



Welcome to **Mathematics II** with Dr Neil Course

Lecture 1

- Information about this course
- 8.1 Using Basic Integration Formulae
- 8.2 Integration by Parts
- 8.3 Trigonometric Integrals

MATH113

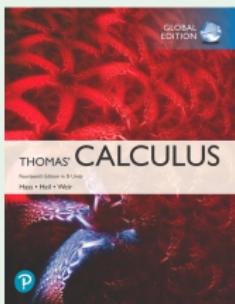
MATH114

MATH215

MATH216

MATH113

MATH114



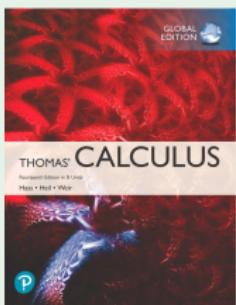
Calculus

MATH215

MATH216

MATH113

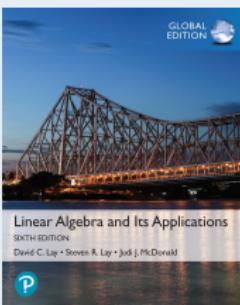
MATH114



Calculus

MATH215

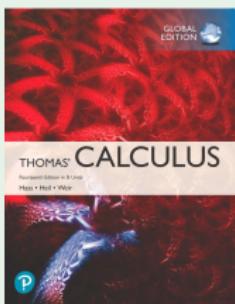
MATH216



Linear Algebra

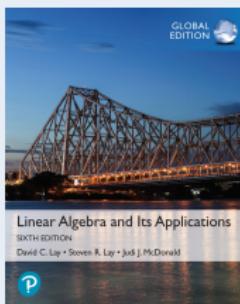
MATH113

MATH114



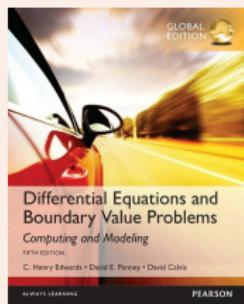
Calculus

MATH215



Linear Algebra

MATH216



Differential Equations

Information about this course

- \approx 12 classes. Friday afternoons 2pm-4:30pm.

14:00

15:00

16:00

Information about this course

- \approx 12 classes. Friday afternoons 2pm-4:30pm.
- 2 lectures with a break between.

lecture

lecture

14:00

15:00

16:00

Information about this course

- \approx 12 classes. Friday afternoons 2pm-4:30pm.
- 2 lectures with a break between.
- Then I will answers your questions.

lecture

lecture

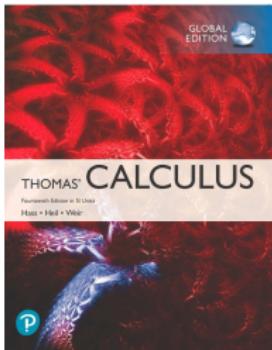
questions

14:00

15:00

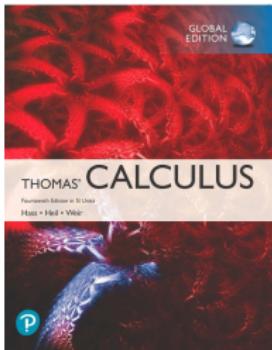
16:00

The Book



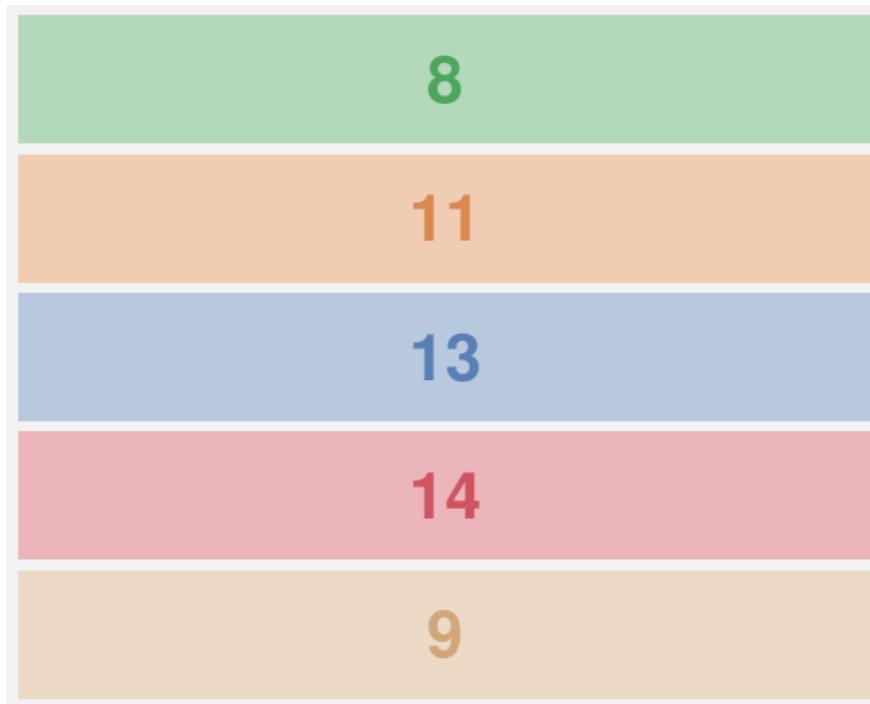
Joel R. Hass, Christopher E. Heil and Maurice D. Weir,
Thomas' Calculus,
14th Edition in SI Units, Pearson.

The Book



Joel R. Hass, Christopher E. Heil and Maurice D. Weir,
Thomas' Calculus,
14th Edition in SI Units, Pearson.

This is a required purchase.
You need to have this book to be
able to do the homework.



8. Techniques of Integration

11

13

14

9

8. Techniques of Integration

11. Vectors and the Geometry of Space

13

14

9

8. Techniques of Integration

11. Vectors and the Geometry of Space

13. Partial Derivatives

14

9

8. Techniques of Integration

11. Vectors and the Geometry of Space

13. Partial Derivatives

14. Multiple Integrals



8. Techniques of Integration

11. Vectors and the Geometry of Space

13. Partial Derivatives

14. Multiple Integrals

9. Infinite Sequences and Series

8. Techniques of Integration

} 2 weeks

11. Vectors and the Geometry of Space

} 2 weeks

13. Partial Derivatives

} 2 weeks

14. Multiple Integrals

} 3 weeks

9. Infinite Sequences and Series

} 3 weeks



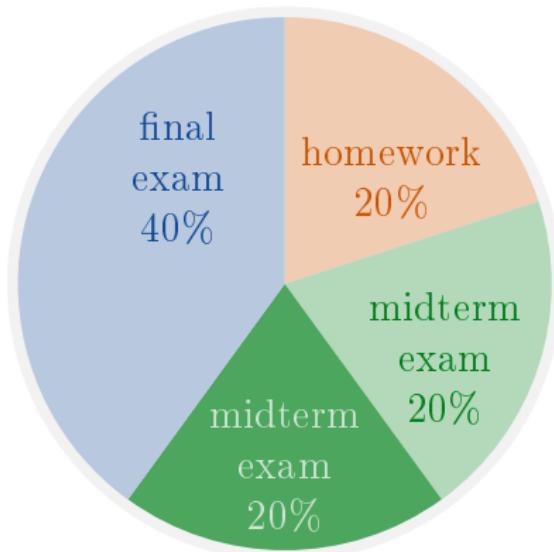
Exams and homework

(This information may change based on the University's decisions)



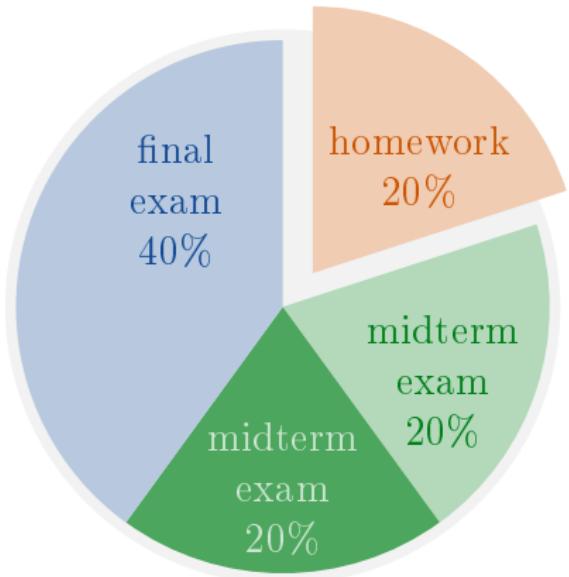
Exams and homework

(This information may change based on the University's decisions)



Exams and homework

(This information may change based on the University's decisions)



using Pearson's
MyLab Math

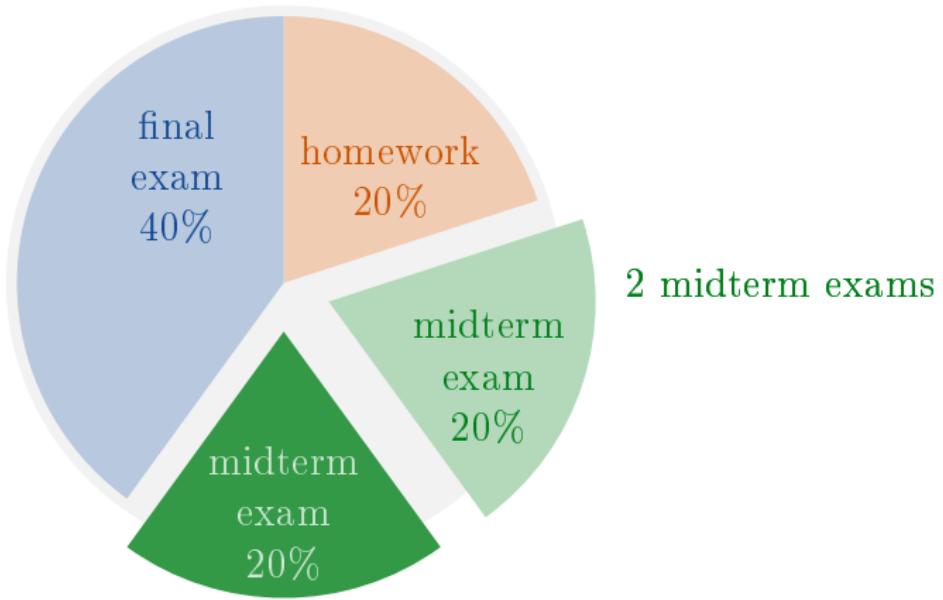
12 pieces of
homework

deadline = one
day before the
final exam

more details in
O'Learn

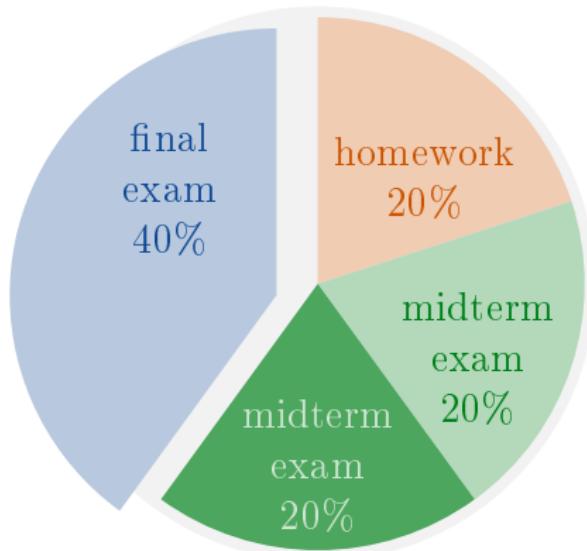
Exams and homework

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Exams and homework

(This information may change based on the University's decisions)



Expectations

If this was a classroom course, the expectation is that for each hour of lectures, you would study an extra 1-2 hours outside of class.

classroom
course

lectures (5 hours)

other study (5-10 hours)

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lectures (5 hours)

other study (5-10 hours)

For an online course, you are still expected to study a total of 10-15 hours each week.

online
course

class
(2.5 hours)

other study (7.5-12.5 hours)

This may include:

- Do the online homework on MyLab;

⋮

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⋮

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- Read the textbook;
- Solve the exercises in the textbook;
- Use the O'Learn Discussion Board;

:

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- Solve the exercises in the textbook;
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:

This may include:

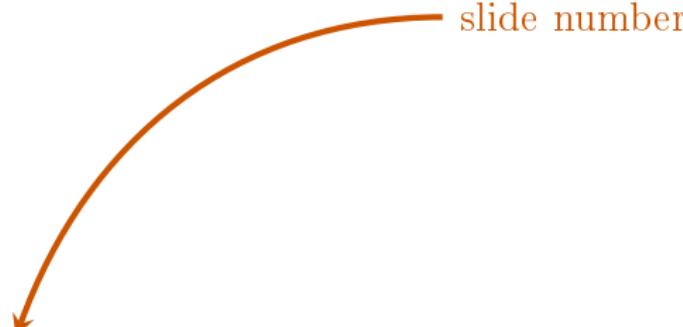
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- Read the textbook;
- Solve the exercises in the textbook;
- Use the O'Learn Discussion Board;
- Read other books?;
- Watch online videos;

:

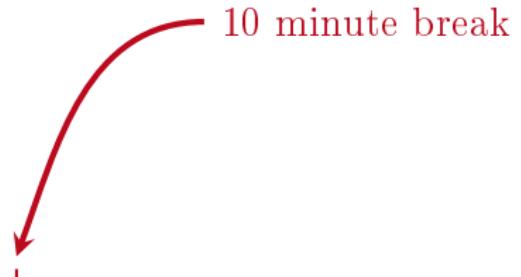
99.9 Section Title



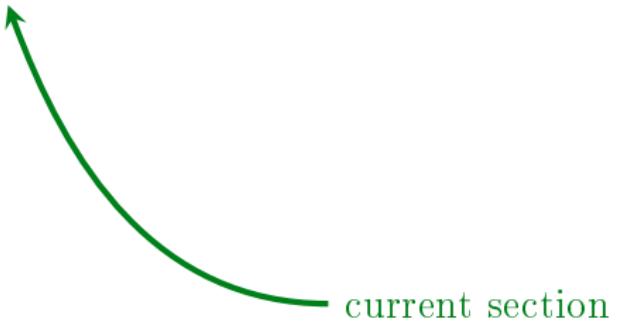
slide number



10 minute break



99.9 Section Title



99.9 Section Title

$$1 + 2 = 3$$



something important





Using Basic Integration Formulae



22 Basic Integration Formulae

Let's start with a little bit of revision

22 Basic Integration Formulae

1 $\int k \, dx = kx + C$ $(k$ is a number)

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2 $\int x^n \, dx = \frac{x^{n+1}}{n+1} + C$

22 Basic Integration Formulae

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2 $\int x^n \, dx = \frac{x^{n+1}}{n+1} + C$

3 $\int \frac{1}{x} \, dx = \ln|x| + C$

4 $\int e^x \, dx = e^x + C$

22 Basic Integration Formulae

1 $\int k \, dx = kx + C$ $(k$ is a number)

2 $\int x^n \, dx = \frac{x^{n+1}}{n+1} + C$

3 $\int \frac{1}{x} \, dx = \ln|x| + C$

4 $\int e^x \, dx = e^x + C$

5 $\int a^x \, dx = \frac{a^x}{\ln a} + C$ $(a > 0, a \neq 1)$

22 Basic Integration Formulae

6 $\int \sin x \, dx = -\cos x + C$

7 $\int \cos x \, dx = \sin x + C$

22 Basic Integration Formulae

$$6 \quad \int \sin x \, dx = -\cos x + C$$

$$7 \quad \int \cos x \, dx = \sin x + C$$

$$8 \quad \int \sec^2 x \, dx = \tan x + C$$

$$9 \quad \int \operatorname{cosec}^2 x \, dx = -\cot x + C$$

$$10 \quad \int \sec x \tan x \, dx = \sec x + C$$

$$11 \quad \int \operatorname{cosec} x \cot x \, dx = -\operatorname{cosec} x + C$$

22 Basic Integration Formulae

$$12 \quad \int \tan x \, dx = \ln |\sec x| + C$$

$$13 \quad \int \cot x \, dx = \ln |\sin x| + C$$

$$14 \quad \int \sec x \, dx = \ln |\sec x + \tan x| + C$$

$$15 \quad \int \operatorname{cosec} x \, dx = -\ln |\operatorname{cosec} x + \cot x| + C$$



22 Basic Integration Formulae

$$16 \quad \int \sinh x \, dx = \cosh x + C$$

$$17 \quad \int \cosh x \, dx = \sinh x + C$$



22 Basic Integration Formulae

$$18 \quad \int \frac{dx}{\sqrt{a^2 - x^2}} = \sin^{-1} \left(\frac{x}{a} \right) + C$$

22 Basic Integration Formulae

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$$19 \quad \int \frac{dx}{a^2 + x^2} = \frac{1}{a} \tan^{-1} \left(\frac{x}{a} \right) + C$$

22 Basic Integration Formulae

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$$19 \quad \int \frac{dx}{a^2 + x^2} = \frac{1}{a} \tan^{-1} \left(\frac{x}{a} \right) + C$$

$$20 \quad \int \frac{dx}{x\sqrt{x^2 - a^2}} = \frac{1}{a} \sec^{-1} \left| \frac{x}{a} \right| + C$$

22 Basic Integration Formulae

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$$21 \quad \int \frac{dx}{\sqrt{a^2 + x^2}} = \sinh^{-1} \left(\frac{x}{a} \right) + C \quad (a > 0)$$

22 Basic Integration Formulae

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$$21 \int \frac{dx}{\sqrt{a^2 + x^2}} = \sinh^{-1} \left(\frac{x}{a} \right) + C \quad (a > 0)$$

$$22 \int \frac{dx}{\sqrt{x^2 - a^2}} = \cosh^{-1} \left(\frac{x}{a} \right) + C \quad (x > a > 0)$$

0.1 Using Basic Integration Formulae

Example

Calculate $\int_3^5 \frac{2x - 3}{\sqrt{x^2 - 3x + 1}} dx.$

0.1 Using Basic Integration Formulae

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Calculate $\int_3^5 \frac{2x - 3}{\sqrt{x^2 - 3x + 1}} dx.$

We use the substitution $u = x^2 - 3x + 1.$

0.1 Using Basic Integration Formulae

Example

Calculate $\int_3^5 \frac{2x - 3}{\sqrt{x^2 - 3x + 1}} dx$.

We use the substitution $u = x^2 - 3x + 1$. Then $du = (2x - 3) dx$

0.1 Using Basic Integration Formulae

Example

Calculate $\int_3^5 \frac{2x - 3}{\sqrt{x^2 - 3x + 1}} dx.$

We use the substitution $u = x^2 - 3x + 1$. Then $du = (2x - 3) dx$ and

$$x = 5 \implies u = x^2 - 3x + 1 = 25 - 15 + 1 = 11$$

$$x = 3 \implies u = x^2 - 3x + 1 = 9 - 9 + 1 = 1.$$

0.1 Using Basic Integration Formulae

Example

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We use the substitution $u = x^2 - 3x + 1$. Then $du = (2x - 3) dx$ and

$$x = 5 \implies u = x^2 - 3x + 1 = 25 - 15 + 1 = 11$$

$$x = 3 \implies u = x^2 - 3x + 1 = 9 - 9 + 1 = 1.$$

Hence

$$\int_3^5 \frac{2x - 3}{\sqrt{x^2 - 3x + 1}} dx = \int_1^{11} u^{-\frac{1}{2}} du = [2\sqrt{u}]_1^{11} = 2(\sqrt{11} - 1).$$

0.1 Using Basic Integration

$$\int \frac{dx}{\sqrt{a^2 - x^2}} = \sin^{-1} \left(\frac{x}{a} \right) + C$$



Example

Find $\int \frac{dx}{\sqrt{8x - x^2}}$.

0.1 Using Basic Integration

$$\int \frac{dx}{\sqrt{a^2 - x^2}} = \sin^{-1} \left(\frac{x}{a} \right) + C$$



Example

Find $\int \frac{dx}{\sqrt{8x - x^2}}$.

This time we will complete the square of $x^2 - 8x$ and use that to simplify the integral:

0.1 Using Basic Integration

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Find $\int \frac{dx}{\sqrt{8x - x^2}}$.

This time we will complete the square of $x^2 - 8x$ and use that to simplify the integral:

$$x^2 - 8x = x^2 - 8x + 16 - 16$$

0.1 Using Basic Integration

$$\int \frac{dx}{\sqrt{a^2 - x^2}} = \sin^{-1} \left(\frac{x}{a} \right) + C$$



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This time we will complete the square of $x^2 - 8x$ and use that to simplify the integral:

$$x^2 - 8x = x^2 - 8x + 16 - 16 = (x^2 - 8x + 16) - 16 = (x - 4)^2 - 16.$$

0.1 Using Basic Integration

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So

$$\int \frac{dx}{\sqrt{8x - x^2}} = \int \frac{dx}{\sqrt{16 - (x - 4)^2}}$$

=

=

=

0.1 Using Basic Integration

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$$\int \frac{dx}{\sqrt{8x - x^2}} = \int \frac{dx}{\sqrt{16 - (x - 4)^2}}$$

$$= \int \frac{du}{\sqrt{16 - u^2}}$$

=

=

0.1 Using Basic Integration

$$\int \frac{dx}{\sqrt{a^2 - x^2}} = \sin^{-1} \left(\frac{x}{a} \right) + C$$



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So

$$\begin{aligned}\int \frac{dx}{\sqrt{8x - x^2}} &= \int \frac{dx}{\sqrt{16 - (x - 4)^2}} \\&= \int \frac{du}{\sqrt{16 - u^2}} \\&= \sin^{-1} \left(\frac{u}{4} \right) + C\end{aligned}$$

=

0.1 Using Basic Integration

$$\int \frac{dx}{\sqrt{a^2 - x^2}} = \sin^{-1} \left(\frac{x}{a} \right) + C$$



Example

Find $\int \frac{dx}{\sqrt{8x - x^2}}$.

This time we will complete the square of $x^2 - 8x$ and use that to simplify the integral:

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So

$$\begin{aligned}\int \frac{dx}{\sqrt{8x - x^2}} &= \int \frac{dx}{\sqrt{16 - (x - 4)^2}} \\&= \int \frac{du}{\sqrt{16 - u^2}} \\&= \sin^{-1} \left(\frac{u}{4} \right) + C \\&= \sin^{-1} \left(\frac{x - 4}{4} \right) + C.\end{aligned}$$

0.1 Using Basic Integration Formulae



Example

Find $\int \cos x \sin 2x + \sin x \cos 2x \, dx$.

0.1 Using Basic Integration Formulae



Example

Find $\int \cos x \sin 2x + \sin x \cos 2x \, dx$.

$$\begin{aligned}\int \cos x \sin 2x + \sin x \cos 2x \, dx &= \int \sin(x + 2x) \, dx \\ &= \int \sin 3x \, dx \\ &= \dots\end{aligned}$$

0.1 Using Basic Integration Formulae



Example

Find $\int_0^{\frac{\pi}{4}} \frac{dx}{1 - \sin x}$.

0.1 Using Basic Integration Formulae



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Find $\int_0^{\frac{\pi}{4}} \frac{dx}{1 - \sin x}$.

Here is a trick for dealing with $\frac{1}{A-B}$:

0.1 Using Basic Integration Formulae



Example

Find $\int_0^{\frac{\pi}{4}} \frac{dx}{1 - \sin x}.$

Here is a trick for dealing with $\frac{1}{A-B}$: Multiply by $\frac{A+B}{A+B}$.

0.1 Using Basic Integration Formulae



Example

Find $\int_0^{\frac{\pi}{4}} \frac{dx}{1 - \sin x}$.

Here is a trick for dealing with $\frac{1}{A-B}$: Multiply by $\frac{A+B}{A+B}$.
Then we get

$$\frac{1}{A-B} = \left(\frac{1}{A-B} \right) \left(\frac{A+B}{A+B} \right) = \frac{A+B}{A^2 - B^2}$$

which is sometimes easier to deal with.

0.1 Using Basic Integration Formulae



$$\int_0^{\frac{\pi}{4}} \frac{dx}{1 - \sin x} = \int_0^{\frac{\pi}{4}} \frac{1}{1 - \sin x} \cdot dx =$$

$$= =$$

$$=$$

$$= =$$

$$= .$$

0.1 Using Basic Integration Formulae



$$\int_0^{\frac{\pi}{4}} \frac{dx}{1 - \sin x} = \int_0^{\frac{\pi}{4}} \frac{1}{1 - \sin x} \cdot \frac{1 + \sin x}{1 + \sin x} dx =$$

$$= =$$

$$=$$

$$= =$$

$$= .$$

0.1 Using Basic Integration Formulae



0.1 Using Basic Integration Formulae



$$\begin{aligned} \int_0^{\frac{\pi}{4}} \frac{dx}{1 - \sin x} &= \int_0^{\frac{\pi}{4}} \frac{1}{1 - \sin x} \cdot \frac{1 + \sin x}{1 + \sin x} dx = \int_0^{\frac{\pi}{4}} \frac{1 + \sin x}{1 - \sin^2 x} dx \\ &= \int_0^{\frac{\pi}{4}} \frac{1 + \sin x}{\cos^2 x} dx = \\ &= \dots \end{aligned}$$

0.1 Using Basic Integration Formulae



$$\begin{aligned} \int_0^{\frac{\pi}{4}} \frac{dx}{1 - \sin x} &= \int_0^{\frac{\pi}{4}} \frac{1}{1 - \sin x} \cdot \frac{1 + \sin x}{1 + \sin x} dx = \int_0^{\frac{\pi}{4}} \frac{1 + \sin x}{1 - \sin^2 x} dx \\ &= \int_0^{\frac{\pi}{4}} \frac{1 + \sin x}{\cos^2 x} dx = \int_0^{\frac{\pi}{4}} \frac{1}{\cos^2 x} + \frac{\sin x}{\cos^2 x} dx \\ &= \int_0^{\frac{\pi}{4}} \sec^2 x + \sec x \tan x dx \\ &= \quad \quad \quad = \quad \quad \quad = \quad \quad . \end{aligned}$$

0.1 Using Basic Integration Formulae



$$\begin{aligned} \int_0^{\frac{\pi}{4}} \frac{dx}{1 - \sin x} &= \int_0^{\frac{\pi}{4}} \frac{1}{1 - \sin x} \cdot \frac{1 + \sin x}{1 + \sin x} dx = \int_0^{\frac{\pi}{4}} \frac{1 + \sin x}{1 - \sin^2 x} dx \\ &= \int_0^{\frac{\pi}{4}} \frac{1 + \sin x}{\cos^2 x} dx = \int_0^{\frac{\pi}{4}} \frac{1}{\cos^2 x} + \frac{\sin x}{\cos^2 x} dx \\ &= \int_0^{\frac{\pi}{4}} \sec^2 x + \sec x \tan x dx \\ &= \left[\tan x + \sec x \right]_0^{\frac{\pi}{4}} = \end{aligned}$$

0.1 Using Basic Integration Formulae



$$\begin{aligned}\int_0^{\frac{\pi}{4}} \frac{dx}{1 - \sin x} &= \int_0^{\frac{\pi}{4}} \frac{1}{1 - \sin x} \cdot \frac{1 + \sin x}{1 + \sin x} dx = \int_0^{\frac{\pi}{4}} \frac{1 + \sin x}{1 - \sin^2 x} dx \\&= \int_0^{\frac{\pi}{4}} \frac{1 + \sin x}{\cos^2 x} dx = \int_0^{\frac{\pi}{4}} \frac{1}{\cos^2 x} + \frac{\sin x}{\cos^2 x} dx \\&= \int_0^{\frac{\pi}{4}} \sec^2 x + \sec x \tan x dx \\&= \left[\tan x + \sec x \right]_0^{\frac{\pi}{4}} = \left(1 + \sqrt{2} \right) - (0 + 1) = \sqrt{2}.\end{aligned}$$

0.1 Using Basic Integration Formulae



Example

Find $\int \frac{3x^2 - 7x}{3x + 2} dx.$

0.1 Using Basic Integration Formulae



Example

Find $\int \frac{3x^2 - 7x}{3x + 2} dx.$

The integrand is an improper fraction because the degree of $3x^2 - 7x$ (2nd degree) is greater than the degree of $3x + 2$ (1st degree).

0.1 Using Basic Integration Formulae

Example

Find $\int \frac{3x^2 - 7x}{3x + 2} dx.$

The integrand is an improper fraction because the degree of $3x^2 - 7x$ (2nd degree) is greater than the degree of $3x + 2$ (1st degree). We want to write our integrand as

$$\frac{3x^2 - 7x}{3x + 2} = (\text{linear function}) + \frac{\text{something}}{3x + 2}$$

[Do you remember how we studied *oblique asymptotes* in MATH113?]

0.1 Using Basic Integration Formulae

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$$\frac{3x^2 - 7x}{3x + 2} = (ax + b) + \frac{\text{something}}{3x + 2}$$

[Do you remember how we studied *oblique asymptotes* in MATH113?]

0.1 Using Basic Integration Formulae

Example

Find $\int \frac{3x^2 - 7x}{3x + 2} dx.$

The integrand is an improper fraction because the degree of $3x^2 - 7x$ (2nd degree) is greater than the degree of $3x + 2$ (1st degree). We want to write our integrand as

$$\frac{3x^2 - 7x}{3x + 2} = (ax + b) + \frac{c}{3x + 2}$$

[Do you remember how we studied *oblique asymptotes* in MATH113?]

0.1 Using Basic Integration Formulae

Example

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0.1 Using Basic Integration Formulae

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0.1 Using Basic Integration Formulae

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0.1 Using Basic Integration Formulae

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$$a = 1, b = -3, c = 6.$$

0.1 Using Basic Integration Formulae

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0.1 Using Basic Integration Formulae

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0.1 Using Basic Integration Formulae

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$$\int \frac{3x^2 - 7x}{3x + 2} dx = \int 2x - 3 + \frac{6}{3x + 2} dx = \frac{x^2}{2} - 3x + 2 \ln |3x + 2| + C.$$

0.1 Using Basic Integration

$$\int \frac{dx}{\sqrt{a^2 - x^2}} = \sin^{-1} \left(\frac{x}{a} \right) + C$$



Example

Find $\int \frac{3x + 2}{\sqrt{1 - x^2}}.$

0.1 Using Basic Integration

$$\int \frac{dx}{\sqrt{a^2 - x^2}} = \sin^{-1} \left(\frac{x}{a} \right) + C$$



Example

Find $\int \frac{3x + 2}{\sqrt{1 - x^2}}$.

First note that

$$\int \frac{3x + 2}{\sqrt{1 - x^2}} = 3 \int \frac{x \, dx}{\sqrt{1 - x^2}} + 2 \int \frac{dx}{\sqrt{1 - x^2}}$$

0.1 Using Basic Integration

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0.1 Using Basic Integration

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So we just need to calculate $\int \frac{x \, dx}{\sqrt{1 - x^2}}$.

0.1 Using Basic Integration Formulae



$$\int \frac{x \, dx}{\sqrt{1 - x^2}}$$

Let $u = 1 - x^2$.

0.1 Using Basic Integration Formulae



$$\int \frac{x \, dx}{\sqrt{1 - x^2}}$$

Let $u = 1 - x^2$. Then $du = -2x \, dx$ and $-\frac{1}{2} du = x \, dx$.

0.1 Using Basic Integration Formulae



$$\int \frac{x \, dx}{\sqrt{1 - x^2}}$$

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$$\int \frac{x \, dx}{\sqrt{1 - x^2}} = \int \frac{-\frac{1}{2} du}{\sqrt{u}} = -\frac{1}{2} \int u^{-\frac{1}{2}} \, du = \dots = -\sqrt{1 - x^2} + C.$$

0.1 Using Basic Integration Formulae



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Therefore

$$\begin{aligned}\int \frac{3x + 2}{\sqrt{1 - x^2}} &= 3 \int \frac{x \, dx}{\sqrt{1 - x^2}} + 2 \int \frac{dx}{\sqrt{1 - x^2}} \\ &= -3\sqrt{1 - x^2} + 2 \sin^{-1} x + C.\end{aligned}$$

0.1 Using Basic Integration Formulae

Example

Find $\int \frac{dx}{(1 + \sqrt{x})^3}$.

0.1 Using Basic Integration Formulae



Example

$$\text{Find } \int \frac{dx}{(1 + \sqrt{x})^3}.$$

I want to make a substitution to make this integral easier, but what u should I choose?

0.1 Using Basic Integration Formulae



Example

$$\text{Find } \int \frac{dx}{(1 + \sqrt{x})^3}.$$

I want to make a substitution to make this integral easier, but what u should I choose?

First guess: $u = \sqrt{x}$.

0.1 Using Basic Integration Formulae

Example

Find $\int \frac{dx}{(1 + \sqrt{x})^3}$.

I want to make a substitution to make this integral easier, but what u should I choose?

First guess: $u = \sqrt{x}$. But then $du = \frac{1}{2\sqrt{x}} dx$ and we would have to deal with this extra $\sqrt{x} = u$ term.

0.1 Using Basic Integration Formulae

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Second guess: Instead let us try $u = 1 + \sqrt{x}$.

0.1 Using Basic Integration Formulae



Example

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I want to make a substitution to make this integral easier, but what u should I choose?

First guess: $u = \sqrt{x}$. But then $du = \frac{1}{2\sqrt{x}} dx$ and we would have to deal with this extra $\sqrt{x} = u$ term.

Second guess: Instead let us try $u = 1 + \sqrt{x}$. Then again we have $du = \frac{1}{2\sqrt{x}} dx$ and $dx = 2\sqrt{x} du = 2(u - 1) du$. Hence

$$\int \frac{dx}{(1 + \sqrt{x})^3} = \int \frac{2(u - 1) du}{u^3} = \int \frac{2}{u^2} - \frac{2}{u^3} du = \dots$$

0.1 Using Basic Integration Formulae

Example

Calculate $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} x^3 \cos x \, dx$.

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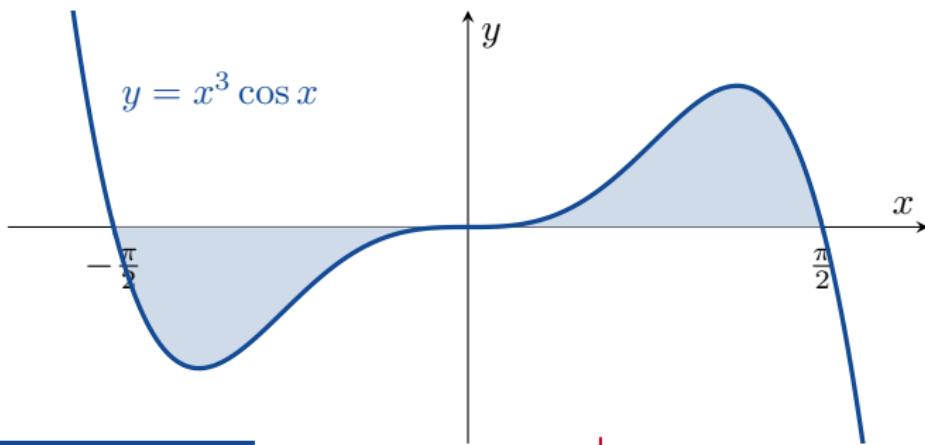
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Integration by Parts

0.2 Integration by Parts



How can we calculate

$$\int x \cos x \, dx$$

or

$$\int x^2 e^x \, dx ?$$

0.2 Integration by Parts



How can we calculate

$$\int x \cos x \, dx$$

or

$$\int x^2 e^x \, dx ?$$

How do we integrate **function × function**?

0.2 Integration by Parts



We know how to differentiate **function** \times **function**. By the product rule we have

$$(\textcolor{brown}{u}\textcolor{green}{v})' = \textcolor{brown}{u}'\textcolor{green}{v} + \textcolor{brown}{u}\textcolor{green}{v}' .$$

0.2 Integration by Parts



We know how to differentiate **function** \times **function**. By the product rule we have

$$\int (\mathbf{u}\mathbf{v})' dx = \int \mathbf{u}'\mathbf{v} dx + \int \mathbf{u}\mathbf{v}' dx.$$

0.2 Integration by Parts



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0.2 Integration by Parts



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$$uv = \int u'v \, dx + \int uv' \, dx.$$

0.2 Integration by Parts

We know how to differentiate **function \times function**. By the product rule we have

$$uv = \int u'v \, dx + \int uv' \, dx.$$

This rearranges to

Theorem (Integration by Parts)

$$\int u(x)v'(x) \, dx = u(x)v(x) - \int u'(x)v(x) \, dx$$

0.2 Integration by Parts

$$\int \textcolor{blue}{u} \textcolor{red}{v}' dx = \textcolor{blue}{u} \textcolor{red}{v} - \int \textcolor{red}{u}' \textcolor{blue}{v} dx$$



Example

Find $\int x \cos x dx$.

0.2 Integration by Parts

$$\int \textcolor{red}{u} \textcolor{green}{v}' dx = \textcolor{red}{u} \textcolor{green}{v} - \int \textcolor{red}{u}' \textcolor{green}{v} dx$$



Example

Find $\int x \cos x dx$.

We need to choose a $\textcolor{red}{u}(x)$ and a $v'(x)$.

0.2 Integration by Parts

$$\int \textcolor{red}{u} \textcolor{green}{v}' dx = \textcolor{red}{u} \textcolor{green}{v} - \int \textcolor{red}{u}' \textcolor{green}{v} dx$$



Example

Find $\int x \cos x dx$.

We need to choose a $\textcolor{red}{u}(x)$ and a $v'(x)$.

Let

$$\textcolor{red}{u} = x \qquad \qquad v' = \cos x$$

Then

$$\int \textcolor{red}{x} \cos x dx = \qquad \qquad - \int \qquad \qquad dx = \qquad \qquad .$$

0.2 Integration by Parts

$$\int \textcolor{red}{u} \textcolor{green}{v}' dx = \textcolor{red}{u} \textcolor{green}{v} - \int \textcolor{red}{u}' \textcolor{green}{v} dx$$



Example

Find $\int x \cos x dx$.

We need to choose a $\textcolor{red}{u}(x)$ and a $v'(x)$.

Let

$$\begin{aligned} u &= x & v' &= \cos x \\ u' &= 1 \end{aligned}$$

Then

$$\int \textcolor{brown}{x} \cos x dx = \quad - \int \quad dx = \quad .$$

0.2 Integration by Parts

$$\int \textcolor{red}{u} \textcolor{green}{v}' dx = \textcolor{red}{u} \textcolor{green}{v} - \int \textcolor{red}{u}' \textcolor{green}{v} dx$$



Example

Find $\int x \cos x dx$.

We need to choose a $u(x)$ and a $v'(x)$.

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0.2 Integration by Parts

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Then

$$\int \textcolor{red}{x} \cos x dx = \textcolor{red}{x} \sin x - \int \quad dx = \quad .$$

0.2 Integration by Parts

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Example

Find $\int x \cos x dx$.

We need to choose a $u(x)$ and a $v'(x)$.

Let

$$\begin{array}{ll} u = x & v' = \cos x \\ u' = 1 & v = \sin x. \end{array}$$

Then

$$\int \textcolor{blue}{x} \cos x dx = \textcolor{blue}{x} \sin x - \int \textcolor{blue}{1} \sin x dx = \quad .$$

0.2 Integration by Parts

$$\int \textcolor{red}{u} \textcolor{green}{v}' dx = \textcolor{red}{u} \textcolor{green}{v} - \int \textcolor{red}{u}' \textcolor{green}{v} dx$$



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We need to choose a $\textcolor{red}{u}(x)$ and a $v'(x)$.

Let

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Then

$$\int \textcolor{red}{x} \cos x dx = \textcolor{red}{x} \sin x - \int 1 \sin x dx = x \sin x + \cos x + C.$$

0.2 Integration by Parts

$$\int \textcolor{blue}{uv'} dx = \textcolor{blue}{uv} - \int \textcolor{blue}{u'}v dx$$



Example

Find $\int \ln x dx$.

0.2 Integration by Parts

$$\int \textcolor{blue}{uv'} dx = \textcolor{blue}{uv} - \int \textcolor{blue}{u'}v dx$$



Example

Find $\int \ln x dx$.

We will consider $\int \ln x \cdot 1 dx$.

0.2 Integration by Parts

$$\int \textcolor{red}{u} \textcolor{green}{v}' dx = \textcolor{red}{u} \textcolor{green}{v} - \int \textcolor{red}{u}' \textcolor{green}{v} dx$$



Example

Find $\int \ln x dx$.

We will consider $\int \ln x \cdot 1 dx$.

Let

$$u = \ln x \quad v' = 1$$

0.2 Integration by Parts

$$\int \textcolor{red}{uv}' dx = \textcolor{red}{uv} - \int \textcolor{red}{u}' \textcolor{green}{v} dx$$



Example

Find $\int \ln x dx$.

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Let

$$u = \ln x \qquad \qquad v' = 1$$

$$u' = \frac{1}{x}$$

0.2 Integration by Parts

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0.2 Integration by Parts

$$\int \textcolor{red}{uv}' dx = \textcolor{red}{uv} - \int \textcolor{red}{u}' \textcolor{green}{v} dx$$



Example

Find $\int \ln x dx$.

We will consider $\int \ln x \cdot 1 dx$.

Let

$$u = \ln x \quad v' = 1$$

$$u' = \frac{1}{x} \quad v = x.$$

Then

$$\int \ln x \cdot 1 dx = \ln x \cdot x - \int \frac{1}{x} \cdot x dx$$

=

= .

0.2 Integration by Parts

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Example

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We will consider $\int \ln x \cdot 1 dx$.

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$$\begin{array}{ll} u = \ln x & v' = 1 \\ u' = \frac{1}{x} & v = x. \end{array}$$

Then

$$\begin{aligned} \int \ln x \cdot 1 dx &= \ln x \cdot x - \int \frac{1}{x} \cdot x dx \\ &= x \ln x - \int 1 dx \\ &= . \end{aligned}$$

0.2 Integration by Parts

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Find $\int \ln x dx$.

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$$\begin{array}{ll} u = \ln x & v' = 1 \\ u' = \frac{1}{x} & v = x. \end{array}$$

Then

$$\begin{aligned} \int \ln x \cdot 1 dx &= \ln x \cdot x - \int \frac{1}{x} \cdot x dx \\ &= x \ln x - \int 1 dx \\ &= x \ln x - x + C. \end{aligned}$$

0.2 Integration by Parts

$$\int \color{red}{uv'} \, dx = \color{red}{uv} - \int \color{red}{u'} \color{green}{v} \, dx$$



Sometimes we have to use integration by parts more than once.

0.2 Integration by Parts

$$\int \textcolor{blue}{u} \textcolor{red}{v}' dx = \textcolor{blue}{u} \textcolor{red}{v} - \int \textcolor{red}{u}' \textcolor{blue}{v} dx$$



Example

Find $\int \textcolor{blue}{x}^2 e^x dx$.

0.2 Integration by Parts

$$\int \textcolor{red}{u} \textcolor{green}{v}' dx = \textcolor{red}{u} \textcolor{green}{v} - \int \textcolor{red}{u}' \textcolor{green}{v} dx$$



Example

Find $\int \textcolor{brown}{x}^2 e^x dx$.

We calculate that

$$\int \textcolor{brown}{x}^2 e^x dx = \textcolor{brown}{x}^2 e^x - 2 \int \textcolor{brown}{x} e^x dx.$$

0.2 Integration by Parts

$$\int \textcolor{blue}{uv'} dx = \textcolor{blue}{uv} - \int \textcolor{blue}{u'}v dx$$



Example

Find $\int \textcolor{blue}{x^2} e^x dx$.

We calculate that

$$\int \textcolor{blue}{x^2} e^x dx = \textcolor{blue}{x^2} e^x - 2 \int \textcolor{blue}{x} e^x dx.$$

But what do we do with $\int \textcolor{blue}{x} e^x dx$?

0.2 Integration by Parts

$$\int \textcolor{blue}{uv'} dx = \textcolor{blue}{uv} - \int \textcolor{blue}{u'}v dx$$



Example

Find $\int \textcolor{blue}{x}^2 e^x dx$.

We calculate that

$$\int \textcolor{blue}{x}^2 e^x dx = \textcolor{blue}{x}^2 e^x - 2 \int \textcolor{blue}{x} e^x dx.$$

But what do we do with $\int \textcolor{blue}{x} e^x dx$?

$$\int \textcolor{blue}{x} e^x dx = \textcolor{blue}{x} e^x - \int \textcolor{blue}{1} e^x dx = x e^x - e^x + C_1.$$

0.2 Integration by Parts

$$\int \textcolor{blue}{uv'} dx = \textcolor{blue}{uv} - \int \textcolor{blue}{u'}v dx$$



Example

Find $\int \textcolor{blue}{x^2} e^x dx$.

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$$\int \textcolor{blue}{x} e^x dx = \textcolor{blue}{x} e^x - \int \textcolor{blue}{1} e^x dx = x e^x - e^x + C_1.$$

Putting it all together, we have

$$\int x^2 e^x dx = x^2 e^x - 2 \int x e^x dx = x^2 e^x - 2x e^x + 2e^x + C.$$

0.2 Integration by Parts

$$\int \textcolor{blue}{uv'} dx = \textcolor{blue}{uv} - \int \textcolor{blue}{u'}v dx$$



Remark

We can use the same technique to calculate $\int x^n e^x dx$.

We would have to do integration by parts n times.

0.2 Integration by Parts

$$\int \textcolor{blue}{u} \textcolor{red}{v}' dx = \textcolor{blue}{u} \textcolor{red}{v} - \int \textcolor{blue}{u}' \textcolor{red}{v} dx$$



Theorem

$$\int u dv = \textcolor{blue}{u} \textcolor{red}{v} - \int v du$$

0.2 Integration by Parts

$$\int u \, dv = uv - \int v \, du$$



Example (Using Integration by Parts Twice)

Calculate $\int e^x \cos x \, dx$.

0.2 Integration by Parts

$$\int u \, dv = uv - \int v \, du$$



Example (Using Integration by Parts Twice)

Calculate $\int e^x \cos x \, dx$.

Let $u = e^x$ and $dv = \cos x \, dx$.

0.2 Integration by Parts

$$\int u \, dv = uv - \int v \, du$$



Example (Using Integration by Parts Twice)

Calculate $\int e^x \cos x \, dx$.

Let $u = e^x$ and $dv = \cos x \, dx$. Then $du = e^x \, dx$ and $v = \sin x$.

0.2 Integration by Parts

$$\int u \, dv = uv - \int v \, du$$



Example (Using Integration by Parts Twice)

Calculate $\int e^x \cos x \, dx$.

Let $u = e^x$ and $dv = \cos x \, dx$. Then $du = e^x \, dx$ and $v = \sin x$.
Hence

$$\int e^x \cos x \, dx = e^x \sin x - \int \sin x \cdot e^x \, dx.$$

0.2 Integration by Parts

$$\int u \, dv = uv - \int v \, du$$



Example (Using Integration by Parts Twice)

Calculate $\int e^x \cos x \, dx$.

Let $u = e^x$ and $dv = \cos x \, dx$. Then $du = e^x \, dx$ and $v = \sin x$.
Hence

$$\int e^x \cos x \, dx = e^x \sin x - \int \sin x \cdot e^x \, dx.$$

But we still have $\int (\text{function}) \times (\text{function}) \, dx$. So we need to use Integration by Parts a second time.

0.2 Integration by Parts

$$\int u \, dv = uv - \int v \, du$$



So far we have

$$\int e^x \cos x \, dx = e^x \sin x - \int e^x \sin x \, dx.$$

0.2 Integration by Parts

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So far we have

$$\int e^x \cos x \, dx = e^x \sin x - \int e^x \sin x \, dx.$$

Let $u = e^x$ and $dv = \sin x \, dx$.

0.2 Integration by Parts

$$\int u \, dv = uv - \int v \, du$$



So far we have

$$\int e^x \cos x \, dx = e^x \sin x - \int e^x \sin x \, dx.$$

Let $u = e^x$ and $dv = \sin x \, dx$. Then $du = e^x \, dx$ and $v = -\cos x$.

0.2 Integration by Parts

$$\int u \, dv = uv - \int v \, du$$



So far we have

$$\int e^x \cos x \, dx = e^x \sin x - \int e^x \sin x \, dx.$$

Let $u = e^x$ and $dv = \sin x \, dx$. Then $du = e^x \, dx$ and $v = -\cos x$.
Hence

$$\begin{aligned}\int e^x \cos x \, dx &= e^x \sin x - \int e^x \sin x \, dx \\ &= e^x \sin x - \left(-e^x \cos x - \int (-\cos x) \cdot e^x \, dx \right)\end{aligned}$$

=

0.2 Integration by Parts

$$\int u \, dv = uv - \int v \, du$$



So far we have

$$\int e^x \cos x \, dx = e^x \sin x - \int e^x \sin x \, dx.$$

Let $u = e^x$ and $dv = \sin x \, dx$. Then $du = e^x \, dx$ and $v = -\cos x$.

Hence

$$\begin{aligned}\int e^x \cos x \, dx &= e^x \sin x - \int e^x \sin x \, dx \\&= e^x \sin x - \left(-e^x \cos x - \int (-\cos x) \cdot e^x \, dx \right) \\&= e^x \sin x + e^x \cos x - \int e^x \cos x \, dx.\end{aligned}$$

0.2 Integration by Parts

$$\int u \, dv = uv - \int v \, du$$



So far we have

$$\int e^x \cos x \, dx = e^x \sin x - \int e^x \sin x \, dx.$$

Let $u = e^x$ and $dv = \sin x \, dx$. Then $du = e^x \, dx$ and $v = -\cos x$.

Hence

$$\begin{aligned}\int e^x \cos x \, dx &= e^x \sin x - \int e^x \sin x \, dx \\&= e^x \sin x - \left(-e^x \cos x - \int (-\cos x) \cdot e^x \, dx \right) \\&= e^x \sin x + e^x \cos x - \int e^x \cos x \, dx.\end{aligned}$$

0.2 Integration by Parts

$$\int u \, dv = uv - \int v \, du$$



$$\int e^x \cos x \, dx = e^x \sin x + e^x \cos x - \int e^x \cos x \, dx.$$

0.2 Integration by Parts

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$$\int e^x \cos x \, dx = e^x \sin x + e^x \cos x - \int e^x \cos x \, dx.$$

It follows that

$$2 \int e^x \cos x \, dx = e^x \sin x + e^x \cos x + C_1$$

0.2 Integration by Parts

$$\int u \, dv = uv - \int v \, du$$



$$\int e^x \cos x \, dx = e^x \sin x + e^x \cos x - \int e^x \cos x \, dx.$$

It follows that

$$2 \int e^x \cos x \, dx = e^x \sin x + e^x \cos x + C_1$$

and hence that

$$\int e^x \cos x \, dx = \frac{e^x \sin x + e^x \cos x}{2} + C.$$

0.2 Integration by Parts

$$\int u \, dv = uv - \int v \, du$$



Example

Obtain a formula that expresses the integral $\int \cos^n x \, dx$ in terms of an integral of a lower power of $\cos x$.

0.2 Integration by Parts

$$\int u \, dv = uv - \int v \, du$$



Example

Obtain a formula that expresses the integral $\int \cos^n x \, dx$ in terms of an integral of a lower power of $\cos x$.

To use integration by parts, we need (function) \times (function). So think of the integral like this:

$$\int \cos^n x \, dx = \int \cos^{n-1} x \cdot \cos x \, dx.$$

0.2 Integration by Parts

$$\int u \, dv = uv - \int v \, du$$



$$\int \cos^n x \, dx = \int \cos^{n-1} x \cdot \cos x \, dx.$$

If $u = \cos^{n-1} x$ and $dv = \cos x \, dx$,

0.2 Integration by Parts

$$\int u \, dv = uv - \int v \, du$$



$$\int \cos^n x \, dx = \int \cos^{n-1} x \cdot \cos x \, dx.$$

If $u = \cos^{n-1} x$ and $dv = \cos x \, dx$, then we have that
 $du = -(n-1) \cos^{n-2} x \sin x \, dx$ and $v = \sin x$.

0.2 Integration by Parts

$$\int u \, dv = uv - \int v \, du$$



$$\int \cos^n x \, dx = \int \cos^{n-1} x \cdot \cos x \, dx.$$

If $u = \cos^{n-1} x$ and $dv = \cos x \, dx$, then we have that
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$$\int \cos^n x \, dx = \int u \, dv = uv - \int v \, du$$

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0.2 Integration by Parts

$$\int u \, dv = uv - \int v \, du$$



$$\int \cos^n x \, dx = \int \cos^{n-1} x \cdot \cos x \, dx.$$

If $u = \cos^{n-1} x$ and $dv = \cos x \, dx$, then we have that
 $du = -(n-1) \cos^{n-2} x \sin x \, dx$ and $v = \sin x$. Hence

$$\begin{aligned}\int \cos^n x \, dx &= \int u \, dv = uv - \int v \, du \\ &= \cos^{n-1} x \sin x + (n-1) \int \sin x \cos^{n-2} x \sin x \, dx\end{aligned}$$

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0.2 Integration by Parts

$$\int u \, dv = uv - \int v \, du$$



$$\int \cos^n x \, dx = \int \cos^{n-1} x \cdot \cos x \, dx.$$

If $u = \cos^{n-1} x$ and $dv = \cos x \, dx$, then we have that
 $du = -(n-1) \cos^{n-2} x \sin x \, dx$ and $v = \sin x$. Hence

$$\begin{aligned}\int \cos^n x \, dx &= \int u \, dv = uv - \int v \, du \\&= \cos^{n-1} x \sin x + (n-1) \int \sin x \cos^{n-2} x \sin x \, dx \\&= \cos^{n-1} x \sin x + (n-1) \int \sin^2 x \cos^{n-2} x \, dx \\&= \\&= \end{aligned}$$

0.2 Integration by Parts

$$\int u \, dv = uv - \int v \, du$$



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If $u = \cos^{n-1} x$ and $dv = \cos x \, dx$, then we have that
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0.2 Integration by Parts

$$\int u \, dv = uv - \int v \, du$$



$$\int \cos^n x \, dx = \int \cos^{n-1} x \cdot \cos x \, dx.$$

If $u = \cos^{n-1} x$ and $dv = \cos x \, dx$, then we have that
 $du = -(n-1) \cos^{n-2} x \sin x \, dx$ and $v = \sin x$. Hence

$$\begin{aligned}\int \cos^n x \, dx &= \int u \, dv = uv - \int v \, du \\&= \cos^{n-1} x \sin x + (n-1) \int \sin x \cos^{n-2} x \sin x \, dx \\&= \cos^{n-1} x \sin x + (n-1) \int \sin^2 x \cos^{n-2} x \, dx \\&= \cos^{n-1} x \sin x + (n-1) \int (1 - \cos^2 x) \cos^{n-2} x \, dx \\&= \cos^{n-1} x \sin x + (n-1) \int \cos^{n-2} x \, dx - (n-1) \int \cos^n x \, dx.\end{aligned}$$

0.2 Integration by Parts

$$\int u \, dv = uv - \int v \, du$$



$$\int \cos^n x \, dx = \cos^{n-1} x \sin x + (n-1) \int \cos^{n-2} x \, dx - (n-1) \int \cos^n x \, dx.$$

0.2 Integration by Parts

$$\int u \, dv = uv - \int v \, du$$



$$\int \cos^n x \, dx = \cos^{n-1} x \sin x + (n-1) \int \cos^{n-2} x \, dx - (n-1) \int \cos^n x \, dx.$$

If we add $(n-1) \int \cos^n x \, dx$ to both sides then we obtain

$$n \int \cos^n x \, dx = \cos^{n-1} x \sin x + (n-1) \int \cos^{n-2} x \, dx.$$

0.2 Integration by Parts

$$\int u \, dv = uv - \int v \, du$$



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If we add $(n-1) \int \cos^n x \, dx$ to both sides then we obtain

$$n \int \cos^n x \, dx = \cos^{n-1} x \sin x + (n-1) \int \cos^{n-2} x \, dx.$$

Finally we divide by n to obtain our answer

$$\int \cos^n x \, dx = \frac{\cos^{n-1} x \sin x}{n} + \frac{n-1}{n} \int \cos^{n-2} x \, dx.$$

0.2 Integration by Parts

$$\int u \, dv = uv - \int v \, du$$



Remark

$$\int \cos^n x \, dx = \frac{\cos^{n-1} x \sin x}{n} + \frac{n-1}{n} \int \cos^{n-2} x \, dx.$$

The formula above is called a *reduction formula* because it replaces it replaces an integral containing $\cos^n x$ with an integral containing a smaller power of $\cos x$.

0.2 Integration by Parts

$$\int u \, dv = uv - \int v \, du$$



Remark

$$\int \cos^n x \, dx = \frac{\cos^{n-1} x \sin x}{n} + \frac{n-1}{n} \int \cos^{n-2} x \, dx.$$

The formula above is called a *reduction formula* because it replaces it replaces an integral containing $\cos^n x$ with an integral containing a smaller power of $\cos x$.

Example ($n = 3$)

$$\begin{aligned}\int \cos^3 x \, dx &= \frac{1}{3} \cos^2 x \sin x + \frac{2}{3} \int \cos x \, dx \\ &= \frac{1}{3} \cos^2 x \sin x + \frac{2}{3} \sin x + C.\end{aligned}$$



Break

We will continue at 3pm



0.2 Integration by Parts



Integration by Parts for Definite Integrals

Theorem

$$\int_a^b \textcolor{brown}{u} v' dx =$$

Integration by Parts for Definite Integrals

Theorem

$$\int_a^b uv' dx = [uv]_a^b - \int_a^b u'v dx$$

0.2 Integration by Parts

$$\int_a^b u v' dx = [uv]_a^b - \int_a^b u' v dx$$



Example

Calculate the area of the region bounded by the curve $y = xe^{-x}$ and the x -axis from $x = 0$ to $x = 4$.

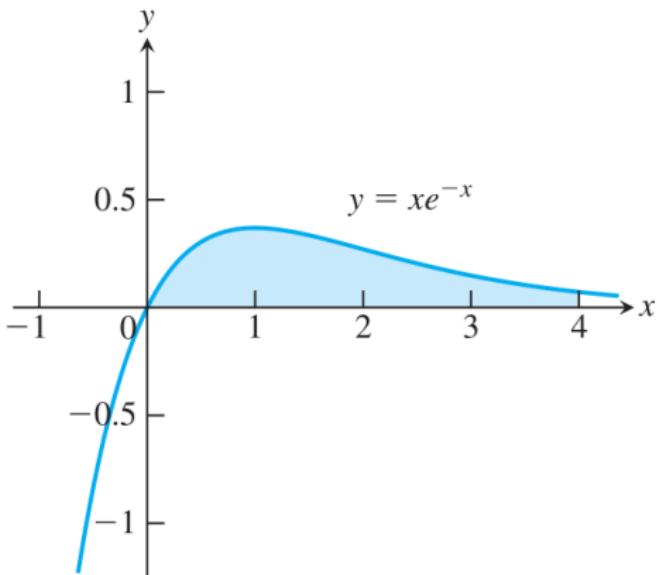
0.2 Integration by Parts

$$\int_a^b uv' dx = [uv]_a^b - \int_a^b u'v dx$$



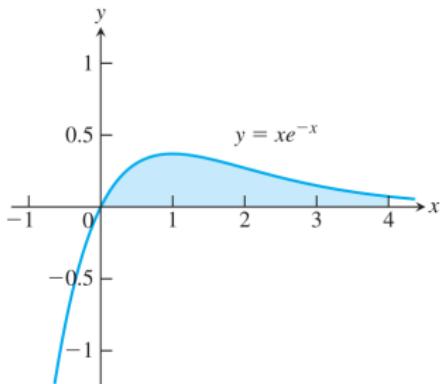
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0.2 Integration by Parts

$$\int_a^b u v' dx = [uv]_a^b - \int_a^b u' v dx$$



We calculate that

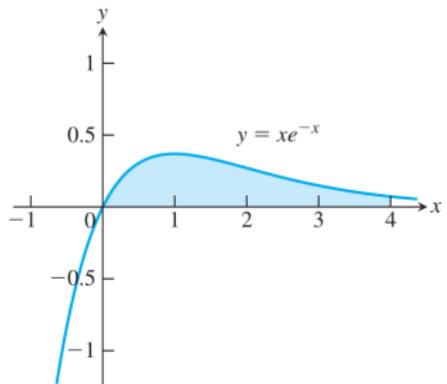
$$\int_0^4 xe^{-x} dx =$$

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0.2 Integration by Parts

$$\int_a^b uv' dx = [uv]_a^b - \int_a^b u'v dx$$



$$u = x$$

$$u' = 1$$

$$v' = e^{-x}$$

$$v = -e^{-x}$$

We calculate that

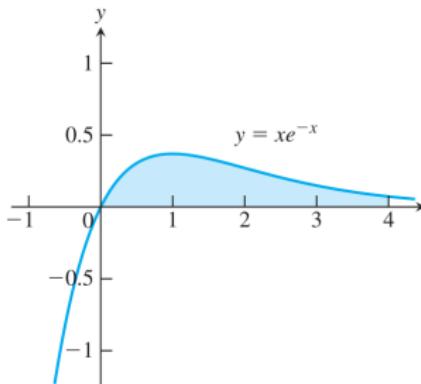
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0.2 Integration by Parts

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We calculate that

$$\int_0^4 x e^{-x} dx = [-xe^{-x}]_0^4 - \int_0^4 1(-e^{-x}) dx$$

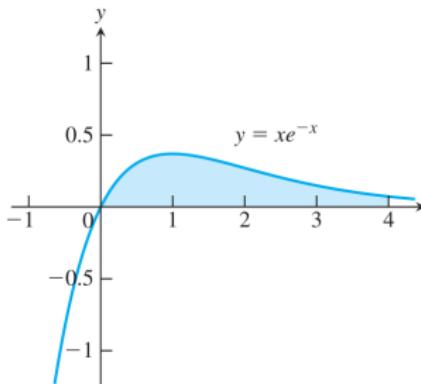
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0.2 Integration by Parts

$$\int_a^b u v' dx = [uv]_a^b - \int_a^b u' v dx$$



$$u = x$$

$$u' = 1$$

$$v' = e^{-x}$$

$$v = -e^{-x}$$

We calculate that

$$\begin{aligned}\int_0^4 x e^{-x} dx &= \left[-xe^{-x} \right]_0^4 - \int_0^4 1(-e^{-x}) dx \\ &= (-4e^{-4} + 0) + [-e^{-x}]_0^4 \\ &= -4e^{-4} + (-e^{-4} + 1) = 1 - 5e^{-4}.\end{aligned}$$

0.2 Integration by Parts

$$\int_a^b u v' dx = [uv]_a^b - \int_a^b u' v dx$$



Example

Find $\int_0^1 \sin^{-1} x dx$.

0.2 Integration by Parts

$$\int_a^b u v' dx = [uv]_a^b - \int_a^b u' v dx$$



Example

Find $\int_0^1 \sin^{-1} x dx$.

Recall that $\frac{d}{dx} \sin^{-1} x = \frac{1}{\sqrt{1-x^2}}$.

0.2 Integration by Parts

$$\int_a^b u v' dx = [uv]_a^b - \int_a^b u' v dx$$



Example

Find $\int_0^1 \sin^{-1} x dx$.

Recall that $\frac{d}{dx} \sin^{-1} x = \frac{1}{\sqrt{1-x^2}}$.

Let $u = \sin^{-1} x$ and $v' = 1$.

0.2 Integration by Parts

$$\int_a^b u v' dx = [uv]_a^b - \int_a^b u' v dx$$



Example

Find $\int_0^1 \sin^{-1} x dx$.

Recall that $\frac{d}{dx} \sin^{-1} x = \frac{1}{\sqrt{1-x^2}}$.

Let $u = \sin^{-1} x$ and $v' = 1$. Then $u' = \frac{1}{\sqrt{1-x^2}}$ and $v = x$.

0.2 Integration by Parts

$$\int_a^b u v' dx = [uv]_a^b - \int_a^b u' v dx$$



Example

Find $\int_0^1 \sin^{-1} x dx$.

Recall that $\frac{d}{dx} \sin^{-1} x = \frac{1}{\sqrt{1-x^2}}$.

Let $u = \sin^{-1} x$ and $v' = 1$. Then $u' = \frac{1}{\sqrt{1-x^2}}$ and $v = x$. It follows that

$$\int_0^1 \sin^{-1} x \cdot 1 dx = [\cancel{x} \sin^{-1} x]_0^1 - \int_0^1 \frac{\cancel{x}}{\sqrt{1-x^2}} dx$$

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0.2 Integration by Parts

$$\int_a^b u v' dx = [uv]_a^b - \int_a^b u' v dx$$



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Recall that $\frac{d}{dx} \sin^{-1} x = \frac{1}{\sqrt{1-x^2}}$.

Let $u = \sin^{-1} x$ and $v' = 1$. Then $u' = \frac{1}{\sqrt{1-x^2}}$ and $v = x$. It follows that

$$\begin{aligned}\int_0^1 \sin^{-1} x \cdot 1 dx &= [x \sin^{-1} x]_0^1 - \int_0^1 \frac{x}{\sqrt{1-x^2}} dx \\&= [x \sin^{-1} x]_0^1 - \left[-\sqrt{1-x^2} \right]_0^1 \\&= \left(\frac{\pi}{2} - 0 \right) - (-0 + 1) \\&= \frac{\pi}{2} - 1.\end{aligned}$$



Trigonometric Integrals

0.3 Trigonometric Integrals



$$\cos \theta = \sin \left(\frac{\pi}{2} - \theta \right)$$

$$\cos^2 \theta + \sin^2 \theta = 1$$

$$1 + \tan^2 \theta = \sec^2 \theta$$

$$1 + \cot^2 \theta = \operatorname{cosec}^2 \theta$$

$$\cos(A + B) = \cos A \cos B - \sin A \sin B$$

$$\sin(A + B) = \sin A \cos B + \cos A \sin B$$

$$\cos 2\theta = \cos^2 \theta - \sin^2 \theta$$

$$\sin 2\theta = 2 \sin \theta \cos \theta$$

$$\cos^2 \theta = \frac{1}{2}(1 + \cos 2\theta)$$

$$\sin^2 \theta = \frac{1}{2}(1 - \cos 2\theta)$$

0.3 Trigonometric Integrals



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0.3 Trigonometric Integrals



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0.3 Trigonometric Integrals



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0.3 Trigonometric Integrals

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0.3 Trigonometric Integrals



How can we find

$$\int \sin^m x \cos^n x dx$$

if $m, n \in \{0, 1, 2, 3, 4, 5, \dots\}$?

0.3 Trigonometric Integrals



$$\int \sin^m x \cos^n x dx$$

We need to look at 3 different cases:

- 1 m is an odd number;
- 2 n is an odd number;
- 3 both m and n are even.

0.3 Trigonometric Integrals



$$\int \sin^m x \cos^n x dx$$

We need to look at 3 different cases:

- 1 m is an odd number;
- 2 n is an odd number;
- 3 both m and n are even.

[If both m and n are odd, then you can choose to use case 1 or case 2.]

0.3 Trigonometric Integrals



$$\int \sin^m x \cos^n x dx$$

- 1 m is an odd number;

0.3 Trigonometric Integrals



$$\int \sin^m x \cos^n x dx$$

- 1 m is an odd number;

The idea is:

- a Use $\sin^2 x = 1 - \cos^2 x$ to change all but one of the sin terms into cos terms;
- b Use the substitution $u = \cos x$.

0.3 Trigonometric Integrals



$$\int \sin^m x \cos^n x dx$$

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- a Use $\sin^2 x = 1 - \cos^2 x$ to change all but one of the sin terms into cos terms;
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Since m is odd, we can write $m = 2k + 1$ for $k \in \mathbb{N}$.

0.3 Trigonometric Integrals

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$$\int \sin^m x \cos^n x dx = \int \sin^{2k+1} x \cos^n x dx$$

0.3 Trigonometric Integrals



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0.3 Trigonometric Integrals



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$$\int \sin^m x \cos^n x dx = \int \sin x (\sin^2 x)^k \cos^n x dx$$

0.3 Trigonometric Integrals



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$$\int \sin^m x \cos^n x dx = \int \sin x (1 - \cos^2 x)^k \cos^n x dx$$

0.3 Trigonometric Integrals



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0.3 Trigonometric Integrals



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$$\int \sin^m x \cos^n x dx = \int \sin x (1 - u^2)^k u^n du$$

0.3 Trigonometric Integrals



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Since m is odd, we can write $m = 2k + 1$ for $k \in \mathbb{N}$.

$$\int \sin^m x \cos^n x dx = \int (1 - u^2)^k u^n (-du)$$

0.3 Trigonometric Integrals



$$\int \sin^m x \cos^n x dx$$

- 2 n is odd;

Same idea, but with cos and sin swapped:

0.3 Trigonometric Integrals



$$\int \sin^m x \cos^n x dx$$

- 2 n is odd;

Same idea, but with cos and sin swapped:

- a Use $\cos^2 x = 1 - \sin^2 x$ to change all but one of the cos terms into sin terms;
- b Use the substitution $u = \sin x$. The final $\cos x$ term will disappear.

0.3 Trigonometric Integrals



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0.3 Trigonometric Integrals

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$$\int \sin^m x \cos^n x dx = \int \sin^m x \cos x (\cos^2 x)^k dx$$

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0.3 Trigonometric Integrals

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- 2 n is odd;

Same idea, but with cos and sin swapped:

- a Use $\cos^2 x = 1 - \sin^2 x$ to change all but one of the cos terms into sin terms;
- b Use the substitution $u = \sin x$. The final $\cos x$ term will disappear.

Since n is odd, we can write $n = 2k + 1$ for $k \in \mathbb{N}$.

$$\begin{aligned}\int \sin^m x \cos^n x dx &= \int \sin^m x \cos x (\cos^2 x)^k dx \\ &= \int \sin^m x \cos x (1 - \sin^2 x)^k dx =\end{aligned}$$

0.3 Trigonometric Integrals



$$\int \sin^m x \cos^n x dx$$

- 2 n is odd;

Same idea, but with cos and sin swapped:

- a Use $\cos^2 x = 1 - \sin^2 x$ to change all but one of the cos terms into sin terms;
- b Use the substitution $u = \sin x$. The final $\cos x$ term will disappear.

Since n is odd, we can write $n = 2k + 1$ for $k \in \mathbb{N}$.

$$\begin{aligned}\int \sin^m x \cos^n x dx &= \int \sin^m x \cos x (\cos^2 x)^k dx \\ &= \int \sin^m x \cos x (1 - \sin^2 x)^k dx = \int u^m (1 - u^2)^k du.\end{aligned}$$

0.3 Trigonometric Integrals



$$\int \sin^m x \cos^n x dx$$

- 3 both m and n are even.

0.3 Trigonometric Integrals



$$\int \sin^m x \cos^n x dx$$

- 3 both m and n are even.

This time the idea is to use

$$\sin^2 x = \frac{1 - \cos 2x}{2} \quad \text{and} \quad \cos^2 x = \frac{1 + \cos 2x}{2}$$

to reduce the powers of sin and cos that we need to deal with.

0.3 Trigonometric Integrals

Example

Find $\int \sin^3 x \cos^2 x dx$.

0.3 Trigonometric Integrals

Example

Find $\int \sin^3 x \cos^2 x dx$.

This is an example of case 1 since 3 is an odd number.

0.3 Trigonometric Integrals

Example

Find $\int \sin^3 x \cos^2 x dx$.

This is an example of case 1 since 3 is an odd number. We are going to be using the substitution $u = \cos x$ and $-du = \sin x dx$.

0.3 Trigonometric Integrals

Example

Find $\int \sin^3 x \cos^2 x dx.$

This is an example of case 1 since 3 is an odd number. We are going to be using the substitution $u = \cos x$ and $-du = \sin x dx.$

$$\int \sin^3 x \cos^2 x dx = \int \sin^2 x \cos^2 x \sin x dx$$

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0.3 Trigonometric Integrals

Example

Find $\int \sin^3 x \cos^2 x dx.$

This is an example of case 1 since 3 is an odd number. We are going to be using the substitution $u = \cos x$ and $-du = \sin x dx.$

$$\begin{aligned}\int \sin^3 x \cos^2 x dx &= \int \sin^2 x \cos^2 x \sin x dx \\ &= \int (1 - \cos^2 x) \cos^2 x \sin x dx\end{aligned}$$

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0.3 Trigonometric Integrals

Example

Find $\int \sin^3 x \cos^2 x dx.$

This is an example of case 1 since 3 is an odd number. We are going to be using the substitution $u = \cos x$ and $-du = \sin x dx.$

$$\begin{aligned}\int \sin^3 x \cos^2 x dx &= \int \sin^2 x \cos^2 x \sin x dx \\&= \int (1 - \cos^2 x) \cos^2 x \sin x dx \\&= \int (1 - u^2) u^2 (-du)\end{aligned}$$

=

=

0.3 Trigonometric Integrals

Example

Find $\int \sin^3 x \cos^2 x dx.$

This is an example of case 1 since 3 is an odd number. We are going to be using the substitution $u = \cos x$ and $-du = \sin x dx.$

$$\begin{aligned}\int \sin^3 x \cos^2 x dx &= \int \sin^2 x \cos^2 x \sin x dx \\&= \int (1 - \cos^2 x) \cos^2 x \sin x dx \\&= \int (1 - u^2) u^2 (-du) \\&= \int u^4 - u^2 du \\&= \frac{u^5}{5} - \frac{u^3}{3} + C = \frac{\cos^5 x}{5} - \frac{\cos^3 x}{3} + C.\end{aligned}$$

0.3 Trigonometric Integrals

Example

Find $\int \cos^5 x dx$.

0.3 Trigonometric Integrals

Example

Find $\int \cos^5 x dx$.

This is an example of case 2 since 5 is an odd number. We are going to be using the substitution $u = \sin x$ and $du = \cos x dx$.

0.3 Trigonometric Integrals

Example

Find $\int \cos^5 x dx$.

This is an example of case 2 since 5 is an odd number. We are going to be using the substitution $u = \sin x$ and $du = \cos x dx$.

$$\int \cos^5 x dx = \int \cos^4 x \cos x dx$$

=

=

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0.3 Trigonometric Integrals

Example

Find $\int \cos^5 x dx.$

This is an example of case 2 since 5 is an odd number. We are going to be using the substitution $u = \sin x$ and $du = \cos x dx.$

$$\begin{aligned}\int \cos^5 x dx &= \int \cos^4 x \cos x dx \\&= \int (\cos^2 x)^2 \cos x dx\end{aligned}$$

=

=

0.3 Trigonometric Integrals

Example

Find $\int \cos^5 x dx$.

This is an example of case 2 since 5 is an odd number. We are going to be using the substitution $u = \sin x$ and $du = \cos x dx$.

$$\begin{aligned}\int \cos^5 x dx &= \int \cos^4 x \cos x dx \\&= \int (\cos^2 x)^2 \cos x dx \\&= \int (1 - \sin^2 x)^2 \cos x dx \\&= \end{aligned}$$

0.3 Trigonometric Integrals

Example

Find $\int \cos^5 x dx$.

This is an example of case 2 since 5 is an odd number. We are going to be using the substitution $u = \sin x$ and $du = \cos x dx$.

$$\begin{aligned}\int \cos^5 x dx &= \int \cos^4 x \cos x dx \\&= \int (\cos^2 x)^2 \cos x dx \\&= \int (1 - \sin^2 x)^2 \cos x dx \\&= \int (1 - u^2)^2 du = \dots\end{aligned}$$

$$\sin^2 x = \frac{1 - \cos 2x}{2} \quad \cos^2 x = \frac{1 + \cos 2x}{2}$$



Example

Find $\int \sin^2 x \cos^4 x dx$.

Both 2 and 4 are even so this is an example of case 3.

$$\sin^2 x = \frac{1 - \cos 2x}{2} \quad \cos^2 x = \frac{1 + \cos 2x}{2}$$



Example

Find $\int \sin^2 x \cos^4 x dx$.

Both 2 and 4 are even so this is an example of case 3.

$$\int \sin^2 x \cos^4 x dx = \int \left(\frac{1 - \cos 2x}{2} \right) \left(\frac{1 + \cos 2x}{2} \right)^2 dx$$

=

=

=

$$\sin^2 x = \frac{1 - \cos 2x}{2} \quad \cos^2 x = \frac{1 + \cos 2x}{2}$$



Example

Find $\int \sin^2 x \cos^4 x dx$.

Both 2 and 4 are even so this is an example of case 3.

$$\begin{aligned}\int \sin^2 x \cos^4 x dx &= \int \left(\frac{1 - \cos 2x}{2} \right) \left(\frac{1 + \cos 2x}{2} \right)^2 dx \\ &= \frac{1}{8} \int (1 - \cos 2x)(1 + 2\cos 2x + \cos^2 2x) dx \\ &= \frac{1}{8} \int 1 + \cos 2x - \cos^2 2x - \cos^3 2x dx \\ &= \end{aligned}$$

$$\sin^2 x = \frac{1 - \cos 2x}{2} \quad \cos^2 x = \frac{1 + \cos 2x}{2}$$



Example

Find $\int \sin^2 x \cos^4 x dx$.

Both 2 and 4 are even so this is an example of case 3.

$$\begin{aligned}\int \sin^2 x \cos^4 x dx &= \int \left(\frac{1 - \cos 2x}{2} \right) \left(\frac{1 + \cos 2x}{2} \right)^2 dx \\ &= \frac{1}{8} \int (1 - \cos 2x)(1 + 2\cos 2x + \cos^2 2x) dx \\ &= \frac{1}{8} \int \underbrace{1 + \cos 2x}_{\text{easy}} - \cos^2 2x - \cos^3 2x dx\end{aligned}$$

=

$$\sin^2 x = \frac{1 - \cos 2x}{2} \quad \cos^2 x = \frac{1 + \cos 2x}{2}$$



Example

Find $\int \sin^2 x \cos^4 x dx$.

Both 2 and 4 are even so this is an example of case 3.

$$\begin{aligned}\int \sin^2 x \cos^4 x dx &= \int \left(\frac{1 - \cos 2x}{2} \right) \left(\frac{1 + \cos 2x}{2} \right)^2 dx \\ &= \frac{1}{8} \int (1 - \cos 2x)(1 + 2\cos 2x + \cos^2 2x) dx \\ &= \frac{1}{8} \int 1 + \cos 2x - \underbrace{\cos^2 2x}_{\text{case 3}} - \cos^3 2x dx\end{aligned}$$

=

$$\sin^2 x = \frac{1 - \cos 2x}{2} \quad \cos^2 x = \frac{1 + \cos 2x}{2}$$



Example

Find $\int \sin^2 x \cos^4 x dx$.

Both 2 and 4 are even so this is an example of case 3.

$$\begin{aligned}\int \sin^2 x \cos^4 x dx &= \int \left(\frac{1 - \cos 2x}{2} \right) \left(\frac{1 + \cos 2x}{2} \right)^2 dx \\ &= \frac{1}{8} \int (1 - \cos 2x)(1 + 2\cos 2x + \cos^2 2x) dx \\ &= \frac{1}{8} \int 1 + \cos 2x - \cos^2 2x - \underbrace{\cos^3 2x}_{\text{case 2}} dx\end{aligned}$$

=

$$\sin^2 x = \frac{1 - \cos 2x}{2} \quad \cos^2 x = \frac{1 + \cos 2x}{2}$$



Example

Find $\int \sin^2 x \cos^4 x dx$.

Both 2 and 4 are even so this is an example of case 3.

$$\begin{aligned}\int \sin^2 x \cos^4 x dx &= \int \left(\frac{1 - \cos 2x}{2} \right) \left(\frac{1 + \cos 2x}{2} \right)^2 dx \\&= \frac{1}{8} \int (1 - \cos 2x)(1 + 2\cos 2x + \cos^2 2x) dx \\&= \frac{1}{8} \int 1 + \cos 2x - \cos^2 2x - \cos^3 2x dx \\&= \frac{1}{8} \left(x + \frac{1}{2} \sin 2x - \int \cos^2 2x dx - \int \cos^3 2x dx \right)\end{aligned}$$

0.3 Trigonometry

$$\sin^2 x = \frac{1 - \cos 2x}{2} \quad \cos^2 x = \frac{1 + \cos 2x}{2}$$



For the orange integral we calculate that

$$\int \cos^2 2x \, dx =$$

=

=

For the green integral we calculate that

$$\int \cos^3 2x \, dx =$$

=

=

0.3 Trigonometry

$$\sin^2 x = \frac{1 - \cos 2x}{2} \quad \cos^2 x = \frac{1 + \cos 2x}{2}$$



For the orange integral we calculate that

$$\int \cos^2 2x \, dx = \frac{1}{2} \int 1 + \cos 4x \, dx$$

=

=

For the green integral we calculate that

$$\int \cos^3 2x \, dx =$$

=

=

0.3 Trigonometric Integrals

$$\sin^2 x = \frac{1 - \cos 2x}{2} \quad \cos^2 x = \frac{1 + \cos 2x}{2}$$



For the orange integral we calculate that

$$\begin{aligned}\int \cos^2 2x \, dx &= \frac{1}{2} \int 1 + \cos 4x \, dx \\ &= \frac{1}{2} \left(x + \frac{1}{4} \sin 4x \right) + C_1 \\ &= \frac{x}{2} + \frac{1}{8} \sin 4x + C_1.\end{aligned}$$

For the green integral we calculate that

$$\int \cos^3 2x \, dx =$$

=

=

0.3 Trigonometric Integrals

$$\sin^2 x = \frac{1 - \cos 2x}{2} \quad \cos^2 x = \frac{1 + \cos 2x}{2}$$



For the orange integral we calculate that

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For the green integral we calculate that

$$\int \cos^3 2x \, dx = \int (1 - \sin^2 2x) \cos 2x \, dx$$

=

=

0.3 Trigonometric Integrals

$$\sin^2 x = \frac{1 - \cos 2x}{2} \quad \cos^2 x = \frac{1 + \cos 2x}{2}$$



For the orange integral we calculate that

$$\begin{aligned}\int \cos^2 2x \, dx &= \frac{1}{2} \int 1 + \cos 4x \, dx \\ &= \frac{1}{2} \left(x + \frac{1}{4} \sin 4x \right) + C_1 \\ &= \frac{x}{2} + \frac{1}{8} \sin 4x + C_1.\end{aligned}$$

For the green integral we calculate that

$$\begin{aligned}\int \cos^3 2x \, dx &= \int (1 - \sin^2 2x) \cos 2x \, dx \\ &= \frac{1}{2} \int (1 - u^2) \, du \\ &= \end{aligned}$$

0.3 Trigonometric Integrals

$$\sin^2 x = \frac{1 - \cos 2x}{2} \quad \cos^2 x = \frac{1 + \cos 2x}{2}$$



For the orange integral we calculate that

$$\begin{aligned}\int \cos^2 2x \, dx &= \frac{1}{2} \int 1 + \cos 4x \, dx \\ &= \frac{1}{2} \left(x + \frac{1}{4} \sin 4x \right) + C_1 \\ &= \frac{x}{2} + \frac{1}{8} \sin 4x + C_1.\end{aligned}$$

For the green integral we calculate that

$$\begin{aligned}\int \cos^3 2x \, dx &= \int (1 - \sin^2 2x) \cos 2x \, dx \\ &= \frac{1}{2} \int (1 - u^2) \, du \\ &= \frac{1}{2} \left(\sin 2x - \frac{1}{3} \sin^3 2x \right) + C_2\end{aligned}$$

0.3 Trigonometry

$$\sin^2 x = \frac{1 - \cos 2x}{2} \quad \cos^2 x = \frac{1 + \cos 2x}{2}$$



Now let's put it all together.

$$\int \sin^2 x \cos^4 x \, dx$$

$$= \frac{1}{8} \left(x + \frac{1}{2} \sin 2x - \int \cos^2 2x \, dx - \int \cos^3 2x \, dx \right)$$

=

=

0.3 Trigonometric Identities

$$\sin^2 x = \frac{1 - \cos 2x}{2} \quad \cos^2 x = \frac{1 + \cos 2x}{2}$$



Now let's put it all together.

$$\int \sin^2 x \cos^4 x \, dx$$

$$= \frac{1}{8} \left(x + \frac{1}{2} \sin 2x - \int \cos^2 2x \, dx - \int \cos^3 2x \, dx \right)$$

$$= \frac{1}{8} \left(x + \frac{1}{2} \sin 2x - \left(\frac{x}{2} + \frac{1}{8} \sin 4x \right) - \frac{1}{2} \left(\sin 2x - \frac{1}{3} \sin^3 2x \right) \right) + C$$

=

0.3 Trigonometric Identities

$$\sin^2 x = \frac{1 - \cos 2x}{2} \quad \cos^2 x = \frac{1 + \cos 2x}{2}$$



Now let's put it all together.

$$\begin{aligned} & \int \sin^2 x \cos^4 x \, dx \\ &= \frac{1}{8} \left(x + \frac{1}{2} \sin 2x - \int \cos^2 2x \, dx - \int \cos^3 2x \, dx \right) \\ &= \frac{1}{8} \left(x + \frac{1}{2} \sin 2x - \left(\frac{x}{2} + \frac{1}{8} \sin 4x \right) - \frac{1}{2} \left(\sin 2x - \frac{1}{3} \sin^3 2x \right) \right) + C \\ &= \frac{x}{16} - \frac{1}{64} \sin 4x + \frac{1}{48} \sin^3 2x + C. \end{aligned}$$

$$\sin^2 x = \frac{1 - \cos 2x}{2} \quad \cos^2 x = \frac{1 + \cos 2x}{2}$$



Example

$$\text{Find } \int_0^{\frac{\pi}{4}} \sqrt{1 + \cos 4x} dx.$$

0.3 Trigonometry

$$\sin^2 x = \frac{1 - \cos 2x}{2} \quad \cos^2 x = \frac{1 + \cos 2x}{2}$$



Example

Find $\int_0^{\frac{\pi}{4}} \sqrt{1 + \cos 4x} dx$.

To get rid of the $\sqrt{}$, we are going to use this .

0.3 Trigonometric Identities

$$\sin^2 x = \frac{1 - \cos 2x}{2} \quad \cos^2 x = \frac{1 + \cos 2x}{2}$$



Example

Find $\int_0^{\frac{\pi}{4}} \sqrt{1 + \cos 4x} dx.$

To get rid of the $\sqrt{}$, we are going to use this . Replacing x by $2x$ and rearranging, this is

$$1 + \cos 4x = 2 \cos^2 2x.$$

$$\sin^2 x = \frac{1 - \cos 2x}{2} \quad \cos^2 x = \frac{1 + \cos 2x}{2}$$



Example

$$\text{Find } \int_0^{\frac{\pi}{4}} \sqrt{1 + \cos 4x} dx.$$

To get rid of the $\sqrt{}$, we are going to use this . Replacing x by $2x$ and rearranging, this is

$$1 + \cos 4x = 2 \cos^2 2x.$$

Hence

$$\int_0^{\frac{\pi}{4}} \sqrt{1 + \cos 4x} dx = \int_0^{\frac{\pi}{4}} \sqrt{2 \cos^2 2x} dx =$$

=

=

0.3 Trigonometry

$$\sin^2 x = \frac{1 - \cos 2x}{2} \quad \cos^2 x = \frac{1 + \cos 2x}{2}$$



Example

Find $\int_0^{\frac{\pi}{4}} \sqrt{1 + \cos 4x} dx$.

To get rid of the $\sqrt{}$, we are going to use this . Replacing x by $2x$ and rearranging, this is

$$1 + \cos 4x = 2 \cos^2 2x.$$

Hence

$$\int_0^{\frac{\pi}{4}} \sqrt{1 + \cos 4x} dx = \int_0^{\frac{\pi}{4}} \sqrt{2 \cos^2 2x} dx = \int_0^{\frac{\pi}{4}} \sqrt{2} \sqrt{\cos^2 2x} dx$$

=

=

0.3 Trigonometry

$$\sin^2 x = \frac{1 - \cos 2x}{2} \quad \cos^2 x = \frac{1 + \cos 2x}{2}$$



Example

Find $\int_0^{\frac{\pi}{4}} \sqrt{1 + \cos 4x} dx$.

To get rid of the $\sqrt{}$, we are going to use this . Replacing x by $2x$ and rearranging, this is

$$1 + \cos 4x = 2 \cos^2 2x.$$

Hence

$$\begin{aligned}\int_0^{\frac{\pi}{4}} \sqrt{1 + \cos 4x} dx &= \int_0^{\frac{\pi}{4}} \sqrt{2 \cos^2 2x} dx = \int_0^{\frac{\pi}{4}} \sqrt{2} \sqrt{\cos^2 2x} dx \\ &= \sqrt{2} \int_0^{\frac{\pi}{4}} |\cos 2x| dx\end{aligned}$$

=

0.3 Trigonometric Identities

Example

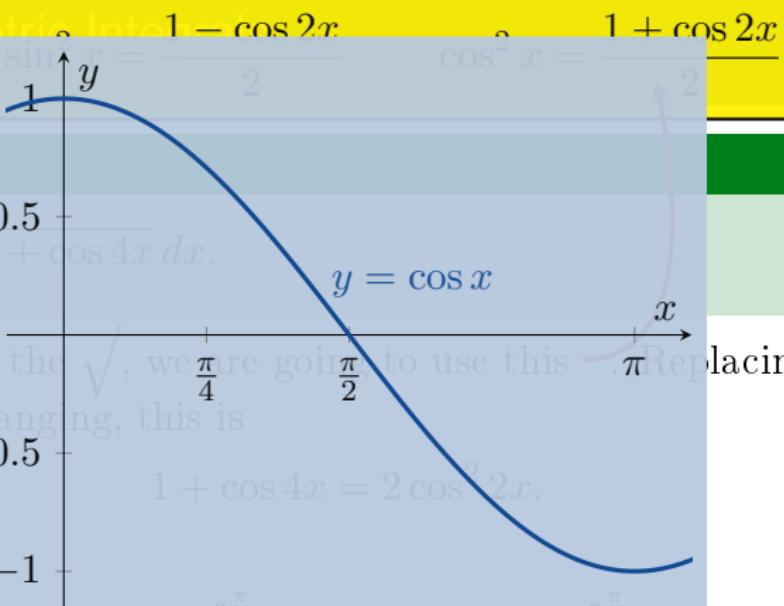
Find $\int_0^{\frac{\pi}{4}} \sqrt{1 + \cos 4x} dx.$

To get rid of the $\sqrt{}$, we're going to use this $y = \cos x$. Replacing x by $2x$ and rearranging, this is

Hence

$$\begin{aligned} \int_0^{\frac{\pi}{4}} \sqrt{1 + \cos 4x} dx &= \int_0^{\frac{\pi}{4}} \sqrt{2 \cos^2 2x} dx = \int_0^{\frac{\pi}{4}} \sqrt{2} \sqrt{\cos^2 2x} dx \\ &= \sqrt{2} \int_0^{\frac{\pi}{4}} |\cos 2x| dx \end{aligned}$$

=



0.3 Trigonometric Identities



Example

Find $\int_0^{\frac{\pi}{4}} \sqrt{1 + \cos 4x} dx$

To get rid of the $\sqrt{}$, we're going to use this $y = \cos 2x$.
 Replacing x by $2x$ and rearranging, this is

$$1 + \cos 4x = 2 \cos^2 2x.$$

Hence

$$\begin{aligned} \int_0^{\frac{\pi}{4}} \sqrt{1 + \cos 4x} dx &= \int_0^{\frac{\pi}{4}} \sqrt{2 \cos^2 2x} dx = \int_0^{\frac{\pi}{4}} \sqrt{2} \sqrt{\cos^2 2x} dx \\ &= \sqrt{2} \int_0^{\frac{\pi}{4}} |\cos 2x| dx \end{aligned}$$

=

0.3 Trigonometric Integrals



Example

Find $\int_0^{\frac{\pi}{4}} \sqrt{1 + \cos 4x} dx$

To get rid of the $\sqrt{}$, we're going to use this $1 + \cos 2x = 2 \cos^2 x$. Replacing x by $2x$ and rearranging, this is

Hence

$$\begin{aligned} \int_0^{\frac{\pi}{4}} \sqrt{1 + \cos 4x} dx &= \int_0^{\frac{\pi}{4}} \sqrt{2 \cos^2 2x} dx = \int_0^{\frac{\pi}{4}} \sqrt{2} \sqrt{\cos^2 2x} dx \\ &= \sqrt{2} \int_0^{\frac{\pi}{4}} |\cos 2x| dx \end{aligned}$$

=

0.3 Trigonometry

$$\sin^2 x = \frac{1 - \cos 2x}{2} \quad \cos^2 x = \frac{1 + \cos 2x}{2}$$



Example

Find $\int_0^{\frac{\pi}{4}} \sqrt{1 + \cos 4x} dx$.

To get rid of the $\sqrt{}$, we are going to use this . Replacing x by $2x$ and rearranging, this is

$$1 + \cos 4x = 2 \cos^2 2x.$$

Hence

$$\begin{aligned}\int_0^{\frac{\pi}{4}} \sqrt{1 + \cos 4x} dx &= \int_0^{\frac{\pi}{4}} \sqrt{2 \cos^2 2x} dx = \int_0^{\frac{\pi}{4}} \sqrt{2} \sqrt{\cos^2 2x} dx \\ &= \sqrt{2} \int_0^{\frac{\pi}{4}} \cos 2x \, dx\end{aligned}$$

=

0.3 Trigonometric Identities

$$\sin^2 x = \frac{1 - \cos 2x}{2} \quad \cos^2 x = \frac{1 + \cos 2x}{2}$$



Example

Find $\int_0^{\frac{\pi}{4}} \sqrt{1 + \cos 4x} dx.$

To get rid of the $\sqrt{}$, we are going to use this . Replacing x by $2x$ and rearranging, this is

$$1 + \cos 4x = 2 \cos^2 2x.$$

Hence

$$\begin{aligned}\int_0^{\frac{\pi}{4}} \sqrt{1 + \cos 4x} dx &= \int_0^{\frac{\pi}{4}} \sqrt{2 \cos^2 2x} dx = \int_0^{\frac{\pi}{4}} \sqrt{2} \sqrt{\cos^2 2x} dx \\&= \sqrt{2} \int_0^{\frac{\pi}{4}} \cos 2x \, dx \\&= \sqrt{2} \left[\frac{\sin 2x}{2} \right]_0^{\frac{\pi}{4}} = \sqrt{2} \left(\frac{1}{2} - 0 \right) = \frac{\sqrt{2}}{2}.\end{aligned}$$

0.3 Trigonometry

$$\sec^2 x = 1 + \tan^2 x \quad \frac{d}{dx} \tan x = \sec^2 x$$



Example

Find $\int \tan^4 x dx$.

0.3 Trigonometric Integrals

$$\sec^2 x = 1 + \tan^2 x \quad \frac{d}{dx} \tan x = \sec^2 x$$



Example

Find $\int \tan^4 x \, dx$.

Solution

$$\begin{aligned}\int \tan^4 x \, dx &= \int \tan^2 x \cdot \tan^2 x \, dx = \int \tan^2 x \cdot (\sec^2 x - 1) \, dx \\&= \int \tan^2 x \sec^2 x \, dx - \int \tan^2 x \, dx \\&= \int \tan^2 x \sec^2 x \, dx - \int (\sec^2 x - 1) \, dx \\&= \int \tan^2 x \sec^2 x \, dx - \int \sec^2 x \, dx + \int dx\end{aligned}$$

In the first integral, we let

$$u = \tan x, \quad du = \sec^2 x \, dx$$

and have

$$\int u^2 \, du = \frac{1}{3} u^3 + C_1.$$

The remaining integrals are standard forms, so

$$\int \tan^4 x \, dx = \frac{1}{3} \tan^3 x - \tan x + x + C.$$



0.3 Trigonometric Integrals

$$\int u \, dv = uv - \int v \, du$$



Example

Find $\int \sec^3 x \, dx$.

0.3 Trigonometric Integrals

$$\int u \, dv = uv - \int v \, du$$



Example

Find $\int \sec^3 x \, dx$.

Think of this as $\int \sec x \sec^2 x \, dx$. We are going to use Integration by Parts.

0.3 Trigonometric Integrals

$$\int u \, dv = uv - \int v \, du$$



Example

Find $\int \sec^3 x \, dx$.

Think of this as $\int \sec x \sec^2 x \, dx$. We are going to use Integration by Parts. Let $u = \sec x$ and $dv = \sec^2 x \, dx$. Then $du = \sec x \tan x \, dx$ and $v = \tan x$.

0.3 Trigonometric Integrals

$$\int u \, dv = uv - \int v \, du$$



Example

Find $\int \sec^3 x \, dx$.

Think of this as $\int \sec x \sec^2 x \, dx$. We are going to use

Integration by Parts. Let $u = \sec x$ and $dv = \sec^2 x \, dx$. Then
 $du = \sec x \tan x \, dx$ and $v = \tan x$. Hence

$$\int \sec x \sec^2 x \, dx = \sec x \tan x - \int \tan x \sec x \tan x \, dx$$

=

=

=

0.3 Trigonometric Integrals

$$\int u \, dv = uv - \int v \, du$$



Example

Find $\int \sec^3 x \, dx$.

Think of this as $\int \sec x \sec^2 x \, dx$. We are going to use

Integration by Parts. Let $u = \sec x$ and $dv = \sec^2 x \, dx$. Then
 $du = \sec x \tan x \, dx$ and $v = \tan x$. Hence

$$\begin{aligned}\int \sec x \sec^2 x \, dx &= \sec x \tan x - \int \tan x \sec x \tan x \, dx \\ &= \sec x \tan x - \int \tan^2 x \sec x \, dx\end{aligned}$$

=

=

0.3 Trigonometric Integrals

$$\int u \, dv = uv - \int v \, du$$



Example

Find $\int \sec^3 x \, dx$.

Think of this as $\int \sec x \sec^2 x \, dx$. We are going to use

Integration by Parts. Let $u = \sec x$ and $dv = \sec^2 x \, dx$. Then
 $du = \sec x \tan x \, dx$ and $v = \tan x$. Hence

$$\begin{aligned}\int \sec x \sec^2 x \, dx &= \sec x \tan x - \int \tan x \sec x \tan x \, dx \\&= \sec x \tan x - \int \tan^2 x \sec x \, dx \\&= \sec x \tan x - \int (\sec^2 x - 1) \sec x \, dx \\&= \end{aligned}$$

0.3 Trigonometric Integrals

$$\int u \, dv = uv - \int v \, du$$



Example

Find $\int \sec^3 x \, dx$.

Think of this as $\int \sec x \sec^2 x \, dx$. We are going to use

Integration by Parts. Let $u = \sec x$ and $dv = \sec^2 x \, dx$. Then
 $du = \sec x \tan x \, dx$ and $v = \tan x$. Hence

$$\begin{aligned}\int \sec x \sec^2 x \, dx &= \sec x \tan x - \int \tan x \sec x \tan x \, dx \\&= \sec x \tan x - \int \tan^2 x \sec x \, dx \\&= \sec x \tan x - \int (\sec^2 x - 1) \sec x \, dx \\&= \sec x \tan x + \int \sec x \, dx - \int \sec^3 x \, dx.\end{aligned}$$

0.3 Trigonometric Integrals



So we have

$$\int \sec^3 x \, dx = \sec x \tan x + \int \sec x \, dx - \int \sec^3 x \, dx.$$

0.3 Trigonometric Integrals

So we have

$$\int \sec^3 x \, dx = \sec x \tan x + \int \sec x \, dx - \int \sec^3 x \, dx.$$

Thus

$$2 \int \sec^3 x \, dx = \sec x \tan x + \int \sec x \, dx$$

0.3 Trigonometric Integrals



So we have

$$\int \sec^3 x \, dx = \sec x \tan x + \int \sec x \, dx - \int \sec^3 x \, dx.$$

Thus

$$2 \int \sec^3 x \, dx = \sec x \tan x + \int \sec x \, dx$$

and

$$\begin{aligned}\int \sec^3 x \, dx &= \frac{1}{2} \sec x \tan x + \frac{1}{2} \int \sec x \, dx \\ &= \frac{1}{2} \sec x \tan x + \frac{1}{2} \ln |\sec x + \tan x| + C.\end{aligned}$$

0.3 Trigonometric

$$\sec^2 x = 1 + \tan^2 x \quad \frac{d}{dx} \tan x = \sec^2 x$$



Example

Find $\int \tan^4 x \sec^4 x \, dx$.

0.3 Trigonometric Integrals

$$\sec^2 x = 1 + \tan^2 x \quad \frac{d}{dx} \tan x = \sec^2 x$$



Example

Find $\int \tan^4 x \sec^4 x \, dx$.

Solution

$$\begin{aligned}\int (\tan^4 x)(\sec^4 x) \, dx &= \int (\tan^4 x)(1 + \tan^2 x)(\sec^2 x) \, dx && \sec^2 x = 1 + \tan^2 x \\&= \int (\tan^4 x + \tan^6 x)(\sec^2 x) \, dx \\&= \int (\tan^4 x)(\sec^2 x) \, dx + \int (\tan^6 x)(\sec^2 x) \, dx \\&= \int u^4 \, du + \int u^6 \, du = \frac{u^5}{5} + \frac{u^7}{7} + C && u = \tan x, \\&= \frac{\tan^5 x}{5} + \frac{\tan^7 x}{7} + C && du = \sec^2 x \, dx\end{aligned}$$



0.3 Trigonometric Integrals

How do we calculate

$$\int \sin mx \sin nx \, dx$$

or

$$\int \sin mx \cos nx \, dx$$

or

$$\int \cos mx \cos nx \, dx$$

?

0.3 Trigonometric Integrals

How do we calculate

$$\int \sin mx \sin nx \, dx$$

or

$$\int \sin mx \cos nx \, dx$$

or

$$\int \cos mx \cos nx \, dx$$

?

It is possible to use integration by parts (twice), but there is an easier way.

0.3 Trigonometric

$$\cos(A + B) = \cos A \cos B - \sin A \sin B$$



$$\cos(mx - nx) - \cos(mx + nx) =$$

0.3 Trigonometric

$$\cos(A + B) = \cos A \cos B - \sin A \sin B$$



$$\begin{aligned}\cos(mx - nx) - \cos(mx + nx) &= \cos mx \cos nx + \sin mx \sin nx \\ &\quad - \cos mx \cos nx + \sin mx \sin nx\end{aligned}$$

0.3 Trigonometric

$$\cos(A + B) = \cos A \cos B - \sin A \sin B$$



$$\begin{aligned}\cos(mx - nx) - \cos(mx + nx) &= \cos mx \cos nx + \sin mx \sin nx \\ &\quad - \cos mx \cos nx + \sin mx \sin nx \\ &= 2 \sin mx \sin nx\end{aligned}$$

0.3 Trigonometric

$$\cos(A + B) = \cos A \cos B - \sin A \sin B$$



$$\begin{aligned}\cos(mx - nx) - \cos(mx + nx) &= \cos mx \cos nx + \sin mx \sin nx \\ &\quad - \cos mx \cos nx + \sin mx \sin nx \\ &= 2 \sin mx \sin nx\end{aligned}$$

Therefore

$$\sin mx \sin nx = \frac{1}{2}(\cos(m - n)x - \cos(m + n)x).$$

0.3 Trigonometric

$$\cos(A + B) = \cos A \cos B - \sin A \sin B$$



$$\begin{aligned}\cos(mx - nx) - \cos(mx + nx) &= \cos mx \cos nx + \sin mx \sin nx \\ &\quad - \cos mx \cos nx + \sin mx \sin nx \\ &= 2 \sin mx \sin nx\end{aligned}$$

Therefore

$$\sin mx \sin nx = \frac{1}{2}(\cos(m - n)x - \cos(m + n)x).$$

Similarly

$$\sin mx \cos nx = \frac{1}{2}(\sin(m - n)x + \sin(m + n)x)$$

and

$$\cos mx \cos nx = \frac{1}{2}(\cos(m - n)x + \cos(m + n)x).$$

0.3 Trigonometric Functions

$$\sin mx \cos nx = \frac{1}{2} (\sin(m-n)x + \sin(m+n)x)$$



Example

Find $\int \sin 3x \cos 5x \, dx$.

0.3 Trigonometric Functions

$$\sin mx \cos nx = \frac{1}{2} (\sin(m-n)x + \sin(m+n)x)$$



Example

Find $\int \sin 3x \cos 5x \, dx$.

Solution From Equation (4) with $m = 3$ and $n = 5$, we get

$$\begin{aligned}\int \sin 3x \cos 5x \, dx &= \frac{1}{2} \int [\sin(-2x) + \sin 8x] \, dx \\ &= \frac{1}{2} \int (\sin 8x - \sin 2x) \, dx \\ &= -\frac{\cos 8x}{16} + \frac{\cos 2x}{4} + C.\end{aligned}$$



0.3 Trigonometric Functions

$$\cos mx \cos nx = \frac{1}{2} (\cos(m-n)x + \cos(m+n)x)$$



Example

Find $\int \cos 3x \cos 2x \, dx$.

$$\cos mx \cos nx = \frac{1}{2} (\cos(m-n)x + \cos(m+n)x)$$



Example

Find $\int \cos 3x \cos 2x dx$.

We have $m = 3$ and $n = 2$. It follows that

$$\begin{aligned}\int \cos 3x \cos 2x dx &= \frac{1}{2} \int \cos(3-2)x dx + \frac{1}{2} \int \cos(3+2)x dx \\ &= \dots\end{aligned}$$



Next Time

- 8.4 Trigonometric Substitutions
- 8.5 Integration of Rational Functions by Partial Fractions
- 8.8 Improper Integrals