Lecture 10

- Inner Product Spaces
- Orthogonality
- Orthogonal Sets and Orthonormal Sets

Inner Product Spaces

In MATH114 Mathematics II we studied the *dot product* of two vectors in $\mathbb{R}^2$ or $\mathbb{R}^3$.

You will recall that if $\mathbf{u} = (u_1, u_2, u_3), \mathbf{v} = (v_1, v_2, v_3) \in \mathbb{R}^3$ then

$$\mathbf{u} \cdot \mathbf{v} = u_1 v_1 + u_2 v_2 + u_3 v_3$$

and

$$\mathbf{u} \cdot \mathbf{u} = \|\mathbf{u}\|^2 .$$

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and

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This week will extend these ideas to real vector spaces.

(We will not concern ourselves with complex numbers this week. Any time you see a scalar k you can assume that it is a real number.)

Inner Product Spaces

Definition

An *inner product* on a (real) vector space V is a function

$$\langle \cdot, \cdot \rangle : V \times V \to \mathbb{R}$$

that satisfies the following axioms for all vectors $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ in V and all scalars k:

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2.
3.
4.

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1. $\langle \mathbf{u}, \mathbf{v} \rangle = \langle \mathbf{v}, \mathbf{u} \rangle$ [Symmetry axiom]
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2. $\langle \mathbf{u} + \mathbf{v}, \mathbf{w} \rangle = \langle \mathbf{u}, \mathbf{w} \rangle + \langle \mathbf{v}, \mathbf{w} \rangle$ [Additivity axiom]
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3. $\langle k\mathbf{u}, \mathbf{v} \rangle = k \langle \mathbf{u}, \mathbf{v} \rangle$ [Homogeneity axiom]
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3. $\langle k\mathbf{u}, \mathbf{v} \rangle = k \langle \mathbf{u}, \mathbf{v} \rangle$ [Homogeneity axiom]
4. $\langle \mathbf{v}, \mathbf{v} \rangle \geq 0$ and $\langle \mathbf{v}, \mathbf{v} \rangle = 0$ if and only if $\mathbf{v} = \mathbf{0}$. [Positivity axiom]

Remark

If we combine

1. $\langle \mathbf{u}, \mathbf{v} \rangle = \langle \mathbf{v}, \mathbf{u} \rangle$ [Symmetry axiom]

and

2. $\langle \mathbf{u} + \mathbf{v}, \mathbf{w} \rangle = \langle \mathbf{u}, \mathbf{w} \rangle + \langle \mathbf{v}, \mathbf{w} \rangle$ [Additivity axiom]

then we can show that

$$\langle \mathbf{u}, \mathbf{v} + \mathbf{w} \rangle = \langle \mathbf{v} + \mathbf{w}, \mathbf{u} \rangle = \langle \mathbf{v}, \mathbf{u} \rangle + \langle \mathbf{w}, \mathbf{u} \rangle = \langle \mathbf{u}, \mathbf{v} \rangle + \langle \mathbf{u}, \mathbf{w} \rangle .$$

Remark

Likewise, we can combine

1. $\langle \mathbf{u}, \mathbf{v} \rangle = \langle \mathbf{v}, \mathbf{u} \rangle$ [Symmetry axiom]

and

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to prove that

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to prove that

$$\langle \mathbf{u}, k\mathbf{v} \rangle = k \langle \mathbf{u}, \mathbf{v} \rangle$$

and then we can prove that

$$\langle \mathbf{0}, \mathbf{v} \rangle = 0 = \langle \mathbf{v}, \mathbf{0} \rangle .$$

(proofs left to you.)

Example

Let $V = \mathbb{R}^n$. The function

$$\langle \mathbf{u}, \mathbf{v} \rangle = \mathbf{u} \cdot \mathbf{v} = u_1 v_1 + u_2 v_2 + \ldots + u_n v_n$$

is an inner product on $\mathbb{R}^n$. I leave it to you to prove that all 4 axioms are satisfied. This inner product is called the *Euclidean inner product* on $\mathbb{R}^n$.

Definition

A (real) vector space with an inner product is called a *(real) inner product space.*

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Example

$\mathbb{R}^n$ with $\langle \mathbf{u}, \mathbf{v} \rangle = \mathbf{u} \cdot \mathbf{v}$ is an inner product space.

Definition

If V is a inner product space, then the *norm* (or *length*) of a vector $\mathbf{v}$ in V is denoted by $\|\mathbf{v}\|$ and is defined by

$$\|\mathbf{v}\| = \sqrt{\langle \mathbf{v}, \mathbf{v} \rangle}.$$

Definition

A vector of norm 1 is called a *unit vector*.

$$\|\mathbf{v}\| = \sqrt{\langle \mathbf{v}, \mathbf{v} \rangle}$$

Theorem

If $\mathbf{u}$ *and* $\mathbf{v}$ *are vectors in a inner product space* V, *and if* k *is a scalar, then:*

1. $\|\mathbf{v}\| \geq 0$
2. $\|\mathbf{v}\| = 0$ *if and only if* $\mathbf{v} = \mathbf{0}$.
3. $\|k\mathbf{v}\| = |k|\,\|\mathbf{v}\|$.

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Proof.

1. $\sqrt{x}$ is always ≥ 0.

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1. $\sqrt{x}$ is always ≥ 0.
2. $\|\mathbf{v}\| = 0 \iff \langle \mathbf{v}, \mathbf{v} \rangle = 0 \iff \mathbf{v} = \mathbf{0}$ by definition.

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Proof.

1. $\sqrt{x}$ is always ≥ 0.
2. $\|\mathbf{v}\| = 0 \iff \langle \mathbf{v}, \mathbf{v} \rangle = 0 \iff \mathbf{v} = \mathbf{0}$ by definition.
3. $\|k\mathbf{v}\|^2 = \langle k\mathbf{v}, k\mathbf{v} \rangle = k \langle \mathbf{v}, k\mathbf{v} \rangle = k^2 \langle \mathbf{v}, \mathbf{v} \rangle = k^2 \|\mathbf{v}\|^2$.

$\square$

$$\|\mathbf{v}\| = \sqrt{\langle \mathbf{v}, \mathbf{v} \rangle}$$

Definition

If V is an inner product space, then the *distance* between two vectors $\mathbf{u}$ and $\mathbf{v}$ in V is

$$\operatorname{dist}(\mathbf{u}, \mathbf{v}) = \|\mathbf{u} - \mathbf{v}\|.$$

Example (A Weighted Inner Product)

Let $\mathbf{u}, \mathbf{v} \in \mathbb{R}^2$. Show that the function

$$\langle \mathbf{u}, \mathbf{v} \rangle = 3u_1v_1 + 2u_2v_2$$

is an inner product on $\mathbb{R}^2$.

We need to show that all four of the axioms are satisfied.

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1. Interchanging $\mathbf{u}$ and $\mathbf{v}$ in the formula does not change the sum on the right side, so $\langle \mathbf{u}, \mathbf{v} \rangle = \langle \mathbf{v}, \mathbf{u} \rangle$.

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2. We have

$$\begin{aligned} \langle \mathbf{u}+\mathbf{v}, \mathbf{w} \rangle &= 3(u_1+v_1)w_1 + 2(u_2+v_2)w_2 \\ &= 3(u_1w_1 + v_1w_1) + 2(u_2w_2 + v_2w_2) \\ &= \ldots = \langle \mathbf{u}, \mathbf{w} \rangle + \langle \mathbf{v}, \mathbf{w} \rangle \end{aligned}$$

3. and that

$$\langle k\mathbf{u}, \mathbf{v} \rangle = 3(ku_1)v_1 + 2(ku_2)v_2 = k(3u_1v_1 + 2u_2v_2) = k \langle \mathbf{u}, \mathbf{v} \rangle .$$

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is an inner product on $\mathbb{R}^2$.

We need to show that all four of the axioms are satisfied.

4. Finally note that

$$\langle \mathbf{v}, \mathbf{v} \rangle = 3v_1^2 + 2v_2^2 \geq 0$$

and we can only get "$= 0$" here if $v_1 = v_2 = 0$.

Therefore $\langle \cdot, \cdot \rangle$ is an inner product. This is called a *weighted Euclidean inner product*. The numbers 3 and 2 are called the *weights*.

Example (Calculating with a Weighted Euclidean Inner Product)

Let $\mathbf{u} = (1, 0) \in \mathbb{R}^2$. Note that if we use the Euclidean inner product, then we have

$$\|\mathbf{u}\| = \sqrt{\mathbf{u} \cdot \mathbf{u}} = \sqrt{1^2 + 0^2} = 1.$$

Example (Calculating with a Weighted Euclidean Inner Product)

Let $\mathbf{u} = (1, 0) \in \mathbb{R}^2$. Note that if we use the Euclidean inner product, then we have

$$\|\mathbf{u}\| = \sqrt{\mathbf{u} \cdot \mathbf{u}} = \sqrt{1^2 + 0^2} = 1.$$

However if we use the weighted Euclidean inner product from the last example, then

$$\|\mathbf{u}\| = \sqrt{\langle \mathbf{u}, \mathbf{u} \rangle} = \sqrt{3u_1^2 + 2u_2^2} = \sqrt{3(1)^2 + 2(0)^2} = \sqrt{3}.$$

Unit Circles and Unit Spheres

Definition

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Example (Unusual Unit Circles in $\mathbb{R}^2$)

1. Sketch the unit circle in $\mathbb{R}^2$ using the Euclidean inner product $\langle \mathbf{u}, \mathbf{v} \rangle = u_1 v_1 + u_2 v_2$.
2. Sketch the unit circle in $\mathbb{R}^2$ using the weighted Euclidean inner product $\langle \mathbf{u}, \mathbf{v} \rangle = \frac{1}{9} u_1 v_1 + \frac{1}{4} u_2 v_2$.

Let $\mathbf{u} = (x, y)$. Then

1. $$1 = \|\mathbf{u}\|^2 = (x, y) \bullet (x, y) = x^2 + y^2$$

   and

2. $$1 = \|\mathbf{u}\|^2 = \langle \mathbf{u}, \mathbf{u} \rangle = \frac{1}{9} u_1 u_1 + \frac{1}{4} u_2 u_2 = \frac{x^2}{9} + \frac{y^2}{4}.$$

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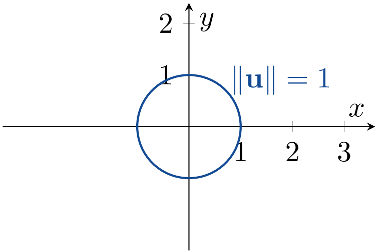

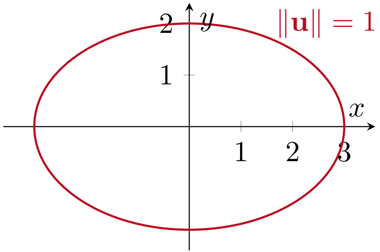

Inner Products Generated by Matrices

Definition

Let A be an invertible $n \times n$ matrix. Then the formula

$$\langle \mathbf{u}, \mathbf{v} \rangle = A\mathbf{u} \cdot A\mathbf{v}$$

defines an inner product on $\mathbb{R}^n$ called the *inner product generated by A*.

Remark

Note that

$$\langle \mathbf{u}, \mathbf{v} \rangle = A\mathbf{u} \cdot A\mathbf{v} = (A\mathbf{v})^T A\mathbf{u} = \mathbf{v}^T A^T A\mathbf{u}.$$

Example

Let $A = \begin{bmatrix} \sqrt{3} & 0 \\ 0 & \sqrt{2} \end{bmatrix}$. Then the inner product on $\mathbb{R}^2$ generated by A is one of the weighted Euclidean inner products that we looked at earlier:

$$\langle \mathbf{u}, \mathbf{v} \rangle = A\mathbf{u} \cdot A\mathbf{v} = \begin{bmatrix} \sqrt{3}u_1 \\ \sqrt{2}u_2 \end{bmatrix} \cdot \begin{bmatrix} \sqrt{3}v_1 \\ \sqrt{2}v_2 \end{bmatrix} = 3u_1v_2 + 2u_1v_2.$$

Note that

$$A^T A = \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}.$$

More Inner Products

Example (The Standard Inner Product on $\mathbb{R}^{n\times n} = M_{nn}$)

If $\mathbf{u} = U$ and $\mathbf{v} = V$ are matrices in the vector space $\mathbb{R}^{n\times n}$, then the formula

$$\langle \mathbf{u}, \mathbf{v} \rangle = \operatorname{tr}(U^T V)$$

defines an inner product on $\mathbb{R}^{n\times n}$ called the *standard inner product* on $\mathbb{R}^{n\times n}$.

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defines an inner product on $\mathbb{R}^{n\times n}$ called the *standard inner product* on $\mathbb{R}^{n\times n}$.

This can be proved by confirming that the four inner product axioms are satisfied. But there is an easier way:

If $\mathbf{u} = U = \begin{bmatrix} u_1 & u_2 \\ u_3 & u_4 \end{bmatrix}$ and $\mathbf{v} = V = \begin{bmatrix} v_1 & v_2 \\ v_3 & v_4 \end{bmatrix}$ then

$$\begin{aligned} \langle \mathbf{u}, \mathbf{v} \rangle = \operatorname{tr}(U^T V) &= \operatorname{tr} \begin{bmatrix} u_1v_1 + u_3v_3 & u_2v_2 + u_4v_4 \\ u_1v_1 + u_3v_3 & u_2v_2 + u_4v_4 \end{bmatrix} \\ &= u_1v_1 + u_2v_2 + u_3v_3 + u_4v_4 \end{aligned}$$

which is just like the dot product in $\mathbb{R}^4$.

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which is just like the dot product in $\mathbb{R}^4$.

For example, if

$$\mathbf{u} = U = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \qquad \text{and} \qquad \mathbf{v} = V = \begin{bmatrix} -1 & 0 \\ 3 & 2 \end{bmatrix}$$

then

$$\langle \mathbf{u}, \mathbf{v} \rangle = \operatorname{tr}(U^T V) = 1(-1) + 2(0) + 3(3) + 4(2) = 16$$

and

$$\|\mathbf{u}\| = \sqrt{\langle \mathbf{u}, \mathbf{u} \rangle} = \sqrt{1^2 + 2^2 + 3^2 + 4^2} = \sqrt{30}.$$

Example (The Standard Inner Product on $\mathbb{P}^n$)

If

$$\mathbf{p} = a_0 + a_1 x + \ldots + a_n x^n$$

and

$$\mathbf{q} = b_0 + b_1 x + \ldots + b_n x^n$$

are polynomials in $\mathbb{P}^n$, then the following formula defines an inner product on $\mathbb{P}^n$ (please verify) that we will call the *standard inner product* on $\mathbb{P}^n$:

$$\langle \mathbf{p}, \mathbf{q} \rangle = a_0 b_0 + a_1 b_1 + \ldots + a_n b_n$$

The norm of a polynomial $\mathbf{p}$ relative to this inner product is

$$\|\mathbf{p}\| = \sqrt{\langle \mathbf{p}, \mathbf{p} \rangle} = \sqrt{a_0^2 + a_1^2 + \ldots + a_n^2}.$$

Example (The Evaluation Inner Product on $\mathbb{P}^n$)

If

$$\mathbf{p} = a_0 + a_1 x + \ldots + a_n x^n$$

and

$$\mathbf{q} = b_0 + b_1 x + \ldots + b_n x^n$$

are polynomials in $\mathbb{P}^n$, and if $x_0, x_1, \ldots, x_n$ are distinct real numbers, then the following formula defines an inner product on $\mathbb{P}^n$ that we call the *evaluation inner product* at $x_0, x_1, \ldots, x_n$:

$$\langle \mathbf{p}, \mathbf{q} \rangle = p(x_0)q(x_0) + p(x_1)q(x_1) + \ldots + p(x_n)q(x_n)$$

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$$\langle \mathbf{p}, \mathbf{q} \rangle = p(x_0)q(x_0) + p(x_1)q(x_1) + \ldots + p(x_n)q(x_n)$$

We can think of this as the dot product of the vector $\big(p(x_0), p(x_1), \ldots, p(x_n)\big)$ with the vector $\big(q(x_0), q(x_1), \ldots, q(x_n)\big)$.

The first three axioms follow from properties of the dot product. For the fourth axiom, we have that

$$\langle \mathbf{p}, \mathbf{p} \rangle = \big[p(x_0)\big]^2 + \big[p(x_1)\big]^2 + \ldots + \big[p(x_n)\big]^2 \geq 0$$

for all polynomials $\mathbf{p}$. We can only have "$= 0$" here if

$$p(x_0) = p(x_1) = \ldots = p(x_n) = 0.$$

However the only polynomial, of nth degree or less, with $n + 1$ roots is $\mathbf{p} = \mathbf{0}$.

$C[a,b] = \{\text{all continuous functions } f : [a,b] \to \mathbb{R}\}$

Example (An Integral Inner Product on $C[a,b]$)

Show that the following function is an inner product on $C[a,b]$:

$$\langle \mathbf{f}, \mathbf{g} \rangle = \int_a^b f(x)g(x)\,dx.$$

1

2

3

4

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2. $\langle \mathbf{f} + \mathbf{g}, \mathbf{h} \rangle = \int_a^b \big(f(x) + g(x)\big)h(x)\,dx =$
   $\int_a^b f(x)h(x)\,dx + \int_a^b g(x)h(x)\,dx = \langle \mathbf{f}, \mathbf{h} \rangle + \langle \mathbf{g}, \mathbf{h} \rangle$.
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2. $\langle \mathbf{f}+\mathbf{g}, \mathbf{h} \rangle = \int_a^b \big(f(x)+g(x)\big)h(x)\,dx =$ $\int_a^b f(x)h(x)\,dx + \int_a^b g(x)h(x)\,dx = \langle \mathbf{f}, \mathbf{h} \rangle + \langle \mathbf{g}, \mathbf{h} \rangle$.
3. $\langle k\mathbf{f}, \mathbf{g} \rangle = \int_a^b kf(x)g(x)\,dx = k\int_a^b f(x)g(x)\,dx = k\langle \mathbf{f}, \mathbf{g} \rangle$.
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1. $\langle \mathbf{f}, \mathbf{g} \rangle = \int_a^b f(x)g(x)\,dx = \int_a^b g(x)f(x)\,dx = \langle \mathbf{g}, \mathbf{f} \rangle$.
2. $\langle \mathbf{f} + \mathbf{g}, \mathbf{h} \rangle = \int_a^b \big(f(x) + g(x)\big)h(x)\,dx =$ $\int_a^b f(x)h(x)\,dx + \int_a^b g(x)h(x)\,dx = \langle \mathbf{f}, \mathbf{h} \rangle + \langle \mathbf{g}, \mathbf{h} \rangle$.
3. $\langle k\mathbf{f}, \mathbf{g} \rangle = \int_a^b kf(x)g(x)\,dx = k\int_a^b f(x)g(x)\,dx = k\,\langle \mathbf{f}, \mathbf{g} \rangle$.
4. $\langle \mathbf{f}, \mathbf{f} \rangle = \int_a^b \big(f(x)\big)^2\,dx \geq 0$ since $\big(f(x)\big)^2 \geq 0$ for all $x \in [a,b]$. Since f is continuous, we can only have "$= 0$" here if $\mathbf{f} = \mathbf{0}$.

Calculating with Inner Products

$$
\begin{aligned}
\langle \mathbf{u} - 2\mathbf{v}, 3\mathbf{u} + 4\mathbf{v} \rangle &= \langle \mathbf{u}, 3\mathbf{u} + 4\mathbf{v} \rangle - \langle 2\mathbf{v}, 3\mathbf{u} + 4\mathbf{v} \rangle \\
&= \\
&= \\
&= \\
&=
\end{aligned}
$$

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&= \langle \mathbf{u}, 3\mathbf{u} \rangle + \langle \mathbf{u}, 4\mathbf{v} \rangle - \langle 2\mathbf{v}, 3\mathbf{u} \rangle - \langle 2\mathbf{v}, 4\mathbf{v} \rangle \\
&= \\
&= \\
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&= 3 \langle \mathbf{u}, \mathbf{u} \rangle + 4 \langle \mathbf{u}, \mathbf{v} \rangle - 6 \langle \mathbf{v}, \mathbf{u} \rangle - 8 \langle \mathbf{v}, \mathbf{v} \rangle \\
&= \\
&=
\end{aligned}$$

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Calculating with Inner Products

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&= 3 \|\mathbf{u}\|^2 - 2 \langle \mathbf{u}, \mathbf{v} \rangle - 8 \|\mathbf{v}\|^2 .
\end{aligned}
$$

Orthogonality

Orthogonal Vectors

Definition

Two vectors $\mathbf{u}$ and $\mathbf{v}$ in a (real) inner product space V are *orthogonal* iff

$$\langle \mathbf{u}, \mathbf{v} \rangle = 0.$$

Example (Orthogonality Depends on the Inner Product)

Let $\mathbf{u} = (1, 1)$ and $\mathbf{v} = (1, -1)$. Note that $\mathbf{u}$ and $\mathbf{v}$ are orthogonal with respect to the Euclidean inner product on $\mathbb{R}^2$ since

$$\mathbf{u} \cdot \mathbf{v} = (1)(1) + (1)(-1) = 0.$$

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However $\mathbf{u}$ and $\mathbf{v}$ are not orthogonal with respect to the weighted Euclidean inner product $\langle \mathbf{u}, \mathbf{v} \rangle = 3u_1v_1 + 2u_2v_2$ since

$$\langle \mathbf{u}, \mathbf{v} \rangle = 3(1)(1) + 2(1)(-1) = 1 \neq 0.$$

Example

The matrices $U = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$ and $V = \begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix}$ are orthogonal with respect to the standard inner product on $\mathbb{R}^{2\times 2}$ since

$$\langle \mathbf{u}, \mathbf{v} \rangle = (1)(0) + (0)(2) + (1)(0) + (1)(0) = 0.$$

Example

Let $\mathbb{P}^2$ have the inner product

$$\langle \mathbf{p}, \mathbf{q} \rangle = \int_{-1}^{1} p(x)q(x)\,dx$$

and let $\mathbf{p} = x$ and $\mathbf{q} = x^2$.

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and let $\mathbf{p} = x$ and $\mathbf{q} = x^2$. Then

$$\|\mathbf{p}\| = \langle \mathbf{p}, \mathbf{p} \rangle^{\frac{1}{2}} = \left[\int_{-1}^{1} x^2\,dx\right]^{\frac{1}{2}} = \sqrt{\frac{2}{3}}$$

$$\|\mathbf{q}\| = \langle \mathbf{q}, \mathbf{q} \rangle^{\frac{1}{2}} = \left[\int_{-1}^{1} x^4\,dx\right]^{\frac{1}{2}} = \sqrt{\frac{2}{5}}$$

$$\langle \mathbf{p}, \mathbf{q} \rangle = \int_{-1}^{1} x^3\,dx = 0.$$

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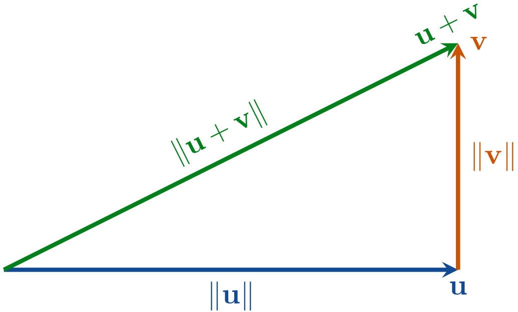
$$\langle \mathbf{p}, \mathbf{q} \rangle = \int_{-1}^{1} x^3\,dx = 0.$$

Because $\langle \mathbf{p}, \mathbf{q} \rangle = 0$, the vectors $\mathbf{p} = x$ and $\mathbf{q} = x^2$ are orthogonal relative to this inner product.

Orthogonality



Theorem (The Pythagorean Theorem)

If $\mathbf{u}$ *and* $\mathbf{v}$ *are orthogonal then*

$$\|\mathbf{u}+\mathbf{v}\|^2 = \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2 .$$

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If $\mathbf{u}$ *and* $\mathbf{v}$ *are orthogonal then*

$$\|\mathbf{u}+\mathbf{v}\|^2 = \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2 .$$

Proof.

If $\langle \mathbf{u}, \mathbf{v} \rangle = 0$ then

$$\begin{aligned} \|\mathbf{u}+\mathbf{v}\|^2 &= \langle \mathbf{u}+\mathbf{v}, \mathbf{u}+\mathbf{v} \rangle = \langle \mathbf{u}, \mathbf{u}+\mathbf{v} \rangle + \langle \mathbf{v}, \mathbf{u}+\mathbf{v} \rangle \\ &= \langle \mathbf{u}, \mathbf{u} \rangle + \langle \mathbf{u}, \mathbf{v} \rangle + \langle \mathbf{v}, \mathbf{u} \rangle + \langle \mathbf{v}, \mathbf{v} \rangle \\ &= \|\mathbf{u}\|^2 + 0 + 0 + \|\mathbf{v}\|^2 . \end{aligned}$$

$\square$

Example

We have seen that $\mathbf{p} = x$ and $\mathbf{q} = x^2$ are orthogonal with respect to the inner product

$$\langle \mathbf{p}, \mathbf{q} \rangle = \int_{-1}^{1} p(x)q(x)\,dx.$$

It follows from the Pythagorean Theorem that

$$\|\mathbf{p} + \mathbf{q}\|^2 = \|\mathbf{p}\|^2 + \|\mathbf{q}\|^2 .$$

Let's check this:

Example

$$\|\mathbf{p}\|^2 + \|\mathbf{q}\|^2 = \left(\sqrt{\frac{2}{3}}\right)^2 + \left(\sqrt{\frac{2}{5}}\right)^2 = \frac{2}{3} + \frac{2}{5} = \frac{16}{15}$$

$$\|\mathbf{p} + \mathbf{q}\|^2 =$$

Example

$$
\begin{aligned}
\|\mathbf{p}\|^2 + \|\mathbf{q}\|^2 &= \left(\sqrt{\frac{2}{3}}\right)^2 + \left(\sqrt{\frac{2}{5}}\right)^2 = \frac{2}{3} + \frac{2}{5} = \frac{16}{15} \\
\|\mathbf{p}+\mathbf{q}\|^2 &= \langle \mathbf{p}+\mathbf{q}, \mathbf{p}+\mathbf{q} \rangle = \int_{-1}^{1} (x+x^2)(x+x^2)\,dx \\
&= \int_{-1}^{1} x^2\,dx + 2\int_{-1}^{1} x^3\,dx + \int_{-1}^{1} x^4\,dx \\
&= \frac{2}{3} + 0 + \frac{2}{5} = \frac{16}{15}.
\end{aligned}
$$

Definition

Let W be a subspace of an inner product space V. The *orthogonal complement* of W is

$$W^{\perp} = \{\mathbf{v} \in V \mid \langle \mathbf{v}, \mathbf{w} \rangle = 0 \text{ for all } \mathbf{w} \in W\}.$$

Theorem

1. *$W^{\perp}$ is also a subspace of V.*
2. $W \cap W^{\perp} = \{\mathbf{0}\}$.

Orthogonality

Definition

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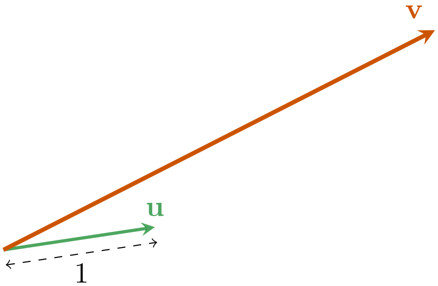
Theorem

1. *$W^\perp$ is also a subspace of V.*
2. $W \cap W^\perp = \{\mathbf{0}\}$.
3. *If V is finite dimensional then $\left(W^\perp\right)^\perp = W$.*

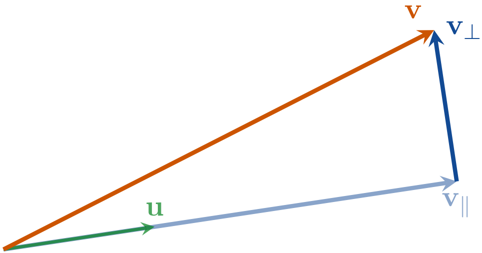
Orthogonal Projection

Let $\mathbf{u}$ be a unit vector and let $\mathbf{v}$ be any nonzero vector in V.



Orthogonal Projection

Let $\mathbf{u}$ be a unit vector and let $\mathbf{v}$ be any nonzero vector in V.



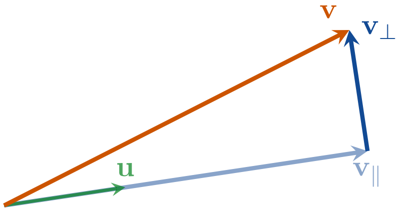
We can write

$$\mathbf{v} = \mathbf{v}_\parallel + \mathbf{v}_\perp$$

where

$$\mathbf{v}_\parallel = \langle \mathbf{u}, \mathbf{v} \rangle \, \mathbf{u} \qquad \text{and} \qquad \mathbf{v}_\perp = \mathbf{v} - \langle \mathbf{u}, \mathbf{v} \rangle \, \mathbf{u}.$$

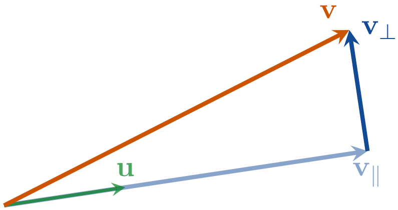
$$\mathbf{v} = \mathbf{v}_{\|} + \mathbf{v}_{\perp} \qquad \mathbf{v}_{\|} = \langle \mathbf{u}, \mathbf{v} \rangle \, \mathbf{u} \qquad \mathbf{v}_{\perp} = \mathbf{v} - \langle \mathbf{u}, \mathbf{v} \rangle \, \mathbf{u}$$



Note that $\mathbf{v}_{\perp}$ is orthogonal to $\mathbf{u}$ because

$$\langle \mathbf{u}, \mathbf{v}_{\perp} \rangle = \langle \mathbf{u}, \mathbf{v} - \langle \mathbf{u}, \mathbf{v} \rangle \, \mathbf{u} \rangle$$

$$\mathbf{v} = \mathbf{v}_{\parallel} + \mathbf{v}_{\perp} \qquad \mathbf{v}_{\parallel} = \langle \mathbf{u}, \mathbf{v} \rangle \, \mathbf{u} \qquad \mathbf{v}_{\perp} = \mathbf{v} - \langle \mathbf{u}, \mathbf{v} \rangle \, \mathbf{u}$$



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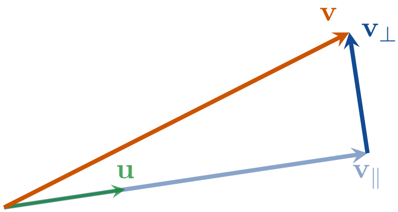

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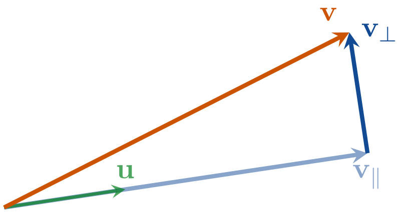
since $\mathbf{u}$ is a unit vector ($\|\mathbf{u}\| = 1 \implies \langle \mathbf{u}, \mathbf{u} \rangle = 1$).

$$\mathbf{v} = \mathbf{v}_{\parallel} + \mathbf{v}_{\perp} \qquad \mathbf{v}_{\parallel} = \langle \mathbf{u}, \mathbf{v} \rangle \, \mathbf{u} \qquad \mathbf{v}_{\perp} = \mathbf{v} - \langle \mathbf{u}, \mathbf{v} \rangle \, \mathbf{u}$$



Note that $\mathbf{v}_{\perp}$ is orthogonal to $\mathbf{u}$ because

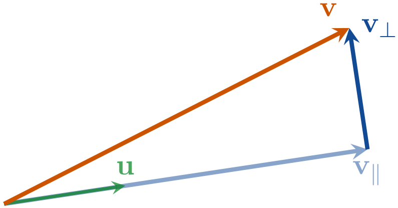
$$\langle \mathbf{u}, \mathbf{v}_{\perp} \rangle = \langle \mathbf{u}, \mathbf{v} - \langle \mathbf{u}, \mathbf{v} \rangle \, \mathbf{u} \rangle = \langle \mathbf{u}, \mathbf{v} \rangle - \langle \mathbf{u}, \mathbf{v} \rangle \langle \mathbf{u}, \mathbf{u} \rangle = \langle \mathbf{u}, \mathbf{v} \rangle - \langle \mathbf{u}, \mathbf{v} \rangle = 0$$

since $\mathbf{u}$ is a unit vector ($\|\mathbf{u}\| = 1 \implies \langle \mathbf{u}, \mathbf{u} \rangle = 1$).

It follows that $\mathbf{v}_{\parallel}$ and $\mathbf{v}_{\perp}$ are orthogonal.

Orthogonality

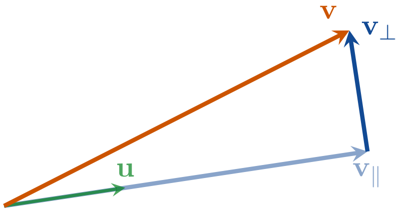
$$\mathbf{v} = \mathbf{v}_{\|} + \mathbf{v}_{\perp} \qquad \mathbf{v}_{\|} = \langle \mathbf{u}, \mathbf{v} \rangle \, \mathbf{u} \qquad \mathbf{v}_{\perp} = \mathbf{v} - \langle \mathbf{u}, \mathbf{v} \rangle \, \mathbf{u}$$



Theorem

1. $\left\|\mathbf{v}_{\|}\right\| \leq \|\mathbf{v}\|$
2. $\|\mathbf{v}_{\perp}\| \leq \|\mathbf{v}\|$

$$\mathbf{v} = \mathbf{v}_\parallel + \mathbf{v}_\perp \qquad \mathbf{v}_\parallel = \langle \mathbf{u}, \mathbf{v} \rangle \, \mathbf{u} \qquad \mathbf{v}_\perp = \mathbf{v} - \langle \mathbf{u}, \mathbf{v} \rangle \, \mathbf{u}$$



Theorem

1. $\|\mathbf{v}_\parallel\| \leq \|\mathbf{v}\|$
2. $\|\mathbf{v}_\perp\| \leq \|\mathbf{v}\|$
3. $\mathbf{v}_\parallel$ *is the unique vector parallel to* $\mathbf{u}$ *which is closest to* $\mathbf{v}$.

$$\mathbf{v} = \mathbf{v}_{\parallel} + \mathbf{v}_{\perp} \qquad \mathbf{v}_{\parallel} = \langle \mathbf{u}, \mathbf{v} \rangle\, \mathbf{u} \qquad \mathbf{v}_{\perp} = \mathbf{v} - \langle \mathbf{u}, \mathbf{v} \rangle\, \mathbf{u}$$

Proof.

By the Pythogorean Theorem,

$$\|\mathbf{v}\|^2 = \left\|\mathbf{v}_{\parallel}\right\|^2 + \|\mathbf{v}_{\perp}\|^2$$

since $\mathbf{v}_{\parallel}$ and $\mathbf{v}_{\perp}$ are orthogonal, and since $\mathbf{v} = \mathbf{v}_{\parallel} + \mathbf{v}_{\perp}$.

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$$\|\mathbf{v}\|^2 = \left\|\mathbf{v}_{\parallel}\right\|^2 + \|\mathbf{v}_{\perp}\|^2$$

since $\mathbf{v}_{\parallel}$ and $\mathbf{v}_{\perp}$ are orthogonal, and since $\mathbf{v} = \mathbf{v}_{\parallel} + \mathbf{v}_{\perp}$. It follows that

$$\left\|\mathbf{v}_{\parallel}\right\| \leq \|\mathbf{v}\| \qquad \text{and} \qquad \|\mathbf{v}_{\perp}\| \leq \|\mathbf{v}\| .$$

$$\mathbf{v} = \mathbf{v}_{\parallel} + \mathbf{v}_{\perp} \qquad \mathbf{v}_{\parallel} = \langle \mathbf{u}, \mathbf{v} \rangle \, \mathbf{u} \qquad \mathbf{v}_{\perp} = \mathbf{v} - \langle \mathbf{u}, \mathbf{v} \rangle \, \mathbf{u}$$

Proof Continued.

For part 3: Let $\alpha\mathbf{u}$ be any other vector which is parallel to $\mathbf{u}$. Then

$$\left\|\mathbf{v} - \alpha\mathbf{u}\right\|^2 = \left\|\mathbf{v}_{\perp} + (\mathbf{v}_{\parallel} - \alpha\mathbf{u})\right\|^2$$

$$\mathbf{v} = \mathbf{v}_{\parallel} + \mathbf{v}_{\perp} \qquad \mathbf{v}_{\parallel} = \langle \mathbf{u}, \mathbf{v} \rangle \, \mathbf{u} \qquad \mathbf{v}_{\perp} = \mathbf{v} - \langle \mathbf{u}, \mathbf{v} \rangle \, \mathbf{u}$$

Proof Continued.

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Proof Continued.

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This distance is smallest when $\alpha = \langle \mathbf{u}, \mathbf{v} \rangle$. Hence $\mathbf{v}_{\|} = \langle \mathbf{u}, \mathbf{v} \rangle \, \mathbf{u}$ is the unique vector parallel to $\mathbf{u}$ which is closest to $\mathbf{v}$. □

Augustin-Louis Cauchy

BORN
21 August 1789

DECEICED
23 May 1857

NATIONALITY
French

Hermann Schwarz

BORN
25 January 1843

DECEICED
30 November 1921

NATIONALITY
German

Theorem (The Cauchy-Schwarz Inequality)

If $\mathbf{u}$ *and* $\mathbf{v}$ *are vectors in a inner product space* V*, then*

$$|\langle \mathbf{u}, \mathbf{v} \rangle| \leq \|\mathbf{u}\| \, \|\mathbf{v}\| \, .$$

This is one of the most important and widely used inequalities in mathematics.

Proof.

We want to prove that

$$|\langle \mathbf{u}, \mathbf{v} \rangle| \leq \|\mathbf{u}\| \, \|\mathbf{v}\| \, .$$

Proof.

We want to prove that

$$|\langle \mathbf{u}, \mathbf{v} \rangle| \le \|\mathbf{u}\| \, \|\mathbf{v}\| \, .$$

We will do this in 3 steps

1. Prove it is true for $\mathbf{u} = \mathbf{0}$;
2. Prove it is true if $\mathbf{u}$ is a unit vector;
3. Prove it is true for any $\mathbf{u} \neq \mathbf{0}$.

We want to prove that $|\langle \mathbf{u}, \mathbf{v} \rangle| \leq \|\mathbf{u}\| \, \|\mathbf{v}\|$

Proof.

First note that if $\mathbf{u} = \mathbf{0}$ then the Cauchy-Schwarz Inequality is true because $\|\mathbf{u}\| = 0$ and $\langle \mathbf{u}, \mathbf{v} \rangle = 0$.

We want to prove that $|\langle \mathbf{u}, \mathbf{v} \rangle| \leq \|\mathbf{u}\| \, \|\mathbf{v}\|$

Proof.

First note that if $\mathbf{u} = \mathbf{0}$ then the Cauchy-Schwarz Inequality is true because $\|\mathbf{u}\| = 0$ and $\langle \mathbf{u}, \mathbf{v} \rangle = 0$.

Next suppose that $\mathbf{u}$ is a unit vector. Write $\mathbf{v} = \mathbf{v}_{\|} + \mathbf{v}_{\perp}$ where $\mathbf{v}_{\|} = \langle \mathbf{u}, \mathbf{v} \rangle \, \mathbf{u}$.

We want to prove that $|\langle \mathbf{u}, \mathbf{v} \rangle| \leq \|\mathbf{u}\| \, \|\mathbf{v}\|$

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Next suppose that $\mathbf{u}$ is a unit vector. Write $\mathbf{v} = \mathbf{v}_{\|} + \mathbf{v}_{\perp}$ where $\mathbf{v}_{\|} = \langle \mathbf{u}, \mathbf{v} \rangle \, \mathbf{u}$. Recall that $\|c\mathbf{u}\| = |c| \, \|\mathbf{u}\|$ for any scalar c.

We want to prove that $|\langle \mathbf{u}, \mathbf{v}\rangle| \le \|\mathbf{u}\|\,\|\mathbf{v}\|$

Proof.

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Next suppose that $\mathbf{u}$ is a unit vector. Write $\mathbf{v} = \mathbf{v}_{\parallel} + \mathbf{v}_{\perp}$ where $\mathbf{v}_{\parallel} = \langle \mathbf{u}, \mathbf{v}\rangle\,\mathbf{u}$. Recall that $\|c\mathbf{u}\| = |c|\,\|\mathbf{u}\|$ for any scalar c. Thus

$$\left\|\mathbf{v}_{\parallel}\right\| = \|\langle \mathbf{u}, \mathbf{v}\rangle\,\mathbf{u}\|$$

We want to prove that $|\langle \mathbf{u}, \mathbf{v}\rangle| \leq \|\mathbf{u}\| \, \|\mathbf{v}\|$

Proof.

First note that if $\mathbf{u} = \mathbf{0}$ then the Cauchy-Schwarz Inequality is true because $\|\mathbf{u}\| = 0$ and $\langle \mathbf{u}, \mathbf{v}\rangle = 0$.

Next suppose that $\mathbf{u}$ is a unit vector. Write $\mathbf{v} = \mathbf{v}_{\|} + \mathbf{v}_{\perp}$ where $\mathbf{v}_{\|} = \langle \mathbf{u}, \mathbf{v}\rangle \, \mathbf{u}$. Recall that $\|c\mathbf{u}\| = |c| \, \|\mathbf{u}\|$ for any scalar c. Thus

$$\left\|\mathbf{v}_{\|}\right\| = \|\langle \mathbf{u}, \mathbf{v}\rangle \, \mathbf{u}\| = |\langle \mathbf{u}, \mathbf{v}\rangle| \, \|\mathbf{u}\|$$

We want to prove that $|\langle \mathbf{u}, \mathbf{v} \rangle| \leq \|\mathbf{u}\| \|\mathbf{v}\|$

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Since $\left\|\mathbf{v}_{\|}\right\| \le \|\mathbf{v}\|$ we have

$$|\langle \mathbf{u}, \mathbf{v} \rangle| \le \|\mathbf{v}\| \, .$$

We want to prove that $|\langle \mathbf{u}, \mathbf{v}\rangle| \leq \|\mathbf{u}\| \, \|\mathbf{v}\|$

Proof Continued.

Now let $\mathbf{u}$ be any nonzero vector and define $\hat{\mathbf{u}} = \frac{\mathbf{u}}{\|\mathbf{u}\|}$. Then

- $\hat{\mathbf{u}}$ is a unit vector;
- $\mathbf{u} = \|\mathbf{u}\| \, \hat{\mathbf{u}}$; and
- $|\langle \hat{\mathbf{u}}, \mathbf{v}\rangle| \leq \|\mathbf{v}\|$.

We want to prove that $|\langle \mathbf{u}, \mathbf{v}\rangle| \le \|\mathbf{u}\| \, \|\mathbf{v}\|$

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Therefore

$$|\langle \mathbf{u}, \mathbf{v}\rangle| = |\langle \|\mathbf{u}\| \, \hat{\mathbf{u}}, \mathbf{v}\rangle|$$

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and we are finished.

□

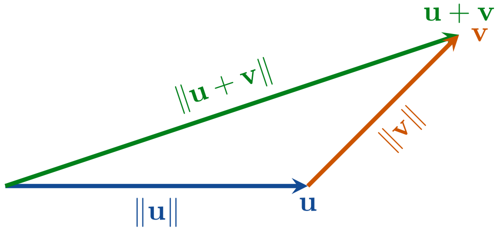
$|\langle \mathbf{u}, \mathbf{v} \rangle| \leq \|\mathbf{u}\| \, \|\mathbf{v}\|$

We can use the Cauchy-Schwarz Inequality to prove the following result:

Theorem (The Triangle Inequality)

For all $\mathbf{u}$, $\mathbf{v}$ *in* V,

$$\|\mathbf{u} + \mathbf{v}\| \leq \|\mathbf{u}\| + \|\mathbf{v}\| .$$



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Using the Cauchy-Schwarz Inequality we calculate that

$$\begin{aligned} \|\mathbf{u} + \mathbf{v}\|^2 &= \langle \mathbf{u} + \mathbf{v}, \mathbf{u} + \mathbf{v} \rangle = \\ &= \\ &= \\ &\leq \\ &\leq \\ &= \end{aligned}$$

$\square$

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&= \langle \mathbf{u}, \mathbf{u} \rangle + \langle \mathbf{u}, \mathbf{v} \rangle + \langle \mathbf{v}, \mathbf{u} \rangle + \langle \mathbf{v}, \mathbf{v} \rangle \\
&= \|\mathbf{u}\|^2 + 2 \langle \mathbf{u}, \mathbf{v} \rangle + \|\mathbf{v}\|^2 \\
&\leq \|\mathbf{u}\|^2 + 2 |\langle \mathbf{u}, \mathbf{v} \rangle| + \|\mathbf{v}\|^2 \\
&\leq \\
&=
\end{aligned}$$

□

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□

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Theorem (The Triangle Inequality)

For all $\mathbf{u}$, $\mathbf{v}$ *in* V,

$$\|\mathbf{u}+\mathbf{v}\| \leq \|\mathbf{u}\| + \|\mathbf{v}\| .$$

Proof.

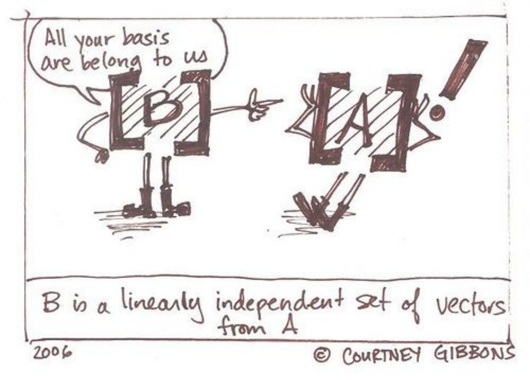
Using the Cauchy-Schwarz Inequality we calculate that

$$\begin{aligned}
\|\mathbf{u}+\mathbf{v}\|^2 = \langle \mathbf{u}+\mathbf{v}, \mathbf{u}+\mathbf{v} \rangle &= \langle \mathbf{u}, \mathbf{u}+\mathbf{v} \rangle + \langle \mathbf{v}, \mathbf{u}+\mathbf{v} \rangle \\
&= \langle \mathbf{u}, \mathbf{u} \rangle + \langle \mathbf{u}, \mathbf{v} \rangle + \langle \mathbf{v}, \mathbf{u} \rangle + \langle \mathbf{v}, \mathbf{v} \rangle \\
&= \|\mathbf{u}\|^2 + 2\langle \mathbf{u}, \mathbf{v} \rangle + \|\mathbf{v}\|^2 \\
&\leq \|\mathbf{u}\|^2 + 2|\langle \mathbf{u}, \mathbf{v} \rangle| + \|\mathbf{v}\|^2 \\
&\leq \|\mathbf{u}\|^2 + 2\|\mathbf{u}\| \, \|\mathbf{v}\| + \|\mathbf{v}\|^2 \\
&= (\|\mathbf{u}\| + \|\mathbf{v}\|)^2 .
\end{aligned}$$

$\square$

Break

We will continue at 3pm



Orthogonal Sets and Orthonormal Sets

Definition

A set of two or more vectors in a real inner product space is called an *orthogonal set* if all pairs of distinct vectors in the set are orthogonal.

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$$\langle \mathbf{u}, \mathbf{v} \rangle = 0$$

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Definition

An orthogonal set in which each vector is a unit vector is called an *orthonormal set*.

So we must have

1. $\langle \mathbf{u}, \mathbf{v} \rangle = 0$ for all $\mathbf{u} \neq \mathbf{v}$; and
2. $\langle \mathbf{u}, \mathbf{u} \rangle = 1$ for all $\mathbf{u} \in V$.

Example (An Orthogonal Set in $\mathbb{R}^3$)

Let

$$\mathbf{v}_1 = (0,1,0), \qquad \mathbf{v}_2 = (1,0,1), \qquad \mathbf{v}_3 = (1,0,-1).$$

Assume that $\mathbb{R}^3$ has the Euclidean inner product (dot product).

Show that $S = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is an orthogonal set.

$$\begin{aligned}
\langle \mathbf{v}_1, \mathbf{v}_2 \rangle &= \mathbf{v}_1 \cdot \mathbf{v}_2 = \\
\langle \mathbf{v}_1, \mathbf{v}_3 \rangle &= \mathbf{v}_1 \cdot \mathbf{v}_3 = \\
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$$\begin{aligned}
\langle \mathbf{v}_1, \mathbf{v}_2 \rangle &= \mathbf{v}_1 \cdot \mathbf{v}_2 = (0)(1) + (1)(0) + (0)(1) = 0 \\
\langle \mathbf{v}_1, \mathbf{v}_3 \rangle &= \mathbf{v}_1 \cdot \mathbf{v}_3 = (0)(1) + (1)(0) + (0)(-1) = 0 \\
\langle \mathbf{v}_2, \mathbf{v}_3 \rangle &= \mathbf{v}_2 \cdot \mathbf{v}_3 = (1)(1) + (0)(0) + (1)(-1) = 0.
\end{aligned}$$

Therefore S is orthogonal.

Recall that if $\mathbf{v} \neq \mathbf{0}$ is any nonzero vector, then $\mathbf{u} = \frac{\mathbf{v}}{\|\mathbf{v}\|}$ is a unit vector because

$$\|\mathbf{u}\| = \left\| \frac{\mathbf{v}}{\|\mathbf{v}\|} \right\| = \left| \frac{1}{\|\mathbf{v}\|} \right| \|\mathbf{v}\| = \frac{\|\mathbf{v}\|}{\|\mathbf{v}\|} = 1.$$

$$\mathbf{v}_1 = (0, 1, 0), \qquad \mathbf{v}_2 = (1, 0, 1), \qquad \mathbf{v}_3 = (1, 0, -1)$$

Example (Constructing an Orthonormal Set)

We have that $S = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is an orthogonal set in $\mathbb{R}^3$ with respect to the Euclidean inner product. Note that

$$\|\mathbf{v}_1\| = \sqrt{0^2 + 1^2 + 0^2} = 1 \qquad \|\mathbf{v}_2\| = \sqrt{1^2 + 0^2 + 1^2} = \sqrt{2}$$

$$\|\mathbf{v}_3\| = \sqrt{1^2 + 0^2 + (-1)^2} = \sqrt{2}.$$

It follows that if

$$\mathbf{u}_1 = \frac{\mathbf{v}_1}{\|\mathbf{v}_1\|} = (0, 1, 0) \qquad \mathbf{u}_2 = \frac{\mathbf{v}_2}{\|\mathbf{v}_2\|} = \left(\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right)$$

$$\mathbf{u}_3 = \frac{\mathbf{v}_3}{\|\mathbf{v}_3\|} = \left(\frac{1}{\sqrt{2}}, 0, -\frac{1}{\sqrt{2}}\right)$$

then $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ is an orthonormal set.

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then $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ is an orthonormal set.

I leave it to you to check that

$$\langle \mathbf{u}_1, \mathbf{u}_2 \rangle = \langle \mathbf{u}_1, \mathbf{u}_3 \rangle = \langle \mathbf{u}_2, \mathbf{u}_3 \rangle = 0$$

and

$$\|\mathbf{u}_1\| = \|\mathbf{u}_2\| = \|\mathbf{u}_3\| = 1.$$

Theorem

If $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is an orthogonal set of nonzero vectors in an inner product space, then S is linearly independent.

Proof.

Suppose that

$$k_1\mathbf{v}_1 + k_2\mathbf{v}_2 + \ldots + k_n\mathbf{v}_n = \mathbf{0}.$$

We must prove that $k_1 = k_2 = \ldots = k_n = 0$.

Orthogonal Sets and Orthonormal Sets

Proof Continued.

For each $\mathbf{v}_i \in S$ we have

$$\begin{aligned} 0 = \langle \mathbf{0}, \mathbf{v}_i \rangle &= \\ &= \\ &= \\ &= \end{aligned}$$

Proof Continued.

For each $\mathbf{v}_i \in S$ we have

$$
\begin{aligned}
0 = \langle \mathbf{0}, \mathbf{v}_i \rangle &= \langle k_1\mathbf{v}_1 + k_2\mathbf{v}_2 + \ldots + k_n\mathbf{v}_n, \mathbf{v}_i \rangle \\
&= \\
&= \\
&=
\end{aligned}
$$

Proof Continued.

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&= k_1 \langle \mathbf{v}_1, \mathbf{v}_i \rangle + k_2 \langle \mathbf{v}_2, \mathbf{v}_i \rangle + \ldots + k_i \langle \mathbf{v}_i, \mathbf{v}_i \rangle + \ldots + k_n \langle \mathbf{v}_n, \mathbf{v}_i \rangle \\
&= \\
&=
\end{aligned}$$

Proof Continued.

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0 = \langle \mathbf{0}, \mathbf{v}_i \rangle &= \langle k_1\mathbf{v}_1 + k_2\mathbf{v}_2 + \ldots + k_n\mathbf{v}_n, \mathbf{v}_i \rangle \\
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&= 0 + 0 + \ldots + k_i \langle \mathbf{v}_i, \mathbf{v}_i \rangle + \ldots + 0 \\
&=
\end{aligned}
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Proof Continued.

For each $\mathbf{v}_i \in S$ we have

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Proof Continued.

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Since $\mathbf{v}_i \neq \mathbf{0}$, we have that $\langle \mathbf{v}_i, \mathbf{v}_i \rangle \neq 0$.

Proof Continued.

For each $\mathbf{v}_i \in S$ we have

$$\begin{aligned} 0 = \langle \mathbf{0}, \mathbf{v}_i \rangle &= \langle k_1\mathbf{v}_1 + k_2\mathbf{v}_2 + \ldots + k_n\mathbf{v}_n, \mathbf{v}_i \rangle \\ &= k_1 \langle \mathbf{v}_1, \mathbf{v}_i \rangle + k_2 \langle \mathbf{v}_2, \mathbf{v}_i \rangle + \ldots + k_i \langle \mathbf{v}_i, \mathbf{v}_i \rangle + \ldots + k_n \langle \mathbf{v}_n, \mathbf{v}_i \rangle \\ &= 0 + 0 + \ldots + k_i \langle \mathbf{v}_i, \mathbf{v}_i \rangle + \ldots + 0 \\ &= k_i \langle \mathbf{v}_i, \mathbf{v}_i \rangle . \end{aligned}$$

Since $\mathbf{v}_i \neq \mathbf{0}$, we have that $\langle \mathbf{v}_i, \mathbf{v}_i \rangle \neq 0$.

Therefore $k_i = 0$ for all i. This proves that S is linearly independent.

$\square$

Orthogonal and Orthonormal Bases

Definition

In an inner product space, a basis consisting of orthonormal vectors is called an *orthonormal basis*, and a basis consisting of orthogonal vectors is called an *orthogonal basis*.

Example

$$\mathbf{e}_1 = (1, 0, 0, \dots, 0),\ \mathbf{e}_2 = (0, 1, 0, \dots, 0),\ \dots,\ \mathbf{e}_n = (0, 0, 0, \dots, 1)$$

is an orthonormal basis in $\mathbb{R}^n$ with the Euclidean inner product.

(Recall that this basis is called the *standard basis* for $\mathbb{R}^n$.)

Example (An Orthonormal Basis for $\mathbb{P}^n$)

Consider the vector space of polynomials of degree $\leq n$ with the inner product

$$\langle \mathbf{p}, \mathbf{q} \rangle = a_0 b_0 + a_1 b_1 + \ldots + a_n b_n$$

where

$$\mathbf{p} = a_0 + a_1 x + \ldots + a_n x^n$$
$$\mathbf{q} = b_0 + b_1 x + \ldots + b_n x^n.$$

I leave it to you to prove that the standard basis

$$S = \{1, x, x^2, x^3, \ldots, x^n\}$$

is an orthonormal basis with respect to this inner product.

Example

In an earlier example we saw that

$$\mathbf{u}_1 = (0, 1, 0), \qquad \mathbf{u}_2 = \left(\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right), \qquad \mathbf{u}_3 = \left(\frac{1}{\sqrt{2}}, 0, -\frac{1}{\sqrt{2}}\right)$$

form an orthonormal set with respect to the Euclidean inner product on $\mathbb{R}^3$.

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By the previous theorem, these three vectors are linearly independent.

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form an orthonormal set with respect to the Euclidean inner product on $\mathbb{R}^3$.

By the previous theorem, these three vectors are linearly independent.

Since $\mathbb{R}^3$ is three-dimensional, $S = \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ is an orthonormal basis for $\mathbb{R}^3$.

Coordinates Relative to Orthonormal Bases

Recall that if $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is a basis for a vector space V, and if

$$\mathbf{u} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \ldots + c_n\mathbf{v}_n$$

then the coordinates of $\mathbf{u}$ relative to this basis is

$$(\mathbf{u})_S = (c_1, c_2, \ldots, c_n).$$

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then the coordinates of $\mathbf{u}$ relative to this basis is

$$(\mathbf{u})_S = (c_1, c_2, \ldots, c_n).$$

If the basis is orthogonal or orthonormal then there is an easy way to find the coefficients $c_1, c_2, \ldots, c_n$.

Orthogonal Sets and Orthonormal Sets

Theorem

If $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ *is an* *orthogonal* *basis for an inner product space* V, *and if* $\mathbf{u}$ *is any vector in* V, *then*

$$\mathbf{u} = \frac{\langle \mathbf{u}, \mathbf{v}_1 \rangle}{\|\mathbf{v}_1\|^2}\mathbf{v}_1 + \frac{\langle \mathbf{u}, \mathbf{v}_2 \rangle}{\|\mathbf{v}_2\|^2}\mathbf{v}_2 + \ldots + \frac{\langle \mathbf{u}, \mathbf{v}_n \rangle}{\|\mathbf{v}_n\|^2}\mathbf{v}_n.$$

Theorem

If $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is an orthogonal basis for an inner product space V, and if $\mathbf{u}$ is any vector in V, then

$$\mathbf{u} = \frac{\langle \mathbf{u}, \mathbf{v}_1 \rangle}{\|\mathbf{v}_1\|^2}\mathbf{v}_1 + \frac{\langle \mathbf{u}, \mathbf{v}_2 \rangle}{\|\mathbf{v}_2\|^2}\mathbf{v}_2 + \ldots + \frac{\langle \mathbf{u}, \mathbf{v}_n \rangle}{\|\mathbf{v}_n\|^2}\mathbf{v}_n.$$

Proof.

Let

$$\mathbf{u} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \ldots + c_n\mathbf{v}_n.$$

We need to show that $c_i = \dfrac{\langle \mathbf{u}, \mathbf{v}_i \rangle}{\|\mathbf{v}_i\|^2}$ for each i.

Proof Continued.

Note that

$$\begin{aligned}
\langle \mathbf{u}, \mathbf{v}_i \rangle &= \langle c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \ldots + c_n\mathbf{v}_n, \mathbf{v}_i \rangle \\
&= c_1\langle \mathbf{v}_1, \mathbf{v}_i \rangle + c_2\langle \mathbf{v}_2, \mathbf{v}_i \rangle + \ldots + c_i\langle \mathbf{v}_i, \mathbf{v}_i \rangle + \ldots + c_n\langle \mathbf{v}_n, \mathbf{v}_i \rangle \\
&= \\
&=
\end{aligned}$$

Proof Continued.

Note that

$$\begin{aligned}
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&= c_1\langle \mathbf{v}_1, \mathbf{v}_i \rangle + c_2\langle \mathbf{v}_2, \mathbf{v}_i \rangle + \ldots + c_i\langle \mathbf{v}_i, \mathbf{v}_i \rangle + \ldots + c_n\langle \mathbf{v}_n, \mathbf{v}_i \rangle \\
&= 0 + 0 + \ldots + c_i\|\mathbf{v}_i\|^2 + \ldots + 0 \\
&=
\end{aligned}$$

Proof Continued.

Note that

$$\begin{aligned}
\langle \mathbf{u}, \mathbf{v}_i \rangle &= \langle c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \ldots + c_n\mathbf{v}_n, \mathbf{v}_i \rangle \\
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&= 0 + 0 + \ldots + c_i\|\mathbf{v}_i\|^2 + \ldots + 0 \\
&= c_i \left\|\mathbf{v}_i\right\|^2 .
\end{aligned}$$

Proof Continued.

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&= 0 + 0 + \ldots + c_i\|\mathbf{v}_i\|^2 + \ldots + 0 \\
&= c_i \|\mathbf{v}_i\|^2 .
\end{aligned}$$

Hence $c_i = \dfrac{\langle \mathbf{u}, \mathbf{v}_i \rangle}{\|\mathbf{v}_i\|^2}$ for each i and we are finished. $\square$

Theorem

If $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ *is an* *<u>orthonormal</u>* *basis for an inner product space* V*, and if* $\mathbf{u}$ *is any vector in* V*, then*

$$\mathbf{u} = \langle \mathbf{u}, \mathbf{v}_1 \rangle \mathbf{v}_1 + \langle \mathbf{u}, \mathbf{v}_2 \rangle \mathbf{v}_2 + \ldots + \langle \mathbf{u}, \mathbf{v}_n \rangle \mathbf{v}_n.$$

Theorem

If $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is an orthonormal basis for an inner product space V, and if $\mathbf{u}$ is any vector in V, then

$$\mathbf{u} = \langle \mathbf{u}, \mathbf{v}_1 \rangle \mathbf{v}_1 + \langle \mathbf{u}, \mathbf{v}_2 \rangle \mathbf{v}_2 + \ldots + \langle \mathbf{u}, \mathbf{v}_n \rangle \mathbf{v}_n.$$

Proof.

Just take the previous formula

$$\mathbf{u} = \frac{\langle \mathbf{u}, \mathbf{v}_1 \rangle}{\|\mathbf{v}_1\|^2}\mathbf{v}_1 + \frac{\langle \mathbf{u}, \mathbf{v}_2 \rangle}{\|\mathbf{v}_2\|^2}\mathbf{v}_2 + \ldots + \frac{\langle \mathbf{u}, \mathbf{v}_n \rangle}{\|\mathbf{v}_n\|^2}\mathbf{v}_n$$

and replace every $\|\mathbf{v}_i\|$ by 1 since each $\mathbf{v}_i$ is a unit vector. $\square$

Remark

So if S is an orthogonal basis then

$$(\mathbf{u})_S = \left(\frac{\langle \mathbf{u}, \mathbf{v}_1 \rangle}{\|\mathbf{v}_1\|^2}, \frac{\langle \mathbf{u}, \mathbf{v}_2 \rangle}{\|\mathbf{v}_2\|^2}, \dots, \frac{\langle \mathbf{u}, \mathbf{v}_n \rangle}{\|\mathbf{v}_n\|^2} \right)$$

and if S is an orthonormal basis then

$$(\mathbf{u})_S = \big(\langle \mathbf{u}, \mathbf{v}_1 \rangle, \langle \mathbf{u}, \mathbf{v}_2 \rangle, \dots, \langle \mathbf{u}, \mathbf{v}_n \rangle \big).$$

Orthogonal Sets and Orthonormal Sets

Example

Let

$$\mathbf{v}_1 = (0, 1, 0), \qquad \mathbf{v}_2 = \left(-\frac{4}{5}, 0, \frac{3}{5}\right), \qquad \mathbf{v}_3 = \left(\frac{3}{5}, 0, \frac{4}{5}\right).$$

I leave it to you to check that $S = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is an orthonormal basis for $\mathbb{R}^3$ with the Euclidean inner product.

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Find $(\mathbf{u})_S$ if $\mathbf{u} = (1,1,1)$.

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Find $(\mathbf{u})_S$ if $\mathbf{u} = (1,1,1)$.

Since

$$\langle \mathbf{u}, \mathbf{v}_1 \rangle = 1, \qquad \langle \mathbf{u}, \mathbf{v}_2 \rangle = -\frac{1}{5}, \qquad \langle \mathbf{u}, \mathbf{v}_3 \rangle = \frac{7}{5},$$

(please check) we have that

$$(\mathbf{u})_S = \left(\langle \mathbf{u}, \mathbf{v}_1 \rangle, \langle \mathbf{u}, \mathbf{v}_2 \rangle, \langle \mathbf{u}, \mathbf{v}_3 \rangle \right) = \left(1, -\frac{1}{5}, \frac{7}{5}\right).$$

Example (An Orthonormal Basis from an Orthogonal Basis)

1. Show that the vectors

$$\mathbf{w}_1 = (0, 2, 0), \qquad \mathbf{w}_2 = (3, 0, 3), \qquad \mathbf{w}_3 = (-4, 0, 4)$$

   form an orthogonal basis for $\mathbb{R}^3$ with the Euclidean inner product.
2.
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2. Normalise each vector above to find an orthonormal basis $S = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$.

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   form an orthogonal basis for $\mathbb{R}^3$ with the Euclidean inner product.
2. Normalise each vector above to find an orthonormal basis $S = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$.
3. Find $(\mathbf{u})_S$ if $\mathbf{u} = (1, 2, 4)$.

$$\mathbf{w}_1 = (0,2,0), \qquad \mathbf{w}_2 = (3,0,3), \qquad \mathbf{w}_3 = (-4,0,4)$$

1. I leave it to you to check that

$$\langle \mathbf{w}_1, \mathbf{w}_2 \rangle = \langle \mathbf{w}_1, \mathbf{w}_3 \rangle = \langle \mathbf{w}_2, \mathbf{w}_3 \rangle = 0.$$

   This shows that $\{\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3\}$ is an orthogonal set.

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Recall that sets of nonzero orthogonal vectors are always linearly independent.

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Recall that sets of nonzero orthogonal vectors are always linearly independent.

Since $\mathbb{R}^3$ is three-dimensional, $\{\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3\}$ must be a basis for $\mathbb{R}^3$.

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   Since $\mathbb{R}^3$ is three-dimensional, $\{\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3\}$ must be a basis for $\mathbb{R}^3$.

2. We calculate

$$\mathbf{v}_1 = \frac{\mathbf{w}_1}{\|\mathbf{w}_1\|} = (0,1,0), \qquad \mathbf{v}_2 = \frac{\mathbf{w}_2}{\|\mathbf{w}_2\|} = \left(\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right)$$

$$\mathbf{v}_3 = \frac{\mathbf{w}_3}{\|\mathbf{w}_3\|} = \left(-\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right).$$

$\mathbf{v}_1 = (0,1,0), \quad \mathbf{v}_2 = \left(\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right), \quad \mathbf{v}_3 = \left(-\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right)$

3. Since

$$\langle \mathbf{u}, \mathbf{v}_1 \rangle = (1,2,4) \bullet (0,1,0) = 2$$
$$\langle \mathbf{u}, \mathbf{v}_2 \rangle = (1,2,4) \bullet \left(\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right) = \frac{5}{\sqrt{2}}$$
$$\langle \mathbf{u}, \mathbf{v}_3 \rangle = (1,2,4) \bullet \left(-\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right) = \frac{3}{\sqrt{2}},$$

we have that

$$(\mathbf{u})_S = \left(2, \frac{5}{\sqrt{2}}, \frac{3}{\sqrt{2}}\right).$$

Orthogonal Projections

Theorem

If W is a finite-dimensional subspace of an inner product space V, then every vector $\mathbf{u}$ in V can be expressed in exactly one way as

$$\mathbf{u} = \mathbf{w}_{\|} + \mathbf{w}_{\perp}$$

where $\mathbf{w}_{\|}$ is in W and $\mathbf{w}_{\perp}$ is in $W^{\perp}$.

Orthogonal Projections

Theorem

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$$\mathbf{u} = \mathbf{w}_{\parallel} + \mathbf{w}_{\perp}$$

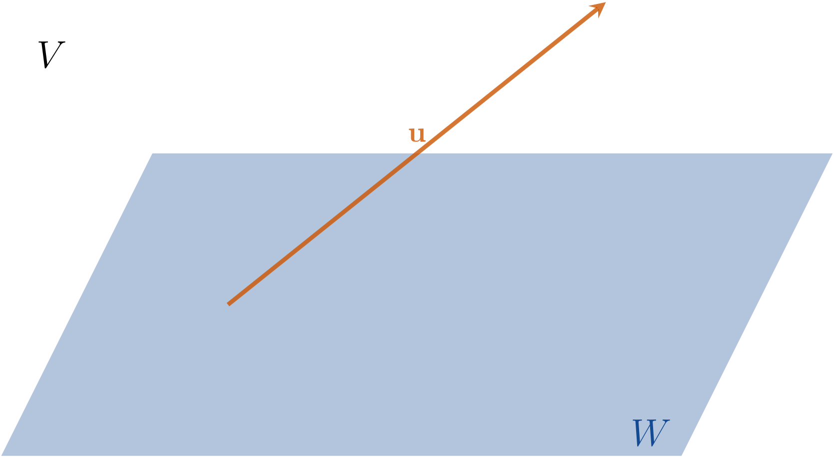
where $\mathbf{w}_{\parallel}$ is in W and $\mathbf{w}_{\perp}$ is in $W^{\perp}$.

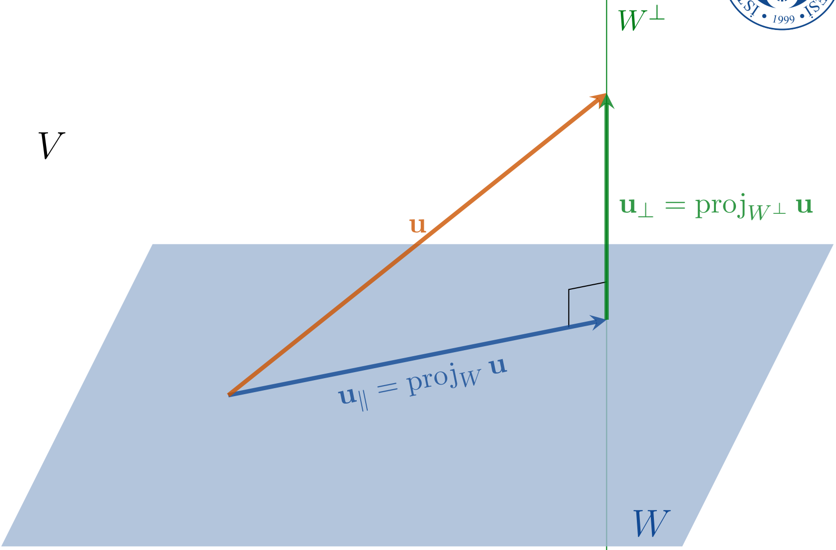
The vectors $\mathbf{w}_{\parallel}$ and $\mathbf{w}_{\perp}$ are often denoted as

$$\mathbf{w}_{\parallel} = \operatorname{proj}_W \mathbf{u} \qquad \text{and} \qquad \mathbf{w}_{\perp} = \operatorname{proj}_{W^{\perp}} \mathbf{u}$$

and are called the *orthogonal projection of* $\mathbf{u}$ *on* W and the *orthogonal projection of* $\mathbf{u}$ *on* $W^{\perp}$, respectively.


V
W


V
$\mathbf{u}$
W



$$\mathbf{u} = \mathrm{proj}_W\,\mathbf{u} + \mathrm{proj}_{W^\perp}\,\mathbf{u}$$

Theorem

Let W be a finite-dimensional subspace of an inner product space V.

1. *If $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_r\}$ is an orthogonal basis for W, and $\mathbf{u}$ is any vector in V, then*

$$\operatorname{proj}_W \mathbf{u} = \frac{\langle \mathbf{u}, \mathbf{v}_1 \rangle}{\|\mathbf{v}_1\|^2}\mathbf{v}_1 + \frac{\langle \mathbf{u}, \mathbf{v}_2 \rangle}{\|\mathbf{v}_2\|^2}\mathbf{v}_2 + \ldots + \frac{\langle \mathbf{u}, \mathbf{v}_n \rangle}{\|\mathbf{v}_r\|^2}\mathbf{v}_r.$$

2. *If $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_r\}$ is an orthonormal basis for W, and $\mathbf{u}$ is any vector in V, then*

$$\operatorname{proj}_W \mathbf{u} = \langle \mathbf{u}, \mathbf{v}_1 \rangle \mathbf{v}_1 + \langle \mathbf{u}, \mathbf{v}_2 \rangle \mathbf{v}_2 + \ldots + \langle \mathbf{u}, \mathbf{v}_r \rangle \mathbf{v}_r.$$

Next Time

- The Gram-Schmidt Process
- Orthogonal Matrices
- Orthogonal Diagonalisation