

# Lecture 5

- 3.4 Repeated Roots of the Characteristic Equation
- 3.5 Reduction of Order
- 3.6 Nonhomogeneous Equations
- 3.7 The Method of Undetermined Coefficients

## Summary

To solve

$$ay'' + by' + cy = 0$$

we need to find two linearly independent solutions,  $y_1(t)$  and  $y_2(t)$ . Then the general solution to the ODE is

$$y(t) = c_1 y_1(t) + c_2 y_2(t).$$

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$$y(t) = c_1 y_1(t) + c_2 y_2(t).$$

First we solve the characteristic equation

$$ar^2 + br + c = 0$$

and find the roots  $r_1$  and  $r_2$ .

## Summary

- 1 If  $r_1, r_2 \in \mathbb{R}$  and  $r_1 \neq r_2$ , then

$$y_1(t) = e^{r_1 t} \quad \text{and} \quad y_2(t) = e^{r_2 t};$$

- 2 If  $r_{1,2} = \lambda \pm i\mu$  ( $\lambda, \mu \in \mathbb{R}$ ), then

$$y_1(t) = e^{\lambda t} \cos \mu t \quad \text{and} \quad y_2(t) = e^{\lambda t} \sin \mu t;$$

- 3 If the roots are repeated, then ?????????????

# Repeated Roots of the Characteristic Equation

### 3.4 Repeated Roots of the Characteristic Equation



Now consider

$$ay'' + by' + cy = 0 \quad (1)$$

where  $b^2 - 4ac = 0$ . Then the only root of

$$ar^2 + br + c = 0$$

is

$$r = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-b \pm \sqrt{0}}{2a} = -\frac{b}{2a}.$$

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We know that  $y_1(t) = e^{-\frac{bt}{2a}}$  is a solution of (1), but how do we find a linearly independent second solution?

## 3.4 Repeated Roots of the Characteristic Equation



### Example

Solve  $y'' + 4y' + 4y = 0$ .



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$$0 = r^2 + 4r + 4 = (r + 2)^2$$

has repeated root  $r_1 = r_2 = -2$ .

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- We know that  $y_1(t)$  is a solution;

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The idea is:

- We know that  $y_1(t)$  is a solution;
- So  $cy_1(t)$  is a solution for any  $c \in \mathbb{R}$ ;
- Maybe  $v(t)y_1(t)$  is a solution for some non-constant function  $v(t)$ .

### 3.4 Repeated Roots of the Characteristic Equation



We consider  $y_2(t) = v(t)y_1(t)$  for some function  $v(t)$  which we don't know yet.

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We consider  $y_2(t) = v(t)y_1(t)$  for some function  $v(t)$  which we don't know yet. Then we calculate that

$$y_2 = ve^{-2t}$$

$$y_2' = v'e^{-2t} - 2ve^{-2t}$$

$$y_2'' = v''e^{-2t} - 4v'e^{-2t} + 4ve^{-2t}$$



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and that

$$\begin{aligned} 0 &= y_2'' + 4y_2' + 4y_2 \\ &= (v''e^{-2t} - 4v'e^{-2t} + 4ve^{-2t}) + 4(v'e^{-2t} - 2ve^{-2t}) + 4(ve^{-2t}) \\ &= e^{-2t} [v'' - 4v' + 4v + 4v' - 8v + 4v] \\ &= v''e^{-2t}. \end{aligned}$$

### 3.4 Repeated Roots of the Characteristic Equation



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$$y_2(t) = te^{-2t}.$$

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$$y_2(t) = te^{-2t}.$$

But are  $y_1(t)$  and  $y_2(t)$  linearly independent? Since

$$W(y_1, y_2)(t) = \begin{vmatrix} e^{-2t} & te^{-2t} \\ -2e^{-2t} & (1-2t)e^{-2t} \end{vmatrix} = e^{-4t} \neq 0,$$

the answer is YES.

## 3.4 Repeated Roots of the Characteristic Equation



Therefore  $y_1(t) = e^{-2t}$  and  $y_2(t) = te^{-2t}$  form a fundamental set of solutions and the general solution is

$$y(t) = c_1e^{-2t} + c_2te^{-2t}.$$

## 3.4 Repeated Roots of the Characteristic Equation



For the general equation  $ay'' + by' + cy = 0$ , we can use the same method:



## 3.4 Repeated Roots of the Characteristic Equation



For the general equation  $ay'' + by' + cy = 0$ , we can use the same method: We have  $y_1(t) = e^{rt} = e^{-\frac{bt}{2a}}$  and we guess that  $y_2(t) = v(t)e^{-\frac{bt}{2a}}$  for some function  $v(t)$ .

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$$0 = ay_2'' + by_2' + cy_2 = \dots = ae^{-\frac{bt}{2a}}v''.$$

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So again we want  $v'' = 0$  and we can choose  $v(t) = t$ . Thus  $y_2(t) = te^{rt} = te^{-\frac{bt}{2a}}$ .

## 3.4 Repeated Roots of the Characteristic Equation



For the general equation  $ay'' + by' + cy = 0$ , we can use the same method: We have  $y_1(t) = e^{rt} = e^{-\frac{bt}{2a}}$  and we guess that  $y_2(t) = v(t)e^{-\frac{bt}{2a}}$  for some function  $v(t)$ . Then we calculate (you fill in the details)

$$0 = ay_2'' + by_2' + cy_2 = \dots = ae^{-\frac{bt}{2a}}v''.$$

So again we want  $v'' = 0$  and we can choose  $v(t) = t$ . Thus  $y_2(t) = te^{rt} = te^{-\frac{bt}{2a}}$ .

I leave it for you to calculate that  $W(e^{rt}, te^{rt}) \neq 0$ . Thus  $e^{rt}$  and  $te^{rt}$  form a fundamental set of solutions to (1).

## 3.4 Repeated Roots of the Characteristic Equation



### Example

Solve

$$\begin{cases} y'' - y' + \frac{1}{4}y = 0 \\ y(0) = 2 \\ y'(0) = \frac{1}{3}. \end{cases}$$

### 3.4 Repeated Roots of the Characteristic Equation



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The characteristic equation

$$0 = r^2 - r + \frac{1}{4} = \left(r - \frac{1}{2}\right)^2$$

has repeated root  $r = \frac{1}{2}$ .

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### Example

Solve

$$\begin{cases} y'' - y' + \frac{1}{4}y = 0 \\ y(0) = 2 \\ y'(0) = \frac{1}{3}. \end{cases}$$

The characteristic equation

$$0 = r^2 - r + \frac{1}{4} = \left(r - \frac{1}{2}\right)^2$$

has repeated root  $r = \frac{1}{2}$ . So we know that the general solution to the ODE is

$$y(t) = c_1 e^{\frac{t}{2}} + c_2 t e^{\frac{t}{2}}.$$

## 3.4 Repeated Roots of the Characteristic Equation



Next we need to look at the initial conditions: Since  $y'(t) = \frac{1}{2}c_1e^{\frac{t}{2}} + c_2e^{\frac{t}{2}} + \frac{1}{2}c_2te^{\frac{t}{2}}$ , we have that

$$\begin{aligned} 2 = y(0) &= c_1 + 0 & \implies & c_1 = 2 \\ \frac{1}{3} = y'(0) &= \frac{1}{2}c_1 + c_2 + 0 & \implies & c_2 = -\frac{2}{3}. \end{aligned}$$



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Therefore the solution to the IVP is

$$y = 2e^{\frac{t}{2}} - \frac{2}{3}te^{\frac{t}{2}}.$$

## 3.4 Repeated Roots of the Characteristic Equation



### Example

Now solve

$$\begin{cases} y'' - y' + \frac{1}{4}y = 0 \\ y(0) = 2 \\ y'(0) = 2 \end{cases}$$

## 3.4 Repeated Roots of the Characteristic Equation



### Example

Now solve

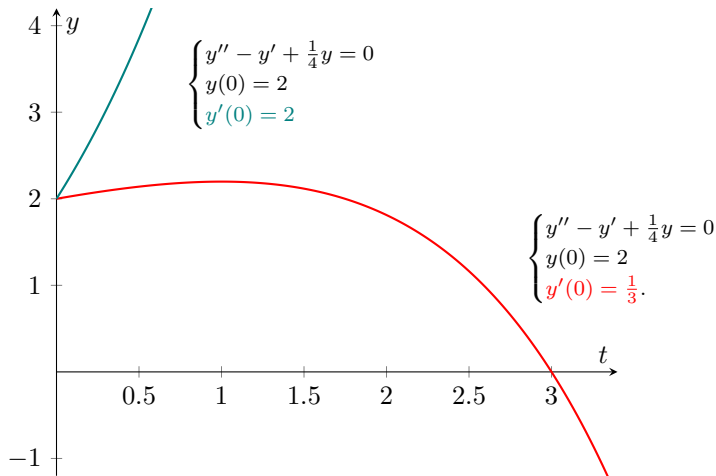
$$\begin{cases} y'' - y' + \frac{1}{4}y = 0 \\ y(0) = 2 \\ y'(0) = 2 \end{cases}$$

You can check that the solution is

$$y = 2e^{\frac{t}{2}} + te^{\frac{t}{2}}.$$

The graph of this solution, and the solution to the previous example, are shown on the next slide.

## 3.4 Repeated Roots of the Characteristic Equation



Note that even though these two functions share the same  $y(0)$  value, and that their  $y'(0)$  value does not differ by much, their behaviour as  $t \rightarrow \infty$  is very different.

## 3.4 Repeated Roots of the Characteristic Equation



### Summary

To solve

$$ay'' + by' + cy = 0$$

we need to find two linearly independent solutions.

- 1** If  $r_1, r_2 \in \mathbb{R}$  and  $r_1 \neq r_2$ , then

$$y_1(t) = e^{r_1 t} \quad \text{and} \quad y_2(t) = e^{r_2 t};$$

- 2** If  $r_{1,2} = \lambda \pm i\mu$  ( $\lambda, \mu \in \mathbb{R}$ ), then

$$y_1(t) = e^{\lambda t} \cos \mu t \quad \text{and} \quad y_2(t) = e^{\lambda t} \sin \mu t;$$

- 3** If  $r_1, r_2 \in \mathbb{R}$  but  $r_1 = r_2$ , then

$$y_1(t) = e^{r_1 t} \quad \text{and} \quad y_2(t) = te^{r_1 t}.$$

# Reduction of Order

## 3.5 Reduction of Order



Consider

$$y'' + p(t)y' + q(t)y = 0. \quad (2)$$

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$$y'' + p(t)y' + q(t)y = 0. \quad (2)$$

Suppose that we know that  $y_1(t)$  is a solution to (2) and suppose that we want to find a second, linearly independent solution.



## 3.5 Reduction of Order



Consider

$$y'' + p(t)y' + q(t)y = 0. \quad (2)$$

Suppose that we know that  $y_1(t)$  is a solution to (2) and suppose that we want to find a second, linearly independent solution.

The main idea in this section is that we guess that

$$y_2(t) = v(t)y_1(t)$$

for some non-constant function  $v(t)$ . If we can find  $v(t)$ , then we have our  $y_2(t)$ .

## 3.5 Reduction of Order



Then we calculate that

$$y_2 = v y_1$$

$$y_2' = v' y_1 + v y_1'$$

$$y_2'' = v'' y_1 + 2v' y_1' + v y_1''$$

and

$$0 = y_2'' + p y_2' + q y_2$$

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and

$$\begin{aligned} 0 &= y_2'' + p y_2' + q y_2 \\ &= (v'' y_1 + 2v' y_1' + v y_1'') + p(t) (v' y_1 + v y_1') + q(t) (v y_1) \\ &= \\ &= \end{aligned}$$

## 3.5 Reduction of Order



Then we calculate that

$$y_2 = vy_1$$

$$y_2' = v'y_1 + vy_1'$$

$$y_2'' = v''y_1 + 2v'y_1' + vy_1''$$

and

$$\begin{aligned} 0 &= y_2'' + py_2' + qy_2 \\ &= (v''y_1 + 2v'y_1' + vy_1'') + p(t)(v'y_1 + vy_1') + q(t)(vy_1) \\ &= v''y_1 + v'(2y_1' + py_1) + \underbrace{v(y_1'' + py_1' + qy_1)}_{=0} \\ &= \end{aligned}$$

## 3.5 Reduction of Order



Then we calculate that

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## 3.5 Reduction of Order



$$0 = v''y_1 + v' (2y_1' + py_1) + 0v$$

### Remark

Note that since  $y_1$  solves the ODE, we must always get “ $0v$ ” here. We can have  $v'$  and  $v''$  terms, but if you do a reduction of order calculation and still have  $v$  terms, then you have made a mistake.

### Remark

$$v''y_1 + v'(2y_1' + py_1) = 0 \quad (3)$$

is actually a first order ODE for  $v'$ .

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$$u'y_1 + u(2y_1' + py_1) = 0. \quad (4)$$



## 3.5 Reduction of Order



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$$u'y_1 + u(2y_1' + py_1) = 0. \quad (4)$$

If we can find  $u(t)$ , then we can find  $v(t) = \int u(t) dt$  and  $y_2(t) = v(t)y_1(t)$ .

## 3.5 Reduction of Order



### Remark

Instead of having to solve a second order ODE to find  $y_2$ , we only need to solve a first order ODE to find  $u(t)$ . Hence the name “Reduction of Order”.

### Remark

The method is

- 1 Guess  $y_2 = vy_1$ .
- 2 Put this into your ODE and find an equation for  $v$ ;
- 3 Set  $u = v'$ ;
- 4 Find  $u$ ;
- 5 Integrate to find  $v$ ;
- 6 Then  $y_2(t) = v(t)y_1(t)$ .

## 3.5 Reduction of Order



### Example

Given that  $y_1(t) = \frac{1}{t}$  is a solution of

$$2t^2y'' + 3ty' - y = 0, \quad t > 0$$

find a linearly independent second solution.

## 3.5 Reduction of Order



$$y_1(t) = \frac{1}{t}$$

Let  $y_2(t) = v(t)y_1(t)$ . Then we have

$$y_2 = vt^{-1}$$

$$y_2' = v't^{-1} - vt^{-2}$$

$$y_2'' = v''t^{-1} - 2v't^{-2} + 2vt^{-3}$$

and

$$0 = 2t^2 y_2'' + 3t y_2' - y_2$$

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## 3.5 Reduction of Order



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$$\begin{aligned} 0 &= 2t^2 y_2'' + 3t y_2' - y_2 \\ &= 2t^2 (v''t^{-1} - 2v't^{-2} + 2vt^{-3}) + 3t (v't^{-1} - vt^{-2}) - vt^{-1} \\ &= \\ &= \end{aligned}$$

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and

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## 3.5 Reduction of Order



$$2tv'' - v' = 0$$

Now let  $u = v'$ .

## 3.5 Reduction of Order



$$2tv'' - v' = 0$$

Now let  $u = v'$ . We need to solve

$$2t \frac{du}{dt} - u = 0.$$

## 3.5 Reduction of Order



$$2tv'' - v' = 0$$

Now let  $u = v'$ . We need to solve

$$2t \frac{du}{dt} - u = 0.$$

This equation is both linear and separable, so we know 2 ways to solve it.

## 3.5 Reduction of Order



$$2t \frac{du}{dt} = u$$

$$\frac{du}{u} = \frac{1}{2} \frac{dt}{t}$$

$$\int \frac{du}{u} = \int \frac{1}{2} \frac{dt}{t}$$

$$\ln |u| = \frac{1}{2} \ln |t| + C$$

$$e^{\ln |u|} = e^{\ln |t|^{\frac{1}{2}}} e^C$$

$$|u| = |t|^{\frac{1}{2}} e^C$$

$$u = \pm e^C t^{\frac{1}{2}} = ct^{\frac{1}{2}}.$$

## 3.5 Reduction of Order



$$u(t) = ct^{\frac{1}{2}}$$

Then we have

$$v(t) = \int u(t) dt = \int ct^{\frac{1}{2}} dt = \frac{2}{3}ct^{\frac{3}{2}} + k$$

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Then we have

$$v(t) = \int u(t) dt = \int ct^{\frac{1}{2}} dt = \frac{2}{3}ct^{\frac{3}{2}} + k$$

and

$$y_2(t) = v(t)t^{-1} = \frac{2}{3}ct^{\frac{1}{2}} + kt^{-1}.$$

## 3.5 Reduction of Order



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Then we have

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and

$$y_2(t) = v(t)t^{-1} = \frac{2}{3}ct^{\frac{1}{2}} + kt^{-1}.$$

Remember that we are trying to find a solution that is linearly independent from  $y_1(t) = t^{-1}$ . The second term in  $y_2(t) = \frac{2}{3}ct^{\frac{1}{2}} + kt^{-1}$  is just a multiple of  $y_1(t)$  – we don't need this. So it is ok to choose  $k = 0$ .

## 3.5 Reduction of Order



$$u(t) = ct^{\frac{1}{2}}$$

Then we have

$$v(t) = \int u(t) dt = \int ct^{\frac{1}{2}} dt = \frac{2}{3}ct^{\frac{3}{2}} + k$$

and

$$y_2(t) = v(t)t^{-1} = \frac{2}{3}ct^{\frac{1}{2}} + kt^{-1}.$$

Remember that we are trying to find a solution that is linearly independent from  $y_1(t) = t^{-1}$ . The second term in  $y_2(t) = \frac{2}{3}ct^{\frac{1}{2}} + kt^{-1}$  is just a multiple of  $y_1(t)$  – we don't need this. So it is ok to choose  $k = 0$ . Hence

$$y_2(t) = \frac{2}{3}ct^{\frac{1}{2}}$$



## 3.5 Reduction of Order



$$y_2(t) = \frac{2}{3}ct^{\frac{1}{2}}$$

Finally, since I like simple functions I choose  $c = \frac{3}{2}$  to get

$$y_2(t) = t^{\frac{1}{2}}.$$

I leave it to you to check that  $W(t^{-1}, t^{\frac{1}{2}})$  is not always zero.

## 3.5 Reduction of Order



### Example

Given that  $y_1(t) = t$  solves

$$t^2 y'' + 2ty' - 2y = 0, \quad t > 0,$$

find a second linearly independent solution  $y_2(t)$ .

## 3.5 Reduction of Order



### Example

Given that  $y_1(t) = t$  solves

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We start with  $y_2(t) = v(t)y_1(t) = v(t)t$ .

## 3.5 Reduction of Order



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## 3.5 Reduction of Order



### Example

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find a second linearly independent solution  $y_2(t)$ .

We start with  $y_2(t) = v(t)y_1(t) = v(t)t$ . Then  $y_2' = v't + v$  and  $y_2'' = v''t + 2v'$ . Substituting into the ODE, we calculate that

$$\begin{aligned} 0 &= t^2 y_2'' + 2ty_2' - 2y_2 \\ &= t^2(v''t + 2v') + 2t(v't + v) - 2vt \\ &= t^3 v'' + v'(2t^2 + 2t^2) + v(2t - 2t) \\ &= t^3 v'' + 4t^2 v' \\ &= t^2(tv'' + 4v'). \end{aligned}$$

## 3.5 Reduction of Order



$$t^2(tv'' + 4v') = 0$$

Letting  $u = v'$ , we obtain the first order ODE

$$t \frac{du}{dt} + 4u = 0.$$

## 3.5 Reduction of Order



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Letting  $u = v'$ , we obtain the first order ODE

$$t \frac{du}{dt} + 4u = 0.$$

We calculate that

$$\begin{aligned} t \frac{du}{dt} &= -4u \\ \frac{du}{u} &= -4 \frac{dt}{t} \\ \int \frac{du}{u} &= -4 \int \frac{dt}{t} \\ \ln |u| &= -4 \ln |t| + C \\ u &= \pm e^C t^{-4} = ct^{-4} \end{aligned}$$

## 3.5 Reduction of Order



$$y_1(t) = t \qquad v' = u \qquad u = ct^{-4}$$

and

$$\begin{aligned} v &= \int u \, dt = \int ct^{-4} \, dt \\ &= -\frac{1}{3}ct^{-3} + k. \end{aligned}$$



## 3.5 Reduction of Order



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Thus

$$y_2(t) = v(t)t = -\frac{1}{3}ct^{-2} + kt.$$

## 3.5 Reduction of Order



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Thus

$$y_2(t) = v(t)t = -\frac{1}{3}ct^{-2} + kt.$$

Choosing  $c = -3$  and  $k = 0$ , we obtain the solution

$$y_2(t) = t^{-2}.$$

## 3.5 Reduction of Order



**Does  $y_2(t) = t^{-2}$  really solve  $t^2 y'' + 2ty' - 2y = 0$ ?**

## 3.5 Reduction of Order



**Does  $y_2(t) = t^{-2}$  really solve  $t^2 y'' + 2ty' - 2y = 0$ ?**

Since  $y_2' = -2t^{-3}$  and  $y_2'' = 6t^{-4}$ , we have that

$$\begin{aligned} t^2 y_2'' + 2ty_2' - 2y_2 &= t^2(6t^{-4}) + 2t(-2t^{-3}) - 2t^{-2} \\ &= 6t^{-2} - 4t^{-2} - 2t^{-2} \\ &= 0. \end{aligned}$$

The answer is YES!!

## 3.5 Reduction of Order



**Are  $y_1(t) = t$  and  $y_2(t) = t^{-2}$  linearly independent?**

**Are  $y_1(t) = t$  and  $y_2(t) = t^{-2}$  linearly independent?**

We have that

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} t & t^{-2} \\ 1 & -2t^{-3} \end{vmatrix} = -2t^{-2} - t^{-2} = -3t^{-2} \neq 0$$

since  $t > 0$ . Therefore  $y_1$  and  $y_2$  are linearly independent.

# Nonhomogeneous Equations

## 3.6 Nonhomogeneous Equations



Consider

$$y'' + p(t)y' + q(t)y = g(t). \quad (5)$$



## 3.6 Nonhomogeneous Equations



Consider

$$y'' + p(t)y' + q(t)y = g(t). \quad (5)$$

The equation

$$y'' + p(t)y' + q(t)y = 0 \quad (6)$$

is called the *homogeneous equation corresponding to* (5).

## 3.6 Nonhomogeneous Equations



$$y'' + p(t)y' + q(t)y = g(t) \quad (5)$$

### Theorem

*The general solution to (5) can be written in the form*

$$y(t) = c_1 y_1(t) + c_2 y_2(t) + Y(t)$$

*where*

## 3.6 Nonhomogeneous Equations



$$y'' + p(t)y' + q(t)y = g(t) \quad (5)$$

### Theorem

*The general solution to (5) can be written in the form*

$$y(t) = c_1 y_1(t) + c_2 y_2(t) + Y(t)$$

*where*

- $y_1$  and  $y_2$  form a fundamental set of solutions to the homogeneous equation corresponding to (5);
- $c_1$  and  $c_2$  are constants; and
- $Y$  is a particular solution to (5).

## 3.6 Nonhomogeneous Equations



To solve  $L[y] = g$

- 1 Find the general solution to  $L[y] = 0$ ;
- 2 Find a particular solution to  $L[y] = g$ ;
- 3 1 + 2

## 3.6 Nonhomogeneous Equations



To solve  $L[y] = g$

- 1 Find the general solution to  $L[y] = 0$ ;
- 2 Find a particular solution to  $L[y] = g$ ;
- 3 1 + 2

We will study 2 methods to do step 2. One method this week and one method next week.

# The Method of Undetermined Coefficients

## 3.7 The Method of Undetermined Coefficients



$$y'' + p(t)y' + q(t)y = g(t) \quad (5)$$

The idea is:

- 1 Look at  $g(t)$
- 2 Make a guess with constants
- 3 Try to find the constants

## 3.7 The Method of Undetermined Coefficients



### Example

Find a particular solution to  $y'' - 3y' - 4y = 3e^{2t}$ .



## 3.7 The Method of Undetermined Coefficients



### Example

Find a particular solution to  $y'' - 3y' - 4y = 3e^{2t}$ .

Here we have  $g(t) = 3e^{2t}$ . We look at this  $g$  and we make a guess:

## 3.7 The Method of Undetermined Coefficients



### Example

Find a particular solution to  $y'' - 3y' - 4y = 3e^{2t}$ .

Here we have  $g(t) = 3e^{2t}$ . We look at this  $g$  and we make a guess:  $g$  includes  $e^{2t}$  so we guess that  $Y(t)$  also includes  $e^{2t}$ . So we guess that  $Y(t) = Ae^{2t}$  for some constant  $A$ .

## 3.7 The Method of Undetermined Coefficients



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Find a particular solution to  $y'' - 3y' - 4y = 3e^{2t}$ .

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We must try to find  $A$ . We calculate that

$$Y(t) = Ae^{2t} \quad Y'(t) = 2Ae^{2t} \quad Y''(t) = 4Ae^{2t}$$

## 3.7 The Method of Undetermined Coefficients



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We must try to find  $A$ . We calculate that

$$Y(t) = Ae^{2t} \quad Y'(t) = 2Ae^{2t} \quad Y''(t) = 4Ae^{2t}$$

and

$$\begin{aligned} 3e^{2t} &= Y'' - 3Y' - 4Y = 4Ae^{2t} - 3(2Ae^{2t}) - 4(Ae^{2t}) \\ &= -6Ae^{2t}. \end{aligned}$$

## 3.7 The Method of Undetermined Coefficients



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We must have  $A = -\frac{1}{2}$ .

## 3.7 The Method of Undetermined Coefficients



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We must have  $A = -\frac{1}{2}$ . Therefore a particular solution is

$$Y(t) = -\frac{1}{2}e^{2t}.$$

## 3.7 The Method of Undetermined Coefficients



### Example

Find a particular solution to  $y'' - 3y' - 4y = 4t^2 - 1$ .

## 3.7 The Method of Undetermined Coefficients



### Example

Find a particular solution to  $y'' - 3y' - 4y = 4t^2 - 1$ .

Since  $g(t) = 4t^2 - 1$  is a 2nd degree polynomial, we guess that  $Y$  is also a second degree polynomial. So we try the ansatz

$$Y(t) = At^2 + Bt + C.$$

I will leave this example for you to finish.



## 3.7 The Method of Undetermined Coefficients



### Example

Find a particular solution to  $y'' - 3y' - 4y = 4t^2 - 1$ .

Since  $g(t) = 4t^2 - 1$  is a 2nd degree polynomial, we guess that  $Y$  is also a second degree polynomial. So we try the ansatz

$$Y(t) = At^2 + Bt + C.$$

I will leave this example for you to finish.

(ansatz = a mathematical guess)

## 3.7 The Method of Undetermined Coefficients



### Example

Find a particular solution to  $y'' - 3y' - 4y = 2 \sin t$ .

## 3.7 The Method of Undetermined Coefficients



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**First guess:**  $Y(t) = A \sin t$ .

## 3.7 The Method of Undetermined Coefficients



### Example

Find a particular solution to  $y'' - 3y' - 4y = 2 \sin t$ .

**First guess:**  $Y(t) = A \sin t$ . Then  $Y' = A \cos t$  and  $Y'' = -A \sin t$ .

## 3.7 The Method of Undetermined Coefficients



### Example

Find a particular solution to  $y'' - 3y' - 4y = 2 \sin t$ .

**First guess:**  $Y(t) = A \sin t$ . Then  $Y' = A \cos t$  and  $Y'' = -A \sin t$ . Hence

$$\begin{aligned} 2 \sin t &= Y'' - 3Y' - 4Y \\ &= (-A \sin t) - 3(A \cos t) - 4(A \sin t) = -5A \sin t - 3A \cos t. \end{aligned}$$

## 3.7 The Method of Undetermined Coefficients



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We can see that we must have

$$\begin{cases} -5A = 2 \\ -3A = 0. \end{cases}$$

## 3.7 The Method of Undetermined Coefficients



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We can see that we must have

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This linear system is inconsistent: It not possible to find a constant  $A$  which satisfies both of these equations. Our first guess failed.

## 3.7 The Method of Undetermined Coefficients



**Second guess:**  $Y(t) = A \sin t + B \cos t$ .



## 3.7 The Method of Undetermined Coefficients



**Second guess:**  $Y(t) = A \sin t + B \cos t$ . Then we calculate that

$$Y' = A \cos t - B \sin t, \quad Y'' = -A \sin t - B \cos t$$

and

$$\begin{aligned} 2 \sin t &= Y'' - 3Y' - 4Y \\ &= (-A \sin t - B \cos t) - 3(A \cos t - B \sin t) - 4(A \sin t + B \cos t) \\ &= (-5A + 3B) \sin t + (-3A - 5B) \cos t. \end{aligned}$$

## 3.7 The Method of Undetermined Coefficients



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So we need  $A$  and  $B$  to satisfy

$$\begin{cases} -5A + 3B = 2 \\ -3A - 5B = 0. \end{cases}$$

## 3.7 The Method of Undetermined Coefficients



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So we need  $A$  and  $B$  to satisfy

$$\begin{cases} -5A + 3B = 2 \\ -3A - 5B = 0. \end{cases}$$

Please check that the solution to this linear system is  $A = -\frac{5}{17}$  and  $B = \frac{3}{17}$ .

## 3.7 The Method of Undetermined Coefficients



**Second guess:**  $Y(t) = A \sin t + B \cos t$ . Then we calculate that

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Please check that the solution to this linear system is  $A = -\frac{5}{17}$  and  $B = \frac{3}{17}$ . Therefore a particular solution is

$$Y(t) = -\frac{5}{17} \sin t + \frac{3}{17} \cos t.$$

## 3.7 The Method of Undetermined Coefficients



### Remark

$\sin$  and  $\cos$  are friends! They always go together. If you see either  $\sin$  or  $\cos$  in  $g(t)$ , then your ansatz needs to contain *both*  $\sin$  and  $\cos$ .

Likewise  $\sinh$  and  $\cosh$  always go together.

## 3.7 The Method of Undetermined Coefficients



### Example

Find a particular solution to  $y'' - 3y' - 4y = -8e^t \cos 2t$ .

## 3.7 The Method of Undetermined Coefficients



### Example

Find a particular solution to  $y'' - 3y' - 4y = -8e^t \cos 2t$ .

We will try the ansatz

$$Y(t) = Ae^t \cos 2t + Be^t \sin 2t.$$

Then

## 3.7 The Method of Undetermined Coefficients



### Example

Find a particular solution to  $y'' - 3y' - 4y = -8e^t \cos 2t$ .

We will try the ansatz

$$Y(t) = Ae^t \cos 2t + Be^t \sin 2t.$$

Then

$$\begin{aligned} Y'(t) &= Ae^t \cos 2t - 2Ae^t \sin 2t + Be^t \sin 2t + 2Be^t \cos 2t \\ &= (A + 2B)e^t \cos 2t + (B - 2A)e^t \sin 2t, \\ Y''(t) &= (A + 2B)e^t \cos 2t - 2(A + 2B)e^t \sin 2t + (B - 2A)e^t \sin 2t \\ &\quad + 2(B - 2A)e^t \cos 2t \\ &= (-3A + 4B)e^t \cos 2t + (-4A - 3B)e^t \sin 2t \end{aligned}$$



## 3.7 The Method of Undetermined Coefficients



and

$$\begin{aligned}-8e^t \cos 2t &= Y'' - 3Y' - 4Y \\ &= (-3A + 4B)e^t \cos 2t + (-4A - 3B)e^t \sin 2t \\ &\quad + (-3A - 6B)e^t \cos 2t + (-3B + 6A)e^t \sin 2t \\ &\quad + (-4A)e^t \cos 2t + (-4B)e^t \sin 2t \\ &= (-10A - 2B)e^t \cos 2t + (2A - 10B)e^t \sin 2t.\end{aligned}$$

## 3.7 The Method of Undetermined Coefficients



and

$$\begin{aligned}-8e^t \cos 2t &= Y'' - 3Y' - 4Y \\ &= (-3A + 4B)e^t \cos 2t + (-4A - 3B)e^t \sin 2t \\ &\quad + (-3A - 6B)e^t \cos 2t + (-3B + 6A)e^t \sin 2t \\ &\quad + (-4A)e^t \cos 2t + (-4B)e^t \sin 2t \\ &= (-10A - 2B)e^t \cos 2t + (2A - 10B)e^t \sin 2t.\end{aligned}$$

Thus we must solve

$$\begin{cases} 10A + 2B = 8 \\ 2A - 10B = 0. \end{cases}$$

## 3.7 The Method of Undetermined Coefficients



and

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Thus we must solve

$$\begin{cases} 10A + 2B = 8 \\ 2A - 10B = 0. \end{cases}$$

Please check that the solution to this linear system is  $A = \frac{10}{13}$  and  $B = \frac{2}{13}$ .

## 3.7 The Method of Undetermined Coefficients



and

$$\begin{aligned}-8e^t \cos 2t &= Y'' - 3Y' - 4Y \\&= (-3A + 4B)e^t \cos 2t + (-4A - 3B)e^t \sin 2t \\&\quad + (-3A - 6B)e^t \cos 2t + (-3B + 6A)e^t \sin 2t \\&\quad + (-4A)e^t \cos 2t + (-4B)e^t \sin 2t \\&= (-10A - 2B)e^t \cos 2t + (2A - 10B)e^t \sin 2t.\end{aligned}$$

Thus we must solve

$$\begin{cases} 10A + 2B = 8 \\ 2A - 10B = 0. \end{cases}$$

Please check that the solution to this linear system is  $A = \frac{10}{13}$  and  $B = \frac{2}{13}$ . Therefore a particular solution is

$$Y(t) = \frac{10}{13}e^t \cos 2t + \frac{2}{13}e^t \sin 2t.$$

## 3.7 The Method of Undetermined Coefficients



### Theorem

$$\left. \begin{array}{l} Y_1 \text{ solves} \\ ay'' + by' + cy = g_1(t) \\ \\ Y_2 \text{ solves} \\ ay'' + by' + cy = g_2(t) \end{array} \right\} \Rightarrow Y_1 + Y_2 \text{ solves} \\ ay'' + by' + cy = g_1(t) + g_2(t)$$

## 3.7 The Method of Undetermined Coefficients



### Example

Find a particular solution to

$$y'' - 3y' - 4y = 3e^{2t} + 2 \sin t - 8e^t \cos 2t. \quad (7)$$

## 3.7 The Method of Undetermined Coefficients



### Example

Find a particular solution to

$$y'' - 3y' - 4y = 3e^{2t} + 2\sin t - 8e^t \cos 2t. \quad (7)$$

We can split this problem up into three easier problems:

$$y'' - 3y' - 4y = 3e^{2t}$$

$$y'' - 3y' - 4y = 2\sin t$$

$$y'' - 3y' - 4y = -8e^t \cos 2t$$

We know particular solutions to these three ODEs.

## 3.7 The Method of Undetermined Coefficients



### Example

Find a particular solution to

$$y'' - 3y' - 4y = 3e^{2t} + 2\sin t - 8e^t \cos 2t. \quad (7)$$

We can split this problem up into three easier problems:

$$y'' - 3y' - 4y = 3e^{2t}$$

$$y'' - 3y' - 4y = 2\sin t$$

$$y'' - 3y' - 4y = -8e^t \cos 2t$$

We know particular solutions to these three ODEs. Therefore

$$Y(t) = -\frac{1}{2}e^{2t} - \frac{5}{17}\sin t + \frac{3}{17}\cos t + \frac{10}{13}e^t \cos 2t + \frac{2}{13}e^t \sin 2t.$$

is a particular solution to (7).



## 3.7 The Method of Undetermined Coefficients



### Remark

To find a particular solution to  $ay'' + by' + cy = g(t)$ , we have been looking at  $g(t)$  and choosing a similar function for  $Y(t)$ .

## 3.7 The Method of Undetermined Coefficients



### Remark

To find a particular solution to  $ay'' + by' + cy = g(t)$ , we have been looking at  $g(t)$  and choosing a similar function for  $Y(t)$ .

This method doesn't always work: There is a difficulty that can occur as we shall see in the next example.

## 3.7 The Method of Undetermined Coefficients



### Example

Find a particular solution to  $y'' + 4y = 3 \cos 2t$ .

**First guess:**  $Y(t) = A \cos 2t + B \sin 2t$ .

## 3.7 The Method of Undetermined Coefficients



### Example

Find a particular solution to  $y'' + 4y = 3 \cos 2t$ .

**First guess:**  $Y(t) = A \cos 2t + B \sin 2t$ .

Then we have that

$$Y' = -2A \sin 2t + 2B \cos 2t$$

$$Y'' = -4A \cos 2t - 4B \sin 2t$$

and

$$\begin{aligned} 3 \cos 2t &= Y'' + 4Y \\ &= (-4A \cos 2t - 4B \sin 2t) + 4(A \cos 2t + B \sin 2t) = 0. \end{aligned}$$

## 3.7 The Method of Undetermined Coefficients



### Example

Find a particular solution to  $y'' + 4y = 3 \cos 2t$ .

**First guess:**  $Y(t) = A \cos 2t + B \sin 2t$ .

Then we have that

$$Y' = -2A \sin 2t + 2B \cos 2t$$

$$Y'' = -4A \cos 2t - 4B \sin 2t$$

and

$$\begin{aligned} 3 \cos 2t &= Y'' + 4Y \\ &= (-4A \cos 2t - 4B \sin 2t) + 4(A \cos 2t + B \sin 2t) = 0. \end{aligned}$$

This is a FAILURE!!! It not possible to choose  $A$  and  $B$  such that

$$3 \cos 2t = 0$$

for all  $t$ .

## 3.7 The Method of Undetermined Coefficients



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We need two functions which, when differentiated, give us  $\cos 2t$  and  $\sin 2t$ . We will try  $t \cos 2t$  and  $t \sin 2t$  because

$$\frac{d}{dt} t \cos 2t = \cos 2t - 2t \sin 2t \text{ and } \frac{d}{dt} t \sin 2t = \sin 2t + 2t \cos 2t.$$

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We have that

$$\begin{aligned} Y' &= A \cos 2t - 2At \sin 2t + B \sin 2t + 2Bt \cos 2t \\ &= (A + 2Bt) \cos 2t + (B - 2At) \sin 2t, \\ Y'' &= 2B \cos 2t - 2(A + 2Bt) \sin 2t - 2A \sin 2t + 2(B - 2At) \cos 2t \\ &= (4B - 4At) \cos 2t + (-4A - 4Bt) \sin 2t \end{aligned}$$

and

$$\begin{aligned} 3 \cos 2t &= Y'' + 4Y \\ &= (4B - 4At) \cos 2t + (-4A - 4Bt) \sin 2t \\ &\quad + 4At \cos 2t + 4Bt \sin 2t \\ &= 4B \cos 2t - 4A \sin 2t. \end{aligned}$$

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$$3 \cos 2t = 4B \cos 2t - 4A \sin 2t$$

Thus

$$\begin{cases} -4A = 0 \\ 4B = 3 \end{cases}$$

which has solution  $A = 0$  and  $B = \frac{3}{4}$ .

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Thus

$$\begin{cases} -4A = 0 \\ 4B = 3 \end{cases}$$

which has solution  $A = 0$  and  $B = \frac{3}{4}$ . Therefore a particular solution is

$$Y(t) = \frac{3}{4}t \sin 2t.$$



# Next Time

- 3.8 Solving Initial Value Problems
- 3.9 The Method of Variation of Parameters
- 3.10 Higher Order Linear ODEs