

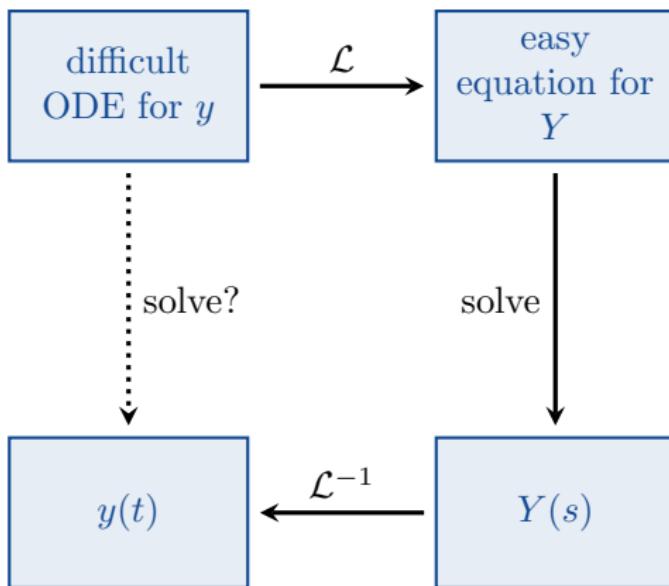
Lecture 8

- 4.3 Solving More Initial Value Problems
- 4.4 Step Functions



Solving More Initial Value Problems

4.3 Solving More Initial Value Problems



4.3 Solving More Initial Value Problems



Theorem

1 $\mathcal{L}[f'](s) = s\mathcal{L}[f](s) - f(0).$

4.3 Solving More Initial Value Problems



Theorem

- 1 $\mathcal{L}[f'](s) = s\mathcal{L}[f](s) - f(0).$
- 2 $\mathcal{L}[f''](s) = s^2\mathcal{L}[f](s) - sf(0) - f'(0).$
- 3 $\mathcal{L}[f'''](s) = s^3\mathcal{L}[f](s) - s^2f(0) - sf'(0) - f''(0).$
- 4 $\mathcal{L}[f^{(n)}](s) = s^n\mathcal{L}[f](s) - s^{n-1}f(0) - s^{n-2}f'(0) - \dots - sf^{(n-2)}(0) - f^{(n-1)}(0).$

4.3 Solving More Initial Value Problems



Example

Use the Laplace Transform to solve

$$\begin{cases} y'' - 3y' + 2y = \cos t \\ y(0) = 0 \\ y'(0) = 0. \end{cases}$$

4.3 Solving More Initial Value Problems



Example

Use the Laplace Transform to solve

$$\begin{cases} y'' - 3y' + 2y = \cos t \\ y(0) = 0 \\ y'(0) = 0. \end{cases}$$

Taking the Laplace Transform of the ODE gives

$$\begin{aligned} \mathcal{L}[y''] - 3\mathcal{L}[y'] + 2\mathcal{L}[y] &= \mathcal{L}[\cos t] \\ (s^2Y - sy(0) - y'(0)) - 3(sY - y(0)) + 2Y &= \frac{s}{s^2 + 1} \\ (s^2 - 3s + 2)Y &= \frac{s}{s^2 + 1} \end{aligned}$$

4.3 Solving More Initial Value Problems



$$Y(s) = \frac{s}{(s^2 + 1)(s^2 - 3s + 2)} = \frac{s}{(s^2 + 1)(s - 2)(s - 1)}$$

=

=

=

=

=

4.3 Solving More Initial Value Problems



$$Y(s) = \frac{s}{(s^2 + 1)(s^2 - 3s + 2)} = \frac{s}{(s^2 + 1)(s - 2)(s - 1)}$$
$$= \frac{As + B}{s^2 + 1} + \frac{C}{s - 2} + \frac{D}{s - 1}$$

=

=

=

=

4.3 Solving More Initial Value Problems



$$\begin{aligned}Y(s) &= \frac{s}{(s^2 + 1)(s^2 - 3s + 2)} = \frac{s}{(s^2 + 1)(s - 2)(s - 1)} \\&= \frac{As + B}{s^2 + 1} + \frac{C}{s - 2} + \frac{D}{s - 1} \\&= \frac{(As + B)(s - 2)(s - 1) + C(s^2 + 1)(s - 1) + D(s^2 + 1)(s - 2)}{(s^2 + 1)(s - 2)(s - 1)}\end{aligned}$$

=

=

=

4.3 Solving More Initial Value Problems



$$\begin{aligned}Y(s) &= \frac{s}{(s^2 + 1)(s^2 - 3s + 2)} = \frac{s}{(s^2 + 1)(s - 2)(s - 1)} \\&= \frac{As + B}{s^2 + 1} + \frac{C}{s - 2} + \frac{D}{s - 1} \\&= \frac{(As + B)(s - 2)(s - 1) + C(s^2 + 1)(s - 1) + D(s^2 + 1)(s - 2)}{(s^2 + 1)(s - 2)(s - 1)} \\&\quad (A = \frac{1}{10}, B = -\frac{3}{10}, C = \frac{2}{5}, D = -\frac{1}{2})\end{aligned}$$

=

=

=

4.3 Solving More Initial Value Problems



$$\begin{aligned}Y(s) &= \frac{s}{(s^2 + 1)(s^2 - 3s + 2)} = \frac{s}{(s^2 + 1)(s - 2)(s - 1)} \\&= \frac{As + B}{s^2 + 1} + \frac{C}{s - 2} + \frac{D}{s - 1} \\&= \frac{(As + B)(s - 2)(s - 1) + C(s^2 + 1)(s - 1) + D(s^2 + 1)(s - 2)}{(s^2 + 1)(s - 2)(s - 1)}\end{aligned}$$

$$(A = \frac{1}{10}, B = -\frac{3}{10}, C = \frac{2}{5}, D = -\frac{1}{2})$$

$$= \frac{\frac{1}{10}s - \frac{3}{10}}{s^2 + 1} + \frac{\frac{2}{5}}{s - 2} - \frac{\frac{1}{2}}{s - 1}$$

=

=

4.3 Solving More Initial Value Problems



$$\begin{aligned}Y(s) &= \frac{s}{(s^2 + 1)(s^2 - 3s + 2)} = \frac{s}{(s^2 + 1)(s - 2)(s - 1)} \\&= \frac{As + B}{s^2 + 1} + \frac{C}{s - 2} + \frac{D}{s - 1} \\&= \frac{(As + B)(s - 2)(s - 1) + C(s^2 + 1)(s - 1) + D(s^2 + 1)(s - 2)}{(s^2 + 1)(s - 2)(s - 1)} \\&\quad (A = \frac{1}{10}, B = -\frac{3}{10}, C = \frac{2}{5}, D = -\frac{1}{2}) \\&= \frac{\frac{1}{10}s - \frac{3}{10}}{s^2 + 1} + \frac{\frac{2}{5}}{s - 2} - \frac{\frac{1}{2}}{s - 1} \\&= \frac{1}{10} \left(\frac{s}{s^2 + 1} \right) - \frac{3}{10} \left(\frac{1}{s^2 + 1} \right) + \frac{2}{5} \left(\frac{1}{s - 2} \right) - \frac{1}{2} \left(\frac{1}{s - 1} \right) \\&=\end{aligned}$$

4.3 Solving More Initial Value Problems



$$\begin{aligned}Y(s) &= \frac{s}{(s^2 + 1)(s^2 - 3s + 2)} = \frac{s}{(s^2 + 1)(s - 2)(s - 1)} \\&= \frac{As + B}{s^2 + 1} + \frac{C}{s - 2} + \frac{D}{s - 1} \\&= \frac{(As + B)(s - 2)(s - 1) + C(s^2 + 1)(s - 1) + D(s^2 + 1)(s - 2)}{(s^2 + 1)(s - 2)(s - 1)} \\&\quad (A = \frac{1}{10}, B = -\frac{3}{10}, C = \frac{2}{5}, D = -\frac{1}{2}) \\&= \frac{\frac{1}{10}s - \frac{3}{10}}{s^2 + 1} + \frac{\frac{2}{5}}{s - 2} - \frac{\frac{1}{2}}{s - 1} \\&= \frac{1}{10} \left(\frac{s}{s^2 + 1} \right) - \frac{3}{10} \left(\frac{1}{s^2 + 1} \right) + \frac{2}{5} \left(\frac{1}{s - 2} \right) - \frac{1}{2} \left(\frac{1}{s - 1} \right) \\&= \frac{1}{10}\mathcal{L}[\cos t] - \frac{3}{10}\mathcal{L}[\sin t] + \frac{2}{5}\mathcal{L}[e^{2t}] - \frac{1}{2}\mathcal{L}[e^t].\end{aligned}$$

4.3 Solving More Initial Value Problems



$$Y(s) = \frac{1}{10}\mathcal{L}[\cos t] - \frac{3}{10}\mathcal{L}[\sin t] + \frac{2}{5}\mathcal{L}[e^{2t}] - \frac{1}{2}\mathcal{L}[e^t]$$

Therefore the solution to the IVP is

$$y(t) = \mathcal{L}^{-1}[Y](t) =$$

4.3 Solving More Initial Value Problems



$$Y(s) = \frac{1}{10}\mathcal{L}[\cos t] - \frac{3}{10}\mathcal{L}[\sin t] + \frac{2}{5}\mathcal{L}[e^{2t}] - \frac{1}{2}\mathcal{L}[e^t]$$

Therefore the solution to the IVP is

$$y(t) = \mathcal{L}^{-1}[Y](t) = \frac{1}{10}\cos t - \frac{3}{10}\sin t + \frac{2}{5}e^{2t} - \frac{1}{2}e^t.$$

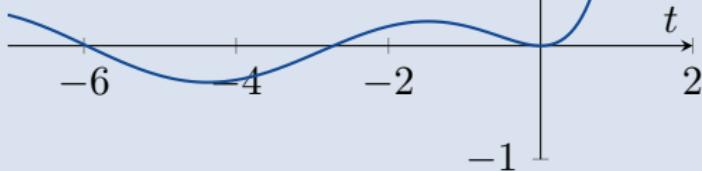
4.3 Solving More Initial Value Problems



$$Y(s) = \frac{1}{10}\mathcal{L}[\cos t] - \frac{3}{10}\mathcal{L}[\sin t] + \frac{2}{5}\mathcal{L}[e^{2t}] - \frac{1}{2}\mathcal{L}[e^t]$$

Therefore the solution to the IVP is

$$y(t) = \mathcal{L}^{-1}[Y](t) = \frac{1}{10} \cos t - \frac{3}{10} \sin t + \frac{2}{5} e^{2t} - \frac{1}{2} e^t.$$



4.3 Solving More Initial Value Problems



Example

Use the Laplace Transform to solve

$$\begin{cases} y'' + 2y' + y = 4e^{-t} \\ y(0) = 2 \\ y'(0) = -1. \end{cases}$$

4.3 Solving More Initial Value Problems



Example

Use the Laplace Transform to solve

$$\begin{cases} y'' + 2y' + y = 4e^{-t} \\ y(0) = 2 \\ y'(0) = -1. \end{cases}$$

$$y'' + 2y' + y = 4e^{-t}$$

4.3 Solving More Initial Value Problems



Example

Use the Laplace Transform to solve

$$\begin{cases} y'' + 2y' + y = 4e^{-t} \\ y(0) = 2 \\ y'(0) = -1. \end{cases}$$

$$\mathcal{L}[y''] + 2\mathcal{L}[y'] + \mathcal{L}[y] = \mathcal{L}[4e^{-t}]$$

4.3 Solving More Initial Value Problems



Example

Use the Laplace Transform to solve

$$\begin{cases} y'' + 2y' + y = 4e^{-t} \\ y(0) = 2 \\ y'(0) = -1. \end{cases}$$

$$\mathcal{L}[y''] + 2\mathcal{L}[y'] + Y = \frac{4}{s+1}$$

4.3 Solving More Initial Value Problems



Example

Use the Laplace Transform to solve

$$\begin{cases} y'' + 2y' + y = 4e^{-t} \\ y(0) = 2 \\ y'(0) = -1. \end{cases}$$

$$\mathcal{L}[y''] + 2(sY - y(0)) + Y = \frac{4}{s+1}$$

4.3 Solving More Initial Value Problems



Example

Use the Laplace Transform to solve

$$\begin{cases} y'' + 2y' + y = 4e^{-t} \\ y(0) = 2 \\ y'(0) = -1. \end{cases}$$

$$(s^2Y - sy(0) - y'(0)) + 2(sY - y(0)) + Y = \frac{4}{s+1}$$

4.3 Solving More Initial Value Problems



Example

Use the Laplace Transform to solve

$$\begin{cases} y'' + 2y' + y = 4e^{-t} \\ y(0) = 2 \\ y'(0) = -1. \end{cases}$$

$$(s^2Y - s\textcolor{red}{y}(0) - \textcolor{red}{y}'(0)) + 2(sY - \textcolor{red}{y}(0)) + Y = \frac{4}{s+1}$$

4.3 Solving More Initial Value Problems



Example

Use the Laplace Transform to solve

$$\begin{cases} y'' + 2y' + y = 4e^{-t} \\ y(0) = 2 \\ y'(0) = -1. \end{cases}$$

$$(s^2Y - 2s + 1) + 2(sY - 2) + Y = \frac{4}{s+1}$$

4.3 Solving More Initial Value Problems



Example

Use the Laplace Transform to solve

$$\begin{cases} y'' + 2y' + y = 4e^{-t} \\ y(0) = 2 \\ y'(0) = -1. \end{cases}$$

$$(s^2 + 2s + 1)Y - 2s + 1 - 4 = \frac{4}{s + 1}$$

4.3 Solving More Initial Value Problems



Example

Use the Laplace Transform to solve

$$\begin{cases} y'' + 2y' + y = 4e^{-t} \\ y(0) = 2 \\ y'(0) = -1. \end{cases}$$

$$(s^2 + 2s + 1)Y = \frac{4}{s+1} + 2s + 3$$

4.3 Solving More Initial Value Problems



Example

Use the Laplace Transform to solve

$$\begin{cases} y'' + 2y' + y = 4e^{-t} \\ y(0) = 2 \\ y'(0) = -1. \end{cases}$$

$$(s+1)^2 Y = \frac{4}{s+1} + 2s + 3$$

4.3 Solving More Initial Value Problems



Example

Use the Laplace Transform to solve

$$\begin{cases} y'' + 2y' + y = 4e^{-t} \\ y(0) = 2 \\ y'(0) = -1. \end{cases}$$

$$(s+1)^2 Y = \frac{2s^2 + 5s + 7}{s+1}$$

4.3 Solving More Initial Value Problems



Example

Use the Laplace Transform to solve

$$\begin{cases} y'' + 2y' + y = 4e^{-t} \\ y(0) = 2 \\ y'(0) = -1. \end{cases}$$

$$Y = \frac{2s^2 + 5s + 7}{(s + 1)^3}$$

4.3 Solving More Initial Value Problems



Example

Use the Laplace Transform to solve

$$\begin{cases} y'' + 2y' + y = 4e^{-t} \\ y(0) = 2 \\ y'(0) = -1. \end{cases}$$

$$y(t) = \mathcal{L}^{-1} \left[\frac{2s^2 + 5s + 7}{(s+1)^3} \right]$$

4.3 Solving More Initial Value Problems



I leave it for you to check that if

$$\frac{2s^2 + 5s + 7}{(s+1)^3} = \frac{A}{s+1} + \frac{B}{(s+1)^2} + \frac{C}{(s+1)^3}$$

then $A = 2$, $B = 1$ and $C = 4$.

4.3 Solving More Initial Value Problems



I leave it for you to check that if

$$\frac{2s^2 + 5s + 7}{(s+1)^3} = \frac{A}{s+1} + \frac{B}{(s+1)^2} + \frac{C}{(s+1)^3}$$

then $A = 2$, $B = 1$ and $C = 4$.

Thus

$$\frac{2s^2 + 5s + 7}{(s+1)^3} = \frac{2}{s+1} + \frac{1}{(s+1)^2} + \frac{4}{(s+1)^3}$$

4.3 Solving More Initial Value Problems



I leave it for you to check that if

$$\frac{2s^2 + 5s + 7}{(s+1)^3} = \frac{A}{s+1} + \frac{B}{(s+1)^2} + \frac{C}{(s+1)^3}$$

then $A = 2$, $B = 1$ and $C = 4$.

Thus

$$\begin{aligned}\frac{2s^2 + 5s + 7}{(s+1)^3} &= \frac{2}{s+1} + \frac{1}{(s+1)^2} + \frac{4}{(s+1)^3} \\ &= 2\left(\frac{1}{s+1}\right) + \left(\frac{1}{(s+1)^2}\right) + 2\left(\frac{2}{(s+1)^3}\right).\end{aligned}$$

4.3 Solving More Initial Value Problems



I leave it for you to check that if

$$\frac{2s^2 + 5s + 7}{(s+1)^3} = \frac{A}{s+1} + \frac{B}{(s+1)^2} + \frac{C}{(s+1)^3}$$

then $A = 2$, $B = 1$ and $C = 4$.

Thus

$$\begin{aligned}\frac{2s^2 + 5s + 7}{(s+1)^3} &= \frac{2}{s+1} + \frac{1}{(s+1)^2} + \frac{4}{(s+1)^3} \\ &= 2\left(\frac{1}{s+1}\right) + \left(\frac{1}{(s+1)^2}\right) + 2\left(\frac{2}{(s+1)^3}\right).\end{aligned}$$

In our table of Laplace Transforms, we find that $\mathcal{L}[e^{-t}] = \frac{1}{s+1}$, $\mathcal{L}[te^{-t}] = \frac{1}{(s+1)^2}$ and $\mathcal{L}[t^2e^{-t}] = \frac{2}{(s+1)^3}$.

4.3 Solving More Initial Value Problems



Therefore the solution to the IVP is

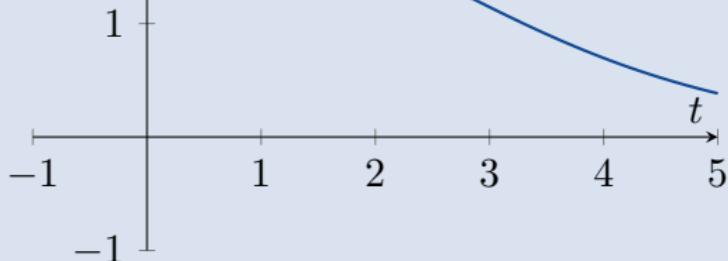
$$\begin{aligned}y(t) &= \mathcal{L}^{-1} \left[\frac{2s^2 + 5s + 7}{(s+1)^3} \right] \\&= 2\mathcal{L}^{-1} \left[\frac{1}{s+1} \right] + \mathcal{L}^{-1} \left[\frac{1}{(s+1)^2} \right] + 2\mathcal{L}^{-1} \left[\frac{2}{(s+1)^3} \right] \\&= 2(e^{-t}) + (te^{-t}) + 2(t^2e^{-t}) \\&= \boxed{(2t^2 + t + 2)e^{-t}.}\end{aligned}$$

4.3 Solving More Initial Value Problems



Therefore the solution to the IVP is

$$\begin{aligned}y(t) &= \mathcal{L}^{-1} \left[\frac{2s^2 + 5s + 7}{(s+1)^3} \right] \\&= 2\mathcal{L}^{-1} \left[\frac{1}{s+1} \right] + \mathcal{L}^{-1} \left[\frac{1}{(s+1)^2} \right] + 2\mathcal{L}^{-1} \left[\frac{2}{(s+1)^3} \right] \\&= 2(e^{-t}) + (te^{-t}) + (2t^2 + t + 2)e^{-t} \\&= (2t^2 + t + 2)e^{-t}.\end{aligned}$$



4.3 Solving More Initial Value Problems



Example

Use the Laplace Transform to solve

$$\begin{cases} y^{(4)} + 2y'' + y = e^{2t} \\ y(0) = 1 \\ y'(0) = 1 \\ y''(0) = 1 \\ y'''(0) = 1. \end{cases}$$

4.3 Solving More Initial Value Problems



$$y^{(4)} + 2y'' + y = e^{2t} \quad y(0) = y'(0) = y''(0) = y^{(3)}(0) = 1$$

4.3 Solving More Initial Value Problems



$$y^{(4)} + 2y'' + y = e^{2t} \quad y(0) = y'(0) = y''(0) = y^{(3)}(0) = 1$$

Taking the Laplace Transform of the ODE gives

$$\mathcal{L}[y^{(4)}] + 2\mathcal{L}[y''] + \mathcal{L}[y] = \mathcal{L}[e^{2t}].$$

4.3 Solving More Initial Value Problems



$$y^{(4)} + 2y'' + y = e^{2t} \quad y(0) = y'(0) = y''(0) = y^{(3)}(0) = 1$$

Taking the Laplace Transform of the ODE gives

$$\mathcal{L}[y^{(4)}] + 2\mathcal{L}[y''] + \mathcal{L}[y] = \mathcal{L}[e^{2t}].$$

Thus

$$\begin{aligned} & \left(s^4 Y(s) - s^3 y(0) - s^2 y'(0) - s y''(0) - y^{(3)}(0) \right) \\ & + 2(s^2 Y(s) - s y(0) - y'(0)) + Y(s) = \frac{1}{s-2} \end{aligned}$$

4.3 Solving More Initial Value Problems



$$y^{(4)} + 2y'' + y = e^{2t} \quad y(0) = y'(0) = y''(0) = y^{(3)}(0) = 1$$

Taking the Laplace Transform of the ODE gives

$$\mathcal{L}[y^{(4)}] + 2\mathcal{L}[y''] + \mathcal{L}[y] = \mathcal{L}[e^{2t}].$$

Thus

$$\begin{aligned} & \left(s^4 Y(s) - s^3 y(0) - s^2 y'(0) - s y''(0) - y^{(3)}(0) \right) \\ & + 2(s^2 Y(s) - s y(0) - y'(0)) + Y(s) = \frac{1}{s-2} \end{aligned}$$

and

$$(s^4 Y(s) - s^3 - s^2 - s - 1) + 2(s^2 Y(s) - s - 1) + Y(s) = \frac{1}{s-2}.$$

4.3 Solving More Initial Value Problems



Thus

$$(s^4 + 2s^2 + 1) Y(s) - s^3 - s^2 - s - 1 - 2s - 2 = \frac{1}{s - 2}.$$

4.3 Solving More Initial Value Problems



Thus

$$(s^4 + 2s^2 + 1) Y(s) - s^3 - s^2 - s - 1 - 2s - 2 = \frac{1}{s - 2}.$$

Hence

$$(s^4 + 2s^2 + 1) Y(s) = \frac{1}{s - 2} + \textcolor{brown}{s^3} + s^2 + \textcolor{brown}{3s} + 3$$

4.3 Solving More Initial Value Problems



Thus

$$(s^4 + 2s^2 + 1) Y(s) - s^3 - s^2 - s - 1 - 2s - 2 = \frac{1}{s - 2}.$$

Hence

$$\begin{aligned}(s^4 + 2s^2 + 1) Y(s) &= \frac{1}{s - 2} + s^3 + s^2 + 3s + 3 \\&= \frac{1}{s - 2} + \frac{s^4 - 2s^3}{s - 2} + \frac{s^3 - 2s^2}{s - 2} \\&\quad + \frac{3s^2 - 6s}{s - 2} + \frac{3s - 6}{s - 2}\end{aligned}$$

4.3 Solving More Initial Value Problems



Thus

$$(s^4 + 2s^2 + 1) Y(s) - s^3 - s^2 - s - 1 - 2s - 2 = \frac{1}{s - 2}.$$

Hence

$$\begin{aligned}(s^4 + 2s^2 + 1) Y(s) &= \frac{1}{s - 2} + s^3 + s^2 + 3s + 3 \\&= \frac{1}{s - 2} + \frac{s^4 - 2s^3}{s - 2} + \frac{s^3 - 2s^2}{s - 2} \\&\quad + \frac{3s^2 - 6s}{s - 2} + \frac{3s - 6}{s - 2} \\&= \frac{s^4 - s^3 + s^2 - 3s - 5}{s - 2}.\end{aligned}$$

4.3 Solving More Initial Value Problems



$$Y(s) = \frac{s^4 - s^3 + s^2 - 3s - 5}{(s-2)(s^4 + 2s^2 + 1)} = \frac{s^4 - s^3 + s^2 - 3s - 5}{(s-2)(s^2 + 1)^2}$$

=

=

4.3 Solving More Initial Value Problems



$$\begin{aligned}Y(s) &= \frac{s^4 - s^3 + s^2 - 3s - 5}{(s-2)(s^4 + 2s^2 + 1)} = \frac{s^4 - s^3 + s^2 - 3s - 5}{(s-2)(s^2 + 1)^2} \\&= \frac{\frac{1}{25}}{s-2} + \frac{\frac{24}{25}s + \frac{23}{25}}{s^2 + 1} + \frac{\frac{9}{5}s + \frac{8}{5}}{(s^2 + 1)^2}\end{aligned}$$

=

4.3 Solving More Initial Value Problems



$$\begin{aligned}Y(s) &= \frac{s^4 - s^3 + s^2 - 3s - 5}{(s-2)(s^4 + 2s^2 + 1)} = \frac{s^4 - s^3 + s^2 - 3s - 5}{(s-2)(s^2 + 1)^2} \\&= \frac{\frac{1}{25}}{s-2} + \frac{\frac{24}{25}s + \frac{23}{25}}{s^2 + 1} + \frac{\frac{9}{5}s + \frac{8}{5}}{(s^2 + 1)^2} \\&= \frac{1}{25} \left(\frac{1}{s-2} \right) + \frac{24}{25} \left(\frac{s}{s^2 + 1} \right) + \frac{23}{25} \left(\frac{1}{s^2 + 1} \right) \\&\quad + \frac{9}{10} \left(\frac{2s}{(s^2 + 1)^2} \right) + \frac{4}{5} \left(\frac{2}{(s^2 + 1)^2} \right).\end{aligned}$$

4.3 Solving More Initial Value Problems



$$\begin{aligned}Y(s) &= \frac{s^4 - s^3 + s^2 - 3s - 5}{(s-2)(s^4 + 2s^2 + 1)} = \frac{s^4 - s^3 + s^2 - 3s - 5}{(s-2)(s^2 + 1)^2} \\&= \frac{\frac{1}{25}}{s-2} + \frac{\frac{24}{25}s + \frac{23}{25}}{s^2 + 1} + \frac{\frac{9}{5}s + \frac{8}{5}}{(s^2 + 1)^2} \\&= \frac{1}{25} \left(\frac{1}{s-2} \right) + \frac{24}{25} \left(\frac{s}{s^2 + 1} \right) + \frac{23}{25} \left(\frac{1}{s^2 + 1} \right) \\&\quad + \frac{9}{10} \left(\frac{2s}{(s^2 + 1)^2} \right) + \frac{4}{5} \left(\frac{2}{(s^2 + 1)^2} \right).\end{aligned}$$

Now $\mathcal{L}[e^{2t}] = \frac{1}{s-2}$, $\mathcal{L}[\cos t] = \frac{s}{s^2+1}$ and $\mathcal{L}[\sin t] = \frac{1}{s^2+1}$. But what do we do with $\frac{2s}{(s^2+1)^2}$ and $\frac{2}{(s^2+1)^2}$?

4.3 Solving More Initial Value Problems



$$\begin{aligned}Y(s) &= \frac{s^4 - s^3 + s^2 - 3s - 5}{(s-2)(s^4 + 2s^2 + 1)} = \frac{s^4 - s^3 + s^2 - 3s - 5}{(s-2)(s^2 + 1)^2} \\&= \frac{\frac{1}{25}}{s-2} + \frac{\frac{24}{25}s + \frac{23}{25}}{s^2 + 1} + \frac{\frac{9}{5}s + \frac{8}{5}}{(s^2 + 1)^2} \\&= \frac{1}{25} \left(\frac{1}{s-2} \right) + \frac{24}{25} \left(\frac{s}{s^2 + 1} \right) + \frac{23}{25} \left(\frac{1}{s^2 + 1} \right) \\&\quad + \frac{9}{10} \left(\frac{2s}{(s^2 + 1)^2} \right) + \frac{4}{5} \left(\frac{2}{(s^2 + 1)^2} \right).\end{aligned}$$

Now $\mathcal{L}[e^{2t}] = \frac{1}{s-2}$, $\mathcal{L}[\cos t] = \frac{s}{s^2+1}$ and $\mathcal{L}[\sin t] = \frac{1}{s^2+1}$. But what do we do with $\frac{2s}{(s^2+1)^2}$ and $\frac{2}{(s^2+1)^2}$?

4.3 Solving More Initial Value Problems



$$\mathcal{L}^{-1} \left[\frac{2s}{(s^2 + 1)^2} \right] = ? \quad \mathcal{L}^{-1} \left[\frac{2}{(s^2 + 1)^2} \right] = ?$$

Remember that $\mathcal{L}[tf(t)] = -F'(s)$.

4.3 Solving More Initial Value Problems



$$\mathcal{L}^{-1} \left[\frac{2s}{(s^2 + 1)^2} \right] = ? \quad \mathcal{L}^{-1} \left[\frac{2}{(s^2 + 1)^2} \right] = ?$$

Remember that $\mathcal{L}[tf(t)] = -F'(s)$. Hence

$$\mathcal{L}[t \sin t] =$$

and

$$\mathcal{L}[t \cos t] =$$

4.3 Solving More Initial Value Problems



$$\mathcal{L}^{-1} \left[\frac{2s}{(s^2 + 1)^2} \right] = ? \quad \mathcal{L}^{-1} \left[\frac{2}{(s^2 + 1)^2} \right] = ?$$

Remember that $\mathcal{L}[tf(t)] = -F'(s)$. Hence

$$\mathcal{L}[t \sin t] = -\frac{d}{ds} \mathcal{L}[\sin t]$$

and

$$\mathcal{L}[t \cos t] =$$

4.3 Solving More Initial Value Problems



$$\mathcal{L}^{-1} \left[\frac{2s}{(s^2 + 1)^2} \right] = ? \quad \mathcal{L}^{-1} \left[\frac{2}{(s^2 + 1)^2} \right] = ?$$

Remember that $\mathcal{L}[tf(t)] = -F'(s)$. Hence

$$\mathcal{L}[t \sin t] = -\frac{d}{ds} \mathcal{L}[\sin t] = -\frac{d}{ds} \left(\frac{1}{s^2 + 1} \right) = \frac{2s}{(s^2 + 1)^2}$$

and

$$\mathcal{L}[t \cos t] =$$

4.3 Solving More Initial Value Problems



$$\mathcal{L}^{-1} \left[\frac{2s}{(s^2 + 1)^2} \right] = ? \quad \mathcal{L}^{-1} \left[\frac{2}{(s^2 + 1)^2} \right] = ?$$

Remember that $\mathcal{L}[tf(t)] = -F'(s)$. Hence

$$\mathcal{L}[t \sin t] = -\frac{d}{ds} \mathcal{L}[\sin t] = -\frac{d}{ds} \left(\frac{1}{s^2 + 1} \right) = \frac{2s}{(s^2 + 1)^2}$$

and

$$\mathcal{L}[t \cos t] = -\frac{d}{ds} \mathcal{L}[\cos t]$$

4.3 Solving More Initial Value Problems



$$\mathcal{L}^{-1} \left[\frac{2s}{(s^2 + 1)^2} \right] = ? \quad \mathcal{L}^{-1} \left[\frac{2}{(s^2 + 1)^2} \right] = ?$$

Remember that $\mathcal{L}[tf(t)] = -F'(s)$. Hence

$$\mathcal{L}[t \sin t] = -\frac{d}{ds} \mathcal{L}[\sin t] = -\frac{d}{ds} \left(\frac{1}{s^2 + 1} \right) = \frac{2s}{(s^2 + 1)^2}$$

and

$$\mathcal{L}[t \cos t] = -\frac{d}{ds} \mathcal{L}[\cos t] = -\frac{d}{ds} \left(\frac{s}{s^2 + 1} \right) = \frac{1}{s^2 + 1} - \frac{2}{(s^2 + 1)^2}.$$

4.3 Solving More Initial Value Problems



$$\mathcal{L}^{-1} \left[\frac{2s}{(s^2 + 1)^2} \right] = ? \quad \mathcal{L}^{-1} \left[\frac{2}{(s^2 + 1)^2} \right] = ?$$

Remember that $\mathcal{L}[tf(t)] = -F'(s)$. Hence

$$\mathcal{L}[t \sin t] = -\frac{d}{ds} \mathcal{L}[\sin t] = -\frac{d}{ds} \left(\frac{1}{s^2 + 1} \right) = \frac{2s}{(s^2 + 1)^2}$$

and

$$\mathcal{L}[t \cos t] = -\frac{d}{ds} \mathcal{L}[\cos t] = -\frac{d}{ds} \left(\frac{s}{s^2 + 1} \right) = \frac{1}{s^2 + 1} - \frac{2}{(s^2 + 1)^2}.$$

It follows that

$$\mathcal{L}[\sin t - t \cos t] = \frac{1}{s^2 + 1} - \left(\frac{1}{s^2 + 1} - \frac{2}{(s^2 + 1)^2} \right) = \frac{2}{(s^2 + 1)^2}.$$

4.3 Solving More Initial Value Problems



$$\mathcal{L}^{-1} \left[\frac{2s}{(s^2 + 1)^2} \right] = ? \quad \mathcal{L}^{-1} \left[\frac{2}{(s^2 + 1)^2} \right] = ?$$

Remember that $\mathcal{L}[tf(t)] = -F'(s)$. Hence

$$\mathcal{L}[t \sin t] = -\frac{d}{ds} \mathcal{L}[\sin t] = -\frac{d}{ds} \left(\frac{1}{s^2 + 1} \right) = \frac{2s}{(s^2 + 1)^2}$$

and

$$\mathcal{L}[t \cos t] = -\frac{d}{ds} \mathcal{L}[\cos t] = -\frac{d}{ds} \left(\frac{s}{s^2 + 1} \right) = \frac{1}{s^2 + 1} - \frac{2}{(s^2 + 1)^2}.$$

It follows that

$$\mathcal{L}[\sin t - t \cos t] = \frac{1}{s^2 + 1} - \left(\frac{1}{s^2 + 1} - \frac{2}{(s^2 + 1)^2} \right) = \frac{2}{(s^2 + 1)^2}.$$

4.3 Solving More Initial Value Problems



$$\mathcal{L}^{-1} \left[\frac{2s}{(s^2 + 1)^2} \right] = t \sin t \quad \mathcal{L}^{-1} \left[\frac{2}{(s^2 + 1)^2} \right] = \sin t - t \cos t$$

Recall that

$$Y(s) = \frac{1}{25} \left(\frac{1}{s-2} \right) + \frac{24}{25} \left(\frac{s}{s^2+1} \right) + \frac{23}{25} \left(\frac{1}{s^2+1} \right) \\ + \frac{9}{10} \left(\frac{2s}{(s^2+1)^2} \right) + \frac{4}{5} \left(\frac{2}{(s^2+1)^2} \right).$$

4.3 Solving More Initial Value Problems



$$\mathcal{L}^{-1} \left[\frac{2s}{(s^2 + 1)^2} \right] = t \sin t \quad \mathcal{L}^{-1} \left[\frac{2}{(s^2 + 1)^2} \right] = \sin t - t \cos t$$

Recall that

$$Y(s) = \frac{1}{25} \left(\frac{1}{s-2} \right) + \frac{24}{25} \left(\frac{s}{s^2+1} \right) + \frac{23}{25} \left(\frac{1}{s^2+1} \right) \\ + \frac{9}{10} \left(\frac{2s}{(s^2+1)^2} \right) + \frac{4}{5} \left(\frac{2}{(s^2+1)^2} \right).$$

Hence

$$y(t) = \frac{1}{25} (e^{2t} + 24 \cos t + 23 \sin t) + \frac{9}{10} t \sin t + \frac{4}{5} (\sin t - t \cos t)$$

is the solution to the IVP.

4.3 Solving More Initial Value Problems

$$\mathcal{L}^{-1} \left[\frac{2s}{(s^2 + 1)^2} \right] = t \sin t$$

$$\mathcal{L}^{-1} \left[\frac{2}{(s^2 + y^2)} \right] = \sin t - t \cos t$$

Recall that

$$Y(s) = \frac{1}{25} \left(\frac{1}{s-2} \right) + \frac{24}{25} \left(\frac{s}{s^2+1} \right) + \frac{9}{25} \left(\frac{2s}{(s^2+1)^2} \right) + \frac{4}{5} \left(\frac{2t}{(s^2+5^2)} \right).$$

Hence

$$y(t) = \frac{1}{25} (e^{2t} + 24 \cos t + 23 \sin t) + \frac{9}{10} t \sin t + \frac{4}{5} (\sin t - t \cos t)$$

is the solution to the IVP.



Step Functions

4.4 Step Functions

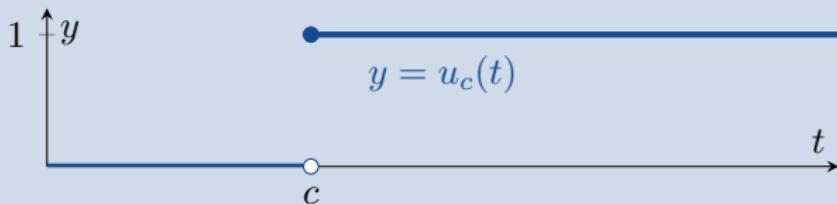


Definition

The *unit step function* $u_c : [0, \infty) \rightarrow \mathbb{R}$ is defined by

$$u_c(t) = \begin{cases} 0 & t < c \\ 1 & t \geq c \end{cases}$$

for $c \geq 0$.



4.4 Step Functions



Example

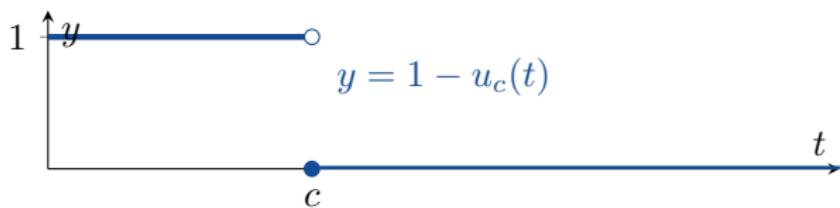
Draw the graph of $y = 1 - u_c(t)$.

4.4 Step Functions



Example

Draw the graph of $y = 1 - u_c(t)$.



4.4 Step Functions



Example

Draw the graph of $y = u_1(t) - u_2(t)$.

4.4 Step Functions



Example

Draw the graph of $y = u_1(t) - u_2(t)$.

Clearly $t = 1$ and $t = 2$ are important points.

4.4 Step Functions



Example

Draw the graph of $y = u_1(t) - u_2(t)$.

Clearly $t = 1$ and $t = 2$ are important points. So we consider the function on the intervals $[0, 1)$, $[1, 2)$ and $[2, \infty)$.

4.4 Step Functions



Example

Draw the graph of $y = u_1(t) - u_2(t)$.

Clearly $t = 1$ and $t = 2$ are important points. So we consider the function on the intervals $[0, 1)$, $[1, 2)$ and $[2, \infty)$.

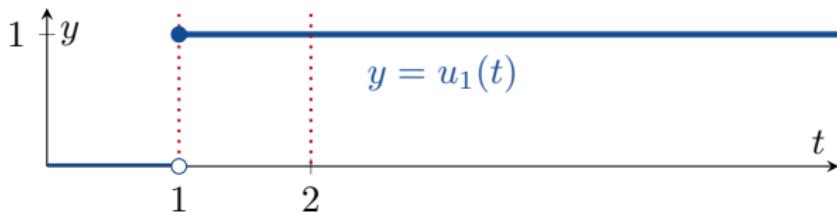
$$u_1(t) - u_2(t) = \begin{cases} u_1(t) - u_2(t) & 0 \leq t < 1 \\ u_1(t) - u_2(t) & 1 \leq t < 2 \\ u_1(t) - u_2(t) & 2 \leq t \end{cases}$$

4.4 Step Functions



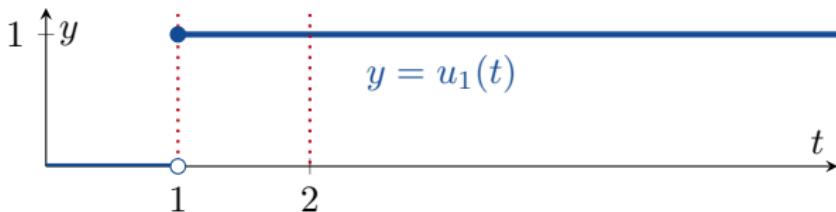
$$u_1(t) - u_2(t) = \begin{cases} u_1(t) - u_2(t) & 0 \leq t < 1 \\ u_1(t) - u_2(t) & 1 \leq t < 2 \\ u_1(t) - u_2(t) & 2 \leq t \end{cases}$$

4.4 Step Functions



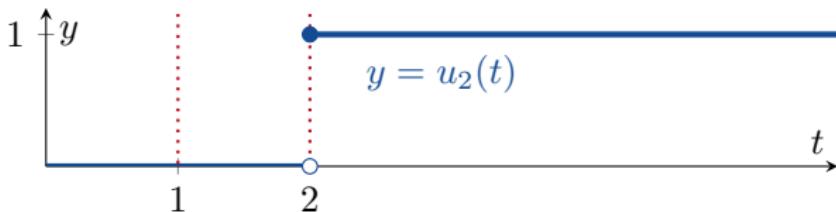
$$u_1(t) - u_2(t) = \begin{cases} \textcolor{red}{u_1(t)} - u_2(t) & 0 \leq t < 1 \\ \textcolor{red}{u_1(t)} - u_2(t) & 1 \leq t < 2 \\ \textcolor{red}{u_1(t)} - u_2(t) & 2 \leq t \end{cases}$$

4.4 Step Functions



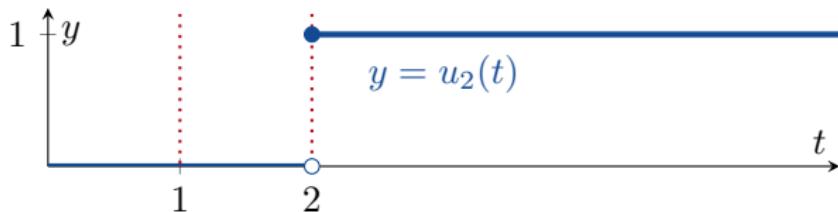
$$u_1(t) - u_2(t) = \begin{cases} 0 - u_2(t) & 0 \leq t < 1 \\ 1 - u_2(t) & 1 \leq t < 2 \\ 1 - u_2(t) & 2 \leq t \end{cases}$$

4.4 Step Functions



$$u_1(t) - u_2(t) = \begin{cases} 0 - u_2(t) & 0 \leq t < 1 \\ 1 - u_2(t) & 1 \leq t < 2 \\ 1 - u_2(t) & 2 \leq t \end{cases}$$

4.4 Step Functions



$$u_1(t) - u_2(t) = \begin{cases} 0 - 0 & 0 \leq t < 1 \\ 1 - 0 & 1 \leq t < 2 \\ 1 - 1 & 2 \leq t \end{cases}$$

4.4 Step Functions

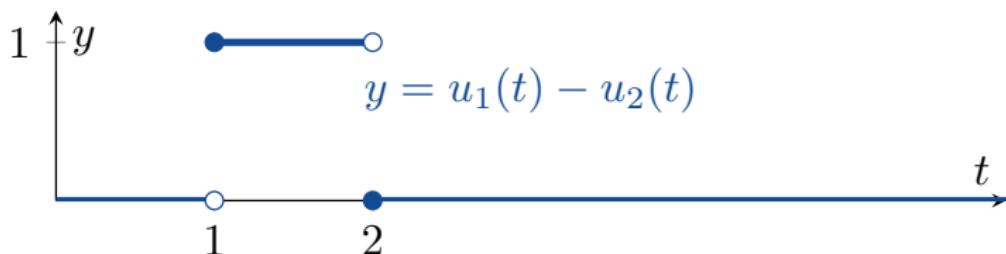


$$u_1(t) - u_2(t) = \begin{cases} 0 & 0 \leq t < 1 \\ 1 & 1 \leq t < 2 \\ 0 & 2 \leq t \end{cases}$$

4.4 Step Functions



$$u_1(t) - u_2(t) = \begin{cases} 0 & 0 \leq t < 1 \\ 1 & 1 \leq t < 2 \\ 0 & 2 \leq t \end{cases}$$



4.4 Step Functions



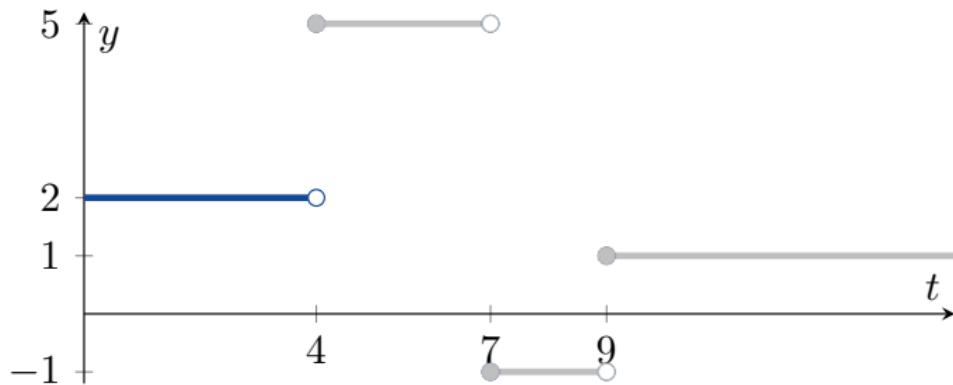
Example

Write the function

$$f(t) = \begin{cases} 2 & 0 \leq t < 4 \\ 5 & 4 \leq t < 7 \\ -1 & 7 \leq t < 9 \\ 1 & 9 \leq t \end{cases}$$

in terms of the unit step function.

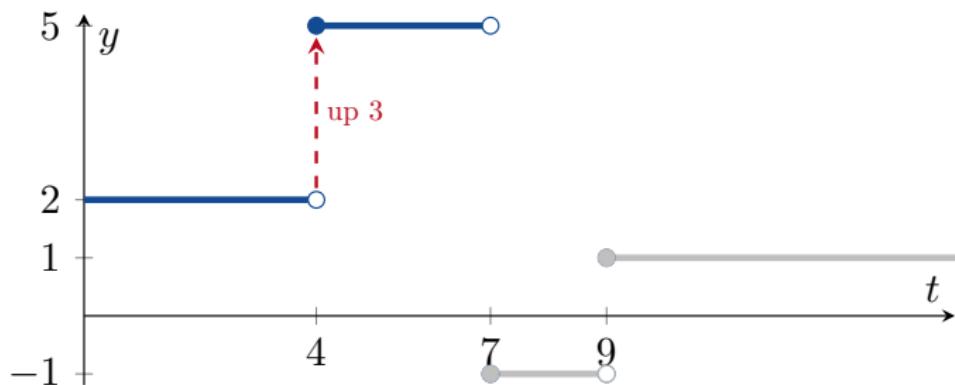
4.4 Step Functions



The function starts at $f(0) = 2$. So we will have

$$f(t) = 2 + (\text{something}).$$

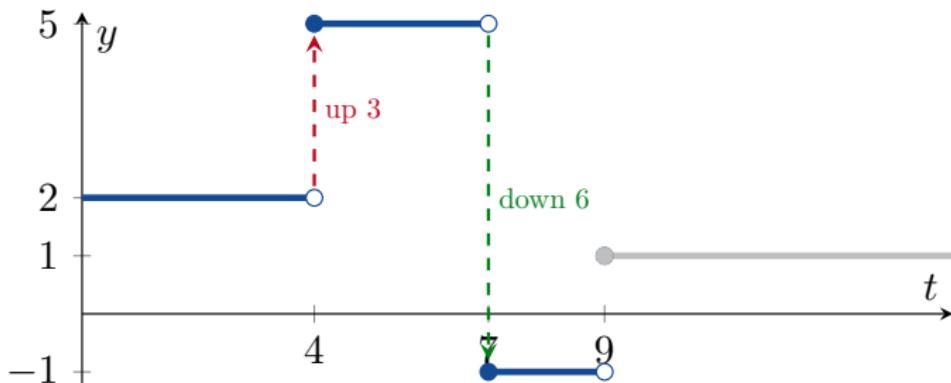
4.4 Step Functions



At $t = 4$, the function jumps from 2 to 5 (it goes “up 3”). So

$$f(t) = 2 + 3u_4(t) + (\text{something}).$$

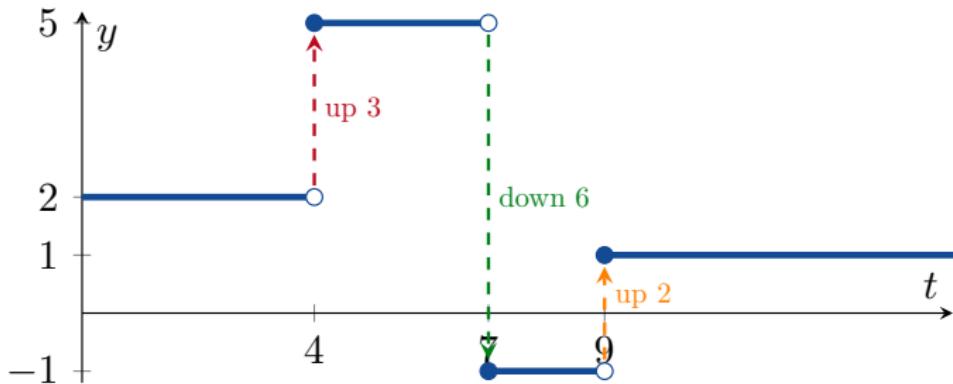
4.4 Step Functions



Then it goes “down 6” when $t = 7$. So

$$f(t) = 2 + 3u_4(t) - 6u_7(t) + (\text{something}).$$

4.4 Step Functions



Finally it goes “up 2” when $t = 9$. Therefore

$$f(t) = 2 + 3u_4(t) - 6u_7(t) + 2u_9(t).$$

4.4 Step Functions



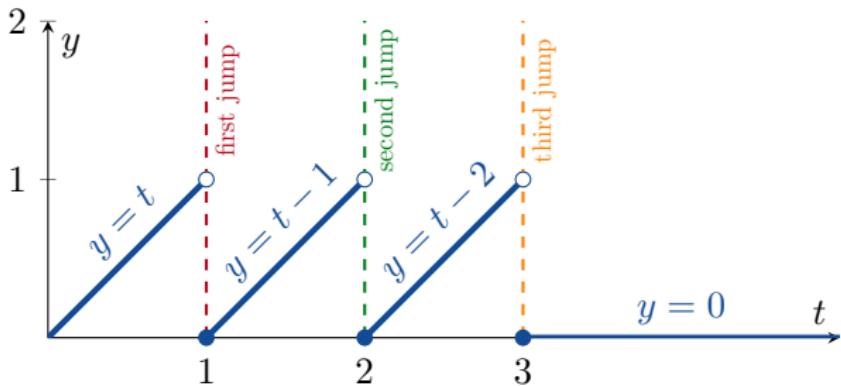
Example

Write the function

$$f(t) = \begin{cases} t & 0 \leq t < 1 \\ t - 1 & 1 \leq t < 2 \\ t - 2 & 2 \leq t < 3 \\ 0 & 3 \leq t \end{cases}$$

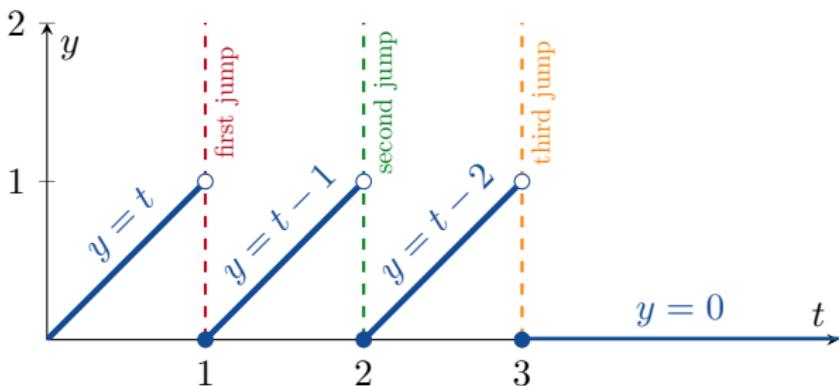
in terms of the unit step function.

4.4 Step Functions



This function starts with $f(t) = t$, then changes when $t = 1$, $t = 2$ and $t = 3$:

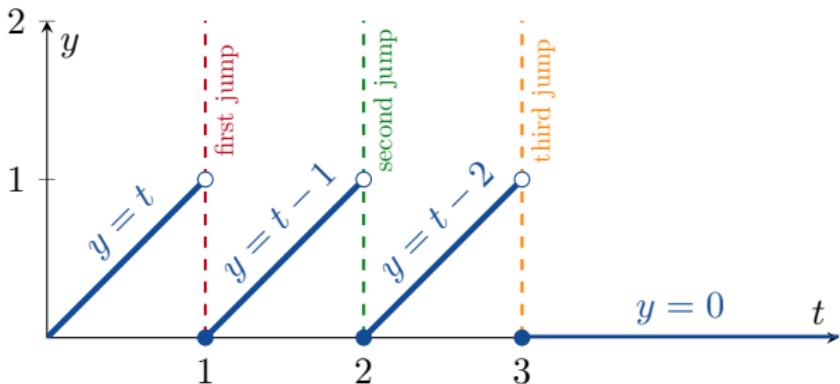
4.4 Step Functions



This function starts with $f(t) = t$, then changes when $t = 1$, $t = 2$ and $t = 3$: So we must have

$$f(t) = t + \begin{pmatrix} \text{first} \\ \text{jump} \end{pmatrix} u_1(t) + \begin{pmatrix} \text{second} \\ \text{jump} \end{pmatrix} u_2(t) + \begin{pmatrix} \text{third} \\ \text{jump} \end{pmatrix} u_3(t).$$

4.4 Step Functions



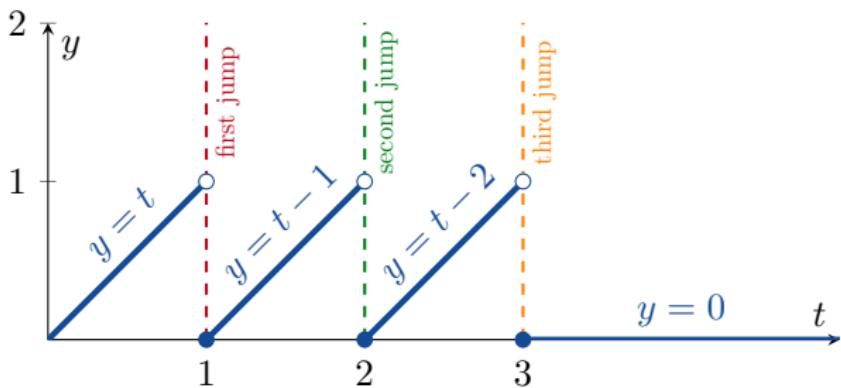
This function starts with $f(t) = t$, then changes when $t = 1$, $t = 2$ and $t = 3$: So we must have

$$f(t) = t + \begin{pmatrix} \text{first} \\ \text{jump} \end{pmatrix} u_1(t) + \begin{pmatrix} \text{second} \\ \text{jump} \end{pmatrix} u_2(t) + \begin{pmatrix} \text{third} \\ \text{jump} \end{pmatrix} u_3(t).$$

At each “jump” we calculate

$$\text{jump} = \begin{pmatrix} \text{function} \\ \text{on right} \end{pmatrix} - \begin{pmatrix} \text{function} \\ \text{on left} \end{pmatrix}.$$

4.4 Step Functions



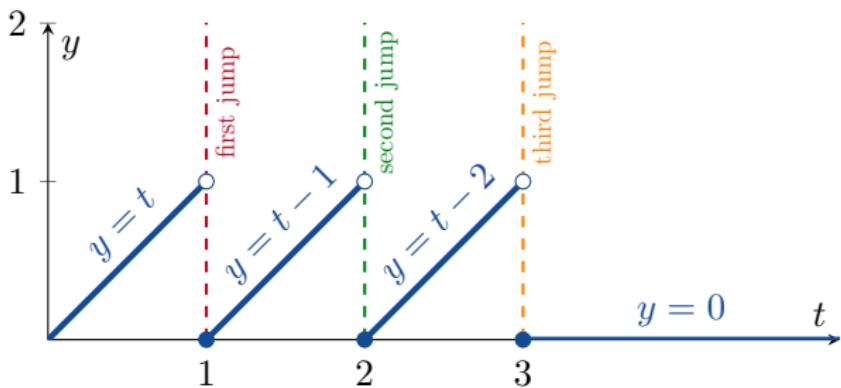
So we have

$$\begin{pmatrix} \text{first} \\ \text{jump} \end{pmatrix} =$$

$$\begin{pmatrix} \text{second} \\ \text{jump} \end{pmatrix} =$$

$$\begin{pmatrix} \text{third} \\ \text{jump} \end{pmatrix} =$$

4.4 Step Functions



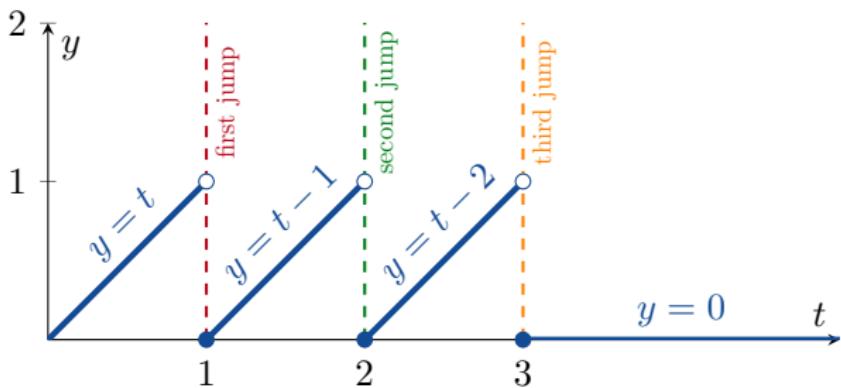
So we have

$$\begin{pmatrix} \text{first} \\ \text{jump} \end{pmatrix} = (t - 1) - t = -1$$

$$\begin{pmatrix} \text{second} \\ \text{jump} \end{pmatrix} =$$

$$\begin{pmatrix} \text{third} \\ \text{jump} \end{pmatrix} =$$

4.4 Step Functions



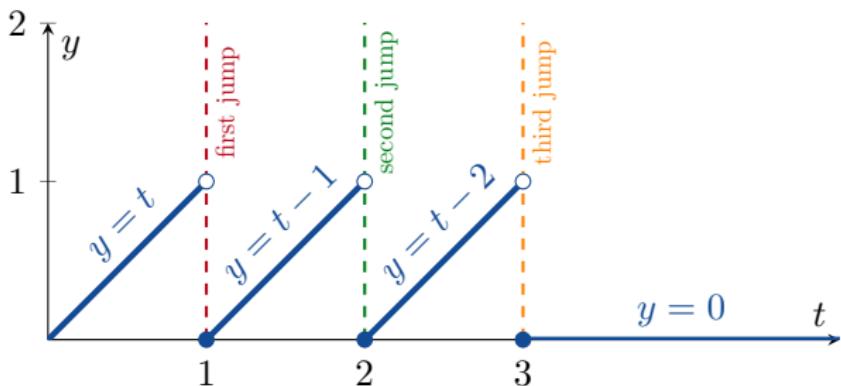
So we have

$$\begin{pmatrix} \text{first} \\ \text{jump} \end{pmatrix} = (t - 1) - t = -1$$

$$\begin{pmatrix} \text{second} \\ \text{jump} \end{pmatrix} = (t - 2) - (t - 1) = -1$$

$$\begin{pmatrix} \text{third} \\ \text{jump} \end{pmatrix} =$$

4.4 Step Functions



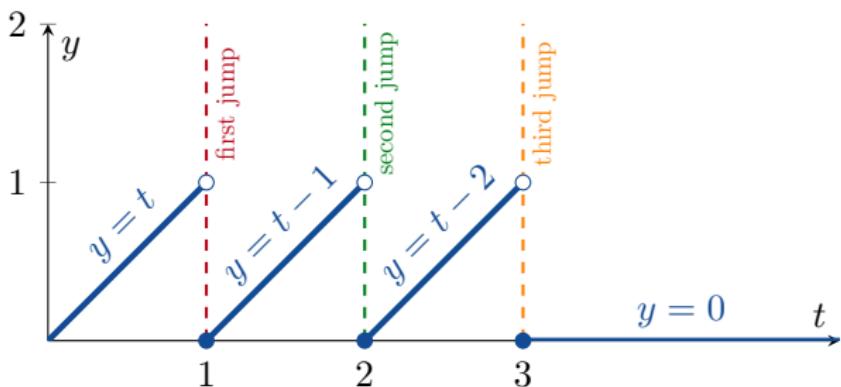
So we have

$$\begin{pmatrix} \text{first} \\ \text{jump} \end{pmatrix} = (t - 1) - t = -1$$

$$\begin{pmatrix} \text{second} \\ \text{jump} \end{pmatrix} = (t - 2) - (t - 1) = -1$$

$$\begin{pmatrix} \text{third} \\ \text{jump} \end{pmatrix} = 0 - (t - 2) = 2 - t$$

4.4 Step Functions



Hence

$$f(t) = t - u_1(t) - u_2(t) + (2-t)u_3(t).$$

4.4 Step Functions



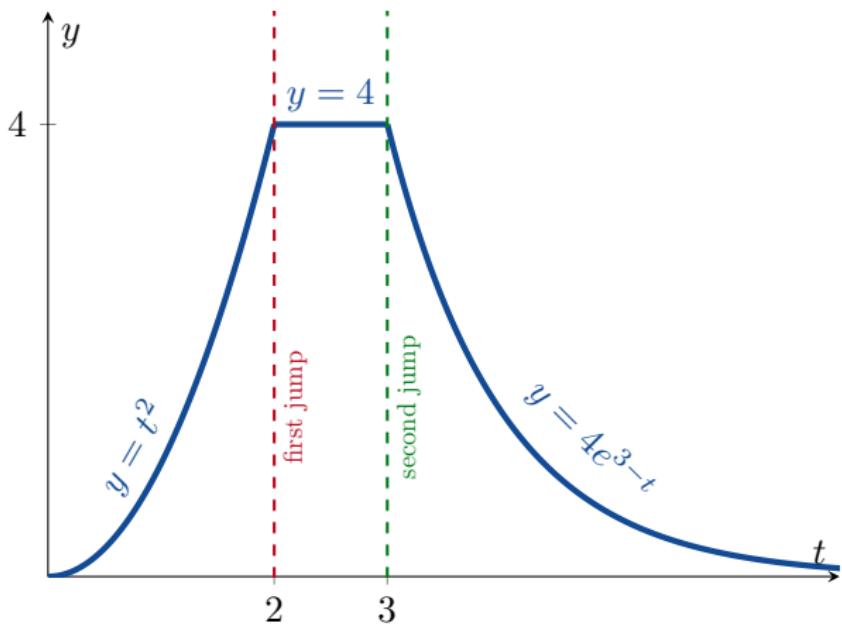
Example

Write the function

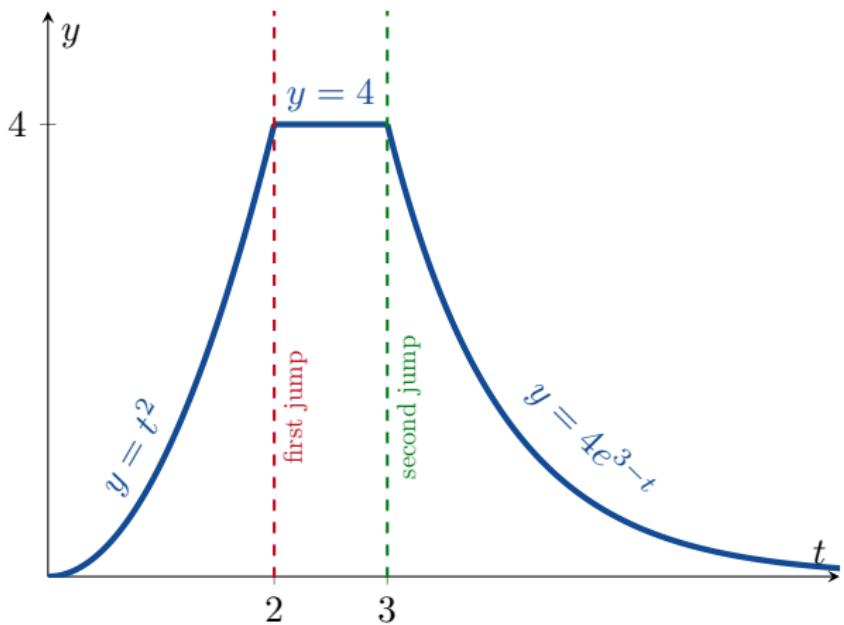
$$f(t) = \begin{cases} t^2 & 0 \leq t < 2 \\ 4 & 2 \leq t < 3 \\ 4e^{t-3} & 3 \leq t \end{cases}$$

in terms of the unit step function.

4.4 Step Functions

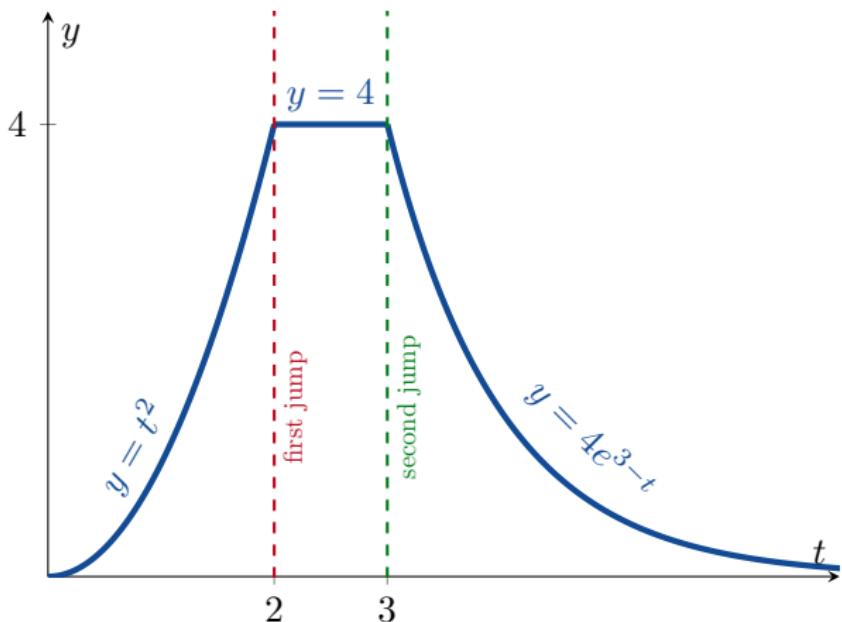


4.4 Step Functions



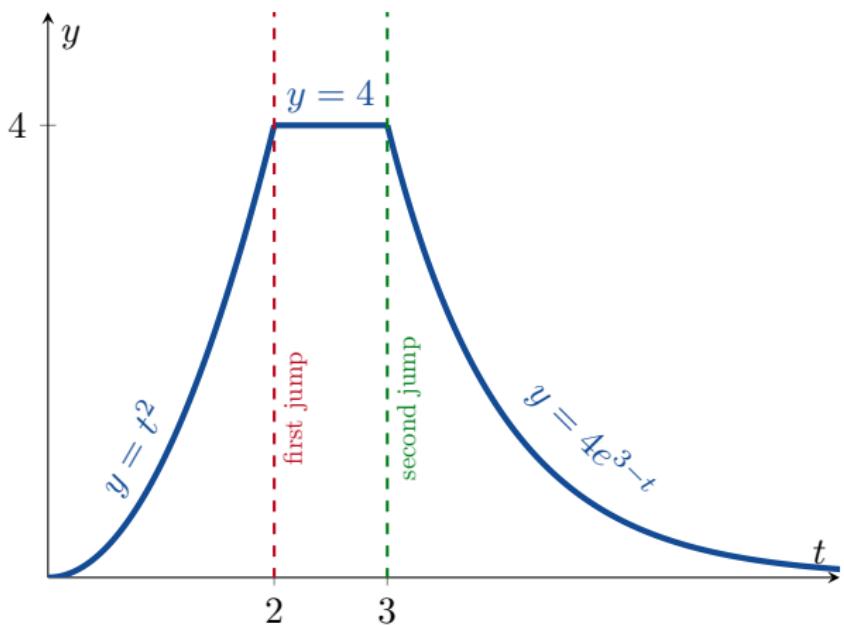
$$f(t) = t^2 + \begin{pmatrix} \text{first} \\ \text{jump} \end{pmatrix} u_2(t) + \begin{pmatrix} \text{second} \\ \text{jump} \end{pmatrix} u_3(t).$$

4.4 Step Functions



$$f(t) = t^2 + (4 - t^2)u_2(t) + \begin{pmatrix} \text{second} \\ \text{jump} \end{pmatrix} u_3(t).$$

4.4 Step Functions



$$f(t) = t^2 + (4 - t^2)u_2(t) + (4e^{t-3} - 4)u_3(t).$$

4.4 Step Functions $\mathcal{L} [f] (s) = \int_0^{\infty} e^{-st} f(t) dt$



What is the Laplace Transform of the unit step function?

4.4 Step Functions $\mathcal{L} [f] (s) = \int_0^\infty e^{-st} f(t) dt$



What is the Laplace Transform of the unit step function?

We calculate that

$$\begin{aligned}\mathcal{L} [u_c] (s) &= \int_0^\infty e^{-st} u_c(t) dt = \\ &= \end{aligned}$$

for $s > 0$.

4.4 Step Functions $\mathcal{L}[f](s) = \int_0^\infty e^{-st} f(t) dt$



What is the Laplace Transform of the unit step function?

We calculate that

$$\begin{aligned}\mathcal{L}[u_c](s) &= \int_0^\infty e^{-st} u_c(t) dt = \int_0^c e^{-st} 0 dt + \int_c^\infty e^{-st} 1 dt \\ &= \quad \quad \quad =\end{aligned}$$

for $s > 0$.

4.4 Step Functions $\mathcal{L} [f] (s) = \int_0^\infty e^{-st} f(t) dt$



What is the Laplace Transform of the unit step function?

We calculate that

$$\begin{aligned}\mathcal{L} [u_c] (s) &= \int_0^\infty e^{-st} \mathbf{u}_c(t) dt = \int_0^c e^{-st} \mathbf{0} dt + \int_c^\infty e^{-st} \mathbf{1} dt \\ &= \int_c^\infty e^{-st} dt =\end{aligned}$$

for $s > 0$.

4.4 Step Functions $\mathcal{L}[f](s) = \int_0^\infty e^{-st} f(t) dt$



What is the Laplace Transform of the unit step function?

We calculate that

$$\begin{aligned}\mathcal{L}[u_c](s) &= \int_0^\infty e^{-st} u_c(t) dt = \int_0^c e^{-st} 0 dt + \int_c^\infty e^{-st} 1 dt \\ &= \int_c^\infty e^{-st} dt = \left[-\frac{1}{s} e^{-st} \right]_c^\infty = \frac{e^{-cs}}{s}\end{aligned}$$

for $s > 0$.

4.4 Step Functions $\mathcal{L}[f](s) = \int_0^\infty e^{-st} f(t) dt$



What is the Laplace Transform of the unit step function?

We calculate that

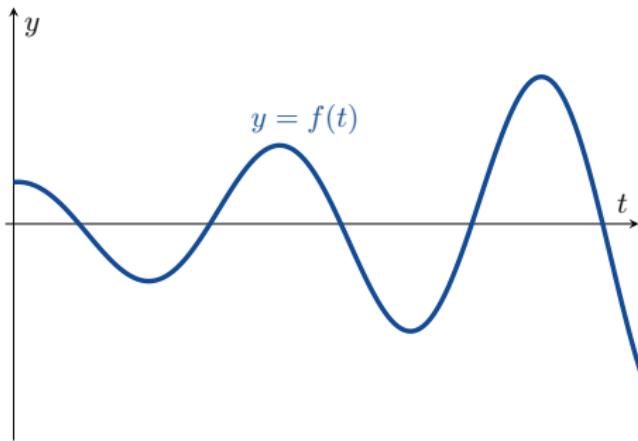
$$\begin{aligned}\mathcal{L}[u_c](s) &= \int_0^\infty e^{-st} u_c(t) dt = \int_0^c e^{-st} 0 dt + \int_c^\infty e^{-st} 1 dt \\ &= \int_c^\infty e^{-st} dt = \left[-\frac{1}{s} e^{-st} \right]_c^\infty = \frac{e^{-cs}}{s}\end{aligned}$$

for $s > 0$.

Theorem

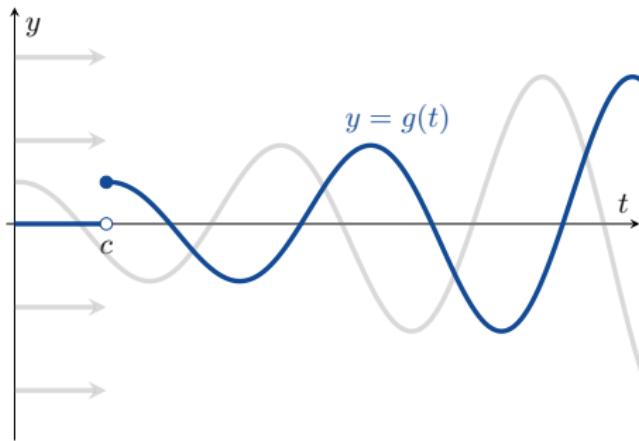
$$\mathcal{L}[u_c](s) = \frac{e^{-cs}}{s}$$

4.4 Step Functions



Now suppose that we have some function $f : [0, \infty) \rightarrow \mathbb{R}$

4.4 Step Functions



Now suppose that we have some function $f : [0, \infty) \rightarrow \mathbb{R}$ and we define a new function $g : [0, \infty) \rightarrow \mathbb{R}$ by

$$g(t) = \begin{cases} 0 & t < c \\ f(t - c) & t \geq c. \end{cases}$$

We can write $g(t) = u_c(t)f(t - c)$.

4.4 Step Functions

What is the Laplace Transform of $g(t) = u_c(t)f(t - c)$?

4.4 Step Functions

What is the Laplace Transform of $g(t) = u_c(t)f(t - c)$?

$$\mathcal{L}[g] = \mathcal{L}[u_c(t)f(t - c)]$$

4.4 Step Functions

What is the Laplace Transform of $g(t) = u_c(t)f(t - c)$?

$$\mathcal{L}[g] = \mathcal{L}[u_c(t)f(t - c)] = \int_0^{\infty} e^{-st} u_c(t) f(t - c) dt$$

4.4 Step Functions

What is the Laplace Transform of $g(t) = u_c(t)f(t - c)$?

$$\begin{aligned}\mathcal{L}[g] &= \mathcal{L}[u_c(t)f(t - c)] = \int_0^{\infty} e^{-st} u_c(t) f(t - c) dt \\ &= \int_c^{\infty} e^{-st} f(t - c) dt.\end{aligned}$$

4.4 Step Functions

What is the Laplace Transform of $g(t) = u_c(t)f(t - c)$?

$$\begin{aligned}\mathcal{L}[g] &= \mathcal{L}[u_c(t)f(t - c)] = \int_0^\infty e^{-st}u_c(t)f(t - c) dt \\ &= \int_c^\infty e^{-st}f(t - c) dt.\end{aligned}$$

Let $u = t - c$.

4.4 Step Functions

What is the Laplace Transform of $g(t) = u_c(t)f(t - c)$?

$$\begin{aligned}\mathcal{L}[g] &= \mathcal{L}[u_c(t)f(t - c)] = \int_0^{\infty} e^{-st}u_c(t)f(t - c) dt \\ &= \int_c^{\infty} e^{-st}f(t - c) dt.\end{aligned}$$

Let $u = t - c$. Then $du = dt$ and $t = c \iff u = 0$.

4.4 Step Functions

What is the Laplace Transform of $g(t) = u_c(t)f(t - c)$?

$$\begin{aligned}\mathcal{L}[g] &= \mathcal{L}[u_c(t)f(t - c)] = \int_0^\infty e^{-st}u_c(t)f(t - c) dt \\ &= \int_c^\infty e^{-st}f(t - c) dt.\end{aligned}$$

Let $u = t - c$. Then $du = dt$ and $t = c \iff u = 0$. Therefore

$$\mathcal{L}[g] = \int_0^\infty e^{-s(u+c)}f(u) du$$

4.4 Step Functions

What is the Laplace Transform of $g(t) = u_c(t)f(t - c)$?

$$\begin{aligned}\mathcal{L}[g] &= \mathcal{L}[u_c(t)f(t - c)] = \int_0^{\infty} e^{-st} u_c(t) f(t - c) dt \\ &= \int_c^{\infty} e^{-st} f(t - c) dt.\end{aligned}$$

Let $u = t - c$. Then $du = dt$ and $t = c \iff u = 0$. Therefore

$$\mathcal{L}[g] = \int_0^{\infty} e^{-s(u+c)} f(u) du = e^{-cs} \int_0^{\infty} e^{-su} f(u) du$$

4.4 Step Functions

What is the Laplace Transform of $g(t) = u_c(t)f(t - c)$?

$$\begin{aligned}\mathcal{L}[g] &= \mathcal{L}[u_c(t)f(t - c)] = \int_0^\infty e^{-st}u_c(t)f(t - c) dt \\ &= \int_c^\infty e^{-st}f(t - c) dt.\end{aligned}$$

Let $u = t - c$. Then $du = dt$ and $t = c \iff u = 0$. Therefore

$$\mathcal{L}[g] = \int_0^\infty e^{-s(u+c)}f(u) du = e^{-cs} \int_0^\infty e^{-su}f(u) du = e^{-cs}\mathcal{L}[f].$$

4.4 Step Functions

What is the Laplace Transform of $g(t) = u_c(t)f(t - c)$?

$$\begin{aligned}\mathcal{L}[g] &= \mathcal{L}[u_c(t)f(t - c)] = \int_0^\infty e^{-st}u_c(t)f(t - c) dt \\ &= \int_c^\infty e^{-st}f(t - c) dt.\end{aligned}$$

Let $u = t - c$. Then $du = dt$ and $t = c \iff u = 0$. Therefore

$$\mathcal{L}[g] = \int_0^\infty e^{-s(u+c)}f(u) du = e^{-cs} \int_0^\infty e^{-su}f(u) du = e^{-cs}\mathcal{L}[f].$$

Theorem

$$\mathcal{L}[u_c(t)f(t - c)](s) = e^{-cs}F(s)$$

4.4 Step Functions



Example

Find the Laplace Transform of

$$f(t) = \begin{cases} t & 0 \leq t < 1 \\ t - 1 & 1 \leq t < 2 \\ t - 2 & 2 \leq t < 3 \\ 0 & 3 \leq t. \end{cases}$$

$$4.4 \quad \mathcal{L} [u_c(t)f(t - c)](s) = e^{-cs}F(s)$$

Since

$$f(t) = t - u_1(t) - u_2(t) + (2-t)u_3(t)$$

$$4.4 \quad \mathcal{L} [u_c(t)f(t - c)](s) = e^{-cs}F(s)$$

Since

$$\begin{aligned}f(t) &= t - u_1(t) - u_2(t) + (2-t)u_3(t) \\&= t - u_1(t) - u_2(t) - u_3(t) - u_3(t)(t-3)\end{aligned}$$

$$4.4 \quad \mathcal{L} [u_c(t)f(t - c)](s) = e^{-cs}F(s)$$

Since

$$\begin{aligned} f(t) &= t - u_1(t) - u_2(t) + (2-t)u_3(t) \\ &= t - u_1(t) - u_2(t) - u_3(t) - u_3(t)(t-3) \end{aligned}$$

we have that

$$\begin{aligned} F(s) &= \mathcal{L}[t] - \mathcal{L}[u_1] - \mathcal{L}[u_2] - \mathcal{L}[u_3] - \mathcal{L}[u_3(t)(t-3)] \\ &= \frac{1}{s^2} - \frac{e^{-s}}{s} - \frac{e^{-2s}}{s} - \frac{e^{-3s}}{s} - \frac{e^{-3s}}{s^2}. \end{aligned}$$

4.4 Step Functions



Example

Find the Laplace Transform of

$$f(t) = \begin{cases} \sin t & 0 \leq t \leq \frac{\pi}{4} \\ \sin t + \cos(t - \frac{\pi}{4}) & \frac{\pi}{4} \leq t. \end{cases}$$

4.4 Step Functions



Note that $f(t) = \sin t + g(t)$ where

$$g(t) = \begin{cases} 0 & 0 \leq t \leq \frac{\pi}{4} \\ \cos(t - \frac{\pi}{4}) & \frac{\pi}{4} \leq t \end{cases} = u_{\frac{\pi}{4}}(t) \cos\left(t - \frac{\pi}{4}\right).$$

4.4 Step Functions



$$\mathcal{L} [u_c(t)f(t - c)](s) = e^{-cs}F(s)$$

Note that $f(t) = \sin t + g(t)$ where

$$g(t) = \begin{cases} 0 & 0 \leq t \leq \frac{\pi}{4} \\ \cos(t - \frac{\pi}{4}) & \frac{\pi}{4} \leq t \end{cases} = u_{\frac{\pi}{4}}(t) \cos\left(t - \frac{\pi}{4}\right).$$

So

$$F(s) = \mathcal{L}[f] = \mathcal{L}[\sin t] + \mathcal{L}\left[u_{\frac{\pi}{4}}(t) \cos\left(t - \frac{\pi}{4}\right)\right]$$

4.4 Step Functions



$$\mathcal{L} [u_c(t)f(t - c)](s) = e^{-cs}F(s)$$

Note that $f(t) = \sin t + g(t)$ where

$$g(t) = \begin{cases} 0 & 0 \leq t \leq \frac{\pi}{4} \\ \cos(t - \frac{\pi}{4}) & \frac{\pi}{4} \leq t \end{cases} = u_{\frac{\pi}{4}}(t) \cos\left(t - \frac{\pi}{4}\right).$$

So

$$\begin{aligned} F(s) &= \mathcal{L}[f] = \mathcal{L}[\sin t] + \mathcal{L}\left[u_{\frac{\pi}{4}}(t)\cos\left(t - \frac{\pi}{4}\right)\right] \\ &= \mathcal{L}[\sin t] + e^{-\frac{\pi s}{4}}\mathcal{L}[\cos t] \end{aligned}$$

4.4 Step Functions



$$\mathcal{L} [u_c(t)f(t - c)](s) = e^{-cs}F(s)$$

Note that $f(t) = \sin t + g(t)$ where

$$g(t) = \begin{cases} 0 & 0 \leq t \leq \frac{\pi}{4} \\ \cos(t - \frac{\pi}{4}) & \frac{\pi}{4} \leq t \end{cases} = u_{\frac{\pi}{4}}(t) \cos\left(t - \frac{\pi}{4}\right).$$

So

$$\begin{aligned} F(s) &= \mathcal{L}[f] = \mathcal{L}[\sin t] + \mathcal{L}\left[u_{\frac{\pi}{4}}(t) \cos\left(t - \frac{\pi}{4}\right)\right] \\ &= \mathcal{L}[\sin t] + e^{-\frac{\pi s}{4}} \mathcal{L}[\cos t] = \frac{1}{s^2 + 1} + e^{-\frac{\pi s}{4}} \frac{s}{s^2 + 1} \end{aligned}$$

4.4 Step Functions



$$\mathcal{L} [u_c(t)f(t - c)](s) = e^{-cs}F(s)$$

Note that $f(t) = \sin t + g(t)$ where

$$g(t) = \begin{cases} 0 & 0 \leq t \leq \frac{\pi}{4} \\ \cos(t - \frac{\pi}{4}) & \frac{\pi}{4} \leq t \end{cases} = u_{\frac{\pi}{4}}(t) \cos\left(t - \frac{\pi}{4}\right).$$

So

$$\begin{aligned} F(s) &= \mathcal{L}[f] = \mathcal{L}[\sin t] + \mathcal{L}\left[u_{\frac{\pi}{4}}(t)\cos\left(t - \frac{\pi}{4}\right)\right] \\ &= \mathcal{L}[\sin t] + e^{-\frac{\pi s}{4}}\mathcal{L}[\cos t] = \frac{1}{s^2 + 1} + e^{-\frac{\pi s}{4}}\frac{s}{s^2 + 1} \\ &= \frac{1 + se^{-\frac{\pi s}{4}}}{s^2 + 1}. \end{aligned}$$

4.4 Step Functions



Example

Find the inverse Laplace Transform of $F(s) = \frac{1-e^{-2s}}{s^2}$.

4.4 Step Functions

Example

Find the inverse Laplace Transform of $F(s) = \frac{1-e^{-2s}}{s^2}$.

$$\begin{aligned}f(t) &= \mathcal{L}^{-1}[F] = \mathcal{L}^{-1}\left[\frac{1}{s^2}\right] - \mathcal{L}^{-1}\left[\frac{e^{-2s}}{s^2}\right] = t - u_2(t)(t-2) \\&= \begin{cases} t & 0 \leq t < 2 \\ 2 & t \geq 2. \end{cases}\end{aligned}$$

4.4 Step Functions



And what is the Laplace Transform of $e^{ct}f(t)$?

4.4 Step Functions



And what is the Laplace Transform of $e^{ct}f(t)$?

$$\mathcal{L}[e^{ct}f(t)] = \int_0^{\infty} e^{-st} e^{ct} f(t) dt = \int_0^{\infty} e^{-(s-c)t} f(t) dt = F(s-c).$$

4.4 Step Functions



And what is the Laplace Transform of $e^{ct}f(t)$?

$$\mathcal{L}[e^{ct}f(t)] = \int_0^{\infty} e^{-st} e^{ct} f(t) dt = \int_0^{\infty} e^{-(s-c)t} f(t) dt = F(s - c).$$

Theorem

$$\mathcal{L}[e^{ct}f(t)] = F(s - c)$$

4.4 Step Functions

$$\mathcal{L} [e^{ct} f(t)] = F(s - c)$$

Example

Find the inverse Laplace Transform of $G(s) = \frac{1}{s^2 - 4s + 5}$.

4.4 Step Functions



$$\mathcal{L} [e^{ct} f(t)] = F(s - c)$$

Example

Find the inverse Laplace Transform of $G(s) = \frac{1}{s^2 - 4s + 5}$.

Note first that

$$G(s) = \frac{1}{s^2 - 4s + 5} = \frac{1}{(s - 2)^2 + 1}.$$

4.4 Step Functions

$$\mathcal{L}[e^{ct}f(t)] = F(s - c)$$

Example

Find the inverse Laplace Transform of $G(s) = \frac{1}{s^2 - 4s + 5}$.

Note first that

$$G(s) = \frac{1}{s^2 - 4s + 5} = \frac{1}{(s - 2)^2 + 1}.$$

If $F(s) = \frac{1}{s^2 + 1}$, then we have $G(s) = F(s - 2)$.

4.4 Step Functions

$$\mathcal{L} [e^{ct} f(t)] = F(s - c)$$

Example

Find the inverse Laplace Transform of $G(s) = \frac{1}{s^2 - 4s + 5}$.

Note first that

$$G(s) = \frac{1}{s^2 - 4s + 5} = \frac{1}{(s - 2)^2 + 1}.$$

If $F(s) = \frac{1}{s^2 + 1}$, then we have $G(s) = F(s - 2)$. But

$$\mathcal{L}^{-1} [F] = \mathcal{L}^{-1} \left[\frac{1}{s^2 + 1} \right] = \sin t.$$

4.4 Step Functions



$$\mathcal{L} [e^{ct} f(t)] = F(s - c)$$

Example

Find the inverse Laplace Transform of $G(s) = \frac{1}{s^2 - 4s + 5}$.

Note first that

$$G(s) = \frac{1}{s^2 - 4s + 5} = \frac{1}{(s - 2)^2 + 1}.$$

If $F(s) = \frac{1}{s^2 + 1}$, then we have $G(s) = F(s - 2)$. But

$$\mathcal{L}^{-1} [F] = \mathcal{L}^{-1} \left[\frac{1}{s^2 + 1} \right] = \sin t.$$

Therefore

$$g(t) = \mathcal{L}^{-1} [G] = \mathcal{L}^{-1} [F(s - 2)]$$

4.4 Step Functions

$$\mathcal{L} [e^{ct} f(t)] = F(s - c)$$

Example

Find the inverse Laplace Transform of $G(s) = \frac{1}{s^2 - 4s + 5}$.

Note first that

$$G(s) = \frac{1}{s^2 - 4s + 5} = \frac{1}{(s - 2)^2 + 1}.$$

If $F(s) = \frac{1}{s^2 + 1}$, then we have $G(s) = F(s - 2)$. But

$$\mathcal{L}^{-1} [F] = \mathcal{L}^{-1} \left[\frac{1}{s^2 + 1} \right] = \sin t.$$

Therefore

$$g(t) = \mathcal{L}^{-1} [G] = \mathcal{L}^{-1} [F(s - 2)] = e^{2t} \mathcal{L}^{-1} [F] = e^{2t} \sin t.$$



4.4 Step Functions

How to find the inverse Laplace Transform of $G(s) = \frac{ms + n}{as^2 + bs + c}$

4.4 Step Functions

How to find the inverse Laplace Transform of $G(s) = \frac{ms + n}{as^2 + bs + c}$



Does $as^2 + bs + c = 0$
have roots in \mathbb{R} ?

4.4 Step Functions

How to find the inverse Laplace Transform of $G(s) = \frac{ms + n}{as^2 + bs + c}$

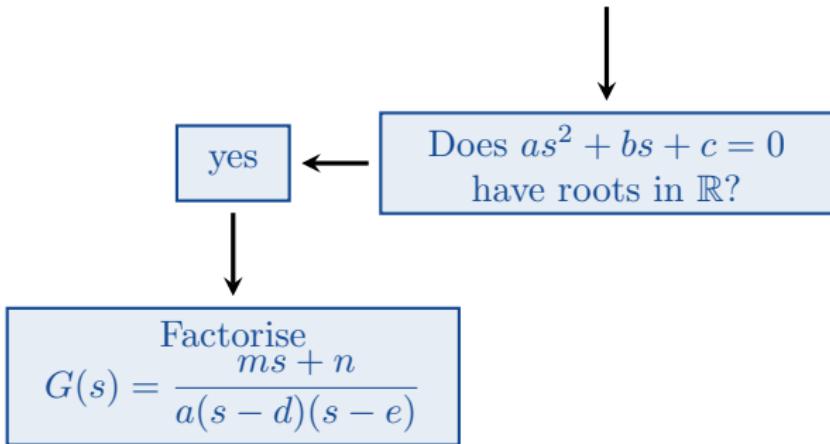
yes

Does $as^2 + bs + c = 0$
have roots in \mathbb{R} ?



4.4 Step Functions

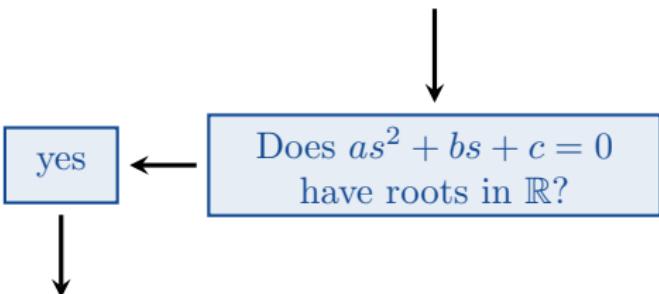
How to find the inverse Laplace Transform of $G(s) = \frac{ms + n}{as^2 + bs + c}$



4.4 Step Functions



How to find the inverse Laplace Transform of $G(s) = \frac{ms + n}{as^2 + bs + c}$



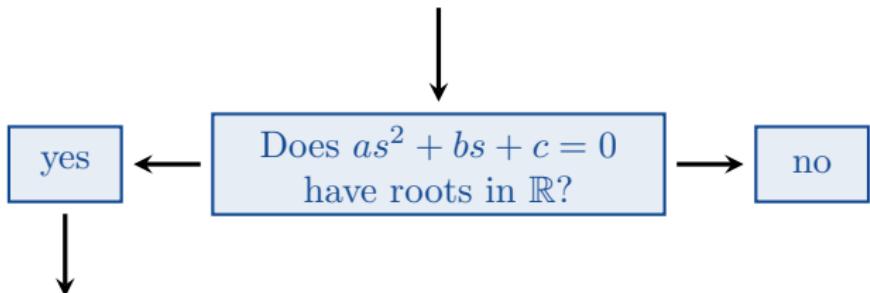
$$G(s) = \frac{ms + n}{a(s - d)(s - e)}$$

$$G(s) = \frac{A}{s - d} + \frac{B}{s - e}$$

4.4 Step Functions



How to find the inverse Laplace Transform of $G(s) = \frac{ms + n}{as^2 + bs + c}$

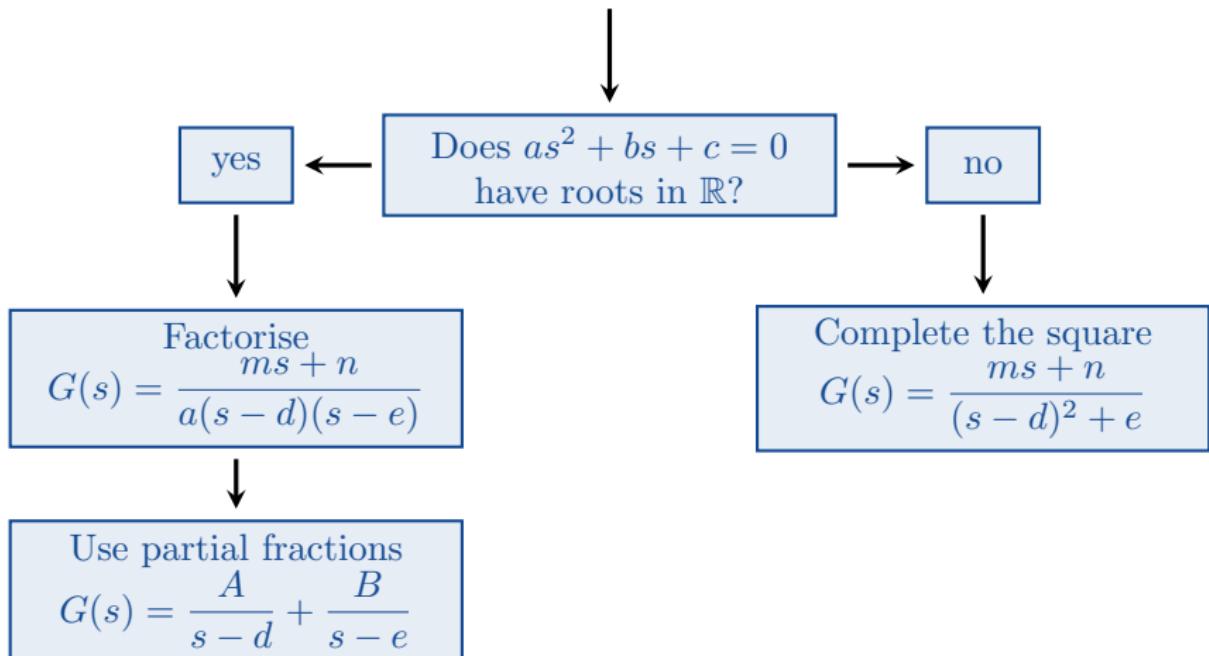


$$G(s) = \frac{ms + n}{a(s - d)(s - e)}$$

$$\text{Use partial fractions}$$
$$G(s) = \frac{A}{s - d} + \frac{B}{s - e}$$

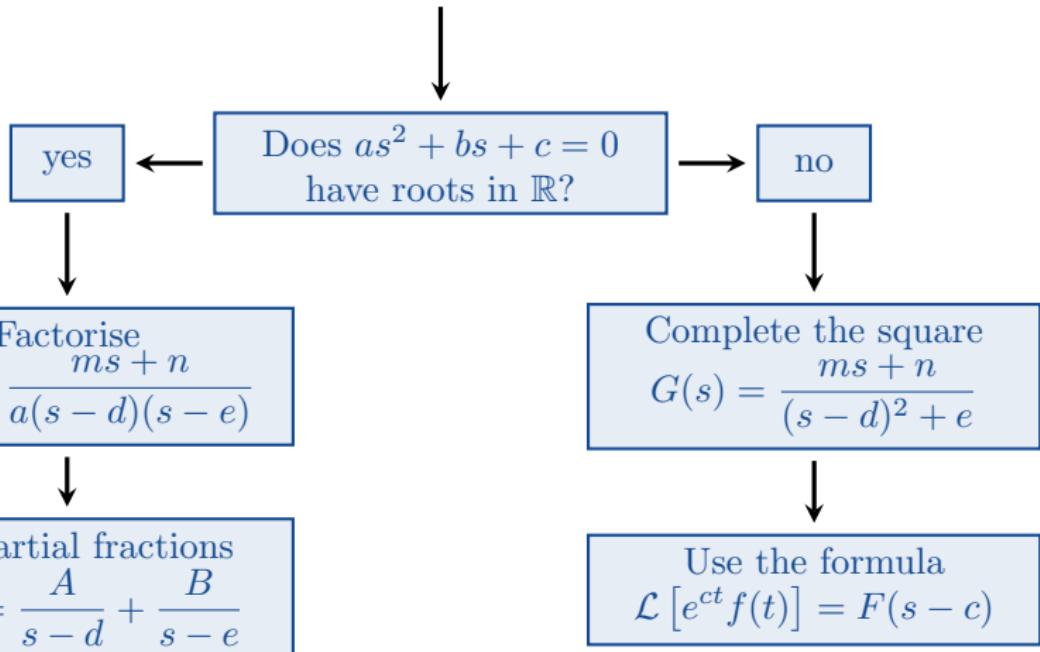
4.4 Step Functions

How to find the inverse Laplace Transform of $G(s) = \frac{ms + n}{as^2 + bs + c}$



4.4 Step Functions

How to find the inverse Laplace Transform of $G(s) = \frac{ms+n}{as^2+bs+c}$



4.4 Step Functions



Example

Find the inverse Laplace Transform of $G(s) = \frac{30s + 440}{s^2 + 32s + 240}$.

4.4 Step Functions



Example

Find the inverse Laplace Transform of $G(s) = \frac{30s + 440}{s^2 + 32s + 240}$.

First note that $s^2 + 32s + 240 = 0$ has roots $s_1 = -12$ and $s_2 = -20$.

4.4 Step Functions



Example

Find the inverse Laplace Transform of $G(s) = \frac{30s + 440}{s^2 + 32s + 240}$.

First note that $s^2 + 32s + 240 = 0$ has roots $s_1 = -12$ and $s_2 = -20$. In fact

$$G(s) = \frac{30s + 440}{s^2 + 32s + 240} = \frac{10}{s + 12} + \frac{20}{s + 20}.$$

4.4 Step Functions



Example

Find the inverse Laplace Transform of $G(s) = \frac{30s + 440}{s^2 + 32s + 240}$.

First note that $s^2 + 32s + 240 = 0$ has roots $s_1 = -12$ and $s_2 = -20$. In fact

$$G(s) = \frac{30s + 440}{s^2 + 32s + 240} = \frac{10}{s + 12} + \frac{20}{s + 20}.$$

I leave this example for you to finish.

4.4 Step Functions



Example

Find the inverse Laplace Transform of $G(s) = \frac{10s + 12}{s^2 + 40s + 420}$.

4.4 Step Functions



Example

Find the inverse Laplace Transform of $G(s) = \frac{10s + 12}{s^2 + 40s + 420}$.

Since the roots of $s^2 + 40s + 420 = 0$ are $s = -20 \pm 2i\sqrt{5}$, we must complete the square.

4.4 Step Functions

Example

Find the inverse Laplace Transform of $G(s) = \frac{10s + 12}{s^2 + 40s + 420}$.

Since the roots of $s^2 + 40s + 420 = 0$ are $s = -20 \pm 2i\sqrt{5}$, we must complete the square. You can check that

$$G(s) = \frac{10s + 12}{s^2 + 40s + 420} = \frac{10s + 12}{(s + 20)^2 + 20}.$$

4.4 Step Functions



Now

$$\begin{aligned}G(s) &= \frac{10s + 12}{(s + 20)^2 + 20} \\&= 10 \left(\frac{s}{(s + 20)^2 + 20} \right) + \frac{12}{\sqrt{20}} \left(\frac{\sqrt{20}}{(s + 20)^2 + 20} \right)\end{aligned}$$

4.4 Step Functions

Now

$$\begin{aligned}
 G(s) &= \frac{10s + 12}{(s + 20)^2 + 20} \\
 &= 10 \left(\frac{s}{(s + 20)^2 + 20} \right) + \frac{12}{\sqrt{20}} \left(\frac{\sqrt{20}}{(s + 20)^2 + 20} \right) \\
 &= 10F(s + 20) + \frac{12}{\sqrt{20}}H(s + 20)
 \end{aligned}$$

where $F(s) = \frac{s}{s^2 + 20}$ and $H(s) = \frac{\sqrt{20}}{s^2 + 20}$.

4.4 Step Functions

Now

$$\begin{aligned}
 G(s) &= \frac{10s + 12}{(s + 20)^2 + 20} \\
 &= 10 \left(\frac{s}{(s + 20)^2 + 20} \right) + \frac{12}{\sqrt{20}} \left(\frac{\sqrt{20}}{(s + 20)^2 + 20} \right) \\
 &= 10F(s + 20) + \frac{12}{\sqrt{20}}H(s + 20)
 \end{aligned}$$

where $F(s) = \frac{s}{s^2 + 20}$ and $H(s) = \frac{\sqrt{20}}{s^2 + 20}$.

Note that

$$f(t) = \mathcal{L}^{-1}[F](t) = \cos \sqrt{20}t$$

and

$$h(t) = \mathcal{L}^{-1}[H](t) = \sin \sqrt{20}t.$$

4.4 Step Functions



$$\mathcal{L} [e^{ct} f(t)] = F(s - c) \quad G(s) = 10F(s + 20) + \frac{12}{20}H(s + 20)$$

Therefore

$$g(t) = 10\mathcal{L}^{-1} [F(s + 20)] + \frac{12}{\sqrt{20}}\mathcal{L}^{-1} [H(s + 20)]$$

=

=

.

4.4 Step Functions



$$\mathcal{L} [e^{ct} f(t)] = F(s - c) \quad G(s) = 10F(s + 20) + \frac{12}{20}H(s + 20)$$

Therefore

$$g(t) = 10\mathcal{L}^{-1} [F(s + 20)] + \frac{12}{\sqrt{20}}\mathcal{L}^{-1} [H(s + 20)]$$

$$= 10e^{-20t}\mathcal{L}^{-1} [F] + \frac{12}{\sqrt{20}}e^{-20t}\mathcal{L}^{-1} [H]$$

=

.

4.4 Step Functions



$$\mathcal{L} [e^{ct} f(t)] = F(s - c) \quad G(s) = 10F(s + 20) + \frac{12}{20}H(s + 20)$$

Therefore

$$\begin{aligned} g(t) &= 10\mathcal{L}^{-1} [F(s + 20)] + \frac{12}{\sqrt{20}}\mathcal{L}^{-1} [H(s + 20)] \\ &= 10e^{-20t}\mathcal{L}^{-1} [F] + \frac{12}{\sqrt{20}}e^{-20t}\mathcal{L}^{-1} [H] \\ &= 10e^{-20t} \cos \sqrt{20}t + \frac{12}{\sqrt{20}}e^{-20t} \sin \sqrt{20}t. \end{aligned}$$



Next Time

- 4.5 ODEs with Discontinuous Forcing Functions
- 4.6 The Convolution Integral