Lecture 8

- Matrices for Linear Transformations
- Similarity

Matrices for Linear Transformations

Matrices for Linear Transformations

Let

- V be an n-dimensional vector space;
- $\mathcal{B}$ is a basis for V;
- W be an m-dimensional vector space;
- $\mathcal{C}$ is a basis for W; and
- $T : V \to W$ be a linear transformation.

a vector in V
(n-dimensional)

a vector in W
(m-dimensional)

$$\mathbf{x} \xrightarrow{T} T(\mathbf{x})$$

coordinates of $\mathbf{x}$
with respect to $\mathcal{B}$
(a vector in $\mathbb{R}^n$)

coordinates of $T(\mathbf{x})$
with respect to $\mathcal{C}$
(a vector in $\mathbb{R}^m$)

Matrices for Linear Transformations

a vector in V
(n-dimensional)

a vector in W
(m-dimensional)

$$\begin{tikzcd} \mathbf{x} \arrow[rr, "T"] \arrow[d, shift right] & & T(\mathbf{x}) \\ {[\mathbf{x}]_{\mathcal{B}}} \arrow[u, shift right] & & \end{tikzcd}$$

coordinates of $\mathbf{x}$
with respect to $\mathcal{B}$
(a vector in $\mathbb{R}^n$)

coordinates of $T(\mathbf{x})$
with respect to $\mathcal{C}$
(a vector in $\mathbb{R}^m$)

Matrices for Linear Transformations

a vector in V
(n-dimensional)

a vector in W
(m-dimensional)

$$\begin{tikzcd}
\mathbf{x} \arrow[rr, "T"] \arrow[d, shift right] & & T(\mathbf{x}) \arrow[d, shift right] \\
{[\mathbf{x}]_{\mathcal{B}}} \arrow[u, shift right] & & {[T(\mathbf{x})]_{\mathcal{C}}} \arrow[u, shift right]
\end{tikzcd}$$

coordinates of $\mathbf{x}$
with respect to $\mathcal{B}$
(a vector in $\mathbb{R}^n$)

coordinates of $T(\mathbf{x})$
with respect to $\mathcal{C}$
(a vector in $\mathbb{R}^m$)

Matrices for Linear Transformations

a vector in V
(n-dimensional)

a vector in W
(m-dimensional)

$$
\begin{tikzcd}
\mathbf{x} \arrow[r, "T"] \arrow[d, shift right] & T(\mathbf{x}) \arrow[d, shift right] \\
{[\mathbf{x}]_{\mathcal{B}}} \arrow[r, "A"] \arrow[u, shift right] & {[T(\mathbf{x})]_{\mathcal{C}}} \arrow[u, shift right]
\end{tikzcd}
$$

coordinates of $\mathbf{x}$
with respect to $\mathcal{B}$
(a vector in $\mathbb{R}^n$)

coordinates of $T(\mathbf{x})$
with respect to $\mathcal{C}$
(a vector in $\mathbb{R}^m$)

Matrices for Linear Transformations

a vector in V (n-dimensional)

a vector in W (m-dimensional)

$$
\begin{tikzcd}[column sep=huge, row sep=huge]
\mathbf{x} \arrow[r, "T"] \arrow[d, shift right] & T(\mathbf{x}) \arrow[d, shift right] \\
{[\mathbf{x}]_{\mathcal{B}}} \arrow[r, "A"] \arrow[u, shift right] & {[T(\mathbf{x})]_{\mathcal{C}}} \arrow[u, shift right]
\end{tikzcd}
$$

coordinates of $\mathbf{x}$ with respect to $\mathcal{B}$ (a vector in $\mathbb{R}^n$)

We want to find a matrix A such that $A[\mathbf{x}]_{\mathcal{B}} = [T(\mathbf{x})]_{\mathcal{C}}$

coordinates of $T(\mathbf{x})$ with respect to $\mathcal{C}$ (a vector in $\mathbb{R}^m$)

Matrices for Linear Transformations

a vector in V
(n-dimensional)

a vector in W
(m-dimensional)

$$
\begin{tikzcd}
\mathbf{x} \arrow[r, "T"] \arrow[d, shift right] & T(\mathbf{x}) \arrow[d, shift right] \\
{[\mathbf{x}]_{\mathcal{B}}} \arrow[r, "A"] \arrow[u, shift right] & {[T(\mathbf{x})]_{\mathcal{C}}} \arrow[u, shift right]
\end{tikzcd}
$$

coordinates of $\mathbf{x}$
with respect to $\mathcal{B}$
(a vector in $\mathbb{R}^n$)

coordinates of $T(\mathbf{x})$
with respect to $\mathcal{C}$
(a vector in $\mathbb{R}^m$)

$$T : V \to W$$

We want to find an $m \times n$ matrix A such that

$$A[\mathbf{x}]_{\mathcal{B}} = [T(\mathbf{x})]_{\mathcal{C}}.$$

Matrices for Linear Transformations

$$T : V \to W$$

We want to find an $m \times n$ matrix A such that

$$A[\mathbf{x}]_{\mathcal{B}} = [T(\mathbf{x})]_{\mathcal{C}}.$$

Let

- $\mathcal{B} = \{\mathbf{v}_1, \mathbf{v}_2, \dots \mathbf{v}_n\}$ be a basis for the n-dimensional vector space V; and
- $\mathcal{C} = \{\mathbf{w}_1, \mathbf{w}_2, \dots \mathbf{w}_m\}$ be a basis for the m-dimensional vector space W.

(This is on page 307 of your textbook.)

$$A[\mathbf{x}]_{\mathcal{B}} = [T(\mathbf{x})]_{\mathcal{C}}$$

We need A to satisfy

$$A[\mathbf{v}_1]_{\mathcal{B}} = [T(\mathbf{v}_1)]_{\mathcal{C}},\ A[\mathbf{v}_s]_{\mathcal{B}} = [T(\mathbf{v}_s)]_{\mathcal{C}},\ \ldots,\ A[\mathbf{v}_n]_{\mathcal{B}} = [T(\mathbf{v}_n)]_{\mathcal{C}}.$$

$$A[\mathbf{x}]_{\mathcal{B}} = [T(\mathbf{x})]_{\mathcal{C}}$$

We need A to satisfy

$A[\mathbf{v}_1]_{\mathcal{B}} = [T(\mathbf{v}_1)]_{\mathcal{C}},\ A[\mathbf{v}_s]_{\mathcal{B}} = [T(\mathbf{v}_s)]_{\mathcal{C}},\ \ldots,\ A[\mathbf{v}_n]_{\mathcal{B}} = [T(\mathbf{v}_n)]_{\mathcal{C}}.$

But $\mathbf{v}_1 = 1\mathbf{v}_1 + 0\mathbf{v}_2 + 0\mathbf{v}_3 + \ldots + 0\mathbf{v}_n$ so

$$[\mathbf{v}_1]_{\mathcal{B}} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

$$A[\mathbf{x}]_{\mathcal{B}} = [T(\mathbf{x})]_{\mathcal{C}}$$

We need A to satisfy

$A[\mathbf{v}_1]_{\mathcal{B}} = [T(\mathbf{v}_1)]_{\mathcal{C}},\ A[\mathbf{v}_s]_{\mathcal{B}} = [T(\mathbf{v}_s)]_{\mathcal{C}},\ \ldots,\ A[\mathbf{v}_n]_{\mathcal{B}} = [T(\mathbf{v}_n)]_{\mathcal{C}}.$

But $\mathbf{v}_1 = 1\mathbf{v}_1 + 0\mathbf{v}_2 + 0\mathbf{v}_3 + \ldots + 0\mathbf{v}_n$ so

$$[\mathbf{v}_1]_{\mathcal{B}} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

which implies that

$$[T(\mathbf{v}_1)]_{\mathcal{C}} = A[\mathbf{v}_1]_{\mathcal{B}} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} =$$

$$A[\mathbf{x}]_{\mathcal{B}} = [T(\mathbf{x})]_{\mathcal{C}}$$

We need A to satisfy

$A[\mathbf{v}_1]_{\mathcal{B}} = [T(\mathbf{v}_1)]_{\mathcal{C}},\ A[\mathbf{v}_s]_{\mathcal{B}} = [T(\mathbf{v}_s)]_{\mathcal{C}},\ \ldots,\ A[\mathbf{v}_n]_{\mathcal{B}} = [T(\mathbf{v}_n)]_{\mathcal{C}}.$

But $\mathbf{v}_1 = 1\mathbf{v}_1 + 0\mathbf{v}_2 + 0\mathbf{v}_3 + \ldots + 0\mathbf{v}_n$ so

$$[\mathbf{v}_1]_{\mathcal{B}} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

which implies that

$$[T(\mathbf{v}_1)]_{\mathcal{C}} = A[\mathbf{v}_1]_{\mathcal{B}} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} = \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix}.$$

$$A[\mathbf{x}]_{\mathcal{B}} = [T(\mathbf{x})]_{\mathcal{C}}$$

Similarly

$$[T(\mathbf{v}_2)]_{\mathcal{C}} = A[\mathbf{v}_2]_{\mathcal{B}} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} = \begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{bmatrix}.$$

$$\vdots$$

$$[T(\mathbf{v}_2)]_{\mathcal{C}} = A[\mathbf{v}_2]_{\mathcal{B}} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} = \begin{bmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{bmatrix}.$$

$$A[\mathbf{x}]_{\mathcal{B}} = [T(\mathbf{x})]_{\mathcal{C}}$$

So the $m \times n$ matrix that we want is

$$A = \begin{bmatrix} [T(\mathbf{v}_1)]_{\mathcal{C}} & [T(\mathbf{v}_2)]_{\mathcal{C}} & \cdots & [T(\mathbf{v}_n)]_{\mathcal{C}} \end{bmatrix}.$$

$$A[\mathbf{x}]_{\mathcal{B}} = [T(\mathbf{x})]_{\mathcal{C}}$$

So the $m \times n$ matrix that we want is

$$A = \begin{bmatrix} [T(\mathbf{v}_1)]_{\mathcal{C}} & [T(\mathbf{v}_2)]_{\mathcal{C}} & \cdots & [T(\mathbf{v}_n)]_{\mathcal{C}} \end{bmatrix}.$$

Definition

This matrix is called the *matrix for T relative to the bases $\mathcal{B}$ and $\mathcal{C}$* and is denoted by $[T]_{\mathcal{C},\mathcal{B}}$.

$$[T]_{\mathcal{C},\mathcal{B}} = \big[[T(\mathbf{v}_1)]_{\mathcal{C}} \quad [T(\mathbf{v}_2)]_{\mathcal{C}} \quad \cdots \quad [T(\mathbf{v}_n)]_{\mathcal{C}} \big]$$

Remark

The linear transformation

$$T : \quad \underset{\text{basis } \mathcal{B}}{V} \quad \to \quad \underset{\text{basis } \mathcal{C}}{W}$$

has matrix

$$[T]_{\mathcal{C},\mathcal{B}}.$$

Matrices for Linear Transformations

$$[T]_{\mathcal{C},\mathcal{B}} = \big[[T(\mathbf{v}_1)]_{\mathcal{C}} \quad [T(\mathbf{v}_2)]_{\mathcal{C}} \quad \cdots \quad [T(\mathbf{v}_n)]_{\mathcal{C}} \big]$$

Remark

The linear transformation

$$T: \quad \underset{\text{basis } \mathcal{B}}{V} \quad \to \quad \underset{\text{basis } \mathcal{C}}{W}$$

has matrix

$$[T]_{\mathcal{C},\mathcal{B}}.$$

We will use the formula

$$[T]_{\mathcal{C},\mathcal{B}}[\mathbf{x}]_{\mathcal{B}} = [T(\mathbf{x})]_{\mathcal{C}}.$$

$$[T]_{\mathcal{C},\mathcal{B}}[\mathbf{x}]_{\mathcal{B}} = [T(\mathbf{x})]_{\mathcal{C}}, \quad [T]_{\mathcal{C},\mathcal{B}} = \begin{bmatrix} [T(\mathbf{v}_1)]_{\mathcal{C}} & [T(\mathbf{v}_2)]_{\mathcal{C}} & \cdots & [T(\mathbf{v}_n)]_{\mathcal{C}} \end{bmatrix}$$

Example

I leave it for you to prove that if $T_A : \mathbb{R}^n \to \mathbb{R}^m$ is a matrix transformation ($T_A(\mathbf{x}) = A\mathbf{x}$) and if $\mathcal{B}$ and $\mathcal{C}$ are the standard bases, then

$$[T_A]_{\mathcal{C},\mathcal{B}} = A.$$

$[T]_{\mathcal{C},\mathcal{B}}[\mathbf{x}]_{\mathcal{B}} = [T(\mathbf{x})]_{\mathcal{C}}, \quad [T]_{\mathcal{C},\mathcal{B}} = \begin{bmatrix} [T(\mathbf{v}_1)]_{\mathcal{C}} & [T(\mathbf{v}_2)]_{\mathcal{C}} & \cdots & [T(\mathbf{v}_n)]_{\mathcal{C}} \end{bmatrix}$

Example

I leave it for you to prove that if $T_A : \mathbb{R}^n \to \mathbb{R}^m$ is a matrix transformation ($T_A(\mathbf{x}) = A\mathbf{x}$) and if $\mathcal{B}$ and $\mathcal{C}$ are the standard bases, then

$$[T_A]_{\mathcal{C},\mathcal{B}} = A.$$

Remark

If we are using the standard bases on both V and W, then we can write the matrix for T as just

$$[T].$$

This is called the *standard matrix* for T.

Example

Let $T : \mathbb{P}^1 \to \mathbb{P}^2$ be the linear transformation defined by

$$T(\mathbf{p}) = x\mathbf{p}.$$

(For example, if $\mathbf{p} = 1 + x$, then $T(\mathbf{p}) = x + x^2$.)

Find the matrix for T with respect to the bases

$$\mathcal{B} = \{\mathbf{v}_1 = 1, \mathbf{v}_2 = x\} \quad \text{and} \quad \mathcal{C} = \{\mathbf{w}_1 = 1, \mathbf{w}_2 = x, \mathbf{w}_3 = x^2\}.$$

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Find the matrix for T with respect to the bases

$$\mathcal{B} = \{\mathbf{v}_1 = 1, \mathbf{v}_2 = x\} \quad \text{and} \quad \mathcal{C} = \{\mathbf{w}_1 = 1, \mathbf{w}_2 = x, \mathbf{w}_3 = x^2\}.$$

First we calculate that

$$\begin{aligned} T(\mathbf{v}_1) &= T(1) = (x)(1) = x = 0\mathbf{w}_1 + 1\mathbf{w}_2 + 0\mathbf{w}_3 \\ T(\mathbf{v}_2) &= T(x) = (x)(x) = x^2 = 0\mathbf{w}_1 + 0\mathbf{w}_2 + 1\mathbf{w}_3. \end{aligned}$$

$$[T]_{\mathcal{C},\mathcal{B}}[\mathbf{x}]_{\mathcal{B}} = [T(\mathbf{x})]_{\mathcal{C}}, \quad [T]_{\mathcal{C},\mathcal{B}} = \begin{bmatrix} [T(\mathbf{v}_1)]_{\mathcal{C}} & [T(\mathbf{v}_2)]_{\mathcal{C}} & \cdots & [T(\mathbf{v}_n)]_{\mathcal{C}} \end{bmatrix}$$

$$T(\mathbf{v}_1) = T(1) = (x)(1) = x = 0\mathbf{w}_1 + 1\mathbf{w}_2 + 0\mathbf{w}_3$$
$$T(\mathbf{v}_2) = T(x) = (x)(x) = x^2 = 0\mathbf{w}_1 + 0\mathbf{w}_2 + 1\mathbf{w}_3.$$

Therefore the coordinates of $T(\mathbf{v}_1)$ and $T(\mathbf{v}_2)$ with respect to $\mathcal{C}$ are

$$[T(\mathbf{v}_1)]_{\mathcal{C}} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \qquad \text{and} \qquad [T(\mathbf{v}_2)]_{\mathcal{C}} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

[1] with respect to

$$[T]_{\mathcal{C},\mathcal{B}}[\mathbf{x}]_{\mathcal{B}} = [T(\mathbf{x})]_{\mathcal{C}}, \quad [T]_{\mathcal{C},\mathcal{B}} = \begin{bmatrix} [T(\mathbf{v}_1)]_{\mathcal{C}} & [T(\mathbf{v}_2)]_{\mathcal{C}} & \cdots & [T(\mathbf{v}_n)]_{\mathcal{C}} \end{bmatrix}$$

$$T(\mathbf{v}_1) = T(1) = (x)(1) = x = 0\mathbf{w}_1 + 1\mathbf{w}_2 + 0\mathbf{w}_3$$
$$T(\mathbf{v}_2) = T(x) = (x)(x) = x^2 = 0\mathbf{w}_1 + 0\mathbf{w}_2 + 1\mathbf{w}_3.$$

Therefore the coordinates of $T(\mathbf{v}_1)$ and $T(\mathbf{v}_2)$ with respect to $\mathcal{C}$ are

$$[T(\mathbf{v}_1)]_{\mathcal{C}} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \quad \text{and} \quad [T(\mathbf{v}_2)]_{\mathcal{C}} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

Hence the matrix for T wrt[1] $\mathcal{B}$ and $\mathcal{C}$ is

$$[T]_{\mathcal{C},\mathcal{B}} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

[1] with respect to

Example

Let $T : \mathbb{R}^2 \to \mathbb{R}^3$ be the linear transformation defined by

$$T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} x_2 \\ -5x_1 + 13x_2 \\ -7x_1 + 16x_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -5 & 13 \\ -7 & 16 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}.$$

Find the matrix for T wrt the bases $\mathcal{B} = \{\mathbf{v}_1, \mathbf{v}_2\}$ for $\mathbb{R}^2$, and $\mathcal{C} = \{\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3\}$ for $\mathbb{R}^3$, where

$$\mathbf{u}_1 = \begin{bmatrix} 3 \\ 1 \end{bmatrix}, \quad \mathbf{u}_2 = \begin{bmatrix} 5 \\ 2 \end{bmatrix}, \quad \mathbf{w}_1 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \quad \mathbf{w}_2 = \begin{bmatrix} -1 \\ 2 \\ 2 \end{bmatrix}, \quad \mathbf{w}_3 = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}.$$

$$T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} x_2 \\ -5x_1 + 13x_2 \\ -7x_1 + 16x_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -5 & 13 \\ -7 & 16 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Note, if we were using the standard bases on $\mathbb{R}^2$ and $\mathbb{R}^3$, then the answer would just be the standard matrix

$$[T] = \begin{bmatrix} 0 & 1 \\ -5 & 13 \\ -7 & 16 \end{bmatrix}.$$

But we are using different bases, so this is not the answer.

$$[T]_{\mathcal{C},\mathcal{B}}[\mathbf{x}]_{\mathcal{B}} = [T(\mathbf{x})]_{\mathcal{C}}, \quad [T]_{\mathcal{C},\mathcal{B}} = \begin{bmatrix} [T(\mathbf{v}_1)]_{\mathcal{C}} & [T(\mathbf{v}_2)]_{\mathcal{C}} & \cdots & [T(\mathbf{v}_n)]_{\mathcal{C}} \end{bmatrix}$$

$$T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} x_2 \\ -5x_1 + 13x_2 \\ -7x_1 + 16x_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -5 & 13 \\ -7 & 16 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$\mathbf{u}_1 = \begin{bmatrix} 3 \\ 1 \end{bmatrix}, \quad \mathbf{u}_2 = \begin{bmatrix} 5 \\ 2 \end{bmatrix}, \quad \mathbf{w}_1 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \quad \mathbf{w}_2 = \begin{bmatrix} -1 \\ 2 \\ 2 \end{bmatrix}, \quad \mathbf{w}_3 = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}$$

We start by calculating

$$T(\mathbf{v}_1) = \begin{bmatrix} 0 & 1 \\ -5 & 13 \\ -7 & 16 \end{bmatrix} \begin{bmatrix} 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \\ -5 \end{bmatrix} = \mathbf{w}_1 - 2\mathbf{w}_3$$

$$T(\mathbf{v}_2) = \begin{bmatrix} 0 & 1 \\ -5 & 13 \\ -7 & 16 \end{bmatrix} \begin{bmatrix} 5 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ -3 \end{bmatrix} = 3\mathbf{w}_1 + \mathbf{w}_2 - \mathbf{w}_3$$

(please check).

$$[T]_{\mathcal{C},\mathcal{B}}[\mathbf{x}]_{\mathcal{B}} = [T(\mathbf{x})]_{\mathcal{C}}, \quad [T]_{\mathcal{C},\mathcal{B}} = \begin{bmatrix} [T(\mathbf{v}_1)]_{\mathcal{C}} & [T(\mathbf{v}_2)]_{\mathcal{C}} & \cdots & [T(\mathbf{v}_n)]_{\mathcal{C}} \end{bmatrix}$$

$$T(\mathbf{v}_1) = \mathbf{w}_1 - 2\mathbf{w}_3 \qquad T(\mathbf{v}_2) = 3\mathbf{w}_1 + \mathbf{w}_2 - \mathbf{w}_3$$

Therefore

$$[T(\mathbf{v}_1)]_{\mathcal{C}} = \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix} \qquad \text{and} \qquad [T(\mathbf{v}_2)]_{\mathcal{C}} = \begin{bmatrix} 3 \\ 1 \\ -1 \end{bmatrix}.$$

Matrices for Linear Transformations

$$[T]_{\mathcal{C},\mathcal{B}}[\mathbf{x}]_{\mathcal{B}} = [T(\mathbf{x})]_{\mathcal{C}}, \quad [T]_{\mathcal{C},\mathcal{B}} = \begin{bmatrix} [T(\mathbf{v}_1)]_{\mathcal{C}} & [T(\mathbf{v}_2)]_{\mathcal{C}} & \cdots & [T(\mathbf{v}_n)]_{\mathcal{C}} \end{bmatrix}$$

$$T(\mathbf{v}_1) = \mathbf{w}_1 - 2\mathbf{w}_3 \qquad\qquad T(\mathbf{v}_2) = 3\mathbf{w}_1 + \mathbf{w}_2 - \mathbf{w}_3$$

Therefore

$$[T(\mathbf{v}_1)]_{\mathcal{C}} = \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix} \qquad \text{and} \qquad [T(\mathbf{v}_2)]_{\mathcal{C}} = \begin{bmatrix} 3 \\ 1 \\ -1 \end{bmatrix}.$$

Hence the matrix for T wrt $\mathcal{B}$ and $\mathcal{C}$ is

$$[T]_{\mathcal{C},\mathcal{B}} = \begin{bmatrix} [T(\mathbf{v}_1)]_{\mathcal{C}} & [T(\mathbf{v}_2)]_{\mathcal{C}} \end{bmatrix} = \begin{bmatrix} 1 & 3 \\ 0 & 1 \\ -2 & -1 \end{bmatrix}.$$

Remark

$$[T] = \begin{bmatrix} 0 & 1 \\ -5 & 13 \\ -7 & 16 \end{bmatrix} \quad \text{and} \quad [T]_{\mathcal{C},\mathcal{B}} = \begin{bmatrix} 1 & 3 \\ 0 & 1 \\ -2 & -1 \end{bmatrix}$$

are two different matrices for the same linear transformation.

Linear Operators

Recall that if $V = W$, then the linear transformation $T : V \to V$ is called a linear operator.

(If we are using the same basis $\mathcal{B}$ on both the domain and the target of T, then) instead of writing $[T]_{\mathcal{B},\mathcal{B}}$ we just write

$$[T]_{\mathcal{B}}.$$

Linear Operators

Recall that if $V = W$, then the linear transformation $T : V \to V$ is called a linear operator.

(If we are using the same basis $\mathcal{B}$ on both the domain and the target of T, then) instead of writing $[T]_{\mathcal{B},\mathcal{B}}$ we just write

$$[T]_{\mathcal{B}}.$$

The formulae are then

$$[T]_{\mathcal{B}} = \begin{bmatrix} [T(\mathbf{v}_1)]_{\mathcal{B}} & [T(\mathbf{v}_2)]_{\mathcal{B}} & \cdots & [T(\mathbf{v}_n)]_{\mathcal{B}} \end{bmatrix}$$

and

$$[T]_{\mathcal{B}}[\mathbf{x}]_{\mathcal{B}} = [T(\mathbf{x})]_{\mathcal{B}}.$$

Identity Operators

Example

Let V be a finite dimensional vector space and let $\mathcal{B} = \{\mathbf{v}_1, \mathbf{v}_2, \ldots \mathbf{v}_n\}$ be a basis for V. Consider the identity operator $I : V \to V$, $I(\mathbf{x}) = \mathbf{x}$.

Identity Operators

Example

Let V be a finite dimensional vector space and let $\mathcal{B} = \{\mathbf{v}_1, \mathbf{v}_2, \dots \mathbf{v}_n\}$ be a basis for V. Consider the identity operator $I : V \to V$, $I(\mathbf{x}) = \mathbf{x}$. Note that

$$I(\mathbf{v}_1) = \mathbf{v}_1, \qquad I(\mathbf{v}_2) = \mathbf{v}_2, \quad \dots, \quad I(\mathbf{v}_n) = \mathbf{v}_n.$$

Identity Operators

Example

Let V be a finite dimensional vector space and let $\mathcal{B} = \{\mathbf{v}_1, \mathbf{v}_2, \ldots \mathbf{v}_n\}$ be a basis for V. Consider the identity operator $I : V \to V$, $I(\mathbf{x}) = \mathbf{x}$. Note that

$$I(\mathbf{v}_1) = \mathbf{v}_1, \qquad I(\mathbf{v}_2) = \mathbf{v}_2, \quad \ldots, \quad I(\mathbf{v}_n) = \mathbf{v}_n.$$

Hence

$$[I]_{\mathcal{B}} = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix} = I_n.$$

Example

Let $T : \mathbb{P}^2 \to \mathbb{P}^2$ be the linear operator defined by

$$T(\mathbf{p}) = \mathbf{p}(3x - 5).$$

For example, if $\mathbf{p} = 2x + x^2$, then

$$T(\mathbf{p}) = \mathbf{p}(3x - 5) = 2(3x - 5) + (3x - 5)^2.$$

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$$T(\mathbf{p}) = \mathbf{p}(3x - 5) = 2(3x - 5) + (3x - 5)^2.$$

1. Find $[T]_{\mathcal{B}}$ where $\mathcal{B} = \{1, x, x^2\}$.
2. Use this to calculate $T(1 + 2x + 3x^2)$.

Matrices for Linear Transformations

1. Since

$$T(1) = 1, \qquad T(x) = (3x - 5), \qquad T(x^2) = (3x-5)^2 = 9x^2 - 30x +$$

we have

$$[T(1)]_{\mathcal{B}} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \qquad [T(x)]_{\mathcal{B}} = \begin{bmatrix} -5 \\ 3 \\ 0 \end{bmatrix}, \qquad [T(x^2)]_{\mathcal{B}} = \begin{bmatrix} 25 \\ -30 \\ 9 \end{bmatrix}.$$

Matrices for Linear Transformations

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$$T(1) = 1, \qquad T(x) = (3x-5), \qquad T(x^2) = (3x-5)^2 = 9x^2-30x+$$

we have

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Hence

$$[T]_{\mathcal{B}} = \begin{bmatrix} 1 & -5 & 25 \\ 0 & 3 & -30 \\ 0 & 0 & 9 \end{bmatrix}.$$

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2. The vector $\mathbf{p} = 1 + 2x + 3x^2$ has coordinate vector

$$[\mathbf{p}]_{\mathcal{B}} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}.$$

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$$[T(\mathbf{p})]_{\mathcal{B}} = [T]_{\mathcal{B}}[\mathbf{p}]_{\mathcal{B}} = \begin{bmatrix} 1 & -5 & 25 \\ 0 & 3 & -30 \\ 0 & 0 & 9 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 66 \\ -84 \\ 27 \end{bmatrix}.$$

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Therefore

$$T(1 + 2x + 3x^2) = 66 - 84x + 27x^2.$$

Compositions and Inverses

Theorem

If $T_1 : U \to V$ and $T_2 : V \to W$ are linear transformations, and if $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$ are bases for U, V, and W, respectively, then

$$[T_2 \circ T_1]_{\mathcal{C},\mathcal{A}} = [T_2]_{\mathcal{C},\mathcal{B}}[T_1]_{\mathcal{B},\mathcal{A}}.$$

Theorem

If $T : V \to V$ is a linear operator, and if $\mathcal{B}$ is a basis for V, then the following are equivalent:

1. *T is one-to-one.*
2. *$[T]_{\mathcal{B}}$ is invertible.*

Theorem

If $T : V \to V$ is a linear operator, and if $\mathcal{B}$ is a basis for V, then the following are equivalent:

1. *T is one-to-one.*
2. *$[T]_{\mathcal{B}}$ is invertible.*

Moreover, when these equivalent conditions hold,

$$[T^{-1}]_{\mathcal{B}} = [T]_{\mathcal{B}}^{-1}$$

Example (Composition)

Let $T_1 : \mathbb{P}^1 \to \mathbb{P}^2$ and $T_2 : \mathbb{P}^2 \to \mathbb{P}^2$ be defined by

$$T_1(\mathbf{p}) = x\mathbf{p} \qquad \text{and} \qquad T_2(\mathbf{p}) = \mathbf{p}(3x-5).$$

Find the matrix of $T_2 \circ T_1$ wrt to the standard bases on $\mathbb{P}^1$ and $\mathbb{P}^2$.

Let

$$\mathcal{B} = \{1, x\} \qquad \text{and} \qquad \mathcal{C} = \{1, x, x^2\}.$$

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$$\mathcal{B} = \{1, x\} \qquad \text{and} \qquad \mathcal{C} = \{1, x, x^2\}.$$

We have seen today that

$$[T_1]_{\mathcal{C},\mathcal{B}} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad \text{and} \qquad [T_2]_{\mathcal{C}} = \begin{bmatrix} 1 & -5 & 25 \\ 0 & 3 & -30 \\ 0 & 0 & 9 \end{bmatrix}.$$

It follows that

$$[T_2 \circ T_1]_{\mathcal{C},\mathcal{B}} = [T_2]_{\mathcal{C}}[T_1]_{\mathcal{C},\mathcal{B}} = \begin{bmatrix} 1 & -5 & 25 \\ 0 & 3 & -30 \\ 0 & 0 & 9 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} -5 & 25 \\ 3 & -30 \\ 0 & 9 \end{bmatrix}$$

Break

We will continue at 3pm

Similarity

Consider the matrix operator $T_A : \mathbb{R}^2 \to \mathbb{R}^2$ where

$$A = \begin{bmatrix} 1 & 1 \\ -2 & 4 \end{bmatrix}.$$

The matrix for T_A relative to the standard basis $\mathcal{B} = \{\mathbf{e}_1, \mathbf{e}_2\}$ for $\mathbb{R}^2$ is

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I want to calculate the matrix for the same operator T_A relative to a different basis.

$$A = \begin{bmatrix} 1 & 1 \\ -2 & 4 \end{bmatrix}$$

Namely, the basis $\mathcal{C} = \{\mathbf{v}_1, \mathbf{v}_2\}$ for $\mathbb{R}^2$ where

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \qquad \text{and} \qquad \mathbf{v}_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}.$$

$$A = \begin{bmatrix} 1 & 1 \\ -2 & 4 \end{bmatrix}$$

Namely, the basis $\mathcal{C} = \{\mathbf{v}_1, \mathbf{v}_2\}$ for $\mathbb{R}^2$ where

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \qquad \text{and} \qquad \mathbf{v}_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}.$$

Since

$$T_A(\mathbf{v}_1) = A\mathbf{v}_1 = \begin{bmatrix} 1 & 1 \\ -2 & 4 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \end{bmatrix} = 2\mathbf{v}_1$$

and

$$T_A(\mathbf{v}_2) = A\mathbf{v}_2 = \begin{bmatrix} 1 & 1 \\ -2 & 4 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 3 \\ 6 \end{bmatrix} = 3\mathbf{v}_2$$

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and

$$T_A(\mathbf{v}_2) = A\mathbf{v}_2 = \begin{bmatrix} 1 & 1 \\ -2 & 4 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 3 \\ 6 \end{bmatrix} = 3\mathbf{v}_2$$

it follows that

$$[T_A(\mathbf{v}_1)]_{\mathcal{C}} = \begin{bmatrix} 2 \\ 0 \end{bmatrix} \qquad \text{and} \qquad [T_A(\mathbf{v}_2)]_{\mathcal{C}} = \begin{bmatrix} 0 \\ 3 \end{bmatrix}.$$

Therefore the matrix for T_A relative to the basis $\mathcal{C}$ is

$$[T_A]_{\mathcal{C}} = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}.$$

Compare these two matrices

$$[T_A]_{\mathcal{B}} = \begin{bmatrix} 1 & 1 \\ -2 & 4 \end{bmatrix} \qquad [T_A]_{\mathcal{C}} = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}.$$

- The determinant of both matrices is 6.
- Both matrices are invertible.
- The trace of both matrices in 5.
- The latter matrix is diagonal and diagonal matrices are easier to deal with.

Definition

If A and B are square matrices, then we say that B is *similar* to A iff there exists an invertible matrix P such that

$$B = P^{-1}AP.$$

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Theorem

$$B \textit{ is similar to } A \quad \Longrightarrow \quad A \textit{ is similar to } B$$

Proof.

Suppose that $B = P^{-1}AP$. Let $Q = P^{-1}$.

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Theorem

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Proof.

Suppose that $B = P^{-1}AP$. Let $Q = P^{-1}$. Then $B = QAQ^{-1}$ which rearranges to $A = Q^{-1}BQ$. Hence A is similar to B. $\square$

Similarity

Theorem

If A and B are similar matrices, then

$$\det(A) = \det(B).$$

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$$\det(A) = \det(B).$$

Proof.

If $B = P^{-1}AP$, then

$$\begin{aligned}\det(B) &= \det(P^{-1}AP) = \det(P^{-1})\det(A)\det(P)\\ &= \frac{1}{\det(P)}\det(A)\det(P) = \det(A).\end{aligned}$$

$\square$

Similarity

Theorem

If A and B are square matrices of the same size, then

$$\mathrm{tr}(AB) = \mathrm{tr}(BA)$$

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If A and B are square matrices of the same size, then

$$\operatorname{tr}(AB) = \operatorname{tr}(BA)$$

Proof.

Recall that $\operatorname{tr}(AB)$ is the sum of the entries on the main diagonal of AB. If A and B are $n \times n$ matrices, then

$$\operatorname{tr}(AB) = \sum_{i=1}^{n} (AB)_{ii} = \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij} b_{ji}.$$

Similarly

$$\operatorname{tr}(BA) = \sum_{i=1}^{n} (BA)_{ii} = \sum_{i=1}^{n} \sum_{j=1}^{n} b_{ij} a_{ji}.$$

I leave if for you to show that these are the same. □

Theorem

If A and B are similar matrices, then

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Similarity

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If A and B are similar matrices, then

$$\operatorname{tr}(A) = \operatorname{tr}(B).$$

Proof.

If $B = P^{-1}AP$, then

$$\operatorname{tr}(B) = \operatorname{tr}(P^{-1}AP) = \operatorname{tr}(PP^{-1}A) = \operatorname{tr}(A).$$

$\square$

A New View of Transition Matrices

Recall that if $\mathcal{B} = \{\mathbf{b}_1, \ldots, \mathbf{b}_n\}$ and $\mathcal{C} = \{\mathbf{c}_1, \ldots, \mathbf{c}_n\}$ are two bases for a vector space V, then the *change-of-coordinates matrix* or *transition matrix* from $\mathcal{B}$ to $\mathcal{C}$, and from $\mathcal{C}$ to $\mathcal{B}$ are the matrices

$$\underset{\mathcal{C}\leftarrow\mathcal{B}}{P} = \begin{bmatrix} [\mathbf{b}_1]_{\mathcal{C}} & [\mathbf{b}_2]_{\mathcal{C}} & \cdots & [\mathbf{b}_n]_{\mathcal{C}} \end{bmatrix},$$

$$\underset{\mathcal{B}\leftarrow\mathcal{C}}{P} = \begin{bmatrix} [\mathbf{c}_1]_{\mathcal{B}} & [\mathbf{c}_2]_{\mathcal{B}} & \cdots & [\mathbf{c}_n]_{\mathcal{B}} \end{bmatrix}$$

which are inverses of each other. Moreover

$$\underset{\mathcal{C}\leftarrow\mathcal{B}}{P}[\mathbf{v}]_{\mathcal{B}} = [\mathbf{v}]_{\mathcal{C}} \quad \text{and} \quad \underset{\mathcal{B}\leftarrow\mathcal{C}}{P}[\mathbf{v}]_{\mathcal{C}} = [\mathbf{v}]_{\mathcal{B}}$$

Similarity

Theorem

If $\mathcal{B} = \{\mathbf{b}_1, \ldots, \mathbf{b}_n\}$ *and* $\mathcal{C} = \{\mathbf{c}_1, \ldots, \mathbf{c}_n\}$ *are bases for a finite-dimensional vector space* V*, and if* $I : V \to V$ *is the identity operator on* V*, then*

$$\underset{\mathcal{C}\leftarrow\mathcal{B}}{P} = [I]_{\mathcal{C},\mathcal{B}} \qquad \text{and} \qquad \underset{\mathcal{B}\leftarrow\mathcal{C}}{P} = [I]_{\mathcal{B},\mathcal{C}}.$$

Proof.

Since $I(\mathbf{v}) = \mathbf{v}$ for all $\mathbf{v} \in V$, we have that

$$[I]_{\mathcal{C},\mathcal{B}} = \begin{bmatrix} [I(\mathbf{b}_1)]_{\mathcal{C}} & [I(\mathbf{b}_2)]_{\mathcal{C}} & \cdots [I(\mathbf{b}_n)]_{\mathcal{C}} \end{bmatrix}$$

$$= \begin{bmatrix} [\mathbf{b}_1]_{\mathcal{C}} & [\mathbf{b}_2]_{\mathcal{C}} & \cdots [\mathbf{b}_n]_{\mathcal{C}} \end{bmatrix} = \underset{\mathcal{C}\leftarrow\mathcal{B}}{P}.$$

Similarly $\underset{\mathcal{B}\leftarrow\mathcal{C}}{P} = [I]_{\mathcal{B},\mathcal{C}}$. $\square$

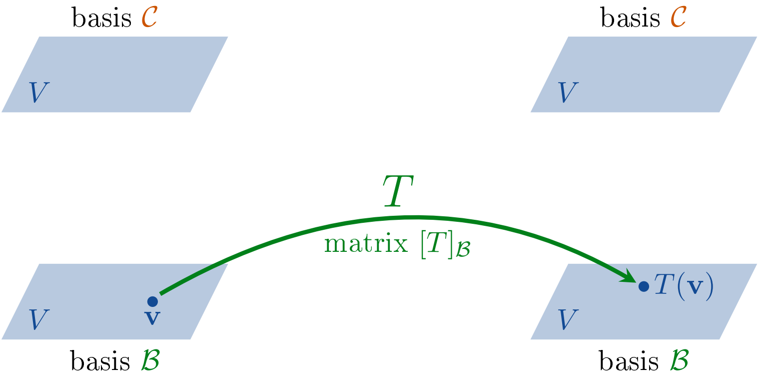
The Effect of Changing Bases on Matrices of Linear Operators

Question: If $\mathcal{B}$ and $\mathcal{C}$ are two bases for a finite-dimensional vector space V, and if $T : V \to V$ is a linear operator, what relationship, if any, exists between the matrices $[T]_{\mathcal{B}}$ and $[T]_{\mathcal{C}}$?

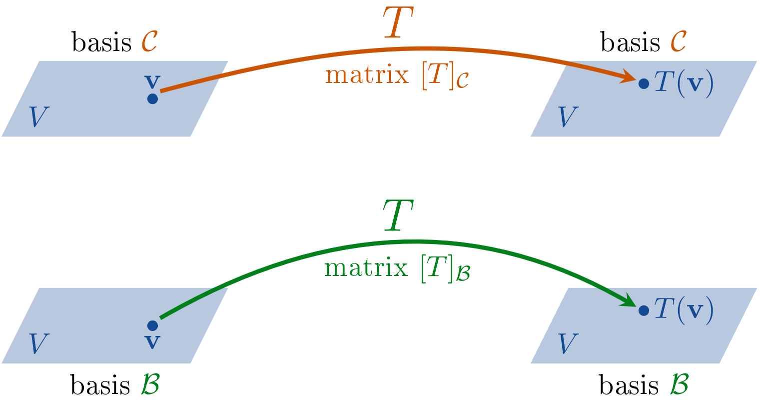
Similarity


basis $\mathcal{C}$
V
basis $\mathcal{C}$
V
V
basis $\mathcal{B}$
V
basis $\mathcal{B}$

Similarity

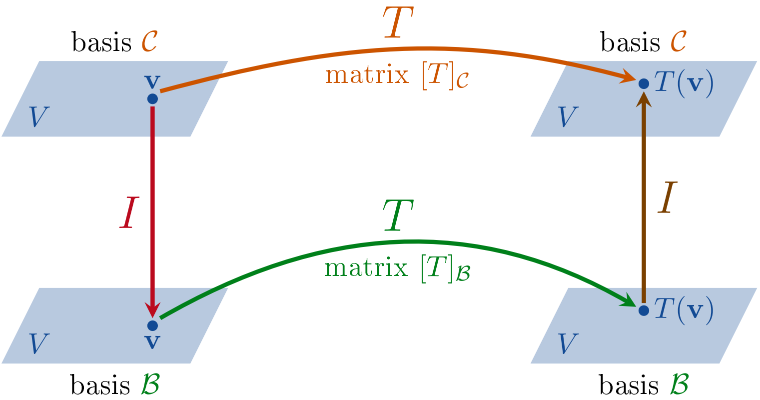

basis $\mathcal{C}$
basis $\mathcal{C}$
V
V
T
matrix $[T]_{\mathcal{B}}$
$T(\mathbf{v})$
V
$\mathbf{v}$
V
basis $\mathcal{B}$
basis $\mathcal{B}$

Similarity

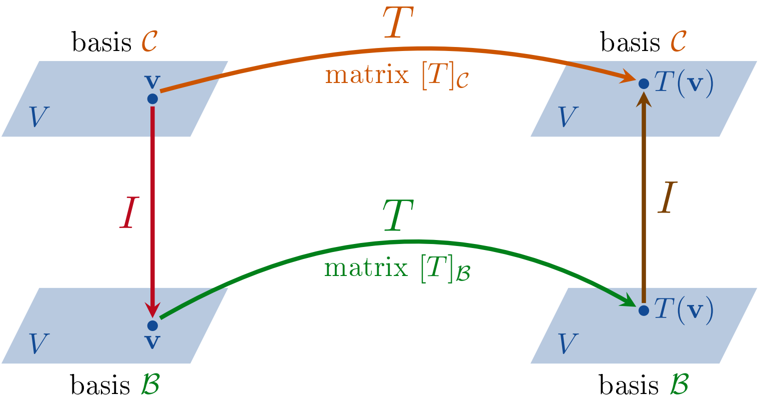


basis $\mathcal{C}$
T
basis $\mathcal{C}$
$\mathbf{v}$
matrix $[T]_{\mathcal{C}}$
$T(\mathbf{v})$
V
V
T
matrix $[T]_{\mathcal{B}}$
$T(\mathbf{v})$
V
$\mathbf{v}$
V
basis $\mathcal{B}$
basis $\mathcal{B}$

Similarity



Note that $T = I \circ T \circ I$.



Note that $T = I \circ T \circ I$. It follows that

$$[T]_\mathcal{C} = [I \circ T \circ I]_\mathcal{C} = [I]_{\mathcal{C},\mathcal{B}}[T]_\mathcal{B}[I]_{\mathcal{B},\mathcal{C}}.$$

$$[T]_{\mathcal{C}} = [I \circ T \circ I]_{\mathcal{C}} = [I]_{\mathcal{C},\mathcal{B}}[T]_{\mathcal{B}}[I]_{\mathcal{B},\mathcal{C}}.$$

Since

$$\underset{\mathcal{C}\leftarrow\mathcal{B}}{P} = [I]_{\mathcal{C},\mathcal{B}} \qquad \text{and} \qquad \underset{\mathcal{B}\leftarrow\mathcal{C}}{P} = [I]_{\mathcal{B},\mathcal{C}}$$

we have

Theorem

Let $T : V \to V$ be a linear operator on a finite-dimensional vector space V, and let $\mathcal{B}$ and $\mathcal{C}$ be bases for V. Then

$$[T]_{\mathcal{C}} = P^{-1}[T]_{\mathcal{B}}P$$

where $P = \underset{\mathcal{B}\leftarrow\mathcal{C}}{P}$ and $P^{-1} = \underset{\mathcal{C}\leftarrow\mathcal{B}}{P}$.

Remark

Therefore, two matrices representing the same linear operator, must be similar.

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Theorem

Two matrices A and B represent the same linear operator $\iff$ *A and B are similar.*

Next Time

- Eigenvalues and Eigenvectors
- Diagonalization
- Complex Vector Spaces