Exercise 19 (Homogeneous Second Order Linear ODEs with constant coefficients). Find the general solution of the following ODEs:

(a) $y'' - 2y' + 2y = 0$
(b) $y'' + 2y' + 2y = 0$
(c) $y'' + 2y' - 8y = 0$
(d) $y'' - 2y' + 6y = 0$
(e) $y'' + 6y' + 13y = 0$
(f) $9y'' + 16y = 0$
(g) $y'' - 2y' + y = 0$
(h) $9y'' + 6y' + y = 0$
(i) $4y'' + 12y' + 9y = 0$
(j) $4y'' - 4y' - 3y = 0$
(k) $y'' - 2y' + 10y = 0$
(l) $y'' - 6y' + 9y = 0$
(m) $4y'' + 17y' + 4y = 0$
(n) $4y'' + 20y' + 25y = 0$
(o) $25y'' - 20y' + 4y = 0$
(p) $2y'' + 2y' + y = 0$

Solve the following IVPs:

(q) $\begin{cases} 9y'' + 6y' + 82y = 0 \\ y(0) = -1 \\ y'(0) = 2 \end{cases}$

(r) $\begin{cases} y'' - 6y' + 9y = 0 \\ y(0) = 0 \\ y'(0) = 2 \end{cases}$

Solution 19.

(a) The characteristic equation is
$$r^2 - 2r + 2 = 0.$$
Thus
$$r = \frac{2 \pm \sqrt{4-8}}{2} = 1 \pm i.$$
Hence we have complex roots with $\lambda = 1$ and $\mu = 1$. The general solution to the ODE is therefore
$$y = c_1 e^t \cos t + c_2 e^t \sin t.$$

(b) $y = c_1 e^{-t} \cos t + c_2 e^{-t} \sin t$

(c) $y = c_1 e^{2t} + c_2 e^{-4t}$

(d) $y = c_1 e^t \cos \sqrt{5}t + c_2 e^t \sin \sqrt{5}t$

(e) $y = c_1 e^{-3t} \cos 2t + c_2 e^{-3t} \sin 2t$

(f) $y = c_1 \cos \frac{4}{3}t + c_2 \sin \frac{4}{3}t$

(g) $y = c_1 e^t + c_2 t e^t$

(h) $y = c_1 e^{-\frac{t}{3}} + c_2 t e^{-\frac{t}{3}}$

(i) $y = c_1 e^{-\frac{3t}{2}} + c_2 t e^{-\frac{3t}{2}}$

(j) $y = c_1 e^{-\frac{t}{2}} + c_2 e^{\frac{3t}{2}}$

(k) $y = c_1 e^t \cos 3t + c_2 e^t \sin 3t$

(l) $y = c_1 e^{3t} + c_2 t e^{3t}$

(m) $y = c_1 e^{-\frac{t}{4}} + c_2 e^{-4t}$

(n) $y = c_1 e^{-\frac{5t}{2}} + c_2 t e^{-\frac{5t}{2}}$

(o) $y = c_1 e^{\frac{2t}{5}} + c_2 t e^{\frac{2t}{5}}$

(p) $y = c_1 e^{-\frac{t}{2}} \cos \frac{t}{2} + c_2 e^{-\frac{t}{2}} \sin \frac{t}{2}$

(q) $y = -e^{-\frac{t}{3}} \cos 3t + \frac{5}{9} e^{-\frac{t}{3}} \sin 3t$

(r) The characteristic equation is
$$0 = r^2 - 6r + 9 = (r-3)^2$$
which implies that we have the repeated root $r = 3$. Therefore the general solution to the ODE is
$$y = c_1 e^{3t} + c_2 t e^{3t}.$$
Since
$$y' = 3c_1 e^{3t} + c_2 e^{3t} + 3c_2 t e^{3t}$$
we have that
$$0 = y(0) = c_1 + 0$$
$$2 = y'(0) = 3c_1 + c_2 + 0,$$
which imples that $c_1 = 0$ and $c_2 = 2$. Therefore the solution to the IVP is
$$y = 2te^{3t}.$$

Exercise 20 (Reduction of Order). In each of the following problems:

(i) Check that y_1 solves the ODE;

(ii) Use the method of reduction of order to find a second, linearly independent solution, y_2 [HINT: Start with $y_2(t) = v(t)y_1(t)$.];

(iii) Check that your y_2 solves the ODE; and

(iv) Calculate the Wronskian of y_1 and y_2.

(a) $t^2y'' + 2ty' - 2y = 0, \quad t > 0; \quad y_1(t) = t$

(b) $t^2y'' - 4ty' + 6y = 0, \quad t > 0; \quad y_1(t) = t^2$

(c) $t^2y'' + 3ty' + y = 0, \quad t > 0; \quad y_1(t) = t^{-1}$

(d) $t^2y'' - t(t+2)y' + (t+2)y = 0, \quad t > 0; \quad y_1(t) = t$

(e) $xy'' - y' + 4x^3y = 0, \quad x > 0; \quad y_1(x) = \sin x^2$

(f) $(x-1)y'' - xy' + y = 0, \quad x > 1; \quad y_1(x) = e^x$

Solution 20.

(a) (i) First we calculate that $y_1' = 1$, $y_1'' = 0$ and that

$$t^2y_1'' + 2ty_1' - 2y_1 = t^2(0) + 2t(1) - 2(t) = 2t - 2t = 0.$$

Hence $y_1(t) = t$ solves the ODE.

(ii) As per the hint, we start with $y_2(t) = v(t)y_1(t) = v(t)t$. Then $y_2' = v't + v$ and $y_2'' = v''t + 2v'$. Substituting into the ODE, we calculate that

$$\begin{aligned} 0 &= t^2y_2'' + 2ty_2' - 2y_2 \\ &= t^2(v''t + 2v') + 2t(v't + v) - 2vt \\ &= t^3v'' + v'(2t^2 + 2t^2) + v(2t - 2t) \\ &= t^3v'' + 4t^2v' \\ &= t^2(tv'' + 4v'). \end{aligned}$$

Letting $u = v'$, we obtain the first order ODE

$$t\frac{du}{dt} + 4u = 0.$$

We calculate that

$$\begin{aligned} t\frac{du}{dt} &= -4u \\ \frac{du}{u} &= -4\frac{dt}{t} \\ \int \frac{du}{u} &= -4\int \frac{dt}{t} \\ \ln|u| &= -4\ln|t| + C \\ u &= \pm e^C t^{-4} = ct^{-4} \end{aligned}$$

and

$$v = \int u \, dt = \int ct^{-4} \, dt = -\frac{1}{3}ct^{-3} + k.$$

Thus $y_2(t) = v(t)t = -\frac{1}{3}ct^{-2} + kt$. Choosing $c = -3$ and $k = 0$, we obtain the solution

$$y_2(t) = t^{-2}.$$

(iii) Since $y_2' = -2t^{-3}$ and $y_2'' = 6t^{-4}$, we have that

$$\begin{aligned} t^2y_2'' + 2ty_2' - 2y_2 &= t^2(6t^{-4}) + 2t(-2t^{-3}) - 2t^{-2} \\ &= 6t^{-2} - 4t^{-2} - 2t^{-2} \\ &= 0 \end{aligned}$$

as required.

(iv) We have that

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} t & t^{-2} \\ 1 & -2t^{-3} \end{vmatrix} = -2t^{-2} - t^{-2} = -3t^{-2} \neq 0.$$

Therefore y_1 and y_2 are linearly independent.

(b) $y_2(t) = t^3$

(c) $y_2(t) = t^{-1}\ln t$

(d) $y_2(t) = te^t$

(e) (This is a tricky one. Don't worry if you didn't solve it.) (i), (iii) and (iv) are omitted.

(ii) Let $y_2(x) = v(x)y_1(x)$. Then $y_2' = v'y_1 + vy_1'$, $y_2'' = v''y_1 + 2v'y_1' + vy_1''$ and

$$\begin{aligned} 0 &= xy_2'' - y_2' + 4x^3y_2 \\ &= xv''y_1 + 2xv'y_1' + xvy_1'' - v'y_1 - vy_1' + 4x^3vy_1 \\ &= xv''y_1 + (2xy_1' - y_1)v' + (xy_1'' - y_1' + 4x^3y_1)v \\ &= xy_1v'' + (2xy_1' - y_1)v' \end{aligned}$$

since y_1 solves the ODE. Let $u = v'$. Then we have the first order ODE

$$u' + \left(2\frac{y_1'}{y_1} - \frac{1}{x}\right)u = 0.$$

Recall that to solve the linear ODE $u' + p(x)u = 0$, we use the integrating factor $\mu(x) = e^{\int p(x)\,dx}$ and calculate that

$$\begin{aligned} u' + pu &= 0 \\ \mu u' + \mu pu &= 0 \\ (\mu u)' &= 0 \\ \mu u &= c \\ u &= \frac{c}{\mu} = ce^{-\int p(x)\,dx}. \end{aligned}$$

It follows that

$$u(x) = ce^{-\int\left(2\frac{y_1'}{y_1} - \frac{1}{x}\right)dx} = ce^{-2\ln y_1 + \ln x} = \frac{cx}{y_1^2} = \frac{cx}{\sin^2 x^2}.$$

Using the substitution $t = x^2$ we calculate that $dt = 2x\,dx$ and

$$\begin{aligned} v(x) &= \int u(x)\,dx = c\int \frac{x}{\sin^2 x^2}\,dx = \frac{c}{2}\int \frac{1}{\sin^2 t}\,dt \\ &= \frac{c}{2}\int \operatorname{cosec}^2 t\,dt = -\frac{c}{2}\cot t + k = -\frac{c}{2}\cot x^2 + k. \end{aligned}$$

Choosing $c = -2$ and $k = 0$ gives $v(x) = \cot x^2$.

Therefore

$$y_2(x) = v(x)y_1(x) = \cot x^2 \sin x^2 = \cos x^2.$$

(f) (ii) Let $y_2(x) = v(x)y_1(x) = ve^x$. Then $y_2' = (v' + v)e^x$, $y_2'' = (v'' + 2v' + v)e^x$ and

$$\begin{aligned} 0 &= (x-1)y'' - xy' + y \\ &= \left[(x-1)(v'' + 2v' + v) - x(v' + v) + v\right]e^x \\ &= \left[(x-1)v'' + (x-2)v'\right]e^x. \end{aligned}$$

Letting $u = v'$ we obtain the first order ODE

$$u' + \left(\frac{x-2}{x-1}\right)u = 0$$

which has solution

$$u(x) = ce^{-x}(x-1).$$

By integrating, we obtain

$$v(x) = \int u(x)\,dx = -cxe^{-x}.$$

Choosing $c = -1$ gives $v(x) = xe^{-x}$. Therefore

$$y_2(x) = v(x)y_1(x) = xe^{-x}e^x = x.$$