

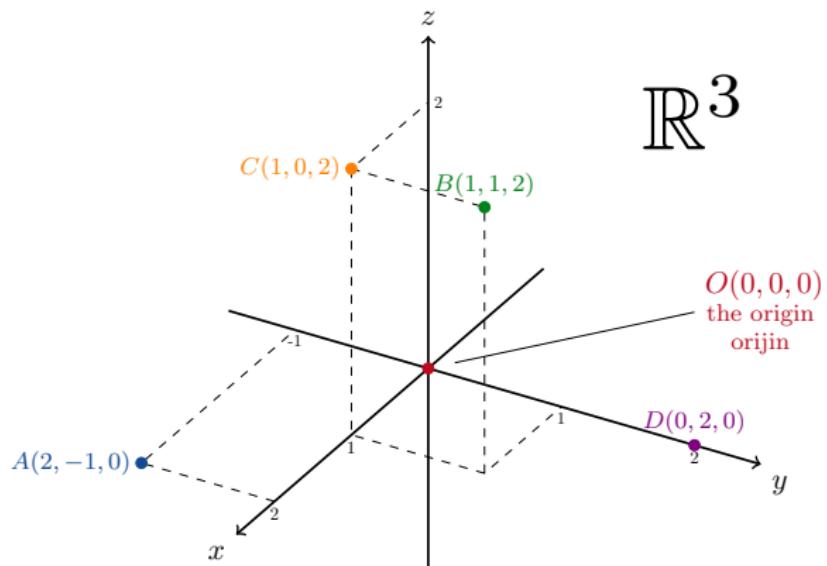
Lecture 3

- 11.1 Three-Dimensional Coordinate Systems
- 11.2 Vectors
- 11.3 The Dot Product

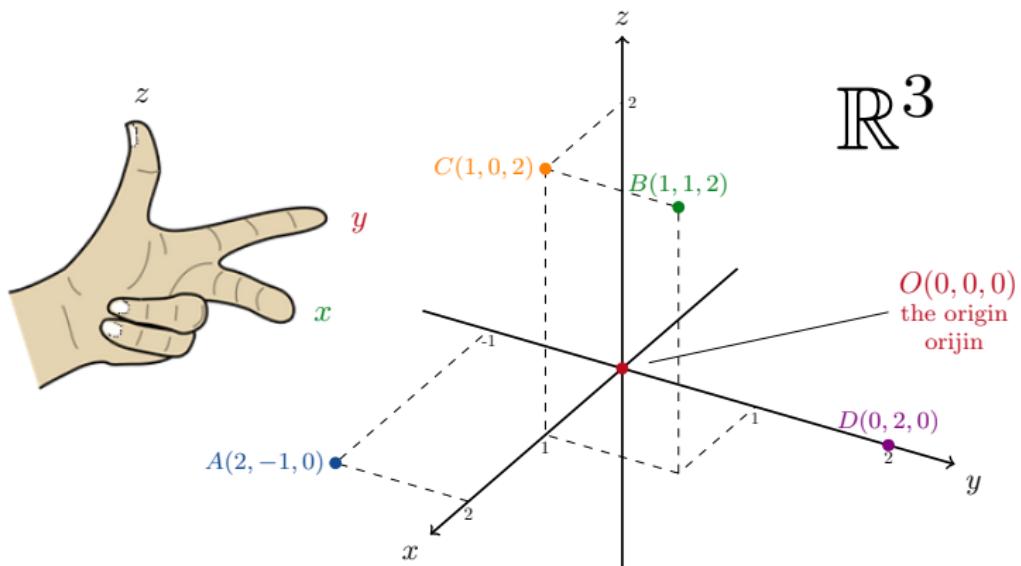


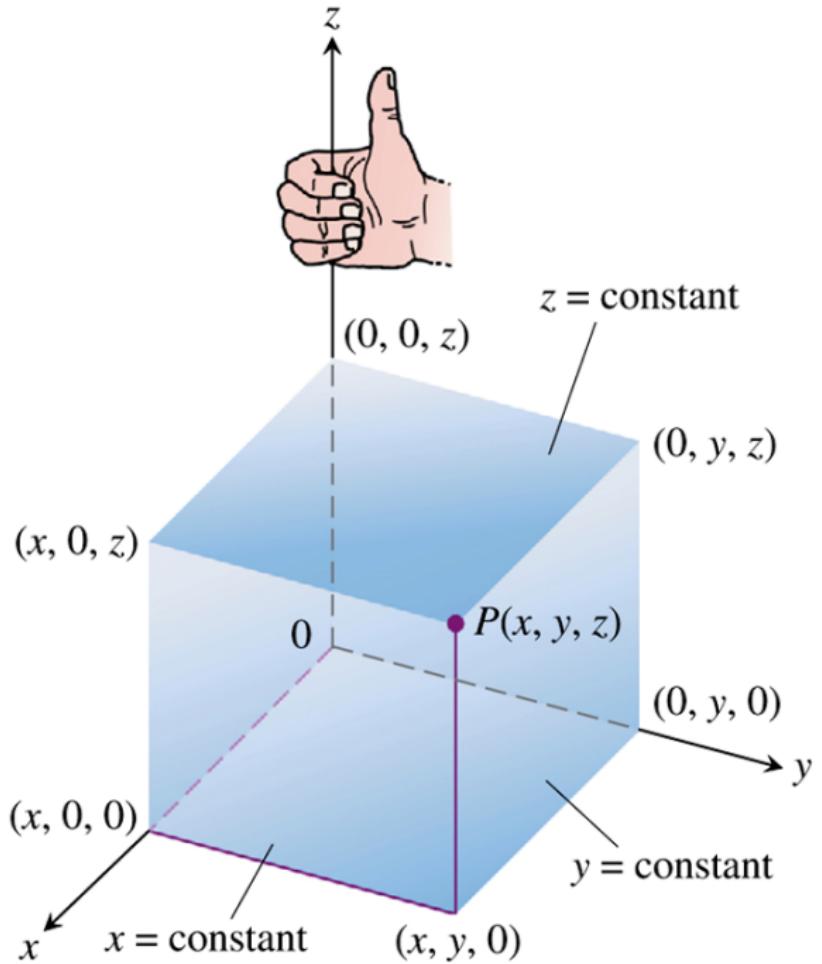
Three-Dimensional Coordinate Systems

11.1 Three-Dimensional Coordinate Systems

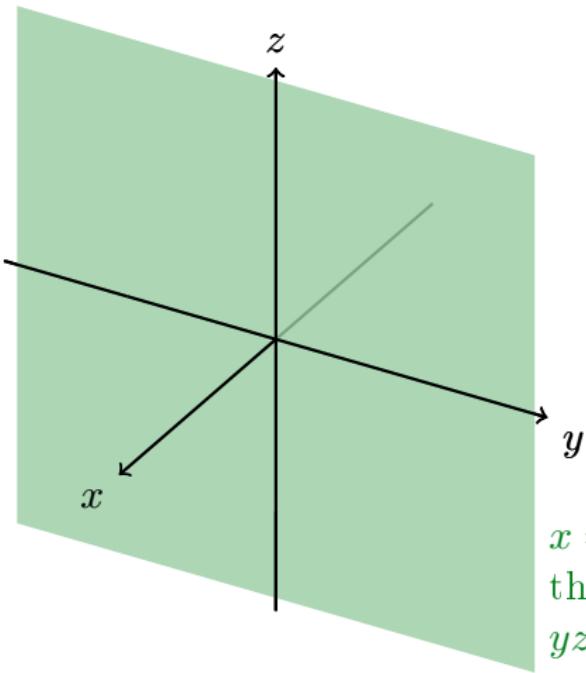


11.1 Three-Dimensional Coordinate Systems



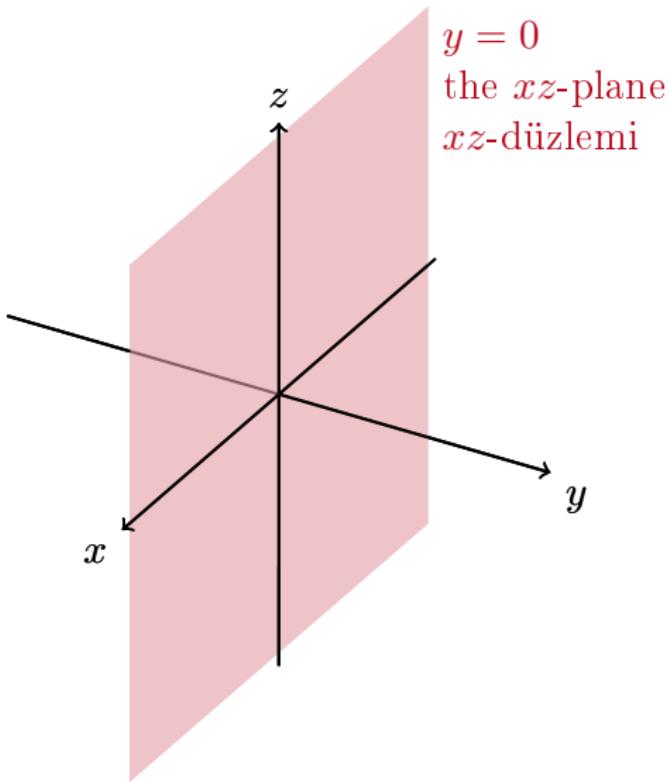


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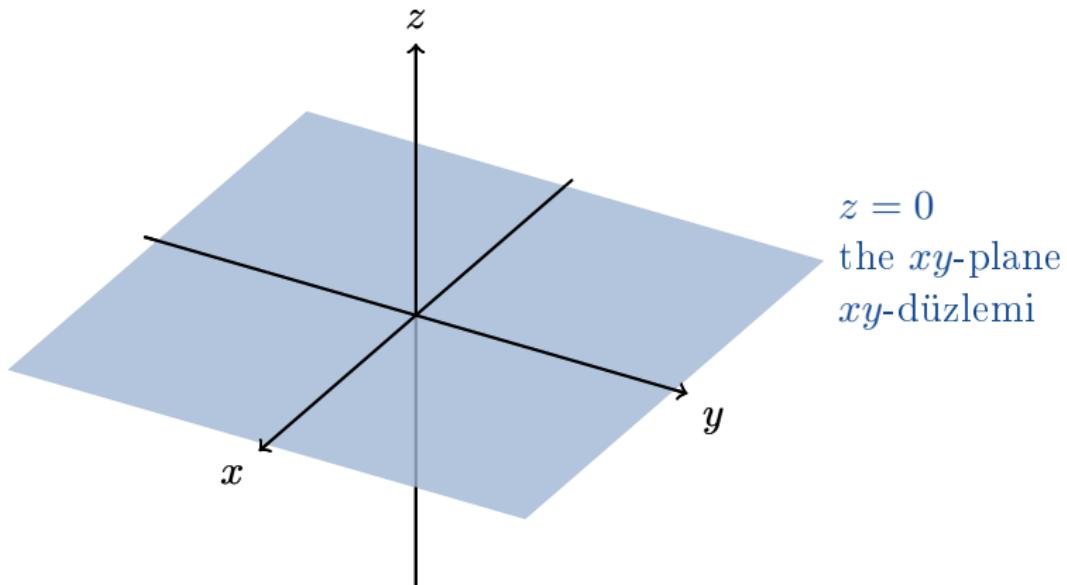


$x = 0$
the *yz*-plane
yz-düzlemi

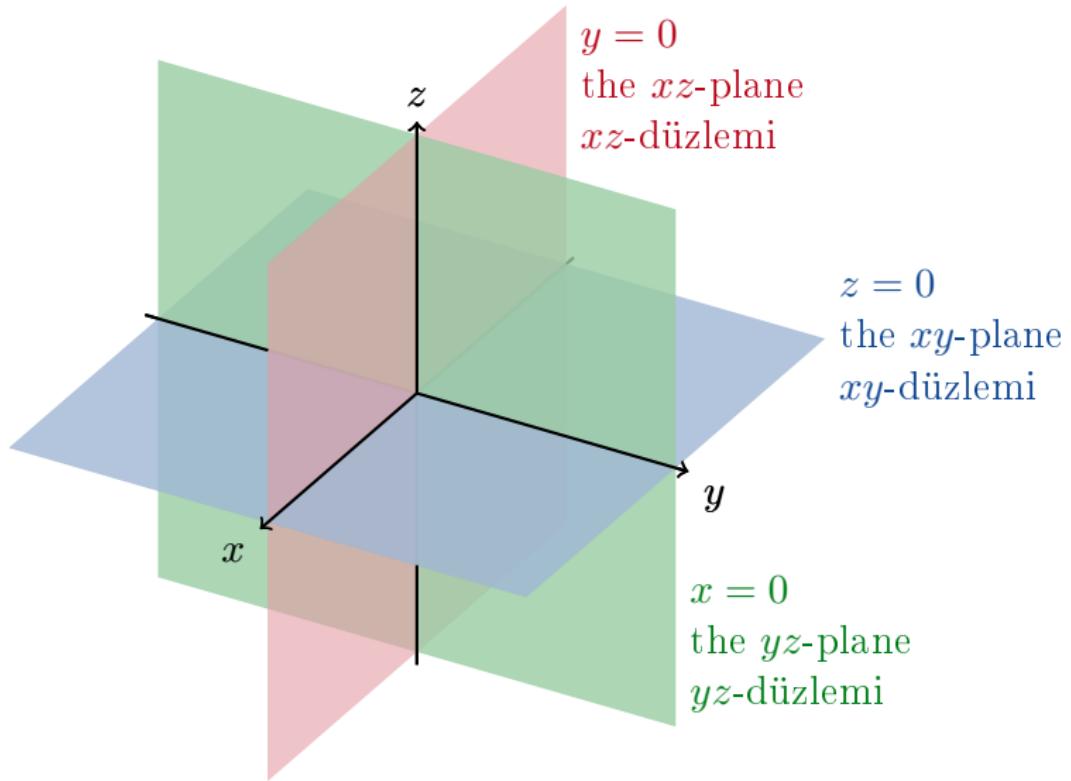
11.1 Three-Dimensional Coordinate Systems



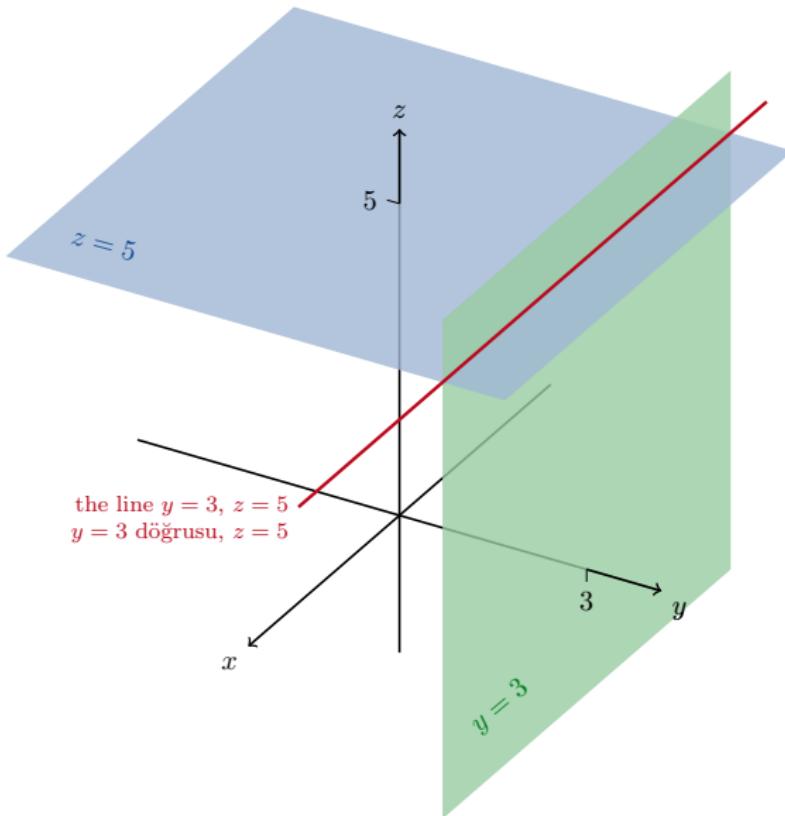
11.1 Three-Dimensional Coordinate Systems



11.1 Three-Dimensional Coordinate Systems



11.1 Three-Dimensional Coordinate Systems



EXAMPLE 1

We interpret these equations and inequalities geometrically.

(a) $z \geq 0$

The half-space consisting of the points on and above the xy -plane.

(b) $x = -3$

The plane perpendicular to the x -axis at $x = -3$. This plane lies parallel to the yz -plane and 3 units behind it.

(c) $z = 0, x \leq 0, y \geq 0$

The second quadrant of the xy -plane.

(d) $x \geq 0, y \geq 0, z \geq 0$

The first octant.

(e) $-1 \leq y \leq 1$

The slab between the planes $y = -1$ and $y = 1$ (planes included).

(f) $y = -2, z = 2$

The line in which the planes $y = -2$ and $z = 2$ intersect. Alternatively, the line through the point $(0, -2, 2)$ parallel to the x -axis. ■

11.1 Three-Dimensional Coordinate Systems



Example

Which points $P(x, y, z)$ satisfy $x^2 + y^2 = 4$ and $z = 3$?

We know that $z = 3$ is a horizontal plane and we recognise that $x^2 + y^2 = 4$ is the equation of a circle of radius 2.

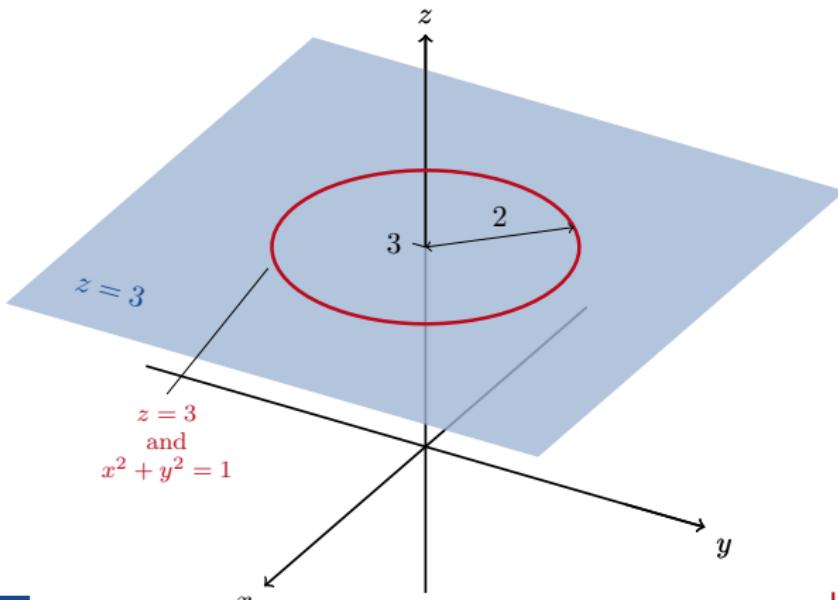
11.1 Three-Dimensional Coordinate Systems



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11.1 Three-Dimensional Coordinate Systems



Distance in \mathbb{R}^3

Definition

The set

$$\{(x, y, z) \mid x, y, z \in \mathbb{R}\}$$

is denoted by \mathbb{R}^3 .

11.1 Three-Dimensional Coordinate Systems



Distance in \mathbb{R}^3

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Definition

The *distance* between $P_1(x_1, y_1, z_1)$ and $P_2(x_2, y_2, z_2)$ is

$$\|P_1P_2\| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}.$$

11.1 Three-Dimensional Space

$$\|P_1P_2\| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}.$$



Example

The distance between $A(2, 1, 5)$ and $B(-2, 3, 0)$ is

11.1 Three-Dimensional Space

$$\|P_1P_2\| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}.$$

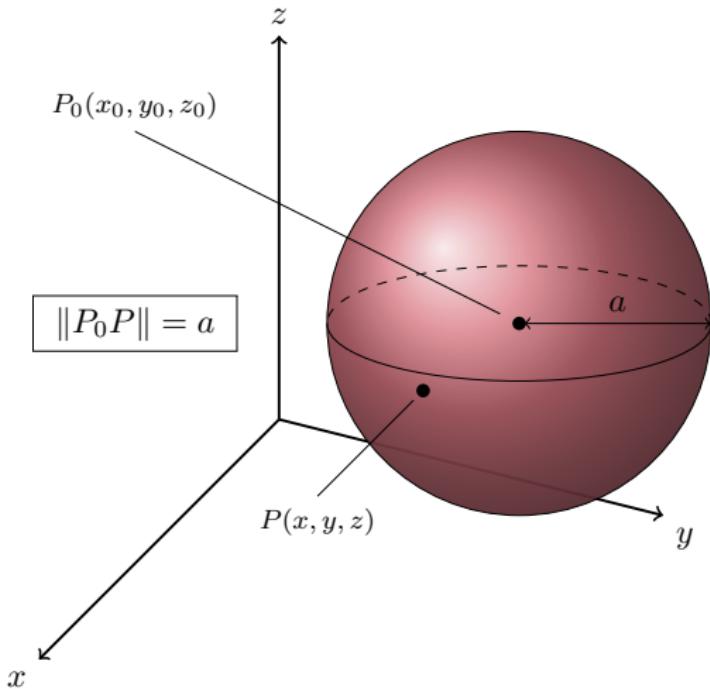


Example

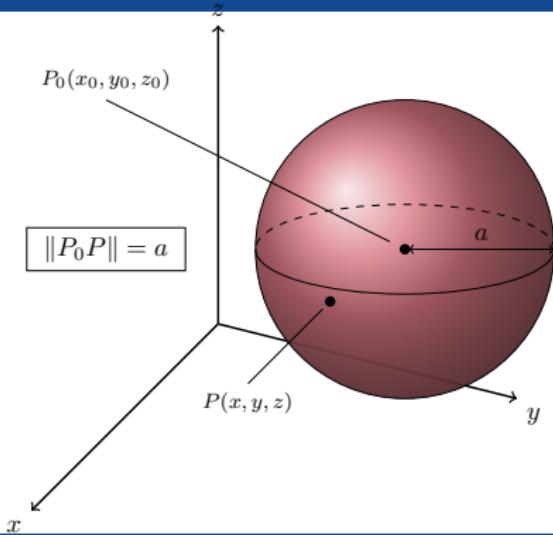
The distance between $A(2, 1, 5)$ and $B(-2, 3, 0)$ is

$$\begin{aligned}\|AB\| &= \sqrt{((-2) - 2)^2 + (3 - 1)^2 + (0 - 5)^2} \\ &= \sqrt{16 + 4 + 25} = \sqrt{45} \\ &= 3\sqrt{5} \approx 6.7.\end{aligned}$$

Spheres



11.1 Three-Dimensional Coordinate Systems



Definition

The *standard equation for a sphere* of radius a centred at $P_0(x_0, y_0, z_0)$ is

$$(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2 = a^2.$$

11.1 Three-Dimensional

$$(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2 = a^2.$$



Example

Find the centre and radius of the sphere

$$x^2 + y^2 + z^2 + 3x - 4z + 1 = 0.$$

11.1 Three-Dimensional

$$(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2 = a^2.$$



Example

Find the centre and radius of the sphere

$$x^2 + y^2 + z^2 + 3x - 4z + 1 = 0.$$

First we need to complete the squares to put this equation into the standard form.



Completing the square

Let's suppose that we want to solve

$$x^2 + 26x = 27.$$



Completing the square

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$$x^2 + 26x = 27.$$

We can think of this in terms of areas of rectangles.

11.1 Three-Dimensional Coordinate Systems



$$\begin{array}{c} \text{+} \\ \text{---} \\ \text{---} \end{array} \quad \begin{array}{l} \text{---} \\ \text{---} \\ \text{---} \end{array}$$

A diagram illustrating a mathematical equation. On the left, there is a blue square with its side length labeled x . To its right is a plus sign. To the right of the plus sign is a green rectangle. A red double-headed arrow below the rectangle indicates its width is 26. To the right of the rectangle is an equals sign followed by the value 27. Red double-headed arrows above and to the right of the rectangle indicate its height is also labeled x .

11.1 Three-Dimensional Coordinate Systems



$$x^2 + 26x$$

The diagram illustrates the equation $x^2 + 26x$ using geometric shapes. A blue square is labeled with side length x and area x^2 . A green rectangle is labeled with width x and length 26 , representing the term $26x$. The total value of the expression is indicated as $= 27$.

11.1 Three-Dimensional Coordinate Systems



$$x^2 + 26x$$

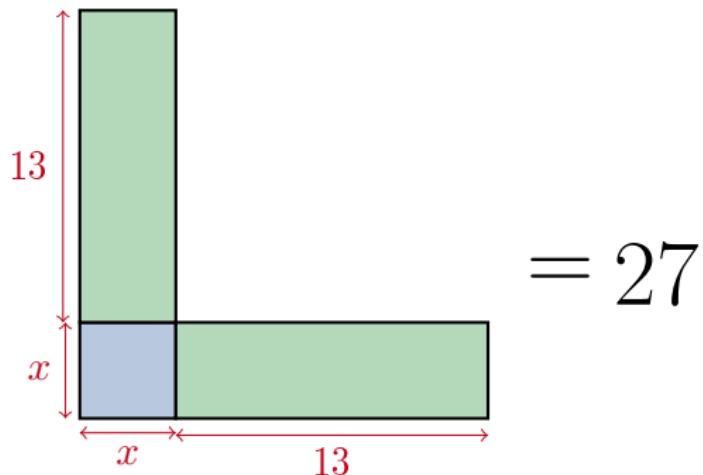
The diagram illustrates the equation $x^2 + 26x$. It consists of two parts separated by a plus sign. The first part is a blue square with side length x , labeled with a double-headed red arrow below it. The second part is a green rectangle with a horizontal width of 26, labeled with a double-headed red arrow below it, and a vertical height of x , labeled with a double-headed red arrow to its right. A vertical dashed line passes through the center of the green rectangle. Below the green rectangle, there is a coordinate system icon with two intersecting lines and arrows at their ends.

11.1 Three-Dimensional Coordinate Systems

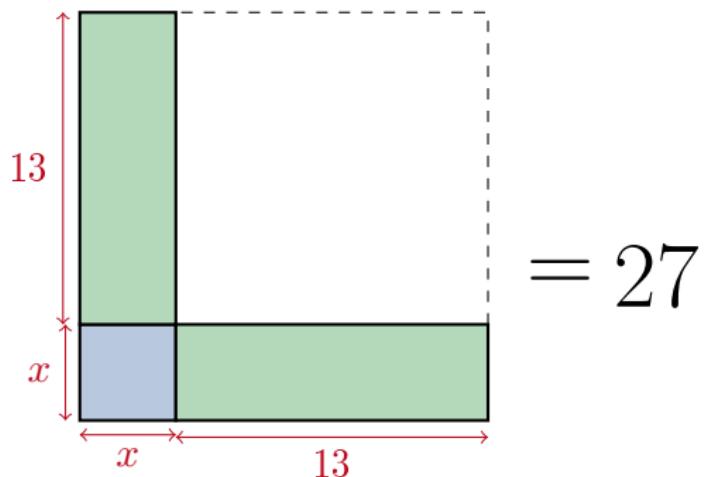


$$\begin{array}{c} \text{blue square: } x \times x \\ \text{green rectangle: } x \times 13 \\ \text{green rectangle: } 13 \times 13 = 27 \end{array}$$

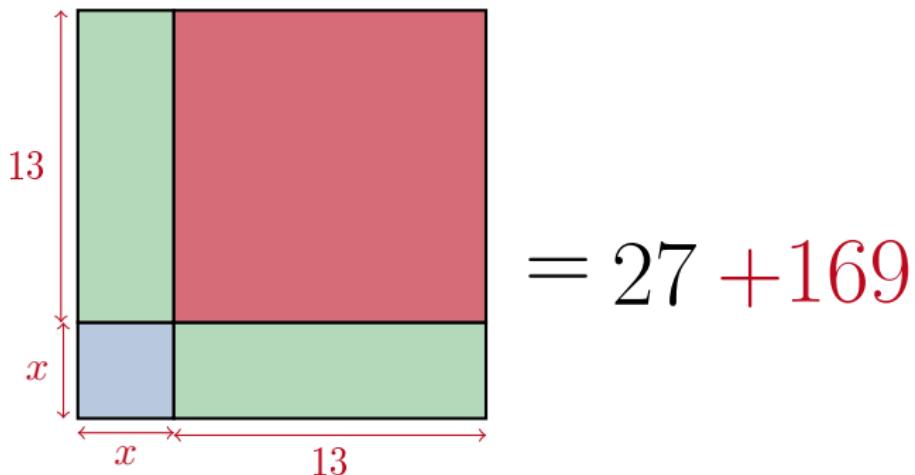
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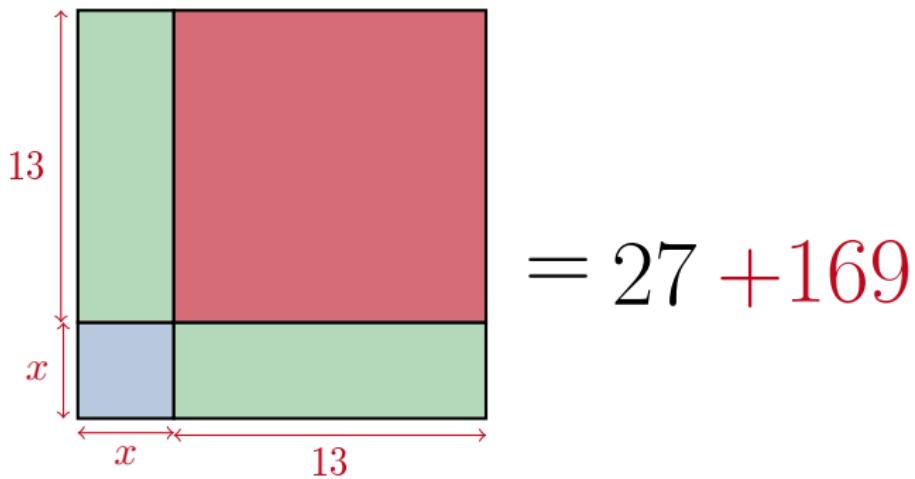
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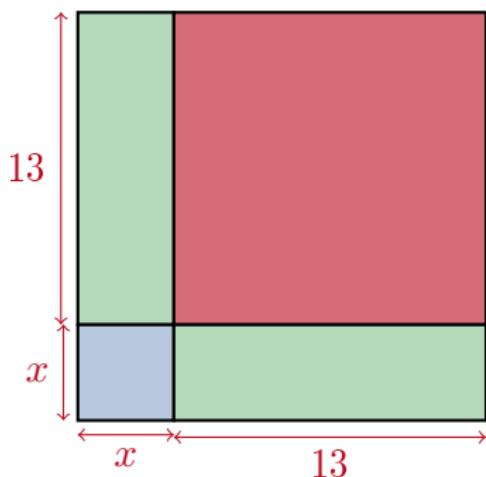


11.1 Three-Dimensional Coordinate Systems



$$(x + 13)^2 = 196$$

11.1 Three-Dimensional Coordinate Systems



$$\left(\frac{26}{2}\right)^2 = 27 + 169$$

$$(x + 13)^2 = 196$$

11.1 Three-Dimer

$$(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2 = a^2.$$



Example

Find the centre and radius of the sphere

$$x^2 + y^2 + z^2 + 3x - 4z + 1 = 0.$$

11.1 Three-Dimer

$$(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2 = a^2.$$



By completing the squares, we find that

$$x^2 + y^2 + z^2 + 3x - 4z + 1 = 0$$

$$(x^2 + 3x) + y^2 + (z^2 - 4z) = -1$$

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$$\left(x^2 + 3x + \frac{9}{4}\right) - \frac{9}{4} + y^2 + (z^2 - 4z + 4) - 4 = -1$$

11.1 Three-Dimensional

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$$\left(x + \frac{3}{2}\right)^2 + y^2 + (z - 2)^2 = \frac{21}{4}.$$

11.1 Three-Dimensional

$$(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2 = a^2.$$



By completing the squares, we find that

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$$\left(x + \frac{3}{2}\right)^2 + y^2 + (z - 2)^2 = \frac{21}{4}.$$

The centre is at $P_0(x_0, y_0, z_0) = P_0(-\frac{3}{2}, 0, 2)$ and the radius is

$$a = \sqrt{\frac{21}{4}} = \frac{\sqrt{3}\sqrt{7}}{2}.$$

11.1 Three-Dimer

$$(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2 = a^2.$$



Example

Find the centre and radius of the sphere

$$x^2 + y^2 + z^2 + 6x - 6y + 6z = 7.$$

11.1 Three-Dimensional

$$(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2 = a^2.$$



Since $(x - b)^2 = x^2 - 2bx + b^2$ we have that

$$x^2 + y^2 + z^2 + 6x - 6y + 6z = 7$$

$$(x^2 + 6x) + (y^2 - 6y) + (z^2 + 6z) = 7$$

$$(x^2 + 6x + 9) - 9 + (y^2 - 6y + 9) - 9 + (z^2 + 6z + 9) - 9 = 7$$

$$(x^2 + 6x + 9) + (y^2 - 6y + 9) + (z^2 + 6z + 9) = 7 + 9$$

$$(x + 3)^2 + (y - 3)^2 + (z + 3)^2 = 16$$

The centre is at $P_0(x_0, y_0, z_0) = P_0(-3, 3, -3)$ and the radius is $a = \sqrt{16} = 4$.

EXAMPLE 5 Here are some geometric interpretations of inequalities and equations involving spheres.

(a) $x^2 + y^2 + z^2 < 4$

The interior of the sphere $x^2 + y^2 + z^2 = 4$.

(b) $x^2 + y^2 + z^2 \leq 4$

The solid ball bounded by the sphere $x^2 + y^2 + z^2 = 4$. Alternatively, the sphere $x^2 + y^2 + z^2 = 4$ together with its interior.

(c) $x^2 + y^2 + z^2 > 4$

The exterior of the sphere $x^2 + y^2 + z^2 = 4$.

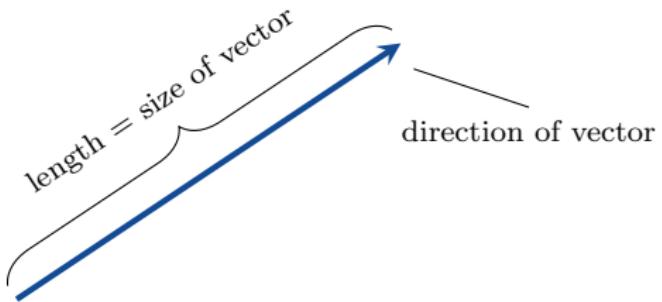
(d) $x^2 + y^2 + z^2 = 4, z \leq 0$

The lower hemisphere cut from the sphere $x^2 + y^2 + z^2 = 4$ by the xy -plane (the plane $z = 0$). ■

11 Vectors 2

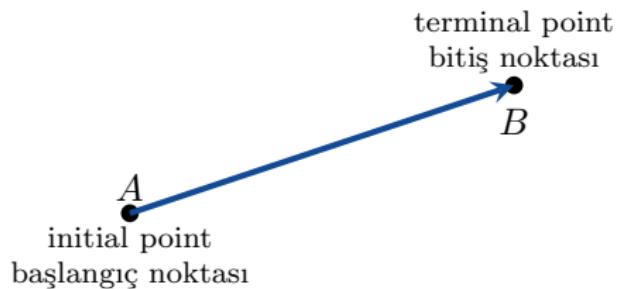
11.2 Vectors

For some quantities (mass, time, distance, ...) we only need a number. For some quantities (velocity, force, ...) we need a number and a direction.



A *vector* is an object which has a size (length) and a direction.

11.2 Vectors

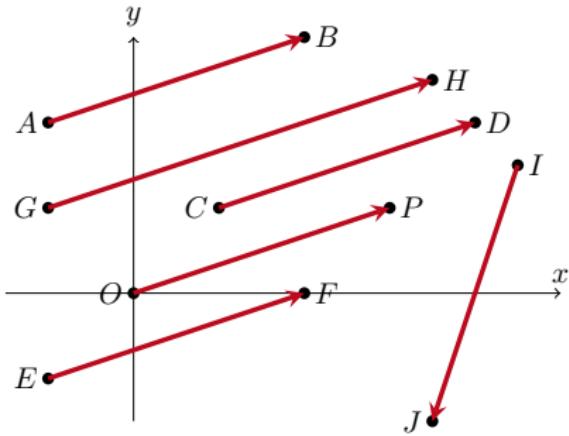


Definition

The vector \overrightarrow{AB} has *initial point* A and *terminal point* B .

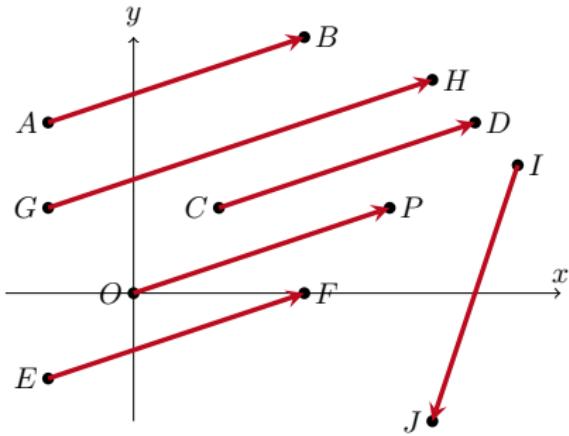
The *length* of \overrightarrow{AB} is written $\|\overrightarrow{AB}\|$ (or $|\overrightarrow{AB}|$).

11.2 Vectors



Two vectors are equal if they have the same length and the same direction.

11.2 Vectors

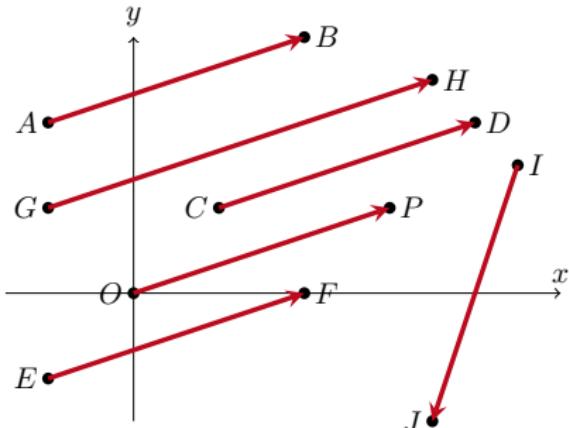


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We can say that

$$\overrightarrow{AB} = \overrightarrow{CD} = \overrightarrow{EF} = \overrightarrow{OP}.$$

11.2 Vectors



Two vectors are equal if they have the same length and the same direction.

We can say that

$$\overrightarrow{AB} = \overrightarrow{CD} = \overrightarrow{EF} = \overrightarrow{OP}.$$

Note that $\overrightarrow{AB} \neq \overrightarrow{GH}$ because the lengths are different, and $\overrightarrow{AB} \neq \overrightarrow{IJ}$ because the directions are different.



Notation

When we use a computer, we use bold letters for vectors: **u**, **v**, **w**,



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When we use a computer, we use bold letters for vectors: \mathbf{u} , \mathbf{v} , \mathbf{w} , . . . When we use a pen, we use underlined letters for vectors: \underline{u} , \underline{v} , \underline{w} , . . .

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If we type $a\mathbf{u} + b\mathbf{v}$ or write $a\underline{u} + b\underline{v}$, then

- a and b are numbers; and
- \mathbf{u} , \mathbf{v} , \underline{u} and \underline{v} are vectors.

11.2 Vectors



Definition

In \mathbb{R}^2 : If \mathbf{v} has initial point $(0, 0)$ and terminal point (v_1, v_2) , then the *component form* of \mathbf{v} is $\mathbf{v} = (v_1, v_2)$.

11.2 Vectors

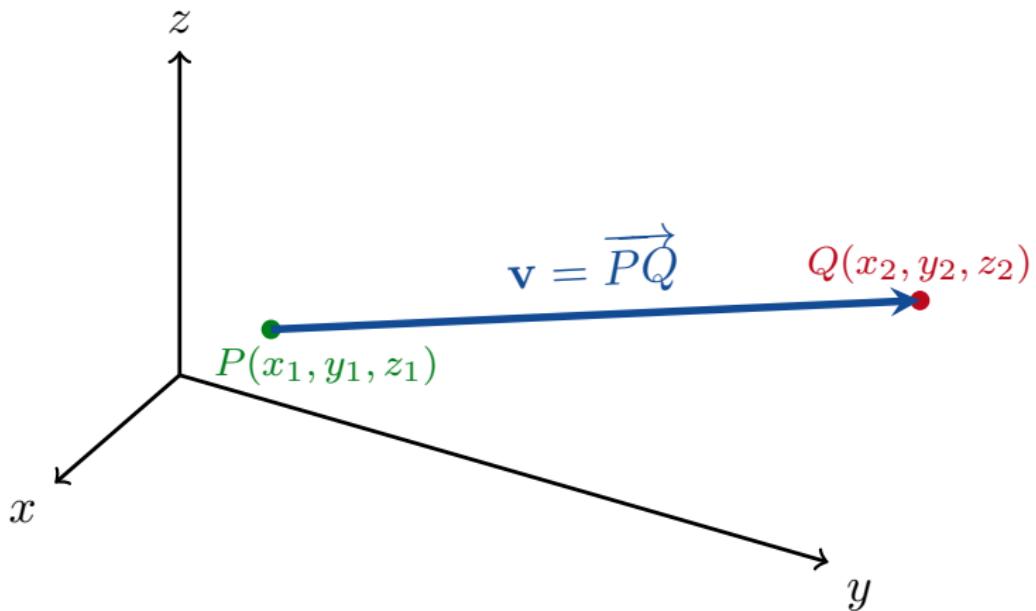


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In \mathbb{R}^3 : If \mathbf{v} has initial point $(0, 0, 0)$ and terminal point (v_1, v_2, v_3) , then the *component form* of \mathbf{v} is $\mathbf{v} = (v_1, v_2, v_3)$.

11.2 Vectors



$$(v_1, v_2, v_3) = \mathbf{v} = (x_2 - x_1, y_2 - y_1, z_2 - z_1)$$

11.2 Vectors

Definition

In \mathbb{R}^2 : The *norm* (or *length*) of $\mathbf{v} = (v_1, v_2)$ is

$$\|\mathbf{v}\| = \sqrt{v_1^2 + v_2^2}$$

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In \mathbb{R}^3 : The *norm* of $\mathbf{v} = \overrightarrow{PQ}$ is

$$\begin{aligned}\|\mathbf{v}\| &= \sqrt{v_1^2 + v_2^2 + v_3^2} \\ &= \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}.\end{aligned}$$

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The vectors $\mathbf{0} = (0, 0)$ and $\mathbf{0} = (0, 0, 0)$ have norm $\|\mathbf{0}\| = 0$. If $\mathbf{v} \neq \mathbf{0}$, then $\|\mathbf{v}\| > 0$.

11.2 Vectors

Example

Find (1) the component form; and (2) the norm of the vector with initial point $P(-3, 4, 1)$ and terminal point $Q(-5, 2, 2)$.

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 $= (-2, -2, 1).$

11.2 Vectors

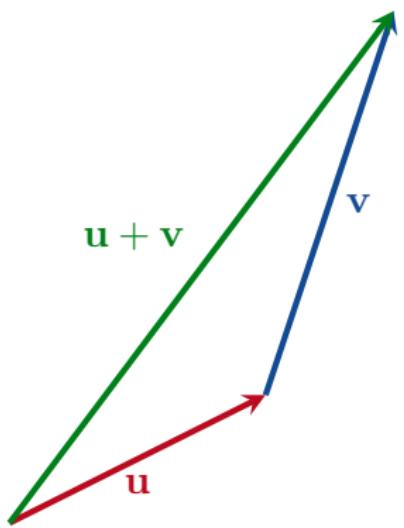
Example

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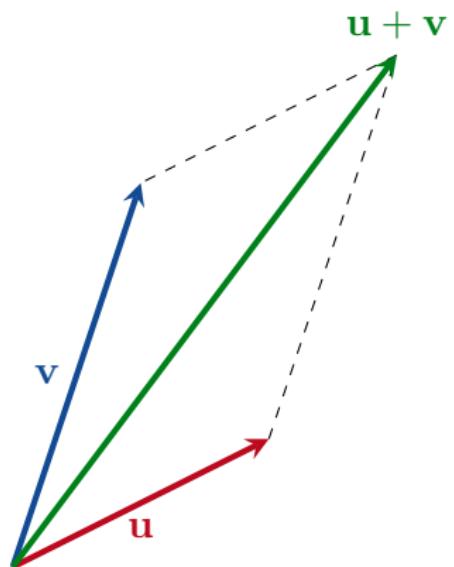
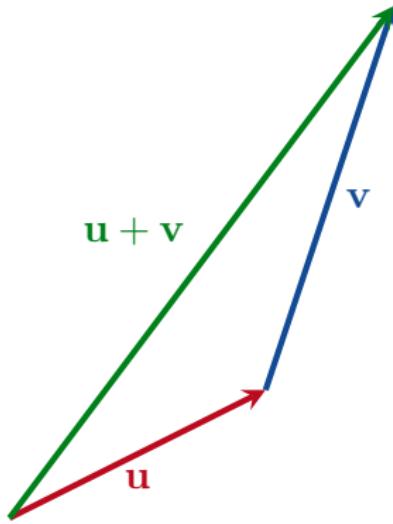
1 $\mathbf{v} = (v_1, v_2, v_3) = Q - P = (-5, 2, 2) - (-3, 4, 1)$
 $= (-2, -2, 1).$

2 $\|\mathbf{v}\| = \sqrt{v_1^2 + v_2^2 + v_3^2} = \sqrt{(-2)^2 + (-2)^2 + 1^2} = \sqrt{9} = 3.$

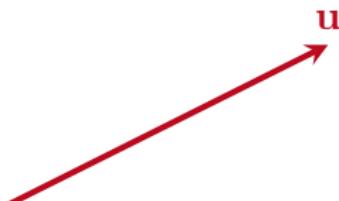
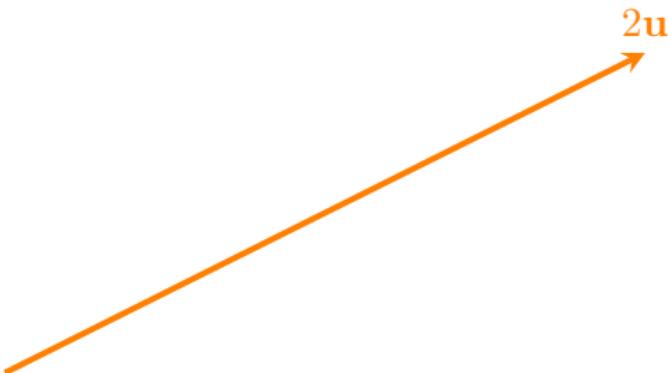
Vector Algebra: Addition



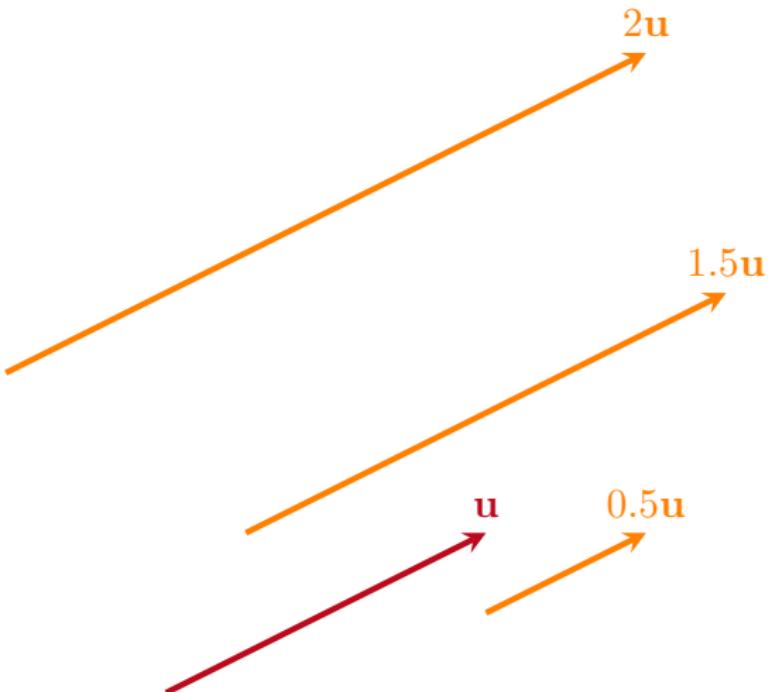
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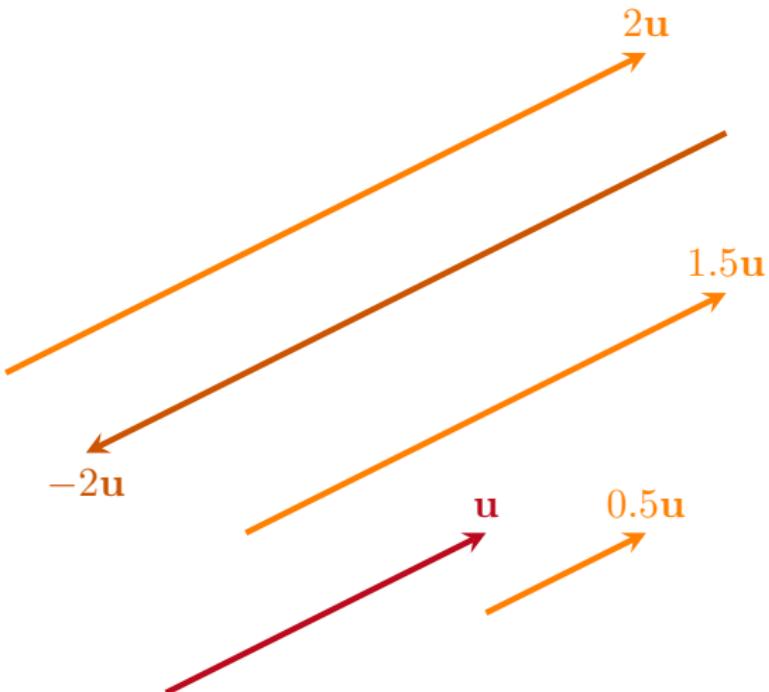
Vector Algebra: Multiplication by a Constant



Vector Algebra: Multiplication by a Constant

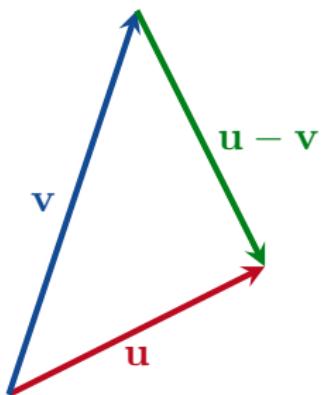


Vector Algebra: Multiplication by a Constant



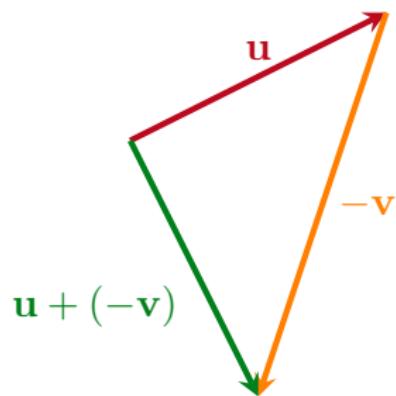
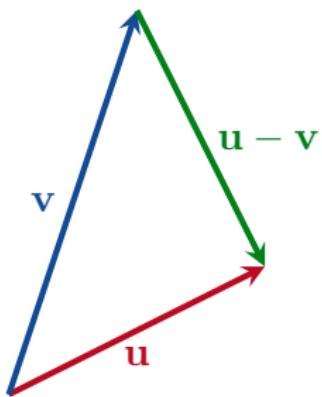
Vector Algebra: Subtraction

$$\mathbf{v} + (\mathbf{u} - \mathbf{v}) = \mathbf{u}$$



Vector Algebra: Subtraction

$$\mathbf{v} + (\mathbf{u} - \mathbf{v}) = \mathbf{u}$$



11.2 Vectors



Let $\mathbf{u} = (u_1, u_2, u_3)$ and $\mathbf{v} = (v_1, v_2, v_3)$ be vectors. Let k be a number.

11.2 Vectors



Let $\mathbf{u} = (u_1, u_2, u_3)$ and $\mathbf{v} = (v_1, v_2, v_3)$ be vectors. Let k be a number. Then

$$\mathbf{u} + \mathbf{v} = (u_1 + v_1, u_2 + v_2, u_3 + v_3)$$

11.2 Vectors



Let $\mathbf{u} = (u_1, u_2, u_3)$ and $\mathbf{v} = (v_1, v_2, v_3)$ be vectors. Let k be a number. Then

$$\mathbf{u} + \mathbf{v} = (u_1 + v_1, u_2 + v_2, u_3 + v_3)$$

and

$$k\mathbf{u} = (ku_1, ku_2, ku_3).$$

11.2 Vectors



Note that

$$\|k\mathbf{u}\| = \|(ku_1, ku_2, ku_3)\|$$

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=

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11.2 Vectors



Note that

$$\begin{aligned}\|k\mathbf{u}\| &= \|(ku_1, ku_2, ku_3)\| \\ &= \sqrt{(ku_1)^2 + (ku_2)^2 + (ku_3)^2}\end{aligned}$$

=

=

=

= .

11.2 Vectors



Note that

$$\begin{aligned}\|k\mathbf{u}\| &= \|(ku_1, ku_2, ku_3)\| \\&= \sqrt{(ku_1)^2 + (ku_2)^2 + (ku_3)^2} \\&= \sqrt{k^2u_1^2 + k^2u_2^2 + k^2u_3^2}\end{aligned}$$

=

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.

11.2 Vectors



Note that

$$\begin{aligned}\|k\mathbf{u}\| &= \|(ku_1, ku_2, ku_3)\| \\&= \sqrt{(ku_1)^2 + (ku_2)^2 + (ku_3)^2} \\&= \sqrt{k^2u_1^2 + k^2u_2^2 + k^2u_3^2} \\&= \sqrt{k^2(u_1^2 + u_2^2 + u_3^2)} \\&= \\&= .\end{aligned}$$

11.2 Vectors



Note that

$$\begin{aligned}\|k\mathbf{u}\| &= \|(ku_1, ku_2, ku_3)\| \\&= \sqrt{(ku_1)^2 + (ku_2)^2 + (ku_3)^2} \\&= \sqrt{k^2u_1^2 + k^2u_2^2 + k^2u_3^2} \\&= \sqrt{k^2(u_1^2 + u_2^2 + u_3^2)} \\&= \sqrt{k^2} \sqrt{u_1^2 + u_2^2 + u_3^2} \\&= \end{aligned}$$

11.2 Vectors



Note that

$$\begin{aligned}\|k\mathbf{u}\| &= \|(ku_1, ku_2, ku_3)\| \\&= \sqrt{(ku_1)^2 + (ku_2)^2 + (ku_3)^2} \\&= \sqrt{k^2u_1^2 + k^2u_2^2 + k^2u_3^2} \\&= \sqrt{k^2(u_1^2 + u_2^2 + u_3^2)} \\&= \sqrt{k^2}\sqrt{u_1^2 + u_2^2 + u_3^2} \\&= |k| \|\mathbf{u}\|.\end{aligned}$$

11.2 Vectors



The vector $-\mathbf{u} = (-1)\mathbf{u}$ has the same length as \mathbf{u} , but points in the opposite direction.

11.2 Vectors



Example

Let $\mathbf{u} = (-1, 3, 1)$ and $\mathbf{v} = (4, 7, 0)$.

Find $2\mathbf{u} + 3\mathbf{v}$, $\mathbf{u} - \mathbf{v}$, and $\left\| \frac{1}{2}\mathbf{u} \right\|$.

11.2 Vectors

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1 $2\mathbf{u} + 3\mathbf{v} = 2(-1, 3, 1) + 3(4, 7, 0) = (-2, 6, 2) + (12, 21, 0) = (10, 27, 2);$

11.2 Vectors

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11.2 Vectors

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3 $\left\| \frac{1}{2}\mathbf{u} \right\| = \frac{1}{2} \left\| \mathbf{u} \right\| = \frac{1}{2} \sqrt{(-1)^2 + 3^2 + 1^2} = \frac{1}{2} \sqrt{11}.$

Properties of Vector Operations

Let \mathbf{u} , \mathbf{v} and \mathbf{w} be vectors. Let a and b be numbers. Then

1 $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u};$

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- 7 $a(b\mathbf{u}) = (ab)\mathbf{u};$

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- 1 $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u};$
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- 3 $\mathbf{u} + \mathbf{0} = \mathbf{u};$
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- 5 $0\mathbf{u} = \mathbf{0};$
- 6 $1\mathbf{u} = \mathbf{u};$
- 7 $a(b\mathbf{u}) = (ab)\mathbf{u};$
- 8 $a(\mathbf{u} + \mathbf{v}) = a\mathbf{u} + a\mathbf{v};$

Properties of Vector Operations

Let \mathbf{u} , \mathbf{v} and \mathbf{w} be vectors. Let a and b be numbers. Then

- 1 $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u};$
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- 3 $\mathbf{u} + \mathbf{0} = \mathbf{u};$
- 4 $\mathbf{u} + (-\mathbf{u}) = \mathbf{0};$
- 5 $0\mathbf{u} = \mathbf{0};$
- 6 $1\mathbf{u} = \mathbf{u};$
- 7 $a(b\mathbf{u}) = (ab)\mathbf{u};$
- 8 $a(\mathbf{u} + \mathbf{v}) = a\mathbf{u} + a\mathbf{v};$
- 9 $(a + b)\mathbf{u} = a\mathbf{u} + b\mathbf{u}.$

11.2 Vectors



Remark

We **can not** multiply vectors. Never never never never write "**uv**".

Unit Vectors

Definition

\mathbf{u} is called a *unit vector* $\iff \|\mathbf{u}\| = 1$.

11.2 Vectors



Example

$\mathbf{u} = (2^{-\frac{1}{2}}, \frac{1}{2}, -\frac{1}{2})$ is a unit vector because

$$\|\mathbf{u}\| = \sqrt{\left(\frac{1}{\sqrt{2}}\right)^2 + \left(\frac{1}{2}\right)^2 + \left(-\frac{1}{2}\right)^2} = \sqrt{\frac{1}{2} + \frac{1}{4} + \frac{1}{4}} = 1.$$

Standard Unit Vectors

In \mathbb{R}^2 : The *standard unit vectors* are $\mathbf{i} = (1, 0)$ and $\mathbf{j} = (0, 1)$.

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Standard Unit Vectors

In \mathbb{R}^2 : The *standard unit vectors* are $\mathbf{i} = (1, 0)$ and $\mathbf{j} = (0, 1)$.

In \mathbb{R}^3 : The *standard unit vectors* are $\mathbf{i} = (1, 0, 0)$, $\mathbf{j} = (0, 1, 0)$ and $\mathbf{k} = (0, 0, 1)$. Any vector $\mathbf{v} \in \mathbb{R}^3$ can be written

$$\begin{aligned}\mathbf{v} &= (v_1, v_2, v_3) = (v_1, 0, 0) + (0, v_2, 0) + (0, 0, v_3) \\ &= v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}.\end{aligned}$$

Normalising a Vector

If $\|\mathbf{v}\| \neq 0$, then $\frac{\mathbf{v}}{\|\mathbf{v}\|}$ is a unit vector because

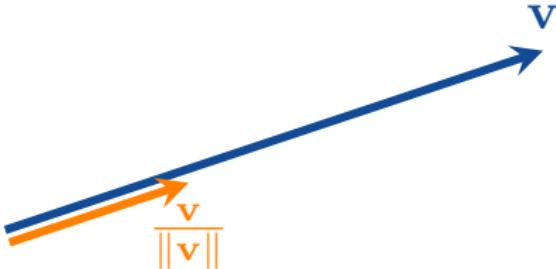
$$\left\| \frac{\mathbf{v}}{\|\mathbf{v}\|} \right\| = \left\| \frac{1}{\|\mathbf{v}\|} \mathbf{v} \right\| = \frac{1}{\|\mathbf{v}\|} \|\mathbf{v}\| = 1.$$

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If $\|\mathbf{v}\| \neq 0$, then $\frac{\mathbf{v}}{\|\mathbf{v}\|}$ is a unit vector because

$$\left\| \frac{\mathbf{v}}{\|\mathbf{v}\|} \right\| = \left\| \frac{1}{\|\mathbf{v}\|} \mathbf{v} \right\| = \frac{1}{\|\mathbf{v}\|} \|\mathbf{v}\| = 1.$$

Clearly $\frac{\mathbf{v}}{\|\mathbf{v}\|}$ and \mathbf{v} point in the same direction.



11.2 Vectors

Example

Find a unit vector \mathbf{u} which points in the same direction as $\overrightarrow{P_1P_2}$, where $P_1(1, 0, 1)$ and $P_2(3, 2, 0)$.

11.2 Vectors

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Find a unit vector \mathbf{u} which points in the same direction as $\overrightarrow{P_1P_2}$, where $P_1(1, 0, 1)$ and $P_2(3, 2, 0)$.

We calculate that

$$\overrightarrow{P_1P_2} = P_2 - P_1 = (3, 2, 0) - (1, 0, 1) = (2, 2, -1) = 2\mathbf{i} + 2\mathbf{j} - \mathbf{k}$$

11.2 Vectors

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and that

$$\left\| \overrightarrow{P_1P_2} \right\| = \sqrt{2^2 + 2^2 + (-1)^2} = 3.$$

11.2 Vectors

Example

Find a unit vector \mathbf{u} which points in the same direction as $\overrightarrow{P_1P_2}$, where $P_1(1, 0, 1)$ and $P_2(3, 2, 0)$.

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and that

$$\left\| \overrightarrow{P_1P_2} \right\| = \sqrt{2^2 + 2^2 + (-1)^2} = 3.$$

The required unit vector is

$$\mathbf{u} = \frac{\overrightarrow{P_1P_2}}{\left\| \overrightarrow{P_1P_2} \right\|} = \frac{2\mathbf{i} + 2\mathbf{j} - \mathbf{k}}{3} = \frac{2}{3}\mathbf{i} + \frac{2}{3}\mathbf{j} - \frac{1}{3}\mathbf{k}.$$

EXAMPLE 5 If $\mathbf{v} = 3\mathbf{i} - 4\mathbf{j}$ is a velocity vector, express \mathbf{v} as a product of its speed times its direction of motion.

Solution Speed is the magnitude (length) of \mathbf{v} :

$$|\mathbf{v}| = \sqrt{(3)^2 + (-4)^2} = \sqrt{9 + 16} = 5.$$

The unit vector $\mathbf{v}/|\mathbf{v}|$ is the direction of \mathbf{v} :

$$\frac{\mathbf{v}}{|\mathbf{v}|} = \frac{3\mathbf{i} - 4\mathbf{j}}{5} = \frac{3}{5}\mathbf{i} - \frac{4}{5}\mathbf{j}.$$

So

$$\mathbf{v} = 3\mathbf{i} - 4\mathbf{j} = 5 \left(\underbrace{\frac{3}{5}\mathbf{i} - \frac{4}{5}\mathbf{j}}_{\substack{\text{Length} \\ \text{Direction of motion} \\ (\text{speed})}} \right).$$



If $\mathbf{v} \neq \mathbf{0}$, then

1. $\frac{\mathbf{v}}{|\mathbf{v}|}$ is a unit vector called the direction of \mathbf{v} ;
2. the equation $\mathbf{v} = |\mathbf{v}| \frac{\mathbf{v}}{|\mathbf{v}|}$ expresses \mathbf{v} as its length times its direction.

EXAMPLE 6 A force of 6 newtons is applied in the direction of the vector $\mathbf{v} = 2\mathbf{i} + 2\mathbf{j} - \mathbf{k}$. Express the force \mathbf{F} as a product of its magnitude and direction.

Solution The force vector has magnitude 6 and direction $\frac{\mathbf{v}}{|\mathbf{v}|}$, so

$$\begin{aligned}\mathbf{F} &= 6 \frac{\mathbf{v}}{|\mathbf{v}|} = 6 \frac{2\mathbf{i} + 2\mathbf{j} - \mathbf{k}}{\sqrt{2^2 + 2^2 + (-1)^2}} = 6 \frac{2\mathbf{i} + 2\mathbf{j} - \mathbf{k}}{3} \\ &= 6 \left(\frac{2}{3}\mathbf{i} + \frac{2}{3}\mathbf{j} - \frac{1}{3}\mathbf{k} \right).\end{aligned}$$



Midpoint of a Line Segment

Vectors are often useful in geometry. For example, the coordinates of the midpoint of a line segment are found by averaging.

The **midpoint** M of the line segment joining points $P_1(x_1, y_1, z_1)$ and $P_2(x_2, y_2, z_2)$ is the point

$$\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}, \frac{z_1 + z_2}{2} \right).$$

To see why, observe (Figure 12.16) that

$$\begin{aligned}\overrightarrow{OM} &= \overrightarrow{OP}_1 + \frac{1}{2}(\overrightarrow{P_1P_2}) = \overrightarrow{OP}_1 + \frac{1}{2}(\overrightarrow{OP}_2 - \overrightarrow{OP}_1) \\ &= \frac{1}{2}(\overrightarrow{OP}_1 + \overrightarrow{OP}_2) \\ &= \frac{x_1 + x_2}{2}\mathbf{i} + \frac{y_1 + y_2}{2}\mathbf{j} + \frac{z_1 + z_2}{2}\mathbf{k}.\end{aligned}$$

EXAMPLE 7 The midpoint of the segment joining $P_1(3, -2, 0)$ and $P_2(7, 4, 4)$ is

$$\left(\frac{3+7}{2}, \frac{-2+4}{2}, \frac{0+4}{2} \right) = (5, 1, 2). \quad \blacksquare$$

11.2 Vectors



Please read the final two examples in this section of the textbook.



Break

We will continue at 3pm



11 The Dot Product 3

11.3 The Dot Product

Definition

In \mathbb{R}^2 , the *dot product* of $\mathbf{u} = (u_1, u_2) = u_1\mathbf{i} + u_2\mathbf{j}$ and $\mathbf{v} = (v_1, v_2) = v_1\mathbf{i} + v_2\mathbf{j}$ is

$$\mathbf{u} \cdot \mathbf{v} = u_1v_1 + u_2v_2.$$

11.3 The Dot Product

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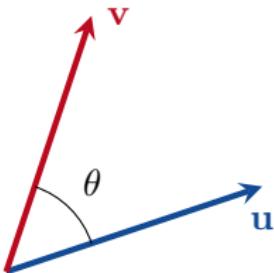
$$\mathbf{u} \cdot \mathbf{v} = u_1v_1 + u_2v_2.$$

Definition

In \mathbb{R}^3 , the *dot product* of $\mathbf{u} = (u_1, u_2, u_3) = u_1\mathbf{i} + u_2\mathbf{j} + u_3\mathbf{k}$ and $\mathbf{v} = (v_1, v_2, v_3) = v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}$ is

$$\mathbf{u} \cdot \mathbf{v} = u_1v_1 + u_2v_2 + u_3v_3.$$

11.3 The Dot Product

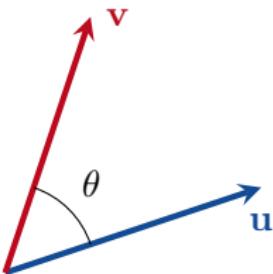


Theorem

The angle between \mathbf{u} and \mathbf{v} is

$$\theta = \cos^{-1} \left(\frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{u}\| \|\mathbf{v}\|} \right).$$

11.3 The Dot Product



Theorem

The angle between \mathbf{u} and \mathbf{v} is

$$\theta = \cos^{-1} \left(\frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{u}\| \|\mathbf{v}\|} \right).$$

This means that

$$\mathbf{u} \cdot \mathbf{v} = \|\mathbf{u}\| \|\mathbf{v}\| \cos \theta.$$

11.3 The Dot Product

Example

$$\begin{aligned}(1, -2, -1) \cdot (-6, 2, -3) &= (1 \times -6) + (-2 \times 2) + (-1 \times -3) \\&= -6 - 4 + 3 = -7.\end{aligned}$$

11.3 The Dot Product

Example

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Example

$$\begin{aligned}\left(\frac{1}{2}\mathbf{i} + 3\mathbf{j} + \mathbf{k}\right) \cdot (4\mathbf{i} - \mathbf{j} + 2\mathbf{k}) &= \left(\frac{1}{2} \times 4\right) + (3 \times -1) + (1 \times 2) \\&= 2 - 3 + 2 = 1.\end{aligned}$$

11.3 The Dot Product

$$\theta = \cos^{-1} \left(\frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{u}\| \|\mathbf{v}\|} \right)$$



Example

Find the angle between $\mathbf{u} = \mathbf{i} - 2\mathbf{j} - 2\mathbf{k}$ and $\mathbf{v} = 6\mathbf{i} + 3\mathbf{j} + 2\mathbf{k}$.

11.3 The Dot Product

$$\theta = \cos^{-1} \left(\frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{u}\| \|\mathbf{v}\|} \right)$$



Example

Find the angle between $\mathbf{u} = \mathbf{i} - 2\mathbf{j} - 2\mathbf{k}$ and $\mathbf{v} = 6\mathbf{i} + 3\mathbf{j} + 2\mathbf{k}$.

Since

$$\begin{aligned}\mathbf{u} \cdot \mathbf{v} &= (1, -2, -2) \cdot (6, 3, 2) = (1 \times 6) + (-2 \times 3) + (-2 \times 2) \\ &= 6 - 6 - 4 = -4,\end{aligned}$$

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$$\|\mathbf{u}\| = \sqrt{1^2 + (-2)^2 + (-2)^2} = \sqrt{9} = 3$$

and

$$\|\mathbf{v}\| = \sqrt{6^2 + 3^2 + 2^2} = \sqrt{49} = 7,$$

11.3 The Dot Product

$$\theta = \cos^{-1} \left(\frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{u}\| \|\mathbf{v}\|} \right)$$



Example

Find the angle between $\mathbf{u} = \mathbf{i} - 2\mathbf{j} - 2\mathbf{k}$ and $\mathbf{v} = 6\mathbf{i} + 3\mathbf{j} + 2\mathbf{k}$.

Since

$$\begin{aligned}\mathbf{u} \cdot \mathbf{v} &= (1, -2, -2) \cdot (6, 3, 2) = (1 \times 6) + (-2 \times 3) + (-2 \times 2) \\ &= 6 - 6 - 4 = -4,\end{aligned}$$

$$\|\mathbf{u}\| = \sqrt{1^2 + (-2)^2 + (-2)^2} = \sqrt{9} = 3$$

and

$$\|\mathbf{v}\| = \sqrt{6^2 + 3^2 + 2^2} = \sqrt{49} = 7,$$

we have that

$$\theta = \cos^{-1} \left(\frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{u}\| \|\mathbf{v}\|} \right) =$$

11.3 The Dot Product

$$\theta = \cos^{-1} \left(\frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{u}\| \|\mathbf{v}\|} \right)$$



Example

Find the angle between $\mathbf{u} = \mathbf{i} - 2\mathbf{j} - 2\mathbf{k}$ and $\mathbf{v} = 6\mathbf{i} + 3\mathbf{j} + 2\mathbf{k}$.

Since

$$\begin{aligned}\mathbf{u} \cdot \mathbf{v} &= (1, -2, -2) \cdot (6, 3, 2) = (1 \times 6) + (-2 \times 3) + (-2 \times 2) \\ &= 6 - 6 - 4 = -4,\end{aligned}$$

$$\|\mathbf{u}\| = \sqrt{1^2 + (-2)^2 + (-2)^2} = \sqrt{9} = 3$$

and

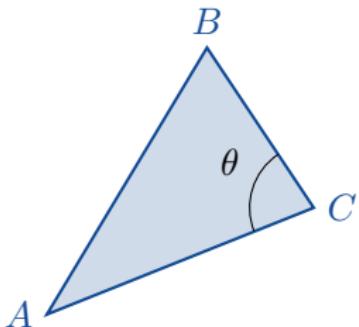
$$\|\mathbf{v}\| = \sqrt{6^2 + 3^2 + 2^2} = \sqrt{49} = 7,$$

we have that

$$\theta = \cos^{-1} \left(\frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{u}\| \|\mathbf{v}\|} \right) = \cos^{-1} \left(-\frac{4}{21} \right) \approx 1.76 \text{ radians} \approx 98.5^\circ.$$

11.3 The Dot Product

$$\theta = \cos^{-1} \left(\frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{u}\| \|\mathbf{v}\|} \right)$$



Example

If $A(0, 0)$, $B(3, 5)$ and $C(5, 2)$, find $\theta = \angle ACB$.

11.3 The Dot Product



θ is the angle between \overrightarrow{CA} and \overrightarrow{CB} .

11.3 The Dot Product

θ is the angle between \overrightarrow{CA} and \overrightarrow{CB} . We calculate that

$$\overrightarrow{CA} = A - C = (0, 0) - (5, 2) = (-5, -2),$$

$$\overrightarrow{CB} = B - C = (3, 5) - (5, 2) = (-2, 3),$$

$$\overrightarrow{CA} \cdot \overrightarrow{CB} = (-5, -2) \cdot (-2, 3) = 4,$$

$$\|\overrightarrow{CA}\| = \sqrt{(-5)^2 + (-2)^2} = \sqrt{29}$$

and

$$\|\overrightarrow{CB}\| = \sqrt{(-2)^2 + 3^2} = \sqrt{13}.$$

11.3 The Dot Product

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and

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Therefore

$$\theta = \cos^{-1} \left(\frac{\overrightarrow{CA} \cdot \overrightarrow{CB}}{\|\overrightarrow{CA}\| \|\overrightarrow{CB}\|} \right) = \cos^{-1} \left(\frac{4}{\sqrt{29}\sqrt{13}} \right)$$

$$\approx 78.1^\circ \approx 1.36 \text{ radians.}$$

11.3 The Dot Product



Definition

\mathbf{u} and \mathbf{v} are *orthogonal* $\iff \mathbf{u} \cdot \mathbf{v} = 0$.

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Remark

Recall that

$$\mathbf{u} \cdot \mathbf{v} = \|\mathbf{u}\| \|\mathbf{v}\| \cos \theta.$$

11.3 The Dot Product

Definition

\mathbf{u} and \mathbf{v} are *orthogonal* $\iff \mathbf{u} \cdot \mathbf{v} = 0$.

Remark

Recall that

$$\mathbf{u} \cdot \mathbf{v} = \|\mathbf{u}\| \|\mathbf{v}\| \cos \theta.$$

Therefore

$$\mathbf{u} \text{ and } \mathbf{v} \text{ are orthogonal} \iff \begin{cases} \mathbf{u} = \mathbf{0} \\ \text{or} \\ \mathbf{v} = \mathbf{0} \\ \text{or} \\ \theta = 90^\circ. \end{cases}$$

11.3 The Dot Product

Example

$\mathbf{u} = (3, -2)$ and $\mathbf{v} = (4, 6)$ are orthogonal because

$$\mathbf{u} \cdot \mathbf{v} = (3, -2) \cdot (4, 6) = (3 \times 4) + (-2 \times 6) = 12 - 12 = 0.$$

11.3 The Dot Product

Example

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Example

$\mathbf{u} = 3\mathbf{i} - 2\mathbf{j} + \mathbf{k}$ and $\mathbf{v} = 2\mathbf{j} + 4\mathbf{k}$ are orthogonal because

$$\mathbf{u} \cdot \mathbf{v} = (3 \times 0) + (-2 \times 2) + (1 \times 4) = 0 - 4 + 4 = 0.$$

11.3 The Dot Product

Example

$\mathbf{u} = (3, -2)$ and $\mathbf{v} = (4, 6)$ are orthogonal because

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$\mathbf{u} = 3\mathbf{i} - 2\mathbf{j} + \mathbf{k}$ and $\mathbf{v} = 2\mathbf{j} + 4\mathbf{k}$ are orthogonal because

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Example

$\mathbf{0}$ is orthogonal to every vector \mathbf{u} because

$$\mathbf{0} \cdot \mathbf{u} = (0, 0, 0) \cdot (u_1, u_2, u_3) = 0u_1 + 0u_2 + 0u_3 = 0.$$

11.3 The Dot Product



Properties of the Dot Product

Let \mathbf{u} , \mathbf{v} and \mathbf{w} be vectors. Let k be a number. Then

1 $\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u};$

11.3 The Dot Product



Properties of the Dot Product

Let \mathbf{u} , \mathbf{v} and \mathbf{w} be vectors. Let k be a number. Then

- 1 $\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u}$;
- 2 $(k\mathbf{u}) \cdot \mathbf{v} = \mathbf{u} \cdot (k\mathbf{v}) = k(\mathbf{u} \cdot \mathbf{v})$;

11.3 The Dot Product



Properties of the Dot Product

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- 3 $\mathbf{u} \cdot (\mathbf{v} + \mathbf{w}) = (\mathbf{u} \cdot \mathbf{v}) + (\mathbf{u} \cdot \mathbf{w})$;

11.3 The Dot Product



Properties of the Dot Product

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- 3 $\mathbf{u} \cdot (\mathbf{v} + \mathbf{w}) = (\mathbf{u} \cdot \mathbf{v}) + (\mathbf{u} \cdot \mathbf{w})$;
- 4 $\mathbf{u} \cdot \mathbf{u} = \|\mathbf{u}\|^2$; and

11.3 The Dot Product

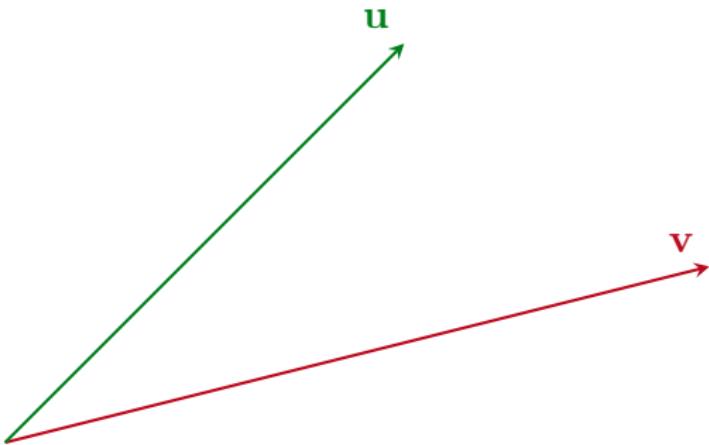


Properties of the Dot Product

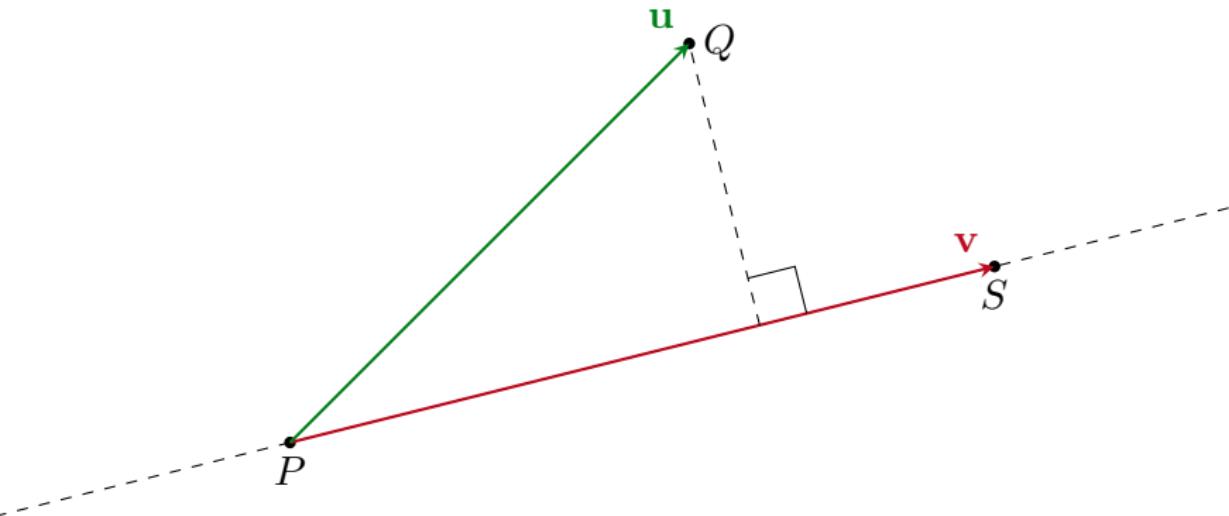
Let \mathbf{u} , \mathbf{v} and \mathbf{w} be vectors. Let k be a number. Then

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- 3 $\mathbf{u} \cdot (\mathbf{v} + \mathbf{w}) = (\mathbf{u} \cdot \mathbf{v}) + (\mathbf{u} \cdot \mathbf{w})$;
- 4 $\mathbf{u} \cdot \mathbf{u} = \|\mathbf{u}\|^2$; and
- 5 $\mathbf{0} \cdot \mathbf{u} = 0$.

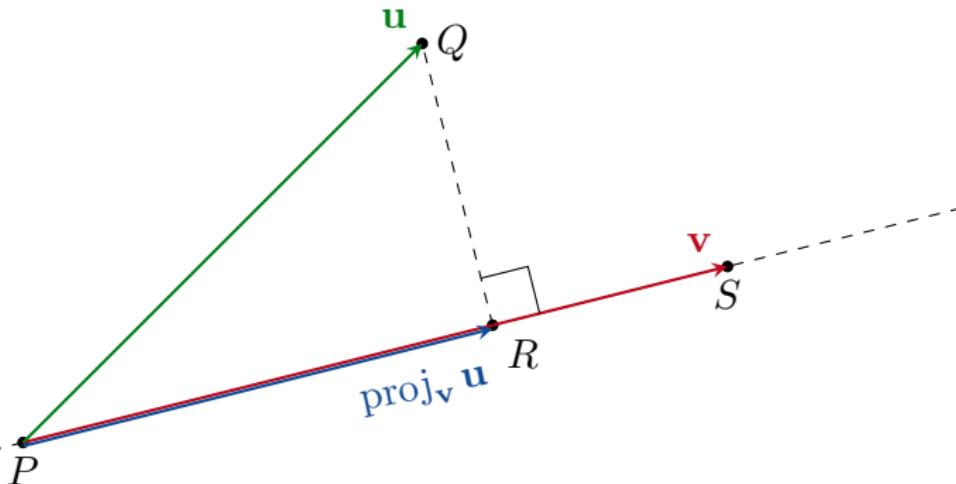
Vector Projections



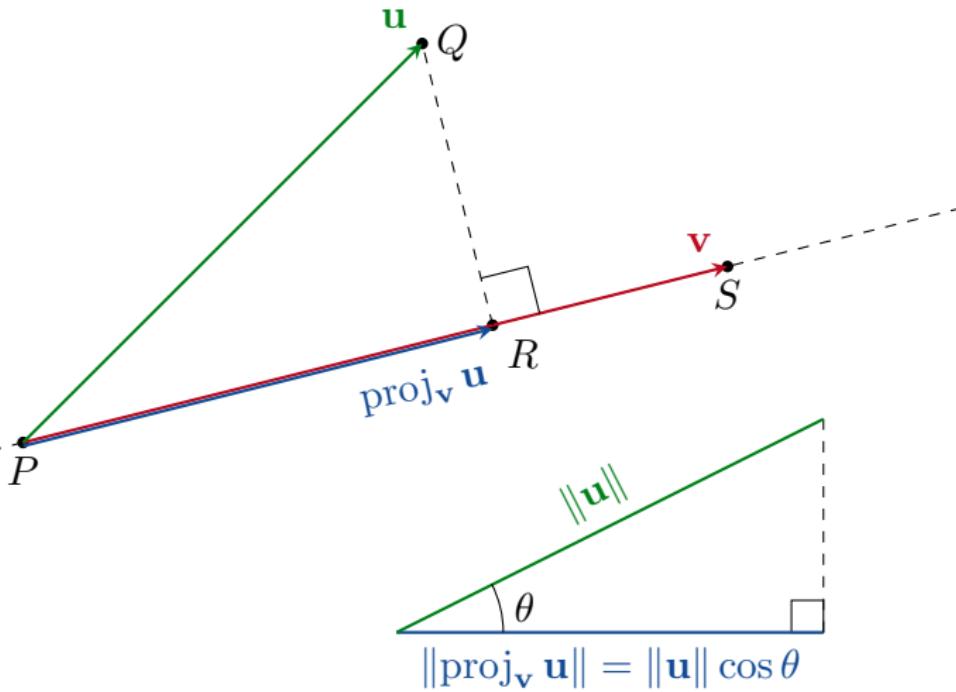
Vector Projections



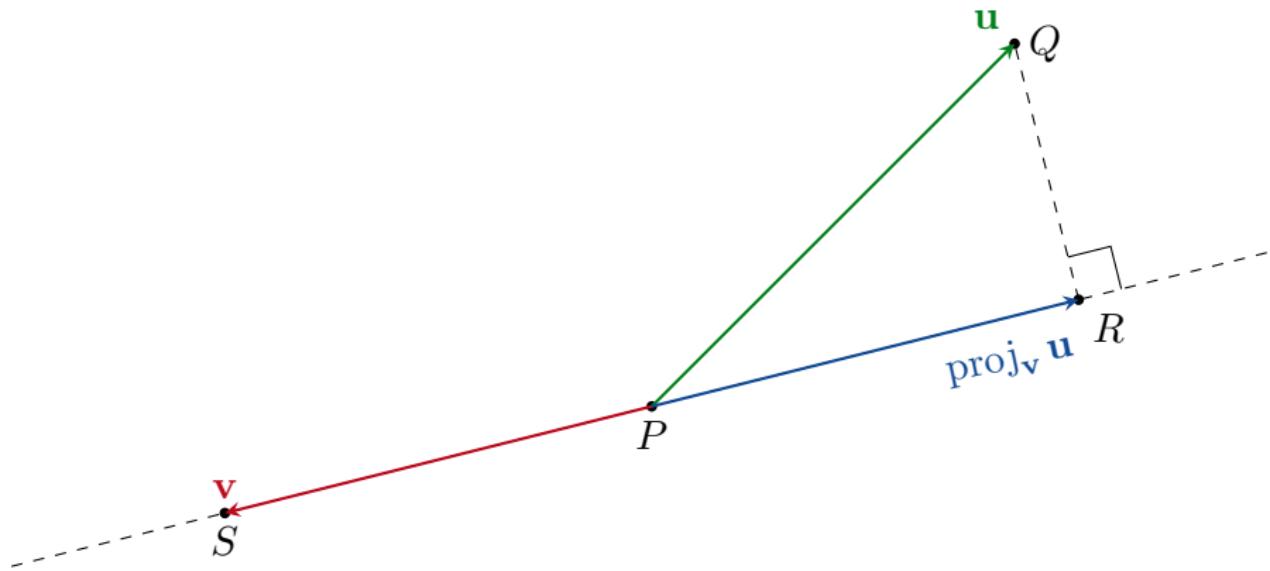
Vector Projections



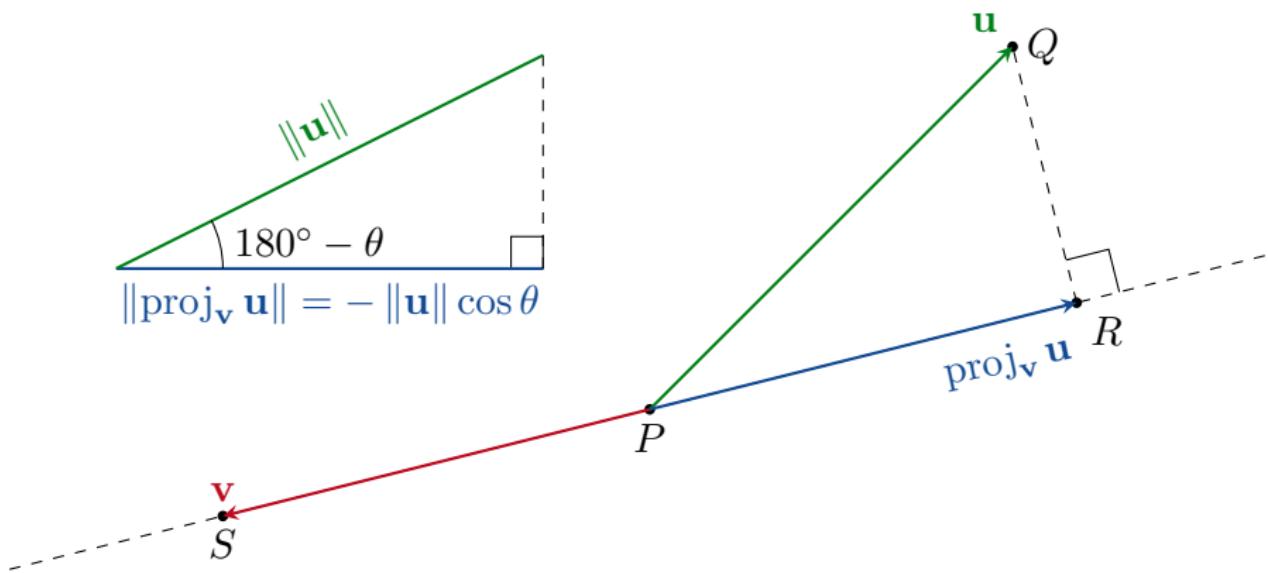
Vector Projections



11.3 The Dot Product



11.3 The Dot Product



11.3 The Dot Product



Definition

The *vector projection* of \mathbf{u} onto \mathbf{v} is the vector

$$\text{proj}_{\mathbf{v}} \mathbf{u} = \overrightarrow{PR}.$$

11.3 The Dot Product

Now

$$\text{proj}_{\mathbf{v}} \mathbf{u} = (\text{length of } \text{proj}_{\mathbf{v}} \mathbf{u}) \begin{pmatrix} \text{a unit vector in} \\ \text{the same} \\ \text{direction as } \mathbf{v} \end{pmatrix}$$

=

=

=

=

11.3 The Dot Product

Now

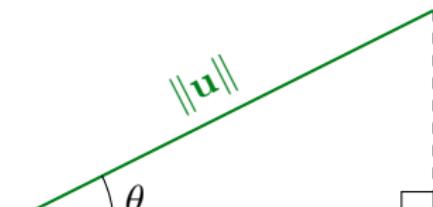
$$\text{proj}_{\mathbf{v}} \mathbf{u} = (\text{length of } \text{proj}_{\mathbf{v}} \mathbf{u}) \begin{pmatrix} \text{a unit vector in} \\ \text{the same} \\ \text{direction as } \mathbf{v} \end{pmatrix}$$

$$= \|\text{proj}_{\mathbf{v}} \mathbf{u}\| \left(\frac{\mathbf{v}}{\|\mathbf{v}\|} \right)$$

=

=

=



$$\|\text{proj}_{\mathbf{v}} \mathbf{u}\| = \|\mathbf{u}\| \cos \theta$$

11.3 The Dot Product

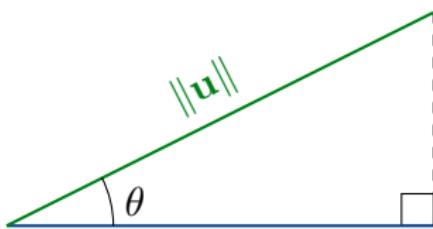
Now

$$\text{proj}_{\mathbf{v}} \mathbf{u} = (\text{length of } \text{proj}_{\mathbf{v}} \mathbf{u}) \begin{pmatrix} \text{a unit vector in} \\ \text{the same} \\ \text{direction as } \mathbf{v} \end{pmatrix}$$

$$= \|\text{proj}_{\mathbf{v}} \mathbf{u}\| \left(\frac{\mathbf{v}}{\|\mathbf{v}\|} \right)$$

$$= \|\mathbf{u}\| (\cos \theta) \left(\frac{\mathbf{v}}{\|\mathbf{v}\|} \right)$$

=

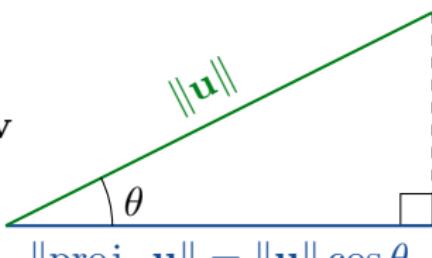


$$= \|\text{proj}_{\mathbf{v}} \mathbf{u}\| = \|\mathbf{u}\| \cos \theta$$

11.3 The Dot Product

Now

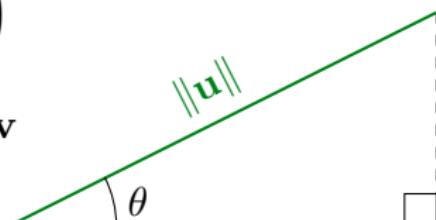
$$\begin{aligned}
 \text{proj}_{\mathbf{v}} \mathbf{u} &= (\text{length of } \text{proj}_{\mathbf{v}} \mathbf{u}) \left(\begin{array}{c} \text{a unit vector in} \\ \text{the same} \\ \text{direction as } \mathbf{v} \end{array} \right) \\
 &= \|\text{proj}_{\mathbf{v}} \mathbf{u}\| \left(\frac{\mathbf{v}}{\|\mathbf{v}\|} \right) \\
 &= \|\mathbf{u}\| (\cos \theta) \left(\frac{\mathbf{v}}{\|\mathbf{v}\|} \right) \\
 &= \left(\frac{\|\mathbf{u}\| \|\mathbf{v}\| \cos \theta}{\|\mathbf{v}\|^2} \right) \mathbf{v} \\
 &= \|\mathbf{proj}_{\mathbf{v}} \mathbf{u}\| = \|\mathbf{u}\| \cos \theta
 \end{aligned}$$



11.3 The Dot Product

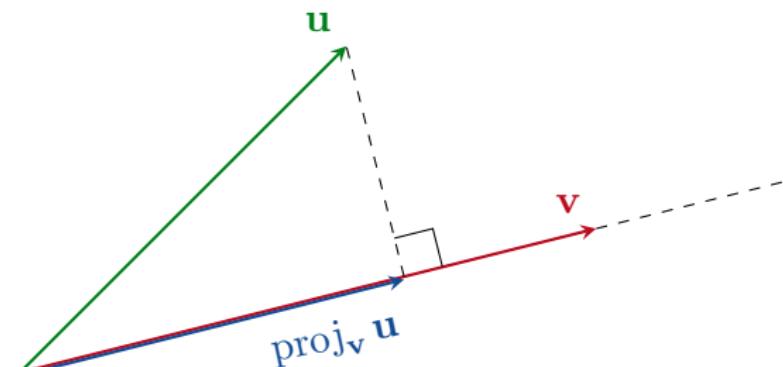
Now

$$\begin{aligned}
 \text{proj}_{\mathbf{v}} \mathbf{u} &= (\text{length of } \text{proj}_{\mathbf{v}} \mathbf{u}) \left(\begin{array}{c} \text{a unit vector in} \\ \text{the same} \\ \text{direction as } \mathbf{v} \end{array} \right) \\
 &= \|\text{proj}_{\mathbf{v}} \mathbf{u}\| \left(\frac{\mathbf{v}}{\|\mathbf{v}\|} \right) \\
 &= \|\mathbf{u}\| (\cos \theta) \left(\frac{\mathbf{v}}{\|\mathbf{v}\|} \right) \\
 &= \left(\frac{\|\mathbf{u}\| \|\mathbf{v}\| \cos \theta}{\|\mathbf{v}\|^2} \right) \mathbf{v} \\
 &= \left(\frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{v}\|^2} \right) \mathbf{v}.
 \end{aligned}$$



Since this is an important formula, we write it as a theorem.

11.3 The Dot Product



Theorem

The vector projection of \mathbf{u} onto \mathbf{v} is

$$\text{proj}_{\mathbf{v}} \mathbf{u} = \left(\frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{v}\|^2} \right) \mathbf{v}.$$

11.3 The Dot Product

$$\text{proj}_{\mathbf{v}} \mathbf{u} = \left(\frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{v}\|^2} \right) \mathbf{v}$$



Example

Find the vector projection of $\mathbf{u} = 6\mathbf{i} + 3\mathbf{j} + 2\mathbf{k}$ onto $\mathbf{v} = \mathbf{i} - 2\mathbf{j} - 2\mathbf{k}$.

11.3 The Dot Product

$$\text{proj}_{\mathbf{v}} \mathbf{u} = \left(\frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{v}\|^2} \right) \mathbf{v}$$



Example

Find the vector projection of $\mathbf{u} = 6\mathbf{i} + 3\mathbf{j} + 2\mathbf{k}$ onto $\mathbf{v} = \mathbf{i} - 2\mathbf{j} - 2\mathbf{k}$.

$$\begin{aligned}\text{proj}_{\mathbf{v}} \mathbf{u} &= \left(\frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{v}\|^2} \right) \mathbf{v} = \left(\frac{6 - 6 - 4}{1 + 4 + 4} \right) (\mathbf{i} - 2\mathbf{j} - 2\mathbf{k}) \\ &= -\frac{4}{9}\mathbf{i} + \frac{8}{9}\mathbf{j} + \frac{8}{9}\mathbf{k}.\end{aligned}$$

11.3 The Dot Product

$$\text{proj}_{\mathbf{v}} \mathbf{u} = \left(\frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{v}\|^2} \right) \mathbf{v}$$



Example

Find the vector projection of $\mathbf{F} = 5\mathbf{i} + 2\mathbf{j}$ onto $\mathbf{v} = \mathbf{i} - 3\mathbf{j}$.

$$\begin{aligned}\text{proj}_{\mathbf{v}} \mathbf{F} &= \left(\frac{\mathbf{F} \cdot \mathbf{v}}{\|\mathbf{v}\|^2} \right) \mathbf{v} = \left(\frac{5 - 6}{1 + 9} \right) (\mathbf{i} - 3\mathbf{j}) \\ &= -\frac{1}{10}\mathbf{i} + \frac{3}{10}\mathbf{j}.\end{aligned}$$

11.3 The Dot Product

$$\text{proj}_{\mathbf{v}} \mathbf{u} = \left(\frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{v}\|^2} \right) \mathbf{v}$$



Example

Verify that the vector $\mathbf{u} - \text{proj}_{\mathbf{v}} \mathbf{u}$ is orthogonal to $\text{proj}_{\mathbf{v}} \mathbf{u}$.

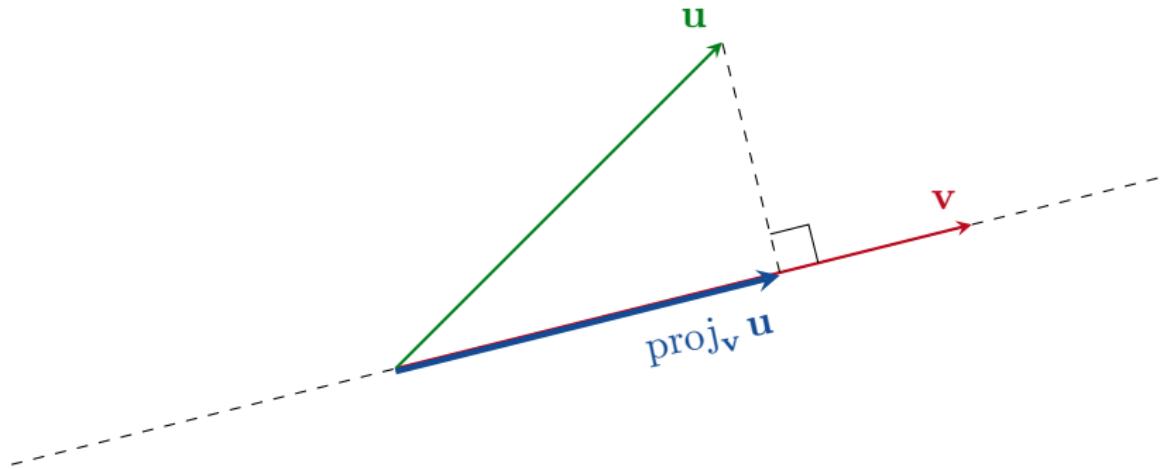
11.3 The Dot Product

$$\text{proj}_v u = \left(\frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{v}\|^2} \right) \mathbf{v}$$



Example

Verify that the vector $\mathbf{u} - \text{proj}_v \mathbf{u}$ is orthogonal to $\text{proj}_v \mathbf{u}$.



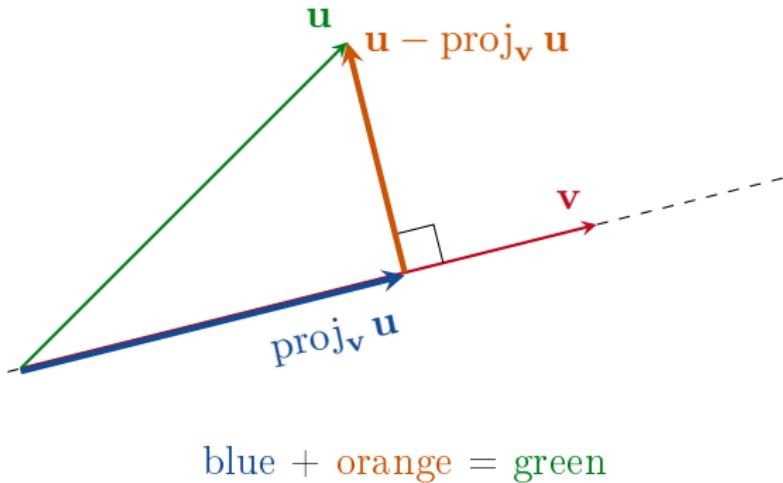
11.3 The Dot Product

$$\text{proj}_{\mathbf{v}} \mathbf{u} = \left(\frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{v}\|^2} \right) \mathbf{v}$$



Example

Verify that the vector $\mathbf{u} - \text{proj}_{\mathbf{v}} \mathbf{u}$ is orthogonal to $\text{proj}_{\mathbf{v}} \mathbf{u}$.



$$\text{blue} + \text{orange} = \text{green}$$

11.3 The Dot Product

$$\text{proj}_{\mathbf{v}} \mathbf{u} = \left(\frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{v}\|^2} \right) \mathbf{v}$$



Clearly

$$\text{proj}_{\mathbf{v}} \mathbf{u} = \left(\frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{v}\|^2} \right) \mathbf{v} = (\text{a number}) \mathbf{v}$$

is parallel to \mathbf{v} .

11.3 The Dot Product

$$\text{proj}_{\mathbf{v}} \mathbf{u} = \left(\frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{v}\|^2} \right) \mathbf{v}$$



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is parallel to \mathbf{v} . So it is enough to show that $\mathbf{u} - \text{proj}_{\mathbf{v}} \mathbf{u}$ is orthogonal to \mathbf{v} .

11.3 The Dot Product

$$\text{proj}_{\mathbf{v}} \mathbf{u} = \left(\frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{v}\|^2} \right) \mathbf{v}$$



Clearly

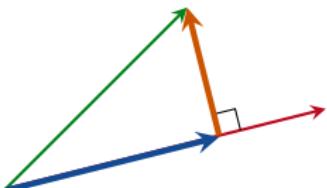
$$\text{proj}_{\mathbf{v}} \mathbf{u} = \left(\frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{v}\|^2} \right) \mathbf{v} = (\text{a number}) \mathbf{v}$$

is parallel to \mathbf{v} . So it is enough to show that $\mathbf{u} - \text{proj}_{\mathbf{v}} \mathbf{u}$ is orthogonal to \mathbf{v} .

Since

$$(\mathbf{u} - \text{proj}_{\mathbf{v}} \mathbf{u}) \cdot \mathbf{v} =$$

$$\begin{aligned} &= \\ &= \\ &= 0 \end{aligned}$$



we have shown that $\mathbf{u} - \text{proj}_{\mathbf{v}} \mathbf{u}$ is orthogonal to \mathbf{v} .

11.3 The Dot Product

$$\text{proj}_{\mathbf{v}} \mathbf{u} = \left(\frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{v}\|^2} \right) \mathbf{v}$$



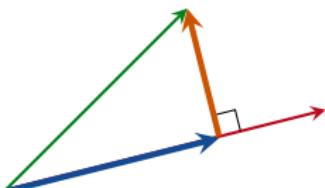
Clearly

$$\text{proj}_{\mathbf{v}} \mathbf{u} = \left(\frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{v}\|^2} \right) \mathbf{v} = (\text{a number}) \mathbf{v}$$

is parallel to \mathbf{v} . So it is enough to show that $\mathbf{u} - \text{proj}_{\mathbf{v}} \mathbf{u}$ is orthogonal to \mathbf{v} .

Since

$$\begin{aligned} (\mathbf{u} - \text{proj}_{\mathbf{v}} \mathbf{u}) \cdot \mathbf{v} &= \mathbf{u} \cdot \mathbf{v} - \left(\frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{v}\|^2} \right) \mathbf{v} \cdot \mathbf{v} \\ &= \mathbf{u} \cdot \mathbf{v} - \frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{v}\|^2} \|\mathbf{v}\|^2 \\ &= \mathbf{u} \cdot \mathbf{v} - \mathbf{u} \cdot \mathbf{v} \\ &= 0 \end{aligned}$$



we have shown that $\mathbf{u} - \text{proj}_{\mathbf{v}} \mathbf{u}$ is orthogonal to \mathbf{v} .



Next Time

- 11.4 The Cross Product
- 11.5 Lines and Planes in Space