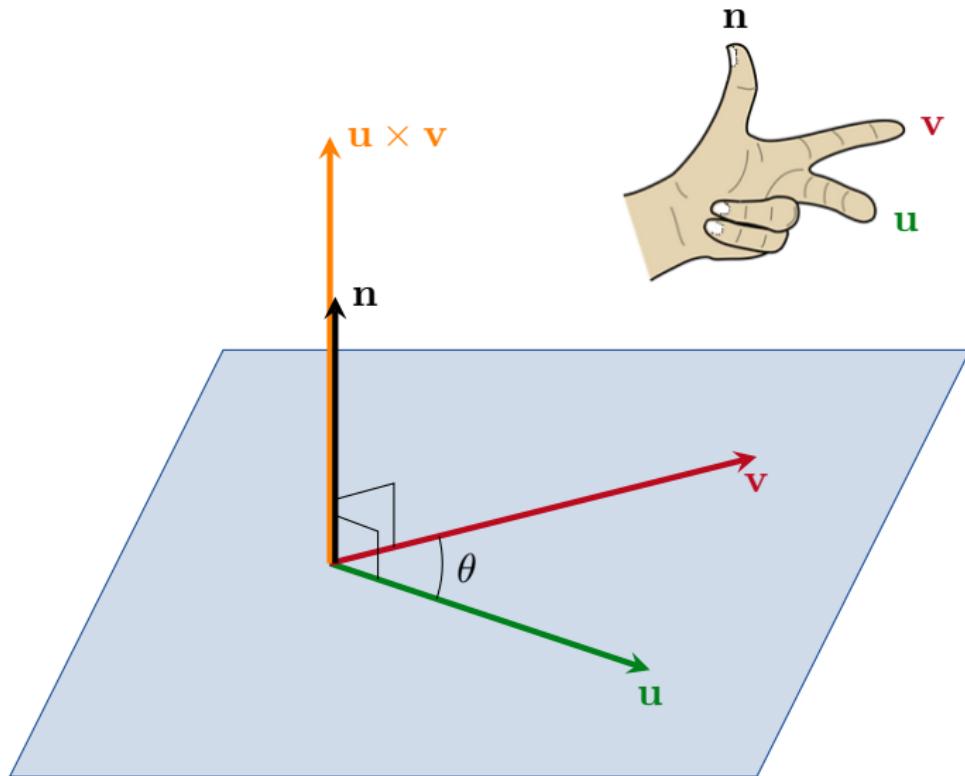


# Lecture 4

- 11.4 The Cross Product
- 11.5 Lines and Planes in Space

# The Cross Product

## 11.4 The Cross Product

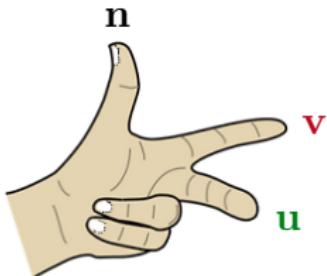


## 11.4 The Cross Product



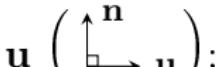
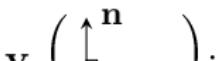
Let  $\mathbf{n}$  be a unit vector which satisfies

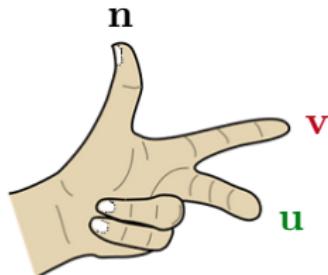
- 1  $\mathbf{n}$  is orthogonal to  $\mathbf{u}$  ( $\begin{smallmatrix} \mathbf{n} \\ \perp \\ \mathbf{u} \end{smallmatrix}$ );
- 2  $\mathbf{n}$  is orthogonal to  $\mathbf{v}$  ( $\begin{smallmatrix} \mathbf{n} \\ \perp \\ \mathbf{v} \end{smallmatrix}$ ); and
- 3 the direction of  $\mathbf{n}$  is chosen using the left-hand rule.



## 11.4 The Cross Product

Let  $\mathbf{n}$  be a unit vector which satisfies

- 1  $\mathbf{n}$  is orthogonal to  $\mathbf{u}$  () ;
- 2  $\mathbf{n}$  is orthogonal to  $\mathbf{v}$  () ; and
- 3 the direction of  $\mathbf{n}$  is chosen using the left-hand rule.



### Definition

The *cross product* of  $\mathbf{u}$  and  $\mathbf{v}$  is

$$\mathbf{u} \times \mathbf{v} = \|\mathbf{u}\| \|\mathbf{v}\| (\sin \theta) \mathbf{n}.$$

## 11.4 The Cross Product



### Remark

- $\mathbf{u} \cdot \mathbf{v}$  is a number.
- $\mathbf{u} \times \mathbf{v}$  is a vector.

$$\mathbf{u} \times \mathbf{v} = \|\mathbf{u}\| \|\mathbf{v}\| (\sin \theta) \mathbf{n}$$

### Remark

$$\begin{pmatrix} \mathbf{u} \text{ and } \mathbf{v} \\ \text{are} \\ \text{parallel} \end{pmatrix} \iff \theta = 0^\circ \text{ or } 180^\circ$$
$$\implies \sin \theta = 0 \implies \mathbf{u} \times \mathbf{v} = \mathbf{0}.$$

## 11.4 The Cross Product



### Properties of the Cross Product

Let  $\mathbf{u}$ ,  $\mathbf{v}$  and  $\mathbf{w}$  be vectors. Let  $r$  and  $s$  be numbers. Then

1  $(r\mathbf{u}) \times (s\mathbf{v}) = (rs)(\mathbf{u} \times \mathbf{v});$

## 11.4 The Cross Product



### Properties of the Cross Product

Let  $\mathbf{u}$ ,  $\mathbf{v}$  and  $\mathbf{w}$  be vectors. Let  $r$  and  $s$  be numbers. Then

- 1  $(r\mathbf{u}) \times (s\mathbf{v}) = (rs)(\mathbf{u} \times \mathbf{v});$
- 2  $\mathbf{u} \times (\mathbf{v} + \mathbf{w}) = (\mathbf{u} \times \mathbf{v}) + (\mathbf{u} \times \mathbf{w});$

## 11.4 The Cross Product



### Properties of the Cross Product

Let  $\mathbf{u}$ ,  $\mathbf{v}$  and  $\mathbf{w}$  be vectors. Let  $r$  and  $s$  be numbers. Then

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- 2  $\mathbf{u} \times (\mathbf{v} + \mathbf{w}) = (\mathbf{u} \times \mathbf{v}) + (\mathbf{u} \times \mathbf{w});$
- 3  $\mathbf{v} \times \mathbf{u} = -\mathbf{u} \times \mathbf{v};$

## 11.4 The Cross Product



### Properties of the Cross Product

Let  $\mathbf{u}$ ,  $\mathbf{v}$  and  $\mathbf{w}$  be vectors. Let  $r$  and  $s$  be numbers. Then

- 1  $(r\mathbf{u}) \times (s\mathbf{v}) = (rs)(\mathbf{u} \times \mathbf{v});$
- 2  $\mathbf{u} \times (\mathbf{v} + \mathbf{w}) = (\mathbf{u} \times \mathbf{v}) + (\mathbf{u} \times \mathbf{w});$
- 3  $\mathbf{v} \times \mathbf{u} = -\mathbf{u} \times \mathbf{v};$
- 4  $(\mathbf{v} + \mathbf{w}) \times \mathbf{u} = (\mathbf{v} \times \mathbf{u}) + (\mathbf{w} \times \mathbf{u});$

## 11.4 The Cross Product



### Properties of the Cross Product

Let  $\mathbf{u}$ ,  $\mathbf{v}$  and  $\mathbf{w}$  be vectors. Let  $r$  and  $s$  be numbers. Then

- 1  $(r\mathbf{u}) \times (s\mathbf{v}) = (rs)(\mathbf{u} \times \mathbf{v});$
- 2  $\mathbf{u} \times (\mathbf{v} + \mathbf{w}) = (\mathbf{u} \times \mathbf{v}) + (\mathbf{u} \times \mathbf{w});$
- 3  $\mathbf{v} \times \mathbf{u} = -\mathbf{u} \times \mathbf{v};$
- 4  $(\mathbf{v} + \mathbf{w}) \times \mathbf{u} = (\mathbf{v} \times \mathbf{u}) + (\mathbf{w} \times \mathbf{u});$
- 5  $\mathbf{0} \times \mathbf{u} = \mathbf{0};$  and

## 11.4 The Cross Product



### Properties of the Cross Product

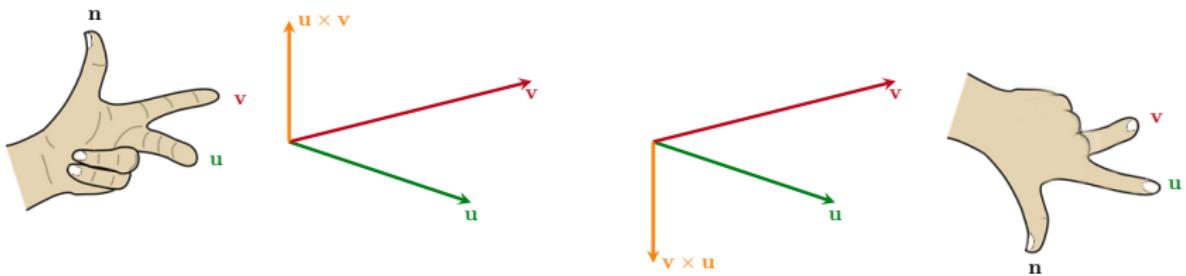
Let  $\mathbf{u}$ ,  $\mathbf{v}$  and  $\mathbf{w}$  be vectors. Let  $r$  and  $s$  be numbers. Then

- 1  $(r\mathbf{u}) \times (s\mathbf{v}) = (rs)(\mathbf{u} \times \mathbf{v});$
- 2  $\mathbf{u} \times (\mathbf{v} + \mathbf{w}) = (\mathbf{u} \times \mathbf{v}) + (\mathbf{u} \times \mathbf{w});$
- 3  $\mathbf{v} \times \mathbf{u} = -\mathbf{u} \times \mathbf{v};$
- 4  $(\mathbf{v} + \mathbf{w}) \times \mathbf{u} = (\mathbf{v} \times \mathbf{u}) + (\mathbf{w} \times \mathbf{u});$
- 5  $\mathbf{0} \times \mathbf{u} = \mathbf{0};$  and
- 6  $\mathbf{u} \times (\mathbf{v} \times \mathbf{w}) = (\mathbf{u} \cdot \mathbf{w})\mathbf{v} - (\mathbf{u} \cdot \mathbf{v})\mathbf{w}.$

## 11.4 The Cross Product



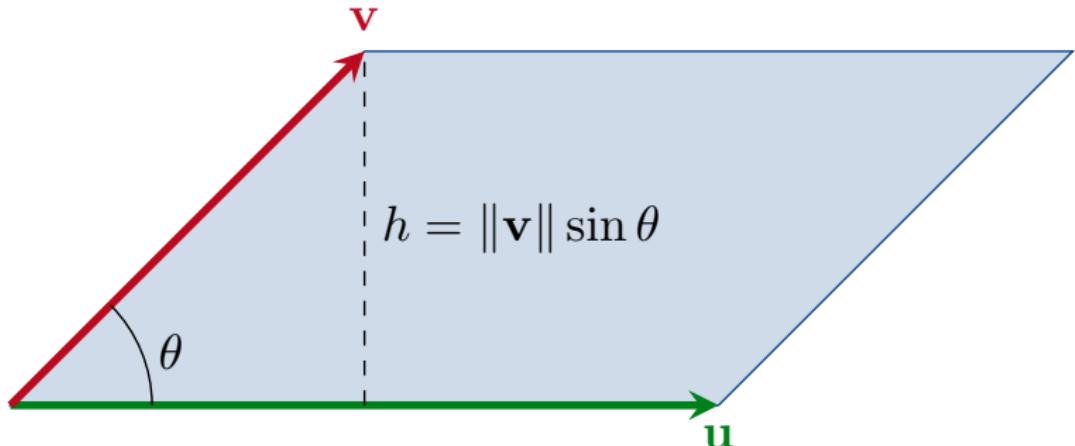
### Property (iii)



$$\mathbf{v} \times \mathbf{u} = -\mathbf{u} \times \mathbf{v}$$

$$\mathbf{u} \times \mathbf{v} = \|\mathbf{u}\| \|\mathbf{v}\| (\sin \theta) \mathbf{n}$$

## Area of a Parallelogram

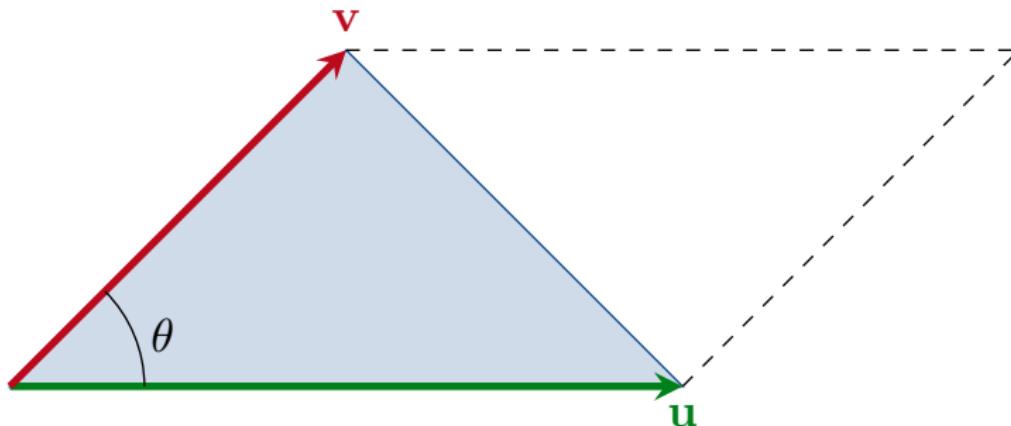


$$\text{area} = (\text{base}) (\text{height}) = \|\mathbf{u}\| \|\mathbf{v}\| \sin \theta = \|\mathbf{u} \times \mathbf{v}\| .$$

## 11.4 The Cross Product



### Area of a Triangle

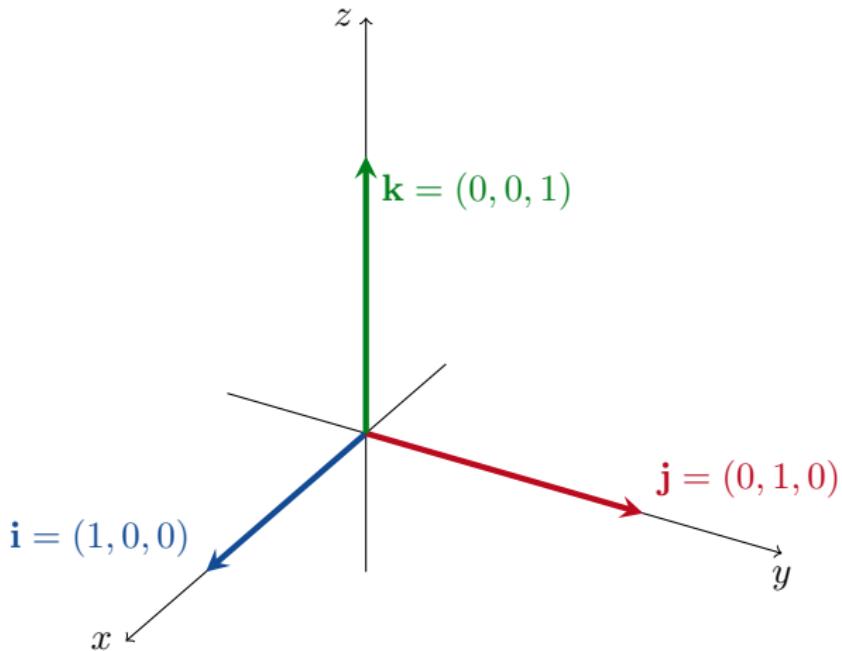


$$\begin{aligned}\text{area of triangle} &= \frac{1}{2} (\text{area of parallelogram}) \\ &= \frac{1}{2} \|\mathbf{u} \times \mathbf{v}\|.\end{aligned}$$

## 11.4 The Cross Product



### A Formula for $\mathbf{u} \times \mathbf{v}$



11.4

$$\mathbf{u} \times \mathbf{v} = \|\mathbf{u}\| \|\mathbf{v}\| (\sin \theta) \mathbf{n}$$



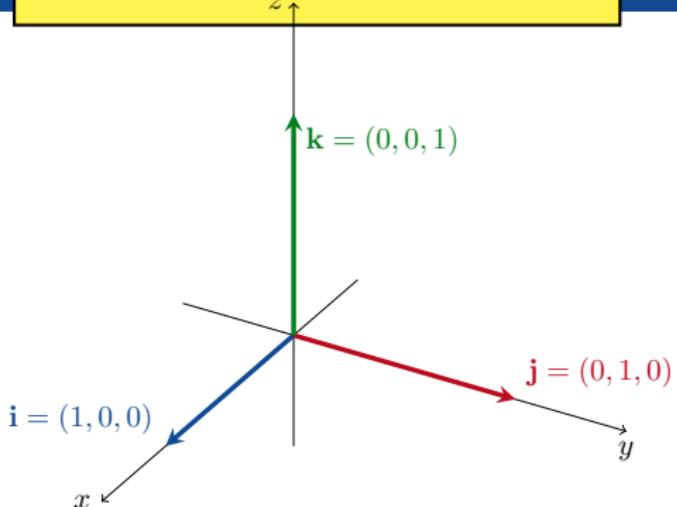
Note first that

$$\mathbf{i} \times \mathbf{i} = \|\mathbf{i}\| \|\mathbf{i}\| \sin 0^\circ \mathbf{n} = \mathbf{0}.$$

Similarly  $\mathbf{j} \times \mathbf{j} = \mathbf{0}$  and  $\mathbf{k} \times \mathbf{k} = \mathbf{0}$  also.

11.4

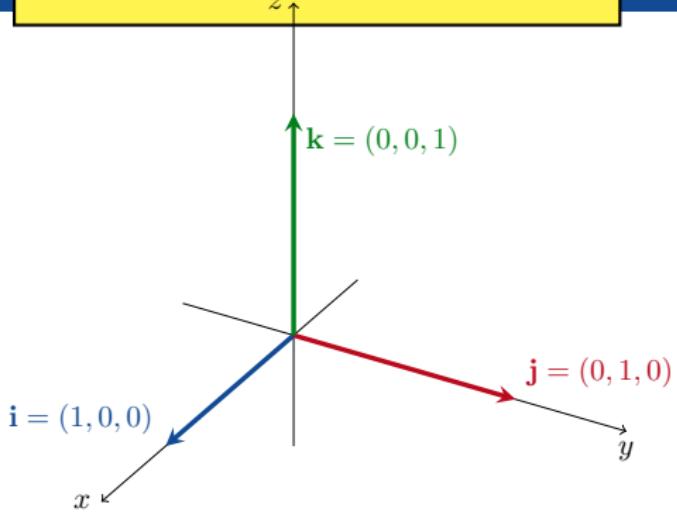
$$\mathbf{u} \times \mathbf{v} = \|\mathbf{u}\| \|\mathbf{v}\| (\sin \theta) \mathbf{n}$$



Next note that  $\mathbf{i} \times \mathbf{j}$  must point in the same direction as  $\mathbf{k}$  by the left-hand rule.

11.4

$$\mathbf{u} \times \mathbf{v} = \|\mathbf{u}\| \|\mathbf{v}\| (\sin \theta) \mathbf{n}$$

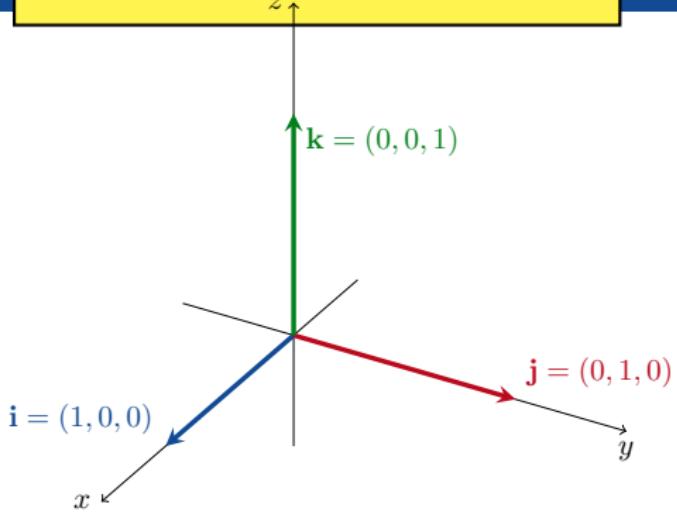


Next note that  $\mathbf{i} \times \mathbf{j}$  must point in the same direction as  $\mathbf{k}$  by the left-hand rule. Thus

$$\mathbf{i} \times \mathbf{j} = \|\mathbf{i}\| \|\mathbf{j}\| \sin 90^\circ \mathbf{k} = \mathbf{k}.$$

11.4

$$\mathbf{u} \times \mathbf{v} = \|\mathbf{u}\| \|\mathbf{v}\| (\sin \theta) \mathbf{n}$$



Next note that  $\mathbf{i} \times \mathbf{j}$  must point in the same direction as  $\mathbf{k}$  by the left-hand rule. Thus

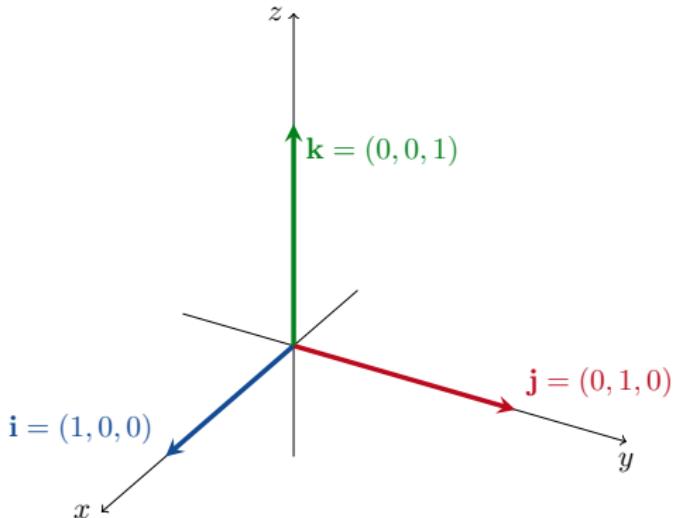
$$\mathbf{i} \times \mathbf{j} = \|\mathbf{i}\| \|\mathbf{j}\| \sin 90^\circ \mathbf{k} = \mathbf{k}.$$

We then immediately also have

$$\mathbf{j} \times \mathbf{i} = -(\mathbf{i} \times \mathbf{j}) = -\mathbf{k}.$$

11.4

$$\mathbf{u} \times \mathbf{v} = \|\mathbf{u}\| \|\mathbf{v}\| (\sin \theta) \mathbf{n}$$



I leave it for you to check that

$$\mathbf{j} \times \mathbf{k} = \mathbf{i}, \quad \mathbf{k} \times \mathbf{j} = -\mathbf{i}, \quad \mathbf{i} \times \mathbf{k} = -\mathbf{j} \quad \text{and} \quad \mathbf{k} \times \mathbf{i} = \mathbf{j}.$$

## 11.4 The Cross Product



Now suppose that  $\mathbf{u} = u_1\mathbf{i} + u_2\mathbf{j} + u_3\mathbf{k}$   
and  $\mathbf{v} = v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}$ .

## 11.4 The Cross Product



Now suppose that  $\mathbf{u} = u_1\mathbf{i} + u_2\mathbf{j} + u_3\mathbf{k}$   
and  $\mathbf{v} = v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}$ . Then we can calculate  
that

$$\mathbf{u} \times \mathbf{v} = (u_1\mathbf{i} + u_2\mathbf{j} + u_3\mathbf{k}) \times (v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k})$$

=

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## 11.4 The Cross Product

Now suppose that  $\mathbf{u} = u_1\mathbf{i} + u_2\mathbf{j} + u_3\mathbf{k}$   
and  $\mathbf{v} = v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}$ . Then we can calculate  
that

$$\begin{aligned}\mathbf{u} \times \mathbf{v} &= (u_1\mathbf{i} + u_2\mathbf{j} + u_3\mathbf{k}) \times (v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}) \\&= u_1v_1\mathbf{i} \times \mathbf{i} + u_1v_2\mathbf{i} \times \mathbf{j} + u_1v_3\mathbf{i} \times \mathbf{k} + u_2v_1\mathbf{j} \times \mathbf{i} + u_2v_2\mathbf{j} \times \mathbf{j} \\&\quad + u_2v_3\mathbf{j} \times \mathbf{k} + u_3v_1\mathbf{k} \times \mathbf{i} + u_3v_2\mathbf{k} \times \mathbf{j} + u_3v_3\mathbf{k} \times \mathbf{k} \\&= \\&= \end{aligned}$$

## 11.4 The Cross Product

$$\begin{aligned}\mathbf{i} \times \mathbf{j} &= \mathbf{k} \\ \mathbf{j} \times \mathbf{i} &= -\mathbf{k} \\ \mathbf{j} \times \mathbf{k} &= \mathbf{i} \\ \mathbf{k} \times \mathbf{j} &= -\mathbf{i} \\ \mathbf{i} \times \mathbf{k} &= -\mathbf{j} \\ \mathbf{k} \times \mathbf{i} &= \mathbf{j}\end{aligned}$$

Now suppose that  $\mathbf{u} = u_1\mathbf{i} + u_2\mathbf{j} + u_3\mathbf{k}$   
and  $\mathbf{v} = v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}$ . Then we can calculate  
that

$$\begin{aligned}\mathbf{u} \times \mathbf{v} &= (u_1\mathbf{i} + u_2\mathbf{j} + u_3\mathbf{k}) \times (v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}) \\ &= u_1v_1\mathbf{i} \times \mathbf{i} + u_1v_2\mathbf{i} \times \mathbf{j} + u_1v_3\mathbf{i} \times \mathbf{k} + u_2v_1\mathbf{j} \times \mathbf{i} + u_2v_2\mathbf{j} \times \mathbf{j} \\ &\quad + u_2v_3\mathbf{j} \times \mathbf{k} + u_3v_1\mathbf{k} \times \mathbf{i} + u_3v_2\mathbf{k} \times \mathbf{j} + u_3v_3\mathbf{k} \times \mathbf{k} \\ &= \\ &= \end{aligned}$$

## 11.4 The Cross Product

Now suppose that  $\mathbf{u} = u_1\mathbf{i} + u_2\mathbf{j} + u_3\mathbf{k}$   
and  $\mathbf{v} = v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}$ . Then we can calculate  
that

$$\begin{aligned}\mathbf{i} \times \mathbf{j} &= \mathbf{k} \\ \mathbf{j} \times \mathbf{i} &= -\mathbf{k} \\ \mathbf{j} \times \mathbf{k} &= \mathbf{i} \\ \mathbf{k} \times \mathbf{j} &= -\mathbf{i} \\ \mathbf{i} \times \mathbf{k} &= -\mathbf{j} \\ \mathbf{k} \times \mathbf{i} &= \mathbf{j}\end{aligned}$$

$$\begin{aligned}\mathbf{u} \times \mathbf{v} &= (u_1\mathbf{i} + u_2\mathbf{j} + u_3\mathbf{k}) \times (v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}) \\ &= u_1v_1\mathbf{i} \times \mathbf{i} + u_1v_2\mathbf{i} \times \mathbf{j} + u_1v_3\mathbf{i} \times \mathbf{k} + u_2v_1\mathbf{j} \times \mathbf{i} + u_2v_2\mathbf{j} \times \mathbf{j} \\ &\quad + u_2v_3\mathbf{j} \times \mathbf{k} + u_3v_1\mathbf{k} \times \mathbf{i} + u_3v_2\mathbf{k} \times \mathbf{j} + u_3v_3\mathbf{k} \times \mathbf{k} \\ &= \mathbf{0} + u_1v_2\mathbf{k} - u_1v_3\mathbf{j} - u_2v_1\mathbf{k} + \mathbf{0} + u_2v_3\mathbf{i} + u_3v_1\mathbf{j} - u_3v_2\mathbf{i} + \mathbf{0} \\ &= \end{aligned}$$

## 11.4 The Cross Product

$$\begin{aligned}\mathbf{i} \times \mathbf{j} &= \mathbf{k} \\ \mathbf{j} \times \mathbf{i} &= -\mathbf{k} \\ \mathbf{j} \times \mathbf{k} &= \mathbf{i} \\ \mathbf{k} \times \mathbf{j} &= -\mathbf{i} \\ \mathbf{i} \times \mathbf{k} &= -\mathbf{j} \\ \mathbf{k} \times \mathbf{i} &= \mathbf{j}\end{aligned}$$

Now suppose that  $\mathbf{u} = u_1\mathbf{i} + u_2\mathbf{j} + u_3\mathbf{k}$   
and  $\mathbf{v} = v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}$ . Then we can calculate  
that

$$\begin{aligned}\mathbf{u} \times \mathbf{v} &= (u_1\mathbf{i} + u_2\mathbf{j} + u_3\mathbf{k}) \times (v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}) \\ &= u_1v_1\mathbf{i} \times \mathbf{i} + u_1v_2\mathbf{i} \times \mathbf{j} + u_1v_3\mathbf{i} \times \mathbf{k} + u_2v_1\mathbf{j} \times \mathbf{i} + u_2v_2\mathbf{j} \times \mathbf{j} \\ &\quad + u_2v_3\mathbf{j} \times \mathbf{k} + u_3v_1\mathbf{k} \times \mathbf{i} + u_3v_2\mathbf{k} \times \mathbf{j} + u_3v_3\mathbf{k} \times \mathbf{k} \\ &= \mathbf{0} + u_1v_2\mathbf{k} - u_1v_3\mathbf{j} - u_2v_1\mathbf{k} + \mathbf{0} + u_2v_3\mathbf{i} + u_3v_1\mathbf{j} - u_3v_2\mathbf{i} + \mathbf{0} \\ &= (u_2v_3 - u_3v_2)\mathbf{i} - (u_1v_3 - u_3v_1)\mathbf{j} + (u_1v_2 - u_2v_1)\mathbf{k}.\end{aligned}$$

## 11.4 The Cross Product



### Theorem

If  $\mathbf{u} = u_1\mathbf{i} + u_2\mathbf{j} + u_3\mathbf{k}$  and  $\mathbf{v} = v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}$ , then

$$\mathbf{u} \times \mathbf{v} = (u_2v_3 - u_3v_2)\mathbf{i} - (u_1v_3 - u_3v_1)\mathbf{j} + (u_1v_2 - u_2v_1)\mathbf{k}.$$

## 11.4 The Cross Product



If you studied matrices and determinants at high school, then you may prefer to use the following symbolic determinant formula instead.

$$\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}.$$

$$\mathbf{u} \times \mathbf{v} = (u_2 v_3 - u_3 v_2) \mathbf{i} - (u_1 v_3 - u_3 v_1) \mathbf{j} + (u_1 v_2 - u_2 v_1) \mathbf{k}$$



## Example

Find  $\mathbf{u} \times \mathbf{v}$  and  $\mathbf{v} \times \mathbf{u}$  if  $\mathbf{u} = 2\mathbf{i} + \mathbf{j} + \mathbf{k}$  and  $\mathbf{v} = -4\mathbf{i} + 3\mathbf{j} + \mathbf{k}$ .

$$\mathbf{u} \times \mathbf{v} = (u_2 v_3 - u_3 v_2) \mathbf{i} - (u_1 v_3 - u_3 v_1) \mathbf{j} + (u_1 v_2 - u_2 v_1) \mathbf{k}$$



### Example

Find  $\mathbf{u} \times \mathbf{v}$  and  $\mathbf{v} \times \mathbf{u}$  if  $\mathbf{u} = 2\mathbf{i} + \mathbf{j} + \mathbf{k}$  and  $\mathbf{v} = -4\mathbf{i} + 3\mathbf{j} + \mathbf{k}$ .

$$\mathbf{u} \times \mathbf{v} = (1 - 3)\mathbf{i} - (2 - -4)\mathbf{j} + (6 - -4)\mathbf{k} = -2\mathbf{i} - 6\mathbf{j} + 10\mathbf{k}$$

$$\mathbf{u} \times \mathbf{v} = (u_2 v_3 - u_3 v_2) \mathbf{i} - (u_1 v_3 - u_3 v_1) \mathbf{j} + (u_1 v_2 - u_2 v_1) \mathbf{k}$$



### Example

Find  $\mathbf{u} \times \mathbf{v}$  and  $\mathbf{v} \times \mathbf{u}$  if  $\mathbf{u} = 2\mathbf{i} + \mathbf{j} + \mathbf{k}$  and  $\mathbf{v} = -4\mathbf{i} + 3\mathbf{j} + \mathbf{k}$ .

$$\mathbf{u} \times \mathbf{v} = (1 - 3)\mathbf{i} - (2 - -4)\mathbf{j} + (6 - -4)\mathbf{k} = -2\mathbf{i} - 6\mathbf{j} + 10\mathbf{k}$$

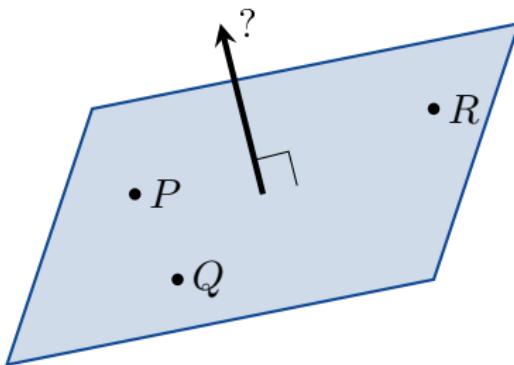
and

$$\mathbf{v} \times \mathbf{u} = -\mathbf{u} \times \mathbf{v} = 2\mathbf{i} + 6\mathbf{j} - 10\mathbf{k}.$$

$$\mathbf{u} \times \mathbf{v} = (u_2 v_3 - u_3 v_2) \mathbf{i} - (u_1 v_3 - u_3 v_1) \mathbf{j} + (u_1 v_2 - u_2 v_1) \mathbf{k}$$

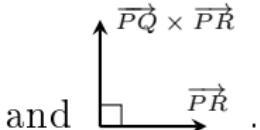
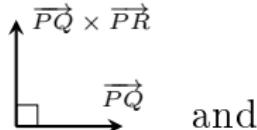
## Example

Find a vector perpendicular to the plane containing the three points  $P(1, -1, 0)$ ,  $Q(2, 1, -1)$  and  $R(-1, 1, 2)$ .



$$\mathbf{u} \times \mathbf{v} = (u_2 v_3 - u_3 v_2) \mathbf{i} - (u_1 v_3 - u_3 v_1) \mathbf{j} + (u_1 v_2 - u_2 v_1) \mathbf{k}$$

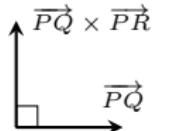
The vector  $\overrightarrow{PQ} \times \overrightarrow{PR}$  is perpendicular to the plane because

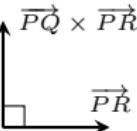


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$$\mathbf{u} \times \mathbf{v} = (u_2 v_3 - u_3 v_2) \mathbf{i} - (u_1 v_3 - u_3 v_1) \mathbf{j} + (u_1 v_2 - u_2 v_1) \mathbf{k}$$

The vector  $\overrightarrow{PQ} \times \overrightarrow{PR}$  is perpendicular to the plane because



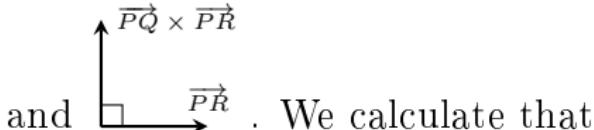
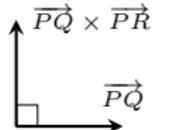
and  . We calculate that

$$\begin{aligned}\overrightarrow{PQ} &= Q - P = (2, 1, -1) - (1, -1, 0) \\ &= (2 - 1, 1 + 1, -1 - 0) = \mathbf{i} + 2\mathbf{j} - \mathbf{k}\end{aligned}$$

$$\begin{aligned}\overrightarrow{PR} &= R - P = (-1, 1, 2) - (1, -1, 0) \\ &= (-1 - 1, 1 + 1, 2 - 0) = -2\mathbf{i} + 2\mathbf{j} + 2\mathbf{k}\end{aligned}$$

$$\mathbf{u} \times \mathbf{v} = (u_2 v_3 - u_3 v_2) \mathbf{i} - (u_1 v_3 - u_3 v_1) \mathbf{j} + (u_1 v_2 - u_2 v_1) \mathbf{k}$$

The vector  $\overrightarrow{PQ} \times \overrightarrow{PR}$  is perpendicular to the plane because



and . We calculate that

$$\begin{aligned}\overrightarrow{PQ} &= Q - P = (2, 1, -1) - (1, -1, 0) \\ &= (2 - 1, 1 + 1, -1 - 0) = \mathbf{i} + 2\mathbf{j} - \mathbf{k}\end{aligned}$$

$$\begin{aligned}\overrightarrow{PR} &= R - P = (-1, 1, 2) - (1, -1, 0) \\ &= (-1 - 1, 1 + 1, 2 - 0) = -2\mathbf{i} + 2\mathbf{j} + 2\mathbf{k}\end{aligned}$$

and

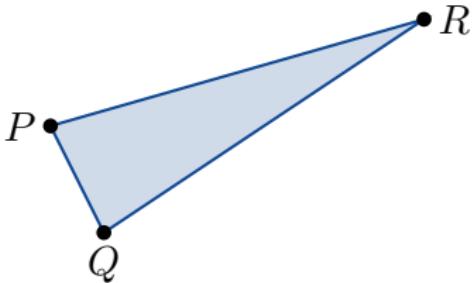
$$\overrightarrow{PQ} \times \overrightarrow{PR} = (4 + 2)\mathbf{i} - (2 - 2)\mathbf{j} + (2 + 4)\mathbf{k} = 6\mathbf{i} + 6\mathbf{k}.$$

## 11.4 The Cross Product

### Example

Find the area of triangle  $PQR$ .

$P(1, -1, 0)$ ,  $Q(2, 1, -1)$  and  $R(-1, 1, 2)$

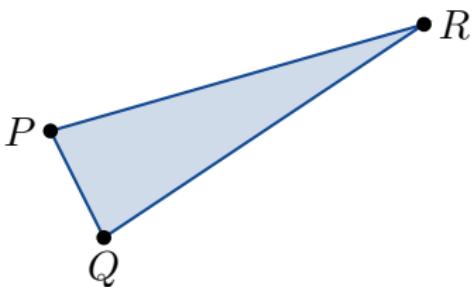


## 11.4 The Cross Product

### Example

Find the area of triangle  $PQR$ .

$P(1, -1, 0)$ ,  $Q(2, 1, -1)$  and  $R(-1, 1, 2)$



The area of the triangle is

$$\begin{aligned}\text{area} &= \frac{1}{2} \left\| \overrightarrow{PQ} \times \overrightarrow{PR} \right\| = \frac{1}{2} \|6\mathbf{i} + 6\mathbf{k}\| \\ &= \frac{1}{2} \sqrt{6^2 + 0^2 + 6^2} = 3\sqrt{2}.\end{aligned}$$

## 11.4 The Cross Product



### Example

Find a unit vector perpendicular to the plane containing  $P$ ,  $Q$  and  $R$ .

$$P(1, -1, 0), Q(2, 1, -1) \text{ and } R(-1, 1, 2)$$

## 11.4 The Cross Product

### Example

Find a unit vector perpendicular to the plane containing  $P$ ,  $Q$  and  $R$ .

$P(1, -1, 0)$ ,  $Q(2, 1, -1)$  and  $R(-1, 1, 2)$

We know that  $\overrightarrow{PQ} \times \overrightarrow{PR}$  is perpendicular to the plane. We just need to normalise this vector to find a unit vector.

## 11.4 The Cross Product



### Example

Find a unit vector perpendicular to the plane containing  $P$ ,  $Q$  and  $R$ .

$P(1, -1, 0)$ ,  $Q(2, 1, -1)$  and  $R(-1, 1, 2)$

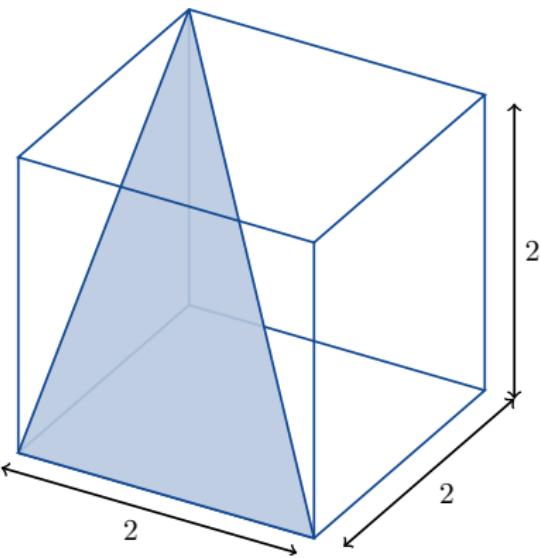
We know that  $\overrightarrow{PQ} \times \overrightarrow{PR}$  is perpendicular to the plane. We just need to normalise this vector to find a unit vector.

$$\mathbf{n} = \frac{\overrightarrow{PQ} \times \overrightarrow{PR}}{\|\overrightarrow{PQ} \times \overrightarrow{PR}\|} = \frac{6\mathbf{i} + 6\mathbf{k}}{6\sqrt{2}} = \frac{1}{\sqrt{2}}\mathbf{i} + \frac{1}{\sqrt{2}}\mathbf{k}.$$

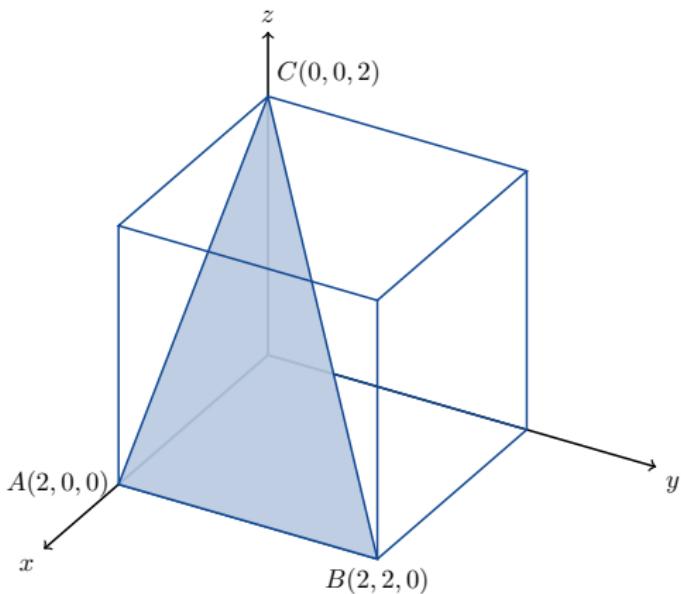
## 11.4 The Cross Product

### Example

A triangle is inscribed inside a cube of side 2 as shown below.  
Use the cross product to find the area of the triangle.



## 11.4 The Cross Product



First we draw coordinate axes and assign coordinates to the vertices of the triangle.

## 11.4 The Cross Product

Then we can calculate

$$\overrightarrow{AB} = B - A = (2, 2, 0) - (2, 0, 0) = (0, 2, 0) = 2\mathbf{j}$$

and

$$\overrightarrow{AC} = C - A = (0, 0, 2) - (2, 0, 0) = (-2, 0, 2) = -2\mathbf{i} + 2\mathbf{k}.$$

## 11.4 The Cross Product

Then we can calculate

$$\overrightarrow{AB} = B - A = (2, 2, 0) - (2, 0, 0) = (0, 2, 0) = 2\mathbf{j}$$

and

$$\overrightarrow{AC} = C - A = (0, 0, 2) - (2, 0, 0) = (-2, 0, 2) = -2\mathbf{i} + 2\mathbf{k}.$$

It follows that

$$\begin{aligned}\overrightarrow{AB} \times \overrightarrow{AC} &= (2\mathbf{j}) \times (-2\mathbf{i} + 2\mathbf{k}) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 2 & 0 \\ -2 & 0 & 2 \end{vmatrix} \\ &= \mathbf{i}(4 - 0) - \mathbf{j}(0 - 0) + \mathbf{k}(0 - -4) = 4\mathbf{i} + 4\mathbf{k}.\end{aligned}$$

## 11.4 The Cross Product

Then we can calculate

$$\overrightarrow{AB} = B - A = (2, 2, 0) - (2, 0, 0) = (0, 2, 0) = 2\mathbf{j}$$

and

$$\overrightarrow{AC} = C - A = (0, 0, 2) - (2, 0, 0) = (-2, 0, 2) = -2\mathbf{i} + 2\mathbf{k}.$$

It follows that

$$\begin{aligned}\overrightarrow{AB} \times \overrightarrow{AC} &= (2\mathbf{j}) \times (-2\mathbf{i} + 2\mathbf{k}) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 2 & 0 \\ -2 & 0 & 2 \end{vmatrix} \\ &= \mathbf{i}(4 - 0) - \mathbf{j}(0 - 0) + \mathbf{k}(0 - -4) = 4\mathbf{i} + 4\mathbf{k}.\end{aligned}$$

Therefore

$$\begin{aligned}\text{area of triangle} &= \frac{1}{2} \left\| \overrightarrow{AB} \times \overrightarrow{AC} \right\| = \frac{1}{2} \sqrt{4^2 + 0^2 + 4^2} \\ &= \frac{1}{2} \sqrt{32} = \frac{1}{2} \sqrt{4} \sqrt{8} = \sqrt{8} = 2\sqrt{2}.\end{aligned}$$

## 11.4 The Cross Product



### The Triple Scalar Product

#### Definition

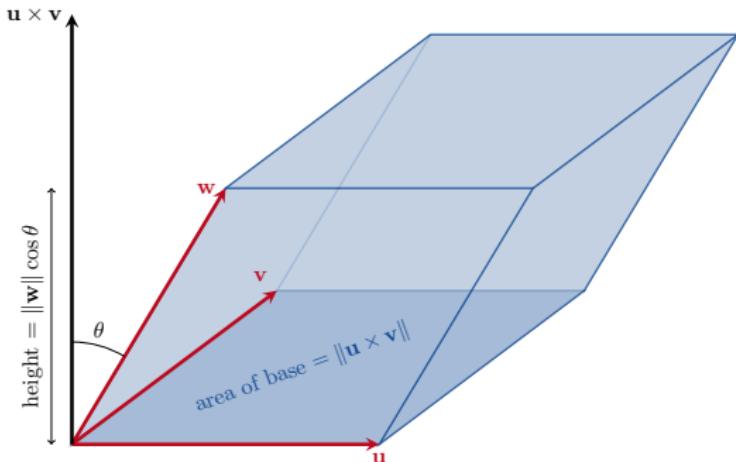
The *triple scalar product* of  $\mathbf{u}$ ,  $\mathbf{v}$  and  $\mathbf{w}$  is

$$(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w}.$$

## 11.4 The Cross Product



### The Volume of a Parallelepiped



$$\text{volume} = (\text{area of base})(\text{height}) = \|\mathbf{u} \times \mathbf{v}\| \|\mathbf{w}\| \cos \theta = |(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w}|$$

## 11.4 The Cross Product



### One Final Comment

We can do the dot product in both  $\mathbb{R}^2$  and  $\mathbb{R}^3$ . But we can only do the cross product in  $\mathbb{R}^3$ . There is no cross product in  $\mathbb{R}^2$ .



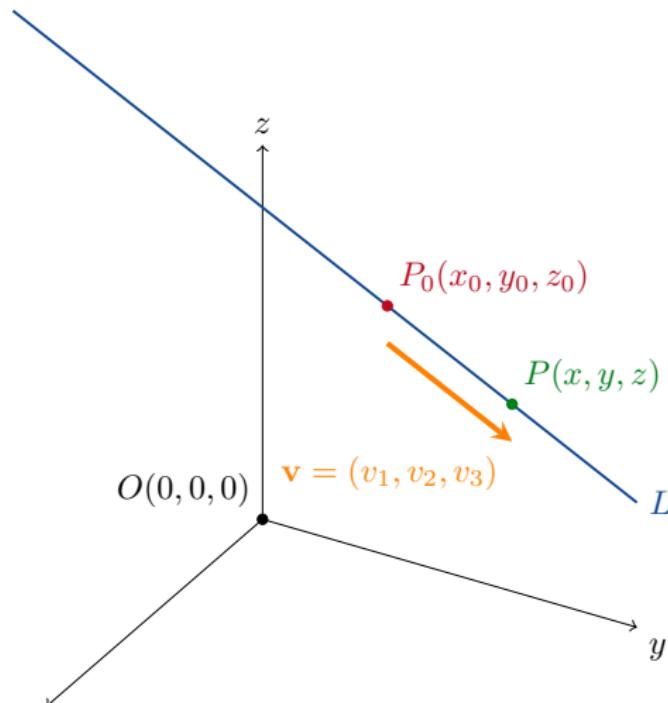
# 1 Lines and Planes in Space 5

## 11.5 Lines and Planes in Space



To describe a line in  $\mathbb{R}^3$ , we need

- a point  $P_0(x_0, y_0, z_0)$  which the line passes through; and
- a vector  $\mathbf{v}$  which gives the direction of the line.



## 11.5 Lines and Planes in Space



Let  $\mathbf{r}_0 = \overrightarrow{OP_0}$  and  $\mathbf{r} = \overrightarrow{OP}$ .

### Definition

The *line L passing through  $P_0(x_0, y_0, z_0)$  parallel to  $\mathbf{v} = (v_1, v_2, v_3)$*  has the vector equation

$$\mathbf{r} = \mathbf{r}_0 + t\mathbf{v}, \quad -\infty < t < \infty.$$

## 11.5 Lines and Planes in Space



This equation is equivalent to

$$(x, y, z) = (x_0, y_0, z_0) + t(v_1, v_2, v_3)$$

or to the set of three equations

$$x = x_0 + tv_1, \quad y = y_0 + tv_2, \quad z = z_0 + tv_3.$$

## 11.5 Lines and Planes in Space



### Definition

The *parametric equations* for the line  $L$  passing through  $P_0(x_0, y_0, z_0)$  parallel to  $\mathbf{v} = (v_1, v_2, v_3)$  are

$$x = x_0 + tv_1, \quad y = y_0 + tv_2, \quad z = z_0 + tv_3.$$

## 11.5 Lines and Planes in Space



### Example

Find parametric equations for the line passing through  $P_0(-2, 0, 4)$  parallel to  $\mathbf{v} = 2\mathbf{i} + 4\mathbf{j} - 2\mathbf{k}$ .

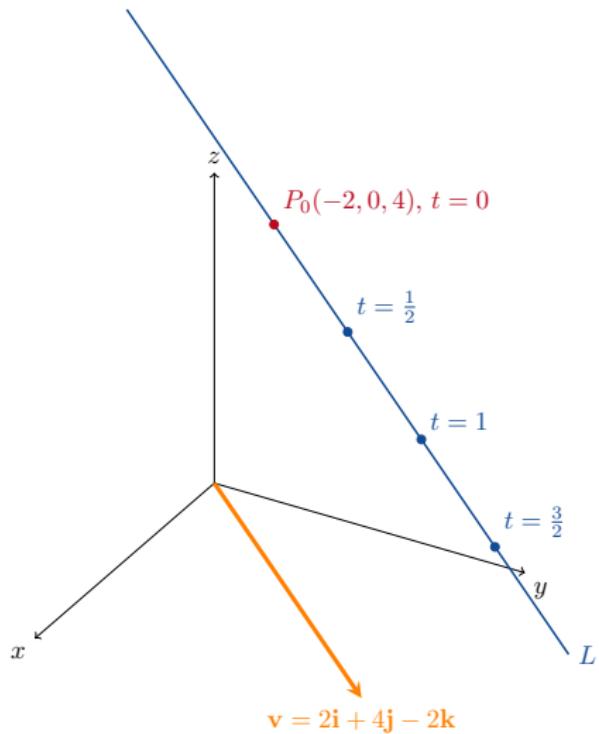
We can write

$$x = -2 + 2t, \quad y = 4t, \quad z = 4 - 2t.$$

## 11.5 Lines and Planes in Space



$$\begin{aligned}x &= -2 + 2t \\y &= 4t \\z &= 4 - 2t\end{aligned}$$



## 11.5 Lines and Planes in Space



### Example

Find parametric equations for the line passing through  $P(-3, 2, -3)$  and  $Q(1, -1, 4)$ .

Choose  $P_0 = P$  and  $\mathbf{v} = \overrightarrow{PQ} = (4, -3, 7) = 4\mathbf{i} - 3\mathbf{j} + 7\mathbf{k}$ . Then we can write

$$x = -3 + 4t, \quad y = 2 - 3t, \quad z = -3 + 7t.$$

## 11.5 Lines and Planes in Space



### Definition

The vector equation

$$\mathbf{r} = \mathbf{r}_0 + t\mathbf{v}, \quad a \leq t \leq b$$

denotes a *line segment*.

## 11.5 Lines and Planes in Space



### Example

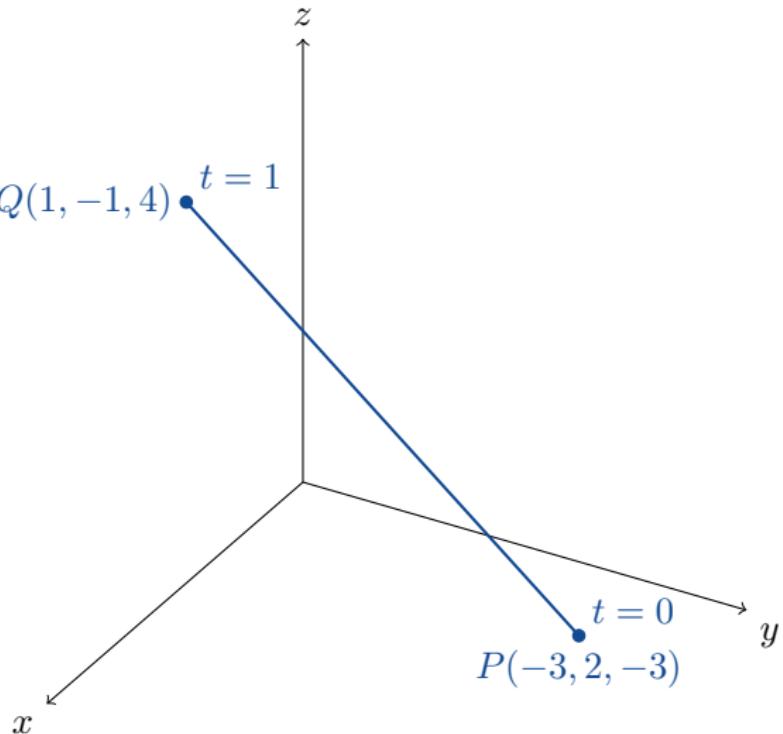
Parametrise the line segment joining  $P(-3, 2, -3)$  and  $Q(1, -1, 4)$ .

We know that  $x = -3 + 4t$ ,  $y = 2 - 3t$  and  $z = -3 + 7t$ . The line passes through  $P$  then  $t = 0$  and passed through  $Q$  when  $t = 1$ . Therefore

$$x = -3 + 4t, \quad y = 2 - 3t, \quad z = -3 + 7t, \quad 0 \leq t \leq 1$$

denotes the line segment from  $P$  to  $Q$ .

## 11.5 Lines and Planes in Space



**EXAMPLE 4** A helicopter is to fly directly from a helipad at the origin in the direction of the point  $(1, 1, 1)$  at a speed of  $60 \text{ m/sec}$ . What is the position of the helicopter after  $10 \text{ sec}$ ?

**Solution** We place the origin at the starting position (helipad) of the helicopter. Then the unit vector

$$\mathbf{u} = \frac{1}{\sqrt{3}}\mathbf{i} + \frac{1}{\sqrt{3}}\mathbf{j} + \frac{1}{\sqrt{3}}\mathbf{k}$$

gives the flight direction of the helicopter. From Equation (4), the position of the helicopter at any time  $t$  is

$$\begin{aligned}\mathbf{r}(t) &= \mathbf{r}_0 + t(\text{speed})\mathbf{u} \\ &= \mathbf{0} + t(60)\left(\frac{1}{\sqrt{3}}\mathbf{i} + \frac{1}{\sqrt{3}}\mathbf{j} + \frac{1}{\sqrt{3}}\mathbf{k}\right) \\ &= 20\sqrt{3}t(\mathbf{i} + \mathbf{j} + \mathbf{k}).\end{aligned}$$

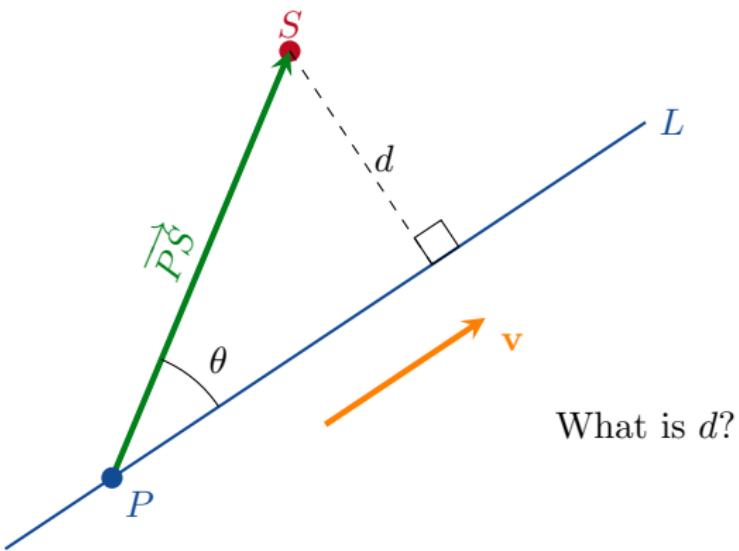
When  $t = 10 \text{ sec}$ ,

$$\begin{aligned}\mathbf{r}(10) &= 200\sqrt{3}(\mathbf{i} + \mathbf{j} + \mathbf{k}) \\ &= \langle 200\sqrt{3}, 200\sqrt{3}, 200\sqrt{3} \rangle.\end{aligned}$$

After  $10 \text{ sec}$  of flight from the origin toward  $(1, 1, 1)$ , the helicopter is located at the point  $(200\sqrt{3}, 200\sqrt{3}, 200\sqrt{3})$  in space. It has traveled a distance of  $(60 \text{ m/sec})(10 \text{ sec}) = 600 \text{ m}$ , which is the length of the vector  $\mathbf{r}(10)$ .



### The Distance from a Point to a Line

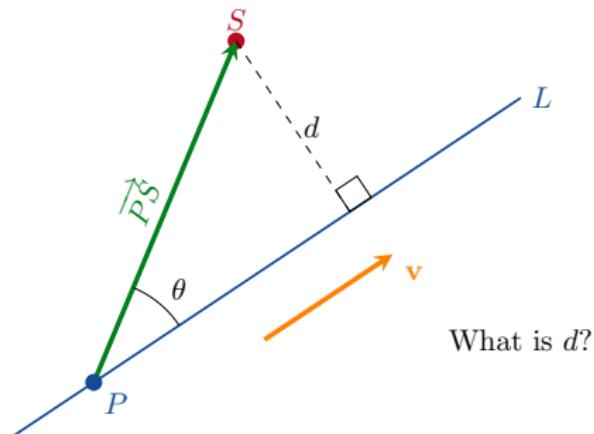


## 11.5 Lines and Planes in Space



Let  $d$  be the shortest distance from the point  $S$  to the line  $L$ .  
We can see from this diagram that

$$d = \|\overrightarrow{PS}\| \sin \theta.$$



## 11.5 Lines and Planes in Space



Let  $d$  be the shortest distance from the point  $S$  to the line  $L$ . We can see from this diagram that

$$d = \|\overrightarrow{PS}\| \sin \theta.$$

But remember that  $\overrightarrow{PS} \times \mathbf{v} = \|\overrightarrow{PS}\| \|\mathbf{v}\| \sin \theta \mathbf{n}$ . Therefore

$$d = \frac{\|\overrightarrow{PS} \times \mathbf{v}\|}{\|\mathbf{v}\|}.$$

## 11.5 Lines and Planes in Space

### Example

Find the distance from the point  $S(1, 1, 5)$  to the line

$$x = 1 + t, \quad y = 3 - t, \quad z = 2t.$$

## 11.5 Lines and Planes in Space

### Example

Find the distance from the point  $S(1, 1, 5)$  to the line

$$x = 1 + t, \quad y = 3 - t, \quad z = 2t.$$

The line passes through the point  $P(1, 3, 0)$  in the direction  
 $\mathbf{v} = \mathbf{i} - \mathbf{j} + 2\mathbf{k}$ .

## 11.5 Lines and Planes in Space

### Example

Find the distance from the point  $S(1, 1, 5)$  to the line

$$x = 1 + t, \quad y = 3 - t, \quad z = 2t.$$

The line passes through the point  $P(1, 3, 0)$  in the direction  $\mathbf{v} = \mathbf{i} - \mathbf{j} + 2\mathbf{k}$ . Thus

$$\overrightarrow{PS} = S - P = (1, 1, 5) - (1, 3, 0) = (0, -2, 5) = -2\mathbf{j} + 5\mathbf{k}$$

and

$$\overrightarrow{PS} \times \mathbf{v} = (-4 + 5)\mathbf{i} - (0 - 5)\mathbf{j} + (0 + 2)\mathbf{k} = \mathbf{i} + 5\mathbf{j} + 2\mathbf{k}.$$

## 11.5 Lines and Planes in Space

### Example

Find the distance from the point  $S(1, 1, 5)$  to the line

$$x = 1 + t, \quad y = 3 - t, \quad z = 2t.$$

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and

$$\overrightarrow{PS} \times \mathbf{v} = (-4 + 5)\mathbf{i} - (0 - 5)\mathbf{j} + (0 + 2)\mathbf{k} = \mathbf{i} + 5\mathbf{j} + 2\mathbf{k}.$$

Therefore

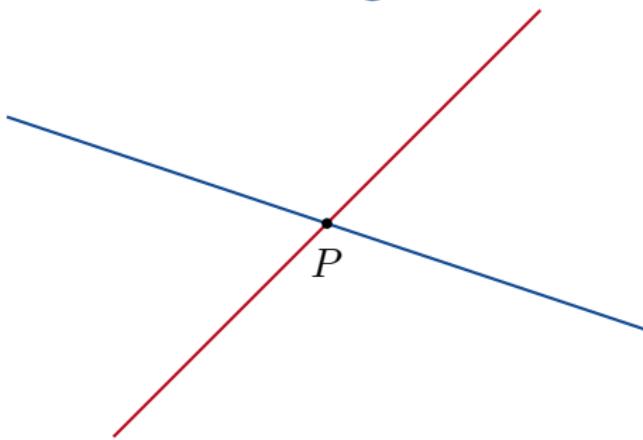
$$d = \frac{\|\overrightarrow{PS} \times \mathbf{v}\|}{\|\mathbf{v}\|} = \frac{\sqrt{1^2 + 5^2 + 2^2}}{\sqrt{1^2 + 1^2 + 2^2}} = \frac{\sqrt{30}}{\sqrt{6}} = \sqrt{5}.$$

# Break

We will continue at 2pm



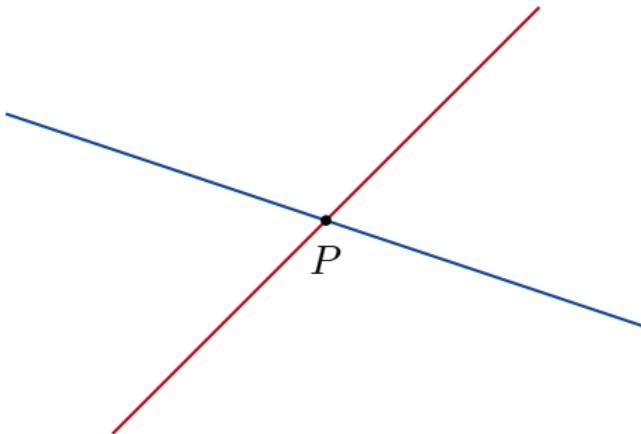
### Intersecting Lines<sup>1</sup>



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<sup>1</sup>not in book

### Intersecting Lines<sup>1</sup>



#### Definition

Two lines intersect at a point  $P$  if and only if  $P$  lies on both lines.

---

<sup>1</sup>not in book

## 11.5 Lines and Planes in Space

### Example

Do the following two lines intersect? If yes, where?

- 1  $x = 7 - t, y = 3 + 3t, z = 2t.$
- 2  $x = -1 + 2s, y = 3s, z = 1 + s.$

## 11.5 Lines and Planes in Space

### Example

Do the following two lines intersect? If yes, where?

- 1  $x = 7 - t, y = 3 + 3t, z = 2t.$
- 2  $x = -1 + 2s, y = 3s, z = 1 + s.$

The two lines intersect if and only if there exist  $s, t \in \mathbb{R}$  such that

$$7 - t = x = -1 + 2s$$

$$3 + 3t = y = 3s$$

$$2t = z = 1 + s$$

## 11.5 Lines and Planes in Space

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Do the following two lines intersect? If yes, where?

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- 2  $x = -1 + 2s, y = 3s, z = 1 + s.$

The two lines intersect if and only if there exist  $s, t \in \mathbb{R}$  such that

$$7 - t = x = -1 + 2s \qquad \Rightarrow \qquad t = 8 - 2s$$

$$3 + 3t = y = 3s$$

$$2t = z = 1 + s$$

## 11.5 Lines and Planes in Space



### Example

Do the following two lines intersect? If yes, where?

- 1  $x = 7 - t, y = 3 + 3t, z = 2t.$
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The two lines intersect if and only if there exist  $s, t \in \mathbb{R}$  such that

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$$2t = z = 1 + s$$

## 11.5 Lines and Planes in Space

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Do the following two lines intersect? If yes, where?

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$$3 + 3t = y = 3s \qquad \Rightarrow \qquad s = t + 1$$

$$2t = z = 1 + s$$

The first equation tells us that  $t = 8 - 2s$ .

## 11.5 Lines and Planes in Space

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Do the following two lines intersect? If yes, where?

- 1  $x = 7 - t, y = 3 + 3t, z = 2t.$
- 2  $x = -1 + 2s, y = 3s, z = 1 + s.$

The two lines intersect if and only if there exist  $s, t \in \mathbb{R}$  such that

$$\begin{aligned} 7 - t &= x = -1 + 2s && \implies t = 8 - 2s \\ 3 + 3t &= y = 3s && \implies s = t + 1 \\ 2t &= z = 1 + s \end{aligned}$$

The first equation tells us that  $t = 8 - 2s$ . Putting this into the second equation gives  $s = t + 1 = (8 - 2s) + 1 = 9 - 2s$  which implies that  $s = 3$  and  $t = 2$ .

## 11.5 Lines and Planes in Space

### Example

Do the following two lines intersect? If yes, where?

- 1  $x = 7 - t, y = 3 + 3t, z = 2t.$
- 2  $x = -1 + 2s, y = 3s, z = 1 + s.$

The two lines intersect if and only if there exist  $s, t \in \mathbb{R}$  such that

$$\begin{aligned} 7 - t &= x = -1 + 2s && \implies t = 8 - 2s \\ 3 + 3t &= y = 3s && \implies s = t + 1 \\ 2t &= z = 1 + s \end{aligned}$$

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## 11.5 Lines and Planes in Space

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Do the following two lines intersect? If yes, where?

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 $2t = 2 \times 2 = 4 = 1 + 3 = 1 + s.$

## 11.5 Lines and Planes in Space

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Do the following two lines intersect? If yes, where?

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The first equation tells us that  $t = 8 - 2s$ . Putting this into the second equation gives  $s = t + 1 = (8 - 2s) + 1 = 9 - 2s$  which implies that  $s = 3$  and  $t = 2$ . We must check the third equation:  $2t = 2 \times 2 = 4 = 1 + 3 = 1 + s$ . Because the third equation is also true, we know that the two lines intersect at  $P(5, 9, 4)$ .

## 11.5 Lines and Planes in Space

### Example

Do the following two lines intersect? If yes, where?

- 1  $x = 1 + t, y = 3t, z = 3 + 3t.$
- 2  $x = -1 + 2s, y = 3s, z = 1 + s.$

## 11.5 Lines and Planes in Space



### Example

Do the following two lines intersect? If yes, where?

- 1  $x = 1 + t, y = 3t, z = 3 + 3t.$
- 2  $x = -1 + 2s, y = 3s, z = 1 + s.$

Can we find  $s, t \in \mathbb{R}$  such that

$$1 + t = x = -1 + 2s$$

$$3t = y = 3s$$

$$3 + 3t = z = 1 + s$$

are all true?

## 11.5 Lines and Planes in Space

### Example

Do the following two lines intersect? If yes, where?

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Can we find  $s, t \in \mathbb{R}$  such that

$$1 + t = x = -1 + 2s$$

$$3t = y = 3s \qquad \qquad \qquad \Rightarrow \quad s = t$$

$$3 + 3t = z = 1 + s$$

are all true?

## 11.5 Lines and Planes in Space

### Example

Do the following two lines intersect? If yes, where?

- 1  $x = 1 + t, y = 3t, z = 3 + 3t.$
- 2  $x = -1 + 2s, y = 3s, z = 1 + s.$

Can we find  $s, t \in \mathbb{R}$  such that

$$1 + t = x = -1 + 2s \quad \Rightarrow \quad 2 + t = 2s \quad \Rightarrow \quad t = 2$$

$$3t = y = 3s \quad \Rightarrow \quad s = t$$

$$3 + 3t = z = 1 + s$$

are all true?

## 11.5 Lines and Planes in Space



### Example

Do the following two lines intersect? If yes, where?

- 1  $x = 1 + t, y = 3t, z = 3 + 3t.$
- 2  $x = -1 + 2s, y = 3s, z = 1 + s.$

Can we find  $s, t \in \mathbb{R}$  such that

$$1 + t = x = -1 + 2s \implies 2 + t = 2s \implies t = 2$$

$$3t = y = 3s \implies s = t$$

$$3 + 3t = z = 1 + s \implies 2 + 3t = 1 + t \implies 2 + 2t = 0 \implies t = -2 \neq 2$$

are all true?

## 11.5 Lines and Planes in Space



### Example

Do the following two lines intersect? If yes, where?

- 1  $x = 1 + t, y = 3t, z = 3 + 3t.$
- 2  $x = -1 + 2s, y = 3s, z = 1 + s.$

Can we find  $s, t \in \mathbb{R}$  such that

$$\begin{array}{ll} 1 + t = x = -1 + 2s & \Rightarrow 2 + t = 2t \Rightarrow t = 2 \\ 3t = y = 3s & \Rightarrow s = t \\ 3 + 3t = z = 1 + s & \Rightarrow 2 + 2t = 0 \Rightarrow t = -2 \neq 2 \end{array}$$

are all true?

Therefore it is not possible to find an  $s$  and a  $t$ . Hence the lines do not intersect.

## 11.5 Lines and Planes in Space



### The Distance Between Two Lines<sup>2</sup>

There are three cases to consider:

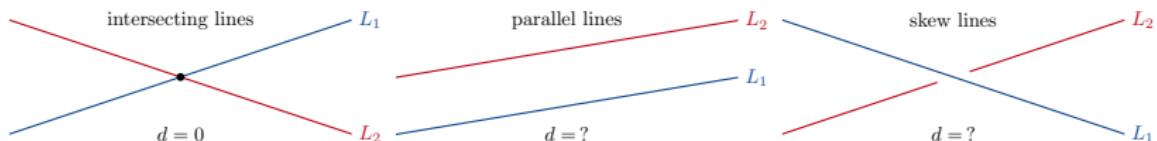
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<sup>2</sup>not in book

# The Distance Between Two Lines<sup>2</sup>

There are three cases to consider:

- the lines intersect;



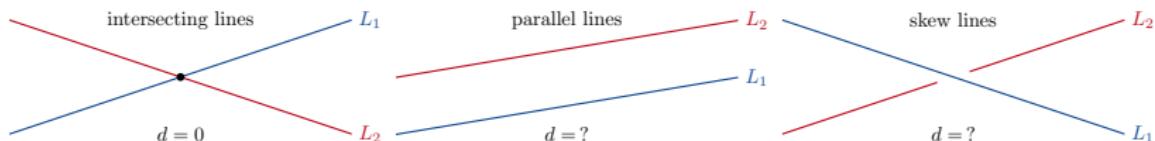
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## The Distance Between Two Lines<sup>2</sup>

There are three cases to consider:

- the lines intersect;
- the lines do not intersect and are parallel ( $\mathbf{v}_1 = k\mathbf{v}_2$  for some  $k \in \mathbb{R}$ ); or

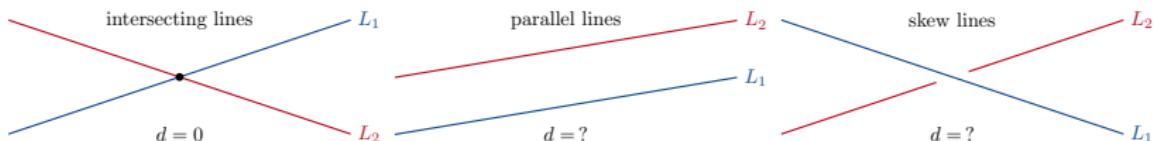


<sup>2</sup>not in book

## The Distance Between Two Lines<sup>2</sup>

There are three cases to consider:

- the lines intersect;
- the lines do not intersect and are parallel ( $\mathbf{v}_1 = k\mathbf{v}_2$  for some  $k \in \mathbb{R}$ ); or
- the lines do not intersect and are skew ( $\mathbf{v}_1 \neq k\mathbf{v}_2$  for all  $k \in \mathbb{R}$ ).



<sup>2</sup>not in book

## 11.5 Lines and Planes in Space



### Intersecting Lines

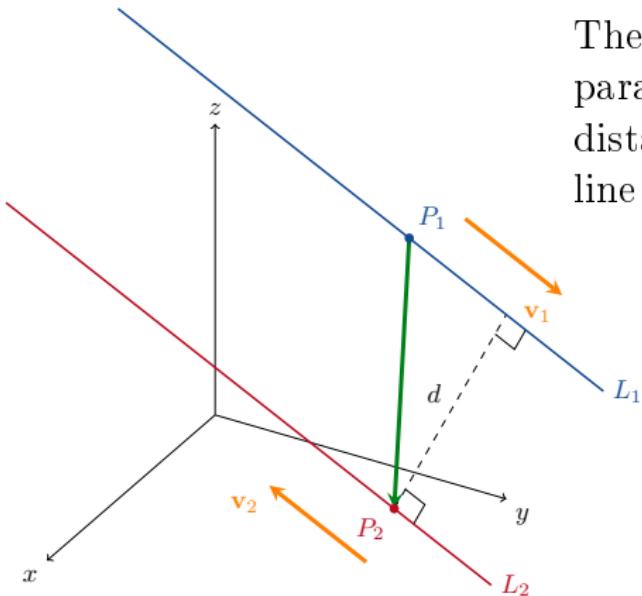
Clearly the distance between intersecting lines is zero. Hence

$$d = 0.$$

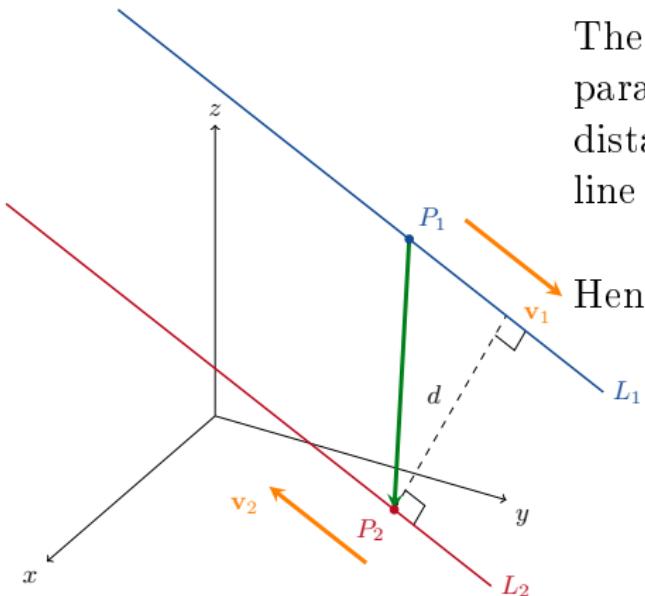
## 11.5 Lines and Planes in Space



### Parallel Lines ( $\mathbf{v}_1 \times \mathbf{v}_2 = 0$ )



The distance between the two parallel lines is the same as the distance between  $P_2$  and the line  $L_1$ .

Parallel Lines ( $\mathbf{v}_1 \times \mathbf{v}_2 = 0$ )

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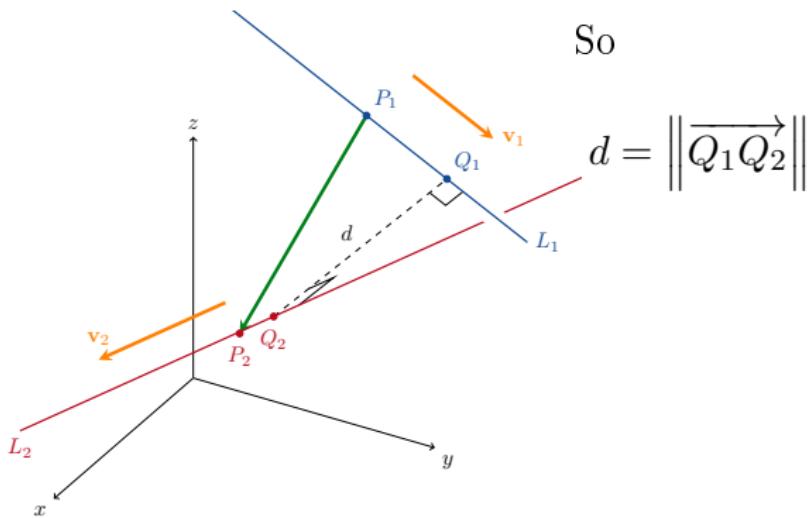
Hence

$$d = \frac{\|\overrightarrow{P_1P_2} \times \mathbf{v}_1\|}{\|\mathbf{v}_1\|}.$$

## 11.5 Lines and Planes in Space



### Skew Lines ( $\mathbf{v}_1 \times \mathbf{v}_2 \neq \mathbf{0}$ )

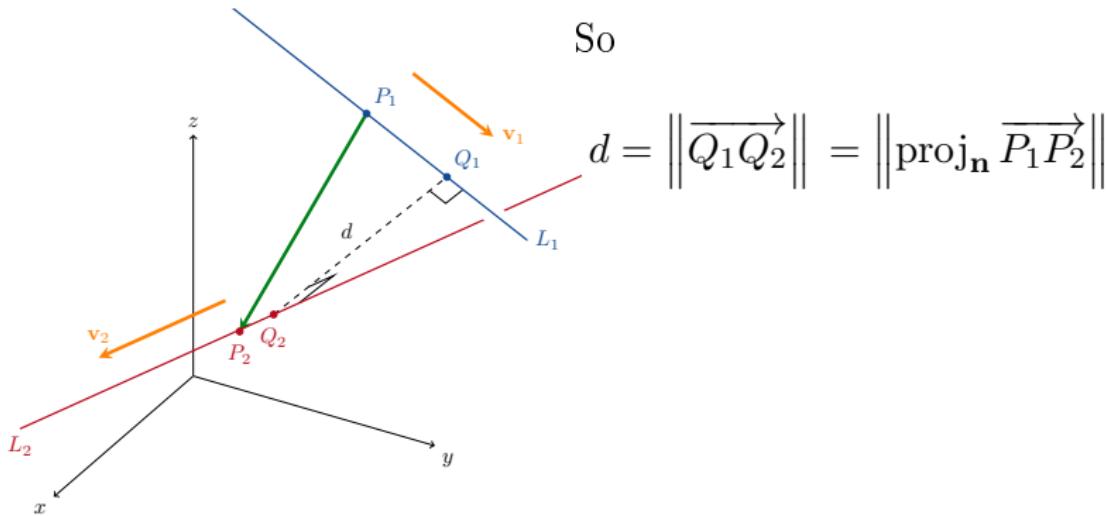


Let  $\mathbf{n} = \mathbf{v}_1 \times \mathbf{v}_2$ . Then  $\mathbf{n}$  is orthogonal to both  $\mathbf{v}_1$  and  $\mathbf{v}_2$ .

## 11.5 Lines and Planes in Space



### Skew Lines ( $\mathbf{v}_1 \times \mathbf{v}_2 \neq \mathbf{0}$ )

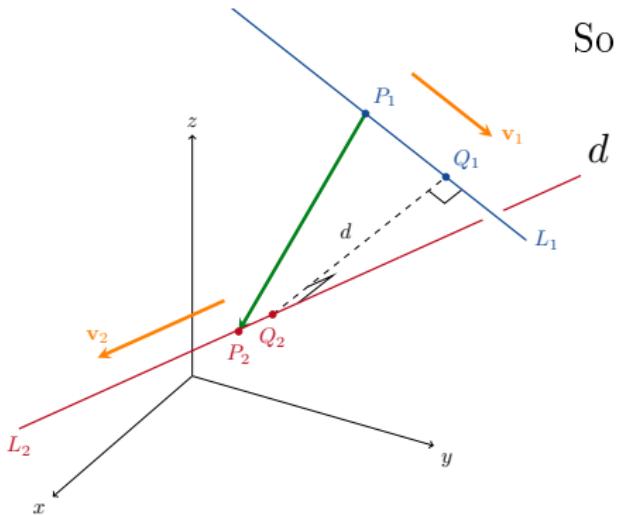


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## 11.5 Lines and Planes in Space



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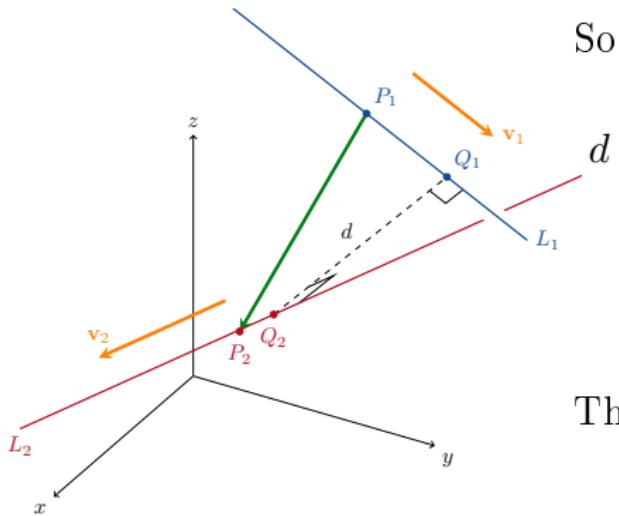
So

$$\begin{aligned} d &= \left\| \overrightarrow{Q_1 Q_2} \right\| = \left\| \text{proj}_{\mathbf{n}} \overrightarrow{P_1 P_2} \right\| \\ &= \frac{\left| \overrightarrow{P_1 P_2} \cdot \mathbf{n} \right|}{\|\mathbf{n}\|}. \end{aligned}$$

Let  $\mathbf{n} = \mathbf{v}_1 \times \mathbf{v}_2$ . Then  $\mathbf{n}$  is orthogonal to both  $\mathbf{v}_1$  and  $\mathbf{v}_2$ .

## 11.5 Lines and Planes in Space

### Skew Lines ( $\mathbf{v}_1 \times \mathbf{v}_2 \neq 0$ )



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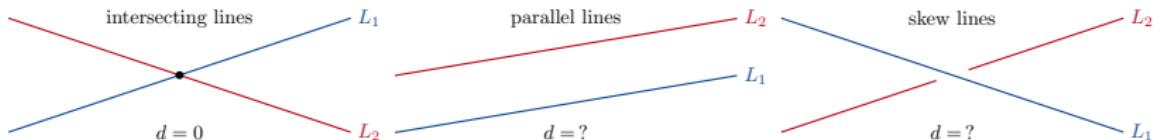
$$\begin{aligned} d &= \left\| \overrightarrow{Q_1 Q_2} \right\| = \left\| \text{proj}_{\mathbf{n}} \overrightarrow{P_1 P_2} \right\| \\ &= \frac{\left| \overrightarrow{P_1 P_2} \cdot \mathbf{n} \right|}{\|\mathbf{n}\|}. \end{aligned}$$

Thus

$$d = \frac{\left| \overrightarrow{P_1 P_2} \cdot (\mathbf{v}_1 \times \mathbf{v}_2) \right|}{\|\mathbf{v}_1 \times \mathbf{v}_2\|}.$$

Let  $\mathbf{n} = \mathbf{v}_1 \times \mathbf{v}_2$ . Then  $\mathbf{n}$  is orthogonal to both  $\mathbf{v}_1$  and  $\mathbf{v}_2$ .

## 11.5 Lines and Planes in Space



- Intersecting Lines:  $d = 0$ .

- Parallel Lines ( $\mathbf{v}_1 \times \mathbf{v}_2 = \mathbf{0}$ ):  $d = \frac{\left\| \overrightarrow{P_1 P_2} \times \mathbf{v}_1 \right\|}{\|\mathbf{v}_1\|}$ .

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## 11.5 Lines and Planes in Space

### Example

Find the distance between the following two lines.

$$\text{line 1: } x = 0, y = -t, z = t,$$

$$\text{line 2: } x = 1 + 2s, y = s, z = -3s.$$

## 11.5 Lines and Planes in Space

### Example

Find the distance between the following two lines.

$$\text{line 1: } x = 0, y = -t, z = t,$$

$$\text{line 2: } x = 1 + 2s, y = s, z = -3s.$$

We have that  $P_1(0, 0, 0)$ ,  $\mathbf{v}_1 = -\mathbf{j} + \mathbf{k}$ ,  $P_2(1, 0, 0)$  and  $\mathbf{v}_2 = 2\mathbf{i} + \mathbf{j} - 3\mathbf{k}$ . Since

$$\mathbf{v}_1 \times \mathbf{v}_2 = 2\mathbf{i} + 2\mathbf{j} + 2\mathbf{k} \neq \mathbf{0},$$

the lines are skew. (Recall that we have  $\mathbf{v}_1 \times \mathbf{v}_2 = \mathbf{0}$  for parallel vectors.)

## 11.5 Lines and Planes in Space

### Example

Find the distance between the following two lines.

line 1:  $x = 0, y = -t, z = t,$

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We have that  $P_1(0, 0, 0)$ ,  $\mathbf{v}_1 = -\mathbf{j} + \mathbf{k}$ ,  $P_2(1, 0, 0)$  and  $\mathbf{v}_2 = 2\mathbf{i} + \mathbf{j} - 3\mathbf{k}$ . Since

$$\mathbf{v}_1 \times \mathbf{v}_2 = 2\mathbf{i} + 2\mathbf{j} + 2\mathbf{k} \neq \mathbf{0},$$

the lines are skew. (Recall that we have  $\mathbf{v}_1 \times \mathbf{v}_2 = \mathbf{0}$  for parallel vectors.) Moreover note that  $\overrightarrow{P_1 P_2} = \mathbf{i}$ . Then we calculate that

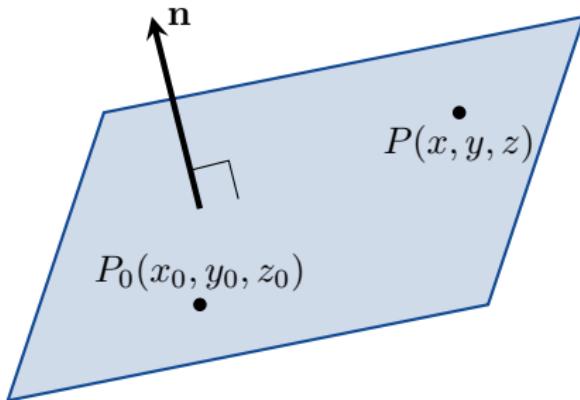
$$\begin{aligned} d &= \frac{\left| \overrightarrow{P_1 P_2} \cdot (\mathbf{v}_1 \times \mathbf{v}_2) \right|}{\|\mathbf{v}_1 \times \mathbf{v}_2\|} = \frac{|(\mathbf{i}) \cdot (2\mathbf{i} + 2\mathbf{j} + 2\mathbf{k})|}{\|2\mathbf{i} + 2\mathbf{j} + 2\mathbf{k}\|} \\ &= \frac{|2 + 0 + 0|}{\sqrt{2^2 + 2^2 + 2^2}} = \frac{1}{\sqrt{3}}. \end{aligned}$$

## An Equation for a Plane in Space

To describe a plane, we need

- a point  $P_0(x_0, y_0, z_0)$  which the plane passes through; and
- a vector  $\mathbf{n} = A\mathbf{i} + B\mathbf{j} + C\mathbf{k}$  which is perpendicular to the plane.

The vector  $\mathbf{n}$  is said to be *normal* to the plane.



## 11.5 Lines and Planes in Space



### Definition

The plane passing through the point  $P_0(x_0, y_0, z_0)$  with normal vector  $\mathbf{n} = A\mathbf{i} + B\mathbf{j} + C\mathbf{k}$  has the vector equation

$$\mathbf{n} \cdot \overrightarrow{P_0P} = 0.$$

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$$\mathbf{n} \cdot \overrightarrow{P_0P} = 0.$$

Writing this equation in coordinates, we have

$$A(x - x_0) + B(y - y_0) + C(z - z_0) = 0$$

or

$$Ax + By + Cz = D$$

where  $D = Ax_0 + By_0 + Cz_0$  is a constant.

## 11.5 Lines and Planes in Space

### Example

Find an equation for the plane passing through  $P_0(-3, 0, 7)$  normal to  $\mathbf{n} = 5\mathbf{i} + 2\mathbf{j} - \mathbf{k}$ .

## 11.5 Lines and Planes in Space



### Example

Find an equation for the plane passing through  $P_0(-3, 0, 7)$  normal to  $\mathbf{n} = 5\mathbf{i} + 2\mathbf{j} - \mathbf{k}$ .

$$A(x - x_0) + B(y - y_0) + C(z - z_0) = 0$$

$$5(x - (-3)) + 2(y - 0) + (-1)(z - 7) = 0$$

$$5x - 15 + 2y - z + 7 = 0$$

$$5x + 2y - z = -22.$$

## 11.5 Lines and Planes in Space



### Remark

The vector  $\mathbf{n} = A\mathbf{i} + B\mathbf{j} + C\mathbf{k}$  is normal to the plane  
 $Ax + By + Cz = D$ .

## 11.5 Lines and Planes in Space

### Remark

The vector  $\mathbf{n} = A\mathbf{i} + B\mathbf{j} + C\mathbf{k}$  is normal to the plane  $Ax + By + Cz = D$ .

### Example

Find a vector normal to the plane  $x + 2y + 3z = 4$ .

## 11.5 Lines and Planes in Space



### Remark

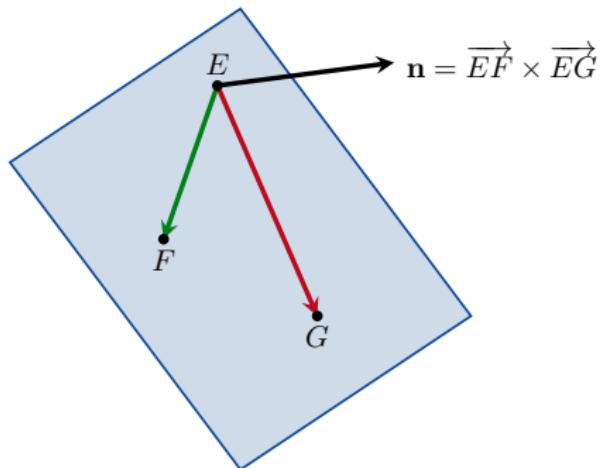
The vector  $\mathbf{n} = A\mathbf{i} + B\mathbf{j} + C\mathbf{k}$  is normal to the plane  $Ax + By + Cz = D$ .

### Example

Find a vector normal to the plane  $x + 2y + 3z = 4$ .

We can immediately write down  $\mathbf{n} = \mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$ .

## 11.5 Lines and Planes in Space



### Example

Find an equation for the plane containing the points  $E(0, 0, 1)$ ,  $F(2, 0, 0)$  and  $G(0, 3, 0)$ .

## 11.5 Lines and Planes in Space



First we need to find a vector normal to the plane. Since  $\overrightarrow{EF} = 2\mathbf{i} - \mathbf{k}$  and  $\overrightarrow{EG} = 3\mathbf{j} - \mathbf{k}$ , we have that

$$\begin{aligned}\mathbf{n} &= \overrightarrow{EF} \times \overrightarrow{EG} = (0 - -3)\mathbf{i} - (-2 - 0)\mathbf{j} + (6 - 0)\mathbf{k} \\ &= 3\mathbf{i} + 2\mathbf{j} + 6\mathbf{k}\end{aligned}$$

is normal to the plane.

## 11.5 Lines and Planes in Space



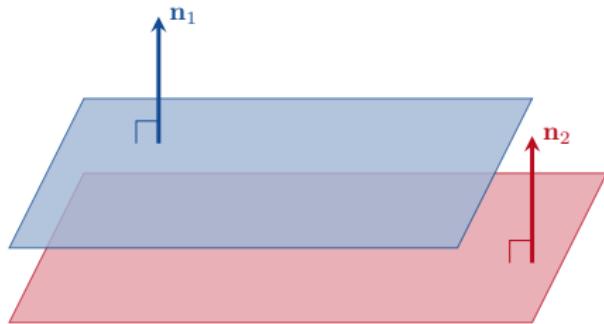
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is normal to the plane. Using  $P_0 = E(0, 0, 1)$ , the equation for the plane is

$$\begin{aligned}3(x - 0) + 2(y - 0) + 6(z - 1) &= 0 \\ 3x + 2y + 6z &= 6.\end{aligned}$$

### Lines of Intersection

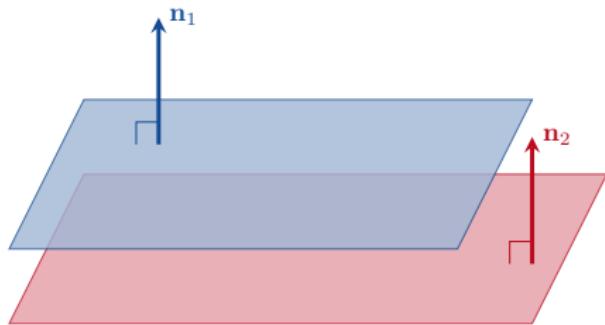


Two planes are parallel  $\iff$   
 $\mathbf{n}_1 = k\mathbf{n}_2$  for some  $k \in \mathbb{R}$ .

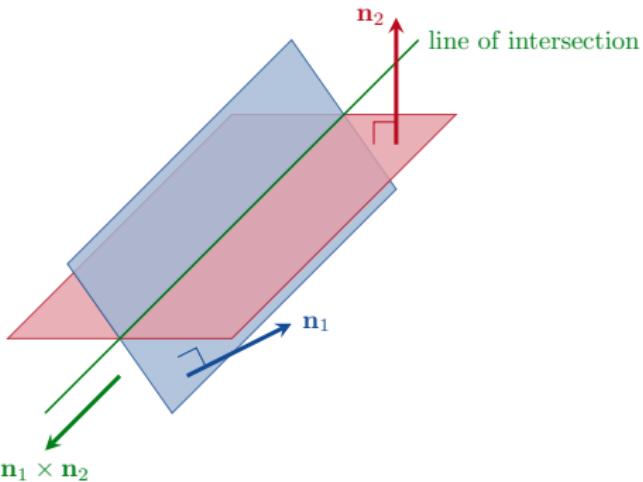
## 11.5 Lines and Planes in Space



### Lines of Intersection



Two planes are parallel  $\iff$   
 $\mathbf{n}_1 = k\mathbf{n}_2$  for some  $k \in \mathbb{R}$ .



Two planes intersect in a line  
 $\iff \mathbf{n}_1 \neq k\mathbf{n}_2$  for all  $k \in \mathbb{R}$ .

## 11.5 Lines and Planes in Space



### Example

Find a vector parallel of the line of intersection of the planes  
 $3x - 6y - 2z = 15$  and  $2x + y - 2z = 5$ .

## 11.5 Lines and Planes in Space



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We can immediately write down  $\mathbf{n}_1 = 3\mathbf{i} - 6\mathbf{j} - 2\mathbf{k}$  and  
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## 11.5 Lines and Planes in Space



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Find a vector parallel of the line of intersection of the planes  
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We can immediately write down  $\mathbf{n}_1 = 3\mathbf{i} - 6\mathbf{j} - 2\mathbf{k}$  and  
 $\mathbf{n}_2 = 2\mathbf{i} + \mathbf{j} - 2\mathbf{k}$ . A vector parallel to the line of intersection is

$$\mathbf{n}_1 \times \mathbf{n}_2 = (12 + 2)\mathbf{i} - (-6 + 4)\mathbf{j} + (3 + 12)\mathbf{k} = 14\mathbf{i} + 2\mathbf{j} + 15\mathbf{k}.$$

## 11.5 Lines and Planes in Space

### Example

Find the point where the line  $x = \frac{8}{3} + 2t$ ,  $y = -2t$ ,  $z = 1 + t$  intersects the plane  $3x + 2y + 6z = 6$ .

## 11.5 Lines and Planes in Space

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We calculate that

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## 11.5 Lines and Planes in Space

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We calculate that

$$3x + 2y + 6z = 6$$

$$3\left(\frac{8}{3} + 2t\right) + 2(-2t) + 6(1 + t) = 6$$

## 11.5 Lines and Planes in Space

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$$\begin{aligned}3x + 2y + 6z &= 6 \\3\left(\frac{8}{3} + 2t\right) + 2(-2t) + 6(1 + t) &= 6 \\8 + 6t - 4t + 6 + 6t &= 6\end{aligned}$$

## 11.5 Lines and Planes in Space

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$$8t = -8$$

$$t = -1.$$

## 11.5 Lines and Planes in Space



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The point of intersection is

$$P(x, y, z)|_{t=-1} =$$

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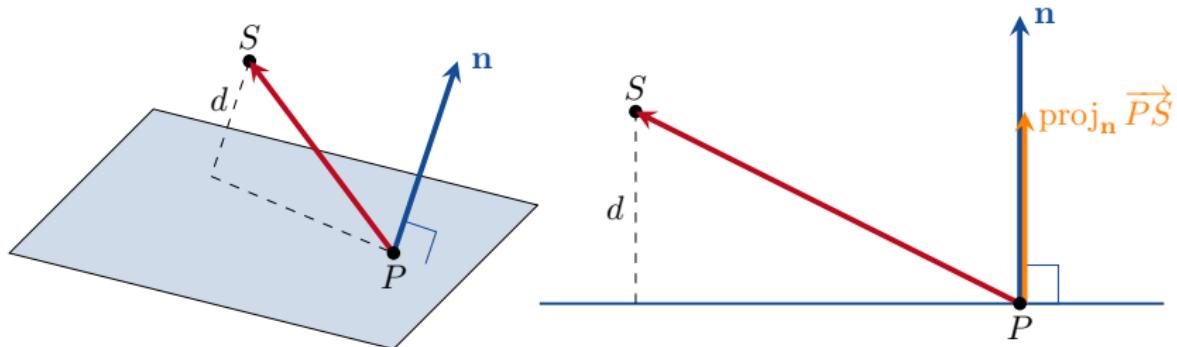
We calculate that

$$\begin{aligned} 3x + 2y + 6z &= 6 \\ 3\left(\frac{8}{3} + 2t\right) + 2(-2t) + 6(1 + t) &= 6 \\ 8 + 6t - 4t + 6 + 6t &= 6 \\ 8t &= -8 \\ t &= -1. \end{aligned}$$

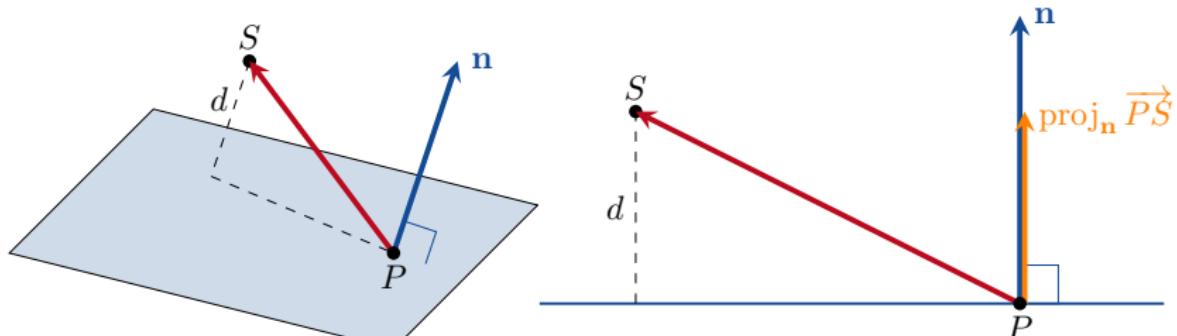
The point of intersection is

$$P(x, y, z)|_{t=-1} = P\left(\frac{8}{3} + 2t, -2t, 1 + t\right)\Big|_{t=-1} = P\left(\frac{2}{3}, 2, 0\right).$$

### The Distance from a Point to a Plane

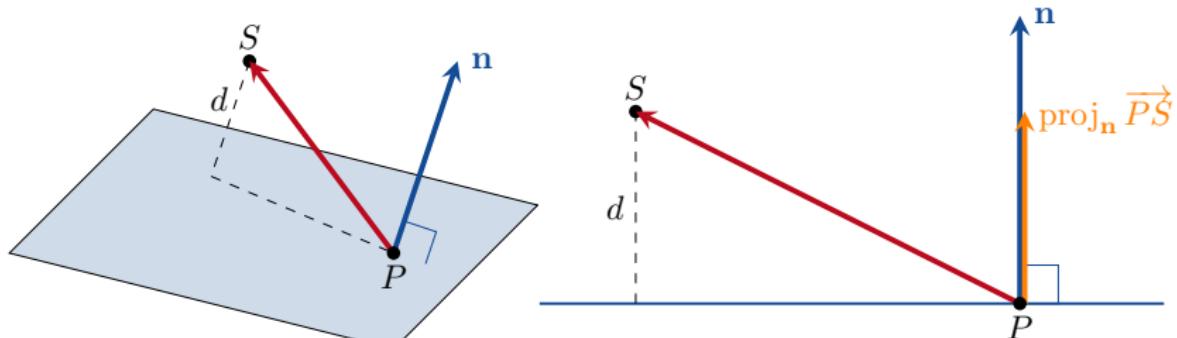


## The Distance from a Point to a Plane



We can see that  $d = \|\text{proj}_{\mathbf{n}} \vec{PS}\|$ .

## The Distance from a Point to a Plane



We can see that  $d = \|\text{proj}_n \overrightarrow{PS}\|$ . Therefore the distance from a point  $S$  to a plane with normal  $\mathbf{n}$  containing the point  $P$  is

$$d = \frac{|\overrightarrow{PS} \cdot \mathbf{n}|}{\|\mathbf{n}\|}.$$

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## Example

Find the distance from the point  $S(1, 2, 3)$  to the plane  
 $x + 2y + 3z = 4$ .

First we need a point in the plane.

$$d = \frac{|\overrightarrow{PS} \cdot \mathbf{n}|}{\|\mathbf{n}\|}$$



## Example

Find the distance from the point  $S(1, 2, 3)$  to the plane  
 $x + 2y + 3z = 4$ .

First we need a point in the plane. Setting  $y = 0$  and  $z = 0$  we must have  $x = 4 - 2y - 3z = 4$ . Therefore  $P(4, 0, 0)$  is in the plane.

$$d = \frac{|\overrightarrow{PS} \cdot \mathbf{n}|}{\|\mathbf{n}\|}$$



## Example

Find the distance from the point  $S(1, 2, 3)$  to the plane  
 $x + 2y + 3z = 4$ .

First we need a point in the plane. Setting  $y = 0$  and  $z = 0$  we must have  $x = 4 - 2y - 3z = 4$ . Therefore  $P(4, 0, 0)$  is in the plane. Clearly  $\mathbf{n} = \mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$ .

$$d = \frac{|\overrightarrow{PS} \cdot \mathbf{n}|}{\|\mathbf{n}\|}$$

## Example

Find the distance from the point  $S(1, 2, 3)$  to the plane  $x + 2y + 3z = 4$ .

First we need a point in the plane. Setting  $y = 0$  and  $z = 0$  we must have  $x = 4 - 2y - 3z = 4$ . Therefore  $P(4, 0, 0)$  is in the plane. Clearly  $\mathbf{n} = \mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$ .

Therefore the required distance is

$$\begin{aligned} d &= \frac{|\overrightarrow{PS} \cdot \mathbf{n}|}{\|\mathbf{n}\|} = \frac{|(-3\mathbf{i} + 2\mathbf{j} + 3\mathbf{k}) \cdot (\mathbf{i} + 2\mathbf{j} + 3\mathbf{k})|}{\|\mathbf{i} + 2\mathbf{j} + 3\mathbf{k}\|} \\ &= \frac{|-3 + 4 + 9|}{\sqrt{1^2 + 2^2 + 3^2}} = \frac{10}{\sqrt{14}}. \end{aligned}$$

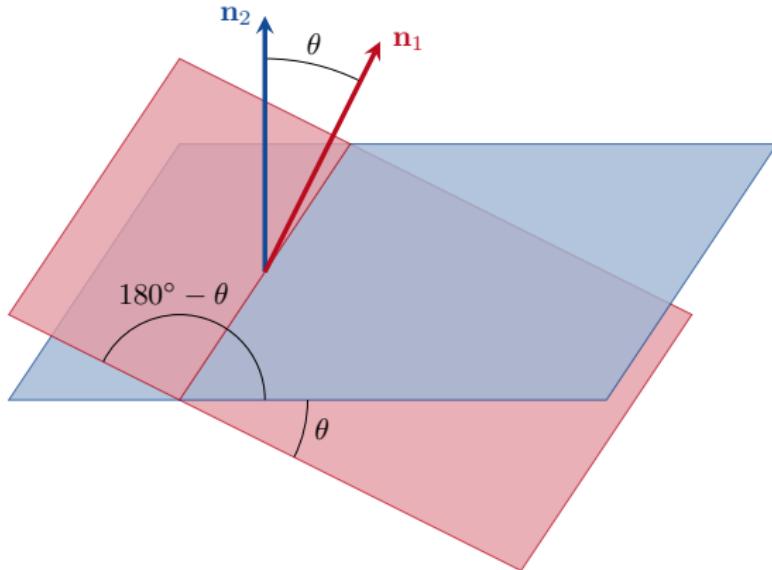
## 11.5 Lines and Planes in Space



Please read Example 11 in the textbook.

## Angles Between Planes

There are two possible angles that can be measured between planes. We are interested in the smaller angle.



## 11.5 Lines and Planes in Space



### Definition

The angle between two planes is defined to be equal to whichever of the following angles is smaller

- the angle between  $\mathbf{n}_1$  and  $\mathbf{n}_2$ ;
- $180^\circ$  minus the angle between  $\mathbf{n}_1$  and  $\mathbf{n}_2$ .

The angle between two planes will always be between  $0^\circ$  and  $90^\circ$ .

## 11.5 Lines and Planes in Space



### Example

Find the angle between the planes  $3x - 6y - 2z = 15$  and  $-2x - y + 2z = 5$ .

## 11.5 Lines and Planes in Space

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Find the angle between the planes  $3x - 6y - 2z = 15$  and  $-2x - y + 2z = 5$ .

We have normal vectors  $\mathbf{n}_1 = 3\mathbf{i} - 6\mathbf{j} - 2\mathbf{k}$  and  $\mathbf{n}_2 = -2\mathbf{i} - \mathbf{j} + 2\mathbf{k}$ . The angle between  $\mathbf{n}_1$  and  $\mathbf{n}_2$  is

$$\theta = \cos^{-1} \left( \frac{\mathbf{n}_1 \cdot \mathbf{n}_2}{\|\mathbf{n}_1\| \|\mathbf{n}_2\|} \right) = \cos^{-1} \left( \frac{-4}{21} \right) \approx 101^\circ.$$

## 11.5 Lines and Planes in Space

### Example

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$$\theta = \cos^{-1} \left( \frac{\mathbf{n}_1 \cdot \mathbf{n}_2}{\|\mathbf{n}_1\| \|\mathbf{n}_2\|} \right) = \cos^{-1} \left( \frac{-4}{21} \right) \approx 101^\circ.$$

Because  $101^\circ > 90^\circ$ , the angle between the two planes is approximately  $180 - 101^\circ = 79^\circ$ .



# Next Time

- 13.1 Functions of Several Variables
- 13.2 Limits and Continuity in Higher Dimensions
- 13.3 Partial Derivatives
- 13.4 The Chain Rule