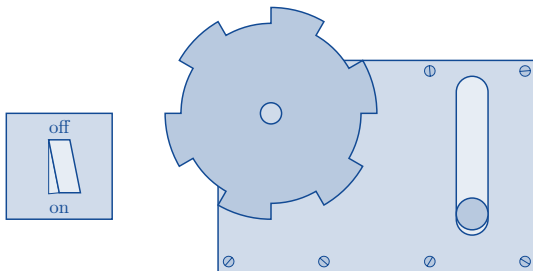


Lecture 9

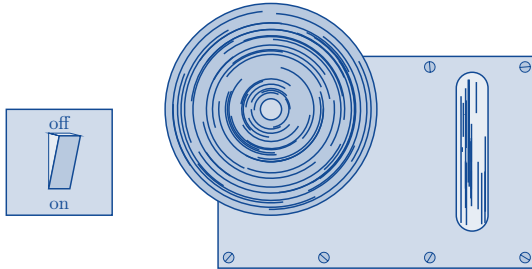
- 4.5 ODEs with Discontinuous Forcing Functions
- 4.6 The Convolution Integral
- 5.1 Introduction
- 5.2 Basic Theory of Systems of First Order Linear Equations

ODEs with Discontinuous Forcing Functions

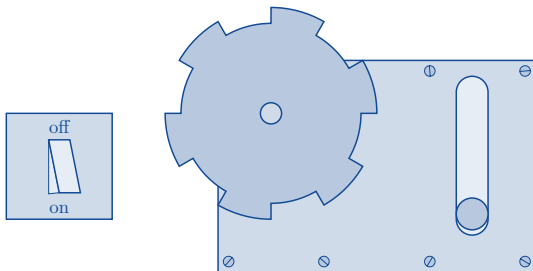
4.5 ODEs with Discontinuous Forcing Functions



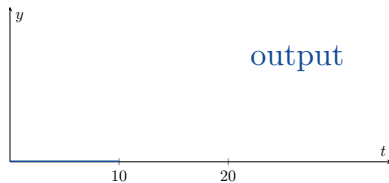
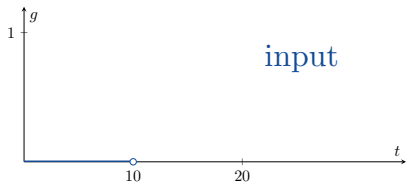
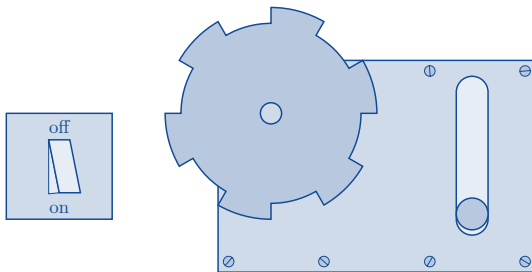
4.5 ODEs with Discontinuous Forcing Functions



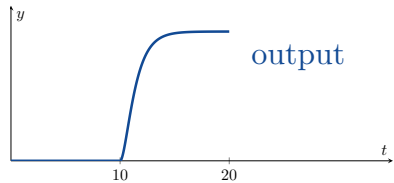
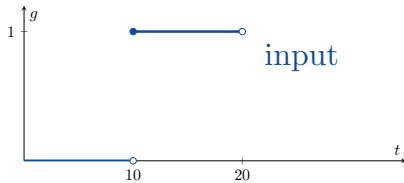
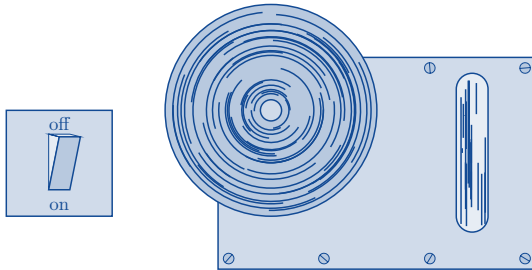
4.5 ODEs with Discontinuous Forcing Functions



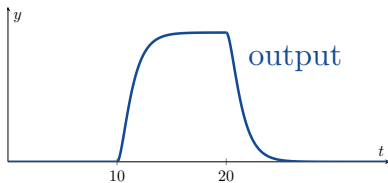
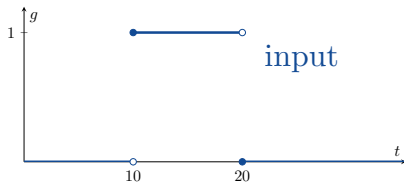
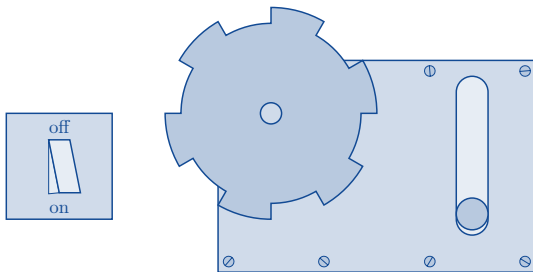
4.5 ODEs with Discontinuous Forcing Functions



4.5 ODEs with Discontinuous Forcing Functions



4.5 ODEs with Discontinuous Forcing Functions



4.5 ODEs with Discontinuous Forcing Functions



Example

Solve

$$\begin{cases} y'' + 4y = f(t) = \begin{cases} 0 & 0 \leq t < 5 \\ \frac{1}{5}(t - 5) & 5 \leq t < 10 \\ 1 & 10 \leq t \end{cases} \\ y(0) = 0 \\ y'(0) = 0. \end{cases}$$

4.5 ODEs with Discontinuous Forcing Functions



Example

Solve

$$\begin{cases} y'' + 4y = f(t) = \begin{cases} 0 & 0 \leq t < 5 \\ \frac{1}{5}(t - 5) & 5 \leq t < 10 \\ 1 & 10 \leq t \end{cases} \\ y(0) = 0 \\ y'(0) = 0. \end{cases}$$

Note that

$$\begin{aligned} f(t) &= 0 + \left(\frac{1}{5}(t - 5) - 0 \right) u_5(t) + \left(1 - \frac{1}{5}(t - 5) \right) u_{10}(t) \\ &= \end{aligned}$$

4.5 ODEs with Discontinuous Forcing Functions



Example

Solve

$$\begin{cases} y'' + 4y = f(t) = \begin{cases} 0 & 0 \leq t < 5 \\ \frac{1}{5}(t - 5) & 5 \leq t < 10 \\ 1 & 10 \leq t \end{cases} \\ y(0) = 0 \\ y'(0) = 0. \end{cases}$$

Note that

$$\begin{aligned} f(t) &= 0 + \left(\frac{1}{5}(t - 5) - 0 \right) u_5(t) + \left(1 - \frac{1}{5}(t - 5) \right) u_{10}(t) \\ &= \frac{1}{5} \left(u_5(t)(t - 5) - u_{10}(t)(t - 10) \right). \end{aligned}$$

4.5 ODEs with Discontinuous Forcing Functions



$$\mathcal{L} [u_c(t)f(t-c)](s) = e^{-cs}F(s) \qquad \mathcal{L} [t] = \frac{1}{s^2}$$

So our IVP is

$$\begin{cases} y'' + 4y = \frac{1}{5} \left(u_5(t)(t-5) - u_{10}(t)(t-10) \right) \\ y(0) = 0 \\ y'(0) = 0. \end{cases}$$

4.5 ODEs with Discontinuous Forcing Functions



$$\mathcal{L} [u_c(t)f(t-c)](s) = e^{-cs}F(s) \qquad \mathcal{L} [t] = \frac{1}{s^2}$$

So our IVP is

$$\begin{cases} y'' + 4y = \frac{1}{5} \left(u_5(t)(t-5) - u_{10}(t)(t-10) \right) \\ y(0) = 0 \\ y'(0) = 0. \end{cases}$$

Taking the Laplace transform of the ODE gives

$$(s^2 + 4)Y = \frac{1}{5} \frac{e^{-5s} - e^{-10s}}{s^2}$$

4.5 ODEs with Discontinuous Forcing Functions



$$\mathcal{L} [u_c(t)f(t-c)](s) = e^{-cs}F(s) \qquad \mathcal{L} [t] = \frac{1}{s^2}$$

So our IVP is

$$\begin{cases} y'' + 4y = \frac{1}{5} \left(u_5(t)(t-5) - u_{10}(t)(t-10) \right) \\ y(0) = 0 \\ y'(0) = 0. \end{cases}$$

Taking the Laplace transform of the ODE gives

$$(s^2 + 4)Y = \frac{1}{5} \frac{e^{-5s} - e^{-10s}}{s^2}$$

and

$$Y = \frac{1}{5} \frac{e^{-5s} - e^{-10s}}{s^2(s^2 + 4)}.$$

4.5 ODEs with Discontinuous Forcing Functions



$$Y = \frac{1}{5} \frac{e^{-5s} - e^{-10s}}{s^2(s^2 + 4)}$$

Let

$$H(s) = \frac{1}{s^2(s^2 + 4)}.$$

Then

$$Y(s) = \frac{1}{5} e^{-5s} H(s) - \frac{1}{5} e^{-10s} H(s).$$

4.5 ODEs with Discontinuous Forcing Functions



$$Y(s) = \frac{1}{5}e^{-5s}H(s) - \frac{1}{5}e^{-10s}H(s)$$

Since

$$\mathcal{L} [u_c(t)h(t - c)] (s) = e^{-cs}H(s)$$

4.5 ODEs with Discontinuous Forcing Functions



$$Y(s) = \frac{1}{5}e^{-5s}H(s) - \frac{1}{5}e^{-10s}H(s)$$

Since

$$\mathcal{L} [u_c(t)h(t-c)](s) = e^{-cs}H(s)$$

we have that

$$u_c(t)h(t-c) = \mathcal{L}^{-1} [e^{-cs}H(s)](t).$$

4.5 ODEs with Discontinuous Forcing Functions



$$Y(s) = \frac{1}{5}e^{-5s}H(s) - \frac{1}{5}e^{-10s}H(s)$$

Since

$$\mathcal{L} [u_c(t)h(t-c)](s) = e^{-cs}H(s)$$

we have that

$$u_c(t)h(t-c) = \mathcal{L}^{-1} [e^{-cs}H(s)](t).$$

If we can find $h(t)$, then we can find $y(t)$.

4.5 ODEs with Discontinuous Forcing Functions



Using partial fractions, we calculate (please check!) that

$$\begin{aligned} H(s) &= \frac{1}{s^2(s^2 + 4)} = \frac{As + B}{s^2} + \frac{Cs + D}{s^2 + 4} \\ &= \frac{As^3 + Bs^2 + 4As + 4B + Cs^3 + Ds^2}{s^2(s^2 + 4)} \\ &= \frac{0s + \frac{1}{4}}{s^2} + \frac{0s - \frac{1}{4}}{s^2 + 4} = \frac{\frac{1}{4}}{s^2} - \frac{\frac{1}{4}}{s^2 + 4}. \end{aligned}$$

4.5 ODEs with Discontinuous Forcing Functions



Using partial fractions, we calculate (please check!) that

$$\begin{aligned} H(s) &= \frac{1}{s^2(s^2 + 4)} = \frac{As + B}{s^2} + \frac{Cs + D}{s^2 + 4} \\ &= \frac{As^3 + Bs^2 + 4As + 4B + Cs^3 + Ds^2}{s^2(s^2 + 4)} \\ &= \frac{0s + \frac{1}{4}}{s^2} + \frac{0s - \frac{1}{4}}{s^2 + 4} = \frac{\frac{1}{4}}{s^2} - \frac{\frac{1}{4}}{s^2 + 4}. \end{aligned}$$

Hence

$$h(t) = \frac{1}{4} \mathcal{L}^{-1} \left[\frac{1}{s^2} \right] - \frac{1}{8} \mathcal{L}^{-1} \left[\frac{2}{s^2 + 4} \right] = \quad .$$

4.5 ODEs with Discontinuous Forcing Functions



Using partial fractions, we calculate (please check!) that

$$\begin{aligned} H(s) &= \frac{1}{s^2(s^2 + 4)} = \frac{As + B}{s^2} + \frac{Cs + D}{s^2 + 4} \\ &= \frac{As^3 + Bs^2 + 4As + 4B + Cs^3 + Ds^2}{s^2(s^2 + 4)} \\ &= \frac{0s + \frac{1}{4}}{s^2} + \frac{0s - \frac{1}{4}}{s^2 + 4} = \frac{\frac{1}{4}}{s^2} - \frac{\frac{1}{4}}{s^2 + 4}. \end{aligned}$$

Hence

$$h(t) = \frac{1}{4} \mathcal{L}^{-1} \left[\frac{1}{s^2} \right] - \frac{1}{8} \mathcal{L}^{-1} \left[\frac{2}{s^2 + 4} \right] = \frac{t}{4} - \frac{1}{8} \sin 2t.$$

4.5 ODEs with Discontinuous Forcing Functions



$$u_c(t)h(t-c)(s) = \mathcal{L}^{-1} [e^{-cs}H(s)]$$

Therefore

$$y(t) = \mathcal{L}^{-1} \left[\frac{1}{5}e^{-5s}H(s) - \frac{1}{5}e^{-10s}H(s) \right]$$

=

=

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4.5 ODEs with Discontinuous Forcing Functions



$$u_c(t)h(t-c)(s) = \mathcal{L}^{-1} [e^{-cs}H(s)]$$

Therefore

$$\begin{aligned} y(t) &= \mathcal{L}^{-1} \left[\frac{1}{5}e^{-5s}H(s) - \frac{1}{5}e^{-10s}H(s) \right] \\ &= \frac{1}{5}u_5(t)h(t-5) - \frac{1}{5}u_{10}(t)h(t-10) \\ &= \end{aligned}$$

4.5 ODEs with Discontinuous Forcing Functions



$$u_c(t)h(t-c)(s) = \mathcal{L}^{-1} [e^{-cs}H(s)]$$

Therefore

$$\begin{aligned}y(t) &= \mathcal{L}^{-1} \left[\frac{1}{5}e^{-5s}H(s) - \frac{1}{5}e^{-10s}H(s) \right] \\&= \frac{1}{5}u_5(t)h(t-5) - \frac{1}{5}u_{10}(t)h(t-10) \\&= u_5(t) \left(\frac{t-5}{20} - \frac{1}{40}\sin(2t-10) \right) \\&\quad - u_{10}(t) \left(\frac{t-10}{20} - \frac{1}{40}\sin(2t-20) \right).\end{aligned}$$

4.5 ODEs with Discontinuous Forcing Functions



Example

Solve

$$\begin{cases} y'' + 3y' + 2y = f(t) = \begin{cases} 1 & 0 \leq t < 10 \\ 0 & 10 \leq t \end{cases} \\ y(0) = 1 \\ y'(0) = 0. \end{cases}$$

4.5 ODEs with Discontinuous Forcing Functions



Example

Solve

$$\begin{cases} y'' + 3y' + 2y = f(t) = \begin{cases} 1 & 0 \leq t < 10 \\ 0 & 10 \leq t \end{cases} \\ y(0) = 1 \\ y'(0) = 0. \end{cases}$$

Since $f(t) = 1 - u_{10}(t)$, the Laplace Transform of the ODE is

$$(s^2 + 3s + 2)Y - (s + 3) = \frac{1 - e^{-10s}}{s}.$$

4.5 ODEs with Discontinuous Forcing Functions



Thus

$$\begin{aligned} Y(s) &= \frac{1 - e^{-10s}}{s(s^2 + 3s + 2)} + \frac{s + 3}{s^2 + 3s + 2} \\ &= \frac{(s^2 + 3s + 1) - e^{-10s}}{s(s^2 + 3s + 2)}. \end{aligned}$$

4.5 ODEs with Discontinuous Forcing Functions



Thus

$$\begin{aligned} Y(s) &= \frac{1 - e^{-10s}}{s(s^2 + 3s + 2)} + \frac{s + 3}{s^2 + 3s + 2} \\ &= \frac{(s^2 + 3s + 1) - e^{-10s}}{s(s^2 + 3s + 2)}. \end{aligned}$$

Let

$$G(s) = \frac{s^2 + 3s + 1}{s(s^2 + 3s + 2)} \quad \text{and} \quad H(s) = \frac{1}{s(s^2 + 3s + 2)}.$$

4.5 ODEs with Discontinuous Forcing Functions



Thus

$$\begin{aligned} Y(s) &= \frac{1 - e^{-10s}}{s(s^2 + 3s + 2)} + \frac{s + 3}{s^2 + 3s + 2} \\ &= \frac{(s^2 + 3s + 1) - e^{-10s}}{s(s^2 + 3s + 2)}. \end{aligned}$$

Let

$$G(s) = \frac{s^2 + 3s + 1}{s(s^2 + 3s + 2)} \quad \text{and} \quad H(s) = \frac{1}{s(s^2 + 3s + 2)}.$$

Then $Y = G(s) - e^{-10s}H(s)$.

4.5 ODEs with Discontinuous Forcing Functions



Thus

$$\begin{aligned} Y(s) &= \frac{1 - e^{-10s}}{s(s^2 + 3s + 2)} + \frac{s + 3}{s^2 + 3s + 2} \\ &= \frac{(s^2 + 3s + 1) - e^{-10s}}{s(s^2 + 3s + 2)}. \end{aligned}$$

Let

$$G(s) = \frac{s^2 + 3s + 1}{s(s^2 + 3s + 2)} \quad \text{and} \quad H(s) = \frac{1}{s(s^2 + 3s + 2)}.$$

Then $Y = G(s) - e^{-10s}H(s)$. If we can find $g(t)$ and $h(t)$, then we can find $y(t)$.

4.5 ODEs with Discontinuous Forcing Functions



Using partial fractions we get

$$G(s) = \frac{A}{s} + \frac{B}{s+1} + \frac{C}{s+2} = \frac{\frac{1}{2}}{s} + \frac{1}{s+1} - \frac{\frac{1}{2}}{s+2}$$

and

$$H(s) = \frac{D}{s} + \frac{E}{s+1} + \frac{F}{s+2} = \frac{\frac{1}{2}}{s} - \frac{1}{s+1} + \frac{\frac{1}{2}}{s+2}$$

(please check!).

4.5 ODEs with Discontinuous Forcing Functions



Using partial fractions we get

$$G(s) = \frac{A}{s} + \frac{B}{s+1} + \frac{C}{s+2} = \frac{\frac{1}{2}}{s} + \frac{1}{s+1} - \frac{\frac{1}{2}}{s+2}$$

and

$$H(s) = \frac{D}{s} + \frac{E}{s+1} + \frac{F}{s+2} = \frac{\frac{1}{2}}{s} - \frac{1}{s+1} + \frac{\frac{1}{2}}{s+2}$$

(please check!). It follows that

$$g(t) = \frac{1}{2} (1 + 2e^{-t} - e^{-2t}) \quad \text{and} \quad h(t) = \frac{1}{2} (1 - 2e^{-t} + e^{-2t}).$$

4.5 ODEs with Discontinuous Forcing Functions



$$g(t) = \frac{1}{2} (1 + 2e^{-t} - e^{-2t})$$

$$h(t) = \frac{1}{2} (1 - 2e^{-t} + e^{-2t})$$

Therefore

$$y(t) = \mathcal{L}^{-1} [Y]$$

$$=$$
$$=$$
$$=$$

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4.5 ODEs with Discontinuous Forcing Functions



$$g(t) = \frac{1}{2} (1 + 2e^{-t} - e^{-2t})$$

$$h(t) = \frac{1}{2} (1 - 2e^{-t} + e^{-2t})$$

Therefore

$$\begin{aligned} y(t) &= \mathcal{L}^{-1} [Y] \\ &= \mathcal{L}^{-1} [G(s) - e^{-10s}H(s)] \\ &= \\ &= \end{aligned}$$

4.5 ODEs with Discontinuous Forcing Functions



$$g(t) = \frac{1}{2} (1 + 2e^{-t} - e^{-2t})$$

$$h(t) = \frac{1}{2} (1 - 2e^{-t} + e^{-2t})$$

Therefore

$$\begin{aligned} y(t) &= \mathcal{L}^{-1} [Y] \\ &= \mathcal{L}^{-1} [G(s) - e^{-10s} H(s)] \\ &= g(t) - u_{10}(t)h(t - 10) \\ &= \end{aligned}$$

4.5 ODEs with Discontinuous Forcing Functions



$$g(t) = \frac{1}{2} (1 + 2e^{-t} - e^{-2t})$$

$$h(t) = \frac{1}{2} (1 - 2e^{-t} + e^{-2t})$$

Therefore

$$\begin{aligned} y(t) &= \mathcal{L}^{-1} [Y] \\ &= \mathcal{L}^{-1} [G(s) - e^{-10s} H(s)] \\ &= g(t) - u_{10}(t)h(t-10) \\ &= \frac{1}{2} (1 + 2e^{-t} - e^{-2t}) - \frac{1}{2} u_{10}(t) (1 - 2e^{-(t-10)} + e^{-2(t-10)}). \end{aligned}$$

4.5 ODEs with Discontinuous Forcing Functions



Example

Solve

$$\begin{cases} y'' + 4y = u_{\pi}(t) - u_{3\pi}(t) \\ y(0) = 0 \\ y'(0) = 0. \end{cases}$$

4.5 ODEs with Discontinuous Forcing Functions



Example

Solve

$$\begin{cases} y'' + 4y = u_{\pi}(t) - u_{3\pi}(t) \\ y(0) = 0 \\ y'(0) = 0. \end{cases}$$

Taking the Laplace Transform of the ODE gives

$$(s^2 + 4)Y(s) = \frac{e^{-\pi s} - e^{-3\pi s}}{s}.$$

4.5 ODEs with Discontinuous Forcing Functions



Example

Solve

$$\begin{cases} y'' + 4y = u_{\pi}(t) - u_{3\pi}(t) \\ y(0) = 0 \\ y'(0) = 0. \end{cases}$$

Taking the Laplace Transform of the ODE gives

$$(s^2 + 4)Y(s) = \frac{e^{-\pi s} - e^{-3\pi s}}{s}.$$

Thus

$$Y(s) = \frac{e^{-\pi s} - e^{-3\pi s}}{s(s^2 + 4)}.$$

4.5 ODEs with Discontinuous Forcing Functions



Example

Solve

$$\begin{cases} y'' + 4y = u_{\pi}(t) - u_{3\pi}(t) \\ y(0) = 0 \\ y'(0) = 0. \end{cases}$$

Taking the Laplace Transform of the ODE gives

$$(s^2 + 4)Y(s) = \frac{e^{-\pi s} - e^{-3\pi s}}{s}.$$

Thus

$$Y(s) = \frac{e^{-\pi s} - e^{-3\pi s}}{s(s^2 + 4)}.$$

Let

$$H(s) = \frac{1}{s(s^2 + 4)}.$$

4.5 ODEs with Discontinuous Forcing Functions



Using partial fractions, we calculate that

$$\begin{aligned} H(s) &= \frac{1}{s(s^2 + 4)} = \frac{A}{s} + \frac{Bs + C}{s^2 + 4} = \frac{\frac{1}{4}}{s} + \frac{-\frac{1}{4}s + 0}{s^2 + 4} \\ &= \frac{1}{4} \left(\frac{1}{s} \right) - \frac{1}{4} \left(\frac{s}{s^2 + 4} \right) = \frac{1}{4} \mathcal{L} [1] - \frac{1}{4} \mathcal{L} [\cos 2t] . \end{aligned}$$

4.5 ODEs with Discontinuous Forcing Functions



Using partial fractions, we calculate that

$$\begin{aligned} H(s) &= \frac{1}{s(s^2 + 4)} = \frac{A}{s} + \frac{Bs + C}{s^2 + 4} = \frac{\frac{1}{4}}{s} + \frac{-\frac{1}{4}s + 0}{s^2 + 4} \\ &= \frac{1}{4} \left(\frac{1}{s} \right) - \frac{1}{4} \left(\frac{s}{s^2 + 4} \right) = \frac{1}{4} \mathcal{L}[1] - \frac{1}{4} \mathcal{L}[\cos 2t]. \end{aligned}$$

It follows that

$$h(t) = \frac{1}{4} - \frac{1}{4} \cos 2t$$

4.5 ODEs with Discontinuous Forcing Functions



Using partial fractions, we calculate that

$$\begin{aligned} H(s) &= \frac{1}{s(s^2 + 4)} = \frac{A}{s} + \frac{Bs + C}{s^2 + 4} = \frac{\frac{1}{4}}{s} + \frac{-\frac{1}{4}s + 0}{s^2 + 4} \\ &= \frac{1}{4} \left(\frac{1}{s} \right) - \frac{1}{4} \left(\frac{s}{s^2 + 4} \right) = \frac{1}{4} \mathcal{L}[1] - \frac{1}{4} \mathcal{L}[\cos 2t]. \end{aligned}$$

It follows that

$$h(t) = \frac{1}{4} - \frac{1}{4} \cos 2t$$

and the solution to the IVP is

$$\begin{aligned} y(t) &= \mathcal{L}^{-1} [e^{-\pi s} H(s)] - \mathcal{L}^{-1} [e^{-3\pi s} H(s)] \\ &= \\ &= \end{aligned}$$

4.5 ODEs with Discontinuous Forcing Functions



Using partial fractions, we calculate that

$$\begin{aligned} H(s) &= \frac{1}{s(s^2 + 4)} = \frac{A}{s} + \frac{Bs + C}{s^2 + 4} = \frac{\frac{1}{4}}{s} + \frac{-\frac{1}{4}s + 0}{s^2 + 4} \\ &= \frac{1}{4} \left(\frac{1}{s} \right) - \frac{1}{4} \left(\frac{s}{s^2 + 4} \right) = \frac{1}{4} \mathcal{L}[1] - \frac{1}{4} \mathcal{L}[\cos 2t]. \end{aligned}$$

It follows that

$$h(t) = \frac{1}{4} - \frac{1}{4} \cos 2t$$

and the solution to the IVP is

$$\begin{aligned} y(t) &= \mathcal{L}^{-1} [e^{-\pi s} H(s)] - \mathcal{L}^{-1} [e^{-3\pi s} H(s)] \\ &= u_{\pi}(t)h(t - \pi) - u_{3\pi}(t)h(t - 3\pi) \\ &= \end{aligned}$$

4.5 ODEs with Discontinuous Forcing Functions



Using partial fractions, we calculate that

$$\begin{aligned} H(s) &= \frac{1}{s(s^2 + 4)} = \frac{A}{s} + \frac{Bs + C}{s^2 + 4} = \frac{\frac{1}{4}}{s} + \frac{-\frac{1}{4}s + 0}{s^2 + 4} \\ &= \frac{1}{4} \left(\frac{1}{s} \right) - \frac{1}{4} \left(\frac{s}{s^2 + 4} \right) = \frac{1}{4} \mathcal{L} [1] - \frac{1}{4} \mathcal{L} [\cos 2t] . \end{aligned}$$

It follows that

$$h(t) = \frac{1}{4} - \frac{1}{4} \cos 2t$$

and the solution to the IVP is

$$\begin{aligned} y(t) &= \mathcal{L}^{-1} [e^{-\pi s} H(s)] - \mathcal{L}^{-1} [e^{-3\pi s} H(s)] \\ &= u_{\pi}(t)h(t - \pi) - u_{3\pi}(t)h(t - 3\pi) \\ &= \frac{1}{4}u_{\pi}(t)(1 - \cos(2t - 2\pi)) - \frac{1}{4}u_{3\pi}(t)(1 - \cos(2t - 6\pi)). \end{aligned}$$

The Convolution Integral

4.6 The Convolution Integral



Let $f : [0, \infty) \rightarrow \mathbb{R}$ and $g : [0, \infty) \rightarrow \mathbb{R}$ be piecewise continuous functions.

Definition

The *convolution* of f and g is

$$(f * g)(t) = \int_0^t f(\tau)g(t - \tau) d\tau.$$

$$(f * g)(t) = \int_0^t f(\tau)g(t - \tau) d\tau$$



Theorem (Properties)

- $f * g = g * f$
- $f * (g * h) = (f * g) * h$
- $f * (g + h) = (f * g) + (f * h)$
- $f * 0 = 0 = 0 * f$

$$(f * g)(t) = \int_0^t f(\tau)g(t - \tau) d\tau$$



Theorem (Properties)

- $f * g = g * f$
- $f * (g * h) = (f * g) * h$
- $f * (g + h) = (f * g) + (f * h)$
- $f * 0 = 0 = 0 * f$

Example

$$(\cos * 1)(t) = \int_0^t \cos \tau \cdot 1 d\tau = [\sin \tau]_0^t = \sin t - \sin 0 = \sin t$$

$$(1 * \cos)(t) =$$

$$(f * g)(t) = \int_0^t f(\tau)g(t - \tau) d\tau$$



Theorem (Properties)

- $f * g = g * f$
- $f * (g * h) = (f * g) * h$
- $f * (g + h) = (f * g) + (f * h)$
- $f * 0 = 0 = 0 * f$

Example

$$(\cos * 1)(t) = \int_0^t \cos \tau \cdot 1 d\tau = [\sin \tau]_0^t = \sin t - \sin 0 = \sin t$$

$$\begin{aligned}(1 * \cos)(t) &= \int_0^t 1 \cdot \cos(t - \tau) d\tau = [-\sin(t - \tau)]_0^t \\ &= -\sin 0 + \sin t = \sin t\end{aligned}$$

$$(f * g)(t) = \int_0^t f(\tau)g(t - \tau) d\tau$$



Theorem (Properties)

- $f * g = g * f$
- $f * (g * h) = (f * g) * h$
- $f * (g + h) = (f * g) + (f * h)$
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Example

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$$\begin{aligned}(1 * \cos)(t) &= \int_0^t 1 \cdot \cos(t - \tau) d\tau = [-\sin(t - \tau)]_0^t \\ &= -\sin 0 + \sin t = \sin t\end{aligned}$$

Note that $f * 1 \neq f$ in general.

$$(f * g)(t) = \int_0^t f(\tau)g(t - \tau) d\tau$$



Example

$$(\sin * \sin)(t) = \int_0^t \sin \tau \sin(t - \tau) d\tau$$

$$=$$
$$=$$
$$=$$
$$=$$
$$=$$

$$(f * g)(t) = \int_0^t f(\tau)g(t - \tau) d\tau$$



Example

$$\begin{aligned}(\sin * \sin)(t) &= \int_0^t \sin \tau \sin(t - \tau) d\tau \\&= \int_0^t \sin \tau (\sin t \cos \tau - \cos t \sin \tau) d\tau \\&= \\&= \\&= \\&= \end{aligned}$$

$$(f * g)(t) = \int_0^t f(\tau)g(t - \tau) d\tau$$



Example

$$\begin{aligned}(\sin * \sin)(t) &= \int_0^t \sin \tau \sin(t - \tau) d\tau \\&= \int_0^t \sin \tau (\sin t \cos \tau - \cos t \sin \tau) d\tau \\&= \sin t \int_0^t \sin \tau \cos \tau d\tau - \cos t \int_0^t \sin^2 \tau d\tau \\&= \\&= \\&= \end{aligned}$$

$$(f * g)(t) = \int_0^t f(\tau)g(t - \tau) d\tau$$



Example

$$\begin{aligned}(\sin * \sin)(t) &= \int_0^t \sin \tau \sin(t - \tau) d\tau \\&= \int_0^t \sin \tau (\sin t \cos \tau - \cos t \sin \tau) d\tau \\&= \sin t \int_0^t \sin \tau \cos \tau d\tau - \cos t \int_0^t \sin^2 \tau d\tau \\&= \sin t \left[-\frac{1}{2} \cos^2 \tau \right]_0^t - \cos t \left[\frac{1}{2} (\tau - \sin \tau \cos \tau) \right]_0^t \\&= \\&= \end{aligned}$$

$$(f * g)(t) = \int_0^t f(\tau)g(t - \tau) d\tau$$



Example

$$\begin{aligned}(\sin * \sin)(t) &= \int_0^t \sin \tau \sin(t - \tau) d\tau \\&= \int_0^t \sin \tau (\sin t \cos \tau - \cos t \sin \tau) d\tau \\&= \sin t \int_0^t \sin \tau \cos \tau d\tau - \cos t \int_0^t \sin^2 \tau d\tau \\&= \sin t \left[-\frac{1}{2} \cos^2 \tau \right]_0^t - \cos t \left[\frac{1}{2} (\tau - \sin \tau \cos \tau) \right]_0^t \\&= \frac{1}{2} \sin t (1 - \cos^2 t) - \frac{1}{2} \cos t (t - \sin t \cos t) \\&= \end{aligned}$$

$$(f * g)(t) = \int_0^t f(\tau)g(t - \tau) d\tau$$



Example

$$\begin{aligned}(\sin * \sin)(t) &= \int_0^t \sin \tau \sin(t - \tau) d\tau \\&= \int_0^t \sin \tau (\sin t \cos \tau - \cos t \sin \tau) d\tau \\&= \sin t \int_0^t \sin \tau \cos \tau d\tau - \cos t \int_0^t \sin^2 \tau d\tau \\&= \sin t \left[-\frac{1}{2} \cos^2 \tau \right]_0^t - \cos t \left[\frac{1}{2} (\tau - \sin \tau \cos \tau) \right]_0^t \\&= \frac{1}{2} \sin t (1 - \cos^2 t) - \frac{1}{2} \cos t (t - \sin t \cos t) \\&= \frac{1}{2} \sin t - \frac{t}{2} \cos t.\end{aligned}$$

$$(f * g)(t) = \int_0^t f(\tau)g(t - \tau) d\tau$$



Example

$$\begin{aligned}(\sin * \sin)(t) &= \int_0^t \sin \tau \sin(t - \tau) d\tau \\&= \int_0^t \sin \tau (\sin t \cos \tau - \cos t \sin \tau) d\tau \\&= \sin t \int_0^t \sin \tau \cos \tau d\tau - \cos t \int_0^t \sin^2 \tau d\tau \\&= \sin t \left[-\frac{1}{2} \cos^2 \tau \right]_0^t - \cos t \left[\frac{1}{2} (\tau - \sin \tau \cos \tau) \right]_0^t \\&= \frac{1}{2} \sin t (1 - \cos^2 t) - \frac{1}{2} \cos t (t - \sin t \cos t) \\&= \frac{1}{2} \sin t - \frac{t}{2} \cos t.\end{aligned}$$

Note that $f * f \geq 0$ is not true in general.

$$(f * g)(t) = \int_0^t f(\tau)g(t - \tau) d\tau$$



Theorem

$$\mathcal{L}[f * g](s) = F(s)G(s)$$

$$(f * g)(t) = \int_0^t f(\tau)g(t - \tau) d\tau$$



Theorem

$$\mathcal{L} [f * g] (s) = F(s)G(s)$$

This means that $\mathcal{L}^{-1} [FG] = f * g$.

$$(f * g)(t) = \int_0^t f(\tau)g(t - \tau) d\tau$$



Example

Find the inverse Laplace Transform of $H(s) = \frac{a}{s^2(s^2 + a^2)}$.

$$(f * g)(t) = \int_0^t f(\tau)g(t - \tau) d\tau$$



Example

Find the inverse Laplace Transform of $H(s) = \frac{a}{s^2(s^2 + a^2)}$.

Note that $H(s) = \left(\frac{1}{s^2}\right) \left(\frac{a}{s^2 + a^2}\right)$.

$$(f * g)(t) = \int_0^t f(\tau)g(t - \tau) d\tau$$



Example

Find the inverse Laplace Transform of $H(s) = \frac{a}{s^2(s^2 + a^2)}$.

Note that $H(s) = \left(\frac{1}{s^2}\right) \left(\frac{a}{s^2 + a^2}\right)$. We know that $\mathcal{L}[t] = \frac{1}{s^2}$ and $\mathcal{L}[\sin at] = \frac{a}{s^2 + a^2}$.

$$(f * g)(t) = \int_0^t f(\tau)g(t - \tau) d\tau$$



Example

Find the inverse Laplace Transform of $H(s) = \frac{a}{s^2(s^2 + a^2)}$.

Note that $H(s) = \left(\frac{1}{s^2}\right) \left(\frac{a}{s^2 + a^2}\right)$. We know that $\mathcal{L}[t] = \frac{1}{s^2}$ and $\mathcal{L}[\sin at] = \frac{a}{s^2 + a^2}$. So

$$h(t) = \mathcal{L}^{-1} \left[\left(\frac{1}{s^2}\right) \left(\frac{a}{s^2 + a^2}\right) \right] =$$

$$=$$

$$=$$

$$(f * g)(t) = \int_0^t f(\tau)g(t - \tau) d\tau$$



Example

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Note that $H(s) = \left(\frac{1}{s^2}\right) \left(\frac{a}{s^2 + a^2}\right)$. We know that $\mathcal{L}[t] = \frac{1}{s^2}$ and $\mathcal{L}[\sin at] = \frac{a}{s^2 + a^2}$. So

$$h(t) = \mathcal{L}^{-1} \left[\left(\frac{1}{s^2}\right) \left(\frac{a}{s^2 + a^2}\right) \right] = \mathcal{L}^{-1} \left[\frac{1}{s^2} \right] * \mathcal{L}^{-1} \left[\frac{a}{s^2 + a^2} \right]$$

=

=

$$(f * g)(t) = \int_0^t f(\tau)g(t - \tau) d\tau$$



Example

Find the inverse Laplace Transform of $H(s) = \frac{a}{s^2(s^2 + a^2)}$.

Note that $H(s) = \left(\frac{1}{s^2}\right) \left(\frac{a}{s^2 + a^2}\right)$. We know that $\mathcal{L}[t] = \frac{1}{s^2}$ and $\mathcal{L}[\sin at] = \frac{a}{s^2 + a^2}$. So

$$\begin{aligned} h(t) &= \mathcal{L}^{-1} \left[\left(\frac{1}{s^2}\right) \left(\frac{a}{s^2 + a^2}\right) \right] = \mathcal{L}^{-1} \left[\frac{1}{s^2} \right] * \mathcal{L}^{-1} \left[\frac{a}{s^2 + a^2} \right] \\ &= t * \sin at = \int_0^t \tau \sin a(t - \tau) d\tau \\ &= \end{aligned}$$

$$(f * g)(t) = \int_0^t f(\tau)g(t - \tau) d\tau$$



Example

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$$\begin{aligned} h(t) &= \mathcal{L}^{-1} \left[\left(\frac{1}{s^2}\right) \left(\frac{a}{s^2 + a^2}\right) \right] = \mathcal{L}^{-1} \left[\frac{1}{s^2} \right] * \mathcal{L}^{-1} \left[\frac{a}{s^2 + a^2} \right] \\ &= t * \sin at = \int_0^t \tau \sin a(t - \tau) d\tau \\ &= \frac{at - \sin at}{a^2}. \end{aligned}$$

$$(f * g)(t) = \int_0^t f(\tau)g(t - \tau) d\tau$$



Example

Solve

$$\begin{cases} y'' + 4y = g(t) \\ y(0) = 3 \\ y'(0) = -1. \end{cases}$$

$$(f * g)(t) = \int_0^t f(\tau)g(t - \tau) d\tau$$



Example

Solve

$$\begin{cases} y'' + 4y = g(t) \\ y(0) = 3 \\ y'(0) = -1. \end{cases}$$

Taking the Laplace Transform of the ODE gives

$$(s^2Y - 3s + 1) + 4Y = G(s)$$

$$(f * g)(t) = \int_0^t f(\tau)g(t - \tau) d\tau$$



Example

Solve

$$\begin{cases} y'' + 4y = g(t) \\ y(0) = 3 \\ y'(0) = -1. \end{cases}$$

Taking the Laplace Transform of the ODE gives

$$(s^2Y - 3s + 1) + 4Y = G(s)$$

which rearranges to

$$\begin{aligned} Y(s) &= \frac{3s - 1}{s^2 + 4} + \frac{G(s)}{s^2 + 4} \\ &= 3 \left(\frac{s}{s^2 + 4} \right) - \frac{1}{2} \left(\frac{2}{s^2 + 4} \right) + \frac{1}{2} \left(\frac{2}{s^2 + 4} \right) G(s). \end{aligned}$$

$$(f * g)(t) = \int_0^t f(\tau)g(t - \tau) d\tau$$



$$Y(s) = 3 \left(\frac{s}{s^2 + 4} \right) - \frac{1}{2} \left(\frac{2}{s^2 + 4} \right) + \frac{1}{2} \left(\frac{2}{s^2 + 4} \right) G(s)$$

Hence the solution to the IVP is

$$\begin{aligned} y(t) &= 3\mathcal{L}^{-1} \left[\frac{s}{s^2 + 4} \right] - \frac{1}{2}\mathcal{L}^{-1} \left[\frac{2}{s^2 + 4} \right] + \frac{1}{2}\mathcal{L}^{-1} \left[\left(\frac{2}{s^2 + 4} \right) G(s) \right] \\ &= \\ &= \end{aligned}$$

$$(f * g)(t) = \int_0^t f(\tau)g(t - \tau) d\tau$$



$$Y(s) = 3 \left(\frac{s}{s^2 + 4} \right) - \frac{1}{2} \left(\frac{2}{s^2 + 4} \right) + \frac{1}{2} \left(\frac{2}{s^2 + 4} \right) G(s)$$

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$$(f * g)(t) = \int_0^t f(\tau)g(t - \tau) d\tau$$



$$Y(s) = 3 \left(\frac{s}{s^2 + 4} \right) - \frac{1}{2} \left(\frac{2}{s^2 + 4} \right) + \frac{1}{2} \left(\frac{2}{s^2 + 4} \right) G(s)$$

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$$(f * g)(t) = \int_0^t f(\tau)g(t - \tau) d\tau$$



Example

Find the inverse Laplace Transform of $\frac{2}{(s-1)(s^2+4)}$.

$$(f * g)(t) = \int_0^t f(\tau)g(t - \tau) d\tau$$



Example

Find the inverse Laplace Transform of $\frac{2}{(s-1)(s^2+4)}$.

$$\mathcal{L}^{-1} \left[\frac{2}{(s-1)(s^2+4)} \right] = \mathcal{L}^{-1} \left[\left(\frac{2}{s^2+4} \right) \left(\frac{1}{s-1} \right) \right]$$

$$(f * g)(t) = \int_0^t f(\tau)g(t - \tau) d\tau$$



Example

Find the inverse Laplace Transform of $\frac{2}{(s-1)(s^2+4)}$.

$$\mathcal{L}^{-1} \left[\frac{2}{(s-1)(s^2+4)} \right] = \mathcal{L}^{-1} \left[\left(\frac{2}{s^2+4} \right) \left(\frac{1}{s-1} \right) \right] = \sin 2t * e^t$$

$$(f * g)(t) = \int_0^t f(\tau)g(t - \tau) d\tau$$



Example

Find the inverse Laplace Transform of $\frac{2}{(s-1)(s^2+4)}$.

$$\begin{aligned}\mathcal{L}^{-1} \left[\frac{2}{(s-1)(s^2+4)} \right] &= \mathcal{L}^{-1} \left[\left(\frac{2}{s^2+4} \right) \left(\frac{1}{s-1} \right) \right] = \sin 2t * e^t \\ &= \int_0^t e^{t-\tau} \sin 2\tau d\tau\end{aligned}$$

$$(f * g)(t) = \int_0^t f(\tau)g(t - \tau) d\tau$$



Example

Find the inverse Laplace Transform of $\frac{2}{(s-1)(s^2+4)}$.

$$\begin{aligned}\mathcal{L}^{-1}\left[\frac{2}{(s-1)(s^2+4)}\right] &= \mathcal{L}^{-1}\left[\left(\frac{2}{s^2+4}\right)\left(\frac{1}{s-1}\right)\right] = \sin 2t * e^t \\ &= \int_0^t e^{t-\tau} \sin 2\tau d\tau = e^t \int_0^t e^{-\tau} \sin 2\tau d\tau\end{aligned}$$

$$(f * g)(t) = \int_0^t f(\tau)g(t - \tau) d\tau$$



Example

Find the inverse Laplace Transform of $\frac{2}{(s-1)(s^2+4)}$.

$$\begin{aligned}\mathcal{L}^{-1} \left[\frac{2}{(s-1)(s^2+4)} \right] &= \mathcal{L}^{-1} \left[\left(\frac{2}{s^2+4} \right) \left(\frac{1}{s-1} \right) \right] = \sin 2t * e^t \\ &= \int_0^t e^{t-\tau} \sin 2\tau d\tau = e^t \int_0^t e^{-\tau} \sin 2\tau d\tau \\ &= e^t \left[\frac{e^{-\tau}}{5} (-\sin 2\tau - 2 \cos 2\tau) \right]_0^t\end{aligned}$$

$$(f * g)(t) = \int_0^t f(\tau)g(t - \tau) d\tau$$



Example

Find the inverse Laplace Transform of $\frac{2}{(s-1)(s^2+4)}$.

$$\begin{aligned}\mathcal{L}^{-1} \left[\frac{2}{(s-1)(s^2+4)} \right] &= \mathcal{L}^{-1} \left[\left(\frac{2}{s^2+4} \right) \left(\frac{1}{s-1} \right) \right] = \sin 2t * e^t \\&= \int_0^t e^{t-\tau} \sin 2\tau d\tau = e^t \int_0^t e^{-\tau} \sin 2\tau d\tau \\&= e^t \left[\frac{e^{-\tau}}{5} (-\sin 2\tau - 2 \cos 2\tau) \right]_0^t \\&= \frac{2}{5}e^t - \frac{1}{5} \sin 2t - \frac{2}{5} \cos 2t.\end{aligned}$$

$$(f * g)(t) = \int_0^t f(\tau)g(t - \tau) d\tau$$



Example

Solve

$$\begin{cases} 4y'' + y = g(t) \\ y(0) = 3 \\ y'(0) = -7. \end{cases}$$

$$(f * g)(t) = \int_0^t f(\tau)g(t - \tau) d\tau$$



Example

Solve

$$\begin{cases} 4y'' + y = g(t) \\ y(0) = 3 \\ y'(0) = -7. \end{cases}$$

$$4y'' + y = g(t)$$

$$(f * g)(t) = \int_0^t f(\tau)g(t - \tau) d\tau$$



Example

Solve

$$\begin{cases} 4y'' + y = g(t) \\ y(0) = 3 \\ y'(0) = -7. \end{cases}$$

$$\mathcal{L}[4y'' + y] = \mathcal{L}[g(t)]$$

$$(f * g)(t) = \int_0^t f(\tau)g(t - \tau) d\tau$$



Example

Solve

$$\begin{cases} 4y'' + y = g(t) \\ y(0) = 3 \\ y'(0) = -7. \end{cases}$$

$$\mathcal{L}[4y'' + y] = \mathcal{L}[g(t)]$$

$$4(s^2Y - sy(0) - y'(0)) + Y = G(s)$$

$$(f * g)(t) = \int_0^t f(\tau)g(t - \tau) d\tau$$



Example

Solve

$$\begin{cases} 4y'' + y = g(t) \\ y(0) = 3 \\ y'(0) = -7. \end{cases}$$

$$\mathcal{L}[4y'' + y] = \mathcal{L}[g(t)]$$

$$4(s^2Y - 3s + 7) + Y = G(s)$$

$$(f * g)(t) = \int_0^t f(\tau)g(t - \tau) d\tau$$



Example

Solve

$$\begin{cases} 4y'' + y = g(t) \\ y(0) = 3 \\ y'(0) = -7. \end{cases}$$

$$\mathcal{L}[4y'' + y] = \mathcal{L}[g(t)]$$

$$(4s^2 + 1)Y - 12s + 28 = G(s)$$

$$(f * g)(t) = \int_0^t f(\tau)g(t - \tau) d\tau$$



Example

Solve

$$\begin{cases} 4y'' + y = g(t) \\ y(0) = 3 \\ y'(0) = -7. \end{cases}$$

$$\mathcal{L}[4y'' + y] = \mathcal{L}[g(t)]$$

$$(4s^2 + 1)Y = 12s - 28 + G(s)$$

$$(f * g)(t) = \int_0^t f(\tau)g(t - \tau) d\tau$$



Example

Solve

$$\begin{cases} 4y'' + y = g(t) \\ y(0) = 3 \\ y'(0) = -7. \end{cases}$$

$$\mathcal{L}[4y'' + y] = \mathcal{L}[g(t)]$$

$$4 \left(s^2 + \frac{1}{4} \right) Y = 12s - 28 + G(s)$$

$$(f * g)(t) = \int_0^t f(\tau)g(t - \tau) d\tau$$



Example

Solve

$$\begin{cases} 4y'' + y = g(t) \\ y(0) = 3 \\ y'(0) = -7. \end{cases}$$

$$\mathcal{L}[4y'' + y] = \mathcal{L}[g(t)]$$

$$Y = \frac{12s}{4(s^2 + \frac{1}{4})} - \frac{28}{4(s^2 + \frac{1}{4})} + \frac{G(s)}{4(s^2 + \frac{1}{4})}$$

$$(f * g)(t) = \int_0^t f(\tau)g(t - \tau) d\tau$$



Example

Solve

$$\begin{cases} 4y'' + y = g(t) \\ y(0) = 3 \\ y'(0) = -7. \end{cases}$$

$$Y = \frac{3s}{s^2 + \frac{1}{4}} - \frac{7}{s^2 + \frac{1}{4}} + G(s) \frac{\frac{1}{4}}{s^2 + \frac{1}{4}}$$

$$(f * g)(t) = \int_0^t f(\tau)g(t - \tau) d\tau$$



Example

Solve

$$\begin{cases} 4y'' + y = g(t) \\ y(0) = 3 \\ y'(0) = -7. \end{cases}$$

$$Y = 3 \left(\frac{s}{s^2 + \frac{1}{4}} \right) - 14 \left(\frac{\frac{1}{2}}{s^2 + \frac{1}{4}} \right) + \frac{1}{2}G(s) \left(\frac{\frac{1}{2}}{s^2 + \frac{1}{4}} \right)$$

$$(f * g)(t) = \int_0^t f(\tau)g(t - \tau) d\tau$$



Example

Solve

$$\begin{cases} 4y'' + y = g(t) \\ y(0) = 3 \\ y'(0) = -7. \end{cases}$$

$$Y = 3\mathcal{L}\left[\cos\frac{t}{2}\right] - 14\left(\frac{\frac{1}{2}}{s^2 + \frac{1}{4}}\right) + \frac{1}{2}G(s)\left(\frac{\frac{1}{2}}{s^2 + \frac{1}{4}}\right)$$

$$(f * g)(t) = \int_0^t f(\tau)g(t - \tau) d\tau$$



Example

Solve

$$\begin{cases} 4y'' + y = g(t) \\ y(0) = 3 \\ y'(0) = -7. \end{cases}$$

$$Y = 3\mathcal{L}\left[\cos \frac{t}{2}\right] - 14\mathcal{L}\left[\sin \frac{t}{2}\right] + \frac{1}{2}G(s)\mathcal{L}\left[\sin \frac{t}{2}\right]$$

$$(f * g)(t) = \int_0^t f(\tau)g(t - \tau) d\tau$$



Example

Solve

$$\begin{cases} 4y'' + y = g(t) \\ y(0) = 3 \\ y'(0) = -7. \end{cases}$$

$$Y = 3\mathcal{L}\left[\cos \frac{t}{2}\right] - 14\mathcal{L}\left[\sin \frac{t}{2}\right] + \frac{1}{2}G(s)\mathcal{L}\left[\sin \frac{t}{2}\right]$$

$$y(t) =$$

$$(f * g)(t) = \int_0^t f(\tau)g(t - \tau) d\tau$$



Example

Solve

$$\begin{cases} 4y'' + y = g(t) \\ y(0) = 3 \\ y'(0) = -7. \end{cases}$$

$$Y = 3\mathcal{L}\left[\cos \frac{t}{2}\right] - 14\mathcal{L}\left[\sin \frac{t}{2}\right] + \frac{1}{2}G(s)\mathcal{L}\left[\sin \frac{t}{2}\right]$$

$$y(t) = 3 \cos \frac{t}{2}$$

$$(f * g)(t) = \int_0^t f(\tau)g(t - \tau) d\tau$$



Example

Solve

$$\begin{cases} 4y'' + y = g(t) \\ y(0) = 3 \\ y'(0) = -7. \end{cases}$$

$$Y = 3\mathcal{L}\left[\cos \frac{t}{2}\right] - 14\mathcal{L}\left[\sin \frac{t}{2}\right] + \frac{1}{2}G(s)\mathcal{L}\left[\sin \frac{t}{2}\right]$$

$$y(t) = 3 \cos \frac{t}{2} - 14 \sin \frac{t}{2}$$

$$(f * g)(t) = \int_0^t f(\tau)g(t - \tau) d\tau$$



Example

Solve

$$\begin{cases} 4y'' + y = g(t) \\ y(0) = 3 \\ y'(0) = -7. \end{cases}$$

$$Y = 3\mathcal{L}\left[\cos \frac{t}{2}\right] - 14\mathcal{L}\left[\sin \frac{t}{2}\right] + \frac{1}{2}G(s)\mathcal{L}\left[\sin \frac{t}{2}\right]$$

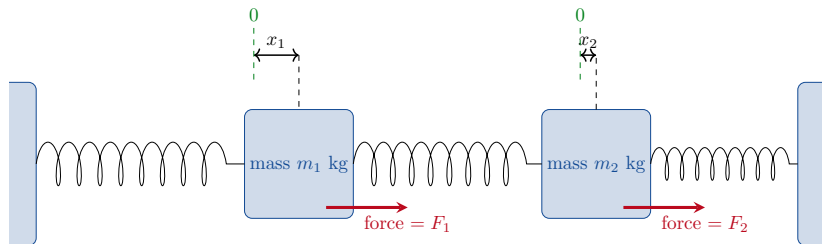
$$y(t) = 3\cos \frac{t}{2} - 14\sin \frac{t}{2} + \frac{1}{2}g(t) * \sin \frac{t}{2}.$$



Systems of First Order Linear Equations

Introduction

5.1 Introduction



Consider the dynamical system shown above. There are two blocks and three springs. Forces F_1 and F_2 act on the blocks as shown.

See <https://tinyurl.com/wm2ogdh>



We expect that the acceleration of the blocks will depend on

- the displacements x_1 and x_2 ;
- the forces F_1 and F_2 ; and
- the masses of the blocks.

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So we expect that:

$$\begin{cases} \frac{d^2 x_1}{dt^2} = f_1(x_1, x_2, F_1, m_1) \\ \frac{d^2 x_2}{dt^2} = f_2(x_1, x_2, F_2, m_2). \end{cases}$$

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This is a system of two ODEs. To find $x_1(t)$ and $x_2(t)$, we would need to solve these equations at the same time.

The most famous system of ODEs is the system of *Predator-Prey* equations:

$$\begin{cases} \frac{dx}{dt} = \alpha x - \beta xy \\ \frac{dy}{dt} = \delta xy - \gamma y \end{cases}$$

where

$x(t)$ = number of prey (e.g. mice)

$y(t)$ = number of predators (e.g. owls),

which originate circa 1925.

5.1 Introduction



It is possible to convert an n th order linear ODE into a system of n first order linear ODEs. Or vice versa.

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \dots + a_1 y' + a_0 y = g(t) \quad \longleftrightarrow \quad \begin{cases} x'_1 = b_{11}x_1 + \dots + b_{1n}x_n + h_1(t) \\ x'_2 = b_{21}x_1 + \dots + b_{2n}x_n + h_2(t) \\ \vdots \\ x'_n = b_{n1}x_1 + \dots + b_{nn}x_n + h_n(t) \end{cases}$$

Example

Write

$$u'' + 0.25u' + u = 0$$

as a system of two first order ODEs.

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Therefore

$$\begin{cases} x_1' = x_2 \\ x_2' = -x_1 - 0.25x_2. \end{cases}$$

Remark

We will need

- matrices,
- eigenvalues,
- eigenvectors,
- the Wronskian,
- linear independence,
- and more

from MATH215 – please either revise your Linear Algebra lecture notes or read your Linear Algebra book or read §7.2-7.3 in the textbook by Boyce and DiPrima.



Basic Theory of Systems of First Order Linear Equations

5.2 Basic Theory of Systems of First Order Linear Equations



$$\begin{cases} x_1' = p_{11}(t)x_1 + p_{12}(t)x_2 + \dots + p_{1n}(t)x_n + g_1(t) \\ x_2' = p_{21}(t)x_1 + p_{22}(t)x_2 + \dots + p_{2n}(t)x_n + g_2(t) \\ \vdots \\ x_n' = p_{n1}(t)x_1 + p_{n2}(t)x_2 + \dots + p_{nn}(t)x_n + g_n(t) \end{cases}$$

is a system of n linear ODEs and n variables: x_1, x_2, \dots, x_n .

5.2 Basic Theory of Systems of First Order Linear Equations



If we write

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, \mathbf{x}' = \begin{bmatrix} x'_1 \\ x'_2 \\ \vdots \\ x'_n \end{bmatrix}, P = \begin{bmatrix} p_{11} & p_{12} & \cdots & p_{1n} \\ p_{21} & p_{22} & \cdots & p_{2n} \\ \vdots & \vdots & & \vdots \\ p_{n1} & p_{n2} & \cdots & p_{nn} \end{bmatrix}, \text{ and } \mathbf{g} = \begin{bmatrix} g_1 \\ g_2 \\ \vdots \\ g_n \end{bmatrix}$$

then we can write this system as

$$\mathbf{x}' = P(t)\mathbf{x} + \mathbf{g}(t).$$

5.2 Basic Theory of Systems of First Order Linear Equations



First we will consider the homogeneous system

$$\mathbf{x}' = P(t)\mathbf{x}.$$

5.2 Basic Theory of Systems of First Order Linear Equations



In Chapters 3 and 4 when we had multiple solutions, we wrote them as $y_1(t)$, $y_2(t)$, \dots . But we are already using x_1 , x_2 , \dots to denote coordinates. So we need a new type of notation.

5.2 Basic Theory of Systems of First Order Linear Equations



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Notation

We use $\mathbf{x}^{(1)}(t)$, $\mathbf{x}^{(2)}(t)$, \dots to denote different vector solutions.

5.2 Basic Theory of Systems of First Order Linear Equations



Recall from Chapter 3 that if $y_1(t)$ and $y_2(t)$ are both solutions to

$$ay'' + by' + cy = 0,$$

then

$$c_1y_1 + c_2y_2$$

is also a solution.

5.2 Basic Theory of Systems of First Order Linear Equations



Theorem

If $\mathbf{x}^{(1)}(t)$ and $\mathbf{x}^{(2)}(t)$ are solutions to $\mathbf{x}' = P(t)\mathbf{x}$, then

$$c_1\mathbf{x}^{(1)} + c_2\mathbf{x}^{(2)}$$

is also a solution for any $c_1, c_2 \in \mathbb{R}$.

5.2 Basic Theory of Systems of First Order Linear Equations



Example

$\mathbf{x}^{(1)} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} e^{3t}$ and $\mathbf{x}^{(2)} = \begin{bmatrix} 1 \\ -2 \end{bmatrix} e^{-t}$ are both solutions to $\mathbf{x}' = \begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix} \mathbf{x}$ (we will see this later).

5.2 Basic Theory of Systems of First Order Linear Equations



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$\mathbf{x}^{(1)} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} e^{3t}$ and $\mathbf{x}^{(2)} = \begin{bmatrix} 1 \\ -2 \end{bmatrix} e^{-t}$ are both solutions to $\mathbf{x}' = \begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix} \mathbf{x}$ (we will see this later). Therefore

$$\mathbf{x}(t) = c_1 \mathbf{x}^{(1)} + c_2 \mathbf{x}^{(2)} = c_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} e^{3t} + c_2 \begin{bmatrix} 1 \\ -2 \end{bmatrix} e^{-t}$$

is also a solution to this system.

5.2 Basic Theory of Systems of First Order Linear Equations



(Suppose that $P(t)$ is an $n \times n$ matrix.)

Theorem

If $\mathbf{x}^{(1)}(t)$, $\mathbf{x}^{(2)}(t)$, \dots , $\mathbf{x}^{(n)}(t)$ are linearly independent solutions to $\mathbf{x}' = P(t)\mathbf{x}$, then every solution to this system can be written as

$$\mathbf{x}(t) = c_1\mathbf{x}^{(1)} + c_2\mathbf{x}^{(2)} + \dots + c_n\mathbf{x}^{(n)}$$

in exactly one way.

5.2 Basic Theory of Systems of First Order Linear Equations



Definition

In this case, we say that $\mathbf{x}^{(1)}(t), \mathbf{x}^{(2)}(t), \dots, \mathbf{x}^{(n)}(t)$ form a *fundamental set of solutions* to $\mathbf{x}' = P(t)\mathbf{x}$.

5.2 Basic Theory of Systems of First Order Linear Equations



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Definition

In this case,

$$\mathbf{x}(t) = c_1\mathbf{x}^{(1)} + c_2\mathbf{x}^{(2)} + \dots + c_n\mathbf{x}^{(n)}$$

is called the *general solution* to $\mathbf{x}' = P(t)\mathbf{x}$.

Next Time

- 5.3 Homogeneous Linear Systems with Constant Coefficients
- 5.4 Complex Eigenvalues
- 5.5 Fundamental Matrices