Lecture 3

- 11.1 Three-Dimensional Coordinate Systems
- 11.2 Vectors
- 11.3 The Dot Product

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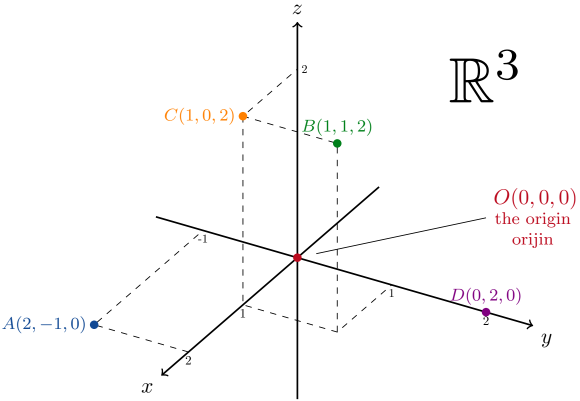
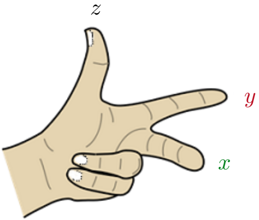
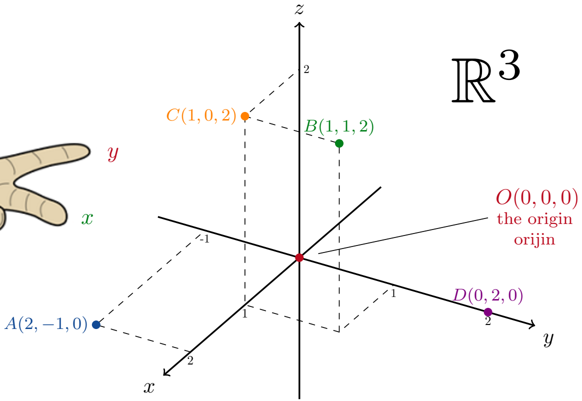
Three-Dimensional Coordinate Systems

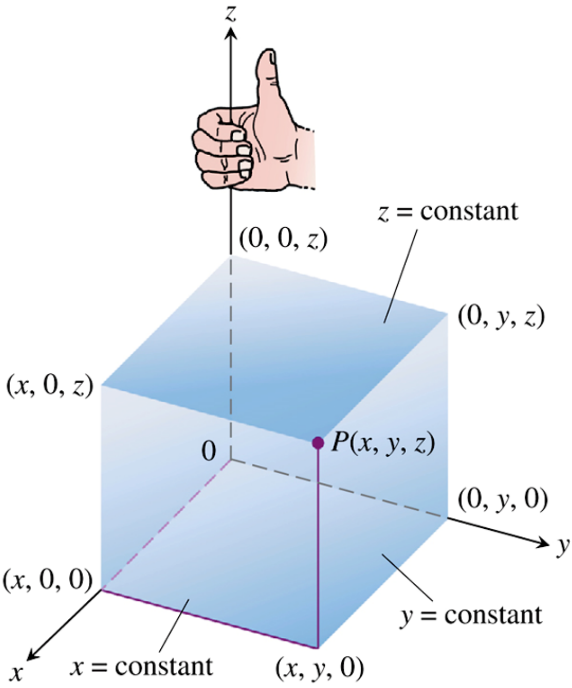


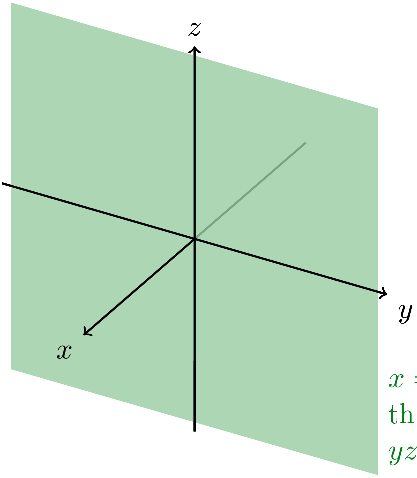
z
2
$\mathbb{R}^3$
C(1, 0, 2)
B(1, 1, 2)
O(0, 0, 0)
the origin
orijin
-1
1
A(2, −1, 0)
2
x
1
D(0, 2, 0)
2
y



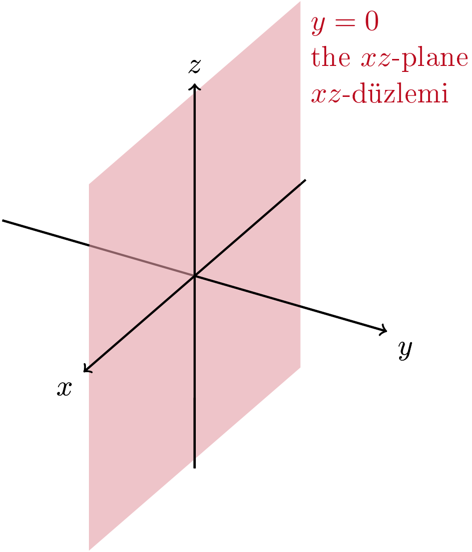


$\mathbb{R}^3$

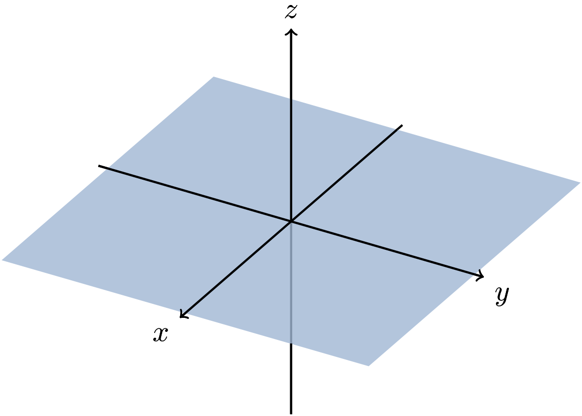

z
z = constant
(0, 0, z)
(0, y, z)
(x, 0, z)
0
P(x, y, z)
(0, y, 0)
y
(x, 0, 0)
y = constant
x
x = constant
(x, y, 0)



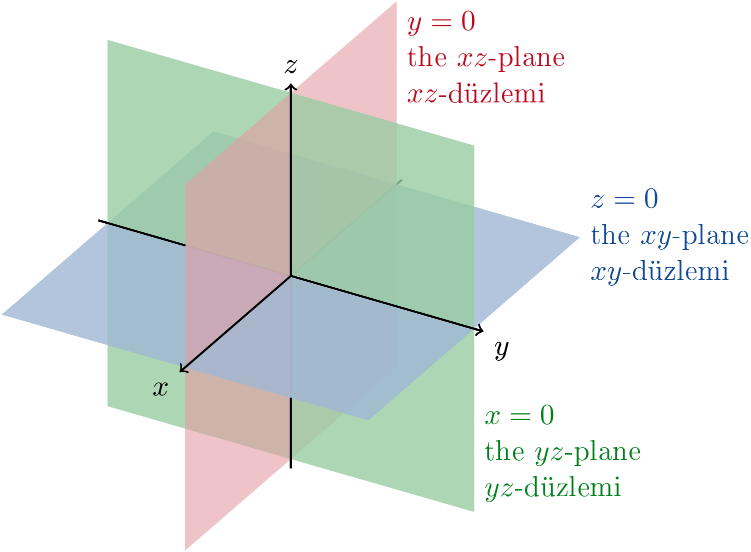
$x = 0$
the yz-plane
yz-düzlemi



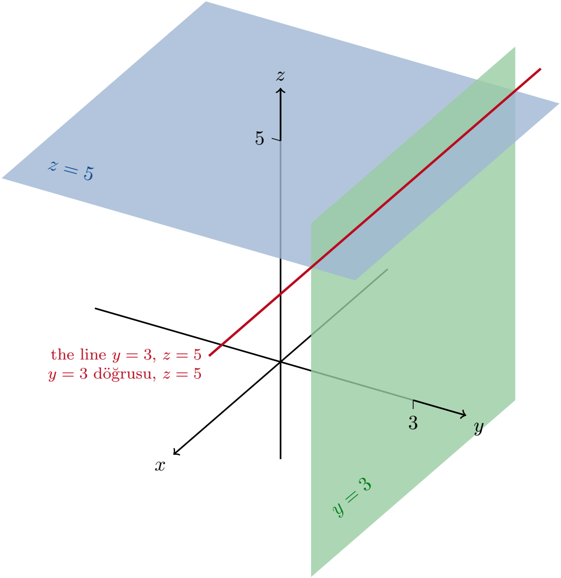
$y = 0$
the xz-plane
xz-düzlemi
z
y
x



$z = 0$
the xy-plane
xy-düzlemi



$y = 0$
the xz-plane
xz-düzlemi
z
$z = 0$
the xy-plane
xy-düzlemi
y
x
$x = 0$
the yz-plane
yz-düzlemi



z
5
$z = 5$
the line $y = 3$, $z = 5$
$y = 3$ döğrusu, $z = 5$
3
y
x
$y = 3$

EXAMPLE 1 We interpret these equations and inequalities geometrically.

(a) $z \geq 0$	The half-space consisting of the points on and above the xy-plane.
(b) $x = -3$	The plane perpendicular to the x-axis at $x = -3$. This plane lies parallel to the yz-plane and 3 units behind it.
(c) $z = 0, x \leq 0, y \geq 0$	The second quadrant of the xy-plane.
(d) $x \geq 0, y \geq 0, z \geq 0$	The first octant.
(e) $-1 \leq y \leq 1$	The slab between the planes $y = -1$ and $y = 1$ (planes included).
(f) $y = -2, z = 2$	The line in which the planes $y = -2$ and $z = 2$ intersect. Alternatively, the line through the point $(0, -2, 2)$ parallel to the x-axis. ■

Example

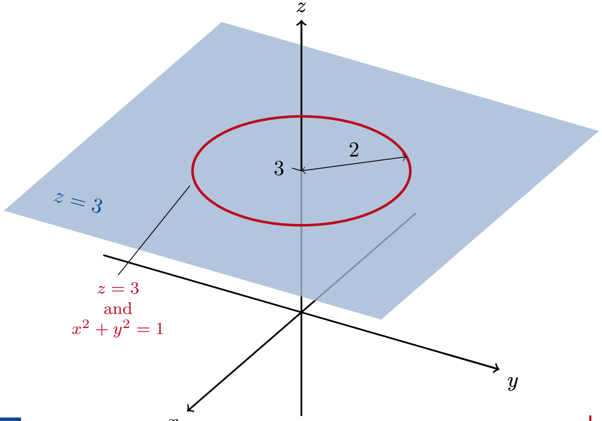
Which points $P(x, y, z)$ satisfy $x^2 + y^2 = 4$ and $z = 3$?

We know that $z = 3$ is a horizontal plane and we recognise that $x^2 + y^2 = 4$ is the equation of a circle of radius 2.

Example

Which points $P(x, y, z)$ satisfy $x^2 + y^2 = 4$ and $z = 3$?

We know that $z = 3$ is a horizontal plane and we recognise that $x^2 + y^2 = 4$ is the equation of a circle of radius 2.



Distance in $\mathbb{R}^3$

Definition

The set

$$\{(x, y, z) \mid x, y, z \in \mathbb{R}\}$$

is denoted by $\mathbb{R}^3$.

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$$\{(x, y, z) \mid x, y, z \in \mathbb{R}\}$$

is denoted by $\mathbb{R}^3$.

Definition

The *distance* between $P_1(x_1, y_1, z_1)$ and $P_2(x_2, y_2, z_2)$ is

$$\|P_1P_2\| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}.$$

$$\|P_1P_2\| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}.$$

Example

The distance between $A(2, 1, 5)$ and $B(-2, 3, 0)$ is

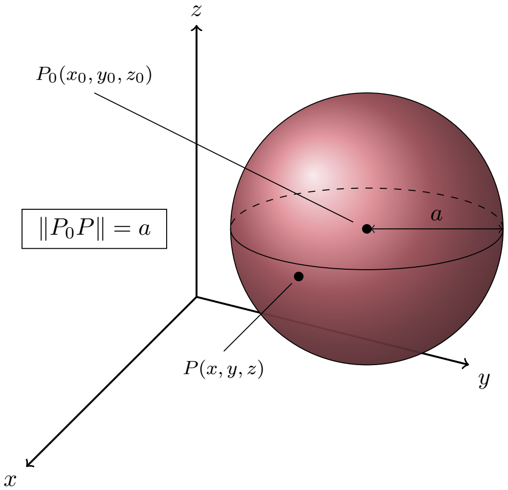
$$\|P_1P_2\| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}.$$

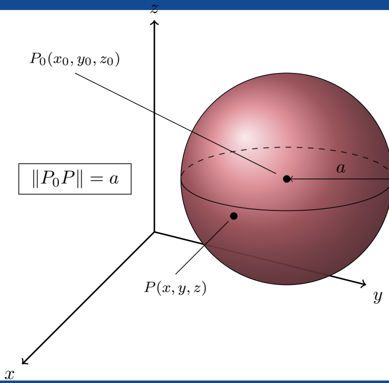
Example

The distance between $A(2, 1, 5)$ and $B(-2, 3, 0)$ is

$$\begin{aligned} \|AB\| &= \sqrt{((-2) - 2)^2 + (3 - 1)^2 + (0 - 5)^2} \\ &= \sqrt{16 + 4 + 25} = \sqrt{45} \\ &= 3\sqrt{5} \approx 6.7. \end{aligned}$$

Spheres





Definition

The *standard equation for a sphere* of radius a centred at $P_0(x_0, y_0, z_0)$ is

$$(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2 = a^2.$$

$$(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2 = a^2.$$

Example

Find the centre and radius of the sphere

$$x^2 + y^2 + z^2 + 3x - 4z + 1 = 0.$$

$$(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2 = a^2.$$

Example

Find the centre and radius of the sphere

$$x^2 + y^2 + z^2 + 3x - 4z + 1 = 0.$$

First we need to complete the squares to put this equation into the standard form.

Completing the square

Let's suppose that we want to solve

$$x^2 + 26x = 27.$$

Completing the square

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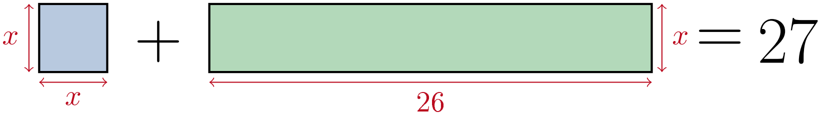
$$x^2 + 26x = 27.$$

We can think of this in terms of areas of rectangles.

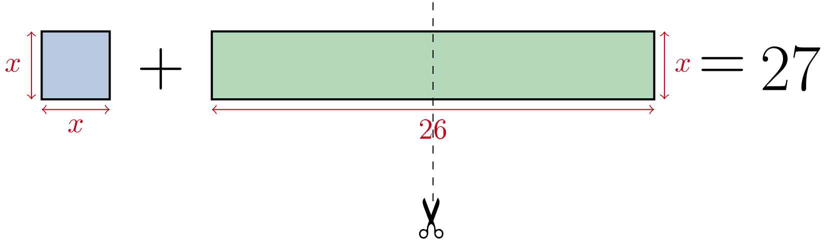
11.1 Three-Dimensional Coordinate Systems



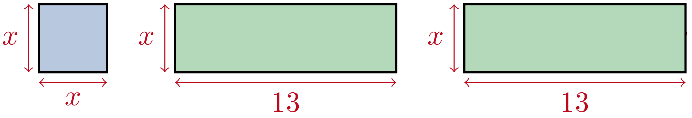
$$x^2 \quad + \quad 26x$$

$$= 27$$

$$x^2 + 26x$$

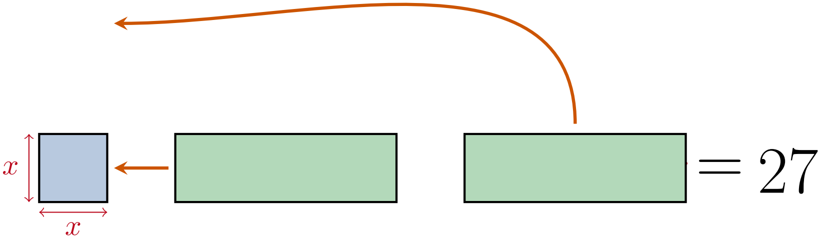


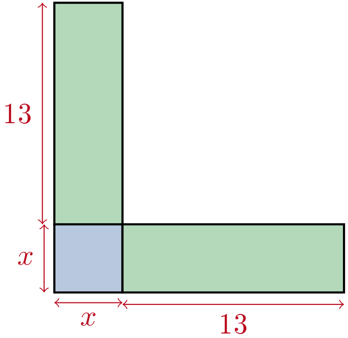
11.1 Three-Dimensional Coordinate Systems



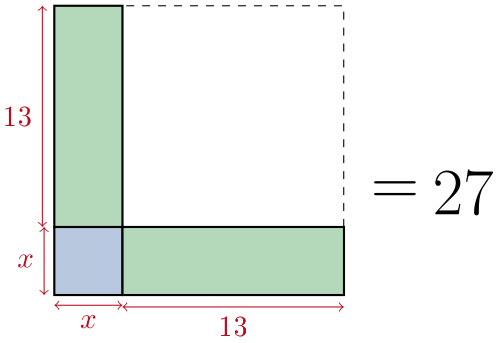
$= 27$

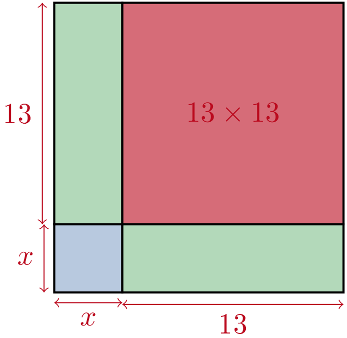
11.1 Three-Dimensional Coordinate Systems



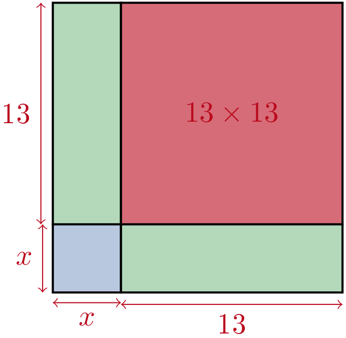


$$= 27$$


13
x
x
13
= 27



$$= 27 + 169$$

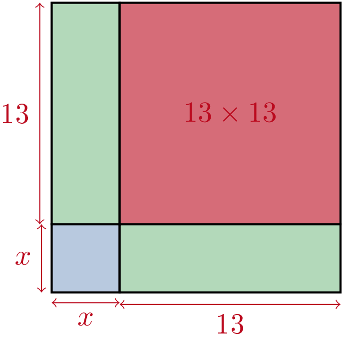


$$= 27 + 169$$

$$(x + 13)^2 = 196$$

$$x^2 + 26x = 27$$

$$\left(\frac{26}{2}\right)^2$$



$$= 27 + 169$$

$$(x + 13)^2 = 196$$

$$(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2 = a^2.$$

Example

Find the centre and radius of the sphere

$$x^2 + y^2 + z^2 + 3x - 4z + 1 = 0.$$

$$(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2 = a^2.$$

By completing the squares, we find that

$$x^2 + y^2 + z^2 + 3x - 4z + 1 = 0$$

$$\left(x^2 + 3x \qquad\right) + y^2 + (z^2 - 4z \qquad) = -1$$

$$(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2 = a^2.$$

By completing the squares, we find that

$$x^2 + y^2 + z^2 + 3x - 4z + 1 = 0$$

$$\left(x^2 + 3x \qquad \right) + y^2 + (z^2 - 4z \qquad) = -1$$

$$\left(\tfrac{3}{2}\right)^2 = \tfrac{9}{4}$$

$$(x-x_0)^2+(y-y_0)^2+(z-z_0)^2=a^2.$$

By completing the squares, we find that

$$x^2+y^2+z^2+3x-4z+1=0$$
$$\left(x^2+3x+\frac{9}{4}\right)+y^2+(z^2-4z\qquad)=-1+\frac{9}{4}$$

$$(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2 = a^2.$$

By completing the squares, we find that

$$x^2 + y^2 + z^2 + 3x - 4z + 1 = 0$$

$$\left(x^2 + 3x + \frac{9}{4}\right) + y^2 + (z^2 - 4z \qquad) = -1 + \frac{9}{4}$$

$$\left(\frac{-4}{2}\right)^2 = (-2)^2 = 4$$

$$(x-x_0)^2+(y-y_0)^2+(z-z_0)^2=a^2.$$

By completing the squares, we find that

$$x^2+y^2+z^2+3x-4z+1=0$$
$$\left(x^2+3x+\frac{9}{4}\right)+y^2+(z^2-4z+4)=-1+\frac{9}{4}+4$$

$$(x-x_0)^2+(y-y_0)^2+(z-z_0)^2=a^2.$$

By completing the squares, we find that

$$\begin{aligned} x^2+y^2+z^2+3x-4z+1 &= 0 \\ \left(x^2+3x+\frac{9}{4}\right)+y^2+(z^2-4z+4) &= -1+\frac{9}{4}+4 \\ &= \frac{21}{4}. \end{aligned}$$

$$(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2 = a^2.$$

By completing the squares, we find that

$$x^2 + y^2 + z^2 + 3x - 4z + 1 = 0$$

$$\left(x^2 + 3x + \frac{9}{4}\right) + y^2 + (z^2 - 4z + 4) = -1 + \frac{9}{4} + 4$$

$$\left(x + \frac{3}{2}\right)^2 + y^2 + (z - 2)^2 = \frac{21}{4}.$$

$$\boxed{(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2 = a^2.}$$

By completing the squares, we find that

$$x^2 + y^2 + z^2 + 3x - 4z + 1 = 0$$

$$\left(x^2 + 3x + \frac{9}{4}\right) + y^2 + (z^2 - 4z + 4) = -1 + \frac{9}{4} + 4$$

$$\left(x + \frac{3}{2}\right)^2 + y^2 + (z - 2)^2 = \frac{21}{4}.$$

$$x_0 = -\frac{3}{2}$$

$$y_0 = 0$$

$$z_0 = 2$$

$$\boxed{(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2 = a^2.}$$

By completing the squares, we find that

$$x^2 + y^2 + z^2 + 3x - 4z + 1 = 0$$

$$\left(x^2 + 3x + \frac{9}{4}\right) + y^2 + (z^2 - 4z + 4) = -1 + \frac{9}{4} + 4$$

$$\left(x + \frac{3}{2}\right)^2 + y^2 + (z - 2)^2 = \frac{21}{4}.$$

$$x_0 = -\frac{3}{2}$$
$$y_0 = 0$$
$$z_0 = 2$$

$$a^2 = \frac{21}{4}$$

$$(x-x_0)^2+(y-y_0)^2+(z-z_0)^2=a^2.$$

By completing the squares, we find that

$$x^2+y^2+z^2+3x-4z+1=0$$

$$\left(x^2+3x+\frac{9}{4}\right)+y^2+(z^2-4z+4)=-1+\frac{9}{4}+4$$

$$\left(x+\frac{3}{2}\right)^2+y^2+(z-2)^2=\frac{21}{4}.$$

$$x_0=-\frac{3}{2}$$
$$y_0=0$$
$$z_0=2$$

$$a^2=\frac{21}{4}$$

The centre is at $P_0(x_0,y_0,z_0)=P_0(-\frac{3}{2},0,2)$ and the radius is $a=\sqrt{\frac{21}{4}}=\frac{\sqrt{3}\sqrt{7}}{2}$.

$$(x-x_0)^2+(y-y_0)^2+(z-z_0)^2=a^2.$$

Example

Find the centre and radius of the sphere

$$x^2+y^2+z^2+6x-6y+6z=7.$$

$$(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2 = a^2.$$

Since $(x - b)^2 = x^2 - 2bx + b^2$ we have that

$$x^2 + y^2 + z^2 + 6x - 6y + 6z = 7$$
$$(x^2 + 6x) + (y^2 - 6y) + (z^2 + 6z) = 7$$

$$(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2 = a^2.$$

Since $(x - b)^2 = x^2 - 2bx + b^2$ we have that

$$\begin{aligned} x^2 + y^2 + z^2 + 6x - 6y + 6z &= 7 \\ (x^2 + 6x) + (y^2 - 6y) + (z^2 + 6z) &= 7 \\ (x^2 + 6x + 9) + (y^2 - 6y + 9) + (z^2 + 6z + 9) &= 7 + 9 + 9 + 9 \end{aligned}$$

$$(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2 = a^2.$$

Since $(x - b)^2 = x^2 - 2bx + b^2$ we have that

$$\begin{aligned}
x^2 + y^2 + z^2 + 6x - 6y + 6z &= 7 \\
(x^2 + 6x) + (y^2 - 6y) + (z^2 + 6z) &= 7 \\
(x^2 + 6x + 9) + (y^2 - 6y + 9) + (z^2 + 6z + 9) &= 7 + 9 + 9 + 9 \\
(x^2 + 6x + 9) + (y^2 - 6y + 9) + (z^2 + 6z + 9) &= 34
\end{aligned}$$

$$(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2 = a^2.$$

Since $(x - b)^2 = x^2 - 2bx + b^2$ we have that

$$\begin{aligned}
x^2 + y^2 + z^2 + 6x - 6y + 6z &= 7 \\
(x^2 + 6x) + (y^2 - 6y) + (z^2 + 6z) &= 7 \\
(x^2 + 6x + 9) + (y^2 - 6y + 9) + (z^2 + 6z + 9) &= 7 + 9 + 9 + 9 \\
(x^2 + 6x + 9) + (y^2 - 6y + 9) + (z^2 + 6z + 9) &= 34 \\
(x + 3)^2 + (y - 3)^2 + (z + 3)^2 &= 34
\end{aligned}$$

$$(x-x_0)^2+(y-y_0)^2+(z-z_0)^2=a^2.$$

Since $(x-b)^2 = x^2 - 2bx + b^2$ we have that

$$\begin{aligned}
x^2+y^2+z^2+6x-6y+6z &= 7\\
(x^2+6x)+(y^2-6y)+(z^2+6z) &= 7\\
(x^2+6x+9)+(y^2-6y+9)+(z^2+6z+9) &= 7+9+9+9\\
(x^2+6x+9)+(y^2-6y+9)+(z^2+6z+9) &= 34\\
(x+3)^2+(y-3)^2+(z+3)^2 &= 34
\end{aligned}$$

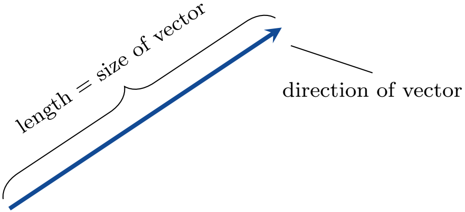
The centre is at $P_0(x_0,y_0,z_0) = P_0(-3,3,-3)$ and the radius is $a=\sqrt{34}$.

EXAMPLE 5 Here are some geometric interpretations of inequalities and equations involving spheres.

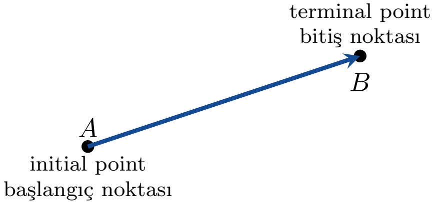
(a) $x^2 + y^2 + z^2 < 4$	The interior of the sphere $x^2 + y^2 + z^2 = 4$.
(b) $x^2 + y^2 + z^2 \leq 4$	The solid ball bounded by the sphere $x^2 + y^2 + z^2 = 4$. Alternatively, the sphere $x^2 + y^2 + z^2 = 4$ together with its interior.
(c) $x^2 + y^2 + z^2 > 4$	The exterior of the sphere $x^2 + y^2 + z^2 = 4$.
(d) $x^2 + y^2 + z^2 = 4, z \leq 0$	The lower hemisphere cut from the sphere $x^2 + y^2 + z^2 = 4$ by the xy-plane (the plane $z = 0$).

11.2 Vectors

For some quantities (mass, time, distance, ...) we only need a number. For some quantities (velocity, force, ...) we need a number and a direction.



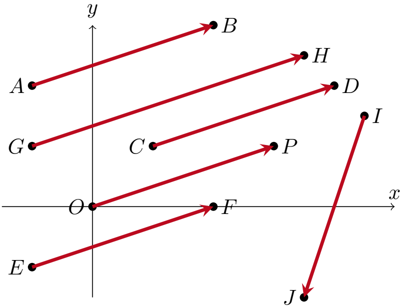
A *vector* is an object which has a size (length) and a direction.



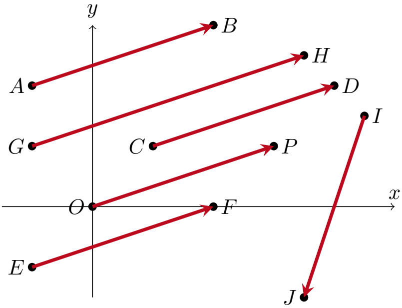
Definition

The vector $\overrightarrow{AB}$ has *initial point* A and *terminal point* B.

The *length* of $\overrightarrow{AB}$ is written $\left\|\overrightarrow{AB}\right\|$ (or $\left|\overrightarrow{AB}\right|$).



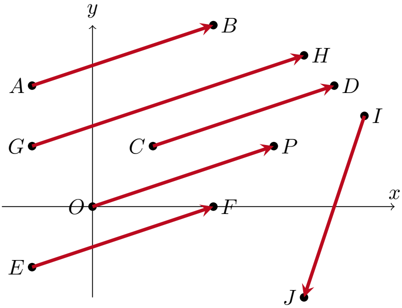
Two vectors are equal if they have the same length and the same direction.



Two vectors are equal if they have the same length and the same direction.

We can say that

$$\overrightarrow{AB} = \overrightarrow{CD} = \overrightarrow{EF} = \overrightarrow{OP}.$$



Two vectors are equal if they have the same length and the same direction.

We can say that

$$\overrightarrow{AB} = \overrightarrow{CD} = \overrightarrow{EF} = \overrightarrow{OP}.$$

Note that $\overrightarrow{AB} \neq \overrightarrow{GH}$ because the lengths are different, and $\overrightarrow{AB} \neq \overrightarrow{IJ}$ because the directions are different.

Notation

When we use a computer, we use bold letters for vectors: $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$, $\ldots$.

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When we use a computer, we use bold letters for vectors: $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$, ... When we use a pen, we use underlined letters for vectors: $\underline{u}$, $\underline{v}$, $\underline{w}$, ...

Notation

When we use a computer, we use bold letters for vectors: $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$, ... When we use a pen, we use underlined letters for vectors: $\underline{u}$, $\underline{v}$, $\underline{w}$, ...

If we type $a\mathbf{u} + b\mathbf{v}$ or write $a\underline{u} + b\underline{v}$, then

- a and b are numbers; and
- $\mathbf{u}$, $\mathbf{v}$, $\underline{u}$ and $\underline{v}$ are vectors.

Definition

In $\mathbb{R}^2$: If $\mathbf{v}$ has initial point $(0,0)$ and terminal point (v_1, v_2), then the *component form* of $\mathbf{v}$ is $\mathbf{v} = (v_1, v_2)$.

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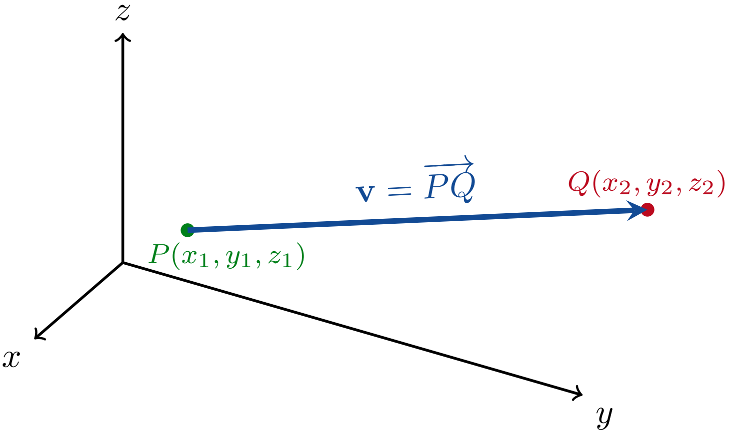
In $\mathbb{R}^3$: If $\mathbf{v}$ has initial point $(0,0,0)$ and terminal point (v_1, v_2, v_3), then the *component form* of $\mathbf{v}$ is $\mathbf{v} = (v_1, v_2, v_3)$.



$$(v_1, v_2, v_3) = \mathbf{v} = (x_2 - x_1, y_2 - y_1, z_2 - z_1)$$

Definition

In $\mathbb{R}^2$: The *norm* (or *length*) of $\mathbf{v} = (v_1, v_2)$ is

$$\|\mathbf{v}\| = \sqrt{v_1^2 + v_2^2}$$

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$$\begin{aligned} \|\mathbf{v}\| &= \sqrt{v_1^2 + v_2^2 + v_3^2} \\ &= \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}. \end{aligned}$$

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The vectors $\mathbf{0} = (0,0)$ and $\mathbf{0} = (0,0,0)$ have norm $\|\mathbf{0}\| = 0$. If $\mathbf{v} \neq \mathbf{0}$, then $\|\mathbf{v}\| > 0$.

Example

Find (1) the component form; and (2) the norm of the vector with initial point $P(-3, 4, 1)$ and terminal point $Q(-5, 2, 2)$.

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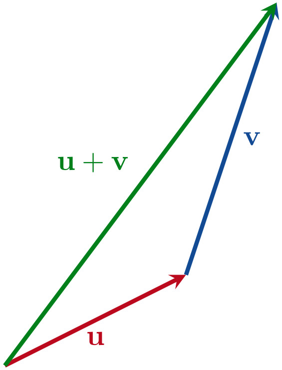
2. $\|\mathbf{v}\| = \sqrt{v_1^2 + v_2^2 + v_3^2} = \sqrt{(-2)^2 + (-2)^2 + 1^2} = \sqrt{9} = 3.$

Vector Algebra: Addition



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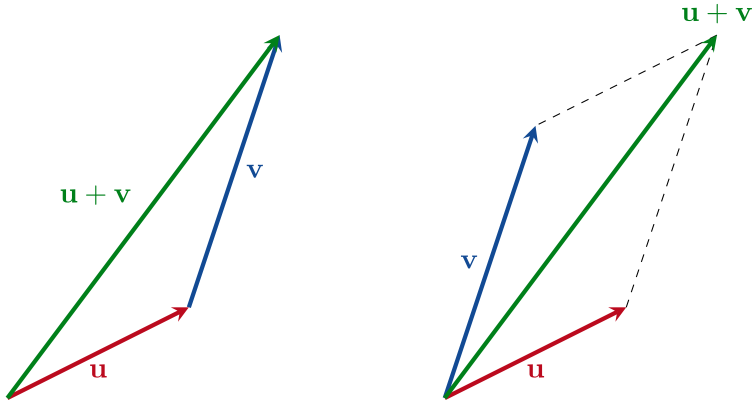
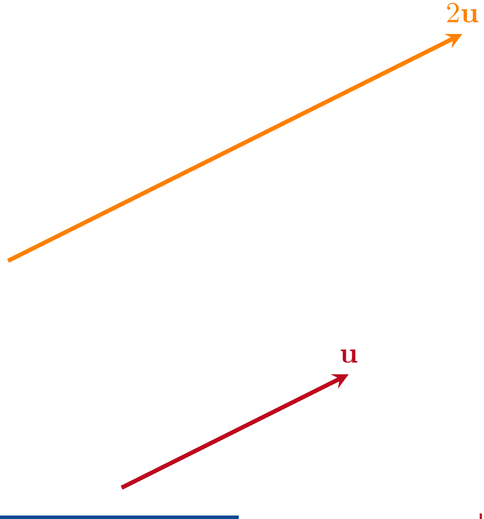


Vector Algebra: Multiplication by a Constant



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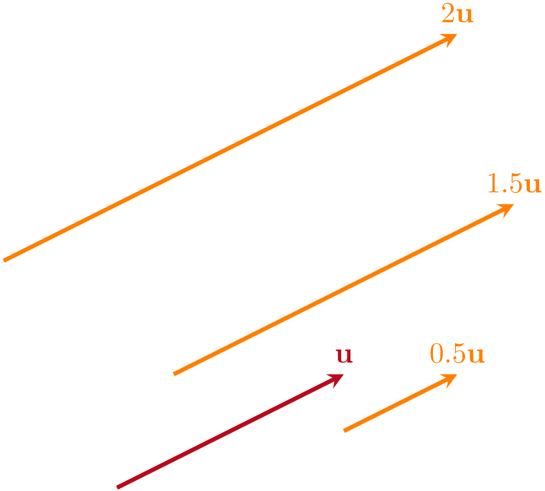

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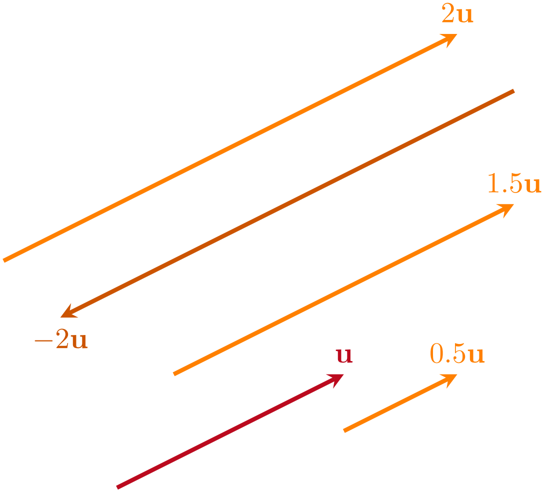
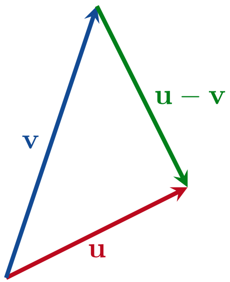


Vector Algebra: Subtraction

$$\mathbf{v} + (\mathbf{u} - \mathbf{v}) = \mathbf{u}$$



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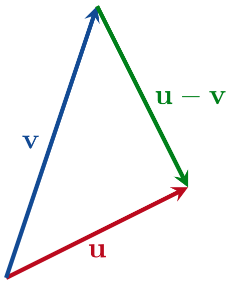

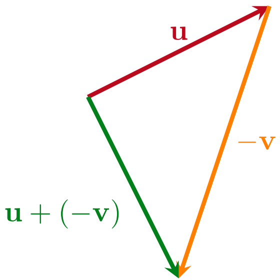

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and

$$k\mathbf{u} = (ku_1, ku_2, ku_3).$$

Note that

$$\begin{aligned}\|k\mathbf{u}\| &= \|(ku_1, ku_2, ku_3)\| \\ &= \\ &= \\ &= \\ &= \\ &= \qquad .\end{aligned}$$

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\|k\mathbf{u}\| &= \|(ku_1, ku_2, ku_3)\| \\
&= \sqrt{(ku_1)^2 + (ku_2)^2 + (ku_3)^2} \\
&= \\
&= \\
&= \\
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The vector $-\mathbf{u} = (-1)\mathbf{u}$ has the same length as $\mathbf{u}$, but points in the opposite direction.

Example

Let $\mathbf{u} = (-1, 3, 1)$ and $\mathbf{v} = (4, 7, 0)$.

Find $2\mathbf{u} + 3\mathbf{v}$, $\mathbf{u} - \mathbf{v}$, and $\left\| \frac{1}{2}\mathbf{u} \right\|$.

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3. $\left\|\frac{1}{2}\mathbf{u}\right\| = \frac{1}{2}\|\mathbf{u}\| = \frac{1}{2}\sqrt{(-1)^2 + 3^2 + 1^2} = \frac{1}{2}\sqrt{11}$.

Properties of Vector Operations

Let $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ be vectors. Let a and b be numbers. Then

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2. $(\mathbf{u}+\mathbf{v})+\mathbf{w}=\mathbf{u}+(\mathbf{v}+\mathbf{w})$;
3. $\mathbf{u}+\mathbf{0}=\mathbf{u}$;
4. $\mathbf{u}+(-\mathbf{u})=\mathbf{0}$;
5. $0\mathbf{u}=\mathbf{0}$;
6. $1\mathbf{u}=\mathbf{u}$;
7. $a(b\mathbf{u})=(ab)\mathbf{u}$;
8. $a(\mathbf{u}+\mathbf{v})=a\mathbf{u}+a\mathbf{v}$;
9. $(a+b)\mathbf{u}=a\mathbf{u}+b\mathbf{u}$.

Remark

We **can not** multiply vectors. Never never never never write "$\mathbf{uv}$".

Unit Vectors

Definition

$\mathbf{u}$ is called a *unit vector* $\iff \|\mathbf{u}\| = 1$.

Example

$\mathbf{u} = (2^{-\frac{1}{2}}, \frac{1}{2}, -\frac{1}{2})$ is a unit vector because

$$\|\mathbf{u}\| = \sqrt{\left(\frac{1}{\sqrt{2}}\right)^2 + \left(\frac{1}{2}\right)^2 + \left(-\frac{1}{2}\right)^2} = \sqrt{\frac{1}{2} + \frac{1}{4} + \frac{1}{4}} = 1.$$

Standard Unit Vectors

In $\mathbb{R}^2$: The *standard unit vectors* are $\mathbf{i} = (1, 0)$ and $\mathbf{j} = (0, 1)$.

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In $\mathbb{R}^3$: The *standard unit vectors* are $\mathbf{i} = (1, 0, 0)$, $\mathbf{j} = (0, 1, 0)$ and $\mathbf{k} = (0, 0, 1)$. Any vector $\mathbf{v} \in \mathbb{R}^3$ can be written

$$\begin{aligned}\mathbf{v} &= (v_1, v_2, v_3) = (v_1, 0, 0) + (0, v_2, 0) + (0, 0, v_3) \\ &= v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}.\end{aligned}$$

Normalising a Vector

If $\|\mathbf{v}\| \neq 0$, then $\frac{\mathbf{v}}{\|\mathbf{v}\|}$ is a unit vector because

$$\left\| \frac{\mathbf{v}}{\|\mathbf{v}\|} \right\| = \left\| \frac{1}{\|\mathbf{v}\|} \mathbf{v} \right\| = \frac{1}{\|\mathbf{v}\|} \|\mathbf{v}\| = 1.$$

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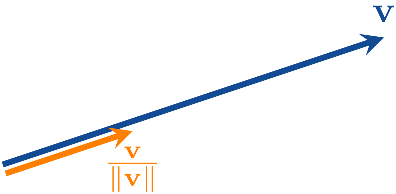
Clearly $\frac{\mathbf{v}}{\|\mathbf{v}\|}$ and $\mathbf{v}$ point in the same direction.



Example

Find a unit vector $\mathbf{u}$ which points in the same direction as $\overrightarrow{P_1P_2}$, where $P_1(1,0,1)$ and $P_2(3,2,0)$.

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We calculate that

$$\overrightarrow{P_1P_2} = P_2 - P_1 = (3,2,0) - (1,0,1) = (2,2,-1) = 2\mathbf{i} + 2\mathbf{j} - \mathbf{k}$$

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$$\left\|\overrightarrow{P_1P_2}\right\| = \sqrt{2^2 + 2^2 + (-1)^2} = 3.$$

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and that

$$\left\|\overrightarrow{P_1P_2}\right\| = \sqrt{2^2 + 2^2 + (-1)^2} = 3.$$

The required unit vector is

$$\mathbf{u} = \frac{\overrightarrow{P_1P_2}}{\left\|\overrightarrow{P_1P_2}\right\|} = \frac{2\mathbf{i} + 2\mathbf{j} - \mathbf{k}}{3} = \frac{2}{3}\mathbf{i} + \frac{2}{3}\mathbf{j} - \frac{1}{3}\mathbf{k}.$$

EXAMPLE 5 If $\mathbf{v} = 3\mathbf{i} - 4\mathbf{j}$ is a velocity vector, express $\mathbf{v}$ as a product of its speed times its direction of motion.

Solution Speed is the magnitude (length) of $\mathbf{v}$:

$$|\mathbf{v}| = \sqrt{(3)^2 + (-4)^2} = \sqrt{9 + 16} = 5.$$

The unit vector $\mathbf{v}/|\mathbf{v}|$ is the direction of $\mathbf{v}$:

$$\frac{\mathbf{v}}{|\mathbf{v}|} = \frac{3\mathbf{i} - 4\mathbf{j}}{5} = \frac{3}{5}\mathbf{i} - \frac{4}{5}\mathbf{j}.$$

So

$$\mathbf{v} = 3\mathbf{i} - 4\mathbf{j} = \underbrace{5}_{\substack{\text{Length} \\ \text{(speed)}}}\underbrace{\left(\frac{3}{5}\mathbf{i} - \frac{4}{5}\mathbf{j}\right)}_{\text{Direction of motion}}.$$ ■

If $\mathbf{v} \neq \mathbf{0}$, then

1. $\dfrac{\mathbf{v}}{|\mathbf{v}|}$ is a unit vector called the direction of $\mathbf{v}$;

2. the equation $\mathbf{v} = |\mathbf{v}|\dfrac{\mathbf{v}}{|\mathbf{v}|}$ expresses $\mathbf{v}$ as its length times its direction.

EXAMPLE 6 A force of 6 newtons is applied in the direction of the vector $\mathbf{v} = 2\mathbf{i} + 2\mathbf{j} - \mathbf{k}$. Express the force $\mathbf{F}$ as a product of its magnitude and direction.

Solution The force vector has magnitude 6 and direction $\dfrac{\mathbf{v}}{|\mathbf{v}|}$, so

$$\mathbf{F} = 6\frac{\mathbf{v}}{|\mathbf{v}|} = 6\frac{2\mathbf{i} + 2\mathbf{j} - \mathbf{k}}{\sqrt{2^2 + 2^2 + (-1)^2}} = 6\frac{2\mathbf{i} + 2\mathbf{j} - \mathbf{k}}{3}$$

$$= 6\left(\frac{2}{3}\mathbf{i} + \frac{2}{3}\mathbf{j} - \frac{1}{3}\mathbf{k}\right).$$

Midpoint of a Line Segment

Vectors are often useful in geometry. For example, the coordinates of the midpoint of a line segment are found by averaging.

The **midpoint** M of the line segment joining points $P_1(x_1, y_1, z_1)$ and $P_2(x_2, y_2, z_2)$ is the point

$$\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}, \frac{z_1 + z_2}{2}\right).$$

To see why, observe (Figure 12.16) that

$$\begin{aligned}\overrightarrow{OM} &= \overrightarrow{OP_1} + \frac{1}{2}(\overrightarrow{P_1P_2}) = \overrightarrow{OP_1} + \frac{1}{2}(\overrightarrow{OP_2} - \overrightarrow{OP_1}) \\ &= \frac{1}{2}(\overrightarrow{OP_1} + \overrightarrow{OP_2}) \\ &= \frac{x_1 + x_2}{2}\mathbf{i} + \frac{y_1 + y_2}{2}\mathbf{j} + \frac{z_1 + z_2}{2}\mathbf{k}.\end{aligned}$$

EXAMPLE 7 The midpoint of the segment joining $P_1(3, -2, 0)$ and $P_2(7, 4, 4)$ is

$$\left(\frac{3 + 7}{2}, \frac{-2 + 4}{2}, \frac{0 + 4}{2}\right) = (5, 1, 2).$$

■

Please read the final two examples in this section of the textbook.

Break

We will continue at 3pm

11.3 The Dot Product

Definition

In $\mathbb{R}^2$, the *dot product* of $\mathbf{u} = (u_1, u_2) = u_1\mathbf{i} + u_2\mathbf{j}$ and $\mathbf{v} = (v_1, v_2) = v_1\mathbf{i} + v_2\mathbf{j}$ is

$$\mathbf{u} \cdot \mathbf{v} = u_1 v_1 + u_2 v_2.$$

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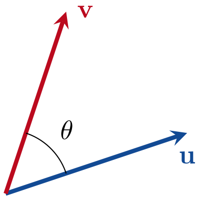
$$\mathbf{u} \bullet \mathbf{v} = u_1 v_1 + u_2 v_2.$$

Definition

In $\mathbb{R}^3$, the *dot product* of $\mathbf{u} = (u_1, u_2, u_3) = u_1\mathbf{i} + u_2\mathbf{j} + u_3\mathbf{k}$ and $\mathbf{v} = (v_1, v_2, v_3) = v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}$ is

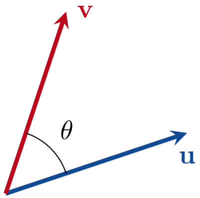
$$\mathbf{u} \bullet \mathbf{v} = u_1 v_1 + u_2 v_2 + u_3 v_3.$$



Theorem

The angle between $\mathbf{u}$ *and* $\mathbf{v}$ *is*

$$\theta = \cos^{-1}\left(\frac{\mathbf{u}\bullet\mathbf{v}}{\|\mathbf{u}\|\,\|\mathbf{v}\|}\right).$$



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This means that

$$\mathbf{u}\cdot\mathbf{v} = \|\mathbf{u}\|\,\|\mathbf{v}\|\cos\theta.$$

Example

$$\begin{aligned}(1,-2,-1)\cdot(-6,2,-3) &= (1\times -6)+(-2\times 2)+(-1\times -3)\\ &= -6-4+3=-7.\end{aligned}$$

Example

$$(1,-2,-1)\bullet(-6,2,-3) = (1\times -6)+(-2\times 2)+(-1\times -3)$$
$$= -6-4+3=-7.$$

Example

$$\left(\tfrac{1}{2}\mathbf{i}+3\mathbf{j}+\mathbf{k}\right)\bullet(4\mathbf{i}-\mathbf{j}+2\mathbf{k}) = \left(\tfrac{1}{2}\times 4\right)+(3\times -1)+(1\times 2)$$
$$= 2-3+2=1.$$

$$\theta = \cos^{-1}\left(\frac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{u}\|\,\|\mathbf{v}\|}\right)$$

Example

Find the angle between $\mathbf{u} = \mathbf{i} - 2\mathbf{j} - 2\mathbf{k}$ and $\mathbf{v} = 6\mathbf{i} + 3\mathbf{j} + 2\mathbf{k}$.

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Since

$$\begin{aligned}\mathbf{u}\cdot\mathbf{v} &= (1,-2,-2)\cdot(6,3,2) = (1\times 6) + (-2\times 3) + (-2\times 2)\\ &= 6 - 6 - 4 = -4,\end{aligned}$$

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$$\|\mathbf{u}\| = \sqrt{1^2 + (-2)^2 + (-2)^2} = \sqrt{9} = 3$$

and

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$$\theta = \cos^{-1}\left(\frac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{u}\|\,\|\mathbf{v}\|}\right) = \cos^{-1}\left(-\frac{4}{21}\right) \approx 1.76 \text{ radians} \approx 98.5^\circ.$$

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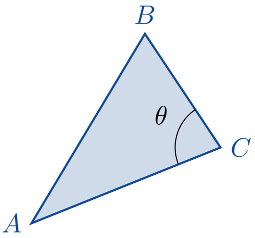

Example

If $A(0,0)$, $B(3,5)$ and $C(5,2)$, find $\theta = \measuredangle ACB$.

θ is the angle between $\overrightarrow{CA}$ and $\overrightarrow{CB}$.

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\overrightarrow{CA} &= A - C = (0,0) - (5,2) = (-5,-2), \\
\overrightarrow{CB} &= B - C = (3,5) - (5,2) = (-2,3), \\
\overrightarrow{CA} \bullet \overrightarrow{CB} &= (-5,-2) \bullet (-2,3) = 4, \\
\left\| \overrightarrow{CA} \right\| &= \sqrt{(-5)^2 + (-2)^2} = \sqrt{29}
\end{aligned}$$

and

$$\left\| \overrightarrow{CB} \right\| = \sqrt{(-2)^2 + 3^2} = \sqrt{13}.$$

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and

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Therefore

$$\theta = \cos^{-1}\left(\frac{\overrightarrow{CA} \bullet \overrightarrow{CB}}{\left\|\overrightarrow{CA}\right\| \left\|\overrightarrow{CB}\right\|}\right) = \cos^{-1}\left(\frac{4}{\sqrt{29}\sqrt{13}}\right)$$
$$\approx 78.1^\circ \approx 1.36 \text{ radians}.$$

Definition

$\mathbf{u}$ and $\mathbf{v}$ are *orthogonal* $\iff$ $\mathbf{u} \bullet \mathbf{v} = 0$.

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Recall that

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Recall that

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Therefore

$$\mathbf{u} \text{ and } \mathbf{v} \text{ are orthogonal} \iff \begin{pmatrix} \mathbf{u} = \mathbf{0} \\ \text{or} \\ \mathbf{v} = \mathbf{0} \\ \text{or} \\ \theta = 90^\circ. \end{pmatrix}$$

Example

$\mathbf{u} = (3, -2)$ and $\mathbf{v} = (4, 6)$ are orthogonal because
$\mathbf{u} \cdot \mathbf{v} = (3, -2) \cdot (4, 6) = (3 \times 4) + (-2 \times 6) = 12 - 12 = 0.$

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Example

$\mathbf{u} = 3\mathbf{i} - 2\mathbf{j} + \mathbf{k}$ and $\mathbf{v} = 2\mathbf{j} + 4\mathbf{k}$ are orthogonal because
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Example

$\mathbf{0}$ is orthogonal to every vector $\mathbf{u}$ because
$\mathbf{0} \cdot \mathbf{u} = (0, 0, 0) \cdot (u_1, u_2, u_3) = 0u_1 + 0u_2 + 0u_3 = 0.$

Properties of the Dot Product

Let $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ be vectors. Let k be a number. Then

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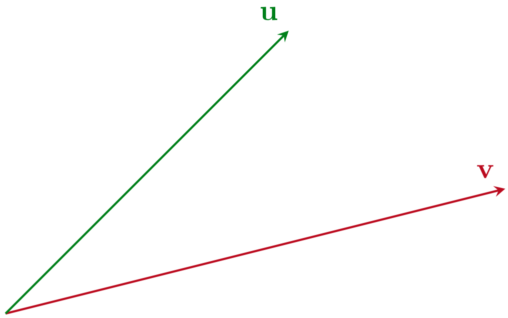
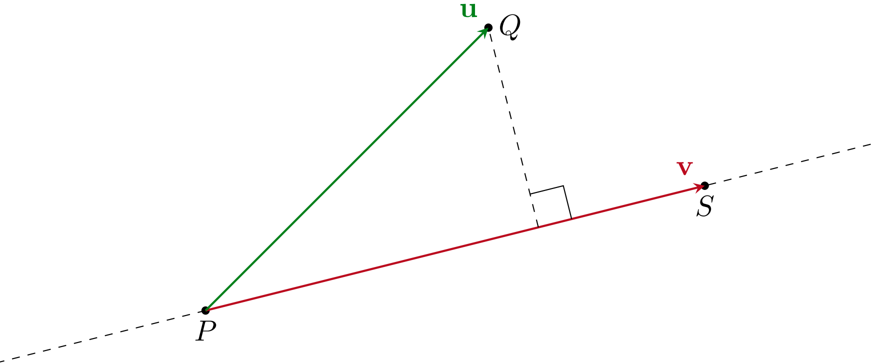
4. $\mathbf{u} \cdot \mathbf{u} = \|\mathbf{u}\|^2$; and
5. $\mathbf{0} \cdot \mathbf{u} = 0$.

Vector Projections



Vector Projections



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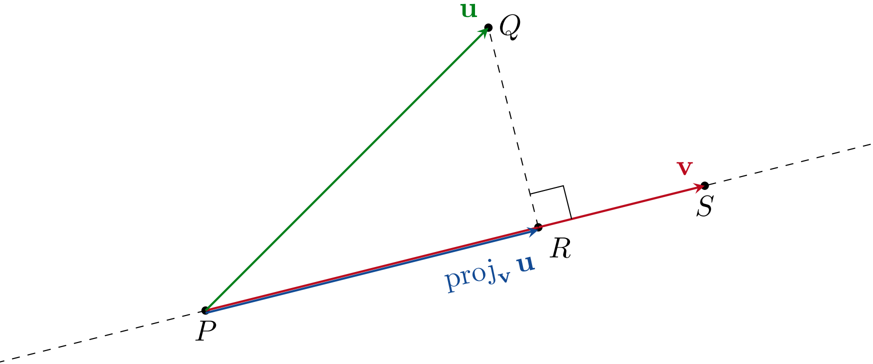

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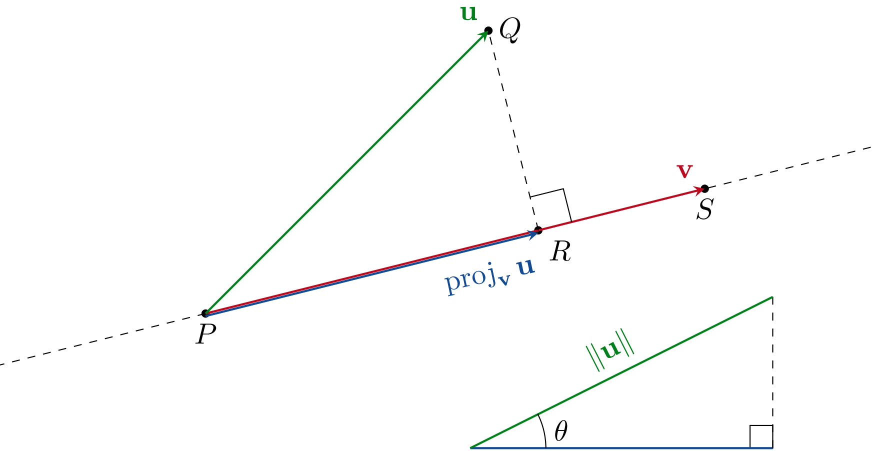
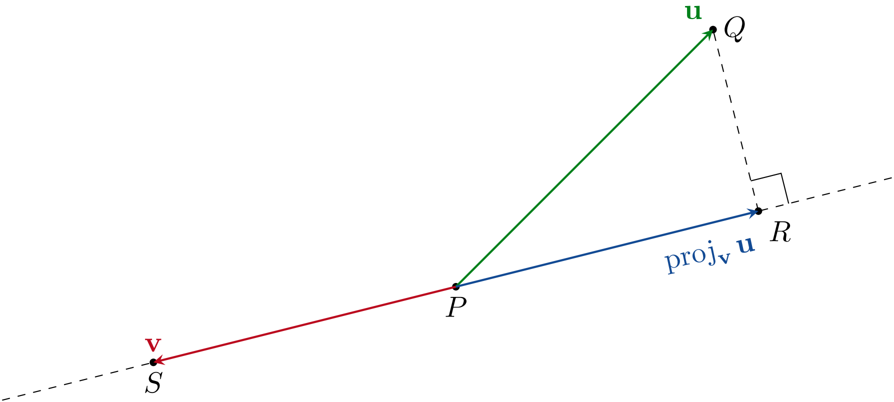
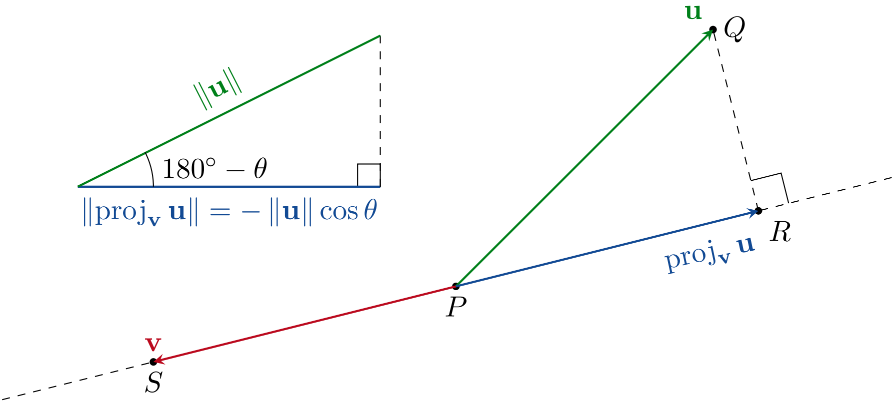


$$\|\text{proj}_{\mathbf{v}}\,\mathbf{u}\| = \|\mathbf{u}\|\cos\theta$$


u
Q
R
$\text{proj}_{\mathbf{v}}\,\mathbf{u}$
P
v
S



$\|\mathbf{u}\|$
$180° - \theta$
$\|\text{proj}_{\mathbf{v}}\,\mathbf{u}\| = -\|\mathbf{u}\|\cos\theta$
$\mathbf{u}$
Q
R
$\text{proj}_{\mathbf{v}}\,\mathbf{u}$
P
$\mathbf{v}$
S

Definition

The *vector projection* of $\mathbf{u}$ onto $\mathbf{v}$ is the vector

$$\operatorname{proj}_{\mathbf{v}} \mathbf{u} = \overrightarrow{PR}.$$

Now

$$\begin{aligned}\operatorname{proj}_{\mathbf{v}} \mathbf{u} &= \left(\text{length of } \operatorname{proj}_{\mathbf{v}} \mathbf{u}\right) \begin{pmatrix} \text{a unit vector in} \\ \text{the same} \\ \text{direction as } \mathbf{v} \end{pmatrix} \\ &= \\ &= \\ &= \\ &= \end{aligned}$$

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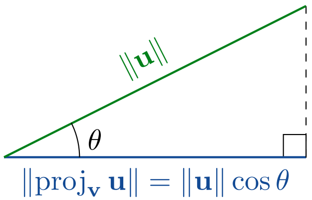
$$= \|\mathrm{proj}_{\mathbf{v}}\,\mathbf{u}\| \left(\frac{\mathbf{v}}{\|\mathbf{v}\|} \right)$$

$$=$$

$$=$$

$$=$$



11.3 The Dot Product

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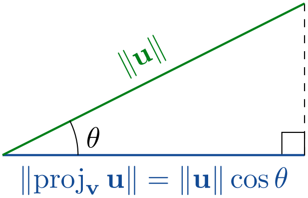
&= \|\operatorname{proj}_{\mathbf{v}} \mathbf{u}\| \left(\frac{\mathbf{v}}{\|\mathbf{v}\|}\right) \\
&= \|\mathbf{u}\| (\cos\theta) \left(\frac{\mathbf{v}}{\|\mathbf{v}\|}\right) \\
&= \\
&=
\end{aligned}$$



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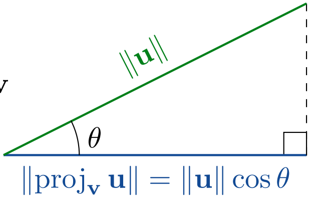
&= \left\|\operatorname{proj}_{\mathbf{v}} \mathbf{u}\right\| \left(\frac{\mathbf{v}}{\|\mathbf{v}\|} \right) \\
&= \|\mathbf{u}\| \left(\cos\theta\right) \left(\frac{\mathbf{v}}{\|\mathbf{v}\|} \right) \\
&= \left(\frac{\|\mathbf{u}\| \, \|\mathbf{v}\| \cos\theta}{\|\mathbf{v}\|^2} \right) \mathbf{v} \\
&=
\end{aligned}$$



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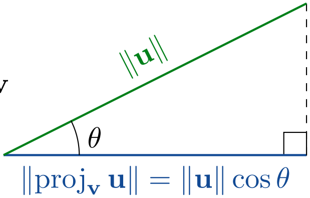
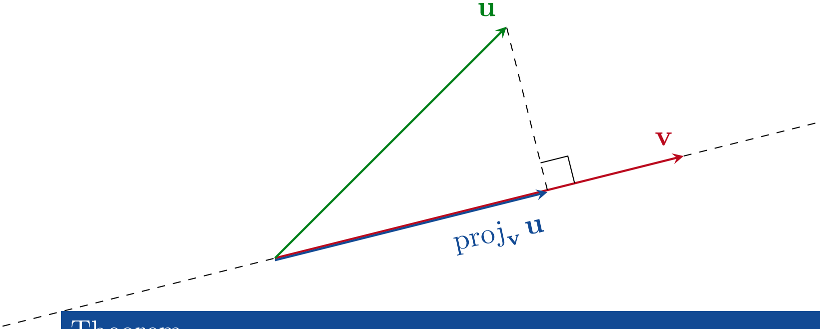
&= \left(\frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{v}\|^2} \right) \mathbf{v}.
\end{aligned}$$



Since this is an important formula, we write it as a theorem.



Theorem

The vector projection of $\mathbf{u}$ *onto* $\mathbf{v}$ *is*

$$\text{proj}_{\mathbf{v}}\,\mathbf{u} = \left(\frac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{v}\|^2}\right)\mathbf{v}.$$

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Example

Find the vector projection of $\mathbf{u} = 6\mathbf{i} + 3\mathbf{j} + 2\mathbf{k}$ onto $\mathbf{v} = \mathbf{i} - 2\mathbf{j} - 2\mathbf{k}$.

$$\text{proj}_{\mathbf{v}}\,\mathbf{u} = \left(\frac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{v}\|^2}\right)\mathbf{v}$$

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$$\begin{aligned}\text{proj}_{\mathbf{v}}\,\mathbf{u} &= \left(\frac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{v}\|^2}\right)\mathbf{v} = \left(\frac{6-6-4}{1+4+4}\right)(\mathbf{i} - 2\mathbf{j} - 2\mathbf{k})\\ &= -\frac{4}{9}\mathbf{i} + \frac{8}{9}\mathbf{j} + \frac{8}{9}\mathbf{k}.\end{aligned}$$

$$\text{proj}_{\mathbf{v}}\,\mathbf{u} = \left(\frac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{v}\|^2}\right)\mathbf{v}$$

Example

Find the vector projection of $\mathbf{F} = 5\mathbf{i} + 2\mathbf{j}$ onto $\mathbf{v} = \mathbf{i} - 3\mathbf{j}$.

$$\begin{aligned}\text{proj}_{\mathbf{v}}\,\mathbf{F} &= \left(\frac{\mathbf{F}\cdot\mathbf{v}}{\|\mathbf{v}\|^2}\right)\mathbf{v} = \left(\frac{5-6}{1+9}\right)(\mathbf{i}-3\mathbf{j}) \\ &= -\frac{1}{10}\mathbf{i} + \frac{3}{10}\mathbf{j}.\end{aligned}$$

$$\text{proj}_{\mathbf{v}}\,\mathbf{u} = \left(\frac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{v}\|^2}\right)\mathbf{v}$$

Example

Verify that the vector $\mathbf{u} - \text{proj}_{\mathbf{v}}\,\mathbf{u}$ is orthogonal to $\text{proj}_{\mathbf{v}}\,\mathbf{u}$.

$$\operatorname{proj}_{\mathbf{v}} \mathbf{u} = \left(\frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{v}\|^2} \right) \mathbf{v}$$

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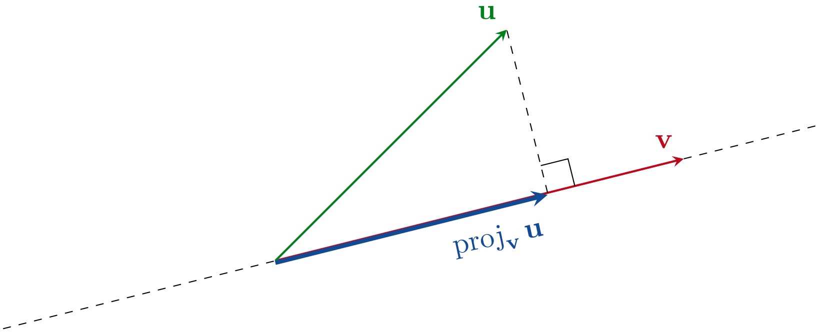

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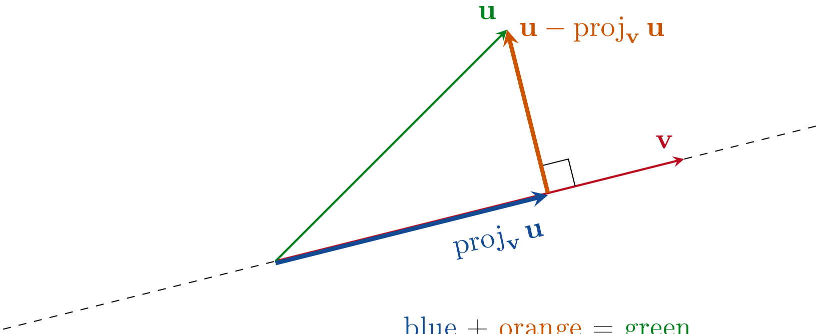

blue + orange = green

$$\operatorname{proj}_{\mathbf{v}} \mathbf{u} = \left(\frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{v}\|^2} \right) \mathbf{v}$$

Clearly

$$\operatorname{proj}_{\mathbf{v}} \mathbf{u} = \left(\frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{v}\|^2} \right) \mathbf{v} = \big(\text{a number}\big) \mathbf{v}$$

is parallel to $\mathbf{v}$.

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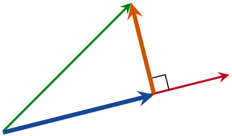

$$\begin{aligned}(\mathbf{u} - \text{proj}_{\mathbf{v}}\,\mathbf{u})\cdot\mathbf{v} &= \\ &= \\ &= \\ &= 0\end{aligned}$$

we have shown that $\mathbf{u} - \text{proj}_{\mathbf{v}}\,\mathbf{u}$ is orthogonal to $\mathbf{v}$.

$$\operatorname{proj}_{\mathbf{v}} \mathbf{u} = \left(\frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{v}\|^2}\right) \mathbf{v}$$

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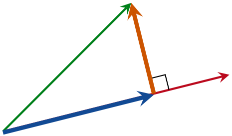
Since



$$\begin{aligned} (\mathbf{u} - \operatorname{proj}_{\mathbf{v}} \mathbf{u}) \cdot \mathbf{v} &= \mathbf{u} \cdot \mathbf{v} - \left(\frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{v}\|^2}\right) \mathbf{v} \cdot \mathbf{v} \\ &= \mathbf{u} \cdot \mathbf{v} - \frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{v}\|^2} \|\mathbf{v}\|^2 \\ &= \mathbf{u} \cdot \mathbf{v} - \mathbf{u} \cdot \mathbf{v} \\ &= 0 \end{aligned}$$

we have shown that $\mathbf{u} - \operatorname{proj}_{\mathbf{v}} \mathbf{u}$ is orthogonal to $\mathbf{v}$.

Next Time

- 11.4 The Cross Product
- 11.5 Lines and Planes in Space