Lecture 6

- 17. Combinatorics: Basic Counting Principles
- 18. Combinatorics: Permutations and Combinations
- 19. Introduction to Probability

Combinatorics: Basic Counting Principles

The Addition Principle

Theorem

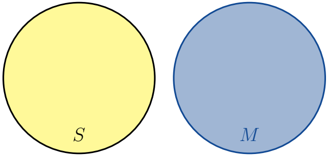
For any two sets A and B,

$$n(A \cup B) = n(A) + n(B) - n(A \cap B)$$

where $n(A)$ denotes the number of elements in set A.

Example

Ali has three yellow shirts and two blue shirts. How many shirts does Ali have?



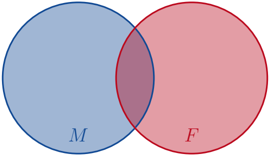
solution: Ali has $3 + 2 = 5$ shirts. If we let S denote the set of Ali's yellow shirts, and M denote the set of Ali's blue shirts, then we have

$$n(S \cup M) = n(S) + n(M) = 3 + 2 = 5.$$

Please note that S and M are discrete sets.

Example

In a college, there are 8 students studying Mathematics, 12 students studying Physics, and 5 students enrolled in a joint Chemistry-Physics-Mathematics program. How many students are studying either Mathematics or Physics?



solution: Let M and F denote the sets of students studying Mathematics and Physics respectively. Then $M \cap F$ will be the set of students who are studying both Mathematics and Physics. The answer is

$$n(M \cup F) = n(M) + n(F) - n(M \cap F) = 8 + 12 - 5 = 15.$$

The Multiplication Principle

Theorem

If operation O_1 can be done n ways, and operation O_2 can be done m ways, then there are

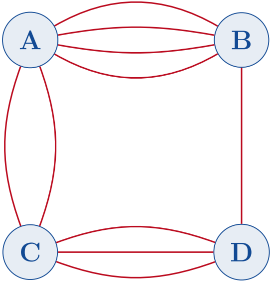
$$n \cdot m$$

possible ways to do O_1 followed by O_2.

Example

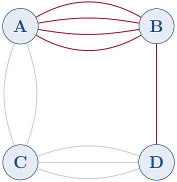
A man has five different shirts, three different ties and two different pairs of trousers. How many different ways can this person wear a shirt, a tie and a pair of trousers combination?

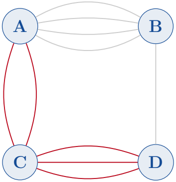
solution: Once the person has chosen a pair of trousers, he has a choice of 5 different shirts. For each shirt there are 3 different ties. Therefore he has $2 \cdot 5 \cdot 3 = 30$ choices of outfit.



Example

Four cities, called Aberdeen (A), Birmingham (B), Coventry (C) and Derby (D), are joined by roads as shown above. By how many different routes can a vehicle travel from A to D, without going back on itself.





solution: A vehicle that wants to travel from A to D must pass through either B or C.

If it passes through B, it can travel from A to B in 4 different ways, then from B to D in only one way – thus it can arrive at D along $4 \cdot 1 = 4$ different routes.

If the vehicle passes through C, it can travel from A to C in 2 different ways, then from C to D in 3 different ways – thus it can arrive at D along $2 \cdot 3 = 6$ different routes.

Hence the total number of different routes from A to D is $4 + 6 = 10$.

Example

How many divisors does 2800 have?

solution: Note first that

$$2800 = 2^4 \cdot 5^2 \cdot 7 = 1 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 5 \cdot 5 \cdot 7.$$

To create a divisor of 2800, we can use between zero and four of the 2s (so there are 4+1 choices of how many 2s to use). We can also use zero, one or both of the 5s (2+1 choices) and we can either use or not use the 7 (1+1 choices). So there are

$$(4+1)(2+1)(1+1) = 30$$

divisors of 2800. In fact, the divisors of 2800 are 1, 2, 4, 5, 7, 8, 10, 14, 16, 20, 25, 28, 35, 40, 50, 56, 70, 80, 100, 112, 140, 175, 200, 280, 350, 400, 560, 700, 1400 and 2800.

In general, suppose that $p_1, p_2, \ldots, p_n$ are prime numbers and suppose that $x = p_1^{k_1} p_2^{k_2} \ldots p_n^{k_n}$. Then x has

$$(k_1 + 1)(k_2 + 1) \ldots (k_n + 1)$$

divisors.

Combinatorics: Permutations and Combinations

Factorials

Definition

The product of the first n natural numbers is called n *factorial* and denoted by $n!$.
We also define the *zero factorial*, $0!$ to be equal to 1.

$$n! = n(n-1)(n-2) \cdot \ldots \cdot 2 \cdot 1$$
$$0! = 1$$
$$n! = n \cdot (n-1)!$$

Example

1. $4! = 4 \cdot 3 \cdot 2 \cdot 1 = 24$
2. $\dfrac{7!}{5!} = \dfrac{7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{5!} = \dfrac{7 \cdot 6 \cdot \cancel{5!}}{\cancel{5!}} = 42$
3. $\dfrac{52!}{5!47!} = \dfrac{52 \cdot 51 \cdot 50 \cdot 49 \cdot 48 \cdot \cancel{47!}}{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 \cdot \cancel{47!}} = 2\,598\,960$

Remark

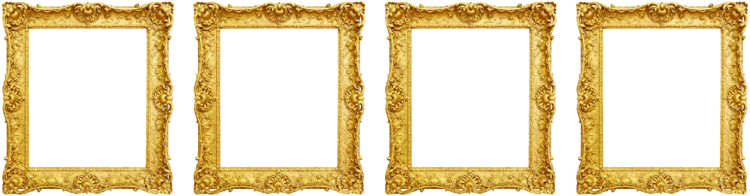
Note that $n!$ grows very rapidly:

$$\begin{aligned} 5! &= 120 \\ 10! &= 3\,628\,800 \\ 15! &= 1\,307\,674\,368\,000 \end{aligned}$$

Permutations

Example

Imagine that you have four pictures to arrange on a wall. How many different ways are there to arrange them?



Permutations

Example

Imagine that you have four pictures to arrange on a wall. How many different ways are there to arrange them?

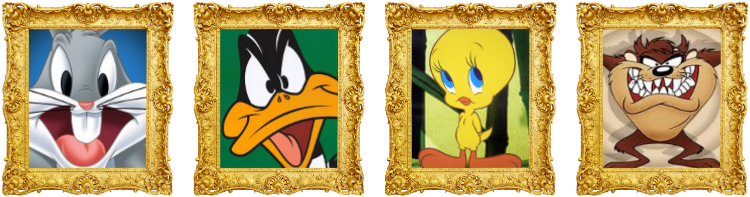
1. 

Permutations

Example

Imagine that you have four pictures to arrange on a wall. How many different ways are there to arrange them?

2.



Permutations

Example

Imagine that you have four pictures to arrange on a wall. How many different ways are there to arrange them?

3. 

Permutations

Example

Imagine that you have four pictures to arrange on a wall. How many different ways are there to arrange them?

4. 

Permutations

Example

Imagine that you have four pictures to arrange on a wall. How many different ways are there to arrange them?

5.    

Permutations

Example

Imagine that you have four pictures to arrange on a wall. How many different ways are there to arrange them?

6.

Permutations

Example

Imagine that you have four pictures to arrange on a wall. How many different ways are there to arrange them?

7.    

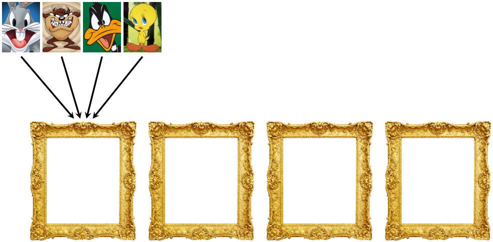
Permutations

Example

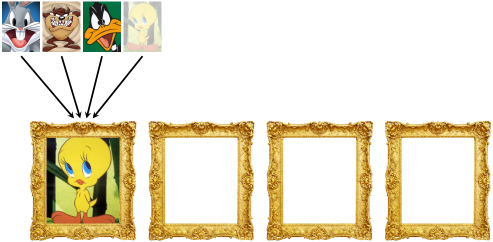
Imagine that you have four pictures to arrange on a wall. How many different ways are there to arrange them?

$\vdots$

18. Combinatorics: Permutations and Combinations

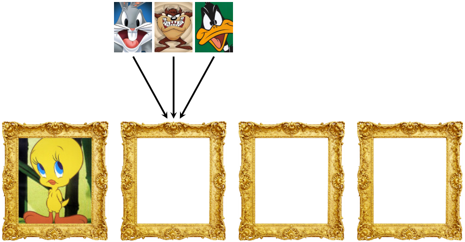


18. Combinatorics: Permutations and Combinations

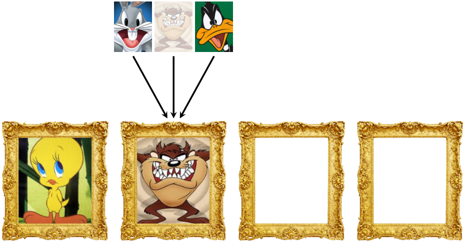


$$4$$

18. Combinatorics: Permutations and Combinations

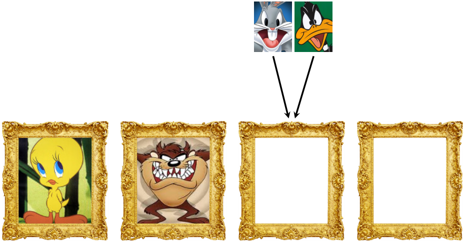


$$4$$



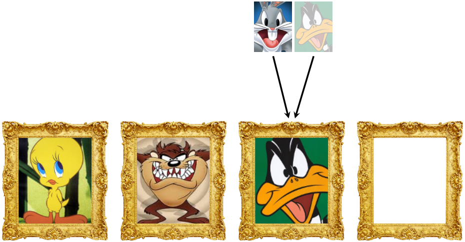
$$4 \cdot 3$$

18. Combinatorics: Permutations and Combinations



$$4 \cdot 3$$

18. Combinatorics: Permutations and Combinations



$$4 \cdot 3 \cdot 2$$

18. Combinatorics: Permutations and Combinations

$$4 \cdot 3 \cdot 2$$

18. Combinatorics: Permutations and Combinations

$$4 \cdot 3 \cdot 2 \cdot 1$$

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$$4 \cdot 3 \cdot 2 \cdot 1 = 4! = 24$$

18. Combinatorics: Permutations and Combinations

$$4 \cdot 3 \cdot 2 \cdot 1 = 4! = 24$$

So there are $4 \cdot 3 \cdot 2 \cdot 1 = 4! = 24$ different ways to hang our pictures.

Definition

A *permutation* of a set of distinct objects is an arrangement of the objects in a specific order without repetition.
The number of permuations of n distinct objects (without repetition) is denoted by ${}_nP_n$ or by $P(n, n)$.

Theorem

$$ {}_{n}P_{n} = n! $$

Example

Now suppose that from your four pictures, you decide to only hang two on your wall. How many ways are there to hang two out of four pictures?

1.  

Example

Now suppose that from your four pictures, you decide to only hang two on your wall. How many ways are there to hang two out of four pictures?

2. 

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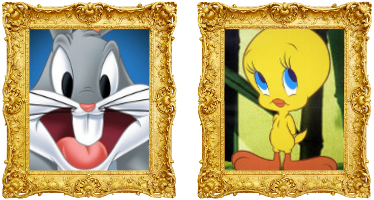
3. 

Example

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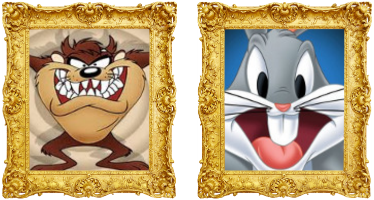
7. 

Example

Now suppose that from your four pictures, you decide to only hang two on your wall. How many ways are there to hang two out of four pictures?

8. 

18. Combinatorics: Permutations and Combinations

Example

Now suppose that from your four pictures, you decide to only hang two on your wall. How many ways are there to hang two out of four pictures?

9.  

Example

Now suppose that from your four pictures, you decide to only hang two on your wall. How many ways are there to hang two out of four pictures?

10.  

Example

Now suppose that from your four pictures, you decide to only hang two on your wall. How many ways are there to hang two out of four pictures?

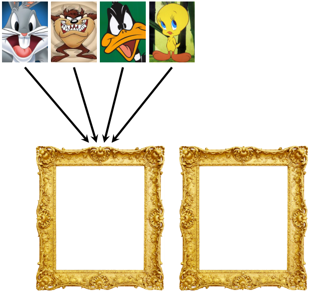
11. 

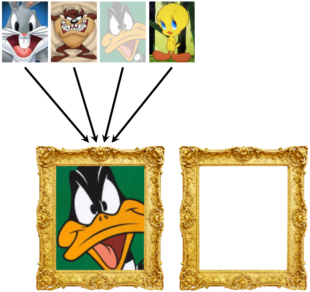
Example

Now suppose that from your four pictures, you decide to only hang two on your wall. How many ways are there to hang two out of four pictures?

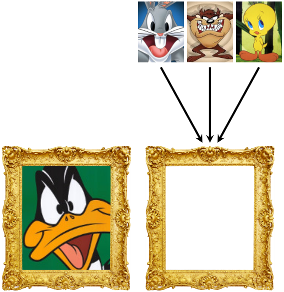
12.

18. Combinatorics: Permutations and Combinations





4



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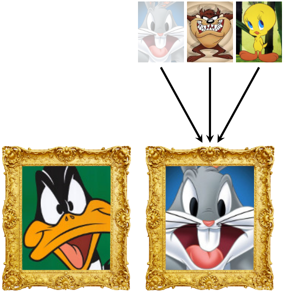
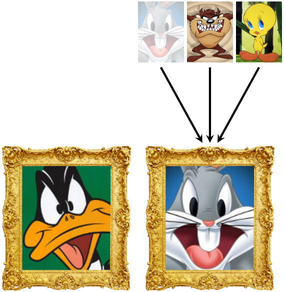


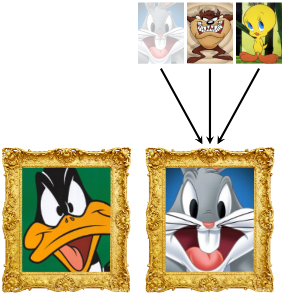
$$4 \cdot 3$$

18. Combinatorics: Permutations and Combinations



$$4 \cdot 3 = 12$$

18. Combinatorics: Permutations and Combinations



$$4 \cdot 3 = 12$$

So there are $4 \cdot 3 = 12$ different ways to hang our pictures.

How can we write this in terms of $n!$?

How can we write this in terms of $n!$?

Note that

$$12 = 4 \cdot 3 = \frac{4 \cdot 3 \cdot 2 \cdot 1}{2 \cdot 1} = \frac{4!}{2!}.$$

18. Combinatorics: Permutations and Combinations

Definition

A permutation of a set of n distinct objects taken r at a time (without repetition) is an arrangement of r of the n objects in a specific order. The numbers of such permutations is denoted by ${}_nP_r$ or by $P(n, r)$.

For example, suppose that we have three objects (labeled a, b and c) and suppose that we take r of the objects. The possible permuations are shown below:

For example, suppose that we have three objects (labeled a, b and c) and suppose that we take r of the objects. The possible permuations are shown below:

$n = 3$		
$r = 1$	$r = 2$	$r = 3$
a	ab	abc
b	ac	acb
c	ba	bac
	bc	bca
	ca	cab
	cb	cba
${}_3P_1 = P(3,1) = 3$	${}_3P_2 = P(3,2) = 6$	${}_3P_3 = P(3,3) = 6$

18. Combinatorics: Permutations and Combinations

Theorem

The number of permuations of n distinct objects taken r at a time (without repetition) is

$$ {}_nP_r = \frac{n!}{(n-r)!} \qquad (1 \leq r \leq n) $$

$$_{n}P_{r} = \frac{n!}{(n-r)!}$$

Example

Find the number of permutations of 16 objects taken 3 at a time.

solution: Using the formula, we calculate that:

$$_{16}P_{3} = \frac{16!}{(16-3)!} = \frac{16 \cdot 15 \cdot 14 \cdot \cancel{13!}}{\cancel{13!}} = 16 \cdot 15 \cdot 14 = 3360.$$

Combinations

Example

To enter the Turkish national lottery (Sayısal Loto 6/49) you must select six numbers from a choice of 49 numbers.
How many different ways are there of choosing 6 objects from 49 objects?

18. Combinatorics: Permutations and Combinations

The answer is not ${}_{49}P_6$ because the order of the numbers does not matter: For example

28 16 9 7 35 47

is the same as

7 9 16 28 35 47.

Definition

A *combination* of a set of n distinct objects taken r at a time (without repetition) is an r-element subset of the set of n objects. The arrangement of the elements in the subset does not matter. We denote the number of combinations by

$$ {}_nC_r \quad \text{or} \quad \binom{n}{r} \quad \text{or} \quad C(n,r). $$

For example, suppose again that we have three objects labeled a, b and c and suppose that we take r of these objects. The possible combinations are shown below:

For example, suppose again that we have three objects labeled a, b and c and suppose that we take r of these objects. The possible combinations are shown below:

$n = 3$		
$r = 1$	$r = 2$	$r = 3$
a b c	ab ac bc	abc
${}_3C_1 = C(3,1) = 3$	${}_3C_2 = C(3,2) = 3$	${}_3C_3 = C(3,3) = 1$

Theorem

The number of combinations of n distinct objects taken r at a time (without repetition) is

$$ {}_nC_r = \frac{n!}{(n-r)! \cdot r!} \qquad (1 \leq r \leq n)$$

$$_nC_r = \frac{n!}{(n-r)! \cdot r!}$$

Example

A collector has 20 different coins. How many different ways can 6 coins be selected?

solution:

$$\begin{aligned} {}_{20}C_6 &= \frac{20!}{(20-6)! \cdot 6!} = \frac{20 \cdot 19 \cdot 18 \cdot 17 \cdot 16 \cdot 15 \cdot \cancel{14!}}{\cancel{14!} \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2} \\ &= \frac{\cancel{20} \cdot 19 \cdot \cancel{18} \cdot 17 \cdot 16 \cdot 15}{\cancel{6} \cdot \cancel{5} \cdot \cancel{4} \cdot \cancel{3} \cdot 2} = \frac{19 \cdot 17 \cdot 16 \cdot 15}{2} \\ &= 38\,760. \end{aligned}$$

$$_{n}C_{r} = \frac{n!}{(n-r)! \cdot r!}$$

Example

We can now answer the lottery question.

There are

$$_{49}C_{6} = \frac{49!}{(49-6)! \cdot 6!} = \frac{49!}{43! \cdot 6!} = 13\,983\,816$$

different ways to choose 6 numbers from a choice of 49 numbers.

18. Combinatorics: Permutations and Combinations

Remember

- Permutations: ${}_nP_r = \dfrac{n!}{(n-r)!}$
- Combinations: ${}_nC_r = \dfrac{n!}{(n-r)! \cdot r!}$

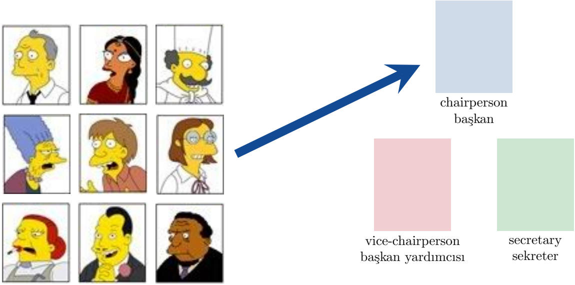
Example

From a group of 9 people,

1. In how many ways can a chairperson, a vice-chairperson and a secretary be elected, assuming that one person can not hold more than one position?
2. In how many ways can we choose a committee of three people?

solution:

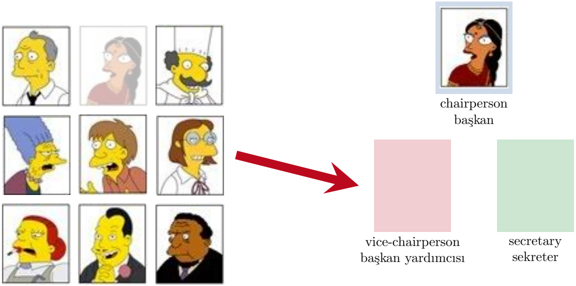
1.


chairperson
başkan
vice-chairperson
başkan yardımcısı
secretary
sekreter

$$9$$

solution:

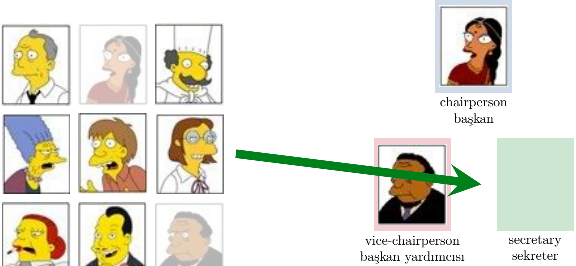
1



$$9 \cdot 8$$

solution:

1



$$9 \cdot 8 \cdot 7$$

solution:

1

$$9 \cdot 8 \cdot 7 = 504$$

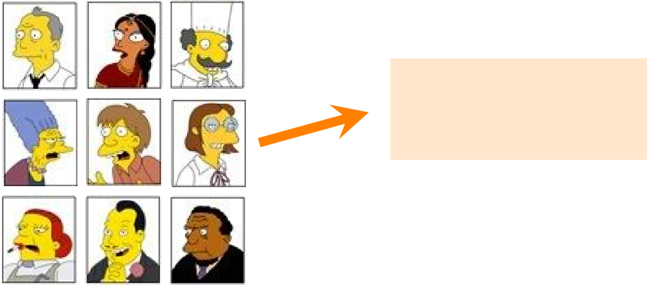
Therefore, this tripartite committee can be formed in $9 \cdot 8 \cdot 7 = 504$ different ways.

Or we can answer this problem with permutations:

$$ {}_9P_3 = \frac{9!}{(9-3)!} = 504. $$

solution:

2

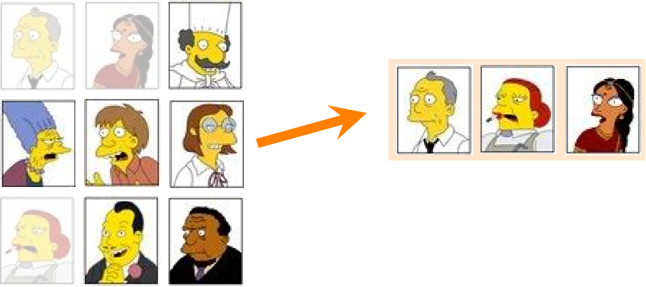


In order to form a communitee of 3 people, the order does not matter!

18. Combinatorics: Permutations and Combinations

solution:

2



In order to form a communitee of 3 people, the order does not matter! So we use combinations:

$$_9C_3 = \frac{9!}{(9-3)!3!} = 84.$$

Remark

Permutations and Combinations are similar in that repetition in selections are not permitted. However, there is one important difference between them:

- In a permutation, the order is important;
- In a combination, the order is irrelevent.

You need to understand when to use ${}_nP_r$ and when to use ${}_nC_r$.

Example

From a standard deck of 52 cards,

1. How many 5-card hands have two kings and three aces?
2. How many 5-card hands have two clubs and three hearts?
3. How many 3-card hands have all cards from the same suit?

solution:

1. **How many 5-card hands have two kings and three aces?**
   We need to use the multiplication princple and combinations. We must select two kings from a total of four (that is ${}_4C_2$) and we must select three aces from a total of four (${}_4C_3$). Therefore, by the multiplication principle we have that

$$\begin{aligned}\text{number of hands} &= {}_4C_2 \cdot {}_4C_3 \\ &= \frac{4!}{(4-2)! \cdot 2!} \cdot \frac{4!}{(4-3)! \cdot 3!} \\ &= 6 \cdot 4 = 24.\end{aligned}$$

2. **How many 5-card hands have two clubs and three hearts?**

   There are ${}_{13}C_2 \cdot {}_{13}C_3 = \frac{13!}{11! \cdot 2!} \cdot \frac{13!}{10! \cdot 3!} = 78 \cdot 208 = 22\,308$ hands with two clubs and three hearts.

3 **How many 3-card hands have all cards from the same suit?**

There are 13 cards in each suit, so the number of 3-card hands having all hearts, say, is

$$ {}_{13}C_3 = \frac{13!}{(13-3)! \cdot 3!} = \frac{13!}{10! \cdot 3!} = 286. $$

Similarly, there are 286 hands having all clubs, 286 hands having all diamonds and 286 hands having all spades. Thus the number of hands having all cards from the same suit is

$$ 4 \cdot {}_{13}C_3 = 4 \cdot 286 = 1144. $$

Example

A bag contains 2 white and 3 red balls. In how many ways can 3 balls be chosen if at least one ball must be white?

solution: We can have either 1 white ball and 2 red balls, or 2 white balls and 1 red ball.
There are ${}_2C_1 \cdot {}_3C_2$ ways to get 1 white and 2 red. There are ${}_2C_2 \cdot {}_3C_1$ ways to get 2 white and 1 red. So the total number of ways is

$${}_2C_1 \cdot {}_3C_2 + {}_2C_2 \cdot {}_3C_1 = (2)(3) + (1)(3) = 9.$$

Example

You have 1 red ball, 1 blue ball, 1 green ball and 3 orange balls. The three orange balls are identical. How many visually different ways are there to arrange the balls in a line?

solution: If you had balls of six different colours, then this is easy: ${}_6P_6 = 6!$. However because you have 3 balls of the same colour, the correct answer will be less than this. For example, if we label the balls r, b, g, o_1, o_2 and o_3 then $r\,g\,o_1\,o_2\,b\,o_3$ and $r\,g\,o_2\,o_3\,b\,o_1$ will look the same, but be counted twice.

But how many different ways are there to rearrange the orange balls, o_1 o_2 o_3? There are ${}_3P_3 = 6$ different ways. This means that when we calculate ${}_6P_6$, we are counting each arrangement 6 times.

Therefore the answer to this problem is

$$\frac{{}_6P_6}{{}_3P_3} = 120.$$

Example

In how many different ways can the letters of the word 'MATHEMATICS' be arranged such that the vowels are consecutive?

solution: The word

M A T H E M A T I C S

has 11 letters including 4 vowels: ‘A’,‘E’,‘A’,‘I’. These 4 vowels must always be consecutive. Hence these 4 vowels can be grouped together and then thought of as a single object. In other words, we may assume that we have only 8 objects,

M T H M T C S AEAI.

18. Combinatorics: Permutations and Combinations

M T H M T C S AEAI .

Of these 8 objects, we have two ’M’s, two ’T’s, one ‘H’, one ‘C’, one ‘S’ and one ‘(AEAI)’. The number of ways to arrange these 8 objects is

$$\frac{{}_8P_8}{{}_2P_2 \cdot {}_2P_2} = \frac{8!}{2! \cdot 2!} = 10080.$$

M T H M T C S AEAI.

Next we must ask how many ways the vowels ‘AEAI’ can be rearranged. The letter ‘A’ occurs twice and the other letters occur once. Hence there are

$$\frac{{}_4P_4}{{}_2P_2} = 12$$

ways to arrange the vowels.

Multiplying these together, we get our answer: There are

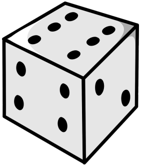
$$10080 \cdot 12 = 120\,960$$

ways.

18. Combinatorics: Permutations and Combinations

18. Combinatorics: Permutations and Combinations

Introduction to Probability



If you roll a single (standard six-sided) die, what are the possible outcomes?

$$S = \{1, 2, 3, 4, 5, 6\}$$

This is called a *sample space*. Each of the numbers in S are called *simple events*.

Now suppose that we want to roll an even number: The subset

$$E = \{2, 4, 6\} \subseteq S$$

is called a *compound event*.

Definition

The set

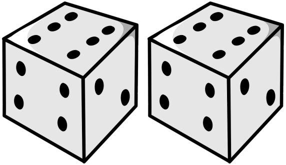
$$S = \{e_1, e_2, \ldots, e_n\}$$

is called a *sample space* for some experiment iff,

- S contains all possible outcomes;
- one and only one of the outcomes in S must occur.

Definition

An *event* E is any subset of S (including the empty set $\varnothing$ and S itself). e_i is a *simple event*. $E = \{e_i\}$ is a *simple event* if E contains only one element. E is a *compound event* if E contains more than one element.



Now suppose that you are rolling 2 dice.

19. Introduction to Probability

		second die					
		⚀	⚁	⚂	⚃	⚄	⚅
first die	⚀	(1,1)	(1,2)	(1,3)	(1,4)	(1,5)	(1,6)
	⚁	(2,1)	(2,2)	(2,3)	(2,4)	(2,5)	(2,6)
	⚂	(3,1)	(3,2)	(3,3)	(3,4)	(3,5)	(3,6)
	⚃	(4,1)	(4,2)	(4,3)	(4,4)	(4,5)	(4,6)
	⚄	(5,1)	(5,2)	(5,3)	(5,4)	(5,5)	(5,6)
	⚅	(6,1)	(6,2)	(6,3)	(6,4)	(6,5)	(6,6)

Example (Two Dice)

If we are rolling two dice, then the sample space is

$$S = \{(a, b) \mid a, b \in \{1, 2, \dots, 6\}\}.$$

What is the event which corresponds to:

1. A total score of 7.
2. A total score of 3.
3. A total score greater than 10.
4. A total score of 2.

solution:

1. $E = \{(1, 6), (2, 5), (3, 4), (4, 3), (5, 2), (6, 1)\}.$
2. $E = \{(1, 2), (2, 1)\}.$
3. $E = \{(5, 6), (6, 5), (6, 6)\}.$
4. $E = \{(1, 1)\}.$

Definition

Let

$$S = \{e_1, e_2, e_e, \ldots, e_n\}$$

be a sample space with n simple events. The *probability of event* e_i is a real number denoted by $P(e_i)$. We must have

1. $P(e_i) \in [0, 1]$ for all i; and
2. $P(e_1) + P(e_2) + P(e_3) + \ldots + P(e_n) = 1$.

Example (A Single Coin)

Suppose that we are flipping a single coin. Then $S = \{H, T\}$. We can assume that

$$P(H) = \frac{1}{2} \qquad \text{and} \qquad P(T) = \frac{1}{2}.$$

Note that

1. $0 \leq P(H) \leq 1$ and $0 \leq P(T) \leq 1$; and
2. $P(H) + P(T) = \frac{1}{2} + \frac{1}{2} = 1$.

If we flip this coin 1000 times, we would expect to get H roughly (but not exactly) 500 times.

Definition

The *probability of an event* E, denoted $P(E)$ must satisfy:

1. If $E = \varnothing$ is the empty set, then $P(E) = 0$;
2. If $E = S$, then $P(E) = 1$.
3. If $E = \{e_i\}$ is a simple event, then $P(E) = P(e_i)$;
4. If E is a compound event, then $P(E)$ must be equal to the sum of the probabilities of all the simple events in E. E.g. if $E = \{a, b, c\}$ then $P(E) = P(a) + P(b) + P(c)$.

Remark

$P(E) = 1$ means that E is certain to occur. $P(E) = 0$ means that E will never occur.

Example (Two coins)

Now suppose that you are flipping two different coins. The sample space is

$$S = \{HH, HT, TH, TT\}.$$

We can assume that

$$P(HH) = P(HT) = P(TH) = P(TT) = \tfrac{1}{4}.$$

1. What is the probability of getting one head and one tail?
2. What is the probability of getting at least one tail?
3. What is the probability of getting at least one head or at least one tail?
4. What is the probability of getting two tails?
5. What is the probability of getting three tails?

solution:

1. **1 head and 1 tail:** We have $E = \{HT, TH\}$ and

$$P(E) = P(HT) + P(TH) = \frac{1}{4} + \frac{1}{4} = \frac{1}{2}.$$

2. **At least one tail:** We have $E = \{HT, TH, TT\}$ and

$$P(E) = P(HT) + P(TH) + P(TT) = \frac{1}{4} + \frac{1}{4} + \frac{1}{4} = \frac{3}{4}.$$

3. **At least one head or at least one tail:** We have $E = \{HH, HT, TH, TT\} = S$ and

$$P(E) = P(S) = 1.$$

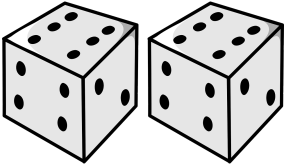
4. **2 tails:** We have $E = \{TT\}$ and $P(E) = \frac{1}{4}$.

5. **3 tails:** It is not possible to get three tails, so $E = \varnothing$ and $P(E) = P(\varnothing) = 0$.

Theorem

If we assume that each simple event in S is equally likely, then the probability of an event E is

$$P(E) = \frac{\textit{number of elements in } E}{\textit{number of elements in } S} = \frac{n(E)}{n(S)}.$$



Example

Suppose that we are rolling two dice and suppose that each simple event is equally likely. Find the probabilities of the following:

1. A total score of 7.
2. A total score of 3.
3. A total score greater than 10.
4. A total score of 2.

solution:

1. **total score = 7:** Since $E = \{(1,6),(2,5),(3,4),(4,3),(5,2),(6,1)\}$, we have that

$$P(E) = \frac{n(E)}{n(S)} = \frac{6}{36} = \frac{1}{6}.$$

2 **total score = 3:** We have that $E = \{(1,2),(2,1)\}$ and that

$$P(E) = \frac{n(E)}{n(S)} = \frac{2}{36} = \frac{1}{18}.$$

3. **total score > 10:** Here $E = \{(5,6),(6,5),(6,6)\}$ and

$$P(E) = \frac{n(E)}{n(S)} = \frac{3}{36} = \frac{1}{12}.$$

4. **total score $= 2$:** Since $E = \{(1,1)\}$, it follows that

$$P(E) = \frac{n(E)}{n(S)} = \frac{1}{36}.$$

Example

You randomly draw five cards from a standard deck of 52 cards. What is the probability of getting two clubs and three hearts?

solution: There are $n(S) = {}_{52}C_5$ possible 5-card hands. As we covered previously, there are $n(E) = {}_{13}C_2 \cdot {}_{13}C_3$ 5-card hands which have two clubs and three hearts. So the probability is

$$P(E) = \frac{n(E)}{n(S)} = \frac{{}_{13}C_2 \cdot {}_{13}C_3}{{}_{52}C_5} = \frac{78 \cdot 208}{2\,598\,960} \approx 0.0062$$

Next Time

- 20. Concepts of Probability
- 21. Conditional Probability
- 22. Probability Trees