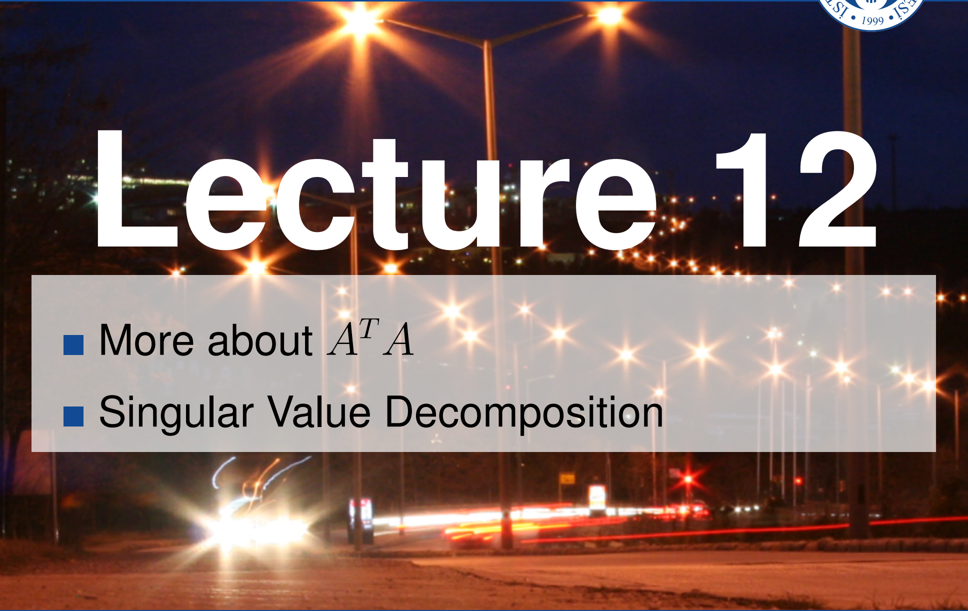


Lecture 12

- More about $A^T A$
- Singular Value Decomposition

More about $A^T A$

Revision from Lecture 3

Note that if A is an $m \times n$ matrix, then A^T is an $n \times m$ matrix, so the products AA^T and A^TA are both square matrices.

Revision from Lecture 3

Note that if A is an $m \times n$ matrix, then A^T is an $n \times m$ matrix, so the products AA^T and $A^T A$ are both square matrices. Moreover, since $(AB)^T = B^T A^T$, we have

$$\left(AA^T\right)^T = (A^T)^T A^T$$

Revision from Lecture 3

Note that if A is an $m \times n$ matrix, then A^T is an $n \times m$ matrix, so the products AA^T and $A^T A$ are both square matrices. Moreover, since $(AB)^T = B^T A^T$, we have

$$\left(AA^T\right)^T = (A^T)^T A^T = AA^T$$

Revision from Lecture 3

Note that if A is an $m \times n$ matrix, then A^T is an $n \times m$ matrix, so the products AA^T and $A^T A$ are both square matrices. Moreover, since $(AB)^T = B^T A^T$, we have

$$\left(AA^T\right)^T = (A^T)^T A^T = AA^T$$

and

$$\left(A^T A\right)^T = A^T (A^T)^T = A^T A$$

which shows that both AA^T and $A^T A$ are symmetric.

Example

Let

$$A = \begin{bmatrix} 1 & -2 & 4 \\ 3 & 0 & -5 \end{bmatrix}.$$

Please check that

$$A^T A = \begin{bmatrix} 1 & 3 \\ -2 & 0 \\ 4 & -5 \end{bmatrix} \begin{bmatrix} 1 & -2 & 4 \\ 3 & 0 & -5 \end{bmatrix} = \begin{bmatrix} 10 & -2 & -11 \\ -2 & 4 & -8 \\ -11 & -8 & 41 \end{bmatrix}$$

and

$$A A^T = \begin{bmatrix} 1 & -2 & 4 \\ 3 & 0 & -5 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ -2 & 0 \\ 4 & -5 \end{bmatrix} = \begin{bmatrix} 21 & -17 \\ -17 & 34 \end{bmatrix}.$$

More about $A^T A$

Theorem

If A is an invertible matrix, then AA^T and $A^T A$ are also invertible.

Proof.

A is invertible $\implies$ A^T is invertible. Recall that the product of two invertible matrices is invertible. $\square$

Six Lemmata about $A^T A$

Let A be an $m \times n$ matrix. I want to state and prove 6 lemmata[1] about $A^T A$.

[1] "lemmata" is the plural of "lemma"

More about $A^T A$

Lemma (1/6)

A and $A^T A$ have the same null space.

More about $A^T A$

Lemma (1/6)

A and $A^T A$ have the same null space.

Proof.

We must show that

$$\mathbf{x}_0 \text{ is a solution of } A\mathbf{x} = \mathbf{0} \iff \mathbf{x}_0 \text{ is a solution of } A^T A\mathbf{x} = \mathbf{0}$$

Lemma (1/6)

A and $A^T A$ have the same null space.

Proof.

We must show that

$$\mathbf{x}_0 \text{ is a solution of } A\mathbf{x} = \mathbf{0} \iff \mathbf{x}_0 \text{ is a solution of } A^T A\mathbf{x} = \mathbf{0}$$

Note that

$$\mathbf{x}_0 \text{ is a solution of } A\mathbf{x} = \mathbf{0} \implies A^T A\mathbf{x}_0 = A^T(A\mathbf{x}_0) = A^T\mathbf{0} = \mathbf{0}.$$

Next we need to prove " $\Longleftarrow$ ".

Proof Continued.

Suppose that $\mathbf{x}_0$ is a solution of $A^T A\mathbf{x} = \mathbf{0}$, which is the same as supposing that $\mathbf{x}_0 \in \text{Nul}(A^T A)$. We need to show that $A\mathbf{x}_0 = \mathbf{0}$ too.

Proof Continued.

Suppose that $\mathbf{x}_0$ is a solution of $A^TA\mathbf{x} = \mathbf{0}$, which is the same as supposing that $\mathbf{x}_0 \in \mathrm{Nul}(A^TA)$. We need to show that $A\mathbf{x}_0 = \mathbf{0}$ too.

Recall that in Lecture 7 we saw that

$$\mathrm{Nul}\, B = (\mathrm{Row}\, B)^{\perp}.$$

This means that $\mathbf{x}_0$ is orthogonal to every vector in the row space of A^TA.

More about A^TA

Proof Continued.

Suppose that $\mathbf{x}_0$ is a solution of $A^TA\mathbf{x} = \mathbf{0}$, which is the same as supposing that $\mathbf{x}_0 \in \mathrm{Nul}(A^TA)$. We need to show that $A\mathbf{x}_0 = \mathbf{0}$ too.

Recall that in Lecture 7 we saw that

$$\mathrm{Nul}\, B = (\mathrm{Row}\, B)^\perp.$$

This means that $\mathbf{x}_0$ is orthogonal to every vector in the row space of A^TA.

But A^TA is symmetric, so $\mathbf{x}_0$ is also orthogonal to every vector in the column space of A^TA.

More about $A^T A$

Proof Continued.

Since the vector $\mathbf{y}_0 = (A^T A)\mathbf{x}_0$ is in $\mathrm{Col}(A^T A)$ we must have

$$0 = \mathbf{x}_0 \cdot \mathbf{y}_0 = \qquad = \qquad = \qquad .$$

Proof Continued.

Since the vector $\mathbf{y}_0 = (A^T A)\mathbf{x}_0$ is in $\mathrm{Col}(A^T A)$ we must have

$$0 = \mathbf{x}_0 \cdot \mathbf{y}_0 = \mathbf{x}_0 \cdot A^T A \mathbf{x}_0 = \qquad\qquad = \qquad\qquad .$$

Proof Continued.

Since the vector $\mathbf{y}_0 = (A^TA)\mathbf{x}_0$ is in $\mathrm{Col}(A^TA)$ we must have

$$0 = \mathbf{x}_0 \cdot \mathbf{y}_0 = \mathbf{x}_0 \cdot A^TA\mathbf{x}_0 = A\mathbf{x}_0 \cdot A\mathbf{x}_0 = \qquad\qquad .$$

More about $A^T A$

Proof Continued.

Since the vector $\mathbf{y}_0 = (A^T A)\mathbf{x}_0$ is in $\mathrm{Col}(A^T A)$ we must have

$$0 = \mathbf{x}_0 \cdot \mathbf{y}_0 = \mathbf{x}_0 \cdot A^T A\mathbf{x}_0 = A\mathbf{x}_0 \cdot A\mathbf{x}_0 = \|A\mathbf{x}_0\|^2 .$$

Proof Continued.

Since the vector $\mathbf{y}_0 = (A^T A)\mathbf{x}_0$ is in $\mathrm{Col}(A^T A)$ we must have

$$0 = \mathbf{x}_0 \cdot \mathbf{y}_0 = \mathbf{x}_0 \cdot A^T A\mathbf{x}_0 = A\mathbf{x}_0 \cdot A\mathbf{x}_0 = \left\|A\mathbf{x}_0\right\|^2.$$

Hence $A\mathbf{x}_0 = \mathbf{0}$ and we are finished. $\square$

More about $A^T A$

Lemma (2/6)

A and $A^T A$ have the same rank

More about $A^T A$

Lemma (2/6)

A and $A^T A$ have the same rank

Proof.

Recall that for any matrix B we have

$$\operatorname{rank} B + \operatorname{nullity} B = \begin{array}{c}\text{number of} \\ \text{columns in } B.\end{array}$$

More about $A^T A$

Lemma (2/6)

A and $A^T A$ have the same rank

Proof.

Recall that for any matrix B we have

$$\operatorname{rank} B + \operatorname{nullity} B = \begin{array}{c}\text{number of}\\ \text{columns in } B.\end{array}$$

Since A and $A^T A$ have the same null space, and since they have the same number of columns we must have that

$$\operatorname{rank} A = \operatorname{rank}(A^T A).$$

□

More about $A^T A$

Lemma (3/6)

A and $A^T A$ have the same row space.

More about $A^T A$

Lemma (3/6)

A and $A^T A$ have the same row space.

Proof.

Because $A^T A$ is symmetric, we have $\mathrm{Row}(A^T A) = \mathrm{Col}(A^T A)$.

Each column of $A^T A$ is a linear combination of the columns in A^T because of the way in which we multiply matrices together (think about it).

More about $A^T A$

Lemma (3/6)

A and $A^T A$ have the same row space.

Proof.

Because $A^T A$ is symmetric, we have $\mathrm{Row}(A^T A) = \mathrm{Col}(A^T A)$.

Each column of $A^T A$ is a linear combination of the columns in A^T because of the way in which we multiply matrices together (think about it).

So every column of $A^T A$ is in $\mathrm{Col}\, A^T = \mathrm{Row}\, A$.

Lemma (3/6)

A and $A^T A$ have the same row space.

Proof.

Because $A^T A$ is symmetric, we have $\mathrm{Row}(A^T A) = \mathrm{Col}(A^T A)$.

Each column of $A^T A$ is a linear combination of the columns in A^T because of the way in which we multiply matrices together (think about it).

So every column of $A^T A$ is in $\mathrm{Col}\, A^T = \mathrm{Row}\, A$. So every row of $A^T A$ is in $\mathrm{Row}(A)$.

More about $A^T A$

Lemma (3/6)

A and $A^T A$ have the same row space.

Proof.

Because $A^T A$ is symmetric, we have $\mathrm{Row}(A^T A) = \mathrm{Col}(A^T A)$.

Each column of $A^T A$ is a linear combination of the columns in A^T because of the way in which we multiply matrices together (think about it).

So every column of $A^T A$ is in $\mathrm{Col}\, A^T = \mathrm{Row}\, A$. So every row of $A^T A$ is in $\mathrm{Row}(A)$. This proves that $\mathrm{Row}(A^T A)$ is a subspace of $\mathrm{Row}\, A$.

Lemma (3/6)

A and $A^T A$ have the same row space.

Proof.

Because $A^T A$ is symmetric, we have $\mathrm{Row}(A^T A) = \mathrm{Col}(A^T A)$.

Each column of $A^T A$ is a linear combination of the columns in A^T because of the way in which we multiply matrices together (think about it).

So every column of $A^T A$ is in $\mathrm{Col}\, A^T = \mathrm{Row}\, A$. So every row of $A^T A$ is in $\mathrm{Row}(A)$. This proves that $\mathrm{Row}(A^T A)$ is a subspace of $\mathrm{Row}\, A$.

But since $A^T A$ and A have the same rank, we must have that $\mathrm{Row}\, A = \mathrm{Row}(A^T A)$. $\square$

More about $A^T A$

Lemma (4/6)

A^T and $A^T A$ have the same column space.

More about $A^T A$

Lemma (4/6)

A^T and $A^T A$ have the same column space.

Proof.

$$\operatorname{Col} A^T = \operatorname{Row} A = \operatorname{Row}(A^T A) = \operatorname{Col}(A^T A).$$

□

More about $A^T A$

Lemma (5/6)

$A^T A$ *is orthogonally diagonalisable.*

More about $A^T A$

Lemma (5/6)

$A^T A$ is orthogonally diagonalisable.

Proof.

$A^T A$ is symmetric and all symmetric matrices are orthogonally diagonalisable. $\square$

Lemma (6/6)

The eigenvalues of $A^T A$ are nonnegative (i.e. $\lambda \geq 0$).

More about $A^T A$

Lemma (6/6)

The eigenvalues of $A^T A$ are nonnegative (i.e. $\lambda \geq 0$).

Proof.

Since $A^T A$ is orthogonally diagonalisable, there is an orthonormal basis for $\mathbb{R}^n$ consisting of eigenvectors of $A^T A$, say

$$\{\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_n\}.$$

More about $A^T A$

Lemma (6/6)

The eigenvalues of $A^T A$ are nonnegative (i.e. $\lambda \geq 0$).

Proof.

Since $A^T A$ is orthogonally diagonalisable, there is an orthonormal basis for $\mathbb{R}^n$ consisting of eigenvectors of $A^T A$, say

$$\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n\}.$$

Let $\lambda_1, \lambda_2, \ldots, \lambda_n$ be the corresponding eigenvalues (these may not be all different).

Lemma (6/6)

The eigenvalues of $A^T A$ are nonnegative (i.e. $\lambda \geq 0$).

Proof.

Since $A^T A$ is orthogonally diagonalisable, there is an orthonormal basis for $\mathbb{R}^n$ consisting of eigenvectors of $A^T A$, say

$$\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n\}.$$

Let $\lambda_1, \lambda_2, \ldots, \lambda_n$ be the corresponding eigenvalues (these may not be all different). For $1 \leq i \leq n$ we have that

$$0 \leq \|A\mathbf{u}_i\|^2 = A\mathbf{u}_i \cdot A\mathbf{u}_i$$

More about $A^T A$

Lemma (6/6)

The eigenvalues of $A^T A$ are nonnegative (i.e. $\lambda \geq 0$).

Proof.

Since $A^T A$ is orthogonally diagonalisable, there is an orthonormal basis for $\mathbb{R}^n$ consisting of eigenvectors of $A^T A$, say

$$\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n\}.$$

Let $\lambda_1, \lambda_2, \ldots, \lambda_n$ be the corresponding eigenvalues (these may not be all different). For $1 \leq i \leq n$ we have that

$$0 \leq \|A\mathbf{u}_i\|^2 = A\mathbf{u}_i \cdot A\mathbf{u}_i = \mathbf{u}_i \cdot A^T A\mathbf{u}_i$$

Lemma (6/6)

The eigenvalues of $A^T A$ are nonnegative (i.e. $\lambda \geq 0$).

Proof.

Since $A^T A$ is orthogonally diagonalisable, there is an orthonormal basis for $\mathbb{R}^n$ consisting of eigenvectors of $A^T A$, say

$$\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n\}.$$

Let $\lambda_1, \lambda_2, \ldots, \lambda_n$ be the corresponding eigenvalues (these may not be all different). For $1 \leq i \leq n$ we have that

$$0 \leq \|A\mathbf{u}_i\|^2 = A\mathbf{u}_i \cdot A\mathbf{u}_i = \mathbf{u}_i \cdot A^T A\mathbf{u}_i = \mathbf{u}_i \cdot \lambda_i \mathbf{u}_i$$

Lemma (6/6)

The eigenvalues of $A^T A$ are nonnegative (i.e. $\lambda \geq 0$).

Proof.

Since $A^T A$ is orthogonally diagonalisable, there is an orthonormal basis for $\mathbb{R}^n$ consisting of eigenvectors of $A^T A$, say

$$\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n\}.$$

Let $\lambda_1, \lambda_2, \ldots, \lambda_n$ be the corresponding eigenvalues (these may not be all different). For $1 \leq i \leq n$ we have that

$$0 \leq \|A\mathbf{u}_i\|^2 = A\mathbf{u}_i \cdot A\mathbf{u}_i = \mathbf{u}_i \cdot A^T A\mathbf{u}_i = \mathbf{u}_i \cdot \lambda_i \mathbf{u}_i = \lambda_i(\mathbf{u}_i \cdot \mathbf{u}_i)$$

Lemma (6/6)

The eigenvalues of $A^T A$ are nonnegative (i.e. $\lambda \geq 0$).

Proof.

Since $A^T A$ is orthogonally diagonalisable, there is an orthonormal basis for $\mathbb{R}^n$ consisting of eigenvectors of $A^T A$, say

$$\{\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_n\}.$$

Let $\lambda_1, \lambda_2, \dots, \lambda_n$ be the corresponding eigenvalues (these may not be all different). For $1 \leq i \leq n$ we have that

$$0 \leq \|A\mathbf{u}_i\|^2 = A\mathbf{u}_i \cdot A\mathbf{u}_i = \mathbf{u}_i \cdot A^T A\mathbf{u}_i = \mathbf{u}_i \cdot \lambda_i \mathbf{u}_i = \lambda_i(\mathbf{u}_i \cdot \mathbf{u}_i) = \lambda_i$$

$\square$

Singular Values

Definition

If A is an $m \times n$ matrix, and if $\lambda_1, \lambda_2, \ldots, \lambda_n$ are the eigenvalues of A^TA, then the numbers

$$\sigma_1 = \sqrt{\lambda_1}, \quad \sigma_2 = \sqrt{\lambda_2}, \quad \ldots, \quad \sigma_n = \sqrt{\lambda_n}$$

are called the *singular values* of A.

σ is the lowercase of the Greek letter sigma, and Σ is capital sigma.

More about $A^T A$

Remark

From now on, we are going to assume that the eigenvalues of $A^T A$ are named so that

$$\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n \geq 0$$

and hence that

$$\sigma_1 \geq \sigma_2 \geq \ldots \geq \sigma_n \geq 0.$$

Example

Find the singular values of the matrix $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}$.

Example

Find the singular values of the matrix $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}$.

First we need to find the eigenvalues of the matrix

$$A^TA = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}.$$

Example

Find the singular values of the matrix $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}$.

First we need to find the eigenvalues of the matrix

$$A^T A = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}.$$

The characteristic equation is

$$\begin{aligned} 0 = \det(\lambda I - A) &= \begin{bmatrix} \lambda - 2 & -1 \\ -1 & \lambda - 2 \end{bmatrix} = (\lambda - 2)^2 - 1 \\ &= \lambda^2 - 4\lambda + 3 = (\lambda - 3)(\lambda - 1). \end{aligned}$$

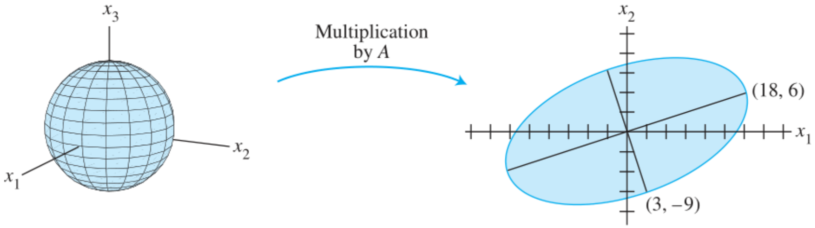
More about $A^T A$

So the eigenvalues are $\lambda_1 = 3$ and $\lambda_2 = 1$. (Note that we put the biggest eigenvalue first.)

So the eigenvalues are $\lambda_1 = 3$ and $\lambda_2 = 1$. (Note that we put the biggest eigenvalue first.)

Therefore the singular values of A are

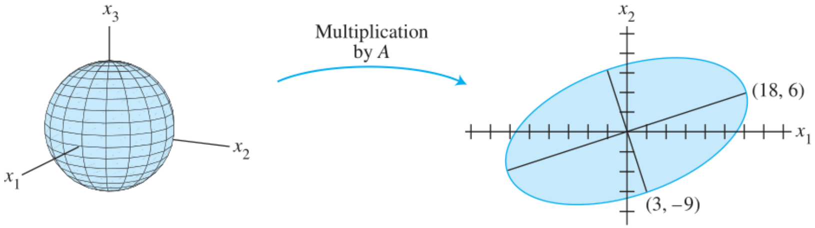
$$\sigma_1 = \sqrt{\lambda_1} = \sqrt{3}, \qquad \sigma_2 = \sqrt{\lambda_2} = 1.$$



Example

Find the singular values of the matrix $A = \begin{bmatrix} 4 & 11 & 14 \\ 8 & 7 & -2 \end{bmatrix}$.

More about $A^T A$



Example

Find the singular values of the matrix $A = \begin{bmatrix} 4 & 11 & 14 \\ 8 & 7 & -2 \end{bmatrix}$.

We need to find the eigenvalues of the following matrix, then take their square roots:

$$A^T A = \begin{bmatrix} 4 & 8 \\ 11 & 7 \\ 14 & -2 \end{bmatrix} \begin{bmatrix} 4 & 11 & 14 \\ 8 & 7 & -2 \end{bmatrix} = \begin{bmatrix} 80 & 100 & 40 \\ 100 & 170 & 140 \\ 40 & 140 & 200 \end{bmatrix}.$$

More about $A^T A$

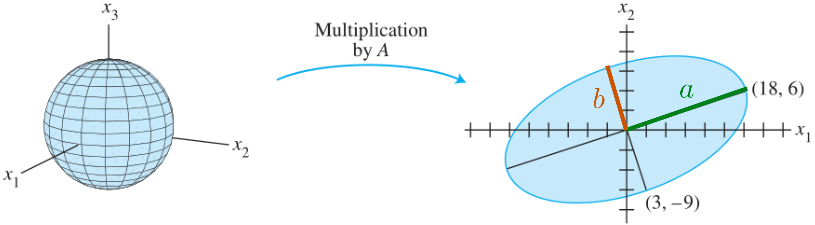
The eigenvalues of $A^T A$ are $\lambda_1 = 360$, $\lambda_2 = 90$ and $\lambda_3 = 0$ (please check!).

More about $A^T A$

The eigenvalues of $A^T A$ are $\lambda_1 = 360$, $\lambda_2 = 90$ and $\lambda_3 = 0$ (please check!). Therefore the singular values of A are

$$\sigma_1 = \sqrt{360} = 6\sqrt{10}, \qquad \sigma_2 = \sqrt{90} = 3\sqrt{10}, \qquad \sigma_3 = \sqrt{0} = 0.$$

More about $A^T A$



Remark

The matrix transformation $T_A : \mathbb{R}^3 \to \mathbb{R}^2$, $\mathbf{x} \mapsto A\mathbf{x}$ maps the unit sphere in $\mathbb{R}^3$ to an ellipse in $\mathbb{R}^2$ as shown above. This ellipse has semimajor axis[2]

$$a = \sqrt{18^2 + 6^2} = \sqrt{360} = \sigma_1$$

and seminor axis

$$b = \sqrt{3^2 + (-9)^2} = \sqrt{90} = \sigma_2.$$

[2] see Thomas' Calculus

More about $A^T A$

Theorem

Suppose $\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n\}$ *is an orthonormal basis of* $\mathbb{R}^n$ *consisting of eigenvectors of* $A^T A$*, arranged so that the corresponding eigenvalues of* $A^T A$ *satisfy*

$$\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n \geq 0,$$

and suppose A *has* r *nonzero singular values.*

More about $A^T A$

Theorem

Suppose $\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n\}$ *is an orthonormal basis of* $\mathbb{R}^n$ *consisting of eigenvectors of* $A^T A$*, arranged so that the corresponding eigenvalues of* $A^T A$ *satisfy*

$$\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n \geq 0,$$

and suppose A *has* r *nonzero singular values.*

Then $\{A\mathbf{u}_1, A\mathbf{u}_2, \ldots, A\mathbf{u}_r\}$ *is an orthogonal basis for* $\operatorname{Col} A$*, and*

$$\operatorname{rank} A = r.$$

More about $A^T A$

Proof.

Suppose that $i \neq j$. Since $\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n\}$ is an orthonormal basis, we know that

$$\mathbf{u}_i \bullet \mathbf{u}_j = 0.$$

Proof.

Suppose that $i \neq j$. Since $\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n\}$ is an orthonormal basis, we know that

$$\mathbf{u}_i \cdot \mathbf{u}_j = 0.$$

Therefore

$$A\mathbf{u}_i \cdot A\mathbf{u}_j = \mathbf{u}_i \cdot A^T A\mathbf{u}_j =$$

Proof.

Suppose that $i \neq j$. Since $\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n\}$ is an orthonormal basis, we know that

$$\mathbf{u}_i \bullet \mathbf{u}_j = 0.$$

Therefore

$$A\mathbf{u}_i \bullet A\mathbf{u}_j = \mathbf{u}_i \bullet A^T A\mathbf{u}_j = \mathbf{u}_i \bullet \lambda_j \mathbf{u}_j =$$

Proof.

Suppose that $i \neq j$. Since $\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n\}$ is an orthonormal basis, we know that

$$\mathbf{u}_i \bullet \mathbf{u}_j = 0.$$

Therefore

$$A\mathbf{u}_i \bullet A\mathbf{u}_j = \mathbf{u}_i \bullet A^T A\mathbf{u}_j = \mathbf{u}_i \bullet \lambda_j \mathbf{u}_j = \lambda_j \left(\mathbf{u}_i \bullet \mathbf{u}_j\right) = 0.$$

Proof.

Suppose that $i \neq j$. Since $\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n\}$ is an orthonormal basis, we know that

$$\mathbf{u}_i \cdot \mathbf{u}_j = 0.$$

Therefore

$$A\mathbf{u}_i \cdot A\mathbf{u}_j = \mathbf{u}_i \cdot A^T A\mathbf{u}_j = \mathbf{u}_i \cdot \lambda_j \mathbf{u}_j = \lambda_j \left(\mathbf{u}_i \cdot \mathbf{u}_j\right) = 0.$$

Hence $\{A\mathbf{u}_1, A\mathbf{u}_2, \ldots, A\mathbf{u}_n\}$ is an orthogonal set.

Proof Continued.

Moreover since the theorem said that A has r nonzero singular values, and since

$$\|A\mathbf{u}_i\| = \sqrt{A\mathbf{u}_i \cdot A\mathbf{u}_i} = \sqrt{\lambda_i}\sqrt{\mathbf{u}_i \cdot \mathbf{u}_i} = \sigma_i,$$

we have that

$$A\mathbf{u}_i \neq \mathbf{0} \qquad \iff \qquad 1 \leq i \leq r.$$

More about $A^T A$

Proof Continued.

Moreover since the theorem said that A has r nonzero singular values, and since

$$\|A\mathbf{u}_i\| = \sqrt{A\mathbf{u}_i \cdot A\mathbf{u}_i} = \sqrt{\lambda_i}\sqrt{\mathbf{u}_i \cdot \mathbf{u}_i} = \sigma_i,$$

we have that

$$A\mathbf{u}_i \neq \mathbf{0} \qquad \iff \qquad 1 \leq i \leq r.$$

Thus

- $A\mathbf{u}_1, A\mathbf{u}_2, \ldots, A\mathbf{u}_r$ are linearly independent vectors; and
- $A\mathbf{u}_1, A\mathbf{u}_2, \ldots, A\mathbf{u}_r$ are in $\operatorname{Col} A$.

Proof Continued.

Finally, let $\mathbf{y}$ be any vector in $\operatorname{Col} A$. Let $\mathbf{x} \in \mathbb{R}^n$ satisfy $A\mathbf{x} = \mathbf{y}$. Since $\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n\}$ is an orthonormal basis for $\mathbb{R}^n$, we can write

$$\mathbf{x} = c_1\mathbf{u}_1 + c_2\mathbf{u}_2 + \ldots + c_n\mathbf{u}_n.$$

More about $A^T A$

Proof Continued.

Finally, let $\mathbf{y}$ be any vector in $\operatorname{Col} A$. Let $\mathbf{x} \in \mathbb{R}^n$ satisfy $A\mathbf{x} = \mathbf{y}$. Since $\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n\}$ is an orthonormal basis for $\mathbb{R}^n$, we can write

$$\mathbf{x} = c_1\mathbf{u}_1 + c_2\mathbf{u}_2 + \ldots + c_n\mathbf{u}_n.$$

But then

$$\begin{aligned}\mathbf{y} &= A\mathbf{x} \\ &= c_1A\mathbf{u}_1 + c_2A\mathbf{u}_2 + \ldots + c_rA\mathbf{u}_r + c_{r+1}A\mathbf{u}_{r+1} + \ldots + c_nA\mathbf{u}_n \\ &= c_1A\mathbf{u}_1 + c_2A\mathbf{u}_2 + \ldots + c_rA\mathbf{u}_r + 0 + \ldots + 0.\end{aligned}$$

Proof Continued.

Finally, let $\mathbf{y}$ be any vector in $\operatorname{Col} A$. Let $\mathbf{x} \in \mathbb{R}^n$ satisfy $A\mathbf{x} = \mathbf{y}$. Since $\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n\}$ is an orthonormal basis for $\mathbb{R}^n$, we can write

$$\mathbf{x} = c_1\mathbf{u}_1 + c_2\mathbf{u}_2 + \ldots + c_n\mathbf{u}_n.$$

But then

$$\begin{aligned}\mathbf{y} &= A\mathbf{x} \\ &= c_1 A\mathbf{u}_1 + c_2 A\mathbf{u}_2 + \ldots + c_r A\mathbf{u}_r + c_{r+1} A\mathbf{u}_{r+1} + \ldots + c_n A\mathbf{u}_n \\ &= c_1 A\mathbf{u}_1 + c_2 A\mathbf{u}_2 + \ldots + c_r A\mathbf{u}_r + 0 + \ldots + 0.\end{aligned}$$

This proves that $\mathbf{y} \in \operatorname{span}\{A\mathbf{u}_1, A\mathbf{u}_2, \ldots, A\mathbf{u}_r\}$ and thus that $\{A\mathbf{u}_1, A\mathbf{u}_2, \ldots, A\mathbf{u}_r\}$ is an orthogonal basis for $\operatorname{Col} A$.

More about $A^T A$

Proof Continued.

Finally, let $\mathbf{y}$ be any vector in $\operatorname{Col} A$. Let $\mathbf{x} \in \mathbb{R}^n$ satisfy $A\mathbf{x} = \mathbf{y}$. Since $\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n\}$ is an orthonormal basis for $\mathbb{R}^n$, we can write

$$\mathbf{x} = c_1\mathbf{u}_1 + c_2\mathbf{u}_2 + \ldots + c_n\mathbf{u}_n.$$

But then

$$\begin{aligned} \mathbf{y} &= A\mathbf{x} \\ &= c_1A\mathbf{u}_1 + c_2A\mathbf{u}_2 + \ldots + c_rA\mathbf{u}_r + c_{r+1}A\mathbf{u}_{r+1} + \ldots + c_nA\mathbf{u}_n \\ &= c_1A\mathbf{u}_1 + c_2A\mathbf{u}_2 + \ldots + c_rA\mathbf{u}_r + 0 + \ldots + 0. \end{aligned}$$

This proves that $\mathbf{y} \in \operatorname{span}\{A\mathbf{u}_1, A\mathbf{u}_2, \ldots, A\mathbf{u}_r\}$ and thus that $\{A\mathbf{u}_1, A\mathbf{u}_2, \ldots, A\mathbf{u}_r\}$ is an orthogonal basis for $\operatorname{Col} A$. We conclude that

$$\operatorname{rank} A = \dim \operatorname{Col} A = r.$$

$\square$

Singular Value Decomposition

Singular Value Decomposition

In Lecture 9 we studied diagonalisation

$$A = PDP^{-1}$$

and in Lecture 11 we studied orthogonal diagonalisation

$$A = PDP^{T}.$$

Unfortunately, as we have learned, not all matrices can be factorised like this.

In Lecture 9 we studied diagonalisation

$$A = PDP^{-1}$$

and in Lecture 11 we studied orthogonal diagonalisation

$$A = PDP^T.$$

Unfortunately, as we have learned, not all matrices can be factorised like this.

There is however, another type of factorisation that is possible for <u>any</u> $m \times n$ matrix A. This is called *Singular Value Decomposition* (SVD) and takes the form

$$A = U\Sigma V^T.$$

$$A = U\Sigma V^T.$$

Remark

Note here that we have two different matrices U and V.

$$A = U\Sigma V^T.$$

Remark

Note here that we have two different matrices U and V.

The matrix Σ will be an $m \times n$ matrix of the form

$$\Sigma = \left[\begin{array}{c|c} D & 0 \\ \hline 0 & 0 \end{array}\right]$$

where D is a diagonal matrix and each 0 is a zero matrix. (If $r = m$ or $r = n$ then some of the zero matrices do not appear.)

Singular Value Decomposition

$$A = U \Sigma V^T.$$

Remark

Note here that we have two different matrices U and V.

The matrix Σ will be an $m \times n$ matrix of the form

$$\Sigma = \begin{array}{cl} \left[\begin{array}{c|c} D & 0 \\ \hline 0 & 0 \end{array}\right] & \begin{array}{l} \}\, r \\ \}\, m-r \end{array} \\ \begin{array}{cc} \underbrace{}_{r} & \underbrace{}_{n-r} \end{array} & \end{array}$$

where D is a diagonal matrix and each 0 is a zero matrix. (If $r = m$ or $r = n$ then some of the zero matrices do not appear.)

Theorem (The Singular Value Decomposition Theorem)

Let A be an $m \times n$ matrix with rank r. Then there exists an $m \times n$ matrix Σ (as on the previous slide) for which the diagonal entries in D are the first r singular values of A,

$$\sigma_1 \geq \sigma_2 \geq \ldots \geq \sigma_r > 0,$$

and there exist an $m \times m$ orthogonal matrix U and an $n \times n$ orthogonal matrix V such that

$$A = U \Sigma V^T.$$

Theorem (The Singular Value Decomposition Theorem)

Let A be an $m \times n$ matrix with rank r. Then there exists an $m \times n$ matrix Σ (as on the previous slide) for which the diagonal entries in D are the first r singular values of A,

$$\sigma_1 \geq \sigma_2 \geq \ldots \geq \sigma_r > 0,$$

and there exist an $m \times m$ orthogonal matrix U and an $n \times n$ orthogonal matrix V such that

$$A = U\Sigma V^T.$$

Any factorisation

$$A = U\Sigma V^T$$

where

- U and V are orthogonal matrices;
- $\Sigma = \begin{bmatrix} D & 0 \\ 0 & 0 \end{bmatrix}$; and
- $D = \begin{bmatrix} \sigma_1 & 0 & \cdots & 0 \\ 0 & \sigma_2 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & \sigma_r \end{bmatrix}$

is called a *singular value decomposition* (or SVD) of A.

Singular Value Decomposition

$$A = U\Sigma V^T = U \begin{bmatrix} D & 0 \\ 0 & 0 \end{bmatrix} V^T$$

Remark

The matrices U and V are not unique.

$$A = U\Sigma V^T = U \begin{bmatrix} D & 0 \\ 0 & 0 \end{bmatrix} V^T$$

Remark

The matrices U and V are not unique.

Remark

D (and hence Σ) is unique, because the diagonal entries of D must be the first r singular values of A.

$$A = U\Sigma V^T = U \begin{bmatrix} D & 0 \\ 0 & 0 \end{bmatrix} V^T$$

Remark

The matrices U and V are not unique.

Remark

D (and hence Σ) is unique, because the diagonal entries of D must be the first r singular values of A.

Definition

The columns of U in a SVD are called *left singular vectors* of A.

The columns of V in a SVD are called *right singular vectors* of A.

Proof of the The Singular Value Decomposition Theorem.

Let $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ be an orthonormal basis of $\mathbb{R}^n$ consisting of eigenvectors of $A^T A$, arranged so that the corresponding eigenvalues of $A^T A$ satisfy

$$\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n \geq 0.$$

Proof of the The Singular Value Decomposition Theorem.

Let $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ be an orthonormal basis of $\mathbb{R}^n$ consisting of eigenvectors of A^TA, arranged so that the corresponding eigenvalues of A^TA satisfy

$$\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n \geq 0.$$

Let

$$V = \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \ldots & \mathbf{v}_n \end{bmatrix}.$$

Proof of the The Singular Value Decomposition Theorem.

Let $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ be an orthonormal basis of $\mathbb{R}^n$ consisting of eigenvectors of $A^T A$, arranged so that the corresponding eigenvalues of $A^T A$ satisfy

$$\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n \geq 0.$$

Let

$$V = \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \ldots & \mathbf{v}_n \end{bmatrix}.$$

Note that

- V is an orthogonal $n \times n$ matrix; and
- this is almost exactly the same as the matrix P that we use for orthogonal diagonalisation. The only difference here is that we require that $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n \geq 0$.

Proof Continued.

By the previous theorem, we know that $\{A\mathbf{v}_1, A\mathbf{v}_2, \ldots, A\mathbf{v}_r\}$ is an orthogonal basis for $\operatorname{Col} A$. We need to normalise these vectors.

[3] We talked briefly about this in the "Dimension" section of Lecture 6

Proof Continued.

By the previous theorem, we know that $\{A\mathbf{v}_1, A\mathbf{v}_2, \ldots, A\mathbf{v}_r\}$ is an orthogonal basis for $\operatorname{Col} A$. We need to normalise these vectors. For each $i \in \{1, 2, \ldots, r\}$ define

$$\mathbf{u}_i = \frac{A\mathbf{v}_i}{\|A\mathbf{v}_i\|} = \frac{A\mathbf{v}_i}{\sigma_i}.$$

Then $\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_r\}$ is an orthonormal basis for $\operatorname{Col} A$.

[3] We talked briefly about this in the "Dimension" section of Lecture 6

Proof Continued.

By the previous theorem, we know that $\{A\mathbf{v}_1, A\mathbf{v}_2, \ldots, A\mathbf{v}_r\}$ is an orthogonal basis for $\operatorname{Col} A$. We need to normalise these vectors. For each $i \in \{1, 2, \ldots, r\}$ define

$$\mathbf{u}_i = \frac{A\mathbf{v}_i}{\|A\mathbf{v}_i\|} = \frac{A\mathbf{v}_i}{\sigma_i}.$$

Then $\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_r\}$ is an orthonormal basis for $\operatorname{Col} A$.

By adding in an extra $m - r$ vectors, we can extend[3] this to be an orthonormal basis for $\mathbb{R}^m$:

$$\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_r, \mathbf{u}_{r+1}, \ldots, \mathbf{u}_m\}.$$

[3] We talked briefly about this in the "Dimension" section of Lecture 6

Proof Continued.

Let

$$Q = \begin{bmatrix} \mathbf{u}_1 & \mathbf{u}_2 & \dots & \mathbf{u}_m \end{bmatrix}.$$

Proof Continued.

Let

$$Q = \begin{bmatrix} \mathbf{u}_1 & \mathbf{u}_2 & \dots & \mathbf{u}_m \end{bmatrix}.$$

Note that

- Q is an orthogonal $m \times m$ matrix.

$$\mathbf{u}_i = \frac{A\mathbf{v}_i}{\sigma_i}$$

Proof Continued.

Moreover, note that

$$AV = A \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \dots & \mathbf{v}_n \end{bmatrix}$$

$$=$$

$$=$$

$$\mathbf{u}_i = \frac{A\mathbf{v}_i}{\sigma_i}$$

Proof Continued.

Moreover, note that

$$\begin{aligned} AV &= A \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \dots & \mathbf{v}_n \end{bmatrix} \\ &= \begin{bmatrix} A\mathbf{v}_1 & A\mathbf{v}_2 & \dots & A\mathbf{v}_r & \mathbf{0} & \cdots & \mathbf{0} \end{bmatrix} \\ &= \end{aligned}$$

$$\mathbf{u}_i = \frac{A\mathbf{v}_i}{\sigma_i}$$

Proof Continued.

Moreover, note that

$$\begin{aligned} AV &= A\begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \ldots & \mathbf{v}_n \end{bmatrix} \\ &= \begin{bmatrix} A\mathbf{v}_1 & A\mathbf{v}_2 & \ldots & A\mathbf{v}_r & \mathbf{0} & \cdots & \mathbf{0} \end{bmatrix} \\ &= \begin{bmatrix} \sigma_1\mathbf{u}_1 & \sigma_2\mathbf{u}_2 & \ldots & \sigma_r\mathbf{u}_r & \mathbf{0} & \cdots & \mathbf{0} \end{bmatrix} \end{aligned}$$

$$\Sigma = \begin{bmatrix} D & 0 \\ 0 & 0 \end{bmatrix}$$

Proof Continued.

and that

$$U\Sigma = \begin{bmatrix} \mathbf{u}_1 & \mathbf{u}_2 & \dots & \mathbf{u}_m \end{bmatrix} \begin{bmatrix} \sigma_1 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & \sigma_2 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & \sigma_r & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \end{bmatrix}$$

$$=$$

$$=$$

$$\Sigma = \begin{bmatrix} D & 0 \\ 0 & 0 \end{bmatrix}$$

Proof Continued.

and that

$$U\Sigma = \begin{bmatrix} \mathbf{u}_1 & \mathbf{u}_2 & \dots & \mathbf{u}_m \end{bmatrix} \begin{bmatrix} \sigma_1 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & \sigma_2 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & \sigma_r & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \end{bmatrix}$$

$$= \begin{bmatrix} \sigma_1\mathbf{u}_1 & \sigma_2\mathbf{u}_2 & \dots & \sigma_r\mathbf{u}_r & \mathbf{0} & \cdots & \mathbf{0} \end{bmatrix}$$

$$=$$

Singular Value Decomposition

$$\Sigma = \begin{bmatrix} D & 0 \\ 0 & 0 \end{bmatrix}$$

Proof Continued.

and that

$$U\Sigma = \begin{bmatrix} \mathbf{u}_1 & \mathbf{u}_2 & \dots & \mathbf{u}_m \end{bmatrix} \begin{bmatrix} \sigma_1 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & \sigma_2 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & \sigma_r & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \end{bmatrix}$$

$$= \begin{bmatrix} \sigma_1\mathbf{u}_1 & \sigma_2\mathbf{u}_2 & \dots & \sigma_r\mathbf{u}_r & \mathbf{0} & \cdots & \mathbf{0} \end{bmatrix}$$

$$= AV.$$

Singular Value Decomposition

$$\Sigma = \begin{bmatrix} D & 0 \\ 0 & 0 \end{bmatrix}$$

Proof Continued.

So we have

$$U\Sigma = AV.$$

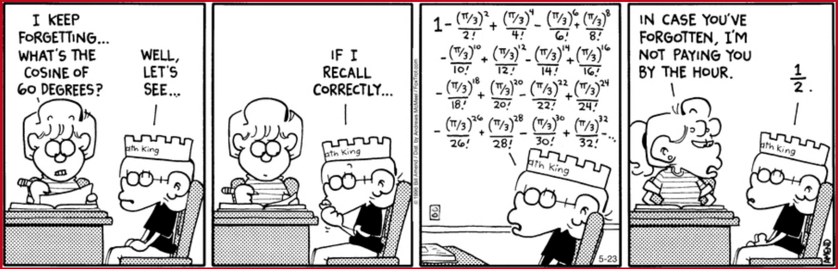
It follows that

$$U\Sigma V^T = AVV^T = A$$

since V is an orthogonal matrix. $\square$

Break

We will continue at 3pm



$$\Sigma = \begin{bmatrix} D & 0 \\ 0 & 0 \end{bmatrix}$$

How to do a Singular Value Decomposition

Let A be an $m \times n$ matrix of rank r.

1. Find[4] an orthonormal basis for $\mathbb{R}^n$ consisting of eigenvectors of $A^T A$. After rearranging as necessary, call this

$$\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$$

where

$$\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n \geq 0.$$

[4] Gram-Schmidt process

$$\Sigma = \begin{bmatrix} D & 0 \\ 0 & 0 \end{bmatrix}$$

How to do a Singular Value Decomposition

Let A be an $m \times n$ matrix of rank r.

1. Find[4] an orthonormal basis for $\mathbb{R}^n$ consisting of eigenvectors of $A^T A$. After rearranging as necessary, call this
$$\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\}$$
where
$$\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_n \geq 0.$$

2. Let $V = \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \dots & \mathbf{v}_n \end{bmatrix}$. This will be an $n \times n$ orthogonal matrix.

[4] Gram-Schmidt process

Singular Value Decomposition

$$\Sigma = \begin{bmatrix} D & 0 \\ 0 & 0 \end{bmatrix}$$

3 Define

$$\mathbf{u}_1 = \frac{A\mathbf{v}_1}{\sigma_1}, \quad \mathbf{u}_2 = \frac{A\mathbf{v}_2}{\sigma_2}, \quad \ldots, \quad \mathbf{u}_r = \frac{A\mathbf{v}_r}{\sigma_r}$$

where $\sigma_i = \sqrt{\lambda_i}$ are the singular values of A.

$$\Sigma = \begin{bmatrix} D & 0 \\ 0 & 0 \end{bmatrix}$$

3. Define

$$\mathbf{u}_1 = \frac{A\mathbf{v}_1}{\sigma_1}, \quad \mathbf{u}_2 = \frac{A\mathbf{v}_2}{\sigma_2}, \quad \dots, \quad \mathbf{u}_r = \frac{A\mathbf{v}_r}{\sigma_r}$$

where $\sigma_i = \sqrt{\lambda_i}$ are the singular values of A.

4. Add in $m - r$ extra vectors so that $\{\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_m\}$ is an orthonormal basis for $\mathbb{R}^m$.

$$\Sigma = \begin{bmatrix} D & 0 \\ 0 & 0 \end{bmatrix}$$

3 Define

$$\mathbf{u}_1 = \frac{A\mathbf{v}_1}{\sigma_1}, \quad \mathbf{u}_2 = \frac{A\mathbf{v}_2}{\sigma_2}, \quad \dots, \quad \mathbf{u}_r = \frac{A\mathbf{v}_r}{\sigma_r}$$

where $\sigma_i = \sqrt{\lambda_i}$ are the singular values of A.

4 Add in $m - r$ extra vectors so that $\{\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_m\}$ is an orthonormal basis for $\mathbb{R}^m$.

5 Let $U = \begin{bmatrix} \mathbf{u}_1 & \mathbf{u}_2 & \dots & \mathbf{u}_m \end{bmatrix}$. This will be an $m \times m$ orthogonal matrix.

6 Write

$$A = U\Sigma V^T$$

where Σ is the $m \times n$ matrix of the form

$$\Sigma = \left[\begin{array}{cccc|c} \sigma_1 & 0 & \cdots & 0 & \\ 0 & \sigma_2 & \cdots & 0 & \\ \vdots & \vdots & & \vdots & 0_{r\times(n-r)} \\ 0 & 0 & \cdots & \sigma_r & \\ \hline \multicolumn{4}{c|}{0_{(m-r)\times n}} & 0_{(m-r)\times(n-r)} \end{array}\right]$$

Example

Find a singular value decomposition of the matrix $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}$.

Example

Find a singular value decomposition of the matrix $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}$.

We saw earlier that the eigenvalues of A^TA are $\lambda_1 = 3$ and $\lambda_2 = 1$, and that the singular values of A are $\sigma_1 = \sqrt{3}$ and $\sigma_2 = 1$.

Example

Find a singular value decomposition of the matrix $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}$.

We saw earlier that the eigenvalues of $A^T A$ are $\lambda_1 = 3$ and $\lambda_2 = 1$, and that the singular values of A are $\sigma_1 = \sqrt{3}$ and $\sigma_2 = 1$. I leave it to you to check that

$$\mathbf{v}_1 = \begin{bmatrix} \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} \end{bmatrix} \qquad \text{and} \qquad \mathbf{v}_2 = \begin{bmatrix} \frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} \end{bmatrix}$$

are eigenvalues corresponding to λ_1 and λ_2, respectively, and that $\{\mathbf{v}_1, \mathbf{v}_2\}$ is an orthonormal basis for $\mathbb{R}^2$.

So we have

$$V = \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \end{bmatrix}.$$

(Note that V orthogonally diagonalises $A^T A$.)

Next define

$$\mathbf{u}_1 = \frac{1}{\sigma_1} A\mathbf{v}_1 = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} \end{bmatrix} = \frac{\sqrt{3}}{3} \begin{bmatrix} \sqrt{2} \\ \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{6}}{3} \\ \frac{\sqrt{6}}{6} \\ \frac{\sqrt{6}}{6} \end{bmatrix}$$

and

$$\mathbf{u}_2 = \frac{1}{\sigma_2} A\mathbf{v}_2 = \frac{1}{1} \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} \end{bmatrix} = \begin{bmatrix} 0 \\ -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} \end{bmatrix}.$$

Next define

$$\mathbf{u}_1 = \frac{1}{\sigma_1} A\mathbf{v}_1 = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} \end{bmatrix} = \frac{\sqrt{3}}{3} \begin{bmatrix} \sqrt{2} \\ \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{6}}{3} \\ \frac{\sqrt{6}}{6} \\ \frac{\sqrt{6}}{6} \end{bmatrix}$$

and

$$\mathbf{u}_2 = \frac{1}{\sigma_2} A\mathbf{v}_2 = \frac{1}{1} \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} \end{bmatrix} = \begin{bmatrix} 0 \\ -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} \end{bmatrix}.$$

Recall that U will be a 3×3 matrix. We have found the first two columns of U, now we need to find the third column.

We need to find a unit vector $\mathbf{u}_3$ such that $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ is an orthonormal basis for $\mathbb{R}^3$.

We need to find a unit vector $\mathbf{u}_3$ such that $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ is an orthonormal basis for $\mathbb{R}^3$. So we need to find a unit vector $\mathbf{u}_3$ that is orthogonal to both $\mathbf{u}_1$ and $\mathbf{u}_2$.

We need to find a unit vector $\mathbf{u}_3$ such that $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ is an orthonormal basis for $\mathbb{R}^3$. So we need to find a unit vector $\mathbf{u}_3$ that is orthogonal to both $\mathbf{u}_1$ and $\mathbf{u}_2$.

So as to make our calculations a little easier, let us instead look for a unit vector $\mathbf{u}_3$ that is orthogonal to both

$$\sqrt{6}\mathbf{u}_1 = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} \qquad \text{and} \qquad \sqrt{2}\mathbf{u}_2 = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}.$$

We need to find a unit vector $\mathbf{u}_3$ such that $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ is an orthonormal basis for $\mathbb{R}^3$. So we need to find a unit vector $\mathbf{u}_3$ that is orthogonal to both $\mathbf{u}_1$ and $\mathbf{u}_2$.

So as to make our calculations a little easier, let us instead look for a unit vector $\mathbf{u}_3$ that is orthogonal to both

$$\sqrt{6}\mathbf{u}_1 = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} \qquad \text{and} \qquad \sqrt{2}\mathbf{u}_2 = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}.$$

So we want to find a unit vector $\mathbf{u}_3$ that is a solution to the homogeneous linear system

$$\begin{bmatrix} 2 & 1 & 1 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

I leave it to you to check that the general solution to

$$\begin{bmatrix} 2 & 1 & 1 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

is

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = t \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}.$$

I leave it to you to check that the general solution to

$$\begin{bmatrix} 2 & 1 & 1 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

is

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = t \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}.$$

Therefore the unit vector that we want is

$$\mathbf{u}_3 = \frac{\begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}}{\left\| \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} \right\|} = \begin{bmatrix} -\frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{bmatrix}.$$

So we have

$$U = \begin{bmatrix} \mathbf{u}_1 & \mathbf{u}_2 & \mathbf{u}_3 \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{6}}{3} & 0 & -\frac{1}{\sqrt{3}} \\ \frac{\sqrt{6}}{6} & -\frac{\sqrt{2}}{2} & \frac{1}{\sqrt{3}} \\ \frac{\sqrt{6}}{6} & \frac{\sqrt{2}}{2} & \frac{1}{\sqrt{3}} \end{bmatrix}.$$

So we have

$$U = \begin{bmatrix} \mathbf{u}_1 & \mathbf{u}_2 & \mathbf{u}_3 \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{6}}{3} & 0 & -\frac{1}{\sqrt{3}} \\ \frac{\sqrt{6}}{6} & -\frac{\sqrt{2}}{2} & \frac{1}{\sqrt{3}} \\ \frac{\sqrt{6}}{6} & \frac{\sqrt{2}}{2} & \frac{1}{\sqrt{3}} \end{bmatrix}.$$

Therefore a singular value decomposition of A is

$$\underbrace{\begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}}_{A} = \underbrace{\begin{bmatrix} \frac{\sqrt{6}}{3} & 0 & -\frac{1}{\sqrt{3}} \\ \frac{\sqrt{6}}{6} & -\frac{\sqrt{2}}{2} & \frac{1}{\sqrt{3}} \\ \frac{\sqrt{6}}{6} & \frac{\sqrt{2}}{2} & \frac{1}{\sqrt{3}} \end{bmatrix}}_{U} \underbrace{\begin{bmatrix} \sqrt{3} & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}}_{\Sigma} \underbrace{\begin{bmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \end{bmatrix}}_{V^T}$$

Example

Find a singular value decomposition of the matrix
$A = \begin{bmatrix} 4 & 11 & 14 \\ 8 & 7 & -2 \end{bmatrix}$.

We saw earlier that the eigenvalues of $A^T A$ are $\lambda_1 = 360$, $\lambda_2 = 90$ and $\lambda_3 = 0$.

Example

Find a singular value decomposition of the matrix
$A = \begin{bmatrix} 4 & 11 & 14 \\ 8 & 7 & -2 \end{bmatrix}$.

We saw earlier that the eigenvalues of $A^T A$ are $\lambda_1 = 360$, $\lambda_2 = 90$ and $\lambda_3 = 0$. I leave it to you to check that

$$\mathbf{v}_1 = \begin{bmatrix} \frac{1}{3} \\ \frac{2}{3} \\ \frac{2}{3} \end{bmatrix}, \qquad \mathbf{v}_2 = \begin{bmatrix} -\frac{2}{3} \\ -\frac{1}{3} \\ \frac{2}{3} \end{bmatrix}, \qquad \mathbf{v}_3 = \begin{bmatrix} \frac{2}{3} \\ -\frac{2}{3} \\ \frac{1}{3} \end{bmatrix}$$

are corresponding orthonormal eigenvectors of $A^T A$.

So we have our first matrix

$$V = \begin{bmatrix} \\ \mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3 \\ \\ \end{bmatrix} = \begin{bmatrix} \frac{1}{3} & -\frac{2}{3} & \frac{2}{3} \\ \frac{2}{3} & -\frac{1}{3} & -\frac{2}{3} \\ \frac{2}{3} & \frac{2}{3} & \frac{1}{3} \end{bmatrix}.$$

The singular values of A are

$$\sigma_1 = 6\sqrt{10}, \qquad \sigma_2 = 3\sqrt{10}, \qquad \sigma_3 = 0.$$

Therefore

$$D = \begin{bmatrix} 6\sqrt{10} & 0 \\ 0 & 3\sqrt{10} \end{bmatrix}.$$

The singular values of A are

$$\sigma_1 = 6\sqrt{10}, \qquad \sigma_2 = 3\sqrt{10}, \qquad \sigma_3 = 0.$$

Therefore

$$D = \begin{bmatrix} 6\sqrt{10} & 0 \\ 0 & 3\sqrt{10} \end{bmatrix}.$$

Recall that the matrix Σ is the same size as A (i.e. 2×3), with D in its upper left and zeros everywhere else:

$$\Sigma = \left[\begin{array}{c|c} D & 0 \end{array}\right] = \left[\begin{array}{cc|c} 6\sqrt{10} & 0 & 0 \\ 0 & 3\sqrt{10} & 0 \end{array}\right].$$

Finally we need to construct U.

$$\sigma_1 = 6\sqrt{10}, \qquad \sigma_2 = 3\sqrt{10}, \qquad \sigma_3 = 0$$

Since A has two nonzero singular values, we have $\operatorname{rank} A = 2$.

$$\sigma_1 = 6\sqrt{10}, \qquad \sigma_2 = 3\sqrt{10}, \qquad \sigma_3 = 0$$

Since A has two nonzero singular values, we have $\operatorname{rank} A = 2$.

We calculate that

$$\mathbf{u}_1 = \frac{1}{\sigma_1} A\mathbf{v}_1 = \frac{1}{6\sqrt{10}} \begin{bmatrix} 4 & 11 & 14 \\ 8 & 7 & -2 \end{bmatrix} \begin{bmatrix} \frac{1}{3} \\ \frac{2}{3} \\ \frac{2}{3} \end{bmatrix} = \frac{1}{6\sqrt{10}} \begin{bmatrix} 18 \\ 6 \end{bmatrix} = \begin{bmatrix} \frac{3}{\sqrt{10}} \\ \frac{1}{\sqrt{10}} \end{bmatrix}$$

$$\mathbf{u}_2 = \frac{1}{\sigma_2} A\mathbf{v}_2 = \frac{1}{3\sqrt{10}} \begin{bmatrix} 4 & 11 & 14 \\ 8 & 7 & -2 \end{bmatrix} \begin{bmatrix} -\frac{2}{3} \\ -\frac{1}{3} \\ \frac{2}{3} \end{bmatrix} = \frac{1}{3\sqrt{10}} \begin{bmatrix} 3 \\ -9 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{10}} \\ -\frac{3}{\sqrt{10}} \end{bmatrix}$$

Since $\{\mathbf{u}_1, \mathbf{u}_2\}$ is already a basis for $\mathbb{R}^2$, we do not need to add any extra vectors in.

So our third and final matrix is

$$U = \begin{bmatrix} \mathbf{u}_1 & \mathbf{u}_2 \end{bmatrix} = \begin{bmatrix} \frac{3}{\sqrt{10}} & \frac{1}{\sqrt{10}} \\ \frac{1}{\sqrt{10}} & -\frac{3}{\sqrt{10}} \end{bmatrix}.$$

So our third and final matrix is

$$U = \begin{bmatrix} \mathbf{u}_1 & \mathbf{u}_2 \end{bmatrix} = \begin{bmatrix} \frac{3}{\sqrt{10}} & \frac{1}{\sqrt{10}} \\ \frac{1}{\sqrt{10}} & -\frac{3}{\sqrt{10}} \end{bmatrix}.$$

Therefore an SVD of A is

$$\begin{bmatrix} 4 & 11 & 14 \\ 8 & 7 & -2 \end{bmatrix} = \begin{bmatrix} \frac{3}{\sqrt{10}} & \frac{1}{\sqrt{10}} \\ \frac{1}{\sqrt{10}} & -\frac{3}{\sqrt{10}} \end{bmatrix} \begin{bmatrix} 6\sqrt{10} & 0 & 0 \\ 0 & 3\sqrt{10} & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{3} & \frac{2}{3} & \frac{2}{3} \\ -\frac{2}{3} & -\frac{1}{3} & \frac{2}{3} \\ \frac{2}{3} & -\frac{2}{3} & \frac{1}{3} \end{bmatrix}$$

$$A \quad = \quad U \quad \Sigma \quad V^T$$

Example (Final Example of This Course)

Find a singular value decomposition of the matrix $A = \begin{bmatrix} 1 & -1 \\ -2 & 2 \\ 2 & -2 \end{bmatrix}$.

Example (Final Example of This Course)

Find a singular value decomposition of the matrix $A = \begin{bmatrix} 1 & -1 \\ -2 & 2 \\ 2 & -2 \end{bmatrix}$.

First we calculate that

$$A^T A = \begin{bmatrix} 1 & -2 & 2 \\ -1 & 2 & -2 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ -2 & 2 \\ 2 & -2 \end{bmatrix} = \begin{bmatrix} 9 & -9 \\ -9 & 9 \end{bmatrix}.$$

Example (Final Example of This Course)

Find a singular value decomposition of the matrix $A = \begin{bmatrix} 1 & -1 \\ -2 & 2 \\ 2 & -2 \end{bmatrix}$.

First we calculate that

$$A^T A = \begin{bmatrix} 1 & -2 & 2 \\ -1 & 2 & -2 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ -2 & 2 \\ 2 & -2 \end{bmatrix} = \begin{bmatrix} 9 & -9 \\ -9 & 9 \end{bmatrix}.$$

The eigenvalues of $A^T A$ are $\lambda_1 = 18$ and $\lambda_2 = 0$ with corresponding eigenvectors

$$\mathbf{x}_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \qquad \text{and} \qquad \mathbf{x}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

Example (Final Example of This Course)

Find a singular value decomposition of the matrix $A = \begin{bmatrix} 1 & -1 \\ -2 & 2 \\ 2 & -2 \end{bmatrix}$.

First we calculate that

$$A^T A = \begin{bmatrix} 1 & -2 & 2 \\ -1 & 2 & -2 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ -2 & 2 \\ 2 & -2 \end{bmatrix} = \begin{bmatrix} 9 & -9 \\ -9 & 9 \end{bmatrix}.$$

The eigenvalues of $A^T A$ are $\lambda_1 = 18$ and $\lambda_2 = 0$ with corresponding eigenvectors

$$\mathbf{x}_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \qquad \text{and} \qquad \mathbf{x}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

These are orthogonal, but are not orthonormal so we need to normalise them:

$$\mathbf{v}_1 = \frac{\mathbf{x}_1}{\|\mathbf{x}_1\|} = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{bmatrix} \qquad \text{and} \qquad \mathbf{v}_2 = \frac{\mathbf{x}_2}{\|\mathbf{x}_2\|} = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}.$$

Thus

$$V = \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}.$$

Singular Value Decomposition

Thus

$$V = \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}.$$

The singular values are $\sigma_1 = \sqrt{\lambda_1} = \sqrt{18} = 3\sqrt{2}$ and $\sigma_2 = \sqrt{\lambda_2} = 0$. Since there is only one nonzero singular value, D will be a 1×1 matrix. The matrix Σ is the same size as A, with D in its upper left corner.

Singular Value Decomposition

Thus

$$V = \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}.$$

The singular values are $\sigma_1 = \sqrt{\lambda_1} = \sqrt{18} = 3\sqrt{2}$ and $\sigma_2 = \sqrt{\lambda_2} = 0$. Since there is only one nonzero singular value, D will be a 1×1 matrix. The matrix Σ is the same size as A, with D in its upper left corner. Hence

$$\Sigma = \begin{bmatrix} D & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 3\sqrt{2} & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

Singular Value Decomposition

To construct U, we will need $A\mathbf{v}_1$ and $A\mathbf{v}_2$:

To construct U, we will need $A\mathbf{v}_1$ and $A\mathbf{v}_2$:

$$A\mathbf{v}_1 = \begin{bmatrix} 1 & -1 \\ -2 & 2 \\ 2 & -2 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{bmatrix} = \begin{bmatrix} \frac{2}{\sqrt{2}} \\ -\frac{4}{\sqrt{2}} \\ \frac{4}{\sqrt{2}} \end{bmatrix}$$

$$A\mathbf{v}_2 = \begin{bmatrix} 1 & -1 \\ -2 & 2 \\ 2 & -2 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

Singular Value Decomposition

To construct U, we will need $A\mathbf{v}_1$ and $A\mathbf{v}_2$:

$$A\mathbf{v}_1 = \begin{bmatrix} 1 & -1 \\ -2 & 2 \\ 2 & -2 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{bmatrix} = \begin{bmatrix} \frac{2}{\sqrt{2}} \\ -\frac{4}{\sqrt{2}} \\ \frac{4}{\sqrt{2}} \end{bmatrix}$$

$$A\mathbf{v}_2 = \begin{bmatrix} 1 & -1 \\ -2 & 2 \\ 2 & -2 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

Then we normalise the first one of these

$$\mathbf{u}_1 = \frac{1}{\sigma_1} A\mathbf{v}_1 = \frac{1}{3\sqrt{2}} \begin{bmatrix} \frac{2}{\sqrt{2}} \\ -\frac{4}{\sqrt{2}} \\ \frac{4}{\sqrt{2}} \end{bmatrix} = \begin{bmatrix} \frac{1}{3} \\ -\frac{2}{3} \\ \frac{2}{3} \end{bmatrix}.$$

$$\mathbf{u}_1 = \begin{bmatrix} \frac{1}{3} \\ -\frac{2}{3} \\ \frac{2}{3} \end{bmatrix}.$$

Since the matrix U needs to be a 3×3 matrix, we need another two orthogonal unit vectors $\mathbf{u}_2$ and $\mathbf{u}_3$.

Singular Value Decomposition

$$\mathbf{u}_1 = \begin{bmatrix} \frac{1}{3} \\ -\frac{2}{3} \\ \frac{2}{3} \end{bmatrix}.$$

Since the matrix U needs to be a 3×3 matrix, we need another two orthogonal unit vectors $\mathbf{u}_2$ and $\mathbf{u}_3$.

Each of $\mathbf{u}_2$ and $\mathbf{u}_3$ must be orthogonal to $\mathbf{u}_1$, but the general solution of

$$0 = 3\mathbf{u}_1 \cdot \mathbf{x} = x_1 - 2x_2 + 2x_3$$

is

$$\mathbf{x} = s\mathbf{w}_2 + t\mathbf{w}_3 = s \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}.$$

I leave it to you to check that each of $\mathbf{w}_2$ and $\mathbf{w}_3$ is orthogonal to $\mathbf{u}_1$.

$$\mathbf{u}_1 = \begin{bmatrix} \frac{1}{3} \\ -\frac{2}{3} \\ \frac{2}{3} \end{bmatrix}.$$

Since the matrix U needs to be a 3×3 matrix, we need another two orthogonal unit vectors $\mathbf{u}_2$ and $\mathbf{u}_3$.

Each of $\mathbf{u}_2$ and $\mathbf{u}_3$ must be orthogonal to $\mathbf{u}_1$, but the general solution of

$$0 = 3\mathbf{u}_1 \bullet \mathbf{x} = x_1 - 2x_2 + 2x_3$$

is

$$\mathbf{x} = s\mathbf{w}_2 + t\mathbf{w}_3 = s \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}.$$

I leave it to you to check that each of $\mathbf{w}_2$ and $\mathbf{w}_3$ is orthogonal to $\mathbf{u}_1$.

However $\mathbf{w}_2$ and $\mathbf{w}_3$ are not orthogonal to each other, so we need to apply the Gram-Schmidt process.

$$\mathbf{u}_1 = \begin{bmatrix} \frac{1}{3} \\ -\frac{2}{3} \\ \frac{2}{3} \end{bmatrix}. \qquad \mathbf{w}_2 = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}, \quad \mathbf{w}_3 = \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}$$

We define

$$\mathbf{z}_2 = \mathbf{w}_2 = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$$

$$\mathbf{z}_3 = \mathbf{w}_3 - \frac{\mathbf{w}_3 \bullet \mathbf{z}_2}{\|\mathbf{z}_2\|^2}\mathbf{z}_2 = \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix} - \frac{(-4)}{5}\begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -\frac{2}{5} \\ \frac{4}{5} \\ 1 \end{bmatrix}$$

$$\mathbf{u}_1 = \begin{bmatrix} \frac{1}{3} \\ -\frac{2}{3} \\ \frac{2}{3} \end{bmatrix}. \qquad \mathbf{w}_2 = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}, \quad \mathbf{w}_3 = \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}$$

We define

$$\mathbf{z}_2 = \mathbf{w}_2 = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$$

$$\mathbf{z}_3 = \mathbf{w}_3 - \frac{\mathbf{w}_3 \bullet \mathbf{z}_2}{\|\mathbf{z}_2\|^2}\mathbf{z}_2 = \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix} - \frac{(-4)}{5}\begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -\frac{2}{5} \\ \frac{4}{5} \\ 1 \end{bmatrix}$$

and then

$$\mathbf{u}_2 = \frac{\mathbf{z}_2}{\|\mathbf{z}_2\|} = \begin{bmatrix} \frac{2}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} \\ 0 \end{bmatrix}, \qquad \mathbf{u}_3 = \frac{\mathbf{z}_3}{\|\mathbf{z}_3\|} = \begin{bmatrix} -\frac{2}{\sqrt{45}} \\ \frac{4}{\sqrt{45}} \\ \frac{5}{\sqrt{45}} \end{bmatrix}.$$

$$\mathbf{u}_1 = \begin{bmatrix} \frac{1}{3} \\ -\frac{2}{3} \\ \frac{2}{3} \end{bmatrix}, \quad \mathbf{u}_2 = \begin{bmatrix} \frac{2}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} \\ 0 \end{bmatrix}, \quad \mathbf{u}_3 = \begin{bmatrix} -\frac{2}{\sqrt{45}} \\ \frac{4}{\sqrt{45}} \\ \frac{5}{\sqrt{45}} \end{bmatrix}.$$

Thus

$$U = \begin{bmatrix} \mathbf{u}_1 & \mathbf{u}_2 & \mathbf{u}_3 \end{bmatrix} = \begin{bmatrix} \frac{1}{3} & \frac{2}{\sqrt{5}} & -\frac{2}{\sqrt{45}} \\ -\frac{2}{3} & \frac{1}{\sqrt{5}} & \frac{4}{\sqrt{45}} \\ \frac{2}{3} & 0 & \frac{5}{\sqrt{45}} \end{bmatrix}.$$

Now we have everything that we need.

Therefore an SVD for A is

$$\begin{bmatrix} 1 & -1 \\ -2 & 2 \\ 2 & -2 \end{bmatrix} = \begin{bmatrix} \frac{1}{3} & \frac{2}{\sqrt{5}} & -\frac{2}{\sqrt{45}} \\ -\frac{2}{3} & \frac{1}{\sqrt{5}} & \frac{4}{\sqrt{45}} \\ \frac{2}{3} & 0 & \frac{5}{\sqrt{45}} \end{bmatrix} \begin{bmatrix} 3\sqrt{2} & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}.$$

$$A \quad = \quad U \quad \Sigma \quad V^T$$

The End