Week 14

- 30. Antiderivatives
- 31. Integration
- 32. The Definite Integral

30 Antiderivatives

Definition

F is an *antiderivative* of f on an interval I if $F'(x) = f(x)$ for all $x \in I$.

Example

$2x$ is the derivative of x^2.
x^2 is an antiderivative of $2x$.

Example

If $g(x) = \cos x$, then an antiderivative of g is

$$G(x) = \sin x$$

because

$$G'(x) = \frac{d}{dx}(\sin x) = \cos x = g(x).$$

Example

If $h(x) = 2x + \cos x$, then $H(x) = x^2 + \sin x$ is an antiderivative of $h(x)$.

Remark

$F(x) = x^2$ is not the only antiderivative of $f(x) = 2x$.

$x^2 + 1$ is an antiderivative of $2x$ because $\frac{d}{dx}\left(x^2+1\right) = 2x$.

$x^2 + 5$ is an antiderivative of $2x$ because $\frac{d}{dx}\left(x^2+5\right) = 2x$.

$x^2 - 1234$ is an antiderivative of $2x$ because $\frac{d}{dx}\left(x^2-1234\right) = 2x$.

Theorem

If F is an antiderivative of f on I, then the general antiderivative of f is

$$F(x) + C$$

where C is a constant.

Example

Find an antiderivative of $f(x) = 3x^2$ that satisfies $F(1) = -1$.

solution: x^3 is an antiderivative of f because $\frac{d}{dx}\left(x^3\right) = 3x^2$. So the general antiderivative of f is

$$F(x) = x^3 + C.$$

Then we calculate that

$$-1 = F(1) = 1^3 + C = 1 + C \implies C = -2.$$

Therefore $F(x) = x^3 - 2$.

30. Antiderivatives

function $f(x)$	derivative $f'(x)$
x^n	nx^{n-1}
$\sin kx$	$k\cos kx$
$\cos kx$	$-k\sin kx$
e^{kx}	ke^{kx}

30. Antiderivatives

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x^n	nx^{n-1}
$\sin kx$	$k\cos kx$
$\cos kx$	$-k\sin kx$
e^{kx}	ke^{kx}

function $f(x)$	general antiderivative $F(x)$
$x^n \ (n \neq -1)$	
$\sin kx$	
$\cos kx$	
e^{kx}	

function $f(x)$	derivative $f'(x)$
x^n	nx^{n-1}
$\sin kx$	$k\cos kx$
$\cos kx$	$-k\sin kx$
e^{kx}	ke^{kx}

function $f(x)$	general antiderivative $F(x)$
$x^n \;(n \neq -1)$	$\frac{x^{n+1}}{n+1} + C$
$\sin kx$	
$\cos kx$	
e^{kx}	

function $f(x)$	derivative $f'(x)$
x^n	nx^{n-1}
$\sin kx$	$k\cos kx$
$\cos kx$	$-k\sin kx$
e^{kx}	ke^{kx}

function $f(x)$	general antiderivative $F(x)$
$x^n \ (n \neq -1)$	$\frac{x^{n+1}}{n+1} + C$
$\sin kx$	$-\frac{1}{k}\cos kx + C$
$\cos kx$	
e^{kx}	

function $f(x)$	derivative $f'(x)$
x^n	nx^{n-1}
$\sin kx$	$k\cos kx$
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$\sin kx$	$-\frac{1}{k}\cos kx + C$
$\cos kx$	$\frac{1}{k}\sin kx + C$
e^{kx}	$\frac{1}{k}e^{kx} + C$

The Sum Rule and the Constant Multiple Rule

Suppose that

- F is an antiderivative of f;
- G is an antiderivative of g;
- $k \in \mathbb{R}$.

The Sum Rule: The general antiderivative of $f + g$ is

$$F(x) + G(x) + C.$$

The Constant Multiple Rule: The general antiderivative of kf is

$$kF(x) + C.$$

Example

Find the general antiderivative of $f(x) = \frac{3}{\sqrt{x}} + \sin 2x$.

solution: We have $f = 3g + h$ where $g(x) = x^{-\frac{1}{2}}$ and $h(x) = \sin 2x$. An antiderivative of g is

$$G(x) = \frac{x^{-\frac{1}{2}+1}}{-\frac{1}{2}+1} = \frac{x^{\frac{1}{2}}}{\frac{1}{2}} = 2\sqrt{x}.$$

An antiderivative of h is

$$H(x) = -\frac{1}{2}\cos 2x.$$

Therefore the general antiderivative of f is

$$F(x) = 6\sqrt{x} - \frac{1}{2}\cos 2x + C.$$

Definition

The general antiderivative of f is also called the *indefinite integral* of f with respect to x, and is denoted by

$$\int f(x)\ dx.$$

30. Antiderivatives

the integral sign
integral işareti

x is the variable of integration
x ise integral değişkeni olarak tanımlanır

$$\int \underbrace{f(x)}_{\substack{\text{the integrand} \\ \text{integralin integrandı}}} \, dx$$

Example

$$\int 2x \; dx = x^2 + C$$

$$\int \cos x \; dx = \sin x + C$$

$$\int (2x + \cos x) \; dx = x^2 + \sin x + C$$

Example

Calculate $\int (x^2 - 2x + 5)\ dx$.

solution 1. Since $\frac{d}{dx}\left(\frac{x^3}{3} - x^2 + 5x\right) = x^2 - 2x + 5$ we have that

$$\int (x^2 - 2x + 5)\ dx = \frac{x^3}{3} - x^2 + 5x + C.$$

solution 2.

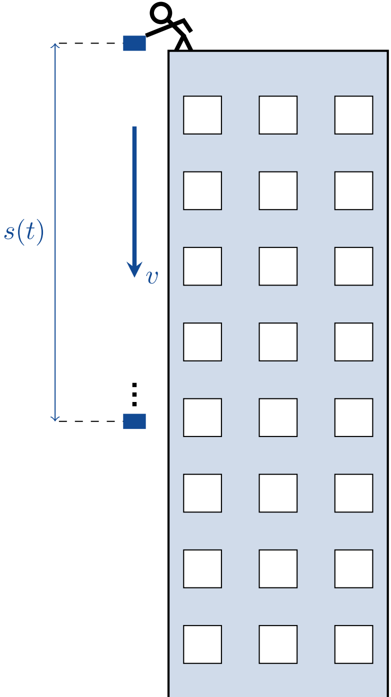
$$\begin{aligned}\int (x^2 - 2x + 5)\ dx &= \int x^2\ dx - \int 2x\ dx + \int 5\ dx \\ &= \left(\frac{x^3}{3} + C_1\right) - \left(x^2 + C_2\right) + \left(5x + C_3\right) \\ &= \left(\frac{x^3}{3} - x^2 + 5x\right) + \left(C_1 - C_2 + C_3\right).\end{aligned}$$

Because we only need one constant, we can define $C := C_1 - C_2 + C_3$. Therefore

$$\int (x^2 - 2x + 5)\ dx = \frac{x^3}{3} - x^2 + 5x + C.$$

Example

You drop a box off the top of a tall building. The acceleration due to gravity is $9.8\,\mathrm{ms}^{-2}$. You can ignore air resistance. How far does the box fall in 5 seconds?


$s(t)$
v

solution: The acceleration is

$$a(t) = 9.8\text{ms}^{-2}$$

downwards. Since

$$\text{acceleration} = \frac{d}{dt}(\text{velocity}),$$

the velocity is an antiderivative of the acceleration. Therefore the velocity is

$$v(t) = 9.8t + C \text{ ms}^{-1}.$$

You let go of the box at time $t = 0$. So $v(0) = 0$. Thus $C = 0$. Hence

$$v(t) = 9.8t \text{ ms}^{-1}.$$

Now velocity $= \frac{d}{dt}(\text{position})$. So the distance fallen is an antiderivative of velocity. Hence

$$s(t) = 4.9t^2 + \tilde{C} \text{ m.}$$

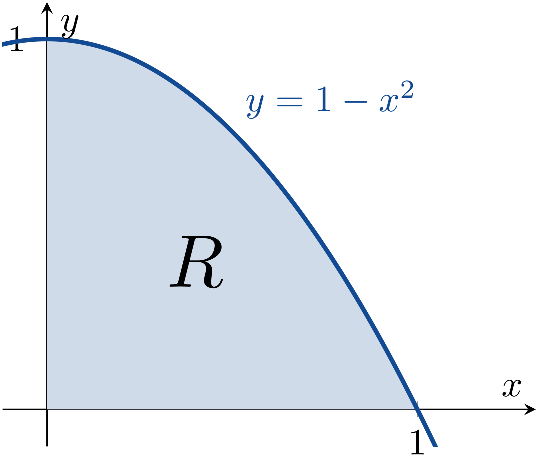
Because you let go of the box at time $t = 0$, we have $s(0) = 0$. Thus $\tilde{C} = 0$. Therefore

$$s(t) = 4.9t^2 \text{ m.}$$

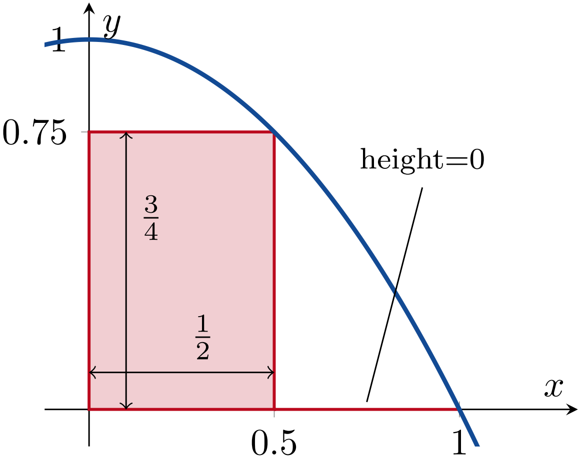
After 5 seconds, the box has fallen

$$s(5) = 4.9 \times 25 = 122.5 \text{ metres.}$$

Integration



Question: What is the area of R?

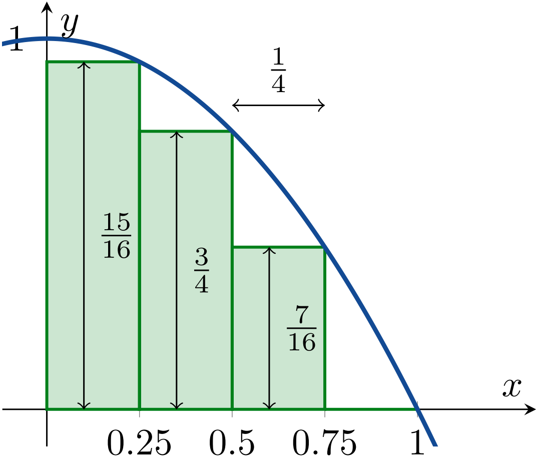


We can use two rectangles to approximate the area of R.

Then we have

$$\begin{aligned}\text{area of } R &\approx \text{area of 2 rectangles} \\ &= \left(\frac{3}{4} \times \frac{1}{2}\right) + \left(0 \times \frac{1}{2}\right) \\ &= \frac{3}{8} = 0.375.\end{aligned}$$

Can we do better than this? Yes! We could use more rectangles.

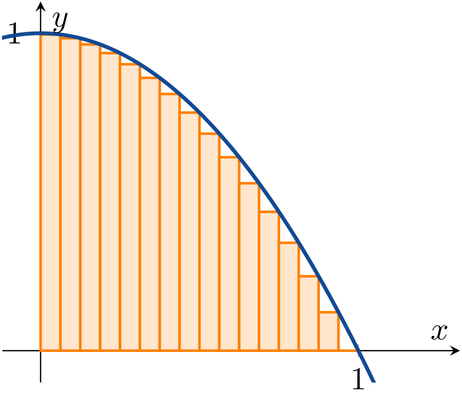


y
1
$\frac{1}{4}$
$\frac{15}{16}$
$\frac{3}{4}$
$\frac{7}{16}$
x
0.25
0.5
0.75
1

We can say that

$$
\begin{aligned}
\text{area of } R &\approx \text{area of 4 rectangles} \\
&= \left(\frac{15}{16} \times \frac{1}{4}\right) + \left(\frac{3}{4} \times \frac{1}{4}\right) \\
&\quad + \left(\frac{7}{16} \times \frac{1}{4}\right) + \left(0 \times \frac{1}{4}\right) \\
&= \frac{17}{32} = 0.53125.
\end{aligned}
$$

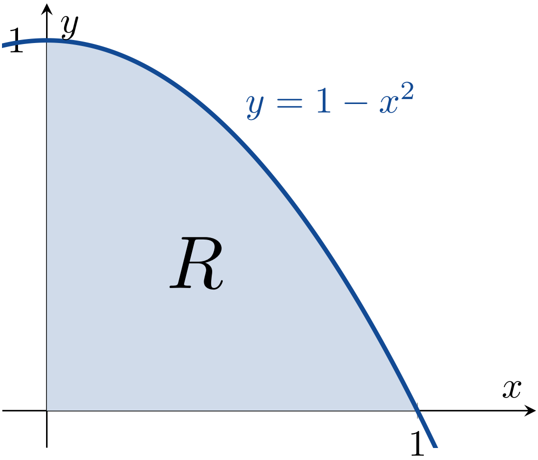
Every time we increase the number of rectangles, the total area of the rectangles gets closer and closer to the area of R.



area of $R \approx$ area of 16 rectangles

$$= 0.63476.$$

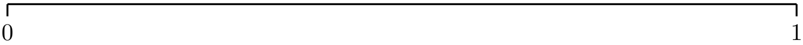
Limits of Finite Sums



31. Integration

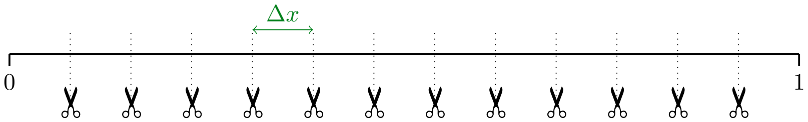
STEP 1: We will cut $[0,1]$ into n pieces of width

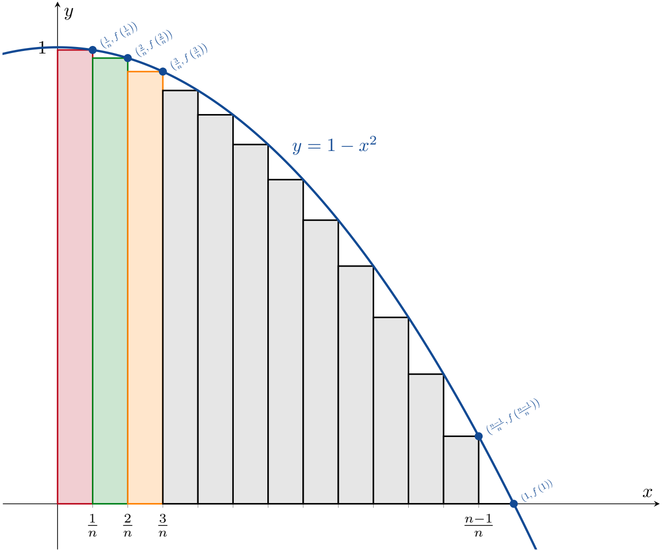
$$\Delta x = \frac{1-0}{n} = \frac{1}{n}.$$



STEP 1: We will cut $[0,1]$ into n pieces of width

$$\Delta x = \frac{1-0}{n} = \frac{1}{n}.$$





STEP 2: We will use n rectangles to approximate the area of R.

STEP 3: Then we will take the limit as $n \to \infty$.

Let $f(x) = 1 - x^2$. Then

- the first rectangle has area $\frac{1}{n} f\left(\frac{1}{n}\right)$;
- the second rectangle has area $\frac{1}{n} f\left(\frac{2}{n}\right)$;
- the third rectangle has area $\frac{1}{n} f\left(\frac{3}{n}\right)$;

and so on.

The area of all n rectangles is

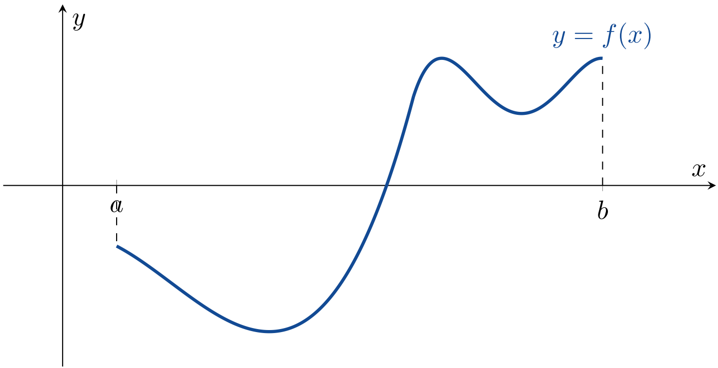
$$\begin{aligned}
\text{area} &= \sum_{k=1}^{n} (\text{area of the } k\text{th rectangle}) = \sum_{k=1}^{n} \frac{1}{n} f\left(\frac{k}{n}\right) \\
&= \sum_{k=1}^{n} \frac{1}{n}\left(1 - \left(\frac{k}{n}\right)^2\right) = \sum_{k=1}^{n}\left(\frac{1}{n} - \frac{k^2}{n^3}\right) \\
&= \sum_{k=1}^{n} \frac{1}{n} - \sum_{k=1}^{n} \frac{k^2}{n^3} \\
&= n\left(\frac{1}{n}\right) - \frac{1}{n^3}\sum_{k=1}^{n} k^2 \\
&= 1 - \frac{1}{n^3}\left(\frac{n(n+1)(2n+1)}{6}\right) \\
&= 1 - \frac{2n^2+3n+1}{6n^2}.
\end{aligned}$$

Taking the limit gives

$$\begin{aligned}\lim_{n\to\infty}\left(\sum_{k=1}^{n}\frac{1}{n}f\left(\frac{k}{n}\right)\right) &= \lim_{n\to\infty}\left(1-\frac{2n^2+3n+1}{6n^2}\right)\\ &= 1-\frac{2}{6}=\frac{2}{3}.\end{aligned}$$

Therefore the area of R is $\frac{2}{3}$.

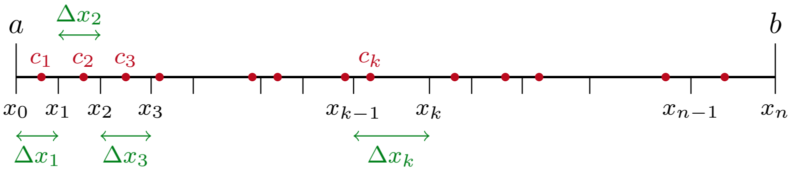
Riemann Sums



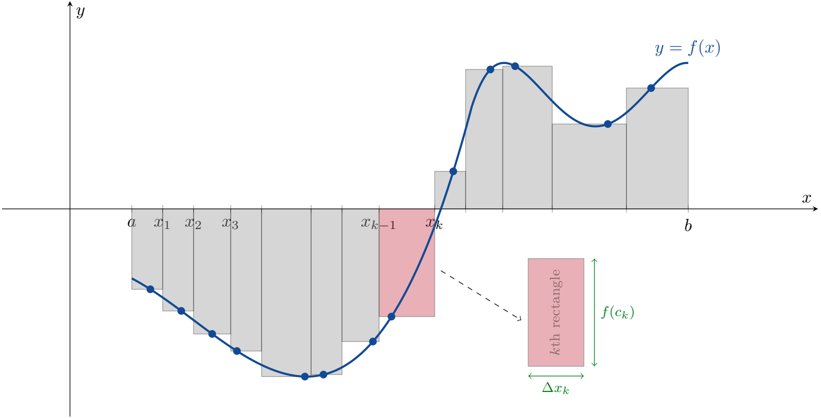
Now let $f[a, b] \to \mathbb{R}$ be a function. We will cut $[a, b]$ into n subintervals (the pieces don't have to all be the same size).

In each subinterval we will choose one point $c_k \in [x_{k-1}, x_k]$.

The width of each subinterval is $\Delta x_k = x_k - x_{k-1}$.



On each subinterval $[x_{k-1}, x_k]$, we draw a rectangle of width Δx_k and height $f(c_k)$.



Note that if $f(c_k) < 0$, then the rectangle on $[x_{k-1}, x_k]$ will have 'negative area' – this is ok.

The total of the n rectangles is

$$\sum_{k=1}^{n} f(c_k)\Delta x_k.$$

This is called a *Riemann Sum for f on* $[a, b]$.

Then we want to take the limit as $n \to \infty$ (or more precisely, we want to take the limit as $\max\{\Delta x_1, \Delta x_2, \ldots, \Delta x_n\} \to 0$). Sometimes this limit exists, sometimes this limit does not exist.

32 The Definite Integral

Definition

<u>If</u> the limit

$$\lim_{n\to\infty} \sum_{k=1}^{n} f(c_k)\Delta x_k$$

exists, then it is called the *definite integral of f over* $[a, b]$. We write

$$\int_a^b f(x)\ dx = \lim_{n\to\infty} \sum_{k=1}^{n} f(c_k)\Delta x_k$$

if the limit exists.

32. The Definite Integral

upper limit of integration
integralin üst sınırı

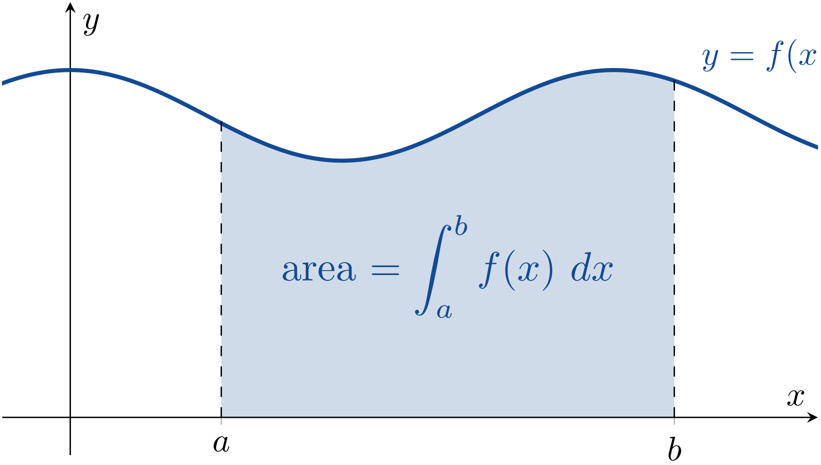
x is the variable of integration
x, integral değişkenidir

integral sign
integral işareti

$$\int_a^b \underbrace{f(x)}_{\substack{\text{the integrand}\\ \text{integralin integrandı}}} \, dx$$

lower limit of integration
integralin alt sınırı

“the integral of f from a to b”



y
$y = f(x$
$\text{area} = \int_a^b f(x)\ dx$
x
a
b

Definition

If $\int_a^b f(x)\,dx$ exists, then we say that f *is integrable on* $[a,b]$.

Example

$f(x) = 1 - x^2$ is integrable on $[0, 1]$ and $\int_0^1 (1 - x^2)\ dx = \frac{2}{3}$.

Remark

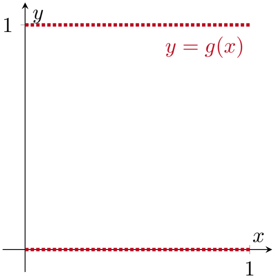
$$\int_a^b f(x)\,dx = \int_a^b f(u)\,du = \int_a^b f(t)\,dt$$

It doesn't matter which letter we use for the *dummy variable*.

Theorem

If f is continuous on $[a, b]$, then f is integrable on $[a, b]$.

If f has finitely many jump discontinuities but is otherwise continuous on $[a, b]$, then f is integrable on $[a, b]$.



Example

Define a function $g : [0,1] \to \mathbb{R}$ by

$$g(x) = \begin{cases} 1 & x \in \mathbb{Q} \\ 0 & x \notin \mathbb{Q}. \end{cases}$$

This function is not integrable on $[0,1]$.

Theorem

Suppose that f and g are integrable. Let k be a number.

Theorem

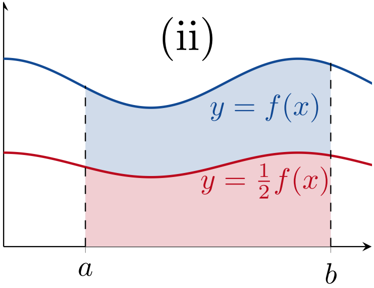
Suppose that f and g are integrable. Let k be a number. Then

1. $\displaystyle\int_b^a f(x)\ dx = -\int_a^b f(x)\ dx;$

Theorem

Suppose that f and g are integrable. Let k be a number. Then

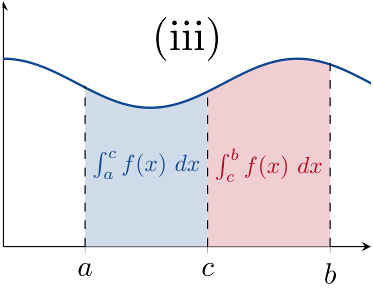
2. $\displaystyle\int_a^b kf(x)\ dx = k\int_a^b f(x)\ dx;$



Theorem

Suppose that f and g are integrable. Let k be a number. Then

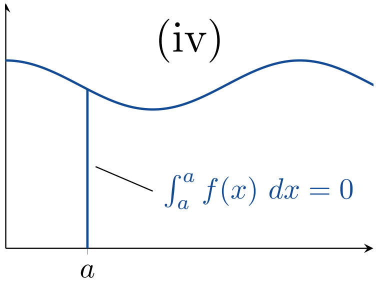
3. $$\int_a^c f(x)\ dx + \int_c^b f(x)\ dx = \int_a^b f(x)\ dx$$



Theorem

Suppose that f and g are integrable. Let k be a number. Then

4. $\displaystyle\int_a^a f(x)\ dx = 0;$



Theorem

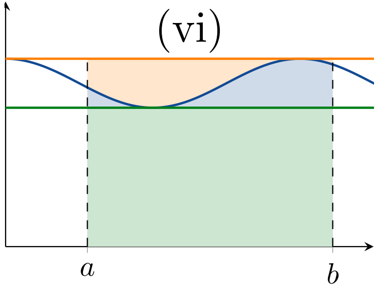
Suppose that f and g are integrable. Let k be a number. Then

5. $$\int_a^b \big(f(x)+g(x)\big)\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx;$$

Theorem

Suppose that f and g are integrable. Let k be a number. Then

6. $(b-a)\min f \leq \int_a^b f(x)\ dx \leq (b-a)\max f$;

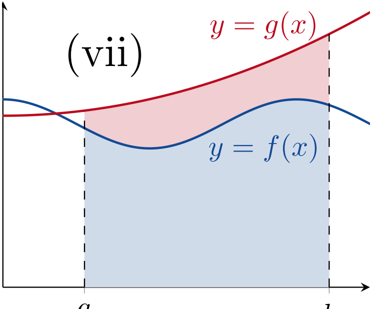


Theorem

Suppose that f and g are integrable. Let k be a number. Then

7. *if $f(x) \leq g(x)$ on $[a, b]$, then*

$$\int_a^b f(x)\ dx \leq \int_a^b g(x)\ dx;$$



Theorem

Suppose that f and g are integrable. Let k be a number. Then

8. *if $g(x) \geq 0$ on $[a, b]$, then*

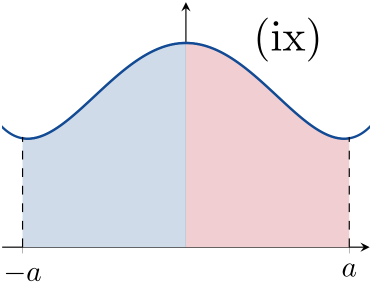
$$\int_a^b g(x)\ dx \geq 0;$$

Theorem

Suppose that f and g are integrable. Let k be a number. Then

9. *if f is an even function, then*

$$\int_{-a}^{a} f(x)\ dx = 2\int_{0}^{a} f(x)\ dx;$$

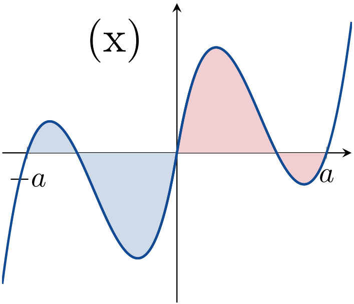


Theorem

Suppose that f and g are integrable. Let k be a number. Then

10 *if f is an odd function, then*

$$\int_{-a}^{a} f(x)\, dx = 0.$$



Example

Suppose that $\displaystyle\int_{-1}^{1} f(x)\ dx = 5$, $\displaystyle\int_{1}^{4} f(x)\ dx = -2$ and $\displaystyle\int_{-1}^{1} h(x)\ dx = 7$. Then

$$\int_{4}^{1} f(x)\ dx = -\int_{1}^{4} f(x)\ dx = 2,$$

$$\begin{aligned}\int_{-1}^{1} \big(2f(x) + 3h(x)\big)\ dx &= 2\int_{-1}^{1} f(x)\ dx + 3\int_{-1}^{1} h(x)\ dx \\ &= 2(5) + 3(7) = 31\end{aligned}$$

and

$$\begin{aligned}\int_{-1}^{4} f(x)\ dx &= \int_{-1}^{1} f(x)\ dx + \int_{1}^{4} f(x)\ dx \\ &= 5 + (-2) = 3.\end{aligned}$$

Example

Show that $\int_0^1 \sqrt{1+\cos x}\, dx \leq \sqrt{2}$.

solution: The maximum value of $\sqrt{1+\cos x}$ on $[0,1]$ is $\sqrt{1+1} = \sqrt{2}$. Therefore

$$\int_0^1 \sqrt{1+\cos x}\, dx \leq (1-0) \max \sqrt{1+\cos x} = 1 \times \sqrt{2}.$$

Example

Calculate $\int_{-2}^{2} (x^3 + x) \; dx$.

solution: Because $(x^3 + x)$ is an odd function, we have that

$$\int_{-2}^{2} (x^3 + x) \; dx = 0.$$

Example

Calculate $\displaystyle\int_{-1}^{1} (1 - x^2)\ dx$.

solution: Because $(1 - x^2)$ is an even function, we have that

$$\int_{-1}^{1} (1 - x^2)\ dx = 2 \int_{0}^{1} (1 - x^2)\ dx = 2 \times \frac{2}{3} = \frac{4}{3}.$$

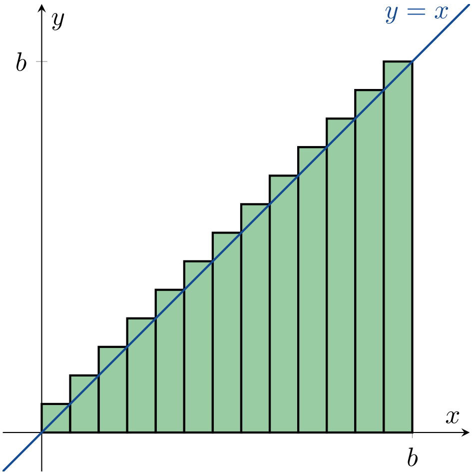
Example

Calculate $\int_0^b x\,dx$ for $b > 0$.

solution 1: We will use a Riemann Sum. First we cut $[0, b]$ in to n pieces using

$$0 < \frac{b}{n} < \frac{2b}{n} < \frac{3b}{n} < \ldots < \frac{(n-1)b}{n} < b$$

and $c_k = \frac{kb}{n}$. Note that $\Delta x_k = \frac{b}{n}$ for all k.



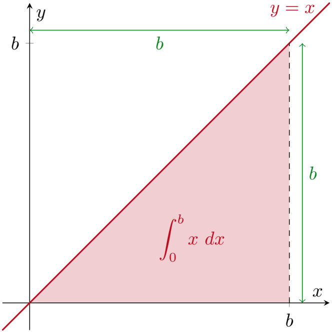
y
$y = x$
b
x
b

Then

$$\sum_{k=1}^{n} f(c_k)\Delta x_k = \sum_{k=1}^{n} \frac{kb}{n}\frac{b}{n} = \frac{b^2}{n^2}\sum_{k=1}^{n} k$$
$$= \frac{b^2}{n^2}\left(\frac{n(n+1)}{2}\right) = \frac{b^2}{2}\left(1+\frac{1}{n}\right).$$

Then

$$\int_0^b x\ dx = \lim_{n\to\infty} \sum_{k=1}^{n} f(c_k)\Delta x_k$$
$$= \lim_{n\to\infty} \frac{b^2}{2}\left(1+\frac{1}{n}\right) = \frac{b^2}{2}.$$



solution 2: Alternately, we can look at the triangle above and say that

$$\int_0^b x\ dx = \text{area of a triangle} = \frac{1}{2} \times b \times b = \frac{b^2}{2}.$$

Example

$$\begin{aligned}\int_a^b x\ dx &= \int_a^0 x\ dx + \int_0^b x\ dx \\ &= -\int_0^a x\ dx + \int_0^b x\ dx \\ &= -\frac{a^2}{2} + \frac{b^2}{2} \\ &= \frac{b^2}{2} - \frac{a^2}{2}.\end{aligned}$$

Next Week

- 33. The Fundamental Theorem of Calculus
- 34. The Substitution Method
- 35. Area Between Curves