03.01.2013 **MAT 371 – Diferansiyel Denklemler – Final Sınavın Çözümleri** N. Course

Question 1 (Green Dye in a Swimming Pool).

English

Your swimming pool contains 150,000 litres of water. It has been contaminated by 5 kg of a dye that leaves a swimmer's skin an unattractive green.

The swimming pool's filtering system can take water from the pool, remove the dye, and return the water to the pool at a rate of 500 litres/minute.

You have invited your friends to a pool party that is scheduled to begin 4 hours later. If the concentration of the dye is less than 0.005 grams/litre, then the swimming pool is safe to swim in.

[25p] Is your swimming pool's filtering system capable of reducing the dye concentration to this level within 4 hours? (Justify your answer).

Türkçe

Yüzme havuzunuz 150.000 litre su almaktadır. Havuza, 5 kg boya karışmış ve havuza girenlerin cildini itici bir yeşile boyamaktadır.

Yüzme havuzunun filtre sistemi 500 litre/dakika hızla havuzdan suyu alıp boyadan temizleyerek suyu havuza geri vermektedir.

Arkadaşlarınızı havuz partisine davet etmiştiniz ve parti 4 saat sonra başlayacak. Eğer boyanın yoğunluğu 0,005 gram/litre'den azsa havuz suyu, içinde yüzmeye elverişli demektir.

[25p] Havuzunuzun filtre sistemi, boya yoğunluğunu 4 saat içinde bu seviyeye indirebilir mi? (Çözmnz ispatlayın.)

Let t denote time in minutes, and let $D(t)$ denote the amount (in grams) of dye in the pool at time t. Clearly $D(0) = 5000$.

The concentration of dye in the pool is thus $\frac{D(t)}{150000}$ grams/litre. Now, every minute the filtering system cleans 500 litres. In other words, every minute the filtering system removes $500 \times \frac{D(t)}{150000} = \frac{D(t)}{300}$ grams of dye from the pool.

Therefore, the required IVP is

$$\begin{cases} \frac{dD}{dt} = -\frac{D(t)}{300} \\ D(0) = 5000. \end{cases}$$

Rearranging $\frac{dD}{dt} = -\frac{D(t)}{300}$ gives $\frac{dD}{D} = -\frac{dt}{300}$. Integrating gives $\log|D| = -\frac{t}{300} + c$, and then rearranging gives $D(t) = Ce^{-\frac{t}{300}}$ (for a different constant C). Finally, we use the initial condition to see that $5000 = D(0) = Ce^0 = C$. Hence $D(t) = 5000e^{-\frac{t}{300}}$.

After 4 hours, $t = 240$ and $D(240) = 5000e^{-\frac{240}{300}} \approx 2247$ grams. This gives a concentration of $\approx \frac{2247}{150000} \approx 0.015$ grams/litre which is too high. So the answer is "NO, CANCEL THE PARTY!!!!!"

Question 2 (Fundamental Sets of Solutions and the Wronskian)**.** Suppose that $p : \mathbb{R} \to \mathbb{R}$ and $q : \mathbb{R} \to \mathbb{R}$ are continuous functions. Let

$$L[y] = y'' + p(t)y' + q(t)y.$$

(a) [8p] Suppose that $y_1(t)$ and $y_2(t)$ are both solutions of $L[y] = 0$. Let $c_1, c_2 \in \mathbb{R}$. Show that

$$y(t) := c_1 y_1(t) + c_2 y_2(t)$$

is also a solution of $L[y] = 0$.

Clearly

$$\begin{aligned} L[y] &= L[c_1 y_1 + c_2 y_2] \\ &= (c_1 y_1 + c_2 y_2)'' + p(t)(c_1 y_1 + c_2 y_2)' + q(t)(c_1 y_1 + c_2 y_2) \\ &= c_1 \left(y_1'' + p(t)y_1' + q(t)y_1\right) + c_2 \left(y_2'' + p(t)y_2' + q(t)y_2\right) \\ &= c_1 L[y_1] + c_2 L[y_2] \\ &= 0 + 0 \\ &= 0. \end{aligned}$$

(b) [8p] Now suppose that $y_1(t) = e^{r_1 t}$ and $y_2(t) = e^{r_2 t}$ are both solutions of $L[y] = 0$. Show that

$$y_1 \text{ and } y_2 \text{ form a fundamental set of solutions of } L[y] = 0 \qquad \Longleftrightarrow \qquad r_1 \neq r_2.$$

[HINT: Start by calculating the Wronskian of y_1 and y_2.]

The Wronskian of y_1 and y_2 is

$$W(y_1, y_2) = \begin{pmatrix} y_1 & y_2 \\ y_1' & y_2' \end{pmatrix} = \begin{pmatrix} e^{r_1 t} & e^{r_2 t} \\ r_1 e^{r_1 t} & r_2 e^{r_2 t} \end{pmatrix} = (r_2 - r_1)e^{(r_1 + r_2)t}$$

Since $e^{(r_1+r_2)t} \neq 0$ for all t, it follows that: $W = 0 \iff r_1 = r_2$. Therefore

$$y_1 \text{ and } y_2 \text{ form a fundamental set of solutions of } L[y] = 0 \iff W \neq 0 \iff r_1 \neq r_2.$$

Now let $v_1(x) = x$ and $v_2(x) = xe^x$.

(c) [4p] Show that v_1 and v_2 are both solutions of

$$x^2 v'' - x(x+2)v' + (x+2)v = 0 \tag{1}$$

for $t > 0$.

Clearly

$$x^2 v_1'' - x(x+2)v_1' + (x+2)v_1 = x^2(0) - x(x+2)(1) + (x+2)x = 0$$

and

$$x^2 v_2'' - x(x+2)v_2' + (x+2)v_2 = x^2(xe^x + 2e^x) - x(x+2)(xe^x + e^x) + (x+2)xe^x = 0.$$

Therefore v_1 and v_2 are both solutions of (1).

(d) [1p] Do v_1 and v_2 form a fundamental set of solutions of (1)?

☑ Yes, ☐ No.

(e) [4p] Justify (explain) your answer to part d.

Again, we calculate the Wronskian (the clue was in the name of the question) to see that

$$W(v_1, v_2)(x) = \begin{pmatrix} v_1 & v_2 \\ v_1' & v_2' \end{pmatrix} = \begin{pmatrix} x & xe^x \\ 1 & xe^x + e^x \end{pmatrix} = x^2 e^x + xe^x - xe^x = x^2 e^x,$$

which is non-zero for all $x \neq 0$. Therefore v_1 and v_2 form a fundamental set of solutions of (1).

Question 3 (Second Order Linear Differential Equations)**.** Find the general solution $y : \mathbb{R} \to \mathbb{R}$ of

$$y'' - 2y' + y = t^2 - 2t + 1 + e^{2t} \cos t \tag{2}$$

First consider the homogeneous equation $y'' - 2y' + y = 0$. The characteristic equation is $r^2 - 2r + 1 = 0$ which has repeated root $r = 1$. Therefore the general solution of $y'' - 2y' + y = 0$ is

$$y(t) = c_1 e^t + c_2 t e^t.$$

Next consider $y'' - 2y' + y = t^2 - 2t + 1$. Trying the ansatz $Y(t) = t^2 + Bt + C$, we see that

$$t^2 - 2t + 1 = Y'' - 2Y' + Y = 2 - 2(2t + B) + (t^2 + Bt + C) = t^2 + (B - 4)t + (2 + C - 2B).$$

We must choose $B = 2$ and $C = 3$. Hence

$$Y(t) = t^2 + 2t + 3.$$

Now consider $y'' - 2y' + y = e^{2t} \cos t$. We try the ansatz $Y(t) = Ae^{2t} \cos t + Be^{2t} \sin t$ and find that

$$\begin{aligned}
e^{2t} \cos t = Y'' - 2Y' + Y \\
&= e^{2t}\big((3B - 4A) \sin t + (3A + 4B) \cos t\big) \\
&\quad - 2e^{2t}\big((2A + B) \cos t - (A - 2B) \sin t\big) \\
&\quad + e^{2t}\big(A \cos t + B \sin t\big) \\
&= e^{2t} \cos t\big(3A + 4B - 4A - 2B + A\big) + e^{2t} \sin t\big(3B - 4A + 2A - 4B + B\big) \\
&= e^{2t} \cos t\big(2B\big) + e^{2t} \sin t\big(2A\big)
\end{aligned}$$

Thus, we need $A = 0$ and $B = \frac{1}{2}$. Hence

$$Y(t) = \frac{1}{2} e^{2t} \sin t$$

Finally we add these 3 solutions together. Therefore, the general solution of (2) is

$$y(t) = c_1 e^t + c_2 t e^t + t^2 + 2t + 3 + \frac{1}{2} e^{2t} \sin t$$

Question 4 (Reduction of Order). Consider

$$x^2\frac{d^2y}{dx^2} + 2x\frac{dy}{dx} - 2y = 0, \qquad x > 0. \tag{3}$$

(a) [2p] Show that $y_1(x) = x$ is a solution of (3).

$$x^2y_1'' + 2xy_1' - 2y_1 = x^2(0) + 2x(1) - 2(x) = 2x - 2x = 0.$$

(b) [17p] Using the method of reduction of order, find a second solution $y_2(x)$ of (3).
[HINT: Start with $y_2(x) = v(x)y_1(x)$.]

As per the hint, we start with $y_2(x) = v(x)x$. Then $y_2' = v'x + v$ and $y_2'' = v''x + 2v'$. Putting these into the differential equation, we get

$$\begin{aligned} 0 &= x^2y_2'' + 2xy_2' - 2y_2 \\ &= (x^3v'' + 2x^2v') + (2x^2v' + 2xv) - 2vx \\ &= x^3v'' + 4x^2v' \\ &= x^2(xv'' + 4v') \\ &= x^2(xu' + 4u) \end{aligned}$$

where $u = v'$. We must solve $x\frac{du}{dx} + 4u = 0$. Rearranging gives $\frac{du}{u} = -4\frac{dx}{x}$. Integrating gives $\log|u| = -4\log|x| + c$. Rearranging then gives $u = Cx^{-4}$ for some constant C.

Next we integrate to find $v(x) = \int u(x)dx = -\frac{1}{3}Cx^{-3} + c$. For simplicity, choose $c = 0$ and $C = -3$ to get $v(x) = x^{-3}$. Then $y_2(x) = v(x)x = x^{-2}$.

(c) [2p] Check that the function $y_2(x)$, that you found in part b, is a solution of (3).

If $y_2(x) = x^{-2}$, then

$$x^2y_2'' + 2xy_2' - 2y_2 = x^2(6x^{-4}) + 2x(-2x^{-3}) - 2(x^{-2}) = 6x^{-2} - 4x^{-2} - 2x^{-2} = 0.$$

(d) [4p] Solve

$$\begin{cases} x^2y'' + 2xy' - 2y = 0, & x > 0 \\ y(1) = 7 \\ y'(1) = -2. \end{cases}$$

The general solution to the ODE is $y(t) = c_1x + c_2x^{-2}$.

Using the initial conditions, we can calculate $7 = y(1) = c_1 + c_2$ and $-2 = y'(1) = c_1 - 2c_2$. Hence $c_1 = 4$ and $c_2 = 3$.

Therefore

$$y(t) = 4x + 3x^{-2} = 4x + \frac{3}{x^2}.$$

Question 5 (Systems of Equations).

(a) [13p] Solve

$$\mathbf{x}' = \begin{pmatrix} 1 & -5 \\ 1 & -3 \end{pmatrix} \mathbf{x}, \qquad \mathbf{x}(0) = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

The eigenvalues of $\begin{pmatrix} 1 & -5 \\ 1 & -3 \end{pmatrix}$ are $r_{1,2} = -1 \pm i$. The eigenvectors are $\xi^{(1,2)} = \begin{pmatrix} 2 \pm i \\ 1 \end{pmatrix}$.
We only need $r_1 = -1 + i$ and $\xi^{(1)} = \begin{pmatrix} 2 \\ 1 \end{pmatrix} + i \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ to find

$$\begin{aligned} \mathbf{x}_1(t) &= \xi^{(1)} e^{r_1 t} \\ &= \left[\begin{pmatrix} 2 \\ 1 \end{pmatrix} + i \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right] e^{-t} \big(\cos t + i \sin t \big) \\ &= e^{-t} \left[\begin{pmatrix} 2\cos t - \sin t \\ \cos t \end{pmatrix} + i \begin{pmatrix} 2\sin t + \cos t \\ \sin t \end{pmatrix} \right]. \end{aligned}$$

Therefore, the general solution of $\mathbf{x}' = \begin{pmatrix} 1 & -5 \\ 1 & -3 \end{pmatrix} \mathbf{x}$ is

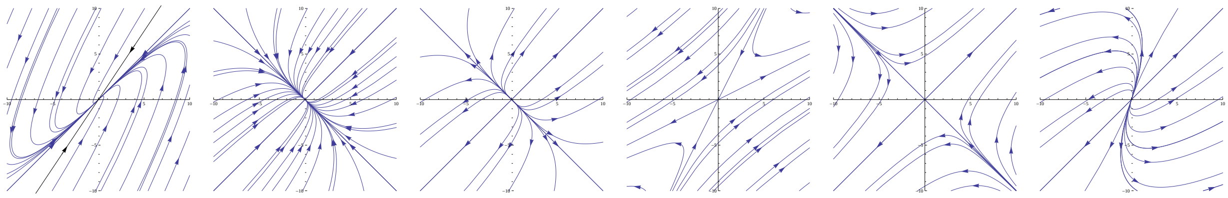
$$\mathbf{x}(t) = c_1 e^{-t} \begin{pmatrix} 2\cos t - \sin t \\ \cos t \end{pmatrix} + c_2 e^{-t} \begin{pmatrix} 2\sin t + \cos t \\ \sin t \end{pmatrix}.$$

Finally, we use the initial condition to calculate that

$$\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \mathbf{x}(0) = c_1 \begin{pmatrix} 2 \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

which tells us that $c_1 = 1$ and $c_2 = -1$. Thus, the answer to this question is

$$\mathbf{x}(t) = e^{-t} \begin{pmatrix} 2\cos t - \sin t \\ \cos t \end{pmatrix} - e^{-t} \begin{pmatrix} 2\sin t + \cos t \\ \sin t \end{pmatrix} = e^{-t} \begin{pmatrix} \cos t - 3\sin t \\ \cos t - \sin t \end{pmatrix}.$$



(i) Stable node (ii) Stable node (iii) Unstable node (iv) Saddle (v) Saddle (vi) Unstable node

Let $A = \begin{pmatrix} 5 & -1 \\ 3 & 1 \end{pmatrix}$. The eigenvalues of A are $r_1 = 4$ and $r_2 = 2$. The corresponding eigenvectors of A are $\xi^{(1)} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $\xi^{(2)} = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$ respectively.

(b) [2p] Which of the graphs (above) is the phase plot of the equation $\mathbf{x}' = A\mathbf{x}$?

[Mark ☑ one box only.]

☐ (i) ☐ (ii) ☐ (iii) ☐ (iv) ☐ (v) ☑ (vi)

(c) [10p] Justify (explain) your answer to part (c).

Since both r_1 and r_2 are strictly positive (> 0), we must have an unstable node. So the phase plot must be either (iii) or (vi).

The phase plot must also have straight lines in the directions of the eigenvectors. So it must be (vi).