# Lecture 4

- 3.1 Homogeneous Equations with Constant Coefficients
- 3.2 Fundamental Sets of Solutions
- 3.3 Complex Roots of the Characteristic Equation

# Exams

- **Midterm Exam**

  Wednesday 11 August, 2pm

  60 minutes.

- **Final Exam**

  Thursday 26 August, 2pm

  90 minutes.

# Second and Higher Order Linear ODEs

In this chapter we will consider equations of the form

$$y'' + p(t)y' + q(t)y = g(t)$$

or

$$P(t)y'' + Q(t)y' + R(t)y = G(t).$$

Such equations are *linear* second order ODEs.

In this chapter we will consider equations of the form

$$y'' + p(t)y' + q(t)y = g(t)$$

or

$$P(t)y'' + Q(t)y' + R(t)y = G(t).$$

Such equations are *linear* second order ODEs.

If $g(t)$ (or $G(t)$) is always zero, then the ODE is called *homogeneous*. Otherwise it is *nonhomogeneous*.

# Homogeneous Equations with Constant Coefficients

First we will consider the equation

$$ay'' + by' + cy = 0 \tag{1}$$

where $a, b, c \in \mathbb{R}$ are constants.

**Example**

Solve $y'' - y = 0$.

**Example**

Solve $y'' - y = 0$.

We want to find a function $y(t)$ which satisfies

$$\frac{d^2y}{dt^2} = y.$$

**Example**

Solve $y'' - y = 0$.

We want to find a function $y(t)$ which satisfies

$$\frac{d^2y}{dt^2} = y.$$

- What about $e^t$?

**Example**

Solve $y'' - y = 0$.

We want to find a function $y(t)$ which satisfies

$$\frac{d^2y}{dt^2} = y.$$

- What about $e^t$? Yes!

**Example**

Solve $y'' - y = 0$.

We want to find a function $y(t)$ which satisfies

$$\frac{d^2y}{dt^2} = y.$$

- What about $e^t$? Yes!
- What about $e^{-t}$?

**Example**

Solve $y'' - y = 0$.

We want to find a function $y(t)$ which satisfies

$$\frac{d^2y}{dt^2} = y.$$

- What about $e^t$? Yes!
- What about $e^{-t}$? Yes!

**Example**

Solve $y'' - y = 0$.

We want to find a function $y(t)$ which satisfies

$$\frac{d^2y}{dt^2} = y.$$

- What about $e^t$? Yes!
- What about $e^{-t}$? Yes!
- And what about $c_1e^t + c_2e^{-t}$?

**Example**

Solve $y'' - y = 0$.

We want to find a function $y(t)$ which satisfies

$$\frac{d^2y}{dt^2} = y.$$

- What about $e^t$? Yes!
- What about $e^{-t}$? Yes!
- And what about $c_1e^t + c_2e^{-t}$? Yes! In fact, this is the general solution to $y'' - y = 0$.

**Example**

Solve

$$\begin{cases} y'' - y = 0 \\ y(0) = 2 \\ y'(0) = -1. \end{cases}$$

First note that this IVP has one $2^{\text{nd}}$ order ODE and two initial conditions.

**Example**

Solve

$$\begin{cases} y'' - y = 0 \\ y(0) = 2 \\ y'(0) = -1. \end{cases}$$

First note that this IVP has one $2^{\text{nd}}$ order ODE and two initial conditions.

We know that $y(t) = c_1 e^t + c_2 e^{-t}$. We are looking for the solution which passes through the point $(0, 2)$ and has slope $-1$ at this point.

### Example

Solve

$$\begin{cases} y'' - y = 0 \\ y(0) = 2 \\ y'(0) = -1. \end{cases}$$

First note that this IVP has one $2^{\text{nd}}$ order ODE and two initial conditions.
We know that $y(t) = c_1 e^t + c_2 e^{-t}$. We are looking for the solution which passes through the point $(0, 2)$ and has slope $-1$ at this point. Using the first initial condition we get that

$$2 = y(0) = c_1 + c_2 \qquad \Longrightarrow \qquad c_1 + c_2 = 2.$$

Next we need to differentiate $y(t)$:

$$y'(t) = \frac{d}{dt}\left(c_1e^t + c_2e^{-t}\right) = c_1e^t - c_2e^{-t}.$$

Next we need to differentiate $y(t)$:

$$y'(t) = \frac{d}{dt}\left(c_1 e^t + c_2 e^{-t}\right) = c_1 e^t - c_2 e^{-t}.$$

Thus

$$-1 = y'(0) = c_1 - c_2 \qquad \Longrightarrow \qquad c_1 - c_2 = -1.$$

Next we need to differentiate $y(t)$:

$$y'(t) = \frac{d}{dt}\left(c_1 e^t + c_2 e^{-t}\right) = c_1 e^t - c_2 e^{-t}.$$

Thus

$$-1 = y'(0) = c_1 - c_2 \qquad \Longrightarrow \qquad c_1 - c_2 = -1.$$

To satisfy these two conditions we must have $c_1 = \frac{1}{2}$ and $c_2 = \frac{3}{2}$.

Next we need to differentiate $y(t)$:

$$y'(t) = \frac{d}{dt}\left(c_1 e^t + c_2 e^{-t}\right) = c_1 e^t - c_2 e^{-t}.$$

Thus

$$-1 = y'(0) = c_1 - c_2 \qquad \Longrightarrow \qquad c_1 - c_2 = -1.$$

To satisfy these two conditions we must have $c_1 = \frac{1}{2}$ and $c_2 = \frac{3}{2}$. Therefore the solution to the IVP is

$$\boxed{y(t) = \frac{1}{2}e^t + \frac{3}{2}e^{-t}.}$$

Now let's go back to

$$ay'' + by' + cy = 0. \tag{1}$$

In the previous example, we used exponential functions in our solution. Maybe we always want exponential solutions?

We guess that $y(t) = e^{rt}$ might be the solution to (1) for some number $r$ that we don't know yet.

We guess that $y(t) = e^{rt}$ might be the solution to (1) for some number $r$ that we don't know yet.
Then we calculate that

$$\begin{aligned} y &= e^{rt} \\ y' &= re^{rt} \\ y'' &= r^2 e^{rt} \end{aligned}$$

and

We guess that $y(t) = e^{rt}$ might be the solution to (1) for some number $r$ that we don't know yet.
Then we calculate that

$$\begin{aligned} y &= e^{rt} \\ y' &= re^{rt} \\ y'' &= r^2 e^{rt} \end{aligned}$$

and

$$0 = ay'' + by' + cy = (ar^2 + br + c)e^{rt}.$$

We guess that $y(t) = e^{rt}$ might be the solution to (1) for some number $r$ that we don't know yet.
Then we calculate that

$$\begin{aligned} y &= e^{rt} \\ y' &= re^{rt} \\ y'' &= r^2 e^{rt} \end{aligned}$$

and

$$0 = ay'' + by' + cy = (ar^2 + br + c)e^{rt}.$$

Since $e^{rt} \neq 0$ for all $t$, we must have that

$$ar^2 + br + c = 0. \tag{2}$$

# 3.1 Homogeneous Equations with Constant Coefficients

$$ay'' + by' + cy = 0 \tag{1}$$

$$ar^2 + br + c = 0 \tag{2}$$

**Definition**

(2) is called the *characteristic equation* of (1).

$$ay'' + by' + cy = 0 \tag{1}$$

$$ar^2 + br + c = 0 \tag{2}$$

**Definition**

(2) is called the *characteristic equation* of (1).

**Theorem**

$$e^{rt} \textit{ solves } (1) \qquad \Longleftrightarrow \qquad r \textit{ solves } (2).$$

$ar^2 + br + c = 0$ has two roots, $r_1$ and $r_2$:

$$r_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

These roots might be

1. real numbers and different ($r_1, r_2 \in \mathbb{R}$ and $r_1 \neq r_2$);
2. complex conjugates ($r_1, r_2 \in \mathbb{C} \setminus \mathbb{R}$, $\overline{r}_1 = r_2$); or
3. real numbers but repeated ($r_1, r_2 \in \mathbb{R}$ and $r_1 = r_2$).

We will study these three cases separately. First we study case 1.

Suppose that $r_1, r_2 \in \mathbb{R}$ and $r_1 \neq r_2$. In other words, suppose that $b^2 - 4ac > 0$.

Suppose that $r_1, r_2 \in \mathbb{R}$ and $r_1 \neq r_2$. In other words, suppose that $b^2 - 4ac > 0$. Then $y_1(t) = e^{r_1 t}$ and $y_2(t) = e^{r_2 t}$ are both solutions to (1).

Suppose that $r_1, r_2 \in \mathbb{R}$ and $r_1 \neq r_2$. In other words, suppose that $b^2 - 4ac > 0$. Then $y_1(t) = e^{r_1 t}$ and $y_2(t) = e^{r_2 t}$ are both solutions to (1). So

$$y(t) = c_1 y_1(t) + c_2 y_2(t) = c_1 e^{r_1 t} + c_2 e^{r_2 t}$$

will also be a solution for any constants $c_1, c_2 \in \mathbb{R}$. This is called the *general solution* to (1).

**Example**

Solve $y'' + 5y' + 6y = 0$.

**Example**

Solve $y'' + 5y' + 6y = 0$.

The first thing that we must do is to write down the characteristic equation.

**Example**

Solve $y'' + 5y' + 6y = 0$.

The first thing that we must do is to write down the characteristic equation. The characteristic equation for this ODE is

$$0 = r^2 + 5r + 6$$

**Example**

Solve $y'' + 5y' + 6y = 0$.

The first thing that we must do is to write down the characteristic equation. The characteristic equation for this ODE is

$$0 = r^2 + 5r + 6 = (r + 2)(r + 3).$$

**Example**

Solve $y'' + 5y' + 6y = 0$.

The first thing that we must do is to write down the characteristic equation. The characteristic equation for this ODE is

$$0 = r^2 + 5r + 6 = (r + 2)(r + 3).$$

The two roots are $r_1 = -2$ and $r_2 = -3$.

**Example**

Solve $y'' + 5y' + 6y = 0$.

The first thing that we must do is to write down the characteristic equation. The characteristic equation for this ODE is

$$0 = r^2 + 5r + 6 = (r + 2)(r + 3).$$

The two roots are $r_1 = -2$ and $r_2 = -3$. Therefore the general solution is

$$\boxed{y(t) = c_1 e^{-2t} + c_2 e^{-3t}.}$$

**Example**

Solve

$$\begin{cases} y'' + 5y' + 6y = 0 \\ y(0) = 2 \\ y'(0) = 3. \end{cases}$$

We already found that $y(t) = c_1 e^{-2t} + c_2 e^{-3t}$ is the general solution to the ODE. We just need to find $c_1$ and $c_2$.

We already found that $y(t) = c_1 e^{-2t} + c_2 e^{-3t}$ is the general solution to the ODE. We just need to find $c_1$ and $c_2$. Since $y'(t) = -2c_1 e^{-2t} - 3c_2 e^{-3t}$ we have that

$$2 = y(0) = c_1 + c_2 \qquad \Longrightarrow \qquad c_1 = 2 - c_2$$

and

We already found that $y(t) = c_1 e^{-2t} + c_2 e^{-3t}$ is the general solution to the ODE. We just need to find $c_1$ and $c_2$. Since $y'(t) = -2c_1 e^{-2t} - 3c_2 e^{-3t}$ we have that

$$2 = y(0) = c_1 + c_2 \qquad \Longrightarrow \qquad c_1 = 2 - c_2$$

and

$$\begin{aligned} 3 = y'(0) = -2c_1 - 3c_2 &= -2(2 - c_2) - 3c_2 = -4 - c_2 \\ \Longrightarrow \qquad c_2 &= -7 \\ \Longrightarrow \qquad c_1 &= 9. \end{aligned}$$

We already found that $y(t) = c_1 e^{-2t} + c_2 e^{-3t}$ is the general solution to the ODE. We just need to find $c_1$ and $c_2$. Since $y'(t) = -2c_1 e^{-2t} - 3c_2 e^{-3t}$ we have that

$$2 = y(0) = c_1 + c_2 \qquad \Longrightarrow \qquad c_1 = 2 - c_2$$

and

$$\begin{aligned} 3 = y'(0) &= -2c_1 - 3c_2 = -2(2 - c_2) - 3c_2 = -4 - c_2 \\ \Longrightarrow \qquad & c_2 = -7 \\ \Longrightarrow \qquad & c_1 = 9. \end{aligned}$$

Therefore the solution to the IVP is

$$\boxed{y(t) = 9e^{-2t} - 7e^{-3t}.}$$

**Example**

Solve

$$\begin{cases} 4y'' - 8y' + 3y = 0 \\ y(0) = 2 \\ y'(0) = \frac{1}{2}. \end{cases}$$

$$4y'' - 8y' + 3y = 0$$

Since the characteristic equation

$$4r^2 - 8r + 3 = 0$$

has roots,

$$r_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{8 \pm \sqrt{64 - 48}}{8} = 1 \pm \frac{1}{2} = \frac{3}{2} \text{ or } \frac{1}{2},$$

it follows that the general solution to the ODE is

$$y(t) = c_1 e^{\frac{3t}{2}} + c_2 e^{\frac{t}{2}}.$$

$$y(t) = c_1 e^{\frac{3t}{2}} + c_2 e^{\frac{t}{2}}$$

Using the initial conditions, we calculate that

$$\left.\begin{aligned} 2 &= y(0) = c_1 + c_2 \\ \frac{1}{2} &= y'(0) = \frac{3}{2}c_1 + \frac{1}{2}c_2 \end{aligned}\right. \qquad \Longrightarrow \qquad c_1 = -\frac{1}{2} \text{ and } c_2 = \frac{5}{2}.$$

$$y(t) = c_1 e^{\frac{3t}{2}} + c_2 e^{\frac{t}{2}}$$

Using the initial conditions, we calculate that

$$\begin{aligned} 2 &= y(0) = c_1 + c_2 \\ \frac{1}{2} &= y'(0) = \frac{3}{2}c_1 + \frac{1}{2}c_2 \end{aligned} \qquad \Longrightarrow \qquad c_1 = -\frac{1}{2} \text{ and } c_2 = \frac{5}{2}.$$

Therefore the solution to the IVP is

$$\boxed{y = -\tfrac{1}{2}e^{\frac{3t}{2}} + \tfrac{5}{2}e^{\frac{t}{2}}.}$$

## Summary

To solve

$$ay'' + by' + cy = 0$$

we need to find two linearly independent solutions.

1. If $r_1, r_2 \in \mathbb{R}$ and $r_1 \neq r_2$, then

$$y_1(t) = e^{r_1 t} \qquad \text{and} \qquad y_2(t) = e^{r_2 t};$$

2. If the roots are complex numbers, then ?????????????
3. If the roots are repeated, then ?????????????

3.2

# Fundamental Sets of Solutions

$$y'' + p(t)y' + q(t)y = 0$$

**Definition**

Let $L = \frac{d^2}{dt^2} + p(t)\frac{d}{dt} + q(t)$.

So

$$L[y] = \frac{d^2y}{dt^2} + p(t)\frac{dy}{dt} + q(t)y = y'' + p(t)y' + q(t)y$$

and we can write the ODE above as just $L[y] = 0$.

**Theorem**

*If $y_1$ and $y_2$ are both solutions of $L[y] = 0$, then $c_1 y_1 + c_2 y_2$ is also a solution to $L[y] = 0$ for all constants $c_1, c_2$.*

## Theorem

*If $y_1$ and $y_2$ are both solutions of $L[y] = 0$, then $c_1 y_1 + c_2 y_2$ is also a solution to $L[y] = 0$ for all constants $c_1, c_2$.*

## Proof.

Since $L[y_1] = 0$ and $L[y_2] = 0$, we have that

$$\begin{aligned}
L[y] &= L[c_1 y_1 + c_2 y_2] \\
&= \frac{d^2}{dt^2}\left(c_1 y_1 + c_2 y_2\right) + p(t)\frac{d}{dt}\left(c_1 y_1 + c_2 y_2\right) + q(t)\left(c_1 y_1 + c_2 y_2\right) \\
&= c_1\left(y_1'' + p(t) y_1' + q(t) y_1\right) + c_2\left(y_2'' + p(t) y_2' + q(t) y_2\right) \\
&= c_1 L[y_1] + c_2 L[y_2] \\
&= 0 + 0 = 0.
\end{aligned}$$

$\square$

Jósef Maria Hoëné-Wronkski
POL, 1776-1853

**Definition**

The *Wronskian* of $y_1(t)$ and $y_2(t)$ is

$$W = W(y_1, y_2)(t) = \begin{vmatrix} y_1(t) & y_2(t) \\ y_1'(t) & y_2'(t) \end{vmatrix}$$

**Theorem**

*Suppose that*

- $y_1$ *and* $y_2$ *both solve* $L[y] = 0$*; and*
- $\exists\ t$ *s.t.* $W(t) \neq 0$.

*Then* $\{c_1 y_1 + c_2 y_2 \,:\, c_1, c_2 \in \mathbb{R}\}$ *contains every solution of* $L[y] = 0$.

**Definition**

Since $y(t) = c_1 y_1(t) + c_2 y_2(t)$ contains every solution to $L[y] = 0$, $y(t)$ is called the *general solution* to $L[y] = 0$.

**Definition**

Since $y(t) = c_1 y_1(t) + c_2 y_2(t)$ contains every solution to $L[y] = 0$, $y(t)$ is called the *general solution* to $L[y] = 0$.

**Definition**

In this case, we say that $y_1$ and $y_2$ form a *fundamental set of solutions* to $L[y] = 0$.

**Example**

Show that $y_1(t) = t^{\frac{1}{2}}$ and $y_2(t) = t^{-1}$ form a fundamental set of solutions to

$$2t^2y'' + 3ty' - y = 0$$

for $t > 0$

**Example**

Show that $y_1(t) = t^{\frac{1}{2}}$ and $y_2(t) = t^{-1}$ form a fundamental set of solutions to

$$2t^2y'' + 3ty' - y = 0$$

for $t > 0$

We must show three things:

1. that $y_1 = t^{\frac{1}{2}}$ is a solution to the ODE;
2. that $y_2 = t^{-1}$ is also a solution to the ODE; and
3. that $y_1$ and $y_2$ are linearly independent ($W \neq 0$ somewhere).

Since

$$
\begin{aligned}
2t^2 y_1'' + 3t y_1' - y_1 &= 2t^2 \left(t^{\frac{1}{2}}\right)'' + 3t \left(t^{\frac{1}{2}}\right)' - t^{\frac{1}{2}} \\
&= 2t^2 \left(-\frac{1}{4} t^{-\frac{3}{2}}\right) + 3t \left(\frac{1}{2} t^{-\frac{1}{2}}\right) - t^{\frac{1}{2}} \\
&= -\frac{1}{2} t^{\frac{1}{2}} + \frac{3}{2} t^{\frac{1}{2}} - t^{\frac{1}{2}} = 0
\end{aligned}
$$

Since

$$\begin{aligned}2t^2y_1'' + 3ty_1' - y_1 &= 2t^2\left(t^{\frac{1}{2}}\right)'' + 3t\left(t^{\frac{1}{2}}\right)' - t^{\frac{1}{2}} \\ &= 2t^2\left(-\frac{1}{4}t^{-\frac{3}{2}}\right) + 3t\left(\frac{1}{2}t^{-\frac{1}{2}}\right) - t^{\frac{1}{2}} \\ &= -\frac{1}{2}t^{\frac{1}{2}} + \frac{3}{2}t^{\frac{1}{2}} - t^{\frac{1}{2}} = 0\end{aligned}$$

and

$$\begin{aligned}2t^2y_2'' + 3ty_2' - y_2 &= 2t^2\left(t^{-1}\right)'' + 3t\left(t^{-1}\right)' - t^{-1} \\ &= 2t^2\left(2t^{-3}\right) + 3t\left(-t^{-2}\right) - t^{-1} \\ &= 4t^{-1} - 3t^{-1} - t^{-1} = 0,\end{aligned}$$

$y_1$ and $y_2$ both solve the ODE.

Moreover since

$$W(y_1, y_2)(t) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} t^{\frac{1}{2}} & t^{-1} \\ \frac{1}{2}t^{-\frac{1}{2}} & -t^{-2} \end{vmatrix} = -t^{-\frac{3}{2}} - \frac{1}{2}t^{-\frac{3}{2}} = -\frac{3}{2}t^{-\frac{3}{2}} \neq 0$$

for all $t > 0$, we have that $y_1$ and $y_2$ are linearly independent.

Moreover since

$$W(y_1, y_2)(t) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} t^{\frac{1}{2}} & t^{-1} \\ \frac{1}{2}t^{-\frac{1}{2}} & -t^{-2} \end{vmatrix} = -t^{-\frac{3}{2}} - \frac{1}{2}t^{-\frac{3}{2}} = -\frac{3}{2}t^{-\frac{3}{2}} \neq 0$$

for all $t > 0$, we have that $y_1$ and $y_2$ are linearly independent.

Therefore $y_1 = t^{\frac{1}{2}}$ and $y_2 = t^{-1}$ form a fundamental set of solutions to this ODE.

# Complex Roots of the Characteristic Equation

Now consider

$$ay'' + by' + cy = 0 \tag{1}$$

where $b^2 - 4ac < 0$.

Now consider

$$ay'' + by' + cy = 0 \tag{1}$$

where $b^2 - 4ac < 0$. The two roots of the characteristic equation are complex conjugates. We denote them by

$$r_1 = \lambda + i\mu \qquad \text{and} \qquad r_2 = \lambda - i\mu$$

where $\lambda, \mu \in \mathbb{R}$.

Now consider

$$ay'' + by' + cy = 0 \tag{1}$$

where $b^2 - 4ac < 0$. The two roots of the characteristic equation are complex conjugates. We denote them by

$$r_1 = \lambda + i\mu \qquad \text{and} \qquad r_2 = \lambda - i\mu$$

where $\lambda, \mu \in \mathbb{R}$. The corresponding solutions are

$$y_1(t) = e^{r_1 t} = e^{(\lambda + i\mu)t} \qquad \text{and} \qquad y_2(t) = e^{r_2 t} = e^{(\lambda - i\mu)t}.$$

But what does $e$ to the power of a complex number mean?

### Definition

$$e^{(\lambda+i\mu)t} = e^{\lambda t}\cos\mu t + ie^{\lambda t}\sin\mu t.$$

**Remark**

Please note that

$$\begin{aligned}
\frac{d}{dt}\left(e^{r_1 t}\right) &= \frac{d}{dt}\left(e^{\lambda t}\cos\mu t + ie^{\lambda t}\sin\mu t\right) \\
&= \\
&= \\
&= \\
&= \\
&=
\end{aligned}$$

**Remark**

Please note that

$$\begin{aligned}
\frac{d}{dt}\left(e^{r_1 t}\right) &= \frac{d}{dt}\left(e^{\lambda t}\cos\mu t + ie^{\lambda t}\sin\mu t\right) \\
&= \lambda e^{\lambda t}\cos\mu t - \mu e^{\lambda t}\sin\mu t + i\lambda e^{\lambda t}\sin\mu t + i\mu e^{\lambda t}\cos\mu t \\
&= \\
&= \\
&= \\
&=
\end{aligned}$$

## Remark

Please note that

$$
\begin{aligned}
\frac{d}{dt}\left(e^{r_1 t}\right) &= \frac{d}{dt}\left(e^{\lambda t}\cos\mu t + ie^{\lambda t}\sin\mu t\right) \\
&= \lambda e^{\lambda t}\cos\mu t - \mu e^{\lambda t}\sin\mu t + i\lambda e^{\lambda t}\sin\mu t + i\mu e^{\lambda t}\cos\mu t \\
&= (\lambda + i\mu)\, e^{\lambda t}\cos\mu t + (i\lambda - \mu)\, e^{\lambda t}\sin\mu t \\
&= \\
&= \\
&=
\end{aligned}
$$

### Remark

Please note that

$$\begin{aligned}
\frac{d}{dt}\left(e^{r_1 t}\right) &= \frac{d}{dt}\Big(e^{\lambda t}\cos\mu t + ie^{\lambda t}\sin\mu t\Big) \\
&= \lambda e^{\lambda t}\cos\mu t - \mu e^{\lambda t}\sin\mu t + i\lambda e^{\lambda t}\sin\mu t + i\mu e^{\lambda t}\cos\mu t \\
&= (\lambda + i\mu)\, e^{\lambda t}\cos\mu t + (i\lambda - \mu)\, e^{\lambda t}\sin\mu t \\
&= (\lambda + i\mu)\, e^{\lambda t}\cos\mu t + i\,(\lambda + i\mu)\, e^{\lambda t}\sin\mu t \\
&= \\
&=
\end{aligned}$$

### Remark

Please note that

$$\begin{aligned}
\frac{d}{dt}\left(e^{r_1 t}\right) &= \frac{d}{dt}\left(e^{\lambda t}\cos\mu t + ie^{\lambda t}\sin\mu t\right)\\
&= \lambda e^{\lambda t}\cos\mu t - \mu e^{\lambda t}\sin\mu t + i\lambda e^{\lambda t}\sin\mu t + i\mu e^{\lambda t}\cos\mu t\\
&= \left(\lambda + i\mu\right)e^{\lambda t}\cos\mu t + \left(i\lambda - \mu\right)e^{\lambda t}\sin\mu t\\
&= \left(\lambda + i\mu\right)e^{\lambda t}\cos\mu t + i\left(\lambda + i\mu\right)e^{\lambda t}\sin\mu t\\
&= \left(\lambda + i\mu\right)\left(e^{\lambda t}\cos\mu t + ie^{\lambda t}\sin\mu t\right)\\
&=
\end{aligned}$$

**Remark**

Please note that

$$\begin{aligned}
\frac{d}{dt}\left(e^{r_1 t}\right) &= \frac{d}{dt}\left(e^{\lambda t}\cos\mu t + ie^{\lambda t}\sin\mu t\right) \\
&= \lambda e^{\lambda t}\cos\mu t - \mu e^{\lambda t}\sin\mu t + i\lambda e^{\lambda t}\sin\mu t + i\mu e^{\lambda t}\cos\mu t \\
&= (\lambda + i\mu)\, e^{\lambda t}\cos\mu t + (i\lambda - \mu)\, e^{\lambda t}\sin\mu t \\
&= (\lambda + i\mu)\, e^{\lambda t}\cos\mu t + i\,(\lambda + i\mu)\, e^{\lambda t}\sin\mu t \\
&= (\lambda + i\mu)\left(e^{\lambda t}\cos\mu t + ie^{\lambda t}\sin\mu t\right) \\
&= r_1 e^{r_1 t}.
\end{aligned}$$

## Real Valued Solutions

The solutions $y_1(t) = e^{(\lambda+i\mu)t}$ and $y_2(t) = e^{(\lambda-i\mu)t}$ are functions $y_1, y_2 : \mathbb{R} \to \mathbb{C}$. But we want solutions $\mathbb{R} \to \mathbb{R}$.

Consider

$$u(t) = \frac{1}{2}\left(y_1(t) + y_2(t)\right)$$

and

$$v(t) = \frac{1}{2i}\left(y_1(t) - y_2(t)\right)$$

Consider

$$
\begin{aligned}
u(t) &= \frac{1}{2}\left(y_1(t) + y_2(t)\right) \\
&= \frac{1}{2}e^{\lambda t}\left(\cos\mu t + i\sin\mu t\right) + \frac{1}{2}e^{\lambda t}\left(\cos\mu t - i\sin\mu t\right) \\
&= e^{\lambda t}\cos\mu t
\end{aligned}
$$

and

$$
\begin{aligned}
v(t) &= \frac{1}{2i}\left(y_1(t) - y_2(t)\right) \\
&= \frac{1}{2i}e^{\lambda t}\left(\cos\mu t + i\sin\mu t\right) - \frac{1}{2i}e^{\lambda t}\left(\cos\mu t - i\sin\mu t\right) \\
&= \frac{1}{2i}2ie^{\lambda t}\sin\mu t = e^{\lambda t}\sin\mu t.
\end{aligned}
$$

Consider

$$\begin{aligned}
u(t) &= \frac{1}{2}\left(y_1(t) + y_2(t)\right) \\
&= \frac{1}{2}e^{\lambda t}\left(\cos \mu t + i \sin \mu t\right) + \frac{1}{2}e^{\lambda t}\left(\cos \mu t - i \sin \mu t\right) \\
&= e^{\lambda t}\cos \mu t
\end{aligned}$$

and

$$\begin{aligned}
v(t) &= \frac{1}{2i}\left(y_1(t) - y_2(t)\right) \\
&= \frac{1}{2i}e^{\lambda t}\left(\cos \mu t + i \sin \mu t\right) - \frac{1}{2i}e^{\lambda t}\left(\cos \mu t - i \sin \mu t\right) \\
&= \frac{1}{2i}2ie^{\lambda t}\sin \mu t = e^{\lambda t}\sin \mu t.
\end{aligned}$$

Note that $u, v : \mathbb{R} \to \mathbb{R}$ both solve (1). But are they linearly independent?

Since

$$
\begin{aligned}
W(u,v)(t) &= \begin{vmatrix} u & v \\ u' & v' \end{vmatrix} \\
&= \begin{vmatrix} e^{\lambda t}\cos\mu t & e^{\lambda t}\sin\mu t \\ \lambda e^{\lambda t}\cos\mu t - \mu e^{\lambda t}\sin\mu t & \lambda e^{\lambda t}\sin\mu t + \mu e^{\lambda t}\cos\mu t \end{vmatrix} \\
&= e^{2\lambda t}\left(\lambda\cos\mu t\sin\mu t + \mu\cos^2\mu t - \lambda\cos\mu t\sin\mu t + \mu\sin^2\mu t\right) \\
&= \mu e^{2\lambda t} \neq 0
\end{aligned}
$$

(because $\mu \neq 0$), the answer is YES.

Since

$$
\begin{aligned}
W\big(u,v\big)(t) &= \begin{vmatrix} u & v \\ u' & v' \end{vmatrix} \\
&= \begin{vmatrix} e^{\lambda t}\cos\mu t & e^{\lambda t}\sin\mu t \\ \lambda e^{\lambda t}\cos\mu t - \mu e^{\lambda t}\sin\mu t & \lambda e^{\lambda t}\sin\mu t + \mu e^{\lambda t}\cos\mu t \end{vmatrix} \\
&= e^{2\lambda t}\left(\lambda\cos\mu t\sin\mu t + \mu\cos^2\mu t - \lambda\cos\mu t\sin\mu t + \mu\sin^2\mu t\right) \\
&= \mu e^{2\lambda t} \neq 0
\end{aligned}
$$

(because $\mu \neq 0$), the answer is YES. Therefore $u(t)$ and $v(t)$ form a fundamental set of solutions to (1).

Since

$$
\begin{aligned}
W(u,v)(t) &= \begin{vmatrix} u & v \\ u' & v' \end{vmatrix} \\
&= \begin{vmatrix} e^{\lambda t}\cos\mu t & e^{\lambda t}\sin\mu t \\ \lambda e^{\lambda t}\cos\mu t - \mu e^{\lambda t}\sin\mu t & \lambda e^{\lambda t}\sin\mu t + \mu e^{\lambda t}\cos\mu t \end{vmatrix} \\
&= e^{2\lambda t}\left(\lambda\cos\mu t\sin\mu t + \mu\cos^2\mu t - \lambda\cos\mu t\sin\mu t + \mu\sin^2\mu t\right) \\
&= \mu e^{2\lambda t} \neq 0
\end{aligned}
$$

(because $\mu \neq 0$), the answer is YES. Therefore $u(t)$ and $v(t)$ form a fundamental set of solutions to (1). The general solution to (1) is therefore

$$y(t) = c_1 u(t) + c_2 v(t) = c_1 e^{\lambda t}\cos\mu t + c_2 e^{\lambda t}\sin\mu t.$$

**Example**

Solve $y'' + y' + y = 0$.

**Example**

Solve $y'' + y' + y = 0$.

The characteristic equation

$$r^2 + r + 1 = 0$$

has roots

$$r = \frac{-1 \pm \sqrt{1-4}}{2} = \frac{-1 \pm \sqrt{(-1)(3)}}{2} = -\frac{1}{2} \pm i\frac{\sqrt{3}}{2}.$$

**Example**

Solve $y'' + y' + y = 0$.

The characteristic equation

$$r^2 + r + 1 = 0$$

has roots

$$r = \frac{-1 \pm \sqrt{1-4}}{2} = \frac{-1 \pm \sqrt{(-1)(3)}}{2} = -\frac{1}{2} \pm i\frac{\sqrt{3}}{2}.$$

So $\lambda = -\frac{1}{2}$ and $\mu = \frac{\sqrt{3}}{2}$.

### Example

Solve $y'' + y' + y = 0$.

The characteristic equation

$$r^2 + r + 1 = 0$$

has roots

$$r = \frac{-1 \pm \sqrt{1-4}}{2} = \frac{-1 \pm \sqrt{(-1)(3)}}{2} = -\frac{1}{2} \pm i\frac{\sqrt{3}}{2}.$$

So $\lambda = -\frac{1}{2}$ and $\mu = \frac{\sqrt{3}}{2}$.
Therefore the general solution is

$$\boxed{y(t) = c_1 e^{-\frac{t}{2}} \cos \frac{\sqrt{3}}{2}t + c_2 e^{-\frac{t}{2}} \sin \frac{\sqrt{3}}{2}t.}$$

**Example**

Solve $y'' + 9y = 0$.

**Example**

Solve $y'' + 9y = 0$.

Since $0 = r^2 + 9 = (r - 3i)(r + 3i)$ we have $r = \pm 3i$ (i.e. $\lambda = 0$ and $\mu = 3$). Therefore the general solution is

$$\boxed{y(t) = c_1 \cos 3t + c_2 \sin 3t.}$$

### Example

Solve

$$\begin{cases} 16y'' - 8y' + 145y = 0 \\ y(0) = -2 \\ y'(0) = 1. \end{cases}$$

**Example**

Solve

$$\begin{cases} 16y'' - 8y' + 145y = 0 \\ y(0) = -2 \\ y'(0) = 1. \end{cases}$$

The characteristic equation $16r^2 - 8r + 145 = 0$ has roots

$$\begin{aligned} r &= \frac{8 \pm \sqrt{64 - 4(16)(145)}}{32} = \frac{8 \pm \sqrt{(64)(1-145)}}{32} \\ &= \frac{8 \pm \sqrt{(-1)(64)(144)}}{32} = \frac{1}{4} \pm 3i. \end{aligned}$$

**Example**

Solve

$$\begin{cases} 16y'' - 8y' + 145y = 0 \\ y(0) = -2 \\ y'(0) = 1. \end{cases}$$

The characteristic equation $16r^2 - 8r + 145 = 0$ has roots

$$\begin{aligned} r = \frac{8 \pm \sqrt{64 - 4(16)(145)}}{32} &= \frac{8 \pm \sqrt{(64)(1-145)}}{32} \\ = \frac{8 \pm \sqrt{(-1)(64)(144)}}{32} &= \frac{1}{4} \pm 3i. \end{aligned}$$

Therefore the general solution to the ODE is

$$y(t) = c_1 e^{\frac{t}{4}} \cos 3t + c_2 e^{\frac{t}{4}} \sin 3t.$$

$$y(t) = c_1 e^{\frac{t}{4}} \cos 3t + c_2 e^{\frac{t}{4}} \sin 3t$$

Finally we calculate that

$$y'(t) = \frac{1}{4} c_1 e^{\frac{t}{4}} \cos 3t - 3c_1 e^{\frac{t}{4}} \sin 3t + \frac{1}{4} c_2 e^{\frac{t}{4}} \sin 3t + 3c_2 e^{\frac{t}{4}} \cos 3t$$

$$y(t) = c_1 e^{\frac{t}{4}} \cos 3t + c_2 e^{\frac{t}{4}} \sin 3t$$

Finally we calculate that

$$y'(t) = \frac{1}{4} c_1 e^{\frac{t}{4}} \cos 3t - 3c_1 e^{\frac{t}{4}} \sin 3t + \frac{1}{4} c_2 e^{\frac{t}{4}} \sin 3t + 3c_2 e^{\frac{t}{4}} \cos 3t$$

and

$$-2 = y(0) = c_1 + 0 \qquad \Longrightarrow \qquad c_1 = -2$$
$$1 = y'(0) = \frac{1}{4} c_1 + 3c_2 = -\frac{1}{2} + 3c_2 \qquad \Longrightarrow \qquad c_2 = \frac{1}{2}.$$

$$y(t) = c_1 e^{\frac{t}{4}} \cos 3t + c_2 e^{\frac{t}{4}} \sin 3t$$

Finally we calculate that

$$y'(t) = \frac{1}{4} c_1 e^{\frac{t}{4}} \cos 3t - 3c_1 e^{\frac{t}{4}} \sin 3t + \frac{1}{4} c_2 e^{\frac{t}{4}} \sin 3t + 3c_2 e^{\frac{t}{4}} \cos 3t$$

and

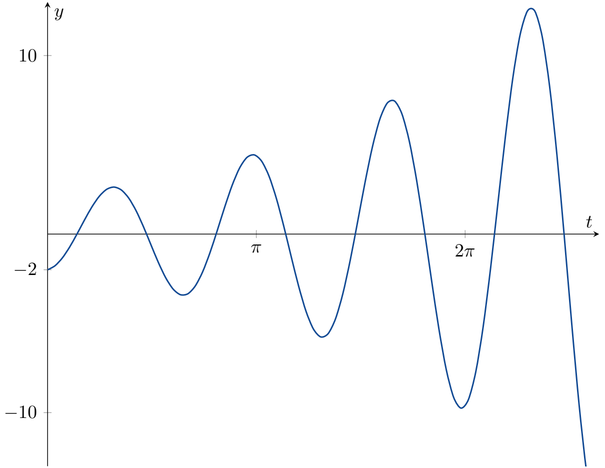
$$\begin{aligned} -2 = y(0) &= c_1 + 0 && \implies && c_1 = -2 \\ 1 = y'(0) &= \frac{1}{4} c_1 + 3c_2 = -\frac{1}{2} + 3c_2 && \implies && c_2 = \frac{1}{2}. \end{aligned}$$

Therefore the solution to the IVP is

$$\boxed{y = -2e^{\frac{t}{4}} \cos 3t + \tfrac{1}{2} e^{\frac{t}{4}} \sin 3t.}$$

# 3.3 Complex Roots of the Characteristic Equation

$$\begin{cases} 16y'' - 8y' + 145y = 0 \\ y(0) = -2 \\ y'(0) = 1. \end{cases}$$



## Summary

To solve

$$ay'' + by' + cy = 0$$

we need to find two linearly independent solutions.

1. If $r_1, r_2 \in \mathbb{R}$ and $r_1 \neq r_2$, then

$$y_1(t) = e^{r_1 t} \qquad \text{and} \qquad y_2(t) = e^{r_2 t};$$

2. If $r_{1,2} = \lambda \pm i\mu$ $(\lambda, \mu \in \mathbb{R})$, then

$$y_1(t) = e^{\lambda t} \cos \mu t \qquad \text{and} \qquad y_2(t) = e^{\lambda t} \sin \mu t;$$

3. If the roots are repeated, then ?????????????

# Next Time

- 3.4 Repeated Roots of the Characteristic Equation
- 3.5 Reduction of Order
- 3.6 Nonhomogeneous Equations
- 3.7 The Method of Undetermined Coefficients