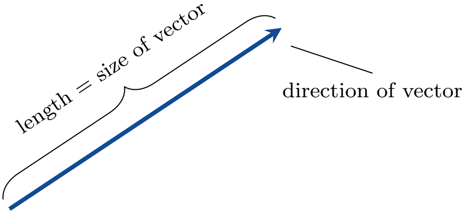
# Lecture 4

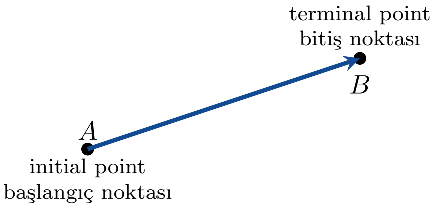
- 11. Vectors
- 12. The Dot Product
- 13. The Cross Product

# 11 Vectors

For some quantities (mass, time, distance, . . . ) we only need a number. For some quantities (velocity, force, . . . ) we need a number and a direction.



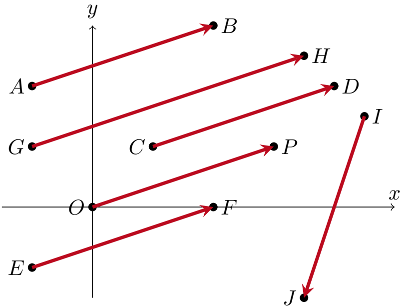

A *vector* is an object which has a size (length) and a direction.




**Definition**

The vector $\overrightarrow{AB}$ has *initial point* $A$ and *terminal point* $B$.

The *length* of $\overrightarrow{AB}$ is written $\left\|\overrightarrow{AB}\right\|$.




Two vectors are equal if they have the same length and the same direction.

We can say that

$$\overrightarrow{AB} = \overrightarrow{CD} = \overrightarrow{EF} = \overrightarrow{OP}.$$

Note that $\overrightarrow{AB} \neq \overrightarrow{GH}$ because the lengths are different, and $\overrightarrow{AB} \neq \overrightarrow{IJ}$ because the directions are different.

## Notation

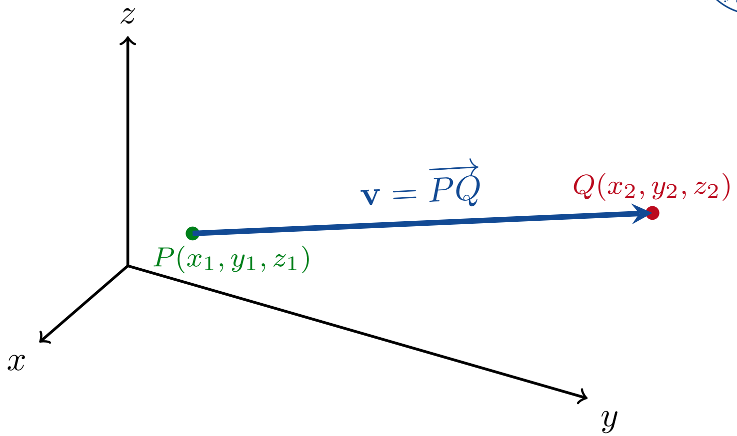
When we use a computer, we use bold letters for vectors: $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$, . . . . When we use a pen, we use underlined letters for vectors: $\underline{u}$, $\underline{v}$, $\underline{w}$, . . . .

If we type $a\mathbf{u} + b\mathbf{v}$ or write $a\underline{u} + b\underline{v}$, then

- $a$ and $b$ are numbers; and
- $\mathbf{u}$, $\mathbf{v}$, $\underline{u}$ and $\underline{v}$ are vectors.

**Definition**

In $\mathbb{R}^2$: If $\mathbf{v}$ has initial point $(0,0)$ and terminal point $(v_1, v_2)$, then the *component form* of $\mathbf{v}$ is $\mathbf{v} = (v_1, v_2)$.
In $\mathbb{R}^3$: If $\mathbf{v}$ has initial point $(0,0,0)$ and terminal point $(v_1, v_2, v_3)$, then the *component form* of $\mathbf{v}$ is $\mathbf{v} = (v_1, v_2, v_3)$.




$$(v_1, v_2, v_3) = \mathbf{v} = (x_2 - x_1, y_2 - y_1, z_2 - z_1)$$

**Definition**

In $\mathbb{R}^2$: The *norm* (or *length*) of $\mathbf{v} = (v_1, v_2)$ is

$$\|\mathbf{v}\| = \sqrt{v_1^2 + v_2^2}$$

In $\mathbb{R}^3$: The *norm* of $\mathbf{v} = \overrightarrow{PQ}$ is

$$\begin{aligned} \|\mathbf{v}\| &= \sqrt{v_1^2 + v_2^2 + v_3^2} \\ &= \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}. \end{aligned}$$

The vectors $\mathbf{0} = (0, 0)$ and $\mathbf{0} = (0, 0, 0)$ have norm $\|\mathbf{0}\| = 0$. If $\mathbf{v} \neq \mathbf{0}$, then $\|\mathbf{v}\| > 0$.

## Example

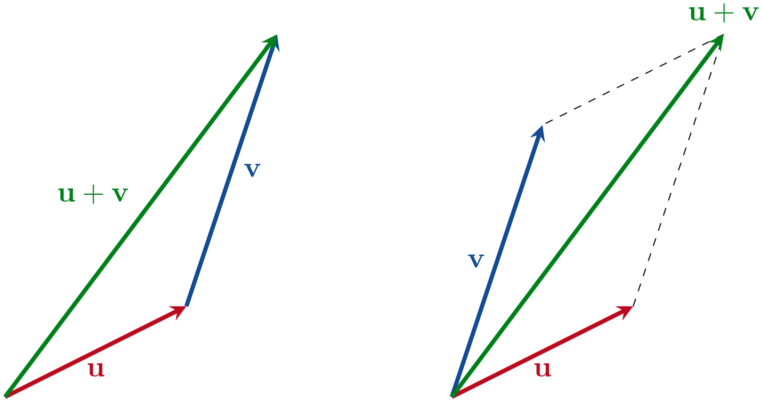
Find (a) the component form; and (b) the norm of the vector with initial point $P(-3,4,1)$ and terminal point $Q(-5,2,2)$.

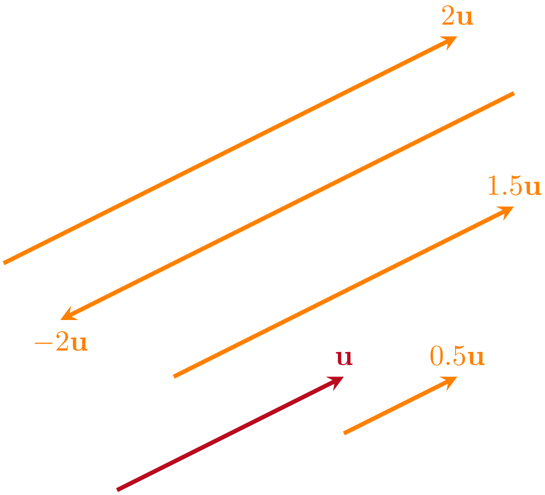
*solution:*

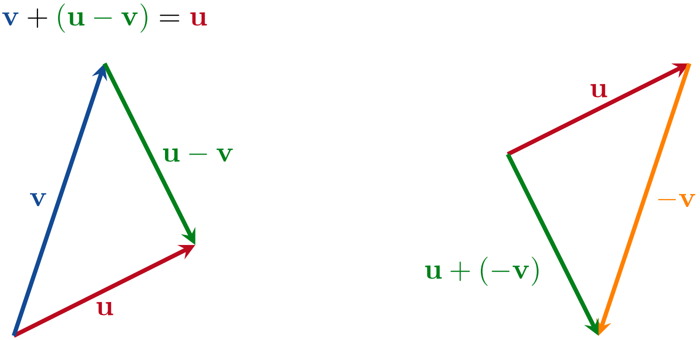
(a $\mathbf{v} = (v_1, v_2, v_3) = Q - P = (-5,2,2) - (-3,4,1)$
$= (-2,-2,1).$

(b $\|\mathbf{v}\| = \sqrt{v_1^2 + v_2^2 + v_3^2} = \sqrt{(-2)^2 + (-2)^2 + 1^2} = \sqrt{9} = 3.$

# Vector Algebra




$2\mathbf{u}$
$1.5\mathbf{u}$
$-2\mathbf{u}$
$\mathbf{u}$
$0.5\mathbf{u}$


$\mathbf{v} + (\mathbf{u} - \mathbf{v}) = \mathbf{u}$
$\mathbf{v}$
$\mathbf{u} - \mathbf{v}$
$\mathbf{u}$
$\mathbf{u}$
$-\mathbf{v}$
$\mathbf{u} + (-\mathbf{v})$

Let $\mathbf{u} = (u_1, u_2, u_3)$ and $\mathbf{v} = (v_1, v_2, v_3)$ be vectors. Let $k$ be a number. Then

$$\mathbf{u} + \mathbf{v} = (u_1 + v_1, u_2 + v_2, u_3 + v_3)$$

and

$$k\mathbf{u} = (ku_1, ku_2, ku_3).$$

Note that

$$\begin{aligned}\|k\mathbf{u}\| &= \|(ku_1, ku_2, ku_3)\| = \sqrt{(ku_1)^2 + (ku_2)^2 + (ku_3)^2} \\ &= \sqrt{k^2u_1^2 + k^2u_2^2 + k^2u_3^2} = \sqrt{k^2(u_1^2 + u_2^2 + u_3^2)} \\ &= \sqrt{k^2}\sqrt{u_1^2 + u_2^2 + u_3^2} = |k|\,\|\mathbf{u}\|\,.\end{aligned}$$

The vector $-\mathbf{u} = (-1)\mathbf{u}$ has the same length as $\mathbf{u}$, but points in the opposite direction.

**Example**

Let $\mathbf{u} = (-1, 3, 1)$ and $\mathbf{v} = (4, 7, 0)$. Find (a) $2\mathbf{u} + 3\mathbf{v}$, (b) $\mathbf{u} - \mathbf{v}$, and (c) $\left\|\frac{1}{2}\mathbf{u}\right\|$.

*solution:*

(a) $2\mathbf{u} + 3\mathbf{v} = 2(-1, 3, 1) + 3(4, 7, 0) = (-2, 6, 2) + (12, 21, 0) = (10, 27, 2)$;

(b) $\mathbf{u} - \mathbf{v} = (-1, 3, 1) - (4, 7, 0) = (-5, -4, 1)$;

(c) $\left\|\frac{1}{2}\mathbf{u}\right\| = \frac{1}{2}\|\mathbf{u}\| = \frac{1}{2}\sqrt{(-1)^2 + 3^2 + 1^2} = \frac{1}{2}\sqrt{11}$.

# Properties of Vector Operations

Let $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ be vectors. Let $a$ and $b$ be numbers. Then

1. $\mathbf{u}+\mathbf{v}=\mathbf{v}+\mathbf{u}$;
2. $(\mathbf{u}+\mathbf{v})+\mathbf{w}=\mathbf{u}+(\mathbf{v}+\mathbf{w})$;
3. $\mathbf{u}+\mathbf{0}=\mathbf{u}$;
4. $\mathbf{u}+(-\mathbf{u})=\mathbf{0}$;
5. $0\mathbf{u}=\mathbf{0}$;
6. $1\mathbf{u}=\mathbf{u}$;
7. $a(b\mathbf{u})=(ab)\mathbf{u}$;
8. $a(\mathbf{u}+\mathbf{v})=a\mathbf{u}+a\mathbf{v}$;
9. $(a+b)\mathbf{u}=a\mathbf{u}+b\mathbf{u}$.

**Remark**

We **can not** multiply vectors. Never never never never write "**uv**".

# Unit Vectors

**Definition**

$\mathbf{u}$ is called a *unit vector* $\iff \|\mathbf{u}\| = 1$.

**Example**

$\mathbf{u} = (2^{-\frac{1}{2}}, \frac{1}{2}, -\frac{1}{2})$ is a unit vector because

$$\|\mathbf{u}\| = \sqrt{\left(\frac{1}{\sqrt{2}}\right)^2 + \left(\frac{1}{2}\right)^2 + \left(-\frac{1}{2}\right)^2} = \sqrt{\frac{1}{2} + \frac{1}{4} + \frac{1}{4}} = 1.$$

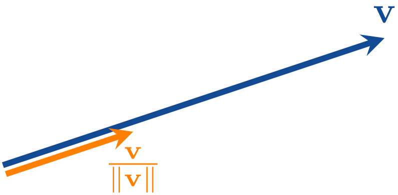
In $\mathbb{R}^2$: The *standard unit vectors* are $\mathbf{i} = (1, 0)$ and $\mathbf{j} = (0, 1)$.
In $\mathbb{R}^3$: The *standard unit vectors* are $\mathbf{i} = (1, 0, 0)$, $\mathbf{j} = (0, 1, 0)$ and $\mathbf{k} = (0, 0, 1)$. Any vector $\mathbf{v} \in \mathbb{R}^3$ can be written

$$\begin{aligned} \mathbf{v} &= (v_1, v_2, v_3) = (v_1, 0, 0) + (0, v_2, 0) + (0, 0, v_3) \\ &= v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}. \end{aligned}$$

If $\|\mathbf{v}\| \neq 0$, then $\frac{\mathbf{v}}{\|\mathbf{v}\|}$ is a unit vector because

$$\left\| \frac{\mathbf{v}}{\|\mathbf{v}\|} \right\| = \left\| \frac{1}{\|\mathbf{v}\|} \mathbf{v} \right\| = \frac{1}{\|\mathbf{v}\|} \|\mathbf{v}\| = 1.$$

Clearly $\frac{\mathbf{v}}{\|\mathbf{v}\|}$ and $\mathbf{v}$ point in the same direction.



## Example

Find a unit vector $\mathbf{u}$ which points in the same direction as $\overrightarrow{P_1P_2}$, where $P_1(1,0,1)$ and $P_2(3,2,0)$.

*solution:*

We calculate that
$\overrightarrow{P_1P_2} = P_2 - P_1 = (3,2,0) - (1,0,1) = (2,2,-1) = 2\mathbf{i} + 2\mathbf{j} - \mathbf{k}$
and that $\left\|\overrightarrow{P_1P_2}\right\| = \sqrt{2^2 + 2^2 + (-1)^2} = 3$. The required unit vector is

$$\mathbf{u} = \frac{\overrightarrow{P_1P_2}}{\left\|\overrightarrow{P_1P_2}\right\|} = \frac{2\mathbf{i} + 2\mathbf{j} - \mathbf{k}}{3} = \frac{2}{3}\mathbf{i} + \frac{2}{3}\mathbf{j} - \frac{1}{3}\mathbf{k}.$$

# 12 The Dot Product

# 12. The Dot Product

**Definition**

In $\mathbb{R}^2$, the *dot product* of $\mathbf{u} = (u_1, u_2) = u_1\mathbf{i} + u_2\mathbf{j}$ and $\mathbf{v} = (v_1, v_2) = v_1\mathbf{i} + v_2\mathbf{j}$ is

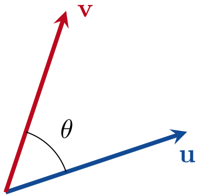
$$\mathbf{u} \bullet \mathbf{v} = u_1 v_1 + u_2 v_2.$$

**Definition**

In $\mathbb{R}^3$, the *dot product* of $\mathbf{u} = (u_1, u_2, u_3) = u_1\mathbf{i} + u_2\mathbf{j} + u_3\mathbf{k}$ and $\mathbf{v} = (v_1, v_2, v_3) = v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}$ is

$$\mathbf{u} \bullet \mathbf{v} = u_1 v_1 + u_2 v_2 + u_3 v_3.$$

# 12. The Dot Product




**Theorem**

*The angle between* $\mathbf{u}$ *and* $\mathbf{v}$ *is*

$$\theta = \cos^{-1}\left(\frac{\mathbf{u}\bullet\mathbf{v}}{\|\mathbf{u}\|\,\|\mathbf{v}\|}\right).$$

**Example**

$$
\begin{aligned}
(1, -2, -1) \bullet (-6, 2, -3) &= (1 \times -6) + (-2 \times 2) + (-1 \times -3) \\
&= -6 - 4 + 3 = -7.
\end{aligned}
$$

**Example**

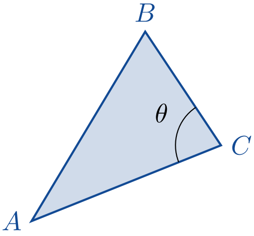
$$\begin{aligned}(\tfrac{1}{2}\mathbf{i} + 3\mathbf{j} + \mathbf{k}) \cdot (4\mathbf{i} - \mathbf{j} + 2\mathbf{k}) &= (\tfrac{1}{2} \times 4) + (3 \times -1) + (1 \times 2) \\ &= 2 - 3 + 2 = 1.\end{aligned}$$

**Example**

Find the angle between $\mathbf{u} = \mathbf{i} - 2\mathbf{j} - 2\mathbf{k}$ and $\mathbf{v} = 6\mathbf{i} + 3\mathbf{j} + 2\mathbf{k}$.

*solution:* Since $\mathbf{u} \cdot \mathbf{v} = (1, -2, -2) \cdot (6, 3, 2) = (1 \times 6) + (-2 \times 3) + (-2 \times 2) = 6 - 6 - 4 = -4$,
$\|\mathbf{u}\| = \sqrt{1^2 + (-2)^2 + (-2)^2} = \sqrt{9} = 3$ and
$\|\mathbf{v}\| = \sqrt{6^2 + 3^2 + 2^2} = \sqrt{49} = 7$, we have that

$$\theta = \cos^{-1}\left(\frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{u}\|\,\|\mathbf{v}\|}\right) = \cos^{-1}\left(-\frac{4}{21}\right) \approx 1.76 \text{ radians} \approx 98.5^\circ.$$




**Example**

If $A(0,0)$, $B(3,5)$ and $C(5,2)$, find $\theta = \measuredangle ACB$.

*solution:* $\theta$ is the angle between $\overrightarrow{CA}$ and $\overrightarrow{CB}$. We calculate that
$\overrightarrow{CA} = A - C = (0,0) - (5,2) = (-5,-2)$,
$\overrightarrow{CB} = B - C = (3,5) - (5,2) = (-2,3)$,
$\overrightarrow{CA} \bullet \overrightarrow{CB} = (-5,-2) \bullet (-2,3) = 4$,
$\left\|\overrightarrow{CA}\right\| = \sqrt{(-5)^2 + (-2)^2} = \sqrt{29}$ and
$\left\|\overrightarrow{CB}\right\| = \sqrt{(-2)^2 + 3^2} = \sqrt{13}$. Therefore

$$\theta = \cos^{-1}\left(\frac{\overrightarrow{CA} \bullet \overrightarrow{CB}}{\left\|\overrightarrow{CA}\right\| \left\|\overrightarrow{CB}\right\|}\right) = \cos^{-1}\left(\frac{4}{\sqrt{29}\sqrt{13}}\right)$$
$$\approx 78.1^\circ \approx 1.36 \text{ radians.}$$

**Definition**

$\mathbf{u}$ and $\mathbf{v}$ are *orthogonal* $\iff$ $\mathbf{u} \cdot \mathbf{v} = 0$.

**Remark**

Note that

$$\mathbf{u} \cdot \mathbf{v} = \|\mathbf{u}\| \, \|\mathbf{v}\| \cos\theta$$

by Theorem 9. Therefore

$$\mathbf{u} \text{ and } \mathbf{v} \text{ are orthogonal} \iff \begin{pmatrix} \mathbf{u} = \mathbf{0} \\ \text{or} \\ \mathbf{v} = \mathbf{0} \\ \text{or} \\ \theta = 90^\circ. \end{pmatrix}$$

### Example

$\mathbf{u} = (3, -2)$ and $\mathbf{v} = (4, 6)$ are orthogonal because
$\mathbf{u} \bullet \mathbf{v} = (3, -2) \bullet (4, 6) = (3 \times 4) + (-2 \times 6) = 12 - 12 = 0.$

**Example**

$\mathbf{u} = 3\mathbf{i} - 2\mathbf{j} + \mathbf{k}$ and $\mathbf{v} = 2\mathbf{j} + 4\mathbf{k}$ are orthogonal because
$\mathbf{u} \cdot \mathbf{v} = (3 \times 0) + (-2 \times 2) + (1 \times 4) = 0 - 4 + 4 = 0.$

**Example**

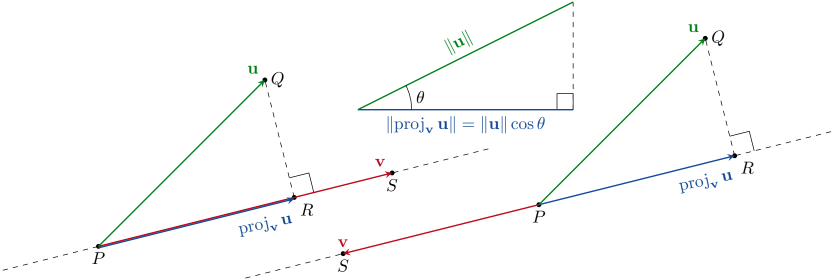
$\mathbf{0}$ is orthogonal to every vector $\mathbf{u}$ because
$\mathbf{0} \bullet \mathbf{u} = (0,0,0) \bullet (u_1, u_2, u_3) = 0u_1 + 0u_2 + 0u_3 = 0.$

## Properties of the Dot Product

Let $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ be vectors. Let $k$ be a number. Then

1. $\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u}$;
2. $(k\mathbf{u}) \cdot \mathbf{v} = \mathbf{u} \cdot (k\mathbf{v}) = k(\mathbf{u} \cdot \mathbf{v})$;
3. $\mathbf{u} \cdot (\mathbf{v} + \mathbf{w}) = (\mathbf{u} \cdot \mathbf{v}) + (\mathbf{u} \cdot \mathbf{w})$;
4. $\mathbf{u} \cdot \mathbf{u} = \|\mathbf{u}\|^2$; and
5. $\mathbf{0} \cdot \mathbf{u} = 0$.

# Vector Projections




## Definition

The *vector projection* of $\mathbf{u}$ onto $\mathbf{v}$ is the vector

$$\operatorname{proj}_{\mathbf{v}} \mathbf{u} = \overrightarrow{PR}.$$

Now

$$\begin{aligned}
\operatorname{proj}_{\mathbf{v}} \mathbf{u} &= \left(\text{length of } \operatorname{proj}_{\mathbf{v}} \mathbf{u}\right) \begin{pmatrix} \text{a unit vector in} \\ \text{the same} \\ \text{direction as } \mathbf{v} \end{pmatrix} \\
&= \left\|\operatorname{proj}_{\mathbf{v}} \mathbf{u}\right\| \left(\frac{\mathbf{v}}{\|\mathbf{v}\|}\right) \\
&= \|\mathbf{u}\| \left(\cos\theta\right) \left(\frac{\mathbf{v}}{\|\mathbf{v}\|}\right) \\
&= \left(\frac{\|\mathbf{u}\| \, \|\mathbf{v}\| \cos\theta}{\|\mathbf{v}\|^2}\right) \mathbf{v} \\
&= \left(\frac{\mathbf{u} \bullet \mathbf{v}}{\|\mathbf{v}\|^2}\right) \mathbf{v}.
\end{aligned}$$

Since this is an important formula, we write it as a theorem.

**Theorem**

*The vector projection of* $\mathbf{u}$ *onto* $\mathbf{v}$ *is*

$$\operatorname{proj}_{\mathbf{v}} \mathbf{u} = \left( \frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{v}\|^2} \right) \mathbf{v}.$$

**Example**

Find the vector projection of $\mathbf{u} = 6\mathbf{i} + 3\mathbf{j} + 2\mathbf{k}$ onto $\mathbf{v} = \mathbf{i} - 2\mathbf{j} - 2\mathbf{k}$.

*solution:*

$$\begin{aligned}\operatorname{proj}_{\mathbf{v}} \mathbf{u} &= \left(\frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{v}\|^2}\right) \mathbf{v} = \left(\frac{6 - 6 - 4}{1 + 4 + 4}\right) (\mathbf{i} - 2\mathbf{j} - 2\mathbf{k}) \\ &= -\frac{4}{9}\mathbf{i} + \frac{8}{9}\mathbf{j} + \frac{8}{9}\mathbf{k}.\end{aligned}$$

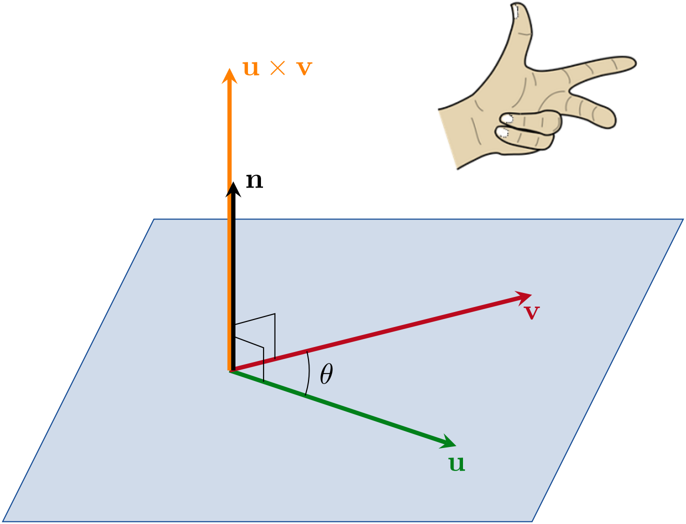
**Example**

Find the vector projection of $\mathbf{F} = 5\mathbf{i} + 2\mathbf{j}$ onto $\mathbf{v} = \mathbf{i} - 3\mathbf{j}$.

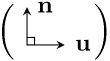
*solution:*

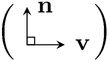
$$\begin{aligned}\operatorname{proj}_{\mathbf{v}} \mathbf{F} &= \left(\frac{\mathbf{F} \cdot \mathbf{v}}{\|\mathbf{v}\|^2}\right)\mathbf{v} = \left(\frac{5-6}{1+9}\right)(\mathbf{i} - 3\mathbf{j}) \\ &= -\frac{1}{10}\mathbf{i} + \frac{3}{10}\mathbf{j}.\end{aligned}$$

# The Cross Product


u × v
n
v
θ
u

Let **n** be a unit vector which satisfies

1. **n** is orthogonal to **u** 

;
2. **n** is orthogonal to **v** 

; and
3. the direction of **n** is chosen using the left-hand rule.

**Definition**

The *cross product* of **u** and **v** is

$$\mathbf{u} \times \mathbf{v} = \|\mathbf{u}\| \, \|\mathbf{v}\| \, (\sin\theta)\mathbf{n}.$$

### Remark

- $\mathbf{u} \cdot \mathbf{v}$ is a number.
- $\mathbf{u} \times \mathbf{v}$ is a vector.

### Remark

$$\begin{pmatrix} \textbf{u} \text{ and } \textbf{v} \\ \text{are} \\ \text{parallel} \end{pmatrix} \iff \theta = 0^\circ \text{ or } 180^\circ$$

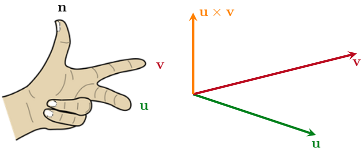
$$\implies \sin\theta = 0 \implies \mathbf{u} \times \mathbf{v} = \mathbf{0}.$$

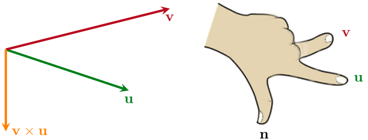
## Properties of the Cross Product

Let $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ be vectors. Let $r$ and $s$ be numbers. Then

1. $(r\mathbf{u}) \times (s\mathbf{v}) = (rs)(\mathbf{u} \times \mathbf{v})$;
2. $\mathbf{u} \times (\mathbf{v} + \mathbf{w}) = (\mathbf{u} \times \mathbf{v}) + (\mathbf{u} \times \mathbf{w})$;
3. $\mathbf{v} \times \mathbf{u} = -\mathbf{u} \times \mathbf{v}$;
4. $(\mathbf{v} + \mathbf{w}) \times \mathbf{u} = (\mathbf{v} \times \mathbf{u}) + (\mathbf{w} \times \mathbf{u})$;
5. $\mathbf{0} \times \mathbf{u} = \mathbf{0}$; and
6. $\mathbf{u} \times (\mathbf{v} \times \mathbf{w}) = (\mathbf{u} \bullet \mathbf{w})\mathbf{v} - (\mathbf{u} \bullet \mathbf{v})\mathbf{w}$.

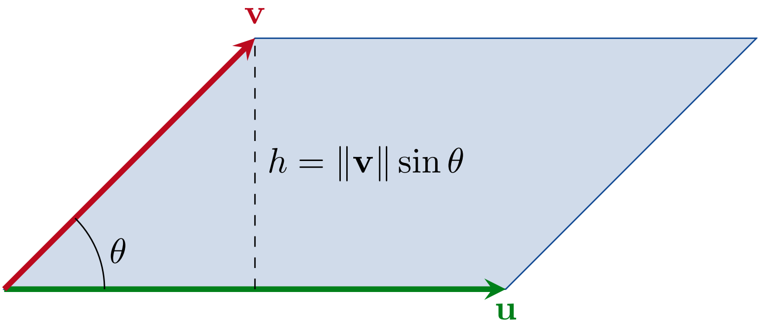
## Property (iii)



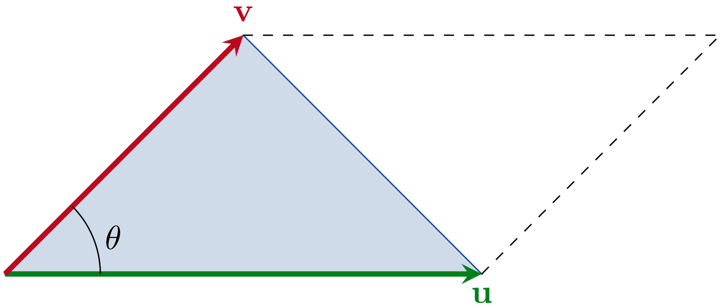

$$\mathbf{v} \times \mathbf{u} = -\mathbf{u} \times \mathbf{v}$$

# Area of a Parallelogram



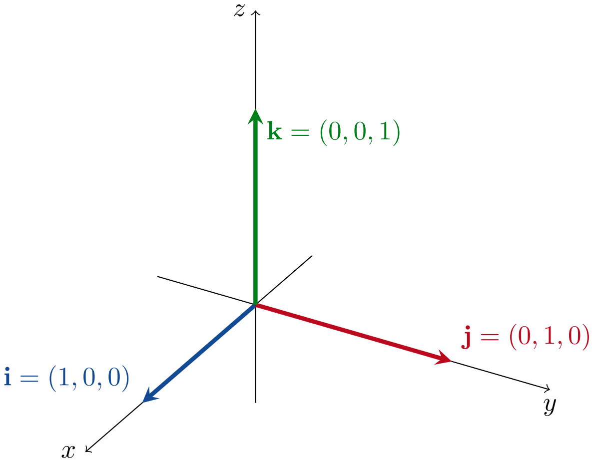

$$\text{area} = (\text{base})\,(\text{height}) = \|\mathbf{u}\|\,\|\mathbf{v}\| \sin\theta = \|\mathbf{u} \times \mathbf{v}\|\,.$$

# Area of a Triangle




$$\begin{aligned}\text{area of triangle} &= \frac{1}{2}\,(\text{area of parallelogram}) \\ &= \frac{1}{2}\,\|\mathbf{u}\times\mathbf{v}\|\,.\end{aligned}$$

# A Formula for $\mathbf{u} \times \mathbf{v}$



Note first that

$$\mathbf{i} \times \mathbf{i} = \|\mathbf{i}\| \, \|\mathbf{i}\| \sin 0^\circ \mathbf{n} = \mathbf{0}.$$

Similarly $\mathbf{j} \times \mathbf{j} = \mathbf{0}$ and $\mathbf{k} \times \mathbf{k} = \mathbf{0}$ also.

Next note that $\mathbf{i} \times \mathbf{j}$ must point in the same direction at $\mathbf{k}$ by the left-hand rule. Thus

$$\mathbf{i} \times \mathbf{j} = \|\mathbf{i}\| \, \|\mathbf{j}\| \sin 90^\circ \mathbf{k} = \mathbf{k}.$$

We then immediately also have

$$\mathbf{j} \times \mathbf{i} = -\left(\mathbf{i} \times \mathbf{j}\right) = -\mathbf{k}.$$

It is left for you to check that

$$\mathbf{j} \times \mathbf{k} = \mathbf{i}, \qquad \mathbf{k} \times \mathbf{j} = -\mathbf{i}, \qquad \mathbf{i} \times \mathbf{k} = -\mathbf{j} \quad \text{and} \quad \mathbf{k} \times \mathbf{i} = \mathbf{j}.$$

Now suppose that $\mathbf{u} = u_1\mathbf{i} + u_2\mathbf{j} + u_3\mathbf{k}$ and $\mathbf{v} = v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}$. Then we can calculate that

$$\begin{aligned}
\mathbf{u} \times \mathbf{v} &= (u_1\mathbf{i} + u_2\mathbf{j} + u_3\mathbf{k}) \times (v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}) \\
&= u_1v_1\mathbf{i} \times \mathbf{i} + u_1v_2\mathbf{i} \times \mathbf{j} + u_1v_3\mathbf{i} \times \mathbf{k} + u_2v_1\mathbf{j} \times \mathbf{i} + u_2v_2\mathbf{j} \times \mathbf{j} \\
&\quad + u_2v_3\mathbf{j} \times \mathbf{k} + u_3v_1\mathbf{k} \times \mathbf{i} + u_3v_2\mathbf{k} \times \mathbf{j} + u_3v_3\mathbf{k} \times \mathbf{k} \\
&= \mathbf{0} + u_1v_2\mathbf{k} - u_1v_3\mathbf{j} - u_2v_1\mathbf{k} + \mathbf{0} + u_2v_3\mathbf{i} + u_3v_1\mathbf{j} - u_3v_2\mathbf{i} + \mathbf{0} \\
&= (u_2v_3 - u_3v_2)\mathbf{i} - (u_1v_3 - u_3v_1)\mathbf{j} + (u_1v_2 - u_2v_1)\mathbf{k}.
\end{aligned}$$

**Theorem**

*If* $\mathbf{u} = u_1\mathbf{i} + u_2\mathbf{j} + u_3\mathbf{k}$ *and* $\mathbf{v} = v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}$, *then*

$$\boxed{\mathbf{u} \times \mathbf{v} = (u_2v_3 - u_3v_2)\mathbf{i} - (u_1v_3 - u_3v_1)\mathbf{j} + (u_1v_2 - u_2v_1)\mathbf{k}}$$

If you studied matrices and determinants at high school, then you may prefer to use the following symbolic determinant formula instead.

$$\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}.$$

**Example**

Find $\mathbf{u} \times \mathbf{v}$ and $\mathbf{v} \times \mathbf{u}$ if $\mathbf{u} = 2\mathbf{i} + \mathbf{j} + \mathbf{k}$ and $\mathbf{v} = -4\mathbf{i} + 3\mathbf{j} + \mathbf{k}$.

*solution:*

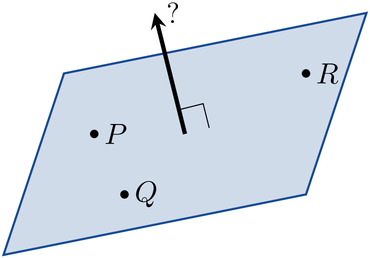
$$\mathbf{u} \times \mathbf{v} = (1 - 3)\mathbf{i} - (2 - -4)\mathbf{j} + (6 - -4)\mathbf{k} = -2\mathbf{i} - 6\mathbf{j} + 10\mathbf{k}$$

and

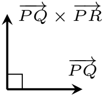
$$\mathbf{v} \times \mathbf{u} = -\mathbf{u} \times \mathbf{v} = 2\mathbf{i} + 6\mathbf{j} - 10\mathbf{k}.$$

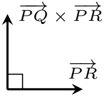
**Example**

Find a vector perpendicular to the plane containing the three points $P(1,-1,0)$, $Q(2,1,-1)$ and $R(-1,1,2)$.



*solution:* The vector $\overrightarrow{PQ} \times \overrightarrow{PR}$ is perpendicular to the plane

because 

and 

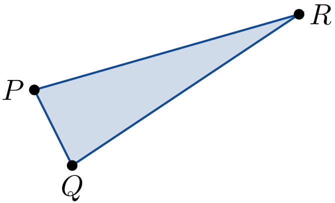
. We calculate that

$$\begin{aligned}
\overrightarrow{PQ} &= Q - P = (2, 1, -1) - (1, -1, 0) \\
&= (2-1, 1+1, -1-0) = \mathbf{i} + 2\mathbf{j} - \mathbf{k} \\
\overrightarrow{PR} &= R - P = (-1, 1, 2) - (1, -1, 0) \\
&= (-1-1, 1+1, 2-0) = -2\mathbf{i} + 2\mathbf{j} + 2\mathbf{k} \\
\overrightarrow{PQ} \times \overrightarrow{PR} &= (4+2)\mathbf{i} - (2-2)\mathbf{j} + (2+4)\mathbf{k} = 6\mathbf{i} + 6\mathbf{k}.
\end{aligned}$$

**Example**

Find the area of triangle $PQR$.

$P(1, -1, 0)$, $Q(2, 1, -1)$ and $R(-1, 1, 2)$



*solution:* The area of the triangle is

$$\begin{aligned} \text{area} &= \frac{1}{2}\left\|\overrightarrow{PQ} \times \overrightarrow{PR}\right\| = \frac{1}{2}\left\|6\mathbf{i} + 6\mathbf{k}\right\| \\ &= \frac{1}{2}\sqrt{6^2 + 0^2 + 6^2} = 3\sqrt{2}. \end{aligned}$$

**Example**

Find a unit vector perpendicular to the plane containing $P$, $Q$ and $R$.

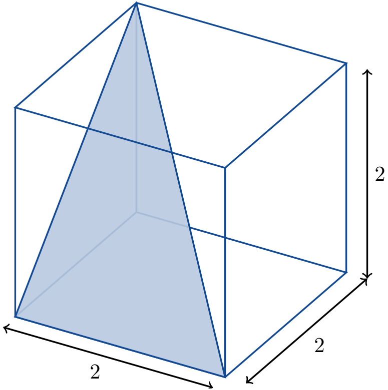
$P(1,-1,0)$, $Q(2,1,-1)$ and $R(-1,1,2)$

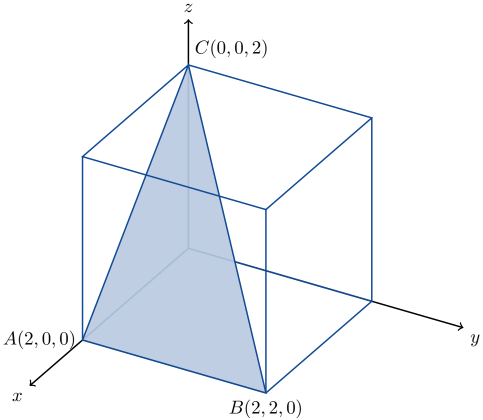
*solution:* We know that $\overrightarrow{PQ} \times \overrightarrow{PR}$ is perpendicular to the plane. We just need to normalise this vector to find a unit vector.

$$\mathbf{n} = \frac{\overrightarrow{PQ} \times \overrightarrow{PR}}{\left\|\overrightarrow{PQ} \times \overrightarrow{PR}\right\|} = \frac{6\mathbf{i} + 6\mathbf{k}}{6\sqrt{2}} = \frac{1}{\sqrt{2}}\mathbf{i} + \frac{1}{\sqrt{2}}\mathbf{k}.$$

## Example

A triangle is inscribed inside a cube of side 2 as shown below. Use the cross product to find the area of the triangle.






*solution:* First we draw coordinate axes and assign coordinates to the vertices of the triangle.

Then we can calculate

$$\overrightarrow{AB} = B - A = (2,2,0) - (2,0,0) = (0,2,0) = 2\mathbf{j}$$

and

$$\overrightarrow{AC} = C - A = (0,0,2) - (2,0,0) = (-2,0,2) = -2\mathbf{i} + 2\mathbf{k}.$$

It follows that

$$\overrightarrow{AB} \times \overrightarrow{AC} = (2\mathbf{j}) \times (-2I \times 2\mathbf{k}) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 2 & 0 \\ -2 & 0 & 2 \end{vmatrix}$$

$$= \mathbf{i}(4-0) - \mathbf{j}(0-0) + \mathbf{k}(0 - -4) = 4\mathbf{i} + 4\mathbf{k}.$$

Therefore

$$\begin{aligned}\text{area of triangle} &= \frac{1}{2}\left\|\overrightarrow{AB} \times \overrightarrow{AC}\right\| = \frac{1}{2}\sqrt{4^2 + 0^2 + 4^2} \\ &= \frac{1}{2}\sqrt{32} = \frac{1}{2}\sqrt{4}\sqrt{8} = \sqrt{8} = 2\sqrt{2}.\end{aligned}$$

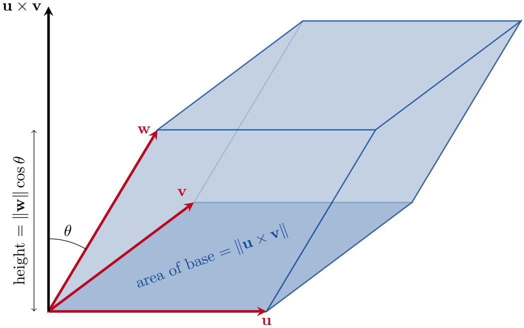
# The Triple Scalar Product

**Definition**

The *triple scalar product* of $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ is

$$(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w}.$$

# The Volume of a Parallelepiped




$$\text{volume} = \big(\text{area of base}\big)\big(\text{height}\big) = \|\mathbf{u} \times \mathbf{v}\|\,\|\mathbf{w}\| \cos\theta = |(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w}|$$

## One Final Comment

We can do the dot product in both $\mathbb{R}^2$ and $\mathbb{R}^3$. But we can only do the cross product in $\mathbb{R}^3$. There is no cross product in $\mathbb{R}^2$.

# Next Time

- 14. Lines
- 15. Planes
- 16. Projections