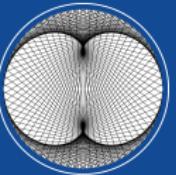


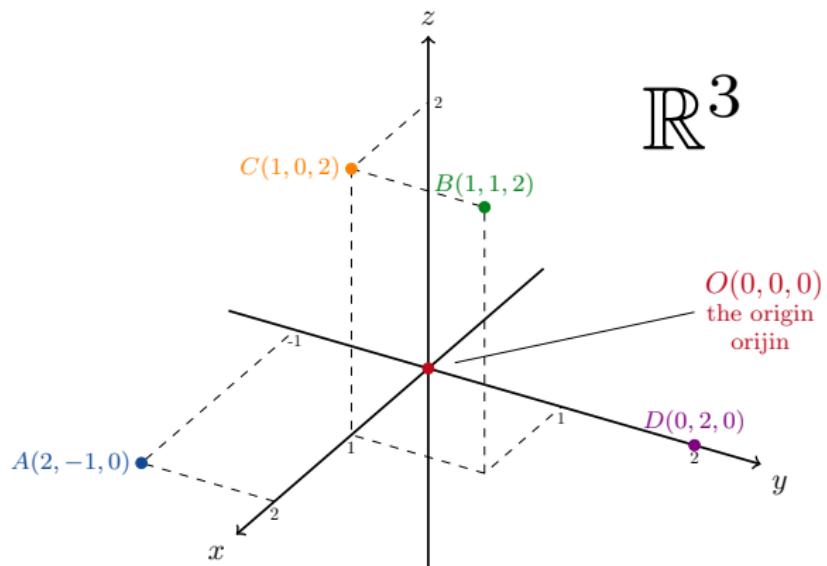
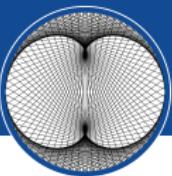
# Lecture 3

- 11.1 Three-Dimensional Coordinate Systems
- 11.2 Vectors
- 11.3 The Dot Product

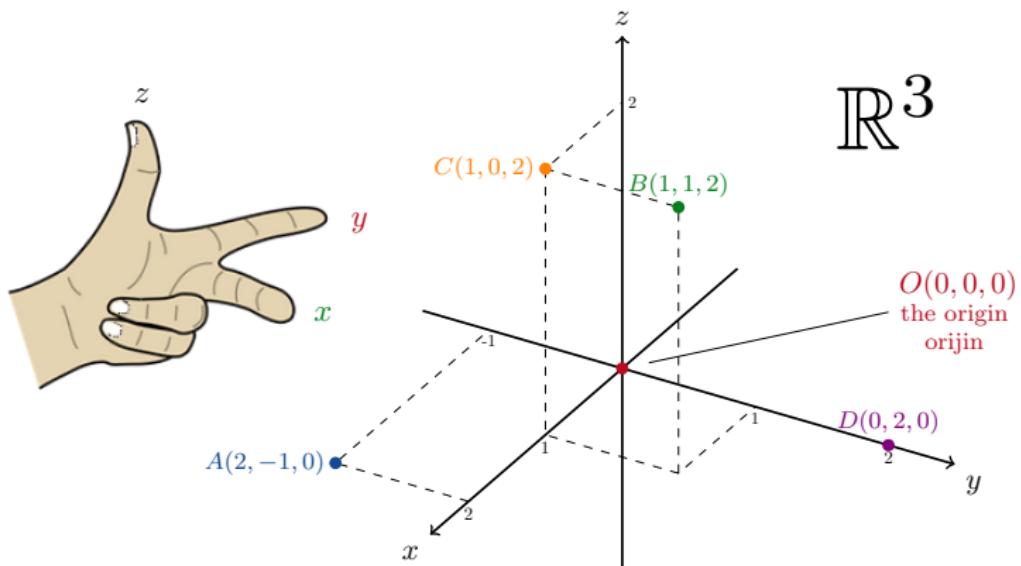
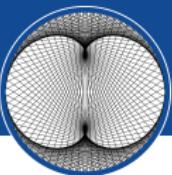


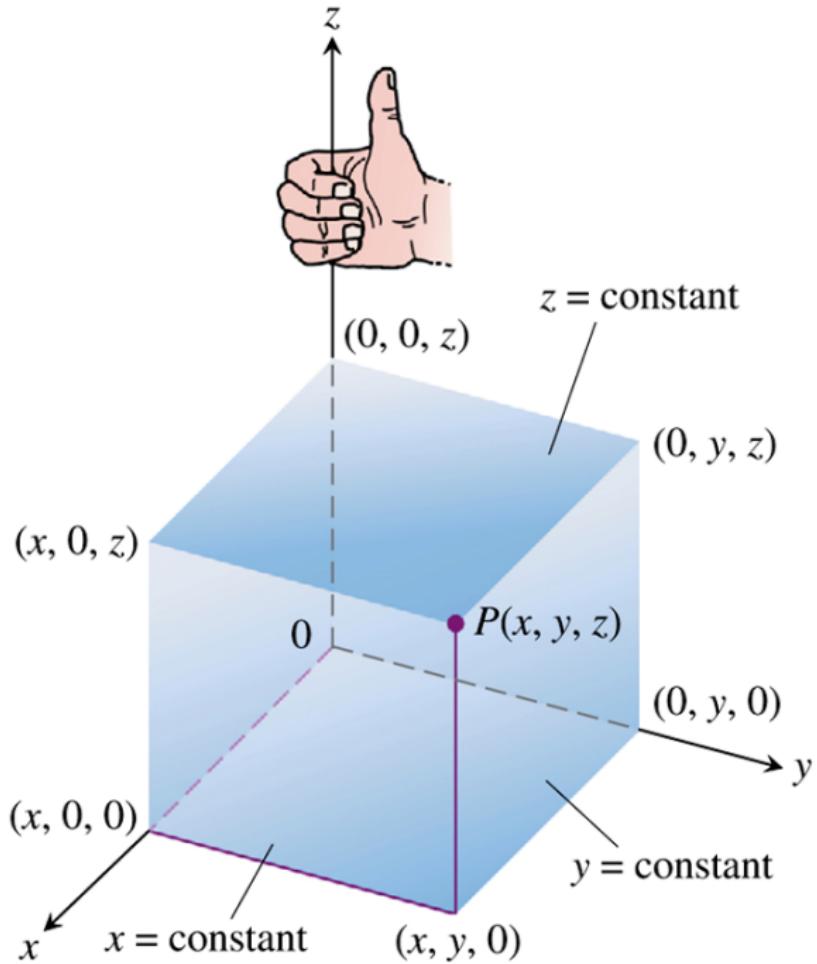
# Three-Dimensional Coordinate Systems

# 11.1 Three-Dimensional Coordinate Systems

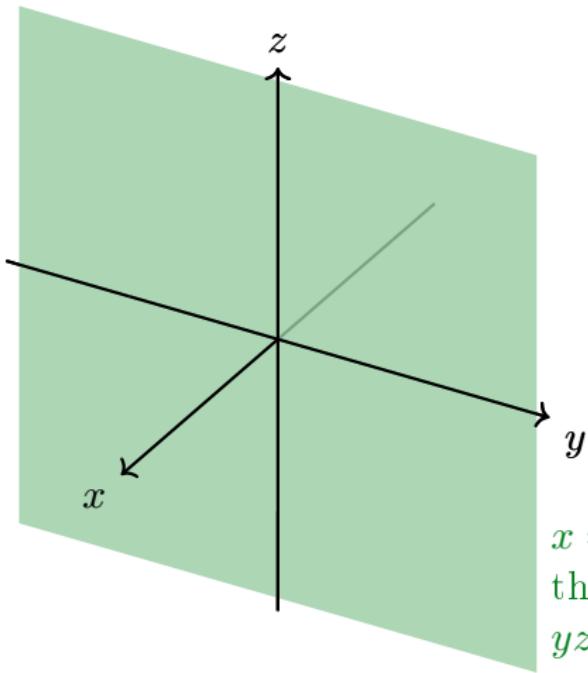
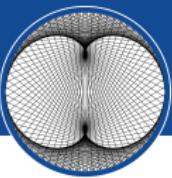


# 11.1 Three-Dimensional Coordinate Systems



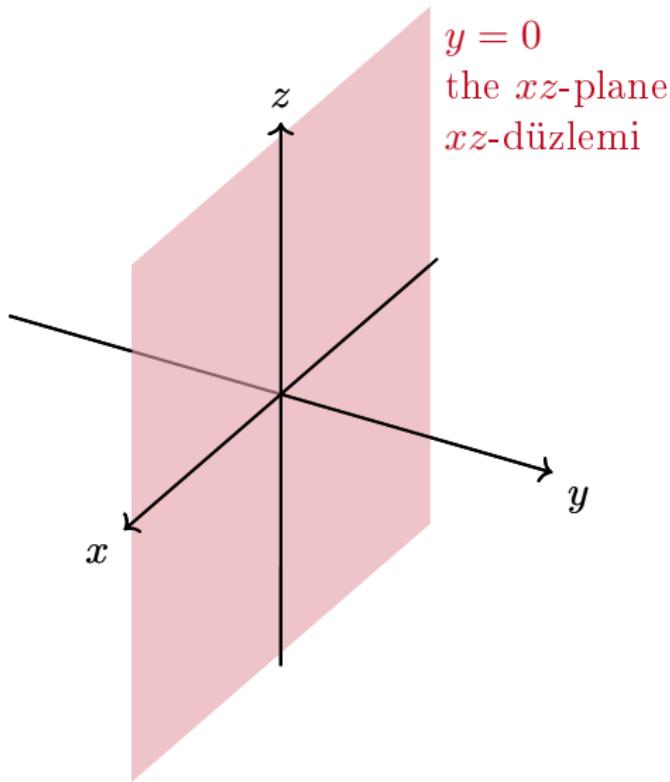
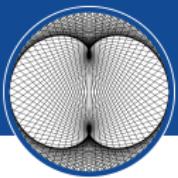


## 11.1 Three-Dimensional Coordinate Systems

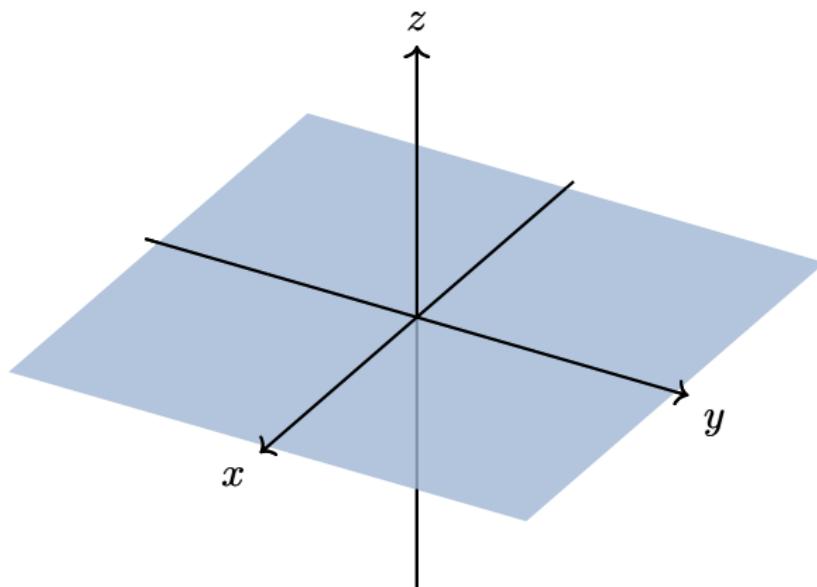
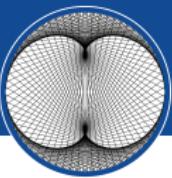


$x = 0$   
the *yz*-plane  
*yz*-düzlemi

## 11.1 Three-Dimensional Coordinate Systems

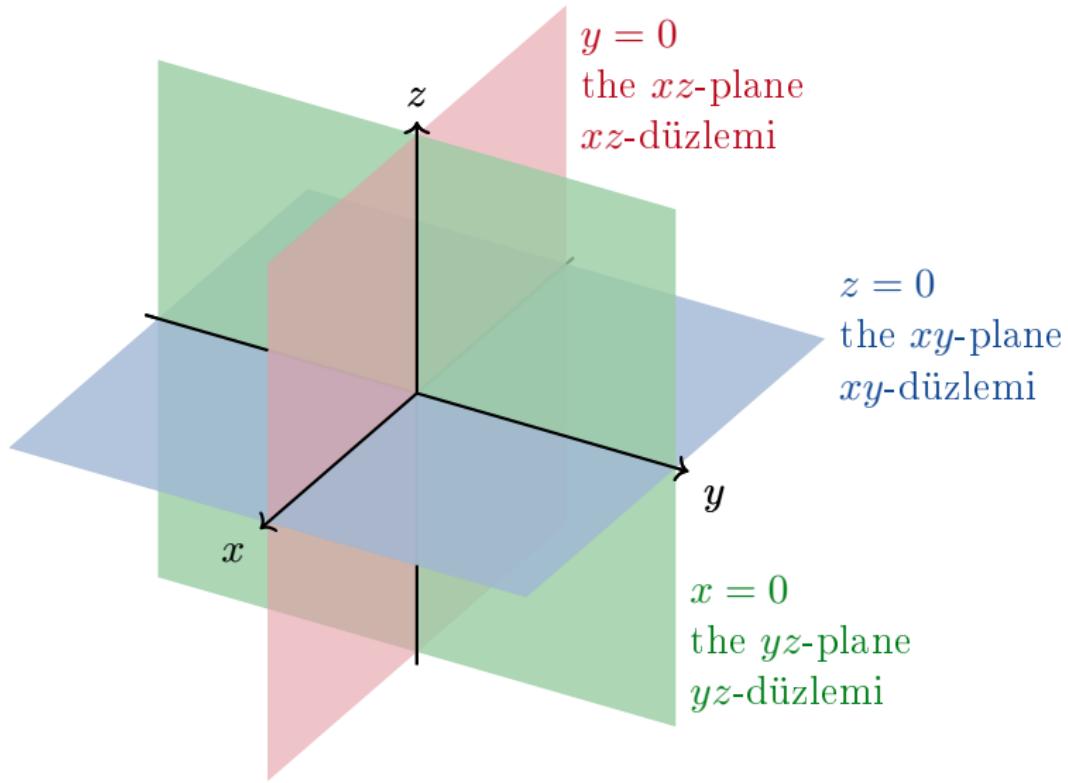
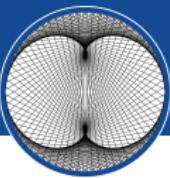


## 11.1 Three-Dimensional Coordinate Systems

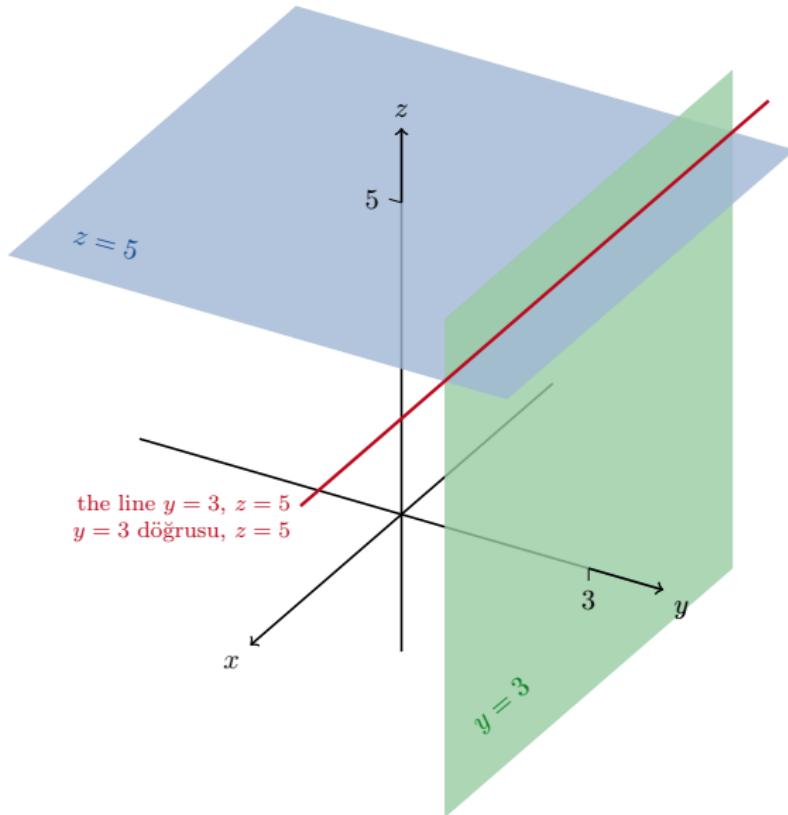
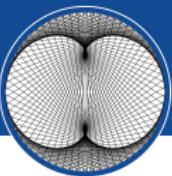


$z = 0$   
the  $xy$ -plane  
*xy*-düzlemi

## 11.1 Three-Dimensional Coordinate Systems



## 11.1 Three-Dimensional Coordinate Systems



**EXAMPLE 1**

We interpret these equations and inequalities geometrically.

(a)  $z \geq 0$

The half-space consisting of the points on and above the  $xy$ -plane.

(b)  $x = -3$

The plane perpendicular to the  $x$ -axis at  $x = -3$ . This plane lies parallel to the  $yz$ -plane and 3 units behind it.

(c)  $z = 0, x \leq 0, y \geq 0$

The second quadrant of the  $xy$ -plane.

(d)  $x \geq 0, y \geq 0, z \geq 0$

The first octant.

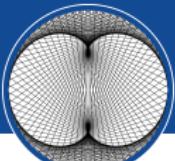
(e)  $-1 \leq y \leq 1$

The slab between the planes  $y = -1$  and  $y = 1$  (planes included).

(f)  $y = -2, z = 2$

The line in which the planes  $y = -2$  and  $z = 2$  intersect. Alternatively, the line through the point  $(0, -2, 2)$  parallel to the  $x$ -axis. ■

## 11.1 Three-Dimensional Coordinate Systems

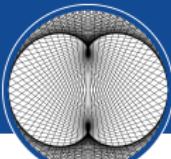


### Example

Which points  $P(x, y, z)$  satisfy  $x^2 + y^2 = 4$  and  $z = 3$ ?

We know that  $z = 3$  is a horizontal plane and we recognise that  $x^2 + y^2 = 4$  is the equation of a circle of radius 2.

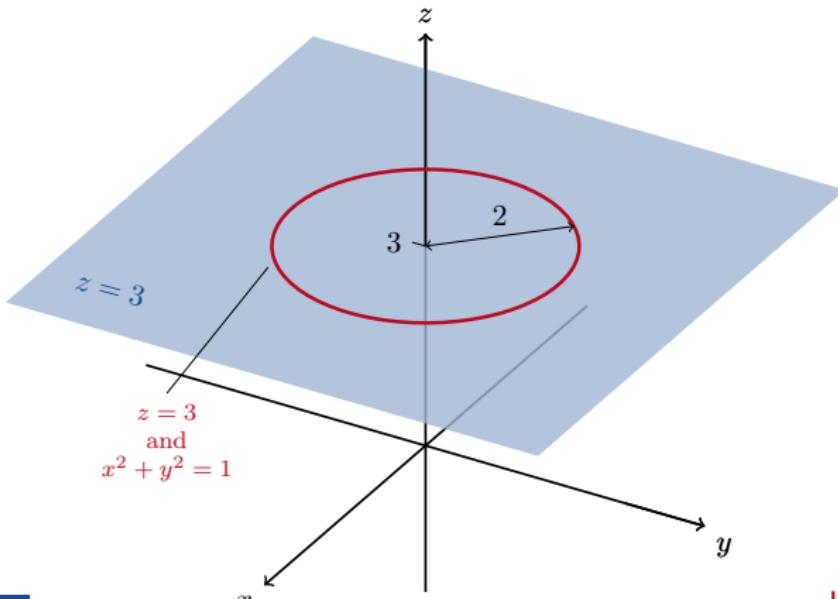
## 11.1 Three-Dimensional Coordinate Systems



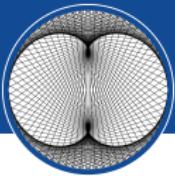
### Example

Which points  $P(x, y, z)$  satisfy  $x^2 + y^2 = 4$  and  $z = 3$ ?

We know that  $z = 3$  is a horizontal plane and we recognise that  $x^2 + y^2 = 4$  is the equation of a circle of radius 2.



## 11.1 Three-Dimensional Coordinate Systems



### Distance in $\mathbb{R}^3$

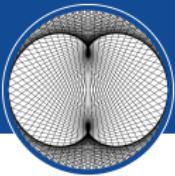
#### Definition

The set

$$\{(x, y, z) \mid x, y, z \in \mathbb{R}\}$$

is denoted by  $\mathbb{R}^3$ .

## 11.1 Three-Dimensional Coordinate Systems



### Distance in $\mathbb{R}^3$

#### Definition

The set

$$\{(x, y, z) \mid x, y, z \in \mathbb{R}\}$$

is denoted by  $\mathbb{R}^3$ .

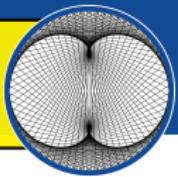
#### Definition

The *distance* between  $P_1(x_1, y_1, z_1)$  and  $P_2(x_2, y_2, z_2)$  is

$$\|P_1P_2\| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}.$$

## 11.1 Three-Dimensional Space

$$\|P_1P_2\| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}.$$

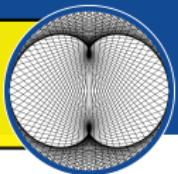


### Example

The distance between  $A(2, 1, 5)$  and  $B(-2, 3, 0)$  is

## 11.1 Three-Dimensional Space

$$\|P_1P_2\| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}.$$

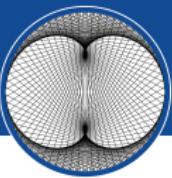


### Example

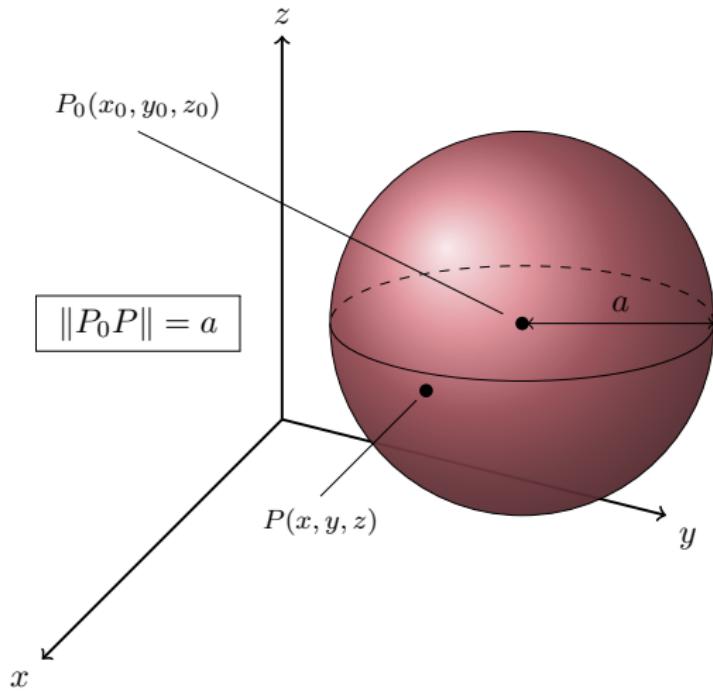
The distance between  $A(2, 1, 5)$  and  $B(-2, 3, 0)$  is

$$\begin{aligned}\|AB\| &= \sqrt{((-2) - 2)^2 + (3 - 1)^2 + (0 - 5)^2} \\ &= \sqrt{16 + 4 + 25} = \sqrt{45} \\ &= 3\sqrt{5} \approx 6.7.\end{aligned}$$

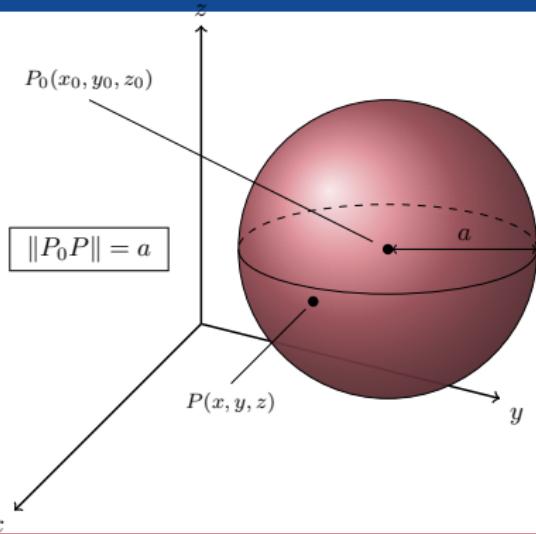
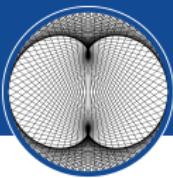
## 11.1 Three-Dimensional Coordinate Systems



### Spheres



## 11.1 Three-Dimensional Coordinate Systems



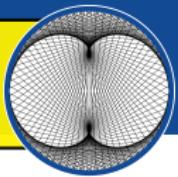
### Definition

The *standard equation for a sphere* of radius  $a$  centred at  $P_0(x_0, y_0, z_0)$  is

$$(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2 = a^2.$$

## 11.1 Three-Dimensional

$$(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2 = a^2.$$



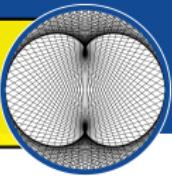
### Example

Find the centre and radius of the sphere

$$x^2 + y^2 + z^2 + 3x - 4z + 1 = 0.$$

## 11.1 Three-Dimensional Geometry

$$(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2 = a^2.$$



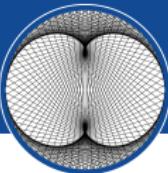
### Example

Find the centre and radius of the sphere

$$x^2 + y^2 + z^2 + 3x - 4z + 1 = 0.$$

First we need to complete the squares to put this equation into the standard form.

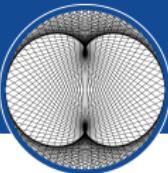
## 11.1 Three-Dimensional Coordinate Systems



### Completing the square

Let's suppose that we want to solve

$$x^2 + 26x = 27.$$



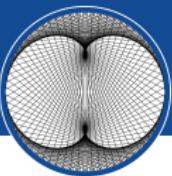
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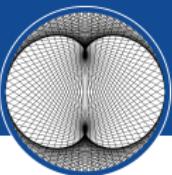
We can think of this in terms of areas of rectangles.

## 11.1 Three-Dimensional Coordinate Systems



$$\begin{array}{c} \text{---} \\ | \\ \text{---} \\ x \\ | \\ \text{---} \\ + \\ \text{---} \\ | \\ \text{---} \\ x \\ | \\ \text{---} \\ 26 \\ | \\ \text{---} \end{array} = 27$$

## 11.1 Three-Dimensional Coordinate Systems

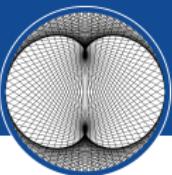


$$x^2 + 26x$$

$$\begin{array}{c} \text{---} \\ | \\ \text{---} \\ x \\ | \\ \text{---} \\ + \\ \text{---} \\ | \\ \text{---} \\ 26 \\ | \\ \text{---} \\ x = 27 \end{array}$$

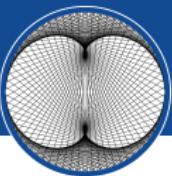
A diagram illustrating the equation  $x^2 + 26x = 27$ . It shows a blue square with side length  $x$  and a green rectangle with width 26 and height  $x$ . The total width of the combined shapes is labeled as 27.

## 11.1 Three-Dimensional Coordinate Systems



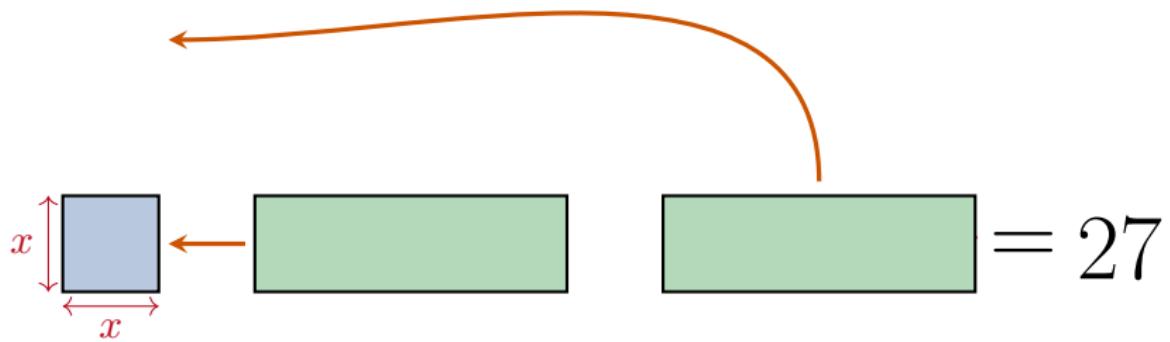
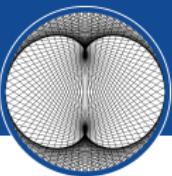
$$x^2 + 26x$$
 $x^2 + 26x = 27$

## 11.1 Three-Dimensional Coordinate Systems

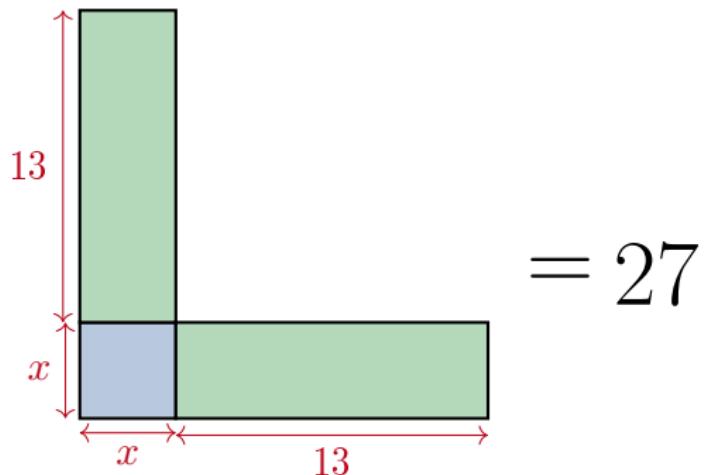
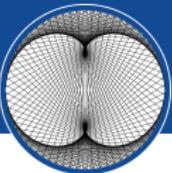


$$\begin{array}{c} \text{blue square: } x \times x \\ \text{green rectangle: } x \times 13 \\ \text{green rectangle: } x \times 13 = 27 \end{array}$$

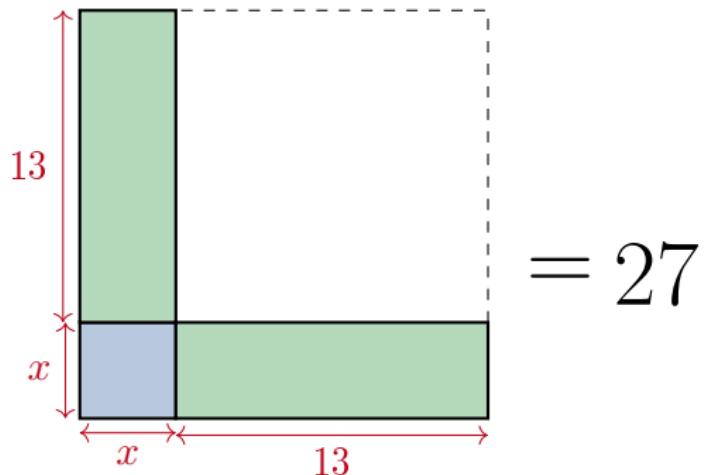
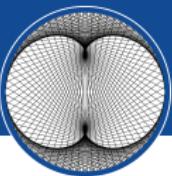
## 11.1 Three-Dimensional Coordinate Systems



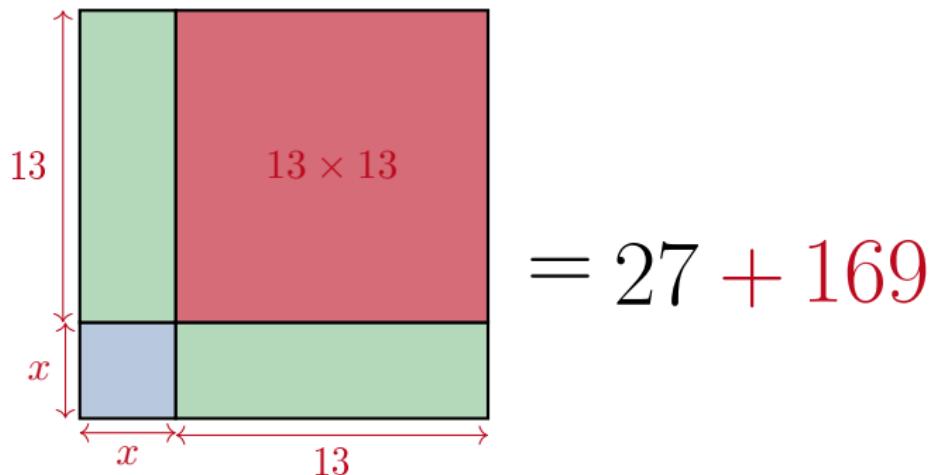
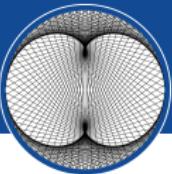
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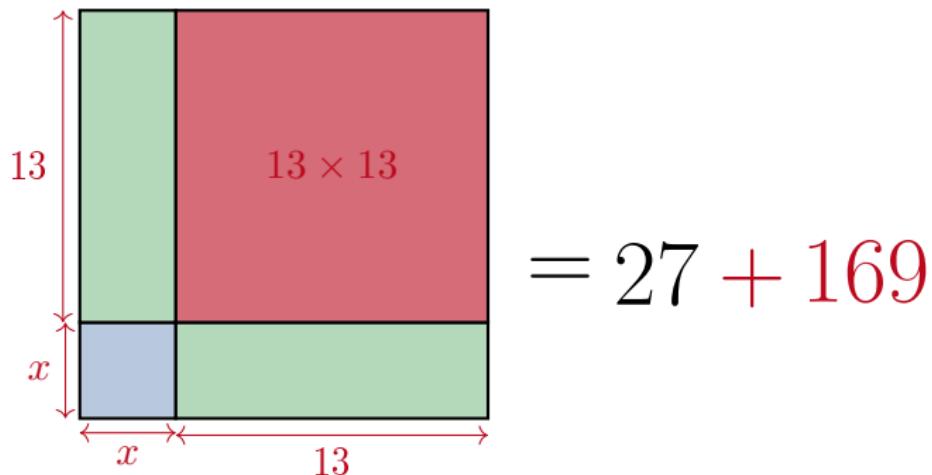
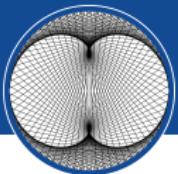
## 11.1 Three-Dimensional Coordinate Systems



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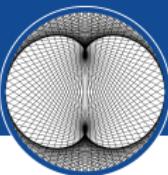


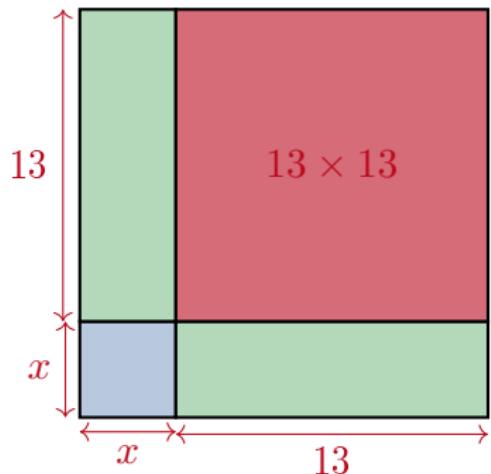
## 11.1 Three-Dimensional Coordinate Systems



$$(x + 13)^2 = 196$$

## 11.1 Three-Dimensional Coordinate Systems

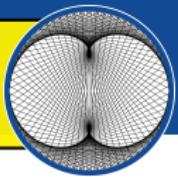


$$\begin{aligned}x^2 + 26x &= 27 \\ \left(\frac{26}{2}\right)^2 \\ &= 27 + 169\end{aligned}$$


$$(x + 13)^2 = 196$$

## 11.1 Three-Dimensional

$$(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2 = a^2.$$



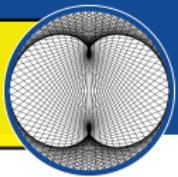
### Example

Find the centre and radius of the sphere

$$x^2 + y^2 + z^2 + 3x - 4z + 1 = 0.$$

## 11.1 Three-Dimer

$$(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2 = a^2.$$



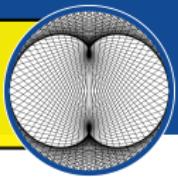
By completing the squares, we find that

$$x^2 + y^2 + z^2 + 3x - 4z + 1 = 0$$

$$\left( x^2 + 3x \quad \right) + y^2 + (z^2 - 4z \quad ) = -1$$

## 11.1 Three-Dimension

$$(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2 = a^2.$$



By completing the squares, we find that

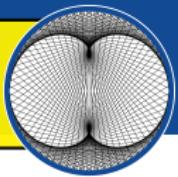
$$x^2 + y^2 + z^2 + 3x - 4z + 1 = 0$$

$$\left( x^2 + \color{brown}{3}x \right) + y^2 + (z^2 - 4z) = -1$$

$\uparrow$   
 $\left(\frac{3}{2}\right)^2 = \frac{9}{4}$

## 11.1 Three-Dimer

$$(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2 = a^2.$$



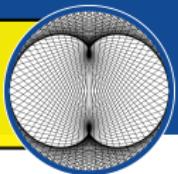
By completing the squares, we find that

$$x^2 + y^2 + z^2 + 3x - 4z + 1 = 0$$

$$\left( x^2 + 3x + \frac{9}{4} \right) + y^2 + (z^2 - 4z ) = -1 + \frac{9}{4}$$

## 11.1 Three-Dimer

$$(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2 = a^2.$$



By completing the squares, we find that

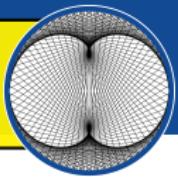
$$x^2 + y^2 + z^2 + 3x - 4z + 1 = 0$$

$$\left( x^2 + 3x + \frac{9}{4} \right) + y^2 + (z^2 - 4z ) = -1 + \frac{9}{4}$$

$$\left( \frac{-4}{2} \right)^2 = (-2)^2 = 4$$

## 11.1 Three-Dimer

$$(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2 = a^2.$$



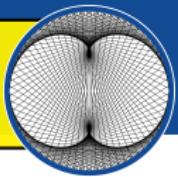
By completing the squares, we find that

$$x^2 + y^2 + z^2 + 3x - 4z + 1 = 0$$

$$\left(x^2 + 3x + \frac{9}{4}\right) + y^2 + (z^2 - 4z + 4) = -1 + \frac{9}{4} + 4$$

## 11.1 Three-Dimensional Surfaces

$$(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2 = a^2.$$



By completing the squares, we find that

$$\begin{aligned}x^2 + y^2 + z^2 + 3x - 4z + 1 &= 0 \\ \left(x^2 + 3x + \frac{9}{4}\right) + y^2 + (z^2 - 4z + 4) &= -1 + \frac{9}{4} + 4 \\ &= \frac{21}{4}.\end{aligned}$$

## 11.1 Three-Dimer

$$(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2 = a^2.$$



By completing the squares, we find that

$$x^2 + y^2 + z^2 + 3x - 4z + 1 = 0$$

$$\left(x^2 + 3x + \frac{9}{4}\right) + y^2 + (z^2 - 4z + 4) = -1 + \frac{9}{4} + 4$$

$$\left(x + \frac{3}{2}\right)^2 + y^2 + (z - 2)^2 = \frac{21}{4}.$$

## 11.1 Three-Dimensional Surfaces

$$(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2 = a^2.$$



By completing the squares, we find that

$$x^2 + y^2 + z^2 + 3x - 4z + 1 = 0$$

$$x_0 = \left( -\frac{3}{2} + 3x + \frac{9}{4} \right) + y^2 + (z^2 - 4z + 4) = -1 + \frac{9}{4} + 4$$

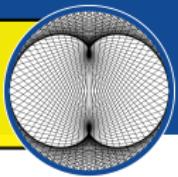
$$y_0 = 0$$

$$z_0 = 2$$

$$\left( x + \frac{3}{2} \right)^2 + y^2 + (z - 2)^2 = \frac{21}{4}.$$

## 11.1 Three-Dimensional Surfaces

$$(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2 = a^2.$$



By completing the squares, we find that

$$x^2 + y^2 + z^2 + 3x - 4z + 1 = 0$$

$$x_0 = \left( -\frac{3}{2} + 3x + \frac{9}{4} \right) + y^2 + (z^2 - 4z + 4) = -1 + \frac{9}{4} + 4$$

$$y_0 = 0$$

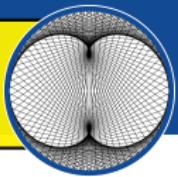
$$z_0 = 2$$

$$\left( x + \frac{3}{2} \right)^2 + y^2 + (z - 2)^2 = \frac{21}{4}.$$

$$a^2 = \frac{21}{4}$$

## 11.1 Three-Dimensional Surfaces

$$(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2 = a^2.$$



By completing the squares, we find that

$$x^2 + y^2 + z^2 + 3x - 4z + 1 = 0$$

$$x_0 = \left( -\frac{3}{2} + 3x + \frac{9}{4} \right) + y^2 + (z^2 - 4z + 4) = -1 + \frac{9}{4} + 4$$

$$y_0 = 0$$

$$z_0 = 2$$

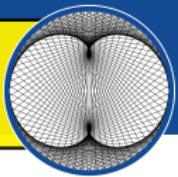
$$\left( x + \frac{3}{2} \right)^2 + y^2 + (z - 2)^2 = \frac{21}{4}.$$

$$a^2 = \frac{21}{4}$$

The centre is at  $P_0(x_0, y_0, z_0) = P_0(-\frac{3}{2}, 0, 2)$  and the radius is  $a = \sqrt{\frac{21}{4}} = \frac{\sqrt{3}\sqrt{7}}{2}$ .

## 11.1 Three-Dimensional

$$(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2 = a^2.$$



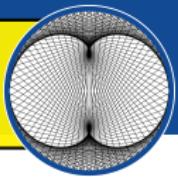
### Example

Find the centre and radius of the sphere

$$x^2 + y^2 + z^2 + 6x - 6y + 6z = 7.$$

## 11.1 Three-Dimer

$$(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2 = a^2.$$



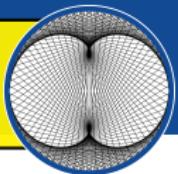
Since  $(x - b)^2 = x^2 - 2bx + b^2$  we have that

$$x^2 + y^2 + z^2 + 6x - 6y + 6z = 7$$

$$(x^2 + 6x) + (y^2 - 6y) + (z^2 + 6z) = 7$$

## 11.1 Three-Dimer

$$(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2 = a^2.$$



Since  $(x - b)^2 = x^2 - 2bx + b^2$  we have that

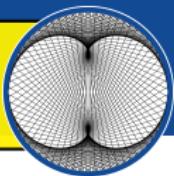
$$x^2 + y^2 + z^2 + 6x - 6y + 6z = 7$$

$$(x^2 + 6x) + (y^2 - 6y) + (z^2 + 6z) = 7$$

$$(x^2 + 6x + 9) + (y^2 - 6y + 9) + (z^2 + 6z + 9) = 7 + 9 + 9 + 9$$

## 11.1 Three-Dimer

$$(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2 = a^2.$$



Since  $(x - b)^2 = x^2 - 2bx + b^2$  we have that

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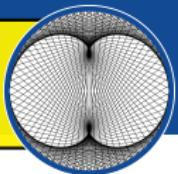
$$(x^2 + 6x) + (y^2 - 6y) + (z^2 + 6z) = 7$$

$$(x^2 + 6x + 9) + (y^2 - 6y + 9) + (z^2 + 6z + 9) = 7 + 9 + 9 + 9$$

$$(x^2 + 6x + 9) + (y^2 - 6y + 9) + (z^2 + 6z + 9) = 34$$

## 11.1 Three-Dimer

$$(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2 = a^2.$$



Since  $(x - b)^2 = x^2 - 2bx + b^2$  we have that

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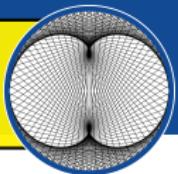
$$(x^2 + 6x + 9) + (y^2 - 6y + 9) + (z^2 + 6z + 9) = 7 + 9 + 9 + 9$$

$$(x^2 + 6x + 9) + (y^2 - 6y + 9) + (z^2 + 6z + 9) = 34$$

$$(x + 3)^2 + (y - 3)^2 + (z + 3)^2 = 34$$

## 11.1 Three-Dimensional Surfaces

$$(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2 = a^2.$$



Since  $(x - b)^2 = x^2 - 2bx + b^2$  we have that

$$x^2 + y^2 + z^2 + 6x - 6y + 6z = 7$$

$$(x^2 + 6x) + (y^2 - 6y) + (z^2 + 6z) = 7$$

$$(x^2 + 6x + 9) + (y^2 - 6y + 9) + (z^2 + 6z + 9) = 7 + 9 + 9 + 9$$

$$(x^2 + 6x + 9) + (y^2 - 6y + 9) + (z^2 + 6z + 9) = 34$$

$$(x + 3)^2 + (y - 3)^2 + (z + 3)^2 = 34$$

The centre is at  $P_0(x_0, y_0, z_0) = P_0(-3, 3, -3)$  and the radius is  $a = \sqrt{34}$ .

**EXAMPLE 5** Here are some geometric interpretations of inequalities and equations involving spheres.

(a)  $x^2 + y^2 + z^2 < 4$

The interior of the sphere  $x^2 + y^2 + z^2 = 4$ .

(b)  $x^2 + y^2 + z^2 \leq 4$

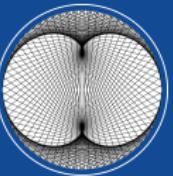
The solid ball bounded by the sphere  $x^2 + y^2 + z^2 = 4$ . Alternatively, the sphere  $x^2 + y^2 + z^2 = 4$  together with its interior.

(c)  $x^2 + y^2 + z^2 > 4$

The exterior of the sphere  $x^2 + y^2 + z^2 = 4$ .

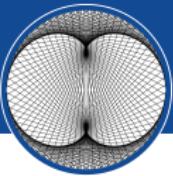
(d)  $x^2 + y^2 + z^2 = 4, z \leq 0$

The lower hemisphere cut from the sphere  $x^2 + y^2 + z^2 = 4$  by the  $xy$ -plane (the plane  $z = 0$ ). ■

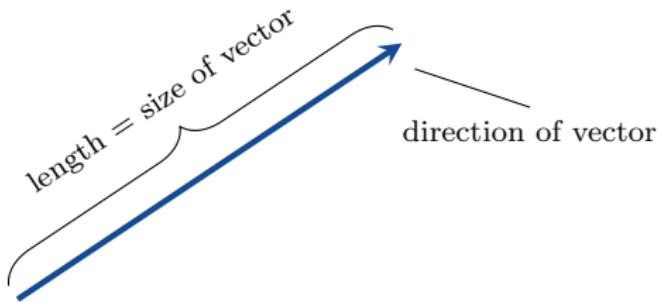


# Vectors

## 11.2 Vectors

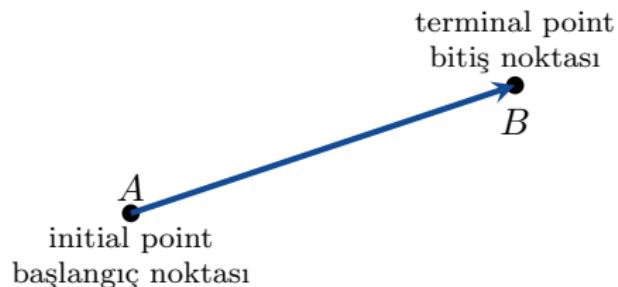
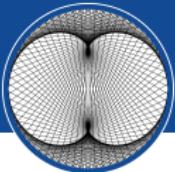


For some quantities (mass, time, distance, ...) we only need a number. For some quantities (velocity, force, ...) we need a number and a direction.



A *vector* is an object which has a size (length) and a direction.

## 11.2 Vectors

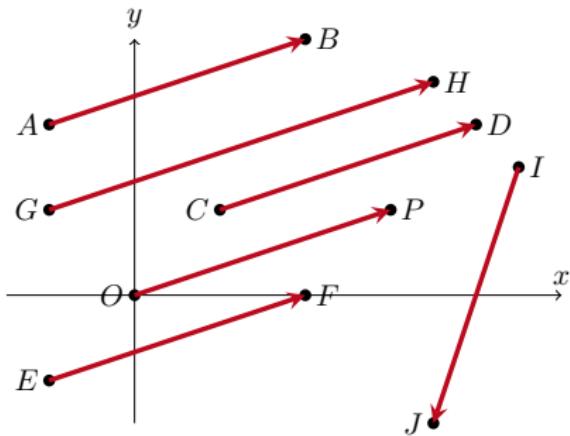
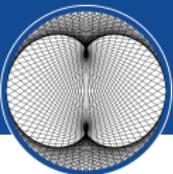


### Definition

The vector  $\overrightarrow{AB}$  has *initial point*  $A$  and *terminal point*  $B$ .

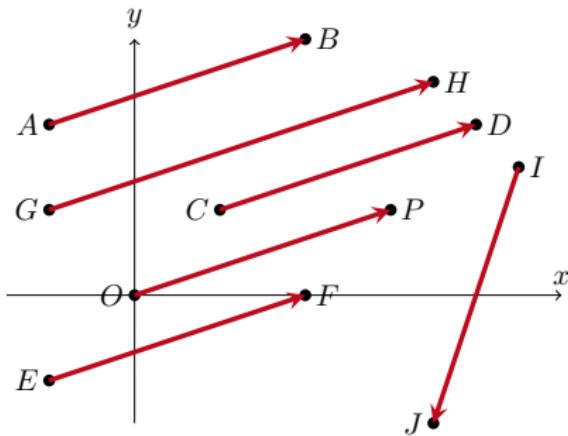
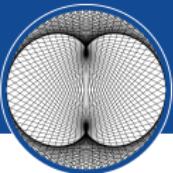
The *length* of  $\overrightarrow{AB}$  is written  $\|\overrightarrow{AB}\|$  (or  $|\overrightarrow{AB}|$ ).

## 11.2 Vectors



Two vectors are equal if they have the same length and the same direction.

## 11.2 Vectors

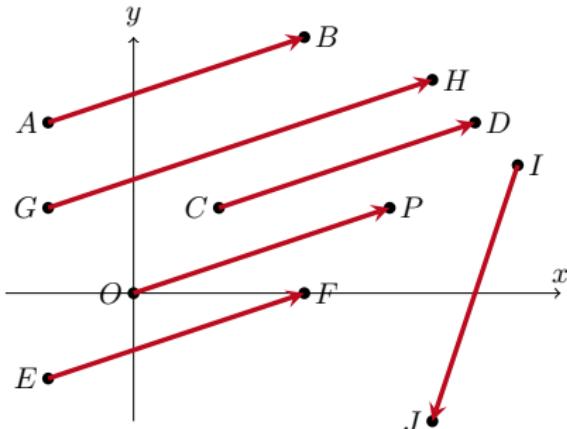
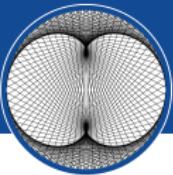


Two vectors are equal if they have the same length and the same direction.

We can say that

$$\overrightarrow{AB} = \overrightarrow{CD} = \overrightarrow{EF} = \overrightarrow{OP}.$$

## 11.2 Vectors

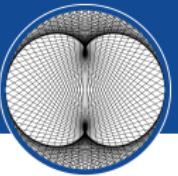


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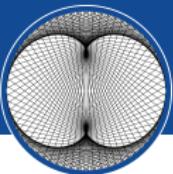
$$\overrightarrow{AB} = \overrightarrow{CD} = \overrightarrow{EF} = \overrightarrow{OP}.$$

Note that  $\overrightarrow{AB} \neq \overrightarrow{GH}$  because the lengths are different, and  $\overrightarrow{AB} \neq \overrightarrow{IJ}$  because the directions are different.



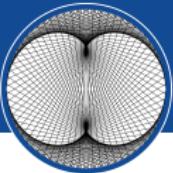
### Notation

When we use a computer, we use bold letters for vectors: **u**, **v**, **w**, . . . .



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When we use a computer, we use bold letters for vectors: **u**, **v**, **w**, .... When we use a pen, we use underlined letters for vectors: u, v, w, ....



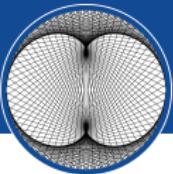
### Notation

When we use a computer, we use bold letters for vectors: **u**, **v**, **w**, .... When we use a pen, we use underlined letters for vectors: u, v, w, ....

If we type  $a\mathbf{u} + b\mathbf{v}$  or write  $a\underline{u} + b\underline{v}$ , then

- $a$  and  $b$  are numbers; and
- **u**, **v**, u and v are vectors.

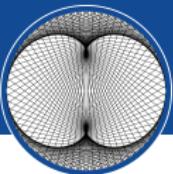
## 11.2 Vectors



### Definition

In  $\mathbb{R}^2$ : If  $\mathbf{v}$  has initial point  $(0, 0)$  and terminal point  $(v_1, v_2)$ , then the *component form* of  $\mathbf{v}$  is  $\mathbf{v} = (v_1, v_2)$ .

## 11.2 Vectors

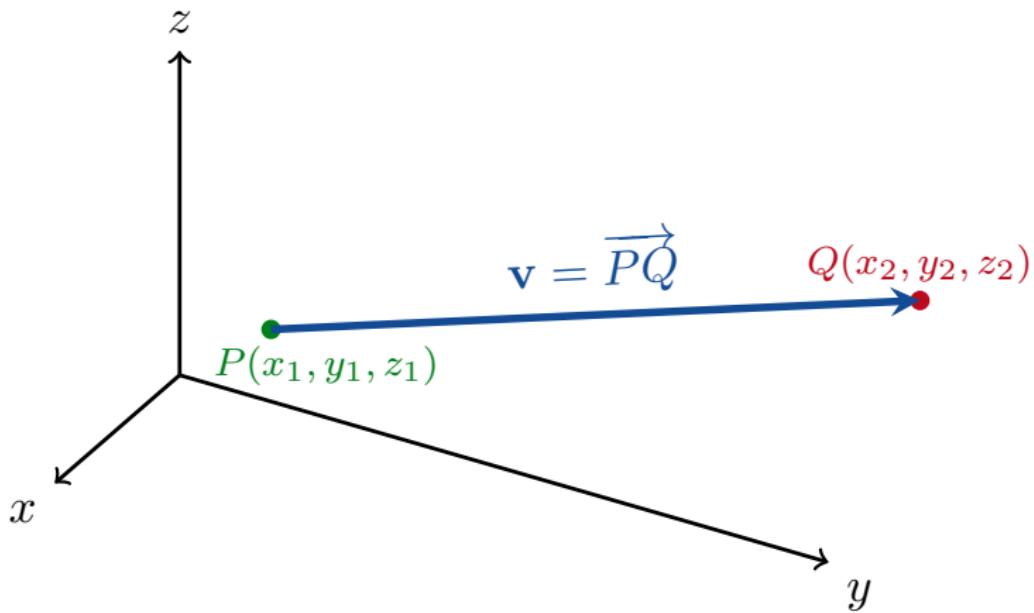
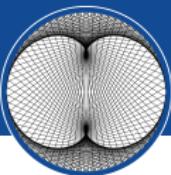


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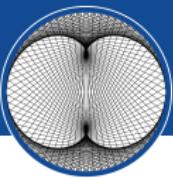
In  $\mathbb{R}^3$ : If  $\mathbf{v}$  has initial point  $(0, 0, 0)$  and terminal point  $(v_1, v_2, v_3)$ , then the *component form* of  $\mathbf{v}$  is  $\mathbf{v} = (v_1, v_2, v_3)$ .

## 11.2 Vectors



$$(v_1, v_2, v_3) = \mathbf{v} = (x_2 - x_1, y_2 - y_1, z_2 - z_1)$$

## 11.2 Vectors

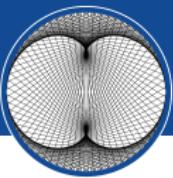


### Definition

In  $\mathbb{R}^2$ : The *norm* (or *length*) of  $\mathbf{v} = (v_1, v_2)$  is

$$\|\mathbf{v}\| = \sqrt{v_1^2 + v_2^2}$$

## 11.2 Vectors



### Definition

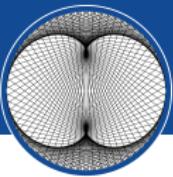
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In  $\mathbb{R}^3$ : The *norm* of  $\mathbf{v} = \overrightarrow{PQ}$  is

$$\begin{aligned}\|\mathbf{v}\| &= \sqrt{v_1^2 + v_2^2 + v_3^2} \\ &= \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}.\end{aligned}$$

## 11.2 Vectors



### Definition

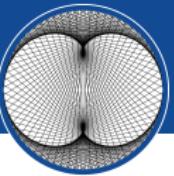
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The vectors  $\mathbf{0} = (0, 0)$  and  $\mathbf{0} = (0, 0, 0)$  have norm  $\|\mathbf{0}\| = 0$ .



## 11.2 Vectors

### Definition

In  $\mathbb{R}^2$ : The *norm* (or *length*) of  $\mathbf{v} = (v_1, v_2)$  is

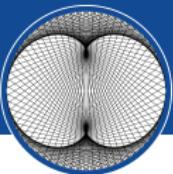
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The vectors  $\mathbf{0} = (0, 0)$  and  $\mathbf{0} = (0, 0, 0)$  have norm  $\|\mathbf{0}\| = 0$ . If  $\mathbf{v} \neq \mathbf{0}$ , then  $\|\mathbf{v}\| > 0$ .

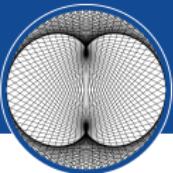
## 11.2 Vectors



### Example

Find (1) the component form; and (2) the norm of the vector with initial point  $P(-3, 4, 1)$  and terminal point  $Q(-5, 2, 2)$ .

## 11.2 Vectors

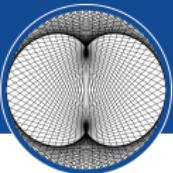


### Example

Find (1) the component form; and (2) the norm of the vector with initial point  $P(-3, 4, 1)$  and terminal point  $Q(-5, 2, 2)$ .

1  $\mathbf{v} = (v_1, v_2, v_3) = Q - P = (-5, 2, 2) - (-3, 4, 1)$   
 $= (-2, -2, 1).$

## 11.2 Vectors

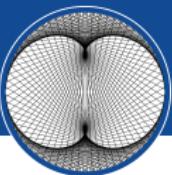


### Example

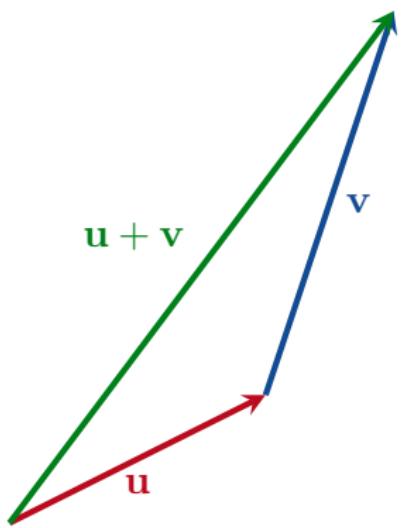
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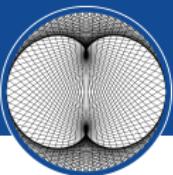
$$\begin{aligned} \text{1 } \mathbf{v} &= (v_1, v_2, v_3) = Q - P = (-5, 2, 2) - (-3, 4, 1) \\ &= (-2, -2, 1). \end{aligned}$$

$$\text{2 } \|\mathbf{v}\| = \sqrt{v_1^2 + v_2^2 + v_3^2} = \sqrt{(-2)^2 + (-2)^2 + 1^2} = \sqrt{9} = 3.$$

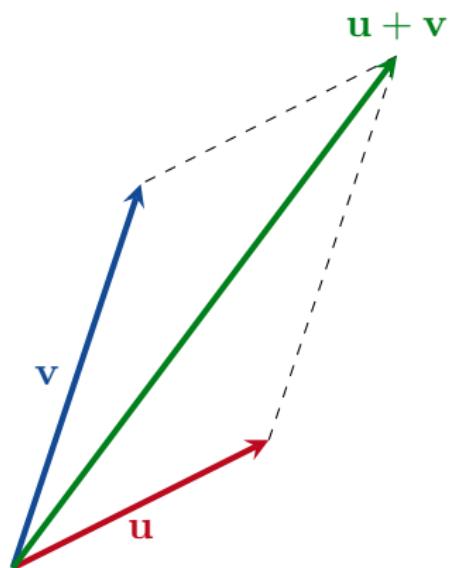
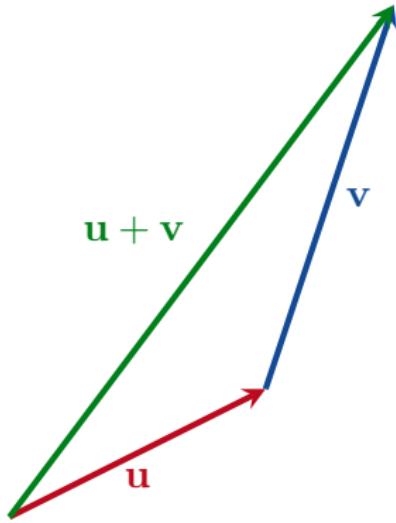


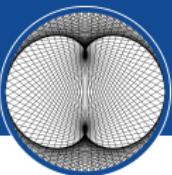
### Vector Algebra: Addition



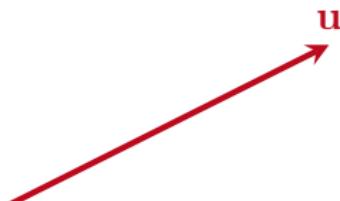
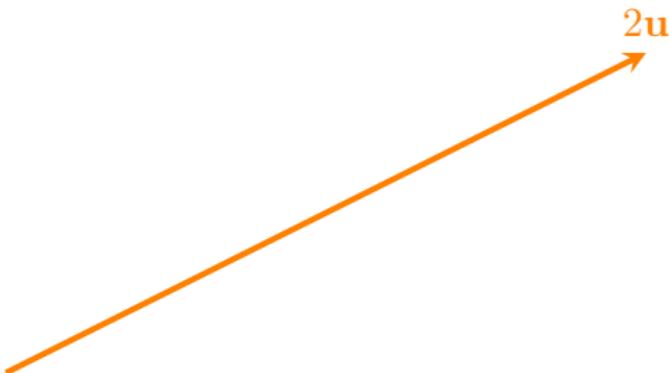


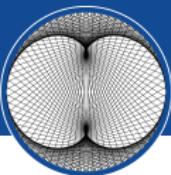
## Vector Algebra: Addition



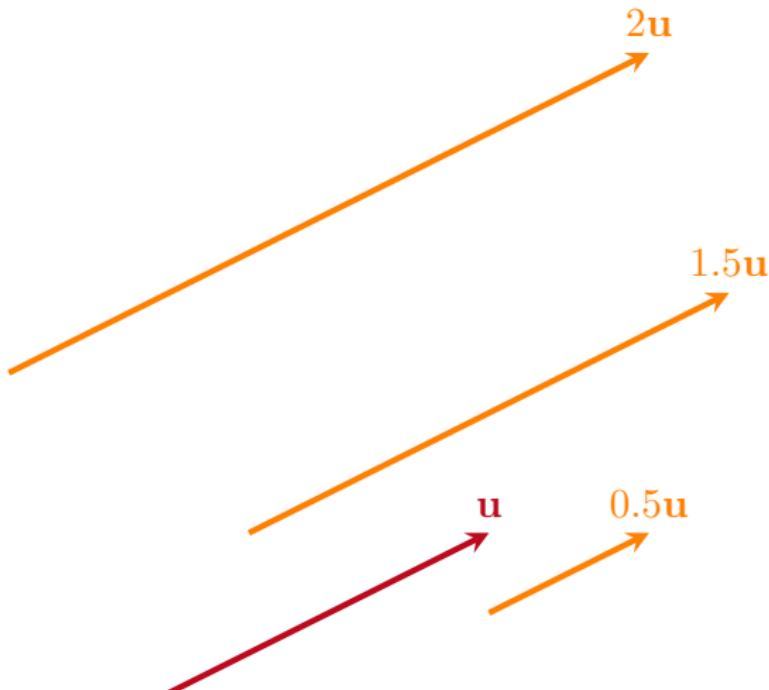


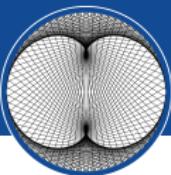
## Vector Algebra: Multiplication by a Constant



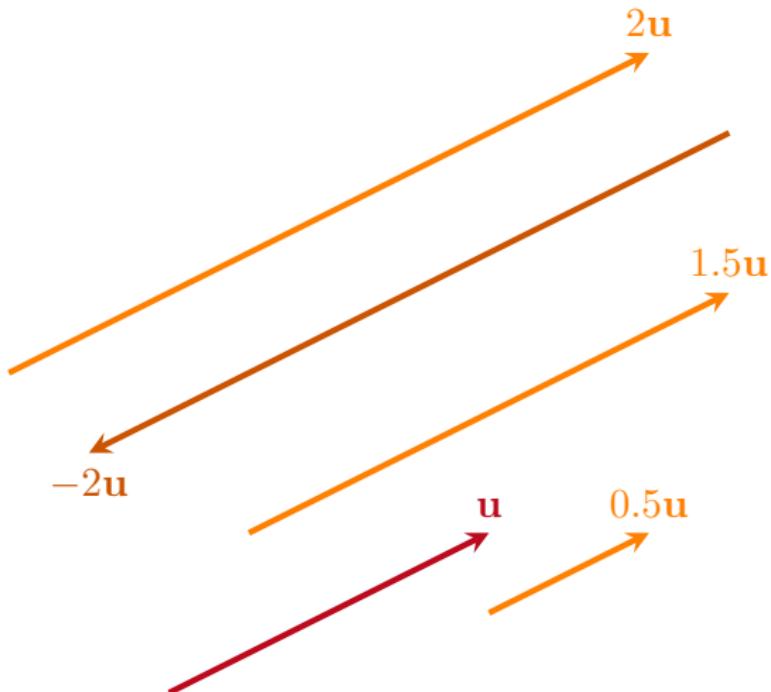


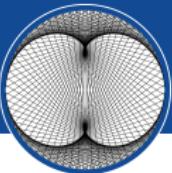
## Vector Algebra: Multiplication by a Constant





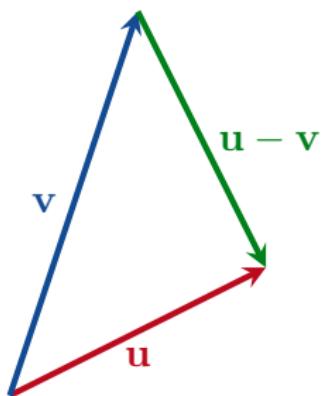
## Vector Algebra: Multiplication by a Constant

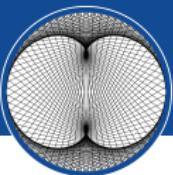




## Vector Algebra: Subtraction

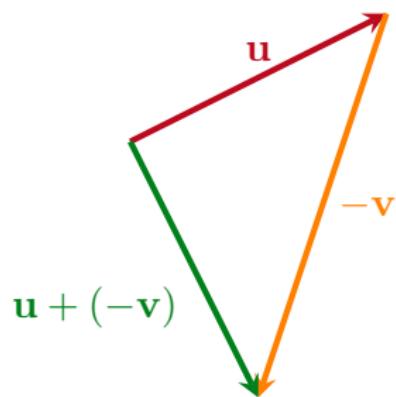
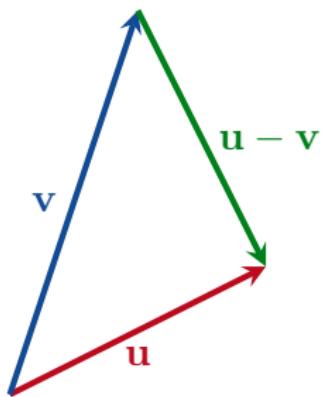
$$\mathbf{v} + (\mathbf{u} - \mathbf{v}) = \mathbf{u}$$



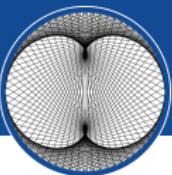


## Vector Algebra: Subtraction

$$\mathbf{v} + (\mathbf{u} - \mathbf{v}) = \mathbf{u}$$

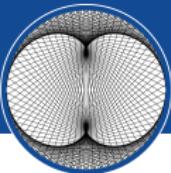


## 11.2 Vectors



Let  $\mathbf{u} = (u_1, u_2, u_3)$  and  $\mathbf{v} = (v_1, v_2, v_3)$  be vectors. Let  $k$  be a number.

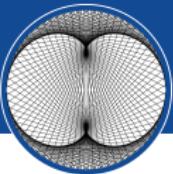
## 11.2 Vectors



Let  $\mathbf{u} = (u_1, u_2, u_3)$  and  $\mathbf{v} = (v_1, v_2, v_3)$  be vectors. Let  $k$  be a number. Then

$$\mathbf{u} + \mathbf{v} = (u_1 + v_1, u_2 + v_2, u_3 + v_3)$$

## 11.2 Vectors



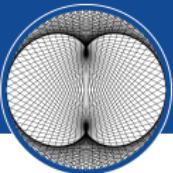
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and

$$k\mathbf{u} = (ku_1, ku_2, ku_3).$$

## 11.2 Vectors



Note that

$$\|k\mathbf{u}\| = \|(ku_1, ku_2, ku_3)\|$$

$$=$$

$$=$$

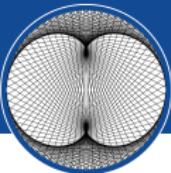
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## 11.2 Vectors



Note that

$$\begin{aligned}\|k\mathbf{u}\| &= \|(ku_1, ku_2, ku_3)\| \\ &= \sqrt{(ku_1)^2 + (ku_2)^2 + (ku_3)^2}\end{aligned}$$

=

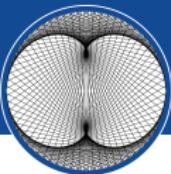
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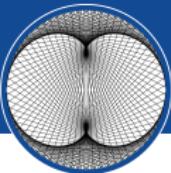
## 11.2 Vectors



Note that

$$\begin{aligned}\|k\mathbf{u}\| &= \|(ku_1, ku_2, ku_3)\| \\&= \sqrt{(ku_1)^2 + (ku_2)^2 + (ku_3)^2} \\&= \sqrt{k^2u_1^2 + k^2u_2^2 + k^2u_3^2} \\&= \\&= \\&= .\end{aligned}$$

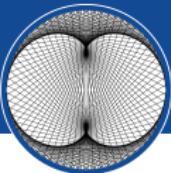
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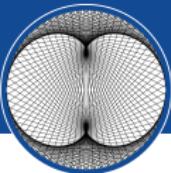
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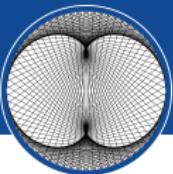
## 11.2 Vectors



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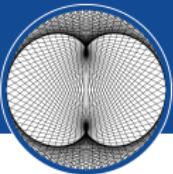
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## 11.2 Vectors



The vector  $-\mathbf{u} = (-1)\mathbf{u}$  has the same length as  $\mathbf{u}$ , but points in the opposite direction.

## 11.2 Vectors

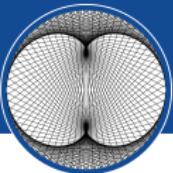


### Example

Let  $\mathbf{u} = (-1, 3, 1)$  and  $\mathbf{v} = (4, 7, 0)$ .

Find  $2\mathbf{u} + 3\mathbf{v}$ ,  $\mathbf{u} - \mathbf{v}$ , and  $\left\| \frac{1}{2}\mathbf{u} \right\|$ .

## 11.2 Vectors



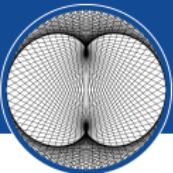
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1  $2\mathbf{u} + 3\mathbf{v} = 2(-1, 3, 1) + 3(4, 7, 0) = (-2, 6, 2) + (12, 21, 0) = (10, 27, 2);$

## 11.2 Vectors



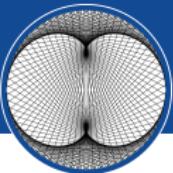
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- 2  $\mathbf{u} - \mathbf{v} = (-1, 3, 1) - (4, 7, 0) = (-5, -4, 1);$

## 11.2 Vectors



### Example

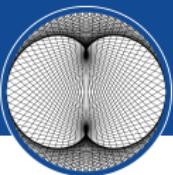
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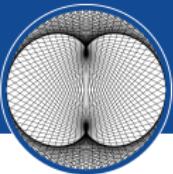
3  $\left\| \frac{1}{2}\mathbf{u} \right\| = \frac{1}{2} \left\| \mathbf{u} \right\| = \frac{1}{2} \sqrt{(-1)^2 + 3^2 + 1^2} = \frac{1}{2} \sqrt{11}.$



### Properties of Vector Operations

Let  $\mathbf{u}$ ,  $\mathbf{v}$  and  $\mathbf{w}$  be vectors. Let  $a$  and  $b$  be numbers. Then

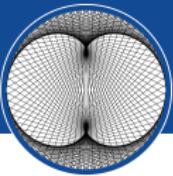
1  $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u};$



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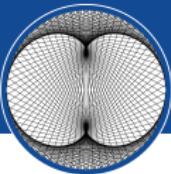
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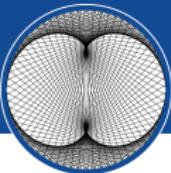
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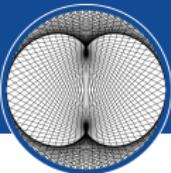
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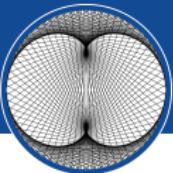
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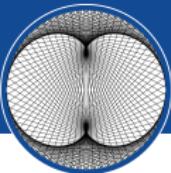
- 1  $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$ ;
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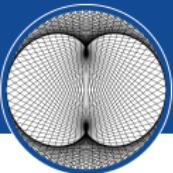
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- 7  $a(b\mathbf{u}) = (ab)\mathbf{u};$
- 8  $a(\mathbf{u} + \mathbf{v}) = a\mathbf{u} + a\mathbf{v};$

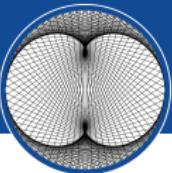


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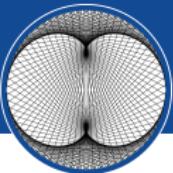
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- 8  $a(\mathbf{u} + \mathbf{v}) = a\mathbf{u} + a\mathbf{v};$
- 9  $(a + b)\mathbf{u} = a\mathbf{u} + b\mathbf{u}.$

## 11.2 Vectors



### Remark

We **can not** multiply vectors. Never never never never write “ $\mathbf{uv}$ ”.

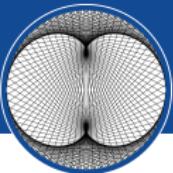


### Unit Vectors

#### Definition

$\mathbf{u}$  is called a *unit vector*  $\iff \|\mathbf{u}\| = 1$ .

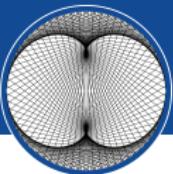
## 11.2 Vectors



### Example

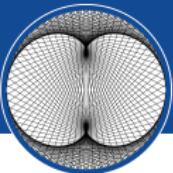
$\mathbf{u} = (2^{-\frac{1}{2}}, \frac{1}{2}, -\frac{1}{2})$  is a unit vector because

$$\|\mathbf{u}\| = \sqrt{\left(\frac{1}{\sqrt{2}}\right)^2 + \left(\frac{1}{2}\right)^2 + \left(-\frac{1}{2}\right)^2} = \sqrt{\frac{1}{2} + \frac{1}{4} + \frac{1}{4}} = 1.$$



### Standard Unit Vectors

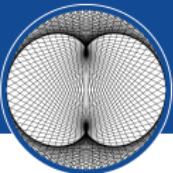
In  $\mathbb{R}^2$ : The *standard unit vectors* are  $\mathbf{i} = (1, 0)$  and  $\mathbf{j} = (0, 1)$ .



### Standard Unit Vectors

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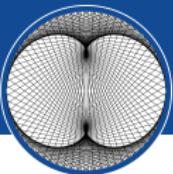


## Standard Unit Vectors

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In  $\mathbb{R}^3$ : The *standard unit vectors* are  $\mathbf{i} = (1, 0, 0)$ ,  $\mathbf{j} = (0, 1, 0)$  and  $\mathbf{k} = (0, 0, 1)$ . Any vector  $\mathbf{v} \in \mathbb{R}^3$  can be written

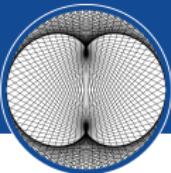
$$\begin{aligned}\mathbf{v} &= (v_1, v_2, v_3) = (v_1, 0, 0) + (0, v_2, 0) + (0, 0, v_3) \\ &= v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}.\end{aligned}$$



### Normalising a Vector

If  $\|\mathbf{v}\| \neq 0$ , then  $\frac{\mathbf{v}}{\|\mathbf{v}\|}$  is a unit vector because

$$\left\| \frac{\mathbf{v}}{\|\mathbf{v}\|} \right\| = \left\| \frac{1}{\|\mathbf{v}\|} \mathbf{v} \right\| = \frac{1}{\|\mathbf{v}\|} \|\mathbf{v}\| = 1.$$

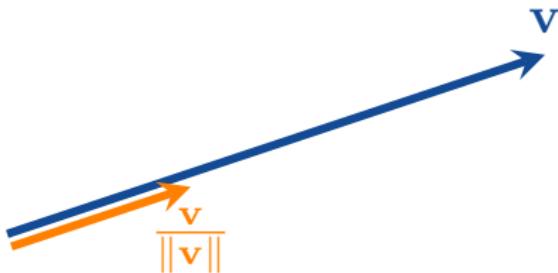


## Normalising a Vector

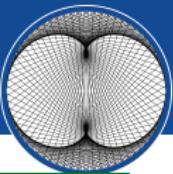
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Clearly  $\frac{\mathbf{v}}{\|\mathbf{v}\|}$  and  $\mathbf{v}$  point in the same direction.



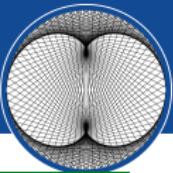
## 11.2 Vectors



### Example

Find a unit vector  $\mathbf{u}$  which points in the same direction as  $\overrightarrow{P_1P_2}$ , where  $P_1(1, 0, 1)$  and  $P_2(3, 2, 0)$ .

## 11.2 Vectors



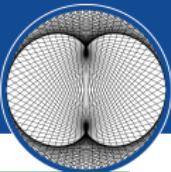
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We calculate that

$$\overrightarrow{P_1P_2} = P_2 - P_1 = (3, 2, 0) - (1, 0, 1) = (2, 2, -1) = 2\mathbf{i} + 2\mathbf{j} - \mathbf{k}$$

## 11.2 Vectors



### Example

Find a unit vector  $\mathbf{u}$  which points in the same direction as  $\overrightarrow{P_1P_2}$ , where  $P_1(1, 0, 1)$  and  $P_2(3, 2, 0)$ .

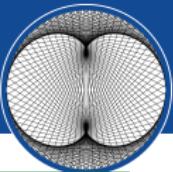
We calculate that

$$\overrightarrow{P_1P_2} = P_2 - P_1 = (3, 2, 0) - (1, 0, 1) = (2, 2, -1) = 2\mathbf{i} + 2\mathbf{j} - \mathbf{k}$$

and that

$$\left\| \overrightarrow{P_1P_2} \right\| = \sqrt{2^2 + 2^2 + (-1)^2} = 3.$$

## 11.2 Vectors



### Example

Find a unit vector  $\mathbf{u}$  which points in the same direction as  $\overrightarrow{P_1P_2}$ , where  $P_1(1, 0, 1)$  and  $P_2(3, 2, 0)$ .

We calculate that

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and that

$$\left\| \overrightarrow{P_1P_2} \right\| = \sqrt{2^2 + 2^2 + (-1)^2} = 3.$$

The required unit vector is

$$\mathbf{u} = \frac{\overrightarrow{P_1P_2}}{\left\| \overrightarrow{P_1P_2} \right\|} = \frac{2\mathbf{i} + 2\mathbf{j} - \mathbf{k}}{3} = \frac{2}{3}\mathbf{i} + \frac{2}{3}\mathbf{j} - \frac{1}{3}\mathbf{k}.$$

**EXAMPLE 5** If  $\mathbf{v} = 3\mathbf{i} - 4\mathbf{j}$  is a velocity vector, express  $\mathbf{v}$  as a product of its speed times its direction of motion.

**Solution** Speed is the magnitude (length) of  $\mathbf{v}$ :

$$|\mathbf{v}| = \sqrt{(3)^2 + (-4)^2} = \sqrt{9 + 16} = 5.$$

The unit vector  $\mathbf{v}/|\mathbf{v}|$  is the direction of  $\mathbf{v}$ :

$$\frac{\mathbf{v}}{|\mathbf{v}|} = \frac{3\mathbf{i} - 4\mathbf{j}}{5} = \frac{3}{5}\mathbf{i} - \frac{4}{5}\mathbf{j}.$$

So

$$\mathbf{v} = 3\mathbf{i} - 4\mathbf{j} = 5 \left( \underbrace{\frac{3}{5}\mathbf{i} - \frac{4}{5}\mathbf{j}}_{\substack{\text{Length} \\ \text{Direction of motion} \\ (\text{speed})}} \right).$$



If  $\mathbf{v} \neq \mathbf{0}$ , then

1.  $\frac{\mathbf{v}}{|\mathbf{v}|}$  is a unit vector called the direction of  $\mathbf{v}$ ;
2. the equation  $\mathbf{v} = |\mathbf{v}| \frac{\mathbf{v}}{|\mathbf{v}|}$  expresses  $\mathbf{v}$  as its length times its direction.

**EXAMPLE 6** A force of 6 newtons is applied in the direction of the vector  $\mathbf{v} = 2\mathbf{i} + 2\mathbf{j} - \mathbf{k}$ . Express the force  $\mathbf{F}$  as a product of its magnitude and direction.

**Solution** The force vector has magnitude 6 and direction  $\frac{\mathbf{v}}{|\mathbf{v}|}$ , so

$$\begin{aligned}\mathbf{F} &= 6 \frac{\mathbf{v}}{|\mathbf{v}|} = 6 \frac{2\mathbf{i} + 2\mathbf{j} - \mathbf{k}}{\sqrt{2^2 + 2^2 + (-1)^2}} = 6 \frac{2\mathbf{i} + 2\mathbf{j} - \mathbf{k}}{3} \\ &= 6 \left( \frac{2}{3}\mathbf{i} + \frac{2}{3}\mathbf{j} - \frac{1}{3}\mathbf{k} \right).\end{aligned}$$



## Midpoint of a Line Segment

Vectors are often useful in geometry. For example, the coordinates of the midpoint of a line segment are found by averaging.

The **midpoint**  $M$  of the line segment joining points  $P_1(x_1, y_1, z_1)$  and  $P_2(x_2, y_2, z_2)$  is the point

$$\left( \frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}, \frac{z_1 + z_2}{2} \right).$$

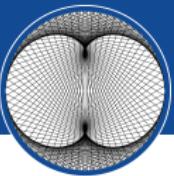
To see why, observe (Figure 12.16) that

$$\begin{aligned}\overrightarrow{OM} &= \overrightarrow{OP}_1 + \frac{1}{2}(\overrightarrow{P_1P_2}) = \overrightarrow{OP}_1 + \frac{1}{2}(\overrightarrow{OP}_2 - \overrightarrow{OP}_1) \\ &= \frac{1}{2}(\overrightarrow{OP}_1 + \overrightarrow{OP}_2) \\ &= \frac{x_1 + x_2}{2}\mathbf{i} + \frac{y_1 + y_2}{2}\mathbf{j} + \frac{z_1 + z_2}{2}\mathbf{k}.\end{aligned}$$

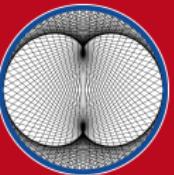
**EXAMPLE 7** The midpoint of the segment joining  $P_1(3, -2, 0)$  and  $P_2(7, 4, 4)$  is

$$\left( \frac{3 + 7}{2}, \frac{-2 + 4}{2}, \frac{0 + 4}{2} \right) = (5, 1, 2). \quad \blacksquare$$

## 11.2 Vectors



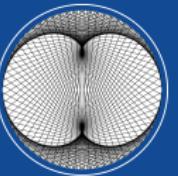
Please read the final two examples in this section of the textbook.



# Break

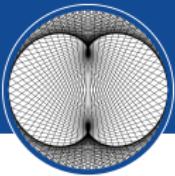
We will continue at 3pm





# The Dot Product

## 11.3 The Dot Product

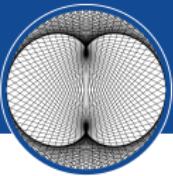


### Definition

In  $\mathbb{R}^2$ , the *dot product* of  $\mathbf{u} = (u_1, u_2) = u_1\mathbf{i} + u_2\mathbf{j}$  and  $\mathbf{v} = (v_1, v_2) = v_1\mathbf{i} + v_2\mathbf{j}$  is

$$\mathbf{u} \cdot \mathbf{v} = u_1v_1 + u_2v_2.$$

## 11.3 The Dot Product



### Definition

In  $\mathbb{R}^2$ , the *dot product* of  $\mathbf{u} = (u_1, u_2) = u_1\mathbf{i} + u_2\mathbf{j}$  and  $\mathbf{v} = (v_1, v_2) = v_1\mathbf{i} + v_2\mathbf{j}$  is

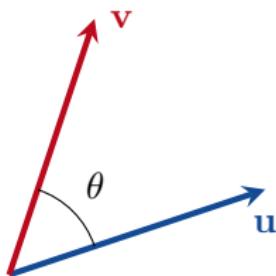
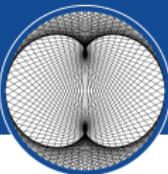
$$\mathbf{u} \cdot \mathbf{v} = u_1v_1 + u_2v_2.$$

### Definition

In  $\mathbb{R}^3$ , the *dot product* of  $\mathbf{u} = (u_1, u_2, u_3) = u_1\mathbf{i} + u_2\mathbf{j} + u_3\mathbf{k}$  and  $\mathbf{v} = (v_1, v_2, v_3) = v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}$  is

$$\mathbf{u} \cdot \mathbf{v} = u_1v_1 + u_2v_2 + u_3v_3.$$

## 11.3 The Dot Product

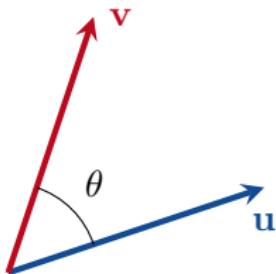
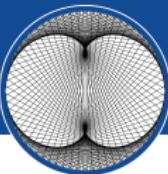


### Theorem

*The angle between  $\mathbf{u}$  and  $\mathbf{v}$  is*

$$\theta = \cos^{-1} \left( \frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{u}\| \|\mathbf{v}\|} \right).$$

## 11.3 The Dot Product



### Theorem

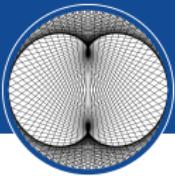
*The angle between  $\mathbf{u}$  and  $\mathbf{v}$  is*

$$\theta = \cos^{-1} \left( \frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{u}\| \|\mathbf{v}\|} \right).$$

This means that

$$\mathbf{u} \cdot \mathbf{v} = \|\mathbf{u}\| \|\mathbf{v}\| \cos \theta.$$

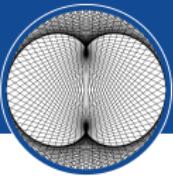
## 11.3 The Dot Product



### Example

$$\begin{aligned}(1, -2, -1) \cdot (-6, 2, -3) &= (1 \times -6) + (-2 \times 2) + (-1 \times -3) \\&= -6 - 4 + 3 = -7.\end{aligned}$$

## 11.3 The Dot Product



### Example

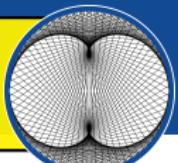
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### Example

$$\begin{aligned}\left(\frac{1}{2}\mathbf{i} + 3\mathbf{j} + \mathbf{k}\right) \cdot (4\mathbf{i} - \mathbf{j} + 2\mathbf{k}) &= \left(\frac{1}{2} \times 4\right) + (3 \times -1) + (1 \times 2) \\&= 2 - 3 + 2 = 1.\end{aligned}$$

## 11.3 The Dot Product

$$\theta = \cos^{-1} \left( \frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{u}\| \|\mathbf{v}\|} \right)$$

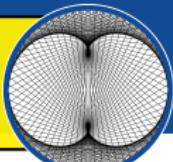


### Example

Find the angle between  $\mathbf{u} = \mathbf{i} - 2\mathbf{j} - 2\mathbf{k}$  and  $\mathbf{v} = 6\mathbf{i} + 3\mathbf{j} + 2\mathbf{k}$ .

## 11.3 The Dot Product

$$\theta = \cos^{-1} \left( \frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{u}\| \|\mathbf{v}\|} \right)$$



### Example

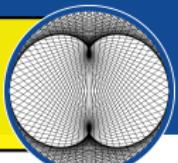
Find the angle between  $\mathbf{u} = \mathbf{i} - 2\mathbf{j} - 2\mathbf{k}$  and  $\mathbf{v} = 6\mathbf{i} + 3\mathbf{j} + 2\mathbf{k}$ .

Since

$$\begin{aligned}\mathbf{u} \cdot \mathbf{v} &= (1, -2, -2) \cdot (6, 3, 2) = (1 \times 6) + (-2 \times 3) + (-2 \times 2) \\ &= 6 - 6 - 4 = -4,\end{aligned}$$

## 11.3 The Dot Product

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### Example

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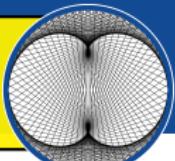
$$\|\mathbf{u}\| = \sqrt{1^2 + (-2)^2 + (-2)^2} = \sqrt{9} = 3$$

and

$$\|\mathbf{v}\| = \sqrt{6^2 + 3^2 + 2^2} = \sqrt{49} = 7,$$

## 11.3 The Dot Product

$$\theta = \cos^{-1} \left( \frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{u}\| \|\mathbf{v}\|} \right)$$



### Example

Find the angle between  $\mathbf{u} = \mathbf{i} - 2\mathbf{j} - 2\mathbf{k}$  and  $\mathbf{v} = 6\mathbf{i} + 3\mathbf{j} + 2\mathbf{k}$ .

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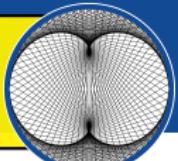
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## 11.3 The Dot Product

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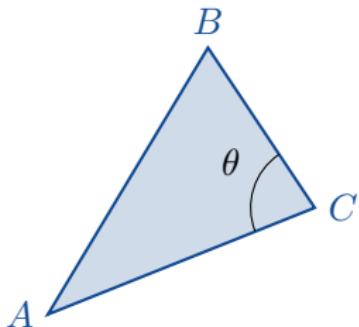
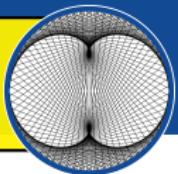
$$\|\mathbf{v}\| = \sqrt{6^2 + 3^2 + 2^2} = \sqrt{49} = 7,$$

we have that

$$\theta = \cos^{-1} \left( \frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{u}\| \|\mathbf{v}\|} \right) = \cos^{-1} \left( -\frac{4}{21} \right) \approx 1.76 \text{ radians} \approx 98.5^\circ.$$

## 11.3 The Dot Product

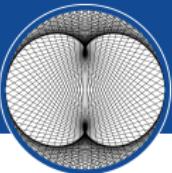
$$\theta = \cos^{-1} \left( \frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{u}\| \|\mathbf{v}\|} \right)$$



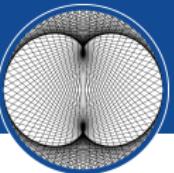
### Example

If  $A(0, 0)$ ,  $B(3, 5)$  and  $C(5, 2)$ , find  $\theta = \angle ACB$ .

## 11.3 The Dot Product



$\theta$  is the angle between  $\overrightarrow{CA}$  and  $\overrightarrow{CB}$ .



## 11.3 The Dot Product

$\theta$  is the angle between  $\overrightarrow{CA}$  and  $\overrightarrow{CB}$ . We calculate that

$$\overrightarrow{CA} = A - C = (0, 0) - (5, 2) = (-5, -2),$$

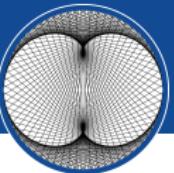
$$\overrightarrow{CB} = B - C = (3, 5) - (5, 2) = (-2, 3),$$

$$\overrightarrow{CA} \cdot \overrightarrow{CB} = (-5, -2) \cdot (-2, 3) = 4,$$

$$\|\overrightarrow{CA}\| = \sqrt{(-5)^2 + (-2)^2} = \sqrt{29}$$

and

$$\|\overrightarrow{CB}\| = \sqrt{(-2)^2 + 3^2} = \sqrt{13}.$$



## 11.3 The Dot Product

$\theta$  is the angle between  $\overrightarrow{CA}$  and  $\overrightarrow{CB}$ . We calculate that

$$\overrightarrow{CA} = A - C = (0, 0) - (5, 2) = (-5, -2),$$

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and

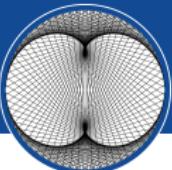
$$\|\overrightarrow{CB}\| = \sqrt{(-2)^2 + 3^2} = \sqrt{13}.$$

Therefore

$$\theta = \cos^{-1} \left( \frac{\overrightarrow{CA} \cdot \overrightarrow{CB}}{\|\overrightarrow{CA}\| \|\overrightarrow{CB}\|} \right) = \cos^{-1} \left( \frac{4}{\sqrt{29}\sqrt{13}} \right)$$

$$\approx 78.1^\circ \approx 1.36 \text{ radians.}$$

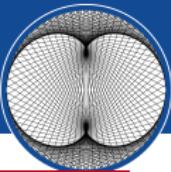
## 11.3 The Dot Product



### Definition

$\mathbf{u}$  and  $\mathbf{v}$  are *orthogonal*  $\iff \mathbf{u} \cdot \mathbf{v} = 0$ .

## 11.3 The Dot Product



### Definition

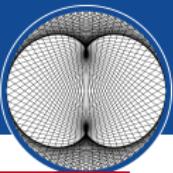
$\mathbf{u}$  and  $\mathbf{v}$  are *orthogonal*  $\iff \mathbf{u} \cdot \mathbf{v} = 0$ .

### Remark

Recall that

$$\mathbf{u} \cdot \mathbf{v} = \|\mathbf{u}\| \|\mathbf{v}\| \cos \theta.$$

## 11.3 The Dot Product



### Definition

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### Remark

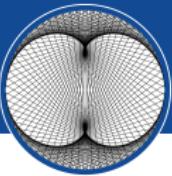
Recall that

$$\mathbf{u} \cdot \mathbf{v} = \|\mathbf{u}\| \|\mathbf{v}\| \cos \theta.$$

Therefore

$$\mathbf{u} \text{ and } \mathbf{v} \text{ are orthogonal} \iff \begin{cases} \mathbf{u} = \mathbf{0} \\ \text{or} \\ \mathbf{v} = \mathbf{0} \\ \text{or} \\ \theta = 90^\circ. \end{cases}$$

## 11.3 The Dot Product

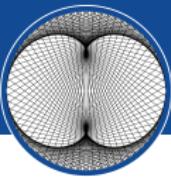


### Example

$\mathbf{u} = (3, -2)$  and  $\mathbf{v} = (4, 6)$  are orthogonal because

$$\mathbf{u} \cdot \mathbf{v} = (3, -2) \cdot (4, 6) = (3 \times 4) + (-2 \times 6) = 12 - 12 = 0.$$

## 11.3 The Dot Product



### Example

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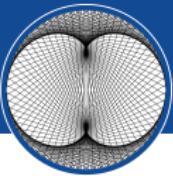
$$\mathbf{u} \cdot \mathbf{v} = (3, -2) \cdot (4, 6) = (3 \times 4) + (-2 \times 6) = 12 - 12 = 0.$$

### Example

$\mathbf{u} = 3\mathbf{i} - 2\mathbf{j} + \mathbf{k}$  and  $\mathbf{v} = 2\mathbf{j} + 4\mathbf{k}$  are orthogonal because

$$\mathbf{u} \cdot \mathbf{v} = (3 \times 0) + (-2 \times 2) + (1 \times 4) = 0 - 4 + 4 = 0.$$

## 11.3 The Dot Product



### Example

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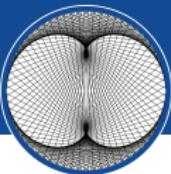
$$\mathbf{u} \cdot \mathbf{v} = (3 \times 0) + (-2 \times 2) + (1 \times 4) = 0 - 4 + 4 = 0.$$

### Example

$\mathbf{0}$  is orthogonal to every vector  $\mathbf{u}$  because

$$\mathbf{0} \cdot \mathbf{u} = (0, 0, 0) \cdot (u_1, u_2, u_3) = 0u_1 + 0u_2 + 0u_3 = 0.$$

## 11.3 The Dot Product

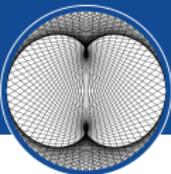


### Properties of the Dot Product

Let  $\mathbf{u}$ ,  $\mathbf{v}$  and  $\mathbf{w}$  be vectors. Let  $k$  be a number. Then

1  $\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u};$

## 11.3 The Dot Product

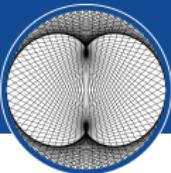


### Properties of the Dot Product

Let  $\mathbf{u}$ ,  $\mathbf{v}$  and  $\mathbf{w}$  be vectors. Let  $k$  be a number. Then

- 1  $\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u}$ ;
- 2  $(k\mathbf{u}) \cdot \mathbf{v} = \mathbf{u} \cdot (k\mathbf{v}) = k(\mathbf{u} \cdot \mathbf{v})$ ;

## 11.3 The Dot Product

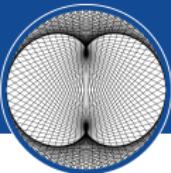


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- 3  $\mathbf{u} \cdot (\mathbf{v} + \mathbf{w}) = (\mathbf{u} \cdot \mathbf{v}) + (\mathbf{u} \cdot \mathbf{w})$ ;

## 11.3 The Dot Product

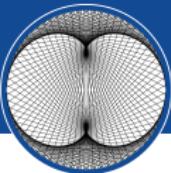


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- 3  $\mathbf{u} \cdot (\mathbf{v} + \mathbf{w}) = (\mathbf{u} \cdot \mathbf{v}) + (\mathbf{u} \cdot \mathbf{w})$ ;
- 4  $\mathbf{u} \cdot \mathbf{u} = \|\mathbf{u}\|^2$ ; and

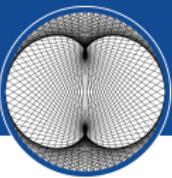
## 11.3 The Dot Product



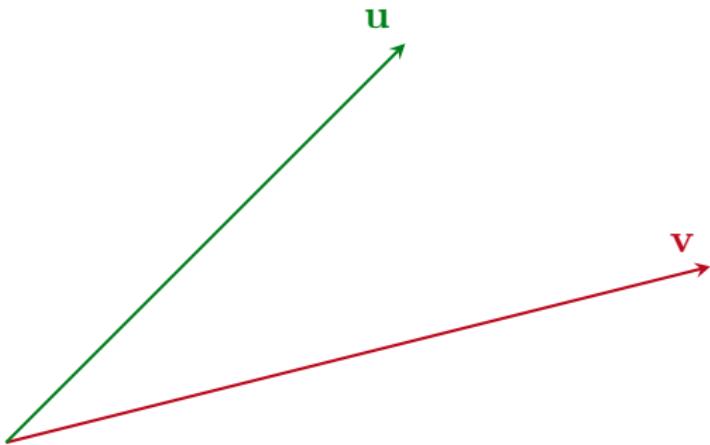
### Properties of the Dot Product

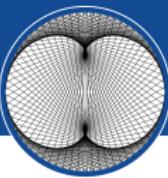
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- 1  $\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u}$ ;
- 2  $(k\mathbf{u}) \cdot \mathbf{v} = \mathbf{u} \cdot (k\mathbf{v}) = k(\mathbf{u} \cdot \mathbf{v})$ ;
- 3  $\mathbf{u} \cdot (\mathbf{v} + \mathbf{w}) = (\mathbf{u} \cdot \mathbf{v}) + (\mathbf{u} \cdot \mathbf{w})$ ;
- 4  $\mathbf{u} \cdot \mathbf{u} = \|\mathbf{u}\|^2$ ; and
- 5  $\mathbf{0} \cdot \mathbf{u} = 0$ .

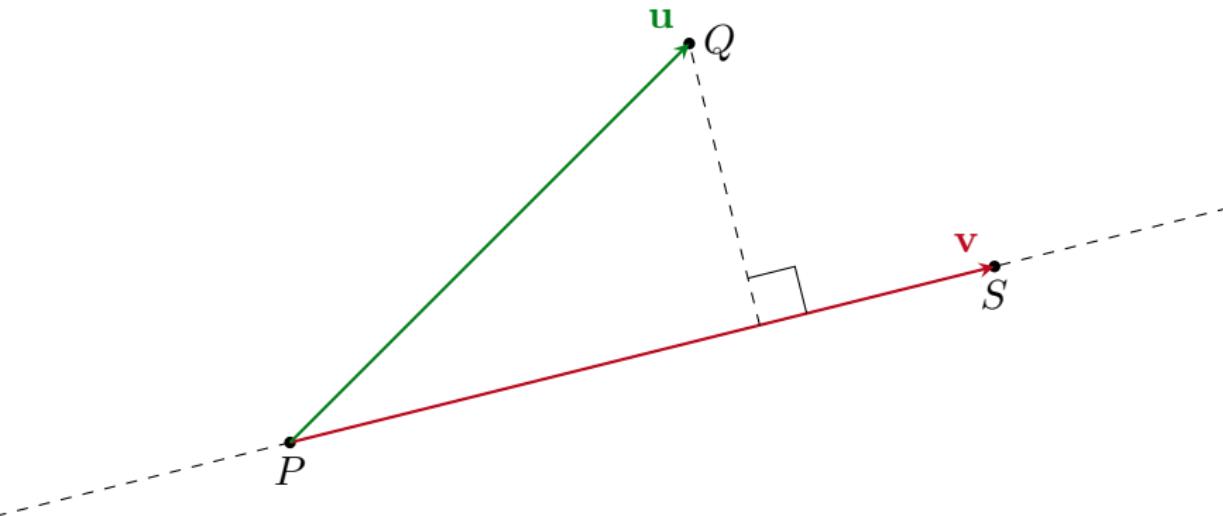


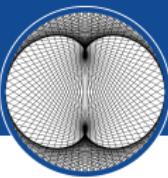
### Vector Projections



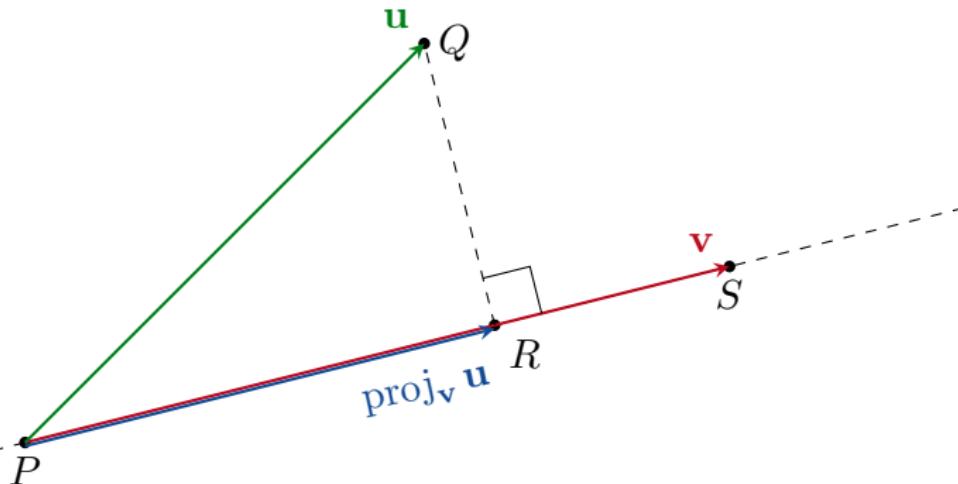


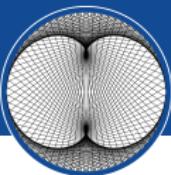
### Vector Projections



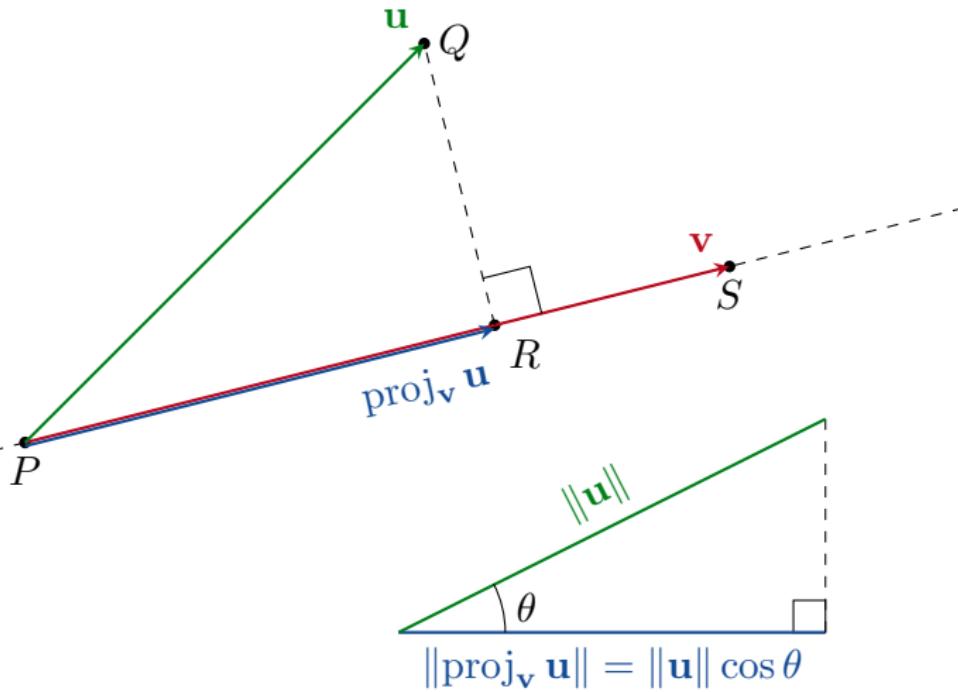


## Vector Projections

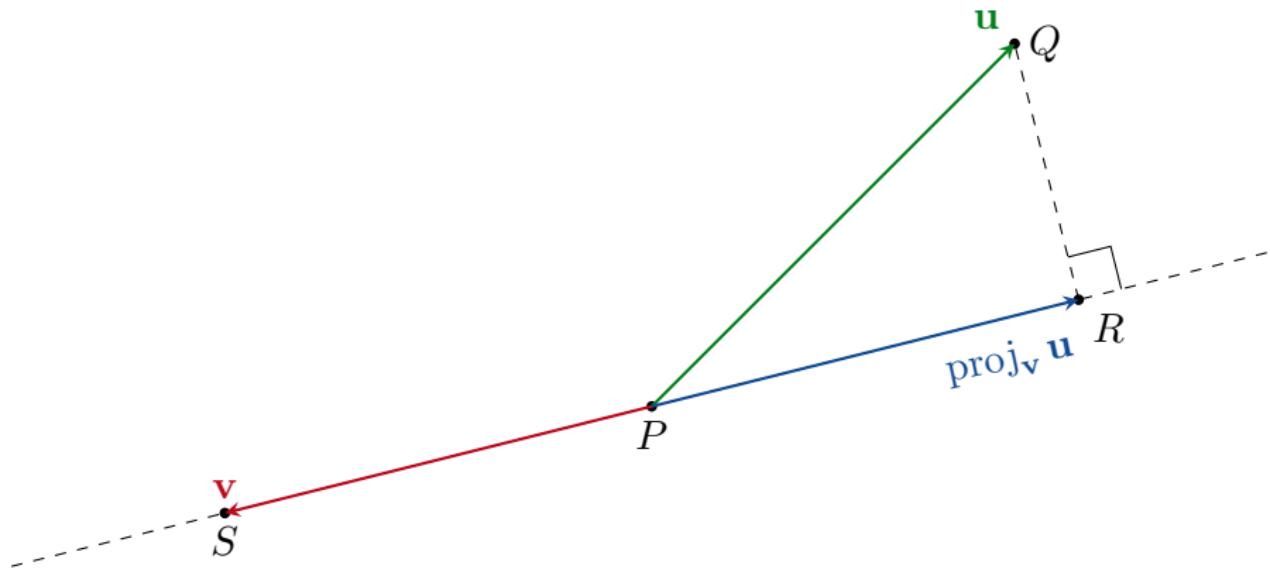
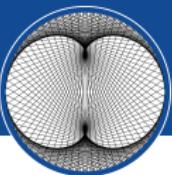




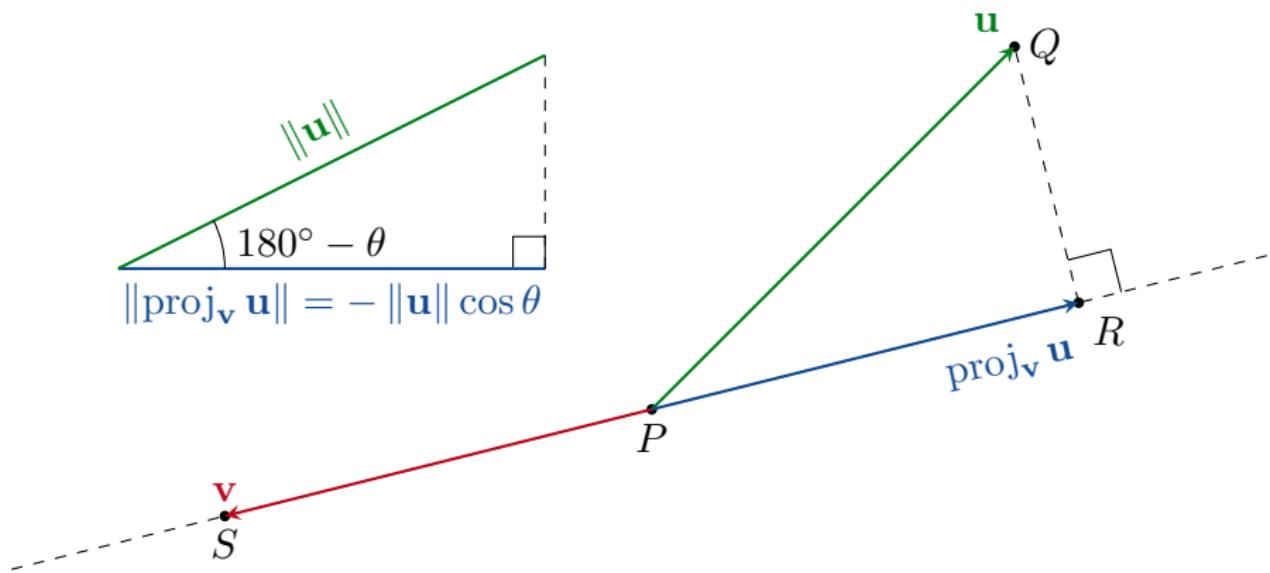
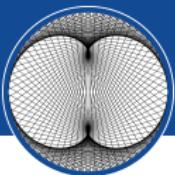
## Vector Projections



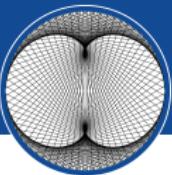
## 11.3 The Dot Product



## 11.3 The Dot Product



## 11.3 The Dot Product

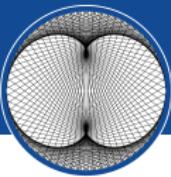


### Definition

The *vector projection* of  $\mathbf{u}$  onto  $\mathbf{v}$  is the vector

$$\text{proj}_{\mathbf{v}} \mathbf{u} = \overrightarrow{PR}.$$

## 11.3 The Dot Product



Now

$$\text{proj}_{\mathbf{v}} \mathbf{u} = (\text{length of } \text{proj}_{\mathbf{v}} \mathbf{u}) \begin{pmatrix} \text{a unit vector in} \\ \text{the same} \\ \text{direction as } \mathbf{v} \end{pmatrix}$$

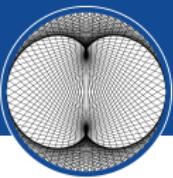
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## 11.3 The Dot Product



Now

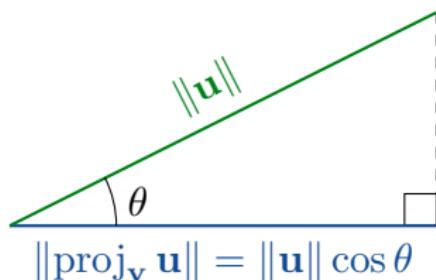
$$\text{proj}_{\mathbf{v}} \mathbf{u} = (\text{length of } \text{proj}_{\mathbf{v}} \mathbf{u}) \begin{pmatrix} \text{a unit vector in} \\ \text{the same} \\ \text{direction as } \mathbf{v} \end{pmatrix}$$

$$= \|\text{proj}_{\mathbf{v}} \mathbf{u}\| \left( \frac{\mathbf{v}}{\|\mathbf{v}\|} \right)$$

=

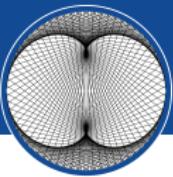
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$$\|\text{proj}_{\mathbf{v}} \mathbf{u}\| = \|\mathbf{u}\| \cos \theta$$

## 11.3 The Dot Product



Now

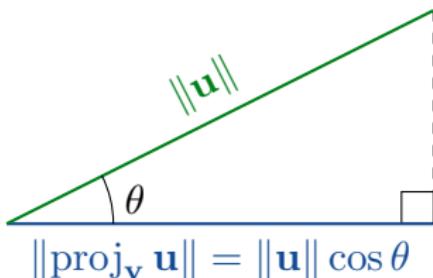
$$\text{proj}_{\mathbf{v}} \mathbf{u} = (\text{length of } \text{proj}_{\mathbf{v}} \mathbf{u}) \begin{pmatrix} \text{a unit vector in} \\ \text{the same} \\ \text{direction as } \mathbf{v} \end{pmatrix}$$

$$= \|\text{proj}_{\mathbf{v}} \mathbf{u}\| \left( \frac{\mathbf{v}}{\|\mathbf{v}\|} \right)$$

$$= \|\mathbf{u}\| (\cos \theta) \left( \frac{\mathbf{v}}{\|\mathbf{v}\|} \right)$$

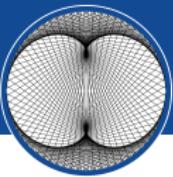
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$$\|\text{proj}_{\mathbf{v}} \mathbf{u}\| = \|\mathbf{u}\| \cos \theta$$

## 11.3 The Dot Product



Now

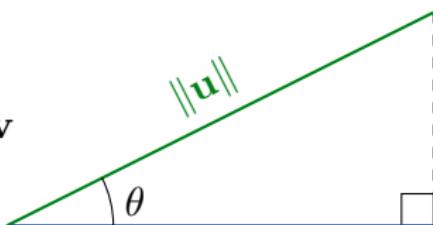
$$\text{proj}_{\mathbf{v}} \mathbf{u} = (\text{length of } \text{proj}_{\mathbf{v}} \mathbf{u}) \begin{pmatrix} \text{a unit vector in} \\ \text{the same} \\ \text{direction as } \mathbf{v} \end{pmatrix}$$

$$= \|\text{proj}_{\mathbf{v}} \mathbf{u}\| \left( \frac{\mathbf{v}}{\|\mathbf{v}\|} \right)$$

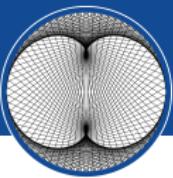
$$= \|\mathbf{u}\| (\cos \theta) \left( \frac{\mathbf{v}}{\|\mathbf{v}\|} \right)$$

$$= \left( \frac{\|\mathbf{u}\| \|\mathbf{v}\| \cos \theta}{\|\mathbf{v}\|^2} \right) \mathbf{v}$$

$$= \|\text{proj}_{\mathbf{v}} \mathbf{u}\| = \|\mathbf{u}\| \cos \theta$$

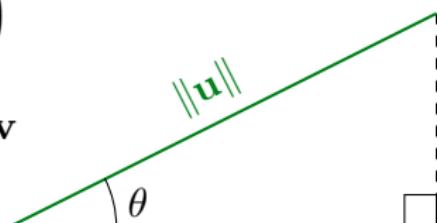


## 11.3 The Dot Product



Now

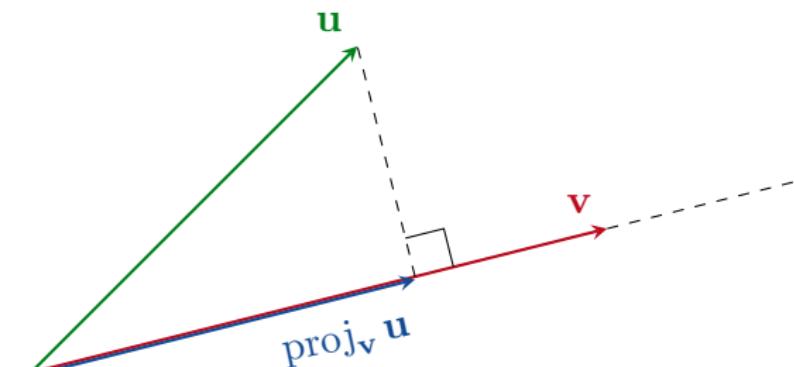
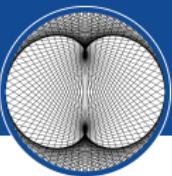
$$\begin{aligned}\text{proj}_{\mathbf{v}} \mathbf{u} &= (\text{length of } \text{proj}_{\mathbf{v}} \mathbf{u}) \left( \begin{array}{c} \text{a unit vector in} \\ \text{the same} \\ \text{direction as } \mathbf{v} \end{array} \right) \\ &= \|\text{proj}_{\mathbf{v}} \mathbf{u}\| \left( \frac{\mathbf{v}}{\|\mathbf{v}\|} \right) \\ &= \|\mathbf{u}\| (\cos \theta) \left( \frac{\mathbf{v}}{\|\mathbf{v}\|} \right) \\ &= \left( \frac{\|\mathbf{u}\| \|\mathbf{v}\| \cos \theta}{\|\mathbf{v}\|^2} \right) \mathbf{v} \\ &= \left( \frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{v}\|^2} \right) \mathbf{v}.\end{aligned}$$



$$\|\text{proj}_{\mathbf{v}} \mathbf{u}\| = \|\mathbf{u}\| \cos \theta$$

Since this is an important formula, we write it as a theorem.

## 11.3 The Dot Product



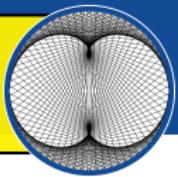
Theorem

*The vector projection of  $\mathbf{u}$  onto  $\mathbf{v}$  is*

$$\text{proj}_{\mathbf{v}} \mathbf{u} = \left( \frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{v}\|^2} \right) \mathbf{v}.$$

## 11.3 The Dot Product

$$\text{proj}_{\mathbf{v}} \mathbf{u} = \left( \frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{v}\|^2} \right) \mathbf{v}$$



### Example

Find the vector projection of  $\mathbf{u} = 6\mathbf{i} + 3\mathbf{j} + 2\mathbf{k}$  onto  $\mathbf{v} = \mathbf{i} - 2\mathbf{j} - 2\mathbf{k}$ .

## 11.3 The Dot Product

$$\text{proj}_{\mathbf{v}} \mathbf{u} = \left( \frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{v}\|^2} \right) \mathbf{v}$$



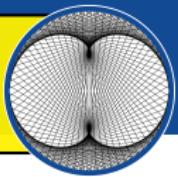
### Example

Find the vector projection of  $\mathbf{u} = 6\mathbf{i} + 3\mathbf{j} + 2\mathbf{k}$  onto  $\mathbf{v} = \mathbf{i} - 2\mathbf{j} - 2\mathbf{k}$ .

$$\begin{aligned}\text{proj}_{\mathbf{v}} \mathbf{u} &= \left( \frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{v}\|^2} \right) \mathbf{v} = \left( \frac{6 - 6 - 4}{1 + 4 + 4} \right) (\mathbf{i} - 2\mathbf{j} - 2\mathbf{k}) \\ &= -\frac{4}{9}\mathbf{i} + \frac{8}{9}\mathbf{j} + \frac{8}{9}\mathbf{k}.\end{aligned}$$

## 11.3 The Dot Product

$$\text{proj}_{\mathbf{v}} \mathbf{u} = \left( \frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{v}\|^2} \right) \mathbf{v}$$



### Example

Find the vector projection of  $\mathbf{F} = 5\mathbf{i} + 2\mathbf{j}$  onto  $\mathbf{v} = \mathbf{i} - 3\mathbf{j}$ .

$$\begin{aligned}\text{proj}_{\mathbf{v}} \mathbf{F} &= \left( \frac{\mathbf{F} \cdot \mathbf{v}}{\|\mathbf{v}\|^2} \right) \mathbf{v} = \left( \frac{5 - 6}{1 + 9} \right) (\mathbf{i} - 3\mathbf{j}) \\ &= -\frac{1}{10}\mathbf{i} + \frac{3}{10}\mathbf{j}.\end{aligned}$$

## 11.3 The Dot Product

$$\text{proj}_{\mathbf{v}} \mathbf{u} = \left( \frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{v}\|^2} \right) \mathbf{v}$$

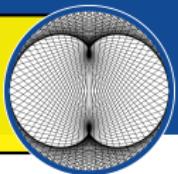


### Example

Verify that the vector  $\mathbf{u} - \text{proj}_{\mathbf{v}} \mathbf{u}$  is orthogonal to  $\text{proj}_{\mathbf{v}} \mathbf{u}$ .

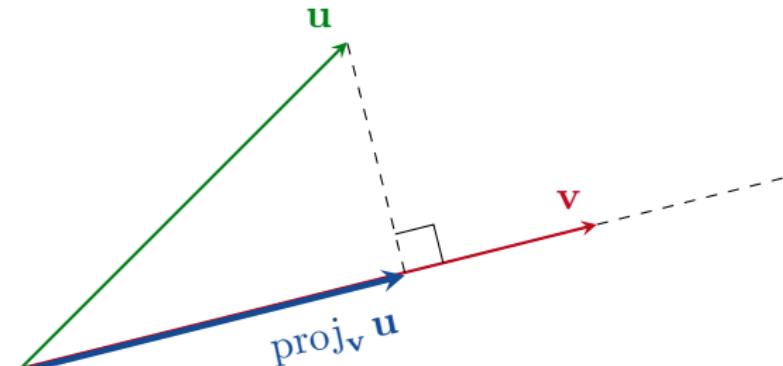
## 11.3 The Dot Product

$$\text{proj}_{\mathbf{v}} \mathbf{u} = \left( \frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{v}\|^2} \right) \mathbf{v}$$



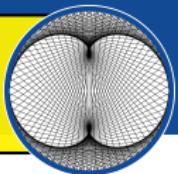
### Example

Verify that the vector  $\mathbf{u} - \text{proj}_{\mathbf{v}} \mathbf{u}$  is orthogonal to  $\text{proj}_{\mathbf{v}} \mathbf{u}$ .



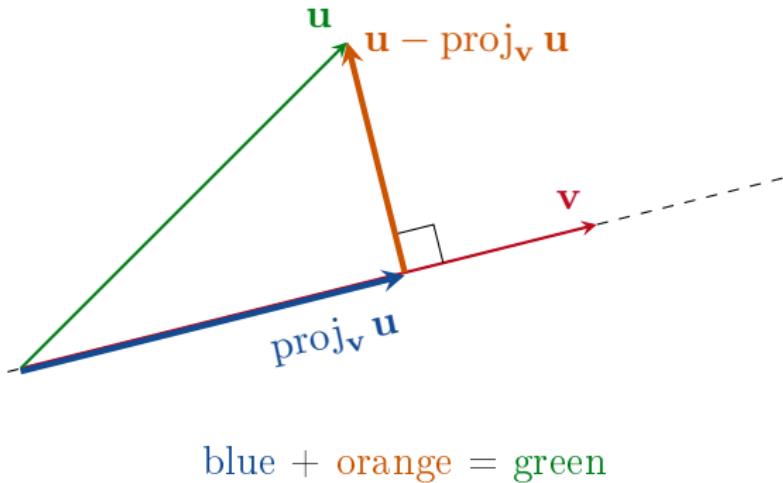
## 11.3 The Dot Product

$$\text{proj}_{\mathbf{v}} \mathbf{u} = \left( \frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{v}\|^2} \right) \mathbf{v}$$



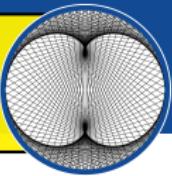
### Example

Verify that the vector  $\mathbf{u} - \text{proj}_{\mathbf{v}} \mathbf{u}$  is orthogonal to  $\text{proj}_{\mathbf{v}} \mathbf{u}$ .



## 11.3 The Dot Product

$$\text{proj}_{\mathbf{v}} \mathbf{u} = \left( \frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{v}\|^2} \right) \mathbf{v}$$



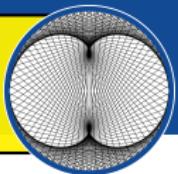
Clearly

$$\text{proj}_{\mathbf{v}} \mathbf{u} = \left( \frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{v}\|^2} \right) \mathbf{v} = (\text{a number}) \mathbf{v}$$

is parallel to  $\mathbf{v}$ .

## 11.3 The Dot Product

$$\text{proj}_{\mathbf{v}} \mathbf{u} = \left( \frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{v}\|^2} \right) \mathbf{v}$$



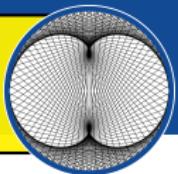
Clearly

$$\text{proj}_{\mathbf{v}} \mathbf{u} = \left( \frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{v}\|^2} \right) \mathbf{v} = (\text{a number}) \mathbf{v}$$

is parallel to  $\mathbf{v}$ . So it is enough to show that  $\mathbf{u} - \text{proj}_{\mathbf{v}} \mathbf{u}$  is orthogonal to  $\mathbf{v}$ .

## 11.3 The Dot Product

$$\text{proj}_{\mathbf{v}} \mathbf{u} = \left( \frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{v}\|^2} \right) \mathbf{v}$$



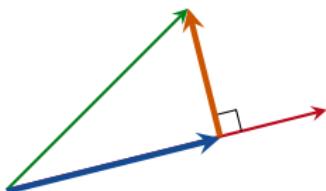
Clearly

$$\text{proj}_{\mathbf{v}} \mathbf{u} = \left( \frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{v}\|^2} \right) \mathbf{v} = (\text{a number}) \mathbf{v}$$

is parallel to  $\mathbf{v}$ . So it is enough to show that  $\mathbf{u} - \text{proj}_{\mathbf{v}} \mathbf{u}$  is orthogonal to  $\mathbf{v}$ .

Since

$$(\mathbf{u} - \text{proj}_{\mathbf{v}} \mathbf{u}) \cdot \mathbf{v} =$$

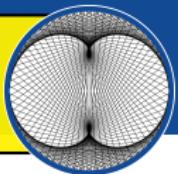


$$\begin{aligned} &= \\ &= \\ &= 0 \end{aligned}$$

we have shown that  $\mathbf{u} - \text{proj}_{\mathbf{v}} \mathbf{u}$  is orthogonal to  $\mathbf{v}$ .

## 11.3 The Dot Product

$$\text{proj}_{\mathbf{v}} \mathbf{u} = \left( \frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{v}\|^2} \right) \mathbf{v}$$



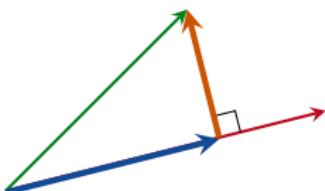
Clearly

$$\text{proj}_{\mathbf{v}} \mathbf{u} = \left( \frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{v}\|^2} \right) \mathbf{v} = (\text{a number}) \mathbf{v}$$

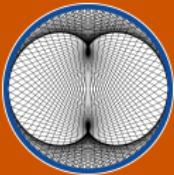
is parallel to  $\mathbf{v}$ . So it is enough to show that  $\mathbf{u} - \text{proj}_{\mathbf{v}} \mathbf{u}$  is orthogonal to  $\mathbf{v}$ .

Since

$$\begin{aligned} (\mathbf{u} - \text{proj}_{\mathbf{v}} \mathbf{u}) \cdot \mathbf{v} &= \mathbf{u} \cdot \mathbf{v} - \left( \frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{v}\|^2} \right) \mathbf{v} \cdot \mathbf{v} \\ &= \mathbf{u} \cdot \mathbf{v} - \frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{v}\|^2} \|\mathbf{v}\|^2 \\ &= \mathbf{u} \cdot \mathbf{v} - \mathbf{u} \cdot \mathbf{v} \\ &= 0 \end{aligned}$$



we have shown that  $\mathbf{u} - \text{proj}_{\mathbf{v}} \mathbf{u}$  is orthogonal to  $\mathbf{v}$ .



# Next Time

- 11.4 The Cross Product
- 11.5 Lines and Planes in Space