2019-20

Differential Equations

Neil Course

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Introduction

1.1 Some Basic Mathematical Models; Direction Fields ✗

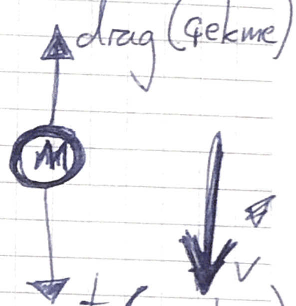
Differential Equations

Chapter 1 - Introduction

Neil Course.

Many problems in engineering, science and the social sciences can be modelled using differential equations. We will start with a few examples.

Example 1 - A falling object.



Suppose an object of mass M is falling.

$v(t)$ ~~denote~~ ~~the~~ the velocity of the object at time t

measure t in seconds, ~~and~~ $v(t)$ in ~~ms~~ ms^{-1} ($=m/s$), gravity (yerçekimi)
and M in kg.

Newton's second law states force = mass × acceleration.

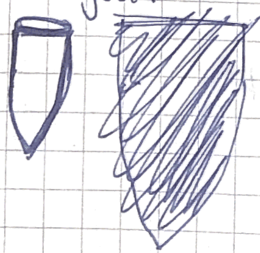
$$= M \cdot \frac{dv}{dt}$$

$\frac{dv}{dt}$ is measured in ms^{-2}.

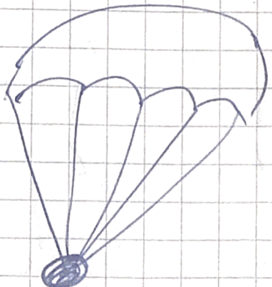
Now force = gravity − drag
$$= Mg - \gamma v$$

(we assume that drag is proportional to velocity)

where $g \approx 9.8\,ms^{-2}$ and γ is a constant depending on the shape of the object.



low γ



high γ.

So

$$M\frac{dv}{dt} = Mg - \gamma v$$

Now suppose $M = 10\text{ kg}$, $\gamma = 2\text{ kg/sec}$. then

(*)

$$\frac{dv}{dt} = 9.8 - \frac{v}{5}.$$

We will solve this equation later. First we will draw a direction field for this equation.

$y = ax$ $\quad \frac{dy}{dx} = a.$ $\quad$ etc. $\quad \frac{dy}{dx} = 0$

~~How to draw~~

~~Example 2~~

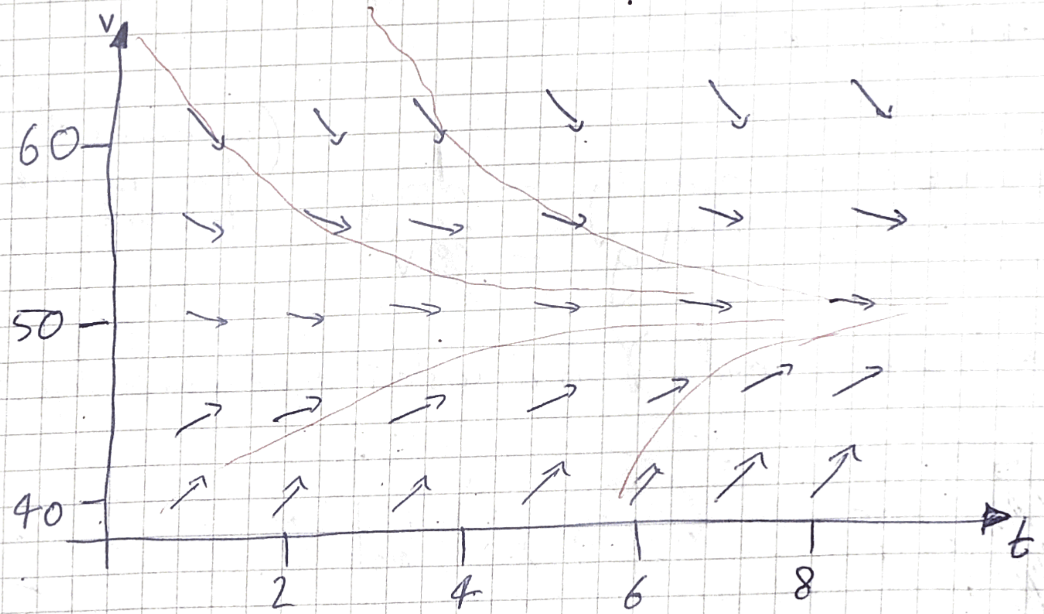
If $v = 40$ then $\frac{dv}{dt} = 1.8$

~~$v = 49$ then $\frac{dv}{dt} = 0$~~

$v = 50$ then $\frac{dv}{dt} = -0.2$.

$v = 60$ then $\frac{dv}{dt} = -2.2$

etc.



We can now guess what the solutions of (*) look like. For high v, the solution decreases (the object slows). For low v, the solution increases (the object increase speed).

Question: What value of v will cause $\frac{dv}{dt} = 0$?

Answer: $v = 5 \times 9.8 = 49\ ms^{-1}$

The constant function $v(t) = 49$ is a solution of $(*)$.
Because it does not change with time, we call this solution an equilibrium solution.

Example 2. Mice and Owls.

Let $p(t)$ denote the population of mice (fareler) in a area, where t is measured in months. We assume that there is plenty of food, so if ~~no one~~ nothing eats the mice, then $p(t)$ will increase at a rate proportional to $p(t)$. We write this as

$$\frac{dp}{dt} = rp.$$

where r is a constant called the rate constant. Suppose $r = 0.5/\text{month}$.

However, suppose ~~some~~ 5 owls (baykuşlar) live in the area, and ~~they eat 3~~ each ~~the~~ owl eats 3 mice every day.

$5 \times 3 = 15$ mice eaten per day
$15 \times 30 = 450$ mice eaten/month

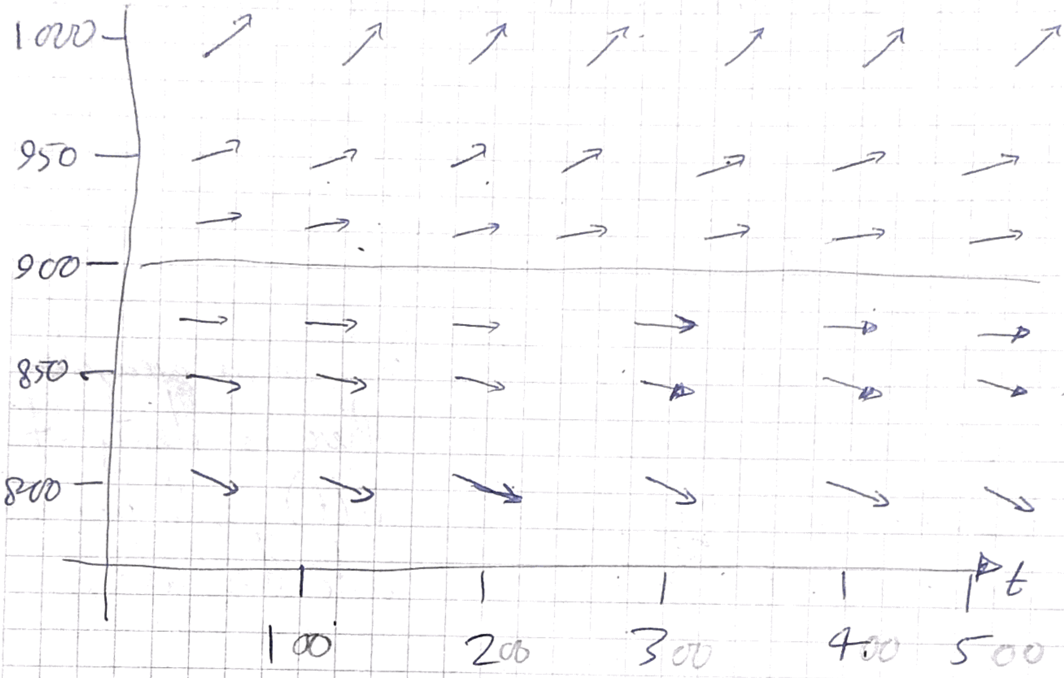
We change our differential equation to

$$\frac{dp}{dt} = \frac{p}{2} - 450 \qquad (**)$$

(assuming there are 30 days in a month).

Direction field.

tinyurl.com/gn4dugh
(page 5)



When $p = 900$, $\frac{dp}{dt} = 0$. So $p(t) = 900$ is equilibrium solution.

If $p > 900$, $\frac{dp}{dt} > 0$, so the mice population increases ~~and increases~~.

If $p < 900$, $\frac{dp}{dt} < 0$, so the mice population decreases.

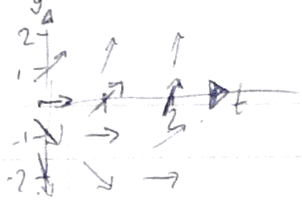
Note. In example 1, other solutions converged to, or were attracted by, the equilibrium solution. In example 2, other solutions diverge from, or are repelled from, the equilibrium solution.

Note: Both of these models have limitations. Example 1 stops being valid as soon as the object hits the ground. Example 2 predicts negative numbers of mice $(p<900)$ or enormous number of mice $(p>900)$ for high t. The model is only suitable for a short time interval.

* more direction fields. $y' = 2t + y$

$y' = 2 - x - y$

$y' = xy$



Solutions of some differential equations.

Both equation we looked at were of the general form

$$\frac{dy}{dt} = ay - b \qquad (***)$$

for constants a and b. We will now see how to solve this type of differential equation.

Mice and Owls:

$$\frac{dp}{dt} = 0.5p - 450 = \frac{p-900}{2} \qquad (**)$$

If $p \neq 900$, rearrange to

$$\frac{dp}{p-900} = \frac{1}{2}\,dt$$

Note that all the terms involving p are on the left, and all terms involving t are on the right. Ignore for ~~the~~ now that we are not really allowed to separate $\frac{dp}{dt}$ into dp and dt. Integrate both sides

$$\int \frac{1}{p-900}\,dp = \int \frac{1}{2}\,dt$$

$$\log|p-900| = \frac{t}{2} + C$$

where C is a constant. Here $\log = \log_e = \ln$. So

$$|p-900| = e^C \cdot e^{t/2}$$

$$p - 900 = \pm e^C e^{t/2}$$

$$p = 900 \pm e^C e^{t/2}$$

$$= 900 + c e^{t/2}$$

where $c = e^C$ is a non-zero constant.

Since we know $p(t) = 900$ is also a solution, we can now allow $c = 0$.

So $p(t) = 900 + ce^{t/2}$ is a solution to (**) for all $c \in \mathbb{R}$.
~~Also~~ there are infinitely many solutions to (**).

Suppose now I tell you that at time $t=0$, there were 850 mice.
That is $p(0) = 850$. We must find c.

$$850 = p(0) = 900 + ce^{0}$$
$$= 900 + c$$

So $c = -50$. Therefore $p(t) = 900 - 50e^{t/2}$
is the desired solution.

The added condition $p(0) = 850$ is called an initial condition
and the whole problem

$$\begin{cases} \dfrac{dp}{dt} = 0.5p - 450 \\ p(0) = 850 \end{cases}$$

is called an initial value problem (IVP).

Now consider the ~~more~~ ~~general~~ ~~(**)~~ differential equation.

$$\frac{dy}{dt} = ay - b \quad (***) \quad \cancel{\text{where } a, b, y_0 \text{ constants}}$$

and the initial condition

$$y(0) = y_0 \quad \cancel{(***)}$$

where a, b, y_0 are constants. We solve this in the same way:

$$\frac{dy}{dt} = ay - b$$

$$\frac{dy}{y - (b/a)} = a\, dt$$

$$\int \frac{dy}{y - b/a} = \int a\, dt$$

$$\log|y - b/a| = at + C \quad \text{constant.}$$

$$|y - b/a| = e^{C} \cdot e^{at}$$

$$y = b/a \pm e^{C} e^{at}$$

$$= b/a + c\, e^{at} \quad (c = \pm e^{C})$$

constant.

the initial condition requires $c = y_0 - b/a$, so

$$y(t) = b/a - (t_0 - b/a)e^{at}$$

is the solution of the IVP

$$\begin{cases} \frac{dy}{dt} = ay - b \\ y(0) = y_0 \end{cases}$$

Note: $y = b/a + ce^{at}$ is called the general solution of (***)

A Falling Object:

An object of mass $M = 10$ kg and drag coefficient $\gamma = 2$ kg/sec is falling:

$$\frac{dv}{dt} = 9.8 - \frac{v}{5}. \qquad (*)$$

Suppose this object is dropped from a height of 300 m.

1. What is its velocity at time t?
2. How long will it take to fall to the ground?
3. How fast will it be moving, when it hits the ground?

"Dropped" tells us $v(0) = 0$. We must solve $(*)$:

$$\frac{dv}{v - 49} = -\frac{1}{5}\,dt.$$

$$\int \frac{dv}{v-49} = \int -\frac{1}{5}\,dt.$$

$$\log|v-49| = -\frac{t}{5} + C$$

So the general solution to $(*)$ is

$$v = 49 + ce^{-t/5}.$$

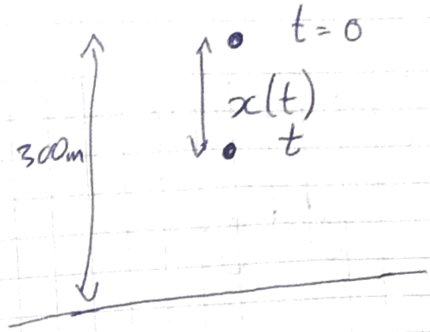
To find c, we calculate

$$0 = v(0) = 49 + ce^{0} = 49 + c. \qquad c = -49.$$

So

$$v(t) = 49\left(1 - e^{-t/5}\right)$$

the solution of the IVP



$$\begin{cases} \dfrac{dv}{dt} = 9.8 - \dfrac{v}{5} \\ v(0) = 0. \end{cases}$$

We must find how far the object has fallen at time t.

et $x(t)$ be the distance fallen. Then

$$\frac{dx}{dt} = v = 49(1 - e^{-t/5})$$

So

$$x(t) = 49t + 245e^{-t/5} + c.$$

Since $x(0) = 0$, $c = -245$. So

$$x(t) = 49t + 245e^{-t/5} - 245.$$

Suppose the object hits the ground at time T. So $x(T) = 300$

$$0 = 49T + 245e^{-T/5} - 545.$$

Using a computer, we can see $T \approx 10.51$ seconds.

3. $v(T) \approx 43.01 \, ms^{-1}$ (≈ 155 kph.)

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Classification of Differential Equations:

Ordinary and Partial Differential Equations:

If only ordinary derivatives appear in ~~th~~ a differential equation, then it is called an ordinary differential equation (ODE). For example

$$\frac{dv}{dt} = 9.8 - \frac{v}{5} \qquad (*)$$

Adi diferansiyel denklemler

and

$$\frac{dp}{dt} = \frac{p}{2} - 450 \qquad (**)$$

are ODEs. If the ~~differential equation contains partial~~ derivatives in a differential equations are partial derivatives, then ~~the~~ it is called a partial differential equation (PDE). For example

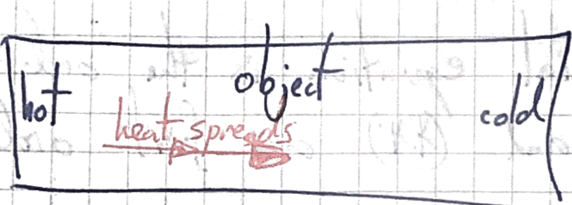
Kısmi türevli diferansiyel denklemler

$$k \frac{\partial^2 u(x,t)}{\partial x^2} = \frac{\partial u(x,t)}{\partial t} \qquad \text{(heat equation)} \quad (\dagger) \quad (k>0).$$

and

$$k \frac{\partial^2 u(x,t)}{\partial x^2} = \frac{\partial^2 u}{\partial t^2}(x,t) \qquad \text{(wave equation)} \quad (\dagger\dagger)$$

are PDEs. ~~α and a are constants.~~



Systems of Diferential Equations:

If there is a single function to be found, then one differential equation is enough. However, if there are two or more unknown functions, then we need a system of equations. For example

$$\frac{dx}{dt} = ax - \alpha xy \qquad \text{(Predator - Prey equations)}$$

$$\frac{dy}{dt} = -cy + \gamma xy$$

$x(t)$ is the population of prey (e.g. mice)
$y(t)$ is the population of predators (e.g. owls).
a, c, α, γ constants.

Remark: This is a more complicated model than the mice and owls model we looked at last week, because now the number of predators (owls) can change.

Order:

The order of a differential equation is the order of the highest derivative. (*) and (**) are first order ODEs.

$$\frac{d^2y}{dt^2} + \frac{dy}{dt} + y = 0$$

is a second order ODE. (†) and (††) are 2nd order

ODEs. ~~The equation~~

$$\cancel{F[t, u(t), u'(t), u''(t), \dots, u^{(n)}(t)] = 0}$$

is an nth order ODE. ~~I will~~ ~~We will usually write~~

~~$y = u(t)$, $y' = u'(t)$, etc,~~ so ~~the~~ we will write this equation as

$$F[t, y, y', y'', \dots, y^{(n)}] = 0. \qquad (1)$$

For example

$$y''' + 2e^t y'' + yy' = t^4$$

is a 3rd order ODE, for $y = y(t)$.

We will assume that it is always possible to solve an ODE for the highest derivative, obtaining

$$\boxed{y^{(n)} = f(t, y, y', \dots, y^{(n-1)})} \qquad (2).$$

We will only study equations of the form (2). This is because equations of the form (1) can lead to ~~multip~~ more than one equation of the form (2), which can add complications. For example

$$(y')^2 + ty' + 4y = 0$$

leads to

$$y' = \frac{-t + \sqrt{t^2 - 16y}}{2} \quad \text{or} \quad y' = \frac{-t - \sqrt{t^2 - 16y}}{2}.$$

similarly for PDEs.

Linear and non-linear equations.

The ODE $F(t, y, y', \ldots, y^{(n)}) = 0$ is called linear if F is a linear functions of $y, y', \ldots, y^{(n)}$ [not of t]. The general linear ODE of order n is

$$a_0(t)\, y^{(n)} + a_1(t)\, y^{(n-1)} + \ldots + a_n(t)\, y = g(t). \quad (3)$$

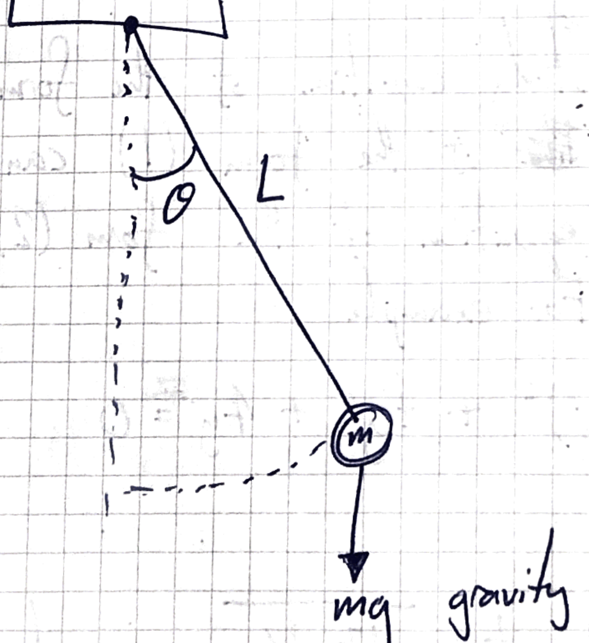
(*) and (**) are linear. (+) and (#) are linear PDEs.
An ~~eq~~ differential equation not of the form (3) is called non-linear.

$$y''' + 2e^t y'' + y'y = t^4$$

is non-linear because of the $y'y$ term. The predator-prey system of equations are non-linear because of the xy terms.

Pendulum

angle θ
length L



The equation

$$\frac{d^2\theta}{dt^2} + \frac{g}{L} \sin\theta = 0. \quad (4).$$

is non-linear, because of the $\sin\theta$ term.
Solving linear equations is easier than solving non-linear equations.
~~but~~ We can often approximate non-linear equations with linear equations.
If θ is small, $\sin\theta \approx \theta$. So we approximate (4) by
$$\frac{d^2\theta}{dt^2} + \frac{g\theta}{L} = 0.$$

This process of approximating a non-linear equation by a linear equation is called linearisation, and can be very useful. Sometimes however, we must study non-linear equations.

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Solutions: A solution of the ODE (2) on the interval $\alpha < t < \beta$, is a function ϕ such that

(i) $\phi', \phi'', \ldots, \phi^{(n)}$ exist; and

(ii) $\phi^{(n)}(t) = f[t, \phi(t), \phi'(t), \ldots, \phi^{(n-1)}(t)]$, $\forall t \in (\alpha, \beta)$.

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For Example: (**) has solution $p = 900 + ce^{t/2}$, $\forall t \in \mathbb{R}$.

$\phi(t) = \cos t$ is a solution of $y'' + y = 0$ $\forall t$.

check. $\phi' = -\sin t$

$\phi'' = -\cos t$.

$\phi'' + \phi = \cos t - \cos t = 0.$ ✓.

Important Questions:

1. Does an equation of the form (2) always have a solution?
answer: no.
We will study when an ~~ODE~~ ODE has a solution. Existence

2. If ~~an ODE~~ a diferential equation has atleast one solution, then how many solutions does it have? What conditions (for example initial conditions) do we need to get a particular solution. Uniqueness

3. Given an equation of the form (2), can we ~~solve it~~ ~~[illegible]~~ solve it, and how?

$$\boxed{\frac{Q \text{ grams}}{1000000 \text{ litres}}}$$

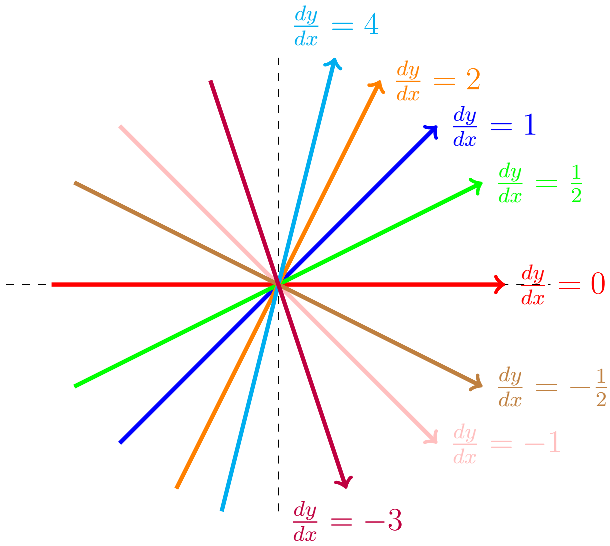
1 litre contains $\frac{Q}{1000000}$ g $\times 300$ L

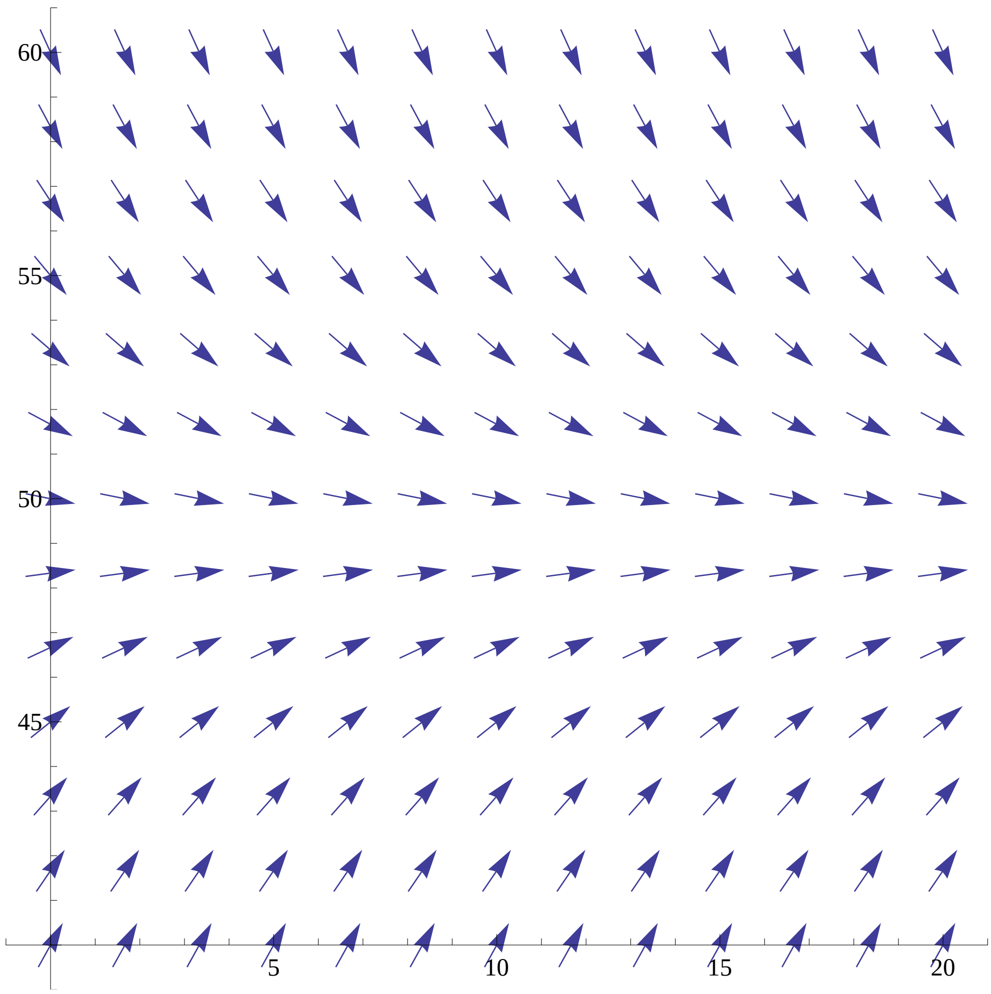
t hours

$$\frac{dQ}{dt} = 3 - \frac{300Q}{1000000}$$

Example 1.1 (A Falling Object).

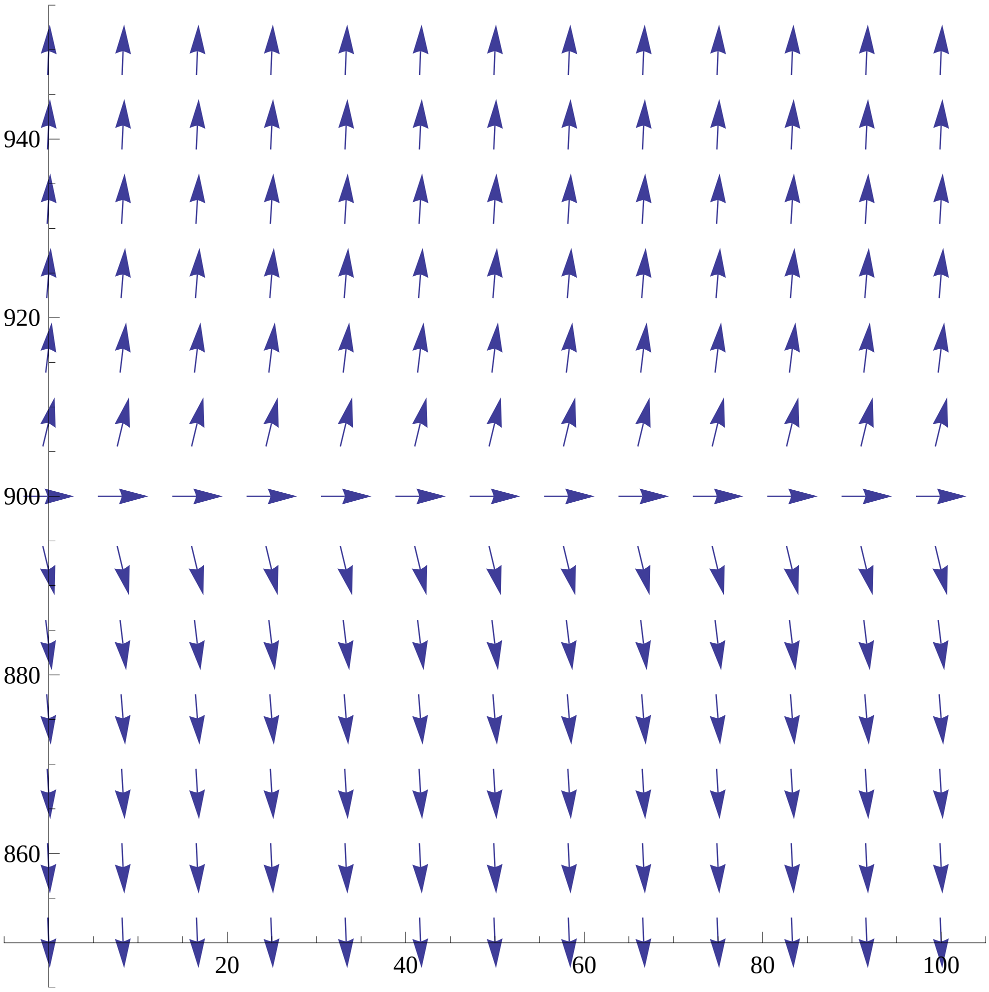
$$\frac{dv}{dt} = 9.8 - \frac{v}{5}$$



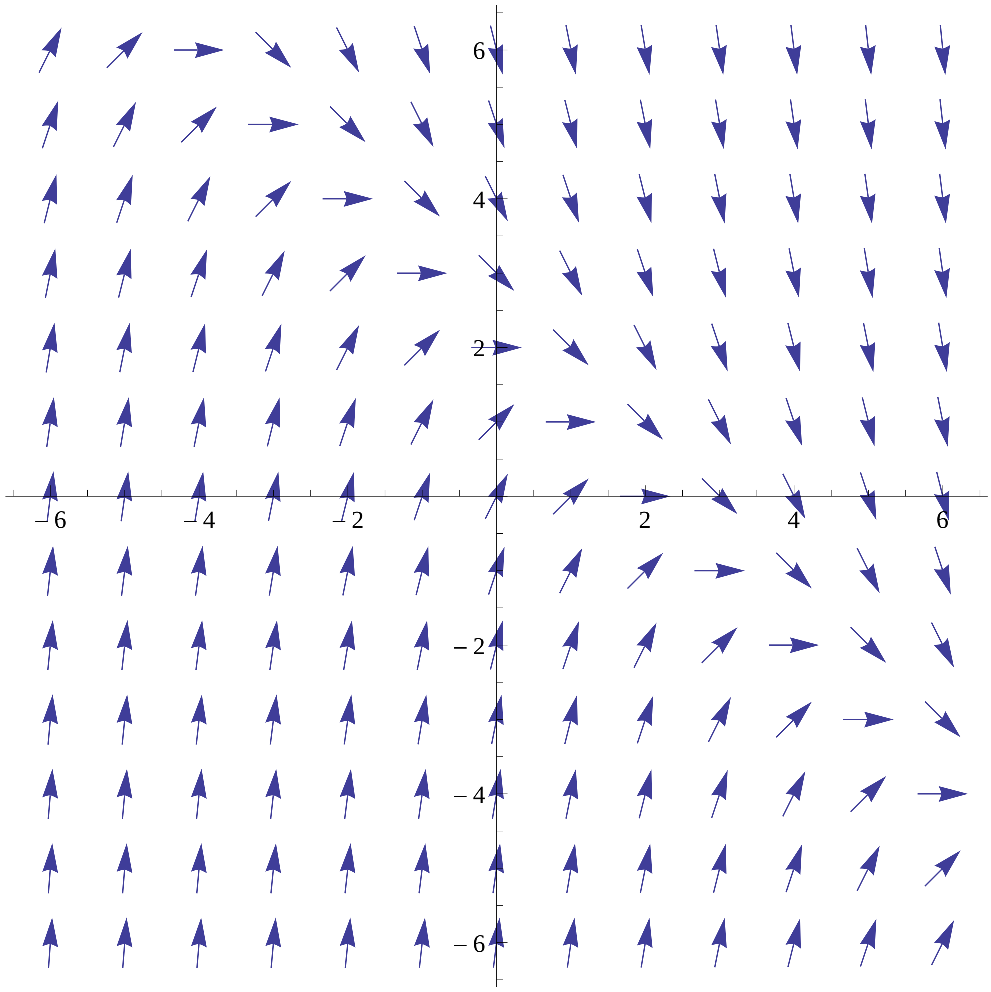


Example 1.2 (Mice and Owls).

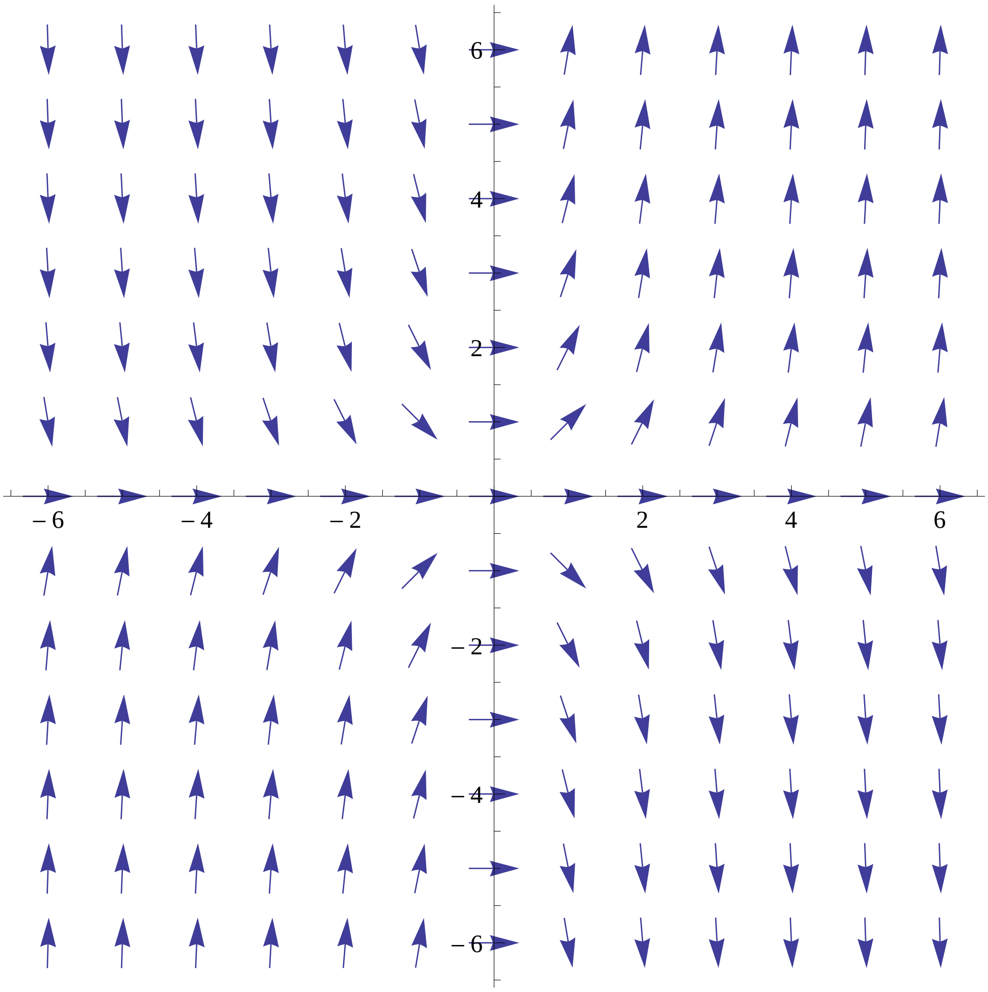
$$\frac{dp}{dt} = \frac{p}{2} - 450$$



Example. Draw a direction field for $\frac{dy}{dx} = 2 - x - y$



Example. Draw a direction field for $\frac{dy}{dx} = xy$



1.2 Solutions of Some Differential Equations ✗

1.3 Classification of Differential Equations ✗

2

First Order Differential Equations

In this chapter, we will consider equations of the form

$$\frac{dy}{dt} = f(t, y). \tag{2.1}$$ {eq:firstorderOD

2.1 Linear Equations

If the function f in (2.1) depends linearly on y (we don't care about t), then (2.1) is a first order ***linear*** ODE. Last week we only talked about equations of the form

$$\frac{dy}{dt} = -ay + b \tag{2.2}$$ {eq:firstorderCC

where the coefficients a and b are constants. We will now consider

$$\frac{dy}{dt} + p(t)y = g(t) \tag{2.3}$$ {eq:firstorderli

where the coefficients $p(t)$ and $g(t)$ are functions of t.

We have seen how to solve (2.2):

$$\begin{aligned}
\frac{dy}{dt} &= -ay + b \\
\int \frac{dy}{y - \frac{b}{a}} &= \int -a\, dt \\
\ln\left| y - \frac{b}{a} \right| &= -at + C \\
&\vdots \\
y &= \frac{b}{a} + ce^{-at}.
\end{aligned}$$

So for example $\frac{dy}{dt} + 2y = 3$ has solution $y = \frac{3}{2} + ce^{-2t}$.

Unfortunately this method can not be used to solve (2.3). So we need a different method – we use a method by Gottfried Leibniz (1646-1716). The idea is

- Find a special function $\mu(t)$ called an integrating factor;

- Multiply the ODE by $\mu(t)$;
- Integrate.

Example 2.1. Use an integrating factor to solve $\frac{dy}{dt} + 2y = 3$.

First we multiply by an unknown function $\mu(t)$:

$$\mu(t)\frac{dy}{dt} + 2\mu(t)y = 3\mu(t).$$

How do we find $\mu(t)$ so that the left-hand side is integrable? Notice that

$$\frac{d}{dt}\left(\mu(t)y\right) = \mu(t)\frac{dy}{dt} + \frac{d\mu}{dt}(t)y.$$

We want to choose $\mu(t)$ such that

$$\frac{d\mu}{dt} = 2\mu.$$

We know how to solve this equation:

$$\begin{aligned}\int \frac{d\mu}{\mu} &= \int 2\,dt \\ \ln|\mu| &= 2t + C \\ \therefore \mu(t) &= ce^{2t}.\end{aligned}$$

We only need to find one $mu(t)$ which works – so we can choose whichever value of c that we wish. I choose $c = 1$. We will use $\mu(t) = e^{2t}$.

Our ODE is then

$$e^{2t}\frac{dy}{dt} + 2e^{2t}y = 2e^{2t}.$$

Because we chose μ carefully, we can use the product rule ($(uv)' = uv' + u'v$) to write this as

$$\frac{d}{dt}\left(e^{2t}y\right) = 3e^{2t}.$$

Integrating gives

$$e^{2t}y = \frac{3}{2}e^{2t} + c.$$

Therefore

$$\boxed{y = \frac{3}{2} + ce^{-2t}.}$$

Remark. For the ODE $\frac{dy}{dt} + 2y = 3$ we use the integrating factor $\mu(t) = e^{2t}$.

Example 2.2. Use an integrating factor to solve $\frac{dy}{dt} + ay = b$.

If we were to repeat the previous method, we would find that we need the integrating factor $\mu(t) = e^{at}$. (Please check!)

Example 2.3. Solve $\frac{dy}{dt} + ay = g(t)$.

The integrating factor depends only on the coefficient of y. So again we use $\mu(t) = e^{at}$. Multiplying the ODE by e^{at} gives

$$e^{at}\frac{dy}{dt} + ae^{at}y = e^{at}g(t).$$

So

$$\frac{d}{dt}\left(e^{a}ty\right) = e^{at}g(t).$$

By integrating, we obtain

$$e^{at}y = \int e^{as}g(s)\,ds + c.$$

Thus

$$\boxed{y = e^{-at}\int e^{as}g(s)\,ds + ce^{-at}} \tag{2.4}$$

Example 2.4. Solve

$$\begin{cases} \frac{dy}{dt} + \frac{1}{2}y = 2 + t \\ y(0) = 2. \end{cases}$$

We multiply the ODE by the integrating factor $e^{\frac{t}{2}}$ to obtain

$$e^{\frac{t}{2}}y' + \frac{1}{2}e^{\frac{t}{2}}y = 2e^{\frac{t}{2}} + te^{\frac{t}{2}}$$

and

$$\frac{d}{dt}\left(e^{\frac{t}{2}}y\right) = 2e^{\frac{t}{2}} + te^{\frac{t}{2}}.$$

Integrating gives us

$$e^{\frac{t}{2}}y = 4e^{\frac{t}{2}} + 2te^{\frac{t}{2}} - 4e^{\frac{t}{2}} + c = 2te^{\frac{t}{2}} + c$$

(where we have used $\int u\frac{dv}{dt} = uv - \int \frac{du}{dt}v$ with $u = t$ and $v = 2e^{\frac{t}{2}}$). Therefore

$$y(t) = 2t + ce^{-\frac{t}{2}}.$$

Now

$$2 = y(0) = 0 + c \qquad \Longrightarrow \qquad c = 2.$$

Therefore the solution to the IVP is

$$\boxed{y(t) = 2t + 2e^{-\frac{t}{2}}.}$$

Example 2.5. Solve $\dfrac{dy}{dt} - 2y = 4 - t$.

Please check that by using $\mu(t) = e^{-2t}$ we obtain $y(t) = -\frac{7}{4} + \frac{t}{2} + ce^{2t}$.

Now consider
$$\frac{dy}{dt} + p(t)y = g(t).$$
We must find the integrating factor.

WARNING: The integrating factor is NOT $e^{p(t)}$.

If we multiply by an unknown function $\mu(t)$, we obtain
$$\mu\frac{dy}{dt} + p(t)\mu y = \mu g(t).$$
As before, then left-hand side looks like
$$\frac{d}{dt}(\mu y) = \mu\frac{dy}{dt} + \frac{d\mu}{dt}y.$$
So we want
$$\frac{d\mu}{dt} = p(t)\mu.$$
We know how to solve this ODE:
$$\begin{aligned}\int \frac{d\mu}{\mu} &= \int p(t)\,dt \\ \ln|\mu| &= \int p(t)\,dt + C \\ \therefore \mu(t) &= c\exp\int p(t)\,dt.\end{aligned}$$
As before, we can choose $c = 1$ to obtain
$$\mu(t) = \exp\int p(t)\,dt = e^{\int p(t)\,dt}. \tag{2.5}$$
Then our ODE becomes
$$\frac{d}{dt}(\mu y) = \mu g(t)$$
and we calculate that
$$\mu y = \int \mu(s)g(s)\,ds + c$$
and
$$y(t) = \frac{\int \mu(s)g(s)\,ds_{+}c}{\mu(t)}.$$
Example 2.6. Solve
$$\begin{cases} ty' + 2y = 4t^2 \\ y(1) = 2. \end{cases}$$
First we must write the equation in the standard form:
$$\frac{dy}{dt} + \frac{2}{t}y = 4t.$$
Here $p(t) = \frac{2}{t}$ and $g(t) = 4t$.

Next we must calculate $\mu(t)$:
$$\mu(t) = \exp\int \frac{2}{t}dt = e^{2\ln|t|} = t^2.$$

Multiplying the ODE by t^2 gives

$$\frac{d}{dt}\left(t^2 y\right) = t^2 \frac{dy}{dt} + 2ty = 4t^3.$$

Integrating gives

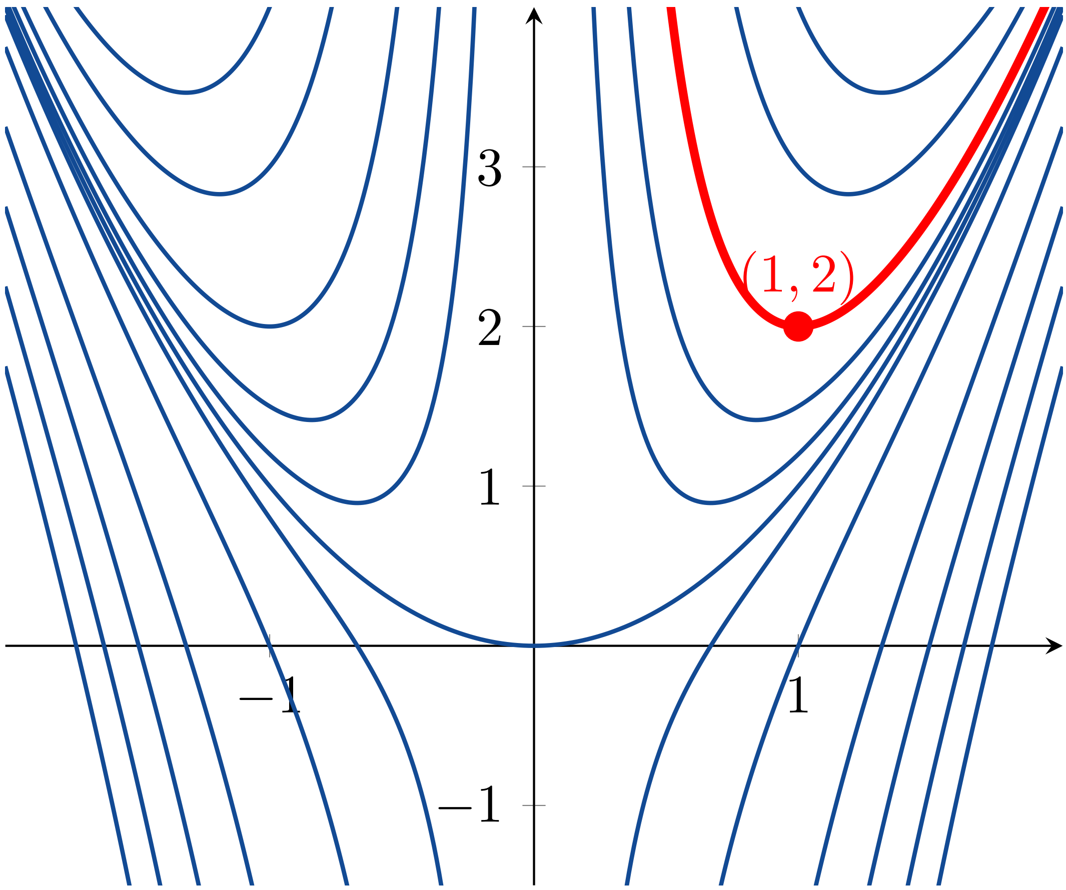
$$t^2 y = t^4 + c.$$

Hence the general solution to the ODE is

$$y(t) = t^2 + \frac{c}{t^2}.$$

To satisfy $y(1) = 2$, we choose $c = 1$. Therefore

$$\boxed{y(t) = t^2 + \frac{1}{t^2} \qquad (t > 0).}$$



Note that

(i). the solution satisfying $y(1) = 2$ is a differentiable function $y : (0, \infty) \to \mathbb{R}$.

(ii). the solution becomes unbounded and asymptotic to the y-axis as $t \searrow 0$. This is because $p(t)$ has a discontinuity at $t = 0$.

(iii). The function $y = t^2 + \frac{1}{t^2}$, $t < 0$ is ***not*** part of the solution to the IVP. The solution to the IVP only exists for $t \in (0, \infty)$.

(iv). Solutions for which $c > 0$ (i.e. $y(1) >$) are asymptotic to the positive y-axis as $t \searrow 0$. But solutions for which $c < 0$ (i.e. $y(1) < 1$) are asymptotic to the negative y-axis as $t \searrow 0$. So there is an initial value ($y(1) = 0$) where the behaviour changes. This is called a ***critical initial value***.

2.2 Separable Equations

The general first order ODE is

$$\frac{dy}{dx} = f(x,y). \tag{2.6}$$ {eq

In the previous section we looked at a special case called "linear equations" – now we will study another special case. Equations (2.6) can ***always*** be written in the form

{eq:separable1}
$$M(x,y) + N(x,y)\frac{dy}{dx} = 0. \tag{2.7}$$

One way would be to write $M = -f$ and $N = 1$, but there may be other ways. ***If*** we can do this so that $M(x)$ is a function only of x and $N(y)$ is a function only of y, then (2.7) becomes

{eq:separable2}
$$M(x) + N(y)\frac{dy}{dx} = 0. \tag{2.8}$$

Definition. A first order ODE is called ***separable*** if it can be written in the form (2.8).

Remark. Note that we can rearrange (2.8) to

$$\underbrace{M(x)\,dx}_{\text{all } x \text{ terms}} = -\underbrace{N(y)\,dy}_{\text{all } y \text{ terms}}.$$

In other words, it is possible to "separate" the variables.

Example 2.7. Consider

$$\frac{dy}{dx} = \frac{x^2}{1-y^2}.$$

(i). Show that this ODE is separable.

(ii). Solve this ODE.

We can rearrange this ODE to

$$-x^2 + (1-y^2)\frac{dy}{dx} = 0.$$

This is of the form (2.8). Therefore this ODE is separable.

Note that $\frac{d}{dx}\left(-\frac{1}{3}x^3\right) = -x^2$ and $\frac{d}{dy}\left(y - \frac{1}{3}y^3\right) = 1 - y^2$. So our ODE is

$$\frac{d}{dx}\left(-\frac{1}{3}x^3\right) + \frac{d}{dy}\left(y - \frac{1}{3}y^3\right)\frac{dy}{dx} = 0$$

Using the Chain Rule, this is

$$\begin{aligned}\frac{d}{dx}\left(-\frac{1}{3}x^3\right) + \frac{d}{dx}\left(y - \frac{1}{3}y^3\right) &= 0\\ \frac{d}{dx}\left(-\frac{1}{3}x^3 + 1 - \frac{1}{3}y^3\right) &= 0.\end{aligned}$$

Therefore

$$-\frac{1}{3}x^3 + 1 - \frac{1}{3}y^3 = C$$

or

$$\boxed{x^3 - 3y + y^3 = c.}$$

The same method can be used to solve any separable equation. Consider

$$M(x) + N(y)y' = 0$$

and suppose that $H_1(x)$ and $H_2(y)$ are functions which satisfy $H_1' = M$ and $H_2' = N$. Then our ODE becomes

$$\begin{aligned} M(x) + N(y)\frac{dy}{dx} &= 0 \\ \frac{dH_1}{dx} + \frac{dH_2}{dy}\frac{dy}{dx} &= 0 \\ \frac{dH_1}{dx} + \frac{dH_2}{dx} &= 0 \end{aligned}$$

by the Chain Rule. Then integrating gives the solution

$$H_1(x) + H_2(y) = c.$$

So to recap: To solve $M(x) + N(y)y' = 0$ we must integrate M wrt x and integrate N wrt y. But this is basically what we were doing in Chapter 1, where we did the following:

$$\begin{aligned} M(x) + N(y)\frac{dy}{dx} &= 0 \\ M(x) &= -N(y)\frac{dy}{dx} \\ M(x)\,ds &= -N(y)\,dy \\ \int M(x)\,ds &= -\int N(y)\,dy + c. \end{aligned}$$

Example 2.8. Solve $\begin{cases} \frac{dy}{dx} = \frac{3x^2+4x+2}{2(y-1)} \\ y(0) = -1 \end{cases}$.

The ODE can be written as

$$2(y-1)\,dy = (3x^2 + 4x + 2)\,dx.$$

Integrating gives

$$y^2 - 2y = x^3 + 2x^2 + 2x + c.$$

To find c, we use the initial condition $y(0) = 1$ and calculate that

$$1 + 2 = 0 + 0 + 0 + c \qquad \Longrightarrow \qquad c = 3.$$

So the solution to the IVP is given by

$$\boxed{y^2 - 2y = x^3 + 2x^2 + 2x + 3.}$$

This is called an ***implicit solution***. Sometimes this is the best that we can do. But in this example, it is possible to solve for y. Since

$$y^2 - 2y - (x^3 + 2x^2 + 2x + 3) = 0$$

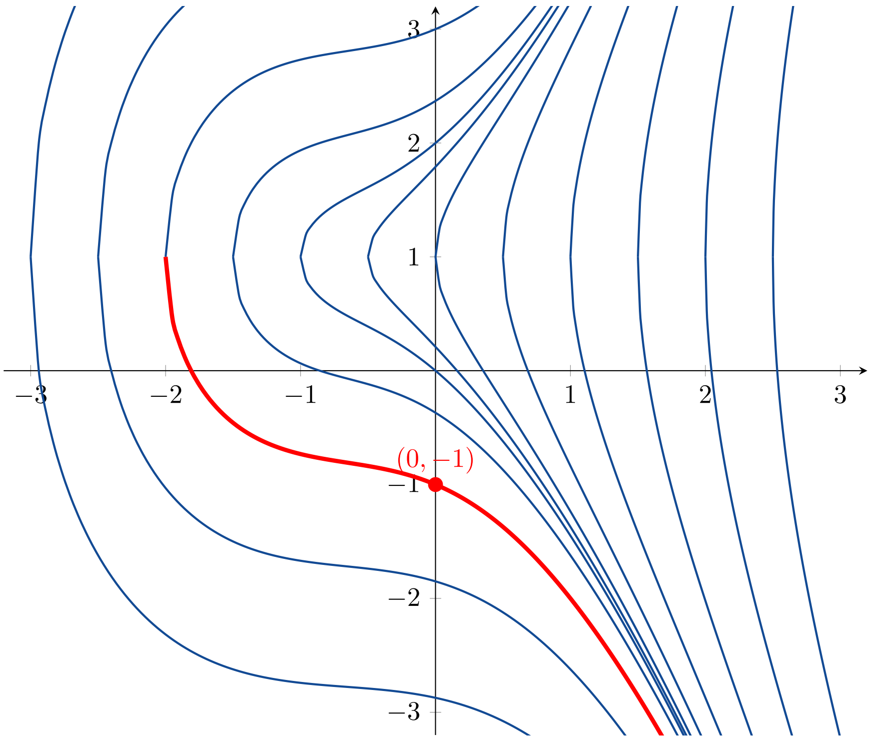
is a quadratic equation, we find that

$$y = 1 \pm \sqrt{x^3 + 2x^2 + 2x + 4}.$$

There are two solutions here, but only only is correct. Which solution satisfies $y(0) = -1$? The answer is the solution with $-$. Therefore the solution to the IVP is

$$\boxed{y = 1 - \sqrt{x^3 + 2x^2 + 2x + 4}.}$$

A solution of the form $y = f(x)$ is called an ***explicit solution***.



Note that the solution satisfying $y(0) = -1$ is a differentiable function $y : (-2, \infty) \to \mathbb{R}$.

Example 2.9. Solve $\begin{cases} \frac{dy}{dx} = \frac{y \cos x}{1+2y^2} \\ y(0) = 1. \end{cases}$.

$$\int \frac{1 + 2y^2}{y} dy = \int \cos x \, dx$$

$$\ln |y| + y^2 = \sin x + c$$

$$y(0) = 1 \qquad \Longrightarrow \qquad \ln 1 + 1^2 = \sin 0 + c \qquad \Longrightarrow \qquad c = 1.$$

$$\boxed{\ln |y| + y^2 = \sin x + 1.}$$

This equation can not be easily solved for y, so we leave it as an implicit solution. What can we say about this solution?

(i). If $y = 0$, the left-hand side is ∞, but the right-hand side is in $[0, 2]$. This means that $y = 0$ is not possible. Since we know that $y(0) = 1$, we must therefore have $y(x) > 0$ for all x in the domain of the solution.

(ii). The solution exists on $(-\infty, \infty)$ (left for you to prove).

2.4 Differences Between Linear and Nonlinear Equations

Theorem 2.1. *Suppose*

- p *and* g *are continuous on* (α, β)*;*
- $t_0 \in (\alpha, \beta)$*; and*
- $y_0 \in \mathbb{R}$.

Then there exists a unique solution to

$$\begin{cases} y' + p(t)y = g(t) \\ y(t_0) = y_0 \end{cases}$$

on (α, β).

Remark. This theorem says that as long as p and g are continuous, the solution keeps existing. To say this another way: The solution can only stop existing at a discontinuity of either p or g.

Theorem 2.2. *Suppose that*

- f *and* $\frac{\partial f}{\partial y}$ *are continuous for all* $\alpha < t < \beta$ *and* $\gamma < y < \delta$*;*
- $t_0 \in (\alpha, \beta)$*; and*
- $y_0 \in (\gamma, \delta)$.

Then in some interval $(t_0 - h, t_0 + h) \subseteq (\alpha, \beta)$*, there exists a unique solution to*

$$\begin{cases} y' = f(t, y) \\ y(t_0) = y_0. \end{cases}$$

Remark. This theorem tells us that two solutions to $y' = f(t, y)$ can not intersect.

To understand why: Suppose that two solutions intersect at the point (t_0, y_0). But then there would be two solutions to

$$\begin{cases} y' = f(t, y) \\ y(t_0) = y_0 \end{cases}$$

and the theorem says that this is not possible.

Solutions to first order ODEs do not intersect !!! (assuming that f and $\frac{\partial f}{\partial y}$ are $\ldots$)

2.5 Autonomous Equations and Population Dynamics ✗

Equations of the form

$$\frac{dy}{dt} = \underbrace{f(y)}_{\text{only } y} \tag{2.9}$$

are called ***autonomous***.

Example 2.10 (Exponential Growth)**.** Let $y(t)$ denote the number of cats in İstanbul.

The simplest model is to assume that the rate of change of y is proportional to y.

$$\frac{dy}{dt} = ry$$

for some constant r. We will assume that $r > 0$. The solution to

$$\begin{cases} y' = ry \\ y(0) = y_0 \end{cases}$$

is $y(t) = y_0 e^{rt}$.

GRAPH

This model predicts that the number of cats in İstanbul will increase exponentially for all time. This can not be true. At some points:

- the food will run out
- there will be no space
- people will get angry

  $\vdots$

So we need a better model.

Example 2.11 (Logistic Growth)**.** Now we replace the constant r with a function $h(y)$.

$$\frac{dy}{dt} = h(y)y$$

We want a function h which satisfies

- $h(y) \approx y$ if r is small;
- $h(y)$ decreases as y grows larger; and
- $h(y) < 0$ for large y.

The simplest such h is $h(y) = r - ay$. So

$$\frac{dy}{dt} = (r - ay)y$$

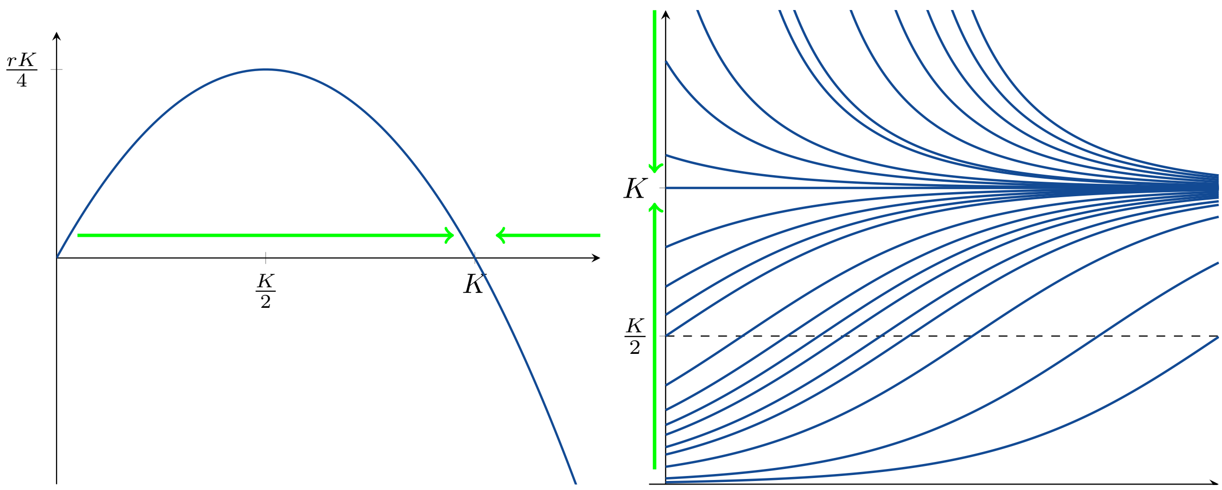
which we will write as

$$\frac{dy}{dt} = r\left(1 - \frac{y}{K}\right)y$$

for $K = \frac{r}{a}$. This is called the ***Logistic Equation***.

First we look for equilibrium solutions – that is solutions with $\frac{dy}{dt} = 0$ for all t.

$$0 = \frac{dy}{dt} = r\left(a - \frac{y}{K}\right)y \qquad \Longrightarrow \qquad y = 0 \quad \text{or} \quad y = K.$$



rK/4
K/2
K
K
K/2

used in Chapter 1, "done properly".

Ex 18/10/11

Example 2 - Logarithmic Logistic Growth

We replace r by a function $h(y)$.

$$\frac{dy}{dt} = h(y)\,y$$

We want to choose h, so that $h(y) \approx r$ when y is small, ~~but~~ $h(y)$ decreases as y grows larger, and $h(y) < 0$ for large y.

The simplest h having these properties is $h(y) = r - ay$.

So

$$\frac{dy}{dt} = (r - ay)\,y \quad \text{the Logistic equation}$$

~~It is often~~

Sometimes we like to write this as

$$\frac{dy}{dt} = r\left(1 - \frac{y}{K}\right)y \qquad (18)$$

$K = r/a$. r is called the intrinsic growth rate.

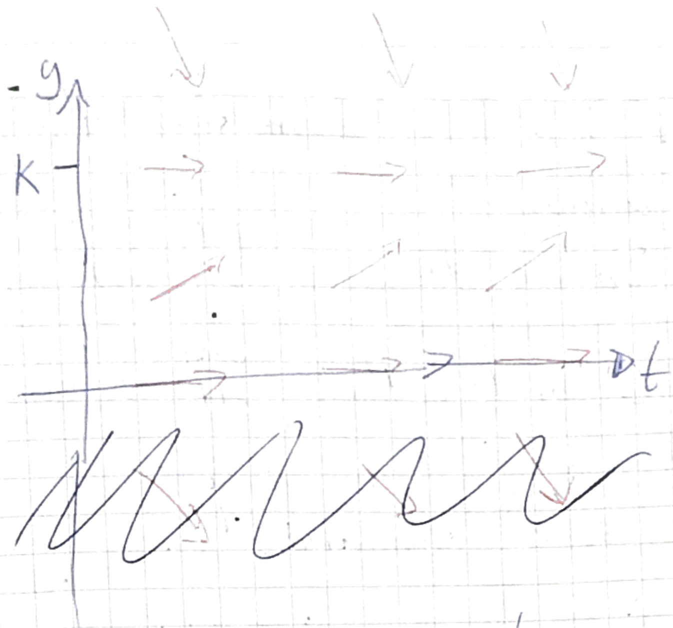
~~We sta~~

First, we look for equilibrium solutions. For such a solution $\frac{dy}{dt} = 0 \ \forall t$. So $0 = r(1 - y/K)\,y$.

So the equilibrium solutions are

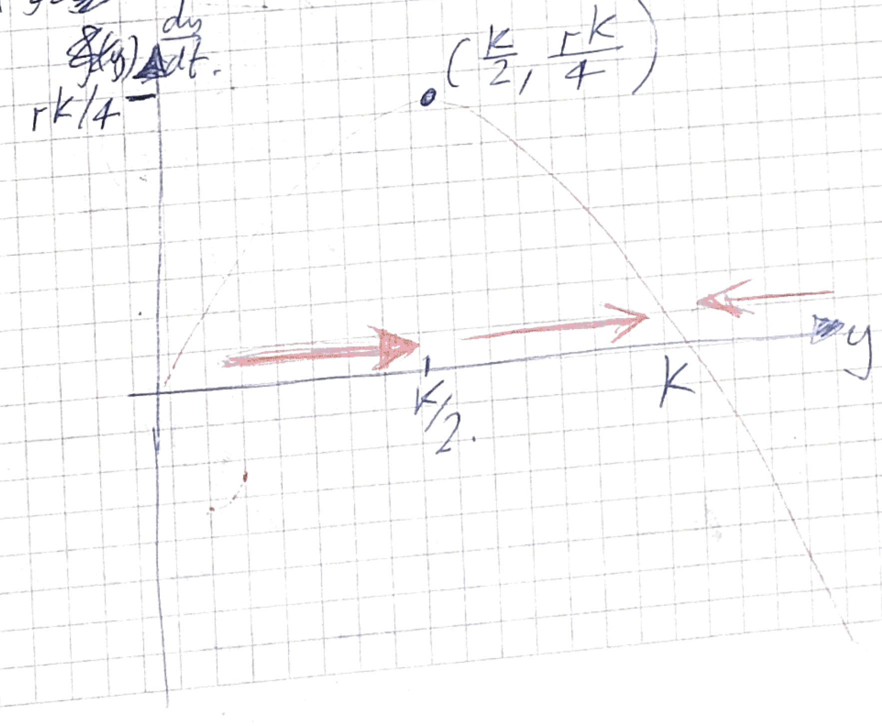
$y = \phi_1(t) = 0$ and $y = \phi_2(t) = K$.

— ~~Direction field on page XX~~



- on a horizontal line, the arrows are the same.
- ~~if $y_0 > 0$, $y(t) \to K$ as $t \to \infty$~~
- ~~if $y_0 < 0$, $y(t) \to -\infty$ as $t \to \infty$~~
- the equilibrium solutions are important
  - solutions ~~converge~~ ~~approach~~ converge to $y = K$ but diverge from $y = 0$.

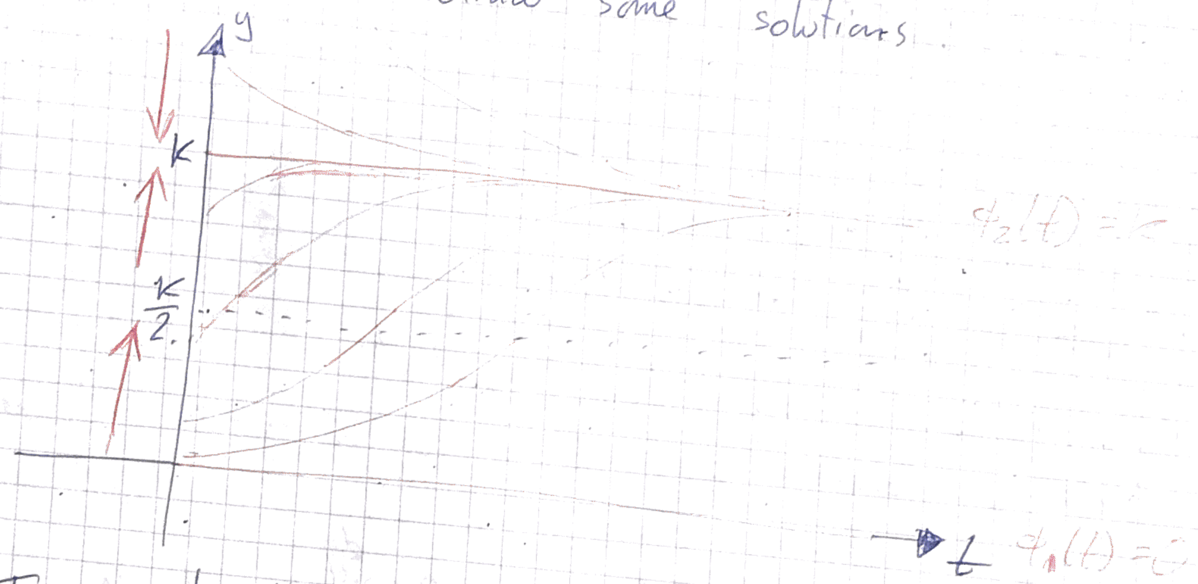
To understand this, we graph $\frac{dy}{dt}$ against y.
Here $\frac{dy}{dt} = r(1 - y/K)y$.



used in Chapter 1, "done properly".

E 181 If $y \in (0, K)$, $\frac{dy}{dt} > 0$. We ~~idea~~ show this by drawing a rightwards arrow.
If $y > K$, $\frac{dy}{dt} < 0$. So we draw a leftwards arrow.
If y is close to 0, or K, then $\frac{dy}{dt}$ is small.

So we can draw some solutions.



21/10/10.

To investigate further, we look at $\frac{d^2y}{dt^2}$.

If $\frac{dy}{dt} = f(y)$, then
$$\frac{d^2y}{dt^2} = \frac{d}{dt}\left[f(y(t))\right] = f'(y)\frac{dy}{dt} = f'(y)\,f(y).$$

The graph of y is concave up (when f and f' ~~have the same sign~~ are both positive / both negative) or when $y'' > 0$

and concave down or when $y''<0$ (when ~~only~~ one of f and ~~Inflection points~~ f' is positive and one is negative).

For (18), solutions are concave up ~~when~~ for $0<y<K/2$, ~~and~~ (where f is positive and f is increasing ($f'>0$)) ~~and~~ Solutions are also concave up for $y>K$ (where ~~o~~ f is negative and decreasing. For $K/2<y<K$, solutions are concave down since f is positive and decreasing ($f'<0$)

Finally, remember theorem 2. ~~This theorem~~ the fundamental existence and uniqueness theorem tells us that ~~no~~ two different solutions never pass through the same point. So while solutions $\to K$ as $t\to\infty$, they are ~~never~~ not ~~get there~~ equal to K ~~...~~ $\forall t\in\mathbb{R}$.

K is ~~an upper~~ called the saturation level.

We have found all this information about the solutions without solving the equation. Sometimes it is more useful to understand the behaviour of solutions than it is to have a formula for the solutions.

Sometimes, we do want ~~the so~~ a formula ~~for the solution~~, so let's solve ~~the~~ equation (18).

used in Chapter 1, "done properly".

If $y \neq 0$ and $y \neq K$, we can write (18) as

$$\frac{dy}{(1 - y/k)\, y} = r\,dt.$$

Using partial fractions,

$$\left(\frac{1}{y} + \frac{1/k}{1 - y/k} \right) dy = r\,dt.$$

Integrating gives

$$\log|y| - \log|1 - y/k| = rt + c.$$

CASE 1: $y_0 \in (0, K)$

We already saw that if $y(0) = y_0 \in (0, k)$, then $y(t) \in (0, k)$ $\forall t$. So we don't need the $|\cdot|$ bars. Therefore

$$\frac{y}{1 - (y/k)} = C e^{rt} \qquad C = e^c$$

$$y(0) = y_0 \Rightarrow C = \frac{y_0}{1 - y_0/k}.$$

Solving for y gives

$$y(t) = \frac{y_0 K}{y_0 + (K - y_0) e^{-rt}}. \qquad (19)$$

[illegible]

CASE 2 : $y_0 > K$.

If $y_0 > K$, ~~I leave it to you to show that (19) is~~ ~~st~~

$$\log|1 - y/K| = \log\left(\frac{y}{K} - 1\right).$$

I leave it to you to show that (19) is still the solution.

Note that ~~Remark~~ (19) also contains the equilibrium solutions $y = \phi_1(t) \equiv 0$ and $y = \phi_2(t) \equiv K$.

If $y_0 = 0$, then $y(t) = 0$ $\forall t$.

If $y_0 > 0$, then

$$\lim_{t\to\infty} y(t) = \frac{y_0 K}{y_0} = K.$$

So for each $y_0 > 0$, the solution approaches $y = \phi_2(t) \equiv K$ as $t \to \infty$. So we say ~~that the solution $\phi_2(t) \equiv K$ is~~ an asymptotically stable solution of (18), ~~or we also say that the point $y = K$ is an asymptotically stable~~ ~~critical~~ ~~critical point.~~ ~~equilibrium solution~~ of (18).

~~However~~, For $y = \phi_1(t) \equiv 0$: Even solutions that start very near, move away as t increases and approach K as $t \to \infty$. We say that $\phi_1(t) \equiv 0$ is an unstable equilibrium solution of (18). ~~$y = 0$ is an~~ ~~unstable critical point.~~ of (18)

used in Chapter 1, "done properly".

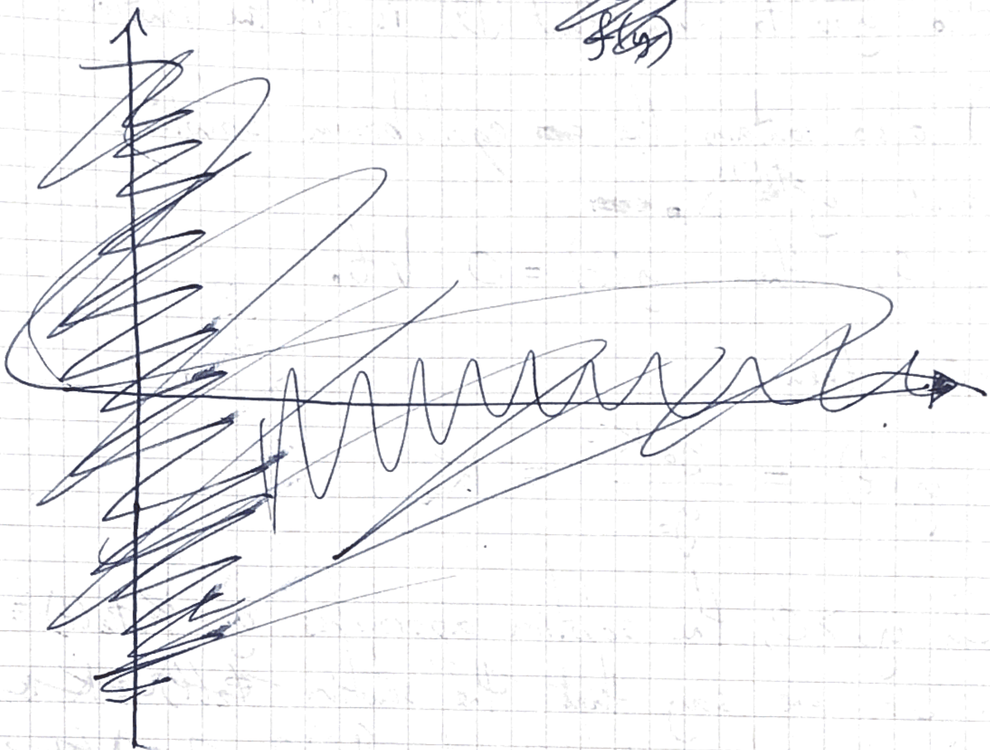
Remarks: x_0 critical point.

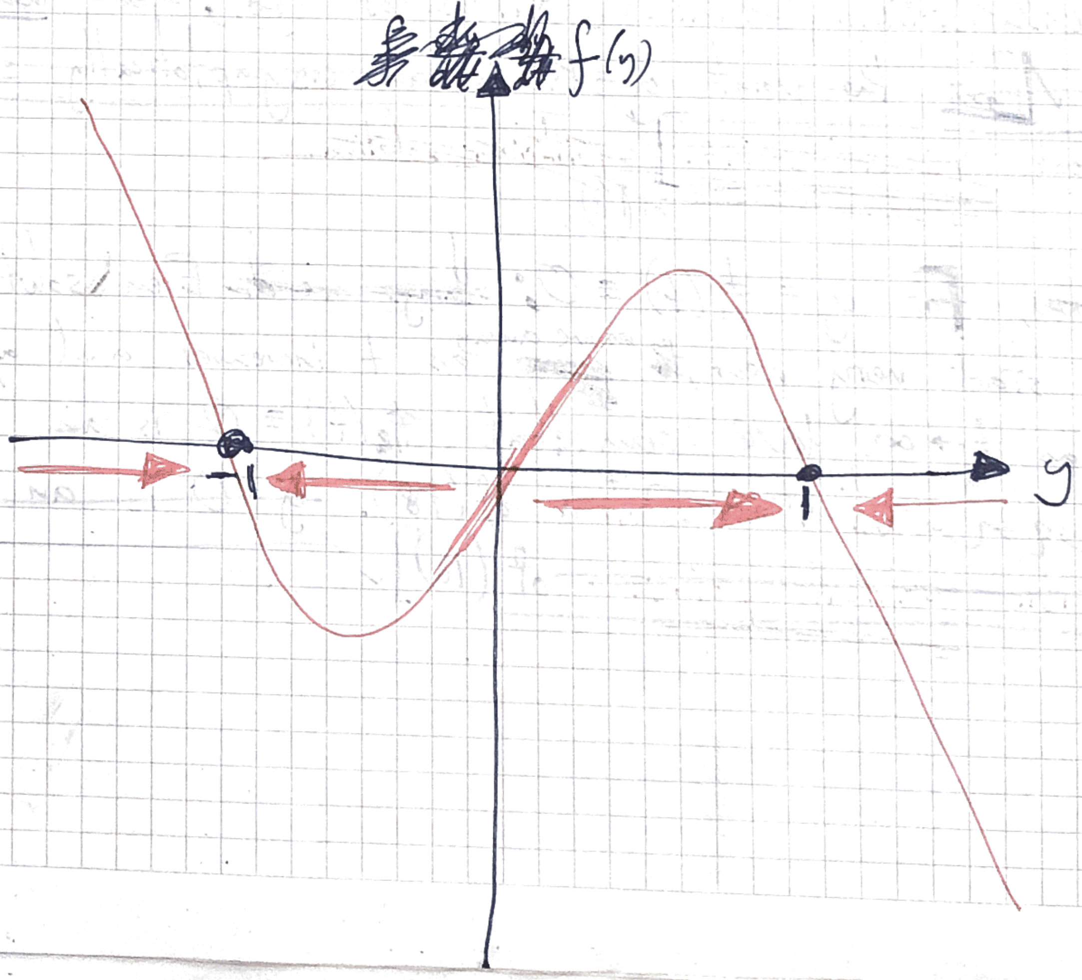
Asy. Stable $\rightarrow x_0 \leftarrow$

Unstable $\leftarrow x_0 \rightarrow$

Semi Stable $\rightarrow x_0 \rightarrow$ or $\leftarrow x_0 \leftarrow$

Example Suppose $\dfrac{dy}{dt} = \overbrace{y(1-y^2)}^{f(y)}$ $\quad -\infty < y_0 < \infty.$





What are the critical points (where $\frac{dy}{dt} = 0$)?

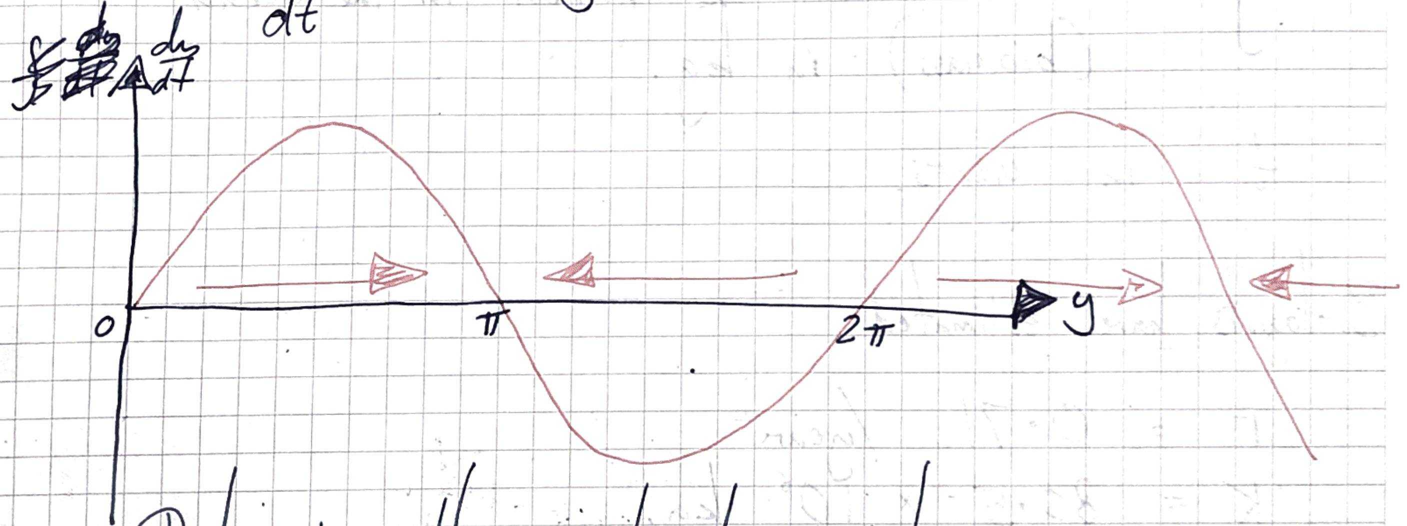
$y = -1, 0, 1.$

Are the critical points asymptotically stable or unstable.

1 and -1 are asymptotically stable ($\rightarrow \cdot \leftarrow$)

0 is unstable. ($\leftarrow \cdot \rightarrow$).

Example: $\frac{dy}{dt} = \sin y \qquad 0 \leq y_0 < \infty$



Determine the critical points.

$y = 0, \pi, 2\pi, 3\pi, \ldots$

Classify them as asymptotically stable or unstable.

$2k\pi$: unstable

$(2k+1)\pi$: asymptotically stable.

$k \in \mathbb{N} \cup \{0\}$

used in Chapter 1, "done properly".

Example.

Now back to the logistic equation.

$$\frac{dy}{dt} = r\left(1 - \frac{y}{K}\right)y$$

This has been used for the population of a type of fish called halibut, that lives in the Pacific Ocean.

(Kalkana benzer yassı balık)

y = mass of all the halibut in the ocean (biomass) in kg.

t in years.

Biologists have estimated

$r = 0.71$ /year
$K = 80.5 \times 10^6$ kg.

Suppose $y_0 = 0.25\,K$.

Find the biomass 2 years later.

~~Find the t such that y(t) = 0.75 K.~~

First we write (19) as

$$\frac{y}{K} = \frac{y_0/K}{(y_0/K) + \left[1 - y_0/K\right] e^{-rt}}$$

So $\dfrac{y(2)}{K} = \dfrac{0.25}{0.25 + 0.75e^{-1.42}} \approx 0.5797.$

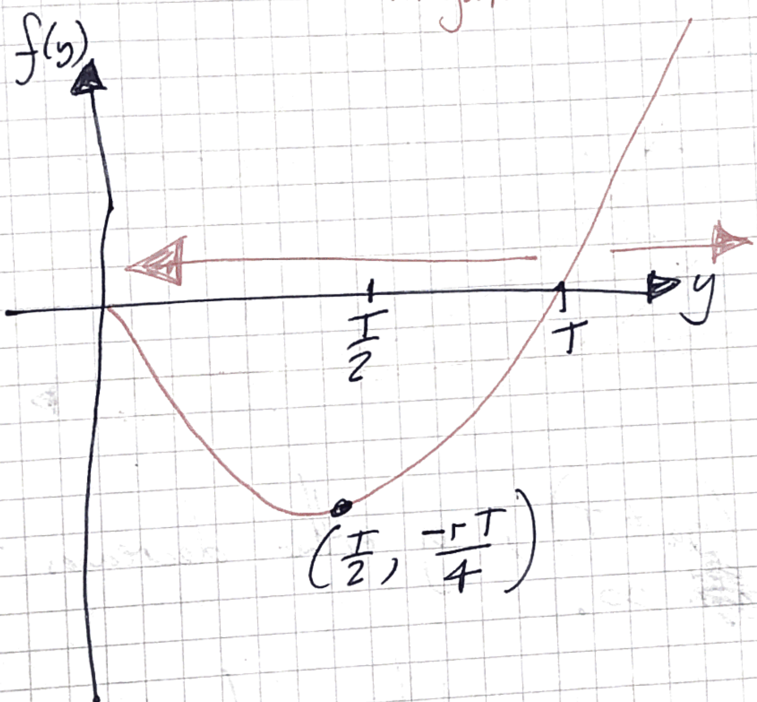
So $y(2) \approx 46.7 \times 10^6$ kg

Example 3 -
A Critical Threshold.

20/10/11.

Now consider $\dfrac{dy}{dt} = -r\left(1 - \frac{y}{T}\right)y$, $y_0 \geq 0$. $(T>0)$

negative.



The critical points are $y=0$ and $y=T$.

If $0 < y < T$, $\frac{\partial y}{\partial t} < 0$ so y decrease.

If $y > T$, $\frac{\partial y}{\partial t} > 0$ so y increases.

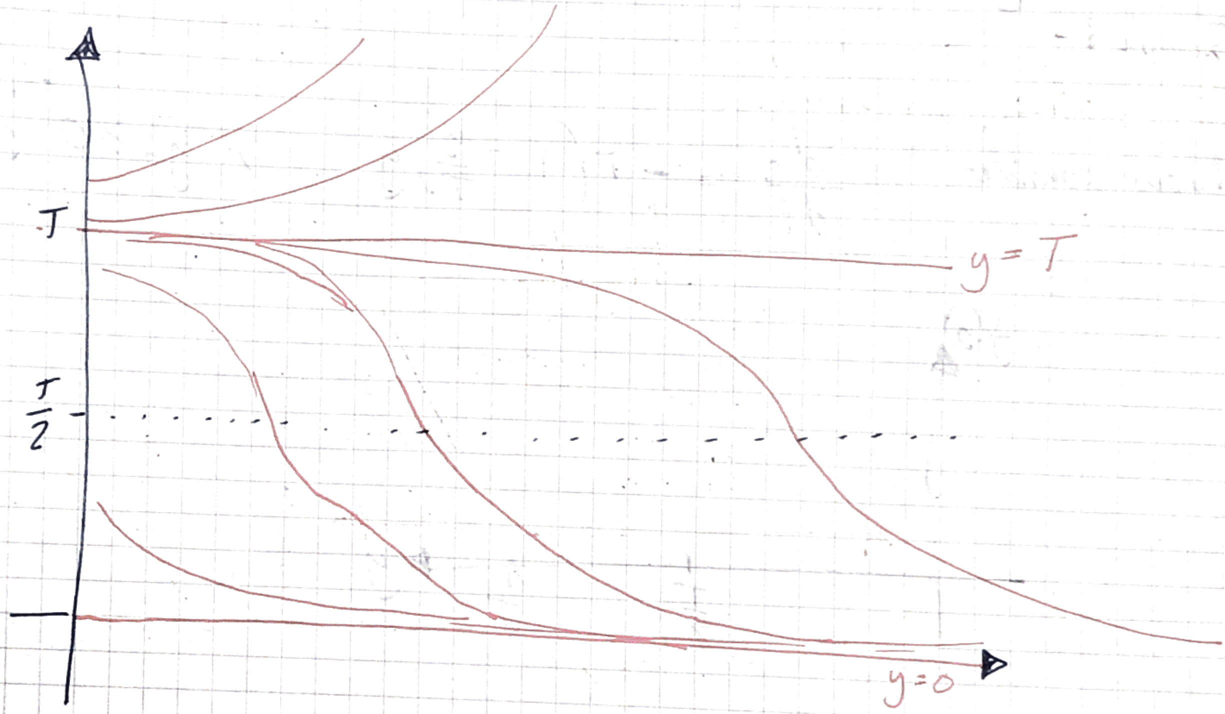
So $y=0$ is asymptotically stable, and $y=T$ is unstable.

Furthermore $f'(y) < 0$ for ~~y~~ $0 < y < T/2$, so the graph is concave up, and $f'(y) > 0$ for $0 < y < T$, so the graph is concave down there.

used in Chapter 1, "done properly".

$f'(y) > 0$ for $y > T$, so the graph is concave up there.

With this information, we can sketch some solutions.



Depending on y_0 ($y_0 \neq T$), y either decreases to 0, or grows to ∞.

T is called a threshold level, below which no growth happens.

The population of some species have the threshold property ~~behaves like this~~.
If there are not enough animals, then the species becomes extinct.

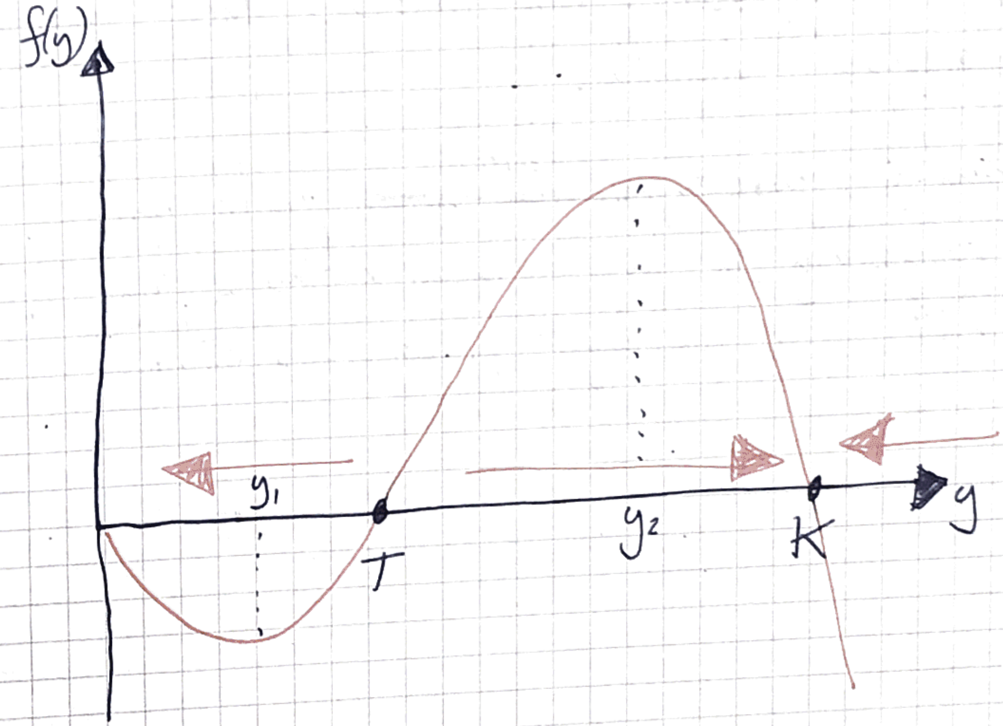
However, we have growth to ∞ again, if $y_0 > T$.
So...

26/10/11

Example 4 – Logistic Growth with a Threshold

Consider $\dfrac{dy}{dt} = -r\underbrace{\left(1-\frac{y}{T}\right)}_{\text{threshold}}\underbrace{\left(1-\frac{y}{K}\right)}_{\text{logistic growth}}y$

where $y>0$ and $0<T<K$.



Critical points: $y=0$, $y=T$, $y=K$.

Equilibrium solutions: $\phi_1(t)\equiv 0$, $\phi_2(t)\equiv T$, $\phi_3(t)\equiv K$.

ϕ_1 and ϕ_3 are asymptotically stable equilibrium solutions.

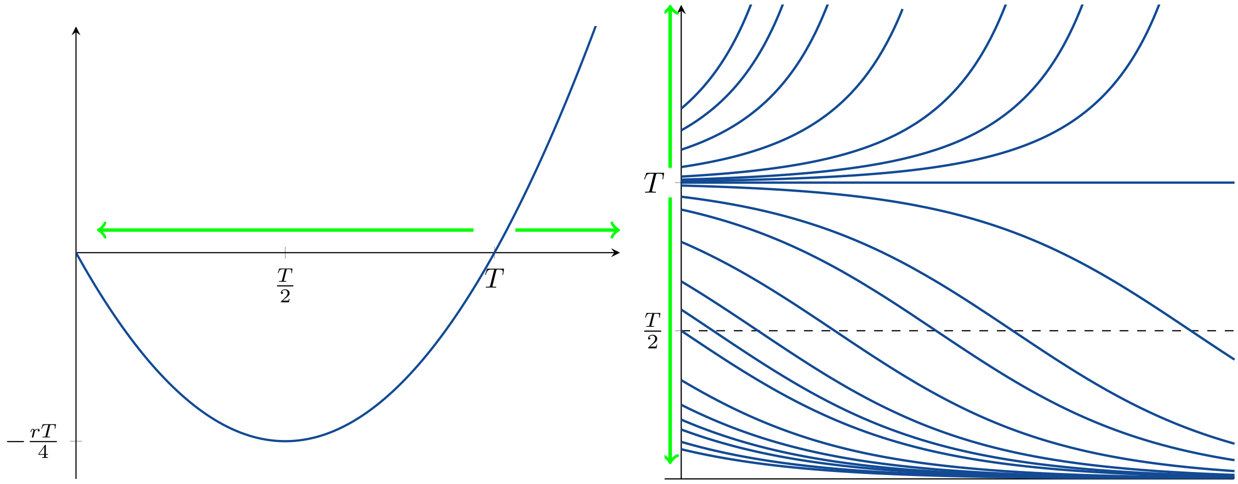
ϕ_2 is an unstable equilibrium solution.

~~This model describes the population of passenger pigeons in USA.~~

figure page 68.

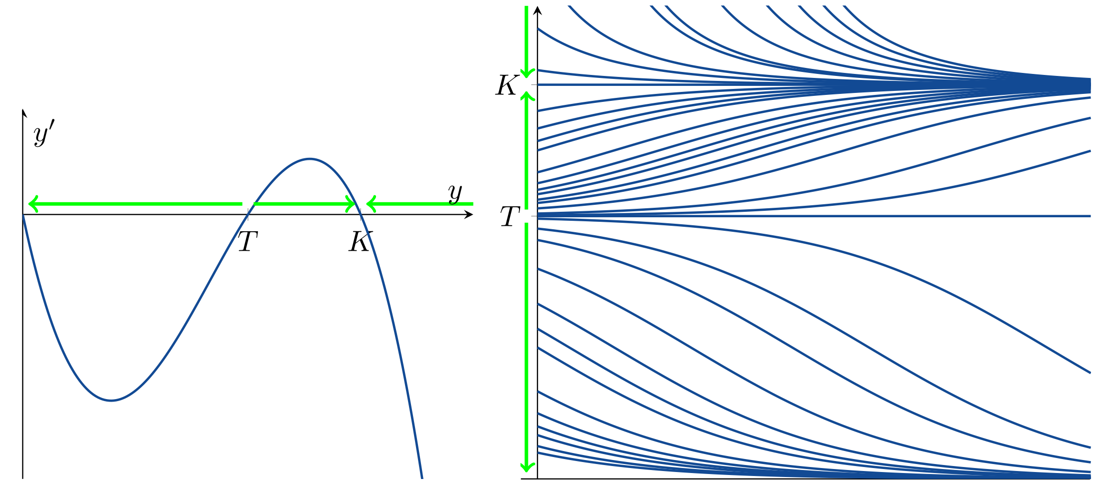
Example 2.12 (A Critical Threshold).

$$\frac{dy}{dt} = -r\left(1 - \frac{y}{T}\right)y$$



Example 2.13 (Logistic Growth with a Threshold). $0 < T < K$, $r > 0$

$$\frac{dy}{dt} = -r\left(1 - \frac{y}{T}\right)\left(1 - \frac{y}{K}\right)y$$



2.6 Exact Equations

We have looked at linear equations and separable equations. Now we will look at another special type of equation.

Example 2.14. Solve $2x + y^2 + 2xyy' = 0$.

This equation is not linear and is not separable. Note that if $\psi(x, y) = x^2 + xy^2$, then $\frac{\partial \psi}{\partial x} = 2x + y^2$ and $\frac{\partial \psi}{\partial y} = 2xy$. So we can write the ODE as

$$\frac{\partial \psi}{\partial x} + \frac{\partial \psi}{\partial y}\frac{dy}{dx} = 0.$$

Since $y(x)$ is a function of x, we also have that

$$\frac{\partial}{\partial x}\Big(\psi\big(x, y(x)\big)\Big) = \frac{\partial \psi}{\partial x} + \frac{\partial \psi}{\partial y}\frac{dy}{dx}$$

by the Chain Rule. So our ODE can be written as

$$\frac{\partial}{\partial x}\left(x^2 + xy^2\right) = 0.$$

Therefore

$$x^2 + xy^2 = c.$$

Remark. The key step was finding $\psi(x, y)$.

Now consider

{eq:exact}
$$M(x,y) + N(x,y)y' = 0. \tag{2.10}$$

Definition. If we can find a function $\psi(x,y)$ such that

$$\frac{\partial \psi}{\partial x} = M \qquad \text{and} \qquad \frac{\partial \psi}{\partial y} = N,$$

then (2.10) is called an ***exact equation***.

If (2.10) is exact, then

$$0 = M(x,y) + N(x,y)y' = \frac{\partial \psi}{\partial x}(x,y) + \frac{\partial \psi}{\partial y}(x,y)\frac{dy}{dx} = \frac{d}{dx}\Big(\psi\big(x, y(x)\big)\Big)$$

which has solution

$$\psi(x,y) = c.$$

Notation.

$$y' = \frac{dy}{dx} \qquad\qquad f_x = \frac{\partial f}{\partial x} \qquad\qquad f_y = \frac{\partial f}{\partial y}$$

{thm:exact}

Theorem 2.3. *Suppose that M, N, M_y and N_x are continuous on the rectangular region $R = \{(x,y)\ :\ \alpha < x < \beta,\ \gamma < y < \delta\}$. Then*

$$M + Ny' = 0 \text{ is exact} \qquad \Longleftrightarrow \qquad M_y = N_x.$$

Example 2.15. Consider

$$(y\cos x + 2xe^y) + (\sin x + x^2e^y - 1)y' = 0.$$

Is this ODE exact? If yes, solve it.

$$\begin{aligned} M &= & M_y &= \cos x + 2xe^y \\ N &= & N_x &= \cos x + 2xe^y \end{aligned}$$

Yes, the ODE is exact. So $\exists\ \psi$ such that

$$\begin{aligned} \psi_x &= y\cos x + 2xe^y \\ \psi_y &= \sin x + x^2e^y - 1. \end{aligned}$$

Integrating the first equation (wrt x) gives

$$\psi = \int \psi_x\,dx = y\sin x + x^2e^y + h(y).$$

Then differentiating (wrt y) gives

$$\psi_y = \sin x + x^2e^y + h'(y).$$

But we already know that $\psi_y = \sin x + x^2e^y - 1$. So $h'(y) = -1$ and $h(y) = -y$. So

$$\psi(x,y) = \sin x + x^2e^y - y.$$

The solution to the ODE is

$$\boxed{\sin x + x^2e^y - y = c.}$$

Example 2.16. Consider
$$ye^{xy} + e^{xy}y' = 0.$$
Is this ODE exact? If yes, solve it.

Not exact.

Example 2.17. Consider
$$\left(4x^3y^3 + \frac{1}{x}\right) + \left(3x^4y^2 + \frac{1}{y}\right)y' = 0.$$
Is this ODE exact? If yes, solve it.

$$\boxed{x^4y^3 + \ln}$$

Example 2.18. Consider
$$(3xy + y^2) + (x^2 + xy)y' = 0.$$
Is this ODE exact? If yes, solve it.

First note that
$$\begin{aligned} M &= 3xy + y^2 & M_y &= 3x + 2y \\ N &= x^2 + xy & N_x &= 2x + y \neq M_y \end{aligned}$$
Since $M_y \neq N_x$, this ODE is not exact. So our method ***will not work***. But we are going to try our method anyway, so that we can see what goes wrong.

Suppose that $\exists\ \psi(x, y)$ such that
$$\begin{aligned} \psi_x &= 3xy + y^2 \\ \psi_y &= x^2 + xy. \end{aligned}$$
Integrating ψ_x with respect to x gives
$$\psi = \frac{3}{2}x^2yxy^2 + h(y).$$
Thus
$$x^2 + xy = \psi_y = \frac{\partial}{\partial y}\left(\frac{3}{2}x^2yxy^2 + h(y)\right) = \frac{3}{2}x^2 + 2xy + h'(y).$$
So we need h to satisfy
$$h'(y) = -\frac{1}{2}x^2 - xy.$$
This is not possible!!! $h(y)$ must be a function of y, but $-\frac{1}{2}x^2 - xy$ depends on both x and y. So it is not possible to find h. So it is not possible to find ψ. Our method failed because $M_y \neq N_x$.

Integrating Factors

It is sometimes possible to convert a differential equation which is not exact into an exact equation by multiplying it by an integrating factor. (Do you remember how we solve linear equations?)

Consider
$$M(x, y)\,dx + N(x, y)\,dy = 0. \tag{2.11}$$ {eq:notexact}

Suppose that (2.11) is not exact. If we multiply by $\mu(x, y)$, we obtain
$$\mu(x, y)M(x, y)\,dx + \mu(x, y)N(x, y)\,dy = 0. \tag{2.12}$$ {eq:notexact2}

By Theorem 2.3, we know that

$$(2.12) \text{ is exact} \qquad \Longleftrightarrow \qquad (\mu M)_y = (\mu N)_x.$$

Now

$$\begin{aligned} (\mu M)_y &= (\mu N)_x \\ \mu_y M + \mu M_y &= \mu_x N + \mu N_x \end{aligned}$$

{eq:exactif}

$$M\mu_y - N\mu_x + (M_y - N_x)\mu = 0. \tag{2.13}$$

If we can find $\mu(x, y)$ which solves (2.13), then (2.12) is exact and we know how to solve exact equations.

But (2.13) is a first order partial differential equation and PDEs are typically not easy to solve. How can we make this easier? Instead of $\mu(x, y)$, we could look for $\mu(x)$. Then $\mu_y = 0$ and (2.13) becomes

$$\begin{aligned} 0 - N\frac{d\mu}{dx} + (M_y - N_x)\mu &= 0 \\ N\frac{d\mu}{dx} &= (M_y - N_x)\mu \end{aligned}$$

{eq:exactifx}

$$\boxed{\frac{d\mu}{dx} = \left(\frac{M_y - N_x}{N}\right)\mu.} \tag{2.14}$$

If $\frac{M_y - N_x}{N}$ is a function only of x, then there is an integrating factor $\mu(x)$. Please note that (2.14) is both linear and separable.

If instead we looked for $\mu(y)$, we would obtain the ODE

{eq:exactify}

$$\boxed{\frac{d\mu}{dy} = \left(\frac{N_x - M_y}{M}\right)\mu.} \tag{2.15}$$

Remark. You are expected to remember (2.14) and (2.15).

Example 2.19. Solve

$$(3xy + y^2) + (x^2 + xy)y' = 0.$$

We know that this equation is not exact. So we will try to find an integrating factor: We have that

$$\begin{aligned} M &= 3xy + y^2 & M_y &= 3x + 2y \\ N &= x^2 + xy & N_x &= 2x + y \neq M_y \end{aligned}$$

$$\frac{M_y - N_x}{N} = \frac{(3x + 2y) - (2x + y)}{x^2 + xy} = \frac{x + y}{x(x + y)} = \frac{1}{x}$$

and

$$\frac{N_x - M_y}{M} = \frac{(2x + y) - (3x + 2y)}{3xy + y^2} = \frac{-x - y}{y(3x + y)}.$$

Note that $\frac{M_y - N_x}{N}$ is a function only of x – so it is possible to find an integrating factor $\mu(x)$. Moreover note that $\frac{N_x - M_y}{M}$ is ***not*** a function only of y – so it is ***not*** possible to find a $\mu(y)$.

We calculate that

$$\begin{aligned}
\frac{d\mu}{dx} &= \left(\frac{M_y - N_x}{N}\right)\mu \\
\frac{d\mu}{dx} &= \frac{\mu}{x} \\
\frac{d\mu}{\mu} &= \frac{dx}{x} \\
\int \frac{d\mu}{\mu} &= \int \frac{dx}{x} \\
\ln|\mu| &= \ln|x| + C \\
\mu &= cx
\end{aligned}$$

and we choose $c = 1$ for simplicity. So $\mu(x) = x$.

Multiplying our original ODE by $\mu(x) = x$ gives

$$(3x^2 y_x y^2) + (x^3 + x^2 y)y' = 0.$$

This ODE is exact ($M_y = 3x^2 + 2xy = N_x$) and we know how to solve exact equations. We must find ψ such that

$$\begin{aligned}
\psi_x &= 3x^2 y + xy^2 \\
\psi_y &= x^3 + x^2 y.
\end{aligned}$$

Integrating ψ_x wrt x gives

$$\psi = x^3 y + \frac{1}{2}x^2 y^2 + h(y).$$

Hence

$$x^3 + x^2 y = \psi_y = \frac{\partial}{\partial y}\left(x^3 y + \frac{1}{2}x^2 y^2 + h(y)\right) = x^3 + x^2 y + h'(y)$$

and we see that we may choose $h(y) = 0$. Therefore

$$\psi = x^3 y + \frac{1}{2}x^2 y^2.$$

So the solution to the ODE is

$$\boxed{x^3 y + \frac{1}{2}x^2 y^2 = c.}$$

Example 2.20. Solve

$$ye^{xy} + \left(\left(\frac{2}{y} + x\right)e^{xy}\right)y' = 0.$$

This ODE is not exact.

$$\begin{aligned}
\frac{M_y - N_x}{N} &= \frac{e^{xy} + xye^{xy} - e^{xy} - (2 + xy)\,e^{xy}}{\left(\frac{2}{y} + x\right)e^{xy}} = \frac{-2}{\frac{2}{y} + x} \\
\frac{N_x - M_y}{M} &= \frac{2e^{xy}}{ye^{xy}} = \frac{2}{y}.
\end{aligned}$$

Since $\frac{N_x - M_y}{M}$ is a function only of y, we look for $\mu(y)$.

–omitted–

Therefore $\mu(y) = y^2$.

–omitted–

Hence the solution is

$$\boxed{y^2 e^{xy} = c.}$$

Substitutions

$$\int 3x^2 \sin x^3 \; dx = \int \sin u \; du$$

$$\boxed{\text{hard ODE}} \xrightarrow{\text{substitution}} \boxed{\text{easy ODE}}$$

Homogeneous Equations

$\frac{dy}{dx} = f(x, y)$ is ***homogeneous*** iff we can write it as $\frac{dy}{dx} = g\left(\frac{y}{x}\right)$. We use the substitution $v(x) = \frac{y}{x}$ or $y = xv(x)$. Note that

$$\frac{dy}{dx} = \frac{d}{dx}\big(xv(x)\big) = v + x\frac{dv}{dx}.$$

Example 2.21.

$$\frac{dy}{dx} = \frac{y - 4x}{x - y}$$

$$x\frac{dv}{dx} = \frac{v^2 - 4}{1 - v}$$

separable

$$\left(-\frac{3}{4(v+2)} - \frac{1}{4(v-2)}\right) dv = \frac{dx}{x}$$

$$\boxed{|y + 2x|^3 \, |y - 2x| = c}$$

Example 2.22.

$$\frac{dy}{dx} = \frac{x^2 + 3y^2}{2xy}$$

$$\boxed{x^2 + y^2 - cx^3 = 0}$$

Bernoulli Equations

An equation of the form

$$y' + p(t)y = q(t)y^n$$

is called a ***Bernoulli equation***. We use the substitution $v(x) = y^{1-n}$.

Example 2.23.

$$\frac{dy}{dx} - \left(\frac{3}{2x}\right) y = 2xy^{-1}$$

$n = -1$ $v = y^2$ $y = v^{\frac{1}{2}}$

$$\frac{dy}{dx} = \frac{dy}{dv}\frac{dv}{dx} = \frac{1}{2}v^{-\frac{1}{2}}\frac{dv}{dx}$$

$$\frac{1}{2}v^{-\frac{1}{2}}\frac{dv}{dx} - \left(\frac{3}{2x}\right) v^{\frac{1}{2}} = 2xv^{-\frac{1}{2}}$$

multiply by $2v^{\frac{1}{2}}$

$$\frac{dv}{dx} - \frac{3}{x}v = 4x$$

linear equation. Use $\mu(x) = x^{-3}$

$$\boxed{y^2 = -4x^2 + Cx^3}$$

Example 2.24.

$$x\frac{dy}{dx} + 6y = 3xy^{\frac{4}{3}}$$

$v = y^{-\frac{1}{3}}$ $y = v^{-3}$

$$\frac{dv}{dx} - \frac{2}{x}v = -1$$

$\mu(x) = x^{-2}$.

$$\boxed{y = \frac{1}{(x + Cx^2)^3}}$$

3

Second Order Linear Differential Equations

In this chapter we will consider equations of the form

$$y'' + p(t)y' + q(t)y = g(t)$$

or

$$P(t)y'' + Q(t)y' + R(t)y = G(t).$$

Such equations are ***linear*** second order ODEs.

If $g(t)$ (or $G(t)$) is always zero, then the ODE is called ***homogeneous***. Otherwise it is ***nonhomogeneous***.

3.1 Homogeneous Equations with Constant Coefficients

First we will consider the equation

$$ay'' + by' + cy = 0 \tag{3.1}$$ {eq:constanthomo

where $a, b, c \in \mathbb{R}$ are constants.

Example 3.1. Solve $y'' - y = 0$.

We want to find a function $y(t)$ which satisfies

$$\frac{d^2y}{dt^2} = y.$$

- What about e^t? Yes!
- What about e^{-t}? Yes!
- And what about $c_1e^t + c_2e^{-t}$? Yes! In fact, this is the general solution to $y'' - y = 0$.

Example 3.2. Solve

$$\begin{cases} y'' - y = 0 \\ y(0) = 2 \\ y'(0) = -1. \end{cases}$$

First note that this IVP has one 2^{nd} order ODE and two initial conditions.

We know that $y(t) = c_1 e^t + c_2 e^{-t}$. We are looking for the solution which passes through the point $(0, 2)$ and has slope -1 at this point. Using the first initial condition we get that

$$2 = y(0) = c_1 + c_2 \qquad \Longrightarrow \qquad c_1 + c_2 = 2.$$

Next we need to differentiate $y(t)$:

$$y'(t) = \frac{d}{dt}\left(c_1 e^t + c_2 e^{-t}\right) = c_1 e^t - c_2 e^{-t}.$$

Thus

$$-1 = y'(0) = c_1 - c_2 \qquad \Longrightarrow \qquad c_1 - c_2 = -1.$$

To satisfy these two conditions we must have $c_1 = \frac{1}{2}$ and $c_2 = \frac{3}{2}$. Therefore the solution to the IVP is

$$\boxed{y(t) = \frac{1}{2}e^t + \frac{3}{2}e^{-t}.}$$

Now let's go back to

$$ay'' + by' + cy = 0. \tag{3.1}$$

In the previous example, we used exponential functions in our solution. Maybe we always want exponential solutions? We guess that $y(t) = e^{rt}$ might be the solution to (3.1) for some number r that we don't know yet.

Then we calculate that

$$\begin{aligned} y &= e^{rt} \\ y' &= re^{rt} \\ y'' &= r^2 e^{rt} \end{aligned}$$

and

$$0 = ay'' + by' + cy = (ar^2 + br_c)e^{rt}.$$

Since $e^{rt} \neq 0$ for all t, we must have that

:characteristic}

$$ar^2 + br + c = 0 \tag{3.2}$$

Definition. (3.2) is called the ***characteristic equation*** of (3.1).

Theorem 3.1.

$$e^{rt} \textit{ solves } (3.1) \qquad \Longleftrightarrow \qquad r \textit{ solves } (3.2).$$

(3.2) has two roots, r_1 and r_2.

$$r_{1,2} = \frac{= b \pm \sqrt{b^2 - 4ac}}{2a}.$$

These roots might be

(i). real numbers and different ($r_1, r_2 \in \mathbb{R}$ and $r_1 \neq r_2$);

(ii). complex conjugates ($r_1, r_2 \in \mathbb{C} \setminus \mathbb{R}$, $\overline{r}_1 = r_2$); or

(iii). real numbers but repeated ($r_1, r_2 \in \mathbb{R}$ and $r_1 = r_2$).

We will study these three cases separately. First we study case (i).

Suppose that $r_1, r_2 \in \mathbb{R}$ and $r_1 \neq r_2$. In other words, suppose that $b^2 - 4ac > 0$. Then $y_1(t) = e^{r_1 t}$ and $y_2(t) = e^{r_2 t}$ are both solutions to (3.1). So

$$y(t) = c_1 y_1(t) + c_2 y_2(t) = c_1 e^{r_1 t} + c_2 e^{r_2 t}$$

will also be a solution for any constants $c_1, c_2 \in \mathbb{R}$. This is called the ***general solution*** to (3.1).

Example 3.3. Solve $y'' + 5y' + 6y = 0$.

The first thing that we must do is to write down the characteristic equation. The characteristic equation for this ODE is

$$0 = r^2 + 5r + 6 = (r+2)(r+3).$$

The two roots are $r_1 = -2$ and $r_2 = -3$. Therefore the general solution is

$$\boxed{y(t) = c_1 e^{-2t} + c_2 e^{-3t}.}$$

Example 3.4. Solve

$$\begin{cases} y'' + 5y' + 6y = 0 \\ y(0) = 2 \\ y'(0) = 3. \end{cases}$$

We already found that $y(t) = c_1 e^{-2t} + c_2 e^{-3t}$ is the general solution to the ODE. We just need to find c_1 and c_2. Since $y'(t) = -2c_1 e^{-2t} - 3c_2 e^{-3t}$ we have that

$$2 = y(0) = c_1 + c_2 \qquad \Longrightarrow \qquad c_1 = 2 - c_2$$

and

$$\begin{aligned} 3 = y'(0) = -2c_1 - 3c_2 = -2(2 - c_2) - 3c_2 = -4 - c_2 \qquad &\Longrightarrow \qquad c_2 = -7 \\ &\Longrightarrow \qquad c_1 = 9. \end{aligned}$$

Therefore the solution to the IVP is

$$\boxed{y(t) = 9e^{-2t} - 7e^{-3t}.}$$

Example 3.5. Solve

$$\begin{cases} 4y'' - 8y' + 3y = 0 \\ y(0) = 2 \\ y'(0) = \frac{1}{2}. \end{cases}$$

Since the characteristic equation

$$4r^2 - 8r + 3 = 0$$

has roots,

$$r_{1,2} = \frac{-b + \pm\sqrt{b" - 4ac}}{2a} = \frac{8 \pm \sqrt{64 - 48}}{8} = 1 \pm \frac{1}{2} = \frac{3}{2} \text{ or } \frac{1}{2},$$

it follows that the general solution to the ODE is

$$y(t) = c_1 e^{\frac{3t}{2}} + c_2 e^{\frac{t}{2}}.$$

Using the initial conditions, we calculate that

$$\begin{aligned} 2 &= y(0) = c_1 + c_2 \\ \frac{1}{2} &= y'(0) = \frac{3}{2}c_1 + \frac{1}{2}c_2 \end{aligned} \qquad \Longrightarrow \qquad c_1 = -\frac{1}{2} \text{ and } c_2 = \frac{5}{2}.$$

Therefore the solution to the IVP is

$$\boxed{y = -\tfrac{1}{2}e^{\frac{3t}{2}} + \tfrac{5}{2}e^{\frac{t}{2}}.}$$

Summary. To solve

$$ay'' + by' + cy = 0$$

we need to find two linearly independent solutions.

(i). If $r_1, r_2 \in \mathbb{R}$ and $r_1 \neq r_2$, then

$$y_1(t) = e^{r_1 t} \qquad \text{and} \qquad y_2(t) = e^{r_2 t};$$

(ii). If the roots are complex numbers, then ?????????????

(iii). If the roots are repeated, then ?????????????

3.2 Fundamental Sets of Solutions

$$y'' + p(t)y' + q(t)y = 0$$

Definition. L

Theorem 3.2. *If y_1 and y_2 are both solutions of $L[y] = 0$, then $c_1y_1 + c_2y_2$ is also a solution to $L[y] = 0$ for all constants c_1, c_2.*

Proof. TO DO □

Definition. The ***Wronskian*** of $y_1(t)$ and $y_2(t)$ is

$$W = W(y_1, y_2)(t) = \begin{vmatrix} y_1(t) & y_2(t) \\ y_1'(t) & y_2'(t) \end{vmatrix}$$

Jósef Maria Hoëné-Wronski
POL, 1776-1853

Theorem 3.3. *Suppose that*

- *y_1 and y_2 both solve $L[y] = 0$; and*
- *$\exists\, t$ s.t. $W(t) \neq 0$.*

Then $\{c_1y_1 + c_2y_2 \,:\, c_1, c_2 \in \mathbb{R}\}$ contains every solution of $L[y] = 0$.

Definition. general solution

Definition. form a fundamental set of solutions

Example 3.6. Show that $y_1 = t^{\frac{1}{2}}$ and $y_2 = t^{-1}$ form a fundamental set of solutions to

$$2t^2y'' + 3ty' - y = 0.$$

3.3 Complex Roots of the Characteristic Equation

Now consider

$$ay'' + by' + cy = 0 \tag{3.1}$$

where $b^2 - 4ac < 0$. The two roots of the characteristic equation are complex conjugates. We denote them by

$$r_1 = \lambda + i\mu \qquad \text{and} \qquad r_2 = \lambda - i\mu$$

where $\lambda, \mu \in \mathbb{R}$. The corresponding solutions are

$$y_1(t) = e^{(\lambda+i\mu)t} \qquad \text{and} \qquad y_2(t) = e^{(\lambda-i\mu)t}.$$

But what does e to the power of a complex number mean?

Definition.

$$e^{(\lambda+i\mu)t} = e^{\lambda t}\cos\mu t + ie^{\lambda t}\sin\mu t.$$

Remark. Please note that

$$\begin{aligned}
\frac{d}{dt}\left(e^{r_1 t}\right) &= \frac{d}{dt}\left(e^{\lambda t}\cos\mu t + ie^{\lambda t}\sin\mu t\right)\\
&= \lambda e^{\lambda t}\cos\mu t - \mu e^{\lambda t}\sin\mu t + i\lambda e^{\lambda t}\sin\mu t + i\mu e^{\lambda t}\cos\mu t\\
&= (\lambda + i\mu)\, e^{\lambda t}\cos\mu t + (i\lambda - \mu)\, e^{\lambda t}\sin\mu t\\
&= (\lambda + i\mu)\, e^{\lambda t}\cos\mu t + i\,(\lambda + i\mu)\, e^{\lambda t}\sin\mu t\\
&= r_1 e^{r_1 t}.
\end{aligned}$$

Real Valued Solutions

The solutions $y_1(t) = e^{(\lambda+i\mu)t}$ and $y_2(t) = e^{(\lambda-i\mu)t}$ are functions $y_1, y_2 : \mathbb{R} \to \mathbb{C}$. But we want solutions $\mathbb{R} \to \mathbb{R}$.

Consider

$$\begin{aligned}
u(t) &= \frac{1}{2}\left(y_1(t) + y_2(t)\right) = \frac{1}{2}e^{\lambda t}\left(\cos\mu t + i\sin\mu t\right) + \frac{1}{2}e^{\lambda t}\left(\cos\mu t - i\sin\mu t\right)\\
&= e^{\lambda t}\cos\mu t
\end{aligned}$$

and

$$\begin{aligned}
v(t) &= \frac{1}{2i}\left(y_1(t) - y_2(t)\right) = \frac{1}{2}e^{\lambda t}\left(\cos\mu t + i\sin\mu t\right) - \frac{1}{2}e^{\lambda t}\left(\cos\mu t - i\sin\mu t\right)\\
&= \frac{1}{2i}2ie^{\lambda t}\sin\mu t = e^{\lambda t}\sin\mu t.
\end{aligned}$$

Note that $u, v : \mathbb{R} \to \mathbb{R}$ both solve (3.1). But are they linearly independent?

Since

$$\begin{aligned}
W(u,v)(t) &= \begin{vmatrix} u & v \\ u' & v' \end{vmatrix} = \begin{vmatrix} e^{\lambda t}\cos\mu t & e^{\lambda t}\sin\mu t \\ \lambda e^{\lambda t}\cos\mu t - \mu e^{\lambda t}\sin\mu t & \lambda e^{\lambda t}\sin\mu t + \mu e^{\lambda t}\cos\mu t \end{vmatrix}\\
&= e^{2\lambda t}\left(\lambda\cos\mu t\sin\mu t + \mu\cos^2\mu t - \lambda\cos\mu t\sin\mu t + \mu\sin^2\mu t\right) = \mu e^{2\lambda t} \neq 0,
\end{aligned}$$

and since $\mu \neq 0$, the answer is YES. Therefore $u(t)$ and $v(t)$ form a fundamental set of solutions to (3.1). The general solution to (3.1) is therefore

$$y(t) = c_1 u(t) + c_2 v(t) = c_1 e^{\lambda t}\cos\mu t + c_2 e^{\lambda t}\sin\mu t.$$

Example 3.7. Solve $y'' + y' + y = 0$.

The characteristic equation
$$r^2 + r + 1 = 0$$
has roots
$$r = \frac{-1 \pm \sqrt{1-4}}{2} = \frac{-1 \pm \sqrt{(-1)(3)}}{2} = -\frac{1}{2} \pm i\frac{\sqrt{3}}{2}.$$
So $\lambda = -\frac{1}{2}$ and $\mu = \frac{\sqrt{3}}{2}$.

$\sqrt{-1} = i$

Therefore the general solution is
$$\boxed{y(t) = c_1 e^{-\frac{t}{2}} \cos \frac{\sqrt{3}}{2} t + c_2 e^{-\frac{t}{2}} \sin \frac{\sqrt{3}}{2} t.}$$

Example 3.8. Solve $y'' + 9y = 0$.

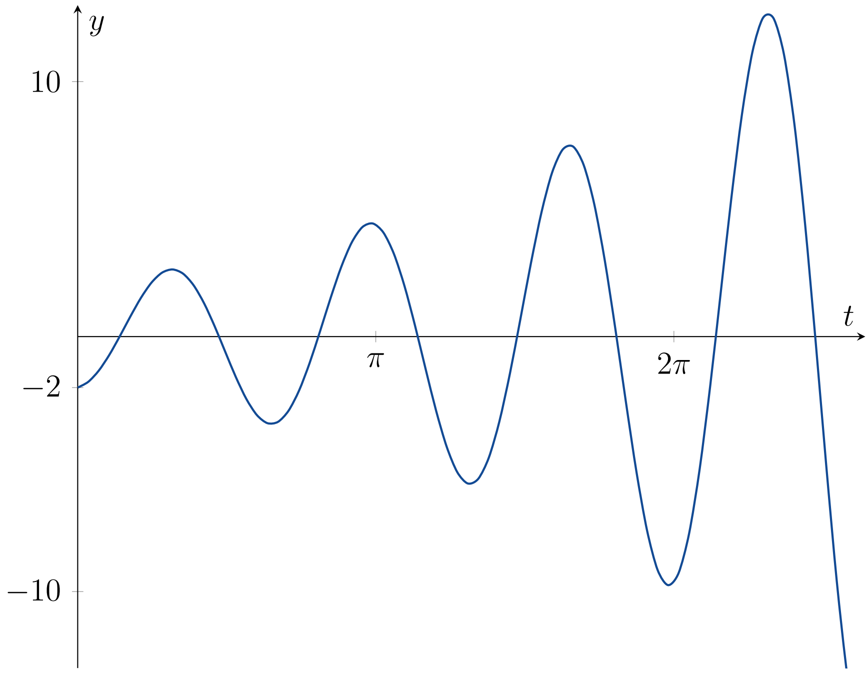
Since $0 = r^2 + 9 = (r - 3i)(r + 3i)$ we have $r = \pm 3i$ (i.e. $\lambda = 0$ and $\mu = 4$). Therefore the general solution is
$$\boxed{y(t) = c_1 \cos 3t + c_2 \sin 3t.}$$

Example 3.9. Solve
$$\begin{cases} 16y'' - 8y' + 145y = 0 \\ y(0) = -2 \\ y'(0) = 1. \end{cases}$$

The characteristic equation
$$16r^2 - 8r + 145$$
has roots
$$r = \frac{8 \pm \sqrt{64 - 4(16)(145)}}{32} = \frac{8 \pm \sqrt{(64)(1-145)}}{32} = \frac{8 \pm \sqrt{(-1)(64)(144)}}{32} = \frac{1}{4} \pm 3i.$$
Therefore the general solution to the ODE is
$$y(t) = c_1 e^{\frac{t}{4}} \cos 3t + c_2 e^{\frac{t}{4}} \sin 3t.$$

Finally we calculate that
$$y'(t) = \frac{1}{4} c_1 e^{\frac{t}{4}} \cos 3t - 3c_1 e^{\frac{t}{4}} \sin 3t + \frac{1}{4} c_2 e^{\frac{t}{4}} \sin 3t + 3c_2 e^{\frac{t}{4}} \cos 3t$$
and
$$\begin{aligned} -2 = y(0) = c_1 + 0 \qquad &\Longrightarrow \qquad c_1 = -2 \\ 1 = y'(0) = \frac{1}{4}c_1 + 3c_2 = -\frac{1}{2} + 3c_2 \qquad &\Longrightarrow \qquad c_2 = \frac{1}{2}. \end{aligned}$$

Therefore the solution to the IVP is
$$\boxed{y = -2e^{\frac{t}{4}} \cos 3t + \tfrac{1}{2} e^{\frac{t}{4}} \sin 3t.}$$



Summary. To solve

$$ay'' + by' + cy = 0$$

we need to find two linearly independent solutions.

(i). If $r_1, r_2 \in \mathbb{R}$ and $r_1 \neq r_2$, then

$$y_1(t) = e^{r_1 t} \qquad \text{and} \qquad y_2(t) = e^{r_2 t};$$

(ii). If $r_{1,2} = \lambda \pm i\mu$ $(\lambda, \mu \in \mathbb{R})$, then

$$y_1(t) = e^{\lambda t} \cos \mu t \qquad \text{and} \qquad y_2(t) = e^{\lambda t} \sin \mu t;$$

(iii). If the roots are repeated, then ?????????????

3.4 Repeated Roots

Now consider

$$ay'' + by' + cy = 0 \tag{3.1}$$

where $b^2 - 4ac = 0$. Then the only root of

$$ar^2 + br + c = 0$$

is

$$r = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = -\frac{b}{2a}.$$

We know that $y_1(t) = e^{-\frac{bt}{2a}}$ is a solution of (3.1), but how do we find a linearly independent second solution?

Example 3.10. Solve $y'' + 4' + 4y = 0$.

The characteristic equation

$$0 = r^2 + 4r + 4 = (r+2)^2$$

has repeated root $r_1 = r_2 = -2$. So one solution is $y_1(t) = e^{-2t}$. To find the general solution, we need to find a second solution.

The idea is:

- We know that $y_1(t)$ is a solution;
- So $cy_1(t)$ is a solution for any $c \in \mathbb{R}$;
- Maybe $v(t)y_1(t)$ is a solution for some non-constant function $v(t)$.

We consider $y_2(t) = v(t)y_1(t)$ for some function $v(t)$ which we don't know yet. Then we calculate that

$$\begin{aligned}
y_2 &= ve^{-2t} \\
y_2' &= v'e^{-2t} - 2ve^{-2t} \\
y_2'' &= v''e^{-2t} - 4ve^{-2t} + 4ve^{-2t}
\end{aligned}$$

and that

$$\begin{aligned}
0 &= y_2'' + 4y_2' + 4y_2 \\
&= \left(v''e^{-2t} - 4ve^{-2t} + 4ve^{-2t}\right) + 4\left(v'e^{-2t} - 2ve^{-2t}\right) + 4\left(ve^{-2t}\right) \\
&= e^{-2t}\left[v'' - 4v' + 4v + 4v' - 8v + 4v\right] \\
&= v''e^{-2t}.
\end{aligned}$$

Since $e^{-2t} \neq 0$, we must have $v'' = 0$. We can choose *any* non-constant function $v(t)$ which satisfies $v'' = 0$. I like easy functions, so I choose $v(t) = t$. Therefore

$$y_2(t) = te^{-2t}.$$

But are $y_1(t)$ and $y_2(t)$ linearly independent? Since

$$W\left(y_1, y_2\right)(t) = \begin{vmatrix} e^{-2t} & te^{-2t} \\ -2e^{-2t} & (1-2t)e^{-2t} \end{vmatrix} = e^{-4t} \neq 0,$$

the answer is YES.

Therefore $y_1(t) = e^{-2t}$ and $y_2(t) = te^{-2t}$ form a fundamental set of solutions and the general solution is

$$y(t) = c_1 e^{-2t} + c_2 te^{-2t}.$$

For the general equation $ay'' + by' + cy = 0$, we can use the same method: We have $y_1(t) = e^{rt} = e^{-\frac{bt}{2a}}$ and we guess that $y_2(t) = v(t)e^{-\frac{bt}{2a}}$ for some function $v(t)$. Then we calculate (you fill in the details)

$$0 = ay_2'' + by_2' + cy_2 = \ldots = ae^{-\frac{bt}{2a}}v''.$$

So again we want $v'' = 0$ and we can choose $v(t) = t$. Thus $y_2(t) = te^{rt} = te^{-\frac{bt}{2a}}$.

I leave it for you to calculate that $W(e^{rt}, te^{rt}) \neq 0$. Thus e^{rt} and te^{rt} form a fundamental set of solutions to (3.1).

Example 3.11. Solve

$$\begin{cases} y'' - y' + \frac{1}{4}y = 0 \\ y(0) = 2 \\ y'(0) = \frac{1}{3}. \end{cases}$$

The characteristic equation

$$0 = r^2 - r + \frac{1}{4} = (r - \frac{1}{2})^2$$

has repeated root $r = \frac{1}{2}$. So we know that the general solution to the ODE is

$$y(t) = c_1 e^{\frac{t}{2}} + c_2 te^{\frac{t}{2}}.$$

Next we need to look at the initial conditions: Since $y'(t) = \frac{1}{2}c_1 e^{\frac{t}{2}} + c_2 e^{\frac{t}{2}} + \frac{1}{2}c_2 te^{\frac{t}{2}}$, we have that

$$\begin{aligned} 2 = y(0) &= c_1 + 0 \qquad &\Longrightarrow \qquad &c_1 = 2 \\ \frac{1}{3} = y'(0) &= \frac{1}{2}c_1 + c_2 + 0 \qquad &\Longrightarrow \qquad &c_2 = -\frac{2}{3}. \end{aligned}$$

Therefore the solution to the IVP is

$$y = 2e^{\frac{t}{2}} - \tfrac{2}{3}te^{\frac{t}{2}}.$$

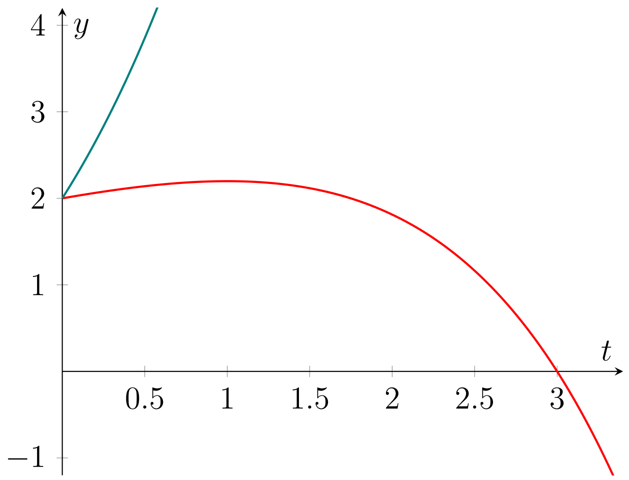
Example 3.12. Now solve

$$\begin{cases} y'' - y' + \frac{1}{4}y = 0 \\ y(0) = 2 \\ y'(0) = 2 \end{cases}$$

You can check that the solution is

$$y = 2e^{\frac{t}{2}} + te^{\frac{t}{2}}.$$

The graph of this solution, and the solution to the previous example, are shown below.



Note that even though these two functions share the same $y(0)$ value, and that their $y'(0)$ value does not differ by much, their behaviour as $t \to \infty$ is very different.

Summary. To solve

$$ay'' + by' + cy = 0$$

we need to find two linearly independent solutions.

(i). If $r_1, r_2 \in \mathbb{R}$ and $r_1 \neq r_2$, then

$$y_1(t) = e^{r_1 t} \qquad \text{and} \qquad y_2(t) = e^{r_2 t};$$

(ii). If $r_{1,2} = \lambda \pm i\mu$ $(\lambda, \mu \in \mathbb{R})$, then

$$y_1(t) = e^{\lambda t} \cos \mu t \qquad \text{and} \qquad y_2(t) = e^{\lambda t} \sin \mu t;$$

(iii). If $r_1, r_2 \in \mathbb{R}$ but $r_1 = r_2$, then

$$y_1(t) = e^{r_1 t} \qquad \text{and} \qquad y_2(t) = te^{r_1 t}.$$

Reduction of Order

Consider
$$y'' + p(t)y' + q(t)y = 0. \tag{3.6}$$
Suppose that we know that $y_1(t)$ is a solution to (3.6) and suppose that we want to find a second, linearly independent solution.

The main idea in this section is that we guess that
$$y_2(t) = v(t)y_1(t)$$
for some non-constant function $v(t)$. If we can find $v(t)$, then we have our $y_2(t)$. Then we calculate that
$$\begin{aligned} y_2 &= vy_1 \\ y_2' &= v'y_1 + vy_1' \\ y_2'' &= v''y_1 + 2v'y_1' + vy_1'' \end{aligned}$$
and
$$\begin{aligned} 0 &= y_2'' + py_2' + qy_2 \\ &= (v''y_1 + 2v'y_1' + vy_1'') + p(t)\,(v'y_1 + vy_1') + q(t)\,(vy_1) \\ &= v''y_1 + v'\,(2y_1' + py_1) + v\underbrace{(y_1'' + py_1' + qy_1)}_{=0} \\ &= v''y_1 + v'\,(2y_1' + py_1)\,. \end{aligned}$$

Remark. Note that since y_1 solves the ODE, we must always get "$0v$" here. We can have v' and v'' terms, but if you do a reduction of order calculation and still have v terms, then you have made a mistake.

Remark.

{eq:reduction}
$$v''y_1 + v'\,(2y_1' + py_1) = 0 \tag{3.3}$$
is actually a first order ODE. To see this, we can use the substitution $u = v'$ then (3.3) becomes
$$u'y_1 + u'\,(2y_1' + py_1) = 0. \tag{3.4}$$
If we can find $u(t)$, then we can find $v(t) = \int u(t)\,dt$ and $y_2(t) = v(t)y_1(t)$.

Remark. Instead of having to solve a second order ODE to find y_2, we only need to solve a first order ODE to find $u(t)$. Hence the name "Reduction of Order".

Example 3.13. Given that $y_1(t) = \frac{1}{t}$ is a solution of
$$2t^2y'' + 3ty' - y = 0, \qquad t > 0$$
find a linearly independent second solution.

Let $y_2(t) = v(t)y_1(t)$. Then we have
$$\begin{aligned} y_2 &= vt^{-1} \\ y_2' &= v't^{-1} - vt^{-2} \\ y_2'' &= v''t^{-1} - 2v't^{-2} + 2vt^{-3} \end{aligned}$$

and

$$\begin{aligned}
0 &= 2t^2 y_2'' + 3ty_2' - y_2 \\
&= 2t^2\left(v''t^{-1} - 2v't^{-2} + 2vt^{-3}\right) + 3t\left(v't^{-1} - vt^{-2}\right) - vt^{-1} \\
&= 2tv'' + (-4+3)v' + (4t^{-1} - 3t^{-1} - t^{-1})v \\
&= 2tv'' - v'.
\end{aligned}$$

Now let $u = v'$. We need to solve

$$2t\frac{du}{dt} - u = 0.$$

This equation is both linear and separable, so we know 2 ways to solve it.

$$\begin{aligned}
2t\frac{du}{dt} &= u \\
\frac{du}{u} &= \frac{1}{2}\frac{dt}{t} \\
\int \frac{du}{u} &= \int \frac{1}{2}\frac{dt}{t} \\
\ln|u| &= \frac{1}{2}\ln|t| + C \\
e^{\ln|u|} &= e^{\ln abst^{\frac{1}{2}}} e^C \\
|u| &= |t|^{\frac{1}{2}} e^C \\
u &= \pm e^C t^{\frac{1}{2}} = ct^{\frac{1}{2}}.
\end{aligned}$$

Then we have

$$v(t) = \int u(t)\,dt = \int ct^{\frac{1}{2}}\,dt = \frac{2}{3}ct^{\frac{3}{2}} + k$$

and

$$y_2(t) = v(t)t^{-1} = \frac{2}{3}ct^{\frac{1}{2}} + kt^{-1}.$$

Remember that we are trying to find a solution that is linearly independent from $y_1(t) = t^{-1}$. The second term in $y_2(t) = \frac{2}{3}ct^{\frac{1}{2}} + kt^{-1}$ is just a multiple of $y_1(t)$ – we don't need this. So it is ok to choose $k = 0$.

$$y_2(t) = \frac{2}{3}ct^{\frac{1}{2}}$$

Finally, since I like simple functions I choose $c = \frac{3}{2}$ to get

$$\boxed{y_2(t) = t^{\frac{1}{2}}.}$$

I leave it to you to check that $W(t^{-1}, t^{\frac{1}{2}})$ is not always zero.

Example 3.14. Given that $y_1(t) = t$ solves

$$t^2y'' + 2ty' - 2y = 0, \quad t > 0,$$

find a second linearly independent solution $y_2(t)$.

We start with $y_2(t) = v(t)y_1(t) = v(t)t$. Then $y_2' = v't+v$ and $y_2'' = v''t+2v'$. Substituting into the ODE, we calculate that

$$\begin{aligned} 0 &= t^2y_2'' + 2ty_2' - 2y_2 \\ &= t^2(v''t + 2v') + 2t(v't + v) - 2vt \\ &= t^3v'' + v'(2t^2 + 2t^2) + v(2t - 2t) \\ &= t^3v'' + 4t^2v' \\ &= t^2(tv'' + 4v'). \end{aligned}$$

Letting $u = v'$, we obtain the first order ODE

$$t\frac{du}{dt} + 4u = 0.$$

We calculate that

$$\begin{aligned} t\frac{du}{dt} &= -4u \\ \frac{du}{u} &= -4\frac{dt}{t} \\ \int \frac{du}{u} &= -4\int \frac{dt}{t} \\ \ln|u| &= -4\ln|t| + C \\ u &= \pm e^C t^{-4} = ct^{-4} \end{aligned}$$

and

$$\begin{aligned} v &= \int u\ dt = \int ct^{-4}\ dt \\ &= -\frac{1}{3}ct^{-3} + k. \end{aligned}$$

Thus $y_2(t) = v(t)t = -\frac{1}{3}ct^{-2} + kt$. Choosing $c = -3$ and $k = 0$, we obtain the solution

$$y_2(t) = t^{-2}.$$

Does $y_2(t) = t^{-2}$ really solve $t^2y'' + 2ty' - 2y = 0$?

Since $y_2' = -2t^{-3}$ and $y_2'' = 6t^{-4}$, we have that

$$\begin{aligned} t^2y_2'' + 2ty_2' - 2y_2 &= t^2(6t^{-4}) + 2t(-2t^{-3}) - 2t^{-2} \\ &= 6t^{-2} - 4t^{-2} - 2t^{-2} \\ &= 0. \end{aligned}$$

YES!!

Are $y_1(t) = t$ and $y_2(t) = t^{-2}$ linearly independent?

We have that

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} t & t^{-2} \\ 1 & -2t^{-3} \end{vmatrix} = -2t^{-2} - t^{-2} = -3t^{-2} \neq 0.$$

Therefore y_1 and y_2 are linearly independent.

3.5 Nonhomogeneous Equations; Method of Undetermined Coefficients

Consider

$$y'' + p(t)y' + q(t)y = g(t). \tag{3.5}$$ {eq:nonhomo}

The equation

$$y'' + p(t)y' + q(t)y = 0 \tag{3.6}$$ {eq:homo}

is called the ***homogeneous equation corresponding to*** (3.5).

Theorem 3.4. *The general solution to* (3.5) *can be written in the form*

$$y(t) = c_1 y_1(t) + c_2(t) + Y(t)$$

where

- y_1 *and* y_2 *form a fundamental set of solutions to the homogeneous equation corresponding to* (3.5)*;*
- c_1 *and* c_2 *are constants; and*
- Y *is a particular solution to* (3.5)*.*

To solve $L[y] = g$

(i). Find the general solution to $L[y] = 0$;

(ii). Find a particular solution to $L[y] = g$;

(iii). $1 + 2$

We will study 2 methods to do step 2. The first method is called...

The Method of Undetermined Coefficients

$$y'' + p(t)y' + q(t)y = g(t) \tag{3.5}$$

The idea is

(i). Look at $g(t)$

(ii). Make a guess with constants

(iii). Try to find the constants

Example 3.15. Find a particular solution to $y'' - 3y' - 4y = 3e^{2t}$.

$$Y(y) = -\frac{1}{2}e^{2t}$$

Example 3.16. Find a particular solution to $y'' - 3y' - 4y = 4t^2 - 1$.
guess: $Y(t) = At^2 + Bt + C$
you finish this example.

Example 3.17. Find a particular solution to $y'' - 3y' - 4y = 2\sin t$.
first guess: $Y(t) = A\sin t$
second guess: $Y(t) = A\sin t + B\cos t$

$$\begin{cases} -5A + 3B = 2 \\ -3A - 5B = 0 \end{cases}$$

$$\boxed{Y(t) = -\frac{5}{17}\sin t + \frac{3}{17}\cos t}$$

Remark. sin and cos. sinh and cosh.

Example 3.18. Find a particular solution to $y'' - 3y' - 4y = -8e^t\cos 2t$.
guess: $Y(t) = Ae^2\cos 2t + Be^t\sin 2t$

$$\begin{cases} 10A + 2B = 8 \\ 2A - 10B = 0 \end{cases}$$

$$\boxed{Y(t) = \frac{10}{13}e^t\cos 2t + \frac{2}{13}e^2\sin 2t}$$

Remark.

$$\left.\begin{matrix} Y_1 \text{ solves } ay'' + by' + cy = g_1(t) \\ Y_2 \text{ solves } ay'' + by' + cy = g_2(t) \end{matrix}\right. \implies Y_1 + Y_2 \text{ solves } ay'' + by' + cy = g_1(t) + g_2(t)$$

Example 3.19. Find a particular solution to

{eq:undeter5}
$$y'' - 3y' - 4y = 3e^{2t} + 2\sin t - 8e^t\cos 2t. \tag{3.7}$$

Split up into

$$\begin{aligned} y'' - 3y' - 4y &= 3e^{2t} \\ y'' - 3y' - 4y &= 2\sin t \\ y'' - 3y' - 4y &= -8e^t\cos 2t \end{aligned}$$

We know particular solutions to these three ODEs. Therefore

$$\boxed{Y(t) = -\frac{1}{2}e^2t - \frac{5}{17}\sin t + \frac{3}{17}\cos t + \frac{10}{13}e^t\cos 2t + \frac{2}{13}e^2\sin 2t}$$

is a particular solution to (3.7).

Remark. one difficulty can occur

Example 3.20. Find a particular solution to $y'' + 4y = 3\cos 2t$.
first guess: $Y(t) = A\cos 2t + B\sin 2t$
FAILURE!!!
why? solve homogeneous equation
RULE:
second guess: $Y(t) = At\cos 2t + Bt\sin 2t$
$A = 0$, $B = \frac{3}{4}$

$$\boxed{Y(t) = \frac{3}{4}t\sin 2t}$$

Example 3.21. Solve

$$\begin{cases} -y'' + 6y' - 16y = 1 + 6e^{3t}\sin(2t) \\ y(0) = \frac{15}{16} \\ y'(0) = -1 \end{cases} \tag{3.8}$$

(exam question from 2013: students had 30 minutes to solve this.)

First consider the homogeneous equation $-y''+6y'-16y=0$. The characteristic equation is $-r^2+6r-16=0$ which has roots $r = 3 \pm i\sqrt{7}$. Therefore the general solution of $-y''+6y'-16y=0$ is

$$y(t) = c_1 e^{3t}\sin(\sqrt{7}t) + c_2 e^{3t}\cos(\sqrt{7}t).$$

Next consider $-y''+6y'-16y=1$. Trying the ansatz $Y(t)=C$, we see that

$$1 = -Y'' + 6Y' - 16Y = -16C.$$

We must choose $C = -\frac{1}{16}$. Hence $Y(t) = -\frac{1}{16}$.

Now consider $-y''+6y'-16y = 6e^{3t}\sin(2t)$. We try the ansatz $Y(t) = Ae^{3t}\cos 2t + Be^{3t}\sin 2t$ and find that

$$\begin{aligned} 6e^{3t}\sin 2t &= -Y'' + 6Y' - 16Y \\ &= -e^{3t}\Big((5A+12B)\cos 2t + (5B-12A)\sin 2t\Big) \\ &\quad + 6e^{3t}\big((3A+2B)\cos 2t + (3B-2A)\sin 2t\big) \\ &\quad - 16e^{3t}\big(A\cos 2t + B\sin 2t\big) \\ &= e^{3t}\cos 2t\big(-5A - 12B + 16A + 12B - 16A\big) \\ &\quad + e^{3t}\sin 2t\big(-5B + 12A + 18B - 12A - 16B\big) \\ &= e^{3t}\cos 2t\big(-5A\big) + e^{3t}\sin 2t\big(-3B\big) \end{aligned}$$

Thus, we need $A=0$ and $B=-2$. Hence

$$Y(t) = -2e^{3t}\sin 2t$$

Next we add these 3 solutions together. Therefore, the general solution of the ODE is

$$y(t) = c_1 e^{3t}\sin(\sqrt{7}t) + c_2 e^{3t}\cos(\sqrt{7}t) - \frac{1}{16} - 2e^{3t}\sin(2t).$$

The final step is to satisfy the initial conditions.

$$\frac{15}{16} = y(0) = 0 + c_2 - \frac{1}{16} - 0 \qquad \Longrightarrow \qquad c_2 = 1.$$

$$\begin{aligned} -1 &= y'(0) \\ &= 3c_1e^{3t}\sin(\sqrt{7}t) + \sqrt{7}c_1e^{3t}\cos(\sqrt{7}t) + 3e^{3t}\cos(\sqrt{7}t) - \sqrt{7}e^{3t}\sin(\sqrt{7}t) \\ &\quad -6e^{3t}\sin(2t) - 4e^{3t}\cos(2t)\Big|_{t=0} \\ &= 0 + \sqrt{7}c_1 + 3 - 0 - 0 - 4 \qquad \Longrightarrow \qquad c_1 = 0. \end{aligned}$$

Therefore, the solution of (3.8) is

$$\boxed{y(t) = e^{3t}\cos(\sqrt{7}t) - \frac{1}{16} - 2e^{3t}\sin(2t).}$$

3.6 Variation of Parameters

Example 3.22. Find a particular solution of

{eq:variation1}
$$y'' + 4y = 3\operatorname{cosec} t \tag{3.9}$$

The homogeneous equation $y'' + 4y = 0$ has general solution $y = c_1 \cos 2t + c_2 \sin 2t$. The idea is:

(i). Replace the constants c_1 and c_2 by functions $u_1(t)$ and $u_2(t)$.

$$y = u_1(t)\cos 2t + u_2(t)\sin 2t$$

(ii). Try to find u_1 and u_2 so that y solves (3.9). There will be lots of u_1 and u_2 that we can use, so we will be free to add an extra condition.

So

$$y = u_1 \cos 2t + u_2 \sin 2t$$
$$y' = -2u_1 \sin 2t + 2u_2 \cos 2t + u_1' \cos 2t + u_2' \sin 2t$$

At this point, it is getting complicated so we will use our chance to add a condition: Suppose that

{eq:variation2}
$$u_1' \cos 2t + u_2' \sin 2t = 0 \tag{3.10}$$

So

$$y' = -2u_1 \sin 2t + 2u_2 \cos 2t$$
$$y'' = -4u_1 \cos 2t - 4u_2 \sin 2t - 2u_1' \sin 2t + 2u_2' \cos 2t$$

Then

{eq:variation3}
$$3\operatorname{cosec} t = y'' + 4y = -2u_1' \sin 2t + 2u_2' \cos 2t \tag{3.11}$$

We want to find u_1 and u_2 which satisfy (3.10) and (3.11).

$$u_2' = -u_1' \frac{\cos 2t}{\sin 2t}$$

$$u_1' = \frac{-3\operatorname{cosec} t \sin 2t}{2} = -3\cos t$$

$$u_2' = \frac{3\cos t \cos 2t}{\sin 2t} = \frac{3\cos t(1-\sin^2 t)}{2\sin t \cos t} = \frac{3}{2}\operatorname{cosec} t - 3\sin t$$

Integrating gives

$$u_1 = -3\sin t$$

$$u_2 = \frac{3}{2}\ln|\operatorname{cosec} t - \cot t| + 3\cos t$$

Therefore

$$\begin{aligned} Y(t) &= -3\sin t\cos 2t + \frac{3}{2}\ln|\operatorname{cosec} t - \cot t|\sin 2t + 3\cos t \sin 2t \\ &= 3\sin t + \frac{3}{2}\ln|\operatorname{cosec} t - \cot t|\sin 2t. \end{aligned}$$

Summary

Suppose that $c_1y_1 + c_2y_2$ is the general solution of $L[y] = 0$.

(i). Guess $Y = u_1(t)y_1 + u_2(t)y_2$;

(ii). Make the extra condition $u_1'y_1 + u_2'y_2 = 0$;

(iii). Put Y into $L[y] = g(t)$;

(iv). Find u_1' and u_2';

(v). Integrate to get u_1 and u_2;

Then Y is a particular solution to $L[y] = g(t)$.

Isn't there an easier way?

Theorem 3.5. *Suppose that $y_1(t)$ and $y_2(t)$ form a fundamental set of solutions of $y' + p(t)y' + q(t)y = 0$. Then a particular solution of $y' + p(t)y' + q(t)y = g(t)$ is given by*

$$Y(t) = -y_1 \int \frac{y_2 g}{W} + y_2 \int \frac{y_1 g}{W} \tag{3.12}$$

where $W = W(y_1, y_2)$ is the Wronskian.

Example 3.23. Solve $y'' - 2y' + y = e^t \ln t$.

$$0 = r^2 - 2r + 1 = (r-1)^2$$

$y_1 = e^t \; y_2 = te^t$

$$W = \begin{vmatrix} e^t & te^t \\ e^t e^t + te^t & \end{vmatrix} = e^{2t}$$

$$\begin{aligned} Y(t) &= -e^t \int \frac{te^t e^t \ln t}{e^{2t}} dt + te^t \int \frac{e^t e^t \ln t}{e^{2t}} dt \\ &= -e^t \int t \ln t \; dt + te^t \int \ln t \; dt \\ &= -e^t \left(\frac{1}{2} t^2 \ln t - \frac{1}{4} t^2 \right) + te^t (t \ln t - t) \\ &= \left(\frac{1}{2} \ln t - \frac{3}{4} \right) t^2 e^t. \end{aligned}$$

Therefore

$$\boxed{y(t) = c_1 e^t + c_2 te^t + \left(\frac{1}{2} \ln t - \frac{3}{4} \right) t^2 e^t.}$$

4

Higher Order Linear Equations

4.2 Homogeneous Equations with Constant Coefficients

Example 4.1. Solve

$$\begin{cases} y^{(4)} + y''' - 7y'' - y' + 6y = 0 \\ y(0) = 1 \\ y'(0) = 0 \\ y''(0) = -2 \\ y'''(0) = -1. \end{cases}$$

The characteristic equation is

$$r^4 + r^3 - 7r^2 - r + 6 = 0$$

which has roots $r_1 = 1$, $r_2 = -1$, $r_3 = 2$ and $r_4 = -3$. So the general solution to the ODE is

$$y(t) = c_1 e^t + c_2 e^{-t} + c_3 e^{2t} + c^4 e^{-3t}.$$

Then

$$\begin{aligned} 1 = y(0) &= c_1 + c_2 + c_3 + c + 4 \\ 0 = y'(0) &= c_1 - c_2 + 2c_3 - 3c_4 \\ -2 = y''(0) &= c_1 + c_2 + 4c_3 + 9c_4 \\ -1 = y'''(0) &= c_1 - c_2 + 8c_3 - 27c_4 \end{aligned} \implies \begin{aligned} c_1 &= \frac{11}{8} \\ c_2 &= \frac{5}{12} \\ c_3 &= -\frac{2}{3} \\ c_4 &= -\frac{1}{8} \end{aligned}$$

Therefore the solution to the IVP is

$$\boxed{y = \frac{11}{8}e^t + \frac{5}{12}e^{-t} - \frac{2}{3}e^{2t} - \frac{1}{8}e^{-3t}.}$$

Example 4.2. Solve

$$\begin{cases} y^{(4)} - y = 0 \\ y(0) = \frac{7}{2} \\ y'(0) = -4 \\ y''(0) = \frac{5}{2} \\ y'''(0) = -2. \end{cases}$$

The characteristic equation

$$0 = r^4 - 1 = (r^2 - 1)(r^2 + 1)$$

has roots $r_1 = 1$, $r_2 = -1$, $r_3 = i$ and $r_4 = -i$. Therefore

$$y = c_1 e^t + c_2 e^{-t} + c_3 \cos t + c_4 \sin t$$

The initial conditions give $c_1 = 0$, $c_2 = 3$, $c_3 = \frac{1}{2}$ and $c_4 = -1$. Therefore

$$\boxed{y = 3e^{-t} + \tfrac{1}{2}\cos t - \sin t}$$

Remark. comment about repeated roots.

6

The Laplace Transform

Recall that $\int_a^\infty f(t)\,dt$ means $\lim_{R\to\infty}\int_a^R f(t)\,dt$.

Example 6.1. Let $c \neq 0$. Then

$$\int_0^\infty e^{ct}\,dt = \lim_{R\to\infty}\int_0^R e^{ct}\,dt = \lim_{R\to\infty}\left[\frac{1}{c}e^{ct}\right]_0^R = \lim_{R\to\infty}\frac{1}{c}\left(e^{cR}-1\right) = \begin{cases}\infty & c>0\\ -\frac{1}{c} & c<0.\end{cases}$$

Example 6.2.

$$\int_1^\infty \frac{1}{t}\,dt = \lim_{R\to\infty}\int_1^R \frac{1}{t}\,dt = \lim_{R\to\infty}[\ln t]_1^R = \lim_{R\to\infty}(\ln R - 0) = \infty$$

Pierre-Simon Laplace
FRA, 1749-1827

6.1 Definition of the Laplace Transform

$\frac{d}{dt}$ changes a function $f(t)$ into a new function $f'(t)$.
$\mathcal{L}$ changes a function $f(t)$ into a new function $F(s)$.

Definition. Suppose that

(i) $A > 0$, $K > 0$, $M > 0$, $a \in \mathbb{R}$;

(ii) f is piecewise continuous on $[0, A]$; and

(iii) $|f(t)| \leq Ke^{at}$ for all $t \geq M$.

The ***Laplace Transform*** of $f : [0,\infty) \to \mathbb{R}$ is a new function defined by

$$F(s) = \mathcal{L}\big[f\big](s) = \int_0^\infty e^{-st} f(t)\,dt.$$

Example 6.3.

$$\mathcal{L}\big[1\big](s) = \int_0^\infty e^{-st}\,dt = -\lim_{R\to\infty}\left[\frac{e^{-st}}{s}\right]_0^R = \frac{1}{s} \qquad \text{if } x > 0.$$

Example 6.4.

$$\mathcal{L}\big[e^{at}\big](s) = \int_0^\infty e^{-st}e^{at}\,dt = \int_0^\infty e^{-(s-a)t}\,dt = \frac{1}{s-a} \qquad \text{if } s > a.$$

The Laplace Transform of $e^{at} : [0,\infty) \to \mathbb{R}$ is $\frac{1}{s-a} : (a,\infty) \to \mathbb{R}$.

Example 6.5. Let

$$f(t) = \begin{cases} 1 & 0 \le t < 1 \\ k & t = 1 \\ 0 & t > 1. \end{cases}$$

Then

$$F(s) = \mathcal{L}\big[f\big](s) = \int_0^\infty e^{-st}f(t)\,dt = \int_0^1 e^{-st}\,dt = \left[-\frac{1}{s}e^{-st}\right]_0^1 = \frac{1-e^{-s}}{s} \qquad \text{if } s > 1.$$

Example 6.6. Now consider $g(t) = \sin at$ $(t \ge 0)$. Using integration by parts, we have

$$G(s) = \mathcal{L}\big[g\big](s) = \int_0^\infty e^{-st}\sin at\,dt = \lim_{R\to\infty}\int_0^R e^{-st}\sin at\,dt$$
$$= \lim_{R\to\infty}\left(\left[-\frac{1}{a}e^{-st}\cos at\right]0^R - \frac{s}{a}\int_0^R e^{-st}\cos at\,dt\right) = \frac{1}{a} - \frac{s}{a}\int_0^\infty e^{-st}\cos at\,dt.$$

Using integration by parts a second time, we have

$$G(s) = \frac{1}{a} - \frac{s^2}{a^2}\int_0^\infty e^{-st}\sin at\,dt = \frac{1}{a} - \frac{s^2}{a^2}G(s).$$

Therefore

$$\mathcal{L}\big[\sin at\big](s) = G(s) = \frac{a}{s^2+a^2} \qquad \text{if } s > 0.$$

Theorem 6.1.

$$\mathcal{L}\big[c_1f_1 + c_2f_2\big] = c_1\mathcal{L}\big[f_1\big] + c_2\mathcal{L}\big[f_2\big].$$

You prove.

Example 6.7. If $h(t) = 5e^{-2t} - 3\sin 4t$ $(t \ge 0)$, then

$$H(s) = \mathcal{L}\big[h\big](s) = 5\mathcal{L}\big[e^{-2t}\big] - 3\mathcal{L}\big[\sin 4t\big] = \frac{5}{s+2} - \frac{12}{s^2+16} \qquad \text{if } s > 0.$$

Theorem 6.2.

$$\mathcal{L}\left[t^n f(t)\right] = (-1)^n\frac{d^nF}{ds^n}$$

You prove this. In Exercise 28(f), you are required to prove this formula with $n = 1$.

Example 6.8.

$$\mathcal{L}\left[t^2\cosh 2t\right] = (-1)^2\frac{d^2}{ds^2}\mathcal{L}\left[\cosh 2t\right] = \frac{d^2}{ds^2}\left(\frac{s}{s^2-2^2}\right) = \ldots = \frac{2s(s^2+12)}{(x^2-4)^3}$$

$f(t)$	$F(s)=\mathcal{L}\big[f\big](s)$	
1	$\frac{1}{s}$	$s>0$
e^{at}	$\frac{1}{s-a}$	$s>a$
$t^n \quad (n\in\mathbb{N})$	$\frac{n!}{s^{n+1}}$	$s>0$
$\sin at$	$\frac{a}{s^2+a^2}$	$s>0$
$\cos at$	$\frac{s}{s^2+a^2}$	$s>0$
$\sinh at$	$\frac{a}{s^2-a^2}$	$s>\lvert a\rvert$
$\cosh at$	$\frac{s}{s^2-a^2}$	$s>\lvert a\rvert$
$e^{at}\sin bt$	$\frac{b}{(s-a)^2+b^2}$	$s>a$
$e^{at}\cos bt$	$\frac{s-a}{(s-a)^2+b^2}$	$s>a$
$t^ne^{at} \quad (n\in\mathbb{N})$	$\frac{n!}{(s-a)^{n+1}}$	$s>a$
$u_c(t)$	$\frac{e^{-cs}}{s}$	$s>0$
$u_c(t)f(t-c)$	$e^{-cs}F(s)$	
$e^{ct}f(t)$	$F(s-c)$	
$f(ct) \quad (c>0)$	$\frac{1}{c}F\left(\frac{s}{c}\right)$	
$\int_0^t f(t-\tau)g(\tau)d\tau$	$F(s)G(s)$	
$t^nf(t)$	$(-1)^nF^{(n)}(s)$	

$$\mathcal{L}\big[f'\big](s)=s\mathcal{L}\big[f\big](s)-f(0)$$

$$\mathcal{L}\big[f''\big](s)=s^2\mathcal{L}\big[f\big](s)-sf(0)-f'(0)$$

$$\mathcal{L}\big[f^{(n)}\big](s)=s^n\mathcal{L}\big[f\big](s)-s^{n-1}f(0)-\ldots-sf^{(n-2)}(0)-f^{(n-1)}(0)$$

Inverse Laplace Transforms

$$\mathcal{L}\big[f\big] = F \qquad \Longleftrightarrow \qquad \mathcal{L}^{-1}\big[F\big] = f.$$

Example 6.9. Find the inverse Laplace Transform of $F(s) = \dfrac{9s^2 - 12s + 216}{s(s^2+36)}$.

Using partial fractions we calculate that

$$\begin{aligned} F(s) &= \frac{9s^2 - 12s + 216}{s(s^2+36)} = \frac{A}{s} + \frac{Bs + C}{s^2+36} = \frac{As^2 + 36A + Bs^s + Cs}{s(s^2+36)} \\ &= 6\left(\frac{1}{s}\right) + 3\left(\frac{s}{s^2+36}\right) - 12\left(\frac{1}{s^2+36}\right) \\ &= 6\left(\frac{1}{s}\right) + 3\left(\frac{s}{s^2+36}\right) - \frac{12}{6}\left(\frac{6}{s^2+36}\right) \\ &= 6\mathcal{L}\big[1\big] + 3\mathcal{L}\big[\cos 6t\big] - 2\mathcal{L}\big[\sin 6t\big]. \end{aligned}$$

$A + B = 9$
$C = -12$
$36A = 216$

$A = 6$
$B = 3$
$C = -12$

and that

$$f(t) = \mathcal{L}^{-1}\big[F\big](t) = 6 + 3\cos 6t - 2\sin 6t.$$

6.2 Solving Initial Value Problems

Theorem 6.3.

(*i*). $\mathcal{L}[f'](s) = s\mathcal{L}[f](s) - f(0)$.

(*ii*). $\mathcal{L}[f''](s) = s^2\mathcal{L}[f](s) - sf(0) - f'(0)$.

(*iii*). $\mathcal{L}[f'''](s) = s^3\mathcal{L}[f](s) - s^2 f(0) - sf'(0) - f''(0)$.

(*iv*). $\mathcal{L}[f^{(n)}](s) = s^n\mathcal{L}[f](s) - s^{n-1}f(0) - s^{n-2}f'(0) - \dots sf^{(n-2)}(0) - f^{(n-1)}(0)$.

Proof.

(i). Using integration-by-parts ($\int uv' = uv - \int u'v$) we calculate that

$$\begin{aligned} \mathcal{L}[f'](s) &= \int_0^\infty e^{-st} f'(t)\,dt = \big[e^{-st}f(t)\big]_0^\infty - \int_0^\infty \left(\frac{d}{dt}e^{-st}\right) f(t)\,dt \\ &= 0 - f(0) - \int_0^\infty -se^{-st}f(t)\,dt = -f(0) + s\int_0^\infty e^{-st}f(t)\,dt \\ &= -f(0) + sF(s) \end{aligned}$$

as required.

(ii). Using (i), but replacing each f by f' we get

$$\mathcal{L}[f''](s) = s\mathcal{L}[f'](s) - f'(0) = s\Big(s\mathcal{L}[f](s) - f(0)\Big) - f'(0) = s^2\mathcal{L}[f](s) - sf(0) - f'(0).$$

You prove (iii) and (iv) $\square$

Example 6.10. Solve

$$\begin{cases} y'' - y' - 2y = 0 \\ y(0) = 1 \\ y'(0) = 0. \end{cases}$$

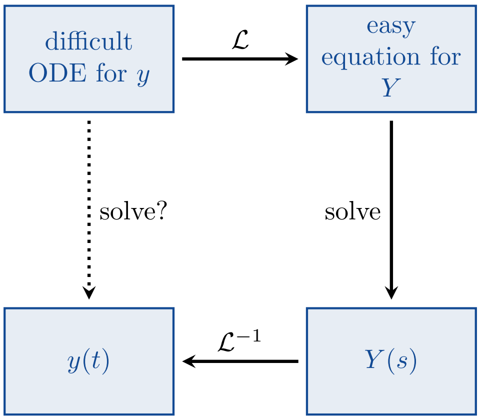
solution 1 (method from Chapter 3): The characteristic equation

$$0 = r^2 - r - 2 = (r-2)(r+1)$$

has roots $r_1 = -1$ and $r_2 = 2$. So $y = c_1 e^{-t} + c_2 e^{2t}$. Using the initial conditions we find that $c_1 = \frac{2}{3}$ and $c_2 = \frac{1}{3}$. Therefore

$$\boxed{y(t) = \frac{2}{3}e^{-t} + \frac{1}{3}e^{2t}.}$$

solution 2 (Laplace Transform):



First we take the Laplace Transform of the ODE

$$\begin{aligned} y'' - y' - 2y &= 0 \\ \mathcal{L}\big[y'' - y' - 2y\big] &= \mathcal{L}\big[0\big] \\ \mathcal{L}\big[y''\big] - \mathcal{L}\big[y'\big] - 2\mathcal{L}\big[y\big] &= 0 \\ \big(s^2Y - sy(0) - y'(0)\big) - (sY - y(0)) - 2Y &= 0 \\ \big(s^2Y - s - 0\big) - (sY - 1) - 2Y &= 0 \\ (s^2 - s - 2)Y + (1 - s) &= 0 \end{aligned}$$

Thus

$$Y(s) = \frac{s-1}{s^2 - s - 2} = \frac{s-1}{(s-2)(s+1)}.$$

Using partial fractions we obtain

$A + B = 1$
$A - 2B = -1$

$$\begin{aligned} Y(s) = \frac{s-1}{(s-2)(s+1)} &= \frac{A}{s-2} + \frac{B}{s+1} = \frac{As + A + Bs - 2B}{(s-2)(s+1)} \\ &= \frac{1}{3}\left(\frac{1}{s-2}\right) + \frac{2}{3}\left(\frac{1}{s+1}\right). \end{aligned}$$

$A = \frac{1}{3}$
$B = \frac{2}{3}$

But recall that $\mathcal{L}\big[e^{2t}\big] = \frac{1}{s-2}$ and $\mathcal{L}\big[e^{-t}\big] = \frac{1}{s+1}$. Therefore

$$y(t) = \mathcal{L}^{-1}\big[Y\big] = \frac{1}{3}\mathcal{L}^{-1}\left[\frac{1}{s-2}\right] + \frac{2}{3}\mathcal{L}^{-1}\left[\frac{1}{s+1}\right] = \boxed{\frac{1}{3}e^{2t} + \frac{2}{3}e^{-t}.}$$

Example 6.11. Solve

$$\begin{cases} y'' + y = \sin 2t \\ y(0) = 2 \\ y'(0) = 1. \end{cases}$$

$$\begin{aligned} y'' + y &= \sin 2t \\ \mathcal{L}[y''] + \mathcal{L}[y] &= \mathcal{L}[\sin 2t] \\ (s^2Y - sy(0) - y'(0)) + Y &= \frac{2}{s^2+4} \\ s^2Y - 2s - 1 + Y &= \frac{2}{s^2+4} \\ (s^2+1)Y &= 2s + 1 + \frac{2}{s^2+4} \end{aligned}$$

$$\begin{aligned} Y &= \frac{2s+1}{s^2+1} + \frac{2}{(s^2+1)(s^2+4)} = \frac{2s+1}{s^2+1} + \frac{As+B}{s^2+1} + \frac{Cs+D}{s^2+4} \\ &= \frac{2s}{s^2+1} + \frac{\frac{5}{3}}{s^2+1} - \frac{\frac{2}{3}}{s^2+4} = 2\mathcal{L}\left[\cos t\right] + \frac{5}{3}\mathcal{L}\left[\sin t\right] - \frac{1}{3}\mathcal{L}\left[\sin 2t\right] \end{aligned}$$

Therefore

$$\boxed{y(t) = 2\cos t + \frac{5}{3}\sin t - \frac{1}{3}\sin 2t.}$$

Example 6.12. Solve

$$\begin{cases} y^{(4)} - y = 0 \\ y(0) = 0 \\ y'(0) = 1 \\ y''(0) = 0 \\ y'''(0) = 0. \end{cases}$$

Using the Laplace Transform we calculate that

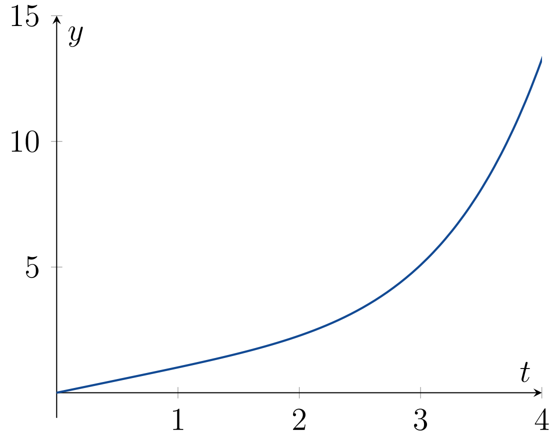
$$\begin{aligned} 0 &= \mathcal{L}\left[y^{(4)}\right] - \mathcal{L}\left[y\right] = \left(s^4Y - s^3y(0) - s^2y'(0) - sy''(0) - y'''(0)\right) - Y \\ &= s^4Y - s^2 - Y = (s^4-1)Y - s^2. \end{aligned}$$

Thus

$$Y(s) = \frac{s^2}{s^4-1} = \frac{s^2}{(s^2-1)(s^2+1)} = \frac{\frac{1}{2}}{s^2-1} + \frac{\frac{1}{2}}{s^2+1}$$

Therefore

$$y = \frac{1}{2}\mathcal{L}^{-1}\left[\frac{1}{s^2-1}\right] + \frac{1}{2}\mathcal{L}^{-1}\left[\frac{1}{s^2+1}\right] = \boxed{\frac{1}{2}\sinh t + \frac{1}{2}\sin t.}$$

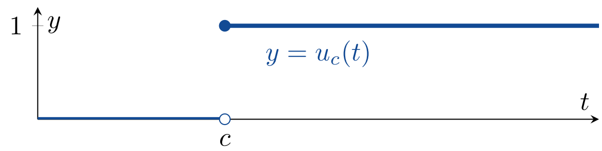

15
10
5
y
t
1
2
3
4

6.3 Step Functions

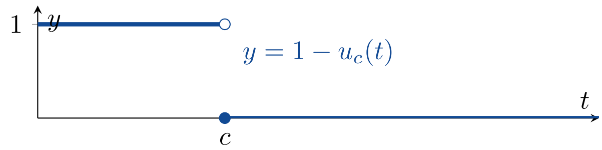
Definition. The ***unit step function*** $u_c : [0, \infty) \to \mathbb{R}$ is defined by

$$u_c(t) = \begin{cases} 0 & t < c \\ 1 & t \geq c \end{cases}$$

for $c \geq 0$.



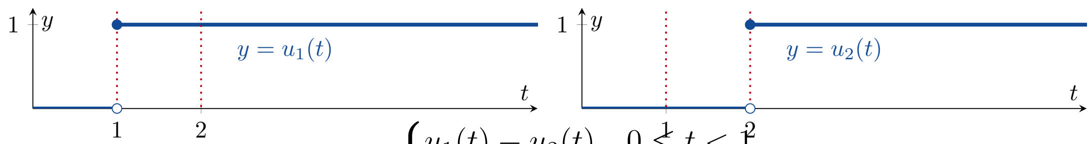
Example 6.13. Draw the graph of $y = 1 - u_c(t)$.



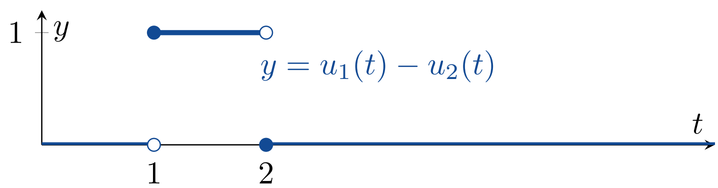
Example 6.14. Draw the graph of $y = u_1(t) - u_2(t)$.

Clearly $t = 1$ and $t = 2$ are important points. So we consider the function on the intervals $[0, 1)$, $[1, 2)$ and $[2, \infty)$.

$$u_1(t) - u_2(t) = \begin{cases} u_1(t) - u_2(t) & 0 \leq t < 1 \\ u_1(t) - u_2(t) & 1 \leq t < 2 \\ u_1(t) - u_2(t) & 2 \leq t \end{cases}$$



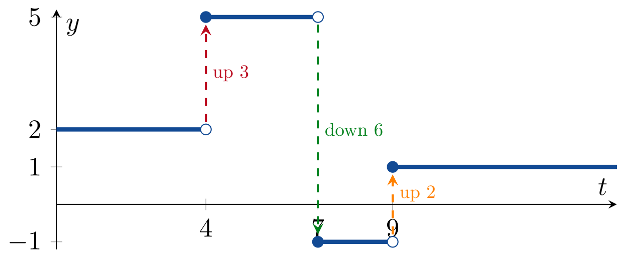
$$\begin{aligned} u_1(t) - u_2(t) &= \begin{cases} u_1(t) - u_2(t) & 0 \leq t < 1 \\ u_1(t) - u_2(t) & 1 \leq t < 2 \\ u_1(t) - u_2(t) & 2 \leq t \end{cases} \\ &= \begin{cases} 0 - 0 & 0 \leq t < 1 \\ 1 - 0 & 1 \leq t < 2 \\ 1 - 1 & 2 \leq t \end{cases} = \begin{cases} 0 & 0 \leq t < 1 \\ 1 & 1 \leq t < 2 \\ 0 & 2 \leq t. \end{cases} \end{aligned}$$



Example 6.15. Write the function

$$f(t) = \begin{cases} 2 & 0 \leq t < 4 \\ 5 & 4 \leq t < 7 \\ -1 & 7 \leq t < 9 \\ 1 & 9 \leq t \end{cases}$$

in terms of the unit step function.

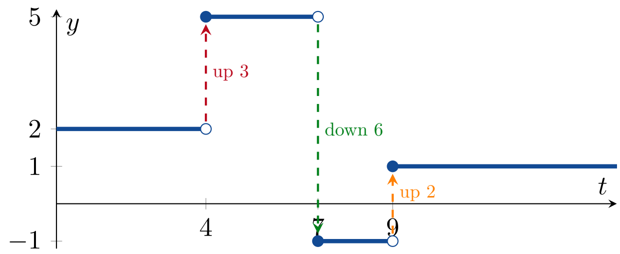


The function starts at $f(0) = 2$. So we will have

$$f(t) = 2 + (\text{something}).$$

At $t = 4$, the function jumps from 2 to 5 (it goes "up 3"). So

$$f(t) = 2 \; {\color{red}+3u_4(t)} + (\text{something}).$$



Then it goes "down 6" when $t = 7$. So

$$f(t) = 2 \; {\color{red}+3u_4(t)} \; {\color{green}-6u_7(t)} + (\text{something}).$$

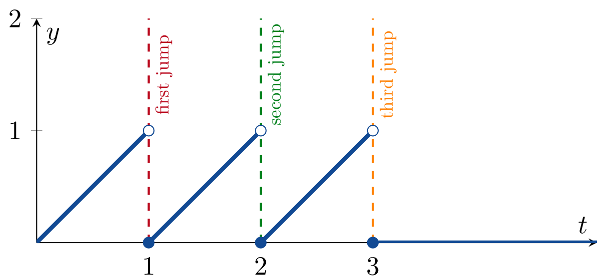
Finally it goes "up 2" when $t = 9$. Therefore

$$f(t) = 2 \; {\color{red}+3u_4(t)} \; {\color{green}-6u_7(t)} \; {\color{orange}+2u_9(t)}.$$

Example 6.16. Write the function

$$f(t) = \begin{cases} t & 0 \le t < 1 \\ t-1 & 1 \le t < 2 \\ t-2 & 2 \le t < 3 \\ 0 & 3 \le t \end{cases}$$

in terms of the unit step function.



This function starts with $f(t) = t$, then changes when $t = 1$, $t = 2$ and $t = 3$: So we must have

$$f(t) = t + \begin{pmatrix}\text{first}\\ \text{jump}\end{pmatrix} u_1(t) + \begin{pmatrix}\text{second}\\ \text{jump}\end{pmatrix} u_2(t) + \begin{pmatrix}\text{third}\\ \text{jump}\end{pmatrix} u_3(t).$$

At each "jump" we calculate

$$\text{jump} = \begin{pmatrix}\text{function}\\ \text{on right}\end{pmatrix} - \begin{pmatrix}\text{function}\\ \text{on left}\end{pmatrix}.$$

So we have

$$\begin{aligned}
\begin{pmatrix}\text{first}\\ \text{jump}\end{pmatrix} &= (t-1) - t = -1\\
\begin{pmatrix}\text{second}\\ \text{jump}\end{pmatrix} &= (t-2) - (t-1) = -1\\
\begin{pmatrix}\text{third}\\ \text{jump}\end{pmatrix} &= 0 - (t-2) = 2 - t
\end{aligned}$$

Hence

$$f(t) = t - u_1(t) - u_2(t) + (2-t)u_3(t).$$

What is the Laplace Transform of the unit step function?

We calculate that

$$\mathcal{L}\left[u_c\right](s) = \int_0^\infty e^{-st} u_c(t)\,dt = \int_c^\infty e^{-st}\,dt = \left[-\frac{1}{s}e^{-st}\right]_c^\infty = \frac{e^{-cs}}{s}$$

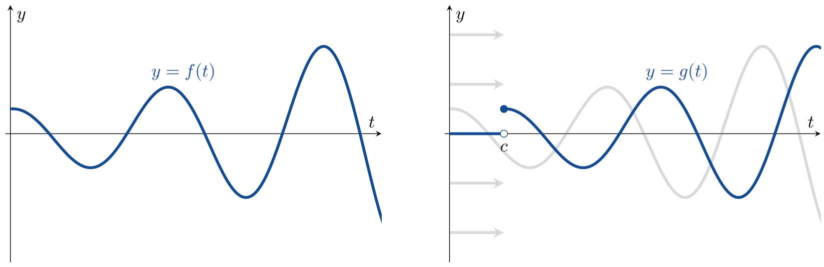
for $s > 0$.

Theorem 6.4.

$$\mathcal{L}\left[u_c\right](s) = \frac{e^{-cs}}{s}$$

Now suppose that we have some function $f : [0, \infty) \to \mathbb{R}$ and we define a new function $g : [0, \infty) \to \mathbb{R}$ by

$$g(t) = \begin{cases} 0 & t < c \\ f(t-c) & t \geq c. \end{cases}$$



We can write $g(t) = u_c(t)f(t-c)$. What is the Laplace Transform of g?

$$\begin{aligned} \mathcal{L}\left[g\right] = \mathcal{L}\left[u_c(t)f(t-c)\right] &= \int_0^\infty e^{-st} u_c(t) f(t-c)\,dt \\ &= \int_c^\infty e^{-st} f(t-c)\,dt. \end{aligned}$$

Let $u = t - c$. Then $du = dt$ and $t = c \iff u = 0$. Therefore

$$\mathcal{L}\left[g\right] = \int_0^\infty e^{-s(u+c)} f(u)\,du = e^{-cs} \int_0^\infty e^{-su} f(u)\,du = e^{-cs}\mathcal{L}\left[f\right].$$

Theorem 6.5.

$$\mathcal{L}\left[u_c(t)f(t-c)\right](s) = e^{-cs}F(s)$$

Example 6.17. Find the Laplace Transform of

$$f(t) = \begin{cases} t & 0 \leq t < 1 \\ t-1 & 1 \leq t < 2 \\ t-2 & 2 \leq t < 3 \\ 0 & 3 \leq t. \end{cases}$$

Since

$$\begin{aligned} f(t) &= t - u_1(t) - u_2(t) + (2-t)u_3(t) \\ &= t - u_1(t) - u_2(t) - u_3(t) - u_3(t)(t-3) \end{aligned}$$

we have that

$$\begin{aligned} F(s) &= \mathcal{L}\left[t\right] - \mathcal{L}\left[u_1\right] - \mathcal{L}\left[u_2\right] - \mathcal{L}\left[u_3\right] - \mathcal{L}\left[u_3(t)(t-3)\right] \\ &= \frac{1}{s^2} - \frac{e^{-s}}{s} - \frac{e^{-2s}}{s} - \frac{e^{-3s}}{s} - \frac{e^{-3s}}{s^2}. \end{aligned}$$

Example 6.18. Find the Laplace Transform of

$$f(t) = \begin{cases} \sin t & 0 \le t \le \frac{\pi}{4} \\ \sin t + \cos(t - \frac{\pi}{4}) & \frac{\pi}{4} \le t. \end{cases}$$

Note that $f(t) = \sin t + g(t)$ where

$$g(t) = \begin{cases} 0 & 0 \le t \le \frac{\pi}{4} \\ \cos(t - \frac{\pi}{4}) & \frac{\pi}{4} \le t \end{cases} = u_{\frac{\pi}{4}}(t)\cos\left(t - \frac{\pi}{4}\right).$$

So

$$\begin{aligned} F(s) &= \mathcal{L}\left[f\right] = \mathcal{L}\left[\sin t\right] + \mathcal{L}\left[u_{\frac{\pi}{4}}(t)\cos\left(t - \frac{\pi}{4}\right)\right] \\ &= \mathcal{L}\left[\sin t\right] + e^{-\frac{\pi s}{4}}\mathcal{L}\left[\cos t\right] = \frac{1}{s^2+1} + e^{-\frac{\pi s}{4}}\frac{s}{s^2+1} \\ &= \frac{1 + se^{-\frac{\pi s}{4}}}{s^2+1}. \end{aligned}$$

Example 6.19. Find the inverse Laplace Transform of $F(s) = \frac{1-e^{-2s}}{s^2}$.

$$\begin{aligned} f(t) &= \mathcal{L}^{-1}\left[F\right] = \mathcal{L}^{-1}\left[\frac{1}{s^2}\right] - \mathcal{L}^{-1}\left[\frac{e^{-2s}}{s^2}\right] = t - u_2(t)(t-2) \\ &= \begin{cases} t & 0 \le t < 2 \\ 2 & t \ge 2. \end{cases} \end{aligned}$$

And what is the Laplace Transform of $e^{ct}f(t)$?

$$\mathcal{L}\left[e^{ct}f(t)\right] = \int_0^\infty e^{-st}e^{ct}f(t)\,dt = \int_0^\infty e^{-(s-c)t}f(t)\,dt = F(s-c).$$

Theorem 6.6.

$$\mathcal{L}\left[e^{ct}f(t)\right] = F(s-c)$$

Example 6.20. Find the inverse Laplace Transform of $G(s) = \frac{1}{s^2-4s+5}$.

Note first that

$$G(s) = \frac{1}{s^2-4s+5} = \frac{1}{(s-2)^2+1}.$$

If $F(s) = \frac{1}{s^2+1}$, then we have $G(s) = F(s-2)$. But $\mathcal{L}^{-1}\left[F\right] = \mathcal{L}^{-1}\left[\frac{1}{s^2+1}\right] = \sin t$. Therefore

$$g(t) = \mathcal{L}^{-1}\left[G\right] = \mathcal{L}^{-1}\left[F(s-2)\right] = e^{2t}\mathcal{L}^{-1}\left[F\right] = e^{2t}\sin t.$$

6.4 ODEs with Discontinuous Forcing Functions

Example 6.21. Solve

$$\begin{cases} y'' + 4y = f(t) = \begin{cases} 0 & 0 \le t < 5 \\ \frac{1}{5}(t-5) & 5 \le t < 10 \\ 1 & 10 \le t \end{cases} \\ y(0) = 0 \\ y'(0) = 0. \end{cases}$$

Note that

$$\begin{aligned} f(t) &= 0 + \left(\frac{1}{5}(t-5) - 0\right) u_5(t) + \left(1 - \frac{1}{5}(t-5)\right) u_{10}(t) \\ &= \frac{1}{5}\left(u_5(t)(t-5) - u_{10}(t)(t-10)\right). \end{aligned}$$

Taking the Laplace transform of the ODE gives

$$(s^2+4)Y = \frac{1}{5}\frac{e^{-5s} - e^{-10s}}{s^2}$$

and

$$Y = \frac{1}{5}\frac{e^{-5s} - e^{-10s}}{s^2(s^2+4)}.$$

Let

$$H(s) = \frac{1}{s^2(s^2+4)}.$$

Then

$$Y(s) = \frac{1}{5}e^{-5s}H(s) - \frac{1}{5}e^{-10s}H(s).$$

Recall that

$$\mathcal{L}\left[u_c(t)h(t-c)\right](s) = e^{-cs}H(s).$$

So

$$u_c(t)h(t-c)(s) = \mathcal{L}^{-1}\left[e^{-cs}H(s)\right].$$

If we can find $h(t)$, then we can find $y(t)$.

Using partial fractions, we calculate (please check!) that

$$\begin{aligned} H(s) &= \frac{1}{s^2(s^2+4)} = \frac{As+B}{s^2} + \frac{Cs+D}{s^2+4} \\ &= \frac{As^3 + Bs^2 + 4As + 4B + Cs^3 + Ds^2}{s^2(s^2+4)} \\ &= \frac{0s + \frac{1}{4}}{s^2} + \frac{0s - \frac{1}{4}}{s^2+4} = \frac{\frac{1}{4}}{s^2} - \frac{\frac{1}{4}}{s^2+4}. \end{aligned}$$

Hence

$$h(t) = \frac{1}{4}\mathcal{L}^{-1}\left[\frac{1}{s^2}\right] - \frac{1}{8}\mathcal{L}^{-1}\left[\frac{2}{s^2+4}\right] = \frac{t}{4} - \frac{1}{8}\sin 2t.$$

Therefore

$$\begin{aligned}
y(t) &= \mathcal{L}^{-1}\left[\frac{1}{5}e^{-5s}H(s) - \frac{1}{5}e^{-10s}H(s)\right] \\
&= \frac{1}{5}u_5(t)h(t-5) - \frac{1}{5}u_{10}(t)h(t-10) \\
&= u_5(t)\left(\frac{t-5}{20} - \frac{1}{40}\sin(2t-10)\right) \\
&\qquad - u_{10}(t)\left(\frac{t-10}{20} - \frac{1}{40}\sin(2t-20)\right).
\end{aligned}$$

Example 6.22. Solve

$$\begin{cases} y'' + 3y' + 2y = f(t) = \begin{cases} 1 & 0 \le t < 10 \\ 0 & 10 \le t \end{cases} \\ y(0) = 1 \\ y'(0) = 0. \end{cases}$$

Since $f(t) = 1 - u_{10}(t)$, the Laplace Transform of the ODE is

$$(s^2 + 3s + 2)Y - (s + 3) = \frac{1 - e^{-10s}}{s}.$$

Thus

$$\begin{aligned} Y(s) &= \frac{1 - e^{-10s}}{s(s^2 + 3s + 2)} + \frac{s + 3}{s^2 + 3s + 2} \\ &= \frac{(s^2 + 3s + 1) - e^{-10s}}{s(s^2 + 3s + 2)}. \end{aligned}$$

Let

$$G(s) = \frac{s^2 + 3s + 1}{s(s^2 + 3s + 2)} \qquad \text{and} \qquad H(s) = \frac{1}{s(s^2 + 3s + 2)}.$$

Then $Y = G(s) - e^{-10s}H(s)$. If we can find $g(t)$ and $h(t)$, then we can find $y(t)$.

Using partial fractions we get

$$G(s) = \frac{A}{s} + \frac{B}{s+1} + \frac{C}{s+2} = \frac{\frac{1}{2}}{s} + \frac{1}{s+1} - \frac{\frac{1}{2}}{s+2}$$

and

$$H(s) = \frac{D}{s} + \frac{E}{s+1} + \frac{F}{s+2} = \frac{\frac{1}{2}}{s} - \frac{1}{s+1} + \frac{\frac{1}{2}}{s+2}$$

(please check!). It follows that

$$g(t) = \frac{1}{2}\left(1 + 2e^{-t} - e^{-2t}\right) \qquad \text{and} \qquad h(t) = \frac{1}{2}\left(1 - 2e^{-t} + e^{-2t}\right).$$

Therefore

$$\begin{aligned} y(t) &= \mathcal{L}^{-1}\left[Y\right] \\ &= \mathcal{L}^{-1}\left[G(s) - e^{-10s}H(s)\right] \\ &= g(t) - u_{10}(t)h(t - 10) \\ &= \frac{1}{2}\left(1 + 2e^{-t} - e^{-2t}\right) - \frac{1}{2}u_{10}(t)\left(1 - 2e^{-(t-10)} + e^{-2(t-10)}\right). \end{aligned}$$

Example 6.23. Solve

$$\begin{cases} y'' + 4y = u_\pi(t) - u_{3\pi}(t) \\ y(0) = 0 \\ y'(0) = 0. \end{cases}$$

Taking the Laplace Transform of the ODE gives

$$(s^2+4)Y(s) = \frac{e^{-\pi s} - e^{-3\pi s}}{s}.$$

Thus

$$Y(s) = \frac{e^{-\pi s} - e^{-3\pi s}}{s(s^2+4)}.$$

Let

$$H(s) = \frac{1}{s(s^2+4)}.$$

Using partial fractions, we calculate that

$$\begin{aligned} H(s) = \frac{1}{s(s^2+4)} &= \frac{A}{s} + \frac{Bs+C}{s^2+4} = \frac{\frac{1}{4}}{s} + \frac{-\frac{1}{4}s+0}{s^2+4} \\ &= \frac{1}{4}\left(\frac{1}{s}\right) - \frac{1}{4}\left(\frac{s}{s^2+4}\right) = \frac{1}{4}\mathcal{L}\left[1\right] - \frac{1}{4}\mathcal{L}\left[\cos 2t\right]. \end{aligned}$$

It follows that

$$h(t) = \frac{1}{4} - \frac{1}{4}\cos 2t$$

and the solution to the IVP is

$$\begin{aligned} y(t) &= \mathcal{L}^{-1}\left[e^{-\pi s}H(s)\right] - \mathcal{L}^{-1}\left[e^{-3\pi s}H(s)\right] \\ &= u_\pi(t)h(t-\pi) - u_{3\pi}(t)h(t-3\pi) \\ &= \frac{1}{4}u_\pi(t)\big(1-\cos(2t-2\pi)\big) - \frac{1}{4}u_{3\pi}(t)\big(1-\cos(2t-6\pi)\big). \end{aligned}$$

6.6 The Convolution Integral

Let $f : [0,\infty) \to \mathbb{R}$ and $g : [0,\infty) \to \mathbb{R}$ be piecewise continuous functions.

Definition. The ***convolution*** of f and g is

$$\big(f * g\big)(t) = \int_0^t f(\tau)g(t-\tau)\,d\tau.$$

Theorem 6.7 (Properties)**.**

- $f * g = g * f$
- $f * (g * h) = (f * g) * h$
- $f * (g + h) = (f * g) + (f * h)$
- $f * 0 = 0 = 0 * f$

Example 6.24.

$$\big(\cos * 1\big)(t) = \int_0^t \cos\tau \cdot 1\,d\tau = [\sin\tau]_0^t = \sin t - \sin 0 = \sin t$$
$$\big(1 * \cos\big)(t) = \int_0^t 1 \cdot \cos(t-\tau)\,d\tau = [-\sin(t-\tau)]_0^t = -\sin 0 + \sin t = \sin t$$

Note that $f * 1 \neq f$ in general.

Example 6.25.

$$\begin{aligned}
\big(\sin * \sin\big)(t) &= \int_0^t \sin\tau\sin(t-\tau)\,d\tau = \int_0^t \sin\tau\left(\sin t\cos\tau - \cos t\sin\tau\right)\,d\tau \\
&= \sin t\int_0^t \sin\tau\cos\tau\,d\tau - \cos t\int_0^t \sin^2\tau\,d\tau \\
&= \sin t\left[-\frac{1}{2}\cos^2\tau\right]_0^t - \cos t\left[\frac{1}{2}(\tau - \sin\tau\cos\tau)\right]_0^t \\
&= \frac{1}{2}\sin t\left(1-\cos^2 t\right) - \frac{1}{2}\cos t\left(t - \sin t\cos t\right) \\
&= \frac{1}{2}\sin t - \frac{t}{2}\cos t.
\end{aligned}$$

Note that $f * f \geq 0$ is ***not*** true in general.

Theorem 6.8.

$$\mathcal{L}\left[f * g\right](s) = F(s)G(s)$$

This means that $\mathcal{L}^{-1}\left[FG\right] = f * g$.

Example 6.26. Find the inverse Laplace Transform of $H(s) = \dfrac{a}{s^2(s^2+a^2)}$.

Note that $H(s) = \left(\frac{1}{s^2}\right)\left(\frac{a}{s^2+a^2}\right)$. We know that $\mathcal{L}\left[t\right] = \frac{1}{s^2}$ and $\mathcal{L}\left[\sin at\right] = \frac{a}{s^2+a^2}$. So

$$\begin{aligned} h(t) &= \mathcal{L}^{-1}\left[\left(\frac{1}{s^2}\right)\left(\frac{a}{s^2+a^2}\right)\right] = \mathcal{L}^{-1}\left[\frac{1}{s^2}\right] * \mathcal{L}^{-1}\left[\frac{a}{s^2+a^2}\right] \\ &= t * \sin at = \int_0^t \tau \sin a(t-\tau)\,d\tau \\ &= \frac{at - \sin at}{a^2}. \end{aligned}$$

Example 6.27. Solve

$$\begin{cases} y'' + 4y = g(t) \\ y(0) = 3 \\ y'(0) = -1. \end{cases}$$

Taking the Laplace Transform of the ODE gives

$$(s^2Y - 3s + 1) + 4Y = G(s)$$

which rearranges to

$$\begin{aligned} Y(s) &= \frac{3s-1}{s^2+4} + \frac{G(s)}{s^2+4} \\ &= 3\left(\frac{s}{s^2+4}\right) - \frac{1}{2}\left(\frac{2}{s^2+4}\right) + \frac{1}{2}\left(\frac{2}{s^2+4}\right)G(s). \end{aligned}$$

Hence the solution to the IVP is

$$\begin{aligned} y(t) &= 3\mathcal{L}^{-1}\left[\frac{s}{s^2+4}\right] - \frac{1}{2}\mathcal{L}^{-1}\left[\frac{2}{s^2+4}\right] + \frac{1}{2}\mathcal{L}^{-1}\left[\left(\frac{2}{s^2+4}\right)G(s)\right] \\ &= 3\cos 2t - \frac{1}{2}\sin 2t + \frac{1}{2}\sin 2t * g(t) \\ &= 3\cos 2t - \frac{1}{2}\sin 2t + \frac{1}{2}\int_0^t \sin 2(t-\tau)g(\tau)\,d\tau. \end{aligned}$$

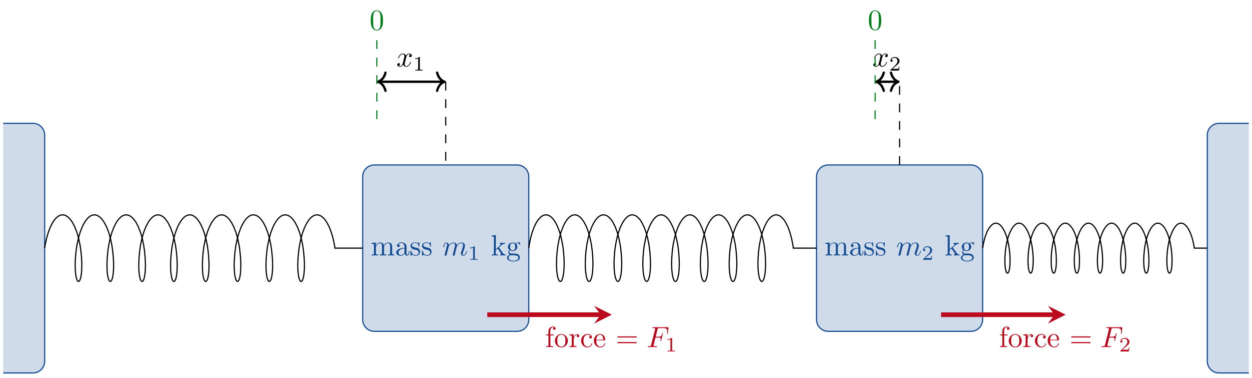
Example 6.28. Find the inverse Laplace Transform of $\dfrac{2}{(s-1)(s^2+4)}$.

$$\begin{aligned} \mathcal{L}^{-1}\left[\frac{2}{(s-1)(s^2+4)}\right] &= \mathcal{L}^{-1}\left[\left(\frac{2}{s^2+4}\right)\left(\frac{1}{s-1}\right)\right] = \sin 2t * e^t \\ &= \int_0^t e^{t-\tau}\sin 2\tau\,d\tau = e^t\int_0^t e^{-\tau}\sin 2\tau\,d\tau \\ &= e^t\left[\frac{e^{-\tau}}{5}(-\sin 2\tau - 2\cos 2\tau)\right]_0^t \\ &= \frac{2}{5}e^t - \frac{1}{5}\sin 2t - \frac{2}{5}\cos 2t. \end{aligned}$$

7

Systems of First Order Linear Equations

7.1 Introduction



Consider the dynamical system shown above. There are two blocks and three springs. Forces F_1 and F_2 act on the blocks as shown.

See https://tinyurl.com/wm2ogdh

We expect that the acceleration of the blocks will depend on

- the displacements x_1 and x_2;
- the forces F_1 and F_2; and
- the masses of the blocks.

So we expect that:

$$\begin{cases} \dfrac{d^2x_1}{dt^2} = f_1(x_1, x_2, F_1, m_1) \\ \dfrac{d^2x_2}{dt^2} = f_2(x_1, x_2, F_2, m_2). \end{cases}$$

This is a system of two ODEs. To find $x_1(t)$ and $x_2(t)$, we would need to solve these equations at the same time.

The most famous system of ODEs is the system of ***Predator-Prey*** equations:

$$\begin{cases} \dfrac{dx}{dt} = \alpha x - \beta xy \\ \dfrac{dy}{dt} = \delta xy - \gamma y \end{cases}$$

where

$$x(t) = \text{number of prey (e.g. mice)}$$
$$y(t) = \text{number of predators (e.g. owls)},$$

which originate circa 1925.

It is possible to convert an nth order linear ODE into a system of n first order linear ODEs. Or vice versa.

$$\begin{aligned} a_n y^{(n)} + a_{n-1} y^{(n-1)} + \dots \\ + a_1 y' + a_0 y = g(t) \end{aligned} \quad \longleftrightarrow \quad \begin{cases} x_1' = b_{11}x_1 + \ldots + b_{1n}x_n + h_1(t) \\ x_2' = b_{21}x_1 + \ldots + b_{2n}x_n + h_2(t) \\ \vdots \\ x_n' = b_{n1}x_1 + \ldots + b_{nn}x_n + h_n(t) \end{cases}$$

Example 7.1. Write

$$u'' + 0.25u' + u = 0$$

as a system of two first order ODEs.

Let $x_1 = u$ and $x_2 = u'$. Then clearly $x_1' = u' = x_2$ and

$$x_2' = u'' = -0.25u' - u = -0.25x_2 - x_1.$$

Therefore

$$\begin{cases} x_1' = x_2 \\ x_2' = -x_1 - 0.25x_2. \end{cases}$$

Remark. We will need

- matrices,
- eigenvalues,
- eigenvectors,
- the Wronskian,
- linear independence,
- and more

from MATH215 – please either revise your linear algebra lecture notes or read §7.2-7.3 in the textbook.

7.4 Basic Theory of Systems of First Order Linear Equations

$$\begin{cases} x_1' = p_{11}(t)x_1 + p_{12}(t)x_2 + \ldots + p_{1n}(t)x_n + g_1(t) \\ x_2' = p_{21}(t)x_1 + p_{22}(t)x_2 + \ldots + p_{2n}(t)x_n + g_2(t) \\ \quad\vdots \\ x_n' = p_{n1}(t)x_1 + p_{n2}(t)x_2 + \ldots + p_{nn}(t)x_n + g_n(t) \end{cases}$$

is a system of n linear ODEs and n variables: x_1, x_2, $\ldots$, x_n.

If we write

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, \ \mathbf{x}' = \begin{bmatrix} x_1' \\ x_2' \\ \vdots \\ x_n' \end{bmatrix}, \ P = \begin{bmatrix} p_{11} & p_{12} & \ldots & p_{1n} \\ p_{21} & p_{22} & \ldots & p_{2n} \\ \vdots & \vdots & & \vdots \\ p_{n1} & p_{n2} & \ldots & p_{nn} \end{bmatrix}, \text{ and } \mathbf{g} = \begin{bmatrix} g_1 \\ g_2 \\ \vdots \\ g_n \end{bmatrix}$$

then we can write this system as

$$\mathbf{x}' = P(t)\mathbf{x} + \mathbf{g}(t).$$

First we will consider the homogeneous system

$$\mathbf{x}' = P(t)\mathbf{x}.$$

In Chapters 3 and 4 when we had multiple solutions, we wrote them as $y_1(t)$, $y_2(t)$, $\ldots$. But we are already using x_1, x_2, $\ldots$ to denote coordinates. So we need a new type of notation.

Notation. We use $\mathbf{x}^{(1)}(t)$, $\mathbf{x}^{(2)}(t)$, $\ldots$ to denote different vector solutions.

Recall from Chapter 3 that if $y_1(t)$ and $y_2(t)$ are both solutions to

$$ay'' + by' + cy = 0,$$

then

$$c_1y_1 + c_2y_2$$

is also a solution.

Theorem 7.1. *If $\mathbf{x}^{(1)}(t)$ and $\mathbf{x}^{(2)}(t)$ are solutions to $\mathbf{x}' = P(t)\mathbf{x}$, then*

$$c_1\mathbf{x}^{(1)} + c_2\mathbf{x}^{(2)}$$

is also a solution for any $c_1, c_2 \in \mathbb{R}$.

Example 7.2. $\mathbf{x}^{(1)} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} e^{3t}$ and $\mathbf{x}^{(2)} = \begin{bmatrix} 1 \\ -2 \end{bmatrix} e^{-t}$ are both solutions to $\mathbf{x}' = \begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix} \mathbf{x}$ (we will see this later). Therefore

$$\mathbf{x}(t) = c_1\mathbf{x}^{(1)} + c_2\mathbf{x}^{(2)} = c_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} e^{3t} + c_2 \begin{bmatrix} 1 \\ -2 \end{bmatrix} e^{-t}$$

is also a solution to this system.

(Suppose that $P(t)$ is an $n \times n$ matrix.)

Theorem 7.2. *If $\mathbf{x}^{(1)}(t)$, $\mathbf{x}^{(2)}(t)$, ..., $\mathbf{x}^{(n)}(t)$ are linearly independent solutions to $\mathbf{x}' = P(t)\mathbf{x}$, then every solution to this system can be written as*

$$\mathbf{x}(t) = c_1\mathbf{x}^{(1)} + c_2\mathbf{x}^{(2)} + \ldots + c_n\mathbf{x}^{(n)}$$

in exactly one way.

Definition. In this case, we say that $\mathbf{x}^{(1)}(t)$, $\mathbf{x}^{(2)}(t)$, ..., $\mathbf{x}^{(n)}(t)$ form a ***fundamental set of solutions*** to $\mathbf{x}' = P(t)\mathbf{x}$.

Definition. In this case,

$$\mathbf{x}(t) = c_1\mathbf{x}^{(1)} + c_2\mathbf{x}^{(2)} + \ldots + c_n\mathbf{x}^{(n)}$$

is called the ***general solution*** to $\mathbf{x}' = P(t)\mathbf{x}$.

7.5 Homogeneous Linear Systems with Constant Coefficients

Consider

$$\mathbf{x}' = A\mathbf{x}$$

where $A \in \mathbb{R}^{n\times n}$.

If $n = 1$, then we just have

$$\frac{dx}{dt} = ax$$

which has general solution $x(t) = ce^{at}$.

For $n > 1$, we guess that

$$\mathbf{x}(t) = \boldsymbol{\xi} e^{rt}$$

is a solution to $\mathbf{x}' = A\mathbf{x}$, for some number $r \in \mathbb{C}$ and some vector $\boldsymbol{\xi} \in \mathbb{C}^n$.

But if $\mathbf{x}(t) = \boldsymbol{\xi} e^{rt}$, then

$$\begin{aligned} r\boldsymbol{\xi} e^{rt} &= \mathbf{x}' = A\mathbf{x} = A\boldsymbol{\xi} e^{rt} \\ r\boldsymbol{\xi} &= A\boldsymbol{\xi} \\ (A - rI)\boldsymbol{\xi} &= \mathbf{0} \end{aligned}$$

where I is the identity matrix. Hence r must be an eigenvalue of A and $\boldsymbol{\xi}$ must be a corresponding eigenvector of A.

Remark. So the idea is:

(i). Find the eigenvalues;

(ii). Find the eigenvectors; then

(iii). Write $\mathbf{x}^{(j)}(t) = \boldsymbol{\xi}^{(j)} e^{r_j t}$.

Example 7.3. Solve

$$\mathbf{x}' = \begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix} \mathbf{x}.$$

First we find the eigenvalues. Since

$$\begin{aligned} 0 = \det(A - rI) &= \begin{vmatrix} 1-r & 1 \\ 4 & 1-r \end{vmatrix} = (1-r)^2 - 4 \\ &= r^2 - 2r - 3 = (r+1)(r-3), \end{aligned}$$

the eigenvalues are $r_1 = 3$ and $r_2 = -1$.

Using the first eigenvalue $r_1 = 3$, we calculate that

$$\mathbf{0} = (A - r_1 I)\boldsymbol{\xi} = \begin{bmatrix} -2 & 1 \\ 4 & -2 \end{bmatrix} \begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix} \qquad \Longrightarrow \qquad 0 = -2\xi_1 + \xi_2.$$

Hence we can choose $\boldsymbol{\xi}^{(1)} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$. Then using the second eigenvalue $r_2 = -1$, we calculate that

$$\mathbf{0} = (A - r_2 I)\boldsymbol{\xi} = \begin{bmatrix} 2 & 1 \\ 4 & 2 \end{bmatrix} \begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix} \qquad \Longrightarrow \qquad 0 = 2\xi_1 + \xi_2.$$

Hence we can choose $\boldsymbol{\xi}^{(2)} = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$. This gives us two solutions:

$$\mathbf{x}^{(1)}(t) = \begin{bmatrix} 1 \\ 2 \end{bmatrix} e^{3t} \qquad \text{and} \qquad \mathbf{x}^{(2)}(t) = \begin{bmatrix} 1 \\ -2 \end{bmatrix} e^{-t}.$$

But are these two solutions linearly independent? To find out, we calculate the Wronskian of $\mathbf{x}^{(1)}(t)$ and $\mathbf{x}^{(2)}(t)$:

$$\mathbf{x}^{(1)}(t) = \begin{bmatrix} 1 \\ 2 \end{bmatrix} e^{3t} = \begin{bmatrix} e^{3t} \\ 2e^{3t} \end{bmatrix} \quad \text{and} \quad \mathbf{x}^{(2)}(t) = \begin{bmatrix} 1 \\ -2 \end{bmatrix} e^{-t} = \begin{bmatrix} e^{-t} \\ -2e^{-t} \end{bmatrix}.$$

$$W\left(\mathbf{x}^{(1)}, \mathbf{x}^{(2)}\right)(t) = \begin{vmatrix} e^{3t} & e^{-t} \\ 2e^{3t} & -2e^{-t} \end{vmatrix} = -4e^{2t} \neq 0.$$

Since $W \neq 0$, we have that $\mathbf{x}^{(1)}(t)$ and $\mathbf{x}^{(2)}(t)$ are linearly independent. So $\mathbf{x}^{(1)}(t)$ and $\mathbf{x}^{(2)}(t)$ form a fundamental set of solutions. Therefore the general solution is

$$\boxed{\mathbf{x}(t) = c_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} e^{3t} + c_2 \begin{bmatrix} 1 \\ -2 \end{bmatrix} e^{-t}.}$$

Example 7.4. Solve

$$\begin{cases} \mathbf{x}' = \begin{bmatrix} 8 & -1 \\ 6 & 1 \end{bmatrix} \mathbf{x} \\ \mathbf{x}(0) = \begin{bmatrix} 1 \\ -2 \end{bmatrix}. \end{cases}$$

The eigenvalues are $r_1 = 7$ and $r_2 = 2$. The corresponding eigenvectors are $\boldsymbol{\xi}^{(1)} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\boldsymbol{\xi}^{(2)} = \begin{bmatrix} 1 \\ 6 \end{bmatrix}$. Therefore the general solution to the ODE is

$$\mathbf{x}(t) = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{7t} + c_2 \begin{bmatrix} 1 \\ 6 \end{bmatrix} e^{2t}.$$

Setting $t = 0$, we have

$$\begin{bmatrix} 1 \\ -2 \end{bmatrix} = \mathbf{x}(0) = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 6 \end{bmatrix} = \begin{bmatrix} c_1 + c_2 \\ c_1 + 6c_2 \end{bmatrix} \implies \begin{cases} c_1 + c_2 = 1 \\ c_1 + 6c_2 = -2 \end{cases} \implies \begin{cases} c_1 = \frac{8}{5} \\ c_2 = -\frac{3}{5}. \end{cases}$$

Therefore the solution to the IVP is

$$\boxed{\mathbf{x}(t) = \frac{8}{5} \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{7t} - \frac{3}{5} \begin{bmatrix} 1 \\ 6 \end{bmatrix} e^{2t}.}$$

Example 7.5. Solve

$$\mathbf{x}' = \begin{bmatrix} -3 & \sqrt{2} \\ \sqrt{2} & -2 \end{bmatrix} \mathbf{x}.$$

The eigenvalues are $r_1 = -1$ and $r_2 = -4$. The corresponding eigenvectors are $\boldsymbol{\xi}^{(1)} = \begin{bmatrix} 1 \\ \sqrt{2} \end{bmatrix}$ and $\boldsymbol{\xi}^{(2)} = \begin{bmatrix} -\sqrt{2} \\ 1 \end{bmatrix}$. Hence the general solution is

$$\boxed{\mathbf{x}(t) = c_1 \begin{bmatrix} 1 \\ \sqrt{2} \end{bmatrix} e^{-t} + c_2 \begin{bmatrix} -\sqrt{2} \\ 1 \end{bmatrix} e^{-4t}.}$$

Remark.

$$\det(A - rI) = 0$$

There are three possibilities for the eigenvalues of A.

(i). All the eigenvalues are real and different;

(ii). Some eigenvalues occur in complex conjugate pairs;

(iii). Some eigenvalues are repeated.

If all the eigenvalues are real and different, then the eigenvectors are linearly independent: So $W\big(\mathbf{x}^{(1)}, \ldots, \mathbf{x}^{(n)}\big)(t) \neq 0$ and $\mathbf{x}^{(1)}(t)$, $\ldots$, $\mathbf{x}^{(n)}(t)$ form a fundamental set of solutions.

If some eigenvalues are repeated, ***but there are n linearly independent eigenvectors***, then this is also true: $\mathbf{x}^{(1)}(t)$, $\ldots$, $\mathbf{x}^{(n)}(t)$ form a fundamental set of solutions.

Example 7.6. Solve

$$\mathbf{x}' = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} \mathbf{x}.$$

The eigenvalues and eigenvectors are

$$r_1 = 2 \qquad\qquad r_2 = -1 \qquad\qquad r_3 = -1$$

$$\boldsymbol{\xi}^{(1)} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \qquad \boldsymbol{\xi}^{(2)} = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \qquad \boldsymbol{\xi}^{(3)} = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$$

which gives us the following three solutions

$$\mathbf{x}^{(1)}(t) = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} e^{2t} \qquad \mathbf{x}^{(2)}(t) = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} e^{-t} \qquad \mathbf{x}^{(3)}(t) = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} e^{-t}.$$

You can check that the Wronskian of $\mathbf{x}^{(1)}$, $\mathbf{x}^{(2)}$ and $\mathbf{x}^{(3)}$ is non-zero. Therefore $\mathbf{x}^{(1)}$, $\mathbf{x}^{(2)}$ and $\mathbf{x}^{(3)}$ form a fundamental set of solutions. The general solution to the ODE is

$$\boxed{\mathbf{x}(t) = c_1 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} e^{2t} + c_2 \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} e^{-t} + c_3 \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} e^{-t}.}$$

Remark. Next we will study systems with complex eigenvalues.

7.6 Complex Eigenvalues

Consider
$$\mathbf{x}' = A\mathbf{x}$$
where $A \in \mathbb{R}^{n\times n}$.

Any complex eigenvalues of A must occur in complex conjugate pairs: If $r_1 = \lambda + i\mu$ is an eigenvalue of A, then $r_2 = \overline{r}_1 = \lambda - i\mu$ is also an eigenvalue of A.

Moreover, if $\boldsymbol{\xi}^{(1)}$ is an eigenvector of A corresponding to r_1, then $\boldsymbol{\xi}^{(2)} = \overline{\boldsymbol{\xi}^{(1)}}$ is an eigenvector of A corresponding to $r_2 = \overline{r}_1$.

Two solutions of $\mathbf{x}' = A\mathbf{x}$ are
$$\mathbf{x}^{(1)}(t) = \boldsymbol{\xi}^{(1)} e^{r_1 t} \qquad \text{and} \qquad \mathbf{x}^{(2)}(t) = \overline{\boldsymbol{\xi}^{(1)}} e^{\overline{r}_1 t}.$$
But $\mathbf{x}^{(1)}, \mathbf{x}^{(2)} : \mathbb{R} \to \mathbb{C}^n$ and we want solutions $: \mathbb{R} \to \mathbb{R}^n$.

If $r_1 = \lambda + i\mu$, and $\boldsymbol{\xi}^{(1)} = \mathbf{a} + i\mathbf{b}$ (λ, $\mu \in \mathbb{R}$, $\mathbf{a}$, $\mathbf{b} \in \mathbb{R}^n$), then
$$\begin{aligned}
\mathbf{x}^{(1)}(t) &= (\mathbf{a} + i\mathbf{b}) e^{(\lambda + i\mu)t} \\
&= (\mathbf{a} + i\mathbf{b}) e^{\lambda t} (\cos \mu t + i \sin \mu t) \\
&= e^{\lambda t} (\mathbf{a} \cos \mu t - \mathbf{b} \sin \mu t) + i e^{\lambda t} (\mathbf{a} \sin \mu t + \mathbf{b} \cos \mu t) \\
&= \mathbf{u}(t) + i\mathbf{v}(t).
\end{aligned}$$
The functions $\mathbf{u}(t)$ and $\mathbf{v}(t)$ will be linearly independent. Furthermore
$$\operatorname{span}\{\mathbf{u}(t), \mathbf{v}(t)\} = \operatorname{span}\{\mathbf{x}^{(1)}, \mathbf{x}^{(2)}\}.$$
So we can include $\mathbf{u}(t)$ and $\mathbf{v}(t)$ in our fundamental set of solutions instead of $\mathbf{x}^{(1)}(t)$ and $\mathbf{x}^{(2)}(t)$.

Example 7.7. Solve
$$\mathbf{x}' = \begin{bmatrix} -\frac{1}{2} & 1 \\ -1 & -\frac{1}{2} \end{bmatrix} \mathbf{x}.$$

We calculate that
$$0 = \det(A - rI) = \begin{vmatrix} -\frac{1}{2} - r & 1 \\ -1 & -\frac{1}{2} - r \end{vmatrix} = r^2 + r + \frac{5}{4}$$
and
$$r_{1,2} = \frac{-1 \pm \sqrt{1-5}}{2} = \frac{-1 \pm 2i}{2} = -\frac{1}{2} \pm i.$$
So we have $r_1 = -\frac{1}{2} + i$ and $r_2 = -\frac{1}{2} - i$. We will use r_1. We do not need r_2.

Since
$$0 = (A - r_1 I)\boldsymbol{\xi}^{(1)} = \begin{bmatrix} -i & 1 \\ -1 & -i \end{bmatrix} \begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix} \qquad \Longrightarrow \begin{cases} -i\xi_1 + \xi_2 = 0 \\ -\xi_1 - i\xi_2 = 0 \end{cases}$$
we can choose
$$\boldsymbol{\xi}^{(1)} = \begin{bmatrix} 1 \\ i \end{bmatrix}.$$

Note that we also have
$$\boldsymbol{\xi}^{(2)} = \overline{\boldsymbol{\xi}^{(1)}} = \overline{\begin{bmatrix} 1 \\ i \end{bmatrix}} = \begin{bmatrix} 1 \\ -i \end{bmatrix},$$
but we don't need $\boldsymbol{\xi}^{(2)}$.

Next we look at $\mathbf{x}^{(1)}(t)$:

$$\begin{aligned}
\mathbf{x}^{(1)}(t) = \boldsymbol{\xi}^{(1)} e^{r_1 t} &= \begin{bmatrix} 1 \\ i \end{bmatrix} e^{-\frac{t}{2}} (\cos t + i \sin t) \\
&= e^{-\frac{t}{2}} \begin{bmatrix} \cos t + i \sin t \\ i \cos t - \sin t \end{bmatrix} \\
&= e^{-\frac{t}{2}} \begin{bmatrix} \cos t \\ -\sin t \end{bmatrix} + i e^{-\frac{t}{2}} \begin{bmatrix} \sin t \\ \cos t \end{bmatrix} \\
&= \mathbf{u}(t) + i \mathbf{v}(t).
\end{aligned}$$

Hence we choose

$$\mathbf{u}(t) = e^{-\frac{t}{2}} \begin{bmatrix} \cos t \\ -\sin t \end{bmatrix} \quad \text{and} \quad \mathbf{v}(t) = e^{-\frac{t}{2}} \begin{bmatrix} \sin t \\ \cos t \end{bmatrix}.$$

But are $\mathbf{u}(t)$ and $\mathbf{v}(t)$ linearly independent? Since

$$\begin{aligned}
W\big(\mathbf{u}(t), \mathbf{v}(t)\big)(t) = \begin{vmatrix} u_1 & v_1 \\ u_2 & v_2 \end{vmatrix} &= \begin{vmatrix} e^{-\frac{t}{2}} \cos t & e^{-\frac{t}{2}} \sin t \\ -e^{-\frac{t}{2}} \sin t & e^{-\frac{t}{2}} \cos t \end{vmatrix} \\
&= e^{-t} \cos^2 t + e^{-t} \sin^2 t = e^{-t} \\
&\neq 0
\end{aligned}$$

the answer is yes. Therefore $\mathbf{u}(t)$ and $\mathbf{v}(t)$ form a fundamental set of solutions.

Therefore the general solution to $\mathbf{x}' = \begin{bmatrix} -\frac{1}{2} & 1 \\ -1 & -\frac{1}{2} \end{bmatrix} \mathbf{x}$ is

$$\boxed{\mathbf{x}(t) = c_1 e^{-\frac{t}{2}} \begin{bmatrix} \cos t \\ -\sin t \end{bmatrix} + c_2 e^{-\frac{t}{2}} \begin{bmatrix} \sin t \\ \cos t \end{bmatrix}.}$$

Remark. Our method is

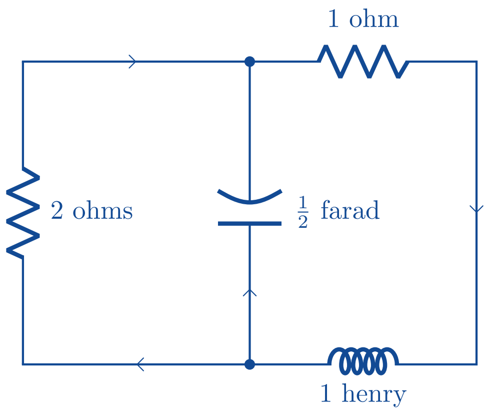
1. Find the eigenvalues;

2. Find the eigenvectors;

3. - If r_j is real, just use the solution $\mathbf{x}^{(j)}(t) = \boldsymbol{\xi}^{(j)} e^{r_j t}$;
   - But if r_j is complex, write

$$\mathbf{x}^{(j)}(t) = \boldsymbol{\xi}^{(j)} e^{r_j t} = \begin{pmatrix} \text{real valued} \\ \text{function} \end{pmatrix} + i \begin{pmatrix} \text{real valued} \\ \text{function} \end{pmatrix}$$

   and use these two functions.



Example 7.8. The electric circuit shown above is described by

$$\begin{cases} I' = -I - V \\ V' = 2I - V \end{cases}$$

where

$$\begin{aligned} I &= \text{the current through the inductor} \\ V &= \text{the voltage drop across the capacitor.} \end{aligned}$$

(Ask an Electrical Engineer.)

Suppose that at time $t = 0$ the current is 2 amperes and the voltage drop is 2 volts. Find $I(t)$ and $V(t)$.

We must solve the IVP

$$\begin{cases} \dfrac{d}{dt}\begin{bmatrix} I \\ V \end{bmatrix} = \begin{bmatrix} -1 & -1 \\ 2 & -1 \end{bmatrix}\begin{bmatrix} I \\ V \end{bmatrix} \\ \begin{bmatrix} I \\ V \end{bmatrix}(0) = \begin{bmatrix} 2 \\ 2 \end{bmatrix}. \end{cases}$$

The eigenvalues of $\begin{bmatrix} -1 & -1 \\ 2 & -1 \end{bmatrix}$ are $r_1 = -1 + i\sqrt{2}$ and $r_2 = -1 - i\sqrt{2}$ (please check). The corresponding eigenvectors are

$$\boldsymbol{\xi}^{(1)} = \begin{bmatrix} 1 \\ -i\sqrt{2} \end{bmatrix} \qquad \text{and} \qquad \boldsymbol{\xi}^{(2)} = \begin{bmatrix} 1 \\ i\sqrt{2} \end{bmatrix}.$$

Then we calculate that

$$\begin{aligned} \mathbf{x}^{(1)}(t) = \boldsymbol{\xi}^{(1)} e^{r_1 t} &= \begin{bmatrix} 1 \\ -i\sqrt{2} \end{bmatrix} e^{(-1+i\sqrt{2})t} \\ &= \begin{bmatrix} 1 \\ -i\sqrt{2} \end{bmatrix} e^{-t}\big(\cos\sqrt{2}t + i\sin\sqrt{2}t\big) \\ &= e^{-t}\begin{bmatrix} \cos\sqrt{2}t + i\sin\sqrt{2}t \\ -i\sqrt{2}\cos\sqrt{2}t + \sqrt{2}\sin\sqrt{2}t \end{bmatrix} \\ &= e^{-t}\begin{bmatrix} \cos\sqrt{2}t \\ \sqrt{2}\sin\sqrt{2}t \end{bmatrix} + ie^{-t}\begin{bmatrix} \sin\sqrt{2}t \\ -\sqrt{2}\cos\sqrt{2}t \end{bmatrix}. \end{aligned}$$

Hence the general solution to the ODE is

$$\begin{bmatrix} I(t) \\ V(t) \end{bmatrix} = c_1 e^{-t} \begin{bmatrix} \cos\sqrt{2}t \\ \sqrt{2}\sin\sqrt{2}t \end{bmatrix} + c_2 e^{-t} \begin{bmatrix} \sin\sqrt{2}t \\ -\sqrt{2}\cos\sqrt{2}t \end{bmatrix}.$$

Using the initial condition, we calculate that

$$\begin{bmatrix} 2 \\ 2 \end{bmatrix} = \begin{bmatrix} I(0) \\ V(0) \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ -\sqrt{2} \end{bmatrix} \qquad \Longrightarrow \qquad \begin{cases} c_1 = 2 \\ c_2 = -\sqrt{2}. \end{cases}$$

Thus

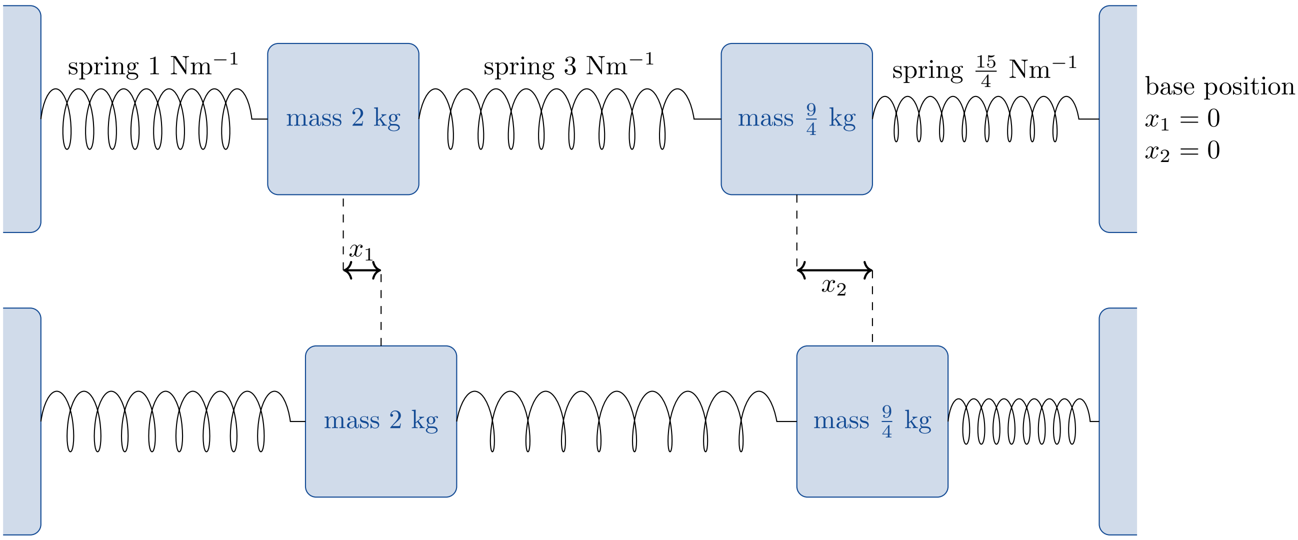
$$\begin{bmatrix} I(t) \\ V(t) \end{bmatrix} = 2\, e^{-t} \begin{bmatrix} \cos\sqrt{2}t \\ \sqrt{2}\sin\sqrt{2}t \end{bmatrix} - \sqrt{2}\, e^{-t} \begin{bmatrix} \sin\sqrt{2}t \\ -\sqrt{2}\cos\sqrt{2}t \end{bmatrix}.$$

So the answers to this problem are

$$\boxed{I(t) = 2e^{-t}\cos\sqrt{2}t - \sqrt{2}e^{-t}\sin\sqrt{2}t}$$

and

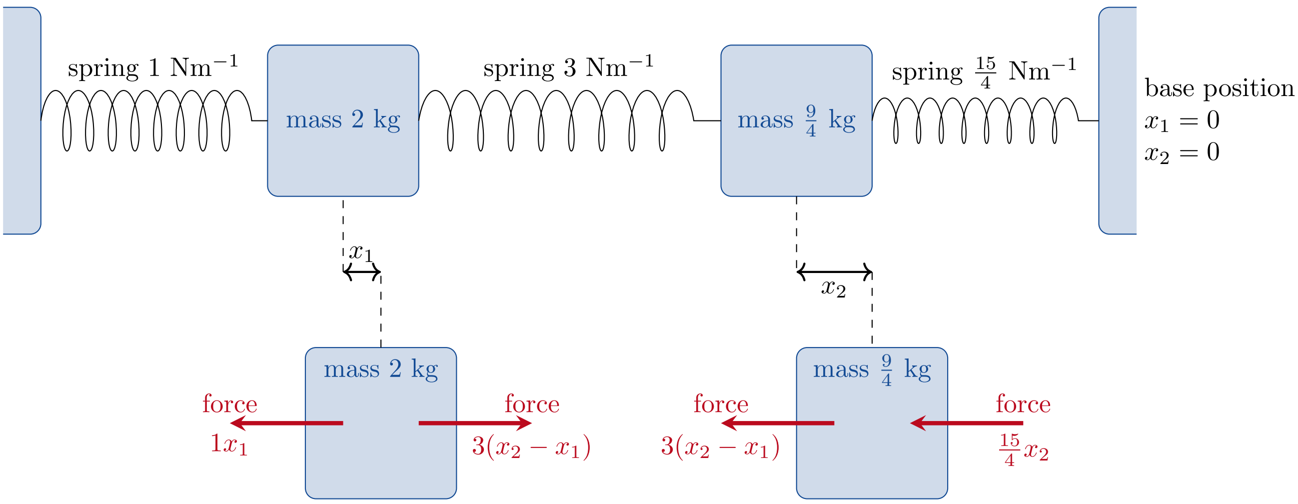
$$\boxed{V(t) = 2\sqrt{2}e^{-t}\sin\sqrt{2}t + 2e^{-t}\cos\sqrt{2}t.}$$



See https://tinyurl.com/wm2ogdh for an animated figure.

Example 7.9. For the dynamical system shown above, find $x_1(t)$ and $x_2(t)$.

As the springs are stretched and compressed, they apply forces on the blocks as shown below (Hooke's Law).



We calculate that

$$2\frac{d^2x_1}{dt^2} = \text{mass} \times \text{acceleration} = \text{force} = -x_1 + 3(x_2 - x_1)$$
$$\frac{9}{4}\frac{d^2x_2}{dt^2} = \text{mass} \times \text{acceleration} = \text{force} = -3(x_2 - x_1) - \frac{15}{4}x_2.$$

This is a system of 2 second order ODEs. We want a system of first order ODEs. Now let $y_1 = x_1$, $y_2 = x_2$, $y_3 = x_1'$ and $y_4 = x_2'$. Then

$$\begin{aligned}
y_1' &= x_1' = y_3 \\
y_2' &= x_2' = y_4 \\
y_3' &= x_1'' = \frac{1}{2}\Big(- x_1 + 3x_2 - 3x_1\Big) = -2y_1 + \frac{3}{2}y_2 \\
y_4' &= x_2'' = \frac{4}{9}\left(-3x_2 + 3x_1 - \frac{15}{4}x_2\right) = \frac{4}{3}y_1 - 3y_2.
\end{aligned}$$

So

$$\mathbf{y}' = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -2 & \frac{3}{2} & 0 & 0 \\ \frac{4}{3} & -3 & 0 & 0 \end{bmatrix} \mathbf{y}.$$

The characteristic polynomial of this matrix is

$$0 = r^4 + 5r^2 + 4 = (r^2+1)(r^2+4).$$

So $r_1 = i$, $r_2 = -i$, $r_3 = 2i$ and $r_4 = -2i$. We will use r_1 and r_3 (we do not need r_2 and r_4).

The corresponding eigenvectors (please check) are

$$\boldsymbol{\xi}^{(1)} = \begin{bmatrix} 3 \\ 2 \\ 3i \\ 2i \end{bmatrix} \qquad \text{and} \qquad \boldsymbol{\xi}^{(3)} = \begin{bmatrix} 3 \\ -4 \\ 6i \\ -8i \end{bmatrix}.$$

It follows that

$$\boldsymbol{\xi}^{(1)} e^{r_1 t} = \begin{bmatrix} 3 \\ 2 \\ 3i \\ 2i \end{bmatrix} (\cos t + i \sin t) = \begin{bmatrix} 3\cos t \\ 2\cos t \\ -3\sin t \\ -2\sin t \end{bmatrix} + i \begin{bmatrix} 3\sin t \\ 2\sin t \\ 3\cos t \\ 2\cos t \end{bmatrix} = \mathbf{u}(t) + i\mathbf{v}(t)$$

and

$$\boldsymbol{\xi}^{(3)} e^{r_3 t} = \begin{bmatrix} 3 \\ -4 \\ 6i \\ -8i \end{bmatrix} (\cos 2t + i \sin 2t) = \begin{bmatrix} 3\cos 2t \\ -4\cos 2t \\ -6\sin 2t \\ 8\sin 2t \end{bmatrix} + i \begin{bmatrix} 3\sin 2t \\ -4\sin 2t \\ 6\cos 2t \\ -8\cos 2t \end{bmatrix} = \mathbf{w}(t) + i\mathbf{z}(t)$$

Therefore the general solution is

$$\begin{aligned} \mathbf{y}(t) &= c_1\mathbf{u}(t) + c_2\mathbf{v}(t) + c_3\mathbf{w}(t) + c_4\mathbf{z}(t) \\ &= c_1 \begin{bmatrix} 3\cos t \\ 2\cos t \\ -3\sin t \\ -2\sin t \end{bmatrix} + c_2 \begin{bmatrix} 3\sin t \\ 2\sin t \\ 3\cos t \\ 2\cos t \end{bmatrix} + c_3 \begin{bmatrix} 3\cos 2t \\ -4\cos 2t \\ -6\sin 2t \\ 8\sin 2t \end{bmatrix} + c_4 \begin{bmatrix} 3\sin 2t \\ -4\sin 2t \\ 6\cos 2t \\ -8\cos 2t \end{bmatrix}. \end{aligned}$$

Example 7.10. Suppose that the above system has initial condition

$$\mathbf{y}(0) = \begin{bmatrix} -1 \\ 4 \\ 1 \\ 1 \end{bmatrix}.$$

Sketch graphs of $y_1(t)$ and $y_2(t)$.

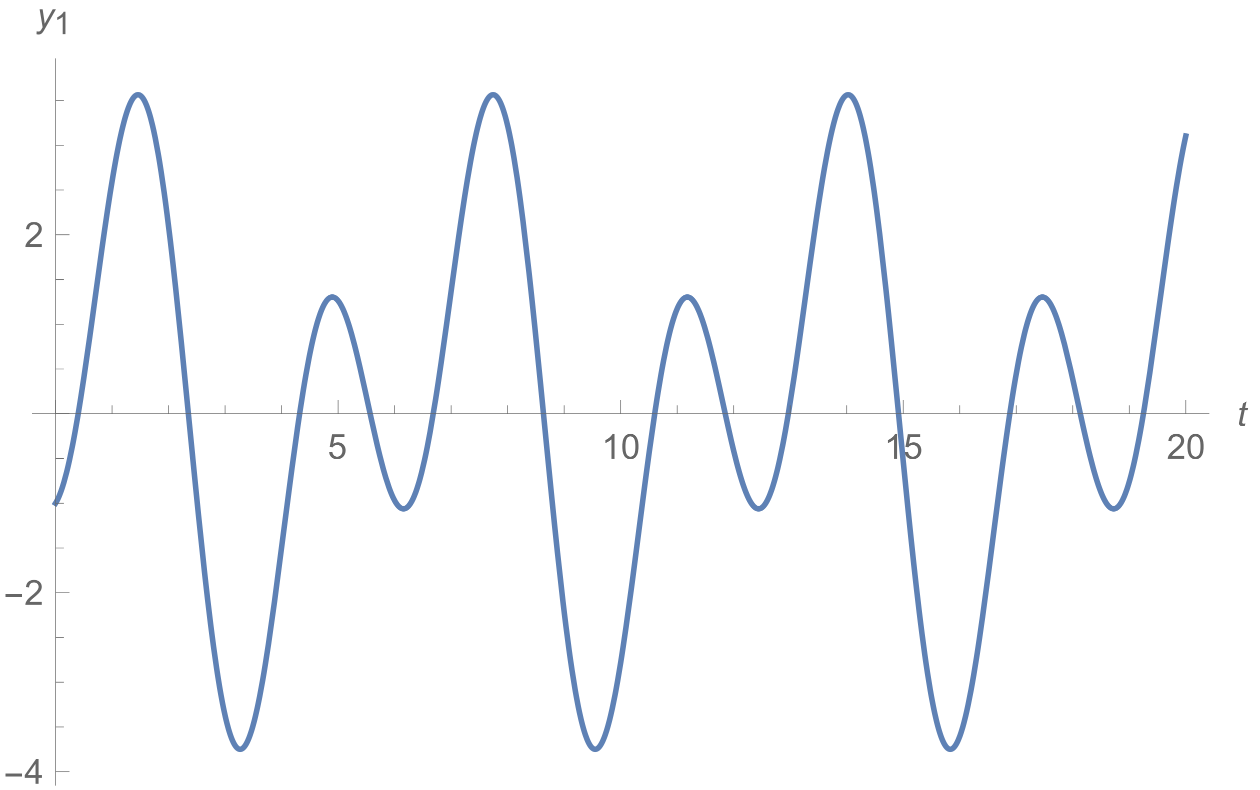
The initial value problem

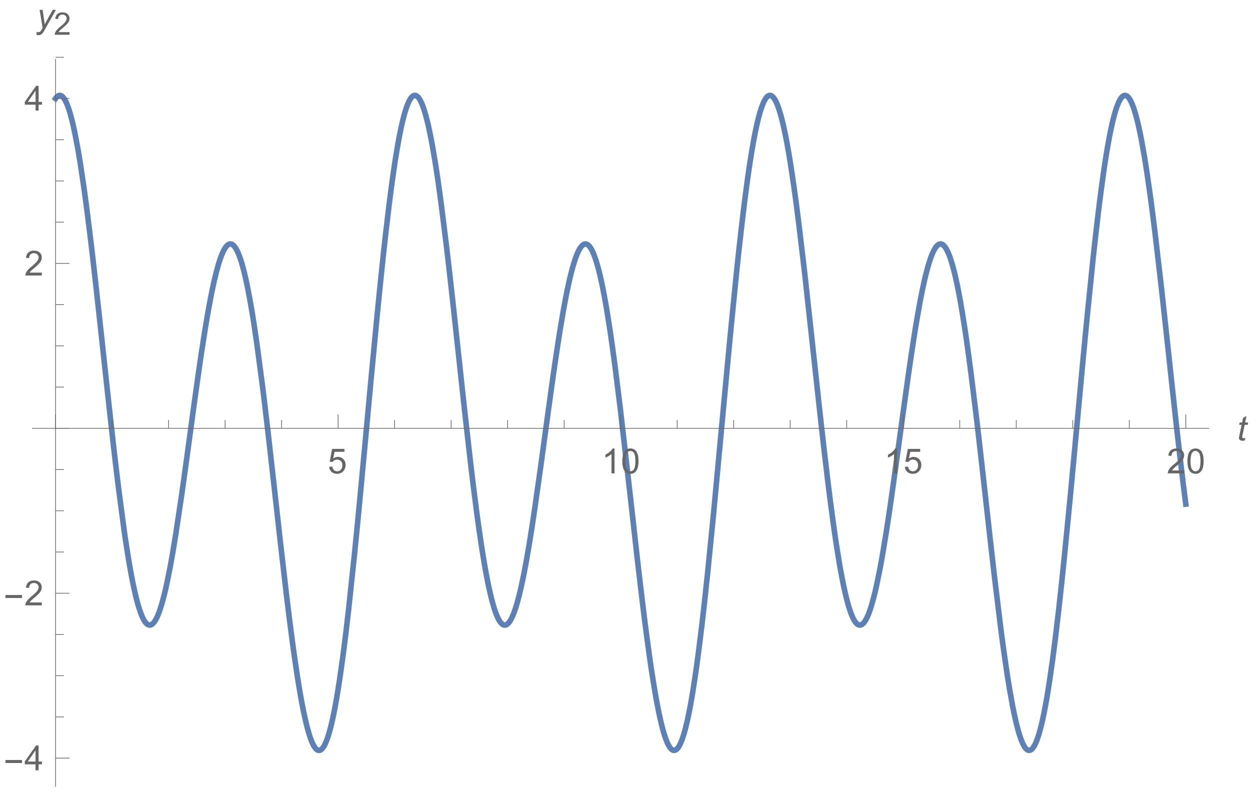
$$\mathbf{y}' = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -2 & \frac{3}{2} & 0 & 0 \\ \frac{4}{3} & -3 & 0 & 0 \end{bmatrix} \mathbf{y}, \qquad \mathbf{y}(0) = \begin{bmatrix} -1 \\ 4 \\ 1 \\ 1 \end{bmatrix}$$

has solution

$$\mathbf{y}(t) = \frac{4}{9}\begin{bmatrix} 3\cos t \\ 2\cos t \\ -3\sin t \\ -2\sin t \end{bmatrix} + \frac{7}{18}\begin{bmatrix} 3\sin t \\ 2\sin t \\ 3\cos t \\ 2\cos t \end{bmatrix} - \frac{7}{9}\begin{bmatrix} 3\cos 2t \\ -4\cos 2t \\ -6\sin 2t \\ 8\sin 2t \end{bmatrix} - \frac{1}{36}\begin{bmatrix} 3\sin 2t \\ -4\sin 2t \\ 6\cos 2t \\ -8\cos 2t \end{bmatrix}.$$

Then we can draw the graphs of y_1 and y_2:





Please see https://tinyurl.com/s7uww7m

7.7 Fundamental Matrices

Now consider
$$\mathbf{x}' = P(t)\mathbf{x}$$
where P is an $n \times n$ matrix. Suppose that $\mathbf{x}^{(1)}$, $\mathbf{x}^{(2)}$, ..., $\mathbf{x}^{(n)}$ are linearly independent solutions to this ODE. In other words, suppose that $\mathbf{x}^{(1)}, \mathbf{x}^{(2)}, \ldots, \mathbf{x}^{(n)}$ form a ***fundamental set of solutions*** to this ODE.

Definition. The matrix
$$\Psi(t) = \begin{bmatrix} \mathbf{x}^{(1)} & \mathbf{x}^{(2)} & \ldots & \mathbf{x}^{(n)} \end{bmatrix} = \begin{bmatrix} x_1^{(1)}(t) & x_1^{(2)}(t) & \ldots & x_1^{(n)}(t) \\ x_2^{(1)}(t) & x_2^{(2)}(t) & \ldots & x_2^{(n)}(t) \\ \vdots & \vdots & & \vdots \\ x_n^{(1)}(t) & x_n^{(2)}(t) & \ldots & x_n^{(n)}(t) \end{bmatrix}$$
is called a ***fundamental matrix*** of $\mathbf{x}' = P(t)\mathbf{x}$.

Example 7.11. Find a fundamental matrix for
$$\mathbf{x}' = \begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix} \mathbf{x}.$$

Recall that
$$\mathbf{x}^{(1)}(t) = \begin{bmatrix} 1 \\ 2 \end{bmatrix} e^{3t} \qquad \text{and} \qquad \mathbf{x}^{(2)}(t) = \begin{bmatrix} 1 \\ -2 \end{bmatrix} e^{-t}$$
form a fundamental set of solutions to this ODE. Therefore
$$\Psi(t) = \begin{bmatrix} e^{3t} & e^{-t} \\ 2e^{3t} & -2e^{-t} \end{bmatrix}$$
is a fundamental matrix of this ODE.

Now, the general solution to
$$\mathbf{x}' = A\mathbf{x}$$
is
$$\mathbf{x}(t) = c_1\mathbf{x}^{(1)}(t) + c_2\mathbf{x}^{(2)}(t) + \ldots + c_n\mathbf{x}^{(n)}(t) = \Psi(t)\mathbf{c}$$
where
$$\mathbf{c} = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} \in \mathbb{R}^n.$$

If we have an initial condition $\mathbf{x}(t_0) = \mathbf{x}^0$, then
$$\Psi(t_0)\mathbf{c} = \mathbf{x}^0.$$

But
$$\begin{array}{c} \mathbf{x}^{(1)}, \ldots, \mathbf{x}^{(n)} \\ \text{are linearly} \\ \text{independent} \end{array} \implies \Psi(t) \text{ is invertible} \implies \mathbf{c} = \Psi^{-1}(t_0)\mathbf{x}^0.$$

Therefore the solution to the IVP

$$\begin{cases} \mathbf{x}' = A\mathbf{x} \\ \mathbf{x}(t_0) = \mathbf{x}^0 \end{cases}$$

is

$$\boxed{\mathbf{x}(t) = \Psi(t)\Psi^{-1}(t_0)\mathbf{x}^0.}$$

Remark. The matrix $\Psi(t)$ solves the differential equation $\Psi' = P(t)\Psi$. (Homework)

Remark. It is possible to find a ***special fundamental matrix***, $\Phi(t)$, which satisfies

$$\Phi(t_0) = \begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \dots & 1 \end{bmatrix} = I.$$

We will use Φ for this special fundamental matrix, and Ψ for any fundamental matrix.

Example 7.12. Consider

$$\mathbf{x}' = \begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix} \mathbf{x}.$$

Find the special fundamental matrix which satisfies $\Phi(0) = I$.

To find the matrix Φ which satisfies

$$\Phi(0) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

we must solve two IVPs:

$$\begin{cases} \mathbf{x}' = \begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix} \mathbf{x} \\ \mathbf{x}(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \end{cases} \qquad \text{and} \qquad \begin{cases} \mathbf{x}' = \begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix} \mathbf{x} \\ \mathbf{x}(0) = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \end{cases}.$$

The general solution to the ODE is

$$\mathbf{x}(t) = c_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} e^{3t} + c_2 \begin{bmatrix} 1 \\ -2 \end{bmatrix} e^{-t}.$$

We calculate that

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \mathbf{x}(0) = c_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ -2 \end{bmatrix} \implies \begin{matrix} c_1 = \frac{1}{2} \\ c_2 = \frac{1}{2} \end{matrix} \implies \mathbf{x}(t) = \begin{bmatrix} \frac{1}{2}e^{3t} + \frac{1}{2}e^{-t} \\ e^{3t} - e^{-t} \end{bmatrix}$$

and

$$\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \mathbf{x}(0) = c_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ -2 \end{bmatrix} \implies \begin{matrix} c_1 = \frac{1}{4} \\ c_2 = -\frac{1}{4} \end{matrix} \implies \mathbf{x}(t) = \begin{bmatrix} \frac{1}{4}e^{3t} - \frac{1}{4}e^{-t} \\ \frac{1}{2}e^{3t} + \frac{1}{2}e^{-t} \end{bmatrix}.$$

Therefore the special fundamental matrix is

$$\Phi(t) = \begin{bmatrix} \frac{1}{2}e^{3t} + \frac{1}{2}e^{-t} & \frac{1}{4}e^{3t} - \frac{1}{4}e^{-t} \\ e^{3t} - e^{-t} & \frac{1}{2}e^{3t} + \frac{1}{2}e^{-t} \end{bmatrix}.$$

What is e^{At}?

Recall that the solution to

$$\begin{cases} x' = ax \ (a \in \mathbb{R}) \\ x(0) = x_0 \end{cases}$$

is

$$x(t) = x_0 e^{at} = x_0 \exp(at)$$

and recall that

$$\exp(at) = 1 + \sum_{n=1}^{\infty} \frac{a^n t^n}{n!}.$$

Now consider

$$\begin{cases} \mathbf{x}' = A\mathbf{x} \\ \mathbf{x}(0) = \mathbf{x}^0 \end{cases}$$

for $A \in \mathbb{R}^{n \times n}$.

Definition.

$$\exp(At) = I + \sum_{n=1}^{\infty} \frac{A^n t^n}{n!} = I + At + \frac{A^2 t^2}{2!} + \frac{A^3 t^3}{3!} + \dots.$$

Note that

$$\begin{aligned}
\frac{d}{dt} \exp(At) &= \frac{d}{dt} \left(I + \sum_{n=1}^{\infty} \frac{A^n t^n}{n!} \right) = 0 + \sum_{n=1}^{\infty} \frac{d}{dt} \left(\frac{A^n t^n}{n!} \right) \\
&= \sum_{n=1}^{\infty} \frac{A^n t^{n-1}}{(n-1)!} = A + \sum_{n=2}^{\infty} \frac{A^n t^{n-1}}{(n-1)!} \\
&= A + \sum_{k=1}^{\infty} \frac{A^{k+1} t^k}{(k)!} \qquad\qquad (k = n-1) \\
&= A \left(I + \sum_{k=1}^{\infty} \frac{A^k t^k}{(k)!} \right) = A \exp(At).
\end{aligned}$$

This means that $\exp(At)$ solves

$$\begin{cases} \big(\exp(At)\big)' = A \exp(At) \\ \exp(At)\big|_{t=0} = I. \end{cases}$$

But remember that Φ solves

$$\begin{cases} \Phi' = A\phi \\ \Phi(0) = I. \end{cases}$$

Therefore

$$\boxed{\Phi(t) = \exp(At).}$$

Example 7.13. Let $A = \begin{bmatrix} 8 & -1 \\ 6 & 1 \end{bmatrix}$. Find $\exp(At)$.

We have previously found that the general solution to $\mathbf{x}' = A\mathbf{x}$ is

$$\mathbf{x}(t) = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{7t} + c_2 \begin{bmatrix} 1 \\ 6 \end{bmatrix} e^{2t}.$$

To satisfy $\mathbf{x}(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ we require $c_1 = \frac{6}{5}$ and $c_2 = -\frac{1}{5}$ – hence

$$\mathbf{x}(t) = \frac{6}{5} \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{7t} - \frac{1}{5} \begin{bmatrix} 1 \\ 6 \end{bmatrix} e^{2t} = \begin{bmatrix} \frac{6}{5}e^{7t} - \frac{1}{5}e^{2t} \\ \frac{6}{5}e^{7t} - \frac{6}{5}e^{2t} \end{bmatrix}.$$

To satisfy $\mathbf{x}(0) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ we require $c_1 = -\frac{1}{5}$ and $c_2 = \frac{1}{5}$ – hence

$$\mathbf{x}(t) = -\frac{1}{5} \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{7t} + \frac{1}{5} \begin{bmatrix} 1 \\ 6 \end{bmatrix} e^{2t} = \begin{bmatrix} -\frac{1}{5}e^{7t} + \frac{1}{5}e^{2t} \\ -\frac{1}{5}e^{7t} + \frac{6}{5}e^{2t} \end{bmatrix}.$$

Therefore the answer is

$$\exp(At) = \Phi(t) = \begin{bmatrix} \frac{6}{5}e^{7t} - \frac{1}{5}e^{2t} & -\frac{1}{5}e^{7t} + \frac{1}{5}e^{2t} \\ \frac{6}{5}e^{7t} - \frac{6}{5}e^{2t} & -\frac{1}{5}e^{7t} + \frac{6}{5}e^{2t} \end{bmatrix}.$$

Diagonalisable Matrices

If

$$D = \begin{bmatrix} r_1 & 0 & 0 & \dots & 0 \\ 0 & r_2 & 0 & \dots & 0 \\ 0 & 0 & r_3 & \dots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \dots & r_n \end{bmatrix}$$

is a diagonal matrix, then it is easy to calculate $\exp(Dt)$. We simply have

$$\exp(Dt) = \begin{bmatrix} e^{r_1 t} & 0 & 0 & \dots & 0 \\ 0 & e^{r_2 t} & 0 & \dots & 0 \\ 0 & 0 & e^{r_3 t} & \dots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \dots & e^{r_n t} \end{bmatrix}.$$

Now consider

$$\mathbf{x}' = A\mathbf{x}$$

for $A \in \mathbb{R}^{n\times n}$. Recall how we diagonalise a matrix: If $\boldsymbol{\xi}^{(1)}, \boldsymbol{\xi}^{(2)}, \dots, \boldsymbol{\xi}^{(n)}$ are the eigenvectors of A, we let

$$T = \begin{bmatrix} \boldsymbol{\xi}^{(1)} & \boldsymbol{\xi}^{(2)} & \dots & \boldsymbol{\xi}^{(n)} \end{bmatrix}.$$

Then

$$\det(T) \neq 0 \quad \Longrightarrow \quad T^{-1} \text{ exists} \quad \Longrightarrow \quad \begin{matrix} T^{-1}AT \\ \text{is diagonal} \end{matrix} \quad \Longrightarrow \quad \begin{matrix} A \text{ is} \\ \text{diagonalisable.} \end{matrix}$$

Example 7.14. Diagonalise

$$A = \begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix}.$$

The eigenvalues are $r_1 = 3$ and $r_2 = -1$. The corresponding eigenvectors are $\boldsymbol{\xi}^{(1)} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $\boldsymbol{\xi}^{(2)} = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$. Thus

$$T = \begin{bmatrix} 1 & 1 \\ 2 & -2 \end{bmatrix} \qquad \text{and} \qquad T^{-1} = \begin{bmatrix} \frac{1}{2} & \frac{1}{4} \\ \frac{1}{2} & -\frac{1}{4} \end{bmatrix}.$$

It follows that

$$D = T^{-1}AT = \begin{bmatrix} 3 & 0 \\ 0 & -1 \end{bmatrix}.$$

Now consider
$$\mathbf{x}' = A\mathbf{x}.$$
Define a new variable $\mathbf{y}$ by
$$\mathbf{x} = T\mathbf{y} \qquad \text{or} \qquad \mathbf{y} = T^{-1}\mathbf{x}.$$
Then we calculate that
$$\begin{aligned} \mathbf{x}' &= A\mathbf{x} \\ T\mathbf{y}' &= AT\mathbf{y} \\ \mathbf{y}' &= T^{-1}AT\mathbf{y} = D\mathbf{y}. \end{aligned}$$
We know that a fundamental matrix for $\mathbf{y}' = D\mathbf{y}$ is
$$\exp(Dt) = \begin{bmatrix} e^{r_1 t} & 0 & \dots & 0 \\ 0 & e^{r_2 t} & \dots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \dots & e^{r_n t} \end{bmatrix}.$$
Therefore a fundamental matrix for $\mathbf{x}' = A\mathbf{x}$ is
$$\Psi = T\exp(Dt) = \begin{bmatrix} \boldsymbol{\xi}^{(1)} e^{r_1 t} & \boldsymbol{\xi}^{(2)} e^{r_2 t} & \dots & \boldsymbol{\xi}^{(n)} e^{r_n t} \end{bmatrix}.$$

Example 7.15. Find a fundamental matrix for
$$\mathbf{x}' = \begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix}\mathbf{x}.$$

Recall that $T = \begin{bmatrix} 1 & 1 \\ 2 & -2 \end{bmatrix}$. Letting $\mathbf{y} = T^{-1}\mathbf{x}$, we have
$$\mathbf{y}' = \begin{bmatrix} 3 & 0 \\ 0 & -1 \end{bmatrix}\mathbf{y}.$$
A fundamental matrix of $\mathbf{y}' = \begin{bmatrix} 3 & 0 \\ 0 & -1 \end{bmatrix}\mathbf{y}$ is
$$\exp(Dt) = e^{Dt} = \begin{bmatrix} e^{3t} & 0 \\ 0 & e^{-t} \end{bmatrix}.$$
Hence
$$\Psi(t) = T\exp(Dt) = \begin{bmatrix} 1 & 1 \\ 2 & -2 \end{bmatrix}\begin{bmatrix} e^{3t} & 0 \\ 0 & e^{-t} \end{bmatrix} = \begin{bmatrix} e^{3t} & e^{-t} \\ 2e^{3t} & -2e^{-t} \end{bmatrix}.$$

7.8 Repeated Eigenvalues

Example 7.16. Find the eigenvalues and eigenvectors of $A = \begin{bmatrix} 1 & -1 \\ 1 & 3 \end{bmatrix}$.

We calculate that

$$0 = \det(A - rI) = \begin{vmatrix} 1-r & -1 \\ 1 & 3-r \end{vmatrix} = r^2 - 4r + 4 = (r-2)^2.$$

Therefore $r_1 = 2 = r_2$. Moreover

$$\mathbf{0} = (A - rI)\boldsymbol{\xi} = \begin{bmatrix} -1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix} \implies \xi_1 + \xi_2 = 0 \implies \boldsymbol{\xi} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

Note that A has only one linearly independent eigenvector.

Example 7.17. Solve

$$\mathbf{x}' = \begin{bmatrix} 1 & -1 \\ 1 & 3 \end{bmatrix} \mathbf{x}.$$

We know that

$$\mathbf{x}^{(1)}(t) = \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{2t}$$

is a solution. But we need two solutions.

Guess 1: I guess that

$$\mathbf{x}^{(2)}(t) = \boldsymbol{\xi} t e^{2t}$$

for some $\boldsymbol{\xi} \in \mathbb{R}^2$. Then we have

$$\begin{aligned} \boldsymbol{\xi} e^{2t} + 2\boldsymbol{\xi} t e^{2t} = \mathbf{x}^{(2)\prime} &= A\mathbf{x}^{(2)} = A\boldsymbol{\xi} t e^{2t} \\ \boldsymbol{\xi} + (2\boldsymbol{\xi} - A\boldsymbol{\xi})\, t &= \mathbf{0} \qquad \forall t \\ &\implies \boldsymbol{\xi} = \mathbf{0}. \end{aligned}$$

This guess did not work.

Guess 2: Now I guess that

$$\mathbf{x}^{(2)}(t) = \boldsymbol{\xi} t e^{2t} + \boldsymbol{\eta} e^{2t}$$

for some $\boldsymbol{\xi}, \boldsymbol{\eta} \in \mathbb{R}^2$. Then we have

$$\boldsymbol{\xi} e^{2t} + 2\boldsymbol{\xi} t e^{2t} + 2\boldsymbol{\eta} e^{2t} = \mathbf{x}^{(2)\prime} = A\mathbf{x}^{(2)} = A\left(\boldsymbol{\xi} t e^{2t} + \boldsymbol{\eta} e^{2t}\right)$$

and

$$(2\boldsymbol{\xi} - A\boldsymbol{\xi})\, t + (\boldsymbol{\xi} + 2\boldsymbol{\eta} - A\boldsymbol{\eta}) = \mathbf{0}.$$

Since this must be true $\forall\, t$, we must have

$$(A - 2I)\boldsymbol{\xi} = \mathbf{0} \qquad \text{and} \qquad (A - 2I)\boldsymbol{\eta} = \boldsymbol{\xi}.$$

Clearly $\boldsymbol{\xi} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$. Then we calculate that

$$\begin{aligned} \begin{bmatrix} -1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \eta_1 \\ \eta_2 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix} &\implies \eta_1 + \eta_2 = -1 \\ &\implies \boldsymbol{\eta} = \begin{bmatrix} k \\ -1-k \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \end{bmatrix} + k \begin{bmatrix} 1 \\ -1 \end{bmatrix} \end{aligned}$$

for some k. So

$$\begin{aligned}\mathbf{x}^{(2)}(t) = \boldsymbol{\xi} t e^{2t} + \boldsymbol{\eta} e^{2t} &= \begin{bmatrix} 1 \\ -1 \end{bmatrix} t e^{2t} + \begin{bmatrix} 0 \\ -1 \end{bmatrix} e^{2t} + k \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{2t} \\ &= \begin{bmatrix} 1 \\ -1 \end{bmatrix} t e^{2t} + \begin{bmatrix} 0 \\ -1 \end{bmatrix} e^{2t} + k\mathbf{x}^{(1)}(t).\end{aligned}$$

Because we already have $\mathbf{x}^{(1)}(t)$, we can choose $k = 0$. So

$$\mathbf{x}^{(2)}(t) = \boldsymbol{\xi} t e^{2t} + \boldsymbol{\eta} e^{2t} = \begin{bmatrix} 1 \\ -1 \end{bmatrix} t e^{2t} + \begin{bmatrix} 0 \\ -1 \end{bmatrix} e^{2t}.$$

The general solution of $\mathbf{x}' = \begin{bmatrix} 1 & -1 \\ 1 & 3 \end{bmatrix} \mathbf{x}$ is therefore

$$\boxed{\mathbf{x}(t) = c_1 \mathbf{x}^{(1)} + c_2 \mathbf{x}^{(2)} = c_1 \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{2t} + c_2 \left(\begin{bmatrix} 1 \\ -1 \end{bmatrix} t e^{2t} + \begin{bmatrix} 0 \\ -1 \end{bmatrix} e^{2t} \right).}$$

Example 7.18. Find a fundamental matrix for

$$\mathbf{x}' = \begin{bmatrix} 1 & -1 \\ 1 & 3 \end{bmatrix} \mathbf{x}.$$

Then find the special fundamental matrix $\Phi(t)$ which satisfies $\Phi(0) = I$.

Since $\mathbf{x}^{(1)}(t) = \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{2t}$ and $\mathbf{x}^{(2)}(t) = \begin{bmatrix} 1 \\ -1 \end{bmatrix} t e^{2t} + \begin{bmatrix} 0 \\ -1 \end{bmatrix} e^{2t}$ we have that

$$\Psi(t) = \begin{bmatrix} \mathbf{x}^{(1)} & \mathbf{x}^{(2)} \end{bmatrix} = \begin{bmatrix} e^{2t} & t e^{2t} \\ -e^{2t} & -t e^{2t} - e^{2t} \end{bmatrix}$$

is a fundamental matrix for this system.

Now

$$\Psi(0) = \begin{bmatrix} 1 & 0 \\ -1 & -1 \end{bmatrix} \qquad \text{and} \qquad \Psi^{-1}(0) = \begin{bmatrix} 1 & 0 \\ -1 & -1 \end{bmatrix}.$$

Therefore

$$\begin{aligned}\exp(At) = \Phi(t) = \Psi(t)\Psi^{-1}(0) &= e^{2t} \begin{bmatrix} 1 & t \\ -1 & -1 - t \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -1 & -1 \end{bmatrix} \\ &= e^{2t} \begin{bmatrix} 1 - t & -t \\ t & 1 + t \end{bmatrix}.\end{aligned}$$

Remark.

$$\mathbf{x}' = A\mathbf{x}$$

For two repeated eigenvalues (but with only one linearly independent eigenvector), the key equations to remember are

$$\boxed{\mathbf{x}^{(2)}(t) = \boldsymbol{\xi} t e^{rt} + \boldsymbol{\eta} e^{rt}} \qquad \text{and} \qquad \boxed{(A - rI)\boldsymbol{\eta} = \boldsymbol{\xi}}.$$

Definition. η is called a ***generalised eigenvector*** of A.

Remark. If you have 2 repeated eigenvalues (but with only one linearly independent eigenvector), the method is:

(i). Find the eigenvalues and eigenvectors;

(ii). The first solution is $\mathbf{x}^{(1)}(t) = \boldsymbol{\xi} e^{rt}$;

(iii). Use $(A - rI)\boldsymbol{\eta} = \boldsymbol{\xi}$ to find a generalised eigenvector $\boldsymbol{\eta}$;

(iv). The second solution is $\mathbf{x}^{(2)}(t) = \boldsymbol{\xi} t e^{rt} + \boldsymbol{\eta} e^{rt}$.

Example 7.19. Solve

$$\begin{cases} \mathbf{x}' = \begin{bmatrix} -\frac{5}{2} & \frac{3}{2} \\ -\frac{3}{2} & \frac{1}{2} \end{bmatrix} \mathbf{x}, \\ \mathbf{x}(0) = \begin{bmatrix} 3 \\ -1 \end{bmatrix}. \end{cases}$$

The only eigenvalue of the matrix is $r = -1$. The corresponding eigenvector is $\boldsymbol{\xi} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. Therefore one solution of the linear system is

$$\mathbf{x}^{(1)}(t) = \boldsymbol{\xi} e^{rt} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{-t}.$$

We need to find a second, linearly independent solution. We will consider the ansatz

$$\mathbf{x}^{(2)}(t) = \boldsymbol{\xi} t e^{-t} + \boldsymbol{\eta} e^{-t}$$

where $\boldsymbol{\xi} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ as above and $\boldsymbol{\eta}$ is a generalised eigenvector solving $(A - rI)\boldsymbol{\eta} = \boldsymbol{\xi}$.

Solving the latter equation,

$$\begin{aligned} (A - rI)\boldsymbol{\eta} &= \boldsymbol{\xi} \\ \begin{bmatrix} -\frac{3}{2} & \frac{3}{2} \\ -\frac{3}{2} & \frac{3}{2} \end{bmatrix} \begin{bmatrix} \eta_1 \\ \eta_2 \end{bmatrix} &= \begin{bmatrix} 1 \\ 1 \end{bmatrix} \\ -\frac{3}{2}\eta_1 + \frac{3}{2}\eta_2 &= 1 \\ -\eta_1 + \eta_2 &= \frac{2}{3} \end{aligned}$$

we can choose $\boldsymbol{\eta} = \begin{bmatrix} 0 \\ \frac{2}{3} \end{bmatrix}$.

Note that we don't need to find ***every*** generalised eigenvector

$$\boldsymbol{\eta} = \begin{bmatrix} k \\ k + \frac{2}{3} \end{bmatrix} = k \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{2}{3} \end{bmatrix} = k\boldsymbol{\xi} + \begin{bmatrix} 0 \\ \frac{2}{3} \end{bmatrix}$$

because we already have $\mathbf{x}^{(1)}(t) = \boldsymbol{\xi} e^{rt}$.

Instead we only need to find ***one*** generalised eigenvector – that means that we can choose any k that we want.

Hence I have chosen $k = 0$ which gives $\boldsymbol{\eta} = \begin{bmatrix} 0 \\ \frac{2}{3} \end{bmatrix}$.

Thus

$$\mathbf{x}^{(2)}(t) = \boldsymbol{\xi} t e^{-t} + \boldsymbol{\eta} e^{-t} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} t e^{-t} + \begin{bmatrix} 0 \\ \frac{2}{3} \end{bmatrix} e^{-t}.$$

Hence the general solution to the linear system is

$$\mathbf{x}(t) = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{-t} + c_2 \left(\begin{bmatrix} 1 \\ 1 \end{bmatrix} t e^{-t} + \begin{bmatrix} 0 \\ \frac{2}{3} \end{bmatrix} e^{-t} \right).$$

The initial condition gives

$$\begin{bmatrix} 3 \\ -1 \end{bmatrix} = \mathbf{x}(0) = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ \frac{2}{3} \end{bmatrix}$$

which implies that $c_1 = 3$ and $c_2 = -6$.

Therefore the solution to the IVP is

$$\boxed{\mathbf{x}(t) = 3 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{-t} - 6 \left(\begin{bmatrix} 1 \\ 1 \end{bmatrix} t e^{-t} + \begin{bmatrix} 0 \\ \frac{2}{3} \end{bmatrix} e^{-t} \right) = \begin{bmatrix} 3 \\ -1 \end{bmatrix} e^{-t} - 6 \begin{bmatrix} 1 \\ 1 \end{bmatrix} t e^{-t}.}$$

Example 7.20. Solve
$$\mathbf{x}' = \begin{bmatrix} 1 & -4 \\ 4 & -7 \end{bmatrix} \mathbf{x}, \qquad \mathbf{x}(0) = \begin{bmatrix} 3 \\ 2 \end{bmatrix}.$$

The only eigenvalue of the matrix is $r = -3$. The corresponding eigenvector is $\boldsymbol{\xi} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. Therefore one solution of the linear system is
$$\mathbf{x}^{(1)}(t) = \boldsymbol{\xi} e^{rt} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{-3t}.$$

Next we need to find a generalised eigenvector $\boldsymbol{\eta}$.
We calculate that
$$\begin{aligned} (A - rI)\boldsymbol{\eta} &= \boldsymbol{\xi} \\ \begin{bmatrix} 4 & -4 \\ 4 & -4 \end{bmatrix} \begin{bmatrix} \eta_1 \\ \eta_2 \end{bmatrix} &= \begin{bmatrix} 1 \\ 1 \end{bmatrix} \\ 4\eta_1 - 4\eta_2 &= 1 \\ -\eta_1 + \eta_2 &= -\frac{1}{4} \\ \eta_2 &= \eta_1 - \frac{1}{4}. \end{aligned}$$
So we can choose any vector $\boldsymbol{\eta}$ that satisfies $\eta_2 = \eta_1 - \frac{1}{4}$. Thus we may choose $\boldsymbol{\eta} = \begin{bmatrix} 0 \\ -\frac{1}{4} \end{bmatrix}$.

Therefore
$$\mathbf{x}^{(2)}(t) = \begin{bmatrix} 1 \\ 1 \end{bmatrix} te^{-3t} + \begin{bmatrix} 0 \\ -\frac{1}{4} \end{bmatrix} e^{-3t}.$$

Hence the general solution to the ODE is
$$\mathbf{x}(t) = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{-3t} + c_2 \left(\begin{bmatrix} 1 \\ 1 \end{bmatrix} te^{-3t} + \begin{bmatrix} 0 \\ -\frac{1}{4} \end{bmatrix} e^{-3t} \right).$$

The initial condition gives
$$\begin{bmatrix} 3 \\ 2 \end{bmatrix} = \mathbf{x}(0) = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ -\frac{1}{4} \end{bmatrix}$$
which implies that $c_1 = 3$ and $c_2 = 4$.

Therefore the solution to the IVP is
$$\begin{aligned} \mathbf{x}(t) &= 3 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{-3t} + 4 \left(\begin{bmatrix} 1 \\ 1 \end{bmatrix} te^{-3t} + \begin{bmatrix} 0 \\ -\frac{1}{4} \end{bmatrix} e^{-3t} \right) \\ &= \begin{bmatrix} 3 \\ 2 \end{bmatrix} e^{-3t} + 4 \begin{bmatrix} 1 \\ 1 \end{bmatrix} te^{-3t} \\ &= \begin{bmatrix} 3 + 4t \\ 2 + 4t \end{bmatrix} e^{-3t}. \end{aligned}$$

7.9 Nonhomogeneous Linear Systems

Consider

q:systemnonhomo}

$$\mathbf{x}' = P(t)\mathbf{x} + \mathbf{g}(t) \tag{7.1}$$

where $P(t)$ and $\mathbf{g}(t)$ are continuous for $\alpha < t < \beta$. The general solution of (7.1) can be written as

$$\mathbf{x}(t) = c_1\mathbf{x}^{(1)} + c_2\mathbf{x}^{(2)} + \ldots + c_n\mathbf{x}^{(n)} + \mathbf{v}(t)$$

where

- $c_1\mathbf{x}^{(1)} + c_2\mathbf{x}^{(2)} + \ldots + c_n\mathbf{x}^{(n)}$ is the general solution to the homogeneous system $\mathbf{x}' = P(t)\mathbf{x}$; and
- $\mathbf{v}(t)$ is a particular solution to (7.1).

Remark. We will study four methods to solve (7.1):

(i). Diagonalisation;

(ii). Undetermined Coefficients;

(iii). Variation of Parameters;

(iv). The Laplace Transform.

Method 1 – Diagonalisation:

Consider

$$\mathbf{x}' = A\mathbf{x} + \mathbf{g}(t).$$

Suppose that

- $A \in \mathbb{R}^{n\times n}$ is diagonalisable;
- $\mathbf{g} : (\alpha, \beta) \to \mathbb{R}^n$;
- $\boldsymbol{\xi}^{(1)}, \ldots, \boldsymbol{\xi}^{(n)}$ are eigenvectors of A; and
- $T = \begin{bmatrix} \boldsymbol{\xi}^{(1)} & \cdots & \boldsymbol{\xi}^{(n)} \end{bmatrix}$.

Then

$$D = T^{-1}AT = \begin{bmatrix} r_1 & 0 & \cdots & 0 \\ 0 & r_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & r_n \end{bmatrix}.$$

Let $\mathbf{y} = T^{-1}\mathbf{x}$. Then $\mathbf{x} = T\mathbf{y}$. It follows that

$$T\mathbf{y}' = \mathbf{x}' = A\mathbf{x} + \mathbf{g}(t) = AT\mathbf{y} + \mathbf{g}(t)$$

and

$$\mathbf{y}' = T^{-1}AT\mathbf{y} + T^{-1}\mathbf{g}(t) = D\mathbf{y} + \mathbf{h}(t) \tag{7.2}$$ {eq:diagonalsyst

where $\mathbf{h} = T^{-1}\mathbf{g}$.

But (7.2) is just the system

$$\begin{cases} y_1' = r_1 y_1 + h_1(t) & \longleftarrow \text{only } y_1 \text{ and } t \\ y_2' = r_2 y_2 + h_2(t) & \longleftarrow \text{only } y_2 \text{ and } t \\ \quad\vdots & \\ y_n' = r_n y_n + h_n(t) & \longleftarrow \text{only } y_n \text{ and } t \end{cases}$$

We can solve each of these n first order linear ODEs individually. The solution to

$$y_j' - r_j y_j = h_j$$

(see Chapter 2) is

$$y_j(t) = e^{r_j t}\int_{t_0}^{t} e^{-r_j s}h(s)\,ds + c_j e^{r_j t}.$$

If we know $\mathbf{y}$, then we know $\mathbf{x} = T\mathbf{y}$.

Example 7.21. Solve

$$\mathbf{x}' = \begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 2e^{-t} \\ 3t \end{bmatrix} = A\mathbf{x} + \mathbf{g}(t).$$

The eigenvalues of $A = \begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix}$ are $r_1 = -3$ and $r_2 = -1$. The eigenvectors are

$$\boldsymbol{\xi}^{(1)} = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \qquad \text{and} \qquad \boldsymbol{\xi}^{(2)} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

So

$$T = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \qquad \text{and} \qquad T^{-1} = \frac{1}{2}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}.$$

Let $\mathbf{y} = T^{-1}\mathbf{x}$. Then

$$\begin{aligned} T\mathbf{y}' = \mathbf{x}' = A\mathbf{x} + \begin{bmatrix} 2e^{-t} \\ 3t \end{bmatrix} &= AT\mathbf{y} + \begin{bmatrix} 2e^{-t} \\ 3t \end{bmatrix} \\ \mathbf{y}' &= T^{-1}AT\mathbf{y} + T^{-1}\begin{bmatrix} 2e^{-t} \\ 3t \end{bmatrix} \\ &= D\mathbf{y} + \frac{1}{2}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 2e^{-t} \\ 3t \end{bmatrix} \\ &= \begin{bmatrix} -3 & 0 \\ 0 & -1 \end{bmatrix}\mathbf{y} + \frac{1}{2}\begin{bmatrix} 2e^{-t} - 3t \\ 2e^{-t} + 3t \end{bmatrix}. \end{aligned}$$

Therefore

$$\begin{cases} y_1' + 3y_1 = e^{-t} - \frac{3}{2}t \\ y_2' + y_2 = e^{-t} + \frac{3}{2}t. \end{cases}$$

You know how to solve first order linear ODEs. The solutions to these two ODEs are

$$\begin{aligned} y_1(t) &= \frac{1}{2}e^{-t} - \frac{t}{2} + \frac{1}{6} + c_1 e^{-3t} \\ y_2(t) &= te^{-t} + \frac{3t}{2} - \frac{3}{2} + c_2 e^{-t}. \end{aligned}$$

Finally we calculate that

$$\begin{aligned} \mathbf{x} = T\mathbf{y} &= \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \\ &= \begin{bmatrix} y_1 + y_2 \\ -y_1 + y_2 \end{bmatrix} \\ &= c_1\begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{-3t} + c_2\begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{-t} + \frac{1}{2}\begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{-t} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} te^{-t} + \begin{bmatrix} 1 \\ 2 \end{bmatrix} t - \frac{1}{3}\begin{bmatrix} 4 \\ 5 \end{bmatrix}. \end{aligned}$$

Example 7.22. Solve

$$\mathbf{x}' = \begin{bmatrix} 1 & \sqrt{3} \\ \sqrt{3} & -1 \end{bmatrix} \mathbf{x} + \begin{bmatrix} e^t \\ \sqrt{3}e^{-t} \end{bmatrix}.$$

The eigenvalues of $\begin{bmatrix} 1 & \sqrt{3} \\ \sqrt{3} & -1 \end{bmatrix}$ are $r_1 = -2$ and $r_2 = 2$. The corresponding eigenvectors are $\boldsymbol{\xi}^{(1)} = \begin{bmatrix} 1 \\ -\sqrt{3} \end{bmatrix}$ and $\boldsymbol{\xi}^{(2)} = \begin{bmatrix} \sqrt{3} \\ 1 \end{bmatrix}$.

Thus

$$T = \begin{bmatrix} 1 & \sqrt{3} \\ -\sqrt{3} & 1 \end{bmatrix}$$

and

$$T^{-1} = \frac{1}{\det T} \begin{bmatrix} 1 & -\sqrt{3} \\ \sqrt{3} & 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{4} & -\frac{\sqrt{3}}{4} \\ \frac{\sqrt{3}}{4} & \frac{1}{4} \end{bmatrix}.$$

Now we must change variables: Let $\mathbf{y} = T^{-1}\mathbf{x}$. Then we have

$$\begin{aligned} \mathbf{y}' = D\mathbf{y} + T^{-1}\mathbf{g} &= \begin{bmatrix} -2 & 0 \\ 0 & 2 \end{bmatrix} \mathbf{y} + \begin{bmatrix} \frac{1}{4} & -\frac{\sqrt{3}}{4} \\ \frac{\sqrt{3}}{4} & \frac{1}{4} \end{bmatrix} \begin{bmatrix} e^t \\ \sqrt{3}e^{-t} \end{bmatrix} \\ &= \begin{bmatrix} -2y_1 \\ 2y_2 \end{bmatrix} + \begin{bmatrix} \frac{1}{4}e^t - \frac{3}{4}e^{-t} \\ \frac{\sqrt{3}}{4}e^t + \frac{\sqrt{3}}{4}e^{-t} \end{bmatrix}. \end{aligned}$$

We know how to solve

$$y_1' + 2y_1 = \frac{1}{4}e^t - \frac{3}{4}e^{-t}$$

and

$$y_2' - 2y_2 = \frac{\sqrt{3}}{4}e^t + \frac{\sqrt{3}}{4}e^{-t}.$$

The solutions are

$$y_1(t) = \frac{1}{12}e^t - \frac{3}{4}e^{-t} + c_1 e^{-2t}$$

and

$$y_2(t) = -\frac{\sqrt{3}}{4}e^t - \frac{\sqrt{3}}{12}e^{-t} + c_2 e^{2t}.$$

So

$$\mathbf{y} = \begin{bmatrix} \frac{1}{12}e^t - \frac{3}{4}e^{-t} + c_1 e^{-2t} \\ -\frac{\sqrt{3}}{4}e^t - \frac{\sqrt{3}}{12}e^{-t} + c_2 e^{2t} \end{bmatrix}.$$

Therefore the general solution to the ODE is

$$\mathbf{x} = T\mathbf{y} = \begin{bmatrix} 1 & \sqrt{3} \\ -\sqrt{3} & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{12}e^t - \frac{3}{4}e^{-t} + c_1 e^{-2t} \\ -\frac{\sqrt{3}}{4}e^t - \frac{\sqrt{3}}{12}e^{-t} + c_2 e^{2t} \end{bmatrix} = \dots$$

Method 2 – Undeterminded Coefficients:

Consider

$$\mathbf{x}' = A\mathbf{x} + \mathbf{g}(t).$$

(Remember Chapter 3?)

The idea is

(i). Find the general solution to $\mathbf{x}' = A\mathbf{x}$.

(ii). Look at $\mathbf{g}(t)$. Make a guess with constants. Find the constants.

(iii). $1 + 2$.

Example 7.23. Solve

$$\mathbf{x}' = \begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 2e^{-t} \\ 3t \end{bmatrix} = A\mathbf{x} + \mathbf{g}(t).$$

1. The solution of $\mathbf{x}' = \begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix} \mathbf{x}$ is

$$\mathbf{x}(t) = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{-t} + c_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{-3t}.$$

2. Since $\mathbf{g}(t) = \begin{bmatrix} 2 \\ 0 \end{bmatrix} e^{-t} + \begin{bmatrix} 0 \\ 3 \end{bmatrix} t$, we try the ansatz

$$\mathbf{x} = \mathbf{v}(t) = \mathbf{a}te^{-t} + \mathbf{b}e^{-t} + \mathbf{c}t + \mathbf{d}.$$

(Note that because $r_1 = -1$ is an eigenvalue of $\begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix}$, we need both te^{-t} and e^{-t}.)

Then we calculate that

$$\mathbf{x}' = A\mathbf{x} + \mathbf{g}$$

$$\mathbf{a}e^{-t} - \mathbf{a}te^{-t} - \mathbf{b}e^{-t} + \mathbf{c} = A\mathbf{a}te^{-t} + A\mathbf{b}e^{-t} + A\mathbf{c}t + A\mathbf{d} + \begin{bmatrix} 2 \\ 0 \end{bmatrix} e^{-t} + \begin{bmatrix} 0 \\ 3 \end{bmatrix} t.$$

- If we look at the te^{-t} terms, we have

$$-\mathbf{a} = A\mathbf{a} \quad \Longrightarrow \quad \mathbf{a} \text{ is an eigenvector} \quad \Longrightarrow \quad \mathbf{a} = \begin{bmatrix} \alpha \\ \alpha \end{bmatrix} \quad \text{for some } \alpha \in \mathbb{R}.$$

- If we look at the e^{-t} terms, we have

$$\mathbf{a} - \mathbf{b} = A\mathbf{b} + \begin{bmatrix} 2 \\ 0 \end{bmatrix}$$

which becomes

$$\begin{bmatrix} \alpha - 2 \\ \alpha \end{bmatrix} = \mathbf{a} - \begin{bmatrix} 2 \\ 0 \end{bmatrix} = (A + I)\mathbf{b} = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} = \begin{bmatrix} -b_1 + b_2 \\ b_1 - b_2 \end{bmatrix}.$$

But this means that

$$\alpha - 2 = -b_1 + b_2 = -(b_1 - b_2) = -\alpha \quad \Longrightarrow \alpha = 1.$$

So $\mathbf{a} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. Then we have that

$$b_1 - b_2 = 1 \qquad \Longrightarrow \qquad \mathbf{b} = \begin{bmatrix} k \\ k-1 \end{bmatrix}$$

for any k. If we choose $k = 0$, we get $\mathbf{b} = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$.

- If we look at the t terms, we have

$$0 = A\mathbf{c} + \begin{bmatrix} 0 \\ 3 \end{bmatrix} \qquad \Longrightarrow \mathbf{c} = A^{-1} \begin{bmatrix} 0 \\ -3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}.$$

- Finally, if we look at the 1 terms, we have

$$\mathbf{c} = A\mathbf{d} \qquad \Longrightarrow \qquad \mathbf{d} = A^{-1}\mathbf{c} = \begin{bmatrix} -\frac{4}{3} \\ -\frac{5}{3} \end{bmatrix}.$$

So

$$\mathbf{v} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} te^{-t} - \begin{bmatrix} 0 \\ 1 \end{bmatrix} e^{-t} + \begin{bmatrix} 1 \\ 2 \end{bmatrix} t - \frac{1}{3} \begin{bmatrix} 4 \\ 5 \end{bmatrix}.$$

3. Therefore the general solution to the ODE is

$$\boxed{\mathbf{x}(t) = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{-t} + c_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{-3t} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} te^{-t} - \begin{bmatrix} 0 \\ 1 \end{bmatrix} e^{-t} + \begin{bmatrix} 1 \\ 2 \end{bmatrix} t - \frac{1}{3} \begin{bmatrix} 4 \\ 5 \end{bmatrix}.}$$

Example 7.24. Solve

$$\mathbf{x}' = \begin{bmatrix} 2 & 3 \\ 4 & 1 \end{bmatrix} \mathbf{x} + \begin{bmatrix} e^t \\ -10t - 3 \end{bmatrix}.$$

The matrix $\begin{bmatrix} 2 & 3 \\ 4 & 1 \end{bmatrix}$ has eigenvalues $r_1 = 5$ and $r_2 = -2$ and eigenvectors $\boldsymbol{\xi}^{(1)} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\boldsymbol{\xi}^{(2)} = \begin{bmatrix} -3 \\ 4 \end{bmatrix}$. Hence the general solution of $\mathbf{x}' = \begin{bmatrix} 2 & 3 \\ 4 & 1 \end{bmatrix} \mathbf{x}$ is

$$\mathbf{x}(t) = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{5t} + c_2 \begin{bmatrix} -3 \\ 4 \end{bmatrix} e^{-2t}.$$

Next we need to find a particular solution to $\mathbf{x}' = \begin{bmatrix} 2 & 3 \\ 4 & 1 \end{bmatrix} \mathbf{x} + \begin{bmatrix} e^t \\ 0 \end{bmatrix}$. Since 1 is not an eigenvector of $\begin{bmatrix} 2 & 3 \\ 4 & 1 \end{bmatrix}$, we try the ansatz $\mathbf{x} = \mathbf{a}e^t$ for some $\mathbf{a} = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} \in \mathbb{R}^2$. Then we calculate that

$$\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} e^t = \mathbf{x}' = \begin{bmatrix} 2 & 3 \\ 4 & 1 \end{bmatrix} \mathbf{x} + \begin{bmatrix} e^t \\ 0 \end{bmatrix} = \begin{bmatrix} 2a_1 + 3a_2 + 1 \\ 4a_1 + a_2 \end{bmatrix} e^t$$

which gives

$$\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} 2a_1 + 3a_2 + 1 \\ 4a_1 + a_2 \end{bmatrix}.$$

Hence $a_1 = 0$ and $a_2 = -\frac{1}{3}$. So $\mathbf{x} = \begin{bmatrix} 0 \\ -\frac{1}{3} \end{bmatrix} e^t$.

Then we need to find a particular solution to $\mathbf{x}' = \begin{bmatrix} 2 & 3 \\ 4 & 1 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 0 \\ -10t - 3 \end{bmatrix}$. We try the ansatz $\mathbf{x} = \mathbf{a}t + \mathbf{b} = \begin{bmatrix} a_1 t + b_1 \\ a_2 t + b_2 \end{bmatrix}$ for some $\mathbf{a}, \mathbf{b} \in \mathbb{R}^2$ and calculate that

$$\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \mathbf{x}' = \begin{bmatrix} 2 & 3 \\ 4 & 1 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 0 \\ t \end{bmatrix} = \begin{bmatrix} 2a_1 t + 2b_1 + 3a_2 t + 3b_2 \\ 4a_1 t + 4b_1 + a_2 t + b_2 - 10t - 3 \end{bmatrix}$$

which leads to

$$\begin{cases} 0 = 2a_1 + 3a_2 \\ a_1 = 2b_1 + 3b_2 \\ 0 = 4a_1 + a_2 - 10 \\ a_2 = 4b_1 + b_2 - 3. \end{cases}$$

The solution to this linear system is $\mathbf{a} = \begin{bmatrix} 3 \\ -2 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. Hence $\mathbf{x} = \begin{bmatrix} 3t \\ 1 - 2t \end{bmatrix}$.

Adding all of these together, we find that the general solution to the given ODE is

$$\boxed{\mathbf{x} = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{5t} + c_2 \begin{bmatrix} -3 \\ 4 \end{bmatrix} e^{-2t} + \begin{bmatrix} 0 \\ -\frac{1}{3} \end{bmatrix} e^t + \begin{bmatrix} 3t \\ 1 - 2t \end{bmatrix}.}$$

Method 3 – Variation of Parameters:

Consider

$$\mathbf{x}' = P(t)\mathbf{x} + \mathbf{g}(t) \tag{7.1}$$

where

- P and $\mathbf{g}$ are continuous for $\alpha < t < \beta$;
- there exists a fundamental matrix $\Psi(t)$ for the homogeneous system $\mathbf{x}' = P(t)\mathbf{x}$.

We know that the general solution to $\mathbf{x}' = P(t)\mathbf{x}$ is $\mathbf{x} = \Psi(t)\mathbf{c}$.

We guess that

$$\mathbf{x} = \Psi(t)\mathbf{u}(t)$$

is a solution to (7.1). Can we find $\mathbf{u}(t)$?

If $\mathbf{x} = \Psi\mathbf{u}$, we can calculate that

$$\cancel{\Psi'\mathbf{u}} + \Psi\mathbf{u}' = \mathbf{x}' = P\mathbf{x} + \mathbf{g} = \cancel{P\Psi\mathbf{u}} + \mathbf{g}. \tag{7.3}$$

{eq:systemnonhom

But remember that

$$\Psi \text{ is a fundamental matrix for } \mathbf{x}' = P(t)\mathbf{x} \implies \Psi \text{ solves } \Psi' = P\Psi.$$

Hence (7.3) becomes

$$\Psi\mathbf{u}' = \mathbf{g}.$$

Therefore

$$\mathbf{u}' = \Psi^{-1}\mathbf{g}$$

and

$$\mathbf{u} = \int \Psi^{-1}\mathbf{g}.$$

Hence

$$\mathbf{x} = \Psi(t)\mathbf{u}(t) = \Psi(t)\int \Psi^{-1}(s)g(s)\,ds.$$

Remark. To solve $\mathbf{x}' = P(t)\mathbf{x} + \mathbf{g}(t)$, the method is

(i). Find a fundamental matrix Ψ for $\mathbf{x}' = P(t)\mathbf{x}$;

(ii). Calculate $\mathbf{x} = \Psi(t)\int \Psi^{-1}(s)g(s)\,ds$.

Example 7.25. Solve

$$\mathbf{x}' = \begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 2e^{-t} \\ 3t \end{bmatrix} = A\mathbf{x} + \mathbf{g}(t).$$

The solution of $\mathbf{x}' = \begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix} \mathbf{x}$ is

$$\mathbf{x}(t) = c_1 \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{-3t} + c_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{-t}.$$

So

$$\Psi(t) = \begin{bmatrix} e^{-3t} & e^{-t} \\ -e^{-3t} & e^{-t} \end{bmatrix}$$

is a fundamental matrix.

Then we calculate that

$$\Psi^{-1}(t) = \frac{1}{2e^{-4t}} \begin{bmatrix} e^{-t} & -e^{-t} \\ e^{-3t} & e^{-3t} \end{bmatrix} = \frac{1}{2} e^{4t} \begin{bmatrix} e^{-t} & -e^{-t} \\ e^{-3t} & e^{-3t} \end{bmatrix} = \begin{bmatrix} \frac{1}{2}e^{3t} & -\frac{1}{2}e^{3t} \\ \frac{1}{2}e^{t} & \frac{1}{2}e^{t} \end{bmatrix}$$

and

$$\begin{aligned} \int \Psi^{-1}(t)\mathbf{g}(t)\, dt &= \int \begin{bmatrix} \frac{1}{2}e^{3t} & -\frac{1}{2}e^{3t} \\ \frac{1}{2}e^{t} & \frac{1}{2}e^{t} \end{bmatrix} \begin{bmatrix} 2e^{-t} \\ 3t \end{bmatrix} dt \\ &= \int \begin{bmatrix} e^{2t} - \frac{3}{2}te^{3t} \\ 1 + \frac{3}{2}te^{t} \end{bmatrix} dt = \begin{bmatrix} \frac{1}{2}e^{2t} - \frac{1}{2}te^{3t} + \frac{1}{6}e^{3t} + c_1 \\ t + \frac{3}{2}te^{t} - \frac{3}{2}e^{t} + c_2 \end{bmatrix}. \end{aligned}$$

Therefore the solution to $\mathbf{x}' = A\mathbf{x} + g$ is

$$\begin{aligned} \mathbf{x} &= \Psi(t) \int \Psi^{-1}(s)\mathbf{g}(s)\, ds \\ &= \begin{bmatrix} e^{-3t} & e^{-t} \\ -e^{-3t} & e^{-t} \end{bmatrix} \begin{bmatrix} \frac{1}{2}e^{2t} - \frac{1}{2}te^{3t} + \frac{1}{6}e^{3t} + c_1 \\ t + \frac{3}{2}te^{t} - \frac{3}{2}e^{t} + c_2 \end{bmatrix} \\ &= c_1 \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{-3t} + c_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{-t} + \frac{1}{2} \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{-t} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} te^{-t} + \begin{bmatrix} 1 \\ 2 \end{bmatrix} t - \frac{1}{3} \begin{bmatrix} 4 \\ 5 \end{bmatrix}. \end{aligned}$$

Example 7.26. Solve
$$\mathbf{x}' = \begin{bmatrix} -4 & 2 \\ 2 & -1 \end{bmatrix} \mathbf{x} + \begin{bmatrix} t^{-1} \\ 2t^{-1} + 4 \end{bmatrix}$$
for $t > 0$.

The eigenvalues of $A = \begin{bmatrix} -4 & 2 \\ 2 & -1 \end{bmatrix}$ are $r_1 = 0$ and $r_2 = -5$; and the eigenvectors are $\xi^{(1)} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $\xi^{(2)} = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$. Thus
$$\Psi(t) = \begin{bmatrix} 1 & -2e^{-5t} \\ 2 & e^{-5t} \end{bmatrix}$$
is a fundamental matrix for $\mathbf{x}' = A\mathbf{x}$.

Using the formula $\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$ we calculate that
$$\Psi^{-1}(t) = \frac{1}{e^{-5t} + 4e^{-5t}} \begin{bmatrix} e^{-5t} & 2e^{-5t} \\ -2 & 1 \end{bmatrix} = \frac{1}{5} \begin{bmatrix} 1 & 2 \\ -2e^{5t} & e^{5t} \end{bmatrix}.$$

Then
$$\begin{aligned} \Psi^{-1}(t)\mathbf{g}(t) &= \frac{1}{5} \begin{bmatrix} 1 & 2 \\ -2e^{5t} & e^{5t} \end{bmatrix} \begin{bmatrix} t^{-1} \\ 2t^{-1} + 4 \end{bmatrix} \\ &= \frac{1}{5} \begin{bmatrix} t^{-1} + 4t^{-1} + 8 \\ -2t^{-1}e^{5t} + 2t^{-1}e^{5t} + 4e^{5t} \end{bmatrix} = \begin{bmatrix} t^{-1} + \frac{8}{5} \\ \frac{4}{5}e^{5t} \end{bmatrix} \end{aligned}$$
and
$$\int \Psi^{-1}(t)\mathbf{g}(t)\,dt = \int \begin{bmatrix} t^{-1} + \frac{8}{5} \\ \frac{4}{5}e^{5t} \end{bmatrix} dt = \begin{bmatrix} \ln t + \frac{8}{5}t + c_1 \\ \frac{4}{25}e^{5t} + c_2 \end{bmatrix}.$$

It follows that
$$\begin{aligned} \mathbf{x}(t) = \Psi(t) \int \Psi^{-1}(s)\mathbf{g}(s)\,ds &= \begin{bmatrix} 1 & -2e^{-5t} \\ 2 & e^{-5t} \end{bmatrix} \begin{bmatrix} \ln t + \frac{8}{5}t + c_1 \\ \frac{4}{25}e^{5t} + c_2 \end{bmatrix} \\ &= \begin{bmatrix} \ln t + \frac{8}{5}t - \frac{8}{25} + c_1 - 2c_2e^{-5t} \\ 2\ln t + \frac{16}{5}t + \frac{4}{25} + 2c_1 + c_2e^{-5t} \end{bmatrix} \\ &= c_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + c_2 \begin{bmatrix} -2 \\ 1 \end{bmatrix} e^{-5t} + \begin{bmatrix} 1 \\ 2 \end{bmatrix} \ln t + \frac{8}{5} \begin{bmatrix} 1 \\ 2 \end{bmatrix} t + \frac{4}{25} \begin{bmatrix} -2 \\ 1 \end{bmatrix}. \end{aligned}$$

Method 4 – The Laplace Transform:

First some notation: If $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$, then $\mathbf{X} = \mathcal{L}\left[\mathbf{x}\right] = \begin{bmatrix} \mathcal{L}\left[x_1\right] \\ \mathcal{L}\left[x_2\right] \\ \vdots \\ \mathcal{L}\left[x_n\right] \end{bmatrix}$.

Recall from Chapter 6 that $\mathcal{L}\left[y'\right]$ satisfies

$$\mathcal{L}\left[y'\right](s) = sY(s) - y(0).$$

It follows that:

Theorem 7.3.

$$\mathcal{L}\left[\mathbf{x}'\right](s) = s\mathbf{X}(s) - \mathbf{x}(0).$$

Example 7.27. Solve

$$\begin{cases} \mathbf{x}' = \begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 2e^{-t} \\ 3t \end{bmatrix} = A\mathbf{x} + \mathbf{g}(t), \\ \mathbf{x}(0) = \mathbf{0}. \end{cases}$$

Taking Laplace Transforms of the ODE gives

$$s\mathbf{X}(s) - \mathbf{x}(0) = A\mathbf{X}(s) + \mathbf{G}(s)$$

where $\mathbf{G}(s) = \mathcal{L}\left[\mathbf{g}\right](s) = \begin{bmatrix} \frac{2}{s+1} \\ \frac{3}{s^2} \end{bmatrix}$.

Thus

$$(sI - A)\mathbf{X} = \mathbf{G}$$

and

$$\mathbf{X} = (sI - A)^{-1}\mathbf{G}$$

where

$$(sI - A)^{-1} = \begin{bmatrix} s+2 & -1 \\ -1 & s+2 \end{bmatrix}^{-1} = \frac{1}{(s+1)(s+3)} \begin{bmatrix} s+2 & 1 \\ 1 & s+2 \end{bmatrix}.$$

So

$$\begin{aligned} \mathbf{X} &= (sI - A)^{-1}\mathbf{G} \\ &= \frac{1}{(s+1)(s+3)} \begin{bmatrix} s+2 & 1 \\ 1 & s+2 \end{bmatrix} \begin{bmatrix} \frac{2}{s+1} \\ \frac{3}{s^2} \end{bmatrix} \\ &= \begin{bmatrix} \frac{2(s+2)}{(s+1)^2(s+3)} + \frac{3}{s^2(s+1)(s+3)} \\ \frac{2}{(s+1)^2(s+3)} + \frac{3(s+2)}{s^2(s+1)(s+3)} \end{bmatrix}. \end{aligned}$$

When we take the inverse Laplace Transform of this, we find our solution

$$\mathbf{x} = \mathcal{L}^{-1}\left[\mathbf{X}\right] = \begin{bmatrix} 2 \\ 1 \end{bmatrix} e^{-t} - \frac{2}{3} \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{-3t} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} te^{-t} + \begin{bmatrix} 1 \\ 2 \end{bmatrix} t - \frac{1}{3} \begin{bmatrix} 4 \\ 5 \end{bmatrix}.$$

Example 7.28. Solve

$$\begin{cases} 2x' + y' - y - t = 0 \\ x' + y' - t^2 = 0 \\ x(0) = 1 \\ y(0) = 0 \end{cases}$$

The ODEs above can be written as

$$\begin{cases} x' = y - t^2 + t \\ y' = -y + 2t^2 - t \end{cases}$$

(please check!).

If we write the problem in terms of matrices (using $\mathbf{x} = \begin{bmatrix} x \\ y \end{bmatrix}$) we have

$$\begin{cases} \mathbf{x}' = A\mathbf{x} + \mathbf{g} = \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix} \mathbf{x} + \begin{bmatrix} t - t^2 \\ 2t^2 - t \end{bmatrix} \\ \mathbf{x}(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}. \end{cases}$$

Taking the Laplace transform of the ODE gives

$$(sI - A)\,\mathbf{X}\,(s) = \mathbf{x}\,(0) + \mathbf{G}(s)$$

$$\begin{bmatrix} s & -1 \\ 0 & s+1 \end{bmatrix} \mathbf{X}\,(s) = \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \begin{bmatrix} \frac{1}{s^2} - \frac{2}{s^3} \\ \frac{4}{s^3} - \frac{1}{s^2} \end{bmatrix}$$

Thus

$$\begin{aligned} \mathbf{X}(s) &= \frac{1}{s\,(s+1)} \begin{bmatrix} s+1 & 1 \\ 0 & s \end{bmatrix} \frac{1}{s^3} \begin{bmatrix} s^3 + s - 2 \\ 4 - s \end{bmatrix} \\ &= \frac{1}{s^4\,(s+1)} \begin{bmatrix} s^4 + s^3 + s^2 - 2s + 2 \\ 4s - s^2 \end{bmatrix}. \end{aligned}$$

Note that

$$\frac{s^4 + s^3 + s^2 - 2s + 2}{s^4\,(s+1)} = \frac{5}{s+1} - 4\frac{1}{s} + 5\frac{1}{s^2} - 4\frac{1}{s^3} + 2\frac{1}{s^4}$$

and

$$\frac{4s - s^2}{s^4\,(s+1)} = -5\frac{1}{s+1} + 5\frac{1}{s} - 5\frac{1}{s^2} + 4\frac{1}{s^3}$$

(please check!).

It follows that

$$\mathcal{L}^{-1}\left(\frac{s^4 + s^3 + s^2 - 2s + 2}{s^4\,(s+1)}\right) = 5e^{-t} - 4 + 5t - 2t^2 + \frac{1}{3}t^3$$

and

$$\mathcal{L}^{-1}\left(\frac{4s - s^2}{s^4\,(s+1)}\right) = -5e^{-t} + 5 - 5t + 2t^2.$$

Therefore the solution to the initial value problem is

$$\boxed{\mathbf{x}\,(t) = \begin{bmatrix} 5e^{-t} - 4 + 5t - 2t^2 + \frac{1}{3}t^3 \\ -5e^{-t} + 5 - 5t + 2t^2 \end{bmatrix}}.$$