Week 7

- 3.8 Solving Initial Value Problems
- 3.9 The Method of Variation of Parameters
- 3.10 Higher Order Linear ODEs

Solving Initial Value Problems

3.8

Remark

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2. Find a particular solution to $ay'' + by' + cy = g(t)$:
    1. if $g(t)$ **does not** solve the homogeneous equation, then your ansatz should look like $g(t)$;
    2. if $g(t)$ **does** solve the homogeneous equation, then "multiply by t" (repeat as necessary);

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4. Find c_1 and c_2.

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You must do step 4 last. If you try to find c_1 and c_2 before doing the other steps, you may get the wrong answer.

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Example

Solve

$$\begin{cases} y'' - y = 2e^t \\ y(0) = 1 \\ y'(0) = 2. \end{cases}$$

Correct Solution:

1. First we consider $y'' - y = 0$. The characteristic equation $r^2 - 1 = 0$ has roots $r_1 = 1$ and $r_2 = -1$. Hence the general solution is $y(t) = c_1 e^t + c_2 e^{-t}$.

2 Next we need to find a particular solution. Since Ae^t solves the homogeneous equation, we must "multiply by t". We try the ansatz $Y(t) = Ate^t$ and we calculate that

$$\begin{aligned} Y' &= Ae^t + Ate^t, \\ Y'' &= 2Ae^t + Ate^t \end{aligned}$$

and

$$\begin{aligned} 2e^t &= Y'' - Y \\ &= 2Ae^t + Ate^t - Ate^t \\ &= 2Ae^t. \end{aligned}$$

We must have $A = 1$. Therefore $Y(t) = te^t$ is a particular solution.

3. Thus
$$y(t) = c_1 e^t + c_2 e^{-t} + te^t$$
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4. Finally we must satisfy the initial conditions. Since
$$y'(t) = c_1 e^t - c_2 e^{-t} + e^t + te^t$$
we have
$$\begin{aligned} 1 &= y(0) = c_1 + c_2 + 0 \\ 2 &= y'(0) = c_1 - c_2 + 1 + 0 \end{aligned}$$
which implies that $c_1 = 1$ and $c_2 = 0$. Therefore the solution to the IVP is
$$\boxed{y(t) = e^t + te^t.}$$

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$$\boxed{y(t) = e^t + te^t.}$$

Incorrect Solution:

1. First we consider $y'' - y = 0$. The characteristic equation $r^2 - 1 = 0$ has roots $r_1 = 1$ and $r_2 = -1$. Hence the general solution is $y(t) = c_1e^t + c_2e^{-t}$.

4. Next we find c_1 and c_2. Since

$$y'(t) = c_1 e^t - c_2 e^{-t}$$

we have

$$\begin{aligned} 1 &= y(0) = c_1 + c_2 \\ 2 &= y'(0) = c_1 - c_2 \end{aligned}$$

which implies that $c_1 = \frac{3}{2}$ and $c_2 = -\frac{1}{2}$. Thus

$$y(t) = \frac{3}{2}e^t - \frac{1}{2}e^{-t}.$$

2 Next we need to find a particular solution. Since Ae^t solves the homogeneous equation, we must "multiply by t". We try the ansatz $Y(t) = Ate^t$ and we calculate that

$$\begin{aligned} Y' &= Ae^t + Ate^t, \\ Y'' &= 2Ae^t + Ate^t \end{aligned}$$

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$y(t) = e^t + te^t$

3. Finally we add our solutions together to get

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which **is WRONG!!!** This function does not satisfy the initial conditions.

Example

Solve

$$\begin{cases} -y'' + 6y' - 16y = 1 + 6e^{3t}\sin(2t) \\ y(0) = \frac{15}{16} \\ y'(0) = -1. \end{cases} \tag{1}$$

(This is an exam question from 2013: Students had 30 minutes to solve this.)

First consider the homogeneous equation $-y'' + 6y' - 16y = 0$.

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$$y(t) = c_1 e^{3t} \sin(\sqrt{7}t) + c_2 e^{3t} \cos(\sqrt{7}t).$$

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Next consider $-y'' + 6y' - 16y = 1$.

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$$1 = -Y'' + 6Y' - 16Y = -16C.$$

We must choose $C = -\frac{1}{16}$. Hence $Y(t) = -\frac{1}{16}$.

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$$\begin{aligned}
6e^{3t}\sin 2t &= -Y'' + 6Y' - 16Y \\
&= -e^{3t}\Big((5A + 12B)\cos 2t + (5B - 12A)\sin 2t\Big) \\
&\quad + 6e^{3t}\big((3A + 2B)\cos 2t + (3B - 2A)\sin 2t\big) \\
&\quad - 16e^{3t}\big(A\cos 2t + B\sin 2t\big) \\
&= e^{3t}\cos 2t\big(-5A - 12B + 16A + 12B - 16A\big) \\
&\quad + e^{3t}\sin 2t\big(-5B + 12A + 18B - 12A - 16B\big) \\
&= e^{3t}\cos 2t\big(-5A\big) + e^{3t}\sin 2t\big(-3B\big).
\end{aligned}$$

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6e^{3t}\sin 2t &= -Y'' + 6Y' - 16Y \\
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&\quad + 6e^{3t}\big((3A+2B)\cos 2t + (3B-2A)\sin 2t\big) \\
&\quad - 16e^{3t}\big(A\cos 2t + B\sin 2t\big) \\
&= e^{3t}\cos 2t\big(-5A - 12B + 16A + 12B - 16A\big) \\
&\quad + e^{3t}\sin 2t\big(-5B + 12A + 18B - 12A - 16B\big) \\
&= e^{3t}\cos 2t\big(-5A\big) + e^{3t}\sin 2t\big(-3B\big).
\end{aligned}$$

Thus, we need $A = 0$ and $B = -2$. Hence

$$Y(t) = -2e^{3t}\sin 2t.$$

Next we add these 3 solutions together. Therefore, the general solution to the ODE is

$$y(t) = c_1 e^{3t} \sin(\sqrt{7}t) + c_2 e^{3t} \cos(\sqrt{7}t) - \frac{1}{16} - 2e^{3t} \sin(2t).$$

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The final step is to choose c_1 and c_2 to satisfy the initial conditions.

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$$\frac{15}{16} = y(0) = 0 + c_2 - \frac{1}{16} - 0 \qquad \Longrightarrow \qquad c_2 = 1.$$

$$\begin{aligned} -1 &= y'(0) \\ &= 3c_1 e^{3t}\sin(\sqrt{7}t) + \sqrt{7}c_1 e^{3t}\cos(\sqrt{7}t) + 3e^{3t}\cos(\sqrt{7}t) \\ &\quad - \sqrt{7}e^{3t}\sin(\sqrt{7}t) - 6e^{3t}\sin(2t) - 4e^{3t}\cos(2t)\Big|_{t=0} \\ &= 0 + \sqrt{7}c_1 + 3 - 0 - 0 - 4 \qquad \Longrightarrow \qquad c_1 = 0. \end{aligned}$$

Therefore, the solution to the IVP is

$$\boxed{y(t) = e^{3t}\cos(\sqrt{7}t) - \frac{1}{16} - 2e^{3t}\sin(2t).}$$

Remark

$$ay'' + by' + cy = g(t)$$

The method of undetermined coefficients works well if $g(t)$ is a nice function: $e^k t$, $\sin kt$, $t^3 + 2t^2 + 3t + 4$, $e^{at} \cosh kt$, ...

However if $g(t)$ is a less nice function, then we may need a different method to find a particular solution.

The Method of Variation of Parameters

Example

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1. Replace the constants c_1 and c_2 by functions $u_1(t)$ and $u_2(t)$:
$$Y(t) = u_1(t) \cos 2t + u_2(t) \sin 2t.$$

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1. Replace the constants c_1 and c_2 by functions $u_1(t)$ and $u_2(t)$:
$$Y(t) = u_1(t) \cos 2t + u_2(t) \sin 2t.$$

2. Try to find u_1 and u_2 so that Y solves (2). There will be lots of u_1 and u_2 that we can use, so we will be free to add an extra condition.

So suppose that

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Then

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$$Y' = u_1' \cos 2t - 2u_1 \sin 2t + u_2' \sin 2t + 2u_2 \cos 2t$$

At this point, it is getting complicated so we will use our chance to add a condition: Suppose that

$$u_1' \cos 2t + u_2' \sin 2t = 0 \tag{3}$$

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So

$$Y' = -2u_1 \sin 2t + 2u_2 \cos 2t$$

and

$$Y'' = -2u_1' \sin 2t - 4u_1 \cos 2t + 2u_2' \cos 2t - 4u_2 \sin 2t.$$

Then

$$\begin{aligned}
3\operatorname{cosec} t &= Y'' + 4Y \\
&= \left(-2u_1' \sin 2t - 4u_1 \cos 2t + 2u_2' \cos 2t - 4u_2 \sin 2t\right) \\
&\quad + 4\left(u_1 \cos 2t + u_2 \sin 2t\right) \\
&= -2u_1' \sin 2t + 2u_2' \cos 2t
\end{aligned}$$

Then

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&\quad + 4\left(u_1 \cos 2t + u_2 \sin 2t\right) \\
&= -2u_1' \sin 2t + 2u_2' \cos 2t
\end{aligned}$$

We want to find $u_1(t)$ and $u_2(t)$ which satisfy

$$\begin{cases} 3\,\mathrm{cosec}\,t = -2u_1' \sin 2t + 2u_2' \cos 2t \\ u_1' \cos 2t + u_2' \sin 2t = 0 \end{cases}$$

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From the latter condition, we have $u_2' = -u_1' \frac{\cos 2t}{\sin 2t}$.

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From the latter condition, we have $u_2' = -u_1' \frac{\cos 2t}{\sin 2t}$. Putting this into the first condition, we calculate that

$$3 \operatorname{cosec} t = -2u_1' \sin 2t + 2\left(-u_1' \frac{\cos 2t}{\sin 2t}\right) \cos 2t$$

$$3 \operatorname{cosec} t \sin 2t = -2u_1' \sin^2 2t - 2u_1' \cos^2 2t = -2u_1'$$

$$u_1' = \frac{-3 \operatorname{cosec} t \sin 2t}{2} = \frac{-3 \sin 2t}{2 \sin t} = -3 \cos t$$

and

$$u_2' = \frac{3 \cos t \cos 2t}{\sin 2t} = \frac{3 \cos t(1 - \sin^2 t)}{2 \sin t \cos t} = \frac{3}{2} \operatorname{cosec} t - 3 \sin t.$$

Integrating gives

$$u_1(t) = \int u_1'(t)\,dt = \int -3\cos t\,dt = -3\sin t$$

$$\begin{aligned} u_2(t) &= \int u_2'(t)\,dt = \int \frac{3}{2}\operatorname{cosec} t - 3\sin t\,dt \\ &= \frac{3}{2}\ln|\operatorname{cosec} t - \cot t| + 3\cos t \end{aligned}$$

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&= \frac{3}{2}\ln|\operatorname{cosec} t - \cot t| + 3\cos t
\end{aligned}
$$

Therefore a particular solution is

$$
\begin{aligned}
Y(t) &= u_1(t)\cos 2t + u_2(t)\sin 2t \\
&= -3\sin t\cos 2t + \frac{3}{2}\ln|\operatorname{cosec} t - \cot t|\sin 2t + 3\cos t\sin 2t \\
&= 3\sin t + \frac{3}{2}\ln|\operatorname{cosec} t - \cot t|\sin 2t.
\end{aligned}
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1. Guess $Y = u_1(t)y_1 + u_2(t)y_2$;
2. Make the extra condition $u_1'y_1 + u_2'y_2 = 0$;
3. Put Y into $L[y] = g(t)$;
4. Find u_1' and u_2';
5. Integrate to get u_1 and u_2;

Then Y is a particular solution to $L[y] = g(t)$.

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Therefore we guess that $Y = u_1(t)e^t + u_2(t)te^t$.

We make the extra condition that

$$\begin{aligned} u_1' y_1 + u_2' y_2 &= 0 \\ u_1' e^t + u_2' t e^t &= 0 \\ u_1' + u_2' t &= 0. \end{aligned}$$

Then we calculate that

$$\begin{aligned} Y' &= u_1' e^t + u_1 e^t + u_2' t e^t + u_2 e^t + u_2 t e^t \\ &= \\ Y'' &= \\ &= \end{aligned}$$

and

$$\begin{aligned} e^t \ln t &= Y'' - 2Y' + Y \\ &= \\ \\ &= \end{aligned}$$

Then we calculate that

$$
\begin{aligned}
Y' &= \cancel{u_1' e^t} + u_1 e^t + \cancel{u_2' t e^t} + u_2 e^t + u_2 t e^t \\
&= \\
Y'' &= \\
&=
\end{aligned}
$$

and

$$
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$$\begin{aligned} Y' &= \cancel{u_1' e^t} + u_1 e^t + \cancel{u_2' t e^t} + u_2 e^t + u_2 t e^t \\ &= u_1 e^t + u_2 e^t + u_2 t e^t, \\ Y'' &= \\ &= \end{aligned}$$

and

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Then we calculate that

$$
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Y' &= \cancel{u_1'e^t} + u_1e^t + \cancel{u_2'te^t} + u_2e^t + u_2te^t \\
&= u_1e^t + u_2e^t + u_2te^t, \\
Y'' &= u_1'e^t + u_1e^t + u_2'e^t + u_2e^t + u_2'te^t + u_2e^t + u_2te^t \\
&=
\end{aligned}
$$

and

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\begin{aligned}
e^t \ln t &= Y'' - 2Y' + Y \\
&= \\
&=
\end{aligned}
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\begin{aligned}
Y' &= \cancel{u_1' e^t} + u_1 e^t + \cancel{u_2' t e^t} + u_2 e^t + u_2 t e^t \\
&= u_1 e^t + u_2 e^t + u_2 t e^t, \\
Y'' &= \cancel{u_1' e^t} + u_1 e^t + u_2' e^t + u_2 e^t + \cancel{u_2' t e^t} + u_2 e^t + u_2 t e^t \\
&=
\end{aligned}
$$

and

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\begin{aligned}
e^t \ln t &= Y'' - 2Y' + Y \\
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\\
&=
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Then we calculate that

$$\begin{aligned} Y' &= \cancel{u_1' e^t} + u_1 e^t + \cancel{u_2' t e^t} + u_2 e^t + u_2 t e^t \\ &= u_1 e^t + u_2 e^t + u_2 t e^t, \\ Y'' &= \cancel{u_1' e^t} + u_1 e^t + u_2' e^t + u_2 e^t + \cancel{u_2' t e^t} + u_2 e^t + u_2 t e^t \\ &= u_1 e^t + u_2' e^t + 2u_2 e^t + u_2 t e^t \end{aligned}$$

and

$$\begin{aligned} e^t \ln t &= Y'' - 2Y' + Y \\ &= \\ &= \end{aligned}$$

Then we calculate that

$$\begin{aligned} Y' &= \cancel{u_1' e^t} + u_1 e^t + \cancel{u_2' t e^t} + u_2 e^t + u_2 t e^t \\ &= u_1 e^t + u_2 e^t + u_2 t e^t, \\ Y'' &= \cancel{u_1' e^t} + u_1 e^t + u_2' e^t + u_2 e^t + \cancel{u_2' t e^t} + u_2 e^t + u_2 t e^t \\ &= u_1 e^t + u_2' e^t + 2u_2 e^t + u_2 t e^t \end{aligned}$$

and

$$\begin{aligned} e^t \ln t &= Y'' - 2Y' + Y \\ &= \left(u_1 e^t + u_2' e^t + 2u_2 e^t + u_2 t e^t\right) - 2\left(u_1 e^t + u_2 e^t + u_2 t e^t\right) \\ &\quad + \left(u_1 e^t + u_2 t e^t\right) \\ &= \end{aligned}$$

Then we calculate that

$$
\begin{aligned}
Y' &= \cancel{u_1' e^t} + u_1 e^t + \cancel{u_2' t e^t} + u_2 e^t + u_2 t e^t \\
&= u_1 e^t + u_2 e^t + u_2 t e^t, \\
Y'' &= \cancel{u_1' e^t} + u_1 e^t + u_2' e^t + u_2 e^t + \cancel{u_2' t e^t} + u_2 e^t + u_2 t e^t \\
&= u_1 e^t + u_2' e^t + 2u_2 e^t + u_2 t e^t
\end{aligned}
$$

and

$$
\begin{aligned}
e^t \ln t &= Y'' - 2Y' + Y \\
&= \left(u_1 e^t + u_2' e^t + 2u_2 e^t + u_2 t e^t\right) - 2\left(u_1 e^t + u_2 e^t + u_2 t e^t\right) \\
&\quad + \left(u_1 e^t + u_2 t e^t\right) \\
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and

$$\begin{aligned} e^t \ln t &= Y'' - 2Y' + Y \\ &= \left(u_1 e^t + u_2' e^t + 2u_2 e^t + u_2 t e^t\right) - 2\left(u_1 e^t + u_2 e^t + u_2 t e^t\right) \\ &\quad + \left(u_1 e^t + u_2 t e^t\right) \\ &= u_2' e^t. \end{aligned}$$

It follows that $u_2' = \ln t$ and thus $u_1' = -u_2' t = -t \ln t$.

Next we integrate to find

$$u_1(t) = \int u_1'(t)\,dt = \int -t\ln t\,dt = -\frac{1}{2}t^2\ln t + \frac{1}{4}t^2$$

and

$$u_2(t) = \int u_2'(t)\,dt = \int \ln t\,dt = t\ln t - t.$$

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and

$$u_2(t) = \int u_2'(t)\,dt = \int \ln t\,dt = t\ln t - t.$$

Therefore a particular solution is

$$\begin{aligned} Y(t) &= u_1(t)e^t + u_2(t)te^t \\ &= \left(-\frac{1}{2}t^2\ln t + \frac{1}{4}t^2\right)e^t + (t\ln t - t)\,te^t \\ &= \left(\frac{1}{2}\ln t - \frac{3}{4}\right)t^2e^t. \end{aligned}$$

Isn't there an easier way?

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Theorem

Suppose that $y_1(t)$ and $y_2(t)$ form a fundamental set of solutions of $y' + p(t)y' + q(t)y = 0$. Then a particular solution of $y' + p(t)y' + q(t)y = g(t)$ is given by

$$Y(t) =$$

3.9 The Method of Variation of Parameters

Isn't there an easier way?

Theorem

Suppose that $y_1(t)$ and $y_2(t)$ form a fundamental set of solutions of $y' + p(t)y' + q(t)y = 0$. Then a particular solution of $y' + p(t)y' + q(t)y = g(t)$ is given by

$$Y(t) = -y_1 \int \frac{y_2 g}{W} + y_2 \int \frac{y_1 g}{W} \tag{4}$$

where $W = W(y_1, y_2)$ is the Wronskian.

Example

Find a particular solution to $y'' - 2y' + y = e^t \ln t$.

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The characteristic equation $0 = r^2 - 2r + 1 = (r-1)^2$ has roots $r_1 = r_2 = 1$. Hence

$$y_1 = e^t \qquad \text{and} \qquad y_2 = te^t$$

form a fundamental set of solutions to the homogeneous equation $y'' - 2y' + y = 0$.

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$$y_1 = e^t \qquad \text{and} \qquad y_2 = te^t$$

form a fundamental set of solutions to the homogeneous equation $y'' - 2y' + y = 0$.
We calculate that

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} e^t & te^t \\ e^t & e^t + te^t \end{vmatrix} = e^{2t} + te^{2t} - te^{2t} = e^{2t}.$$

$$y_1 = e^t \qquad y_2 = te^t \qquad g = e^t \ln t \qquad W = e^{2t}$$

It follows that

$$\begin{aligned} Y(t) &= -y_1 \int \frac{y_2 g}{W} + y_2 \int \frac{y_1 g}{W} \\ &= \\ &= \\ &= \\ &= \end{aligned}$$

is a particular solution to the ODE.

$$y_1 = e^t \qquad y_2 = te^t \qquad g = e^t \ln t \qquad W = e^{2t}$$

It follows that

$$\begin{aligned} Y(t) &= -y_1 \int \frac{y_2 g}{W} + y_2 \int \frac{y_1 g}{W} \\ &= -e^t \int \frac{te^t e^t \ln t}{e^{2t}} dt + te^t \int \frac{e^t e^t \ln t}{e^{2t}} dt \\ &= \\ &= \\ &= \end{aligned}$$

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is a particular solution to the ODE.

$$y_1 = e^t \qquad y_2 = te^t \qquad g = e^t \ln t \qquad W = e^{2t}$$

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$$\begin{aligned} Y(t) &= -y_1 \int \frac{y_2 g}{W} + y_2 \int \frac{y_1 g}{W} \\ &= -e^t \int \frac{te^t e^t \ln t}{e^{2t}} dt + te^t \int \frac{e^t e^t \ln t}{e^{2t}} dt \\ &= -e^t \int t \ln t \, dt + te^t \int \ln t \, dt \\ &= -e^t \left(\frac{1}{2} t^2 \ln t - \frac{1}{4} t^2 \right) + te^t (t \ln t - t) \\ &= \end{aligned}$$

is a particular solution to the ODE.

$$y_1 = e^t \qquad y_2 = te^t \qquad g = e^t \ln t \qquad W = e^{2t}$$

It follows that

$$\begin{aligned}
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&= -e^t \int \frac{te^t e^t \ln t}{e^{2t}} dt + te^t \int \frac{e^t e^t \ln t}{e^{2t}} dt \\
&= -e^t \int t \ln t \; dt + te^t \int \ln t \; dt \\
&= -e^t \left(\frac{1}{2} t^2 \ln t - \frac{1}{4} t^2 \right) + te^t (t \ln t - t) \\
&= \left(\frac{1}{2} \ln t - \frac{3}{4} \right) t^2 e^t
\end{aligned}$$

is a particular solution to the ODE.

Higher Order Linear ODEs

We can use the same ideas to solve higher order linear ODEs.

Example

Solve

$$\begin{cases} y^{(4)} + y''' - 7y'' - y' + 6y = 0 \\ y(0) = 1 \\ y'(0) = 0 \\ y''(0) = -2 \\ y'''(0) = -1. \end{cases}$$

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Solve

$$\begin{cases} y^{(4)} + y''' - 7y'' - y' + 6y = 0 \\ y(0) = 1 \\ y'(0) = 0 \\ y''(0) = -2 \\ y'''(0) = -1. \end{cases}$$

The characteristic equation is

$$r^4 + r^3 - 7r^2 - r + 6 = 0$$

which has roots $r_1 = 1$, $r_2 = -1$, $r_3 = 2$ and $r_4 = -3$.

Example

Solve

$$\begin{cases} y^{(4)} + y''' - 7y'' - y' + 6y = 0 \\ y(0) = 1 \\ y'(0) = 0 \\ y''(0) = -2 \\ y'''(0) = -1. \end{cases}$$

The characteristic equation is

$$r^4 + r^3 - 7r^2 - r + 6 = 0$$

which has roots $r_1 = 1$, $r_2 = -1$, $r_3 = 2$ and $r_4 = -3$. So the general solution to the ODE is

$$y(t) = c_1 e^t + c_2 e^{-t} + c_3 e^{2t} + c_4 e^{-3t}.$$

Then

$$
\begin{aligned}
1 &= y(0) = c_1 + c_2 + c_3 + c + 4 \\
0 &= y'(0) = c_1 - c_2 + 2c_3 - 3c_4 \\
-2 &= y''(0) = c_1 + c_2 + 4c_3 + 9c_4 \\
-1 &= y'''(0) = c_1 - c_2 + 8c_3 - 27c_4
\end{aligned}
$$

Then

$$\begin{aligned} 1 &= y(0) = c_1 + c_2 + c_3 + c + 4 \\ 0 &= y'(0) = c_1 - c_2 + 2c_3 - 3c_4 \\ -2 &= y''(0) = c_1 + c_2 + 4c_3 + 9c_4 \\ -1 &= y'''(0) = c_1 - c_2 + 8c_3 - 27c_4 \end{aligned} \implies \begin{aligned} c_1 &= \frac{11}{8} \\ c_2 &= \frac{5}{12} \\ c_3 &= -\frac{2}{3} \\ c_4 &= -\frac{1}{8} \end{aligned}$$

Then

$$\begin{aligned} 1 &= y(0) = c_1 + c_2 + c_3 + c + 4 \\ 0 &= y'(0) = c_1 - c_2 + 2c_3 - 3c_4 \\ -2 &= y''(0) = c_1 + c_2 + 4c_3 + 9c_4 \\ -1 &= y'''(0) = c_1 - c_2 + 8c_3 - 27c_4 \end{aligned} \implies \begin{aligned} c_1 &= \frac{11}{8} \\ c_2 &= \frac{5}{12} \\ c_3 &= -\frac{2}{3} \\ c_4 &= -\frac{1}{8} \end{aligned}$$

Therefore the solution to the IVP is

$$\boxed{y = \frac{11}{8}e^t + \frac{5}{12}e^{-t} - \frac{2}{3}e^{2t} - \frac{1}{8}e^{-3t}.}$$

Example

Solve

$$y^{(4)} - y = e^t$$

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The characteristic equation

$$0 = r^4 - 1 = (r^2 - 1)(r^2 + 1)$$

has roots $r_1 = 1$, $r_2 = -1$, $r_3 = i$ and $r_4 = -i$.

Example

Solve
$$y^{(4)} - y = e^t$$

The characteristic equation

$$0 = r^4 - 1 = (r^2 - 1)(r^2 + 1)$$

has roots $r_1 = 1$, $r_2 = -1$, $r_3 = i$ and $r_4 = -i$. Therefore

$$y = c_1 e^t + c_2 e^{-t} + c_3 \cos t + c_4 \sin t$$

is the general solution of the homogenous equation $y^{(4)} - y = 0$.

$$y^{(4)} - y = e^t$$

Next we need to find a particular solution.

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$$\begin{aligned} Y' &= Ae^t + Ate^t \\ Y'' &= Ae^t + Ae^t + Ate^t = 2Ae^t + Ate^t \\ Y''' &= 2Ae^t + Ae^t + Ate^t = 3Ae^t + Ate^t \\ Y^{(4)} &= 3Ae^t + Ae^t + Ate^t = 4Ae^t + Ate^t \end{aligned}$$

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and

$$e^t = Y^{(4)} - Y = 4Ae^t + Ate^t - Ate^t = 4Ae^t \qquad \Longrightarrow \qquad A = \tfrac{1}{4}.$$

Therefore $Y(t) = \frac{1}{4}te^t$ is a particular solution to the ODE.

The general solution to the ODE is therefore

$$\boxed{y(t) = c_1 e^t + c_2 e^{-t} + c_3 \cos t + c_4 \sin t + \frac{1}{4} t e^t.}$$

Remark

Any time the characteristic equation has a repeated root, just multiply by t.

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Any time the characteristic equation has a repeated root, just multiply by t. E.g. if the roots are $r_1 = 7$, $r_2 = 7$, $r_3 = 7$, $r_4 = 7$, $r_5 = 7$ and $r_6 = 8$, then the general solution is

$$y(t) = c_1 e^{7t} + c_2 t e^{7t} + c_3 t^2 e^{7t} + c_4 t^3 e^{7t} + c_5 t^4 e^{7t} + c_6 e^{8t}.$$

Example (Going backwards)

Find a linear, homogeneous ODEs with constant coefficients, which has general solution
$y(t) = c_1 e^t + c_2 t e^t + c_3 e^{2t} \sin t + c_4 e^{2t} \cos t + c_5 e^{2t} t \sin t + c_6 e^{2t} t \cos t.$

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The first two terms correspond to a double root $r = 1$.

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The first two terms correspond to a double root $r = 1$. The last four terms correspond to a double complex root $r = 2 \pm i$.

Example (Going backwards)

Find a linear, homogeneous ODEs with constant coefficients, which has general solution
$y(t) = c_1e^t + c_2te^t + c_3e^{2t}\sin t + c_4e^{2t}\cos t + c_5e^{2t}t\sin t + c_6e^{2t}t\cos t.$

The first two terms correspond to a double root $r = 1$. The last four terms correspond to a double complex root $r = 2 \pm i$. Consequently, the characteristic equation is

$$\begin{aligned} 0 &= (r-1)^2(r-2-i)^2(r-2+i)^2 \\ &= (r-1)^2(r^2-4r+5)^2 \\ &= r^6 - 10r^5 + 43r^4 - 100r^3 + 131r^2 - 90r + 25. \end{aligned}$$

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The first two terms correspond to a double root $r = 1$. The last four terms correspond to a double complex root $r = 2 \pm i$. Consequently, the characteristic equation is

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Then, a differential equation is

$$\frac{d^6 y}{dt^6} - 10\frac{d^5 y}{dt^5} + 43\frac{d^4 y}{dt^4} - 100\frac{d^3 y}{dt^3} + 131\frac{d^2 y}{dt^2} - 90\frac{dy}{dt} + 25y = 0.$$

Next Week

- 4.1 Definition of the Laplace Transform
- 4.2 Solving Initial Value Problems