OKAN ÜNİVERSİTESİ
FEN EDEBİYAT FAKÜLTESİ
MATEMATİK BÖLÜMÜ

2013.05.23 **MAT 234 – Matematik IV – Final Sınavın Çözümleri** N. Course

Question 1 (Symbolic Logic and Proof by Contrpositive).

(a) [4×2p] Mark the following statements as *true* or *false*?

$(P \implies Q) = (Q \implies P)$ ☐ *true* ☑ *false*

$(P \wedge \neg P) = \text{true}$ ☐ *true* ☑ *false*

$\neg(P \wedge Q) = (\neg P \wedge \neg Q)$ ☐ *true* ☑ *false*

$\neg(P \implies Q) = (P \wedge \neg Q)$ ☑ *true* ☐ *false*

(b) [6p] Prove that $(P \implies Q) = (\neg Q \implies \neg P)$.

P	Q	$P \implies Q$	$\neg Q$	$\neg P$	$\neg Q \implies \neg P$
T	T	T	F	F	T
T	F	F	T	F	F
F	T	T	F	T	T
F	F	T	T	T	T

(c) [8p] Let $a, b \in \mathbb{Z}$. Use **proof by contrapositive** to prove that

$$a + b \geq 15 \qquad \implies \qquad a \geq 8 \text{ or } b \geq 8.$$

The contrapositive is

$$a < 8 \text{ and } b < 8 \qquad \implies \qquad a + b < 15.$$

This is the statement that we must prove. $\boxed{4}$
Suppose $a, b \in \mathbb{Z}$, $a < 8$ and $b < 8$. Then $a \leq 7$ and $b \leq 7$. It follows that

$$a + b \leq 7 + 7 \leq 14 < 15$$

and we are done. $\boxed{4}$

(d) [3p] We say that a sequence (a_n) is *bounded* iff, there exists $M \geq 0$ such that for all $n \in \mathbb{N}$, we have $|a_n| \leq M$.
Give the definition of "(a_n) is *not bounded*".

$(\forall M \geq 0)(\exists n \in \mathbb{N})(|a_n| > M)$. $\boxed{\text{1 point for each part}}$

— OR —

We say that a sequence (a_n) is not bounded iff, for all $M \geq 0$, there exists $n \in \mathbb{N}$ such that $|a_n| > M$.

Question 2 (Cauchy Sequences).

(a) [5p] Give the definition of a *Cauchy sequence*.

A sequence (a_n) is called a Cauchy sequence iff, $\underline{\forall}\ \underline{\varepsilon > 0}\ \underline{\exists}\ \underline{N = N(\varepsilon) \in \mathbb{N}}$ such that

$$\underline{n, m > N}\ \underline{\implies}\ \underline{|a_n - a_m| < \varepsilon}.$$

$\boxed{-1 \text{ points for each underlined piece missing/incorrect in answer.}}$

(b) [8p] Let $b_n = 10^{-n} - 100$, for all $n \in \mathbb{N}$. **Use the definition** that you wrote in part (a) to show that (b_n) is a Cauchy sequence.

Let $\varepsilon > 0$. $\boxed{2}$ Choose $N > -\log_{10} \varepsilon$. $\boxed{2}$ Then

$$\begin{aligned} n > m > N \implies |b_n - b_m| &= \left|(10^{-n} - 100) - (10^{-m} - 100)\right| \boxed{1} \\ &= \left|10^{-n} - 10^{-m}\right| \\ &= 10^{-m} - 10^{-n} \\ &= 10^{-m}(1 - 10^{m-n}) \\ &< 10^{-m} \\ &< 10^{-N} \boxed{2} \\ &< \varepsilon. \end{aligned}$$

Therefore (b_n) is a Cauchy sequence. $\boxed{1}$

(c) [12p] Show that

$$(x_n) \text{ is a convergent sequence} \quad \implies \quad (x_n) \text{ is a Cauchy sequence.}$$

Let $\varepsilon > 0$ $\boxed{2}$. Let $x_n \to x$ as $n \to \infty$. Then $\exists\ N \in \mathbb{N}$ such that

$$n > N \implies |x_n - x| < \frac{\varepsilon}{2}. \boxed{4}$$

But then

$$\begin{aligned} n, m > N \implies |x_n - x_m| &= |x_n - x + x - x_m| \ \boxed{2} \\ &\le |x_n - x| + |x - x_m| \\ &\qquad \text{(by the triangle inequality)} \boxed{2} \\ &= |x_n - x| + |x_m - x| \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\ &= \varepsilon. \ \boxed{1} \end{aligned}$$

Therefore (x_n) is a Cauchy sequence. $\boxed{1}$

Question 3 (The Proof of The Alternating Series Test).

(a) [5p] Give the definition of a *convergent series*.

Consider the series $\sum_{n=1}^{\infty} b_n$. Define $s_n = \sum_{k=1}^{n} b_k$. We say that the series $\sum_{n=1}^{\infty} b_n$ converges iff the sequence $(s_n)_{n=1}^{\infty}$ converges.

For parts (b) – (f), suppose that

- (a_n) is a sequence of real numbers;
- $a_n > 0 \ \forall n$;
- (a_n) is decreasing [i.e. $a_n \geq a_{n+1} \ \forall n$];
- $a_n \to 0$ as $n \to \infty$;
- $s_n = \sum_{k=1}^{n} (-1)^{k+1} a_k = a_1 - a_2 + a_3 - a_4 + a_5 - a_6 + \ldots + (-1)^{n+1} a_n$.

(b) [4p] Show that $s_{2n+2} - s_{2n} \geq 0$ for all n.
[In other words: Show that (s_{2n}) is an increasing sequence.]

Let $n \in \mathbb{N}$. Clearly

$$\begin{aligned} s_{2n+2} - s_{2n} &= (a_1 - a_2 + a_3 - a_4 + a_5 - a_6 + \ldots - a_{2n} + a_{2n+1} - a_{2n+2}) \\ &\qquad - (a_1 - a_2 + a_3 - a_4 + a_5 - a_6 + \ldots - a_{2n}) \\ &= a_{2n+1} - a_{2n+2} \boxed{2} \\ &\geq 0 \end{aligned}$$

since (a_n) is a decreasing sequence. $\boxed{2}$

(c) [4p] Show that $s_{2n} \leq a_1$ for all n.
[This proves that (s_{2n}) is bounded above.]

We can see that

$$\begin{aligned} s_{2n} &= a_1 - a_2 + a_3 - a_4 + a_5 - a_6 + \ldots - a_{2n} \\ &= a_1 - (a_2 - a_3) - (a_4 - a_5) - (a_6 - a_7) - \ldots - (a_{2n-2} - a_{2n-1}) - a_{2n} \\ &\leq a_1, \end{aligned}$$

since (a_n) is a decreasing sequence and since $a_{2n} > 0$.

(d) [4p] Show that (s_{2n}) is convergent.

By parts (b) and (c), (s_{2n}) is increasing $\boxed{1}$ and bounded above $\boxed{1}$. Therefore, by a theorem from the course $\boxed{1}$, (s_{2n}) is convergent $\boxed{1}$.

(e) [4p] Let $s = \lim_{n\to\infty} s_{2n}$. Show that $s_{2n+1} \to s$ as $n \to \infty$ also.

$$s_{2n+1} = s_{2n} + a_{2n+1} \to s + 0 = s$$

as $n \to \infty$.

(f) [4p] Show that $\sum_{k=1}^{\infty} (-1)^{k+1} a_k$ converges.

By part (e), it follows that $s_n \to s$ as $n \to \infty$. Therefore (s_n) is a convergent sequence, and hence $\sum_{k=1}^{\infty} (-1)^{k+1} a_k$ is a convergent series.

Question 4 (Series)**.** Decide if each of the following series converges or diverges. Justify (prove) your answers.

(a) [8p] $\displaystyle\sum_{n=1}^{\infty} \sin\left(\frac{\pi n}{2}\right)\cos\left(\frac{\pi}{n}\right).$

(b) [8p] $\displaystyle\sum_{n=1}^{\infty} n! e^{-n}.$

(c) [9p] $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}\,(\sqrt{n}+1)}.$

[You may use any theorem/lemma/test/example/etc. from the course, but you must say which one you are using.]

2 pts for "converges/diverges" without justification.
2 pts for saying which test is being used (as long as there is some proof given).
Remaining 4/5 pts for accuracy of proof.

(a) For n sufficiently large, $\cos\frac{\pi}{n} > \frac{1}{2}$. So we can see that

$$|a_{2n-1}| := \left|\sin\left(\frac{\pi(2n-1)}{2}\right)\cos\left(\frac{\pi}{2n-1}\right)\right| = \cos\left(\frac{\pi}{2n-1}\right) > \frac{1}{2}$$

for large n. Therefore $a_n \not\to 0$ as $n \to \infty$. It follows that $\sum_{n=1}^{\infty} \sin\left(\frac{\pi n}{2}\right)\cos\left(\frac{\pi}{n}\right)$ diverges by the Divergence Test.

(b) Since

$$\frac{a_{n+1}}{a_n} = \frac{(n+1)!e^{-(n+1)}}{n!e^{-n}} = (n+1)e^{-1} \to \infty$$

as $n \to \infty$, it follows that $\sum_{n=1}^{\infty} n!e^{-n}$ diverges by the Ratio Test.

— OR —

Use Divergence Test again.

(c) Since

$$\int_1^{\infty} \frac{dx}{\sqrt{x}(\sqrt{x}+1)} = \lim_{R\to\infty}\int_1^{R} \frac{dx}{\sqrt{x}(\sqrt{x}+1)} = \lim_{R\to\infty}\int_2^{\sqrt{R}+1}\frac{du}{u}$$
$$= \lim_{R\to\infty} \log(\sqrt{R}+1) - \log 2 = \infty,$$

it follows that $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}(\sqrt{n}+1)}$ diverges by the Integral Test.

— OR —

Since

$$\frac{1}{\sqrt{n}\,(\sqrt{n}+1)} = \frac{1}{n+\sqrt{n}} \geq \frac{1}{n+n} = \frac{1}{2n} = \frac{\frac{1}{2}}{n}$$

and since we know that $\sum_{n=1}^{\infty}\frac{1}{n}$ diverges, it follows that $\sum_{n=1}^{\infty}\frac{1}{\sqrt{n}(\sqrt{n}+1)}$ also diverges by the Comparison Test.

Question 5 (Power Series and Taylor Series).

(a) [5p] Let $\sum_{n=0}^{\infty} a_n x^n$ be a power series. Give the definition of the *radius of convergence* of $\sum_{n=0}^{\infty} a_n x^n$.

> If $\sum_{n=0}^{\infty} a_n x^n$ converges $\forall\,|x| < R$ and diverges $\forall\,|x| > R$, then R is called the radius of convergence of $\sum_{n=0}^{\infty} a_n x^n$.

Consider the power series

$$\sum_{n=0}^{\infty} \frac{x^n}{2^n (n+1)^2}. \tag{1}$$

(b) [7p] Find the radius of convergence of (1).

> For this power series, $a_n = \frac{1}{2^n(n+1)^2}$ and
>
> $$\left|\frac{a_n}{a_{n+1}}\right| = \frac{2^{n+1}(n+2)(n+2)}{2^n(n+1)(n+1)} = 2\frac{(1+\frac{2}{n})(1+\frac{2}{n})}{(1+\frac{1}{n})(1+\frac{1}{n})} \to 2\frac{(1+0)(1+0)}{(1+0)(1+0)} = 2$$
>
> as $n \to \infty$ $\boxed{4}$ $\boxed{\text{-1 point if candidate omits absolute value signs}}$. By a theorem from the course $\boxed{1}$, the radius of convergence of (1) is $R = 2$ $\boxed{2}$.

(c) [13p] Calculate the Taylor Series for $f(x) = \cos x$, **centred at** $a = \pi$.

[HINT: You may assume without proof that $\left|\frac{f^n(c)}{n!}(x-\pi)^n\right| \to 0$ as $n \to \infty$ for all $c, x \in \mathbb{R}$.]

> Since
>
> $$\frac{d^n}{dx^n}\cos x = \begin{cases} \cos x & n = 0,4,8,\dots \\ -\sin x & n = 1,5,9,\dots \\ -\cos x & n = 2,6,10,\dots \\ \sin x & n = 3,7,11,\dots, \end{cases}$$
>
> we can see that
>
> $$f^n(\pi) = \begin{cases} -1 & n = 0,4,8,\dots \\ 0 & n = 1,3,5,7,9,\dots \\ 1 & n = 2,6,10,\dots \boxed{4} \end{cases}$$
>
> By Taylor's Theorem (and by the hint), we have
>
> $$\begin{aligned} \cos x &= f(\pi) + f'(\pi)(x-\pi) + \frac{f''(\pi)(x-\pi)^2}{2!} + \frac{f'''(\pi)(x-\pi)^3}{3!} + \frac{f^{(4)}(\pi)(x-\pi)^4}{4!} + \dots \boxed{4} \\ &= -1 + \frac{(x-\pi)^2}{2!} - \frac{(x-\pi)^4}{4!} + \frac{(x-\pi)^6}{6!} - \frac{(x-\pi)^8}{8!} + \frac{(x-\pi)^{10}}{10!} - \frac{(x-\pi)^{12}}{12!} \\ &\quad + \frac{(x-\pi)^{14}}{14!} - \frac{(x-\pi)^{16}}{16!} + \frac{(x-\pi)^{18}}{18!} + \dots \boxed{5} \\ &= -1 + \frac{(x-\pi)^2}{2} - \frac{(x-\pi)^4}{24} + \frac{(x-\pi)^6}{720} - \frac{(x-\pi)^8}{40320} + \frac{(x-\pi)^{10}}{3628800} - \frac{(x-\pi)^{12}}{479001600} \\ &\quad + \frac{(x-\pi)^{14}}{87178291200} - \frac{(x-\pi)^{16}}{20922789888000} + \frac{(x-\pi)^{18}}{6402373705728000} + \dots \boxed{\text{optional}} \end{aligned}$$