begin{array}{cccc} 1 & 0 & -5 & 1 \\ 0 & 1 & 1 & 4 \\ 0 & 0 & 0 & 0 \end{array} \right] \quad \left\{ \begin{array}{rcl} x_1 & - 5x_3 & = 1 \\ x_2 + x_3 & = 4 \\ 0 & & 0 = 0. \end{array} \right.$$

Columns 1 and 2 are the pivot columns.

Therefore x_1 and x_2 are basic variables. x_3 is a free variable.

The general solution to the linear system can be written as

$$\left\{ \begin{array}{l} x_1 = 1 + 5x_3 \\ x_2 = 4 - x_3 \\ x_3 \text{ is free.} \end{array} \right.$$

Solutions of Linear Systems



$$\begin{cases} x_1 = 1 + 5x_3 \\ x_2 = 4 - x_3 \\ x_3 \text{ is free.} \end{cases}$$

The statement “ x_3 is free” means that you can choose any value for x_3 . After you choose x_3 , the values of x_1 and x_2 are determined by the formulae.

Solutions of Linear Systems



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The statement “ x_3 is free” means that you can choose any value for x_3 . After you choose x_3 , the values of x_1 and x_2 are determined by the formulae.

- e.g. if you choose $x_3 = 0$, then you get the solution $(1, 4, 0)$;
- e.g. if you choose $x_3 = 1$, then you get $(6, 3, 1)$.

Solutions of Linear Systems



Example

Find the general solution of the linear system whose augmented matrix has been reduced to

$$\begin{bmatrix} 1 & 6 & 2 & -5 & -2 & -4 \\ 0 & 0 & 2 & -8 & -1 & 3 \\ 0 & 0 & 0 & 0 & 1 & 7 \end{bmatrix}.$$

Solutions of Linear Systems



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$$\begin{bmatrix} 1 & 6 & 2 & -5 & -2 & -4 \\ 0 & 0 & 2 & -8 & -1 & 3 \\ 0 & 0 & 0 & 0 & 1 & 7 \end{bmatrix}.$$

This matrix is in REF, but we need it in RREF so we will have to do some row reduction.

Solutions of Linear Systems



$$\left[\begin{array}{cccccc} 1 & 6 & 2 & -5 & -2 & -4 \\ 0 & 0 & 2 & -8 & -1 & 3 \\ 0 & 0 & 0 & 0 & 1 & 7 \end{array} \right] \xrightarrow{\frac{R_3+R_2}{2R_3+R_1}} \left[\begin{array}{cccccc} 1 & 6 & 2 & -5 & 0 & 10 \\ 0 & 0 & 2 & -8 & 0 & 10 \\ 0 & 0 & 0 & 0 & 1 & 7 \end{array} \right]$$

Solutions of Linear Systems



$$\left[\begin{array}{cccccc} 1 & 6 & 2 & -5 & -2 & -4 \\ 0 & 0 & 2 & -8 & -1 & 3 \\ 0 & 0 & 0 & 0 & 1 & 7 \end{array} \right] \xrightarrow{\frac{R_3+R_2}{2R_3+R_1}} \left[\begin{array}{cccccc} 1 & 6 & 2 & -5 & 0 & 10 \\ 0 & 0 & 2 & -8 & 0 & 10 \\ 0 & 0 & 0 & 0 & 1 & 7 \end{array} \right]$$
$$\xrightarrow{\frac{1}{2}R_2} \left[\begin{array}{cccccc} 1 & 6 & 2 & -5 & 0 & 10 \\ 0 & 0 & 1 & -4 & 0 & 5 \\ 0 & 0 & 0 & 0 & 1 & 7 \end{array} \right]$$

Solutions of Linear Systems



$$\begin{bmatrix} 1 & 6 & 2 & -5 & -2 & -4 \\ 0 & 0 & 2 & -8 & -1 & 3 \\ 0 & 0 & 0 & 0 & 1 & 7 \end{bmatrix} \xrightarrow{\frac{R_3+R_2}{2R_3+R_1}} \begin{bmatrix} 1 & 6 & 2 & -5 & 0 & 10 \\ 0 & 0 & 2 & -8 & 0 & 10 \\ 0 & 0 & 0 & 0 & 1 & 7 \end{bmatrix}$$
$$\xrightarrow{\frac{1}{2}R_2} \begin{bmatrix} 1 & 6 & 2 & -5 & 0 & 10 \\ 0 & 0 & 1 & -4 & 0 & 5 \\ 0 & 0 & 0 & 0 & 1 & 7 \end{bmatrix} \xrightarrow{-2R_2+R_1} \begin{bmatrix} 1 & 6 & 0 & 3 & 0 & 0 \\ 0 & 0 & 1 & -4 & 0 & 5 \\ 0 & 0 & 0 & 0 & 1 & 7 \end{bmatrix}.$$

Solutions of Linear Systems

$$\left[\begin{array}{cccccc} 1 & 6 & 0 & 3 & 0 & 0 \\ 0 & 0 & 1 & -4 & 0 & 5 \\ 0 & 0 & 0 & 0 & 1 & 7 \end{array} \right]$$

This augmented matrix has 6 columns. That means that the linear system has 5 variables.

$$\left\{ \begin{array}{l} x_1 + 6x_2 + 3x_4 = 0 \\ x_3 - 4x_4 = 5 \\ x_5 = 7. \end{array} \right.$$

Solutions of Linear Systems



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$$\left\{ \begin{array}{l} x_1 + 6x_2 + 3x_4 = 0 \\ x_3 - 4x_4 = 5 \\ x_5 = 7. \end{array} \right.$$

The pivot columns are the first, third and fifth columns. So the basic variables are x_1 , x_3 and x_5 . The free variables are x_2 and x_4 .

Solutions of Linear Systems



$$\begin{cases} x_1 + 6x_2 + 3x_4 = 0 \\ x_3 - 4x_4 = 5 \\ x_5 = 7. \end{cases}$$

The general solution of the linear system is

$$\begin{cases} x_1 = -6x_2 - 3x_4 \\ x_2 \text{ is free} \\ x_3 = 5 + 4x_4 \\ x_4 \text{ is free} \\ x_5 = 7. \end{cases}$$

Solutions of Linear Systems



Theorem (Existence and Uniqueness)

A linear system is consistent if and only if the rightmost column of the augmented matrix is not a pivot column

Solutions of Linear Systems

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A linear system is consistent if and only if the rightmost column of the augmented matrix is not a pivot column - that is, if and only if an echelon form of the augmented matrix has no row that looks like

$$[0 \ \cdots \ 0 \ b]$$

for $b \neq 0$.

Solutions of Linear Systems



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A linear system is consistent if and only if the rightmost column of the augmented matrix is not a pivot column - that is, if and only if an echelon form of the augmented matrix has no row that looks like

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for $b \neq 0$.

If a linear system is consistent, then the solution set contains either

- 1** *a unique solution (when there are no free variables); or*
- 2** *infinitely many solutions (when there is at least one free variable).*

USING ROW REDUCTION TO SOLVE A LINEAR SYSTEM

1. Write the augmented matrix of the system.
2. Use the row reduction algorithm to obtain an equivalent augmented matrix in echelon form. Decide whether the system is consistent. If there is no solution, stop; otherwise, go to the next step.
3. Continue row reduction to obtain the reduced echelon form.
4. Write the system of equations corresponding to the matrix obtained in step 3.
5. Rewrite each nonzero equation from step 4 so that its one basic variable is expressed in terms of any free variables appearing in the equation.

Example

How many solutions does the following linear system have?

$$\begin{cases} 3x_2 - 6x_3 + 6x_4 + 4x_5 = -5 \\ 3x_1 - 7x_2 + 8x_3 - 5x_4 + 8x_5 = 9 \\ 3x_1 - 9x_2 + 12x_3 - 9x_4 + 6x_5 = 15. \end{cases}$$

Solutions of Linear Systems

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First we write our augmented matrix

$$\left[\begin{array}{cccccc} 0 & 3 & -6 & 6 & 4 & -5 \\ 3 & -7 & 8 & -5 & 8 & 9 \\ 3 & -9 & 12 & -9 & 6 & 15 \end{array} \right].$$

Then we will reduce it to REF to see if the linear system is consistent.

Solutions of Linear Systems



$$\left[\begin{array}{cccccc} 0 & 3 & -6 & 6 & 4 & -5 \\ 3 & -7 & 8 & -5 & 8 & 9 \\ 3 & -9 & 12 & -9 & 6 & 15 \end{array} \right] \xrightarrow{R_1 \leftrightarrow R_3} \left[\begin{array}{cccccc} 3 & -9 & 12 & -9 & 6 & 15 \\ 3 & -7 & 8 & -5 & 8 & 9 \\ 0 & 3 & -6 & 6 & 4 & -5 \end{array} \right]$$

Solutions of Linear Systems



$$\left[\begin{array}{cccccc} 0 & 3 & -6 & 6 & 4 & -5 \\ 3 & -7 & 8 & -5 & 8 & 9 \\ 3 & -9 & 12 & -9 & 6 & 15 \end{array} \right] \xrightarrow{R_1 \leftrightarrow R_3} \left[\begin{array}{cccccc} 3 & -9 & 12 & -9 & 6 & 15 \\ 3 & -7 & 8 & -5 & 8 & 9 \\ 0 & 3 & -6 & 6 & 4 & -5 \end{array} \right]$$
$$\xrightarrow{-R_1 + R_2} \left[\begin{array}{cccccc} 3 & -9 & 12 & -9 & 6 & 15 \\ 0 & 2 & -4 & 4 & 2 & -6 \\ 0 & 3 & -6 & 6 & 4 & -5 \end{array} \right]$$

Solutions of Linear Systems

$$\begin{bmatrix} 0 & 3 & -6 & 6 & 4 & -5 \\ 3 & -7 & 8 & -5 & 8 & 9 \\ 3 & -9 & 12 & -9 & 6 & 15 \end{bmatrix} \xrightarrow{R_1 \leftrightarrow R_3} \begin{bmatrix} 3 & -9 & 12 & -9 & 6 & 15 \\ 3 & -7 & 8 & -5 & 8 & 9 \\ 0 & 3 & -6 & 6 & 4 & -5 \end{bmatrix}$$

$$\xrightarrow{-R_1 + R_2} \begin{bmatrix} 3 & -9 & 12 & -9 & 6 & 15 \\ 0 & 2 & -4 & 4 & 2 & -6 \\ 0 & 3 & -6 & 6 & 4 & -5 \end{bmatrix}$$

$$\xrightarrow{-\frac{3}{2}R_2 + R_3} \begin{bmatrix} 3 & -9 & 12 & -9 & 6 & 15 \\ 0 & 2 & -4 & 4 & 2 & -6 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{bmatrix}$$

This matrix is now in REF.

Solutions of Linear Systems



$$\begin{bmatrix} 3 & -9 & 12 & -9 & 6 & 15 \\ 0 & 2 & -4 & 4 & 2 & -6 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{bmatrix}$$

- The pivot columns are the first, second and fifth columns.

Solutions of Linear Systems



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- The pivot columns are the first, second and fifth columns.
- Since the sixth column is not a pivot column, the linear system is consistent. So either there is one unique solution or there are infinitely many solutions.

Solutions of Linear Systems



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- The basic variables are x_1 , x_2 and x_5 .

Solutions of Linear Systems



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- The free variables are x_3 and x_4 .

Solutions of Linear Systems



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- The pivot columns are the first, second and fifth columns.
- Since the sixth column is not a pivot column, the linear system is consistent. So either there is one unique solution or there are infinitely many solutions.
- The basic variables are x_1 , x_2 and x_5 .
- The free variables are x_3 and x_4 .
- Since there are free variables, there are infinitely many solutions to the linear system.

Homogeneous Linear Systems

Definition

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Homogeneous Linear Systems

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A system of linear equations is said to be *homogeneous* if the constant terms are all zero; that is, if the system has the form

$$\left\{ \begin{array}{l} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \dots + a_{1n}x_n = 0 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \dots + a_{2n}x_n = 0 \\ \vdots \qquad \vdots \qquad \vdots \qquad \vdots \qquad \vdots \\ a_{m1}x_1 + a_{m2}x_2 + a_{m3}x_3 + \dots + a_{mn}x_n = 0. \end{array} \right.$$

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$$\left\{ \begin{array}{ccccccc} a_{11}x_1 & + & a_{12}x_2 & + & a_{13}x_3 & + & \dots + a_{1n}x_n = 0 \\ a_{21}x_1 & + & a_{22}x_2 & + & a_{23}x_3 & + & \dots + a_{2n}x_n = 0 \\ \vdots & & \vdots & & \vdots & & \vdots \\ a_{m1}x_1 & + & a_{m2}x_2 & + & a_{m3}x_3 & + & \dots + a_{mn}x_n = 0. \end{array} \right.$$

Remark

Every homogeneous linear system is consistent because $(0, 0, 0, \dots, 0)$ is always a solution.

Definition

The solution $(0, 0, 0, \dots, 0)$ is called the *trivial solution*.

Solutions of Linear Systems



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If there are other solutions, they are called *nontrivial solutions*.

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Theorem

A homogeneous linear system has a nontrivial solution if and only if it has atleast one free variable.

Solutions of Linear Systems



Example

Use Gauss-Jordan elimination to solve the homogeneous linear system

$$\begin{cases} x_1 + 3x_2 - 2x_3 & + 2x_5 = 0 \\ 2x_1 + 6x_2 - 5x_3 - 2x_4 + 4x_5 - 3x_6 = 0 \\ & 5x_3 + 10x_4 & + 15x_6 = 0 \\ 2x_1 + 6x_2 & + 8x_4 + 4x_5 + 18x_6 = 0. \end{cases}$$

Solutions of Linear Systems



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The augmented matrix is

$$\left[\begin{array}{ccccccc} 1 & 3 & -2 & 0 & 2 & 0 & 0 \\ 2 & 6 & -5 & -2 & 4 & -3 & 0 \\ 0 & 0 & 5 & 10 & 0 & 15 & 0 \\ 2 & 6 & 0 & 8 & 4 & 18 & 0 \end{array} \right].$$

Solutions of Linear Systems

The augmented matrix is

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can be row-reduced to

$$\begin{bmatrix} 1 & 3 & 0 & 4 & 2 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

I leave it for you to check this (exercise).

Solutions of Linear Systems

The corresponding linear system is

$$\left\{ \begin{array}{l} x_1 + 3x_2 + 4x_4 + 2x_5 = 0 \\ x_3 + 2x_4 = 0 \\ x_6 = 0 \end{array} \right.$$

which has solution

$$\left\{ \begin{array}{l} x_1 = -3x_2 - 4x_4 - 2x_5 \\ x_2 \text{ is free} \\ x_3 = -2x_4 \\ x_4 \text{ is free} \\ x_5 \text{ is free} \\ x_6 = 0. \end{array} \right.$$

This linear system has infinitely many solutions.

Solutions of Linear Systems



Theorem

If a homogeneous linear system has n variables, and if the RREF of its augmented matrix has r nonzero rows, then the system has $n - r$ free variables.

Solutions of Linear Systems



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If a homogeneous linear system has n variables, and if the RREF of its augmented matrix has r nonzero rows, then the system has $n - r$ free variables.

Theorem

A homogeneous linear system with more variables than equations has infinitely many solutions.



Introduction to Matrices

Introduction to Matrices



We have been using rectangular arrays of numbers, called *augmented matrices*, to abbreviate linear systems.

Now I want to discuss what a *matrix* is and what we can do with them.

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The numbers in a matrix are called *entries*.

Introduction to Matrices



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Definition

The numbers in a matrix are called *entries*.

The plural of matrix is *matrices*.

Introduction to Matrices



Example

These are matrices:

$$\begin{bmatrix} 1 & 2 \\ 3 & 0 \\ -1 & 4 \end{bmatrix}, \quad [2 \ 1 \ 0 \ -3], \quad \begin{bmatrix} e & \pi & -\sqrt{2} \\ 0 & \frac{1}{2} & 1 \\ 0 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 1 \\ 3 \end{bmatrix}, \quad [4].$$

Introduction to Matrices



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Definition

The *size* of a matrix is

number of rows \times number of column.

Introduction to Matrices



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3×2

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Introduction to Matrices



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1×4

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Introduction to Matrices

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3×3

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The *size* of a matrix is

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Introduction to Matrices



Example

These are matrices:

$$\begin{bmatrix} 1 & 2 \\ 3 & 0 \\ -1 & 4 \end{bmatrix}, \quad [2 \ 1 \ 0 \ -3], \quad \begin{bmatrix} e & \pi & -\sqrt{2} \\ 0 & \frac{1}{2} & 1 \\ 0 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 1 \\ 3 \end{bmatrix}, \quad [4].$$

2×1

Definition

The *size* of a matrix is

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Introduction to Matrices

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These are matrices:

$$\begin{bmatrix} 1 & 2 \\ 3 & 0 \\ -1 & 4 \end{bmatrix}, \quad [2 \ 1 \ 0 \ -3], \quad \begin{bmatrix} e & \pi & -\sqrt{2} \\ 0 & \frac{1}{2} & 1 \\ 0 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 1 \\ 3 \end{bmatrix}, \quad [4].$$

1×1

Definition

The *size* of a matrix is

number of rows \times number of column.

Introduction to Matrices



$$\begin{bmatrix} 2 & 1 & 0 & -3 \end{bmatrix} \quad \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

Definition

A matrix with only one row is called a *row vector*.

Definition

A matrix with only one column is called a *column vector*.

Notation

We will use

- capital letters, A, B, C, \dots to denote matrices; and
- lowercase letters, a, b, c, \dots to denote numbers.

Introduction to Matrices



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Notation

The entry with occurs in row i and column j of a matrix A will be denoted by a_{ij} .

Introduction to Matrices



Thus a general 3×4 matrix might be written as

$$\begin{matrix} 3 \text{ rows} & \left[\begin{array}{cccc} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \end{array} \right] \\ & \quad \uparrow \quad \uparrow \quad \uparrow \\ & \quad 4 \text{ columns} \end{matrix}$$

Introduction to Matrices

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row 2
column 4

“a two four”

not “a twenty four”

Introduction to Matrices



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and a general $m \times n$ matrix might be written as

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1j} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2j} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots & & \vdots \\ a_{i1} & a_{i2} & \cdots & a_{ij} & \cdots & a_{in} \\ \vdots & \vdots & & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mj} & \cdots & a_{mn} \end{bmatrix}.$$

Introduction to Matrices

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row 1

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row 2

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1j} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2j} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots & & \vdots \\ a_{i1} & a_{i2} & \cdots & a_{ij} & \cdots & a_{in} \\ \vdots & \vdots & & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mj} & \cdots & a_{mn} \end{bmatrix}.$$

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Thus a general 3×4 matrix might be written as

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \end{bmatrix}$$

and a general $m \times n$ matrix might be written as

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1j} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2j} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots & & \vdots \\ a_{i1} & a_{i2} & \cdots & a_{ij} & \cdots & a_{in} \\ \vdots & \vdots & & \vdots & & \vdots \\ \text{row } m & a_{m1} & a_{m2} & \cdots & a_{mj} & \cdots & a_{mn} \end{bmatrix}.$$

Introduction to Matrices



Thus a general 3×4 matrix might be written as

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column 1

Introduction to Matrices

Thus a general 3×4 matrix might be written as

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \end{bmatrix}$$

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column 2

Introduction to Matrices

Thus a general 3×4 matrix might be written as

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \end{bmatrix}$$

and a general $m \times n$ matrix might be written as

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1j} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2j} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots & & \vdots \\ a_{i1} & a_{i2} & \cdots & a_{ij} & \cdots & a_{in} \\ \vdots & \vdots & & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mj} & \cdots & a_{mn} \end{bmatrix}.$$

column j

Introduction to Matrices

Thus a general 3×4 matrix might be written as

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \end{bmatrix}$$

and a general $m \times n$ matrix might be written as

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1j} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2j} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots & & \vdots \\ a_{i1} & a_{i2} & \cdots & a_{ij} & \cdots & a_{in} \\ \vdots & \vdots & & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mj} & \cdots & a_{mn} \end{bmatrix}.$$

column n

Introduction to Matrices



Thus a general 3×4 matrix might be written as

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \end{bmatrix}$$

and a general $m \times n$ matrix might be written as

$$\begin{array}{c} \text{row } i \\ \hline \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1j} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2j} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots & & \vdots \\ a_{i1} & a_{i2} & \cdots & a_{ij} & \cdots & a_{in} \\ \vdots & \vdots & & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mj} & \cdots & a_{mn} \end{bmatrix} \\ \text{column } j \end{array}$$

Introduction to Matrices



$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1j} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2j} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots & & \vdots \\ a_{i1} & a_{i2} & \cdots & a_{ij} & \cdots & a_{in} \\ \vdots & \vdots & & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mj} & \cdots & a_{mn} \end{bmatrix}.$$

Notation

We can also write $A = [a_{ij}]_{m \times n}$

Introduction to Matrices

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1j} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2j} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots & & \vdots \\ a_{i1} & a_{i2} & \cdots & a_{ij} & \cdots & a_{in} \\ \vdots & \vdots & & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mj} & \cdots & a_{mn} \end{bmatrix}.$$

Notation

We can also write $A = [a_{ij}]_{m \times n}$ or just $A = [a_{ij}]$.

Introduction to Matrices



$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1j} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2j} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots & & \vdots \\ a_{i1} & a_{i2} & \cdots & a_{ij} & \cdots & a_{in} \\ \vdots & \vdots & & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mj} & \cdots & a_{mn} \end{bmatrix}.$$

Notation

We can also write $A = [a_{ij}]_{m \times n}$ or just $A = [a_{ij}]$.

The entry in row i and column j of a matrix A can be denoted by

$$a_{ij} = (A)_{ij}$$

also.

Introduction to Matrices



Example

Let $A = \begin{bmatrix} 2 & -3 \\ 7 & 0 \end{bmatrix}$ be a matrix. Then

$$(A)_{11} = 2, \quad (A)_{12} = -3, \quad (A)_{21} = 7 \quad \text{and} \quad (A)_{22} = 0.$$

Introduction to Matrices



Notation

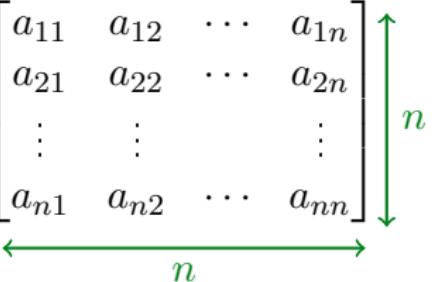
For vectors, that is $1 \times n$ and $m \times 1$ matrices, it is common practice to denote them by bold lowercase letters instead of capital letters. E.g.

$$\mathbf{a} = [a_1 \quad a_2 \quad a_3 \quad \cdots \quad a_n] \quad \text{and} \quad \mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}.$$

Introduction to Matrices

Definition

A matrix which has the same number of rows and columns, n say, is called a *square matrix of order n* .

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$$


Introduction to Matrices

Definition

A matrix which has the same number of rows and columns, n say, is called a *square matrix of order n* .

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$$

Definition

The entries $a_{11}, a_{22}, \dots, a_{nn}$ are on the *main diagonal* of a square matrix.

Operations on Matrices

Definition

Two matrices $A = [a_{ij}]$ and $B = [b_{ij}]$, are *equal* if

- they have the same size; and
- the corresponding entries are equal ($a_{ij} = b_{ij}$ for all i and j).

Definition (addition and subtraction)

If A and B have the same size, then we can define the sum $A + B$ and the difference $A - B$ in the obvious way.

Introduction to Matrices

Example

Let

$$A = \begin{bmatrix} 2 & 1 & 0 & 3 \\ -1 & 0 & 2 & 4 \\ 4 & -2 & 7 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} -4 & 3 & 5 & 1 \\ 2 & 2 & 0 & -1 \\ 3 & 2 & -4 & 5 \end{bmatrix} \text{ and } C = \begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}.$$

Then

$$A+B = \begin{bmatrix} -2 & 4 & 5 & 4 \\ 1 & 2 & 2 & 3 \\ 7 & 0 & 3 & 5 \end{bmatrix} \quad \text{and} \quad A-B = \begin{bmatrix} 6 & -2 & -5 & 2 \\ -3 & -2 & 2 & 5 \\ 1 & -4 & 11 & -5 \end{bmatrix}.$$

Introduction to Matrices

Example

Let

$$A = \begin{bmatrix} 2 & 1 & 0 & 3 \\ -1 & 0 & 2 & 4 \\ 4 & -2 & 7 & 0 \end{bmatrix}, B = \begin{bmatrix} -4 & 3 & 5 & 1 \\ 2 & 2 & 0 & -1 \\ 3 & 2 & -4 & 5 \end{bmatrix} \text{ and } C = \begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}.$$

Then

$$A+B = \begin{bmatrix} -2 & 4 & 5 & 4 \\ 1 & 2 & 2 & 3 \\ 7 & 0 & 3 & 5 \end{bmatrix} \quad \text{and} \quad A-B = \begin{bmatrix} 6 & -2 & -5 & 2 \\ -3 & -2 & 2 & 5 \\ 1 & -4 & 11 & -5 \end{bmatrix}.$$

$A+C$, $B+C$, $A-C$ and $B-C$ are all undefined because the sizes are different.

Introduction to Matrices



And we can multiply a matrix by a number in the obvious way.

Introduction to Matrices

And we can multiply a matrix by a number in the obvious way.

Example

If

$$A = \begin{bmatrix} 2 & 3 & 4 \\ 1 & 3 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 2 & 7 \\ -1 & 3 & -5 \end{bmatrix} \text{ and } C = \begin{bmatrix} 9 & -6 & -3 \\ 3 & 0 & 12 \end{bmatrix},$$

then

$$2A = \begin{bmatrix} 4 & 6 & 8 \\ 2 & 6 & 2 \end{bmatrix}, \quad -B = (-1)B = \begin{bmatrix} 0 & -2 & -7 \\ 1 & -3 & 5 \end{bmatrix}$$

and

$$\frac{1}{3}C = \begin{bmatrix} 3 & -2 & -1 \\ 1 & 0 & 4 \end{bmatrix}.$$

The Zero Matrix

Definition

An $m \times n$ matrix whose entries are all zero is called a *zero matrix* and is written as 0 or $0_{m \times n}$.

Example (Some zero matrices)

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \quad \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad [0]$$

Let A , B , and C be matrices of the same size, and let r and s be scalars.

a. $A + B = B + A$

d. $r(A + B) = rA + rB$

b. $(A + B) + C = A + (B + C)$

e. $(r + s)A = rA + sA$

c. $A + 0 = A$

f. $r(sA) = (rs)A$

Introduction to Matrices



Theorem

Let A , B and C be matrices of the same size. Let r and s be numbers. Then

1 $A + B = B + A$

Introduction to Matrices



Theorem

Let A , B and C be matrices of the same size. Let r and s be numbers. Then

- 1 $A + B = B + A$
- 2 $(A + B) + C = A + (B + C)$

Introduction to Matrices



Theorem

Let A , B and C be matrices of the same size. Let r and s be numbers. Then

- 1 $A + B = B + A$
- 2 $(A + B) + C = A + (B + C)$
- 3 $A + 0 = A$ and $A - 0 = A$

Introduction to Matrices



Theorem

Let A , B and C be matrices of the same size. Let r and s be numbers. Then

- 1 $A + B = B + A$
- 2 $(A + B) + C = A + (B + C)$
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Introduction to Matrices



Theorem

Let A , B and C be matrices of the same size. Let r and s be numbers. Then

- 1 $A + B = B + A$
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- 4 $r(A + B) = rA + rB$
- 5 $(r + s)A = rA + sA$

Introduction to Matrices



Theorem

Let A , B and C be matrices of the same size. Let r and s be numbers. Then

- 1 $A + B = B + A$
- 2 $(A + B) + C = A + (B + C)$
- 3 $A + 0 = A$ and $A - 0 = A$
- 4 $r(A + B) = rA + rB$
- 5 $(r + s)A = rA + sA$
- 6 $r(sA) = (rs)A.$

Introduction to Matrices



Definition

If A_1, A_2, \dots, A_r are matrices of the same size, and if c_1, c_2, \dots, c_r are numbers, then an expression of the form

$$c_1A_1 + c_2A_2 + \dots + c_rA_r$$

is called a *linear combination* of A_1, A_2, \dots, A_r with coefficients c_1, c_2, \dots, c_r .

Matrix Multiplication

Multiplying two matrices together is more complicated.

Matrix Multiplication

Multiplying two matrices together is more complicated.

$$A \underset{m \times r}{\cdot} B \underset{r \times n}{=} AB \underset{m \times n}{}$$

Definition

If A is an $m \times r$ matrix and B is an $r \times n$ matrix, then AB will be an $m \times n$ matrix.

Matrix Multiplication

Multiplying two matrices together is more complicated.

$$A \cdot B = AB$$

$m \times r$ $r \times n$ $m \times n$

Definition

If A is an $m \times r$ matrix and B is an $r \times n$ matrix, then AB will be an $m \times n$ matrix.

To find the entry in the i th row and j th column of AB , we must multiply each entry from row i of A with the corresponding entry from column j of B , then sum them.

Introduction to Matrices



Example

Multiply the following two matrices together:

$$A = \begin{bmatrix} 1 & 2 & 4 \\ 2 & 6 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 4 & 1 & 4 & 3 \\ 0 & -1 & 3 & 1 \\ 2 & 7 & 5 & 2 \end{bmatrix}.$$

Introduction to Matrices



Example

Multiply the following two matrices together:

$$A = \begin{bmatrix} 1 & 2 & 4 \\ 2 & 6 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 4 & 1 & 4 & 3 \\ 0 & -1 & 3 & 1 \\ 2 & 7 & 5 & 2 \end{bmatrix}.$$

A is a 2×3 matrix and B is a 3×4 matrix. So the product AB is a 2×4 matrix.

Introduction to Matrices



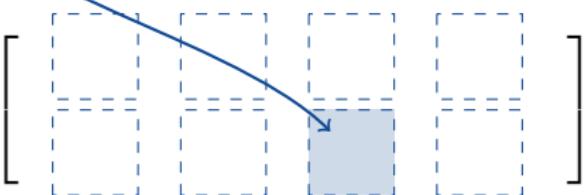
$$\begin{bmatrix} 1 & 2 & 4 \\ 2 & 6 & 0 \end{bmatrix} \begin{bmatrix} 4 & 1 & 4 & 3 \\ 0 & -1 & 3 & 1 \\ 2 & 7 & 5 & 2 \end{bmatrix} = \begin{bmatrix} \quad & \quad & \quad & \quad \\ \quad & \quad & \quad & \quad \\ \quad & \quad & \quad & \quad \\ \quad & \quad & \quad & \quad \end{bmatrix}$$

$2 \times 3 \qquad\qquad 3 \times 4 \qquad\qquad 2 \times 4$

Introduction to Matrices

Let's say that I want to calculate the entry in the 2nd row and the 3rd column.

$$\begin{bmatrix} 1 & 2 & 4 \\ 2 & 6 & 0 \end{bmatrix} \begin{bmatrix} 4 & 1 & 4 & 3 \\ 0 & -1 & 3 & 1 \\ 2 & 7 & 5 & 2 \end{bmatrix} =$$



A diagram illustrating matrix multiplication. Two matrices are multiplied together. The result is shown as a grid of four dashed boxes. A blue arrow points from the text "Let's say that I want to calculate the entry in the 2nd row and the 3rd column." to the third column of the first dashed box. This indicates that the entry at the intersection of the second row and third column of the first matrix and the first column of the second matrix is being calculated. The third column of the first matrix and the first column of the second matrix are highlighted with a light blue color.

Introduction to Matrices

I need row
2 of the
first matrix,

$$\begin{bmatrix} 1 & 2 & 4 \\ 2 & 6 & 0 \end{bmatrix} \begin{bmatrix} 4 & 1 & 4 & 3 \\ 0 & -1 & 3 & 1 \\ 2 & 7 & 5 & 2 \end{bmatrix} = \begin{bmatrix} \quad & \quad & \quad & \quad \\ \quad & \quad & \quad & \quad \\ \quad & \quad & \quad & \quad \\ \quad & \quad & \quad & \quad \end{bmatrix}$$

A blue curved arrow points from the text "I need row 2 of the first matrix," to the second row of the first matrix [2 6 0]. Another blue curved arrow points from the text "Let's say that I want to calculate the entry in the 2nd row and the 3rd column." to the entry 5 in the second row and third column of the product matrix.

Let's say that I want to
calculate the entry in the 2nd
row and the 3rd column.

Introduction to Matrices

I need row
2 of the
first matrix,

and I need
column 3 of
the second
matrix.

$$\begin{bmatrix} 1 & 2 & 4 \\ 2 & 6 & 0 \end{bmatrix} \begin{bmatrix} 4 & 1 & 4 & 3 \\ 0 & -1 & 3 & 1 \\ 2 & 7 & 5 & 2 \end{bmatrix} = \begin{bmatrix} \quad & \quad & \quad & \quad \\ \quad & \quad & \quad & \quad \\ \quad & \quad & \quad & \quad \\ \quad & \quad & \quad & \quad \end{bmatrix}$$

A blue arrow points from the second row of the first matrix to the third column of the second matrix. A blue bracket underlines the second row of the first matrix and the third column of the second matrix. A blue curved arrow points from the text "I need row 2 of the first matrix," to the second row of the first matrix. Another blue curved arrow points from the text "and I need column 3 of the second matrix." to the third column of the second matrix.

Let's say that I want to
calculate the entry in the 2nd
row and the 3rd column.

Introduction to Matrices



$$\begin{bmatrix} 1 & 2 & 4 \\ 2 & 6 & 0 \end{bmatrix} \begin{bmatrix} 4 & 1 & 4 & 3 \\ 0 & -1 & 3 & 1 \\ 2 & 7 & 5 & 2 \end{bmatrix} = \begin{bmatrix} \quad & \quad & \quad & \quad \\ \quad & \quad & \quad & \quad \\ \quad & \quad & \quad & \quad \\ \quad & \quad & \quad & \quad \end{bmatrix}$$

$$(2 \cdot 4) + (6 \cdot 3) + (0 \cdot 5) = 26$$

Introduction to Matrices



$$\begin{bmatrix} 1 & 2 & 4 \\ 2 & 6 & 0 \end{bmatrix} \begin{bmatrix} 4 & 1 & 4 & 3 \\ 0 & -1 & 3 & 1 \\ 2 & 7 & 5 & 2 \end{bmatrix} = \begin{bmatrix} \quad & \quad & \quad & \quad \\ \quad & \quad & \quad & \quad \end{bmatrix}$$

26



$$(2 \cdot 4) + (6 \cdot 3) + (0 \cdot 5) = 26$$

Introduction to Matrices



$$\begin{bmatrix} 1 & 2 & 4 \\ 2 & 6 & 0 \end{bmatrix} \begin{bmatrix} 4 & 1 & 4 & 3 \\ 0 & -1 & 3 & 1 \\ 2 & 7 & 5 & 2 \end{bmatrix} = \begin{bmatrix} \text{---} & \text{---} & \text{---} & \text{---} \\ \text{---} & \text{---} & \text{---} & \text{---} \\ \text{---} & \text{---} & \text{---} & 26 \\ \text{---} & \text{---} & \text{---} & \text{---} \end{bmatrix}$$

row 1
column 4

Introduction to Matrices



Introduction to Matrices



$$\begin{bmatrix} 1 & 2 & 4 \\ 2 & 6 & 0 \end{bmatrix} \begin{bmatrix} 4 & 1 & 4 & 3 \\ 0 & -1 & 3 & 1 \\ 2 & 7 & 5 & 2 \end{bmatrix} = \begin{bmatrix} \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & 26 \\ \dots & \dots & \dots & 13 \end{bmatrix}$$

$$(1 \cdot 3) + (2 \cdot 1) + (4 \cdot 2) = 13$$

Introduction to Matrices



$$\begin{bmatrix} 1 & 2 & 4 \\ 2 & 6 & 0 \end{bmatrix} \begin{bmatrix} 4 & 1 & 4 & 3 \\ 0 & -1 & 3 & 1 \\ 2 & 7 & 5 & 2 \end{bmatrix} = \begin{bmatrix} 12 & 27 & 30 & 13 \\ 8 & -4 & 26 & 12 \end{bmatrix}$$

I leave the other 6 entries for you to check.

Introduction to Matrices



Remember, we can only multiply two matrices together if the inside size numbers are equal.

$$A \cdot B = AB$$

$m \times r$ $r \times n$

inside

Introduction to Matrices



Remember, we can only multiply two matrices together if the inside size numbers are equal.

$$A \cdot B = AB$$

The diagram shows two matrices, A and B, represented by rectangles. Matrix A has dimensions $m \times r$ indicated by orange arrows pointing up from its top and left sides. Matrix B has dimensions $r \times n$ indicated by green arrows pointing up from its top and left sides. The intersection of the dimensions, where they are equal, is highlighted with a green box labeled "inside". Below the matrices, the product AB is shown, with its dimension $m \times n$ indicated by an orange arrow pointing up from its top side. The word "outside" is written in orange at the bottom center of the diagram.

Then the outside size numbers give us the size of the product AB .

Introduction to Matrices



EXAMPLE 4 If A is a 3×5 matrix and B is a 5×2 matrix, what are the sizes of AB and BA , if they are defined?

Introduction to Matrices

EXAMPLE 4 If A is a 3×5 matrix and B is a 5×2 matrix, what are the sizes of AB and BA , if they are defined?

SOLUTION Since A has 5 columns and B has 5 rows, the product AB is defined and is a 3×2 matrix:

$$\begin{array}{c} A \\ \left[\begin{array}{ccccc} * & * & * & * & * \\ * & * & * & * & * \\ * & * & * & * & * \end{array} \right] \end{array} \begin{array}{c} B \\ \left[\begin{array}{cc} * & * \\ * & * \\ * & * \\ * & * \\ * & * \end{array} \right] \end{array} = \begin{array}{c} AB \\ \left[\begin{array}{cc} * & * \\ * & * \\ * & * \end{array} \right] \end{array}$$

3×5 5×2 3×2

↑ Match ↑
Size of AB

The product BA is *not* defined because the 2 columns of B do not match the 3 rows of A . ■

Ask the audience 1/3

Suppose that A , B and C are matrices with the following sizes:

$$\begin{array}{ccc} A & B & C \\ 3 \times 4 & 4 \times 7 & 7 \times 3. \end{array}$$

Which of the following is correct?

- 1 AB is a 3×7 matrix.
 BA is a 4×4 matrix.

- 3 AB is a 3×7 matrix.
 BA is undefined.

- 2 AB is a 3×7 matrix.
 BA is a 7×3 matrix.

- 4 AB is undefined.
 BA is a 4×4 matrix.

Ask the audience 1/3

Suppose that A , B and C are matrices with the following sizes:

$$\begin{array}{ccc} A & B & C \\ 3 \times 4 & 4 \times 7 & 7 \times 3 \end{array}$$

Which of the following is correct?

- 1 AB is a 3×7 matrix.
 BA is a 4×4 matrix.

- 3 AB is a 3×7 matrix.
 BA is undefined.

- 2 AB is a 3×7 matrix.
 BA is a 7×3 matrix.

- 4 AB is undefined.
 BA is a 4×4 matrix.

Ask the audience 2/3

Suppose that A , B and C are matrices with the following sizes:

$$\begin{array}{ccc} A & B & C \\ 3 \times 4 & 4 \times 7 & 7 \times 3 \end{array}$$

Which of the following is correct?

- 1 BC is a 4×3 matrix.
 CB is undefined.
- 3 BC is undefined.
 CB is undefined.
- 2 BC is undefined.
 CB is a 3×4 matrix.
- 4 BC is undefined.
 CB is a 7×7 matrix.

Ask the audience 2/3

Suppose that A , B and C are matrices with the following sizes:

$$\begin{array}{ccc} A & B & C \\ 3 \times 4 & 4 \times 7 & 7 \times 3 \end{array}$$

Which of the following is correct?

- 1 BC is a 4×3 matrix.
 CB is undefined.
- 3 BC is undefined.
 CB is undefined.
- 2 BC is undefined.
 CB is a 3×4 matrix.
- 4 BC is undefined.
 CB is a 7×7 matrix.

Ask the audience 3/3

Suppose that A , B and C are matrices with the following sizes:

$$\begin{array}{ccc} A & B & C \\ 3 \times 4 & 4 \times 7 & 7 \times 3 \end{array}$$

Which of the following is correct?

1 AC is undefined.
 CA is a 7×4 matrix.

3 AC is a 3×3 matrix.
 CA is a 7×4 matrix.

2 AC is undefined.
 CA is undefined.

4 I didn't understand
any of this.

Ask the audience 3/3

Suppose that A , B and C are matrices with the following sizes:

$$\begin{array}{ccc} A & B & C \\ 3 \times 4 & 4 \times 7 & 7 \times 3. \end{array}$$

Which of the following is correct?

1 AC is undefined.
 CA is a 7×4 matrix.

3 AC is a 3×3 matrix.
 CA is a 7×4 matrix.

2 AC is undefined.
 CA is undefined.

4 I didn't understand
any of this.

In general, if $A = [a_{ij}]$ is an $m \times r$ matrix and $B = [b_{ij}]$ is an $r \times n$ matrix, then, as illustrated by the shading in the following display,

$$AB = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1r} \\ a_{21} & a_{22} & \cdots & a_{2r} \\ \vdots & \vdots & & \vdots \\ a_{i1} & a_{i2} & \cdots & a_{ir} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mr} \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1j} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2j} & \cdots & b_{2n} \\ \vdots & \vdots & & \vdots & & \vdots \\ b_{r1} & b_{r2} & \cdots & b_{rj} & \cdots & b_{rn} \end{bmatrix} \quad (4)$$

the entry $(AB)_{ij}$ in row i and column j of AB is given by

$$(AB)_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + a_{i3}b_{3j} + \cdots + a_{ir}b_{rj} \quad (5)$$

Formula (5) is called the **row-column rule** for matrix multiplication.

Break

We will continue at 3pm



The Identity Matrix

Definition

A square matrix with 1's on the main diagonal and 0's everywhere else is called an *identity matrix*, and is denoted by I or I_n .

Example

$$I = I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad I = I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad I = I_4 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Introduction to Matrices



Note that if A is a 3×2 matrix, then

$$AI_3 = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Introduction to Matrices



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Introduction to Matrices



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Introduction to Matrices



Note that if A is a 3×2 matrix, then

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and

$$I_2A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix} = A.$$

Properties of Matrix Multiplication

Theorem

Let A , B and C be matrices and let r be a number. If the sizes of the matrices are correct, then

$$1 \quad A(BC) = (AB)C$$

Properties of Matrix Multiplication

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Let A , B and C be matrices and let r be a number. If the sizes of the matrices are correct, then

- 1 $A(BC) = (AB)C$
- 2 $A(B + C) = AB + AC$

Properties of Matrix Multiplication

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Properties of Matrix Multiplication

Theorem

Let A , B and C be matrices and let r be a number. If the sizes of the matrices are correct, then

- 1 $A(BC) = (AB)C$
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Properties of Matrix Multiplication

Theorem

Let A , B and C be matrices and let r be a number. If the sizes of the matrices are correct, then

- 1 $A(BC) = (AB)C$
- 2 $A(B + C) = AB + AC$
- 3 $(B + C)A = BA + CA$
- 4 $r(AB) = (rA)B = A(rB)$
- 5 $IA = A = AI$

Introduction to Matrices

Example

Let $A = \begin{bmatrix} 5 & 1 \\ 3 & -2 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 0 \\ 4 & 3 \end{bmatrix}$. Calculate AB and BA .

I leave it for you to check that

$$AB = \begin{bmatrix} 5 & 1 \\ 3 & -2 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 4 & 3 \end{bmatrix} =$$

and

$$BA = \begin{bmatrix} 2 & 0 \\ 4 & 3 \end{bmatrix} \begin{bmatrix} 5 & 1 \\ 3 & -2 \end{bmatrix} =$$

Introduction to Matrices

Example

Let $A = \begin{bmatrix} 5 & 1 \\ 3 & -2 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 0 \\ 4 & 3 \end{bmatrix}$. Calculate AB and BA .

I leave it for you to check that

$$AB = \begin{bmatrix} 5 & 1 \\ 3 & -2 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 4 & 3 \end{bmatrix} = \begin{bmatrix} 14 & 3 \\ -2 & -6 \end{bmatrix},$$

and

$$BA = \begin{bmatrix} 2 & 0 \\ 4 & 3 \end{bmatrix} \begin{bmatrix} 5 & 1 \\ 3 & -2 \end{bmatrix} =$$

Introduction to Matrices



Example

Let $A = \begin{bmatrix} 5 & 1 \\ 3 & -2 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 0 \\ 4 & 3 \end{bmatrix}$. Calculate AB and BA .

I leave it for you to check that

$$AB = \begin{bmatrix} 5 & 1 \\ 3 & -2 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 4 & 3 \end{bmatrix} = \begin{bmatrix} 14 & 3 \\ -2 & -6 \end{bmatrix},$$

and

$$BA = \begin{bmatrix} 2 & 0 \\ 4 & 3 \end{bmatrix} \begin{bmatrix} 5 & 1 \\ 3 & -2 \end{bmatrix} = \begin{bmatrix} 10 & 2 \\ 29 & -2 \end{bmatrix}.$$

Note that $AB \neq BA$.

Introduction to Matrices



numbers

$$ab = ba$$

matrices

in general, $AB \neq BA$

Introduction to Matrices



numbers	matrices
$ab = ba$	in general, $AB \neq BA$
$ab = ac \implies a = 0$ or $b = c$	$AB = AC \not\implies A = 0$ or $B = C$

Introduction to Matrices



numbers	matrices
$ab = ba$	in general, $AB \neq BA$
$ab = ac \implies a = 0$ or $b = c$	$AB = AC \not\implies A = 0$ or $B = C$
$ab = 0 \implies a = 0$ or $b = 0$	$AB = 0 \not\implies A = 0$ or $B = 0$

Introduction to Matrices

Example

Let

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 2 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 1 \\ 3 & 4 \end{bmatrix}, \quad C = \begin{bmatrix} 2 & 5 \\ 3 & 4 \end{bmatrix}.$$

Clearly $B \neq C$ and $A \neq 0$. I leave it for you to check that

$$AB = \begin{bmatrix} 3 & 4 \\ 6 & 8 \end{bmatrix} = AC.$$

Introduction to Matrices

Example

Let

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 2 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 1 \\ 3 & 4 \end{bmatrix}, \quad C = \begin{bmatrix} 2 & 5 \\ 3 & 4 \end{bmatrix}.$$

Clearly $B \neq C$ and $A \neq 0$. I leave it for you to check that

$$AB = \begin{bmatrix} 3 & 4 \\ 6 & 8 \end{bmatrix} = AC.$$

Example

Note that

$$\begin{bmatrix} 0 & 1 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 3 & 7 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix},$$

but neither matrix on the left is the zero matrix.

Transpose of a Matrix

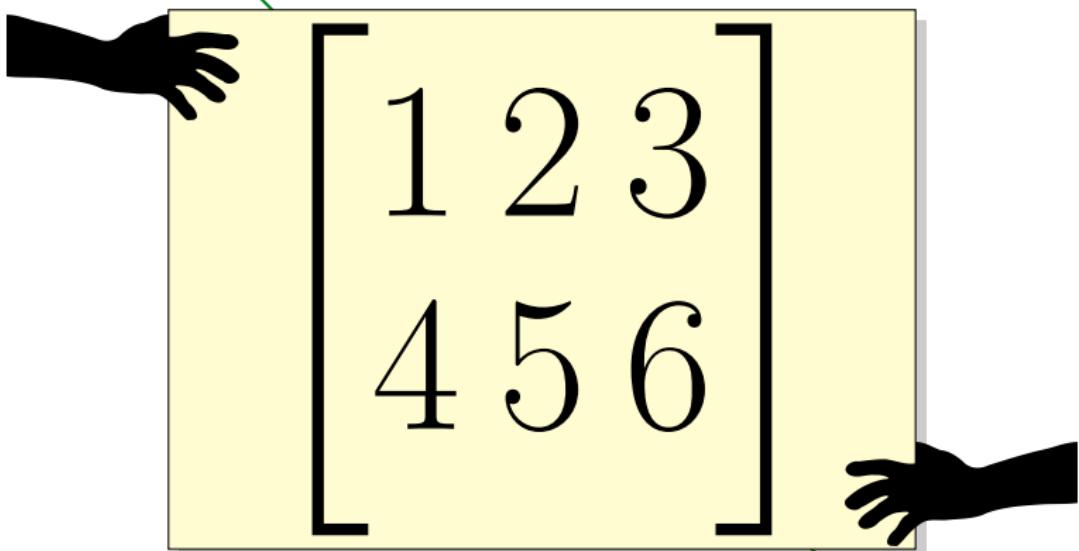
Definition

If A is any $m \times n$ matrix, then the transpose of A , denoted by A^T , is defined to be the $n \times m$ matrix that results by interchanging the rows and columns of A ; that is, the first column of A^T is the first row of A , the second column of A^T is the second row of A , and so forth.

$$\begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}$$

Introduction to Matrices



Introduction to Matrices

Example

If $A = \begin{bmatrix} 2 & 3 \\ 1 & 4 \\ 5 & 6 \end{bmatrix}$, then $A^T = \begin{bmatrix} 2 & 1 & 5 \\ 3 & 4 & 6 \end{bmatrix}$.

Example

If $C = \begin{bmatrix} 1 & 3 & 5 \end{bmatrix}$, then $C^T = \begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix}$.

Example

If $D = [4]$, then $D^T = [4]$.

Remark

Note that since row i of A has the same entries as column i of A^T , and column j of A has the same entries as row j of A^T , we have the formula

$$(A^T)_{ij} = A_{ji}.$$

Trace of a Matrix

Definition

If A is a square matrix, then the *trace of A* , denoted $\text{tr}(A)$, is defined to be the sum of the entries on the main diagonal.

(The trace of A is not defined if A is not a square matrix.)

Introduction to Matrices



Example

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$B = \begin{bmatrix} -1 & 2 & 7 & 0 \\ 9 & 5 & 1 & 1 \\ 0 & 2 & 7 & 0 \\ 4 & 4 & 4 & 0 \end{bmatrix}$$

$$\text{tr}(A) = a_{11} + a_{22} + a_{33} \quad \text{tr}(B) = -1 + 5 + 7 + 0 = 11$$



The Inverse of a Matrix

The Inverse of a Matrix



In the real numbers, every number $a \neq 0$ has a multiplicative inverse called a^{-1} . For example, if $a = 4$, then we have $4^{-1} = \frac{1}{4}$ which satisfies

$$4 \cdot 4^{-1} = 1 \quad \text{and} \quad 4^{-1} \cdot 4 = 1.$$

We want to generalise this idea to square matrices.

The Inverse of a Matrix



Definition

Let A be a square matrix. If there exists a matrix B of the same size which satisfies

$$AB = I \quad \text{and} \quad BA = I$$

then we say that A is *invertible* and B is called the *inverse* of A .

If such a B does not exist, then A is called *singular*.

The Inverse of a Matrix

Remark

Note that

$$\begin{array}{c} B \text{ is the} \\ \text{inverse of } A \end{array} \implies AB = I = BA \implies \begin{array}{c} A \text{ is the} \\ \text{inverse of } B \end{array}$$

So A and B are inverses of each other.

If A is invertible, then its inverse B is invertible also.

The Inverse of a Matrix

$$AB = I = BA$$



Example

Let

$$A = \begin{bmatrix} 2 & -5 \\ 1 & 3 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 3 & 5 \\ 1 & 2 \end{bmatrix}.$$

The Inverse of a Matrix

$$AB = I = BA$$



Example

Let

$$A = \begin{bmatrix} 2 & -5 \\ 1 & 3 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 3 & 5 \\ 1 & 2 \end{bmatrix}.$$

Then

$$AB = \begin{bmatrix} 2 & -5 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} 3 & 5 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

and

$$BA = \begin{bmatrix} 3 & 5 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 2 & -5 \\ 1 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

Therefore A and B are invertible and each is the inverse of the other.

The Inverse of a Matrix

$$AB = I = BA$$



A square matrix with a row or column of zeros is always singular.

The Inverse of a Matrix

$$AB = I = BA$$



A square matrix with a row or column of zeros is always singular.

Example

Consider $A = \begin{bmatrix} 1 & 4 & 0 \\ 2 & 5 & 0 \\ 3 & 6 & 0 \end{bmatrix}$. I want to show that such a B does not exist.

The Inverse of a Matrix

$$AB = I = BA$$



A square matrix with a row or column of zeros is always singular.

Example

Consider $A = \begin{bmatrix} 1 & 4 & 0 \\ 2 & 5 & 0 \\ 3 & 6 & 0 \end{bmatrix}$. I want to show that such a B does not exist. We calculate that

$$BA = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \begin{bmatrix} 1 & 4 & 0 \\ 2 & 5 & 0 \\ 3 & 6 & 0 \end{bmatrix}$$

The Inverse of a Matrix

$$AB = I = BA$$



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Example

Consider $A = \begin{bmatrix} 1 & 4 & 0 \\ 2 & 5 & 0 \\ 3 & 6 & 0 \end{bmatrix}$. I want to show that such a B does not exist. We calculate that

$$BA = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \begin{bmatrix} 1 & 4 & 0 \\ 2 & 5 & 0 \\ 3 & 6 & 0 \end{bmatrix} = \begin{bmatrix} a + 2b + 3c & 4a + 5b + 6c & 0 \\ d + 2e + 3f & 4d + 5e + 6f & 0 \\ g + 2h + 3i & 4g + 5h + 6i & 0 \end{bmatrix}.$$

The 3rd column will always be full of zeros. It is not possible to choose a, \dots, i to make $(BA)_{33} = 1$.

The Inverse of a Matrix

$$AB = I = BA$$



A square matrix with a row or column of zeros is always singular.

Example

Consider $A = \begin{bmatrix} 1 & 4 & 0 \\ 2 & 5 & 0 \\ 3 & 6 & 0 \end{bmatrix}$. I want to show that such a B does not exist. We calculate that

$$BA = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \begin{bmatrix} 1 & 4 & 0 \\ 2 & 5 & 0 \\ 3 & 6 & 0 \end{bmatrix} = \begin{bmatrix} a + 2b + 3c & 4a + 5b + 6c & 0 \\ d + 2e + 3f & 4d + 5e + 6f & 0 \\ g + 2h + 3i & 4g + 5h + 6i & 0 \end{bmatrix}.$$

The 3rd column will always be full of zeros. It is not possible to choose a, \dots, i to make $(BA)_{33} = 1$. So A does not have an inverse. Therefore A is singular.

The Inverse of a Matrix

$$AB = I = BA$$



Can an invertible matrix have more than one inverse?

The Inverse of a Matrix

$$AB = I = BA$$



Can an invertible matrix have more than one inverse? **No.**

Theorem

If B and C are both inverses of A , then $B = C$.

The Inverse of a Matrix

$$AB = I = BA$$



Can an invertible matrix have more than one inverse? **No.**

Theorem

If B and C are both inverses of A , then $B = C$.

Proof.

Suppose that

$$AB = I = BA \quad \text{and} \quad AC = I = CA.$$

The Inverse of a Matrix

$$AB = I = BA$$



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Theorem

If B and C are both inverses of A , then $B = C$.

Proof.

Suppose that

$$AB = I = BA \quad \text{and} \quad AC = I = CA.$$

Then we have that

$$B = BI$$

The Inverse of a Matrix

$$AB = I = BA$$



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Theorem

If B and C are both inverses of A , then $B = C$.

Proof.

Suppose that

$$AB = I = BA \quad \text{and} \quad AC = I = CA.$$

Then we have that

$$B = BI = B(AC)$$

The Inverse of a Matrix

$$AB = I = BA$$



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Suppose that

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Then we have that

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The Inverse of a Matrix

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If B and C are both inverses of A , then $B = C$.

Proof.

Suppose that

$$AB = I = BA \quad \text{and} \quad AC = I = CA.$$

Then we have that

$$B = BI = B(AC) = (BA)C = IC = C.$$



The Inverse of a Matrix



Definition

An The inverse of A is denoted by A^{-1} . Thus

$$AA^{-1} = I = A^{-1}A.$$

The Inverse of a Matrix



Theorem

The matrix

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

is invertible if and only if $ad - bc \neq 0$, in which case its inverse is given by

$$A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

We will generalise this to bigger square matrices later in the course.

$$ad - bc \neq 0 \quad A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

Definition

The number $ad - bc$ is called the *determinant* of $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and is denoted by

$$\det(A) = ad - bc$$

or by

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc.$$

Theorem

$$\det(A) = ad - bc \neq 0 \quad A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$



Example

Is $B = \begin{bmatrix} 6 & 1 \\ 5 & 2 \end{bmatrix}$ invertible? If so, find its inverse.

Theorem

$$\det(A) = ad - bc \neq 0 \quad A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$



Example

Is $B = \begin{bmatrix} 6 & 1 \\ 5 & 2 \end{bmatrix}$ invertible? If so, find its inverse.

The determinant of B is

$$\det(B) = 6 \cdot 2 - 1 \cdot 5 = 7 \neq 0.$$

Therefore B is invertible.

The I

$$\det(A) = ad - bc \neq 0 \quad A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$



Example

Is $B = \begin{bmatrix} 6 & 1 \\ 5 & 2 \end{bmatrix}$ invertible? If so, find its inverse.

The determinant of B is

$$\det(B) = 6 \cdot 2 - 1 \cdot 5 = 7 \neq 0.$$

Therefore B is invertible. Its inverse is

$$B^{-1} = \frac{1}{7} \begin{bmatrix} 2 & -1 \\ -5 & 6 \end{bmatrix} = \begin{bmatrix} \frac{2}{7} & -\frac{1}{7} \\ -\frac{5}{7} & \frac{6}{7} \end{bmatrix}.$$

The I

$$\det(A) = ad - bc \neq 0 \quad A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$



Example

Is $C = \begin{bmatrix} -1 & 2 \\ 3 & -6 \end{bmatrix}$ invertible? If so, find its inverse.

The |

$$\det(A) = ad - bc \neq 0 \quad A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$



Example

Is $C = \begin{bmatrix} -1 & 2 \\ 3 & -6 \end{bmatrix}$ invertible? If so, find its inverse.

The determinant of C is

$$\det(C) = (-1) \cdot (-6) - 2 \cdot 3 = 0.$$

Therefore C is not invertible. C is singular.

The Inverse of a Matrix

$$B = \begin{bmatrix} 6 & 1 \\ 5 & 2 \end{bmatrix} \quad B^{-1} = \begin{bmatrix} \frac{2}{7} & -\frac{1}{7} \\ -\frac{5}{7} & \frac{6}{7} \end{bmatrix}$$

Inverse matrices can be used to solve linear systems.

Example

Solve $\begin{cases} 6x_1 + x_2 = 2 \\ 5x_1 + 2x_2 = 3. \end{cases}$

The Inverse of a Matrix

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Example

Solve $\begin{cases} 6x_1 + x_2 = 2 \\ 5x_1 + 2x_2 = 3. \end{cases}$

We can write this as

$$\begin{bmatrix} 6x_1 + x_2 \\ 5x_1 + 2x_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

The Inverse of a Matrix

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Solve $\begin{cases} 6x_1 + x_2 = 2 \\ 5x_1 + 2x_2 = 3. \end{cases}$

We can write this as

$$\begin{bmatrix} 6x_1 + x_2 \\ 5x_1 + 2x_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

or as

$$\begin{bmatrix} 6 & 1 \\ 5 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}.$$

The Inverse of a Matrix

$$B = \begin{bmatrix} 6 & 1 \\ 5 & 2 \end{bmatrix} \quad B^{-1} = \begin{bmatrix} \frac{2}{7} & -\frac{1}{7} \\ -\frac{5}{7} & \frac{6}{7} \end{bmatrix}$$

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Example

Solve $\begin{cases} 6x_1 + x_2 = 2 \\ 5x_1 + 2x_2 = 3. \end{cases}$

We can write this as

$$\begin{bmatrix} 6x_1 + x_2 \\ 5x_1 + 2x_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

or as

$$\begin{bmatrix} 6 & 1 \\ 5 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}.$$

Multiplying both sides on the left by B^{-1} gives

$$\begin{bmatrix} \frac{2}{7} & -\frac{1}{7} \\ -\frac{5}{7} & \frac{6}{7} \end{bmatrix} \begin{bmatrix} 6 & 1 \\ 5 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \frac{2}{7} & -\frac{1}{7} \\ -\frac{5}{7} & \frac{6}{7} \end{bmatrix} \begin{bmatrix} 2 \\ 3 \end{bmatrix} =$$

The Inverse of a Matrix

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Inverse matrices can be used to solve linear systems.

Example

Solve $\begin{cases} 6x_1 + x_2 = 2 \\ 5x_1 + 2x_2 = 3. \end{cases}$

We can write this as

$$\begin{bmatrix} 6x_1 + x_2 \\ 5x_1 + 2x_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

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$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \frac{2}{7} & -\frac{1}{7} \\ -\frac{5}{7} & \frac{6}{7} \end{bmatrix} \begin{bmatrix} 6 & 1 \\ 5 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \frac{2}{7} & -\frac{1}{7} \\ -\frac{5}{7} & \frac{6}{7} \end{bmatrix} \begin{bmatrix} 2 \\ 3 \end{bmatrix} =$$

The Inverse of a Matrix

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Inverse matrices can be used to solve linear systems.

Example

Solve $\begin{cases} 6x_1 + x_2 = 2 \\ 5x_1 + 2x_2 = 3. \end{cases}$

We can write this as

$$\begin{bmatrix} 6x_1 + x_2 \\ 5x_1 + 2x_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

or as

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Multiplying both sides on the left by B^{-1} gives

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Theorem of Matrix

Remark

In general, the linear system

$$\left\{ \begin{array}{l} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m. \end{array} \right.$$

can be written as the matrix equation

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Theorem of Matrix

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or just

$$A\mathbf{x} = \mathbf{b}.$$

The Inverse of a Matrix



Remark

Clearly, if A is a square matrix and is invertible, then the solution to

$$A\mathbf{x} = \mathbf{b}$$

is

$$\mathbf{x} = A^{-1}\mathbf{b}.$$

The Inverse of a Matrix



Theorem

If A and B are invertible matrices of the same size, then AB is invertible and

$$(AB)^{-1} = B^{-1}A^{-1}.$$

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We calculate that

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Similarly

$$(B^{-1}A^{-1})(AB) = B^{-1}(A^{-1}A)B = B^{-1}IB = B^{-1}B = I.$$

Therefore AB is invertible and $B^{-1}A^{-1}$ is the inverse of AB . □

The Inverse of a Matrix



Remark

This is also true if we have more than 2 matrices. For example

$$(ABC)^{-1} = C^{-1}B^{-1}A^{-1},$$

$$(ABCD)^{-1} = D^{-1}C^{-1}B^{-1}A^{-1},$$

⋮

The Inverse of a Matrix

Example

Let

$$A = \begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 3 & 2 \\ 2 & 2 \end{bmatrix}.$$

I leave it for you to check the following:

$$AB = \begin{bmatrix} 7 & 6 \\ 9 & 8 \end{bmatrix} \qquad A^{-1} = \begin{bmatrix} 3 & -2 \\ -1 & 1 \end{bmatrix}$$

$$(AB)^{-1} = \begin{bmatrix} \frac{4}{9} & -\frac{3}{2} \\ -\frac{9}{2} & \frac{7}{2} \end{bmatrix} \qquad B^{-1} = \begin{bmatrix} 1 & -1 \\ -1 & \frac{3}{2} \end{bmatrix}$$

$$B^{-1}A^{-1} = \begin{bmatrix} 1 & -1 \\ -1 & \frac{3}{2} \end{bmatrix} \begin{bmatrix} 3 & -2 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} \frac{4}{9} & -\frac{3}{2} \\ -\frac{9}{2} & \frac{7}{2} \end{bmatrix}$$

Powers of a Matrix

Definition

Let A is a square matrix and let $k \in \mathbb{N}$. We define

$$A^0 = I \quad \text{and} \quad A^k = \underbrace{A \cdot A \cdot \dots \cdot A}_k.$$

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Note that

$$A^r A^s = A^{r+s} \quad \text{and} \quad (A^r)^s = A^{rs}$$

if $r, s \geq 0$ as expected.

The Inverse of a Matrix



Theorem

Let A be an invertible matrix. Let $n \in \mathbb{N} \cup \{0\}$ and $k \neq 0$. Then:

- 1 A^{-1} is invertible and $(A^{-1})^{-1} = A$.

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- 2 A^n is invertible and $(A^n)^{-1} = A^{-n} = (A^{-1})^n$.
- 3 kA is invertible and $(kA)^{-1} = k^{-1}A^{-1}$.

I will prove part 3. I leave parts 1 and 2 for you to prove.

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Proof of Part 3.

We have that

$$(kA)(k^{-1}A^{-1}) = k^{-1}(kA)A^{-1} = (k^1k)(AA^{-1}) = (1)(I) = I.$$

Similarly $(k^{-1}A^{-1})(kA) = I$. □

Theorem 1

Example

Let $A = \begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix}$ and $A^{-1} = \begin{bmatrix} 3 & -2 \\ -1 & 1 \end{bmatrix}$. Then

$$\begin{aligned} A^{-3} &= A^{-1}A^{-1}A^{-1} = \begin{bmatrix} 3 & -2 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 3 & -2 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 3 & -2 \\ -1 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 41 & -30 \\ -15 & 11 \end{bmatrix}. \end{aligned}$$

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Moreover

$$A^3 = AAA = \begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix} = \begin{bmatrix} 11 & 30 \\ 15 & 41 \end{bmatrix}$$

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and

$$(A^3)^{-1} = \frac{1}{(11 \cdot 41 - 30 \cdot 15)} \begin{bmatrix} 41 & -30 \\ -15 & 11 \end{bmatrix} = \begin{bmatrix} 41 & -30 \\ -15 & 11 \end{bmatrix} = (A^{-1})^3$$

The Inverse of a Matrix



Remark

If a and b are numbers then $ab = ba$, so we can calculate

$$(a + b)^2 = a^2 + ab + ba + b^2 = a^2 + ab + ab + b^2 = a^2 + 2ab + b^2.$$

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$$(a + b)^2 = a^2 + ab + ba + b^2 = a^2 + ab + ab + b^2 = a^2 + 2ab + b^2.$$

But remember that $AB \neq BA$ in general for matrices. So the best that we can do is

$$(A + B)^2 = A^2 + AB + BA + B^2.$$

Properties of the Transpose

Theorem

(If the sizes are correct, then)

- 1 $(A^T)^T = A$
- 2 $(A + B)^T = A^T + B^T$
- 3 $(kA)^T = kA^T$

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Theorem

If A is an invertible matrix, then A^T is also invertible and

$$(A^T)^{-1} = (A^{-1})^T.$$

Next Time

- An algorithm for finding A^{-1}
- More About Invertible Matrices
- The Invertible Matrix Theorem
- Diagonal, Triangular, and Symmetric Matrices
- Some Applications of Linear Systems.