Lecture 8

- 4.3 Solving More Initial Value Problems
- 4.4 Step Functions

Solving More Initial Value Problems

4.3 Solving More Initial Value Problems

$$\begin{tikzcd}[column sep=huge, row sep=huge]
\boxed{\text{difficult ODE for } y} \arrow[r, "\mathcal{L}"] \arrow[d, dotted, "\text{solve?}"] & \boxed{\text{easy equation for } Y} \arrow[d, "\text{solve}"'] \\
\boxed{y(t)} & \boxed{Y(s)} \arrow[l, "\mathcal{L}^{-1}"']
\end{tikzcd}$$

Theorem

1. $\mathcal{L}[f'](s) = s\mathcal{L}[f](s) - f(0)$.

4.3 Solving More Initial Value Problems

Theorem

1. $\mathcal{L}[f'](s) = s\mathcal{L}[f](s) - f(0)$.
2. $\mathcal{L}[f''](s) = s^2\mathcal{L}[f](s) - sf(0) - f'(0)$.
3. $\mathcal{L}[f'''](s) = s^3\mathcal{L}[f](s) - s^2 f(0) - sf'(0) - f''(0)$.
4. $$\mathcal{L}[f^{(n)}](s) = s^n\mathcal{L}[f](s) - s^{n-1}f(0) - s^{n-2}f'(0) - \dots$$
$$-sf^{(n-2)}(0) - f^{(n-1)}(0).$$

Example

Use the Laplace Transform to solve

$$\begin{cases} y'' - 3y' + 2y = \cos t \\ y(0) = 0 \\ y'(0) = 0. \end{cases}$$

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Taking the Laplace Transform of the ODE gives

$$\begin{aligned} \mathcal{L}[y''] - 3\mathcal{L}[y'] + 2\mathcal{L}[y] &= \mathcal{L}[\cos t] \\ \big(s^2Y - sy(0) - y'(0)\big) - 3\big(sY - y(0)\big) + 2Y &= \frac{s}{s^2+1} \\ (s^2 - 3s + 2)Y &= \frac{s}{s^2+1} \end{aligned}$$

$$Y(s) = \frac{s}{(s^2+1)(s^2-3s+2)} = \frac{s}{(s^2+1)(s-2)(s-1)}$$

$$=$$

$$=$$

$$=$$

$$=$$

$$=$$

$$
\begin{aligned}
Y(s) &= \frac{s}{(s^2+1)(s^2-3s+2)} = \frac{s}{(s^2+1)(s-2)(s-1)} \\
&= \frac{As+B}{s^2+1} + \frac{C}{s-2} + \frac{D}{s-1} \\
&= \\
&= \\
&= \\
&=
\end{aligned}
$$

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\\
&= \\
\\
&= \\
\\
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&\quad \left(A = \tfrac{1}{10},\ B = -\tfrac{3}{10},\ C = \tfrac{2}{5},\ D = -\tfrac{1}{2}\right) \\
&= \\
&= \\
&=
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$$
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&= \frac{\frac{1}{10}s - \frac{3}{10}}{s^2+1} + \frac{\frac{2}{5}}{s-2} - \frac{\frac{1}{2}}{s-1} \\
&= \\
&=
\end{aligned}
$$

$$
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&= \frac{\frac{1}{10}s - \frac{3}{10}}{s^2+1} + \frac{\frac{2}{5}}{s-2} - \frac{\frac{1}{2}}{s-1} \\
&= \frac{1}{10}\left(\frac{s}{s^2+1}\right) - \frac{3}{10}\left(\frac{1}{s^2+1}\right) + \frac{2}{5}\left(\frac{1}{s-2}\right) - \frac{1}{2}\left(\frac{1}{s-1}\right) \\
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&= \frac{1}{10}\mathcal{L}\left[\cos t\right] - \frac{3}{10}\mathcal{L}\left[\sin t\right] + \frac{2}{5}\mathcal{L}\left[e^{2t}\right] - \frac{1}{2}\mathcal{L}\left[e^{t}\right].
\end{aligned}
$$

$$Y(s) = \frac{1}{10}\mathcal{L}\left[\cos t\right] - \frac{3}{10}\mathcal{L}\left[\sin t\right] + \frac{2}{5}\mathcal{L}\left[e^{2t}\right] - \frac{1}{2}\mathcal{L}\left[e^{t}\right]$$

Therefore the solution to the IVP is

$$y(t) = \mathcal{L}^{-1}\left[Y\right](t) =$$

$$Y(s) = \frac{1}{10}\mathcal{L}\left[\cos t\right] - \frac{3}{10}\mathcal{L}\left[\sin t\right] + \frac{2}{5}\mathcal{L}\left[e^{2t}\right] - \frac{1}{2}\mathcal{L}\left[e^{t}\right]$$

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$$\boxed{y(t) = \mathcal{L}^{-1}\left[Y\right](t) = \frac{1}{10}\cos t - \frac{3}{10}\sin t + \frac{2}{5}e^{2t} - \frac{1}{2}e^{t}.}$$

$$Y(s) = \frac{1}{10}\mathcal{L}[\cos t] - \frac{3}{10}\mathcal{L}[\sin t] + \frac{2}{5}\mathcal{L}\left[e^{2t}\right] - \frac{1}{2}\mathcal{L}\left[e^{t}\right]$$

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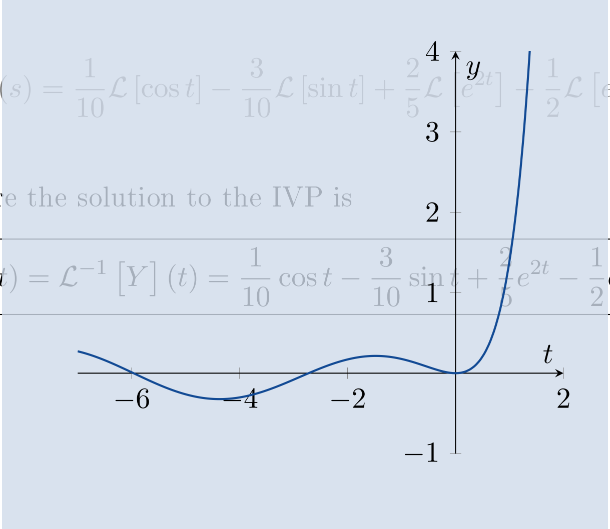

Example

Use the Laplace Transform to solve

$$\begin{cases} y'' + 2y' + y = 4e^{-t} \\ y(0) = 2 \\ y'(0) = -1. \end{cases}$$

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Example

Use the Laplace Transform to solve

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$$y'' \quad + 2 \quad y' \quad + \quad y \quad = \quad 4e^{-t}$$

Example

Use the Laplace Transform to solve

$$\begin{cases} y'' + 2y' + y = 4e^{-t} \\ y(0) = 2 \\ y'(0) = -1. \end{cases}$$

$$\mathcal{L}\left[y''\right] + 2\mathcal{L}\left[y'\right] + \mathcal{L}\left[y\right] = \mathcal{L}\left[4e^{-t}\right]$$

Example

Use the Laplace Transform to solve

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$$\mathcal{L}\left[y''\right] + 2\mathcal{L}\left[y'\right] + Y = \frac{4}{s+1}$$

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Use the Laplace Transform to solve

$$\begin{cases} y'' + 2y' + y = 4e^{-t} \\ y(0) = 2 \\ y'(0) = -1. \end{cases}$$

$$\mathcal{L}\left[y''\right] + 2(sY - y(0)) + Y = \frac{4}{s+1}$$

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Use the Laplace Transform to solve

$$\begin{cases} y'' + 2y' + y = 4e^{-t} \\ y(0) = 2 \\ y'(0) = -1. \end{cases}$$

$$\left(s^2Y - sy(0) - y'(0)\right) + 2(sY - y(0)) + Y = \frac{4}{s+1}$$

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Example

Use the Laplace Transform to solve

$$\begin{cases} y'' + 2y' + y = 4e^{-t} \\ y(0) = 2 \\ y'(0) = -1. \end{cases}$$

$$(s^2Y - 2s + 1) + 2(sY - 2) + Y = \frac{4}{s+1}$$

Example

Use the Laplace Transform to solve

$$\begin{cases} y'' + 2y' + y = 4e^{-t} \\ y(0) = 2 \\ y'(0) = -1. \end{cases}$$

$$(s^2 + 2s + 1)Y - 2s + 1 - 4 = \frac{4}{s+1}$$

Example

Use the Laplace Transform to solve

$$\begin{cases} y'' + 2y' + y = 4e^{-t} \\ y(0) = 2 \\ y'(0) = -1. \end{cases}$$

$$(s^2 + 2s + 1)Y = \frac{4}{s+1} + 2s + 3$$

Example

Use the Laplace Transform to solve

$$\begin{cases} y'' + 2y' + y = 4e^{-t} \\ y(0) = 2 \\ y'(0) = -1. \end{cases}$$

$$(s+1)^2 Y = \frac{4}{s+1} + 2s + 3$$

Example

Use the Laplace Transform to solve

$$\begin{cases} y'' + 2y' + y = 4e^{-t} \\ y(0) = 2 \\ y'(0) = -1. \end{cases}$$

$$(s+1)^2 Y = \frac{2s^2 + 5s + 7}{s+1}$$

Example

Use the Laplace Transform to solve

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Example

Use the Laplace Transform to solve

$$\begin{cases} y'' + 2y' + y = 4e^{-t} \\ y(0) = 2 \\ y'(0) = -1. \end{cases}$$

$$y(t) = \mathcal{L}^{-1}\left[\frac{2s^2 + 5s + 7}{(s+1)^3}\right]$$

I leave it for you to check that if

$$\frac{2s^2 + 5s + 7}{(s+1)^3} = \frac{A}{s+1} + \frac{B}{(s+1)^2} + \frac{C}{(s+1)^3}$$

then $A = 2$, $B = 1$ and $C = 4$.

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Thus

$$\frac{2s^2+5s+7}{(s+1)^3} = \frac{2}{s+1} + \frac{1}{(s+1)^2} + \frac{4}{(s+1)^3}$$

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$$\begin{aligned}\frac{2s^2+5s+7}{(s+1)^3} &= \frac{2}{s+1} + \frac{1}{(s+1)^2} + \frac{4}{(s+1)^3} \\ &= 2\left(\frac{1}{s+1}\right) + \left(\frac{1}{(s+1)^2}\right) + 2\left(\frac{2}{(s+1)^3}\right).\end{aligned}$$

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$$\frac{2s^2+5s+7}{(s+1)^3} = \frac{A}{s+1} + \frac{B}{(s+1)^2} + \frac{C}{(s+1)^3}$$

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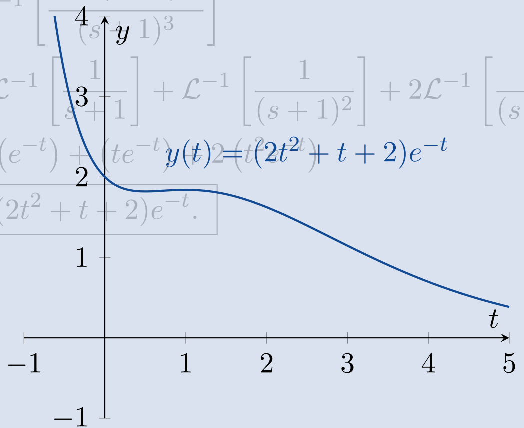
In our table of Laplace Transforms, we find that $\mathcal{L}\left[e^{-t}\right] = \frac{1}{s+1}$, $\mathcal{L}\left[te^{-t}\right] = \frac{1}{(s+1)^2}$ and $\mathcal{L}\left[t^2e^{-t}\right] = \frac{2}{(s+1)^3}$.

Therefore the solution to the IVP is

$$\begin{aligned} y(t) &= \mathcal{L}^{-1}\left[\frac{2s^2+5s+7}{(s+1)^3}\right] \\ &= 2\mathcal{L}^{-1}\left[\frac{1}{s+1}\right] + \mathcal{L}^{-1}\left[\frac{1}{(s+1)^2}\right] + 2\mathcal{L}^{-1}\left[\frac{2}{(s+1)^3}\right] \\ &= 2\left(e^{-t}\right) + \left(te^{-t}\right) + 2\left(t^2e^{-t}\right) \\ &= \boxed{\ (2t^2+t+2)e^{-t}.\ } \end{aligned}$$

Therefore the solution to the IVP is

$$
\begin{aligned}
y(t) &= \mathcal{L}^{-1}\left[\frac{2s^2+5s+7}{(s+1)^3}\right] \\
&= 2\mathcal{L}^{-1}\left[\frac{1}{s+1}\right] + \mathcal{L}^{-1}\left[\frac{1}{(s+1)^2}\right] + 2\mathcal{L}^{-1}\left[\frac{2}{(s+1)^3}\right] \\
&= 2\left(e^{-t}\right) + \left(te^{-t}\right) + 2\left(t^2e^{-t}\right) \\
&= \boxed{(2t^2+t+2)e^{-t}.}
\end{aligned}
$$



Example

Use the Laplace Transform to solve

$$\begin{cases} y^{(4)} + 2y'' + y = e^{2t} \\ y(0) = 1 \\ y'(0) = 1 \\ y''(0) = 1 \\ y'''(0) = 1. \end{cases}$$

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$$y^{(4)} + 2y'' + y = e^{2t} \qquad\qquad y(0) = y'(0) = y''(0) = y^{(3)}(0) = 1$$

$$y^{(4)} + 2y'' + y = e^{2t} \qquad\qquad y(0) = y'(0) = y''(0) = y^{(3)}(0) = 1$$

Taking the Laplace Transform of the ODE gives

$$\mathcal{L}\left[y^{(4)}\right] + 2\mathcal{L}\left[y''\right] + \mathcal{L}\left[y\right] = \mathcal{L}\left[e^{2t}\right].$$

$$y^{(4)} + 2y'' + y = e^{2t} \qquad y(0) = y'(0) = y''(0) = y^{(3)}(0) = 1$$

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$$\mathcal{L}\left[y^{(4)}\right] + 2\mathcal{L}\left[y''\right] + \mathcal{L}\left[y\right] = \mathcal{L}\left[e^{2t}\right].$$

Thus

$$\left(s^4Y(s) - s^3y(0) - s^2y'(0) - sy''(0) - y^{(3)}(0)\right)$$
$$+2\left(s^2Y(s) - sy(0) - y'(0)\right) + Y(s) = \frac{1}{s-2}$$

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$$y^{(4)} + 2y'' + y = e^{2t} \qquad y(0) = y'(0) = y''(0) = y^{(3)}(0) = 1$$

Taking the Laplace Transform of the ODE gives

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Thus

$$\begin{aligned}\Big(s^4Y(s) - s^3y(0) - s^2y'(0) - sy''(0) - y^{(3)}(0)\Big)& \\ +2\big(s^2Y(s) - sy(0) - y'(0)\big) + Y(s) &= \frac{1}{s-2}\end{aligned}$$

and

$$\big(s^4Y(s) - s^3 - s^2 - s - 1\big) + 2\big(s^2Y(s) - s - 1\big) + Y(s) = \frac{1}{s-2}.$$

Thus

$$\left(s^4 + 2s^2 + 1\right) Y(s) - s^3 - s^2 - s - 1 - 2s - 2 = \frac{1}{s-2}.$$

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Thus

$$\left(s^4 + 2s^2 + 1\right) Y(s) - s^3 - s^2 - s - 1 - 2s - 2 = \frac{1}{s-2}.$$

Hence

$$\left(s^4 + 2s^2 + 1\right) Y(s) = \frac{1}{s-2} + s^3 + s^2 + 3s + 3$$

Thus

$$\left(s^4 + 2s^2 + 1\right) Y(s) - s^3 - s^2 - s - 1 - 2s - 2 = \frac{1}{s-2}.$$

Hence

$$\begin{aligned}
\left(s^4 + 2s^2 + 1\right) Y(s) &= \frac{1}{s-2} + s^3 + s^2 + 3s + 3 \\
&= \frac{1}{s-2} + \frac{s^4 - 2s^3}{s-2} + \frac{s^3 - 2s^2}{s-2} \\
&\qquad + \frac{3s^2 - 6s}{s-2} + \frac{3s - 6}{s-2}
\end{aligned}$$

Thus

$$\left(s^4+2s^2+1\right)Y(s)-s^3-s^2-s-1-2s-2=\frac{1}{s-2}.$$

Hence

$$\begin{aligned}\left(s^4+2s^2+1\right)Y(s)&=\frac{1}{s-2}+s^3+s^2+3s+3\\&=\frac{1}{s-2}+\frac{s^4-2s^3}{s-2}+\frac{s^3-2s^2}{s-2}\\&\qquad+\frac{3s^2-6s}{s-2}+\frac{3s-6}{s-2}\\&=\frac{s^4-s^3+s^2-3s-5}{s-2}.\end{aligned}$$

$$Y(s) = \frac{s^4 - s^3 + s^2 - 3s - 5}{(s-2)\left(s^4 + 2s^2 + 1\right)} = \frac{s^4 - s^3 + s^2 - 3s - 5}{(s-2)\left(s^2+1\right)^2}$$

$$=$$

$$=$$

$$
\begin{aligned}
Y(s) &= \frac{s^4 - s^3 + s^2 - 3s - 5}{(s-2)\left(s^4 + 2s^2 + 1\right)} = \frac{s^4 - s^3 + s^2 - 3s - 5}{(s-2)\left(s^2+1\right)^2} \\
&= \frac{\frac{1}{25}}{s-2} + \frac{\frac{24}{25}s + \frac{23}{25}}{s^2+1} + \frac{\frac{9}{5}s + \frac{8}{5}}{\left(s^2+1\right)^2} \\
&=
\end{aligned}
$$

$$
\begin{aligned}
Y(s) &= \frac{s^4 - s^3 + s^2 - 3s - 5}{(s-2)\left(s^4 + 2s^2 + 1\right)} = \frac{s^4 - s^3 + s^2 - 3s - 5}{(s-2)\left(s^2+1\right)^2} \\
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&= \frac{1}{25}\left(\frac{1}{s-2}\right) + \frac{24}{25}\left(\frac{s}{s^2+1}\right) + \frac{23}{25}\left(\frac{1}{s^2+1}\right) \\
&\qquad + \frac{9}{10}\left(\frac{2s}{\left(s^2+1\right)^2}\right) + \frac{4}{5}\left(\frac{2}{\left(s^2+1\right)^2}\right).
\end{aligned}
$$

$$
\begin{aligned}
Y(s) &= \frac{s^4 - s^3 + s^2 - 3s - 5}{(s-2)\left(s^4 + 2s^2 + 1\right)} = \frac{s^4 - s^3 + s^2 - 3s - 5}{(s-2)\left(s^2+1\right)^2} \\
&= \frac{\frac{1}{25}}{s-2} + \frac{\frac{24}{25}s + \frac{23}{25}}{s^2+1} + \frac{\frac{9}{5}s + \frac{8}{5}}{\left(s^2+1\right)^2} \\
&= \frac{1}{25}\left(\frac{1}{s-2}\right) + \frac{24}{25}\left(\frac{s}{s^2+1}\right) + \frac{23}{25}\left(\frac{1}{s^2+1}\right) \\
&\qquad + \frac{9}{10}\left(\frac{2s}{\left(s^2+1\right)^2}\right) + \frac{4}{5}\left(\frac{2}{\left(s^2+1\right)^2}\right).
\end{aligned}
$$

Now $\mathcal{L}\left[e^{2t}\right] = \frac{1}{s-2}$, $\mathcal{L}\left[\cos t\right] = \frac{s}{s^2+1}$ and $\mathcal{L}\left[\sin t\right] = \frac{1}{s^2+1}$. But what do we do with $\frac{2s}{\left(s^2+1\right)^2}$ and $\frac{2}{\left(s^2+1\right)^2}$?

$$
\begin{aligned}
Y(s) &= \frac{s^4 - s^3 + s^2 - 3s - 5}{(s-2)\left(s^4 + 2s^2 + 1\right)} = \frac{s^4 - s^3 + s^2 - 3s - 5}{(s-2)\left(s^2+1\right)^2} \\
&= \frac{\frac{1}{25}}{s-2} + \frac{\frac{24}{25}s + \frac{23}{25}}{s^2+1} + \frac{\frac{9}{5}s + \frac{8}{5}}{\left(s^2+1\right)^2} \\
&= \frac{1}{25}\left(\frac{1}{s-2}\right) + \frac{24}{25}\left(\frac{s}{s^2+1}\right) + \frac{23}{25}\left(\frac{1}{s^2+1}\right) \\
&\qquad + \frac{9}{10}\left(\frac{2s}{\left(s^2+1\right)^2}\right) + \frac{4}{5}\left(\frac{2}{\left(s^2+1\right)^2}\right).
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$$\mathcal{L}^{-1}\left[\frac{2s}{\left(s^2+1\right)^2}\right] = ? \qquad \mathcal{L}^{-1}\left[\frac{2}{\left(s^2+1\right)^2}\right] = ?$$

Remember that $\mathcal{L}\left[tf(t)\right] = -F'(s)$.

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$$\mathcal{L}^{-1}\left[\frac{2s}{(s^2+1)^2}\right] = ? \qquad \mathcal{L}^{-1}\left[\frac{2}{(s^2+1)^2}\right] = ?$$

Remember that $\mathcal{L}\left[tf(t)\right] = -F'(s)$. Hence

$$\mathcal{L}\left[t\sin t\right] =$$

and

$$\mathcal{L}\left[t\cos t\right] =$$

4.3 Solving More Initial Value Problems

$$\mathcal{L}^{-1}\left[\frac{2s}{\left(s^2+1\right)^2}\right] = ? \qquad \mathcal{L}^{-1}\left[\frac{2}{\left(s^2+1\right)^2}\right] = ?$$

Remember that $\mathcal{L}\left[tf(t)\right] = -F'(s)$. Hence

$$\mathcal{L}\left[t\sin t\right] = -\frac{d}{ds}\mathcal{L}\left[\sin t\right]$$

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$$\mathcal{L}\left[\sin t - t\cos t\right] = \frac{1}{s^2+1} - \left(\frac{1}{s^2+1} - \frac{2}{(s^2+1)^2}\right) = \frac{2}{(s^2+1)^2}.$$

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Recall that

$$Y(s) = \frac{1}{25}\left(\frac{1}{s-2}\right) + \frac{24}{25}\left(\frac{s}{s^2+1}\right) + \frac{23}{25}\left(\frac{1}{s^2+1}\right) + \frac{9}{10}\left(\frac{2s}{\left(s^2+1\right)^2}\right) + \frac{4}{5}\left(\frac{2}{\left(s^2+1\right)^2}\right).$$

4.3 Solving More Initial Value Problems

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Hence

$$\boxed{y(t) = \frac{1}{25}\left(e^{2t} + 24\cos t + 23\sin t\right) + \frac{9}{10}t\sin t + \frac{4}{5}(\sin t - t\cos t)}$$

is the solution to the IVP.

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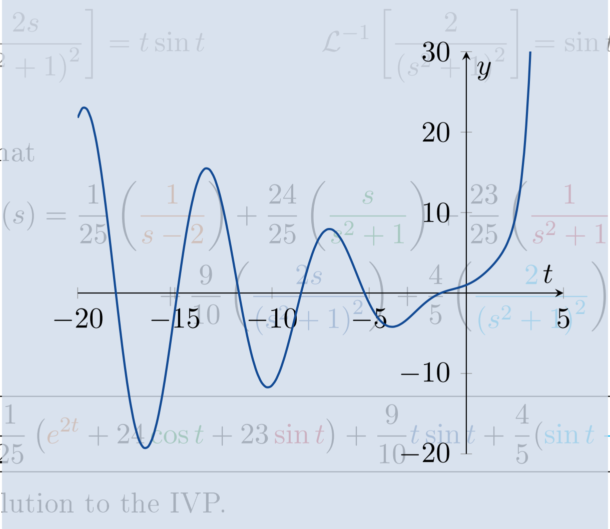


4.4

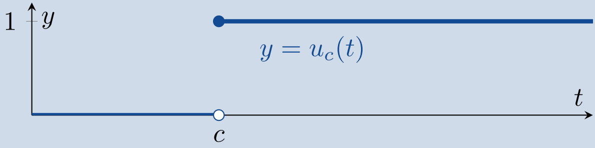
Step Functions

Definition

The *unit step function* $u_c : [0, \infty) \to \mathbb{R}$ is defined by

$$u_c(t) = \begin{cases} 0 & t < c \\ 1 & t \geq c \end{cases}$$

for $c \geq 0$.



Example

Draw the graph of $y = 1 - u_c(t)$.

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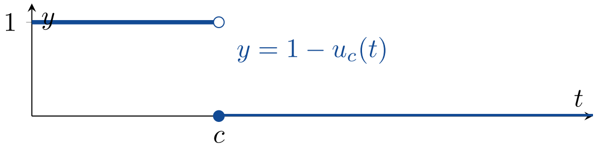

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$$u_1(t) - u_2(t) = \begin{cases} u_1(t) - u_2(t) & 0 \leq t < 1 \\ u_1(t) - u_2(t) & 1 \leq t < 2 \\ u_1(t) - u_2(t) & 2 \leq t \end{cases}$$

4.4 Step Functions

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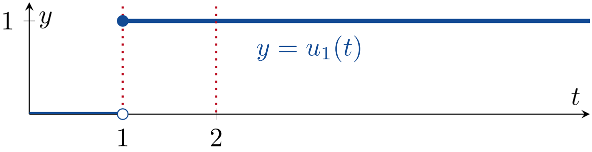

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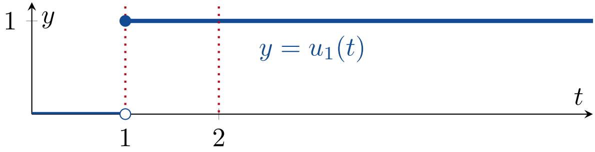
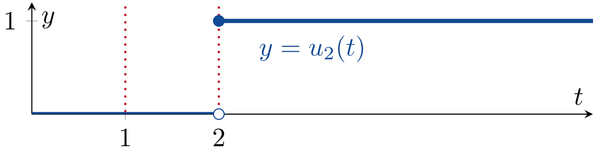


$$u_1(t) - u_2(t) = \begin{cases} 0 - u_2(t) & 0 \le t < 1 \\ 1 - u_2(t) & 1 \le t < 2 \\ 1 - u_2(t) & 2 \le t \end{cases}$$

4.4 Step Functions



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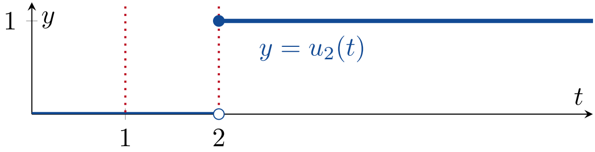

$$u_1(t) - u_2(t) = \begin{cases} 0 - 0 & 0 \leq t < 1 \\ 1 - 0 & 1 \leq t < 2 \\ 1 - 1 & 2 \leq t \end{cases}$$

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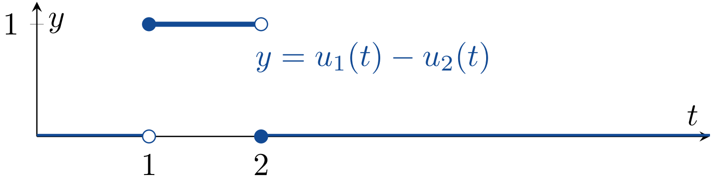
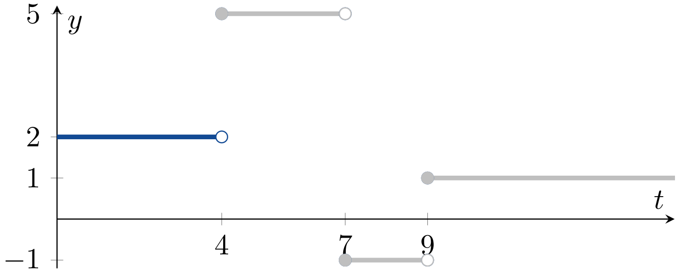


Example

Write the function

$$f(t) = \begin{cases} 2 & 0 \leq t < 4 \\ 5 & 4 \leq t < 7 \\ -1 & 7 \leq t < 9 \\ 1 & 9 \leq t \end{cases}$$

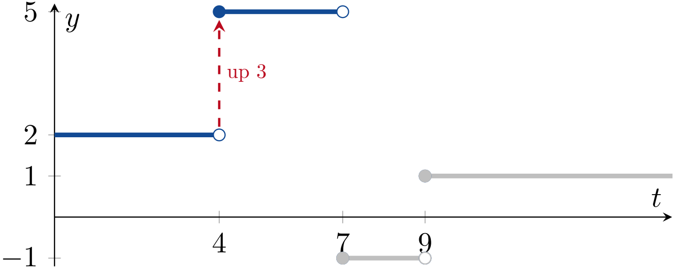
in terms of the unit step function.



The function starts at $f(0) = 2$. So we will have

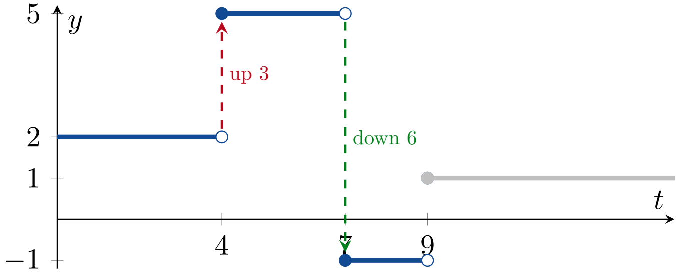
$$f(t) = 2 + (\text{something}).$$

4.4 Step Functions



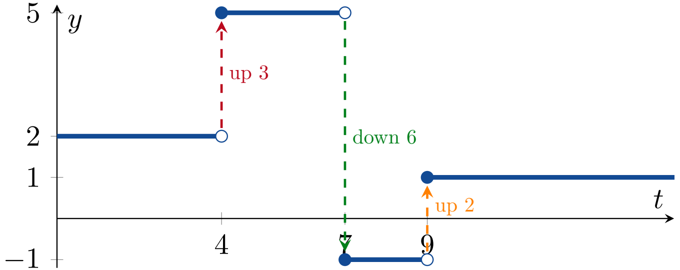
At $t = 4$, the function jumps from 2 to 5 (it goes "up 3"). So

$$f(t) = 2 + 3u_4(t) + (\text{something}).$$



Then it goes "down 6" when $t = 7$. So

$$f(t) = 2 + 3u_4(t) - 6u_7(t) + (\text{something}).$$



Finally it goes "up 2" when $t = 9$. Therefore

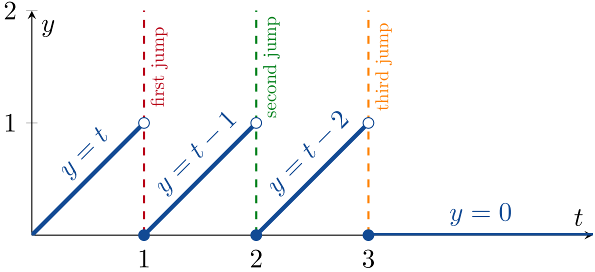
$$f(t) = 2 + 3u_4(t) - 6u_7(t) + 2u_9(t).$$

Example

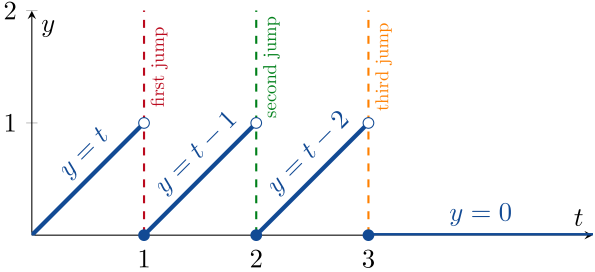
Write the function

$$f(t) = \begin{cases} t & 0 \leq t < 1 \\ t-1 & 1 \leq t < 2 \\ t-2 & 2 \leq t < 3 \\ 0 & 3 \leq t \end{cases}$$

in terms of the unit step function.

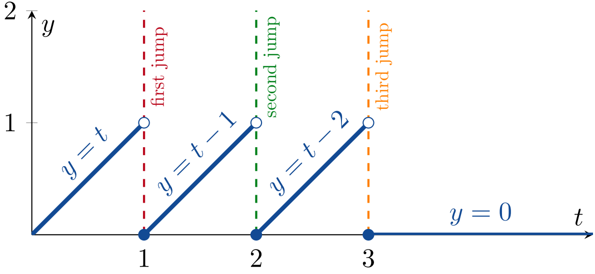


This function starts with $f(t) = t$, then changes when $t = 1$, $t = 2$ and $t = 3$:



This function starts with $f(t) = t$, then changes when $t = 1$, $t = 2$ and $t = 3$: So we must have

$$f(t) = t + \begin{pmatrix} \text{first} \\ \text{jump} \end{pmatrix} u_1(t) + \begin{pmatrix} \text{second} \\ \text{jump} \end{pmatrix} u_2(t) + \begin{pmatrix} \text{third} \\ \text{jump} \end{pmatrix} u_3(t).$$



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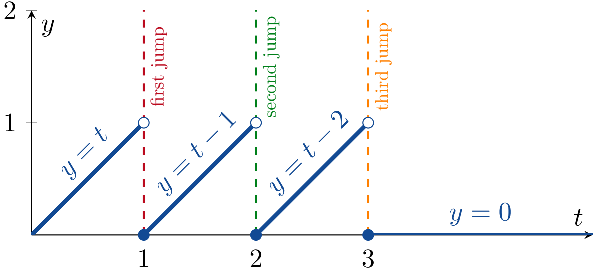
At each "jump" we calculate

$$\text{jump} = \begin{pmatrix} \text{function} \\ \text{on right} \end{pmatrix} - \begin{pmatrix} \text{function} \\ \text{on left} \end{pmatrix}.$$



So we have

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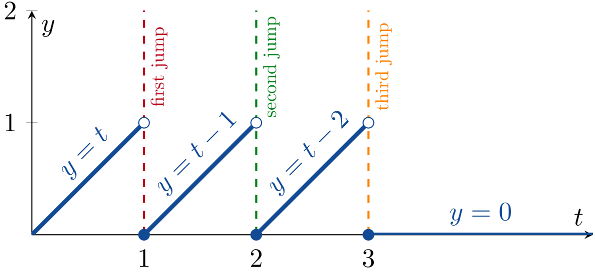

So we have

$$\begin{pmatrix}\text{first}\\ \text{jump}\end{pmatrix} = (t-1) - t = -1$$

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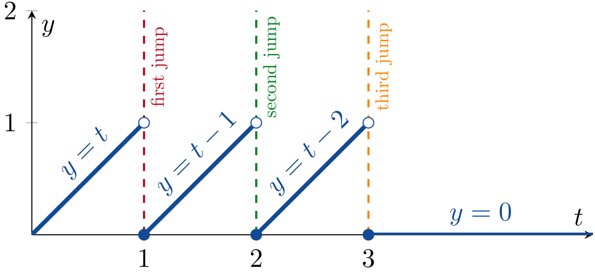

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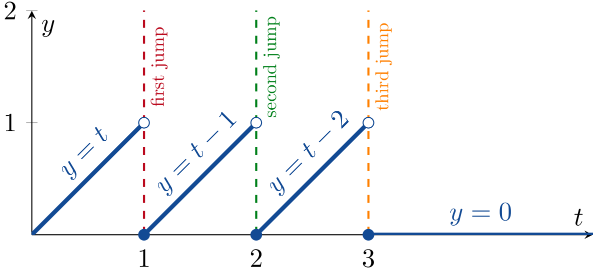

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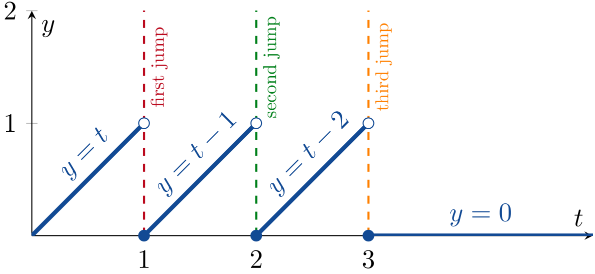
$$\begin{pmatrix}\text{third}\\\text{jump}\end{pmatrix} = 0 - (t-2) = 2-t$$



Hence

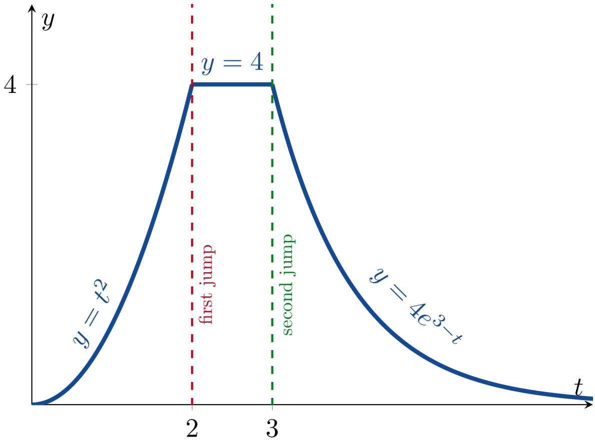
$$f(t) = t - u_1(t) - u_2(t) + (2 - t)u_3(t).$$

Example

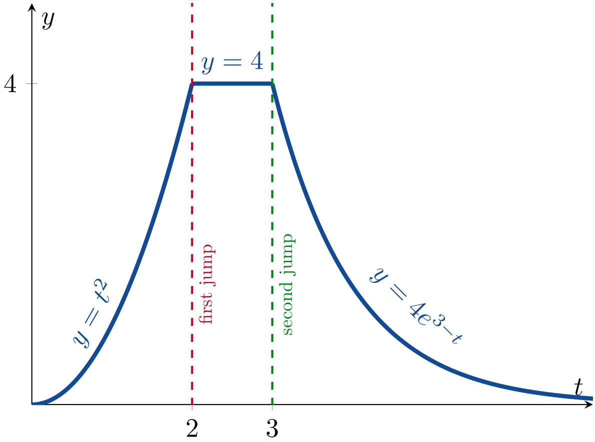
Write the function

$$f(t) = \begin{cases} t^2 & 0 \le t < 2 \\ 4 & 2 \le t < 3 \\ 4e^{t-3} & 3 \le t \end{cases}$$

in terms of the unit step function.

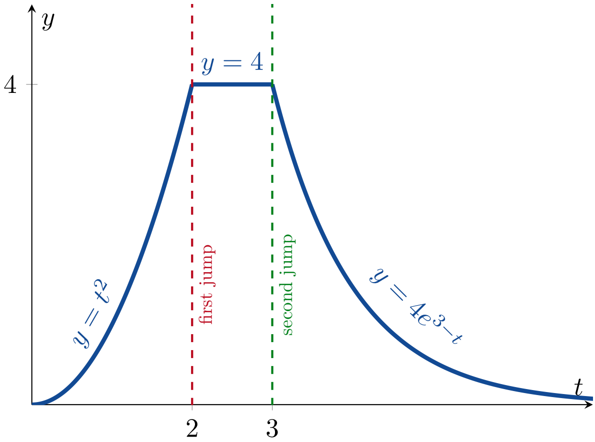


y
4
$y = 4$
$y = t^2$
first jump
second jump
$y = 4e^{3-t}$
t
2
3



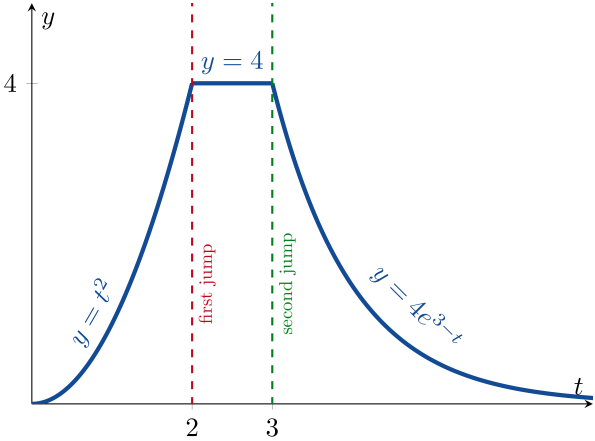
$$f(t) = t^2 + \begin{pmatrix} \text{first} \\ \text{jump} \end{pmatrix} u_2(t) + \begin{pmatrix} \text{second} \\ \text{jump} \end{pmatrix} u_3(t).$$

4.4 Step Functions



$$f(t) = t^2 + (4 - t^2)u_2(t) + \begin{pmatrix}\text{second}\\ \text{jump}\end{pmatrix} u_3(t).$$

4.4 Step Functions



$$f(t) = t^2 + \left(4 - t^2\right)u_2(t) + \left(4e^{t-3} - 4\right)u_3(t).$$

What is the Laplace Transform of the unit step function?

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We calculate that

$$\mathcal{L}\left[u_c\right](s) = \int_0^\infty e^{-st} u_c(t)\,dt =$$

$$= \qquad\qquad =$$

for $s > 0$.

What is the Laplace Transform of the unit step function?

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for $s > 0$.

What is the Laplace Transform of the unit step function?

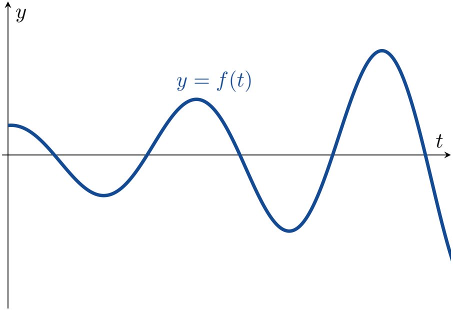
We calculate that

$$\mathcal{L}\left[u_c\right](s) = \int_0^\infty e^{-st} u_c(t)\,dt = \int_0^c e^{-st} 0\,dt + \int_c^\infty e^{-st} 1\,dt$$
$$= \int_c^\infty e^{-st}\,dt = \left[-\frac{1}{s}e^{-st}\right]_c^\infty = \frac{e^{-cs}}{s}$$

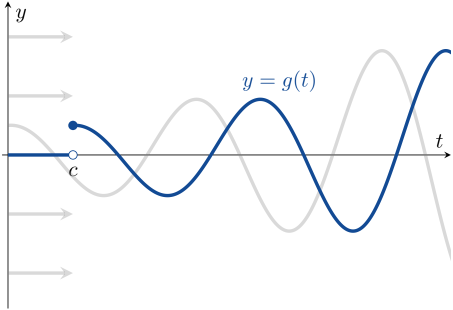
for $s > 0$.

Theorem

$$\mathcal{L}\left[u_c\right](s) = \frac{e^{-cs}}{s}$$



Now suppose that we have some function $f : [0, \infty) \to \mathbb{R}$



Now suppose that we have some function $f : [0, \infty) \to \mathbb{R}$ and we define a new function $g : [0, \infty) \to \mathbb{R}$ by

$$g(t) = \begin{cases} 0 & t < c \\ f(t-c) & t \geq c. \end{cases}$$

We can write $g(t) = u_c(t) f(t-c)$.

4.4 Step Functions

What is the Laplace Transform of $g(t) = u_c(t) f(t - c)$?

4.4 Step Functions

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$$\mathcal{L}\left[g\right] = \mathcal{L}\left[u_c(t)f(t-c)\right]$$

4.4 Step Functions

What is the Laplace Transform of $g(t) = u_c(t)f(t-c)$?

$$\mathcal{L}\left[g\right] = \mathcal{L}\left[u_c(t)f(t-c)\right] = \int_0^\infty e^{-st}u_c(t)f(t-c)\,dt$$

What is the Laplace Transform of $g(t) = u_c(t) f(t-c)$?

$$\begin{aligned}\mathcal{L}\left[g\right] = \mathcal{L}\left[u_c(t) f(t-c)\right] &= \int_0^\infty e^{-st} u_c(t) f(t-c)\, dt \\ &= \int_c^\infty e^{-st} f(t-c)\, dt.\end{aligned}$$

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Let $u = t - c$.

What is the Laplace Transform of $g(t) = u_c(t)f(t-c)$?

$$
\begin{aligned}
\mathcal{L}\left[g\right] = \mathcal{L}\left[u_c(t)f(t-c)\right] &= \int_0^\infty e^{-st}u_c(t)f(t-c)\,dt \\
&= \int_c^\infty e^{-st}f(t-c)\,dt.
\end{aligned}
$$

Let $u = t - c$. Then $du = dt$ and $t = c \iff u = 0$.

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Let $u = t - c$. Then $du = dt$ and $t = c \iff u = 0$. Therefore

$$\mathcal{L}\left[g\right] = \int_0^\infty e^{-s(u+c)}f(u)\,du$$

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4.4 Step Functions

What is the Laplace Transform of $g(t) = u_c(t) f(t-c)$?

$$\mathcal{L}\left[g\right] = \mathcal{L}\left[u_c(t)f(t-c)\right] = \int_0^\infty e^{-st} u_c(t) f(t-c)\, dt$$
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Let $u = t - c$. Then $du = dt$ and $t = c \iff u = 0$. Therefore

$$\mathcal{L}\left[g\right] = \int_0^\infty e^{-s(u+c)} f(u)\, du = e^{-cs} \int_0^\infty e^{-su} f(u)\, du = e^{-cs} \mathcal{L}\left[f\right].$$

Theorem

$$\mathcal{L}\left[u_c(t)f(t-c)\right](s) = e^{-cs} F(s)$$

Example

Find the Laplace Transform of

$$f(t) = \begin{cases} t & 0 \leq t < 1 \\ t-1 & 1 \leq t < 2 \\ t-2 & 2 \leq t < 3 \\ 0 & 3 \leq t. \end{cases}$$

Since

$$f(t) = t - u_1(t) - u_2(t) + (2-t)u_3(t)$$

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we have that

$$\begin{aligned} F(s) &= \mathcal{L}\left[t\right] - \mathcal{L}\left[u_1\right] - \mathcal{L}\left[u_2\right] - \mathcal{L}\left[u_3\right] - \mathcal{L}\left[u_3(t)(t-3)\right] \\ &= \frac{1}{s^2} - \frac{e^{-s}}{s} - \frac{e^{-2s}}{s} - \frac{e^{-3s}}{s} - \frac{e^{-3s}}{s^2}. \end{aligned}$$

Example

Find the Laplace Transform of

$$f(t) = \begin{cases} \sin t & 0 \leq t \leq \frac{\pi}{4} \\ \sin t + \cos(t - \frac{\pi}{4}) & \frac{\pi}{4} \leq t. \end{cases}$$

Note that $f(t) = \sin t + g(t)$ where

$$g(t) = \begin{cases} 0 & 0 \leq t \leq \frac{\pi}{4} \\ \cos(t - \frac{\pi}{4}) & \frac{\pi}{4} \leq t \end{cases} = u_{\frac{\pi}{4}}(t) \cos\left(t - \frac{\pi}{4}\right).$$

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So

$$F(s) = \mathcal{L}\left[f\right] = \mathcal{L}\left[\sin t\right] + \mathcal{L}\left[u_{\frac{\pi}{4}}(t)\cos\left(t - \frac{\pi}{4}\right)\right]$$

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Find the inverse Laplace Transform of $F(s) = \frac{1-e^{-2s}}{s^2}$.

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$$f(t) = \mathcal{L}^{-1}\left[F\right] = \mathcal{L}^{-1}\left[\frac{1}{s^2}\right] - \mathcal{L}^{-1}\left[\frac{e^{-2s}}{s^2}\right] = t - u_2(t)(t-2)$$

$$= \begin{cases} t & 0 \leq t < 2 \\ 2 & t \geq 2. \end{cases}$$

And what is the Laplace Transform of $e^{ct}f(t)$?

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$$\mathcal{L}\left[e^{ct}f(t)\right] = \int_0^\infty e^{-st}e^{ct}f(t)\,dt = \int_0^\infty e^{-(s-c)t}f(t)\,dt = F(s-c).$$

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Theorem

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Example

Find the inverse Laplace Transform of $G(s) = \dfrac{1}{s^2 - 4s + 5}$.

4.4 Step Functions

$$\mathcal{L}\left[e^{ct}f(t)\right] = F(s-c)$$

Example

Find the inverse Laplace Transform of $G(s) = \dfrac{1}{s^2 - 4s + 5}$.

Note first that

$$G(s) = \frac{1}{s^2 - 4s + 5} = \frac{1}{(s-2)^2 + 1}.$$

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If $F(s) = \frac{1}{s^2+1}$, then we have $G(s) = F(s-2)$.

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$$\mathcal{L}^{-1}\left[F\right] = \mathcal{L}^{-1}\left[\frac{1}{s^2+1}\right] = \sin t.$$

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Therefore

$$g(t) = \mathcal{L}^{-1}\left[G\right] = \mathcal{L}^{-1}\left[F(s-2)\right]$$

$$\mathcal{L}\left[e^{ct}f(t)\right] = F(s-c)$$

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Therefore

$$g(t) = \mathcal{L}^{-1}\left[G\right] = \mathcal{L}^{-1}\left[F(s-2)\right] = e^{2t}\mathcal{L}^{-1}\left[F\right] = e^{2t}\sin t.$$

4.4 Step Functions

How to find the inverse Laplace Transform of $G(s) = \dfrac{ms+n}{as^2+bs+c}$

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$\downarrow$

Does $as^2+bs+c=0$ have roots in $\mathbb{R}$?

4.4 Step Functions

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Factorise

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Complete the square
$$G(s) = \frac{ms+n}{(s-d)^2+e}$$

How to find the inverse Laplace Transform of $G(s) = \dfrac{ms+n}{as^2+bs+c}$

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no

Complete the square
$$G(s) = \frac{ms+n}{(s-d)^2+e}$$

Use the formula
$$\mathcal{L}\left[e^{ct} f(t)\right] = F(s-c)$$

Example

Find the inverse Laplace Transform of $G(s) = \dfrac{30s + 440}{s^2 + 32s + 240}$.

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Example

Find the inverse Laplace Transform of $G(s) = \dfrac{30s + 440}{s^2 + 32s + 240}$.

First note that $s^2 + 32s + 240 = 0$ has roots $s_1 = -12$ and $s_2 = -20$. In fact

$$G(s) = \frac{30s + 440}{s^2 + 32s + 240} = \frac{10}{s + 12} + \frac{20}{s + 20}.$$

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I leave this example for you to finish.

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Find the inverse Laplace Transform of $G(s) = \dfrac{10s + 12}{s^2 + 40s + 420}$.

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Find the inverse Laplace Transform of $G(s) = \dfrac{10s + 12}{s^2 + 40s + 420}$.

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Example

Find the inverse Laplace Transform of $G(s) = \dfrac{10s+12}{s^2+40s+420}$.

Since the roots of $s^2 + 40s + 420 = 0$ are $s = -20 \pm 2i\sqrt{5}$, we must complete the square. You can check that

$$G(s) = \frac{10s+12}{s^2+40s+420} = \frac{10s+12}{(s+20)^2+20}.$$

Now

$$
\begin{aligned}
G(s) &= \frac{10s + 12}{(s+20)^2 + 20} \\
&= 10\left(\frac{s}{(s+20)^2+20}\right) + \frac{12}{\sqrt{20}}\left(\frac{\sqrt{20}}{(s+20)^2+20}\right)
\end{aligned}
$$

Now

$$
\begin{aligned}
G(s) &= \frac{10s+12}{(s+20)^2+20} \\
&= 10\left(\frac{s}{(s+20)^2+20}\right) + \frac{12}{\sqrt{20}}\left(\frac{\sqrt{20}}{(s+20)^2+20}\right) \\
&= 10F(s+20) + \frac{12}{\sqrt{20}}H(s+20)
\end{aligned}
$$

where $F(s) = \frac{s}{s^2+20}$ and $H(s) = \frac{\sqrt{20}}{s^2+20}$.

Now

$$G(s) = \frac{10s + 12}{(s+20)^2 + 20}$$
$$= 10\left(\frac{s}{(s+20)^2+20}\right) + \frac{12}{\sqrt{20}}\left(\frac{\sqrt{20}}{(s+20)^2+20}\right)$$
$$= 10F(s+20) + \frac{12}{\sqrt{20}}H(s+20)$$

where $F(s) = \frac{s}{s^2+20}$ and $H(s) = \frac{\sqrt{20}}{s^2+20}$.

Note that

$$f(t) = \mathcal{L}^{-1}\left[F\right](t) = \cos\sqrt{20}t$$

and

$$h(t) = \mathcal{L}^{-1}\left[H\right](t) = \sin\sqrt{20}t.$$

4.4 Step Functions

$$\mathcal{L}\left[e^{ct}f(t)\right] = F(s-c) \qquad\qquad G(s) = 10F(s+20) + \frac{12}{\sqrt{20}}H(s+20)$$

Therefore

$$g(t) = 10\mathcal{L}^{-1}\left[F(s+20)\right] + \frac{12}{\sqrt{20}}\mathcal{L}^{-1}\left[H(s+20)\right]$$

$$=$$

$$= \qquad .$$

$$\mathcal{L}\left[e^{ct}f(t)\right] = F(s-c) \qquad\qquad G(s) = 10F(s+20) + \frac{12}{\sqrt{20}}H(s+20)$$

Therefore

$$\begin{aligned} g(t) &= 10\mathcal{L}^{-1}\left[F(s+20)\right] + \frac{12}{\sqrt{20}}\mathcal{L}^{-1}\left[H(s+20)\right] \\ &= 10e^{-20t}\mathcal{L}^{-1}\left[F\right] + \frac{12}{\sqrt{20}}e^{-20t}\mathcal{L}^{-1}\left[H\right] \\ &= \qquad\qquad\qquad\qquad\qquad\qquad . \end{aligned}$$

$$\mathcal{L}\left[e^{ct}f(t)\right] = F(s-c) \qquad G(s) = 10F(s+20) + \frac{12}{\sqrt{20}}H(s+20)$$

Therefore

$$\begin{aligned} g(t) &= 10\mathcal{L}^{-1}\left[F(s+20)\right] + \frac{12}{\sqrt{20}}\mathcal{L}^{-1}\left[H(s+20)\right] \\ &= 10e^{-20t}\mathcal{L}^{-1}\left[F\right] + \frac{12}{\sqrt{20}}e^{-20t}\mathcal{L}^{-1}\left[H\right] \\ &= 10e^{-20t}\cos\sqrt{20}t + \frac{12}{\sqrt{20}}e^{-20t}\sin\sqrt{20}t. \end{aligned}$$

Next Time

- 4.5 ODEs with Discontinuous Forcing Functions
- 4.6 The Convolution Integral