

# Lecture 9

- 24. Limits
- 25. Continuity
- 26. Differentiation



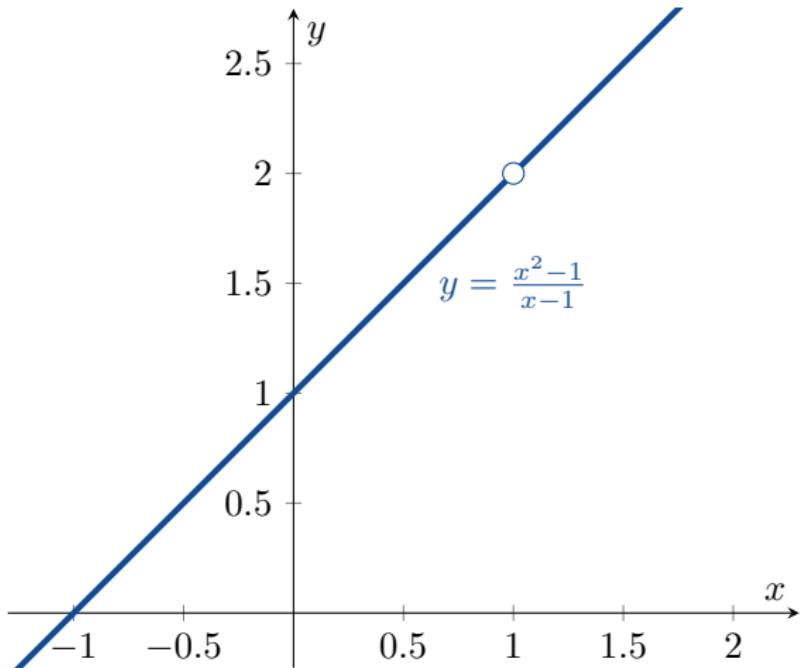
# Limits

## 24. Limits

Consider the function  $f(x) = \frac{x^2 - 1}{x - 1}$ ,  $f : (-\infty, 1) \cup (1, \infty) \rightarrow \mathbb{R}$ .

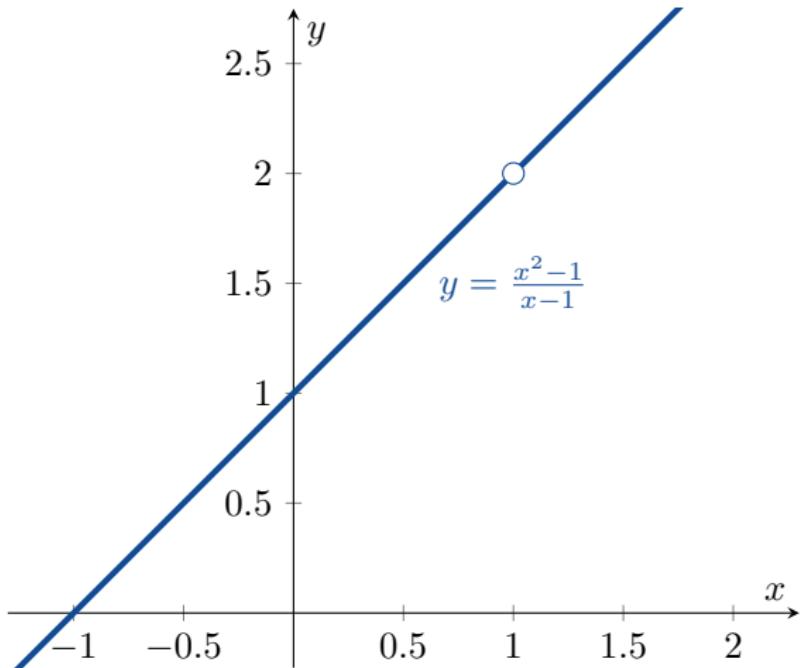
## 24. Limits

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## 24. Limits

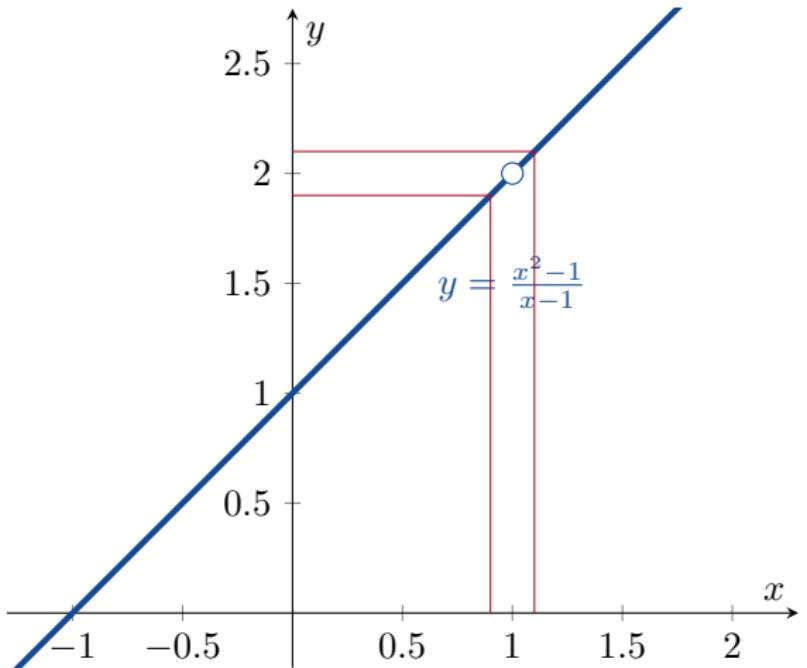
Consider the function  $f(x) = \frac{x^2-1}{x-1}$ ,  $f : (-\infty, 1) \cup (1, \infty) \rightarrow \mathbb{R}$ .



*Question:* How does  $f$  behave when  $x$  is close to 1?

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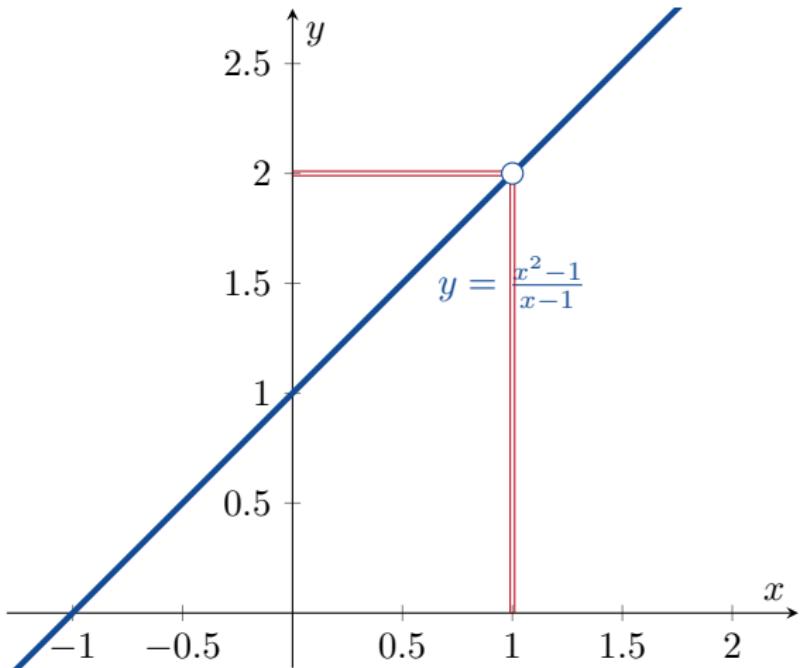


$x$	$f(x)$
0.9	1.9
1.1	2.1

*Question:* How does  $f$  behave when  $x$  is close to 1?

## 24. Limits

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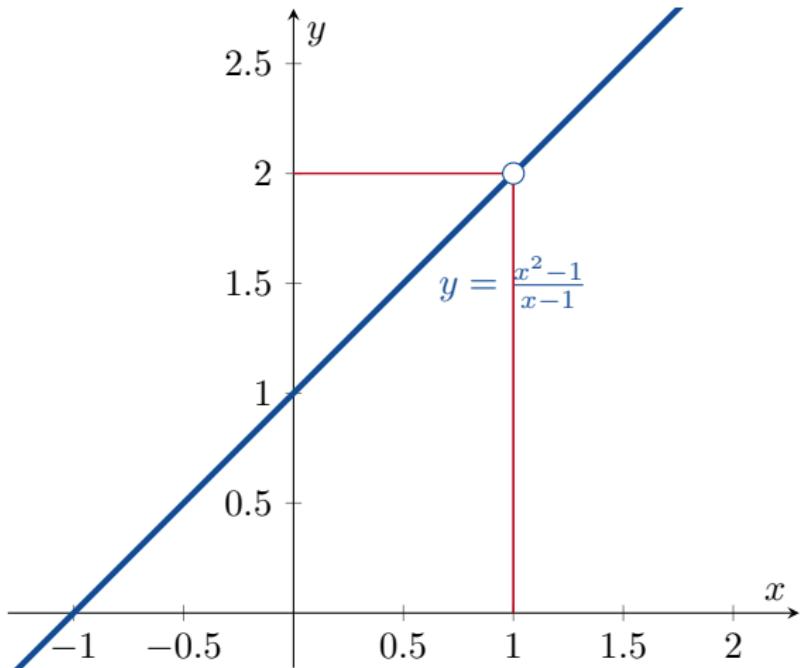


$x$	$f(x)$
0.9	1.9
1.1	2.1
0.99	1.99
1.01	2.01

*Question:* How does  $f$  behave when  $x$  is close to 1?

## 24. Limits

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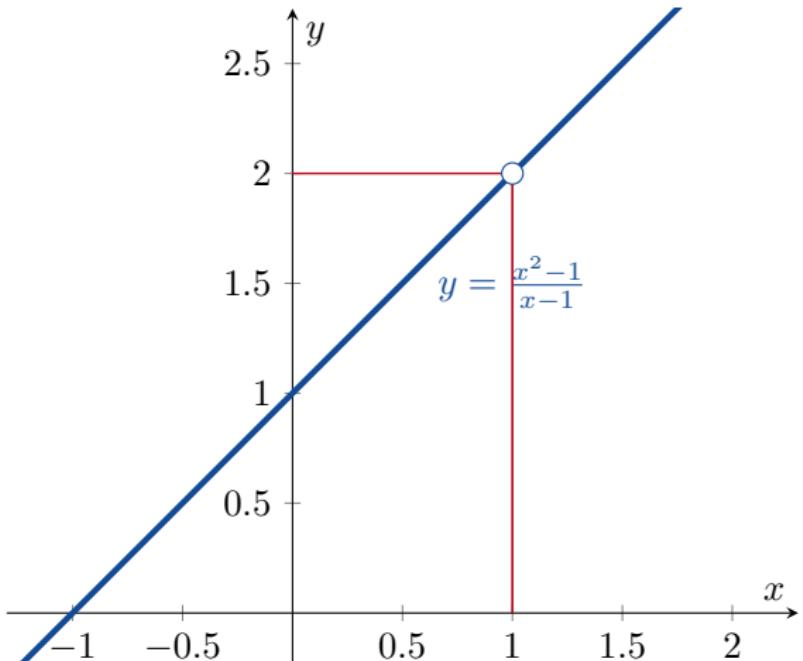


$x$	$f(x)$
0.9	1.9
1.1	2.1
0.99	1.99
1.01	2.01
0.999	1.999
1.001	2.001

Question: How does  $f$  behave when  $x$  is close to 1?

## 24. Limits

Consider the function  $f(x) = \frac{x^2-1}{x-1}$ ,  $f : (-\infty, 1) \cup (1, \infty) \rightarrow \mathbb{R}$ .



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0.99	1.99
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0.999	1.999
1.001	2.001

“If  $x$  is close to 1, then  $f(x)$  is close to 2.”

## 24. Limits



*“If  $x$  is close to 1, then  $f(x)$  is close to 2.”*

Mathematically, we write this as

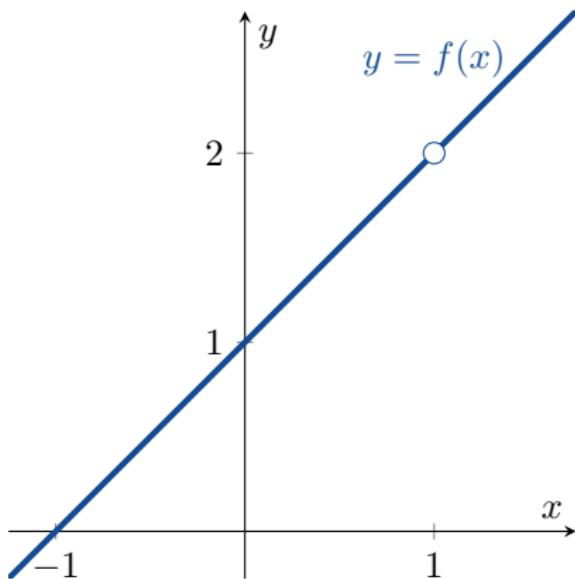
$$\lim_{x \rightarrow 1} f(x) = 2$$

and read it as “the limit, as  $x$  tends to 1, of  $f(x)$  is equal to 2”.

## 24. Limits

### Example

$$f(x) = \frac{x^2 - 1}{x - 1}$$

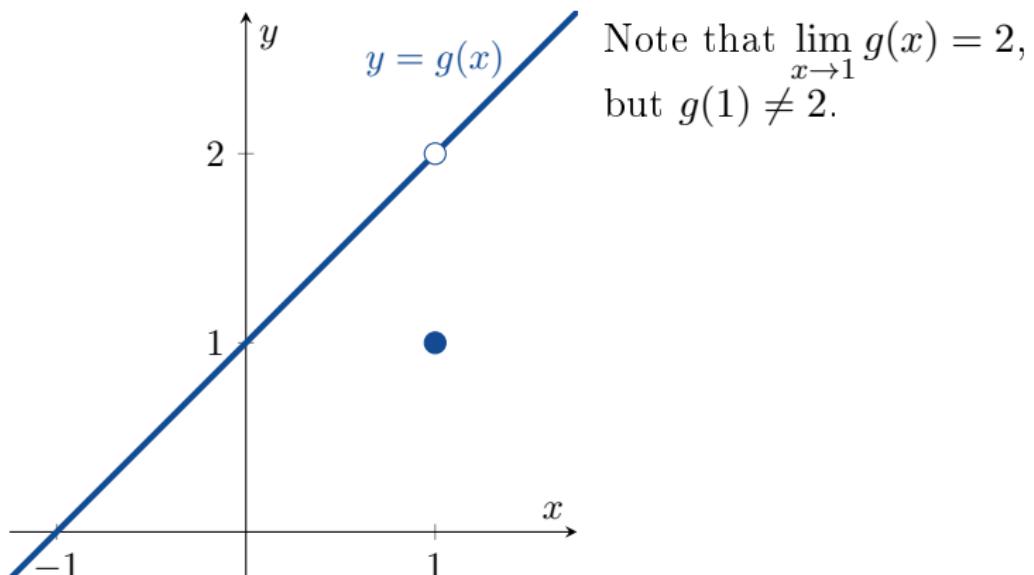


Note that  $\lim_{x \rightarrow 1} f(x) = 2$ ,  
but  $f$  is not defined at  $x = 1$ .

## 24. Limits

## Example

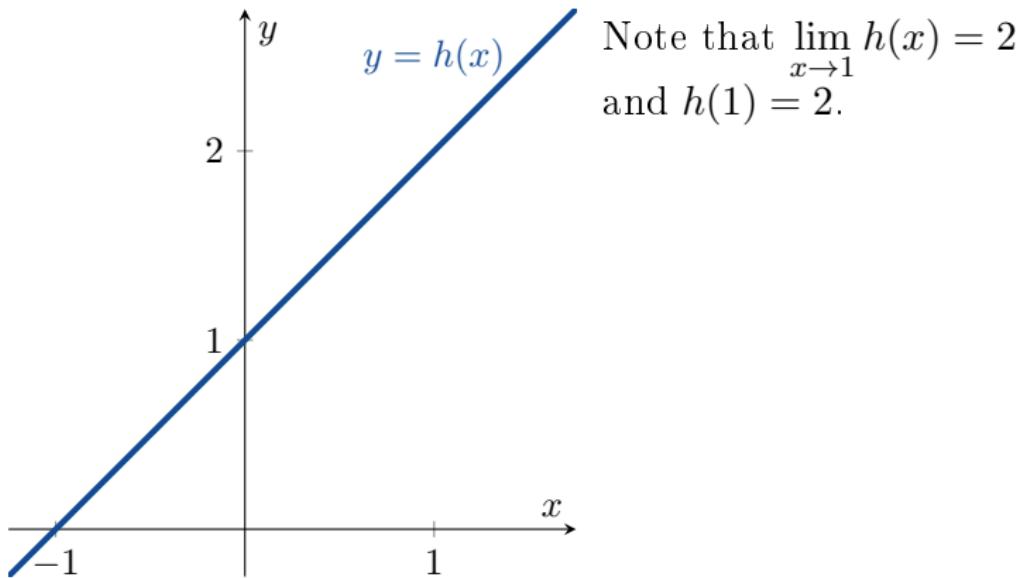
$$g(x) = \begin{cases} \frac{x^2-1}{x-1} & x \neq 1 \\ 1 & x = 1 \end{cases}$$



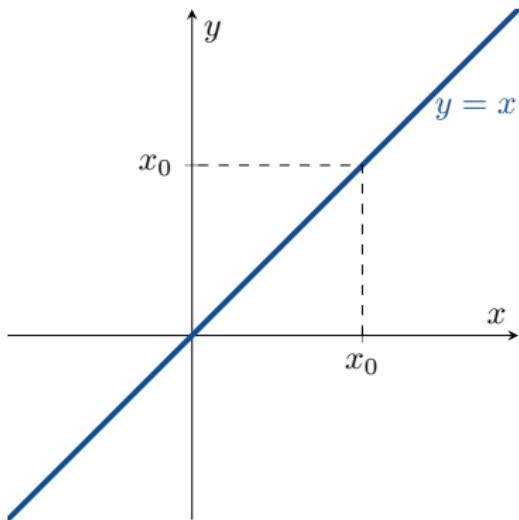
## 24. Limits

### Example

$$h(x) = x + 1$$



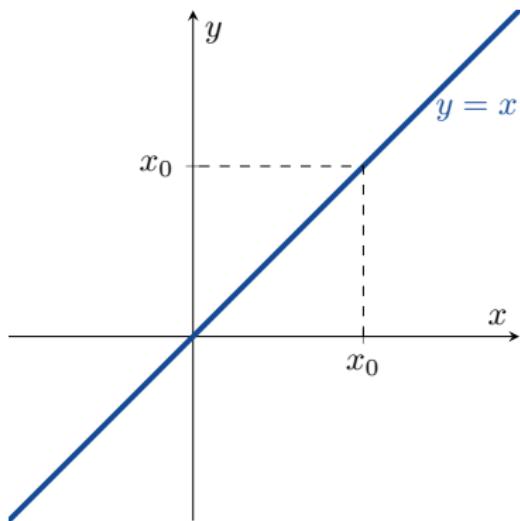
## 24. Limits



Example (The Identity Function)

$$f(x) = x$$

## 24. Limits

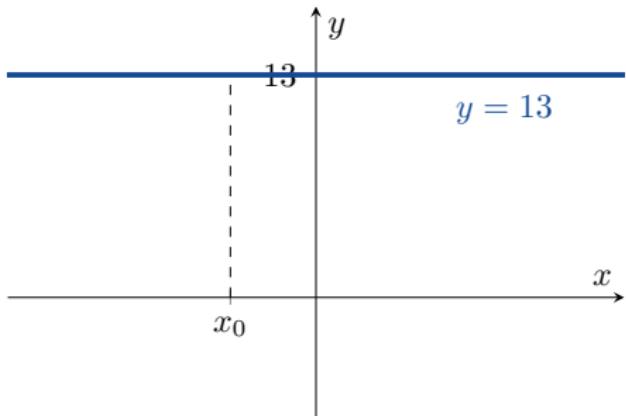


Example (The Identity Function)

$$f(x) = x$$

$$\lim_{x \rightarrow x_0} f(x) = \lim_{x \rightarrow x_0} x = x_0$$

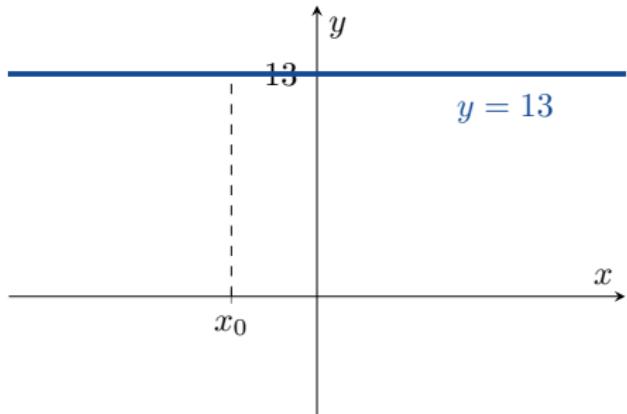
## 24. Limits



Example (A Constant Function)

$$f(x) = 13$$

## 24. Limits



Example (A Constant Function)

$$f(x) = 13$$

$$\lim_{x \rightarrow x_0} f(x) = \lim_{x \rightarrow x_0} 13 = 13$$

## 24. Limits

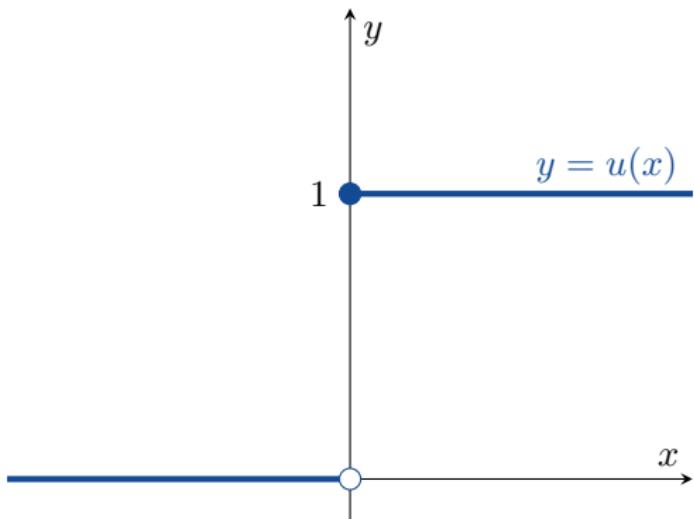


### Example (Sometimes Limits Do Not Exist)

Consider the functions

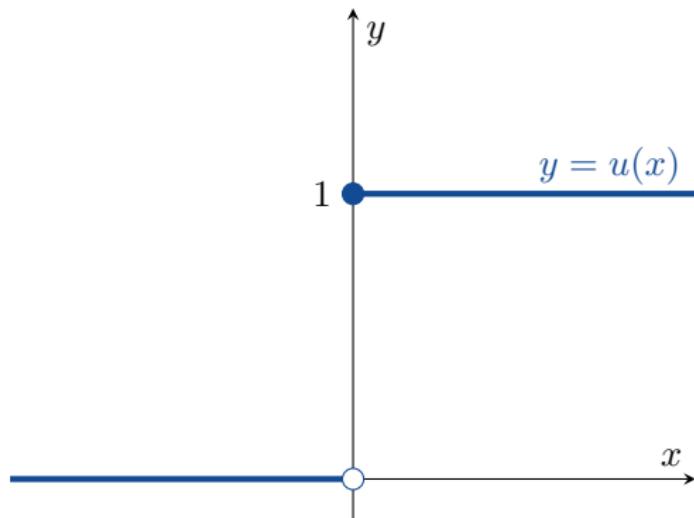
$$u(x) = \begin{cases} 0 & x < 0 \\ 1 & x \geq 0 \end{cases} \quad \text{and} \quad v(x) = \begin{cases} 0 & x \leq 0 \\ \sin \frac{1}{x} & x > 0. \end{cases}$$

## 24. Limits



Note that  $\lim_{x \rightarrow 0} u(x)$  does not exist.

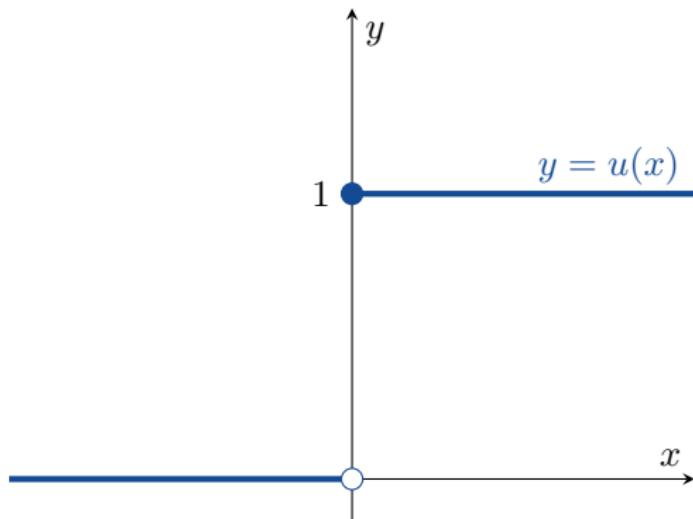
## 24. Limits



Note that  $\lim_{x \rightarrow 0} u(x)$  does not exist. To understand why, we consider  $x$  close to 0:

- If  $x$  is close to 0 and  $x < 0$ , then  $u(x) = 0$ .
- If  $x$  is close to 0 and  $x > 0$ , then  $u(x) = 1$ .

## 24. Limits

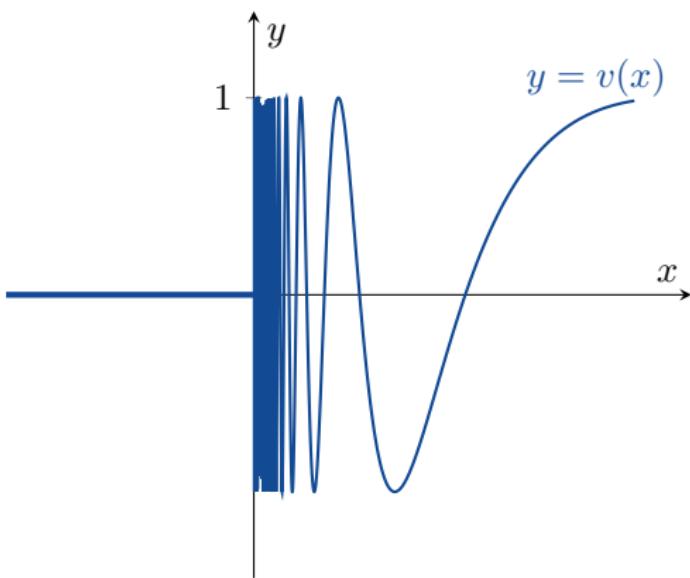


Note that  $\lim_{x \rightarrow 0} u(x)$  does not exist. To understand why, we consider  $x$  close to 0:

- If  $x$  is close to 0 and  $x < 0$ , then  $u(x) = 0$ .
- If  $x$  is close to 0 and  $x > 0$ , then  $u(x) = 1$ .

Because 0 is not close to 1, the limit as  $x \rightarrow 0$  can not exist.

## 24. Limits



Moreover  $\lim_{x \rightarrow 0} v(x)$  does not exist because  $v(x)$  oscillates up and down too quickly if  $x > 0$  and  $x \rightarrow 0$ .

## 24. Limits



### Theorem (The Limit Laws)

Suppose that

- $L, M, c, k \in \mathbb{R}$ ;
- $f$  and  $g$  are functions;
- $\lim_{x \rightarrow c} f(x) = L$ ; and
- $\lim_{x \rightarrow c} g(x) = M$ .

Then

## 24. Limits



### Theorem (The Limit Laws)

Suppose that

- $L, M, c, k \in \mathbb{R}$ ;
- $f$  and  $g$  are functions;
- $\lim_{x \rightarrow c} f(x) = L$ ; and
- $\lim_{x \rightarrow c} g(x) = M$ .

Then

- 1 Sum Rule:

$$\lim_{x \rightarrow c} (f(x) + g(x)) = L + M;$$

## 24. Limits

### Theorem (The Limit Laws)

Suppose that

- $L, M, c, k \in \mathbb{R}$ ;
- $f$  and  $g$  are functions;
- $\lim_{x \rightarrow c} f(x) = L$ ; and
- $\lim_{x \rightarrow c} g(x) = M$ .

Then

#### 2 Difference Rule:

$$\lim_{x \rightarrow c} (f(x) - g(x)) = L - M;$$

## 24. Limits

### Theorem (The Limit Laws)

Suppose that

- $L, M, c, k \in \mathbb{R}$ ;
- $f$  and  $g$  are functions;
- $\lim_{x \rightarrow c} f(x) = L$ ; and
- $\lim_{x \rightarrow c} g(x) = M$ .

Then

#### 3 Constant Multiple Rule:

$$\lim_{x \rightarrow c} (kf(x)) = kL;$$

## 24. Limits



### Theorem (The Limit Laws)

Suppose that

- $L, M, c, k \in \mathbb{R}$ ;
- $f$  and  $g$  are functions;
- $\lim_{x \rightarrow c} f(x) = L$ ; and
- $\lim_{x \rightarrow c} g(x) = M$ .

Then

- 4 Product Rule:

$$\lim_{x \rightarrow c} (f(x)g(x)) = LM;$$

## 24. Limits

### Theorem (The Limit Laws)

Suppose that

- $L, M, c, k \in \mathbb{R}$ ;
- $f$  and  $g$  are functions;
- $\lim_{x \rightarrow c} f(x) = L$ ; and
- $\lim_{x \rightarrow c} g(x) = M$ .

Then

- 5 Quotient Rule: if  $M \neq 0$ , then

$$\lim_{x \rightarrow c} \left( \frac{f(x)}{g(x)} \right) = \frac{L}{M};$$

## 24. Limits

### Theorem (The Limit Laws)

Suppose that

- $L, M, c, k \in \mathbb{R}$ ;
- $f$  and  $g$  are functions;
- $\lim_{x \rightarrow c} f(x) = L$ ; and
- $\lim_{x \rightarrow c} g(x) = M$ .

Then

- 6 Power Rule: if  $n \in \mathbb{N}$ , then

$$\lim_{x \rightarrow c} (f(x))^n = L^n;$$

## 24. Limits

### Theorem (The Limit Laws)

Suppose that

- $L, M, c, k \in \mathbb{R}$ ;
- $f$  and  $g$  are functions;
- $\lim_{x \rightarrow c} f(x) = L$ ; and
- $\lim_{x \rightarrow c} g(x) = M$ .

Then

- 7 Root Rule: if  $n \in \mathbb{N}$  and  $\sqrt[n]{L}$  exists, then

$$\lim_{x \rightarrow c} \sqrt[n]{f(x)} = \sqrt[n]{L} = L^{\frac{1}{n}}.$$

## 24. Limits



### Example

$$\text{Find } \lim_{x \rightarrow 2} (x^3 + 4x^2 - 3).$$

## 24. Limits

### Example

Find  $\lim_{x \rightarrow 5} \frac{x^4 + x^2 - 1}{x^2 + 5}$ .

*solution:*

$$\lim_{x \rightarrow 5} \frac{x^4 + x^2 - 1}{x^2 + 5} = \frac{\lim_{x \rightarrow 5}(x^4 + x^2 - 1)}{\lim_{x \rightarrow 5}(x^2 + 5)}$$

(quotient rule)

$$= \frac{\lim_{x \rightarrow 5} x^4 + \lim_{x \rightarrow 5} x^2 - \lim_{x \rightarrow 5} 1}{\lim_{x \rightarrow 5} x^2 + \lim_{x \rightarrow 5} 5}$$

(sum and difference rules)

$$= \frac{5^4 + 5^2 - 1}{5^2 + 5}$$

(power rule)

$$= \frac{649}{30}.$$

## 24. Limits



### Theorem (Limits of Polynomial Functions)

If  $P(x) = a_nx^n + a_{n-1}x^{n-1} + \dots + a_1x + a_0$  is a polynomial function, then

$$\lim_{x \rightarrow c} P(x) = P(c).$$

## 24. Limits



### Theorem (Limits of Rational Functions)

If  $P(x)$  and  $Q(x)$  are polynomial functions and if  $Q(c) \neq 0$ , then

$$\lim_{x \rightarrow c} \frac{P(x)}{Q(x)} = \frac{P(c)}{Q(c)}.$$

## 24. Limits



Example

$$\lim_{x \rightarrow -1} \frac{x^3 + 4x^2 - 3}{x^2 + 5} = \frac{(-1)^3 + 4(-1)^2 - 3}{(-1)^2 + 5} = \frac{0}{6} = 0.$$



# Eliminating Zero Denominators Algebraically

$$\lim_{x \rightarrow c} \frac{P(x)}{Q(x)}$$

What can we do if  $Q(c) = 0$ ?

## 24. Limits



### Example

$$\lim_{x \rightarrow 1} \frac{x^2 + x - 2}{x^2 - x}$$

If we just put in  $x = 1$ , we would get “ $\frac{0}{0}$ ” and we never never never want “ $\frac{0}{0}$ ”.

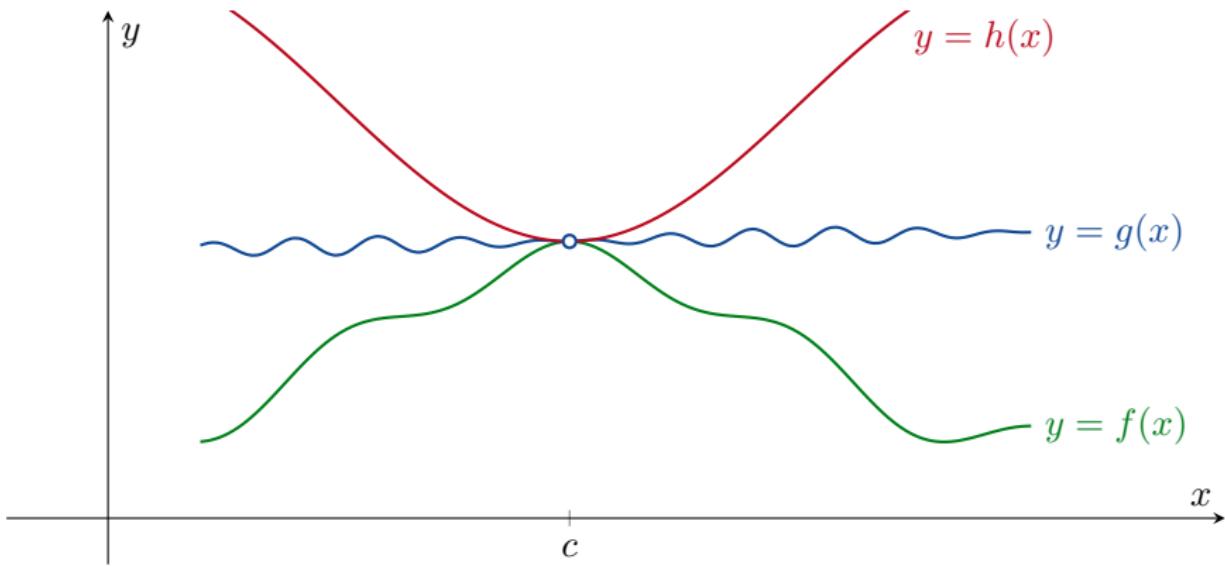
Instead, we try to factor  $x^2 + x - 2$  and  $x^2 - x$ . If  $x \neq 1$ , we have that

$$\frac{x^2 + x - 2}{x^2 - x} = \frac{(x - 1)(x + 2)}{x(x - 1)} = \frac{x + 2}{x}.$$

So

$$\lim_{x \rightarrow 1} \frac{x^2 + x - 2}{x^2 - x} = \lim_{x \rightarrow 1} \frac{x + 2}{x} = \frac{1 + 2}{1} = 3.$$

## The Sandwich Theorem



## 24. Limits



### Theorem (The Sandwich Theorem)

Suppose that

- $f(x) \leq g(x) \leq h(x)$  for all  $x$  “close” to  $c$  ( $x \neq c$ ); and
- $\lim_{x \rightarrow c} f(x) = \lim_{x \rightarrow c} h(x) = L$ .

Then

$$\lim_{x \rightarrow c} g(x) = L$$

also.

## 24. Limits

### Example

The inequality

$$1 - \frac{x^2}{6} < \frac{x \sin x}{2 - 2 \cos x} < 1$$

holds for all  $x$  close to 0 ( $x \neq 0$ ). Calculate  $\lim_{x \rightarrow 0} \frac{x \sin x}{2 - 2 \cos x}$ .

*solution:* Since  $\lim_{x \rightarrow 0} 1 - \frac{x^2}{6} = 1 - \frac{0}{6} = 1$  and  $\lim_{x \rightarrow 0} 1 = 1$ , it follows by the Sandwich Theorem that  $\lim_{x \rightarrow 0} \frac{x \sin x}{2 - 2 \cos x} = 1$ .

## Theorem

If

- $f(x) \leq g(x)$  for all  $x$  close to  $c$  ( $x \neq c$ );
- $\lim_{x \rightarrow c} f(x)$  exists; and
- $\lim_{x \rightarrow c} g(x)$  exists,

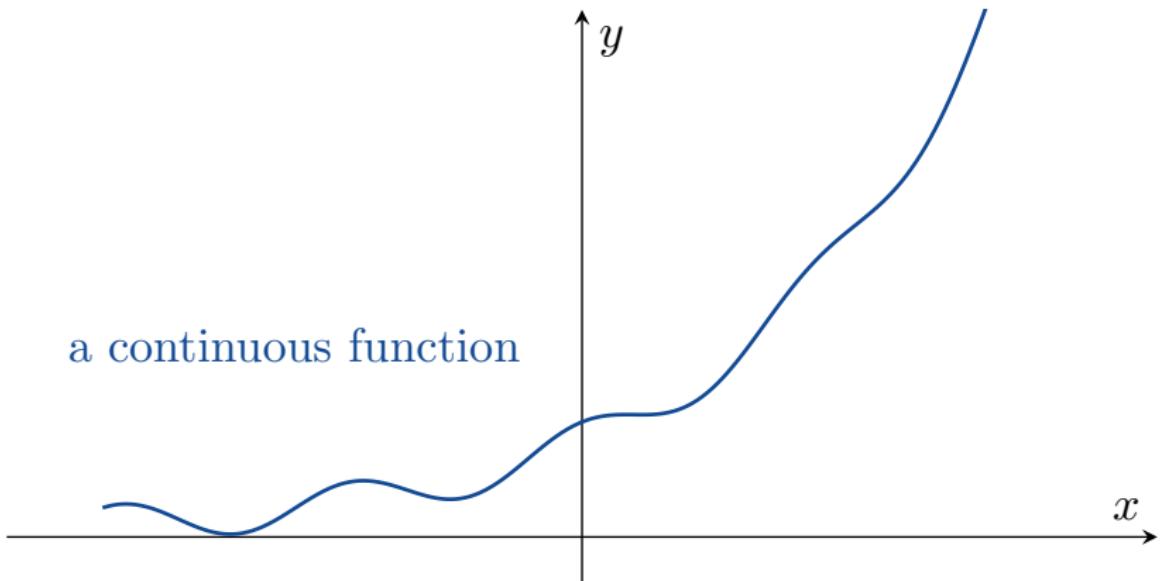
then

$$\lim_{x \rightarrow c} f(x) \leq \lim_{x \rightarrow c} g(x).$$

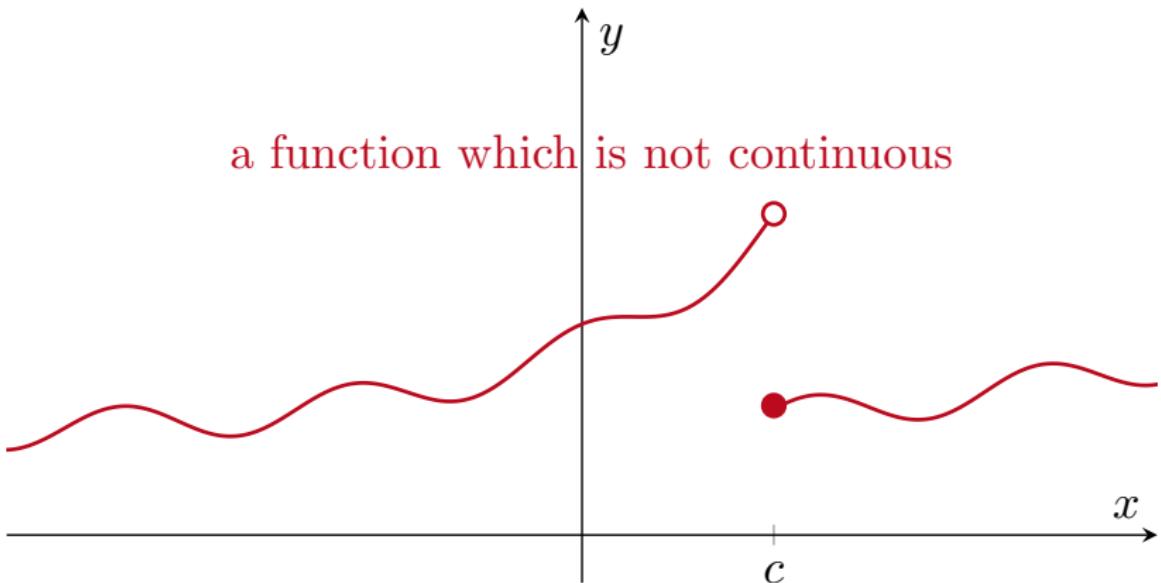


# Continuity

## 25. Continuity



## 25. Continuity



## 25. Continuity



### Definition

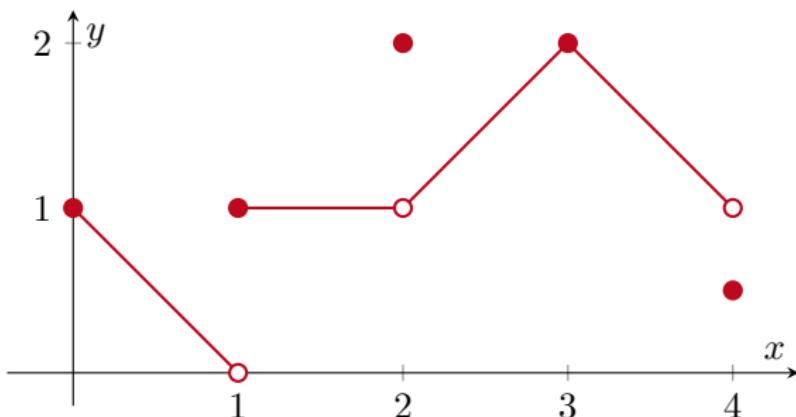
The function  $f : D \rightarrow \mathbb{R}$  is *continuous at  $c \in D$*  if

- $f(c)$  exists;
- $\lim_{x \rightarrow c} f(x)$  exists; and
- $\lim_{x \rightarrow c} f(x) = f(c)$ .

### Definition

If  $f$  is not continuous at  $c$ , we say that  $f$  is *discontinuous at  $c$*  – we say that  $c$  is a *point of discontinuity* of  $f$ .

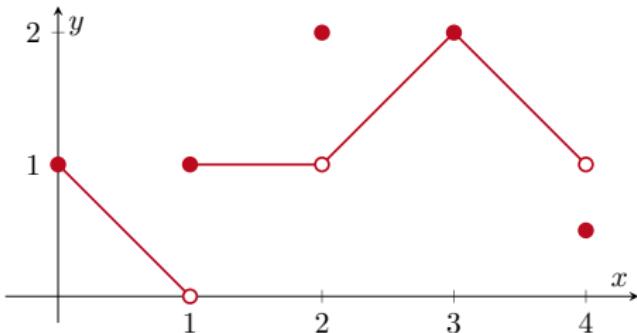
## 25. Continuity



### Example

Consider the function  $f : [0, 4] \rightarrow \mathbb{R}$  above. Where is  $f$  continuous? Where is  $f$  discontinuous?

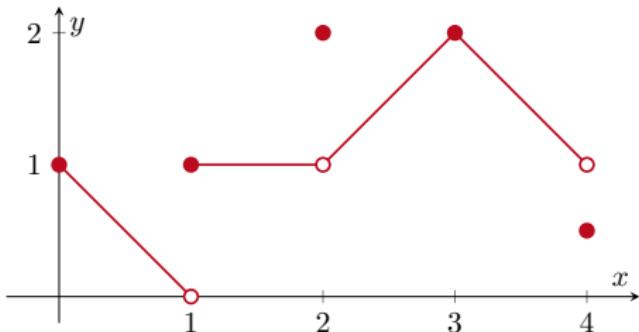
## 25. Continuity



*solution:*

$c$	Is $f$ continuous at $c$ ?	Why?
0	Yes	because $\lim_{x \rightarrow 0} f(x) = 1 = f(0)$
$(0, 1)$	Yes	because $\lim_{x \rightarrow c} f(x) = f(c)$
1	No	because $\lim_{x \rightarrow 1} f(x)$ does not exist

## 25. Continuity



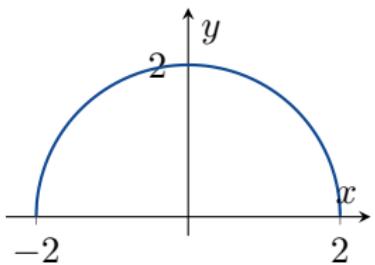
*solution:*

$c$	Is $f$ continuous at $c$ ?	Why?
(1, 2)	Yes	because $\lim_{x \rightarrow c} f(x) = f(c)$
2	No	because $\lim_{x \rightarrow 2} f(x) = 1 \neq 2 = f(2)$
(2, 4)	Yes	because $\lim_{x \rightarrow c} f(x) = f(c)$
4	No	because $\lim_{x \rightarrow 4} f(x) = 1 \neq \frac{1}{2} = f(4)$

## 25. Continuity

Example

$$f : [-2, 2] \rightarrow \mathbb{R}, f(x) = \sqrt{4 - x^2}$$

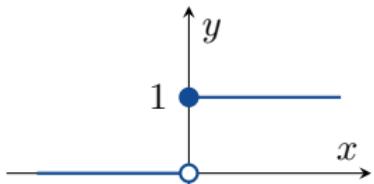


$f$  is continuous at every  $c \in [-2, 2]$ .

## 25. Continuity

### Example

$$g : \mathbb{R} \rightarrow \mathbb{R}, g(x) = \begin{cases} 0 & x < 0 \\ 1 & x \geq 0 \end{cases}$$

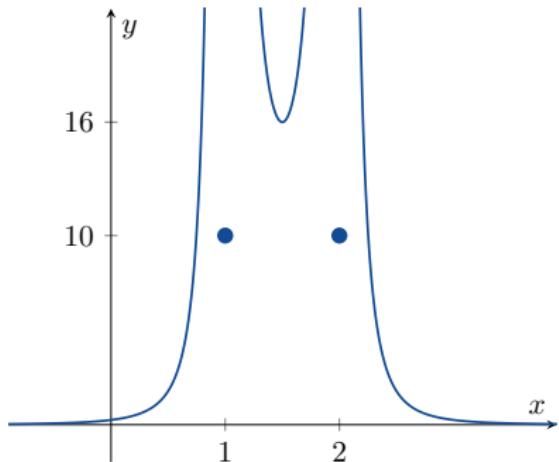


$g$  has a point of discontinuity at  $c = 0$ .  $g$  is continuous at every point  $c \neq 0$ .

## 25. Continuity

### Example

$$h : \mathbb{R} \rightarrow \mathbb{R}, h(x) = \begin{cases} \frac{1}{(x-1)^2(x-2)^2} & x \neq 1 \text{ or } 2 \\ 10 & x = 1 \text{ or } 2 \end{cases}$$



$h$  is continuous on  $(-\infty, 1)$ ,  $(1, 2)$  and  $(2, \infty)$ .  $h$  has a points of discontinuity at  $c = 1$  and  $c = 2$ .

# Continuous Functions

## Definition

$f : D \rightarrow \mathbb{R}$  is a *continuous function* if it is continuous at every  $c \in D$ .

## 25. Continuity



### Theorem

If  $f$  and  $g$  are continuous at  $c$ , then  $f + g$ ,  $f - g$ ,  $kf$  ( $k \in \mathbb{R}$ ),  $fg$ ,  $\frac{f}{g}$  (if  $g(c) \neq 0$ ) and  $f^n$  ( $n \in \mathbb{N}$ ) are all continuous at  $c$ . If  $\sqrt[n]{f}$  is defined on  $(c - \delta, c + \delta)$ , then  $\sqrt[n]{f}$  is also continuous at  $c$  ( $n \in \mathbb{N}$ ).

## 25. Continuity



### Example

Every polynomial

$$P(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_2 x^2 + a_1 x + a_0$$

is continuous.

## 25. Continuity



### Example

If

- $P$  and  $Q$  are polynomials; and
- $Q(c) \neq 0$ ,

then  $\frac{P(x)}{Q(x)}$  is continuous at  $c$ .

## 25. Continuity



### Example

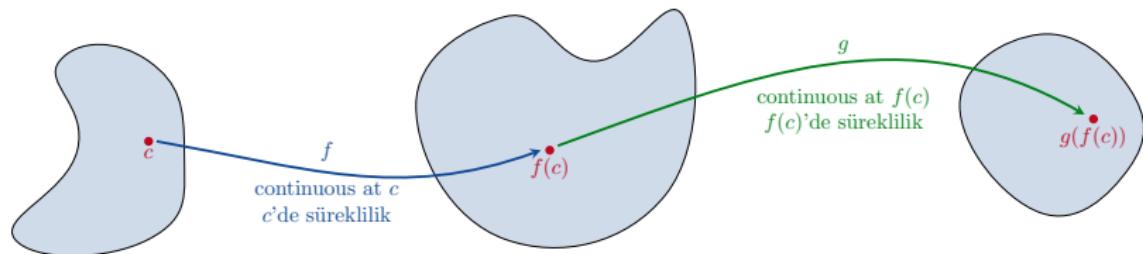
$\sin x$  and  $\cos x$  are continuous.

# Composites

$$g \circ f(x)$$

$g \circ f(x)$  means  $g(f(x))$ .

## 25. Continuity



### Theorem

If

- $f$  is continuous at  $c$ ; and
- $g$  is continuous at  $f(c)$ ,

then  $g \circ f$  is continuous at  $c$ .

## 25. Continuity

### Example

Show that  $h(x) = \sqrt{x^2 - 2x - 5}$  is continuous on its domain.

*solution:* The function  $g(t) = \sqrt{t}$  is continuous by Theorem 24. The function  $f(x) = x^2 - 2x - 5$  is continuous because all polynomials are continuous. Therefore  $h(x) = g \circ f(x)$  is continuous.

## 25. Continuity



### Example

Show that  $\frac{x^{\frac{2}{3}}}{1+x^4}$  is continuous.

*solution:*  $x^{\frac{2}{3}}$  and  $1 + x^4$  are continuous. Because  $1 + x^4 \neq 0$  for all  $x$ , we have that  $\frac{x^{\frac{2}{3}}}{1+x^4}$  is continuous.

## 25. Continuity

### Theorem

If

- $g(x)$  is continuous at  $x = b$ ; and
- $\lim_{x \rightarrow c} f(x) = b$ ,

then

$$\lim_{x \rightarrow c} g(f(x)) = g\left(\lim_{x \rightarrow c} f(x)\right).$$

## 25. Continuity

### Example

$$\begin{aligned}& \lim_{x \rightarrow \frac{\pi}{2}} \cos \left[ 2x + \sin \left( \frac{3\pi}{2} + x \right) \right] \\&= \cos \left[ \lim_{x \rightarrow \frac{\pi}{2}} \left( 2x + \sin \left( \frac{3\pi}{2} + x \right) \right) \right] \\&= \cos \left[ \lim_{x \rightarrow \frac{\pi}{2}} (2x) + \lim_{x \rightarrow \frac{\pi}{2}} \left( \sin \left( \frac{3\pi}{2} + x \right) \right) \right] \\&= \cos \left[ \pi + \sin \left( \lim_{x \rightarrow \frac{\pi}{2}} \left( \frac{3\pi}{2} + x \right) \right) \right] \\&= \cos [\pi + \sin 2\pi] = \cos [\pi + 0] = -1.\end{aligned}$$

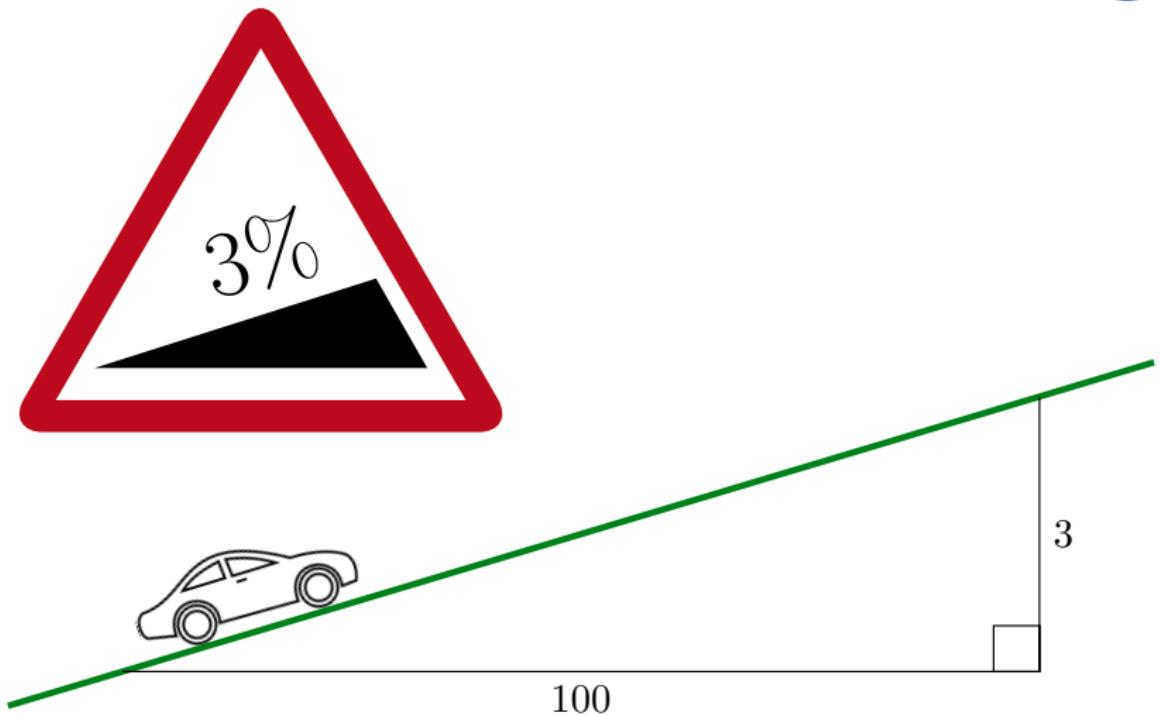


# Differentiation

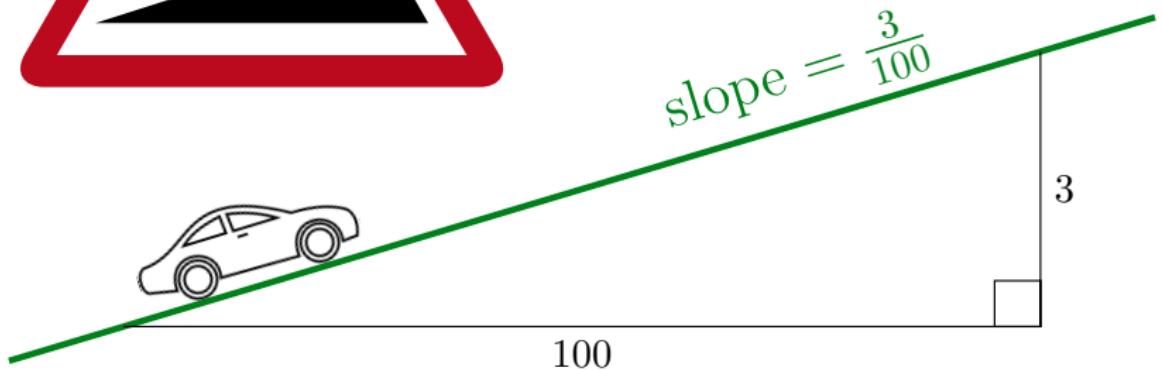
## 26. Differentiation



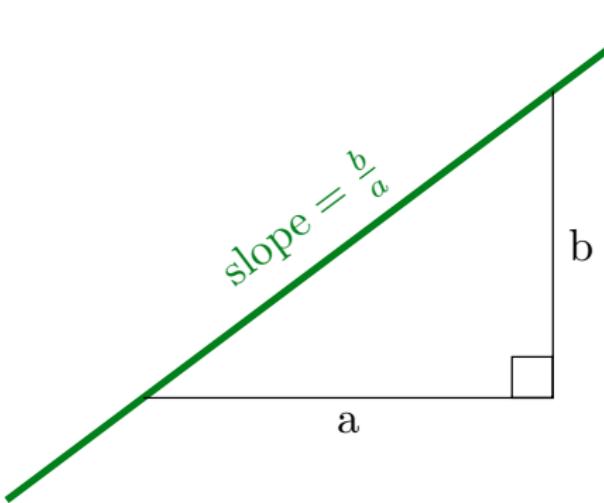
## 26. Differentiation



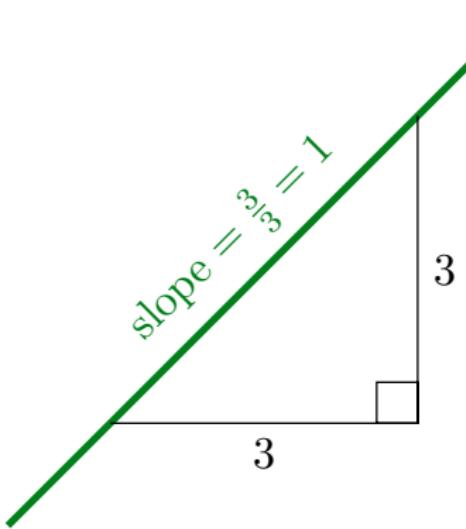
## 26. Differentiation



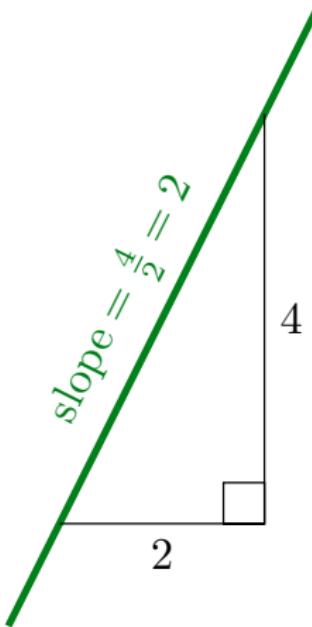
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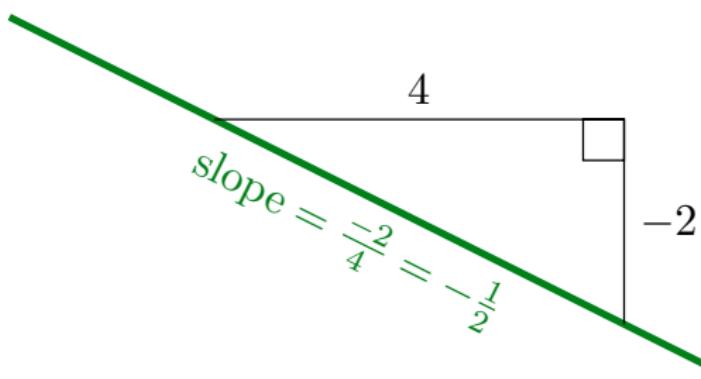
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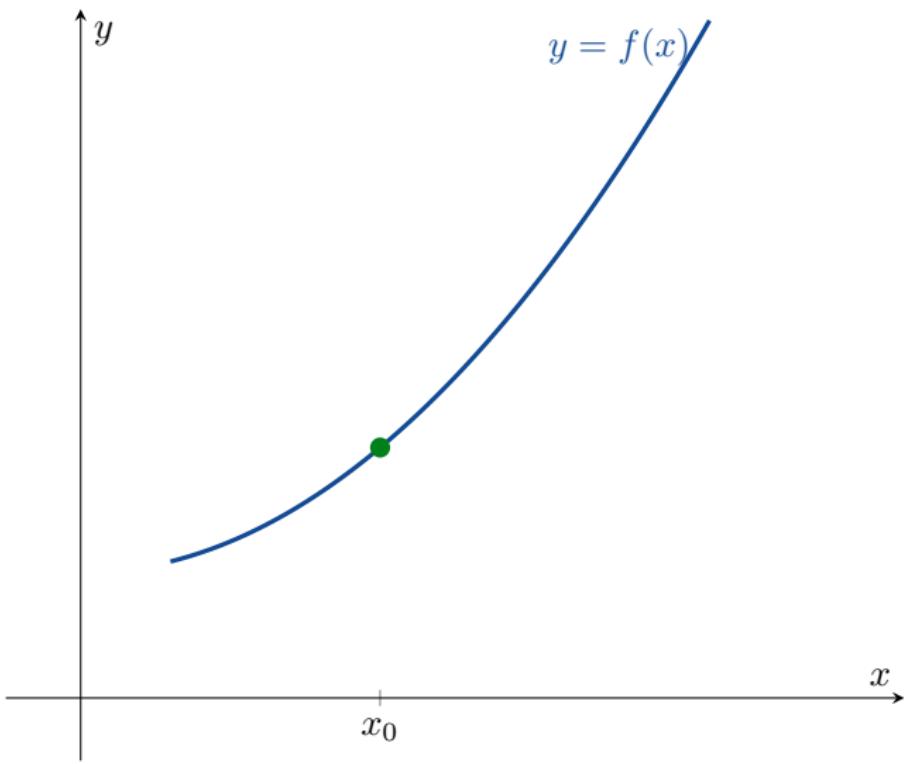
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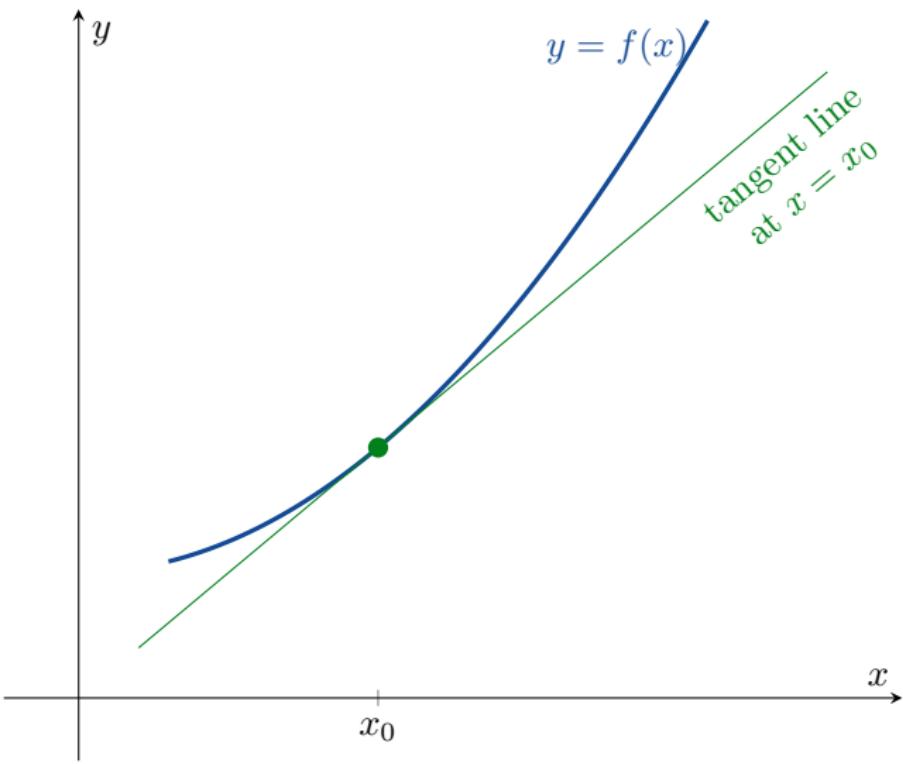
## 26. Differentiation



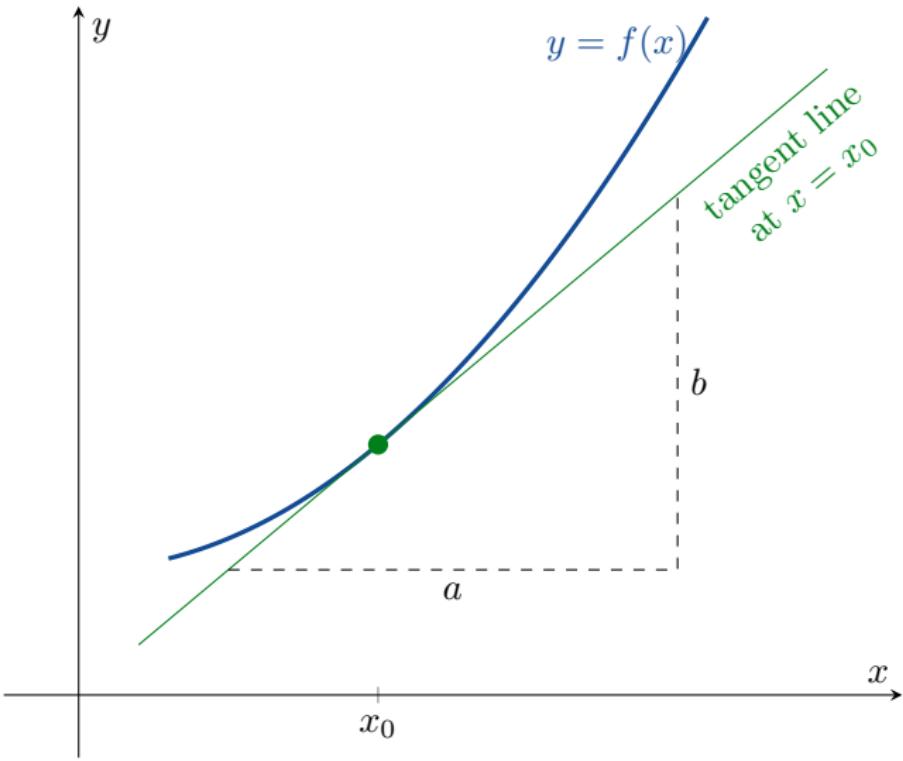
## 26. Differentiation



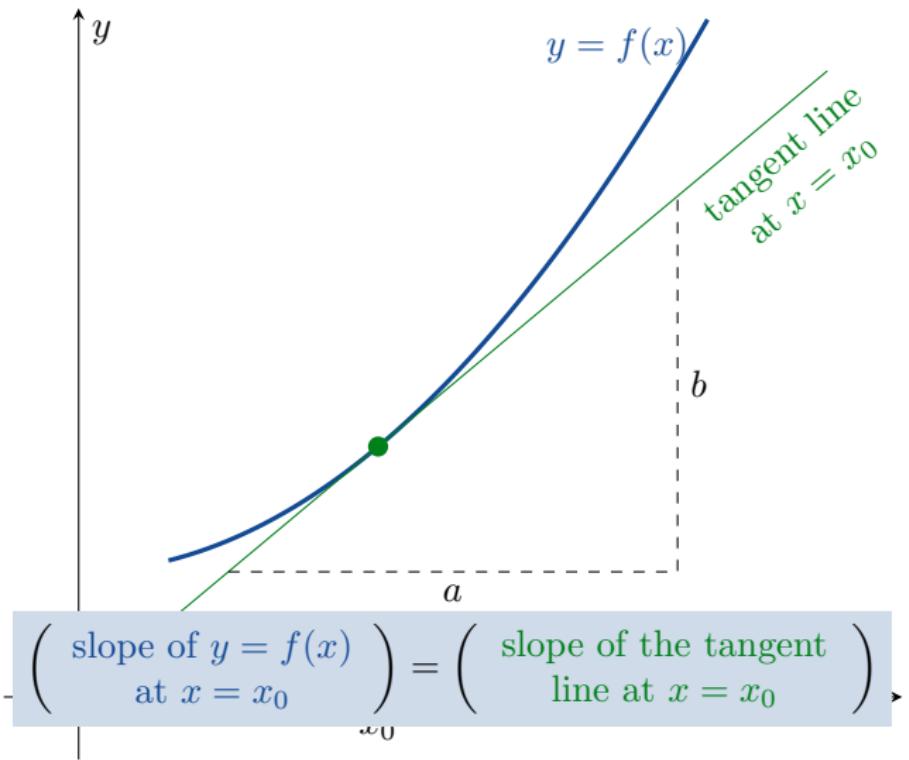
## 26. Differentiation



## 26. Differentiation



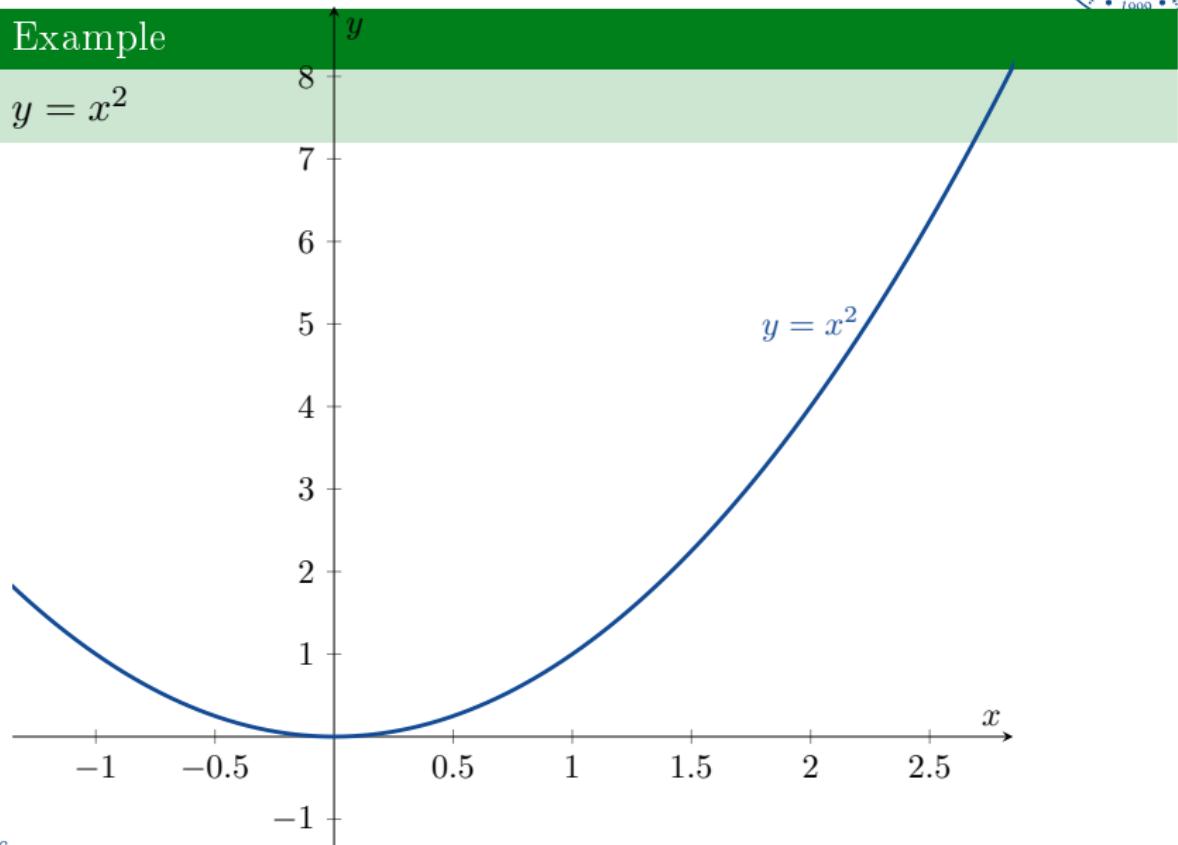
## 26. Differentiation



## 26. Differentiation

Example

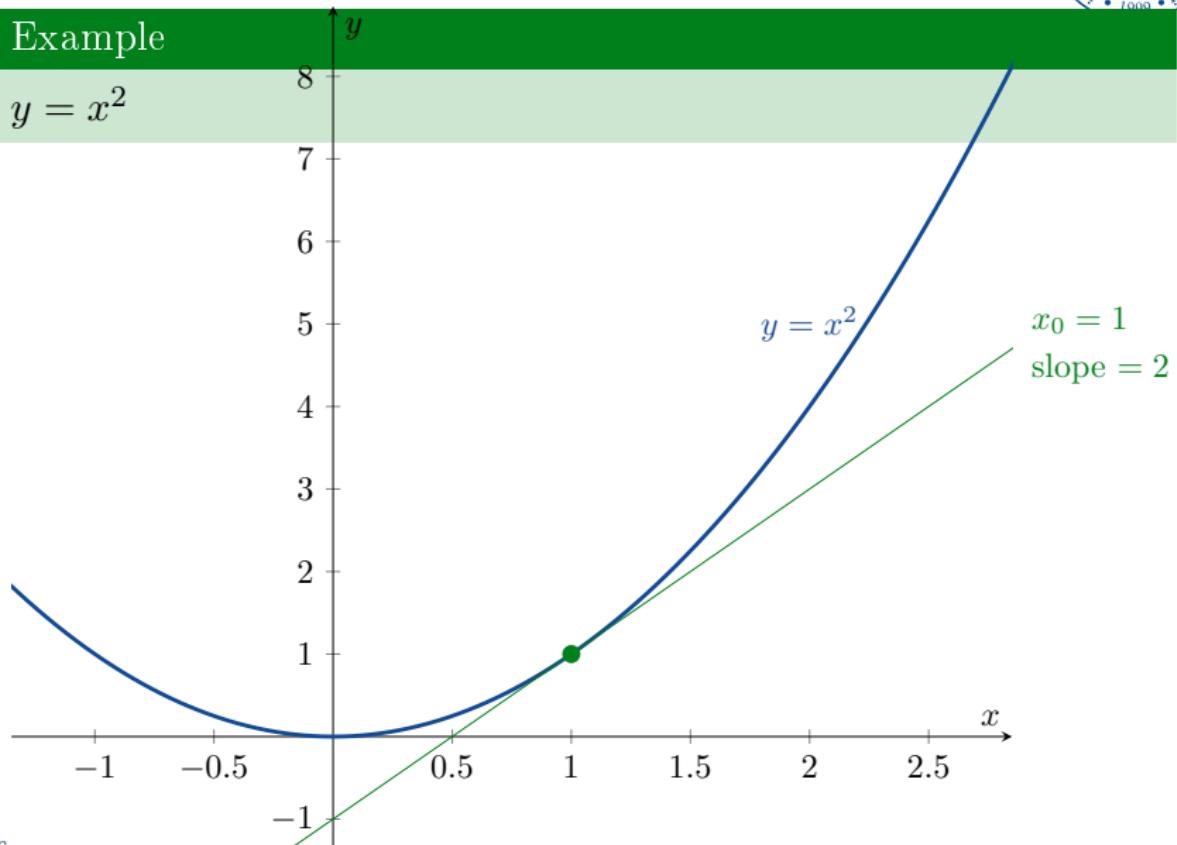
$$y = x^2$$



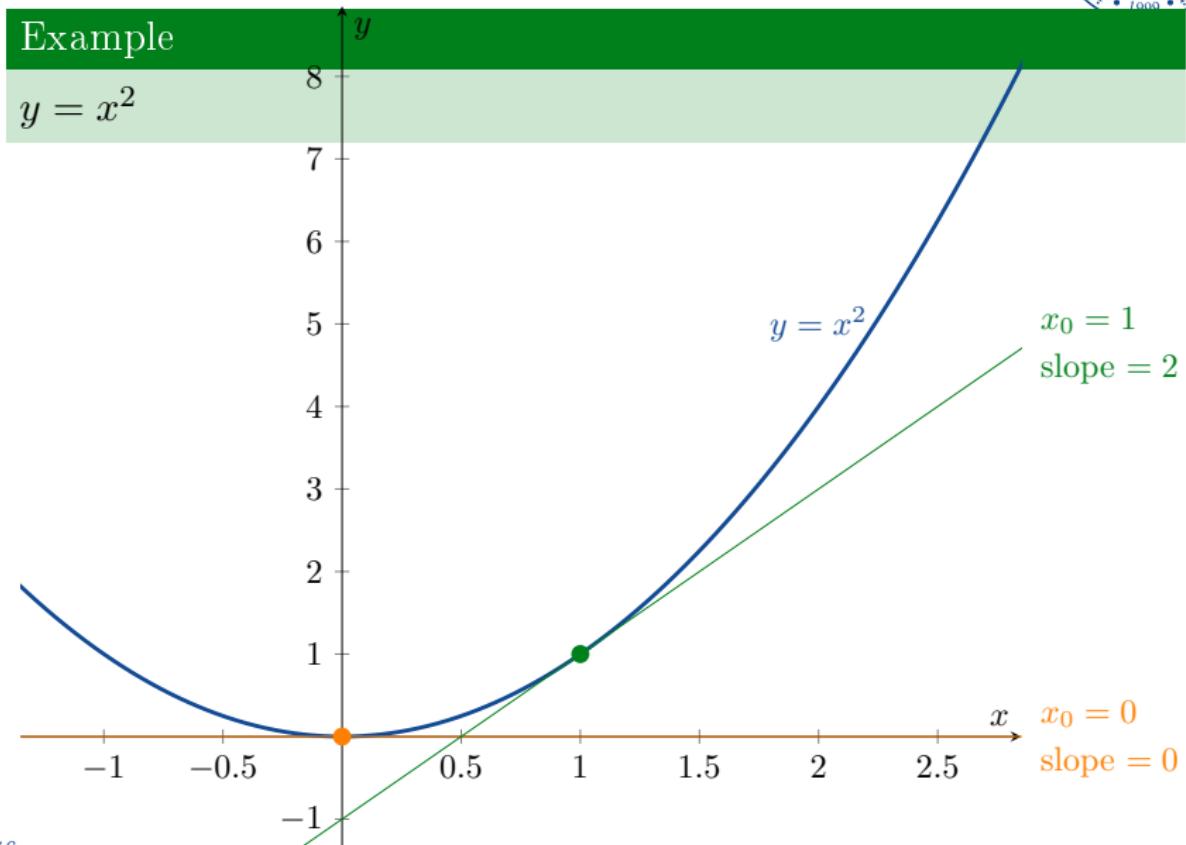
## 26. Differentiation

Example

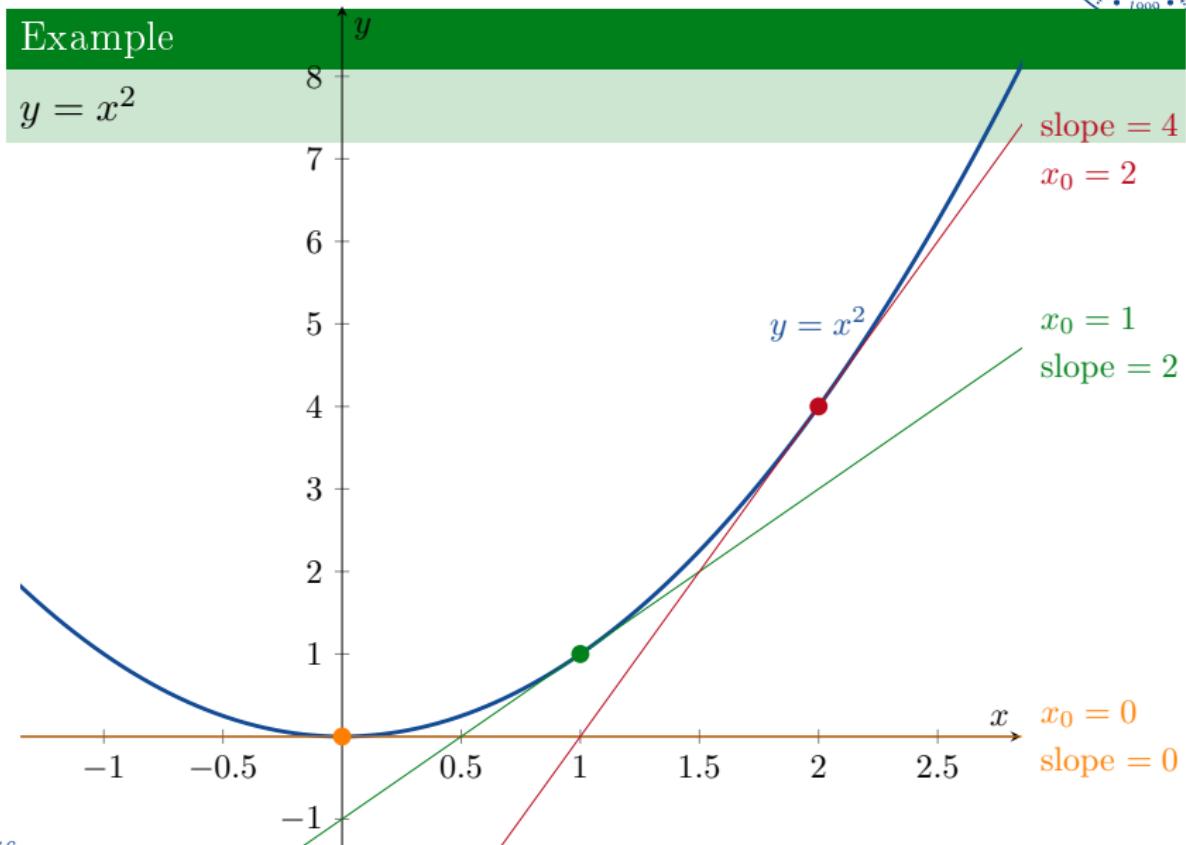
$$y = x^2$$



## 26. Differentiation



## 26. Differentiation

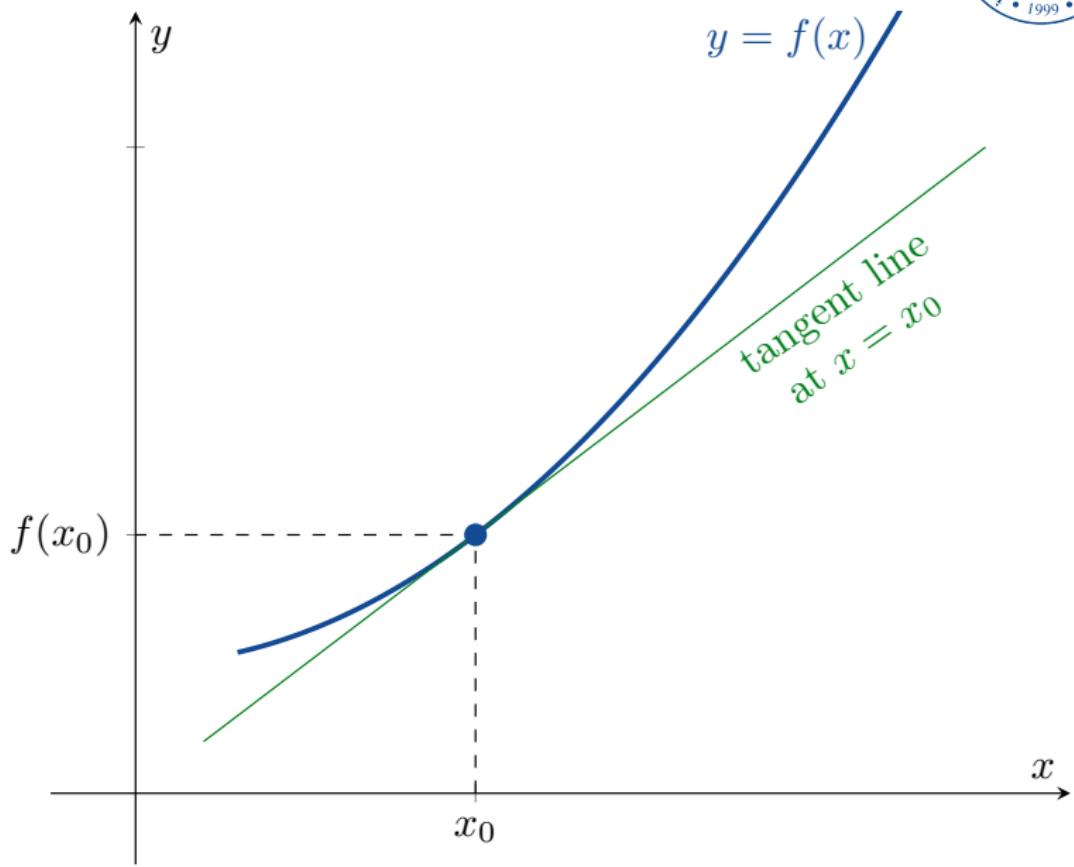


## 26. Differentiation

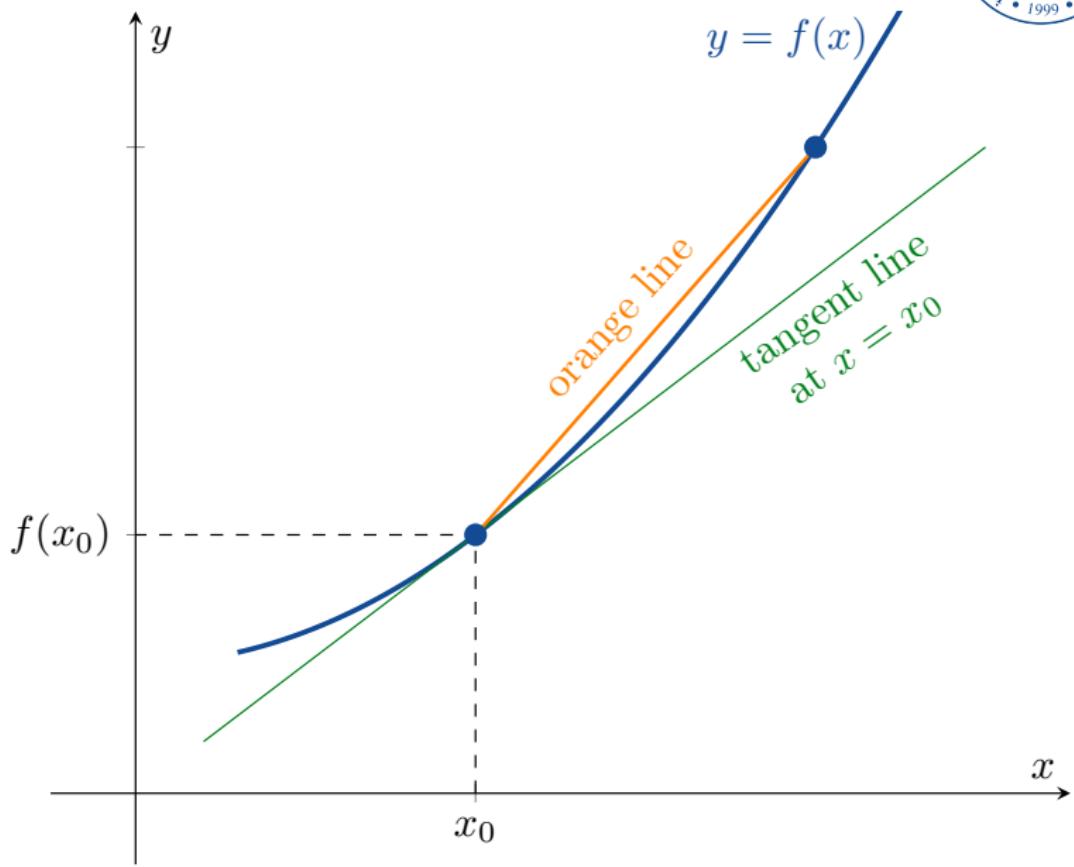


How can we calculate the slope of the tangent line?

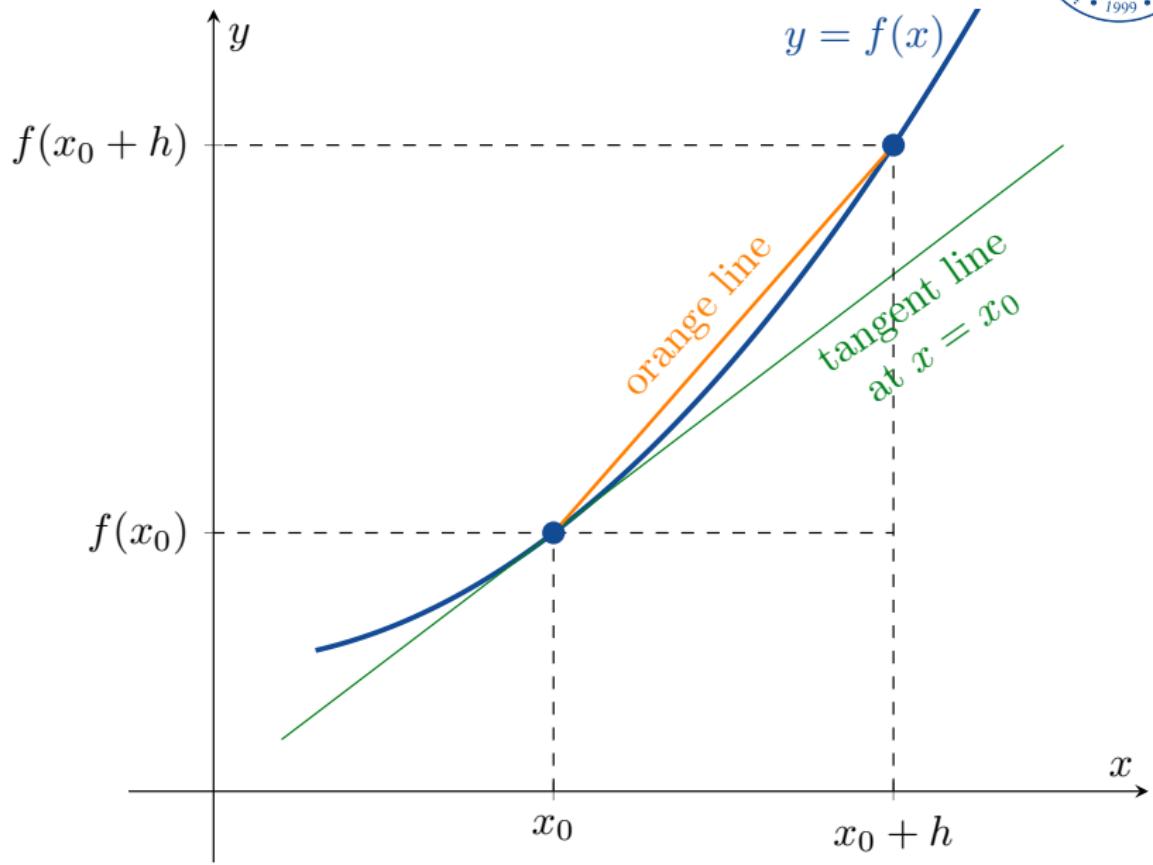
## 26. Differentiation



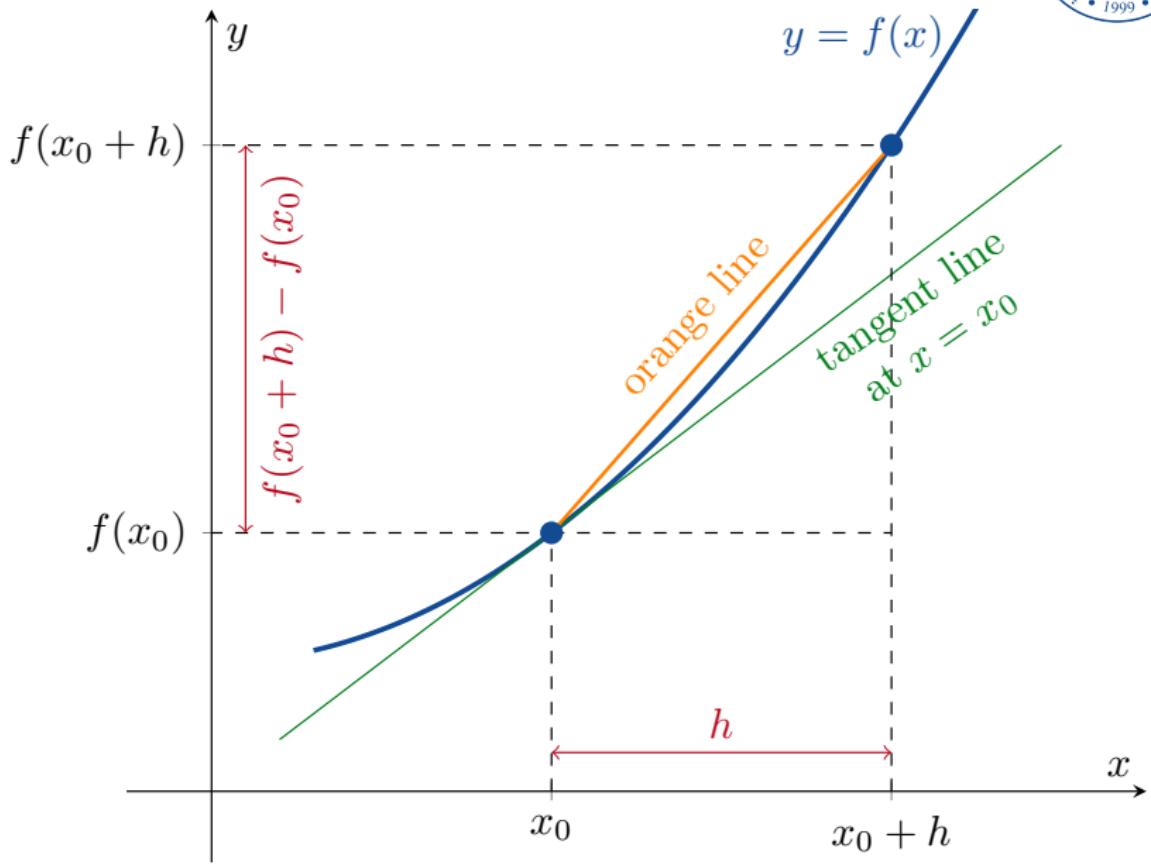
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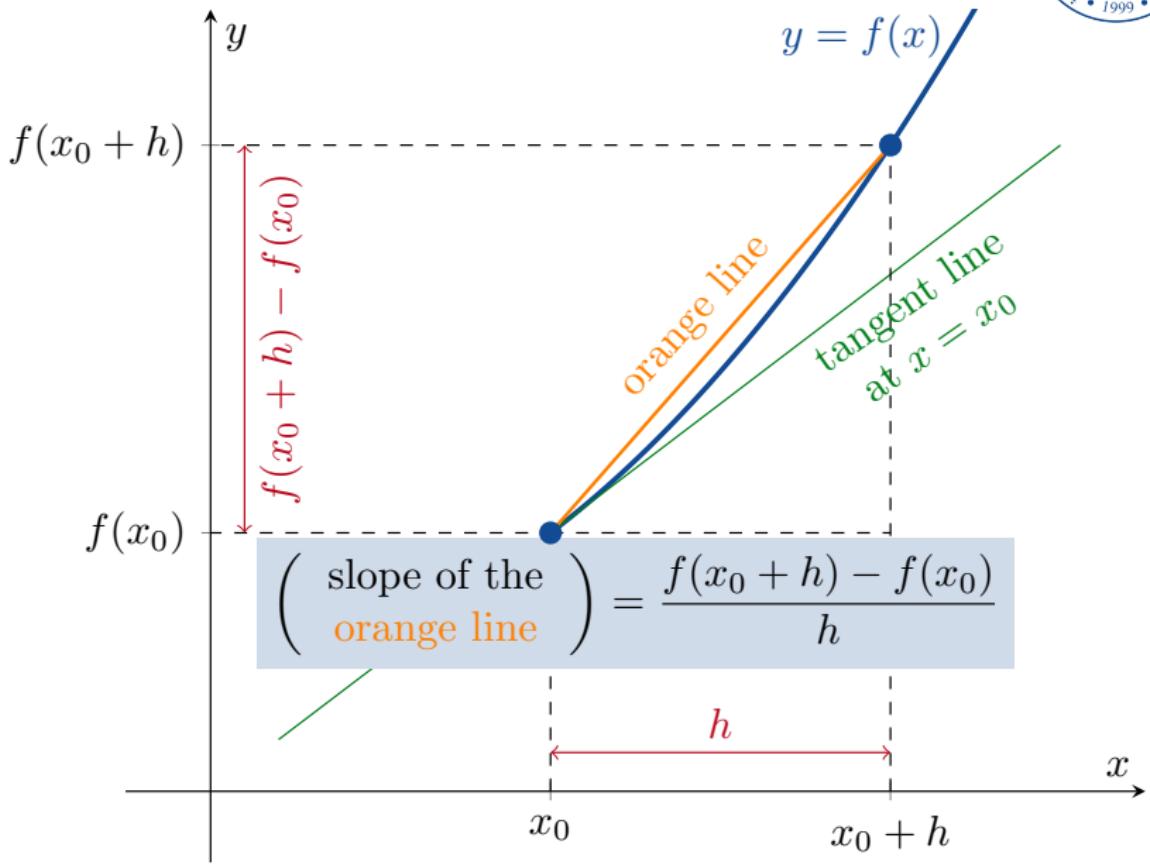
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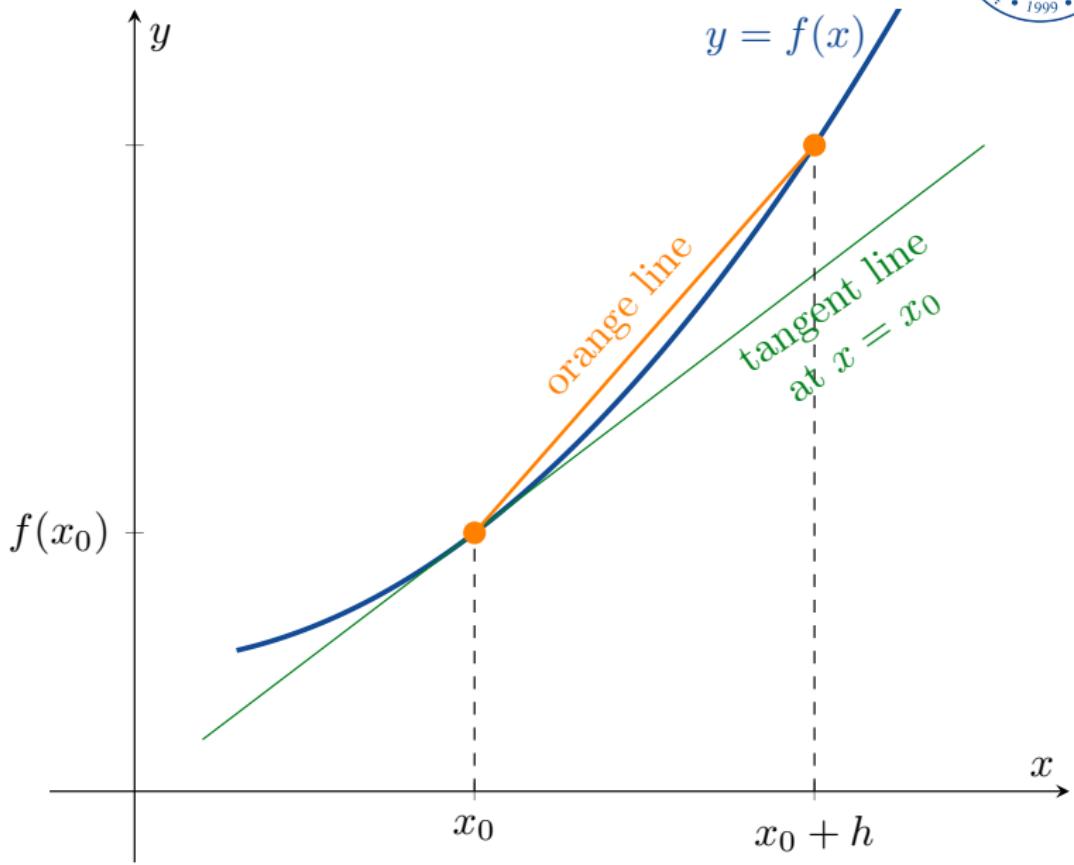
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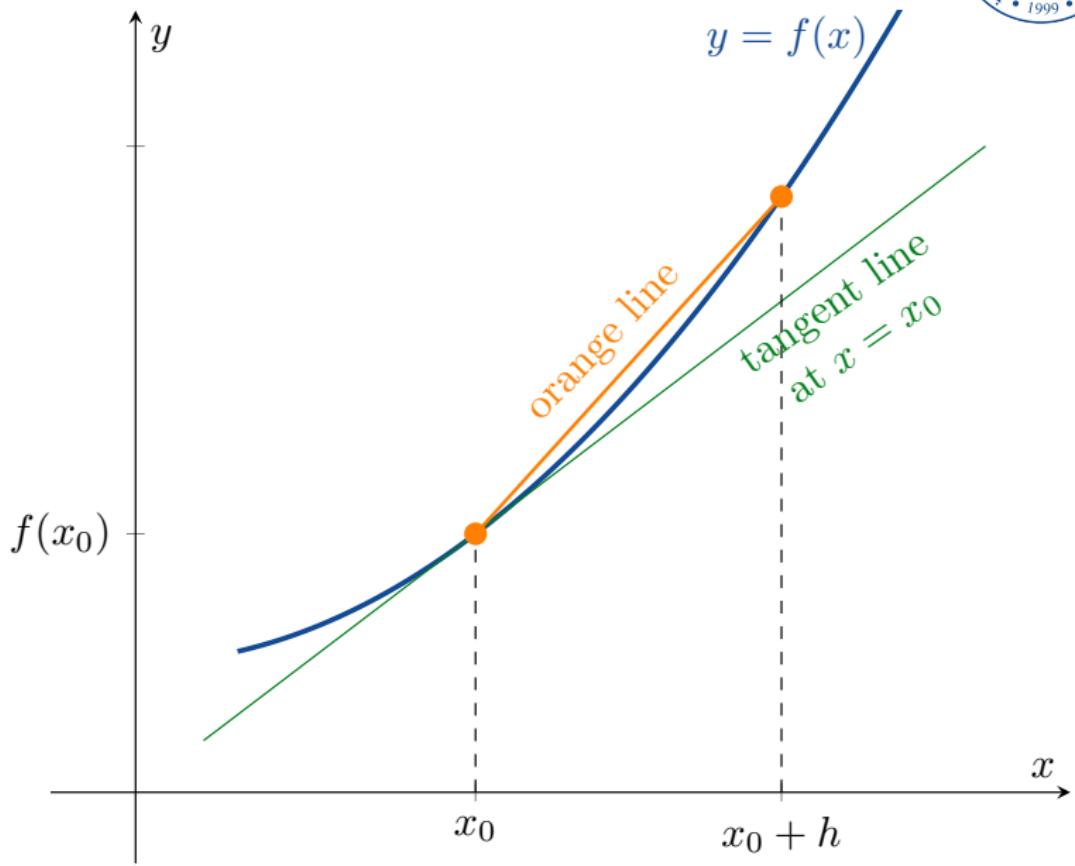
## 26. Differentiation



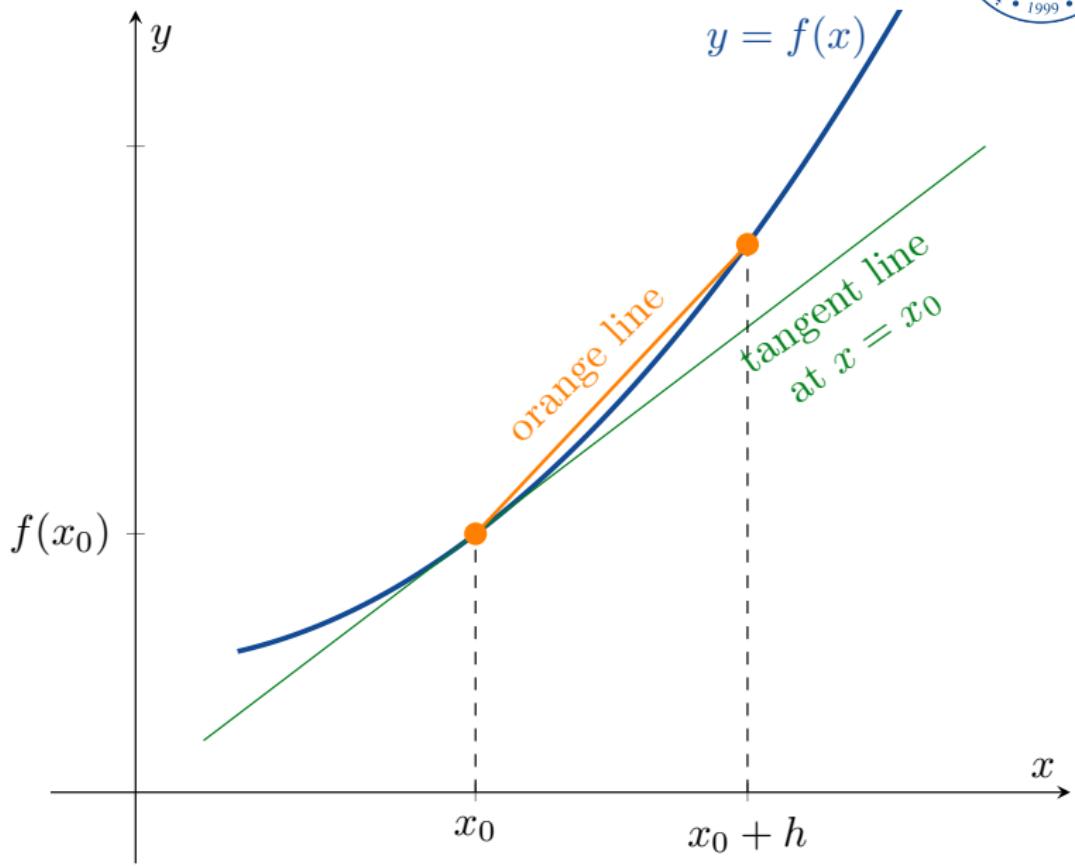
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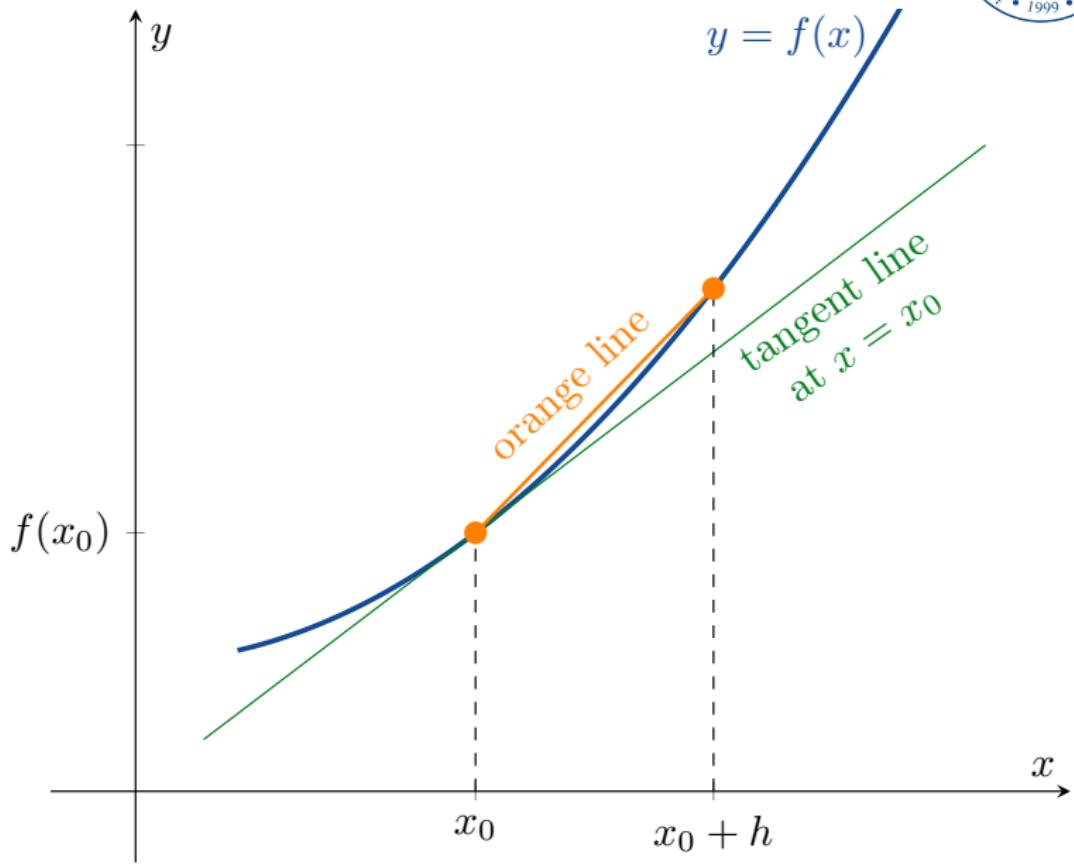
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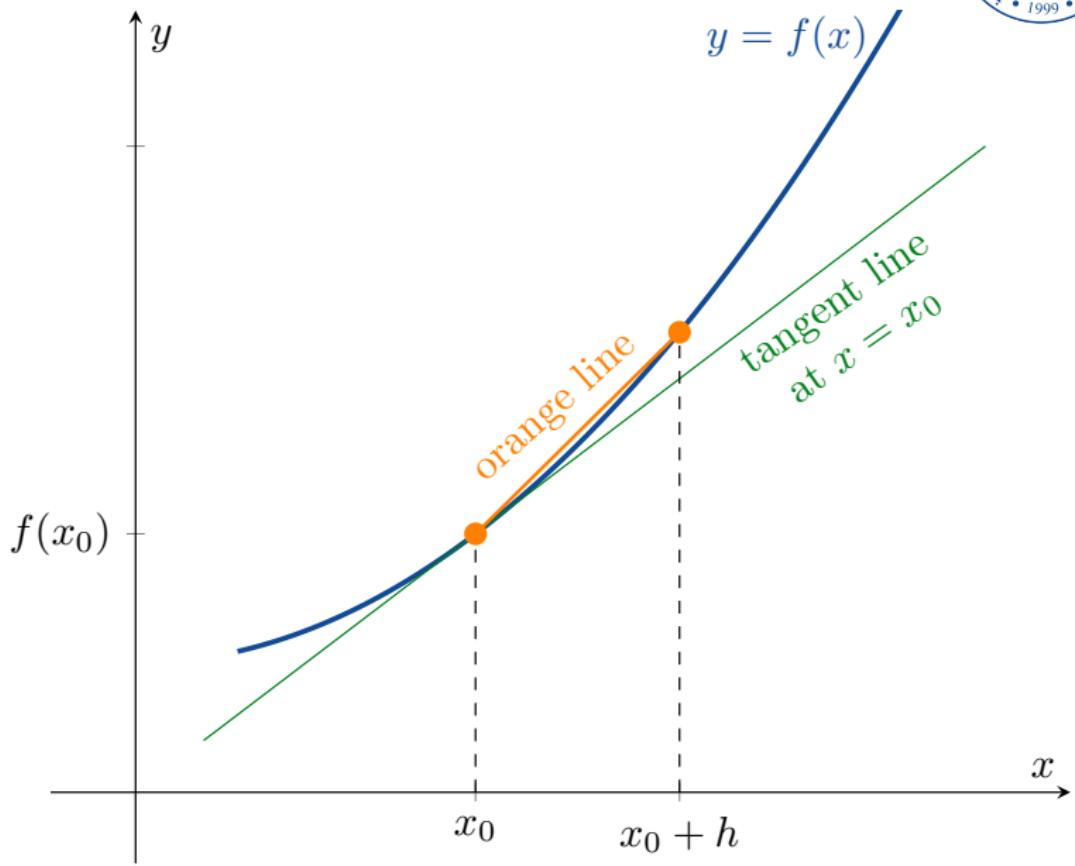
## 26. Differentiation



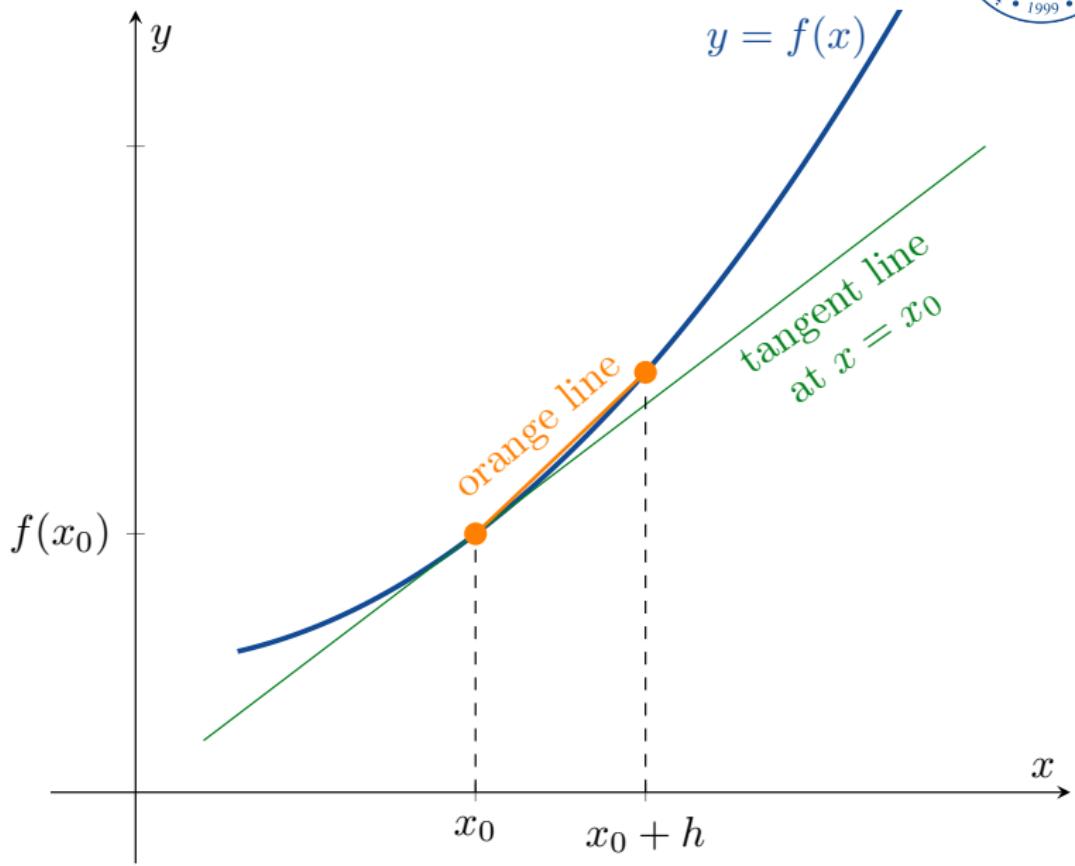
## 26. Differentiation



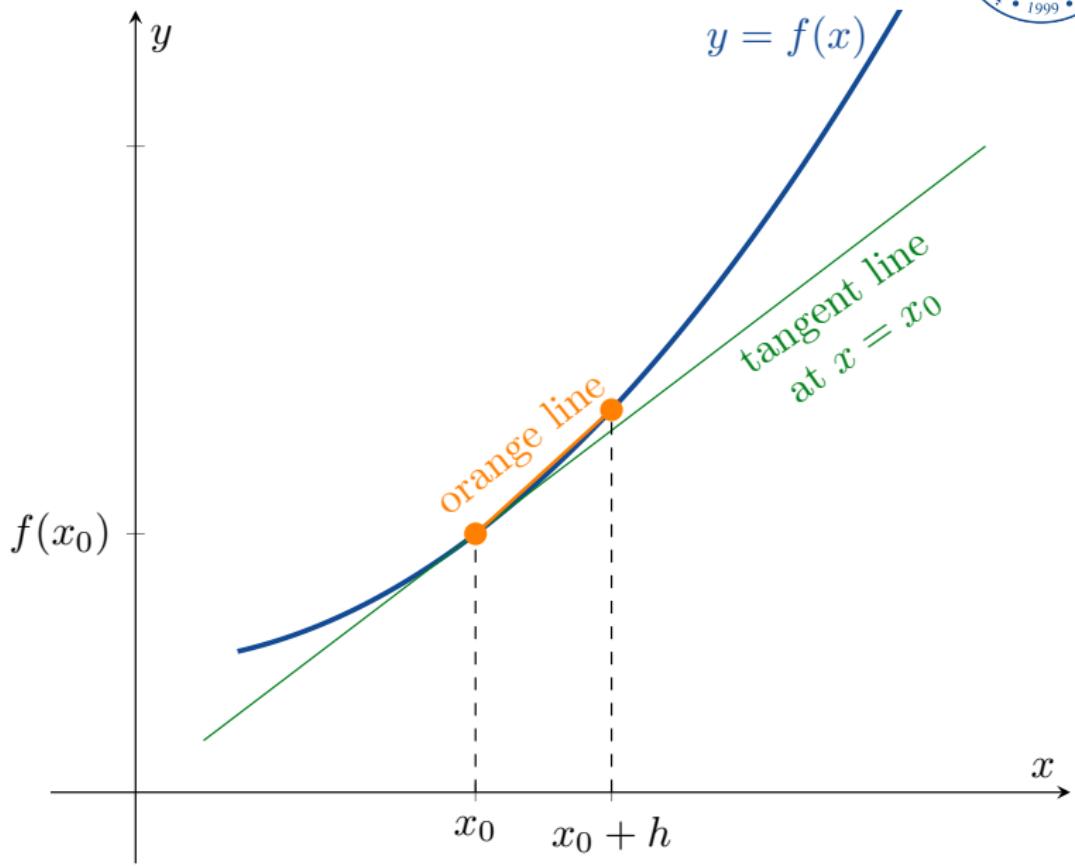
## 26. Differentiation



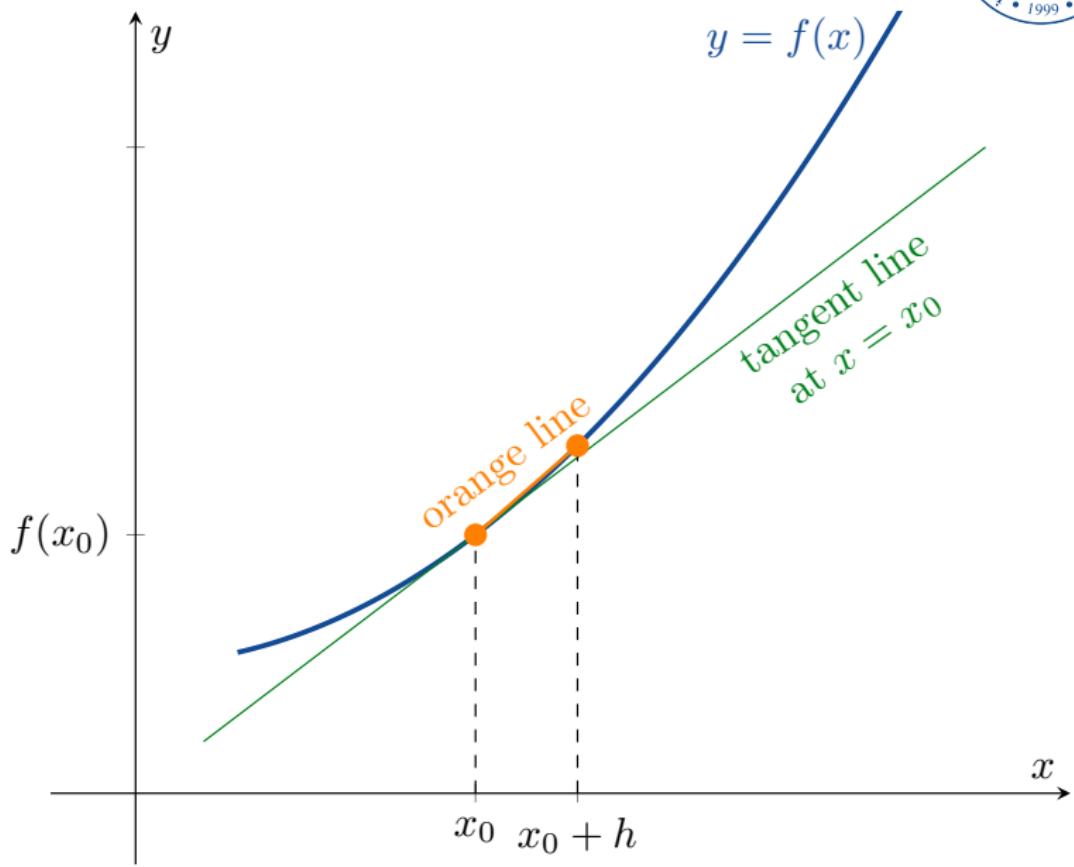
## 26. Differentiation



## 26. Differentiation



## 26. Differentiation



## 26. Differentiation



If  $h$  is very very small, then

$$\left( \begin{array}{l} \text{slope of the} \\ \text{tangent line} \end{array} \right) \approx \left( \begin{array}{l} \text{slope of the} \\ \text{orange line} \end{array} \right) = \frac{f(x_0 + h) - f(x_0)}{h}$$

# The Derivative of $f$

## Definition

The *derivative of a function f at a point  $x_0$*  is

$$f'(x_0) = \lim_{h \rightarrow 0} \frac{f(x_0 + h) - f(x_0)}{h}$$

if the limit exists.

( $f'$  is pronounced “ $f$  prime”)

## 26. Differentiation

### Example

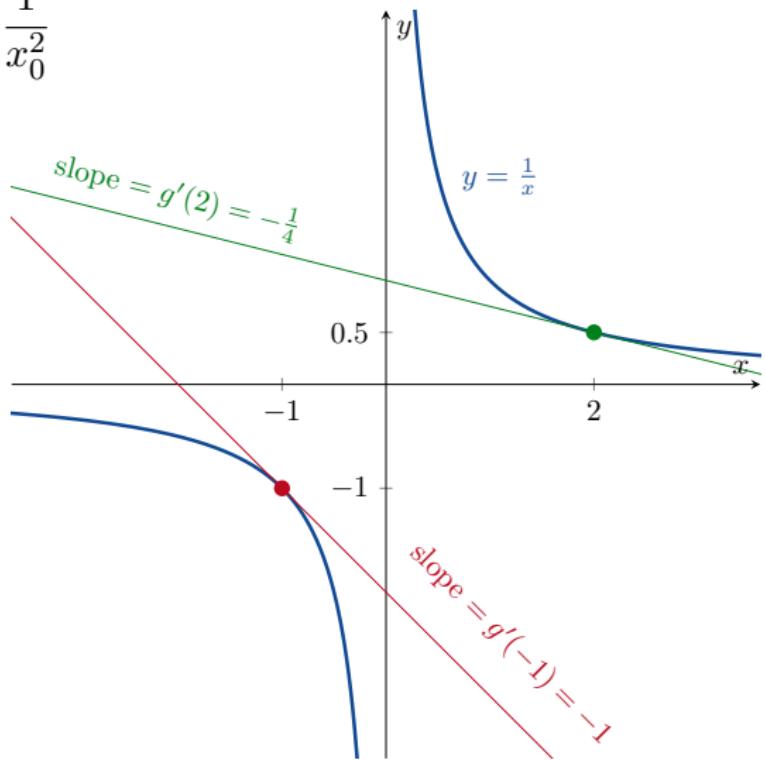
Consider the function  $g(x) = \frac{1}{x}$ ,  $x \neq 0$ .

If  $x_0 \neq 0$ , then

$$\begin{aligned} g'(x_0) &= \lim_{h \rightarrow 0} \frac{g(x_0 + h) - g(x_0)}{h} \\ &= \lim_{h \rightarrow 0} \frac{\frac{1}{x_0+h} - \frac{1}{x_0}}{h} \\ &= \lim_{h \rightarrow 0} \frac{\left( \frac{x_0}{x_0(x_0+h)} - \frac{x_0+h}{x_0(x_0+h)} \right)}{h} \\ &= \lim_{h \rightarrow 0} \frac{x_0 - x_0 - h}{hx_0(x_0 + h)} \\ &= \lim_{h \rightarrow 0} \frac{-1}{x_0(x_0 + h)} = -\frac{1}{x_0^2}. \end{aligned}$$

## 26. Differentiation

$$g'(x_0) = -\frac{1}{x_0^2}$$



## 26. Differentiation



### Definition

If  $f'(x_0)$  exists, we say that  $f$  is differentiable at  $x_0$ .

## 26. Differentiation



### Definition

Let  $f : D \rightarrow \mathbb{R}$  be a function. If  $f$  is differentiable at every  $x_0 \in D$ , we say that  $f$  is *differentiable*.

## 26. Differentiation

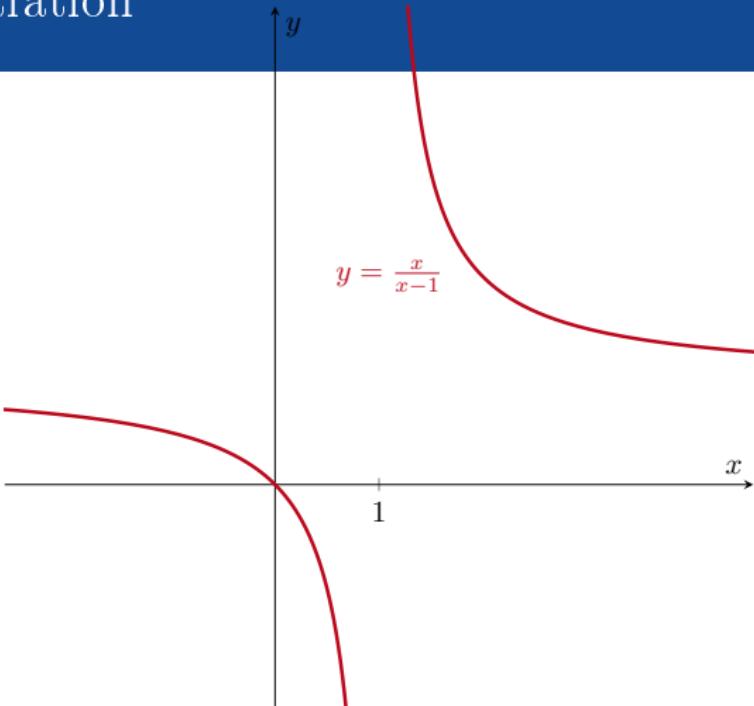


If  $f : D \rightarrow \mathbb{R}$  is differentiable, then we have a new function  
 $f' : D \rightarrow \mathbb{R}$ .

### Definition

$f'$  is called the *derivative* of  $f$ .

## 26. Differentiation



Example

$$\text{Differentiate } f(x) = \frac{x}{x-1}.$$

## 26. Differentiation

*solution:* First note that  $f(x + h) = \frac{x+h}{x+h-1}$ . Therefore

$$\begin{aligned}
 f'(x) &= \lim_{h \rightarrow 0} \frac{f(x + h) - f(x)}{h} \\
 &= \lim_{h \rightarrow 0} \frac{\frac{x+h}{x+h-1} - \frac{x}{x-1}}{h} \\
 &= \lim_{h \rightarrow 0} \frac{1}{h} \left( \frac{(x+h)(x-1) - x(x+h-1)}{(x-1)(x+h-1)} \right) \\
 &= \lim_{h \rightarrow 0} \frac{1}{h} \left( \frac{-h}{(x-1)(x+h-1)} \right) \\
 &= \lim_{h \rightarrow 0} \frac{-1}{(x-1)(x+h-1)} \\
 &= \frac{-1}{(x-1)(x+0-1)} \\
 &= \frac{-1}{(x-1)^2}.
 \end{aligned}$$

## Notations

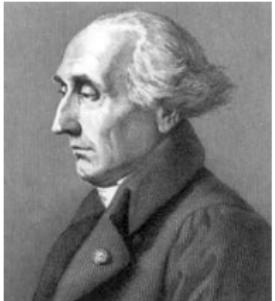
There are many ways to write the derivative of  $y = f(x)$ .

$$f'(x) = y' = \frac{dy}{dx} = \frac{df}{dx} = \frac{d}{dx}f(x) = \dot{y} = \dot{f}(x)$$



“the derivative of  $y$  with respect to  $x$ ”

## 26. Differentiation



Sir Isaac Newton  
UK, 1642-1726

Gottfried Leibniz  
GER, 1646-1716

Joseph-Louis Lagrange  
ITA, 1736-1813

Calculus was started by two men who hated each other: Sir Isaac Newton used  $f'$  and  $\dot{y}$ . Gottfried Leibniz used  $\frac{df}{dx}$  and  $\frac{dy}{dx}$ .

The  $f'$  and  $y'$  notation came later from Joseph-Louis Lagrange.

## 26. Differentiation



If we want the derivative of  $y = f(x)$  at the point  $x = x_0$ , we can write

$$f'(x_0) = \frac{dy}{dx} \Big|_{x=x_0} = \frac{df}{dx} \Big|_{x=x_0} = \frac{d}{dx} f(x) \Big|_{x=x_0}$$



“the derivative of  $y$  with respect to  $x$  at  $x = x_0$ ”

## 26. Differentiation



For example, if  $u(x) = \frac{1}{x}$ , then

$$u'(4) = \left. \frac{d}{dx} \left( \frac{1}{x} \right) \right|_{x=4} = \left. \frac{-1}{x^2} \right|_{x=4} = \frac{-1}{4^2} = \frac{-1}{16}.$$

## 26. Differentiation



### Example

Show that  $f(x) = |x|$  is differentiable on  $(-\infty, 0)$  and on  $(0, \infty)$ , but is not differentiable at  $x = 0$ .

## 26. Differentiation



*solution:* If  $x > 0$  then

$$\frac{df}{dx} = \frac{d}{dx}(|x|) = \frac{d}{dx}(x) = \lim_{h \rightarrow 0} \frac{(x+h)-x}{h} = \lim_{h \rightarrow 0} 1 = 1.$$

## 26. Differentiation



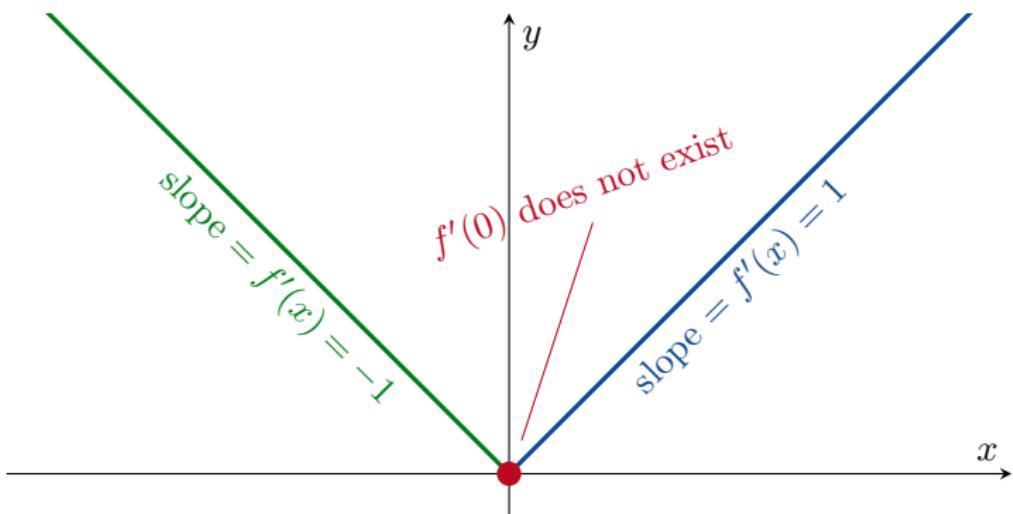
Similarly, if  $x < 0$  then

$$\begin{aligned}\frac{df}{dx} &= \frac{d}{dx}(|x|) = \frac{d}{dx}(-x) = \lim_{h \rightarrow 0} \frac{(-x - h) - (-x)}{h} \\ &= \lim_{h \rightarrow 0} -1 = -1.\end{aligned}$$

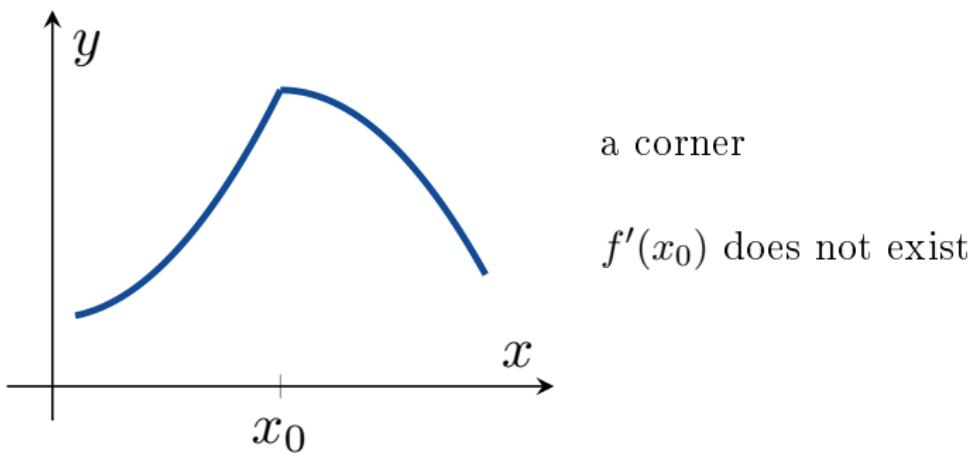
Therefore  $f$  is differentiable on  $(-\infty, 0)$  and on  $(0, \infty)$ .

## 26. Differentiation

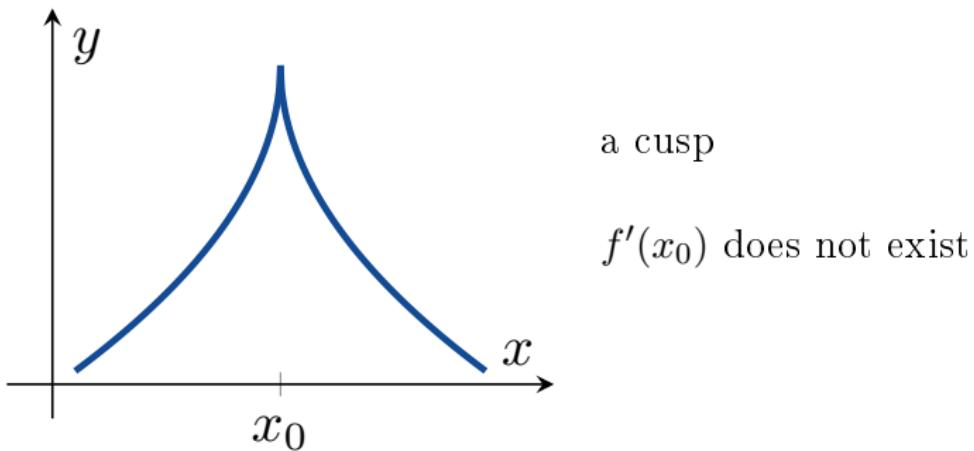
Since  $\lim_{h \rightarrow 0} \frac{|0+h| - |0|}{h} = \lim_{h \rightarrow 0} \frac{|h|}{h} = \lim_{h \rightarrow 0} (\pm 1)$  does not exist,  $f$  is not differentiable at 0.



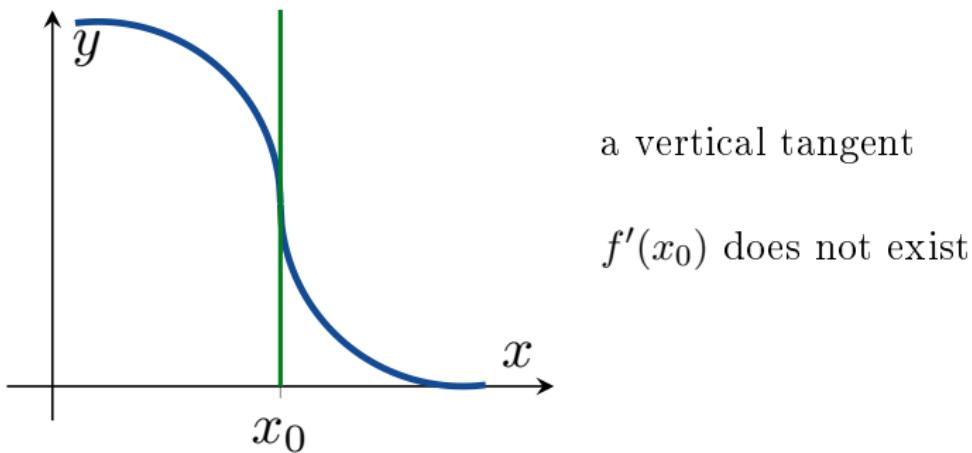
# When Does a Function Not Have a Derivative at a Point?



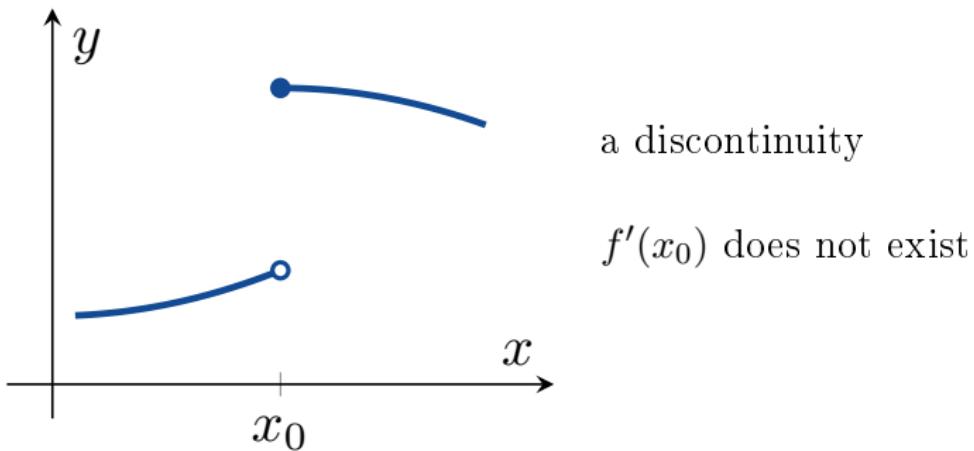
### When Does a Function Not Have a Derivative at a Point?



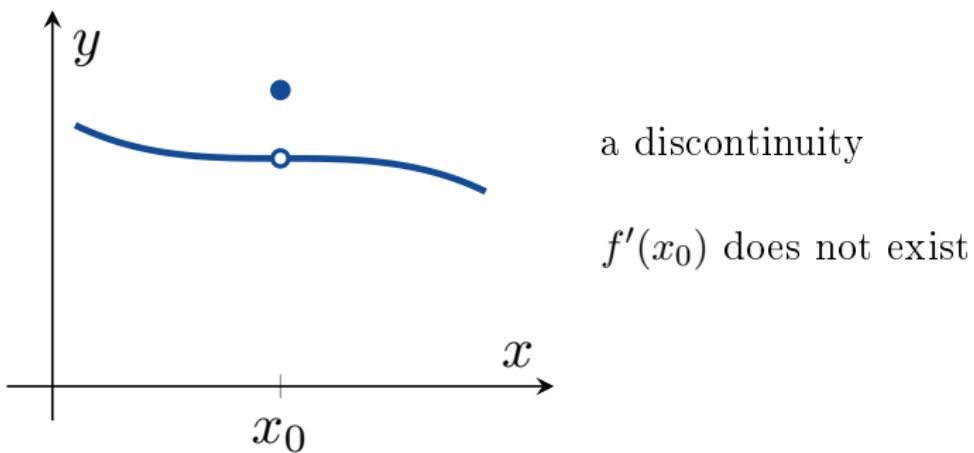
### When Does a Function Not Have a Derivative at a Point?



### When Does a Function Not Have a Derivative at a Point?



### When Does a Function Not Have a Derivative at a Point?



## 26. Differentiation



Theorem

$$\left( \begin{array}{c} f \text{ has a derivative} \\ \text{at } x = x_0 \end{array} \right) \implies \left( \begin{array}{c} f \text{ is continuous} \\ \text{at } x = x_0 \end{array} \right)$$



# Next Time

- 27. Differentiation Rules
- 28. Derivatives of Trigonometric Functions
- 29. The Chain Rule