Week 3

- 2.3 Differences Between Linear and Nonlinear Equations
- 2.4 Autonomous Equations and Population Dynamics

Differences Between Linear and Nonlinear Equations

Theorem

Suppose

- p *and* g *are continuous on* (α, β)*;*
- $t_0 \in (\alpha, \beta)$*; and*
- $y_0 \in \mathbb{R}$.

Then there exists a unique solution to

$$\begin{cases} y' + p(t)y = g(t) \\ y(t_0) = y_0 \end{cases}$$

on (α, β).

$$\begin{cases} y' + p(t)y = g(t) \\ y(t_0) = y_0 \end{cases}$$

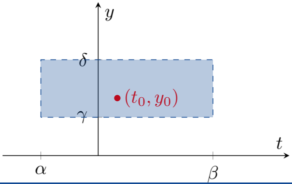
Remark

This theorem says that as long as p and g are continuous, the solution keeps existing. To say this another way: The solution can only stop existing at a discontinuity of either p or g.

Theorem

Suppose that

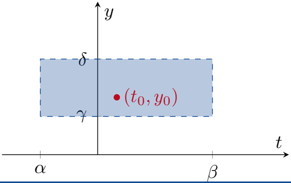
- f *and* $\frac{\partial f}{\partial y}$ *are continuous for all* $\alpha < t < \beta$ *and* $\gamma < y < \delta$;
- $t_0 \in (\alpha, \beta)$; *and*
- $y_0 \in (\gamma, \delta)$.



Theorem

Suppose that

- f *and* $\frac{\partial f}{\partial y}$ *are continuous for all* $\alpha < t < \beta$ *and* $\gamma < y < \delta$;
- $t_0 \in (\alpha, \beta)$; *and*
- $y_0 \in (\gamma, \delta)$.



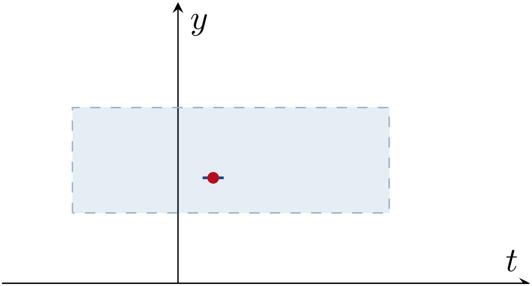
Theorem

Suppose that

- f *and* $\frac{\partial f}{\partial y}$ *are continuous for all* $\alpha < t < \beta$ *and* $\gamma < y < \delta$;
- $t_0 \in (\alpha, \beta)$; *and*
- $y_0 \in (\gamma, \delta)$.

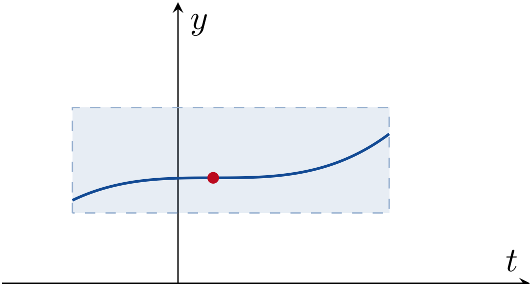
Then in some interval $(t_0 - h, t_0 + h) \subseteq (\alpha, \beta)$, *there exists a unique solution to*

$$\begin{cases} y' = f(t, y) \\ y(t_0) = y_0. \end{cases}$$



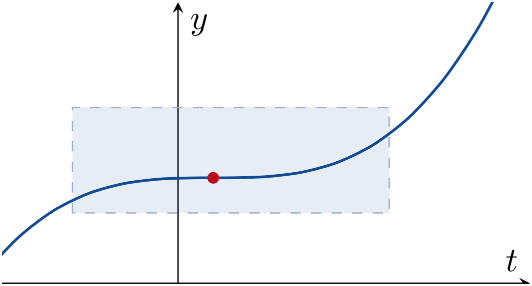
Remark

This theorem tells us that “a little bit” of the solution exists. This theorem does not tell us if we only have this little bit of solution or if the solution exists further.



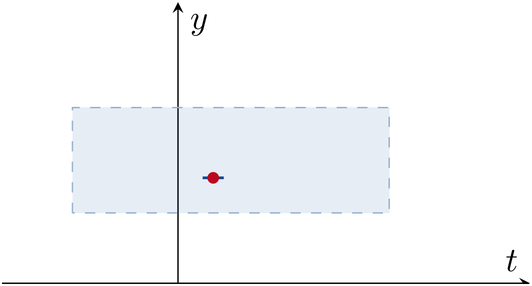
Remark

This theorem tells us that "a little bit" of the solution exists. This theorem does not tell us if we only have this little bit of solution or if the solution exists further.



Remark

This theorem tells us that “a little bit” of the solution exists. This theorem does not tell us if we only have this little bit of solution or if the solution exists further.



Remark

This theorem tells us that “a little bit” of the solution exists. This theorem does not tell us if we only have this little bit of solution or if the solution exists further.

Remark

This theorem tells us that two solutions to $y' = f(t, y)$ can not intersect.

Remark

This theorem tells us that two solutions to $y' = f(t, y)$ can not intersect.

To understand why: Suppose that two solutions intersect at the point (t_0, y_0). But then there would be two solutions to

$$\begin{cases} y' = f(t, y) \\ y(t_0) = y_0 \end{cases}$$

and the theorem says that this is not possible.

Remark

This theorem tells us that two solutions to $y' = f(t, y)$ can not intersect.

To understand why: Suppose that two solutions intersect at the point (t_0, y_0). But then there would be two solutions to

$$\begin{cases} y' = f(t, y) \\ y(t_0) = y_0 \end{cases}$$

and the theorem says that this is not possible.

Solutions to first order ODEs do not intersect !!! (assuming that f and $\frac{\partial f}{\partial y}$ are ...)

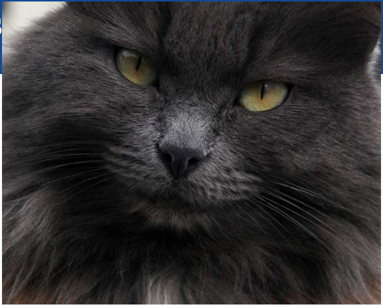
Autonomous Equations and Population Dynamics

Equations of the form

$$\frac{dy}{dt} = \underbrace{f(y)}_{\text{only } y} \tag{1}$$

are called *autonomous*.

2.5 Autonomous … ion Dyna…



Example (Exponential Growth)

Let $y(t)$ denote the number of cats in İstanbul.
The simplest model is to assume that the rate of change of y is proportional to y.

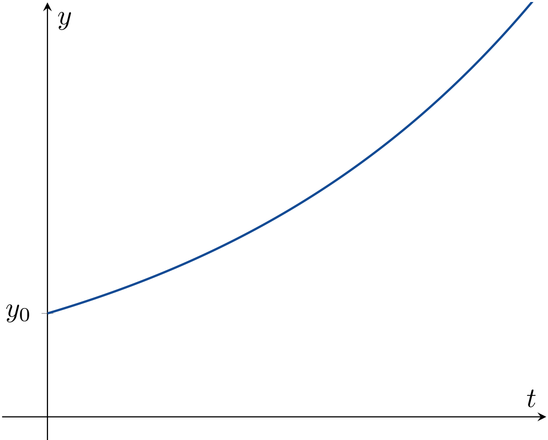
$$\frac{dy}{dt} = ry$$

for some constant r. We will assume that $r > 0$.

The solution to

$$\begin{cases} y' = ry \\ y(0) = y_0 \end{cases}$$

is $y(t) = y_0 e^{rt}$.



This model is good for small y, but it predicts that the number of cats in İstanbul will increase exponentially for all time. This can not be true.

This model is good for small y, but it predicts that the number of cats in İstanbul will increase exponentially for all time. This can not be true. At some point

- the food will run out
- there will be no space
- people will get angry

$$\vdots$$

This model is good for small y, but it predicts that the number of cats in İstanbul will increase exponentially for all time. This can not be true. At some point

- the food will run out
- there will be no space
- people will get angry

$$\vdots$$

So we need a better model.

Example (Logistic Growth)

Now we replace the constant r with a function $h(y)$.

$$\frac{dy}{dt} = h(y)y.$$

Example (Logistic Growth)

Now we replace the constant r with a function $h(y)$.

$$\frac{dy}{dt} = h(y)y.$$

We want a function h which satisfies

- $h(y) \approx r$ if y is small;
- $h(y)$ decreases as y grows larger; and
- $h(y) < 0$ for large y.

The simplest such h is $h(y) = r - ay$.

The simplest such h is $h(y) = r - ay$. So

$$\frac{dy}{dt} = (r - ay)y$$

The simplest such h is $h(y) = r - ay$. So

$$\frac{dy}{dt} = (r - ay)y$$

which we will write as

$$\frac{dy}{dt} = r\left(1 - \frac{y}{K}\right)y$$

for $K = \frac{r}{a}$. This is called the *Logistic Equation*.

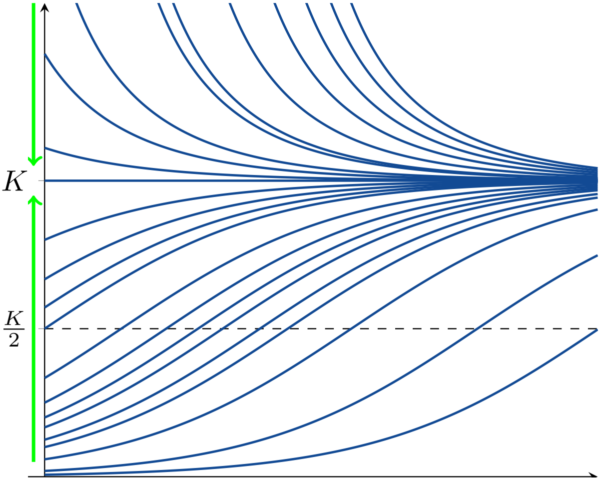
First we look for equilibrium solutions – that is solutions with $\frac{dy}{dt} = 0$ for all t.

$$0 = \frac{dy}{dt} = r\left(1 - \frac{y}{K}\right)y \qquad \Longrightarrow \qquad y = 0 \ \text{ or } \ y = K.$$

First we look for equilibrium solutions – that is solutions with $\frac{dy}{dt} = 0$ for all t.

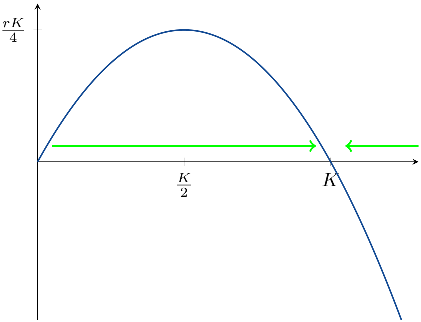
$$0 = \frac{dy}{dt} = r\left(1 - \frac{y}{K}\right)y \qquad \Longrightarrow \qquad y = 0 \;\text{ or }\; y = K.$$

The equilibrium solutions are important. If we look at some more solutions, we can see that the other solutions converge to $y = K$, but diverge from $y = 0$.



K
$\frac{K}{2}$

To understand this behaviour, we graph $\frac{dy}{dt}$ against y.



Note that

- $\frac{dy}{dt} > 0 \implies y$ is increasing; and
- $\frac{dy}{dt} < 0 \implies y$ is decreasing; and

We can show this on the graph by drawing green arrows.

To investigate further, we look at $\frac{d^2y}{dt^2}$:

To investigate further, we look at $\frac{d^2y}{dt^2}$: If $\frac{dy}{dt} = f(y)$, then

$$\frac{d^2y}{dt^2} = \frac{d}{dt}\Big(f\big(y(t)\big)\Big) = f'(y)\frac{dy}{dt} = f'(y)f(y).$$

To investigate further, we look at $\frac{d^2y}{dt^2}$: If $\frac{dy}{dt} = f(y)$, then

$$\frac{d^2y}{dt^2} = \frac{d}{dt}\Big(f\big(y(t)\big)\Big) = f'(y)\frac{dy}{dt} = f'(y)f(y).$$

The solution $y(t)$ is concave up (or) when $y'' > 0$ (i.e. when both f and f' are both positive or both negative).

2.5 Autonomous Equations and Population Dyna

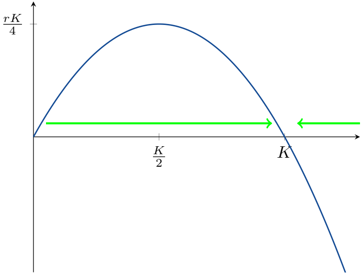
To investigate further, we look at $\frac{d^2y}{dt^2}$: If $\frac{dy}{dt} = f(y)$, then

$$\frac{d^2y}{dt^2} = \frac{d}{dt}\Big(f\big(y(t)\big)\Big) = f'(y)\frac{dy}{dt} = f'(y)f(y).$$

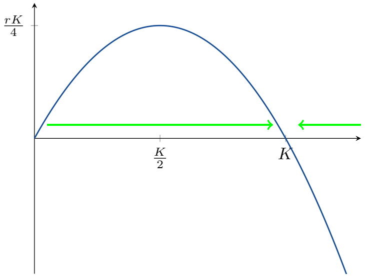
The solution $y(t)$ is concave up (or) when $y'' > 0$ (i.e. when both f and f' are both positive or both negative).

The solution $y(t)$ is concave down (or) when $y'' < 0$ (i.e. when one of f and f' is positive and one is negative).

Look again at the graph of $f(y) = r\left(1 - \frac{y}{K}\right)y$ against y.



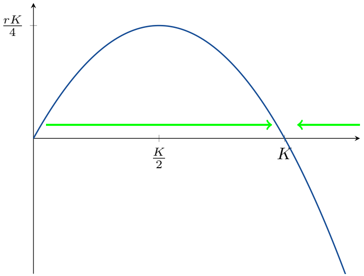
Look again at the graph of $f(y) = r\left(1 - \frac{y}{K}\right)y$ against y.



We can see that

- $y \in (0, \frac{K}{2}) \implies f > 0$ and $f' > 0 \implies y(t)$ is increasing and concave up;

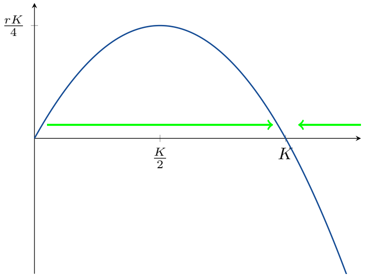
Look again at the graph of $f(y) = r\left(1 - \frac{y}{K}\right)y$ against y.



We can see that

- $y \in (0, \frac{K}{2}) \implies f > 0$ and $f' > 0 \implies y(t)$ is increasing and concave up;
- $y \in (\frac{K}{2}, K) \implies f > 0$ and $f' < 0 \implies y(t)$ is increasing and concave down;

Look again at the graph of $f(y) = r\left(1 - \frac{y}{K}\right)y$ against y.

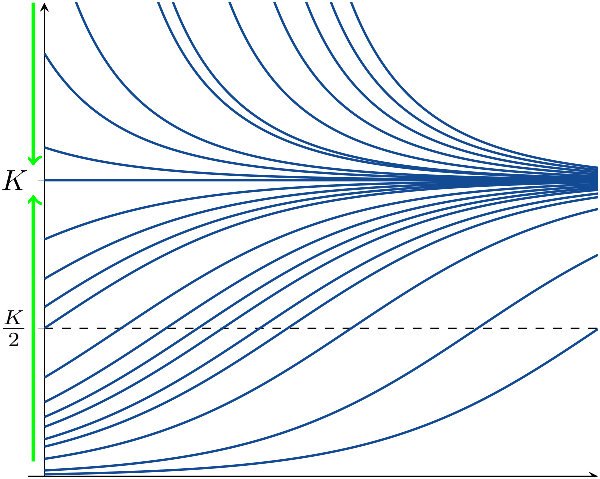


We can see that

- $y \in (0, \frac{K}{2}) \implies f > 0$ and $f' > 0 \implies y(t)$ is increasing and concave up;
- $y \in (\frac{K}{2}, K) \implies f > 0$ and $f' < 0 \implies y(t)$ is increasing and concave down;
- $y \in (K, \infty) \implies f < 0$ and $f' < 0 \implies y(t)$ is decreasing and concave up;

Moreover, remember that a theorem from earlier told us that two solutions can not intersect.

Moreover, remember that a theorem from earlier told us that two solutions can not intersect. Hence the solutions look like this:



Because solutions converge to $y = K$, we say that $y = K$ is an *asymptotically stable equilibrium solution* or an *asymptotically stable critical point*.

Because solutions converge to $y = K$, we say that $y = K$ is an *asymptotically stable equilibrium solution* or an *asymptotically stable critical point*.

Because solutions diverge from $y = 0$, we say that $y = 0$ is an *unstable equilibrium solution* or an *unstable critical point*.

2.5 Autonomous Equations and Population Dyna

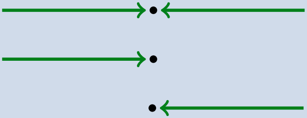
Definition

Equilibrium solutions can be

2.5 Autonomous Equations and Population Dyna

Definition

Equilibrium solutions can be

	asymptotically stable

2.5 Autonomous Equations and Population Dyna

Definition

Equilibrium solutions can be

$\longrightarrow\bullet\longleftarrow$ $\longrightarrow\bullet$ $\bullet\longleftarrow$	asymptotically stable
$\longleftarrow\bullet\longrightarrow$ $\longleftarrow\bullet$ $\bullet\longrightarrow$	unstable

2.5 Autonomous Equations and Population Dyna

Definition

Equilibrium solutions can be

$\longrightarrow\bullet\longleftarrow$ $\longrightarrow\bullet$ $\bullet\longleftarrow$	asymptotically stable
$\longleftarrow\bullet\longrightarrow$ $\longleftarrow\bullet$ $\bullet\longrightarrow$	unstable
$\longleftarrow\bullet\longleftarrow$ $\longrightarrow\bullet\longrightarrow$	semistable

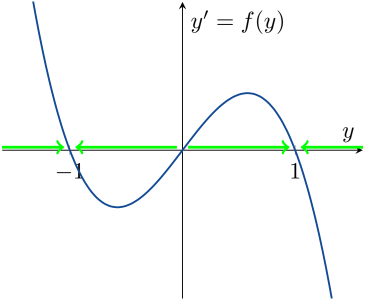
Example

Find all of the critical points of

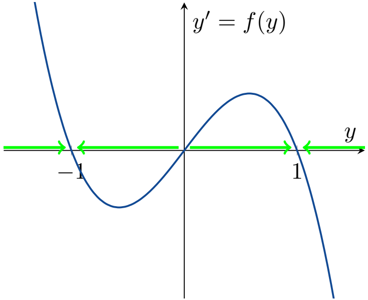
$$\frac{dy}{dt} = \underbrace{y(1-y^2)}_{f(y)} \qquad (-\infty < y_0 < \infty)$$

and classify each as asymptotically stable, unstable or semistable.

$$\frac{dy}{dt} = y(1 - y^2)$$

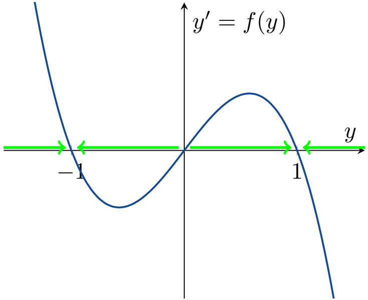


$$\frac{dy}{dt} = y(1 - y^2)$$



The critical points are $y = -1, 0, 1$.

$$\frac{dy}{dt} = y(1 - y^2)$$



The critical points are $y = -1, 0, 1$.

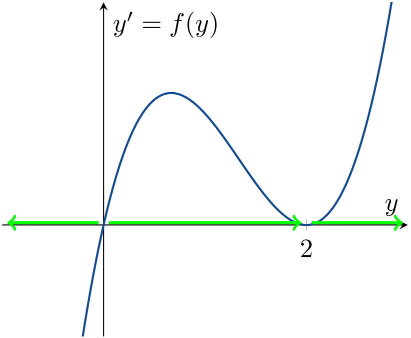
- $y = -1$ is asymptotically stable;
- $y = 0$ is unstable; and
- $y = 1$ is asymptotically stable.

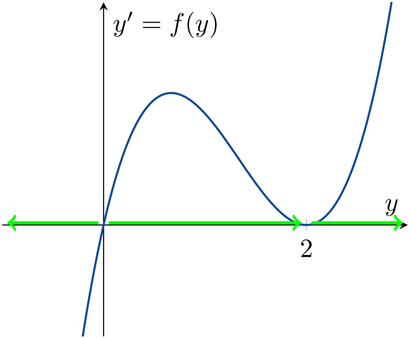
Example

Find all of the critical points of

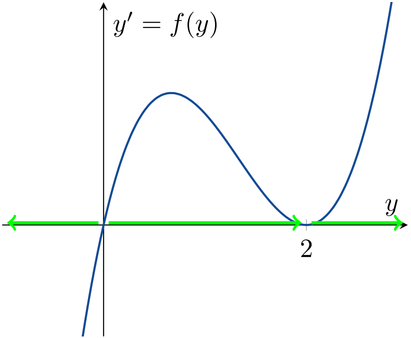
$$\frac{dy}{dt} = \underbrace{y(y-2)^2}_{f(y)} \qquad (-\infty < y_0 < \infty)$$

and classify each as asymptotically stable, unstable or semistable.


$y' = f(y)$
y
2



The critical points are $y = 0$ and 2.



The critical points are $y = 0$ and 2.

- $y = 0$ is unstable; and
- $y = 2$ is semistable.

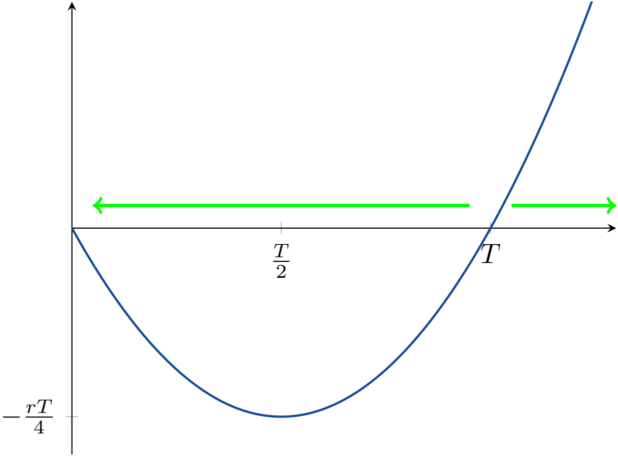
Example (A Critical Threshold)

Now suppose that we can model the number of cats in İstanbul by

$$\frac{dy}{dt} = -r\left(1 - \frac{y}{T}\right)y$$

where $T > 0$ and $r > 0$.

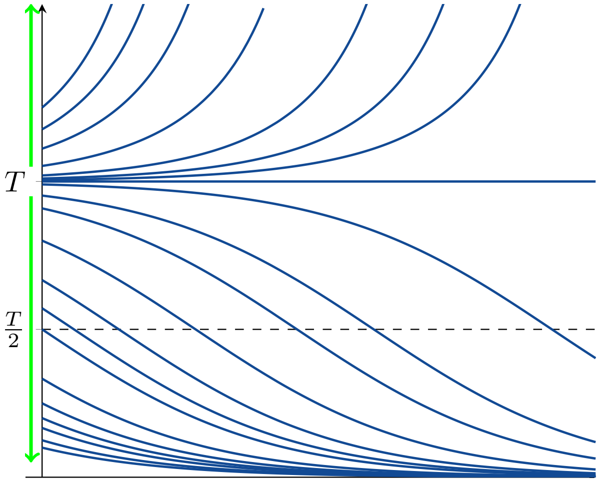
$$\frac{dy}{dt} = -r\left(1 - \frac{y}{T}\right)y$$



The critical points/equilibrium solutions are $y = 0$ and $y = T$.

- $y = 0$ is asymptotically stable; and
- $y = T$ is unstable.

With this information we can sketch some solutions



T
$\frac{T}{2}$

Depending on y_0 ($y_0 \neq T$), we either have $y \to 0$ or $y \to \infty$.

Depending on y_0 ($y_0 \neq T$), we either have $y \to 0$ or $y \to \infty$.

The number T is called a *threshold level*, below which no growth happens.

Depending on y_0 ($y_0 \neq T$), we either have $y \to 0$ or $y \to \infty$.

The number T is called a *threshold level*, below which no growth happens.

The population of some species have the threshold property: If there are not enough individuals, then the species becomes extinct.

Depending on y_0 ($y_0 \neq T$), we either have $y \to 0$ or $y \to \infty$.

The number T is called a *threshold level*, below which no growth happens.

The population of some species have the threshold property: If there are not enough individuals, then the species becomes extinct.

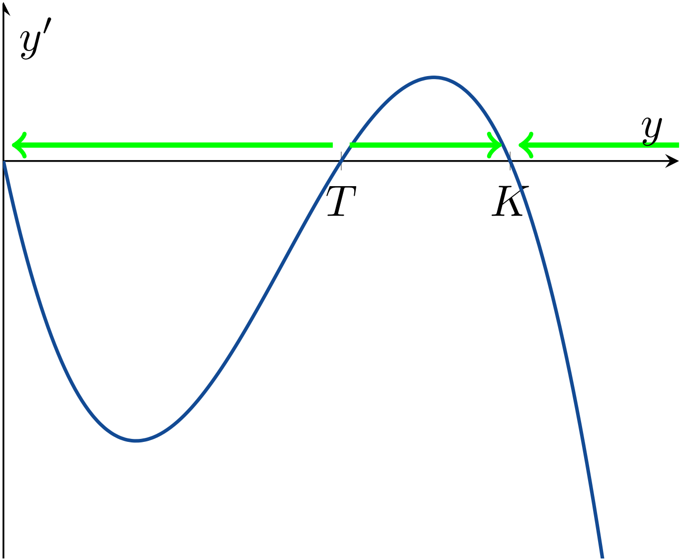
This model predicts that the number of cats in İstanbul will increase to ∞ (if $y_0 > T$), so we need a more advanced model.

Example (Logistic Growth with a Threshold)

Now consider

$$\frac{dy}{dt} = -r\left(1 - \frac{y}{T}\right)\left(1 - \frac{y}{K}\right)y$$

for $0 < T < K$ and $r > 0$.

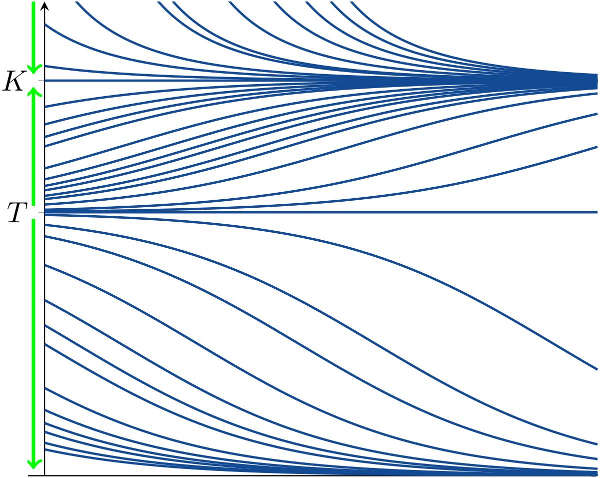


y'
y
T
K

The critical points/equilibrium solutions are $y = 0$, $y = T$ and $y = K$.

- $y = 0$ is asymptotically stable;
- $y = T$ is unstable; and
- $y = K$ is asymptotically stable.

Solutions look like this:



This is an equation which has been used by biologists to model certain populations of animals.

Next Week

- 2.5 Exact Equations
- 2.6 Substitutions