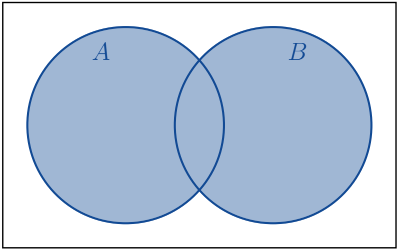
Week 7

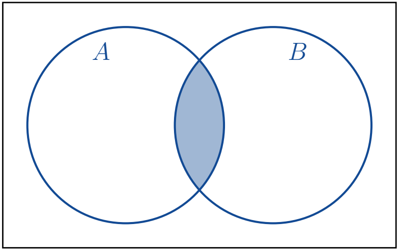
- 20. Concepts of Probability
- 21. Conditional Probability
- 22. Probability Trees

Concepts of Probability

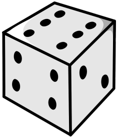
Union and Intersection



$A \cup B$



$A \cap B$



Example (One die)

The sample space for rolling a single die is

$$S = \{1, 2, 3, 4, 5, 6\}.$$

Assume that each of these simple events are equally likely.

1. What is the probability of rolling a number which is even and greater than 3?
2. What is the probability of rolling a number which is even or greater than 3?

$$S = \{1, 2, 3, 4, 5, 6\}$$

solution: Let

$$A = \text{even numbers} = \{2, 4, 6\}$$

and

$$B = \text{numbers greater than } 3 = \{4, 5, 6\}.$$

1. **And:** Since $A \cap B = \{4, 6\}$, we have that

$$P(A \cap B) = \frac{2}{6} = \frac{1}{3}.$$

2. **Or:** Since $A \cup B = \{2, 4, 5, 6\}$, we have that

$$P(A \cup B) = \frac{4}{6} = \frac{2}{3}.$$

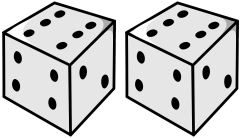
Remark

Please recall the Addition Principle which stated that

$$n(A \cup B) = n(A) + n(B) - n(A \cap B).$$

Theorem

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$



Example (Two dice)

You roll two dice. What is the probability that:

1. the sum is either 5 or 10?
2. either the sum is greater than 9, or both dice show the same number?

solution:

1. **5 or 10:** Let

$$\begin{aligned} A &= \text{the sum is } 5 \\ &= \{(1,4),(2,3),(3,2),(4,1)\} \end{aligned}$$

and

$$\begin{aligned} B &= \text{the sum is } 10 \\ &= \{(4,6),(5,5),(6,4)\}. \end{aligned}$$

Since $A \cap B = \varnothing$, we have that

$$P(A \cup B) = P(A) + P(B) = \frac{4}{36} + \frac{3}{36} = \frac{7}{36}.$$

2 **> 9 or same number:** Let

$$
\begin{aligned}
C &= \text{the sum is greater than 9} \\
&= \{(4,6),(5,5),(5,6),(6,4),(6,5),(6,6)\}
\end{aligned}
$$

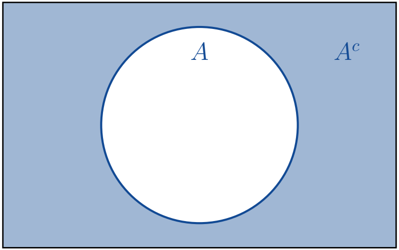
and

$$
\begin{aligned}
D &= \text{both dice show the same number} \\
&= \{(1,1),(2,2),(3,3),(4,4),(5,5),(6,6)\}.
\end{aligned}
$$

Note that $C \cap D = \{(5,5),(6,6)\}$. Therefore

$$
\begin{aligned}
P(C \cup D) &= P(C) + P(D) - P(C \cap D) \\
&= \frac{6}{36} + \frac{6}{36} - \frac{2}{36} = \frac{10}{36} = \frac{5}{18}.
\end{aligned}
$$

Complements



A^c

Theorem

$$P(E) = 1 - P(E^c).$$

Sometimes it is easier to calculate $1 - P(E^c)$, than to calculate $P(E)$ directly.

20. Concepts of Probability

Example (Whiteboard Markers)

A box containing 45 whiteboard markers is delivered to Istanbul Okan University. Nine of the markers are red. The remaining markers are black.

Your teacher is given 10 markers at random. He will be happy if one or more of his markers is red.

What is the probability that your teacher will be happy?

solution: Let

$$E = \text{one or more of the markers is red.}$$

Then

$$E^c = \text{all 10 markers are black.}$$

Since

$$P(E^c) = \frac{n(E^c)}{n(S)} = \frac{{}_{36}C_{10}}{{}_{45}C_{10}},$$

we have that

$$P(E) = 1 - P(E^c) = 1 - \frac{{}_{36}C_{10}}{{}_{45}C_{10}} \approx 0.92.$$

Example

In a class of 30 students, what is the probability that at least two students have the same birthday? (Same month and day. Ignore 29 February)

solution: We assume that there are 365 days in a year and that each day is equally likely. We have

$$n(S) = 365^{30}$$

by the Multiplication Principle. Let

$$E = 2 \text{ or more people have the same birthday.}$$

Then

$$E^c = \text{all 30 students have different birthdays.}$$

We calculate that

$$n(E^c) = 365 \cdot 364 \cdot 363 \cdot \ldots \cdot 336 = \frac{365!}{335!} = {}_{365}P_{30}\,,$$

$$P(E^c) = \frac{n(E^c)}{n(S)} = \frac{365!}{335! \cdot 365^{30}}$$

and

$$P(E) = 1 - P(E^c) = 1 - \frac{365!}{335! \cdot 365^{30}} \approx 0.706$$

2.1 Conditional Probability

Sometimes the probability of an event will depend on another event. For example, suppose that

$$A = \text{Ali has cancer}$$

and

$$B = \text{Ali is a smoker.}$$

Clearly the probability that A occurs depends on B.

21. Conditional Probability

Definition

The *conditional probability* of A given B is

$$P(A|B) = \begin{pmatrix} \text{the probability of A, if we} \\ \text{already know that B occurs} \end{pmatrix}.$$

21. Conditional Probability

Theorem

$$P(A|B) = \frac{P(A \cap B)}{P(B)}.$$

Example (Marbles)

A bag contains red and blue marbles. Two marbles are drawn without replacement.

The probability of selecting a red marble and then a blue marble is 0.28. The probability of selecting a red marble on the first draw is 0.5.

What is the probability of selecting a blue marble on the second draw, given that the first marble drawn was red?

solution: Let

$$R = \text{the first marble is red}$$

and

$$B = \text{the second marble is blue.}$$

The required probability is

$$P(B|R) = \frac{P(B \cap R)}{P(R)} = \frac{0.28}{0.5} = 0.56$$

Example (One die)

Your friend says that when she rolled a die, she rolled an odd number. What is the probability that your friend rolled a 3?

solution: Let

$$A = \text{your friend rolled a } 3$$

and

$$B = \text{your friend rolled an odd number.}$$

Then $P(A) = \frac{1}{6}$, $P(A \cap B) = P(A) = \frac{1}{6}$ and $P(B) = \frac{1}{2}$. Hence

$$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{\frac{1}{6}}{\frac{1}{2}} = \frac{1}{3}.$$

Remark

Given that $P(A|B) = \frac{P(A \cap B)}{P(B)}$, we have that

$$P(A \cap B) = P(B)P(A|B).$$

Similarly,

$$P(A \cap B) = P(B \cap A) = P(A)P(B|A).$$

Therefore:

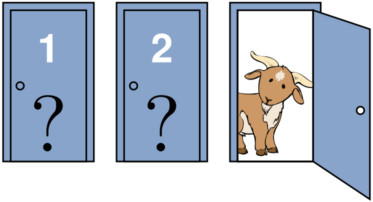
Theorem (Product Rule)

$$P(A \cap B) = P(A)P(B|A) = P(B)P(A|B)$$

and

$$P(A|B) = \frac{P(A)P(B|A)}{P(B)}.$$

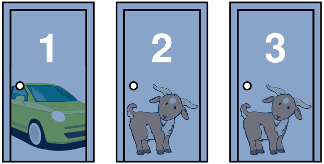
The Monty Hall Problem

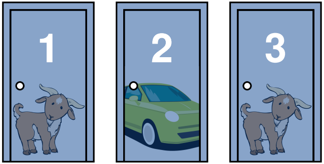


Suppose you're on a TV game show, and you're given the choice of three doors: Behind one door is a car; behind the others, goats. You pick a door, say No. 1, and the host, who knows what's behind the doors, opens another door, say No. 3, which has a goat. He then says to you, "Do you want to pick door No. 2?" Is it to your advantage to switch your choice?

Monty Hall
USA, 1921-2017.

The *Monty Hall Problem* is one that confuses a lot of people that haven't studied Probability.


1
2
3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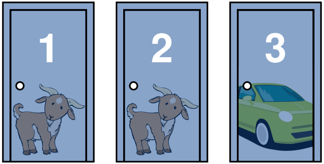
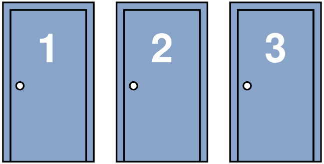
2
3


1
2
3



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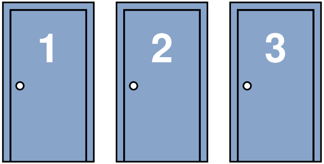

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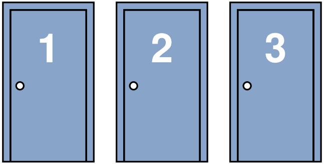

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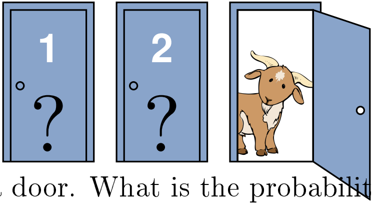
CORRECT



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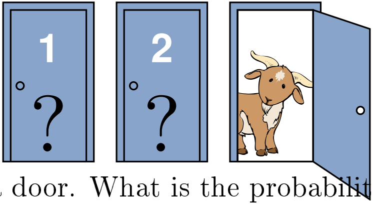
CORRECT

2. Then the host opens another door and shows you a goat. Now there are two closed doors: Behind one is a car and behind the other is a goat.



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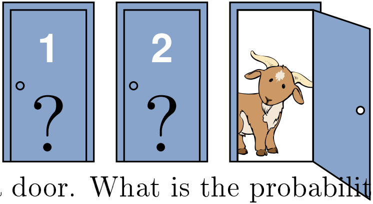
3. What is the probability that the car is behind the door that you chose?



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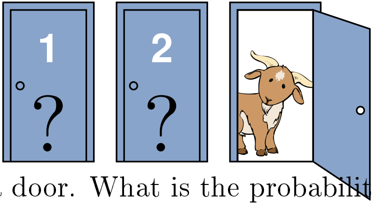
CORRECT

2. Then the host opens another door and shows you a goat. Now there are two closed doors: Behind one is a car and behind the other is a goat.
3. What is the probability that the car is behind the door that you chose? Two closed doors. One Car. So clearly now $P(\text{car}) = \frac{1}{2}$ right?



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WRONG!!!

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To explain, let us look at all the possible outcomes if you choose door number 1 first.

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behind door 1	behind door 2	behind door 3	outcome if you don't switch	outcome if you switch
car	goat	goat		

What? Why?
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behind door 1	behind door 2	behind door 3	outcome if you don't switch	outcome if you switch
car	goat	goat	win	lose

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car	goat	goat	win	lose
goat	car	goat	lose	win

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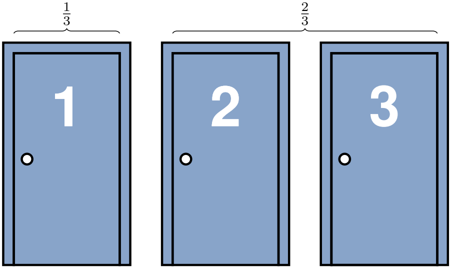
behind door 1	behind door 2	behind door 3	outcome if you don't switch	outcome if you switch
car	goat	goat	win	lose
goat	car	goat	lose	win
goat	goat	car		

What? Why?
To explain, let us look at all the possible outcomes if you choose door number 1 first.

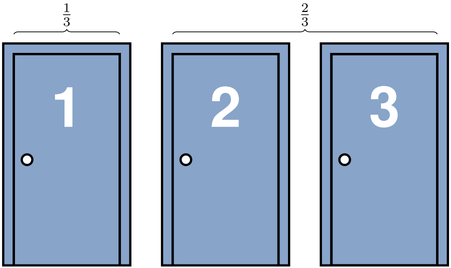
behind door 1	behind door 2	behind door 3	outcome if you don't switch	outcome if you switch
car	goat	goat	win	lose
goat	car	goat	lose	win
goat	goat	car	lose	win

We can see from the table that if you don't switch your choice, then you have a $\frac{1}{3}$ chance of winning the car, but if you do switch then you have $\frac{2}{3}$ chance of winning it. These are the probabilities if you choose door number 1, but of course we would get the same results if you choose door 2 or 3 first.

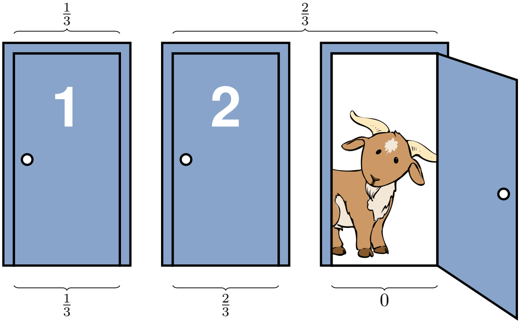
Another way



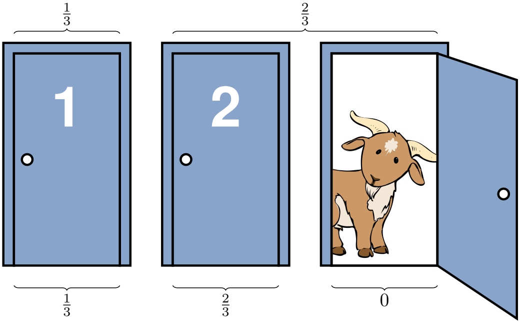
Let's think of this another way. You initially choose door number 1. At the start, the chance of the car being behind door one is $\frac{1}{3}$ and the chance of the car being behind doors 2 or 3 is $\frac{2}{3}$.



Let's imagine that the host doesn't open a door, he just says you can change your choice for *both* of the other two doors. Would you switch then?

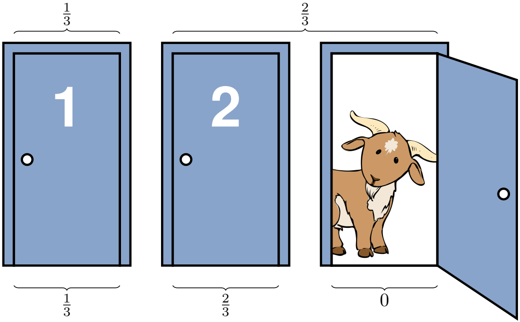


We know that atleast one of doors 2 and 3 hides a goat.



We know that atleast one of doors 2 and 3 hides a goat.

Remember that the host knows where the car is: He doesn't open a door at random, he always opens a door with a goat. So he isn't really giving you any extra information.



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Remember that the host knows where the car is: He doesn't open a door at random, he always opens a door with a goat. So he isn't really giving you any extra information.

The probabilities don't change to $\frac{1}{2}$, $\frac{1}{2}$, they are still $\frac{1}{3}$, $\frac{2}{3}$.

Using Conditional Probabilities

Let

$$\begin{aligned} C &= \text{door number 1 has a car behind it} \\ C^c &= \text{door number 1 does not have a car behind it} \\ &= \text{door number 1 has a goat behind it} \end{aligned}$$

and

$$E = \text{the host has opened a door with a goat behind it.}$$

Using Conditional Probabilities

Let

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and

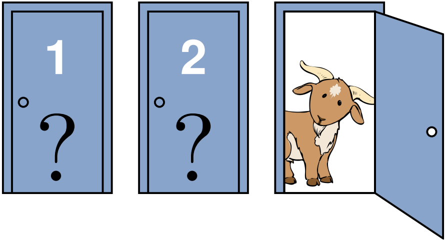
$$E = \text{the host has opened a door with a goat behind it.}$$

We have that $P(C) = \frac{1}{3}$ and $P(C^c) = 1 - P(C) = \frac{2}{3}$. Moreover, $P(E|C) = 1$ and $P(E|C^c) = 1$ because the host always opens a door with a goat.

Then we can calculate that

$$\begin{aligned} P(C|E) &= \frac{P(C)P(E|C)}{P(E)} \\ &= \frac{P(C)P(E|C)}{P(E\cap C) + P(E \cap C^c)} \\ &= \frac{P(C)P(E|C)}{P(C)P(E|C) + P(C^c)P(E|C^c)} \\ &= \frac{\frac{1}{3}\cdot 1}{\frac{1}{3}\cdot 1 + \frac{2}{3}\cdot 1} = \frac{1}{3}. \end{aligned}$$

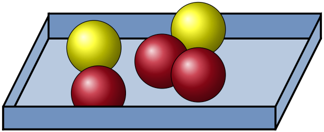
This means that it doesn't matter if the host opens a door or not, the probability that the car is behind door number 1 is always $\frac{1}{3}$.



Conclusion

You should always switch door if you want to win the car.

Probability Trees

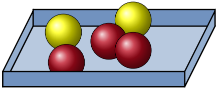


Example (5 balls in a box)

A box contains 3 red and 2 yellow balls. Two balls are randomly drawn without replacement. What is the probability that the second ball is yellow?

solution: First we draw a *probability tree*.

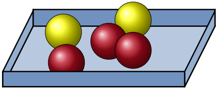
first ball **second ball**

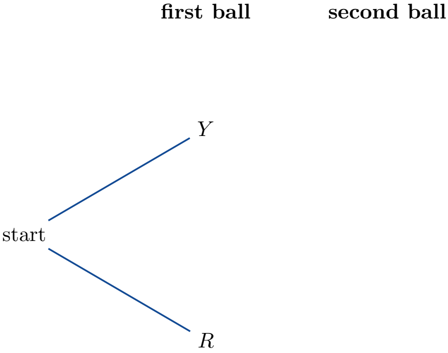


start

22. Probability Trees

solution: First we draw a *probability tree*.





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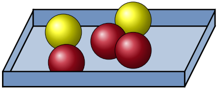

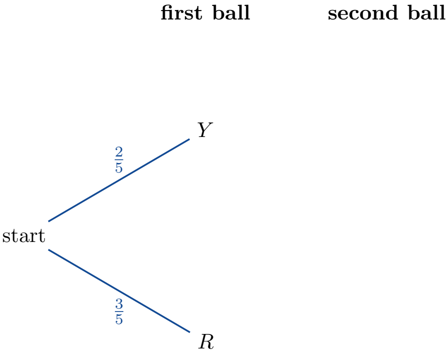

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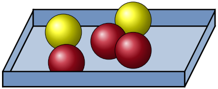

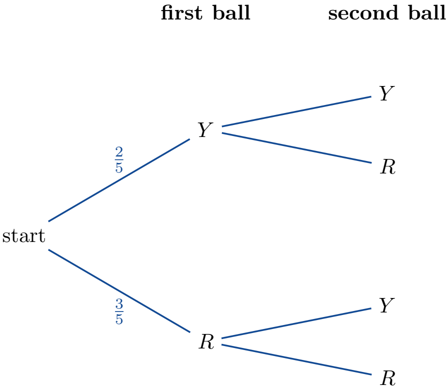

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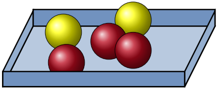

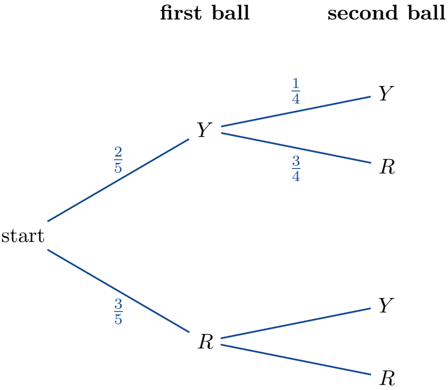

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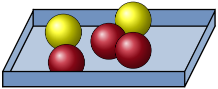

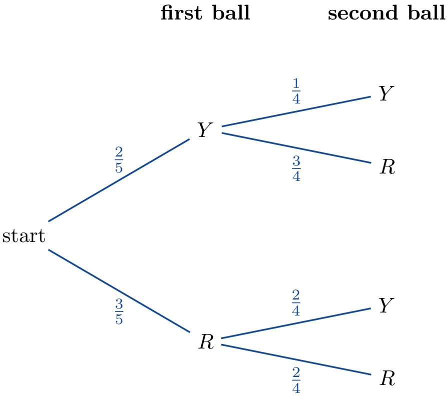

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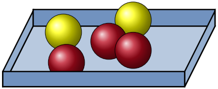

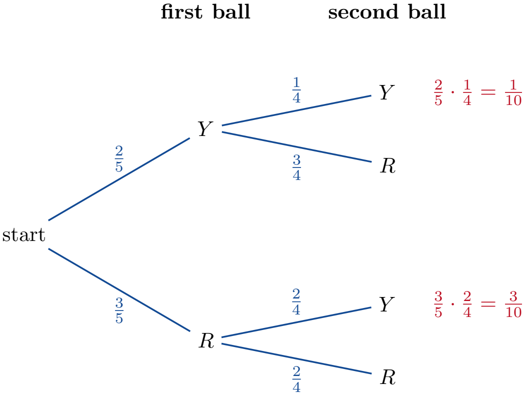

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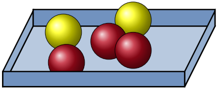
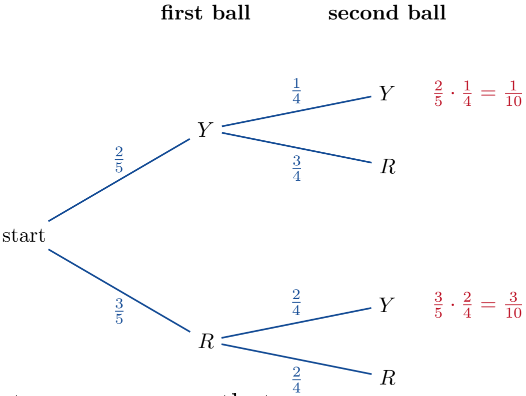
solution: First we draw a *probability tree*.





From this probability tree, we can see that

$$P(YY) + P(RY) = \frac{1}{10} + \frac{3}{10} = \frac{4}{10} = \frac{2}{5}.$$

The probability that the second ball is yellow is $\frac{2}{5}$.

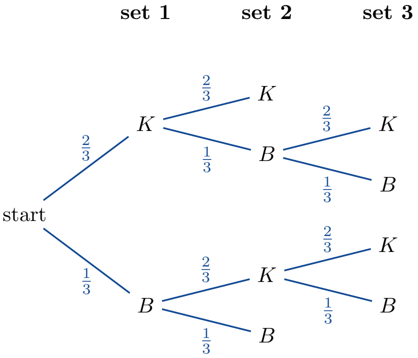
Example (Tennis)

Jeremy and Keir are playing tennis. The first player to win 2 sets wins the match. In each set, the probability that Keir wins that set is $\frac{2}{3}$. Find the probability that:

1. Boris wins the match.
2. 3 sets are played.
3. The player who wins the first set, wins the match.

22. Probability Trees

solution:



$\frac{2}{3} \cdot \frac{2}{3} = \frac{4}{9}$

$\frac{2}{3} \cdot \frac{1}{3} \cdot \frac{2}{3} = \frac{4}{27}$

$\frac{2}{3} \cdot \frac{1}{3} \cdot \frac{1}{3} = \frac{2}{27}$

$\frac{1}{3} \cdot \frac{2}{3} \cdot \frac{2}{3} = \frac{4}{27}$

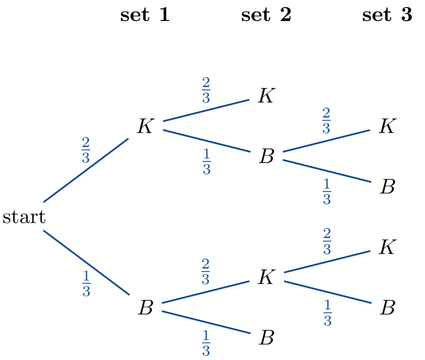
$\frac{1}{3} \cdot \frac{2}{3} \cdot \frac{1}{3} = \frac{2}{27}$

$\frac{1}{3} \cdot \frac{1}{3} = \frac{1}{9}$

1

22. Probability Trees

solution:



$\frac{2}{3}\cdot\frac{2}{3}=\frac{4}{9}$

$\frac{2}{3}\cdot\frac{1}{3}\cdot\frac{2}{3}=\frac{4}{27}$

$\frac{2}{3}\cdot\frac{1}{3}\cdot\frac{1}{3}=\frac{2}{27}$

$\frac{1}{3}\cdot\frac{2}{3}\cdot\frac{2}{3}=\frac{4}{27}$

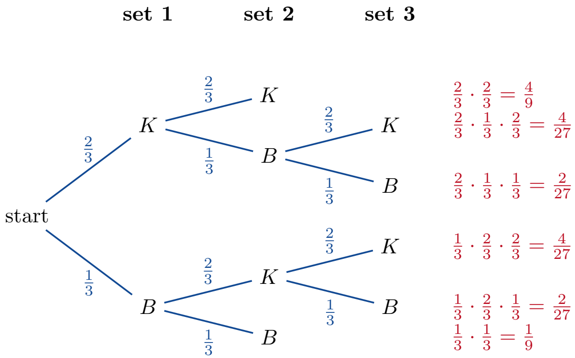
$\frac{1}{3}\cdot\frac{2}{3}\cdot\frac{1}{3}=\frac{2}{27}$

$\frac{1}{3}\cdot\frac{1}{3}=\frac{1}{9}$

1. **Boris wins:** We calculate that

$$P(KBB)+P(BKB)+P(BB)=\frac{2}{27}+\frac{2}{27}+\frac{1}{9}=\frac{7}{27}.$$

solution:

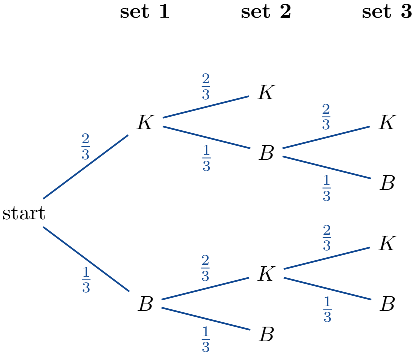


2 **3 sets are played:** Now

$$P(KBK) + P(KBB) + P(BKK) + P(BKB)$$
$$= \frac{4}{27} + \frac{2}{27} + \frac{4}{27} + \frac{2}{27} = \frac{4}{9}.$$

22. Probability Trees

solution:



$\frac{2}{3}\cdot\frac{2}{3}=\frac{4}{9}$

$\frac{2}{3}\cdot\frac{1}{3}\cdot\frac{2}{3}=\frac{4}{27}$

$\frac{2}{3}\cdot\frac{1}{3}\cdot\frac{1}{3}=\frac{2}{27}$

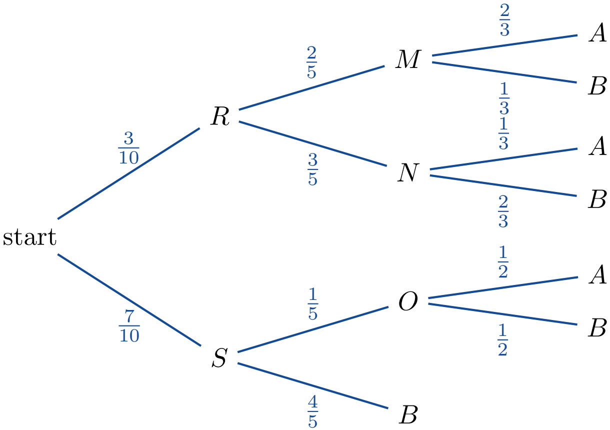
$\frac{1}{3}\cdot\frac{2}{3}\cdot\frac{2}{3}=\frac{4}{27}$

$\frac{1}{3}\cdot\frac{2}{3}\cdot\frac{1}{3}=\frac{2}{27}$

$\frac{1}{3}\cdot\frac{1}{3}=\frac{1}{9}$

3. **Player who wins first set wins match:** Finally

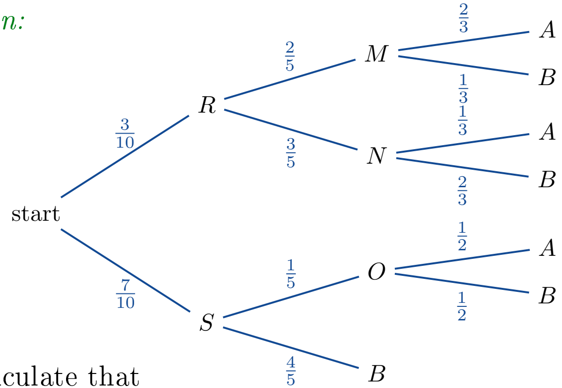
$$P(KK)+P(KBK)+P(BKB)+P(BB)$$
$$=\frac{4}{9}+\frac{4}{27}+\frac{2}{27}+\frac{1}{9}=\frac{7}{9}.$$



Example

Calculate $P(B)$.

solution:



We calculate that

$$
\begin{aligned}
P(B) &= P(RMB) + P(RNB) + P(SOB) + P(SB) \\
&= \left(\frac{3}{10} \cdot \frac{2}{5} \cdot \frac{1}{3}\right) + \left(\frac{3}{10} \cdot \frac{3}{5} \cdot \frac{2}{3}\right) + \left(\frac{7}{10} \cdot \frac{1}{5} \cdot \frac{1}{2}\right) + \left(\frac{7}{10} \cdot \frac{4}{5}\right) \\
&= \frac{6}{150} + \frac{18}{150} + \frac{7}{100} + \frac{28}{50} = \frac{79}{100}.
\end{aligned}
$$

Example

Ron Weasley has a bag with 7 blue sweets and 3 red sweets in it.

He takes a sweet at random from the bag, then puts it back in the bag.

Then he picks a sweet at random from the bag and eats it.

Finally he picks a third sweet at random.

Draw a probability tree to represent this situation.

7 blue sweets and 3 red sweets

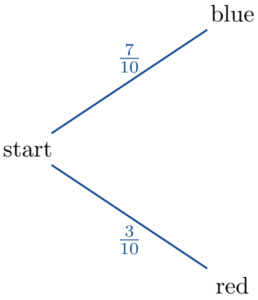
solution: **sweet 1** **sweet 2** **sweet 3**

start

7 blue sweets and 3 red sweets

solution:

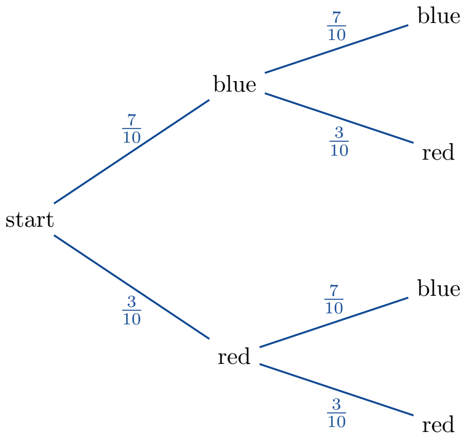
sweet 1 **sweet 2** **sweet 3**



7 blue sweets and 3 red sweets

solution:

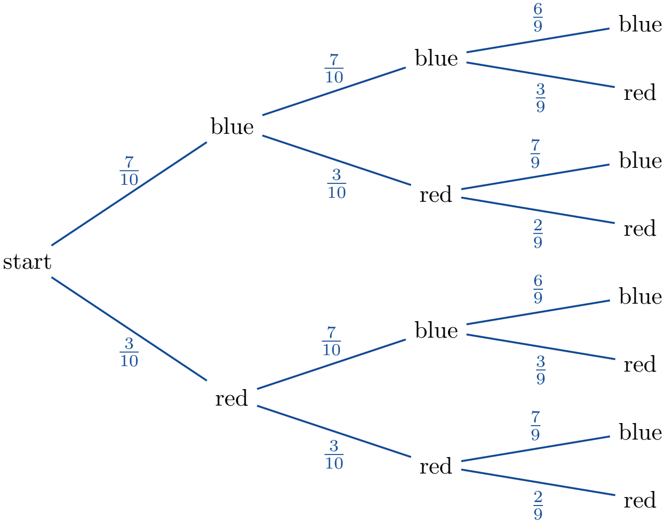
sweet 1 **sweet 2** **sweet 3**



7 blue sweets and 3 red sweets

solution:

sweet 1 **sweet 2** **sweet 3**



Next Week

- 23. Graph Theory