

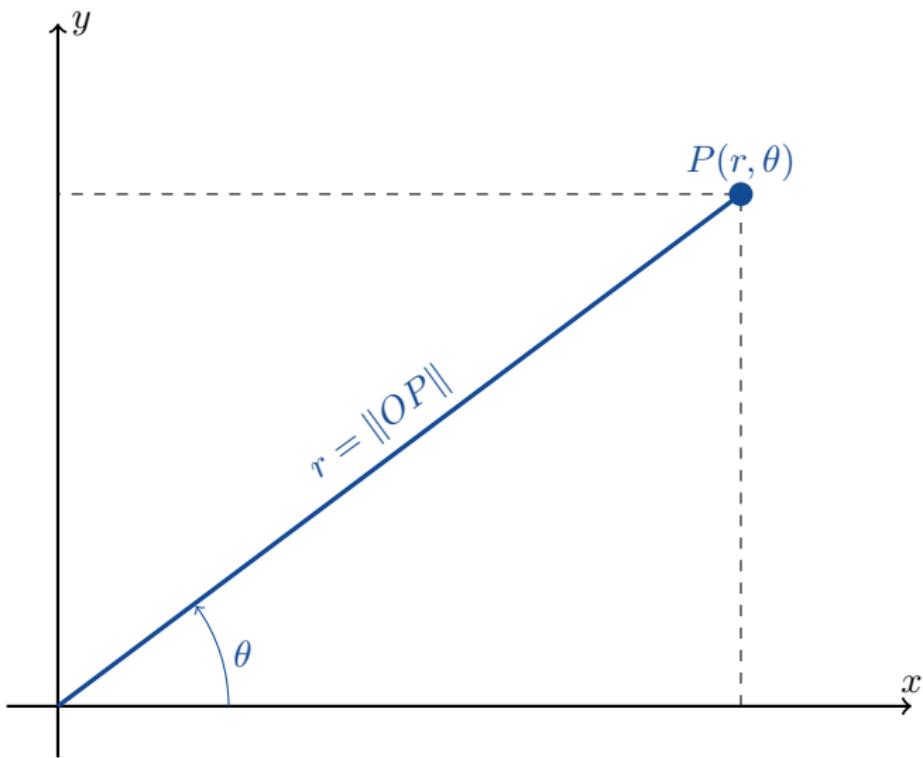
Lecture 3

- 8. Polar Coordinates
- 9. Conic Sections
- 10. Three Dimensional Cartesian Coordinates



Polar Coordinates

8. Polar Coordinates



8. Polar Coordinates

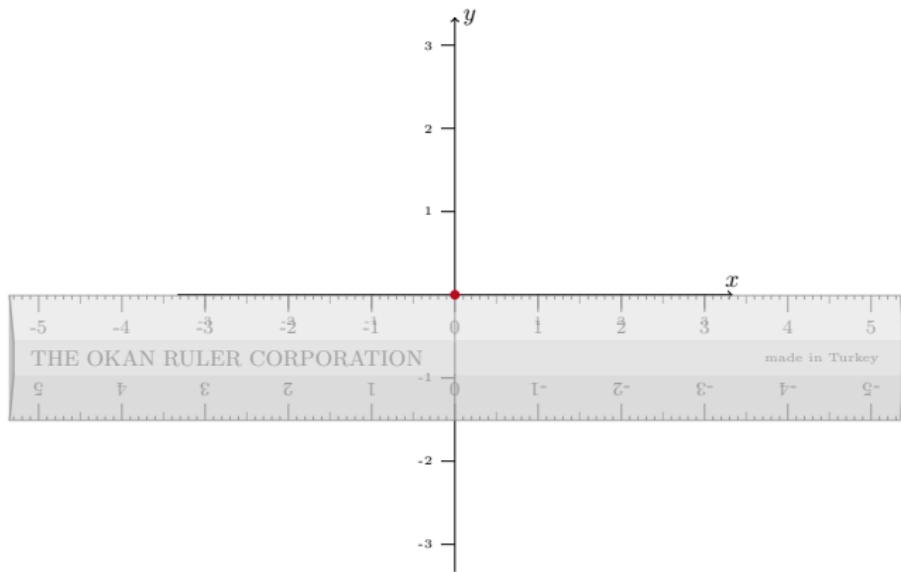


anticlockwise = positive angle
saat yönünün tersi = pozitif açı

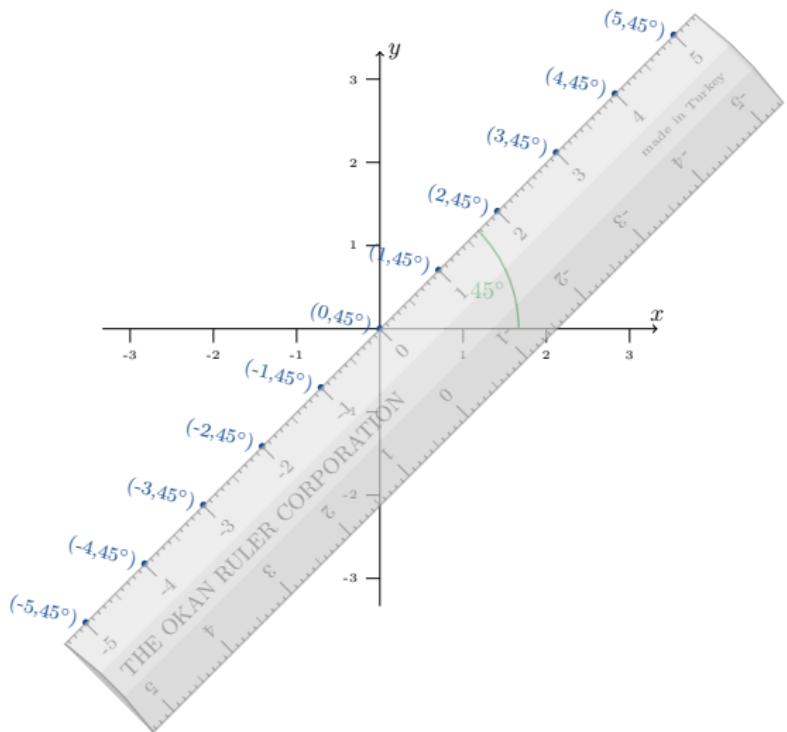


clockwise = negative angle
saat yönünde = negatif açı

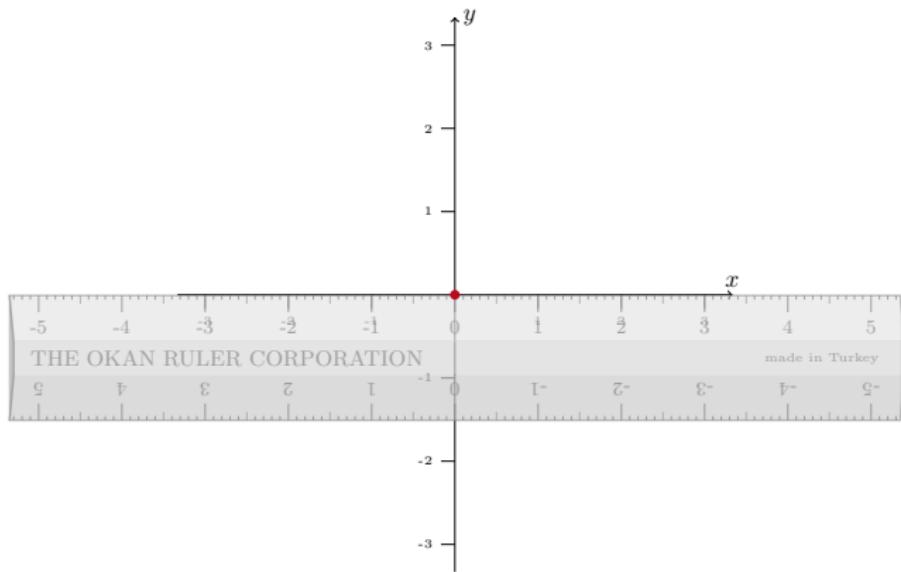
8. Polar Coordinates



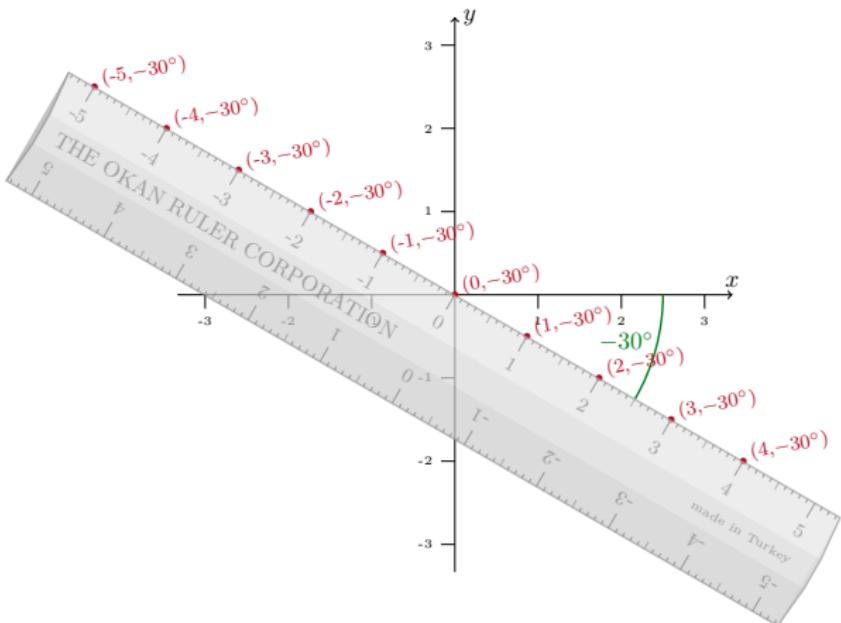
8. Polar Coordinates



8. Polar Coordinates

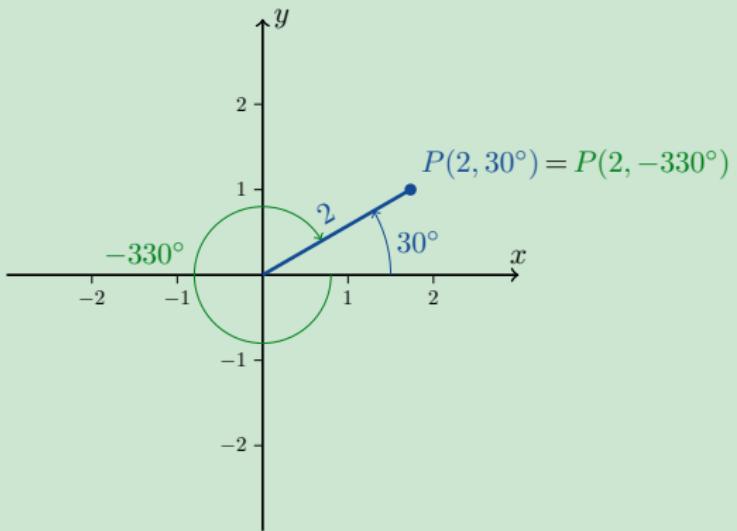


8. Polar Coordinates



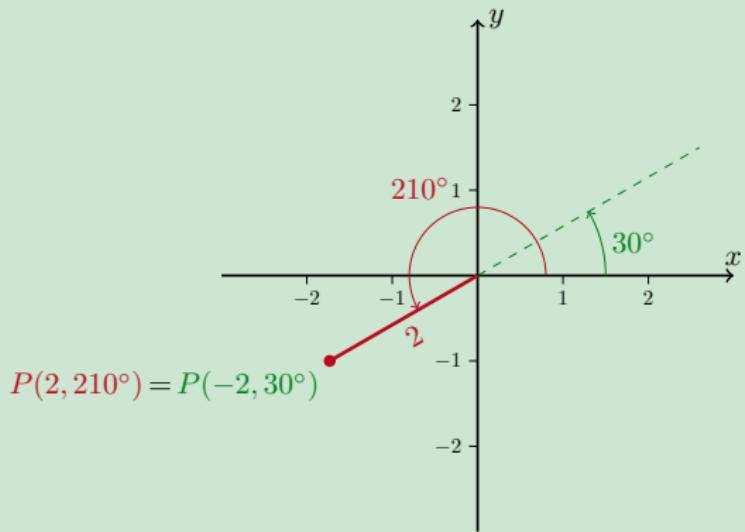
8. Polar Coordinates

Example



8. Polar Coordinates

Example



8. Polar Coordinates

Example

Find all the polar coordinates of $P(2, 30^\circ)$.

solution: We can have either $r = 2$ or $r = -2$. For $r = 2$, we can have

$$\theta = 30^\circ, 30^\circ \pm 360^\circ, 30^\circ \pm 720^\circ, 30^\circ \pm 1080^\circ, \dots$$

For $r = -2$, we can have

$$\theta = 210^\circ, 210^\circ \pm 360^\circ, 210^\circ \pm 720^\circ, 210^\circ \pm 1080^\circ, \dots$$

Therefore

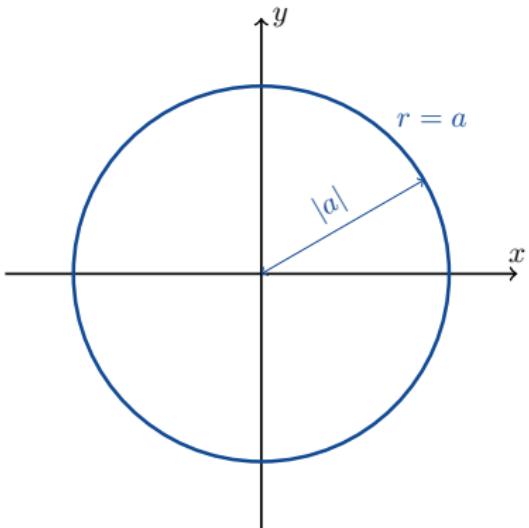
$$P(2, 30^\circ) = P(2, (30 + 360n)^\circ) = P(-2, (210 + 360m)^\circ)$$

for all $m, n \in \mathbb{Z}$.

8. Polar Coordinates

Example

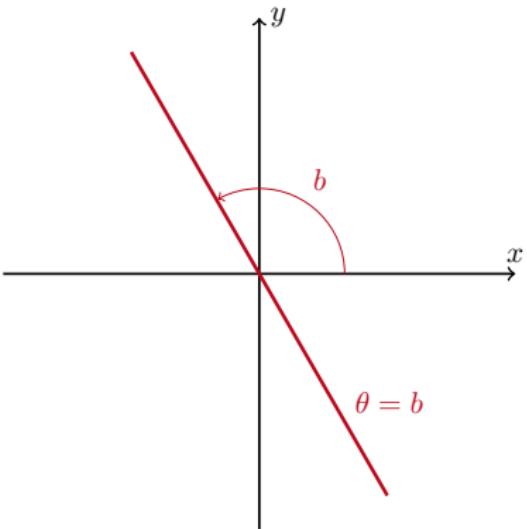
Draw the graph of $r = a$.



8. Polar Coordinates

Example

Draw the graph of $\theta = b$.



8. Polar Coordinates



Example

- 1 $r = 1$ and $r = -1$ are both equations for a circle of radius 1 centred at the origin.
- 2 $\theta = 30^\circ$, $\theta = 210^\circ$ and $\theta = -150^\circ$ are all equations for the same line.

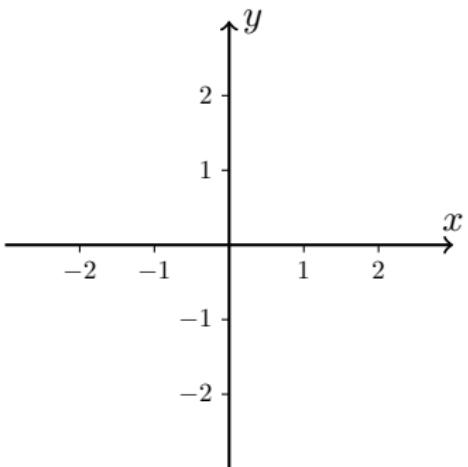
8. Polar Coordinates



Example

Draw the sets of points whose polar coordinates satisfy the following: $1 \leq r \leq 2$ and $0 \leq \theta \leq 90^\circ$.

solution:

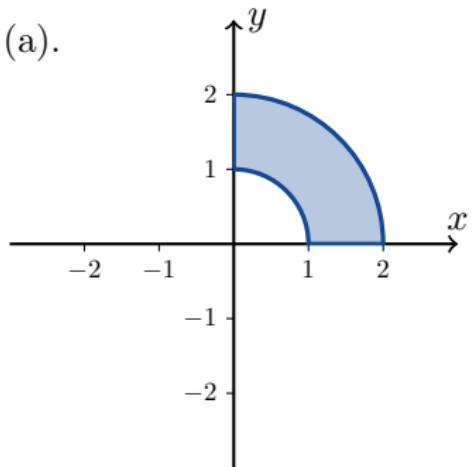


8. Polar Coordinates

Example

Draw the sets of points whose polar coordinates satisfy the following: $1 \leq r \leq 2$ and $0 \leq \theta \leq 90^\circ$.

solution:



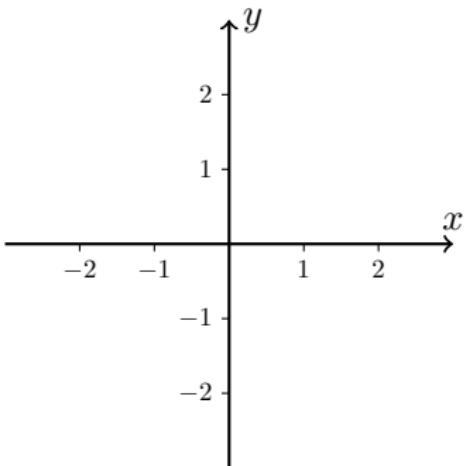
8. Polar Coordinates



Example

Draw the sets of points whose polar coordinates satisfy the following: $-3 \leq r \leq 2$ and $\theta = 45^\circ$.

solution:

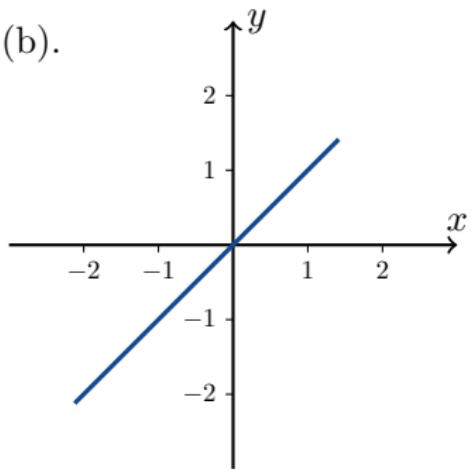


8. Polar Coordinates

Example

Draw the sets of points whose polar coordinates satisfy the following: $-3 \leq r \leq 2$ and $\theta = 45^\circ$.

solution:



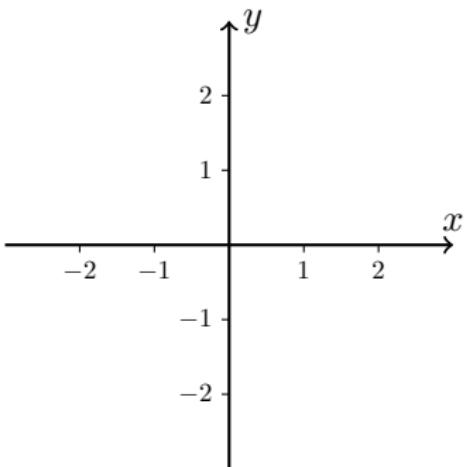
8. Polar Coordinates



Example

Draw the sets of points whose polar coordinates satisfy the following: $r \leq 0$ and $\theta = 60^\circ$.

solution:



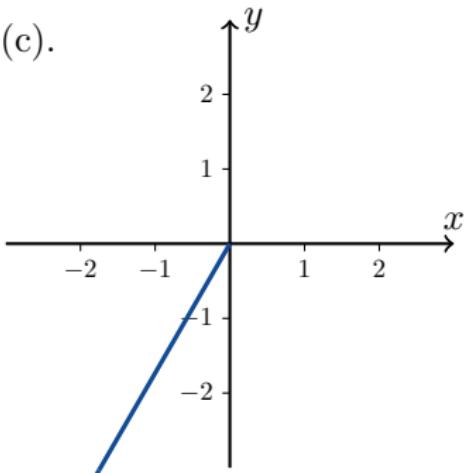
8. Polar Coordinates



Example

Draw the sets of points whose polar coordinates satisfy the following: $r \leq 0$ and $\theta = 60^\circ$.

solution:



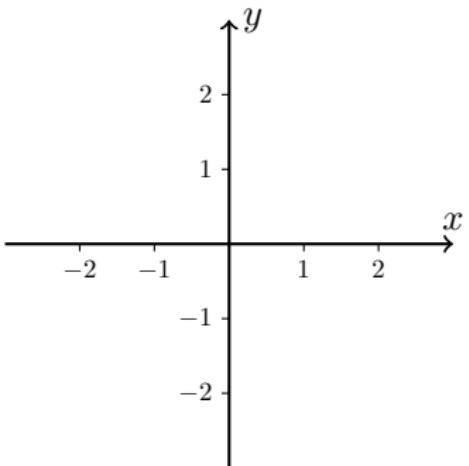
8. Polar Coordinates



Example

Draw the sets of points whose polar coordinates satisfy the following: $120^\circ \leq \theta \leq 150^\circ$.

solution:

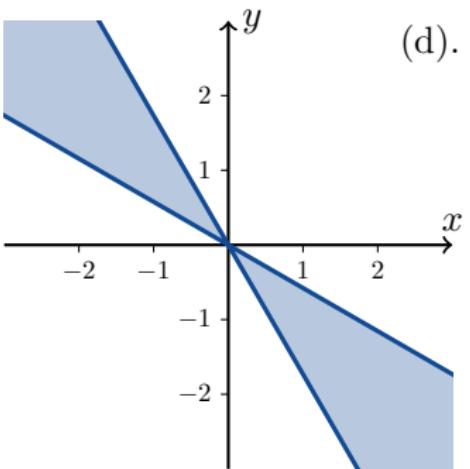


8. Polar Coordinates

Example

Draw the sets of points whose polar coordinates satisfy the following: $120^\circ \leq \theta \leq 150^\circ$.

solution:



8. Polar Coordinates

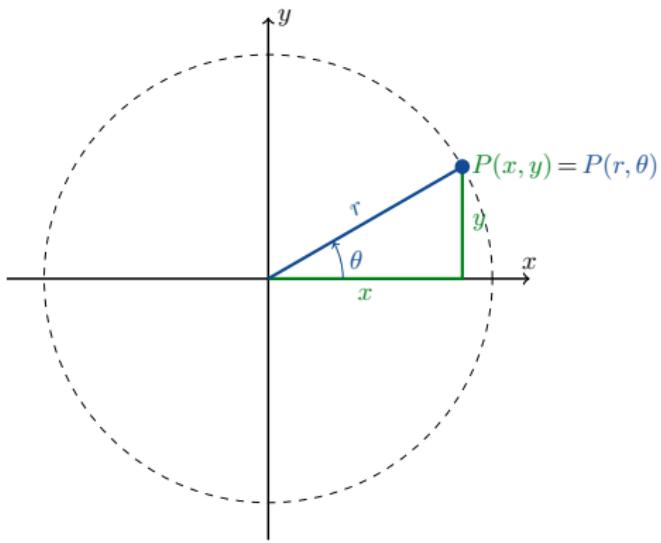
Relating Polar and Cartesian Coordinates

$$x = r \cos \theta$$

$$x^2 + y^2 = r^2$$

$$y = r \sin \theta$$

$$\tan \theta = \frac{y}{x}$$



8. Polar Coordinates



$$x = r \cos \theta \quad y = r \sin \theta \quad x^2 + y^2 = r^2 \quad \tan \theta = \frac{y}{x}$$

Example

Convert the polar coordinates $(r, \theta) = (-3, 90^\circ)$ into Cartesian coordinates.

solution:

$$(x, y) = (r \cos \theta, r \sin \theta) = (-3 \cos 90^\circ, -3 \sin 90^\circ) = (0, -3).$$

8. Polar Coordinates

Example

Find polar coordinates for the Cartesian coordinates $(x, y) = (5, -12)$.

solution: Choosing $r > 0$, we calculate that

$$r = \sqrt{x^2 + y^2} = \sqrt{25 + 144} = \sqrt{169} = 13.$$

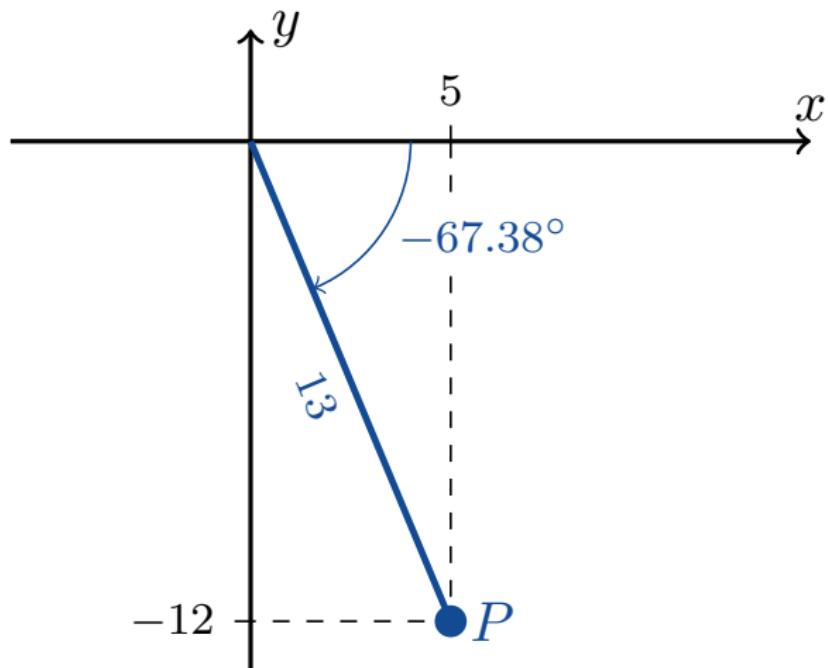
To find θ we use the equation $y = r \sin \theta$ to calculate that

$$\theta = \sin^{-1} \frac{y}{r} = \sin^{-1} \frac{-12}{13} \approx -67.38^\circ.$$

Therefore

$$(r, \theta) = (13, -67.38^\circ).$$

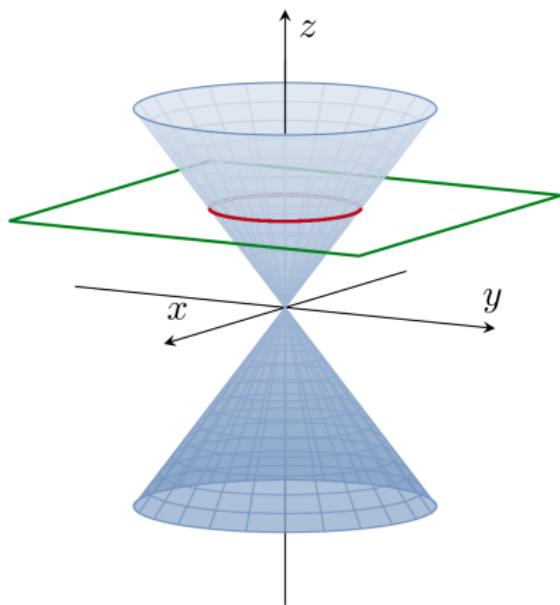
8. Polar Coordinates



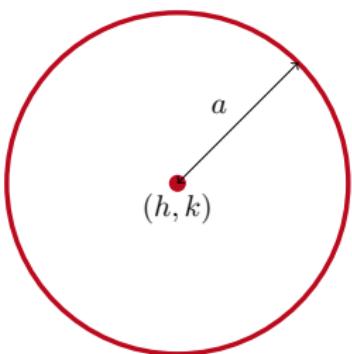
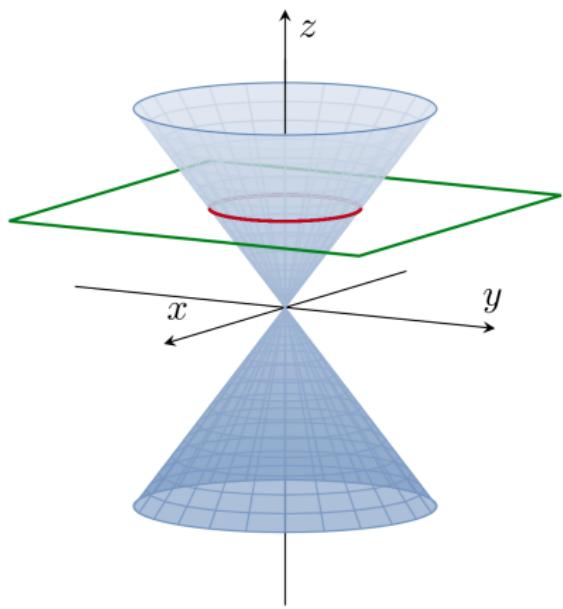


Conic Sections

9. Conic Sections



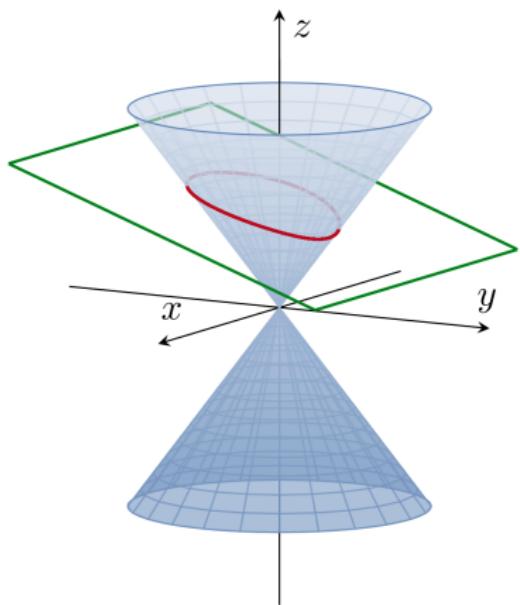
9. Conic Sections



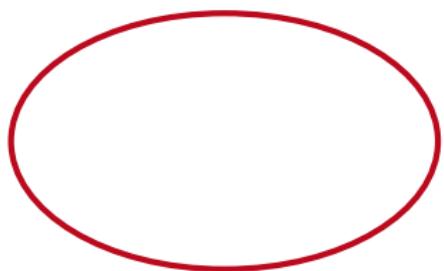
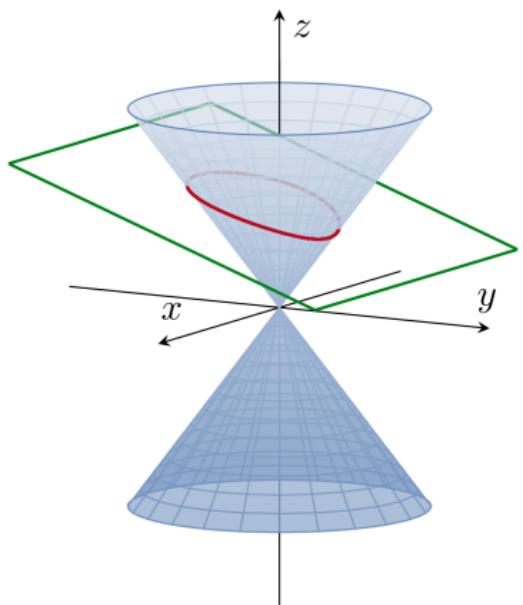
a circle

$$(x - h)^2 + (y - k)^2 = a^2$$

9. Conic Sections

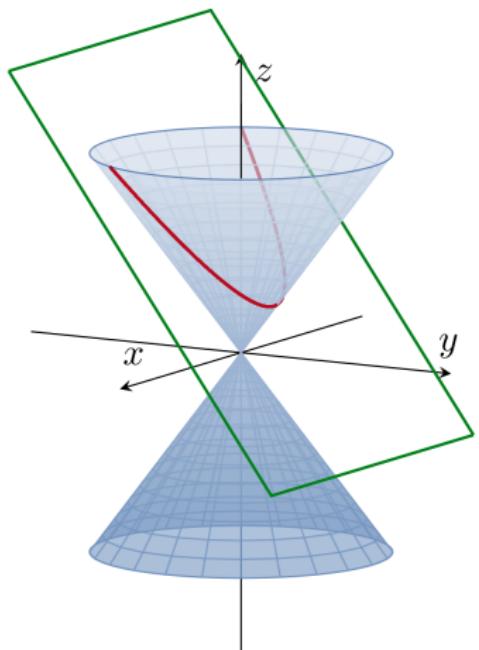


9. Conic Sections

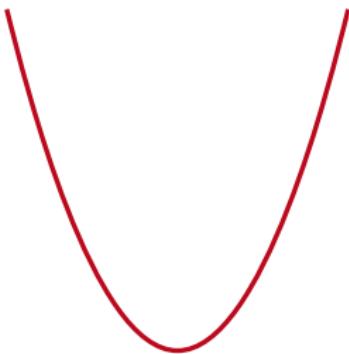
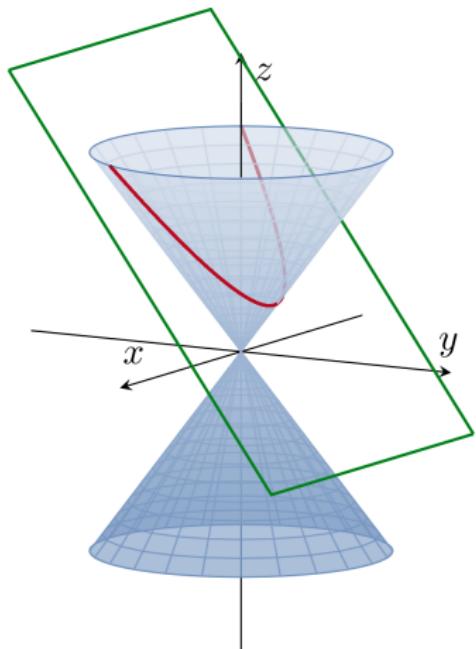


an ellipse

9. Conic Sections

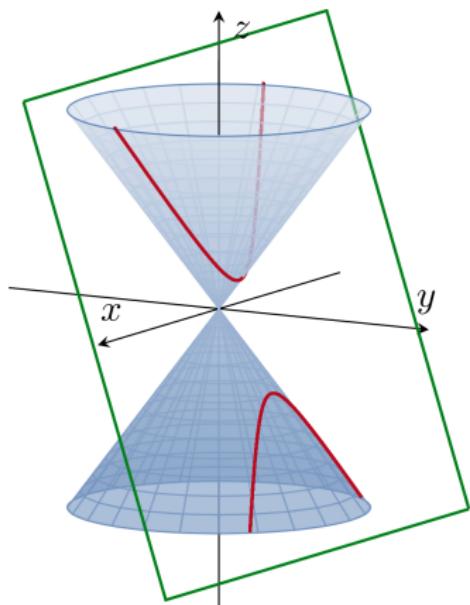


9. Conic Sections

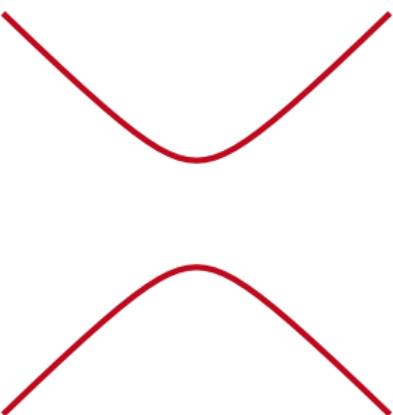
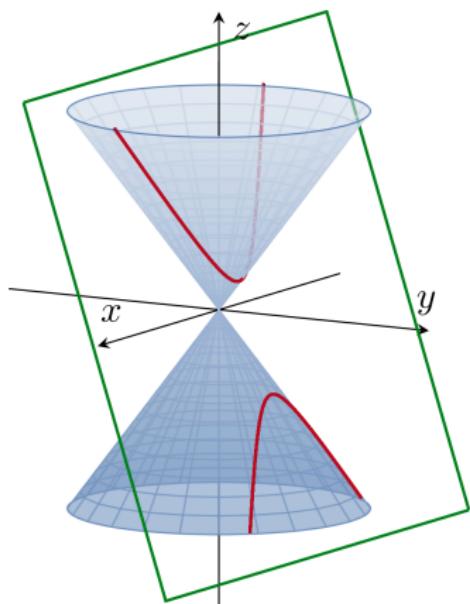


a parabola

9. Conic Sections

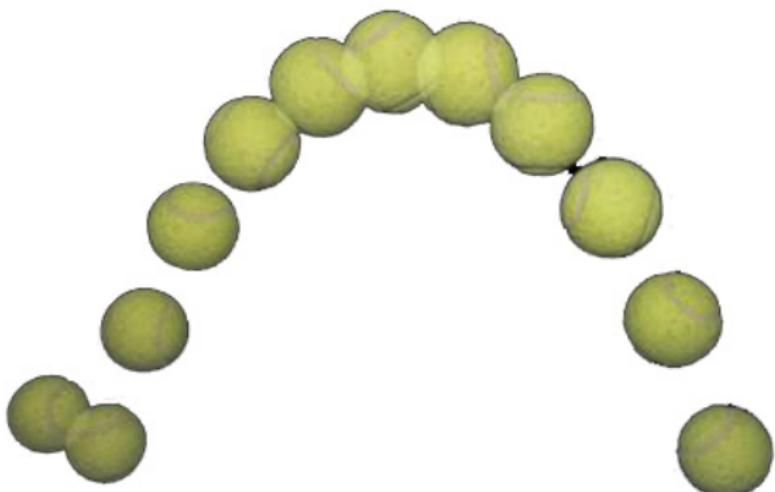


9. Conic Sections



a hyperbola

Parabolas



9. Conic Sections



Clifton suspension bridge, Bristol, UK.

The cables of a suspension bridges hang in a shape which is almost (but not exactly) a parabola.

9. Conic Sections

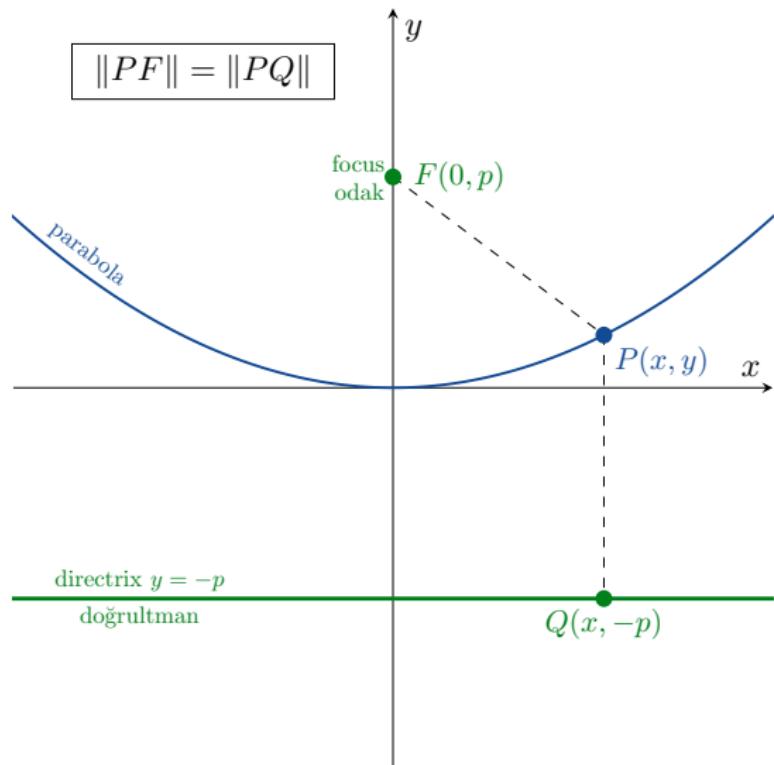


9. Conic Sections



To describe a parabola, we need a point called a *focus* and a line called a *directrix*.

9. Conic Sections



9. Conic Sections



Definition

A point $P(x, y)$ lies on the *parabola* if and only if

$$\|PF\| = \|PQ\|.$$

9. Conic Sections

Now

$$\begin{aligned}\|PF\| &= \text{distance between } P(x, y) \text{ and } F(0, p) \\ &= \sqrt{(x - 0)^2 + (y - p)^2} = \sqrt{x^2 + (y - p)^2}\end{aligned}$$

and

$$\begin{aligned}\|PQ\| &= \text{distance between } P(x, y) \text{ and } Q(x, -p) \\ &= \sqrt{(x - x)^2 + (y + p)^2} = \sqrt{(y + p)^2} = y + p.\end{aligned}$$

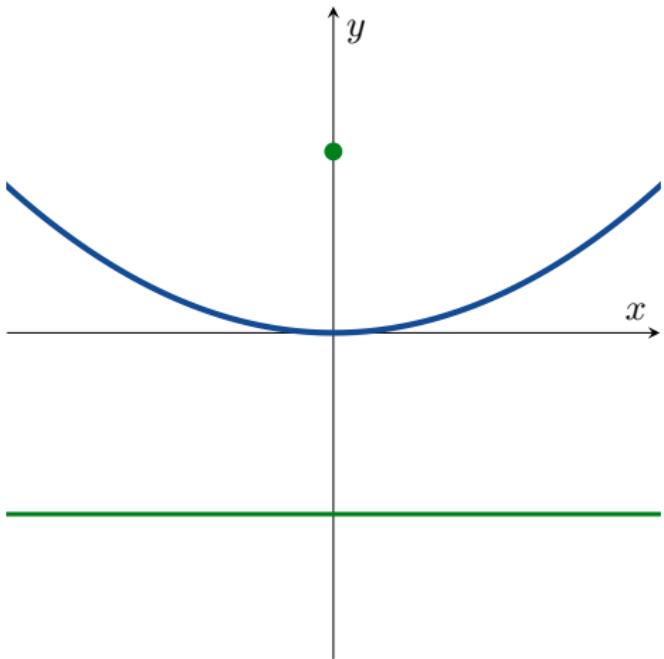
Therefore

$$\begin{aligned}\|PF\| &= \|PQ\| \\ \sqrt{x^2 + (y - p)^2} &= y + p \\ x^2 + (y - p)^2 &= (y + p)^2 \\ x^2 + y^2 - 2py + p^2 &= y^2 + 2py + p^2 \\ x^2 - 2py &= 2py \\ \boxed{x^2 = 4py}\end{aligned}$$

9. Conic Sections



equation: $x^2 = 4py$
focus: $F(0, p)$
directrix: $y = -p$



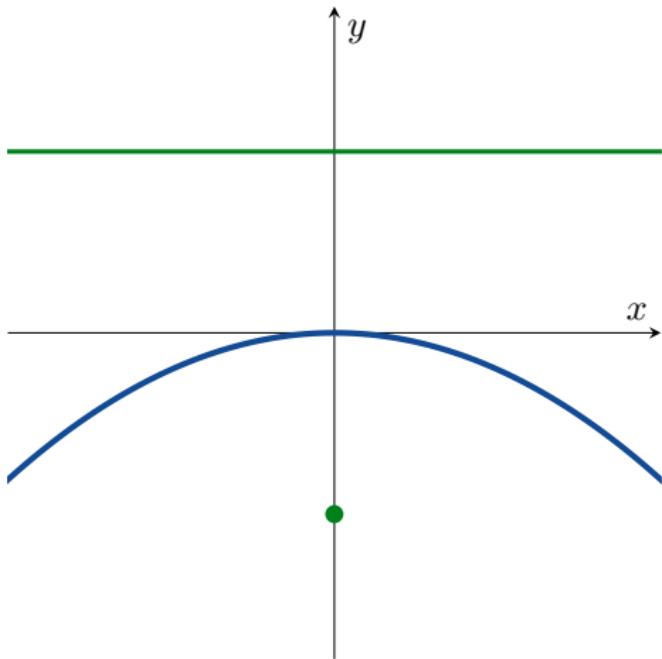
9. Conic Sections



equation: $x^2 = -4py$

focus: $F(0, -p)$

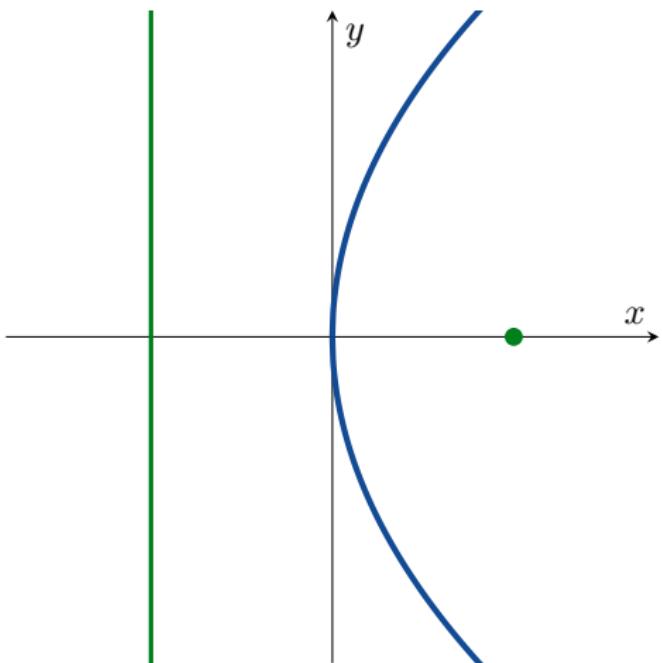
directrix: $y = p$



9. Conic Sections



equation: $y^2 = 4px$
focus: $F(p, 0)$
directrix: $x = -p$



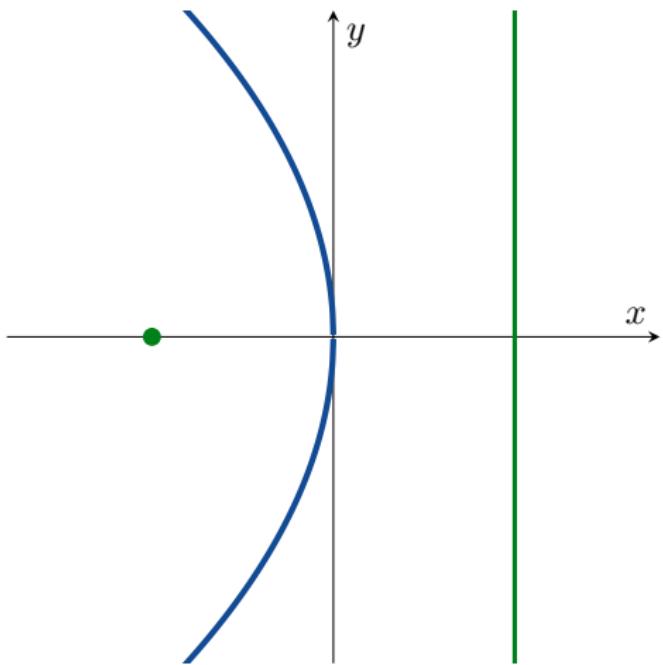
9. Conic Sections



equation: $y^2 = -4px$

focus: $F(-p, 0)$

directrix: $x = p$



9. Conic Sections



Example

Find the focus and directrix of the parabola $y^2 = 10x$.

solution: Our equation $y^2 = 10x$ looks like $y^2 = 4px$ with $p = \frac{10}{4} = 2.5$. Therefore the focus is at the point $F(2.5, 0)$ and the directrix is the line $x = -2.5$.

9. Conic Sections

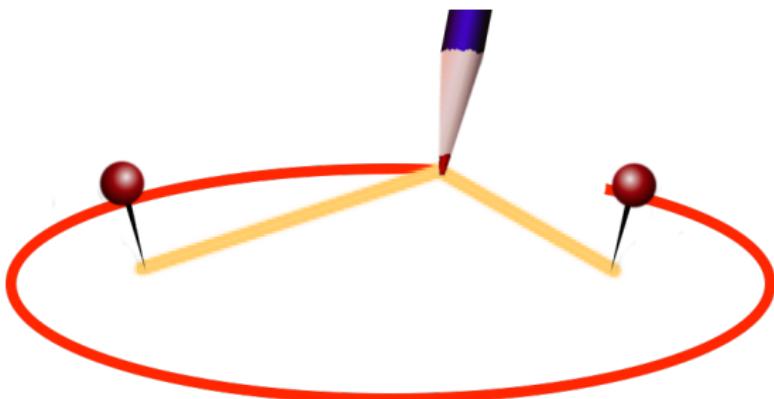


Example

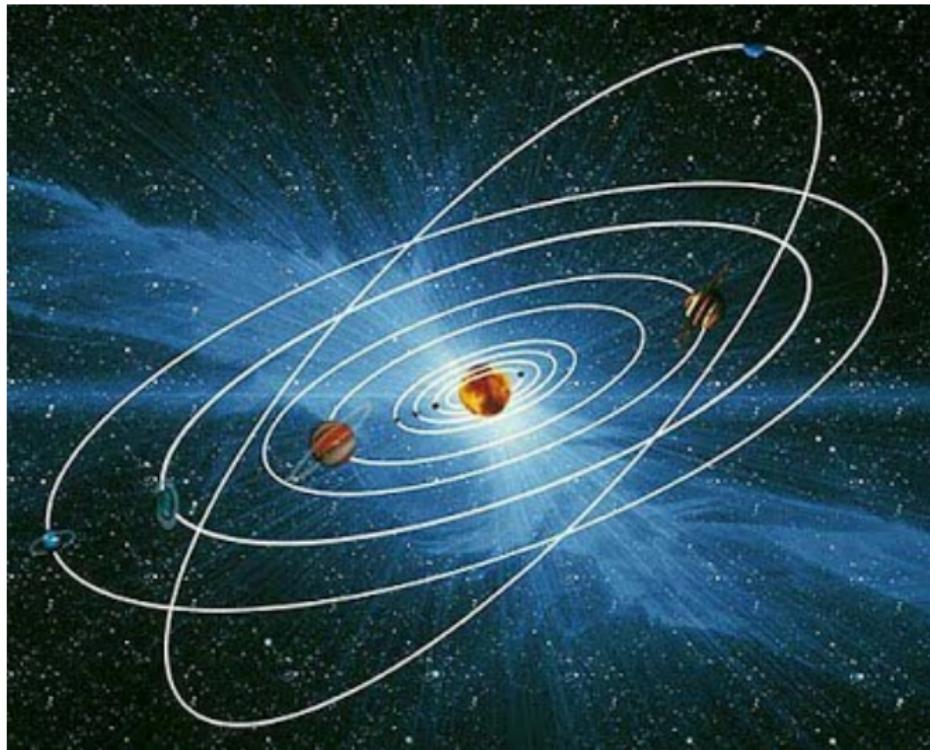
Find the equation for the parabola which has focus $F(0, -10)$ and directrix $y = 10$.

solution: Clearly $p = 10$ and $x^2 = -4py$. Therefore the answer is $x^2 = -40y$.

Ellipses



9. Conic Sections

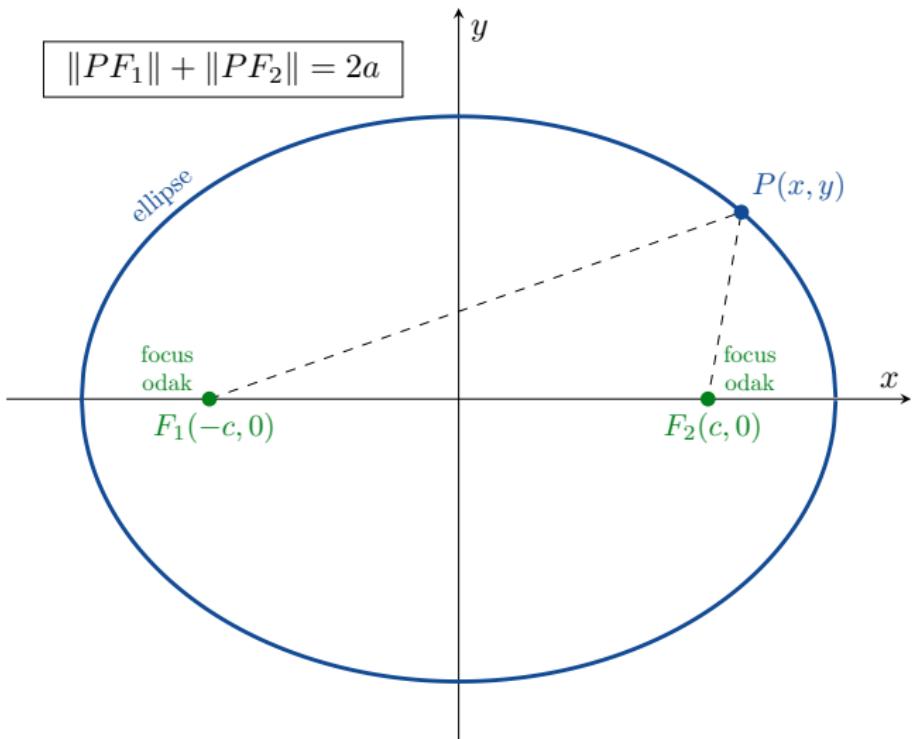


9. Conic Sections



Tycho Brahe Planetarium, Copenhagen, Denmark.

9. Conic Sections



To describe an ellipse, we need two *foci*.

9. Conic Sections

Definition

A point $P(x, y)$ is on the *ellipse* if and only if

$$\|PF_1\| + \|PF_2\| = 2a.$$

So

$$\sqrt{(x + c)^2 + y^2} + \sqrt{(x - c)^2 + y^2} = 2a.$$

This rearranges to

$$\frac{x^2}{a^2} + \frac{y^2}{a^2 - c^2} = 1.$$

If we set $b = \sqrt{a^2 - c^2}$, then we have

$$\boxed{\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1} \quad (0 < b < a).$$

9. Conic Sections

equation:

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \quad (0 < b < a)$$

centre-to-focus distance:

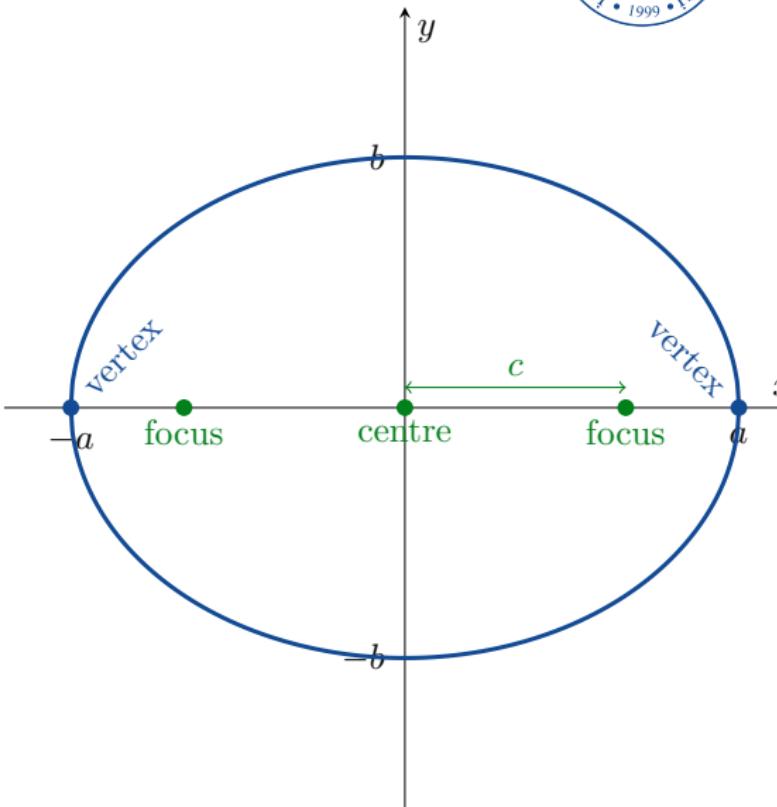
$$c = \sqrt{a^2 - b^2}$$

foci:

$$F_1(-c, 0) \quad \& \quad F_2(c, 0)$$

vertices :

$$(-a, 0) \quad \& \quad (a, 0)$$



9. Conic Sections

equation:

$$\frac{x^2}{b^2} + \frac{y^2}{a^2} = 1 \quad (0 < b < a)$$

centre-to-focus distance:

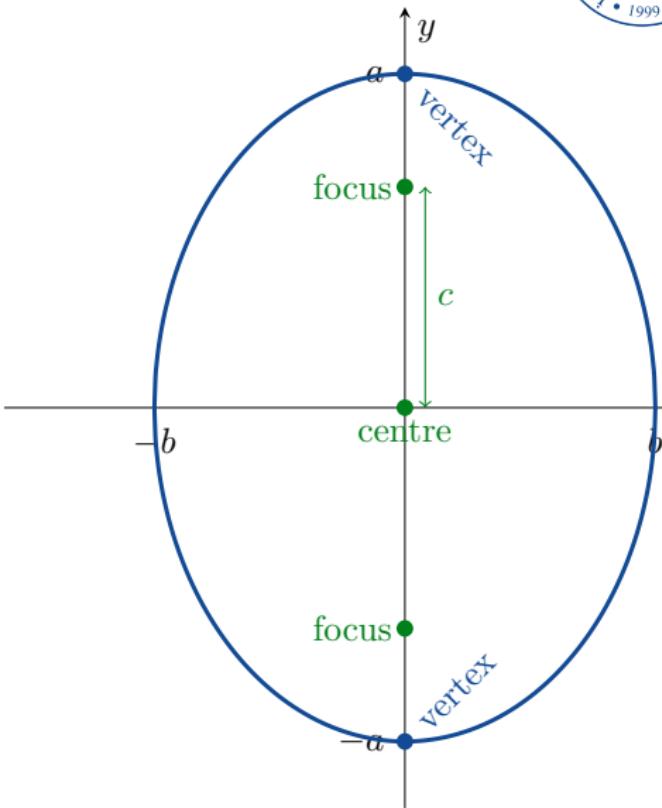
$$c = \sqrt{a^2 - b^2}$$

foci:

$$F_1(0, -c) \quad \& \quad F_2(0, c)$$

vertices :

$$(0, -a) \quad \& \quad (0, a)$$



9. Conic Sections



Example

The ellipse $\frac{x^2}{16} + \frac{y^2}{9} = 1$ has

- $a = 4$ and $b = 3$;
- centre-to-focus distance $c = \sqrt{a^2 - b^2} = \sqrt{16 - 9} = \sqrt{7}$;
- centre at $(0, 0)$;
- foci at $(-\sqrt{7}, 0)$ and $(\sqrt{7}, 0)$; and
- vertices at $(-4, 0)$ and $(4, 0)$.

9. Conic Sections



Example

The ellipse $\frac{x^2}{16} + \frac{y^2}{25} = 1$ has

- $a = 5$ and $b = 4$;
- centre-to-focus distance
 $c = \sqrt{a^2 - b^2} = \sqrt{25 - 16} = \sqrt{9} = 3$;
- centre at $(0, 0)$;
- foci at $(0, -3)$ and $(0, 3)$; and
- vertices at $(0, -5)$ and $(0, 5)$.

Hyperbolas

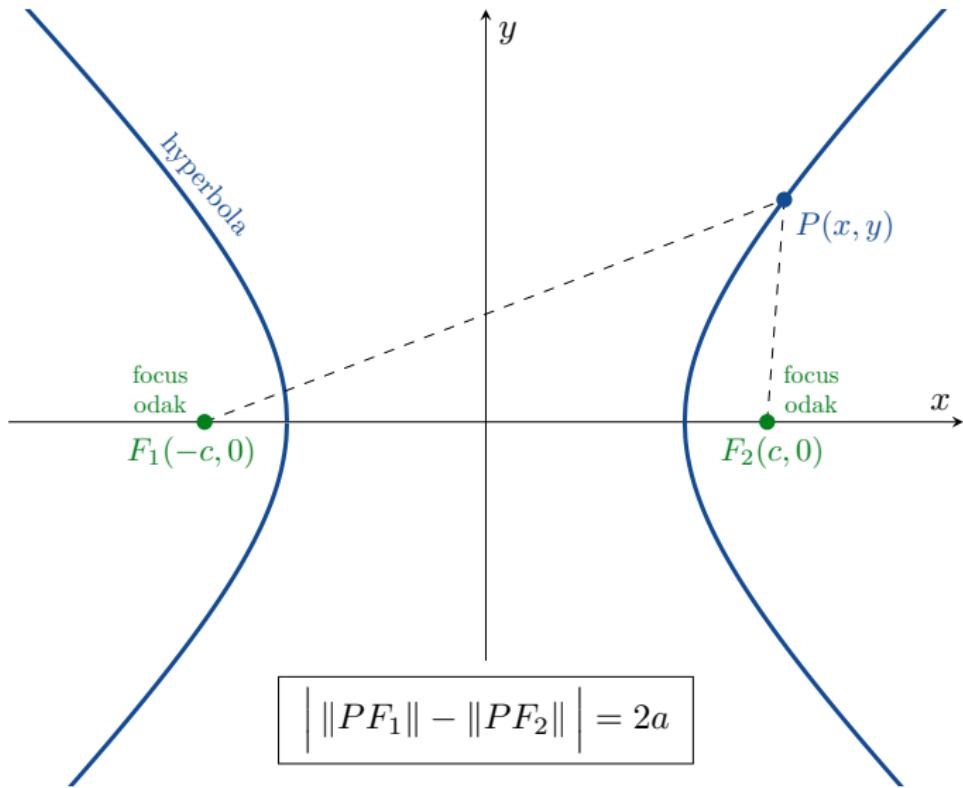


9. Conic Sections



Twin Arch 138, Ichinomiya City, Japan.

9. Conic Sections



9. Conic Sections



To describe a hyperbola, we again need two foci.

Definition

A point $P(x, y)$ is on the *hyperbola* if and only if

$$\left| \|PF_1\| - \|PF_2\| \right| = 2a.$$

9. Conic Sections



So

$$\sqrt{(x + c)^2 + y^2} - \sqrt{(x - c)^2 + y^2} = \pm 2a.$$

This rearranges to

$$\frac{x^2}{a^2} + \frac{y^2}{a^2 - c^2} = 1$$

where $c > a > 0$. If we set $b = \sqrt{c^2 - a^2}$, then

$$\boxed{\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1.}$$

9. Conic Sections

equation:

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

centre-to-focus distance:

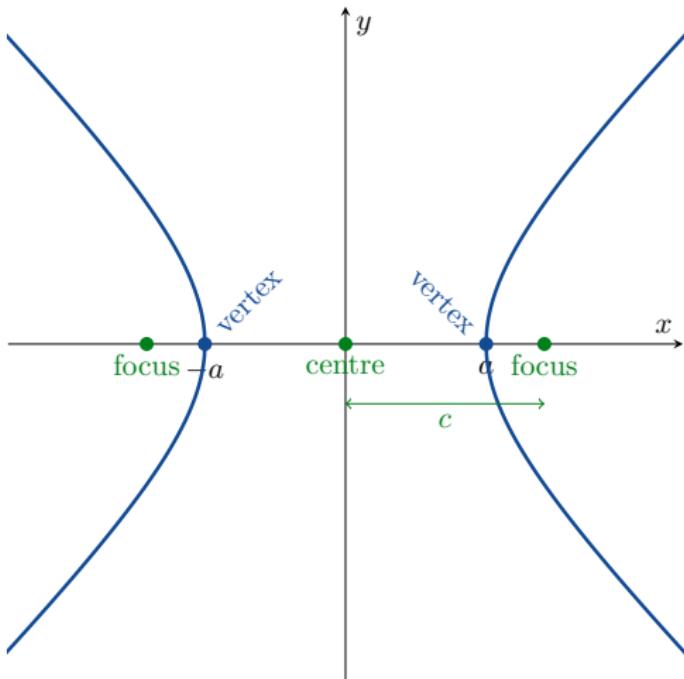
$$c = \sqrt{a^2 + b^2}$$

foci:

$$F_1(-c, 0) \quad \& \quad F_2(c, 0)$$

vertices :

$$(-a, 0) \quad \& \quad (a, 0)$$



9. Conic Sections

equation:

$$\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$$

centre-to-focus distance:

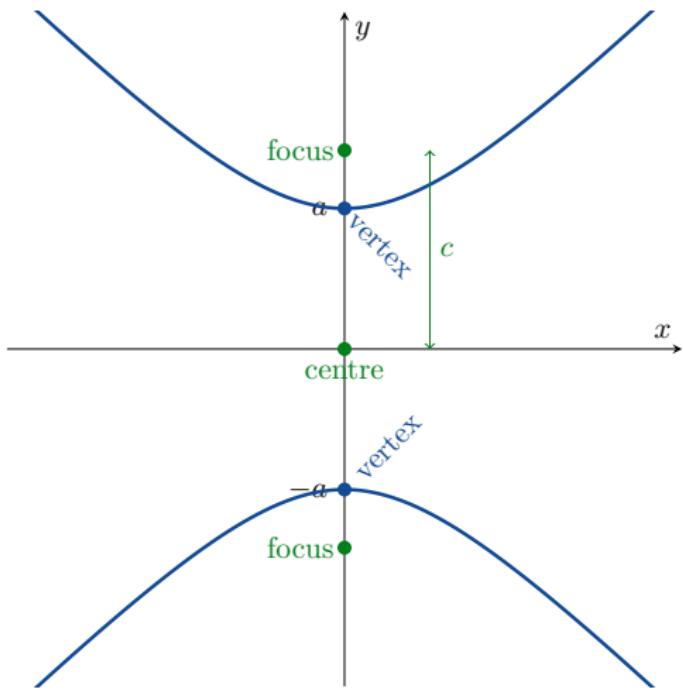
$$c = \sqrt{a^2 + b^2}$$

foci:

$$F_1(0, -c) \quad \& \quad F_2(0, c)$$

vertices :

$$(0, -a) \quad \& \quad (0, a)$$



9. Conic Sections

Example

The hyperbola $\frac{x^2}{4} - \frac{y^2}{5} = 1$ has

- $a = 2$ and $b = \sqrt{5}$;
- centre at $(0, 0)$;
- centre-to-focus distance $c = \sqrt{a^2 + b^2} = \sqrt{4 + 5} = 3$;
- foci at $(-3, 0)$ and $(3, 0)$; and
- vertices at $(-2, 0)$ and $(2, 0)$.

9. Conic Sections



Example

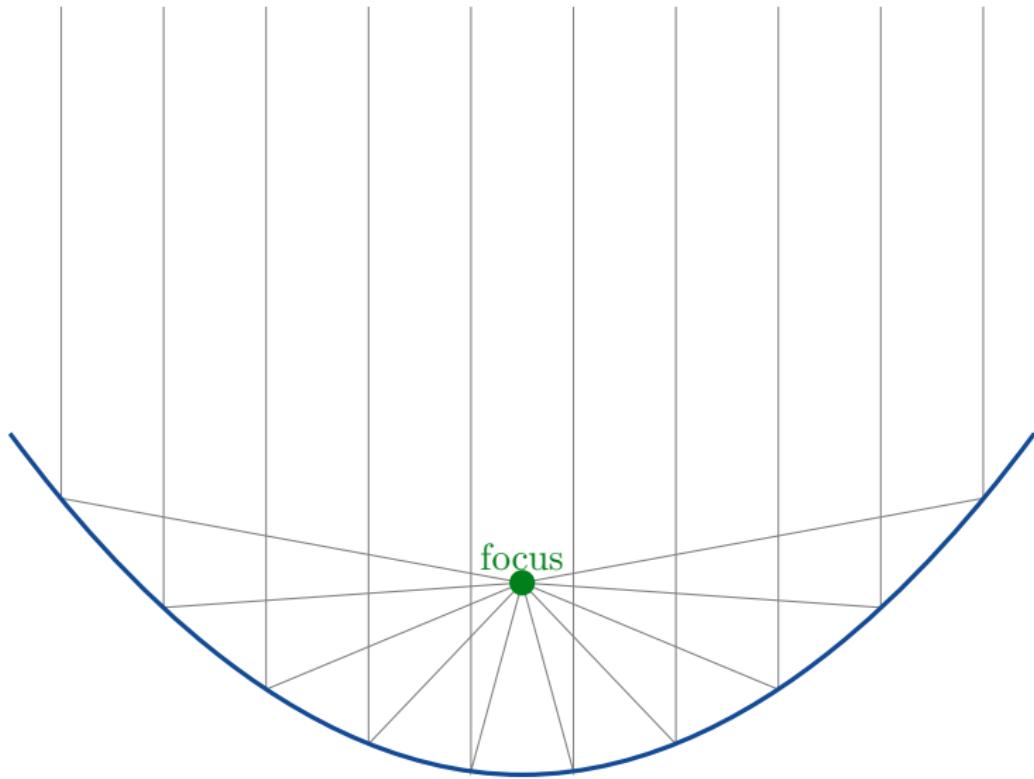
The hyperbola $\frac{y^2}{9} - \frac{x^2}{16} = 1$ has

- $a = 3$ and $b = 4$;
- centre at $(0, 0)$;
- centre-to-focus distance $c = \sqrt{a^2 + b^2} = \sqrt{9 + 16} = 5$;
- foci at $(0, -5)$ and $(0, 5)$; and
- vertices at $(0, -3)$ and $(0, 3)$.

Reflective Properties

Parabolas, ellipses and hyperbolas are useful in architecture and engineering because of their reflective properties.

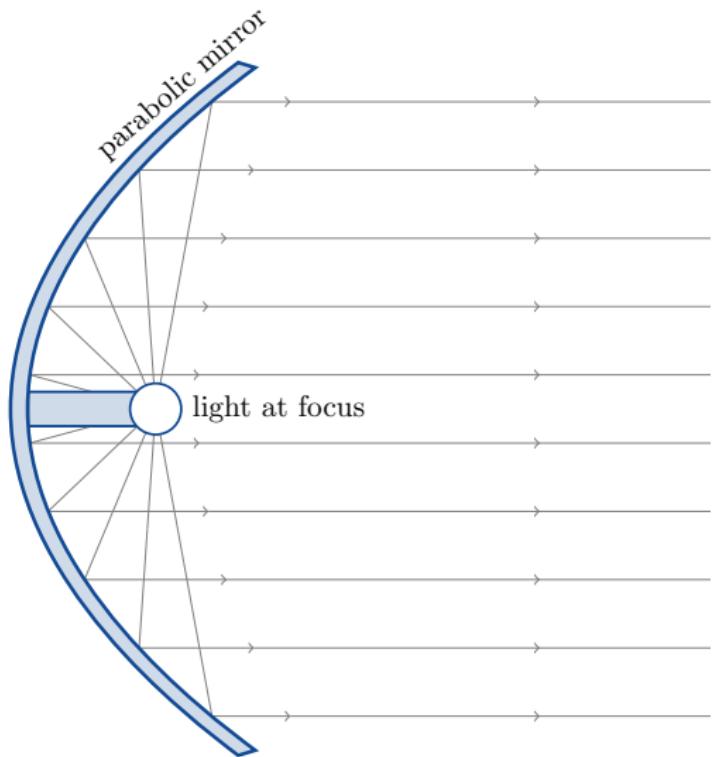
9. Conic Sections



9. Conic Sections



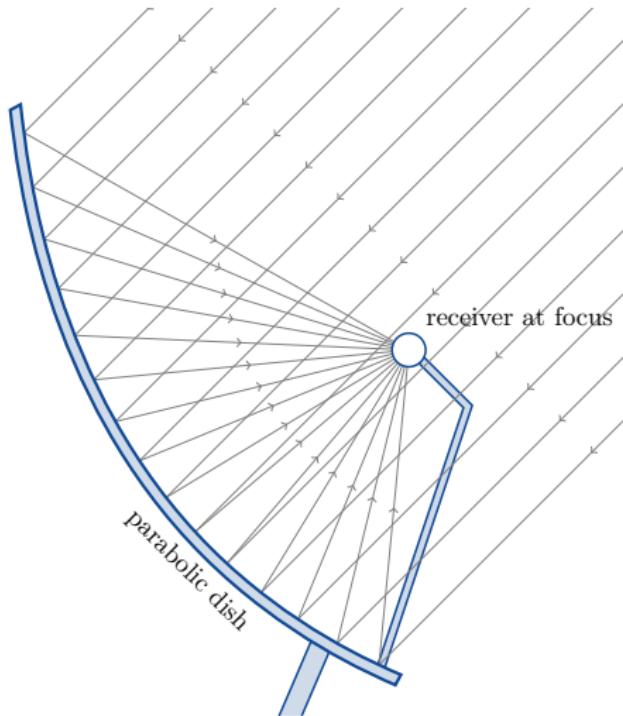
9. Conic Sections



9. Conic Sections



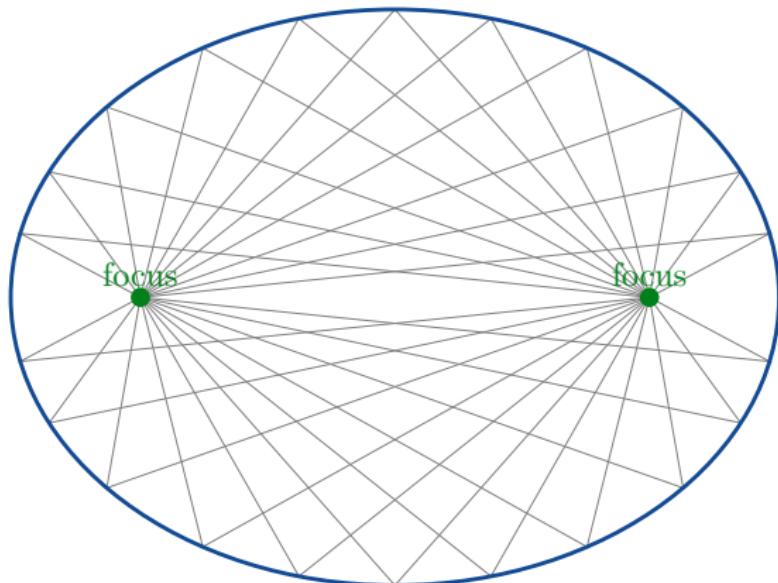
9. Conic Sections



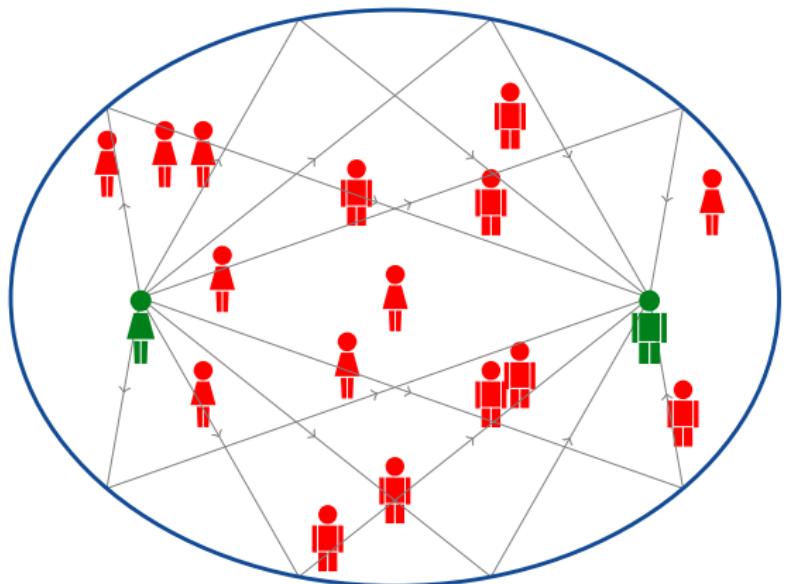
9. Conic Sections



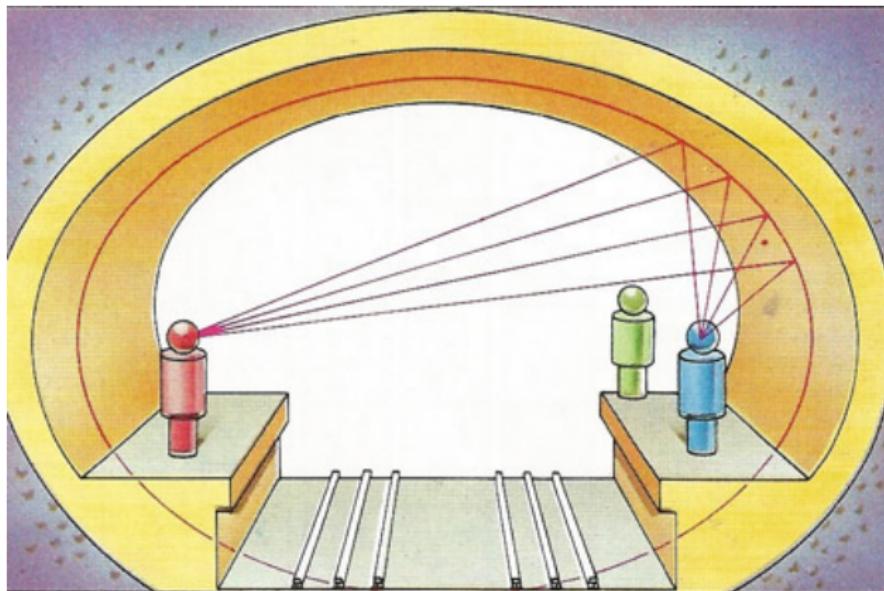
9. Conic Sections



9. Conic Sections



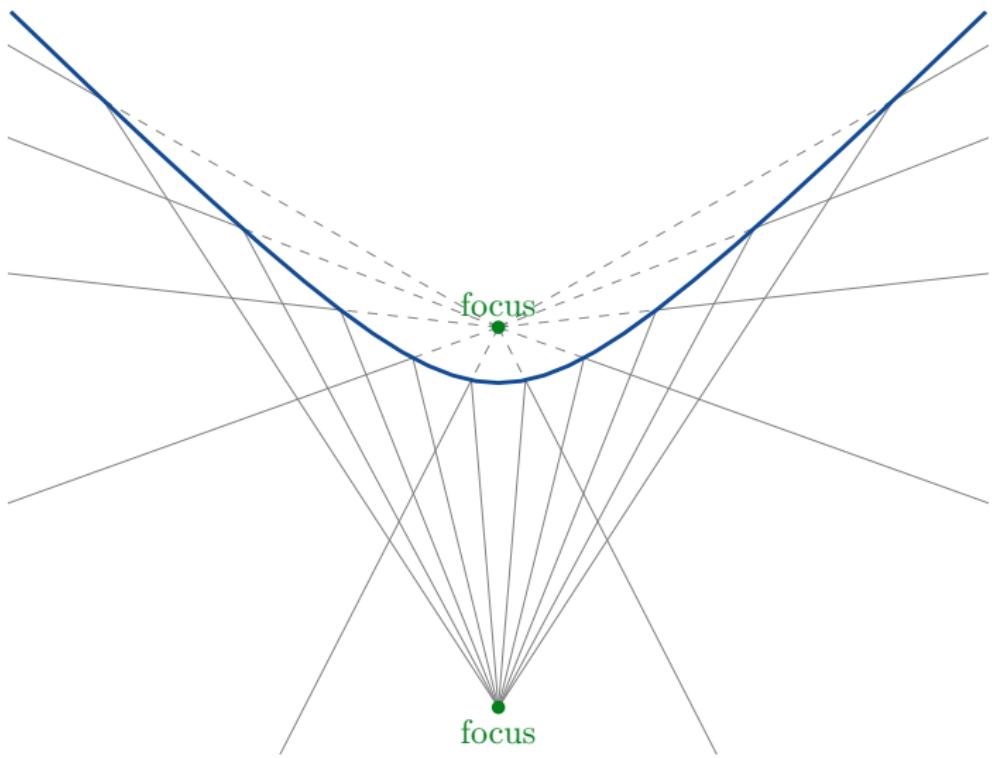
9. Conic Sections



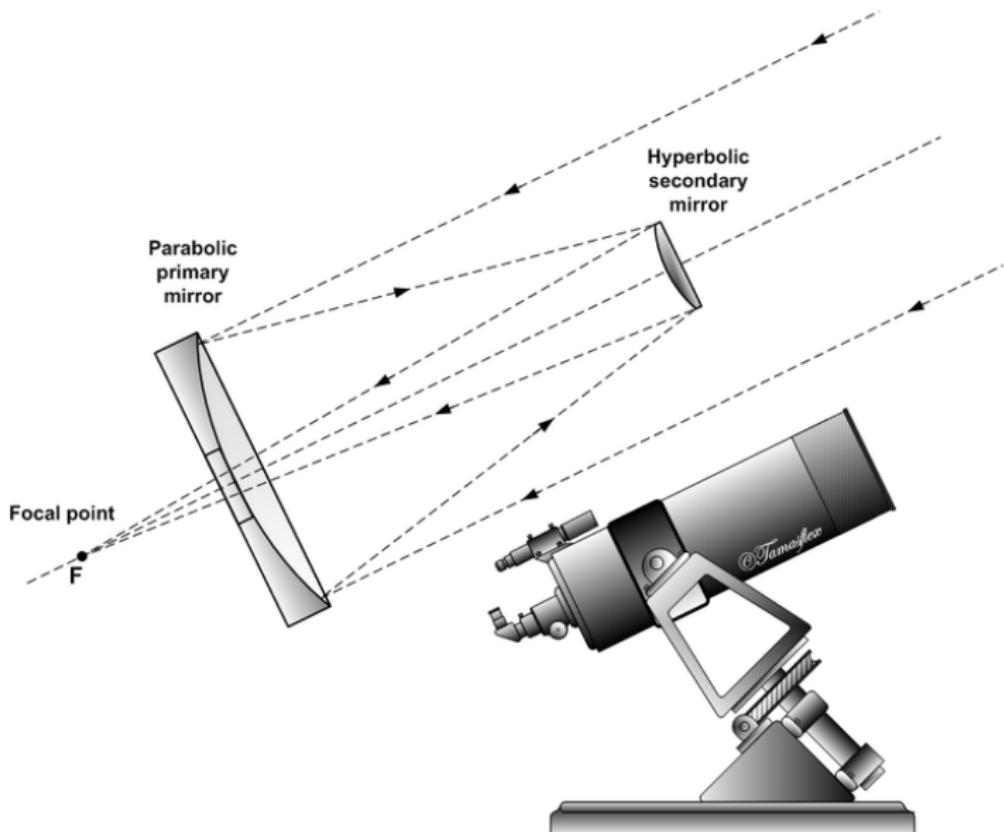
9. Conic Sections



9. Conic Sections



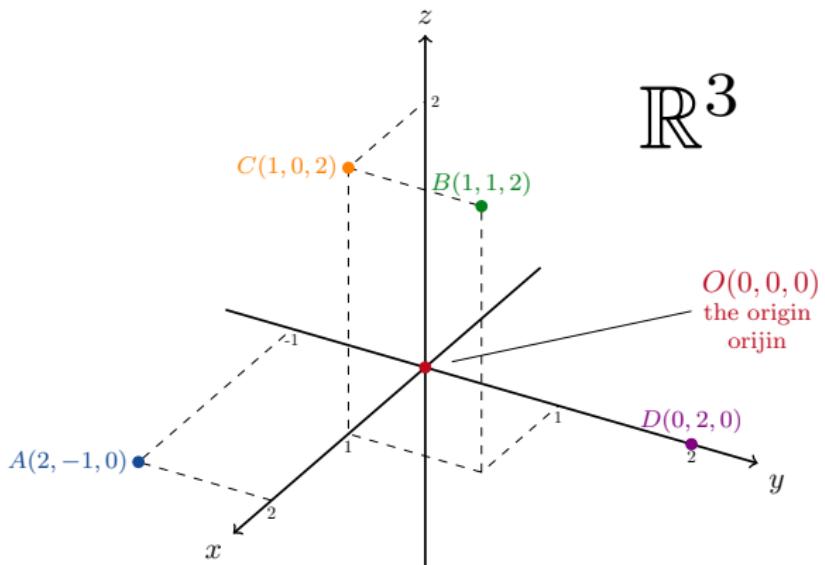
9. Conic Sections



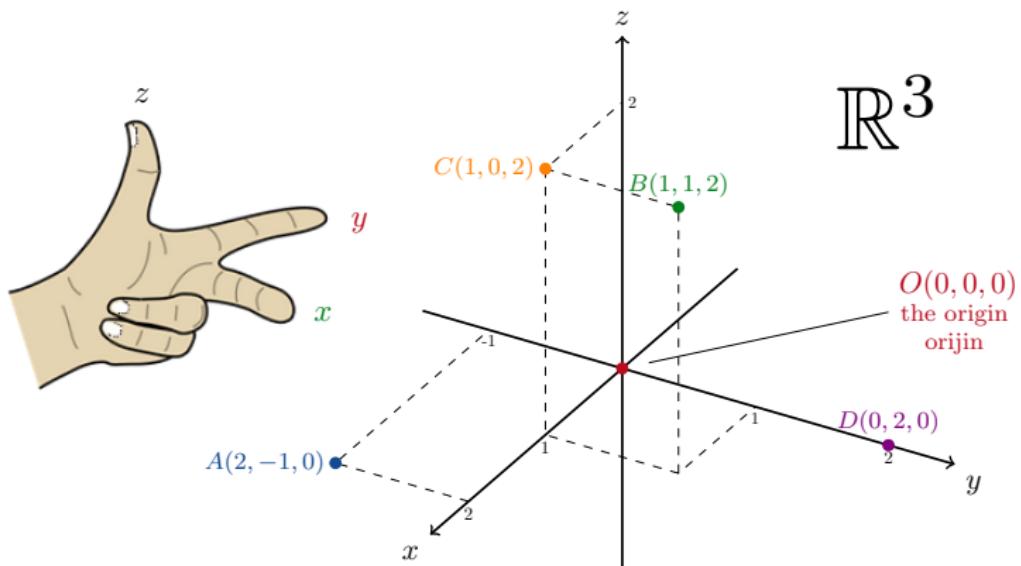


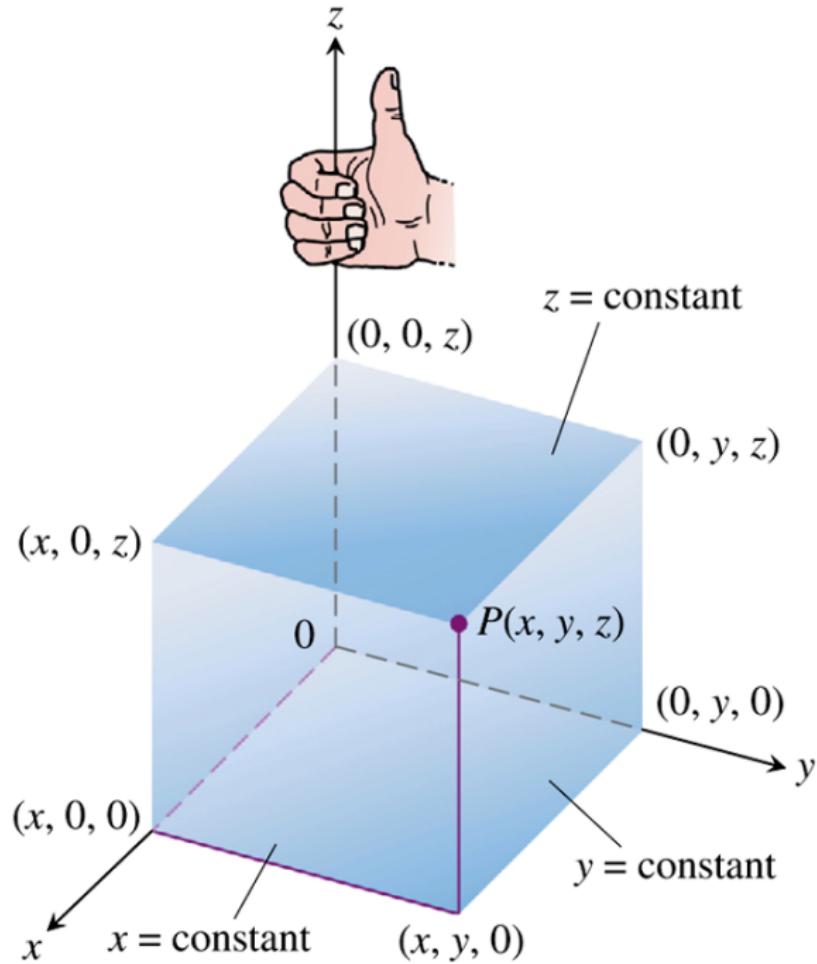
Three Dimensional Cartesian Coordinates

10. Three Dimensional Cartesian Coordinates

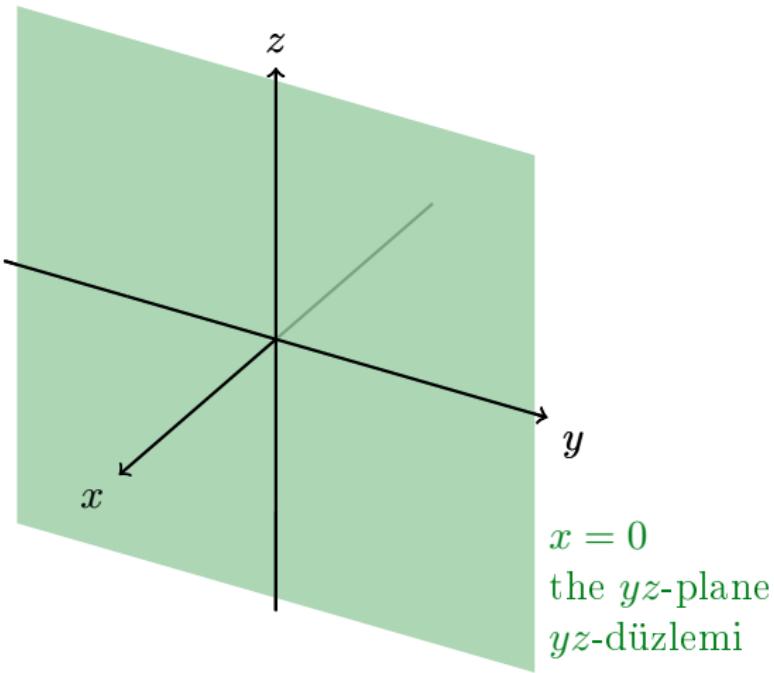


10. Three Dimensional Cartesian Coordinates



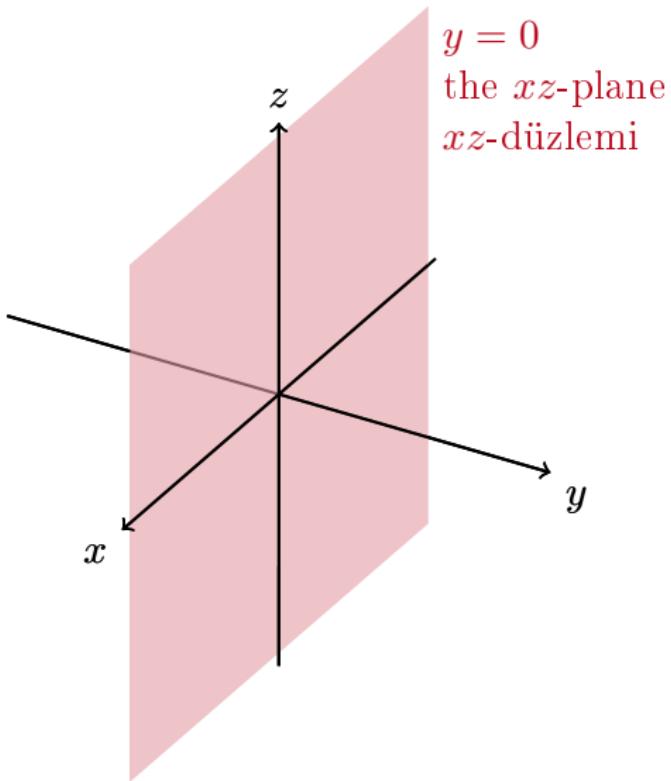


10. Three Dimensional Cartesian Coordinates

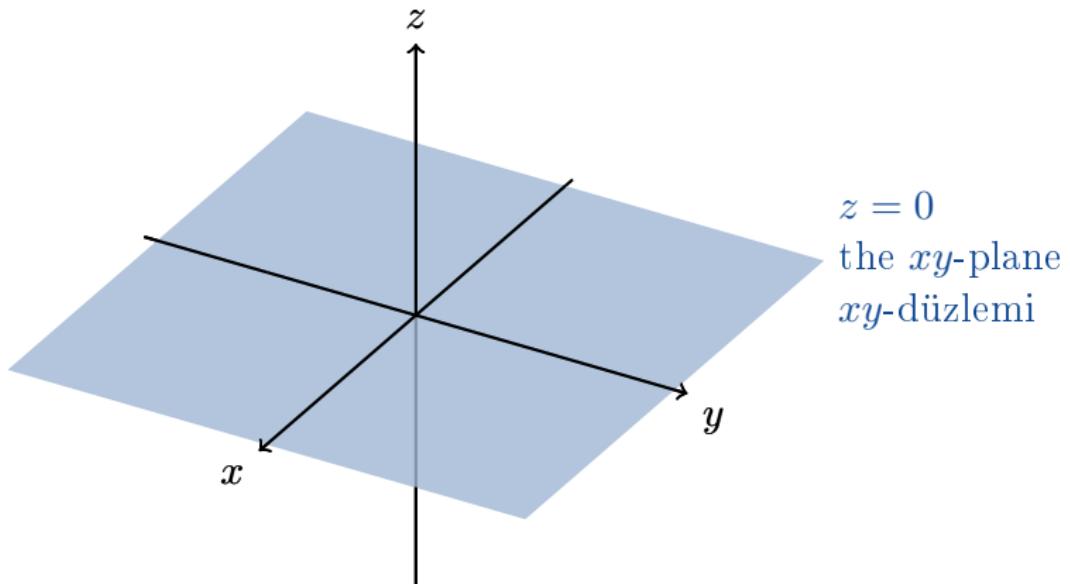


$x = 0$
the yz -plane
 yz -düzlemi

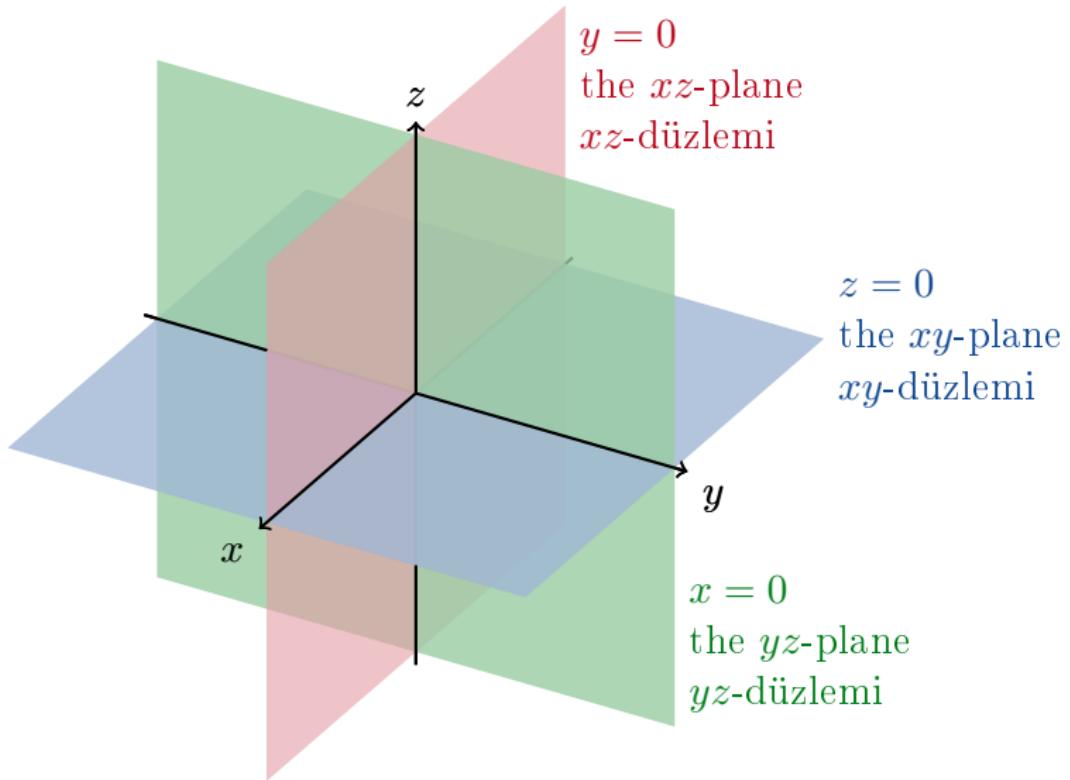
10. Three Dimensional Cartesian Coordinates



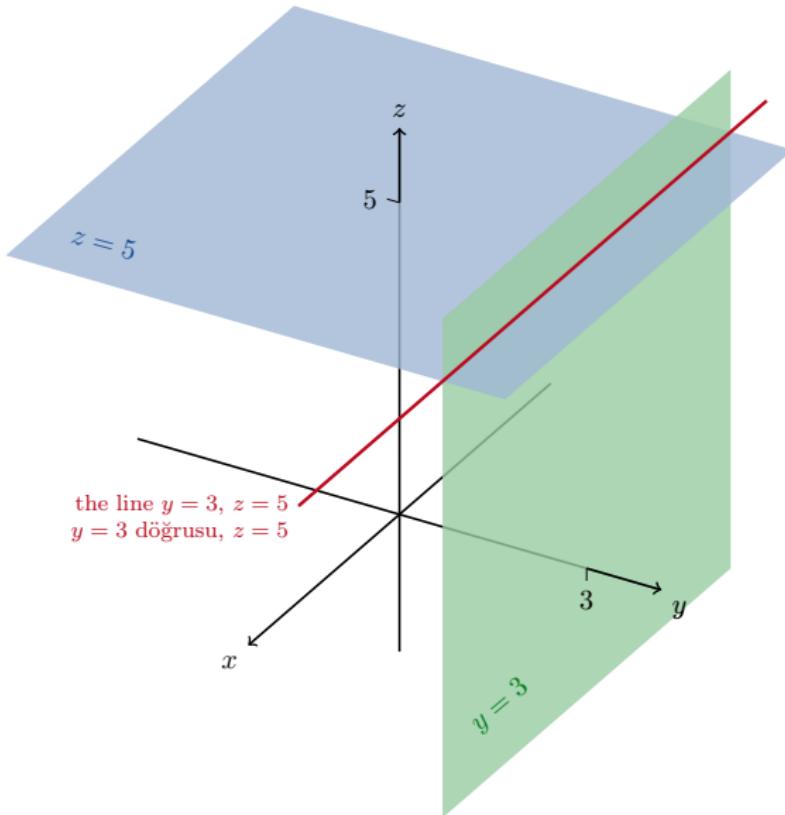
10. Three Dimensional Cartesian Coordinates



10. Three Dimensional Cartesian Coordinates



10. Three Dimensional Cartesian Coordinates



EXAMPLE 1 We interpret these equations and inequalities geometrically.

- (a) $z \geq 0$ The half-space consisting of the points on and above the xy -plane.
- (b) $x = -3$ The plane perpendicular to the x -axis at $x = -3$. This plane lies parallel to the yz -plane and 3 units behind it.
- (c) $z = 0, x \leq 0, y \geq 0$ The second quadrant of the xy -plane.
- (d) $x \geq 0, y \geq 0, z \geq 0$ The first octant.
- (e) $-1 \leq y \leq 1$ The slab between the planes $y = -1$ and $y = 1$ (planes included).
- (f) $y = -2, z = 2$ The line in which the planes $y = -2$ and $z = 2$ intersect. Alternatively, the line through the point $(0, -2, 2)$ parallel to the x -axis. ■

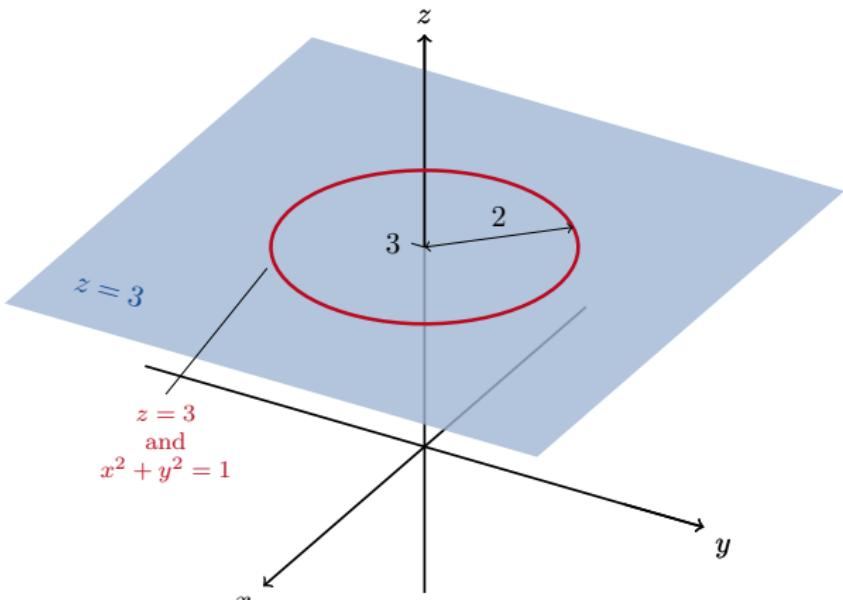
10. Three Dimensional Cartesian Coordinates



Example

Which points $P(x, y, z)$ satisfy $x^2 + y^2 = 4$ and $z = 3$?

We know that $z = 3$ is a horizontal plane and we recognise that $x^2 + y^2 = 4$ is the equation of a circle of radius 2.



10. Three Dimensional Cartesian Coordinates



Distance in \mathbb{R}^3

Definition

The set

$$\{(x, y, z) \mid x, y, z \in \mathbb{R}\}$$

is denoted by \mathbb{R}^3 .

10. Three Dimensional Cartesian Coordinates



Definition

The *distance* between $P_1(x_1, y_1, z_1)$ and $P_2(x_2, y_2, z_2)$ is

$$\|P_1P_2\| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}.$$

10.

$$\|P_1P_2\| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}.$$



Example

The distance between $A(2, 1, 5)$ and $B(-2, 3, 0)$ is

10.

$$\|P_1P_2\| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}.$$



Example

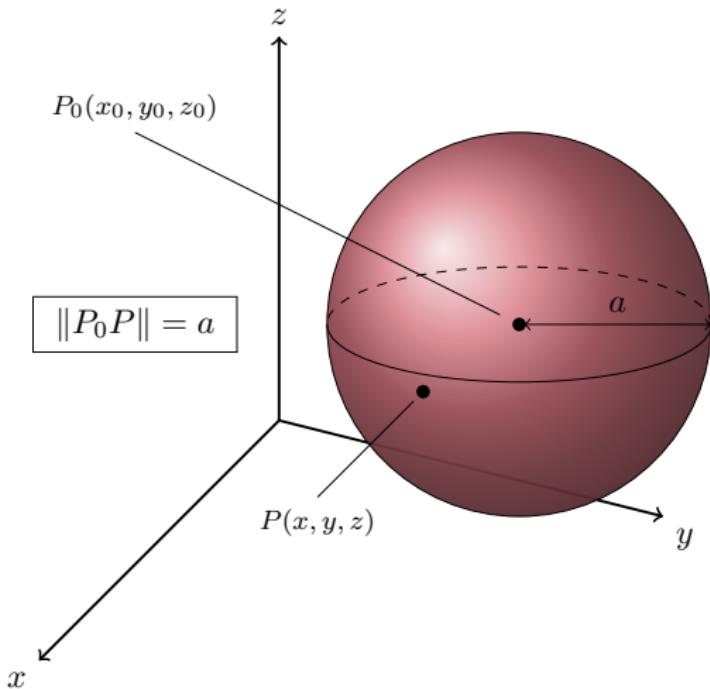
The distance between $A(2, 1, 5)$ and $B(-2, 3, 0)$ is

$$\begin{aligned}\|AB\| &= \sqrt{((-2) - 2)^2 + (3 - 1)^2 + (0 - 5)^2} \\ &= \sqrt{16 + 4 + 25} = \sqrt{45} \\ &= 3\sqrt{5} \approx 6.7.\end{aligned}$$

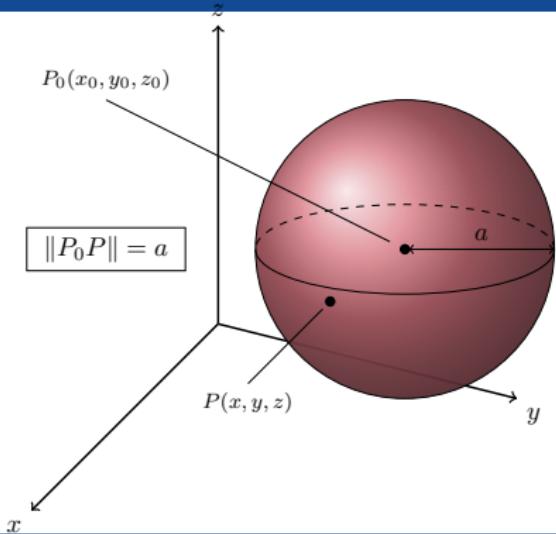
10. Three Dimensional Cartesian Coordinates



Spheres



10. Three Dimensional Cartesian Coordinates



Definition

The *standard equation for a sphere* of radius a centred at $P_0(x_0, y_0, z_0)$ is

$$(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2 = a^2.$$

10.

$$(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2 = a^2.$$



Example

Find the centre and radius of the sphere

$$x^2 + y^2 + z^2 + 3x - 4z + 1 = 0.$$

10.

$$(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2 = a^2.$$



Example

Find the centre and radius of the sphere

$$x^2 + y^2 + z^2 + 3x - 4z + 1 = 0.$$

First we need to put this equation into the standard form.

10.

$$(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2 = a^2.$$



Since $(x - b)^2 = x^2 - 2bx + b^2$ we have that

$$x^2 + y^2 + z^2 + 3x - 4z + 1 = 0$$

$$(x^2 + 3x) + y^2 + (z^2 - 4z) = -1$$

10.

$$(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2 = a^2.$$



Since $(x - b)^2 = x^2 - 2bx + b^2$ we have that

$$x^2 + y^2 + z^2 + 3x - 4z + 1 = 0$$

$$(x^2 + 3x) + y^2 + (z^2 - 4z) = -1$$

$$\left(x^2 + 3x + \frac{9}{4}\right) - \frac{9}{4} + y^2 + (z^2 - 4z + 4) - 4 = -1$$

10.

$$(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2 = a^2.$$



Since $(x - b)^2 = x^2 - 2bx + b^2$ we have that

$$x^2 + y^2 + z^2 + 3x - 4z + 1 = 0$$

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$$\left(x^2 + 3x + \frac{9}{4}\right) - \frac{9}{4} + y^2 + (z^2 - 4z + 4) - 4 = -1$$

$$\left(x^2 + 3x + \frac{9}{4}\right) + y^2 + (z^2 - 4z + 4) = -1 + \frac{9}{4} + 4$$

10.

$$(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2 = a^2.$$



Since $(x - b)^2 = x^2 - 2bx + b^2$ we have that

$$x^2 + y^2 + z^2 + 3x - 4z + 1 = 0$$

$$(x^2 + 3x) + y^2 + (z^2 - 4z) = -1$$

$$\left(x^2 + 3x + \frac{9}{4}\right) - \frac{9}{4} + y^2 + (z^2 - 4z + 4) - 4 = -1$$

$$\left(x^2 + 3x + \frac{9}{4}\right) + y^2 + (z^2 - 4z + 4) = -1 + \frac{9}{4} + 4$$

$$\left(x + \frac{3}{2}\right)^2 + y^2 + (z - 2)^2 = \frac{21}{4}.$$

10.

$$(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2 = a^2.$$



Since $(x - b)^2 = x^2 - 2bx + b^2$ we have that

$$x^2 + y^2 + z^2 + 3x - 4z + 1 = 0$$

$$(x^2 + 3x) + y^2 + (z^2 - 4z) = -1$$

$$\left(x^2 + 3x + \frac{9}{4}\right) - \frac{9}{4} + y^2 + (z^2 - 4z + 4) - 4 = -1$$

$$\left(x^2 + 3x + \frac{9}{4}\right) + y^2 + (z^2 - 4z + 4) = -1 + \frac{9}{4} + 4$$

$$\left(x + \frac{3}{2}\right)^2 + y^2 + (z - 2)^2 = \frac{21}{4}.$$

The centre is at $P_0(x_0, y_0, z_0) = P_0(-\frac{3}{2}, 0, 2)$ and the radius is

$$a = \sqrt{\frac{21}{4}} = \frac{\sqrt{3}\sqrt{7}}{2}.$$

10.

$$(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2 = a^2.$$



Example

Find the centre and radius of the sphere

$$x^2 + y^2 + z^2 + 6x - 6y + 6z = 7.$$

10.

$$(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2 = a^2.$$



Since $(x - b)^2 = x^2 - 2bx + b^2$ we have that

$$x^2 + y^2 + z^2 + 6x - 6y + 6z = 7$$

$$(x^2 + 6x) + (y^2 - 6y) + (z^2 + 6z) = 7$$

$$(x^2 + 6x + 9) - 9 + (y^2 - 6y + 9) - 9 + (z^2 + 6z + 9) - 9 = 7$$

$$(x^2 + 6x + 9) + (y^2 - 6y + 9) + (z^2 + 6z + 9) = 7 + 9$$

$$(x + 3)^2 + (y - 3)^2 + (z + 3)^2 = 16$$

The centre is at $P_0(x_0, y_0, z_0) = P_0(-3, 3, -3)$ and the radius is $a = \sqrt{16} = 4$.

EXAMPLE 5 Here are some geometric interpretations of inequalities and equations involving spheres.

(a) $x^2 + y^2 + z^2 < 4$

The interior of the sphere $x^2 + y^2 + z^2 = 4$.

(b) $x^2 + y^2 + z^2 \leq 4$

The solid ball bounded by the sphere $x^2 + y^2 + z^2 = 4$. Alternatively, the sphere $x^2 + y^2 + z^2 = 4$ together with its interior.

(c) $x^2 + y^2 + z^2 > 4$

The exterior of the sphere $x^2 + y^2 + z^2 = 4$.

(d) $x^2 + y^2 + z^2 = 4, z \leq 0$

The lower hemisphere cut from the sphere $x^2 + y^2 + z^2 = 4$ by the xy -plane (the plane $z = 0$). ■



Next Time

- 11. Vectors
- 12. The Dot Product
- 13. The Cross Product