

Lecture 9

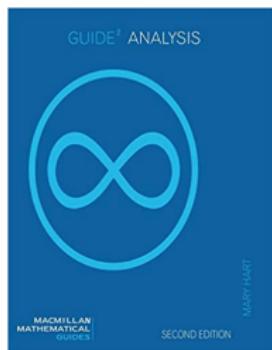
■ 9.1 Sequences

Sequences

9.1 Sequences



A better book



Mary Hart,
Guide to Analysis,
MacMillan.

9.1 Sequences



A(n *infinite*) *sequence* is an endless list of real numbers

$$a_1, a_2, a_3, a_4, \dots$$

in a particular order.

9.1 Sequences



A(n *infinite*) *sequence* is an endless list of real numbers

$$a_1, a_2, a_3, a_4, \dots$$

in a particular order. Each of the a_j represents a number. These are the *terms* or the sequence. For example, the sequence

$$2, 4, 6, 8, 10, 12, 14, 16, \dots, 2n, \dots$$

has first term $a_1 = 2$, second term $a_2 = 4$ and n^{th} term $a_n = 2n$.

9.1 Sequences



We write $(a_n)_{n=1}^{\infty}$ – or sometimes just (a_n) – to denote to the sequence

$$a_1, a_2, a_3, a_4, a_5, a_6, a_7, \dots$$

9.1 Sequences



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If we remove the first four terms, we would get the sequence

$$a_5, a_6, a_7, a_8, a_9, a_{10}, a_{11}, \dots$$

which we denote by $(a_n)_{n=5}^{\infty}$.

9.1 Sequences

Example

Let $b_n = \frac{1}{n}$ for all $n \in \mathbb{N}$. Then $(b_n)_{n=1}^{\infty}$ is the sequence

$$1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \frac{1}{6}, \dots$$

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Example

The sequence $((-1)^n \frac{1}{n})_{n=1}^{\infty}$ is

$$-1, \frac{1}{2}, -\frac{1}{3}, \frac{1}{4}, -\frac{1}{5}, \frac{1}{6}, \dots$$

9.1 Sequences

Example

Let $x_n = \cos n\pi$ for all $n \in \mathbb{N}$. Then $(x_n)_{n=1}^{\infty}$ is the sequence

$$-1, 1, -1, 1, -1, 1, -1, 1, \dots$$

9.1 Sequences

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$$-1, 1, -1, 1, -1, 1, -1, 1, \dots$$

Example

The sequence $\left(\frac{1}{n^2}\right)_{n=5}^{\infty}$ is

$$\frac{1}{25}, \frac{1}{36}, \frac{1}{49}, \frac{1}{64}, \frac{1}{81}, \dots$$

9.1 Sequences

Example (The Fibonacci Numbers)

Let $a_1 = 1$, $a_2 = 1$ and $a_n = a_{n-2} + a_{n-1}$ for all $n > 2$.

$$1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, \dots$$

9.1 Sequences

Example (The Fibonacci Numbers)

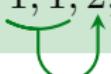
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2
1 1

9.1 Sequences

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3	2
1	1

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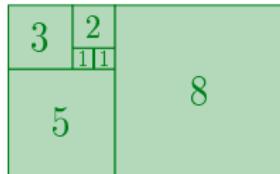



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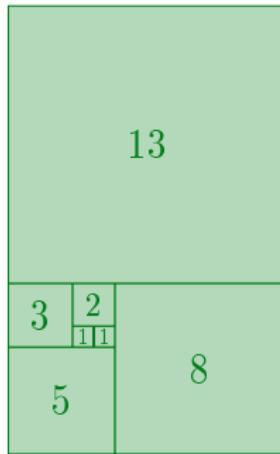



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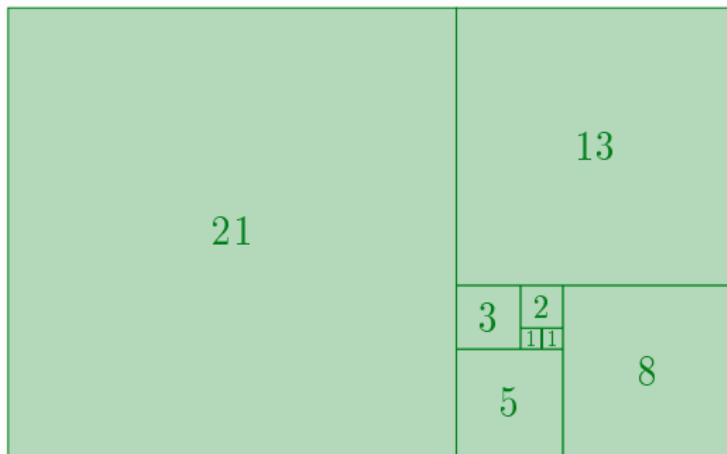


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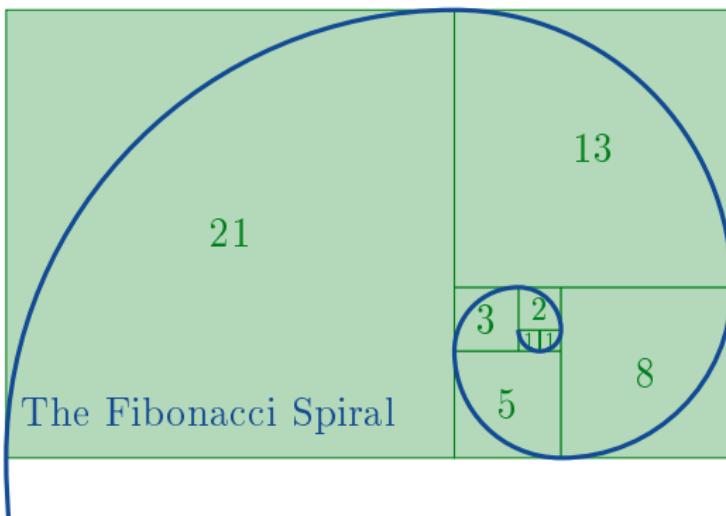


9.1 Sequences

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9.1 Sequences



Let (a_n) be a sequence. Note that for every number $n \in \mathbb{N}$, we have a number $a_n \in \mathbb{R}$. So we have a function $\mathbb{N} \rightarrow \mathbb{R}$. We use this idea to formally define a sequence:

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Definition

A *sequence* of real numbers is a function $a : \mathbb{N} \rightarrow \mathbb{R}$. We could write $a_n := a(n)$ if we wanted to.

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Definition

$\mathbb{R}^{\mathbb{N}} := \{f : \mathbb{N} \rightarrow \mathbb{R}\} = \{\text{all sequences of real numbers}\}.$

9.1 Sequences



Remark

One notation for the sequence $(\frac{1}{n})_{n=1}^{\infty}$ is

$$\left(1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \frac{1}{6}, \frac{1}{7}, \frac{1}{8}, \frac{1}{9}, \dots\right).$$

Just as \mathbb{R}^3 is the set of all vectors (x, y, z) , we might expect the set of all sequences to be denoted \mathbb{R}^{∞}

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Just as \mathbb{R}^3 is the set of all vectors (x, y, z) , we might expect the set of all sequences to be denoted \mathbb{R}^{∞} – but what is “ ∞ ”? ∞ has many different types. The notation $\mathbb{R}^{\mathbb{N}}$ is more precise.

More generally, B^A denotes the set of all functions from A to B , but we won’t need this in this course.

9.1 Sequences



Remark

Even though sequences are really functions, we will usually think of them as lists of numbers.

For sequences, the important things are:

- the order in which the numbers appear;

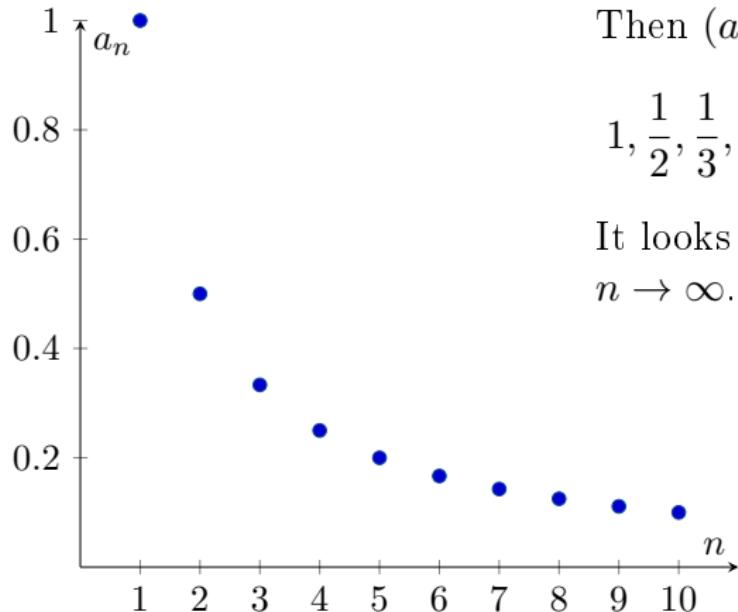
and

- the behaviour of the terms as $n \rightarrow \infty$.

9.1 Sequences

Example

Let $a_n = \frac{1}{n}$ for all $n \in \mathbb{N}$.



Then $(a_n)_{n=1}^{\infty}$ is the sequence

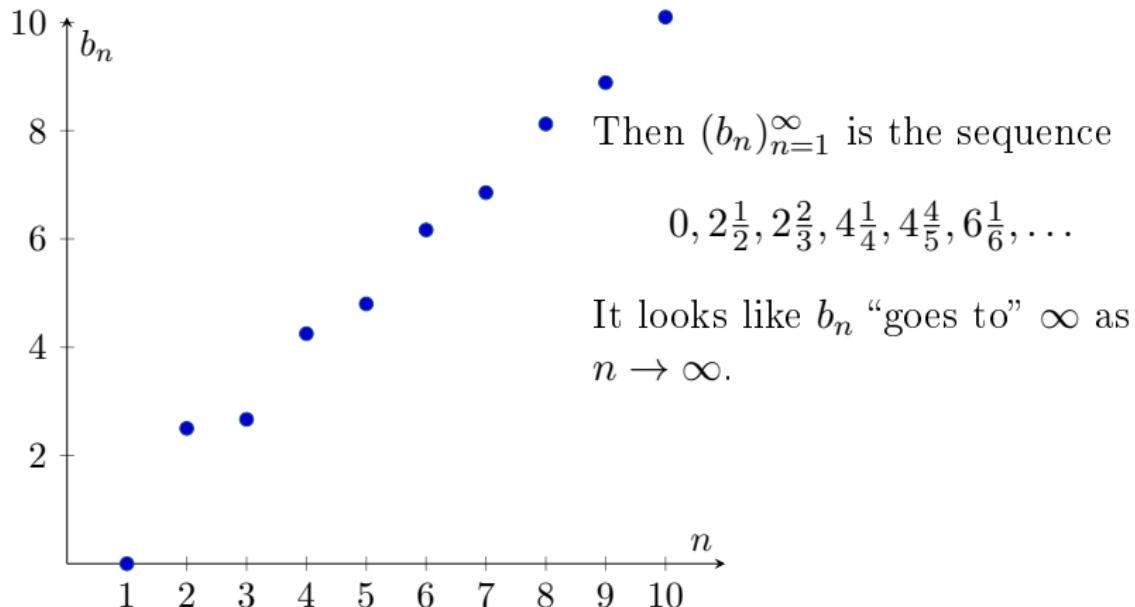
$$1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \frac{1}{6}, \frac{1}{7}, \frac{1}{8}, \frac{1}{9}, \frac{1}{10}, \dots$$

It looks like a_n “goes to” 0 as $n \rightarrow \infty$.

9.1 Sequences

Example

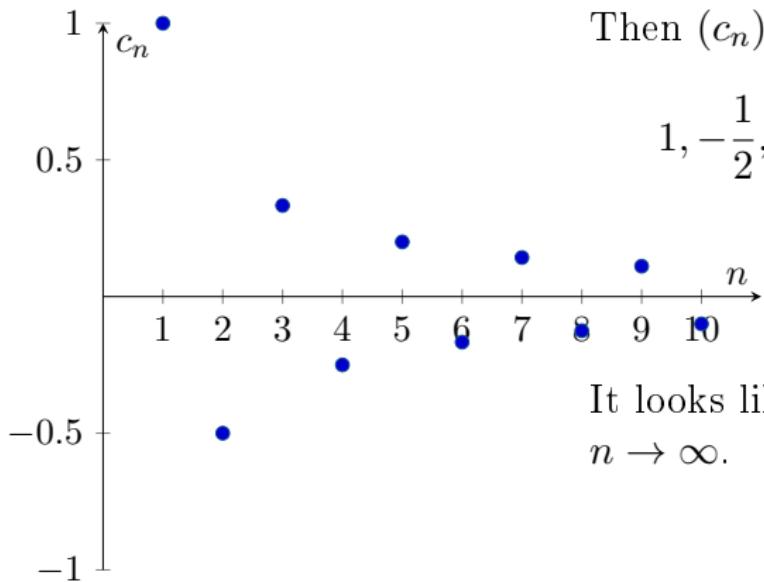
Let $b_n = n + (-1)^n \frac{1}{n}$ for all $n \in \mathbb{N}$.



9.1 Sequences

Example

Let $c_n = \frac{(-1)^{n+1}}{n}$ for all $n \in \mathbb{N}$.



Then $(c_n)_{n=1}^{\infty}$ is the sequence

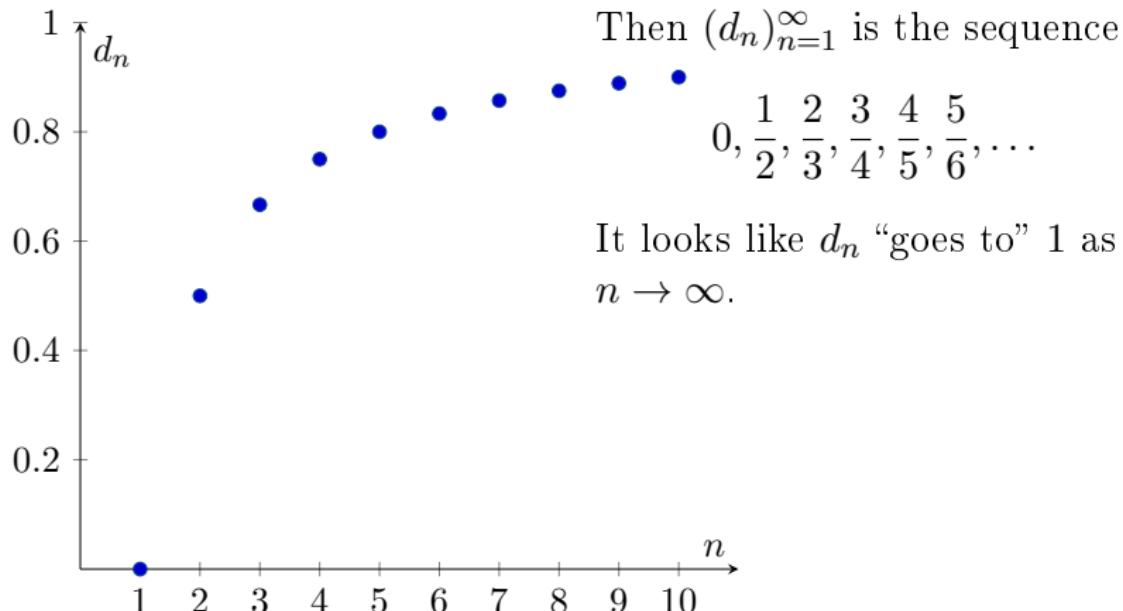
$$1, -\frac{1}{2}, \frac{1}{3}, -\frac{1}{4}, \frac{1}{5}, -\frac{1}{6}, \dots$$

It looks like c_n “goes to” 0 as $n \rightarrow \infty$.

9.1 Sequences

Example

Let $d_n = 1 - \frac{1}{n}$ for all $n \in \mathbb{N}$.



9.1 Sequences



Definition

The *floor function*, $\lfloor \cdot \rfloor : \mathbb{R} \rightarrow \mathbb{Z}$, is defined by

$$\lfloor x \rfloor = \max\{p \in \mathbb{Z} : p \leq x\}.$$

For example

$$\lfloor 3.79 \rfloor = 3$$

$$\lfloor 4 \rfloor = 4$$

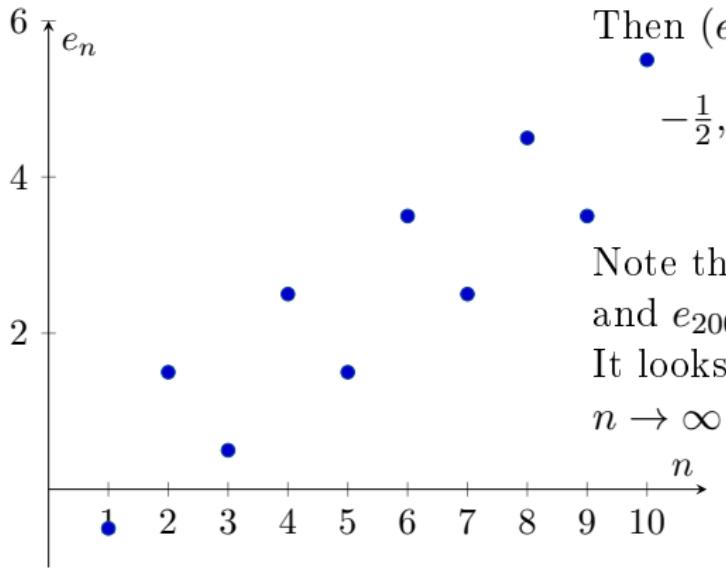
$$\lfloor -3.79 \rfloor = -4$$

$$\lfloor -4 \rfloor = -4$$

9.1 Sequences

Example

Let $e_n = \left\lfloor \frac{n}{2} \right\rfloor + \frac{(-1)^n}{2}$ for all $n \in \mathbb{N}$.



Then $(e_n)_{n=1}^{\infty}$ is the sequence

$$-\frac{1}{2}, \frac{3}{2}, \frac{1}{2}, \frac{5}{2}, \frac{3}{2}, \frac{7}{2}, \frac{5}{2}, \frac{9}{2}, \dots$$

Note that $e_{2000} = 1000 + \frac{1}{2}$ and $e_{2000000} = 1000000 + \frac{1}{2}$. It looks like e_n “goes to” ∞ as $n \rightarrow \infty$.

9.1 Sequences



Remark

In these last five examples, we have said “looks like” and “goes to” a lot. But what does this mean mathematically? We need to be more precise.

What does “goes to ∞ ” really mean?

9.1 Sequences



Remark

In these last five examples, we have said “looks like” and “goes to” a lot. But what does this mean mathematically? We need to be more precise.

What does “goes to ∞ ” really mean?

It doesn’t mean “gets bigger” because $d_n = 1 - \frac{1}{n}$ gets bigger, but we think that d_n “goes to” 1.

9.1 Sequences

Remark

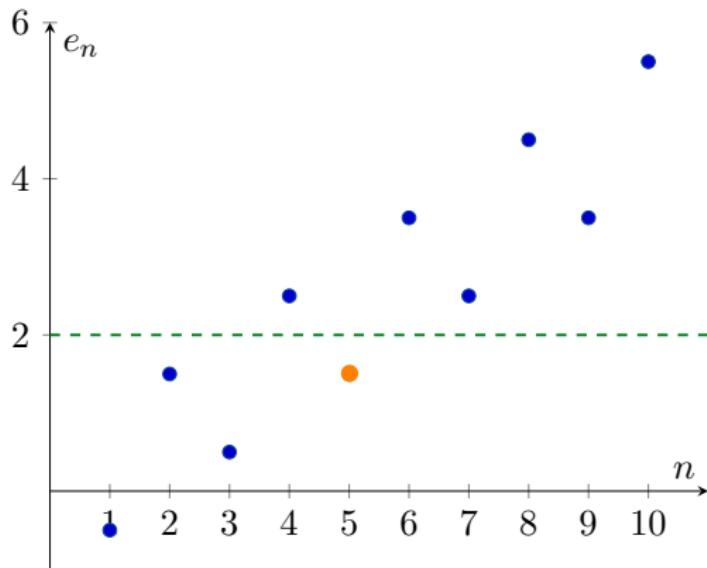
In these last five examples, we have said “looks like” and “goes to” a lot. But what does this mean mathematically? We need to be more precise.

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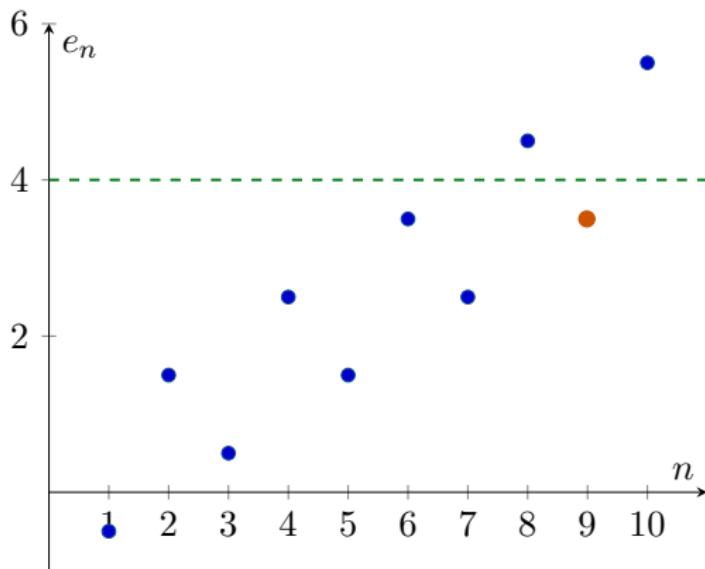
Furthermore, we think that $e_n = \lfloor \frac{n}{2} \rfloor + \frac{(-1)^n}{2}$ “goes to” ∞ , but e_n gets bigger, smaller, bigger, smaller, bigger, smaller, bigger, smaller,

9.1 Sequences



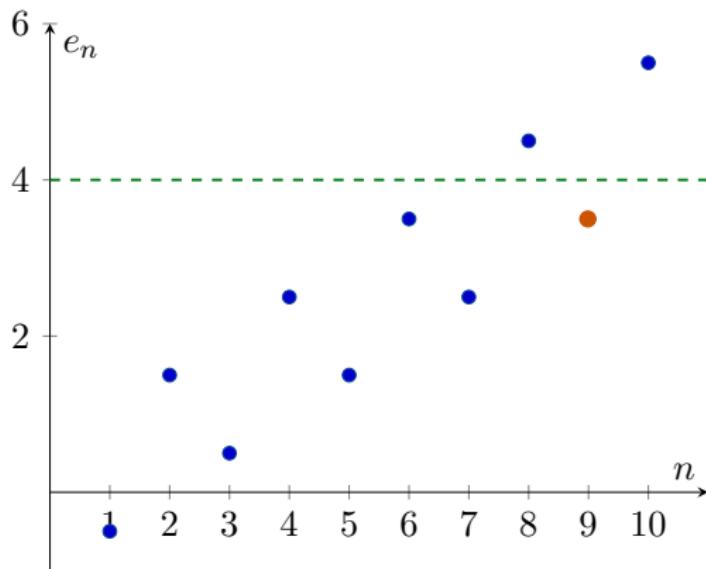
Notice that if we draw a green line at height 2, then 4 points at the start of the sequence are underneath the line and all of the rest of the sequence is above the line.

9.1 Sequences



If we draw a green line at height 4, then a finite number of points at the start of the sequence are underneath the line and all of the rest of the sequence is above the line.

9.1 Sequences



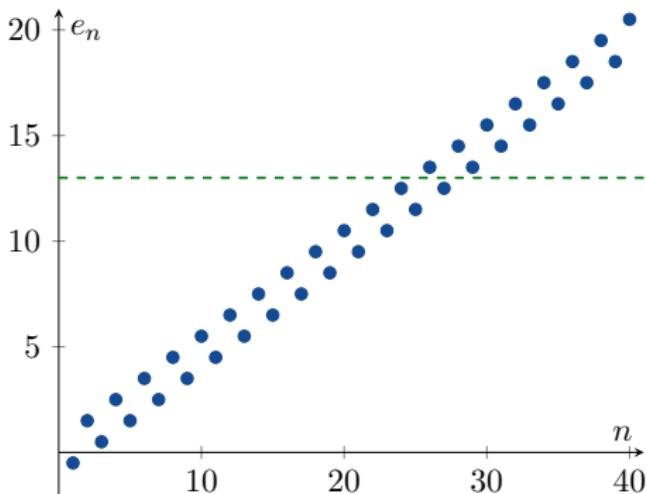
If we draw a green line at height 4, then a finite number of points at the start of the sequence are underneath the line and all of the rest of the sequence is above the line.

Now we are getting somewhere.

9.1 Sequences

\forall = “for all”

\exists = “there exists”

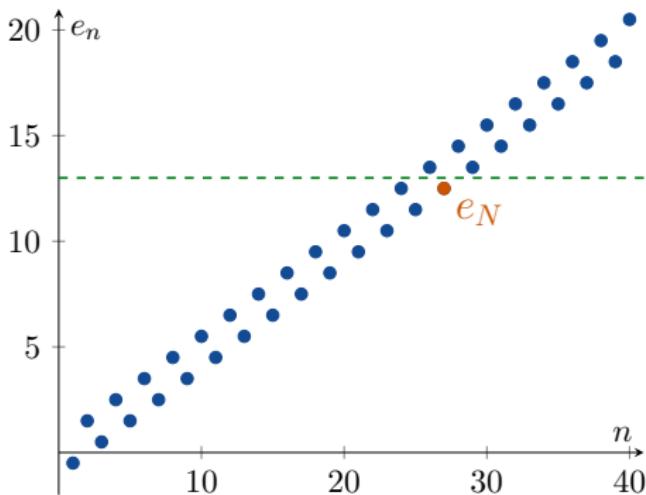


In general, if I choose any number $A \in \mathbb{R}$ and draw a green line at height A , then there will be a finite number of points underneath the line and an infinite number of points above the line.

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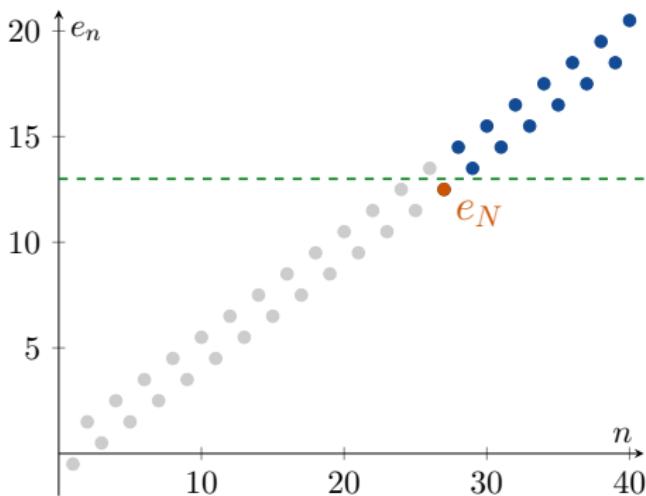


One of the points under the green line must be the last one.
Call this point e_N .

9.1 Sequences

\forall = “for all”

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One of the points under the green line must be the last one.
Call this point e_N . This means that

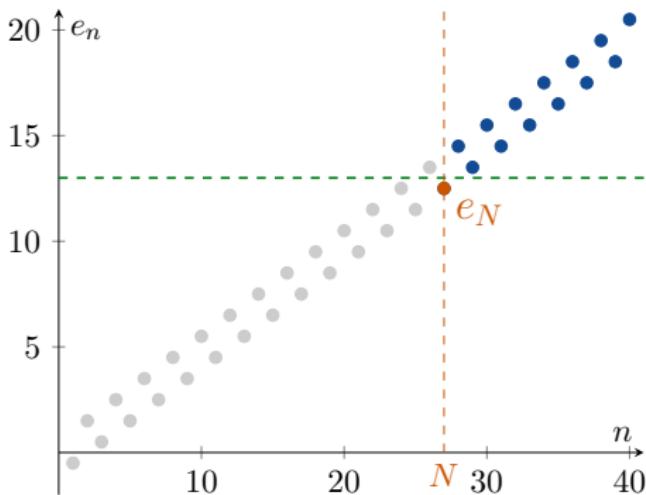
$$e_{N+1}, e_{N+2}, e_{N+3}, e_{N+4}, e_{N+5}, \dots$$

are all above the green line.

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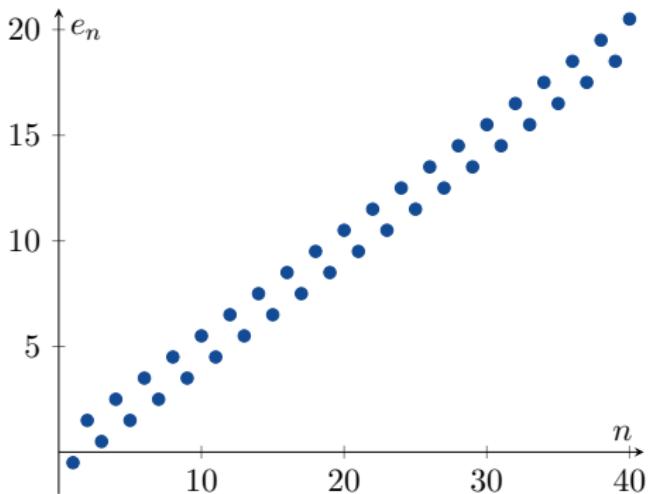
$$e_{N+1}, e_{N+2}, e_{N+3}, e_{N+4}, e_{N+5}, \dots$$

are all above the green line. In other words, $\exists N \in \mathbb{N}$ such that $e_n > A$ for all $n > N$.

9.1 Sequences

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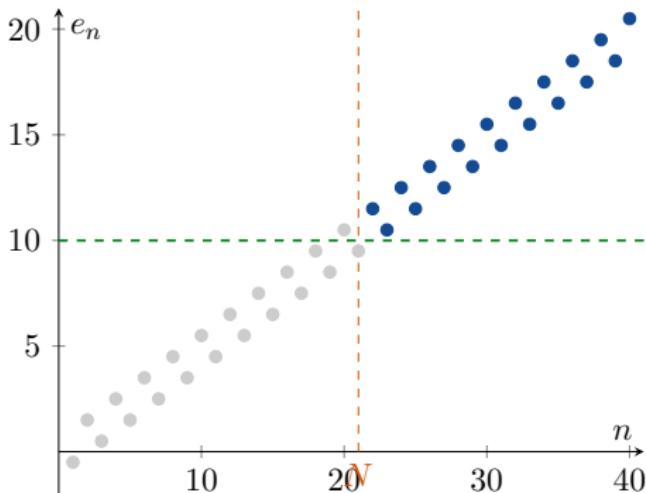


Obviously the number N will depend on A . We will write $N = N(A)$ so that we remember this.

9.1 Sequences

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If we choose $A = 10$, then note that

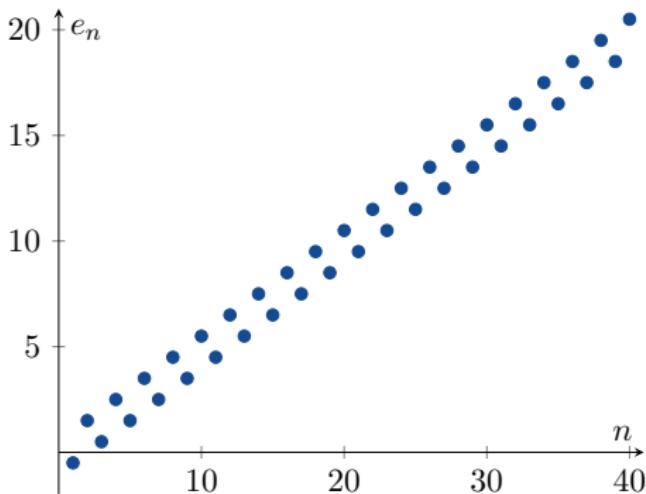
$$n > 21 \implies e_n = \left\lfloor \frac{n}{2} \right\rfloor + \frac{(-1)^n}{2} \geq \left\lfloor \frac{22}{2} \right\rfloor + \frac{(-1)^n}{2} = 11 \pm \frac{1}{2} > 10 = A$$

which means that we can choose $N(10) = 21$.

9.1 Sequences

\forall = “for all”

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In fact, we don't have to choose the “best” N – any N which works is good enough. So if we wanted to, we could choose $N(10) = 1000000$ and the calculation above still works. If $n > 1000000$, then $e_n > 10$ (check it!!!).

9.1 Sequences



If we choose $A = 100$, then $e_n > 100 = A$ for all $n > 201$ (you check!), so we can choose $N(100) = 201$.

9.1 Sequences

If we choose $A = 100$, then $e_n > 100 = A$ for all $n > 201$ (you check!), so we can choose $N(100) = 201$.

In general, for any given $A > 0$, we can always find an $N = N(A)$ for the sequence (e_n) . If we choose

$N = \text{"the smallest integer such that } N > 2A + 3\text{"}$,

then

$$\begin{aligned} n > N &\implies n \geq 2A + 4 \\ &\implies e_n = \left\lfloor \frac{n}{2} \right\rfloor + \frac{(-1)^n}{2} \geq \lfloor A + 2 \rfloor - \frac{1}{2} > A. \end{aligned}$$

9.1 Sequences



Definition

A sequence of real numbers (a_n) *diverges to infinity* iff for all $A > 0$, there exists $N = N(A) \in \mathbb{N}$ such that

$$n > N \implies a_n > A.$$

We write " $a_n \rightarrow \infty$ as $n \rightarrow \infty$ " or " $\lim_{n \rightarrow \infty} a_n = \infty$ " in this case.

9.1 Sequences

$$\forall A > 0, \exists N \in \mathbb{N} \text{ s.t. } n > N \implies a_n > A$$



Example

Let $a_n = \sqrt{n}$ for all $n \in \mathbb{N}$. Show that $a_n \rightarrow \infty$ as $n \rightarrow \infty$.

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Let $A > 0$. Choose $N \in \mathbb{N}$ such that $N \geq A^2$. Then for all $n \in \mathbb{N}$,

$$n > N \implies a_n = \sqrt{n} > \sqrt{N} \geq A.$$

Therefore $a_n \rightarrow \infty$ as $n \rightarrow \infty$.

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$$\forall A > 0, \exists N \in \mathbb{N} \text{ s.t. } n > N \implies b_n > A$$



Example

Let $b_n = \ln n$ for all $n \in \mathbb{N}$. Show that $b_n \rightarrow \infty$ as $n \rightarrow \infty$.

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$$\forall A > 0, \exists N \in \mathbb{N} \text{ s.t. } n > N \implies b_n > A$$



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Let $A > 0$. Choose $N \in \mathbb{N}$ such that _____ . Then for all $n \in \mathbb{N}$,

$$n > N \implies b_n = \ln n > \underline{\hspace{2cm}} A.$$

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$$n > N \implies$$

Therefore $b_n \rightarrow \infty$ as n

scrap paper

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all

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scrap paper

$$\ln n > A$$

$$n > e^A$$

all

We can choose $N \geq e^A$

9.1 Sequences

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Example

Let $b_n = \ln n$ for all $n \in \mathbb{N}$. Show that $b_n \rightarrow \infty$ as $n \rightarrow \infty$.

Let $A > 0$. Choose $N \in \mathbb{N}$ such that $N \geq e^A$. Then for all $n \in \mathbb{N}$,

$$n > N \implies b_n = \ln n > \underline{\ln N \geq \ln e^A = A}.$$

Therefore $b_n \rightarrow \infty$ as $n \rightarrow \infty$.

9.1 Sequences

$$\forall A > 0, \exists N \in \mathbb{N} \text{ s.t. } n > N \implies c_n > A$$



Example

Let

$$c_n = \begin{cases} \frac{n^2\sqrt{n}+n^2+1}{n^2-43} & n \geq 7 \\ 0 & 1 \leq n \leq 6 \end{cases}$$

for all $n \in \mathbb{N}$. Show that $c_n \rightarrow \infty$ as $n \rightarrow \infty$.

9.1 Sequences

$$\forall A > 0, \exists N \in \mathbb{N} \text{ s.t. } n > N \implies c_n > A$$



Example

Let

$$c_n = \begin{cases} \frac{n^2\sqrt{n} + n^2 + 1}{n^2 - 43} & n \geq 7 \\ 0 & 1 \leq n \leq 6 \end{cases}$$

for all $n \in \mathbb{N}$. Show that $c_n \rightarrow \infty$ as $n \rightarrow \infty$.

First note that if $n \geq 7$ then

$$c_n = \frac{n^2\sqrt{n} + n^2 + 1}{n^2 - 43} > \frac{n^2\sqrt{n}}{n^2 - 43} > \frac{n^2\sqrt{n}}{n^2} = \sqrt{n}.$$

9.1 Sequences

$$\forall A > 0, \exists N \in \mathbb{N} \text{ s.t. } n > N \implies c_n > A$$



Example

Let

$$c_n = \begin{cases} \frac{n^2\sqrt{n} + n^2 + 1}{n^2 - 43} & n \geq 7 \\ 0 & 1 \leq n \leq 6 \end{cases}$$

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First note that if $n \geq 7$ then

$$c_n = \frac{n^2\sqrt{n} + n^2 + 1}{n^2 - 43} > \frac{n^2\sqrt{n}}{n^2 - 43} > \frac{n^2\sqrt{n}}{n^2} = \sqrt{n}.$$

Let $A > 0$. Choose $N \in \mathbb{N}$ such that $N \geq \max\{A^2, 7\}$. Then for all $n \in \mathbb{N}$,

$$n > N \implies c_n > \sqrt{n} > \sqrt{N} \geq A.$$

Therefore $c_n \rightarrow \infty$ as $n \rightarrow \infty$.

9.1 Sequences



Remark

Remember that we don't need to find the “best” or smallest N . We only need to find one which works.

Remark

In two of the last three examples it was easy to find an N . In the previous example, we used an inequality first so that finding an N was easier.

9.1 Sequences

Theorem

Let (a_n) and (b_n) be two sequences of real numbers such that

$$a_n \geq b_n$$

for all $n > N_0 \in \mathbb{N}$. If $b_n \rightarrow \infty$ as $n \rightarrow \infty$, then $a_n \rightarrow \infty$ as $n \rightarrow \infty$.

9.1 Sequences

Theorem

Let (a_n) and (b_n) be two sequences of real numbers such that

$$a_n \geq b_n$$

for all $n > N_0 \in \mathbb{N}$. If $b_n \rightarrow \infty$ as $n \rightarrow \infty$, then $a_n \rightarrow \infty$ as $n \rightarrow \infty$.

Proof.

Let $A > 0$. Since $b_n \rightarrow \infty$ as $n \rightarrow \infty$, $\exists N_1 \in \mathbb{N}$ such that

$$n > N_1 \implies b_n > A.$$

9.1 Sequences

Theorem

Let (a_n) and (b_n) be two sequences of real numbers such that

$$a_n \geq b_n$$

for all $n > N_0 \in \mathbb{N}$. If $b_n \rightarrow \infty$ as $n \rightarrow \infty$, then $a_n \rightarrow \infty$ as $n \rightarrow \infty$.

Proof.

Let $A > 0$. Since $b_n \rightarrow \infty$ as $n \rightarrow \infty$, $\exists N_1 \in \mathbb{N}$ such that

$$n > N_1 \implies b_n > A.$$

Choose $N = \max\{N_0, N_1\}$.

9.1 Sequences

Theorem

Let (a_n) and (b_n) be two sequences of real numbers such that

$$a_n \geq b_n$$

for all $n > N_0 \in \mathbb{N}$. If $b_n \rightarrow \infty$ as $n \rightarrow \infty$, then $a_n \rightarrow \infty$ as $n \rightarrow \infty$.

Proof.

Let $A > 0$. Since $b_n \rightarrow \infty$ as $n \rightarrow \infty$, $\exists N_1 \in \mathbb{N}$ such that

$$n > N_1 \implies b_n > A.$$

Choose $N = \max\{N_0, N_1\}$. Then

$$n > N \implies a_n \geq b_n > A.$$

9.1 Sequences

Example

Let $a_n = n^2 + n \cos n\pi$ and $b_n = \frac{1}{2}n^2$.

Let $A > 0$. Choose $N \geq \sqrt{2A}$. Then

$$n > N \implies b_n = \frac{1}{2}n^2 > \frac{1}{2}N^2 \geq A.$$

Hence $b_n \rightarrow \infty$ as $n \rightarrow \infty$.

9.1 Sequences

Example

Let $a_n = n^2 + n \cos n\pi$ and $b_n = \frac{1}{2}n^2$.

Let $A > 0$. Choose $N \geq \sqrt{2A}$. Then

$$n > N \implies b_n = \frac{1}{2}n^2 > \frac{1}{2}N^2 \geq A.$$

Hence $b_n \rightarrow \infty$ as $n \rightarrow \infty$.

Moreover, if $n \geq 2$ then

$$a_n = n^2 + n \cos n\pi = n^2 + n(-1)^n \geq n^2 - n \geq n^2 - \frac{1}{2}n^2 = \frac{1}{2}n^2 = b_n.$$

Therefore $a_n \rightarrow \infty$ as $n \rightarrow \infty$ by the theorem.

9.1 Sequences



Example

Let $a_n := \frac{n^2 + \sqrt{n}}{n + \cos n}$. Show that $a_n \rightarrow \infty$ as $n \rightarrow \infty$.

If $n \geq 2$, then

$$a_n > \frac{n^2}{n + \cos n} \geq \frac{n^2}{n + 1} > \frac{n^2}{n + n} = \frac{1}{2}n.$$

Now let $b_n = \frac{1}{2}n$. Since $b_n \rightarrow \infty$ as $n \rightarrow \infty$ (you check!!!) and since $a_n > b_n$ for all $n \geq 2$, it follows by the theorem that $a_n \rightarrow \infty$ as $n \rightarrow \infty$.

9.1 Sequences



Definition

A sequence of real numbers a_n *diverges to minus infinity* ($a_n \rightarrow -\infty$ as $n \rightarrow \infty$) iff for all $A > 0$, there exists $N = N(A) \in \mathbb{N}$ such that

$$n > N \implies a_n < -A.$$

9.1 Sequences



Remark

$$a_n \rightarrow -\infty \text{ as } n \rightarrow \infty \iff -a_n \rightarrow \infty \text{ as } n \rightarrow \infty.$$

(you prove)

9.1 Sequences

Theorem

Let (a_n) and (b_n) be two sequences of real numbers such that

$$a_n \leq b_n$$

for all $n > N_0 \in \mathbb{N}$. If $b_n \rightarrow -\infty$ as $n \rightarrow \infty$, then $a_n \rightarrow -\infty$ as $n \rightarrow \infty$.

9.1 Sequences

Theorem

Let (a_n) and (b_n) be two sequences of real numbers such that

$$a_n \leq b_n$$

for all $n > N_0 \in \mathbb{N}$. If $b_n \rightarrow -\infty$ as $n \rightarrow \infty$, then $a_n \rightarrow -\infty$ as $n \rightarrow \infty$.

Proof.

Since $a_n \leq b_n \forall n \geq N_0$, it follows that $-a_n \geq -b_n \forall n \geq N_0$.

Thus

$$\begin{aligned} b_n \rightarrow -\infty \text{ as } n \rightarrow \infty &\implies -b_n \rightarrow \infty \text{ as } n \rightarrow \infty \\ &\implies -a_n \rightarrow \infty \text{ as } n \rightarrow \infty \\ &\implies a_n \rightarrow -\infty \text{ as } n \rightarrow \infty. \end{aligned}$$



9.1 Sequences

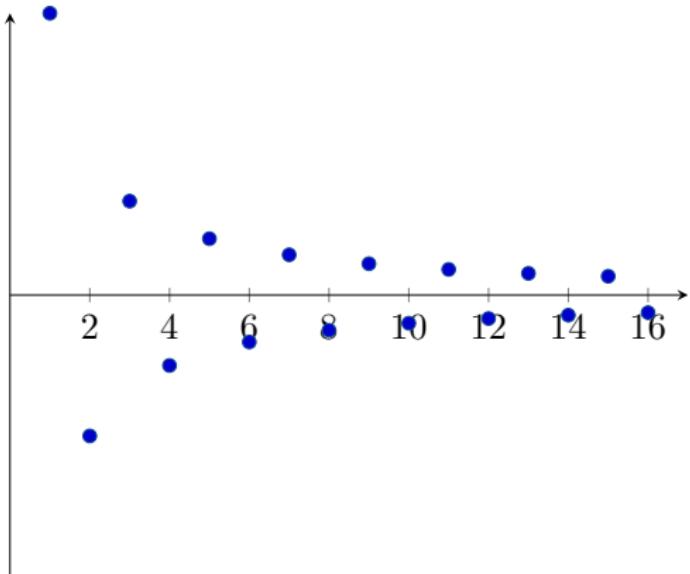
Now let $y_n = \frac{(-1)^{n+1}}{n}$ for all $n \in \mathbb{N}$. Then $(y_n)_{n=1}^{\infty}$ is the sequence

$$1, -\frac{1}{2}, \frac{1}{3}, -\frac{1}{4}, \frac{1}{5}, -\frac{1}{6}, \frac{1}{7}, -\frac{1}{8}, \frac{1}{9}, \dots$$

9.1 Sequences

Now let $y_n = \frac{(-1)^{n+1}}{n}$ for all $n \in \mathbb{N}$. Then $(y_n)_{n=1}^{\infty}$ is the sequence

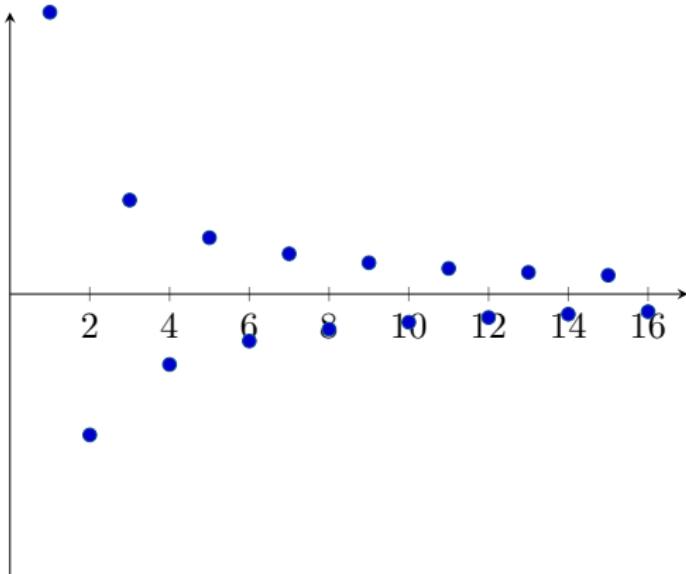
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9.1 Sequences

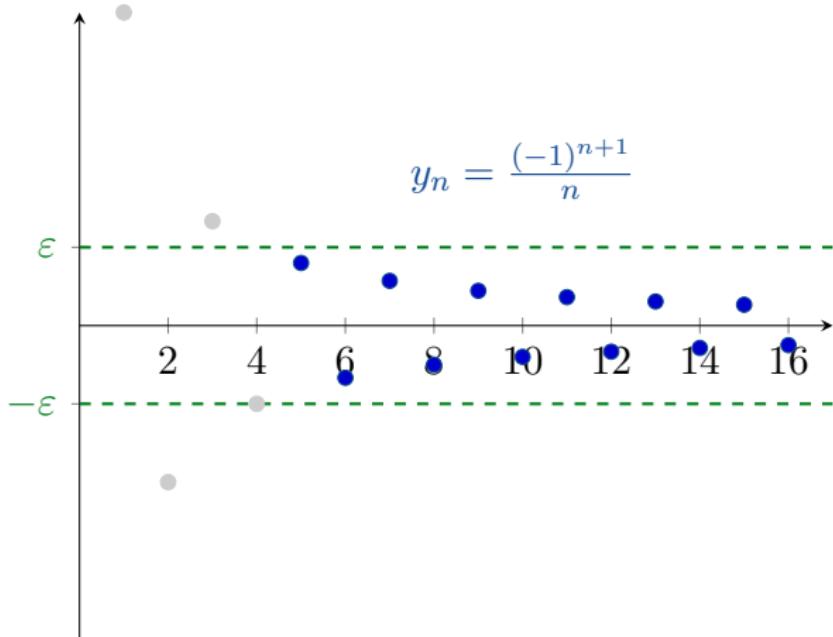
Now let $y_n = \frac{(-1)^{n+1}}{n}$ for all $n \in \mathbb{N}$. Then $(y_n)_{n=1}^{\infty}$ is the sequence

$$1, -\frac{1}{2}, \frac{1}{3}, -\frac{1}{4}, \frac{1}{5}, -\frac{1}{6}, \frac{1}{7}, -\frac{1}{8}, \frac{1}{9}, \dots$$



It “looks like” y_n “goes to” 0 as $n \rightarrow \infty$. But what does this mean? How can we be more precise?

9.1 Sequences



If we draw green lines at heights ε and $-\varepsilon$, then (apart from a finite number of points) the sequence will be between the two green lines.

9.1 Sequences



$$y_n = \frac{(-1)^{n+1}}{n}$$

E.g. Let $\varepsilon = \frac{1}{100}$. Then $-\frac{1}{100} < y_n < \frac{1}{100}$ for all $n > 100$.

9.1 Sequences



$$y_n = \frac{(-1)^{n+1}}{n}$$

E.g. Let $\varepsilon = \frac{1}{100}$. Then $-\frac{1}{100} < y_n < \frac{1}{100}$ for all $n > 100$.

In general: If we choose $N \geq \frac{1}{\varepsilon}$, then

$$n > N \implies -\varepsilon < y_n < \varepsilon.$$

In other words, if $N \geq \frac{1}{\varepsilon}$ then

$$n > N \implies |y_n| < \varepsilon.$$

9.1 Sequences

Definition

A sequence of real numbers a_n *tends to zero* ($a_n \rightarrow 0$ as $n \rightarrow \infty$) iff for all $\varepsilon > 0$, there exists $N = N(\varepsilon) \in \mathbb{N}$ such that

$$n > N \implies |a_n| < \varepsilon.$$

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A sequence of real numbers a_n *tends to zero* ($a_n \rightarrow 0$ as $n \rightarrow \infty$) iff for all $\varepsilon > 0$, there exists $N = N(\varepsilon) \in \mathbb{N}$ such that

$$n > N \implies |a_n| < \varepsilon.$$

Remark

In Mathematics; we usually use ε for arbitrarily small numbers and a capital letter (e.g. A) for arbitrarily large numbers.

9.1 Sequences

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$$n > N \implies |a_n| < \varepsilon.$$

Remark

In Mathematics; we usually use ε for arbitrarily small numbers and a capital letter (e.g. A) for arbitrarily large numbers.

Definition

A sequence of real numbers a_n is called a *null sequence* iff $a_n \rightarrow 0$ as $n \rightarrow \infty$.

9.1 Sequences

$$\forall \varepsilon > 0, \exists N \in \mathbb{N} \text{ s.t. } n > N \implies |a_n| < \varepsilon$$



Example

Let $a_n = n^{-7}$ for all $n \in \mathbb{N}$. Show that (a_n) is a null sequence.

We have to show that $a_n \rightarrow 0$ as $n \rightarrow \infty$.

9.1 Sequences

$$\forall \varepsilon > 0, \exists N \in \mathbb{N} \text{ s.t. } n > N \implies |a_n| < \varepsilon$$



Example

Let $a_n = n^{-7}$ for all $n \in \mathbb{N}$. Show that (a_n) is a null sequence.

We have to show that $a_n \rightarrow 0$ as $n \rightarrow \infty$.

Let $\varepsilon > 0$. Choose . Then for all $n \in \mathbb{N}$,

$$n > N \implies |a_n| = \left| \frac{1}{n^7} \right| \text{ } < \text{ } \varepsilon.$$

Therefore (a_n) is a null sequence.

9.1 Sequences

$$\forall \varepsilon > 0, \exists N \in \mathbb{N} \text{ s.t. } n > N \implies |a_n| < \varepsilon$$



Example

Let $a_n = n^{-7}$ for all $n \in$

We have to show that a_n

Let $\varepsilon > 0$. Choose

$$n > N \implies$$

Therefore (a_n) is a null

$$\left| \frac{1}{n^7} \right| < \varepsilon$$

9.1 Sequences

$$\forall \varepsilon > 0, \exists N \in \mathbb{N} \text{ s.t. } n > N \implies |a_n| < \varepsilon$$



Example

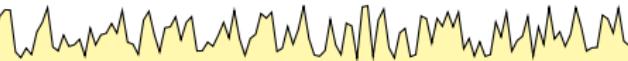
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Therefore (a_n) is a null



scrap paper

$$\left| \frac{1}{n^7} \right| < \varepsilon$$

$$\frac{1}{n^7} < \varepsilon$$

$$n^7 > \frac{1}{\varepsilon}$$



9.1 Sequences

$$\forall \varepsilon > 0, \exists N \in \mathbb{N} \text{ s.t. } n > N \implies |a_n| < \varepsilon$$



Example

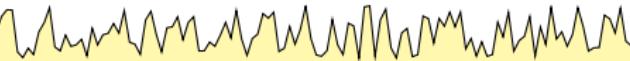
Let $a_n = n^{-7}$ for all $n \in$

We have to show that a_n

Let $\varepsilon > 0$. Choose

$$n > N \implies$$

Therefore (a_n) is a null



scrap paper

$$\left| \frac{1}{n^7} \right| < \varepsilon$$

$$\frac{1}{n^7} < \varepsilon$$

$$n^7 > \frac{1}{\varepsilon}$$

$$n > \frac{1}{\varepsilon^{\frac{1}{7}}}$$

We can choose $N \geq \varepsilon^{-\frac{1}{7}}$



9.1 Sequences

$$\forall \varepsilon > 0, \exists N \in \mathbb{N} \text{ s.t. } n > N \implies |a_n| < \varepsilon$$



Example

Let $a_n = n^{-7}$ for all $n \in \mathbb{N}$. Show that (a_n) is a null sequence.

We have to show that $a_n \rightarrow 0$ as $n \rightarrow \infty$.

Let $\varepsilon > 0$. Choose $N \geq \varepsilon^{-\frac{1}{7}}$. Then for all $n \in \mathbb{N}$,

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Let $\varepsilon > 0$. Choose $N \geq \varepsilon^{-\frac{1}{7}}$. Then for all $n \in \mathbb{N}$,

$$n > N \implies |a_n| = \left| \frac{1}{n^7} \right| = \frac{1}{n^7} < \underline{\hspace{2cm}} \varepsilon.$$

Therefore (a_n) is a null sequence.

9.1 Sequences

$$\forall \varepsilon > 0, \exists N \in \mathbb{N} \text{ s.t. } n > N \implies |a_n| < \varepsilon$$



Example

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We have to show that $a_n \rightarrow 0$ as $n \rightarrow \infty$.

Let $\varepsilon > 0$. Choose $N \geq \varepsilon^{-\frac{1}{7}}$. Then for all $n \in \mathbb{N}$,

$$n > N \implies |a_n| = \left| \frac{1}{n^7} \right| = \frac{1}{n^7} < \frac{1}{N^7} \leq \varepsilon.$$

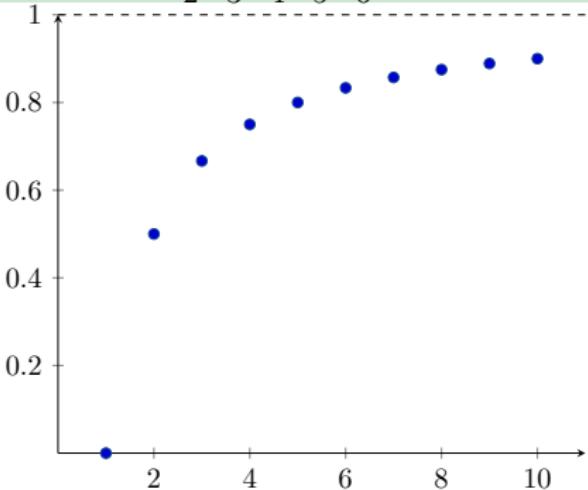
Therefore (a_n) is a null sequence.

9.1 Sequences

Example

Let $z_n = 1 - \frac{1}{n}$ for all $n \in \mathbb{N}$. Then (z_n) is the sequence

$$0, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \frac{5}{6}, \dots$$

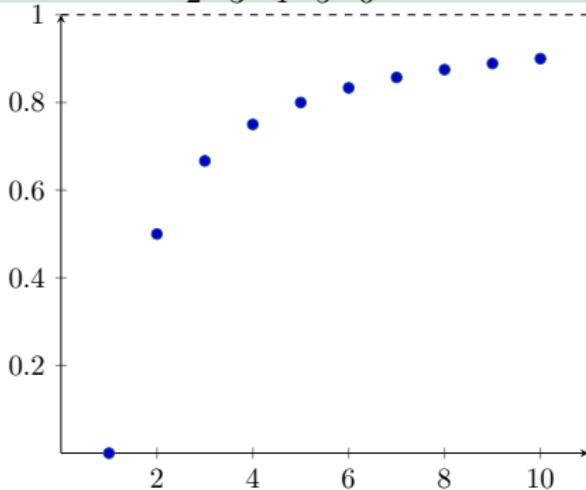


9.1 Sequences

Example

Let $z_n = 1 - \frac{1}{n}$ for all $n \in \mathbb{N}$. Then (z_n) is the sequence

$$0, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \frac{5}{6}, \dots$$



It “looks like” z_n “goes to” 1 as $n \rightarrow \infty$. But this is Mathematics, so we need to be precise.

9.1 Sequences



Definition

A sequence of real numbers a_n *tends to* l ($a_n \rightarrow l$ as $n \rightarrow \infty$) iff for all $\varepsilon > 0$, there exists $N = N(\varepsilon) \in \mathbb{N}$ such that

$$n > N \implies |a_n - l| < \varepsilon.$$

9.1 Sequences

Definition

A sequence of real numbers a_n *tends to* l ($a_n \rightarrow l$ as $n \rightarrow \infty$) iff for all $\varepsilon > 0$, there exists $N = N(\varepsilon) \in \mathbb{N}$ such that

$$n > N \implies |a_n - l| < \varepsilon.$$

Remark

We can also write $\lim_{n \rightarrow \infty} a_n = l$ if a_n tends to l .

9.1 Sequences



Example

Let $u_n = \begin{cases} 7 & n \geq 7 \\ n & 1 \leq n \leq 6. \end{cases}$ Show that $u_n \rightarrow 7$ as $n \rightarrow \infty$.

Let $\varepsilon > 0$. Choose $N = 6$. Then

$$n > N \implies n \geq 7 \implies u_n = 7 \implies |u_n - 7| = 0 < \varepsilon.$$

Therefore $u_n \rightarrow 7$ as $n \rightarrow \infty$.

9.1 Sequences

Example

Let $v_n = \frac{n^2+n+1}{2n^2+1}$ for all $n \in \mathbb{N}$. Show that $v_n \rightarrow \frac{1}{2}$ as $n \rightarrow \infty$.

Let $\varepsilon > 0$. First note that

$$v_n - \frac{1}{2} = \left(\frac{n^2+n+1}{2n^2+1} \right) - \frac{1}{2} = \left(\frac{n^2+n+1}{2n^2+1} \right) - \left(\frac{n^2+\frac{1}{2}}{2n^2+1} \right) = \frac{2n+1}{2(2n^2+1)}.$$

So

$$\left| v_n - \frac{1}{2} \right| < \frac{2n+1}{4n^2} \leq \frac{2n+n}{4n^2} = \frac{3}{4n}.$$

Now choose $N > \frac{3}{4\varepsilon}$. Then for all $n \in \mathbb{N}$,

$$n > N \implies \left| v_n - \frac{1}{2} \right| < \frac{3}{4n} < \frac{3}{4N} < \varepsilon.$$

Therefore $v_n \rightarrow \frac{1}{2}$ as $n \rightarrow \infty$.

9.1 Sequences

Example

Define a sequence (p_n) by

$$p_1 = 3$$

$$p_2 = 3.1$$

$$p_3 = 3.14$$

$$p_4 = 3.141$$

$$p_5 = 3.1415$$

⋮

p_n = the first n digits of π

⋮

Show that $p_n \rightarrow \pi$ as $n \rightarrow \infty$.

9.1 Sequences

First note that

$$|p_1 - \pi| = 0.141592\dots < 1 = 10^0$$

$$|p_2 - \pi| = 0.041592\dots < 0.1 = 10^{-1}$$

$$|p_3 - \pi| = 0.001592\dots < 0.01 = 10^{-2}$$

⋮

$$|p_n - \pi| < 10^{1-n}$$

⋮

9.1 Sequences

First note that

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⋮

$$|p_n - \pi| < 10^{1-n}$$

⋮

Let $\varepsilon > 0$. Choose $N > 1 - \log_{10} \varepsilon$.

9.1 Sequences

First note that

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$$|p_n - \pi| < 10^{1-n}$$

⋮

Let $\varepsilon > 0$. Choose $N > 1 - \log_{10} \varepsilon$. Then for all $n \in \mathbb{N}$,

$$n > N \implies |p_n - \pi| < 10^{1-n} < 10^{1-N} < 10^{1-(1-\log_{10} \varepsilon)} = \varepsilon.$$

9.1 Sequences

First note that

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$$n > N \implies |p_n - \pi| < 10^{1-n} < 10^{1-N} < 10^{1-(1-\log_{10} \varepsilon)} = \varepsilon.$$

Therefore $p_n \rightarrow \pi$ as $n \rightarrow \infty$.

9.1 Sequences



Theorem

A sequence of real numbers cannot have more than one limit.

9.1 Sequences

Proof.

Let (a_n) be a sequence.

CASE 1: Suppose first that $a_n \rightarrow l \in \mathbb{R}$ and $a_n \rightarrow m \in \mathbb{R}$ as $n \rightarrow \infty$. We will use proof by contradiction to prove that $l = m$: Assume that $l \neq m$. Then $l - m \neq 0$ and $|l - m| > 0$. Let $\varepsilon = \frac{1}{2} |l - m| > 0$.

9.1 Sequences

Proof.

Let (a_n) be a sequence.

CASE 1: Suppose first that $a_n \rightarrow l \in \mathbb{R}$ and $a_n \rightarrow m \in \mathbb{R}$ as $n \rightarrow \infty$. We will use proof by contradiction to prove that $l = m$: Assume that $l \neq m$. Then $l - m \neq 0$ and $|l - m| > 0$. Let $\varepsilon = \frac{1}{2} |l - m| > 0$.

Since $a_n \rightarrow l$ as $n \rightarrow \infty$, $\exists N_0 \in \mathbb{N}$ such that

$$n > N_0 \implies |a_n - l| < \varepsilon.$$

Similarly, since $a_n \rightarrow m$ as $n \rightarrow \infty$, $\exists N_1 \in \mathbb{N}$ such that

$$n > N_1 \implies |a_n - m| < \varepsilon.$$

9.1 Sequences



Let $N = \max\{N_0, N_1\}$. Then $\forall n > N$ we have that

$$\begin{aligned}|l - m| &= |l - a_n + a_n - m| \leq |l - a_n| + |a_n - m| \\&= |a_n - l| + |a_n - m| < \varepsilon + \varepsilon = |l - m|\end{aligned}$$

by the triangle inequality.

9.1 Sequences



Let $N = \max\{N_0, N_1\}$. Then $\forall n > N$ we have that

$$\begin{aligned}|l - m| &= |l - a_n + a_n - m| \leq |l - a_n| + |a_n - m| \\&= |a_n - l| + |a_n - m| < \varepsilon + \varepsilon = |l - m|\end{aligned}$$

by the triangle inequality.

But $|l - m| < |l - m|$ is a contradiction.

9.1 Sequences

Let $N = \max\{N_0, N_1\}$. Then $\forall n > N$ we have that

$$\begin{aligned}|l - m| &= |l - a_n + a_n - m| \leq |l - a_n| + |a_n - m| \\&= |a_n - l| + |a_n - m| < \varepsilon + \varepsilon = |l - m|\end{aligned}$$

by the triangle inequality.

But $|l - m| < |l - m|$ is a contradiction. Since $l \neq m$ leads to a contradiction, we must have $l = m$. This means that a sequence cannot have two different finite limits.

9.1 Sequences



CASE 2: Moreover, if $a_n \rightarrow l$ as $n \rightarrow \infty$, then $\exists N \in \mathbb{N}$ such that

$$n > N \implies |a_n - l| < 1 \implies l - 1 < a_n < l + 1.$$

Hence $a_n \not\rightarrow \infty$ and $a_n \not\rightarrow -\infty$ as $n \rightarrow \infty$. Therefore a sequence cannot have both a finite limit and an infinite limit.

9.1 Sequences

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9.1 Sequences

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CASE 3: Finally, I leave it for you to prove that a_n cannot tend to both ∞ and $-\infty$.

Therefore a sequence cannot have two different limits. □

9.1 Sequences

$$\forall \varepsilon > 0, \exists N \in \mathbb{N} \text{ s.t. } n > N \implies |a_n| < \varepsilon$$



Example

Let (z_n) be the sequence

$$1, 0, 1, 0, 1, 0, 1, 0, 1, 0, 1, 0, \dots$$

Show that $z_n \not\rightarrow 0$ as $n \rightarrow \infty$.

9.1 Sequences

$$\forall \varepsilon > 0, \exists N \in \mathbb{N} \text{ s.t. } n > N \implies |a_n| < \varepsilon$$



Example

Let (z_n) be the sequence

$$1, 0, 1, 0, 1, 0, 1, 0, 1, 0, 1, 0, \dots$$

Show that $z_n \not\rightarrow 0$ as $n \rightarrow \infty$.

Choose $\varepsilon = \frac{1}{2}$.

9.1 Sequences

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Example

Let (z_n) be the sequence

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Show that $z_n \not\rightarrow 0$ as $n \rightarrow \infty$.

Choose $\varepsilon = \frac{1}{2}$. Let N be any natural number.

9.1 Sequences

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Example

Let (z_n) be the sequence

$$1, 0, 1, 0, 1, 0, 1, 0, 1, 0, 1, 0, \dots$$

Show that $z_n \not\rightarrow 0$ as $n \rightarrow \infty$.

Choose $\varepsilon = \frac{1}{2}$. Let N be any natural number. If N is odd, choose $n = N + 2$. If N is even, choose $n = N + 1$. Then clearly $n > N$.

9.1 Sequences

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Example

Let (z_n) be the sequence

$$1, 0, 1, 0, 1, 0, 1, 0, 1, 0, 1, 0, \dots$$

Show that $z_n \not\rightarrow 0$ as $n \rightarrow \infty$.

Choose $\varepsilon = \frac{1}{2}$. Let N be any natural number. If N is odd, choose $n = N + 2$. If N is even, choose $n = N + 1$. Then clearly $n > N$. Since n is odd, we have that

$$|z_n| = 1 \geq \frac{1}{2} = \varepsilon.$$

Therefore $z_n \not\rightarrow 0$ as $n \rightarrow \infty$.

9.1 Sequences

Definition

A sequence of real numbers (a_n) is called a *convergent sequence* iff $\exists l \in \mathbb{R}$ such that $a_n \rightarrow l$ as $n \rightarrow \infty$.

Remark

We know from a theorem that a convergent sequence has only one limit.

9.1 Sequences

Definition

A sequence of real numbers (a_n) is called a *convergent sequence* iff $\exists l \in \mathbb{R}$ such that $a_n \rightarrow l$ as $n \rightarrow \infty$.

Remark

We know from a theorem that a convergent sequence has only one limit.

Definition

A sequence which is not convergent is called a *divergent sequence*.

9.1 Sequences

Definition

If $a_n \rightarrow \infty$ as $n \rightarrow \infty$, we say that (a_n) *diverges to infinity* (sonsuzda iraksar).

Definition

If $a_n \rightarrow -\infty$ as $n \rightarrow \infty$, we say that (a_n) *diverges to minus infinity*.

9.1 Sequences

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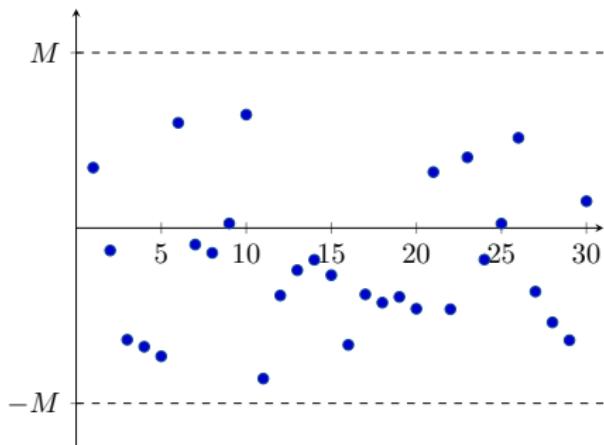
Definition

If $a_n \rightarrow -\infty$ as $n \rightarrow \infty$, we say that (a_n) *diverges to minus infinity*.

Example

Let $a_n = (-1)^n$, $b_n = (-1)^n n$ and $c_n = n^2$. Then (a_n) , (b_n) and (c_n) are divergent sequences. (a_n) and (b_n) do not have a finite limit or an infinite limit. (c_n) diverges to infinity.

9.1 Sequences



Definition

A sequence of real numbers (a_n) is called a *bounded sequence* (sınırlı dizi) iff $\exists M > 0$ such that $|a_n| \leq M$ for all $n \in \mathbb{N}$.

9.1 Sequences



Theorem

Every convergent sequence is bounded.

Proof.

Let (a_n) be a convergent sequence.

9.1 Sequences

Theorem

Every convergent sequence is bounded.

Proof.

Let (a_n) be a convergent sequence. Then $\exists a \in \mathbb{R}$ such that $a_n \rightarrow a$ as $n \rightarrow \infty$.

9.1 Sequences

Theorem

Every convergent sequence is bounded.

Proof.

Let (a_n) be a convergent sequence. Then $\exists a \in \mathbb{R}$ such that $a_n \rightarrow a$ as $n \rightarrow \infty$. So $\exists N \in \mathbb{N}$ such that

$$n > N \implies |a_n - a| < 1 \implies |a_n| < |a| + 1.$$

9.1 Sequences



Theorem

Every convergent sequence is bounded.

Proof.

Let (a_n) be a convergent sequence. Then $\exists a \in \mathbb{R}$ such that $a_n \rightarrow a$ as $n \rightarrow \infty$. So $\exists N \in \mathbb{N}$ such that

$$n > N \implies |a_n - a| < 1 \implies |a_n| < |a| + 1.$$

Now let $M := \max\{|a_1|, |a_2|, |a_3|, \dots, |a_N|, |a| + 1\}$.

9.1 Sequences



Theorem

Every convergent sequence is bounded.

Proof.

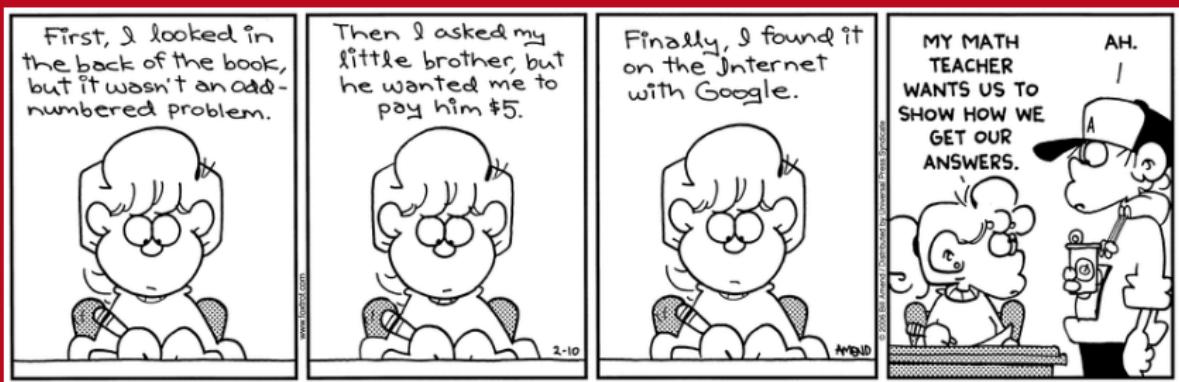
Let (a_n) be a convergent sequence. Then $\exists a \in \mathbb{R}$ such that $a_n \rightarrow a$ as $n \rightarrow \infty$. So $\exists N \in \mathbb{N}$ such that

$$n > N \implies |a_n - a| < 1 \implies |a_n| < |a| + 1.$$

Now let $M := \max\{|a_1|, |a_2|, |a_3|, \dots, |a_N|, |a| + 1\}$. Then $|a_n| \leq M$ for all $n \in \mathbb{N}$. Therefore (a_n) is a bounded sequence. □

Break

We will continue at 3pm



Rules

- The Sum Rule
- The Constant Multiple Rule
- The Product Rule
- The Quotient Rule
- The Sandwich Rule

9.1 Sequences

Lemma (Sum Rule)

Suppose that $a_n \rightarrow a \in \mathbb{R}$ and $b_n \rightarrow b \in \mathbb{R}$ as $n \rightarrow \infty$. Then $a_n + b_n \rightarrow a + b$ as $n \rightarrow \infty$.

9.1 Sequences

Lemma (Sum Rule)

Suppose that $a_n \rightarrow a \in \mathbb{R}$ and $b_n \rightarrow b \in \mathbb{R}$ as $n \rightarrow \infty$. Then $a_n + b_n \rightarrow a + b$ as $n \rightarrow \infty$.

Proof.

Let $\varepsilon > 0$. Since $a_n \rightarrow a \in \mathbb{R}$ and $b_n \rightarrow b \in \mathbb{R}$ as $n \rightarrow \infty$, $\exists N \in \mathbb{N}$ such that

$$n > N \implies a - \frac{\varepsilon}{2} < a_n < a + \frac{\varepsilon}{2} \quad \text{and} \quad b - \frac{\varepsilon}{2} < b_n < b + \frac{\varepsilon}{2}.$$

Adding these inequalities together, we see that

$$n > N \implies a + b - \varepsilon < a_n + b_n < a + b + \varepsilon.$$

Therefore $a_n + b_n \rightarrow a + b$ as $n \rightarrow \infty$. □

9.1 Sequences



Lemma (Constant Multiple Rule)

Suppose that $k \in \mathbb{R}$ and $a_n \rightarrow a \in \mathbb{R}$ as $n \rightarrow \infty$. Then $ka_n \rightarrow ka$ as $n \rightarrow \infty$.

9.1 Sequences

Lemma (Constant Multiple Rule)

Suppose that $k \in \mathbb{R}$ and $a_n \rightarrow a \in \mathbb{R}$ as $n \rightarrow \infty$. Then $ka_n \rightarrow ka$ as $n \rightarrow \infty$.

(you prove)

9.1 Sequences

Lemma (Product Rule)

Suppose that $a_n \rightarrow a \in \mathbb{R}$ and $b_n \rightarrow b \in \mathbb{R}$ as $n \rightarrow \infty$. Then $a_n b_n \rightarrow ab$ as $n \rightarrow \infty$.

Proof.

Let $\varepsilon > 0$. First

$b_n \rightarrow b$ as $n \rightarrow \infty \implies (b_n)$ is convergent $\implies (b_n)$ is bounded by a theorem. So $\exists M > 0$ such that $|b_n| \leq M \ \forall n \in \mathbb{N}$. Note that $\frac{\varepsilon}{M+|a|} > 0$.

9.1 Sequences



Since $a_n \rightarrow a$ and $b_n \rightarrow b$ as $n \rightarrow \infty$, $\exists N \in \mathbb{N}$ such that

$$n > N \implies |a_n - a| < \frac{\varepsilon}{M + |a|} \quad \text{and} \quad |b_n - b| < \frac{\varepsilon}{M + |a|}.$$

9.1 Sequences



Since $a_n \rightarrow a$ and $b_n \rightarrow b$ as $n \rightarrow \infty$, $\exists N \in \mathbb{N}$ such that

$$n > N \implies |a_n - a| < \frac{\varepsilon}{M + |a|} \quad \text{and} \quad |b_n - b| < \frac{\varepsilon}{M + |a|}.$$

But then

$$\begin{aligned} n > N \implies |a_n b_n - ab| &= |a_n b_n - ab_n + ab_n - ab| \\ &\leq |a_n b_n - ab_n| + |ab_n - ab| \\ &= |a_n - a| |b_n| + |a| |b_n - b| \\ &< \frac{\varepsilon}{M + |a|} M + |a| \frac{\varepsilon}{M + |a|} = \varepsilon. \end{aligned}$$

Therefore $a_n b_n \rightarrow ab$ as $n \rightarrow \infty$. □

9.1 Sequences



Lemma (Quotient Rule)

Suppose that $a_n \rightarrow a \in \mathbb{R}$ and $b_n \rightarrow b \in \mathbb{R}$ as $n \rightarrow \infty$. Suppose that $b \neq 0$. Then $\frac{a_n}{b_n} \rightarrow \frac{a}{b}$ as $n \rightarrow \infty$.

(proof omitted)

9.1 Sequences

Example

Let $a_n = \frac{n^5 + 7n^3 + 5n^2 + 8}{5n^5 + 3n^4 + 27}$.

Then

$$\begin{aligned}a_n &= \frac{n^5 + 7n^3 + 5n^2 + 8}{5n^5 + 3n^4 + 27} = \frac{1 + 7n^{-2} + 5n^{-3} + 8n^{-5}}{5 + 3n^{-1} + 27n^{-5}} \\&\rightarrow \frac{1 + 0 + 0 + 0}{5 + 0 + 0} = \frac{1}{5}\end{aligned}$$

as $n \rightarrow \infty$.

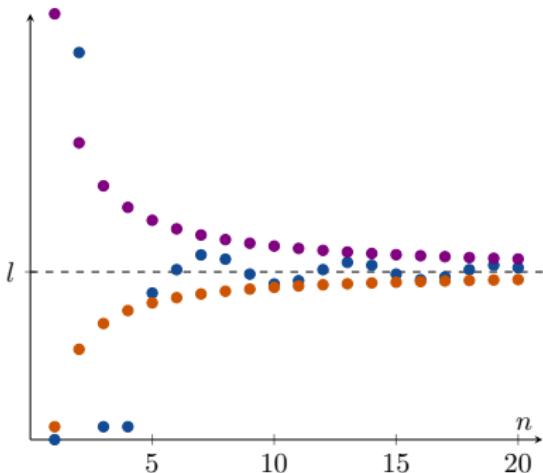
9.1 Sequences

Theorem (The Sandwich Rule)

Let (a_n) , (b_n) and (c_n) be three sequences of real numbers such that

$$a_n \leq b_n \leq c_n$$

for all $n > N_0 \in \mathbb{N}$.



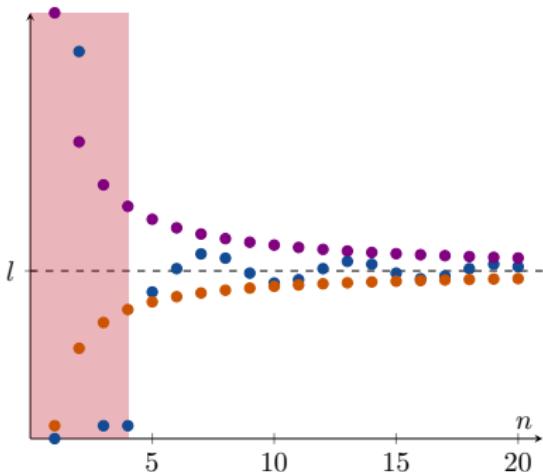
9.1 Sequences

Theorem (The Sandwich Rule)

Let (a_n) , (b_n) and (c_n) be three sequences of real numbers such that

$$a_n \leq b_n \leq c_n$$

for all $n > N_0 \in \mathbb{N}$. If $a_n \rightarrow l$ and $c_n \rightarrow l$ as $n \rightarrow \infty$, then $b_n \rightarrow l$ as $n \rightarrow \infty$ also.



9.1 Sequences



Proof.

Let $\varepsilon > 0$.

9.1 Sequences

Proof.

Let $\varepsilon > 0$. Since $a_n \rightarrow l$ as $n \rightarrow \infty$, $\exists N_1 \in \mathbb{N}$ such that

$$n > N_1 \implies l - \varepsilon < a_n < l + \varepsilon.$$

9.1 Sequences

Proof.

Let $\varepsilon > 0$. Since $a_n \rightarrow l$ as $n \rightarrow \infty$, $\exists N_1 \in \mathbb{N}$ such that

$$n > N_1 \implies l - \varepsilon < a_n < l + \varepsilon.$$

Since $c_n \rightarrow l$ as $n \rightarrow \infty$, $\exists N_2 \in \mathbb{N}$ such that

$$n > N_2 \implies l - \varepsilon < c_n < l + \varepsilon.$$

9.1 Sequences

Proof.

Let $\varepsilon > 0$. Since $a_n \rightarrow l$ as $n \rightarrow \infty$, $\exists N_1 \in \mathbb{N}$ such that

$$n > N_1 \implies l - \varepsilon < a_n < l + \varepsilon.$$

Since $c_n \rightarrow l$ as $n \rightarrow \infty$, $\exists N_2 \in \mathbb{N}$ such that

$$n > N_2 \implies l - \varepsilon < c_n < l + \varepsilon.$$

Let $N = \max\{N_0, N_1, N_2\}$.

9.1 Sequences

Proof.

Let $\varepsilon > 0$. Since $a_n \rightarrow l$ as $n \rightarrow \infty$, $\exists N_1 \in \mathbb{N}$ such that

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$$n > N_2 \implies l - \varepsilon < c_n < l + \varepsilon.$$

Let $N = \max\{N_0, N_1, N_2\}$. Then

$$n > N \implies l - \varepsilon < a_n \leq b_n \leq c_n < l + \varepsilon.$$

9.1 Sequences

Proof.

Let $\varepsilon > 0$. Since $a_n \rightarrow l$ as $n \rightarrow \infty$, $\exists N_1 \in \mathbb{N}$ such that

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Since $c_n \rightarrow l$ as $n \rightarrow \infty$, $\exists N_2 \in \mathbb{N}$ such that

$$n > N_2 \implies l - \varepsilon < c_n < l + \varepsilon.$$

Let $N = \max\{N_0, N_1, N_2\}$. Then

$$n > N \implies l - \varepsilon < a_n \leq b_n \leq c_n < l + \varepsilon.$$

Therefore $b_n \rightarrow l$ as $n \rightarrow \infty$.



9.1 Sequences

Theorem

Let (c_n) be a sequence of real numbers such that

$$c_n \geq 0$$

for all $n > N_0 \in \mathbb{N}$. Suppose that $c_n \rightarrow c$ as $n \rightarrow \infty$. Then $c \geq 0$.

(proof omitted)

9.1 Sequences

Theorem

Let (c_n) be a sequence of real numbers such that

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for all $n > N_0 \in \mathbb{N}$. Suppose that $c_n \rightarrow c$ as $n \rightarrow \infty$. Then $c \geq 0$.

(proof omitted)

Corollary

Let (a_n) and (b_n) be sequences of real numbers such that

$$a_n \leq b_n$$

for all $n > N_0 \in \mathbb{N}$. Suppose that $a_n \rightarrow a$ and $b_n \rightarrow b$ as $n \rightarrow \infty$. Then $a \leq b$.

9.1 Sequences

Remark

So

$$a_n \leq b_n \implies \lim_{n \rightarrow \infty} a_n \leq \lim_{n \rightarrow \infty} b_n$$

if the limits exist. But is

$$“a_n < b_n \implies \lim_{n \rightarrow \infty} a_n < \lim_{n \rightarrow \infty} b_n.”$$

true?

9.1 Sequences

Remark

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true? The answer is NO!!!!

9.1 Sequences

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if the limits exist. But is

$$a_n < b_n \implies \lim_{n \rightarrow \infty} a_n < \lim_{n \rightarrow \infty} b_n.$$

true? The answer is NO!!!!

Example

Let $a_n = \frac{1}{n^2}$ and $b_n = \frac{1}{n}$. Then $a_n = \frac{1}{n^2} < \frac{1}{n} = b_n$ for all $n > 1$,
but

$$\lim_{n \rightarrow \infty} a_n = 0 = \lim_{n \rightarrow \infty} b_n.$$

9.1 Sequences



Remark

Be careful when taking limits of inequalities!

9.1 Sequences



Remark

Be careful when taking limits of inequalities!

Remark

$$a_n < b_n \implies \lim_{n \rightarrow \infty} a_n \leq \lim_{n \rightarrow \infty} b_n$$

if the limits exist.

9.1 Sequences

Theorem

Let (a_n) be a sequence. If $a_n \rightarrow \infty$ as $n \rightarrow \infty$, then $\frac{1}{a_n} \rightarrow 0$ as $n \rightarrow \infty$.

9.1 Sequences

Theorem

Let (a_n) be a sequence. If $a_n \rightarrow \infty$ as $n \rightarrow \infty$, then $\frac{1}{a_n} \rightarrow 0$ as $n \rightarrow \infty$.

Proof.

Let $\varepsilon > 0$. Then let $A = \frac{1}{\varepsilon} > 0$. Since $a_n \rightarrow \infty$ as $n \rightarrow \infty$, $\exists N \in \mathbb{N}$ such that

$$n > N \implies a_n > A.$$

So

$$n > N \implies 0 < \frac{1}{a_n} < \frac{1}{A} = \varepsilon.$$

Therefore $\frac{1}{a_n} \rightarrow 0$ as $n \rightarrow \infty$. □

9.1 Sequences

Example

Since

$$0 \leftarrow -\frac{1}{n} \leq \frac{\cos n}{n} \leq \frac{1}{n} \rightarrow 0$$

as $n \rightarrow \infty$, it follows by the Sandwich Rule that $\frac{\cos n}{n} \rightarrow 0$ as $n \rightarrow \infty$.

9.1 Sequences

Example

Since

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as $n \rightarrow \infty$, it follows by the Sandwich Rule that $\frac{\cos n}{n} \rightarrow 0$ as $n \rightarrow \infty$.

Example

$$\frac{n^5 + n^4 \cos n + 6}{4n^5 + n^3 + \cos n} = \frac{1 + \frac{\cos n}{n} + 6n^{-5}}{4 + \frac{1}{n} + \frac{\cos n}{n^5}} \rightarrow \frac{1 + 0 + 0}{4 + 0 + 0} = \frac{1}{4}$$

as $n \rightarrow \infty$.

Limits of Standard Sequences

- n^α
- a^n
- $a^{\frac{1}{n}}$
- $n^{\frac{1}{n}}$
- $n!$
- n^n

9.1 Sequences



Lemma

Let $\alpha \in \mathbb{R}$. Then

$$n^\alpha \rightarrow \begin{cases} \infty & \text{if } \alpha > 0 \\ 1 & \text{if } \alpha = 0 \\ 0 & \text{if } \alpha < 0 \end{cases}$$

as $n \rightarrow \infty$.

9.1 Sequences

Lemma

Let $\alpha \in \mathbb{R}$. Then

$$n^\alpha \rightarrow \begin{cases} \infty & \text{if } \alpha > 0 \\ 1 & \text{if } \alpha = 0 \\ 0 & \text{if } \alpha < 0 \end{cases}$$

as $n \rightarrow \infty$.

Proof.

CASE 1 ($\alpha > 0$): Let $A > 0$. Choose N such that $\alpha \ln N \geq \ln A$. Then

$$n > N \implies n^\alpha > N^\alpha = e^{\ln N^\alpha} = e^{\alpha \ln N} \geq e^{\ln A} = A.$$

So $n^\alpha \rightarrow \infty$ as $n \rightarrow \infty$.



9.1 Sequences

Lemma

Let $\alpha \in \mathbb{R}$. Then

$$n^\alpha \rightarrow \begin{cases} \infty & \text{if } \alpha > 0 \\ 1 & \text{if } \alpha = 0 \\ 0 & \text{if } \alpha < 0 \end{cases}$$

as $n \rightarrow \infty$.

Proof.

CASE 2 ($\alpha = 0$): Clearly $n^\alpha = n^0 = 1 \ \forall n \in \mathbb{N}$. So (n^α) is the sequence

$$1, 1, 1, 1, 1, 1, \dots$$

which must converge to 1.



9.1 Sequences

Lemma

Let $\alpha \in \mathbb{R}$. Then

$$n^\alpha \rightarrow \begin{cases} \infty & \text{if } \alpha > 0 \\ 1 & \text{if } \alpha = 0 \\ 0 & \text{if } \alpha < 0 \end{cases}$$

as $n \rightarrow \infty$.

Proof.

CASE 3 ($\alpha < 0$): Let $\beta = -\alpha > 0$. Then $n^\beta \rightarrow \infty$ as $n \rightarrow \infty$ by Case 1. Therefore

$$n^\alpha = \frac{1}{n^\beta} \rightarrow 0$$

as $n \rightarrow \infty$, by a theorem from earlier.



9.1 Sequences



Lemma

Let $a \in \mathbb{R}$. Then

$$a^n \rightarrow \begin{cases} \infty & \text{if } a > 1 \\ 1 & \text{if } a = 1 \\ 0 & \text{if } |a| < 1 \end{cases}$$

as $n \rightarrow \infty$, and a^n does not have a limit if $a \leq -1$.

9.1 Sequences

Lemma

Let $a \in \mathbb{R}$. Then

$$a^n \rightarrow \begin{cases} \infty & \text{if } a > 1 \\ 1 & \text{if } a = 1 \\ 0 & \text{if } |a| < 1 \end{cases}$$

as $n \rightarrow \infty$, and a^n does not have a limit if $a \leq -1$.

Proof.

CASE 1 ($a > 1$): Let $h = a - 1 > 0$. Then

$$\begin{aligned} a^n &= (1 + h)^n = 1 + nh + \frac{n(n-1)}{2!}h^2 + \frac{n(n-1)(n-2)}{3!}h^3 + \dots + h^n \\ &\geq 1 + nh \rightarrow \infty \end{aligned}$$

as $n \rightarrow \infty$. It follows that $a^n \rightarrow \infty$ as $n \rightarrow \infty$, by a theorem from earlier.

9.1 Sequences

Lemma

Let $a \in \mathbb{R}$. Then

$$a^n \rightarrow \begin{cases} \infty & \text{if } a > 1 \\ 1 & \text{if } a = 1 \\ 0 & \text{if } |a| < 1 \end{cases}$$

as $n \rightarrow \infty$, and a^n does not have a limit if $a \leq -1$.

Proof.

CASE 2 ($a = 1$): Since $a^n = 1 \forall n$, we must have that $a^n \rightarrow 1$ as $n \rightarrow \infty$.



9.1 Sequences

Lemma

Let $a \in \mathbb{R}$. Then

$$a^n \rightarrow \begin{cases} \infty & \text{if } a > 1 \\ 1 & \text{if } a = 1 \\ 0 & \text{if } |a| < 1 \end{cases}$$

as $n \rightarrow \infty$, and a^n does not have a limit if $a \leq -1$.

Proof.

CASE 3 ($0 < a < 1$): Let $b = \frac{1}{a} > 1$. Then $b^n \rightarrow \infty$ as $n \rightarrow \infty$, by Case 1. Therefore $a^n = \left(\frac{1}{b}\right)^n = \frac{1}{b^n} \rightarrow 0$ as $n \rightarrow \infty$, by a theorem from earlier.



9.1 Sequences

Lemma

Let $a \in \mathbb{R}$. Then

$$a^n \rightarrow \begin{cases} \infty & \text{if } a > 1 \\ 1 & \text{if } a = 1 \\ 0 & \text{if } |a| < 1 \end{cases}$$

as $n \rightarrow \infty$, and a^n does not have a limit if $a \leq -1$.

Proof.

CASE 4 ($a = 0$): Another easy case. Since $a^n = 0 \forall n$, we have that $a^n \rightarrow 0$ as $n \rightarrow \infty$.



9.1 Sequences

Lemma

Let $a \in \mathbb{R}$. Then

$$a^n \rightarrow \begin{cases} \infty & \text{if } a > 1 \\ 1 & \text{if } a = 1 \\ 0 & \text{if } |a| < 1 \end{cases}$$

as $n \rightarrow \infty$, and a^n does not have a limit if $a \leq -1$.

Proof.

CASE 5 ($-1 < a < 0$): Since $0 < |a| < 1$, we have that

$$0 \leftarrow -|a|^n = -|a^n| \leq a^n \leq |a^n| = |a|^n \rightarrow 0$$

as $n \rightarrow \infty$. By the Sandwich Rule, $a^n \rightarrow 0$ as $n \rightarrow \infty$.



9.1 Sequences

Lemma

Let $a \in \mathbb{R}$. Then

$$a^n \rightarrow \begin{cases} \infty & \text{if } a > 1 \\ 1 & \text{if } a = 1 \\ 0 & \text{if } |a| < 1 \end{cases}$$

as $n \rightarrow \infty$, and a^n does not have a limit if $a \leq -1$.

Proof.

CASE 6 ($a \leq -1$): Now we have $a^n = (-1)^n |a|^n$. Since $|a|^n \geq 1$, $a^n \leq -1$ if n is odd and $a^n \geq 1$ if n is even. Therefore a^n cannot tend to any finite or infinite limit as $n \rightarrow \infty$. □

9.1 Sequences



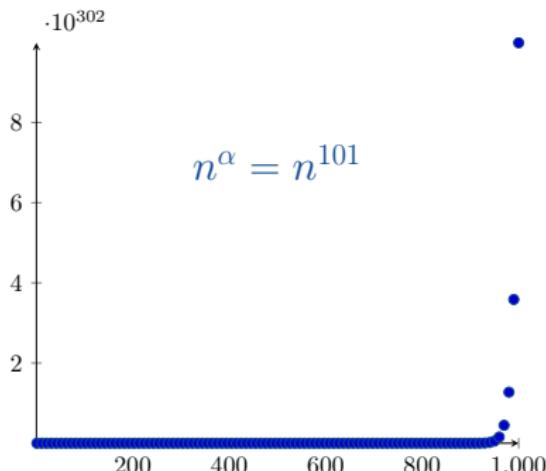
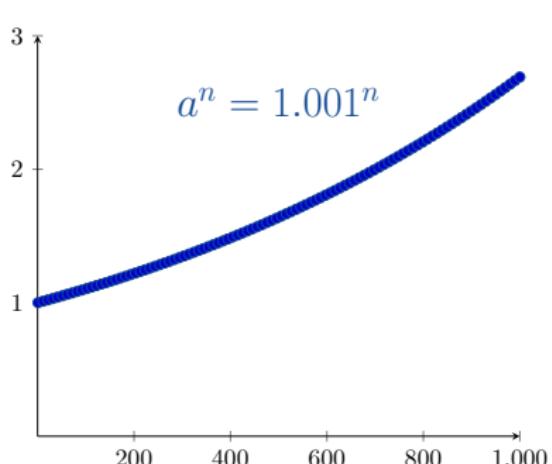
Now suppose that $a > 1$ and $\alpha > 0$. We know that $a^n \rightarrow \infty$ and $n^\alpha \rightarrow \infty$ as $n \rightarrow \infty$.

QUESTION: $\frac{a^n}{n^\alpha} \rightarrow ?$ as $n \rightarrow \infty$

9.1 Sequences

Example

Let $a = 1.001$ and $\alpha = 101$.

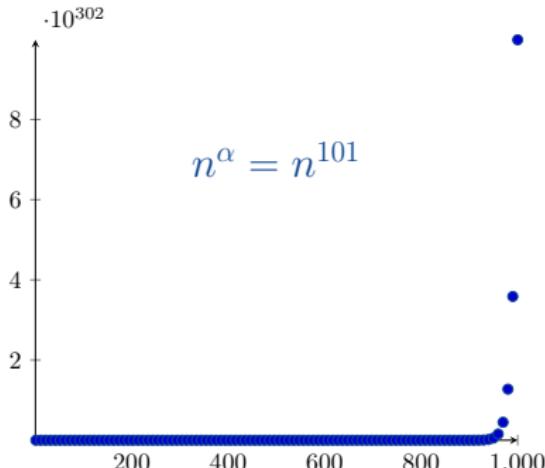
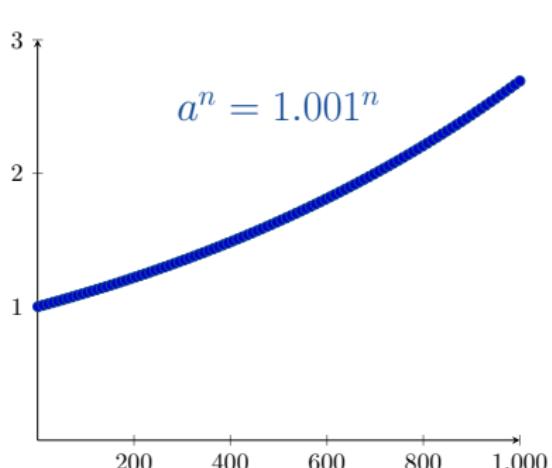


Note that $1.001^{1000} \approx 2.7$, but $1000^{101} = 10^{303}$.

9.1 Sequences

Example

Let $a = 1.001$ and $\alpha = 101$.



Note that $1.001^{1000} \approx 2.7$, but $1000^{101} = 10^{303}$.

So what will happen to $\frac{1.001^n}{n^{101}}$ as $n \rightarrow \infty$?

9.1 Sequences



It might surprise you to learn that

$$\frac{1.001^n}{n^{101}} \rightarrow \infty$$

as $n \rightarrow \infty$.

9.1 Sequences



It might surprise you to learn that

$$\frac{1.001^n}{n^{101}} \rightarrow \infty$$

as $n \rightarrow \infty$.

Lemma

Let $a > 1$ and $\alpha > 0$. Then

$$\frac{a^n}{n^\alpha} \rightarrow \infty$$

as $n \rightarrow \infty$.

9.1 Sequences



Proof.

Let $p \in \mathbb{N}$ and $p \geq \alpha$. Then

$$\frac{a^n}{n^\alpha} \geq \frac{a^n}{n^p}$$

for all $n \in \mathbb{N}$.

We want to prove that $\frac{a^n}{n^p} \rightarrow \infty$ as $n \rightarrow \infty$. For general p , the notation in the proof is complicated – so we will only prove it for $p = 2$.

9.1 Sequences



Since $a > 1$, we have that $h := a - 1 > 0$.

9.1 Sequences

Since $a > 1$, we have that $h := a - 1 > 0$. So

$$\begin{aligned}
 n \geq 4 \implies \frac{a^n}{n^2} &= \frac{(1+h)^n}{n^2} \\
 &= \frac{1}{n^2} \left(1 + nh + \frac{n(n-1)}{2!} h^2 + \frac{n(n-1)(n-2)}{3!} h^3 + \dots + h^n \right) \\
 &> \frac{n(n-1)(n-2)}{3!n^2} h^3 \\
 &= \frac{(n-1)(n-2)}{6n} h^3 \\
 &> \frac{\left(\frac{1}{2}n\right)\left(\frac{1}{2}n\right)}{6n} h^3 \\
 &= \frac{nh^3}{24} \rightarrow \infty
 \end{aligned}$$

as $n \rightarrow \infty$.

9.1 Sequences



A similar argument shows that $\frac{a^n}{n^p} \rightarrow \infty$ as $n \rightarrow \infty$ for all $p \in \mathbb{N}$.

Then a theorem from earlier tells us that $\frac{a^n}{n^\alpha} \rightarrow \infty$ as $n \rightarrow \infty$. □

9.1 Sequences



Corollary

Let $a > 1$ and $\alpha > 0$. Then $\frac{n^\alpha}{a^n} \rightarrow 0$ as $n \rightarrow \infty$.

Corollary

Let $\alpha > 0$ and $|b| < 1$. Then $n^\alpha b^n \rightarrow 0$ as $n \rightarrow \infty$.

9.1 Sequences



Lemma

Let $a > 0$. Then $a^{\frac{1}{n}} \rightarrow 1$ as $n \rightarrow \infty$.

9.1 Sequences

Lemma

Let $a > 0$. Then $a^{\frac{1}{n}} \rightarrow 1$ as $n \rightarrow \infty$.

Proof.

CASE 1 ($a > 1$): Let $h_n = a^{\frac{1}{n}} - 1 > 0$. Then

$$a = (1 + h_n)^n = 1 + nh_n + \frac{n(n-1)}{2!}h_n^2 + \dots + h_n^n > nh_n.$$

So

$$0 < h_n < \frac{a}{n} \rightarrow 0$$

as $n \rightarrow \infty$. It follows by the Sandwich Rule that $h_n \rightarrow 0$ as $n \rightarrow \infty$. Therefore $a^{\frac{1}{n}} = 1 + h_n \rightarrow 1$ as $n \rightarrow \infty$.



9.1 Sequences

Lemma

Let $a > 0$. Then $a^{\frac{1}{n}} \rightarrow 1$ as $n \rightarrow \infty$.

Proof.

CASE 2 ($a = 1$): Clearly $a^{\frac{1}{n}} = 1 \forall n$. Hence $a^{\frac{1}{n}} \rightarrow 1$ as $n \rightarrow \infty$.

CASE 3 ($0 < a < 1$): Let $b = \frac{1}{a} > 1$. Then

$$a^{\frac{1}{n}} = \left(\frac{1}{b}\right)^{\frac{1}{n}} \rightarrow \frac{1}{b} = 1$$

as $n \rightarrow \infty$.

Therefore $a^{\frac{1}{n}} \rightarrow 1$ as $n \rightarrow \infty \forall a > 0$.

□

9.1 Sequences

Lemma

$$n^{\frac{1}{n}} \rightarrow 1 \text{ as } n \rightarrow \infty.$$

9.1 Sequences

Lemma

$$n^{\frac{1}{n}} \rightarrow 1 \text{ as } n \rightarrow \infty.$$

Proof.

Let $k_n := n^{\frac{1}{n}} - 1$. If $n > 1$, then $n^{\frac{1}{n}} > 1$ and $k_n > 0$. So

$$\begin{aligned} n \geq 2 \implies n &= (1 + k_n)^n = 1 + nk_n + \frac{n(n-1)}{2!}k_n^2 + \dots + k_n^n \\ &> \frac{n(n-1)}{2!}k_n^2. \end{aligned}$$

Thus

$$0 < k_n < \sqrt{\frac{2}{n-1}}$$

for all $n \geq 2$. By the Sandwich Rule, $k_n \rightarrow 0$ as $n \rightarrow \infty$.

Therefore $n^{\frac{1}{n}} \rightarrow 1$ as $n \rightarrow \infty$. □

9.1 Sequences

Lemma

Let $a \in \mathbb{R}$. Then $\frac{a^n}{n!} \rightarrow 0$ as $n \rightarrow \infty$.

9.1 Sequences

Lemma

Let $a \in \mathbb{R}$. Then $\frac{a^n}{n!} \rightarrow 0$ as $n \rightarrow \infty$.

Proof.

Let N be the smallest number in \mathbb{N} such that $N \geq 2|a|$. Then

$$p \in \mathbb{N}, p \geq N \implies \frac{|a|}{p} \leq \frac{|a|}{N} \leq \frac{1}{2}.$$

So

$$\begin{aligned} n > N &\implies 0 \leq \left| \frac{a^n}{n!} \right| = \left(\frac{|a|}{n} \right) \left(\frac{|a|}{n-1} \right) \left(\frac{|a|}{n-2} \right) \cdots \left(\frac{|a|}{N+1} \right) \left(\frac{|a|^N}{N!} \right) \\ &\leq \left(\frac{1}{2} \right) \left(\frac{1}{2} \right) \left(\frac{1}{2} \right) \cdots \left(\frac{1}{2} \right) \left(\frac{|a|^N}{N!} \right) \\ &= \left(\frac{1}{2} \right)^{n-N} \frac{|a|^N}{N!} \rightarrow 0 \end{aligned}$$

as $n \rightarrow \infty$. Therefore $\frac{a^n}{n!} \rightarrow 0$ as $n \rightarrow \infty$, by the Sandwich Rule. □

9.1 Sequences



Lemma

$$\frac{n!}{n^n} \rightarrow 0 \text{ as } n \rightarrow \infty.$$

9.1 Sequences

Lemma

$$\frac{n!}{n^n} \rightarrow 0 \text{ as } n \rightarrow \infty.$$

Proof.

Since

$$\begin{aligned} 0 < \frac{n!}{n^n} &= \frac{n(n-1)(n-2)\cdots(2)(1)}{nnn\cdots nn} \\ &= 1\left(1 - \frac{1}{n}\right)\left(1 - \frac{2}{n}\right)\cdots\left(\frac{2}{n}\frac{1}{n}\right) \leq \frac{1}{n} \rightarrow 0 \end{aligned}$$

as $n \rightarrow \infty$, the result follows by the Sandwich Rule. □

9.1 Sequences

$$\alpha > 0 \implies n^\alpha \rightarrow \infty$$

$$\alpha = 0 \implies n^\alpha \rightarrow 1$$

$$\alpha < 0 \implies n^\alpha \rightarrow 0$$

Summary

9.1 Sequences



$$\alpha > 0 \implies n^\alpha \rightarrow \infty$$

$$\alpha = 0 \implies n^\alpha \rightarrow 1$$

$$\alpha < 0 \implies n^\alpha \rightarrow 0$$

$$a > 1 \implies a^n \rightarrow \infty$$

$$a = 1 \implies a^n \rightarrow 1$$

$$|a| < 1 \implies a^n \rightarrow 0$$

$a < -1 \implies a^n$ does not have a limit

9.1 Sequences

$$\alpha > 0 \implies n^\alpha \rightarrow \infty$$

$$\alpha = 0 \implies n^\alpha \rightarrow 1$$

$$\alpha < 0 \implies n^\alpha \rightarrow 0$$

$$n^{\frac{1}{n}} \rightarrow 1$$

$$a > 0 \implies a^{\frac{1}{n}} \rightarrow 1$$

$$a > 1 \implies a^n \rightarrow \infty$$

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9.1 Sequences

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$a < -1 \implies a^n$ does not have a limit

$$a > 0 \implies a^{\frac{1}{n}} \rightarrow 1$$

$$a > 1, \alpha > 0 \implies \frac{a^n}{n^\alpha} \rightarrow \infty$$

$$\frac{a^n}{n!} \rightarrow 0$$

$$\frac{n!}{n^n} \rightarrow 0$$

9.1 Sequences



Example

$$\frac{n^7 7^n + n^5 5^n}{3^n + 8^n}$$

9.1 Sequences



Example

$$\frac{n^7 7^n + n^5 5^n}{3^n + 8^n} = \frac{n^7 \left(\frac{7}{8}\right)^n + n^5 \left(\frac{5}{8}\right)^n}{\left(\frac{3}{8}\right)^n + 1}$$

9.1 Sequences

Example

$$\frac{n^7 7^n + n^5 5^n}{3^n + 8^n} = \frac{n^7 \left(\frac{7}{8}\right)^n + n^5 \left(\frac{5}{8}\right)^n}{\left(\frac{3}{8}\right)^n + 1} \rightarrow \frac{0+0}{0+1} = 0$$

as $n \rightarrow \infty$.

9.1 Sequences

Example

$$\frac{n^7 7^n + n^5 5^n}{3^n + 8^n} = \frac{n^7 \left(\frac{7}{8}\right)^n + n^5 \left(\frac{5}{8}\right)^n}{\left(\frac{3}{8}\right)^n + 1} \rightarrow \frac{0 + 0}{0 + 1} = 0$$

as $n \rightarrow \infty$.

Example

$$\frac{n! + 8^n}{7^n + n!}$$

9.1 Sequences

Example

$$\frac{n^7 7^n + n^5 5^n}{3^n + 8^n} = \frac{n^7 \left(\frac{7}{8}\right)^n + n^5 \left(\frac{5}{8}\right)^n}{\left(\frac{3}{8}\right)^n + 1} \rightarrow \frac{0 + 0}{0 + 1} = 0$$

as $n \rightarrow \infty$.

Example

$$\frac{n! + 8^n}{7^n + n!} = \frac{1 + \frac{8^n}{n!}}{\frac{7^n}{n!} + 1}$$

9.1 Sequences

Example

$$\frac{n^7 7^n + n^5 5^n}{3^n + 8^n} = \frac{n^7 \left(\frac{7}{8}\right)^n + n^5 \left(\frac{5}{8}\right)^n}{\left(\frac{3}{8}\right)^n + 1} \rightarrow \frac{0+0}{0+1} = 0$$

as $n \rightarrow \infty$.

Example

$$\frac{n! + 8^n}{7^n + n!} = \frac{1 + \frac{8^n}{n!}}{\frac{7^n}{n!} + 1} \rightarrow \frac{1+0}{0+1} = 1$$

as $n \rightarrow \infty$.

9.1 Sequences

Example

Let $r > 0$ and $a_n = (4^{10} + r^n)^{\frac{1}{n}}$. Calculate $\lim_{n \rightarrow \infty} a_n$.

9.1 Sequences

Example

Let $r > 0$ and $a_n = (4^{10} + r^n)^{\frac{1}{n}}$. Calculate $\lim_{n \rightarrow \infty} a_n$.

CASE 1 ($0 < r \leq 1$):

CASE 2 ($r > 1$):

9.1 Sequences

Example

Let $r > 0$ and $a_n = (4^{10} + r^n)^{\frac{1}{n}}$. Calculate $\lim_{n \rightarrow \infty} a_n$.

CASE 1 ($0 < r \leq 1$): Since

$$1 \leftarrow (4^{10})^{\frac{1}{n}} \leq a_n \leq (4^{10} + 1)^{\frac{1}{n}} \rightarrow 1$$

as $n \rightarrow \infty$, it follows by the Sandwich Rule that if $0 < r \leq 1$, we have that $\lim_{n \rightarrow \infty} a_n = 1$.

CASE 2 ($r > 1$):

9.1 Sequences

Example

Let $r > 0$ and $a_n = (4^{10} + r^n)^{\frac{1}{n}}$. Calculate $\lim_{n \rightarrow \infty} a_n$.

CASE 1 ($0 < r \leq 1$): Since

$$1 \leftarrow (4^{10})^{\frac{1}{n}} \leq a_n \leq (4^{10} + 1)^{\frac{1}{n}} \rightarrow 1$$

as $n \rightarrow \infty$, it follows by the Sandwich Rule that if $0 < r \leq 1$, we have that $\lim_{n \rightarrow \infty} a_n = 1$.

CASE 2 ($r > 1$): In this case $r^n \rightarrow \infty$ as $n \rightarrow \infty$. So $\exists N \in \mathbb{N}$ such that $r^n > 4^{10}$ for all $n > N$. So

$$n > N \implies r = (r^n)^{\frac{1}{n}} < (4^{10} + r^n)^{\frac{1}{n}} < (r^n + r^n)^{\frac{1}{n}} = 2^{\frac{1}{n}}r \rightarrow r$$

as $n \rightarrow \infty$. By the Sandwich Rule, $\lim_{n \rightarrow \infty} a_n = r$ if $r > 1$.

9.1 Sequences



Monotonic Sequences

9.1 Sequences



Definition

A sequence (a_n) is called an *increasing sequence* (artan dizi) iff

$$a_n \leq a_{n+1}$$

for all $n \in \mathbb{N}$.

(Note: Thomas's Calculus calls this a “nondecreasing sequence”.)

9.1 Sequences

Definition

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$$a_n \leq a_{n+1}$$

for all $n \in \mathbb{N}$.

(Note: Thomas's Calculus calls this a “nondecreasing sequence”.)

Definition

A sequence (a_n) is called a *strictly increasing sequence* iff

$$a_n < a_{n+1}$$

for all $n \in \mathbb{N}$.

9.1 Sequences

Definition

A sequence (a_n) is called a *decreasing sequence* (azalan dizi) iff

$$a_n \geq a_{n+1}$$

for all $n \in \mathbb{N}$.

Definition

A sequence (a_n) is called a *strictly decreasing sequence* iff

$$a_n > a_{n+1}$$

for all $n \in \mathbb{N}$.

9.1 Sequences

Definition

A sequence (a_n) is called a *decreasing sequence* (azalan dizi) iff

$$a_n \geq a_{n+1}$$

for all $n \in \mathbb{N}$.

Definition

A sequence (a_n) is called a *strictly decreasing sequence* iff

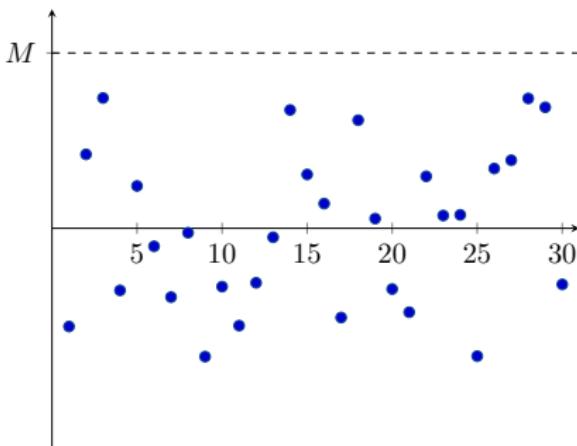
$$a_n > a_{n+1}$$

for all $n \in \mathbb{N}$.

Definition

A sequence (a_n) is called a *monotonic sequence* (monoton dizi) iff it is either an increasing sequence or a decreasing sequence.

9.1 Sequences



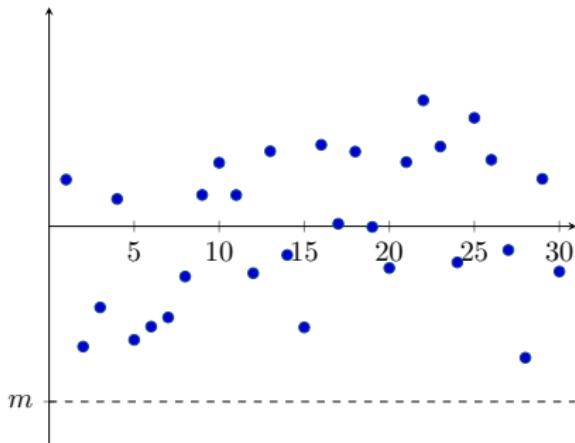
Definition

A sequence (a_n) is said to be *bounded above* (üstten sınırlı) iff $\exists M \in \mathbb{R}$ such that

$$a_n \leq M$$

for all $n \in \mathbb{N}$. The number M is called an *upper bound* (üst sınırıdır) for (a_n) .

9.1 Sequences



Definition

A sequence (a_n) is said to be *bounded below* (alttan sınırlı) iff $\exists m \in \mathbb{R}$ such that

$$a_n \geq m$$

for all $n \in \mathbb{N}$. The number m is called a *lower bound* (alt sınırdır) for (a_n) .

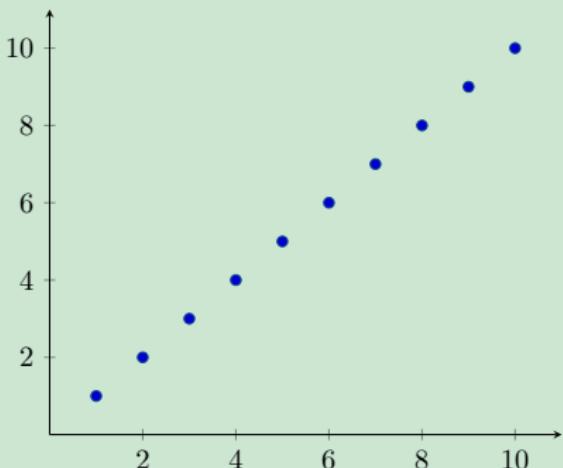
9.1 Sequences

Example

Let $b_n = n$ for all $n \in \mathbb{N}$.

Then (b_n) is

- increasing;
- strictly increasing;
- monotonic;
- bounded below
 $(b_n \geq 0 \ \forall n)$;
- not bounded above.



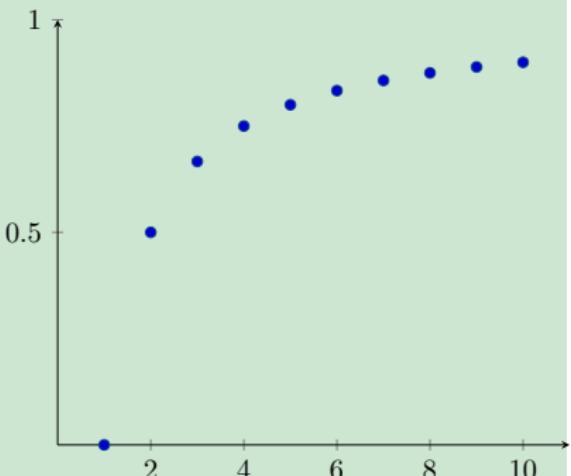
9.1 Sequences

Example

Let $c_n = 1 - \frac{1}{n}$ for all $n \in \mathbb{N}$.

Then (c_n) is

- increasing;
- strictly increasing;
- monotonic;
- bounded above
 $(c_n \leq 1 \ \forall n)$;
- bounded below
 $(c_n \geq 0 \ \forall n)$.



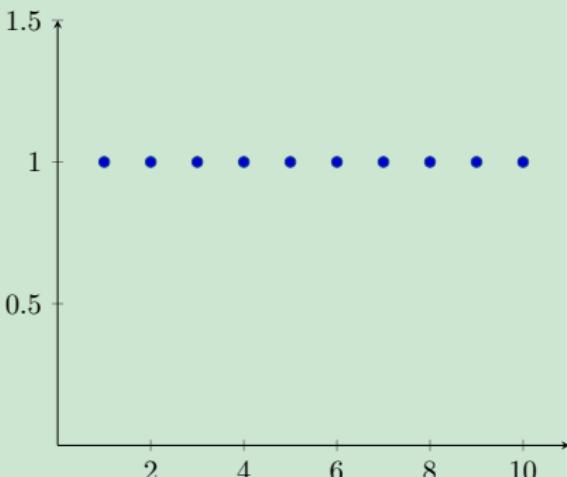
9.1 Sequences

Example

Let $d_n = 1$ for all $n \in \mathbb{N}$.

Then (d_n) is

- increasing;
- not strictly increasing;
- decreasing;
- not strictly decreasing;
- monotonic;
- bounded below
 $(d_n \geq 0 \ \forall n)$;
- bounded above
 $(d_n \leq 567 \ \forall n)$.



9.1 Sequences



Theorem

Let (a_n) be an increasing sequence.

- 1 If (a_n) is bounded above, then (a_n) converges.
- 2 If (a_n) is not bounded above, then $a_n \rightarrow \infty$ as $n \rightarrow \infty$.

9.1 Sequences



Theorem

Let (a_n) be an increasing sequence.

- 1 If (a_n) is bounded above, then (a_n) converges.
- 2 If (a_n) is not bounded above, then $a_n \rightarrow \infty$ as $n \rightarrow \infty$.

This is an important theorem. We need some more theory before we can prove this theorem.

9.1 Sequences

Definition

Let $S \subseteq \mathbb{R}$ be a set. We say that S is *bounded above* iff $\exists M \in \mathbb{R}$ such that

$$x \leq M$$

for all $x \in S$. M is called an *upper bound* for S .

Definition

Let $S \subseteq \mathbb{R}$ be a set. We say that S is *bounded below* iff $\exists m \in \mathbb{R}$ such that

$$x \geq m$$

for all $x \in S$. m is called a *lower bound* for S .

9.1 Sequences



Example

Let $S = \{1, 2, 3, 4\}$. Then $x \leq 7$ for all $x \in S$. So 7 is an upper bound for S .

9.1 Sequences



Example

Let $S = \{1, 2, 3, 4\}$. Then $x \leq 7$ for all $x \in S$. So 7 is an upper bound for S .

Note that 5 is also an upper bound for S .

9.1 Sequences



Example

Let $S = \{1, 2, 3, 4\}$. Then $x \leq 7$ for all $x \in S$. So 7 is an upper bound for S .

Note that 5 is also an upper bound for S . So is 4.

9.1 Sequences



Example

Let $S = \{1, 2, 3, 4\}$. Then $x \leq 7$ for all $x \in S$. So 7 is an upper bound for S .

Note that 5 is also an upper bound for S . So is 4. In fact, 4 is the least upper bound for S .

9.1 Sequences

Definition

Let $S \subseteq \mathbb{R}$. The *supremum* of S , $\sup S$, is the least upper bound (en küçük üst sınır) for S .

If S is empty, we define $\sup S = -\infty$.

If S is not bounded above, we define $\sup S = \infty$.

9.1 Sequences

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If S is empty, we define $\sup S = -\infty$.

If S is not bounded above, we define $\sup S = \infty$.

Example

$$\sup\{1, 2, 3\} = 3$$

$$\sup[0, 1] = 1$$

$$\sup(0, 1) = 1$$

$$\sup \mathbb{Z} = \infty$$

$$\sup \emptyset = -\infty$$

9.1 Sequences

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Example

$$\sup\{1, 2, 3\} = 3$$

$$\sup[0, 1] = 1$$

$$\sup(0, 1) = 1$$

$$\sup \mathbb{Z} = \infty$$

$$\sup \emptyset = -\infty$$

$$\max\{1, 2, 3\} = 3$$

$$\max[0, 1] = 1$$

$\max(0, 1)$ does not exist

$\max \mathbb{Z}$ does not exist

$\max \emptyset$ does not exist

9.1 Sequences

Definition

Let $S \subseteq \mathbb{R}$. The *infimum* of S , $\inf S$, is the greatest lower bound (en büyük alt sınır) for S .

If S is empty, we define $\inf S = \infty$.

If S is not bounded above, we define $\inf S = -\infty$.

9.1 Sequences

Definition

Let $S \subseteq \mathbb{R}$. The *infimum* of S , $\inf S$, is the greatest lower bound (en büyük alt sınır) for S .

If S is empty, we define $\inf S = \infty$.

If S is not bounded above, we define $\inf S = -\infty$.

Example

$$\inf\{-1, 0, 7, 11\} = -1$$

$$\inf(0, 1] = 0$$

$$\inf \mathbb{Z} = -\infty$$

$$\inf \mathbb{N} = 1$$

9.1 Sequences

Definition

Let $S \subseteq \mathbb{R}$. The *infimum* of S , $\inf S$, is the greatest lower bound (en büyük alt sınır) for S .

If S is empty, we define $\inf S = \infty$.

If S is not bounded above, we define $\inf S = -\infty$.

Example

$$\inf\{-1, 0, 7, 11\} = -1$$

$$\inf(0, 1] = 0$$

$$\inf \mathbb{Z} = -\infty$$

$$\inf \mathbb{N} = 1$$

$$\min\{-1, 0, 7, 11\} = -1$$

$$\min(0, 1] \text{ does not exist}$$

$$\min \mathbb{Z} \text{ does not exist}$$

$$\min \mathbb{N} = 1$$

9.1 Sequences

Lemma

Let $S \subseteq \mathbb{R}$ and $S \neq \emptyset$. Then

$$\sup S = \alpha \iff \begin{array}{l} \text{(i) } x \leq \alpha \quad \forall x \in S; \text{ and} \\ \text{(ii) } \forall \varepsilon > 0 \quad \exists x_0 \in S \text{ such that } \alpha - \varepsilon < x_0 \leq \alpha. \end{array}$$

9.1 Sequences

Lemma

Let $S \subseteq \mathbb{R}$ and $S \neq \emptyset$. Then

$$\sup S = \alpha \iff \begin{array}{l} \text{(i)} \ x \leq \alpha \quad \forall x \in S; \text{ and} \\ \text{(ii)} \ \forall \varepsilon > 0 \quad \exists x_0 \in S \text{ such that } \alpha - \varepsilon < x_0 \leq \alpha. \end{array}$$

Proof.

“ \iff ”

(i) $\implies \alpha$ is an upper bound for S .

(ii) $\implies \alpha - \varepsilon$ is not an upper bound for $S \quad \forall \varepsilon > 0$.

Therefore α is the least upper bound.

“ \implies ”

$\sup S = \alpha \implies \alpha$ is the least upper bound \implies (i) and (ii)
are true. □

9.1 Sequences



Completeness Axiom

Every non-empty set of real numbers, which is bounded above, has a supremum.

9.1 Sequences



Completeness Axiom

Every non-empty set of real numbers, which is bounded above, has a supremum.

Now we can prove the important theorem:

Theorem

Let (a_n) be an increasing sequence.

- 1** *If (a_n) is bounded above, then (a_n) converges.*
- 2** *If (a_n) is not bounded above, then $a_n \rightarrow \infty$ as $n \rightarrow \infty$.*

9.1 Sequences



Proof of Theorem 75.

- (i) Let $S = \{a_n : n \in \mathbb{N}\}$. If (a_n) is bounded above, then S is bounded above.

9.1 Sequences



Proof of Theorem 75.

- (i) Let $S = \{a_n : n \in \mathbb{N}\}$. If (a_n) is bounded above, then S is bounded above. By the completeness axiom, S has a supremum.

9.1 Sequences



Proof of Theorem 75.

(i) Let $S = \{a_n : n \in \mathbb{N}\}$. If (a_n) is bounded above, then S is bounded above. By the completeness axiom, S has a supremum.

Let $\alpha = \sup S < \infty$.

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Since (a_n) is increasing,

$$n > N \implies \alpha - \varepsilon < a_n < \alpha \implies |a_n - \alpha| < \varepsilon.$$

Therefore $a_n \rightarrow \alpha$ as $n \rightarrow \infty$.

9.1 Sequences



Proof of Theorem 75 continued.

(ii) Let $A > 0$.

9.1 Sequences



Proof of Theorem 75 continued.

- (ii) Let $A > 0$. If (a_n) is not bounded above, then A is not an upper bound for (a_n) .

9.1 Sequences



Proof of Theorem 75 continued.

- (ii) Let $A > 0$. If (a_n) is not bounded above, then A is not an upper bound for (a_n) . Hence $\exists a_N$ such that $a_N > A$.

9.1 Sequences



Proof of Theorem 75 continued.

(ii) Let $A > 0$. If (a_n) is not bounded above, then A is not an upper bound for (a_n) . Hence $\exists a_N$ such that $a_N > A$.

Since (a_n) is increasing,

$$n > N \implies a_n \geq a_N > A.$$

Therefore $a_n \rightarrow \infty$ as $n \rightarrow \infty$.



9.1 Sequences

Corollary

Let (a_n) be an increasing sequence. Then

$$(a_n) \text{ is convergent} \iff (a_n) \text{ is bounded above.}$$

Corollary

Let (a_n) be an decreasing sequence. Then

$$(a_n) \text{ is convergent} \iff (a_n) \text{ is bounded below.}$$

9.1 Sequences



Theorem (The Monotonic Sequence Theorem)

Every bounded monotonic sequence converges.



Next Time

- 9.2 Infinite Series
- 9.3 The Integral Test
- 9.4 Comparison Tests
- 9.5 Absolute Convergence; The Ratio and Root Tests