Math 216

Final Exam : Solutions

1. Solve the following initial value problem.

$$\mathring{\mathbf{x}} = \begin{bmatrix} 2 & -5 \\ -4 & 1 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 2 \\ e^{3t} \end{bmatrix}, \quad \mathbf{x}(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

Solution : Let us calculate the characteristic polynomial.

$$\det\left(\begin{bmatrix} 2-\lambda & -5 \\ -4 & 1-\lambda \end{bmatrix}\right) = \lambda^2 - 3\lambda - 18 = (\lambda - 6)(\lambda + 3).$$

Thus, the eigenvalues are $\{6, -3\}$. Corresponding eigenvectors can be calculated as follows.

$$\begin{aligned} \mathbf{0} &= \begin{bmatrix} 2-6 & -5 \\ -4 & 1-6 \end{bmatrix} \mathbf{q}_1 = \begin{bmatrix} -4 & -5 \\ -4 & -5 \end{bmatrix} \mathbf{q}_1 \Longrightarrow \mathbf{q}_1 = \begin{bmatrix} 5 \\ -4 \end{bmatrix}, \\ \mathbf{0} &= \begin{bmatrix} 2+3 & -5 \\ -4 & 1+3 \end{bmatrix} \mathbf{q}_2 = \begin{bmatrix} 5 & -5 \\ -4 & 4 \end{bmatrix} \mathbf{q}_2 \Longrightarrow \mathbf{q}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}. \end{aligned}$$

The general solution of the homogeneous equation can be written as follows.

$$\mathbf{x}_H(t) = \begin{bmatrix} 5 \\ -4 \end{bmatrix} e^{6t} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{-3t}.$$

Note that a fundamental matrix solution is

$$\mathbf{W}(t) = \begin{bmatrix} 5 & 1 \\ -4 & 1 \end{bmatrix} \begin{bmatrix} e^{6t} & 0 \\ 0 & e^{-3t} \end{bmatrix} \Longrightarrow \mathbf{W}^{-1}(0) = \frac{1}{9} \begin{bmatrix} 1 & -1 \\ 4 & 5 \end{bmatrix}.$$

Consequently, $e^{\mathbf{A}t}$ can be calculated as follows.

$$\begin{aligned} e^{\mathbf{A}t} &= \mathbf{W}(t)\mathbf{W}^{-1}(0) = \frac{1}{9} \begin{bmatrix} 5e^{6t} & e^{-3t} \\ -4e^{6t} & e^{-3t} \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 4 & 5 \end{bmatrix}, \\ &= \frac{1}{9} \begin{bmatrix} 5e^{6t} + 4e^{-3t} & 5e^{-3t} - 5e^{6t} \\ -4e^{6t} + 4e^{-3t} & 4e^{6t} + 5e^{-3t} \end{bmatrix}. \end{aligned}$$

Then, the solution of the initial value problem is

$$\mathbf{x}(t) = e^{\mathbf{A}t} \left\{ \mathbf{x}(0) + \int_0^t e^{-\mathbf{A}\tau} \begin{bmatrix} 2 \\ e^{3\tau} \end{bmatrix} d\tau \right\}.$$

Since the inverse of $e^{\mathbf{A}t}$ is $e^{-\mathbf{A}t}$, we get

$$\begin{aligned} \mathbf{x}(t) &= \frac{1}{9} \begin{bmatrix} 5e^{6t} + 4e^{-3t} & 5e^{-3t} - 5e^{6t} \\ -4e^{6t} + 4e^{-3t} & 4e^{6t} + 5e^{-3t} \end{bmatrix} \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \frac{1}{9} \int_0^t \begin{bmatrix} 5e^{-6\tau} + 4e^{3\tau} & 5e^{3\tau} - 5e^{-6\tau} \\ -4e^{-6\tau} + 4e^{3\tau} & 4e^{-6\tau} + 5e^{3\tau} \end{bmatrix} \begin{bmatrix} 2 \\ e^{3\tau} \end{bmatrix} d\tau \right\}, \\ &= \frac{1}{9} \begin{bmatrix} 5e^{6t} + 4e^{-3t} & 5e^{-3t} - 5e^{6t} \\ -4e^{6t} + 4e^{-3t} & 4e^{6t} + 5e^{-3t} \end{bmatrix} \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \frac{1}{9} \int_0^t \begin{bmatrix} 10e^{-6\tau} + 8e^{3\tau} + 5e^{6\tau} - 5e^{-3\tau} \\ -8e^{-6\tau} + 8e^{3\tau} + 4e^{-3\tau} + 5e^{6\tau} \end{bmatrix} d\tau \right\}, \\ &= \frac{1}{9} \begin{bmatrix} 5e^{6t} + 4e^{-3t} & 5e^{-3t} - 5e^{6t} \\ -4e^{6t} + 4e^{-3t} & 4e^{6t} + 5e^{-3t} \end{bmatrix} \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \frac{1}{9} \begin{bmatrix} -\frac{10}{6}e^{-6\tau} + \frac{16}{6}e^{3\tau} + \frac{5}{6}e^{6\tau} + \frac{10}{6}e^{-3\tau} \\ \frac{8}{6}e^{-6\tau} + \frac{16}{6}e^{3\tau} - \frac{8}{6}e^{-3\tau} + \frac{5}{6}e^{6\tau} \end{bmatrix}_0^t \right\}, \\ &= \frac{1}{81} \begin{bmatrix} 5e^{6t} + 4e^{-3t} & 5e^{-3t} - 5e^{6t} \\ -4e^{6t} + 4e^{-3t} & 4e^{6t} + 5e^{-3t} \end{bmatrix} \begin{bmatrix} -\frac{10}{6}e^{-6t} + \frac{16}{6}e^{3t} + \frac{5}{6}e^{6t} + \frac{10}{6}e^{-3t} + \frac{11}{2} \\ \frac{8}{6}e^{-6t} + \frac{16}{6}e^{3t} - \frac{8}{6}e^{-3t} + \frac{5}{6}e^{6t} - \frac{7}{2} \end{bmatrix}, \\ &= \frac{1}{81} \begin{bmatrix} 45e^{6t} + \frac{135}{6}e^{3t} + \frac{9}{2}e^{-3t} + \frac{54}{6} \\ -36e^{6t} - \frac{27}{6}e^{3t} + \frac{9}{2}e^{-3t} + \frac{216}{6} \end{bmatrix} = \frac{1}{9} \begin{bmatrix} 5e^{6t} + \frac{5}{2}e^{3t} + \frac{1}{2}e^{-3t} + 1 \\ -4e^{6t} - \frac{1}{2}e^{3t} + \frac{1}{2}e^{-3t} + 4 \end{bmatrix}. \end{aligned}$$