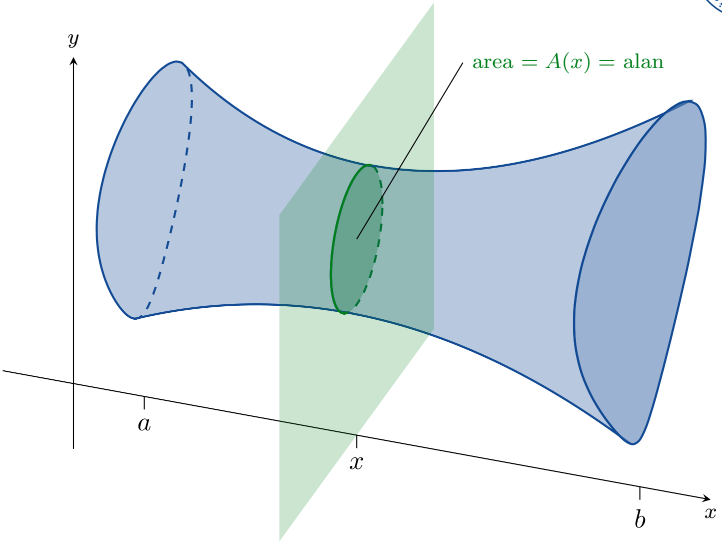
Lecture 10

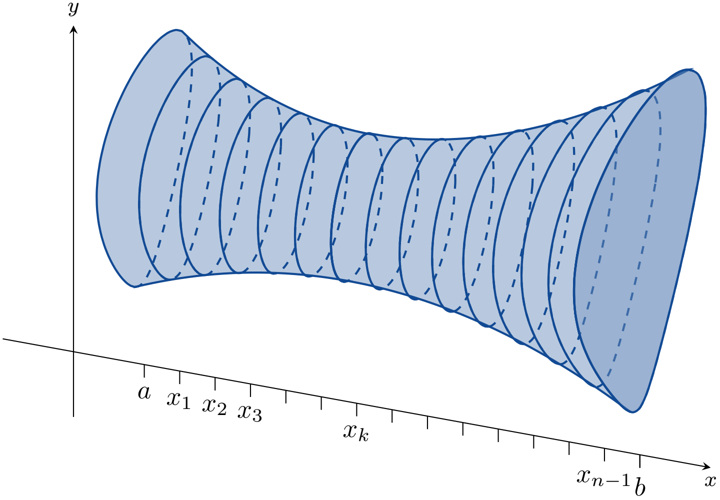
- 6.1 Volumes Using Cross-Sections
- 6.2 Volumes Using Cylindrical Shells
- 6.3 Arc Length
- 6.4 Areas of Surfaces of Revolution

Volumes Using Cross-Sections

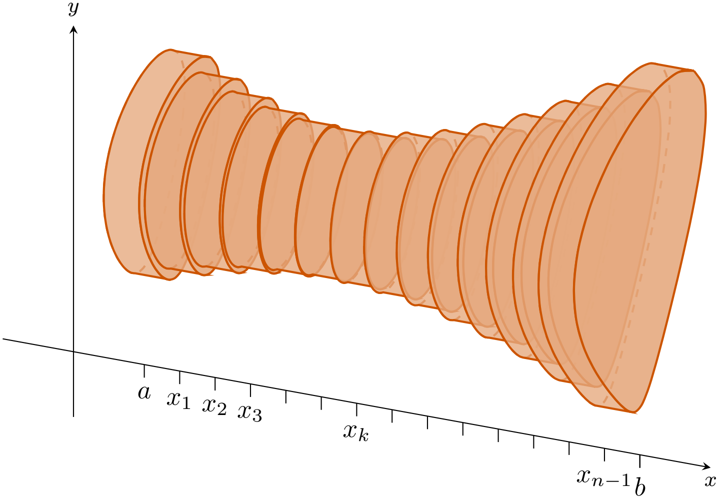
6.1 Volumes Using Cross-Sections



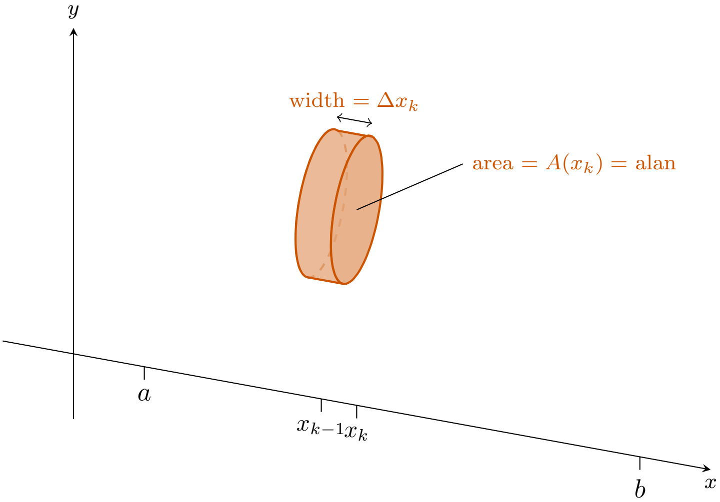
6.1 Volumes Using Cross-Sections



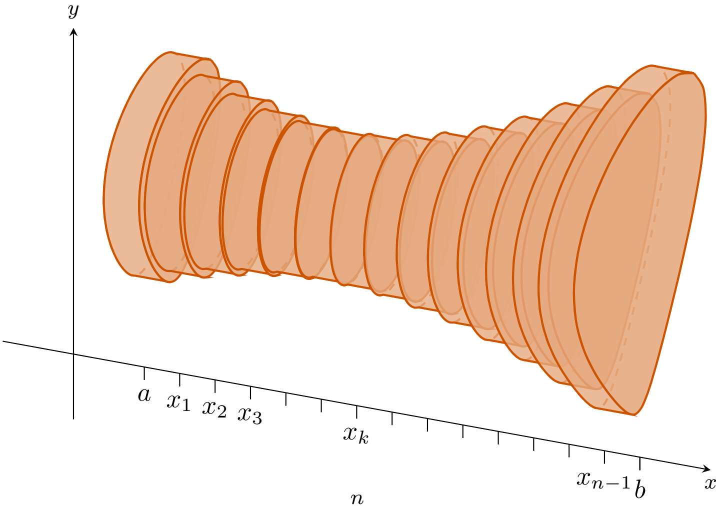
6.1 Volumes Using Cross-Sections



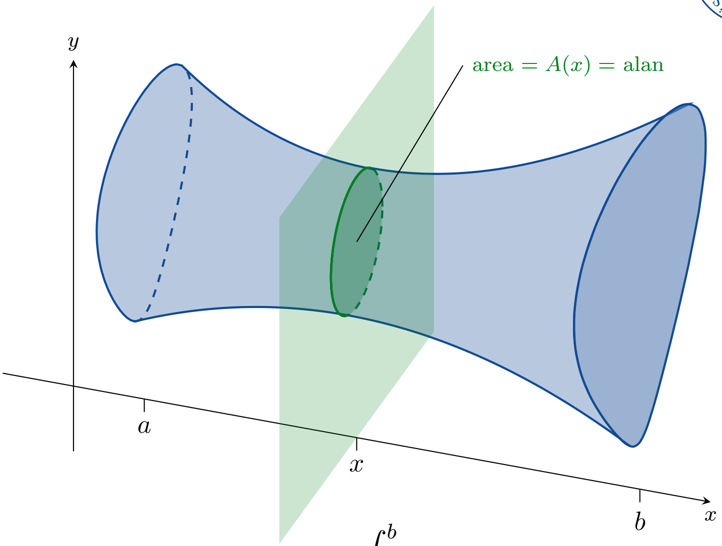
6.1 Volumes Using Cross-Sections



$$\text{volume} = (\text{area})(\text{width}) = A(x_k)\Delta x_k$$



$$\text{volume} = \sum_{k=1}^{n} A(x_k)\Delta x_k$$



$$\text{volume} = \int_a^b A(x)\,dx$$

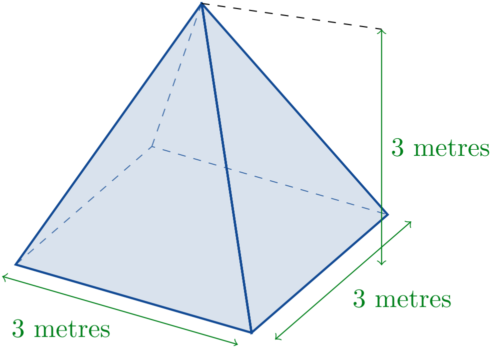
Calculating the Volume of a Solid

1. *Sketch the solid and a typical cross-section.*
2. *Find a formula for* $A(x)$, the area of a typical cross-section.
3. *Find the limits of integration.*
4. *Integrate* $A(x)$ to find the volume.

$$\text{volume} = \int_a^b A(x)\,dx$$

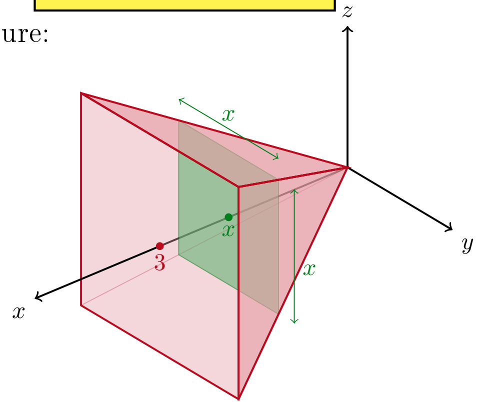
Example

A 3 metres tall pyramid has a square 3 metres × 3 metres base, as shown below. The cross-section x metres from the vertex is an $x\,\mathrm{m} \times x\,\mathrm{m}$ square. Find the volume of the pyramid.



$$\text{volume} = \int_a^b A(x)\,dx$$

1. Draw a picture:



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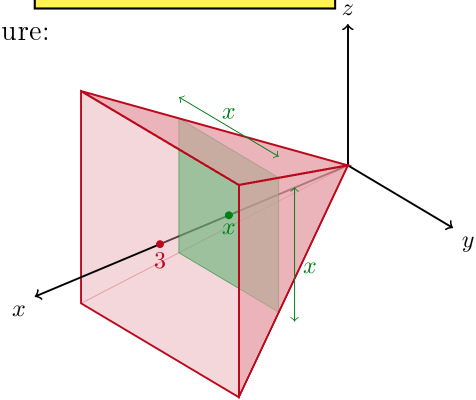

2. Find a formula for $A(x)$: $A(x) = x^2$.

$$\text{volume} = \int_a^b A(x)\,dx$$

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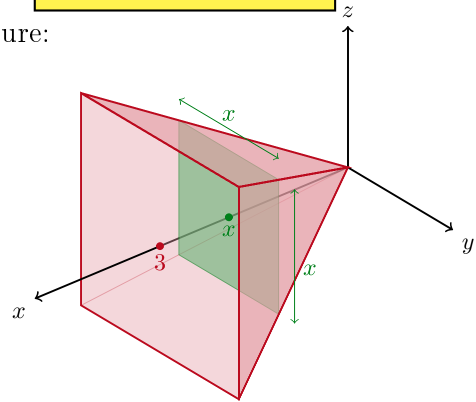

2. Find a formula for $A(x)$: $A(x) = x^2$.
3. Find the limits of integration: $0 \leq x \leq 3$.

$$\text{volume} = \int_a^b A(x)\,dx$$

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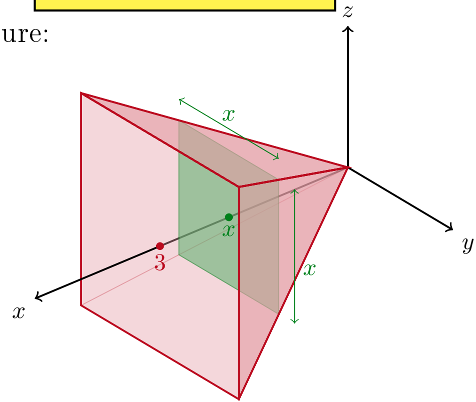
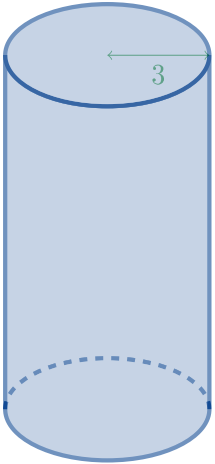


2. Find a formula for $A(x)$: $A(x) = x^2$.
3. Find the limits of integration: $0 \leq x \leq 3$.
4. Integrate:

$$\text{volume} = \int_a^b A(x)\ dx = \int_0^3 x^2\ dx = \left[\frac{1}{3}x^3\right]_0^3 = 9\,\mathrm{m}^3.$$

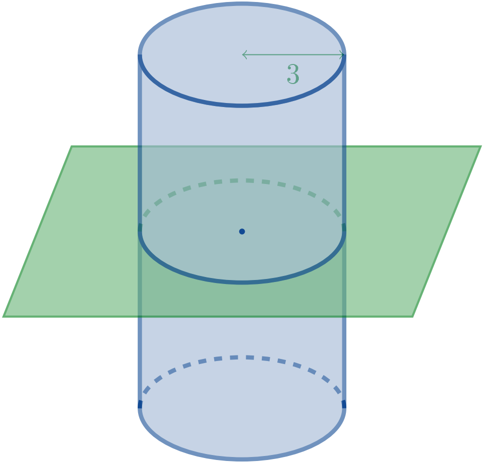
Example

A curved wedge is cut from a cylinder of radius 3 by two planes.



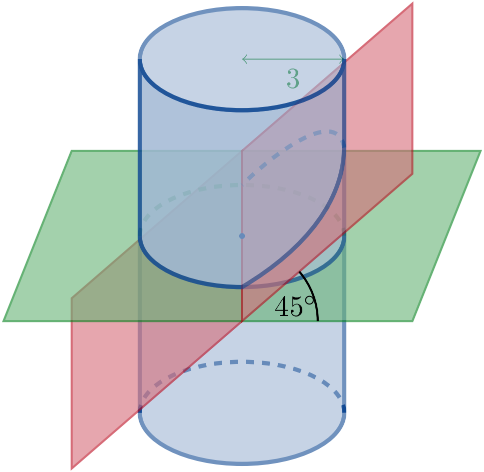
Example

A curved wedge is cut from a cylinder of radius 3 by two planes. The first plane is perpendicular to the axis of the cylinder.



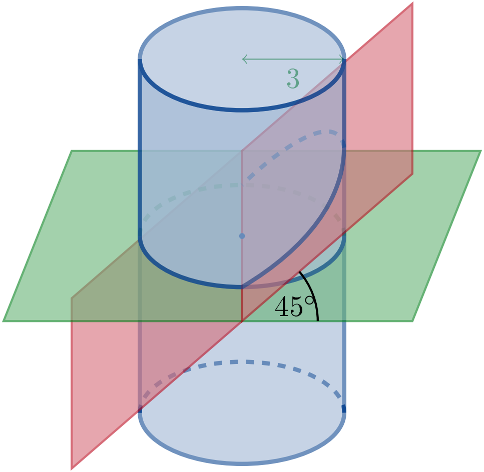
Example

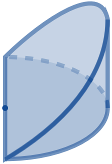
A curved wedge is cut from a cylinder of radius 3 by two planes. The first plane is perpendicular to the axis of the cylinder. The second plane crosses the first plane with an angle of $45° = \frac{\pi}{4}$ at the centre of the cylinder.



Example

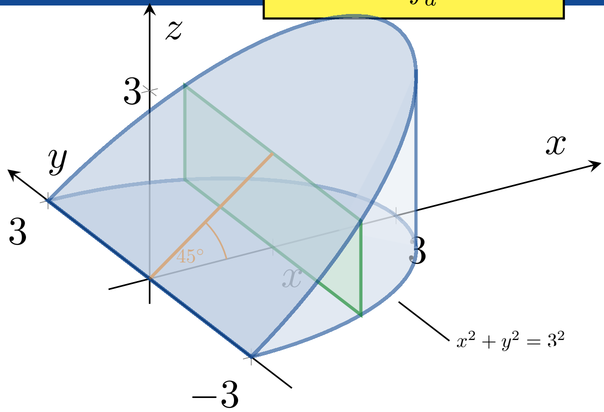
A curved wedge is cut from a cylinder of radius 3 by two planes. The first plane is perpendicular to the axis of the cylinder. The second plane crosses the first plane with an angle of $45° = \frac{\pi}{4}$ at the centre of the cylinder. Find the volume of the wedge.





the wedge

$$\text{volume} = \int_a^b A(x)\,dx$$



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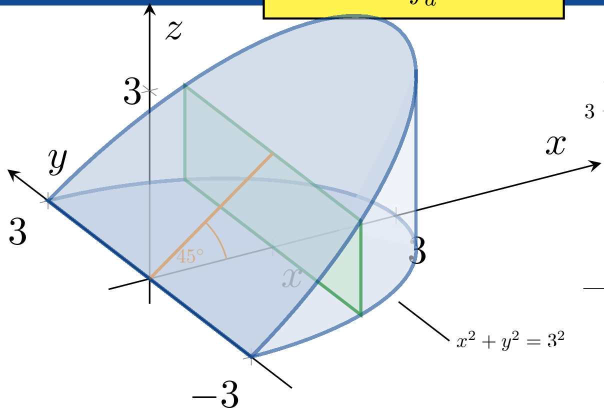

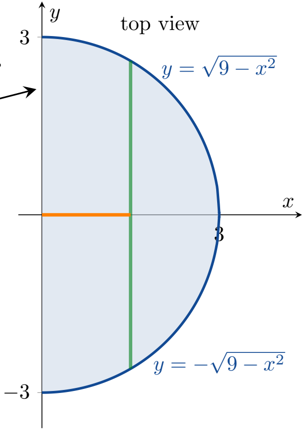

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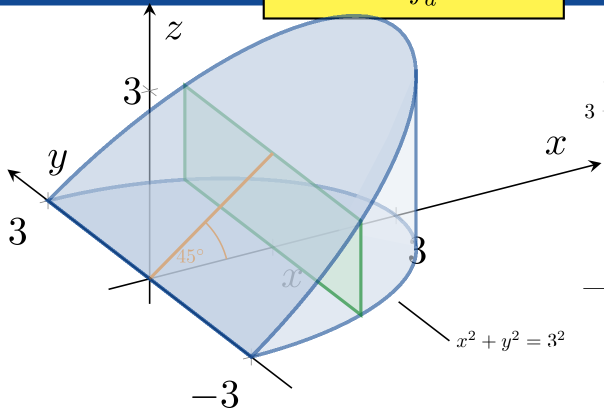

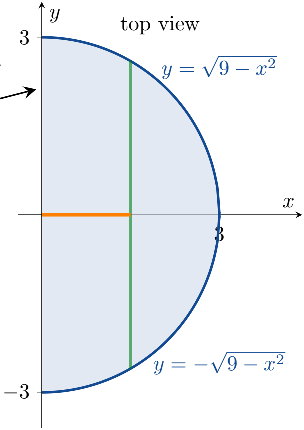

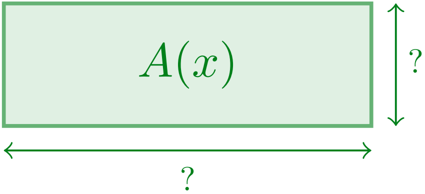

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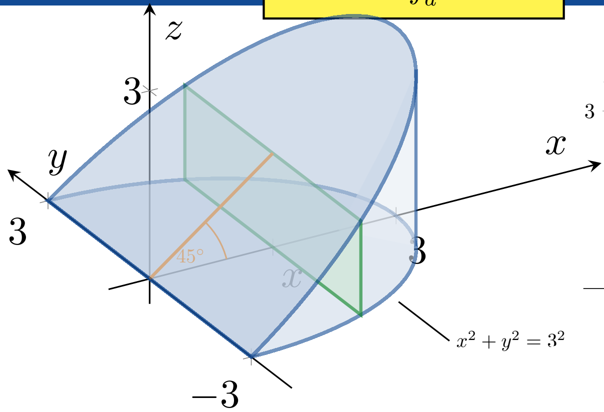

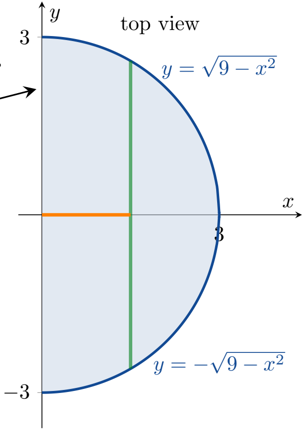

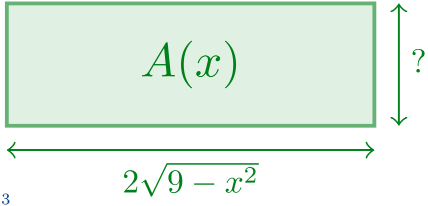

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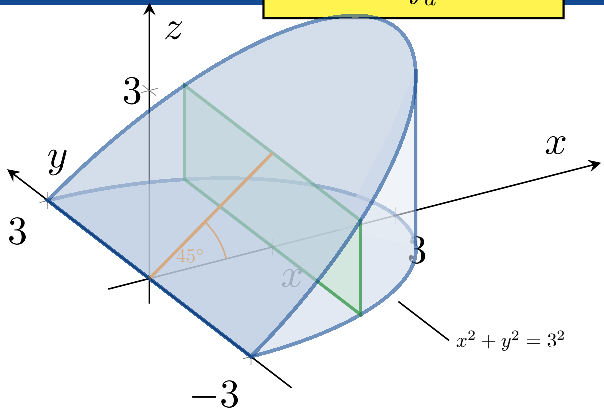

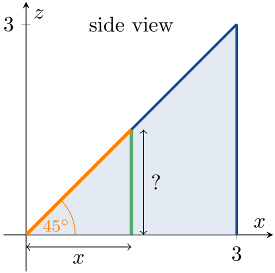

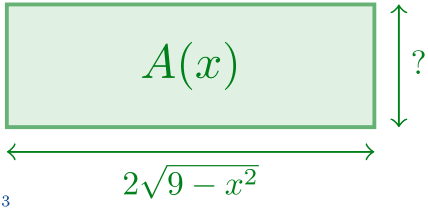

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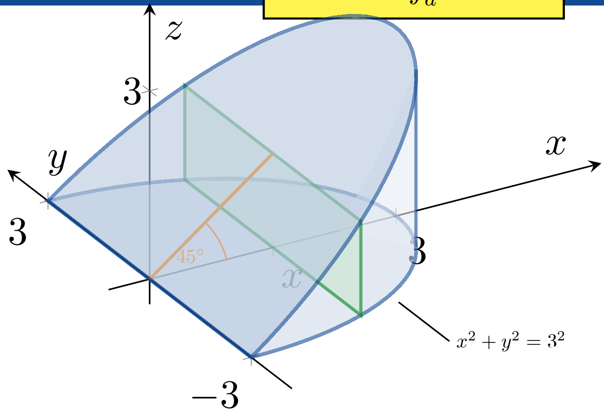

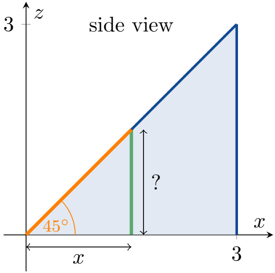

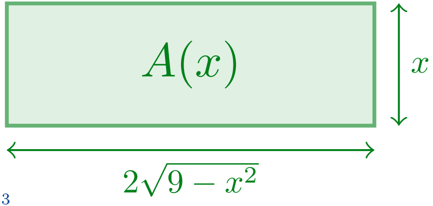

$$A(x) = 2x\sqrt{9-x^2}$$

$$\boxed{\text{volume} = \int_a^b A(x)\,dx}$$

The cross-sectional area is

$$A(x) = 2x\sqrt{9 - x^2}$$

for $0 \leq x \leq 3$. Therefore

$$\text{volume} = \int_0^3 2x\sqrt{9 - x^2}\,dx$$

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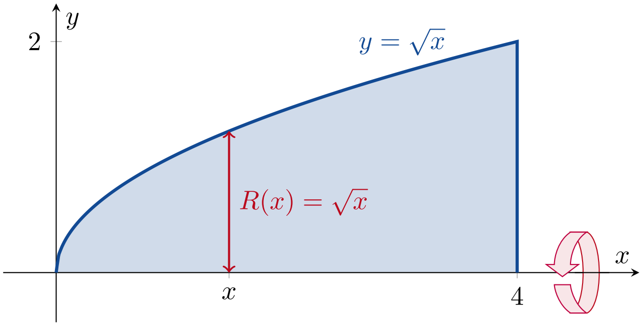
We need to make a substitution. Let $u = 9 - x^2$. Then $du = -2x\ dx$ and

$$\begin{aligned}
\text{volume} &= \int_{x=0}^{x=3} -u^{\frac{1}{2}}\ du = \left[-\frac{2}{3}u^{\frac{3}{2}}\right]_{x=0}^{x=3} \\
&= \left[-\frac{2}{3}(9 - x^2)^{\frac{3}{2}}\right]_{x=0}^{x=3} = 0 - \frac{2}{3}(9)^{\frac{3}{2}} \\
&= 18.
\end{aligned}$$

$$\text{volume} = \int_a^b A(x)\,dx$$

Solids of Revolution: The Disk Method



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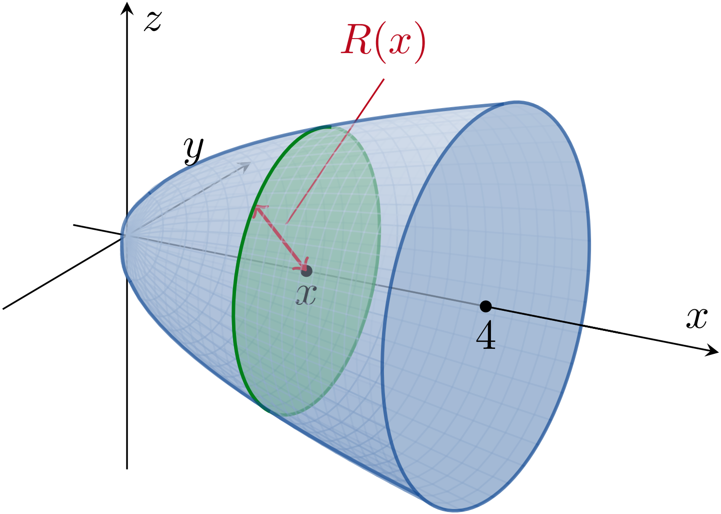

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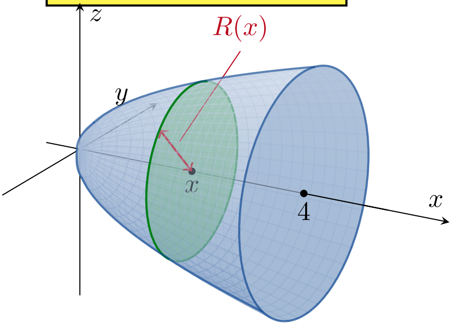

Definition

The solid generated by rotating a plane region about a line in the plane is called a *solid of revolution*.

$$\boxed{\text{volume} = \int_a^b A(x)\,dx}$$

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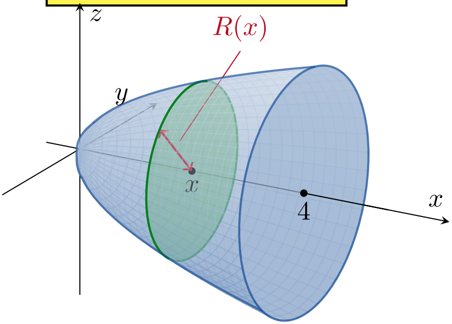

Definition

The solid generated by rotating a plane region about a line in the plane is called a *solid of revolution*.

$$\boxed{\text{volume} = \int_a^b A(x)\,dx = \int_a^b \pi\Big(R(x)\Big)^2\,dx}$$

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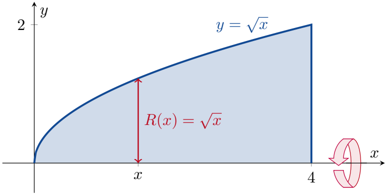

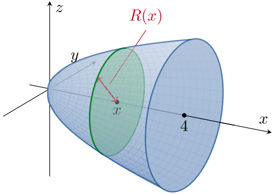

Example

The region between the curve $y = \sqrt{x}$ and the x-axis, for $0 \le x \le 4$, is rotated about the x-axis to generate a sold. Find its volume.

$$\text{volume} = \int_a^b \pi \Big(R(x)\Big)^2\,dx =$$

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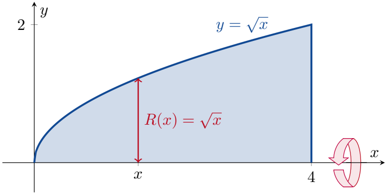

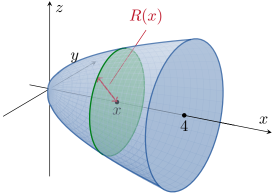

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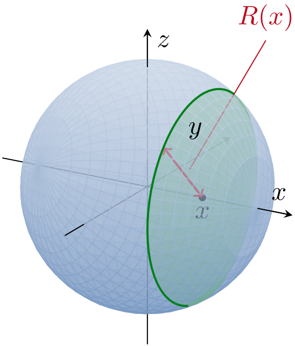
$$\text{volume} = \int_a^b \pi \Big(R(x)\Big)^2 \, dx = \int_0^4 \pi \Big(\sqrt{x}\Big)^2 \, dx$$

$$= \pi \int_0^4 x \, dx = \pi \left[\frac{1}{2}x^2\right]_0^4 = 8\pi.$$

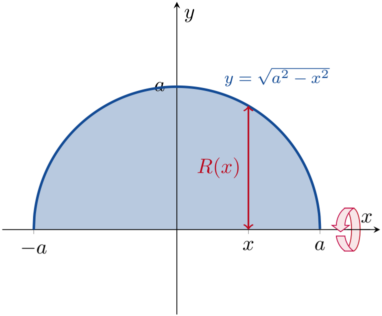
$$\text{volume} = \int_a^b \pi\Big(R(x)\Big)^2\,dx$$

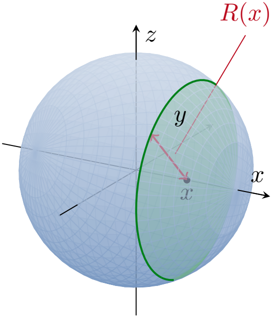


Example

Find the volume of a sphere of radius a.

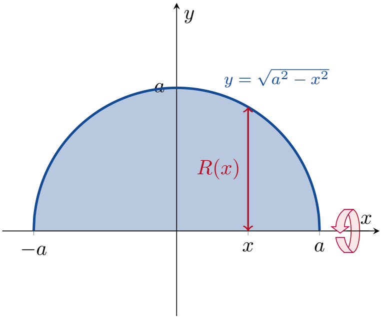
$$\text{volume} = \int_a^b \pi \Big(R(x) \Big)^2 \, dx$$

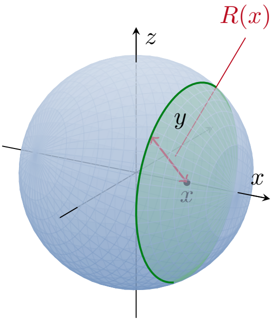




To generate a sphere, we rotate the area between $y = \sqrt{a^2 - x^2}$ and the x-axis about the x-axis.

$$\text{volume} = \int_a^b \pi \Big(R(x)\Big)^2 \, dx$$

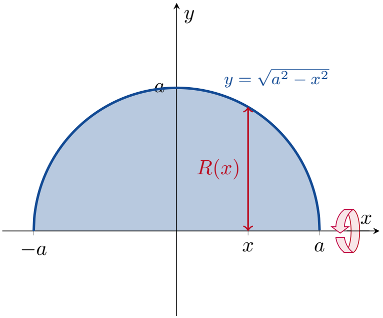


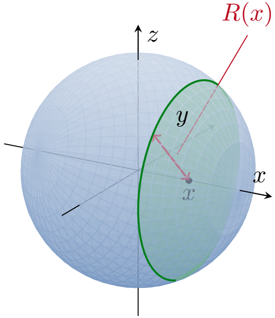


To generate a sphere, we rotate the area between $y = \sqrt{a^2 - x^2}$ and the x-axis about the x-axis. Its volume is

$$\text{volume} = \int_{-a}^{a} \pi \Big(R(x)\Big)^2 \, dx = \int_{-a}^{a} \pi \Big(\sqrt{a^2 - x^2}\Big)^2 \, dx$$

$$\text{volume} = \int_a^b \pi \Big(R(x)\Big)^2 \, dx$$

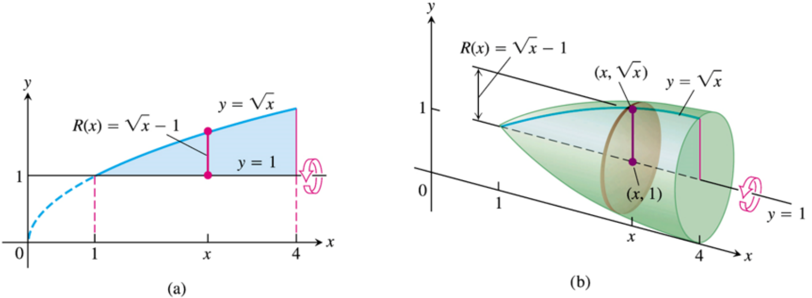




To generate a sphere, we rotate the area between $y = \sqrt{a^2 - x^2}$ and the x-axis about the x-axis. Its volume is

$$\begin{aligned}\text{volume} &= \int_{-a}^{a} \pi \Big(R(x)\Big)^2 \, dx = \int_{-a}^{a} \pi \Big(\sqrt{a^2 - x^2}\Big)^2 \, dx \\ &= \pi \int_{-a}^{a} \left(a^2 - x^2\right) \, dx = \pi \left[a^2 x - \frac{x^3}{3}\right]_{-a}^{a} = \frac{4}{3}\pi a^3.\end{aligned}$$

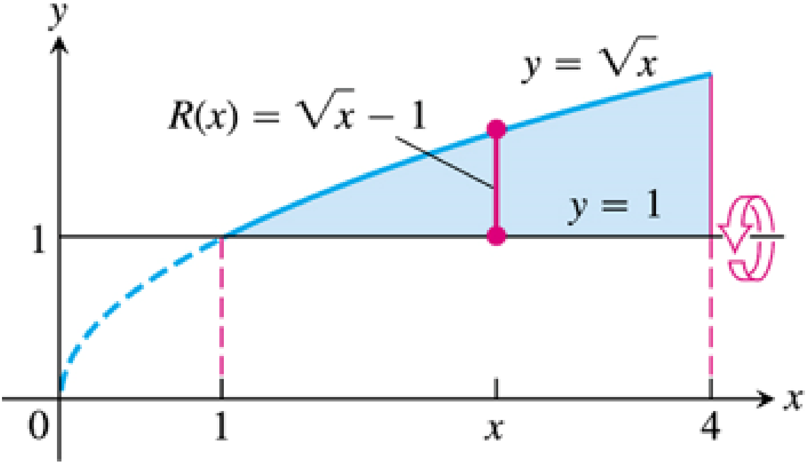
$$\text{volume} = \int_a^b \pi \Big(R(x)\Big)^2 \, dx$$



Example

Find the volume of the solid generated by revolving the region bounded by $y = \sqrt{x}$ and the lines $y = 1$ and $x = 4$, about the line $y = 1$.

$$\text{volume} = \int_a^b \pi \Big(R(x)\Big)^2 \, dx$$



EXAMPLE 6 Find the volume of the solid generated by revolving the region bounded by $y = \sqrt{x}$ and the lines $y = 1$, $x = 4$ about the line $y = 1$.

Solution We draw figures showing the region, a typical radius, and the generated solid (Figure 6.10). The volume is

$$V = \int_1^4 \pi \left[R(x) \right]^2 dx$$

$$= \int_1^4 \pi \left[\sqrt{x} - 1 \right]^2 dx \qquad \text{Radius } R(x) = \sqrt{x} - 1 \text{ for rotation around } y = 1.$$

$$= \pi \int_1^4 \left[x - 2\sqrt{x} + 1 \right] dx \qquad \text{Expand integrand.}$$

$$= \pi \left[\frac{x^2}{2} - 2 \cdot \frac{2}{3} x^{3/2} + x \right]_1^4 = \frac{7\pi}{6}. \qquad \text{Integrate.}$$

Volume by Disks for Rotation About the y-Axis

$$\boxed{\text{volume} = \int_c^d A(y)\,dy = \int_c^d \pi\Big(R(y)\Big)^2\,dy}$$

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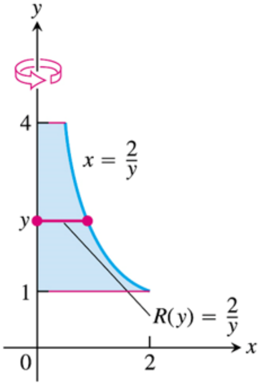
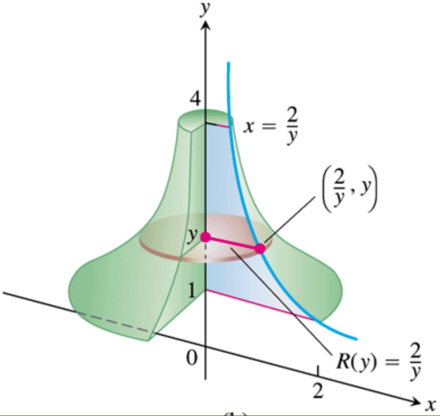
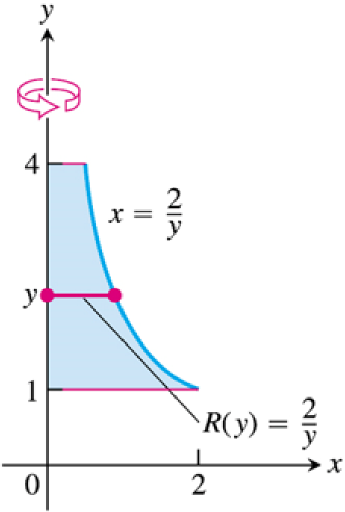




Example

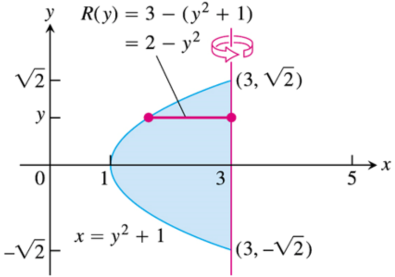
Find the volume of the solid generated by revolving the region between the y-axis and the curve $x = \frac{2}{y}$, $1 \le y \le 4$ about the y-axis.

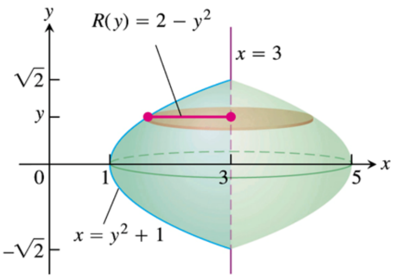
$$\text{volume} = \int_c^d \pi \Big(R(y)\Big)^2 \, dy$$



$$\begin{aligned}\text{volume} &= \int_1^4 \pi \left(\frac{2}{y}\right)^2 \, dy \\ &= \ldots\end{aligned}$$

$$\text{volume} = \int_c^d \pi\Big(R(y)\Big)^2\,dy$$

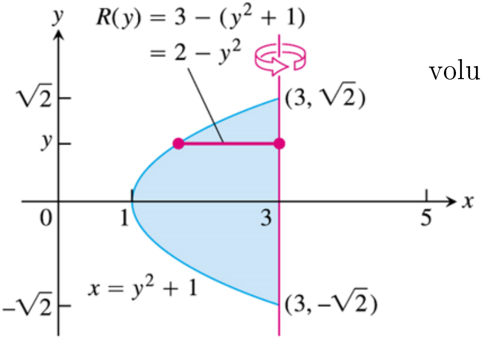




Example

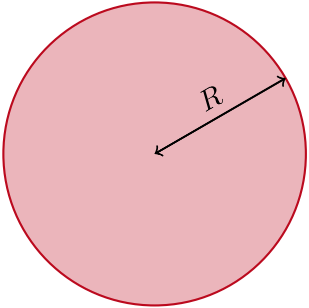
Find the volume of the solid generated by revolving the region between the parabola $x = y^2 + 1$ and the line $x = 3$ about the line $x = 3$.

$$\text{volume} = \int_c^d \pi \Big(R(y)\Big)^2 \, dy$$



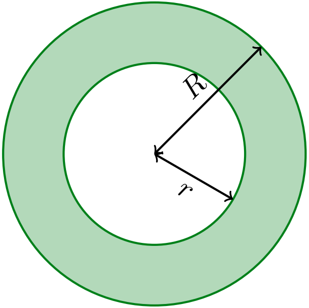
$$\text{volume} = \int_{-\sqrt{2}}^{\sqrt{2}} \pi \left(2 - y^2\right)^2 \, dy$$
$$= \ldots$$

Solids of Revolution: The Washer Method



a disk

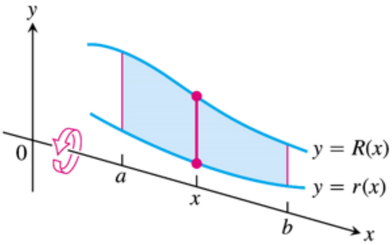
area $= \pi R^2$

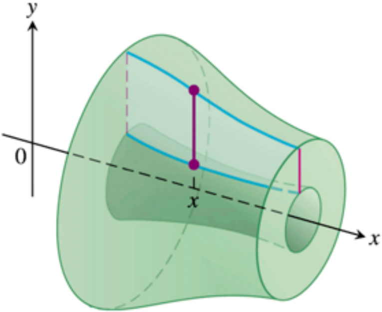


a washer

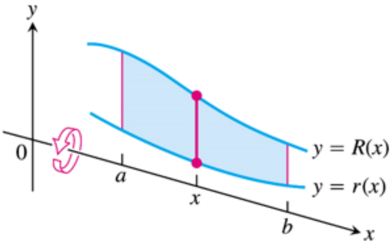
area $= \pi R^2 - \pi r^2$

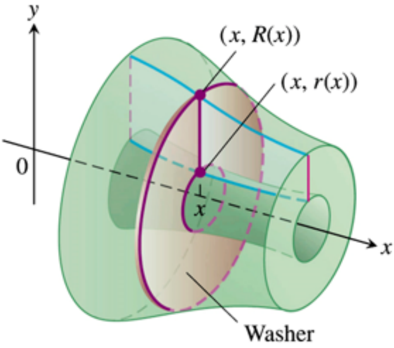
Volume by Washers for Rotation About the x-Axis



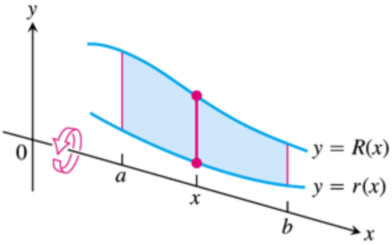


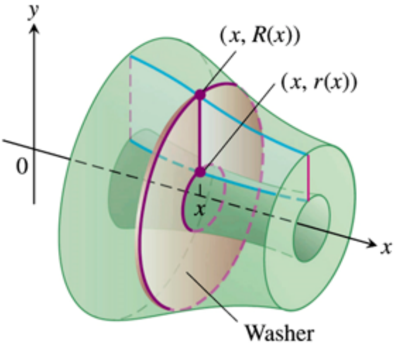
Volume by Washers for Rotation About the x-Axis





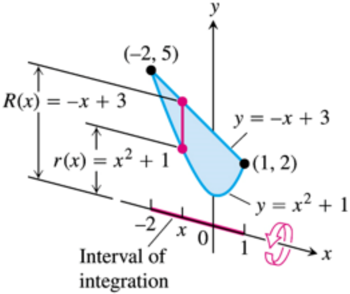
Volume by Washers for Rotation About the x-Axis

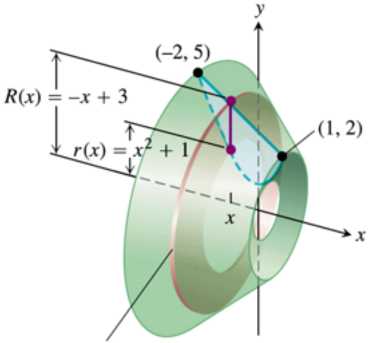




$$\boxed{\text{volume} = \int_a^b A(x)\,dx = \int_a^b \pi\Big(\big(R(x)\big)^2 - \big(r(x)\big)^2\Big)\,dx.}$$

$$\text{volume} = \int_a^b \pi\left(\left(R(x)\right)^2 - \left(r(x)\right)^2\right) dx$$



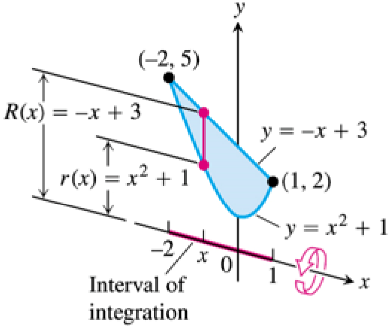


Washer cross section
Outer radius: $R(x) = -x + 3$
Inner radius: $r(x) = x^2 + 1$

Example

The region bounded by the curve $y = x^2 + 1$ and the line $y = -x + 3$ is revolved about the x-axis to generate a solid. Find the volume of the solid.

$$\text{volume} = \int_a^b \pi\Big(\big(R(x)\big)^2 - \big(r(x)\big)^2\Big)\,dx$$

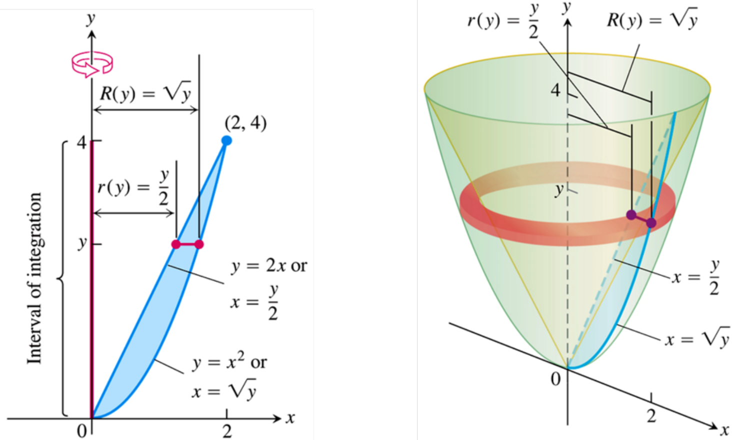


$$\begin{aligned}
\text{volume} &= \int_{-2}^{1} \pi\Big((-x+3)^2 - \big(x^2+1\big)^2\Big)\,dx \\
&= \pi\int_{-2}^{1} 8 - 6x - x^2 - x^4\,dx \\
&= \pi\left[8x - 3x^2 - \frac{x^3}{3} - \frac{x^5}{5}\right]_{-2}^{1} = \frac{117\pi}{5}.
\end{aligned}$$

Volume by Washers for Rotation About the y-Axis

$$\boxed{\text{volume} = \int_c^d A(y)\,dy = \int_c^d \pi\Big(\big(R(y)\big)^2 - \big(r(y)\big)^2\Big)\,dy.}$$

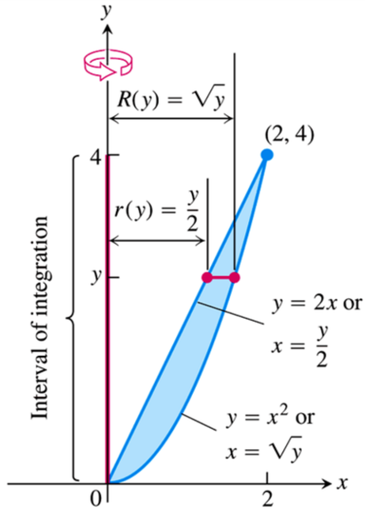
$$\text{volume} = \int_c^d \pi\left(\left(R(y)\right)^2 - \left(r(y)\right)^2\right) dy$$



Example

The region bounded by the parabola $y = x^2$ and the line $y = 2x$ in the first quadrant is revolved about the y-axis to generate a solid. Find the volume of the solid.

$$\text{volume} = \int_c^d \pi\left(\left(R(y)\right)^2 - \left(r(y)\right)^2\right) dy$$



$$\text{volume} = \int_0^4 \pi\left((\sqrt{y})^2 - \left(\frac{y}{2}\right)^2\right) dy$$
$$= \ldots$$

Volumes Using Cylindrical Shells

$$\text{volume} = \int_c^d \pi\Big(\big(R(y)\big)^2 - \big(r(y)\big)^2\Big)\,dy$$

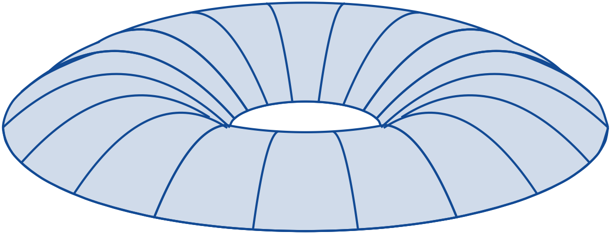
The Disk and Washer methods.

$$\text{volume} = \int_c^d \pi\Big(\big(R(y)\big)^2 - \big(r(y)\big)^2\Big)\,dy$$

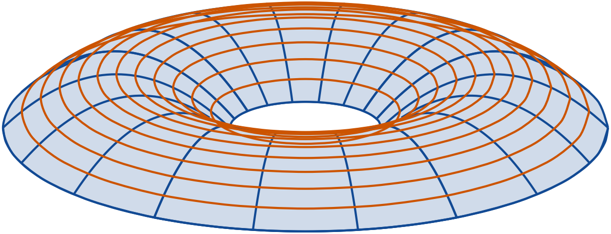
The Disk and Washer methods.

The Shell method.

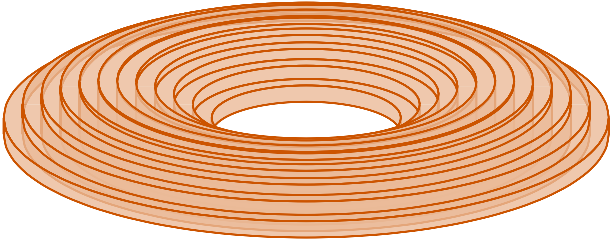
$$\text{volume} = \int_c^d \pi\Big(\big(R(y)\big)^2 - \big(r(y)\big)^2\Big)\,dy$$



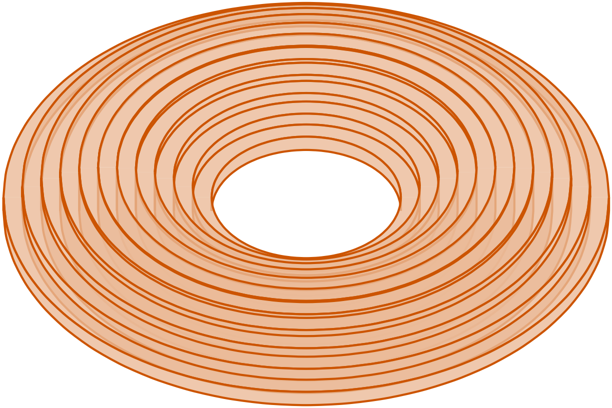
$$\text{volume} = \int_c^d \pi\Big(\big(R(y)\big)^2 - \big(r(y)\big)^2\Big)\,dy$$



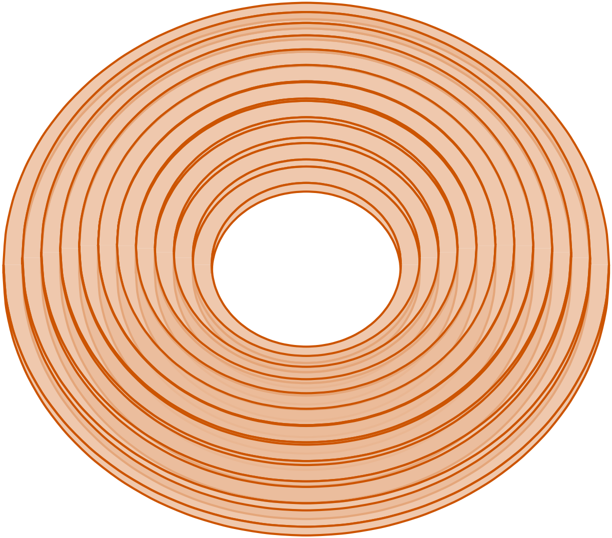
$$\text{volume} = \int_c^d \pi\Big(\big(R(y)\big)^2 - \big(r(y)\big)^2\Big)\,dy$$



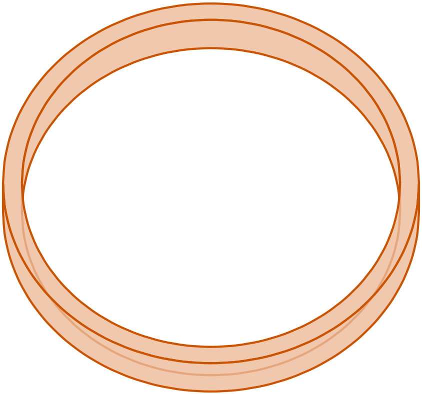
$$\text{volume} = \int_c^d \pi\Big(\big(R(y)\big)^2 - \big(r(y)\big)^2\Big)\,dy$$



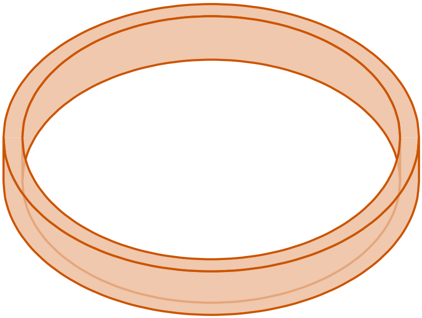
$$\text{volume} = \int_c^d \pi\Big(\big(R(y)\big)^2 - \big(r(y)\big)^2\Big)\,dy$$



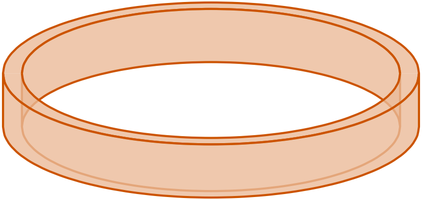
$$\text{volume} = \int_c^d \pi\Big(\big(R(y)\big)^2 - \big(r(y)\big)^2\Big)\, dy$$



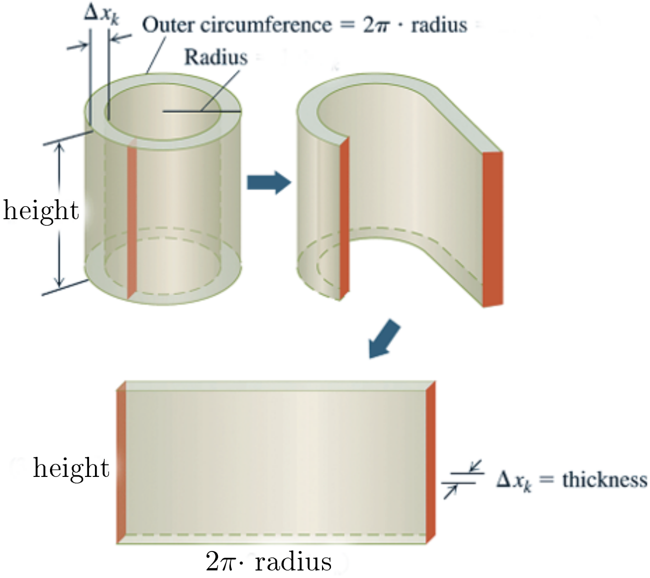
$$\text{volume} = \int_c^d \pi\Big(\big(R(y)\big)^2 - \big(r(y)\big)^2\Big)\,dy$$



$$\text{volume} = \int_c^d \pi\Big(\big(R(y)\big)^2 - \big(r(y)\big)^2\Big)\,dy$$



$$\text{volume} = \int_c^d \pi\left(\left(R(y)\right)^2 - \left(r(y)\right)^2\right) dy$$



$$\text{volume} = \int_c^d \pi\Big(\big(R(y)\big)^2 - \big(r(y)\big)^2\Big)\,dy$$

The volume of one of these shells is approximately

$$2\pi\binom{\text{shell}}{\text{radius}}\binom{\text{shell}}{\text{height}}\Delta x_k$$

$$\text{volume} = \int_c^d \pi\Big(\big(R(y)\big)^2 - \big(r(y)\big)^2\Big)\,dy$$

The volume of one of these shells is approximately

$$2\pi\binom{\text{shell}}{\text{radius}}\binom{\text{shell}}{\text{height}}\Delta x_k$$

So the volume of all of the shells is the Riemann sum

$$\sum_{k=1}^{n} 2\pi\binom{\text{shell}}{\text{radius}}\binom{\text{shell}}{\text{height}}\Delta x_k$$

The volume of one of these shells is approximately

$$2\pi\binom{\text{shell}}{\text{radius}}\binom{\text{shell}}{\text{height}}\Delta x_k$$

So the volume of all of the shells is the Riemann sum

$$\sum_{k=1}^{n} 2\pi\binom{\text{shell}}{\text{radius}}\binom{\text{shell}}{\text{height}}\Delta x_k$$

So the volume of the solid object is

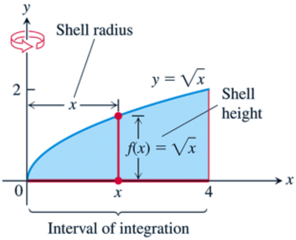
$$\boxed{\text{volume} = \int_a^b 2\pi\binom{\text{shell}}{\text{radius}}\binom{\text{shell}}{\text{height}}\,dx.}$$

Shell Formula for Revolution About a Vertical Line

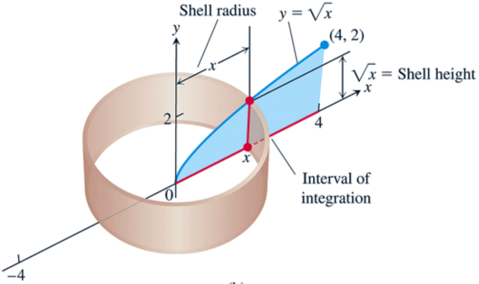
The volume of the solid generated by revolving the region between the x-axis and the graph of a continuous function $y = f(x) \geq 0, L \leq a \leq x \leq b$, about a vertical line $x = L$ is

$$V = \int_a^b 2\pi \begin{pmatrix} \text{shell} \\ \text{radius} \end{pmatrix} \begin{pmatrix} \text{shell} \\ \text{height} \end{pmatrix} dx.$$

$$\text{volume} = \int_a^b 2\pi \begin{pmatrix}\text{shell}\\ \text{radius}\end{pmatrix} \begin{pmatrix}\text{shell}\\ \text{height}\end{pmatrix} dx$$



(a)

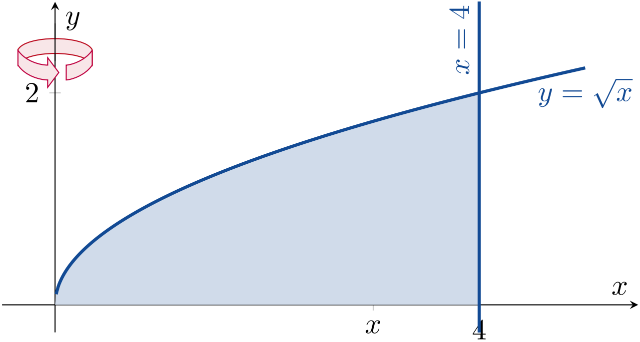


(b)

Example

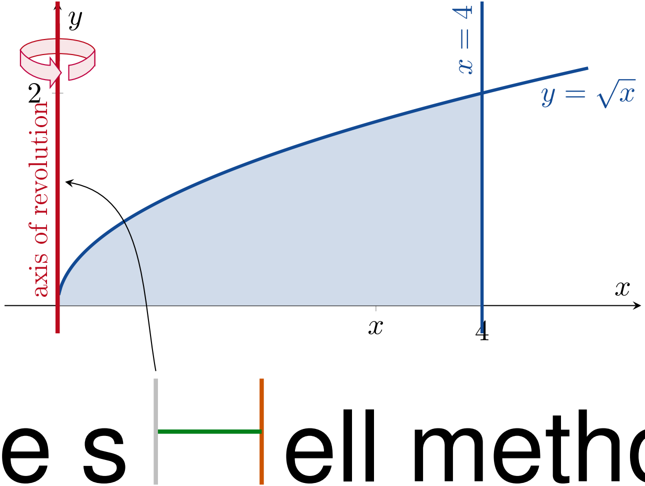
The region bounded by the curve $y = \sqrt{x}$, the x-axis and the line $x = 4$ is revolved about the y-axis to generate a solid. Find the volume of the solid.

$$\text{volume} = \int_a^b 2\pi \begin{pmatrix}\text{shell}\\\text{radius}\end{pmatrix}\begin{pmatrix}\text{shell}\\\text{height}\end{pmatrix} dx$$



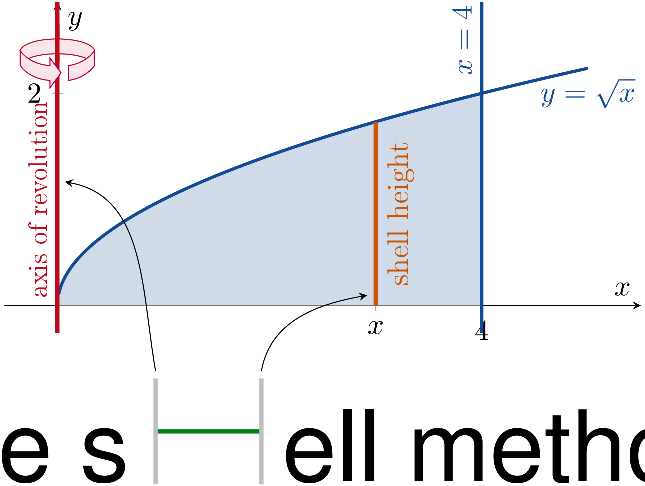
the s|—|ell method

$$\text{volume} = \int_a^b 2\pi \left(\begin{smallmatrix}\text{shell}\\ \text{radius}\end{smallmatrix}\right)\left(\begin{smallmatrix}\text{shell}\\ \text{height}\end{smallmatrix}\right)\,dx$$



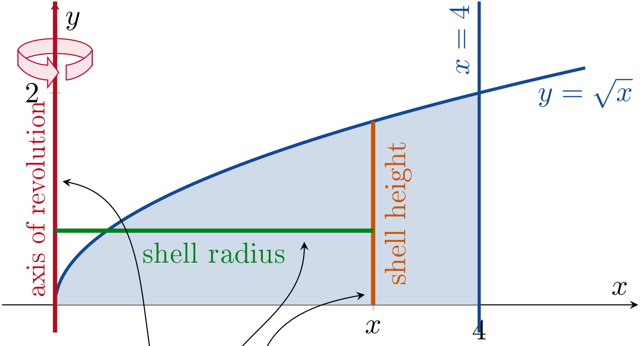
the shell method

$$\text{volume} = \int_a^b 2\pi \begin{pmatrix}\text{shell}\\ \text{radius}\end{pmatrix}\begin{pmatrix}\text{shell}\\ \text{height}\end{pmatrix} dx$$



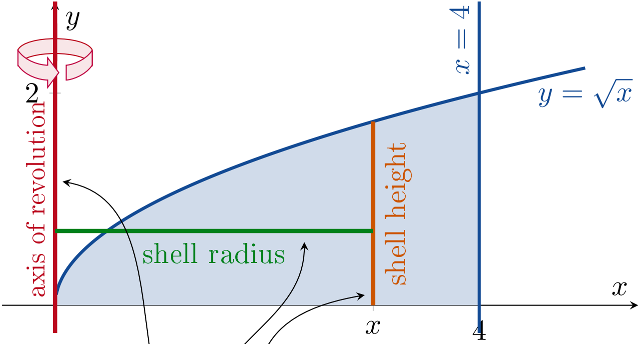
the s⊢ell method

$$\text{volume} = \int_a^b 2\pi \left(\begin{smallmatrix}\text{shell}\\\text{radius}\end{smallmatrix}\right)\left(\begin{smallmatrix}\text{shell}\\\text{height}\end{smallmatrix}\right) dx$$



the s├┤ell method

$$\text{volume} = \int_a^b 2\pi \left(\begin{array}{c}\text{shell}\\ \text{radius}\end{array}\right)\left(\begin{array}{c}\text{shell}\\ \text{height}\end{array}\right) dx$$



$$0 \le x \le 4$$

$$\text{shell height} = \sqrt{x}$$

$$\text{shell radius} = x$$

the shell method

$$\text{volume} = \int_a^b 2\pi \binom{\text{shell}}{\text{radius}} \binom{\text{shell}}{\text{height}} \, dx$$

$$0 \leq x \leq 4$$

$$\text{shell height} = \sqrt{x}$$

$$\text{shell radius} = x$$

Therefore

$$\text{volume} = \int_a^b 2\pi \binom{\text{shell}}{\text{radius}} \binom{\text{shell}}{\text{height}} \, dx$$

$$=$$

$$= \qquad = \qquad = \qquad .$$

$$\text{volume} = \int_a^b 2\pi \binom{\text{shell}}{\text{radius}} \binom{\text{shell}}{\text{height}} \, dx$$

$$0 \leq x \leq 4$$
$$\text{shell height} = \sqrt{x}$$
$$\text{shell radius} = x$$

Therefore

$$\begin{aligned} \text{volume} &= \int_a^b 2\pi \binom{\text{shell}}{\text{radius}} \binom{\text{shell}}{\text{height}} \, dx \\ &= \int_0^4 2\pi (x) \left(\sqrt{x}\right) dx \\ &= \qquad = \qquad = \qquad . \end{aligned}$$

$$\text{volume} = \int_a^b 2\pi \binom{\text{shell}}{\text{radius}} \binom{\text{shell}}{\text{height}}\, dx$$

$$0 \le x \le 4$$
$$\text{shell height} = \sqrt{x}$$
$$\text{shell radius} = x$$

Therefore

$$\begin{aligned}\text{volume} &= \int_a^b 2\pi \binom{\text{shell}}{\text{radius}} \binom{\text{shell}}{\text{height}}\, dx \\ &= \int_0^4 2\pi (x)\,(\sqrt{x})\, dx \\ &= 2\pi \int_0^4 x^{\frac{3}{2}}\, dx = 2\pi \left[\frac{2}{5} x^{\frac{5}{2}}\right]_0^4 = \frac{128\pi}{5}.\end{aligned}$$

$$\text{volume} = \int_a^b 2\pi \left(\begin{smallmatrix}\text{shell}\\ \text{radius}\end{smallmatrix}\right)\left(\begin{smallmatrix}\text{shell}\\ \text{height}\end{smallmatrix}\right) dx$$

Remark

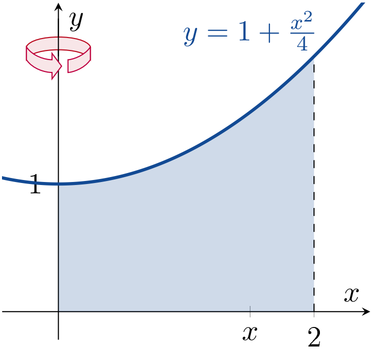
When you do the shell method, it doesn't really matter if you get the shell radius and the shell height mixed up because

- you are just going to multiply them together;
- so you still get the correct answer; and
- you will only lose a few points in an exam.

$$\text{volume} = \int_a^b 2\pi \left(\begin{smallmatrix}\text{shell}\\ \text{radius}\end{smallmatrix}\right)\left(\begin{smallmatrix}\text{shell}\\ \text{height}\end{smallmatrix}\right) dx$$

Example (Page 348, Exercise 1)

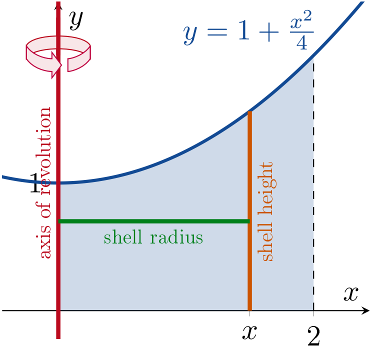
The region bounded by the curve $y = 1 + \frac{x^2}{4}$ and the x-axis, for $0 \le x \le 2$, is revolved about the y-axis to generate a solid. Find the volume of the solid.



$$\text{volume} = \int_a^b 2\pi \left(\begin{smallmatrix}\text{shell}\\ \text{radius}\end{smallmatrix}\right)\left(\begin{smallmatrix}\text{shell}\\ \text{height}\end{smallmatrix}\right) dx$$

Example (Page 348, Exercise 1)

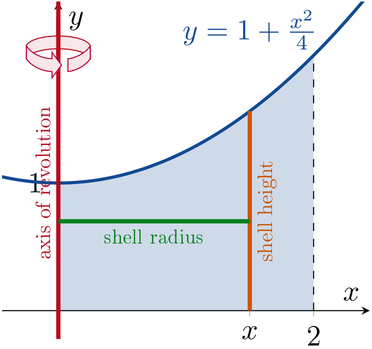
The region bounded by the curve $y = 1 + \frac{x^2}{4}$ and the x-axis, for $0 \le x \le 2$, is revolved about the y-axis to generate a solid. Find the volume of the solid.



$$\text{volume} = \int_a^b 2\pi \left(\begin{smallmatrix}\text{shell}\\ \text{radius}\end{smallmatrix}\right)\left(\begin{smallmatrix}\text{shell}\\ \text{height}\end{smallmatrix}\right) dx$$

Example (Page 348, Exercise 1)

The region bounded by the curve $y = 1 + \frac{x^2}{4}$ and the x-axis, for $0 \leq x \leq 2$, is revolved about the y-axis to generate a solid. Find the volume of the solid.

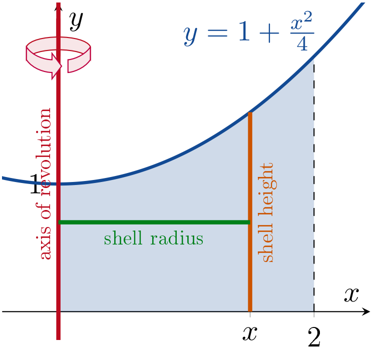


$0 \leq x \leq 2$

$$\text{volume} = \int_a^b 2\pi \left(\begin{smallmatrix}\text{shell}\\ \text{radius}\end{smallmatrix}\right)\left(\begin{smallmatrix}\text{shell}\\ \text{height}\end{smallmatrix}\right) dx$$

Example (Page 348, Exercise 1)

The region bounded by the curve $y = 1 + \frac{x^2}{4}$ and the x-axis, for $0 \le x \le 2$, is revolved about the y-axis to generate a solid. Find the volume of the solid.



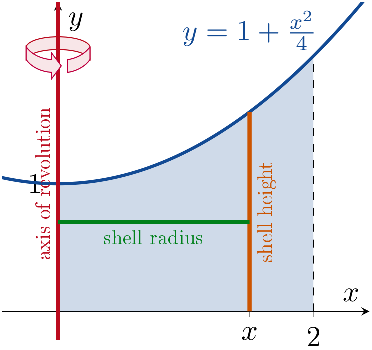
$0 \le x \le 2$

shell height $= 1 + \frac{x^2}{4}$

$$\text{volume} = \int_a^b 2\pi \left(\substack{\text{shell}\\\text{radius}}\right)\left(\substack{\text{shell}\\\text{height}}\right) dx$$

Example (Page 348, Exercise 1)

The region bounded by the curve $y = 1 + \frac{x^2}{4}$ and the x-axis, for $0 \leq x \leq 2$, is revolved about the y-axis to generate a solid. Find the volume of the solid.

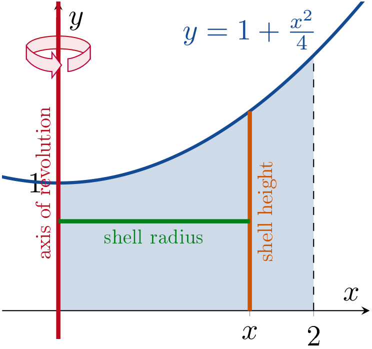


$$0 \leq x \leq 2$$
$$\text{shell height} = 1 + \frac{x^2}{4}$$
$$\text{shell radius} = x$$

$$\text{volume} = \int_a^b 2\pi \left(\begin{smallmatrix}\text{shell}\\ \text{radius}\end{smallmatrix}\right)\left(\begin{smallmatrix}\text{shell}\\ \text{height}\end{smallmatrix}\right) dx$$

Example (Page 348, Exercise 1)

The region bounded by the curve $y = 1 + \frac{x^2}{4}$ and the x-axis, for $0 \le x \le 2$, is revolved about the y-axis to generate a solid. Find the volume of the solid.



$0 \le x \le 2$

shell height $= 1 + \frac{x^2}{4}$

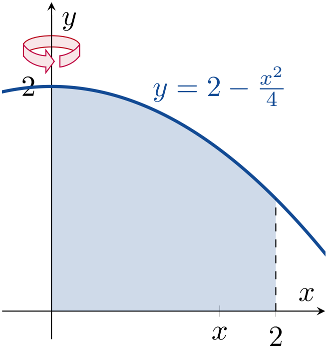
shell radius $= x$

$$\text{volume} = \int_0^2 2\pi\,(x)\left(1 + \frac{x^2}{4}\right) dx$$
$$= \ldots$$

$$\text{volume} = \int_a^b 2\pi \binom{\text{shell}}{\text{radius}} \binom{\text{shell}}{\text{height}}\, dx$$

Example (Page 348, Exercise 2)

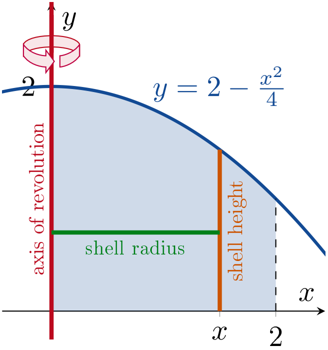
The region shown below is revolved about the y-axis to generate a solid. Find the volume of the solid.



$$\text{volume} = \int_a^b 2\pi \binom{\text{shell}}{\text{radius}} \binom{\text{shell}}{\text{height}} \, dx$$

Example (Page 348, Exercise 2)

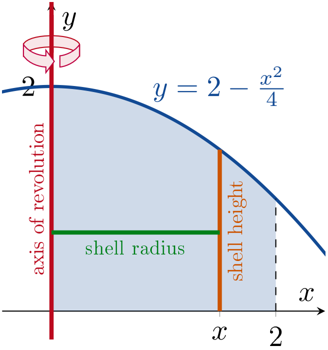
The region shown below is revolved about the y-axis to generate a solid. Find the volume of the solid.



$$\text{volume} = \int_a^b 2\pi \binom{\text{shell}}{\text{radius}} \binom{\text{shell}}{\text{height}}\, dx$$

Example (Page 348, Exercise 2)

The region shown below is revolved about the y-axis to generate a solid. Find the volume of the solid.

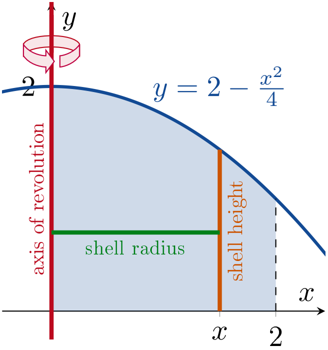


$$0 \le x \le 2$$
$$\text{shell height} = 2 - \frac{x^2}{4}$$
$$\text{shell radius} = x$$

$$\text{volume} = \int_a^b 2\pi \left(\begin{smallmatrix}\text{shell}\\ \text{radius}\end{smallmatrix}\right)\left(\begin{smallmatrix}\text{shell}\\ \text{height}\end{smallmatrix}\right)\,dx$$

Example (Page 348, Exercise 2)

The region shown below is revolved about the y-axis to generate a solid. Find the volume of the solid.



$$0 \le x \le 2$$
$$\text{shell height} = 2 - \frac{x^2}{4}$$
$$\text{shell radius} = x$$

$$\begin{aligned}\text{volume} &= \int_0^2 2\pi\,(x)\left(2 - \frac{x^2}{4}\right)\,dx\\ &= \dots\end{aligned}$$

$$\text{volume} = \int_a^b 2\pi \binom{\text{shell}}{\text{radius}} \binom{\text{shell}}{\text{height}} \, dx$$

We can use the

the shell method

when we rotate about a vertical line.

$$\boxed{V = \int_a^b 2\pi \binom{\text{shell}}{\text{radius}} \binom{\text{shell}}{\text{height}} \, dx.}$$

$$\text{volume} = \int_a^b 2\pi \left(\begin{smallmatrix}\text{shell}\\ \text{radius}\end{smallmatrix}\right)\left(\begin{smallmatrix}\text{shell}\\ \text{height}\end{smallmatrix}\right) dx$$

We can use the

the sHell method

when we rotate about a vertical line.

$$\boxed{V = \int_a^b 2\pi \left(\begin{smallmatrix}\text{shell}\\ \text{radius}\end{smallmatrix}\right)\left(\begin{smallmatrix}\text{shell}\\ \text{height}\end{smallmatrix}\right) dx.}$$

And we can use the

the sHell method

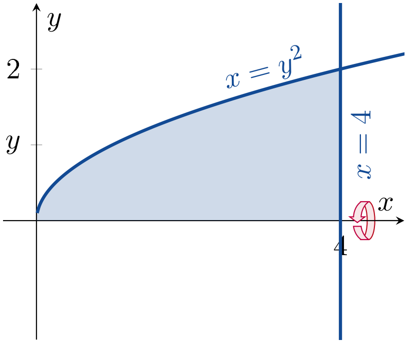
when we rotate about a horizontal line.

$$\boxed{V = \int_a^b 2\pi \left(\begin{smallmatrix}\text{shell}\\ \text{radius}\end{smallmatrix}\right)\left(\begin{smallmatrix}\text{shell}\\ \text{height}\end{smallmatrix}\right) d{\color{red}y}.}$$

$$\text{volume} = \int_a^b 2\pi \binom{\text{shell}}{\text{radius}} \binom{\text{shell}}{\text{height}}\, dy$$

Example

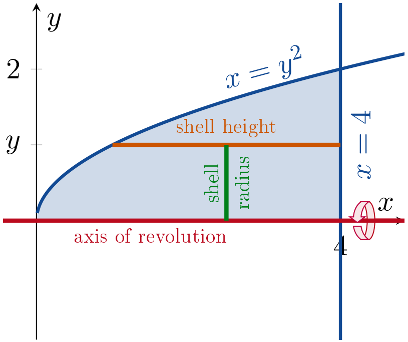
The region bounded by the curve $y = \sqrt{x}$, the x-axis and the line $x = 4$ is revolved about the x-axis to generate a solid. Find the volume of the solid.



$$\text{volume} = \int_a^b 2\pi \binom{\text{shell}}{\text{radius}} \binom{\text{shell}}{\text{height}} \, dy$$

Example

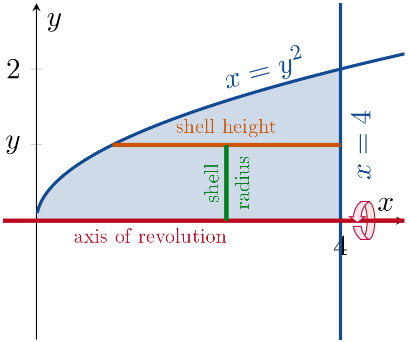
The region bounded by the curve $y = \sqrt{x}$, the x-axis and the line $x = 4$ is revolved about the x-axis to generate a solid. Find the volume of the solid.



$$\text{volume} = \int_a^b 2\pi \binom{\text{shell}}{\text{radius}} \binom{\text{shell}}{\text{height}} \, dy$$

Example

The region bounded by the curve $y = \sqrt{x}$, the x-axis and the line $x = 4$ is revolved about the x-axis to generate a solid. Find the volume of the solid.

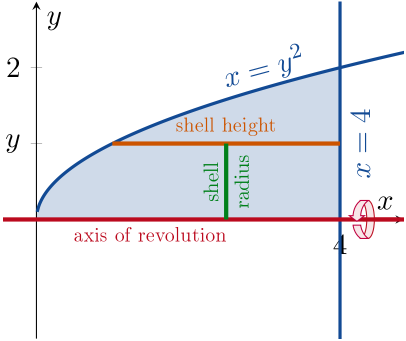


$0 \le y \le 2$

$$\text{volume} = \int_a^b 2\pi \binom{\text{shell}}{\text{radius}} \binom{\text{shell}}{\text{height}} \, dy$$

Example

The region bounded by the curve $y = \sqrt{x}$, the x-axis and the line $x = 4$ is revolved about the x-axis to generate a solid. Find the volume of the solid.

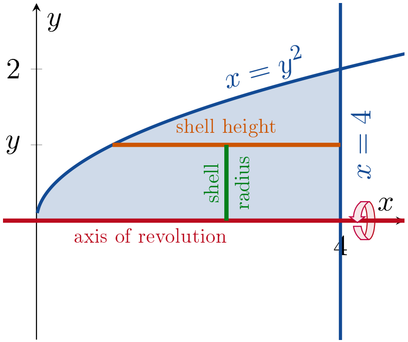


$$0 \le y \le 2$$
$$\text{shell height} = 4 - y^2$$

$$\text{volume} = \int_a^b 2\pi \left(\begin{smallmatrix}\text{shell}\\ \text{radius}\end{smallmatrix}\right)\left(\begin{smallmatrix}\text{shell}\\ \text{height}\end{smallmatrix}\right) dy$$

Example

The region bounded by the curve $y = \sqrt{x}$, the x-axis and the line $x = 4$ is revolved about the x-axis to generate a solid. Find the volume of the solid.

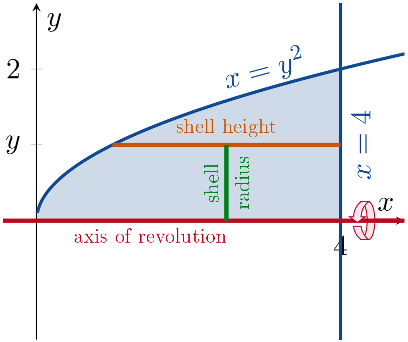


$$0 \le y \le 2$$
$$\text{shell height} = 4 - y^2$$
$$\text{shell radius} = y$$

$$\text{volume} = \int_a^b 2\pi \left(\begin{smallmatrix}\text{shell}\\ \text{radius}\end{smallmatrix}\right)\left(\begin{smallmatrix}\text{shell}\\ \text{height}\end{smallmatrix}\right) dy$$

Example

The region bounded by the curve $y = \sqrt{x}$, the x-axis and the line $x = 4$ is revolved about the x-axis to generate a solid. Find the volume of the solid.



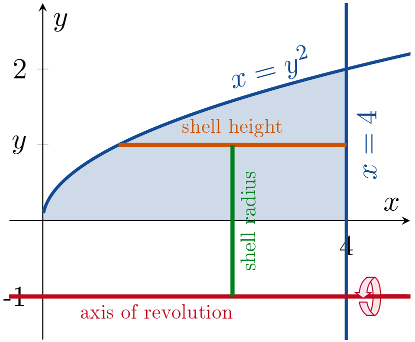
$$0 \le y \le 2$$
$$\text{shell height} = 4 - y^2$$
$$\text{shell radius} = y$$

$$\text{volume} = \int_0^2 2\pi\,(y)\left(4 - y^2\right)\,dy$$
$$= \ldots$$

$$\text{volume} = \int_a^b 2\pi \begin{pmatrix}\text{shell}\\ \text{radius}\end{pmatrix} \begin{pmatrix}\text{shell}\\ \text{height}\end{pmatrix} dy$$

Example

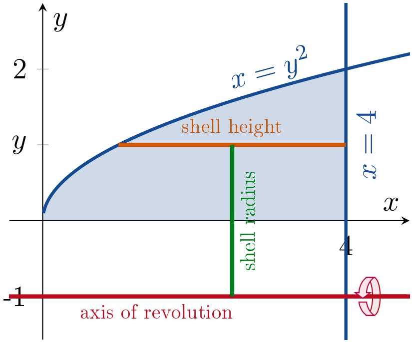
The region bounded by the curve $y = \sqrt{x}$, the x-axis and the line $x = 4$ is revolved about the line $y = -1$ to generate a solid. Find the volume of the solid.



$$\text{volume} = \int_a^b 2\pi \left(\begin{smallmatrix}\text{shell}\\ \text{radius}\end{smallmatrix}\right)\left(\begin{smallmatrix}\text{shell}\\ \text{height}\end{smallmatrix}\right) dy$$

Example

The region bounded by the curve $y = \sqrt{x}$, the x-axis and the line $x = 4$ is revolved about the line $y = -1$ to generate a solid. Find the volume of the solid.



$$0 \le y \le 2$$

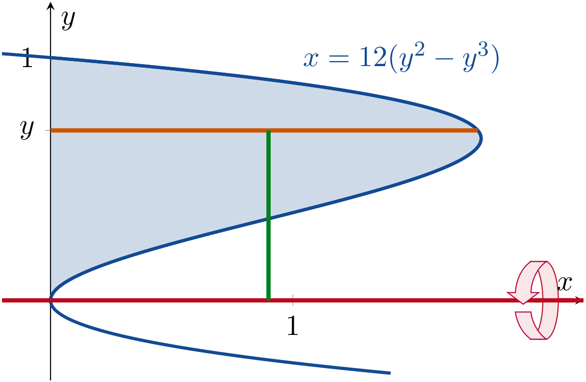
$$\text{shell height} = 4 - y^2$$

$$\text{shell radius} = y - (-1) = y + 1$$

$$\text{volume} = \int_0^2 2\pi\,(y+1)\left(4 - y^2\right)\,dy$$

$$= \dots$$

6.2 Volumes Using Cylindrical Shells

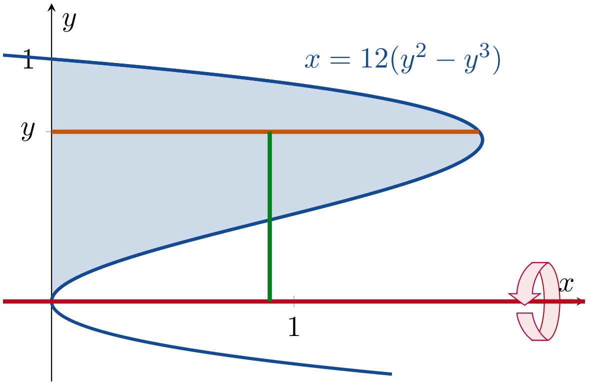


1. shell height $= 12y^3$
   shell radius $= 12y^2$

2. shell height $= y$
   shell radius $= 12(y^2 - y^3)$

3. shell height $= 12(y^2 - y^3)$
   shell radius $= y$

4. shell height $= 12y^2$
   shell radius $= 12y^3$

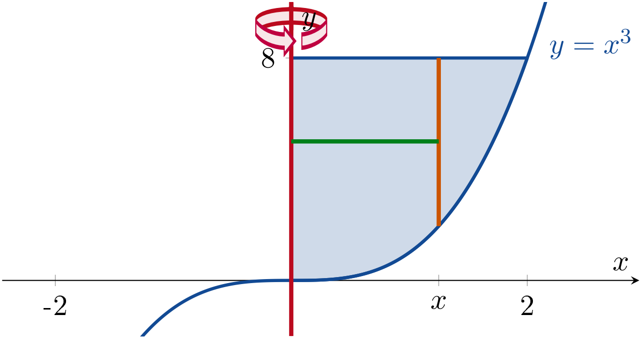


1. shell height $= 12y^3$
   shell radius $= 12y^2$

2. shell height $= y$
   shell radius $= 12(y^2 - y^3)$

3. shell height $= 12(y^2 - y^3)$
   shell radius $= y$

4. shell height $= 12y^2$
   shell radius $= 12y^3$

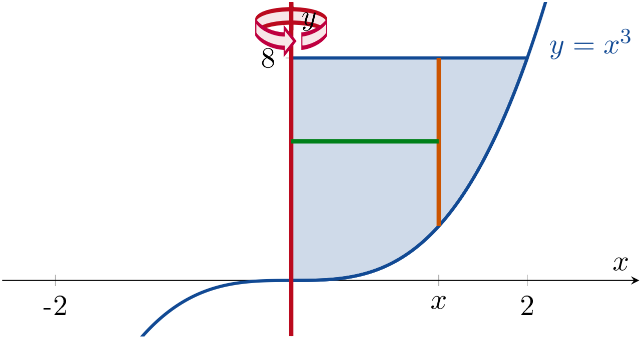


1. shell height $= 8 - x^2$
   shell radius $= x$

2. shell height $= y^{\frac{1}{3}}$
   shell radius $= 8 - y$

3. shell height $= 8 - x^2$
   shell radius $= x + 2$

4. shell height $= y^{\frac{1}{3}}$
   shell radius $= y$

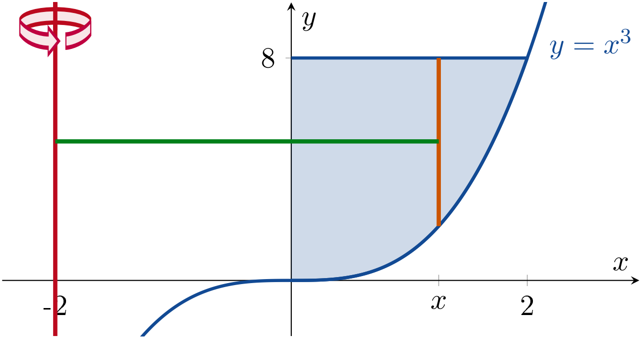


1. shell height $= 8 - x^2$
shell radius $= x$

2. shell height $= y^{\frac{1}{3}}$
shell radius $= 8 - y$

3. shell height $= 8 - x^2$
shell radius $= x + 2$

4. shell height $= y^{\frac{1}{3}}$
shell radius $= y$

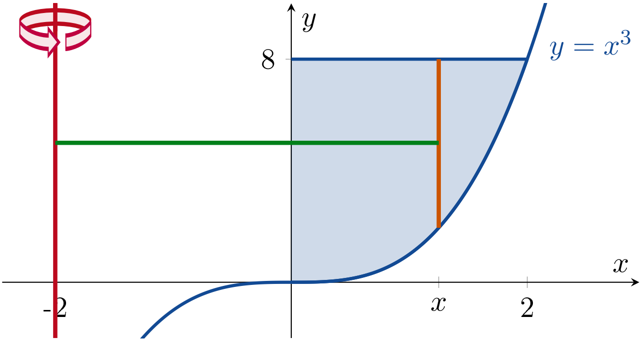


1. shell height $= 8 - x^2$
   shell radius $= x$

2. shell height $= y^{\frac{1}{3}}$
   shell radius $= 8 - y$

3. shell height $= 8 - x^2$
   shell radius $= x + 2$

4. shell height $= y^{\frac{1}{3}}$
   shell radius $= y$

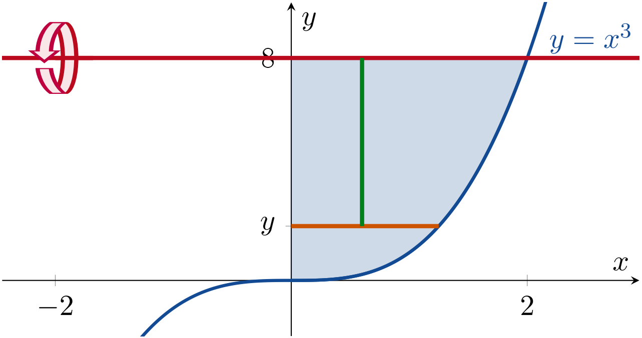


1. shell height $= 8 - x^2$
   shell radius $= x$

2. shell height $= y^{\frac{1}{3}}$
   shell radius $= 8 - y$

3. shell height $= 8 - x^2$
   shell radius $= x + 2$

4. shell height $= y^{\frac{1}{3}}$
   shell radius $= y$

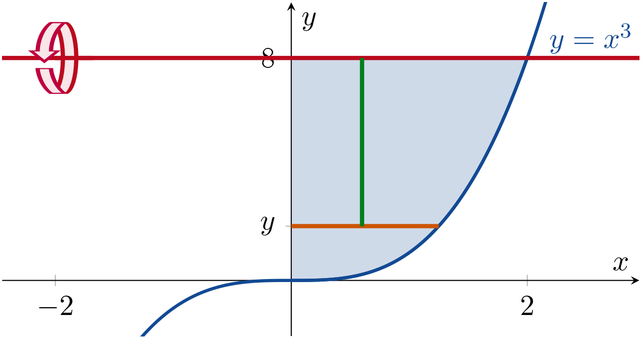


1. shell height $= 8 - x^2$
   shell radius $= x$

2. shell height $= y^{\frac{1}{3}}$
   shell radius $= 8 - y$

3. shell height $= 8 - x^2$
   shell radius $= x + 2$

4. shell height $= y^{\frac{1}{3}}$
   shell radius $= y$



1 shell height $= 8 - x^2$
shell radius $= x$

3 shell height $= 8 - x^2$
shell radius $= x + 2$

2 shell height $= y^{\frac{1}{3}}$
shell radius $= 8 - y$

4 shell height $= y^{\frac{1}{3}}$
shell radius $= y$

Break

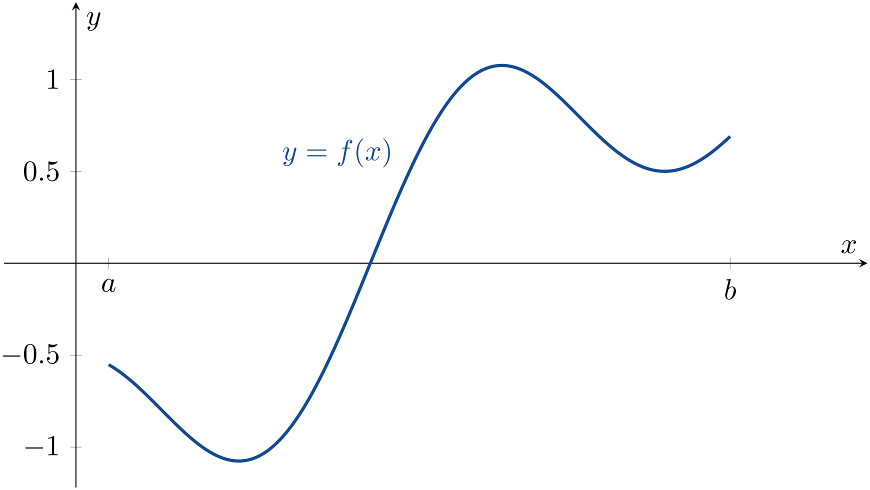
We will continue at 2pm

KEEP
CALM
AND
PASS
CALCULUS

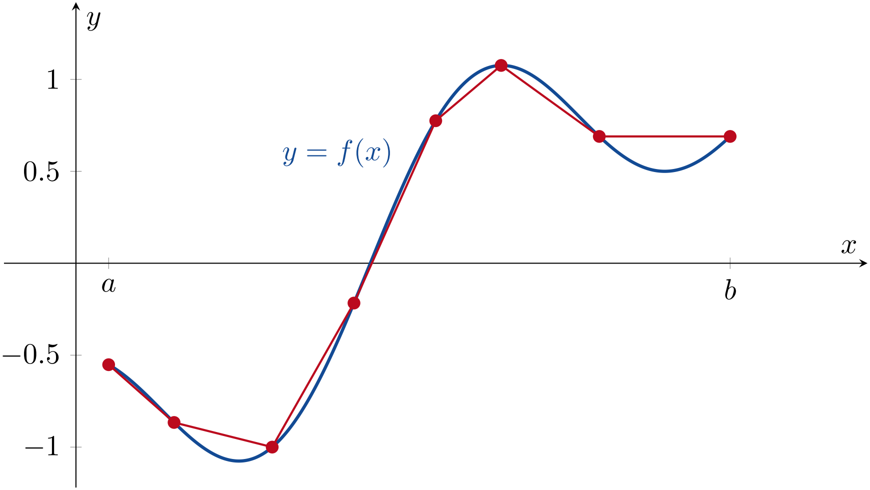
6.3

Arc Length

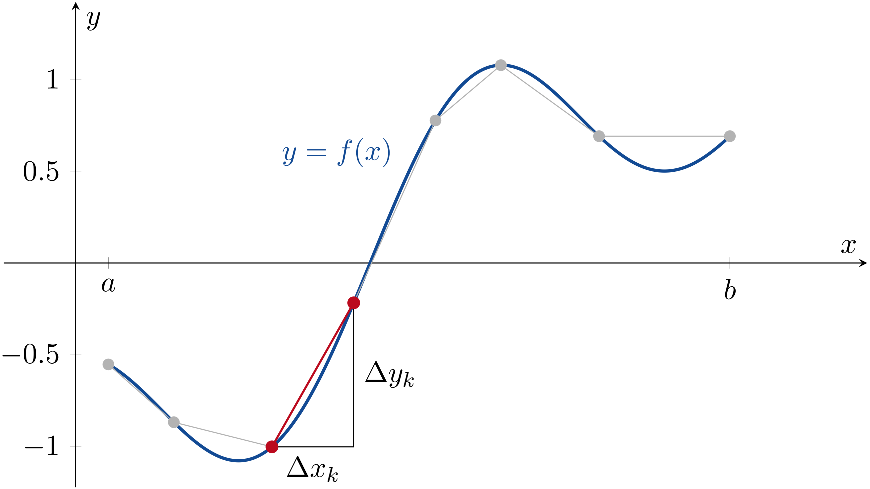
What is the length of this curve?



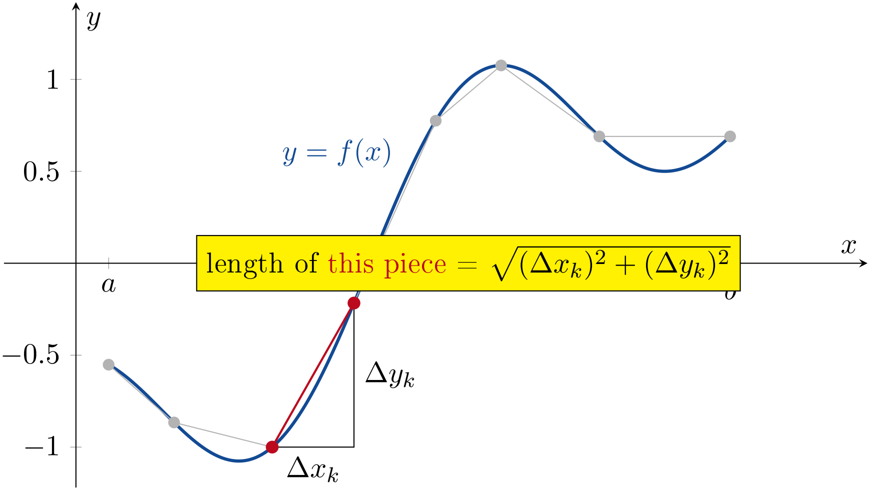
What is the length of this curve?



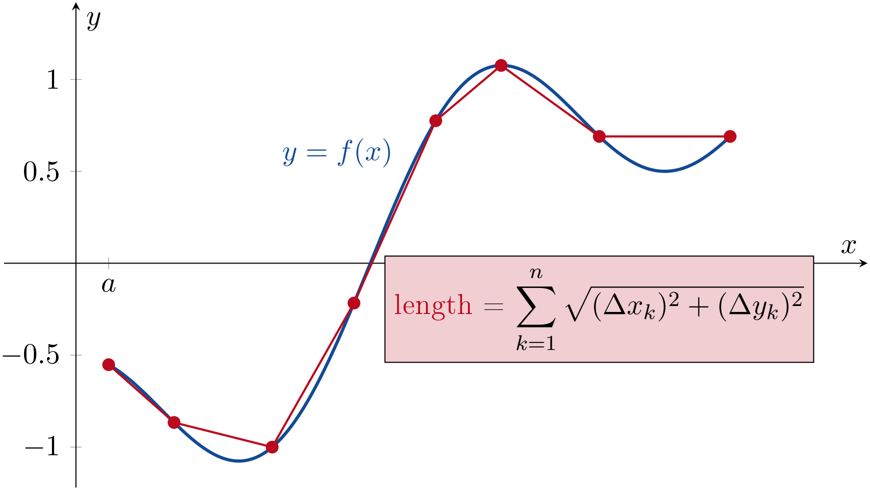
What is the length of this curve?



What is the length of this curve?

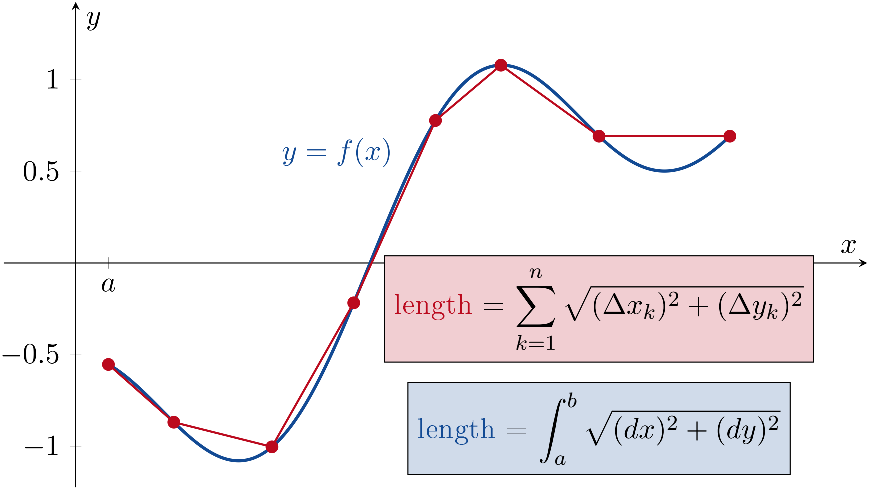


What is the length of this curve?



$$\text{length} = \sum_{k=1}^{n} \sqrt{(\Delta x_k)^2 + (\Delta y_k)^2}$$

What is the length of this curve?



$$\text{length} = \sum_{k=1}^{n} \sqrt{(\Delta x_k)^2 + (\Delta y_k)^2}$$

$$\text{length} = \int_a^b \sqrt{(dx)^2 + (dy)^2}$$

OK, so

$$\text{length} = \int_a^b \sqrt{(dx)^2 + (dy)^2}$$

but what is $\sqrt{(dx)^2 + (dy)^2}$?

OK, so

$$\text{length} = \int_a^b \sqrt{(dx)^2 + (dy)^2}$$

but what is $\sqrt{(dx)^2 + (dy)^2}$?

$$\sqrt{(dx)^2 + (dy)^2} = \sqrt{(dx)^2 \left(1 + \frac{(dy)^2}{(dx)^2}\right)}$$

$$=$$

$$=$$

OK, so

$$\text{length} = \int_a^b \sqrt{(dx)^2 + (dy)^2}$$

but what is $\sqrt{(dx)^2 + (dy)^2}$?

$$\sqrt{(dx)^2 + (dy)^2} = \sqrt{(dx)^2 \left(1 + \frac{(dy)^2}{(dx)^2}\right)}$$

$$= \sqrt{(dx)^2 \left(1 + \left(\frac{dy}{dx}\right)^2\right)}$$

$$=$$

OK, so

$$\text{length} = \int_a^b \sqrt{(dx)^2 + (dy)^2}$$

but what is $\sqrt{(dx)^2 + (dy)^2}$?

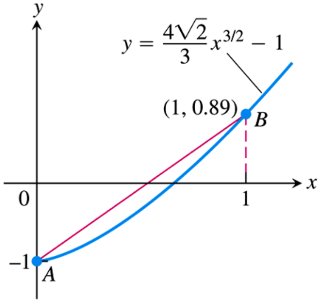
$$\begin{aligned}\sqrt{(dx)^2 + (dy)^2} &= \sqrt{(dx)^2 \left(1 + \frac{(dy)^2}{(dx)^2}\right)} \\ &= \sqrt{(dx)^2 \left(1 + \left(\frac{dy}{dx}\right)^2\right)} \\ &= \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx\end{aligned}$$

Definition

If f' is continuous on $[a, b]$, then the *length* of the curve $y = f(x)$ from $x = a$ to $x = b$ is

$$\text{length} = \int_a^b \sqrt{1 + \big(f'(x)\big)^2}\, dx = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx.$$

$$\text{length} = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$



Example

Find the length of the graph of $y = \frac{4\sqrt{2}}{3}x^{\frac{3}{2}} - 1$ for $0 \leq x \leq 1$.

$$\text{length} = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

We calculate that

$$\begin{aligned} y &= \frac{4\sqrt{2}}{3} x^{\frac{3}{2}} - 1 \\ \frac{dy}{dx} &= \\ \left(\frac{dy}{dx}\right)^2 &= \\ \text{length} &= \int_0^1 \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx = \end{aligned}$$

$$\text{length} = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

We calculate that

$$y = \frac{4\sqrt{2}}{3} x^{\frac{3}{2}} - 1$$

$$\frac{dy}{dx} = \frac{4\sqrt{2}}{3} \cdot \frac{3}{2} x^{\frac{1}{2}} = 2\sqrt{2} x^{\frac{1}{2}}$$

$$\left(\frac{dy}{dx}\right)^2 =$$

$$\text{length} = \int_0^1 \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx =$$

$$\text{length} = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

We calculate that

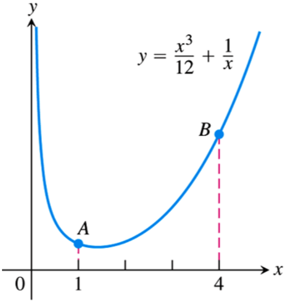
$$
\begin{aligned}
y &= \frac{4\sqrt{2}}{3} x^{\frac{3}{2}} - 1 \\
\frac{dy}{dx} &= \frac{4\sqrt{2}}{3} \cdot \frac{3}{2} x^{\frac{1}{2}} = 2\sqrt{2} x^{\frac{1}{2}} \\
\left(\frac{dy}{dx}\right)^2 &= \left(2\sqrt{2} x^{\frac{1}{2}}\right)^2 = 8x \\
\text{length} &= \int_0^1 \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx =
\end{aligned}
$$

$$\text{length} = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

We calculate that

$$\begin{aligned} y &= \frac{4\sqrt{2}}{3} x^{\frac{3}{2}} - 1 \\ \frac{dy}{dx} &= \frac{4\sqrt{2}}{3} \cdot \frac{3}{2} x^{\frac{1}{2}} = 2\sqrt{2} x^{\frac{1}{2}} \\ \left(\frac{dy}{dx}\right)^2 &= \left(2\sqrt{2} x^{\frac{1}{2}}\right)^2 = 8x \\ \text{length} &= \int_0^1 \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx = \int_0^1 \sqrt{1+8x}\, dx = \dots \end{aligned}$$

$$\text{length} = \int_a^b \sqrt{1 + \left(f'(x)\right)^2}\, dx$$



Example

Find the length of the graph of $f(x) = \dfrac{x^3}{12} + \dfrac{1}{x}$ for $1 \leq x \leq 4$.

$$\text{length} = \int_a^b \sqrt{1 + \left(f'(x)\right)^2}\,dx$$

We calculate that

$$f(x) = \frac{x^3}{12} + \frac{1}{x}$$

$$f'(x) =$$

$$1 + \left(f'(x)\right)^2 =$$

$$\text{length} = \int_1^4 \sqrt{1 + \left(f'(x)\right)^2}\,dx =$$

$$\text{length} = \int_a^b \sqrt{1 + \left(f'(x)\right)^2}\, dx$$

We calculate that

$$f(x) = \frac{x^3}{12} + \frac{1}{x}$$
$$f'(x) = \frac{x^2}{4} - \frac{1}{x^2}$$

$$1 + \left(f'(x)\right)^2 =$$

$$\text{length} = \int_1^4 \sqrt{1 + \left(f'(x)\right)^2}\, dx =$$

$$\text{length} = \int_a^b \sqrt{1 + \left(f'(x)\right)^2}\, dx$$

We calculate that

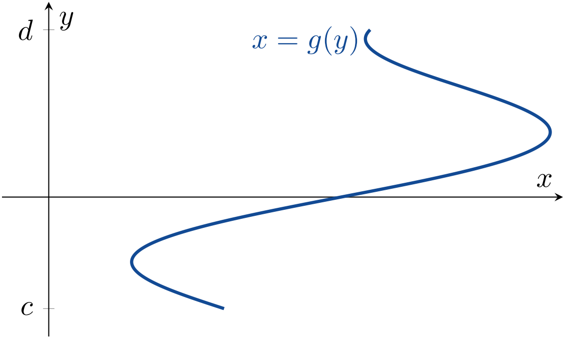
$$\begin{aligned}
f(x) &= \frac{x^3}{12} + \frac{1}{x} \\
f'(x) &= \frac{x^2}{4} - \frac{1}{x^2} \\
1 + \left(f'(x)\right)^2 &= 1 + \left(\frac{x^2}{4} - \frac{1}{x^2}\right)^2 = 1 + \left(\frac{x^4}{16} - \frac{1}{2} + \frac{1}{x^4}\right) \\
&= \frac{x^4}{16} + \frac{1}{2} + \frac{1}{x^4} = \left(\frac{x^2}{4} + \frac{1}{x^2}\right)^2 \\
\text{length} &= \int_1^4 \sqrt{1 + \left(f'(x)\right)^2}\, dx =
\end{aligned}$$

$$\text{length} = \int_a^b \sqrt{1 + \left(f'(x)\right)^2}\, dx$$

We calculate that

$$
\begin{aligned}
f(x) &= \frac{x^3}{12} + \frac{1}{x} \\
f'(x) &= \frac{x^2}{4} - \frac{1}{x^2} \\
1 + \left(f'(x)\right)^2 &= 1 + \left(\frac{x^2}{4} - \frac{1}{x^2}\right)^2 = 1 + \left(\frac{x^4}{16} - \frac{1}{2} + \frac{1}{x^4}\right) \\
&= \frac{x^4}{16} + \frac{1}{2} + \frac{1}{x^4} = \left(\frac{x^2}{4} + \frac{1}{x^2}\right)^2 \\
\text{length} &= \int_1^4 \sqrt{1 + \left(f'(x)\right)^2}\, dx = \int_1^4 \left(\frac{x^2}{4} + \frac{1}{x^2}\right) dx = \ldots
\end{aligned}
$$

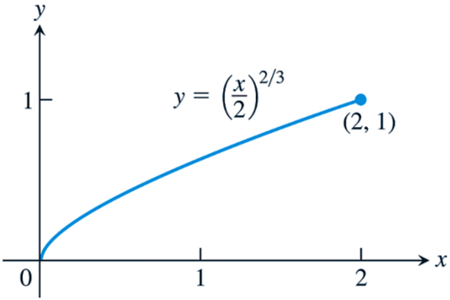
6.3 Arc Length



Definition

If g' is continuous on $[c, d]$, then the *length* of the curve $x = g(y)$ from $y = c$ to $y = d$ is

$$\text{length} = \int_c^d \sqrt{1 + \big(g'(x)\big)^2}\, dy = \int_c^d \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy.$$



Example

Find the length of the graph of $y = \left(\frac{x}{2}\right)^{\frac{2}{3}}$ for $0 \le x \le 2$.

Note first that if $y = \left(\frac{x}{2}\right)^{\frac{2}{3}}$ then

$$\frac{dy}{dx} = \frac{2}{3}\left(\frac{x}{2}\right)^{-\frac{1}{3}}\left(\frac{1}{2}\right) = \frac{1}{3}\left(\frac{2}{x}\right)^{\frac{1}{3}}$$

is not defined at $x = 0$.

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Instead, we need to use

$$\text{length} = \int_c^d \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy.$$

$$\text{length} = \int_c^d \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy$$

First we calculate that

$$y = \left(\frac{x}{2}\right)^{\frac{2}{3}} \qquad \implies \qquad y^{\frac{3}{2}} = \frac{x}{2} \qquad \implies \qquad x = 2y^{\frac{3}{2}}.$$

$$\text{length} = \int_c^d \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy$$

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Then we differentiate

$$\frac{dx}{dy} = 2\left(\frac{3}{2}\right) y^{\frac{1}{2}} = 3y^{\frac{1}{2}}.$$

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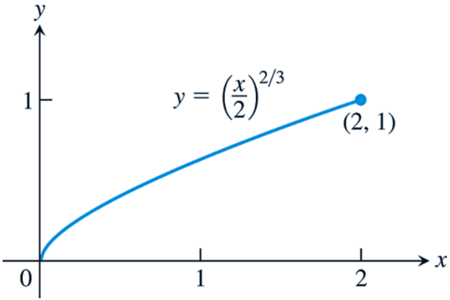
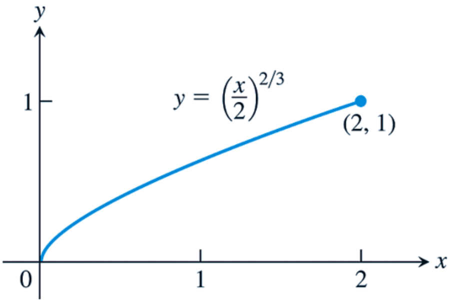
Then we differentiate

$$\frac{dx}{dy} = 2\left(\frac{3}{2}\right) y^{\frac{1}{2}} = 3y^{\frac{1}{2}}.$$

This function is continuous on $[0, 1]$.



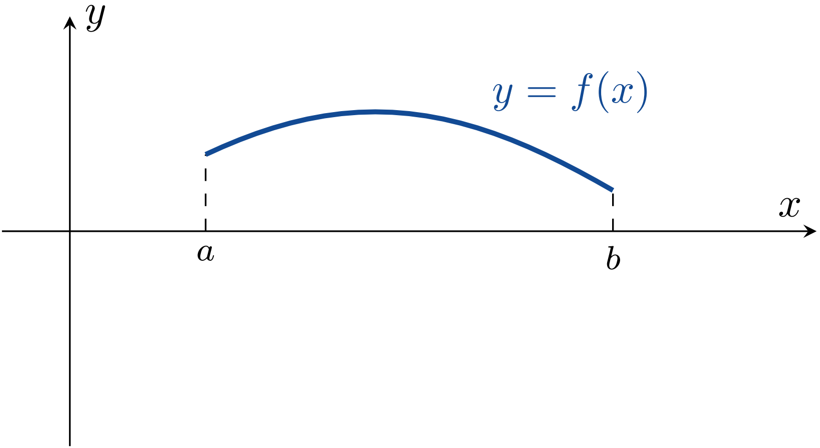
$$\text{length} = \int_c^d \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy$$

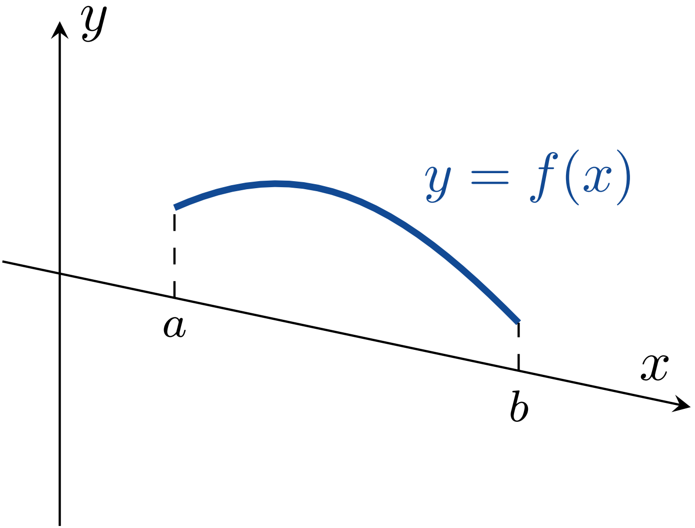


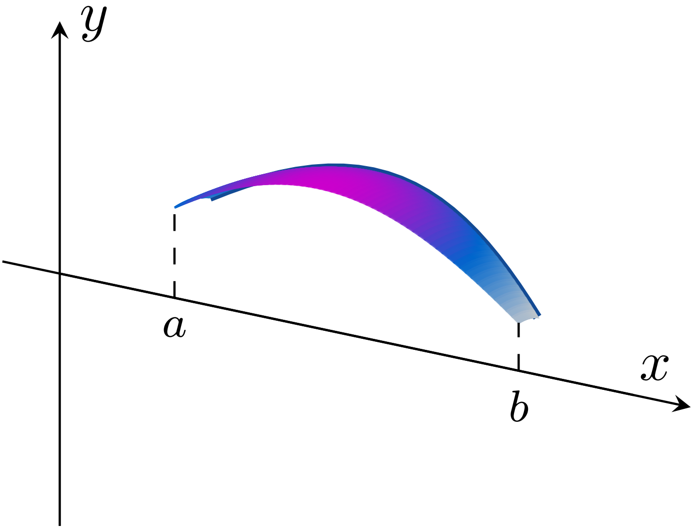
Therefore

$$\text{length} = \int_c^d \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy = \int_0^1 \sqrt{1 + 9y}\, dy = \dots$$

6.4 Areas of Surfaces of Revolution

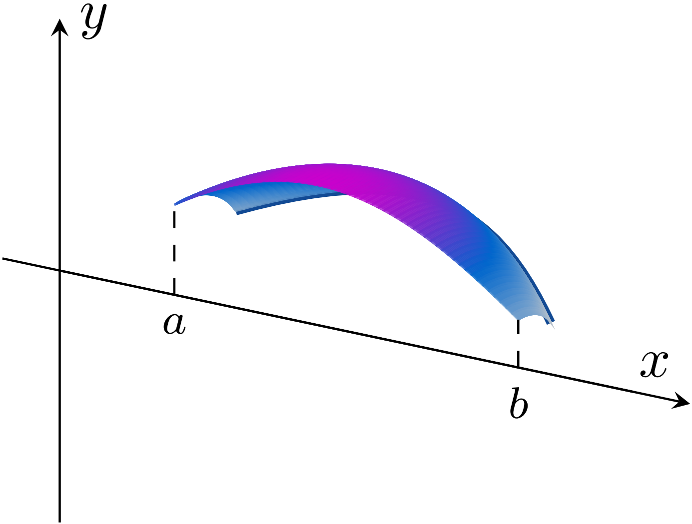

y
$y = f(x)$
x
a
b

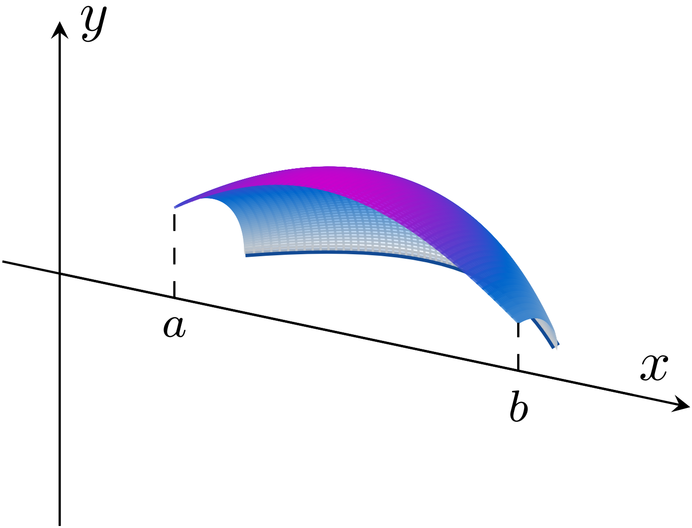

y
$y = f(x)$
a
x
b



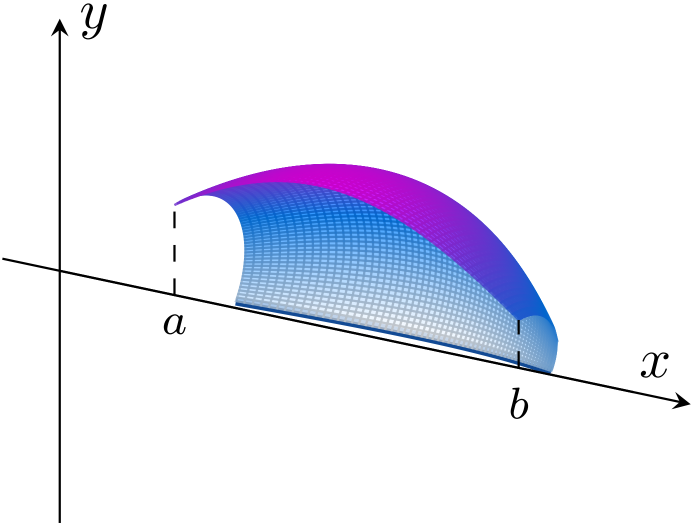
y
a
b
x

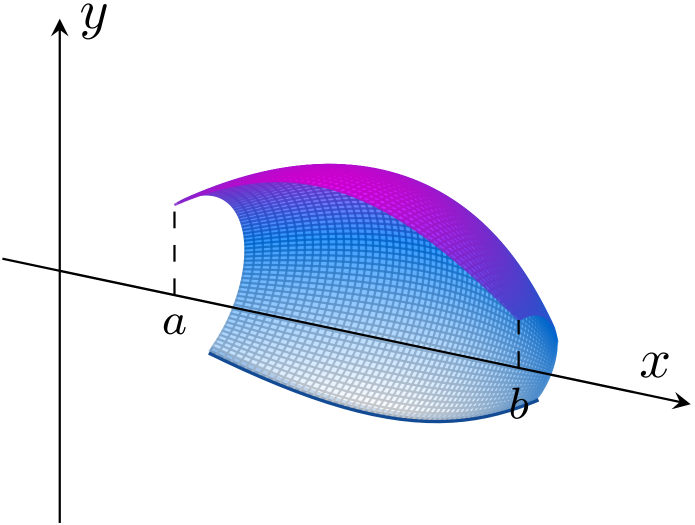
6.4 Areas of Surfaces of Revolution





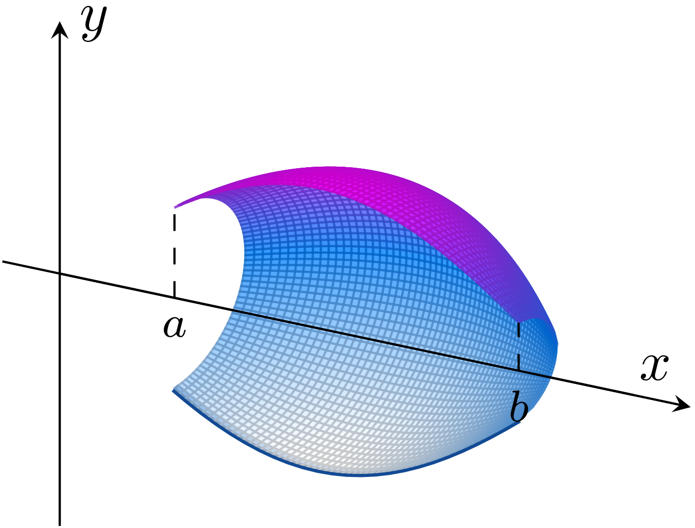
y
a
b
x


y
x
a
b

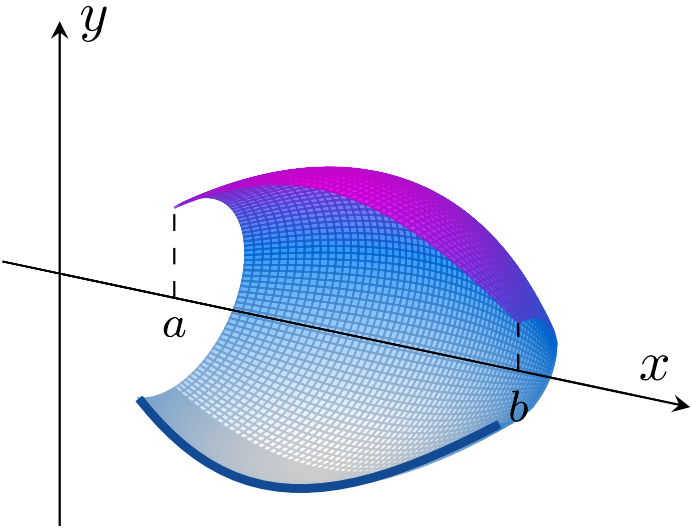


y
a
b
x

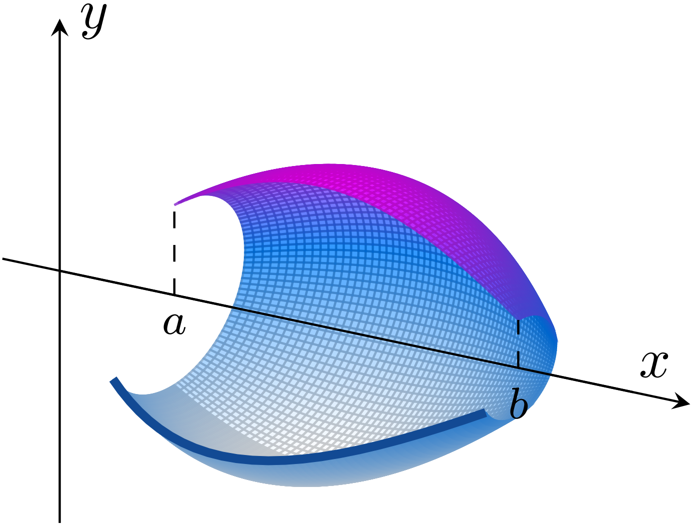
6.4 Areas of Surfaces of Revolution

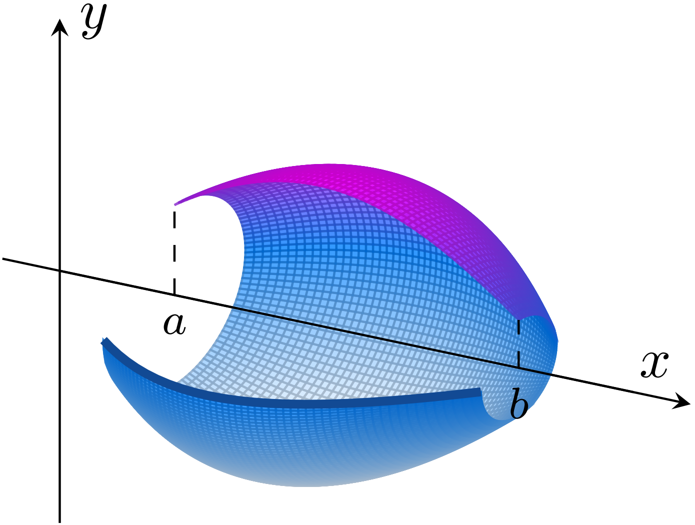


6.4 Areas of Surfaces of Revolution

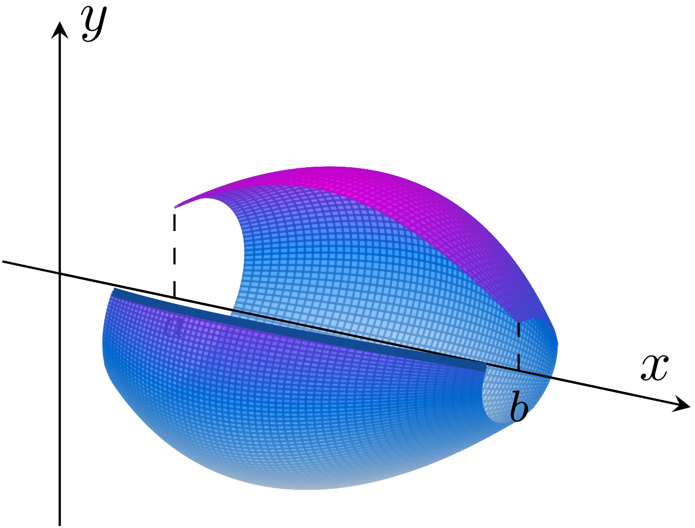


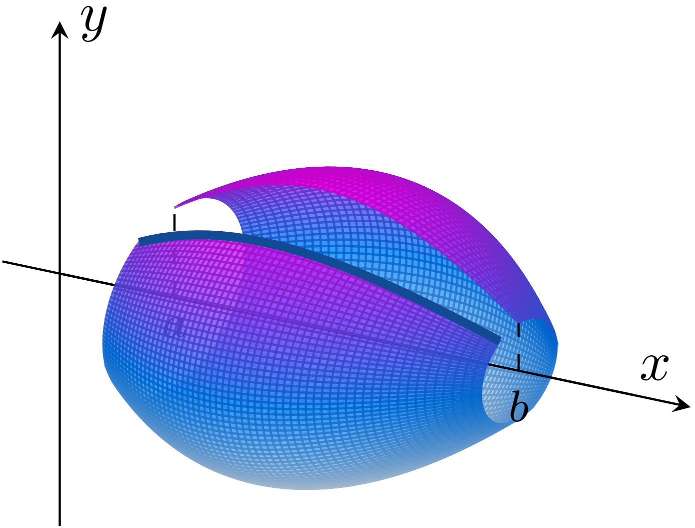
6.4 Areas of Surfaces of Revolution



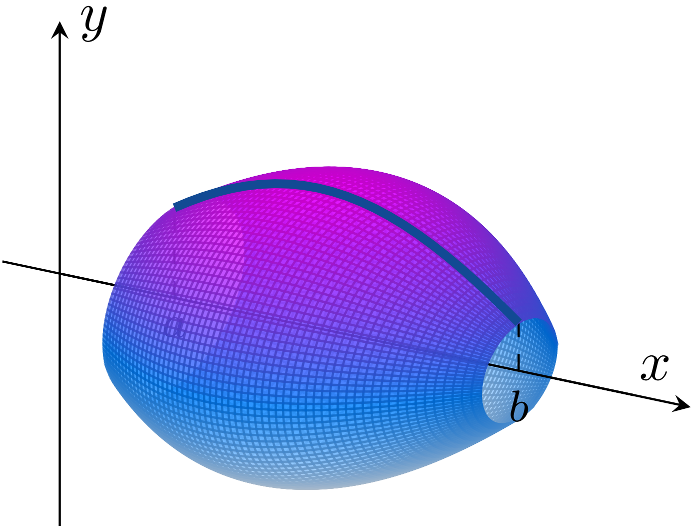

y
x
a
b

6.4 Areas of Surfaces of Revolution





y
x
a
b



y
x
a
b

$$\text{length} = \int_c^d \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy$$

Revolution About the x-Axis

This time I will just tell you the formula without showing how to derive it.

$$\text{length} = \int_c^d \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy$$

Revolution About the x-Axis

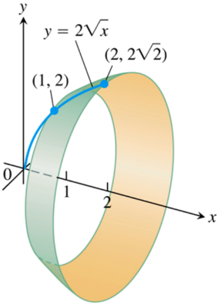
This time I will just tell you the formula without showing how to derive it.

Definition

If the function $f(x) \geq 0$ is continuously differentiable on $[a, b]$, the *area of the surface* generated by revolving the graph $y = f(x)$ about the x-axis is

$$\boxed{\text{surface area} = \int_a^b 2\pi y \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx.}$$

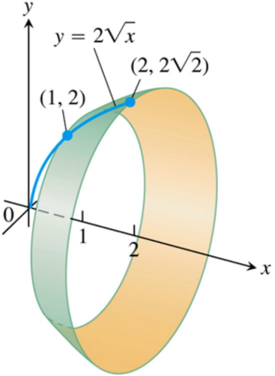
$$\text{surface area} = \int_a^b 2\pi y \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$



Example

Find the area of the surface generated by revolving the curve $y = 2\sqrt{x}$, $1 \le x \le 2$, about the x-axis.

$$\text{surface area} = \int_a^b 2\pi y \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$



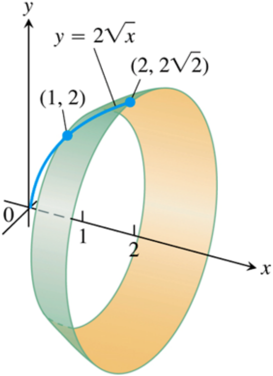
$$y = 2\sqrt{x}$$

$$\frac{dy}{dx} = \frac{1}{\sqrt{x}}$$

$$1 + \left(\frac{dy}{dx}\right)^2 = 1 + \left(\frac{1}{\sqrt{x}}\right)^2$$

$$= 1 + \frac{1}{x} = \frac{x+1}{x}$$

$$\text{surface area} = \int_a^b 2\pi y \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$



$$y = 2\sqrt{x}$$

$$\frac{dy}{dx} = \frac{1}{\sqrt{x}}$$

$$1 + \left(\frac{dy}{dx}\right)^2 = 1 + \left(\frac{1}{\sqrt{x}}\right)^2$$

$$= 1 + \frac{1}{x} = \frac{x+1}{x}$$

$$\text{surface area} = \int_a^b 2\pi y \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx = \int_1^2 2\pi \cdot 2\sqrt{x} \cdot \frac{\sqrt{x+1}}{\sqrt{x}}\, dx = \dots$$

$$\text{surface area} = \int_a^b 2\pi y \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

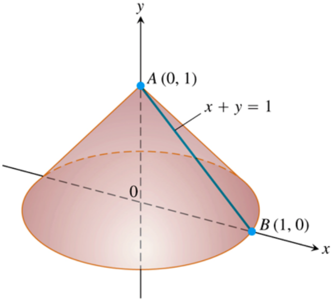
Revolution About the y-Axis

Definition

If the function $x = g(y) \geq 0$ is continuously differentiable on $[c, d]$, the *area of the surface* generated by revolving the graph $x = g(y)$ about the y-axis is

$$\boxed{\text{surface area} = \int_c^d 2\pi x \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy.}$$

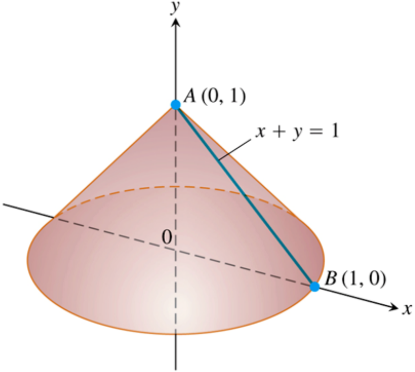
$$\text{surface area} = \int_c^d 2\pi x \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy$$



Example

The line segment $x = 1 - y$, $0 \leq y \leq 1$, is revolved about the y-axis to generate the cone shown above. Find its surface area.

$$\text{surface area} = \int_c^d 2\pi x \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy$$

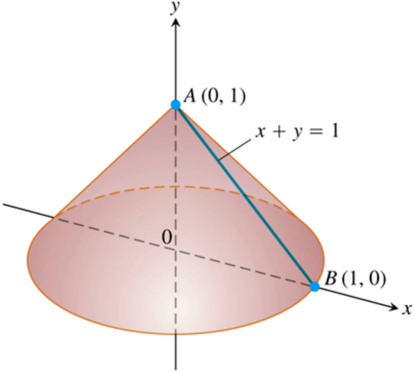


$$x = 1 - y$$

$$\frac{dx}{dy} = -1$$

$$1 + \left(\frac{dx}{dy}\right)^2 = 1 + (-1)^2 = 2$$

$$\text{surface area} = \int_c^d 2\pi x \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy$$



$$x = 1 - y$$

$$\frac{dx}{dy} = -1$$

$$1 + \left(\frac{dx}{dy}\right)^2 = 1 + (-1)^2 = 2$$

$$\text{surface area} = \int_c^d 2\pi x \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy = \int_0^1 2\pi(1-y)\sqrt{2}\, dy = \ldots$$

6.4 Areas of Surfaces of Revolution

volumes by cross sections

$$V = \int_a^b A(x)\,dx$$

the shell method

$$V = \int_a^b 2\pi \binom{\text{shell}}{\text{radius}} \binom{\text{shell}}{\text{height}}\,dx$$

the disk method

$$V = \int_a^b \pi \Big(R(x)\Big)^2\,dx$$

arc length

$$L = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\,dx$$

the washer method

$$V = \int_a^b \pi \Big(\big(R(x)\big)^2 - \big(r(x)\big)^2\Big)\,dx$$

surfaces of revolution

$$S = \int_a^b 2\pi y \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\,dx$$

Next Time

- 7.1 Inverse Functions and Their Derivatives
- 7.2 Natural Logarithms
- 7.3 Exponential Functions