Differential Equations

Chapter 6 and Chapter 7

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2019-20

Contents

6

The Laplace Transform

Recall that $\int_a^\infty f(t)\,dt$ means $\lim_{R\to\infty}\int_a^R f(t)\,dt$.

Example 6.1. Let $c \neq 0$. Then

$$\int_0^\infty e^{ct}\,dt = \lim_{R\to\infty}\int_0^R e^{ct}\,dt = \lim_{R\to\infty}\left[\frac{1}{c}e^{ct}\right]_0^R = \lim_{R\to\infty}\frac{1}{c}\left(e^{cR}-1\right) = \begin{cases}\infty & c>0\\ -\frac{1}{c} & c<0.\end{cases}$$

Example 6.2.

$$\int_1^\infty \frac{1}{t}\,dt = \lim_{R\to\infty}\int_1^R \frac{1}{t}\,dt = \lim_{R\to\infty}\left[\ln t\right]_1^R = \lim_{R\to\infty}(\ln R - 0) = \infty$$

Pierre-Simon Laplace
FRA, 1749-1827

6.1 Definition of the Laplace Transform

$\frac{d}{dt}$ changes a function $f(t)$ into a new function $f'(t)$.
$\mathcal{L}$ changes a function $f(t)$ into a new function $F(s)$.

Definition. Suppose that

(i) $A>0$, $K>0$, $M>0$, $a\in\mathbb{R}$;

(ii) f is piecewise continuous on $[0,A]$; and

(iii) $|f(t)| \leq Ke^{at}$ for all $t \geq M$.

The ***Laplace Transform*** of $f:[0,\infty)\to\mathbb{R}$ is a new function defined by

$$F(s) = \mathcal{L}\big[f\big](s) = \int_0^\infty e^{-st}f(t)\,dt.$$

Example 6.3.

$$\mathcal{L}\big[1\big](s) = \int_0^\infty e^{-st}\,dt = -\lim_{R\to\infty}\left[\frac{e^{-st}}{s}\right]_0^R = \frac{1}{s} \qquad \text{if } x>0.$$

Example 6.4.

$$\mathcal{L}\big[e^{at}\big](s) = \int_0^\infty e^{-st}e^{at}\,dt = \int_0^\infty e^{-(s-a)t}\,dt = \frac{1}{s-a} \qquad \text{if } s > a.$$

The Laplace Transform of $e^{at} : [0,\infty) \to \mathbb{R}$ is $\frac{1}{s-a} : (a,\infty) \to \mathbb{R}$.

Example 6.5. Let

$$f(t) = \begin{cases} 1 & 0 \le t < 1 \\ k & t = 1 \\ 0 & t > 1. \end{cases}$$

Then

$$F(s) = \mathcal{L}\big[f\big](s) = \int_0^\infty e^{-st}f(t)\,dt = \int_0^1 e^{-st}\,dt = \left[-\frac{1}{s}e^{-st}\right]_0^1 = \frac{1-e^{-s}}{s} \qquad \text{if } s > 1.$$

Example 6.6. Now consider $g(t) = \sin at$ $(t \ge 0)$. Using integration by parts, we have

$$\begin{aligned} G(s) = \mathcal{L}\big[g\big](s) &= \int_0^\infty e^{-st}\sin at\,dt = \lim_{R\to\infty}\int_0^R e^{-st}\sin at\,dt \\ &= \lim_{R\to\infty}\left(\left[-\frac{1}{a}e^{-st}\cos at\right]0^R - \frac{s}{a}\int_0^R e^{-st}\cos at\,dt\right) = \frac{1}{a} - \frac{s}{a}\int_0^\infty e^{-st}\cos at\,dt. \end{aligned}$$

Using integration by parts a second time, we have

$$G(s) = \frac{1}{a} - \frac{s^2}{a^2}\int_0^\infty e^{-st}\sin at\,dt = \frac{1}{a} - \frac{s^2}{a^2}G(s).$$

Therefore

$$\mathcal{L}\big[\sin at\big](s) = G(s) = \frac{a}{s^2+a^2} \qquad \text{if } s > 0.$$

Theorem 6.1.

$$\mathcal{L}\big[c_1f_1 + c_2f_2\big] = c_1\mathcal{L}\big[f_1\big] + c_2\mathcal{L}\big[f_2\big].$$

You prove.

Example 6.7. If $h(t) = 5e^{-2t} - 3\sin 4t$ $(t \ge 0)$, then

$$H(s) = \mathcal{L}\big[h\big](s) = 5\mathcal{L}\big[e^{-2t}\big] - 3\mathcal{L}\big[\sin 4t\big] = \frac{5}{s+2} - \frac{12}{s^2+16} \qquad \text{if } s > 0.$$

Theorem 6.2.

$$\mathcal{L}\left[t^nf(t)\right] = (-1)^n\frac{d^nF}{ds^n}$$

You prove this. In Exercise 28(f), you are required to prove this formula with $n = 1$.

Example 6.8.

$$\mathcal{L}\left[t^2\cosh 2t\right] = (-1)^2\frac{d^2}{ds^2}\mathcal{L}\left[\cosh 2t\right] = \frac{d^2}{ds^2}\left(\frac{s}{s^2-2^2}\right) = \ldots = \frac{2s(s^2+12)}{(x^2-4)^3}$$

$f(t)$	$F(s)=\mathcal{L}\big[f\big](s)$	
1	$\frac{1}{s}$	$s>0$
e^{at}	$\frac{1}{s-a}$	$s>a$
$t^n \quad (n\in\mathbb{N})$	$\frac{n!}{s^{n+1}}$	$s>0$
$\sin at$	$\frac{a}{s^2+a^2}$	$s>0$
$\cos at$	$\frac{s}{s^2+a^2}$	$s>0$
$\sinh at$	$\frac{a}{s^2-a^2}$	$s>\lvert a\rvert$
$\cosh at$	$\frac{s}{s^2-a^2}$	$s>\lvert a\rvert$
$e^{at}\sin bt$	$\frac{b}{(s-a)^2+b^2}$	$s>a$
$e^{at}\cos bt$	$\frac{s-a}{(s-a)^2+b^2}$	$s>a$
$t^n e^{at} \quad (n\in\mathbb{N})$	$\frac{n!}{(s-a)^{n+1}}$	$s>a$
$u_c(t)$	$\frac{e^{-cs}}{s}$	$s>0$
$u_c(t)f(t-c)$	$e^{-cs}F(s)$	
$e^{ct}f(t)$	$F(s-c)$	
$f(ct) \quad (c>0)$	$\frac{1}{c}F\left(\frac{s}{c}\right)$	
$\int_0^t f(t-\tau)g(\tau)d\tau$	$F(s)G(s)$	
$t^n f(t)$	$(-1)^n F^{(n)}(s)$	

$$\mathcal{L}\big[f'\big](s)=s\mathcal{L}\big[f\big](s)-f(0)$$

$$\mathcal{L}\big[f''\big](s)=s^2\mathcal{L}\big[f\big](s)-sf(0)-f'(0)$$

$$\mathcal{L}\big[f^{(n)}\big](s)=s^n\mathcal{L}\big[f\big](s)-s^{n-1}f(0)-\ldots-sf^{(n-2)}(0)-f^{(n-1)}(0)$$

Inverse Laplace Transforms

$$\mathcal{L}\big[f\big] = F \qquad \iff \qquad \mathcal{L}^{-1}\big[F\big] = f.$$

Example 6.9. Find the inverse Laplace Transform of $F(s) = \dfrac{9s^2 - 12s + 216}{s(s^2+36)}$.

Using partial fractions we calculate that

$A + B = 9$
$C = -12$
$36A = 216$

$A = 6$
$B = 3$
$C = -12$

$$\begin{aligned} F(s) &= \frac{9s^2 - 12s + 216}{s(s^2+36)} = \frac{A}{s} + \frac{Bs + C}{s^2 + 36} = \frac{As^2 + 36A + Bs^s + Cs}{s(s^2+36)} \\ &= 6\left(\frac{1}{s}\right) + 3\left(\frac{s}{s^2+36}\right) - 12\left(\frac{1}{s^2+36}\right) \\ &= 6\left(\frac{1}{s}\right) + 3\left(\frac{s}{s^2+36}\right) - \frac{12}{6}\left(\frac{6}{s^2+36}\right) \\ &= 6\mathcal{L}\big[1\big] + 3\mathcal{L}\big[\cos 6t\big] - 2\mathcal{L}\big[\sin 6t\big]. \end{aligned}$$

and that

$$f(t) = \mathcal{L}^{-1}\big[F\big](t) = 6 + 3\cos 6t - 2\sin 6t.$$

6.2 Solving Initial Value Problems

Theorem 6.3.

(*i*). $\mathcal{L}[f'](s) = s\mathcal{L}[f](s) - f(0)$.

(*ii*). $\mathcal{L}[f''](s) = s^2\mathcal{L}[f](s) - sf(0) - f'(0)$.

(*iii*). $\mathcal{L}[f'''](s) = s^3\mathcal{L}[f](s) - s^2f(0) - sf'(0) - f''(0)$.

(*iv*). $\mathcal{L}[f^{(n)}](s) = s^n\mathcal{L}[f](s) - s^{n-1}f(0) - s^{n-2}f'(0) - \dots sf^{(n-2)}(0) - f^{(n-1)}(0)$.

Proof.

(i). Using integration-by-parts ($\int uv' = uv - \int u'v$) we calculate that

$$\begin{aligned} \mathcal{L}[f'](s) &= \int_0^\infty e^{-st} f'(t)\,dt = \big[e^{-st}f(t)\big]_0^\infty - \int_0^\infty \left(\frac{d}{dt}e^{-st}\right) f(t)\,dt \\ &= 0 - f(0) - \int_0^\infty -se^{-st}f(t)\,dt = -f(0) + s\int_0^\infty e^{-st}f(t)\,dt \\ &= -f(0) + sF(s) \end{aligned}$$

as required.

(ii). Using (i), but replacing each f by f' we get

$$\mathcal{L}[f''](s) = s\mathcal{L}[f'](s) - f'(0) = s\Big(s\mathcal{L}[f](s) - f(0)\Big) - f'(0) = s^2\mathcal{L}[f](s) - sf(0) - f'(0).$$

You prove (iii) and (iv) $\square$

Example 6.10. Solve

$$\begin{cases} y'' - y' - 2y = 0 \\ y(0) = 1 \\ y'(0) = 0. \end{cases}$$

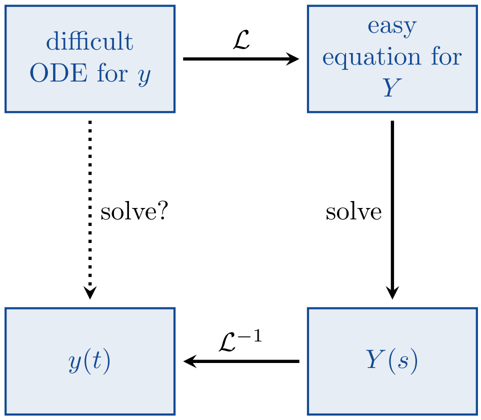
solution 1 (method from Chapter 3): The characteristic equation

$$0 = r^2 - r - 2 = (r-2)(r+1)$$

has roots $r_1 = -1$ and $r_2 = 2$. So $y = c_1 e^{-t} + c_2 e^{2t}$. Using the initial conditions we find that $c_1 = \frac{2}{3}$ and $c_2 = \frac{1}{3}$. Therefore

$$\boxed{y(t) = \frac{2}{3}e^{-t} + \frac{1}{3}e^{2t}.}$$

solution 2 (Laplace Transform):



First we take the Laplace Transform of the ODE

$$\begin{aligned} y'' - y' - 2y &= 0 \\ \mathcal{L}[y'' - y' - 2y] &= \mathcal{L}[0] \\ \mathcal{L}[y''] - \mathcal{L}[y'] - 2\mathcal{L}[y] &= 0 \\ \big(s^2Y - sy(0) - y'(0)\big) - (sY - y(0)) - 2Y &= 0 \\ \big(s^2Y - s - 0\big) - (sY - 1) - 2Y &= 0 \\ (s^2 - s - 2)Y + (1-s) &= 0 \end{aligned}$$

Thus

$$Y(s) = \frac{s-1}{s^2 - s - 2} = \frac{s-1}{(s-2)(s+1)}.$$

Using partial fractions we obtain

$$\begin{aligned} Y(s) &= \frac{s-1}{(s-2)(s+1)} = \frac{A}{s-2} + \frac{B}{s+1} = \frac{As + A + Bs - 2B}{(s-2)(s+1)} \\ &= \frac{1}{3}\left(\frac{1}{s-2}\right) + \frac{2}{3}\left(\frac{1}{s+1}\right). \end{aligned}$$

$A + B = 1$
$A - 2B = -1$

$A = \frac{1}{3}$
$B = \frac{2}{3}$

But recall that $\mathcal{L}[e^{2t}] = \frac{1}{s-2}$ and $\mathcal{L}[e^{-t}] = \frac{1}{s+1}$. Therefore

$$y(t) = \mathcal{L}^{-1}[Y] = \frac{1}{3}\mathcal{L}^{-1}\left[\frac{1}{s-2}\right] + \frac{2}{3}\mathcal{L}^{-1}\left[\frac{1}{s+1}\right] = \boxed{\frac{1}{3}e^{2t} + \frac{2}{3}e^{-t}.}$$

Example 6.11. Solve

$$\begin{cases} y'' + y = \sin 2t \\ y(0) = 2 \\ y'(0) = 1. \end{cases}$$

$$\begin{aligned} y'' + y &= \sin 2t \\ \mathcal{L}\big[y''\big] + \mathcal{L}\big[y\big] &= \mathcal{L}\big[\sin 2t\big] \\ (s^2Y - sy(0) - y'(0)) + Y &= \frac{2}{s^2+4} \\ s^2Y - 2s - 1 + Y &= \frac{2}{s^2+4} \\ (s^2+1)Y &= 2s + 1 + \frac{2}{s^2+4} \end{aligned}$$

$$\begin{aligned} Y &= \frac{2s+1}{s^2+1} + \frac{2}{(s^2+1)(s^2+4)} = \frac{2s+1}{s^2+1} + \frac{As+B}{s^2+1} + \frac{Cs+D}{s^2+4} \\ &= \frac{2s}{s^2+1} + \frac{\frac{5}{3}}{s^2+1} - \frac{\frac{2}{3}}{s^2+4} = 2\mathcal{L}\left[\cos t\right] + \frac{5}{3}\mathcal{L}\left[\sin t\right] - \frac{1}{3}\mathcal{L}\left[\sin 2t\right] \end{aligned}$$

Therefore

$$\boxed{y(t) = 2\cos t + \frac{5}{3}\sin t - \frac{1}{3}\sin 2t.}$$

Example 6.12. Solve

$$\begin{cases} y^{(4)} - y = 0 \\ y(0) = 0 \\ y'(0) = 1 \\ y''(0) = 0 \\ y'''(0) = 0. \end{cases}$$

Using the Laplace Transform we calculate that

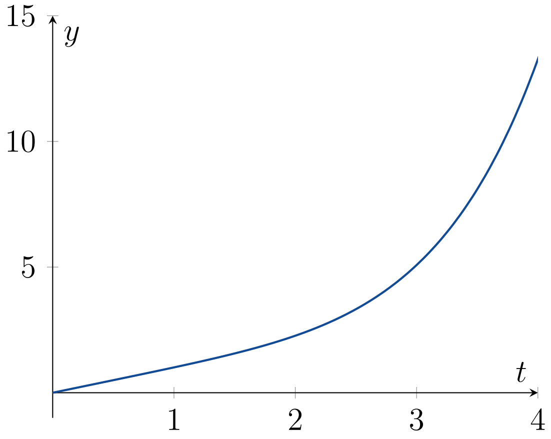
$$\begin{aligned} 0 = \mathcal{L}\left[y^{(4)}\right] - \mathcal{L}\left[y\right] &= \left(s^4Y - s^3y(0) - s^2y'(0) - sy''(0) - y'''(0)\right) - Y \\ &= s^4Y - s^2 - Y = (s^4-1)Y - s^2. \end{aligned}$$

Thus

$$Y(s) = \frac{s^2}{s^4-1} = \frac{s^2}{(s^2-1)(s^2+1)} = \frac{\frac{1}{2}}{s^2-1} + \frac{\frac{1}{2}}{s^2+1}$$

Therefore

$$y = \frac{1}{2}\mathcal{L}^{-1}\left[\frac{1}{s^2-1}\right] + \frac{1}{2}\mathcal{L}^{-1}\left[\frac{1}{s^2+1}\right] = \boxed{\frac{1}{2}\sinh t + \frac{1}{2}\sin t.}$$

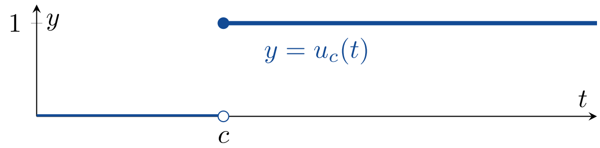

15
10
5
y
t
1
2
3
4

6.3 Step Functions

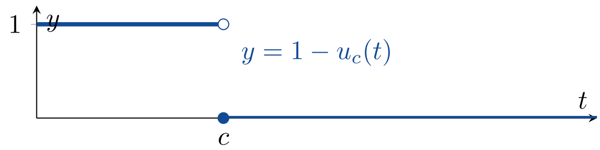
Definition. The ***unit step function*** $u_c : [0, \infty) \to \mathbb{R}$ is defined by

$$u_c(t) = \begin{cases} 0 & t < c \\ 1 & t \geq c \end{cases}$$

for $c \geq 0$.



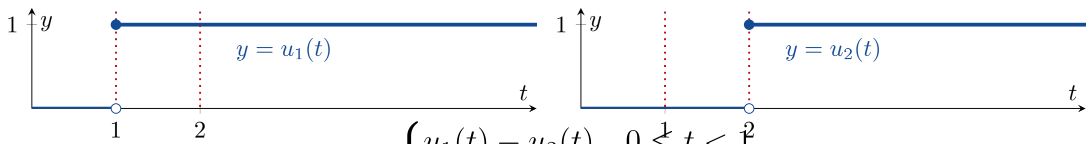
Example 6.13. Draw the graph of $y = 1 - u_c(t)$.



Example 6.14. Draw the graph of $y = u_1(t) - u_2(t)$.

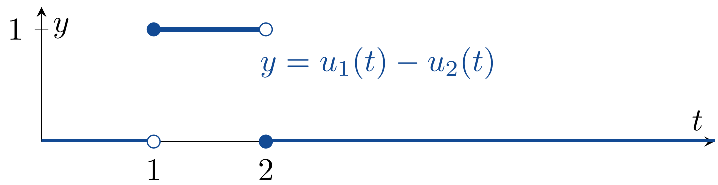
Clearly $t = 1$ and $t = 2$ are important points. So we consider the function on the intervals $[0, 1)$, $[1, 2)$ and $[2, \infty)$.

$$u_1(t) - u_2(t) = \begin{cases} u_1(t) - u_2(t) & 0 \leq t < 1 \\ u_1(t) - u_2(t) & 1 \leq t < 2 \\ u_1(t) - u_2(t) & 2 \leq t \end{cases}$$



$$u_1(t) - u_2(t) = \begin{cases} u_1(t) - u_2(t) & 0 \leq t < 1 \\ u_1(t) - u_2(t) & 1 \leq t < 2 \\ u_1(t) - u_2(t) & 2 \leq t \end{cases}$$

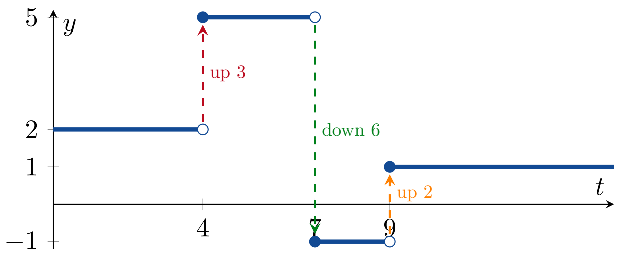
$$= \begin{cases} 0 - 0 & 0 \leq t < 1 \\ 1 - 0 & 1 \leq t < 2 \\ 1 - 1 & 2 \leq t \end{cases} = \begin{cases} 0 & 0 \leq t < 1 \\ 1 & 1 \leq t < 2 \\ 0 & 2 \leq t. \end{cases}$$



Example 6.15. Write the function

$$f(t) = \begin{cases} 2 & 0 \leq t < 4 \\ 5 & 4 \leq t < 7 \\ -1 & 7 \leq t < 9 \\ 1 & 9 \leq t \end{cases}$$

in terms of the unit step function.

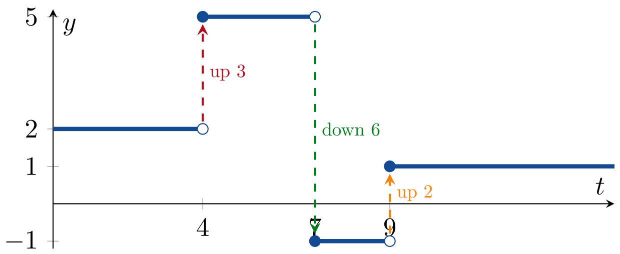


The function starts at $f(0) = 2$. So we will have

$$f(t) = 2 + (\text{something}).$$

At $t = 4$, the function jumps from 2 to 5 (it goes "up 3"). So

$$f(t) = 2 \; +3u_4(t) + (\text{something}).$$



Then it goes "down 6" when $t = 7$. So

$$f(t) = 2 \; +3u_4(t) \; -6u_7(t) + (\text{something}).$$

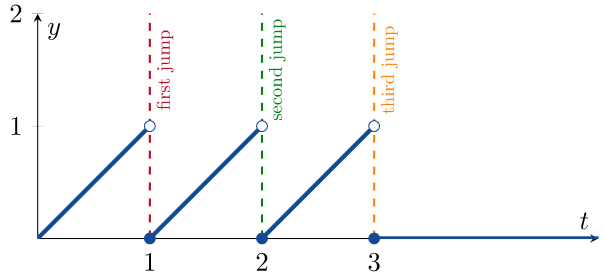
Finally it goes "up 2" when $t = 9$. Therefore

$$f(t) = 2 \; +3u_4(t) \; -6u_7(t) \; +2u_9(t).$$

Example 6.16. Write the function

$$f(t) = \begin{cases} t & 0 \le t < 1 \\ t-1 & 1 \le t < 2 \\ t-2 & 2 \le t < 3 \\ 0 & 3 \le t \end{cases}$$

in terms of the unit step function.



This function starts with $f(t) = t$, then changes when $t = 1$, $t = 2$ and $t = 3$: So we must have

$$f(t) = t + \begin{pmatrix} \text{first} \\ \text{jump} \end{pmatrix} u_1(t) + \begin{pmatrix} \text{second} \\ \text{jump} \end{pmatrix} u_2(t) + \begin{pmatrix} \text{third} \\ \text{jump} \end{pmatrix} u_3(t).$$

At each “jump” we calculate

$$\text{jump} = \begin{pmatrix}\text{function}\\ \text{on right}\end{pmatrix} - \begin{pmatrix}\text{function}\\ \text{on left}\end{pmatrix}.$$

So we have

$$\begin{aligned}\begin{pmatrix}\text{first}\\ \text{jump}\end{pmatrix} &= (t-1) - t = -1\\ \begin{pmatrix}\text{second}\\ \text{jump}\end{pmatrix} &= (t-2) - (t-1) = -1\\ \begin{pmatrix}\text{third}\\ \text{jump}\end{pmatrix} &= 0 - (t-2) = 2-t\end{aligned}$$

Hence

$$f(t) = t - u_1(t) - u_2(t) + (2-t)u_3(t).$$

What is the Laplace Transform of the unit step function?

We calculate that

$$\mathcal{L}\left[u_c\right](s) = \int_0^\infty e^{-st} u_c(t)\,dt = \int_c^\infty e^{-st}\,dt = \left[-\frac{1}{s}e^{-st}\right]_c^\infty = \frac{e^{-cs}}{s}$$

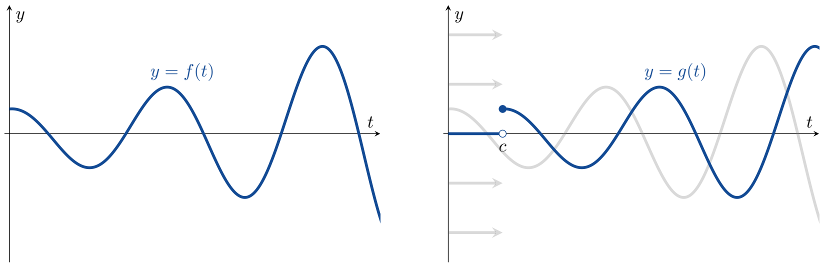
for $s > 0$.

Theorem 6.4.

$$\mathcal{L}\left[u_c\right](s) = \frac{e^{-cs}}{s}$$

Now suppose that we have some function $f : [0, \infty) \to \mathbb{R}$ and we define a new function $g : [0, \infty) \to \mathbb{R}$ by

$$g(t) = \begin{cases} 0 & t < c \\ f(t-c) & t \geq c. \end{cases}$$



We can write $g(t) = u_c(t)f(t-c)$. What is the Laplace Transform of g?

$$\begin{aligned} \mathcal{L}\left[g\right] = \mathcal{L}\left[u_c(t)f(t-c)\right] &= \int_0^\infty e^{-st} u_c(t) f(t-c)\,dt \\ &= \int_c^\infty e^{-st} f(t-c)\,dt. \end{aligned}$$

Let $u = t - c$. Then $du = dt$ and $t = c \iff u = 0$. Therefore

$$\mathcal{L}\left[g\right] = \int_0^\infty e^{-s(u+c)} f(u)\,du = e^{-cs}\int_0^\infty e^{-su} f(u)\,du = e^{-cs}\mathcal{L}\left[f\right].$$

Theorem 6.5.

$$\mathcal{L}\left[u_c(t)f(t-c)\right](s) = e^{-cs}F(s)$$

Example 6.17. Find the Laplace Transform of

$$f(t) = \begin{cases} t & 0 \leq t < 1 \\ t-1 & 1 \leq t < 2 \\ t-2 & 2 \leq t < 3 \\ 0 & 3 \leq t. \end{cases}$$

Since

$$\begin{aligned} f(t) &= t - u_1(t) - u_2(t) + (2-t)u_3(t) \\ &= t - u_1(t) - u_2(t) - u_3(t) - u_3(t)(t-3) \end{aligned}$$

we have that

$$\begin{aligned} F(s) &= \mathcal{L}\left[t\right] - \mathcal{L}\left[u_1\right] - \mathcal{L}\left[u_2\right] - \mathcal{L}\left[u_3\right] - \mathcal{L}\left[u_3(t)(t-3)\right] \\ &= \frac{1}{s^2} - \frac{e^{-s}}{s} - \frac{e^{-2s}}{s} - \frac{e^{-3s}}{s} - \frac{e^{-3s}}{s^2}. \end{aligned}$$

Example 6.18. Find the Laplace Transform of

$$f(t) = \begin{cases} \sin t & 0 \le t \le \frac{\pi}{4} \\ \sin t + \cos(t - \frac{\pi}{4}) & \frac{\pi}{4} \le t. \end{cases}$$

Note that $f(t) = \sin t + g(t)$ where

$$g(t) = \begin{cases} 0 & 0 \le t \le \frac{\pi}{4} \\ \cos(t - \frac{\pi}{4}) & \frac{\pi}{4} \le t \end{cases} = u_{\frac{\pi}{4}}(t)\cos\left(t - \frac{\pi}{4}\right).$$

So

$$\begin{aligned} F(s) &= \mathcal{L}\left[f\right] = \mathcal{L}\left[\sin t\right] + \mathcal{L}\left[u_{\frac{\pi}{4}}(t)\cos\left(t - \frac{\pi}{4}\right)\right] \\ &= \mathcal{L}\left[\sin t\right] + e^{-\frac{\pi s}{4}}\mathcal{L}\left[\cos t\right] = \frac{1}{s^2+1} + e^{-\frac{\pi s}{4}}\frac{s}{s^2+1} \\ &= \frac{1 + se^{-\frac{\pi s}{4}}}{s^2+1}. \end{aligned}$$

Example 6.19. Find the inverse Laplace Transform of $F(s) = \frac{1-e^{-2s}}{s^2}$.

$$\begin{aligned} f(t) &= \mathcal{L}^{-1}\left[F\right] = \mathcal{L}^{-1}\left[\frac{1}{s^2}\right] - \mathcal{L}^{-1}\left[\frac{e^{-2s}}{s^2}\right] = t - u_2(t)(t-2) \\ &= \begin{cases} t & 0 \le t < 2 \\ 2 & t \ge 2. \end{cases} \end{aligned}$$

And what is the Laplace Transform of $e^{ct}f(t)$?

$$\mathcal{L}\left[e^{ct}f(t)\right] = \int_0^\infty e^{-st}e^{ct}f(t)\,dt = \int_0^\infty e^{-(s-c)t}f(t)\,dt = F(s-c).$$

Theorem 6.6.

$$\mathcal{L}\left[e^{ct}f(t)\right] = F(s-c)$$

Example 6.20. Find the inverse Laplace Transform of $G(s) = \frac{1}{s^2-4s+5}$.

Note first that

$$G(s) = \frac{1}{s^2-4s+5} = \frac{1}{(s-2)^2+1}.$$

If $F(s) = \frac{1}{s^2+1}$, then we have $G(s) = F(s-2)$. But $\mathcal{L}^{-1}\left[F\right] = \mathcal{L}^{-1}\left[\frac{1}{s^2+1}\right] = \sin t$. Therefore

$$g(t) = \mathcal{L}^{-1}\left[G\right] = \mathcal{L}^{-1}\left[F(s-2)\right] = e^{2t}\mathcal{L}^{-1}\left[F\right] = e^{2t}\sin t.$$

6.4 ODEs with Discontinuous Forcing Functions

Example 6.21. Solve

$$\begin{cases} y'' + 4y = f(t) = \begin{cases} 0 & 0 \le t < 5 \\ \frac{1}{5}(t-5) & 5 \le t < 10 \\ 1 & 10 \le t \end{cases} \\ y(0) = 0 \\ y'(0) = 0. \end{cases}$$

Note that

$$\begin{aligned} f(t) &= 0 + \left(\frac{1}{5}(t-5) - 0\right) u_5(t) + \left(1 - \frac{1}{5}(t-5)\right) u_{10}(t) \\ &= \frac{1}{5}\left(u_5(t)(t-5) - u_{10}(t)(t-10)\right). \end{aligned}$$

Taking the Laplace transform of the ODE gives

$$(s^2+4)Y = \frac{1}{5}\frac{e^{-5s} - e^{-10s}}{s^2}$$

and

$$Y = \frac{1}{5}\frac{e^{-5s} - e^{-10s}}{s^2(s^2+4)}.$$

Let

$$H(s) = \frac{1}{s^2(s^2+4)}.$$

Then

$$Y(s) = \frac{1}{5}e^{-5s}H(s) - \frac{1}{5}e^{-10s}H(s).$$

Recall that

$$\mathcal{L}\left[u_c(t)h(t-c)\right](s) = e^{-cs}H(s).$$

So

$$u_c(t)h(t-c)(s) = \mathcal{L}^{-1}\left[e^{-cs}H(s)\right].$$

If we can find $h(t)$, then we can find $y(t)$.

Using partial fractions, we calculate (please check!) that

$$\begin{aligned} H(s) &= \frac{1}{s^2(s^2+4)} = \frac{As+B}{s^2} + \frac{Cs+D}{s^2+4} \\ &= \frac{As^3 + Bs^2 + 4As + 4B + Cs^3 + Ds^2}{s^2(s^2+4)} \\ &= \frac{0s + \frac{1}{4}}{s^2} + \frac{0s - \frac{1}{4}}{s^2+4} = \frac{\frac{1}{4}}{s^2} - \frac{\frac{1}{4}}{s^2+4}. \end{aligned}$$

Hence

$$h(t) = \frac{1}{4}\mathcal{L}^{-1}\left[\frac{1}{s^2}\right] - \frac{1}{8}\mathcal{L}^{-1}\left[\frac{2}{s^2+4}\right] = \frac{t}{4} - \frac{1}{8}\sin 2t.$$

Therefore

$$\begin{aligned} y(t) &= \mathcal{L}^{-1}\left[\frac{1}{5}e^{-5s}H(s) - \frac{1}{5}e^{-10s}H(s)\right] \\ &= \frac{1}{5}u_5(t)h(t-5) - \frac{1}{5}u_{10}(t)h(t-10) \\ &= u_5(t)\left(\frac{t-5}{20} - \frac{1}{40}\sin(2t-10)\right) \\ &\quad - u_{10}(t)\left(\frac{t-10}{20} - \frac{1}{40}\sin(2t-20)\right). \end{aligned}$$

Example 6.22. Solve

$$\begin{cases} y'' + 3y' + 2y = f(t) = \begin{cases} 1 & 0 \le t < 10 \\ 0 & 10 \le t \end{cases} \\ y(0) = 1 \\ y'(0) = 0. \end{cases}$$

Since $f(t) = 1 - u_{10}(t)$, the Laplace Transform of the ODE is

$$(s^2 + 3s + 2)Y - (s+3) = \frac{1 - e^{-10s}}{s}.$$

Thus

$$\begin{aligned} Y(s) &= \frac{1 - e^{-10s}}{s(s^2 + 3s + 2)} + \frac{s+3}{s^2 + 3s + 2} \\ &= \frac{(s^2 + 3s + 1) - e^{-10s}}{s(s^2 + 3s + 2)}. \end{aligned}$$

Let

$$G(s) = \frac{s^2 + 3s + 1}{s(s^2 + 3s + 2)} \qquad \text{and} \qquad H(s) = \frac{1}{s(s^2 + 3s + 2)}.$$

Then $Y = G(s) - e^{-10s}H(s)$. If we can find $g(t)$ and $h(t)$, then we can find $y(t)$.

Using partial fractions we get

$$G(s) = \frac{A}{s} + \frac{B}{s+1} + \frac{C}{s+2} = \frac{\frac{1}{2}}{s} + \frac{1}{s+1} - \frac{\frac{1}{2}}{s+2}$$

and

$$H(s) = \frac{D}{s} + \frac{E}{s+1} + \frac{F}{s+2} = \frac{\frac{1}{2}}{s} - \frac{1}{s+1} + \frac{\frac{1}{2}}{s+2}$$

(please check!). It follows that

$$g(t) = \frac{1}{2}\left(1 + 2e^{-t} - e^{-2t}\right) \qquad \text{and} \qquad h(t) = \frac{1}{2}\left(1 - 2e^{-t} + e^{-2t}\right).$$

Therefore

$$\begin{aligned} y(t) &= \mathcal{L}^{-1}\left[Y\right] \\ &= \mathcal{L}^{-1}\left[G(s) - e^{-10s}H(s)\right] \\ &= g(t) - u_{10}(t)h(t-10) \\ &= \frac{1}{2}\left(1 + 2e^{-t} - e^{-2t}\right) - \frac{1}{2}u_{10}(t)\left(1 - 2e^{-(t-10)} + e^{-2(t-10)}\right). \end{aligned}$$

Example 6.23. Solve

$$\begin{cases} y'' + 4y = u_\pi(t) - u_{3\pi}(t) \\ y(0) = 0 \\ y'(0) = 0. \end{cases}$$

Taking the Laplace Transform of the ODE gives

$$(s^2+4)Y(s) = \frac{e^{-\pi s} - e^{-3\pi s}}{s}.$$

Thus

$$Y(s) = \frac{e^{-\pi s} - e^{-3\pi s}}{s(s^2+4)}.$$

Let

$$H(s) = \frac{1}{s(s^2+4)}.$$

Using partial fractions, we calculate that

$$\begin{aligned} H(s) = \frac{1}{s(s^2+4)} &= \frac{A}{s} + \frac{Bs+C}{s^2+4} = \frac{\frac{1}{4}}{s} + \frac{-\frac{1}{4}s+0}{s^2+4} \\ &= \frac{1}{4}\left(\frac{1}{s}\right) - \frac{1}{4}\left(\frac{s}{s^2+4}\right) = \frac{1}{4}\mathcal{L}\left[1\right] - \frac{1}{4}\mathcal{L}\left[\cos 2t\right]. \end{aligned}$$

It follows that

$$h(t) = \frac{1}{4} - \frac{1}{4}\cos 2t$$

and the solution to the IVP is

$$\begin{aligned} y(t) &= \mathcal{L}^{-1}\left[e^{-\pi s}H(s)\right] - \mathcal{L}^{-1}\left[e^{-3\pi s}H(s)\right] \\ &= u_\pi(t)h(t-\pi) - u_{3\pi}(t)h(t-3\pi) \\ &= \frac{1}{4}u_\pi(t)\big(1-\cos(2t-2\pi)\big) - \frac{1}{4}u_{3\pi}(t)\big(1-\cos(2t-6\pi)\big). \end{aligned}$$

6.6 The Convolution Integral

Let $f : [0, \infty) \to \mathbb{R}$ and $g : [0, \infty) \to \mathbb{R}$ be piecewise continuous functions.

Definition. The ***convolution*** of f and g is

$$(f * g)(t) = \int_0^t f(\tau) g(t - \tau)\, d\tau.$$

Theorem 6.7 (Properties)**.**

- $f * g = g * f$
- $f * (g * h) = (f * g) * h$
- $f * (g + h) = (f * g) + (f * h)$
- $f * 0 = 0 = 0 * f$

Example 6.24.

$$\big(\cos * 1\big)(t) = \int_0^t \cos\tau \cdot 1\, d\tau = [\sin\tau]_0^t = \sin t - \sin 0 = \sin t$$
$$\big(1 * \cos\big)(t) = \int_0^t 1 \cdot \cos(t - \tau)\, d\tau = [-\sin(t - \tau)]_0^t = -\sin 0 + \sin t = \sin t$$

Note that $f * 1 \neq f$ in general.

Example 6.25.

$$\begin{aligned}
\big(\sin * \sin\big)(t) &= \int_0^t \sin\tau \sin(t - \tau)\, d\tau = \int_0^t \sin\tau\,(\sin t \cos\tau - \cos t \sin\tau)\; d\tau \\
&= \sin t \int_0^t \sin\tau\cos\tau\, d\tau - \cos t \int_0^t \sin^2\tau\, d\tau \\
&= \sin t \left[-\frac{1}{2}\cos^2\tau\right]_0^t - \cos t \left[\frac{1}{2}(\tau - \sin\tau\cos\tau)\right]_0^t \\
&= \frac{1}{2}\sin t\,\big(1 - \cos^2 t\big) - \frac{1}{2}\cos t\,(t - \sin t\cos t) \\
&= \frac{1}{2}\sin t - \frac{t}{2}\cos t.
\end{aligned}$$

Note that $f * f \geq 0$ is ***not*** true in general.

Theorem 6.8.

$$\mathcal{L}\big[f * g\big](s) = F(s)G(s)$$

This means that $\mathcal{L}^{-1}\big[FG\big] = f * g$.

Example 6.26. Find the inverse Laplace Transform of $H(s) = \dfrac{a}{s^2(s^2 + a^2)}$.

Note that $H(s) = \left(\frac{1}{s^2}\right)\left(\frac{a}{s^2+a^2}\right)$. We know that $\mathcal{L}\left[t\right] = \frac{1}{s^2}$ and $\mathcal{L}\left[\sin at\right] = \frac{a}{s^2+a^2}$. So

$$\begin{aligned}
h(t) &= \mathcal{L}^{-1}\left[\left(\frac{1}{s^2}\right)\left(\frac{a}{s^2+a^2}\right)\right] = \mathcal{L}^{-1}\left[\frac{1}{s^2}\right] * \mathcal{L}^{-1}\left[\frac{a}{s^2+a^2}\right] \\
&= t * \sin at = \int_0^t \tau \sin a(t-\tau)\, d\tau \\
&= \frac{at - \sin at}{a^2}.
\end{aligned}$$

Example 6.27. Solve

$$\begin{cases} y'' + 4y = g(t) \\ y(0) = 3 \\ y'(0) = -1. \end{cases}$$

Taking the Laplace Transform of the ODE gives

$$(s^2 Y - 3s + 1) + 4Y = G(s)$$

which rearranges to

$$\begin{aligned}
Y(s) &= \frac{3s-1}{s^2+4} + \frac{G(s)}{s^2+4} \\
&= 3\left(\frac{s}{s^2+4}\right) - \frac{1}{2}\left(\frac{2}{s^2+4}\right) + \frac{1}{2}\left(\frac{2}{s^2+4}\right) G(s).
\end{aligned}$$

Hence the solution to the IVP is

$$\begin{aligned}
y(t) &= 3\mathcal{L}^{-1}\left[\frac{s}{s^2+4}\right] - \frac{1}{2}\mathcal{L}^{-1}\left[\frac{2}{s^2+4}\right] + \frac{1}{2}\mathcal{L}^{-1}\left[\left(\frac{2}{s^2+4}\right) G(s)\right] \\
&= 3\cos 2t - \frac{1}{2}\sin 2t + \frac{1}{2}\sin 2t * g(t) \\
&= 3\cos 2t - \frac{1}{2}\sin 2t + \frac{1}{2}\int_0^t \sin 2(t-\tau) g(\tau)\, d\tau.
\end{aligned}$$

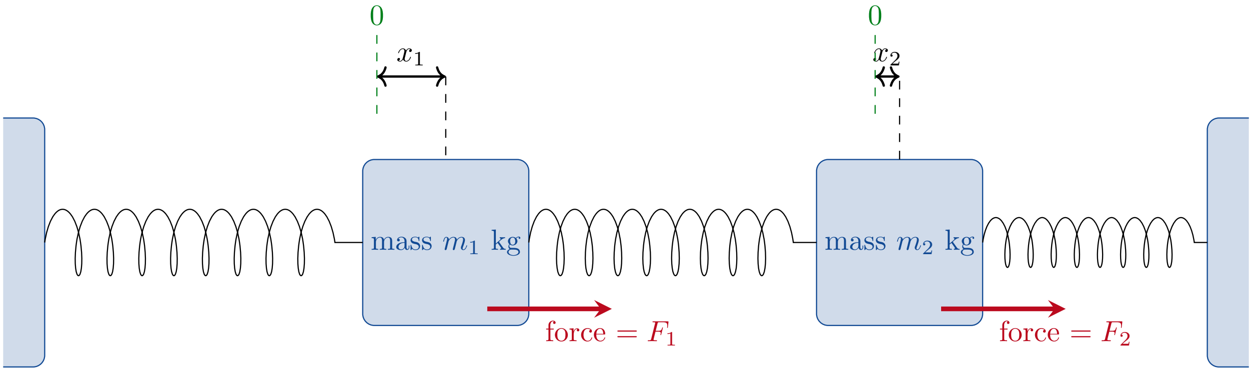
Example 6.28. Find the inverse Laplace Transform of $\dfrac{2}{(s-1)(s^2+4)}$.

$$\begin{aligned}
\mathcal{L}^{-1}\left[\frac{2}{(s-1)(s^2+4)}\right] &= \mathcal{L}^{-1}\left[\left(\frac{2}{s^2+4}\right)\left(\frac{1}{s-1}\right)\right] = \sin 2t * e^t \\
&= \int_0^t e^{t-\tau} \sin 2\tau\, d\tau = e^t \int_0^t e^{-\tau} \sin 2\tau\, d\tau \\
&= e^t \left[\frac{e^{-\tau}}{5}\left(-\sin 2\tau - 2\cos 2\tau\right)\right]_0^t \\
&= \frac{2}{5}e^t - \frac{1}{5}\sin 2t - \frac{2}{5}\cos 2t.
\end{aligned}$$

7

Systems of First Order Linear Equations

7.1 Introduction



Consider the dynamical system shown above. There are two blocks and three springs. Forces F_1 and F_2 act on the blocks as shown.

See https://tinyurl.com/wm2ogdh
We expect that the acceleration of the blocks will depend on

- the displacements x_1 and x_2;
- the forces F_1 and F_2; and
- the masses of the blocks.

So we expect that:

$$\begin{cases} \dfrac{d^2x_1}{dt^2} = f_1(x_1, x_2, F_1, m_1) \\ \dfrac{d^2x_2}{dt^2} = f_2(x_1, x_2, F_2, m_2). \end{cases}$$

This is a system of two ODEs. To find $x_1(t)$ and $x_2(t)$, we would need to solve these equations at the same time.

The most famous system of ODEs is the system of ***Predator-Prey*** equations:

$$\begin{cases} \dfrac{dx}{dt} = \alpha x - \beta xy \\ \dfrac{dy}{dt} = \delta xy - \gamma y \end{cases}$$

where

$$\begin{aligned} x(t) &= \text{number of prey (e.g. mice)} \\ y(t) &= \text{number of predators (e.g. owls),} \end{aligned}$$

which originate circa 1925.

It is possible to convert an nth order linear ODE into a system of n first order linear ODEs. Or vice versa.

$$\begin{gathered} a_n y^{(n)} + a_{n-1} y^{(n-1)} + \dots \\ + a_1 y' + a_0 y = g(t) \end{gathered} \quad \longleftrightarrow \quad \begin{cases} x_1' = b_{11}x_1 + \dots + b_{1n}x_n + h_1(t) \\ x_2' = b_{21}x_1 + \dots + b_{2n}x_n + h_2(t) \\ \vdots \\ x_n' = b_{n1}x_1 + \dots + b_{nn}x_n + h_n(t) \end{cases}$$

Example 7.1. Write

$$u'' + 0.25u' + u = 0$$

as a system of two first order ODEs.

Let $x_1 = u$ and $x_2 = u'$. Then clearly $x_1' = u' = x_2$ and

$$x_2' = u'' = -0.25u' - u = -0.25x_2 - x_1.$$

Therefore

$$\begin{cases} x_1' = x_2 \\ x_2' = -x_1 - 0.25x_2. \end{cases}$$

Remark. We will need

- matrices,
- eigenvalues,
- eigenvectors,
- the Wronskian,
- linear independence,
- and more

from MATH215 – please either revise your linear algebra lecture notes or read §7.2-7.3 in the textbook.

7.4 Basic Theory of Systems of First Order Linear Equations

$$\begin{cases} x_1' = p_{11}(t)x_1 + p_{12}(t)x_2 + \ldots + p_{1n}(t)x_n + g_1(t) \\ x_2' = p_{21}(t)x_1 + p_{22}(t)x_2 + \ldots + p_{2n}(t)x_n + g_2(t) \\ \quad \vdots \\ x_n' = p_{n1}(t)x_1 + p_{n2}(t)x_2 + \ldots + p_{nn}(t)x_n + g_n(t) \end{cases}$$

is a system of n linear ODEs and n variables: x_1, x_2, $\ldots$, x_n.

If we write

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, \ \mathbf{x}' = \begin{bmatrix} x_1' \\ x_2' \\ \vdots \\ x_n' \end{bmatrix}, \ P = \begin{bmatrix} p_{11} & p_{12} & \ldots & p_{1n} \\ p_{21} & p_{22} & \ldots & p_{2n} \\ \vdots & \vdots & & \vdots \\ p_{n1} & p_{n2} & \ldots & p_{nn} \end{bmatrix}, \text{ and } \mathbf{g} = \begin{bmatrix} g_1 \\ g_2 \\ \vdots \\ g_n \end{bmatrix}$$

then we can write this system as

$$\mathbf{x}' = P(t)\mathbf{x} + \mathbf{g}(t).$$

First we will consider the homogeneous system

$$\mathbf{x}' = P(t)\mathbf{x}.$$

In Chapters 3 and 4 when we had multiple solutions, we wrote them as $y_1(t)$, $y_2(t)$, $\ldots$. But we are already using x_1, x_2, $\ldots$ to denote coordinates. So we need a new type of notation.

Notation. We use $\mathbf{x}^{(1)}(t)$, $\mathbf{x}^{(2)}(t)$, $\ldots$ to denote different vector solutions.

Recall from Chapter 3 that if $y_1(t)$ and $y_2(t)$ are both solutions to

$$ay'' + by' + cy = 0,$$

then

$$c_1y_1 + c_2y_2$$

is also a solution.

Theorem 7.1. *If $\mathbf{x}^{(1)}(t)$ and $\mathbf{x}^{(2)}(t)$ are solutions to $\mathbf{x}' = P(t)\mathbf{x}$, then*

$$c_1\mathbf{x}^{(1)} + c_2\mathbf{x}^{(2)}$$

is also a solution for any $c_1, c_2 \in \mathbb{R}$.

Example 7.2. $\mathbf{x}^{(1)} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} e^{3t}$ and $\mathbf{x}^{(2)} = \begin{bmatrix} 1 \\ -2 \end{bmatrix} e^{-t}$ are both solutions to $\mathbf{x}' = \begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix} \mathbf{x}$ (we will see this later). Therefore

$$\mathbf{x}(t) = c_1\mathbf{x}^{(1)} + c_2\mathbf{x}^{(2)} = c_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} e^{3t} + c_2 \begin{bmatrix} 1 \\ -2 \end{bmatrix} e^{-t}$$

is also a solution to this system.

(Suppose that $P(t)$ is an $n \times n$ matrix.)

Theorem 7.2. *If* $\mathbf{x}^{(1)}(t)$, $\mathbf{x}^{(2)}(t)$, $\dots$, $\mathbf{x}^{(n)}(t)$ *are linearly independent solutions to* $\mathbf{x}' = P(t)\mathbf{x}$, *then every solution to this system can be written as*

$$\mathbf{x}(t) = c_1\mathbf{x}^{(1)} + c_2\mathbf{x}^{(2)} + \dots + c_n\mathbf{x}^{(n)}$$

in exactly one way.

Definition. In this case, we say that $\mathbf{x}^{(1)}(t)$, $\mathbf{x}^{(2)}(t)$, $\dots$, $\mathbf{x}^{(n)}(t)$ form a ***fundamental set of solutions*** to $\mathbf{x}' = P(t)\mathbf{x}$.

Definition. In this case,

$$\mathbf{x}(t) = c_1\mathbf{x}^{(1)} + c_2\mathbf{x}^{(2)} + \dots + c_n\mathbf{x}^{(n)}$$

is called the ***general solution*** to $\mathbf{x}' = P(t)\mathbf{x}$.

7.5 Homogeneous Linear Systems with Constant Coefficients

Consider

$$\mathbf{x}' = A\mathbf{x}$$

where $A \in \mathbb{R}^{n\times n}$.

If $n = 1$, then we just have

$$\frac{dx}{dt} = ax$$

which has general solution $x(t) = ce^{at}$.

For $n > 1$, we guess that

$$\mathbf{x}(t) = \boldsymbol{\xi} e^{rt}$$

is a solution to $\mathbf{x}' = A\mathbf{x}$, for some number $r \in \mathbb{C}$ and some vector $\boldsymbol{\xi} \in \mathbb{C}^n$.

But if $\mathbf{x}(t) = \boldsymbol{\xi} e^{rt}$, then

$$\begin{aligned} r\boldsymbol{\xi} e^{rt} &= \mathbf{x}' = A\mathbf{x} = A\boldsymbol{\xi} e^{rt} \\ r\boldsymbol{\xi} &= A\boldsymbol{\xi} \\ (A - rI)\boldsymbol{\xi} &= \mathbf{0} \end{aligned}$$

where I is the identity matrix. Hence r must be an eigenvalue of A and $\boldsymbol{\xi}$ must be a corresponding eigenvector of A.

Remark. So the idea is:

(i). Find the eigenvalues;

(ii). Find the eigenvectors; then

(iii). Write $\mathbf{x}^{(j)}(t) = \boldsymbol{\xi}^{(j)} e^{r_j t}$.

Example 7.3. Solve

$$\mathbf{x}' = \begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix} \mathbf{x}.$$

First we find the eigenvalues. Since

$$\begin{aligned} 0 = \det(A - rI) &= \begin{vmatrix} 1-r & 1 \\ 4 & 1-r \end{vmatrix} = (1-r)^2 - 4 \\ &= r^2 - 2r - 3 = (r+1)(r-3), \end{aligned}$$

the eigenvalues are $r_1 = 3$ and $r_2 = -1$.

Using the first eigenvalue $r_1 = 3$, we calculate that

$$\mathbf{0} = (A - r_1 I)\boldsymbol{\xi} = \begin{bmatrix} -2 & 1 \\ 4 & -2 \end{bmatrix} \begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix} \qquad \Longrightarrow \qquad 0 = -2\xi_1 + \xi_2.$$

Hence we can choose $\boldsymbol{\xi}^{(1)} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$. Then using the second eigenvalue $r_2 = -1$, we calculate that

$$\mathbf{0} = (A - r_2 I)\boldsymbol{\xi} = \begin{bmatrix} 2 & 1 \\ 4 & 2 \end{bmatrix} \begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix} \qquad \Longrightarrow \qquad 0 = 2\xi_1 + \xi_2.$$

Hence we can choose $\boldsymbol{\xi}^{(2)} = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$. This gives us two solutions:

$$\mathbf{x}^{(1)}(t) = \begin{bmatrix} 1 \\ 2 \end{bmatrix} e^{3t} \qquad \text{and} \qquad \mathbf{x}^{(2)}(t) = \begin{bmatrix} 1 \\ -2 \end{bmatrix} e^{-t}.$$

But are these two solutions linearly independent? To find out, we calculate the Wronskian of $\mathbf{x}^{(1)}(t)$ and $\mathbf{x}^{(2)}(t)$:

$$\mathbf{x}^{(1)}(t) = \begin{bmatrix} 1 \\ 2 \end{bmatrix} e^{3t} = \begin{bmatrix} e^{3t} \\ 2e^{3t} \end{bmatrix} \quad \text{and} \quad \mathbf{x}^{(2)}(t) = \begin{bmatrix} 1 \\ -2 \end{bmatrix} e^{-t} = \begin{bmatrix} e^{-t} \\ -2e^{-t} \end{bmatrix}.$$

$$W\left(\mathbf{x}^{(1)}, \mathbf{x}^{(2)}\right)(t) = \begin{vmatrix} e^{3t} & e^{-t} \\ 2e^{3t} & -2e^{-t} \end{vmatrix} = -4e^{2t} \neq 0.$$

Since $W \neq 0$, we have that $\mathbf{x}^{(1)}(t)$ and $\mathbf{x}^{(2)}(t)$ are linearly independent. So $\mathbf{x}^{(1)}(t)$ and $\mathbf{x}^{(2)}(t)$ form a fundamental set of solutions. Therefore the general solution is

$$\boxed{\mathbf{x}(t) = c_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} e^{3t} + c_2 \begin{bmatrix} 1 \\ -2 \end{bmatrix} e^{-t}.}$$

Example 7.4. Solve

$$\begin{cases} \mathbf{x}' = \begin{bmatrix} 8 & -1 \\ 6 & 1 \end{bmatrix} \mathbf{x} \\ \mathbf{x}(0) = \begin{bmatrix} 1 \\ -2 \end{bmatrix}. \end{cases}$$

The eigenvalues are $r_1 = 7$ and $r_2 = 2$. The corresponding eigenvectors are $\boldsymbol{\xi}^{(1)} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\boldsymbol{\xi}^{(2)} = \begin{bmatrix} 1 \\ 6 \end{bmatrix}$. Therefore the general solution to the ODE is

$$\mathbf{x}(t) = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{7t} + c_2 \begin{bmatrix} 1 \\ 6 \end{bmatrix} e^{2t}.$$

Setting $t = 0$, we have

$$\begin{bmatrix} 1 \\ -2 \end{bmatrix} = \mathbf{x}(0) = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 6 \end{bmatrix} = \begin{bmatrix} c_1 + c_2 \\ c_1 + 6c_2 \end{bmatrix} \implies \begin{cases} c_1 + c_2 = 1 \\ c_1 + 6c_2 = -2 \end{cases} \implies \begin{cases} c_1 = \frac{8}{5} \\ c_2 = -\frac{3}{5}. \end{cases}$$

Therefore the solution to the IVP is

$$\boxed{\mathbf{x}(t) = \frac{8}{5} \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{7t} - \frac{3}{5} \begin{bmatrix} 1 \\ 6 \end{bmatrix} e^{2t}.}$$

Example 7.5. Solve

$$\mathbf{x}' = \begin{bmatrix} -3 & \sqrt{2} \\ \sqrt{2} & -2 \end{bmatrix} \mathbf{x}.$$

The eigenvalues are $r_1 = -1$ and $r_2 = -4$. The corresponding eigenvectors are $\boldsymbol{\xi}^{(1)} = \begin{bmatrix} 1 \\ \sqrt{2} \end{bmatrix}$ and $\boldsymbol{\xi}^{(2)} = \begin{bmatrix} -\sqrt{2} \\ 1 \end{bmatrix}$. Hence the general solution is

$$\boxed{\mathbf{x}(t) = c_1 \begin{bmatrix} 1 \\ \sqrt{2} \end{bmatrix} e^{-t} + c_2 \begin{bmatrix} -\sqrt{2} \\ 1 \end{bmatrix} e^{-4t}.}$$

Remark.

$$\det(A - rI) = 0$$

There are three possibilities for the eigenvalues of A.

(i). All the eigenvalues are real and different;

(ii). Some eigenvalues occur in complex conjugate pairs;

(iii). Some eigenvalues are repeated.

If all the eigenvalues are real and different, then the eigenvectors are linearly independent: So $W\big(\mathbf{x}^{(1)}, \dots, \mathbf{x}^{(n)}\big)(t) \neq 0$ and $\mathbf{x}^{(1)}(t)$, $\dots$, $\mathbf{x}^{(n)}(t)$ form a fundamental set of solutions.

If some eigenvalues are repeated, ***but there are n linearly independent eigenvectors***, then this is also true: $\mathbf{x}^{(1)}(t)$, $\dots$, $\mathbf{x}^{(n)}(t)$ form a fundamental set of solutions.

Example 7.6. Solve

$$\mathbf{x}' = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} \mathbf{x}.$$

The eigenvalues and eigenvectors are

$$r_1 = 2 \qquad\qquad r_2 = -1 \qquad\qquad r_3 = -1$$

$$\boldsymbol{\xi}^{(1)} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \qquad \boldsymbol{\xi}^{(2)} = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \qquad \boldsymbol{\xi}^{(3)} = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$$

which gives us the following three solutions

$$\mathbf{x}^{(1)}(t) = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} e^{2t} \qquad \mathbf{x}^{(2)}(t) = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} e^{-t} \qquad \mathbf{x}^{(3)}(t) = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} e^{-t}.$$

You can check that the Wronskian of $\mathbf{x}^{(1)}$, $\mathbf{x}^{(2)}$ and $\mathbf{x}^{(3)}$ is non-zero. Therefore $\mathbf{x}^{(1)}$, $\mathbf{x}^{(2)}$ and $\mathbf{x}^{(3)}$ form a fundamental set of solutions. The general solution to the ODE is

$$\boxed{\mathbf{x}(t) = c_1 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} e^{2t} + c_2 \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} e^{-t} + c_3 \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} e^{-t}.}$$

Remark. Next we will study systems with complex eigenvalues.

7.6 Complex Eigenvalues

Consider
$$\mathbf{x}' = A\mathbf{x}$$
where $A \in \mathbb{R}^{n\times n}$.

Any complex eigenvalues of A must occur in complex conjugate pairs: If $r_1 = \lambda + i\mu$ is an eigenvalue of A, then $r_2 = \bar{r}_1 = \lambda - i\mu$ is also an eigenvalue of A.

Moreover, if $\boldsymbol{\xi}^{(1)}$ is an eigenvector of A corresponding to r_1, then $\boldsymbol{\xi}^{(2)} = \overline{\boldsymbol{\xi}^{(1)}}$ is an eigenvector of A corresponding to $r_2 = \bar{r}_1$.

Two solutions of $\mathbf{x}' = A\mathbf{x}$ are
$$\mathbf{x}^{(1)}(t) = \boldsymbol{\xi}^{(1)} e^{r_1 t} \qquad \text{and} \qquad \mathbf{x}^{(2)}(t) = \overline{\boldsymbol{\xi}^{(1)}} e^{\bar{r}_1 t}.$$

But $\mathbf{x}^{(1)}, \mathbf{x}^{(2)} : \mathbb{R} \to \mathbb{C}^n$ and we want solutions $: \mathbb{R} \to \mathbb{R}^n$.

If $r_1 = \lambda + i\mu$, and $\boldsymbol{\xi}^{(1)} = \mathbf{a} + i\mathbf{b}$ ($\lambda,\, \mu \in \mathbb{R}$, $\mathbf{a},\, \mathbf{b} \in \mathbb{R}^n$), then
$$\begin{aligned}
\mathbf{x}^{(1)}(t) &= (\mathbf{a} + i\mathbf{b}) e^{(\lambda + i\mu)t} \\
&= (\mathbf{a} + i\mathbf{b}) e^{\lambda t} (\cos \mu t + i \sin \mu t) \\
&= e^{\lambda t} (\mathbf{a} \cos \mu t - \mathbf{b} \sin \mu t) + i e^{\lambda t} (\mathbf{a} \sin \mu t + \mathbf{b} \cos \mu t) \\
&= \mathbf{u}(t) + i\mathbf{v}(t).
\end{aligned}$$

The functions $\mathbf{u}(t)$ and $\mathbf{v}(t)$ will be linearly independent. Furthermore
$$\operatorname{span}\{\mathbf{u}(t), \mathbf{v}(t)\} = \operatorname{span}\{\mathbf{x}^{(1)}, \mathbf{x}^{(2)}\}.$$

So we can include $\mathbf{u}(t)$ and $\mathbf{v}(t)$ in our fundamental set of solutions instead of $\mathbf{x}^{(1)}(t)$ and $\mathbf{x}^{(2)}(t)$.

Example 7.7. Solve
$$\mathbf{x}' = \begin{bmatrix} -\frac{1}{2} & 1 \\ -1 & -\frac{1}{2} \end{bmatrix} \mathbf{x}.$$

We calculate that
$$0 = \det(A - rI) = \begin{vmatrix} -\frac{1}{2} - r & 1 \\ -1 & -\frac{1}{2} - r \end{vmatrix} = r^2 + r + \frac{5}{4}$$
and
$$r_{1,2} = \frac{-1 \pm \sqrt{1-5}}{2} = \frac{-1 \pm 2i}{2} = -\frac{1}{2} \pm i.$$
So we have $r_1 = -\frac{1}{2} + i$ and $r_2 = -\frac{1}{2} - i$. We will use r_1. We do not need r_2.

Since
$$0 = (A - r_1 I)\boldsymbol{\xi}^{(1)} = \begin{bmatrix} -i & 1 \\ -1 & -i \end{bmatrix} \begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix} \quad \Longrightarrow \quad \begin{cases} -i\xi_1 + \xi_2 = 0 \\ -\xi_1 - i\xi_2 = 0 \end{cases}$$
we can choose
$$\boldsymbol{\xi}^{(1)} = \begin{bmatrix} 1 \\ i \end{bmatrix}.$$

Note that we also have
$$\boldsymbol{\xi}^{(2)} = \overline{\boldsymbol{\xi}^{(1)}} = \overline{\begin{bmatrix} 1 \\ i \end{bmatrix}} = \begin{bmatrix} 1 \\ -i \end{bmatrix},$$
but we don't need $\boldsymbol{\xi}^{(2)}$.

Next we look at $\mathbf{x}^{(1)}(t)$:

$$\begin{aligned}\mathbf{x}^{(1)}(t) = \boldsymbol{\xi}^{(1)} e^{r_1 t} &= \begin{bmatrix}1\\ i\end{bmatrix} e^{-\frac{t}{2}} (\cos t + i \sin t)\\ &= e^{-\frac{t}{2}} \begin{bmatrix}\cos t + i\sin t\\ i\cos t - \sin t\end{bmatrix}\\ &= e^{-\frac{t}{2}} \begin{bmatrix}\cos t\\ -\sin t\end{bmatrix} + i e^{-\frac{t}{2}} \begin{bmatrix}\sin t\\ \cos t\end{bmatrix}\\ &= \mathbf{u}(t) + i\mathbf{v}(t).\end{aligned}$$

Hence we choose

$$\mathbf{u}(t) = e^{-\frac{t}{2}} \begin{bmatrix}\cos t\\ -\sin t\end{bmatrix} \quad \text{and} \quad \mathbf{v}(t) = e^{-\frac{t}{2}} \begin{bmatrix}\sin t\\ \cos t\end{bmatrix}.$$

But are $\mathbf{u}(t)$ and $\mathbf{v}(t)$ linearly independent? Since

$$\begin{aligned}W\big(\mathbf{u}(t), \mathbf{v}(t)\big)(t) = \begin{vmatrix}u_1 & v_1\\ u_2 & v_2\end{vmatrix} &= \begin{vmatrix}e^{-\frac{t}{2}}\cos t & e^{-\frac{t}{2}}\sin t\\ -e^{-\frac{t}{2}}\sin t & e^{-\frac{t}{2}}\cos t\end{vmatrix}\\ &= e^{-t}\cos^2 t + e^{-t}\sin^2 t = e^{-t}\\ &\neq 0\end{aligned}$$

the answer is yes. Therefore $\mathbf{u}(t)$ and $\mathbf{v}(t)$ form a fundamental set of solutions.

Therefore the general solution to $\mathbf{x}' = \begin{bmatrix}-\frac{1}{2} & 1\\ -1 & -\frac{1}{2}\end{bmatrix}\mathbf{x}$ is

$$\boxed{\mathbf{x}(t) = c_1 e^{-\frac{t}{2}} \begin{bmatrix}\cos t\\ -\sin t\end{bmatrix} + c_2 e^{-\frac{t}{2}} \begin{bmatrix}\sin t\\ \cos t\end{bmatrix}.}$$

Remark. Our method is

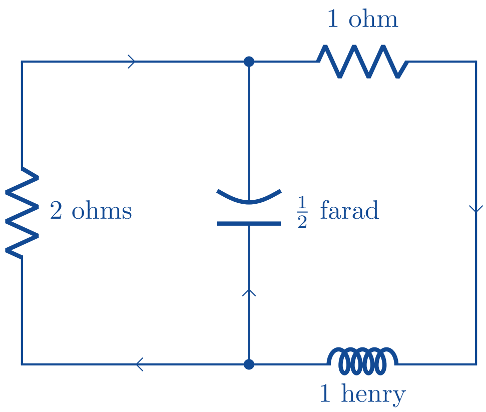
1. Find the eigenvalues;

2. Find the eigenvectors;

3.
   - If r_j is real, just use the solution $\mathbf{x}^{(j)}(t) = \boldsymbol{\xi}^{(j)} e^{r_j t}$;
   - But if r_j is complex, write

$$\mathbf{x}^{(j)}(t) = \boldsymbol{\xi}^{(j)} e^{r_j t} = \begin{pmatrix}\text{real valued}\\ \text{function}\end{pmatrix} + i\begin{pmatrix}\text{real valued}\\ \text{function}\end{pmatrix}$$

   and use these two functions.



Example 7.8. The electric circuit shown above is described by

$$\begin{cases} I' = -I - V \\ V' = 2I - V \end{cases}$$

where

$$\begin{aligned} I &= \text{the current through the inductor} \\ V &= \text{the voltage drop across the capacitor.} \end{aligned}$$

(Ask an Electrical Engineer.)

Suppose that at time $t = 0$ the current is 2 amperes and the voltage drop is 2 volts. Find $I(t)$ and $V(t)$.

We must solve the IVP

$$\begin{cases} \dfrac{d}{dt}\begin{bmatrix} I \\ V \end{bmatrix} = \begin{bmatrix} -1 & -1 \\ 2 & -1 \end{bmatrix}\begin{bmatrix} I \\ V \end{bmatrix} \\ \begin{bmatrix} I \\ V \end{bmatrix}(0) = \begin{bmatrix} 2 \\ 2 \end{bmatrix}. \end{cases}$$

The eigenvalues of $\begin{bmatrix} -1 & -1 \\ 2 & -1 \end{bmatrix}$ are $r_1 = -1 + i\sqrt{2}$ and $r_2 = -1 - i\sqrt{2}$ (please check). The corresponding eigenvectors are

$$\boldsymbol{\xi}^{(1)} = \begin{bmatrix} 1 \\ -i\sqrt{2} \end{bmatrix} \qquad \text{and} \qquad \boldsymbol{\xi}^{(2)} = \begin{bmatrix} 1 \\ i\sqrt{2} \end{bmatrix}.$$

Then we calculate that

$$\begin{aligned} \mathbf{x}^{(1)}(t) = \boldsymbol{\xi}^{(1)} e^{r_1 t} &= \begin{bmatrix} 1 \\ -i\sqrt{2} \end{bmatrix} e^{(-1+i\sqrt{2})t} \\ &= \begin{bmatrix} 1 \\ -i\sqrt{2} \end{bmatrix} e^{-t}\big(\cos\sqrt{2}t + i\sin\sqrt{2}t\big) \\ &= e^{-t}\begin{bmatrix} \cos\sqrt{2}t + i\sin\sqrt{2}t \\ -i\sqrt{2}\cos\sqrt{2}t + \sqrt{2}\sin\sqrt{2}t \end{bmatrix} \\ &= e^{-t}\begin{bmatrix} \cos\sqrt{2}t \\ \sqrt{2}\sin\sqrt{2}t \end{bmatrix} + ie^{-t}\begin{bmatrix} \sin\sqrt{2}t \\ -\sqrt{2}\cos\sqrt{2}t \end{bmatrix}. \end{aligned}$$

Hence the general solution to the ODE is

$$\begin{bmatrix} I(t) \\ V(t) \end{bmatrix} = c_1 e^{-t} \begin{bmatrix} \cos\sqrt{2}t \\ \sqrt{2}\sin\sqrt{2}t \end{bmatrix} + c_2 e^{-t} \begin{bmatrix} \sin\sqrt{2}t \\ -\sqrt{2}\cos\sqrt{2}t \end{bmatrix}.$$

Using the initial condition, we calculate that

$$\begin{bmatrix} 2 \\ 2 \end{bmatrix} = \begin{bmatrix} I(0) \\ V(0) \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ -\sqrt{2} \end{bmatrix} \qquad \Longrightarrow \qquad \begin{cases} c_1 = 2 \\ c_2 = -\sqrt{2}. \end{cases}$$

Thus

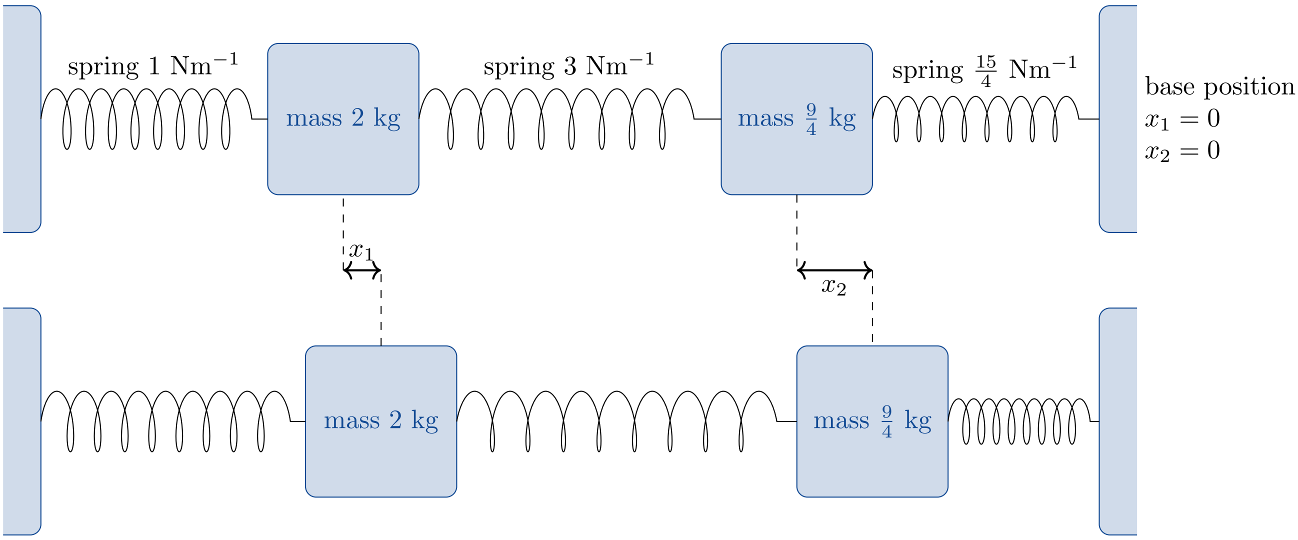
$$\begin{bmatrix} I(t) \\ V(t) \end{bmatrix} = 2\, e^{-t} \begin{bmatrix} \cos\sqrt{2}t \\ \sqrt{2}\sin\sqrt{2}t \end{bmatrix} - \sqrt{2}\, e^{-t} \begin{bmatrix} \sin\sqrt{2}t \\ -\sqrt{2}\cos\sqrt{2}t \end{bmatrix}.$$

So the answers to this problem are

$$\boxed{I(t) = 2e^{-t}\cos\sqrt{2}t - \sqrt{2}e^{-t}\sin\sqrt{2}t}$$

and

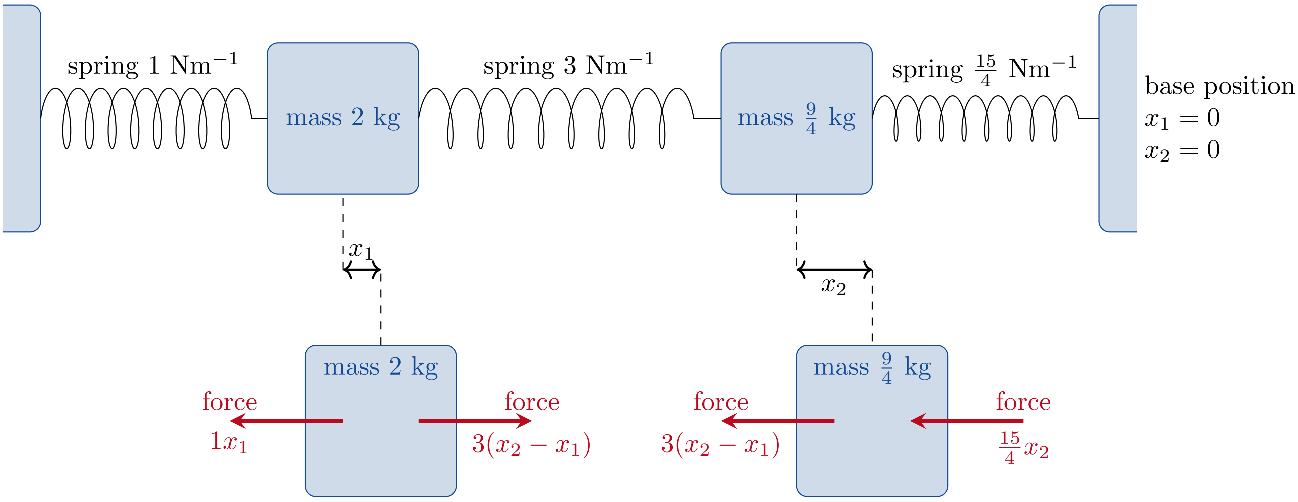
$$\boxed{V(t) = 2\sqrt{2}e^{-t}\sin\sqrt{2}t + 2e^{-t}\cos\sqrt{2}t.}$$



See https://tinyurl.com/wm2ogdh for an animated figure.

Example 7.9. For the dynamical system shown above, find $x_1(t)$ and $x_2(t)$.

As the springs are stretched and compressed, they apply forces on the blocks as shown below (Hooke's Law).



We calculate that

$$\begin{aligned} 2\frac{d^2x_1}{dt^2} &= \text{mass} \times \text{acceleration} = \text{force} = -x_1 + 3(x_2 - x_1) \\ \frac{9}{4}\frac{d^2x_2}{dt^2} &= \text{mass} \times \text{acceleration} = \text{force} = -3(x_2 - x_1) - \frac{15}{4}x_2. \end{aligned}$$

This is a system of 2 second order ODEs. We want a system of first order ODEs.
Now let $y_1 = x_1$, $y_2 = x_2$, $y_3 = x_1'$ and $y_4 = x_2'$. Then

$$\begin{aligned} y_1' &= x_1' = y_3 \\ y_2' &= x_2' = y_4 \\ y_3' &= x_1'' = \frac{1}{2}\Big(-x_1 + 3x_2 - 3x_1 \Big) = -2y_1 + \frac{3}{2}y_2 \\ y_4' &= x_2'' = \frac{4}{9}\left(-3x_2 + 3x_1 - \frac{15}{4}x_2 \right) = \frac{4}{3}y_1 - 3y_2. \end{aligned}$$

So

$$\mathbf{y}' = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -2 & \frac{3}{2} & 0 & 0 \\ \frac{4}{3} & -3 & 0 & 0 \end{bmatrix} \mathbf{y}.$$

The characteristic polynomial of this matrix is

$$0 = r^4 + 5r^2 + 4 = (r^2+1)(r^2+4).$$

So $r_1 = i$, $r_2 = -i$, $r_3 = 2i$ and $r_4 = -2i$. We will use r_1 and r_3 (we do not need r_2 and r_4).

The corresponding eigenvectors (please check) are

$$\boldsymbol{\xi}^{(1)} = \begin{bmatrix} 3 \\ 2 \\ 3i \\ 2i \end{bmatrix} \qquad \text{and} \qquad \boldsymbol{\xi}^{(3)} = \begin{bmatrix} 3 \\ -4 \\ 6i \\ -8i \end{bmatrix}.$$

It follows that

$$\boldsymbol{\xi}^{(1)} e^{r_1 t} = \begin{bmatrix} 3 \\ 2 \\ 3i \\ 2i \end{bmatrix} (\cos t + i \sin t) = \begin{bmatrix} 3\cos t \\ 2\cos t \\ -3\sin t \\ -2\sin t \end{bmatrix} + i \begin{bmatrix} 3\sin t \\ 2\sin t \\ 3\cos t \\ 2\cos t \end{bmatrix} = \mathbf{u}(t) + i\mathbf{v}(t)$$

and

$$\boldsymbol{\xi}^{(3)} e^{r_3 t} = \begin{bmatrix} 3 \\ -4 \\ 6i \\ -8i \end{bmatrix} (\cos 2t + i \sin 2t) = \begin{bmatrix} 3\cos 2t \\ -4\cos 2t \\ -6\sin 2t \\ 8\sin 2t \end{bmatrix} + i \begin{bmatrix} 3\sin 2t \\ -4\sin 2t \\ 6\cos 2t \\ -8\cos 2t \end{bmatrix} = \mathbf{w}(t) + i\mathbf{z}(t)$$

Therefore the general solution is

$$\begin{aligned} \mathbf{y}(t) &= c_1\mathbf{u}(t) + c_2\mathbf{v}(t) + c_3\mathbf{w}(t) + c_4\mathbf{z}(t) \\ &= c_1 \begin{bmatrix} 3\cos t \\ 2\cos t \\ -3\sin t \\ -2\sin t \end{bmatrix} + c_2 \begin{bmatrix} 3\sin t \\ 2\sin t \\ 3\cos t \\ 2\cos t \end{bmatrix} + c_3 \begin{bmatrix} 3\cos 2t \\ -4\cos 2t \\ -6\sin 2t \\ 8\sin 2t \end{bmatrix} + c_4 \begin{bmatrix} 3\sin 2t \\ -4\sin 2t \\ 6\cos 2t \\ -8\cos 2t \end{bmatrix}. \end{aligned}$$

Example 7.10. Suppose that the above system has initial condition

$$\mathbf{y}(0) = \begin{bmatrix} -1 \\ 4 \\ 1 \\ 1 \end{bmatrix}.$$

Sketch graphs of $y_1(t)$ and $y_2(t)$.

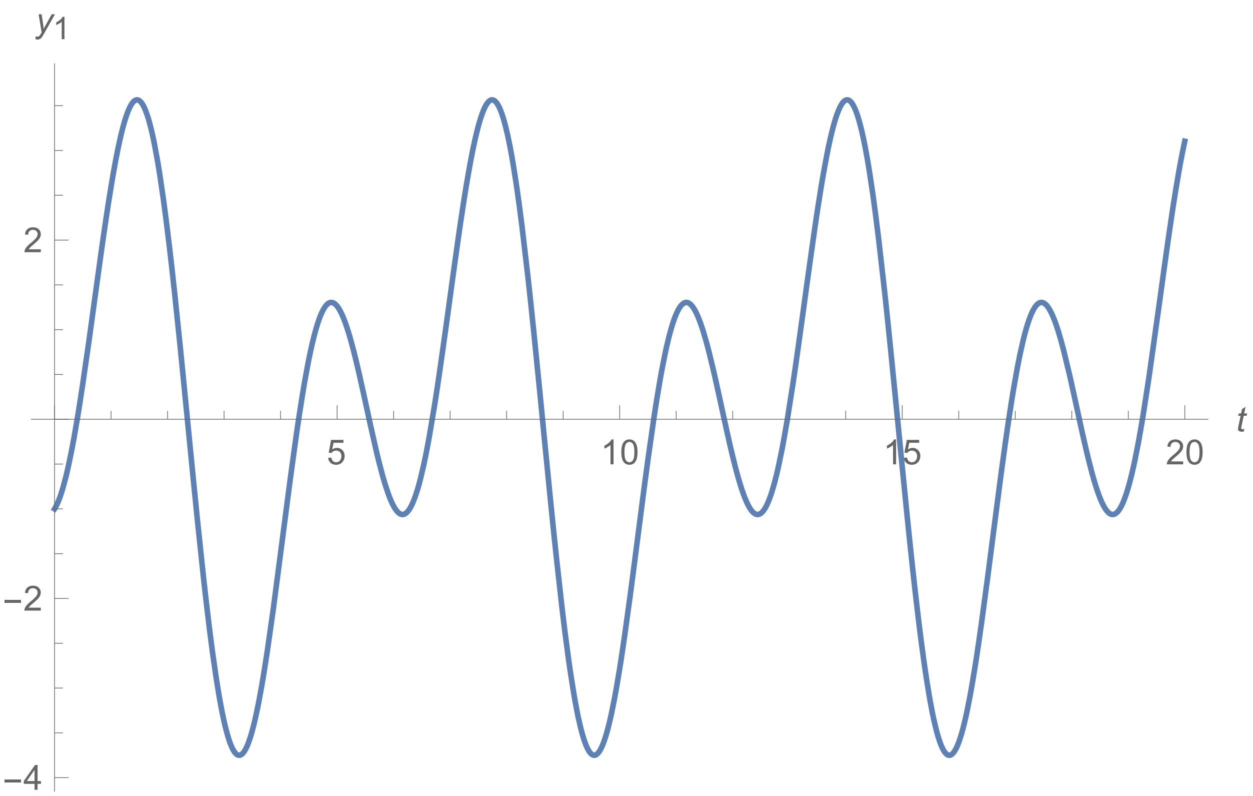
The initial value problem

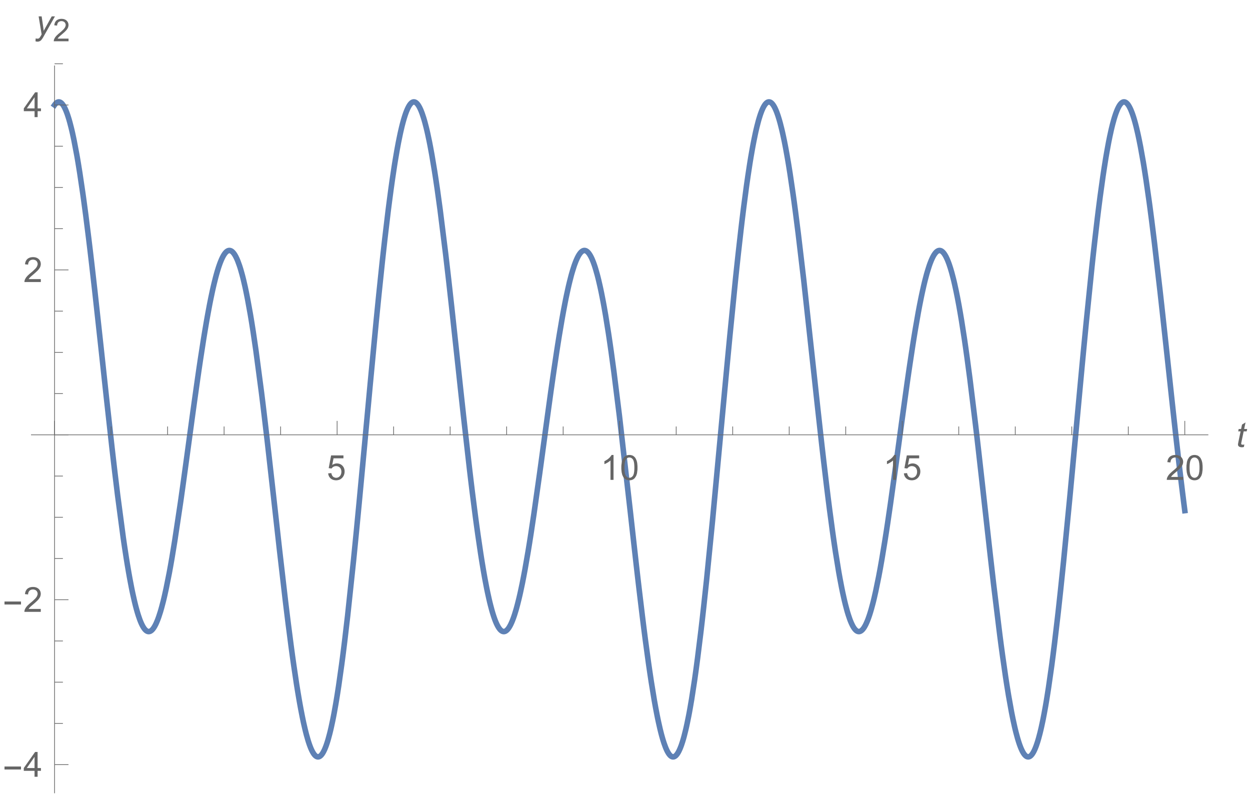
$$\mathbf{y}' = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -2 & \frac{3}{2} & 0 & 0 \\ \frac{4}{3} & -3 & 0 & 0 \end{bmatrix} \mathbf{y}, \qquad \mathbf{y}(0) = \begin{bmatrix} -1 \\ 4 \\ 1 \\ 1 \end{bmatrix}$$

has solution

$$\mathbf{y}(t) = \frac{4}{9}\begin{bmatrix} 3\cos t \\ 2\cos t \\ -3\sin t \\ -2\sin t \end{bmatrix} + \frac{7}{18}\begin{bmatrix} 3\sin t \\ 2\sin t \\ 3\cos t \\ 2\cos t \end{bmatrix} - \frac{7}{9}\begin{bmatrix} 3\cos 2t \\ -4\cos 2t \\ -6\sin 2t \\ 8\sin 2t \end{bmatrix} - \frac{1}{36}\begin{bmatrix} 3\sin 2t \\ -4\sin 2t \\ 6\cos 2t \\ -8\cos 2t \end{bmatrix}.$$

Then we can draw the graphs of y_1 and y_2:





Please see https://tinyurl.com/s7uww7m

7.7 Fundamental Matrices

Now consider

$$\mathbf{x}' = P(t)\mathbf{x}$$

where P is an $n \times n$ matrix. Suppose that $\mathbf{x}^{(1)}$, $\mathbf{x}^{(2)}$, $\ldots$, $\mathbf{x}^{(n)}$ are linearly independent solutions to this ODE. In other words, suppose that $\mathbf{x}^{(1)}$, $\mathbf{x}^{(2)}$, $\ldots$, $\mathbf{x}^{(n)}$ form a ***fundamental set of solutions*** to this ODE.

Definition. The matrix

$$\Psi(t) = \begin{bmatrix} \mathbf{x}^{(1)} & \mathbf{x}^{(2)} & \ldots & \mathbf{x}^{(n)} \end{bmatrix} = \begin{bmatrix} x_1^{(1)}(t) & x_1^{(2)}(t) & \ldots & x_1^{(n)}(t) \\ x_2^{(1)}(t) & x_2^{(2)}(t) & \ldots & x_2^{(n)}(t) \\ \vdots & \vdots & & \vdots \\ x_n^{(1)}(t) & x_n^{(2)}(t) & \ldots & x_n^{(n)}(t) \end{bmatrix}$$

is called a ***fundamental matrix*** of $\mathbf{x}' = P(t)\mathbf{x}$.

Example 7.11. Find a fundamental matrix for

$$\mathbf{x}' = \begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix} \mathbf{x}.$$

Recall that

$$\mathbf{x}^{(1)}(t) = \begin{bmatrix} 1 \\ 2 \end{bmatrix} e^{3t} \qquad \text{and} \qquad \mathbf{x}^{(2)}(t) = \begin{bmatrix} 1 \\ -2 \end{bmatrix} e^{-t}$$

form a fundamental set of solutions to this ODE. Therefore

$$\Psi(t) = \begin{bmatrix} e^{3t} & e^{-t} \\ 2e^{3t} & -2e^{-t} \end{bmatrix}$$

is a fundamental matrix of this ODE.

Now, the general solution to

$$\mathbf{x}' = A\mathbf{x}$$

is

$$\mathbf{x}(t) = c_1\mathbf{x}^{(1)}(t) + c_2\mathbf{x}^{(2)}(t) + \ldots + c_n\mathbf{x}^{(n)}(t) = \Psi(t)\mathbf{c}$$

where

$$\mathbf{c} = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} \in \mathbb{R}^n.$$

If we have an initial condition $\mathbf{x}(t_0) = \mathbf{x}^0$, then

$$\Psi(t_0)\mathbf{c} = \mathbf{x}^0.$$

But

$$\begin{matrix} \mathbf{x}^{(1)}, \ldots, \mathbf{x}^{(n)} \\ \text{are linearly} \\ \text{independent} \end{matrix} \quad \Longrightarrow \quad \Psi(t) \text{ is invertible} \quad \Longrightarrow \quad \mathbf{c} = \Psi^{-1}(t_0)\mathbf{x}^0.$$

Therefore the solution to the IVP

$$\begin{cases} \mathbf{x}' = A\mathbf{x} \\ \mathbf{x}(t_0) = \mathbf{x}^0 \end{cases}$$

is

$$\boxed{\mathbf{x}(t) = \Psi(t)\Psi^{-1}(t_0)\mathbf{x}^0.}$$

Remark. The matrix $\Psi(t)$ solves the differential equation $\Psi' = P(t)\Psi$. (Homework)

Remark. It is possible to find a ***special fundamental matrix***, $\Phi(t)$, which satisfies

$$\Phi(t_0) = \begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \dots & 1 \end{bmatrix} = I.$$

We will use Φ for this special fundamental matrix, and Ψ for any fundamental matrix.

Example 7.12. Consider

$$\mathbf{x}' = \begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix} \mathbf{x}.$$

Find the special fundamental matrix which satisfies $\Phi(0) = I$.

To find the matrix Φ which satisfies

$$\Phi(0) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

we must solve two IVPs:

$$\begin{cases} \mathbf{x}' = \begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix} \mathbf{x} \\ \mathbf{x}(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \end{cases} \quad \text{and} \quad \begin{cases} \mathbf{x}' = \begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix} \mathbf{x} \\ \mathbf{x}(0) = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \end{cases}.$$

The general solution to the ODE is

$$\mathbf{x}(t) = c_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} e^{3t} + c_2 \begin{bmatrix} 1 \\ -2 \end{bmatrix} e^{-t}.$$

We calculate that

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \mathbf{x}(0) = c_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ -2 \end{bmatrix} \implies \begin{matrix} c_1 = \dfrac{1}{2} \\ c_2 = \dfrac{1}{2} \end{matrix} \implies \mathbf{x}(t) = \begin{bmatrix} \frac{1}{2}e^{3t} + \frac{1}{2}e^{-t} \\ e^{3t} - e^{-t} \end{bmatrix}$$

and

$$\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \mathbf{x}(0) = c_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ -2 \end{bmatrix} \implies \begin{matrix} c_1 = \dfrac{1}{4} \\ c_2 = -\dfrac{1}{4} \end{matrix} \implies \mathbf{x}(t) = \begin{bmatrix} \frac{1}{4}e^{3t} - \frac{1}{4}e^{-t} \\ \frac{1}{2}e^{3t} + \frac{1}{2}e^{-t} \end{bmatrix}.$$

Therefore the special fundamental matrix is

$$\Phi(t) = \begin{bmatrix} \frac{1}{2}e^{3t} + \frac{1}{2}e^{-t} & \frac{1}{4}e^{3t} - \frac{1}{4}e^{-t} \\ e^{3t} - e^{-t} & \frac{1}{2}e^{3t} + \frac{1}{2}e^{-t} \end{bmatrix}.$$

What is e^{At}?

Recall that the solution to

$$\begin{cases} x' = ax \ (a \in \mathbb{R}) \\ x(0) = x_0 \end{cases}$$

is

$$x(t) = x_0 e^{at} = x_0 \exp(at)$$

and recall that

$$\exp(at) = 1 + \sum_{n=1}^{\infty} \frac{a^n t^n}{n!}.$$

Now consider

$$\begin{cases} \mathbf{x}' = A\mathbf{x} \\ \mathbf{x}(0) = \mathbf{x}^0 \end{cases}$$

for $A \in \mathbb{R}^{n \times n}$.

Definition.

$$\exp(At) = I + \sum_{n=1}^{\infty} \frac{A^n t^n}{n!} = I + At + \frac{A^2 t^2}{2!} + \frac{A^3 t^3}{3!} + \dots .$$

Note that

$$\begin{aligned} \frac{d}{dt} \exp(At) &= \frac{d}{dt}\left(I + \sum_{n=1}^{\infty} \frac{A^n t^n}{n!} \right) = 0 + \sum_{n=1}^{\infty} \frac{d}{dt}\left(\frac{A^n t^n}{n!} \right) \\ &= \sum_{n=1}^{\infty} \frac{A^n t^{n-1}}{(n-1)!} = A + \sum_{n=2}^{\infty} \frac{A^n t^{n-1}}{(n-1)!} \\ &= A + \sum_{k=1}^{\infty} \frac{A^{k+1} t^k}{(k)!} \qquad\qquad (k = n-1) \\ &= A\left(I + \sum_{k=1}^{\infty} \frac{A^k t^k}{(k)!} \right) = A \exp(At). \end{aligned}$$

This means that $\exp(At)$ solves

$$\begin{cases} \big(\exp(At)\big)' = A \exp(At) \\ \exp(At)\big|_{t=0} = I. \end{cases}$$

But remember that Φ solves

$$\begin{cases} \Phi' = A\phi \\ \Phi(0) = I. \end{cases}$$

Therefore

$$\boxed{\Phi(t) = \exp(At).}$$

Diagonalisable Matrices

If

$$D = \begin{bmatrix} r_1 & 0 & 0 & \dots & 0 \\ 0 & r_2 & 0 & \dots & 0 \\ 0 & 0 & r_3 & \dots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \dots & r_n \end{bmatrix}$$

is a diagonal matrix, then it is easy to calculate $\exp(Dt)$. We simply have

$$\exp(Dt) = \begin{bmatrix} e^{r_1 t} & 0 & 0 & \dots & 0 \\ 0 & e^{r_2 t} & 0 & \dots & 0 \\ 0 & 0 & e^{r_3 t} & \dots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \dots & e^{r_n t} \end{bmatrix}.$$

Now consider

$$\mathbf{x}' = A\mathbf{x}$$

for $A \in \mathbb{R}^{n \times n}$. Recall how we diagonalise a matrix: If $\boldsymbol{\xi}^{(1)}, \boldsymbol{\xi}^{(2)}, \dots, \boldsymbol{\xi}^{(n)}$ are the eigenvectors of A, we let

$$T = \begin{bmatrix} \boldsymbol{\xi}^{(1)} & \boldsymbol{\xi}^{(2)} & \dots & \boldsymbol{\xi}^{(n)} \end{bmatrix}.$$

Then

$$\det(T) \neq 0 \implies T^{-1} \text{ exists} \implies \begin{matrix} T^{-1}AT \\ \text{is diagonal} \end{matrix} \implies \begin{matrix} A \text{ is} \\ \text{diagonalisable.} \end{matrix}$$

Example 7.13. Diagonalise

$$A = \begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix}.$$

The eigenvalues are $r_1 = 3$ and $r_2 = -1$. The corresponding eigenvectors are $\boldsymbol{\xi}^{(1)} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $\boldsymbol{\xi}^{(2)} = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$. Thus

$$T = \begin{bmatrix} 1 & 1 \\ 2 & -2 \end{bmatrix} \qquad \text{and} \qquad T^{-1} = \begin{bmatrix} \frac{1}{2} & \frac{1}{4} \\ \frac{1}{2} & -\frac{1}{4} \end{bmatrix}.$$

It follows that

$$D = T^{-1}AT = \begin{bmatrix} 3 & 0 \\ 0 & -1 \end{bmatrix}.$$

Now consider

$$\mathbf{x}' = A\mathbf{x}.$$

Define a new variable $\mathbf{y}$ by

$$\mathbf{x} = T\mathbf{y} \qquad \text{or} \qquad \mathbf{y} = T^{-1}\mathbf{x}.$$

Then we calculate that

$$\begin{aligned} \mathbf{x}' &= A\mathbf{x} \\ T\mathbf{y}' &= AT\mathbf{y} \\ \mathbf{y}' &= T^{-1}AT\mathbf{y} = D\mathbf{y}. \end{aligned}$$

We know that a fundamental matrix for $\mathbf{y}' = D\mathbf{y}$ is

$$\exp(Dt) = \begin{bmatrix} e^{r_1 t} & 0 & \dots & 0 \\ 0 & e^{r_2 t} & \dots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \dots & e^{r_n t} \end{bmatrix}.$$

Therefore a fundamental matrix for $\mathbf{x}' = A\mathbf{x}$ is

$$\Psi = T\exp(Dt) = \begin{bmatrix} \boldsymbol{\xi}^{(1)} e^{r_1 t} & \boldsymbol{\xi}^{(2)} e^{r_2 t} & \dots & \boldsymbol{\xi}^{(n)} e^{r_n t} \end{bmatrix}.$$

Example 7.14. Find a fundamental matrix for

$$\mathbf{x}' = \begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix} \mathbf{x}.$$

Recall that $T = \begin{bmatrix} 1 & 1 \\ 2 & -2 \end{bmatrix}$. Letting $\mathbf{y} = T^{-1}\mathbf{x}$, we have

$$\mathbf{y}' = \begin{bmatrix} 3 & 0 \\ 0 & -1 \end{bmatrix} \mathbf{y}.$$

A fundamental matrix of $\mathbf{y}' = \begin{bmatrix} 3 & 0 \\ 0 & -1 \end{bmatrix} \mathbf{y}$ is

$$\exp(Dt) = e^{Dt} = \begin{bmatrix} e^{3t} & 0 \\ 0 & e^{-t} \end{bmatrix}.$$

Hence

$$\Psi(t) = T\exp(Dt) = \begin{bmatrix} 1 & 1 \\ 2 & -2 \end{bmatrix} \begin{bmatrix} e^{3t} & 0 \\ 0 & e^{-t} \end{bmatrix} = \begin{bmatrix} e^{3t} & e^{-t} \\ 2e^{3t} & -2e^{-t} \end{bmatrix}.$$

7.8 Repeated Eigenvalues

Example 7.15. Find the eigenvalues and eigenvectors of $A = \begin{bmatrix} 1 & -1 \\ 1 & 3 \end{bmatrix}$.

We calculate that

$$0 = \det(A - rI) = \begin{vmatrix} 1-r & -1 \\ 1 & 3-r \end{vmatrix} = r^2 - 4r + 4 = (r-2)^2.$$

Therefore $r_1 = 2 = r_2$. Moreover

$$\mathbf{0} = (A - rI)\boldsymbol{\xi} = \begin{bmatrix} -1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix} \implies \xi_1 + \xi_2 = 0 \implies \boldsymbol{\xi} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

Note that A has only one linearly independent eigenvector.

Example 7.16. Solve

$$\mathbf{x}' = \begin{bmatrix} 1 & -1 \\ 1 & 3 \end{bmatrix} \mathbf{x}.$$

We know that

$$\mathbf{x}^{(1)}(t) = \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{2t}$$

is a solution. But we need two solutions.

Guess 1: I guess that

$$\mathbf{x}^{(2)}(t) = \boldsymbol{\xi} t e^{2t}$$

for some $\boldsymbol{\xi} \in \mathbb{R}^2$. Then we have

$$\begin{aligned} \boldsymbol{\xi} e^{2t} + 2\boldsymbol{\xi} t e^{2t} = \mathbf{x}^{(2)\prime} = A\mathbf{x}^{(2)} &= A\boldsymbol{\xi} t e^{2t} \\ \boldsymbol{\xi} + (2\boldsymbol{\xi} - A\boldsymbol{\xi})\, t = \mathbf{0} &\qquad \forall t \\ \implies \boldsymbol{\xi} &= \mathbf{0}. \end{aligned}$$

This guess did not work.

Guess 2: Now I guess that

$$\mathbf{x}^{(2)}(t) = \boldsymbol{\xi} t e^{2t} + \boldsymbol{\eta} e^{2t}$$

for some $\boldsymbol{\xi}, \boldsymbol{\eta} \in \mathbb{R}^2$. Then we have

$$\boldsymbol{\xi} e^{2t} + 2\boldsymbol{\xi} t e^{2t} + 2\boldsymbol{\eta} e^{2t} = \mathbf{x}^{(2)\prime} = A\mathbf{x}^{(2)} = A\left(\boldsymbol{\xi} t e^{2t} + \boldsymbol{\eta} e^{2t}\right)$$

and

$$(2\boldsymbol{\xi} - A\boldsymbol{\xi})\, t + (\boldsymbol{\xi} + 2\boldsymbol{\eta} - A\boldsymbol{\eta}) = \mathbf{0}.$$

Since this must be true $\forall\, t$, we must have

$$(A - 2I)\boldsymbol{\xi} = \mathbf{0} \qquad \text{and} \qquad (A - 2I)\boldsymbol{\eta} = \boldsymbol{\xi}.$$

Clearly $\boldsymbol{\xi} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$. Then we calculate that

$$\begin{aligned} \begin{bmatrix} -1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \eta_1 \\ \eta_2 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix} &\implies \eta_1 + \eta_2 = -1 \\ &\implies \boldsymbol{\eta} = \begin{bmatrix} k \\ -1-k \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \end{bmatrix} + k \begin{bmatrix} 1 \\ -1 \end{bmatrix} \end{aligned}$$

for some k. So

$$\begin{aligned}\mathbf{x}^{(2)}(t) = \boldsymbol{\xi} te^{2t} + \boldsymbol{\eta} e^{2t} &= \begin{bmatrix} 1 \\ -1 \end{bmatrix} te^{2t} + \begin{bmatrix} 0 \\ -1 \end{bmatrix} e^{2t} + k \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{2t} \\ &= \begin{bmatrix} 1 \\ -1 \end{bmatrix} te^{2t} + \begin{bmatrix} 0 \\ -1 \end{bmatrix} e^{2t} + k\mathbf{x}^{(1)}(t).\end{aligned}$$

Because we already have $\mathbf{x}^{(1)}(t)$, we can choose $k = 0$. So

$$\mathbf{x}^{(2)}(t) = \boldsymbol{\xi} te^{2t} + \boldsymbol{\eta} e^{2t} = \begin{bmatrix} 1 \\ -1 \end{bmatrix} te^{2t} + \begin{bmatrix} 0 \\ -1 \end{bmatrix} e^{2t}.$$

The general solution of $\mathbf{x}' = \begin{bmatrix} 1 & -1 \\ 1 & 3 \end{bmatrix} \mathbf{x}$ is therefore

$$\boxed{\mathbf{x}(t) = c_1 \mathbf{x}^{(1)} + c_2 \mathbf{x}^{(2)} = c_1 \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{2t} + c_2 \left(\begin{bmatrix} 1 \\ -1 \end{bmatrix} te^{2t} + \begin{bmatrix} 0 \\ -1 \end{bmatrix} e^{2t} \right).}$$

Example 7.17. Find a fundamental matrix for

$$\mathbf{x}' = \begin{bmatrix} 1 & -1 \\ 1 & 3 \end{bmatrix} \mathbf{x}.$$

Then find the special fundamental matrix $\Phi(t)$ which satisfies $\Phi(0) = I$.

Since $\mathbf{x}^{(1)}(t) = \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{2t}$ and $\mathbf{x}^{(2)}(t) = \begin{bmatrix} 1 \\ -1 \end{bmatrix} te^{2t} + \begin{bmatrix} 0 \\ -1 \end{bmatrix} e^{2t}$ we have that

$$\Psi(t) = \begin{bmatrix} \mathbf{x}^{(1)} & \mathbf{x}^{(2)} \end{bmatrix} = \begin{bmatrix} e^{2t} & te^{2t} \\ -e^{2t} & -te^{2t} - e^{2t} \end{bmatrix}$$

is a fundamental matrix for this system.

Now

$$\Psi(0) = \begin{bmatrix} 1 & 0 \\ -1 & -1 \end{bmatrix} \qquad \text{and} \qquad \Psi^{-1}(0) = \begin{bmatrix} 1 & 0 \\ -1 & -1 \end{bmatrix}.$$

Therefore

$$\begin{aligned}\exp(At) = \Phi(t) = \Psi(t)\Psi^{-1}(0) &= e^{2t} \begin{bmatrix} 1 & t \\ -1 & -1-t \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -1 & -1 \end{bmatrix} \\ &= e^{2t} \begin{bmatrix} 1-t & -t \\ t & 1+t \end{bmatrix}.\end{aligned}$$

Remark.

$$\mathbf{x}' = A\mathbf{x}$$

For two repeated eigenvalues (but with only one linearly independent eigenvector), the key equations to remember are

$$\boxed{\mathbf{x}^{(2)}(t) = \boldsymbol{\xi} te^{rt} + \boldsymbol{\eta} e^{rt}} \qquad \text{and} \qquad \boxed{(A - rI)\boldsymbol{\eta} = \boldsymbol{\xi}}.$$

Definition. η is called a ***generalised eigenvector*** of A.

Remark. If you have 2 repeated eigenvalues (but with only one linearly independent eigenvector), the method is:

(i). Find the eigenvalues and eigenvectors;

(ii). The first solution is $\mathbf{x}^{(1)}(t) = \boldsymbol{\xi} e^{rt}$;

(iii). Use $(A - rI)\boldsymbol{\eta} = \boldsymbol{\xi}$ to find a generalised eigenvector $\boldsymbol{\eta}$;

(iv). The second solution is $\mathbf{x}^{(2)}(t) = \boldsymbol{\xi} te^{rt} + \boldsymbol{\eta} e^{rt}$.

Example 7.18. Solve

$$\begin{cases} \mathbf{x}' = \begin{bmatrix} -\frac{5}{2} & \frac{3}{2} \\ -\frac{3}{2} & \frac{1}{2} \end{bmatrix} \mathbf{x}, \\ \mathbf{x}(0) = \begin{bmatrix} 3 \\ -1 \end{bmatrix}. \end{cases}$$

The only eigenvalue of the matrix is $r = -1$. The corresponding eigenvector is $\boldsymbol{\xi} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. Therefore one solution of the linear system is

$$\mathbf{x}^{(1)}(t) = \boldsymbol{\xi} e^{rt} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{-t}.$$

We need to find a second, linearly independent solution. We will consider the ansatz

$$\mathbf{x}^{(2)}(t) = \boldsymbol{\xi} te^{-t} + \boldsymbol{\eta} e^{-t}$$

where $\boldsymbol{\xi} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ as above and $\boldsymbol{\eta}$ is a generalised eigenvector solving $(A - rI)\boldsymbol{\eta} = \boldsymbol{\xi}$.

Solving the latter equation,

$$\begin{aligned} (A - rI)\boldsymbol{\eta} &= \boldsymbol{\xi} \\ \begin{bmatrix} -\frac{3}{2} & \frac{3}{2} \\ -\frac{3}{2} & \frac{3}{2} \end{bmatrix} \begin{bmatrix} \eta_1 \\ \eta_2 \end{bmatrix} &= \begin{bmatrix} 1 \\ 1 \end{bmatrix} \\ -\frac{3}{2}\eta_1 + \frac{3}{2}\eta_2 &= 1 \\ -\eta_1 + \eta_2 &= \frac{2}{3} \end{aligned}$$

we can choose $\boldsymbol{\eta} = \begin{bmatrix} 0 \\ \frac{2}{3} \end{bmatrix}$.

Note that we don't need to find ***every*** generalised eigenvector

$$\boldsymbol{\eta} = \begin{bmatrix} k \\ k + \frac{2}{3} \end{bmatrix} = k \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{2}{3} \end{bmatrix} = k\boldsymbol{\xi} + \begin{bmatrix} 0 \\ \frac{2}{3} \end{bmatrix}$$

because we already have $\mathbf{x}^{(1)}(t) = \boldsymbol{\xi} e^{rt}$.

Instead we only need to find ***one*** generalised eigenvector – that means that we can choose any k that we want.

Hence I have chosen $k = 0$ which gives $\boldsymbol{\eta} = \begin{bmatrix} 0 \\ \frac{2}{3} \end{bmatrix}$.

Thus
$$\mathbf{x}^{(2)}(t) = \boldsymbol{\xi} t e^{-t} + \boldsymbol{\eta} e^{-t} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} t e^{-t} + \begin{bmatrix} 0 \\ \frac{2}{3} \end{bmatrix} e^{-t}.$$

Hence the general solution to the linear system is

$$\mathbf{x}(t) = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{-t} + c_2 \left(\begin{bmatrix} 1 \\ 1 \end{bmatrix} t e^{-t} + \begin{bmatrix} 0 \\ \frac{2}{3} \end{bmatrix} e^{-t} \right).$$

The initial condition gives

$$\begin{bmatrix} 3 \\ -1 \end{bmatrix} = \mathbf{x}(0) = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ \frac{2}{3} \end{bmatrix}$$

which implies that $c_1 = 3$ and $c_2 = -6$.

Therefore the solution to the IVP is

$$\boxed{\mathbf{x}(t) = 3 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{-t} - 6 \left(\begin{bmatrix} 1 \\ 1 \end{bmatrix} t e^{-t} + \begin{bmatrix} 0 \\ \frac{2}{3} \end{bmatrix} e^{-t} \right) = \begin{bmatrix} 3 \\ -1 \end{bmatrix} e^{-t} - 6 \begin{bmatrix} 1 \\ 1 \end{bmatrix} t e^{-t}.}$$

Example 7.19. Solve
$$\mathbf{x}' = \begin{bmatrix} 1 & -4 \\ 4 & -7 \end{bmatrix} \mathbf{x}, \qquad \mathbf{x}(0) = \begin{bmatrix} 3 \\ 2 \end{bmatrix}.$$

The only eigenvalue of the matrix is $r = -3$. The corresponding eigenvector is $\boldsymbol{\xi} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. Therefore one solution of the linear system is

$$\mathbf{x}^{(1)}(t) = \boldsymbol{\xi} e^{rt} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{-3t}.$$

Next we need to find a generalised eigenvector $\boldsymbol{\eta}$.
We calculate that
$$\begin{aligned} (A - rI)\boldsymbol{\eta} &= \boldsymbol{\xi} \\ \begin{bmatrix} 4 & -4 \\ 4 & -4 \end{bmatrix} \begin{bmatrix} \eta_1 \\ \eta_2 \end{bmatrix} &= \begin{bmatrix} 1 \\ 1 \end{bmatrix} \\ 4\eta_1 - 4\eta_2 &= 1 \\ -\eta_1 + \eta_2 &= -\frac{1}{4} \\ \eta_2 &= \eta_1 - \frac{1}{4}. \end{aligned}$$
So we can choose any vector $\boldsymbol{\eta}$ that satisfies $\eta_2 = \eta_1 - \frac{1}{4}$. Thus we may choose $\boldsymbol{\eta} = \begin{bmatrix} 0 \\ -\frac{1}{4} \end{bmatrix}$.

Therefore
$$\mathbf{x}^{(2)}(t) = \begin{bmatrix} 1 \\ 1 \end{bmatrix} te^{-3t} + \begin{bmatrix} 0 \\ -\frac{1}{4} \end{bmatrix} e^{-3t}.$$

Hence the general solution to the ODE is

$$\mathbf{x}(t) = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{-3t} + c_2 \left(\begin{bmatrix} 1 \\ 1 \end{bmatrix} te^{-3t} + \begin{bmatrix} 0 \\ -\frac{1}{4} \end{bmatrix} e^{-3t} \right).$$

The initial condition gives

$$\begin{bmatrix} 3 \\ 2 \end{bmatrix} = \mathbf{x}(0) = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ -\frac{1}{4} \end{bmatrix}$$

which implies that $c_1 = 3$ and $c_2 = 4$.

Therefore the solution to the IVP is

$$\begin{aligned} \mathbf{x}(t) &= 3 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{-3t} + 4 \left(\begin{bmatrix} 1 \\ 1 \end{bmatrix} te^{-3t} + \begin{bmatrix} 0 \\ -\frac{1}{4} \end{bmatrix} e^{-3t} \right) \\ &= \begin{bmatrix} 3 \\ 2 \end{bmatrix} e^{-3t} + 4 \begin{bmatrix} 1 \\ 1 \end{bmatrix} te^{-3t} \\ &= \begin{bmatrix} 3 + 4t \\ 2 + 4t \end{bmatrix} e^{-3t}. \end{aligned}$$

7.9 Nonhomogeneous Linear Systems

Consider

$$\mathbf{x}' = P(t)\mathbf{x} + \mathbf{g}(t) \tag{7.1}$$

where $P(t)$ and $\mathbf{g}(t)$ are continuous for $\alpha < t < \beta$. The general solution of (7.1) can be written as

$$\mathbf{x}(t) = c_1\mathbf{x}^{(1)} + c_2\mathbf{x}^{(2)} + \ldots + c_n\mathbf{x}^{(n)} + \mathbf{v}(t)$$

where

- $c_1\mathbf{x}^{(1)} + c_2\mathbf{x}^{(2)} + \ldots + c_n\mathbf{x}^{(n)}$ is the general solution to the homogeneous system $\mathbf{x}' = P(t)\mathbf{x}$; and
- $\mathbf{v}(t)$ is a particular solution to (7.1).

Remark. We will study four methods to solve (7.1):

(i). Diagonalisation;

(ii). Undetermined Coefficients;

(iii). Variation of Parameters;

(iv). The Laplace Transform.

Method 1 – Diagonalisation:

Consider

$$\mathbf{x}' = A\mathbf{x} + \mathbf{g}(t).$$

Suppose that

- $A \in \mathbb{R}^{n\times n}$ is diagonalisable;
- $\mathbf{g} : (\alpha, \beta) \to \mathbb{R}^n$;
- $\boldsymbol{\xi}^{(1)}, \ldots, \boldsymbol{\xi}^{(n)}$ are eigenvectors of A; and
- $T = \begin{bmatrix} \boldsymbol{\xi}^{(1)} & \cdots & \boldsymbol{\xi}^{(n)} \end{bmatrix}$.

Then

$$D = T^{-1}DT = \begin{bmatrix} r_1 & 0 & \cdots & 0 \\ 0 & r_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & r_n \end{bmatrix}.$$

Let $\mathbf{y} = T^{-1}\mathbf{x}$. Then $\mathbf{x} = T\mathbf{y}$. It follows that

$$T\mathbf{y}' = \mathbf{x}' = A\mathbf{x} + \mathbf{g}(t) = AT\mathbf{y} + \mathbf{g}(t)$$

and

$$\mathbf{y}' = T^{-1}AT\mathbf{y} + T^{-1}\mathbf{g}(t) = D\mathbf{y} + \mathbf{h}(t) \tag{7.2}$$

where $\mathbf{h} = T^{-1}\mathbf{g}$.

But (7.2) is just the system

$$\begin{cases} y_1' = r_1y_1 + h_1(t) & \longleftarrow \text{only } y_1 \text{ and } t \\ y_2' = r_2y_2 + h_2(t) & \longleftarrow \text{only } y_2 \text{ and } t \\ \quad\vdots & \\ y_n' = r_ny_n + h_n(t) & \longleftarrow \text{only } y_n \text{ and } t \end{cases}$$

We can solve each of these n first order linear ODEs individually. The solution to

$$y_j' - r_jy_j = h_j$$

(see Chapter 2) is

$$y_j(t) = e^{r_jt}\int_{t_0}^{t} e^{-r_js}h(s)\,ds + c_je^{r_jt}.$$

If we know $\mathbf{y}$, then we know $\mathbf{x} = T\mathbf{y}$.

Example 7.20. Solve

$$\mathbf{x}' = \begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 2e^{-t} \\ 3t \end{bmatrix} = A\mathbf{x} + \mathbf{g}(t).$$

The eigenvalues of $A = \begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix}$ are $r_1 = -3$ and $r_2 = -1$. The eigenvectors are

$$\boldsymbol{\xi}^{(1)} = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \qquad \text{and} \qquad \boldsymbol{\xi}^{(2)} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

So

$$T = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \qquad \text{and} \qquad T^{-1} = \frac{1}{2} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}.$$

Let $\mathbf{y} = T^{-1}\mathbf{x}$. Then

$$\begin{aligned} T\mathbf{y}' = \mathbf{x}' = A\mathbf{x} + \begin{bmatrix} 2e^{-t} \\ 3t \end{bmatrix} &= AT\mathbf{y} + \begin{bmatrix} 2e^{-t} \\ 3t \end{bmatrix} \\ \mathbf{y}' &= T^{-1}AT\mathbf{y} + T^{-1} \begin{bmatrix} 2e^{-t} \\ 3t \end{bmatrix} \\ &= D\mathbf{y} + \frac{1}{2} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 2e^{-t} \\ 3t \end{bmatrix} \\ &= \begin{bmatrix} -3 & 0 \\ 0 & -1 \end{bmatrix} \mathbf{y} + \frac{1}{2} \begin{bmatrix} 2e^{-t} - 3t \\ 2e^{-t} + 3t \end{bmatrix}. \end{aligned}$$

Therefore

$$\begin{cases} y_1' + 3y_1 = e^{-t} - \frac{3}{2}t \\ y_2' + y_2 = e^{-t} + \frac{3}{2}t. \end{cases}$$

You know how to solve first order linear ODEs. The solutions to these two ODEs are

$$\begin{aligned} y_1(t) &= \frac{1}{2}e^{-t} - \frac{t}{2} + \frac{1}{6} + c_1 e^{-3t} \\ y_2(t) &= te^{-t} + \frac{3t}{2} - \frac{3}{2} + c_2 e^{-t}. \end{aligned}$$

Finally we calculate that

$$\begin{aligned} \mathbf{x} = T\mathbf{y} &= \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \\ &= \begin{bmatrix} y_1 + y_2 \\ -y_1 + y_2 \end{bmatrix} \\ &= c_1 \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{-3t} + c_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{-t} + \frac{1}{2} \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{-t} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} te^{-t} + \begin{bmatrix} 1 \\ 2 \end{bmatrix} t - \frac{1}{3} \begin{bmatrix} 4 \\ 5 \end{bmatrix}. \end{aligned}$$

Example 7.21. Solve

$$\mathbf{x}' = \begin{bmatrix} 1 & \sqrt{3} \\ \sqrt{3} & -1 \end{bmatrix} \mathbf{x} + \begin{bmatrix} e^t \\ \sqrt{3}e^{-t} \end{bmatrix}.$$

The eigenvalues of $\begin{bmatrix} 1 & \sqrt{3} \\ \sqrt{3} & -1 \end{bmatrix}$ are $r_1 = -2$ and $r_2 = 2$. The corresponding eigenvectors are $\boldsymbol{\xi}^{(1)} = \begin{bmatrix} 1 \\ -\sqrt{3} \end{bmatrix}$ and $\boldsymbol{\xi}^{(2)} = \begin{bmatrix} \sqrt{3} \\ 1 \end{bmatrix}$.

Thus

$$T = \begin{bmatrix} 1 & \sqrt{3} \\ -\sqrt{3} & 1 \end{bmatrix}$$

and

$$T^{-1} = \frac{1}{\det T} \begin{bmatrix} 1 & -\sqrt{3} \\ \sqrt{3} & 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{4} & -\frac{\sqrt{3}}{4} \\ \frac{\sqrt{3}}{4} & \frac{1}{4} \end{bmatrix}.$$

Now we must change variables: Let $\mathbf{y} = T^{-1}\mathbf{x}$. Then we have

$$\begin{aligned} \mathbf{y}' = D\mathbf{y} + T^{-1}\mathbf{g} &= \begin{bmatrix} -2 & 0 \\ 0 & 2 \end{bmatrix} \mathbf{y} + \begin{bmatrix} \frac{1}{4} & -\frac{\sqrt{3}}{4} \\ \frac{\sqrt{3}}{4} & \frac{1}{4} \end{bmatrix} \begin{bmatrix} e^t \\ \sqrt{3}e^{-t} \end{bmatrix} \\ &= \begin{bmatrix} -2y_1 \\ 2y_2 \end{bmatrix} + \begin{bmatrix} \frac{1}{4}e^t - \frac{3}{4}e^{-t} \\ \frac{\sqrt{3}}{4}e^t + \frac{\sqrt{3}}{4}e^{-t} \end{bmatrix}. \end{aligned}$$

We know how to solve

$$y_1' + 2y_1 = \frac{1}{4}e^t - \frac{3}{4}e^{-t}$$

and

$$y_2' - 2y_2 = \frac{\sqrt{3}}{4}e^t + \frac{\sqrt{3}}{4}e^{-t}.$$

The solutions are

$$y_1(t) = \frac{1}{12}e^t - \frac{3}{4}e^{-t} + c_1 e^{-2t}$$

and

$$y_2(t) = -\frac{\sqrt{3}}{4}e^t - \frac{\sqrt{3}}{12}e^{-t} + c_2 e^{2t}.$$

So

$$\mathbf{y} = \begin{bmatrix} \frac{1}{12}e^t - \frac{3}{4}e^{-t} + c_1 e^{-2t} \\ -\frac{\sqrt{3}}{4}e^t - \frac{\sqrt{3}}{12}e^{-t} + c_2 e^{2t} \end{bmatrix}.$$

Therefore the general solution to the ODE is

$$\mathbf{x} = T\mathbf{y} = \begin{bmatrix} 1 & \sqrt{3} \\ -\sqrt{3} & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{12}e^t - \frac{3}{4}e^{-t} + c_1 e^{-2t} \\ -\frac{\sqrt{3}}{4}e^t - \frac{\sqrt{3}}{12}e^{-t} + c_2 e^{2t} \end{bmatrix} = \ldots$$

Method 2 – Undeterminded Coefficients:

Consider

$$\mathbf{x}' = A\mathbf{x} + \mathbf{g}(t).$$

(Remember Chapter 3?)

The idea is

(i). Find the general solution to $\mathbf{x}' = A\mathbf{x}$.

(ii). Look at $\mathbf{g}(t)$. Make a guess with constants. Find the constants.

(iii). $1 + 2$.

Example 7.22. Solve

$$\mathbf{x}' = \begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 2e^{-t} \\ 3t \end{bmatrix} = A\mathbf{x} + \mathbf{g}(t).$$

1. The solution of $\mathbf{x}' = \begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix} \mathbf{x}$ is

$$\mathbf{x}(t) = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{-t} + c_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{-3t}.$$

2. Since $\mathbf{g}(t) = \begin{bmatrix} 2 \\ 0 \end{bmatrix} e^{-t} + \begin{bmatrix} 0 \\ 3 \end{bmatrix} t$, we try the ansatz

$$\mathbf{x} = \mathbf{v}(t) = \mathbf{a}te^{-t} + \mathbf{b}e^{-t} + \mathbf{c}t + \mathbf{d}.$$

(Note that because $r_1 = -1$ is an eigenvalue of $\begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix}$, we need both te^{-t} and e^{-t}.)

Then we calculate that

$$\mathbf{x}' = A\mathbf{x} + \mathbf{g}$$

$$\mathbf{a}e^{-t} - \mathbf{a}te^{-t} - \mathbf{b}e^{-t} + \mathbf{c} = A\mathbf{a}te^{-t} + A\mathbf{b}e^{-t} + A\mathbf{c}t + A\mathbf{d} + \begin{bmatrix} 2 \\ 0 \end{bmatrix} e^{-t} + \begin{bmatrix} 0 \\ 3 \end{bmatrix} t.$$

- If we look at the te^{-t} terms, we have

$$-\mathbf{a} = A\mathbf{a} \quad \Longrightarrow \quad \mathbf{a} \text{ is an eigenvector} \quad \Longrightarrow \quad \mathbf{a} = \begin{bmatrix} \alpha \\ \alpha \end{bmatrix} \quad \text{for some } \alpha \in \mathbb{R}.$$

- If we look at the e^{-t} terms, we have

$$\mathbf{a} - \mathbf{b} = A\mathbf{b} + \begin{bmatrix} 2 \\ 0 \end{bmatrix}$$

which becomes

$$\begin{bmatrix} \alpha - 2 \\ \alpha \end{bmatrix} = \mathbf{a} - \begin{bmatrix} 2 \\ 0 \end{bmatrix} = (A + I)\mathbf{b} = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} = \begin{bmatrix} -b_1 + b_2 \\ b_1 - b_2 \end{bmatrix}.$$

But this means that

$$\alpha - 2 = -b_1 + b_2 = -(b_1 - b_2) = -\alpha \qquad \Longrightarrow \alpha = 1.$$

So $\mathbf{a} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. Then we have that

$$b_1 - b_2 = 1 \quad \implies \quad \mathbf{b} = \begin{bmatrix} k \\ k-1 \end{bmatrix}$$

for any k. If we choose $k = 0$, we get $\mathbf{b} = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$.

- If we look at the t terms, we have

$$0 = A\mathbf{c} + \begin{bmatrix} 0 \\ 3 \end{bmatrix} \quad \implies \mathbf{c} = A^{-1} \begin{bmatrix} 0 \\ -3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}.$$

- Finally, if we look at the 1 terms, we have

$$\mathbf{c} = A\mathbf{d} \quad \implies \quad \mathbf{d} = A^{-1}\mathbf{c} = \begin{bmatrix} -\frac{4}{3} \\ -\frac{5}{3} \end{bmatrix}.$$

So

$$\mathbf{v} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} te^{-t} - \begin{bmatrix} 0 \\ 1 \end{bmatrix} e^{-t} + \begin{bmatrix} 1 \\ 2 \end{bmatrix} t - \frac{1}{3} \begin{bmatrix} 4 \\ 5 \end{bmatrix}.$$

3. Therefore the general solution to the IVP is

$$\boxed{\mathbf{x}(t) = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{-t} + c_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{-3t} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} te^{-t} - \begin{bmatrix} 0 \\ 1 \end{bmatrix} e^{-t} + \begin{bmatrix} 1 \\ 2 \end{bmatrix} t - \frac{1}{3} \begin{bmatrix} 4 \\ 5 \end{bmatrix}.}$$

Example 7.23. Solve

$$\mathbf{x}' = \begin{bmatrix} 2 & 3 \\ 4 & 1 \end{bmatrix} \mathbf{x} + \begin{bmatrix} e^t \\ -10t-3 \end{bmatrix}.$$

The matrix $\begin{bmatrix} 2 & 3 \\ 4 & 1 \end{bmatrix}$ has eigenvalues $r_1 = 5$ and $r_2 = -2$ and eigenvectors $\boldsymbol{\xi}^{(1)} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\boldsymbol{\xi}^{(2)} = \begin{bmatrix} -3 \\ 4 \end{bmatrix}$. Hence the general solution of $\mathbf{x}' = \begin{bmatrix} 2 & 3 \\ 4 & 1 \end{bmatrix} \mathbf{x}$ is

$$\mathbf{x}(t) = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{5t} + c_2 \begin{bmatrix} -3 \\ 4 \end{bmatrix} e^{-2t}.$$

Next we need to find a particular solution to $\mathbf{x}' = \begin{bmatrix} 2 & 3 \\ 4 & 1 \end{bmatrix} \mathbf{x} + \begin{bmatrix} e^t \\ 0 \end{bmatrix}$. Since 1 is not an eigenvector of $\begin{bmatrix} 2 & 3 \\ 4 & 1 \end{bmatrix}$, we try the ansatz $\mathbf{x} = \mathbf{a}e^t$ for some $\mathbf{a} = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} \in \mathbb{R}^2$. Then we calculate that

$$\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} e^t = \mathbf{x}' = \begin{bmatrix} 2 & 3 \\ 4 & 1 \end{bmatrix} \mathbf{x} + \begin{bmatrix} e^t \\ 0 \end{bmatrix} = \begin{bmatrix} 2a_1 + 3a_2 + 1 \\ 4a_1 + a_2 \end{bmatrix} e^t$$

which gives

$$\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} 2a_1 + 3a_2 + 1 \\ 4a_1 + a_2 \end{bmatrix}.$$

Hence $a_1 = 0$ and $a_2 = -\frac{1}{3}$. So $\mathbf{x} = \begin{bmatrix} 0 \\ -\frac{1}{3} \end{bmatrix} e^t$.

Then we need to find a particular solution to $\mathbf{x}' = \begin{bmatrix} 2 & 3 \\ 4 & 1 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 0 \\ -10t-3 \end{bmatrix}$. We try the ansatz $\mathbf{x} = \mathbf{a}t + \mathbf{b} = \begin{bmatrix} a_1 t + b_1 \\ a_2 t + b_2 \end{bmatrix}$ for some $\mathbf{a}, \mathbf{b} \in \mathbb{R}^2$ and calculate that

$$\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \mathbf{x}' = \begin{bmatrix} 2 & 3 \\ 4 & 1 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 0 \\ t \end{bmatrix} = \begin{bmatrix} 2a_1 t + 2b_1 + 3a_2 t + 3b_2 \\ 4a_1 t + 4b_1 + a_2 t + b_2 - 10t - 3 \end{bmatrix}$$

which leads to

$$\begin{cases} 0 = 2a_1 + 3a_2 \\ a_1 = 2b_1 + 3b_2 \\ 0 = 4a_1 + a_2 - 10t \\ a_2 = 4b_1 + b_2 - 3. \end{cases}$$

The solution to this linear system is $\mathbf{a} = \begin{bmatrix} 3 \\ -2 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. Hence $\mathbf{x} = \begin{bmatrix} 3t \\ 1-2t \end{bmatrix}$.

Adding all of these together, we find that the general solution to the given ODE is

$$\boxed{\mathbf{x} = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{5t} + c_2 \begin{bmatrix} -3 \\ 4 \end{bmatrix} e^{-2t} + \begin{bmatrix} 0 \\ -\frac{1}{3} \end{bmatrix} e^t + \begin{bmatrix} 3t \\ 1-2t \end{bmatrix}.}$$

Method 3 – Variation of Parameters:

Consider

$$\mathbf{x}' = P(t)\mathbf{x} + \mathbf{g}(t) \tag{7.1}$$

where

- P and $\mathbf{g}$ are continuous for $\alpha < t < \beta$;
- there exists a fundamental matrix $\Psi(t)$ for the homogeneous system $\mathbf{x}' = P(t)\mathbf{x}$.

We know that the general solution to $\mathbf{x}' = P(t)\mathbf{x}$ is $\mathbf{x} = \Psi(t)\mathbf{c}$.

We guess that

$$\mathbf{x} = \Psi(t)\mathbf{u}(t)$$

is a solution to (7.1). Can we find $\mathbf{u}(t)$?

If $\mathbf{x} = \Psi\mathbf{u}$, we can calculate that

$$\cancel{\Psi'}\mathbf{u} + \Psi\mathbf{u}' = \mathbf{x}' = P\mathbf{x} + \mathbf{g} = \cancel{P\Psi}\mathbf{u} + \mathbf{g}. \tag{7.3}$$

But remember that

$$\Psi \text{ is a fundamental matrix for } \mathbf{x}' = P(t)\mathbf{x} \implies \Psi \text{ solves } \Psi' = P\Psi.$$

Hence (7.3) becomes

$$\Psi\mathbf{u}' = \mathbf{g}.$$

Therefore

$$\mathbf{u}' = \Psi^{-1}\mathbf{g}$$

and

$$\mathbf{u} = \int \Psi^{-1}\mathbf{g}.$$

Hence

$$\mathbf{x} = \Psi(t)\mathbf{u}(t) = \Psi(t)\int \Psi^{-1}(s)g(s)\,ds.$$

Remark. To solve $\mathbf{x}' = P(t)\mathbf{x} + \mathbf{g}(t)$, the method is

(i). Find a fundamental matrix Ψ for $\mathbf{x}' = P(t)\mathbf{x}$;

(ii). Calculate $\mathbf{x} = \Psi(t)\int \Psi^{-1}(s)g(s)\,ds$.

Example 7.24. Solve

$$\mathbf{x}' = \begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 2e^{-t} \\ 3t \end{bmatrix} = A\mathbf{x} + \mathbf{g}(t).$$

The solution of $\mathbf{x}' = \begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix} \mathbf{x}$ is

$$\mathbf{x}(t) = c_1 \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{-3t} + c_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{-t}.$$

So

$$\Psi(t) = \begin{bmatrix} e^{-3t} & e^{-t} \\ -e^{-3t} & e^{-t} \end{bmatrix}$$

is a fundamental matrix.

Then we calculate that

$$\Psi^{-1}(t) = \frac{1}{2e^{-4t}} \begin{bmatrix} e^{-t} & -e^{-t} \\ e^{-3t} & e^{-3t} \end{bmatrix} = \frac{1}{2} e^{4t} \begin{bmatrix} e^{-t} & -e^{-t} \\ e^{-3t} & e^{-3t} \end{bmatrix} = \begin{bmatrix} \frac{1}{2}e^{3t} & -\frac{1}{2}e^{3t} \\ \frac{1}{2}e^{t} & \frac{1}{2}e^{t} \end{bmatrix}$$

and

$$\begin{aligned} \int \Psi^{-1}(t)\mathbf{g}(t)\,dt &= \int \begin{bmatrix} \frac{1}{2}e^{3t} & -\frac{1}{2}e^{3t} \\ \frac{1}{2}e^{t} & \frac{1}{2}e^{t} \end{bmatrix} \begin{bmatrix} 2e^{-t} \\ 3t \end{bmatrix} dt \\ &= \int \begin{bmatrix} e^{2t} - \frac{3}{2}te^{3t} \\ 1 + \frac{3}{2}te^{t} \end{bmatrix} dt = \begin{bmatrix} \frac{1}{2}e^{2t} - \frac{1}{2}te^{3t} + \frac{1}{6}e^{3t} + c_1 \\ t + \frac{3}{2}te^{t} - \frac{3}{2}e^{t} + c_2 \end{bmatrix}. \end{aligned}$$

Therefore the solution to $\mathbf{x}' = A\mathbf{x} + g$ is

$$\begin{aligned} \mathbf{x} &= \Psi(t) \int \Psi^{-1}(s)\mathbf{g}(s)\,ds \\ &= \begin{bmatrix} e^{-3t} & e^{-t} \\ -e^{-3t} & e^{-t} \end{bmatrix} \begin{bmatrix} \frac{1}{2}e^{2t} - \frac{1}{2}te^{3t} + \frac{1}{6}e^{3t} + c_1 \\ t + \frac{3}{2}te^{t} - \frac{3}{2}e^{t} + c_2 \end{bmatrix} \\ &= c_1 \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{-3t} + c_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{-t} + \frac{1}{2} \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{-t} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} te^{-t} + \begin{bmatrix} 1 \\ 2 \end{bmatrix} t - \frac{1}{3} \begin{bmatrix} 4 \\ 5 \end{bmatrix}. \end{aligned}$$

Example 7.25. Solve

$$\mathbf{x}' = \begin{bmatrix} -4 & 2 \\ 2 & -1 \end{bmatrix} \mathbf{x} + \begin{bmatrix} t^{-1} \\ 2t^{-1} + 4 \end{bmatrix}$$

for $t > 0$.

The eigenvalues of $A = \begin{bmatrix} -4 & 2 \\ 2 & -1 \end{bmatrix}$ are $r_1 = 0$ and $r_2 = -5$; and the eigenvectors are $\xi^{(1)} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $\xi^{(2)} = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$. Thus

$$\Psi(t) = \begin{bmatrix} 1 & -2e^{-5t} \\ 2 & e^{-5t} \end{bmatrix}$$

is a fundamental matrix for $\mathbf{x}' = A\mathbf{x}$.

Using the formula $\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$ we calculate that

$$\Psi^{-1}(t) = \frac{1}{e^{-5t} + 4e^{-5t}} \begin{bmatrix} e^{-5t} & 2e^{-5t} \\ -2 & 1 \end{bmatrix} = \frac{1}{5} \begin{bmatrix} 1 & 2 \\ -2e^{5t} & e^{5t} \end{bmatrix}.$$

Then

$$\begin{aligned} \Psi^{-1}(t)\mathbf{g}(t) &= \frac{1}{5} \begin{bmatrix} 1 & 2 \\ -2e^{5t} & e^{5t} \end{bmatrix} \begin{bmatrix} t^{-1} \\ 2t^{-1} + 4 \end{bmatrix} \\ &= \frac{1}{5} \begin{bmatrix} t^{-1} + 4t^{-1} + 8 \\ -2t^{-1}e^{5t} + 2t^{-1}e^{5t} + 4e^{5t} \end{bmatrix} = \begin{bmatrix} t^{-1} + \frac{8}{5} \\ \frac{4}{5}e^{5t} \end{bmatrix} \end{aligned}$$

and

$$\int \Psi^{-1}(t)\mathbf{g}(t)\, dt = \int \begin{bmatrix} t^{-1} + \frac{8}{5} \\ \frac{4}{5}e^{5t} \end{bmatrix} dt = \begin{bmatrix} \ln t + \frac{8}{5}t + c_1 \\ \frac{4}{25}e^{5t} + c_2 \end{bmatrix}.$$

It follows that

$$\begin{aligned} \mathbf{x}(t) = \Psi(t) \int \Psi^{-1}(s)\mathbf{g}(s)\, ds &= \begin{bmatrix} 1 & -2e^{-5t} \\ 2 & e^{-5t} \end{bmatrix} \begin{bmatrix} \ln t + \frac{8}{5}t + c_1 \\ \frac{4}{25}e^{5t} + c_2 \end{bmatrix} \\ &= \begin{bmatrix} \ln t + \frac{8}{5}t - \frac{8}{25} + c_1 - 2c_2e^{-5t} \\ 2\ln t + \frac{16}{5}t + \frac{4}{25} + 2c_1 + c_2e^{-5t} \end{bmatrix} \\ &= c_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + c_2 \begin{bmatrix} -2 \\ 1 \end{bmatrix} e^{-5t} + \begin{bmatrix} 1 \\ 2 \end{bmatrix} \ln t + \frac{8}{5} \begin{bmatrix} 1 \\ 2 \end{bmatrix} t + \frac{4}{25} \begin{bmatrix} -2 \\ 1 \end{bmatrix}. \end{aligned}$$

Method 4 – The Laplace Transform:

First some notation: If $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$, then $\mathbf{X} = \mathcal{L}\left[\mathbf{x}\right] = \begin{bmatrix} \mathcal{L}\left[x_1\right] \\ \mathcal{L}\left[x_2\right] \\ \vdots \\ \mathcal{L}\left[x_n\right] \end{bmatrix}$.

Recall from Chapter 6 that $\mathcal{L}\left[y'\right]$ satisfies

$$\mathcal{L}\left[y'\right](s) = sY(s) - y(0).$$

It follows that:

Theorem 7.3.

$$\mathcal{L}\left[\mathbf{x}'\right](s) = s\mathbf{X}(s) - \mathbf{x}(0).$$

Example 7.26. Solve

$$\begin{cases} \mathbf{x}' = \begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 2e^{-t} \\ 3t \end{bmatrix} = A\mathbf{x} + \mathbf{g}(t), \\ \mathbf{x}(0) = \mathbf{0}. \end{cases}$$

Taking Laplace Transforms of the ODE gives

$$s\mathbf{X}(s) - \mathbf{x}(0) = A\mathbf{X}(s) + \mathbf{G}(s)$$

where $\mathbf{G}(s) = \mathcal{L}\left[\mathbf{g}\right](s) = \begin{bmatrix} \frac{2}{s+1} \\ \frac{3}{s^2} \end{bmatrix}$.

Thus

$$(sI - A)\mathbf{X} = \mathbf{G}$$

and

$$\mathbf{X} = (sI - A)^{-1}\mathbf{G}$$

where

$$(sI - A)^{-1} = \begin{bmatrix} s+2 & -1 \\ -1 & s+2 \end{bmatrix}^{-1} = \frac{1}{(s+1)(s+3)} \begin{bmatrix} s+2 & 1 \\ 1 & s+2 \end{bmatrix}.$$

So

$$\begin{aligned} \mathbf{X} &= (sI - A)^{-1}\mathbf{G} \\ &= \frac{1}{(s+1)(s+3)} \begin{bmatrix} s+2 & 1 \\ 1 & s+2 \end{bmatrix} \begin{bmatrix} \frac{2}{s+1} \\ \frac{3}{s^2} \end{bmatrix} \\ &= \begin{bmatrix} \frac{2(s+2)}{(s+1)^2(s+3)} + \frac{3}{s^2(s+1)(s+3)} \\ \frac{2}{(s+1)^2(s+3)} + \frac{3(s+2)}{s^2(s+1)(s+3)} \end{bmatrix}. \end{aligned}$$

When we take the inverse Laplace Transform of this, we find our solution

$$\mathbf{x} = \mathcal{L}^{-1}\left[\mathbf{X}\right] = \begin{bmatrix} 2 \\ 1 \end{bmatrix} e^{-t} - \frac{2}{3}\begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{-3t} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} te^{-t} + \begin{bmatrix} 1 \\ 2 \end{bmatrix} t - \frac{1}{3}\begin{bmatrix} 4 \\ 5 \end{bmatrix}.$$

Example 7.27. Solve

$$\begin{cases} 2x' + y' - y - t = 0 \\ x' + y' - t^2 = 0 \\ x(0) = 1 \\ y(0) = 0 \end{cases}$$

The ODEs above can be written as

$$\begin{cases} x' = y - t^2 + t \\ y' = -y + 2t^2 - t \end{cases}$$

(please check!).

If we write the problem in terms of matrices (using $\mathbf{x} = \begin{bmatrix} x \\ y \end{bmatrix}$) we have

$$\begin{cases} \mathbf{x}' = A\mathbf{x} + \mathbf{g} = \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix} \mathbf{x} + \begin{bmatrix} t - t^2 \\ 2t^2 - t \end{bmatrix} \\ \mathbf{x}(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}. \end{cases}$$

Taking the Laplace transform of the ODE gives

$$(sI - A)\,\mathbf{X}(s) = \mathbf{x}(0) + \mathbf{G}(s)$$
$$\begin{bmatrix} s & -1 \\ 0 & s+1 \end{bmatrix} \mathbf{X}(s) = \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \begin{bmatrix} \frac{1}{s^2} - \frac{2}{s^3} \\ \frac{4}{s^3} - \frac{1}{s^2} \end{bmatrix}$$

Thus

$$\begin{aligned} \mathbf{X}(s) &= \frac{1}{s(s+1)} \begin{bmatrix} s+1 & 1 \\ 0 & s \end{bmatrix} \frac{1}{s^3} \begin{bmatrix} s^3 + s - 2 \\ 4 - s \end{bmatrix} \\ &= \frac{1}{s^4(s+1)} \begin{bmatrix} s^4 + s^3 + s^2 - 2s + 2 \\ 4s - s^2 \end{bmatrix}. \end{aligned}$$

Note that

$$\frac{s^4 + s^3 + s^2 - 2s + 2}{s^4(s+1)} = \frac{5}{s+1} - 4\frac{1}{s} + 5\frac{1}{s^2} - 4\frac{1}{s^3} + 2\frac{1}{s^4}$$

and

$$\frac{4s - s^2}{s^4(s+1)} = -5\frac{1}{s+1} + 5\frac{1}{s} - 5\frac{1}{s^2} + 4\frac{1}{s^3}$$

(please check!).

It follows that

$$\mathcal{L}^{-1}\left(\frac{s^4 + s^3 + s^2 - 2s + 2}{s^4(s+1)}\right) = 5e^{-t} - 4 + 5t - 2t^2 + \frac{1}{3}t^3$$

and

$$\mathcal{L}^{-1}\left(\frac{4s - s^2}{s^4(s+1)}\right) = -5e^{-t} + 5 - 5t + 2t^2.$$

Therefore the solution to the initial value problem is

$$\boxed{\mathbf{x}(t) = \begin{bmatrix} 5e^{-t} - 4 + 5t - 2t^2 + \frac{1}{3}t^3 \\ -5e^{-t} + 5 - 5t + 2t^2 \end{bmatrix}}.$$