

Lecture 8

- 5.1 Area and Estimating with Finite Sums
- 5.2 Sigma Notation and Limits of Finite Sums
- 5.3 The Definite Integral



Area and Estimating with Finite Sums

5.1 Area and Estimating with Finite Sums



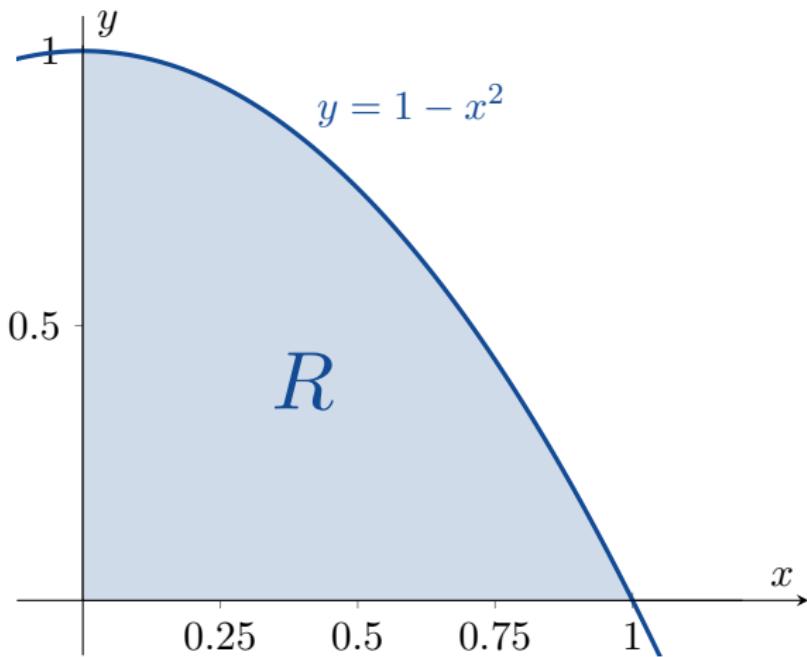
I am just going to give a quick recap of this section.

Please read more details in the textbook.

5.1 Area and Estimating with Finite Sums

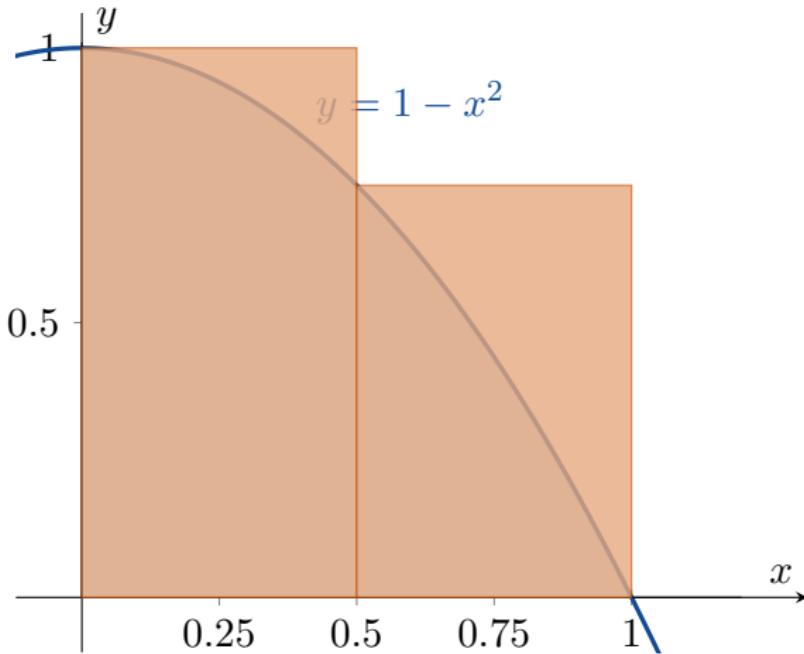


What is the area of R ?



5.1 Area and Estimating with Finite Sums

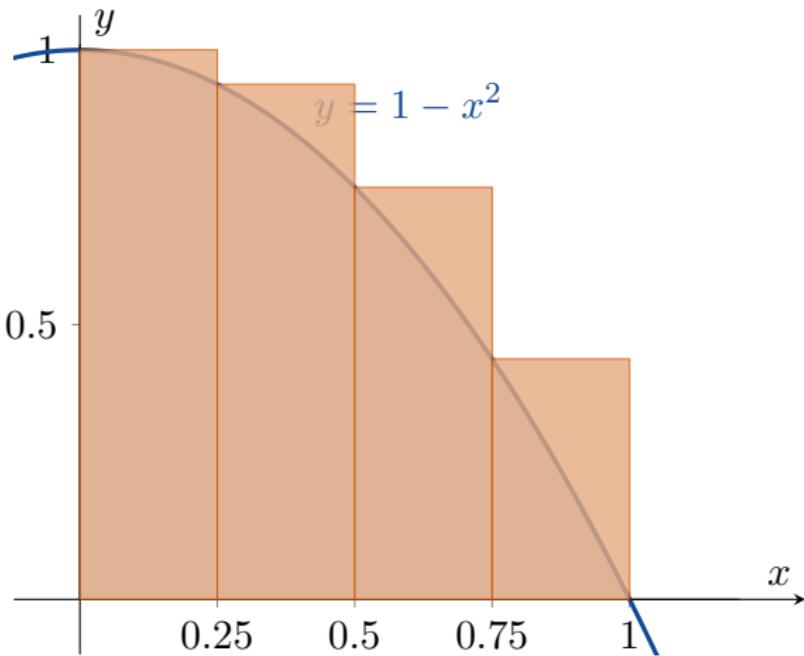
Upper Sum ($n = 2$)



$$1 \cdot \frac{1}{2} + \frac{3}{4} \cdot \frac{1}{2} = \frac{7}{8} = 0.875$$

5.1 Area and Estimating with Finite Sums

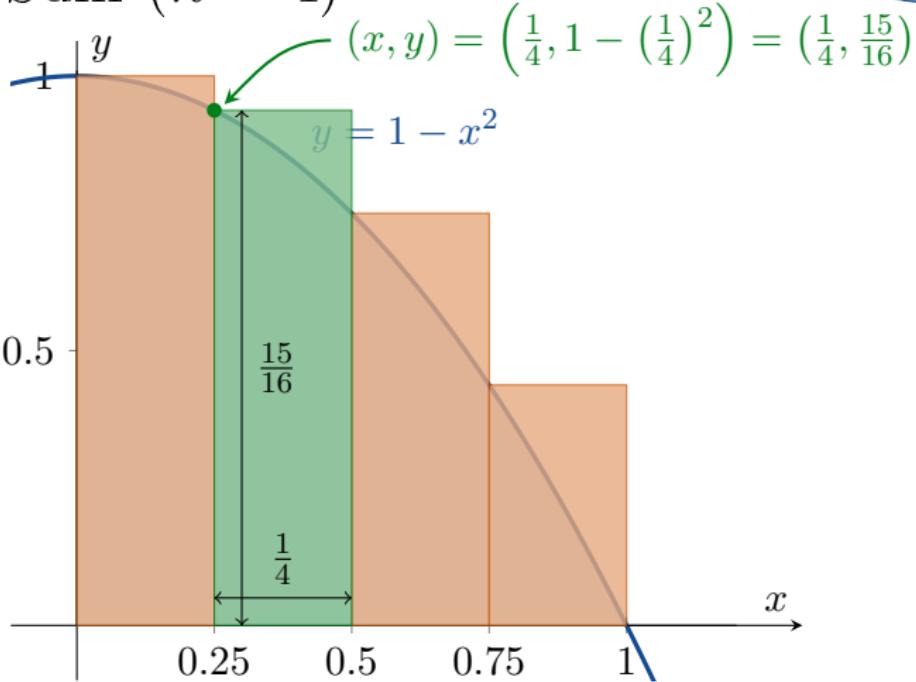
Upper Sum ($n = 4$)



$$1 \cdot \frac{1}{4} + \frac{15}{16} \cdot \frac{1}{4} + \frac{3}{4} \cdot \frac{1}{4} + \frac{7}{16} \cdot \frac{1}{4} = \frac{25}{32} = 0.78125$$

5.1 Area and Estimating with Finite Sums

Upper Sum ($n = 4$)

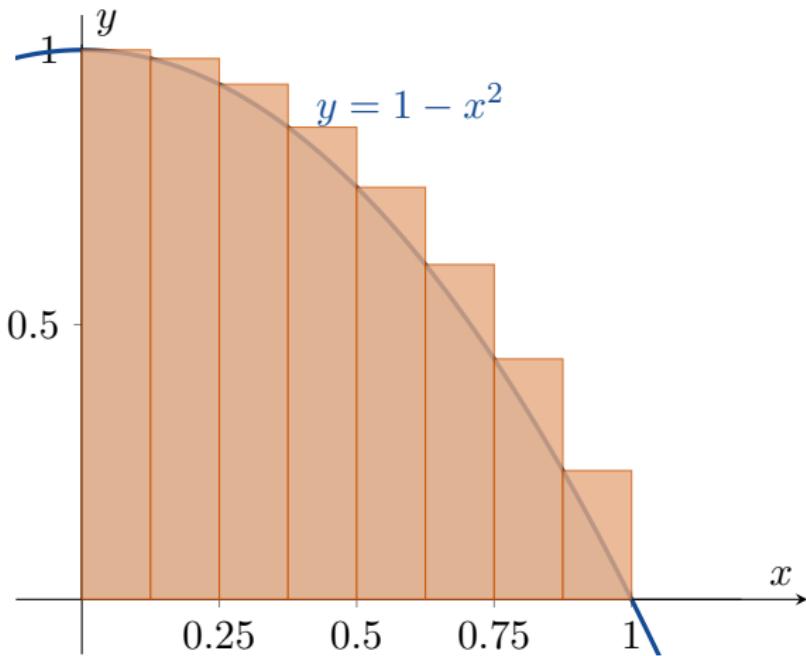


$$1 \cdot \frac{1}{4} + \frac{15}{16} \cdot \frac{1}{4} + \frac{3}{4} \cdot \frac{1}{4} + \frac{7}{16} \cdot \frac{1}{4} = \frac{25}{32} = 0.78125$$

5.1 Area and Estimating with Finite Sums



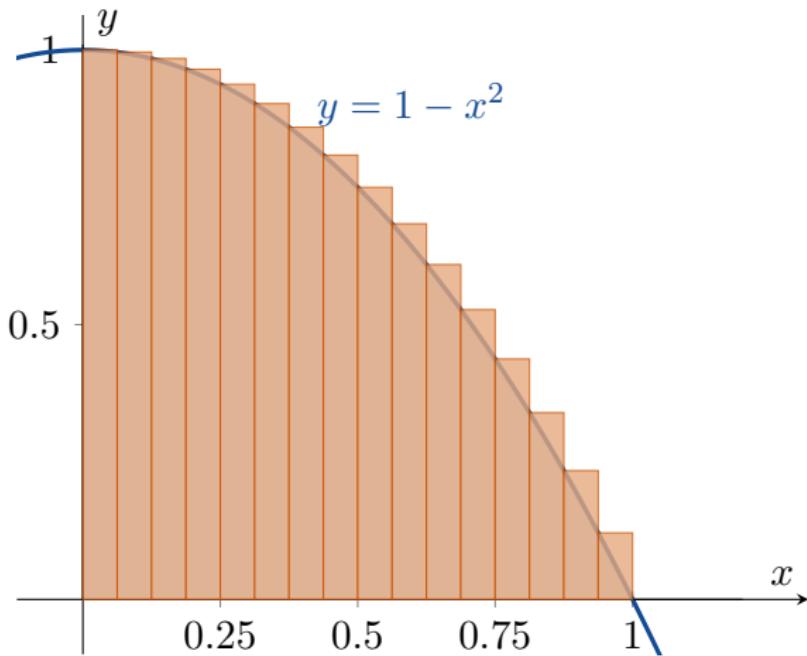
Upper Sum ($n = 8$)



5.1 Area and Estimating with Finite Sums



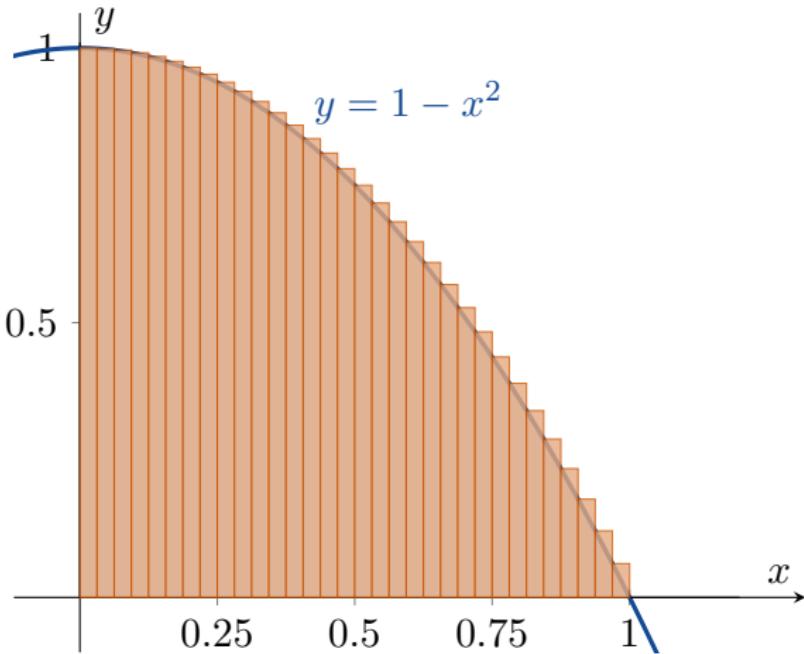
Upper Sum ($n = 16$)



5.1 Area and Estimating with Finite Sums

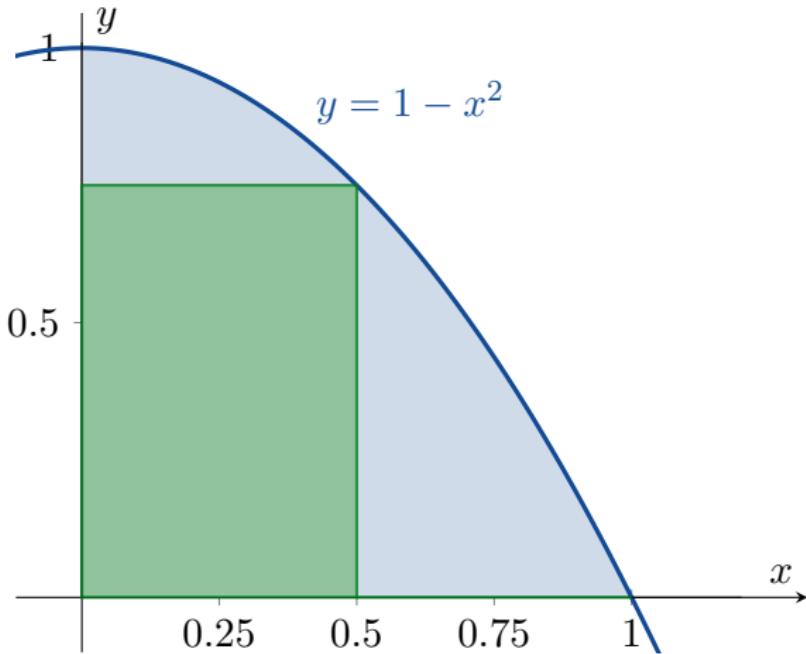


Upper Sum ($n = 32$)



5.1 Area and Estimating with Finite Sums

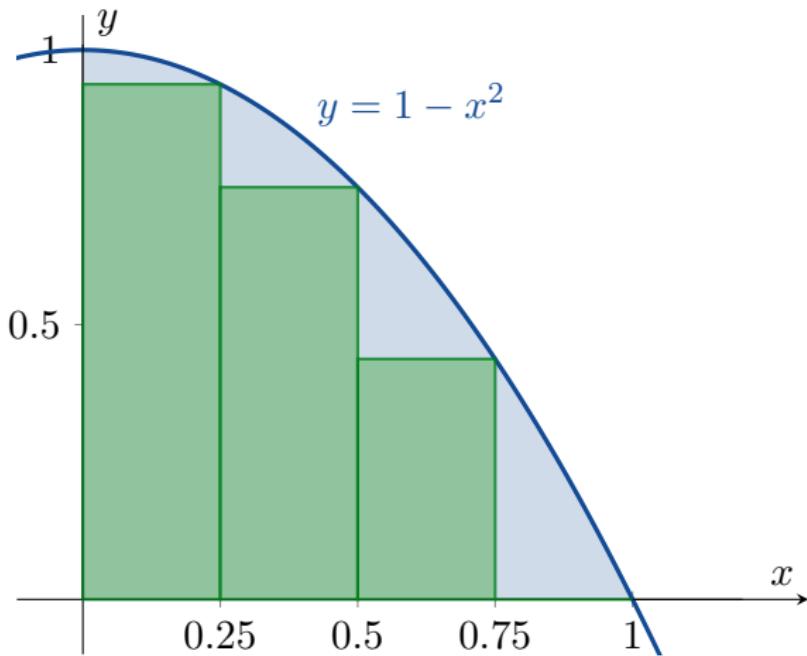
Lower Sum ($n = 2$)



$$\frac{3}{4} \cdot \frac{1}{2} + 0 \cdot \frac{1}{2} = \frac{3}{8} = 0.375$$

5.1 Area and Estimating with Finite Sums

Lower Sum ($n = 4$)

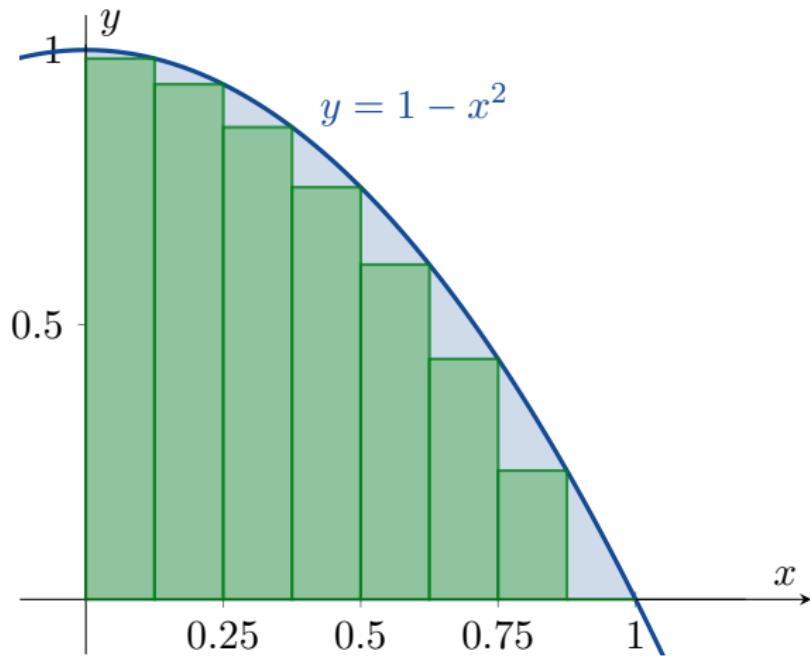


$$\frac{15}{16} \cdot \frac{1}{4} + \frac{3}{4} \cdot \frac{1}{4} + \frac{7}{16} \cdot \frac{1}{4} + 0 \cdot \frac{1}{4} = \frac{17}{32} = 0.53125$$

5.1 Area and Estimating with Finite Sums



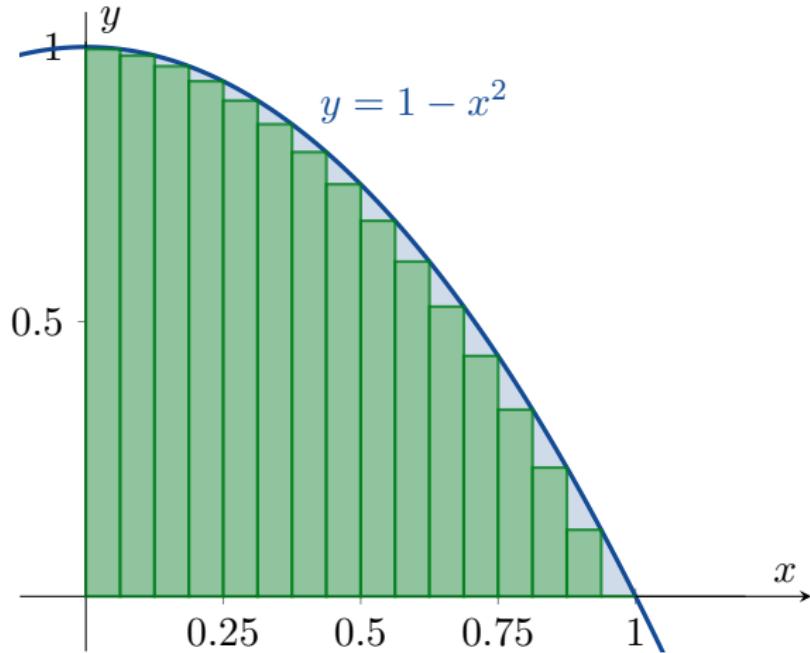
Lower Sum ($n = 8$)



5.1 Area and Estimating with Finite Sums



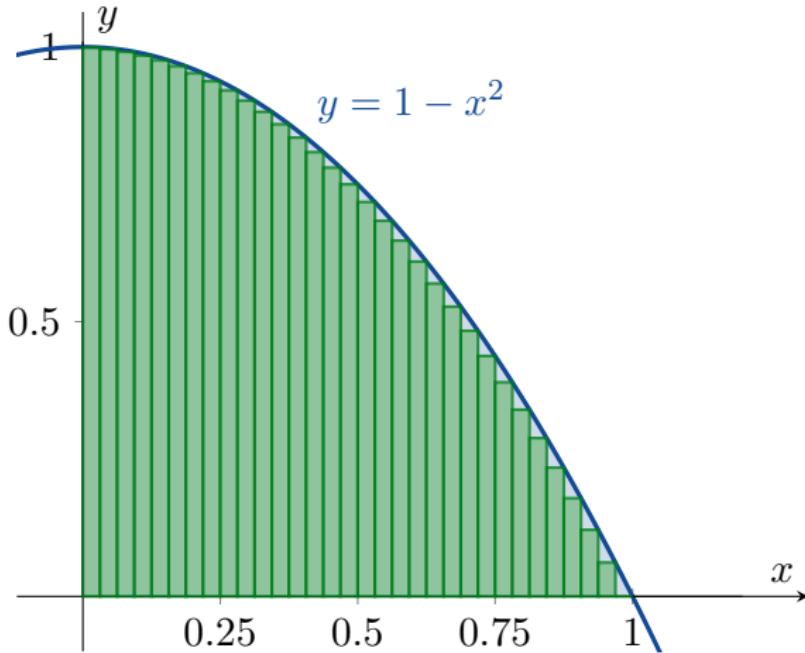
Lower Sum ($n = 16$)



5.1 Area and Estimating with Finite Sums



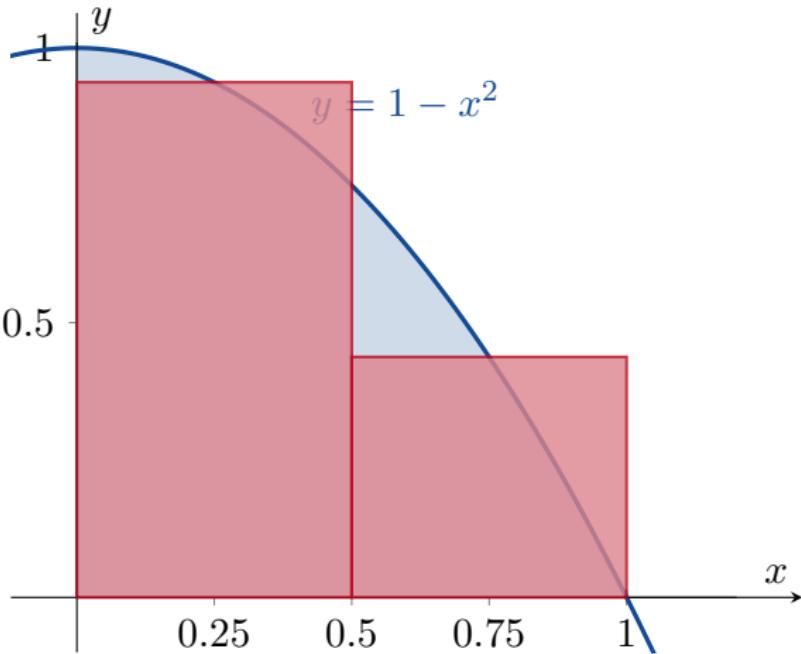
Lower Sum ($n = 32$)



5.1 Area and Estimating with Finite Sums



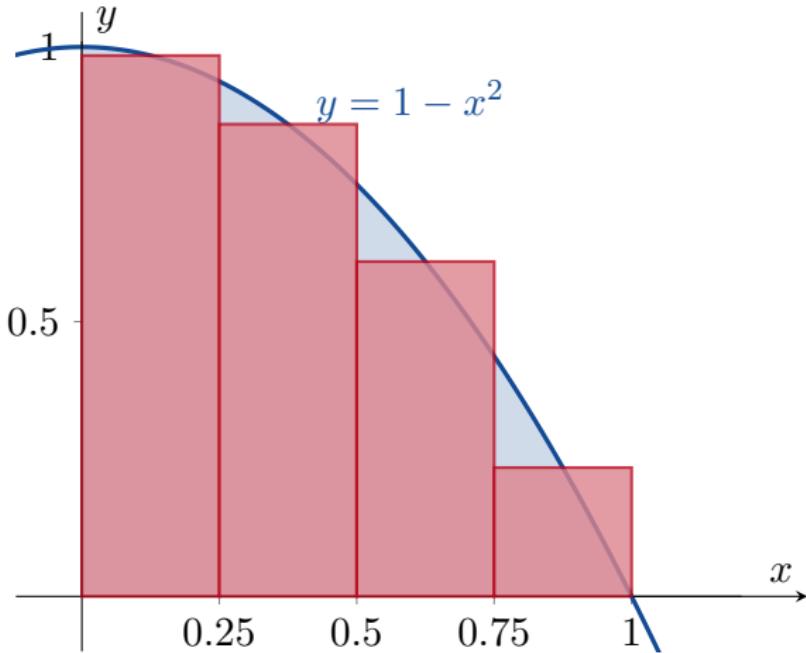
Midpoint Sum ($n = 2$)



5.1 Area and Estimating with Finite Sums



Midpoint Sum ($n = 4$)

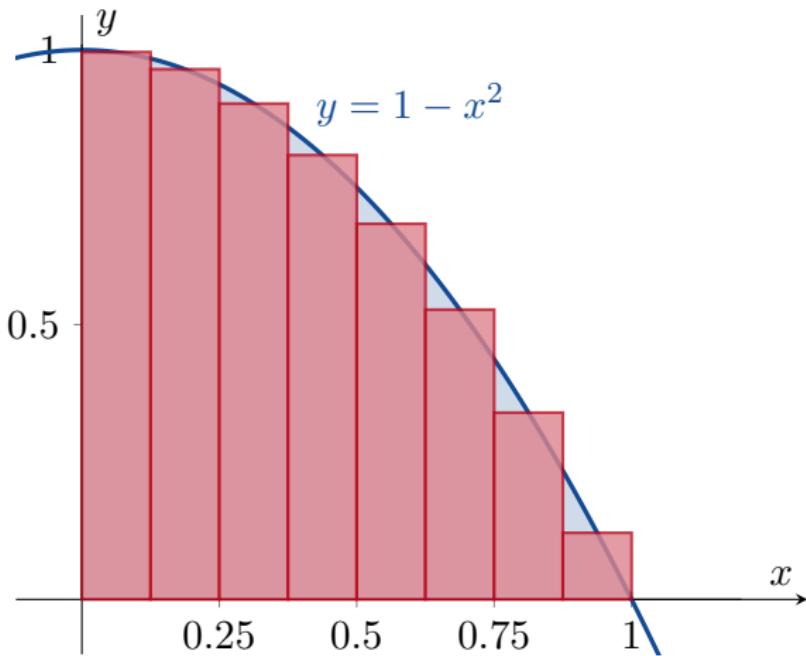


$$\frac{63}{64} \cdot \frac{1}{4} + \frac{55}{64} \cdot \frac{1}{4} + \frac{39}{64} \cdot \frac{1}{4} + \frac{15}{64} \cdot \frac{1}{4} = \frac{172}{256} = 0.671875$$

5.1 Area and Estimating with Finite Sums



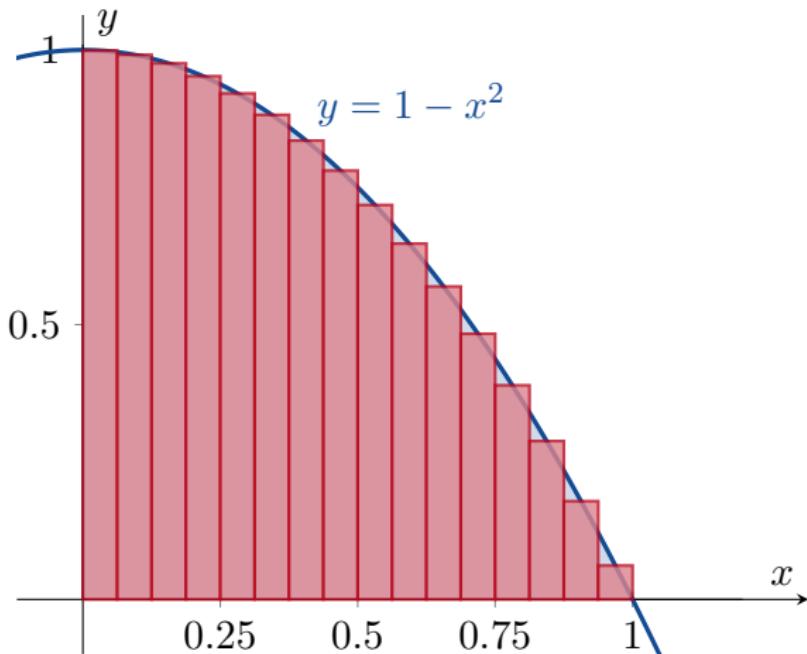
Midpoint Sum ($n = 8$)



5.1 Area and Estimating with Finite Sums



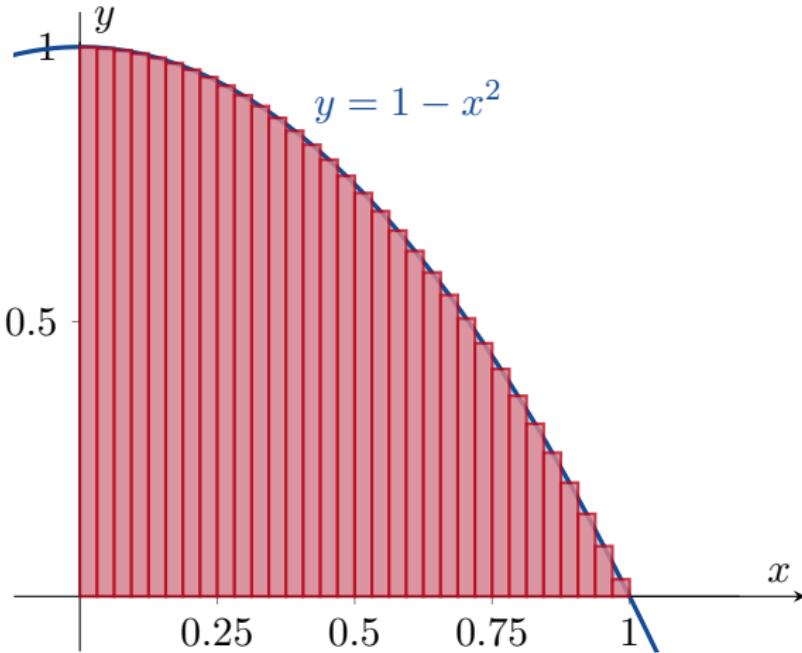
Midpoint Sum ($n = 16$)



5.1 Area and Estimating with Finite Sums



Midpoint Sum ($n = 32$)



Number of subintervals	Lower sum	Midpoint sum	Upper sum
2	0.375	0.6875	0.875
4	0.53125	0.671875	0.78125
16	0.634765625	0.6669921875	0.697265625
50	0.6566	0.6667	0.6766
100	0.66165	0.666675	0.67165
1000	0.6661665	0.66666675	0.6671665

5.1 Area and Estimating with Finite Sums



Remark

As n increases, the estimates get closer and closer to the real area of R . We want to take the limit as $n \rightarrow \infty$, but we don't yet have enough notation.



Sigma Notation and Limits of Finite Sums

5.2 Sigma Notation and Limits of Finite Sums



$$\sum_{k=1}^n a_k = a_1 + a_2 + a_3 + a_4 + a_5 + a_6 + a_7 + a_8 + \dots + a_{n-1} + a_n$$

5.2 Sigma Notation and Limits of Finite Sums



$$\sum_{k=1}^n a_k = a_1 + a_2 + a_3 + a_4 + a_5 + a_6 + a_7 + a_8 + \dots + a_{n-1} + a_n$$

$$\sum_{k=1}^n a_k$$

5.2 Sigma Notation and Limits of Finite Sums



$$\sum_{k=1}^n a_k = a_1 + a_2 + a_3 + a_4 + a_5 + a_6 + a_7 + a_8 + \dots + a_{n-1} + a_n$$

the Greek
letter Sigma

$$\sum_{k=1}^n a_k$$

5.2 Sigma Notation and Limits of Finite Sums



$$\sum_{k=1}^n a_k = a_1 + a_2 + a_3 + a_4 + a_5 + a_6 + a_7 + a_8 + \dots + a_{n-1} + a_n$$

the Greek
letter Sigma

$$\sum_{k=1}^n a_k$$

the sum starts
at $k = 1$

5.2 Sigma Notation and Limits of Finite Sums



$$\sum_{k=1}^n a_k = a_1 + a_2 + a_3 + a_4 + a_5 + a_6 + a_7 + a_8 + \dots + a_{n-1} + a_n$$

the Greek
letter Sigma

$$\sum_{k=1}^n a_k$$

the sum finishes
at $k = n$

the sum starts
at $k = 1$

5.2 Sigma Notation and Limits of Finite Sums



$$\sum_{k=1}^n a_k = a_1 + a_2 + a_3 + a_4 + a_5 + a_6 + a_7 + a_8 + \dots + a_{n-1} + a_n$$

the Greek
letter Sigma

$$\sum_{k=1}^n a_k$$

the sum finishes
at $k = n$

a_k is a formula for
the k^{th} term.

the sum starts
at $k = 1$

5.2 Sigma Notation and Limits of Finite Sums



Example

$$1^2 + 2^2 + 3^2 + 4^2 + 5^2 + 6^2 + 7^2 + 8^2 + 9^2 + 10^2 + 11^2 = \sum_{k=1}^{11} k^2$$

$$f(1) + f(2) + f(3) + \dots + f(99) + f(100) = \sum_{k=1}^{100} f(k)$$

$$\sum_{k=1}^5 k = 1 + 2 + 3 + 4 + 5 = 15$$

5.2 Sigma Notation and Limits of Finite Sums



Example

$$\sum_{k=1}^3 (-1)^k k = (-1)(1) + (-1)^2(2) + (-1)^3(3) = -1 + 2 - 3 = -2$$

$$\sum_{k=1}^2 \frac{k}{k+1} = \frac{1}{1+1} + \frac{2}{2+1} = \frac{1}{2} + \frac{2}{3} = \frac{7}{6}$$

$$\sum_{k=4}^5 \frac{k^2}{k-1} = \frac{4^2}{4-1} + \frac{5^2}{5-1} = \frac{16}{3} + \frac{25}{4} = \frac{139}{12}$$

Algebra Rules for Finite Sums

1. Sum Rule:

$$\sum_{k=1}^n (a_k + b_k) = \sum_{k=1}^n a_k + \sum_{k=1}^n b_k$$

2. Difference Rule:

$$\sum_{k=1}^n (a_k - b_k) = \sum_{k=1}^n a_k - \sum_{k=1}^n b_k$$

3. Constant Multiple Rule:

$$\sum_{k=1}^n c a_k = c \cdot \sum_{k=1}^n a_k \quad (\text{Any number } c)$$

4. Constant Value Rule:

$$\sum_{k=1}^n c = n \cdot c \quad (\text{Any number } c)$$

EXAMPLE 3

We demonstrate the use of the algebra rules.

(a) $\sum_{k=1}^n (3k - k^2) = 3 \sum_{k=1}^n k - \sum_{k=1}^n k^2$

Difference Rule and Constant Multiple Rule

(b) $\sum_{k=1}^n (-a_k) = \sum_{k=1}^n (-1) \cdot a_k = -1 \cdot \sum_{k=1}^n a_k = -\sum_{k=1}^n a_k$

Constant Multiple Rule

(c)
$$\begin{aligned}\sum_{k=1}^3 (k + 4) &= \sum_{k=1}^3 k + \sum_{k=1}^3 4 \\ &= (1 + 2 + 3) + (3 \cdot 4) \\ &= 6 + 12 = 18\end{aligned}$$

Sum Rule

Constant Value Rule

(d) $\sum_{k=1}^n \frac{1}{n} = n \cdot \frac{1}{n} = 1$

Constant Value Rule
($1/n$ is constant)

5.2 Sigma Notation and Limits of Finite Sums



Example

I want to find a formula for $1 + 2 + 3 + \dots + n$.

5.2 Sigma Notation and Limits of Finite Sums



Example

I want to find a formula for $1 + 2 + 3 + \dots + n$.

Note that

$$2(1+2+3+4+5+\dots+(n-1)+n)$$

=

=

=

5.2 Sigma Notation and Limits of Finite Sums



Example

I want to find a formula for $1 + 2 + 3 + \dots + n$.

Note that

$$\begin{aligned} & 2(1+2+3+4+5+\dots+(n-1)+n) \\ &= 1 + 2 + 3 + 4 + 5 + \dots + (n-1) + n \\ &\quad + n + (n-1) + (n-2) + (n-3) + (n-4) + \dots + 2 + 1 \\ &= \\ &= \end{aligned}$$

5.2 Sigma Notation and Limits of Finite Sums



Example

I want to find a formula for $1 + 2 + 3 + \dots + n$.

Note that

$$2(1+2+3+4+5+\dots+(n-1)+n)$$

$$\begin{aligned} &= 1 + 2 + 3 + 4 + 5 + \dots + (n-1) + n \\ &\quad + n + (n-1) + (n-2) + (n-3) + (n-4) + \dots + 2 + 1 \end{aligned}$$

$$= (n+1) + (n+1) + (n+1) + (n+1) + (n+1) + \dots + (n+1) + (n+1)$$

=

5.2 Sigma Notation and Limits of Finite Sums



Example

I want to find a formula for $1 + 2 + 3 + \dots + n$.

Note that

$$\begin{aligned} & 2(1+2+3+4+5+\dots+(n-1)+n) \\ &= 1 + 2 + 3 + 4 + 5 + \dots + (n-1) + n \\ &\quad + n + (n-1) + (n-2) + (n-3) + (n-4) + \dots + 2 + 1 \\ &= (n+1) + (n+1) + (n+1) + (n+1) + (n+1) + \dots + (n+1) + (n+1) \\ &= n(n+1). \end{aligned}$$

5.2 Sigma Notation and Limits of Finite Sums



Example

I want to find a formula for $1 + 2 + 3 + \dots + n$.

Note that

$$\begin{aligned} & 2(1+2+3+4+5+\dots+(n-1)+n) \\ &= 1 + 2 + 3 + 4 + 5 + \dots + (n-1) + n \\ &\quad + n + (n-1) + (n-2) + (n-3) + (n-4) + \dots + 2 + 1 \\ &= (n+1) + (n+1) + (n+1) + (n+1) + (n+1) + \dots + (n+1) + (n+1) \\ &= n(n+1). \end{aligned}$$

Therefore

$$\sum_{k=1}^n k = \frac{n(n+1)}{2}.$$

5.2 Sigma Notation and Limits of Finite Sums



$$\sum_{k=1}^n k = \frac{n(n+1)}{2}$$

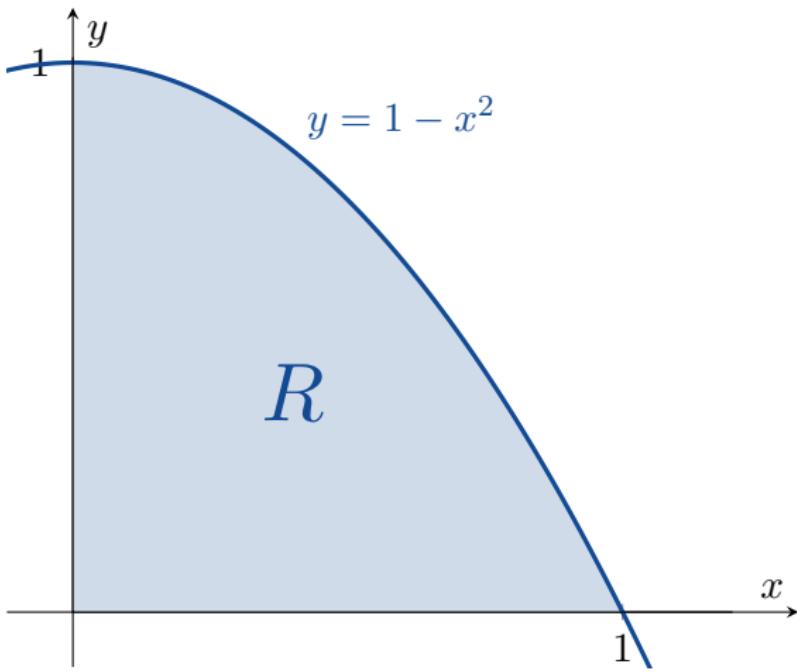
Similarly (but more difficult) we can find that

$$\sum_{k=1}^n k^2 = \frac{n(n+1)(2n+1)}{6}$$

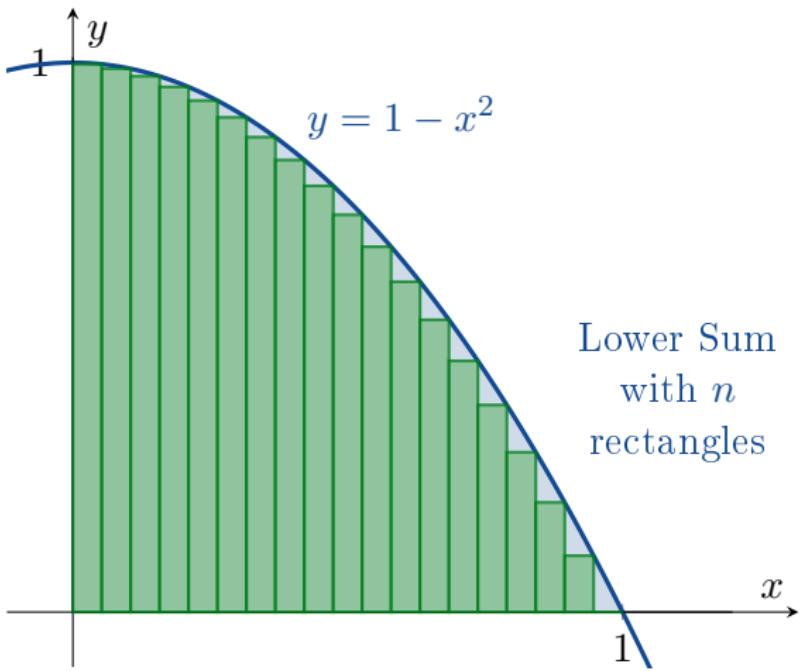
and

$$\sum_{k=1}^n k^3 = \left(\frac{n(n+1)}{2} \right)^2.$$

Limits of Finite Sums



Limits of Finite Sums



5.2 Sigma Notation and Limits of Finite Sums



STEP 1: We will cut $[0, 1]$ into n pieces of width

$$\Delta x = \frac{1 - 0}{n} = \frac{1}{n}.$$

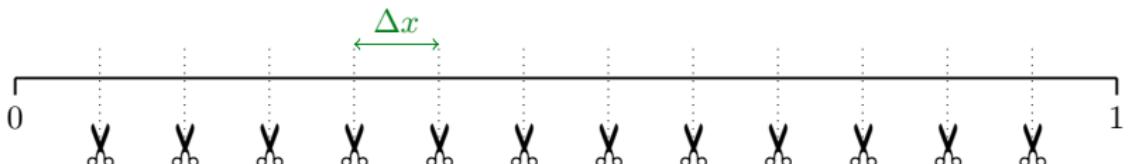


5.2 Sigma Notation and Limits of Finite Sums



STEP 1: We will cut $[0, 1]$ into n pieces of width

$$\Delta x = \frac{1 - 0}{n} = \frac{1}{n}.$$



5.2 Sigma Notation and Limits of Finite Sums



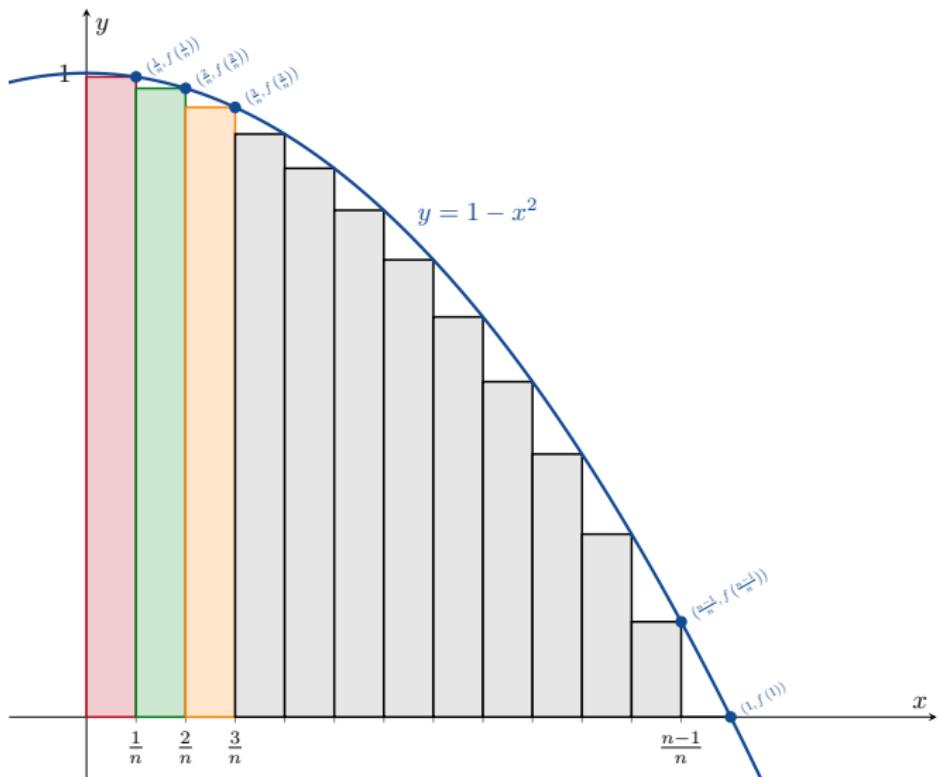
STEP 1: We will cut $[0, 1]$ into n pieces of width

$$\Delta x = \frac{1 - 0}{n} = \frac{1}{n}.$$

$$\Delta x = \frac{1}{n}$$




5.2 Sigma Notation and Limits of Finite Sums



STEP 2: We will use n rectangles to approximate the area of R .

5.2 Sigma Notation and Limits of Finite Sums



STEP 3: Then we will take the limit as $n \rightarrow \infty$.

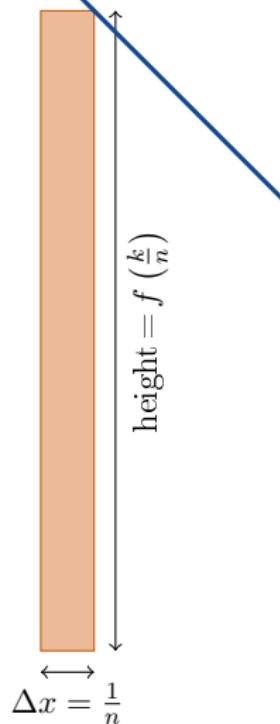
5.2 Sigma Notation and Limits of Finite Sums



Let $f(x) = 1 - x^2$. Then

- the first rectangle has area $\frac{1}{n}f\left(\frac{1}{n}\right)$;
- the second rectangle has area $\frac{1}{n}f\left(\frac{2}{n}\right)$;
- the third rectangle has area $\frac{1}{n}f\left(\frac{3}{n}\right)$;

and so on.



5.2 Sigma Notation

$$f(x) = 1 - x^2 \quad \sum_{k=1}^n k^2 = \frac{n(n+1)(2n+1)}{6}$$



The area of all n rectangles is

$$\text{area} = \sum_{k=1}^n (\text{area of the } k\text{th rectangle}) = \sum_{k=1}^n \frac{1}{n} f\left(\frac{k}{n}\right)$$

=

=

=

=

=

5.2 Sigma Notation

$$f(x) = 1 - x^2 \quad \sum_{k=1}^n k^2 = \frac{n(n+1)(2n+1)}{6}$$



The area of all n rectangles is

$$\text{area} = \sum_{k=1}^n (\text{area of the } k\text{th rectangle}) = \sum_{k=1}^n \frac{1}{n} f\left(\frac{k}{n}\right)$$

$$= \sum_{k=1}^n \frac{1}{n} \left(1 - \left(\frac{k}{n}\right)^2\right) = \sum_{k=1}^n \left(\frac{1}{n} - \frac{k^2}{n^3}\right)$$

=

=

=

=

5.2 Sigma Notation

$$f(x) = 1 - x^2 \quad \sum_{k=1}^n k^2 = \frac{n(n+1)(2n+1)}{6}$$



The area of all n rectangles is

$$\text{area} = \sum_{k=1}^n (\text{area of the } k\text{th rectangle}) = \sum_{k=1}^n \frac{1}{n} f\left(\frac{k}{n}\right)$$

$$= \sum_{k=1}^n \frac{1}{n} \left(1 - \left(\frac{k}{n}\right)^2\right) = \sum_{k=1}^n \left(\frac{1}{n} - \frac{k^2}{n^3}\right)$$

$$= \sum_{k=1}^n \frac{1}{n} - \sum_{k=1}^n \frac{k^2}{n^3}$$

=

=

=

5.2 Sigma Notation

$$f(x) = 1 - x^2 \quad \sum_{k=1}^n k^2 = \frac{n(n+1)(2n+1)}{6}$$



The area of all n rectangles is

$$\text{area} = \sum_{k=1}^n (\text{area of the } k\text{th rectangle}) = \sum_{k=1}^n \frac{1}{n} f\left(\frac{k}{n}\right)$$

$$= \sum_{k=1}^n \frac{1}{n} \left(1 - \left(\frac{k}{n}\right)^2\right) = \sum_{k=1}^n \left(\frac{1}{n} - \frac{k^2}{n^3}\right)$$

$$= \sum_{k=1}^n \frac{1}{n} - \sum_{k=1}^n \frac{k^2}{n^3}$$

$$= n \left(\frac{1}{n}\right) - \frac{1}{n^3} \sum_{k=1}^n k^2$$

=

=

5.2 Sigma Notation

$$f(x) = 1 - x^2 \quad \sum_{k=1}^n k^2 = \frac{n(n+1)(2n+1)}{6}$$



The area of all n rectangles is

$$\begin{aligned} \text{area} &= \sum_{k=1}^n (\text{area of the } k\text{th rectangle}) = \sum_{k=1}^n \frac{1}{n} f\left(\frac{k}{n}\right) \\ &= \sum_{k=1}^n \frac{1}{n} \left(1 - \left(\frac{k}{n}\right)^2\right) = \sum_{k=1}^n \left(\frac{1}{n} - \frac{k^2}{n^3}\right) \\ &= \sum_{k=1}^n \frac{1}{n} - \sum_{k=1}^n \frac{k^2}{n^3} \\ &= n \left(\frac{1}{n}\right) - \frac{1}{n^3} \sum_{k=1}^n k^2 \\ &= 1 - \frac{1}{n^3} \left(\frac{n(n+1)(2n+1)}{6}\right) \\ &= \end{aligned}$$

5.2 Sigma Notation

$$f(x) = 1 - x^2 \quad \sum_{k=1}^n k^2 = \frac{n(n+1)(2n+1)}{6}$$



The area of all n rectangles is

$$\begin{aligned}\text{area} &= \sum_{k=1}^n (\text{area of the } k\text{th rectangle}) = \sum_{k=1}^n \frac{1}{n} f\left(\frac{k}{n}\right) \\ &= \sum_{k=1}^n \frac{1}{n} \left(1 - \left(\frac{k}{n}\right)^2\right) = \sum_{k=1}^n \left(\frac{1}{n} - \frac{k^2}{n^3}\right) \\ &= \sum_{k=1}^n \frac{1}{n} - \sum_{k=1}^n \frac{k^2}{n^3} \\ &= n \left(\frac{1}{n}\right) - \frac{1}{n^3} \sum_{k=1}^n k^2 \\ &= 1 - \frac{1}{n^3} \left(\frac{n(n+1)(2n+1)}{6}\right) \\ &= 1 - \frac{2n^2 + 3n + 1}{6n^2}.\end{aligned}$$

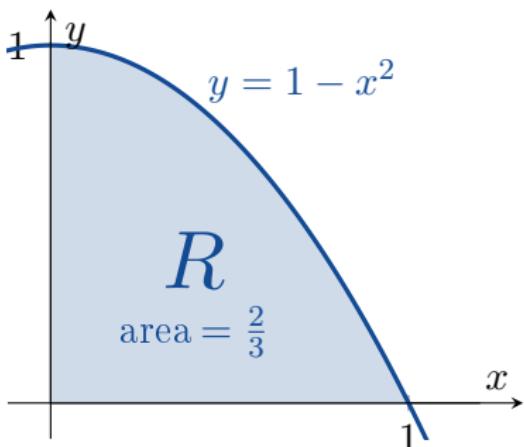
5.2 Sigma Notation and Limits of Finite Sums



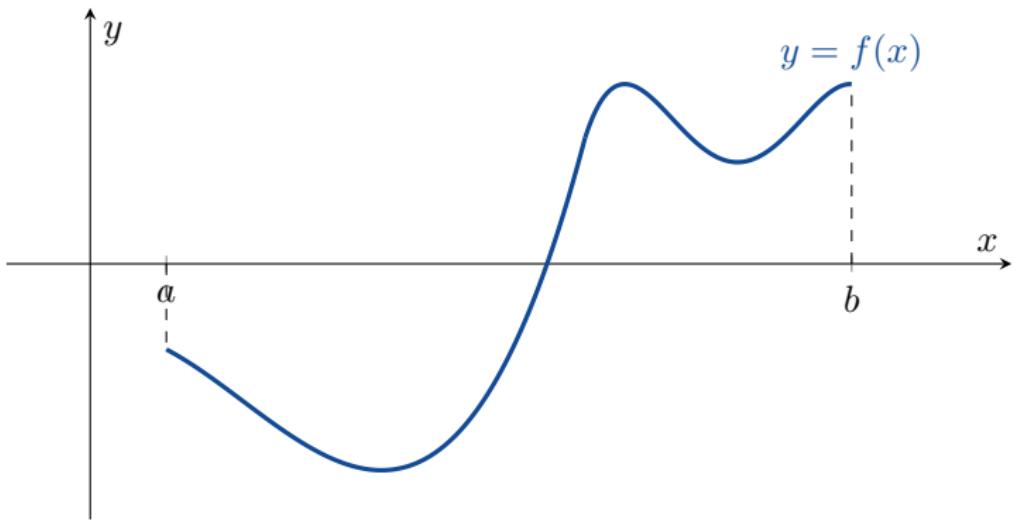
Taking the limit gives

$$\begin{aligned}\lim_{n \rightarrow \infty} \left(\sum_{k=1}^n \frac{1}{n} f\left(\frac{k}{n}\right) \right) &= \lim_{n \rightarrow \infty} \left(1 - \frac{2n^2 + 3n + 1}{6n^2} \right) \\ &= 1 - \frac{2}{6} = \frac{2}{3}.\end{aligned}$$

Therefore the area of R is $\frac{2}{3}$.



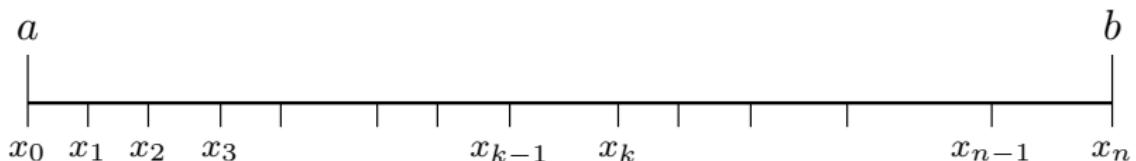
Riemann Sums



5.2 Sigma Notation and Limits of Finite Sums



Now let $f : [a, b] \rightarrow \mathbb{R}$ be a function. We will cut $[a, b]$ into n subintervals (the pieces don't have to all be the same size).

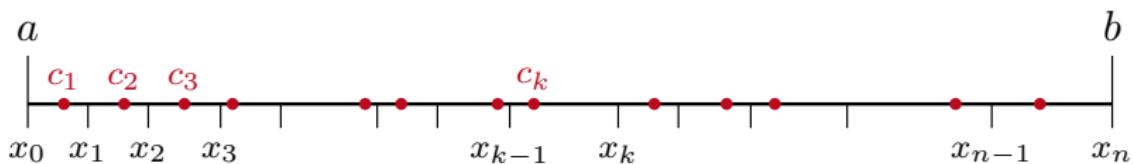


5.2 Sigma Notation and Limits of Finite Sums



Now let $f : [a, b] \rightarrow \mathbb{R}$ be a function. We will cut $[a, b]$ into n subintervals (the pieces don't have to all be the same size).

In each subinterval we will choose one point $c_k \in [x_{k-1}, x_k]$.



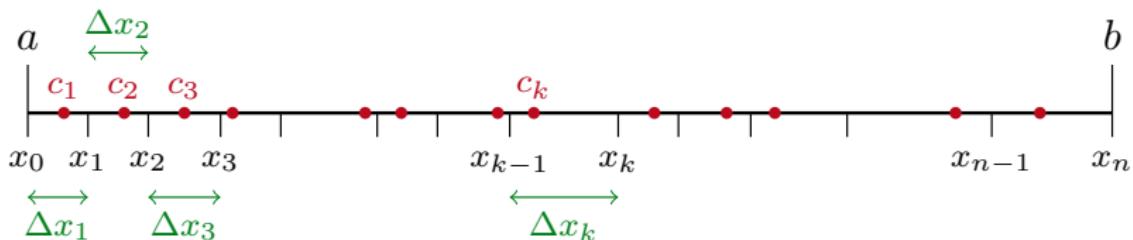
5.2 Sigma Notation and Limits of Finite Sums



Now let $f : [a, b] \rightarrow \mathbb{R}$ be a function. We will cut $[a, b]$ into n subintervals (the pieces don't have to all be the same size).

In each subinterval we will choose one point $c_k \in [x_{k-1}, x_k]$.

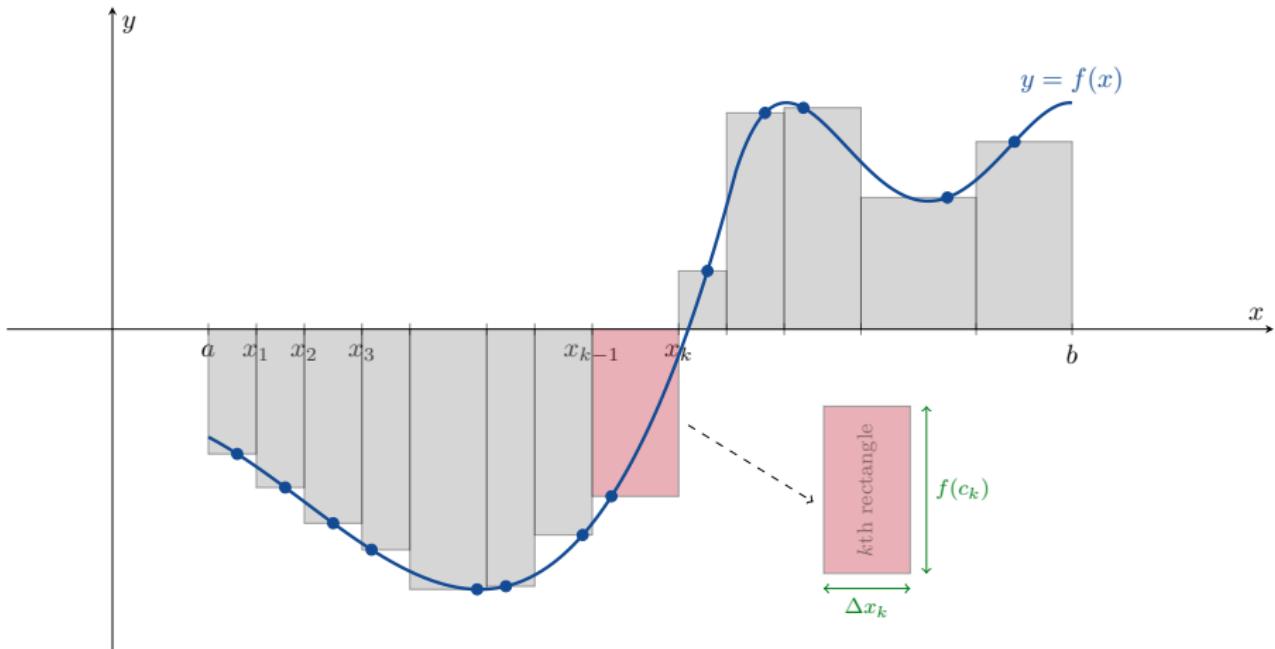
The width of each subinterval is $\Delta x_k = x_k - x_{k-1}$.



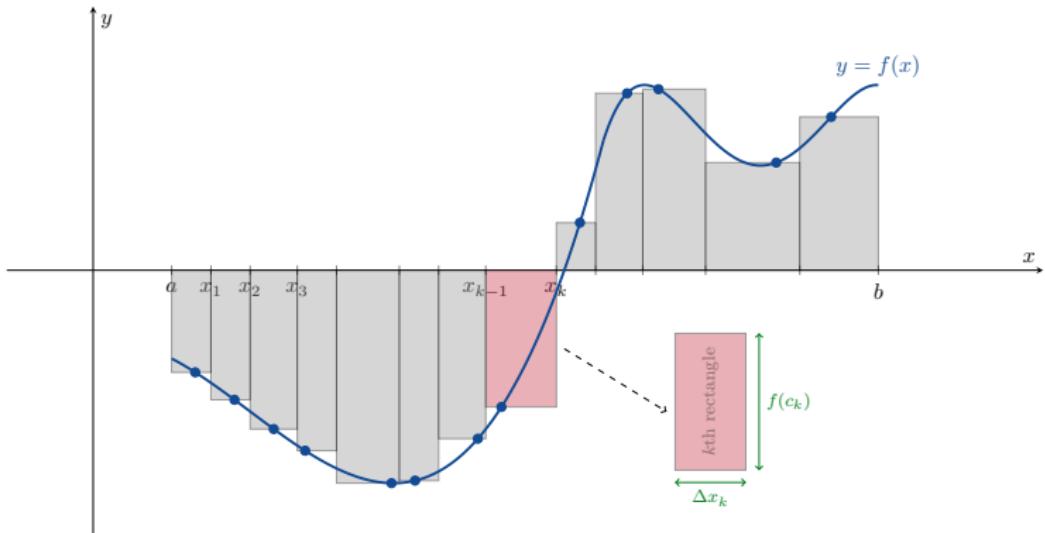
5.2 Sigma Notation and Limits of Finite Sums



On each subinterval $[x_{k-1}, x_k]$, we draw a rectangle of width Δx_k and height $f(c_k)$.

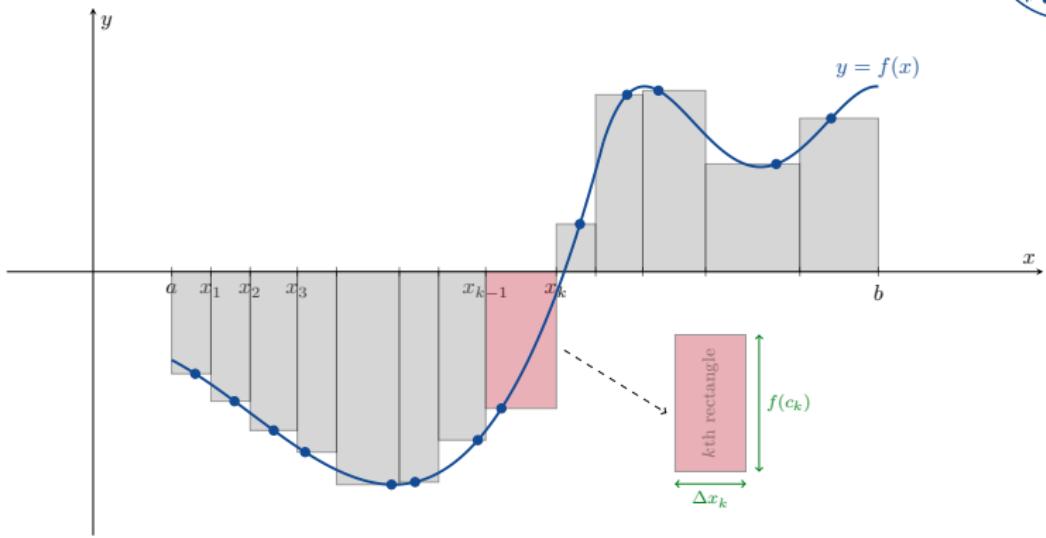


5.2 Sigma Notation and Limits of Finite Sums



Note that if $f(c_k) < 0$, then the rectangle on $[x_{k-1}, x_k]$ will have 'negative area' – this is ok.

5.2 Sigma Notation and Limits of Finite Sums



The total area of the n rectangles is

$$\sum_{k=1}^n f(c_k) \Delta x_k.$$

This is called a *Riemann Sum for f on $[a, b]$* .

5.2 Sigma Notation and Limits of Finite Sums



$$\sum_{k=1}^n f(c_k) \Delta x_k.$$

Then we want to take the limit as $n \rightarrow \infty$ (or more precisely, we want to take the limit as $\|P\| = \max\{\Delta x_1, \Delta x_2, \dots, \Delta x_n\} \rightarrow 0$).

5.2 Sigma Notation and Limits of Finite Sums



$$\sum_{k=1}^n f(c_k) \Delta x_k.$$

Then we want to take the limit as $n \rightarrow \infty$ (or more precisely, we want to take the limit as $\|P\| = \max\{\Delta x_1, \Delta x_2, \dots, \Delta x_n\} \rightarrow 0$).

Remark

Sometimes this limit exists, sometimes this limit does not exist.



Break

We will continue at 3pm





The Definite Integral

Definition of the Definite Integral

Definition

If the limit

$$\lim_{n \rightarrow \infty} \sum_{k=1}^n f(c_k) \Delta x_k$$

exists, then it is called the *definite integral of f over [a, b]*. We write

$$\int_a^b f(x) \, dx = \lim_{n \rightarrow \infty} \sum_{k=1}^n f(c_k) \Delta x_k$$

if the limit exists.

Definition of the Definite Integral

Definition

If the limit

$$\lim_{n \rightarrow \infty} \sum_{k=1}^n f(c_k) \Delta x_k$$

exists, then it is called the *definite integral of f over [a, b]*. We write

$$\int_a^b f(x) dx = \lim_{n \rightarrow \infty} \sum_{k=1}^n f(c_k) \Delta x_k$$

if the limit exists.

Remark

We need this limit to exist and be the same for all Riemann Sums that we can create.

DEFINITION Let $f(x)$ be a function defined on a closed interval $[a, b]$. We say that a number J is the **definite integral of f over $[a, b]$** and that J is the limit of the Riemann sums $\sum_{k=1}^n f(c_k) \Delta x_k$ if the following condition is satisfied:

Given any number $\varepsilon > 0$ there is a corresponding number $\delta > 0$ such that for every partition $P = \{x_0, x_1, \dots, x_n\}$ of $[a, b]$ with $\|P\| < \delta$ and any choice of c_k in $[x_{k-1}, x_k]$, we have

$$\left| \sum_{k=1}^n f(c_k) \Delta x_k - J \right| < \varepsilon.$$

5.3 The Definite Integral



$$\int_a^b f(x) \, dx$$

“the integral of f from a to b ”

“ a 'dan b 'ye f 'nin integrali”

5.3 The Definite Integral



integral sign
integral işaretti

$$\int_a^b f(x) \, dx$$

“the integral of f from a to b ”

“ a 'dan b 'ye f 'nin integrali”

5.3 The Definite Integral



integral sign
integral işaretti

$$\int_a^b f(x) dx$$

lower limit of integration
integralin alt sınırı

“the integral of f from a to b ”

“ a 'dan b 'ye f 'nin integrali”

5.3 The Definite Integral

upper limit of integration

integralin üst sınırı

integral sign
integral işaretti

$$\int_a^b f(x) dx$$

lower limit of integration

integralin alt sınırı

“the integral of f from a to b ”

“ a 'dan b 'ye f 'nin integrali”

5.3 The Definite Integral

upper limit of integration

integralin üst sınırı

integral sign
integral işaretti

$$\int_a^b$$

$f(x)$

the integrand

integralin integrandi

lower limit of integration

integralin alt sınırı

“the integral of f from a to b ”

“ a 'dan b 'ye f 'nin integrali”

5.3 The Definite Integral



upper limit of integration
integralin üst sınırı

integral sign
integral işaretti

$$\int_a^b f(x) \, dx$$

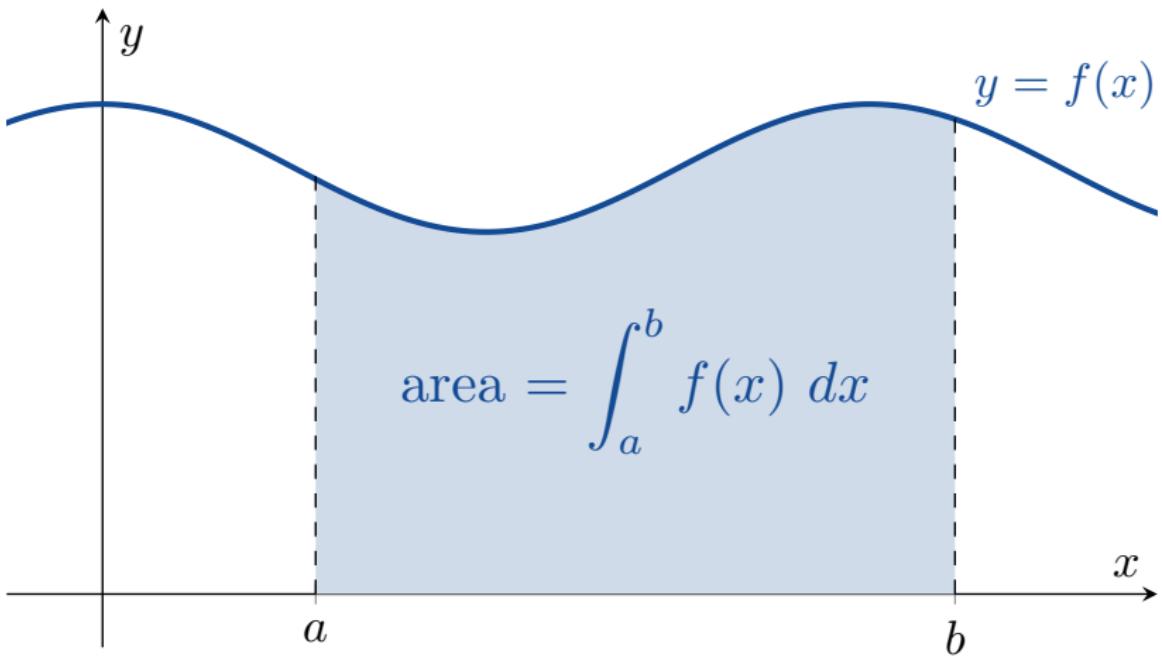
x is the variable of integration
 x , integral değişkenidir

lower limit of integration
integralin alt sınırı

“the integral of f from a to b ”

“ a 'dan b 'ye f 'nin integrali”

5.3 The Definite Integral



5.3 The Definite Integral



Definition

If $\int_a^b f(x) dx$ exists, then we say that f is *integrable* on $[a, b]$.

5.3 The Definite Integral

Definition

If $\int_a^b f(x) dx$ exists, then we say that f is *integrable on $[a, b]$* .

Example

$f(x) = 1 - x^2$ is integrable on $[0, 1]$ and $\int_0^1 (1 - x^2) dx = \frac{2}{3}$.

5.3 The Definite Integral

Definition

If $\int_a^b f(x) dx$ exists, then we say that f is *integrable on $[a, b]$* .

Example

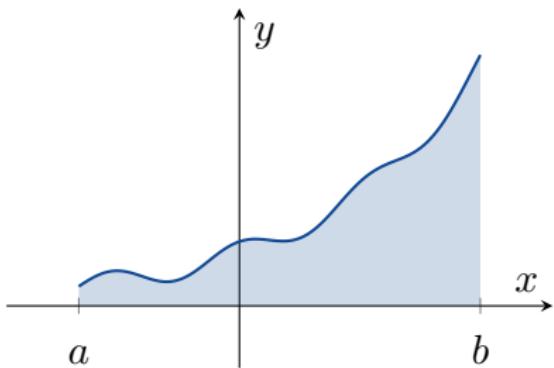
$f(x) = 1 - x^2$ is integrable on $[0, 1]$ and $\int_0^1 (1 - x^2) dx = \frac{2}{3}$.

Remark

$$\int_a^b f(\textcolor{red}{x}) dx = \int_a^b f(\textcolor{red}{u}) du = \int_a^b f(\textcolor{red}{t}) dt$$

It doesn't matter which letter we use for the *dummy variable*.

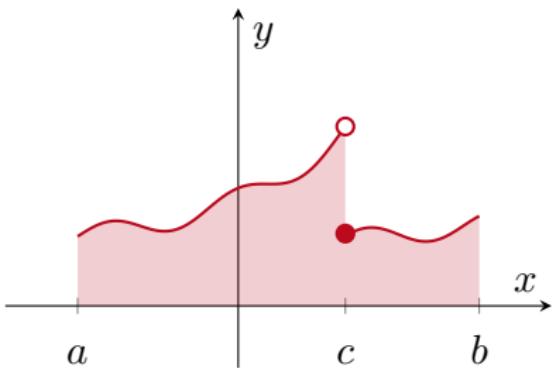
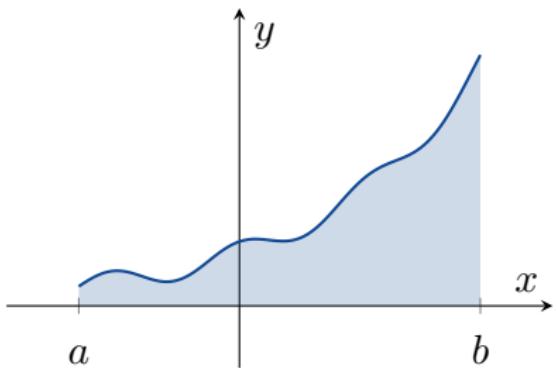
5.3 The Definite Integral



Theorem

If f is continuous on $[a, b]$, then f is integrable on $[a, b]$.

5.3 The Definite Integral

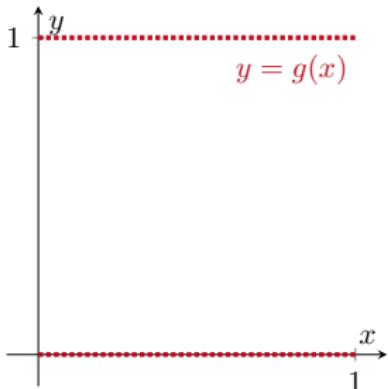


Theorem

If f is continuous on $[a, b]$, then f is integrable on $[a, b]$.

If f has finitely many jump discontinuities but is otherwise continuous on $[a, b]$, then f is integrable on $[a, b]$.

5.3 The Definite Integral



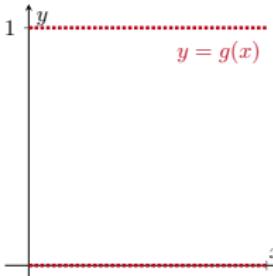
Example (Not all functions are integrable)

Define a function $g : [0, 1] \rightarrow \mathbb{R}$ by

$$g(x) = \begin{cases} 1 & x \in \mathbb{Q} \\ 0 & x \notin \mathbb{Q}. \end{cases}$$

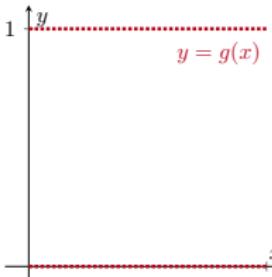
This function is not integrable on $[0, 1]$.

5.3 The Definite Integral



When we set up a Riemann sum, we choose the points $c_k \in [x_{k-1}, x_k]$ where we calculate the height of each rectangle.

5.3 The Definite Integral

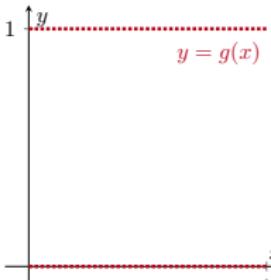


When we set up a Riemann sum, we choose the points $c_k \in [x_{k-1}, x_k]$ where we calculate the height of each rectangle.

If we choose $c_k \in \mathbb{Q}$, then we always have $g(c_k) = 1$ and thus

$$\sum_{k=1}^n g(c_k) \Delta x_k = \sum_{k=1}^n 1 \cdot \Delta x_k = 1.$$

5.3 The Definite Integral



When we set up a Riemann sum, we choose the points $c_k \in [x_{k-1}, x_k]$ where we calculate the height of each rectangle.

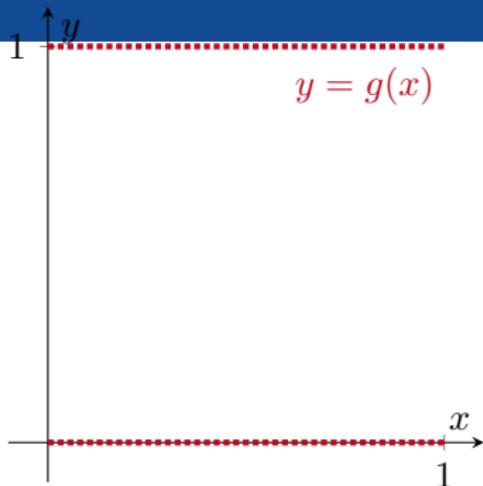
If we choose $c_k \in \mathbb{Q}$, then we always have $g(c_k) = 1$ and thus

$$\sum_{k=1}^n g(c_k) \Delta x_k = \sum_{k=1}^n 1 \cdot \Delta x_k = 1.$$

However if we choose $c_k \in \mathbb{R} \setminus \mathbb{Q}$, then we always have $g(c_k) = 0$ and thus

$$\sum_{k=1}^n g(c_k) \Delta x_k = \sum_{k=1}^n 0 \cdot \Delta x_k = 0.$$

5.3 The Definite Integral

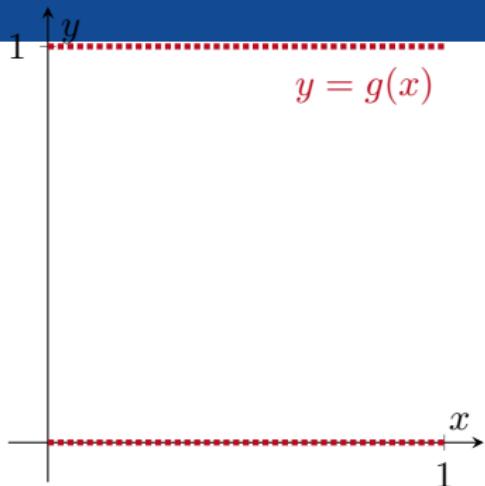


Since

$$\lim_{n \rightarrow \infty} 1 \neq \lim_{n \rightarrow \infty} 0,$$

there does not exist a common limit of the Riemann sums.

5.3 The Definite Integral



Since

$$\lim_{n \rightarrow \infty} 1 \neq \lim_{n \rightarrow \infty} 0,$$

there does not exist a common limit of the Riemann sums.

Therefore $g(x) = \begin{cases} 1 & x \in \mathbb{Q} \\ 0 & x \notin \mathbb{Q}. \end{cases}$ is not integrable.

Properties of Definite Integrals

Theorem

Suppose that f and g are integrable. Let k be a number.

Properties of Definite Integrals

Theorem

Suppose that f and g are integrable. Let k be a number. Then

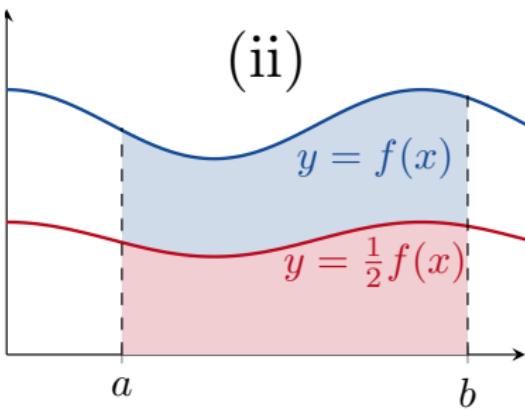
1 $\int_b^a f(x) \, dx = - \int_a^b f(x) \, dx;$

Properties of Definite Integrals

Theorem

Suppose that f and g are integrable. Let k be a number. Then

2 $\int_a^b kf(x) \, dx = k \int_a^b f(x) \, dx;$

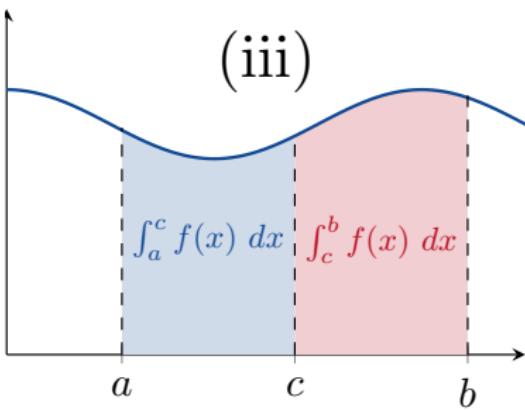


Properties of Definite Integrals

Theorem

Suppose that f and g are integrable. Let k be a number. Then

3 $\int_a^c f(x) \, dx + \int_c^b f(x) \, dx = \int_a^b f(x) \, dx$

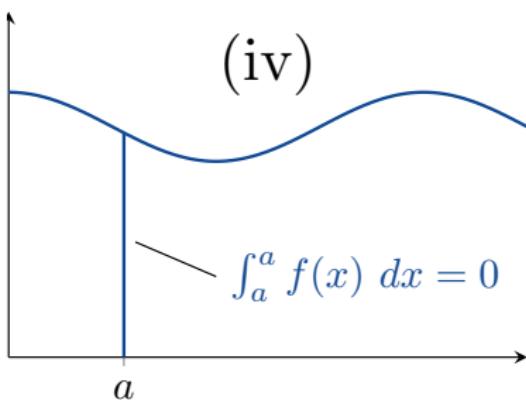


Properties of Definite Integrals

Theorem

Suppose that f and g are integrable. Let k be a number. Then

4 $\int_a^a f(x) \, dx = 0;$



Properties of Definite Integrals

Theorem

Suppose that f and g are integrable. Let k be a number. Then

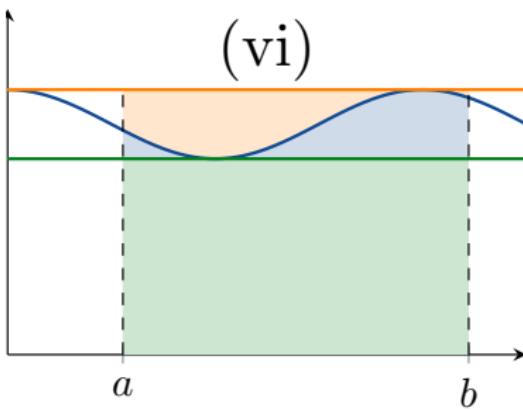
$$5 \quad \int_a^b (f(x) + g(x)) \, dx = \int_a^b f(x) \, dx + \int_a^b g(x) \, dx;$$

Properties of Definite Integrals

Theorem

Suppose that f and g are integrable. Let k be a number. Then

6 $(b - a) \min f \leq \int_a^b f(x) dx \leq (b - a) \max f;$



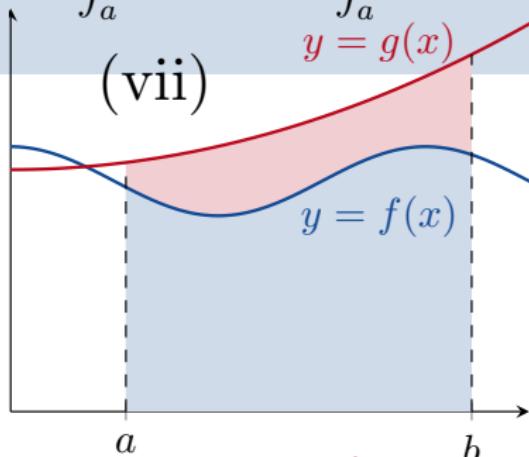
Properties of Definite Integrals

Theorem

Suppose that f and g are integrable. Let k be a number. Then

- 7 if $f(x) \leq g(x)$ on $[a, b]$, then

$$\int_a^b f(x) \, dx \leq \int_a^b g(x) \, dx;$$



Properties of Definite Integrals

Theorem

Suppose that f and g are integrable. Let k be a number. Then

- 8 if $g(x) \geq 0$ on $[a, b]$, then

$$\int_a^b g(x) \, dx \geq 0;$$

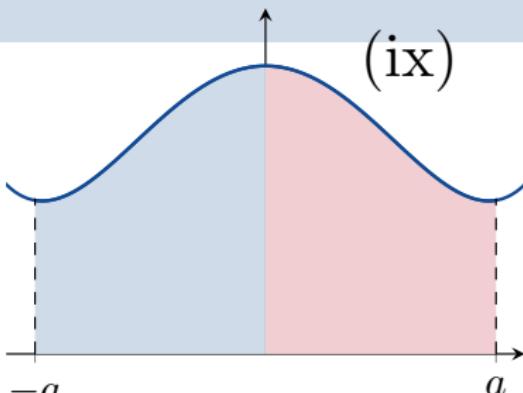
Properties of Definite Integrals

Theorem

Suppose that f and g are integrable. Let k be a number. Then

- 9 if f is an even function, then

$$\int_{-a}^a f(x) \, dx = 2 \int_0^a f(x) \, dx;$$



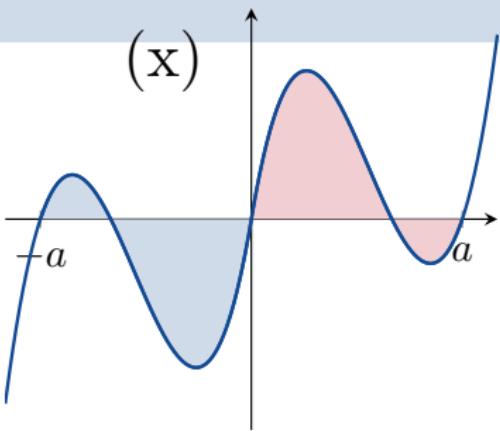
Properties of Definite Integrals

Theorem

Suppose that f and g are integrable. Let k be a number. Then

- 10 if f is an odd function, then

$$\int_{-a}^a f(x) \, dx = 0.$$



5.3 The Definite Integral

Example

Suppose that

$$\int_{-1}^1 f(x) \, dx = 5, \int_1^4 f(x) \, dx = -2 \text{ and } \int_{-1}^1 h(x) \, dx = 7.$$

5.3 The Definite Integral

Example

Suppose that

$$\int_{-1}^1 f(x) \, dx = 5, \int_1^4 f(x) \, dx = -2 \text{ and } \int_{-1}^1 h(x) \, dx = 7.$$

Then

$$\int_4^1 f(x) \, dx = - \int_1^4 f(x) \, dx = 2,$$

5.3 The Definite Integral

Example

Suppose that

$$\int_{-1}^1 f(x) \, dx = 5, \int_1^4 f(x) \, dx = -2 \text{ and } \int_{-1}^1 h(x) \, dx = 7.$$

Then

$$\int_4^1 f(x) \, dx = - \int_1^4 f(x) \, dx = 2,$$

$$\begin{aligned}\int_{-1}^1 (2f(x) + 3h(x)) \, dx &= 2 \int_{-1}^1 f(x) \, dx + 3 \int_{-1}^1 h(x) \, dx \\ &= 2(5) + 3(7) = 31\end{aligned}$$

5.3 The Definite Integral

Example

Suppose that

$$\int_{-1}^1 f(x) \, dx = 5, \quad \int_1^4 f(x) \, dx = -2 \text{ and } \int_{-1}^1 h(x) \, dx = 7.$$

Then

$$\int_4^1 f(x) \, dx = - \int_1^4 f(x) \, dx = 2,$$

$$\begin{aligned}\int_{-1}^1 (2f(x) + 3h(x)) \, dx &= 2 \int_{-1}^1 f(x) \, dx + 3 \int_{-1}^1 h(x) \, dx \\ &= 2(5) + 3(7) = 31\end{aligned}$$

and

$$\begin{aligned}\int_{-1}^4 f(x) \, dx &= \int_{-1}^1 f(x) \, dx + \int_1^4 f(x) \, dx \\ &= 5 + (-2) = 3.\end{aligned}$$

5.3 The Definite Integral

Example

Show that $\int_0^1 \sqrt{1 + \cos x} dx \leq \sqrt{2}$.

5.3 The Definite Integral

Example

Show that $\int_0^1 \sqrt{1 + \cos x} dx \leq \sqrt{2}$.

The maximum value of $\sqrt{1 + \cos x}$ on $[0, 1]$ is
 $\sqrt{1 + 1} = \sqrt{2}$.

5.3 The Definite Integral

Example

Show that $\int_0^1 \sqrt{1 + \cos x} dx \leq \sqrt{2}$.

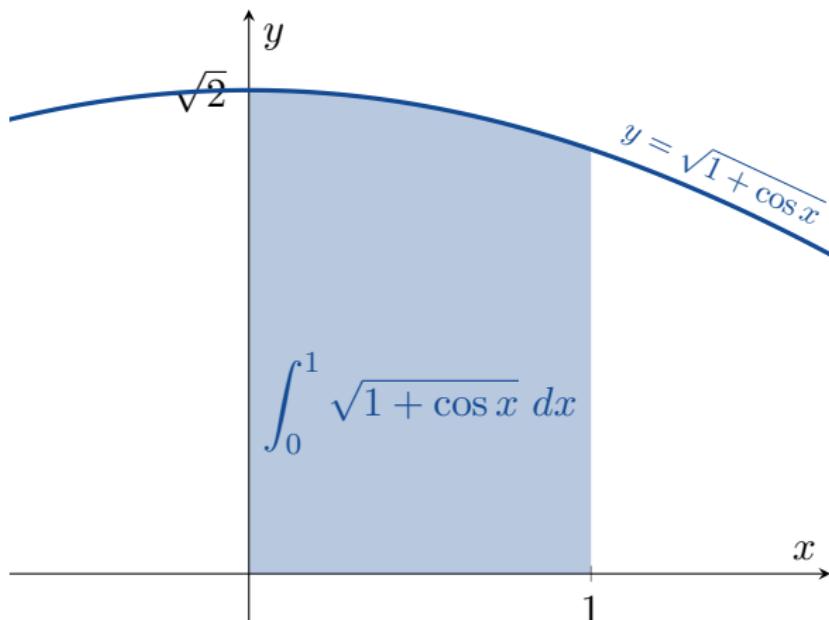
The maximum value of $\sqrt{1 + \cos x}$ on $[0, 1]$ is $\sqrt{1 + 1} = \sqrt{2}$. Therefore

$$\int_0^1 \sqrt{1 + \cos x} dx \leq (1 - 0) \max \sqrt{1 + \cos x} = 1 \times \sqrt{2}.$$

5.3 The Definite Integral

Example

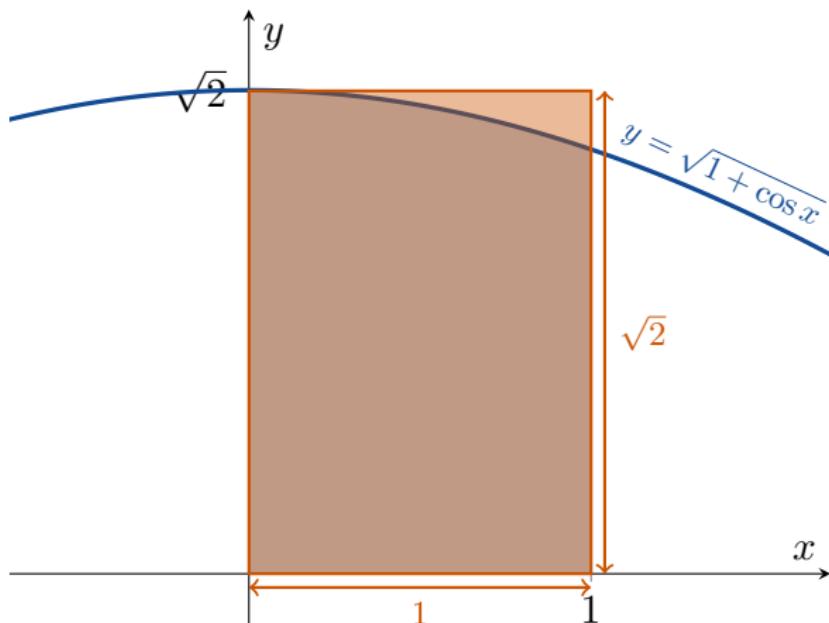
Show that $\int_0^1 \sqrt{1 + \cos x} dx \leq \sqrt{2}$.



5.3 The Definite Integral

Example

Show that $\int_0^1 \sqrt{1 + \cos x} dx \leq \sqrt{2}$.



5.3 The Definite Integral



Example

Calculate $\int_{-2}^2 (x^3 + x) \, dx$.

5.3 The Definite Integral



Example

Calculate $\int_{-2}^2 (x^3 + x) \, dx$.

Because $(x^3 + x)$ is an odd function, we have that

$$\int_{-2}^2 (x^3 + x) \, dx = 0.$$

5.3 The Definite Integral



Example

Calculate $\int_{-1}^1 (1 - x^2) \, dx$.

5.3 The Definite Integral



Example

Calculate $\int_{-1}^1 (1 - x^2) dx$.

Because $(1 - x^2)$ is an even function, we have that

$$\int_{-1}^1 (1 - x^2) dx = 2 \int_0^1 (1 - x^2) dx = 2 \times \frac{2}{3} = \frac{4}{3}.$$

5.3 The Definite Integral



Example

Calculate $\int_0^b x \, dx$ for $b > 0$.

solution 1: We will use a Riemann Sum.

5.3 The Definite Integral



Example

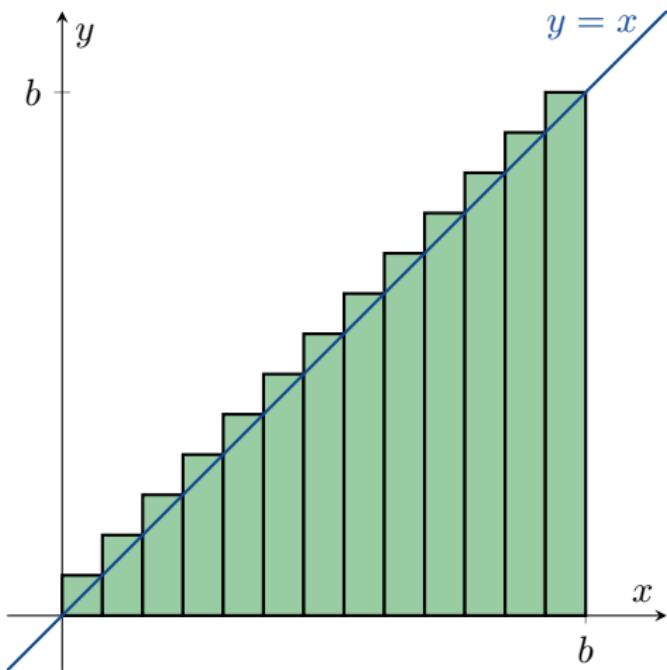
Calculate $\int_0^b x \, dx$ for $b > 0$.

solution 1: We will use a Riemann Sum. First we cut $[0, b]$ in to n pieces using

$$0 < \frac{b}{n} < \frac{2b}{n} < \frac{3b}{n} < \dots < \frac{(n-1)b}{n} < b$$

and $c_k = \frac{kb}{n}$. Note that $\Delta x_k = \frac{b}{n}$ for all k .

5.3 The Definite Integral



5.3 The Definite Integral



Then

$$\begin{aligned}\sum_{k=1}^n f(c_k) \Delta x_k &= \sum_{k=1}^n \frac{kb}{n} \frac{b}{n} = \frac{b^2}{n^2} \sum_{k=1}^n k \\ &= \frac{b^2}{n^2} \left(\frac{n(n+1)}{2} \right) = \frac{b^2}{2} \left(1 + \frac{1}{n} \right).\end{aligned}$$

5.3 The Definite Integral



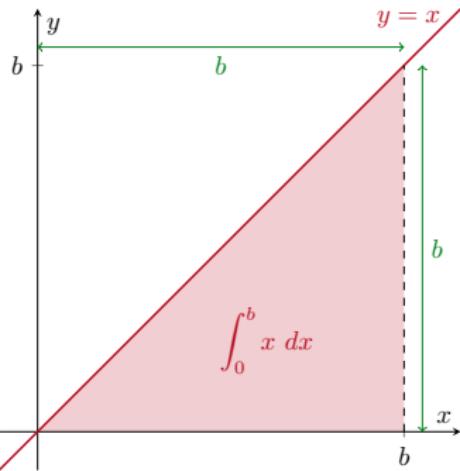
Then

$$\begin{aligned}\sum_{k=1}^n f(c_k)\Delta x_k &= \sum_{k=1}^n \frac{kb}{n} \frac{b}{n} = \frac{b^2}{n^2} \sum_{k=1}^n k \\ &= \frac{b^2}{n^2} \left(\frac{n(n+1)}{2} \right) = \frac{b^2}{2} \left(1 + \frac{1}{n} \right).\end{aligned}$$

Therefore

$$\begin{aligned}\int_0^b x \, dx &= \lim_{n \rightarrow \infty} \sum_{k=1}^n f(c_k)\Delta x_k \\ &= \lim_{n \rightarrow \infty} \frac{b^2}{2} \left(1 + \frac{1}{n} \right) = \frac{b^2}{2}.\end{aligned}$$

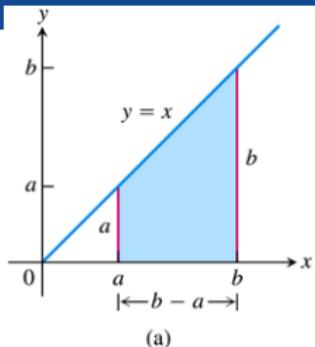
5.3 The Definite Integral



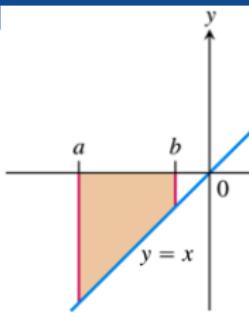
solution 2: Alternately, we can look at the triangle above and say that

$$\int_0^b x \, dx = \text{area of a triangle} = \frac{1}{2} \times b \times b = \frac{b^2}{2}.$$

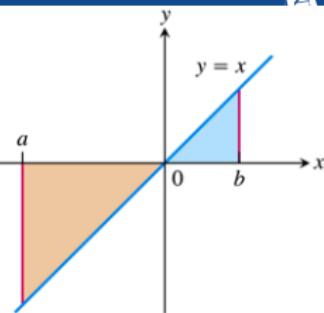
5.3 The Definite Integral



(a)



(b)

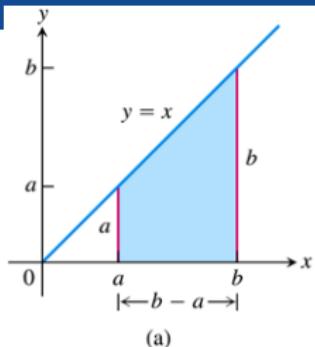


(c)

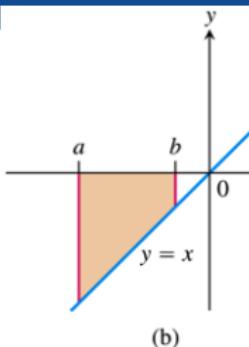
Example

$$\int_a^b x \, dx = \int_a^0 x \, dx + \int_0^b x \, dx$$

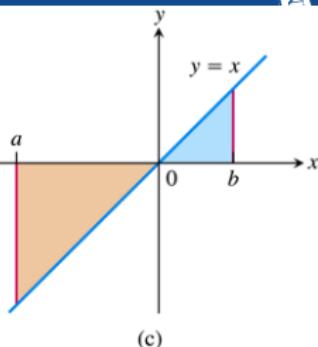
5.3 The Definite Integral



(a)



(b)

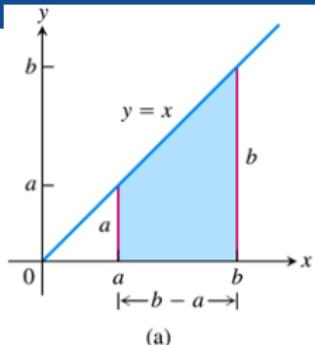


(c)

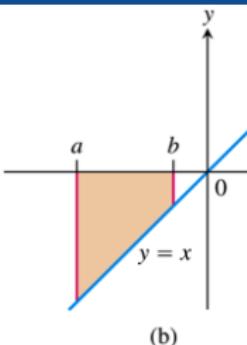
Example

$$\begin{aligned}\int_a^b x \, dx &= \int_a^0 x \, dx + \int_0^b x \, dx \\ &= -\int_0^a x \, dx + \int_0^b x \, dx\end{aligned}$$

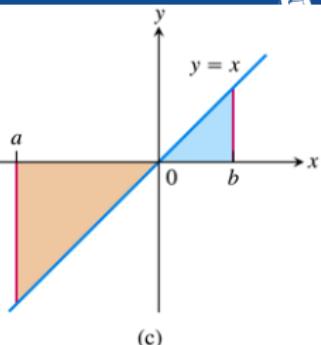
5.3 The Definite Integral



(a)



(b)



(c)

Example

$$\begin{aligned}\int_a^b x \, dx &= \int_a^0 x \, dx + \int_0^b x \, dx \\ &= -\int_0^a x \, dx + \int_0^b x \, dx \\ &= -\frac{a^2}{2} + \frac{b^2}{2} = \frac{b^2}{2} - \frac{a^2}{2}.\end{aligned}$$



Next Time

- 5.4 The Fundamental Theorem of Calculus
- 5.5 Indefinite Integrals and the Substitution Method
- 5.6 Substitution and Area Between Curves