Piecewise continuous functions.

Consider the following equation,

$$f(t) = \begin{cases} 1 & for\ 0 \le t < 1 \\ k & for\ t = 1 \\ 0 & for\ t \ge 0. \end{cases}$$

The Laplace Transform of this fuction is

$$\mathcal{L}(f(t)) = F(s) = \int_0^\infty e^{-st} f(t)dt = \int_0^1 e^{-st} f(t)dt = -\frac{1}{s}[e^{-st}]_0^1 = \frac{1-e^{-s}}{s}.$$

Note that the integral is independent of the value of k. In the sequel, we use **unit step function** u_c in order to represent such functions. The unit step function is defined as follows.

$$u_c(t) = \begin{cases} 0 & for\ t < c \\ 1 & for\ t \ge c. \end{cases}$$

Consequently, $f(t)$ given above can be written as follows. $f(t) = 1 - u_1(t)$. Note that $\mathcal{L}(u_1(t)) = \frac{e^{-cs}}{s}$. Let us give few more examples.

- Let

$$f(t) = \begin{cases} \sin(t) & for\ 0 \le t < \frac{\pi}{4} \\ \sin(t) + \cos(t - \pi/4) & for\ t \ge \frac{\pi}{4}. \end{cases}$$

  This implies that $f(t) = \sin(t) + u_{\pi/4}(t)\cos(t - \pi/4)$. Note that the Laplace Transform of this function is

$$\mathcal{L}(f(t)) = \mathcal{L}(\sin(t)) + \mathcal{L}(u_{\pi/4}(t)\cos(t - \pi/4)) = \frac{1}{s^2+1} + \frac{se^{-\pi s/4}}{s^2+1}.$$

- Similarly, if

$$F(s) = \frac{1-e^{-2s}}{s^2},$$

  then it follows that $f(t) = t - u_2(t)(t-2)$.
- Consider the following piecewise continuous function

$$f(t) = \begin{cases} 0 & for\ 0 \le t < 3 \\ -1 & for\ 3 \le t < 5 \\ 2 & for\ 5 \le t < 7 \\ 1 & for\ t \ge 7. \end{cases}$$

  Note that $f(t)$ can be written as $f(t) = -u_3(t) + 3u_5(t) - u_7(t)$. Furthermore,

$$\mathcal{L}(f(t)) = \frac{3e^{-5s} - e^{-3s} - e^{-7s}}{s}.$$

- Suppose that

$$f(t) = \begin{cases} t & for\ 0 \le t < 1 \\ t-1 & for\ 1 \le t < 2 \\ t-2 & for\ 2 \le t < 3 \\ 0 & for\ t \ge 3. \end{cases}$$

  then $f(t)$ can be written as $f(t) = t - u_1(t) - u_2(t) - u_3(t)(t-3) - u_3(t)$. Furthermore,

$$\mathcal{L}(f(t)) = \frac{1 - se^{-s} - se^{-2s} - (s+1)e^{-3s}}{s^2}. \qquad (1)$$

- Given the following piecewise continuous function

$$f(t) = \begin{cases} 0 & for\ 0 \le t < 2 \\ t-4 & for\ 2 \le t < 3 \\ 1 & for\ 3 \le t. \end{cases}$$

it can be expressed as $f(t) = u_2(t)(t-4) - u_3(t)(t-5)$. This equation can also be written as follows.

$$f(t) = u_2(t)(t-2) - 2u_2(t) - u_3(t)(t-3) - 2u_3(t).$$

Then, it follows that

$$\mathcal{L}(f(t)) = \frac{e^{-2s}(1-2s) - (2s+1)e^{-3s}}{s^2}.$$

- Let us consider the following function

$$f(t) = \begin{cases} 0 & for\ 0 \le t < 1 \\ t^2 - 2t + 3 & for\ t \ge 1. \end{cases}$$

This function can also be written as follows.

$$f(t) = \begin{cases} 0 & for\ 0 \le t < 1 \\ (t-1)^2 + 2 & for\ t \ge 1. \end{cases}$$

Using the unit step function we have $f(t) = u_1(t)[(t-1)^2 + 2]$. Then, it follows that

$$\mathcal{L}(f(t)) = \frac{2e^{-s}(1+s^2)}{s^3}. \tag{2}$$

Inverse Laplace Transforms of piecewise continuous functions can be obtained in a similar way. Here are some examples.

- Consider the following Laplace Transform.

$$F(s) = \frac{e^{-3s}}{s^2 + s - 2}. \tag{3}$$

Let $H(s) = \frac{1}{s^2+s-2}$. Then, we get

$$H(s) = \frac{1}{s^2 + s - 2} = \frac{A}{s+2} + \frac{B}{s-1}. \tag{4}$$

where $A = -1/3$ and $B = 1/3$. This implies that

$$\mathcal{L}^{-1}(H(s)) = \frac{e^t - e^{-2t}}{3}. \tag{5}$$

Consequently, we get

$$\mathcal{L}^{-1}(F(s)) = u_3(t)\left(\frac{e^{(t-3)} - e^{-2(t-3)}}{3}\right). \tag{6}$$

- Consider the following Laplace Transform.

$$F(s) = \frac{2(s-1)e^{-s}}{s^2 - 2s + 2}. \tag{7}$$

Let

$$H(s) = \frac{2(s-1)}{s^2 - 2s + 2} = \frac{2(s-1)}{(s-1)^2 + 1}. \tag{8}$$

This implies that $\mathcal{L}^{-1}(H(s)) = 2e^t\cos(t)$.

Consequently, we get $\mathcal{L}^{-1}(F(s)) = 2u_1(t)e^{(t-1)}\cos(t-1)$.

In practical applications, differential equations often have piecewise continuous or discontinuous forcing functions. Simple examples are temperature control in a room or in a refregirator which are accomplished by an on-off control. The techniques described above can be used in order to get the Laplace Transform of the solutions for such differential equations. Then, one can use the inverse Laplace Transform to obtain the solution.

- Consider the following initial value problem.

$$y'' + 4y = f(t) = \begin{cases} 0 & 0 \le t < 5 \\ (t-5)/5 & 5 \le t < 10 \\ 1 & t \ge 10, \end{cases}$$

where $y(0) = y'(0) = 0$. Note that $f(t)$ can also be written as follows.

$$f(t) = \frac{u_5(t)(t-5) - u_{10}(t)(t-10)}{5}$$

Therefore, the Laplace Transform of the differential equation is

$$(s^2+4)Y(s) = \frac{1}{5}\frac{e^{-5s} - e^{-10s}}{s^2}.$$

Let

$$H(s) = \frac{1}{5}\frac{1}{s^2(s^2+4)}.$$

Using partial fraction expansion, we get

$$H(s) = \frac{1}{5}\left(\frac{A}{s} + \frac{B}{s^2} + \frac{Cs+D}{(s^2+4)}\right).$$

where $A = 0$, $B = 1/4$, $C = 0$, and $D = -1/4$. This implies that inverse Laplace Transform $h(t)$ of $H(s)$ is as follows.

$$h(t) = \left(\frac{t}{20} - \frac{1}{40}\sin(2t)\right).$$

Consequently, inverse Laplace Transform $y(t)$ of $Y(s)$ is
$y(t) = u_5(t)h(t-5) - u_{10}(t)h(t-10)$ or equivalently

$$y(t) = u_5(t)\left(\frac{t-5}{20} - \frac{1}{40}\sin(2(t-5))\right) - u_{10}(t)\left(\frac{t-10}{20} - \frac{1}{40}\sin(2(t-10))\right).$$

- Let us find the solution of the following initial value problem

$$y'' + 3y' + 2y = f(t) = \begin{cases} 1 & 0 \le t < 10 \\ 0 & t \ge 10, \end{cases}$$

where $y(0) = 1$, and $y'(0) = 0$. Note that a short way of writing $f(t)$ is as follows.

$$f(t) = 1 - u_{10}(t)$$

Therefore, the Laplace Transform of the differential equation is

$$(s^2+3s+2)Y(s) - (s+3) = \frac{1-e^{-10s}}{s}.$$

Rearranging the terms we get

$$Y(s) = \frac{1-e^{-10s}}{s(s^2+3s+2)} + \frac{(s+3)}{(s^2+3s+2)} = \frac{(s^2+3s+1) - e^{-10s}}{s(s^2+3s+2)}.$$

Let us define H(s) and $G(s)$as follows

$$H(s) = \frac{s^2+3s+1}{s(s^2+3s+2)} \quad \textit{and}\ G(s) = \frac{1}{s(s^2+3s+2)}.$$

Using partial fraction expansion of $H(s)$, we get

$$H(s) = \frac{A}{s} + \frac{B}{s+1} + \frac{C}{(s+2)}.$$

where $A = 1/2$, $B = 1$, and $C = -1/2$. Similarly, partial fraction expansion of $G(s)$ yields

$$G(s) = \frac{D}{s} + \frac{E}{s+1} + \frac{F}{(s+2)}.$$

where $D = 1/2$, $E = -1$, and $F = 1/2$. This implies that the inverse Laplace Transforms of $h(t)$ and $g(t)$ of $H(s)$ and $G(s)$ are respectively as follows.

$$h(t) = \frac{1}{2}(1 + 2e^{-t} - e^{-2t}) \ \ and\ g(t) = \frac{1}{2}(1 - 2e^{-t} + e^{-2t}).$$

Consequently, inverse Laplace Transform $y(t)$ of $Y(s)$ is
$y(t) = h(t) - u_{10}(t)g(t - 10)$ or equivalently

$$y(t) = \frac{1}{2}(1 + 2e^{-t} - e^{-2t}) - u_{10}(t)\frac{1}{2}(1 - 2e^{-(t-10)} + e^{-2(t-10)}).$$

- Consider the following initial value problem.

$$y'' + 4y = u_{\pi}(t) - u_{3\pi}(t), \tag{9}$$

where $y(0) = y'(0) = 0$. Note that the Laplace Transform of the differential equation is

$$(s^2 + 4)Y(s) = \frac{e^{-\pi s} - e^{-3\pi s}}{s}.$$

Let

$$H(s) = \frac{1}{s(s^2 + 4)}.$$

Using partial fraction expansion, we get

$$H(s) = \left(\frac{A}{s} + \frac{Bs + C}{(s^2 + 4)}\right).$$

where $A = 1/4$, $B = -1/4$, and $C = 0$. This implies that inverse Laplace Transform $h(t)$ of $H(s)$ is as follows.

$$h(t) = \left(\frac{1}{4} - \frac{1}{4}\cos(2t)\right).$$

Consequently, inverse Laplace Transform $y(t)$ of $Y(s)$ is
$y(t) = u_{\pi}(t)h(t - \pi) - u_{3\pi}(t)h(t - 3\pi)$ or equivalently

$$y(t) = \frac{1}{4}u_{\pi}(t)\left(1 - \cos(2(t - \pi))\right) - \frac{1}{4}u_{3\pi}(t)\left(1 - \cos(2(t - 3\pi))\right).$$