Hint for 27(j)-(k): Note that since $\mathcal{L}[tf(t)] = (-1)\frac{dF}{ds}$, we have that $-\mathcal{L}^{-1}\left[\frac{dF}{ds}\right] = tf(t)$ and thus $f(t) = -\frac{1}{t}\mathcal{L}^{-1}\left[\frac{dF}{ds}\right]$.

Exercise 28 (The Laplace Transform)**.** Use the definition $\mathcal{L}[f](s) = \int_0^\infty e^{-st} f(t)\,dt$ to prove that the following identities are true. The first one is done for you.

(ω) $\mathcal{L}[1](s) = \frac{1}{s}$

$$\boxed{\mathcal{L}[1](s) = \int_0^\infty e^{-st}(1)\,dt = \lim_{A\to\infty}\int_0^A e^{-st}(1)\,dt = \lim_{A\to\infty}\left[-\frac{e^{-st}}{s}\right]_0^A = \lim_{A\to\infty}\left(-\frac{e^{-sA}}{s} + \frac{e^0}{s}\right) = \frac{1}{s}.}$$

(a) $\mathcal{L}[t^2](s) = \frac{2}{s^3}$ for $s > 0$

(b) $\mathcal{L}[\cos at](s) = \frac{s}{s^2+a^2}$ for $s > 0$

(c) $\mathcal{L}[\sinh at](s) = \frac{a}{s^2-a^2}$ for $s > a$

(d) $\mathcal{L}[\cosh at](s) = \frac{s}{s^2-a^2}$ for $s > a$

(e) $\mathcal{L}[f(ct)](s) = \frac{1}{c}\mathcal{L}[f]\left(\frac{s}{c}\right)$

(f) $\frac{d}{ds}\mathcal{L}[f](s) = -\mathcal{L}[tf(t)](s)$

Solution 28.

(a) Using integration by parts, we calculate that if $s > 0$ then

$$\begin{aligned}
\mathcal{L}[t^2](s) &= \int_0^\infty e^{-st}t^2\,dt \\
&= \left[-t^2\frac{e^{-st}}{s}\right]_0^\infty - \int_0^\infty -\frac{e^{-st}}{s}\,2t\,dt \\
&= \frac{2}{s}\int_0^\infty e^{-st}t\,dt \\
&= \frac{2}{s}\left(\left[-t\frac{e^{-st}}{s}\right]_0^\infty - \int_0^\infty -\frac{e^{-st}}{s}\,1\,dt\right) \\
&= \frac{2}{s^2}\int_0^\infty e^{-st}\,dt \\
&= \frac{2}{s^2}\left[-\frac{e^{-st}}{s}\right]_0^\infty \\
&= \frac{2}{s^3}
\end{aligned}$$

where the notation $[\,\cdot\,]_0^\infty$ means $\lim_{A\to\infty}[\,\cdot\,]_0^A$.

(b) omitted

(c) omitted

(d) For brevity of notation, let $L = \mathcal{L}[\cosh at](s)$. Again using integration by parts, we have that

$$\begin{aligned}
L &= \int_0^\infty e^{-st}\cosh at\,dt \\
&= \left[e^{-st}\frac{1}{a}\sinh at\right]_0^\infty - \int_0^\infty -se^{-st}\frac{1}{a}\sinh at\,dt \\
&= \frac{s}{a}\int_0^\infty e^{-st}\sinh at\,dt \\
&= \frac{s}{a}\left(\left[e^{-st}\frac{1}{a}\cosh at\right]_0^\infty - \int_0^\infty -se^{-st}\frac{1}{a}\cosh at\,dt\right) \\
&= \frac{s}{a}\left(-\frac{1}{a} + \frac{s}{a}\int_0^\infty e^{-st}\cosh at\,dt\right) \\
&= -\frac{s}{a^2} + \frac{s^2}{a^2}L.
\end{aligned}$$

Rearranging this equation gives

$$\begin{aligned}
\frac{s^2}{a^2}L - L &= \frac{s}{a^2} \\
s^2L - a^2L &= s \\
L &= \frac{s}{s^2-a^2}
\end{aligned}$$

as required, if $s > a$.

(e) Let $\xi = ct$. Then $d\xi = c\,dt$ and $dt = \frac{1}{c}\,d\xi$. Thus

$$\begin{aligned}
\mathcal{L}[f(ct)](s) &= \int_0^\infty e^{-st}f(ct)\,dt \\
&= \int_0^\infty e^{-s\frac{\xi}{c}}f(\xi)\,\frac{1}{c}\,d\xi \\
&= \frac{1}{c}\int_0^\infty e^{-\frac{s}{c}\xi}f(\xi)\,d\xi \\
&= \frac{1}{c}\mathcal{L}[f]\left(\frac{s}{c}\right)
\end{aligned}$$

as required.

(f) We calculate that

$$\begin{aligned}
\frac{d}{ds}\mathcal{L}[f](s) &= \frac{d}{ds}\int_0^\infty e^{-st}f(t)\,dt \\
&= \int_0^\infty \frac{d}{ds}e^{-st}f(t)\,dt \\
&= \int_0^\infty -te^{-st}f(t)\,dt \\
&= -\mathcal{L}[tf(t)](s)
\end{aligned}$$

as required.

Exercise 29 (The Laplace Transform). Use the Laplace Transform to solve the following initial value problems:

(a) $\begin{cases} x'' + 4x = 0 \\ x(0) = 5 \\ x'(0) = 0 \end{cases}$

(b) $\begin{cases} x'' - x' - 2x = 0 \\ x(0) = 0 \\ x'(0) = 2 \end{cases}$

(c) $\begin{cases} x'' + 9x = 1 \\ x(0) = 0 \\ x'(0) = 0 \end{cases}$

(d) $\begin{cases} x'' + 6x' + 25x = 0 \\ x(0) = 2 \\ x'(0) = 3 \end{cases}$

(e) $\begin{cases} x'' - 6x' + 8x = 2 \\ x(0) = 0 \\ x'(0) = 0 \end{cases}$

(f) $\begin{cases} x'' - 4x = 3t \\ x(0) = 0 \\ x'(0) = 0 \end{cases}$

(g) $\begin{cases} x'' + 4x' + 8x = e^{-t} \\ x(0) = 0 \\ x'(0) = 0 \end{cases}$

(h) $\begin{cases} x^{(4)} + 8x'' + 16x = 0 \\ x(0) = x'(0) = x''(0) = 0 \\ x^{(3)}(0) = 1 \end{cases}$

(i) $\begin{cases} x^{(4)} + 2x'' + x = e^{2t} \\ x(0) = x'(0) = x''(0) = x^{(3)}(0) = 1 \end{cases}$

(j) $\begin{cases} x^{(3)} + 4x'' + 5x' + 2x = 10\cos t \\ x(0) = x'(0) = 0 \\ x''(0) = 3 \end{cases}$

(k) $\begin{cases} x'' + 4x' + 13x = te^{-t} \\ x(0) = 0 \\ x'(0) = 2 \end{cases}$

(l) $\begin{cases} x'' + x = \sin 2t \\ x(\frac{\pi}{2}) = 2 \\ x'(\frac{\pi}{2}) = 0 \end{cases}$

Solution 29.

(a) We calculate that

$$\begin{aligned} \mathcal{L}\left[x'' + 4x\right] &= \mathcal{L}[0] \\ \left[s^2F(s) - sx(0) - x'(0)\right] + 4F(s) &= 0 \\ \left(s^2+4\right)F(s) - 5s &= 0 \\ F(s) &= \frac{5s}{(s^2+4)} \\ x(t) &= \mathcal{L}^{-1}\left[\frac{5s}{(s^2+4)}\right] = 5\cos 2t \end{aligned}$$

Therefore the solution to the IVP is $x(t) = 5\cos 2t$.

(b)

$$\begin{aligned} \mathcal{L}\left[x'' - x' - 2x\right] &= \mathcal{L}[0] \\ \left[s^2F(s) - sx(0) - x'(0)\right] - \left[sF(s) - x(0)\right] - 2F(s) &= 0 \\ \left(s^2 - s - 2\right)F(s) - 2 &= 0 \\ F(s) &= \frac{2}{(s^2 - s - 2)} \\ x(t) &= \mathcal{L}^{-1}\left[\frac{2}{3(s-2)} - \frac{2}{3\,(s+1)}\right] \\ x(t) &= \frac{2}{3}e^{2t} - \frac{2}{3}e^{-t} \end{aligned}$$

(c)

$$\begin{aligned} \mathcal{L}\left[x'' + 9x\right] &= \mathcal{L}[1] \\ \left[s^2F(s) - sx(0) - x'(0)\right] + 9F(s) &= \frac{1}{s} \\ \left(s^2+9\right)F(s) &= \frac{1}{s} \\ F(s) &= \frac{1}{s\,(s^2+9)} \\ x(t) &= \mathcal{L}^{-1}\left[\frac{1}{9s} - \frac{s}{9\,(s^2+9)}\right] \\ x(t) &= \frac{1}{9} - \frac{1}{9}\cos 3t \end{aligned}$$

(d)

$$\begin{aligned}
\mathcal{L}\left[x''+6x'+25x\right] &= \mathcal{L}\left[0\right]\\
\left[s^2F(s)-sx(0)-x'(0)\right]+6\left[sF(s)-x(0)\right]+25F(s) &= 0\\
\left(s^2+6s+25\right)F(s)-2s-3-12 &= 0\\
F(s) &= \frac{2s+15}{s^2+6s+25}\\
x(t) &= \mathcal{L}^{-1}\left[\frac{2s+15}{(s+3)^2+16}\right]=\\
x(t) &= \mathcal{L}^{-1}\left[\frac{2(s+3)}{(s+3)^2+16}+\frac{9}{(s+3)^2+16}\right]\\
x(t) &= 2e^{-3t}\cos 4t+\frac{9}{4}e^{-3t}\sin 4t
\end{aligned}$$

(e)

$$\begin{aligned}
\mathcal{L}\left[x''-6x'+8x\right] &= \mathcal{L}[2]\\
\left[s^2F(s)-sx(0)-x'(0)\right]-6\left[sF(s)-x(0)\right]+8F(s) &= \frac{2}{s}\\
\left(s^2-6s+8\right)F(s) &= \frac{2}{s}\\
F(s) &= \frac{2}{s\left(s^2-6s+8\right)}\\
x(t) &= \mathcal{L}^{-1}\left[\frac{1}{4(s-4)}-\frac{1}{2(s-2)}+\frac{1}{4s}\right]\\
x(t) &= \frac{1}{4}e^{4t}-\frac{1}{2}e^{2t}+\frac{1}{4}
\end{aligned}$$

(f)

$$\begin{aligned}
\mathcal{L}\left[x''-4x\right] &= \mathcal{L}\left[3t\right]\\
\left[s^2F(s)-sx(0)-x'(0)\right]-4F(s) &= \frac{3}{s^2}\\
\left(s^2-4\right)F(s) &= \frac{3}{s^2}\\
F(s) &= \frac{3}{s^2\left(s^2-4\right)}\\
x(t) &= \mathcal{L}^{-1}\left[\frac{3}{16(s-2)}-\frac{3}{4s^2}-\frac{3}{16(s+2)}\right]\\
x(t) &= \frac{3}{16}e^{2t}-\frac{3}{4}t-\frac{3}{16}e^{-2t}\\
x(t) &= \frac{1}{8}\left(-6t+3\sinh 2t\right)
\end{aligned}$$

(g)

$$\begin{aligned}
\mathcal{L}\left[x''+4x'+8x\right] &= \mathcal{L}\left[e^{-t}\right]\\
\left[s^2F(s)-sx(0)-x'(0)\right]+4\left[sF(s)-x(0)\right]+8F(s) &= \frac{1}{s+1}\\
\left(s^2+4s+8\right)F(s) &= \frac{1}{s+1}\\
F(s) &= \frac{1}{(s+1)\left(s^2+4s+8\right)}\\
x(t) &= \mathcal{L}^{-1}\left[\frac{1}{5}\frac{1}{s+1}-\frac{1}{5}\frac{s+3}{\left(s^2+4s+8\right)}\right]\\
x(t) &= \mathcal{L}^{-1}\left[\frac{1}{5}\frac{1}{s+1}-\frac{1}{5}\frac{s+2}{(s+2)^2+4}-\frac{1}{10}\frac{2}{(s+2)^2+4}\right]\\
x(t) &= \frac{1}{5}e^{-t}-\frac{1}{5}e^{-2t}\cos 2t-\frac{1}{10}e^{-2t}\sin 2t
\end{aligned}$$

(h)

$$\begin{aligned}
\mathcal{L}\left[x^{(4)}+8x''+16x\right] &= \mathcal{L}\left[0\right]\\
\left[s^4F(s)-s^3x(0)-s^2x'(0)-sx''(0)-x^{(3)}(0)\right]+8\left[s^2F(s)-sx(0)-x'(0)\right]+16F(s) &= 0\\
\left(s^4+8s^2+16\right)F(s)-1 &= 0\\
\left(s^4+8s^2+16\right)F(s) &= 1
\end{aligned}$$

This implies that

$$\begin{aligned}
F(s) &= \frac{1}{s^4+8s^2+16}\\
x(t) &= \mathcal{L}^{-1}\left[\frac{1}{(s^2+4)^2}\right]\\
x(t) &= \frac{1}{8}\left(\sin 2t - t\cos 2t\right).
\end{aligned}$$

(i)

$$\begin{aligned}
\mathcal{L}\left[x^{(4)}+2x''+x\right] &= \mathcal{L}\left[e^{2t}\right]\\
\left[s^{(4)}F(s)-s^3x(0)-s^2x'(0)-sx''(0)-x^{(3)}(0)\right]+2\left[s^2F(s)-sx(0)-x'(0)\right]+F(s) &= \frac{1}{s-2}\\
\left(s^4+2s^2+1\right)F(s)-s^3-s^2-s-1-2s-2 &= \frac{1}{s-2}
\end{aligned}$$

$$\begin{aligned}
\left(s^4+2s^2+1\right)F(s) &= \frac{1}{s-2}+s^3+s^2+3s+3=\frac{-3s+s^2-s^3+s^4-5}{s-2}\\
F(s) &= \frac{-3s+s^2-s^3+s^4-5}{(s-2)\left(s^4+2s^2+1\right)}\\
x(t) &= \mathcal{L}^{-1}\left[\frac{-3s+s^2-s^3+s^4-5}{(s-2)\left(s^2+1\right)^2}\right]\\
x(t) &= \mathcal{L}^{-1}\left[\frac{1}{25}\frac{1}{s-2}+\frac{1}{25}\frac{24s+23}{s^2+1}+\frac{1}{5}\frac{9s+8}{\left(s^2+1\right)^2}\right]\\
x(t) &= \frac{1}{25}\left(e^{2t}+24\cos t+23\sin t\right)+\frac{9}{10}t\sin t+\frac{4}{5}(\sin t-t\cos t)
\end{aligned}$$

Therefore $x(t)=\frac{1}{50}\left[2e^{2t}+(48-40t)\cos t+(45t+86)\sin t\right]$ is the solution to the IVP.

(j)

$$\begin{aligned}
\mathcal{L}\left[x^{(3)}+4x''+5x'+2x\right] &= \mathcal{L}\left[10\cos t\right]\\
\left[s^3F(s)-s^2x(0)-sx'(0)-x''(0)\right]+4\left[s^2F(s)-sx(0)-x'(0)\right]+5\left[sF(s)-x(0)\right]+2F(s) &= \frac{10s}{s^2+1}\\
\left(s^3+4s^2+5s+2\right)F(s)-3 &= \frac{10s}{s^2+1}
\end{aligned}$$

$$\begin{aligned}
\left(s^3+4s^2+5s+2\right)F(s) &= \frac{10s}{s^2+1}+3=\frac{3s^2+10s+3}{s^2+1}\\
F(s) &= \frac{3s^2+10s+3}{\left(s^2+1\right)\left(s^3+4s^2+5s+2\right)}\\
x(t) &= \mathcal{L}^{-1}\left[\frac{3s^2+10s+3}{\left(s^2+1\right)\left(s^3+4s^2+5s+2\right)}\right]\\
x(t) &= \mathcal{L}^{-1}\left[\frac{2}{s+1}-\frac{2}{(s+1)^2}-\frac{1}{s+2}-\frac{s}{s^2+1}+\frac{2}{s^2+1}\right]\\
x(t) &= 2e^{-t}-2te^{-t}-e^{-2t}-\cos t+2\sin t
\end{aligned}$$

(k)

$$\begin{aligned}
\mathcal{L}\left[x''+4x'+13x\right] &= \mathcal{L}\left[te^{-t}\right]\\
\left[s^2F(s)-sx(0)-x'(0)\right]+4\left[sF(s)-x(0)\right]+13F(s) &= \frac{1}{(s+1)^2}\\
\left(s^2+4s+13\right)F(s)-2 &= \frac{1}{(s+1)^2}\\
\left(s^2+4s+13\right)F(s) &= \frac{1}{(s+1)^2}+2=\frac{4s+2s^2+3}{(s+1)^2}\\
F(s) &= \frac{4s+2s^2+3}{(s+1)^2\left(s^2+4s+13\right)}
\end{aligned}$$

and

$$\begin{aligned}
\frac{4s+2s^2+3}{(s+1)^2(s^2+4s+13)} &= \frac{A}{s+1}+\frac{B}{(s+1)^2}+\frac{Cs+D}{s^2+4s+13} \\
&\Rightarrow A=\frac{-1}{50}, B=\frac{1}{10}, C=\frac{1}{50}, D=\frac{98}{50} \\
x(t) &= \mathcal{L}^{-1}\left[-\frac{1}{50}\frac{1}{s+1}+\frac{1}{10}\frac{1}{(s+1)^2}+\frac{1}{50}\frac{s+2}{(s+2)^2+9}+\frac{32}{50}\frac{3}{(s+2)^2+9}\right] \\
x(t) &= -\frac{1}{50}e^{-t}+\frac{1}{10}te^{-t}+\frac{1}{50}e^{-2t}\cos 3t+\frac{32}{50}e^{-2t}\sin 3t
\end{aligned}$$

(l) If we use the substitution $k=t-\frac{\pi}{2}$ then we have

$$\begin{aligned}
x''+x &= \sin(2k+\pi) \\
x''+x &= -\sin 2k \\
\mathcal{L}[x''+x] &= -\mathcal{L}[\sin 2k] \\
s^2F(s)-sx(0)-x'(0)+F(s) &= -\frac{2}{s^2+4} \\
(s^2+1)F(s)-2s &= -\frac{2}{s^2+4} \\
(s^2+1)F(s) &= -\frac{2}{s^2+4}+2s=\frac{2s^3+8s-2}{s^2+4} \\
F(s) &= \frac{2s^3+8s-2}{(s^2+1)(s^2+4)}
\end{aligned}$$

By using partial fractions we obtain

$$\begin{aligned}
F(s) &= \frac{2s^3+8s-2}{(s^2+1)(s^2+4)}=\frac{As+B}{s^2+1}+\frac{Cs+D}{s^2+4} \\
2s^3+8s-2 &= 4B+D+As^3+Bs^2+Cs^3+s^2D+4As+Cs \\
A &= 2, B=-\frac{2}{3}, C=0, D=\frac{2}{3} \\
F(s) &= \frac{2s^3+8s-2}{(s^2+1)(s^2+4)}=\frac{2s}{s^2+1}-\frac{2}{3}\frac{1}{s^2+1}+\frac{1}{3}\frac{2}{s^2+4} \\
x(k) &= 2\cos k-\frac{2}{3}\sin k+\frac{1}{3}\sin 2k \\
x(t) &= 2\cos\left(t-\frac{\pi}{2}\right)-\frac{2}{3}\sin\left(t-\frac{\pi}{2}\right)+\frac{1}{3}\sin(2t-\pi) \\
x(t) &= -\frac{2}{3}\cos t+2\sin t-\frac{1}{3}\sin 2t
\end{aligned}$$