2019–20 **MATH216 Mathematics IV – Solutions to Exercise Sheet 9** N. Course

Exercise 32 (Systems of First Order Linear Equations). Transform each of the following equations into a system of first order linear ODEs.

(a) $u'' + 0.5u' + 2u = 0$

(b) $u'' + 0.5u' + 2u = 3\sin t$

(c) $t^2u'' + tu' + (t^2 - 0.25)u = 0$

(d) $u^{(4)} - u = 0$

Transform each of the following systems into a single second order ODE for x_1.

(e) $\begin{cases} x_1' = 3x_1 - 2x_2 \\ x_2' = 2x_1 - 2x_2 \end{cases}$

(f) $\begin{cases} x_1' = 1.25x_1 + 0.75x_2 \\ x_2' = 0.75x_1 + 1.25x_2 \end{cases}$

(g) $\begin{cases} x_1' = x_1 - 2x_2 \\ x_2' = 3x_1 - 4x_2 \end{cases}$

(h) $\begin{cases} x_1' = 2x_2 \\ x_2' = -2x_1 \end{cases}$

Solution 32.

(a) $\begin{cases} x_1' = x_2 \\ x_2' = -2x_1 - 0.5x_2 \end{cases}$

(b) $\begin{cases} x_1' = x_2 \\ x_2' = -2x_1 - 0.5x_2 + 3\sin t \end{cases}$

(c) $\begin{cases} x_1' = x_2 \\ x_2' = -(1 - 0.25t^{-2})x_1 - t^{-1}x_2 \end{cases}$

(d) $\begin{cases} x_1' = x_2 \\ x_2' = x_3 \\ x_3' = x_4 \\ x_4' = x_1 \end{cases}$

(e) $x_1'' - x_1' - 2x_1 = 0$

(f) $x_1'' - 2.5x_1 + x_1 = 0$

(g) First we solve the first equation for x_2:

$$x_2 = \frac{1}{2}x_1 - \frac{1}{2}x_1'.$$

Then we substitute into the second equation to find

$$\begin{aligned} x_2' &= 3x_1 - 4x_2 \\ \left(\frac{1}{2}x_1 - \frac{1}{2}x_1'\right)' &= 3x_1 - 4\left(\frac{1}{2}x_1 - \frac{1}{2}x_1'\right) \\ \frac{1}{2}x_1' - \frac{1}{2}x_1'' &= 3x_1 - 2x_1 - 2x_1' \\ 0 &= \frac{1}{2}x_1'' - \frac{5}{2}x_1' + x_1 \\ 0 &= x_1'' - 5x_1' + 2x_1 \end{aligned}$$

(h) $x_1'' + 4x_1 = 0$

Exercise 33 (Fundamental Matrices).

(a) Suppose that $\Psi(t)$ is a fundamental matrix for $\mathbf{x}' = A\mathbf{x}$, where $A \in \mathbb{R}^{n\times n}$. Show that $\Psi' = A\Psi$.

(b) Let $A = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}$. Find $e^{At} = \exp(At)$.

For each of the following;

(i) Find a fundamental matrix $\Psi(t)$ for the system; and

(ii) Find the special fundamental matrix $\Phi(t)$ which satisfies $\Phi(0) = I$.

The first one is done for you.

(ω) $\mathbf{x}' = \begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix}\mathbf{x}$

(c) $\mathbf{x}' = \begin{bmatrix} 3 & -2 \\ 2 & -2 \end{bmatrix}\mathbf{x}$

(d) $\mathbf{x}' = \begin{bmatrix} -\frac{3}{4} & \frac{1}{2} \\ \frac{1}{8} & -\frac{3}{4} \end{bmatrix}\mathbf{x}$

(e) $\mathbf{x}' = \begin{bmatrix} 2 & -5 \\ 1 & -2 \end{bmatrix}\mathbf{x}$

(f) $\mathbf{x}' = \begin{bmatrix} 1 & 1 \\ 4 & -2 \end{bmatrix}\mathbf{x}$

(g) $\mathbf{x}' = \begin{bmatrix} 2 & -1 \\ 3 & -2 \end{bmatrix}\mathbf{x}$

(h) $\mathbf{x}' = \begin{bmatrix} -1 & -4 \\ 1 & -1 \end{bmatrix}\mathbf{x}$

(i) $\mathbf{x}' = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 1 & -1 \\ -8 & -5 & -3 \end{bmatrix}\mathbf{x}$

(ω) (i) The general solution to this system is $\mathbf{x}(t) = c_1 \begin{bmatrix}1\\2\end{bmatrix} e^{3t} + c_2 \begin{bmatrix}1\\-2\end{bmatrix} e^{-t}$. Therefore $\Psi(t) = \begin{bmatrix} e^{3t} & e^{-t} \\ 2e^{3t} & -2e^{-t}\end{bmatrix}$ is a fundamental matrix for this system.

(ii) We must solve $\begin{cases} \mathbf{x}' = \begin{bmatrix}1 & 1\\4 & 1\end{bmatrix}\mathbf{x} \\ \mathbf{x}(0) = \begin{bmatrix}1\\0\end{bmatrix}\end{cases}$ and $\begin{cases} \mathbf{x}' = \begin{bmatrix}1 & 1\\4 & 1\end{bmatrix}\mathbf{x} \\ \mathbf{x}(0) = \begin{bmatrix}0\\1\end{bmatrix}\end{cases}$. Using the general solution from above, we calculate that

$$\begin{bmatrix}1\\0\end{bmatrix} = \mathbf{x}(0) = c_1 \begin{bmatrix}1\\2\end{bmatrix} + c_2 \begin{bmatrix}1\\-2\end{bmatrix} \implies c_1 = c_2 = \frac{1}{2} \implies \mathbf{x}^{(1)}(t) = \begin{bmatrix}\frac{1}{2}e^{3t} + \frac{1}{2}e^{-t} \\ e^{3t} - e^{-t}\end{bmatrix}$$

and

$$\begin{bmatrix}0\\1\end{bmatrix} = \mathbf{x}(0) = c_1 \begin{bmatrix}1\\2\end{bmatrix} + c_2 \begin{bmatrix}1\\-2\end{bmatrix} \implies c_1 = \frac{1}{4},\ c_2 = -\frac{1}{4} \implies \mathbf{x}^{(2)}(t) = \begin{bmatrix}\frac{1}{4}e^{3t} - \frac{1}{4}e^{-t} \\ \frac{1}{2}e^{3t} + \frac{1}{2}e^{-t}\end{bmatrix}$$

Therefore

$$\Phi(t) = \begin{bmatrix}\frac{1}{2}e^{3t} + \frac{1}{2}e^{-t} & \frac{1}{4}e^{3t} - \frac{1}{4}e^{-t} \\ e^{3t} - e^{-t} & \frac{1}{2}e^{3t} + \frac{1}{2}e^{-t}\end{bmatrix}.$$

Solution 33.

(a) We have that

$$A\Psi = A\begin{bmatrix}\mathbf{x}^{(1)} & \mathbf{x}^{(2)} & \dots & \mathbf{x}^{(n)}\end{bmatrix} = \begin{bmatrix}A\mathbf{x}^{(1)} & A\mathbf{x}^{(2)} & \dots & A\mathbf{x}^{(n)}\end{bmatrix} = \begin{bmatrix}\mathbf{x}^{(1)\prime} & \mathbf{x}^{(2)\prime} & \dots & \mathbf{x}^{(n)\prime}\end{bmatrix} = \Psi'$$

since $\mathbf{x}^{(k)}$ solves $A\mathbf{x}^{(k)} = x^{(k)\prime}$ for each k.

(b) Recall first that $e^{At} = \Phi(t)$ where Φ is the special fundamental matrix which satisfies $\Phi(0) = I$. Since the eigenvalues of A are $r_1 = 1$ and $r_2 = 2$; and the eigenvectors are $\xi^{(1)} = \begin{bmatrix}1\\0\end{bmatrix}$ and $\xi^{(2)} = \begin{bmatrix}1\\1\end{bmatrix}$, the general solution of $\mathbf{x}' = A\mathbf{x}$ is $\mathbf{x}(t) = c_1 \begin{bmatrix}1\\0\end{bmatrix} e^t + c_2 \begin{bmatrix}1\\1\end{bmatrix} e^{2t}$. Then we calculate that

$$\begin{bmatrix}1\\0\end{bmatrix} = \mathbf{x}(0) = c_1 \begin{bmatrix}1\\0\end{bmatrix} + c_2 \begin{bmatrix}1\\1\end{bmatrix} \implies c_1 = 1,\ c_2 = 0 \implies \mathbf{x}^{(1)} = \begin{bmatrix}e^t\\0\end{bmatrix}$$

and

$$\begin{bmatrix}0\\1\end{bmatrix} = \mathbf{x}(0) = c_1 \begin{bmatrix}1\\0\end{bmatrix} + c_2 \begin{bmatrix}1\\1\end{bmatrix} \implies c_1 = -1,\ c_2 = 1 \implies \mathbf{x}^{(2)} = \begin{bmatrix}e^{2t} - e^t\\e^{2t}\end{bmatrix}.$$

Therefore

$$e^{At} = \Phi(t) = \begin{bmatrix}e^{2t} & e^t - e^{2t} \\ 0 & e^t\end{bmatrix}.$$

(c) (i) omitted (ii) $\Phi(t) = \begin{bmatrix}-\frac{1}{3}e^{-t} + \frac{4}{3}e^{2t} & \frac{2}{3}e^{-t} - \frac{2}{3}e^{2t} \\ -\frac{2}{3}e^{-t} + \frac{2}{3}e^{2t} & \frac{4}{3}e^{-t} - \frac{1}{3}e^{2t}\end{bmatrix}$

(d) (i) This matrix has eigenvalues $r_1 = -\frac{1}{2}$ and $r_2 = -1$; and corresponding eigenvectors $\xi^{(1)} = \begin{bmatrix}2\\1\end{bmatrix}$ and $\xi^{(2)} = \begin{bmatrix}-2\\1\end{bmatrix}$. Therefore the general solution is $\mathbf{x}(t) = c_1 \begin{bmatrix}2\\1\end{bmatrix} e^{-\frac{t}{2}} + c_2 \begin{bmatrix}-2\\1\end{bmatrix} e^{-t}$ and a fundamental matrix is $\Psi(t) = \begin{bmatrix}2e^{-\frac{t}{2}} & -2e^{-t} \\ e^{-\frac{t}{2}} & e^{-t}\end{bmatrix}$.

(ii) Since

$$\begin{bmatrix}1\\0\end{bmatrix} = c_1 \begin{bmatrix}2\\1\end{bmatrix} e^{-\frac{0}{2}} + c_2 \begin{bmatrix}-2\\1\end{bmatrix} e^{-0} \implies c_1 = \frac{1}{4},\ c_2 = -\frac{1}{4} \implies \mathbf{x}^{(1)}(t) = \begin{bmatrix}\frac{1}{2}e^{-\frac{t}{2}} + \frac{1}{2}e^{-t} \\ \frac{1}{4}e^{-\frac{t}{2}} - \frac{1}{4}e^{-t}\end{bmatrix}$$

and

$$\begin{bmatrix}0\\1\end{bmatrix} = c_1 \begin{bmatrix}2\\1\end{bmatrix} e^{-\frac{0}{2}} + c_2 \begin{bmatrix}-2\\1\end{bmatrix} e^{-0} \implies c_1 = c_2 = \frac{1}{2} \implies \mathbf{x}^{(1)}(t) = \begin{bmatrix}e^{-\frac{t}{2}} - e^{-t} \\ \frac{1}{2}e^{-\frac{t}{2}} + \frac{1}{2}e^{-t}\end{bmatrix}$$

we have that

$$\Phi(t) = \begin{bmatrix}\frac{1}{2}e^{-\frac{t}{2}} + \frac{1}{2}e^{-t} & e^{-\frac{t}{2}} - e^{-t} \\ \frac{1}{4}e^{-\frac{t}{2}} - \frac{1}{4}e^{-t} & \frac{1}{2}e^{-\frac{t}{2}} + \frac{1}{2}e^{-t}\end{bmatrix}.$$

(e) (i) This matrix has eigenvalues $r_1 = i$ and $r_2 = -i$; and corresponding eigenvectors $\xi^{(1)} = \begin{bmatrix}2+i\\1\end{bmatrix}$ and $\xi^{(2)} = \begin{bmatrix}2-i\\1\end{bmatrix}$. Since

$$\begin{aligned}\mathbf{x}^{(1)}(t) = \xi^{(1)}e^{r_1 t} &= \begin{bmatrix}2+i\\1\end{bmatrix}(\cos t + i\sin t) \\ &= \begin{bmatrix}2\cos t - \sin t\\ \cos t\end{bmatrix} + i\begin{bmatrix}\cos t + 2\sin t\\ \sin t\end{bmatrix},\end{aligned}$$

we have general solution

$$\mathbf{x}(t) = c_1 \begin{bmatrix} 2\cos t - \sin t \\ \cos t \end{bmatrix} + c_2 \begin{bmatrix} \cos t + 2\sin t \\ \sin t \end{bmatrix}$$

and fundamental matrix

$$\Psi(t) = \begin{bmatrix} 2\cos t - \sin t & \cos t + 2\sin t \\ \cos t & \sin t \end{bmatrix}.$$

(ii) Then we calculate that

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \mathbf{x}(0) = c_1 \begin{bmatrix} 2 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 0 \end{bmatrix} \implies c_1 = 0,\ c_2 = 1 \implies \mathbf{x}(t) = \begin{bmatrix} \cos t + 2\sin t \\ \sin t \end{bmatrix}$$

and

$$\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \mathbf{x}(0) = c_1 \begin{bmatrix} 2 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 0 \end{bmatrix} \implies c_1 = 1,\ c_2 = -2 \implies \mathbf{x}(t) = \begin{bmatrix} -5\sin t \\ \cos t - 2\sin t \end{bmatrix}.$$

Therefore

$$\Phi(t) = \begin{bmatrix} \cos t + 2\sin t & -5\sin t \\ \sin t & \cos t - 2\sin t \end{bmatrix}.$$

(f) (i) omitted (ii) $\Phi(t) = \begin{bmatrix} \frac{1}{5}e^{-3t} + \frac{4}{5}e^{2t} & -\frac{1}{5}e^{-3t} + \frac{1}{5}e^{2t} \\ -\frac{4}{5}e^{-3t} + \frac{4}{5}e^{2t} & \frac{4}{5}e^{-3t} + \frac{1}{5}e^{2t} \end{bmatrix}$

(g) (i) omitted (ii) $\Phi(t) = \begin{bmatrix} \frac{3}{2}e^{t} - \frac{1}{2}e^{-t} & -\frac{1}{2}e^{t} + \frac{1}{2}e^{-t} \\ \frac{3}{2}e^{t} - \frac{3}{2}e^{-t} & -\frac{1}{2}e^{t} + \frac{3}{2}e^{-t} \end{bmatrix}$

(h) (i) omitted (ii) $\Phi(t) = \begin{bmatrix} e^{-t}\cos 2t & -2e^{-t}\sin 2t \\ \frac{1}{2}e^{-t}\sin 2t & e^{-t}\cos 2t \end{bmatrix}$

(g) (i) omitted (ii) $\Phi(t) = \begin{bmatrix} -2e^{-2t} + 3e^{-t} & -e^{-2t} + e^{-t} & -e^{-2t} + e^{-t} \\ \frac{5}{2}e^{-2t} - 4e^{-t} + \frac{3}{2}e^{2t} & \frac{5}{4}e^{-2t} - \frac{4}{3}e^{-t} + \frac{13}{12}e^{2t} & \frac{5}{4}e^{-2t} - \frac{4}{3}e^{-t} + \frac{1}{12}e^{2t} \\ \frac{7}{2}e^{-2t} - 2e^{-t} - \frac{3}{2}e^{2t} & \frac{7}{4}e^{-2t} - \frac{2}{3}e^{-t} - \frac{13}{12}e^{2t} & \frac{7}{4}e^{-2t} - \frac{2}{3}e^{-t} - \frac{1}{12}e^{2t} \end{bmatrix}$