

Lecture 4

- 11. Vectors
- 12. The Dot Product
- 13. The Cross Product

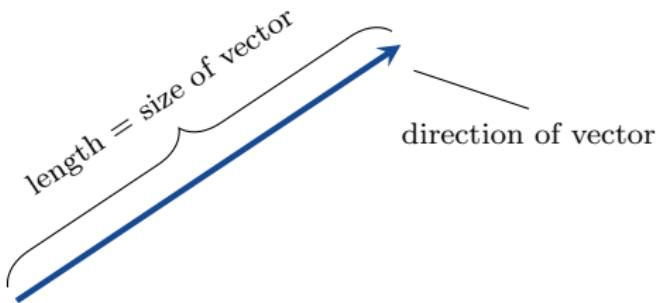


Vectors

11. Vectors

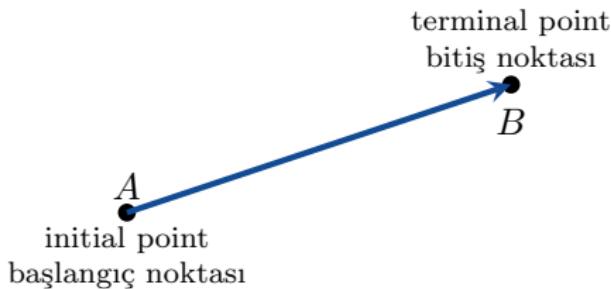


For some quantities (mass, time, distance, ...) we only need a number. For some quantities (velocity, force, ...) we need a number and a direction.



A *vector* is an object which has a size (length) and a direction.

11. Vectors

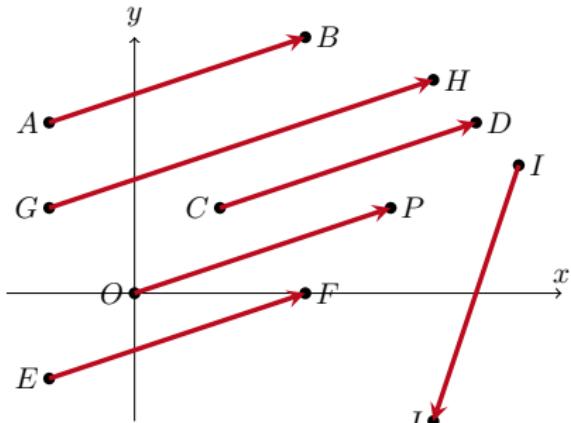


Definition

The vector \overrightarrow{AB} has *initial point* A and *terminal point* B .

The *length* of \overrightarrow{AB} is written $\|\overrightarrow{AB}\|$.

11. Vectors



Two vectors are equal if they have the same length and the same direction.

We can say that

$$\overrightarrow{AB} = \overrightarrow{CD} = \overrightarrow{EF} = \overrightarrow{OP}.$$

Note that $\overrightarrow{AB} \neq \overrightarrow{GH}$ because the lengths are different, and $\overrightarrow{AB} \neq \overrightarrow{IJ}$ because the directions are different.

Notation

When we use a computer, we use bold letters for vectors: \mathbf{u} , \mathbf{v} , \mathbf{w} , When we use a pen, we use underlined letters for vectors: \underline{u} , \underline{v} , \underline{w} ,

If we type $a\mathbf{u} + b\mathbf{v}$ or write $a\underline{u} + b\underline{v}$, then

- a and b are numbers; and
- \mathbf{u} , \mathbf{v} , \underline{u} and \underline{v} are vectors.

11. Vectors

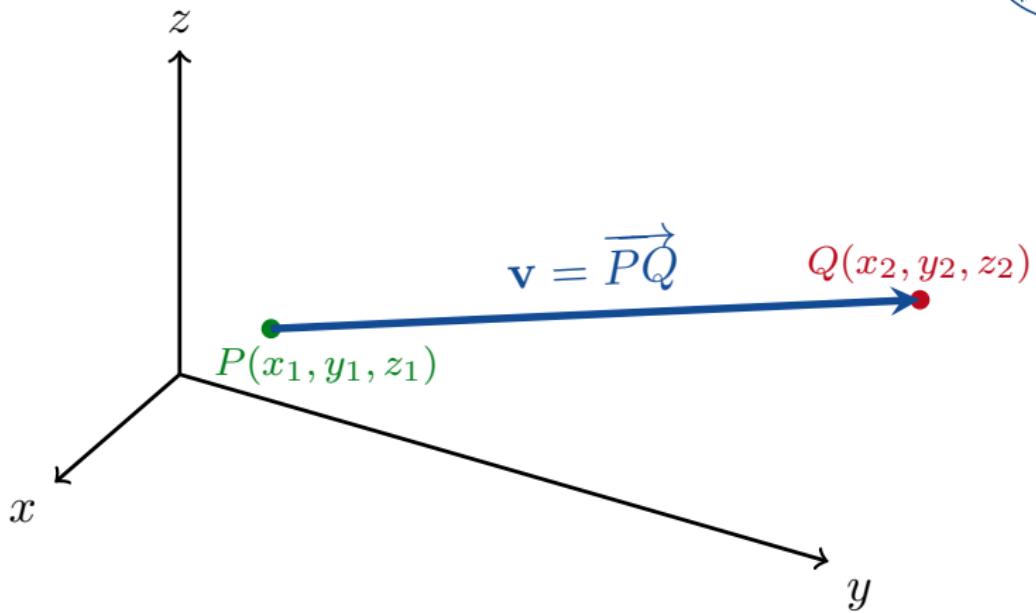


Definition

In \mathbb{R}^2 : If \mathbf{v} has initial point $(0, 0)$ and terminal point (v_1, v_2) , then the *component form* of \mathbf{v} is $\mathbf{v} = (v_1, v_2)$.

In \mathbb{R}^3 : If \mathbf{v} has initial point $(0, 0, 0)$ and terminal point (v_1, v_2, v_3) , then the *component form* of \mathbf{v} is $\mathbf{v} = (v_1, v_2, v_3)$.

11. Vectors



$$(v_1, v_2, v_3) = \mathbf{v} = (x_2 - x_1, y_2 - y_1, z_2 - z_1)$$

11. Vectors



Definition

In \mathbb{R}^2 : The *norm* (or *length*) of $\mathbf{v} = (v_1, v_2)$ is

$$\|\mathbf{v}\| = \sqrt{v_1^2 + v_2^2}$$

In \mathbb{R}^3 : The *norm* of $\mathbf{v} = \overrightarrow{PQ}$ is

$$\begin{aligned}\|\mathbf{v}\| &= \sqrt{v_1^2 + v_2^2 + v_3^2} \\ &= \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}.\end{aligned}$$

The vectors $\mathbf{0} = (0, 0)$ and $\mathbf{0} = (0, 0, 0)$ have norm $\|\mathbf{0}\| = 0$. If $\mathbf{v} \neq \mathbf{0}$, then $\|\mathbf{v}\| > 0$.

Example

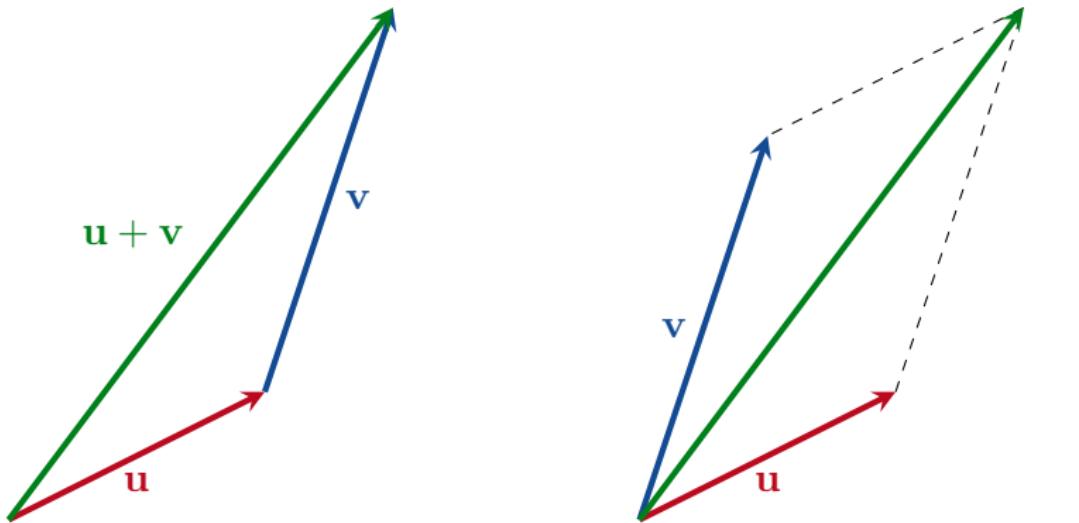
Find (a) the component form; and (b) the norm of the vector with initial point $P(-3, 4, 1)$ and terminal point $Q(-5, 2, 2)$.

solution:

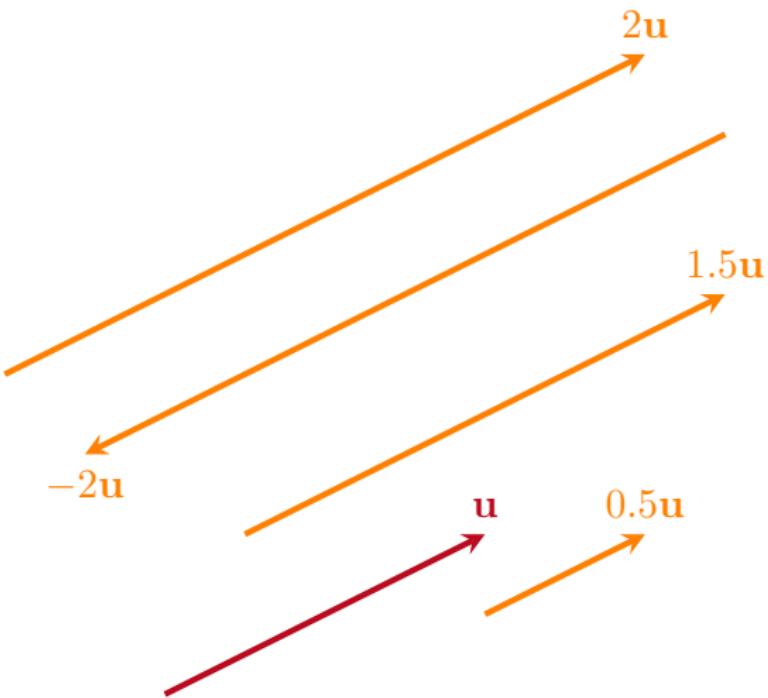
$$\begin{aligned}\text{(a)} \quad \mathbf{v} &= (v_1, v_2, v_3) = Q - P = (-5, 2, 2) - (-3, 4, 1) \\ &= (-2, -2, 1).\end{aligned}$$

$$\text{(b)} \quad \|\mathbf{v}\| = \sqrt{v_1^2 + v_2^2 + v_3^2} = \sqrt{(-2)^2 + (-2)^2 + 1^2} = \sqrt{9} = 3.$$

Vector Algebra



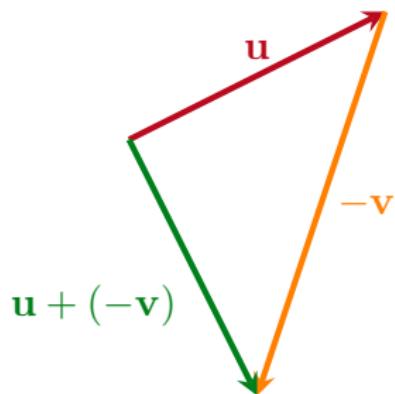
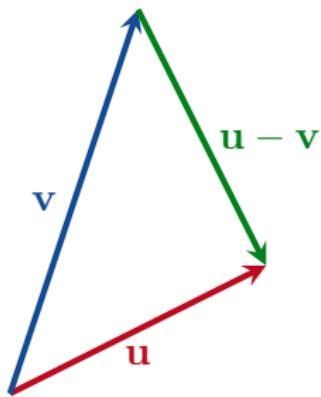
11. Vectors



11. Vectors



$$\mathbf{v} + (\mathbf{u} - \mathbf{v}) = \mathbf{u}$$



11. Vectors



Let $\mathbf{u} = (u_1, u_2, u_3)$ and $\mathbf{v} = (v_1, v_2, v_3)$ be vectors. Let k be a number. Then

$$\mathbf{u} + \mathbf{v} = (u_1 + v_1, u_2 + v_2, u_3 + v_3)$$

and

$$k\mathbf{u} = (ku_1, ku_2, ku_3).$$

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Note that

$$\begin{aligned}\|k\mathbf{u}\| &= \|(ku_1, ku_2, ku_3)\| = \sqrt{(ku_1)^2 + (ku_2)^2 + (ku_3)^2} \\ &= \sqrt{k^2u_1^2 + k^2u_2^2 + k^2u_3^2} = \sqrt{k^2(u_1^2 + u_2^2 + u_3^2)} \\ &= \sqrt{k^2} \sqrt{u_1^2 + u_2^2 + u_3^2} = |k| \|\mathbf{u}\|.\end{aligned}$$

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The vector $-\mathbf{u} = (-1)\mathbf{u}$ has the same length as \mathbf{u} , but points in the opposite direction.

11. Vectors



Example

Let $\mathbf{u} = (-1, 3, 1)$ and $\mathbf{v} = (4, 7, 0)$. Find (a) $2\mathbf{u} + 3\mathbf{v}$, (b) $\mathbf{u} - \mathbf{v}$, and (c) $\left\| \frac{1}{2}\mathbf{u} \right\|$.

solution:

(a) $2\mathbf{u} + 3\mathbf{v} = 2(-1, 3, 1) + 3(4, 7, 0) = (-2, 6, 2) + (12, 21, 0) = (10, 27, 2);$

(b) $\mathbf{u} - \mathbf{v} = (-1, 3, 1) - (4, 7, 0) = (-5, -4, 1);$

(c) $\left\| \frac{1}{2}\mathbf{u} \right\| = \frac{1}{2} \left\| \mathbf{u} \right\| = \frac{1}{2} \sqrt{(-1)^2 + 3^2 + 1^2} = \frac{1}{2} \sqrt{11}.$

Properties of Vector Operations

Let \mathbf{u} , \mathbf{v} and \mathbf{w} be vectors. Let a and b be numbers. Then

- 1 $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u};$
- 2 $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w});$
- 3 $\mathbf{u} + \mathbf{0} = \mathbf{u};$
- 4 $\mathbf{u} + (-\mathbf{u}) = \mathbf{0};$
- 5 $0\mathbf{u} = \mathbf{0};$
- 6 $1\mathbf{u} = \mathbf{u};$
- 7 $a(b\mathbf{u}) = (ab)\mathbf{u};$
- 8 $a(\mathbf{u} + \mathbf{v}) = a\mathbf{u} + a\mathbf{v};$
- 9 $(a + b)\mathbf{u} = a\mathbf{u} + b\mathbf{u}.$

11. Vectors



Remark

We **can not** multiply vectors. Never never never never write " \mathbf{uv} ".

Unit Vectors

Definition

\mathbf{u} is called a *unit vector* $\iff \|\mathbf{u}\| = 1$.

11. Vectors



Example

$\mathbf{u} = (2^{-\frac{1}{2}}, \frac{1}{2}, -\frac{1}{2})$ is a unit vector because

$$\|\mathbf{u}\| = \sqrt{\left(\frac{1}{\sqrt{2}}\right)^2 + \left(\frac{1}{2}\right)^2 + \left(-\frac{1}{2}\right)^2} = \sqrt{\frac{1}{2} + \frac{1}{4} + \frac{1}{4}} = 1.$$

11. Vectors



In \mathbb{R}^2 : The *standard unit vectors* are $\mathbf{i} = (1, 0)$ and $\mathbf{j} = (0, 1)$.

In \mathbb{R}^3 : The *standard unit vectors* are $\mathbf{i} = (1, 0, 0)$, $\mathbf{j} = (0, 1, 0)$ and $\mathbf{k} = (0, 0, 1)$. Any vector $\mathbf{v} \in \mathbb{R}^3$ can be written

$$\begin{aligned}\mathbf{v} &= (v_1, v_2, v_3) = (v_1, 0, 0) + (0, v_2, 0) + (0, 0, v_3) \\ &= v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}.\end{aligned}$$

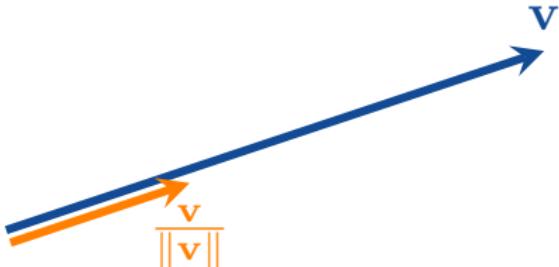
11. Vectors



If $\|\mathbf{v}\| \neq 0$, then $\frac{\mathbf{v}}{\|\mathbf{v}\|}$ is a unit vector because

$$\left\| \frac{\mathbf{v}}{\|\mathbf{v}\|} \right\| = \left\| \frac{1}{\|\mathbf{v}\|} \mathbf{v} \right\| = \frac{1}{\|\mathbf{v}\|} \|\mathbf{v}\| = 1.$$

Clearly $\frac{\mathbf{v}}{\|\mathbf{v}\|}$ and \mathbf{v} point in the same direction.



11. Vectors



Example

Find a unit vector \mathbf{u} which points in the same direction as $\overrightarrow{P_1P_2}$, where $P_1(1, 0, 1)$ and $P_2(3, 2, 0)$.

solution:

We calculate that

$$\overrightarrow{P_1P_2} = P_2 - P_1 = (3, 2, 0) - (1, 0, 1) = (2, 2, -1) = 2\mathbf{i} + 2\mathbf{j} - \mathbf{k}$$

and that $\|\overrightarrow{P_1P_2}\| = \sqrt{2^2 + 2^2 + (-1)^2} = 3$. The required unit vector is

$$\mathbf{u} = \frac{\overrightarrow{P_1P_2}}{\|\overrightarrow{P_1P_2}\|} = \frac{2\mathbf{i} + 2\mathbf{j} - \mathbf{k}}{3} = \frac{2}{3}\mathbf{i} + \frac{2}{3}\mathbf{j} - \frac{1}{3}\mathbf{k}.$$



The Dot Product

12. The Dot Product

Definition

In \mathbb{R}^2 , the *dot product* of $\mathbf{u} = (u_1, u_2) = u_1\mathbf{i} + u_2\mathbf{j}$ and $\mathbf{v} = (v_1, v_2) = v_1\mathbf{i} + v_2\mathbf{j}$ is

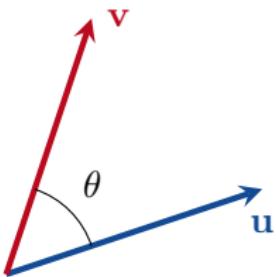
$$\mathbf{u} \cdot \mathbf{v} = u_1v_1 + u_2v_2.$$

Definition

In \mathbb{R}^3 , the *dot product* of $\mathbf{u} = (u_1, u_2, u_3) = u_1\mathbf{i} + u_2\mathbf{j} + u_3\mathbf{k}$ and $\mathbf{v} = (v_1, v_2, v_3) = v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}$ is

$$\mathbf{u} \cdot \mathbf{v} = u_1v_1 + u_2v_2 + u_3v_3.$$

12. The Dot Product



Theorem

The angle between \mathbf{u} and \mathbf{v} is

$$\theta = \cos^{-1} \left(\frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{u}\| \|\mathbf{v}\|} \right).$$

12. The Dot Product



Example

$$\begin{aligned}(1, -2, -1) \cdot (-6, 2, -3) &= (1 \times -6) + (-2 \times 2) + (-1 \times -3) \\&= -6 - 4 + 3 = -7.\end{aligned}$$

12. The Dot Product



Example

$$\begin{aligned} \left(\frac{1}{2}\mathbf{i} + 3\mathbf{j} + \mathbf{k}\right) \cdot (4\mathbf{i} - \mathbf{j} + 2\mathbf{k}) &= \left(\frac{1}{2} \times 4\right) + (3 \times -1) + (1 \times 2) \\ &= 2 - 3 + 2 = 1. \end{aligned}$$

12. The Dot Product

Example

Find the angle between $\mathbf{u} = \mathbf{i} - 2\mathbf{j} - 2\mathbf{k}$ and $\mathbf{v} = 6\mathbf{i} + 3\mathbf{j} + 2\mathbf{k}$.

solution: Since $\mathbf{u} \cdot \mathbf{v} = (1, -2, -2) \cdot (6, 3, 2) =$

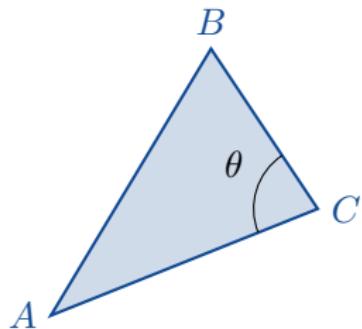
$$(1 \times 6) + (-2 \times 3) + (-2 \times 2) = 6 - 6 - 4 = -4,$$

$$\|\mathbf{u}\| = \sqrt{1^2 + (-2)^2 + (-2)^2} = \sqrt{9} = 3 \text{ and}$$

$$\|\mathbf{v}\| = \sqrt{6^2 + 3^2 + 2^2} = \sqrt{49} = 7, \text{ we have that}$$

$$\theta = \cos^{-1} \left(\frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{u}\| \|\mathbf{v}\|} \right) = \cos^{-1} \left(-\frac{4}{21} \right) \approx 1.76 \text{ radians} \approx 98.5^\circ.$$

12. The Dot Product



Example

If $A(0, 0)$, $B(3, 5)$ and $C(5, 2)$, find $\theta = \angle ACB$.

12. The Dot Product

solution: θ is the angle between \overrightarrow{CA} and \overrightarrow{CB} . We calculate that

$$\overrightarrow{CA} = A - C = (0, 0) - (5, 2) = (-5, -2),$$

$$\overrightarrow{CB} = B - C = (3, 5) - (5, 2) = (-2, 3),$$

$$\overrightarrow{CA} \cdot \overrightarrow{CB} = (-5, -2) \cdot (-2, 3) = 4,$$

$$\|\overrightarrow{CA}\| = \sqrt{(-5)^2 + (-2)^2} = \sqrt{29} \text{ and}$$

$$\|\overrightarrow{CB}\| = \sqrt{(-2)^2 + 3^2} = \sqrt{13}. \text{ Therefore}$$

$$\theta = \cos^{-1} \left(\frac{\overrightarrow{CA} \cdot \overrightarrow{CB}}{\|\overrightarrow{CA}\| \|\overrightarrow{CB}\|} \right) = \cos^{-1} \left(\frac{4}{\sqrt{29}\sqrt{13}} \right)$$

$$\approx 78.1^\circ \approx 1.36 \text{ radians.}$$

12. The Dot Product

Definition

\mathbf{u} and \mathbf{v} are *orthogonal* $\iff \mathbf{u} \cdot \mathbf{v} = 0$.

Remark

Note that

$$\mathbf{u} \cdot \mathbf{v} = \|\mathbf{u}\| \|\mathbf{v}\| \cos \theta$$

by Theorem 9. Therefore

$$\mathbf{u} \text{ and } \mathbf{v} \text{ are orthogonal} \iff \begin{cases} \mathbf{u} = \mathbf{0} \\ \text{or} \\ \mathbf{v} = \mathbf{0} \\ \text{or} \\ \theta = 90^\circ. \end{cases}$$

12. The Dot Product



Example

$\mathbf{u} = (3, -2)$ and $\mathbf{v} = (4, 6)$ are orthogonal because

$$\mathbf{u} \cdot \mathbf{v} = (3, -2) \cdot (4, 6) = (3 \times 4) + (-2 \times 6) = 12 - 12 = 0.$$

12. The Dot Product



Example

$\mathbf{u} = 3\mathbf{i} - 2\mathbf{j} + \mathbf{k}$ and $\mathbf{v} = 2\mathbf{j} + 4\mathbf{k}$ are orthogonal because
 $\mathbf{u} \cdot \mathbf{v} = (3 \times 0) + (-2 \times 2) + (1 \times 4) = 0 - 4 + 4 = 0$.

12. The Dot Product



Example

0 is orthogonal to every vector **u** because

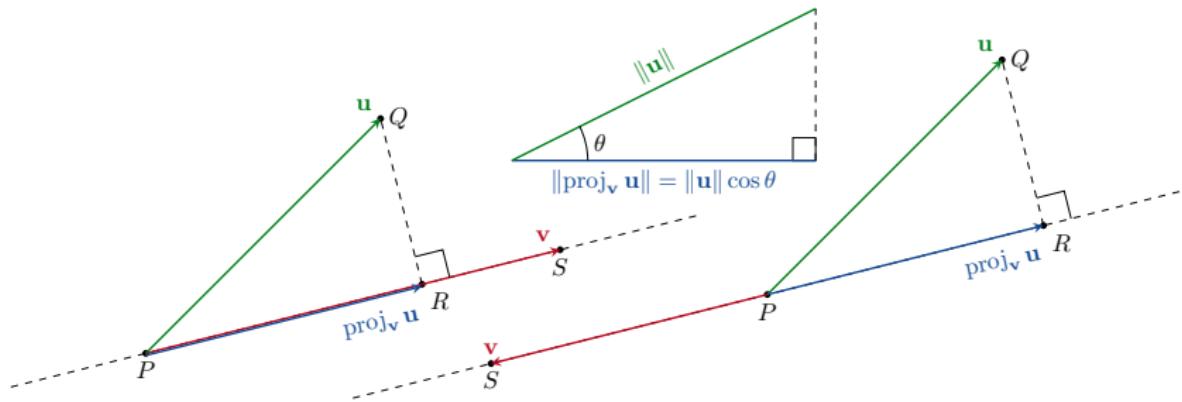
$$\mathbf{0} \cdot \mathbf{u} = (0, 0, 0) \cdot (u_1, u_2, u_3) = 0u_1 + 0u_2 + 0u_3 = 0.$$

Properties of the Dot Product

Let \mathbf{u} , \mathbf{v} and \mathbf{w} be vectors. Let k be a number. Then

- 1 $\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u}$;
- 2 $(k\mathbf{u}) \cdot \mathbf{v} = \mathbf{u} \cdot (k\mathbf{v}) = k(\mathbf{u} \cdot \mathbf{v})$;
- 3 $\mathbf{u} \cdot (\mathbf{v} + \mathbf{w}) = (\mathbf{u} \cdot \mathbf{v}) + (\mathbf{u} \cdot \mathbf{w})$;
- 4 $\mathbf{u} \cdot \mathbf{u} = \|\mathbf{u}\|^2$; and
- 5 $\mathbf{0} \cdot \mathbf{u} = 0$.

Vector Projections



Definition

The *vector projection* of \mathbf{u} onto \mathbf{v} is the vector

$$\text{proj}_{\mathbf{v}} \mathbf{u} = \overrightarrow{PR}.$$

12. The Dot Product



Now

$$\begin{aligned}\text{proj}_{\mathbf{v}} \mathbf{u} &= (\text{length of } \text{proj}_{\mathbf{v}} \mathbf{u}) \begin{pmatrix} \text{a unit vector in} \\ \text{the same} \\ \text{direction as } \mathbf{v} \end{pmatrix} \\ &= \|\text{proj}_{\mathbf{v}} \mathbf{u}\| \left(\frac{\mathbf{v}}{\|\mathbf{v}\|} \right) \\ &= \|\mathbf{u}\| (\cos \theta) \left(\frac{\mathbf{v}}{\|\mathbf{v}\|} \right) \\ &= \left(\frac{\|\mathbf{u}\| \|\mathbf{v}\| \cos \theta}{\|\mathbf{v}\|^2} \right) \mathbf{v} \\ &= \left(\frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{v}\|^2} \right) \mathbf{v}.\end{aligned}$$

Since this is an important formula, we write it as a theorem.

12. The Dot Product



Theorem

The vector projection of \mathbf{u} onto \mathbf{v} is

$$\text{proj}_{\mathbf{v}} \mathbf{u} = \left(\frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{v}\|^2} \right) \mathbf{v}.$$

12. The Dot Product

Example

Find the vector projection of $\mathbf{u} = 6\mathbf{i} + 3\mathbf{j} + 2\mathbf{k}$ onto $\mathbf{v} = \mathbf{i} - 2\mathbf{j} - 2\mathbf{k}$.

solution:

$$\begin{aligned}\text{proj}_{\mathbf{v}} \mathbf{u} &= \left(\frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{v}\|^2} \right) \mathbf{v} = \left(\frac{6 - 6 - 4}{1 + 4 + 4} \right) (\mathbf{i} - 2\mathbf{j} - 2\mathbf{k}) \\ &= -\frac{4}{9}\mathbf{i} + \frac{8}{9}\mathbf{j} + \frac{8}{9}\mathbf{k}.\end{aligned}$$

12. The Dot Product



Example

Find the vector projection of $\mathbf{F} = 5\mathbf{i} + 2\mathbf{j}$ onto $\mathbf{v} = \mathbf{i} - 3\mathbf{j}$.

solution:

$$\begin{aligned}\text{proj}_{\mathbf{v}} \mathbf{F} &= \left(\frac{\mathbf{F} \cdot \mathbf{v}}{\|\mathbf{v}\|^2} \right) \mathbf{v} = \left(\frac{5 - 6}{1 + 9} \right) (\mathbf{i} - 3\mathbf{j}) \\ &= -\frac{1}{10}\mathbf{i} + \frac{3}{10}\mathbf{j}.\end{aligned}$$

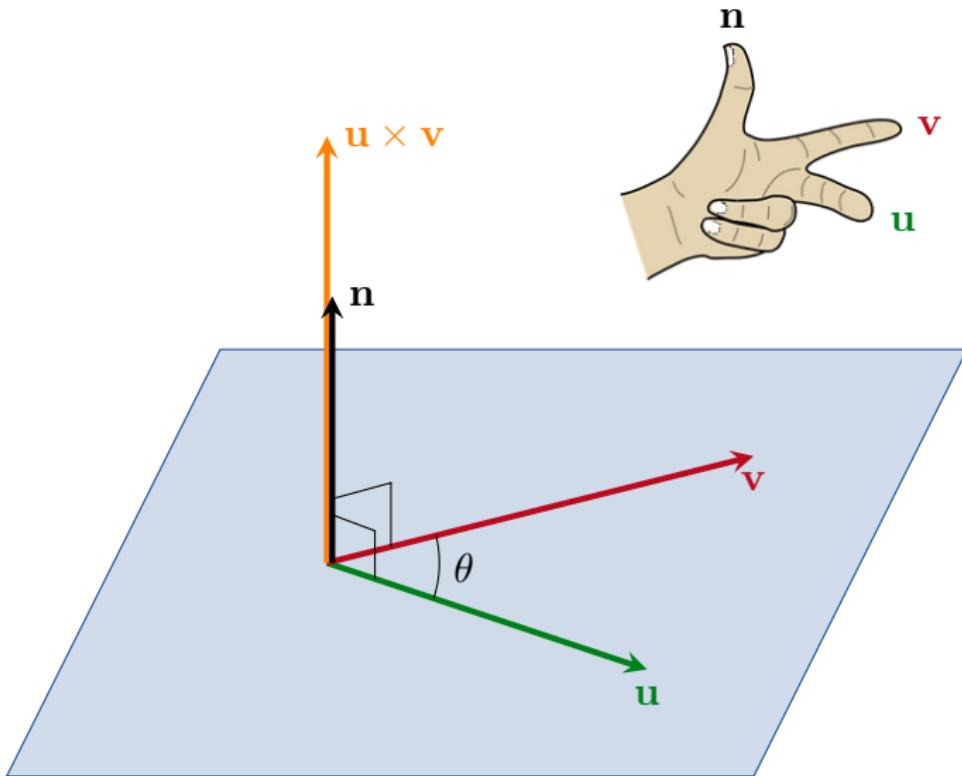


The Cross Product

13. The Cross Product

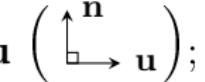
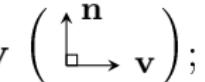


13. The Cross Product



13. The Cross Product

Let \mathbf{n} be a unit vector which satisfies

- 1 \mathbf{n} is orthogonal to \mathbf{u} () ;
- 2 \mathbf{n} is orthogonal to \mathbf{v} () ; and
- 3 the direction of \mathbf{n} is chosen using the left-hand rule.

Definition

The *cross product* of \mathbf{u} and \mathbf{v} is

$$\mathbf{u} \times \mathbf{v} = \|\mathbf{u}\| \|\mathbf{v}\| (\sin \theta) \mathbf{n}.$$

13. The Cross Product



Remark

- $\mathbf{u} \cdot \mathbf{v}$ is a number.
- $\mathbf{u} \times \mathbf{v}$ is a vector.

13. The Cross Product



Remark

$$\begin{pmatrix} \mathbf{u} \text{ and } \mathbf{v} \\ \text{are} \\ \text{parallel} \end{pmatrix} \iff \theta = 0^\circ \text{ or } 180^\circ$$
$$\implies \sin \theta = 0 \implies \mathbf{u} \times \mathbf{v} = \mathbf{0}.$$

13. The Cross Product



Properties of the Cross Product

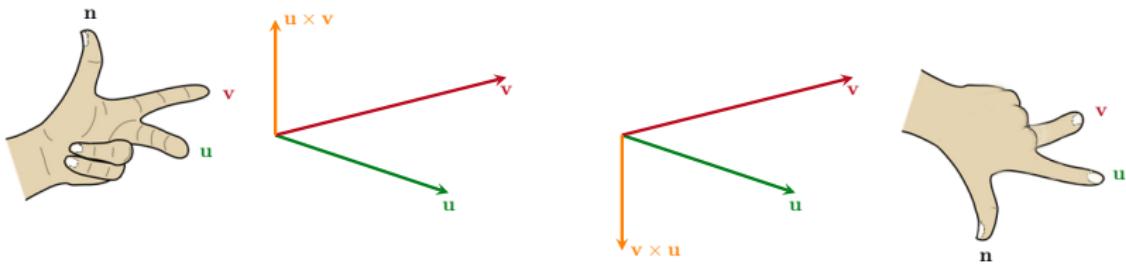
Let \mathbf{u} , \mathbf{v} and \mathbf{w} be vectors. Let r and s be numbers. Then

- 1 $(r\mathbf{u}) \times (s\mathbf{v}) = (rs)(\mathbf{u} \times \mathbf{v});$
- 2 $\mathbf{u} \times (\mathbf{v} + \mathbf{w}) = (\mathbf{u} \times \mathbf{v}) + (\mathbf{u} \times \mathbf{w});$
- 3 $\mathbf{v} \times \mathbf{u} = -\mathbf{u} \times \mathbf{v};$
- 4 $(\mathbf{v} + \mathbf{w}) \times \mathbf{u} = (\mathbf{v} \times \mathbf{u}) + (\mathbf{w} \times \mathbf{u});$
- 5 $\mathbf{0} \times \mathbf{u} = \mathbf{0};$ and
- 6 $\mathbf{u} \times (\mathbf{v} \times \mathbf{w}) = (\mathbf{u} \cdot \mathbf{w})\mathbf{v} - (\mathbf{u} \cdot \mathbf{v})\mathbf{w}.$

13. The Cross Product



Property (iii)

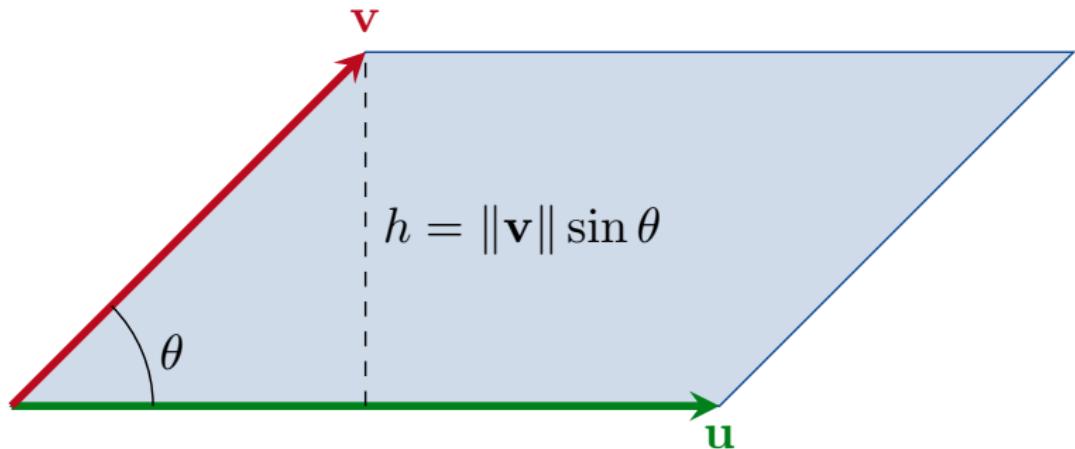


$$\mathbf{v} \times \mathbf{u} = -\mathbf{u} \times \mathbf{v}$$

13. The Cross Product

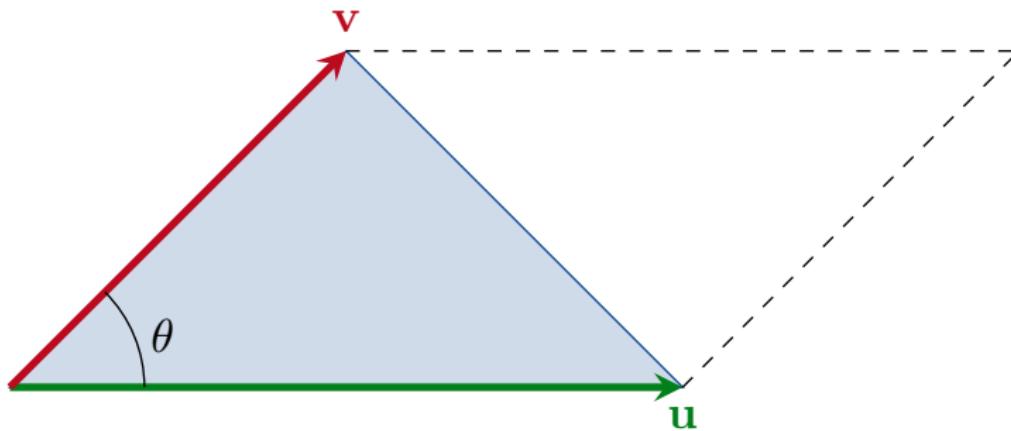


Area of a Parallelogram



$$\text{area} = (\text{base})(\text{height}) = \|\mathbf{u}\| \|\mathbf{v}\| \sin \theta = \|\mathbf{u} \times \mathbf{v}\|.$$

Area of a Triangle

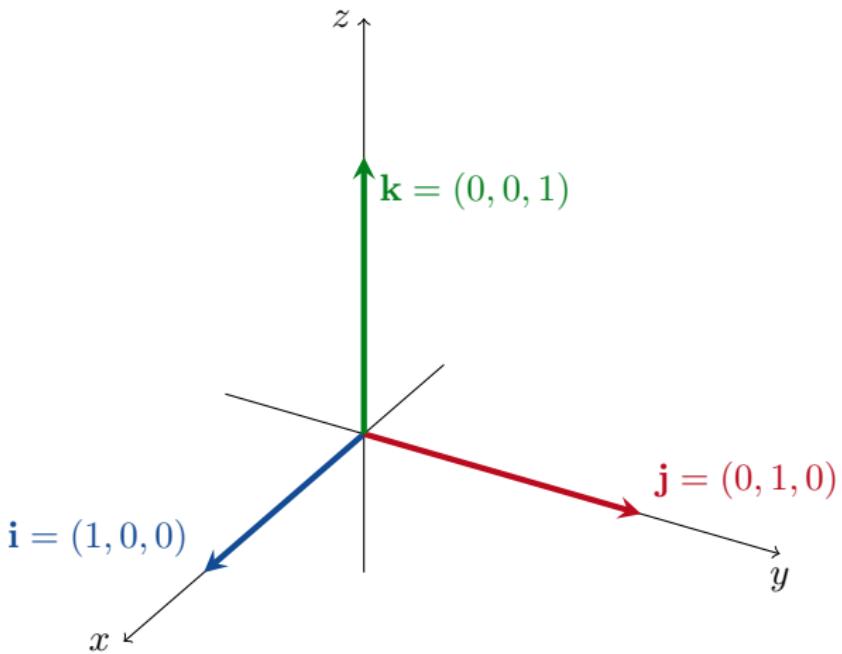


$$\begin{aligned}\text{area of triangle} &= \frac{1}{2} (\text{area of parallelogram}) \\ &= \frac{1}{2} \|\mathbf{u} \times \mathbf{v}\|.\end{aligned}$$

13. The Cross Product



A Formula for $\mathbf{u} \times \mathbf{v}$



13. The Cross Product



Note first that

$$\mathbf{i} \times \mathbf{i} = \|\mathbf{i}\| \|\mathbf{i}\| \sin 0^\circ \mathbf{n} = \mathbf{0}.$$

Similarly $\mathbf{j} \times \mathbf{j} = \mathbf{0}$ and $\mathbf{k} \times \mathbf{k} = \mathbf{0}$ also.

13. The Cross Product



Next note that $\mathbf{i} \times \mathbf{j}$ must point in the same direction as \mathbf{k} by the left-hand rule. Thus

$$\mathbf{i} \times \mathbf{j} = \|\mathbf{i}\| \|\mathbf{j}\| \sin 90^\circ \mathbf{k} = \mathbf{k}.$$

We then immediately also have

$$\mathbf{j} \times \mathbf{i} = -(\mathbf{i} \times \mathbf{j}) = -\mathbf{k}.$$

It is left for you to check that

$$\mathbf{j} \times \mathbf{k} = \mathbf{i}, \quad \mathbf{k} \times \mathbf{j} = -\mathbf{i}, \quad \mathbf{i} \times \mathbf{k} = -\mathbf{j} \quad \text{and} \quad \mathbf{k} \times \mathbf{i} = \mathbf{j}.$$

13. The Cross Product



Now suppose that $\mathbf{u} = u_1\mathbf{i} + u_2\mathbf{j} + u_3\mathbf{k}$ and $\mathbf{v} = v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}$. Then we can calculate that

$$\begin{aligned}\mathbf{u} \times \mathbf{v} &= (u_1\mathbf{i} + u_2\mathbf{j} + u_3\mathbf{k}) \times (v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}) \\&= u_1v_1\mathbf{i} \times \mathbf{i} + u_1v_2\mathbf{i} \times \mathbf{j} + u_1v_3\mathbf{i} \times \mathbf{k} + u_2v_1\mathbf{j} \times \mathbf{i} + u_2v_2\mathbf{j} \times \mathbf{j} \\&\quad + u_2v_3\mathbf{j} \times \mathbf{k} + u_3v_1\mathbf{k} \times \mathbf{i} + u_3v_2\mathbf{k} \times \mathbf{j} + u_3v_3\mathbf{k} \times \mathbf{k} \\&= \mathbf{0} + u_1v_2\mathbf{k} - u_1v_3\mathbf{j} - u_2v_1\mathbf{k} + \mathbf{0} + u_2v_3\mathbf{i} + u_3v_1\mathbf{j} - u_3v_2\mathbf{i} + \mathbf{0} \\&= (u_2v_3 - u_3v_2)\mathbf{i} - (u_1v_3 - u_3v_1)\mathbf{j} + (u_1v_2 - u_2v_1)\mathbf{k}.\end{aligned}$$

13. The Cross Product



Theorem

If $\mathbf{u} = u_1\mathbf{i} + u_2\mathbf{j} + u_3\mathbf{k}$ and $\mathbf{v} = v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}$, then

$$\mathbf{u} \times \mathbf{v} = (u_2v_3 - u_3v_2)\mathbf{i} - (u_1v_3 - u_3v_1)\mathbf{j} + (u_1v_2 - u_2v_1)\mathbf{k}$$

13. The Cross Product



If you studied matrices and determinants at high school, then you may prefer to use the following symbolic determinant formula instead.

$$\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}.$$

13. The Cross Product



Example

Find $\mathbf{u} \times \mathbf{v}$ and $\mathbf{v} \times \mathbf{u}$ if $\mathbf{u} = 2\mathbf{i} + \mathbf{j} + \mathbf{k}$ and $\mathbf{v} = -4\mathbf{i} + 3\mathbf{j} + \mathbf{k}$.

solution:

$$\mathbf{u} \times \mathbf{v} = (1 - 3)\mathbf{i} - (2 - -4)\mathbf{j} + (6 - -4)\mathbf{k} = -2\mathbf{i} - 6\mathbf{j} + 10\mathbf{k}$$

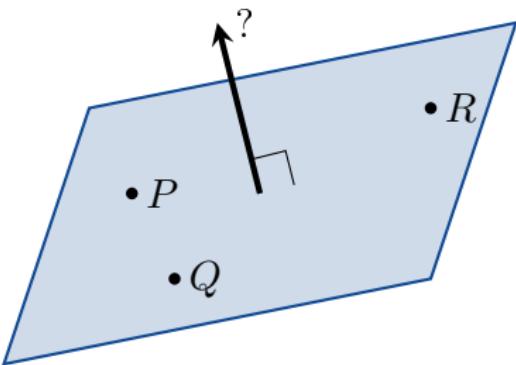
and

$$\mathbf{v} \times \mathbf{u} = -\mathbf{u} \times \mathbf{v} = 2\mathbf{i} + 6\mathbf{j} - 10\mathbf{k}.$$

13. The Cross Product

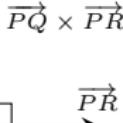
Example

Find a vector perpendicular to the plane containing the three points $P(1, -1, 0)$, $Q(2, 1, -1)$ and $R(-1, 1, 2)$.



13. The Cross Product

solution: The vector $\overrightarrow{PQ} \times \overrightarrow{PR}$ is perpendicular to the plane

because  and  . We calculate that

$$\begin{aligned}\overrightarrow{PQ} &= Q - P = (2, 1, -1) - (1, -1, 0) \\ &= (2 - 1, 1 + 1, -1 - 0) = \mathbf{i} + 2\mathbf{j} - \mathbf{k}\end{aligned}$$

$$\begin{aligned}\overrightarrow{PR} &= R - P = (-1, 1, 2) - (1, -1, 0) \\ &= (-1 - 1, 1 + 1, 2 - 0) = -2\mathbf{i} + 2\mathbf{j} + 2\mathbf{k}\end{aligned}$$

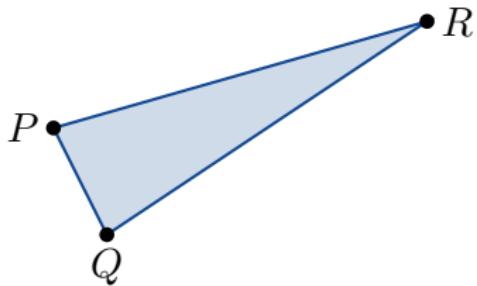
$$\overrightarrow{PQ} \times \overrightarrow{PR} = (4 + 2)\mathbf{i} - (2 - 2)\mathbf{j} + (2 + 4)\mathbf{k} = 6\mathbf{i} + 6\mathbf{k}.$$

13. The Cross Product

Example

Find the area of triangle PQR .

$P(1, -1, 0)$, $Q(2, 1, -1)$ and $R(-1, 1, 2)$



13. The Cross Product



solution: The area of the triangle is

$$\begin{aligned}\text{area} &= \frac{1}{2} \left\| \overrightarrow{PQ} \times \overrightarrow{PR} \right\| = \frac{1}{2} \|6\mathbf{i} + 6\mathbf{k}\| \\ &= \frac{1}{2} \sqrt{6^2 + 0^2 + 6^2} = 3\sqrt{2}.\end{aligned}$$

13. The Cross Product



Example

Find a unit vector perpendicular to the plane containing P , Q and R .

$P(1, -1, 0)$, $Q(2, 1, -1)$ and $R(-1, 1, 2)$

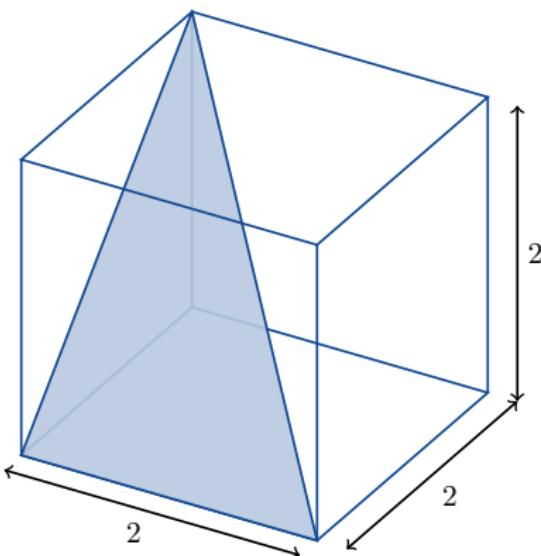
solution: We know that $\overrightarrow{PQ} \times \overrightarrow{PR}$ is perpendicular to the plane. We just need to normalise this vector to find a unit vector.

$$\mathbf{n} = \frac{\overrightarrow{PQ} \times \overrightarrow{PR}}{\|\overrightarrow{PQ} \times \overrightarrow{PR}\|} = \frac{6\mathbf{i} + 6\mathbf{k}}{6\sqrt{2}} = \frac{1}{\sqrt{2}}\mathbf{i} + \frac{1}{\sqrt{2}}\mathbf{k}.$$

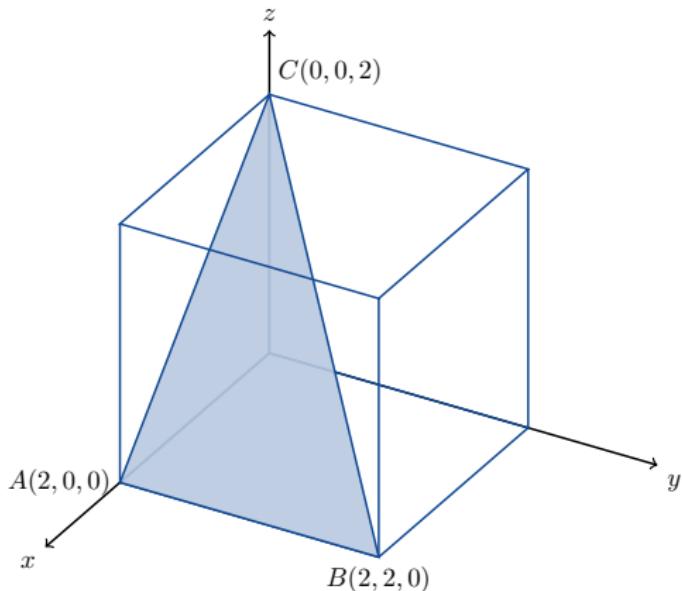
13. The Cross Product

Example

A triangle is inscribed inside a cube of side 2 as shown below.
Use the cross product to find the area of the triangle.



13. The Cross Product



solution: First we draw coordinate axes and assign coordinates to the vertices of the triangle.

13. The Cross Product

Then we can calculate

$$\overrightarrow{AB} = B - A = (2, 2, 0) - (2, 0, 0) = (0, 2, 0) = 2\mathbf{j}$$

and

$$\overrightarrow{AC} = C - A = (0, 0, 2) - (2, 0, 0) = (-2, 0, 2) = -2\mathbf{i} + 2\mathbf{k}.$$

It follows that

$$\begin{aligned}\overrightarrow{AB} \times \overrightarrow{AC} &= (2\mathbf{j}) \times (-2\mathbf{i} + 2\mathbf{k}) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 2 & 0 \\ -2 & 0 & 2 \end{vmatrix} \\ &= \mathbf{i}(4 - 0) - \mathbf{j}(0 - 0) + \mathbf{k}(0 - -4) = 4\mathbf{i} + 4\mathbf{k}.\end{aligned}$$

13. The Cross Product



Therefore

$$\begin{aligned}\text{area of triangle} &= \frac{1}{2} \left\| \overrightarrow{AB} \times \overrightarrow{AC} \right\| = \frac{1}{2} \sqrt{4^2 + 0^2 + 4^2} \\ &= \frac{1}{2} \sqrt{32} = \frac{1}{2} \sqrt{4} \sqrt{8} = \sqrt{8} = 2\sqrt{2}.\end{aligned}$$



The Triple Scalar Product

Definition

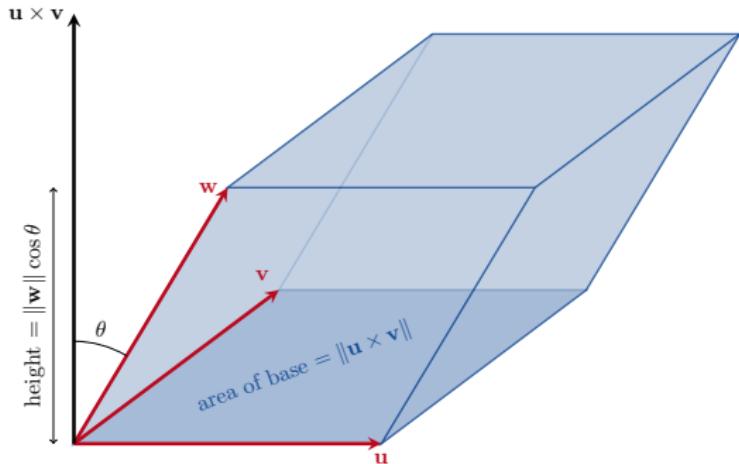
The *triple scalar product* of \mathbf{u} , \mathbf{v} and \mathbf{w} is

$$(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w}.$$

13. The Cross Product



The Volume of a Parallelepiped



$$\text{volume} = (\text{area of base})(\text{height}) = \|\mathbf{u} \times \mathbf{v}\| \|\mathbf{w}\| \cos \theta = |(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w}|$$

13. The Cross Product



One Final Comment

We can do the dot product in both \mathbb{R}^2 and \mathbb{R}^3 . But we can only do the cross product in \mathbb{R}^3 . There is no cross product in \mathbb{R}^2 .



Next Time

- 14. Lines
- 15. Planes
- 16. Projections