Lecture 7

- 4.3 Monotonic Functions and the First Derivative Test
- 4.4 Concavity and Curve Sketching
- 4.5 Applied Optimisation
- 4.7 Antiderivatives

Monotonic Functions and the First Derivative Test

4.3

4.3 Monotonic Functions and the First Derivative Test

Theorem

Suppose that

- $f : [a, b] \to \mathbb{R}$ *is continuous; and*
- f *is differentiable on* (a, b).

1. *If* $f'(x) > 0$ *for all* $x \in (a, b)$, *then* f *is increasing on* $[a, b]$.
2. *If* $f'(x) < 0$ *for all* $x \in (a, b)$, *then* f *is decreasing on* $[a, b]$.

Example

Let $f(x) = x^3 - 12x - 5$.

1. Find the critical points of f.
2. Identify the intervals where f is increasing and the intervals where f is decreasing.

Clearly f is continuous and differentiable everywhere.

$$0 = f'(x) = 3x^2 - 12 = 3(x^2 - 4) = 3(x-2)(x+2)$$
$$\implies \quad x = -2 \text{ or } 2.$$

The critical points are $x = -2$ and $x = 2$.

These critical points cut $(-\infty, \infty)$ into 3 open intervals: $(-\infty, -2)$, $(-2, 2)$ and $(2, \infty)$.

4.3 Monotonic Functions and the First Derivative Test

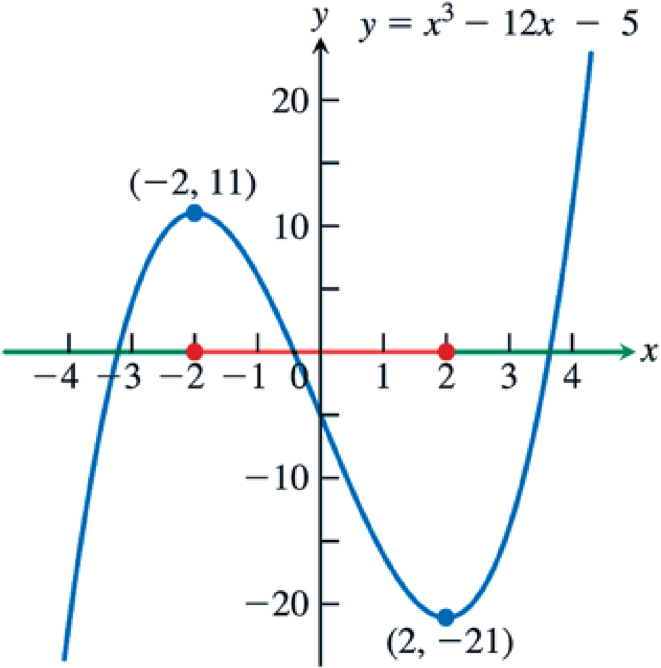
Interval	$(-\infty, -2)$	$(-2, 2)$	$(2, \infty)$
Calculate $f'(x_0)$ at one point			
f' is			
f is			

Interval	$(-\infty, -2)$	$(-2, 2)$	$(2, \infty)$
Calculate $f'(x_0)$ at one point	$f'(-3) = 15$		
f' is	> 0		
f is	increasing		

Interval	$(-\infty, -2)$	$(-2, 2)$	$(2, \infty)$
Calculate $f'(x_0)$ at one point	$f'(-3) = 15$	$f'(0) = -12$	
f' is	> 0	< 0	
f is	increasing	decreasing	

Interval	$(-\infty, -2)$	$(-2, 2)$	$(2, \infty)$
Calculate $f'(x_0)$ at one point	$f'(-3) = 15$	$f'(0) = -12$	$f'(3) = 15$
f' is	> 0	< 0	> 0
f is	increasing	decreasing	increasing

Therefore f is increasing on $(-\infty, -2)$ and on $(2, \infty)$, and f is decreasing on $(-2, 2)$.


y
$y = x^3 - 12x - 5$
20
10
−10
−20
(−2, 11)
(2, −21)
x
−4
−3
−2
−1
0
1
2
3
4

The First Derivative Test For Local Extrema

Theorem (The First Derivative Test For Local Extrema)

Suppose that

- $f : [a, b] \to \mathbb{R}$ *is continuous;*
- c *is a critical point of* f*; and*
- f *is differentiable on both* $(c - \delta, c)$ *and* $(c, c + \delta)$ *for some* $\delta > 0$.

on the left of c	*on the right of* c	*at* c

The First Derivative Test For Local Extrema

Theorem (The First Derivative Test For Local Extrema)

on the left of c	*on the right of c*	*at c*
$f' < 0$	$f' > 0$	f *has a local minimum*

The First Derivative Test For Local Extrema

Theorem (The First Derivative Test For Local Extrema)

on the left of c	*on the right of c*	*at c*
$f' < 0$	$f' > 0$	f *has a local minimum*
$f' > 0$	$f' < 0$	f *has a local maximum*

The First Derivative Test For Local Extrema

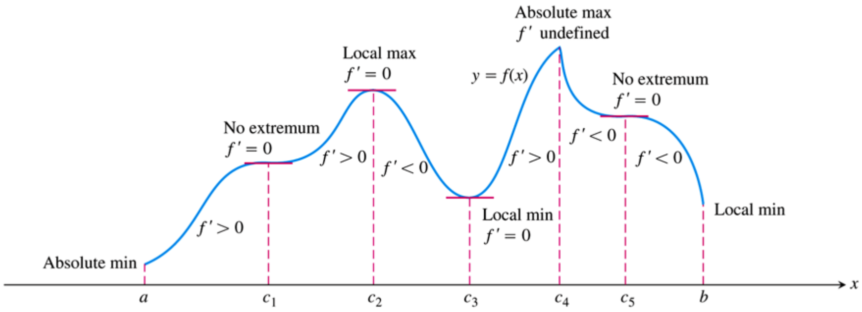
Theorem (The First Derivative Test For Local Extrema)

on the left of c	*on the right of c*	*at c*
$f' < 0$	$f' > 0$	*f has a local minimum*
$f' > 0$	$f' < 0$	*f has a local maximum*
$f' > 0$	$f' > 0$	*f does not have a local extremem*

The First Derivative Test For Local Extrema

Theorem (The First Derivative Test For Local Extrema)

on the left of c	*on the right of c*	*at c*
$f' < 0$	$f' > 0$	*f has a local minimum*
$f' > 0$	$f' < 0$	*f has a local maximum*
$f' > 0$	$f' > 0$	*f does not have a local extremem*
$f' < 0$	$f' < 0$	*f does not have a local extremem*


Absolute max
f' undefined
Local max
$f' = 0$
$y = f(x)$
No extremum
$f' = 0$
No extremum
$f' = 0$
$f' > 0$
$f' < 0$
$f' > 0$
$f' < 0$
$f' < 0$
Local min
$f' = 0$
Local min
$f' > 0$
Absolute min
a
c_1
c_2
c_3
c_4
c_5
b
x

Example

Let $f(x) = x^{\frac{1}{3}}(x-4) = x^{\frac{4}{3}} - 4x^{\frac{1}{3}}$.

1. Find the critical points of f.
2. Identify the intervals on which f is increasing/decreasing.
3. Find the extreme values of f.

Example

Let $f(x) = x^{\frac{1}{3}}(x-4) = x^{\frac{4}{3}} - 4x^{\frac{1}{3}}$.

1. Find the critical points of f.
2. Identify the intervals on which f is increasing/decreasing.
3. Find the extreme values of f.

f is continuous everywhere because $x^{\frac{1}{3}}$ and $(x-4)$ are continuous functions. We can calculate that

$$f'(x) = \frac{d}{dx}\left(x^{\frac{4}{3}} - 4x^{\frac{1}{3}}\right) = \frac{4}{3}x^{\frac{1}{3}} - \frac{4}{3}x^{-\frac{2}{3}}$$
$$= \frac{4}{3}x^{-\frac{2}{3}}(x-1) = \frac{4(x-1)}{3x^{\frac{2}{3}}}.$$

$f'(x)$ does not exist if $x=0$. $f'(x)=0$ if and only if $x=1$. The critical points of f are $x=0$ and $x=1$.

Using the critical points, we "cut" $(-\infty, \infty)$ into three subintervals: $(-\infty, 0)$, $(0, 1)$ and $(1, \infty)$.

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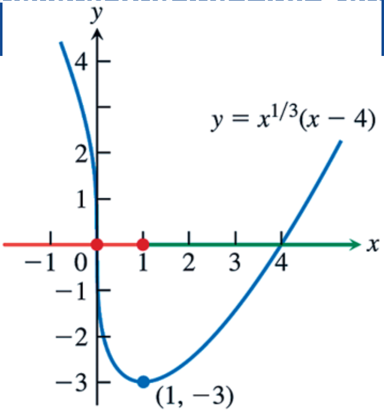
$(-\infty, 0)$	$(0, 1)$	$(1, \infty)$
$f' < 0$	$f' < 0$	$f' > 0$
(e.g. $f'(-1) = -\frac{8}{3}$)		
f is decreasing	f is decreasing	f is increasing

Using the critical points, we "cut" $(-\infty, \infty)$ into three subintervals: $(-\infty, 0)$, $(0, 1)$ and $(1, \infty)$.

$(-\infty, 0)$	$(0, 1)$	$(1, \infty)$
$f' < 0$	$f' < 0$	$f' > 0$
(e.g. $f'(-1) = -\frac{8}{3}$)		
f is decreasing	f is decreasing	f is increasing

We can see from this table that $x = 1$ is a local minimum and $x = 0$ is not an extremum.



So

$$\min_{x\in\mathbb{R}} f(x) = f(1) = 1^{\frac{1}{3}}(1-4) = -3.$$

Note that f does not have an absolute maximum.

Note that $\lim_{x\to 0} f'(x) = -\infty$. Therefore the graph of $y = f(x)$ has a vertical tangent at $x = 0$.

EXAMPLE 3 Within the interval $0 \leq x \leq 2\pi$, find the critical points of

$$f(x) = \sin^2 x - \sin x - 1.$$

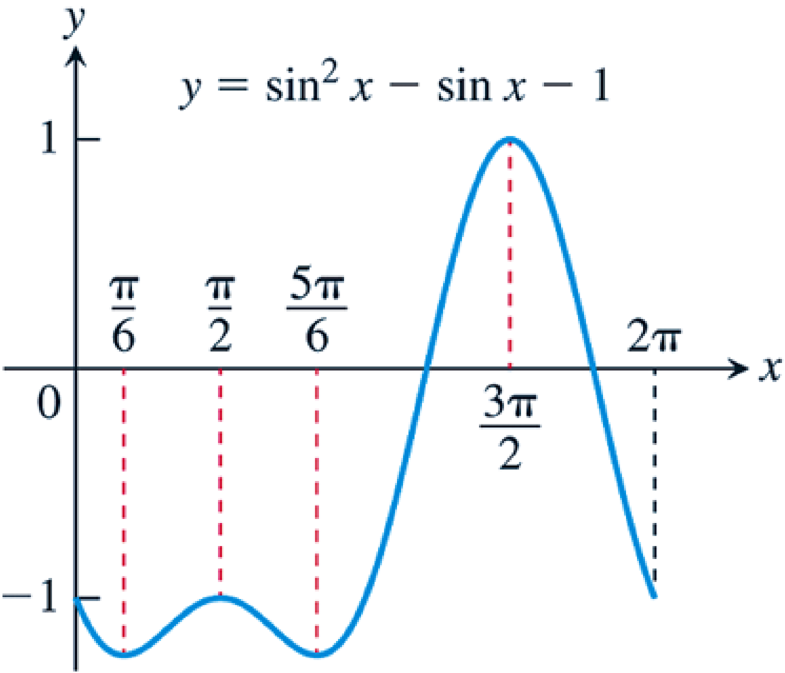
Identify the open intervals on which f is increasing and decreasing. Find the function's local and absolute extreme values.

Solution The function f is continuous over $[0, 2\pi]$ and differentiable over $(0, 2\pi)$, so the critical points occur at the zeros of f' in $(0, 2\pi)$. We find

$$f'(x) = 2 \sin x \cos x - \cos x = (2 \sin x - 1)(\cos x).$$

The first derivative is zero if and only if $\sin x = \frac{1}{2}$ or $\cos x = 0$. So the critical points of f in $(0, 2\pi)$ are $x = \pi/6$, $x = 5\pi/6$, $x = \pi/2$, and $x = 3\pi/2$. They partition $[0, 2\pi]$ into open intervals as follows.

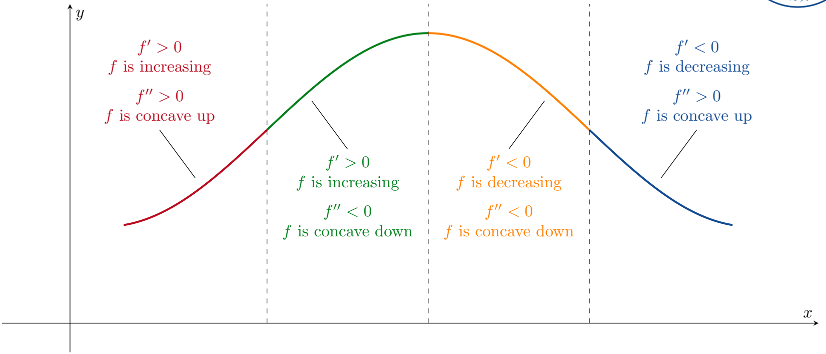
Interval	$\left(0, \frac{\pi}{6}\right)$	$\left(\frac{\pi}{6}, \frac{\pi}{2}\right)$	$\left(\frac{\pi}{2}, \frac{5\pi}{6}\right)$	$\left(\frac{5\pi}{6}, \frac{3\pi}{2}\right)$	$\left(\frac{3\pi}{2}, 2\pi\right)$
Sign of f'	$-$	$+$	$-$	$+$	$-$
Behavior of f	dec	inc	dec	increasing	decreasing



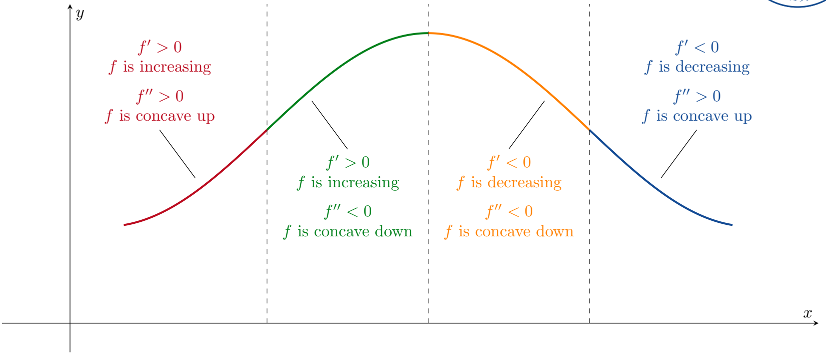
y
$y = \sin^2 x - \sin x - 1$
1
$\frac{\pi}{6}$
$\frac{\pi}{2}$
$\frac{5\pi}{6}$
2π
x
0
$\frac{3\pi}{2}$
−1

Concavity and Curve Sketching

4.4 Concavity and Curve Sketching



4.4 Concavity and Curve Sketching



Definition

$y = f(x)$ is

1. *concave up* if f' is increasing; and
2. *concave down* if f' is decreasing.

Theorem (The Second Derivative Test for Concavity)

Suppose that $f : I \to \mathbb{R}$ is twice differentiable.

1. *If $f'' > 0$ on I, then f is concave up on I.*
2. *If $f'' < 0$ on I, then f is concave down on I.*

Example

Consider $y = x^3$. Then $y' = 3x^2$ and $y'' = 6x$.

$(-\infty, 0)$	$(0, \infty)$
$y'' < 0$	$y'' > 0$
$y = x^3$ is concave down	$y = x^3$ is concave up

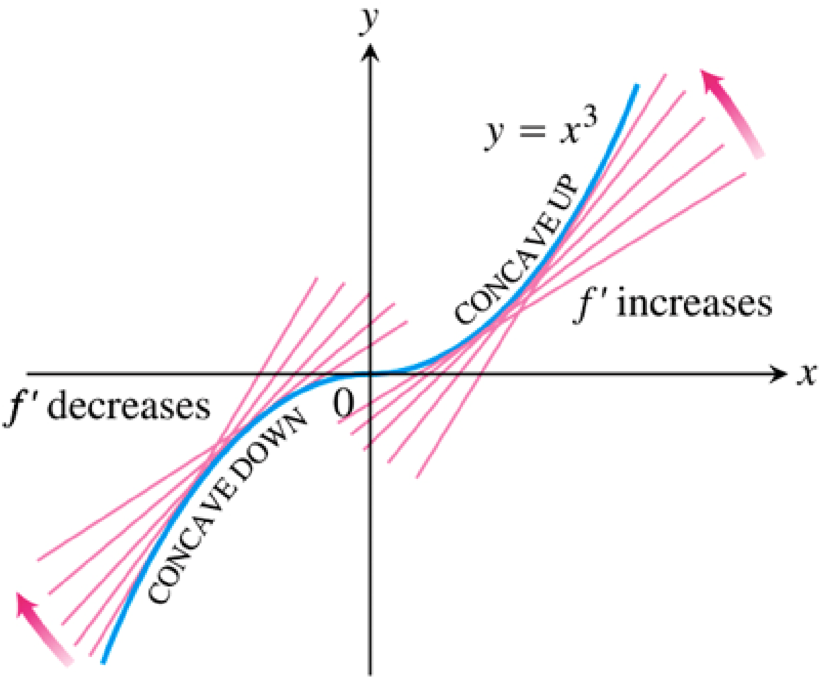
Example

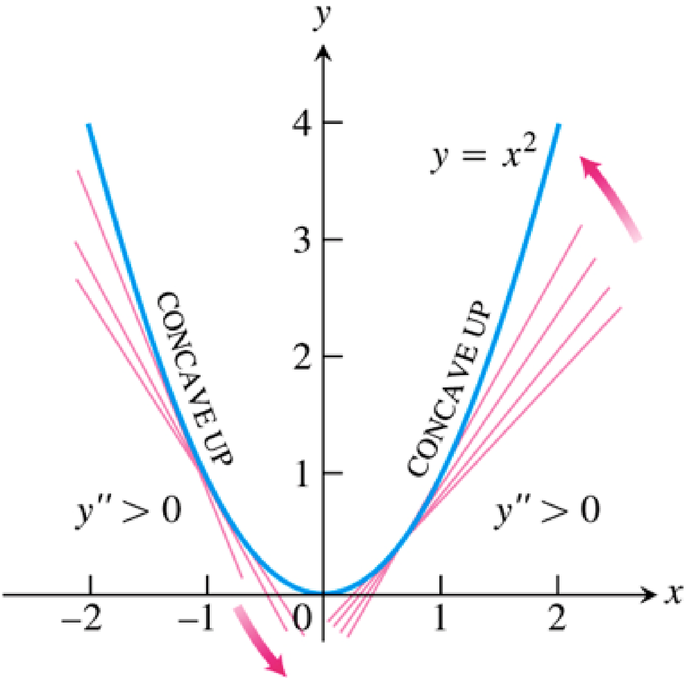
Consider $y = x^3$. Then $y' = 3x^2$ and $y'' = 6x$.

$(-\infty, 0)$	$(0, \infty)$
$y'' < 0$	$y'' > 0$
$y = x^3$ is concave down	$y = x^3$ is concave up

Example

Consider $y = x^2$. Since $y' = 2x$ and $y'' = 2$, we have that $y'' > 0$ everywhere. Therefore $y = x^2$ is concave up everywhere.


y
x
$y = x^3$
CONCAVE UP
f' increases
f' decreases
0
CONCAVE DOWN



y
x
4
3
2
1
0
−2
−1
1
2
$y = x^2$
CONCAVE UP
CONCAVE UP
$y'' > 0$
$y'' > 0$

Example

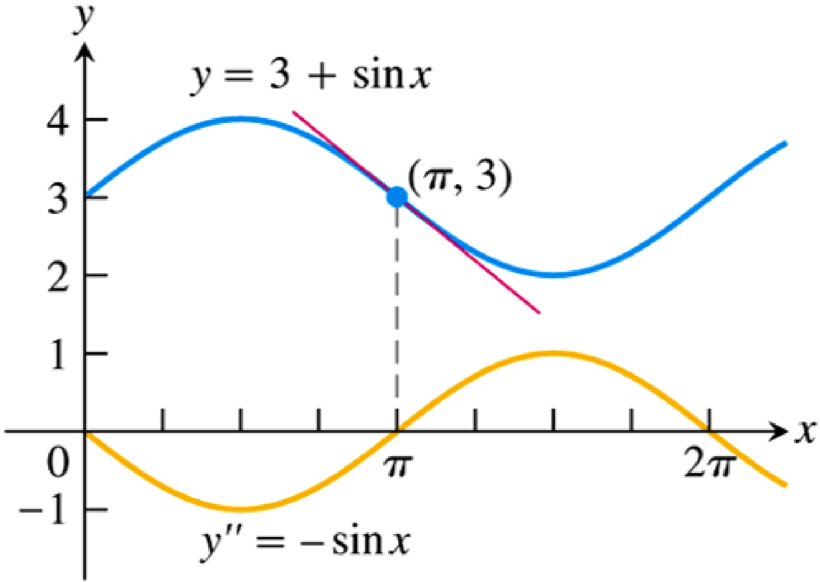
Determine the concavity of $y = 3 + \sin x$ on $[0, 2\pi]$.

Example

Determine the concavity of $y = 3 + \sin x$ on $[0, 2\pi]$.

First we calculate that $y' = \cos x$ and $y'' = -\sin x$.

$(0, \pi)$	$(\pi, 2\pi)$
$y'' < 0$	$y'' > 0$
$y = 3 + \sin x$ is concave down	$y = 3 + \sin x$ is concave up


y
$y = 3 + \sin x$
4
3
2
1
0
−1
$(\pi, 3)$
π
2π
x
$y'' = -\sin x$

Points of Inflection

Definition

$(c, f(c))$ is a *point of inflection* of $y = f(x)$ if

- $y = f(x)$ has a tangent line at $x = c$; and
- the concavity of $y = f(x)$ changes at $x = c$.

Points of Inflection

Definition

$(c, f(c))$ is a *point of inflection* of $y = f(x)$ if

- $y = f(x)$ has a tangent line at $x = c$; and
- the concavity of $y = f(x)$ changes at $x = c$.

Remark

$(c, f(c))$ is a point of inflection $\Longrightarrow$

- $f''(c) = 0$;

  or
- $f''(c)$ does not exist.

EXAMPLE 3 Determine the concavity and find the inflection points of the function

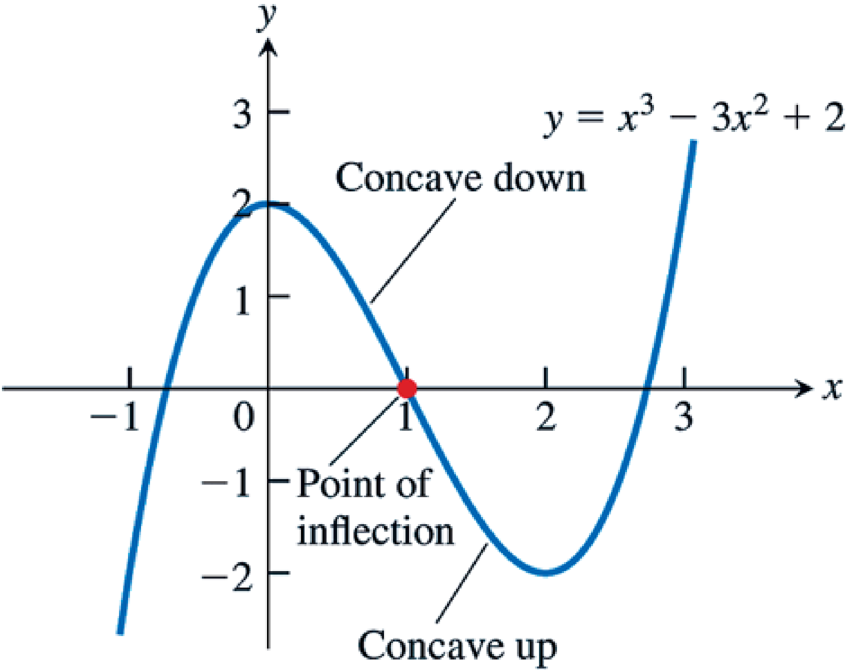
$$f(x) = x^3 - 3x^2 + 2.$$

Solution We start by computing the first and second derivatives.

$$f'(x) = 3x^2 - 6x, \qquad f''(x) = 6x - 6.$$

To determine concavity, we look at the sign of the second derivative $f''(x) = 6x - 6$. The sign is negative when $x < 1$, is 0 at $x = 1$, and is positive when $x > 1$. It follows that the graph of f is concave down on $(-\infty, 1)$, is concave up on $(1, \infty)$, and has an inflection point at the point $(1, 0)$ where the concavity changes.

The graph of f is shown in Figure 4.27. Notice that we did not need to know the shape of this graph ahead of time in order to determine its concavity. ■


y
3
2
1
−1
−2
$y = x^3 - 3x^2 + 2$
Concave down
−1
0
1
2
3
x
Point of
inflection
Concave up

Example

Find all the points of inflection of $f(x) = x^{\frac{5}{3}}$.

Since $f'(x) = \frac{5}{3}x^{\frac{2}{3}}$ and

$$f''(x) = \frac{d}{dx}\left(\frac{5}{3}x^{\frac{2}{3}}\right) = \frac{10}{9}x^{-\frac{1}{3}} = \frac{10}{9\sqrt[3]{x}},$$

Example

Find all the points of inflection of $f(x) = x^{\frac{5}{3}}$.

Since $f'(x) = \frac{5}{3}x^{\frac{2}{3}}$ and

$$f''(x) = \frac{d}{dx}\left(\frac{5}{3}x^{\frac{2}{3}}\right) = \frac{10}{9}x^{-\frac{1}{3}} = \frac{10}{9\sqrt[3]{x}},$$

we can say that

- if $x < 0$, then $f''(x) < 0$;
- $f''(0)$ does not exist; and
- if $x > 0$, then $f''(x) > 0$.

Example

Find all the points of inflection of $f(x) = x^{\frac{5}{3}}$.

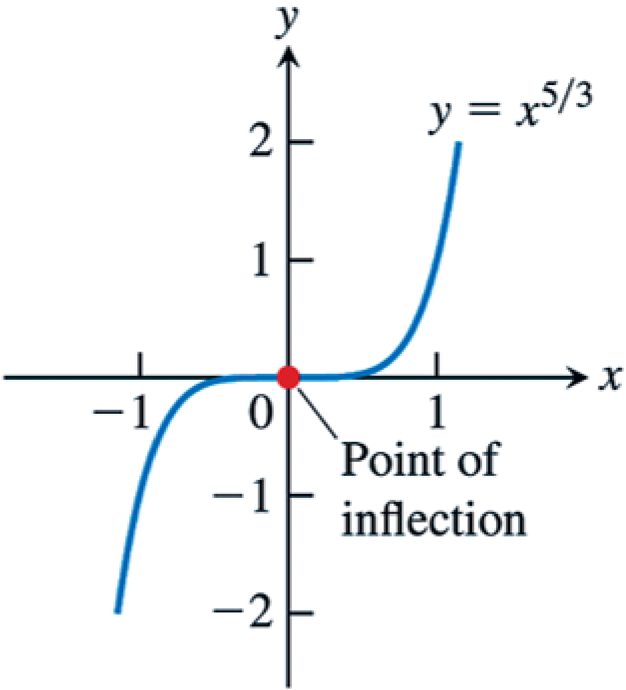
Since $f'(x) = \frac{5}{3}x^{\frac{2}{3}}$ and

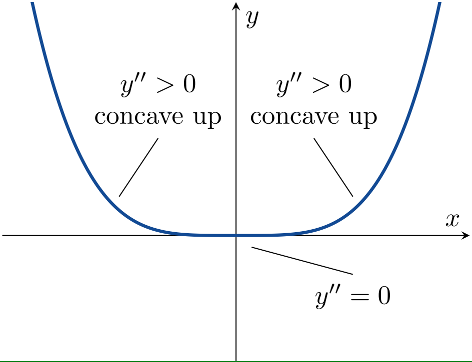
$$f''(x) = \frac{d}{dx}\left(\frac{5}{3}x^{\frac{2}{3}}\right) = \frac{10}{9}x^{-\frac{1}{3}} = \frac{10}{9\sqrt[3]{x}},$$

we can say that

- if $x < 0$, then $f''(x) < 0$;
- $f''(0)$ does not exist; and
- if $x > 0$, then $f''(x) > 0$.

Therefore $(0, 0)$ is an point of inflection of $y = x^{\frac{5}{3}}$.

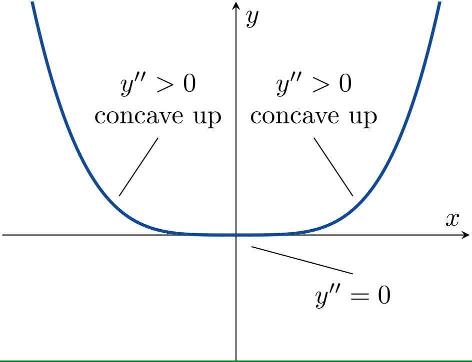

y
$y = x^{5/3}$
2
1
−1
0
1
x
Point of inflection
−1
−2



Example

Let $y = x^4$. Then $y' = 4x^3$ and $y = 12x^2$.

4.4 Concavity and Curve Sketching



Example

Let $y = x^4$. Then $y' = 4x^3$ and $y = 12x^2$. Note that $y'' = 0$ at $x = 0$, but the concavity of the graph does not change. Hence $(0, 0)$ is not a point of inflection of $y = x^4$.

Example

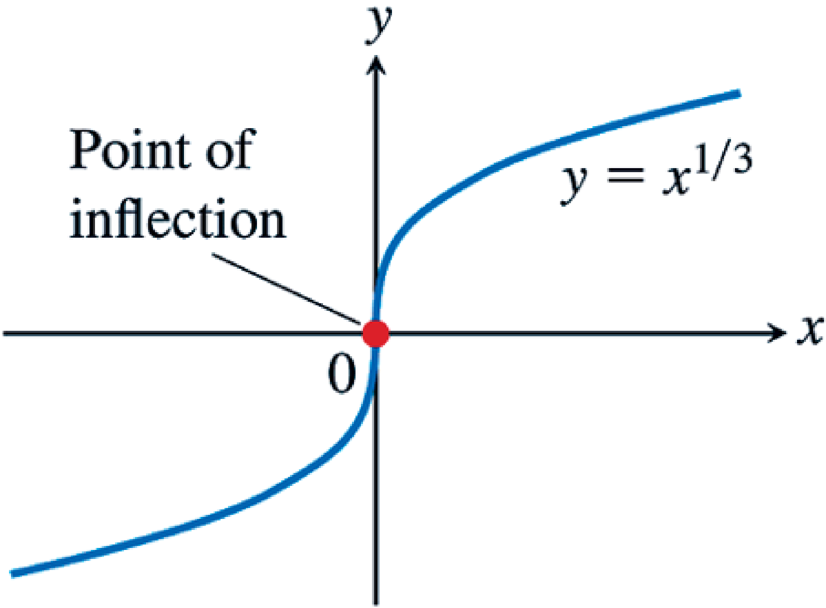
Let $y = x^{\frac{1}{3}}$. Then $y' = \frac{1}{3}x^{-\frac{2}{3}}$ and $y'' = -\frac{2}{9}x^{-\frac{5}{3}}$. Note that y'' does not exist at $x = 0$.

Example

Let $y = x^{\frac{1}{3}}$. Then $y' = \frac{1}{3}x^{-\frac{2}{3}}$ and $y'' = -\frac{2}{9}x^{-\frac{5}{3}}$. Note that y'' does not exist at $x = 0$.

$(-\infty, 0)$	$(0, \infty)$
$y'' > 0$	$y'' < 0$
$y = x^{\frac{1}{3}}$ is concave up	$y = x^{\frac{1}{3}}$ is concave down

$(0, 0)$ is a point of inflection of $y = x^{\frac{1}{3}}$.


y
Point of
inflection
$y = x^{1/3}$
x
0

The Second Derivative Test for Local Extrema

Theorem (The Second Derivative Test for Local Extrema)

Suppose that

- f'' *is continuous on* (a, b)*; and*
- $c \in (a, b)$.

The Second Derivative Test for Local Extrema

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Suppose that

- f'' *is continuous on* (a, b); *and*
- $c \in (a, b)$.

1. *If* $f'(c) = 0$ *and* $f''(c) < 0$, *then* f *has a local maximum at* $x = c$.

The Second Derivative Test for Local Extrema

Theorem (The Second Derivative Test for Local Extrema)

Suppose that

- f'' *is continuous on* (a, b)*; and*
- $c \in (a, b)$.

1. *If* $f'(c) = 0$ *and* $f''(c) < 0$*, then* f *has a local maximum at* $x = c$.
2. *If* $f'(c) = 0$ *and* $f''(c) > 0$*, then* f *has a local minimum at* $x = c$.

The Second Derivative Test for Local Extrema

Theorem (The Second Derivative Test for Local Extrema)

Suppose that

- f'' *is continuous on* (a, b); *and*
- $c \in (a, b)$.

1. *If* $f'(c) = 0$ *and* $f''(c) < 0$, *then* f *has a local maximum at* $x = c$.
2. *If* $f'(c) = 0$ *and* $f''(c) > 0$, *then* f *has a local minimum at* $x = c$.
3. *If* $f'(c) = 0$ *and* $f''(c) = 0$, *then we don't know – we need to use a different theorem.*

Example

Let $f(x) = x^4 - 4x^3 + 10$.

1. Find where the local extrema are.
2. Find the intervals where f is increasing/decreasing.
3. Find the intervals where f is concave up/concave down.
4. Sketch the general shape of $y = f(x)$.
5. Plot some points which satisfy $y = f(x)$.
6. Graph $y = f(x)$.

- $f(x) = x^4 - 4x^3 + 10$ is continuous because it is a polynomial.
- The domain of f is $(-\infty, \infty)$.
- Clearly $f'(x) = 4x^3 - 12x^2$.
- The domain of f' is also $(-\infty, \infty)$.
- To find the critical points, we need to solve $f'(x) = 0$.

- $f(x) = x^4 - 4x^3 + 10$ is continuous because it is a polynomial.
- The domain of f is $(-\infty, \infty)$.
- Clearly $f'(x) = 4x^3 - 12x^2$.
- The domain of f' is also $(-\infty, \infty)$.
- To find the critical points, we need to solve $f'(x) = 0$.

$$0 = f'(x) = 4x^3 - 12x^2 = 4x^2(x - 3) \implies x = 0 \text{ or } x = 3.$$

Interval	$(-\infty, 0)$	$(0, 3)$	$(3, \infty)$
f' is	$f' < 0$	$f' < 0$	$f' > 0$
f is	decreasing	decreasing	increasing

Interval	$(-\infty, 0)$	$(0, 3)$	$(3, \infty)$
f' is	$f' < 0$	$f' < 0$	$f' > 0$
f is	decreasing	decreasing	increasing

1. By the First Derivative Test, $x = 3$ is a local minimum and $x = 0$ is not an extrema.

Interval	$(-\infty, 0)$	$(0, 3)$	$(3, \infty)$
f' is	$f' < 0$	$f' < 0$	$f' > 0$
f is	decreasing	decreasing	increasing

1. By the First Derivative Test, $x = 3$ is a local minimum and $x = 0$ is not an extrema.
2. f is decreasing on $(-\infty, 0]$ and on $[0, 3]$. f is increasing on $[3, \infty)$.

3. Next we need to solve $f''(x) = 0$.

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$$0 = f''(x) = 12x^2 - 24x \implies x = 0 \text{ or } x = 2.$$

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$$0 = f''(x) = 12x^2 - 24x \implies x = 0 \text{ or } x = 2.$$

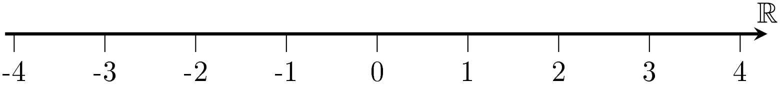
Interval	$(-\infty, 0)$	$(0, 2)$	$(2, \infty)$
f'' is	$f'' > 0$	$f'' < 0$	$f'' > 0$
f is	concave up	concave down	concave up

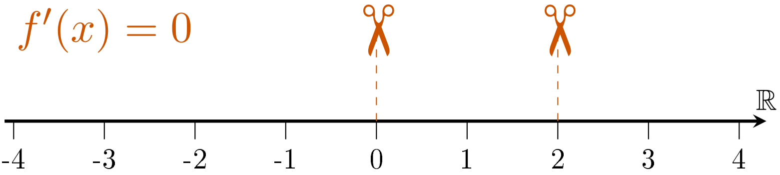
3 Next we need to solve $f''(x) = 0$.

$$0 = f''(x) = 12x^2 - 24x \implies x = 0 \text{ or } x = 2.$$

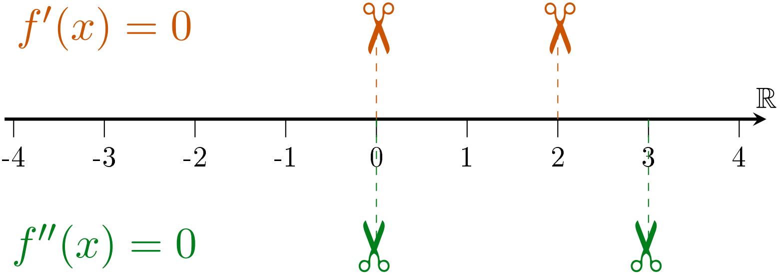
Interval	$(-\infty, 0)$	$(0, 2)$	$(2, \infty)$
f'' is	$f'' > 0$	$f'' < 0$	$f'' > 0$
f is	concave up	concave down	concave up

f is concave up on $(-\infty, 0)$ and on $(2, \infty)$. f is concave down on $(0, 2)$.

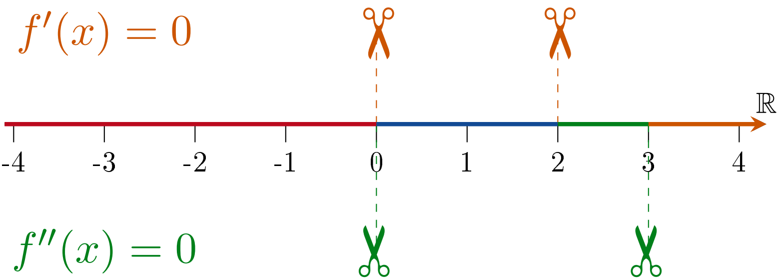

ℝ
-4
-3
-2
-1
0
1
2
3
4


$f'(x) = 0$
ℝ
-4
-3
-2
-1
0
1
2
3
4

4.4 Concavity and Curve Sketching



4.4 Concavity and Curve Sketching



We need to consider 4 intervals: $(-\infty, 0)$, $(0, 2)$, $(2, 3)$, $(3, \infty)$.

4.4 Concavity and Curve Sketching

4 Putting the previous two tables together, we obtain

$(-\infty, 0)$	$(0, 2)$	$(2, 3)$	$(3, \infty)$
decreasing	decreasing	decreasing	increasing
concave up	concave down	concave up	concave up
			

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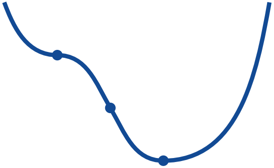
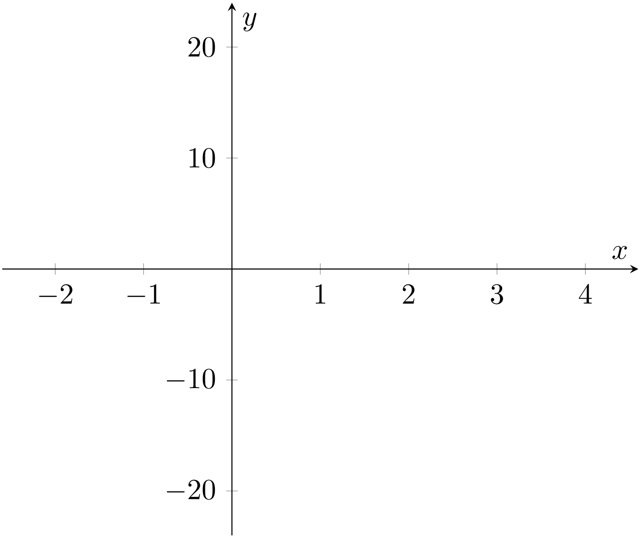
$(-\infty, 0)$	$(0, 2)$	$(2, 3)$	$(3, \infty)$
decreasing	decreasing	decreasing	increasing
concave up	concave down	concave up	concave up
			

Therefore the general shape of f is



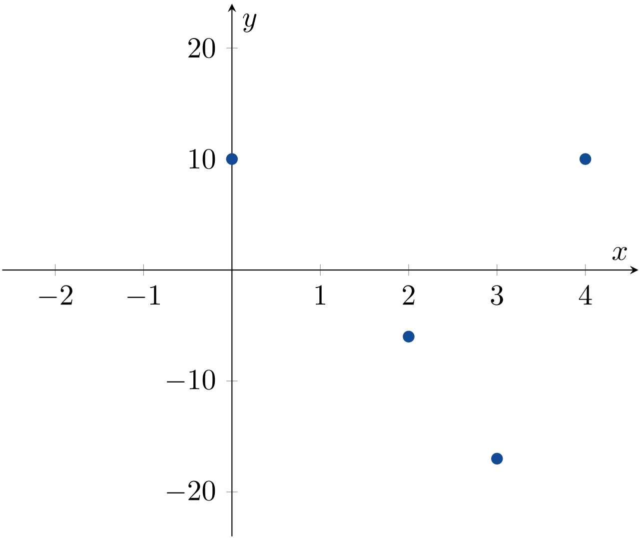
5 We calculate some (x, y) points.

x	y
0	10
2	-6
3	-17
4	10



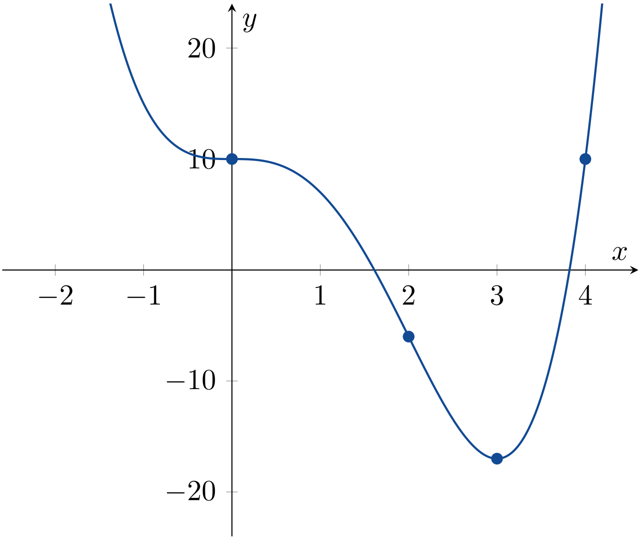
5 We calculate some (x, y) points. Then we plot these points.

x	y
0	10
2	-6
3	-17
4	10



6 Finally we can graph $y = x^4 - 4x^3 + 10$.

x	y
0	10
2	-6
3	-17
4	10



Procedure for Graphing $y = f(x)$

1. Identify the domain of f and any symmetries the curve may have.
2. Find the derivatives y' and y''.
3. Find the critical points of f, if any, and identify the function's behavior at each one.
4. Find where the curve is increasing and where it is decreasing.
5. Find the points of inflection, if any occur, and determine the concavity of the curve.
6. Identify any asymptotes that may exist.
7. Plot key points, such as the intercepts and the points found in Steps 3–5, and sketch the curve together with any asymptotes that exist.

Example

Sketch the graph of $f(x) = \dfrac{(x+1)^2}{1+x^2}$.

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You can check that

$$f'(x) = \frac{2(1-x^2)}{(1+x^2)^2}$$

and

$$f''(x) = \frac{4x(x^2-3)}{(1+x^2)^3}.$$

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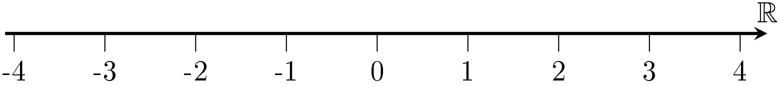
Moreover

$$f'(x) = 0 \qquad \Longrightarrow \qquad x = \pm 1$$

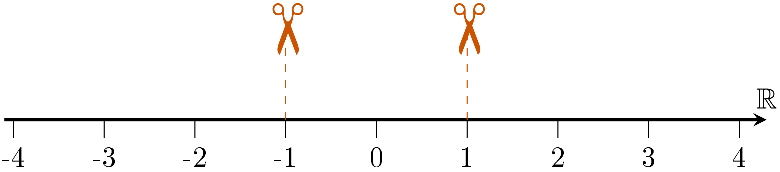
and

$$f''(x) = 0 \qquad \Longrightarrow \qquad x = 0 \text{ or } \pm\sqrt{3}.$$

4.4 Concavity and Curve Sketching



$f'(x) = 0$



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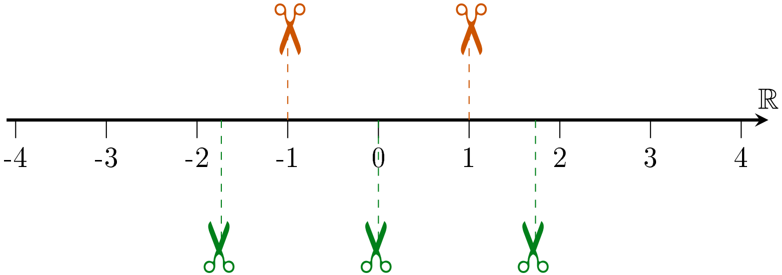
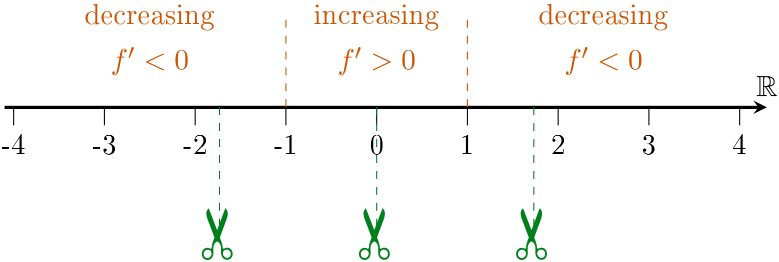
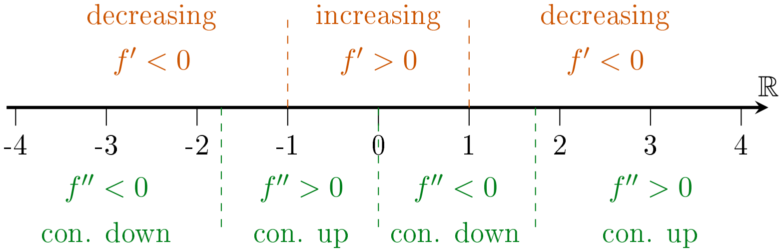


$f''(x) = 0$


decreasing
$f' < 0$
increasing
$f' > 0$
decreasing
$f' < 0$
$\mathbb{R}$
-4
-3
-2
-1
0
1
2
3
4

$f''(x) = 0$


decreasing
$f' < 0$
increasing
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decreasing
$f' < 0$
$\mathbb{R}$
-4
-3
-2
-1
0
1
2
3
4
$f'' < 0$
con. down
$f'' > 0$
con. up
$f'' < 0$
con. down
$f'' > 0$
con. up

4.4 Concavity and Curve Sketching

$(-\infty, -\sqrt{3})$	$(-\sqrt{3}, -1)$	$(-1, 0)$	$(0, 1)$	$(1, \sqrt{3})$	$(\sqrt{3}, \infty)$
decrea.	decrea.	increa.	increa.	decrea.	decrea.
con. down	con. up	con. up	con. down	con. down	con. up
					

Asymptotes:

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Since

$$\lim_{x\to\pm\infty} f(x) = \lim_{x\to\pm\infty} \frac{(x+1)^2}{1+x^2} = \lim_{x\to\pm\infty} \frac{x^2+2x+1}{1+x^2}$$
$$= \lim_{x\to\pm\infty} \frac{1+\frac{2}{x}+\frac{1}{x^2}}{\frac{1}{x^2}+1} = 1,$$

the line $y = 1$ is a horizontal asymptote.

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\begin{aligned}
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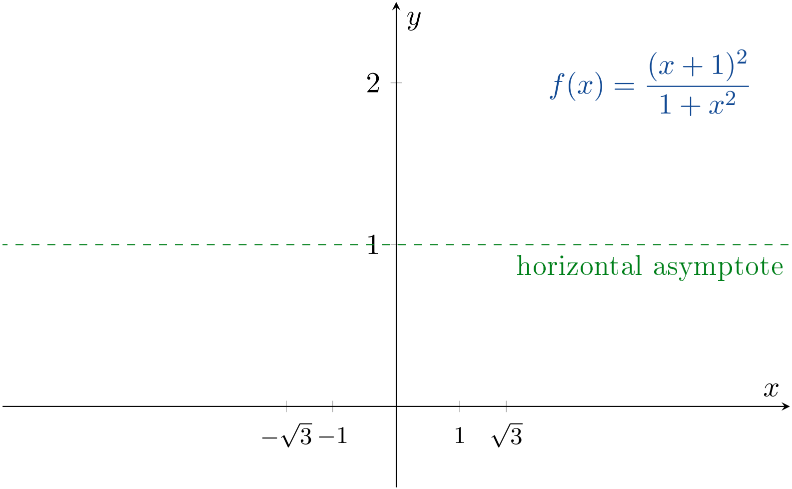
&= \lim_{x\to\pm\infty} \frac{1+\frac{2}{x}+\frac{1}{x^2}}{\frac{1}{x^2}+1} = 1,
\end{aligned}
$$

the line $y = 1$ is a horizontal asymptote.

There are no vertical asymptotes because $f(x)$ is defined everywhere.

Now we have enough information to sketch the graph



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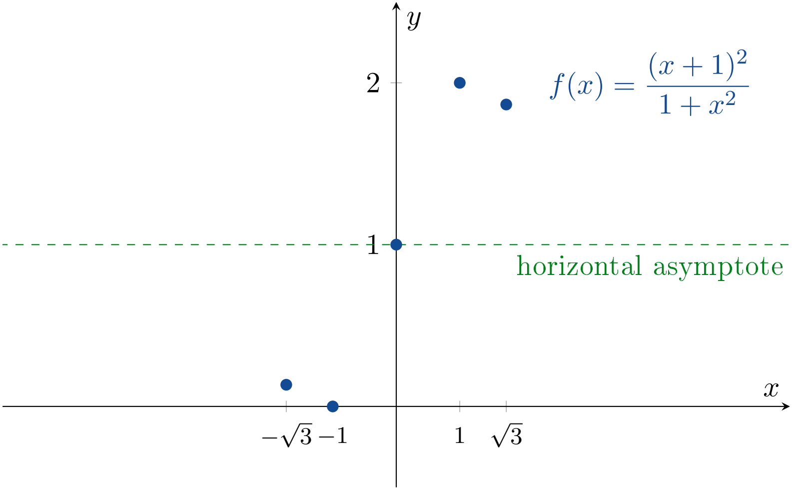
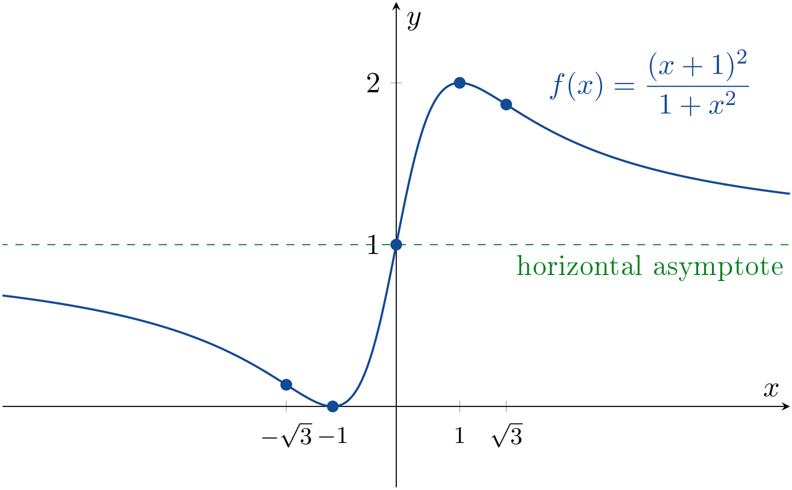


Now we have enough information to sketch the graph



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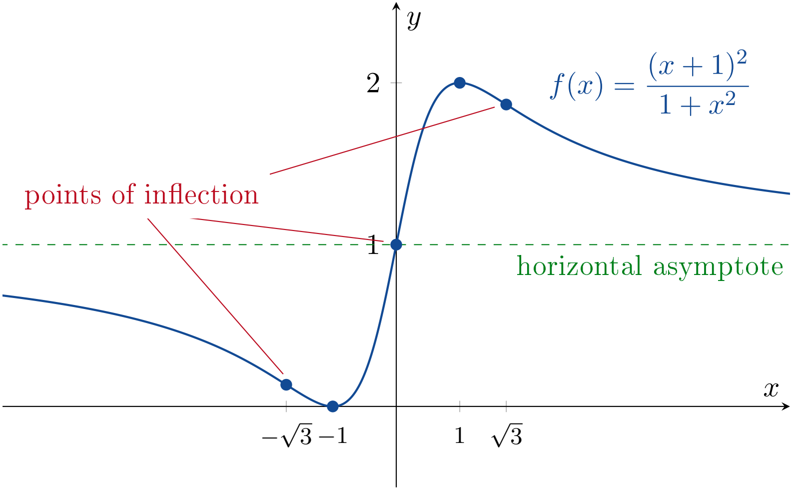

Example

Sketch the graph of $f(x) = \dfrac{x^2+4}{2x}$.

The domain is everything except $x = 0$. f is an odd function.

Note that

$$f(x) = \frac{x^2+4}{2x} = \frac{x}{2} + \frac{2}{x},$$

$$f'(x) = \frac{1}{2} - \frac{2}{x^2} = \frac{x^2-4}{2x^2} \qquad \text{and} \qquad f''(x) = \frac{4}{x^3}.$$

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So we need to consider 4 intervals: $(-\infty, -2)$, $(-2, 0)$, $(0, 2)$ and $(2, \infty)$.

4.4 Concavity and Curve Sketching

$(-\infty, -2)$	$(-2, 0)$	$(0, 2)$	$(2, \infty)$
increasing	decreasing	decreasing	increasing
concave down	concave down	concave up	concave up

Asymptotes: Since

$$\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} \frac{x}{2} + \frac{2}{x} = \infty$$

and

$$\lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} \frac{x}{2} + \frac{2}{x} = -\infty,$$

the line $x = 0$ is a vertical asymptote.

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and

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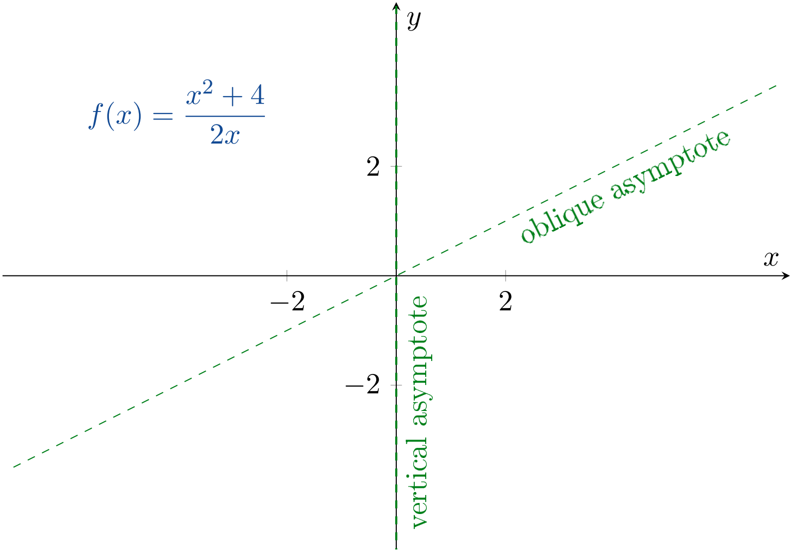
the line $x = 0$ is a vertical asymptote.

Since $f(x) = \dfrac{x}{2} + \dfrac{2}{x}$ and since $\displaystyle\lim_{x\to\pm\infty} \frac{2}{x} = 0$, the line $y = \dfrac{x}{2}$ is an oblique asymptote.

Now we have enough information to sketch the graph.



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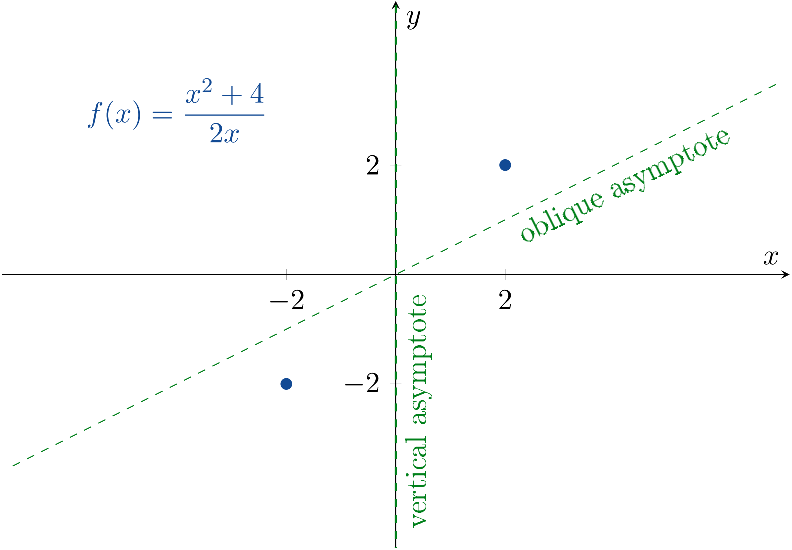

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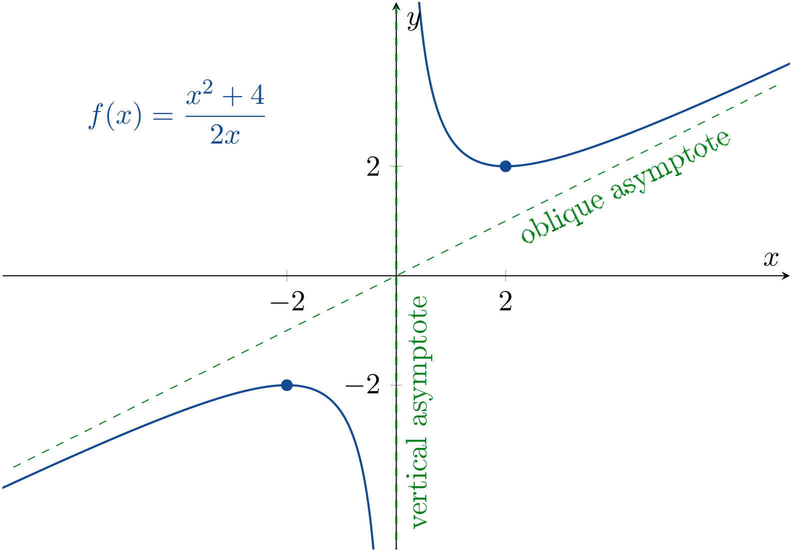


Please read Example 11 in your textbook.

Break

We will continue at 2pm

"The next part of this recipe will involve some calculus."

4.5

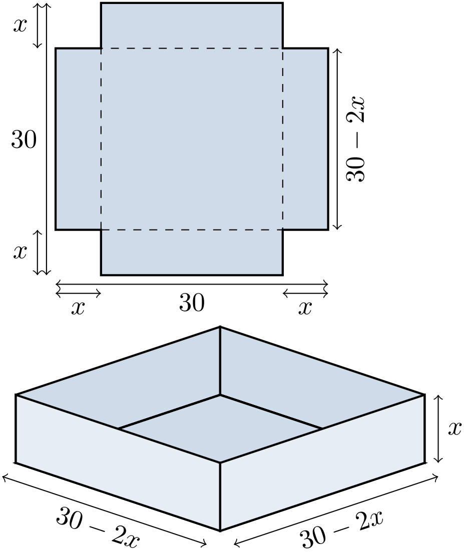
Applied Optimisation

Example

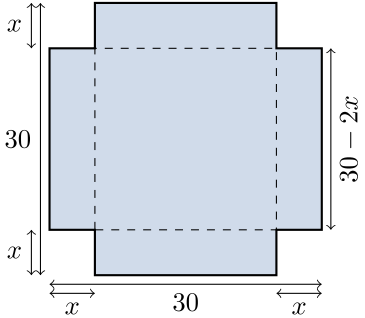
An open-top box is to be made by cutting $x\,\text{cm} \times x\,\text{cm}$ squares from the corners of a $30\,\text{cm} \times 30\,\text{cm}$ piece of metal and bending the sides up.

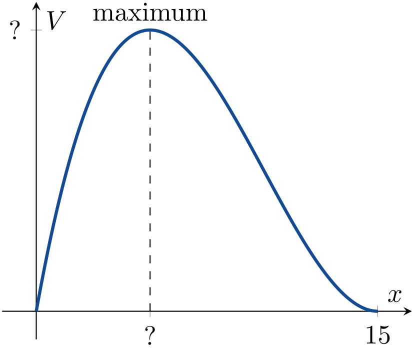
How large should the squares cut from the corners be to make the box hold as much as possible?

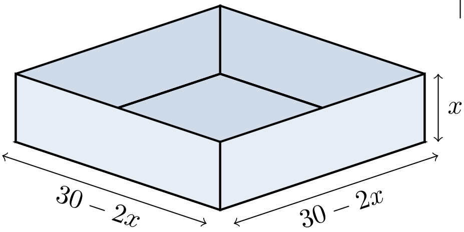
4.5 Applied Optimisation

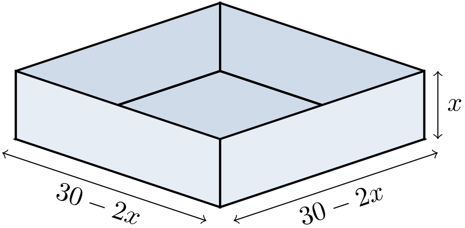


4.5 Applied Optimisation









The volume of the box will be

$$V(x) = x(30 - 2x)^2.$$

Note that the domain of V is $[0, 15]$.

$V(x) = x(30 - 2x)$

We calculate that

$$\begin{aligned} 0 = \frac{dV}{dx} &= (x)'(30-2x)^2 + (x)\big((30-2x)^2\big)' \\ &= (1)(30-2x)^2 + (x)2(30-2x)(-2) \\ &= (30-2x)\big((30-2x) - 4x\big) \\ &= (30-2x)(30-6x) = 12(15-x)(5-x). \end{aligned}$$

Therefore $x = 5$ or $x = 15$.

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Therefore $x = 5$ or $x = 15$. Since $15 \notin (0,15)$, the only critical point of V is $x = 5$.

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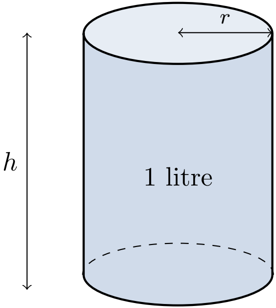
Therefore $x = 5$ or $x = 15$. Since $15 \notin (0, 15)$, the only critical point of V is $x = 5$.

To make the largest possible box, we should choose $x = 5$. Such a box will have a volume of

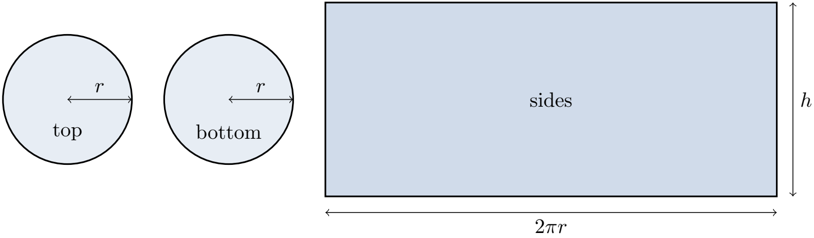
$$V(5) = 5(30-10)^2 = 2000\,\mathrm{cm}^3 = 2\,\text{litres}.$$



Example

You are designing a 1 litre cylindrical drinks can. You will use the same metal and the same thickness of metal for the top, bottom and sides. What dimensions will use the least metal?

We will use cm. Suppose that the radius of the can is r cm and the height of the can is h cm.



Then the volume of the can is

$$\pi r^2 h = 1000 \text{ cm}^3$$

and the surface area of the can is

$$A = 2\pi r^2 + 2\pi r h.$$

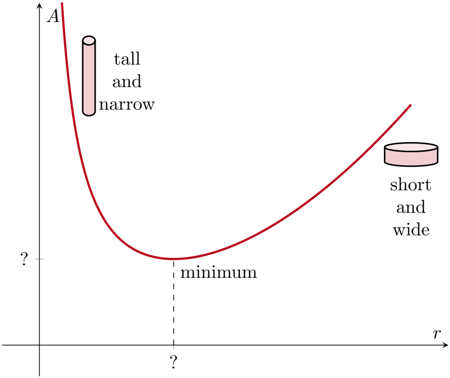
We want to make A as small as we can.

Since

$$\pi r^2 h = 1000 \implies h = \frac{1000}{\pi r^2}$$

we have that

$$A(r) = 2\pi r^2 + 2\pi r h = 2\pi r^2 + 2\pi r\left(\frac{1000}{\pi r^2}\right) = 2\pi r^2 + \frac{2000}{r}.$$

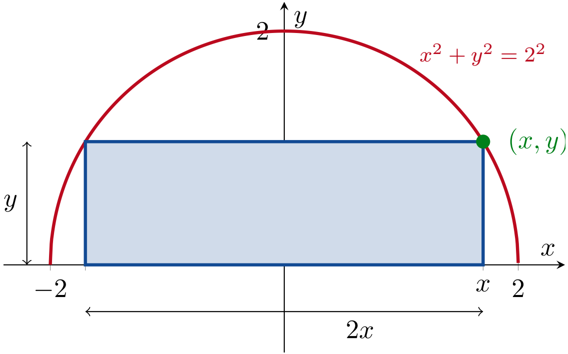


Then we calculate that

$$
\begin{aligned}
0 = \frac{dA}{dx} &= 4\pi r - \frac{2000}{r^2} \\
4\pi r &= \frac{2000}{r^2} \\
4\pi r^3 &= 2000 \\
r &= \sqrt[3]{\frac{2000}{4\pi}} = \sqrt[3]{\frac{500}{\pi}} \approx 5.42 \text{ cm}
\end{aligned}
$$

and

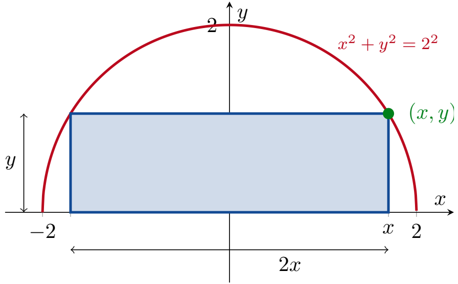
$$h = \frac{1000}{\pi r^2} = 2\sqrt[3]{\frac{500}{\pi}} \approx 10.84 \text{ cm}.$$



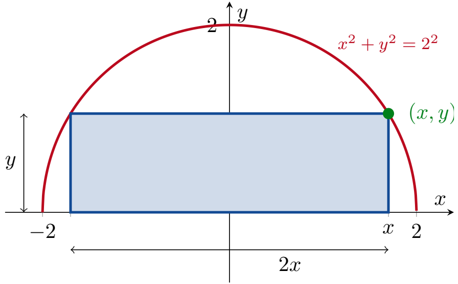
Example

A rectangle is to be inscribed in a semicircle of radius 2 as shown above.

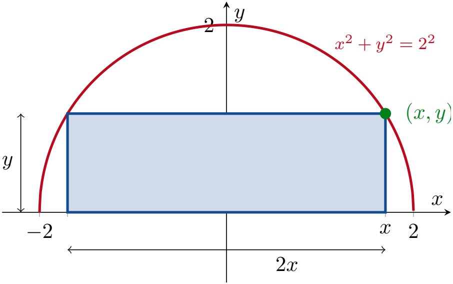
What is the largest possible area of the rectangle?



Consider a rectangle with a vertex at the point (x, y). The area of this rectangle is clearly $A = 2xy$.



Consider a rectangle with a vertex at the point (x, y). The area of this rectangle is clearly $A = 2xy$. Since the point (x, y) lies on the circle $x^2 + y^2 = 2^2$, we must have $y = \sqrt{4 - x^2}$.



Consider a rectangle with a vertex at the point (x, y). The area of this rectangle is clearly $A = 2xy$. Since the point (x, y) lies on the circle $x^2 + y^2 = 2^2$, we must have $y = \sqrt{4 - x^2}$. Hence the area of the rectangle is

$$A(x) = 2x\sqrt{4 - x^2}.$$

We want to find $\max_{x \in [0,2]} A(x)$.

By differentiating A, we see that

$$\begin{aligned} 0 = \frac{dA}{dx} &= \frac{d}{dx}\left(2x\sqrt{4-x^2}\right) \\ &= 2\sqrt{4-x^2} + 2x\left(\frac{-2x}{2\sqrt{4-x^2}}\right) \\ &= 2\sqrt{4-x^2} - \frac{2x^2}{\sqrt{4-x^2}}. \end{aligned}$$

Multiplying by $\sqrt{4-x^2}$ gives

$$0 = 2(4-x^2) - 2x^2 = 8 - 4x^2 = 4(2-x^2)$$

which implies that $x = \pm\sqrt{2}$.

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$$\max_{x\in[0,2]} A(x) = A\left(\sqrt{2}\right) = 2\sqrt{2}\sqrt{4-2} = 2\sqrt{2}\sqrt{2} = 4.$$

Please read three more examples in your textbook.

4.7 Antiderivatives

Definition

F is an *antiderivative* of f on an interval I if $F'(x) = f(x)$ for all $x \in I$.

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x^2 is an antiderivative of $2x$.

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Example

$2x$ is the derivative of x^2.
x^2 is an antiderivative of $2x$.

Example

$G(x) = \sin x$ is an antiderivative of $g(x) = \cos x$ because

$$G'(x) = \frac{d}{dx}(\sin x) = \cos x = g(x).$$

Example

$H(x) = x^2 + \sin x$ is an antiderivative of $h(x) = 2x + \cos x$ because $H'(x) = h(x)$.

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4.7 Antiderivatives

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$x^2 - 1234$ is an antiderivative of $2x$ because $\frac{d}{dx}\left(x^2 - 1234\right) = 2x$.

Theorem

If F is an antiderivative of f on I, then the general antiderivative of f is

$$F(x) + C$$

where C is a constant.

Example

Find an antiderivative of $f(x) = 3x^2$ that satisfies $F(1) = -1$.

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Therefore $F(x) = x^3 - 2$.

function $f(x)$	derivative $f'(x)$
x^n	nx^{n-1}
$\sin kx$	$k \cos kx$
$\cos kx$	$-k \sin kx$
$\tan kx$	$k \sec^2 kx$

4.7 Antiderivatives

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function $f(x)$	general antiderivative $F(x)$
$x^n \ (n \neq -1)$	
$\cos kx$	
$\sin kx$	
$\sec^2 kx$	

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function $f(x)$	general antiderivative $F(x)$
$x^n \ (n \neq -1)$	$\frac{x^{n+1}}{n+1} + C$
$\cos kx$	
$\sin kx$	
$\sec^2 kx$	

function $f(x)$	derivative $f'(x)$
x^n	nx^{n-1}
$\sin kx$	$k\cos kx$
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function $f(x)$	general antiderivative $F(x)$
$x^n \ (n \neq -1)$	$\frac{x^{n+1}}{n+1} + C$
$\cos kx$	

$$\frac{d}{dx}\left(\frac{x^{n+1}}{n+1} + C\right) = \frac{(n+1)x^n}{n+1} = x^n$$

function $f(x)$	derivative $f'(x)$
x^n	nx^{n-1}
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$\sin kx$	
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4.7 Antiderivatives

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4.7 Antiderivatives

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function $f(x)$	general antiderivative $F(x)$
$x^n \ (n \neq -1)$	$\frac{x^{n+1}}{n+1} + C$
$\cos kx$	$\frac{1}{k}\sin kx + C$
$\sin kx$	$-\frac{1}{k}\cos kx + C$
$\sec^2 kx$	$\frac{1}{k}\tan kx + C$

EXAMPLE 3 Find the general antiderivative of each of the following functions.

(a) $f(x) = x^5$ **(b)** $g(x) = \dfrac{1}{\sqrt{x}}$ **(c)** $h(x) = \sin 2x$

(d) $i(x) = \cos \dfrac{x}{2}$

Solution In each case, we can use one of the formulas listed in Table 4.2.

(a) $F(x) = \dfrac{x^6}{6} + C$ Formula 1 with $n = 5$

(b) $g(x) = x^{-1/2}$, so

$$G(x) = \frac{x^{1/2}}{1/2} + C = 2\sqrt{x} + C$$ Formula 1 with $n = -1/2$

(c) $H(x) = \dfrac{-\cos 2x}{2} + C$ Formula 2 with $k = 2$

(d) $I(x) = \dfrac{\sin(x/2)}{1/2} + C = 2\sin\dfrac{x}{2} + C$ Formula 3 with $k = 1/2$

The Sum Rule and the Constant Multiple Rule

Suppose that

- F is an antiderivative of f;
- G is an antiderivative of g;
- $k \in \mathbb{R}$.

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- F is an antiderivative of f;
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The Sum Rule: The general antiderivative of $f + g$ is

$$F(x) + G(x) + C.$$

The Sum Rule and the Constant Multiple Rule

Suppose that

- F is an antiderivative of f;
- G is an antiderivative of g;
- $k \in \mathbb{R}$.

The Sum Rule: The general antiderivative of $f + g$ is

$$F(x) + G(x) + C.$$

The Constant Multiple Rule: The general antiderivative of kf is

$$kF(x) + C.$$

Example

Find the general antiderivative of $f(x) = \frac{3}{\sqrt{x}} + \sin 2x$.

We have $f = 3g + h$ where $g(x) = x^{-\frac{1}{2}}$ and $h(x) = \sin 2x$.

Example

Find the general antiderivative of $f(x) = \frac{3}{\sqrt{x}} + \sin 2x$.

We have $f = 3g + h$ where $g(x) = x^{-\frac{1}{2}}$ and $h(x) = \sin 2x$. An antiderivative of g is

$$G(x) = \frac{x^{-\frac{1}{2}+1}}{-\frac{1}{2}+1} = \frac{x^{\frac{1}{2}}}{\frac{1}{2}} = 2\sqrt{x}.$$

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An antiderivative of h is

$$H(x) = -\frac{1}{2}\cos 2x.$$

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An antiderivative of h is

$$H(x) = -\frac{1}{2}\cos 2x.$$

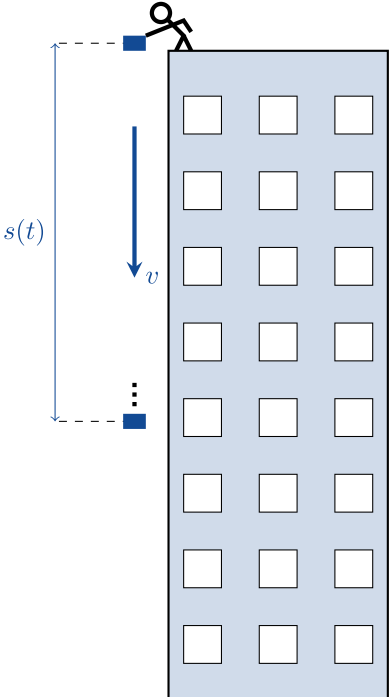
Therefore the general antiderivative of f is

$$F(x) = 6\sqrt{x} - \frac{1}{2}\cos 2x + C.$$

Example

You drop a box off the top of a tall building. The acceleration due to gravity is $9.8\,\mathrm{ms}^{-2}$. You can ignore air resistance.

How far does the box fall in 5 seconds?


$s(t)$
v

The acceleration is

$$a(t) = 9.8\,\mathrm{ms}^{-2}$$

downwards. Since

$$\text{acceleration} = \frac{d}{dt}(\text{velocity}),$$

the velocity is an antiderivative of the acceleration. Therefore the velocity is

$$v(t) = 9.8t + C\,\mathrm{ms}^{-1}.$$

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$$v(t) = 9.8t + C\,\mathrm{ms}^{-1}.$$

You let go of the box at time $t = 0$. So $v(0) = 0$. Thus $C = 0$. Hence

$$v(t) = 9.8t\,\mathrm{ms}^{-1}.$$

Now

$$\text{velocity} = \frac{d}{dt}(\text{displacement}).$$

So the distance fallen is an antiderivative of velocity. Hence

$$s(t) = 4.9t^2 + \tilde{C}\,\text{m}.$$

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So the distance fallen is an antiderivative of velocity. Hence

$$s(t) = 4.9t^2 + \tilde{C}\,\text{m}.$$

Because you let go of the box at time $t = 0$, we have $s(0) = 0$. Thus $\tilde{C} = 0$. Therefore

$$s(t) = 4.9t^2\,\text{m}.$$

Now

$$\text{velocity} = \frac{d}{dt}(\text{displacement}).$$

So the distance fallen is an antiderivative of velocity. Hence

$$s(t) = 4.9t^2 + \tilde{C}\,\text{m}.$$

Because you let go of the box at time $t = 0$, we have $s(0) = 0$. Thus $\tilde{C} = 0$. Therefore

$$s(t) = 4.9t^2\,\text{m}.$$

After 5 seconds, the box has fallen

$$s(5) = 4.9 \times 25 = 122.5\,\text{metres}.$$

Indefinite Integrals

Definition

The general antiderivative of f is also called the *indefinite integral* of f with respect to x, and is denoted by

$$\int f(x)\ dx.$$

4.7 Antiderivatives

the integral sign
integral işareti

x is the variable of integration
x ise integral değişkeni olarak tanımlanır

$$\int \underbrace{f(x)}_{\substack{\text{the integrand} \\ \text{integralin integrandı}}} \, dx$$

Example

$$\int 2x\ dx = x^2 + C$$

$$\int \cos x\ dx = \sin x + C$$

$$\int (2x + \cos x)\ dx = x^2 + \sin x + C$$

Example

Calculate $\displaystyle\int (x^2 - 2x + 5)\ dx$.

Example

Calculate $\int (x^2 - 2x + 5)\ dx$.

solution 1. Since $\frac{d}{dx}\left(\frac{x^3}{3} - x^2 + 5x\right) = x^2 - 2x + 5$ we have that

$$\int (x^2 - 2x + 5)\ dx = \frac{x^3}{3} - x^2 + 5x + C.$$

solution 2.

$$\begin{aligned}\int (x^2 - 2x + 5)\ dx &= \int x^2\ dx - \int 2x\ dx + \int 5\ dx \\ &= \left(\frac{x^3}{3} + C_1\right) - \left(x^2 + C_2\right) + (5x + C_3) \\ &= \left(\frac{x^3}{3} - x^2 + 5x\right) + (C_1 - C_2 + C_3)\,.\end{aligned}$$

solution 2.

$$\begin{aligned}\int (x^2 - 2x + 5)\ dx &= \int x^2\ dx - \int 2x\ dx + \int 5\ dx \\ &= \left(\frac{x^3}{3} + C_1\right) - \left(x^2 + C_2\right) + \left(5x + C_3\right) \\ &= \left(\frac{x^3}{3} - x^2 + 5x\right) + \left(C_1 - C_2 + C_3\right).\end{aligned}$$

Because we only need one constant, we can define $C := C_1 - C_2 + C_3$. Therefore

$$\int (x^2 - 2x + 5)\ dx = \frac{x^3}{3} - x^2 + 5x + C.$$

Next Time

- 5.1 Area and Estimating with Finite Sums
- 5.2 Sigma Notation and Limits of Finite Sums
- 5.3 The Definite Integral