Lecture 11

- 9.2 Infinite Series
- 9.3 The Integral Test
- 9.4 Comparison Tests
- 9.5 Absolute Convergence; The Ratio and Root Tests

9.3 Infinite Series

Let $(a_n)_{n=1}^{\infty}$ be a sequence:

$$a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8, a_9, a_{10}, a_{11}, a_{12}, a_{13}, a_{14}, a_{15}, a_{16}, \ldots$$

Then

$$\sum_{k=1}^{\infty} a_k = a_1 + a_2 + a_3 + a_4 + a_5 + a_6 + a_7 + a_8 + a_9 + a_{10} + a_{11} + \ldots$$

is a *series*.

9.3 Infinite Series

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Then

$$\sum_{k=1}^{\infty} a_k = a_1 + a_2 + a_3 + a_4 + a_5 + a_6 + a_7 + a_8 + a_9 + a_{10} + a_{11} + \ldots$$

is a *series*.

Definition

Let (a_n) be a sequence of real numbers. Let

$$s_n := \sum_{k=1}^{n} a_k = a_1 + a_2 + a_3 + a_4 + a_5 + a_6 + a_7 + \ldots + a_n.$$

Then s_n is called a *partial sum* of the series $\sum_{k=1}^{\infty} a_k$.

9.3 Infinite Series

Definition

We say that $\sum_{k=1}^{\infty} a_k$ *converges* iff (s_n) converges.

Definition

If $s_n \to s$ as $n \to \infty$, we say that s is the *sum of the series* and we write

$$\sum_{k=1}^{\infty} a_k = s.$$

Definition

If $\sum_{k=1}^{\infty} a_k$ does not converge, then we say that $\sum_{k=1}^{\infty} a_k$ *diverges*.

Remark

Sometimes the notation "$\sum_{k=1}^{\infty} a_k$" means $a_1 + a_2 + a_3 + a_4 + a_5 + a_6 + a_7 + \ldots$ which might converge or might diverge. Sometimes the notation "$\sum_{k=1}^{\infty} a_k$" means the sum of the series, i.e.

$$\sum_{k=1}^{\infty} a_k = \lim_{n \to \infty} \sum_{k=1}^{n} a_k = \lim_{n \to \infty} s_n = s.$$

You need to be able to understand what I mean every time I write "$\sum_{k=1}^{\infty} a_k$".

Example

$$\sum_{k=1}^{\infty} 1 = 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1 + \ldots$$

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Note that

$$s_n = \sum_{k=1}^{n} 1 = 1+1+\ldots+ = n \to \infty$$

as $n \to \infty$.

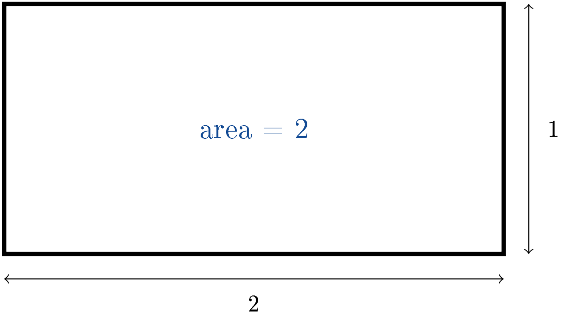
Example

$$\sum_{k=1}^{\infty} 1 = 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1 + \dots$$

Note that

$$s_n = \sum_{k=1}^{n} 1 = 1 + 1 + \ldots + = n \to \infty$$

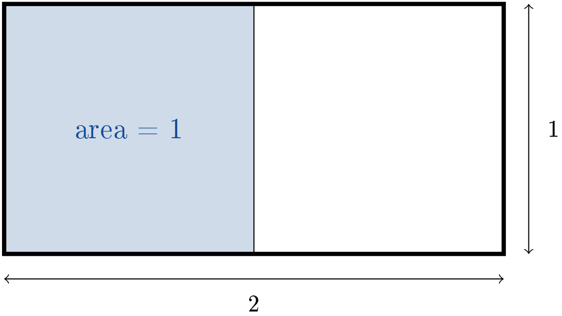
as $n \to \infty$. Therefore $\sum\limits_{k=1}^{\infty} 1$ diverges.



Example

Looking at the area of the rectangle above, we can see that

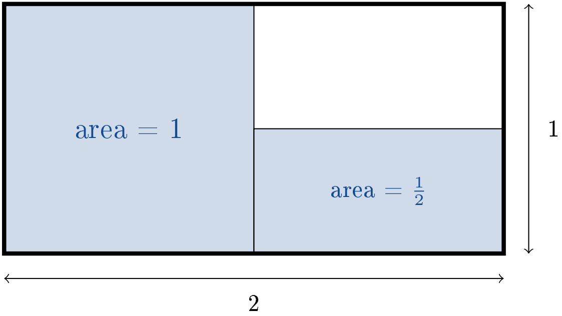
$$2 =$$



Example

Looking at the area of the rectangle above, we can see that

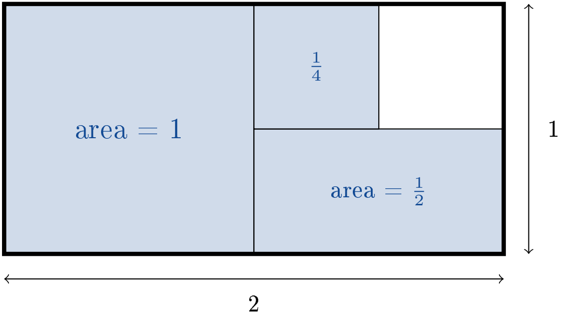
$$2 = 1 +$$



Example

Looking at the area of the rectangle above, we can see that

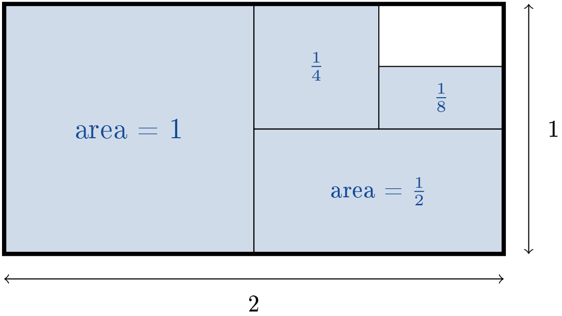
$$2 = 1 + \frac{1}{2} +$$



Example

Looking at the area of the rectangle above, we can see that

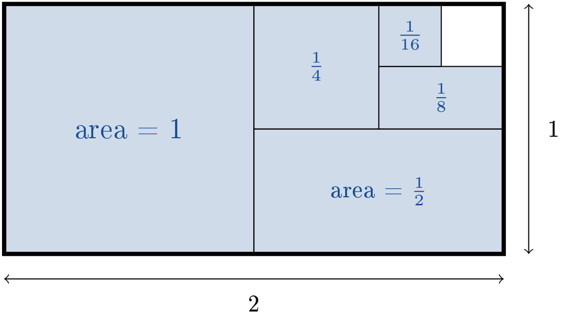
$$2 = 1 + \frac{1}{2} + \frac{1}{4} +$$



Example

Looking at the area of the rectangle above, we can see that

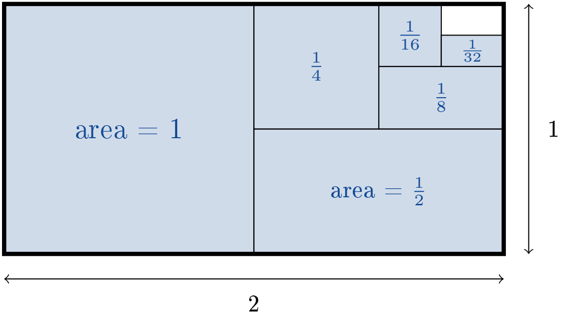
$$2 = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} +$$



Example

Looking at the area of the rectangle above, we can see that

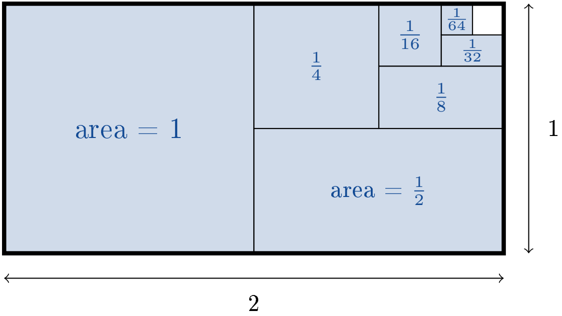
$$2 = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} +$$



Example

Looking at the area of the rectangle above, we can see that

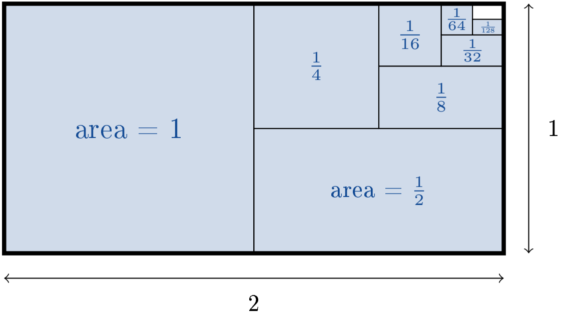
$$2 = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} +$$



Example

Looking at the area of the rectangle above, we can see that

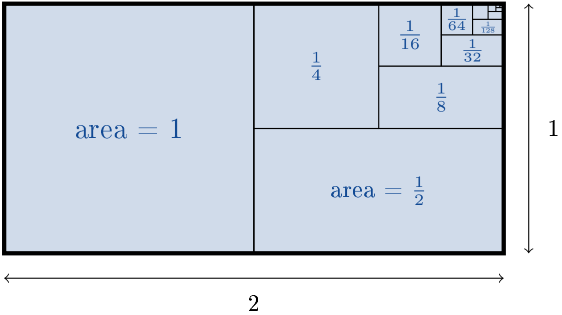
$$2 = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \frac{1}{64} +$$



Example

Looking at the area of the rectangle above, we can see that

$$2 = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \frac{1}{64} + \frac{1}{128} +$$



Example

Looking at the area of the rectangle above, we can see that

$$2 = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \frac{1}{64} + \frac{1}{128} + \frac{1}{256} + \ldots$$

Geometric Series

Now consider the series

$$1 + x + x^2 + x^3 + x^4 + x^5 + x^6 + \dots$$

for $x \in \mathbb{R}$.

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$$s_n = 1 + x + x^2 + x^3 + \dots + x^{n-1}.$$

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$$s_n = 1 + x + x^2 + x^3 + \dots + x^{n-1}.$$

Then we can see that

$$\begin{aligned}(1-x)s_n &= s_n - xs_n \\ &= \left(1 + x + x^2 + x^3 + x^4 + \dots + x^{n-1}\right) \\ &\qquad - \left(x + x^2 + x^3 + x^4 + \dots + x^{n-1} + x^n\right)\end{aligned}$$

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Geometric Series

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Then we can see that

$$\begin{aligned}(1-x)s_n &= s_n - xs_n \\ &= \left(1 + \cancel{x} + \cancel{x^2} + \cancel{x^3} + \cancel{x^4} + \dots + \cancel{x^{n-1}}\right) \\ &\qquad - \left(\cancel{x} + \cancel{x^2} + \cancel{x^3} + \cancel{x^4} + \dots + \cancel{x^{n-1}} + x^n\right) \\ &= 1 - x^n.\end{aligned}$$

If $x \neq 1$, then

$$s_n = \frac{1 - x^n}{1 - x}.$$

Now

- If $x = 1$, then $s_n = 1 + 1 + 1 + 1 + \ldots + 1 = n \to \infty$ as $n \to \infty$. So

$$x = 1 \implies \sum_{k=0}^{\infty} x^k \text{ diverges.}$$

Now

- If $x = 1$, then $s_n = 1 + 1 + 1 + 1 + \ldots + 1 = n \to \infty$ as $n \to \infty$. So

$$x = 1 \implies \sum_{k=0}^{\infty} x^k \text{ diverges.}$$

- If $|x| < 1$, then $s_n = \frac{1-x^n}{1-x} \to \frac{1}{1-x}$ as $n \to \infty$. So

$$|x| < 1 \implies \sum_{k=0}^{\infty} x^k \text{ converges and } \sum_{k=0}^{\infty} x^k = \frac{1}{1-x}.$$

- If $x = -1$, then $s_n = \frac{1-(-1)^n}{2}$ and (s_n) does not have a limit as $n \to \infty$. So

$$x = -1 \implies \sum_{k=0}^{\infty} x^k \text{ diverges.}$$

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$$x = -1 \implies \sum_{k=0}^{\infty} x^k \text{ diverges.}$$

- If $|x| > 1$, then $|s_n| = \frac{|x^n-1|}{|x-1|} \geq \frac{|x|^n-1}{|x|+1} \to \infty$ as $n \to \infty$. So

$$|x| > 1 \implies \sum_{k=0}^{\infty} x^k \text{ diverges.}$$

- If $x = -1$, then $s_n = \frac{1-(-1)^n}{2}$ and (s_n) does not have a limit as $n \to \infty$. So

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- If $|x| > 1$, then $|s_n| = \frac{|x^n - 1|}{|x-1|} \geq \frac{|x|^n - 1}{|x|+1} \to \infty$ as $n \to \infty$. So

$$|x| > 1 \implies \sum_{k=0}^{\infty} x^k \text{ diverges.}$$

Theorem

$$\sum_{k=0}^{\infty} x^k \begin{cases} \textit{converges if } |x| < 1 \\ \textit{diverges if } |x| \geq 1. \end{cases}$$

Moreover, if $|x| < 1$ *then* $\sum_{k=0}^{\infty} x^k = \frac{1}{1-x}$.

$$\sum_{k=0}^{\infty} x^k = \frac{1}{1-x}$$

Example

$$\begin{aligned} 7 + \frac{7}{3} + \frac{7}{27} + \frac{7}{81} + \frac{7}{243} + \dots \\ &= 7\left(1 + \frac{1}{3} + \frac{1}{27} + \frac{1}{81} + \frac{1}{243} + \dots\right) \\ &= 7\left(\frac{1}{1-\frac{1}{3}}\right) \\ &= \frac{21}{2} = 10.5. \end{aligned}$$

EXAMPLE 4 Express the repeating decimal $5.232323\ldots$ as the ratio of two integers.

Solution From the definition of a decimal number, we get a geometric series

$$
\begin{aligned}
5.232323\ldots &= 5 + \frac{23}{100} + \frac{23}{(100)^2} + \frac{23}{(100)^3} + \cdots \\
&= 5 + \frac{23}{100}\underbrace{\left(1 + \frac{1}{100} + \left(\frac{1}{100}\right)^2 + \cdots\right)}_{1/(1-0.01)} \qquad a = 1,\ r = 1/100 \\
&= 5 + \frac{23}{100}\left(\frac{1}{0.99}\right) = 5 + \frac{23}{99} = \frac{518}{99}
\end{aligned}
$$

■

Example

Does $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$ converge or diverge?

Example

Does $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$ converge or diverge?

If we write $a_n = \frac{1}{n(n+1)}$ in partial fractions, we get
$a_n = \frac{1}{n} - \frac{1}{n+1}$. So

$$\begin{aligned}
a_1 &= 1 - \frac{1}{2} \\
a_2 &= \frac{1}{2} - \frac{1}{3} \\
a_3 &= \frac{1}{3} - \frac{1}{4} \\
&\vdots \\
a_{n-1} &= \frac{1}{n-1} - \frac{1}{n} \\
a_n &= \frac{1}{n} - \frac{1}{n+1}.
\end{aligned}$$

Thus

$$\begin{aligned} s_n &= a_1 + a_2 + a_3 + a_4 + \ldots + a_{n-1} + a_n \\ &= \left(1 - \tfrac{1}{2}\right) + \left(\tfrac{1}{2} - \tfrac{1}{3}\right) + \left(\tfrac{1}{3} - \tfrac{1}{4}\right) + \left(\tfrac{1}{4} - \tfrac{1}{5}\right) + \ldots \\ &\qquad + \left(\tfrac{1}{n-1} - \tfrac{1}{n}\right) + \left(\tfrac{1}{n} - \tfrac{1}{n+1}\right) \\ \\ &= \end{aligned}$$

Thus

$$
\begin{aligned}
s_n &= a_1 + a_2 + a_3 + a_4 + \ldots + a_{n-1} + a_n \\
&= \left(1 - \tfrac{1}{2}\right) + \left(\tfrac{1}{2} - \tfrac{1}{3}\right) + \left(\tfrac{1}{3} - \tfrac{1}{4}\right) + \left(\tfrac{1}{4} - \tfrac{1}{5}\right) + \ldots \\
&\qquad + \left(\frac{1}{n-1} - \frac{1}{n}\right) + \left(\frac{1}{n} - \frac{1}{n+1}\right) \\
&= 1 - \frac{1}{2} + \frac{1}{2} - \frac{1}{3} + \frac{1}{3} - \frac{1}{4} + \frac{1}{4} - \frac{1}{5} + \ldots + \frac{1}{n-1} - \frac{1}{n} + \frac{1}{n} - \frac{1}{n+1} \\
&=
\end{aligned}
$$

Thus

$$
\begin{aligned}
s_n &= a_1 + a_2 + a_3 + a_4 + \ldots + a_{n-1} + a_n \\
&= \left(1 - \tfrac{1}{2}\right) + \left(\tfrac{1}{2} - \tfrac{1}{3}\right) + \left(\tfrac{1}{3} - \tfrac{1}{4}\right) + \left(\tfrac{1}{4} - \tfrac{1}{5}\right) + \ldots \\
&\qquad + \left(\frac{1}{n-1} - \frac{1}{n}\right) + \left(\frac{1}{n} - \frac{1}{n+1}\right) \\
&= 1 - \tfrac{1}{2} + \tfrac{1}{2} - \tfrac{1}{3} + \tfrac{1}{3} - \tfrac{1}{4} + \tfrac{1}{4} - \tfrac{1}{5} + \ldots + \tfrac{1}{n-1} - \tfrac{1}{n} + \tfrac{1}{n} - \tfrac{1}{n+1} \\
&= 1 - \tfrac{1}{n+1}
\end{aligned}
$$

Thus

$$\begin{aligned}
s_n &= a_1 + a_2 + a_3 + a_4 + \ldots + a_{n-1} + a_n \\
&= \left(1 - \tfrac{1}{2}\right) + \left(\tfrac{1}{2} - \tfrac{1}{3}\right) + \left(\tfrac{1}{3} - \tfrac{1}{4}\right) + \left(\tfrac{1}{4} - \tfrac{1}{5}\right) + \ldots \\
&\qquad + \left(\tfrac{1}{n-1} - \tfrac{1}{n}\right) + \left(\tfrac{1}{n} - \tfrac{1}{n+1}\right) \\
&= 1 - \tfrac{1}{2} + \tfrac{1}{2} - \tfrac{1}{3} + \tfrac{1}{3} - \tfrac{1}{4} + \tfrac{1}{4} - \tfrac{1}{5} + \ldots + \tfrac{1}{n-1} - \tfrac{1}{n} + \tfrac{1}{n} - \tfrac{1}{n+1} \\
&= 1 - \tfrac{1}{n+1} \\
&\to 1
\end{aligned}$$

as $n \to \infty$.

Thus

$$\begin{aligned} s_n &= a_1 + a_2 + a_3 + a_4 + \ldots + a_{n-1} + a_n \\ &= \left(1 - \tfrac{1}{2}\right) + \left(\tfrac{1}{2} - \tfrac{1}{3}\right) + \left(\tfrac{1}{3} - \tfrac{1}{4}\right) + \left(\tfrac{1}{4} - \tfrac{1}{5}\right) + \ldots \\ &\qquad + \left(\tfrac{1}{n-1} - \tfrac{1}{n}\right) + \left(\tfrac{1}{n} - \tfrac{1}{n+1}\right) \\ &= 1 - \tfrac{1}{2} + \tfrac{1}{2} - \tfrac{1}{3} + \tfrac{1}{3} - \tfrac{1}{4} + \tfrac{1}{4} - \tfrac{1}{5} + \ldots + \tfrac{1}{n-1} - \tfrac{1}{n} + \tfrac{1}{n} - \tfrac{1}{n+1} \\ &= 1 - \tfrac{1}{n+1} \\ &\to 1 \end{aligned}$$

as $n \to \infty$.

Therefore $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$ converges and $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = 1$.

Theorem (The Sum Rule)

If $\sum_{k=1}^{\infty} a_k = s$ *and* $\sum_{k=1}^{\infty} b_k = t$ *are convergent series, then* $\sum_{k=1}^{\infty} (a_k + b_k) = s + t$ *is convergent.*

Theorem (The Sum Rule)

If $\sum_{k=1}^{\infty} a_k = s$ *and* $\sum_{k=1}^{\infty} b_k = t$ *are convergent series, then* $\sum_{k=1}^{\infty} (a_k + b_k) = s + t$ *is convergent.*

Proof.

Let $s_n = a_1 + a_2 + a_3 + \ldots a_n$ and $t_n = b_1 + b_2 + b_3 + \ldots + b_n$ be the partial sums of $\sum_{k=1}^{\infty} a_k$ and $\sum_{k=1}^{\infty} b_k$ respectively.
Then $s_n \to s$ and $t_n \to t$ as $n \to \infty$.

continued.

So

$$\sum_{k=1}^{n}(a_k + b_k) = (a_1 + b_1) + (a_2 + b_2) + (a_3 + b_3) + \ldots + (a_n + b_n)$$
$$= s_n + t_n \to s + t$$

as $n \to \infty$.

continued.

So

$$\sum_{k=1}^{n}(a_k+b_k) = (a_1+b_1)+(a_2+b_2)+(a_3+b_3)+\ldots+(a_n+b_n)$$
$$= s_n + t_n \to s+t$$

as $n \to \infty$. Therefore $\sum_{k=1}^{\infty}(a_k+b_k)$ converges and

$$\sum_{k=1}^{\infty}(a_k+b_k) = \sum_{k=1}^{\infty} a_k + \sum_{k=1}^{\infty} b_k.$$

□

So if two series are convergent, we can add them.

Theorem (The Constant Multiple Rule)

If $\sum_{k=1}^{\infty} a_k = s$ *is a convergent series, then* $\sum_{k=1}^{\infty} ca_k = cs$ *is convergent for any number* $c \in \mathbb{R}$.

(you prove)

Theorem (The Divergence Test / Iraksaklık Testi)

If $a_n \not\to 0$ *as* $n \to \infty$, *then* $\sum_{n=1}^{\infty} a_n$ *diverges.*

Theorem (The Divergence Test / Iraksaklık Testi)

If $a_n \not\to 0$ as $n \to \infty$, then $\displaystyle\sum_{n=1}^{\infty} a_n$ diverges.

Proof.

Let $s_n = a_1 + a_2 + a_3 + \ldots + a_n$. We will use proof by contrapositive: Suppose that $\displaystyle\sum_{k=1}^{\infty} a_k$ converges.

Proof continued.

Then $s_n \to s$ as $n \to \infty$.

Proof continued.

Then $s_n \to s$ as $n \to \infty$. But then $s_{n-1} \to s$ as $n \to \infty$ also. Hence

$$a_n = s_n - s_{n-1}$$

Proof continued.

Then $s_n \to s$ as $n \to \infty$. But then $s_{n-1} \to s$ as $n \to \infty$ also. Hence

$$a_n = s_n - s_{n-1} \to s - s = 0$$

as $n \to \infty$.

Proof continued.

Then $s_n \to s$ as $n \to \infty$. But then $s_{n-1} \to s$ as $n \to \infty$ also. Hence

$$a_n = s_n - s_{n-1} \to s - s = 0$$

as $n \to \infty$. So

$$\sum_{k=1}^{\infty} a_k \text{ converges } \implies a_n \to 0 \text{ as } n \to \infty.$$

Proof continued.

Then $s_n \to s$ as $n \to \infty$. But then $s_{n-1} \to s$ as $n \to \infty$ also. Hence

$$a_n = s_n - s_{n-1} \to s - s = 0$$

as $n \to \infty$. So

$$\sum_{k=1}^{\infty} a_k \text{ converges} \implies a_n \to 0 \text{ as } n \to \infty.$$

Therefore

$$a_n \not\to 0 \text{ as } n \to \infty \implies \sum_{k=1}^{\infty} a_k \text{ diverges.}$$

$\square$

Corollary

If $\sum_{k=1}^{\infty} a_k$ *converges, then* $a_n \to 0$ *as* $n \to \infty$.

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If $\sum_{k=1}^{\infty} a_k$ *converges, then* $a_n \to 0$ *as* $n \to \infty$.

Remark

$$\sum_{k=1}^{\infty} a_k \text{ converges} \implies a_n \to 0 \text{ as } n \to \infty.$$

But

$$a_n \to 0 \text{ as } n \to \infty \;\not\Longrightarrow\; \sum_{k=1}^{\infty} a_k \text{ converges.}$$

Be careful!!!

Example

Let $a_n = \frac{1}{n}$ for all $n \in \mathbb{N}$. Then $a_n \to 0$ as $n \to \infty$.

Does $\displaystyle\sum_{k=1}^{\infty} \frac{1}{n}$ converge or diverge?

Example

Let $a_n = \frac{1}{n}$ for all $n \in \mathbb{N}$. Then $a_n \to 0$ as $n \to \infty$.

Does $\displaystyle\sum_{k=1}^{\infty} \frac{1}{n}$ converge or diverge?

Let

$$s_n = a_1 + a_2 + a_3 + \ldots + a_n = 1 + \tfrac{1}{2} + \tfrac{1}{3} + \ldots + \tfrac{1}{n}.$$

Clearly (s_n) is an increasing sequence.

Example

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Let

$$s_n = a_1 + a_2 + a_3 + \ldots + a_n = 1 + \tfrac{1}{2} + \tfrac{1}{3} + \ldots + \tfrac{1}{n}.$$

Clearly (s_n) is an increasing sequence. Since

$$\begin{aligned} s_{2^n} = 1 + \tfrac{1}{2} + \left(\tfrac{1}{3} + \tfrac{1}{4}\right) + \left(\tfrac{1}{5} + \tfrac{1}{6} + \tfrac{1}{7} + \tfrac{1}{8}\right) + \ldots \\ + \left(\tfrac{1}{2^{n-1}+1} + \ldots + \tfrac{1}{2^n}\right) \end{aligned}$$

Example

Let $a_n = \frac{1}{n}$ for all $n \in \mathbb{N}$. Then $a_n \to 0$ as $n \to \infty$.

Does $\displaystyle\sum_{k=1}^{\infty} \frac{1}{n}$ converge or diverge?

Let

$$s_n = a_1 + a_2 + a_3 + \ldots + a_n = 1 + \tfrac{1}{2} + \tfrac{1}{3} + \ldots + \tfrac{1}{n}.$$

Clearly (s_n) is an increasing sequence. Since

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Therefore $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$ diverges.

Lemma

$$\sum_{n=1}^{\infty} \frac{1}{n} \textit{ diverges.}$$

Example

Does $\sum_{k=1}^{\infty} b_n = \sum_{k=1}^{\infty} \left(\frac{3n+1}{5n+1}\right)^4$ converge or diverge?

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Since

$$b_n = \left(\frac{3n+1}{5n+1}\right)^4 \to \left(\frac{3}{5}\right)^4 \neq 0$$

as $n \to \infty$, it follows that $\sum_{n=1}^{\infty} \left(\frac{3n+1}{5n+1}\right)^4$ diverges by the Divergence Test.

Example

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Let $a_n = \frac{1}{\sqrt{n}}$ and $b_n = \frac{1}{n}$. Then

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for all $n \in \mathbb{N}$.

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$$s_n \geq t_n$$

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$$s_n \geq t_n$$

for all $n \in \mathbb{N}$. Since $t_n \to \infty$ as $n \to \infty$, we have that $s_n \to \infty$ as $n \to \infty$ also. Therefore $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ diverges.

9.4

The Integral Test

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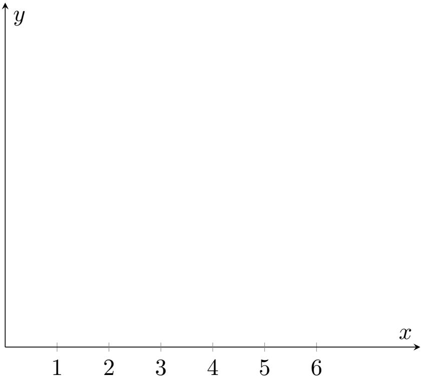

Let $f : [1, \infty) \to [0, \infty)$ be a function which is

- continuous;
- decreasing ($x_1 < x_2 \implies f(x_1) \geq f(x_2)$); and
- positive ($f(x) \geq 0 \ \forall x \in [1, \infty)$).

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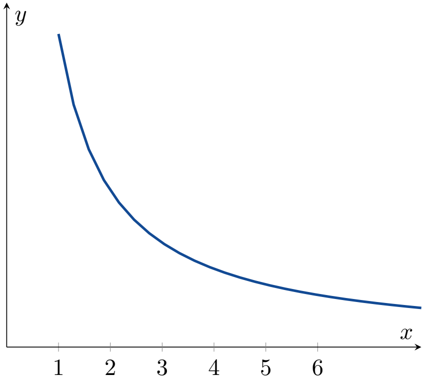

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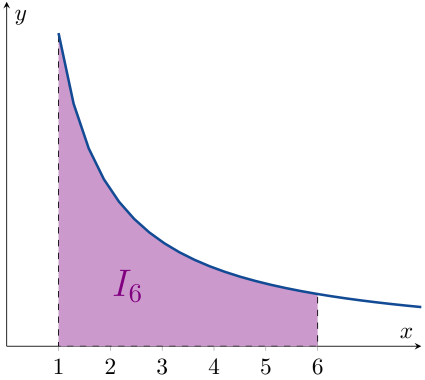
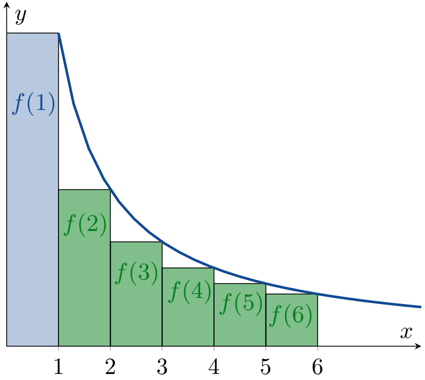


Define

$$I_n := \int_1^n f(x)\ dx$$

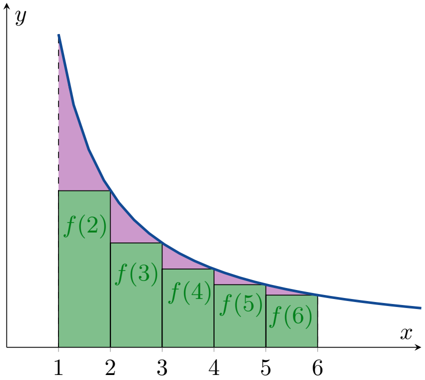


Define

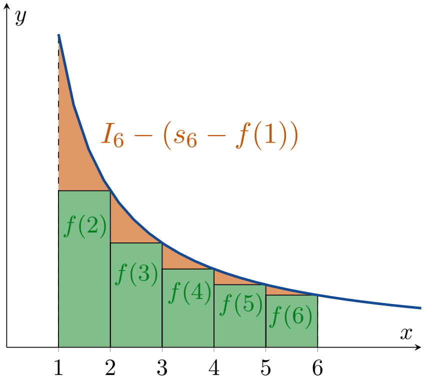
$$I_n := \int_1^n f(x)\ dx$$

and

$$s_n := f(1) + f(2) + f(3) + \ldots + f(n) = \sum_{k=1}^{n} f(k).$$

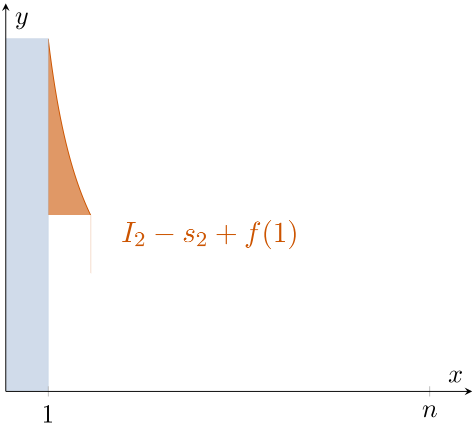


Notice that $f(2) + f(3) + f(4) + \ldots + f(n) \leq I_n$.

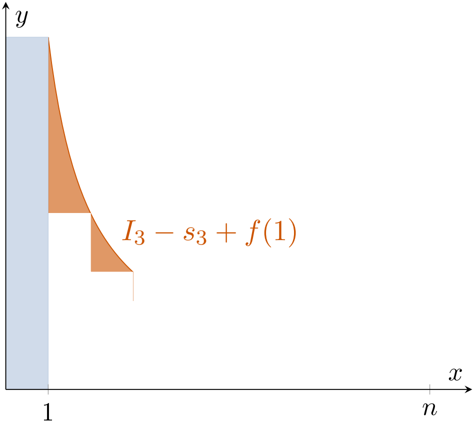


Notice that $f(2) + f(3) + f(4) + \ldots + f(n) \leq I_n$. The difference is

$$I_n - f(2) - f(3) - f(3) - \ldots - f(n) = I_n - (s_n - f(1)) = I_n - s_n + f(1).$$



As n increases, $I_n - s_n + f(1)$ increases. Since $f(1)$ is a constant, we have that $(I_n - s_n)$ is an increasing sequence.



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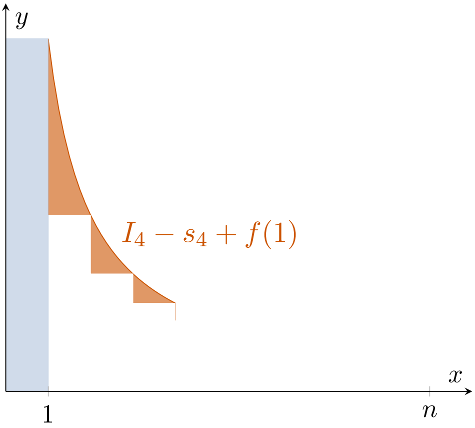

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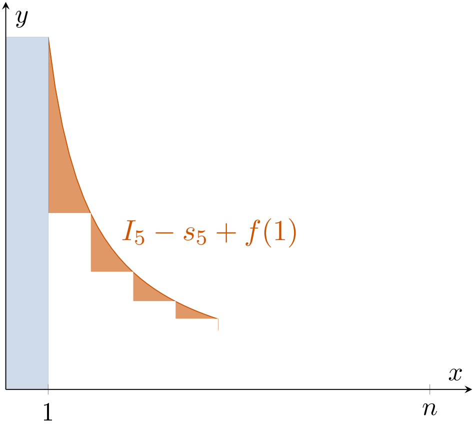

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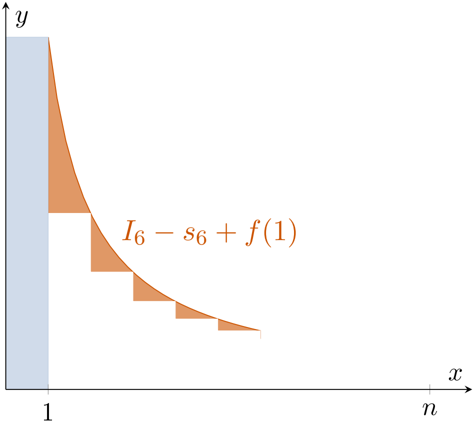

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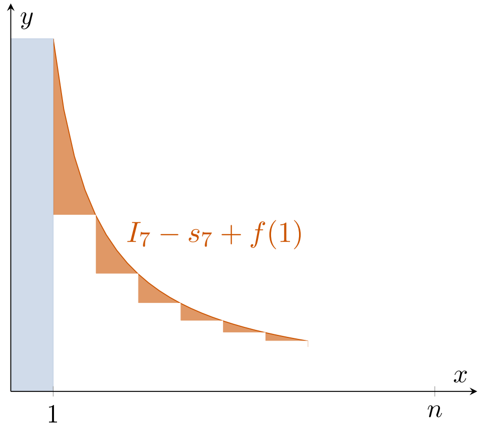

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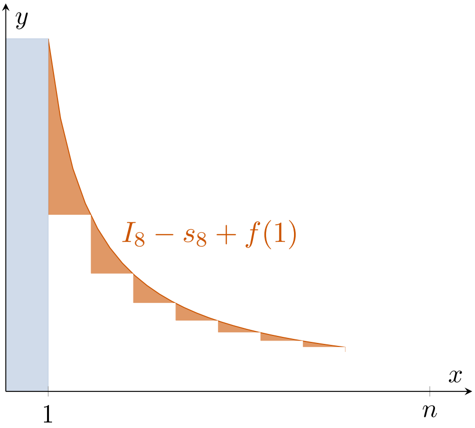

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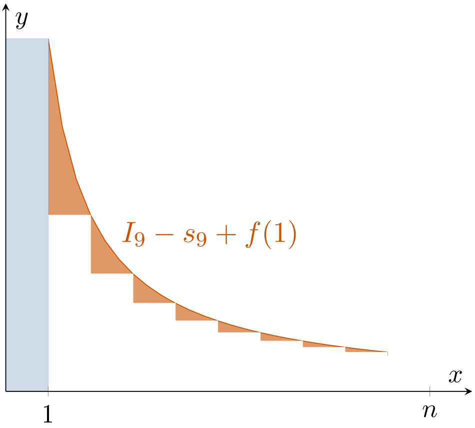

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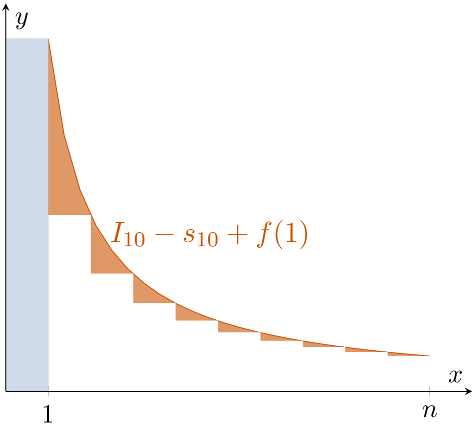

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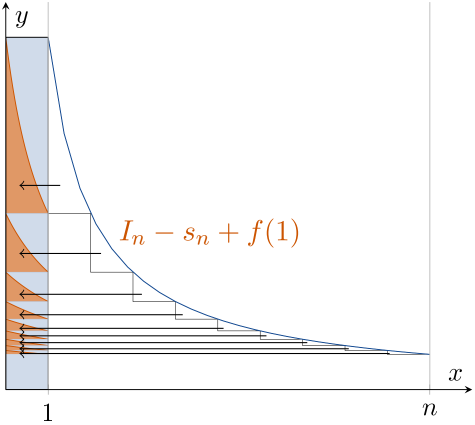

We can see from the picture above that $I_n - s_n + f(1) \leq f(1)$.
Therefore $I_n - s_n \leq 0$ for all $n \in \mathbb{N}$.

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Lemma

Let $f : [1, \infty) \to [0, \infty)$ be a positive, decreasing, continuous function. Let $s_n = f(1) + f(2) + f(3) + \ldots + f(n)$ and

$$I_n := \int_1^n f(x)\, dx \text{ for all } n \in \mathbb{N}.$$

Then $(I_n - s_n)$ is convergent.

Theorem (The Integral Test / İntegral Testi)

Let $f : [1, \infty) \to [0, \infty)$ be a positive, decreasing, continuous function.

1. *If* $\displaystyle\int_1^\infty f(x)\ dx < \infty$, *then* $\displaystyle\sum_{n=1}^\infty f(n)$ *converges.*
2. *If* $\displaystyle\int_1^\infty f(x)\ dx = \infty$, *then* $\displaystyle\sum_{n=1}^\infty f(n)$ *diverges.*

Proof.

Let s_n and I_n be as defined above. Let $c_n = s_n - I_n$. By the previous lemma, we know that $c_n \to c$ as $n \to \infty$.

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Since $f(x) > 0$ for all $x \in [1, \infty)$, (s_n) and (I_n) are both increasing sequences. Either

1. (I_n) is bounded above; or
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CASE 1: If (I_n) is increasing and bounded above, then (I_n) must converge, $I_n \to I$ as $n \to \infty$. But then $s_n = c_n + I_n \to c + I$ as $n \to \infty$. So $\sum_{n=1}^{\infty} f(n)$ converges.

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CASE 2: If (I_n) is increasing and not bounded above, then $I_n \to \infty$ as $n \to \infty$ and we have that $s_n = c_n + I_n \to \infty$ as $n \to \infty$ also. So $\sum_{n=1}^{\infty} f(n)$ diverges. $\square$

Example

For which $\alpha > 0$ does $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^{\alpha}}$ converge?

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Let $f(x) = \frac{1}{x^\alpha}$ for some $\alpha > 0$. Then f is continuous, decreasing and positive $\forall x \geq 1$. So

$$I_n = \int_1^n f(x)\, dx = \int_1^n \frac{1}{x^\alpha} dx = \begin{cases} \frac{1}{\alpha-1}\left[-\frac{1}{x^{\alpha-1}}\right]_1^n & \text{if } \alpha \neq 1 \\ \left[\ln x\right]_1^n & \text{if } \alpha = 1 \end{cases}$$

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- Suppose that $\alpha > 1$. Then

$$I_n = \frac{1}{\alpha - 1}\left(1 - \frac{1}{n^{\alpha-1}}\right) \to \frac{1}{\alpha - 1} < \infty$$

as $n \to \infty$. So $\sum_{n=1}^{\infty} \frac{1}{n^\alpha}$ converges by the Integral Test.

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- Suppose that $\alpha = 1$. Then

$$I_n = \ln n - \ln 1 = \ln n \to \infty$$

as $n \to \infty$. So $\sum_{n=1}^{\infty} \frac{1}{n^\alpha}$ diverges by the Integral Test.

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- Suppose that $0 < \alpha < 1$. Then

$$I_n = \frac{1}{1-\alpha}\left(n^{1-\alpha} - 1\right) \to \infty$$

as $n \to \infty$. So $\sum_{n=1}^{\infty} \frac{1}{n^\alpha}$ diverges by the Integral Test.

Theorem

The series $\sum_{n=1}^{\infty} \frac{1}{n^{\alpha}}$ *diverges for* $0 < \alpha \leq 1$ *and converges for* $\alpha > 1$.

Example

Consider $\sum_{n=2}^{\infty} \frac{1}{n \ln n}$. (Q: Why am I starting at $n = 2$?)

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Use the Integral Test to decide if this series converges or diverges.

Let $f(x) = \frac{1}{x \ln x}$ for $x \geq 2$. Then $f : [2, \infty) \to \mathbb{R}$ is continuous, decreasing and positive. Moreover, for $n \geq 2$,

$$\begin{aligned}
\int_2^{\infty} f(x)\,dx &= \lim_{n\to\infty} \int_2^n f(x)\,dx \\
&= \lim_{n\to\infty} \int_2^n \frac{1}{x \ln x} dx = \lim_{n\to\infty} \Big[\ln(\ln x) \Big]_2^n \\
&= \lim_{n\to\infty} (\ln \ln n - \ln \ln 2) = \infty.
\end{aligned}$$

Example

Consider $\sum_{n=2}^{\infty} \frac{1}{n \ln n}$. (Q: Why am I starting at $n = 2$?)

Use the Integral Test to decide if this series converges or diverges.

Let $f(x) = \frac{1}{x \ln x}$ for $x \geq 2$. Then $f : [2, \infty) \to \mathbb{R}$ is continuous, decreasing and positive. Moreover, for $n \geq 2$,

$$\begin{aligned} \int_2^{\infty} f(x)\, dx &= \lim_{n\to\infty} \int_2^n f(x)\, dx \\ &= \lim_{n\to\infty} \int_2^n \frac{1}{x \ln x} dx = \lim_{n\to\infty} \Big[\ln(\ln x) \Big]_2^n \\ &= \lim_{n\to\infty} (\ln \ln n - \ln \ln 2) = \infty. \end{aligned}$$

Therefore $\sum_{n=2}^{\infty} \frac{1}{n \log n}$ diverges by the Integral Test.

EXAMPLE 5 Determine the convergence or divergence of the series.

(a) $\sum_{n=1}^{\infty} ne^{-n^2}$

Solutions

(a) We apply the Integral Test and find that

$$\int_1^{\infty} \frac{x}{e^{x^2}}\,dx = \frac{1}{2}\int_1^{\infty} \frac{du}{e^u} \qquad u = x^2,\ du = 2x\,dx$$

$$= \lim_{b\to\infty}\left[-\frac{1}{2}e^{-u}\right]_1^b$$

$$= \lim_{b\to\infty}\left(-\frac{1}{2e^b} + \frac{1}{2e}\right) = \frac{1}{2e}.$$

Since the integral converges, the series also converges.

EXAMPLE 5 Determine the convergence or divergence of the series.

(b) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2^{\ln n}}$

(b) Again applying the Integral Test,

$$\int_1^{\infty} \frac{dx}{2^{\ln x}} = \int_0^{\infty} \frac{e^u\, du}{2^u} \qquad u = \ln x,\ x = e^u,\ dx = e^u\, du$$

$$= \int_0^{\infty} \left(\frac{e}{2}\right)^u du$$

$$= \lim_{b\to\infty} \frac{1}{\ln\left(\frac{e}{2}\right)} \left(\left(\frac{e}{2}\right)^b - 1\right) = \infty. \qquad (e/2) > 1$$

The improper integral diverges, so the series diverges also.

9.5 Comparison Tests

We continue with two more tests for convergence.

Theorem (The Comparison Test / Karşılaştırma Testi)

Let $\sum_{n=1}^{\infty} a_n$ *and* $\sum_{n=1}^{\infty} b_n$ *be two series of non-negative real numbers (i.e.* $a_n \geq 0$ *and* $b_n \geq 0$ *for all* n*).*

Suppose that

1. $0 \leq a_n \leq K b_n$ *for all* $n \in \mathbb{N}$ *and for some* $K > 0$*; and*
2. $\sum_{n=1}^{\infty} b_n$ *converges.*

Then $\sum_{n=1}^{\infty} a_n$ *converges.*

$$0 \leq a_n \leq K b_n$$

Proof.

Let $s_n = a_1 + a_2 + a_3 + \ldots + a_n$ and $t_n = b_1 + b_2 + b_3 + \ldots + b_n$ be the partial sums.

Since $a_k \geq 0$ and $b_k \geq 0$ for all $k \in \mathbb{N}$, (s_n) and (t_n) are increasing sequences.

$$\boxed{0 \leq a_n \leq K b_n}$$

Proof.

Let $s_n = a_1 + a_2 + a_3 + \ldots + a_n$ and $t_n = b_1 + b_2 + b_3 + \ldots + b_n$ be the partial sums.

Since $a_k \geq 0$ and $b_k \geq 0$ for all $k \in \mathbb{N}$, (s_n) and (t_n) are increasing sequences.

Since $\sum_{n=1}^{\infty} b_n$ converges, $\exists t \in \mathbb{R}$ such that $t_n \to t$ as $n \to \infty$. So

$$\begin{aligned} s_n &= a_1 + a_2 + a_3 + \ldots + a_n \\ &\leq K b_1 + K b_2 + K b_3 + \ldots + K b_n = K t_n \leq K t \end{aligned}$$

for all $n \in \mathbb{N}$.

$$\boxed{0 \le a_n \le K b_n}$$

Proof.

Let $s_n = a_1 + a_2 + a_3 + \ldots + a_n$ and $t_n = b_1 + b_2 + b_3 + \ldots + b_n$ be the partial sums.

Since $a_k \ge 0$ and $b_k \ge 0$ for all $k \in \mathbb{N}$, (s_n) and (t_n) are increasing sequences.

Since $\sum_{n=1}^{\infty} b_n$ converges, $\exists t \in \mathbb{R}$ such that $t_n \to t$ as $n \to \infty$. So

$$\begin{aligned} s_n &= a_1 + a_2 + a_3 + \ldots + a_n \\ &\le K b_1 + K b_2 + K b_3 + \ldots + K b_n = K t_n \le K t \end{aligned}$$

for all $n \in \mathbb{N}$. So (s_n) is an increasing sequence which is bounded above.

$$0 \le a_n \le Kb_n$$

Proof.

Let $s_n = a_1 + a_2 + a_3 + \ldots + a_n$ and $t_n = b_1 + b_2 + b_3 + \ldots + b_n$ be the partial sums.

Since $a_k \ge 0$ and $b_k \ge 0$ for all $k \in \mathbb{N}$, (s_n) and (t_n) are increasing sequences.

Since $\sum_{n=1}^{\infty} b_n$ converges, $\exists t \in \mathbb{R}$ such that $t_n \to t$ as $n \to \infty$. So

$$\begin{aligned} s_n &= a_1 + a_2 + a_3 + \ldots + a_n \\ &\le Kb_1 + Kb_2 + Kb_3 + \ldots + Kb_n = Kt_n \le Kt \end{aligned}$$

for all $n \in \mathbb{N}$. So (s_n) is an increasing sequence which is bounded above. Therefore (s_n) is convergent by a theorem from last week. $\square$

Corollary

Let $\sum_{n=1}^{\infty} a_n$ *and* $\sum_{n=1}^{\infty} c_n$ *be two series of non-negative real numbers.*

Suppose that

1. $a_n \geq kc_n \geq 0$ *for all* $n \in \mathbb{N}$ *and for some* $k > 0$*; and*
2. $\sum_{n=1}^{\infty} c_n$ *diverges.*

Then $\sum_{n=1}^{\infty} a_n$ *diverges.*

Proof.

Let $K = \frac{1}{k}$. Then $c_n \leq Ka_n$ for all $n \in \mathbb{N}$. By the Comparison Test we have that

$$\sum_{n=1}^{\infty} a_n \text{ converges} \implies \sum_{n=1}^{\infty} c_n \text{ converges.}$$

By proof by contrapositive, we have that

$$\sum_{n=1}^{\infty} c_n \text{ diverges} \implies \sum_{n=1}^{\infty} a_n \text{ diverges.}$$

$\square$

Corollary

Let $\sum_{n=1}^{\infty} a_n$ *and* $\sum_{n=1}^{\infty} b_n$ *be two series of non-negative real numbers. Suppose that*

1. $0 \leq a_n \leq K b_n$ *for all* $n \geq N_0$ *for some* $N_0 \in \mathbb{N}$ *and* $K > 0$*; and*
2. $\sum_{n=1}^{\infty} b_n$ *converges.*

Then $\sum_{n=1}^{\infty} a_n$ *also converges.*

Corollary

Let $\sum_{n=1}^{\infty} a_n$ *and* $\sum_{n=1}^{\infty} b_n$ *be two series of non-negative real numbers. Suppose that*

1. $0 \leq a_n \leq K b_n$ *for all* $n \geq N_0$ *for some* $N_0 \in \mathbb{N}$ *and* $K > 0$; *and*
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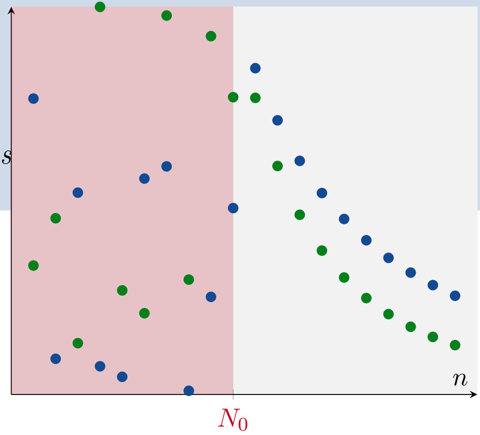
Then $\sum_{n=1}^{\infty} a_n$ *also converges.*



Example

Does $\sum_{n=1}^{\infty} \frac{2}{4n-1}$ converge or diverge?

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Since

$$\frac{2}{4n-1} \geq \frac{2}{4n} = \frac{1}{2n} = \left(\frac{1}{2}\right)\left(\frac{1}{n}\right) = K\left(\frac{1}{n}\right)$$

and since $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$ diverges,

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Since

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and since $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$ diverges, it follows by the Comparison Test that $\displaystyle\sum_{n=1}^{\infty} \frac{2}{4n-1}$ also diverges.

Theorem (The Limit Comparison Test / Limit Karşılaştırma Testi)

Let $\sum_{n=1}^{\infty} a_n$ *and* $\sum_{n=1}^{\infty} b_n$ *be two series of strictly positive real numbers (i.e.* $a_n > 0$ *and* $b_n > 0 \ \forall\ n$*).*

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Let $\sum_{n=1}^{\infty} a_n$ *and* $\sum_{n=1}^{\infty} b_n$ *be two series of strictly positive real numbers (i.e.* $a_n > 0$ *and* $b_n > 0 \ \forall\ n$*).*

Suppose that

1. $\frac{a_n}{b_n} \to l$ *as* $n \to \infty$;
2. $l \in \mathbb{R}$; *and*
3. $l \neq 0$.

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Let $\displaystyle\sum_{n=1}^{\infty} a_n$ *and* $\displaystyle\sum_{n=1}^{\infty} b_n$ *be two series of strictly positive real numbers* *(i.e.* $a_n > 0$ *and* $b_n > 0 \ \forall\ n$*).*

Suppose that

1. $\dfrac{a_n}{b_n} \to l$ *as* $n \to \infty$*;*
2. $l \in \mathbb{R}$*; and*
3. $l \neq 0$.

Then either

- $\displaystyle\sum_{n=1}^{\infty} a_n$ *and* $\displaystyle\sum_{n=1}^{\infty} b_n$ *both converge; or*
- $\displaystyle\sum_{n=1}^{\infty} a_n$ *and* $\displaystyle\sum_{n=1}^{\infty} b_n$ *both diverge.*

Proof.

Since $a_n > 0$, $b_n > 0$, $l \neq 0$ and $\frac{a_n}{b_n} \to l$ as $n \to \infty$, we must have that $l > 0$.

Proof.

Since $a_n > 0$, $b_n > 0$, $l \neq 0$ and $\frac{a_n}{b_n} \to l$ as $n \to \infty$, we must have that $l > 0$.

So $\exists N \in \mathbb{N}$ such that

$$\begin{aligned} n > N \implies & \left| \frac{a_n}{b_n} - l \right| < \frac{l}{2} \\ \implies & \frac{l}{2} < \frac{a_n}{b_n} < \frac{3l}{2} \\ \implies & \frac{l}{2} b_n < a_n < \frac{3l}{2} b_n. \end{aligned}$$

Proof continued.

Now

- $\sum_{n=1}^{\infty} b_n$ converges $\implies \sum_{n=1}^{\infty} a_n$ converges, by the corollary above, since $0 < a_n < \left(\frac{3l}{2}\right) b_n$ for all $n > N$;

Proof continued.

Now

- $\sum_{n=1}^{\infty} b_n$ converges $\implies \sum_{n=1}^{\infty} a_n$ converges, by the corollary above, since $0 < a_n < \left(\frac{3l}{2}\right) b_n$ for all $n > N$; and
- $\sum_{n=1}^{\infty} a_n$ converges $\implies \sum_{n=1}^{\infty} b_n$ converges, by the corollary above, since $0 < b_n < \left(\frac{2}{l}\right) a_n$ for all $n > N$.

Proof continued.

Now

- $\sum_{n=1}^{\infty} b_n$ converges $\implies \sum_{n=1}^{\infty} a_n$ converges, by the corollary above, since $0 < a_n < \left(\frac{3l}{2}\right) b_n$ for all $n > N$; and
- $\sum_{n=1}^{\infty} a_n$ converges $\implies \sum_{n=1}^{\infty} b_n$ converges, by the corollary above, since $0 < b_n < \left(\frac{2}{l}\right) a_n$ for all $n > N$.

So the two series both converge, or both diverge. □

Break

We will continue at 2pm

Lemma

$$\sum_{n=1}^{\infty} \frac{1}{n^2} \text{ converges.}$$

Lemma

$\sum_{n=1}^{\infty} \frac{1}{n^2}$ *converges.*

Proof.

Let $a_n = \frac{1}{n^2}$ and $b_n = \frac{1}{n(n+1)}$. Earlier we showed that

$$\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \frac{1}{n(n+1)} \text{ converges.}$$

Lemma

$$\sum_{n=1}^{\infty} \frac{1}{n^2} \text{ converges.}$$

Proof.

Let $a_n = \frac{1}{n^2}$ and $b_n = \frac{1}{n(n+1)}$. Earlier we showed that $\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \frac{1}{n(n+1)}$ converges. Note that $\forall n \in \mathbb{N}$, $a_n > 0$ and $b_n > 0$. Moreover

$$\frac{a_n}{b_n} = \frac{n(n+1)}{n^2} = 1 + \frac{1}{n} \to 1$$

as $n \to \infty$.

Lemma

$\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2}$ *converges.*

Proof.

Let $a_n = \frac{1}{n^2}$ and $b_n = \frac{1}{n(n+1)}$. Earlier we showed that $\displaystyle\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \frac{1}{n(n+1)}$ converges. Note that $\forall n \in \mathbb{N}$, $a_n > 0$ and $b_n > 0$. Moreover

$$\frac{a_n}{b_n} = \frac{n(n+1)}{n^2} = 1 + \frac{1}{n} \to 1$$

as $n \to \infty$. It follows by the Limit Comparison Test that $\displaystyle\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{1}{n^2}$ also converges. $\square$

Lemma

$\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^{\alpha}}$ *converges for all* $\alpha \geq 2$.

Lemma

$$\sum_{n=1}^{\infty} \frac{1}{n^\alpha} \textit{ converges for all } \alpha \geq 2.$$

Proof.

Let $a_n = \frac{1}{n^\alpha}$ where $\alpha \geq 2$ and $b_n = \frac{1}{n^2}$. Then $\forall n \in \mathbb{N}$, $a_n > 0$, $b_n > 0$ and

$$0 < a_n = \frac{1}{n^\alpha} \leq \frac{1}{n^2} = b_n.$$

Since $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{1}{n^2}$ converges, it follows by the Comparison Test that $\sum_{n=1}^{\infty} a_n$ also converges. □

Lemma

$\sum_{n=1}^{\infty} \frac{1}{n^{\alpha}}$ *diverges for all* $\alpha \leq 1$.

Lemma

$\sum_{n=1}^{\infty} \frac{1}{n^\alpha}$ diverges for all $\alpha \leq 1$.

Proof.

Let $a_n = \frac{1}{n}$ and $b_n = \frac{1}{n^\alpha}$ where $\alpha \leq 1$. Then $\forall n \in \mathbb{N}$, $a_n > 0$, $b_n > 0$ and

$$0 < a_n = \frac{1}{n} \leq \frac{1}{n^\alpha} = b_n.$$

Since $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges, it follows by the corollary above that $\sum_{n=1}^{\infty} b_n$ also diverges. $\square$

Lemma

$\displaystyle\sum_{n=1}^{\infty} \sin\frac{1}{n}$ *diverges.*

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Proof.

Let $a_n = \sin \frac{1}{n}$ and $b_n = \frac{1}{n}$. Then $\forall n \in \mathbb{N}$, $0 < \frac{1}{n} \leq 1 < \frac{\pi}{2}$. So $\sin \frac{1}{n} > 0$ for all $n \in \mathbb{N}$. Hence $\forall n \in \mathbb{N}$, $a_n > 0$, $b_n > 0$ and

$$\frac{a_n}{b_n} = \frac{\sin \frac{1}{n}}{\frac{1}{n}} \to 1$$

as $n \to \infty$. Since $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges, it follows by the Limit Comparison Test that $\sum_{n=1}^{\infty} a_n$ also diverges. $\square$

There are more examples in the textbook.

Absolute Convergence; The Ratio and Root Tests

Theorem (The Ratio Test / Oran Testi)

Let $\sum_{n=1}^{\infty} a_n$ *be a series of strictly positive real numbers (i.e.* $a_n > 0$ *for all* n*). Suppose that*

$$\frac{a_{n+1}}{a_n} \to l \in \mathbb{R} \cup \{\infty\}$$

as $n \to \infty$*.*

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$$\frac{a_{n+1}}{a_n} \to l \in \mathbb{R} \cup \{\infty\}$$

as $n \to \infty$.

1. *If* $l < 1$, *then* $\sum_{n=1}^{\infty} a_n$ *converges.*
2. *If* $l > 1$, *then* $\sum_{n=1}^{\infty} a_n$ *diverges.*
3. *If* $l = \infty$, *then* $\sum_{n=1}^{\infty} a_n$ *diverges.*

Proof.

CASE 1 ($l < 1$): Let $k \in (l, 1)$. Then $k - l > 0$.

Proof.

CASE 1 ($l < 1$): Let $k \in (l, 1)$. Then $k - l > 0$. Since $\frac{a_{n+1}}{a_n} \to l$ as $n \to \infty$, $\exists N \in \mathbb{N}$ such that

$$n > N \implies \left|\frac{a_{n+1}}{a_n} - l\right| < k - l \implies \frac{a_{n+1}}{a_n} < k \implies a_{n+1} < ka_n.$$

Proof.

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Thus

$$\begin{aligned} n > N + 1 \implies 0 < a_n < k a_{n-1} < k^2 a_{n-2} < \dots \\ < k^{n-N-1} a_{N+1} = k^n \left(\frac{a_{N+1}}{k^{N+1}} \right). \end{aligned}$$

Proof.

CASE 1 ($l < 1$): Let $k \in (l, 1)$. Then $k - l > 0$. Since $\frac{a_{n+1}}{a_n} \to l$ as $n \to \infty$, $\exists N \in \mathbb{N}$ such that

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Now $\frac{a_{N+1}}{k^{N+1}}$ is a constant, so $0 < a_n < k^n C$ for all $n > N + 1$.

Proof.

CASE 1 ($l < 1$): Let $k \in (l, 1)$. Then $k - l > 0$. Since $\frac{a_{n+1}}{a_n} \to l$ as $n \to \infty$, $\exists N \in \mathbb{N}$ such that

$$n > N \implies \left| \frac{a_{n+1}}{a_n} - l \right| < k - l \implies \frac{a_{n+1}}{a_n} < k \implies a_{n+1} < ka_n.$$

Thus

$$\begin{aligned} n > N + 1 \implies 0 < a_n < ka_{n-1} < k^2 a_{n-2} < \dots \\ < k^{n-N-1} a_{N+1} = k^n \left(\frac{a_{N+1}}{k^{N+1}} \right). \end{aligned}$$

Now $\frac{a_{N+1}}{k^{N+1}}$ is a constant, so $0 < a_n < k^n C$ for all $n > N + 1$.

We know that $\sum_{k=1}^{\infty} k^n$ converges, since $0 < k < 1$. By the Comparison Test, $\sum_{n=1}^{\infty} a_n$ also converges.

Proof continued.

CASE 2 ($l > 1$): Since $\frac{a_{n+1}}{a_n} \to l$ as $n \to \infty$, $\exists M$ such that

$$n > M \implies \frac{a_{n+1}}{a_n} > 1 \implies a_{n+1} > a_n.$$

Proof continued.

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So

$$n > M + 1 \implies a_n > a_{n-1} > a_{n-2} > \ldots > a_{M+1}.$$

So $a_n \not\to 0$ as $n \to \infty$.

Proof continued.

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CASE 3 ($l = \infty$): I leave this for you to prove.

Proof continued.

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So $a_n \not\to 0$ as $n \to \infty$.

CASE 3 ($l = \infty$): I leave this for you to prove.

Therefore $\sum_{n=1}^{\infty} a_n$ diverges. $\square$

Remark

If $l = 1$, the Ratio Test tells us nothing.

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For example, let $a_n = \frac{1}{n}$ and $b_n = \frac{1}{n^2}$. We know that $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges and $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges.

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$$\frac{a_{n+1}}{a_n} = \frac{\frac{1}{n+1}}{\frac{1}{n}} = \frac{n}{n+1} = \frac{1}{1+\frac{1}{n}} \to 1$$

as $n \to \infty$ and

$$\frac{b_{n+1}}{b_n} = \frac{\frac{1}{(n+1)^2}}{\frac{1}{n^2}} = \frac{n^2}{(n+1)^2} = \frac{1}{(1+\frac{1}{n})^2} \to 1$$

as $n \to \infty$.

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$$\frac{a_{n+1}}{a_n} = \frac{\frac{1}{n+1}}{\frac{1}{n}} = \frac{n}{n+1} = \frac{1}{1+\frac{1}{n}} \to 1$$

as $n \to \infty$ and

$$\frac{b_{n+1}}{b_n} = \frac{\frac{1}{(n+1)^2}}{\frac{1}{n^2}} = \frac{n^2}{(n+1)^2} = \frac{1}{(1+\frac{1}{n})^2} \to 1$$

as $n \to \infty$.

If we get $\frac{a_{n+1}}{a_n} \to 1$ as $n \to \infty$, then we cannot use the Ratio Test – we have to use a different test to see if $\sum_{n=1}^{\infty} a_n$ converges or diverges.

Remark

Moreover, note that if $a_n = \frac{1}{n}$, then $\frac{a_{n+1}}{a_n} < 1$ for all $n \in \mathbb{N}$.

Please remember that when we use the Ratio Test, we look at $\lim_{n\to\infty} \frac{a_{n+1}}{a_n}$, not at $\frac{a_{n+1}}{a_n}$.

Example

Does $\sum_{n=1}^{\infty} \frac{(2n)!}{7^n (n!)^2}$ converge or diverge?

Let $z_n = \frac{(2n)!}{7^n(n!)^2}$. Then $z_n > 0$ for all $n \in \mathbb{N}$

Example

Does $\sum_{n=1}^{\infty} \frac{(2n)!}{7^n (n!)^2}$ converge or diverge?

Let $z_n = \frac{(2n)!}{7^n (n!)^2}$. Then $z_n > 0$ for all $n \in \mathbb{N}$ and

$$\frac{z_{n+1}}{z_n} = \frac{(2n+2)!}{7^{n+1}((n+1)!)^2} \cdot \frac{7^n (n!)^2}{(2n)!} = \frac{(2n+2)(2n+1)}{7(n+1)^2}$$

$$= \frac{(2+\frac{2}{n})(2+\frac{1}{n})}{7(1+\frac{1}{n})(1+\frac{1}{n})} \to \frac{4}{7} < 1$$

as $n \to \infty$.

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as $n \to \infty$. By the Ratio Test, $\sum_{n=1}^{\infty} \frac{(2n)!}{7^n (n!)^2}$ converges.

Example

Does $\displaystyle\sum_{n=1}^{\infty} n^2 e^{-n(n+1)}$ converge or diverge?

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$$\frac{y_{n+1}}{y_n} = \frac{(n+1)^2 e^{-(n+1)(n+2)}}{n^2 e^{-n(n+1)}} = \left(1 + \frac{1}{n}\right)^2 e^{-2(n+1)} \to 0 < 1$$

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$$\sum_{k=0}^{\infty} x^k \begin{cases} \text{converges if } |x| < 1 \\ \text{diverges if } |x| \geq 1. \end{cases}$$

When some of the terms in a series are positive and some are negative then the series may or may not converge.

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When some of the terms in a series are positive and some are negative then the series may or may not converge.

For example, the geometric series

$$5 - \frac{5}{4} + \frac{5}{16} - \frac{5}{64} + \frac{5}{256} - \dots$$

converges because $x = -\frac{1}{4}$ and $\left|-\frac{1}{4}\right| < 1$.

However, the geometric series

$$1 - \frac{5}{4} + \frac{25}{16} - \frac{125}{64} + \frac{625}{256} - \dots$$

diverges because $x = -\frac{5}{4}$ and $\left|-\frac{5}{4}\right| \geq 1$.

Definition

Let $\displaystyle\sum_{n=1}^{\infty} a_n$ be a series of real numbers. If $\displaystyle\sum_{n=1}^{\infty} |a_n|$ is convergent, then we say that $\displaystyle\sum_{n=1}^{\infty} a_n$ is *absolutely convergent*.

(We can also say that $\displaystyle\sum_{k=1}^{\infty} a_k$ *converges absolutely* in this case.)

Theorem

Every absolutely convergent series is convergent.

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Remark

The theorem says that

$$\sum_{n=1}^{\infty} |a_n| \text{ converges} \quad \Longrightarrow \quad \sum_{n=1}^{\infty} a_n \text{ converges.}$$

Proof.

Let $\sum_{k=1}^{\infty} a_k$ be absolutely convergent. Then $\sum_{n=1}^{\infty} |a_n|$ converges.

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Some of the a_n might be ≥ 0 and some might be < 0. We want to separate these two types of a_n. Define

$$b_n := \begin{cases} a_n & \text{if } a_n \geq 0 \\ 0 & \text{if } a_n < 0 \end{cases} \qquad \text{and} \qquad c_n := \begin{cases} 0 & \text{if } a_n \geq 0 \\ -a_n & \text{if } a_n < 0. \end{cases}$$

Note that $b_n \geq 0 \ \forall n$, $c_n \geq 0 \ \forall n$ and $a_n = b_n - c_n$.

Proof continued.

Let

$$\begin{aligned} s_n &= |a_1| + |a_2| + |a_3| + \ldots + |a_n|, \\ t_n &= a_1 + a_2 + a_3 + \ldots + a_n, \\ r_n &= b_1 + b_2 + b_3 + \ldots + b_n, \\ u_n &= c_1 + c_2 + c_3 + \ldots + c_n. \end{aligned}$$

Now

$$\sum_{n=1}^{\infty} |a_n| \text{ converges} \implies s_n \to s \text{ as } n \to \infty.$$

We want to prove that (t_n) converges.

Proof continued.

Let

$$s_n = |a_1| + |a_2| + |a_3| + \ldots + |a_n|,$$
$$t_n = a_1 + a_2 + a_3 + \ldots + a_n,$$
$$r_n = b_1 + b_2 + b_3 + \ldots + b_n,$$
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Now

$$\sum_{n=1}^{\infty} |a_n| \text{ converges } \implies s_n \to s \text{ as } n \to \infty.$$

We want to prove that (t_n) converges.

Since $|a_k| \geq 0$ for all $k \in \mathbb{N}$, (s_n) is increasing. So $s_n \leq s$ for all $n \in \mathbb{N}$.

Proof continued.

Hence

$$r_n = b_1 + b_2 + b_3 + \ldots + b_n \leq |a_1| + |a_2| + |a_3| + \ldots + |a_n| = s_n \leq s$$

for all $n \in \mathbb{N}$. Since $b_k \geq 0$ for all $k \in \mathbb{N}$, (r_k) is an increasing sequence which is bounded above. So $r_n \to r$ as $n \to \infty$.

Proof continued.

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Similarly $u_n \to u$ as $n \to \infty$.

Therefore $t_n = r_n - u_n \to r - u$ as $n \to \infty$.

So $\displaystyle\sum_{k=1}^{\infty} a_k$ converges. □

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For example, consider $a_n = \frac{(-1)^{n+1}}{n}$. The series $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} - \frac{1}{8} + \ldots$ is convergent, but the series $\sum_{n=1}^{\infty} \left| \frac{(-1)^{n+1}}{n} \right| = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \ldots$ is divergent.

Corollary (The Triangle Inequality)

If $\sum_{k=1}^{\infty} a_k$ *is absolutely convergent, then*

$$\left|\sum_{n=1}^{\infty} a_n\right| \leq \sum_{n=1}^{\infty} |a_n| .$$

(you prove)

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Is $\sum_{n=1}^{\infty}(-1)^{n+1}\sin^2\frac{1}{n}$ absolutely convergent?

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First note that $\left|(-1)^{n+1}\sin^2\frac{1}{n}\right| = \sin^2\frac{1}{n}$. Let $a_n = \sin^2\frac{1}{n}$ and $b_n = \frac{1}{n^2}$ for all $n \in \mathbb{N}$. Then $a_n > 0$ and $b_n > 0$ for all n, and

$$\frac{a_n}{b_n} = \left(\frac{\sin\frac{1}{n}}{\frac{1}{n}}\right)\left(\frac{\sin\frac{1}{n}}{\frac{1}{n}}\right) \to 1 \cdot 1 = 1$$

as $n \to \infty$.

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$$\frac{a_n}{b_n} = \left(\frac{\sin\frac{1}{n}}{\frac{1}{n}}\right)\left(\frac{\sin\frac{1}{n}}{\frac{1}{n}}\right) \to 1 \cdot 1 = 1$$

as $n \to \infty$. We know that $\sum_{n=1}^{\infty}\frac{1}{n^2}$ converges. By the Limit Comparison Test, $\sum_{n=1}^{\infty}\sin^2\frac{1}{n}$ also converges.

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$$\frac{a_n}{b_n} = \left(\frac{\sin\frac{1}{n}}{\frac{1}{n}}\right)\left(\frac{\sin\frac{1}{n}}{\frac{1}{n}}\right) \to 1 \cdot 1 = 1$$

as $n \to \infty$. We know that $\sum_{n=1}^{\infty}\frac{1}{n^2}$ converges. By the Limit Comparison Test, $\sum_{n=1}^{\infty}\sin^2\frac{1}{n}$ also converges.

Hence $\sum_{n=1}^{\infty}\left|(-1)^{n+1}\sin^2\frac{1}{n}\right|$ converges and therefore $\sum_{n=1}^{\infty}(-1)^{n+1}\sin^2\frac{1}{n}$ converges absolutely.

Theorem (The Ratio Test v2)

Let $\sum_{k=1}^{\infty} a_k$ *be a series of non-zero real numbers (i.e.* $a_n \neq 0\ \forall n$*). Suppose that*

$$\left| \frac{a_{n+1}}{a_n} \right| \to l \in \mathbb{R}$$

as $n \to \infty$*.*

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1. *If* $l < 1$*, then* $\sum_{k=1}^{\infty} a_k$ *is absolutely convergent.*
2. *If* $l > 1$*, then* $\sum_{k=1}^{\infty} a_k$ *is divergent.*

Proof.

Let $b_n = |a_n|$. Then $b_n > 0 \ \forall n$ and

$$\frac{b_{n+1}}{b_n} = \frac{|a_{n+1}|}{|a_n|} = \left|\frac{a_{n+1}}{a_n}\right| \to l$$

as $n \to \infty$. Then we can use the Ratio Test to see that

$$l < 1 \implies \sum_{n=1}^{\infty} b_n \text{ converges} \implies \sum_{n=1}^{\infty} a_n \text{ converges absolutely.}$$

Proof.

Let $b_n = |a_n|$. Then $b_n > 0\ \forall n$ and

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$$l < 1 \implies \sum_{n=1}^{\infty} b_n \text{ converges} \implies \sum_{n=1}^{\infty} a_n \text{ converges absolutely.}$$

If $l > 1$, then $\exists N$ such that $\frac{b_{n+1}}{b_n} > 1\ \forall n > N$. Hence $b_n > b_{N+1}\ \forall n > N+1$. Therefore $b_n \not\to 0$ as $n \to \infty$. So $a_n \not\to 0$ as $n \to \infty$ and thus $\displaystyle\sum_{k=1}^{\infty} a_k$ diverges by the Divergence Test. □

Corollary

If $a_n \neq 0\ \forall n$ *and* $\left|\frac{a_{n+1}}{a_n}\right| \to \infty$ *as* $n \to \infty$, *then* $\sum_{k=1}^{\infty} a_k$ *diverges.*

Corollary

If $a_n \neq 0\ \forall n$ *and* $\left|\dfrac{a_{n+1}}{a_n}\right| \to \infty$ *as* $n \to \infty$, *then* $\displaystyle\sum_{k=1}^{\infty} a_k$ *diverges.*

Remark

If $l = 1$, then the Ratio Test v2 tells us nothing.

EXAMPLE 2 Investigate the convergence of the following series.

(a) $\displaystyle\sum_{n=0}^{\infty} \frac{2^n + 5}{3^n}$ **(b)** $\displaystyle\sum_{n=1}^{\infty} \frac{(2n)!}{n!n!}$

Solution We apply the Ratio Test to each series.

(a) For the series $\sum_{n=0}^{\infty}(2^n + 5)/3^n$,

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{(2^{n+1} + 5)/3^{n+1}}{(2^n + 5)/3^n} = \frac{1}{3} \cdot \frac{2^{n+1} + 5}{2^n + 5} = \frac{1}{3} \cdot \left(\frac{2 + 5 \cdot 2^{-n}}{1 + 5 \cdot 2^{-n}}\right) \to \frac{1}{3} \cdot \frac{2}{1} = \frac{2}{3}.$$

The series converges absolutely (and thus converges) because $\rho = 2/3$ is less than 1. This does *not* mean that $2/3$ is the sum of the series. In fact,

$$\sum_{n=0}^{\infty} \frac{2^n + 5}{3^n} = \sum_{n=0}^{\infty}\left(\frac{2}{3}\right)^n + \sum_{n=0}^{\infty} \frac{5}{3^n} = \frac{1}{1 - (2/3)} + \frac{5}{1 - (1/3)} = \frac{21}{2}.$$

(b) If $a_n = \dfrac{(2n)!}{n!n!}$, then $a_{n+1} = \dfrac{(2n + 2)!}{(n + 1)!(n + 1)!}$ and

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{n!n!(2n + 2)(2n + 1)(2n)!}{(n + 1)!(n + 1)!(2n)!}$$

$$= \frac{(2n + 2)(2n + 1)}{(n + 1)(n + 1)} = \frac{4n + 2}{n + 1} \to 4.$$

The series diverges because $\rho = 4$ is greater than 1.

Theorem (The Root Test / Kök Testi)

Suppose that

$$\sqrt[n]{|a_n|} \to l$$

as $n \to \infty$.

Theorem (The Root Test / Kök Testi)

Suppose that

$$\sqrt[n]{|a_n|} \to l$$

as $n \to \infty$.

1. $l < 1 \implies \sum_{k=1}^{\infty} a_k$ *converges absolutely;*
2. $l > 1$ *(or* $l = \infty$*)* $\implies \sum_{k=1}^{\infty} a_k$ *diverges.*

Theorem (The Root Test / Kök Testi)

Suppose that

$$\sqrt[n]{|a_n|} \to l$$

as $n \to \infty$.

1. $l < 1 \implies \sum_{k=1}^{\infty} a_k$ *converges absolutely;*
2. $l > 1$ *(or* $l = \infty$*)* $\implies \sum_{k=1}^{\infty} a_k$ *diverges.*

(proof in textbook)

Remark

If $l = 1$, then the Root Test tells us nothing.

EXAMPLE 3 Consider again the series with terms $a_n = \begin{cases} n/2^n, & n \text{ odd} \\ 1/2^n, & n \text{ even}. \end{cases}$

Does $\sum a_n$ converge?

Solution We apply the Root Test, finding that

$$\sqrt[n]{|a_n|} = \begin{cases} \sqrt[n]{n}/2, & n \text{ odd} \\ 1/2, & n \text{ even}. \end{cases}$$

Therefore,

$$\frac{1}{2} \leq \sqrt[n]{|a_n|} \leq \frac{\sqrt[n]{n}}{2}.$$

Since $\sqrt[n]{n} \to 1$ (Section 10.1, Theorem 5), we have $\lim_{n\to\infty} \sqrt[n]{|a_n|} = 1/2$ by the Sandwich Theorem. The limit is less than 1, so the series converges absolutely by the Root Test. ■

EXAMPLE 4 Which of the following series converge, and which diverge?

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{n^2}{2^n}$ **(b)** $\displaystyle\sum_{n=1}^{\infty} \frac{2^n}{n^3}$ **(c)** $\displaystyle\sum_{n=1}^{\infty} \left(\frac{1}{1+n}\right)^n$

Solution We apply the Root Test to each series, noting that each series has positive terms.

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{n^2}{2^n}$ converges because $\displaystyle\sqrt[n]{\frac{n^2}{2^n}} = \frac{\sqrt[n]{n^2}}{\sqrt[n]{2^n}} = \frac{\left(\sqrt[n]{n}\right)^2}{2} \to \frac{1^2}{2} < 1.$

(b) $\displaystyle\sum_{n=1}^{\infty} \frac{2^n}{n^3}$ diverges because $\displaystyle\sqrt[n]{\frac{2^n}{n^3}} = \frac{2}{\left(\sqrt[n]{n}\right)^3} \to \frac{2}{1^3} > 1.$

(c) $\displaystyle\sum_{n=1}^{\infty} \left(\frac{1}{1+n}\right)^n$ converges because $\displaystyle\sqrt[n]{\left(\frac{1}{1+n}\right)^n} = \frac{1}{1+n} \to 0 < 1.$ ■

Next Time

- 9.6 Alternating Series and Conditional Convergence
- 9.7 Power Series
- 9.8 Taylor and Maclaurin Series