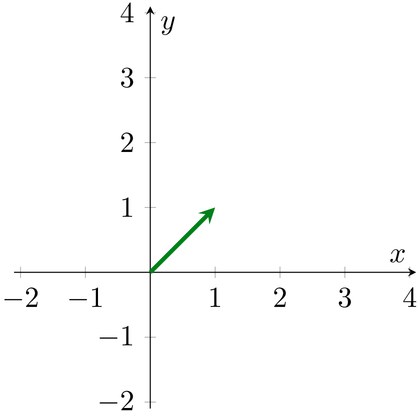
Lecture 9

- Eigenvalues and Eigenvectors
- Diagonalisation
- Complex Vector Spaces

Eigenvalues and Eigenvectors

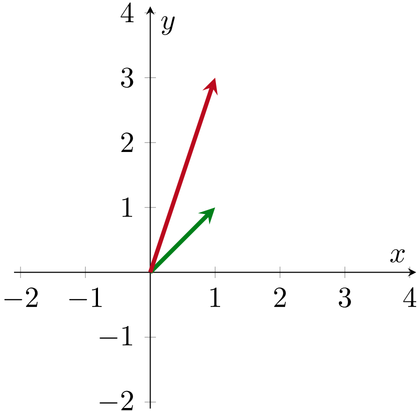
"Matrices do things to vectors"

$$\begin{bmatrix} 3 & -2 \\ -1 & 4 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} =$$



“Matrices do things to vectors”

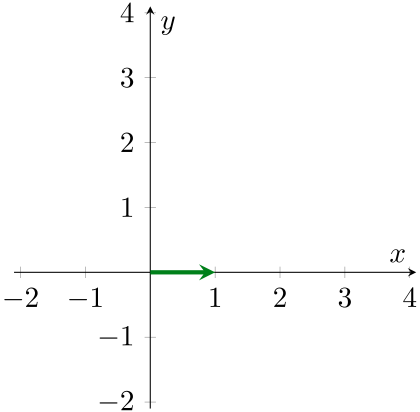
$$\begin{bmatrix} 3 & -2 \\ -1 & 4 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$



"Matrices do things to vectors"

$$\begin{bmatrix} 3 & -2 \\ -1 & 4 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

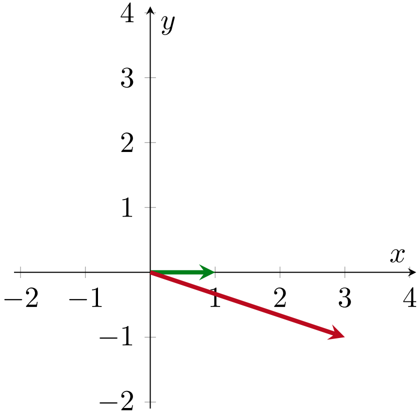
$$\begin{bmatrix} 3 & -2 \\ -1 & 4 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} =$$



"Matrices do things to vectors"

$$\begin{bmatrix} 3 & -2 \\ -1 & 4 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

$$\begin{bmatrix} 3 & -2 \\ -1 & 4 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 3 \\ -1 \end{bmatrix}$$

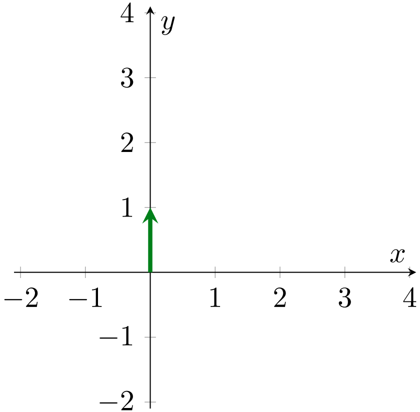


"Matrices do things to vectors"

$$\begin{bmatrix} 3 & -2 \\ -1 & 4 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

$$\begin{bmatrix} 3 & -2 \\ -1 & 4 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 3 \\ -1 \end{bmatrix}$$

$$\begin{bmatrix} 3 & -2 \\ -1 & 4 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} =$$

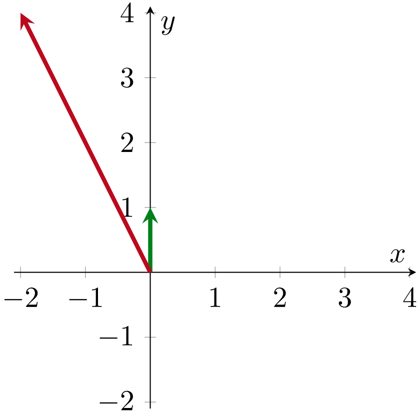


"Matrices do things to vectors"

$$\begin{bmatrix} 3 & -2 \\ -1 & 4 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

$$\begin{bmatrix} 3 & -2 \\ -1 & 4 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 3 \\ -1 \end{bmatrix}$$

$$\begin{bmatrix} 3 & -2 \\ -1 & 4 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -2 \\ 4 \end{bmatrix}$$



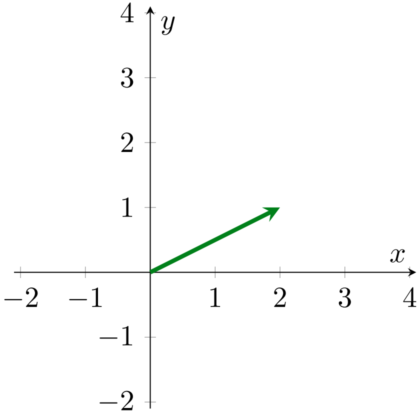
"Matrices do things to vectors"

$$\begin{bmatrix} 3 & -2 \\ -1 & 4 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

$$\begin{bmatrix} 3 & -2 \\ -1 & 4 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 3 \\ -1 \end{bmatrix}$$

$$\begin{bmatrix} 3 & -2 \\ -1 & 4 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -2 \\ 4 \end{bmatrix}$$

$$\begin{bmatrix} 3 & -2 \\ -1 & 4 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix} =$$



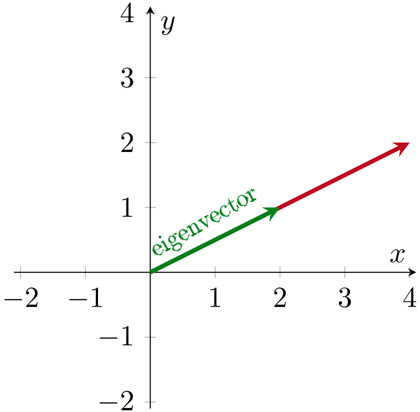
"Matrices do things to vectors"

$$\begin{bmatrix} 3 & -2 \\ -1 & 4 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

$$\begin{bmatrix} 3 & -2 \\ -1 & 4 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 3 \\ -1 \end{bmatrix}$$

$$\begin{bmatrix} 3 & -2 \\ -1 & 4 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -2 \\ 4 \end{bmatrix}$$

$$\begin{bmatrix} 3 & -2 \\ -1 & 4 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \end{bmatrix}$$



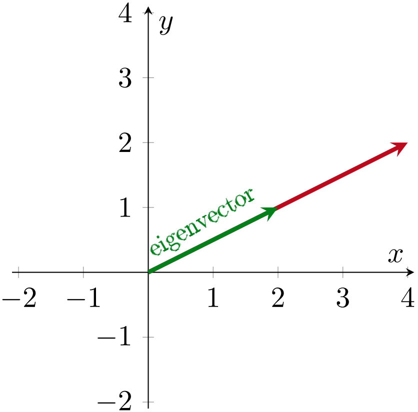
“Matrices do things to vectors”

$$\begin{bmatrix} 3 & -2 \\ -1 & 4 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

$$\begin{bmatrix} 3 & -2 \\ -1 & 4 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 3 \\ -1 \end{bmatrix}$$

$$\begin{bmatrix} 3 & -2 \\ -1 & 4 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -2 \\ 4 \end{bmatrix}$$

$$\begin{bmatrix} 3 & -2 \\ -1 & 4 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \end{bmatrix} = 2 \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$



Definition

If A is an $n \times n$ matrix, then a nonzero vector $\mathbf{x}$ in $\mathbb{R}^n$ is called an *eigenvector* of A if $A\mathbf{x}$ is a scalar multiple of $\mathbf{x}$; i.e. if

$$A\mathbf{x} = \lambda\mathbf{x}$$

for some scalar λ[1].

[1] lambda

Definition

If A is an $n \times n$ matrix, then a nonzero vector $\mathbf{x}$ in $\mathbb{R}^n$ is called an *eigenvector* of A if $A\mathbf{x}$ is a scalar multiple of $\mathbf{x}$; i.e. if

$$A\mathbf{x} = \lambda\mathbf{x}$$

for some scalar λ[1]. The scalar λ is called an *eigenvalue* of A, and $\mathbf{x}$ is said to be an *eigenvector corresponding to* λ.

[1] lambda

Remark

The zero vector $\mathbf{0}$ can not be an eigenvector. Only non-zero vectors can be eigenvectors.

Remark

Some books use

$$A\mathbf{x} = \lambda\mathbf{x}$$

some use

$$(\lambda I - A)\mathbf{x} = \mathbf{0}$$

and some use

$$(A - \lambda I)\mathbf{x} = \mathbf{0}.$$

Example

Let $A = \begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix}$, $\mathbf{u} = \begin{bmatrix} 6 \\ -5 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 3 \\ -2 \end{bmatrix}$. Are $\mathbf{u}$ and $\mathbf{v}$ eigenvectors of A?

$$A\mathbf{u} =$$

$$A\mathbf{v} =$$

Example

Let $A = \begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix}$, $\mathbf{u} = \begin{bmatrix} 6 \\ -5 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 3 \\ -2 \end{bmatrix}$. Are $\mathbf{u}$ and $\mathbf{v}$ eigenvectors of A?

$$A\mathbf{u} = \begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix} \begin{bmatrix} 6 \\ -5 \end{bmatrix} = \begin{bmatrix} -24 \\ 20 \end{bmatrix} = -4 \begin{bmatrix} 6 \\ -5 \end{bmatrix} = -4\mathbf{u}$$

Thus $\mathbf{u}$ is an eigenvector of A corresponding to the eigenvalue $\lambda = -4$.

$$A\mathbf{v} =$$

Example

Let $A = \begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix}$, $\mathbf{u} = \begin{bmatrix} 6 \\ -5 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 3 \\ -2 \end{bmatrix}$. Are $\mathbf{u}$ and $\mathbf{v}$ eigenvectors of A?

$$A\mathbf{u} = \begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix} \begin{bmatrix} 6 \\ -5 \end{bmatrix} = \begin{bmatrix} -24 \\ 20 \end{bmatrix} = -4 \begin{bmatrix} 6 \\ -5 \end{bmatrix} = -4\mathbf{u}$$

Thus $\mathbf{u}$ is an eigenvector of A corresponding to the eigenvalue $\lambda = -4$.

$$A\mathbf{v} = \begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix} \begin{bmatrix} 3 \\ -2 \end{bmatrix} = \begin{bmatrix} -9 \\ 11 \end{bmatrix} \neq \lambda \begin{bmatrix} 3 \\ -2 \end{bmatrix}$$

But $\mathbf{v}$ is not an eigenvector of A because $A\mathbf{v}$ is not a multiple of $\mathbf{v}$.

Note that

$$\det(B) \neq 0 \qquad \Longrightarrow \qquad \begin{array}{c}\text{the only solution}\\ \text{to } B\mathbf{x} = \mathbf{0} \text{ is the}\\ \text{trivial solution}\end{array}$$

Note that

$$\det(B) \neq 0 \quad \Longrightarrow \quad \begin{array}{c}\text{the only solution}\\ \text{to } B\mathbf{x} = \mathbf{0} \text{ is the}\\ \text{trivial solution}\end{array}$$

So

$$\begin{array}{c}\text{there is a nonzero}\\ \text{vector } \mathbf{x} \text{ such that}\\ (\lambda I - A)\mathbf{x} = \mathbf{0}\end{array} \quad \Longrightarrow \quad \det(\lambda I - A) = 0.$$

Note that

$$\det(B) \neq 0 \quad \Longrightarrow \quad \begin{array}{c}\text{the only solution}\\ \text{to } B\mathbf{x} = \mathbf{0} \text{ is the}\\ \text{trivial solution}\end{array}$$

So

$$\begin{array}{c}\text{there is a nonzero}\\ \text{vector } \mathbf{x} \text{ such that}\\ (\lambda I - A)\mathbf{x} = \mathbf{0}\end{array} \quad \Longrightarrow \quad \det(\lambda I - A) = 0.$$

Theorem

If A is an $n \times n$ matrix, then λ is an eigenvalue of A if and only if it satisfies the equation

$$\det(\lambda I - A) = 0.$$

This is called the characteristic equation of A.

Remark

$$\det(\lambda I - A) = 0 \qquad \iff \qquad \det(A - \lambda I) = 0$$

because $\det(-B) = (-1)^n \det(B)$ and $-0 = 0$.

$\det(\lambda I - A) = 0$

Example

Find all the eigenvalues of

$$A = \begin{bmatrix} 3 & 0 \\ 8 & -1 \end{bmatrix}.$$

$\det(\lambda I - A) = 0$

Example

Find all the eigenvalues of

$$A = \begin{bmatrix} 3 & 0 \\ 8 & -1 \end{bmatrix}.$$

Using the formula, we calculate that

$$0 = \det(\lambda I - A) =$$
$$=$$

Example

Find all the eigenvalues of

$$A = \begin{bmatrix} 3 & 0 \\ 8 & -1 \end{bmatrix}.$$

Using the formula, we calculate that

$$0 = \det(\lambda I - A) = \det\left(\begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} - \begin{bmatrix} 3 & 0 \\ 8 & -1 \end{bmatrix} \right) = \begin{vmatrix} \lambda - 3 & 0 \\ -8 & \lambda + 1 \end{vmatrix}$$

$$=$$

$$\det(\lambda I - A) = 0$$

Example

Find all the eigenvalues of

$$A = \begin{bmatrix} 3 & 0 \\ 8 & -1 \end{bmatrix}.$$

Using the formula, we calculate that

$$0 = \det(\lambda I - A) = \det\left(\begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} - \begin{bmatrix} 3 & 0 \\ 8 & -1 \end{bmatrix}\right) = \begin{vmatrix} \lambda - 3 & 0 \\ -8 & \lambda + 1 \end{vmatrix}$$

$$= (\lambda - 3)(\lambda + 3) - (0)(-8) = (\lambda - 3)(\lambda + 1).$$

$\det(\lambda I - A) = 0$

Example

Find all the eigenvalues of

$$A = \begin{bmatrix} 3 & 0 \\ 8 & -1 \end{bmatrix}.$$

Using the formula, we calculate that

$$0 = \det(\lambda I - A) = \det\left(\begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} - \begin{bmatrix} 3 & 0 \\ 8 & -1 \end{bmatrix}\right) = \begin{vmatrix} \lambda - 3 & 0 \\ -8 & \lambda + 1 \end{vmatrix}$$
$$= (\lambda - 3)(\lambda + 3) - (0)(-8) = (\lambda - 3)(\lambda + 1).$$

Thus the eigenvalues of A are $\lambda = 3$ and $\lambda = -1$.

$\det(\lambda I - A) = 0$

Remark

The formula $\det(\lambda I - A) = 0$ leads to the characteristic equation

$$\lambda^n + c_1\lambda^{n-1} + \ldots + c_{n-1}\lambda + c_n = 0.$$

The polynomial

$$p(\lambda) = \lambda^n + c_1\lambda^{n-1} + \ldots + c_{n-1}\lambda + c_n$$

is called the *characteristic polynomial* of the $n \times n$ matrix A.

$\det(\lambda I - A) = 0$

Remark

The formula $\det(\lambda I - A) = 0$ leads to the characteristic equation

$$\lambda^n + c_1\lambda^{n-1} + \ldots + c_{n-1}\lambda + c_n = 0.$$

The polynomial

$$p(\lambda) = \lambda^n + c_1\lambda^{n-1} + \ldots + c_{n-1}\lambda + c_n$$

is called the *characteristic polynomial* of the $n \times n$ matrix A.

For example, the characteristic polynomial of $A = \begin{bmatrix} 3 & 0 \\ 8 & -1 \end{bmatrix}$ is

$$p(\lambda) = (\lambda - 3)(\lambda + 1) = \lambda^2 - 2\lambda - 3$$

which is a polynomial of degree 2.

$$\det(\lambda I - A) = 0$$

Example

Find the eigenvalues of

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 4 & -17 & 8 \end{bmatrix}.$$

$\det(\lambda I - A) = 0$

Example

Find the eigenvalues of

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 4 & -17 & 8 \end{bmatrix}.$$

The characteristic polynomial of A is

$$\det(\lambda I - A) = \det \begin{bmatrix} \lambda & -1 & 0 \\ 0 & \lambda & -1 \\ -4 & 17 & \lambda - 8 \end{bmatrix} = \lambda^3 - 8\lambda^2 + 17\lambda - 4.$$

The characteristic equation of A is

$$\lambda^3 - 8\lambda^2 + 17\lambda - 4 = 0.$$

$$\lambda^3 - 8\lambda^2 + 17\lambda - 4 = 0$$

First we will look for integer solutions of this equation.

$$\lambda^3 - 8\lambda^2 + 17\lambda - 4 = 0$$

First we will look for integer solutions of this equation. Note that integer solutions must be divisors of the -4 term. So we need to try 1, -1, 2, -2, 4 and -4 until we find one.

$$\lambda^3 - 8\lambda^2 + 17\lambda - 4 = 0$$

First we will look for integer solutions of this equation. Note that integer solutions must be divisors of the -4 term. So we need to try 1, -1, 2, -2, 4 and -4 until we find one.

$\lambda = 1$ $\qquad \lambda^3 - 8\lambda^2 + 17\lambda - 4 = 1 - 8 + 17 - 4 = 6$ ✗

$$\lambda^3 - 8\lambda^2 + 17\lambda - 4 = 0$$

First we will look for integer solutions of this equation. Note that integer solutions must be divisors of the -4 term. So we need to try 1, -1, 2, -2, 4 and -4 until we find one.

$\lambda = 1$ $\quad \lambda^3 - 8\lambda^2 + 17\lambda - 4 = 1 - 8 + 17 - 4 = 6$ ✗

$\lambda = -1$ $\quad \lambda^3 - 8\lambda^2 + 17\lambda - 4 = -1 - 8 - 17 - 4 = -30$ ✗

$$\lambda^3 - 8\lambda^2 + 17\lambda - 4 = 0$$

First we will look for integer solutions of this equation. Note that integer solutions must be divisors of the -4 term. So we need to try 1, -1, 2, -2, 4 and -4 until we find one.

$\lambda = 1$ $\quad \lambda^3 - 8\lambda^2 + 17\lambda - 4 = 1 - 8 + 17 - 4 = 6$ ✗

$\lambda = -1$ $\quad \lambda^3 - 8\lambda^2 + 17\lambda - 4 = -1 - 8 - 17 - 4 = -30$ ✗

$\lambda = 2$ $\quad \lambda^3 - 8\lambda^2 + 17\lambda - 4 = 8 - 32 + 34 - 4 = 6$ ✗

$$\lambda^3 - 8\lambda^2 + 17\lambda - 4 = 0$$

First we will look for integer solutions of this equation. Note that integer solutions must be divisors of the -4 term. So we need to try 1, -1, 2, -2, 4 and -4 until we find one.

$\lambda = 1$ $\quad \lambda^3 - 8\lambda^2 + 17\lambda - 4 = 1 - 8 + 17 - 4 = 6$ ✗

$\lambda = -1$ $\quad \lambda^3 - 8\lambda^2 + 17\lambda - 4 = -1 - 8 - 17 - 4 = -30$ ✗

$\lambda = 2$ $\quad \lambda^3 - 8\lambda^2 + 17\lambda - 4 = 8 - 32 + 34 - 4 = 6$ ✗

$\lambda = -2$ $\quad \lambda^3 - 8\lambda^2 + 17\lambda - 4 = -8 - 32 - 34 - 4 = -78$ ✗

$$\lambda^3 - 8\lambda^2 + 17\lambda - 4 = 0$$

First we will look for integer solutions of this equation. Note that integer solutions must be divisors of the -4 term. So we need to try 1, -1, 2, -2, 4 and -4 until we find one.

$\lambda = 1$	$\lambda^3 - 8\lambda^2 + 17\lambda - 4 = 1 - 8 + 17 - 4 = 6$	✗
$\lambda = -1$	$\lambda^3 - 8\lambda^2 + 17\lambda - 4 = -1 - 8 - 17 - 4 = -30$	✗
$\lambda = 2$	$\lambda^3 - 8\lambda^2 + 17\lambda - 4 = 8 - 32 + 34 - 4 = 6$	✗
$\lambda = -2$	$\lambda^3 - 8\lambda^2 + 17\lambda - 4 = -8 - 32 - 34 - 4 = -78$	✗
$\lambda = 4$	$\lambda^3 - 8\lambda^2 + 17\lambda - 4 = 64 - 128 + 68 - 4 = 0$	✓

So $\lambda = 4$ is an eigenvalue. Now we need to look for more.

Now that we know that $\lambda = 4$ is an eigenvalue we can calculate

$$\begin{aligned} 0 &= \lambda^3 - 8\lambda^2 + 17\lambda - 4 = (\lambda - 4)(\lambda^2 + b\lambda + c) \\ &= \\ &= \end{aligned}$$

Now that we know that $\lambda = 4$ is an eigenvalue we can calculate

$$\begin{aligned}0 = \lambda^3 - 8\lambda^2 + 17\lambda - 4 &= (\lambda - 4)(\lambda^2 + b\lambda + c)\\ &= \lambda^3 + b\lambda^2 + c\lambda - 4\lambda^2 - 4b\lambda - 4c\\ &= \lambda^3 + (b-4)\lambda^2 + (c-4b)\lambda - 4c\end{aligned}$$

Now that we know that $\lambda = 4$ is an eigenvalue we can calculate

$$\begin{aligned} 0 = \lambda^3 - 8\lambda^2 + 17\lambda - 4 &= (\lambda - 4)(\lambda^2 + b\lambda + c) \\ &= \lambda^3 + b\lambda^2 + c\lambda - 4\lambda^2 - 4b\lambda - 4c \\ &= \lambda^3 + (b-4)\lambda^2 + (c-4b)\lambda - 4c \end{aligned}$$

which implies that $b = -4$ and $c = 1$.

Now that we know that $\lambda = 4$ is an eigenvalue we can calculate

$$\begin{aligned} 0 = \lambda^3 - 8\lambda^2 + 17\lambda - 4 &= (\lambda - 4)(\lambda^2 + b\lambda + c) \\ &= \lambda^3 + b\lambda^2 + c\lambda - 4\lambda^2 - 4b\lambda - 4c \\ &= \lambda^3 + (b-4)\lambda^2 + (c - 4b)\lambda - 4c \end{aligned}$$

which implies that $b = -4$ and $c = 1$. Hence

$$0 = (\lambda - 4)(\lambda^2 - 4\lambda + 1).$$

Now we need to find the roots of

$$\lambda^2 - 4\lambda + 1 = 0.$$

Now we need to find the roots of

$$\lambda^2 - 4\lambda + 1 = 0.$$

Using the formula

$$\lambda = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

we find

$$\lambda = 2 \pm \sqrt{3}.$$

Now we need to find the roots of

$$\lambda^2 - 4\lambda + 1 = 0.$$

Using the formula

$$\lambda = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

we find

$$\lambda = 2 \pm \sqrt{3}.$$

Therefore the eigenvalues of A are

$$\lambda = 4, \qquad \lambda = 2 + \sqrt{3}, \quad \text{and} \quad \lambda = 2 - \sqrt{3}.$$

$$\det(\lambda I - A) = 0$$

Example

Find the eigenvalues of the upper triangular matrix

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ 0 & a_{22} & a_{23} & a_{24} \\ 0 & 0 & a_{33} & a_{34} \\ 0 & 0 & 0 & a_{44} \end{bmatrix}.$$

$\det(\lambda I - A) = 0$

Recall that the determinant of a triangular matrix is the product of the entries on the main diagonal. Hence

$$0 = \det(\lambda I - A) = \begin{vmatrix} \lambda - a_{11} & -a_{12} & -a_{13} & -a_{14} \\ 0 & \lambda - a_{22} & -a_{23} & -a_{24} \\ 0 & 0 & \lambda - a_{33} & -a_{34} \\ 0 & 0 & 0 & \lambda - a_{44} \end{vmatrix}$$

$$= (\lambda - a_{11})(\lambda - a_{22})(\lambda - a_{33})(\lambda - a_{44}).$$

$$\det(\lambda I - A) = 0$$

Recall that the determinant of a triangular matrix is the product of the entries on the main diagonal. Hence

$$0 = \det(\lambda I - A) = \begin{vmatrix} \lambda - a_{11} & -a_{12} & -a_{13} & -a_{14} \\ 0 & \lambda - a_{22} & -a_{23} & -a_{24} \\ 0 & 0 & \lambda - a_{33} & -a_{34} \\ 0 & 0 & 0 & \lambda - a_{44} \end{vmatrix}$$
$$= (\lambda - a_{11})(\lambda - a_{22})(\lambda - a_{33})(\lambda - a_{44}).$$

Therefore the eigenvalues of A are

$$\lambda = a_{11}, \quad \lambda = a_{22}, \quad \lambda = a_{33}, \quad \lambda = a_{44}$$

which are the entries on the main diagonal.

Theorem

If A is an $n \times n$ triangular matrix (upper triangular, lower triangular, or diagonal), then the eigenvalues of A are the entries on the main diagonal of A.

Example

The eigenvalues of

$$A = \begin{bmatrix} \frac{1}{2} & 0 & 0 \\ -1 & \frac{2}{3} & 0 \\ 5 & -8 & -\frac{1}{4} \end{bmatrix}$$

are $\lambda = \frac{1}{2}$, $\lambda = \frac{2}{3}$ and $\lambda = -\frac{1}{4}$.

Theorem

If A is an $n \times n$ matrix, the following statements are equivalent.

1. *λ is an eigenvalue of A.*
2. *λ is a solution of the characteristic equation* $\det(\lambda I - A) = 0$.
3. *The system of equations $(\lambda I - A)\mathbf{x} = 0$ has nontrivial solutions.*
4. *There is a nonzero vector $\mathbf{x}$ such that $A\mathbf{x} = \lambda\mathbf{x}$.*

$\det(\lambda I - A) = 0 \qquad (\lambda I - A)\mathbf{x} = \mathbf{0}$

Finding Eigenvectors

To find eigenvectors we use the formula

$$(\lambda I - A)\mathbf{x} = \mathbf{0}.$$

$$\det(\lambda I - A) = \underbrace{0}_{\text{number zero}} \qquad (\lambda I - A)\mathbf{x} = \underbrace{\mathbf{0}}_{\text{zero vector}}$$

Finding Eigenvectors

To find eigenvectors we use the formula

$$(\lambda I - A)\mathbf{x} = \mathbf{0}.$$

$$\det(\lambda I - A) = 0 \qquad (\lambda I - A)\mathbf{x} = \mathbf{0}$$

Example

The matrix

$$A = \begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix}$$

has eigenvalues $\lambda = -4$ and $\lambda = 7$. Find the corresponding eigenvectors.

$$\det(\lambda I - A) = 0 \qquad (\lambda I - A)\mathbf{x} = \mathbf{0}$$

$$A = \begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix}$$

First let us look at $\lambda = -4$.

$$\det(\lambda I - A) = 0 \qquad (\lambda I - A)\mathbf{x} = \mathbf{0}$$

$$A = \begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix}$$

First let us look at $\lambda = -4$. Then we have

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix} = (\lambda I - A)\mathbf{x} = \left(\begin{bmatrix} -4 & 0 \\ 0 & -4 \end{bmatrix} - \begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix} \right) \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -5 & -6 \\ -5 & -6 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

$$\det(\lambda I - A) = 0 \qquad (\lambda I - A)\mathbf{x} = \mathbf{0}$$

$$A = \begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix}$$

First let us look at $\lambda = -4$. Then we have

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix} = (\lambda I - A)\mathbf{x} = \left(\begin{bmatrix} -4 & 0 \\ 0 & -4 \end{bmatrix} - \begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix} \right) \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -5 & -6 \\ -5 & -6 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

which implies that

$$0 = -5x - 6y.$$

$$\det(\lambda I - A) = 0 \qquad (\lambda I - A)\mathbf{x} = \mathbf{0}$$

$$A = \begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix}$$

First let us look at $\lambda = -4$. Then we have

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix} = (\lambda I - A)\mathbf{x} = \left(\begin{bmatrix} -4 & 0 \\ 0 & -4 \end{bmatrix} - \begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix} \right) \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -5 & -6 \\ -5 & -6 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

which implies that

$$0 = -5x - 6y.$$

We can choose any x and y (apart from $x = y = 0$) which satisfy this equation. For example we can choose

$$\mathbf{x} = \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 6 \\ -5 \end{bmatrix}.$$

This is an eigenvector of A corresponding to $\lambda = -4$.

$$\det(\lambda I - A) = 0 \qquad (\lambda I - A)\mathbf{x} = \mathbf{0}$$

$$A = \begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix}$$

Next we consider $\lambda = 7$. We calculate that

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix} = (\lambda I - A)\mathbf{x} = \begin{bmatrix} 6 & -6 \\ -5 & 5 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 6x - 6y \\ 5x - 5y \end{bmatrix}$$

$$\det(\lambda I - A) = 0 \qquad (\lambda I - A)\mathbf{x} = \mathbf{0}$$

$$A = \begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix}$$

Next we consider $\lambda = 7$. We calculate that

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix} = (\lambda I - A)\mathbf{x} = \begin{bmatrix} 6 & -6 \\ -5 & 5 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 6x - 6y \\ 5x - 5y \end{bmatrix}$$

which implies that

$$x = y.$$

$$\det(\lambda I - A) = 0 \qquad (\lambda I - A)\mathbf{x} = \mathbf{0}$$

$$A = \begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix}$$

Next we consider $\lambda = 7$. We calculate that

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix} = (\lambda I - A)\mathbf{x} = \begin{bmatrix} 6 & -6 \\ -5 & 5 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 6x - 6y \\ 5x - 5y \end{bmatrix}$$

which implies that

$$x = y.$$

I choose

$$\mathbf{x} = \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

This is an eigenvector of A corresponding to $\lambda = 7$.

$$\det(\lambda I - A) = 0 \qquad (\lambda I - A)\mathbf{x} = \mathbf{0}$$

Eigenspaces

Let A be a square matrix and let λ be an eigenvalue of A.

Definition

The *eigenspace* of A corresponding to λ is

$$\begin{aligned}\mathrm{Nul}(\lambda I - A) &= \{\text{all solutions of } (\lambda I - A)\mathbf{x} = \mathbf{0}\} \\ &= \{\text{all eigenvectors corresponding to } \lambda\} \cup \{\mathbf{0}\}.\end{aligned}$$

$$\det(\lambda I - A) = 0 \qquad (\lambda I - A)\mathbf{x} = \mathbf{0}$$

Example

Find bases for the eigenspaces of

$$A = \begin{bmatrix} -1 & 3 \\ 2 & 0 \end{bmatrix}.$$

First we need to find the eigenvalues.

$$\det(\lambda I - A) = 0 \qquad (\lambda I - A)\mathbf{x} = \mathbf{0}$$

Example

Find bases for the eigenspaces of

$$A = \begin{bmatrix} -1 & 3 \\ 2 & 0 \end{bmatrix}.$$

First we need to find the eigenvalues. Since

$$0 = \det(\lambda I - A) = \begin{vmatrix} \lambda + 1 & -3 \\ -2 & \lambda \end{vmatrix} = \lambda(\lambda + 1) - 6 = (\lambda - 2)(\lambda + 3),$$

the eigenvalues of A are $\lambda = 2$ and $\lambda = -3$. This means that there are two eigenspaces of A, one for each eigenvalue.

$$\det(\lambda I - A) = 0 \qquad (\lambda I - A)\mathbf{x} = \mathbf{0}$$

We have

$$\mathbf{0} = (\lambda I - A)\mathbf{x} = \begin{bmatrix} \lambda + 1 & -3 \\ -2 & \lambda \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}.$$

$$\det(\lambda I - A) = 0 \qquad (\lambda I - A)\mathbf{x} = \mathbf{0}$$

We have

$$\mathbf{0} = (\lambda I - A)\mathbf{x} = \begin{bmatrix} \lambda + 1 & -3 \\ -2 & \lambda \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}.$$

Consider first $\lambda = 2$. We have

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 3 & -3 \\ -2 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3x - 3y \\ -2x + 2y \end{bmatrix} \quad \Longrightarrow \quad x = y.$$

So every eigenvector corresponding to $\lambda = 2$ can be written as

$$\mathbf{x} = t \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

for some $t \neq 0$.

$\det(\lambda I - A) = 0 \qquad (\lambda I - A)\mathbf{x} = \mathbf{0}$

We have

$$\mathbf{0} = (\lambda I - A)\mathbf{x} = \begin{bmatrix} \lambda + 1 & -3 \\ -2 & \lambda \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}.$$

Consider first $\lambda = 2$. We have

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 3 & -3 \\ -2 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3x - 3y \\ -2x + 2y \end{bmatrix} \implies x = y.$$

So every eigenvector corresponding to $\lambda = 2$ can be written as

$$\mathbf{x} = t \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

for some $t \neq 0$. It follows that

$$\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

is a basis for the eigenspace corresponding to $\lambda = 2$.

$\det(\lambda I - A) = 0 \qquad (\lambda I - A)\mathbf{x} = \mathbf{0}$

If $\lambda = -3$ then

$$\mathbf{0} = (\lambda I - A)\mathbf{x} = \begin{bmatrix} \lambda + 1 & -3 \\ -2 & \lambda \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

becomes

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} -2 & -3 \\ -2 & -3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -2x - 3y \\ -2x - 3y \end{bmatrix} \implies 2x = -3y.$$

This means that every eigenvector corresponding to $\lambda = -3$ can be written as

$$\mathbf{x} = s \begin{bmatrix} -3 \\ 2 \end{bmatrix}$$

for some $s \neq 0$ and that

$$\begin{bmatrix} -3 \\ 2 \end{bmatrix}$$

is a basis for the eigenspace corresponding to $\lambda = -3$.

Example

Find bases for the eigenspaces of

$$A = \begin{bmatrix} 0 & 0 & -2 \\ 1 & 2 & 1 \\ 1 & 0 & 3 \end{bmatrix}.$$

Please check that the characteristic equation of A is

$$0 = \lambda^3 - 5\lambda^2 + 8\lambda - 4 = (\lambda - 1)(\lambda - 2)^2.$$

The eigenvalues of A are $\lambda = 1$ and $\lambda = 2$. So there are two eigenspaces.

Now $\mathbf{x} \neq 0$ is an eigenvector of A iff

$$(\lambda I - A)\mathbf{x} = \mathbf{0}$$

$$\begin{bmatrix} \lambda & 0 & 2 \\ -1 & \lambda - 2 & -1 \\ -1 & 0 & \lambda - 3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

Eigenvalues and Eigenvectors

Now $\mathbf{x} \neq 0$ is an eigenvector of A iff

$$(\lambda I - A)\mathbf{x} = \mathbf{0}$$

$$\begin{bmatrix} \lambda & 0 & 2 \\ -1 & \lambda - 2 & -1 \\ -1 & 0 & \lambda - 3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

Consider the case where $\lambda = 2$. Then we have

$$\begin{bmatrix} 2 & 0 & 2 \\ -1 & 0 & -1 \\ -1 & 0 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

Now $\mathbf{x} \neq 0$ is an eigenvector of A iff

$$(\lambda I - A)\mathbf{x} = \mathbf{0}$$

$$\begin{bmatrix} \lambda & 0 & 2 \\ -1 & \lambda - 2 & -1 \\ -1 & 0 & \lambda - 3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

Consider the case where $\lambda = 2$. Then we have

$$\begin{bmatrix} 2 & 0 & 2 \\ -1 & 0 & -1 \\ -1 & 0 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

The augmented matrix is

$$\begin{bmatrix} 2 & 0 & 2 & 0 \\ -1 & 0 & -1 & 0 \\ -1 & 0 & -1 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \implies \begin{cases} x = -z \\ y \text{ is free} \\ z \text{ is free.} \end{cases}$$

$$x = -z$$

This means that the eigenvectors of A corresponding to $\lambda = 2$ are the nonzero vectors of the form

$$\mathbf{x} = z \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} + y \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}.$$

$$x = -z$$

This means that the eigenvectors of A corresponding to $\lambda = 2$ are the nonzero vectors of the form

$$\mathbf{x} = z \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} + y \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}.$$

Hence

$$\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

form a basis for the eigenspace corresponding to $\lambda = 2$. (This eigenspace is 2 dimensional.)

If $\lambda = 1$ then

$$\begin{bmatrix} \lambda & 0 & 2 \\ -1 & \lambda - 2 & -1 \\ -1 & 0 & \lambda - 3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

becomes

$$\begin{bmatrix} 1 & 0 & 2 \\ -1 & -1 & -1 \\ -1 & 0 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

which has solution

$$\mathbf{x} = t \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}$$

(please check).

Hence

$$\begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}$$

is a basis for the eigenspace corresponding to $\lambda = 1$. (This eigenspace is 1 dimensional.)

Eigenvalues and Invertibility

Theorem

A square matrix A is invertible if and only if $\lambda = 0$ is not an eigenvalue of A.

Eigenvalues and Invertibility

Theorem

A square matrix A is invertible if and only if $\lambda = 0$ is not an eigenvalue of A.

Proof.

Let A be an $n \times n$ matrix. Note that $\lambda = 0$ is a solution of the characteristic equation

$$\lambda^n + c_1\lambda^{n-1} + \ldots c_{n-1}\lambda + c_n = 0$$

if and only if $c_n = 0$.

Eigenvalues and Invertibility

Theorem

A square matrix A is invertible if and only if $\lambda = 0$ is not an eigenvalue of A.

Proof.

Let A be an $n \times n$ matrix. Note that $\lambda = 0$ is a solution of the characteristic equation

$$\lambda^n + c_1\lambda^{n-1} + \dots c_{n-1}\lambda + c_n = 0$$

if and only if $c_n = 0$. So we want to prove that

$$A \text{ is invertible} \qquad \Longleftrightarrow \qquad c_n \neq 0.$$

Proof continued.

But note that if we set $\lambda = 0$ in

$$\det(\lambda I - A) = \lambda^n + c_1\lambda^{n-1} + \ldots c_{n-1}\lambda + c_n$$

then we obtain

$$c_n = \det(-A) = (-1)^n \det(A).$$

Proof continued.

But note that if we set $\lambda = 0$ in

$$\det(\lambda I - A) = \lambda^n + c_1\lambda^{n-1} + \ldots c_{n-1}\lambda + c_n$$

then we obtain

$$c_n = \det(-A) = (-1)^n \det(A).$$

So clearly $\det(A) \neq 0$ if and only if $c_n \neq 0$. $\square$

Example

The matrix

$$A = \begin{bmatrix} 0 & 0 & -2 \\ 1 & 2 & 1 \\ 1 & 0 & 3 \end{bmatrix}$$

has eigenvalues $\lambda = 1 \neq 0$ and $\lambda = 2 \neq 0$. Therefore A must be invertible.

I leave it for you to check that $\det(A) \neq 0$.

Theorem

Let A be a square matrix. Then $\mathrm{Nul}(A) = \{\mathbf{0}\}$ if and only if $\lambda = 0$ is not an eigenvalue of A.

Theorem

Let A be a square matrix. Then $\mathrm{Nul}(A) = \{\mathbf{0}\}$ *if and only if* $\lambda = 0$ *is not an eigenvalue of* A.

Proof.

By the previous theorem, $\lambda = 0$ is not an eigenvalue of A iff A is invertible.
But we know by the Invertible Matrix Theorem (lecture 3) that A is invertible iff $A\mathbf{x} = \mathbf{0}$ has only the trivial solution. □

Theorem

Let A be a square matrix. Then $\mathrm{Nul}(A) = \{\mathbf{0}\}$ if and only if $\lambda = 0$ is not an eigenvalue of A.

Proof.

By the previous theorem, $\lambda = 0$ is not an eigenvalue of A iff A is invertible.
But we know by the Invertible Matrix Theorem (lecture 3) that A is invertible iff $A\mathbf{x} = \mathbf{0}$ has only the trivial solution. $\square$

Example

The matrix $A = \begin{bmatrix} 0 & 0 & -2 \\ 1 & 2 & 1 \\ 1 & 0 & 3 \end{bmatrix}$ has null space $\mathrm{Nul}(A) = \{\mathbf{0}\}$ because the eigenvalues of A are $\lambda = 1 \neq 0$ and $\lambda = 2 \neq 0$.

Eigenvalues of Linear Operators

These ideas extend to vector spaces.

Definition

If $T : V \to V$ is a linear operator on a vector space V, then a nonzero vector $\mathbf{x}$ in V is called an *eigenvector* of T if $T(\mathbf{x})$ is a scalar multiple of $\mathbf{x}$; that is,

$$T(x) = \lambda x$$

for some scalar λ. The scalar λ is called an eigenvalue of T, and $\mathbf{x}$ is said to be an eigenvector corresponding to λ.

Similarity (again)

Recall that two square matrices A and B are *similar* iff there exists an invertible matrix P such that $B = P^{-1}AP$.

Similarity (again)

Recall that two square matrices A and B are *similar* iff there exists an invertible matrix P such that $B = P^{-1}AP$.

Theorem

If A and B are similar, then they have the same characteristic polynomial

Similarity (again)

Recall that two square matrices A and B are *similar* iff there exists an invertible matrix P such that $B = P^{-1}AP$.

Theorem

If A and B are similar, then they have the same characteristic polynomial and hence the same eigenvalues (with the same multiplicities).

Proof.

Suppose that $B = P^{-1}AP$. Then

$$P^{-1}\big(A - \lambda I\big)P = P^{-1}AP - \lambda P^{-1}P = P^{-1}AP - \lambda I = B - \lambda I.$$

Proof.

Suppose that $B = P^{-1}AP$. Then

$$P^{-1}\big(A - \lambda I\big)P = P^{-1}AP - \lambda P^{-1}P = P^{-1}AP - \lambda I = B - \lambda I.$$

It follows that

$$\begin{aligned}\det(B - \lambda I) &= \det(P^{-1})\det(A - \lambda I)\det(P) \\ &= \frac{1}{\det(P)}\det(A - \lambda I)\det(P) = \det(A - \lambda I).\end{aligned}$$

□

Remark

$$\begin{pmatrix} A \text{ and } B \\ \text{are similar} \end{pmatrix} \quad \begin{array}{c} \Longrightarrow \\ \not\Longleftarrow \end{array} \quad \begin{pmatrix} A \text{ and } B \text{ have the} \\ \text{same eigenvalues} \end{pmatrix}$$

For example, the matrices $\begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}$ and $\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$ have the same eigenvalues but are not similar.

Remark

Similarity is not the same as row equivalence. Row operations can change a matrix's eigenvalues.

Diagonalisation

Definition

A square matrix A is said to be *diagonalisable* if and only if it is similar to some diagonal matrix; that is, if there exists an invertible matrix P such that $P^{-1}AP$ is diagonal.

In this case, we say that the matrix P *diagonalises* A.

Theorem

If A is an $n \times n$ matrix, then

$$A \text{ is diagonalisable} \iff A \text{ has } n \text{ linearly independent eigenvectors}$$

Diagonalisation

Theorem

If A is an $n \times n$ matrix, then

$$\begin{array}{ccc} \textit{A is diagonalisable} & \iff & \textit{A has n linearly independent eigenvectors} \\ & & \Uparrow \\ & & \textit{A has n distinct eigenvalues.} \end{array}$$

How to Diagonalise a Matrix

Let A be an $n \times n$ matrix.

1. Check if A is diagonalisable by searching for n linearly independent eigenvectors. Call these $\mathbf{p}_1, \mathbf{p}_2, \ldots, \mathbf{p}_n$.

How to Diagonalise a Matrix

Let A be an $n \times n$ matrix.

1. Check if A is diagonalisable by searching for n linearly independent eigenvectors. Call these $\mathbf{p}_1, \mathbf{p}_2, \dots, \mathbf{p}_n$.
2. Let $P = \begin{bmatrix} \\ \mathbf{p}_1 & \mathbf{p}_2 & \cdots & \mathbf{p}_n \\ \\ \end{bmatrix}$.

How to Diagonalise a Matrix

Let A be an $n \times n$ matrix.

1. Check if A is diagonalisable by searching for n linearly independent eigenvectors. Call these $\mathbf{p}_1, \mathbf{p}_2, \ldots, \mathbf{p}_n$.
2. Let $P = \begin{bmatrix} \\ \mathbf{p}_1 & \mathbf{p}_2 & \cdots & \mathbf{p}_n \\ \\ \end{bmatrix}$.
3. Calculate $P^{-1}AP$.

Break

We will continue at 3pm

How to Diagonalise a Matrix

Let A be an $n \times n$ matrix.

1. Check if A is diagonalisable by searching for n linearly independent eigenvectors. Call these $\mathbf{p}_1, \mathbf{p}_2, \ldots, \mathbf{p}_n$.
2. Let $P = \begin{bmatrix} \mathbf{p}_1 & \mathbf{p}_2 & \cdots & \mathbf{p}_n \end{bmatrix}$.
3. Calculate $P^{-1}AP$.

Example

Find a matrix P which diagonalises $A = \begin{bmatrix} 0 & 0 & -2 \\ 1 & 2 & 1 \\ 1 & 0 & 3 \end{bmatrix}$.

Earlier today we found that the eigenvalues of A are $\lambda = 2$ and $\lambda = 1$. We also found that

$$\mathbf{p}_1 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \quad \text{and} \quad \mathbf{p}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

form a basis for the eigenspace corresponding to $\lambda = 2$; and

$$\mathbf{p}_3 = \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}$$

forms a basis for the eigenspace corresponding to $\lambda = 1$.

These three eigenvectors are linearly independent, so the matrix

$$P = \begin{bmatrix} \mathbf{p}_1 & \mathbf{p}_2 & \mathbf{p}_3 \end{bmatrix} = \begin{bmatrix} -1 & 0 & -2 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}$$

diagonalises A.

These three eigenvectors are linearly independent, so the matrix

$$P = \begin{bmatrix} \mathbf{p}_1 & \mathbf{p}_2 & \mathbf{p}_3 \end{bmatrix} = \begin{bmatrix} -1 & 0 & -2 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}$$

diagonalises A.

To check this, we can calculate (please check) that

$$P^{-1}AP = \begin{bmatrix} 1 & 0 & 2 \\ 1 & 1 & 1 \\ -1 & 0 & -1 \end{bmatrix} \begin{bmatrix} 0 & 0 & -2 \\ 1 & 2 & 1 \\ 1 & 0 & 3 \end{bmatrix} \begin{bmatrix} -1 & 0 & -2 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Remark

Note that the entries on the main diagonal of

$$D = P^{-1}AP = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

are the eigenvalues of A.

Remark

Note that the entries on the main diagonal of

$$D = P^{-1}AP = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

are the eigenvalues of A.

Remark

When we wrote P, we used the two eigenvectors corresponding to $\lambda = 2$ in the first two columns, and the eigenvector corresponding to $\lambda = 1$ in the third column. This is why we get $2, 2$ first, then 1, on the main diagonal of D.

Remark

$$P = \begin{bmatrix} -1 & 0 & -2 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}$$

is not the only matrix which diagonalises A.

Remark

$$P = \begin{bmatrix} -1 & 0 & -2 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}$$

is not the only matrix which diagonalises A. If we swap the second and third columns of P to obtain

$$Q = \begin{bmatrix} -1 & -2 & 0 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

Remark

$$P = \begin{bmatrix} -1 & 0 & -2 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}$$

is not the only matrix which diagonalises A. If we swap the second and third columns of P to obtain

$$Q = \begin{bmatrix} -1 & -2 & 0 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

then we get

$$D = Q^{-1}AQ = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}.$$

Example

Show that $A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 2 & 0 \\ -3 & 5 & 2 \end{bmatrix}$ is not diagonalisable.

Example

Show that $A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 2 & 0 \\ -3 & 5 & 2 \end{bmatrix}$ is not diagonalisable.

The characteristic equation of A is

$$0 = \det(\lambda I - A) = \begin{vmatrix} \lambda - 1 & 0 & 0 \\ -1 & \lambda - 2 & 0 \\ 3 & -5 & \lambda - 2 \end{vmatrix} = (\lambda - 1)(\lambda - 2)^2.$$

So the eigenvalues of A are $\lambda = 1$ and $\lambda = 2$.

Example

Show that $A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 2 & 0 \\ -3 & 5 & 2 \end{bmatrix}$ is not diagonalisable.

The characteristic equation of A is

$$0 = \det(\lambda I - A) = \begin{vmatrix} \lambda - 1 & 0 & 0 \\ -1 & \lambda - 2 & 0 \\ 3 & -5 & \lambda - 2 \end{vmatrix} = (\lambda - 1)(\lambda - 2)^2.$$

So the eigenvalues of A are $\lambda = 1$ and $\lambda = 2$.

Example

Show that $A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 2 & 0 \\ -3 & 5 & 2 \end{bmatrix}$ is not diagonalisable.

Since A is a triangular matrix, we can see straight away that the eigenvalues of A are $\lambda = 1$ and $\lambda = 2$.

I leave it for you to check that bases for the eigenspaces of A are

$$\lambda = 1: \ \mathbf{p}_1 = \begin{bmatrix} 1 \\ -1 \\ 8 \end{bmatrix}; \qquad \lambda = 2: \ \mathbf{p}_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

Since A is a 3×3 matrix but with only two linearly independent eigenvectors, A is not diagonalisable.

Example

Is $A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 4 & -17 & 8 \end{bmatrix}$ diagonalisable?

Example

Is $A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 4 & -17 & 8 \end{bmatrix}$ diagonalisable?

We saw earlier that the eigenvalues of this matrix are $\lambda = 4$, $\lambda = 2 + \sqrt{3}$ and $\lambda = 2 - \sqrt{3}$. Since A has 3 distinct eigenvalues, A is diagonalisable.

EXAMPLE 4 Diagonalizability of Triangular Matrices

From Theorem 5.1.2, the eigenvalues of a triangular matrix are the entries on its main diagonal. Thus, a triangular matrix with distinct entries on the main diagonal is diagonalizable. For example,

$$A = \begin{bmatrix} -1 & 2 & 4 & 0 \\ 0 & 3 & 1 & 7 \\ 0 & 0 & 5 & 8 \\ 0 & 0 & 0 & -2 \end{bmatrix}$$

is a diagonalizable matrix with eigenvalues $\lambda_1 = -1$, $\lambda_2 = 3$, $\lambda_3 = 5$, $\lambda_4 = -2$.

Eigenvalues of Powers of a Matrix

Suppose that we know the eigenvalues and eigenvectors of A. What are the eigenvalues and eigenvectors of A^k?

Eigenvalues of Powers of a Matrix

Suppose that we know the eigenvalues and eigenvectors of A. What are the eigenvalues and eigenvectors of A^k?

Suppose that

$$A\mathbf{x} = \lambda\mathbf{x}.$$

Eigenvalues of Powers of a Matrix

Suppose that we know the eigenvalues and eigenvectors of A. What are the eigenvalues and eigenvectors of A^k?

Suppose that

$$A\mathbf{x} = \lambda\mathbf{x}.$$

Then

$$A^2\mathbf{x} = A(A\mathbf{x}) = A(\lambda\mathbf{x}) =$$

Eigenvalues of Powers of a Matrix

Suppose that we know the eigenvalues and eigenvectors of A. What are the eigenvalues and eigenvectors of A^k?

Suppose that

$$A\mathbf{x} = \lambda\mathbf{x}.$$

Then

$$A^2\mathbf{x} = A(A\mathbf{x}) = A(\lambda\mathbf{x}) = \lambda(A\mathbf{x}) = \lambda(\lambda\mathbf{x}) = \lambda^2\mathbf{x}.$$

Eigenvalues of Powers of a Matrix

Suppose that we know the eigenvalues and eigenvectors of A. What are the eigenvalues and eigenvectors of A^k?

Suppose that

$$A\mathbf{x} = \lambda\mathbf{x}.$$

Then

$$A^2\mathbf{x} = A(A\mathbf{x}) = A(\lambda\mathbf{x}) = \lambda(A\mathbf{x}) = \lambda(\lambda\mathbf{x}) = \lambda^2\mathbf{x}.$$

So λ^2 is an eigenvalue of A^2 with corresponding eigenvector $\mathbf{x}$.

Theorem

Let $k \in \mathbb{N}$. If λ is an eigenvalue of A, and $\mathbf{x}$ is a corresponding eigenvector, then λ^k is an eigenvalue of A^k and $\mathbf{x}$ is a corresponding eigenvector.

Example

The matrix $A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 2 & 0 \\ -3 & 5 & 2 \end{bmatrix}$ has eigenvalues $\lambda = 1$ and $\lambda = 2$,

and corresponding eigenvectors $\mathbf{p}_1 = \begin{bmatrix} 1 \\ -1 \\ 8 \end{bmatrix}$ and $\mathbf{p}_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$.

Find the eigenvalues and eigenvectors of A^7.

Example

The matrix $A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 2 & 0 \\ -3 & 5 & 2 \end{bmatrix}$ has eigenvalues $\lambda = 1$ and $\lambda = 2$,

and corresponding eigenvectors $\mathbf{p}_1 = \begin{bmatrix} 1 \\ -1 \\ 8 \end{bmatrix}$ and $\mathbf{p}_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$.

Find the eigenvalues and eigenvectors of A^7.

The eigenvalues of A^7 are $\lambda = 1^7 = 1$ and $\lambda = 2^7 = 128$.

Example

The matrix $A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 2 & 0 \\ -3 & 5 & 2 \end{bmatrix}$ has eigenvalues $\lambda = 1$ and $\lambda = 2$,

and corresponding eigenvectors $\mathbf{p}_1 = \begin{bmatrix} 1 \\ -1 \\ 8 \end{bmatrix}$ and $\mathbf{p}_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$.

Find the eigenvalues and eigenvectors of A^7.

The eigenvalues of A^7 are $\lambda = 1^7 = 1$ and $\lambda = 2^7 = 128$. The eigenvectors $\mathbf{p}_1$ and $\mathbf{p}_2$ of A are also eigenvectors of A^7, corresponding to the eigenvalues $\lambda = 1$ and $\lambda = 128$ respectively.

Computing Powers of a Matrix

Diagonalising a matrix A makes calculating A^k much easier.

Computing Powers of a Matrix

Diagonalising a matrix A makes calculating A^k much easier.
Suppose that A is a diagonalisable matrix. Then

$$P^{-1}AP = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix} = D.$$

Computing Powers of a Matrix

Diagonalising a matrix A makes calculating A^k much easier. Suppose that A is a diagonalisable matrix. Then

$$P^{-1}AP = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix} = D.$$

It follows that

$$(P^{-1}AP)^2 = \begin{bmatrix} \lambda_1^2 & 0 & \cdots & 0 \\ 0 & \lambda_2^2 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & \lambda_n^2 \end{bmatrix} = D^2.$$

However

$$D^2 = (P^{-1}AP)^2 = P^{-1}APP^{-1}AP =$$

However

$$D^2 = (P^{-1}AP)^2 = P^{-1}APP^{-1}AP = P^{-1}AIAP = P^{-1}A^2P.$$

However

$$D^2 = (P^{-1}AP)^2 = P^{-1}APP^{-1}AP = P^{-1}AIAP = P^{-1}A^2P.$$

Indeed

$$P^{-1}A^kP = D^k = \begin{bmatrix} \lambda_1^k & 0 & \cdots & 0 \\ 0 & \lambda_2^k & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & \lambda_n^k \end{bmatrix}$$

for any $k \in \mathbb{N}$.

However

$$D^2 = (P^{-1}AP)^2 = P^{-1}APP^{-1}AP = P^{-1}AIAP = P^{-1}A^2P.$$

Indeed

$$P^{-1}A^kP = D^k = \begin{bmatrix} \lambda_1^k & 0 & \cdots & 0 \\ 0 & \lambda_2^k & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & \lambda_n^k \end{bmatrix}$$

for any $k \in \mathbb{N}$. We can rearrange this to

Theorem

$$A^k = PD^kP^{-1}.$$

$$A^k = PD^kP^{-1}.$$

Example

Find A^{13} if $\begin{bmatrix} 0 & 0 & -2 \\ 1 & 2 & 1 \\ 1 & 0 & 3 \end{bmatrix}$.

We showed earlier that A is diagonalised by

$$P = \begin{bmatrix} -1 & 0 & -2 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix} \qquad \text{and that} \qquad D = P^{-1}AP = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

$$A^k = PD^kP^{-1}.$$

Example

Find A^{13} if $\begin{bmatrix} 0 & 0 & -2 \\ 1 & 2 & 1 \\ 1 & 0 & 3 \end{bmatrix}$.

We showed earlier that A is diagonalised by

$$P = \begin{bmatrix} -1 & 0 & -2 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix} \qquad \text{and that} \qquad D = P^{-1}AP = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

It follows that

$$A^{13} = PD^{13}P^{-1} = \begin{bmatrix} -1 & 0 & -2 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2^{13} & 0 & 0 \\ 0 & 2^{13} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 2 \\ 1 & 1 & 1 \\ -1 & 0 & -1 \end{bmatrix}$$

$$A^k = PD^kP^{-1}.$$

Example

Find A^{13} if $\begin{bmatrix} 0 & 0 & -2 \\ 1 & 2 & 1 \\ 1 & 0 & 3 \end{bmatrix}$.

We showed earlier that A is diagonalised by

$$P = \begin{bmatrix} -1 & 0 & -2 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix} \qquad \text{and that} \qquad D = P^{-1}AP = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

It follows that

$$A^{13} = PD^{13}P^{-1} = \begin{bmatrix} -8190 & 0 & -16382 \\ 8191 & 8192 & 8191 \\ 8191 & 0 & 16383 \end{bmatrix}.$$

Remark

A is diagonalisable	$\Longleftarrow$ $\not\Longrightarrow$	A has n distinct eigenvalues.

Remark

$$A \text{ is diagonalisable} \quad \begin{array}{c}\Longleftarrow \\ \not\Longrightarrow\end{array} \quad A \text{ has } n \text{ distinct eigenvalues.}$$

Example

Consider

$$I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad \text{and} \qquad J = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}.$$

Both I and J only have one eigenvalue $\lambda = 1$.

Diagonalisation

Remark

$$A \text{ is diagonalisable} \quad \begin{array}{c}\Longleftarrow \\ \not\Longrightarrow\end{array} \quad A \text{ has } n \text{ distinct eigenvalues.}$$

Example

Consider

$$I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad \text{and} \qquad J = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}.$$

Both I and J only have one eigenvalue $\lambda = 1$. However I has 3 linearly independent eigenvectors whereas J has only one (check!). Therefore I is diagonalisable but J is not.

Complex Vector Spaces

Complex Eigenvalues

Example

Find the eigenvalues of $A = \begin{bmatrix} -2 & -1 \\ 5 & 2 \end{bmatrix}$.

Complex Eigenvalues

Example

Find the eigenvalues of $A = \begin{bmatrix} -2 & -1 \\ 5 & 2 \end{bmatrix}$.

$$0 = \det(\lambda I - A) = \begin{vmatrix} \lambda + 2 & 1 \\ -5 & \lambda - 2 \end{vmatrix} = (\lambda + 2)(\lambda - 2) + 5 = \lambda^2 + 1.$$

The eigenvalues of A are $\lambda = i$ and $\lambda = -i$.

Vectors in $\mathbb{C}^n$

Definition

If n is a positive integer, then a *complex n-tuple* is a sequence of n complex numbers

$$(v_1, v_2, \ldots, v_n).$$

The set of all complex n-tuples is called *complex n-space* and is denoted by $\mathbb{C}^n$.

Scalars are complex numbers, and the operations of addition, subtraction, and scalar multiplication are performed componentwise.

Example

For example

$$\mathbf{u} = (1+i, -4i, 2+3i), \quad \mathbf{v} = (0, i, 5), \quad \mathbf{w} = \left(6 - \sqrt{2}i, 9 + \frac{1}{2}i, \pi i\right)$$

are vectors in $\mathbb{C}^3$.

Example

For example

$$\mathbf{u} = (1+i, -4i, 2+3i), \quad \mathbf{v} = (0, i, 5), \quad \mathbf{w} = \left(6 - \sqrt{2}i, 9 + \frac{1}{2}i, \pi i\right)$$

are vectors in $\mathbb{C}^3$.

Every vector

$$\mathbf{v} = (v_1, v_2, \ldots, v_n) = (a_1 + b_1 i, a_2 + b_2 i, \ldots, a_n + b_n i)$$

can be split into *real* and *imaginary parts* as

$$\begin{aligned} \mathbf{v} &= (a_1, a_2, \ldots, a_n) + (b_1, b_2, \ldots, b_n)i \\ &= \mathrm{Re}(\mathbf{v}) + \mathrm{Im}(\mathbf{v})i. \end{aligned}$$

The vector

$$\overline{\mathbf{v}} = (\bar{v}_1, \bar{v}_2, \ldots, \bar{v}_n) = (a_1 - b_1 i, a_2 - b_2 i, \ldots, a_n - b_n i)$$

is called the *complex conjugate* of $\mathbf{v}$

The vector

$$\overline{\mathbf{v}} = (\bar{v}_1, \bar{v}_2, \ldots, \bar{v}_n) = (a_1 - b_1 i, a_2 - b_2 i, \ldots, a_n - b_n i)$$

is called the *complex conjugate* of $\mathbf{v}$ and can be written as

$$\overline{\mathbf{v}} = \mathrm{Re}(\mathbf{v}) - \mathrm{Im}(\mathbf{v})i.$$

The vector

$$\overline{\mathbf{v}} = (\bar{v}_1, \bar{v}_2, \ldots, \bar{v}_n) = (a_1 - b_1 i, a_2 - b_2 i, \ldots, a_n - b_n i)$$

is called the *complex conjugate* of $\mathbf{v}$ and can be written as

$$\overline{\mathbf{v}} = \mathrm{Re}(\mathbf{v}) - \mathrm{Im}(\mathbf{v}) i.$$

Remark

We can think of $\mathbb{R}^n$ as a subset (not a subspace) of $\mathbb{C}^n$.

$$\mathbf{v} \in \mathbb{R}^n \qquad \Longleftrightarrow \qquad \overline{\mathbf{v}} = \mathbf{v}.$$

Example

Let $\mathbf{v} = (3+i, -2i, 5) \in \mathbb{C}^3$. Then

- $\overline{\mathbf{v}} = (3-i, 2i, 5)$
- $\mathrm{Re}(\mathbf{v}) = (3, 0, 5)$
- $\mathrm{Im}(\mathbf{v}) = (1, -2, 0)$.

Matrices in $\mathbb{C}^{m \times n}$

If A is a *complex matrix* is a matrix whose entries may be either real numbers or complex numbers.

$\mathrm{Re}(A)$ and $\mathrm{Im}(A)$ are the matrices formed from the real and imaginary parts of the entries of A.

$\overline{A}$ is the matrix formed by taking the complex conjugate of each entry in A.

Example

Let $A = \begin{bmatrix} 1+i & -i \\ 4 & 6-2i \end{bmatrix}$. Then

- $\overline{A} = \begin{bmatrix} 1-i & i \\ 4 & 6+2i \end{bmatrix}$
- $\mathrm{Re}(A) = \begin{bmatrix} 1 & 0 \\ 4 & 6 \end{bmatrix}$
- $\mathrm{Im}(A) = \begin{bmatrix} 1 & -1 \\ 0 & -2 \end{bmatrix}$
- $\det(A) = \begin{vmatrix} 1+i & -i \\ 4 & 6-2i \end{vmatrix} = (1+i)(6-2i) - (-i)(4) = 8+8i.$

Theorem

If $\mathbf{u}$ *and* $\mathbf{v}$ *are vectors in* $\mathbb{C}^n$*, and if* k *is a scalar, then:*

- $\overline{\overline{\mathbf{v}}} = \mathbf{v}$
- $\overline{k\mathbf{v}} = \overline{k}\,\overline{\mathbf{v}}$
- $\overline{\mathbf{u} + \mathbf{v}} = \overline{\mathbf{u}} + \overline{\mathbf{v}}$

Theorem

If A *is an* $m \times r$ *complex matrix,* B *is a* $r \times n$ *complex matrix, and* k *is a scalar, then:*

1. $\overline{\overline{A}} = A$
2. $\overline{A^T} = \overline{A}^T$
3. $\overline{AB} = \overline{A}\,\overline{B}$
4. $\overline{kA} = \overline{k}\,\overline{A}.$

Eigenvalues and Eigenvectors of a Real Matrix

We have seen that sometimes real matrices can have complex eigenvalues.

Theorem

Suppose that

- A *is a real* $n \times n$ *matrix, i.e.* $A \in \mathbb{R}^{n \times n}$
- λ *is an eigenvalue of* A
- $\mathbf{x}$ *is an eigenvector of* A *corresponding to* λ.

Theorem

Suppose that

- A *is a real* $n \times n$ *matrix, i.e.* $A \in \mathbb{R}^{n\times n}$
- λ *is an eigenvalue of* A
- $\mathbf{x}$ *is an eigenvector of* A *corresponding to* λ.

Then $\overline{\lambda}$ *is also an eigenvalue of* A *with corresponding eigenvector* $\overline{\mathbf{x}}$.

Theorem

Suppose that

- A *is a real* $n \times n$ *matrix, i.e.* $A \in \mathbb{R}^{n\times n}$
- λ *is an eigenvalue of* A
- $\mathbf{x}$ *is an eigenvector of* A *corresponding to* λ.

Then $\overline{\lambda}$ *is also an eigenvalue of* A *with corresponding eigenvector* $\overline{\mathbf{x}}$.

Proof.

Suppose that

$$A\mathbf{x} = \lambda\mathbf{x}.$$

Since A is a real matrix we have that $\overline{A} = A$.

Theorem

Suppose that

- A *is a real* $n \times n$ *matrix, i.e.* $A \in \mathbb{R}^{n\times n}$
- λ *is an eigenvalue of* A
- $\mathbf{x}$ *is an eigenvector of* A *corresponding to* λ.

Then $\overline{\lambda}$ *is also an eigenvalue of* A *with corresponding eigenvector* $\overline{\mathbf{x}}$.

Proof.

Suppose that

$$A\mathbf{x} = \lambda\mathbf{x}.$$

Since A is a real matrix we have that $\overline{A} = A$. It follows that

$$A\overline{\mathbf{x}} = \overline{\overline{A}\mathbf{x}} = \overline{A\mathbf{x}} = \overline{\lambda\mathbf{x}} = \overline{\lambda}\overline{\mathbf{x}}.$$

$\square$

Example

Find the eigenvalues and bases for the eigenspaces of

$$A = \begin{bmatrix} -2 & -1 \\ 5 & 2 \end{bmatrix}.$$

Example

Find the eigenvalues and bases for the eigenspaces of

$$A = \begin{bmatrix} -2 & -1 \\ 5 & 2 \end{bmatrix}.$$

As before, we calculate that

$$0 = \det(\lambda I - A) = \begin{vmatrix} \lambda + 2 & 1 \\ -5 & \lambda - 2 \end{vmatrix} = \lambda^2 + 1 = (\lambda - i)(\lambda + i)$$

which implies that the eigenvalues are $\lambda = i$ and $\lambda = -i$. Note that i and $-i$ are complex conjugates.

Next we need to find the eigenvectors.

Since

$$\mathbf{0} = (\lambda I - A)\mathbf{x} = \begin{bmatrix} \lambda + 2 & 1 \\ -5 & \lambda - 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

for the eigenvalue $\lambda = i$ we have

$$\begin{bmatrix} i + 2 & 1 \\ -5 & i - 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

Since

$$\mathbf{0} = (\lambda I - A)\mathbf{x} = \begin{bmatrix} \lambda + 2 & 1 \\ -5 & \lambda - 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

for the eigenvalue $\lambda = i$ we have

$$\begin{bmatrix} i + 2 & 1 \\ -5 & i - 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

The first line says

$$(i + 2)x + y = 0$$

from which (if we choose $x = 1$) we obtain

$$\mathbf{x} = \begin{bmatrix} 1 \\ -2 - i \end{bmatrix}.$$

$$\begin{bmatrix} i+2 & 1 \\ -5 & i-2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

Please note that the second line in the above equation is the same as the first. Why?

$$\begin{bmatrix} i+2 & 1 \\ -5 & i-2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

Please note that the second line in the above equation is the same as the first. Why? If we take the first line and multiply by $(i-2)$ then we get

$$(i+2)x + y \quad = \qquad 0$$

$$\begin{bmatrix} i+2 & 1 \\ -5 & i-2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

Please note that the second line in the above equation is the same as the first. Why? If we take the first line and multiply by $(i-2)$ then we get

$$(i-2)\big((i+2)x+y\big) = (i-2)\big(0\big)$$

$$\begin{bmatrix} i+2 & 1 \\ -5 & i-2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

Please note that the second line in the above equation is the same as the first. Why? If we take the first line and multiply by $(i-2)$ then we get

$$\begin{aligned} (i-2)\big((i+2)x + y\big) &= (i-2)\big(0\big) \\ (i-2)(i+2)x + (i-2)y &= 0 \end{aligned}$$

$$\begin{bmatrix} i+2 & 1 \\ -5 & i-2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

Please note that the second line in the above equation is the same as the first. Why? If we take the first line and multiply by $(i-2)$ then we get

$$\begin{aligned} (i-2)\big((i+2)x+y\big) &= (i-2)\big(0\big) \\ (i-2)(i+2)x+(i-2)y &= 0 \\ -5x+(i-2)y &= 0. \end{aligned}$$

To find an eigenvector corresponding to $\lambda = -i$ we could use the same method.

To find an eigenvector corresponding to $\lambda = -i$ we could use the same method. But of course there is an easier way: We just want the vector

$$\overline{\mathbf{x}} = \overline{\begin{bmatrix} 1 \\ -2-i \end{bmatrix}} = \begin{bmatrix} \overline{1} \\ \overline{-2-i} \end{bmatrix} = \begin{bmatrix} 1 \\ -2+i \end{bmatrix}.$$

Eigenvalues of 2×2 matrices

Consider $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. The characteristic polynomial of this matrix is

$$\begin{aligned} \det(\lambda I - A) &= \begin{vmatrix} \lambda - a & -b \\ -c & \lambda - d \end{vmatrix} \\ &= (\lambda - a)(\lambda - d) - bc \\ &= \lambda^2 - (a+d)\lambda + (ad - bc) \\ &= \end{aligned}$$

Eigenvalues of 2×2 matrices

Consider $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. The characteristic polynomial of this matrix is

$$\begin{aligned} \det(\lambda I - A) &= \begin{vmatrix} \lambda - a & -b \\ -c & \lambda - d \end{vmatrix} \\ &= (\lambda - a)(\lambda - d) - bc \\ &= \lambda^2 - (a + d)\lambda + (ad - bc) \\ &= \end{aligned}$$

Eigenvalues of 2×2 matrices

Consider $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. The characteristic polynomial of this matrix is

$$\begin{aligned} \det(\lambda I - A) &= \begin{vmatrix} \lambda - a & -b \\ -c & \lambda - d \end{vmatrix} \\ &= (\lambda - a)(\lambda - d) - bc \\ &= \lambda^2 - (a + d)\lambda + (ad - bc) \\ &= \lambda^2 - \operatorname{tr}(A)\lambda + \det(A). \end{aligned}$$

Eigenvalues of 2×2 matrices

Consider $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. The characteristic polynomial of this matrix is

$$\begin{aligned} \det(\lambda I - A) &= \begin{vmatrix} \lambda - a & -b \\ -c & \lambda - d \end{vmatrix} \\ &= (\lambda - a)(\lambda - d) - bc \\ &= \lambda^2 - (a + d)\lambda + (ad - bc) \\ &= \lambda^2 - \operatorname{tr}(A)\lambda + \det(A). \end{aligned}$$

So the characteristic equation of A is

$$\boxed{\lambda^2 - \operatorname{tr}(A)\lambda + \det(A) = 0.}$$

Now recall that the roots of

$$ax^2 + bx + c = 0$$

are given by

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

$$\lambda^2 - \operatorname{tr}(A)\lambda + \det(A) = 0$$

Now recall that the roots of

$$ax^2 + bx + c = 0$$

are given by

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

There are three cases:

1. $b^2 - 4ac > 0 \implies$ there are two real roots;
2. $b^2 - 4ac = 0 \implies$ there is one repeated root;
3. $b^2 - 4ac < 0 \implies$ there are complex conjugate roots.

(We have $a = 1$, $b = -\operatorname{tr}(A)$ and $c = \det(A)$.)

$\lambda^2 - \mathrm{tr}(A)\lambda + \det(A) = 0$

Theorem

If $A \in \mathbb{R}^{2\times 2}$*, then the characteristic equation of* A *is*

$$\lambda^2 - \mathrm{tr}(A)\lambda + \det(A) = 0$$

and

1. A *has two distinct real eigenvalues if* $\mathrm{tr}(A)^2 - 4\det(A) > 0$*;*
2. A *has one repeated real eigenvalue if* $\mathrm{tr}(A)^2 - 4\det(A) = 0$*; and*
3. A *has two complex conjugate eigenvalues if* $\mathrm{tr}(A)^2 - 4\det(A) < 0$.

$$\lambda^2 - \mathrm{tr}(A)\lambda + \det(A) = 0$$

Example

Find the eigenvalues of $A = \begin{bmatrix} 2 & 2 \\ -1 & 5 \end{bmatrix}$.

We have $\mathrm{tr}(A) = 7$ and $\det(A) = 12$. Since $\mathrm{tr}(A)^2 > 4\det(A)$ we know that we will get two distinct real eigenvalues.

$$\lambda^2 - \operatorname{tr}(A)\lambda + \det(A) = 0$$

Example

Find the eigenvalues of $A = \begin{bmatrix} 2 & 2 \\ -1 & 5 \end{bmatrix}$.

We have $\operatorname{tr}(A) = 7$ and $\det(A) = 12$. Since $\operatorname{tr}(A)^2 > 4\det(A)$ we know that we will get two distinct real eigenvalues. The characteristic equation is

$$0 = \lambda^2 - 7\lambda + 12 =$$

$$\lambda^2 - \mathrm{tr}(A)\lambda + \det(A) = 0$$

Example

Find the eigenvalues of $A = \begin{bmatrix} 2 & 2 \\ -1 & 5 \end{bmatrix}$.

We have $\mathrm{tr}(A) = 7$ and $\det(A) = 12$. Since $\mathrm{tr}(A)^2 > 4\det(A)$ we know that we will get two distinct real eigenvalues. The characteristic equation is

$$0 = \lambda^2 - 7\lambda + 12 = (\lambda - 4)(\lambda - 3).$$

The eigenvalues of A are $\lambda = 4$ and $\lambda = 3$.

$$\lambda^2 - \mathrm{tr}(A)\lambda + \det(A) = 0$$

Example

Find the eigenvalues of $A = \begin{bmatrix} 0 & -1 \\ 1 & 2 \end{bmatrix}$.

$$\lambda^2 - \mathrm{tr}(A)\lambda + \det(A) = 0$$

Example

Find the eigenvalues of $A = \begin{bmatrix} 0 & -1 \\ 1 & 2 \end{bmatrix}$.

We have $\mathrm{tr}(A) = 2$ and $\det(A) = 1$. This is an example with $\mathrm{tr}(A)^2 = 4\det(A)$.

$$\lambda^2 - \mathrm{tr}(A)\lambda + \det(A) = 0$$

Example

Find the eigenvalues of $A = \begin{bmatrix} 0 & -1 \\ 1 & 2 \end{bmatrix}$.

We have $\mathrm{tr}(A) = 2$ and $\det(A) = 1$. This is an example with $\mathrm{tr}(A)^2 = 4\det(A)$. So the characteristic equation is

$$0 = \lambda^2 - 2\lambda + 1 = (\lambda - 1)^2.$$

$\lambda = 1$ is the only eigenvalue of A.

$$\lambda^2 - \mathrm{tr}(A)\lambda + \det(A) = 0$$

Example

Find the eigenvalues of $A = \begin{bmatrix} 2 & 3 \\ -3 & 2 \end{bmatrix}$.

$$\lambda^2 - \operatorname{tr}(A)\lambda + \det(A) = 0$$

Example

Find the eigenvalues of $A = \begin{bmatrix} 2 & 3 \\ -3 & 2 \end{bmatrix}$.

We have $\operatorname{tr}(A) = 4$ and $\det(A) = 13$. So $\operatorname{tr}(A)^2 < 4\det(A)$.

$\lambda^2 - \mathrm{tr}(A)\lambda + \det(A) = 0$

Example

Find the eigenvalues of $A = \begin{bmatrix} 2 & 3 \\ -3 & 2 \end{bmatrix}$.

We have $\mathrm{tr}(A) = 4$ and $\det(A) = 13$. So $\mathrm{tr}(A)^2 < 4\det(A)$. The characteristic equation is

$$0 = \lambda^2 - 4\lambda + 13.$$

The solutions to this quadratic equation are

$$\lambda = \frac{4 \pm \sqrt{(-4)^2 - 4(13)}}{2} = \frac{4 \pm \sqrt{-36}}{2} = \frac{4 \pm 6i}{2} = 2 \pm 3i.$$

The eigenvalues of A are $\lambda = 2 + 3i$ and $\lambda = 2 - 3i$.

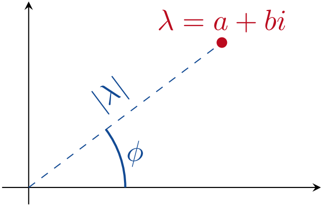
Symmetric Matrices have Real Eigenvalues

Theorem

If A is a real symmetric matrix, then A has real eigenvalues.

(proof omitted)

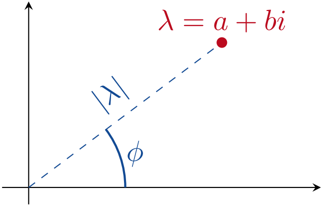
An Interpretation of Complex Eigenvalues



Theorem

The eigenvalues of the real matrix $C = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$ *are* $\lambda = a \pm bi$.

An Interpretation of Complex Eigenvalues



Theorem

The eigenvalues of the real matrix $C = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$ *are* $\lambda = a \pm bi$. *If* a *and* b *are not both zero, then this matrix can be factored as*

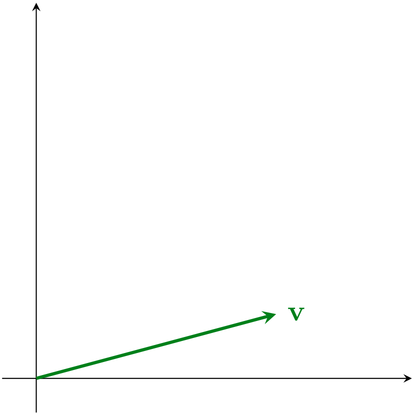
$$\begin{bmatrix} a & -b \\ b & a \end{bmatrix} = \begin{bmatrix} |\lambda| & 0 \\ 0 & |\lambda| \end{bmatrix} \begin{bmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{bmatrix}$$

where ϕ *is the argument of* $a + bi$.

$$C = \begin{bmatrix} a & -b \\ b & a \end{bmatrix} = \begin{bmatrix} |\lambda| & 0 \\ 0 & |\lambda| \end{bmatrix} \begin{bmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{bmatrix}$$

Remark

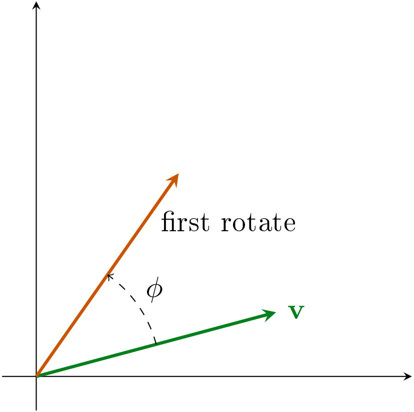
Geometrically, this means that the matrix C can be viewed as a rotation through an angle of ϕ followed by a scaling with factor $|\lambda|$.



$$C = \begin{bmatrix} a & -b \\ b & a \end{bmatrix} = \begin{bmatrix} |\lambda| & 0 \\ 0 & |\lambda| \end{bmatrix} \begin{bmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{bmatrix}$$

Remark

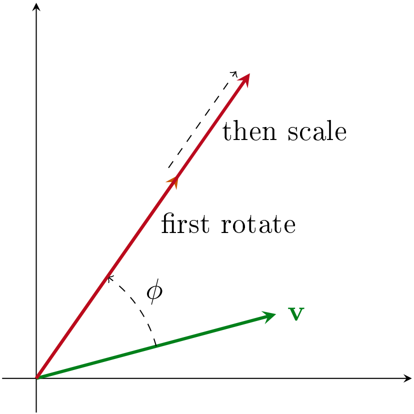
Geometrically, this means that the matrix C can be viewed as a rotation through an angle of ϕ followed by a scaling with factor $|\lambda|$.



$$C = \begin{bmatrix} a & -b \\ b & a \end{bmatrix} = \begin{bmatrix} |\lambda| & 0 \\ 0 & |\lambda| \end{bmatrix} \begin{bmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{bmatrix}$$

Remark

Geometrically, this means that the matrix C can be viewed as a rotation through an angle of ϕ followed by a scaling with factor $|\lambda|$.



Next Time

- Inner Product Spaces
- Orthogonality
- Orthogonal Sets and Orthonormal Sets