

# Lecture 7

- Rank and Nullity
- The Fundamental Matrix Spaces
- Linear Transformations
- Composition and Inverse Transformations
- Isomorphisms



# Rank and Nullity

# Rank and Nullity

Recall, to find a basis for the row space of a matrix  $A$ , we reduce it to REF, then take the rows with a pivot.

$$A = \begin{bmatrix} 1 & -3 & 4 & -2 & 5 & 4 \\ 2 & -6 & 9 & -1 & 8 & 2 \\ 2 & -6 & 9 & -1 & 9 & 7 \\ -1 & 3 & -4 & 2 & -5 & -4 \end{bmatrix} \sim R = \begin{bmatrix} 1 & -3 & 4 & -2 & 5 & 4 \\ 0 & 0 & 1 & 3 & -2 & -6 \\ 0 & 0 & 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

For example

$$\mathbf{r}_1 = [ 1 \quad -3 \quad 4 \quad -2 \quad 5 \quad 4 ]$$

$$\mathbf{r}_2 = [ 0 \quad 0 \quad 1 \quad 3 \quad -2 \quad -6 ]$$

$$\mathbf{r}_3 = [ 0 \quad 0 \quad 0 \quad 0 \quad 1 \quad 5 ]$$

form a basis for Row  $A$  = Row  $R$ . Since  $R$  has three pivots, the dimension of Row  $A$  is 3.

# Rank and Nullity



And to find a basis for a column space, we take the pivot columns.

$$A = \begin{bmatrix} 1 & -3 & 4 & -2 & 5 & 4 \\ 2 & -6 & 9 & -1 & 8 & 2 \\ 2 & -6 & 9 & -1 & 9 & 7 \\ -1 & 3 & -4 & 2 & -5 & -4 \end{bmatrix} \sim R = \begin{bmatrix} 1 & -3 & 4 & -2 & 5 & 4 \\ 0 & 0 & 1 & 3 & -2 & -6 \\ 0 & 0 & 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

The pivot columns of  $A$  are also the **first**, **third** and **fifth** columns.  
Hence

$$\left\{ \mathbf{c}_1 = \begin{bmatrix} 1 \\ 2 \\ 2 \\ -1 \end{bmatrix}, \mathbf{c}_3 = \begin{bmatrix} 4 \\ 9 \\ 9 \\ -4 \end{bmatrix}, \mathbf{c}_5 = \begin{bmatrix} 5 \\ 8 \\ 9 \\ -5 \end{bmatrix} \right\}$$

is a basis for  $\text{Col } A$ . Since  $A$  has three pivot columns, the dimension of  $\text{Col } A$  is 3.

# Rank and Nullity

## Theorem

*The row space and the column space of a matrix  $A$  have the same dimension.*

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## Rank and Nullity

### Definition

The *rank* of a matrix  $A$  is

$$\text{rank } A = \dim (\text{Col } A) = \dim (\text{Row } A).$$

## Theorem

*The row space and the column space of a matrix  $A$  have the same dimension.*

## Rank and Nullity

### Definition

The *rank* of a matrix  $A$  is

$$\text{rank } A = \dim (\text{Col } A) = \dim (\text{Row } A).$$

### Definition

The *nullity* of  $A$  is

$$\text{nullity } A = \dim (\text{Nul } A).$$

# Rank and Nullity

## Example

Find the rank and nullity of the matrix

$$A = \begin{bmatrix} -1 & 2 & 0 & 4 & 5 & -3 \\ 3 & -7 & 2 & 0 & 1 & 4 \\ 2 & -5 & 2 & 4 & 6 & 1 \\ 4 & -9 & 2 & -4 & -4 & 7 \end{bmatrix}.$$

# Rank and Nullity

## Example

Find the rank and nullity of the matrix

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Please check that  $A$  is row equivalent to the RREF matrix

$$R = \begin{bmatrix} 1 & 0 & -4 & -28 & -37 & 13 \\ 0 & 1 & -2 & -12 & -16 & 5 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

# Rank and Nullity

The matrix

$$R = \begin{bmatrix} 1 & 0 & -4 & -28 & -37 & 13 \\ 0 & 1 & -2 & -12 & -16 & 5 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

has two pivots. So both Row  $A$  and Col  $A$  are two dimensional.  
Hence  $\text{rank } A = 2$ .

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To find the nullity of  $A$ , we need to find the dimension of the solution space of  $A\mathbf{x} = \mathbf{0}$ .

# Rank and Nullity

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has two pivots. So both Row  $A$  and Col  $A$  are two dimensional.  
Hence  $\text{rank } A = 2$ .

To find the nullity of  $A$ , we need to find the dimension of the solution space of  $A\mathbf{x} = \mathbf{0}$ . We can see from  $R$  that the general solution of  $A\mathbf{x} = \mathbf{0}$  is

$$\begin{cases} x_1 = 4x_3 + 28x_4 + 37x_5 - 13x_6 \\ x_2 = 2x_3 + 12x_4 + 16x_5 - 5x_6 \\ x_3, x_4, x_5, x_6 \text{ are free.} \end{cases}$$

# Rank and Nullity



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In vector form, the general solution is

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} = r \begin{bmatrix} 4 \\ 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + s \begin{bmatrix} 28 \\ 12 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} 37 \\ 16 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} + u \begin{bmatrix} -13 \\ -5 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

for any  $r, s, t, u \in \mathbb{R}$ .

# Rank and Nullity



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for any  $r, s, t, u \in \mathbb{R}$ . These four vectors (on the right hand side) form a basis for  $\text{Nul } A$ . Hence nullity  $A = 4$ .

# Rank and Nullity



Note that

$$\text{rank } A = \text{number of pivot positions}$$

and

$$\text{nullity } A = \text{number of free variables.}$$

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## Theorem

*If  $A$  is an  $m \times n$  matrix (i.e. with  $n$  columns), then*

$$\text{rank } A + \text{nullity } A = n.$$



## Example

The matrix

$$A = \begin{bmatrix} -1 & 2 & 0 & 4 & 5 & -3 \\ 3 & -7 & 2 & 0 & 1 & 4 \\ 2 & -5 & 2 & 4 & 6 & 1 \\ 4 & -9 & 2 & -4 & -4 & 7 \end{bmatrix}$$

has 6 columns. Therefore

$$\text{rank } A + \text{nullity } A = 6.$$



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has 6 columns. Therefore

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In the previous example we showed that  $\text{rank } A = 2$  and  $\text{nullity } A = 4$ .

$$\text{rank } A + \text{nullity } A = \text{number of columns}$$



## Ask the audience 1/2

Suppose that  $A$  is a  $5 \times 7$  matrix of rank 3. Find the nullity of  $A$ .

1 nullity  $A = 1$

3 nullity  $A = 3$

2 nullity  $A = 2$

4 nullity  $A = 4$

$$\text{rank } A + \text{nullity } A = \text{number of columns}$$



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## Ask the audience 2/2

Suppose that the solution space of  $B\mathbf{x} = \mathbf{0}$  is two dimensional.  
If  $B$  is a  $5 \times 7$ , find the rank of  $B$ .

1 rank  $B = 4$

3 rank  $B = 6$

2 rank  $B = 5$

4 rank  $B = 7$



## Ask the audience 2/2

Suppose that the solution space of  $B\mathbf{x} = \mathbf{0}$  is two dimensional.  
If  $B$  is a  $5 \times 7$ , find the rank of  $B$ .

1  $\text{rank } B = 4$

3  $\text{rank } B = 6$

2  $\text{rank } B = 5$

4  $\text{rank } B = 7$

## Theorem

*If  $A\mathbf{x} = \mathbf{b}$  is a consistent linear system of  $m$  equations in  $n$  unknowns, and if  $A$  has rank  $r$ , then the general solution of the system contains  $n - r$  free variables (parameters).*



# The Fundamental Matrix Spaces

# The Fundamental Matrix Spaces



Consider the following six vector spaces:

- the row space of  $A$
- the column space of  $A$
- the null space of  $A$
- the row space of  $A^T$
- the column space of  $A^T$
- the null space of  $A^T$

# The Fundamental Matrix Spaces



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When we take the transpose of a matrix, rows become columns and columns become rows. That means that

$$\text{Row } A = \text{Col } A^T$$

and

$$\text{Col } A = \text{Row } A^T.$$

# The Fundamental Matrix Spaces



## Definition

The *fundamental spaces* of a matrix  $A$  are

- 1 the row space of  $A$
- 2 the column space of  $A$
- 3 the null space of  $A$
- 4 the null space of  $A^T$ .

We will talk about how these four vector spaces are related.

# The Fundamental Matrix Spaces



## Theorem

*If  $A$  is any matrix, then*

$$\text{rank } A = \text{rank } A^T.$$

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*If  $A$  is any matrix, then*

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## Proof.

$$\begin{aligned}\text{rank } A &= \dim(\text{row space of } A) \\ &= \dim(\text{column space of } A^T) = \text{rank } A^T.\end{aligned}$$



# The Fundamental Matrix Spaces



If  $A$  is an  $m \times n$  matrix, then  $A^T$  is an  $n \times m$  matrix.

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$$\text{rank } A^T + \text{nullity } A^T = \textcolor{red}{m}.$$

If  $\text{rank } A = r$ , then

$$\dim (\text{Row } A) = r$$

$$\dim (\text{Col } A) = r$$

$$\dim (\text{Nul } A) = n - r$$

$$\dim (\text{Nul } A^T) = m - r.$$

# The Fundamental Matrix Spaces

Now suppose that  $\mathbf{x}$  is a solution of

$$\begin{bmatrix} 1 & 3 & -2 & 0 & 2 & 0 \\ 2 & 6 & -5 & -2 & 4 & -3 \\ 0 & 0 & 5 & 10 & 0 & 15 \\ 2 & 6 & 0 & 8 & 4 & 18 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

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Note that

$$\mathbf{r}_1 \cdot \mathbf{x} = (1, 3, -2, 0, 2, 0) \cdot (x_1, x_2, x_3, x_4, x_5, x_6)$$

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$$= [1 \ 3 \ -2 \ 0 \ 2 \ 0] \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} =$$

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Note that

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$$= [1 \quad 3 \quad -2 \quad 0 \quad 2 \quad 0] \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} = 0.$$

So

$\mathbf{x}$  solves  $A\mathbf{x} = \mathbf{0} \implies \mathbf{x}$  is orthogonal to  $\mathbf{r}_1$ .

# The Fundamental Matrix Spaces



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Similarly  $\mathbf{r}_2 \cdot \mathbf{x} = 0$ ,  $\mathbf{r}_3 \cdot \mathbf{x} = 0$  and  $\mathbf{r}_4 \cdot \mathbf{x} = 0$ .

# The Fundamental Matrix Spaces



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Similarly  $\mathbf{r}_2 \cdot \mathbf{x} = 0$ ,  $\mathbf{r}_3 \cdot \mathbf{x} = 0$  and  $\mathbf{r}_4 \cdot \mathbf{x} = 0$ .

## Theorem

*If  $A$  is an  $m \times n$  matrix, then the null space of  $A$  (i.e. the solution space of  $A\mathbf{x} = \mathbf{0}$ ) consists of all vectors in  $\mathbb{R}^n$  that are orthogonal to every row vector of  $A$ .*

# The Fundamental Matrix Spaces

## Definition

If  $W$  is a subspace of  $\mathbb{R}^n$ , then the set of all vectors in  $\mathbb{R}^n$  that are orthogonal to every vector in  $W$  is called the orthogonal complement of  $W$  and is denoted by the symbol  $W^\perp$ .

In other words,

$$W^\perp = \{\mathbf{v} \in V \mid \mathbf{v} \cdot \mathbf{w} = 0 \text{ for all } \mathbf{w} \in W\}.$$

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## Theorem

*If  $W$  is a subspace of  $\mathbb{R}^n$ , then:*

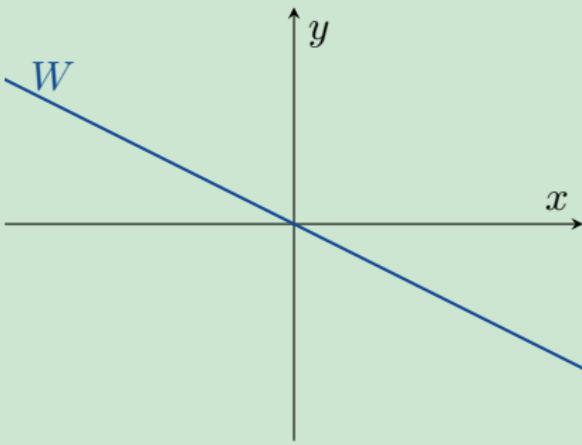
- 1  $W^\perp$  is also a subspace of  $\mathbb{R}^n$ .
- 2  $W \cap W^\perp = \{\mathbf{0}\}$ .
- 3  $(W^\perp)^\perp = W$ .

# The Fundamental Matrix Spaces



## Example

In  $\mathbb{R}^2$ :

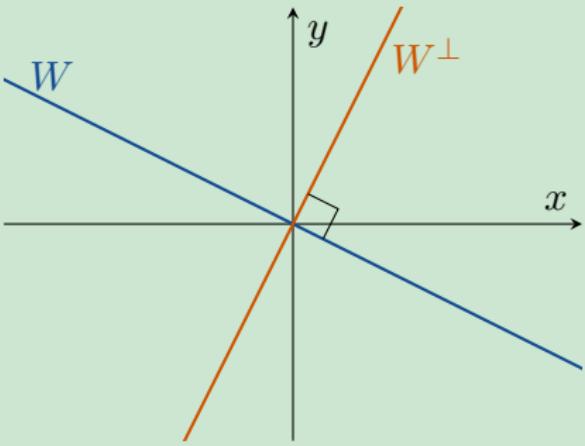


# The Fundamental Matrix Spaces



## Example

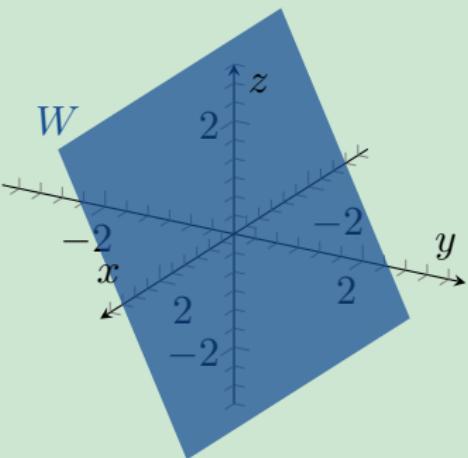
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# The Fundamental Matrix Spaces

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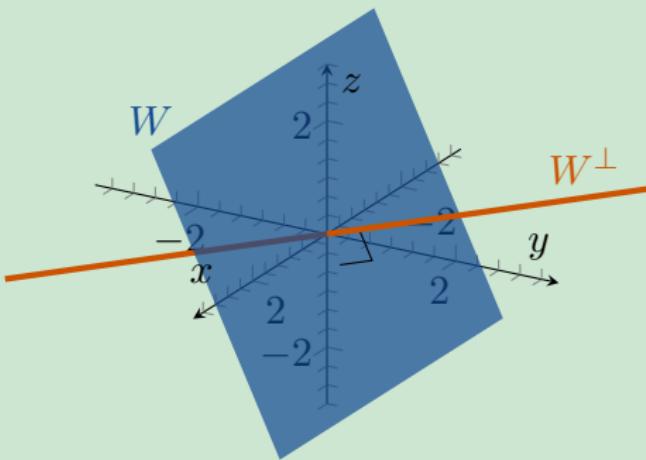
In  $\mathbb{R}^3$ :



# The Fundamental Matrix Spaces

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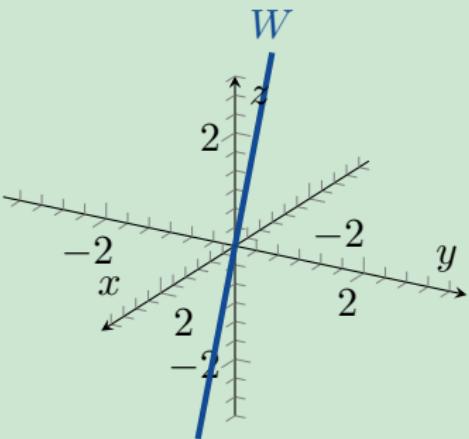
In  $\mathbb{R}^3$ :



# The Fundamental Matrix Spaces

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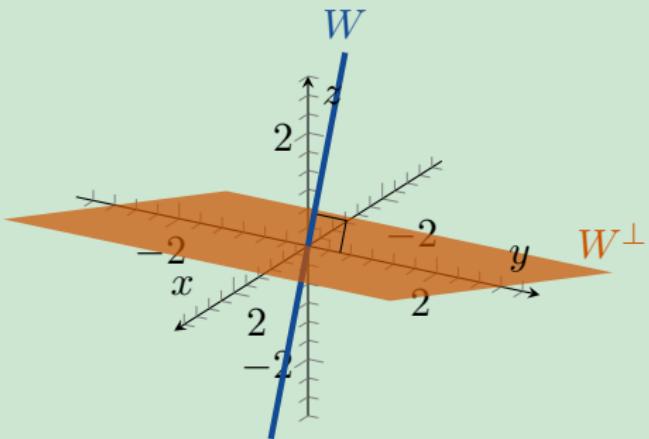
In  $\mathbb{R}^3$ :



# The Fundamental Matrix Spaces

## Example

In  $\mathbb{R}^3$ :



# The Fundamental Matrix Spaces



## Remark

Note that

$$V^\perp = \{\mathbf{0}\}$$

because only the zero vector is orthogonal to every vector in  $V$ ;  
and hence

$$\{\mathbf{0}\}^\perp = V.$$

# The Fundamental Matrix Spaces



## Theorem

Let  $A$  be an  $m \times n$  matrix. Then

- 1 The null space of  $A$  is the orthogonal complement of the row space of  $A$ , in  $\mathbb{R}^n$ .

$$\text{Nul } A = (\text{Row } A)^\perp \quad \text{and} \quad \text{Row } A = (\text{Nul } A)^\perp.$$

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- 2 The null space of  $A^T$  is the orthogonal complement of the column space of  $A$ , in  $\mathbb{R}^m$ .

$$\text{Nul } A^T = (\text{Col } A)^\perp \quad \text{and} \quad \text{Col } A = (\text{Nul } A^T)^\perp.$$

# The Fundamental Matrix Spaces

## Theorem

Let  $A$  be an  $n \times n$  matrix. The following statements are equivalent:

- 1  $A$  is invertible. 9
- 2  $A\mathbf{x} = \mathbf{0}$  has only the trivial solution. 10
- 3 The reduced row echelon form of  $A$  is  $I_n$ . 11
- 4  $A$  is expressible as a product of elementary matrices. 12
- 5  $A\mathbf{x} = \mathbf{b}$  is consistent for every  $n \times 1$  matrix  $\mathbf{b}$ . 13
- 6  $A\mathbf{x} = \mathbf{b}$  has exactly one solution for every  $n \times 1$  matrix  $\mathbf{b}$ . 14
- 7  $\det(A) \neq 0$ . 15
- 8 16

# The Fundamental Matrix Spaces

## Theorem

Let  $A$  be an  $n \times n$  matrix. The following statements are equivalent:

- |   |  |    |   |
|---|--|----|---|
| 1 | $A$ is invertible.   | 9  | The row vectors of $A$ are linearly independent.            |
| 2 | $A\mathbf{x} = \mathbf{0}$ has only the trivial solution.  | 10 | The column vectors of $A$ span $\mathbb{R}^n$ .             |
| 3 | The reduced row echelon form of $A$ is $I_n$ .   | 11 | The row vectors of $A$ span $\mathbb{R}^n$ .                |
| 4 | $A$ is expressible as a product of elementary matrices.  | 12 | The column vectors of $A$ form a basis for $\mathbb{R}^n$ . |
| 5 | $A\mathbf{x} = \mathbf{b}$ is consistent for every $n \times 1$ matrix $\mathbf{b}$ .            | 13 | The row vectors of $A$ form a basis for $\mathbb{R}^n$ .    |
| 6 | $A\mathbf{x} = \mathbf{b}$ has exactly one solution for every $n \times 1$ matrix $\mathbf{b}$ . | 14 |   |
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- 5  $A\mathbf{x} = \mathbf{b}$  is consistent for every  $n \times 1$  matrix  $\mathbf{b}$ .
- 6  $A\mathbf{x} = \mathbf{b}$  has exactly one solution for every  $n \times 1$  matrix  $\mathbf{b}$ .
- 7  $\det(A) \neq 0$ .
- 8 The column vectors of  $A$  are linearly independent.
- 9 The row vectors of  $A$  are linearly independent.
- 10 The column vectors of  $A$  span  $\mathbb{R}^n$ .
- 11 The row vectors of  $A$  span  $\mathbb{R}^n$ .
- 12 The column vectors of  $A$  form a basis for  $\mathbb{R}^n$ .
- 13 The row vectors of  $A$  form a basis for  $\mathbb{R}^n$ .
- 14  $\text{rank } A = n$ .
- 15  $\text{nullity } A = 0$ .
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- 17

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- 7**  $\det(A) \neq 0$ .
- 8** The column vectors of  $A$  are linearly independent.
- 9** The row vectors of  $A$  are linearly independent.
- 10** The column vectors of  $A$  span  $\mathbb{R}^n$ .
- 11** The row vectors of  $A$  span  $\mathbb{R}^n$ .
- 12** The column vectors of  $A$  form a basis for  $\mathbb{R}^n$ .
- 13** The row vectors of  $A$  form a basis for  $\mathbb{R}^n$ .
- 14**  $\text{rank } A = n$ .
- 15**  $\text{nullity } A = 0$ .
- 16**  $(\text{Nul } A)^\perp = \mathbb{R}^n$ .
- 17**  $(\text{Row } A)^\perp = \{\mathbf{0}\}$ .



# Linear Transformations

## Definition

Let  $V$  and  $W$  be vector spaces. A function  $T : V \rightarrow W$  is called a *linear transformation* iff

- 1  $T(\mathbf{u} + \mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v})$
- 2  $T(k\mathbf{u}) = kT(\mathbf{u})$ ; and

for all vectors  $\mathbf{u}, \mathbf{v} \in V$  and all scalars  $k$ .

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## Definition

If  $V = W$ , then the linear transformation  $T : V \rightarrow V$  is called a *linear operator* on  $V$ .

$$T(\mathbf{u} + \mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v}) \text{ and } T(k\mathbf{u}) = kT(\mathbf{u})$$



## Remark

I have tried to avoid any mention of linear transformations up to now. Your textbook introduces them in section 1.8.

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$$T(k_1\mathbf{v}_1 + k_2\mathbf{v}_2) = k_1T(\mathbf{v}_1) + k_2T(\mathbf{v}_2)$$

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$$T(k_1\mathbf{v}_1 + k_2\mathbf{v}_2) = k_1T(\mathbf{v}_1) + k_2T(\mathbf{v}_2)$$

or more generally that

$$T(k_1\mathbf{v}_1 + k_2\mathbf{v}_2 + \dots + k_r\mathbf{v}_r) = k_1T(\mathbf{v}_1) + k_2T(\mathbf{v}_2) + \dots + k_rT(\mathbf{v}_r).$$

$$T(\mathbf{u} + \mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v}) \text{ and } T(k\mathbf{u}) = kT(\mathbf{u})$$



## Theorem

If  $T : V \rightarrow W$  is a linear transformation, then

$$T(\mathbf{0}) = \mathbf{0}.$$

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## Proof.

Let  $\mathbf{u} \in V$  be any vector. Since  $0\mathbf{u} = \mathbf{0}$ , it follows that

$$T(\mathbf{0}) = T(0\mathbf{u}) = 0T(\mathbf{u}) = \mathbf{0}.$$



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## Example (Matrix Transformations)

Let  $A$  be an  $m \times n$  matrix. We can define a function  
 $T_A : \mathbb{R}^n \rightarrow \mathbb{R}^m$  by

$$T_A(\mathbf{x}) = A\mathbf{x}.$$

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Note that if  $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$  are any vectors, and  $k$  is any number, then

$$T_A(\mathbf{u} + \mathbf{v}) = A(\mathbf{u} + \mathbf{v}) = A\mathbf{u} + A\mathbf{v} = T_A(\mathbf{u}) + T_A(\mathbf{v})$$

and

$$T_A(k\mathbf{u}) = A(k\mathbf{u}) = k(A\mathbf{u}) = kT_A(\mathbf{u}).$$

Therefore  $T_A$  is a linear transformation.

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Functions of the form  $T_A : \mathbb{R}^n \rightarrow \mathbb{R}^m$  are called *matrix transformations*.

$$T(\mathbf{u} + \mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v}) \text{ and } T(k\mathbf{u}) = kT(\mathbf{u})$$



### Example (The Zero Transformation)

Let  $V$  and  $W$  be any two vector spaces. Define a function  $T : V \rightarrow W$  by  $T(\mathbf{v}) = \mathbf{0}$  for all  $\mathbf{v} \in V$ .

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for all  $\mathbf{u}$ ,  $\mathbf{v}$  and  $k$ . Hence  $T$  is linear.

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for all  $\mathbf{u}, \mathbf{v}$  and  $k$ . Hence  $T$  is linear.

### Example (The Identity Operator)

Let  $V$  be any vector space. The function  $I : V \rightarrow V$  defined by  $I(\mathbf{v}) = \mathbf{v}$  is called the *identity operator* on  $V$ . I leave it to you to show that  $I$  is linear.

$$T(\mathbf{u} + \mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v}) \text{ and } T(k\mathbf{u}) = kT(\mathbf{u})$$



### Example (A Linear Transformation from $\mathbb{P}^n$ to $\mathbb{P}^{n+1}$ )

Let  $\mathbf{p} = a_0 + a_1x + \dots + a_nx^n$  be a polynomial in  $\mathbb{P}^n$ . Define  $T : \mathbf{P}^n \rightarrow \mathbb{P}^{n+1}$  by  $T(\mathbf{p}) = x\mathbf{p}$ . That is

$$T(\mathbf{p}) = a_0x + a_1x^2 + \dots + a_nx^{n+1}.$$

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This function is a linear transformation because

$$T(\mathbf{p}_1 + \mathbf{p}_2) = x(\mathbf{p}_1 + \mathbf{p}_2) = x\mathbf{p}_1 + x\mathbf{p}_2 = T(\mathbf{p}_1) + T(\mathbf{p}_2)$$

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## Example

Let  $\mathbb{R}^{n \times n} = M_{nn}$  be the vector space of  $n \times n$  matrices.

- 1 Is  $T_1 : \mathbb{R}^{n \times n} \rightarrow \mathbb{R}^{n \times n}$ ,  $T_1(A) = A^T$  linear?
- 2 Is  $T_2 : \mathbb{R}^{n \times n} \rightarrow \mathbb{R}$ ,  $T_2(A) = \det(A)$  linear?

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- 1 Since

$$T_1(A + B) = (A + B)^T = A^T + B^T = T_1(A) + T_1(B)$$

and

$$T_2(kA) = (kA)^T = kA^T = kT_2(A)$$

the answer is YES.

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and

$$T_2(kA) = (kA)^T = kA^T = kT_2(A)$$

the answer is YES.

- 2 Recall from Lecture 4 that

$$T_2(kA) = \det(kA) = k^n \det(A) = k^n T_2(A).$$

If  $n \geq 2$ , then we have  $T_2(kA) \neq kT_2(A)$ . So the answer is NO (if  $n \geq 2$ ).

$$T(\mathbf{u} + \mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v}) \text{ and } T(k\mathbf{u}) = kT(\mathbf{u})$$



Recall that Linear Transformations always have  $T(\mathbf{0}) = \mathbf{0}$ .

### Example (A Nonlinear Transformation)

Let  $\mathbf{b} \in \mathbb{R}^2$  such that  $\mathbf{b} \neq \mathbf{0}$ . Define  $T : \mathbb{R}^2 \rightarrow \mathbb{R}^2$  by  $T(\mathbf{x}) = \mathbf{x} + \mathbf{b}$ . Note that  $T$  is not linear because

$$T(\mathbf{0}) = \mathbf{b} \neq \mathbf{0}.$$

$$T(\mathbf{u} + \mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v}) \text{ and } T(k\mathbf{u}) = kT(\mathbf{u})$$



### Example

$\frac{d}{dx} : C^1(-\infty, \infty) \rightarrow F(-\infty, \infty)$  is a linear operator because

$$\frac{d}{dx}(f + g) = \frac{d}{dx}(f) + \frac{d}{dx}(g)$$

and

$$\frac{d}{dx}(kf) = k \frac{d}{dx}(f).$$

## Finding Linear Transformations from Images of Basis Vectors

### Theorem

Let  $T : V \rightarrow W$  be a linear transformation, where  $V$  is finite-dimensional. If  $S = \{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\}$  is a basis for  $V$ , then the image of any vector  $\mathbf{v} \in V$  can be expressed as

$$T(\mathbf{v}) = c_1 T(\mathbf{v}_1) + c_2 T(\mathbf{v}_2) + \dots + c_n T(\mathbf{v}_n)$$

where  $c_1, c_2, \dots, c_n$  are the coefficients required to express  $\mathbf{v}$  as a linear combination of the vectors in the basis  $S$ .

## ► EXAMPLE 10 Computing with Images of Basis Vectors

Consider the basis  $S = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$  for  $\mathbb{R}^3$ , where

$$\mathbf{v}_1 = (1, 1, 1), \quad \mathbf{v}_2 = (1, 1, 0), \quad \mathbf{v}_3 = (1, 0, 0)$$

Let  $T : \mathbb{R}^3 \rightarrow \mathbb{R}^2$  be the linear transformation for which

$$T(\mathbf{v}_1) = (1, 0), \quad T(\mathbf{v}_2) = (2, -1), \quad T(\mathbf{v}_3) = (4, 3)$$

Find a formula for  $T(x_1, x_2, x_3)$ , and then use that formula to compute  $T(2, -3, 5)$ .

**Solution** We first need to express  $\mathbf{x} = (x_1, x_2, x_3)$  as a linear combination of  $\mathbf{v}_1, \mathbf{v}_2$ , and  $\mathbf{v}_3$ . If we write

$$(x_1, x_2, x_3) = c_1(1, 1, 1) + c_2(1, 1, 0) + c_3(1, 0, 0)$$

then on equating corresponding components, we obtain

$$\begin{aligned}c_1 + c_2 + c_3 &= x_1 \\c_1 + c_2 &= x_2 \\c_1 &= x_3\end{aligned}$$

which yields  $c_1 = x_3, c_2 = x_2 - x_3, c_3 = x_1 - x_2$ , so

$$\begin{aligned}(x_1, x_2, x_3) &= x_3(1, 1, 1) + (x_2 - x_3)(1, 1, 0) + (x_1 - x_2)(1, 0, 0) \\&= x_3\mathbf{v}_1 + (x_2 - x_3)\mathbf{v}_2 + (x_1 - x_2)\mathbf{v}_3\end{aligned}$$

Thus

$$\begin{aligned}T(x_1, x_2, x_3) &= x_3T(\mathbf{v}_1) + (x_2 - x_3)T(\mathbf{v}_2) + (x_1 - x_2)T(\mathbf{v}_3) \\&= x_3(1, 0) + (x_2 - x_3)(2, -1) + (x_1 - x_2)(4, 3) \\&= (4x_1 - 2x_2 - x_3, 3x_1 - 4x_2 + x_3)\end{aligned}$$

From this formula we obtain

$$T(2, -3, 5) = (9, 23)$$

## Kernal and Range

### Definition

If  $T : V \rightarrow W$  is a linear transformation, then the set of vectors in  $V$  that  $T$  maps into 0 is called the kernel of  $T$  and is denoted by  $\ker(T)$ .

$$\ker(T) = \{\mathbf{v} \in V \mid T(\mathbf{v}) = \mathbf{0}\}$$

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### Definition

The set of all vectors in  $W$  that are images under  $T$  of at least one vector in  $V$  is called the range of  $T$  and is denoted by  $R(T)$ .

$$R(T) = \{\mathbf{w} \in W \mid \exists \mathbf{v} \in V \text{ such that } T(\mathbf{v}) = \mathbf{w}\}$$

# Linear Transformations



## Example (Matrix Transformations)

If  $T_A : \mathbb{R}^n \rightarrow \mathbb{R}^m$  is a matrix transformation (i.e.  $T_A(\mathbf{x}) = A\mathbf{x}$ ), then

$$\ker(T_A) = \text{Nul } A \quad \text{and} \quad R(T_A) = \text{Col}(A).$$

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## Example (Zero Transformation)

Let  $T : V \rightarrow W$  is the zero transformation. Since  $T$  maps every vector in  $V$  to  $\mathbf{0}$ , it follows that

$$\ker(T) = V \quad \text{and} \quad R(T) = \{\mathbf{0}\}.$$

### ► EXAMPLE 15 Kernel and Range of the Identity Operator

Let  $I: V \rightarrow V$  be the identity operator. Since  $I(\mathbf{v}) = \mathbf{v}$  for all vectors in  $V$ , *every* vector in  $V$  is the image of some vector (namely, itself); thus  $R(I) = V$ . Since the *only* vector that  $I$  maps into  $\mathbf{0}$  is  $\mathbf{0}$ , it follows that  $\ker(I) = \{\mathbf{0}\}$ .

# Linear Transformations



## Example

Let  $T : \mathbb{R}^3 \rightarrow \mathbb{R}^3$  be the orthogonal projection onto the  $xy$ -plane (i.e.  $T(x, y, z) = (x, y, 0)$ ).

# Linear Transformations



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Let  $T : \mathbb{R}^3 \rightarrow \mathbb{R}^3$  be the orthogonal projection onto the  $xy$ -plane (i.e.  $T(x, y, z) = (x, y, 0)$ ). Then

$$\begin{aligned}(x, y, z) \in \ker(T) &\implies (0, 0, 0) = T(x, y, z) = (x, y, 0) \\ &\implies x = y = 0.\end{aligned}$$

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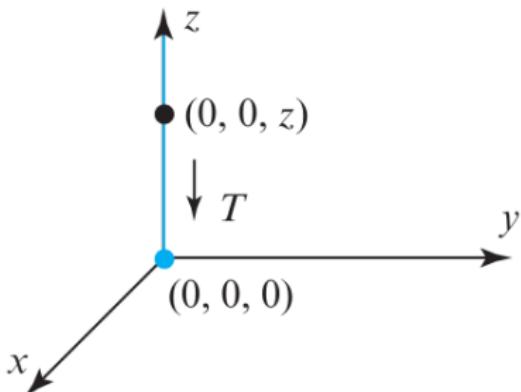
$$\begin{aligned}(x, y, z) \in \ker(T) &\implies (0, 0, 0) = T(x, y, z) = (x, y, 0) \\ &\implies x = y = 0.\end{aligned}$$

Moreover  $R(T)$  is the set of points of the form  $(x, y, 0)$ .

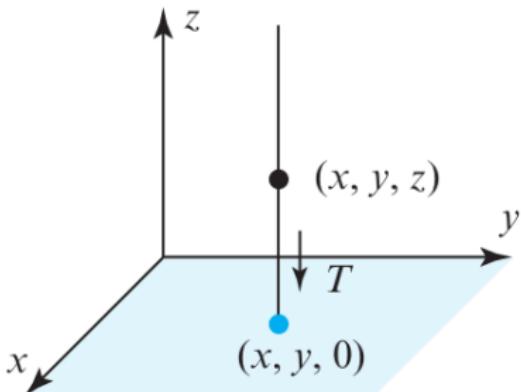
# Linear Transformations

## Example

Let  $T : \mathbb{R}^3 \rightarrow \mathbb{R}^3$  be the orthogonal projection onto the  $xy$ -plane (i.e.  $T(x, y, z) = (x, y, 0)$ ). Then



(a)  $\ker(T)$  is the  $z$ -axis.



(b)  $R(T)$  is the entire  $xy$ -plane

### ► EXAMPLE 17 Kernel and Range of a Rotation

Let  $T: R^2 \rightarrow R^2$  be the linear operator that rotates each vector in the  $xy$ -plane through the angle  $\theta$  (Figure 8.1.3). Since *every* vector in the  $xy$ -plane can be obtained by rotating some vector through the angle  $\theta$ , it follows that  $R(T) = R^2$ . Moreover, the only vector that rotates into  $\mathbf{0}$  is  $\mathbf{0}$ , so  $\ker(T) = \{\mathbf{0}\}$ .

## Properties of Kernel and Range

### Theorem

If  $T : V \rightarrow W$  is a linear transformation, then:

- 1 The kernel of  $T$  is a subspace of  $V$ .
- 2 The range of  $T$  is a subspace of  $W$ .

## Rank and Nullity

Let  $T : V \rightarrow W$  be a linear transformation.

### Definition

If the range of  $T$  is finite-dimensional, then its dimension is called the *rank of  $T$* .

### Definition

If the kernel of  $T$  is finite-dimensional, then its dimension is called the nullity of  $T$ .

## Theorem

*If  $T : V \rightarrow W$  is a linear transformation from a finite-dimensional vector space  $V$  to a vector space  $W$ , then the range of  $T$  is finite-dimensional, and*

$$\text{rank } T + \text{nullity } T = \dim(V).$$

# Break

We will continue at 3pm



*"The next part of this recipe will involve some calculus."*

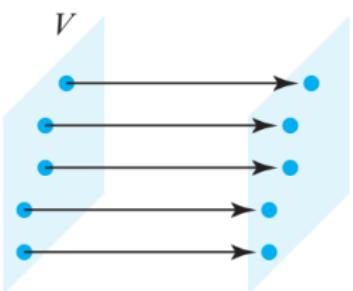


# Composition and Inverse Transforma- tions

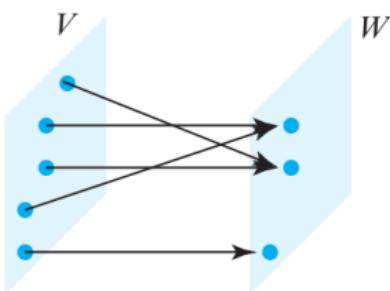
## One-to-One and Onto

### Definition

If  $T : V \rightarrow W$  is a linear transformation from a vector space  $V$  to a vector space  $W$ , then  $T$  is said to be *one-to-one* if  $T$  maps distinct vectors in  $V$  into distinct vectors in  $W$ .



One-to-one. Distinct vectors in  $V$  have distinct images in  $W$ .

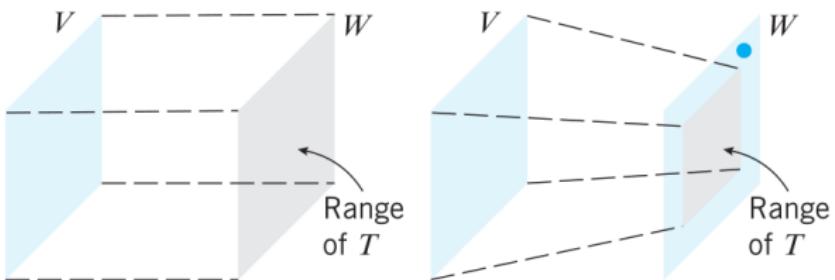


Not one-to-one. There exist distinct vectors in  $V$  with the same image.

# Composition and Inverse Transformations

## Definition

If  $T : V \rightarrow W$  is a linear transformation from a vector space  $V$  to a vector space  $W$ , then  $T$  is said to be *onto* if every vector in  $W$  is the image of at least one vector in  $V$ .



Onto  $W$ . Every vector in  $W$  is the image of some vector in  $V$ .

Not onto  $W$ . Not every vector in  $W$  is the image of some vector in  $V$ .

## Theorem

If  $T : V \rightarrow W$  is a linear transformation, then the following statements are equivalent:

- 1  $T$  is one-to-one.
- 2  $\ker(T) = \{\mathbf{0}\}$ .

# Composition and Inverse Transformations



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## Proof.

**1  $\implies$  2:** We know that  $T(\mathbf{0}) = \mathbf{0}$  because  $T$  is linear. But if  $T$  is one-to-one, then  $\mathbf{0}$  is the only vector that maps to  $\mathbf{0}$ . Hence  $\ker(T) = \{\mathbf{0}\}$ .

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**1  $\iff$  2:** Suppose that  $\ker(T) = \{\mathbf{0}\}$ . Let  $\mathbf{u}, \mathbf{v}$  be two different vectors in  $V$ . So  $\mathbf{u} - \mathbf{v} \neq \mathbf{0}$  which implies that  $T(\mathbf{u} - \mathbf{v}) \neq \mathbf{0}$ .

# Composition and Inverse Transformations



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**1  $\iff$  2:** Suppose that  $\ker(T) = \{\mathbf{0}\}$ . Let  $\mathbf{u}, \mathbf{v}$  be two different vectors in  $V$ . So  $\mathbf{u} - \mathbf{v} \neq \mathbf{0}$  which implies that  $T(\mathbf{u} - \mathbf{v}) \neq \mathbf{0}$ . But then

$$T(\mathbf{u}) - T(\mathbf{v}) = T(\mathbf{u} - \mathbf{v}) \neq \mathbf{0}$$

# Composition and Inverse Transformations



## Theorem

If  $T : V \rightarrow W$  is a linear transformation, then the following statements are equivalent:

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$$T(\mathbf{u}) - T(\mathbf{v}) = T(\mathbf{u} - \mathbf{v}) \neq \mathbf{0} \implies T(\mathbf{u}) \neq T(\mathbf{v}).$$



## Theorem

If  $V$  and  $W$  are finite-dimensional vector spaces with the same dimension, and if  $T : V \rightarrow W$  is a linear transformation, then the following statements are equivalent:

- 1  $T$  is one-to-one.
- 2  $\ker(T) = \{\mathbf{0}\}$ .
- 3  $T$  is onto [i.e.,  $R(T) = W$ ].

# Composition and Inverse Transformations



## Example (Matrix Transformations)

Consider  $T_A : \mathbb{R}^n \rightarrow \mathbb{R}^m$ ,  $T_A(\mathbf{x}) = A\mathbf{x}$  where  $A$  is an  $m \times n$  matrix.

# Composition and Inverse Transformations



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# Composition and Inverse Transformations



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# Composition and Inverse Transformations



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- If  $m < n$ , then  $T_A$  is not one-to-one.
- If  $m > n$ , then  $T_A$  is not onto.
- If  $m = n$ , then

$T_A$  is both one-to-one and onto  $\iff A$  is invertible

# Composition and Inverse Transformations



## Example

The linear transformation  $T_1 : \mathbb{P}^3 \rightarrow \mathbb{R}^4$ ,

$$T_1(a + bx + cx^2 + dx^3) = (a, b, c, d)$$

is both one-to-one and onto because  $\ker(T_1) = \{\mathbf{0}\}$  (you prove).

# Composition and Inverse Transformations



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## Example

The linear transformation  $T_2 : \mathbb{R}^{2 \times 2} \rightarrow \mathbb{R}^4$ ,

$$T_2 \left( \begin{bmatrix} a & b \\ c & d \end{bmatrix} \right) = (a, b, c, d)$$

is both one-to-one and onto because  $\ker(T_2) = \{\mathbf{0}\}$  (you prove).

# Composition and Inverse Transformations



## Example

Let  $T : \mathbb{P}^n \rightarrow \mathbb{P}^{n+1}$  be the linear transformation

$$T(\mathbf{p}) = x\mathbf{p}.$$

If  $\mathbf{p} \neq \mathbf{q}$ , then  $x\mathbf{p} \neq x\mathbf{q}$ . Therefore  $T$  is one-to-one.

# Composition and Inverse Transformations



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# Composition and Inverse Transformations



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## Example

$\frac{d}{dx} : C^1(-\infty, \infty) \rightarrow F(-\infty, \infty)$  is not one-to-one because, for example,

$$\frac{d}{dx}(x^2) = \frac{d}{dx}(x^2 + 1).$$

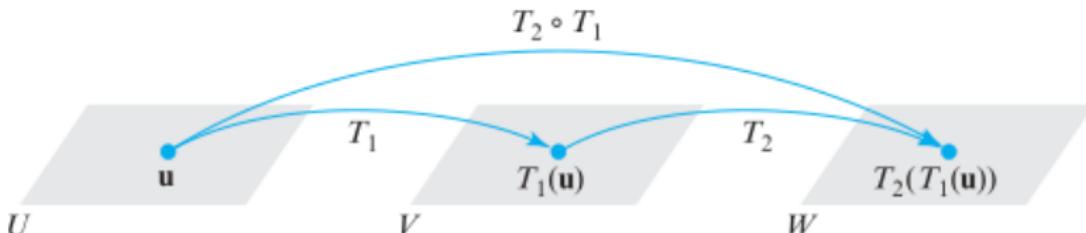
## Compositions of Linear Transformations

### Definition

If  $T_1 : U \rightarrow V$  and  $T_2 : V \rightarrow W$  are linear transformations, then the composition of  $T_2$  with  $T_1$ , denoted by  $T_2 \circ T_1$ , is the function defined by the formula

$$(T_2 \circ T_1)(\mathbf{u}) = T_2(T_1(\mathbf{u}))$$

where  $\mathbf{u}$  is a vector in  $U$ .



# Composition and Inverse Transformations



## Theorem

*If  $T_1 : U \rightarrow V$  and  $T_2 : V \rightarrow W$  are linear transformations, then  $(T_2 \circ T_1) : U \rightarrow W$  is also a linear transformation.*

## ► EXAMPLE 6 Composition of Linear Transformations

Let  $T_1: P_1 \rightarrow P_2$  and  $T_2: P_2 \rightarrow P_2$  be the linear transformations given by the formulas

$$T_1(p(x)) = xp(x) \quad \text{and} \quad T_2(p(x)) = p(2x + 4)$$

Then the composition  $(T_2 \circ T_1): P_1 \rightarrow P_2$  is given by the formula

$$(T_2 \circ T_1)(p(x)) = T_2(T_1(p(x))) = T_2(xp(x)) = (2x + 4)p(2x + 4)$$

In particular, if  $p(x) = c_0 + c_1x$ , then

$$\begin{aligned}(T_2 \circ T_1)(p(x)) &= (T_2 \circ T_1)(c_0 + c_1x) = (2x + 4)(c_0 + c_1(2x + 4)) \\ &= c_0(2x + 4) + c_1(2x + 4)^2\end{aligned}$$

### ► EXAMPLE 7 Composition with the Identity Operator

If  $T: V \rightarrow V$  is any linear operator, and if  $I: V \rightarrow V$  is the identity operator (Example 3 of Section 8.1), then for all vectors  $\mathbf{v}$  in  $V$ , we have

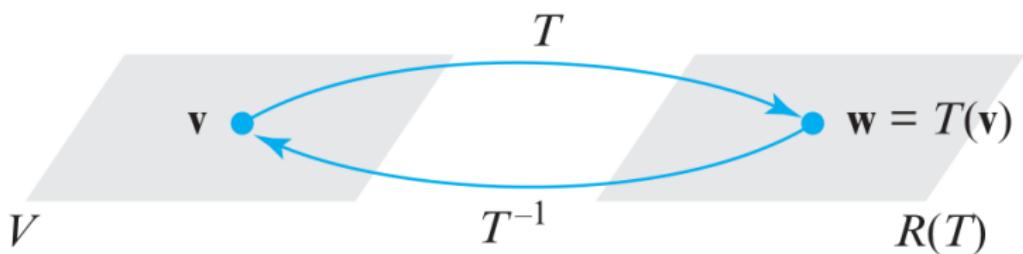
$$(T \circ I)(\mathbf{v}) = T(I(\mathbf{v})) = T(\mathbf{v})$$

$$(I \circ T)(\mathbf{v}) = I(T(\mathbf{v})) = T(\mathbf{v})$$

It follows that  $T \circ I$  and  $I \circ T$  are the same as  $T$ ; that is,

$$T \circ I = T \quad \text{and} \quad I \circ T = T \quad \blacktriangleleft \quad (2)$$

## Inverse Linear Transformations



If  $T : V \rightarrow W$  is a linear transformation, then its *inverse*  $T^{-1} : R(T) \rightarrow V$  is also a linear transformation.

$$T(\mathbf{v}) = \mathbf{w} \iff \mathbf{v} = T^{-1}(\mathbf{w}).$$

### ► EXAMPLE 8 An Inverse Transformation

We showed in Example 3 of this section that the linear transformation  $T: P_n \rightarrow P_{n+1}$  given by

$$T(\mathbf{p}) = T(p(x)) = xp(x)$$

is one-to-one but not onto. The fact that it is not onto can be seen explicitly from the formula

$$T(c_0 + c_1x + \cdots + c_nx^n) = c_0x + c_1x^2 + \cdots + c_nx^{n+1} \quad (6)$$

which makes it clear that the range of  $T$  consists of all polynomials in  $P_{n+1}$  that have zero constant term. Since  $T$  is one-to-one it has an inverse, and from (6) this inverse is given by the formula

$$T^{-1}(c_0x + c_1x^2 + \cdots + c_nx^{n+1}) = c_0 + c_1x + \cdots + c_nx^n$$

For example, in the case where  $n \geq 3$ ,

$$T^{-1}(2x - x^2 + 5x^3 + 3x^4) = 2 - x + 5x^2 + 3x^3$$

# Composition and Inverse Transformations



## Theorem

If  $T_1 : U \rightarrow V$  and  $T_2 : V \rightarrow W$  are one-to-one linear transformations, then:

- 1  $T_2 \circ T_1$  is one-to-one.
- 2  $(T_2 \circ T_1)^{-1} = T_1^{-1} \circ T_2^{-1}$ .



# Isomorphisms

# Isomorphisms

## Definition

A linear transformation  $T : V \rightarrow W$  that is both one-to-one and onto is called an *isomorphism*.

# Isomorphisms

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A linear transformation  $T : V \rightarrow W$  that is both one-to-one and onto is called an *isomorphism*.

## Definition

If there exists an isomorphism  $V \rightarrow W$ , then we say that  $V$  and  $W$  are *isomorphic*.

## Remark

$T : V \rightarrow W$   
is an isomorphism  $\implies T^{-1} : W \rightarrow V$   
is also an isomorphism

(you prove)

## What does isomorphic really mean?

Consider the following isomorphism  $T : \mathbb{P}^2 \rightarrow \mathbb{R}^3$

$$a_0 + a_1x + a_2x^2 \quad \begin{array}{c} \xrightarrow{T} \\ \xleftarrow{T^{-1}} \end{array} \quad (a_0, a_1, a_2)$$

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Consider the following isomorphism  $T : \mathbb{P}^2 \rightarrow \mathbb{R}^3$

$$a_0 + a_1x + a_2x^2 \quad \begin{array}{c} \xrightarrow{T} \\ \xleftarrow{T^{-1}} \end{array} \quad (a_0, a_1, a_2)$$

On the left we have a polynomial function. On the right we have a vector. But the numbers  $a_0$ ,  $a_1$  and  $a_2$  are the same.

# Isomorphisms

The behaviour under vector addition and scalar multiplication are also the same:

$$\begin{array}{ccc} (a_0 + a_1x + a_2x^2) + (b_0 + b_1x + b_2x^2) & \xrightleftharpoons[T]{T^{-1}} & (a_0, a_1, a_2) + (b_0, b_1, b_2) \\ (a_0 + b_0) + (a_1 + b_1)x + (a_2 + b_2)x^2 & \xrightleftharpoons[T]{T^{-1}} & (a_0 + b_0, a_1 + b_1, a_2 + b_2) \end{array}$$

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 \\ 
 (a_0 + b_0) + (a_1 + b_1)x + (a_2 + b_2)x^2 & \xleftrightarrow[T]{T^{-1}} & (a_0 + b_0, a_1 + b_1, a_2 + b_2)
 \end{array}$$

and

$$\begin{array}{ccc}
 k(a_0 + a_1x + a_2x^2) & \xrightarrow{T} & k(a_0, a_1, a_2) \\
 \\ 
 ka_0 + ka_1x + ka_2x^2 & \xleftrightarrow[T]{T^{-1}} & (ka_0, ka_1, ka_2)
 \end{array}$$

# Isomorphisms

## Remark

In the Greek language,

- ‘morphē’ means ‘form’; and
- ‘iso’ means ‘identical’.

# Isomorphisms

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- ‘morphe’ means ‘form’; and
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## An Important Theorem

Theorem (★ ★ ★ ★ ★ )

*Every real  $n$ -dimensional vector space is isomorphic to  $\mathbb{R}^n$ .*

# Isomorphisms

## Theorem

If  $S = \{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\}$  is a basis for a vector space  $V$ , then the coordinate map

$$T : \mathbf{u} \mapsto (\mathbf{u})_S$$

is an isomorphism between  $V$  and  $\mathbb{R}^n$ .

# Isomorphisms

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## Example (The Natural Isomorphism Between $\mathbb{P}^{n-1}$ and $\mathbb{R}^n$ )

The linear transformation  $T : \mathbb{P}^{n-1} \rightarrow \mathbb{R}^n$ ,

$$T : a_0 + a_1x + a_2X^2 + \dots + a_{n-1}x^{n-1} \mapsto (a_0, a_1, a_2, \dots, a_{n-1})$$

is an isomorphism between  $\mathbb{P}^{n-1}$  and  $\mathbb{R}^n$ . This is called the *natural isomorphism* between these vector spaces.

# Isomorphisms

## Example (The Natural Isomorphism Between $\mathbb{R}^{2 \times 2}$ and $\mathbb{R}^4$ )

The coordinate map

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \mapsto (a, b, c, d)$$

defines an isomorphism between  $\mathbb{R}^{2 \times 2} = M_{22}$  and  $\mathbb{R}^4$ . We call this the *natural isomorphism between  $\mathbb{R}^{2 \times 2}$  and  $\mathbb{R}^4$* .

## Working with Isomorphisms

### Example

Are the following polynomials linearly independent?

$$p_1 = 1 + 2x - 3x^2 + 4x^3 + x^5$$

$$p_2 = 1 + 3x - 4x^2 + 6x^3 + 5x^4 + 4x^5$$

$$p_3 = 3 + 8x - 11x^2 - 16x^3 + 10x^4 + 9x^5$$

# Isomorphisms



We will use the fact that  $\mathbb{P}^5$  is isomorphic to  $\mathbb{R}^6$ .

# Isomorphisms



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The following two questions are equivalent: Are the following polynomials linearly independent?

$$\begin{aligned}\mathbf{p}_1 &= 1 + 2x - 3x^2 + 4x^3 + x^5 \\ \mathbf{p}_2 &= 1 + 3x - 4x^2 + 6x^3 + 5x^4 + 4x^5 \\ \mathbf{p}_3 &= 3 + 8x - 11x^2 - 16x^3 + 10x^4 + 9x^5\end{aligned}$$

Are the following vectors linearly independent?

$$\begin{pmatrix} 1, & 2, & -3, & 4, & 0 & 1 \\ 1, & 3, & -4, & 6, & 5, & 4 \\ 3, & 8, & -11, & -16, & 10, & 9 \end{pmatrix}$$

# Isomorphisms



Another equivalent question is: Is the row space of the following matrix 3 dimensional?

$$A = \begin{bmatrix} 1 & 2 & -3 & 4 & 0 & 1 \\ 1 & 3 & -4 & 6 & 5 & 4 \\ 3 & 8 & -11 & 16 & 10 & 9 \end{bmatrix}$$

# Isomorphisms

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I leave it to you to check that  $A$  is row equivalent to

$$R = \begin{bmatrix} 1 & 2 & -3 & 4 & 0 & 1 \\ 0 & 1 & -1 & 2 & 5 & 3 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

We can see that the answer to all three questions is **NO**.

# Isomorphisms



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5 December 2018 [16:00-17:10]

MATH215, Second Exam

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4. (a) 10 points Let  $L : \mathbb{R}^3 \rightarrow \mathbb{R}^4$  be a linear transformation defined by  $L\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2y + 3z \\ x + y - 2z \\ 4x + y \\ 3x - y - z \end{pmatrix}$ . Find the matrix representation of  $L$ .

- (b) 15 points Define  $T : \mathbb{P}_2 \rightarrow \mathbb{R}^2$  by  $T(\mathbf{p}) = \begin{bmatrix} \mathbf{p}(0) \\ \mathbf{p}(1) \end{bmatrix}$ . Find a polynomial  $\mathbf{p} \in \mathbb{P}_2$  which is a basis for kernel of  $T$ .

**Solution:**

$$\ker T = \{\mathbf{p} : \mathbf{p} \in \mathbb{P}_2 \text{ and } T(\mathbf{p}) = \mathbf{0}\}$$

$$\mathbf{p}(t) = a + bt + ct^2 \Rightarrow T(\mathbf{p}(t)) = \begin{bmatrix} \mathbf{p}(0) \\ \mathbf{p}(1) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} \mathbf{p}(0) \\ \mathbf{p}(1) \end{bmatrix} = \begin{bmatrix} a \\ a + b + c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{array}{l} a = 0 \\ b + c = 0 \end{array} \Rightarrow \mathbf{p}(t) = -ct + ct^2$$

$$\ker T = \{\mathbf{p} : \mathbf{p}(t) = (-t + t^2)c, c \in \mathbb{R}\} = \text{Span}\{-t + t^2\}$$

$$\mathbf{p}(t) = -t + t^2$$



# Next Time

- Matrices for Linear Transformations
- Similarity