Lecture 10

- 9.1 Sequences

9.1

Sequences

A better book

Mary Hart,
Guide to Analysis,
MacMillan.

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in a particular order.

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in a particular order. Each of the a_j represents a number. These are the *terms* or the sequence. For example, the sequence

$$2, 4, 6, 8, 10, 12, 14, 16, \ldots, 2n, \ldots$$

has first term $a_1 = 2$, second term $a_2 = 4$ and n^{th} term $a_n = 2n$.

We write $(a_n)_{n=1}^{\infty}$ – or sometimes just (a_n) – to denote to the sequence

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If we remove the first four terms, we would get the sequence

$$a_5, a_6, a_7, a_8, a_9, a_{10}, a_{11}, \ldots$$

which we denote by $(a_n)_{n=5}^{\infty}$.

Example

Let $b_n = \frac{1}{n}$ for all $n \in \mathbb{N}$. Then $(b_n)_{n=1}^{\infty}$ is the sequence

$$1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \frac{1}{6}, \ldots$$

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Example

The sequence $\left((-1)^n \frac{1}{n}\right)_{n=1}^{\infty}$ is

$$-1, \frac{1}{2}, -\frac{1}{3}, \frac{1}{4}, -\frac{1}{5}, \frac{1}{6}, \ldots$$

Example

Let $x_n = \cos n\pi$ for all $n \in \mathbb{N}$. Then $(x_n)_{n=1}^{\infty}$ is the sequence

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$$-1, 1, -1, 1, -1, 1, -1, 1, -1, 1, \ldots$$

Example

The sequence $\left(\frac{1}{n^2}\right)_{n=5}^{\infty}$ is

$$\frac{1}{25}, \frac{1}{36}, \frac{1}{49}, \frac{1}{64}, \frac{1}{81}, \ldots$$

Example (The Fibonacci Numbers)

Let $a_1 = 1$, $a_2 = 1$ and $a_n = a_{n-2} + a_{n-1}$ for all $n > 2$.

$$1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, \ldots$$

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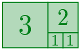

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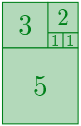

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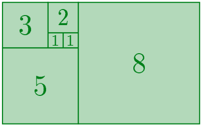

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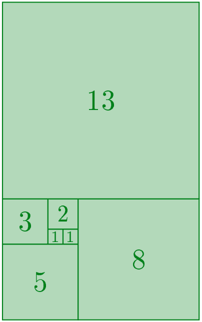

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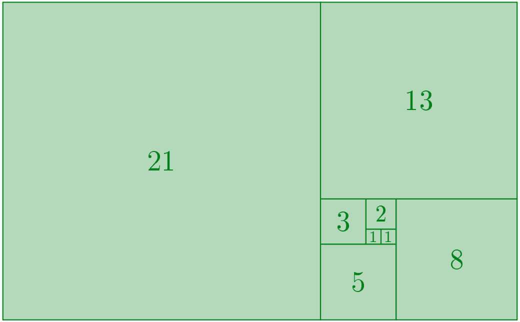

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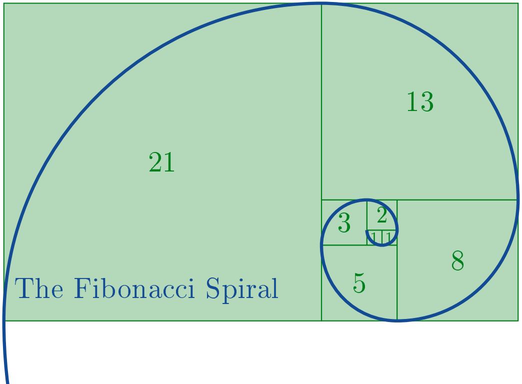

Let (a_n) be a sequence. Note that for every number $n \in \mathbb{N}$, we have a number $a_n \in \mathbb{R}$. So we have a function $\mathbb{N} \to \mathbb{R}$. We use this idea to formally define a sequence:

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Definition

A *sequence* of real numbers is a function $a : \mathbb{N} \to \mathbb{R}$. We could write $a_n := a(n)$ if we wanted to.

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Definition

$\mathbb{R}^{\mathbb{N}} := \{f : \mathbb{N} \to \mathbb{R}\} = \{\text{all sequences of real numbers}\}$.

Remark

One notation for the sequence $\left(\frac{1}{n}\right)_{n=1}^{\infty}$ is

$$\left(1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \frac{1}{6}, \frac{1}{7}, \frac{1}{8}, \frac{1}{9}, \dots\right).$$

Just as $\mathbb{R}^3$ is the set of all vectors (x, y, z), we might expect the set of all sequences to be denoted $\mathbb{R}^\infty$

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More generally, B^A denotes the set of all functions from A to B, but we won't need this in this course.

Remark

Even though sequences are really functions, we will usually think of them as lists of numbers.

For sequences, the important things are:

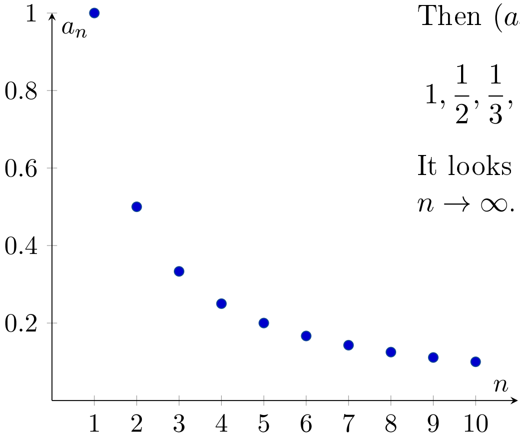
- the order in which the numbers appear;

and

- the behaviour of the terms as $n \to \infty$.

Example

Let $a_n = \frac{1}{n}$ for all $n \in \mathbb{N}$.



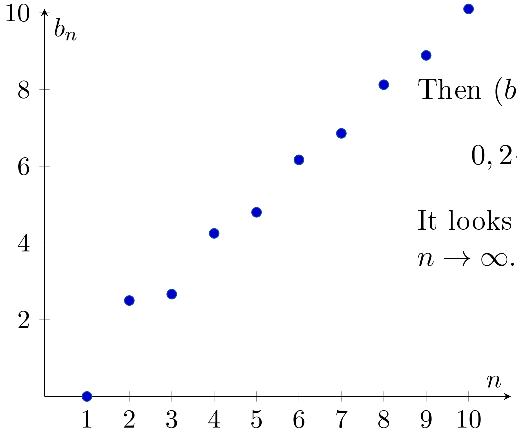
Then $(a_n)_{n=1}^{\infty}$ is the sequence

$$1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \frac{1}{6}, \frac{1}{7}, \frac{1}{8}, \frac{1}{9}, \frac{1}{10}, \ldots$$

It looks like a_n "goes to" 0 as $n \to \infty$.

Example

Let $b_n = n + (-1)^n \frac{1}{n}$ for all $n \in \mathbb{N}$.



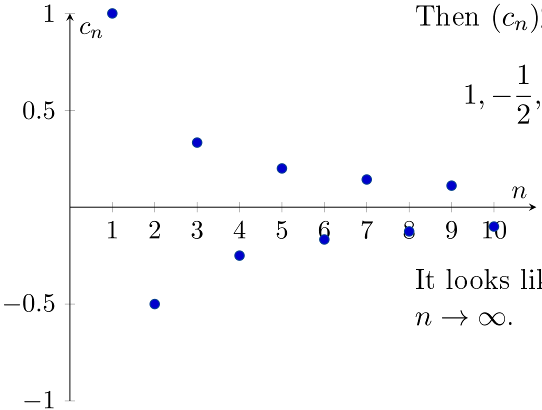
Then $(b_n)_{n=1}^{\infty}$ is the sequence

$$0, 2\tfrac{1}{2}, 2\tfrac{2}{3}, 4\tfrac{1}{4}, 4\tfrac{4}{5}, 6\tfrac{1}{6}, \ldots$$

It looks like b_n "goes to" ∞ as $n \to \infty$.

Example

Let $c_n = \dfrac{(-1)^{n+1}}{n}$ for all $n \in \mathbb{N}$.



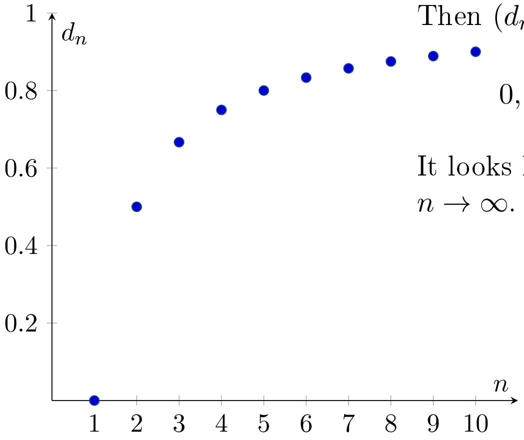
Then $(c_n)_{n=1}^{\infty}$ is the sequence

$$1, -\frac{1}{2}, \frac{1}{3}, -\frac{1}{4}, \frac{1}{5}, -\frac{1}{6}, \ldots$$

It looks like c_n "goes to" 0 as $n \to \infty$.

Example

Let $d_n = 1 - \frac{1}{n}$ for all $n \in \mathbb{N}$.



Then $(d_n)_{n=1}^{\infty}$ is the sequence

$$0, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \frac{5}{6}, \dots$$

It looks like d_n "goes to" 1 as $n \to \infty$.

Definition

The *floor function*, $\lfloor \cdot \rfloor : \mathbb{R} \to \mathbb{Z}$, is defined by

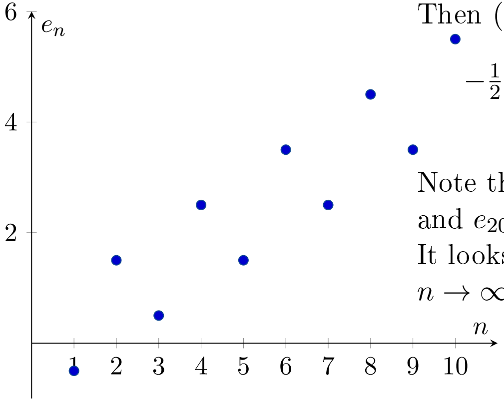
$$\lfloor x \rfloor = \max\{p \in \mathbb{Z} : p \leq x\}.$$

For example

$$\begin{aligned} \lfloor 3.79 \rfloor &= 3 \\ \lfloor 4 \rfloor &= 4 \\ \lfloor -3.79 \rfloor &= -4 \\ \lfloor -4 \rfloor &= -4 \end{aligned}$$

Example

Let $e_n = \left\lfloor \frac{n}{2} \right\rfloor + \frac{(-1)^n}{2}$ for all $n \in \mathbb{N}$.



Then $(e_n)_{n=1}^{\infty}$ is the sequence

$$-\tfrac{1}{2}, \tfrac{3}{2}, \tfrac{1}{2}, \tfrac{5}{2}, \tfrac{3}{2}, \tfrac{7}{2}, \tfrac{5}{2}, \tfrac{9}{2}, \ldots$$

Note that $e_{2000} = 1000 + \frac{1}{2}$ and $e_{2000000} = 1000000 + \frac{1}{2}$. It looks like e_n "goes to" ∞ as $n \to \infty$.

Remark

In these last five examples, we have said "looks like" and "goes to" a lot. But what does this mean mathematically? We need to be more precise.

What does "goes to ∞" really mean?

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What does "goes to ∞" really mean?

It doesn't mean "gets bigger" because $d_n = 1 - \frac{1}{n}$ gets bigger, but we think that d_n "goes to" 1.

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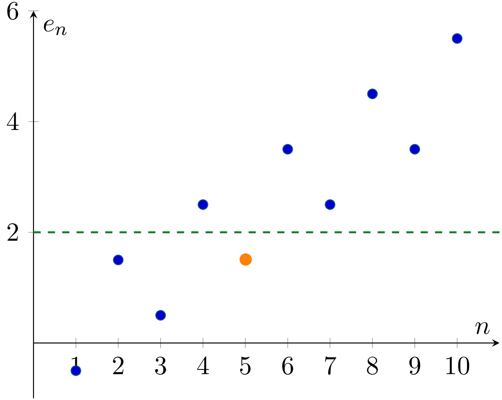
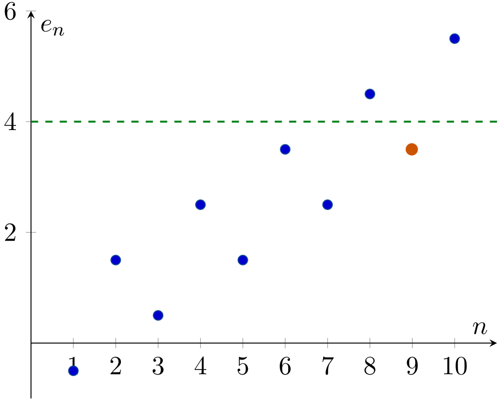
What does "goes to ∞" really mean?

It doesn't mean "gets bigger" because $d_n = 1 - \frac{1}{n}$ gets bigger, but we think that d_n "goes to" 1.

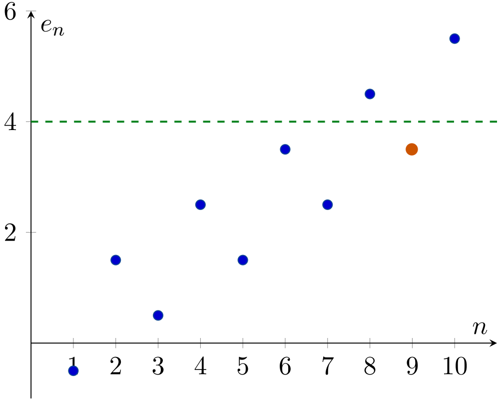
Furthermore, we think that $e_n = \left\lfloor \frac{n}{2} \right\rfloor + \frac{(-1)^n}{2}$ "goes to" ∞, but e_n gets bigger, smaller, bigger, smaller, bigger, smaller, bigger, smaller, ...



Notice that if we draw a green line at height 2, then 4 points at the start of the sequence are underneath the line and all of the rest of the sequence is above the line.



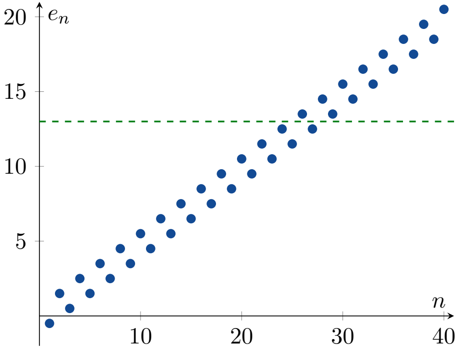
If we draw a green line at height 4, then a finite number of points at the start of the sequence are underneath the line and all of the rest of the sequence is above the line.



If we draw a green line at height 4, then a finite number of points at the start of the sequence are underneath the line and all of the rest of the sequence is above the line.

Now we are getting somewhere.

$\forall$ = "for all" $\exists$ = "there exists"



In general, if I choose any number $A \in \mathbb{R}$ and draw a green line at height A, then there will be a finite number of points underneath the line and an infinite number of points above the line.

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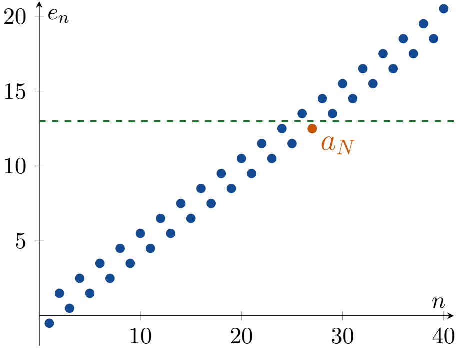

One of the points under the green line must be the last one.
Call this point e_N.

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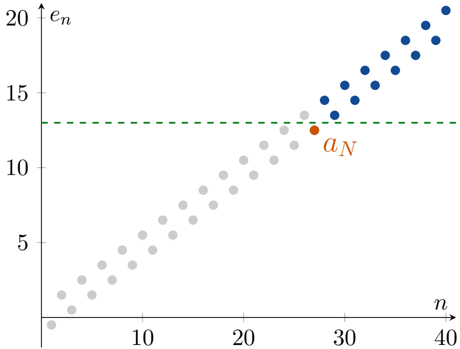

One of the points under the green line must be the last one. Call this point e_N. This means that

$$e_{N+1}, e_{N+2}, e_{N+3}, e_{N+4}, e_{N+5}, \ldots$$

are all above the green line.

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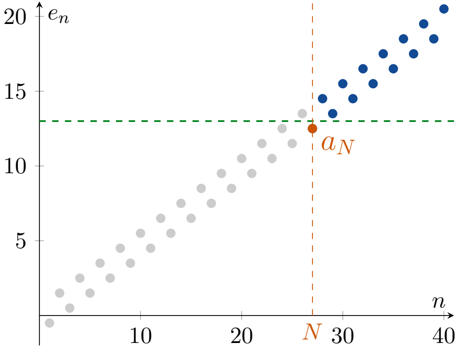

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$$e_{N+1}, e_{N+2}, e_{N+3}, e_{N+4}, e_{N+5}, \ldots$$

are all above the green line. In other words, $\exists\ N \in \mathbb{N}$ such that $e_n > A$ for all $n > N$.

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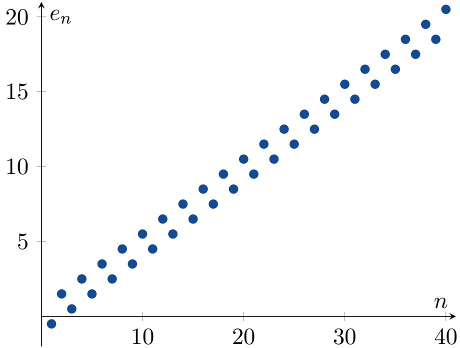

Obviously the number N will depend on A. We will write $N = N(A)$ so that we remember this.

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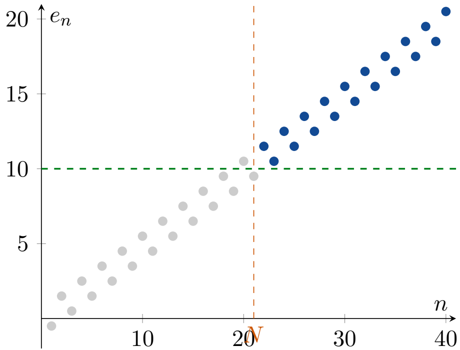

If we choose $A = 10$, then note that

$$n > 21 \implies e_n = \left\lfloor \frac{n}{2} \right\rfloor + \frac{(-1)^n}{2} \geq \left\lfloor \frac{22}{2} \right\rfloor + \frac{(-1)^n}{2} = 11 \pm \frac{1}{2} > 10 = A$$

which means that we can choose $N(10) = 21$.

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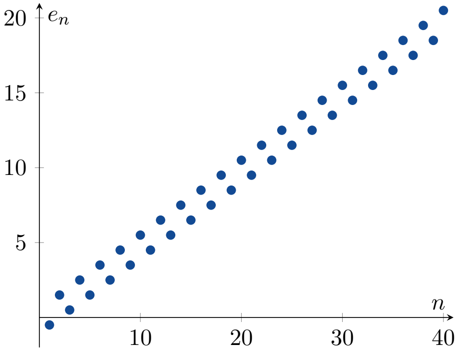

In fact, we don't have to choose the "best" N – any N which works is good enough. So if we wanted to, we could choose $N(10) = 1000000$ and the calculation above still works. If $n > 1000000$, then $e_n > 10$ (check it!!!).

If we choose $A = 100$, then $e_n > 100 = A$ for all $n > 201$ (you check!), so we can choose $N(100) = 201$.

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In general, for any given $A > 0$, we can always find an $N = N(A)$ for the sequence (e_n). If we choose

$$N = \text{“the smallest integer such that } N > 2A + 3\text{”},$$

then

$$\begin{aligned} n > N \quad &\implies \quad n \geq 2A + 4 \\ &\implies \quad e_n = \left\lfloor \frac{n}{2} \right\rfloor + \frac{(-1)^n}{2} \geq \lfloor A + 2 \rfloor - \frac{1}{2} > A. \end{aligned}$$

Definition

A sequence of real numbers (a_n) *diverges to infinity* iff for all $A > 0$, there exists $N = N(A) \in \mathbb{N}$ such that

$$n > N \implies a_n > A.$$

We write "$a_n \to \infty$ as $n \to \infty$" or "$\lim_{n\to\infty} a_n = \infty$" in this case.

$\forall A > 0, \exists N \in \mathbb{N}$ s.t. $n > N \implies a_n > A$

Example

Let $a_n = \sqrt{n}$ for all $n \in \mathbb{N}$. Show that $a_n \to \infty$ as $n \to \infty$.

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$$n > N \implies a_n = \sqrt{n} > \sqrt{N} \geq A.$$

Therefore $a_n \to \infty$ as $n \to \infty$.

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scrap paper

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scrap paper

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We can choose $N \geq e^A$

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Let $b_n = \ln n$ for all $n \in \mathbb{N}$. Show that $b_n \to \infty$ as $n \to \infty$.

Let $A > 0$. Choose $N \in \mathbb{N}$ such that $\underline{\quad N \geq e^A \quad}$. Then for all $n \in \mathbb{N}$,

$$n > N \implies b_n = \ln n > \underline{\qquad\qquad\qquad} A.$$

Therefore $b_n \to \infty$ as $n \to \infty$.

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$$n > N \implies b_n = \ln n > \underline{\ln N \geq \ln e^A =} A.$$

Therefore $b_n \to \infty$ as $n \to \infty$.

$\forall A > 0, \exists N \in \mathbb{N}$ s.t. $n > N \implies c_n > A$

Example

Let

$$c_n = \begin{cases} \frac{n^2\sqrt{n}+n^2+1}{n^2-43} & n \geq 7 \\ 0 & 1 \leq n \leq 6 \end{cases}$$

for all $n \in \mathbb{N}$. Show that $c_n \to \infty$ as $n \to \infty$.

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First note that if $n \geq 7$ then

$$c_n = \frac{n^2\sqrt{n}+n^2+1}{n^2-43} > \frac{n^2\sqrt{n}}{n^2-43} > \frac{n^2\sqrt{n}}{n^2} = \sqrt{n}.$$

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First note that if $n \geq 7$ then

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Let $A > 0$. Choose $N \in \mathbb{N}$ such that $N \geq \max\{A^2, 7\}$. Then for all $n \in \mathbb{N}$,

$$n > N \implies c_n > \sqrt{n} > \sqrt{N} \geq A.$$

Therefore $c_n \to \infty$ as $n \to \infty$.

Remark

Remember that we don't need to find the "best" or smallest N. We only need to find one which works.

Remark

In two of the last three examples it was easy to find an N. In the previous example, we used an inequality first so that finding an N was easier.

Theorem

Let (a_n) and (b_n) be two sequences of real numbers such that

$$a_n \geq b_n$$

for all $n > N_0 \in \mathbb{N}$. If $b_n \to \infty$ as $n \to \infty$, then $a_n \to \infty$ as $n \to \infty$.

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Proof.

Let $A > 0$. Since $b_n \to \infty$ as $n \to \infty$, $\exists\ N_1 \in \mathbb{N}$ such that

$$n > N_1 \implies b_n > A.$$

Theorem

Let (a_n) and (b_n) be two sequences of real numbers such that

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Proof.

Let $A > 0$. Since $b_n \to \infty$ as $n \to \infty$, $\exists\ N_1 \in \mathbb{N}$ such that

$$n > N_1 \implies b_n > A.$$

Choose $N = \max\{N_0, N_1\}$.

Theorem

Let (a_n) and (b_n) be two sequences of real numbers such that

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Proof.

Let $A > 0$. Since $b_n \to \infty$ as $n \to \infty$, $\exists\ N_1 \in \mathbb{N}$ such that

$$n > N_1 \implies b_n > A.$$

Choose $N = \max\{N_0, N_1\}$. Then

$$n > N \implies a_n \geq b_n > A.$$

Therefore $a_n \to \infty$ as $n \to \infty$. $\square$

Example

Let $a_n = n^2 + n\cos n\pi$ and $b_n = \frac{1}{2}n^2$.

Let $A > 0$. Choose $N \geq \sqrt{2A}$. Then

$$n > N \implies b_n = \tfrac{1}{2}n^2 > \tfrac{1}{2}N^2 \geq A.$$

Hence $b_n \to \infty$ as $n \to \infty$.

Example

Let $a_n = n^2 + n\cos n\pi$ and $b_n = \frac{1}{2}n^2$.

Let $A > 0$. Choose $N \geq \sqrt{2A}$. Then

$$n > N \implies b_n = \tfrac{1}{2}n^2 > \tfrac{1}{2}N^2 \geq A.$$

Hence $b_n \to \infty$ as $n \to \infty$.

Moreover, if $n \geq 2$ then

$$a_n = n^2 + n\cos n\pi = n^2 + n(-1)^n \geq n^2 - n \geq n^2 - \tfrac{1}{2}n^2 = \tfrac{1}{2}n^2 = b_n.$$

Therefore $a_n \to \infty$ as $n \to \infty$ by the theorem.

Example

Let $a_n := \frac{n^2+\sqrt{n}}{n+\cos n}$. Show that $a_n \to \infty$ as $n \to \infty$.

If $n \geq 2$, then

$$a_n < \frac{n^2}{n+\cos n} \geq \frac{n^2}{n+1} > \frac{n^2}{n+n} = \tfrac{1}{2}n.$$

Now let $b_n = \frac{1}{2}n$. Since $b_n \to \infty$ as $n \to \infty$ (you check!!!) and since $a_n > b_n$ for all $n \geq 2$, if follows by the theorem that $a_n \to \infty$ as $n \to \infty$.

Definition

A sequence of real numbers a_n *diverges to minus infinity* ($a_n \to -\infty$ as $n \to \infty$) iff for all $A > 0$, there exists $N = N(A) \in \mathbb{N}$ such that

$$n > N \implies a_n < -A.$$

Remark

Prove that

$$a_n \to -\infty \text{ as } n \to \infty \iff -a_n \to \infty \text{ as } n \to \infty.$$

(you prove)

Theorem

Let (a_n) and (b_n) be two sequences of real numbers such that

$$a_n \leq b_n$$

for all $n > N_0 \in \mathbb{N}$. If $b_n \to -\infty$ as $n \to \infty$, then $a_n \to -\infty$ as $n \to \infty$.

Theorem

Let (a_n) and (b_n) be two sequences of real numbers such that

$$a_n \leq b_n$$

for all $n > N_0 \in \mathbb{N}$. If $b_n \to -\infty$ as $n \to \infty$, then $a_n \to -\infty$ as $n \to \infty$.

Proof.

Since $a_n \leq b_n \ \forall \ n \geq N_0$, it follows that $-a_n \geq -b_n \ \forall \ n \geq N_0$. Thus

$$\begin{aligned} b_n \to -\infty \text{ as } n \to \infty \quad &\Longrightarrow \quad -b_n \to \infty \text{ as } n \to \infty \\ &\Longrightarrow \quad -a_n \to \infty \text{ as } n \to \infty \\ &\Longrightarrow \quad a_n \to -\infty \text{ as } n \to \infty. \end{aligned}$$

$\square$

9.1 Sequences

Now let $y_n = \frac{(-1)^{n+1}}{n}$ for all $n \in \mathbb{N}$. Then $(y_n)_{n=1}^{\infty}$ is the sequence

$$1, -\tfrac{1}{2}, \tfrac{1}{3}, -\tfrac{1}{4}, \tfrac{1}{5}, -\tfrac{1}{6}, \tfrac{1}{7}, -\tfrac{1}{8}, \tfrac{1}{9}, \ldots$$

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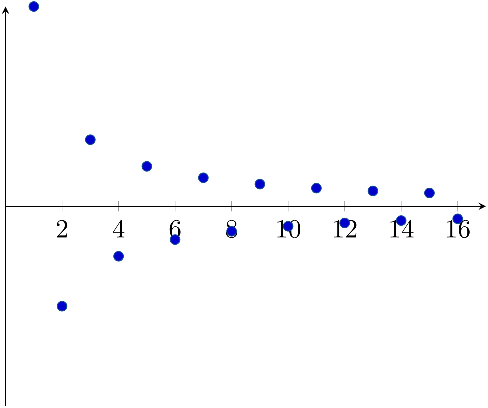

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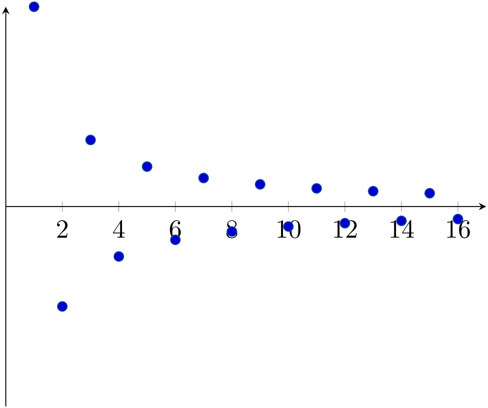
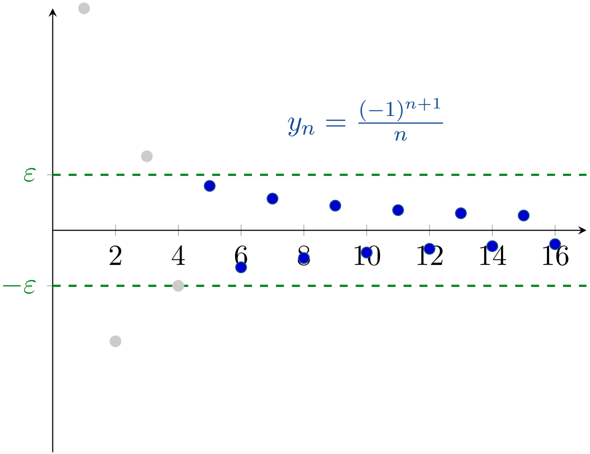


It "looks like" y_n "goes to" 0 as $n \to \infty$. But what does this mean? How can we be more precise?



If we draw green lines at heights ε and $-\varepsilon$, then (apart from a finite number of points) the sequence will be between the two green lines.

$$y_n = \frac{(-1)^{n+1}}{n}$$

E.g. Let $\varepsilon = \frac{1}{100}$. Then $-\frac{1}{100} < y_n < \frac{1}{100}$ for all $n > 100$.

$$y_n = \frac{(-1)^{n+1}}{n}$$

E.g. Let $\varepsilon = \frac{1}{100}$. Then $-\frac{1}{100} < y_n < \frac{1}{100}$ for all $n > 100$.

In general: If we choose $N \geq \frac{1}{\varepsilon}$, then

$$n > N \implies -\varepsilon < y_n < \varepsilon.$$

In other words, if $N \geq \frac{1}{\varepsilon}$ then

$$n > N \implies |y_n| < \varepsilon.$$

Definition

A sequence of real numbers a_n *tends to zero* ($a_n \to 0$ as $n \to \infty$) iff for all $\varepsilon > 0$, there exists $N = N(\varepsilon) \in \mathbb{N}$ such that

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A sequence of real numbers a_n *tends to zero* ($a_n \to 0$ as $n \to \infty$) iff for all $\varepsilon > 0$, there exists $N = N(\varepsilon) \in \mathbb{N}$ such that

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Remark

In Mathematics; we usually use ε for arbitrarily small numbers and a capital letter (e.g. A) for arbitrarily large numbers.

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$$n > N \implies |a_n| < \varepsilon.$$

Remark

In Mathematics; we usually use ε for arbitrarily small numbers and a capital letter (e.g. A) for arbitrarily large numbers.

Definition

A sequence of real numbers a_n is called a *null sequence* iff $a_n \to 0$ as $n \to \infty$.

$$\forall \varepsilon > 0,\ \exists N \in \mathbb{N} \text{ s.t. } n > N \implies |a_n| < \varepsilon$$

Example

Let $a_n = n^{-7}$ for all $n \in \mathbb{N}$. Show that (a_n) is a null sequence.

We have to show that $a_n \to 0$ as $n \to \infty$.

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Let $\varepsilon > 0$. Choose __________. Then for all $n \in \mathbb{N}$,

$$n > N \implies |a_n| = \left|\frac{1}{n^7}\right| \underline{\hspace{2cm}} < \underline{\hspace{2cm}} \varepsilon.$$

Therefore (a_n) is a null sequence.

$\forall \varepsilon > 0, \exists N \in \mathbb{N}$ s.t. $n > N \implies |a_n| < \varepsilon$

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scrap paper

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Let $a_n = n^{-7}$ for all $n \in$ e.

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Example

Let $a_n = n^{-7}$ for all $n \in$ … e.

We have to show that a_n …

Let $\varepsilon > 0$. Choose ______

$n > N \implies$

Therefore (a_n) is a null …

scrap paper

$$\left|\frac{1}{n^7}\right| < \varepsilon$$

$$\frac{1}{n^7} < \varepsilon$$

$$n^7 > \frac{1}{\varepsilon}$$

$$n > \frac{1}{\varepsilon^{\frac{1}{7}}}$$

$\forall \varepsilon > 0, \exists N \in \mathbb{N}$ s.t. $n > N \implies |a_n| < \varepsilon$

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scrap paper

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$$n > \frac{1}{\varepsilon^{\frac{1}{7}}}$$

We can choose $N \geq \varepsilon^{-\frac{1}{7}}$

$\forall \varepsilon > 0, \exists N \in \mathbb{N}$ s.t. $n > N \implies |a_n| < \varepsilon$

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Let $a_n = n^{-7}$ for all $n \in \mathbb{N}$. Show that (a_n) is a null sequence.

We have to show that $a_n \to 0$ as $n \to \infty$.

Let $\varepsilon > 0$. Choose $\underline{\quad N \geq \varepsilon^{-\frac{1}{7}} \quad}$. Then for all $n \in \mathbb{N}$,

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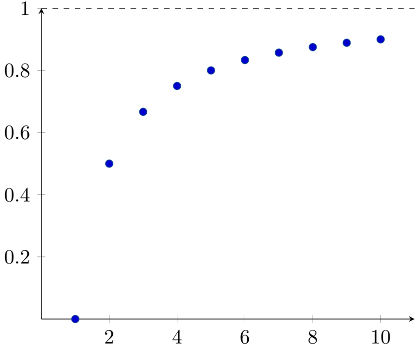
$$n > N \implies |a_n| = \left| \frac{1}{n^7} \right| \underline{= \frac{1}{n^7}} < \underline{\frac{1}{N^7} \leq} \varepsilon.$$

Therefore (a_n) is a null sequence.

Example

Let $z_n = 1 - \frac{1}{n}$ for all $n \in \mathbb{N}$. Then (z_n) is the sequence

$$0, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \frac{5}{6}, \ldots$$



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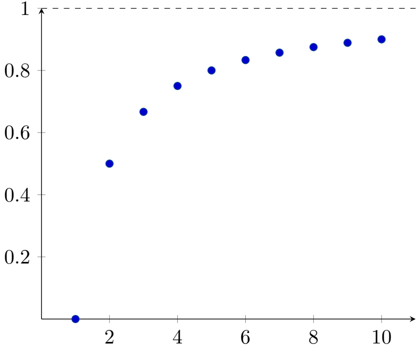

It "looks like" z_n "goes to" 1 as $n \to \infty$. But this is Mathematics, so we need to be precise.

Definition

A sequence of real numbers a_n *tends to* l ($a_n \to l$ as $n \to \infty$) iff for all $\varepsilon > 0$, there exists $N = N(\varepsilon) \in \mathbb{N}$ such that

$$n > N \implies |a_n - l| < \varepsilon.$$

Definition

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$$n > N \implies |a_n - l| < \varepsilon.$$

Remark

We can also write $\lim_{n\to\infty} a_n = l$ if a_n tends to l.

Example

Let $u_n = \begin{cases} 7 & n \geq 7 \\ n & 1 \leq n \leq 6. \end{cases}$ Show that $u_n \to 7$ as $n \to \infty$.

Let $\varepsilon > 0$. Choose $N = 6$. Then

$$n > N \implies n \geq 7 \implies u_n = 7 \implies |u_n - 7| = 0 < \varepsilon.$$

Therefore $u_n \to 7$ as $n \to \infty$.

Example

Let $v_n = \frac{n^2+n+1}{2n^2+1}$ for all $n \in \mathbb{N}$. Show that $v_n \to \frac{1}{2}$ as $n \to \infty$.

Let $\varepsilon > 0$. First note that

$$v_n - \frac{1}{2} = \left(\frac{n^2+n+1}{2n^2+1}\right) - \frac{1}{2} = \left(\frac{n^2+n+1}{2n^2+1}\right) - \left(\frac{n^2+\frac{1}{2}}{2n^2+1}\right) = \frac{2n+1}{2(2n^2+1)}.$$

So

$$\left|v_n - \tfrac{1}{2}\right| < \frac{2n+1}{4n^2} \leq \frac{2n+n}{4n^2} = \frac{3}{4n}.$$

Now choose $N > \frac{3}{4\varepsilon}$. Then for all $n \in \mathbb{N}$,

$$n > N \implies \left|v_n - \tfrac{1}{2}\right| < \frac{3}{4n} < \frac{3}{4N} < \varepsilon.$$

Therefore $v_n \to \frac{1}{2}$ as $n \to \infty$.

Example

Define a sequence (p_n) by

$$\begin{aligned} p_1 &= 3 \\ p_2 &= 3.1 \\ p_3 &= 3.14 \\ p_4 &= 3.141 \\ p_5 &= 3.1415 \\ &\vdots \\ p_n &= \text{the first } n \text{ digits of } \pi \\ &\vdots \end{aligned}$$

Show that $p_n \to \pi$ as $n \to \infty$.

First note that

$$
\begin{aligned}
|p_1 - \pi| &= 0.141592\ldots < 1 = 10^0 \\
|p_2 - \pi| &= 0.041592\ldots < 0.1 = 10^{-1} \\
|p_3 - \pi| &= 0.001592\ldots < 0.01 = 10^{-2} \\
&\vdots \\
|p_n - \pi| &< 10^{1-n} \\
&\vdots
\end{aligned}
$$

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Let $\varepsilon > 0$. Choose $N > 1 - \log_{10} \varepsilon$.

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&\vdots
\end{aligned}
$$

Let $\varepsilon > 0$. Choose $N > 1 - \log_{10}\varepsilon$. Then for all $n \in \mathbb{N}$,

$$n > N \implies |p_n - \pi| < 10^{1-n} < 10^{1-N} < 10^{1-(1-\log_{10}\varepsilon)} = \varepsilon.$$

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$$n > N \implies |p_n - \pi| < 10^{1-n} < 10^{1-N} < 10^{1-(1-\log_{10}\varepsilon)} = \varepsilon.$$

Therefore $p_n \to \pi$ as $n \to \infty$.

Theorem

A sequence of real numbers cannot have more than one limit.

Proof.

Let (a_n) be a sequence.

CASE 1: Suppose first that $a_n \to l \in \mathbb{R}$ and $a_n \to m \in \mathbb{R}$ as $n \to \infty$. We will use proof by contradiction to prove that $l = m$: Assume that $l \neq m$. Then $l - m \neq 0$ and $|l - m| > 0$. Let $\varepsilon = \frac{1}{2}|l - m| > 0$.

Proof.

Let (a_n) be a sequence.

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Since $a_n \to l$ as $n \to \infty$, $\exists N_0 \in \mathbb{N}$ such that

$$n > N_0 \implies |a_n - l| < \varepsilon.$$

Similarly, since $a_n \to m$ as $n \to \infty$, $\exists N_2 \in \mathbb{N}$ such that

$$n > N_0 \implies |a_n - m| < \varepsilon.$$

.

Let $N = \max\{N_0, N_1\}$. Then $\forall\ n > N$ we have that

$$\begin{aligned} |l - m| &= |l - a_n + a_n - m| \leq |l - a_n| + |a_n - m| \\ &= |a_n - l| + |a_n - m| < \varepsilon + \varepsilon = |l - m| \end{aligned}$$

by the triangle inequality.

.

Let $N = \max\{N_0, N_1\}$. Then $\forall\ n > N$ we have that

$$\begin{aligned}|l-m| &= |l-a_n+a_n-m| \leq |l-a_n| + |a_n-m| \\ &= |a_n-l| + |a_n-m| < \varepsilon + \varepsilon = |l-m|\end{aligned}$$

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But $|l-m| < |l-m|$ is a contradiction.

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&= |a_n - l| + |a_n - m| < \varepsilon + \varepsilon = |l - m|
\end{aligned}
$$

by the triangle inequality.

But $|l - m| < |l - m|$ is a contradiction. Since $l \neq m$ leads to a contradiction, we must have $l = m$. This means that a sequence cannot have two different finite limits.

.

CASE 2: Moreover, if $a_n \to l$ as $n \to \infty$, then $\exists N \in \mathbb{N}$ such that

$$n > N \implies |a_n - l| < 1 \implies l - 1 < a_n < l + 1.$$

Hence $a_n \not\to \infty$ and $a_n \not\to -\infty$ as $n \to \infty$. Therefore a sequence cannot have both a finite limit and an infinite limit.

CASE 2: Moreover, if $a_n \to l$ as $n \to \infty$, then $\exists N \in \mathbb{N}$ such that

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Hence $a_n \not\to \infty$ and $a_n \not\to -\infty$ as $n \to \infty$. Therefore a sequence cannot have both a finite limit and an infinite limit.

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CASE 3: Finally, I leave it for you to prove that a_n cannot tend to both ∞ and $-\infty$.

Therefore a sequence cannot have two different limits. $\square$

$$\forall \varepsilon > 0,\ \exists N \in \mathbb{N} \text{ s.t. } n > N \implies |a_n| < \varepsilon$$

Example

Let (z_n) be the sequence

$$1, 0, 1, 0, 1, 0, 1, 0, 1, 0, 1, 0, \ldots$$

Show that $z_n \not\to 0$ as $n \to \infty$.

$$\forall \varepsilon > 0, \exists N \in \mathbb{N} \text{ s.t. } n > N \implies |a_n| < \varepsilon$$

Example

Let (z_n) be the sequence

$$1, 0, 1, 0, 1, 0, 1, 0, 1, 0, 1, 0, \ldots$$

Show that $z_n \not\to 0$ as $n \to \infty$.

Choose $\varepsilon = \frac{1}{2}$.

$$\forall \varepsilon > 0, \exists N \in \mathbb{N} \text{ s.t. } n > N \implies |a_n| < \varepsilon$$

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Choose $\varepsilon = \frac{1}{2}$. Let N be any natural number.

Example

Let (z_n) be the sequence

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Show that $z_n \not\to 0$ as $n \to \infty$.

Choose $\varepsilon = \frac{1}{2}$. Let N be any natural number. If N is odd, choose $n = N + 2$. If N is even, choose $n = N + 1$. Then clearly $n > N$.

$\forall \varepsilon > 0, \exists N \in \mathbb{N}$ s.t. $n > N \implies |a_n| < \varepsilon$

Example

Let (z_n) be the sequence

$$1, 0, 1, 0, 1, 0, 1, 0, 1, 0, 1, 0, \ldots$$

Show that $z_n \not\to 0$ as $n \to \infty$.

Choose $\varepsilon = \frac{1}{2}$. Let N be any natural number. If N is odd, choose $n = N + 2$. If N is even, choose $n = N + 1$. Then clearly $n > N$. Since n is odd, we have that

$$|z_n| = 1 \geq \frac{1}{2} = \varepsilon.$$

Therefore $z_n \not\to 0$ as $n \to \infty$.

Definition

A sequence of real numbers (a_n) is called a *convergent sequence* iff $\exists\ l \in \mathbb{R}$ such that $a_n \to l$ as $n \to \infty$.

Remark

We know from a theorem that a convergent sequence has only one limit.

Definition

A sequence of real numbers (a_n) is called a *convergent sequence* iff $\exists\ l \in \mathbb{R}$ such that $a_n \to l$ as $n \to \infty$.

Remark

We know from a theorem that a convergent sequence has only one limit.

Definition

A sequence which is not convergent is called a *divergent sequence.*

Definition

If $a_n \to \infty$ as $n \to \infty$, we say that (a_n) *diverges to infinity* (sonsuza ıraksar).

Definition

If $a_n \to -\infty$ as $n \to \infty$, we say that (a_n) *diverges to minus infinity*.

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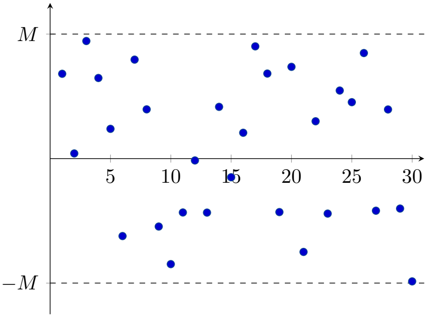
Definition

If $a_n \to -\infty$ as $n \to \infty$, we say that (a_n) *diverges to minus infinity*.

Example

Let $a_n = (-1)^n$, $b_n = (-1)^n n$ and $c_n = n^2$. Then (a_n), (b_n) and (c_n) are divergent sequences. (a_n) and (b_n) do not have a finite limit or an infinite limit. (c_n) diverges to infinity.



Definition

A sequence of real numbers (a_n) is called a *bounded sequence* (sınırlı dizi) iff $\exists\ M > 0$ such that $|a_n| \leq M$ for all $n \in \mathbb{N}$.

Theorem

Every convergent sequence is bounded.

Proof.

Let (a_n) be a convergent sequence.

Theorem

Every convergent sequence is bounded.

Proof.

Let (a_n) be a convergent sequence. Then $\exists\ a \in \mathbb{R}$ such that $a_n \to a$ as $n \to \infty$.

Theorem

Every convergent sequence is bounded.

Proof.

Let (a_n) be a convergent sequence. Then $\exists\ a \in \mathbb{R}$ such that $a_n \to a$ as $n \to \infty$. So $\exists\ N \in \mathbb{N}$ such that

$$n > N \implies |a_n - a| < 1 \implies |a_n| < |a| + 1.$$

Theorem

Every convergent sequence is bounded.

Proof.

Let (a_n) be a convergent sequence. Then $\exists\ a \in \mathbb{R}$ such that $a_n \to a$ as $n \to \infty$. So $\exists\ N \in \mathbb{N}$ such that

$$n > N \implies |a_n - a| < 1 \implies |a_n| < |a| + 1.$$

Now let $M := \max\{|a_1|, |a_2|, |a_3|, \dots, |a_N|, |a| + 1\}$.

Theorem

Every convergent sequence is bounded.

Proof.

Let (a_n) be a convergent sequence. Then $\exists\ a \in \mathbb{R}$ such that $a_n \to a$ as $n \to \infty$. So $\exists\ N \in \mathbb{N}$ such that

$$n > N \implies |a_n - a| < 1 \implies |a_n| < |a| + 1.$$

Now let $M := \max\{|a_1|, |a_2|, |a_3|, \ldots, |a_N|, |a| + 1\}$. Then $|a_n| \leq M$ for all $n \in \mathbb{N}$. Therefore (a_n) is a bounded sequence. $\square$

Break

We will continue at 2pm

Rules

- The Sum Rule
- The Constant Multiple Rule
- The Product Rule
- The Quotient Rule
- The Sandwich Rule

Lemma (Sum Rule)

Suppose that $a_n \to a \in \mathbb{R}$ *and* $b_n \to b \in \mathbb{R}$ *as* $n \to \infty$. *Then* $a_n + b_n \to a + b$ *as* $n \to \infty$.

Lemma (Sum Rule)

Suppose that $a_n \to a \in \mathbb{R}$ *and* $b_n \to b \in \mathbb{R}$ *as* $n \to \infty$. *Then* $a_n + b_n \to a + b$ *as* $n \to \infty$.

Proof.

Let $\varepsilon > 0$. Since $a_n \to a \in \mathbb{R}$ and $b_n \to b \in \mathbb{R}$ as $n \to \infty$, $\exists N \in \mathbb{N}$ such that

$$n > N \implies a - \tfrac{\varepsilon}{2} < a_n < a + \tfrac{\varepsilon}{2} \qquad \text{and} \qquad b - \tfrac{\varepsilon}{2} < b_n < b + \tfrac{\varepsilon}{2}.$$

Adding these inequalities together, we see that

$$n > N \implies a + b - \varepsilon < a_n + b_n < a + b + \varepsilon.$$

Therefore $a_n + b_n \to a + b$ as $n \to \infty$. $\square$

Lemma (Constant Multiple Rule)

Suppose that $k \in \mathbb{R}$ *and* $a_n \to a \in \mathbb{R}$ *as* $n \to \infty$. *Then* $ka_n \to ka$ *as* $n \to \infty$.

Lemma (Constant Multiple Rule)

Suppose that $k \in \mathbb{R}$ and $a_n \to a \in \mathbb{R}$ as $n \to \infty$. Then $ka_n \to ka$ as $n \to \infty$.

(you prove)

Lemma (Product Rule)

Suppose that $a_n \to a \in \mathbb{R}$ *and* $b_n \to b \in \mathbb{R}$ *as* $n \to \infty$. *Then* $a_n b_n \to ab$ *as* $n \to \infty$.

Proof.

Let $\varepsilon > 0$. First

$$b_n \to b \text{ as } n \to \infty \implies (b_n) \text{ is convergent } \implies (b_n) \text{ is bounded}$$

by a theorem. So $\exists M > 0$ such that $|b_n| \leq M \ \forall n \in \mathbb{N}$. Note that $\frac{\varepsilon}{M+|a|} > 0$.

.

Since $a_n \to a$ and $b_n \to b$ as $n \to \infty$, $\exists N \in \mathbb{N}$ such that

$$n > N \implies |a_n - a| < \frac{\varepsilon}{M + |a|} \qquad \text{and} \qquad |b_n - b| < \frac{\varepsilon}{M + |a|}.$$

.

Since $a_n \to a$ and $b_n \to b$ as $n \to \infty$, $\exists N \in \mathbb{N}$ such that

$$n > N \implies |a_n - a| < \frac{\varepsilon}{M + |a|} \qquad \text{and} \qquad |b_n - b| < \frac{\varepsilon}{M + |a|}.$$

But then

$$\begin{aligned} n > N \implies |a_n b_n - ab| &= |a_n b_n - ab_n + ab_n - ab| \\ &\leq |a_n b_n - ab_n| + |ab_n - ab| \\ &= |a_n - a|\,|b_n| + |a|\,|b_n - b| \\ &< \frac{\varepsilon}{M + |a|} M + |a| \frac{\varepsilon}{M + |a|} = \varepsilon. \end{aligned}$$

Therefore $a_n b_n \to ab$ as $n \to \infty$. $\square$

Lemma (Quotient Rule)

Suppose that $a_n \to a \in \mathbb{R}$ *and* $b_n \to b \in \mathbb{R}$ *as* $n \to \infty$. *Suppose that* $b \neq 0$. *Then* $\frac{a_n}{b_n} \to \frac{a}{b}$ *as* $n \to \infty$.

(proof omitted)

Example

Let $a_n = \dfrac{n^5 + 7n^3 + 5n^2 + 8}{5n^5 + 3n^4 + 27}$.

Then

$$\begin{aligned} a_n = \frac{n^5 + 7n^3 + 5n^2 + 8}{5n^5 + 3n^4 + 27} &= \frac{1 + 7n^{-2} + 5n^{-3} + 8n^{-5}}{5 + 3n^{-1} + 27n^{-5}} \\ \to \frac{1 + 0 + 0 + 0}{5 + 0 + 0} &= \frac{1}{5} \end{aligned}$$

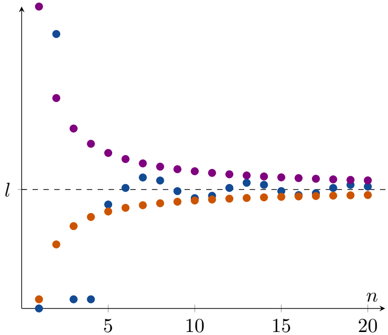
as $n \to \infty$.

Theorem (The Sandwich Rule)

Let (a_n), (b_n) *and* (c_n) *be three sequences of real numbers such that*

$$a_n \leq b_n \leq c_n$$

for all $n > N_0 \in \mathbb{N}$.

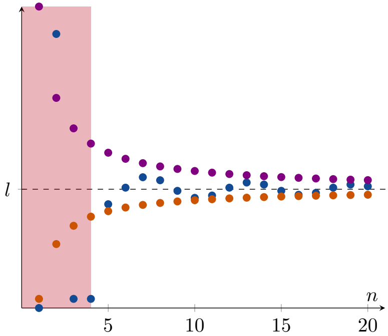


Theorem (The Sandwich Rule)

Let (a_n), (b_n) and (c_n) be three sequences of real numbers such that

$$a_n \leq b_n \leq c_n$$

for all $n > N_0 \in \mathbb{N}$. If $a_n \to l$ and $c_n \to l$ as $n \to \infty$, then $b_n \to l$ as $n \to \infty$ also.



Proof.

Let $\varepsilon > 0$.

Proof.

Let $\varepsilon > 0$. Since $a_n \to l$ as $n \to \infty$, $\exists N_1 \in \mathbb{N}$ such that

$$n > N_1 \implies l - \varepsilon < a_n < l + \varepsilon.$$

Proof.

Let $\varepsilon > 0$. Since $a_n \to l$ as $n \to \infty$, $\exists N_1 \in \mathbb{N}$ such that

$$n > N_1 \implies l - \varepsilon < a_n < l + \varepsilon.$$

Since $c_n \to l$ as $n \to \infty$, $\exists N_2 \in \mathbb{N}$ such that

$$n > N_2 \implies l - \varepsilon < c_n < l + \varepsilon.$$

Proof.

Let $\varepsilon > 0$. Since $a_n \to l$ as $n \to \infty$, $\exists N_1 \in \mathbb{N}$ such that

$$n > N_1 \implies l - \varepsilon < a_n < l + \varepsilon.$$

Since $c_n \to l$ as $n \to \infty$, $\exists N_2 \in \mathbb{N}$ such that

$$n > N_2 \implies l - \varepsilon < c_n < l + \varepsilon.$$

Let $N = \max\{N_0, N_1, N_2\}$.

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Let $\varepsilon > 0$. Since $a_n \to l$ as $n \to \infty$, $\exists N_1 \in \mathbb{N}$ such that

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$$n > N_2 \implies l - \varepsilon < c_n < l + \varepsilon.$$

Let $N = \max\{N_0, N_1, N_2\}$. Then

$$n > N \implies l - \varepsilon < a_n \leq b_n \leq c_n < l + \varepsilon.$$

Proof.

Let $\varepsilon > 0$. Since $a_n \to l$ as $n \to \infty$, $\exists N_1 \in \mathbb{N}$ such that

$$n > N_1 \implies l - \varepsilon < a_n < l + \varepsilon.$$

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$$n > N_2 \implies l - \varepsilon < c_n < l + \varepsilon.$$

Let $N = \max\{N_0, N_1, N_2\}$. Then

$$n > N \implies l - \varepsilon < a_n \leq b_n \leq c_n < l + \varepsilon.$$

Therefore $b_n \to l$ as $n \to \infty$. $\square$

Theorem

Let (c_n) be a sequence of real numbers such that

$$c_n \geq 0$$

for all $n > N_0 \in \mathbb{N}$. Suppose that $c_n \to c$ as $n \to \infty$. Then $c \geq 0$.

(proof omitted)

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for all $n > N_0 \in \mathbb{N}$. Suppose that $c_n \to c$ as $n \to \infty$. Then $c \geq 0$.

(proof omitted)

Corollary

Let (a_n) and (b_n) be sequences of real numbers such that

$$a_n \leq b_n$$

for all $n > N_0 \in \mathbb{N}$. Suppose that $a_n \to a$ and $b_n \to b$ as $n \to \infty$. Then $a \leq b$.

Remark

So

$$a_n \leq b_n \implies \lim_{n\to\infty} a_n \leq \lim_{n\to\infty} b_n$$

if the limits exist. But is

$$\text{“}a_n < b_n \implies \lim_{n\to\infty} a_n < \lim_{n\to\infty} b_n.\text{”}$$

true?

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$$\text{“}a_n < b_n \implies \lim_{n\to\infty} a_n < \lim_{n\to\infty} b_n.\text{”}$$

true? The answer is NO!!!!

Example

Let $a_n = \frac{1}{n}$ and $b_n = \frac{1}{n^2}$. Then $a_n < b_n$ for all $n > 1$, but

$$\lim_{n\to\infty} a_n = 0 = \lim_{n\to\infty} b_n.$$

Remark

Be careful when taking limits of inequalities!

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$$a_n < b_n \implies \lim_{n\to\infty} a_n \leq \lim_{n\to\infty} b_n$$

if the limits exist.

Theorem

Let (a_n) be a sequence. If $a_n \to \infty$ as $n \to \infty$, then $\frac{1}{a_n} \to 0$ as $n \to \infty$.

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Proof.

Let $\varepsilon > 0$. Then let $A = \frac{1}{\varepsilon} > 0$. Since $a_n \to \infty$ as $n \to \infty$, $\exists N \in \mathbb{N}$ such that

$$n > N \implies a_n > A.$$

So

$$n > N \implies 0 < \frac{1}{a_n} < \frac{1}{A} = \varepsilon.$$

Therefore $\frac{1}{a_n} \to 0$ as $n \to \infty$. □

Example

Since

$$0 \leftarrow -\frac{1}{n} \leq \frac{\cos n}{n} \leq \frac{1}{n} \rightarrow 0$$

as $n \to \infty$, it follows by the Sandwich Rule that $\frac{\cos n}{n} \to 0$ as $n \to \infty$.

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as $n \to \infty$, it follows by the Sandwich Rule that $\frac{\cos n}{n} \to 0$ as $n \to \infty$.

Example

$$\frac{n^5 + n^4 \cos n + 6}{4n^5 + n^3 + \cos n} = \frac{1 + \frac{\cos n}{n} + 6n^{-5}}{4 + \frac{1}{n} + \frac{\cos n}{n^5}} \rightarrow \frac{1+0+0}{4+0+0} = \frac{1}{4}$$

as $n \to \infty$.

Limits of Standard Sequences

- n^{α}
- a^n
- $a^{\frac{1}{n}}$
- $n^{\frac{1}{n}}$
- $n!$
- n^n

Lemma

Let $\alpha \in \mathbb{R}$. *Then*

$$n^{\alpha} \to \begin{cases} \infty & \text{if } \alpha > 0 \\ 1 & \text{if } \alpha = 0 \\ 0 & \text{if } \alpha < 0 \end{cases}$$

as $n \to \infty$.

Lemma

Let $\alpha \in \mathbb{R}$. *Then*

$$n^\alpha \to \begin{cases} \infty & \textit{if } \alpha > 0 \\ 1 & \textit{if } \alpha = 0 \\ 0 & \textit{if } \alpha < 0 \end{cases}$$

as $n \to \infty$.

Proof.

CASE 1 ($\alpha > 0$): Let $A > 0$. Choose N such that $\alpha \ln N \geq \ln A$. Then

$$n > N \implies n^\alpha > N^\alpha = e^{\ln N^\alpha} = e^{\alpha \ln N} \geq e^{\ln A} = A.$$

So $n^\alpha \to \infty$ as $n \to \infty$.

Lemma

Let $\alpha \in \mathbb{R}$. Then

$$n^\alpha \to \begin{cases} \infty & \text{if } \alpha > 0 \\ 1 & \text{if } \alpha = 0 \\ 0 & \text{if } \alpha < 0 \end{cases}$$

as $n \to \infty$.

Proof.

CASE 2 ($\alpha = 0$): Clearly $n^\alpha = n^0 = 1\ \forall n \in \mathbb{N}$. So (n^α) is the sequence

$$1, 1, 1, 1, 1, 1, \ldots$$

which must converge to 1.

$\square$

Lemma

Let $\alpha \in \mathbb{R}$. *Then*

$$n^{\alpha} \to \begin{cases} \infty & \textit{if } \alpha > 0 \\ 1 & \textit{if } \alpha = 0 \\ 0 & \textit{if } \alpha < 0 \end{cases}$$

as $n \to \infty$.

Proof.

CASE 3 ($\alpha < 0$): Let $\beta = -\alpha > 0$. Then $n^{\beta} \to \infty$ as $n \to \infty$ by Case 1. Therefore

$$n^{\alpha} = \frac{1}{n^{\beta}} \to 0$$

as $n \to \infty$, by a theorem from earlier.

□

Lemma

Let $a \in \mathbb{R}$. Then

$$a^n \to \begin{cases} \infty & \textit{if } a > 1 \\ 1 & \textit{if } a = 1 \\ 0 & \textit{if } |a| < 1 \end{cases}$$

as $n \to \infty$, and a^n does not have a limit if $a \leq -1$.

Lemma

Let $a \in \mathbb{R}$. Then

$$a^n \to \begin{cases} \infty & \text{if } a > 1 \\ 1 & \text{if } a = 1 \\ 0 & \text{if } |a| < 1 \end{cases}$$

as $n \to \infty$, and a^n does not have a limit if $a \leq -1$.

Proof.

CASE 1 ($a > 1$): Let $h = a - 1 > 0$. Then

$$\begin{aligned} a^n &= (1+h)^n = 1 + nh + \tfrac{n(n-1)}{2!}h^2 + \tfrac{n(n-1)(n-2)}{3!}h^3 + \ldots + h^n \\ &\geq 1 + nh \to \infty \end{aligned}$$

as $n \to \infty$. It follows that $a^n \to \infty$ as $n \to \infty$, by a theorem from earlier.

$\square$

Lemma

Let $a \in \mathbb{R}$. Then

$$a^n \to \begin{cases} \infty & \text{if } a > 1 \\ 1 & \text{if } a = 1 \\ 0 & \text{if } |a| < 1 \end{cases}$$

as $n \to \infty$, and a^n does not have a limit if $a \leq -1$.

Proof.

CASE 2 ($a = 1$): Since $a^n = 1\ \forall n$, we must have that $a^n \to 1$ as $n \to \infty$.

$\square$

Lemma

Let $a \in \mathbb{R}$. Then

$$a^n \to \begin{cases} \infty & \text{if } a > 1 \\ 1 & \text{if } a = 1 \\ 0 & \text{if } |a| < 1 \end{cases}$$

as $n \to \infty$, and a^n does not have a limit if $a \leq -1$.

Proof.

CASE 3 ($0 < a < 1$): Let $b = \frac{1}{a} > 1$. Then $b^n \to \infty$ as $n \to \infty$, by Case 1. Therefore $a^n = \left(\frac{1}{b}\right)^n = \frac{1}{b^n} \to 0$ as $n \to \infty$, by a theorem from earlier.

□

Lemma

Let $a \in \mathbb{R}$. Then

$$a^n \to \begin{cases} \infty & \text{if } a > 1 \\ 1 & \text{if } a = 1 \\ 0 & \text{if } |a| < 1 \end{cases}$$

as $n \to \infty$, and a^n does not have a limit if $a \leq -1$.

Proof.

CASE 4 ($a = 0$): Another easy case. Since $a^n = 0 \ \forall n$, we have that $a^n \to 0$ as $n \to \infty$.

□

Lemma

Let $a \in \mathbb{R}$. Then

$$a^n \to \begin{cases} \infty & \text{if } a > 1 \\ 1 & \text{if } a = 1 \\ 0 & \text{if } |a| < 1 \end{cases}$$

as $n \to \infty$, and a^n does not have a limit if $a \leq -1$.

Proof.

CASE 5 ($-1 < a < 0$): Since $0 < |a| < 1$, we have that

$$0 \leftarrow -|a|^n = -|a^n| \leq a^n \leq |a^n| = |a|^n \to 0$$

as $n \to \infty$. By the Sandwich Rule, $a^n \to 0$ as $n \to \infty$.

□

Lemma

Let $a \in \mathbb{R}$. Then

$$a^n \to \begin{cases} \infty & \text{if } a > 1 \\ 1 & \text{if } a = 1 \\ 0 & \text{if } |a| < 1 \end{cases}$$

as $n \to \infty$, and a^n does not have a limit if $a \leq -1$.

Proof.

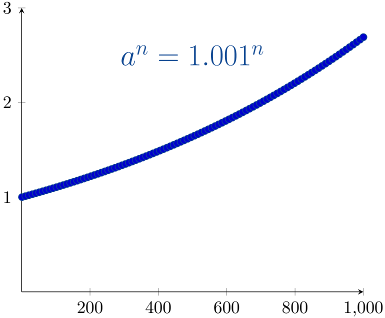
CASE 6 ($a \leq -1$): Now we have $a^n = (-1)^n |a|^n$. Since $|a|^n \geq 1$, $a^n \leq -1$ if n is odd and $a^n \geq 1$ if n is even. Therefore a^n cannot tend to any finite or infinite limit as $n \to \infty$. $\square$

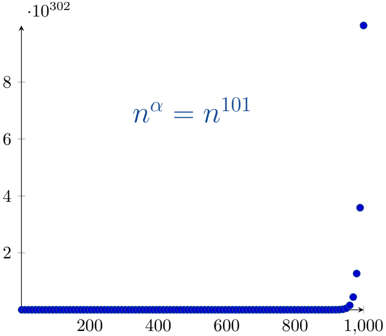
Now suppose that $a > 1$ and $\alpha > 0$. We know that $a^n \to \infty$ and $n^\alpha \to \infty$ as $n \to \infty$.

QUESTION: $\dfrac{a^n}{n^\alpha} \to ?$ as $n \to \infty$

Example

Let $a = 1.001$ and $\alpha = 101$.

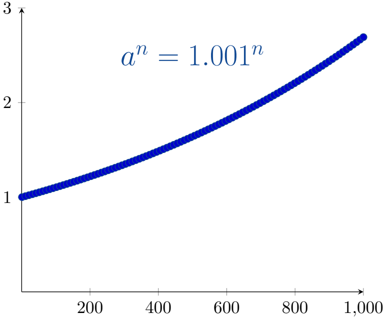


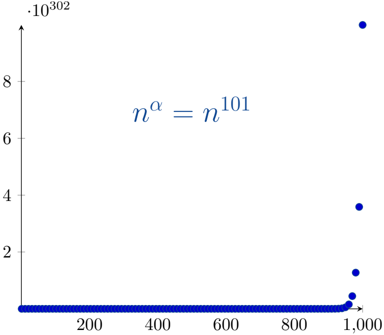


Note that $1.001^{1000} \approx 2.7$, but $1000^{101} = 10^{303}$.

Example

Let $a = 1.001$ and $\alpha = 101$.





Note that $1.001^{1000} \approx 2.7$, but $1000^{101} = 10^{303}$.

So what will happen to $\dfrac{1.001^n}{n^{101}}$ as $n \to \infty$?

It might surprise you to learn that

$$\frac{1.001^n}{n^{101}} \to \infty$$

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Lemma

Let $a > 1$ and $\alpha > 0$. Then

$$\frac{a^n}{n^{\alpha}} \to \infty$$

as $n \to \infty$.

Proof.

Let $p \in \mathbb{N}$ and $p \geq \alpha$. Then

$$\frac{a^n}{n^\alpha} \geq \frac{a^n}{n^p}$$

for all $n \in \mathbb{N}$.

We want to prove that $\frac{a^n}{n^p} \to \infty$ as $n \to \infty$. For general p, the notation in the proof is complicated – so we will only prove it for $p = 2$.

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Since $a > 1$, we have that $h := a - 1 > 0$.

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Since $a > 1$, we have that $h := a - 1 > 0$. So

$$\begin{aligned} n \geq 4 \quad \implies \quad \frac{a^n}{n^2} &= \frac{(1+h)^n}{n^2} \\ &= \tfrac{1}{n^2}\left(1 + nh + \tfrac{n(n-1)}{2!}h^2 + \tfrac{n(n-1)(n-2)}{3!}h^3 + \ldots + h^n\right) \\ &> \frac{n(n-1)(n-2)}{3!n^2}h^3 \\ &= \frac{(n-1)(n-2)}{6n}h^3 \\ &> \frac{(\frac{1}{2}n)(\frac{1}{2}n)}{6n}h^3 \\ &= \frac{nh^3}{24} \to \infty \end{aligned}$$

as $n \to \infty$.

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A similar argument shows that $\frac{a^n}{n^p} \to \infty$ as $n \to \infty$ for all $p \in \mathbb{N}$.

Then a theorem from earlier tells us that $\frac{a^n}{n^\alpha} \to \infty$ as $n \to \infty$. $\square$

Corollary

Let $a > 1$ *and* $\alpha > 0$. *Then* $\frac{n^\alpha}{a^n} \to 0$ *as* $n \to \infty$.

Corollary

Let $\alpha > 0$ *and* $|b| < 1$. *Then* $n^\alpha b^n \to 0$ *as* $n \to \infty$.

Lemma

Let $a > 0$. *Then* $a^{\frac{1}{n}} \to 1$ *as* $n \to \infty$.

Lemma

Let $a > 0$. Then $a^{\frac{1}{n}} \to 1$ as $n \to \infty$.

Proof.

CASE 1 ($a > 1$): Let $h_n = a^{\frac{1}{n}} - 1 > 0$. Then

$$a = (1 + h_n)^n = 1 + nh_n + \frac{n(n-1)}{2!}h_n^2 + \ldots + h_n^n > nh_n.$$

So

$$0 < h_n < \frac{a}{n} \to 0$$

as $n \to \infty$. It follows by the Sandwich Rule that $h_n \to 0$ as $n \to \infty$. Therefore $a^{\frac{1}{n}} = 1 + h_n \to 1$ as $n \to \infty$.

□

Lemma

Let $a > 0$. Then $a^{\frac{1}{n}} \to 1$ as $n \to \infty$.

Proof.

CASE 2 ($a = 1$): Clearly $a^{\frac{1}{n}} = 1 \; \forall n$. Hence $a^{\frac{1}{n}} \to 1$ as $n \to \infty$.

CASE 3 ($0 < a < 1$): Let $b = \frac{1}{a} > 1$. Then

$$a^{\frac{1}{n}} = \left(\frac{1}{b}\right)^{\frac{1}{n}} \to \frac{1}{1} = 1$$

as $n \to \infty$.

Therefore $a^{\frac{1}{n}} \to 1$ as $n \to \infty \; \forall a > 0$. $\square$

Lemma

$n^{\frac{1}{n}} \to 1 \ as \ n \to \infty.$

Lemma

$n^{\frac{1}{n}} \to 1$ *as* $n \to \infty$.

Proof.

Let $k_n := n^{\frac{1}{n}} - 1$. If $n > 1$, then $n^{\frac{1}{n}} > 1$ and $k_n > 0$. So

$$\begin{aligned} n \geq 2 \quad \Longrightarrow \quad n &= (1 + k_n)^n = 1 + nk_n + \frac{n(n-1)}{2!}k_n^2 + \ldots + k_n^n \\ &> \frac{n(n-1)}{2!}k_n^2. \end{aligned}$$

Thus

$$0 < k_n < \sqrt{\frac{2}{n-1}}$$

for all $n \geq 2$. By the Sandwich Rule, $k_n \to 0$ as $n \to \infty$. Therefore $n^{\frac{1}{n}} \to 1$ as $n \to \infty$. $\square$

Lemma

Let $a \in \mathbb{R}$. *Then* $\frac{a^n}{n!} \to 0$ *as* $n \to \infty$.

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Proof.

Let N be the smallest number in $\mathbb{N}$ such that $N \geq 2|a|$. Then

$$p \in \mathbb{N},\ p \geq N \implies \frac{|a|}{p} \leq \frac{|a|}{N} \leq \frac{1}{2}.$$

So

$$\begin{aligned} n > N \implies 0 \leq \left|\frac{a^n}{n!}\right| &= \left(\tfrac{|a|}{n}\right)\left(\tfrac{|a|}{n-1}\right)\left(\tfrac{|a|}{n-2}\right)\cdots\left(\tfrac{|a|}{N+1}\right)\left(\tfrac{|a|^N}{N!}\right) \\ &\leq \left(\frac{1}{2}\right)\left(\frac{1}{2}\right)\left(\frac{1}{2}\right)\cdots\left(\frac{1}{2}\right)\left(\frac{|a|^N}{N!}\right) \\ &= \left(\frac{1}{2}\right)^{n-N}\frac{|a|^N}{N!} \to 0 \end{aligned}$$

as $n \to \infty$. Therefore $\frac{a^n}{n!} \to 0$ as $n \to \infty$, by the Sandwich Rule. □

Lemma

$\frac{n!}{n^n} \to 0$ *as* $n \to \infty$.

Lemma

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Proof.

Since

$$0 < \frac{n!}{n^n} = \frac{n(n-1)(n-2)\cdots(2)(1)}{nnn\cdots nn}$$
$$= 1(1-\tfrac{1}{n})(1-\tfrac{2}{n})\cdots\tfrac{2}{n}\tfrac{1}{n} \leq \tfrac{1}{n} \to 0$$

as $n \to \infty$, the result follows by the Sandwich Rule. $\square$

$$\alpha > 0 \implies n^\alpha \to \infty$$
$$\alpha = 0 \implies n^\alpha \to 1$$
$$\alpha < 0 \implies n^\alpha \to 0$$

Summary

$$\alpha > 0 \implies n^\alpha \to \infty$$

$$\alpha = 0 \implies n^\alpha \to 1$$

$$\alpha < 0 \implies n^\alpha \to 0$$

$$a > 1 \implies a^n \to \infty$$

$$a = 1 \implies a^n \to 1$$

$$|a| < 1 \implies a^n \to 0$$

$$a < -1 \implies a^n \text{ does not have a limit}$$

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$$a > 1, \alpha > 0 \implies \frac{a^n}{n^\alpha} \to \infty$$

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Example

$$\frac{n^7 7^n + n^5 5^n}{3^n + 8^n}$$

Example

$$\frac{n^7 7^n + n^5 5^n}{3^n + 8^n} = \frac{n^7\left(\frac{7}{8}\right)^n + n^5\left(\frac{5}{8}\right)^n}{\left(\frac{3}{8}\right)^n + 1}$$

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as $n \to \infty$.

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as $n \to \infty$.

Example

$$\frac{n! + 8^n}{7^n + n!} = \frac{1 + \frac{8^n}{n!}}{\frac{7^n}{n!} + 1}$$

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$$\frac{n! + 8^n}{7^n + n!} = \frac{1 + \frac{8^n}{n!}}{\frac{7^n}{n!} + 1} \to \frac{1+0}{0+1} = 1$$

as $n \to \infty$.

Example

Let $r > 0$ and $a_n = \left(4^{10} + r^n\right)^{\frac{1}{n}}$. Calculate $\lim\limits_{n\to\infty} a_n$.

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CASE 2 ($r > 1$):

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CASE 1 $(0 < r \leq 1)$: Since

$$1 \leftarrow (4^{10})^{\frac{1}{n}} \leq a_n \leq (4^{10} + 1)^{\frac{1}{n}} \rightarrow 1$$

as $n \to \infty$, it follows by the Sandwich Rule that if $0 < r \leq 1$, we have that $\lim_{n\to\infty} a_n = 1$.

CASE 2 $(r > 1)$:

Example

Let $r > 0$ and $a_n = \left(4^{10} + r^n\right)^{\frac{1}{n}}$. Calculate $\lim_{n\to\infty} a_n$.

CASE 1 ($0 < r \leq 1$): Since

$$1 \leftarrow (4^{10})^{\frac{1}{n}} \leq a_n \leq (4^{10} + 1)^{\frac{1}{n}} \to 1$$

as $n \to \infty$, it follows by the Sandwich Rule that if $0 < r \leq 1$, we have that $\lim_{n\to\infty} a_n = 1$.

CASE 2 ($r > 1$): In this case $r^n \to \infty$ as $n \to \infty$. So $\exists\ N \in \mathbb{N}$ such that $r^n > 4^{10}$ for all $n > N$. So

$$n > N \implies r = (r^n)^{\frac{1}{n}} < (4^{10} + r^n)^{\frac{1}{n}} < (r^n + r^n)^{\frac{1}{n}} = 2^{\frac{1}{n}} r \to r$$

as $n \to \infty$. By the Sandwich Rule, $\lim_{n\to\infty} a_n = r$ if $r > 1$.

Monotonic Sequences

Definition

A sequence (a_n) is called an *increasing sequence* (artan dizi) iff

$$a_n \leq a_{n+1}$$

for all $n \in \mathbb{N}$.

(Note: Thomas's Calculus calls this a "nondecreasing sequence".)

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A sequence (a_n) is called a *strictly increasing sequence* iff

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A sequence (a_n) is called a *strictly decreasing sequence* iff

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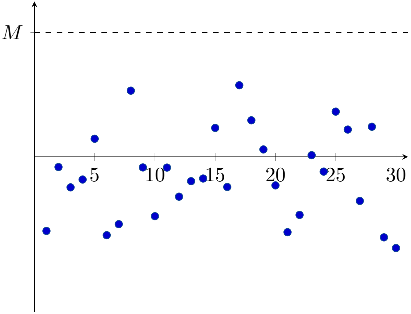
for all $n \in \mathbb{N}$.

Definition

A sequence (a_n) is called a *monotonic sequence* (monoton dizi) iff it is either an increasing sequence or a decreasing sequence.

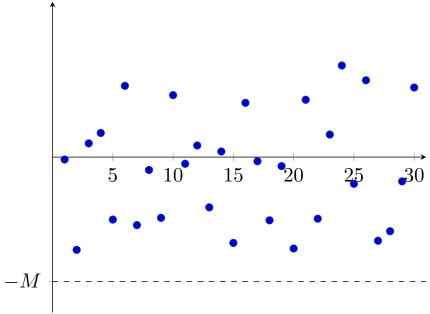


Definition

A sequence (a_n) is said to be *bounded above* (üstten sınırlı) iff $\exists\ M \in \mathbb{R}$ such that

$$a_n \leq M$$

for all $n \in \mathbb{N}$. The number M is called an *upper bound* (üst sınırdır) for (a_n).



Definition

A sequence (a_n) is said to be *bounded below* (alttan sınırlı) iff $\exists\ m \in \mathbb{R}$ such that

$$a_n \geq m$$

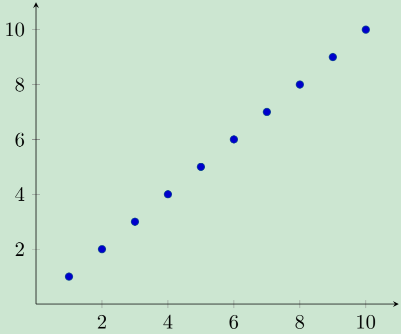
for all $n \in \mathbb{N}$. The number m is called a *lower bound* (alt sınırdır) for (a_n).

Example

Let $b_n = n$ for all $n \in \mathbb{N}$.

Then (b_n) is

- increasing;
- strictly increasing;
- monotonic;
- bounded below ($b_n \geq 0 \ \forall n$);
- not bounded above.

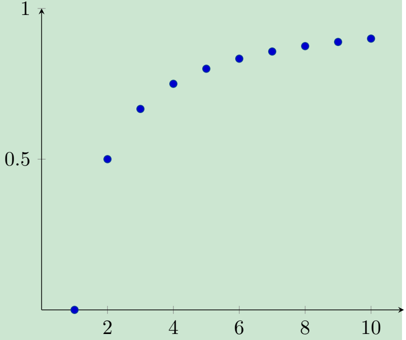


Example

Let $c_n = 1 - \frac{1}{n}$ for all $n \in \mathbb{N}$.

Then (c_n) is

- increasing;
- strictly increasing;
- monotonic;
- bounded above $(c_n \leq 1 \ \forall n)$;
- bounded below $(c_n \geq 0 \ \forall n)$.

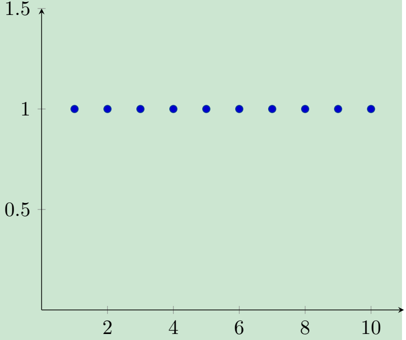


Example

Let $d_n = 1$ for all $n \in \mathbb{N}$.

Then (d_n) is

- increasing;
- not strictly increasing;
- decreasing;
- not strictly decreasing;
- monotonic;
- bounded below ($b_n \geq 0 \ \forall n$);
- bounded above ($b_n \leq 567 \ \forall n$).



Theorem

Let (a_n) be an increasing sequence.

1. *If (a_n) is bounded above, then (a_n) converges.*
2. *If (a_n) is not bounded above, then $a_n \to \infty$ as $n \to \infty$.*

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This is an important theorem. We need some more theory before we can prove this theorem.

Definition

Let $S \subseteq \mathbb{R}$ be a set. We say that S is *bounded above* iff $\exists\ M \in \mathbb{R}$ such that

$$x \leq M$$

for all $x \in S$. M is called an *upper bound* for S.

Definition

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for all $x \in S$. m is called a *lower bound* for S.

Example

Let $S = \{1, 2, 3, 4\}$. Then $x \leq 7$ for all $x \in S$. So 7 is an upper bound for S.

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Note that 5 is also an upper bound for S. So is 4. In fact, 4 is the least upper bound for S.

Definition

Let $S \subseteq \mathbb{R}$. The *supremum* of S, $\sup S$, is the least upper bound (en küçük üst sınır) for S.

If S is empty, we define $\sup S = -\infty$.
If S is not bounded above, we define $\sup S = \infty$.

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Example

$$\begin{aligned} \sup\{1,2,3\} &= 3 \\ \sup[0,1] &= 1 \\ \sup(0,1) &= 1 \\ \sup \mathbb{Z} &= \infty \\ \sup \varnothing &= -\infty \end{aligned}$$

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Example

$$\sup\{1,2,3\} = 3 \qquad \max\{1,2,3\} = 3$$
$$\sup[0,1] = 1 \qquad \max[0,1] = 1$$
$$\sup(0,1) = 1 \qquad \max(0,1) \text{ does not exist}$$
$$\sup \mathbb{Z} = \infty \qquad \max \mathbb{Z} \text{ does not exist}$$
$$\sup \varnothing = -\infty \qquad \max \varnothing \text{ does not exist}$$

Definition

Let $S \subseteq \mathbb{R}$. The *infinum* of S, $\inf S$, is the greatest lower bound (en büyük alt sınır) for S.

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Example

$$\begin{aligned} \inf\{-1,0,7,11\} &= -1 \\ \inf(0,1] &= 0 \\ \inf \mathbb{Z} &= -\infty \\ \inf \mathbb{N} &= 1 \end{aligned}$$

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Example

$$\inf\{-1, 0, 7, 11\} = -1 \qquad \min\{-1, 0, 7, 11\} = -1$$
$$\inf(0, 1] = 0 \qquad \min(0, 1] \text{ does not exist}$$
$$\inf \mathbb{Z} = -\infty \qquad \min \mathbb{Z} \text{ does not exist}$$
$$\inf \mathbb{N} = 1 \qquad \min \mathbb{N} = 1$$

Lemma

Let $S \subseteq \mathbb{R}$ *and* $S \neq \varnothing$. *Then*

$$\sup S = \alpha \iff \begin{array}{l} \text{(i) } x \leq \alpha \quad \forall x \in S; \textit{ and} \\ \text{(ii) } \forall \varepsilon > 0 \quad \exists x_0 \in S \textit{ such that } \alpha - \varepsilon < x_0 \leq \alpha. \end{array}$$

Lemma

Let $S \subseteq \mathbb{R}$ and $S \neq \varnothing$. Then

$$\sup S = \alpha \iff \begin{array}{l} \text{(i) } x \leq \alpha \quad \forall x \in S; \textit{ and} \\ \text{(ii) } \forall \varepsilon > 0 \quad \exists x_0 \in S \textit{ such that } \alpha - \varepsilon < x_0 \leq \alpha. \end{array}$$

Proof.

“ $\Longleftarrow$ ”
(i) $\implies$ α is an upper bound for S.
(ii) $\implies$ $\alpha - \varepsilon$ is not an upper bound for S $\forall\ \varepsilon > 0$.
Therefore α is the least upper bound.
“ $\implies$ ”
$\sup S = \alpha$ $\implies$ α is the least upper bound $\implies$ (i) and (ii) are true. □

Completeness Axiom

Every non-empty set of real numbers, which is bounded above, has a supremum.

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Now we can prove the important theorem:

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Let (a_n) be an increasing sequence.

1. *If (a_n) is bounded above, then (a_n) converges.*
2. *If (a_n) is not bounded above, then $a_n \to \infty$ as $n \to \infty$.*

Proof of Theorem 75.

(i) Let $S = \{a_n : n \in \mathbb{N}\}$. If (a_n) is bounded above, then S is bounded above.

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Let $\alpha = \sup S < \infty$.

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Let $\alpha = \sup S < \infty$. Let $\varepsilon > 0$. Then $\alpha - \varepsilon$ is not an upper bound of S. So $\exists\ a_N \in S$ such that $\alpha - \varepsilon < a_N < \alpha$.

Since (a_n) is increasing,

$$n > N \implies \alpha - \varepsilon < a_n < \alpha \implies |a_n - \alpha| < \varepsilon.$$

Therefore $a_n \to \alpha$ as $n \to \infty$.

Proof of Theorem 75 continued.

(ii) Let $A > 0$.

Proof of Theorem 75 continued.

(ii) Let $A > 0$. If (a_n) is not bounded above, then A is not an upper bound for (a_n).

Proof of Theorem 75 continued.

(ii) Let $A > 0$. If (a_n) is not bounded above, then A is not an upper bound for (a_n). Hence $\exists\ a_N$ such that $a_N > A$.

Proof of Theorem 75 continued.

(ii) Let $A > 0$. If (a_n) is not bounded above, then A is not an upper bound for (a_n). Hence $\exists\ a_N$ such that $a_N > A$.

Since (a_n) is increasing,

$$n > N \implies a_n \geq a_N > A.$$

Therefore $a_n \to \infty$ as $n \to \infty$.

$\square$

Corollary

Let (a_n) be an increasing sequence. Then

$$(a_n) \text{ is convergent} \iff (a_n) \text{ is bounded above.}$$

Corollary

Let (a_n) be a decreasing sequence. Then

$$(a_n) \text{ is convergent} \iff (a_n) \text{ is bounded below.}$$

Theorem (The Monotonic Sequence Theorem)

Every bounded monotonic sequence converges.

Next Time

- 9.2 Infinite Series
- 9.3 The Integral Test
- 9.4 Comparison Tests
- 9.5 Absolute Convergence; The Ratio and Root Tests