

# Lecture 4

- Introduction to Determinants
- Evaluating Determinants by Row Reduction
- Properties of Determinants
- Cramer's Rule



# Introduction to Determinants

# Introduction to Determinants



Recall that a  $2 \times 2$  matrix

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

is invertible if and only if the special number  $\det(A) = ad - bc$  is nonzero. This special number is called the *determinant* of the matrix.

The determinant of a  $A$  is written as  $\det(A)$  or as  $\begin{vmatrix} a & b \\ c & d \end{vmatrix}$ .

Now it is time to generalise this idea to all square matrices.

# Introduction to Determinants



## Remark

In today's lecture, all the matrices will be square matrices unless I specifically say otherwise.

(Remember that only square matrices have determinants.)

# Introduction to Determinants



$1 \times 1$  matrices are easy, we just define

$$\det [a_{11}] = a_{11}.$$

## Definition

If  $A$  is a square matrix, then the *minor of entry*  $a_{ij}$  is denoted by  $M_{ij}$  and is defined to be the determinant of the submatrix that remains after the  $i$ th row and  $j$ th column are deleted from  $A$ .

# Introduction to Determinants



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## Definition

The number

$$C_{ij} = (-1)^{i+j} M_{ij}$$

is called *the cofactor of entry*  $a_{ij}$ .

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

$$C_{ij} = (-1)^{i+j} M_{ij}$$

## Example

Let

$$A = \begin{bmatrix} 3 & 1 & -4 \\ 2 & 5 & 6 \\ 1 & 4 & 8 \end{bmatrix}.$$

Find

- 1 the minor of entry  $a_{11}$ ;
- 2 the cofactor of entry  $a_{11}$ ;
- 3 the minor of entry  $a_{32}$ ; and
- 4 the cofactor of entry  $a_{32}$ .

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

$$C_{ij} = (-1)^{i+j} M_{ij}$$

$$A = \begin{bmatrix} 3 & 1 & -4 \\ 2 & 5 & 6 \\ 1 & 4 & 8 \end{bmatrix}$$

- 1 For the minor of entry  $a_{11}$ , we need to cross out the **first** row and the **first** column.

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

$$C_{ij} = (-1)^{i+j} M_{ij}$$

$$A = \begin{bmatrix} 3 & 1 & -4 \\ 2 & 5 & 6 \\ 1 & 4 & 8 \end{bmatrix}$$

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- For the minor of entry  $a_{11}$ , we need to cross out the **first** row and the **first** column. That leaves us with a  $2 \times 2$  submatrix. The determinant of this submatrix is

$$M_{11} = \begin{vmatrix} 5 & 6 \\ 4 & 8 \end{vmatrix} = 5 \cdot 8 - 6 \cdot 4 = 16.$$

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

$$C_{ij} = (-1)^{i+j} M_{ij}$$

$$A = \begin{bmatrix} 3 & 1 & -4 \\ 2 & 5 & 6 \\ 1 & 4 & 8 \end{bmatrix}$$

- 1 For the minor of entry  $a_{11}$ , we need to cross out the **first** row and the **first** column. That leaves us with a  $2 \times 2$  submatrix. The determinant of this submatrix is

$$M_{11} = \begin{vmatrix} 5 & 6 \\ 4 & 8 \end{vmatrix} = 5 \cdot 8 - 6 \cdot 4 = 16.$$

- 2 The cofactor of entry  $a_{11}$  is then

$$C_{11} = (-1)^{1+1} M_{11} = (-1)^2 \cdot 16 = 16.$$

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

$$C_{ij} = (-1)^{i+j} M_{ij}$$

- 3 The minor of entry  $a_{32}$  is

$$M_{32} = \begin{vmatrix} 3 & 1 & -4 \\ 2 & 5 & 6 \\ 1 & 4 & 8 \end{vmatrix}$$

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

$$C_{ij} = (-1)^{i+j} M_{ij}$$

- 3 The minor of entry  $a_{32}$  is

$$M_{32} = \begin{vmatrix} 3 & 1 & -4 \\ 2 & 5 & 6 \\ 1 & 4 & 8 \end{vmatrix} = \begin{vmatrix} 3 & -4 \\ 2 & 6 \end{vmatrix}$$

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

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- 3 The minor of entry  $a_{32}$  is

$$M_{32} = \begin{vmatrix} 3 & 1 & -4 \\ 2 & 5 & 6 \\ 1 & 4 & 8 \end{vmatrix} = \begin{vmatrix} 3 & -4 \\ 2 & 6 \end{vmatrix} = 3 \cdot 6 - (-4) \cdot 2 = 26$$

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

$$C_{ij} = (-1)^{i+j} M_{ij}$$



- 3 The minor of entry  $a_{32}$  is

$$M_{32} = \begin{vmatrix} 3 & 1 & -4 \\ 2 & 5 & 6 \\ 1 & 4 & 8 \end{vmatrix} = \begin{vmatrix} 3 & -4 \\ 2 & 6 \end{vmatrix} = 3 \cdot 6 - (-4) \cdot 2 = 26$$

- 4 and the cofactor of entry  $a_{32}$  is then

$$C_{32} = (-1)^{3+2} M_{32} = (-1)^3 \cdot 26 = -26.$$

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

$$C_{ij} = (-1)^{i+j} M_{ij}$$



### Remark

We always have  $C_{ij} = +M_{ij}$  or  $C_{ij} = -M_{ij}$ .

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

$$C_{ij} = (-1)^{i+j} M_{ij}$$



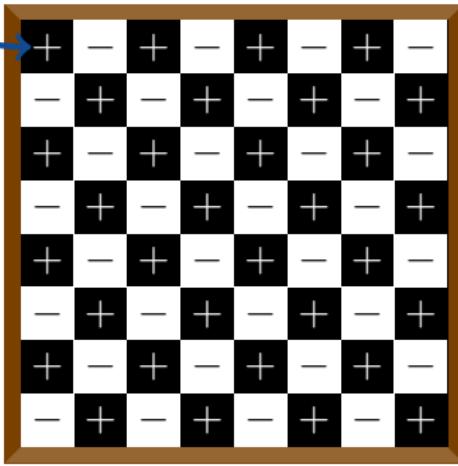
### Remark

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$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

$$C_{ij} = (-1)^{i+j} M_{ij}$$

top left is always +

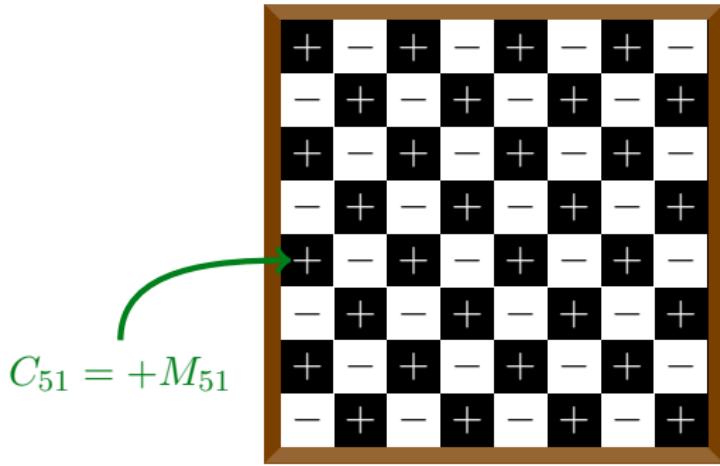


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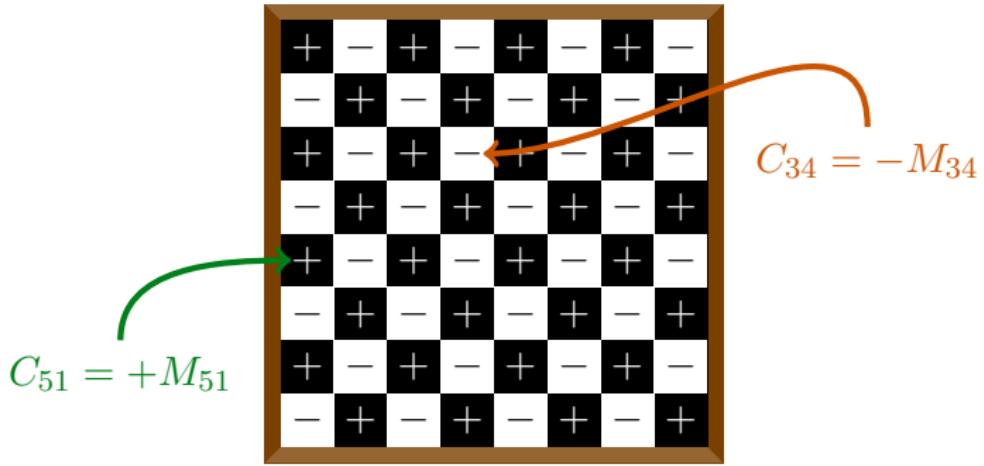


### Remark

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$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

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### Remark

We always have  $C_{ij} = +M_{ij}$  or  $C_{ij} = -M_{ij}$ .

# Introduction to Determinants



## Example

The checkerboard pattern for a  $2 \times 2$  matrix  $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$  is



This means that

$$\begin{aligned} C_{11} &= M_{11} = a_{22} & C_{12} &= -M_{12} = -a_{21} \\ C_{21} &= -M_{21} = -a_{12} & C_{22} &= M_{22} = a_{11}. \end{aligned}$$

$$\begin{array}{ll} C_{11} = M_{11} = a_{22} & C_{12} = -M_{12} = -a_{21} \\ C_{21} = -M_{21} = -a_{12} & C_{22} = M_{22} = a_{11}. \end{array}$$



## Remark

Note that

$$\det(A) = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21}$$

=

=

=

=

$$\begin{array}{ll} C_{11} = M_{11} = a_{22} & C_{12} = -M_{12} = -a_{21} \\ C_{21} = -M_{21} = -a_{12} & C_{22} = M_{22} = a_{11}. \end{array}$$



## Remark

Note that

$$\begin{aligned}\det(A) &= \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21} \\ &= a_{11}C_{11} + a_{12}C_{12} \\ &= \\ &= \\ &= \end{aligned}$$

$$\begin{array}{ll} C_{11} = M_{11} = a_{22} & C_{12} = -M_{12} = -a_{21} \\ C_{21} = -M_{21} = -a_{12} & C_{22} = M_{22} = a_{11}. \end{array}$$



## Remark

Note that

$$\begin{aligned}\det(A) &= \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21} \\ &= a_{11}C_{11} + a_{12}C_{12} \\ &= a_{21}C_{21} + a_{22}C_{22} \\ &= a_{11}C_{11} + a_{21}C_{21} \\ &= a_{12}C_{12} + a_{22}C_{22}.\end{aligned}$$

These last four lines are called *cofactor expansions* of  $\det(A)$ .

## The Definition of the Determinant

### Theorem

*If  $A$  is an  $n \times n$  matrix, then regardless of which row or column of  $A$  is chosen, the number obtained by multiplying the entries in that row or column by the corresponding cofactors and adding the resulting products is always the same.*

# Introduction to Determinants

## Definition

If  $A$  is an  $n \times n$  matrix, then the number obtained by multiplying the entries in any row or column of  $A$  by the corresponding cofactors and adding the resulting products is called the *determinant* of  $A$ , and the sums themselves are called *cofactor expansions of  $A$* . That is,

$$\det(A) = \underbrace{a_{1j}C_{1j} + a_{2j}C_{2j} + \dots + a_{nj}C_{nj}}_{\text{cofactor expansion along the } j\text{th column}}$$

and

$$\det(A) = \underbrace{a_{i1}C_{i1} + a_{i2}C_{i2} + \dots + a_{in}C_{in}}_{\text{cofactor expansion along the } i\text{th row}}.$$

# Introduction to Determinants



## Remark

$$\begin{bmatrix} 3 & 1 & 0 \\ -2 & -4 & 3 \\ 5 & 4 & -2 \end{bmatrix} \quad \text{and} \quad \begin{pmatrix} 3 & 1 & 0 \\ -2 & -4 & 3 \\ 5 & 4 & -2 \end{pmatrix}$$

are matrices, but

$$\left| \begin{array}{ccc} 3 & 1 & 0 \\ -2 & -4 & 3 \\ 5 & 4 & -2 \end{array} \right|$$

is a number.

# Introduction to Determinants



## Example

Find the determinant of

$$A = \begin{bmatrix} 3 & 1 & 0 \\ -2 & -4 & 3 \\ 5 & 4 & -2 \end{bmatrix}$$

by cofactor expansion along the **first row**.

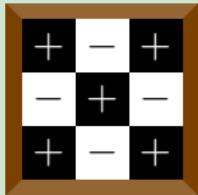
# Introduction to Determinants



## Example

Find the determinant of

$$A = \begin{bmatrix} 3 & 1 & 0 \\ -2 & -4 & 3 \\ 5 & 4 & -2 \end{bmatrix}$$



by cofactor expansion along the **first row**.

$$\det(A) = \begin{vmatrix} 3 & 1 & 0 \\ -2 & -4 & 3 \\ 5 & 4 & -2 \end{vmatrix} = + - +$$

=

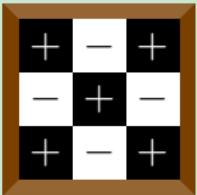
# Introduction to Determinants



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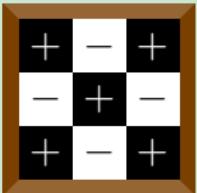
$$\det(A) = \begin{vmatrix} 3 & 1 & 0 \\ -2 & -4 & 3 \\ 5 & 4 & -2 \end{vmatrix} = +3 - 1 + 0 =$$

# Introduction to Determinants

## Example

Find the determinant of

$$A = \begin{bmatrix} 3 & 1 & 0 \\ -2 & -4 & 3 \\ 5 & 4 & -2 \end{bmatrix}$$



by cofactor expansion along the **first row**.

$$\det(A) = \begin{vmatrix} 3 & 1 & 0 \\ -2 & -4 & 3 \\ 5 & 4 & -2 \end{vmatrix} = +3 \begin{vmatrix} -4 & 3 \\ 4 & -2 \end{vmatrix} - 0$$

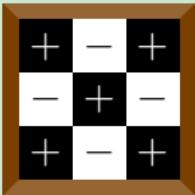
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# Introduction to Determinants

## Example

Find the determinant of

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=

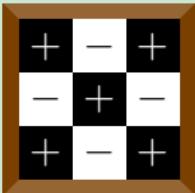
# Introduction to Determinants



## Example

Find the determinant of

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$$\det(A) = \begin{vmatrix} 3 & 1 & 0 \\ -2 & -4 & 3 \\ 5 & 4 & -2 \end{vmatrix} = +3 \begin{vmatrix} -4 & 3 \\ 4 & -2 \end{vmatrix} - 1 \begin{vmatrix} -2 & 3 \\ 5 & -2 \end{vmatrix} + 0 \begin{vmatrix} -2 & -4 \\ 5 & 4 \end{vmatrix}$$

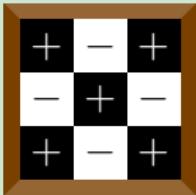
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# Introduction to Determinants

## Example

Find the determinant of

$$A = \begin{bmatrix} 3 & 1 & 0 \\ -2 & -4 & 3 \\ 5 & 4 & -2 \end{bmatrix}$$



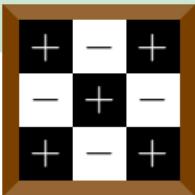
by cofactor expansion along the **first row**.

$$\begin{aligned}\det(A) &= \begin{vmatrix} 3 & 1 & 0 \\ -2 & -4 & 3 \\ 5 & 4 & -2 \end{vmatrix} = +3 \begin{vmatrix} -4 & 3 \\ 4 & -2 \end{vmatrix} - 1 \begin{vmatrix} -2 & 3 \\ 5 & -2 \end{vmatrix} + 0 \begin{vmatrix} -2 & -4 \\ 5 & 4 \end{vmatrix} \\ &= 3(-4) - 1(-11) + 0(12) = -1.\end{aligned}$$

# Introduction to Determinants

## Example

Find the determinant of the same matrix by cofactor expansion along the first column.



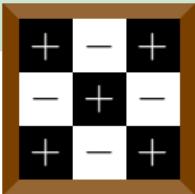
$$\det(A) = \begin{vmatrix} 3 & 1 & 0 \\ -2 & -4 & 3 \\ 5 & 4 & -2 \end{vmatrix} =$$
$$=$$

# Introduction to Determinants



## Example

Find the determinant of the same matrix by cofactor expansion along the first column.



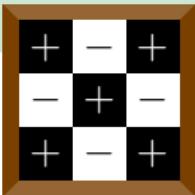
$$\det(A) = \begin{vmatrix} 3 & 1 & 0 \\ -2 & -4 & 3 \\ 5 & 4 & -2 \end{vmatrix} = +3 \quad -(-2) \quad +5 \\ =$$

# Introduction to Determinants



## Example

Find the determinant of the same matrix by cofactor expansion along the first column.



$$\det(A) = \begin{vmatrix} 3 & 1 & 0 \\ -2 & -4 & 3 \\ 5 & 4 & -2 \end{vmatrix} = +3 \begin{vmatrix} -4 & 3 \\ 4 & -2 \end{vmatrix} - (-2) \begin{vmatrix} 1 & 0 \\ 4 & -2 \end{vmatrix} + 5 \begin{vmatrix} 1 & 0 \\ -4 & 3 \end{vmatrix}$$

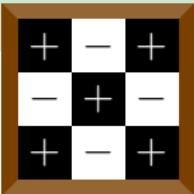
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# Introduction to Determinants



## Example

Find the determinant of the same matrix by cofactor expansion along the first column.



$$\begin{aligned}\det(A) &= \begin{vmatrix} 3 & 1 & 0 \\ -2 & -4 & 3 \\ 5 & 4 & -2 \end{vmatrix} = +3 \begin{vmatrix} -4 & 3 \\ 4 & -2 \end{vmatrix} - (-2) \begin{vmatrix} 1 & 0 \\ 4 & -2 \end{vmatrix} + 5 \begin{vmatrix} 1 & 0 \\ -4 & 3 \end{vmatrix} \\ &= 3(-4) + 2(-2) + 5(3) = -1\end{aligned}$$

again.

# Introduction to Determinants



Sometimes it is easier to use cofactor expansion along some rows or columns than others.

## Example (A Smart Choice of Row or Column)

Calculate the determinant of

$$A = \begin{bmatrix} 1 & 0 & 0 & -1 \\ 3 & 1 & 2 & 2 \\ 1 & 0 & -2 & 1 \\ 2 & 0 & 0 & 1 \end{bmatrix}.$$

# Introduction to Determinants



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## Example (A Smart Choice of Row or Column)

Calculate the determinant of

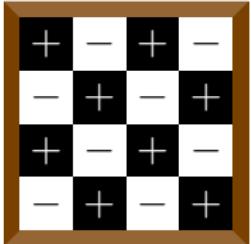
$$A = \begin{bmatrix} 1 & 0 & 0 & -1 \\ 3 & 1 & 2 & 2 \\ 1 & 0 & -2 & 1 \\ 2 & 0 & 0 & 1 \end{bmatrix}.$$

Look at the 2nd column. It has the most zeros. It will be easiest for us if we choose this column.

# Introduction to Determinants



$$A = \begin{bmatrix} 1 & 0 & 0 & -1 \\ 3 & 1 & 2 & 2 \\ 1 & 0 & -2 & 1 \\ 2 & 0 & 0 & 1 \end{bmatrix}$$

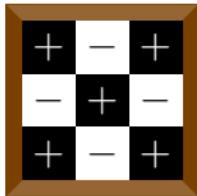


We calculate that

$$\det(A) = -0 + 1 \cdot \begin{vmatrix} 1 & 0 & -1 \\ 1 & -2 & 1 \\ 2 & 0 & 1 \end{vmatrix} - 0 + 0 = \begin{vmatrix} 1 & 0 & -1 \\ 1 & -2 & 1 \\ 2 & 0 & 1 \end{vmatrix}.$$

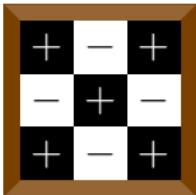
Now which row or column should I choose in this  $3 \times 3$  determinant?

# Introduction to Determinants



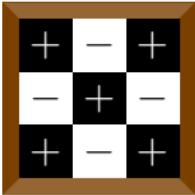
$$\det(A) = \begin{vmatrix} 1 & 0 & -1 \\ 1 & -2 & 1 \\ 2 & 0 & 1 \end{vmatrix} =$$
$$=$$

# Introduction to Determinants



$$\det(A) = \begin{vmatrix} 1 & 0 & -1 \\ 1 & -2 & 1 \\ 2 & 0 & 1 \end{vmatrix} = -0 + (-2) \begin{vmatrix} 1 & -1 \\ 2 & 1 \end{vmatrix} - 0 \\ =$$

# Introduction to Determinants



$$\begin{aligned}\det(A) &= \begin{vmatrix} 1 & 0 & -1 \\ 1 & -2 & 1 \\ 2 & 0 & 1 \end{vmatrix} = -0 + (-2) \begin{vmatrix} 1 & -1 \\ 2 & 1 \end{vmatrix} - 0 \\ &= (-2)(3) = -6.\end{aligned}$$

# Introduction to Determinants

## Example (Determinant of a Lower Triangular Matrix)

Calculate

$$\begin{vmatrix} a_{11} & 0 & 0 & 0 \\ a_{21} & a_{22} & 0 & 0 \\ a_{31} & a_{32} & a_{33} & 0 \\ a_{41} & a_{42} & a_{43} & a_{44} \end{vmatrix}.$$

# Introduction to Determinants



Example (Determinant of a Lower Triangular Matrix)

Calculate

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I am going to be using cofactor expansion along the first rows:

$$\begin{vmatrix} a_{11} & 0 & 0 & 0 \\ a_{21} & a_{22} & 0 & 0 \\ a_{31} & a_{32} & a_{33} & 0 \\ a_{41} & a_{42} & a_{43} & a_{44} \end{vmatrix} = a_{11} \begin{vmatrix} a_{22} & 0 & 0 \\ a_{32} & a_{33} & 0 \\ a_{42} & a_{43} & a_{44} \end{vmatrix} =$$

=

# Introduction to Determinants



## Example (Determinant of a Lower Triangular Matrix)

Calculate

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$$\begin{vmatrix} a_{11} & 0 & 0 & 0 \\ a_{21} & a_{22} & 0 & 0 \\ a_{31} & a_{32} & a_{33} & 0 \\ a_{41} & a_{42} & a_{43} & a_{44} \end{vmatrix} = a_{11} \begin{vmatrix} a_{22} & 0 & 0 \\ a_{32} & a_{33} & 0 \\ a_{42} & a_{43} & a_{44} \end{vmatrix} = a_{11}a_{22} \begin{vmatrix} a_{33} & 0 \\ a_{43} & a_{44} \end{vmatrix}$$

=

# Introduction to Determinants



Example (Determinant of a Lower Triangular Matrix)

Calculate

$$\begin{vmatrix} a_{11} & 0 & 0 & 0 \\ a_{21} & a_{22} & 0 & 0 \\ a_{31} & a_{32} & a_{33} & 0 \\ a_{41} & a_{42} & a_{43} & a_{44} \end{vmatrix}.$$

I am going to be using cofactor expansion along the first rows:

$$\begin{vmatrix} a_{11} & 0 & 0 & 0 \\ a_{21} & a_{22} & 0 & 0 \\ a_{31} & a_{32} & a_{33} & 0 \\ a_{41} & a_{42} & a_{43} & a_{44} \end{vmatrix} = a_{11} \begin{vmatrix} a_{22} & 0 & 0 \\ a_{32} & a_{33} & 0 \\ a_{42} & a_{43} & a_{44} \end{vmatrix} = a_{11}a_{22} \begin{vmatrix} a_{33} & 0 \\ a_{43} & a_{44} \end{vmatrix}$$
$$= a_{11}a_{22}a_{33}a_{44}.$$

# Introduction to Determinants

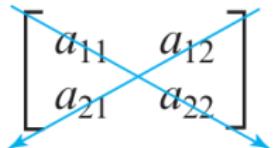


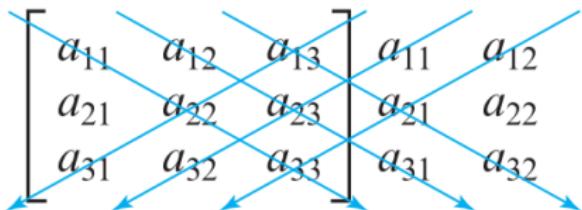
## Theorem

*If  $A$  is an  $n \times n$  triangular matrix (upper triangular, lower triangular, or diagonal), then*

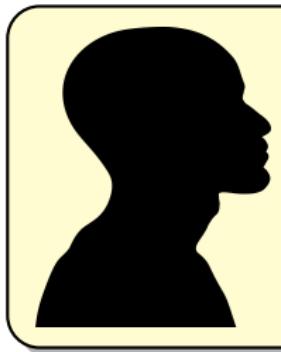
$$\det(A) = a_{11}a_{22}\dots a_{nn}.$$

## Rule of Sarrus

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$


$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$


Determinants of  $2 \times 2$  and  $3 \times 3$  matrices can be evaluated very efficiently using the pattern shown above.



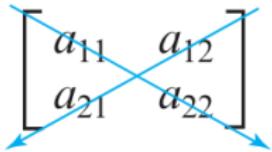
Pierre Frédéric Sarrus

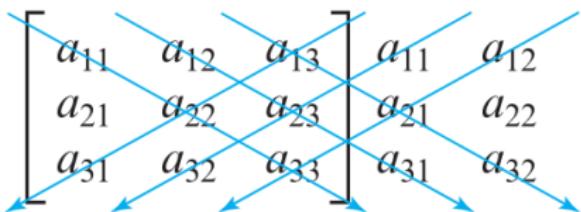
BORN  
10 March 1798

DECEASED  
20 November 1861

NATIONALITY  
French

## Rule of Sarrus

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$


$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$


Determinants of  $2 \times 2$  and  $3 \times 3$  matrices can be evaluated very efficiently using the pattern shown above.

We sum the products of the rightwards arrows, then subtract the products on the leftward arrows.

# Introduction to Determinants



## Remark

The rule of Sarrus only works for  $2 \times 2$  and  $3 \times 3$  matrices.

Don't try to use it for bigger matrices.

## EXAMPLE 7 A Technique for Evaluating $2 \times 2$ and $3 \times 3$ Determinants

$$\begin{vmatrix} 3 & 1 \\ 4 & -2 \end{vmatrix} = \begin{vmatrix} 3 & 1 \\ 4 & -2 \end{vmatrix} = (3)(-2) - (1)(4) = -10$$

$$\begin{vmatrix} 1 & 2 & 3 \\ -4 & 5 & 6 \\ 7 & -8 & 9 \end{vmatrix} = \begin{vmatrix} 1 & 2 & 3 \\ -4 & 5 & 6 \\ 7 & -8 & 9 \end{vmatrix} = [45 + 84 + 96] - [105 - 48 - 72] = 240 \quad \blacktriangleleft$$



# Evaluating Determinants by Row Reduction

# Evaluating Determinants by Row Reduction



$$\begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 7 & 8 & 9 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 3 & 4 & 5 & 6 & 7 & 8 \\ 9 & 0 & 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 & 9 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & 3 & 0 & 5 \\ 7 & 8 & 9 & 0 & 1 \\ 3 & 4 & 5 & 0 & 7 \\ 9 & 0 & 1 & 0 & 3 \\ 5 & 6 & 7 & 0 & 9 \end{bmatrix}$$

## Theorem

If  $A$  has a row of zeros, or a column of zeros, then  $\det(A) = 0$ .

# Evaluating Determinants by Row Reduction



$$\begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 7 & 8 & 9 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 3 & 4 & 5 & 6 & 7 & 8 \\ 9 & 0 & 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 & 9 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & 3 & 0 & 5 \\ 7 & 8 & 9 & 0 & 1 \\ 3 & 4 & 5 & 0 & 7 \\ 9 & 0 & 1 & 0 & 3 \\ 5 & 6 & 7 & 0 & 9 \end{bmatrix}$$

## Theorem

If  $A$  has a row of zeros, or a column of zeros, then  $\det(A) = 0$ .

## Proof.

Let's just do this for the left matrix above.

# Evaluating Determinants by Row Reduction



$$\begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 7 & 8 & 9 & 0 & 1 & 2 \\ \textcolor{red}{0} & \textcolor{red}{0} & \textcolor{red}{0} & \textcolor{red}{0} & \textcolor{red}{0} & \textcolor{red}{0} \\ 3 & 4 & 5 & 6 & 7 & 8 \\ 9 & 0 & 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 & 9 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & 3 & \textcolor{red}{0} & 5 \\ 7 & 8 & 9 & \textcolor{red}{0} & 1 \\ 3 & 4 & 5 & \textcolor{red}{0} & 7 \\ 9 & 0 & 1 & \textcolor{red}{0} & 3 \\ 5 & 6 & 7 & \textcolor{red}{0} & 9 \end{bmatrix}$$

## Theorem

If  $A$  has a row of zeros, or a column of zeros, then  $\det(A) = 0$ .

## Proof.

Let's just do this for the left matrix above. If we use cofactor expansion along the 3rd row, then we get

$$\det(A) = \textcolor{red}{0}C_{31} + \textcolor{red}{0}C_{32} + \textcolor{red}{0}C_{33} + \textcolor{red}{0}C_{34} + \textcolor{red}{0}C_{35} + \textcolor{red}{0}C_{36} = 0.$$



# Evaluating Determinants by Row Reduction



Theorem

$$\det(A) = \det(A^T).$$

# Evaluating Determinants by Row Reduction



## Theorem

$$\det(A) = \det(A^T).$$

## Proof.

Recall that transposing a matrix swaps its columns to rows and rows to columns.

So cofactor expansion along the  $i$ th row of  $A$  is exactly the same as cofactor expansion along the  $i$ th column of  $A^T$ . Hence, both  $A$  and  $A^T$  must have the same determinant. □

# Evaluating Determinants by Row Reduction



Next I want to explain how the elementary row operations affect the determinant.

# Evaluating Determinants by Row Reduction



Next I want to explain how the elementary row operations affect the determinant.

- 1 What happens if we multiply a row by a number  $k$ ?

$$\begin{vmatrix} ka_{11} & ka_{12} & ka_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} =$$

=

=

# Evaluating Determinants by Row Reduction



Next I want to explain how the elementary row operations affect the determinant.

- 1 What happens if we multiply a row by a number  $k$ ?

$$\begin{vmatrix} ka_{11} & ka_{12} & ka_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = ka_{11}C_{11} + ka_{12}C_{12} + ka_{13}C_{13}$$

=

=

# Evaluating Determinants by Row Reduction



Next I want to explain how the elementary row operations affect the determinant.

- 1 What happens if we multiply a row by a number  $k$ ?

$$\begin{vmatrix} ka_{11} & ka_{12} & ka_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = ka_{11}C_{11} + ka_{12}C_{12} + ka_{13}C_{13}$$
$$= k(a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13})$$

=

# Evaluating Determinants by Row Reduction



Next I want to explain how the elementary row operations affect the determinant.

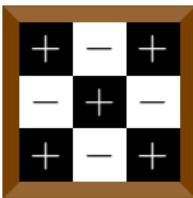
- 1 What happens if we multiply a row by a number  $k$ ?

$$\begin{aligned} \left| \begin{array}{ccc} ka_{11} & ka_{12} & ka_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{array} \right| &= ka_{11}C_{11} + ka_{12}C_{12} + ka_{13}C_{13} \\ &= k(a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13}) \\ &= k \left| \begin{array}{ccc} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{array} \right|. \end{aligned}$$

# Evaluating Determinants by Row Reduction



$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$



A 3x3 grid with alternating plus (+) and minus (-) signs. The pattern is as follows:

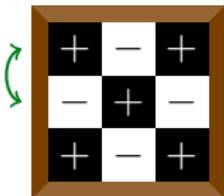
+	-	+
-	+	-
+	-	+

- 2 What happens if we swap two rows?

# Evaluating Determinants by Row Reduction



$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$



- 2 What happens if we swap two rows?

$$\begin{vmatrix} a_{21} & a_{22} & a_{23} \\ a_{11} & a_{12} & a_{13} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = - \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

# Evaluating Determinants by Row Reduction



$$\begin{array}{ccc} \curvearrowright & \longrightarrow & \\ \left[ \begin{array}{ccc} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{array} \right] & & \left[ \begin{array}{ccc} a_{31} & a_{32} & a_{33} \\ a_{21} & a_{22} & a_{23} \\ a_{11} & a_{12} & a_{13} \end{array} \right] \end{array}$$

# Evaluating Determinants by Row Reduction



$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \xrightarrow{\text{Row Swap}} \begin{bmatrix} a_{31} & a_{32} & a_{33} \\ a_{21} & a_{22} & a_{23} \\ a_{11} & a_{12} & a_{13} \end{bmatrix}$$

A 3x3 matrix with a brown border. The elements are arranged as follows:

+	-	+
-	+	-
+	-	+

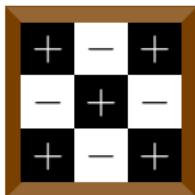
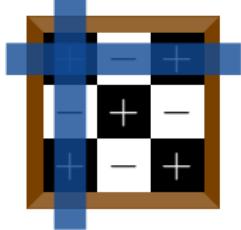
A 3x3 matrix with a brown border, showing the result of swapping the first and third rows from the initial matrix.

+	-	+
-	+	-
+	-	+

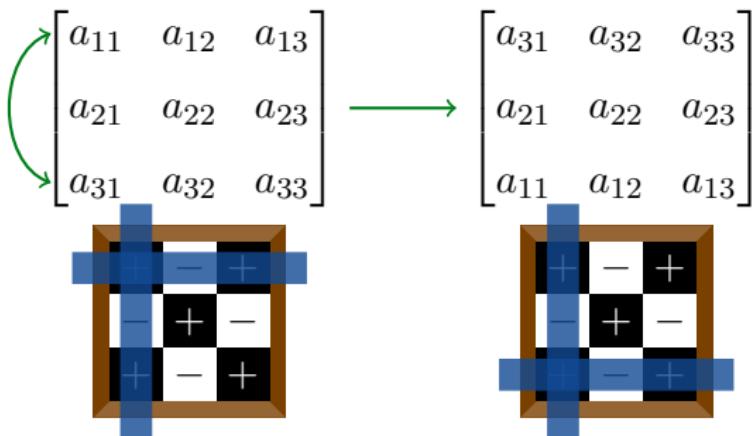
# Evaluating Determinants by Row Reduction



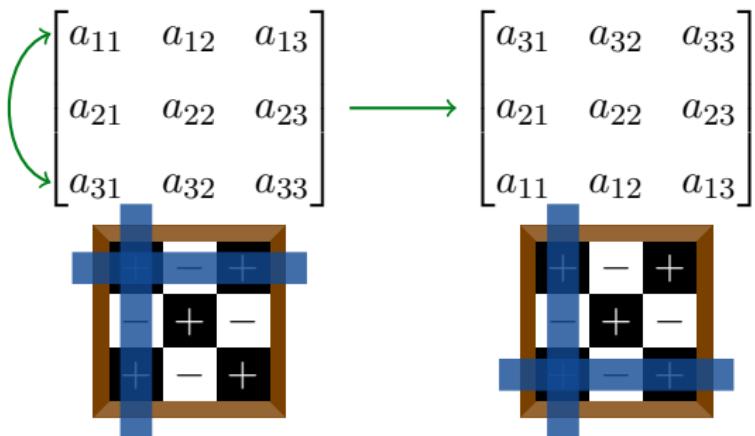
$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \xrightarrow{\text{Row Swap}} \begin{bmatrix} a_{31} & a_{32} & a_{33} \\ a_{21} & a_{22} & a_{23} \\ a_{11} & a_{12} & a_{13} \end{bmatrix}$$



# Evaluating Determinants by Row Reduction



# Evaluating Determinants by Row Reduction



We still get

$$\begin{vmatrix} a_{31} & a_{32} & a_{33} \\ a_{21} & a_{22} & a_{23} \\ a_{11} & a_{12} & a_{13} \end{vmatrix} = - \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

# Evaluating Determinants by Row Reduction



- 3 And what happens if add a multiple of one row to another row?

$$\begin{vmatrix} a_{11} + ka_{21} & a_{12} + ka_{22} & a_{13} + ka_{23} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

=

=

=

# Evaluating Determinants by Row Reduction



- 3 And what happens if add a multiple of one row to another row?

$$\begin{vmatrix} a_{11} + ka_{21} & a_{12} + ka_{22} & a_{13} + ka_{23} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

$$= (a_{11} + ka_{21})C_{11} + (a_{12} + ka_{22})C_{12} + (a_{13} + ka_{23})C_{13}$$

=

=

# Evaluating Determinants by Row Reduction



- 3 And what happens if add a multiple of one row to another row?

$$\begin{vmatrix} a_{11} + ka_{21} & a_{12} + ka_{22} & a_{13} + ka_{23} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

$$= (a_{11} + ka_{21})C_{11} + (a_{12} + ka_{22})C_{12} + (a_{13} + ka_{23})C_{13}$$

$$= a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13} + k(a_{21}C_{11} + a_{22}C_{12} + a_{23}C_{13})$$

=

# Evaluating Determinants by Row Reduction



- 3 And what happens if add a multiple of one row to another row?

$$\begin{aligned} & \left| \begin{array}{ccc} a_{11} + ka_{21} & a_{12} + ka_{22} & a_{13} + ka_{23} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{array} \right| \\ &= (a_{11} + ka_{21})C_{11} + (a_{12} + ka_{22})C_{12} + (a_{13} + ka_{23})C_{13} \\ &= a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13} + k(a_{21}C_{11} + a_{22}C_{12} + a_{23}C_{13}) \\ &= \left| \begin{array}{ccc} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{array} \right| + k(a_{21}C_{11} + a_{22}C_{12} + a_{23}C_{13}). \end{aligned}$$

But what is  $a_{21}C_{11} + a_{22}C_{12} + a_{23}C_{13}$ ?

# Evaluating Determinants by Row Reduction



$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

I leave it for you check that

$$\begin{aligned} a_{21}C_{11} + a_{22}C_{12} + a_{23}C_{13} &= \textcolor{green}{a_{21}}(a_{22}a_{33} - a_{23}a_{32}) \\ &\quad - \textcolor{green}{a_{22}}(a_{21}a_{33} - a_{23}a_{31}) \\ &\quad + a_{23}(a_{21}a_{32} - a_{22}a_{31}) \\ &= \end{aligned}$$

# Evaluating Determinants by Row Reduction



$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

I leave it for you check that

$$\begin{aligned} a_{21}C_{11} + a_{22}C_{12} + a_{23}C_{13} &= a_{21}(a_{22}a_{33} - a_{23}a_{32}) \\ &\quad - a_{22}(a_{21}a_{33} - a_{23}a_{31}) \\ &\quad + a_{23}(a_{21}a_{32} - a_{22}a_{31}) \\ &= 0. \end{aligned}$$

Therefore

$$\begin{vmatrix} a_{11} + ka_{21} & a_{12} + ka_{22} & a_{13} + ka_{23} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}.$$

# Evaluating Determinants by Row Reduction



## Theorem

- 1 If one row of  $A$  is multiplied by a number  $k$  to produce a matrix  $B$ , then

$$\det(B) = k \det(A).$$

(This is also true for columns.)

# Evaluating Determinants by Row Reduction



## Theorem

- 1 If one row of  $A$  is multiplied by a number  $k$  to produce a matrix  $B$ , then

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- 2 If two rows of  $A$  are swapped to produce  $B$ , then

$$\det(B) = -\det(A).$$

(This is also true for columns.)

# Evaluating Determinants by Row Reduction



## Theorem

- 1 If one row of  $A$  is multiplied by a number  $k$  to produce a matrix  $B$ , then

$$\det(B) = k \det(A).$$

- 2 If two rows of  $A$  are swapped to produce  $B$ , then

$$\det(B) = -\det(A).$$

- 3 If a multiple of one row of  $A$  is added to another row to produce  $B$ , then

$$\det(B) = \det(A).$$

(This is also true for columns.)

# Evaluating Determinants by Row Reduction



Example (Determinants of Elementary Matrices)

$$I_4 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \xrightarrow{3R_2} B = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\det(I_4) = 1$$

$$\det(B) = 3$$

# Evaluating Determinants by Row Reduction



Example (Determinants of Elementary Matrices)

$$I_4 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \xrightarrow{R_1 \leftrightarrow R_3} C = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\det(I_4) = 1$$

$$\det(C) = -1$$

# Evaluating Determinants by Row Reduction



Example (Determinants of Elementary Matrices)

$$I_4 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \xrightarrow{7R_4 + R_1} D = \begin{bmatrix} 1 & 0 & 0 & 7 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\det(I_4) = 1$$

$$\det(D) = 1$$

# Evaluating Determinants by Row Reduction



## Theorem

*If  $A$  is a square matrix with two proportional rows or two proportional columns, then  $\det(A) = 0$ .*

# Evaluating Determinants by Row Reduction



## Theorem

If  $A$  is a square matrix with two proportional rows or two proportional columns, then  $\det(A) = 0$ .

## Example

$$\begin{vmatrix} -1 & 4 \\ -2 & 8 \end{vmatrix} = 0, \quad \begin{vmatrix} 1 & -2 & 7 \\ -4 & 8 & 5 \\ 2 & -4 & 3 \end{vmatrix} = 0, \quad \begin{vmatrix} 3 & -1 & 5 & -5 \\ 6 & -2 & 5 & 2 \\ 5 & 8 & 1 & 4 \\ -9 & 3 & -15 & 15 \end{vmatrix} = 0.$$

# Evaluating Determinants by Row Reduction



Why?

# Evaluating Determinants by Row Reduction



Why? Because the elementary row operation  $3R_1 + R_4 \rightarrow R_4$  doesn't change the determinant and a matrix with a row of zeros always has zero determinant.

$$\left| \begin{array}{cccc} 3 & -1 & 5 & -5 \\ 6 & -2 & 5 & 2 \\ 5 & 8 & 1 & 4 \\ -9 & 3 & -15 & 15 \end{array} \right| \xrightarrow{3R_1+R_4} \left| \begin{array}{cccc} 3 & -1 & 5 & -5 \\ 6 & -2 & 5 & 2 \\ 5 & 8 & 1 & 4 \\ 0 & 0 & 0 & 0 \end{array} \right| = 0.$$

# Evaluating Determinants by Row Reduction



## Example (Using Row Reduction to Calculate $\det(A)$ )

Calculate the determinant of

$$A = \begin{bmatrix} 0 & 1 & 5 \\ 3 & -6 & 9 \\ 2 & 6 & 1 \end{bmatrix}.$$

We will reduce  $A$  to REF (which will be upper triangular), then calculate its determinant.

# Evaluating Determinants by Row Reduction



$$\det(A) = \begin{vmatrix} 0 & 1 & 5 \\ 3 & -6 & 9 \\ 2 & 6 & 1 \end{vmatrix} \xrightarrow{R_1 \leftrightarrow R_2} - \begin{vmatrix} 3 & -6 & 9 \\ 0 & 1 & 5 \\ 2 & 6 & 1 \end{vmatrix} \quad (\text{multiply by } -1)$$

( )

( )

( )

# Evaluating Determinants by Row Reduction



$$\det(A) = \begin{vmatrix} 0 & 1 & 5 \\ 3 & -6 & 9 \\ 2 & 6 & 1 \end{vmatrix} \xrightarrow{R_1 \leftrightarrow R_2} - \begin{vmatrix} 3 & -6 & 9 \\ 0 & 1 & 5 \\ 2 & 6 & 1 \end{vmatrix} \quad (\text{multiply by } -1)$$
$$\xrightarrow{\frac{1}{3}R_1} -3 \begin{vmatrix} 1 & -2 & 3 \\ 0 & 1 & 5 \\ 2 & 6 & 1 \end{vmatrix} \quad (\text{multiply by } 3)$$
$$()$$

$$()$$

# Evaluating Determinants by Row Reduction



$$\det(A) = \begin{vmatrix} 0 & 1 & 5 \\ 3 & -6 & 9 \\ 2 & 6 & 1 \end{vmatrix} \xrightarrow{R_1 \leftrightarrow R_2} - \begin{vmatrix} 3 & -6 & 9 \\ 0 & 1 & 5 \\ 2 & 6 & 1 \end{vmatrix} \quad (\text{multiply by } -1)$$
$$\xrightarrow{\frac{1}{3}R_1} -3 \begin{vmatrix} 1 & -2 & 3 \\ 0 & 1 & 5 \\ 2 & 6 & 1 \end{vmatrix} \quad (\text{multiply by } 3)$$
$$\xrightarrow{-2R_1+R_3} -3 \begin{vmatrix} 1 & -2 & 3 \\ 0 & 1 & 5 \\ 0 & 10 & -5 \end{vmatrix} \quad (\text{no change})$$
$$()$$

# Evaluating Determinants by Row Reduction



$$\det(A) = \begin{vmatrix} 0 & 1 & 5 \\ 3 & -6 & 9 \\ 2 & 6 & 1 \end{vmatrix} \xrightarrow{R_1 \leftrightarrow R_2} - \begin{vmatrix} 3 & -6 & 9 \\ 0 & 1 & 5 \\ 2 & 6 & 1 \end{vmatrix} \quad (\text{multiply by } -1)$$
$$\xrightarrow{\frac{1}{3}R_1} -3 \begin{vmatrix} 1 & -2 & 3 \\ 0 & 1 & 5 \\ 2 & 6 & 1 \end{vmatrix} \quad (\text{multiply by } 3)$$
$$\xrightarrow{-2R_1+R_3} -3 \begin{vmatrix} 1 & -2 & 3 \\ 0 & 1 & 5 \\ 0 & 10 & -5 \end{vmatrix} \quad (\text{no change})$$
$$\xrightarrow{-10R_2+R_3} -3 \begin{vmatrix} 1 & -2 & 3 \\ 0 & 1 & 5 \\ 0 & 0 & -55 \end{vmatrix} \quad (\text{no change})$$

# Evaluating Determinants by Row Reduction



$$\det(A) = \begin{vmatrix} 0 & 1 & 5 \\ 3 & -6 & 9 \\ 2 & 6 & 1 \end{vmatrix} \xrightarrow{R_1 \leftrightarrow R_2} - \begin{vmatrix} 3 & -6 & 9 \\ 0 & 1 & 5 \\ 2 & 6 & 1 \end{vmatrix} \quad (\text{multiply by } -1)$$
$$\xrightarrow{\frac{1}{3}R_1} -3 \begin{vmatrix} 1 & -2 & 3 \\ 0 & 1 & 5 \\ 2 & 6 & 1 \end{vmatrix} \quad (\text{multiply by } 3)$$
$$\xrightarrow{-2R_1+R_3} -3 \begin{vmatrix} 1 & -2 & 3 \\ 0 & 1 & 5 \\ 0 & 10 & -5 \end{vmatrix} \quad (\text{no change})$$
$$\xrightarrow{-10R_2+R_3} -3 \begin{vmatrix} 1 & -2 & 3 \\ 0 & 1 & 5 \\ 0 & 0 & -55 \end{vmatrix} \quad (\text{no change})$$
$$-3(1 \cdot 1 \cdot -55) = 165.$$

# Evaluating Determinants by Row Reduction



## Example

Calculate the determinant of

$$A = \begin{bmatrix} 2 & -8 & 6 & 8 \\ 3 & -9 & 5 & 10 \\ -3 & 0 & 1 & -2 \\ 1 & -4 & 0 & 6 \end{bmatrix}.$$

# Evaluating Determinants by Row Reduction



$$\begin{vmatrix} 2 & -8 & 6 & 8 \\ 3 & -9 & 5 & 10 \\ -3 & 0 & 1 & -2 \\ 1 & -4 & 0 & 6 \end{vmatrix} = 2 \begin{vmatrix} 1 & -4 & 3 & 4 \\ 3 & -9 & 5 & 10 \\ -3 & 0 & 1 & -2 \\ 1 & -4 & 0 & 6 \end{vmatrix} = 2 \begin{vmatrix} 1 & -4 & 3 & 4 \\ 0 & 3 & -4 & -2 \\ 0 & -12 & 10 & 10 \\ 0 & 0 & -3 & 2 \end{vmatrix}$$
$$= 2 \begin{vmatrix} 1 & -4 & 3 & 4 \\ 0 & 3 & -4 & -2 \\ 0 & 0 & -6 & 2 \\ 0 & 0 & -3 & 2 \end{vmatrix} = 2 \begin{vmatrix} 1 & -4 & 3 & 4 \\ 0 & 3 & -4 & -2 \\ 0 & 0 & -6 & 2 \\ 0 & 0 & 0 & 1 \end{vmatrix}$$
$$= 2(1 \cdot 3 \cdot (-6) \cdot 1) = -36.$$

# Evaluating Determinants by Row Reduction



Example (Row Reduction AND Cofactor Expansion)

Evaluate  $\det(A)$  where

$$A = \begin{bmatrix} 3 & 5 & -2 & 6 \\ 1 & 2 & -1 & 1 \\ 2 & 4 & 1 & 5 \\ 3 & 7 & 5 & 3 \end{bmatrix}.$$

# Evaluating Determinants by Row Reduction



$$\det(A) = \begin{vmatrix} 3 & 5 & -2 & 6 \\ 1 & 2 & -1 & 1 \\ 2 & 4 & 1 & 5 \\ 3 & 7 & 5 & 3 \end{vmatrix} \xrightarrow{\begin{array}{l} -3R_2+R_1 \\ -2R_2+R_3 \\ -3R_2+R_4 \end{array}} \begin{vmatrix} 0 & -1 & 1 & 3 \\ 1 & 2 & -1 & 1 \\ 0 & 0 & 3 & 3 \\ 0 & 1 & 8 & 0 \end{vmatrix}$$

=

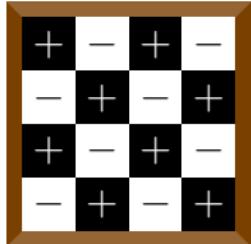
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# Evaluating Determinants by Row Reduction



$$\det(A) = \begin{vmatrix} 3 & 5 & -2 & 6 \\ 1 & 2 & -1 & 1 \\ 2 & 4 & 1 & 5 \\ 3 & 7 & 5 & 3 \end{vmatrix} \xrightarrow{\begin{array}{l} -3R_2+R_1 \\ -2R_2+R_3 \\ -3R_2+R_4 \end{array}} \begin{vmatrix} 0 & -1 & 1 & 3 \\ 1 & 2 & -1 & 1 \\ 0 & 0 & 3 & 3 \\ 0 & 1 & 8 & 0 \end{vmatrix}$$



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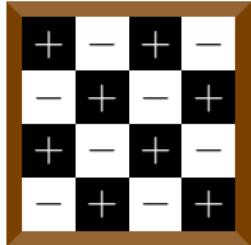
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# Evaluating Determinants by Row Reduction



$$\det(A) = \begin{vmatrix} 3 & 5 & -2 & 6 \\ 1 & 2 & -1 & 1 \\ 2 & 4 & 1 & 5 \\ 3 & 7 & 5 & 3 \end{vmatrix} \xrightarrow{\begin{array}{l} -3R_2+R_1 \\ -2R_2+R_3 \\ -3R_2+R_4 \end{array}} \begin{vmatrix} 0 & -1 & 1 & 3 \\ 1 & 2 & -1 & 1 \\ 0 & 0 & 3 & 3 \\ 0 & 1 & 8 & 0 \end{vmatrix}$$



$$= -1 C_{21} = - \begin{vmatrix} -1 & 1 & 3 \\ 0 & 3 & 3 \\ 1 & 8 & 0 \end{vmatrix}$$

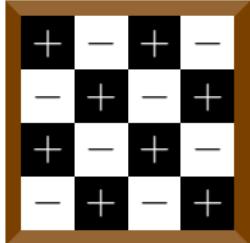
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# Evaluating Determinants by Row Reduction



$$\det(A) = \begin{vmatrix} 3 & 5 & -2 & 6 \\ 1 & 2 & -1 & 1 \\ 2 & 4 & 1 & 5 \\ 3 & 7 & 5 & 3 \end{vmatrix} \xrightarrow{\begin{array}{l} -3R_2+R_1 \\ -2R_2+R_3 \\ -3R_2+R_4 \end{array}} \begin{vmatrix} 0 & -1 & 1 & 3 \\ 1 & 2 & -1 & 1 \\ 0 & 0 & 3 & 3 \\ 0 & 1 & 8 & 0 \end{vmatrix}$$



$$= -1C_{21} = - \begin{vmatrix} -1 & 1 & 3 \\ 0 & 3 & 3 \\ 1 & 8 & 0 \end{vmatrix}$$

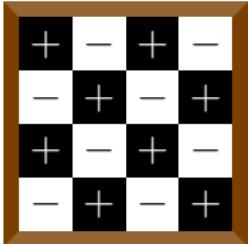
$$\xrightarrow{R_1+R_3} - \begin{vmatrix} -1 & 1 & 3 \\ 0 & 3 & 3 \\ 0 & 9 & 3 \end{vmatrix}$$

=

# Evaluating Determinants by Row Reduction

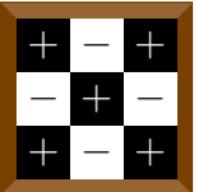


$$\det(A) = \begin{vmatrix} 3 & 5 & -2 & 6 \\ 1 & 2 & -1 & 1 \\ 2 & 4 & 1 & 5 \\ 3 & 7 & 5 & 3 \end{vmatrix} \xrightarrow{\begin{array}{l} -3R_2+R_1 \\ -2R_2+R_3 \\ -3R_2+R_4 \end{array}} \begin{vmatrix} 0 & -1 & 1 & 3 \\ 1 & 2 & -1 & 1 \\ 0 & 0 & 3 & 3 \\ 0 & 1 & 8 & 0 \end{vmatrix}$$



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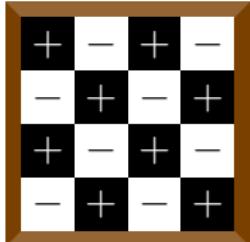


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# Evaluating Determinants by Row Reduction



$$\det(A) = \begin{vmatrix} 3 & 5 & -2 & 6 \\ 1 & 2 & -1 & 1 \\ 2 & 4 & 1 & 5 \\ 3 & 7 & 5 & 3 \end{vmatrix} \xrightarrow{\begin{array}{l} -3R_2+R_1 \\ -2R_2+R_3 \\ -3R_2+R_4 \end{array}} \begin{vmatrix} 0 & -1 & 1 & 3 \\ 1 & 2 & -1 & 1 \\ 0 & 0 & 3 & 3 \\ 0 & 1 & 8 & 0 \end{vmatrix}$$



$$= -1C_{21} = - \begin{vmatrix} -1 & 1 & 3 \\ 0 & 3 & 3 \\ 1 & 8 & 0 \end{vmatrix}$$

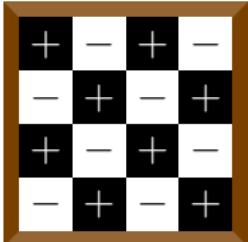
$$\xrightarrow{R_1+R_3} - \begin{vmatrix} -1 & 1 & 3 \\ 0 & 3 & 3 \\ 0 & 9 & 3 \end{vmatrix} \quad \begin{array}{|ccc|} \hline + & - & + \\ - & + & - \\ + & - & + \\ \hline \end{array}$$

$$= -(-1) \begin{vmatrix} 3 & 3 \\ 9 & 3 \end{vmatrix}$$

# Evaluating Determinants by Row Reduction



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$$= -1C_{21} = - \begin{vmatrix} -1 & 1 & 3 \\ 0 & 3 & 3 \\ 1 & 8 & 0 \end{vmatrix}$$

$$\xrightarrow{R_1+R_3} - \begin{vmatrix} -1 & 1 & 3 \\ 0 & 3 & 3 \\ 0 & 9 & 3 \end{vmatrix} \quad \begin{array}{|ccc|} \hline + & - & + \\ - & + & - \\ + & - & + \\ - & + & - \\ \hline \end{array}$$

$$= -(-1) \begin{vmatrix} 3 & 3 \\ 9 & 3 \end{vmatrix} = -18.$$

# Evaluating Determinants by Row Reduction



Example (Exam question from 2017)

Suppose that  $\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = 7$ .

Find  $\begin{vmatrix} d & e & f \\ a & b & c \\ g & h & i \end{vmatrix}$  and  $\begin{vmatrix} a & b & c \\ 2d+a & 2e+b & 2f+c \\ g & h & i \end{vmatrix}$ .

# Evaluating Determinants by Row Reduction



Example (Exam question from 2017)

Suppose that  $\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = 7$ .

Find  $\begin{vmatrix} d & e & f \\ a & b & c \\ g & h & i \end{vmatrix}$  and  $\begin{vmatrix} a & b & c \\ 2d+a & 2e+b & 2f+c \\ g & h & i \end{vmatrix}$ .

First

$$\begin{vmatrix} d & e & f \\ a & b & c \\ g & h & i \end{vmatrix} = -7$$

because swapping two rows multiplies the determinant by  $-1$ .

# Evaluating Determinants by Row Reduction



Multiplying a row by a constant  $k$ , also multiplies the determinant by  $k$ . So

$$\begin{vmatrix} a & b & c \\ 2d & 2e & 2f \\ g & h & i \end{vmatrix} = 2 \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = 14.$$

Since adding one row to another does not change the determinant, we have

$$\begin{vmatrix} a & b & c \\ 2d+a & 2e+b & 2f+c \\ g & h & i \end{vmatrix} = \begin{vmatrix} a & b & c \\ 2d & 2e & 2f \\ g & h & i \end{vmatrix} = 14$$

also.

# Break

We will continue at 3pm





# Properties of Determinants

# Properties of Determinants



- What is  $\det(kA)$ ?
- What is  $\det(A + B)$ ?
- What is  $\det(AB)$ ?

# Properties of Determinants



First note that

$$\begin{vmatrix} ka_{11} & ka_{12} & ka_{13} \\ ka_{21} & ka_{22} & ka_{23} \\ ka_{31} & ka_{32} & ka_{33} \end{vmatrix} = k \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ ka_{21} & ka_{22} & ka_{23} \\ ka_{31} & ka_{32} & ka_{33} \end{vmatrix} =$$

=

# Properties of Determinants



First note that

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=

# Properties of Determinants



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$$= kk \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}.$$

# Properties of Determinants

First note that

$$\begin{aligned}
 \begin{vmatrix} ka_{11} & ka_{12} & ka_{13} \\ ka_{21} & ka_{22} & ka_{23} \\ ka_{31} & ka_{32} & ka_{33} \end{vmatrix} &= k \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ ka_{21} & ka_{22} & ka_{23} \\ ka_{31} & ka_{32} & ka_{33} \end{vmatrix} = kk \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ ka_{31} & ka_{32} & ka_{33} \end{vmatrix} \\
 &= kk \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}.
 \end{aligned}$$

Theorem

If  $A$  is an  $n \times n$  matrix and if  $k$  is a number, then

$$\det(kA) = k^n \det(A).$$

# Properties of Determinants



Example (  $\det(A + B) \neq \det(A) + \det(B)$  )

Consider

$$A = \begin{bmatrix} 1 & 2 \\ 2 & 5 \end{bmatrix}, \quad B = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix} \quad \text{and} \quad A + B = \begin{bmatrix} 4 & 3 \\ 3 & 8 \end{bmatrix}.$$

Note that  $\det(A) = 1$ ,  $\det(B) = 8$  and  $\det(A + B) = 23$ .

Therefore

$$\det(A + B) \neq \det(A) + \det(B).$$

# Properties of Determinants



In general  $\det(A + B) \neq \det(A) + \det(B)$ .

However there is one special case to consider: When two matrices are the same except for one row.

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$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} a_{11} & a_{12} \\ b_{21} & b_{22} \end{bmatrix}.$$

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Note that these matrices have the same first row and only the second row differs. In this special case we have

$$\begin{aligned}\det(A) + \det(B) &= (a_{11}a_{22} - a_{12}a_{21}) + (a_{11}b_{22} - a_{12}b_{21}) \\ &= a_{11}(a_{22} + b_{22}) - a_{12}(a_{21} + b_{21}) \\ &= \begin{vmatrix} a_{11} & a_{12} \\ a_{22} + b_{22} & a_{21} + b_{21} \end{vmatrix}.\end{aligned}$$

# Properties of Determinants



## Theorem

Let  $A$ ,  $B$ , and  $C$  be  $n \times n$  matrices that differ only in a single row, say the  $r$ th row, and assume that the  $r$ th row of  $C$  can be obtained by adding the  $r$ th rows of  $A$  and  $B$ . Then

$$\det(C) = \det(A) + \det(B).$$

(The same result holds for columns.)

# Properties of Determinants

## Theorem

Let  $A$ ,  $B$ , and  $C$  be  $n \times n$  matrices that differ only in a single row, say the  $r$ th row, and assume that the  $r$ th row of  $C$  can be obtained by adding the  $r$ th rows of  $A$  and  $B$ . Then

$$\det(C) = \det(A) + \det(B).$$

(The same result holds for columns.)

## Example

Please check that

$$\begin{vmatrix} 1 & 7 & 5 \\ 2 & 0 & 3 \\ 1+0 & 4+1 & 7+(-1) \end{vmatrix} = \begin{vmatrix} 1 & 7 & 5 \\ 2 & 0 & 3 \\ 1 & 4 & 7 \end{vmatrix} + \begin{vmatrix} 1 & 7 & 5 \\ 2 & 0 & 3 \\ 0 & 1 & -1 \end{vmatrix}.$$

# Properties of Determinants



Theorem (Most important theorem of today's lecture)

*A square matrix  $A$  is invertible if and only if  $\det(A) \neq 0$ .*

# Properties of Determinants



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Example

Is  $A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 0 & 1 \\ 2 & 4 & 6 \end{bmatrix}$  invertible?

# Properties of Determinants



Theorem (Most important theorem of today's lecture)

*A square matrix  $A$  is invertible if and only if  $\det(A) \neq 0$ .*

Example

Is  $A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 0 & 1 \\ 2 & 4 & 6 \end{bmatrix}$  invertible?

Since row 1 and row 3 are proportional ( $R_3 = 2R_1$ ), we have  $\det(A) = 0$ . Therefore  $A$  is not invertible.  $A$  is singular.

# Properties of Determinants



Theorem

$$\det(AB) = \det(A) \det(B).$$

# Properties of Determinants

## Theorem

$$\det(AB) = \det(A) \det(B).$$

## Example

Let

$$A = \begin{bmatrix} 6 & 1 \\ 3 & 2 \end{bmatrix}, \quad B = \begin{bmatrix} 4 & 3 \\ 1 & 2 \end{bmatrix}, \quad \text{and} \quad AB = \begin{bmatrix} 25 & 20 \\ 14 & 13 \end{bmatrix}.$$

Note that

$$\det(A) \det(B) = 9 \cdot 5 = 45 = \det(AB).$$

# Properties of Determinants

$$\det(AB) = \det(A) \det(B)$$



Recall that

$$A^{-1}A = I.$$

So

$$\det(A^{-1}) \underbrace{\det(A)}_{\neq 0} = \det(I) = 1.$$

# Properties of Determinants

$$\det(AB) = \det(A) \det(B)$$



Recall that

$$A^{-1}A = I.$$

So

$$\det(A^{-1}) \underbrace{\det(A)}_{\neq 0} = \det(I) = 1.$$

## Theorem

*If  $A$  is invertible, then*

$$\det(A^{-1}) = \frac{1}{\det(A)}.$$

# Properties of Determinants

$$\det(AB) = \det(A) \det(B)$$



Because

$$\det(AA) = \det(A) \det(A)$$

$$\det(AAA) = \det(A) \det(A) \det(A)$$

$$\det(AAAA) = \det(A) \det(A) \det(A) \det(A)$$

⋮

we have

# Properties of Determinants

$$\det(AB) = \det(A) \det(B)$$



Because

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⋮

we have

Theorem

$$\det(A^n) = (\det(A))^n$$

for any  $n \in \mathbb{N}$ .

# Properties of Determinants



Example (Exam question from 2018)

Compute  $\det(2A^3)$  where  $A = \begin{bmatrix} 1 & 0 & 2 \\ 1 & 1 & 4 \\ 1 & 3 & 3 \end{bmatrix}$ .

# Properties of Determinants



Example (Exam question from 2018)

Compute  $\det(2A^3)$  where  $A = \begin{bmatrix} 1 & 0 & 2 \\ 1 & 1 & 4 \\ 1 & 3 & 3 \end{bmatrix}$ .

Since

$$\det(A) = \begin{vmatrix} 1 & 0 & 2 \\ 1 & 1 & 4 \\ 1 & 3 & 3 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 2 \\ 0 & 1 & 2 \\ 0 & 3 & 1 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 2 \\ 0 & 1 & 2 \\ 0 & 0 & -5 \end{vmatrix} = -5,$$

# Properties of Determinants

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we have that

$$\det(2A^3) = 2^3 \det(A^3)$$

# Properties of Determinants

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# Properties of Determinants

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we have that

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# Properties of Determinants



## Example

Let  $A$ ,  $B$  and  $C$  be  $4 \times 4$  matrices with  $\det(A) = 1$ ,  $\det(B) = 2$  and  $\det(C) = 3$ . Find  $\det(3A^{-1}B^TC^2)$ .

$$\det(3A^{-1}B^TC^2) = 3^4 \det(A^{-1}) \det(B^T) \det(C^2)$$

=

=

# Properties of Determinants



## Example

Let  $A$ ,  $B$  and  $C$  be  $4 \times 4$  matrices with  $\det(A) = 1$ ,  $\det(B) = 2$  and  $\det(C) = 3$ . Find  $\det(3A^{-1}B^TC^2)$ .

$$\begin{aligned}\det(3A^{-1}B^TC^2) &= 3^4 \det(A^{-1}) \det(B^T) \det(C^2) \\ &= 3^4 \frac{1}{\det(A)} \det(B) (\det(C))^2 \\ &= \dots\end{aligned}$$

## Adjoint of a Matrix

Let  $A$  be an  $n \times n$  square matrix and let  $C_{ij}$  denote the cofactors of  $A$ .

### Definition

The matrix

$$\begin{bmatrix} C_{11} & C_{12} & \dots & C_{1n} \\ C_{21} & C_{22} & \dots & C_{2n} \\ \vdots & \vdots & & \vdots \\ C_{n1} & C_{n2} & \dots & C_{nn} \end{bmatrix}$$

is called the *matrix of cofactors from  $A$* .

## Adjoint of a Matrix

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is called the *matrix of cofactors from  $A$* .

### Definition

The transpose of the matrix of cofactors from  $A$  is called the *adjoint of  $A$*  and is denoted by  $\text{adj}(A)$ .

# Properties of Determinants

## Example

Find the adjoint of  $A = \begin{bmatrix} 3 & 2 & -1 \\ 1 & 6 & 3 \\ 2 & -4 & 0 \end{bmatrix}$ .

# Properties of Determinants



## Example

Find the adjoint of  $A = \begin{bmatrix} 3 & 2 & -1 \\ 1 & 6 & 3 \\ 2 & -4 & 0 \end{bmatrix}$ .

- 1 Find the cofactors of  $A$ .
- 2 Write the matrix of cofactors.
- 3 Take the transpose.

# Properties of Determinants



## Example

Find the adjoint of  $A = \begin{bmatrix} 3 & 2 & -1 \\ 1 & 6 & 3 \\ 2 & -4 & 0 \end{bmatrix}$ .

- 1 Find the cofactors of  $A$ .

$$C_{11} = 12 \quad C_{12} = 6 \quad C_{13} = -16$$

$$C_{21} = 4 \quad C_{22} = 2 \quad C_{23} = 16$$

$$C_{31} = 12 \quad C_{32} = -10 \quad C_{33} = 16$$

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# Properties of Determinants



## Example

Find the adjoint of  $A = \begin{bmatrix} 3 & 2 & -1 \\ 1 & 6 & 3 \\ 2 & -4 & 0 \end{bmatrix}$ .

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# Properties of Determinants



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- 2 Write the matrix of cofactors.

$$\begin{bmatrix} 12 & 6 & -16 \\ 4 & 2 & 16 \\ 12 & -10 & 16 \end{bmatrix}$$

- 3 Take the transpose.

$$\text{adj}(A) = \begin{bmatrix} 12 & 4 & 12 \\ 6 & 2 & -10 \\ -16 & 16 & 16 \end{bmatrix}$$

# Properties of Determinants



## Theorem

*If  $A$  is an invertible matrix, then*

$$A^{-1} = \frac{1}{\det(A)} \operatorname{adj}(A).$$

# Properties of Determinants

$$A^{-1} = \frac{1}{\det(A)} \text{adj}(A)$$



## Example

Use this theorem to find the inverse of  $A = \begin{bmatrix} 3 & 2 & -1 \\ 1 & 6 & 3 \\ 2 & -4 & 0 \end{bmatrix}$ .

In the last example we found that  $\text{adj}(A) = \begin{bmatrix} 12 & 4 & 12 \\ 6 & 2 & -10 \\ -16 & 16 & 16 \end{bmatrix}$ .

So we just need to find  $\det(A)$  then use the theorem.

$$A^{-1} = \frac{1}{\det(A)} \text{adj}(A)$$



## Example

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So we just need to find  $\det(A)$  then use the theorem.

Using cofactor expansion along the first row, we calculate that

$$\det(A) = 12C_{11} + 2C_{12} + (-1)C_{13} = 36 + 12 + 16 = 64.$$

$$A^{-1} = \frac{1}{\det(A)} \text{adj}(A)$$



## Example

Use this theorem to find the inverse of  $A = \begin{bmatrix} 3 & 2 & -1 \\ 1 & 6 & 3 \\ 2 & -4 & 0 \end{bmatrix}$ .

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Hence

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# Properties of Determinants

$$A^{-1} = \frac{1}{\det(A)} \text{adj}(A)$$



## Example

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Hence

$$A^{-1} = \frac{1}{\det(A)} \text{adj}(A) = \frac{1}{64} \begin{bmatrix} 12 & 4 & 12 \\ 6 & 2 & -10 \\ -16 & 16 & 16 \end{bmatrix} = \begin{bmatrix} \frac{12}{64} & \frac{4}{64} & \frac{12}{64} \\ \frac{6}{64} & \frac{2}{64} & -\frac{10}{64} \\ -\frac{16}{64} & \frac{16}{64} & \frac{16}{64} \end{bmatrix}.$$

**EXAMPLE 3** Find the inverse of the matrix  $A = \begin{bmatrix} 2 & 1 & 3 \\ 1 & -1 & 1 \\ 1 & 4 & -2 \end{bmatrix}$ .

**SOLUTION** The nine cofactors are

$$\begin{aligned} C_{11} &= + \begin{vmatrix} -1 & 1 \\ 4 & -2 \end{vmatrix} = -2, & C_{12} &= - \begin{vmatrix} 1 & 1 \\ 1 & -2 \end{vmatrix} = 3, & C_{13} &= + \begin{vmatrix} 1 & -1 \\ 1 & 4 \end{vmatrix} = 5 \\ C_{21} &= - \begin{vmatrix} 1 & 3 \\ 4 & -2 \end{vmatrix} = 14, & C_{22} &= + \begin{vmatrix} 2 & 3 \\ 1 & -2 \end{vmatrix} = -7, & C_{23} &= - \begin{vmatrix} 2 & 1 \\ 1 & 4 \end{vmatrix} = -7 \\ C_{31} &= + \begin{vmatrix} 1 & 3 \\ -1 & 1 \end{vmatrix} = 4, & C_{32} &= - \begin{vmatrix} 2 & 3 \\ 1 & 1 \end{vmatrix} = 1, & C_{33} &= + \begin{vmatrix} 2 & 1 \\ 1 & -1 \end{vmatrix} = -3 \end{aligned}$$

The adjugate matrix is the *transpose* of the matrix of cofactors. [For instance,  $C_{12}$  goes in the (2, 1) position.] Thus

$$\text{adj } A = \begin{bmatrix} -2 & 14 & 4 \\ 3 & -7 & 1 \\ 5 & -7 & -3 \end{bmatrix}$$

We could compute  $\det A$  directly, but the following computation provides a check on the calculations above *and* produces  $\det A$ :

$$(\text{adj } A) \cdot A = \begin{bmatrix} -2 & 14 & 4 \\ 3 & -7 & 1 \\ 5 & -7 & -3 \end{bmatrix} \begin{bmatrix} 2 & 1 & 3 \\ 1 & -1 & 1 \\ 1 & 4 & -2 \end{bmatrix} = \begin{bmatrix} 14 & 0 & 0 \\ 0 & 14 & 0 \\ 0 & 0 & 14 \end{bmatrix} = 14I$$

Since  $(\text{adj } A)A = 14I$ , Theorem 8 shows that  $\det A = 14$  and

$$A^{-1} = \frac{1}{14} \begin{bmatrix} -2 & 14 & 4 \\ 3 & -7 & 1 \\ 5 & -7 & -3 \end{bmatrix} = \begin{bmatrix} -1/7 & 1 & 2/7 \\ 3/14 & -1/2 & 1/14 \\ 5/14 & -1/2 & -3/14 \end{bmatrix}$$

■



# Cramer's Rule

# Cramer's Rule



Suppose we have an  $n \times n$  matrix and an  $n \times 1$  vector.

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}.$$

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I want to replace the first column of  $A$  by  $\mathbf{b}$  and call this new matrix  $A_1$ .

$$A_1 = \begin{bmatrix} b_1 & a_{12} & \cdots & a_{1n} \\ b_2 & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ b_n & a_{n2} & \cdots & a_{nn} \end{bmatrix}.$$

## Cramer's Rule

Similarly I want to replace the second column of  $A$  by  $\mathbf{b}$  and call this new matrix  $A_2$ .

$$A_2 = \begin{bmatrix} a_{11} & \textcolor{brown}{b}_1 & \cdots & a_{1n} \\ a_{21} & \textcolor{brown}{b}_2 & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & \textcolor{brown}{b}_n & \cdots & a_{nn} \end{bmatrix}.$$

## Cramer's Rule

Similarly I want to replace the second column of  $A$  by  $\mathbf{b}$  and call this new matrix  $A_2$ .

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⋮

And I want to replace the  $n$ th column of  $A$  by  $\mathbf{b}$  and call this new matrix  $A_n$ .

$$A_n = \begin{bmatrix} a_{11} & a_{12} & \cdots & \textcolor{brown}{b}_1 \\ a_{21} & a_{22} & \cdots & \textcolor{brown}{b}_2 \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & \textcolor{brown}{b}_n \end{bmatrix}.$$

# Cramer's Rule



## Theorem (Cramer's Rule)

If

$$A\mathbf{x} = \mathbf{b}$$

is a system of  $n$  linear equations in  $n$  unknowns and if  $\det(A) \neq 0$ , then this linear system has a unique solution which is given by

$$x_1 = \frac{\det(A_1)}{\det(A)}, \quad x_2 = \frac{\det(A_2)}{\det(A)}, \quad \dots, \quad x_n = \frac{\det(A_n)}{\det(A)}.$$

$$x_1 = \frac{\det(A_1)}{\det(A)}, x_2 = \frac{\det(A_2)}{\det(A)}, \dots, x_n = \frac{\det(A_n)}{\det(A)}.$$

## Example

Use Cramer's Rule to solve  $\begin{cases} 3x_1 - 2x_2 = 6 \\ -5x_1 + 4x_2 = 8 \end{cases}$ .

$$x_1 = \frac{\det(A_1)}{\det(A)}, x_2 = \frac{\det(A_2)}{\det(A)}, \dots, x_n = \frac{\det(A_n)}{\det(A)}.$$

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We have  $A\mathbf{x} = \mathbf{b}$  where

$$A = \begin{bmatrix} 3 & -2 \\ -5 & 4 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 6 \\ 8 \end{bmatrix}, \quad A_1 = \begin{bmatrix} 6 & -2 \\ 8 & 4 \end{bmatrix} \text{ and } A_2 = \begin{bmatrix} 3 & 6 \\ -5 & 8 \end{bmatrix}.$$

$$x_1 = \frac{\det(A_1)}{\det(A)}, x_2 = \frac{\det(A_2)}{\det(A)}, \dots, x_n = \frac{\det(A_n)}{\det(A)}.$$

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Since  $\det(A) = 2 \neq 0$ , the linear system has a unique solution.

$$x_1 = \frac{\det(A_1)}{\det(A)}, x_2 = \frac{\det(A_2)}{\det(A)}, \dots, x_n = \frac{\det(A_n)}{\det(A)}.$$

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Use Cramer's Rule to solve  $\begin{cases} 3x_1 - 2x_2 = 6 \\ -5x_1 + 4x_2 = 8 \end{cases}$ .

We have  $A\mathbf{x} = \mathbf{b}$  where

$$A = \begin{bmatrix} 3 & -2 \\ -5 & 4 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 6 \\ 8 \end{bmatrix}, \quad A_1 = \begin{bmatrix} 6 & -2 \\ 8 & 4 \end{bmatrix} \text{ and } A_2 = \begin{bmatrix} 3 & 6 \\ -5 & 8 \end{bmatrix}.$$

Since  $\det(A) = 2 \neq 0$ , the linear system has a unique solution.

By Cramer's Rule, this solution is

$$x_1 = \frac{\det(A_1)}{\det(A)} = \frac{24 + 16}{2} = 20$$

$$x_2 = \frac{\det(A_2)}{\det(A)} = \frac{24 + 30}{2} = 27.$$

## Cramer's Rule

$$x_1 = \frac{\det(A_1)}{\det(A)}, x_2 = \frac{\det(A_2)}{\det(A)}, \dots, x_n = \frac{\det(A_n)}{\det(A)}.$$



Cramer's Rule is useful if you only want to find some variables and you don't care about the rest.

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Cramer's Rule is useful if you only want to find some variables and you don't care about the rest.

### Example

Find  $x_3$  if  $\begin{cases} x_1 + 2x_3 = 6 \\ -3x_1 + 4x_2 + 6x_3 = 30 \\ -x_1 - 2x_2 + 3x_3 = 8. \end{cases}$

## Cramer's Rule

$$x_1 = \frac{\det(A_1)}{\det(A)}, x_2 = \frac{\det(A_2)}{\det(A)}, \dots, x_n = \frac{\det(A_n)}{\det(A)}.$$

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### Example

Find  $x_3$  if  $\begin{cases} x_1 + 2x_3 = 6 \\ -3x_1 + 4x_2 + 6x_3 = 30 \\ -x_1 - 2x_2 + 3x_3 = 8. \end{cases}$

Here we have  $A\mathbf{x} = \mathbf{b}$  with

$$A = \begin{bmatrix} 1 & 0 & 2 \\ -3 & 4 & 6 \\ -1 & -2 & 3 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 6 \\ 30 \\ 8 \end{bmatrix} \quad \text{and} \quad A_3 = \begin{bmatrix} 1 & 0 & 6 \\ -3 & 4 & 30 \\ -1 & -2 & 8 \end{bmatrix}.$$

I leave it for you to check that  $\det(A) = 44$  and  $\det(A_3) = 152$ .

## Cramer's Rule

$$x_1 = \frac{\det(A_1)}{\det(A)}, x_2 = \frac{\det(A_2)}{\det(A)}, \dots, x_n = \frac{\det(A_n)}{\det(A)}.$$

Cramer's Rule is useful if you only want to find some variables and you don't care about the rest.

### Example

Find  $x_3$  if  $\begin{cases} x_1 + 2x_3 = 6 \\ -3x_1 + 4x_2 + 6x_3 = 30 \\ -x_1 - 2x_2 + 3x_3 = 8. \end{cases}$

Here we have  $A\mathbf{x} = \mathbf{b}$  with

$$A = \begin{bmatrix} 1 & 0 & 2 \\ -3 & 4 & 6 \\ -1 & -2 & 3 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 6 \\ 30 \\ 8 \end{bmatrix} \quad \text{and} \quad A_3 = \begin{bmatrix} 1 & 0 & 6 \\ -3 & 4 & 30 \\ -1 & -2 & 8 \end{bmatrix}.$$

I leave it for you to check that  $\det(A) = 44$  and  $\det(A_3) = 152$ . Since  $\det(A) \neq 0$ , the linear system has a unique solution. By Cramer's Rule, we calculate that  $x_3 = \frac{\det(A_3)}{\det(A)} = \frac{152}{44} = \frac{38}{11}$ .

$$x_1 = \frac{\det(A_1)}{\det(A)}, x_2 = \frac{\det(A_2)}{\det(A)}, \dots, x_n = \frac{\det(A_n)}{\det(A)}.$$

## Example

Consider the following system in which  $s$  is an unspecified parameter. Determine the values of  $s$  for which the system has a unique solution, and use Cramer's rule to find the solution.

$$\begin{cases} 3sx_1 - 2x_2 = 4 \\ -6x_1 + sx_2 = 1. \end{cases}$$

$$x_1 = \frac{\det(A_1)}{\det(A)}, x_2 = \frac{\det(A_2)}{\det(A)}, \dots, x_n = \frac{\det(A_n)}{\det(A)}.$$

## Example

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$$\begin{cases} 3sx_1 - 2x_2 = 4 \\ -6x_1 + sx_2 = 1. \end{cases}$$

We have  $A\mathbf{x} = \mathbf{b}$  with

$$A = \begin{bmatrix} 3s & -2 \\ -6 & s \end{bmatrix}, \quad A_1 = \begin{bmatrix} 4 & -2 \\ 1 & s \end{bmatrix} \quad \text{and} \quad A_2 = \begin{bmatrix} 3s & 4 \\ -6 & 1 \end{bmatrix}.$$

Since

$$\det(A) = 3s^2 - 12 = 3(s+2)(s-2),$$

the linear system has a unique solution when  $s \neq \pm 2$ .

$$x_1 = \frac{\det(A_1)}{\det(A)}, x_2 = \frac{\det(A_2)}{\det(A)}, \dots, x_n = \frac{\det(A_n)}{\det(A)}.$$

$$A = \begin{bmatrix} 3s & -2 \\ -6 & s \end{bmatrix}, \quad A_1 = \begin{bmatrix} 4 & -2 \\ 1 & s \end{bmatrix} \quad \text{and} \quad A_2 = \begin{bmatrix} 3s & 4 \\ -6 & 1 \end{bmatrix}$$

By Cramer's Rule, this solution is

$$x_1 = \frac{\det(A_1)}{\det(A)} = \frac{4x + 2}{3(s+2)(s-2)}$$

$$x_2 = \frac{\det(A_2)}{\det(A)} = \frac{3s + 24}{3(s+2)(s-2)} = \frac{s + 8}{(s+2)(s-2)}.$$

for  $s \neq \pm 2$ .



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FORENAME: \_\_\_\_\_

SURNAME: \_\_\_\_\_

STUDENT NO: \_\_\_\_\_

DEPARTMENT: \_\_\_\_\_

TEACHER:  Neil Course     Vasfi Eldem     M.Tuba Gülpınar     Hasan Özkes

SIGNATURE: \_\_\_\_\_

Question	Points	Score
1	25	
2	25	
3	25	
4	25	
Total:	100	

- The time limit is 75 minutes.
- Any attempts at cheating or plagiarizing and assisting of such actions in any form would result in getting an automatic zero (0) from the exam. Disciplinary action will also be taken in accordance with the regulations of the Council of Higher Education.
- Give your answers in exact form (for ex-

ample  $\frac{\pi}{3}$  or  $5\sqrt{3}$ ), except as noted in particular problems.

- Calculators, mobile phones, smart watches, etc. are not allowed.
- In order to receive credit, you must **show all of your work**. If you do not indicate the way in which you solved a problem, you may get little or no credit for it, even

if your answer is correct.

- Place a box around your answer to each question.
- Use a **BLUE ball-point pen** to fill the cover sheet. Please make sure that your exam is complete.
- Do not write in the table above.

- 
1. **25 points** For which values of  $k$  does the system  $x + 2y + 6z = 2$   
 $y + 2kz = 0$       have  
 $kx + 2z = 1$

- (a) no solution.
- (b) infinitely many solutions.
- (c) a unique solution.

1. [25 points] For which values of  $k$  does the system
- $$\begin{array}{l} x + 2y + 6z = 2 \\ y + 2kz = 0 \\ kx + 2z = 1 \end{array}$$
- have

- (a) no solution.
- (b) infinitely many solutions.
- (c) a unique solution.

**Solution:** Let us transform the augmented matrix to row echelon form.

$$\left[ \begin{array}{cccc} 1 & 2 & 6 & 2 \\ 0 & 1 & 2k & 0 \\ k & 0 & 2 & 1 \end{array} \right] \sim \left[ \begin{array}{cccc} 1 & 2 & 6 & 2 \\ 0 & 1 & 2k & 0 \\ 0 & -2k & 2-6k & 1-2k \end{array} \right] \sim \left[ \begin{array}{cccc} 1 & 2 & 6 & 2 \\ 0 & 1 & 2k & 0 \\ 0 & 0 & 4k^2-6k+2 & 1-2k \end{array} \right] \sim \left[ \begin{array}{cccc} 1 & 2 & 6 & 2 \\ 0 & 1 & 2k & 0 \\ 0 & 0 & 2(2k-1)(k-1) & 1-2k \end{array} \right]$$

If  $k = 1$ , then we obtain  $\left[ \begin{array}{cccc} 1 & 2 & 6 & 2 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & -1 \end{array} \right]$ . Therefore, the system has no solution.

If  $k = \frac{1}{2}$ , then we obtain  $\left[ \begin{array}{cccc} 1 & 2 & 6 & 2 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right]$ . Therefore, the system has infinitely many solutions.

If  $k \neq 1$  and  $k \neq \frac{1}{2}$ , then we obtain  $\left[ \begin{array}{cccc} 1 & 2 & 6 & 2 \\ 0 & 1 & 2k & 0 \\ 0 & 0 & 1 & -\frac{1}{2(k-1)} \end{array} \right]$ . Therefore, the system have a unique solution.



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2. (a) 15 points Calculate the determinant of the matrix  $A = \begin{bmatrix} 1 & 2 & 3 & -1 \\ 0 & 0 & 4 & -2 \\ 3 & -2 & 1 & 1 \\ 2 & 0 & 1 & 1 \end{bmatrix}$ .

**Solution:**

$$\begin{vmatrix} 1 & 2 & 3 & -1 \\ 0 & 0 & 4 & -2 \\ 3 & -2 & 1 & 1 \\ 2 & 0 & 1 & 1 \end{vmatrix} = \begin{vmatrix} 1 & 2 & 3 & -1 \\ 0 & 0 & 4 & -2 \\ 4 & 0 & 4 & 0 \\ 2 & 0 & 1 & 1 \end{vmatrix} = 2(-1)^{1+2} \begin{vmatrix} 0 & 4 & -2 \\ 4 & 4 & 0 \\ 2 & 1 & 1 \end{vmatrix} + 0 + 0 + 0$$

$$= (-2) \begin{vmatrix} 0 & 4 & -2 \\ 0 & 2 & -2 \\ 2 & 1 & 1 \end{vmatrix} = (-2) \left[ (2)(-1)^{3+1} \begin{vmatrix} 4 & -2 \\ 2 & -2 \end{vmatrix} \right] = (-4)(-8 + 4) = 16$$

- (b) 10 points Suppose that  $A\mathbf{x} = \mathbf{b}$  where  $A = \begin{bmatrix} 1 & 1 & 3 & -1 \\ 0 & 2 & 4 & -2 \\ 3 & 0 & 1 & 1 \\ 2 & 4 & 1 & 1 \end{bmatrix}$ ,  $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$ ,  $\mathbf{b} = \begin{bmatrix} 2 \\ 0 \\ -2 \\ 0 \end{bmatrix}$  and  $\det A = (-2)$ . Use Cramer's rule to find  $x_2$ .

**Solution:**

$$x_2 = \frac{|A_2|}{|A|} = \frac{\begin{vmatrix} 1 & 2 & 3 & -1 \\ 0 & 0 & 4 & -2 \\ 3 & -2 & 1 & 1 \\ 2 & 0 & 1 & 1 \end{vmatrix}}{-2} = \frac{16}{-2} = -8$$

3. (a) [10 points] Find the adjoint matrix ( $AdjA$ ) of  $A = \begin{bmatrix} 1 & 2 & 6 \\ 0 & 1 & 6 \\ 0 & 0 & 2 \end{bmatrix}$ .

**Solution:**

$$\begin{aligned} C_{11} &= (-1)^{1+1} \begin{vmatrix} 1 & 6 \\ 0 & 2 \end{vmatrix} = 2 & C_{21} &= (-1)^{2+1} \begin{vmatrix} 2 & 6 \\ 0 & 2 \end{vmatrix} = -4 & C_{31} &= (-1)^{3+1} \begin{vmatrix} 2 & 6 \\ 1 & 6 \end{vmatrix} = 6 \\ C_{12} &= (-1)^{1+2} \begin{vmatrix} 0 & 6 \\ 0 & 2 \end{vmatrix} = 0 & C_{22} &= (-1)^{2+2} \begin{vmatrix} 1 & 6 \\ 0 & 2 \end{vmatrix} = 2 & C_{32} &= (-1)^{3+2} \begin{vmatrix} 1 & 6 \\ 0 & 6 \end{vmatrix} = -6 \\ C_{13} &= (-1)^{1+3} \begin{vmatrix} 0 & 1 \\ 0 & 0 \end{vmatrix} = 0 & C_{23} &= (-1)^{2+3} \begin{vmatrix} 1 & 2 \\ 0 & 0 \end{vmatrix} = 0 & C_{33} &= (-1)^{3+3} \begin{vmatrix} 1 & 2 \\ 0 & 1 \end{vmatrix} = 1 \end{aligned}$$

$$AdjA = \begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix}^T = \begin{bmatrix} 2 & 0 & 0 \\ -4 & 2 & 0 \\ 6 & -6 & 1 \end{bmatrix}^T = \begin{bmatrix} 2 & -4 & 6 \\ 0 & 2 & -6 \\ 0 & 0 & 1 \end{bmatrix}$$

- (b) 15 points Find the inverse of the matrix  $A = \begin{bmatrix} 1 & 2 & 6 \\ 0 & 1 & 6 \\ 0 & 0 & 2 \end{bmatrix}$ .

**Solution:** First Way:

$$A^{-1} = \frac{1}{\det A} \text{Adj} A = \frac{1}{2} \begin{bmatrix} 2 & -4 & 6 \\ 0 & 2 & -6 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -2 & 3 \\ 0 & 1 & -3 \\ 0 & 0 & 0.5 \end{bmatrix}$$

Second Way:

$$\left[ \begin{array}{ccc|ccc} 1 & 2 & 6 & 1 & 0 & 0 \\ 0 & 1 & 6 & 0 & 1 & 0 \\ 0 & 0 & 2 & 0 & 0 & 1 \end{array} \right] \sim \left[ \begin{array}{ccc|ccc} 1 & 0 & -6 & 1 & -2 & 0 \\ 0 & 1 & 6 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0.5 \end{array} \right] \sim \left[ \begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -2 & 3 \\ 0 & 1 & 0 & 0 & 1 & -3 \\ 0 & 0 & 1 & 0 & 0 & 0.5 \end{array} \right] \quad [A|I] \sim [I|A^{-1}]$$



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4. (a) **10 points** Suppose an  $n \times n$  matrix  $A$  satisfies the equation  $A^2 - 2A + I = 0$ . Show that  $A^3 = 3A - 2I$ .

**Solution:**

$$A^2 = 2A - I \Rightarrow A^3 = 2A^2 - A = 2(2A - I) - A = 3A - 2I$$

- (b) **15 points** Let  $A$ ,  $B$  and  $C$  be  $3 \times 3$  matrices with  $\det A = -3$ ,  $\det B = 4$  and  $\det C = 2$ . Compute  $\det(2A^2B^{-2}C^T)$ .

**Solution:**

$$\det(2A^2B^{-2}C^T) = 2^3(\det A)^2 \frac{1}{(\det B)^2} (\det C) = 8(-3)^2 \frac{1}{4^2} 2 = 9$$



# Next Time

- Vector Spaces and Subspaces
- Null Spaces, Column Spaces, and Linear Transformations
- Linearly Independent Sets; Bases