

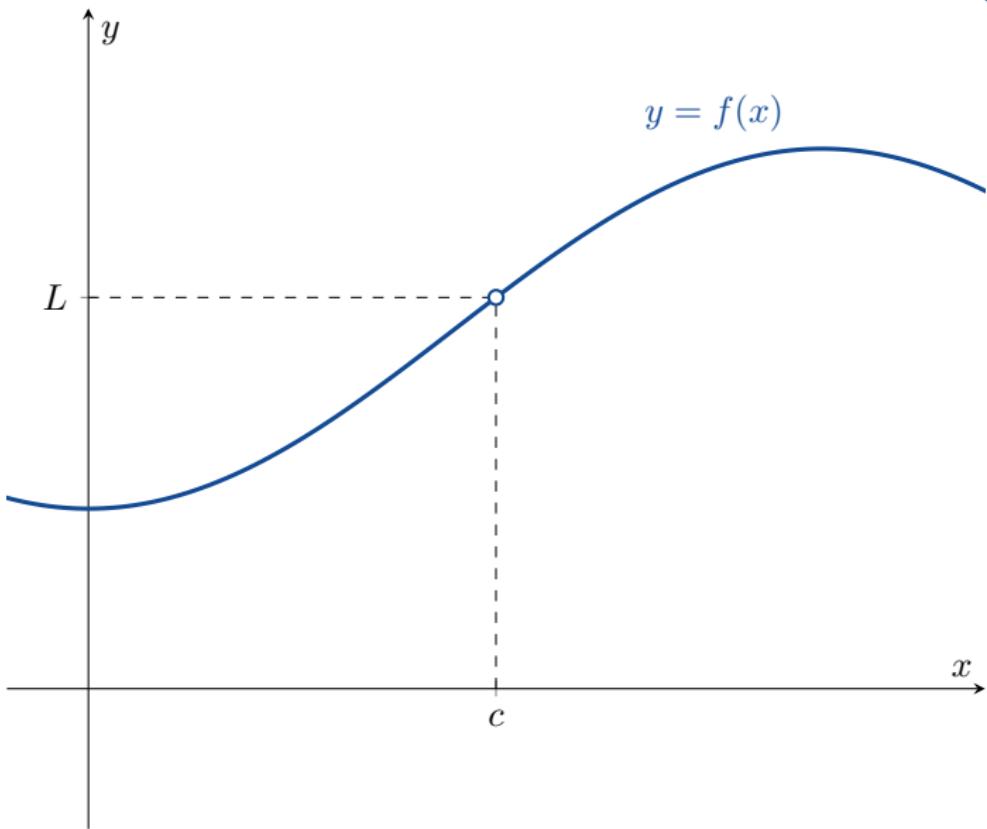
Lecture 3

- 2.4 One-Sided Limits
- 2.5 Continuity
- 2.6 Limits Involving Infinity; Asymptotes of Graphs

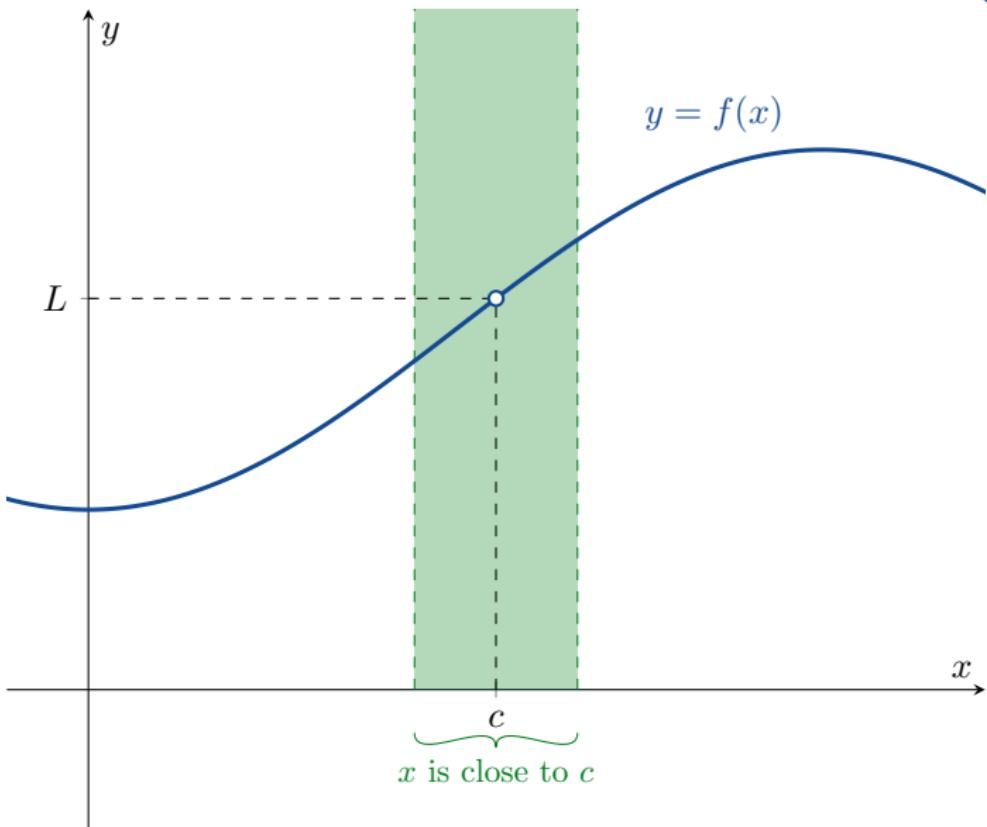


One-Sided Limits

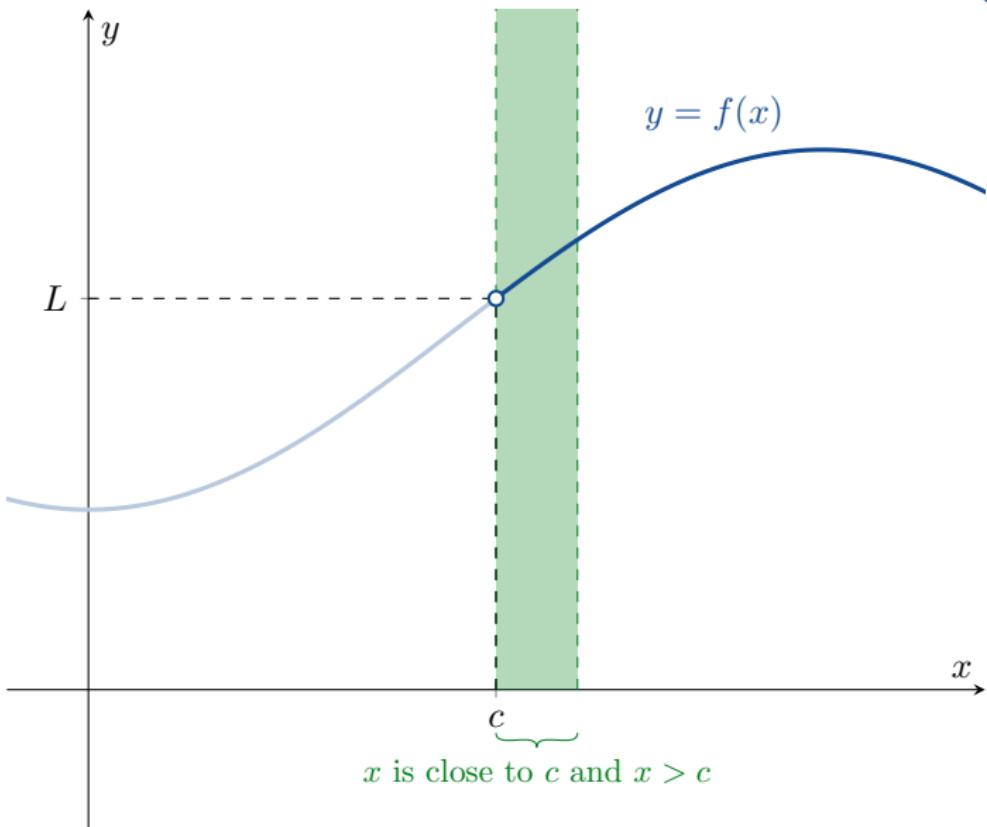
2.4 One-Sided Limits



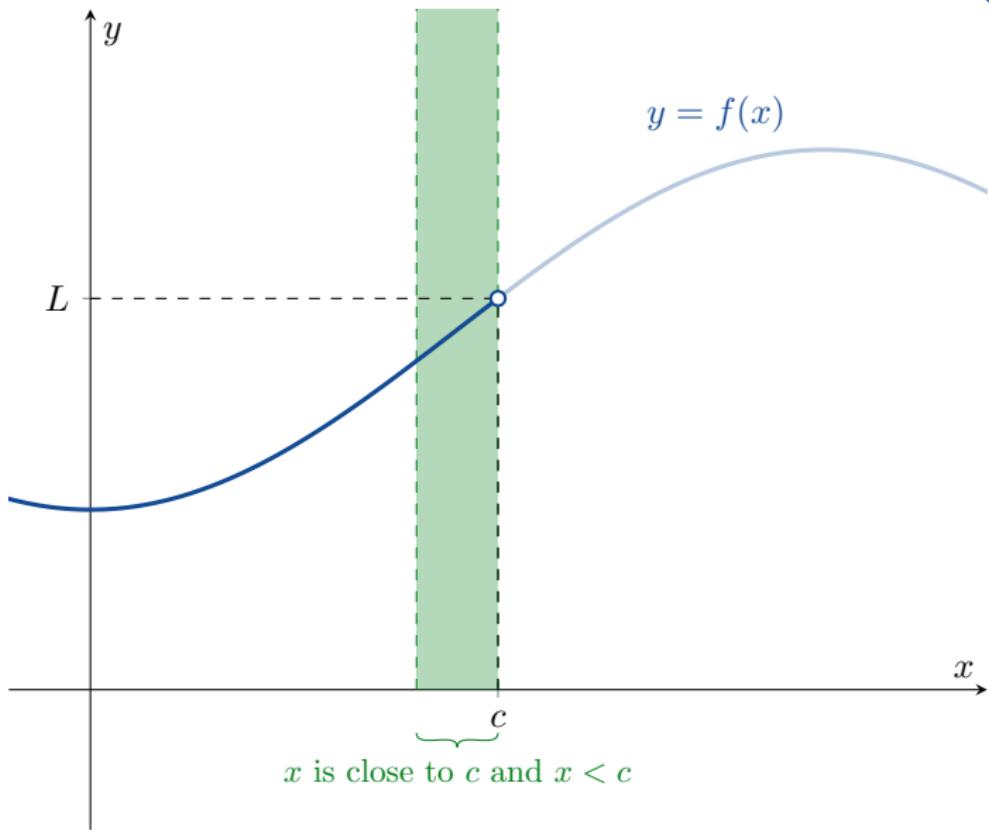
2.4 One-Sided Limits



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2.4 One-Sided Limits



$\lim_{x \rightarrow c^+} f(x)$ is called the *right-hand limit* of $f(x)$ at c .

This means, the limit of $f(x)$ if we only look at values of x on the right of c ($x > c$).

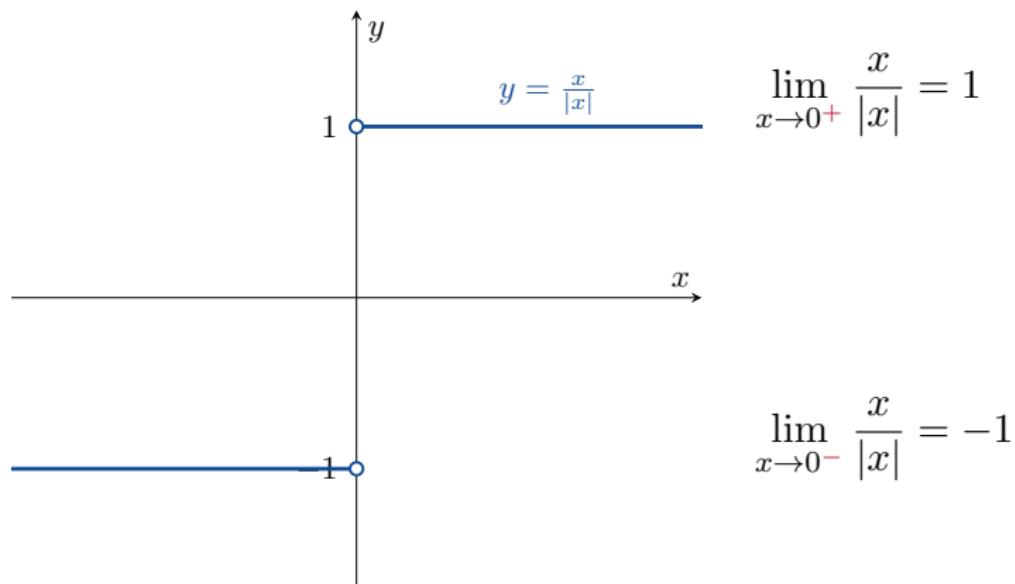
$\lim_{x \rightarrow c^-} f(x)$ is called the *left-hand limit* of $f(x)$ at c .

This means, the limit of $f(x)$ if we only look at values of x on the left of c ($x < c$).

2.4 One-Sided Limits

Example

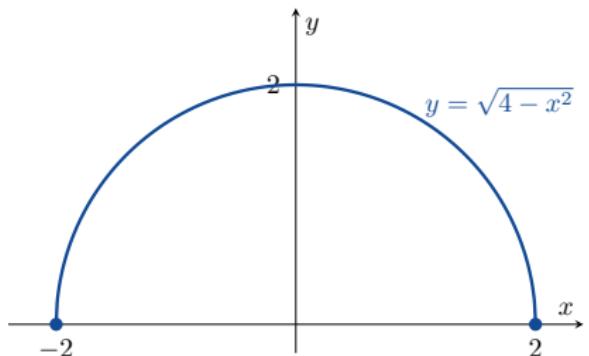
$$y = \frac{x}{|x|}$$



2.4 One-Sided Limits

Example

$$f : [-2, 2] \rightarrow \mathbb{R}, f(x) = \sqrt{4 - x^2}.$$



$$\lim_{x \rightarrow 2^-} \sqrt{4 - x^2} = 0$$

$$\lim_{x \rightarrow 2^+} \sqrt{4 - x^2} \text{ doesn't exist}$$

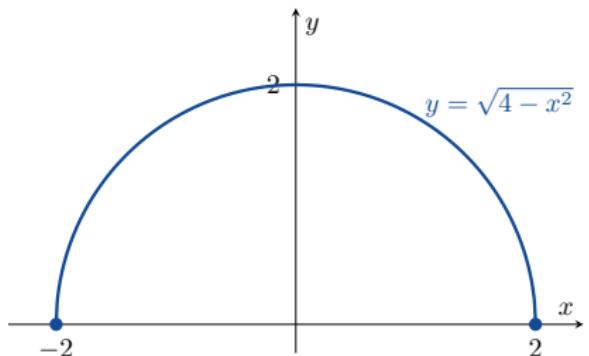
$$\lim_{x \rightarrow -2^+} \sqrt{4 - x^2} = 0$$

$$\lim_{x \rightarrow -2^-} \sqrt{4 - x^2} \text{ doesn't exist}$$

2.4 One-Sided Limits

Example

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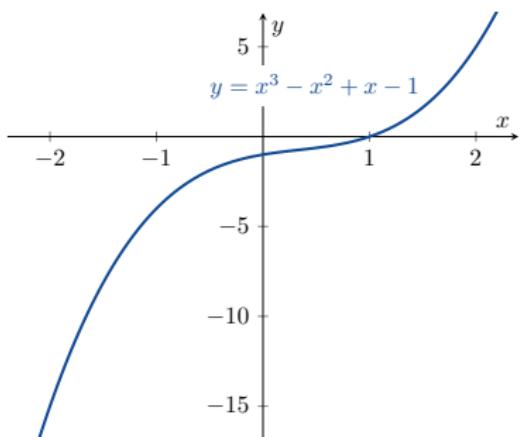
$$\lim_{x \rightarrow -2^-} \sqrt{4 - x^2} \text{ doesn't exist}$$

Note that $\lim_{x \rightarrow c} \sqrt{4 - x^2}$, $\lim_{x \rightarrow c^+} \sqrt{4 - x^2}$ and $\lim_{x \rightarrow c^-} \sqrt{4 - x^2}$ all exist for all $c \in (-2, 2)$.

2.4 One-Sided Limits

Example

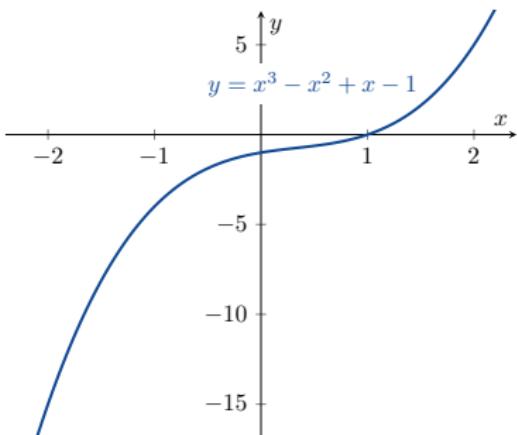
$$y = x^3 - x^2 + x - 1$$



2.4 One-Sided Limits

Example

$$y = x^3 - x^2 + x - 1$$

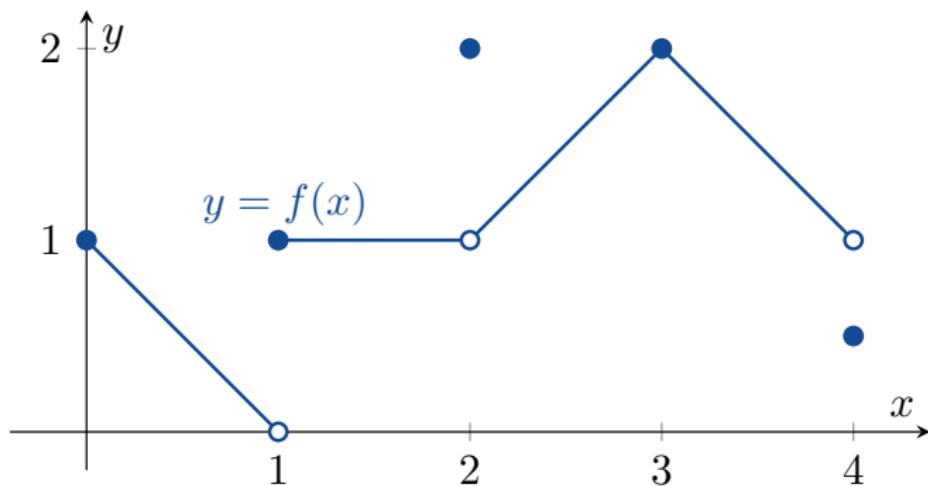


$\lim_{x \rightarrow c} (x^3 - x^2 + x - 1)$ exists for all $c \in (-\infty, \infty)$

$\lim_{x \rightarrow c^+} (x^3 - x^2 + x - 1)$ exists for all $c \in (-\infty, \infty)$

$\lim_{x \rightarrow c^-} (x^3 - x^2 + x - 1)$ exists for all $c \in (-\infty, \infty)$

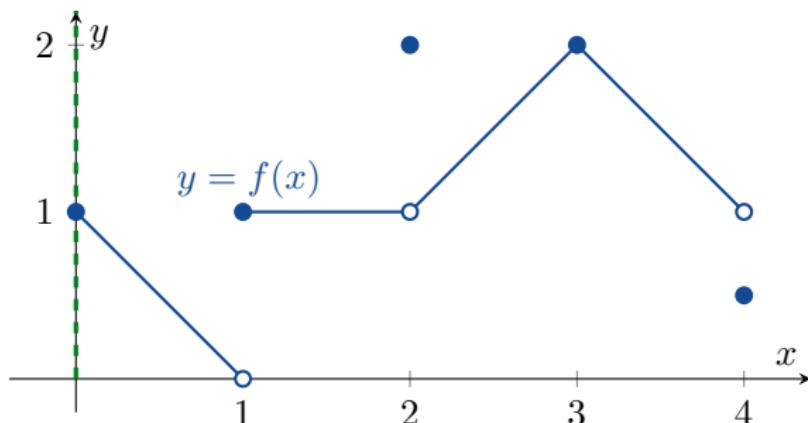
2.4 One-Sided Limits



Example

Consider the function $f : [0, 4] \rightarrow \mathbb{R}$ with graph shown above.

2.4 One-Sided Limits

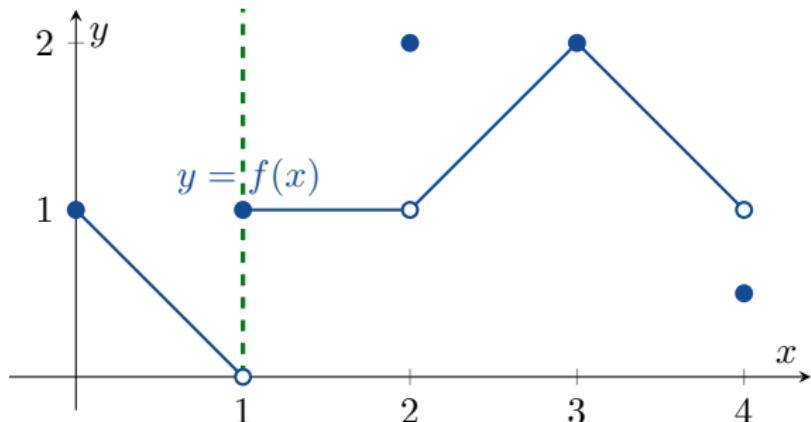


At $x = 0$: $\lim_{x \rightarrow 0^-} f(x)$ does not exist

$$\lim_{x \rightarrow 0^+} f(x) = 1$$

$$\lim_{x \rightarrow 0} f(x) = 1.$$

2.4 One-Sided Limits



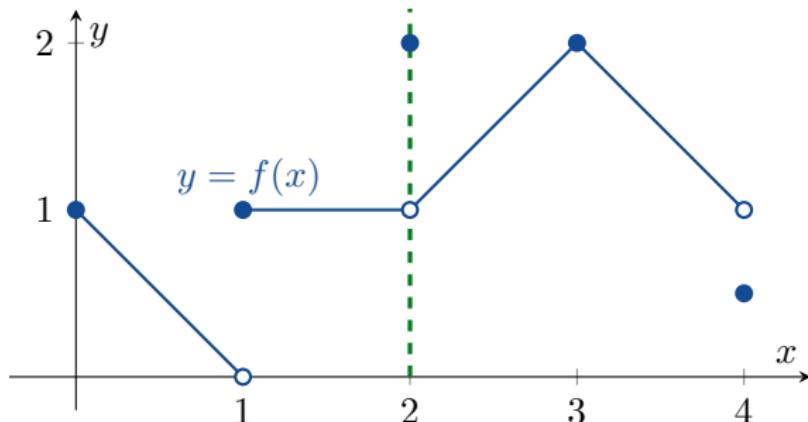
At $x = 1$:

$$\lim_{x \rightarrow 1^-} f(x) = 0$$

$$\lim_{x \rightarrow 1^+} f(x) = 1$$

$\lim_{x \rightarrow 1} f(x)$ does not exist.

2.4 One-Sided Limits

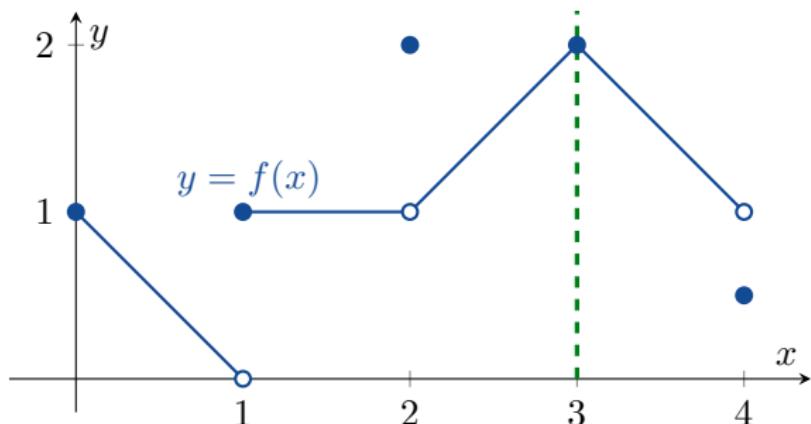


At $x = 2$: $\lim_{x \rightarrow 2^-} f(x) = 1$

$$\lim_{x \rightarrow 2^+} f(x) = 1$$

$$\lim_{x \rightarrow 2} f(x) = 1.$$

2.4 One-Sided Limits

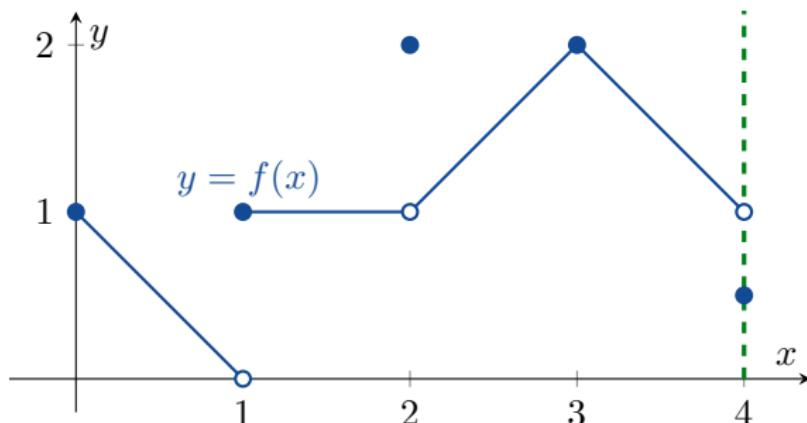


At $x = 3$: $\lim_{x \rightarrow 3^-} f(x) = 2$

$$\lim_{x \rightarrow 3^+} f(x) = 2$$

$$\lim_{x \rightarrow 3} f(x) = 2.$$

2.4 One-Sided Limits



At $x = 4$:

$$\lim_{x \rightarrow 4^-} f(x) = 1$$

$$\lim_{x \rightarrow 4^+} f(x) \text{ does not exist}$$

$$\lim_{x \rightarrow 4} f(x) = 1.$$

2.4 One-Sided Limits



Precise Definitions of One-Sided Limits

Definition (Right-Hand Limit)

Definition (Left-Hand Limit)

2.4 One-Sided Limits



Precise Definitions of One-Sided Limits

Definition (Right-Hand Limit)

We write $\lim_{x \rightarrow c^+} f(x) = L$ iff for all $\varepsilon > 0$, there exists $\delta > 0$ such that

$$c < x < c + \delta \quad \Rightarrow \quad |f(x) - L| < \varepsilon.$$

Definition (Left-Hand Limit)

2.4 One-Sided Limits



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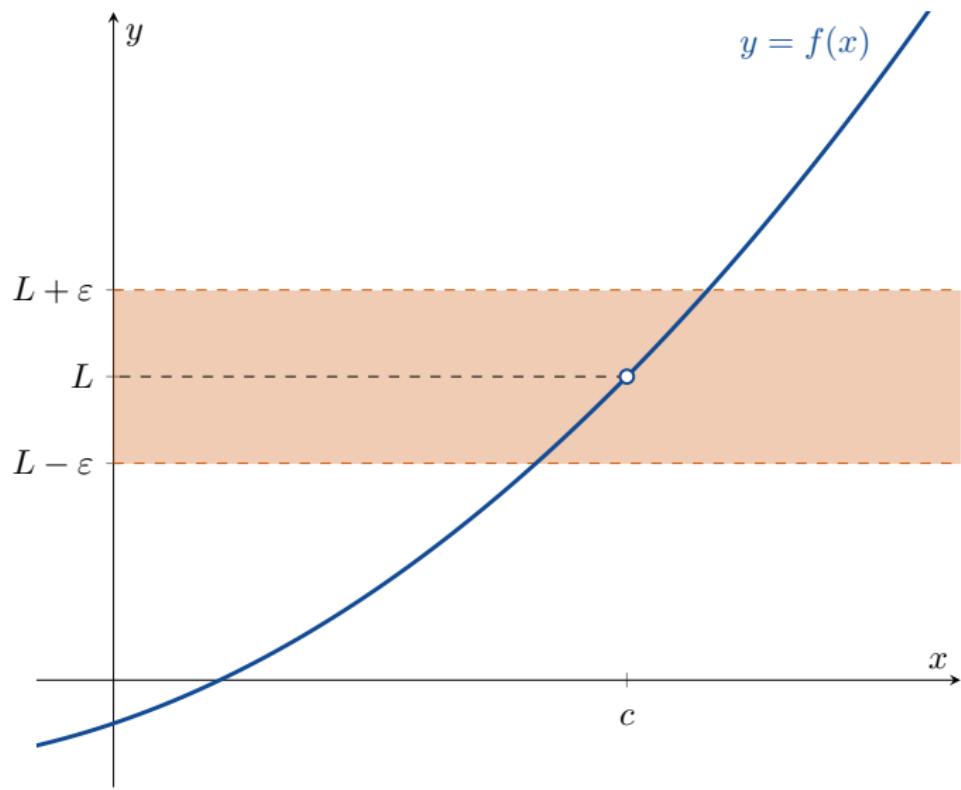
$$c < x < c + \delta \quad \Rightarrow \quad |f(x) - L| < \varepsilon.$$

Definition (Left-Hand Limit)

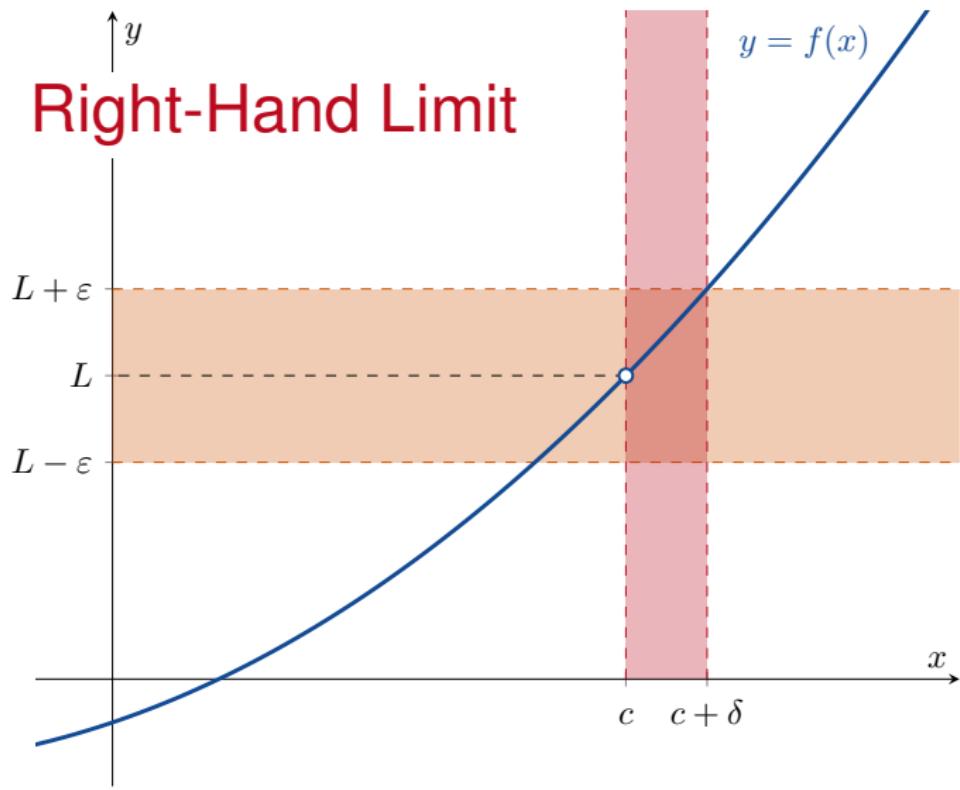
We write $\lim_{x \rightarrow c^-} f(x) = L$ iff for all $\varepsilon > 0$, there exists $\delta > 0$ such that

$$c - \delta < x < c \quad \Rightarrow \quad |f(x) - L| < \varepsilon.$$

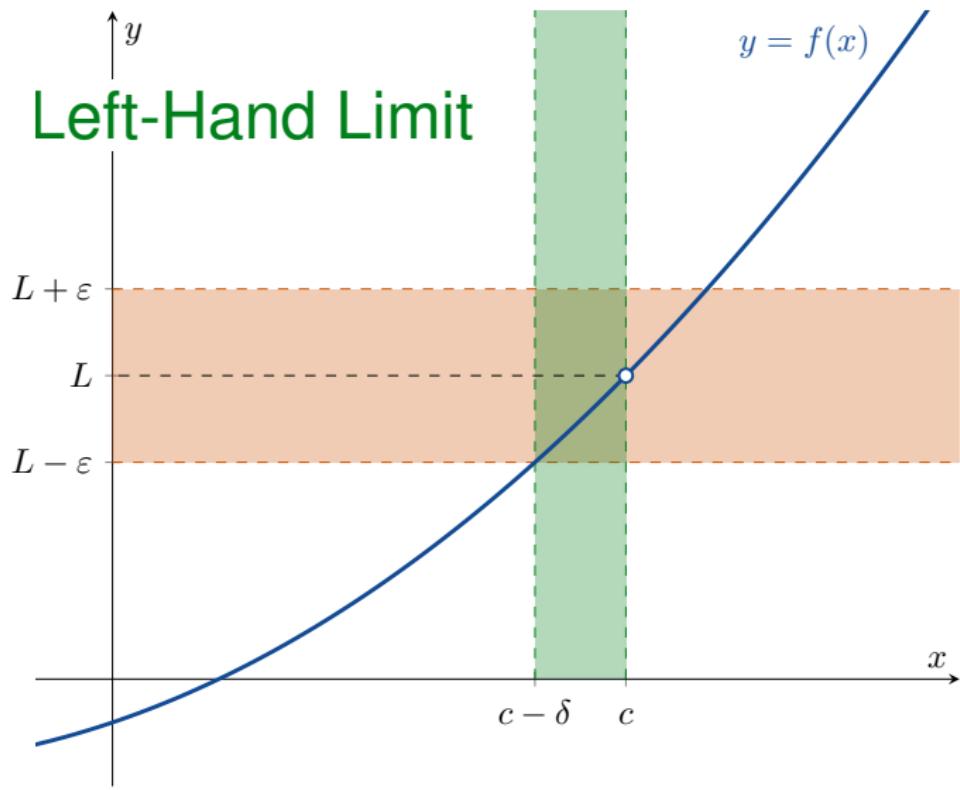
2.4 One-Sided Limits



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2.4 One-Sided Limits



$$\forall \varepsilon > 0, \exists \delta > 0 \text{ s.t. } c < x < c + \delta \implies |f(x) - L| < \varepsilon$$

Example

Prove that $\lim_{x \rightarrow 0^+} \sqrt{x} = 0$.

Let $\varepsilon > 0$. Choose $\delta = \dots$. Then

$$0 < x < \delta \implies |\sqrt{x} - 0| \quad .$$

2.4

$$\forall \varepsilon > 0, \exists \delta > 0 \text{ s.t. } c < x < c + \delta \implies |f(x) - L| < \varepsilon$$



Example

Prove that $\lim_{x \rightarrow 0^+} \sqrt{x} = 0$.

Let $\varepsilon > 0$. Choose $\delta =$

$$0 < x < \delta$$

=

$$|\sqrt{x} - 0| < \varepsilon$$

2.4

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Example

Prove that $\lim_{x \rightarrow 0^+} \sqrt{x} = 0$.

Let $\varepsilon > 0$. Choose $\delta =$

$$0 < x < \delta$$

$$-\varepsilon < \sqrt{x} < \varepsilon$$

$$\sqrt{x} < \varepsilon$$

$$x < \varepsilon^2$$

= choose $\delta = \varepsilon^2$.

$$\forall \varepsilon > 0, \exists \delta > 0 \text{ s.t. } c < x < c + \delta \implies |f(x) - L| < \varepsilon$$

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Let $\varepsilon > 0$. Choose $\delta = \varepsilon^2$. Then

$$0 < x < \delta \implies |\sqrt{x} - 0| = \sqrt{x} < \sqrt{\delta} = \varepsilon.$$

2.4 One-Sided Limits



Theorem

$$\lim_{x \rightarrow x^-} f(x) = L = \lim_{x \rightarrow c^+} f(x) \iff \lim_{x \rightarrow c} f(x) = L$$

You prove.

2.4 One-Sided Limits



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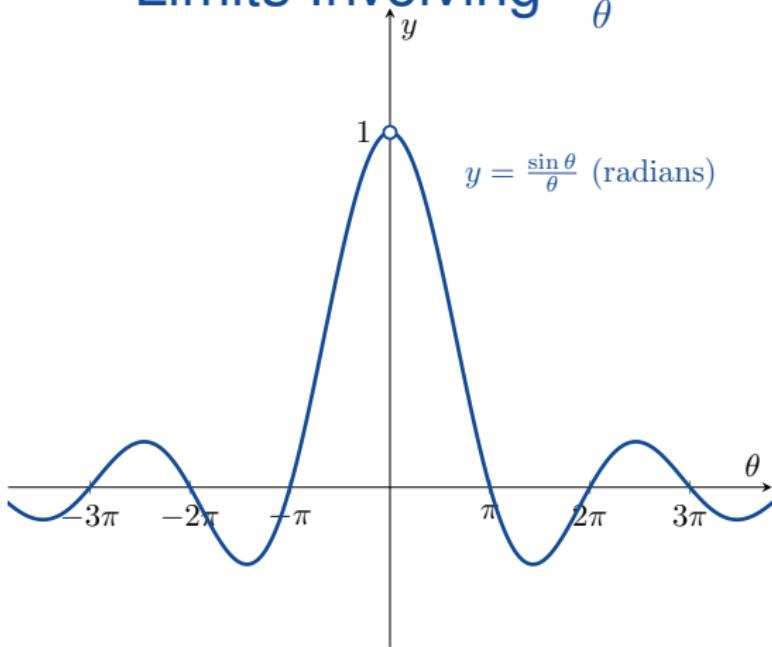
$$\lim_{x \rightarrow x^-} f(x) \neq \lim_{x \rightarrow c^+} f(x) \implies \lim_{x \rightarrow c} f(x) \text{ does not exist.}$$

You prove.

2.4 One-Sided Limits



Limits Involving $\frac{\sin \theta}{\theta}$

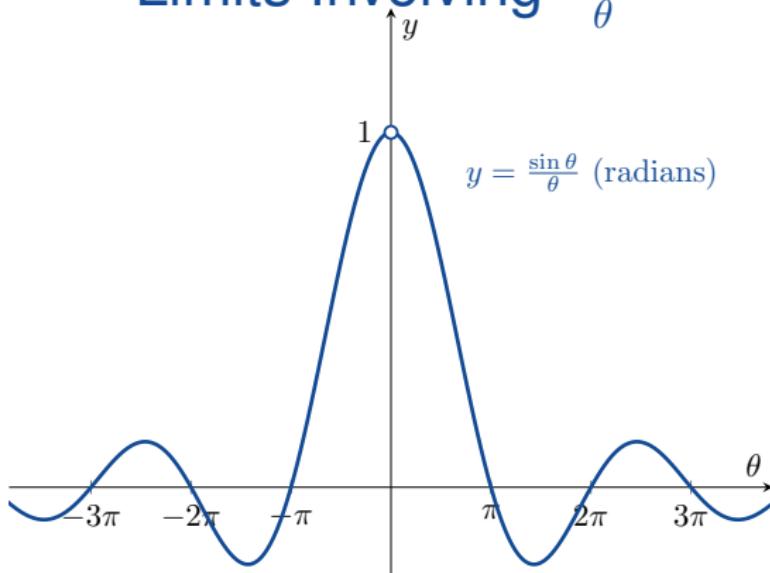


$$y = \frac{\sin \theta}{\theta} \text{ (radians)}$$

2.4 One-Sided Limits



Limits Involving $\frac{\sin \theta}{\theta}$



Theorem

$$\lim_{\theta \rightarrow 0} \frac{\sin \theta}{\theta} = 1 \quad (\theta \text{ in radians})$$

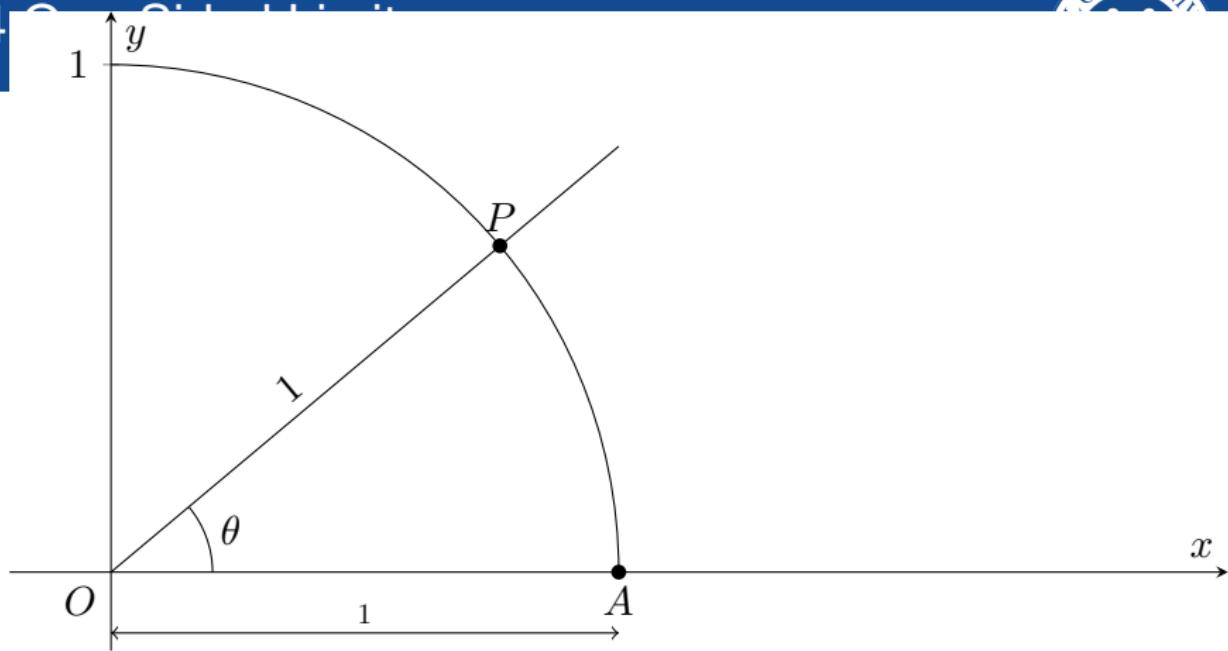
2.4 One-Sided Limits



$$\lim_{\theta \rightarrow 0} \frac{\sin \theta}{\theta} = 1$$

We will need to use this limit in Lecture 5.

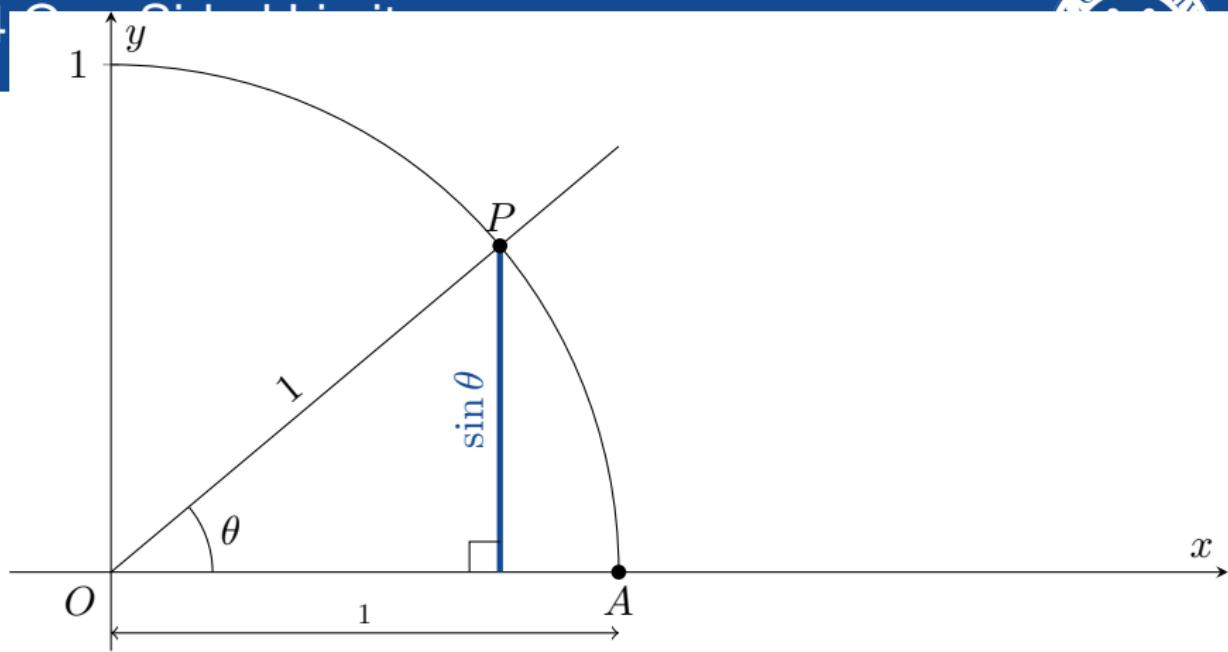
2.4



Proof.

We have that

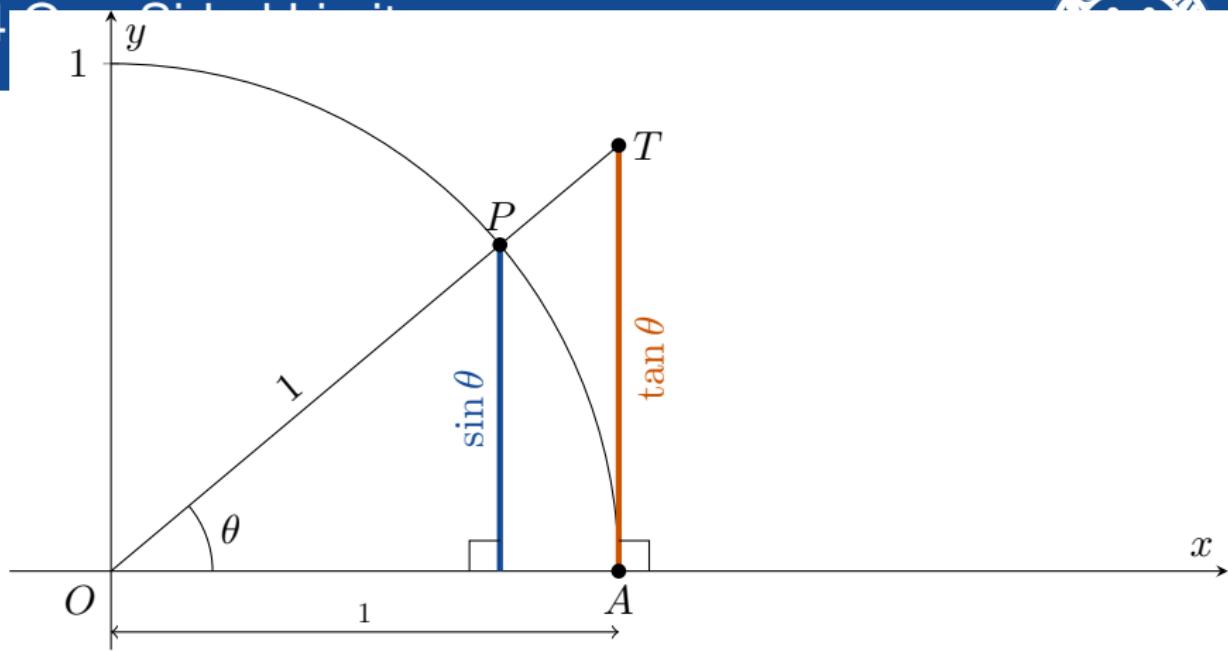
2.4



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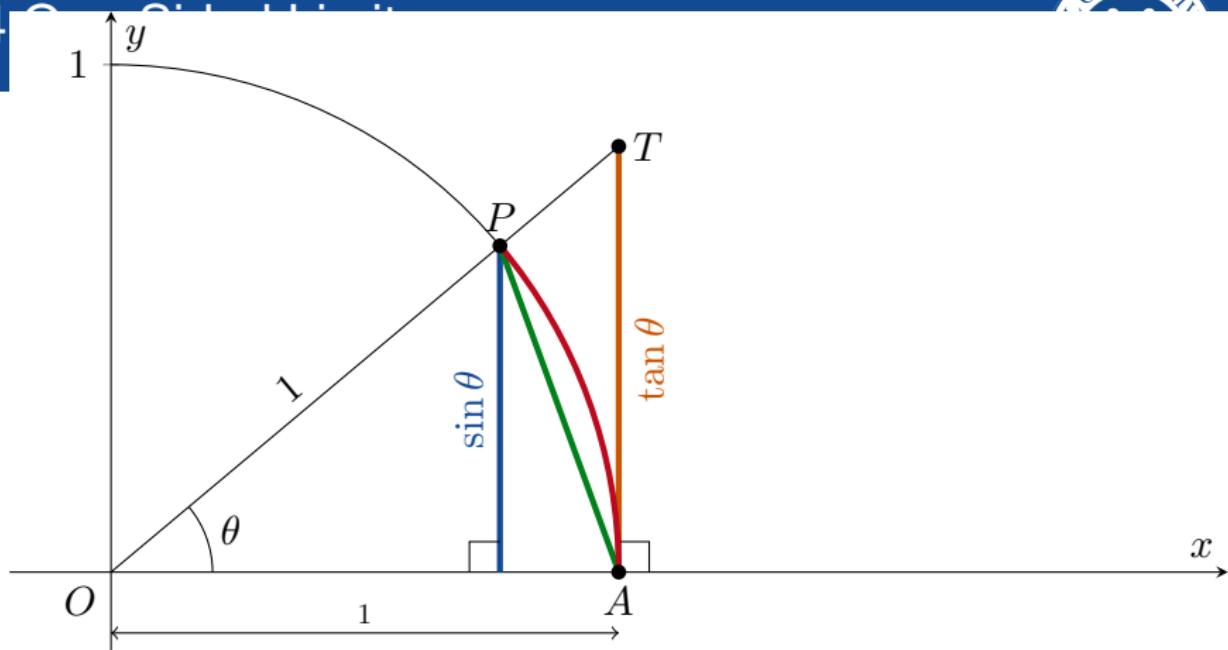
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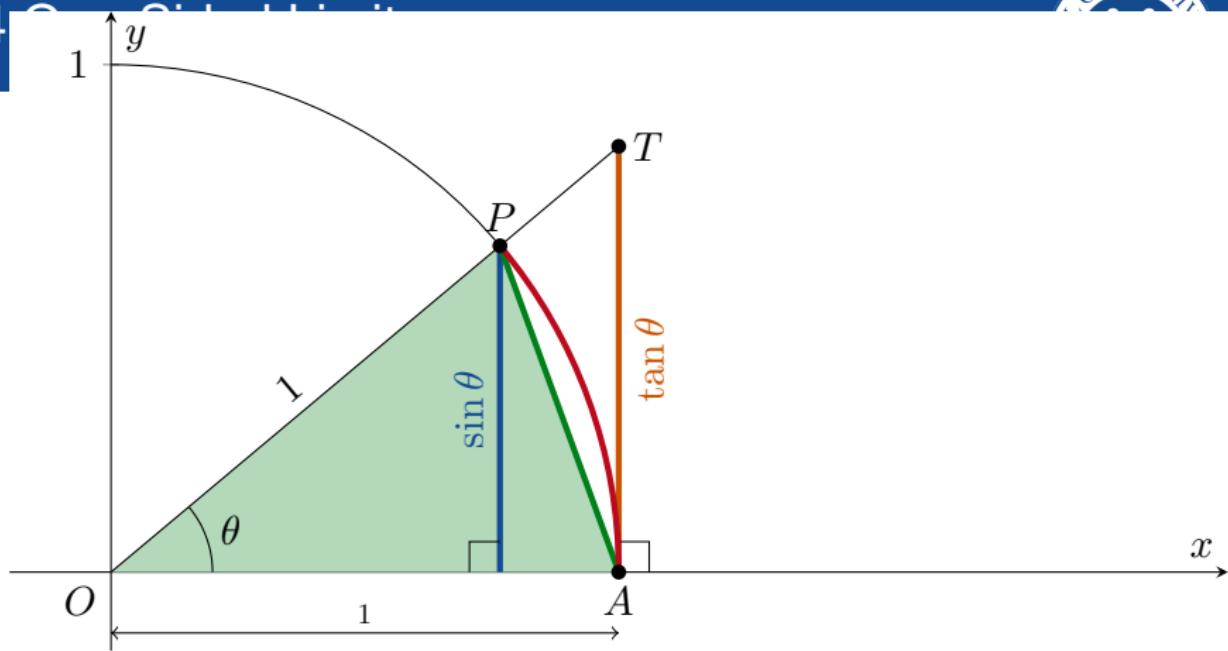


Proof.

We have that

$$\text{area of } \triangle OAP \quad \text{area of sector } OAP \quad \text{area of } \triangle OAT.$$

2.4



Proof.

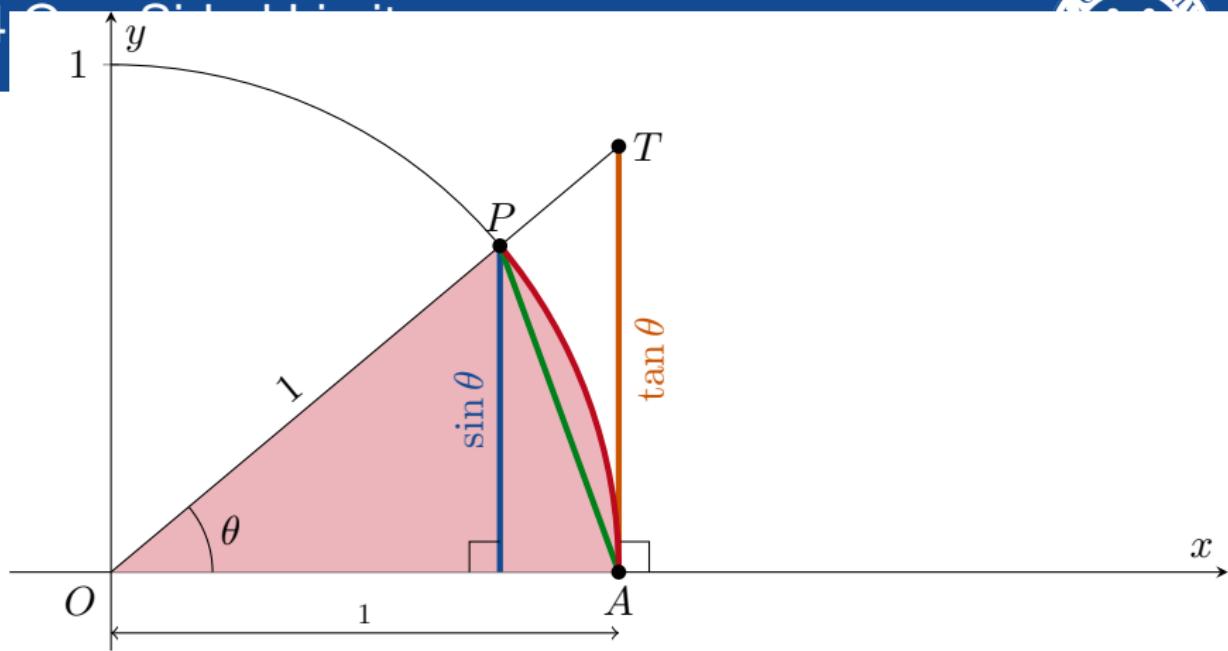
We have that

area of ΔOAP

area of sector OAP

area of ΔOAT .

2.4

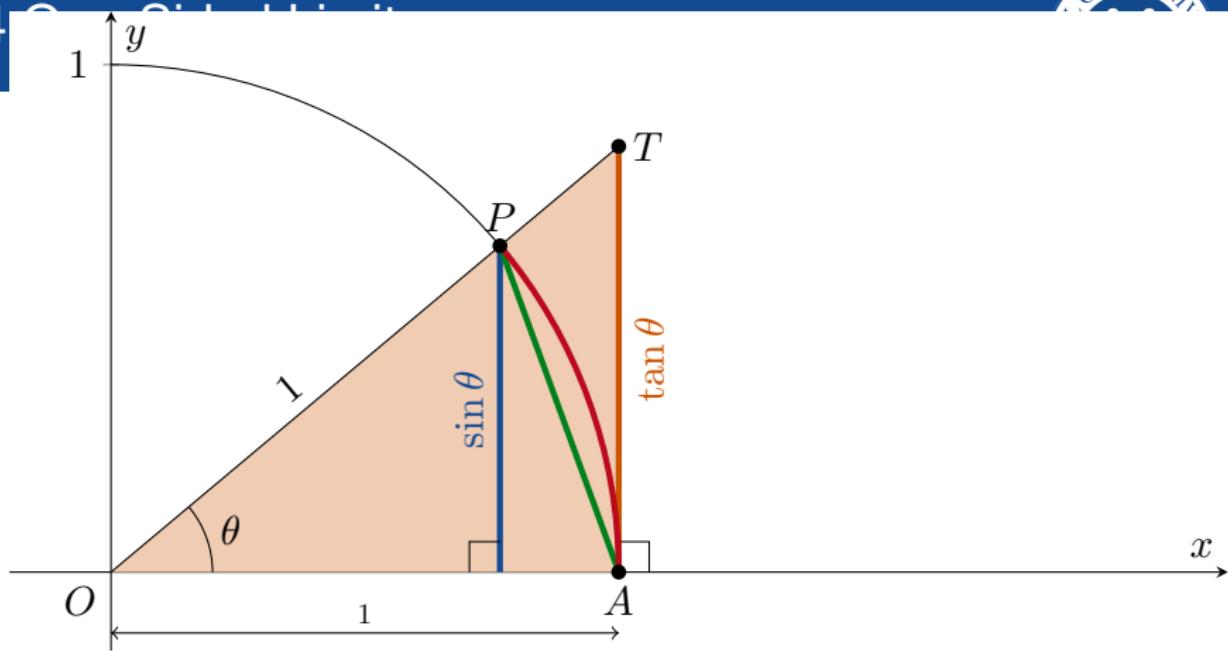


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$$\text{area of } \triangle OAP < \text{area of sector } OAP < \text{area of } \triangle OAT.$$

2.4

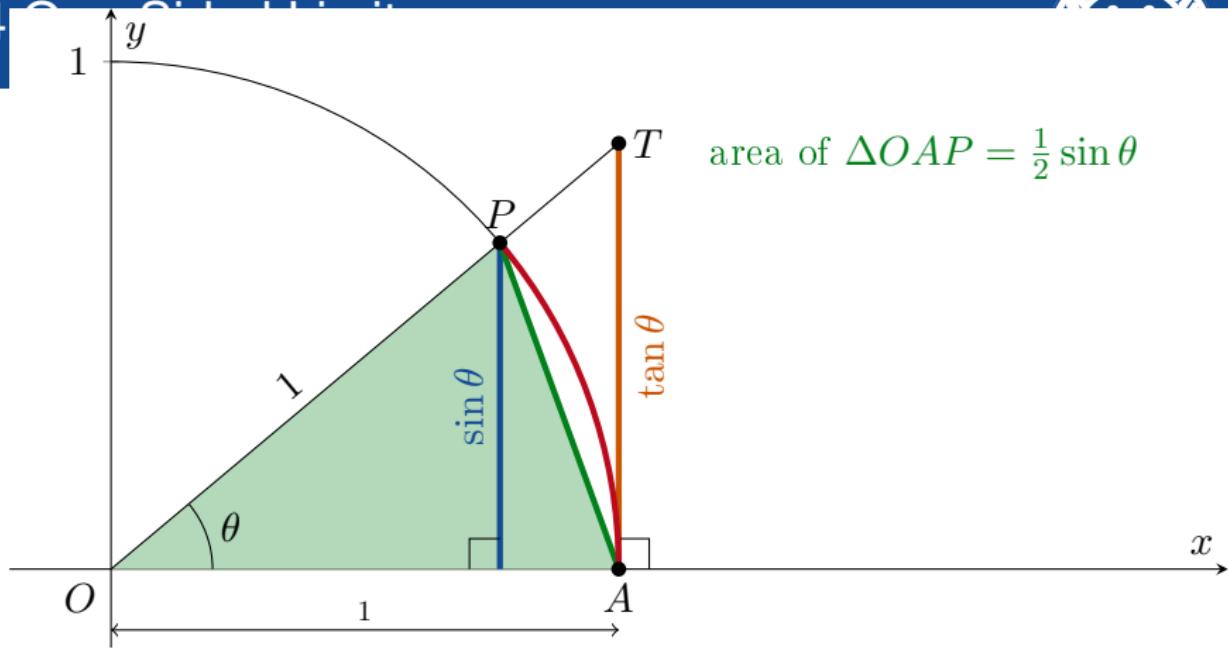


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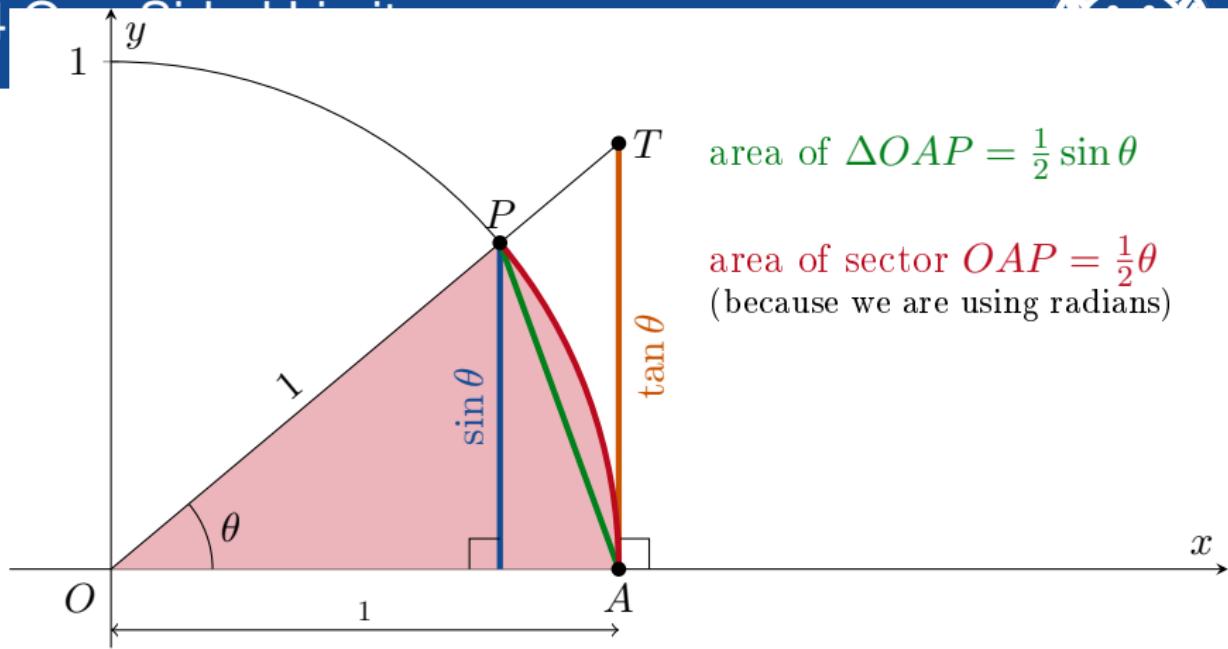


Proof.

We have that

$$\text{area of } \Delta OAP < \text{area of sector } OAP < \text{area of } \Delta OAT.$$

2.4



$$\text{area of } \triangle OAP = \frac{1}{2} \sin \theta$$

$$\text{area of sector } OAP = \frac{1}{2}\theta$$

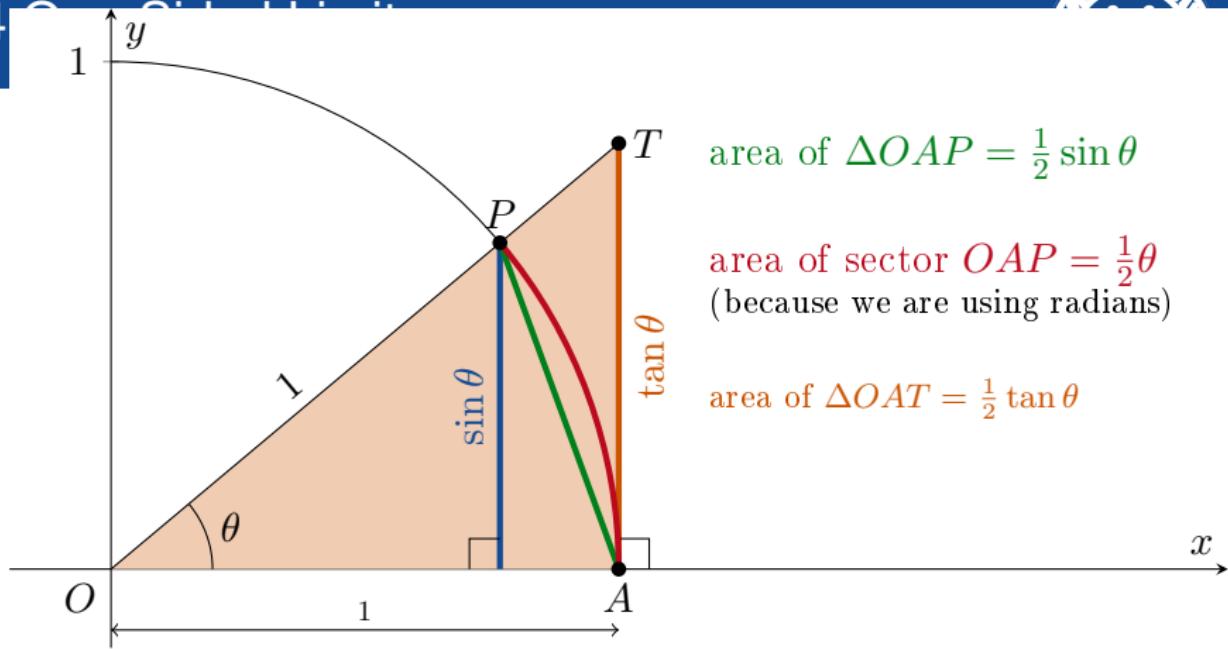
(because we are using radians)

Proof.

We have that

$$\text{area of } \triangle OAP < \text{area of sector } OAP < \text{area of } \triangle OAT.$$

2.4

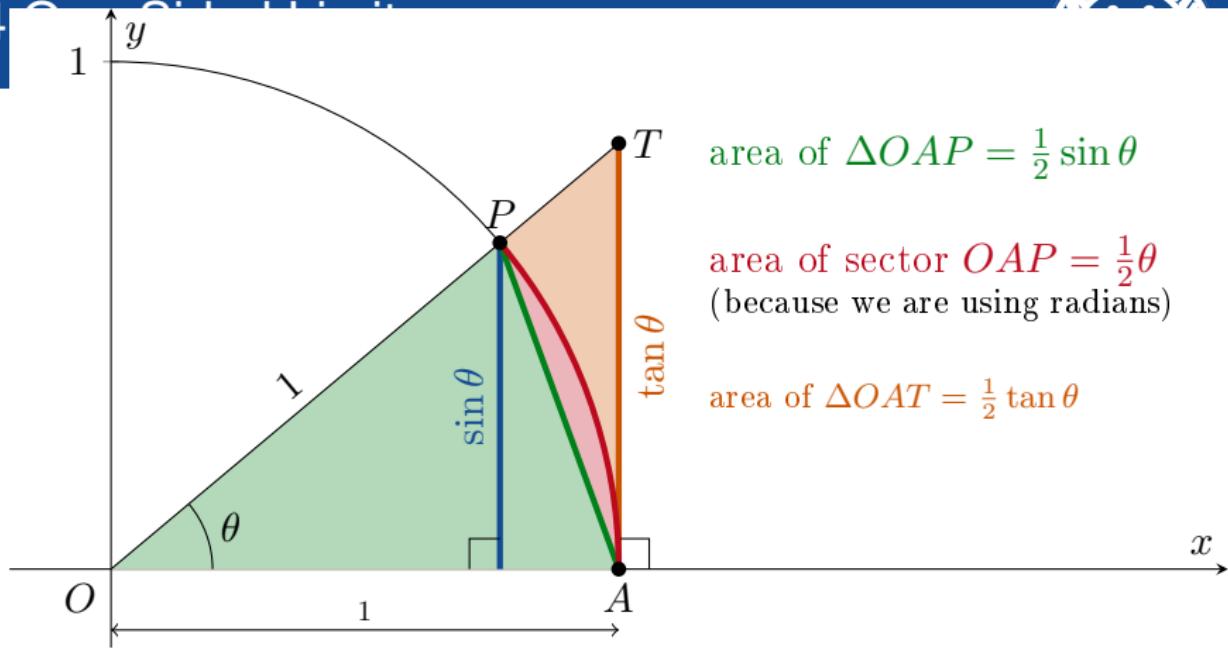


Proof.

We have that

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2.4



$$\text{area of } \triangle OAP = \frac{1}{2} \sin \theta$$

$$\text{area of sector } OAP = \frac{1}{2} \theta \quad (\text{because we are using radians})$$

$$\text{area of } \triangle OAT = \frac{1}{2} \tan \theta$$

Proof.

We have that

$$\frac{1}{2} \sin \theta < \frac{1}{2} \theta < \frac{1}{2} \tan \theta.$$

2.4 One-Sided Limits

Proof continued.

Therefore

$$\frac{1}{2} \sin \theta < \frac{1}{2} \theta < \frac{1}{2} \tan \theta$$

$$\sin \theta < \theta < \tan \theta$$

$$1 < \frac{\theta}{\sin \theta} < \frac{1}{\cos \theta}$$

and

$$1 > \frac{\sin \theta}{\theta} > \cos \theta.$$

2.4 One-Sided Limits

Proof continued.

Therefore

$$\frac{1}{2} \sin \theta < \frac{1}{2} \theta < \frac{1}{2} \tan \theta$$

$$\sin \theta < \theta < \tan \theta$$

$$1 < \frac{\theta}{\sin \theta} < \frac{1}{\cos \theta}$$

and

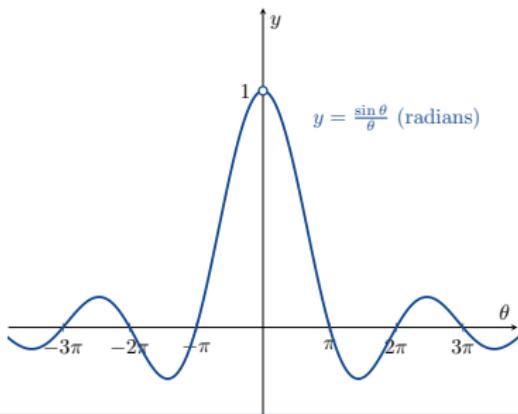
$$1 > \frac{\sin \theta}{\theta} > \cos \theta.$$

Since $\lim_{\theta \rightarrow 0^+} \cos \theta = 1$, it follows by the Sandwich Theorem that

$$\lim_{\theta \rightarrow 0^+} \frac{\sin \theta}{\theta} = 1$$

also.

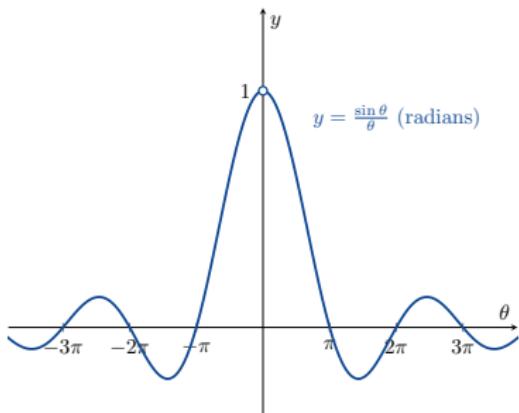
2.4 One-Sided Limits



Proof continued.

We also need to do the left-hand limit:

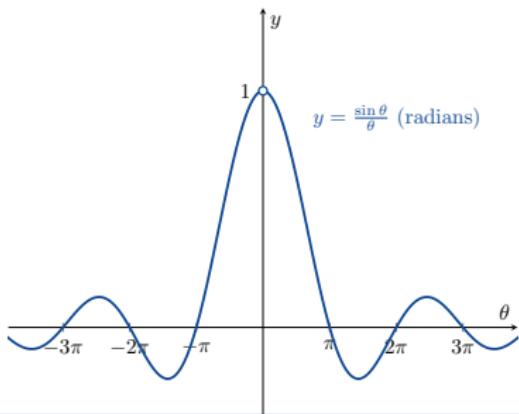
2.4 One-Sided Limits



Proof continued.

We also need to do the left-hand limit: Since $\sin \theta$ and θ are both odd functions, it follows that $\frac{\sin \theta}{\theta}$ is an even function.

2.4 One-Sided Limits



Proof continued.

We also need to do the left-hand limit: Since $\sin \theta$ and θ are both odd functions, it follows that $\frac{\sin \theta}{\theta}$ is an even function. By symmetry, we must also have that

$$\lim_{\theta \rightarrow 0^-} \frac{\sin \theta}{\theta} = 1.$$

2.4 One-Sided Limits



Proof continued.

$$\lim_{\theta \rightarrow 0^-} \frac{\sin \theta}{\theta} = 1 = \lim_{\theta \rightarrow 0^+} \frac{\sin \theta}{\theta}$$

Because the left-hand limit and the right-hand limit are equal, we must also have

$$\lim_{\theta \rightarrow 0} \frac{\sin \theta}{\theta} = 1.$$



EXAMPLE 5 Show that (a) $\lim_{y \rightarrow 0} \frac{\cos y - 1}{y} = 0$ and (b) $\lim_{x \rightarrow 0} \frac{\sin 2x}{5x} = \frac{2}{5}$.

Solution

(a) Using the half-angle formula $\cos y = 1 - 2 \sin^2(y/2)$, we calculate

$$\begin{aligned}\lim_{h \rightarrow 0} \frac{\cos y - 1}{y} &= \lim_{h \rightarrow 0} -\frac{2 \sin^2(y/2)}{y} \\&= -\lim_{\theta \rightarrow 0} \frac{\sin \theta}{\theta} \sin \theta && \text{Let } \theta = y/2. \\&= -(1)(0) = 0. && \text{Eq. (1) and Example 11a} \\&&& \text{in Section 2.2}\end{aligned}$$

(b) Equation (1) does not apply to the original fraction. We need a $2x$ in the denominator, not a $5x$. We produce it by multiplying numerator and denominator by $2/5$:

$$\begin{aligned}\lim_{x \rightarrow 0} \frac{\sin 2x}{5x} &= \lim_{x \rightarrow 0} \frac{(2/5) \cdot \sin 2x}{(2/5) \cdot 5x} \\&= \frac{2}{5} \lim_{x \rightarrow 0} \frac{\sin 2x}{2x} && \text{Eq. (1) applies with} \\&&& \theta = 2x. \\&= \frac{2}{5}(1) = \frac{2}{5}.\end{aligned}$$



EXAMPLE 6 Find $\lim_{t \rightarrow 0} \frac{\tan t \sec 2t}{3t}$.

Solution From the definition of $\tan t$ and $\sec 2t$, we have

$$\begin{aligned}\lim_{t \rightarrow 0} \frac{\tan t \sec 2t}{3t} &= \lim_{t \rightarrow 0} \frac{1}{3} \cdot \frac{1}{t} \cdot \frac{\sin t}{\cos t} \cdot \frac{1}{\cos 2t} \\&= \frac{1}{3} \lim_{t \rightarrow 0} \frac{\sin t}{t} \cdot \frac{1}{\cos t} \cdot \frac{1}{\cos 2t} \\&= \frac{1}{3}(1)(1)(1) = \frac{1}{3}.\end{aligned}$$

Eq. (1) and Example 11b
in Section 2.2



EXAMPLE 7 Show that for nonzero constants A and B .

$$\lim_{\theta \rightarrow 0} \frac{\sin A\theta}{\sin B\theta} = \frac{A}{B}.$$

Solution

$$\lim_{\theta \rightarrow 0} \frac{\sin A\theta}{\sin B\theta} = \lim_{\theta \rightarrow 0} \frac{\sin A\theta}{A\theta} \frac{A\theta}{\sin B\theta} \frac{B\theta}{B\theta} \frac{1}{B}$$

$$= \lim_{\theta \rightarrow 0} \frac{\sin A\theta}{A\theta} \frac{B\theta}{\sin B\theta} \frac{A}{B}$$

$$= \lim_{\theta \rightarrow 0} (1)(1) \frac{A}{B}$$

$$= \frac{A}{B}.$$

Multiply and divide by $A\theta$ and $B\theta$.

$$\lim_{u \rightarrow 0} \frac{\sin u}{u} = 1, \text{ with } u = A\theta$$

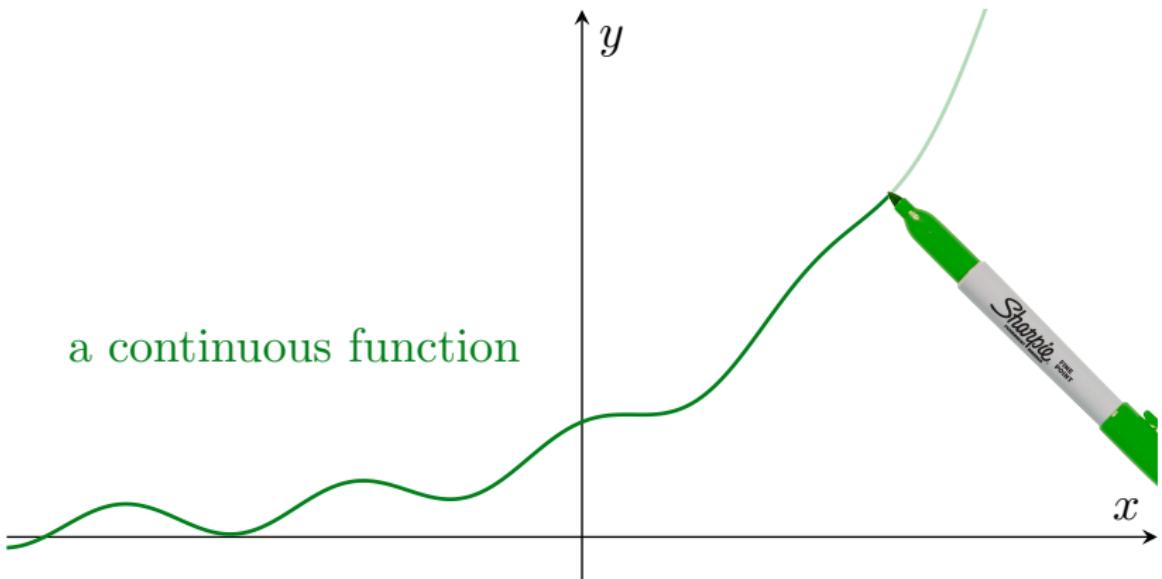
$$\lim_{v \rightarrow 0} \frac{v}{\sin v} = 1, \text{ with } v = B\theta$$



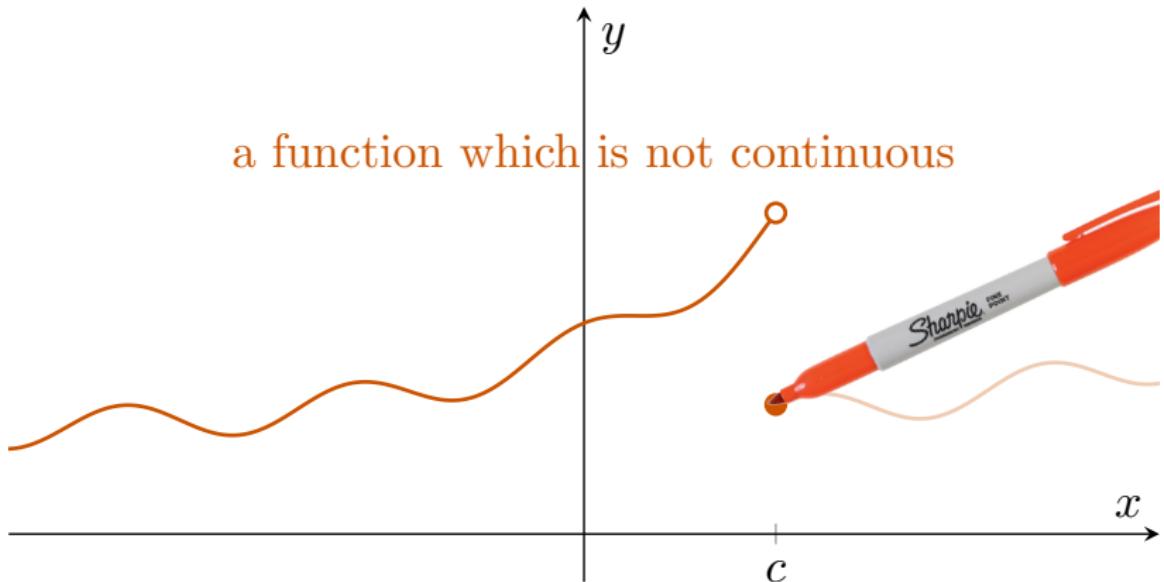


Continuity

2.5 Continuity



2.5 Continuity



2.5 Continuity



Definition

The function $f : D \rightarrow \mathbb{R}$ is *continuous at $c \in D$* iff

- $f(c)$ exists;
- $\lim_{x \rightarrow c} f(x)$ exists; and
- $\lim_{x \rightarrow c} f(x) = f(c)$.

2.5 Continuity



Definition

The function $f : D \rightarrow \mathbb{R}$ is *continuous from the right* at $c \in D$ iff

- $f(c)$ exists;
- $\lim_{x \rightarrow c^+} f(x)$ exists; and
- $\lim_{x \rightarrow c^+} f(x) = f(c)$.

2.5 Continuity



Definition

The function $f : D \rightarrow \mathbb{R}$ is *continuous from the left* at $c \in D$ iff

- $f(c)$ exists;
- $\lim_{x \rightarrow c^-} f(x)$ exists; and
- $\lim_{x \rightarrow c^-} f(x) = f(c)$.

2.5 Continuity

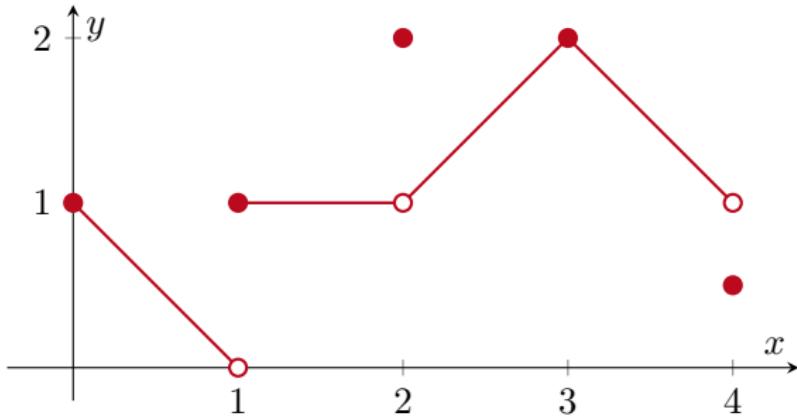


Definition

If f is not continuous at c , we say that f is *discontinuous* at c .

We say that c is a *point of discontinuity* of f .

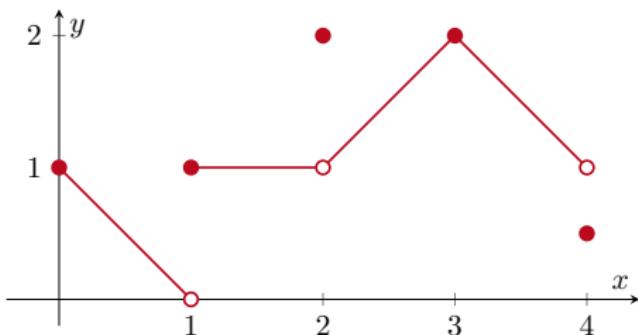
2.5 Continuity



Example

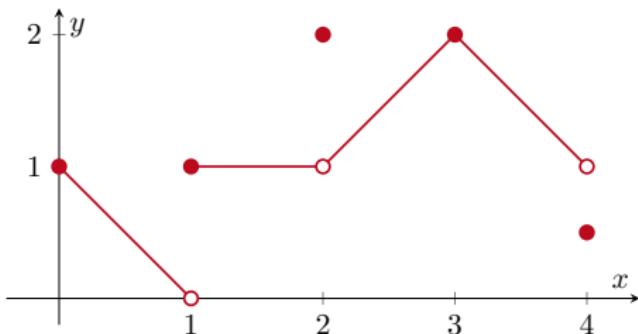
Consider the function $f : [0, 4] \rightarrow \mathbb{R}$ above. Where is f continuous? Where is f discontinuous?

2.5 Continuity



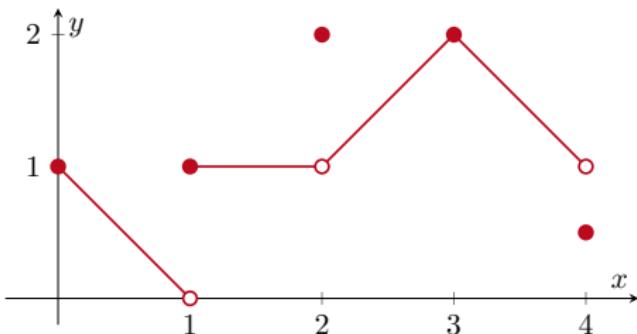
c	Is f continuous at c ?	Why?
0		
$(0, 1)$		
1		

2.5 Continuity



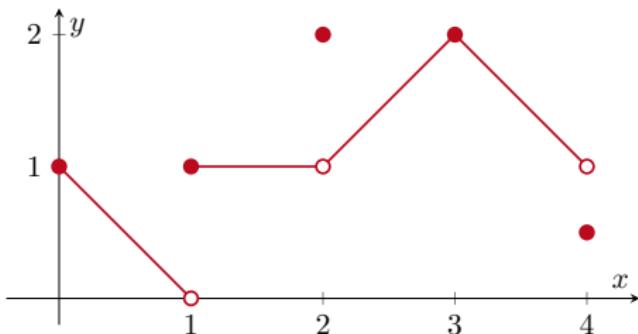
c	Is f continuous at c ?	Why?
0	Yes	because $\lim_{x \rightarrow 0} f(x) = 1 = f(0)$
$(0, 1)$		
1		

2.5 Continuity



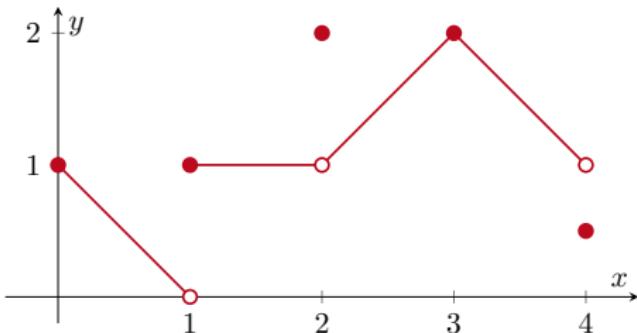
c	Is f continuous at c ?	Why?
0	Yes	because $\lim_{x \rightarrow 0} f(x) = 1 = f(0)$
$(0, 1)$	Yes	because $\lim_{x \rightarrow c} f(x) = f(c)$
1		

2.5 Continuity



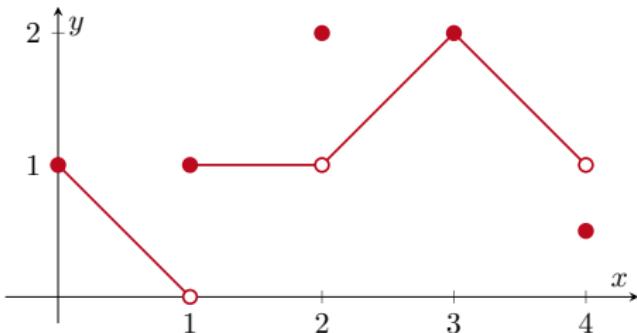
c	Is f continuous at c ?	Why?
0	Yes	because $\lim_{x \rightarrow 0} f(x) = 1 = f(0)$
$(0, 1)$	Yes	because $\lim_{x \rightarrow c} f(x) = f(c)$
1	No	because $\lim_{x \rightarrow 1} f(x)$ does not exist

2.5 Continuity



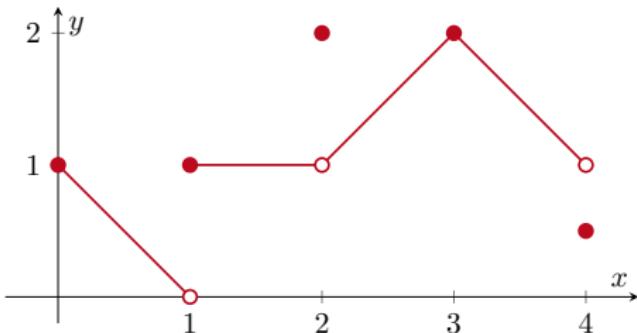
c	Is f continuous at c ?	Why?
(1, 2)		
2		
(2, 4)		
4		

2.5 Continuity



c	Is f continuous at c ?	Why?
(1, 2)	Yes	because $\lim_{x \rightarrow c} f(x) = f(c)$
2	No	because $\lim_{x \rightarrow 2} f(x) = 1 \neq 2 = f(2)$
(2, 4)		
4		

2.5 Continuity

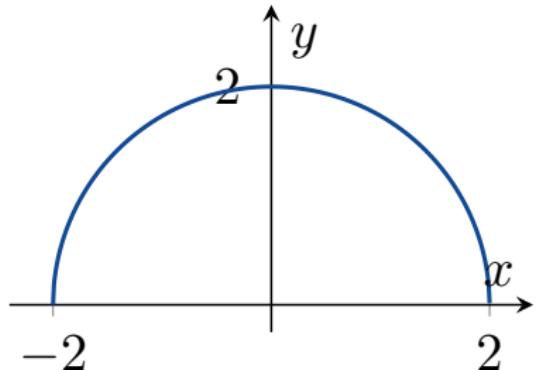


c	Is f continuous at c ?	Why?
(1, 2)	Yes	because $\lim_{x \rightarrow c} f(x) = f(c)$
2	No	because $\lim_{x \rightarrow 2} f(x) = 1 \neq 2 = f(2)$
(2, 4)	Yes	because $\lim_{x \rightarrow c} f(x) = f(c)$
4	No	because $\lim_{x \rightarrow 4} f(x) = 1 \neq \frac{1}{2} = f(4)$

2.5 Continuity

Example

$$f : [-2, 2] \rightarrow \mathbb{R}, f(x) = \sqrt{4 - x^2}$$

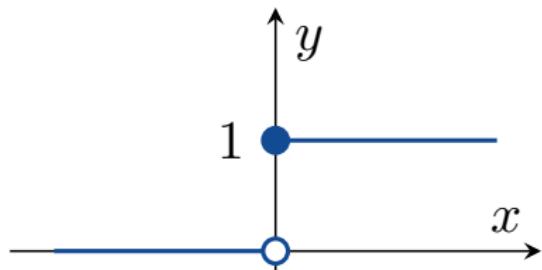


f is continuous at every $c \in [-2, 2]$.

2.5 Continuity

Example

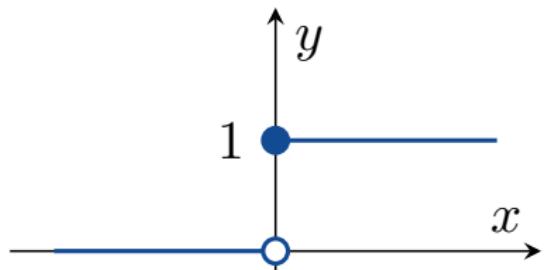
$$g : \mathbb{R} \rightarrow \mathbb{R}, g(x) = \begin{cases} 0 & x < 0 \\ 1 & x \geq 0 \end{cases}$$



2.5 Continuity

Example

$$g : \mathbb{R} \rightarrow \mathbb{R}, g(x) = \begin{cases} 0 & x < 0 \\ 1 & x \geq 0 \end{cases}$$

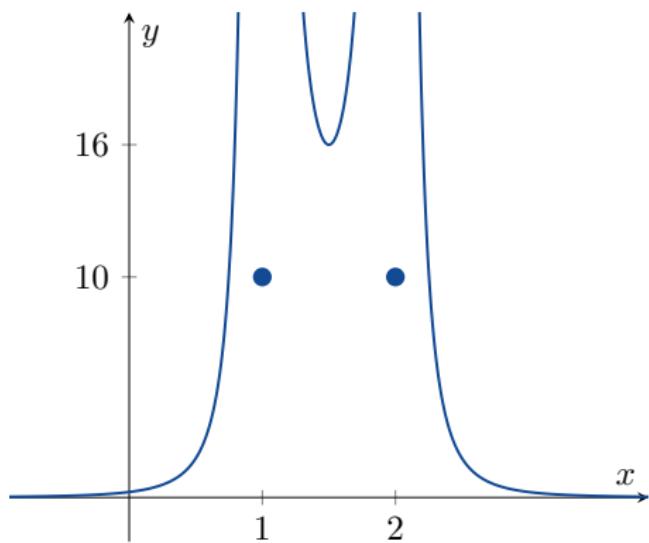


g has a point of discontinuity at $c = 0$. g is continuous at every point $c \neq 0$.

2.5 Continuity

Example

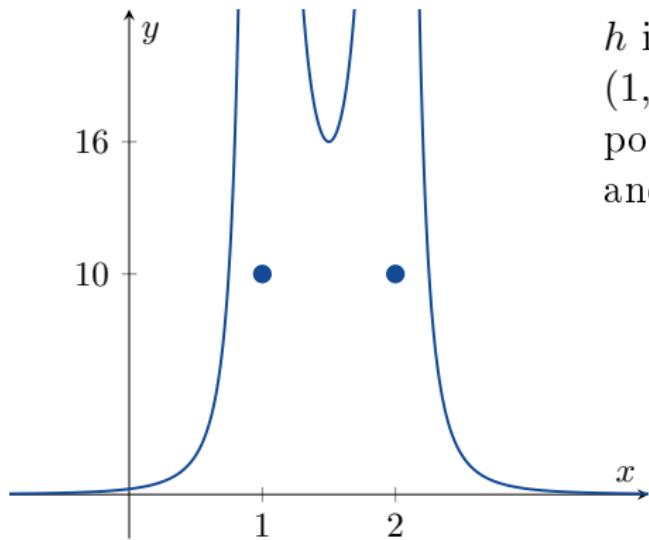
$$h : \mathbb{R} \rightarrow \mathbb{R}, h(x) = \begin{cases} \frac{1}{(x-1)^2(x-2)^2} & x \neq 1 \text{ or } 2 \\ 10 & x = 1 \text{ or } 2 \end{cases}$$



2.5 Continuity

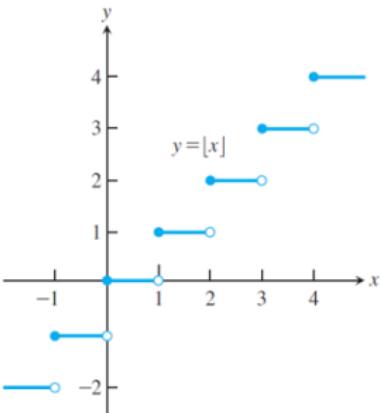
Example

$$h : \mathbb{R} \rightarrow \mathbb{R}, h(x) = \begin{cases} \frac{1}{(x-1)^2(x-2)^2} & x \neq 1 \text{ or } 2 \\ 10 & x = 1 \text{ or } 2 \end{cases}$$



h is continuous on $(-\infty, 1)$, $(1, 2)$ and $(2, \infty)$. h has a points of discontinuity at $c = 1$ and $c = 2$.

2.5 Continuity



EXAMPLE 4 The function $y = \lfloor x \rfloor$ introduced in Section 1.1 is graphed in Figure 2.39. It is discontinuous at every integer n , because the left-hand and right-hand limits are not equal as $x \rightarrow n$:

$$\lim_{x \rightarrow n^-} \lfloor x \rfloor = n - 1 \quad \text{and} \quad \lim_{x \rightarrow n^+} \lfloor x \rfloor = n.$$

Since $\lfloor n \rfloor = n$, the greatest integer function is right-continuous at every integer n (but not left-continuous).

The greatest integer function is continuous at every real number other than the integers. For example,

$$\lim_{x \rightarrow 1.5} \lfloor x \rfloor = 1 = \lfloor 1.5 \rfloor.$$

In general, if $n - 1 < c < n$, n an integer, then

$$\lim_{x \rightarrow c} \lfloor x \rfloor = n - 1 = \lfloor c \rfloor.$$

2.5 Continuity



Continuous Functions

Definition

$f : D \rightarrow \mathbb{R}$ is a *continuous function* if it is continuous at every $c \in D$.

2.5 Continuity



Theorem

Suppose that f and g are continuous at c .

2.5 Continuity



Theorem

Suppose that f and g are continuous at c .

Then $f + g$, $f - g$, kf ($k \in \mathbb{R}$), fg , $\frac{f}{g}$ (if $g(c) \neq 0$) and f^n ($n \in \mathbb{N}$) are all continuous at c .

2.5 Continuity



Theorem

Suppose that f and g are continuous at c .

Then $f + g$, $f - g$, kf ($k \in \mathbb{R}$), fg , $\frac{f}{g}$ (if $g(c) \neq 0$) and f^n ($n \in \mathbb{N}$) are all continuous at c .

If $\sqrt[n]{f}$ is defined on $(c - \delta, c + \delta)$, then $\sqrt[n]{f}$ is also continuous at c ($n \in \mathbb{N}$).

2.5 Continuity



Example

Every polynomial

$$P(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_2 x^2 + a_1 x + a_0$$

is continuous.

2.5 Continuity



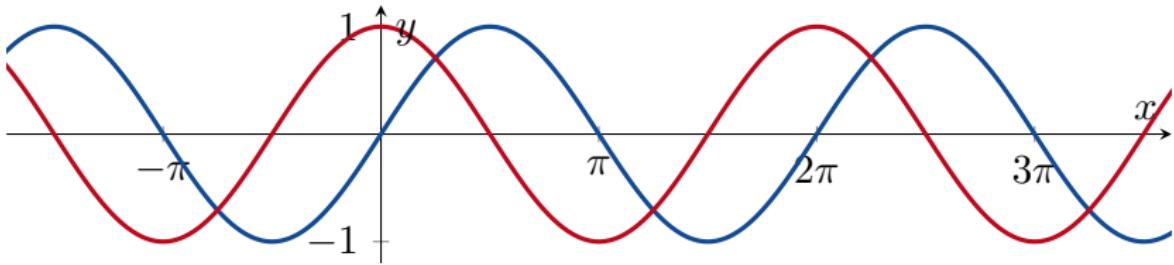
Example

If

- P and Q are polynomials; and
- $Q(c) \neq 0$,

then $\frac{P(x)}{Q(x)}$ is continuous at c .

2.5 Continuity



Example

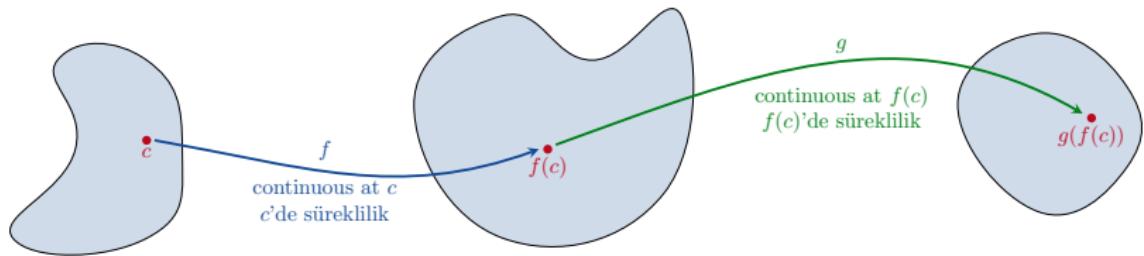
$\sin x$ and $\cos x$ are continuous.

Composites of Continuous Functions

$$g \circ f(x)$$

$g \circ f(x)$ means $g(f(x))$.

2.5 Continuity

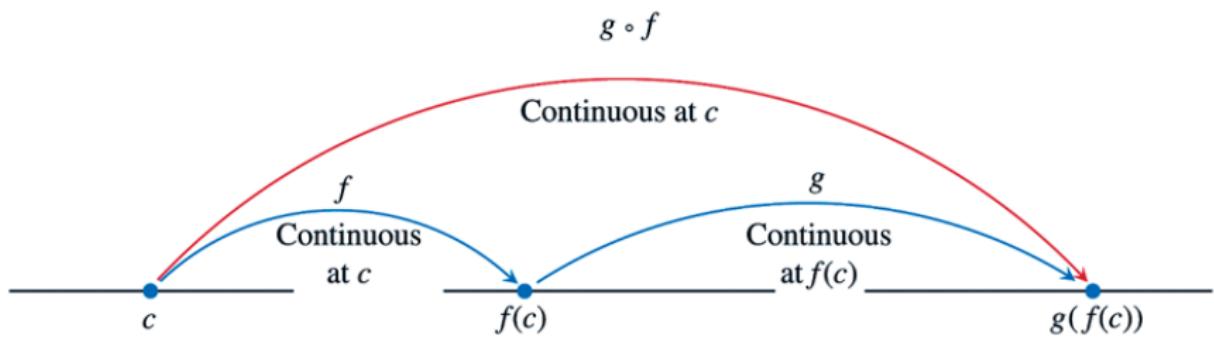


Theorem

If

- f is continuous at c ; and
- g is continuous at $f(c)$,

then $g \circ f$ is continuous at c .



2.5 Continuity



Example

Show that $h(x) = \sqrt{x^2 - 2x - 5}$ is continuous on its natural domain.

2.5 Continuity



Example

Show that $h(x) = \sqrt{x^2 - 2x - 5}$ is continuous on its natural domain.

- The function $g(t) = \sqrt{t}$ is continuous.

2.5 Continuity



Example

Show that $h(x) = \sqrt{x^2 - 2x - 5}$ is continuous on its natural domain.

- The function $g(t) = \sqrt{t}$ is continuous.
- The function $f(x) = x^2 - 2x - 5$ is continuous because all polynomials are continuous.

2.5 Continuity



Example

Show that $h(x) = \sqrt{x^2 - 2x - 5}$ is continuous on its natural domain.

- The function $g(t) = \sqrt{t}$ is continuous.
- The function $f(x) = x^2 - 2x - 5$ is continuous because all polynomials are continuous.

Therefore $h(x) = g \circ f(x)$ is continuous.

2.5 Continuity

Example

Show that $\frac{x^{\frac{2}{3}}}{1+x^4}$ is continuous.

2.5 Continuity

Example

Show that $\frac{x^{\frac{2}{3}}}{1+x^4}$ is continuous.

- $x^{\frac{2}{3}}$ is continuous.
- $1 + x^4$ is continuous.

2.5 Continuity

Example

Show that $\frac{x^{\frac{2}{3}}}{1+x^4}$ is continuous.

- $x^{\frac{2}{3}}$ is continuous.
- $1 + x^4$ is continuous.
- $1 + x^4 \neq 0$ for all x

2.5 Continuity

Example

Show that $\frac{x^{\frac{2}{3}}}{1+x^4}$ is continuous.

- $x^{\frac{2}{3}}$ is continuous.
- $1 + x^4$ is continuous.
- $1 + x^4 \neq 0$ for all x

Hence $\frac{x^{\frac{2}{3}}}{1+x^4}$ is continuous.

2.5 Continuity



Please read part (c) of Example 8 in your textbook.

2.5 Continuity



Example

Where is $h(x) = \left| \frac{x \sin x}{x^2 + 2} \right|$ continuous?

2.5 Continuity



Example

Where is $h(x) = \left| \frac{x \sin x}{x^2 + 2} \right|$ continuous?

- $x^2 + 2$ is continuous and > 0 everywhere.

2.5 Continuity



Example

Where is $h(x) = \left| \frac{x \sin x}{x^2 + 2} \right|$ continuous?

- $x^2 + 2$ is continuous and > 0 everywhere.
- $\sin x$ is continuous everywhere.

2.5 Continuity



Example

Where is $h(x) = \left| \frac{x \sin x}{x^2 + 2} \right|$ continuous?

- $x^2 + 2$ is continuous and > 0 everywhere.
- $\sin x$ is continuous everywhere.
- x is continuous everywhere.

2.5 Continuity



Example

Where is $h(x) = \left| \frac{x \sin x}{x^2 + 2} \right|$ continuous?

- $x^2 + 2$ is continuous and > 0 everywhere.
- $\sin x$ is continuous everywhere.
- x is continuous everywhere.
- $|y|$ is continuous everywhere.

2.5 Continuity



Example

Where is $h(x) = \left| \frac{x \sin x}{x^2 + 2} \right|$ continuous?

- $x^2 + 2$ is continuous and > 0 everywhere.
- $\sin x$ is continuous everywhere.
- x is continuous everywhere.
- $|y|$ is continuous everywhere.

Therefore $h(x)$ is continuous everywhere.

Limits of Continuous Functions

Theorem

If

- $g(x)$ is continuous at $x = b$; and
- $\lim_{x \rightarrow c} f(x) = b$,

then

$$\lim_{x \rightarrow c} g(f(x)) = g\left(\lim_{x \rightarrow c} f(x)\right).$$

You can read the proof in the textbook.

Limits of Continuous Functions

Theorem

If

- $g(x)$ is continuous at $x = b$; and
- $\lim_{x \rightarrow c} f(x) = b$,

then

$$\lim_{x \rightarrow c} g(f(x)) = g\left(\lim_{x \rightarrow c} f(x)\right).$$


You can read the proof in the textbook.

2.5 Continuity

Example

$$\lim_{x \rightarrow \frac{\pi}{2}} \cos \left[2x + \sin \left(\frac{3\pi}{2} + x \right) \right]$$

=

=

=

=

2.5 Continuity

Example

$$\begin{aligned}& \lim_{x \rightarrow \frac{\pi}{2}} \cos \left[2x + \sin \left(\frac{3\pi}{2} + x \right) \right] \\&= \cos \left[\lim_{x \rightarrow \frac{\pi}{2}} \left(2x + \sin \left(\frac{3\pi}{2} + x \right) \right) \right]\end{aligned}$$

=

=

=

2.5 Continuity

Example

$$\begin{aligned}& \lim_{x \rightarrow \frac{\pi}{2}} \cos \left[2x + \sin \left(\frac{3\pi}{2} + x \right) \right] \\&= \cos \left[\lim_{x \rightarrow \frac{\pi}{2}} \left(2x + \sin \left(\frac{3\pi}{2} + x \right) \right) \right] \\&= \cos \left[\lim_{x \rightarrow \frac{\pi}{2}} (2x) + \lim_{x \rightarrow \frac{\pi}{2}} \left(\sin \left(\frac{3\pi}{2} + x \right) \right) \right] \\&= \cos \left[\pi + \right] \\&= \end{aligned}$$

2.5 Continuity

Example

$$\begin{aligned}& \lim_{x \rightarrow \frac{\pi}{2}} \cos \left[2x + \sin \left(\frac{3\pi}{2} + x \right) \right] \\&= \cos \left[\lim_{x \rightarrow \frac{\pi}{2}} \left(2x + \sin \left(\frac{3\pi}{2} + x \right) \right) \right] \\&= \cos \left[\lim_{x \rightarrow \frac{\pi}{2}} (2x) + \lim_{x \rightarrow \frac{\pi}{2}} \left(\sin \left(\frac{3\pi}{2} + x \right) \right) \right] \\&= \cos \left[\pi + \sin \left(\lim_{x \rightarrow \frac{\pi}{2}} \left(\frac{3\pi}{2} + x \right) \right) \right] \\&= \end{aligned}$$

2.5 Continuity

Example

$$\begin{aligned}& \lim_{x \rightarrow \frac{\pi}{2}} \cos \left[2x + \sin \left(\frac{3\pi}{2} + x \right) \right] \\&= \cos \left[\lim_{x \rightarrow \frac{\pi}{2}} \left(2x + \sin \left(\frac{3\pi}{2} + x \right) \right) \right] \\&= \cos \left[\lim_{x \rightarrow \frac{\pi}{2}} (2x) + \lim_{x \rightarrow \frac{\pi}{2}} \left(\sin \left(\frac{3\pi}{2} + x \right) \right) \right] \\&= \cos \left[\pi + \sin \left(\lim_{x \rightarrow \frac{\pi}{2}} \left(\frac{3\pi}{2} + x \right) \right) \right] \\&= \cos [\pi + \sin 2\pi] = \cos [\pi + 0] = -1.\end{aligned}$$

2.5 Continuity

Example

$$\begin{aligned}\lim_{x \rightarrow \frac{\pi}{2}} \tan \left[\frac{5x}{2} - \pi \cos \left(\frac{\pi}{2} - x \right) \right] \\ = \tan \left[\lim_{x \rightarrow \frac{\pi}{2}} \left(\frac{5x}{2} - \pi \cos \left(\frac{\pi}{2} - x \right) \right) \right]\end{aligned}$$

=

=

=

2.5 Continuity

Example

$$\begin{aligned}
 & \lim_{x \rightarrow \frac{\pi}{2}} \tan \left[\frac{5x}{2} - \pi \cos \left(\frac{\pi}{2} - x \right) \right] \\
 &= \tan \left[\lim_{x \rightarrow \frac{\pi}{2}} \left(\frac{5x}{2} - \pi \cos \left(\frac{\pi}{2} - x \right) \right) \right] \\
 &= \tan \left[\lim_{x \rightarrow \frac{\pi}{2}} \left(\frac{5x}{2} \right) - \pi \lim_{x \rightarrow \frac{\pi}{2}} \left(\cos \left(\frac{\pi}{2} - x \right) \right) \right] \\
 &= \tan \left[\frac{5\pi}{4} - \pi \cos \left(\lim_{x \rightarrow \frac{\pi}{2}} \left(\frac{\pi}{2} - x \right) \right) \right] \\
 &= \tan \left[\frac{5\pi}{4} - \pi \cos 0 \right] = \tan \left[\frac{5\pi}{4} - \pi \right] = \tan \frac{\pi}{4} = 1.
 \end{aligned}$$

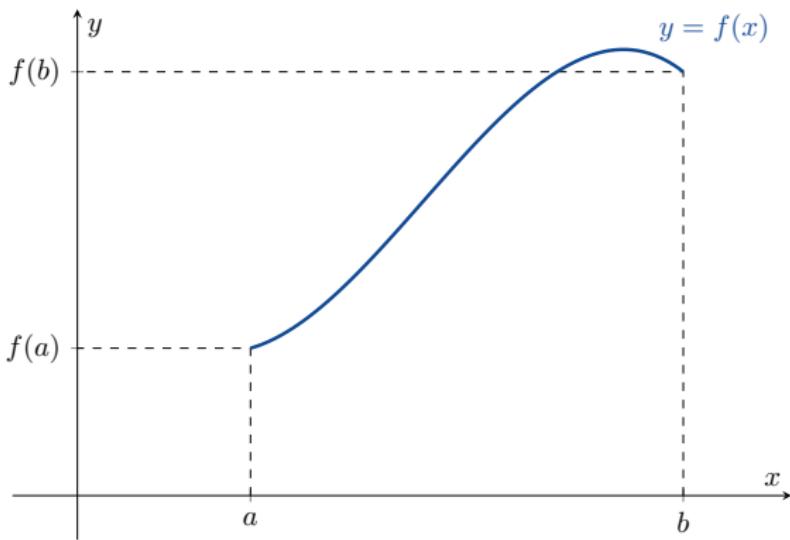


Break

We will continue at 2pm



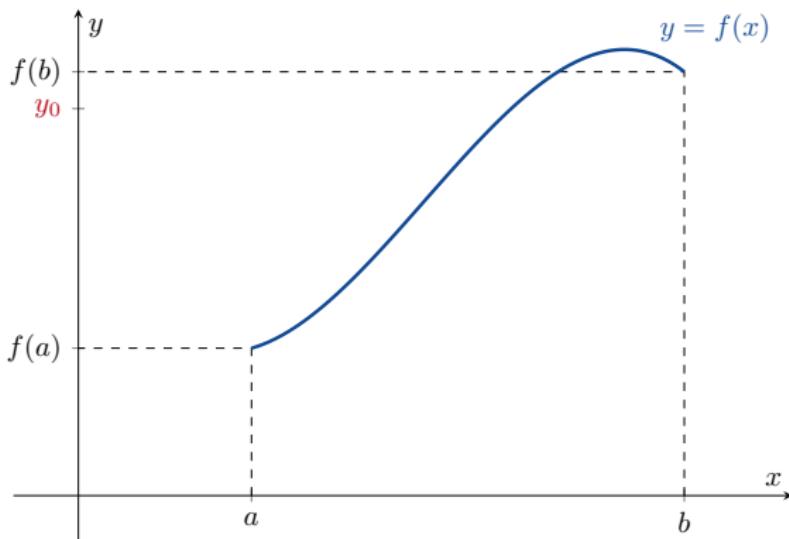
2.5 Continuity



Theorem (The Intermediate Value Theorem)

If f is continuous on a closed interval $[a, b]$,

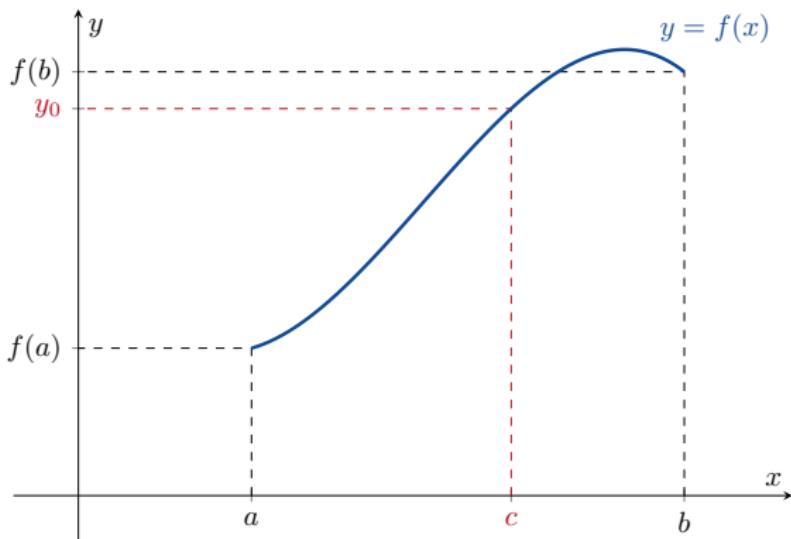
2.5 Continuity



Theorem (The Intermediate Value Theorem)

If f is continuous on a closed interval $[a, b]$, and if y_0 is any value between $f(a)$ and $f(b)$,

2.5 Continuity



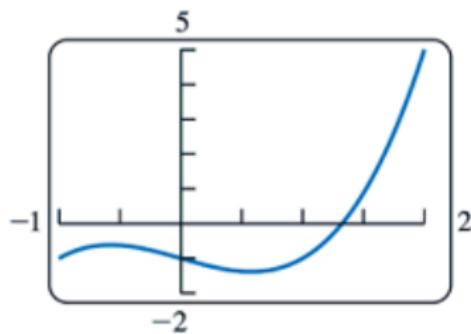
Theorem (The Intermediate Value Theorem)

If f is continuous on a closed interval $[a, b]$, and if y_0 is any value between $f(a)$ and $f(b)$, then there exists $c \in [a, b]$ such that $y_0 = f(c)$.

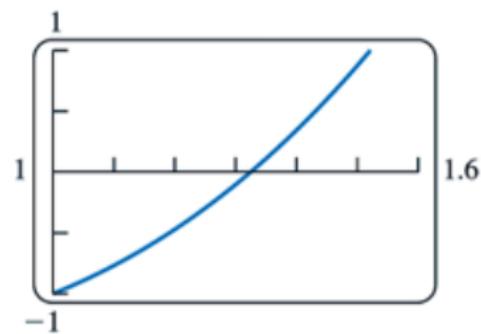
EXAMPLE 10 Show that there is a root of the equation $x^3 - x - 1 = 0$ between 1 and 2.

Solution Let $f(x) = x^3 - x - 1$. Since $f(1) = 1 - 1 - 1 = -1 < 0$ and $f(2) = 2^3 - 2 - 1 = 5 > 0$, we see that $y_0 = 0$ is a value between $f(1)$ and $f(2)$. Since f is a polynomial, it is continuous, and the Intermediate Value Theorem says there is a zero of f between 1 and 2. Figure 2.45 shows the result of zooming in to locate the root near $x = 1.32$.

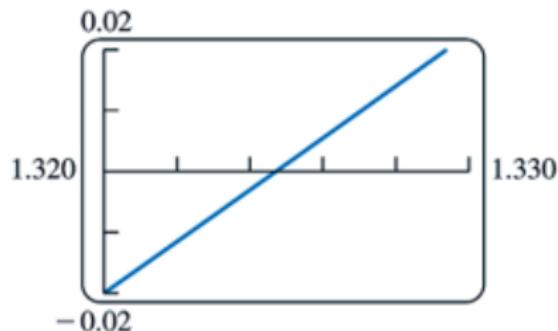




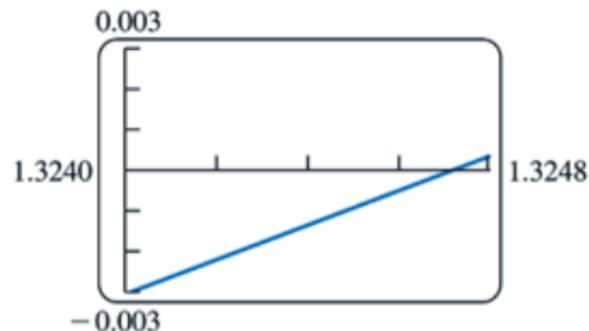
(a)



(b)

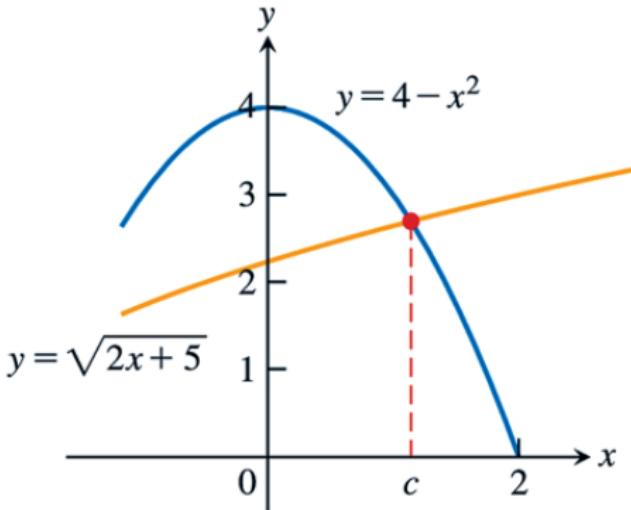


(c)



(d)

2.5 Continuity



Example

Use the Intermediate Value Theorem to prove that the equation

$$\sqrt{2x + 5} = 4 - x^2$$

has a solution.

2.5 Continuity



This is the same as showing that

$$f(x) = \sqrt{2x + 5} + x^2 - 4 = 0$$

has a solution.

2.5 Continuity



This is the same as showing that

$$f(x) = \sqrt{2x + 5} + x^2 - 4 = 0$$

has a solution. Note that

- f is continuous on $[-\frac{5}{2}, \infty)$.
- $f(0) = \sqrt{5} + 0 - 4 \approx -1.76$.
- $f(2) = \sqrt{9} + 4 - 4 = 3$.

2.5 Continuity

This is the same as showing that

$$f(x) = \sqrt{2x + 5} + x^2 - 4 = 0$$

has a solution. Note that

- f is continuous on $[-\frac{5}{2}, \infty)$.
- $f(0) = \sqrt{5} + 0 - 4 \approx -1.76$.
- $f(2) = \sqrt{9} + 4 - 4 = 3$.

Since $f(0) \leq 0 \leq f(2)$,

2.5 Continuity



This is the same as showing that

$$f(x) = \sqrt{2x + 5} + x^2 - 4 = 0$$

has a solution. Note that

- f is continuous on $[-\frac{5}{2}, \infty)$.
- $f(0) = \sqrt{5} + 0 - 4 \approx -1.76$.
- $f(2) = \sqrt{9} + 4 - 4 = 3$.

Since $f(0) \leq 0 \leq f(2)$, it follows by the Intermediate Value Theorem that there exists $c \in [0, 2]$ such that $f(c) = 0$.

2.5 Continuity



Continuous Extension to a Point

2.5 Continuity

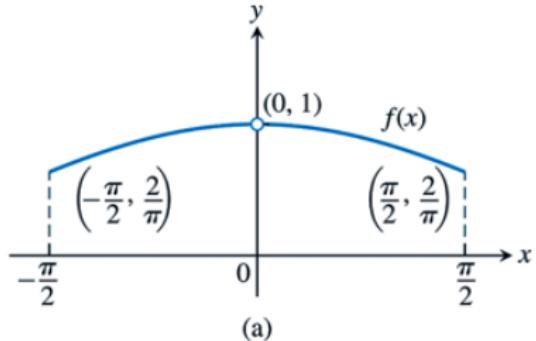


Continuous Extension to a Point

or “filling in holes”.

2.5 Continuity

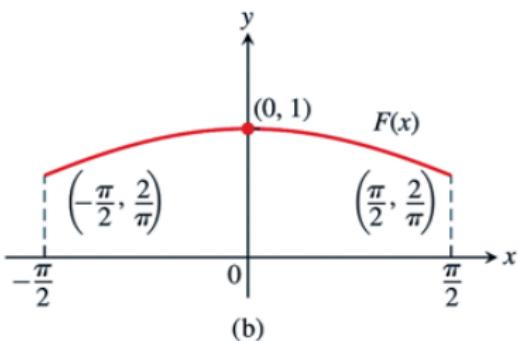
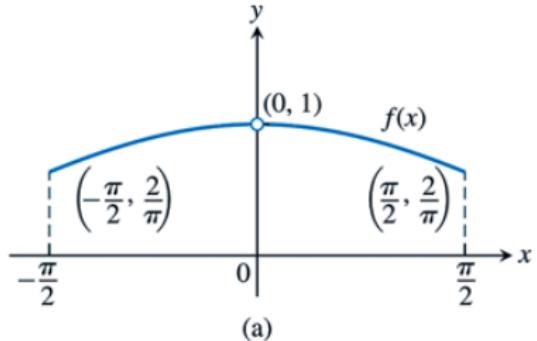
Consider the function $f : (-\infty, 0) \cup (0, \infty) \rightarrow \mathbb{R}$, $f(x) = \frac{\sin x}{x}$.



How can we “fill in” the hole at $(0, 1)$?

2.5 Continuity

Consider the function $f : (-\infty, 0) \cup (0, \infty) \rightarrow \mathbb{R}$, $f(x) = \frac{\sin x}{x}$.

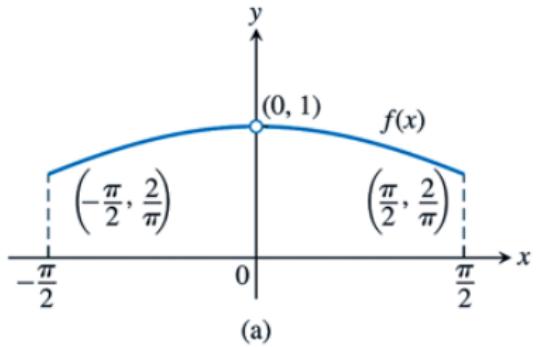


How can we “fill in” the hole at $(0, 1)$?

How can we define a new function $F : \mathbb{R} \rightarrow \mathbb{R}$ such that

- $F = f$ on the domain of f ; and
- F is continuous?

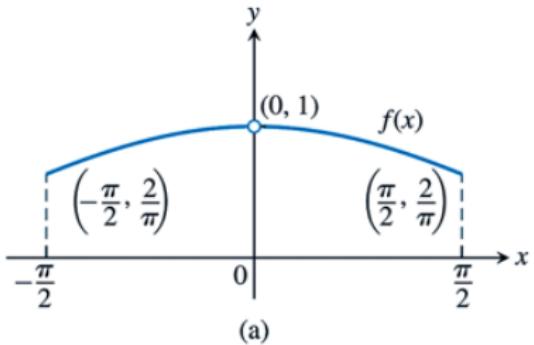
2.5 Continuity



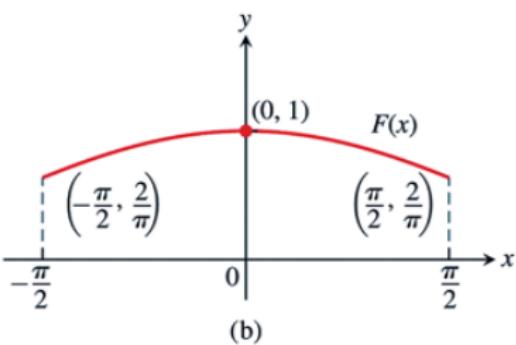
Since

$$\lim_{x \rightarrow 0} f(x) = 1,$$

2.5 Continuity



(a)



(b)

Since

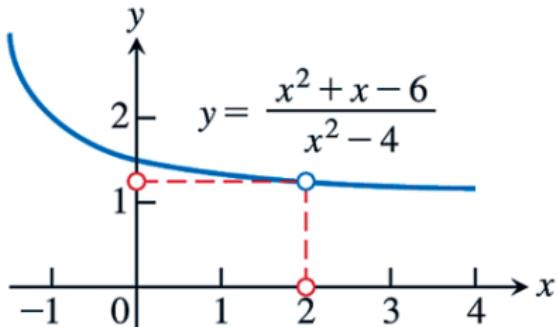
$$\lim_{x \rightarrow 0} f(x) = 1,$$

we define

$$F(x) = \begin{cases} \frac{\sin x}{x} & x \neq 0 \\ 1 & x = 0. \end{cases}$$

We say that F is the *continuous extension of f* to $x = 0$.

2.5 Continuity



(a)

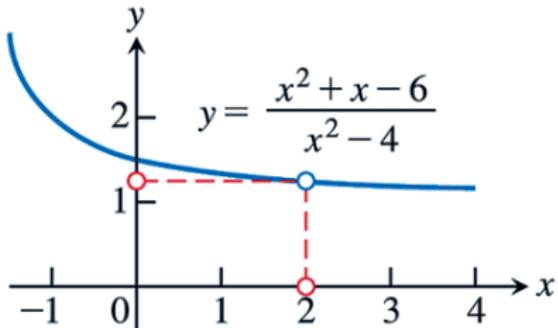
Example

Show that

$$f(x) = \frac{x^2 + x - 6}{x^2 - 4}, \quad x \neq 2$$

has a continuous extension to $x = 0$ and find that extension.

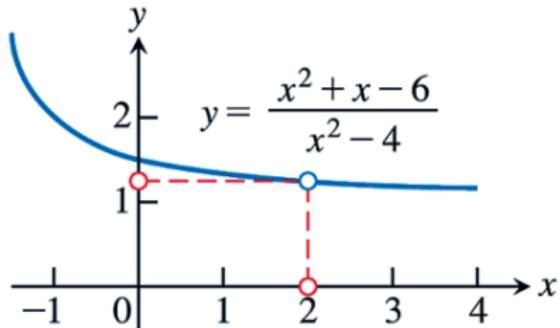
2.5 Continuity



(a)

Note that $f(2)$ is not defined.

2.5 Continuity

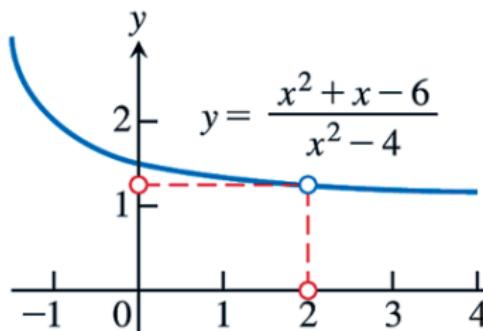


(a)

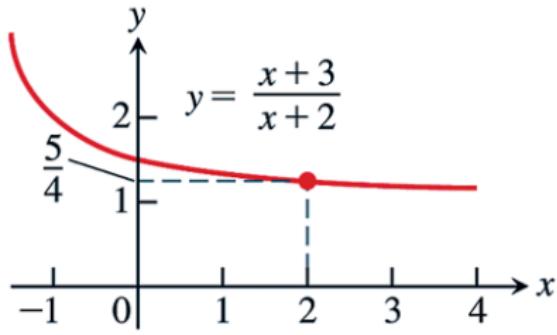
Note that $f(2)$ is not defined. However

$$f(x) = \frac{x^2 + x - 6}{x^2 - 4} = \frac{(x-2)(x+3)}{(x-2)(x+2)} = \frac{x+3}{x+2}.$$

2.5 Continuity



(a)

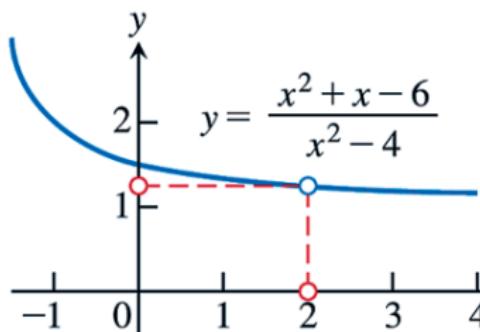


(b)

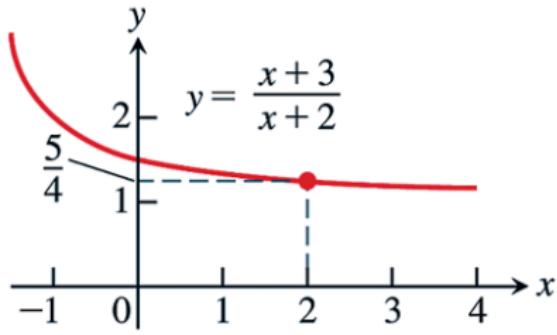
So we define $F : \mathbb{R} \rightarrow \mathbb{R}$ by

$$F(x) = \frac{x+3}{x+2}.$$

2.5 Continuity



(a)



(b)

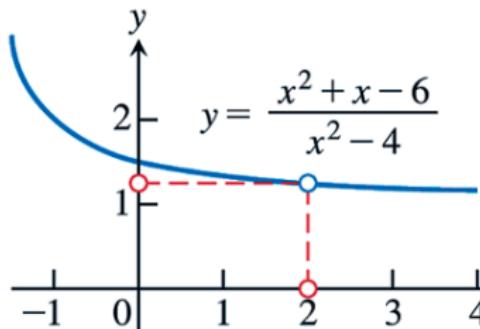
So we define $\textcolor{red}{F} : \mathbb{R} \rightarrow \mathbb{R}$ by

$$\textcolor{red}{F}(x) = \frac{x+3}{x+2}.$$

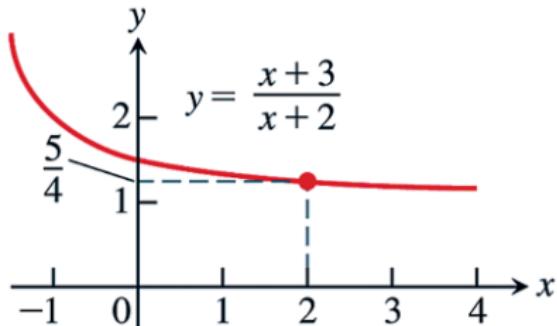
Then

- $\textcolor{red}{F}$ is continuous; and
- $\textcolor{red}{F}(x) = f(x)$ for all $x \neq 2$.

2.5 Continuity



(a)



(b)

$F(x) = \frac{x+3}{x+2}$ is the continuous extension of $f(x) = \frac{x^2 + x - 6}{x^2 - 4}$ to $x = 2$.

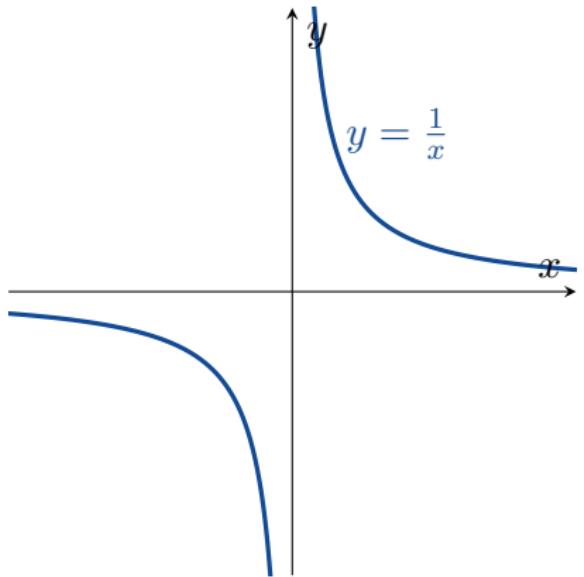


Limits Involving Infinity; Asymptotes of Graphs

2.6 Limits Involving Infinity; Asymptotes of Graphs



Finite Limits as $x \rightarrow \pm\infty$

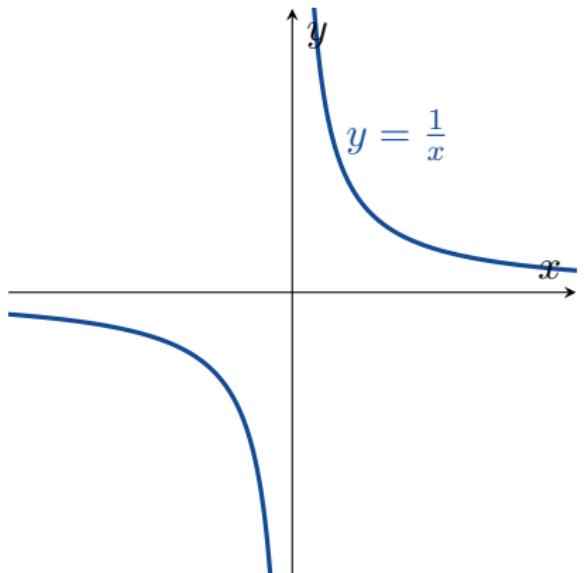


Question: If $x > 0$ and x gets bigger and bigger and bigger, what happens to $\frac{1}{x}$?

2.6 Limits Involving Infinity; Asymptotes of Graphs



Finite Limits as $x \rightarrow \pm\infty$



Question: If $x > 0$ and x gets bigger and bigger and bigger, what happens to $\frac{1}{x}$?

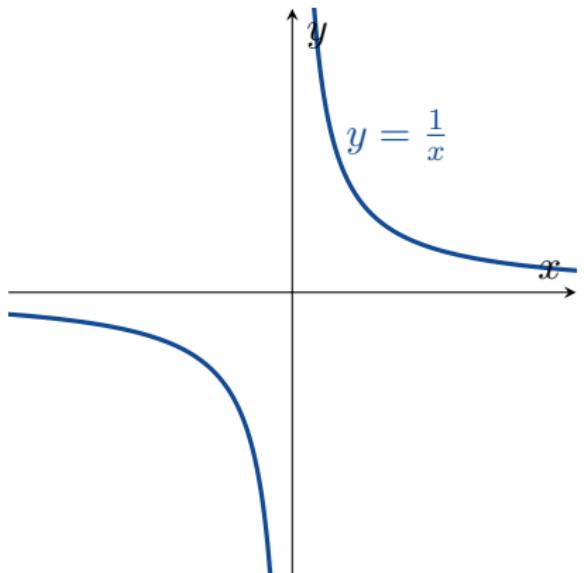
Answer: $\frac{1}{x}$ gets closer and closer and closer to 0. We want to write this as

$$\lim_{x \rightarrow \infty} \frac{1}{x} = 0.$$

2.6 Limits Involving Infinity; Asymptotes of Graphs



Finite Limits as $x \rightarrow \pm\infty$



Question: If $x > 0$ and x gets bigger and bigger and bigger, what happens to $\frac{1}{x}$?

Answer: $\frac{1}{x}$ gets closer and closer and closer to 0. We want to write this as

$$\lim_{x \rightarrow \infty} \frac{1}{x} = 0.$$

Let's be more precise.

2.6 Limits Involving Infinity; Asymptotes of Graphs



Definition

We write $\lim_{x \rightarrow \infty} f(x) = L$ iff for all $\varepsilon > 0$, there exists a number M such that

$$x > M \implies |f(x) - L| < \varepsilon.$$

Definition

2.6 Limits Involving Infinity; Asymptotes of Graphs



Definition

We write $\lim_{x \rightarrow \infty} f(x) = L$ iff for all $\varepsilon > 0$, there exists a number M such that

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Definition

2.6 Limits Involving Infinity; Asymptotes of Graphs



Definition

We write $\lim_{x \rightarrow \infty} f(x) = L$ iff for all $\varepsilon > 0$, there exists a number M such that

$$x > M \implies |f(x) - L| < \varepsilon.$$

Definition

We write $\lim_{x \rightarrow -\infty} f(x) = L$ iff for all $\varepsilon > 0$, there exists a number N such that

$$x < N \implies |f(x) - L| < \varepsilon.$$

2.6 Limits Involving Infinity; Asymptotes of Graphs



Please think

ε = a very small number

δ = a very small number

M = a very large number

N = a very large negative number

2.6 Limits Involving Infinity; Asymptotes of Graphs



Example

Prove that $\lim_{x \rightarrow \infty} \frac{1}{x} = 0$ and $\lim_{x \rightarrow -\infty} \frac{1}{x} = 0$.

Let $\varepsilon > 0$. Choose $M = \dots$ and $N = \dots$. Then

$$x > M \implies$$

and

$$x > N \implies$$

2.6 Limits Involving Infinity; Asymptotes of Graphs



Example

Prove that $\lim_{x \rightarrow \infty} \frac{1}{x} = 0$ and

Let $\varepsilon > 0$. Choose $M =$

$$x > M$$

=

$$\left| \frac{1}{x} - 0 \right| < \varepsilon$$

$$\frac{1}{x} < \varepsilon$$

$$\frac{1}{\varepsilon} < x$$

and

$$x > N$$

\implies

Choose $M = \frac{1}{\varepsilon}$.

2.6 Limits Involving Infinity; Asymptotes of Graphs



Example

Prove that $\lim_{x \rightarrow \infty} \frac{1}{x} = 0$ and

Let $\varepsilon > 0$. Choose $M =$

$$x > M$$

=

$$\left| \frac{1}{x} - 0 \right| < \varepsilon$$

$$-\frac{1}{x} < \varepsilon$$

$$\frac{1}{\varepsilon} < x$$

and

$$x > N$$

\implies

Choose $N = -\frac{1}{\varepsilon}$.

2.6 Limits Involving Infinity; Asymptotes of Graphs



Example

Prove that $\lim_{x \rightarrow \infty} \frac{1}{x} = 0$ and $\lim_{x \rightarrow -\infty} \frac{1}{x} = 0$.

Let $\varepsilon > 0$. Choose $M = \frac{1}{\varepsilon}$ and $N = -\frac{1}{\varepsilon}$. Then

$$x > M \implies$$

and

$$x > N \implies$$

2.6 Limits Involving Infinity; Asymptotes of Graphs



Example

Prove that $\lim_{x \rightarrow \infty} \frac{1}{x} = 0$ and $\lim_{x \rightarrow -\infty} \frac{1}{x} = 0$.

Let $\varepsilon > 0$. Choose $M = \frac{1}{\varepsilon}$ and $N = -\frac{1}{\varepsilon}$. Then

$$x > M \implies \left| \frac{1}{x} - 0 \right| = \frac{1}{x} < \frac{1}{M} = \varepsilon$$

and

$$x > N \implies \left| \frac{1}{x} - 0 \right| = -\frac{1}{x} < -\frac{1}{N} = \varepsilon.$$

2.6 Limits Involving Infinity; Asymptotes of Graphs



Example

Prove that $\lim_{x \rightarrow \infty} k = k$ for any $k \in \mathbb{R}$.

I leave this for you to do.

2.6 Limits Involving Infinity; Asymptotes of Graphs



Theorem

All of the limit laws (sum rule, difference rule, constant multiple rule, . . .) are also true for $\lim_{x \rightarrow \infty}$ and $\lim_{x \rightarrow -\infty}$.

2.6 Limits Involving Infinity; Asymptotes of Graphs



Theorem

All of the limit laws (sum rule, difference rule, constant multiple rule, . . .) are also true for $\lim_{x \rightarrow \infty}$ and $\lim_{x \rightarrow -\infty}$.

Example

$$\begin{aligned}\lim_{x \rightarrow \infty} \left(5 + \frac{1}{x} \right) &= \lim_{x \rightarrow \infty} 5 + \lim_{x \rightarrow \infty} \frac{1}{x} \\&\quad (\text{sum rule}) \\&= 5 + 0 = 5.\end{aligned}$$

2.6 Limits Involving Infinity; Asymptotes of Graphs



Example

$$\begin{aligned}\lim_{x \rightarrow -\infty} \frac{\pi\sqrt{3}}{x^2} &= \lim_{x \rightarrow -\infty} \left(\pi\sqrt{3} \frac{1}{x} \frac{1}{x} \right) \\&= \left(\lim_{x \rightarrow -\infty} \pi\sqrt{3} \right) \left(\lim_{x \rightarrow -\infty} \frac{1}{x} \right) \left(\lim_{x \rightarrow -\infty} \frac{1}{x} \right) \\&\quad (\text{product rule}) \\&= \pi\sqrt{3} \times 0 \times 0 = 0.\end{aligned}$$

2.6 Limits Involving Infinity; Asymptotes of Graphs



Example (Limits at Infinity of Rational Functions)

$$\text{Find } \lim_{x \rightarrow \infty} \frac{5x^2 + 8x - 3}{3x^2 + 2}.$$

Please note that the answer is not " $\frac{\infty}{\infty}$ ". You can expect to receive zero points in the exam if you write " $\frac{\infty}{\infty}$ ".

2.6 Limits Involving Infinity; Asymptotes of Graphs



Example (Limits at Infinity of Rational Functions)

$$\text{Find } \lim_{x \rightarrow \infty} \frac{5x^2 + 8x - 3}{3x^2 + 2}.$$

Please note that the answer is not “ $\frac{\infty}{\infty}$ ”. You can expect to receive zero points in the exam if you write “ $\frac{\infty}{\infty}$ ”.

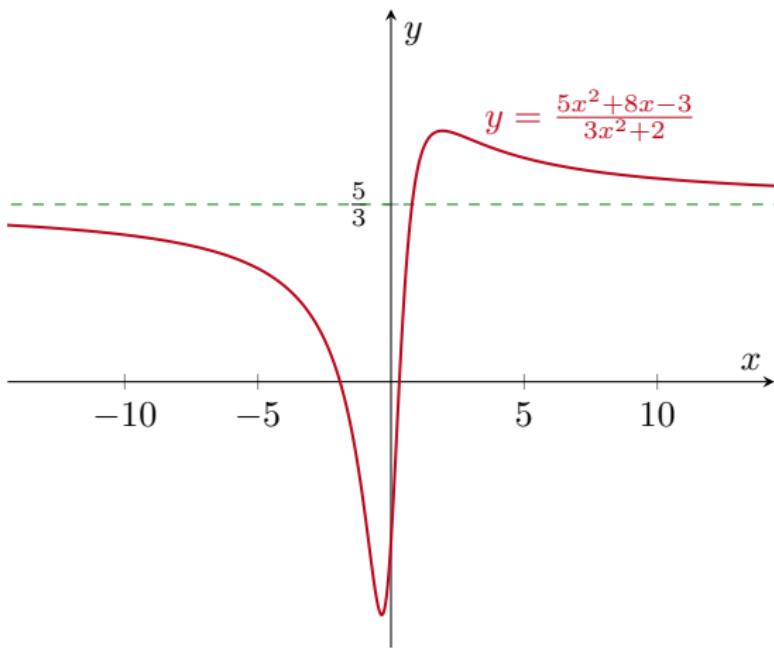
Instead we calculate that

$$\lim_{x \rightarrow \infty} \frac{5x^2 + 8x - 3}{3x^2 + 2} = \lim_{x \rightarrow \infty} \frac{5 + \frac{8}{x} - \frac{3}{x^2}}{3 + \frac{2}{x^2}} = \frac{5 + 0 - 0}{3 + 0} = \frac{5}{3}.$$

2.6 Limits Involving Infinity; Asymptotes of Graphs

Example (Limits at Infinity of Rational Functions)

Find $\lim_{x \rightarrow \infty} \frac{5x^2 + 8x - 3}{3x^2 + 2}$.



2.6 Limits Involving Infinity; Asymptotes of Graphs



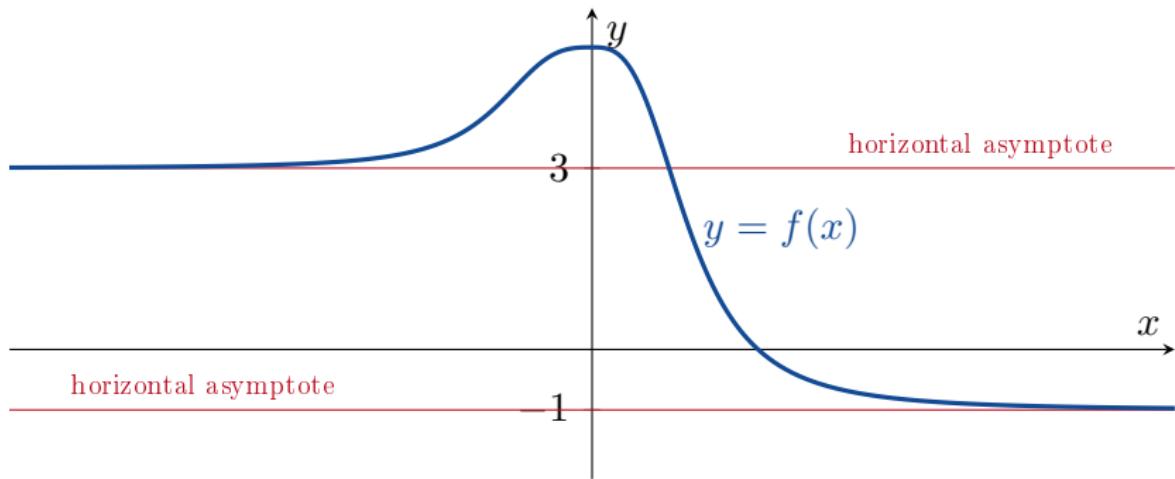
Example

$$\lim_{x \rightarrow -\infty} \frac{11x + 2}{2x^3 - 1} = \lim_{x \rightarrow -\infty} \frac{\frac{11}{x^2} + \frac{2}{x^3}}{2 - \frac{1}{x^3}} = \frac{0 + 0}{2 - 0} = 0.$$

2.6 Limits Involving Infinity; Asymptotes of Graphs



Horizontal Asymptotes



If $y = f(x)$ gets “closer and closer” to a horizontal line as $x \rightarrow \infty$ or $x \rightarrow -\infty$, then that line is called a **horizontal asymptote** of $y = f(x)$.

2.6 Limits Involving Infinity; Asymptotes of Graphs



Definition

A line $y = b$ is a *horizontal asymptote* of $y = f(x)$ if either

$$\lim_{x \rightarrow \infty} f(x) = b \quad \text{or} \quad \lim_{x \rightarrow -\infty} f(x) = b.$$

2.6 Limits Involving Infinity; Asymptotes of Graphs



Definition

A line $y = b$ is a *horizontal asymptote* of $y = f(x)$ if either

$$\lim_{x \rightarrow \infty} f(x) = b \quad \text{or} \quad \lim_{x \rightarrow -\infty} f(x) = b.$$

For example, $y = \frac{5}{2}$ is a horizontal asymptote of $\frac{5x^2 + 8x - 3}{3x^2 + 2}$.

2.6 Limits Involving Infinity; Asymptotes of Graphs



Example

Find the horizontal asymptotes of $y = \frac{x^3 - 2}{|x|^3 + 1}$.

2.6 Limits Involving Infinity; Asymptotes of Graphs



Example

Find the horizontal asymptotes of $y = \frac{x^3 - 2}{|x|^3 + 1}$.

If $x > 0$, then

$$\lim_{x \rightarrow \infty} \frac{x^3 - 2}{|x|^3 + 1} = \lim_{x \rightarrow \infty} \frac{x^3 - 2}{x^3 + 1} = \lim_{x \rightarrow \infty} \frac{1 - \frac{2}{x^3}}{1 + \frac{1}{x^3}} = \frac{1 - 0}{1 + 0} = 1$$

2.6 Limits Involving Infinity; Asymptotes of Graphs



Example

Find the horizontal asymptotes of $y = \frac{x^3 - 2}{|x|^3 + 1}$.

If $x > 0$, then

$$\lim_{x \rightarrow \infty} \frac{x^3 - 2}{|x|^3 + 1} = \lim_{x \rightarrow \infty} \frac{x^3 - 2}{x^3 + 1} = \lim_{x \rightarrow \infty} \frac{1 - \frac{2}{x^3}}{1 + \frac{1}{x^3}} = \frac{1 - 0}{1 + 0} = 1$$

and if $x < 0$ then

$$\lim_{x \rightarrow -\infty} \frac{x^3 - 2}{|x|^3 + 1} = \lim_{x \rightarrow -\infty} \frac{x^3 - 2}{(-x)^3 + 1} = \lim_{x \rightarrow -\infty} \frac{1 - \frac{2}{x^3}}{-1 + \frac{1}{x^3}} = \frac{1 - 0}{-1 + 0} = -1.$$

2.6 Limits Involving Infinity; Asymptotes of Graphs



Example

Find the horizontal asymptotes of $y = \frac{x^3 - 2}{|x|^3 + 1}$.

If $x > 0$, then

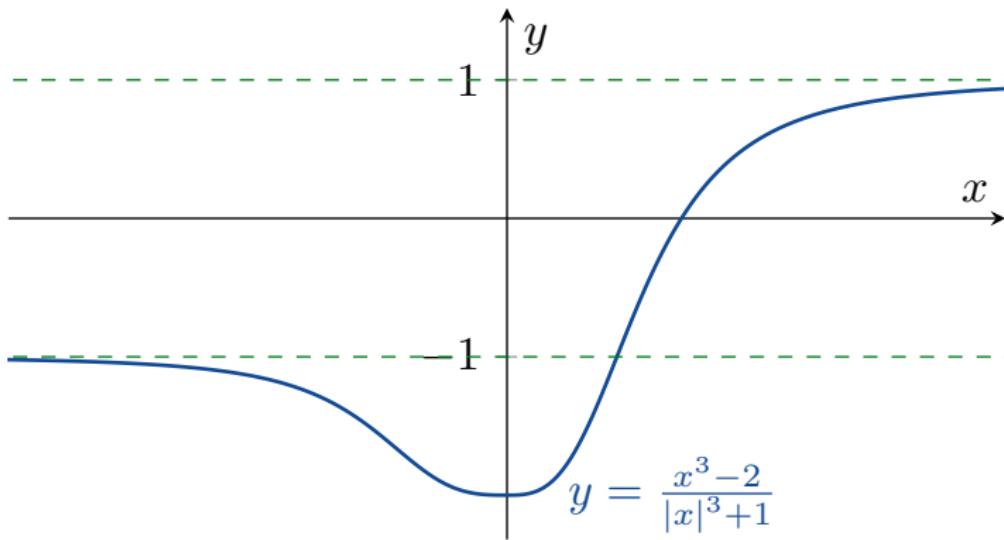
$$\lim_{x \rightarrow \infty} \frac{x^3 - 2}{|x|^3 + 1} = \lim_{x \rightarrow \infty} \frac{x^3 - 2}{x^3 + 1} = \lim_{x \rightarrow \infty} \frac{1 - \frac{2}{x^3}}{1 + \frac{1}{x^3}} = \frac{1 - 0}{1 + 0} = 1$$

and if $x < 0$ then

$$\lim_{x \rightarrow -\infty} \frac{x^3 - 2}{|x|^3 + 1} = \lim_{x \rightarrow -\infty} \frac{x^3 - 2}{(-x)^3 + 1} = \lim_{x \rightarrow -\infty} \frac{1 - \frac{2}{x^3}}{-1 + \frac{1}{x^3}} = \frac{1 - 0}{-1 + 0} = -1.$$

Therefore $y = 1$ and $y = -1$ are horizontal asymptotes of this function.

2.6 Limits Involving Infinity; Asymptotes of Graphs



2.6 Limits Involving Infinity; Asymptotes of Graphs



Example

$$\text{Find } \lim_{x \rightarrow \infty} \sin \frac{1}{x}.$$

Example

$$\text{Find } \lim_{x \rightarrow \infty} x \sin \frac{1}{x}.$$

2.6 Limits Involving Infinity; Asymptotes of Graphs



Example

$$\text{Find } \lim_{x \rightarrow \infty} \sin \frac{1}{x}.$$

We need to do a substitution here: Let $t = \frac{1}{x}$.

Example

$$\text{Find } \lim_{x \rightarrow \infty} x \sin \frac{1}{x}.$$

2.6 Limits Involving Infinity; Asymptotes of Graphs



Example

Find $\lim_{x \rightarrow \infty} \sin \frac{1}{x}$.

We need to do a substitution here: Let $t = \frac{1}{x}$. Note that

$$x \rightarrow \infty \iff t \rightarrow 0^+.$$

Example

Find $\lim_{x \rightarrow \infty} x \sin \frac{1}{x}$.

2.6 Limits Involving Infinity; Asymptotes of Graphs



Example

Find $\lim_{x \rightarrow \infty} \sin \frac{1}{x}$.

We need to do a substitution here: Let $t = \frac{1}{x}$. Note that

$$x \rightarrow \infty \iff t \rightarrow 0^+.$$

So

$$\lim_{x \rightarrow \infty} \sin \frac{1}{x} = \lim_{t \rightarrow 0^+} \sin t = 0.$$

Example

Find $\lim_{x \rightarrow \infty} x \sin \frac{1}{x}$.

2.6 Limits Involving Infinity; Asymptotes of Graphs



Example

Find $\lim_{x \rightarrow \infty} \sin \frac{1}{x}$.

We need to do a substitution here: Let $t = \frac{1}{x}$. Note that

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So

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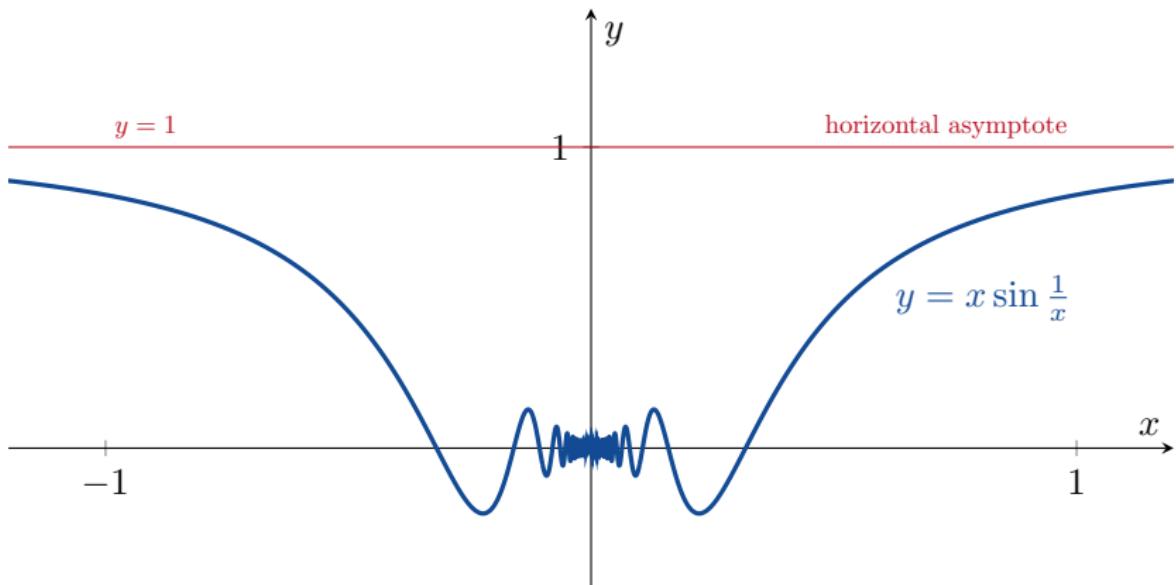
Example

Find $\lim_{x \rightarrow \infty} x \sin \frac{1}{x}$.

Again, let $t = \frac{1}{x}$. Then

$$\lim_{x \rightarrow \infty} x \sin \frac{1}{x} = \lim_{t \rightarrow 0^+} \frac{\sin t}{t} = 1.$$

2.6 Limits Involving Infinity; Asymptotes of Graphs



EXAMPLE 6 Find $\lim_{x \rightarrow 0^+} x \left\lfloor \frac{1}{x} \right\rfloor$.

Solution We let $t = 1/x$ so that

$$\lim_{x \rightarrow 0^+} x \left\lfloor \frac{1}{x} \right\rfloor = \lim_{t \rightarrow \infty} \frac{1}{t} \lfloor t \rfloor$$

From the graph in Figure 2.55, we see that $t - 1 \leq \lfloor t \rfloor \leq t$, which gives

$$1 - \frac{1}{t} \leq \frac{1}{t} \lfloor t \rfloor \leq 1 \quad \text{Multiply inequalities by } \frac{1}{t} > 0.$$

It follows from the Sandwich Theorem that

$$\lim_{t \rightarrow \infty} \frac{1}{t} \lfloor t \rfloor = 1,$$

so 1 is the value of the limit we seek.

2.6 Limits Involving Infinity; Asymptotes of Graphs



Example

Use the Sandwich Theorem to calculate

$$\lim_{x \rightarrow \infty} \left(2 + \frac{\sin x}{x} \right).$$

Since $-1 \leq \sin x \leq 1$, we have that

$$0 \leq \left| \frac{\sin x}{x} \right| \leq \left| \frac{1}{x} \right|.$$

2.6 Limits Involving Infinity; Asymptotes of Graphs



Example

Use the Sandwich Theorem to calculate

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$$0 \leq \left| \frac{\sin x}{x} \right| \leq \left| \frac{1}{x} \right|.$$

Because $\lim_{x \rightarrow \infty} \left| \frac{1}{x} \right| = 0$, it follows by the Sandwich Theorem that

$$\lim_{x \rightarrow \infty} \frac{\sin x}{x} = 0.$$

2.6 Limits Involving Infinity; Asymptotes of Graphs



Example

Use the Sandwich Theorem to calculate

$$\lim_{x \rightarrow \infty} \left(2 + \frac{\sin x}{x} \right).$$

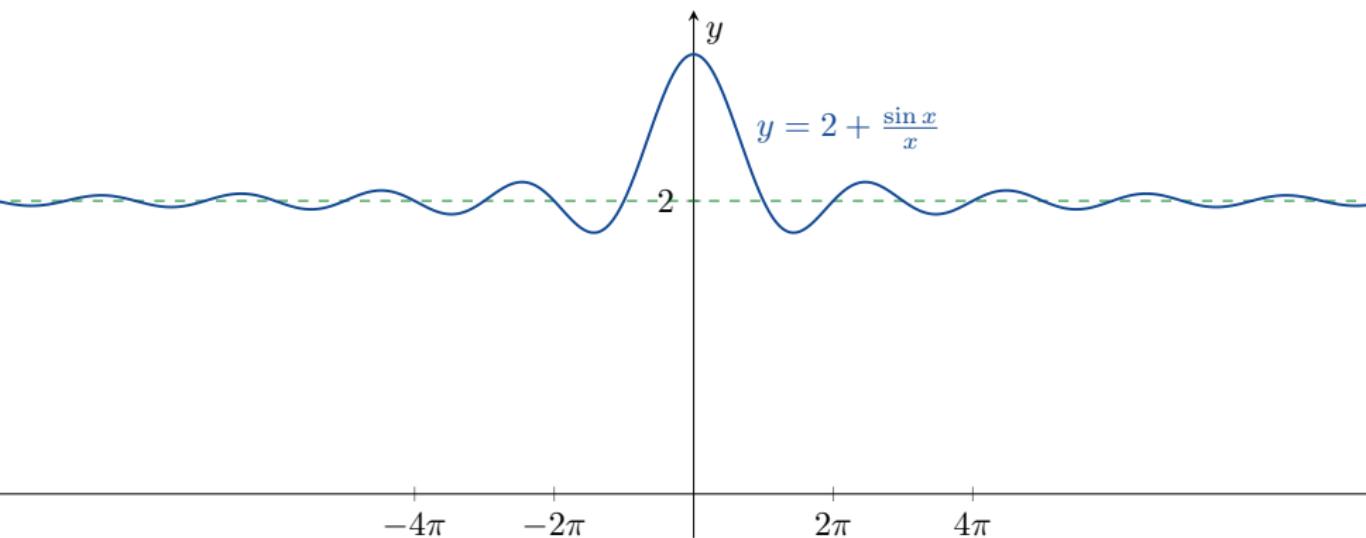
Since $-1 \leq \sin x \leq 1$, we have that

$$0 \leq \left| \frac{\sin x}{x} \right| \leq \left| \frac{1}{x} \right|.$$

Because $\lim_{x \rightarrow \infty} \left| \frac{1}{x} \right| = 0$, it follows by the Sandwich Theorem that $\lim_{x \rightarrow \infty} \frac{\sin x}{x} = 0$. Therefore

$$\lim_{x \rightarrow \infty} \left(2 + \frac{\sin x}{x} \right) = 2 + 0 = 2.$$

2.6 Limits Involving Infinity; Asymptotes of Graphs



2.6 Limits Involving Infinity; Asymptotes of Graphs



Remark

There is one more trick for limits. Because

$$(a - b)(a + b) = a^2 - b^2,$$

it follows that

$$a - b = \frac{a^2 - b^2}{a + b}.$$

This can be useful if the limit contains a $\sqrt{}$.

2.6 Limits Involving Infinity; Asymptotes of Graphs



Remark

There is one more trick for limits. Because

$$(a - b)(a + b) = a^2 - b^2,$$

it follows that

$$a - b = \frac{a^2 - b^2}{a + b}.$$

This can be useful if the limit contains a $\sqrt{}$.

Example

Calculate $\lim_{x \rightarrow \infty} \left(x - \sqrt{x^2 + 16} \right)$.

2.6 Limits Involving Infinity; Asymptotes of Graphs



$$\lim_{x \rightarrow \infty} \left(x - \sqrt{x^2 + 16} \right) = \lim_{x \rightarrow \infty} \left(x - \sqrt{x^2 + 16} \right) \left(\frac{x + \sqrt{x^2 + 16}}{x + \sqrt{x^2 + 16}} \right)$$

=

=

=

=

2.6 Limits Involving Infinity; Asymptotes of Graphs



$$\begin{aligned}\lim_{x \rightarrow \infty} \left(x - \sqrt{x^2 + 16} \right) &= \lim_{x \rightarrow \infty} \left(x - \sqrt{x^2 + 16} \right) \left(\frac{x + \sqrt{x^2 + 16}}{x + \sqrt{x^2 + 16}} \right) \\&= \lim_{x \rightarrow \infty} \frac{x^2 - (x^2 + 16)}{x + \sqrt{x^2 + 16}} \\&= \\&= \\&= \end{aligned}$$

2.6 Limits Involving Infinity; Asymptotes of Graphs



$$\lim_{x \rightarrow \infty} \left(x - \sqrt{x^2 + 16} \right) = \lim_{x \rightarrow \infty} \left(x - \sqrt{x^2 + 16} \right) \left(\frac{x + \sqrt{x^2 + 16}}{x + \sqrt{x^2 + 16}} \right)$$

$$= \lim_{x \rightarrow \infty} \frac{x^2 - (x^2 + 16)}{x + \sqrt{x^2 + 16}}$$

$$= \lim_{x \rightarrow \infty} \frac{-16}{x + \sqrt{x^2 + 16}}$$

=

=

2.6 Limits Involving Infinity; Asymptotes of Graphs



$$\begin{aligned}\lim_{x \rightarrow \infty} \left(x - \sqrt{x^2 + 16} \right) &= \lim_{x \rightarrow \infty} \left(x - \sqrt{x^2 + 16} \right) \left(\frac{x + \sqrt{x^2 + 16}}{x + \sqrt{x^2 + 16}} \right) \\&= \lim_{x \rightarrow \infty} \frac{x^2 - (x^2 + 16)}{x + \sqrt{x^2 + 16}} \\&= \lim_{x \rightarrow \infty} \frac{-16}{x + \sqrt{x^2 + 16}} \\&= \lim_{x \rightarrow \infty} \frac{\frac{-16}{x}}{1 + \sqrt{1 + \frac{16}{x^2}}} \\&= \end{aligned}$$

2.6 Limits Involving Infinity; Asymptotes of Graphs

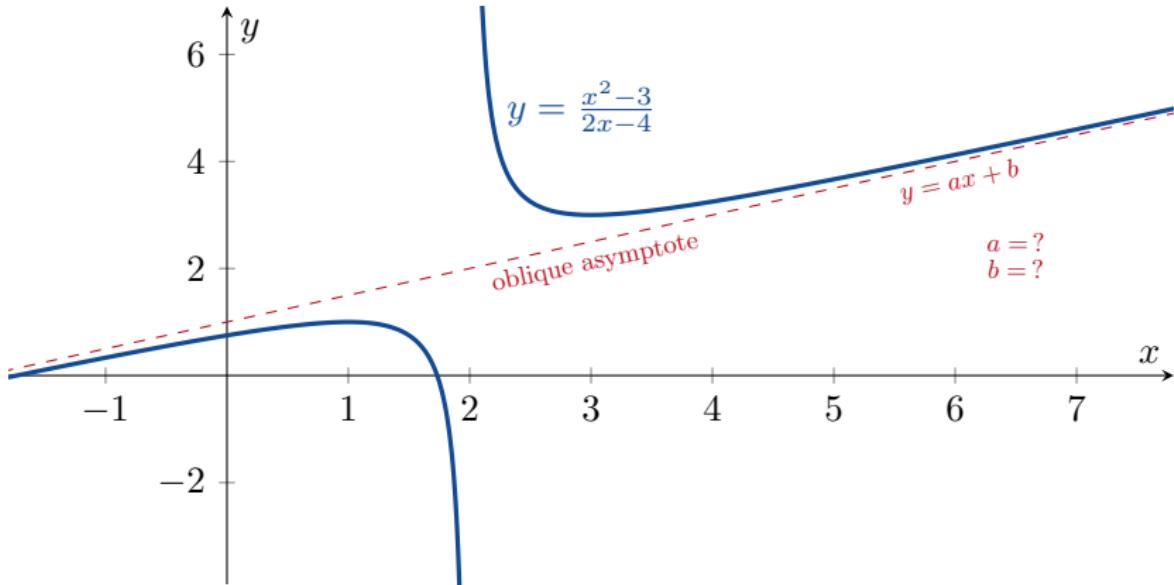


$$\begin{aligned}\lim_{x \rightarrow \infty} \left(x - \sqrt{x^2 + 16} \right) &= \lim_{x \rightarrow \infty} \left(x - \sqrt{x^2 + 16} \right) \left(\frac{x + \sqrt{x^2 + 16}}{x + \sqrt{x^2 + 16}} \right) \\&= \lim_{x \rightarrow \infty} \frac{x^2 - (x^2 + 16)}{x + \sqrt{x^2 + 16}} \\&= \lim_{x \rightarrow \infty} \frac{-16}{x + \sqrt{x^2 + 16}} \\&= \lim_{x \rightarrow \infty} \frac{\frac{-16}{x}}{1 + \sqrt{1 + \frac{16}{x^2}}} \\&= \frac{0}{1 + \sqrt{1 + 0}} = 0.\end{aligned}$$

2.6 Limits Involving Infinity; Asymptotes of Graphs



Oblique Asymptotes

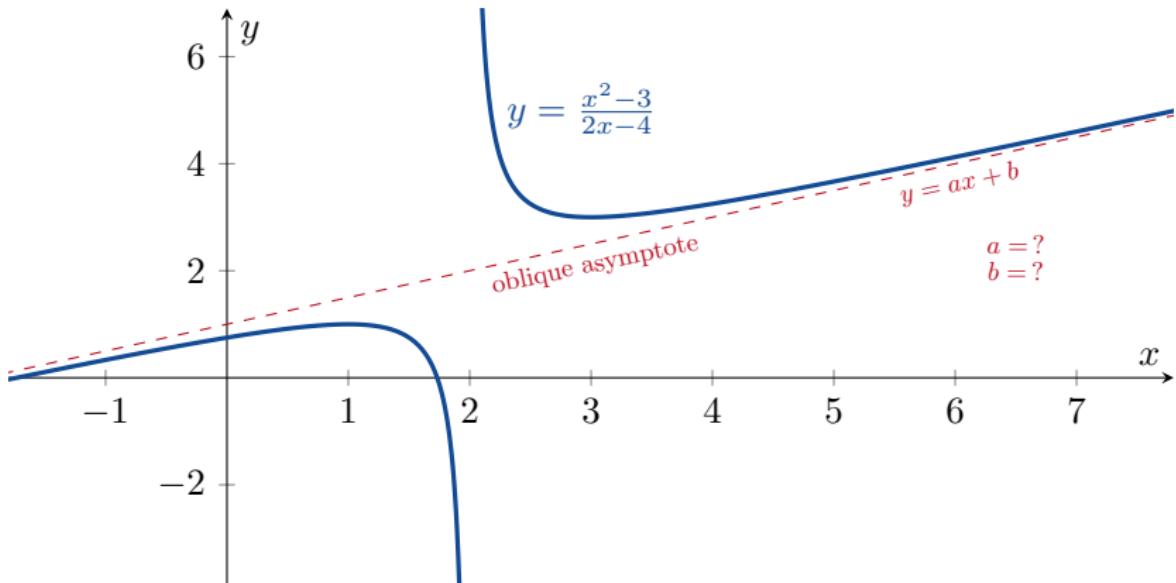


Sometimes the graph of a function approaches a sloped line as $x \rightarrow \infty$ or $x \rightarrow -\infty$. This is called an *oblique asymptote*.

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Example

Find the oblique asymptote of $y = \frac{x^2 - 3}{2x - 4}$.



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Solution 1: We need to find a and b in

$$y = \frac{x^2 - 3}{2x - 4} = (\textcolor{red}{ax + b}) + \left(\frac{\textcolor{green}{c}}{2x - 4} \right).$$

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Solution 1: We need to find a and b in

$$y = \frac{x^2 - 3}{2x - 4} = (\textcolor{red}{ax + b}) + \left(\frac{\textcolor{green}{c}}{2x - 4} \right).$$

Note that $\lim_{x \rightarrow \pm\infty} \frac{\textcolor{green}{c}}{2x - 4} = 0$. So $y = \frac{x^2 - 3}{2x - 4}$ and $y = ax + b$ get closer and closer as $x \rightarrow \pm\infty$.

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So

$$\frac{x^2 - 3}{2x - 4} = (\textcolor{red}{ax + b}) + \left(\frac{\textcolor{green}{c}}{2x - 4} \right)$$

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So

$$\begin{aligned}\frac{x^2 - 3}{2x - 4} &= (ax + b) + \left(\frac{c}{2x - 4} \right) \\ &= (ax + b) \left(\frac{2x - 4}{2x - 4} \right) + \frac{c}{2x - 4}\end{aligned}$$

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So

$$\begin{aligned}\frac{x^2 - 3}{2x - 4} &= (\cancel{ax + b}) + \left(\frac{\cancel{c}}{\cancel{2x - 4}} \right) \\ &= (\cancel{ax + b}) \left(\frac{2x - 4}{2x - 4} \right) + \frac{\cancel{c}}{\cancel{2x - 4}} \\ &= \frac{2ax^2 - 4ax + 2bx - 4b}{2x - 4} + \frac{c}{2x - 4}\end{aligned}$$

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So

$$\begin{aligned}\frac{x^2 - 3}{2x - 4} &= (ax + b) + \left(\frac{c}{2x - 4} \right) \\&= (ax + b) \left(\frac{2x - 4}{2x - 4} \right) + \frac{c}{2x - 4} \\&= \frac{2ax^2 - 4ax + 2bx - 4b}{2x - 4} + \frac{c}{2x - 4} \\&= \frac{2ax^2 + (-4a + 2b)x + (c - 4b)}{2x - 4}\end{aligned}$$

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So

$$\begin{aligned}\frac{x^2 - 3}{2x - 4} &= (ax + b) + \left(\frac{c}{2x - 4} \right) \\&= (ax + b) \left(\frac{2x - 4}{2x - 4} \right) + \frac{c}{2x - 4} \quad \Rightarrow \quad \begin{cases} 2a = 1 \\ -4a + 2b = 0 \\ c - 4b = -3 \end{cases} \\&= \frac{2ax^2 - 4ax + 2bx - 4b}{2x - 4} + \frac{c}{2x - 4} \\&= \frac{2ax^2 + (-4a + 2b)x + (c - 4b)}{2x - 4}\end{aligned}$$

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So

$$\begin{aligned}\frac{x^2 - 3}{2x - 4} &= (ax + b) + \left(\frac{c}{2x - 4} \right) \\&= (ax + b) \left(\frac{2x - 4}{2x - 4} \right) + \frac{c}{2x - 4} \qquad \Rightarrow \qquad \begin{cases} 2a = 1 \\ -4a + 2b = 0 \\ c - 4b = -3 \end{cases} \\&= \frac{2ax^2 - 4ax + 2bx - 4b}{2x - 4} + \frac{c}{2x - 4} \\&= \frac{2ax^2 + (-4a + 2b)x + (c - 4b)}{2x - 4} \qquad \Downarrow \\&\qquad\qquad\qquad \begin{cases} a = \frac{1}{2} \\ b = 1 \\ c = 1 \end{cases}\end{aligned}$$

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So

$$\begin{aligned}\frac{x^2 - 3}{2x - 4} &= (ax + b) + \left(\frac{c}{2x - 4} \right) \\&= (ax + b) \left(\frac{2x - 4}{2x - 4} \right) + \frac{c}{2x - 4} \quad \Rightarrow \quad \begin{cases} 2a = 1 \\ -4a + 2b = 0 \\ c - 4b = -3 \end{cases} \\&= \frac{2ax^2 - 4ax + 2bx - 4b}{2x - 4} + \frac{c}{2x - 4} \\&= \frac{2ax^2 + (-4a + 2b)x + (c - 4b)}{2x - 4} \quad \Downarrow \\&\quad \begin{cases} a = \frac{1}{2} \\ b = 1 \\ c = 1 \end{cases}\end{aligned}$$

Therefore $y = \frac{x}{2} + 1$ is the oblique asymptote of $y = \frac{x^2 - 3}{2x - 4}$.

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Solution 2: We need to divide $(x^2 - 3)$ by $(2x - 4)$ using long division

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Solution 2: We need to divide $(x^2 - 3)$ by $(2x - 4)$ using long division

$$\begin{array}{r} \frac{\frac{1}{2}x + 1}{2x - 4) \overline{x^2 - 3}} \\ \underline{-x^2 + 2x} \\ \hline 2x - 3 \\ \underline{-2x + 4} \\ \hline 1 \end{array}$$

to obtain the oblique asymptote $y = \frac{1}{2}x + 1$.

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In general, we want to write $y = f(x)$ as

$$y = f(x) = (\text{a linear function}) + \begin{pmatrix} \text{a remainder} \\ \text{function which} \\ \text{has limit}=0 \text{ as} \\ x \rightarrow \pm\infty \end{pmatrix}.$$

Then the **linear function** will be the oblique asymptote that we require.

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Example (page 114, exercise 108)

Find the oblique asymptote of $y = \frac{x^3 + 1}{x^2}$.

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Example (page 114, exercise 108)

Find the oblique asymptote of $y = \frac{x^3 + 1}{x^2}$.

Note that

$$y = \frac{x^3 + 1}{x^2} = \frac{x^3}{x^2} + \frac{1}{x^2} = \cancel{x} + \frac{1}{x^2}.$$

Since $\lim_{x \rightarrow \pm\infty} \frac{1}{x^2} = 0$, the oblique asymptote must be $\textcolor{red}{y} = \textcolor{red}{x}$.

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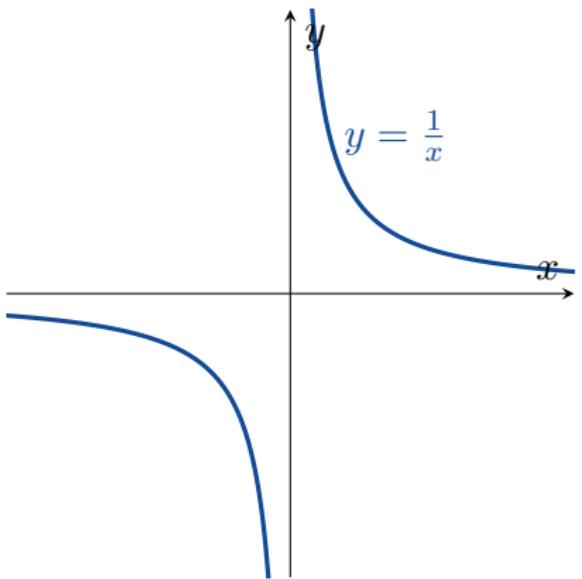


We will do more oblique asymptotes in Lecture 7.

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Infinite Limits

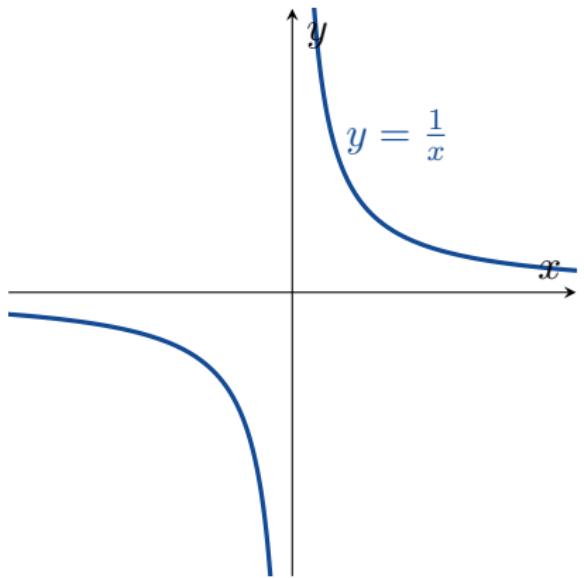


Question: What happens to $\frac{1}{x}$ when $x \rightarrow 0^+$?

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Infinite Limits



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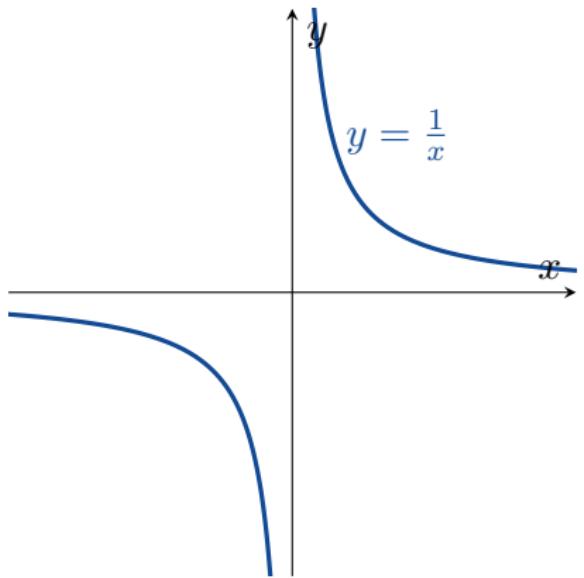
Answer: $\frac{1}{x}$ gets bigger and bigger. We want to write this as

$$\lim_{x \rightarrow 0^+} \frac{1}{x} = \infty.$$

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Infinite Limits



Question: What happens to $\frac{1}{x}$ when $x \rightarrow 0^+$?

Answer: $\frac{1}{x}$ gets bigger and bigger. We want to write this as

$$\lim_{x \rightarrow 0^+} \frac{1}{x} = \infty.$$

Let's be more precise.

2.6 Limits Involving Infinity; Asymptotes of Graphs



Definition

We write $\lim_{x \rightarrow c} f(x) = \infty$

Definition

We write $\lim_{x \rightarrow c} f(x) = -\infty$

2.6 Limits Involving Infinity; Asymptotes of Graphs



Definition

We write $\lim_{x \rightarrow c} f(x) = \infty$ iff for all $B > 0$, there exists $\delta > 0$ such that

$$0 < |x - c| < \delta \implies f(x) > B.$$

Definition

We write $\lim_{x \rightarrow c} f(x) = -\infty$

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Definition

We write $\lim_{x \rightarrow c} f(x) = \infty$ iff for all $B > 0$, there exists $\delta > 0$ such that

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Definition

We write $\lim_{x \rightarrow c} f(x) = -\infty$ iff for all $B > 0$, there exists $\delta > 0$ such that

$$0 < |x - c| < \delta \implies f(x) < -B.$$

2.6 Limits Involving Infinity; Asymptotes of Graphs



There are similar definitions for $x \rightarrow c^+$ and $x \rightarrow c^-$: I leave these for you to write down.

2.6 Limits Involving Infinity; Asymptotes of Graphs



Example

Prove that $\lim_{x \rightarrow 0} \frac{1}{x^2} = \infty$.

Let $B > 0$.

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Example

Prove that $\lim_{x \rightarrow 0} \frac{1}{x^2} = \infty$.

Let $B > 0$. Choose $\delta = \frac{1}{\sqrt{B}}$.

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Example

Prove that $\lim_{x \rightarrow 0} \frac{1}{x^2} = \infty$.

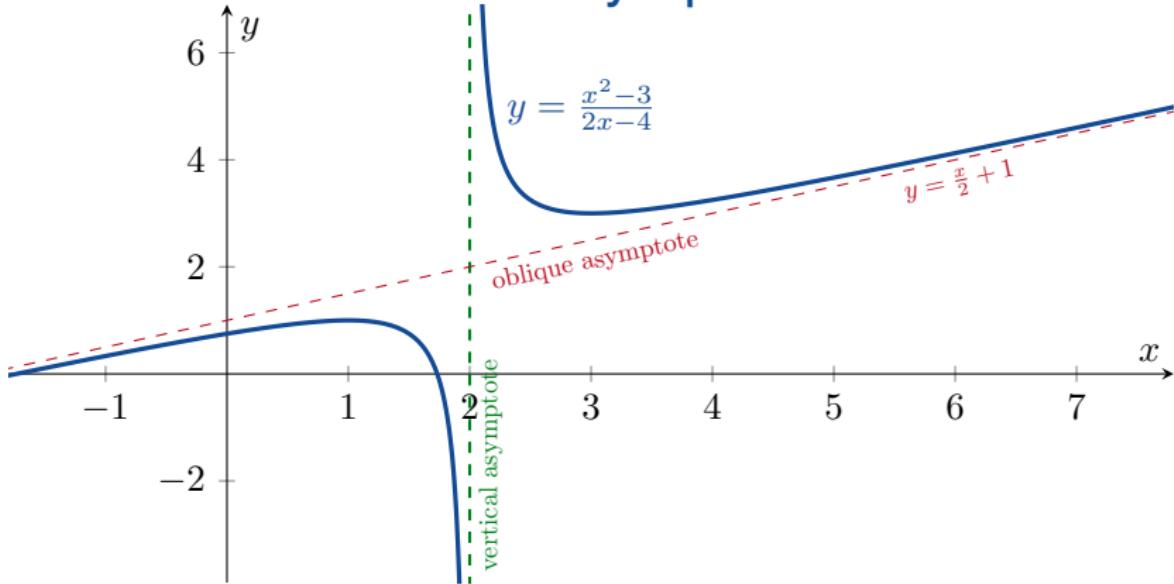
Let $B > 0$. Choose $\delta = \frac{1}{\sqrt{B}}$. Then

$$0 < |x - 0| < \delta \implies x < \delta \implies \frac{1}{x^2} > \frac{1}{\delta^2} = B.$$

2.6 Limits Involving Infinity; Asymptotes of Graphs



Vertical Asymptotes



Sometimes the graph of a function approaches the vertical line $x = a$ as $x \rightarrow a$ or $x \rightarrow a^+$ or $x \rightarrow a^-$. This is called an *vertical asymptote*.

2.6 Limits Involving Infinity; Asymptotes of Graphs



Definition

A line $x = a$ is a *vertical asymptote* of $y = f(x)$ if either

$$\lim_{x \rightarrow a^+} f(x) = \pm\infty \quad \text{or} \quad \lim_{x \rightarrow a^-} f(x) = \pm\infty.$$

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Example

Find the horizontal and vertical asymptotes of $y = \frac{x+3}{x+2}$.

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Example

Find the horizontal and vertical asymptotes of $y = \frac{x+3}{x+2}$.

This function is defined if $x \neq -2$. So we are interested in four limits: $x \rightarrow \infty$, $x \rightarrow -\infty$, $x \rightarrow -2^+$ and $x \rightarrow -2^-$.

2.6 Limits Involving Infinity; Asymptotes of Graphs



Example

Find the horizontal and vertical asymptotes of $y = \frac{x+3}{x+2}$.

This function is defined if $x \neq -2$. So we are interested in four limits: $x \rightarrow \infty$, $x \rightarrow -\infty$, $x \rightarrow -2^+$ and $x \rightarrow -2^-$.

I leave it for you to check that

$$\lim_{x \rightarrow \infty} \frac{x+3}{x+2} = 1 \quad \lim_{x \rightarrow -2^+} \frac{x+3}{x+2} = \infty$$

$$\lim_{x \rightarrow -\infty} \frac{x+3}{x+2} = 1 \quad \lim_{x \rightarrow -2^-} \frac{x+3}{x+2} = -\infty.$$

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Example

Find the horizontal and vertical asymptotes of $y = \frac{x+3}{x+2}$.

This function is defined if $x \neq -2$. So we are interested in four limits: $x \rightarrow \infty$, $x \rightarrow -\infty$, $x \rightarrow -2^+$ and $x \rightarrow -2^-$.

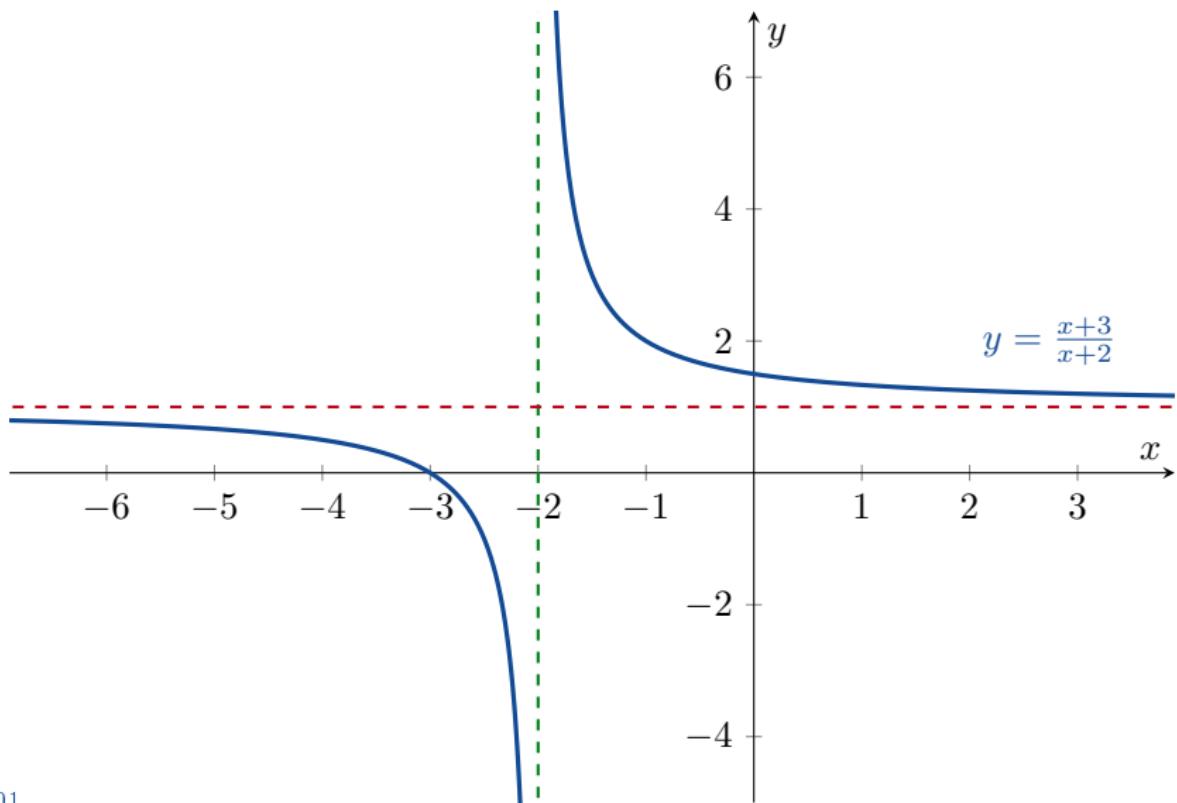
I leave it for you to check that

$$\lim_{x \rightarrow \infty} \frac{x+3}{x+2} = 1 \quad \lim_{x \rightarrow -2^+} \frac{x+3}{x+2} = \infty$$

$$\lim_{x \rightarrow -\infty} \frac{x+3}{x+2} = 1 \quad \lim_{x \rightarrow -2^-} \frac{x+3}{x+2} = -\infty.$$

Therefore $y = 1$ is a horizontal asymptote and $x = -2$ is a vertical asymptote.

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2.6 Limits Involving Infinity; Asymptotes of Graphs



Example

Find the horizontal and vertical asymptotes of $y = -\frac{8}{x^2 - 4}$.

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Example

Find the horizontal and vertical asymptotes of $y = -\frac{8}{x^2 - 4}$.

I leave it for you to check that

$$\lim_{x \rightarrow -2^+} -\frac{8}{x^2 - 4} = \infty$$

$$\lim_{x \rightarrow \infty} -\frac{8}{x^2 - 4} = 0$$

$$\lim_{x \rightarrow -2^-} -\frac{8}{x^2 - 4} = -\infty$$

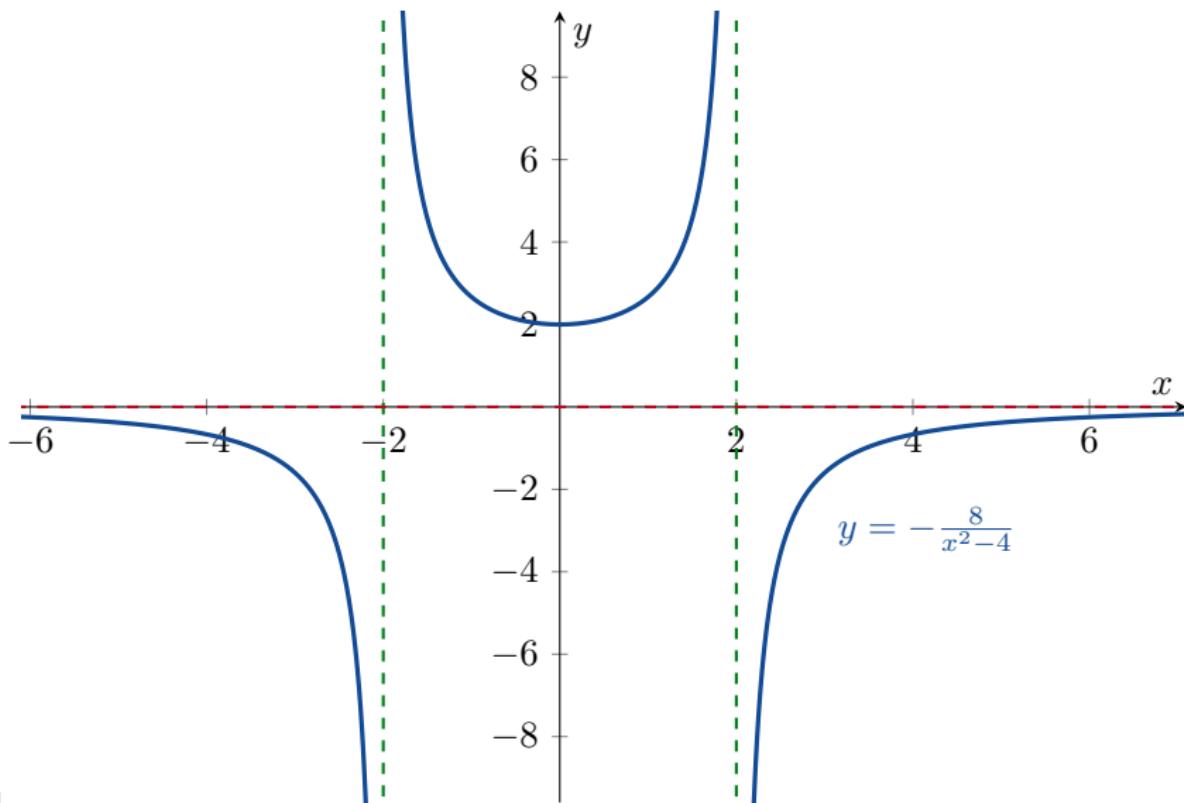
$$\lim_{x \rightarrow -\infty} -\frac{8}{x^2 - 4} = 0$$

$$\lim_{x \rightarrow 2^+} -\frac{8}{x^2 - 4} = -\infty$$

$$\lim_{x \rightarrow 2^-} -\frac{8}{x^2 - 4} = \infty.$$

Therefore $y = 0$ is a horizontal asymptote. $x = -2$ and $x = 2$ are vertical asymptotes.

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2.6 Limits Involving Infinity; Asymptotes of Graphs



Please read Example 17 and page 111 in your textbook.



Next Time

- 3.1 Tangents and the Derivative at a Point
- 3.2 The Derivative as a Function
- 3.3 Differentiation Rules
- 3.4 The Derivative as a Rate of Change