Lecture 3

- An algorithm for finding A^{-1}
- The Invertible Matrix Theorem
- Diagonal, Triangular, and Symmetric Matrices
- Some Applications of Linear Algebra.

An algorithm for finding A^{-1}

Recall that a square matrix A is called *invertible* if there exists a matrix A^{-1} of the same size such that

$$AA^{-1} = I \qquad \text{and} \qquad A^{-1}A = I.$$

A matrix which is not invertible is called *singular*.

Elementary Matrices

Definition

An *elementary matrix* is one that is obtained by performing a single elementary row operation on I.

An algorithm for finding A^{-1}

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

identity matrix

An algorithm for finding A^{-1}

$$\underset{\text{elementary matrix}}{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -4 & 0 & 1 \end{bmatrix}} \xleftarrow{-3R_1 + R_3} \underset{\text{identity matrix}}{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}$$

An algorithm for finding A^{-1}

elementary matrix　　　　　　　　　　elementary matrix

$$\begin{tikzcd} \begin{bmatrix}1&0&0\\0&1&0\\-4&0&1\end{bmatrix} & & \begin{bmatrix}0&1&0\\1&0&0\\0&0&1\end{bmatrix} \\ & \begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix} \arrow[ul, "-3R_1+R_3"'] \arrow[ur, "R_1 \leftrightarrow R_2"] & \end{tikzcd}$$

identity matrix

An algorithm for finding A^{-1}

elementary matrix

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -4 & 0 & 1 \end{bmatrix} \xleftarrow{-3R_1 + R_3} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{R_1 \leftrightarrow R_2} \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

elementary matrix

identity matrix

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{5R_3} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 5 \end{bmatrix}$$

elementary matrix

An algorithm for finding A^{-1}

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{-3R_1 + R_3} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -4 & 0 & 1 \end{bmatrix}$$

identity matrix — elementary matrix

1 elementary row
operation away
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An algorithm for finding A^{-1}

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identity matrix

elementary matrix

not an elementary matrix

1 elementary row operation away from I

2 elementary row operations away from I

Example

Let

$$E_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -4 & 0 & 1 \end{bmatrix}, \quad E_2 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad E_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 5 \end{bmatrix},$$

$$A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}.$$

Calculate E_1A, E_2A and E_3A.

An algorithm for finding A^{-1}

I leave it for you to check that

$$E_1 A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -4 & 0 & 1 \end{bmatrix} \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} = \begin{bmatrix} a & b & c \\ d & e & f \\ g-4a & h-4b & i-4c \end{bmatrix}$$

$$E_2 A = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} = \begin{bmatrix} d & e & f \\ a & b & c \\ g & h & i \end{bmatrix}$$

$$E_3 A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 5 \end{bmatrix} \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} = \begin{bmatrix} a & b & c \\ d & e & f \\ 5g & 5h & 5i \end{bmatrix}.$$

An algorithm for finding A^{-1}

$$E_1 A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -4 & 0 & 1 \end{bmatrix} \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} = \begin{bmatrix} a & b & c \\ d & e & f \\ g-4a & h-4b & i-4c \end{bmatrix}$$

But note that

$$I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{-4R_1+R_3} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -4 & 0 & 1 \end{bmatrix} = E_1$$

and

$$A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \xrightarrow{-4R_1+R_3} \begin{bmatrix} a & b & c \\ d & e & f \\ g-4a & h-4b & i-4c \end{bmatrix} = E_1 A.$$

An algorithm for finding A^{-1}

Similarly (please check)

$$I \xrightarrow{R_1 \leftrightarrow R_2} E_2 \qquad \text{and} \qquad A \xrightarrow{R_1 \leftrightarrow R_2} E_2 A$$

$$I \xrightarrow{5R_3} E_3 \qquad \text{and} \qquad A \xrightarrow{5R_3} E_3 A$$

An algorithm for finding A^{-1}

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Remark

Multiplying (on the left) by an elementary matrix is the same as doing the equivalent elementary row operation.

An algorithm for finding A^{-1}

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Remark

Multiplying (on the left) by an elementary matrix is the same as doing the equivalent elementary row operation.

Remark

Since every elementary row operation is reversible, every elementary matrix is invertible.

An algorithm for finding A^{-1}

Theorem

An $n \times n$ matrix A is invertible if and only if A is row equivalent to I_n.

An algorithm for finding A^{-1}

Theorem

An $n \times n$ matrix A is invertible if and only if A is row equivalent to I_n.

If A is invertible, then any sequence of elementary row operations that reduces A to I_n also transforms I_n into A^{-1}.

(proof in book)

The Algorithm

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2. Use Gauss-Jordan Elimination to reduce this augmented matrix to RREF.
3. If A is invertible, then you will obtain $[I\ A^{-1}]$. If you don't get this, then you know that A is singular.

An algorithm for finding A^{-1}

Example

Find the inverse of the matrix $A = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 0 & 3 \\ 4 & -3 & 8 \end{bmatrix}$, if it exists.

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We start with

$$[A\ I] = \begin{bmatrix} 0 & 1 & 2 & 1 & 0 & 0 \\ 1 & 0 & 3 & 0 & 1 & 0 \\ 4 & -3 & 8 & 0 & 0 & 1 \end{bmatrix}$$

and we need to row reduce this to RREF.

An algorithm for finding A^{-1}

$$[A\ I] = \begin{bmatrix} 0 & 1 & 2 & 1 & 0 & 0 \\ 1 & 0 & 3 & 0 & 1 & 0 \\ 4 & -3 & 8 & 0 & 0 & 1 \end{bmatrix} \xrightarrow{R_1 \leftrightarrow R_2} \begin{bmatrix} 1 & 0 & 3 & 0 & 1 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 \\ 4 & -3 & 8 & 0 & 0 & 1 \end{bmatrix}$$

$$\xrightarrow{-4R_1+R_3} \begin{bmatrix} 1 & 0 & 3 & 0 & 1 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 \\ 0 & -3 & -4 & 0 & -4 & 1 \end{bmatrix} \xrightarrow{3R_2+R_3} \begin{bmatrix} 1 & 0 & 3 & 0 & 1 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 \\ 0 & 0 & 2 & 3 & -4 & 1 \end{bmatrix}$$

$$\xrightarrow{-2R_1} \begin{bmatrix} -2 & 0 & -6 & 0 & -2 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 \\ 0 & 0 & 2 & 3 & -4 & 1 \end{bmatrix} \xrightarrow[-R_3+R_2]{3R_3+R_1} \begin{bmatrix} -2 & 0 & 0 & 9 & -14 & 3 \\ 0 & 1 & 0 & -2 & 4 & -1 \\ 0 & 0 & 2 & 3 & -4 & 1 \end{bmatrix}$$

$$\xrightarrow[\frac{1}{2}R_3]{-\frac{1}{2}R_1} \begin{bmatrix} 1 & 0 & 0 & -\frac{9}{2} & 7 & -\frac{3}{2} \\ 0 & 1 & 0 & -2 & 4 & -1 \\ 0 & 0 & 1 & \frac{3}{2} & -2 & \frac{1}{2} \end{bmatrix} = [I\ A^{-1}].$$

An algorithm for finding A^{-1}

Since

$$\begin{bmatrix} 0 & 1 & 2 & 1 & 0 & 0 \\ 1 & 0 & 3 & 0 & 1 & 0 \\ 4 & -3 & 8 & 0 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & -\frac{9}{2} & 7 & -\frac{3}{2} \\ 0 & 1 & 0 & -2 & 4 & -1 \\ 0 & 0 & 1 & \frac{3}{2} & -2 & \frac{1}{2} \end{bmatrix}$$

we have that

$$A^{-1} = \begin{bmatrix} -\frac{9}{2} & 7 & -\frac{3}{2} \\ -2 & 4 & -1 \\ \frac{3}{2} & -2 & \frac{1}{2} \end{bmatrix}.$$

An algorithm for finding A^{-1}

Let's just check our answer to make sure that we didn't make a mistake in our calculation:

$$AA^{-1} = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 0 & 3 \\ 4 & -3 & 8 \end{bmatrix} \begin{bmatrix} -\frac{9}{2} & 7 & -\frac{3}{2} \\ -2 & 4 & -1 \\ \frac{3}{2} & -2 & \frac{1}{2} \end{bmatrix}$$

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An algorithm for finding A^{-1}

Example

Does $A = \begin{bmatrix} 1 & -2 & 3 \\ 0 & 1 & 4 \\ -2 & 5 & -2 \end{bmatrix}$ have an inverse?

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$$\begin{bmatrix} 1 & -2 & 3 & 1 & 0 & 0 \\ 0 & 1 & 4 & 0 & 1 & 0 \\ -2 & 5 & -2 & 0 & 0 & 1 \end{bmatrix} \xrightarrow{2R_1+R_3} \begin{bmatrix} 1 & -2 & 3 & 1 & 0 & 0 \\ 0 & 1 & 4 & 0 & 1 & 0 \\ 0 & 1 & 4 & 2 & 0 & 1 \end{bmatrix}$$

$$\xrightarrow{-R_2+R_3} \begin{bmatrix} 1 & -2 & 3 & 1 & 0 & 0 \\ 0 & 1 & 4 & 0 & 1 & 0 \\ 0 & 0 & 0 & 2 & -1 & 1 \end{bmatrix}$$

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$$\xrightarrow{-R_2+R_3} \begin{bmatrix} 1 & -2 & 3 & 1 & 0 & 0 \\ 0 & 1 & 4 & 0 & 1 & 0 \\ 0 & 0 & 0 & 2 & -1 & 1 \end{bmatrix}$$

Since the first three entries in R_3 are zeros, we cannot row reduce A to I. This means that A does not have an inverse.

Example

Find the inverse of $A = \begin{bmatrix} 1 & -2 \\ 2 & 4 \end{bmatrix}$ by using this algorithm.

$$\begin{bmatrix} 1 & -2 & 1 & 0 \\ 2 & 4 & 0 & 1 \end{bmatrix} \xrightarrow{-2R_1+R_2} \begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 8 & -2 & 1 \end{bmatrix}$$

$$\xrightarrow{4R_1} \begin{bmatrix} 4 & -8 & 4 & 0 \\ 0 & 8 & -2 & 1 \end{bmatrix}$$

$$\xrightarrow{R_2+R_1} \begin{bmatrix} 4 & 0 & 2 & 1 \\ 0 & 8 & -2 & 1 \end{bmatrix}$$

$$\xrightarrow[\frac{1}{4}R_1]{\frac{1}{8}R_2} \begin{bmatrix} 1 & 0 & \frac{1}{2} & \frac{1}{4} \\ 0 & 1 & -\frac{1}{4} & \frac{1}{8} \end{bmatrix}$$

So $A^{-1} = \begin{bmatrix} \frac{1}{2} & \frac{1}{4} \\ -\frac{1}{4} & \frac{1}{8} \end{bmatrix}$.

Another View of Matrix Inversion

Let e_j denote the j^{th} column of the identity matrix I_n so

$$e_j = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{bmatrix},$$

where the 1 appears in the j^{th} entry of e_j.

An algorithm for finding A^{-1}

We can view the row reduction of $[A\ I] \to [I\ A^{-1}]$ as simultaneously solving the n equations

$$Ax = e_1,\ Ax = e_2\ ,\dots,\ Ax = e_n.$$

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Indeed, if we denote x_j (as a column vector) as the solution of $Ax = e_j$, for $j = 1, 2, \dots, n$, then $A^{-1} = [x_1\ x_2\ \cdots\ x_n]$.

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Indeed, if we denote x_j (as a column vector) as the solution of $Ax = e_j$, for $j = 1, 2, \ldots, n$, then $A^{-1} = [x_1\ x_2\ \cdots\ x_n]$.

Remark

Sometimes we only need the elements in one of the columns, say the j^{th} column, of A^{-1}. In this case, we need only to row reduce $[A\ e_j]$.

An algorithm for finding A^{-1}

Example

Find the second column of the inverse of

$$A = \begin{bmatrix} 2 & 3 \\ 1 & 4 \end{bmatrix}.$$

An algorithm for finding A^{-1}

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Find the second column of the inverse of

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Using our formula, we can easily see that

$$A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & 1 \end{bmatrix} = \begin{bmatrix} \frac{4}{5} & -\frac{3}{5} \\ -\frac{1}{5} & \frac{2}{5} \end{bmatrix}$$

so the answer to this question should be $\begin{bmatrix} -\frac{3}{5} \\ \frac{2}{5} \end{bmatrix}$.

An algorithm for finding A^{-1}

$$A = \begin{bmatrix} 2 & 3 \\ 1 & 4 \end{bmatrix}$$

We will row reduce $\begin{bmatrix} 2 & 3 & 0 \\ 1 & 4 & 1 \end{bmatrix}$ since $\mathbf{e}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ is the second column of I.

An algorithm for finding A^{-1}

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$$\begin{bmatrix} 2 & 3 & 0 \\ 1 & 4 & 1 \end{bmatrix} \xrightarrow{R_1 \leftrightarrow R_2} \begin{bmatrix} 1 & 4 & 1 \\ 2 & 3 & 0 \end{bmatrix} \xrightarrow{-2R_1 + R_2} \begin{bmatrix} 1 & 4 & 1 \\ 0 & -5 & -2 \end{bmatrix}$$

$$\xrightarrow{5R_1} \begin{bmatrix} 5 & 20 & 5 \\ 0 & -5 & -2 \end{bmatrix} \xrightarrow{4R_2 + R_1} \begin{bmatrix} 5 & 0 & -3 \\ 0 & -5 & -2 \end{bmatrix}$$

$$\xrightarrow[\frac{1}{5}R_1]{-\frac{1}{5}R_2} \begin{bmatrix} 1 & 0 & -\frac{3}{5} \\ 0 & 1 & \frac{2}{5} \end{bmatrix}.$$

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$$\xrightarrow[\frac{1}{5}R_1]{-\frac{1}{5}R_2} \begin{bmatrix} 1 & 0 & -\frac{3}{5} \\ 0 & 1 & \frac{2}{5} \end{bmatrix}.$$

So the second column of A^{-1} is the vector $\begin{bmatrix} -\frac{3}{5} \\ \frac{2}{5} \end{bmatrix}$.

The Invertible Matrix Theorem

Theorem (The Invertible Matrix Theorem)

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8. *The columns of A form a linearly independent set;*

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9. *The linear transformation $\mathbf{x} \mapsto A\mathbf{x}$ is one-to-one;*

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10. *The columns of A span $\mathbb{R}^n$;*

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9. *The linear transformation $\mathbf{x} \mapsto A\mathbf{x}$ is one-to-one;*
10. *The columns of A span $\mathbb{R}^n$;*
11. *The linear transformation $\mathbf{x} \mapsto A\mathbf{x}$ maps $\mathbb{R}^n$ onto $\mathbb{R}^n$;*

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10. *The columns of A span $\mathbb{R}^n$;*
11. *The linear transformation $\mathbf{x} \mapsto A\mathbf{x}$ maps $\mathbb{R}^n$ onto $\mathbb{R}^n$;*
12. *There is an $n \times n$ matrix C such that $CA = I$;*

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12. *There is an $n \times n$ matrix C such that $CA = I$;*
13. *There is an $n \times n$ matrix D such that $AD = I$;*

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Let A be a square $n \times n$ matrix. The following statements are equivalent (i.e. for a given A, they are either all true, or all false):

1. A *is invertible;*
2. A *is row equivalent to* I*;*
3. A *is expressible as a product of elementary matrices;*
4. A *has* n *pivot positions;*
5. $A\mathbf{x} = \mathbf{0}$ *has only the trivial solution;*
6. $A\mathbf{x} = \mathbf{b}$ *is consistent;*
7. $A\mathbf{x} = \mathbf{b}$ *has exactly one solution;*
8. *The columns of* A *form a linearly independent set;*
9. *The linear transformation* $\mathbf{x} \mapsto A\mathbf{x}$ *is one-to-one;*
10. *The columns of* A *span* $\mathbb{R}^n$*;*
11. *The linear transformation* $\mathbf{x} \mapsto A\mathbf{x}$ *maps* $\mathbb{R}^n$ *onto* $\mathbb{R}^n$*;*
12. *There is an* $n \times n$ *matrix* C *such that* $CA = I$*;*
13. *There is an* $n \times n$ *matrix* D *such that* $AD = I$*;*
14. A^T *is an invertible matrix.*

Remark

If A is invertible, then statements 2-14 are all true.

If A is singular, then statements 2-14 are all false.

The Invertible Matrix Theorem

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Properties 1, 12 and 13 were

1. A is invertible;
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Properties 1, 12 and 13 were

1. A is invertible;
12. There is an $n \times n$ matrix C such that $CA = I$;
13. There is an $n \times n$ matrix D such that $AD = I$.

This means that we don't need to prove both

$$AA^{-1} = I \qquad \text{and} \qquad A^{-1}A = I,$$

we only need to satisfy one of these.

The Invertible Matrix Theorem

Example

Use the Invertible Matrix Theorem to decide if
$A = \begin{bmatrix} 1 & 0 & -2 \\ 3 & 1 & -2 \\ -5 & -1 & 9 \end{bmatrix}$ is invertible.

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$$\sim \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 4 \\ 0 & 0 & 3 \end{bmatrix},$$

we can see that A has 3 pivot positions.

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$$\sim \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 4 \\ 0 & 0 & 3 \end{bmatrix},$$

we can see that A has 3 pivot positions. Hence A is invertible by the Invertible Matrix Theorem.

Example

Does the linear system

$$\begin{cases} x_1 \phantom{{}+x_2} - 2x_3 = 0 \\ 3x_1 + x_2 - 2x_3 = 0 \\ -5x_1 - x_2 + 9x_3 = 0 \end{cases}$$

have any nontrivial solutions?

Recall that $A = \begin{bmatrix} 1 & 0 & -2 \\ 3 & 1 & -2 \\ -5 & -1 & 9 \end{bmatrix}$ is invertible.

Example

Does the linear system

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have any nontrivial solutions?

Recall that $A = \begin{bmatrix} 1 & 0 & -2 \\ 3 & 1 & -2 \\ -5 & -1 & 9 \end{bmatrix}$ is invertible.

By the theorem, the linear system $A\mathbf{x} = 0$ has only the trivial solution. So the answer is "no".

Number of Solutions of a Linear System

In Lecture 1 I said that

Theorem

A linear system has either

1. *zero solutions; or*
2. *exactly one solution; or*
3. *infinitely many solutions.*

There are no other possibilities.

Now it is time to prove this.

The Invertible Matrix Theorem

Proof.

Consider the linear system $A\mathbf{x} = \mathbf{b}$. Exactly one of the following must be true:

a. $A\mathbf{x} = \mathbf{b}$ has no solutions;
b. $A\mathbf{x} = \mathbf{b}$ has exactly one solution; or
c. $A\mathbf{x} = \mathbf{b}$ has more than one solution.

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We only need to prove that $A\mathbf{x} = \mathbf{b}$ has infinitely many solutions in case c.

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We only need to prove that $A\mathbf{x} = \mathbf{b}$ has infinitely many solutions in case c.

In other words, we are going to prove that

$$\text{there are 2 different solutions} \implies \text{there are } \infty \text{ solutions.}$$

The Invertible Matrix Theorem

Proof continued.

Suppose that $\mathbf{x}_1$ and $\mathbf{x}_2$ be two <u>different</u> solutions.

The Invertible Matrix Theorem

Proof continued.

Suppose that $\mathbf{x}_1$ and $\mathbf{x}_2$ be two <u>different</u> solutions. So we are assuming that

- $A\mathbf{x}_1 = \mathbf{b}$,
- $A\mathbf{x}_2 = \mathbf{b}$ and
- $\mathbf{x}_1 \neq \mathbf{x}_2$

The Invertible Matrix Theorem

Proof continued.

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Let $\mathbf{x}_0 = \mathbf{x}_1 - \mathbf{x}_2 \neq \mathbf{0}$. Then

$$A\mathbf{x}_0 = A(\mathbf{x}_1 - \mathbf{x}_2) = A\mathbf{x}_1 - A\mathbf{x}_2 = \mathbf{b} - \mathbf{b} = \mathbf{0}.$$

The Invertible Matrix Theorem

Proof continued.

Suppose that $\mathbf{x}_1$ and $\mathbf{x}_2$ be two different solutions. So we are assuming that

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Now let $k \in \mathbb{R}$ be any number. Then

$$A(\mathbf{x}_1 + k\mathbf{x}_0) =$$

The Invertible Matrix Theorem

Proof continued.

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Now let $k \in \mathbb{R}$ be any number. Then

$$A(\mathbf{x}_1 + k\mathbf{x}_0) = A\mathbf{x}_1 + kA\mathbf{x}_0 = \mathbf{b} + \mathbf{0} = \mathbf{b}.$$

The Invertible Matrix Theorem

Proof continued.

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So $(\mathbf{x}_1 + k\mathbf{x}_0)$ is a solution for any k.

Proof continued.

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So $(\mathbf{x}_1 + k\mathbf{x}_0)$ is a solution for any k. So $A\mathbf{x} = \mathbf{b}$ has infinitely many solutions. $\square$

Solving Linear Systems by Matrix Inversion

Theorem

If A is an invertible $n \times n$ matrix, then for each $n \times 1$ matrix $\mathbf{b}$, the linear system $A\mathbf{x} = \mathbf{b}$ has exactly one solution,

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Since $A(A^{-1}\mathbf{b}) = \mathbf{b}$, it follows that $A^{-1}\mathbf{b}$ is a solution of $A\mathbf{x} = \mathbf{b}$. We need to show that $A^{-1}\mathbf{b}$ is the only solution.

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□

Example

Solve $\begin{cases} \quad\quad\;\; x_2 + 2x_3 = 1 \\ x_1 \quad\quad\;\; + 3x_3 = 2 \\ 4x_1 - 3x_2 + 8x_3 = 3 \end{cases}.$

We can write this as

$$A\mathbf{x} = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 0 & 3 \\ 4 & -3 & 8 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \mathbf{b}.$$

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Earlier we found that the inverse of A is

$$A^{-1} = \begin{bmatrix} -\frac{9}{2} & 7 & -\frac{3}{2} \\ -2 & 4 & -1 \\ \frac{3}{2} & -2 & \frac{1}{2} \end{bmatrix}.$$

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Therefore the solution is

$$\mathbf{x} = A^{-1}\mathbf{b} = \begin{bmatrix} -\frac{9}{2} & 7 & -\frac{3}{2} \\ -2 & 4 & -1 \\ \frac{3}{2} & -2 & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 5 \\ 3 \\ -1 \end{bmatrix}.$$

Example (Solving 2 Linear Systems at Once)

Solve $\begin{cases} x_1 + 2x_2 + 3x_3 = 4 \\ 2x_1 + 5x_2 + 3x_3 = 5 \\ x_1 + 8x_3 = 9 \end{cases}$ and $\begin{cases} x_1 + 2x_2 + 3x_3 = 1 \\ 2x_1 + 5x_2 + 3x_3 = 6 \\ x_1 + 8x_3 = -6 \end{cases}$.

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Since the two systems have the same coefficient matrix, we can write one augmented matrix which includes both systems:

$$\begin{bmatrix} 1 & 2 & 3 & 4 & 1 \\ 2 & 5 & 3 & 5 & 6 \\ 1 & 0 & 8 & 9 & -6 \end{bmatrix}.$$

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After using Gauss-Jordan Elimination (please check), we obtain:

$$\begin{bmatrix} 1 & 0 & 0 & 1 & 2 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & -1 \end{bmatrix}.$$

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After using Gauss-Jordan Elimination (please check), we obtain:

$$\begin{bmatrix} 1 & 0 & 0 & 1 & 2 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & -1 \end{bmatrix}.$$

So the solutions are $\begin{cases} x_1 = 1 \\ x_2 = 0 \\ x_3 = 1 \end{cases}$ and $\begin{cases} x_1 = 2 \\ x_2 = 1 \\ x_3 = -1 \end{cases}$.

The Invertible Matrix Theorem

Theorem

Let A and B be square matrices of the same size. If AB is invertible, then A and B must also be invertible.

The Invertible Matrix Theorem

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Let A and B be square matrices of the same size. If AB is invertible, then A and B must also be invertible.

Proof.

Suppose that AB is invertible.

The Invertible Matrix Theorem tells us that

B is invertible $\iff$ $B\mathbf{x} = \mathbf{0}$ has only the trivial solution.

First we will use this to prove that B is invertible. Then we will prove that A is also invertible.

Proof continued.

Suppose that $\mathbf{x}$ is a solution to $B\mathbf{x} = \mathbf{0}$. Then

$$(AB)\mathbf{x} = A(B\mathbf{x}) = A\mathbf{0} = \mathbf{0}.$$

Proof continued.

Suppose that $\mathbf{x}$ is a solution to $B\mathbf{x} = \mathbf{0}$. Then

$$(AB)\mathbf{x} = A(B\mathbf{x}) = A\mathbf{0} = \mathbf{0}.$$

Since AB is invertible, this implies that $\mathbf{x} = \mathbf{0}$. Hence the trivial solution is the only solution to $B\mathbf{x} = \mathbf{0}$. Therefore B must be invertible.

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Since AB is invertible, this implies that $\mathbf{x} = \mathbf{0}$. Hence the trivial solution is the only solution to $B\mathbf{x} = \mathbf{0}$. Therefore B must be invertible.

Now since both AB and B^{-1} are invertible matrices, it follows that

$$A = AI = A(BB^{-1}) = (AB)B^{-1}$$

is the product of two invertible matrices and hence is also invertible. □

Diagonal, Triangular, and Symmetric Matrices

Remark

Your textbook doesn't have a section on this. Instead these ideas are spread through various sections and exercises.

I think that it makes sense to introduce these concepts now so that you are familar with them when we need them later in the course.

Diagonal Matrices

Definition

A square matrix in which all the entries off the main diagonal are zero is called a *diagonal matrix*.

Example

$$\begin{bmatrix} 2 & 0 \\ 0 & -5 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} 6 & 0 & 0 & 0 \\ 0 & -4 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 8 \end{bmatrix}, \quad \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

are diagonal matrices.

Diagonal, Triangular, and Symmetric Matrices

A general $n \times n$ diagonal matrix D can be written as

$$D = \begin{bmatrix} d_1 & 0 & \cdots & 0 \\ 0 & d_2 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & d_n \end{bmatrix}.$$

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D is invertible if and only if $d_k \neq 0$ for all k; in this case its inverse is

$$D^{-1} = \begin{bmatrix} \frac{1}{d_1} & 0 & \cdots & 0 \\ 0 & \frac{1}{d_2} & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & \frac{1}{d_n} \end{bmatrix}.$$

(Check what you get if you multiply D and D^{-1} together.)

Powers of diagonal matrices are easy to calculate. I leave it for you to check that if

$$D = \begin{bmatrix} d_1 & 0 & \cdots & 0 \\ 0 & d_2 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & d_n \end{bmatrix}$$

and if $k \in \mathbb{N}$, then

$$D^k = \begin{bmatrix} d_1^k & 0 & \cdots & 0 \\ 0 & d_2^k & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & d_n^k \end{bmatrix}.$$

Example

If

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & 2 \end{bmatrix},$$

then

$$A^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -\frac{1}{3} & 0 \\ 0 & 0 & \frac{1}{2} \end{bmatrix},$$

$$A^5 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -243 & 0 \\ 0 & 0 & 32 \end{bmatrix} \qquad \text{and} \qquad A^{-5} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -\frac{1}{243} & 0 \\ 0 & 0 & \frac{1}{32} \end{bmatrix}.$$

It is easy to calculate the product of two matrices if one is a diagonal matrix.

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$$\begin{bmatrix} d_1 & 0 & 0 \\ 0 & d_2 & 0 \\ 0 & 0 & d_3 \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \end{bmatrix} =$$

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It is easy to calculate the product of two matrices if one is a diagonal matrix.

$$\begin{bmatrix} d_1 & 0 & 0 \\ 0 & d_2 & 0 \\ 0 & 0 & d_3 \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \end{bmatrix} = \begin{bmatrix} d_1a_{11} & d_1a_{12} & d_1a_{13} & d_1a_{14} \\ d_2a_{21} & d_2a_{22} & d_2a_{23} & d_2a_{24} \\ d_3a_{31} & d_3a_{32} & d_3a_{33} & d_3a_{34} \end{bmatrix}$$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \end{bmatrix} \begin{bmatrix} d_1 & 0 & 0 & 0 \\ 0 & d_2 & 0 & 0 \\ 0 & 0 & d_3 & 0 \\ 0 & 0 & 0 & d_4 \end{bmatrix} = \begin{bmatrix} d_1a_{11} & d_2a_{12} & d_3a_{13} & d_4a_{14} \\ d_1a_{21} & d_2a_{22} & d_3a_{23} & d_4a_{24} \\ d_1a_{31} & d_2a_{32} & d_3a_{33} & d_4a_{34} \end{bmatrix}$$

Break

We will continue at 3pm

Triangular Matrices

$$\underbrace{\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ 0 & a_{22} & a_{23} & a_{24} \\ 0 & 0 & a_{33} & a_{34} \\ 0 & 0 & 0 & a_{44} \end{bmatrix}}_{\text{upper triangular } 4\times 4}$$

Definition

A square matrix in which all the entries below the main diagonal are zero is called *upper triangular*.

Triangular Matrices

$$\underbrace{\begin{bmatrix} a_{11} & 0 & 0 & 0 \\ a_{21} & a_{22} & 0 & 0 \\ a_{31} & a_{32} & a_{33} & 0 \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix}}_{\text{lower triangular } 4\times 4}$$

Definition

A square matrix in which all the entries above the main diagonal are zero is called *lower triangular*.

Triangular Matrices

$$\underbrace{\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ 0 & a_{22} & a_{23} & a_{24} \\ 0 & 0 & a_{33} & a_{34} \\ 0 & 0 & 0 & a_{44} \end{bmatrix}}_{\text{upper triangular } 4\times 4} \quad \text{or} \quad \underbrace{\begin{bmatrix} a_{11} & 0 & 0 & 0 \\ a_{21} & a_{22} & 0 & 0 \\ a_{31} & a_{32} & a_{33} & 0 \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix}}_{\text{lower triangular } 4\times 4}$$

Definition

A matrix that is either upper triangular or lower triangular is called *triangular*.

Diagonal, Triangular, and Symmetric Matrices

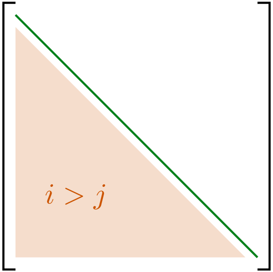
Remark

Note that diagonal matrices are both upper triangular and lower triangular.

Remark

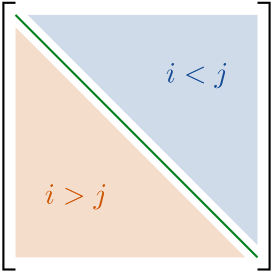
A square matrix in row echelon form is upper triangular since it has zeros below the main diagonal.

Diagonal, Triangular, and Symmetric Matrices



A square matrix $A = [a_{ij}]$ is

- *upper triangular* $\iff$ $a_{ij} = 0$ for all $i > j$;



A square matrix $A = [a_{ij}]$ is

- *upper triangular* $\iff$ $a_{ij} = 0$ for all $i > j$;
- *lower triangular* $\iff$ $a_{ij} = 0$ for all $i < j$.

Diagonal, Triangular, and Symmetric Matrices

Let

$$L = \text{a lower triangular matrix}$$
$$U = \text{an upper triangular matrix}$$

Theorem

1. $L^T = U$
2. $U^T = L$

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3. $L_1 L_2 = L$
4. $U_1 U_2 = U.$

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Theorem

1. $L^T = U$
2. $U^T = L$
3. $L_1 L_2 = L$
4. $U_1 U_2 = U$.
5. *A triangular matrix is invertible iff its diagonal entries are all nonzero.*

Diagonal, Triangular, and Symmetric Matrices

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$$L = \text{a lower triangular matrix}$$
$$U = \text{an upper triangular matrix}$$

Theorem

1. $L^T = U$
2. $U^T = L$
3. $L_1 L_2 = L$
4. $U_1 U_2 = U.$
5. *A triangular matrix is invertible iff its diagonal entries are all nonzero.*
6. L^{-1} *(if it exists) is lower triangular.*
7. U^{-1} *(if it exists) is upper triangular.*

Example

Consider the upper triangular matrices

$$A = \begin{bmatrix} 1 & 3 & -1 \\ 0 & 2 & 4 \\ 0 & 0 & 5 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 3 & -2 & 2 \\ 0 & 0 & -1 \\ 0 & 0 & 1 \end{bmatrix}.$$

Since all the entries on the main diagonal of A are nonzero, A must be invertible. Since B has a 0 on its main diagonal, B is singular.

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Since all the entries on the main diagonal of A are nonzero, A must be invertible. Since B has a 0 on its main diagonal, B is singular.

The theorem tells us that A^{-1}, AB and BA will also be upper triangular. I leave it for you to check that

$$A^{-1} = \begin{bmatrix} 1 & -\frac{3}{2} & \frac{7}{5} \\ 0 & \frac{1}{2} & -\frac{2}{5} \\ 0 & 0 & \frac{1}{5} \end{bmatrix}, \ AB = \begin{bmatrix} 3 & -2 & -2 \\ 0 & 0 & 2 \\ 0 & 0 & 5 \end{bmatrix}, \ BA = \begin{bmatrix} 3 & 5 & -1 \\ 0 & 0 & -5 \\ 0 & 0 & 5 \end{bmatrix}.$$

Symmetric Matrices

Definition

A square matrix A is called *symmetric* if $A = A^T$.

Example

$$\begin{bmatrix} 7 & -3 \\ -3 & 5 \end{bmatrix}, \qquad \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 4 & 5 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} d_1 & 0 & 0 & 0 \\ 0 & d_2 & 0 & 0 \\ 0 & 0 & d_3 & 0 \\ 0 & 0 & 0 & d_4 \end{bmatrix}$$

are symmetric matrices.

Diagonal, Triangular, and Symmetric Matrices

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 4 & 5 \end{bmatrix}$$

Diagonal, Triangular, and Symmetric Matrices

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 4 & 5 \end{bmatrix}$$

Diagonal, Triangular, and Symmetric Matrices

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Diagonal, Triangular, and Symmetric Matrices

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 4 & 5 \end{bmatrix}$$

Remark

The matrix $A = [a_{ij}]$ is symmetric iff

$$a_{ij} = a_{ji}$$

for all i and j.

Theorem

Let A and B be symmetric matrices with the same size, and let k be a number. Then

1. A^T *is symmetric;*
2. $A + B$ *and* $A - B$ *are symmetric;*
3. kA *is symmetric.*

Remark

It is not true, in general, that the product of two symmetric matrices is symmetric.

Since

$$(AB)^T = B^T A^T = BA$$

(if A and B are symmetric) we have $(AB)^T = AB$ if and only if $AB = BA$. Thus...

Diagonal, Triangular, and Symmetric Matrices

Remark

It is not true, in general, that the product of two symmetric matrices is symmetric.

Since
$$(AB)^T = B^T A^T = BA$$
(if A and B are symmetric) we have $(AB)^T = AB$ if and only if $AB = BA$. Thus...

Theorem

The product of two symmetric matrices A and B is symmetric if and only if A and B commute (i.e. if $AB = BA$).

Example

Note that

$$\begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} -4 & 1 \\ 1 & 0 \end{bmatrix} = \underbrace{\begin{bmatrix} -2 & 1 \\ -5 & 2 \end{bmatrix}}_{\text{not symmetric}}$$

and

$$\begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} -4 & 3 \\ 3 & -1 \end{bmatrix} = \underbrace{\begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix}}_{\text{symmetric}}.$$

Example

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$$\begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} -4 & 1 \\ 1 & 0 \end{bmatrix} = \underbrace{\begin{bmatrix} -2 & 1 \\ -5 & 2 \end{bmatrix}}_{\text{not symmetric}}$$

and

$$\begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} -4 & 3 \\ 3 & -1 \end{bmatrix} = \underbrace{\begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix}}_{\text{symmetric}}.$$

So the latter pair of symmetric commute, but the first pair do not.

Invertibility of Symmetric Matrices

Note that the matrix

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix}$$

is symmetric but not invertible (because it has a zero on its main diagonal).

Invertibility of Symmetric Matrices

Note that the matrix

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix}$$

is symmetric but not invertible (because it has a zero on its main diagonal).

Theorem

If A is an invertible symmetric matrix, then A^{-1} is symmetric.

Proof.

$$A = A^T \implies (A^{-1})^T = (A^T)^{-1} = A^{-1} \implies A^{-1} \text{ is symmetric.}$$

□

AA^T and A^TA

Note that if A is an $m \times n$ matrix, then A^T is an $n \times m$ matrix, so the products AA^T and A^TA are both square matrices.

AA^T and A^TA

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$$\left(AA^T\right)^T = (A^T)^T A^T$$

AA^T and A^TA

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AA^T and A^TA

Note that if A is an $m \times n$ matrix, then A^T is an $n \times m$ matrix, so the products AA^T and A^TA are both square matrices. Moreover, since $(AB)^T = B^TA^T$, we have

$$\left(AA^T\right)^T = (A^T)^T A^T = AA^T$$

and

$$\left(A^TA\right)^T = A^T(A^T)^T = A^TA$$

which shows that both AA^T and A^TA are symmetric.

Example

Let

$$A = \begin{bmatrix} 1 & -2 & 4 \\ 3 & 0 & -5 \end{bmatrix}.$$

Please check that

$$A^T A = \begin{bmatrix} 1 & 3 \\ -2 & 0 \\ 4 & -5 \end{bmatrix} \begin{bmatrix} 1 & -2 & 4 \\ 3 & 0 & -5 \end{bmatrix} = \begin{bmatrix} 10 & -2 & -11 \\ -2 & 4 & -8 \\ -11 & -8 & 41 \end{bmatrix}$$

and

$$AA^T = \begin{bmatrix} 1 & -2 & 4 \\ 3 & 0 & -5 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ -2 & 0 \\ 4 & -5 \end{bmatrix} = \begin{bmatrix} 21 & -17 \\ -17 & 34 \end{bmatrix}.$$

Theorem

If A is an invertible matrix, then AA^T and A^TA are also invertible.

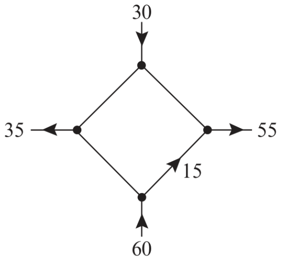
Proof.

A is invertible $\implies$ A^T is invertible. Recall that the product of two invertible matrices is invertible. $\square$

That's enough about AA^T and A^TA for now. We will come back to them later in the course.

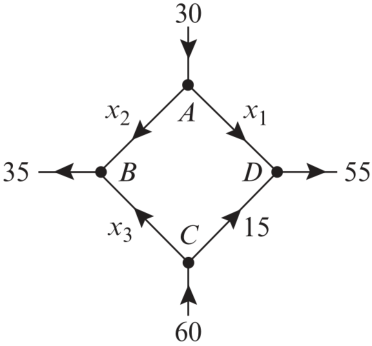
Some Applications of Linear Algebra

Network Analysis



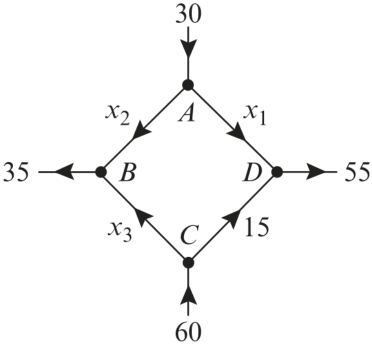
Example

Consider a network with four nodes in which the flow rate and direction of flow in certain branches are known. Find the flow rates and directions of flow in the remaining branches.

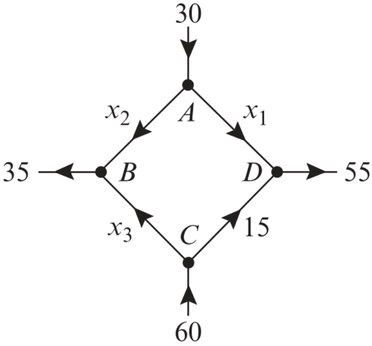


- At node A we have $x_1 + x_2 = 30$;

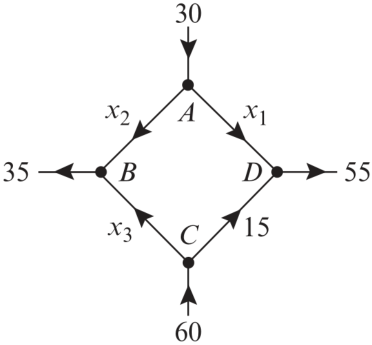
Some Applications of Linear Algebra



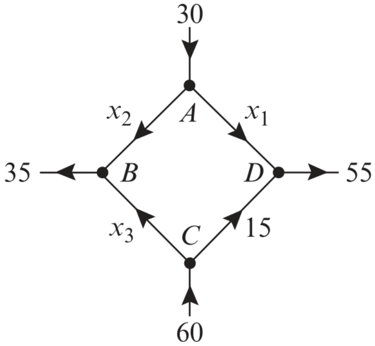
- At node A we have $x_1 + x_2 = 30$;
- At node B we have $x_2 + x_3 = 35$;



- At node A we have $x_1 + x_2 = 30$;
- At node B we have $x_2 + x_3 = 35$;
- At node C we have $x_3 + 15 = 60$;

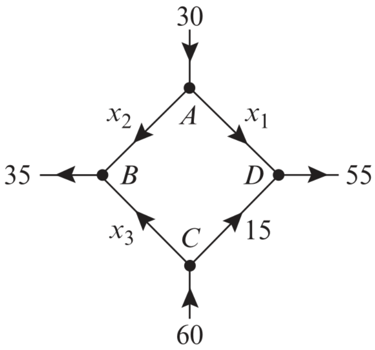


- At node A we have $x_1 + x_2 = 30$;
- At node B we have $x_2 + x_3 = 35$;
- At node C we have $x_3 + 15 = 60$; and
- At node D we have $x_1 + 15 = 55$.



So we have a linear system

$$
\begin{cases}
x_1 + x_2 & = 30 \\
\quad\quad x_2 + x_3 & = 35 \\
\quad\quad\quad\quad x_3 + 15 & = 60 \\
x_1 \quad\quad\quad\quad + 15 & = 55.
\end{cases}
$$



So we have a linear system

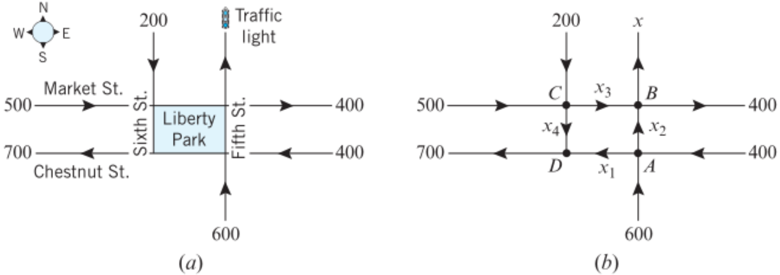
$$\begin{cases} x_1 + x_2 = 30 \\ x_2 + x_3 = 35 \\ x_3 + 15 = 60 \\ x_1 + 15 = 55. \end{cases} \quad \Longrightarrow \quad \begin{cases} x_1 = 40 \\ x_2 = -10 \\ x_3 = 45. \end{cases}$$

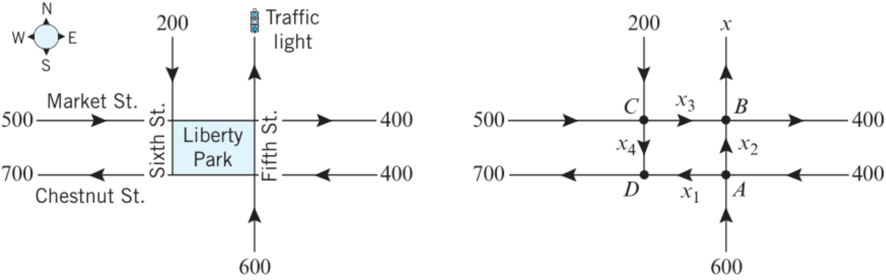
EXAMPLE 2 Design of Traffic Patterns

The network in Figure 1.9.3 shows a proposed plan for the traffic flow around a new park that will house the Liberty Bell in Philadelphia, Pennsylvania. The plan calls for a computerized traffic light at the north exit on Fifth Street, and the diagram indicates the average number of vehicles per hour that are expected to flow in and out of the streets that border the complex. All streets are one-way.

(a) How many vehicles per hour should the traffic light let through to ensure that the average number of vehicles per hour flowing into the complex is the same as the average number of vehicles flowing out?

(b) Assuming that the traffic light has been set to balance the total flow in and out of the complex, what can you say about the average number of vehicles per hour that will flow along the streets that border the complex?





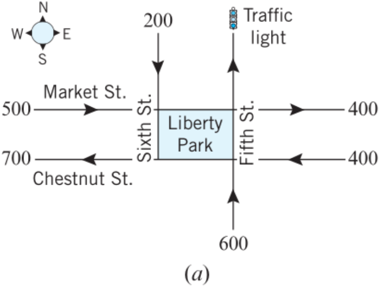
(a) (b)

Solution (a) If, as indicated in Figure 1.9.3b, we let x denote the number of vehicles per hour that the traffic light must let through, then the total number of vehicles per hour that flow in and out of the complex will be

Flowing in: $500 + 400 + 600 + 200 = 1700$

Flowing out: $x + 700 + 400$

Equating the flows in and out shows that the traffic light should let $x = 600$ vehicles per hour pass through.



(a) (b)

Solution (b) To avoid traffic congestion, the flow in must equal the flow out at each intersection. For this to happen, the following conditions must be satisfied:

Intersection	Flow In		Flow Out
A	$400 + 600$	$=$	$x_1 + x_2$
B	$x_2 + x_3$	$=$	$400 + x$
C	$500 + 200$	$=$	$x_3 + x_4$
D	$x_1 + x_4$	$=$	700

Thus, with $x = 600$, as computed in part (a), we obtain the following linear system:

$$\begin{aligned} x_1 + x_2 \qquad\qquad &= 1000 \\ x_2 + x_3 \qquad &= 1000 \\ x_3 + x_4 &= \ \ 700 \\ x_1 \qquad\qquad + x_4 &= \ \ 700 \end{aligned}$$

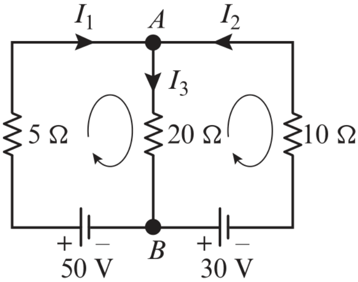
We leave it for you to show that the system has infinitely many solutions and that these are given by the parametric equations

$$x_1 = 700 - t, \quad x_2 = 300 + t, \quad x_3 = 700 - t, \quad x_4 = t \tag{1}$$

However, the parameter t is not completely arbitrary here, since there are physical constraints to be considered. For example, the average flow rates must be nonnegative since we have assumed the streets to be one-way, and a negative flow rate would indicate a flow in the wrong direction. This being the case, we see from (1) that t can be any real number that satisfies $0 \leq t \leq 700$, which implies that the average flow rates along the streets will fall in the ranges

$$0 \leq x_1 \leq 700, \quad 300 \leq x_2 \leq 1000, \quad 0 \leq x_3 \leq 700, \quad 0 \leq x_4 \leq 700 \blacktriangleleft$$

Electric Circuits



▲ Figure 1.9.9

▶ EXAMPLE 4 **A Circuit with Three Closed Loops**

Determine the currents I_1, I_2, and I_3 in the circuit shown in Figure 1.9.9.

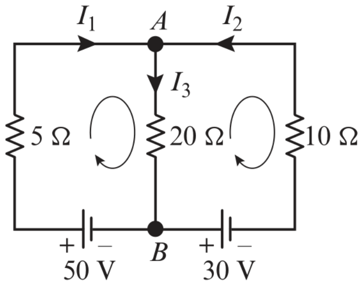


Figure 1.9.9

Using Ohm's Law, Kirchhoff's Current Law and Kirchoff's Voltage Law, it is possible to write down a linear system for I_1, I_2 and I_3.

I'll leave this example for you to think about.

Balancing Chemical Equations

Example

Balance the chemical equation

$$\underset{\text{hydrochloric acid}}{\mathrm{HCl}} + \underset{\text{sodium phosphate}}{\mathrm{Na_3PO_4}} \longrightarrow \underset{\text{phosphoric acid}}{\mathrm{H_3PO_4}} + \underset{\text{sodium chloride}}{\mathrm{NaCl}}$$

Balancing Chemical Equations

Example

Balance the chemical equation

$$\mathrm{HCl} + \mathrm{Na_3PO_4} \longrightarrow \mathrm{H_3PO_4} + \mathrm{NaCl}$$

hydrochloric acid
sodium phosphate
phosphoric acid
sodium chloride

We need to find natural numbers x_1, x_2, x_3, x_4 such that

$$x_1\,\mathrm{HCl} + x_2\,\mathrm{Na_3PO_4} \longrightarrow x_3\,\mathrm{H_3PO_4} + x_4\,\mathrm{NaCl}$$

is balanced (same number of each atom on each side).

$$x_1\,\mathrm{HCl} + x_2\,\mathrm{Na_3PO_4} \longrightarrow x_3\,\mathrm{H_3PO_4} + x_4\,\mathrm{NaCl}$$

$$1x_1 = 3x_3 \qquad \text{(Hydrogen H)}$$

(Chlorine Cl)

(Sodium Na)

(Phosphorus P)

(Oxygen O)

$$x_1\,\mathrm{HCl} + x_2\,\mathrm{Na_3PO_4} \longrightarrow x_3\,\mathrm{H_3PO_4} + x_4\,\mathrm{NaCl}$$

$$\begin{aligned} 1x_1 &= 3x_3 && \text{(Hydrogen H)} \\ 1x_1 &= 1x_4 && \text{(Chlorine Cl)} \\ & && \text{(Sodium Na)} \\ & && \text{(Phosphorus P)} \\ & && \text{(Oxygen O)} \end{aligned}$$

$$x_1\,\mathrm{HCl} + x_2\,\mathrm{Na_3PO_4} \longrightarrow x_3\,\mathrm{H_3PO_4} + x_4\,\mathrm{NaCl}$$

$$\begin{aligned}
1x_1 &= 3x_3 && \text{(Hydrogen H)}\\
1x_1 &= 1x_4 && \text{(Chlorine Cl)}\\
3x_2 &= 1x_4 && \text{(Sodium Na)}\\
& && \text{(Phosphorus P)}\\
& && \text{(Oxygen O)}
\end{aligned}$$

$$x_1\,\mathrm{HCl} + x_2\,\mathrm{Na_3PO_4} \longrightarrow x_3\,\mathrm{H_3PO_4} + x_4\,\mathrm{NaCl}$$

$$
\begin{aligned}
1x_1 &= 3x_3 && \text{(Hydrogen H)}\\
1x_1 &= 1x_4 && \text{(Chlorine Cl)}\\
3x_2 &= 1x_4 && \text{(Sodium Na)}\\
1x_2 &= 1x_3 && \text{(Phosphorus P)}\\
& && \text{(Oxygen O)}
\end{aligned}
$$

$$x_1\,\mathrm{HCl} + x_2\,\mathrm{Na_3PO_4} \longrightarrow x_3\,\mathrm{H_3PO_4} + x_4\,\mathrm{NaCl}$$

$$\begin{aligned}
1x_1 &= 3x_3 && \text{(Hydrogen H)}\\
1x_1 &= 1x_4 && \text{(Chlorine Cl)}\\
3x_2 &= 1x_4 && \text{(Sodium Na)}\\
1x_2 &= 1x_3 && \text{(Phosphorus P)}\\
4x_2 &= 4x_3 && \text{(Oxygen O)}
\end{aligned}$$

So we have a linear system

$$\begin{cases} x_1 \qquad\quad - 3x_3 \qquad\quad = 0 \\ x_1 \qquad\qquad\qquad\quad - x_4 = 0 \\ \qquad 3x_2 \qquad\qquad - x_4 = 0 \\ \qquad\; x_2 - \;\; x_3 \qquad\quad = 0 \\ \qquad 4x_2 - 4x_3 \qquad\quad = 0 \end{cases}$$

So we have a linear system

$$\begin{cases} x_1 \qquad\quad - 3x_3 \qquad\quad = 0 \\ x_1 \qquad\qquad\qquad\quad - x_4 = 0 \\ \qquad 3x_2 \qquad\quad\;\; - x_4 = 0 \\ \qquad\; x_2 - \;\; x_3 \qquad\quad = 0 \\ \qquad 4x_2 - 4x_3 \qquad\quad = 0 \end{cases}$$

which has solution

$$\begin{cases} x_1 = x_4 \\ x_2 = \frac{1}{3}x_4 \\ x_3 = \frac{1}{3}x_4 \\ x_4 \text{ is free.} \end{cases}$$

So we have a linear system

$$\begin{cases} x_1 \quad\quad - 3x_3 \quad\quad = 0 \\ x_1 \quad\quad\quad\quad - x_4 = 0 \\ \quad 3x_2 \quad\quad - x_4 = 0 \\ \quad x_2 - \ \ x_3 \quad\quad = 0 \\ \quad 4x_2 - 4x_3 \quad\quad = 0 \end{cases}$$

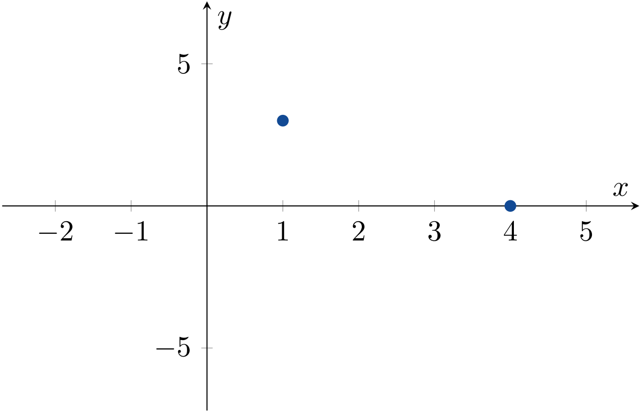
which has solution

$$\begin{cases} x_1 = x_4 \\ x_2 = \frac{1}{3}x_4 \\ x_3 = \frac{1}{3}x_4 \\ x_4 \text{ is free.} \end{cases}$$

Since we want natural numbers, we choose $x_4 = 3$. Then we have $x_1 = 3$, $x_2 = 1$ and $x_3 = 1$. The balanced equation is

$$3\,\mathrm{HCl} + \mathrm{Na_3PO_4} \longrightarrow \mathrm{H_3PO_4} + 3\,\mathrm{NaCl}.$$

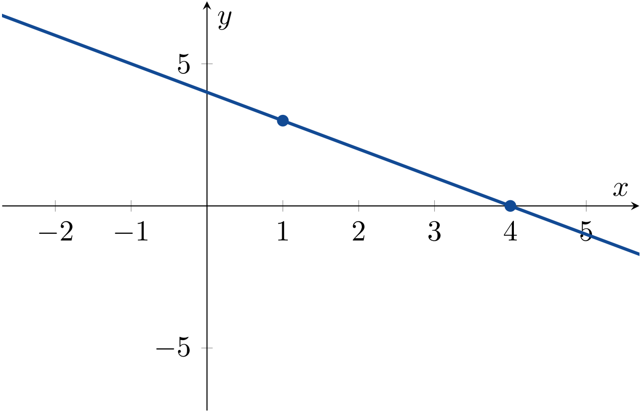
Polynomial Interpolation



If I have 2 points, I can find a unique line through them.

$y = a_0 + a_1 x$

Polynomial Interpolation



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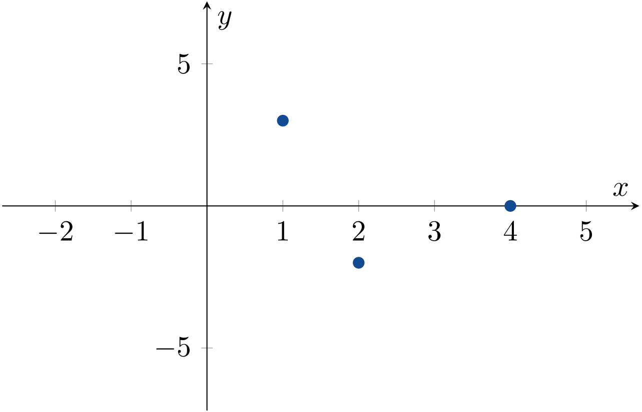
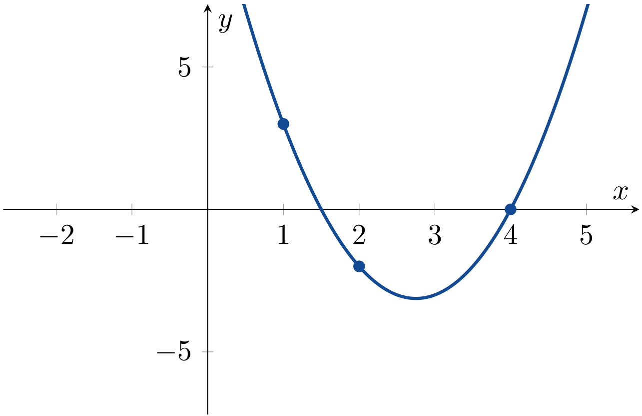


If I have 3 points, I can find a unique quadratic polynomial through them. $y = a_0 + a_1 x + a_2 x^2$

Polynomial Interpolation



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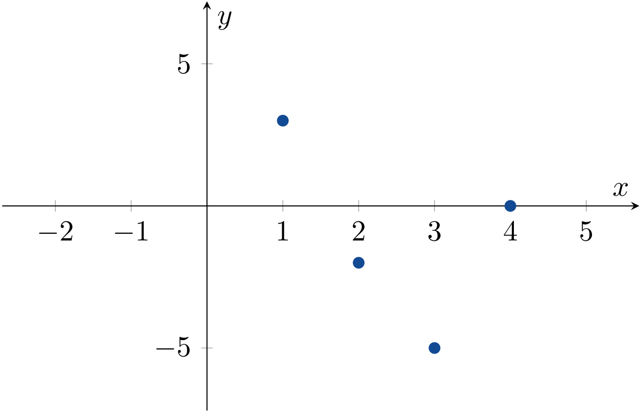
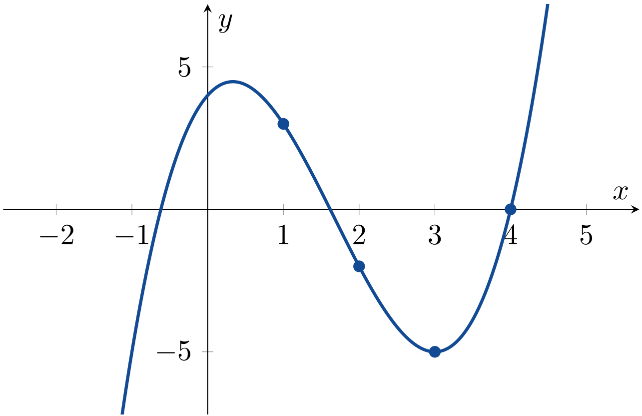
Polynomial Interpolation



If I have 4 points, I can find a unique cubic polynomial through them. $y = a_0 + a_1 x + a_2 x^2 + a_3 x^3$

Polynomial Interpolation



If I have 4 points, I can find a unique cubic polynomial through them. $y = a_0 + a_1 x + a_2 x^2 + a_3 x^3$

Example

Find a cubic polynomial whose graph passes through the points

$$(1,3), \qquad (2,-2), \qquad (3,-5), \qquad (4,0).$$

We are looking for a function

$$y = a_0 + a_1 x + a_2 x^2 + a_3 x^3$$

which passes through these four points.

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At the point $(x, y) = (1, 3)$ we have

$$a_0 + a_1 + a_2 + a_3 = 3 \qquad (1,3)$$

$$(2,-2)$$

$$(3,-5)$$

$$(4,0)$$

Example

Find a cubic polynomial whose graph passes through the points

$$(1, 3), \qquad (2, -2), \qquad (3, -5), \qquad (4, 0).$$

We are looking for a function

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which passes through these four points.

At the point $(x, y) = (1, 3)$ we have

$$a_0 + a_1 + a_2 + a_3 = 3 \qquad (1, 3)$$

Similarly

$$\begin{aligned} a_0 + a_1 2 + a_2 4 + a_3 8 &= -2 && (2, -2) \\ a_0 + a_1 3 + a_2 9 + a_3 27 &= -5 && (3, -5) \\ a_0 + a_1 4 + a_2 16 + a_3 64 &= 0. && (4, 0) \end{aligned}$$

So we have a linear system with augmented matrix

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 3 \\ 1 & 2 & 4 & 8 & -2 \\ 1 & 3 & 9 & 27 & -5 \\ 1 & 4 & 16 & 64 & 0 \end{bmatrix}$$

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which we can row reduce to

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Hence our function is

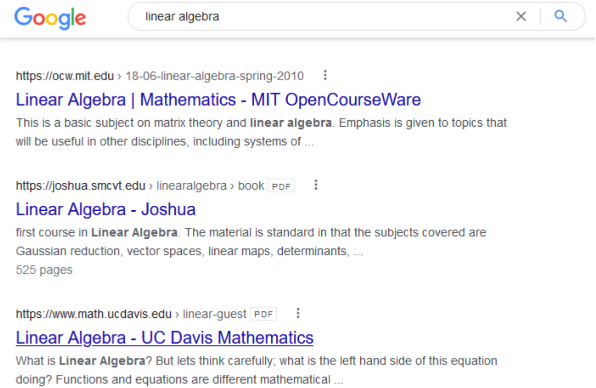
$$y = 4 + 3x - 5x^2 + x^3.$$

There are more examples in your textbook. See sections 1.6 and 1.10.

Google

Search engines, such as Google, rely on linear algebra.

After you finish this course, I encourage you to read section 10.2 in your textbook to understand how Google's PageRank works.



Next Time

- Introduction to Determinants
- Evaluating Determinants by Row Reduction
- Properties of Determinants
- Cramer's Rule