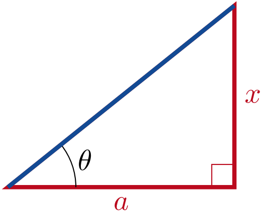
Lecture 2

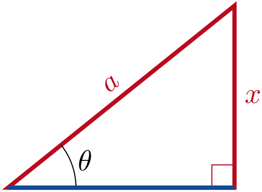
- 8.4 Trigonometric Substitutions
- 8.5 Integration of Rational Functions by Partial Fractions
- 8.8 Improper Integrals

Trigonometric Substitutions

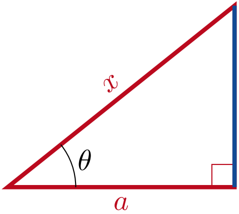
8.4



θ
x
a

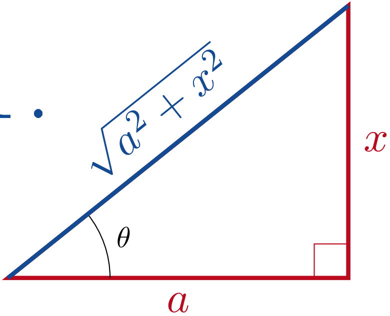


a
θ
x



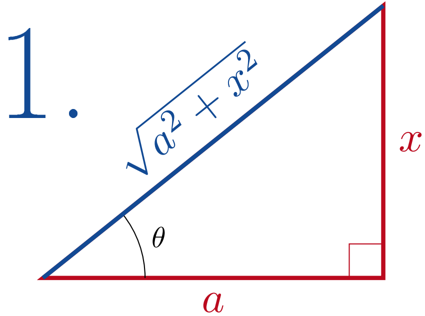
x
θ
a

1.



$$x = a\tan\theta$$

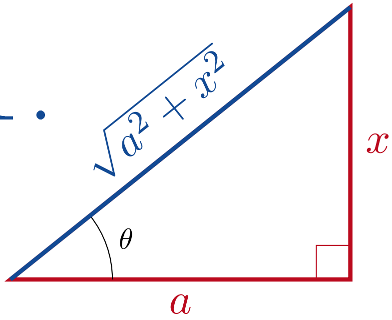
$$a^2 + x^2 = \qquad = \qquad = \qquad .$$



$$x = a\tan\theta$$

$$a^2 + x^2 = a^2 + a^2\tan^2\theta = \qquad = \qquad .$$

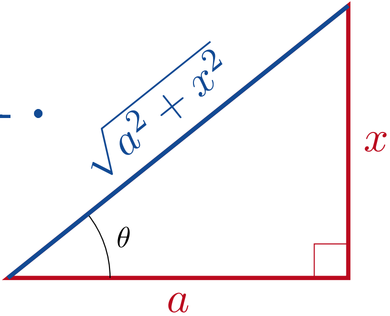
1.



$$x = a\tan\theta$$

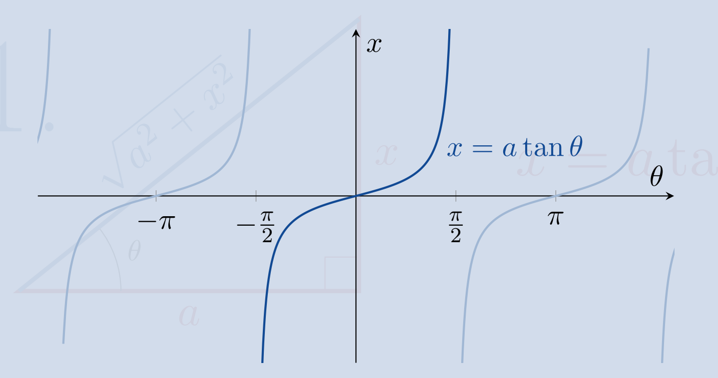
$$a^2 + x^2 = a^2 + a^2\tan^2\theta = a^2(1+\tan^2\theta) = \qquad\qquad .$$

1.

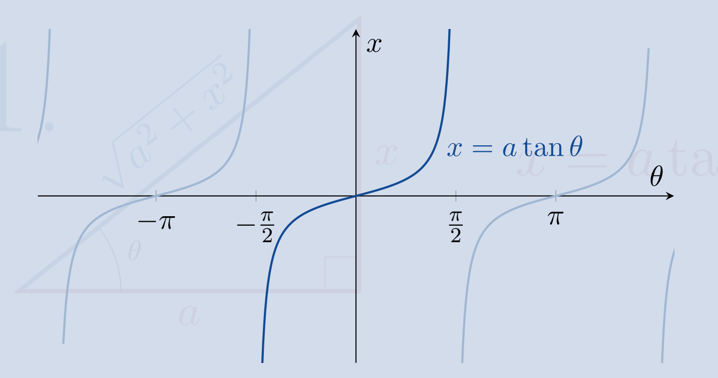


$$x = a\tan\theta$$

$$a^2 + x^2 = a^2 + a^2\tan^2\theta = a^2(1+\tan^2\theta) = a^2\sec^2\theta.$$

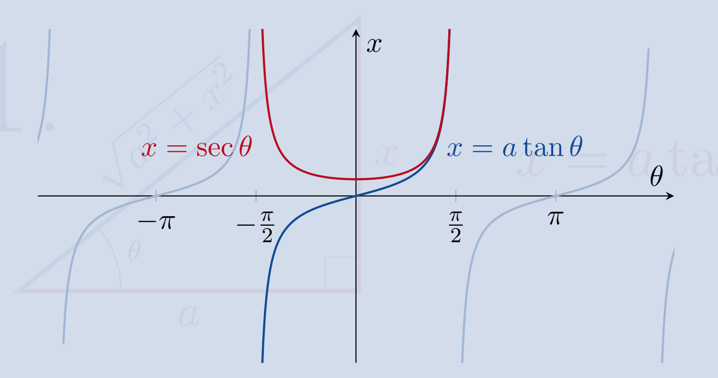


$$a^2 + x^2 = a^2 + a^2\tan^2\theta = a^2(1+\tan^2\theta) = a^2\sec^2\theta.$$



$$a^2 + x^2 = a^2 + a^2 \tan^2\theta = a^2(1 + \tan^2\theta) = a^2 \sec^2\theta.$$

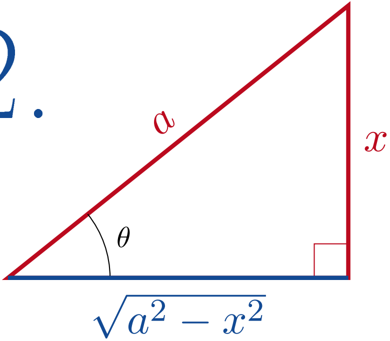
$$\boxed{-\frac{\pi}{2} < \theta < \frac{\pi}{2}.}$$



$$a^2 + x^2 = a^2 + a^2\tan^2\theta = a^2(1+\tan^2\theta) = a^2\sec^2\theta.$$

$$\boxed{\sqrt{a^2+x^2} = a\sec\theta \qquad -\frac{\pi}{2} < \theta < \frac{\pi}{2}.}$$

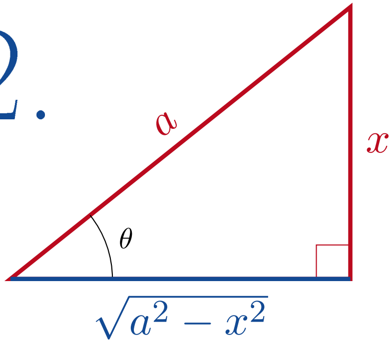
2.



$$x = a \sin \theta$$

$$a^2 - x^2 = \qquad\qquad = \qquad\qquad = \qquad\qquad .$$

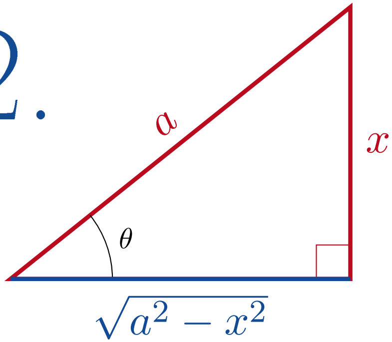
2.



$$x = a\sin\theta$$

$$a^2 - x^2 = a^2 - a^2\sin^2\theta = \qquad\qquad = \qquad\qquad .$$

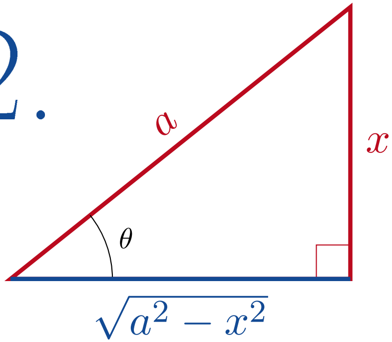
2.



$$x = a \sin\theta$$

$$a^2 - x^2 = a^2 - a^2 \sin^2\theta = a^2(1 - \sin^2\theta) = \qquad .$$

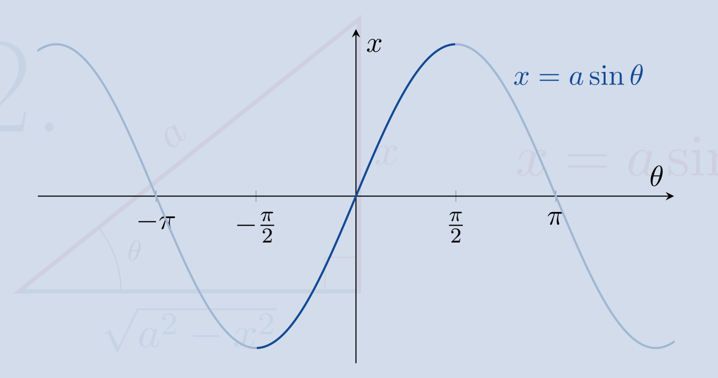
2.



$$x = a \sin\theta$$

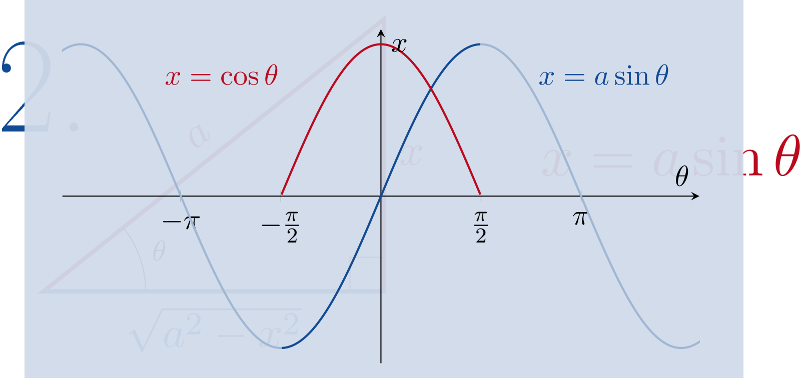
$$a^2 - x^2 = a^2 - a^2 \sin^2\theta = a^2(1 - \sin^2\theta) = a^2 \cos^2\theta.$$

2. $x = a \sin\theta$



$$a^2 - x^2 = a^2 - a^2\sin^2\theta = a^2(1-\sin^2\theta) = a^2\cos^2\theta.$$

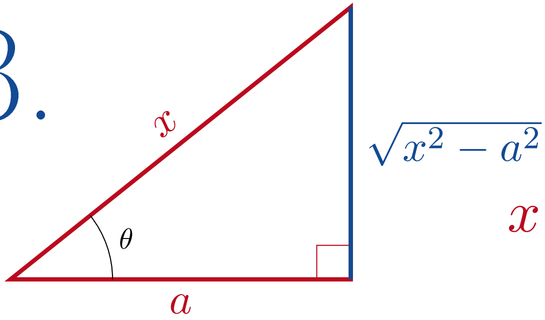
$$-\frac{\pi}{2} < \theta < \frac{\pi}{2}.$$



$$a^2 - x^2 = a^2 - a^2\sin^2\theta = a^2(1 - \sin^2\theta) = a^2\cos^2\theta.$$

$$\boxed{\sqrt{a^2 - x^2} = a\cos x \qquad -\frac{\pi}{2} < \theta < \frac{\pi}{2}.}$$

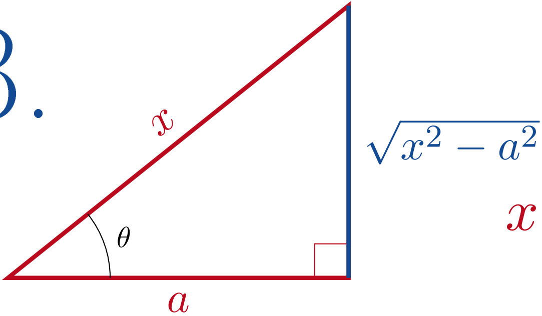
3.



$$x = a \sec \theta$$

$$x^2 - a^2 = \qquad = \qquad = \qquad .$$

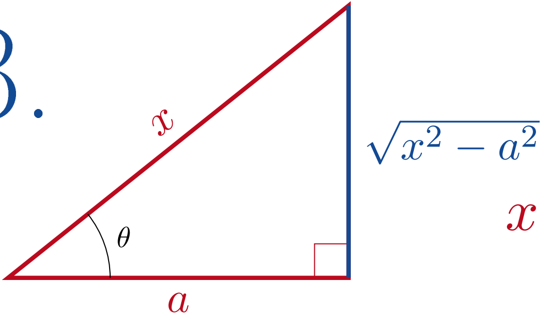
3.



$$x = a\sec\theta$$

$$x^2 - a^2 = a^2\sec^2\theta - a^2 = \qquad\qquad = \qquad\qquad .$$

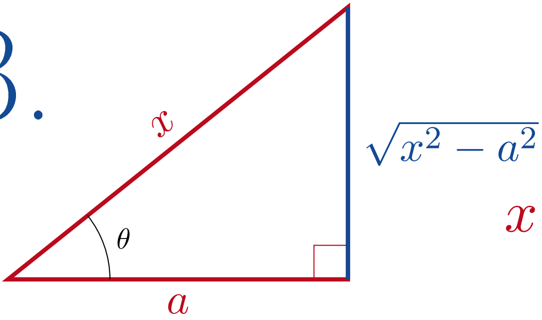
3.



$$x = a \sec \theta$$

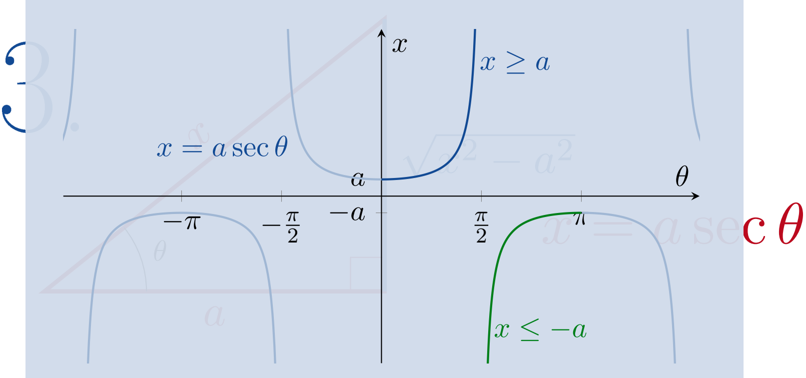
$$x^2 - a^2 = a^2 \sec^2 \theta - a^2 = a^2(\sec^2 \theta - 1) = \qquad\qquad .$$

3.



$$x = a\sec\theta$$

$$x^2 - a^2 = a^2\sec^2\theta - a^2 = a^2(\sec^2\theta - 1) = a^2\tan^2\theta.$$



$$x^2 - a^2 = a^2\sec^2\theta - a^2 = a^2(\sec^2\theta - 1) = a^2\tan^2\theta.$$

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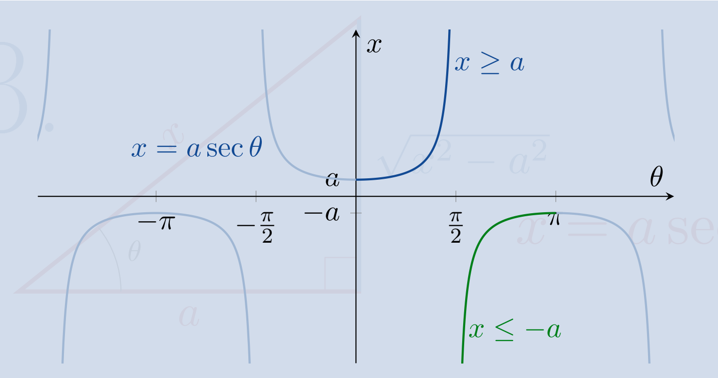
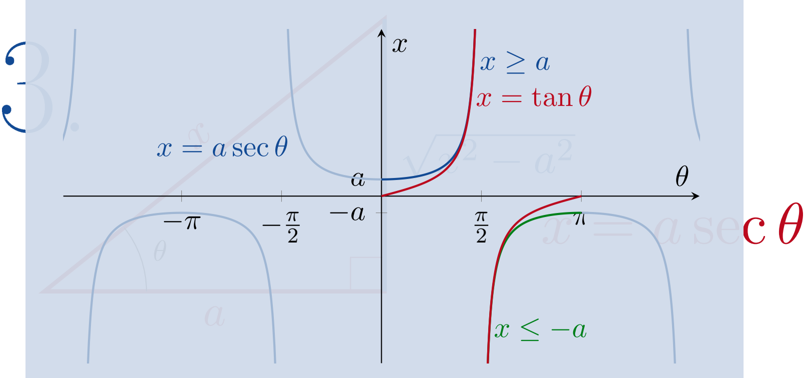


$$x^2 - a^2 = a^2 \sec^2\theta - a^2 = a^2(\sec^2\theta - 1) = a^2\tan^2\theta.$$

$$\begin{cases} 0 \le \theta < \frac{\pi}{2} & x \ge a \\ \frac{\pi}{2} < \theta \le \pi & x \le -a \end{cases}.$$



$$x^2 - a^2 = a^2\sec^2\theta - a^2 = a^2(\sec^2\theta - 1) = a^2\tan^2\theta.$$

$$\boxed{\sqrt{x^2 - a^2} = a\,|\tan x| \qquad \begin{cases} 0 \leq \theta < \frac{\pi}{2} & x \geq a \\ \frac{\pi}{2} < \theta \leq \pi & x \leq -a \end{cases}.}$$

$x = a\tan\theta$	$x = a\sin\theta$	$x = a\sec\theta$
$\sqrt{a^2+x^2} = a\sec\theta$	$\sqrt{a^2-x^2} = a\cos\theta$	$\sqrt{x^2-a^2} = a\lvert\tan\theta\rvert$
$-\frac{\pi}{2} < \theta < \frac{\pi}{2}$	$-\frac{\pi}{2} < \theta < \frac{\pi}{2}$	$\begin{cases} 0 \leq \theta < \frac{\pi}{2} & x \geq a \\ \frac{\pi}{2} < \theta \leq \pi & x \leq -a \end{cases}$

$$x = a\tan\theta \qquad x = a\sin\theta \qquad x = a\sec\theta$$
$$\sqrt{a^2+x^2} = a\sec\theta \qquad \sqrt{a^2-x^2} = a\cos\theta \qquad \sqrt{x^2-a^2} = a\left|\tan\theta\right|$$
$$-\tfrac{\pi}{2} < \theta < \tfrac{\pi}{2} \qquad -\tfrac{\pi}{2} < \theta < \tfrac{\pi}{2} \qquad \begin{cases} 0 \le \theta < \frac{\pi}{2} & x \ge a \\ \frac{\pi}{2} < \theta \le \pi & x \le -a \end{cases}$$

Example

Find $\displaystyle\int \frac{dx}{\sqrt{4+x^2}}$.

$x = a\tan\theta$	$x = a\sin\theta$	$x = a\sec\theta$
$\sqrt{a^2+x^2} = a\sec\theta$	$\sqrt{a^2-x^2} = a\cos\theta$	$\sqrt{x^2-a^2} = a\left\|\tan\theta\right\|$
$-\frac{\pi}{2} < \theta < \frac{\pi}{2}$	$-\frac{\pi}{2} < \theta < \frac{\pi}{2}$	$\begin{cases} 0 \le \theta < \frac{\pi}{2} & x \ge a \\ \frac{\pi}{2} < \theta \le \pi & x \le -a \end{cases}$

Example

Find $\displaystyle\int \frac{dx}{\sqrt{4+x^2}}$.

Let $x = 2\tan\theta$.

$$x = a\tan\theta \qquad \sqrt{a^2+x^2} = a\sec\theta \qquad -\tfrac{\pi}{2} < \theta < \tfrac{\pi}{2}$$

$$x = a\sin\theta \qquad \sqrt{a^2-x^2} = a\cos\theta \qquad -\tfrac{\pi}{2} < \theta < \tfrac{\pi}{2}$$

$$x = a\sec\theta \qquad \sqrt{x^2-a^2} = a\,|\tan\theta| \qquad \begin{cases} 0 \le \theta < \frac{\pi}{2} & x \ge a \\ \frac{\pi}{2} < \theta \le \pi & x \le -a \end{cases}$$

Example

Find $\displaystyle\int \frac{dx}{\sqrt{4+x^2}}$.

Let $x = 2\tan\theta$. Then $dx = 2\sec^2\theta\,d\theta$ and $\sqrt{4+x^2} = 2\sec\theta$.

$$x = a\tan\theta \qquad \sqrt{a^2+x^2} = a\sec\theta \qquad -\tfrac{\pi}{2} < \theta < \tfrac{\pi}{2}$$

$$x = a\sin\theta \qquad \sqrt{a^2-x^2} = a\cos\theta \qquad -\tfrac{\pi}{2} < \theta < \tfrac{\pi}{2}$$

$$x = a\sec\theta \qquad \sqrt{x^2-a^2} = a\left|\tan\theta\right| \qquad \begin{cases} 0 \leq \theta < \frac{\pi}{2} & x \geq a \\ \frac{\pi}{2} < \theta \leq \pi & x \leq -a \end{cases}$$

Example

Find $\displaystyle\int \frac{dx}{\sqrt{4+x^2}}$.

Let $x = 2\tan\theta$. Then $dx = 2\sec^2\theta\,d\theta$ and $\sqrt{4+x^2} = 2\sec\theta$. Therefore

$$\int \frac{dx}{\sqrt{4+x^2}} = \int \frac{2\sec^2\theta\,d\theta}{2\sec\theta}$$

$$=$$

$$=$$

$$= \qquad .$$

$x = a\tan\theta$	$x = a\sin\theta$	$x = a\sec\theta$
$\sqrt{a^2+x^2} = a\sec\theta$	$\sqrt{a^2-x^2} = a\cos\theta$	$\sqrt{x^2-a^2} = a\left\|\tan\theta\right\|$
$-\frac{\pi}{2} < \theta < \frac{\pi}{2}$	$-\frac{\pi}{2} < \theta < \frac{\pi}{2}$	$\begin{cases} 0 \le \theta < \frac{\pi}{2} & x \ge a \\ \frac{\pi}{2} < \theta \le \pi & x \le -a \end{cases}$

Example

Find $\displaystyle\int \frac{dx}{\sqrt{4+x^2}}$.

Let $x = 2\tan\theta$. Then $dx = 2\sec^2\theta\,d\theta$ and $\sqrt{4+x^2} = 2\sec\theta$. Therefore

$$\begin{aligned}\int \frac{dx}{\sqrt{4+x^2}} &= \int \frac{2\sec^2\theta\,d\theta}{2\sec\theta} \\ &= \int \sec\theta\,d\theta \\ &= \\ &= \qquad\qquad\qquad .\end{aligned}$$

$$x = a\tan\theta \qquad x = a\sin\theta \qquad x = a\sec\theta$$
$$\sqrt{a^2+x^2} = a\sec\theta \qquad \sqrt{a^2-x^2} = a\cos\theta \qquad \sqrt{x^2-a^2} = a\left|\tan\theta\right|$$
$$-\tfrac{\pi}{2} < \theta < \tfrac{\pi}{2} \qquad -\tfrac{\pi}{2} < \theta < \tfrac{\pi}{2} \qquad \begin{cases} 0 \leq \theta < \frac{\pi}{2} & x \geq a \\ \frac{\pi}{2} < \theta \leq \pi & x \leq -a \end{cases}$$

Example

Find $\displaystyle\int \frac{dx}{\sqrt{4+x^2}}$.

Let $x = 2\tan\theta$. Then $dx = 2\sec^2\theta\, d\theta$ and $\sqrt{4+x^2} = 2\sec\theta$. Therefore

$$\begin{aligned} \int \frac{dx}{\sqrt{4+x^2}} &= \int \frac{2\sec^2\theta\, d\theta}{2\sec\theta} \\ &= \int \sec\theta\, d\theta \\ &= \ln\left|\sec\theta + \tan\theta\right| + C \\ &= \qquad\qquad\qquad . \end{aligned}$$

$$x = a\tan\theta \qquad \sqrt{a^2+x^2} = a\sec\theta \qquad -\tfrac{\pi}{2} < \theta < \tfrac{\pi}{2}$$

$$x = a\sin\theta \qquad \sqrt{a^2-x^2} = a\cos\theta \qquad -\tfrac{\pi}{2} < \theta < \tfrac{\pi}{2}$$

$$x = a\sec\theta \qquad \sqrt{x^2-a^2} = a\left|\tan\theta\right| \qquad \begin{cases} 0 \le \theta < \frac{\pi}{2} & x \ge a \\ \frac{\pi}{2} < \theta \le \pi & x \le -a \end{cases}$$

Example

Find $\displaystyle\int \frac{dx}{\sqrt{4+x^2}}$.

Let $x = 2\tan\theta$. Then $dx = 2\sec^2\theta\, d\theta$ and $\sqrt{4+x^2} = 2\sec\theta$. Therefore

$$\begin{aligned} \int \frac{dx}{\sqrt{4+x^2}} &= \int \frac{2\sec^2\theta\, d\theta}{2\sec\theta} \\ &= \int \sec\theta\, d\theta \\ &= \ln\left|\sec\theta + \tan\theta\right| + C \\ &= \ln\left|\frac{\sqrt{4+x^2}}{2} + \frac{x}{2}\right| + C. \end{aligned}$$

$$x = a\tan\theta \qquad x = a\sin\theta \qquad x = a\sec\theta$$

$$\sqrt{a^2+x^2} = a\sec\theta \qquad \sqrt{a^2-x^2} = a\cos\theta \qquad \sqrt{x^2-a^2} = a\left|\tan\theta\right|$$

$$-\tfrac{\pi}{2} < \theta < \tfrac{\pi}{2} \qquad -\tfrac{\pi}{2} < \theta < \tfrac{\pi}{2} \qquad \begin{cases} 0 \leq \theta < \frac{\pi}{2} & x \geq a \\ \frac{\pi}{2} < \theta \leq \pi & x \leq -a \end{cases}$$

Example

Calculate $\displaystyle\int_0^{\frac{3}{2}} \frac{dx}{\sqrt{9-x^2}}$.

$$x = a\tan\theta \qquad x = a\sin\theta \qquad x = a\sec\theta$$
$$\sqrt{a^2+x^2} = a\sec\theta \qquad \sqrt{a^2-x^2} = a\cos\theta \qquad \sqrt{x^2-a^2} = a\left|\tan\theta\right|$$
$$-\tfrac{\pi}{2} < \theta < \tfrac{\pi}{2} \qquad -\tfrac{\pi}{2} < \theta < \tfrac{\pi}{2} \qquad \begin{cases} 0 \le \theta < \frac{\pi}{2} & x \ge a \\ \frac{\pi}{2} < \theta \le \pi & x \le -a \end{cases}$$

Example

Calculate $\displaystyle\int_0^{\frac{3}{2}} \frac{dx}{\sqrt{9-x^2}}$.

Let $x = 3\sin\theta$.

$$x = a\tan\theta \qquad \sqrt{a^2+x^2} = a\sec\theta \qquad -\tfrac{\pi}{2} < \theta < \tfrac{\pi}{2}$$

$$x = a\sin\theta \qquad \sqrt{a^2-x^2} = a\cos\theta \qquad -\tfrac{\pi}{2} < \theta < \tfrac{\pi}{2}$$

$$x = a\sec\theta \qquad \sqrt{x^2-a^2} = a\,|\tan\theta| \qquad \begin{cases} 0 \leq \theta < \frac{\pi}{2} & x \geq a \\ \frac{\pi}{2} < \theta \leq \pi & x \leq -a \end{cases}$$

Example

Calculate $\displaystyle\int_0^{\frac{3}{2}} \frac{dx}{\sqrt{9-x^2}}$.

Let $x = 3\sin\theta$. $dx = 3\cos\theta\,d\theta$ and $\sqrt{9-x^2} = 3\cos\theta$.

$$x = a\tan\theta \qquad \sqrt{a^2+x^2} = a\sec\theta \qquad -\tfrac{\pi}{2} < \theta < \tfrac{\pi}{2}$$

$$x = a\sin\theta \qquad \sqrt{a^2-x^2} = a\cos\theta \qquad -\tfrac{\pi}{2} < \theta < \tfrac{\pi}{2}$$

$$x = a\sec\theta \qquad \sqrt{x^2-a^2} = a\left|\tan\theta\right| \qquad \begin{cases} 0 \leq \theta < \frac{\pi}{2} & x \geq a \\ \frac{\pi}{2} < \theta \leq \pi & x \leq -a \end{cases}$$

Example

Calculate $\displaystyle\int_0^{\frac{3}{2}} \frac{dx}{\sqrt{9-x^2}}$.

Let $x = 3\sin\theta$. $dx = 3\cos\theta\, d\theta$ and $\sqrt{9-x^2} = 3\cos\theta$.
Moreover $x = 0 \implies \theta = \sin^{-1}\frac{x}{3} = \sin^{-1} 0 = 0$
and $x = \frac{3}{2} \implies \theta = \sin^{-1}\frac{x}{3} = \sin^{-1}\frac{1}{2} = \frac{\pi}{6}$.

$x = a\tan\theta$	$x = a\sin\theta$	$x = a\sec\theta$
$\sqrt{a^2+x^2} = a\sec\theta$	$\sqrt{a^2-x^2} = a\cos\theta$	$\sqrt{x^2-a^2} = a\lvert\tan\theta\rvert$
$-\frac{\pi}{2} < \theta < \frac{\pi}{2}$	$-\frac{\pi}{2} < \theta < \frac{\pi}{2}$	$\begin{cases} 0 \le \theta < \frac{\pi}{2} & x \ge a \\ \frac{\pi}{2} < \theta \le \pi & x \le -a \end{cases}$

Example

Calculate $\displaystyle\int_0^{\frac{3}{2}} \frac{dx}{\sqrt{9-x^2}}$.

Let $x = 3\sin\theta$. $dx = 3\cos\theta\,d\theta$ and $\sqrt{9-x^2} = 3\cos\theta$.
Moreover $x = 0 \implies \theta = \sin^{-1}\frac{x}{3} = \sin^{-1} 0 = 0$
and $x = \frac{3}{2} \implies \theta = \sin^{-1}\frac{x}{3} = \sin^{-1}\frac{1}{2} = \frac{\pi}{6}$. Therefore

$$\int_0^{\frac{3}{2}} \frac{dx}{\sqrt{9-x^2}} = \int_0^{\frac{\pi}{6}} \frac{3\cos\theta\,d\theta}{3\cos\theta} = \int_0^{\frac{\pi}{6}} d\theta = \frac{\pi}{6}.$$

$$x = a\tan\theta \qquad \sqrt{a^2+x^2} = a\sec\theta \qquad -\tfrac{\pi}{2} < \theta < \tfrac{\pi}{2}$$

$$x = a\sin\theta \qquad \sqrt{a^2-x^2} = a\cos\theta \qquad -\tfrac{\pi}{2} < \theta < \tfrac{\pi}{2}$$

$$x = a\sec\theta \qquad \sqrt{x^2-a^2} = a\left|\tan\theta\right| \qquad \begin{cases} 0 \leq \theta < \frac{\pi}{2} & x \geq a \\ \frac{\pi}{2} < \theta \leq \pi & x \leq -a \end{cases}$$

Example

Calculate $\displaystyle\int_0^{\frac{3}{2}} \frac{dx}{\sqrt{9-x^2}}$.

Let $x = 3\sin\theta$. $dx = 3\cos\theta\, d\theta$ and $\sqrt{9-x^2} = 3\cos\theta$.
Moreover $x = 0 \implies \theta = \sin^{-1}\frac{x}{3} = \sin^{-1} 0 = 0$
and $x = \frac{3}{2} \implies \theta = \sin^{-1}\frac{x}{3} = \sin^{-1}\frac{1}{2} = \frac{\pi}{6}$. Therefore

$$\int_0^{\frac{3}{2}} \frac{dx}{\sqrt{9-x^2}} = \int_0^{\frac{\pi}{6}} \frac{3\cos\theta\, d\theta}{3\cos\theta} = \int_0^{\frac{\pi}{6}} d\theta = \frac{\pi}{6}.$$

Or

$$\int_0^{\frac{3}{2}} \frac{dx}{\sqrt{9-x^2}} = \left[\sin^{-1}\frac{x}{3}\right]_0^{\frac{3}{2}} = \frac{\pi}{6} - 0 = \frac{\pi}{6}.$$

EXAMPLE 2 Here we find an expression for the inverse hyperbolic sine function in terms of the natural logarithm. Following the same procedure as in Example 1, we find that

$$\int \frac{dx}{\sqrt{a^2 + x^2}} = \int \sec\theta \, d\theta \qquad x = a\tan\theta,\ dx = a\sec^2\theta \, d\theta$$

$$= \ln|\sec\theta + \tan\theta| + C$$

$$= \ln\left|\frac{\sqrt{a^2 + x^2}}{a} + \frac{x}{a}\right| + C \qquad \text{Fig. 8.2}$$

From Table 7.11, $\sinh^{-1}(x/a)$ is also an antiderivative of $1/\sqrt{a^2 + x^2}$, so the two antiderivatives differ by a constant, giving

$$\sinh^{-1}\frac{x}{a} = \ln\left|\frac{\sqrt{a^2 + x^2}}{a} + \frac{x}{a}\right| + C.$$

Setting $x = 0$ in this last equation, we find $0 = \ln|1| + C$, so $C = 0$. Since $\sqrt{a^2 + x^2} > |x|$, we conclude that

$$\boxed{\sinh^{-1}\frac{x}{a} = \ln\left(\frac{\sqrt{a^2 + x^2}}{a} + \frac{x}{a}\right)}$$

EXAMPLE 3 Evaluate

$$\int \frac{x^2\,dx}{\sqrt{9 - x^2}}.$$

Solution We set

$$x = 3\sin\theta, \qquad dx = 3\cos\theta\,d\theta, \qquad -\frac{\pi}{2} < \theta < \frac{\pi}{2}$$

$$9 - x^2 = 9 - 9\sin^2\theta = 9(1 - \sin^2\theta) = 9\cos^2\theta.$$

Then

$$\int \frac{x^2\,dx}{\sqrt{9-x^2}} = \int \frac{9\sin^2\theta \cdot 3\cos\theta\,d\theta}{|3\cos\theta|}$$

$$= 9\int \sin^2\theta\,d\theta \qquad \cos\theta > 0 \text{ for } -\frac{\pi}{2} < \theta < \frac{\pi}{2}$$

$$= 9\int \frac{1-\cos 2\theta}{2}\,d\theta$$

$$= \frac{9}{2}\left(\theta - \frac{\sin 2\theta}{2}\right) + C$$

$$= \frac{9}{2}(\theta - \sin\theta\cos\theta) + C \qquad \sin 2\theta = 2\sin\theta\cos\theta$$

$$= \frac{9}{2}\left(\sin^{-1}\frac{x}{3} - \frac{x}{3}\cdot\frac{\sqrt{9-x^2}}{3}\right) + C \qquad \text{From Fig. 8.5}$$

$$= \frac{9}{2}\sin^{-1}\frac{x}{3} - \frac{x}{2}\sqrt{9-x^2} + C.$$

EXAMPLE 4 Evaluate

$$\int \frac{dx}{\sqrt{25x^2 - 4}}, \qquad x > \frac{2}{5}.$$

Solution We first rewrite the radical as

$$\begin{aligned}\sqrt{25x^2 - 4} &= \sqrt{25\left(x^2 - \frac{4}{25}\right)} \\ &= 5\sqrt{x^2 - \left(\frac{2}{5}\right)^2} \qquad \sqrt{x^2 - a^2} \text{ with } a = \frac{2}{5}\end{aligned}$$

to put the radicand in the form $x^2 - a^2$. We then substitute

$$x = \frac{2}{5}\sec\theta, \qquad dx = \frac{2}{5}\sec\theta\tan\theta\, d\theta, \qquad 0 < \theta < \frac{\pi}{2}.$$

We then get

$$x^2 - \left(\frac{2}{5}\right)^2 = \frac{4}{25}\sec^2\theta - \frac{4}{25} = \frac{4}{25}(\sec^2\theta - 1) = \frac{4}{25}\tan^2\theta$$

and

$$\sqrt{x^2 - \left(\frac{2}{5}\right)^2} = \frac{2}{5}|\tan\theta| = \frac{2}{5}\tan\theta. \qquad \begin{array}{l}\tan\theta > 0 \text{ for} \\ 0 < \theta < \pi/2\end{array}$$

With these substitutions, we have

$$\int \frac{dx}{\sqrt{25x^2 - 4}} = \int \frac{dx}{5\sqrt{x^2 - (4/25)}} = \int \frac{(2/5)\sec\theta\tan\theta\, d\theta}{5 \cdot (2/5)\tan\theta}$$

$$= \frac{1}{5}\int \sec\theta\, d\theta = \frac{1}{5}\ln|\sec\theta + \tan\theta| + C$$

$$= \frac{1}{5}\ln\left|\frac{5x}{2} + \frac{\sqrt{25x^2 - 4}}{2}\right| + C. \qquad \text{From Fig. 8.6}$$

8.5 Integration of Rational Functions by Partial Fractions

$$\frac{2}{x+1} + \frac{3}{x-3} = \frac{\text{something?}}{\text{something?}}$$

$$\frac{2}{x+1} + \frac{3}{x-3} = \frac{2 \qquad\qquad\qquad}{(x+1) \qquad\qquad\qquad} + \frac{3 \qquad\qquad\qquad}{(x-3) \qquad\qquad\qquad}$$

$$\frac{2}{x+1} + \frac{3}{x-3} = \frac{2(x-3)}{(x+1)(x-3)} + \frac{3}{(x-3)}$$

$$\frac{2}{x+1}+\frac{3}{x-3}=\frac{2(x-3)}{(x+1)(x-3)}+\frac{3(x+1)}{(x-3)(x+1)}$$

$$\begin{aligned}\frac{2}{x+1}+\frac{3}{x-3} &= \frac{2(x-3)}{(x+1)(x-3)}+\frac{3(x+1)}{(x-3)(x+1)}\\ &= \frac{2x-6}{x^2-2x-3}+\frac{3x+3}{x^2-2x-3}\end{aligned}$$

$$\begin{aligned}\frac{2}{x+1}+\frac{3}{x-3} &= \frac{2(x-3)}{(x+1)(x-3)}+\frac{3(x+1)}{(x-3)(x+1)}\\ &= \frac{2x-6}{x^2-2x-3}+\frac{3x+3}{x^2-2x-3}\\ &= \frac{5x-3}{x^2-2x-3}.\end{aligned}$$

$$\begin{aligned}\frac{2}{x+1}+\frac{3}{x-3} &= \frac{2(x-3)}{(x+1)(x-3)}+\frac{3(x+1)}{(x-3)(x+1)}\\ &= \frac{2x-6}{x^2-2x-3}+\frac{3x+3}{x^2-2x-3}\\ &= \frac{5x-3}{x^2-2x-3}.\end{aligned}$$

But how do we do the opposite?

$$\frac{13x+1}{x^2-9}=\frac{\text{something?}}{x-3}+\frac{\text{something?}}{x+3}.$$

$$\frac{5x-3}{x^2-2x-3} = \frac{A}{x+1} + \frac{B}{x-3}.$$

Pretend for a moment that we don't know that $A = 2$ and $B = 3$. How can we find A and B?

$$\frac{5x-3}{x^2-2x-3} = \frac{A}{x+1} + \frac{B}{x-3}.$$

Pretend for a moment that we don't know that $A = 2$ and $B = 3$. How can we find A and B?

$$\begin{aligned} \frac{5x-3}{x^2-2x-3} &= \frac{A}{(x+1)\qquad\qquad} + \frac{B}{(x-3)\qquad\qquad} \\ &= \\ &= \qquad\qquad\qquad\qquad . \end{aligned}$$

$$\frac{5x-3}{x^2-2x-3} = \frac{A}{x+1} + \frac{B}{x-3}.$$

Pretend for a moment that we don't know that $A = 2$ and $B = 3$. How can we find A and B?

$$\begin{aligned}\frac{5x-3}{x^2-2x-3} &= \frac{A(x-3)}{(x+1)(x-3)} + \frac{B(x+1)}{(x-3)(x+1)} \\ &= \\ &= \qquad\qquad .\end{aligned}$$

$$\frac{5x-3}{x^2-2x-3} = \frac{A}{x+1} + \frac{B}{x-3}.$$

Pretend for a moment that we don't know that $A = 2$ and $B = 3$. How can we find A and B?

$$\begin{aligned}\frac{5x-3}{x^2-2x-3} &= \frac{A(x-3)}{(x+1)(x-3)} + \frac{B(x+1)}{(x-3)(x+1)} \\ &= \frac{Ax-3A}{x^2-2x-3} + \frac{Bx+B}{x^2-2x-3} \\ &= \qquad\qquad\qquad\qquad .\end{aligned}$$

$$\frac{5x-3}{x^2-2x-3} = \frac{A}{x+1} + \frac{B}{x-3}.$$

Pretend for a moment that we don't know that $A = 2$ and $B = 3$. How can we find A and B?

$$\begin{aligned}
\frac{5x-3}{x^2-2x-3} &= \frac{A(x-3)}{(x+1)(x-3)} + \frac{B(x+1)}{(x-3)(x+1)} \\
&= \frac{Ax-3A}{x^2-2x-3} + \frac{Bx+B}{x^2-2x-3} \\
&= \frac{(A+B)x+(B-3A)}{x^2-2x-3}.
\end{aligned}$$

$$\frac{5x-3}{x^2-2x-3} = \frac{A}{x+1} + \frac{B}{x-3}.$$

Pretend for a moment that we don't know that $A = 2$ and $B = 3$. How can we find A and B?

$$\begin{aligned}
\frac{5x-3}{x^2-2x-3} &= \frac{A(x-3)}{(x+1)(x-3)} + \frac{B(x+1)}{(x-3)(x+1)} \\
&= \frac{Ax-3A}{x^2-2x-3} + \frac{Bx+B}{x^2-2x-3} \\
&= \frac{(A+B)x+(B-3A)}{x^2-2x-3}.
\end{aligned}$$

$$\frac{5x-3}{x^2-2x-3} = \frac{A}{x+1} + \frac{B}{x-3}.$$

Pretend for a moment that we don't know that $A = 2$ and $B = 3$. How can we find A and B?

$$\begin{aligned}\frac{5x-3}{x^2-2x-3} &= \frac{A(x-3)}{(x+1)(x-3)} + \frac{B(x+1)}{(x-3)(x+1)} \\ &= \frac{Ax-3A}{x^2-2x-3} + \frac{Bx+B}{x^2-2x-3} \\ &= \frac{(A+B)x+(B-3A)}{x^2-2x-3}.\end{aligned}$$

Hence

$$\begin{cases} A+B=5 \\ B-3A=-3 \end{cases}$$

$$\frac{5x-3}{x^2-2x-3} = \frac{A}{x+1} + \frac{B}{x-3}.$$

Pretend for a moment that we don't know that $A = 2$ and $B = 3$. How can we find A and B?

$$\begin{aligned}
\frac{5x-3}{x^2-2x-3} &= \frac{A(x-3)}{(x+1)(x-3)} + \frac{B(x+1)}{(x-3)(x+1)} \\
&= \frac{Ax-3A}{x^2-2x-3} + \frac{Bx+B}{x^2-2x-3} \\
&= \frac{(A+B)x+(B-3A)}{x^2-2x-3}.
\end{aligned}$$

Hence

$$\begin{cases} A+B=5 \\ B-3A=-3 \end{cases} \implies \begin{cases} A=2 \\ B=3. \end{cases}$$

We can use *partial fractions* on $\dfrac{f(x)}{g(x)}$

We can use *partial fractions* on $\dfrac{f(x)}{g(x)}$ if

- $\left(\begin{array}{c} \text{the degree} \\ \text{of } f(x) \end{array} \right) < \left(\begin{array}{c} \text{the degree} \\ \text{of } g(x) \end{array} \right)$;

We can use *partial fractions* on $\dfrac{f(x)}{g(x)}$ if

- $\left(\begin{array}{c}\text{the degree} \\ \text{of } f(x)\end{array}\right) < \left(\begin{array}{c}\text{the degree} \\ \text{of } g(x)\end{array}\right)$; and
- we can factorise $g(x)$.

Example

Write $\dfrac{x^2+x+2}{(x+1)(x^2+1)}$ in partial fractions.

Example

Write $\dfrac{x^2+x+2}{(x+1)(x^2+1)}$ in partial fractions.

$$\frac{x^2+x+2}{(x+1)(x^2+1)} = \frac{A}{x+1} + \frac{Bx+C}{x^2+1}$$

$$=$$

$$=$$

$$=$$

$$= \qquad .$$

Example

Write $\dfrac{x^2+x+2}{(x+1)(x^2+1)}$ in partial fractions.

$$\begin{aligned}
\frac{x^2+x+2}{(x+1)(x^2+1)} &= \frac{A}{x+1} + \frac{Bx+C}{x^2+1} \\
&= \frac{A(x^2+1)}{(x+1)(x^2+1)} + \frac{(Bx+C)(x+1)}{(x^2+1)(x+1)} \\
&= \\
&= \\
&= \qquad\qquad .
\end{aligned}$$

Example

Write $\dfrac{x^2+x+2}{(x+1)(x^2+1)}$ in partial fractions.

$$\begin{aligned}
\frac{x^2+x+2}{(x+1)(x^2+1)} &= \frac{A}{x+1} + \frac{Bx+C}{x^2+1} \\
&= \frac{A(x^2+1)}{(x+1)(x^2+1)} + \frac{(Bx+C)(x+1)}{(x^2+1)(x+1)} \\
&= \frac{Ax^2+A+Bx^2+Bx+Cx+C}{(x+1)(x^2+1)} \\
&= \\
&= \qquad\qquad .
\end{aligned}$$

Example

Write $\dfrac{x^2+x+2}{(x+1)(x^2+1)}$ in partial fractions.

$$\begin{aligned}
\frac{x^2+x+2}{(x+1)(x^2+1)} &= \frac{A}{x+1} + \frac{Bx+C}{x^2+1} \\
&= \frac{A(x^2+1)}{(x+1)(x^2+1)} + \frac{(Bx+C)(x+1)}{(x^2+1)(x+1)} \\
&= \frac{Ax^2+A+Bx^2+Bx+Cx+C}{(x+1)(x^2+1)} \\
&= \frac{(A+B)x^2+(B+C)x+(A+C)}{(x+1)(x^2+1)} \\
&= \qquad\qquad\qquad .
\end{aligned}$$

8.5 Integration of Rational Functions by Partial Fractions

Example

Write $\dfrac{x^2+x+2}{(x+1)(x^2+1)}$ in partial fractions.

$$\frac{x^2+x+2}{(x+1)(x^2+1)} = \frac{A}{x+\ \cdots}$$

$$= \frac{A\ \cdots\ Bx+C)(x+1)}{(x+\ \cdots\ x^2+1)(x+1)}$$

$$= \frac{Ax^2+A+Bx^2+Bx+Cx+C}{(x+1)(x^2+1)}$$

$$= \frac{(A+B)x^2+(B+C)x+(A+C)}{(x+1)(x^2+1)}$$

$$= \qquad\qquad .$$

$$A+B=1$$
$$B+C=1$$
$$A+C=2$$

8.5 Integration of Rational Functions by Partial Fractions

Example

Write $\dfrac{x^2+x+2}{(x+1)(x^2+1)}$ in partial fractions.

$$A+B=1$$
$$B+C=1$$
$$A+C=2$$

$$A=1$$
$$B=0$$
$$C=1$$

$$\frac{x^2+x+2}{(x+1)(x^2+1)} = \frac{A}{x+\ \cdots}$$

$$= \frac{A\ \cdots\ 1)}{(x+\ \cdots\ 1)}$$

$$= \frac{Ax^2+A+Bx^2+\ \cdots}{(x+1)(x^2+1)}$$

$$= \frac{(A+B)x^2+(B+C)x+(A+C)}{(x+1)(x^2+1)}$$

$$= \qquad .$$

Example

Write $\dfrac{x^2+x+2}{(x+1)(x^2+1)}$ in partial fractions.

$$\frac{x^2+x+2}{(x+1)(x^2+1)} = \frac{A}{x+1} + \frac{Bx+C}{x^2+1}$$

$$= \frac{A(x^2+1)}{(x+1)(x^2+1)} + \frac{(\quad 1)}{\quad 1)}$$

$$= \frac{Ax^2+A+Bx^2+\ldots}{(x+1)(x^2+1)}$$

$$= \frac{(A+B)x^2+(B+C)x+(A+C)}{(x+1)(x^2+1)}$$

$$= \frac{1}{x+1} + \frac{1}{x^2+1}.$$

$$A = 1$$
$$B = 0$$
$$C = 1$$

Example

Write $\dfrac{3x^3 + 2x^2 + x}{(x+3)^4}$ in partial fractions.

Example

Write $\dfrac{3x^3+2x^2+x}{(x+3)^4}$ in partial fractions.

$$\frac{3x^3+2x^2+x}{(x+3)^4} = \frac{A}{x+3} + \frac{B}{(x+3)^2} + \frac{C}{(x+3)^3} + \frac{D}{(x+3)^4} = \dots$$

Example

Write $\dfrac{3x^3+2x^2+x}{(x+3)^4}$ in partial fractions.

$$\frac{3x^3+2x^2+x}{(x+3)^4} = \frac{A}{x+3} + \frac{B}{(x+3)^2} + \frac{C}{(x+3)^3} + \frac{D}{(x+3)^4} = \dots$$

Example

Write $\dfrac{71}{(x+3)(x^2+2x+3)^2}$ in partial fractions.

Example

Write $\dfrac{3x^3+2x^2+x}{(x+3)^4}$ in partial fractions.

$$\frac{3x^3+2x^2+x}{(x+3)^4} = \frac{A}{x+3} + \frac{B}{(x+3)^2} + \frac{C}{(x+3)^3} + \frac{D}{(x+3)^4} = \dots$$

Example

Write $\dfrac{71}{(x+3)(x^2+2x+3)^2}$ in partial fractions.

$$\begin{aligned}\frac{71}{(x+3)(x^2+2x+3)^2} &= \frac{A}{x+3} + \frac{Bx+C}{(x^2+2x+3)} + \frac{Dx+E}{(x^2+2x+3)^2} \\ &= \dots\end{aligned}$$

Method of Partial Fractions When $f(x)/g(x)$ Is Proper

1. Let $x - r$ be a linear factor of $g(x)$. Suppose that $(x - r)^m$ is the highest power of $x - r$ that divides $g(x)$. Then, to this factor, assign the sum of the m partial fractions:

$$\frac{A_1}{(x - r)} + \frac{A_2}{(x - r)^2} + \cdots + \frac{A_m}{(x - r)^m}.$$

Do this for each distinct linear factor of $g(x)$.

2. Let $x^2 + px + q$ be an irreducible quadratic factor of $g(x)$ so that $x^2 + px + q$ has no real roots. Suppose that $(x^2 + px + q)^n$ is the highest power of this factor that divides $g(x)$. Then, to this factor, assign the sum of the n partial fractions:

$$\frac{B_1x + C_1}{(x^2 + px + q)} + \frac{B_2x + C_2}{(x^2 + px + q)^2} + \cdots + \frac{B_nx + C_n}{(x^2 + px + q)^n}.$$

Do this for each distinct quadratic factor of $g(x)$.

3. Set the original fraction $f(x)/g(x)$ equal to the sum of all these partial fractions. Clear the resulting equation of fractions and arrange the terms in decreasing powers of x.

4. Equate the coefficients of corresponding powers of x and solve the resulting equations for the undetermined coefficients.

Example

Use partial fractions to find $\int \frac{x^2+4x+1}{(x-1)(x+1)(x+3)}\,dx$.

Since

$$\frac{x^2+4x+1}{(x-1)(x+1)(x+3)}$$

$$= \frac{A}{x-1} + \frac{B}{x+1} + \frac{C}{x+3}$$

$$=$$

$$=$$

$$=$$

$$=$$

Since

$$
\begin{aligned}
\frac{x^2+4x+1}{(x-1)(x+1)(x+3)} &= \frac{A}{x-1}+\frac{B}{x+1}+\frac{C}{x+3} \\
&= \frac{A(x+1)(x+3)+B(x-1)(x+3)+C(x-1)(x+1)}{(x-1)(x+1)(x+3)} \\
&= \\
&= \\
&=
\end{aligned}
$$

Since

$$
\begin{aligned}
&\frac{x^2+4x+1}{(x-1)(x+1)(x+3)} \\
&\qquad = \frac{A}{x-1} + \frac{B}{x+1} + \frac{C}{x+3} \\
&\qquad = \frac{A(x+1)(x+3) + B(x-1)(x+3) + C(x-1)(x+1)}{(x-1)(x+1)(x+3)} \\
&\qquad = \frac{A(x^2+4x+3) + B(x^2+2x-3) + C(x^2-1)}{(x-1)(x+1)(x+3)} \\
&\qquad = \\
&\qquad =
\end{aligned}
$$

Since

$$\frac{x^2+4x+1}{(x-1)(x+1)(x+3)}$$

$$= \frac{A}{x-1} + \frac{B}{x+1} + \frac{C}{x+3}$$

$$= \frac{A(x+1)(x+3) + B(x-1)(x+3) + C(x-1)(x+1)}{(x-1)(x+1)(x+3)}$$

$$= \frac{A(x^2+4x+3) + B(x^2+2x-3) + C(x^2-1)}{(x-1)(x+1)(x+3)}$$

$$= \frac{(A+B+C)x^2 + (4A+2B)x + (3A-3B-C)}{(x-1)(x+1)(x+3)}$$

$$=$$

Since

$$\frac{x^2+4x+1}{(x-1)(x+1)(x+3)}$$

$$= \frac{A}{x-1} + \frac{B}{x+1} + \cdots$$

$$= \frac{A(x+1)(x+3) \cdots C(x-1)(x+1)}{\cdots}$$

$$= \frac{A(x^2+4x+3) \cdots (x^2-1)}{(x-1)(x+1)(x+3)}$$

$$= \frac{(A+B+C)x^2 + (4A+2B)x + (3A-3B-C)}{(x-1)(x+1)(x+3)}$$

$$=$$

$$\begin{aligned} A+B+C &= 1 \\ 4A+2B &= 4 \\ 3A-3B-C &= 1 \end{aligned}$$

8.5 Integration of Rational Functions by Partial Fractions

Since

$$\frac{x^2+4x+1}{(x-1)(x+1)(x+3)}$$

$$= \frac{A}{x-1} + \frac{B}{x+1} + \ldots$$

$$= \frac{A(x+1)(x+3) \ldots 1)}{\ldots}$$

$$= \frac{A(x^2+4x+3) \ldots}{(x-1)(x+1)(x+3)}$$

$$= \frac{(A+B+C)x^2 + (4A+2B)x + (3A-3B-C)}{(x-1)(x+1)(x+3)}$$

$$=$$

$$\begin{aligned} A+B+C &= 1 \\ 4A+2B &= 4 \\ 3A-3B-C &= 1 \end{aligned}$$

$$A = \frac{3}{4} \qquad B = \frac{1}{2} \qquad C = -\frac{1}{4}$$

8.5 Integration of Rational Functions by Partial Fractions

Since

$$\frac{x^2+4x+1}{(x-1)(x+1)(x+3)}$$

$$= \frac{A}{x-1} + \frac{B}{x+1} + \frac{C}{x+3}$$

$$= \frac{A(x+1)(x+3) + B(x-1)(x+3) + \quad 1)}{(x-1)(x+1)(x+3)}$$

$$= \frac{A(x^2+4x+3) + B(x^2+2x-3) + C}{(x-1)(x+1)(x+3)}$$

$$= \frac{(A+B+C)x^2 + (4A+2B)x + (3A-3B-C)}{(x-1)(x+1)(x+3)}$$

$$= \frac{\frac{3}{4}}{x-1} + \frac{\frac{1}{2}}{x+1} + \frac{-\frac{1}{4}}{x+3}$$

$$A = \frac{3}{4}$$

$$B = \frac{1}{2}$$

$$C = -\frac{1}{4}$$

$$\frac{x^2+4x+1}{(x-1)(x+1)(x+3)} = \frac{\frac{3}{4}}{x-1} + \frac{\frac{1}{2}}{x+1} + \frac{-\frac{1}{4}}{x+3}$$

We have that

$$\begin{aligned}\int \frac{x^2+4x+1}{(x-1)(x+1)(x+3)}\,dx \\ &= \int \frac{\frac{3}{4}}{x-1} + \frac{\frac{1}{2}}{x+1} + \frac{-\frac{1}{4}}{x+3}\,dx \\ &= \\ &= \qquad .\end{aligned}$$

$$\frac{x^2+4x+1}{(x-1)(x+1)(x+3)} = \frac{\frac{3}{4}}{x-1} + \frac{\frac{1}{2}}{x+1} + \frac{-\frac{1}{4}}{x+3}$$

We have that

$$\begin{aligned}
\int \frac{x^2+4x+1}{(x-1)(x+1)(x+3)}\, dx \\
&= \int \frac{\frac{3}{4}}{x-1} + \frac{\frac{1}{2}}{x+1} + \frac{-\frac{1}{4}}{x+3}\, dx \\
&= \frac{3}{4}\int \frac{dx}{x-1} + \frac{1}{2}\int \frac{dx}{x+1} - \frac{1}{4}\int \frac{dx}{x+2} \\
&= \qquad\qquad\qquad\qquad .
\end{aligned}$$

$$\frac{x^2+4x+1}{(x-1)(x+1)(x+3)} = \frac{\frac{3}{4}}{x-1} + \frac{\frac{1}{2}}{x+1} + \frac{-\frac{1}{4}}{x+3}$$

We have that

$$\begin{aligned}
\int \frac{x^2+4x+1}{(x-1)(x+1)(x+3)}\,dx \\
&= \int \frac{\frac{3}{4}}{x-1} + \frac{\frac{1}{2}}{x+1} + \frac{-\frac{1}{4}}{x+3}\,dx \\
&= \frac{3}{4}\int \frac{dx}{x-1} + \frac{1}{2}\int \frac{dx}{x+1} - \frac{1}{4}\int \frac{dx}{x+2} \\
&= \frac{3}{4}\ln|x-1| + \frac{1}{2}\ln|x+1| - \frac{1}{4}\ln|x+2| + K.
\end{aligned}$$

(I already used C)

EXAMPLE 2 Use partial fractions to evaluate

$$\int \frac{6x + 7}{(x + 2)^2}\,dx.$$

Solution First we express the integrand as a sum of partial fractions with undetermined coefficients.

$$\frac{6x + 7}{(x + 2)^2} = \frac{A}{x + 2} + \frac{B}{(x + 2)^2} \qquad \text{Two terms because } (x + 2) \text{ is squared}$$

$$\begin{aligned} 6x + 7 &= A(x + 2) + B && \text{Multiply both sides by } (x + 2)^2. \\ &= Ax + (2A + B) \end{aligned}$$

Equating coefficients of corresponding powers of x gives

$$A = 6 \quad \text{and} \quad 2A + B = 12 + B = 7, \quad \text{or} \quad A = 6 \quad \text{and} \quad B = -5.$$

Therefore,

$$\begin{aligned} \int \frac{6x + 7}{(x + 2)^2}\,dx &= \int \left(\frac{6}{x + 2} - \frac{5}{(x + 2)^2}\right) dx \\ &= 6\int \frac{dx}{x + 2} - 5\int (x + 2)^{-2}\,dx \\ &= 6 \ln |x + 2| + 5(x + 2)^{-1} + C. \end{aligned}$$

■

EXAMPLE 3 Use partial fractions to evaluate

$$\int \frac{2x^3 - 4x^2 - x - 3}{x^2 - 2x - 3}\, dx.$$

Solution First we divide the denominator into the numerator to get a polynomial plus a proper fraction.

$$\begin{array}{r}
2x \phantom{{}- 4x^2 - x - 3} \\
x^2 - 2x - 3 \overline{\big)\, 2x^3 - 4x^2 - x - 3} \\
\underline{2x^3 - 4x^2 - 6x} - 3 \\
5x - 3
\end{array}$$

Then we write the improper fraction as a polynomial plus a proper fraction.

$$\frac{2x^3 - 4x^2 - x - 3}{x^2 - 2x - 3} = 2x + \frac{5x - 3}{x^2 - 2x - 3}$$

We found the partial fraction decomposition of the fraction on the right in the opening example, so

$$\begin{aligned}
\int \frac{2x^3 - 4x^2 - x - 3}{x^2 - 2x - 3}\, dx &= \int 2x\, dx + \int \frac{5x - 3}{x^2 - 2x - 3}\, dx \\
&= \int 2x\, dx + \int \frac{2}{x + 1}\, dx + \int \frac{3}{x - 3}\, dx \\
&= x^2 + 2 \ln |x + 1| + 3 \ln |x - 3| + C.
\end{aligned}$$

EXAMPLE 4 Use partial fractions to evaluate

$$\int \frac{-2x + 4}{(x^2 + 1)(x - 1)^2}\, dx.$$

Solution The denominator has an irreducible quadratic factor $x^2 + 1$ as well as a repeated linear factor $(x - 1)^2$, so we write

$$\frac{-2x + 4}{(x^2 + 1)(x - 1)^2} = \frac{Ax + B}{x^2 + 1} + \frac{C}{x - 1} + \frac{D}{(x - 1)^2}. \tag{2}$$

Clearing the equation of fractions gives

$$\begin{aligned} -2x + 4 &= (Ax + B)(x - 1)^2 + C(x - 1)(x^2 + 1) + D(x^2 + 1) \\ &= (A + C)x^3 + (-2A + B - C + D)x^2 \\ &\quad + (A - 2B + C)x + (B - C + D). \end{aligned}$$

Equating coefficients of like terms gives

Coefficients of x^3:	$0 = A + C$
Coefficients of x^2:	$0 = -2A + B - C + D$
Coefficients of x^1:	$-2 = A - 2B + C$
Coefficients of x^0:	$4 = B - C + D$

We solve these equations simultaneously to find the values of A, B, C, and D:

$$-4 = -2A, \qquad A = 2 \qquad \text{Subtract fourth equation from second.}$$

$$C = -A = -2 \qquad \text{From the first equation}$$

$$B = (A + C + 2)/2 = 1 \qquad \text{From the third equation and } C = -A$$

$$D = 4 - B + C = 1. \qquad \text{From the fourth equation}$$

We substitute these values into Equation (2), obtaining

$$\frac{-2x + 4}{(x^2 + 1)(x - 1)^2} = \frac{2x + 1}{x^2 + 1} - \frac{2}{x - 1} + \frac{1}{(x - 1)^2}.$$

Finally, using the expansion above we can integrate:

$$\begin{aligned}
\int \frac{-2x + 4}{(x^2 + 1)(x - 1)^2}\, dx &= \int \left(\frac{2x + 1}{x^2 + 1} - \frac{2}{x - 1} + \frac{1}{(x - 1)^2} \right) dx \\
&= \int \left(\frac{2x}{x^2 + 1} + \frac{1}{x^2 + 1} - \frac{2}{x - 1} + \frac{1}{(x - 1)^2} \right) dx \\
&= \ln(x^2 + 1) + \tan^{-1} x - 2 \ln |x - 1| - \frac{1}{x - 1} + C. \quad \blacksquare
\end{aligned}$$

EXAMPLE 5 Use partial fractions to evaluate

$$\int \frac{dx}{x(x^2+1)^2}.$$

Solution The form of the partial fraction decomposition is

$$\frac{1}{x(x^2+1)^2} = \frac{A}{x} + \frac{Bx+C}{x^2+1} + \frac{Dx+E}{(x^2+1)^2}.$$

Multiplying by $x(x^2+1)^2$, we have

$$\begin{aligned}
1 &= A(x^2+1)^2 + (Bx+C)x(x^2+1) + (Dx+E)x \\
&= A(x^4+2x^2+1) + B(x^4+x^2) + C(x^3+x) + Dx^2 + Ex \\
&= (A+B)x^4 + Cx^3 + (2A+B+D)x^2 + (C+E)x + A.
\end{aligned}$$

If we equate coefficients, we get the system

$$A+B=0, \qquad C=0, \qquad 2A+B+D=0, \qquad C+E=0, \qquad A=1.$$

Solving this system gives $A=1$, $B=-1$, $C=0$, $D=-1$, and $E=0$. Thus, Solving this system gives $A = 1$, $B = -1$, $C = 0$, $D = -1$, and $E = 0$. Thus,

$$\int \frac{dx}{x(x^2+1)^2} = \int \left[\frac{1}{x} + \frac{-x}{x^2+1} + \frac{-x}{(x^2+1)^2} \right] dx$$

$$= \int \frac{dx}{x} - \int \frac{x\,dx}{x^2+1} - \int \frac{x\,dx}{(x^2+1)^2}$$

$$= \int \frac{dx}{x} - \frac{1}{2}\int \frac{du}{u} - \frac{1}{2}\int \frac{du}{u^2}$$

$u = x^2 + 1$,
$du = 2x\,dx$

$$= \ln|x| - \frac{1}{2}\ln|u| + \frac{1}{2u} + K$$

$$= \ln|x| - \frac{1}{2}\ln(x^2+1) + \frac{1}{2(x^2+1)} + K$$

$$= \ln\frac{|x|}{\sqrt{x^2+1}} + \frac{1}{2(x^2+1)} + K.$$

Remark

When we have

$$\frac{f(x)}{(x-r_1)(x-r_2)\cdots(x-r_n)},$$

where $r_1, r_2, \ldots, r_n$ are all different, there is a quicker way to find partial fractions.

EXAMPLE 6 Find A, B, and C in the partial fraction expansion

$$\frac{x^2+1}{(x-1)(x-2)(x-3)} = \frac{A}{x-1} + \frac{B}{x-2} + \frac{C}{x-3}. \tag{3}$$

Solution If we multiply both sides of Equation (3) by $(x-1)$ to get

$$\frac{x^2+1}{(x-2)(x-3)} = A + \frac{B(x-1)}{x-2} + \frac{C(x-1)}{x-3}$$

and set $x = 1$, the resulting equation gives the value of A:

$$\frac{(1)^2+1}{(1-2)(1-3)} = A + 0 + 0,$$

$$A = 1.$$

In exactly the same way, we can multiply both sides by $(x-2)$ and then substitute in $x = 2$. This gives

$$\frac{(2)^2+1}{(2-1)(2-3)} = B.$$

So $B = -5$. Finally, we multiply both sides by $(x-3)$ and then substitute in $x = 3$, which yields

$$\frac{(3)^2+1}{(3-1)(3-2)} = C,$$

and $C = 5$. ∎

Example

Find $\displaystyle\int \frac{x+4}{x^3+3x^2-10x}\,dx.$

Example

Find $\displaystyle\int \frac{x+4}{x^3+3x^2-10x}\,dx.$

First we have

$$\frac{x+4}{x^3+3x^2-10x} = \frac{x+4}{x(x-2)(x+5)}$$

$$=$$

Example

Find $\displaystyle\int \frac{x+4}{x^3+3x^2-10x}\,dx$.

First we have

$$\begin{aligned}\frac{x+4}{x^3+3x^2-10x} &= \frac{x+4}{x(x-2)(x+5)}\\ &= \frac{A}{x}+\frac{B}{x-2}+\frac{C}{x+5}.\end{aligned}$$

Example

Find $\displaystyle\int \frac{x+4}{x^3+3x^2-10x}\,dx.$

First we have

$$\begin{aligned}\frac{x+4}{x^3+3x^2-10x} &= \frac{x+4}{x(x-2)(x+5)} \\ &= \frac{A}{x} + \frac{B}{x-2} + \frac{C}{x+5}.\end{aligned}$$

1. multiply by x, then set $x = 0$;
2. multiply by $(x-2)$, then set $x = 2$;
3. multiply by $(x+5)$, then set $x = -5$.

$$\boxed{\frac{x+4}{x(x-2)(x+5)} = \frac{A}{x} + \frac{B}{x-2} + \frac{C}{x+5}}$$

1. multiply by x, then set $x = 0$;

$$\begin{aligned}
\frac{x+4}{(x-2)(x+5)} &= A + \frac{Bx}{x-2} + \frac{Cx}{x+5} \\
\frac{4}{(-2)(5)} &= A + 0 + 0 \\
-\frac{2}{5} &= A
\end{aligned}$$

$$\frac{x+4}{x(x-2)(x+5)} = \frac{A}{x} + \frac{B}{x-2} + \frac{C}{x+5}$$

2 multiply by $(x-2)$, then set $x=2$;

$$\begin{aligned}
\frac{x+4}{x(x+5)} &= \frac{A(x-2)}{x} + B + \frac{C(x-2)}{x+5} \\
\frac{2+4}{(2)(7)} &= 0 + B + 0 \\
\frac{3}{7} &= B
\end{aligned}$$

$$\frac{x+4}{x(x-2)(x+5)} = \frac{A}{x} + \frac{B}{x-2} + \frac{C}{x+5}$$

3 multiply by $(x+5)$, then set $x=-5$.

$$\begin{aligned} \frac{x+4}{x(x-2)} &= \frac{A(x+5)}{x} + \frac{B(x+5)}{x-2} + C \\ \frac{-5+4}{(-5)(-7)} &= 0 + 0 + C \\ -\frac{1}{35} &= C \end{aligned}$$

Therefore

$$\frac{x+4}{x(x-2)(x+5)} = \frac{-\frac{2}{5}}{x} + \frac{\frac{3}{7}}{x-2} + \frac{-\frac{1}{35}}{x+5}$$

Therefore

$$\frac{x+4}{x(x-2)(x+5)} = \frac{-\frac{2}{5}}{x} + \frac{\frac{3}{7}}{x-2} + \frac{-\frac{1}{35}}{x+5}$$

and thus

$$\int \frac{x+4}{x(x-2)(x+5)}\,dx = -\frac{2}{5}\ln|x| + \frac{3}{7}\ln|x-2| - \frac{1}{35}\ln|x+5| + C.$$

Remark

We can also use differentiation to find partial fractions.

EXAMPLE 7 Find A, B, and C in the equation

$$\frac{x-1}{(x+1)^3} = \frac{A}{x+1} + \frac{B}{(x+1)^2} + \frac{C}{(x+1)^3}$$

by clearing fractions, differentiating the result, and substituting $x = -1$.

Solution We first clear fractions:

$$x - 1 = A(x+1)^2 + B(x+1) + C.$$

Substituting $x = -1$ shows $C = -2$. We then differentiate both sides with respect to x, obtaining

$$1 = 2A(x+1) + B.$$

Substituting $x = -1$ shows $B = 1$. We differentiate again to get $0 = 2A$, which shows $A = 0$. Hence,

$$\frac{x-1}{(x+1)^3} = \frac{1}{(x+1)^2} - \frac{2}{(x+1)^3}.$$

■

Remark

Sometimes we can just try putting in small numbers $x = 0$, $x = \pm 1$, $x = \pm 2$, etc. to find the coefficients A, B, C, $\ldots$

EXAMPLE 8 Find A, B, and C in the expression

$$\frac{x^2+1}{(x-1)(x-2)(x-3)} = \frac{A}{x-1} + \frac{B}{x-2} + \frac{C}{x-3}$$

by assigning numerical values to x.

Solution Clear fractions to get

$$x^2 + 1 = A(x-2)(x-3) + B(x-1)(x-3) + C(x-1)(x-2).$$

Then let $x = 1, 2, 3$ successively to find A, B, and C:

$$\begin{aligned}
x = 1: \qquad (1)^2 + 1 &= A(-1)(-2) + B(0) + C(0)\\
2 &= 2A\\
A &= 1\\
x = 2: \qquad (2)^2 + 1 &= A(0) + B(1)(-1) + C(0)\\
5 &= -B\\
B &= -5\\
x = 3: \qquad (3)^2 + 1 &= A(0) + B(0) + C(2)(1)\\
10 &= 2C\\
C &= 5.
\end{aligned}$$

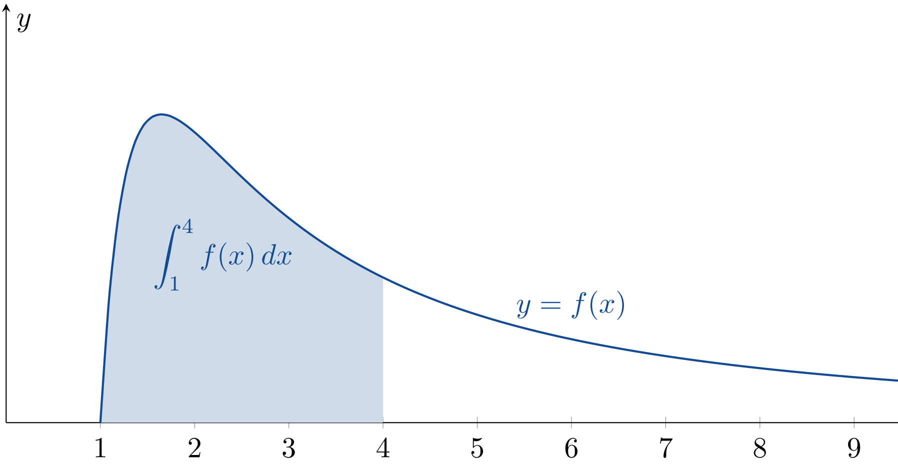
Conclusion:

$$\frac{x^2+1}{(x-1)(x-2)(x-3)} = \frac{1}{x-1} - \frac{5}{x-2} + \frac{5}{x-3}.$$

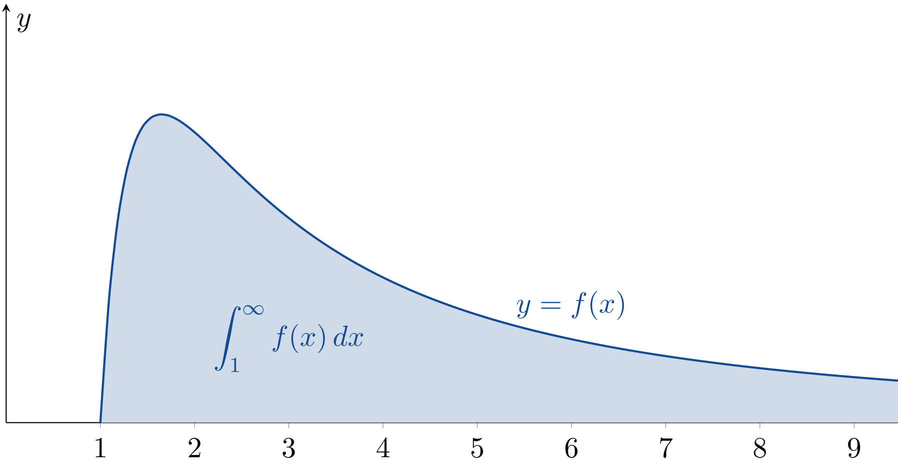
Break

We will continue at 2pm

8.8 Improper Integrals



y
$\int_1^4 f(x)\,dx$
$y = f(x)$
1
2
3
4
5
6
7
8
9


y
$\int_1^\infty f(x)\,dx$
$y = f(x)$
1
2
3
4
5
6
7
8
9

We need to use limits.

Example

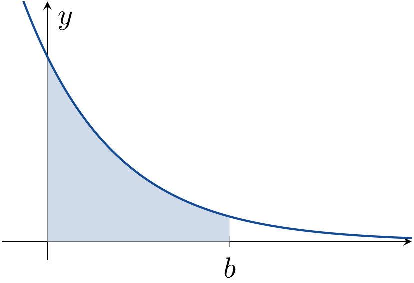
Calculate $\displaystyle\int_0^\infty e^{-\frac{x}{2}}\,dx.$

Example

Calculate $\displaystyle\int_0^\infty e^{-\frac{x}{2}}\,dx$.

Step 1:

$$\int_0^b e^{-\frac{x}{2}}\,dx = ?$$

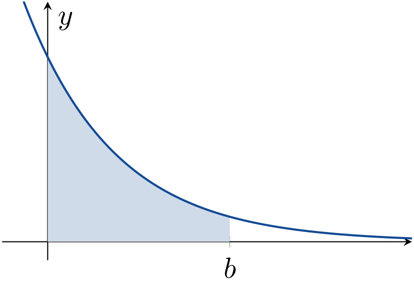


Example

Calculate $\displaystyle\int_0^{\infty} e^{-\frac{x}{2}}\,dx$.

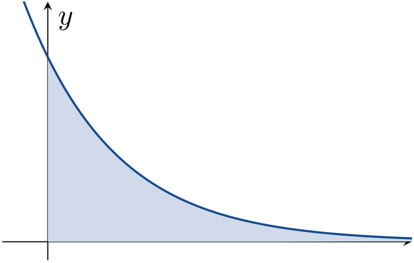
Step 1:

$$\int_0^b e^{-\frac{x}{2}}\,dx = ?$$



Step 2:

$$\int_0^{\infty} e^{-\frac{x}{2}}\,dx = \lim_{b\to\infty} \int_0^b e^{-\frac{x}{2}}\,dx$$



Since

$$\int_0^b e^{-\frac{x}{2}}\,dx = \left[-2e^{-\frac{x}{2}}\right]_0^b = -2e^{-\frac{b}{2}} + 2,$$

Since

$$\int_0^b e^{-\frac{x}{2}}\,dx = \left[-2e^{-\frac{x}{2}}\right]_0^b = -2e^{-\frac{b}{2}} + 2,$$

we have that

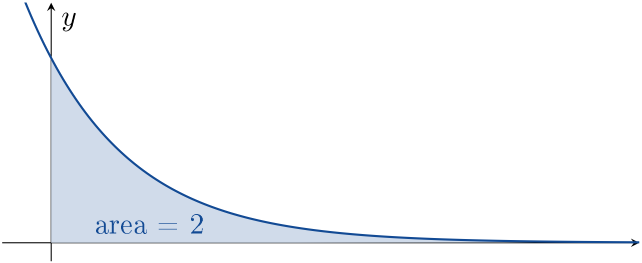
$$\int_0^{\infty} e^{-\frac{x}{2}}\,dx = \lim_{b\to\infty}\int_0^b e^{-\frac{x}{2}}\,dx = \lim_{b\to\infty}\left(-2e^{-\frac{b}{2}} + 2\right) = 2.$$

Since

$$\int_0^b e^{-\frac{x}{2}}\,dx = \left[-2e^{-\frac{x}{2}}\right]_0^b = -2e^{-\frac{b}{2}} + 2,$$

we have that

$$\int_0^\infty e^{-\frac{x}{2}}\,dx = \lim_{b\to\infty} \int_0^b e^{-\frac{x}{2}}\,dx = \lim_{b\to\infty} \left(-2e^{-\frac{b}{2}} + 2\right) = 2.$$



DEFINITION Integrals with infinite limits of integration are **improper integrals of Type I**.

1. If $f(x)$ is continuous on $[a, \infty)$, then

$$\int_a^{\infty} f(x)\,dx = \lim_{b \to \infty} \int_a^b f(x)\,dx.$$

2. If $f(x)$ is continuous on $(-\infty, b]$, then

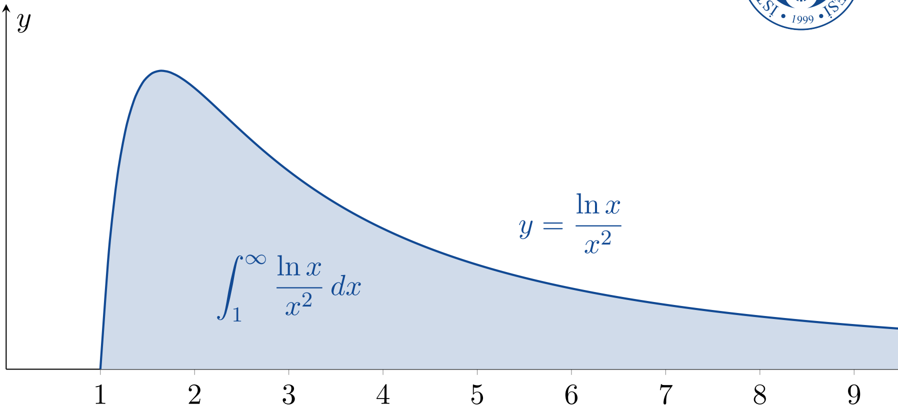
$$\int_{-\infty}^{b} f(x)\,dx = \lim_{a \to -\infty} \int_a^b f(x)\,dx.$$

3. If $f(x)$ is continuous on $(-\infty, \infty)$, then

$$\int_{-\infty}^{\infty} f(x)\,dx = \int_{-\infty}^{c} f(x)\,dx + \int_c^{\infty} f(x)\,dx,$$

   where c is any real number.

In each case, if the limit exists and is finite, we say that the improper integral **converges** and that the limit is the **value** of the improper integral. If the limit fails to exist, the improper integral **diverges**.



Example

Is the area under the curve $y = \frac{\ln x}{x^2}$, from $x = 1$ to $x = \infty$, finite? Is so, what is its value?.

Since

$$\int_1^b \frac{\ln x}{x^2}\,dx = \left[(\ln x)\left(-\frac{1}{x}\right)\right]_1^b - \int_1^b \left(-\frac{1}{x}\right)\left(\frac{1}{x}\right)\,dx$$

$$= -\frac{\ln b}{b} - \left[\frac{1}{x}\right]_1^b = -\frac{\ln b}{b} - \frac{1}{b} + 1,$$

Since

$$\int_1^b \frac{\ln x}{x^2}\,dx = \left[(\ln x)\left(-\frac{1}{x}\right)\right]_1^b - \int_1^b \left(-\frac{1}{x}\right)\left(\frac{1}{x}\right)\,dx$$

$$= -\frac{\ln b}{b} - \left[\frac{1}{x}\right]_1^b = -\frac{\ln b}{b} - \frac{1}{b} + 1,$$

we have that

$$\int_1^\infty \frac{\ln x}{x^2}\,dx = \lim_{b\to\infty}\int_1^b \frac{\ln x}{x^2}\,dx = \lim_{b\to\infty}\left(-\frac{\ln b}{b} - \frac{1}{b} + 1\right)$$

$$=$$

$$=$$

$$.$$

Since

$$\int_1^b \frac{\ln x}{x^2}\,dx = \left[(\ln x)\left(-\frac{1}{x}\right)\right]_1^b - \int_1^b \left(-\frac{1}{x}\right)\left(\frac{1}{x}\right)dx$$

$$= -\frac{\ln b}{b} - \left[\frac{1}{x}\right]_1^b = -\frac{\ln b}{b} - \frac{1}{b} + 1,$$

we have that

$$\int_1^\infty \frac{\ln x}{x^2}\,dx = \lim_{b\to\infty}\int_1^b \frac{\ln x}{x^2}\,dx = \lim_{b\to\infty}\left(-\frac{\ln b}{b} - \frac{1}{b} + 1\right)$$

$$= \lim_{b\to\infty}\left(-\frac{\ln b}{b}\right) - 0 + 1$$

$$=$$

.

Since

$$\int_1^b \frac{\ln x}{x^2}\,dx = \left[(\ln x)\left(-\frac{1}{x}\right)\right]_1^b - \int_1^b \left(-\frac{1}{x}\right)\left(\frac{1}{x}\right)dx$$

$$= -\frac{\ln b}{b} - \left[\frac{1}{x}\right]_1^b = -\frac{\ln b}{b} - \frac{1}{b} + 1,$$

we have that

$$\int_1^\infty \frac{\ln x}{x^2}\,dx = \lim_{b\to\infty}\int_1^b \frac{\ln x}{x^2}\,dx = \lim_{b\to\infty}\left(-\frac{\ln b}{b} - \frac{1}{b} + 1\right)$$

$$= \lim_{b\to\infty}\left(-\frac{\ln b}{b}\right) - 0 + 1$$

$$= \lim_{b\to\infty}\left(-\frac{\frac{1}{b}}{1}\right) + 1 \qquad \text{(l'Hôpital's Rule)}$$

.

Since

$$\int_1^b \frac{\ln x}{x^2}\,dx = \left[(\ln x)\left(-\frac{1}{x}\right)\right]_1^b - \int_1^b \left(-\frac{1}{x}\right)\left(\frac{1}{x}\right)\,dx$$
$$= -\frac{\ln b}{b} - \left[\frac{1}{x}\right]_1^b = -\frac{\ln b}{b} - \frac{1}{b} + 1,$$

we have that

$$\int_1^\infty \frac{\ln x}{x^2}\,dx = \lim_{b\to\infty} \int_1^b \frac{\ln x}{x^2}\,dx = \lim_{b\to\infty}\left(-\frac{\ln b}{b} - \frac{1}{b} + 1\right)$$
$$= \lim_{b\to\infty}\left(-\frac{\ln b}{b}\right) - 0 + 1$$
$$= \lim_{b\to\infty}\left(-\frac{\frac{1}{b}}{1}\right) + 1 \qquad \text{(l'Hôpital's Rule)}$$
$$= 0 + 1 = 1.$$

Therefore the integral converges and the area has finite value 1.

EXAMPLE 2 Evaluate

$$\int_{-\infty}^{\infty} \frac{dx}{1 + x^2}.$$

Solution According to the definition (Part 3), we can choose $c = 0$ and write

$$\int_{-\infty}^{\infty} \frac{dx}{1 + x^2} = \int_{-\infty}^{0} \frac{dx}{1 + x^2} + \int_{0}^{\infty} \frac{dx}{1 + x^2}.$$

Next we evaluate each improper integral on the right side of the equation above.

$$\begin{aligned} \int_{-\infty}^{0} \frac{dx}{1 + x^2} &= \lim_{a \to -\infty} \int_{a}^{0} \frac{dx}{1 + x^2} \\ &= \lim_{a \to -\infty} \tan^{-1} x \Big]_{a}^{0} \\ &= \lim_{a \to -\infty} (\tan^{-1} 0 - \tan^{-1} a) = 0 - \left(-\frac{\pi}{2}\right) = \frac{\pi}{2} \end{aligned}$$

$$\int_0^{\infty} \frac{dx}{1+x^2} = \lim_{b\to\infty} \int_0^b \frac{dx}{1+x^2}$$

$$= \lim_{b\to\infty} \tan^{-1} x \Big]_0^b$$

$$= \lim_{b\to\infty} (\tan^{-1} b - \tan^{-1} 0) = \frac{\pi}{2} - 0 = \frac{\pi}{2}$$

Thus,

$$\int_{-\infty}^{\infty} \frac{dx}{1+x^2} = \frac{\pi}{2} + \frac{\pi}{2} = \pi.$$

Since $1/(1+x^2) > 0$, the improper integral can be interpreted as the (finite) area beneath the curve and above the x-axis (Figure 8.15). ■

Remark

$$\int_{-\infty}^{\infty} f(x)\,dx = \int_{-\infty}^{c} f(x)\,dx + \int_{c}^{\infty} f(x)\,dx.$$

Remark

$$\int_{-\infty}^{\infty} f(x)\,dx = \int_{-\infty}^{c} f(x)\,dx + \int_{c}^{\infty} f(x)\,dx.$$

This is not the same as $\lim_{b\to\infty} \int_{-b}^{b} f(x)\,dx$!!!

Remark

$$\int_{-\infty}^{\infty} f(x)\,dx = \int_{-\infty}^{c} f(x)\,dx + \int_{c}^{\infty} f(x)\,dx.$$

This is not the same as $\displaystyle\lim_{b\to\infty}\int_{-b}^{b} f(x)\,dx$!!!

For example, $\displaystyle\int_0^{\infty} \frac{2x\,dx}{x^2+1}$ diverges

Remark

$$\int_{-\infty}^{\infty} f(x)\,dx = \int_{-\infty}^{c} f(x)\,dx + \int_{c}^{\infty} f(x)\,dx.$$

This is not the same as $\lim\limits_{b\to\infty} \int_{-b}^{b} f(x)\,dx$!!!

For example, $\displaystyle\int_{0}^{\infty} \frac{2x\,dx}{x^2+1}$ diverges and hence $\displaystyle\int_{-\infty}^{\infty} \frac{2x\,dx}{x^2+1}$ diverges.

Remark

$$\int_{-\infty}^{\infty} f(x)\,dx = \int_{-\infty}^{c} f(x)\,dx + \int_{c}^{\infty} f(x)\,dx.$$

This is not the same as $\lim_{b\to\infty} \int_{-b}^{b} f(x)\,dx$!!!

For example, $\int_0^{\infty} \frac{2x\,dx}{x^2+1}$ diverges and hence $\int_{-\infty}^{\infty} \frac{2x\,dx}{x^2+1}$ diverges. However

$$\lim_{b\to\infty} \int_{-b}^{b} \frac{2x\,dx}{x^2+1} = 0.$$

(Left for you to prove.)

EXAMPLE 3 For what values of p does the integral $\int_1^\infty dx/x^p$ converge? When the integral does converge, what is its value?

Solution If $p \neq 1$,

$$\int_1^b \frac{dx}{x^p} = \frac{x^{-p+1}}{-p+1}\bigg]_1^b = \frac{1}{1-p}(b^{-p+1} - 1) = \frac{1}{1-p}\left(\frac{1}{b^{p-1}} - 1\right).$$

Thus,

$$\int_1^\infty \frac{dx}{x^p} = \lim_{b\to\infty}\int_1^b \frac{dx}{x^p}$$

$$= \lim_{b\to\infty}\left[\frac{1}{1-p}\left(\frac{1}{b^{p-1}} - 1\right)\right] = \begin{cases} \dfrac{1}{p-1}, & p > 1 \\ \infty, & p < 1 \end{cases}$$

because

$$\lim_{b\to\infty}\frac{1}{b^{p-1}} = \begin{cases} 0, & p > 1 \\ \infty, & p < 1. \end{cases}$$

Therefore, the integral converges to the value $1/(p-1)$ if $p > 1$ and it diverges if $p < 1$.

If $p = 1$, the integral also diverges:

$$\begin{aligned}\int_1^\infty \frac{dx}{x^p} = \int_1^\infty \frac{dx}{x} &= \lim_{b\to\infty}\int_1^b \frac{dx}{x}\\ &= \lim_{b\to\infty} \ln x \Big]_1^b\\ &= \lim_{b\to\infty} (\ln b - \ln 1) = \infty.\end{aligned}$$

If $p = 1$, the integral also diverges:

$$\begin{aligned}\int_1^\infty \frac{dx}{x^p} = \int_1^\infty \frac{dx}{x} &= \lim_{b\to\infty}\int_1^b \frac{dx}{x} \\ &= \lim_{b\to\infty} \ln x\Big]_1^b \\ &= \lim_{b\to\infty}(\ln b - \ln 1) = \infty.\end{aligned}$$

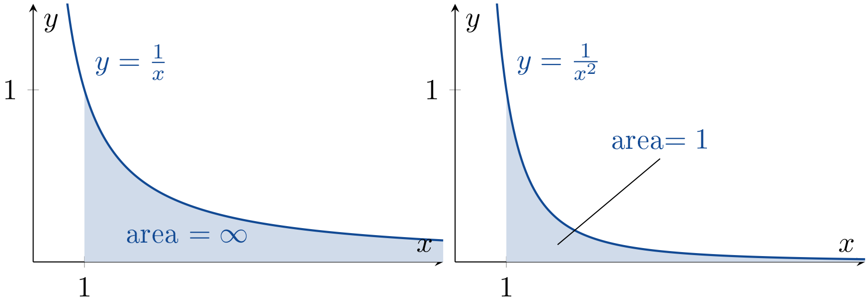
Theorem

$$\int_1^\infty \frac{dx}{x^p} \begin{cases} \textit{converges if } p > 1, \\ \textit{diverges if } p \le 1. \end{cases}$$

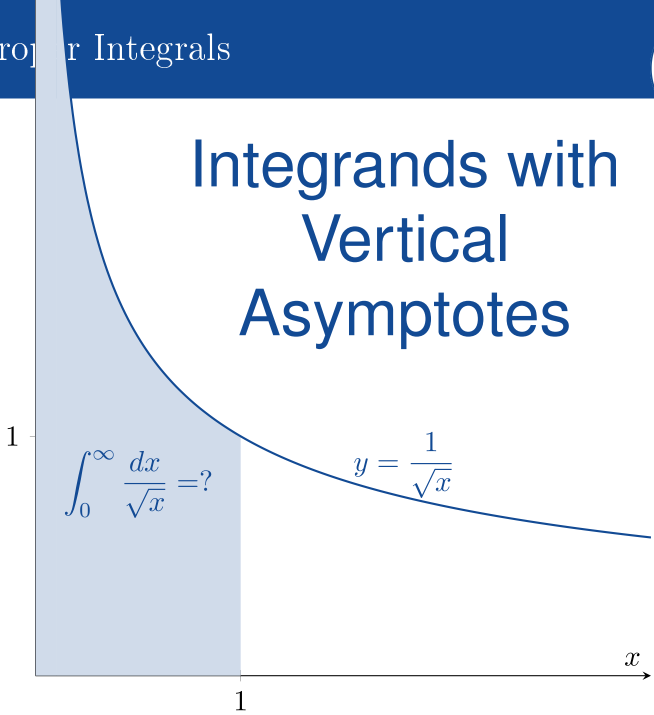
Remark

In particular, please remember that

$$\int_1^\infty \frac{dx}{x} \quad \text{diverges} \qquad \text{and} \qquad \int_1^\infty \frac{dx}{x^2} \quad \text{converges.}$$

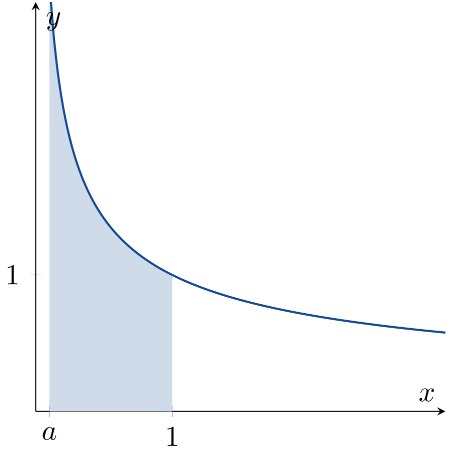


Integrands with Vertical Asymptotes

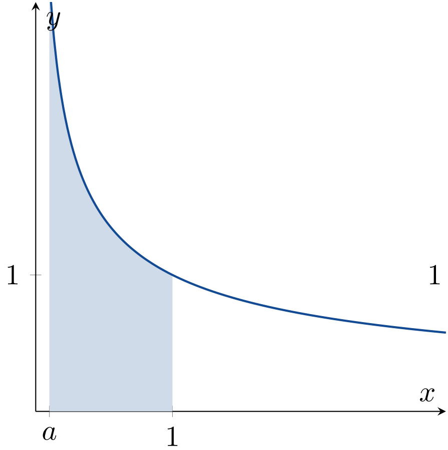


Step 1:

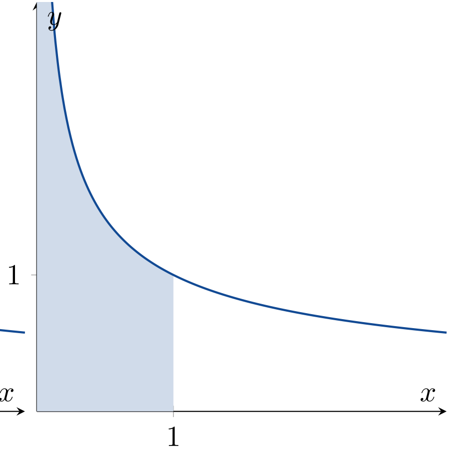
$$\int_a^1 \frac{dx}{\sqrt{x}} =?$$



Step 1:
$$\int_a^1 \frac{dx}{\sqrt{x}} = ?$$



Step 2:
$$\int_0^1 \frac{dx}{\sqrt{x}} = \lim_{a\to 0^+} \int_a^1 \frac{dx}{\sqrt{x}}$$



Since

$$\int_a^1 \frac{dx}{\sqrt{x}} = \left[2\sqrt{x}\right]_a^1 = 2 - 2\sqrt{a},$$

Since

$$\int_a^1 \frac{dx}{\sqrt{x}} = \left[2\sqrt{x}\right]_a^1 = 2 - 2\sqrt{a},$$

we have that

$$\int_0^1 \frac{dx}{\sqrt{x}} = \lim_{a \to 0^+} \int_a^1 \frac{dx}{\sqrt{x}} = \lim_{a \to 0^+} \left(2 - 2\sqrt{a}\right) = 2.$$

DEFINITION Integrals of functions that become infinite at a point within the interval of integration are **improper integrals of Type II**.

1. If $f(x)$ is continuous on $(a, b]$ and discontinuous at a, then

$$\int_a^b f(x)\,dx = \lim_{c \to a^+} \int_c^b f(x)\,dx.$$

2. If $f(x)$ is continuous on $[a, b)$ and discontinuous at b, then

$$\int_a^b f(x)\,dx = \lim_{c \to b^-} \int_a^c f(x)\,dx.$$

3. If $f(x)$ is discontinuous at c, where $a < c < b$, and continuous on $[a, c) \cup (c, b]$, then

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx.$$

In each case, if the limit exists and is finite, we say the improper integral **converges** and that the limit is the **value** of the improper integral. If the limit does not exist, the integral **diverges**.

EXAMPLE 4 Investigate the convergence of

$$\int_0^1 \frac{1}{1-x}\, dx.$$

Solution The integrand $f(x) = 1/(1-x)$ is continuous on $[0, 1)$ but is discontinuous at $x = 1$ and becomes infinite as $x \to 1^-$ (Figure 8.17). We evaluate the integral as

$$\begin{aligned}\lim_{b \to 1^-} \int_0^b \frac{1}{1-x}\, dx &= \lim_{b \to 1^-} \Big[-\ln|1-x|\Big]_0^b \\ &= \lim_{b \to 1^-} [-\ln(1-b) + 0] = \infty.\end{aligned}$$

The limit is infinite, so the integral diverges. ■

EXAMPLE 5 Evaluate

$$\int_0^3 \frac{dx}{(x-1)^{2/3}}.$$

Solution The integrand has a vertical asymptote at $x = 1$ and is continuous on $[0, 1)$ and $(1, 3]$ (Figure 8.18). Thus, by Part 3 of the definition above,

$$\int_0^3 \frac{dx}{(x-1)^{2/3}} = \int_0^1 \frac{dx}{(x-1)^{2/3}} + \int_1^3 \frac{dx}{(x-1)^{2/3}}.$$

Next, we evaluate each improper integral on the right-hand side of this equation.

$$\begin{aligned}
\int_0^1 \frac{dx}{(x-1)^{2/3}} &= \lim_{b\to 1^-}\int_0^b \frac{dx}{(x-1)^{2/3}} \\
&= \lim_{b\to 1^-} 3(x-1)^{1/3}\Big]_0^b \\
&= \lim_{b\to 1^-} \left[3(b-1)^{1/3} + 3\right] = 3
\end{aligned}$$

$$
\begin{aligned}
\int_1^3 \frac{dx}{(x-1)^{2/3}} &= \lim_{c\to 1^+}\int_c^3 \frac{dx}{(x-1)^{2/3}} \\
&= \lim_{c\to 1^+} 3(x-1)^{1/3}\Big]_c^3 \\
&= \lim_{c\to 1^+} \left[3(3-1)^{1/3} - 3(c-1)^{1/3}\right] = 3\sqrt[3]{2}
\end{aligned}
$$

We conclude that

$$\int_0^3 \frac{dx}{(x-1)^{2/3}} = 3 + 3\sqrt[3]{2}.$$

Remark

Sometimes we cannot evaluate an improper integral, but we can still determine whether it converges or diverges.

Remark

Sometimes we cannot evaluate an improper integral, but we can still determine whether it converges or diverges.

Example

Does $\int_1^\infty e^{-x^2}\,dx$ converge or diverge?

We can not calculate $\int_1^b e^{-x^2}\,dx$ because it is nonelementary. But we can answer this example another way.

Since $e^{-x^2} > 0$, we know that $I(b) = \int_1^b e^{-x^2}\,dx$ is an increasing function of b.

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So either

- $\lim_{b\to\infty} \int_1^b e^{-x^2}\,dx = \infty$; or
- $\lim_{b\to\infty} \int_1^b e^{-x^2}\,dx$ is a finite number.

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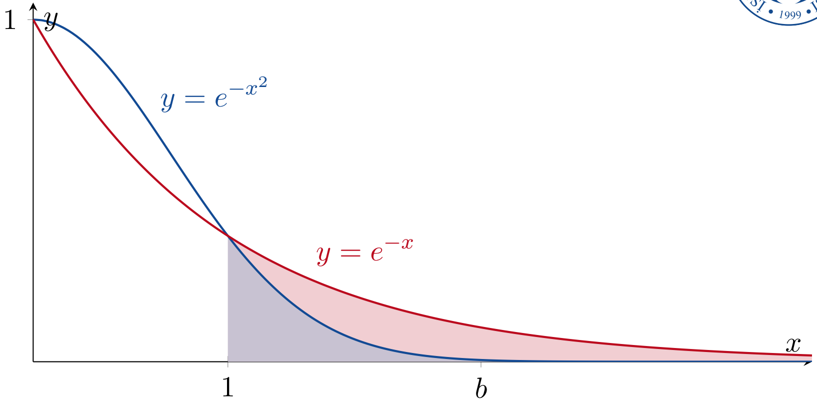
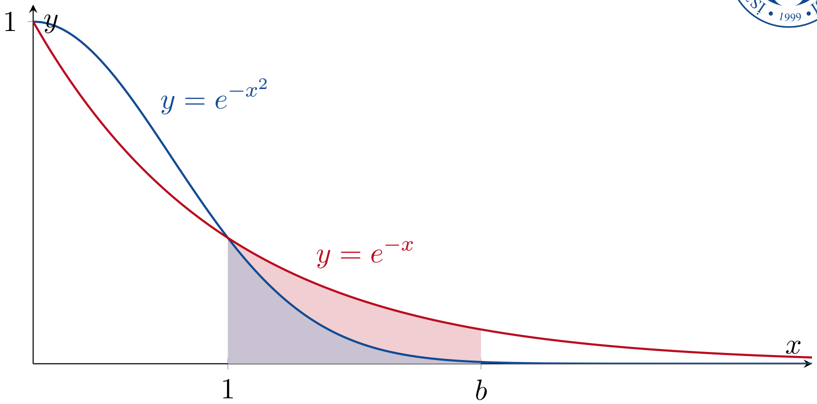
So either

- $\lim\limits_{b\to\infty} \int_1^b e^{-x^2}\,dx = \infty$; or
- $\lim\limits_{b\to\infty} \int_1^b e^{-x^2}\,dx$ is a finite number.

I am going to prove to you that $\lim\limits_{b\to\infty} \int_1^b e^{-x^2}\,dx$ is finite.



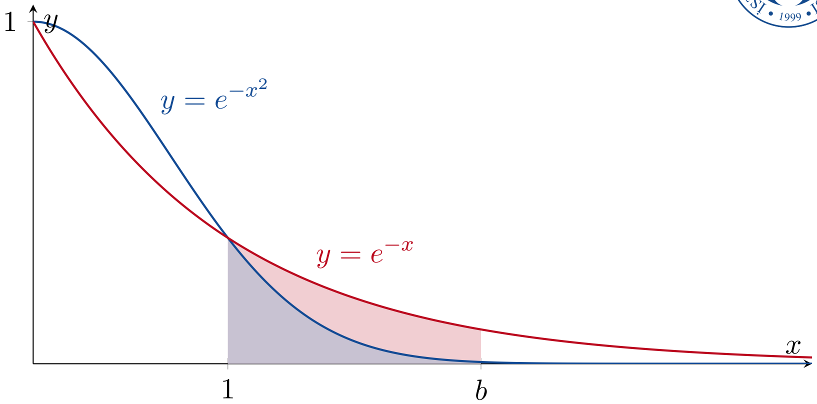
Note that $e^{-x^2} \le e^{-x}$ for all $x \ge 1$.



Note that $e^{-x^2} \leq e^{-x}$ for all $x \geq 1$. So

$$\int_1^b e^{-x^2}\,dx \leq \int_1^b e^{-x}\,dx$$

for any $b > 1$.



Note that $e^{-x^2} \leq e^{-x}$ for all $x \geq 1$. So

$$\int_1^b e^{-x^2}\,dx \leq \int_1^b e^{-x}\,dx = -e^{-b} + e^{-1} < e^{-1} \approx 0.36788$$

for any $b > 1$.

Therefore

$$\int_1^\infty e^{-x^2}\,dx = \lim_{b\to\infty}\int_1^b e^{-x^2}\,dx$$

converges to a finite value.

Theorem (Direct Comparison Test)

Let $f : [a, \infty) \to \mathbb{R}$ and $g : [a, \infty) \to \mathbb{R}$ be continuous functions. Suppose that

$$0 \leq f(x) \leq g(x)$$

for all $x \in [a, \infty)$.

Theorem (Direct Comparison Test)

Let $f : [a, \infty) \to \mathbb{R}$ and $g : [a, \infty) \to \mathbb{R}$ be continuous functions. Suppose that

$$0 \leq f(x) \leq g(x)$$

for all $x \in [a, \infty)$. Then

1. $\displaystyle\int_a^\infty g(x)\,dx$ *converges* $\quad\Longrightarrow\quad$ $\displaystyle\int_a^\infty f(x)\,dx$ *converges;*

Theorem (Direct Comparison Test)

Let $f : [a, \infty) \to \mathbb{R}$ and $g : [a, \infty) \to \mathbb{R}$ be continuous functions. Suppose that

$$0 \le f(x) \le g(x)$$

for all $x \in [a, \infty)$. Then

1. $\displaystyle\int_a^\infty g(x)\,dx$ *converges* $\implies$ $\displaystyle\int_a^\infty f(x)\,dx$ *converges;*
2. $\displaystyle\int_a^\infty f(x)\,dx$ *diverges* $\implies$ $\displaystyle\int_a^\infty g(x)\,dx$ *diverges.*

Theorem (Direct Comparison Test)

Let $f : [a,\infty) \to \mathbb{R}$ and $g : [a,\infty) \to \mathbb{R}$ be continuous functions. Suppose that

$$0 \leq f(x) \leq g(x)$$

for all $x \in [a,\infty)$. Then

1. $\displaystyle\int_a^\infty g(x)\,dx$ *converges* $\implies$ $\displaystyle\int_a^\infty f(x)\,dx$ *converges;*
2. $\displaystyle\int_a^\infty f(x)\,dx$ *diverges* $\implies$ $\displaystyle\int_a^\infty g(x)\,dx$ *diverges.*

(you can read the proof in the book)

EXAMPLE 7 These examples illustrate how we use Theorem 2.

(a) $\displaystyle\int_1^\infty \frac{\sin^2 x}{x^2}\,dx$ converges because

$$0 \le \frac{\sin^2 x}{x^2} \le \frac{1}{x^2} \quad \text{on} \quad [1, \infty) \quad \text{and} \quad \int_1^\infty \frac{1}{x^2}\,dx \quad \text{converges.}$$

Example 3

(b) $\displaystyle\int_1^\infty \frac{1}{\sqrt{x^2 - 0.1}}\,dx$ diverges because

$$\frac{1}{\sqrt{x^2 - 0.1}} \ge \frac{1}{x} \quad \text{on} \quad [1, \infty) \quad \text{and} \quad \int_1^\infty \frac{1}{x}\,dx \quad \text{diverges.}$$

Example 3

(c) $\displaystyle\int_0^{\pi/2} \frac{\cos x}{\sqrt{x}}\,dx$ converges because

$$0 \le \frac{\cos x}{\sqrt{x}} \le \frac{1}{\sqrt{x}} \quad \text{on} \quad \left[0, \frac{\pi}{2}\right],$$

$0 \le \cos x \le 1$ on $\left[0, \frac{\pi}{2}\right]$

and

$$\begin{aligned}
\int_0^{\pi/2} \frac{dx}{\sqrt{x}} &= \lim_{a \to 0^+} \int_a^{\pi/2} \frac{dx}{\sqrt{x}} \\
&= \lim_{a \to 0^+} \sqrt{4x}\,\Big]_a^{\pi/2} \\
&= \lim_{a \to 0^+} \left(\sqrt{2\pi} - \sqrt{4a}\right) = \sqrt{2\pi} \quad \text{converges.}
\end{aligned}$$

$2\sqrt{x} = \sqrt{4x}$

Theorem (Limit Comparison Test)

Suppose that

- $f : [a, \infty) \to \mathbb{R}$ *and* $g : [a, \infty) \to \mathbb{R}$ *are continuous;*
- $f > 0$ *and* $g > 0$*;*
- $\displaystyle\lim_{x\to\infty} \frac{f(x)}{g(x)}$ *exists and* $\displaystyle 0 < \lim_{x\to\infty} \frac{f(x)}{g(x)} < \infty$.

Theorem (Limit Comparison Test)

Suppose that

- $f : [a, \infty) \to \mathbb{R}$ *and* $g : [a, \infty) \to \mathbb{R}$ *are continuous;*
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Theorem (Limit Comparison Test)

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- $\displaystyle\lim_{x\to\infty} \frac{f(x)}{g(x)}$ *exists and* $\displaystyle 0 < \lim_{x\to\infty} \frac{f(x)}{g(x)} < \infty$.

Then

$$\int_a^\infty f(x)\,dx \qquad \text{and} \qquad \int_a^\infty d(x)\,dx$$

both converge or both diverge.

Example

Show that $\displaystyle\int_1^\infty \frac{dx}{1+x^2}$ diverges, by comparing it with $\displaystyle\int_1^\infty \frac{1}{x^2}\,dx$.

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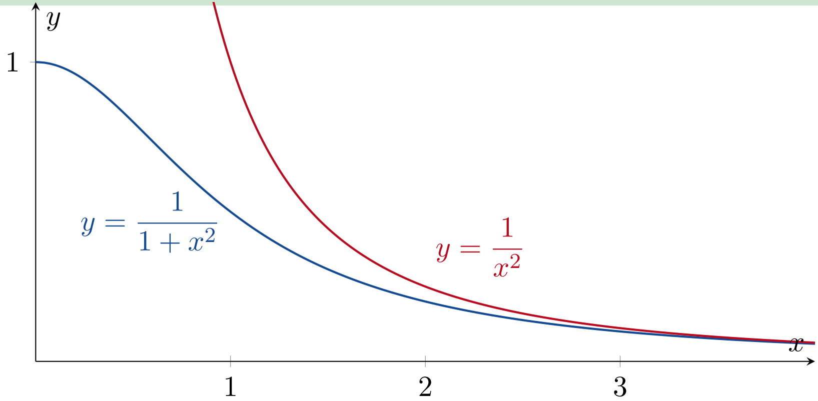


Solution The functions $f(x) = 1/x^2$ and $g(x) = 1/(1 + x^2)$ are positive and continuous on $[1, \infty)$. Also,

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = \lim_{x\to\infty} \frac{1/x^2}{1/(1 + x^2)} = \lim_{x\to\infty} \frac{1 + x^2}{x^2}$$

$$= \lim_{x\to\infty} \left(\frac{1}{x^2} + 1\right) = 0 + 1 = 1,$$

which is a positive finite limit (Figure 8.20). Therefore, $\int_1^\infty \frac{dx}{1 + x^2}$ converges because $\int_1^\infty \frac{dx}{x^2}$ converges.

The integrals converge to different values, however:

$$\int_1^\infty \frac{dx}{x^2} = \frac{1}{2 - 1} = 1 \qquad \text{Example 3}$$

and

$$\int_1^\infty \frac{dx}{1 + x^2} = \lim_{b\to\infty} \int_1^b \frac{dx}{1 + x^2} = \lim_{b\to\infty} \left[\tan^{-1} b - \tan^{-1} 1\right] = \frac{\pi}{2} - \frac{\pi}{4} = \frac{\pi}{4}. \quad \blacksquare$$

EXAMPLE 9 Investigate the convergence of $\int_1^\infty \frac{1 - e^{-x}}{x}\,dx$.

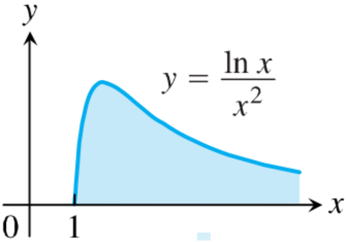
Solution The integrand suggests a comparison of $f(x) = (1 - e^{-x})/x$ with $g(x) = 1/x$. However, we cannot use the Direct Comparison Test because $f(x) \le g(x)$ and the integral of $g(x)$ *diverges*. On the other hand, using the Limit Comparison Test we find that

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = \lim_{x\to\infty} \left(\frac{1 - e^{-x}}{x}\right)\left(\frac{x}{1}\right) = \lim_{x\to\infty} (1 - e^{-x}) = 1,$$

which is a positive finite limit. Therefore, $\int_1^\infty \frac{1 - e^{-x}}{x}\,dx$ diverges because $\int_1^\infty \frac{dx}{x}$ diverges. Approximations to the improper integral are given in Table 8.5. Note that the values do not appear to approach any fixed limiting value as $b \to \infty$. ■

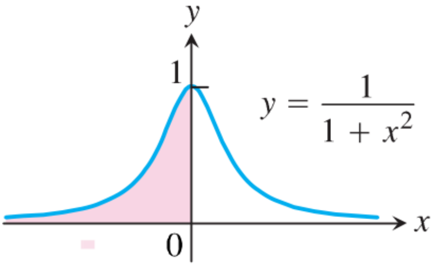
Upper limit

$$\int_1^{\infty} \frac{\ln x}{x^2}\,dx = \lim_{b\to\infty} \int_1^{b} \frac{\ln x}{x^2}\,dx$$



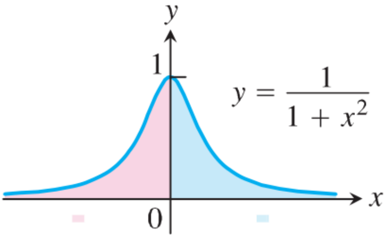
Lower limit

$$\int_{-\infty}^{0} \frac{dx}{1 + x^2} = \lim_{a \to -\infty} \int_{a}^{0} \frac{dx}{1 + x^2}$$



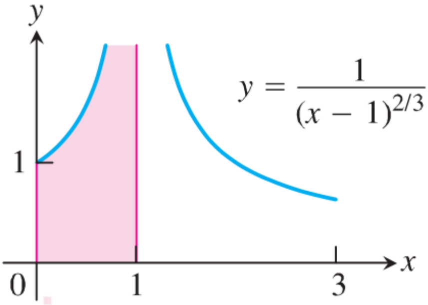
Both limits

$$\int_{-\infty}^{\infty} \frac{dx}{1+x^2} = \lim_{b \to -\infty} \int_{b}^{0} \frac{dx}{1+x^2} + \lim_{c \to \infty} \int_{0}^{c} \frac{dx}{1+x^2}$$



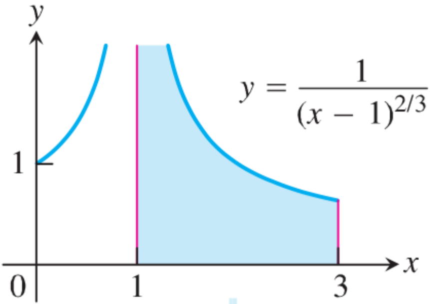
Upper endpoint

$$\int_0^1 \frac{dx}{(x-1)^{2/3}} = \lim_{b\to 1^-} \int_0^b \frac{dx}{(x-1)^{2/3}}$$



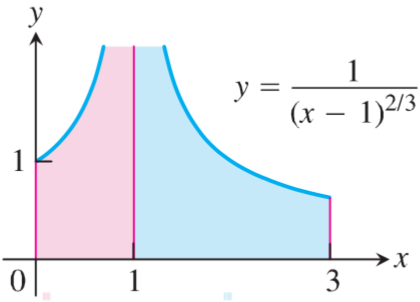
Lower endpoint

$$\int_1^3 \frac{dx}{(x-1)^{2/3}} = \lim_{d\to 1^+} \int_d^3 \frac{dx}{(x-1)^{2/3}}$$



Interior point

$$\int_0^3 \frac{dx}{(x-1)^{2/3}} = \int_0^1 \frac{dx}{(x-1)^{2/3}} + \int_1^3 \frac{dx}{(x-1)^{2/3}}$$



Next Time

- 11.1 Three-Dimensional Coordinate Systems
- 11.2 Vectors
- 11.3 The Dot Product