Lecture 5

- 3.4 Repeated Roots of the Characteristic Equation
- 3.5 Reduction of Order
- 3.6 Nonhomogeneous Equations
- 3.7 The Method of Undetermined Coefficients

Summary

To solve

$$ay'' + by' + cy = 0$$

we need to find two linearly independent solutions, $y_1(t)$ and $y_2(t)$. Then the general solution to the ODE is

$$y(t) = c_1 y_1(t) + c_2 y_2(t).$$

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$$y(t) = c_1 y_1(t) + c_2 y_2(t).$$

First we solve the characteristic equation

$$ar^2 + br + c = 0$$

and find the roots r_1 and r_2.

Summary

1. If $r_1, r_2 \in \mathbb{R}$ and $r_1 \neq r_2$, then

$$y_1(t) = e^{r_1 t} \qquad \text{and} \qquad y_2(t) = e^{r_2 t};$$

2. If $r_{1,2} = \lambda \pm i\mu$ $(\lambda, \mu \in \mathbb{R})$, then

$$y_1(t) = e^{\lambda t} \cos \mu t \qquad \text{and} \qquad y_2(t) = e^{\lambda t} \sin \mu t;$$

3. If the roots are repeated, then ??????????????

Repeated Roots of the Characteristic Equation

3.4

3.4 Repeated Roots of the Characteristic Equation

Now consider

$$ay'' + by' + cy = 0 \tag{1}$$

where $b^2 - 4ac = 0$. Then the only root of

$$ar^2 + br + c = 0$$

is

$$r = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-b \pm \sqrt{0}}{2a} = -\frac{b}{2a}.$$

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We know that $y_1(t) = e^{-\frac{bt}{2a}}$ is a solution of (1), but how do we find a linearly independent second solution?

Example

Solve $y'' + 4y' + 4y = 0$.

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The characteristic equation

$$0 = r^2 + 4r + 4 = (r + 2)^2$$

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The idea is:

- We know that $y_1(t)$ is a solution;

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The idea is:

- We know that $y_1(t)$ is a solution;
- So $cy_1(t)$ is a solution for any $c \in \mathbb{R}$;
- Maybe $v(t)y_1(t)$ is a solution for some non-constant function $v(t)$.

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and that

$$\begin{aligned} 0 &= y_2'' + 4y_2' + 4y_2 \\ &= \left(v''e^{-2t} - 4v'e^{-2t} + 4ve^{-2t}\right) + 4\left(v'e^{-2t} - 2ve^{-2t}\right) + 4\left(ve^{-2t}\right) \\ &= e^{-2t}\left[v'' - 4v' + 4v + 4v' - 8v + 4v\right] \\ &= v''e^{-2t}. \end{aligned}$$

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$$y_2(t) = te^{-2t}.$$

But are $y_1(t)$ and $y_2(t)$ linearly independent? Since

$$W\left(y_1, y_2\right)(t) = \begin{vmatrix} e^{-2t} & te^{-2t} \\ -2e^{-2t} & (1-2t)e^{-2t} \end{vmatrix} = e^{-4t} \neq 0,$$

the answer is YES.

Therefore $y_1(t) = e^{-2t}$ and $y_2(t) = te^{-2t}$ form a fundamental set of solutions and the general solution is

$$y(t) = c_1 e^{-2t} + c_2 t e^{-2t}.$$

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For the general equation $ay'' + by' + cy = 0$, we can use the same method: We have $y_1(t) = e^{rt} = e^{-\frac{bt}{2a}}$ and we guess that $y_2(t) = v(t)e^{-\frac{bt}{2a}}$ for some function $v(t)$.

3.4 Repeated Roots of the Characteristic Equation

For the general equation $ay'' + by' + cy = 0$, we can use the same method: We have $y_1(t) = e^{rt} = e^{-\frac{bt}{2a}}$ and we guess that $y_2(t) = v(t)e^{-\frac{bt}{2a}}$ for some function $v(t)$. Then we calculate (you fill in the details)

$$0 = ay_2'' + by_2' + cy_2 = \ldots = ae^{-\frac{bt}{2a}}v''.$$

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$$0 = ay_2'' + by_2' + cy_2 = \ldots = ae^{-\frac{bt}{2a}}v''.$$

So again we want $v'' = 0$ and we can choose $v(t) = t$. Thus $y_2(t) = te^{rt} = te^{-\frac{bt}{2a}}$.

I leave it for you to calculate that $W(e^{rt}, te^{rt}) \neq 0$. Thus e^{rt} and te^{rt} form a fundamental set of solutions to (1).

Example

Solve

$$\begin{cases} y'' - y' + \frac{1}{4}y = 0 \\ y(0) = 2 \\ y'(0) = \frac{1}{3}. \end{cases}$$

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has repeated root $r = \frac{1}{2}$.

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The characteristic equation

$$0 = r^2 - r + \frac{1}{4} = \left(r - \frac{1}{2}\right)^2$$

has repeated root $r = \frac{1}{2}$. So we know that the general solution to the ODE is

$$y(t) = c_1 e^{\frac{t}{2}} + c_2 t e^{\frac{t}{2}}.$$

Next we need to look at the initial conditions: Since $y'(t) = \frac{1}{2}c_1 e^{\frac{t}{2}} + c_2 e^{\frac{t}{2}} + \frac{1}{2}c_2 t e^{\frac{t}{2}}$, we have that

$$\begin{aligned} 2 = y(0) &= c_1 + 0 \qquad &\implies \qquad c_1 &= 2 \\ \frac{1}{3} = y'(0) &= \frac{1}{2}c_1 + c_2 + 0 \qquad &\implies \qquad c_2 &= -\frac{2}{3}. \end{aligned}$$

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$$\begin{aligned} 2 = y(0) = c_1 + 0 \qquad &\Longrightarrow \qquad c_1 = 2 \\ \frac{1}{3} = y'(0) = \frac{1}{2}c_1 + c_2 + 0 \qquad &\Longrightarrow \qquad c_2 = -\frac{2}{3}. \end{aligned}$$

Therefore the solution to the IVP is

$$y = 2e^{\frac{t}{2}} - \tfrac{2}{3} t e^{\frac{t}{2}}.$$

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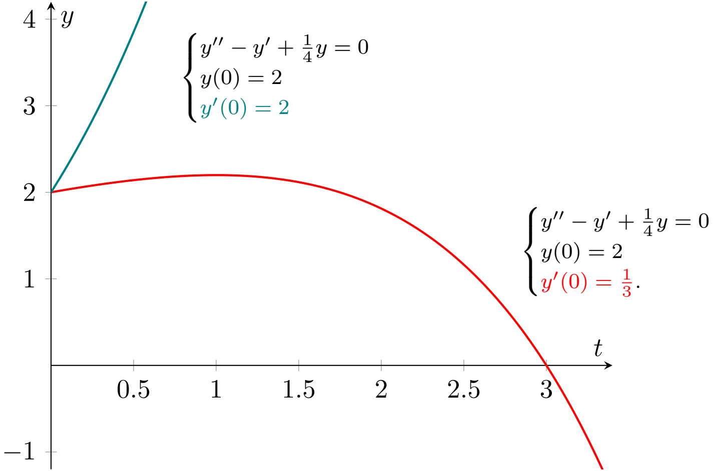
You can check that the solution is

$$y = 2e^{\frac{t}{2}} + te^{\frac{t}{2}}.$$

The graph of this solution, and the solution to the previous example, are shown on the next slide.



Note that even though these two functions share the same $y(0)$ value, and that their $y'(0)$ value does not differ by much, their behaviour as $t \to \infty$ is very different.

Summary

To solve

$$ay'' + by' + cy = 0$$

we need to find two linearly independent solutions.

1. If $r_1, r_2 \in \mathbb{R}$ and $r_1 \neq r_2$, then

$$y_1(t) = e^{r_1 t} \qquad \text{and} \qquad y_2(t) = e^{r_2 t};$$

2. If $r_{1,2} = \lambda \pm i\mu$ $(\lambda, \mu \in \mathbb{R})$, then

$$y_1(t) = e^{\lambda t} \cos \mu t \qquad \text{and} \qquad y_2(t) = e^{\lambda t} \sin \mu t;$$

3. If $r_1, r_2 \in \mathbb{R}$ but $r_1 = r_2$, then

$$y_1(t) = e^{r_1 t} \qquad \text{and} \qquad y_2(t) = te^{r_1 t}.$$

Reduction of Order

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$$y'' + p(t)y' + q(t)y = 0. \tag{2}$$

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The main idea in this section is that we guess that

$$y_2(t) = v(t)y_1(t)$$

for some non-constant function $v(t)$. If we can find $v(t)$, then we have our $y_2(t)$.

Then we calculate that

$$\begin{aligned} y_2 &= vy_1 \\ y_2' &= v'y_1 + vy_1' \\ y_2'' &= v''y_1 + 2v'y_1' + vy_1'' \end{aligned}$$

and

$$\begin{aligned} 0 &= y_2'' + py_2' + qy_2 \\ &= \\ &= \\ &= \qquad . \end{aligned}$$

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y_2 &= vy_1 \\
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and

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0 &= y_2'' + py_2' + qy_2 \\
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&= v''y_1 + v'\left(2y_1' + py_1\right).
\end{aligned}$$

$$0 = v'' y_1 + v' \left(2y_1' + p y_1 \right) + 0v$$

Remark

Note that since y_1 solves the ODE, we must always get "$0v$" here. We can have v' and v'' terms, but if you do a reduction of order calculation and still have v terms, then you have made a mistake.

Remark

$$v''y_1 + v'\left(2y_1' + py_1\right) = 0 \tag{3}$$

is actually a first order ODE for v'.

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If we can find $u(t)$, then we can find $v(t) = \int u(t)\,dt$ and $y_2(t) = v(t)y_1(t)$.

Remark

Instead of having to solve a second order ODE to find y_2, we only need to solve a first order ODE to find $u(t)$. Hence the name "Reduction of Order".

Remark

The method is

1. Guess $y_2 = vy_1$.
2. Put this into your ODE and find an equation for v;
3. Set $u = v'$;
4. Find u;
5. Integrate to find v;
6. Then $y_2(t) = v(t)y_1(t)$.

Example

Given that $y_1(t) = \frac{1}{t}$ is a solution of

$$2t^2y'' + 3ty' - y = 0, \qquad t > 0$$

find a linearly independent second solution.

$$y_1(t) = \frac{1}{t}$$

Let $y_2(t) = v(t)y_1(t)$. Then we have

$$\begin{aligned} y_2 &= vt^{-1} \\ y_2' &= v't^{-1} - vt^{-2} \\ y_2'' &= v''t^{-1} - 2v't^{-2} + 2vt^{-3} \end{aligned}$$

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and

$$\begin{aligned} 0 &= 2t^2 y_2'' + 3ty_2' - y_2 \\ &= 2t^2\left(v''t^{-1} - 2v't^{-2} + 2vt^{-3}\right) + 3t\left(v't^{-1} - vt^{-2}\right) - vt^{-1} \\ &= 2tv'' + (-4+3)v' + (4t^{-1} - 3t^{-1} - t^{-1})v \\ &= \qquad\qquad . \end{aligned}$$

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This equation is both linear and separable, so we know 2 ways to solve it.

$$
\begin{aligned}
2t\frac{du}{dt} &= u \\
\frac{du}{u} &= \frac{1}{2}\frac{dt}{t} \\
\int \frac{du}{u} &= \int \frac{1}{2}\frac{dt}{t} \\
\ln|u| &= \frac{1}{2}\ln|t| + C \\
e^{\ln|u|} &= e^{\ln|t|^{\frac{1}{2}}} e^{C} \\
|u| &= |t|^{\frac{1}{2}}\, e^{C} \\
u &= \pm e^{C} t^{\frac{1}{2}} = c t^{\frac{1}{2}}.
\end{aligned}
$$

$$u(t) = ct^{\frac{1}{2}}$$

Then we have

$$v(t) = \int u(t)\,dt = \int ct^{\frac{1}{2}}\,dt = \frac{2}{3}ct^{\frac{3}{2}} + k$$

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Remember that we are trying to find a solution that is linearly independent from $y_1(t) = t^{-1}$. The second term in $y_2(t) = \frac{2}{3}ct^{\frac{1}{2}} + kt^{-1}$ is just a multiple of $y_1(t)$ – we don't need this. So it is ok to choose $k = 0$.

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Finally, since I like simple functions I choose $c = \frac{3}{2}$ to get

$$\boxed{y_2(t) = t^{\frac{1}{2}}.}$$

I leave it to you to check that $W(t^{-1}, t^{\frac{1}{2}})$ is not always zero.

Example

Given that $y_1(t) = t$ solves

$$t^2 y'' + 2ty' - 2y = 0, \quad t > 0,$$

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$$t^2 y'' + 2ty' - 2y = 0, \quad t > 0,$$

find a second linearly independent solution $y_2(t)$.

We start with $y_2(t) = v(t)y_1(t) = v(t)t$. Then $y_2' = v't + v$ and $y_2'' = v''t + 2v'$. Substituting into the ODE, we calculate that

$$\begin{aligned} 0 &= t^2 y_2'' + 2ty_2' - 2y_2 \\ &= t^2(v''t + 2v') + 2t(v't + v) - 2vt \\ &= t^3 v'' + v'(2t^2 + 2t^2) + v(2t - 2t) \\ &= t^3 v'' + 4t^2 v' \\ &= t^2(tv'' + 4v'). \end{aligned}$$

$$t^2(tv'' + 4v') = 0$$

Letting $u = v'$, we obtain the first order ODE

$$t\frac{du}{dt} + 4u = 0.$$

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Letting $u = v'$, we obtain the first order ODE

$$t\frac{du}{dt} + 4u = 0.$$

We calculate that

$$\begin{aligned} t\frac{du}{dt} &= -4u \\ \frac{du}{u} &= -4\frac{dt}{t} \\ \int \frac{du}{u} &= -4\int \frac{dt}{t} \\ \ln|u| &= -4\ln|t| + C \\ u &= \pm e^C t^{-4} = ct^{-4} \end{aligned}$$

$$y_1(t) = t \qquad v' = u \qquad u = ct^{-4}$$

and

$$\begin{aligned} v &= \int u \; dt = \int ct^{-4} \; dt \\ &= -\frac{1}{3}ct^{-3} + k. \end{aligned}$$

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Thus

$$y_2(t) = v(t)t = -\frac{1}{3}ct^{-2} + kt.$$

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Thus

$$y_2(t) = v(t)t = -\frac{1}{3} ct^{-2} + kt.$$

Choosing $c = -3$ and $k = 0$, we obtain the solution

$$y_2(t) = t^{-2}.$$

Does $y_2(t) = t^{-2}$ **really solve** $t^2y'' + 2ty' - 2y = 0$**?**

Does $y_2(t) = t^{-2}$ really solve $t^2y'' + 2ty' - 2y = 0$?

Since $y_2' = -2t^{-3}$ and $y_2'' = 6t^{-4}$, we have that

$$\begin{aligned} t^2y_2'' + 2ty_2' - 2y_2 &= t^2(6t^{-4}) + 2t(-2t^{-3}) - 2t^{-2} \\ &= 6t^{-2} - 4t^{-2} - 2t^{-2} \\ &= 0. \end{aligned}$$

The answer is YES!!

Are $y_1(t) = t$ **and** $y_2(t) = t^{-2}$ **linearly independent?**

Are $y_1(t) = t$ and $y_2(t) = t^{-2}$ linearly independent?

We have that

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} t & t^{-2} \\ 1 & -2t^{-3} \end{vmatrix} = -2t^{-2} - t^{-2} = -3t^{-2} \neq 0$$

since $t > 0$. Therefore y_1 and y_2 are linearly independent.

Nonhomogeneous Equations

Consider

$$y'' + p(t)y' + q(t)y = g(t). \tag{5}$$

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The equation

$$y'' + p(t)y' + q(t)y = 0 \tag{6}$$

is called the *homogeneous equation corresponding to* (5).

$$y'' + p(t)y' + q(t)y = g(t) \tag{5}$$

Theorem

The general solution to (5) can be written in the form

$$y(t) = c_1 y_1(t) + c_2 y_2(t) + Y(t)$$

where

$$y'' + p(t)y' + q(t)y = g(t) \tag{5}$$

Theorem

The general solution to (5) can be written in the form

$$y(t) = c_1 y_1(t) + c_2 y_2(t) + Y(t)$$

where

- y_1 *and* y_2 *form a fundamental set of solutions to the homogeneous equation corresponding to (5);*
- c_1 *and* c_2 *are constants; and*
- Y *is a particular solution to (5).*

To solve $L[y] = g$

1. Find the general solution to $L[y] = 0$;
2. Find a particular solution to $L[y] = g$;
3. $1 + 2$

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We will study 2 methods to do step 2. One method this week and one method next week.

3.7 The Method of Undetermined Coefficients

$$y'' + p(t)y' + q(t)y = g(t) \tag{5}$$

The idea is:

1. Look at $g(t)$
2. Make a guess with constants
3. Try to find the constants

Example

Find a particular solution to $y'' - 3y' - 4y = 3e^{2t}$.

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Here we have $g(t) = 3e^{2t}$. We look at this g and we make a guess: g includes e^{2t} so we guess that $Y(t)$ also includes e^{2t}. So we guess that $Y(t) = Ae^{2t}$ for some constant A.

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We must try to find A. We calculate that

$$Y(t) = Ae^{2t} \qquad Y'(t) = 2Ae^{2t} \qquad Y''(t) = 4Ae^{2t}$$

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We must try to find A. We calculate that

$$Y(t) = Ae^{2t} \qquad Y'(t) = 2Ae^{2t} \qquad Y''(t) = 4Ae^{2t}$$

and

$$\begin{aligned} 3e^{2t} &= Y'' - 3Y' - 4Y = 4Ae^{2t} - 3\left(2Ae^{2t}\right) - 4\left(Ae^{2t}\right) \\ &= -6Ae^{2t}. \end{aligned}$$

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We must have $A = -\frac{1}{2}$.

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We must have $A = -\frac{1}{2}$. Therefore a particular solution is

$$\boxed{Y(t) = -\frac{1}{2}e^{2t}.}$$

Example

Find a particular solution to $y'' - 3y' - 4y = 4t^2 - 1$.

Example

Find a particular solution to $y'' - 3y' - 4y = 4t^2 - 1$.

Since $g(t) = 4t^2 - 1$ is a 2nd degree polynomial, we guess that Y is also a second degree polynomial. So we try the ansatz

$$Y(t) = At^2 + Bt + C.$$

I will leave this example for you to finish.

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I will leave this example for you to finish.

(ansatz = a mathematical guess)

Example

Find a particular solution to $y'' - 3y' - 4y = 2\sin t$.

Example

Find a particular solution to $y'' - 3y' - 4y = 2\sin t$.

First guess: $Y(t) = A\sin t$.

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$$\begin{aligned} 2\sin t &= Y'' - 3Y' - 4Y \\ &= (-A\sin t) - 3(A\cos t) - 4(A\sin t) = -5A\sin t - 3A\cos t. \end{aligned}$$

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$$\begin{cases} -5A = 2 \\ -3A = 0. \end{cases}$$

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We can see that we must have

$$\begin{cases} -5A = 2 \\ -3A = 0. \end{cases}$$

This linear system is inconsistent: It not possible to find a constant A which satisfies both of these equations. Our first guess failed.

Second guess: $Y(t) = A\sin t + B\cos t$.

Second guess: $Y(t) = A\sin t + B\cos t$. Then we calculate that

$$Y' = A\cos t - B\sin t, \qquad Y'' = -A\sin t - B\cos t$$

and

$$\begin{aligned} 2\sin t &= Y'' - 3Y' - 4Y \\ &= (-A\sin t - B\cos t) - 3(A\cos t - B\sin t) - 4(A\sin t + B\cos t) \\ &= (-5A + 3B)\sin t + (-3A - 5B)\cos t. \end{aligned}$$

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So we need A and B to satisfy

$$\begin{cases} -5A + 3B = 2 \\ -3A - 5B = 0. \end{cases}$$

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Please check that the solution to this linear system is $A = -\frac{5}{17}$ and $B = \frac{3}{17}$.

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Please check that the solution to this linear system is $A = -\frac{5}{17}$ and $B = \frac{3}{17}$. Therefore a particular solution is

$$\boxed{Y(t) = -\frac{5}{17}\sin t + \frac{3}{17}\cos t.}$$

Remark

sin and cos are friends! They always go together. If you see either sin or cos in $g(t)$, then your ansatz needs to contain *both* sin and cos.
Likewise sinh and cosh always go together.

Example

Find a particular solution to $y'' - 3y' - 4y = -8e^t \cos 2t$.

Example

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We will try the ansatz

$$Y(t) = Ae^t \cos 2t + Be^t \sin 2t.$$

Then

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We will try the ansatz

$$Y(t) = Ae^t \cos 2t + Be^t \sin 2t.$$

Then

$$\begin{aligned} Y'(t) &= Ae^t \cos 2t - 2Ae^t \sin 2t + Be^t \sin 2t + 2Be^t \cos 2t \\ &= (A + 2B)e^t \cos 2t + (B - 2A)e^t \sin 2t, \\ Y''(t) &= (A + 2B)e^t \cos 2t - 2(A + 2B)e^t \sin 2t + (B - 2A)e^t \sin 2t \\ &\qquad + 2(B - 2A)e^t \cos 2t \\ &= (-3A + 4B)e^t \cos 2t + (-4A - 3B)e^t \sin 2t \end{aligned}$$

and

$$\begin{aligned} -8e^t\cos 2t &= Y''-3Y'-4Y \\ &= (-3A+4B)e^t\cos 2t + (-4A-3B)e^t\sin 2t \\ &\quad + (-3A-6B)e^t\cos 2t + (-3B+6A)e^t\sin 2t \\ &\quad + (-4A)e^t\cos 2t + (-4B)e^t\sin 2t \\ &= (-10A-2B)e^t\cos 2t + (2A-10B)e^t\sin 2t. \end{aligned}$$

and

$$\begin{aligned}
-8e^t \cos 2t &= Y'' - 3Y' - 4Y \\
&= (-3A + 4B)e^t \cos 2t + (-4A - 3B)e^t \sin 2t \\
&\quad + (-3A - 6B)e^t \cos 2t + (-3B + 6A)e^t \sin 2t \\
&\quad + (-4A)e^t \cos 2t + (-4B)e^t \sin 2t \\
&= (-10A - 2B)e^t \cos 2t + (2A - 10B)e^t \sin 2t.
\end{aligned}$$

Thus we must solve

$$\begin{cases} 10A + 2B = 8 \\ 2A - 10B = 0. \end{cases}$$

and

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Please check that the solution to this linear system is $A = \frac{10}{13}$ and $B = \frac{2}{13}$.

and

$$
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-8e^t \cos 2t &= Y''-3Y'-4Y \\
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&\quad + (-4A)e^t \cos 2t + (-4B)e^t \sin 2t \\
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\end{aligned}
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Thus we must solve

$$
\begin{cases} 10A + 2B = 8 \\ 2A - 10B = 0. \end{cases}
$$

Please check that the solution to this linear system is $A = \frac{10}{13}$ and $B = \frac{2}{13}$. Therefore a particular solution is

$$
\boxed{Y(t) = \frac{10}{13}e^t \cos 2t + \frac{2}{13}e^t \sin 2t.}
$$

Theorem

$$\left.\begin{array}{c} Y_1 \;\; solves \\ ay'' + by' + cy = g_1(t) \\ \\ Y_2 \;\; solves \\ ay'' + by' + cy = g_2(t) \end{array}\right\} \implies \begin{array}{c} Y_1 + Y_2 \;\; solves \\ ay'' + by' + cy = g_1(t) + g_2(t) \end{array}$$

Example

Find a particular solution to

$$y'' - 3y' - 4y = 3e^{2t} + 2\sin t - 8e^{t}\cos 2t. \tag{7}$$

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We can split this problem up into three easier problems:

$$y'' - 3y' - 4y = 3e^{2t}$$
$$y'' - 3y' - 4y = 2\sin t$$
$$y'' - 3y' - 4y = -8e^{t}\cos 2t$$

We know particular solutions to these three ODEs.

Example

Find a particular solution to

$$y'' - 3y' - 4y = 3e^{2t} + 2\sin t - 8e^{t}\cos 2t. \tag{7}$$

We can split this problem up into three easier problems:

$$\begin{aligned} y'' - 3y' - 4y &= 3e^{2t} \\ y'' - 3y' - 4y &= 2\sin t \\ y'' - 3y' - 4y &= -8e^{t}\cos 2t \end{aligned}$$

We know particular solutions to these three ODEs. Therefore

$$\boxed{Y(t) = -\frac{1}{2}e^{2t} - \frac{5}{17}\sin t + \frac{3}{17}\cos t + \frac{10}{13}e^{t}\cos 2t + \frac{2}{13}e^{t}\sin 2t.}$$

is a particular solution to (7).

Remark

To find a particular solution to $ay'' + by' + cy = g(t)$, we have been looking at $g(t)$ and choosing a similar function for $Y(t)$.

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To find a particular solution to $ay'' + by' + cy = g(t)$, we have been looking at $g(t)$ and choosing a similar function for $Y(t)$.

This method doesn't always work: There is a difficulty that can occur as we shall see in the next example.

Example

Find a particular solution to $y'' + 4y = 3\cos 2t$.

First guess: $Y(t) = A\cos 2t + B\sin 2t$.

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First guess: $Y(t) = A\cos 2t + B\sin 2t$.
Then we have that

$$\begin{aligned} Y' &= -2A\sin 2t + 2B\cos 2t \\ Y'' &= -4A\cos 2t - 4B\sin 2t \end{aligned}$$

and

$$\begin{aligned} 3\cos 2t &= Y'' + 4Y \\ &= (-4A\cos 2t - 4B\sin 2t) + 4(A\cos 2t + B\sin 2t) = 0. \end{aligned}$$

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and

$$\begin{aligned} 3\cos 2t &= Y'' + 4Y \\ &= (-4A\cos 2t - 4B\sin 2t) + 4(A\cos 2t + B\sin 2t) = 0. \end{aligned}$$

This is a FAILURE!!! It not possible to choose A and B such that

$$3\cos 2t = 0$$

for all t.

Why did this happen? Why didn't our usual method work?

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$$0 = r^2 + 4 = (r + 2i)(r - 2i).$$

So $r = \pm 2i$.

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Since $\cos 2t$ and $\sin 2t$ appear in the general solution to the homogeneous equation, *we <u>can not</u> use* $\cos 2t$ *and* $\sin 2t$ *in a particular solution.*

HINT: If in doubt, multiply by t.

We need two functions which, when differentiated, give us $\cos 2t$ and $\sin 2t$ terms.

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We will try $t\cos 2t$ and $t\sin 2t$.

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Note that

$$\frac{d}{dt}t\cos 2t = \cos 2t - 2t\sin 2t$$

and

$$\frac{d}{dt}t\sin 2t = \sin 2t + 2t\cos 2t.$$

Second guess: $Y(t) = At\cos 2t + Bt\sin 2t.$

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We have that

$$\begin{aligned} Y' &= A\cos 2t - 2At\sin 2t + B\sin 2t + 2Bt\cos 2t \\ &= (A+2Bt)\cos 2t + (B-2At)\sin 2t, \\ Y'' &= 2B\cos 2t - 2(A+2Bt)\sin 2t - 2A\sin 2t + 2(B-2At)\cos 2t \\ &= (4B-4At)\cos 2t + (-4A-4Bt)\sin 2t \end{aligned}$$

and

$$\begin{aligned} 3\cos 2t &= Y'' + 4Y \\ &= (4B-4At)\cos 2t + (-4A-4Bt)\sin 2t \\ &\quad + 4At\cos 2t + 4Bt\sin 2t \\ &= 4B\cos 2t - 4A\sin 2t. \end{aligned}$$

$$3\cos 2t = 4B\cos 2t - 4A\sin 2t$$

Thus

$$\begin{cases} -4A = 0 \\ 4B = 3 \end{cases}$$

which has solution $A = 0$ and $B = \frac{3}{4}$.

$$3\cos 2t = 4B\cos 2t - 4A\sin 2t$$

Thus

$$\begin{cases} -4A = 0 \\ 4B = 3 \end{cases}$$

which has solution $A = 0$ and $B = \frac{3}{4}$. Therefore a particular solution is

$$\boxed{Y(t) = \frac{3}{4}t\sin 2t.}$$

Next Time

- 3.8 Solving Initial Value Problems
- 3.9 The Method of Variation of Parameters
- 3.10 Higher Order Linear ODEs