

Lecture 9

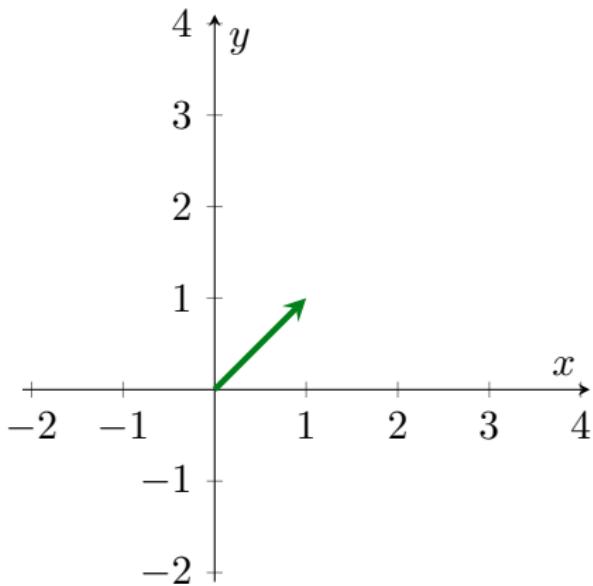
- Eigenvalues and Eigenvectors
- Diagonalisation
- Complex Vector Spaces



Eigenvalues and Eigenvectors

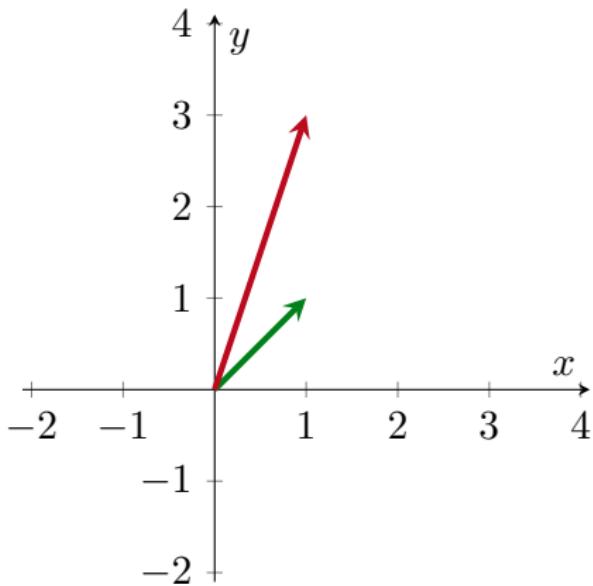
“Matrices do things to vectors”

$$\begin{bmatrix} 3 & -2 \\ -1 & 4 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} =$$



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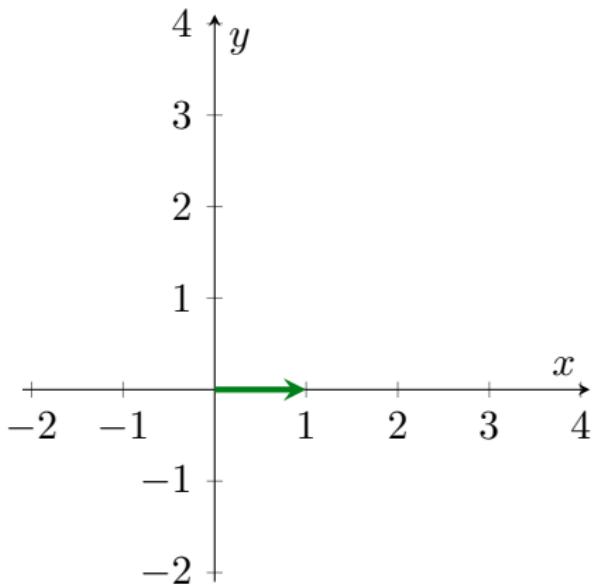
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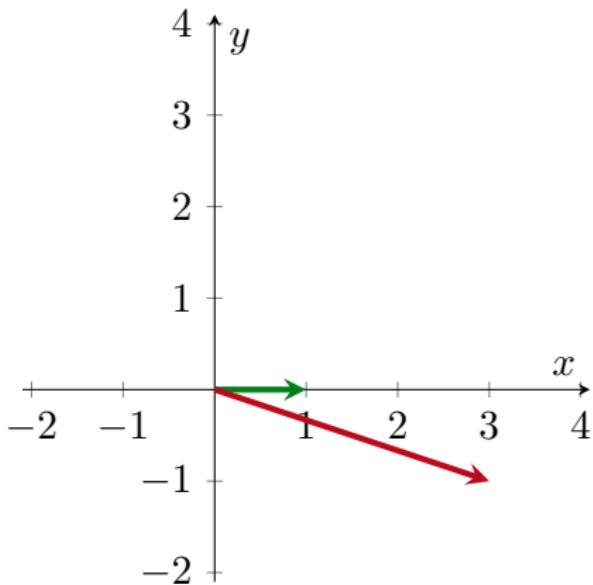
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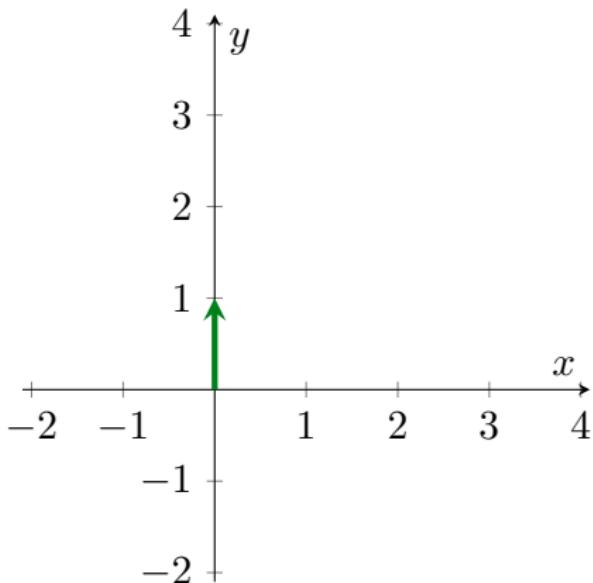


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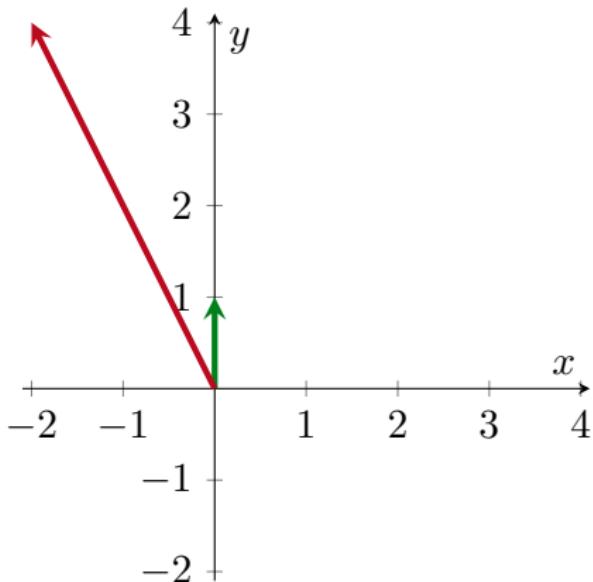


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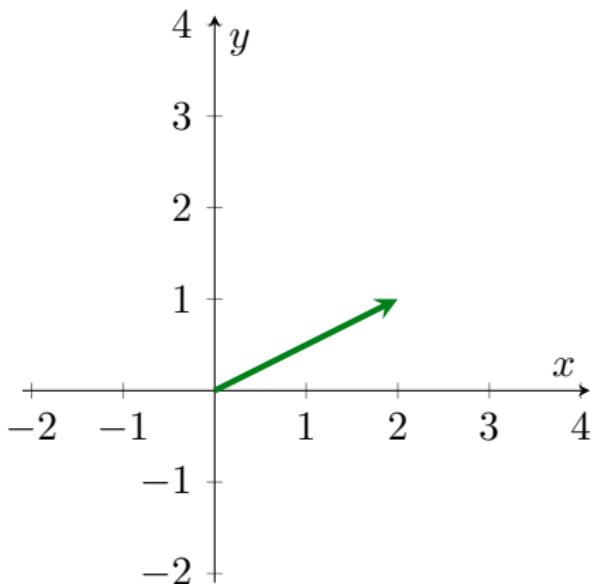
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Eigenvalues and Eigenvectors



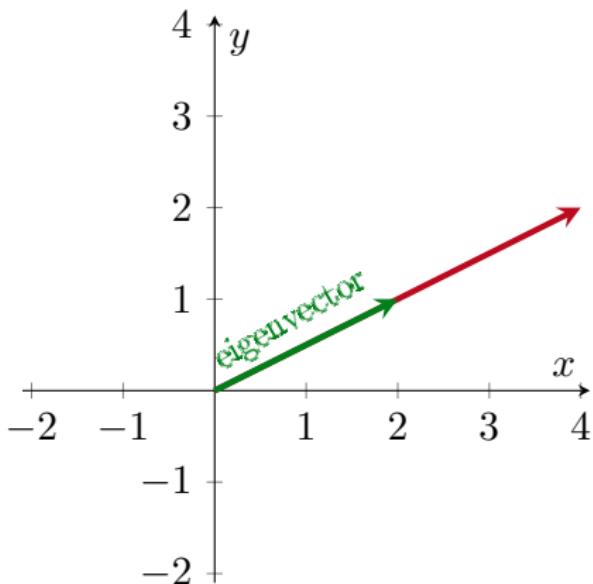
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Eigenvalues and Eigenvectors



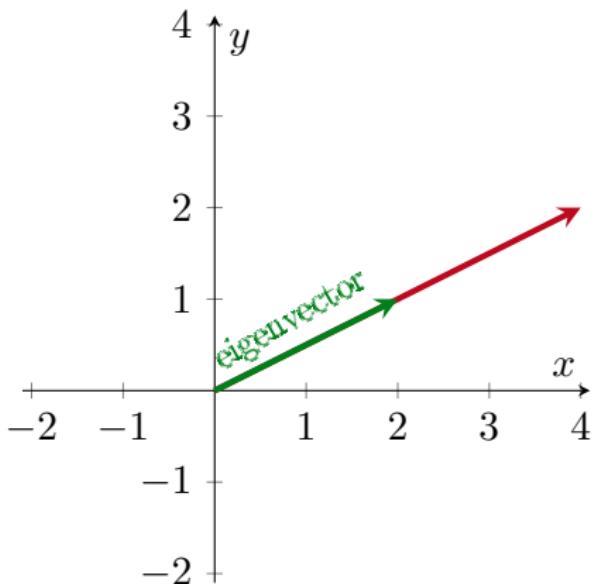
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Eigenvalues and Eigenvectors



Definition

If A is an $n \times n$ matrix, then a nonzero vector \mathbf{x} in \mathbb{R}^n is called an *eigenvector* of A if $A\mathbf{x}$ is a scalar multiple of \mathbf{x} ; i.e. if

$$A\mathbf{x} = \lambda\mathbf{x}$$

for some scalar λ^1 .

¹lambda

Eigenvalues and Eigenvectors



Definition

If A is an $n \times n$ matrix, then a nonzero vector \mathbf{x} in \mathbb{R}^n is called an *eigenvector* of A if $A\mathbf{x}$ is a scalar multiple of \mathbf{x} ; i.e. if

$$A\mathbf{x} = \lambda\mathbf{x}$$

for some scalar λ^1 . The scalar λ is called an *eigenvalue* of A , and \mathbf{x} is said to be an *eigenvector corresponding to λ* .

¹lambda

Eigenvalues and Eigenvectors



Remark

The zero vector $\mathbf{0}$ can not be an eigenvector. Only non-zero vectors can be eigenvectors.

Remark

Some books use

$$A\mathbf{x} = \lambda\mathbf{x}$$

some use

$$(\lambda I - A)\mathbf{x} = \mathbf{0}$$

and some use

$$(A - \lambda I)\mathbf{x} = \mathbf{0}.$$

Eigenvalues and Eigenvectors

Example

Let $A = \begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix}$, $\mathbf{u} = \begin{bmatrix} 6 \\ -5 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 3 \\ -2 \end{bmatrix}$. Are \mathbf{u} and \mathbf{v} eigenvectors of A ?

$$A\mathbf{u} =$$

$$A\mathbf{v} =$$

Eigenvalues and Eigenvectors



Example

Let $A = \begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix}$, $\mathbf{u} = \begin{bmatrix} 6 \\ -5 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 3 \\ -2 \end{bmatrix}$. Are \mathbf{u} and \mathbf{v} eigenvectors of A ?

$$A\mathbf{u} = \begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix} \begin{bmatrix} 6 \\ -5 \end{bmatrix} = \begin{bmatrix} -24 \\ 20 \end{bmatrix} = -4 \begin{bmatrix} 6 \\ -5 \end{bmatrix} = -4\mathbf{u}$$

Thus \mathbf{u} is an eigenvector of A corresponding to the eigenvalue $\lambda = -4$.

$$A\mathbf{v} =$$

Eigenvalues and Eigenvectors



Example

Let $A = \begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix}$, $\mathbf{u} = \begin{bmatrix} 6 \\ -5 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 3 \\ -2 \end{bmatrix}$. Are \mathbf{u} and \mathbf{v} eigenvectors of A ?

$$A\mathbf{u} = \begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix} \begin{bmatrix} 6 \\ -5 \end{bmatrix} = \begin{bmatrix} -24 \\ 20 \end{bmatrix} = -4 \begin{bmatrix} 6 \\ -5 \end{bmatrix} = -4\mathbf{u}$$

Thus \mathbf{u} is an eigenvector of A corresponding to the eigenvalue $\lambda = -4$.

$$A\mathbf{v} = \begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix} \begin{bmatrix} 3 \\ -2 \end{bmatrix} = \begin{bmatrix} -9 \\ 11 \end{bmatrix} \neq \lambda \begin{bmatrix} 3 \\ -2 \end{bmatrix}$$

But \mathbf{v} is not an eigenvector of A because $A\mathbf{v}$ is not a multiple of \mathbf{v} .

Eigenvalues and Eigenvectors

Note that

$$\det(B) \neq 0$$



the only solution
to $B\mathbf{x} = \mathbf{0}$ is the
trivial solution

Eigenvalues and Eigenvectors

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$$\det(B) \neq 0 \quad \implies$$

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So

there is a nonzero
vector \mathbf{x} such that

$$(\lambda I - A)\mathbf{x} = \mathbf{0}$$

$$\implies \det(\lambda I - A) = 0.$$

Eigenvalues and Eigenvectors

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So

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vector \mathbf{x} such that

$$(\lambda I - A)\mathbf{x} = \mathbf{0}$$

$$\Rightarrow \det(\lambda I - A) = 0.$$

Theorem

If A is an $n \times n$ matrix, then λ is an eigenvalue of A if and only if it satisfies the equation

$$\det(\lambda I - A) = 0.$$

This is called the characteristic equation of A .

$$\det(\lambda I - A) = 0$$



Remark

$$\det(\lambda I - A) = 0 \iff \det(A - \lambda I) = 0$$

because $\det(-B) = (-1)^n \det(B)$ and $-0 = 0$.

$$\det(\lambda I - A) = 0$$



Example

Find all the eigenvalues of

$$A = \begin{bmatrix} 3 & 0 \\ 8 & -1 \end{bmatrix}.$$

$$\det(\lambda I - A) = 0$$



Example

Find all the eigenvalues of

$$A = \begin{bmatrix} 3 & 0 \\ 8 & -1 \end{bmatrix}.$$

Using the formula, we calculate that

$$0 = \det(\lambda I - A) =$$

=

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Example

Find all the eigenvalues of

$$A = \begin{bmatrix} 3 & 0 \\ 8 & -1 \end{bmatrix}.$$

Using the formula, we calculate that

$$0 = \det(\lambda I - A) = \det \left(\begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} - \begin{bmatrix} 3 & 0 \\ 8 & -1 \end{bmatrix} \right) = \begin{vmatrix} \lambda - 3 & 0 \\ -8 & \lambda + 1 \end{vmatrix}$$
$$=$$

$$\det(\lambda I - A) = 0$$



Example

Find all the eigenvalues of

$$A = \begin{bmatrix} 3 & 0 \\ 8 & -1 \end{bmatrix}.$$

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$$\begin{aligned} 0 &= \det(\lambda I - A) = \det \left(\begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} - \begin{bmatrix} 3 & 0 \\ 8 & -1 \end{bmatrix} \right) = \begin{vmatrix} \lambda - 3 & 0 \\ -8 & \lambda + 1 \end{vmatrix} \\ &= (\lambda - 3)(\lambda + 1) - (0)(-8) = (\lambda - 3)(\lambda + 1). \end{aligned}$$

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Thus the eigenvalues of A are $\lambda = 3$ and $\lambda = -1$.

$$\det(\lambda I - A) = 0$$



Remark

The formula $\det(\lambda I - A) = 0$ leads to the characteristic equation

$$\lambda^{\textcolor{red}{n}} + c_1\lambda^{n-1} + \dots + c_{n-1}\lambda + c_n = 0.$$

The polynomial

$$p(\lambda) = \lambda^{\textcolor{red}{n}} + c_1\lambda^{n-1} + \dots + c_{n-1}\lambda + c_n$$

is called the *characteristic polynomial* of the $\textcolor{red}{n} \times \textcolor{red}{n}$ matrix A .

$$\det(\lambda I - A) = 0$$



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For example, the characteristic polynomial of $A = \begin{bmatrix} 3 & 0 \\ 8 & -1 \end{bmatrix}$ is

$$p(\lambda) = (\lambda - 3)(\lambda + 1) = \lambda^{\textcolor{red}{2}} - 2\lambda - 3$$

which is a polynomial of degree $\textcolor{red}{2}$.

$$\det(\lambda I - A) = 0$$



Example

Find the eigenvalues of

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 4 & -17 & 8 \end{bmatrix}.$$

$$\det(\lambda I - A) = 0$$



Example

Find the eigenvalues of

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 4 & -17 & 8 \end{bmatrix}.$$

The characteristic polynomial of A is

$$\det(\lambda I - A) = \det \begin{bmatrix} \lambda & -1 & 0 \\ 0 & \lambda & -1 \\ -4 & 17 & \lambda - 8 \end{bmatrix} = \lambda^3 - 8\lambda^2 + 17\lambda - 4.$$

The characteristic equation of A is

$$\lambda^3 - 8\lambda^2 + 17\lambda - 4 = 0.$$

Eigenvalues and Eigenvectors



$$\lambda^3 - 8\lambda^2 + 17\lambda - 4 = 0$$

First we will look for integer solutions of this equation.

Eigenvalues and Eigenvectors



$$\lambda^3 - 8\lambda^2 + 17\lambda - 4 = 0$$

First we will look for integer solutions of this equation. Note that integer solutions must be divisors of the -4 term. So we need to try $1, -1, 2, -2, 4$ and -4 until we find one.

Eigenvalues and Eigenvectors



$$\lambda^3 - 8\lambda^2 + 17\lambda - 4 = 0$$

First we will look for integer solutions of this equation. Note that integer solutions must be divisors of the -4 term. So we need to try $1, -1, 2, -2, 4$ and -4 until we find one.

$$\lambda = 1$$

$$\lambda^3 - 8\lambda^2 + 17\lambda - 4 = 1 - 8 + 17 - 4 = 6$$

X

Eigenvalues and Eigenvectors



$$\lambda^3 - 8\lambda^2 + 17\lambda - 4 = 0$$

First we will look for integer solutions of this equation. Note that integer solutions must be divisors of the -4 term. So we need to try $1, -1, 2, -2, 4$ and -4 until we find one.

$$\lambda = 1 \qquad \lambda^3 - 8\lambda^2 + 17\lambda - 4 = 1 - 8 + 17 - 4 = 6 \quad \textcolor{red}{X}$$

$$\lambda = -1 \qquad \lambda^3 - 8\lambda^2 + 17\lambda - 4 = -1 - 8 - 17 - 4 = -30 \quad \textcolor{red}{X}$$

Eigenvalues and Eigenvectors

$$\lambda^3 - 8\lambda^2 + 17\lambda - 4 = 0$$

First we will look for integer solutions of this equation. Note that integer solutions must be divisors of the -4 term. So we need to try $1, -1, 2, -2, 4$ and -4 until we find one.

$$\lambda = 1 \quad \lambda^3 - 8\lambda^2 + 17\lambda - 4 = 1 - 8 + 17 - 4 = 6 \quad \text{X}$$

$$\lambda = -1 \quad \lambda^3 - 8\lambda^2 + 17\lambda - 4 = -1 - 8 - 17 - 4 = -30 \quad \text{X}$$

$$\lambda = 2 \quad \lambda^3 - 8\lambda^2 + 17\lambda - 4 = 8 - 32 + 34 - 4 = 6 \quad \text{X}$$

Eigenvalues and Eigenvectors



$$\lambda^3 - 8\lambda^2 + 17\lambda - 4 = 0$$

First we will look for integer solutions of this equation. Note that integer solutions must be divisors of the -4 term. So we need to try $1, -1, 2, -2, 4$ and -4 until we find one.

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$$\lambda = -2 \quad \lambda^3 - 8\lambda^2 + 17\lambda - 4 = -8 - 32 - 34 - 4 = -78 \quad \text{X}$$

Eigenvalues and Eigenvectors

$$\lambda^3 - 8\lambda^2 + 17\lambda - 4 = 0$$

First we will look for integer solutions of this equation. Note that integer solutions must be divisors of the -4 term. So we need to try $1, -1, 2, -2, 4$ and -4 until we find one.

$$\lambda = 1 \quad \lambda^3 - 8\lambda^2 + 17\lambda - 4 = 1 - 8 + 17 - 4 = 6 \quad \text{X}$$

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$$\lambda = -2 \quad \lambda^3 - 8\lambda^2 + 17\lambda - 4 = -8 - 32 - 34 - 4 = -78 \quad \text{X}$$

$$\lambda = 4 \quad \lambda^3 - 8\lambda^2 + 17\lambda - 4 = 64 - 128 + 68 - 4 = 0 \quad \checkmark$$

So $\lambda = 4$ is an eigenvalue. Now we need to look for more.

Eigenvalues and Eigenvectors



Now that we know that $\lambda = 4$ is an eigenvalue we can calculate

$$0 = \lambda^3 - 8\lambda^2 + 17\lambda - 4 = (\lambda - 4)(\lambda^2 + b\lambda + c)$$

=

=

Eigenvalues and Eigenvectors



Now that we know that $\lambda = 4$ is an eigenvalue we can calculate

$$\begin{aligned}0 &= \lambda^3 - 8\lambda^2 + 17\lambda - 4 = (\lambda - 4)(\lambda^2 + b\lambda + c) \\&= \lambda^3 + b\lambda^2 + c\lambda - 4\lambda^2 - 4b\lambda - 4c \\&= \lambda^3 + (b - 4)\lambda^2 + (c - 4b)\lambda - 4c\end{aligned}$$

Eigenvalues and Eigenvectors



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which implies that $b = -4$ and $c = 1$.

Eigenvalues and Eigenvectors



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which implies that $b = -4$ and $c = 1$. Hence

$$0 = (\lambda - 4)(\lambda^2 - 4\lambda + 1).$$

Eigenvalues and Eigenvectors



Now we need to find the roots of

$$\lambda^2 - 4\lambda + 1 = 0.$$

Eigenvalues and Eigenvectors

Now we need to find the roots of

$$\lambda^2 - 4\lambda + 1 = 0.$$

Using the formula

$$\lambda = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

we find

$$\lambda = 2 \pm \sqrt{3}.$$

Eigenvalues and Eigenvectors



Now we need to find the roots of

$$\lambda^2 - 4\lambda + 1 = 0.$$

Using the formula

$$\lambda = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

we find

$$\lambda = 2 \pm \sqrt{3}.$$

Therefore the eigenvalues of A are

$$\lambda = 4, \quad \lambda = 2 + \sqrt{3}, \quad \text{and} \quad \lambda = 2 - \sqrt{3}.$$

$$\det(\lambda I - A) = 0$$



Example

Find the eigenvalues of the upper triangular matrix

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ 0 & a_{22} & a_{23} & a_{24} \\ 0 & 0 & a_{33} & a_{34} \\ 0 & 0 & 0 & a_{44} \end{bmatrix}.$$

Eigenvalues and Eigenvectors

$$\det(\lambda I - A) = 0$$



Recall that the determinant of a triangular matrix is the product of the entries on the main diagonal. Hence

$$0 = \det(\lambda I - A) = \begin{vmatrix} \lambda - a_{11} & -a_{12} & -a_{13} & -a_{14} \\ 0 & \lambda - a_{22} & -a_{23} & -a_{24} \\ 0 & 0 & \lambda - a_{33} & -a_{34} \\ 0 & 0 & 0 & \lambda - a_{44} \end{vmatrix}$$
$$= (\lambda - a_{11})(\lambda - a_{22})(\lambda - a_{33})(\lambda - a_{44}).$$

$$\det(\lambda I - A) = 0$$



Recall that the determinant of a triangular matrix is the product of the entries on the main diagonal. Hence

$$0 = \det(\lambda I - A) = \begin{vmatrix} \lambda - a_{11} & -a_{12} & -a_{13} & -a_{14} \\ 0 & \lambda - a_{22} & -a_{23} & -a_{24} \\ 0 & 0 & \lambda - a_{33} & -a_{34} \\ 0 & 0 & 0 & \lambda - a_{44} \end{vmatrix}$$

$$= (\lambda - a_{11})(\lambda - a_{22})(\lambda - a_{33})(\lambda - a_{44}).$$

Therefore the eigenvalues of A are

$$\lambda = a_{11}, \quad \lambda = a_{22}, \quad \lambda = a_{33}, \quad \lambda = a_{44}$$

which are the entries on the main diagonal.

Theorem

If A is an $n \times n$ triangular matrix (upper triangular, lower triangular, or diagonal), then the eigenvalues of A are the entries on the main diagonal of A .

Eigenvalues and Eigenvectors



Example

The eigenvalues of

$$A = \begin{bmatrix} \frac{1}{2} & 0 & 0 \\ -1 & \frac{2}{3} & 0 \\ 5 & -8 & -\frac{1}{4} \end{bmatrix}$$

are $\lambda = \frac{1}{2}$, $\lambda = \frac{2}{3}$ and $\lambda = -\frac{1}{4}$.

Eigenvalues and Eigenvectors



Theorem

If A is an $n \times n$ matrix, the following statements are equivalent.

- 1 λ is an eigenvalue of A .
- 2 λ is a solution of the characteristic equation
$$\det(\lambda I - A) = 0.$$
- 3 The system of equations $(\lambda I - A)\mathbf{x} = 0$ has nontrivial solutions.
- 4 There is a nonzero vector \mathbf{x} such that $A\mathbf{x} = \lambda\mathbf{x}$.

$$\det(\lambda I - A) = 0 \quad (\lambda I - A)\mathbf{x} = \mathbf{0}$$



Finding Eigenvectors

To find eigenvectors we use the formula

$$(\lambda I - A)\mathbf{x} = \mathbf{0}.$$

$$\det(\lambda I - A) = 0$$

number zero

$$(\lambda I - A)\mathbf{x} = \mathbf{0}$$

zero vector

Finding Eigenvectors

To find eigenvectors we use the formula

$$(\lambda I - A)\mathbf{x} = \mathbf{0}.$$

$$\det(\lambda I - A) = 0 \quad (\lambda I - A)\mathbf{x} = \mathbf{0}$$



Example

The matrix

$$A = \begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix}$$

has eigenvalues $\lambda = -4$ and $\lambda = 7$. Find the corresponding eigenvectors.

$$\det(\lambda I - A) = 0 \quad (\lambda I - A)\mathbf{x} = \mathbf{0}$$



$$A = \begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix}$$

First let us look at $\lambda = -4$.

$$\det(\lambda I - A) = 0 \quad (\lambda I - A)\mathbf{x} = \mathbf{0}$$



$$A = \begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix}$$

First let us look at $\lambda = -4$. Then we have

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix} = (\lambda I - A)\mathbf{x} = \left(\begin{bmatrix} -4 & 0 \\ 0 & -4 \end{bmatrix} - \begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix} \right) \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -5 & -6 \\ -5 & -6 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

$$\det(\lambda I - A) = 0 \quad (\lambda I - A)\mathbf{x} = \mathbf{0}$$



$$A = \begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix}$$

First let us look at $\lambda = -4$. Then we have

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix} = (\lambda I - A)\mathbf{x} = \left(\begin{bmatrix} -4 & 0 \\ 0 & -4 \end{bmatrix} - \begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix} \right) \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -5 & -6 \\ -5 & -6 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

which implies that

$$0 = -5x - 6y.$$

$$\det(\lambda I - A) = 0 \quad (\lambda I - A)\mathbf{x} = \mathbf{0}$$



$$A = \begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix}$$

First let us look at $\lambda = -4$. Then we have

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix} = (\lambda I - A)\mathbf{x} = \left(\begin{bmatrix} -4 & 0 \\ 0 & -4 \end{bmatrix} - \begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix} \right) \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -5 & -6 \\ -5 & -6 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

which implies that

$$0 = -5x - 6y.$$

We can choose any x and y (apart from $x = y = 0$) which satisfy this equation. For example we can choose

$$\mathbf{x} = \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 6 \\ -5 \end{bmatrix}.$$

This is an eigenvector of A corresponding to $\lambda = -4$.

$$\det(\lambda I - A) = 0 \quad (\lambda I - A)\mathbf{x} = \mathbf{0}$$



$$A = \begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix}$$

Next we consider $\lambda = 7$. We calculate that

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix} = (\lambda I - A)\mathbf{x} = \begin{bmatrix} 6 & -6 \\ -5 & 5 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 6x - 6y \\ 5x - 5y \end{bmatrix}$$

$$\det(\lambda I - A) = 0 \quad (\lambda I - A)\mathbf{x} = \mathbf{0}$$



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which implies that

$$x = y.$$

$$\det(\lambda I - A) = 0 \quad (\lambda I - A)\mathbf{x} = \mathbf{0}$$

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which implies that

$$x = y.$$

I choose

$$\mathbf{x} = \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

This is an eigenvector of A corresponding to $\lambda = 7$.

$$\det(\lambda I - A) = 0 \quad (\lambda I - A)\mathbf{x} = \mathbf{0}$$



Eigenspaces

Let A be a square matrix and let λ be an eigenvalue of A .

Definition

The *eigenspace* of A corresponding to λ is

$$\begin{aligned}\text{Nul}(\lambda I - A) &= \{\text{all solutions of } (\lambda I - A)\mathbf{x} = \mathbf{0}\} \\ &= \{\text{all eigenvectors corresponding to } \lambda\} \cup \{\mathbf{0}\}.\end{aligned}$$

$$\det(\lambda I - A) = 0 \quad (\lambda I - A)\mathbf{x} = \mathbf{0}$$



Example

Find bases for the eigenspaces of

$$A = \begin{bmatrix} -1 & 3 \\ 2 & 0 \end{bmatrix}.$$

First we need to find the eigenvalues.

$$\det(\lambda I - A) = 0 \quad (\lambda I - A)\mathbf{x} = \mathbf{0}$$



Example

Find bases for the eigenspaces of

$$A = \begin{bmatrix} -1 & 3 \\ 2 & 0 \end{bmatrix}.$$

First we need to find the eigenvalues. Since

$$0 = \det(\lambda I - A) = \begin{vmatrix} \lambda + 1 & -3 \\ -2 & \lambda \end{vmatrix} = \lambda(\lambda + 1) - 6 = (\lambda - 2)(\lambda + 3),$$

the eigenvalues of A are $\lambda = 2$ and $\lambda = -3$. This means that there are two eigenspaces of A , one for each eigenvalue.

$$\det(\lambda I - A) = 0 \quad (\lambda I - A)\mathbf{x} = \mathbf{0}$$



We have

$$\mathbf{0} = (\lambda I - A)\mathbf{x} = \begin{bmatrix} \lambda + 1 & -3 \\ -2 & \lambda \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}.$$

$$\det(\lambda I - A) = 0 \quad (\lambda I - A)\mathbf{x} = \mathbf{0}$$



We have

$$\mathbf{0} = (\lambda I - A)\mathbf{x} = \begin{bmatrix} \lambda + 1 & -3 \\ -2 & \lambda \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}.$$

Consider first $\lambda = 2$. We have

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 3 & -3 \\ -2 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3x - 3y \\ -2x + 2y \end{bmatrix} \implies x = y.$$

So every eigenvector corresponding to $\lambda = 2$ can be written as

$$\mathbf{x} = t \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

for some $t \neq 0$.

$$\det(\lambda I - A) = 0 \quad (\lambda I - A)\mathbf{x} = \mathbf{0}$$



We have

$$\mathbf{0} = (\lambda I - A)\mathbf{x} = \begin{bmatrix} \lambda + 1 & -3 \\ -2 & \lambda \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}.$$

Consider first $\lambda = 2$. We have

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 3 & -3 \\ -2 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3x - 3y \\ -2x + 2y \end{bmatrix} \implies x = y.$$

So every eigenvector corresponding to $\lambda = 2$ can be written as

$$\mathbf{x} = t \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

for some $t \neq 0$. It follows that

$$\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

is a basis for the eigenspace corresponding to $\lambda = 2$.

$$\det(\lambda I - A) = 0 \quad (\lambda I - A)\mathbf{x} = \mathbf{0}$$



If $\lambda = -3$ then

$$\mathbf{0} = (\lambda I - A)\mathbf{x} = \begin{bmatrix} \lambda + 1 & -3 \\ -2 & \lambda \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

becomes

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} -2 & -3 \\ -2 & -3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -2x - 3y \\ -2x - 3y \end{bmatrix} \implies 2x = -3y.$$

This means that every eigenvector corresponding to $\lambda = -3$ can be written as

$$\mathbf{x} = s \begin{bmatrix} -3 \\ 2 \end{bmatrix}$$

for some $s \neq 0$ and that

$$\begin{bmatrix} -3 \\ 2 \end{bmatrix}$$

is a basis for the eigenspace corresponding to $\lambda = -3$.

Eigenvalues and Eigenvectors



Example

Find bases for the eigenspaces of

$$A = \begin{bmatrix} 0 & 0 & -2 \\ 1 & 2 & 1 \\ 1 & 0 & 3 \end{bmatrix}.$$

Please check that the characteristic equation of A is

$$0 = \lambda^3 - 5\lambda^2 + 8\lambda - 4 = (\lambda - 1)(\lambda - 2)^2.$$

The eigenvalues of A are $\lambda = 1$ and $\lambda = 2$. So there are two eigenspaces.

Eigenvalues and Eigenvectors

Now $\mathbf{x} \neq 0$ is an eigenvector of A iff

$$(\lambda I - A)\mathbf{x} = \mathbf{0}$$

$$\begin{bmatrix} \lambda & 0 & 2 \\ -1 & \lambda - 2 & -1 \\ -1 & 0 & \lambda - 3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

Eigenvalues and Eigenvectors

Now $\mathbf{x} \neq 0$ is an eigenvector of A iff

$$(\lambda I - A)\mathbf{x} = \mathbf{0}$$

$$\begin{bmatrix} \lambda & 0 & 2 \\ -1 & \lambda - 2 & -1 \\ -1 & 0 & \lambda - 3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

Consider the case where $\lambda = 2$. Then we have

$$\begin{bmatrix} 2 & 0 & 2 \\ -1 & 0 & -1 \\ -1 & 0 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

Eigenvalues and Eigenvectors

Now $\mathbf{x} \neq 0$ is an eigenvector of A iff

$$(\lambda I - A)\mathbf{x} = \mathbf{0}$$

$$\begin{bmatrix} \lambda & 0 & 2 \\ -1 & \lambda - 2 & -1 \\ -1 & 0 & \lambda - 3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

Consider the case where $\lambda = 2$. Then we have

$$\begin{bmatrix} 2 & 0 & 2 \\ -1 & 0 & -1 \\ -1 & 0 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

The augmented matrix is

$$\left[\begin{array}{cccc} 2 & 0 & 2 & 0 \\ -1 & 0 & -1 & 0 \\ -1 & 0 & -1 & 0 \end{array} \right] \sim \left[\begin{array}{cccc} 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right] \implies \begin{cases} x = -z \\ y \text{ is free} \\ z \text{ is free.} \end{cases}$$

Eigenvalues and Eigenvectors



$$x = -z$$

This means that the eigenvectors of A corresponding to $\lambda = 2$ are the nonzero vectors of the form

$$\mathbf{x} = z \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} + y \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}.$$

Eigenvalues and Eigenvectors



$$x = -z$$

This means that the eigenvectors of A corresponding to $\lambda = 2$ are the nonzero vectors of the form

$$\mathbf{x} = z \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} + y \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}.$$

Hence

$$\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

form a basis for the eigenspace corresponding to $\lambda = 2$. (This eigenspace is 2 dimensional.)

Eigenvalues and Eigenvectors

If $\lambda = 1$ then

$$\begin{bmatrix} \lambda & 0 & 2 \\ -1 & \lambda - 2 & -1 \\ -1 & 0 & \lambda - 3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

becomes

$$\begin{bmatrix} 1 & 0 & 2 \\ -1 & -1 & -1 \\ -1 & 0 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

which has solution

$$\mathbf{x} = t \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}$$

(please check).

Eigenvalues and Eigenvectors



Hence

$$\begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}$$

is a basis for the eigenspace corresponding to $\lambda = 1$. (This eigenspace is 1 dimensional.)

Eigenvalues and Invertibility

Theorem

A square matrix A is invertible if and only if $\lambda = 0$ is not an eigenvalue of A .

Eigenvalues and Invertibility

Theorem

A square matrix A is invertible if and only if $\lambda = 0$ is not an eigenvalue of A .

Proof.

Let A be an $n \times n$ matrix. Note that $\lambda = 0$ is a solution of the characteristic equation

$$\lambda^n + c_1\lambda^{n-1} + \dots + c_{n-1}\lambda + c_n = 0$$

if and only if $c_n = 0$.

Eigenvalues and Invertibility

Theorem

A square matrix A is invertible if and only if $\lambda = 0$ is not an eigenvalue of A .

Proof.

Let A be an $n \times n$ matrix. Note that $\lambda = 0$ is a solution of the characteristic equation

$$\lambda^n + c_1\lambda^{n-1} + \dots + c_{n-1}\lambda + \textcolor{red}{c_n} = 0$$

if and only if $\textcolor{red}{c_n} = 0$. So we want to prove that

$$A \text{ is invertible} \iff c_n \neq 0.$$

Eigenvalues and Eigenvectors



Proof continued.

But note that if we set $\lambda = 0$ in

$$\det(\lambda I - A) = \lambda^n + c_1\lambda^{n-1} + \dots + c_{n-1}\lambda + c_n$$

then we obtain

$$c_n = \det(-A) = (-1)^n \det(A).$$

Eigenvalues and Eigenvectors



Proof continued.

But note that if we set $\lambda = 0$ in

$$\det(\lambda I - A) = \lambda^n + c_1\lambda^{n-1} + \dots + c_{n-1}\lambda + c_n$$

then we obtain

$$c_n = \det(-A) = (-1)^n \det(A).$$

So clearly $\det(A) \neq 0$ if and only if $c_n \neq 0$. □

Eigenvalues and Eigenvectors



Example

The matrix

$$A = \begin{bmatrix} 0 & 0 & -2 \\ 1 & 2 & 1 \\ 1 & 0 & 3 \end{bmatrix}$$

has eigenvalues $\lambda = 1 \neq 0$ and $\lambda = 2 \neq 0$. Therefore A must be invertible.

I leave it for you to check that $\det(A) \neq 0$.

Eigenvalues and Eigenvectors



Theorem

Let A be a square matrix. Then $\text{Nul}(A) = \{\mathbf{0}\}$ if and only if $\lambda = 0$ is not an eigenvalue of A .

Eigenvalues and Eigenvectors



Theorem

Let A be a square matrix. Then $\text{Nul}(A) = \{\mathbf{0}\}$ if and only if $\lambda = 0$ is not an eigenvalue of A .

Proof.

By the previous theorem, $\lambda = 0$ is not an eigenvalue of A iff A is invertible.

But we know by the Invertible Matrix Theorem (lecture 3) that A is invertible iff $A\mathbf{x} = \mathbf{0}$ has only the trivial solution. □

Eigenvalues and Eigenvectors



Theorem

Let A be a square matrix. Then $\text{Nul}(A) = \{\mathbf{0}\}$ if and only if $\lambda = 0$ is not an eigenvalue of A .

Proof.

By the previous theorem, $\lambda = 0$ is not an eigenvalue of A iff A is invertible.

But we know by the Invertible Matrix Theorem (lecture 3) that A is invertible iff $A\mathbf{x} = \mathbf{0}$ has only the trivial solution. □

Example

The matrix $A = \begin{bmatrix} 0 & 0 & -2 \\ 1 & 2 & 1 \\ 1 & 0 & 3 \end{bmatrix}$ has null space $\text{Nul}(A) = \{\mathbf{0}\}$

because the eigenvalues of A are $\lambda = 1 \neq 0$ and $\lambda = 2 \neq 0$.

Eigenvalues of Linear Operators

These ideas extend to vector spaces.

Definition

If $T : V \rightarrow V$ is a linear operator on a vector space V , then a nonzero vector \mathbf{x} in V is called an *eigenvector* of T if $T(\mathbf{x})$ is a scalar multiple of \mathbf{x} ; that is,

$$T(x) = \lambda x$$

for some scalar λ . The scalar λ is called an eigenvalue of T , and \mathbf{x} is said to be an eigenvector corresponding to λ .

Similarity (again)

Recall that two square matrices A and B are *similar* iff there exists an invertible matrix P such that $B = P^{-1}AP$.

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Theorem

If A and B are similar, then they have the same characteristic polynomial

Similarity (again)

Recall that two square matrices A and B are *similar* iff there exists an invertible matrix P such that $B = P^{-1}AP$.

Theorem

If A and B are similar, then they have the same characteristic polynomial and hence the same eigenvalues (with the same multiplicities).

Eigenvalues and Eigenvectors



Proof.

Suppose that $B = P^{-1}AP$. Then

$$P^{-1}(A - \lambda I)P = P^{-1}AP - \lambda P^{-1}P = P^{-1}AP - \lambda I = B - \lambda I.$$

Eigenvalues and Eigenvectors



Proof.

Suppose that $B = P^{-1}AP$. Then

$$P^{-1}(A - \lambda I)P = P^{-1}AP - \lambda P^{-1}P = P^{-1}AP - \lambda I = B - \lambda I.$$

It follows that

$$\begin{aligned}\det(B - \lambda I) &= \det(P^{-1}) \det(A - \lambda I) \det(P) \\ &= \frac{1}{\det(P)} \det(A - \lambda I) \det(P) = \det(A - \lambda I).\end{aligned}$$



Eigenvalues and Eigenvectors

Remark

$(\begin{array}{l} A \text{ and } B \\ \text{are similar} \end{array}) \quad \not\iff \quad (A \text{ and } B \text{ have the} \\ \text{same eigenvalues})$

For example, the matrices $\begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}$ and $\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$ have the same eigenvalues but are not similar.

Remark

Similarity is not the same as row equivalence. Row operations can change a matrix's eigenvalues.



Diagonalisation

Definition

A square matrix A is said to be *diagonalisable* if and only if it is similar to some diagonal matrix; that is, if there exists an invertible matrix P such that $P^{-1}AP$ is diagonal.

In this case, we say that the matrix P *diagonalises* A .

Diagonalisation



Theorem

If A is an $n \times n$ matrix, then

*A is
diagonalisable*

\iff

*A has n linearly
independent
eigenvectors*

Diagonalisation



Theorem

If A is an $n \times n$ matrix, then

A is
diagonalisable

\iff

A has n linearly
independent
eigenvectors



A has n distinct
eigenvalues.

How to Diagonalise a Matrix

Let A be an $n \times n$ matrix.

- 1 Check if A is diagonalisable by searching for n linearly independent eigenvectors. Call these $\mathbf{p}_1, \mathbf{p}_2, \dots, \mathbf{p}_n$.

How to Diagonalise a Matrix

Let A be an $n \times n$ matrix.

- 1 Check if A is diagonalisable by searching for n linearly independent eigenvectors. Call these $\mathbf{p}_1, \mathbf{p}_2, \dots, \mathbf{p}_n$.

- 2 Let $P = \begin{bmatrix} \mathbf{p}_1 & \mathbf{p}_2 & \cdots & \mathbf{p}_n \end{bmatrix}$.

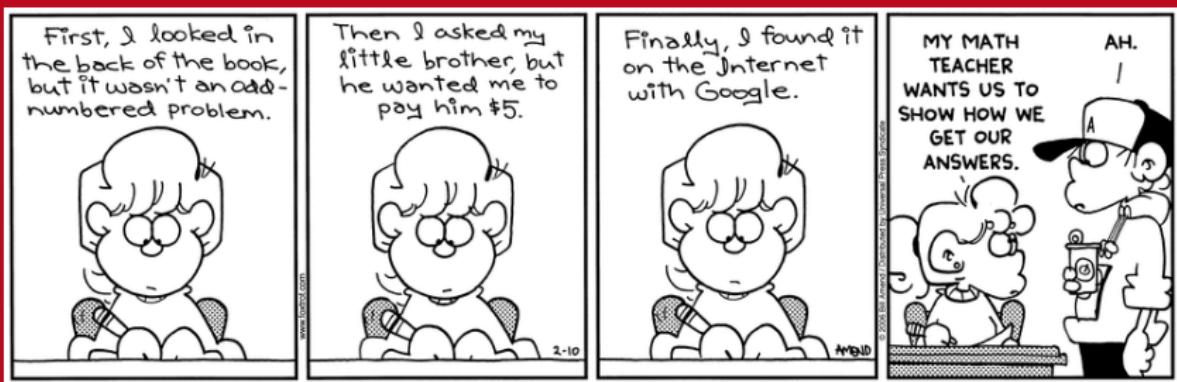
How to Diagonalise a Matrix

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- 2 Let $P = \begin{bmatrix} \mathbf{p}_1 & \mathbf{p}_2 & \cdots & \mathbf{p}_n \end{bmatrix}$.
- 3 Calculate $P^{-1}AP$.

Break

We will continue at 3pm



How to Diagonalise a Matrix

Let A be an $n \times n$ matrix.

- 1 Check if A is diagonalisable by searching for n linearly independent eigenvectors. Call these $\mathbf{p}_1, \mathbf{p}_2, \dots, \mathbf{p}_n$.
- 2 Let $P = \begin{bmatrix} \mathbf{p}_1 & \mathbf{p}_2 & \cdots & \mathbf{p}_n \end{bmatrix}$.
- 3 Calculate $P^{-1}AP$.

Diagonalisation



Example

Find a matrix P which diagonalises $A = \begin{bmatrix} 0 & 0 & -2 \\ 1 & 2 & 1 \\ 1 & 0 & 3 \end{bmatrix}$.

Earlier today we found that the eigenvalues of A are $\lambda = 2$ and $\lambda = 1$. We also found that

$$\mathbf{p}_1 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \quad \text{and} \quad \mathbf{p}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

form a basis for the eigenspace corresponding to $\lambda = 2$; and

$$\mathbf{p}_3 = \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}$$

forms a basis for the eigenspace corresponding to $\lambda = 1$.

Diagonalisation



These three eigenvectors are linearly independent, so the matrix

$$P = \begin{bmatrix} \mathbf{p}_1 & \mathbf{p}_2 & \mathbf{p}_3 \end{bmatrix} = \begin{bmatrix} -1 & 0 & -2 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}$$

diagonalises A .

Diagonalisation



These three eigenvectors are linearly independent, so the matrix

$$P = \begin{bmatrix} \mathbf{p}_1 & \mathbf{p}_2 & \mathbf{p}_3 \end{bmatrix} = \begin{bmatrix} -1 & 0 & -2 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}$$

diagonalises A .

To check this, we can calculate (please check) that

$$P^{-1}AP = \begin{bmatrix} 1 & 0 & 2 \\ 1 & 1 & 1 \\ -1 & 0 & -1 \end{bmatrix} \begin{bmatrix} 0 & 0 & -2 \\ 1 & 2 & 1 \\ 1 & 0 & 3 \end{bmatrix} \begin{bmatrix} -1 & 0 & -2 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Diagonalisation

Remark

Note that the entries on the **main diagonal** of

$$D = P^{-1}AP = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

are the eigenvalues of A .

Diagonalisation

Remark

Note that the entries on the **main diagonal** of

$$D = P^{-1}AP = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

are the eigenvalues of A .

Remark

When we wrote P , we used the two eigenvectors corresponding to $\lambda = 2$ in the first two columns, and the eigenvector corresponding to $\lambda = 1$ in the third column. This is why we get **2, 2** first, then **1**, on the main diagonal of D .

Diagonalisation

Remark

$$P = \begin{bmatrix} -1 & 0 & -2 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}$$

is not the only matrix which diagonalises A .

Diagonalisation



Remark

$$P = \begin{bmatrix} -1 & \textcolor{green}{0} & \textcolor{brown}{-2} \\ 0 & \textcolor{green}{1} & \textcolor{brown}{1} \\ 1 & \textcolor{green}{0} & \textcolor{brown}{1} \end{bmatrix}$$

A red curved arrow points from the value 0 in the first row, second column to the value -2 in the first row, third column.

is not the only matrix which diagonalises A . If we swap the second and third columns of P to obtain

$$Q = \begin{bmatrix} -1 & \textcolor{brown}{-2} & \textcolor{green}{0} \\ 0 & \textcolor{brown}{1} & \textcolor{green}{1} \\ 1 & \textcolor{brown}{1} & \textcolor{green}{0} \end{bmatrix}$$

Diagonalisation

Remark

$$P = \begin{bmatrix} -1 & 0 & -2 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}$$

is not the only matrix which diagonalises A . If we swap the second and third columns of P to obtain

$$Q = \begin{bmatrix} -1 & -2 & 0 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

then we get

$$D = Q^{-1}AQ = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}.$$

Diagonalisation



Example

Show that $A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 2 & 0 \\ -3 & 5 & 2 \end{bmatrix}$ is not diagonalisable.

Diagonalisation



Example

Show that $A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 2 & 0 \\ -3 & 5 & 2 \end{bmatrix}$ is not diagonalisable.

The characteristic equation of A is

$$0 = \det(\lambda I - A) = \begin{vmatrix} \lambda - 1 & 0 & 0 \\ -1 & \lambda - 2 & 0 \\ 3 & -5 & \lambda - 2 \end{vmatrix} = (\lambda - 1)(\lambda - 2)^2.$$

So the eigenvalues of A are $\lambda = 1$ and $\lambda = 2$.

Diagonalisation



Example

Show that $A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 2 & 0 \\ -3 & 5 & 2 \end{bmatrix}$ is not diagonalisable.

The characteristic equation of A is

$$0 = \det(\lambda I - A) = \begin{vmatrix} \lambda - 1 & 0 & 0 \\ -1 & \lambda - 2 & 0 \\ 3 & -5 & \lambda - 2 \end{vmatrix} = (\lambda - 1)(\lambda - 2)^2.$$

So the eigenvalues of A are $\lambda = 1$ and $\lambda = 2$.

Diagonalisation



Example

Show that $A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 2 & 0 \\ -3 & 5 & 2 \end{bmatrix}$ is not diagonalisable.

Since A is a triangular matrix, we can see straight away that the eigenvalues of A are $\lambda = 1$ and $\lambda = 2$.

Diagonalisation



I leave it for you to check that bases for the eigenspaces of A are

$$\lambda = 1 : \mathbf{p}_1 = \begin{bmatrix} 1 \\ -1 \\ 8 \end{bmatrix}; \quad \lambda = 2 : \mathbf{p}_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

Since A is a 3×3 matrix but with only two linearly independent eigenvectors, A is not diagonalisable.

Diagonalisation



Example

Is $A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 4 & -17 & 8 \end{bmatrix}$ diagonalisable?

Diagonalisation



Example

Is $A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 4 & -17 & 8 \end{bmatrix}$ diagonalisable?

We saw earlier that the eigenvalues of this matrix are $\lambda = 4$, $\lambda = 2 + \sqrt{3}$ and $\lambda = 2 - \sqrt{3}$. Since A has 3 distinct eigenvalues, A is diagonalisable.

► EXAMPLE 4 Diagonalizability of Triangular Matrices

From Theorem 5.1.2, the eigenvalues of a triangular matrix are the entries on its main diagonal. Thus, a triangular matrix with distinct entries on the main diagonal is diagonalizable. For example,

$$A = \begin{bmatrix} -1 & 2 & 4 & 0 \\ 0 & 3 & 1 & 7 \\ 0 & 0 & 5 & 8 \\ 0 & 0 & 0 & -2 \end{bmatrix}$$

is a diagonalizable matrix with eigenvalues $\lambda_1 = -1$, $\lambda_2 = 3$, $\lambda_3 = 5$, $\lambda_4 = -2$. ◀

Eigenvalues of Powers of a Matrix

Suppose that we know the eigenvalues and eigenvectors of A .
What are the eigenvalues and eigenvectors of A^k ?

Eigenvalues of Powers of a Matrix

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Eigenvalues of Powers of a Matrix

Suppose that we know the eigenvalues and eigenvectors of A .
What are the eigenvalues and eigenvectors of A^k ?

Suppose that

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Then

$$A^2\mathbf{x} = A(A\mathbf{x}) = A(\lambda\mathbf{x}) =$$

Eigenvalues of Powers of a Matrix

Suppose that we know the eigenvalues and eigenvectors of A .
What are the eigenvalues and eigenvectors of A^k ?

Suppose that

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Eigenvalues of Powers of a Matrix

Suppose that we know the eigenvalues and eigenvectors of A .
What are the eigenvalues and eigenvectors of A^k ?

Suppose that

$$A\mathbf{x} = \lambda\mathbf{x}.$$

Then

$$A^2\mathbf{x} = A(A\mathbf{x}) = A(\lambda\mathbf{x}) = \lambda(A\mathbf{x}) = \lambda(\lambda\mathbf{x}) = \lambda^2\mathbf{x}.$$

So λ^2 is an eigenvalue of A^2 with corresponding eigenvector \mathbf{x} .

Theorem

Let $k \in \mathbb{N}$. If λ is an eigenvalue of A , and \mathbf{x} is a corresponding eigenvector, then λ^k is an eigenvalue of A^k and \mathbf{x} is a corresponding eigenvector.

Diagonalisation



Example

The matrix $A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 2 & 0 \\ -3 & 5 & 2 \end{bmatrix}$ has eigenvalues $\lambda = 1$ and $\lambda = 2$,

and corresponding eigenvectors $\mathbf{p}_1 = \begin{bmatrix} 1 \\ -1 \\ 8 \end{bmatrix}$ and $\mathbf{p}_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$.

Find the eigenvalues and eigenvectors of A^7 .

Diagonalisation



Example

The matrix $A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 2 & 0 \\ -3 & 5 & 2 \end{bmatrix}$ has eigenvalues $\lambda = 1$ and $\lambda = 2$,

and corresponding eigenvectors $\mathbf{p}_1 = \begin{bmatrix} 1 \\ -1 \\ 8 \end{bmatrix}$ and $\mathbf{p}_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$.

Find the eigenvalues and eigenvectors of A^7 .

The eigenvalues of A^7 are $\lambda = 1^7 = 1$ and $\lambda = 2^7 = 128$.

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The eigenvalues of A^7 are $\lambda = 1^7 = 1$ and $\lambda = 2^7 = 128$. The eigenvectors \mathbf{p}_1 and \mathbf{p}_2 of A are also eigenvectors of A^7 , corresponding to the eigenvalues $\lambda = 1$ and $\lambda = 128$ respectively.

Computing Powers of a Matrix

Diagonalising a matrix A makes calculating A^k much easier.

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Suppose that A is a diagonalisable matrix. Then

$$P^{-1}AP = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix} = D.$$

Computing Powers of a Matrix

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It follows that

$$(P^{-1}AP)^2 = \begin{bmatrix} \lambda_1^2 & 0 & \cdots & 0 \\ 0 & \lambda_2^2 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & \lambda_n^2 \end{bmatrix} = D^2.$$

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Indeed

$$P^{-1}A^kP = D^k = \begin{bmatrix} \lambda_1^k & 0 & \cdots & 0 \\ 0 & \lambda_2^k & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & \lambda_n^k \end{bmatrix}$$

for any $k \in \mathbb{N}$.

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for any $k \in \mathbb{N}$. We can rearrange this to

Theorem

$$A^k = PD^kP^{-1}.$$

$$A^k = P D^k P^{-1}.$$



Example

Find A^{13} if $A = \begin{bmatrix} 0 & 0 & -2 \\ 1 & 2 & 1 \\ 1 & 0 & 3 \end{bmatrix}$.

We showed earlier that A is diagonalised by

$$P = \begin{bmatrix} -1 & 0 & -2 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix} \quad \text{and that} \quad D = P^{-1}AP = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

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It follows that

$$A^{13} = PD^{13}P^{-1} = \begin{bmatrix} -1 & 0 & -2 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2^{13} & 0 & 0 \\ 0 & 2^{13} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 2 \\ 1 & 1 & 1 \\ -1 & 0 & -1 \end{bmatrix}$$

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It follows that

$$A^{13} = PD^{13}P^{-1} = \begin{bmatrix} -8190 & 0 & -16382 \\ 8191 & 8192 & 8191 \\ 8191 & 0 & 16383 \end{bmatrix}.$$

Diagonalisation

Remark

$$A \text{ is diagonalisable} \iff A \text{ has } n \text{ distinct eigenvalues.}$$

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Example

Consider

$$I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{and} \quad J = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}.$$

Both I and J only have one eigenvalue $\lambda = 1$.

Diagonalisation

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Both I and J only have one eigenvalue $\lambda = 1$. However I has 3 linearly independent eigenvectors whereas J has only one (check!). Therefore I is diagonalisable but J is not.



Complex Vector Spaces

Complex Eigenvalues

Example

Find the eigenvalues of $A = \begin{bmatrix} -2 & -1 \\ 5 & 2 \end{bmatrix}$.

Complex Eigenvalues

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$$0 = \det(\lambda I - A) = \begin{vmatrix} \lambda + 2 & 1 \\ -5 & \lambda - 2 \end{vmatrix} = (\lambda + 2)(\lambda - 2) + 5 = \lambda^2 + 1.$$

The eigenvalues of A are $\lambda = i$ and $\lambda = -i$.

Vectors in \mathbb{C}^n

Definition

If n is a positive integer, then a *complex n-tuple* is a sequence of n complex numbers

$$(v_1, v_2, \dots, v_n).$$

The set of all complex n-tuples is called *complex n-space* and is denoted by \mathbb{C}^n .

Scalars are complex numbers, and the operations of addition, subtraction, and scalar multiplication are performed componentwise.

Complex Vector Spaces

Example

For example

$$\mathbf{u} = (1+i, -4i, 2+3i), \quad \mathbf{v} = (0, i, 5), \quad \mathbf{w} = \left(6 - \sqrt{2}i, 9 + \frac{1}{2}i, \pi i\right)$$

are vectors in \mathbb{C}^3 .

Complex Vector Spaces

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$$\mathbf{u} = (1+i, -4i, 2+3i), \quad \mathbf{v} = (0, i, 5), \quad \mathbf{w} = \left(6 - \sqrt{2}i, 9 + \frac{1}{2}i, \pi i\right)$$

are vectors in \mathbb{C}^3 .

Every vector

$$\mathbf{v} = (v_1, v_2, \dots, v_n) = (a_1 + b_1i, a_2 + b_2i, \dots, a_n + b_ni)$$

can be split into *real* and *imaginary parts* as

$$\begin{aligned}\mathbf{v} &= (a_1, a_2, \dots, a_n) + (b_1, b_2, \dots, b_n)i \\ &= \text{Re}(\mathbf{v}) + \text{Im}(\mathbf{v})i.\end{aligned}$$

Complex Vector Spaces



The vector

$$\bar{\mathbf{v}} = (\bar{v}_1, \bar{v}_2, \dots, \bar{v}_n) = (a_1 - b_1 i, a_2 - b_2 i, \dots, a_n - b_n i)$$

is called the *complex conjugate* of \mathbf{v}

Complex Vector Spaces



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$$\bar{\mathbf{v}} = \text{Re}(\mathbf{v}) - \text{Im}(\mathbf{v})i.$$

Complex Vector Spaces



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is called the *complex conjugate* of \mathbf{v} and can be written as

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Remark

We can think of \mathbb{R}^n as a subset (not a subspace) of \mathbb{C}^n .

$$\mathbf{v} \in \mathbb{R}^n \iff \bar{\mathbf{v}} = \mathbf{v}.$$

Complex Vector Spaces



Example

Let $\mathbf{v} = (3 + i, -2i, 5) \in \mathbb{C}^3$. Then

- $\bar{\mathbf{v}} = (3 - i, 2i, 5)$
- $\text{Re}(\mathbf{v}) = (3, 0, 5)$
- $\text{Im}(\mathbf{v}) = (1, -2, 0)$.

Matrices in $\mathbb{C}^{m \times n}$

A *complex matrix* is a matrix A whose entries are complex numbers.

$\text{Re}(A)$ and $\text{Im}(A)$ are the matrices formed from the real and imaginary parts of the entries of A .

\overline{A} is the matrix formed by taking the complex conjugate of each entry in A .

Complex Vector Spaces



Example

Let $A = \begin{bmatrix} 1+i & -i \\ 4 & 6-2i \end{bmatrix}$. Then

- $\bar{A} = \begin{bmatrix} 1-i & i \\ 4 & 6+2i \end{bmatrix}$
- $\text{Re}(A) = \begin{bmatrix} 1 & 0 \\ 4 & 6 \end{bmatrix}$
- $\text{Im}(A) = \begin{bmatrix} 1 & -1 \\ 0 & -2 \end{bmatrix}$
- $\det(A) = \begin{vmatrix} 1+i & -i \\ 4 & 6-2i \end{vmatrix} = (1+i)(6-2i) - (-i)(4) = 8 + 8i.$

Complex Vector Spaces

Theorem

If \mathbf{u} and \mathbf{v} are vectors in \mathbb{C}^n , and if k is a scalar, then:

- $\overline{\overline{\mathbf{v}}} = \mathbf{v}$
- $\overline{k\mathbf{v}} = \bar{k}\bar{\mathbf{v}}$
- $\overline{\mathbf{u} + \mathbf{v}} = \bar{\mathbf{u}} + \bar{\mathbf{v}}$

Theorem

If A is an $m \times r$ complex matrix, B is a $r \times n$ complex matrix, and k is a scalar, then:

- 1 $\overline{\overline{A}} = A$
- 2 $\overline{A^T} = \bar{A}^T$
- 3 $\overline{AB} = \bar{A}\bar{B}$
- 4 $\overline{kA} = \bar{k}\bar{A}$.



Eigenvalues and Eigenvectors of a Real Matrix

We have seen that sometimes real matrices can have complex eigenvalues.

Theorem

Suppose that

- A is a real $n \times n$ matrix, i.e. $A \in \mathbb{R}^{n \times n}$
- λ is an eigenvalue of A
- \mathbf{x} is an eigenvector of A corresponding to λ .

Complex Vector Spaces

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Proof.

Suppose that

$$A\mathbf{x} = \lambda\mathbf{x}.$$

Since A is a real matrix we have that $\bar{A} = A$.

Complex Vector Spaces

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Proof.

Suppose that

$$A\mathbf{x} = \lambda\mathbf{x}.$$

Since A is a real matrix we have that $\bar{A} = A$. It follows that

$$A\bar{\mathbf{x}} = \overline{A\mathbf{x}} = \overline{\lambda\mathbf{x}} = \bar{\lambda}\bar{\mathbf{x}} = \bar{\lambda}\bar{\mathbf{x}}.$$



Example

Find the eigenvalues and bases for the eigenspaces of

$$A = \begin{bmatrix} -2 & -1 \\ 5 & 2 \end{bmatrix}.$$

Complex Vector Spaces

Example

Find the eigenvalues and bases for the eigenspaces of

$$A = \begin{bmatrix} -2 & -1 \\ 5 & 2 \end{bmatrix}.$$

As before, we calculate that

$$0 = \det(\lambda I - A) = \begin{vmatrix} \lambda + 2 & 1 \\ -5 & \lambda - 2 \end{vmatrix} = \lambda^2 + 1 = (\lambda - i)(\lambda + i)$$

which implies that the eigenvalues are $\lambda = i$ and $\lambda = -i$. Note that i and $-i$ are complex conjugates.

Next we need to find the eigenvectors.

Complex Vector Spaces

Since

$$\mathbf{0} = (\lambda I - A)\mathbf{x} = \begin{bmatrix} \lambda + 2 & 1 \\ -5 & \lambda - 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

for the eigenvalue $\lambda = i$ we have

$$\begin{bmatrix} i + 2 & 1 \\ -5 & i - 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

Complex Vector Spaces

Since

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$$\begin{bmatrix} i + 2 & 1 \\ -5 & i - 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

The first line says

$$(i + 2)x + y = 0$$

from which (if we choose $x = 1$) we obtain

$$\mathbf{x} = \begin{bmatrix} 1 \\ -2 - i \end{bmatrix}.$$

Complex Vector Spaces



$$\begin{bmatrix} i+2 & 1 \\ -5 & i-2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

Please note that the second line in the above equation is the same as the first. Why?

Complex Vector Spaces



$$\begin{bmatrix} i+2 & 1 \\ -5 & i-2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

Please note that the second line in the above equation is the same as the first. Why? If we take the first line and multiply by $(i-2)$ then we get

$$(i+2)x + y = 0$$

Complex Vector Spaces



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$$(i-2)((i+2)x + y) = (i-2)(0)$$

Complex Vector Spaces



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Please note that the second line in the above equation is the same as the first. Why? If we take the first line and multiply by $(i-2)$ then we get

$$(i-2)((i+2)x + y) = (i-2)(0)$$
$$(i-2)(i+2)x + (i-2)y = 0$$

Complex Vector Spaces



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Please note that the second line in the above equation is the same as the first. Why? If we take the first line and multiply by $(i-2)$ then we get

$$\begin{aligned} (i-2)((i+2)x + y) &= (i-2)(0) \\ (i-2)(i+2)x + (i-2)y &= 0 \\ -5x + (i-2)y &= 0. \end{aligned}$$

Complex Vector Spaces



To find an eigenvector corresponding to $\lambda = -i$ we could use the same method.

Complex Vector Spaces



To find an eigenvector corresponding to $\lambda = -i$ we could use the same method. But of course there is an easier way: We just want the vector

$$\bar{\mathbf{x}} = \begin{bmatrix} 1 \\ -2-i \end{bmatrix} = \begin{bmatrix} \bar{1} \\ \bar{-2-i} \end{bmatrix} = \begin{bmatrix} 1 \\ -2+i \end{bmatrix}.$$

Eigenvalues of 2×2 matrices

Consider $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. The characteristic polynomial of this matrix is

$$\begin{aligned}\det(\lambda I - A) &= \begin{vmatrix} \lambda - a & -b \\ -c & \lambda - d \end{vmatrix} \\ &= (\lambda - a)(\lambda - d) - bc \\ &= \lambda^2 - (a + d)\lambda + (ad - bc) \\ &= \end{aligned}$$

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So the characteristic equation of A is

$$\boxed{\lambda^2 - \text{tr}(A)\lambda + \det(A) = 0.}$$

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Now recall that the roots of

$$ax^2 + bx + c = 0$$

are given by

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

$$\lambda^2 - \text{tr}(A)\lambda + \det(A) = 0$$



Now recall that the roots of

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are given by

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

There are three cases:

- 1** $b^2 - 4ac > 0 \implies$ there are two real roots;
- 2** $b^2 - 4ac = 0 \implies$ there is one repeated root;
- 3** $b^2 - 4ac < 0 \implies$ there are complex conjugate roots.

(We have $a = 1$, $b = -\text{tr}(A)$ and $c = \det(A)$.)

$$\lambda^2 - \text{tr}(A)\lambda + \det(A) = 0$$



Theorem

If $A \in \mathbb{R}^{2 \times 2}$, then the characteristic equation of A is

$$\lambda^2 - \text{tr}(A)\lambda + \det(A) = 0$$

and

- 1 A has two distinct real eigenvalues if $\text{tr}(A)^2 - 4\det(A) > 0$;
- 2 A has one repeated real eigenvalue if $\text{tr}(A)^2 - 4\det(A) = 0$; and
- 3 A has two complex conjugate eigenvalues if $\text{tr}(A)^2 - 4\det(A) < 0$.

$$\lambda^2 - \text{tr}(A)\lambda + \det(A) = 0$$



Example

Find the eigenvalues of $A = \begin{bmatrix} 2 & 2 \\ -1 & 5 \end{bmatrix}$.

We have $\text{tr}(A) = 7$ and $\det(A) = 12$. Since $\text{tr}(A)^2 > 4 \det(A)$ we know that we will get two distinct real eigenvalues.

$$\lambda^2 - \text{tr}(A)\lambda + \det(A) = 0$$



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$$0 = \lambda^2 - 7\lambda + 12 = (\lambda - 4)(\lambda - 3).$$

The eigenvalues of A are $\lambda = 4$ and $\lambda = 3$.

$$\lambda^2 - \text{tr}(A)\lambda + \det(A) = 0$$



Example

Find the eigenvalues of $A = \begin{bmatrix} 0 & -1 \\ 1 & 2 \end{bmatrix}$.

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Example

Find the eigenvalues of $A = \begin{bmatrix} 0 & -1 \\ 1 & 2 \end{bmatrix}$.

We have $\text{tr}(A) = 2$ and $\det(A) = 1$. This is an example with $\text{tr}(A)^2 = 4 \det(A)$.

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Example

Find the eigenvalues of $A = \begin{bmatrix} 0 & -1 \\ 1 & 2 \end{bmatrix}$.

We have $\text{tr}(A) = 2$ and $\det(A) = 1$. This is an example with $\text{tr}(A)^2 = 4 \det(A)$. So the characteristic equation is

$$0 = \lambda^2 - 2\lambda + 1 = (\lambda - 1)^2.$$

$\lambda = 1$ is the only eigenvalue of A .

$$\lambda^2 - \text{tr}(A)\lambda + \det(A) = 0$$



Example

Find the eigenvalues of $A = \begin{bmatrix} 2 & 3 \\ -3 & 2 \end{bmatrix}$.

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Example

Find the eigenvalues of $A = \begin{bmatrix} 2 & 3 \\ -3 & 2 \end{bmatrix}$.

We have $\text{tr}(A) = 4$ and $\det(A) = 13$. So $\text{tr}(A)^2 < 4 \det(A)$.

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Example

Find the eigenvalues of $A = \begin{bmatrix} 2 & 3 \\ -3 & 2 \end{bmatrix}$.

We have $\text{tr}(A) = 4$ and $\det(A) = 13$. So $\text{tr}(A)^2 < 4\det(A)$. The characteristic equation is

$$0 = \lambda^2 - 4\lambda + 13.$$

The solutions to this quadratic equation are

$$\lambda = \frac{4 \pm \sqrt{(-4)^2 - 4(13)}}{2} = \frac{4 \pm \sqrt{-36}}{2} = \frac{4 \pm 6i}{2} = 2 \pm 3i.$$

The eigenvalues of A are $\lambda = 2 + 3i$ and $\lambda = 2 - 3i$.

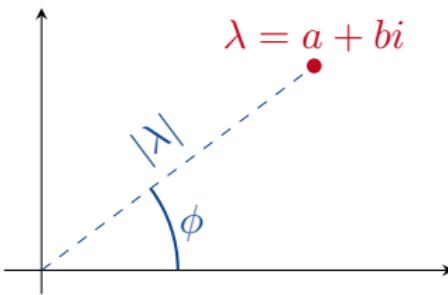
Symmetric Matrices have Real Eigenvalues

Theorem

If A is a real symmetric matrix, then A has real eigenvalues.

(proof omitted)

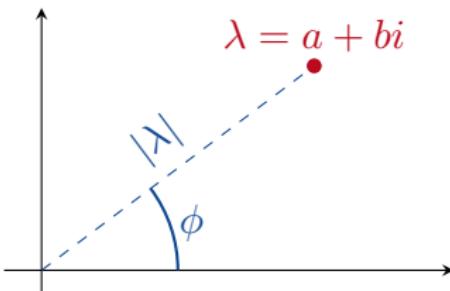
An Interpretation of Complex Eigenvalues



Theorem

The eigenvalues of the real matrix $C = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$ are $\lambda = a \pm bi$.

An Interpretation of Complex Eigenvalues



Theorem

The eigenvalues of the real matrix $C = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$ are $\lambda = a \pm bi$.
If a and b are not both zero, then this matrix can be factored as

$$\begin{bmatrix} a & -b \\ b & a \end{bmatrix} = \begin{bmatrix} |\lambda| & 0 \\ 0 & |\lambda| \end{bmatrix} \begin{bmatrix} \cos \phi & -\sin \phi \\ \sin \phi & \cos \phi \end{bmatrix}$$

where ϕ is the argument of $a + bi$.

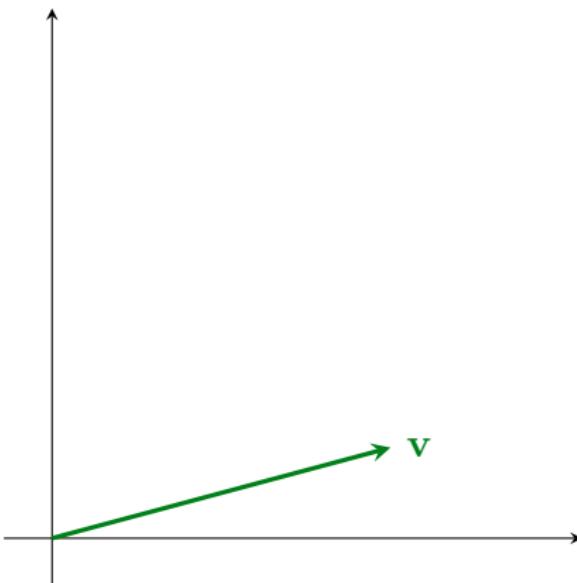
Complex Vectors

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Remark

Geometrically, this means that the matrix C can be viewed as a rotation through an angle of ϕ followed by a scaling with factor $|\lambda|$.



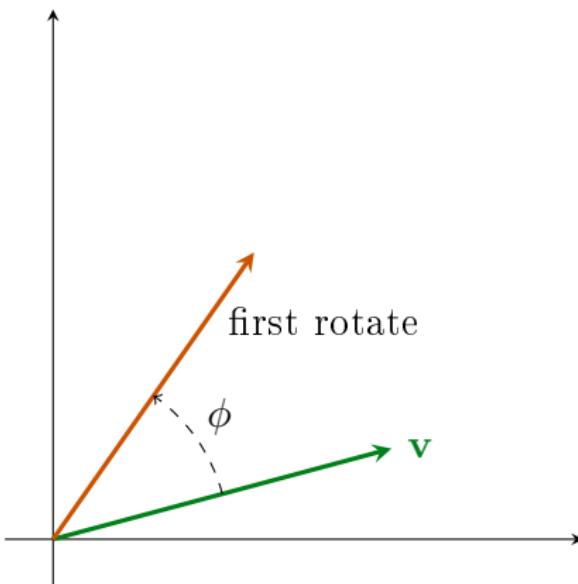
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Remark

Geometrically, this means that the matrix C can be viewed as a rotation through an angle of ϕ followed by a scaling with factor $|\lambda|$.



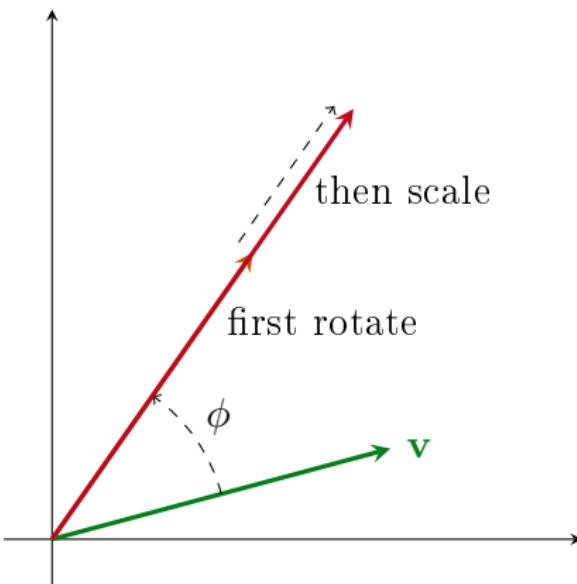
Complex Vectors

$$C = \begin{bmatrix} a & -b \\ b & a \end{bmatrix} = \begin{bmatrix} |\lambda| & 0 \\ 0 & |\lambda| \end{bmatrix} \begin{bmatrix} \cos \phi & -\sin \phi \\ \sin \phi & \cos \phi \end{bmatrix}$$



Remark

Geometrically, this means that the matrix C can be viewed as a rotation through an angle of ϕ followed by a scaling with factor $|\lambda|$.





Next Time

- Inner Product Spaces
- Orthogonality
- Orthogonal Sets and Orthonormal Sets