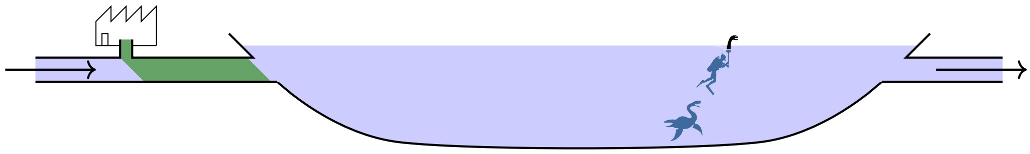
Exercise 1 (Pollution in a Lake).



English

A lake contains 1,000,000 litres of water. At time $t = 0$, the water is pure. Water, containing 0.02 grams/litre of toxic waste, flows into the lake at a rate of 250 litres/hour. Water also flows out of the lake at the same rate, so the amount of water in the lake is always exactly 1,000,000 litres.

Assume that the toxic waste is uniformly distributed throughout the lake. So if there are 2000 grams of toxic waste in the lake, then every litre of water in the lake contains 0.002 grams of toxic waste. Let t be time measured in hours. Let $S(t)$ be the amount (in grams) of toxic waste in the lake at time t.

Türkçe

Bir gölde 1.000.000 litre su vardır. Zaman $t = 0$ olduğunda su saf durumdadır. 1 litrede 0,02 gram zehirli atık yoğunluğu olan akarsu, göle saatte 250 litre hızla akmaktadır. Gölden su aynı hızla da boşalmaktadır, böylece göldeki su her zaman tam olarak 1.000.000 litredir.

Zehirli atıkların göle eşit olarak dağılmış olduğunu varsayın. Yani gölde 2000 gram zehirli atık varsa, göldeki her litre su 0,002 gram zehirli atık içermektedir. t saatle ölçülen zamandır. $S(t)$ ise t zamanında gölde (gram olarak) bulunan zehirli atığın miktarıdır.

(a) How much toxic waste (in grams) enters the lake every hour?

(b) How much toxic waste (as a function of $S(t)$) leaves the lake every hour?

(c) Write an initial value problem for $S(t)$.

[HINT: $\begin{cases} \frac{dS}{dt} = \text{(amount of toxic waste entering the lake per hour)} - \text{(amount of toxic waste leaving the lake per hour)} \\ S(0) = ??? \end{cases}$.]

(d) Solve the initial value problem that you wrote in part (c).

(e) How much toxic waste is in the lake after 1 year?

(f) Sketch a graph of $S(t)$ versus t.

Solution 1.

(a) There are 250 litres/hour $\times$ 0.02 grams/litre = 5 grams/hour of toxic waste flowing into the lake.

(b) The concentration of toxic waste in the lake is $\frac{S(t)}{1000000} = 10^{-6}S(t)$ grams/litre, so 250 litres/hour $\times 10^{-6}S(t)$ grams/litre $= 2.5\times 10^{-4}S(t)$ grams/hour of toxic waste flows out of the lake.

(c) $\begin{cases} S' = 2.5(2 - 10^{-4}S), \\ S(0) = 0. \end{cases}$

(d)

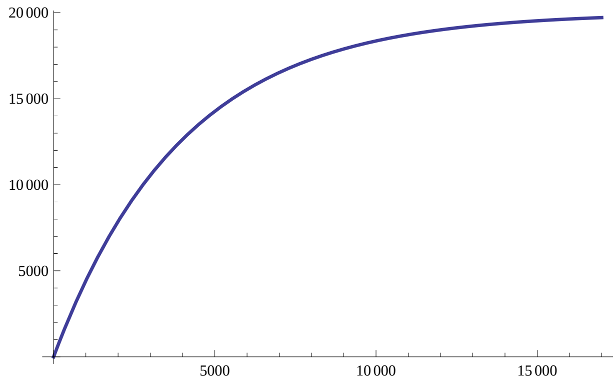
$$\begin{aligned} \frac{dS}{dt} &= 2.5(2 - 10^{-4}S) \\ \frac{dS}{2 - 10^{-4}S} &= 2.5dt \\ \int \frac{dS}{2 - 10^{-4}S} &= \int 2.5dt \\ -10^4 \log\left|2 - 10^{-4}S\right| &= 2.5t + C_1 \\ 2 - 10^{-4}S &= \pm e^{-\frac{2.5t + C_1}{10^4}} = C_2 e^{-2.5t/10^4} \\ S(t) &= 10^4\left(2 - C_2 e^{-2.5t/10^4}\right) \end{aligned}$$

or

$$\begin{aligned} \frac{dS}{dt} &= 2.5(2 - 10^{-4}S) = -2.5\times 10^{-4}(S - 2\times 10^4) \\ \frac{dS}{S - 2\times 10^4} &= -2.5\times 10^{-4}dt \\ \int \frac{dS}{S - 2\times 10^4} &= \int -2.5\times 10^{-4}dt \\ \log\left|S - 2\times 10^4\right| &= -2.5\times 10^{-4}t + C_1 \\ S - 2\times 10^4 &= \pm e^{-\frac{2.5t}{10^4} + C_1} = C_3 e^{-2.5t/10^4} \\ S(t) &= 10^4\left(2 - C_2 e^{-2.5t/10^4}\right) \end{aligned}$$

Finally $S(0) = 0 \implies C_2 = 2$. So $S(t) = 2\times 10^4(1 - e^{-2.5t/10^4})$.

(e) After 1 year, $t = 365\times 24 = 8760$ hours and $S(8760) \approx 17762g$.



(f)

Exercise 2 (A cup of coffee).

English

Newton's law of cooling states that; the temperature of an object changes at a rate proportional to the difference between its temperature and that of its surroundings.

Suppose that the temperature of your cup of coffee obeys Newton's law of cooling, and suppose that the temperature of your room is 20°C.

Türkçe

Newton'ın soğuma kanunu der ki, bir nesnenin ısısı, o nesnenin sıcaklığıyla nesnenin içinde bulunduğu çevrenin sıcaklığı arasındaki farkla orantılı olarak değişir.

Varsayın ki kahvenizin ısısı Newton'ın soğuma kanununa uymakta ve varsayın ki odanızın sıcaklığı 20C°.

(a) Write a differential equation for the temperature of your coffee. (You must explain why your differential equation is valid.)

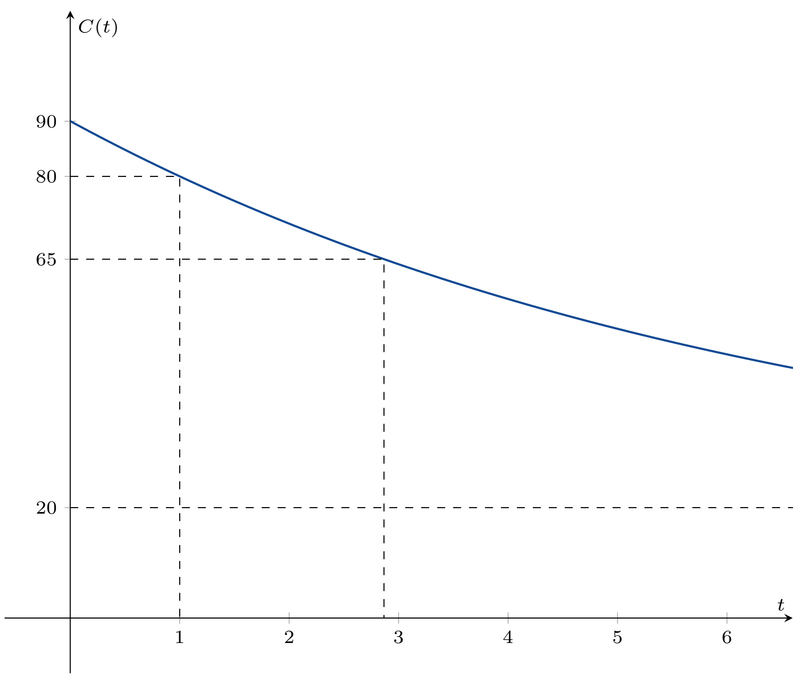
(b) If the coffee has a temperature of 90°C when freshly poured (at time $t = 0$), and 1 minute later has cooled to 80°C, determine how long it takes for the coffee to cool to a temperature of 65°C. (Justify your answer.)

Solution 2.

(a) Let $C(t)$ denote the temperature of your cup of coffee in degrees Centigrade. We are told that $\frac{DC}{dt}$ is proportional to $(20-C)$. Hence $\dfrac{dC}{dt} = r(20-C)$ for some constant r. Since hot coffee cools and very cold coffee warms up, we must have $r > 0$.

(b) The solution to the ODE is $C(t) = 20 + ce^{-rt}$ for some constant c. Using $C(0) = 90$ we find $c = 70$. Thus $C(t) = 20 + 70e^{-rt}$. We are also told that $80 = C(1) = 20 + 70e^{-r}$. Thus $r = \ln \frac{7}{6} \approx 0.154151$. Then we calculate that

$$\begin{aligned} 65 &= C(T) = 20 + 70e^{-rT} \\ \frac{45}{70} &= e^{-rT} \\ \ln \frac{9}{14} &= (\ln \frac{6}{7})T \\ T &= \frac{\ln \frac{9}{14}}{\ln \frac{6}{7}} \approx 2.86623 \text{ minutes.} \end{aligned}$$



Exercise 3 (Direction Fields). Draw direction fields for the following differential equations.

(a) $\dfrac{dy}{dt} = t - y$

(b) $\dfrac{dy}{dt} = 3 - 2y$

(c) $\dfrac{dy}{dt} = y + t$

(d) $\dfrac{dy}{dt} = -ty$

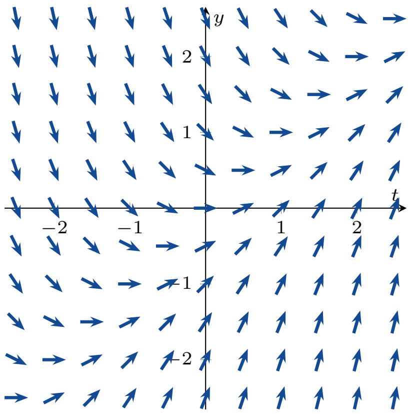
(e) $\dfrac{dy}{dt} = y^2$

(f) $\dfrac{dy}{dt} = -y(2 - y)$

(g) $\dfrac{dy}{dt} = t^2$

(h) $\dfrac{dy}{dt} = 0$

Solution 3.

(a)


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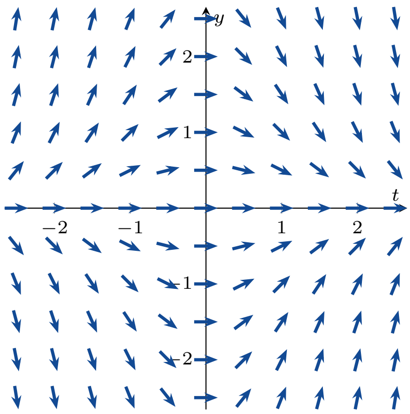

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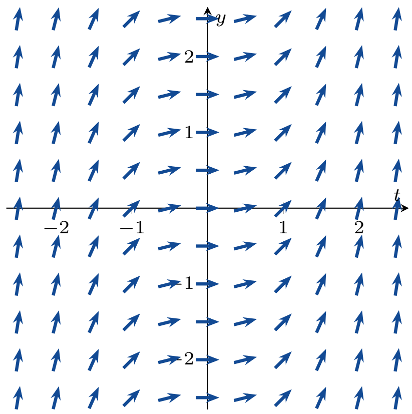

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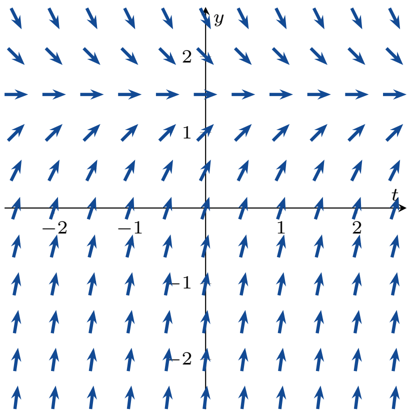

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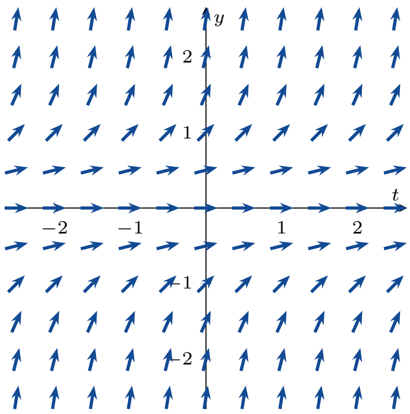

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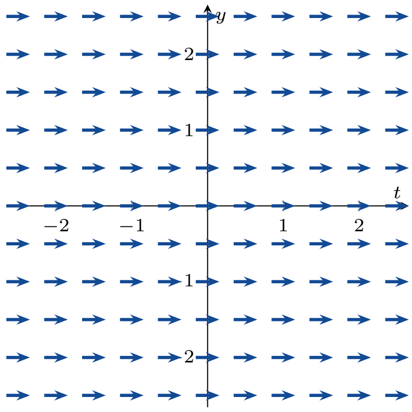

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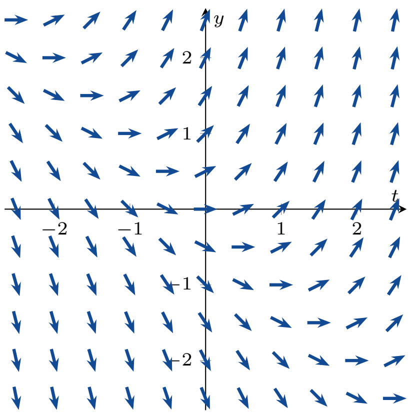

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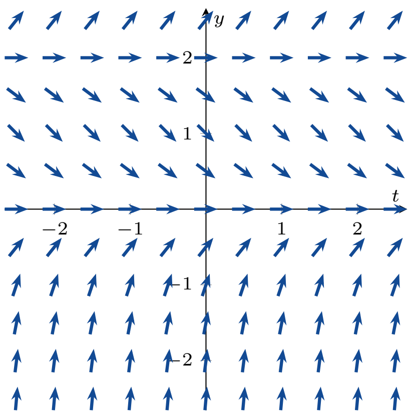

Exercise 4 (Classification). For each of the following differential equations; give the order of the equation and state whether the equation is linear or non-linear. The first one is done for you.

(ω) $y\frac{d^2y}{dt^2} - t\frac{d^3y}{dt^3} = \frac{dy}{dt}$ (3rd order, non-linear)

(a) $t^2\frac{d^2y}{dt^2} + t\frac{dy}{dt} + 2y = \sin t$

(b) $(1+y^2)\frac{d^2y}{dt^2} + t\frac{dy}{dt} + y = e^t$

(c) $\frac{dy}{dt} + ty^2 = 0$

(d) $\frac{d^4y}{dt^4} + \frac{d^3y}{dt^3} + \frac{d^2y}{dt^2} + \frac{dy}{dt} + y = 1$

(e) $\frac{d^2y}{dt^2} + \sin(t+y) = \sin t$

(f) $t\frac{dy}{dt} + (\cos^2 t)y = t^3 + \frac{d^4y}{dt^4}$

(g) $\frac{dy}{dt} + ty = 0$

(h) $\frac{d^3y}{dt^3} + \sin t = y$

(i) $t^2\frac{d^2y}{dt^2} + t\frac{dy}{dt} + 2y = \sin y$

(j) $\frac{d^3y}{dt^3} - \frac{d^2y}{dt^2} + \frac{dy}{dt} - y = \frac{d^4y}{dt^4} + e^y$.

(k) $y''' + y(y')^3 = 2x$.

(l) $y'' + 2e^{3x}y' + 2y = (x^2+5)^3$.

(m) $\frac{dy}{dx} + 3x^2y = 0$.

(n) $x''(t) - x^2t^2 = 0$.

Solution 4.

(a) first order, linear

(b) second order, non-linear

(c) first order, non-linear

(d) fourth order, linear

(e) second order, non-linear

(f) fourth order, linear

(g) first order, linear

(h) third order, linear

(i) second order, non-linear

(j) fourth order, non-linear

(k) third order, non-linear

(l) second order, linear

(m) first order, linear

(n) second order, non-linear

Exercise 5 (Solutions). Show that the following functions are the solutions of the given differential equations. The first one is done for you.

(ω) $y'' + y' = 1; \qquad y(t) = t.$
solution: Since $y' = 1$ and $y'' = 0$, we have that $y'' + y' = 0 + 1 = 1$ as required.

(a) $x'' - 2x' + x = 0; \qquad x(t) = te^t.$

(b) $y'' + 4y = 0; \qquad y(x) = \cos 2x$

Solution 5.

(a) Note that

$$\begin{aligned} x' &= (1+t)\,e^t \text{ and } x'' = (2+t)\,e^t \Longrightarrow \\ x'' - 2x' + x &= (2+t)\,e^t - 2\,(1+t)\,e^t + te^t = 0. \end{aligned}$$

(b) y'' can be calculated as follows.

$$\begin{aligned} y' &= -2\sin 2x \text{ and } y'' = -4\cos 2x \Longrightarrow \\ y'' + 4y &= -4\cos 2x + 4\cos 2x = 0. \end{aligned}$$

Exercise 6 (Linear Equations). Solve the following ODEs:

(a) $y' + y = 5$

(b) $y' + y = te^{-t} + 1$

(c) $y' - 3y = 4e^t$

(d) $ty' - y = t^2e^{-t}$

(e) $y' + 2ty = 2te^{-t^2}$

(f) $y' + y - 5\sin 2t = 0$

Solution 6.

(a) Note that e^x is the integrating factor. Thus, we get

$$\begin{aligned} \frac{d}{dx}\left(ye^x\right) &= 5e^x \Longrightarrow ye^x = \int 5e^x\,dx + C = 5e^x + C \Longrightarrow \\ y &= Ce^{-x} + 5. \end{aligned}$$

(b) $y = ce^{-t} + 1 + \frac{t^2e^{-t}}{2}$

(c) $y = ce^{3t} - 2e^t$

(d) $y = -te^{-t} + ct$

(e) $y = t^2e^{-t^2} + ce^{-t^2}$

(f) $y = ce^{-t} + \sin 2t - 2\cos 2t$

Exercise 7 (Initial Value Problems). Solve the following IVPs:

(a) $\begin{cases} \frac{dy}{dt} - y = 2te^{2t} \\ y(0) = 1 \end{cases}$

(b) $\begin{cases} y' + 3y = 12 \\ y(0) = 6 \end{cases}$

(c) $\begin{cases} y' + \left(\frac{2}{t}\right)y = \frac{\cos t}{t^2} \\ y(\pi) = 0 \end{cases}$

Solution 7.

(a) Multiplying the ODE by the integrating factor $\mu(t) = e^{-t}$ gives

$$e^{-t}\frac{dy}{dt} - e^{-t}y = 2te^{t}.$$

Hence

$$\frac{d}{dt}\left(e^{-t}y\right) = 2te^{t}.$$

Then

$$e^{-t}y = \int 2te^{t}\ dt = 2te^{t} - \int 2e^{t}\ dt = 2te^{t} - 2e^{t} + C$$

so

$$y = Ce^{t} + 2(t-1)e^{2t}.$$

Finally we use $y(0) = 1$ to find that $C = 3$. Therefore

$$y = 3e^{t} + 2(t-1)e^{2t}.$$

(b) The integrating factor is e^{3x}. Then, we get

$$\begin{aligned} e^{3x}y' + 3ye^{3x} &= 12e^{3x} \Longrightarrow \frac{d}{dx}\left(e^{3x}y\right) = 12e^{3x} \Longrightarrow e^{3x}y = 4e^{3x} + C \\ y &= 4 + Ce^{-3x}. \end{aligned}$$

Since $y(0) = 6$, we get $6 = y(0) = 4 + Ce^{-3(0)} \Longrightarrow C = 2 \Longrightarrow y = 4 + 2e^{-3x}$.

(c) $y(t) = \dfrac{\sin t}{t^2}$.