Lecture 2

- 4. Intervals
- 5. Cartesian Coordinates
- 6. Functions
- 7. Sigma Notation

4

Intervals

4. Intervals

Definition

A subset of $\mathbb{R}$ is called an *interval* if

1. it contains atleast 2 numbers; and
2. it doesn't have any holes in it.

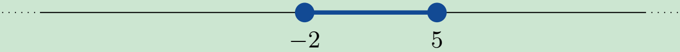
Example

The set $\{x \mid x \text{ is a real number and } x > 6\}$ is an interval.

Because 6 is not in this set, we use $\circ$ at 6.

Example

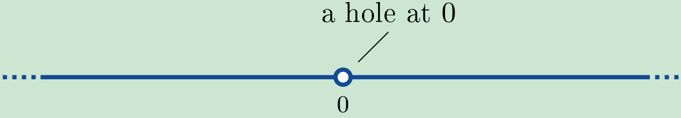
The set of all real numbers x such that $-2 \leq x \leq 5$ is an interval.



Because -2 and 5 are in this set, we use ● at -2 and 5.

Example

The set $\{x \mid x \in \mathbb{R} \text{ and } x \neq 0\}$ is not an interval.



A finite interval is

- *closed* if it contains both its endpoints;
- *half-open* if it contains one of its endpoints;
- *open* if it does not contain its endpoints;

4. Intervals

Notation	Set	Type	Picture
(a,b)	$\{x \mid a < x < b\}$	open	
$[a,b]$	$\{x \mid a \leq x \leq b\}$	closed	
$[a,b)$	$\{x \mid a \leq x < b\}$	half open	
$(a,b]$	$\{x \mid a < x \leq b\}$	half open	

An infinite interval is

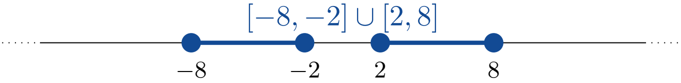
- *closed* if it contains a finite endpoint;
- *open* if it is not closed.

There is one exception to this rule: The whole real line is called both open and closed.

4. Intervals

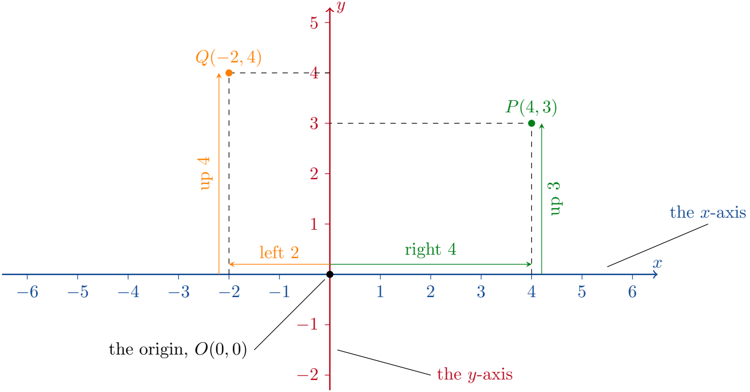
Notation	Set	Type	Picture
(a, ∞)	$\{x \mid a < x\}$	open	
$[a, \infty)$	$\{x \mid a \leq x\}$	closed	
$(-\infty, b)$	$\{x \mid x < b\}$	open	
$(-\infty, b]$	$\{x \mid x \leq b\}$	closed	
$(-\infty, \infty)$	$\mathbb{R}$	both open and closed	

We can combine two (or more) intervals with the notation $\cup$. For example, $[-8,-2] \cup [2,8]$ is called the *union* of $[-8,-2]$ and $[2,8]$ and is shown below.



Cartesian Coordinates

5. Cartesian Coordinates



Definition

The set
$$\{(x, y) | x, y \in \mathbb{R}\}$$
is denoted by $\mathbb{R}^2$.

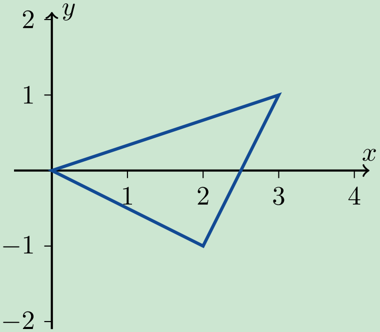
Definition

The point $O(0, 0)$ is called the *origin*.

Example

Let $A(2,-1)$ and $B(3,1)$be points in $\mathbb{R}^2$. Draw the triangle OAB.

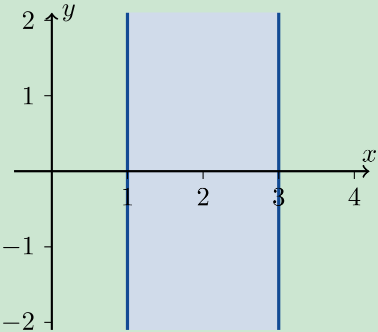
solution:



Example

Draw the region of points which satisfy $1 \leq x \leq 3$.

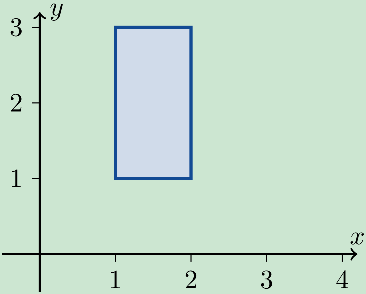
solution:

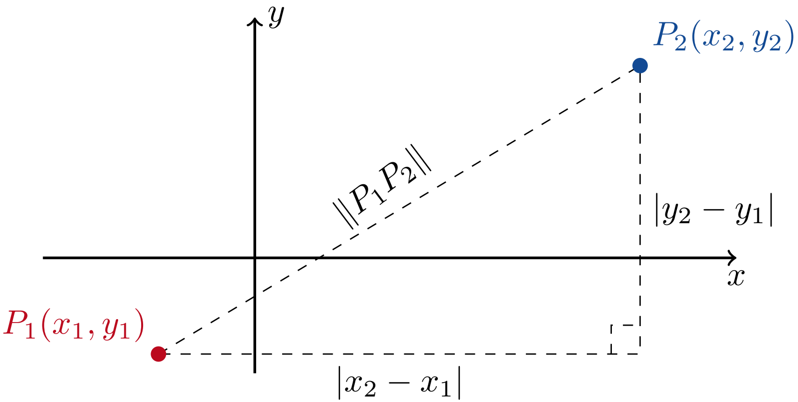


Example

Draw the region of points which satisfy $1 \leq x \leq 2$ and $1 \leq y \leq 3$.

solution:





y
$P_2(x_2, y_2)$
$\|P_1P_2\|$
$|y_2 - y_1|$
x
$P_1(x_1, y_1)$
$|x_2 - x_1|$

Definition

The *distance* between $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$ is

$$\|P_1P_2\| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}.$$

Example

The distance between $A(1, 3)$ and $B(4, -1)$ is

$$\|AB\| = \sqrt{(4-1)^2 + (-1-3)^2} = \sqrt{3^2 + (-4)^2} = \sqrt{25} = 5.$$

6 Functions

$$y = f(x)$$

"y is equal to f of x"

6. Functions

dependent variable

$$y = f(x)$$

function

independent variable

"y is equal to f of x"

Definition

A *function* from a set D to a set Y is a rule that assigns a unique element of Y to each element of D.

Definition

The set D of all possible values of x is called the *domain* of f.

Definition

The set Y is called the *target* of f.

Definition

The set of all possible values of $f(x)$ is called the *range* of f.

6. Functions

If f is a function with domain D and target Y, we can write

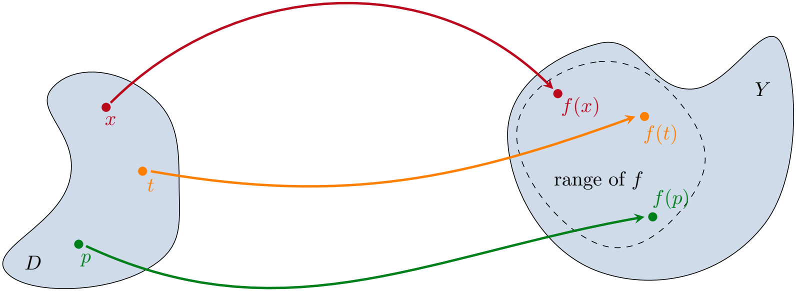
$$\underbrace{f : D}_{\text{domain}} \to \underbrace{Y}_{\text{target}}$$

Example

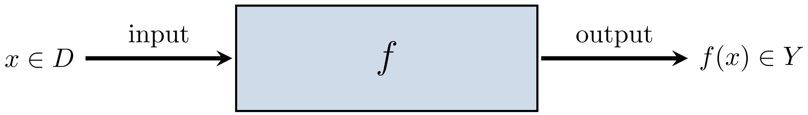
$f : \mathbb{R} \to \mathbb{R},\ f(x) = x^2.$

Example

$f : (-\infty, \infty) \to [0, \infty),\ f(x) = x^2.$


x
t
p
D
$f(x)$
$f(t)$
range of f
$f(p)$
Y

6. Functions



function	domain (x)	range (y)
$y = x^2$	$(-\infty, \infty)$	
$y = \frac{1}{x}$	$\{x \mid x \in \mathbb{R}, x \neq 0\}$	
$y = \sqrt{x}$	$[0, \infty)$	
$y = \sqrt{4 - x}$		
$y = \sqrt{1 - x^2}$		

function	domain (x)	range (y)
$y = x^2$	$(-\infty, \infty)$	$[0, \infty)$
$y = \frac{1}{x}$	$\{x \mid x \in \mathbb{R}, x \neq 0\}$	
$y = \sqrt{x}$	$[0, \infty)$	
$y = \sqrt{4 - x}$		
$y = \sqrt{1 - x^2}$		

function	domain (x)	range (y)
$y = x^2$	$(-\infty, \infty)$	$[0, \infty)$
$y = \frac{1}{x}$	$\{x \mid x \in \mathbb{R}, x \neq 0\}$	$\{y \mid y \in \mathbb{R}, y \neq 0\}$
$y = \sqrt{x}$	$[0, \infty)$	
$y = \sqrt{4 - x}$		
$y = \sqrt{1 - x^2}$		

function	domain (x)	range (y)
$y = x^2$	$(-\infty, \infty)$	$[0, \infty)$
$y = \frac{1}{x}$	$\{x \mid x \in \mathbb{R}, x \neq 0\}$	$\{y \mid y \in \mathbb{R}, y \neq 0\}$
$y = \sqrt{x}$	$[0, \infty)$	$[0, \infty)$
$y = \sqrt{4 - x}$		
$y = \sqrt{1 - x^2}$		

function	domain (x)	range (y)
$y = x^2$	$(-\infty, \infty)$	$[0, \infty)$
$y = \frac{1}{x}$	$\{x \mid x \in \mathbb{R}, x \neq 0\}$	$\{y \mid y \in \mathbb{R}, y \neq 0\}$
$y = \sqrt{x}$	$[0, \infty)$	$[0, \infty)$
$y = \sqrt{4 - x}$	$(-\infty, 4]$	
$y = \sqrt{1 - x^2}$		

function	domain (x)	range (y)
$y = x^2$	$(-\infty, \infty)$	$[0, \infty)$
$y = \frac{1}{x}$	$\{x \mid x \in \mathbb{R}, x \neq 0\}$	$\{y \mid y \in \mathbb{R}, y \neq 0\}$
$y = \sqrt{x}$	$[0, \infty)$	$[0, \infty)$
$y = \sqrt{4 - x}$	$(-\infty, 4]$	$[0, \infty)$
$y = \sqrt{1 - x^2}$		

function	domain (x)	range (y)
$y = x^2$	$(-\infty, \infty)$	$[0, \infty)$
$y = \frac{1}{x}$	$\{x \mid x \in \mathbb{R}, x \neq 0\}$	$\{y \mid y \in \mathbb{R}, y \neq 0\}$
$y = \sqrt{x}$	$[0, \infty)$	$[0, \infty)$
$y = \sqrt{4 - x}$	$(-\infty, 4]$	$[0, \infty)$
$y = \sqrt{1 - x^2}$	$[-1, 1]$	

function	domain (x)	range (y)
$y = x^2$	$(-\infty, \infty)$	$[0, \infty)$
$y = \frac{1}{x}$	$\{x \mid x \in \mathbb{R}, x \neq 0\}$	$\{y \mid y \in \mathbb{R}, y \neq 0\}$
$y = \sqrt{x}$	$[0, \infty)$	$[0, \infty)$
$y = \sqrt{4 - x}$	$(-\infty, 4]$	$[0, \infty)$
$y = \sqrt{1 - x^2}$	$[-1, 1]$	$[0, 1]$

function	domain (x)	range (y)
$y = x^2$	$(-\infty, \infty)$	$[0, \infty)$
$y = \frac{1}{x}$	$\{x \mid x \in \mathbb{R}, x \neq 0\}$	$\{y \mid y \in \mathbb{R}, y \neq 0\}$
$y = \sqrt{x}$	$[0, \infty)$	$[0, \infty)$
$y = \sqrt{4 - x}$	$(-\infty, 4]$	$[0, \infty)$
$y = \sqrt{1 - x^2}$	$[-1, 1]$	$[0, 1]$
$y = x^2$	$[1, 2]$	
$y = x^2$	$[2, \infty)$	
$y = x^2$	$(-\infty, -2]$	

function	domain (x)	range (y)
$y = x^2$	$(-\infty, \infty)$	$[0, \infty)$
$y = \frac{1}{x}$	$\{x \mid x \in \mathbb{R}, x \neq 0\}$	$\{y \mid y \in \mathbb{R}, y \neq 0\}$
$y = \sqrt{x}$	$[0, \infty)$	$[0, \infty)$
$y = \sqrt{4 - x}$	$(-\infty, 4]$	$[0, \infty)$
$y = \sqrt{1 - x^2}$	$[-1, 1]$	$[0, 1]$
$y = x^2$	$[1, 2]$	$[1, 4]$
$y = x^2$	$[2, \infty)$	
$y = x^2$	$(-\infty, -2]$	

function	domain (x)	range (y)
$y = x^2$	$(-\infty, \infty)$	$[0, \infty)$
$y = \frac{1}{x}$	$\{x \mid x \in \mathbb{R}, x \neq 0\}$	$\{y \mid y \in \mathbb{R}, y \neq 0\}$
$y = \sqrt{x}$	$[0, \infty)$	$[0, \infty)$
$y = \sqrt{4 - x}$	$(-\infty, 4]$	$[0, \infty)$
$y = \sqrt{1 - x^2}$	$[-1, 1]$	$[0, 1]$
$y = x^2$	$[1, 2]$	$[1, 4]$
$y = x^2$	$[2, \infty)$	$[4, \infty)$
$y = x^2$	$(-\infty, -2]$	

function	domain (x)	range (y)
$y = x^2$	$(-\infty, \infty)$	$[0, \infty)$
$y = \frac{1}{x}$	$\{x \mid x \in \mathbb{R}, x \neq 0\}$	$\{y \mid y \in \mathbb{R}, y \neq 0\}$
$y = \sqrt{x}$	$[0, \infty)$	$[0, \infty)$
$y = \sqrt{4 - x}$	$(-\infty, 4]$	$[0, \infty)$
$y = \sqrt{1 - x^2}$	$[-1, 1]$	$[0, 1]$
$y = x^2$	$[1, 2]$	$[1, 4]$
$y = x^2$	$[2, \infty)$	$[4, \infty)$
$y = x^2$	$(-\infty, -2]$	$[4, \infty)$

function	domain (x)	range (y)
$y = x^2$	$(-\infty, \infty)$	$[0, \infty)$
$y = \frac{1}{x}$	$\{x \mid x \in \mathbb{R}, x \neq 0\}$	$\{y \mid y \in \mathbb{R}, y \neq 0\}$
$y = \sqrt{x}$	$[0, \infty)$	$[0, \infty)$
$y = \sqrt{4 - x}$	$(-\infty, 4]$	$[0, \infty)$
$y = \sqrt{1 - x^2}$	$[-1, 1]$	$[0, 1]$
$y = x^2$	$[1, 2]$	$[1, 4]$
$y = x^2$	$[2, \infty)$	$[4, \infty)$
$y = x^2$	$(-\infty, -2]$	$[4, \infty)$
$y = 1 + x^2$	$[1, 3)$	
$y = 1 - \sqrt{x}$	$[0, \infty)$	

function	domain (x)	range (y)
$y = x^2$	$(-\infty, \infty)$	$[0, \infty)$
$y = \frac{1}{x}$	$\{x \mid x \in \mathbb{R}, x \neq 0\}$	$\{y \mid y \in \mathbb{R}, y \neq 0\}$
$y = \sqrt{x}$	$[0, \infty)$	$[0, \infty)$
$y = \sqrt{4 - x}$	$(-\infty, 4]$	$[0, \infty)$
$y = \sqrt{1 - x^2}$	$[-1, 1]$	$[0, 1]$
$y = x^2$	$[1, 2]$	$[1, 4]$
$y = x^2$	$[2, \infty)$	$[4, \infty)$
$y = x^2$	$(-\infty, -2]$	$[4, \infty)$
$y = 1 + x^2$	$[1, 3)$	$[2, 10)$
$y = 1 - \sqrt{x}$	$[0, \infty)$	

function	domain (x)	range (y)
$y = x^2$	$(-\infty, \infty)$	$[0, \infty)$
$y = \frac{1}{x}$	$\{x \mid x \in \mathbb{R}, x \neq 0\}$	$\{y \mid y \in \mathbb{R}, y \neq 0\}$
$y = \sqrt{x}$	$[0, \infty)$	$[0, \infty)$
$y = \sqrt{4 - x}$	$(-\infty, 4]$	$[0, \infty)$
$y = \sqrt{1 - x^2}$	$[-1, 1]$	$[0, 1]$
$y = x^2$	$[1, 2]$	$[1, 4]$
$y = x^2$	$[2, \infty)$	$[4, \infty)$
$y = x^2$	$(-\infty, -2]$	$[4, \infty)$
$y = 1 + x^2$	$[1, 3)$	$[2, 10)$
$y = 1 - \sqrt{x}$	$[0, \infty)$	$(-\infty, 1]$

Graphs of Functions

Definition

The *graph* of f is the set containing all the points (x, y) which satisty $y = f(x)$.

Example

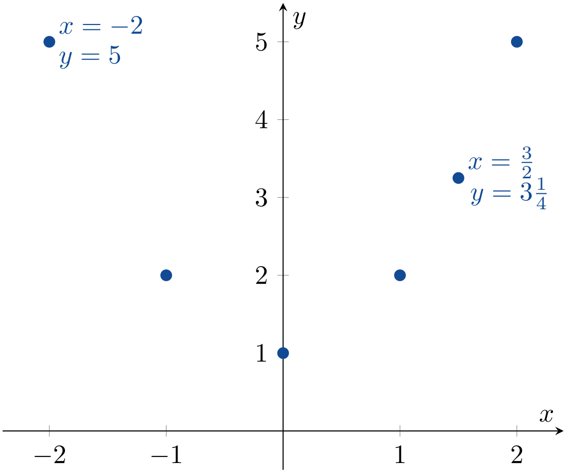
Graph the function $y = 1 + x^2$ over the interval $[-2, 2]$.

solution:

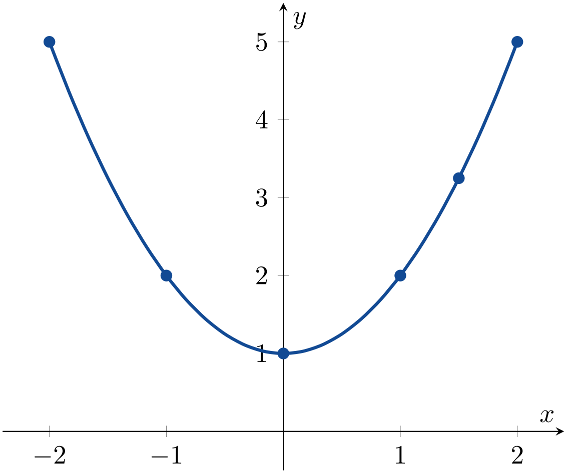
1. Make a table of (x, y) points which satisfy $y = 1 + x^2$.

x	y
-2	5
-1	2
0	1
1	2
$\frac{3}{2}$	$\frac{13}{4} = 3\frac{1}{4}$
2	5

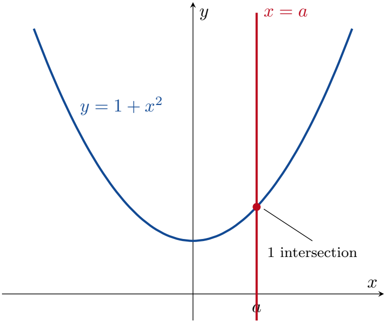
2. Plot these points.

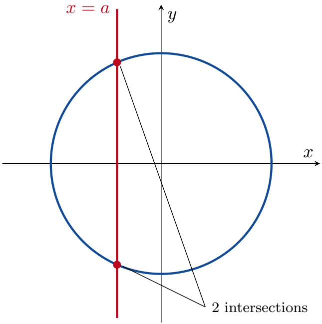


3. Draw a smooth curve through these points.



The Vertical Line Test





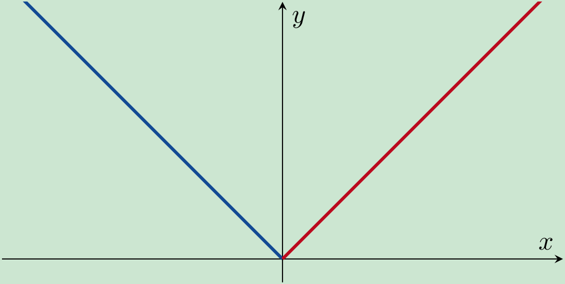
Not every curve that you draw is a graph of a function.

A function can have only one value $f(x)$ for each $x \in D$. This means that a vertical line can intersect the graph of a function at most once.
A circle can not be the graph of a function because some vertical lines intersect the circle at two points.
If $a \in D$, then the vertical line $x = a$ will intersect the graph of $f : D \to Y$ only at the point $(a, f(a))$.

Piecewise-Defined Functions

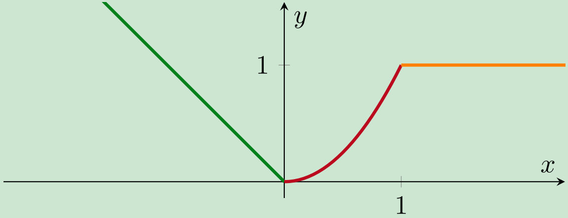
Example

$$|x| = \begin{cases} x & x \geq 0 \\ -x & x < 0 \end{cases}$$



Example

$$f(x) = \begin{cases} -x & x < 0 \\ x^2 & 0 \leq x \leq 1 \\ 1 & x > 1 \end{cases}$$

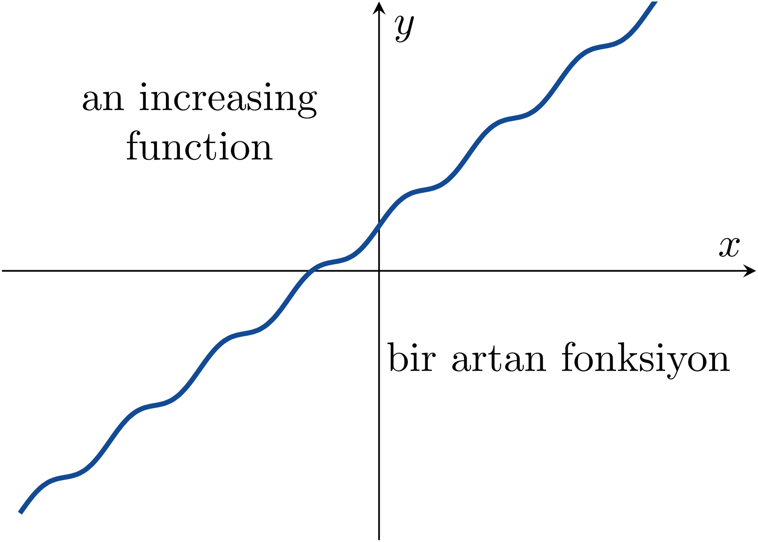


Increasing and Decreasing Functions

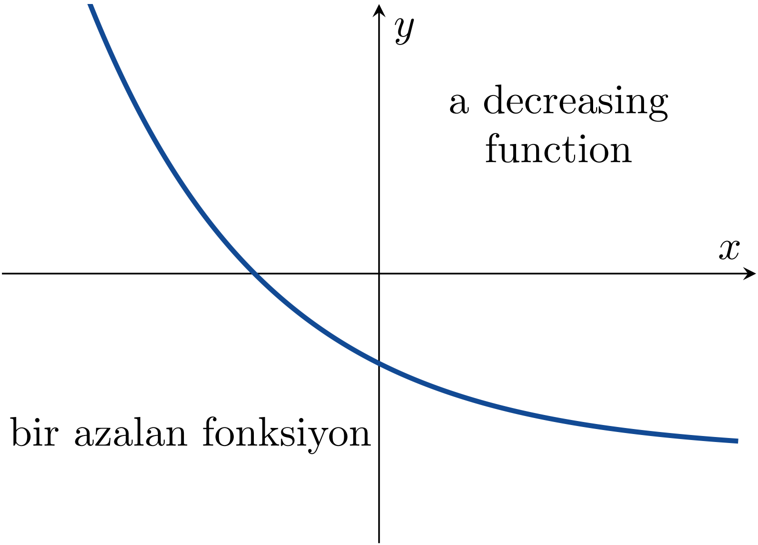
Definition

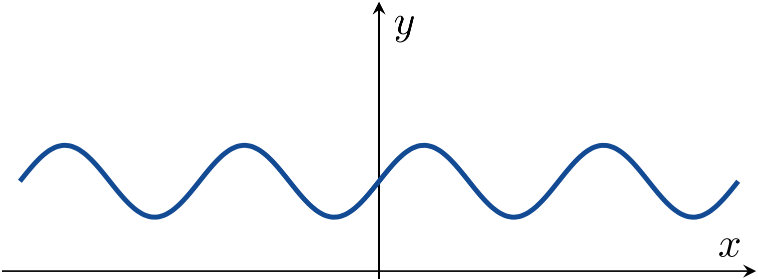
Let I be an interval. Let $f : I \to \mathbb{R}$ be a function.

1. f is called *increasing on I* if
$$f(x_1) < f(x_2)$$
for all $x_1, x_2 \in I$ which satisfy $x_1 < x_2$;
2. f is called *decreasing on I* if
$$f(x_1) > f(x_2)$$
for all $x_1, x_2 \in I$ which satisfy $x_1 < x_2$.



y
x
an increasing function
bir artan fonksiyon


y
a decreasing function
x
bir azalan fonksiyon



not increasing
not decreasing

artmıyor
azalmıyor

Even Functions and Odd Functions

Recall that

- $2, 4, 6, 8, 10, \dots$ are even numbers; and
- $1, 3, 5, 7, 9, \dots$ are odd numbers.

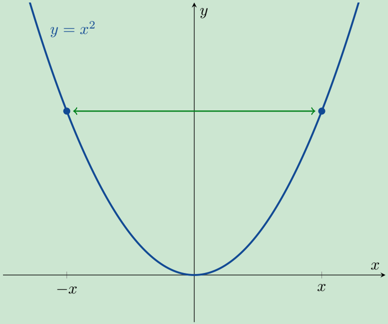
Definition

1. $f : D \to \mathbb{R}$ is an *even function* if $f(-x) = f(x)$ for all $x \in D$;
2. $f : D \to \mathbb{R}$ is an *odd function* if $f(-x) = -f(x)$ for all $x \in D$.

Example

$f(x) = x^2$ is an even function because

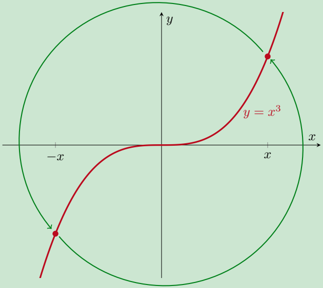
$$f(-x) = (-x)^2 = x^2 = f(x).$$



Example

$f(x) = x^3$ is an odd function because

$$f(-x) = (-x)^3 = -x^3 = -f(x).$$



Example

Is $f(x) = x^2 + 1$ even, odd or neither?

solution: Since

$$f(-x) = (-x)^2 + 1 = x^2 + 1 = f(x),$$

f is an even function.

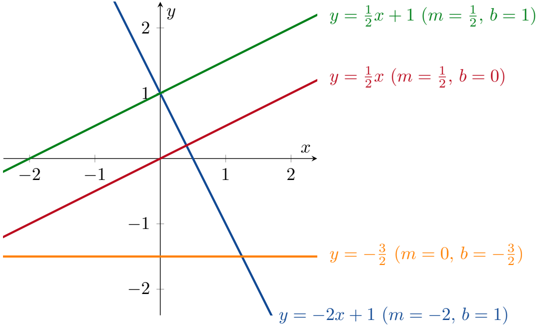
Example

Is $g(x) = x + 1$ even, odd or neither?

solution: Since $g(-2) = -2 + 1 = -1$ and $g(2) = 3$, we have $g(-2) \neq g(2)$ and $g(-2) \neq -g(2)$. Hence g is neither even nor odd.

Linear Functions

$$f(x) = mx + b$$
$$(m, b \in \mathbb{R})$$

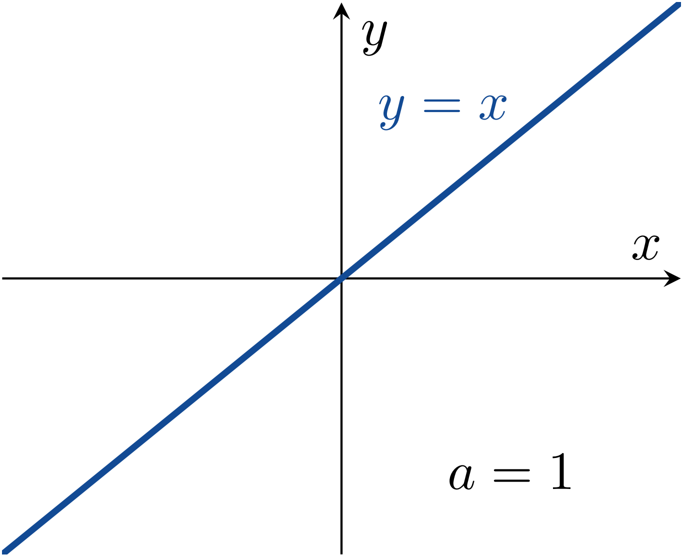


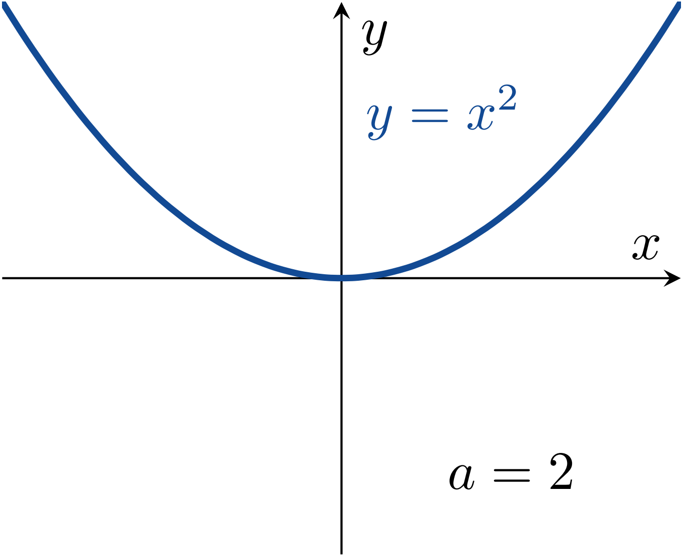
Power Functions

$$f(x) = x^a$$

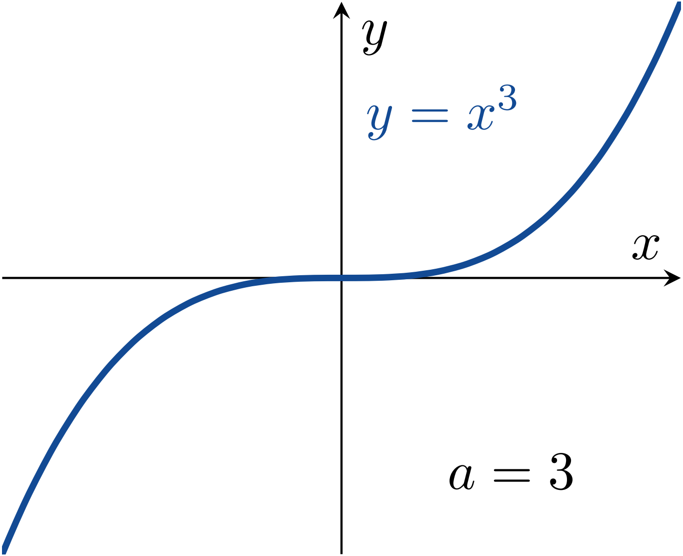
$(a \in \mathbb{R})$

"x to the power of a"

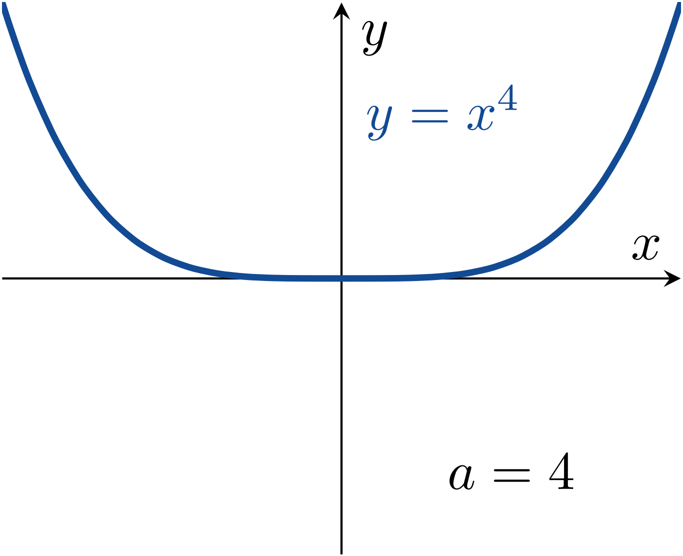

y
$y = x$
x
$a = 1$



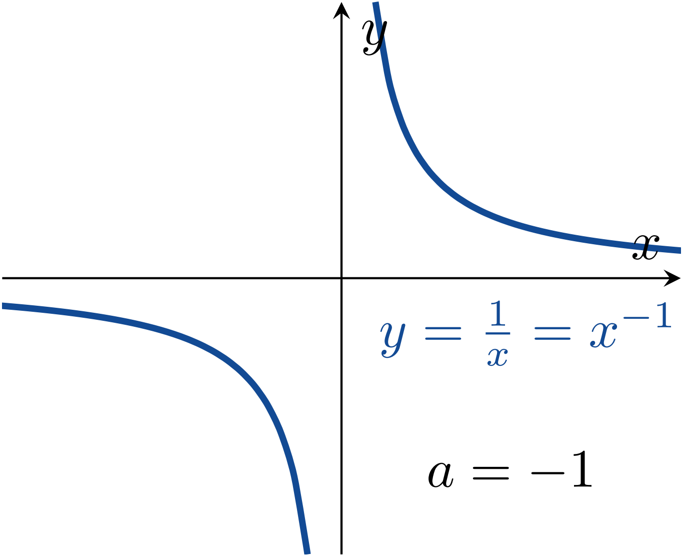
$a = 2$



$$a = 3$$

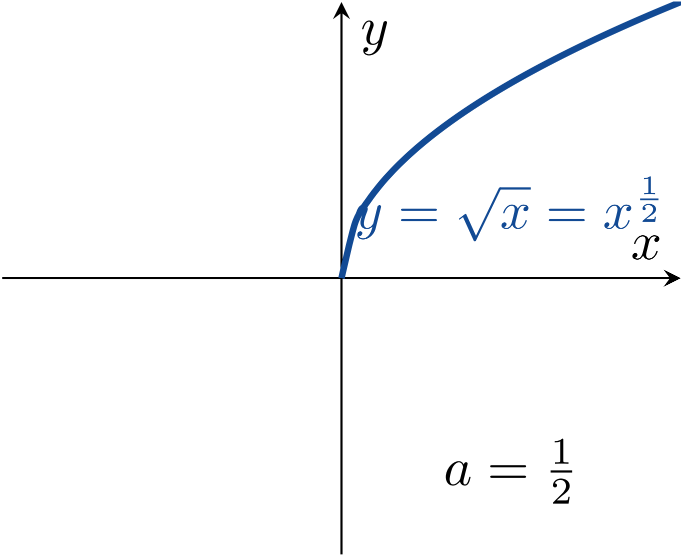


$$a = 4$$

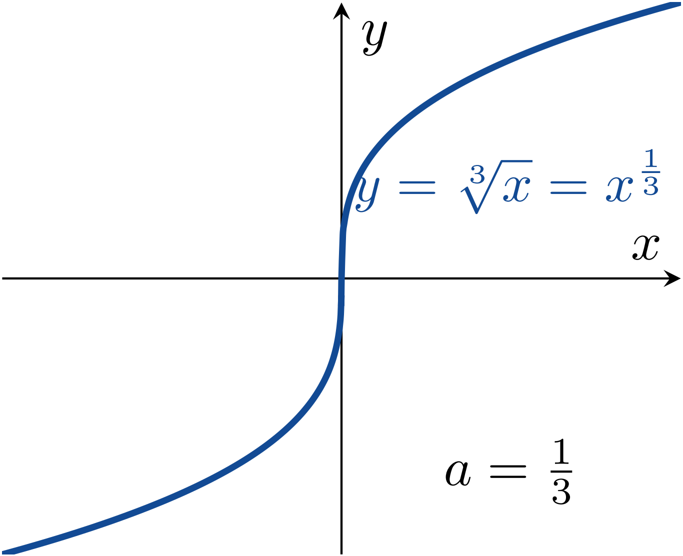


$$y = \frac{1}{x} = x^{-1}$$

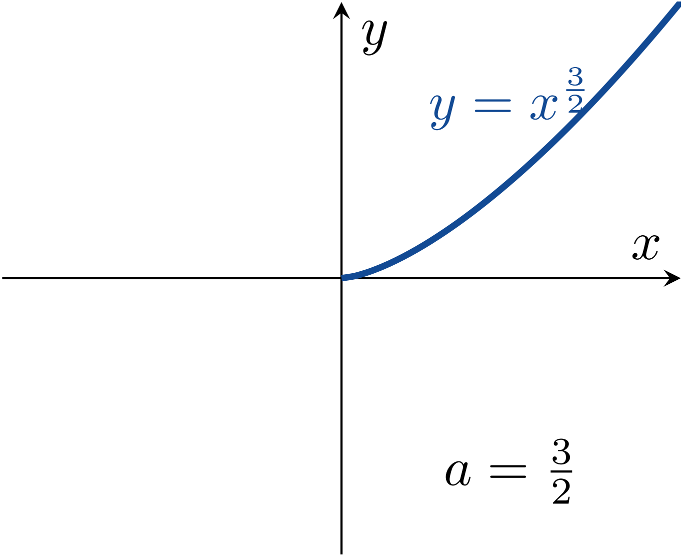
$$a = -1$$



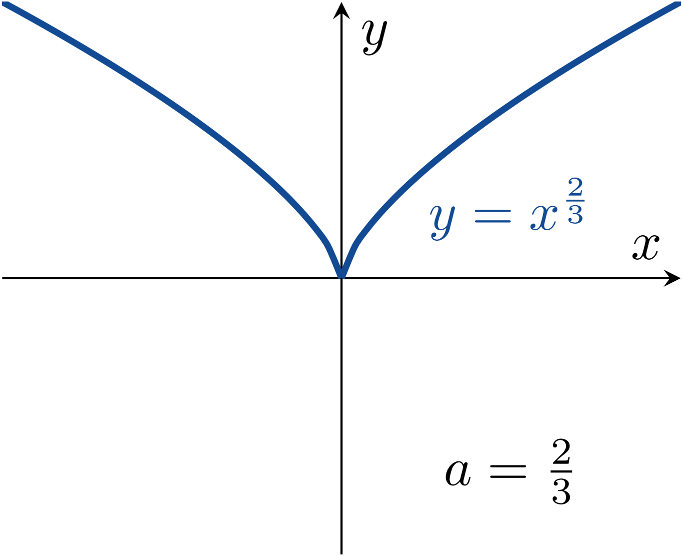
$$a = \frac{1}{2}$$



y
$y = \sqrt[3]{x} = x^{\frac{1}{3}}$
x
$a = \frac{1}{3}$



$$a = \frac{3}{2}$$



$$a = \frac{2}{3}$$

Polynomials

$$p(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_2 x^2 + a_1 x + a_0$$

$$(n \in \mathbb{N} \cup \{0\},\ a_j \in \mathbb{R}).$$

The domain of a polynomial is always $(-\infty, \infty)$. If $n > 0$ and $a_n \neq 0$, then n is called the *degree* of $p(x)$.

Rational Functions

$$f(x) = \frac{p(x)}{q(x)}$$

polynomial

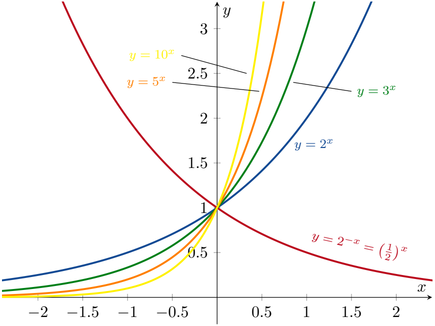
rational function

Example

$$f(x) = \frac{2x^3 - 3}{7x + 4}$$

Exponential Functions

$$f(x) = a^x$$
$$(a \in \mathbb{R},\ a > 0,\ a \neq 1)$$



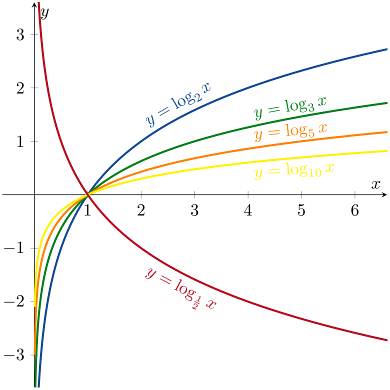
The domain of an exponential function is $(-\infty, \infty)$.

Logarithmic Functions

$$y = \log_a x \iff x = a^y$$

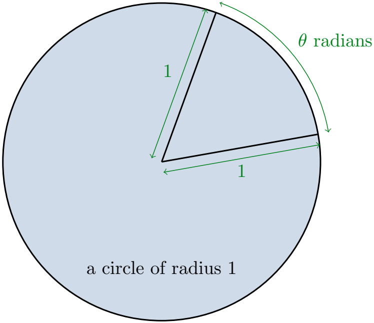
$$(a \in \mathbb{R},\ a > 0,\ a \neq 1)$$

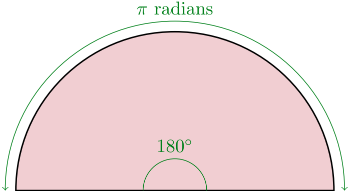
"log base a of x"



Angles

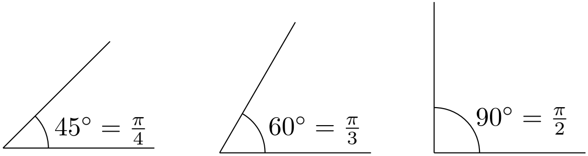
There are two ways to measure angles. Using degrees or using radians.





We have that

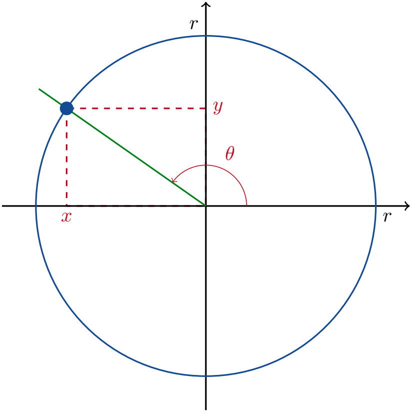
$$\pi \text{ radians} = 180 \text{ degrees}$$
$$1 \text{ radian} = \frac{180}{\pi} \text{ degrees}$$
$$1 \text{ degree} = \frac{\pi}{180} \text{ radians.}$$



Remark

In Calculus, we use radians!!!! If you see an angle in Part IV of this course, it will be in radians. Calculus doesn't work with degrees!!

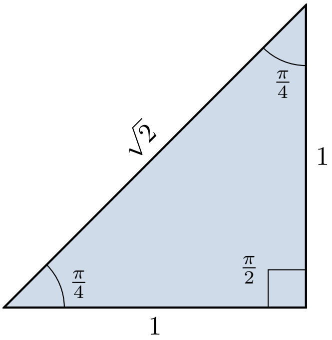
Trigonometric Functions



sine	$\sin\theta = \dfrac{y}{r}$
cosine	$\cos\theta = \dfrac{x}{r}$
tangent	$\tan\theta = \dfrac{\sin\theta}{\cos\theta}$
secant	$\sec\theta = \dfrac{1}{\cos\theta}$
cosecant	$\operatorname{cosec}\theta = \csc\theta = \dfrac{1}{\sin\theta}$
cotangent	$\cot\theta = \dfrac{1}{\tan\theta}$

Remark

Note that $\tan\theta$ and $\sec\theta$ are only defined if $\cos\theta \neq 0$; and $\operatorname{cosec}\theta$ and $\cot\theta$ are only defined if $\sin\theta \neq 0$.



$$\sin 45^\circ = \sin\frac{\pi}{4} = \frac{1}{\sqrt{2}}$$

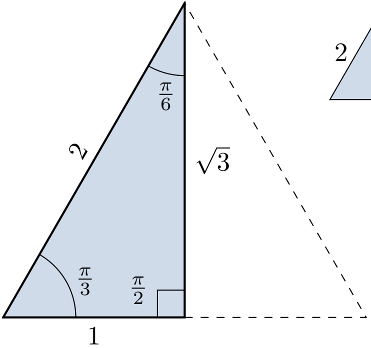
$$\cos 45^\circ = \cos\frac{\pi}{4} = \frac{1}{\sqrt{2}}$$

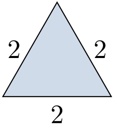
$$\tan 45^\circ = \tan\frac{\pi}{4} = 1$$

$$\sec 45^\circ = \sec\frac{\pi}{4} = \sqrt{2}$$

$$\operatorname{cosec} 45^\circ = \operatorname{cosec}\frac{\pi}{4} = \sqrt{2}$$

$$\cot 45^\circ = \cot\frac{\pi}{4} = 1$$





$$\sin 60^\circ = \sin \frac{\pi}{3} = \frac{\sqrt{3}}{2}$$

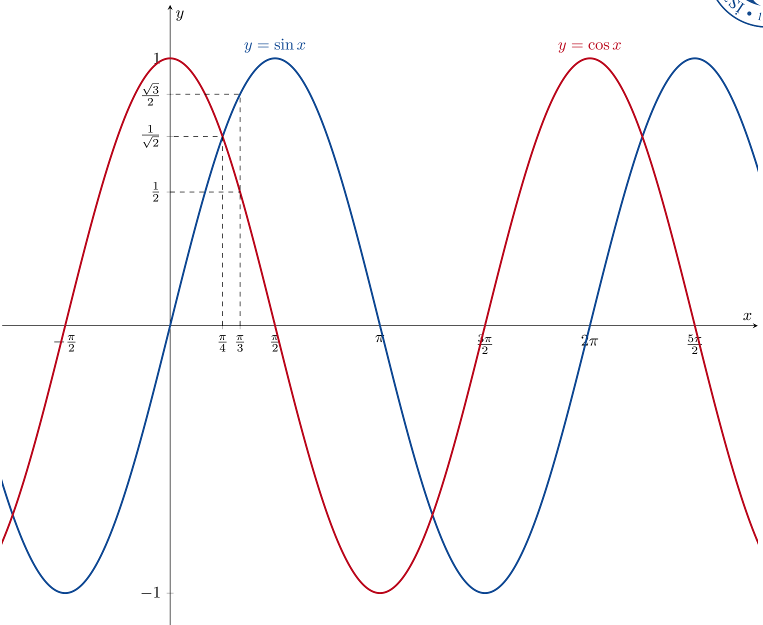
$$\cos 60^\circ = \cos \frac{\pi}{3} = \frac{1}{2}$$

$$\tan 60^\circ = \tan \frac{\pi}{3} = \sqrt{3}$$

$$\sec 60^\circ = \sec \frac{\pi}{3} = 2$$

$$\operatorname{cosec} 60^\circ = \operatorname{cosec} \frac{\pi}{3} = \frac{2}{\sqrt{3}}$$

$$\cot 60^\circ = \cot \frac{\pi}{3} = \frac{1}{\sqrt{3}}$$


y
$y = \sin x$
$y = \cos x$
1
$\frac{\sqrt{3}}{2}$
$\frac{1}{\sqrt{2}}$
$\frac{1}{2}$
-1
x
$-\frac{\pi}{2}$
$\frac{\pi}{4}$
$\frac{\pi}{3}$
$\frac{\pi}{2}$
π
$\frac{3\pi}{2}$
2π
$\frac{5\pi}{2}$

7 Sigma Notation

7. Sigma Notation

$$\sum_{k=1}^{n} a_k = a_1 + a_2 + a_3 + a_4 + a_5 + a_6 + a_7 + a_8 + \ldots + a_{n-1} + a_n$$

7. Sigma Notation

$$\sum_{k=1}^{n} a_k = a_1 + a_2 + a_3 + a_4 + a_5 + a_6 + a_7 + a_8 + \ldots + a_{n-1} + a_n$$

$$\sum_{k=1}^{n} a_k$$

$$\sum_{k=1}^{n} a_k = a_1 + a_2 + a_3 + a_4 + a_5 + a_6 + a_7 + a_8 + \ldots + a_{n-1} + a_n$$

the Greek
letter Sigma

$$\sum_{k=1}^{n} a_k$$

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$$\sum_{k=1}^{n} a_k = a_1 + a_2 + a_3 + a_4 + a_5 + a_6 + a_7 + a_8 + \ldots + a_{n-1} + a_n$$

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$$\sum_{k=1}^{n} a_k$$

the sum starts at $k = 1$

7. Sigma Notation

$$\sum_{k=1}^{n} a_k = a_1 + a_2 + a_3 + a_4 + a_5 + a_6 + a_7 + a_8 + \ldots + a_{n-1} + a_n$$

the Greek letter Sigma

$$\sum_{k=1}^{n} a_k$$

the sum finishes at $k = n$

the sum starts at $k = 1$

7. Sigma Notation

Example

$$1^2 + 2^2 + 3^2 + 4^2 + 5^2 + 6^2 + 7^2 + 8^2 + 9^2 + 10^2 + 11^2 = \sum_{k=1}^{11} k^2$$

$$f(1) + f(2) + f(3) + \ldots + f(99) + f(100) = \sum_{k=1}^{100} f(k)$$

$$\sum_{k=1}^{5} k = 1 + 2 + 3 + 4 + 5 = 15$$

Example

$$\sum_{k=1}^{3}(-1)^k k = (-1)(1) + (-1)^2(2) + (-1)^3(3) = -1 + 2 - 3 = -2$$

$$\sum_{k=1}^{2}\frac{k}{k+1} = \frac{1}{1+1} + \frac{2}{2+1} = \frac{1}{2} + \frac{2}{3} = \frac{7}{6}$$

$$\sum_{k=4}^{5}\frac{k^2}{k-1} = \frac{4^2}{4-1} + \frac{5^2}{5-1} = \frac{16}{3} + \frac{25}{4} = \frac{139}{12}$$

Example

I want to find a formula for $1 + 2 + 3 + \ldots + n$.

Example

I want to find a formula for $1 + 2 + 3 + \ldots + n$.

Note that

$$
\begin{aligned}
2\big(1+2+3+4+5+\ldots+(n-1)+n\big) \\
&= \quad 1 + 2 + 3 + 4 + 5 + \ldots + (n-1) + n \\
&\quad + n + (n-1) + (n-2) + (n-3) + (n-4) + \ldots + 2 + 1 \\
&= (n+1) + (n+1) + (n+1) + (n+1) + (n+1) + \ldots + (n+1) + (n+1) \\
&= n(n+1).
\end{aligned}
$$

Example

I want to find a formula for $1+2+3+\ldots+n$.

Note that

$$\begin{aligned} &2\big(1+2+3+4+5+\ldots+(n-1)+n\big) \\ &\quad = \begin{array}{ccccccccccccccc} & 1 & + & 2 & + & 3 & + & 4 & + & 5 & + & \ldots & + & (n-1) & + & n \\ + & n & + & (n-1) & + & (n-2) & + & (n-3) & + & (n-4) & + & \ldots & + & 2 & + & 1 \end{array} \\ &\quad = (n+1) + (n+1) + (n+1) + (n+1) + (n+1) + \ldots + (n+1) + (n+1) \\ &\quad = n(n+1). \end{aligned}$$

Therefore

$$\boxed{\sum_{k=1}^{n} k = \frac{n(n+1)}{2}.}$$

Similarly (but more difficult) we can find that

$$\boxed{\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}}$$

and

$$\boxed{\sum_{k=1}^{n} k^3 = \left(\frac{n(n+1)}{2}\right)^2.}$$

Next Time

- 8. Polar Coordinates
- 9. Conic Sections
- 10. Three Dimensional Cartesian Coordinates