

Lecture 3

- 2.4 Autonomous Equations and Population Dynamics
- 2.5 Exact Equations
- 2.6 Substitutions



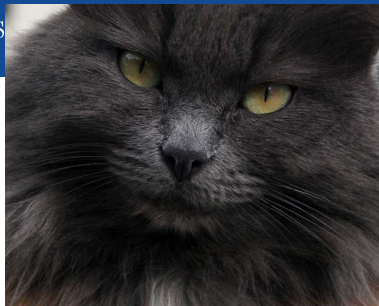
Autonomous Equations and Population Dynamics



Equations of the form

$$\frac{dy}{dt} = \underbrace{f(y)}_{\text{only } y} \quad (1)$$

are called *autonomous*.



Example (Exponential Growth)

Let $y(t)$ denote the number of cats in İstanbul.
The simplest model is to assume that the rate of change of y is proportional to y .

$$\frac{dy}{dt} = ry$$

for some constant r . We will assume that $r > 0$.

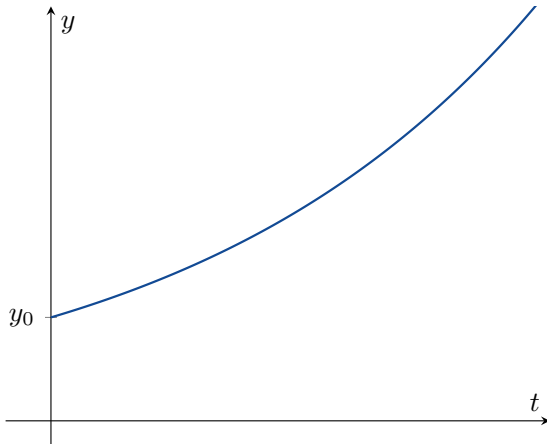
2.4 Autonomous Equations and Population Dynamics



The solution to

$$\begin{cases} y' = ry \\ y(0) = y_0 \end{cases}$$

is $y(t) = y_0 e^{rt}$.



2.4 Autonomous Equations and Population Dynamics



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- the food will run out
- there will be no space
- people will get angry

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- the food will run out
- there will be no space
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So we need a better model.



Example (Logistic Growth)

Now we replace the constant r with a function $h(y)$.

$$\frac{dy}{dt} = h(y)y.$$



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$$\frac{dy}{dt} = h(y)y.$$

We want a function h which satisfies

- $h(y) \approx r$ if y is small;
- $h(y)$ decreases as y grows larger; and
- $h(y) < 0$ for large y .



The simplest such h is $h(y) = r - ay$.



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$$\frac{dy}{dt} = (r - ay)y$$



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$$\frac{dy}{dt} = (r - ay)y$$

which we will write as

$$\frac{dy}{dt} = r \left(1 - \frac{y}{K}\right) y$$

for $K = \frac{r}{a}$. This is called the *Logistic Equation*.

2.4 Autonomous Equations and Population Dynamics



First we look for equilibrium solutions – that is solutions with $\frac{dy}{dt} = 0$ for all t .

$$0 = \frac{dy}{dt} = r \left(1 - \frac{y}{K} \right) y \quad \implies \quad y = 0 \quad \text{or} \quad y = K.$$

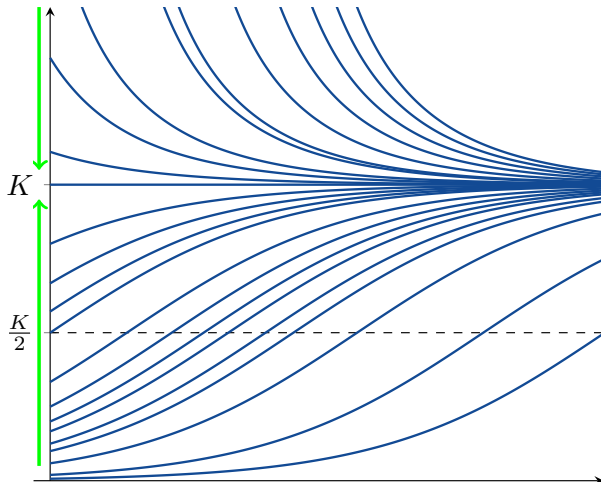


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$$0 = \frac{dy}{dt} = r \left(1 - \frac{y}{K}\right) y \quad \implies \quad y = 0 \quad \text{or} \quad y = K.$$

The equilibrium solutions are important. If we look at some more solutions, we can see that the other solutions converge to $y = K$, but diverge from $y = 0$.

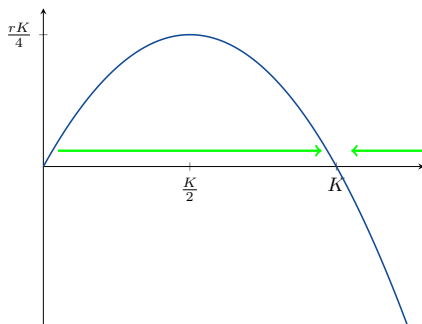
2.4 Autonomous Equations and Population Dynamics



2.4 Autonomous Equations and Population Dynamics



To understand this behaviour, we graph $\frac{dy}{dt}$ against y .



Note that

- $\frac{dy}{dt} > 0 \implies y$ is increasing; and
- $\frac{dy}{dt} < 0 \implies y$ is decreasing; and

We can show this on the graph by drawing green arrows.

2.4 Autonomous Equations and Population Dynamics



To investigate further, we look at $\frac{d^2y}{dt^2}$:

2.4 Autonomous Equations and Population Dynamics



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2.4 Autonomous Equations and Population Dynamics





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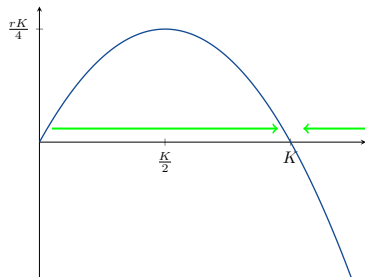
The solution $y(t)$ is concave up ( or ) when $y'' > 0$
(i.e. when both f and f' are both positive or both negative).

The solution $y(t)$ is concave down ( or ) when
 $y'' < 0$ (i.e. when one of f and f' is positive and one is negative).

2.4 Autonomous Equations and Population Dynamics



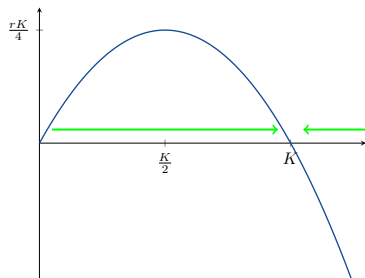
Look again at the graph of $f(y) = r \left(1 - \frac{y}{K}\right) y$ against y .



2.4 Autonomous Equations and Population Dynamics



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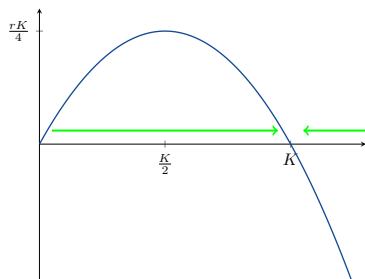
We can see that

- $y \in (0, \frac{K}{2}) \implies f > 0$ and $f' > 0 \implies y(t)$ is increasing and concave up;

2.4 Autonomous Equations and Population Dynamics



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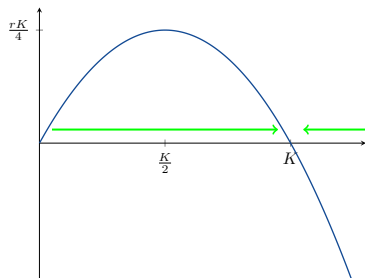
We can see that

- $y \in (0, \frac{K}{2}) \implies f > 0$ and $f' > 0 \implies y(t)$ is increasing and concave up;
- $y \in (\frac{K}{2}, K) \implies f > 0$ and $f' < 0 \implies y(t)$ is increasing and concave down;

2.4 Autonomous Equations and Population Dynamics



Look again at the graph of $f(y) = r \left(1 - \frac{y}{K}\right) y$ against y .



We can see that

- $y \in (0, \frac{K}{2}) \implies f > 0$ and $f' > 0 \implies y(t)$ is increasing and concave up;
- $y \in (\frac{K}{2}, K) \implies f > 0$ and $f' < 0 \implies y(t)$ is increasing and concave down;
- $y \in (K, \infty) \implies f < 0$ and $f' < 0 \implies y(t)$ is decreasing and concave up;

2.4 Autonomous Equations and Population Dynamics

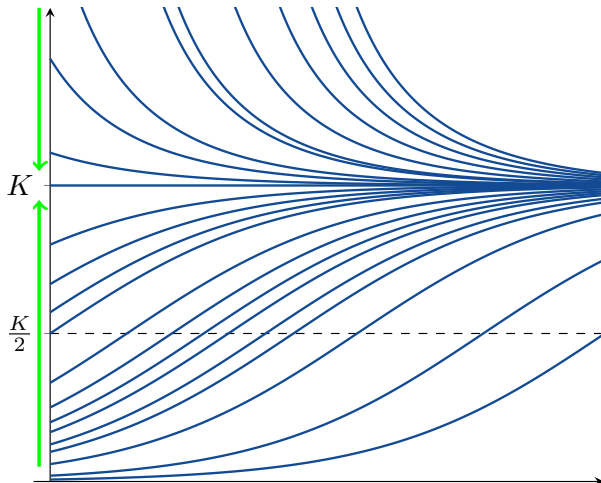


Moreover, remember that a theorem from earlier told us that two solutions can not intersect.

2.4 Autonomous Equations and Population Dynamics



Moreover, remember that a theorem from earlier told us that two solutions can not intersect. Hence the solutions look like this:





Because solutions converge to $y = K$, we say that $y = K$ is an *asymptotically stable equilibrium solution* or an *asymptotically stable critical point*.



Because solutions converge to $y = K$, we say that $y = K$ is an *asymptotically stable equilibrium solution* or an *asymptotically stable critical point*.

Because solutions diverge from $y = 0$, we say that $y = 0$ is an *unstable equilibrium solution* or an *unstable critical point*.



Definition

Equilibrium solutions can be

2.4 Autonomous Equations and Population Dynamics



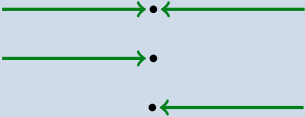
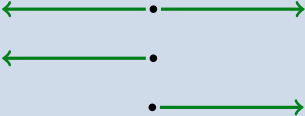
Definition

Equilibrium solutions can be

<p>A phase line diagram showing an equilibrium point (black dot) with green arrows pointing towards it from both sides, indicating asymptotic stability.</p>	asymptotically stable

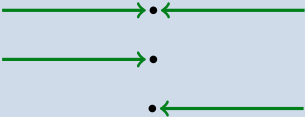
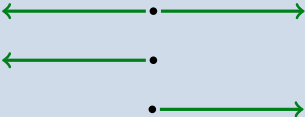
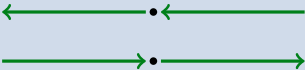
Definition

Equilibrium solutions can be

	asymptotically stable
	unstable

Definition

Equilibrium solutions can be

	asymptotically stable
	unstable
	semistable



Example

Find all of the critical points of

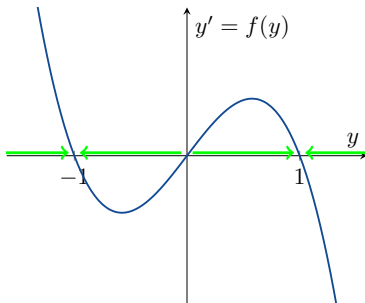
$$\frac{dy}{dt} = \underbrace{y(1 - y^2)}_{f(y)} \quad (-\infty < y_0 < \infty)$$

and classify each as asymptotically stable, unstable or semistable.

2.4 Autonomous Equations and Population Dynamics



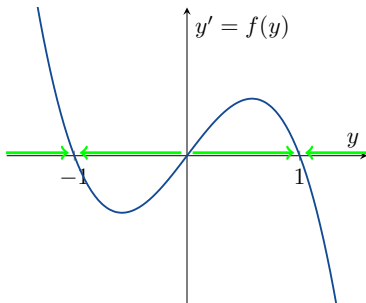
$$\frac{dy}{dt} = y(1 - y^2)$$



2.4 Autonomous Equations and Population Dynamics

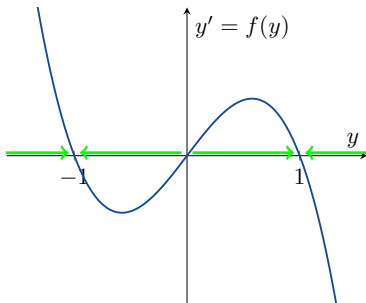


$$\frac{dy}{dt} = y(1 - y^2)$$



The critical points are $y = -1, 0, 1$.

$$\frac{dy}{dt} = y(1 - y^2)$$



The critical points are $y = -1, 0, 1$.

- $y = -1$ is asymptotically stable;
- $y = 0$ is unstable; and
- $y = 1$ is asymptotically stable.

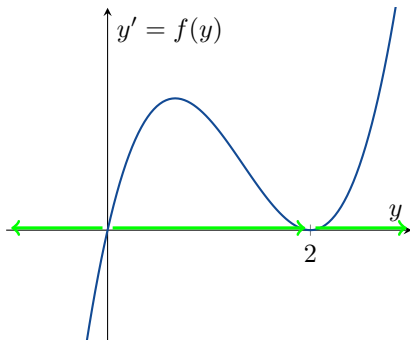
Example

Find all of the critical points of

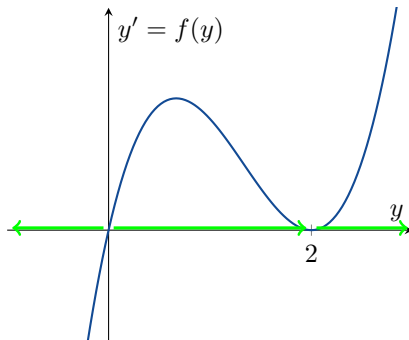
$$\frac{dy}{dt} = \underbrace{y(y-2)^2}_{f(y)} \quad (-\infty < y_0 < \infty)$$

and classify each as asymptotically stable, unstable or semistable.

2.4 Autonomous Equations and Population Dynamics

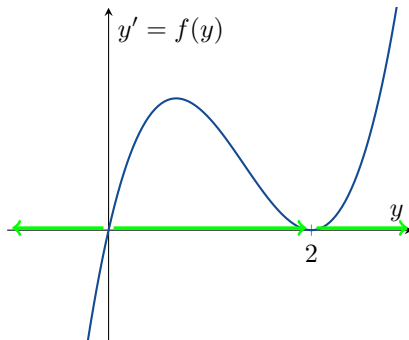


2.4 Autonomous Equations and Population Dynamics



The critical points are $y = 0$ and 2 .

2.4 Autonomous Equations and Population Dynamics



The critical points are $y = 0$ and 2 .

- $y = 0$ is unstable; and
- $y = 2$ is semistable.



Example

Consider the autonomous differential equation

$$\frac{dy}{dt} = f(y) = y^4 - 5y^3 + 6y^2. \quad (2)$$

- 1 Find all of the critical points of (2).
- 2 Sketch the graph of $f(y)$ versus y .
- 3 Determine whether each critical point is asymptotically stable, unstable or semistable.
- 4 Sketch 10 (or more) different solutions of (2).

(This is an exam question from 2013: Students had 30 minutes to solve this.)



1

$$\frac{dy}{dt} = f(y) = y^4 - 5y^3 + 6y^2 = y^2(y - 2)(y - 3).$$

The critical points are $y = 0$, $y = 2$ and $y = 3$.

2.4 Autonomous Equations and Population Dynamics



$$\frac{dy}{dt} = f(y) = y^4 - 5y^3 + 6y^2 = y^2(y - 2)(y - 3)$$

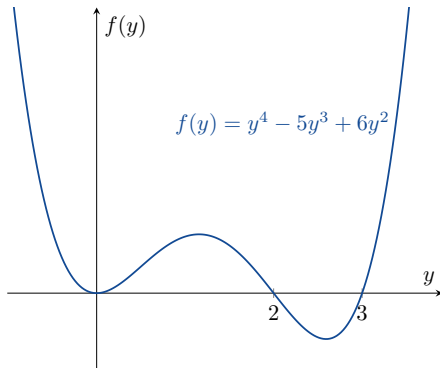
2

2.4 Autonomous Equations and Population Dynamics



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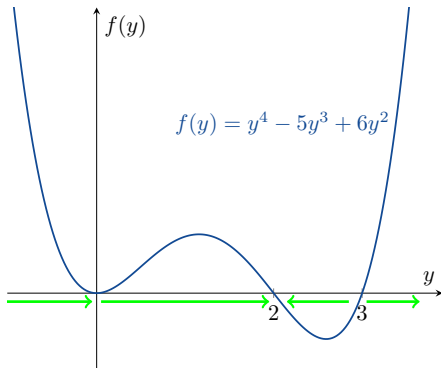


2.4 Autonomous Equations and Population Dynamics



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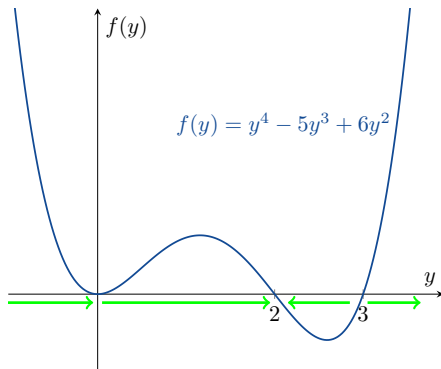


2.4 Autonomous Equations and Population Dynamics



$$\frac{dy}{dt} = f(y) = y^4 - 5y^3 + 6y^2 = y^2(y - 2)(y - 3)$$

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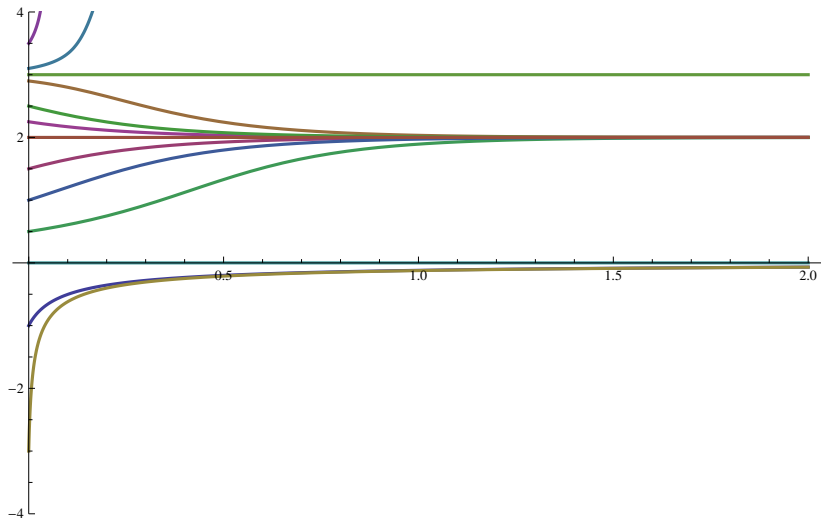


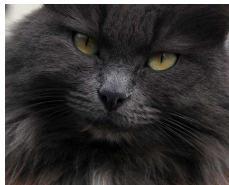
- 3 $y = 0$ is semistable, $y = 2$ is asymptotically stable and $y = 3$ is unstable.

2.4 Autonomous Equations and Population Dynamics



4





Example (A Critical Threshold)

Now suppose that we can model the number of cats in İstanbul by

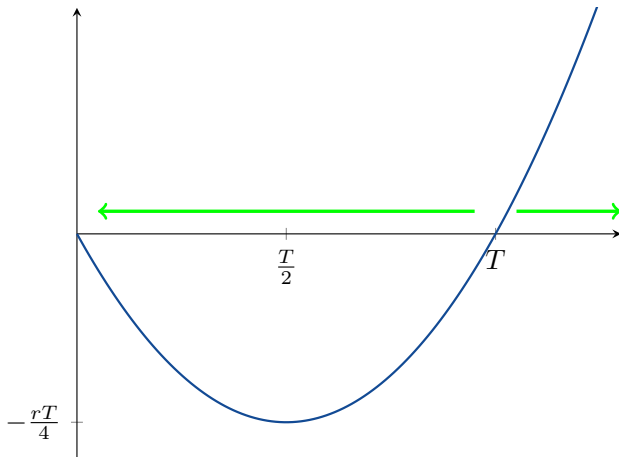
$$\frac{dy}{dt} = -r \left(1 - \frac{y}{T}\right) y$$

where $T > 0$ and $r > 0$.

2.4 Autonomous Equations and Population Dynamics



$$\frac{dy}{dt} = -r \left(1 - \frac{y}{T}\right) y$$



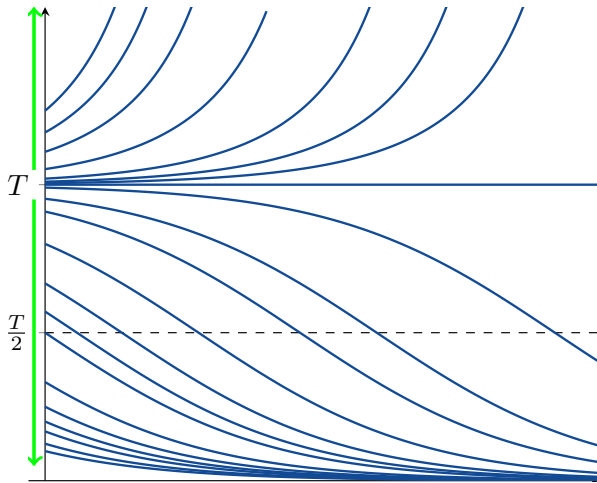


The critical points/equilibrium solutions are $y = 0$ and $y = T$.

- $y = 0$ is asymptotically stable; and
- $y = T$ is unstable.

With this information we can sketch some solutions

2.4 Autonomous Equations and Population Dynamics



2.4 Autonomous Equations and Population Dyna



Depending on y_0 ($y_0 \neq T$), we either have $y \rightarrow 0$ or $y \rightarrow \infty$.

2.4 Autonomous Equations and Population Dynamics



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The number T is called a *threshold level*, below which no growth happens.

2.4 Autonomous Equations and Population Dynamics



Depending on y_0 ($y_0 \neq T$), we either have $y \rightarrow 0$ or $y \rightarrow \infty$.

The number T is called a *threshold level*, below which no growth happens.

The population of some species have the threshold property: If there are not enough individuals, then the species becomes extinct.

2.4 Autonomous Equations and Population Dynamics



Depending on y_0 ($y_0 \neq T$), we either have $y \rightarrow 0$ or $y \rightarrow \infty$.

The number T is called a *threshold level*, below which no growth happens.

The population of some species have the threshold property: If there are not enough individuals, then the species becomes extinct.

This model predicts that the number of cats in İstanbul will increase to ∞ (if $y_0 > T$), so we need a more advanced model.



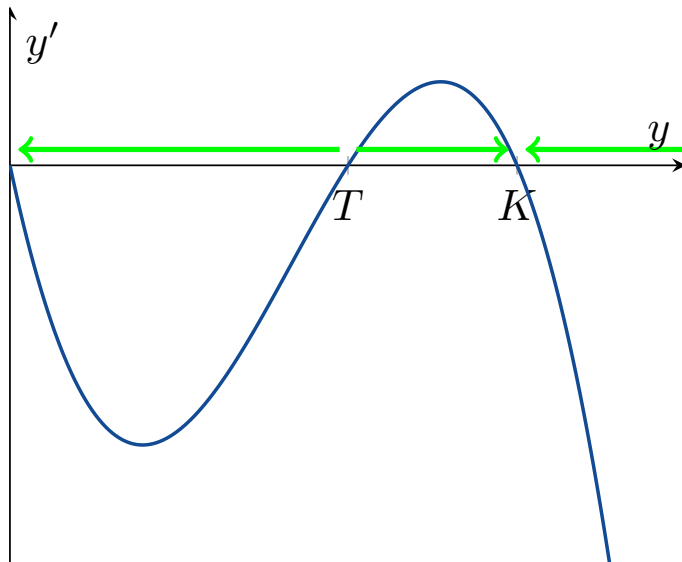
Example (Logistic Growth with a Threshold)

Now consider

$$\frac{dy}{dt} = -r \left(1 - \frac{y}{T}\right) \left(1 - \frac{y}{K}\right) y$$

for $0 < T < K$ and $r > 0$.

2.4 Autonomous Equations and Population Dynamics





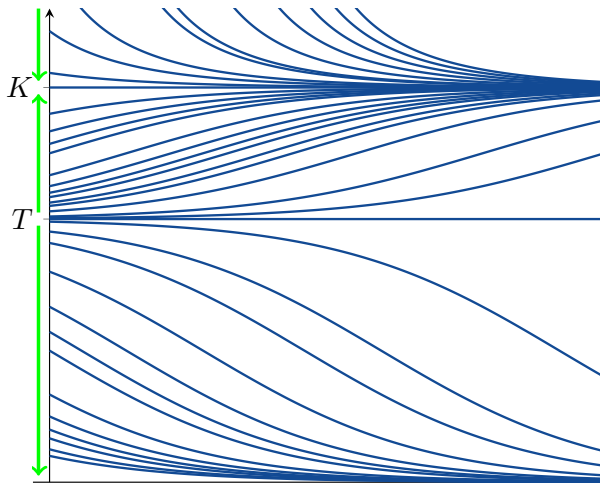
The critical points/equilibrium solutions are $y = 0$, $y = T$ and $y = K$.

- $y = 0$ is asymptotically stable;
- $y = T$ is unstable; and
- $y = K$ is asymptotically stable.

2.4 Autonomous Equations and Population Dynamics



Solutions look like this:



This is an equation which has been used by biologists to model certain populations of animals.

Exact Equations

2.5 Exact Equations



Previously we have looked at linear equations and separable equations. Now we will look at another special type of equation.

2.5 Exact Equations



Example

Solve $2x + y^2 + 2xyy' = 0$.

This equation is not linear and is not separable.

2.5 Exact Equations



Example

Solve $2x + y^2 + 2xyy' = 0$.

This equation is not linear and is not separable.

Note that if $\psi(x, y) = x^2 + xy^2$, then $\frac{\partial \psi}{\partial x} = 2x + y^2$ and $\frac{\partial \psi}{\partial y} = 2xy$.

2.5 Exact Equations



Example

Solve $2x + y^2 + 2xyy' = 0$.

This equation is not linear and is not separable.

Note that if $\psi(x, y) = x^2 + xy^2$, then $\frac{\partial \psi}{\partial x} = 2x + y^2$ and $\frac{\partial \psi}{\partial y} = 2xy$. So we can write the ODE as

$$\frac{\partial \psi}{\partial x} + \frac{\partial \psi}{\partial y} \frac{dy}{dx} = 0.$$

2.5 Exact Equations



Since $y(x)$ is a function of x , we also have that

$$\frac{\partial}{\partial x} \left(\psi(x, y(x)) \right) = \frac{\partial \psi}{\partial x} + \frac{\partial \psi}{\partial y} \frac{dy}{dx}$$

by the Chain Rule.

2.5 Exact Equations



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$$\frac{\partial}{\partial x} (x^2 + xy^2) = 0.$$

Therefore

$$x^2 + xy^2 = c.$$

2.5 Exact Equations



Remark

The key step was finding $\psi(x, y)$.

2.5 Exact Equations



Now consider

$$M(x, y) + N(x, y)y' = 0. \quad (3)$$

Definition

If we can find a function $\psi(x, y)$ such that

$$\frac{\partial \psi}{\partial x} = M \quad \text{and} \quad \frac{\partial \psi}{\partial y} = N,$$

then (3) is called an *exact equation*.

2.5 Exact Equations



Now consider

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Definition

If we can find a function $\psi(x, y)$ such that

$$\frac{\partial \psi}{\partial x} = M \quad \text{and} \quad \frac{\partial \psi}{\partial y} = N,$$

then (3) is called an *exact equation*.

If (3) is exact, then

$$0 = M(x, y) + N(x, y)y' = \frac{\partial \psi}{\partial x}(x, y) + \frac{\partial \psi}{\partial y}(x, y) \frac{dy}{dx} = \frac{d}{dx} \left(\psi(x, y(x)) \right)$$

which has solution

$$\psi(x, y) = c.$$

2.5 Exact Equations



Remark

To solve an exact equation:

- 1 Find $\psi(x, y)$;
- 2 Write $\psi(x, y) = c$.

2.5 Exact Equations



Notation

$$y' = \frac{dy}{dx}$$

$$f_x = \frac{\partial f}{\partial x}$$

$$f_y = \frac{\partial f}{\partial y}$$

2.5 Exact Equations



Notation

$$y' = \frac{dy}{dx} \qquad f_x = \frac{\partial f}{\partial x} \qquad f_y = \frac{\partial f}{\partial y}$$

Theorem

Suppose that M , N , M_y and N_x are continuous on the rectangular region $R = \{(x, y) : \alpha < x < \beta, \gamma < y < \delta\}$.

2.5 Exact Equations



Notation

$$y' = \frac{dy}{dx} \qquad f_x = \frac{\partial f}{\partial x} \qquad f_y = \frac{\partial f}{\partial y}$$

Theorem

Suppose that M , N , M_y and N_x are continuous on the rectangular region $R = \{(x, y) : \alpha < x < \beta, \gamma < y < \delta\}$. Then

$$M + Ny' = 0 \text{ is exact} \qquad \Longleftrightarrow \qquad M_y = N_x.$$

2.5 Exact Equations



Example

Consider

$$(y \cos x + 2xe^y) + (\sin x + x^2e^y - 1)y' = 0.$$

Is this ODE exact? If yes, solve it.

2.5 Exact Equations



Example

Consider

$$(y \cos x + 2xe^y) + (\sin x + x^2e^y - 1)y' = 0.$$

Is this ODE exact? If yes, solve it.

$$M = y \cos x + 2xe^y$$

$$M_y =$$

$$N = \sin x + x^2e^y - 1$$

$$N_x =$$

2.5 Exact Equations



Example

Consider

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Is this ODE exact? If yes, solve it.

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$$M_y = \cos x + 2xe^y$$

$$N = \sin x + x^2e^y - 1$$

$$N_x =$$

2.5 Exact Equations



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$$M_y = \cos x + 2xe^y$$

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2.5 Exact Equations



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$$M = y \cos x + 2xe^y$$

$$M_y = \cos x + 2xe^y$$

$$N = \sin x + x^2e^y - 1$$

$$N_x = \cos x + 2xe^y$$

Yes, the ODE is exact.

2.5 Exact Equations



Example

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$$(y \cos x + 2xe^y) + (\sin x + x^2e^y - 1)y' = 0.$$

Is this ODE exact? If yes, solve it.

$$M = y \cos x + 2xe^y$$

$$M_y = \cos x + 2xe^y$$

$$N = \sin x + x^2e^y - 1$$

$$N_x = \cos x + 2xe^y$$

Yes, the ODE is exact. So $\exists \psi$ such that

$$\psi_x = M = y \cos x + 2xe^y$$

$$\psi_y = N = \sin x + x^2e^y - 1.$$

2.5 Exact Equations



$$\psi_x = y \cos x + 2xe^y$$

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2.5 Exact Equations



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Integrating the first equation (wrt x) gives

$$\psi = \int \psi_x dx = y \sin x + x^2e^y + h(y).$$

2.5 Exact Equations



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2.5 Exact Equations



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But we already know that $\psi_y = \sin x + x^2e^y - 1$.

2.5 Exact Equations



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2.5 Exact Equations



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2.5 Exact Equations



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$$\psi(x, y) = y \sin x + x^2e^y - y.$$

The solution to the ODE is

$$\boxed{y \sin x + x^2e^y - y = c.}$$

2.5 Exact Equations



Example

Consider

$$ye^{xy} + e^{xy}y' = 0.$$

Is this ODE exact? If yes, solve it.

2.5 Exact Equations



Example

Consider

$$ye^{xy} + e^{xy}y' = 0.$$

Is this ODE exact? If yes, solve it.

We have

$$\begin{aligned} M &= ye^{xy} & M_y &= e^{xy} + xye^{xy} \\ N &= e^{xy} & N_x &= ye^{xy}. \end{aligned}$$

2.5 Exact Equations



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Consider

$$ye^{xy} + e^{xy}y' = 0.$$

Is this ODE exact? If yes, solve it.

We have

$$\begin{aligned} M &= ye^{xy} & M_y &= e^{xy} + xye^{xy} \\ N &= e^{xy} & N_x &= ye^{xy}. \end{aligned}$$

Since $M_y \neq N_x$, the ODE is not exact.

2.5 Exact Equations



Example

Consider

$$\left(4x^3y^3 + \frac{1}{x}\right) + \left(3x^4y^2 + \frac{1}{y}\right) y' = 0.$$

Is this ODE exact? If yes, solve it.

2.5 Exact Equations



Example

Consider

$$\left(4x^3y^3 + \frac{1}{x}\right) + \left(3x^4y^2 + \frac{1}{y}\right)y' = 0.$$

Is this ODE exact? If yes, solve it.

I leave this one to you to solve. Please check that the solution is

$$x^4y^3 + \ln|x| + \ln|y| = c.$$

2.5 Exact Equations



Example

Consider

$$1 + (1 + 2y + 3y^2) y' = 0.$$

Is this ODE exact? If yes, solve it.

2.5 Exact Equations



Example

Consider

$$1 + (1 + 2y + 3y^2) y' = 0.$$

Is this ODE exact? If yes, solve it.

First note that

$$M = 1$$

$$M_y = 0$$

$$N = 1 + 2y + 3y^2$$

$$N_x = 0 = M_y$$

2.5 Exact Equations



Example

Consider

$$1 + (1 + 2y + 3y^2) y' = 0.$$

Is this ODE exact? If yes, solve it.

First note that

$$\begin{aligned} M &= 1 & M_y &= 0 \\ N &= 1 + 2y + 3y^2 & N_x &= 0 = M_y \end{aligned}$$

Yes, the ODE is exact. So $\exists \psi$ such that

$$\begin{aligned} \psi_x &= 1 \\ \psi_y &= 1 + 2y + 3y^2. \end{aligned}$$

2.5 Exact Equations



Example

Consider

$$1 + (1 + 2y + 3y^2) y' = 0.$$

Is this ODE exact? If yes, solve it.

First note that

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Yes, the ODE is exact. So $\exists \psi$ such that

$$\begin{aligned} \psi_x &= 1 \\ \psi_y &= 1 + 2y + 3y^2. \end{aligned}$$

We can start with $\psi_x = 1$ or with $\psi_y = 1 + 2y + 3y^2$.

2.5 Exact Equations



$$\psi_x = 1$$

$$\psi_y = 1 + 2y + 3y^2$$

$$\psi_x = 1$$

$$\psi = \int 1 dx = x + h(y)$$

$$\psi_y = h'(y)$$

$$h'(y) = 1 + 2y + 3y^2$$

$$h(y) = y + y^2 + y^3$$

$$\psi = x + y + y^2 + y^3$$

2.5 Exact Equations



$$\psi_x = 1$$

$$\psi_y = 1 + 2y + 3y^2$$

$$\psi_x = 1$$

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$$h(y) = y + y^2 + y^3$$

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$$\psi_y = 1 + 2y + 3y^2$$

$$\psi = \int 1 + 2y + 3y^2 \, dy$$

$$= y + y^2 + y^3 + h(x)$$

$$\psi_x = h'(x)$$

$$h'(x) = 1$$

$$h(x) = x$$

$$\psi = x + y + y^2 + y^3$$

2.5 Exact Equations



$$\psi_x = 1$$

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$$\psi_y = h'(y)$$

$$h'(y) = 1 + 2y + 3y^2$$

$$h(y) = y + y^2 + y^3$$

$$\psi = x + y + y^2 + y^3$$

$$\psi_y = 1 + 2y + 3y^2$$

$$\psi = \int 1 + 2y + 3y^2 dy$$

$$= y + y^2 + y^3 + h(x)$$

$$\psi_x = h'(x)$$

$$h'(x) = 1$$

$$h(x) = x$$

$$\psi = x + y + y^2 + y^3$$

Therefore the solution is $\boxed{x + y + y^2 + y^3 = c.}$

2.5 Exact Equations



Example

Consider

$$(3xy + y^2) + (x^2 + xy)y' = 0.$$

Is this ODE exact? If yes, solve it.

2.5 Exact Equations



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$$(3xy + y^2) + (x^2 + xy)y' = 0.$$

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First note that

$$\begin{aligned} M &= 3xy + y^2 & M_y &= 3x + 2y \\ N &= x^2 + xy & N_x &= 2x + y \neq M_y \end{aligned}$$

2.5 Exact Equations



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Consider

$$(3xy + y^2) + (x^2 + xy)y' = 0.$$

Is this ODE exact? If yes, solve it.

First note that

$$\begin{aligned} M &= 3xy + y^2 & M_y &= 3x + 2y \\ N &= x^2 + xy & N_x &= 2x + y \neq M_y \end{aligned}$$

Since $M_y \neq N_x$, this ODE is not exact. So our method to solve an exact equation *will not work*.

Example

Consider

$$(3xy + y^2) + (x^2 + xy)y' = 0.$$

Is this ODE exact? If yes, solve it.

First note that

$$\begin{aligned} M &= 3xy + y^2 & M_y &= 3x + 2y \\ N &= x^2 + xy & N_x &= 2x + y \neq M_y \end{aligned}$$

Since $M_y \neq N_x$, this ODE is not exact. So our method to solve an exact equation *will not work*. But we are going to try our method anyway, so that we can see what goes wrong.

2.5 Exact Equations



Suppose that $\exists \psi(x, y)$ such that

$$\psi_x = 3xy + y^2$$

$$\psi_y = x^2 + xy.$$

2.5 Exact Equations



Suppose that $\exists \psi(x, y)$ such that

$$\psi_x = 3xy + y^2$$

$$\psi_y = x^2 + xy.$$

Integrating ψ_x with respect to x gives

$$\psi = \frac{3}{2}x^2y + xy^2 + h(y).$$

2.5 Exact Equations



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$$\psi_x = 3xy + y^2$$

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Integrating ψ_x with respect to x gives

$$\psi = \frac{3}{2}x^2y + xy^2 + h(y).$$

Thus

$$x^2 + xy = \psi_y = \frac{\partial}{\partial y} \left(\frac{3}{2}x^2y + xy^2 + h(y) \right) = \frac{3}{2}x^2 + 2xy + h'(y).$$

2.5 Exact Equations



Suppose that $\exists \psi(x, y)$ such that

$$\psi_x = 3xy + y^2$$

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Integrating ψ_x with respect to x gives

$$\psi = \frac{3}{2}x^2y + xy^2 + h(y).$$

Thus

$$x^2 + xy = \psi_y = \frac{\partial}{\partial y} \left(\frac{3}{2}x^2y + xy^2 + h(y) \right) = \frac{3}{2}x^2 + 2xy + h'(y).$$

So we need h to satisfy

$$h'(y) = -\frac{1}{2}x^2 - xy.$$

2.5 Exact Equations



$$h'(y) = -\frac{1}{2}x^2 - xy$$

This is not possible!!! $h(y)$ must be a function of y , but $-\frac{1}{2}x^2 - xy$ depends on both x and y .

2.5 Exact Equations



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This is not possible!!! $h(y)$ must be a function of y , but $-\frac{1}{2}x^2 - xy$ depends on both x and y . So it is not possible to find h .

2.5 Exact Equations



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2.5 Exact Equations



$$h'(y) = -\frac{1}{2}x^2 - xy$$

This is not possible!!! $h(y)$ must be a function of y , but $-\frac{1}{2}x^2 - xy$ depends on both x and y . So it is not possible to find h . So it is not possible to find ψ . Our method failed because $M_y \neq N_x$.



Integrating Factors

It is sometimes possible to convert a differential equation which is not exact into an exact equation by multiplying it by an integrating factor. (Do you remember how we solve linear equations?)

2.5 Exact Equations



Consider

$$M(x, y) dx + N(x, y) dy = 0. \quad (4)$$

Suppose that (4) is not exact.

2.5 Exact Equations



Consider

$$M(x, y) dx + N(x, y) dy = 0. \quad (4)$$

Suppose that (4) is not exact. If we multiply by $\mu(x, y)$, we obtain

$$\mu(x, y)M(x, y) dx + \mu(x, y)N(x, y) dy = 0. \quad (5)$$

2.5 Exact Equations



Consider

$$M(x, y) dx + N(x, y) dy = 0. \quad (4)$$

Suppose that (4) is not exact. If we multiply by $\mu(x, y)$, we obtain

$$\mu(x, y)M(x, y) dx + \mu(x, y)N(x, y) dy = 0. \quad (5)$$

By 11, we know that

$$(5) \text{ is exact} \quad \Longleftrightarrow \quad (\mu M)_y = (\mu N)_x.$$

2.5 Exact Equations



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Suppose that (4) is not exact. If we multiply by $\mu(x, y)$, we obtain

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By 11, we know that

$$(5) \text{ is exact} \quad \Longleftrightarrow \quad (\mu M)_y = (\mu N)_x.$$

Now

$$\begin{aligned} (\mu M)_y &= (\mu N)_x \\ \mu_y M + \mu M_y &= \mu_x N + \mu N_x \\ M\mu_y - N\mu_x + (M_y - N_x)\mu &= 0. \end{aligned} \quad (6)$$

If we can find $\mu(x, y)$ which solves (6), then (5) is exact and we know how to solve exact equations.

2.5 Exact Equations



But (6) is a first order partial differential equation and PDEs are typically not easy to solve.

2.5 Exact Equations



But (6) is a first order partial differential equation and PDEs are typically not easy to solve. How can we make this easier? Instead of $\mu(x, y)$, we could look for $\mu(x)$.

2.5 Exact Equations



But (6) is a first order partial differential equation and PDEs are typically not easy to solve. How can we make this easier? Instead of $\mu(x, y)$, we could look for $\mu(x)$. Then $\mu_y = 0$ and (6) becomes

$$0 - N \frac{d\mu}{dx} + (M_y - N_x)\mu = 0$$

2.5 Exact Equations



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2.5 Exact Equations



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$$0 - N \frac{d\mu}{dx} + (M_y - N_x)\mu = 0$$

$$N \frac{d\mu}{dx} = (M_y - N_x)\mu$$

$$\boxed{\frac{d\mu}{dx} = \left(\frac{M_y - N_x}{N} \right) \mu.} \quad (7)$$

2.5 Exact Equations



But (6) is a first order partial differential equation and PDEs are typically not easy to solve. How can we make this easier? Instead of $\mu(x, y)$, we could look for $\mu(x)$. Then $\mu_y = 0$ and (6) becomes

$$0 - N \frac{d\mu}{dx} + (M_y - N_x)\mu = 0$$

$$N \frac{d\mu}{dx} = (M_y - N_x)\mu$$

$$\boxed{\frac{d\mu}{dx} = \left(\frac{M_y - N_x}{N} \right) \mu.} \quad (7)$$

If $\frac{M_y - N_x}{N}$ is a function only of x , then there is an integrating factor $\mu(x)$. Please note that (7) is both linear and separable.

If instead we looked for $\mu(y)$, we would obtain the ODE

$$\boxed{\frac{d\mu}{dy} = \left(\frac{N_x - M_y}{M} \right) \mu.} \quad (8)$$

Remark

If we were having classroom exams, you would be expected to remember (7) and (8).

2.5 Exact Equations



Example

Solve

$$(3xy + y^2) + (x^2 + xy)y' = 0.$$

2.5 Exact Equations



$$(3xy + y^2) + (x^2 + xy)y' = 0$$

We know that this equation is not exact. So we will try to find an integrating factor:

2.5 Exact Equations



$$(3xy + y^2) + (x^2 + xy)y' = 0$$

We know that this equation is not exact. So we will try to find an integrating factor: We have that

$$\begin{aligned} M &= 3xy + y^2 & M_y &= 3x + 2y \\ N &= x^2 + xy & N_x &= 2x + y \neq M_y \end{aligned}$$

2.5 Exact Equations



$$(3xy + y^2) + (x^2 + xy)y' = 0$$

We know that this equation is not exact. So we will try to find an integrating factor: We have that

$$\begin{aligned} M &= 3xy + y^2 & M_y &= 3x + 2y \\ N &= x^2 + xy & N_x &= 2x + y \neq M_y \end{aligned}$$

So

$$\frac{M_y - N_x}{N} =$$

and

$$\frac{N_x - M_y}{M} =$$

2.5 Exact Equations



$$(3xy + y^2) + (x^2 + xy)y' = 0$$

We know that this equation is not exact. So we will try to find an integrating factor: We have that

$$\begin{aligned} M &= 3xy + y^2 & M_y &= 3x + 2y \\ N &= x^2 + xy & N_x &= 2x + y \neq M_y \end{aligned}$$

So

$$\frac{M_y - N_x}{N} = \frac{(3x + 2y) - (2x + y)}{x^2 + xy} = \frac{x + y}{x(x + y)} = \frac{1}{x}$$

and

$$\frac{N_x - M_y}{M} =$$

2.5 Exact Equations



$$(3xy + y^2) + (x^2 + xy)y' = 0$$

We know that this equation is not exact. So we will try to find an integrating factor: We have that

$$\begin{aligned} M &= 3xy + y^2 & M_y &= 3x + 2y \\ N &= x^2 + xy & N_x &= 2x + y \neq M_y \end{aligned}$$

So

$$\frac{M_y - N_x}{N} = \frac{(3x + 2y) - (2x + y)}{x^2 + xy} = \frac{x + y}{x(x + y)} = \frac{1}{x}$$

and

$$\frac{N_x - M_y}{M} = \frac{(2x + y) - (3x + 2y)}{3xy + y^2} = \frac{-x - y}{y(3x + y)}.$$

2.5 Exact Equations



Note that $\frac{M_y - N_x}{N}$ is a function only of x – so it is possible to find an integrating factor $\mu(x)$. Moreover note that $\frac{N_x - M_y}{M}$ is *not* a function only of y – so it is *not* possible to find a $\mu(y)$.

2.5 Exact Equations



We calculate that

$$\frac{d\mu}{dx} = \left(\frac{M_y - N_x}{N} \right) \mu$$

$$\frac{d\mu}{dx} = \frac{\mu}{x}$$

$$\frac{d\mu}{\mu} = \frac{dx}{x}$$

$$\int \frac{d\mu}{\mu} = \int \frac{dx}{x}$$

$$\ln |\mu| = \ln |x| + C$$

$$\mu = cx$$

and we choose $c = 1$ for simplicity. So $\mu(x) = x$.

2.5 Exact Equations



$$(3xy + y^2) + (x^2 + xy)y' = 0$$

Multiplying our original ODE by $\mu(x) = x$ gives

$$(3x^2y + xy^2) + (x^3 + x^2y)y' = 0.$$

2.5 Exact Equations



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Multiplying our original ODE by $\mu(x) = x$ gives

$$(3x^2y + xy^2) + (x^3 + x^2y)y' = 0.$$

This ODE is exact ($M_y = 3x^2 + 2xy = N_x$) and we know how to solve exact equations.

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Multiplying our original ODE by $\mu(x) = x$ gives

$$(3x^2y + xy^2) + (x^3 + x^2y)y' = 0.$$

This ODE is exact ($M_y = 3x^2 + 2xy = N_x$) and we know how to solve exact equations. We must find ψ such that

$$\psi_x = 3x^2y + xy^2$$

$$\psi_y = x^3 + x^2y.$$

2.5 Exact Equations



$$\begin{aligned}\psi_x &= 3x^2y + xy^2 \\ \psi_y &= x^3 + x^2y\end{aligned}$$

Integrating ψ_x wrt x gives

$$\psi = x^3y + \frac{1}{2}x^2y^2 + h(y).$$

2.5 Exact Equations



$$\begin{aligned}\psi_x &= 3x^2y + xy^2 \\ \psi_y &= x^3 + x^2y\end{aligned}$$

Integrating ψ_x wrt x gives

$$\psi = x^3y + \frac{1}{2}x^2y^2 + h(y).$$

Hence

$$x^3 + x^2y = \psi_y = \frac{\partial}{\partial y} \left(x^3y + \frac{1}{2}x^2y^2 + h(y) \right) = x^3 + x^2y + h'(y)$$

2.5 Exact Equations



$$\begin{aligned}\psi_x &= 3x^2y + xy^2 \\ \psi_y &= x^3 + x^2y\end{aligned}$$

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$$\psi = x^3y + \frac{1}{2}x^2y^2 + h(y).$$

Hence

$$x^3 + x^2y = \psi_y = \frac{\partial}{\partial y} \left(x^3y + \frac{1}{2}x^2y^2 + h(y) \right) = x^3 + x^2y + h'(y)$$

and we see that we may choose $h(y) = 0$.

2.5 Exact Equations



$$\begin{aligned}\psi_x &= 3x^2y + xy^2 \\ \psi_y &= x^3 + x^2y\end{aligned}$$

Integrating ψ_x wrt x gives

$$\psi = x^3y + \frac{1}{2}x^2y^2 + h(y).$$

Hence

$$x^3 + x^2y = \psi_y = \frac{\partial}{\partial y} \left(x^3y + \frac{1}{2}x^2y^2 + h(y) \right) = x^3 + x^2y + h'(y)$$

and we see that we may choose $h(y) = 0$. Therefore

$$\psi = x^3y + \frac{1}{2}x^2y^2.$$

2.5 Exact Equations



$$\begin{aligned}\psi_x &= 3x^2y + xy^2 \\ \psi_y &= x^3 + x^2y\end{aligned}$$

Integrating ψ_x wrt x gives

$$\psi = x^3y + \frac{1}{2}x^2y^2 + h(y).$$

Hence

$$x^3 + x^2y = \psi_y = \frac{\partial}{\partial y} \left(x^3y + \frac{1}{2}x^2y^2 + h(y) \right) = x^3 + x^2y + h'(y)$$

and we see that we may choose $h(y) = 0$. Therefore

$$\psi = x^3y + \frac{1}{2}x^2y^2.$$

So the solution to the ODE is

$$\boxed{x^3y + \frac{1}{2}x^2y^2 = c.}$$

2.5 Exact Equations



Example

Solve

$$ye^{xy} + \left(\left(\frac{2}{y} + x \right) e^{xy} \right) y' = 0.$$

This ODE is not exact (you check!).

2.5 Exact Equations



Example

Solve

$$ye^{xy} + \left(\left(\frac{2}{y} + x \right) e^{xy} \right) y' = 0.$$

This ODE is not exact (you check!).

$$\frac{M_y - N_x}{N} = \frac{e^{xy} + xy e^{xy} - e^{xy} - (2 + xy) e^{xy}}{\left(\frac{2}{y} + x \right) e^{xy}} = \frac{-2}{\frac{2}{y} + x}$$

$$\frac{N_x - M_y}{M} = \frac{2e^{xy}}{ye^{xy}} = \frac{2}{y}.$$

Since $\frac{N_x - M_y}{M}$ is a function only of y , we look for $\mu(y)$.

2.5 Exact Equations



$$\frac{d\mu}{dy} = \frac{N_x - M_y}{M} \mu = \frac{2e^{xy}}{ye^{xy}} \mu = \frac{2}{y} \mu$$

2.5 Exact Equations



$$\frac{d\mu}{dy} = \frac{N_x - M_y}{M} \mu = \frac{2e^{xy}}{ye^{xy}} \mu = \frac{2}{y} \mu$$

-
-
- (you complete this calculation)
-
-

Therefore $\mu(y) = y^2$.

2.5 Exact Equations



Multiplying our ODE by y^2 gives

$$y^3 e^{xy} + ((2y + xy^2) e^{xy}) y' = 0.$$

Multiplying our ODE by y^2 gives

$$y^3 e^{xy} + ((2y + xy^2) e^{xy}) y' = 0.$$

-
-
- (you complete this calculation)
-
-

Hence the solution is

$$\boxed{y^2 e^{xy} = c.}$$

Substitutions

2.6 Substitutions



Recall how we calculate an integral such as $\int 3x^2 \sin x^3 \, dx$.

Recall how we calculate an integral such as $\int 3x^2 \sin x^3 dx$. We use a substitution, in this case $u = x^3$, to turn a difficult integral into an easy integral:

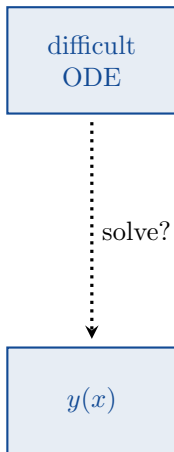
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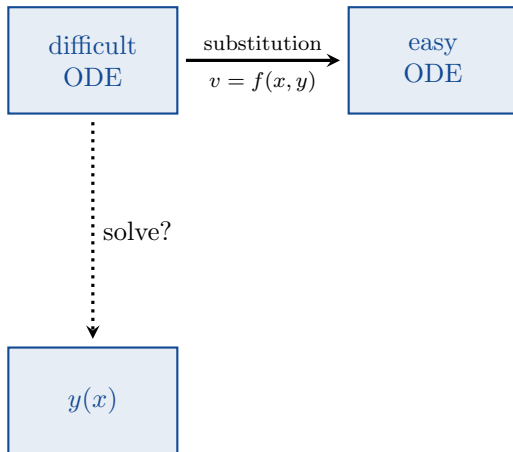
$$\underbrace{\int 3x^2 \sin x^3 dx}_{\text{difficult}} = \underbrace{\int \sin u du}_{\text{easy}}.$$

Sometimes we can use the same idea to solve ODEs.

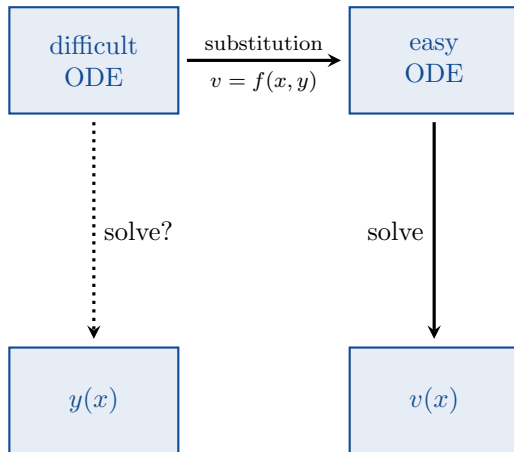
2.6 Substitutions



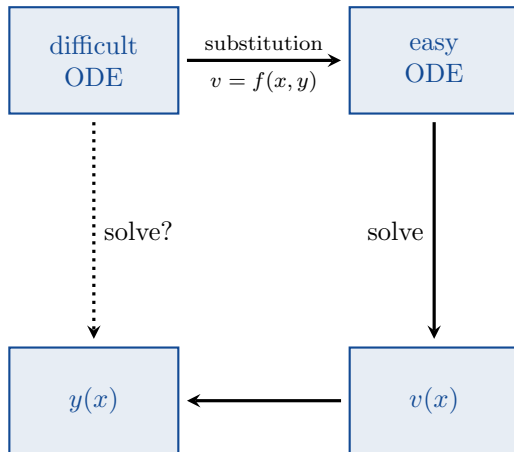
2.6 Substitutions



2.6 Substitutions



2.6 Substitutions



2.6 Substitutions



We will use substitutions to solve two types of first order ODE:

- Homogeneous Equations;
- Bernoulli Equations.

Homogeneous Equations

Definition

The first order ODE $\frac{dy}{dx} = f(x, y)$ is called *homogeneous* iff we can write it as

$$\frac{dy}{dx} = g\left(\frac{y}{x}\right).$$

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For example, the following ODEs are homogeneous:

$$\frac{dy}{dx} = \cos\left(\frac{y}{x}\right)$$

$$\frac{dy}{dx} = \left(\frac{y}{x}\right)^3 + \frac{y}{x}$$

$$\frac{dy}{dx} = \cos\left(\frac{x}{y}\right)$$

$$\frac{dy}{dx} = \frac{\frac{y}{x} - 4}{1 - \frac{y}{x}}$$

2.6 Substitutions



For a homogeneous equation, we use the substitution

$$v(x) = \frac{y}{x}.$$

2.6 Substitutions



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Note that $y = xv(x)$ and

$$\frac{dy}{dx} = \frac{d}{dx}(xv(x)) = v + x \frac{dv}{dx}.$$

2.6 Substitutions



Example

Solve $\frac{dy}{dx} = \frac{y - 4x}{x - y}$.

2.6 Substitutions



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If we substitute in $v = \frac{y}{x}$ we get

$$\frac{dy}{dx} = \frac{v - 4}{1 - v}.$$

2.6 Substitutions



$$\frac{dy}{dx} = \frac{v-4}{1-v}$$

But remember that $\frac{dy}{dx} = v + x \frac{dv}{dx}$.

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2.6 Substitutions



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$$v + x \frac{dv}{dx} = \frac{v-4}{1-v}$$

and

$$x \frac{dv}{dx} = \frac{v-4}{1-v} - v$$

2.6 Substitutions



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But remember that $\frac{dy}{dx} = v + x \frac{dv}{dx}$. Hence

$$v + x \frac{dv}{dx} = \frac{v-4}{1-v}$$

and

$$x \frac{dv}{dx} = \frac{v-4}{1-v} - v = \frac{v-4}{1-v} - \frac{v-v^2}{1-v} = \frac{v^2-4}{1-v}$$

2.6 Substitutions



Note that

$$x \frac{dv}{dx} = \frac{v^2 - 4}{1 - v}$$

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is a separable equation. You know how to solve separable equations – the following should be revision for you. We rearrange to

$$\left(\frac{1 - v}{v^2 - 4} \right) dv = \frac{dx}{x}$$

$$\left(-\frac{3}{4(v + 2)} - \frac{1}{4(v - 2)} \right) dv = \frac{dx}{x}$$

2.6 Substitutions



$$\left(-\frac{3}{4(v+2)} - \frac{1}{4(v-2)}\right) dv = \frac{dx}{x}$$

then integrate to find

$$-\frac{3}{4} \ln |v+2| - \frac{1}{4} \ln |v-2| = \ln |x| + k$$

$$\ln |v+2|^3 + \ln |v-2| = \ln |x|^{-4} - 4k$$

$$|v+2|^3 |v-2| = c |x|^{-4} \quad (c = \pm e^{-4k})$$

$$|x|^4 |v+2|^3 |v-2| = c$$

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Now we have an equation for v . The final step is to find an equation for y .

$$|vx + 2x|^3 |vx - 2x| = c.$$

If we substitute $y = vx$ into this equation, we find the solution

$$\boxed{|y + 2x|^3 |y - 2x| = c.}$$

Remark

To solve a homogeneous equation:

- 1 Substitute $v = \frac{y}{x}$ (and $\frac{dy}{dx} = v + x\frac{dv}{dx}$);
- 2 Solve a separable equation;
- 3 Substitute $y = vx$.

2.6 Substitutions



Example

Solve $\frac{dy}{dx} = \frac{x^2 + 3y^2}{2xy}$.

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First we rearrange

$$\frac{dy}{dx} = \frac{1 + 3\frac{y^2}{x^2}}{2\frac{y}{x}}$$

and substitute $v = \frac{y}{x}$ and $\frac{dy}{dx} = v + x\frac{dv}{dx}$ to get

$$v + x\frac{dv}{dx} = \frac{1 + 3v^2}{2v}.$$

2.6 Substitutions



Rearranging gives

$$x \frac{dv}{dx} = \frac{1 + 3v^2}{2v} - v = \frac{1 + 3v^2 - 2v^2}{2v} = \frac{1 + v^2}{2v}.$$

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This is a separable equation which we can solve:

$$\begin{aligned}\frac{2v dv}{1 + v^2} &= \frac{dx}{x} \\ \int \frac{2v dv}{1 + v^2} &= \int \frac{dx}{x} \\ \ln |1 + v^2| &= \ln |x| + k \\ 1 + v^2 &= cx \\ 1 + v^2 - cx &= 0.\end{aligned}$$

2.6 Substitutions



Substituting $v = \frac{y}{x}$ then gives

$$1 + \frac{y^2}{x^2} - cx = 0$$

and

$$x^2 + y^2 - cx^3 = 0.$$

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Definition

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2.6 Substitutions



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Solve $\frac{dy}{dx} - \left(\frac{3}{2x}\right)y = 2xy^{-1}.$

2.6 Substitutions



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Note first that this ODE has $n = -1$. Therefore we will use the substitution $v = y^{1-n} = y^{1-(-1)} = y^2$.

2.6 Substitutions



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$$\text{Solve } \frac{dy}{dx} - \left(\frac{3}{2x}\right)y = 2xy^{-1}.$$

Note first that this ODE has $n = -1$. Therefore we will use the substitution $v = y^{1-n} = y^{1-(-1)} = y^2$. This means that $y = v^{\frac{1}{2}}$ and

$$\frac{dy}{dx} = \frac{dy}{dv} \frac{dv}{dx} = \frac{1}{2}v^{-\frac{1}{2}} \frac{dv}{dx}.$$

2.6 Substitutions



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$$\frac{1}{2}v^{-\frac{1}{2}} \frac{dv}{dx} - \left(\frac{3}{2x}\right)v^{\frac{1}{2}} = 2xv^{-\frac{1}{2}}.$$

2.6 Substitutions



Multiplying by $2v^{\frac{1}{2}}$ gives

$$\frac{dv}{dx} - \frac{3}{x}v = 4x$$

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2.6 Substitutions



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2.6 Substitutions



Multiplying by $2v^{\frac{1}{2}}$ gives

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which is a linear equation. You know how to solve linear equations, so the following should be revision for you. We multiply by the integrating factor

$$\mu(x) = e^{\int -\frac{3}{x} dx} = e^{-3 \ln|x|} = \dots = x^{-3}$$

to get

$$x^{-3} \frac{dv}{dx} - 3x^{-4}v = 4x^{-2}$$

which is

$$\frac{d}{dx} (x^{-3}v) = 4x^{-2}.$$



Integrating gives

$$\begin{aligned}x^{-3}v &= -4x^{-1} + C \\ v &= -4x^2 + Cx^3.\end{aligned}$$

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But $v = y^2$, so the solution is

$$\boxed{y^2 = -4x^2 + Cx^3.}$$

Remark

To solve a Bernoulli equation:

- 1 Substitute $v = y^{1-n}$;
- 2 Solve a linear equation;
- 3 Substitute $y^{1-n} = v$.

2.6 Substitutions



Example

Solve $x \frac{dy}{dx} + 6y = 3xy^{\frac{4}{3}}$.

Example

Solve $x \frac{dy}{dx} + 6y = 3xy^{\frac{4}{3}}$.

Note that this time we have $n = \frac{4}{3}$ and $v = y^{1-n} = y^{-\frac{1}{3}}$. Hence $y = v^{-3}$ and

$$\frac{dy}{dx} = \frac{dy}{dv} \frac{dv}{dx} = -3v^{-4} \frac{dv}{dx}.$$

2.6 Substitutions



Thus our ODE becomes

$$-3xv^{-4}\frac{dv}{dx} + 6v^{-3} = 3xv^{-4}$$

$$-x\frac{dv}{dx} + 2v = x$$

$$\frac{dv}{dx} - \frac{2}{x}v = -1.$$

2.6 Substitutions



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$$v = x + Cx^2.$$

Thus our ODE becomes

$$\begin{aligned}-3xv^{-4}\frac{dv}{dx} + 6v^{-3} &= 3xv^{-4} \\ -x\frac{dv}{dx} + 2v &= x \\ \frac{dv}{dx} - \frac{2}{x}v &= -1.\end{aligned}$$

This is a linear equation which we can solve using the integrating factor $\mu(x) = x^{-2}$. Please check that its solution is

$$v = x + Cx^2.$$

Finally we use $v = y^{-\frac{1}{3}}$ to find that

$$y = \frac{1}{(x + Cx^2)^3}.$$

Next Time

- 3.1 Homogeneous Equations with Constant Coefficients
- 3.2 Fundamental Sets of Solutions
- 3.3 Complex Roots of the Characteristic Equation