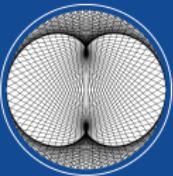


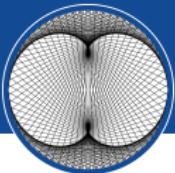
Lecture 10

- 9.2 Infinite Series
- 9.3 The Integral Test
- 9.4 Comparison Tests
- 9.5 Absolute Convergence; The Ratio and Root Tests



Infinite Series

9.2 Infinite Series



Let $(a_n)_{n=1}^{\infty}$ be a sequence:

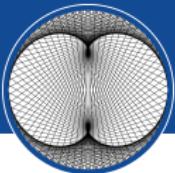
$$a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8, a_9, a_{10}, a_{11}, a_{12}, a_{13}, a_{14}, a_{15}, a_{16}, \dots$$

Then

$$\sum_{k=1}^{\infty} a_k = a_1 + a_2 + a_3 + a_4 + a_5 + a_6 + a_7 + a_8 + a_9 + a_{10} + a_{11} + \dots$$

is a *series*.

9.2 Infinite Series



Let $(a_n)_{n=1}^{\infty}$ be a sequence:

$$a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8, a_9, a_{10}, a_{11}, a_{12}, a_{13}, a_{14}, a_{15}, a_{16}, \dots$$

Then

$$\sum_{k=1}^{\infty} a_k = a_1 + a_2 + a_3 + a_4 + a_5 + a_6 + a_7 + a_8 + a_9 + a_{10} + a_{11} + \dots$$

is a *series*.

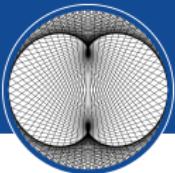
Definition

Let (a_n) be a sequence of real numbers. Let

$$s_n := \sum_{k=1}^n a_k = a_1 + a_2 + a_3 + a_4 + a_5 + a_6 + a_7 + \dots + a_n.$$

Then s_n is called a *partial sum* of the series $\sum_{k=1}^{\infty} a_k$.

9.2 Infinite Series



Definition

We say that $\sum_{k=1}^{\infty} a_k$ converges iff (s_n) converges.

Definition

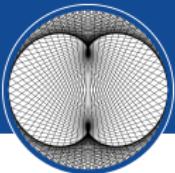
If $s_n \rightarrow s$ as $n \rightarrow \infty$, we say that s is the *sum of the series* and we write

$$\sum_{k=1}^{\infty} a_k = s.$$

Definition

If $\sum_{k=1}^{\infty} a_k$ does not converge, then we say that $\sum_{k=1}^{\infty} a_k$ diverges.

9.2 Infinite Series



Remark

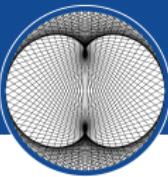
Sometimes the notation “ $\sum_{k=1}^{\infty} a_k$ ” means

$a_1 + a_2 + a_3 + a_4 + a_5 + a_6 + a_7 + \dots$ which might converge or might diverge. Sometimes the notation “ $\sum_{k=1}^{\infty} a_k$ ” means the sum of the series, i.e.

$$\sum_{k=1}^{\infty} a_k = \lim_{n \rightarrow \infty} \sum_{k=1}^n a_k = \lim_{n \rightarrow \infty} s_n = s.$$

You need to be able to understand what I mean every time I write “ $\sum_{k=1}^{\infty} a_k$ ”.

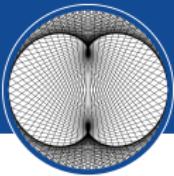
9.2 Infinite Series



Example

$$\sum_{k=1}^{\infty} 1 = 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1 + \dots$$

9.2 Infinite Series



Example

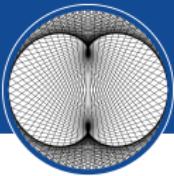
$$\sum_{k=1}^{\infty} 1 = 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1 + \dots$$

Note that

$$s_n = \sum_{k=1}^n 1 = 1 + 1 + \dots + 1 = n \rightarrow \infty$$

as $n \rightarrow \infty$.

9.2 Infinite Series



Example

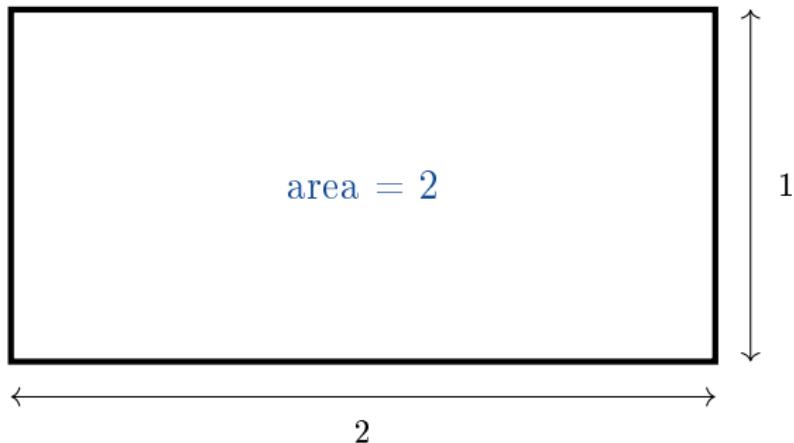
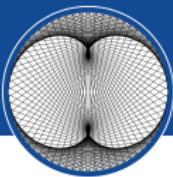
$$\sum_{k=1}^{\infty} 1 = 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1 + \dots$$

Note that

$$s_n = \sum_{k=1}^n 1 = 1 + 1 + \dots + 1 = n \rightarrow \infty$$

as $n \rightarrow \infty$. Therefore $\sum_{k=1}^{\infty} 1$ diverges.

9.2 Infinite Series

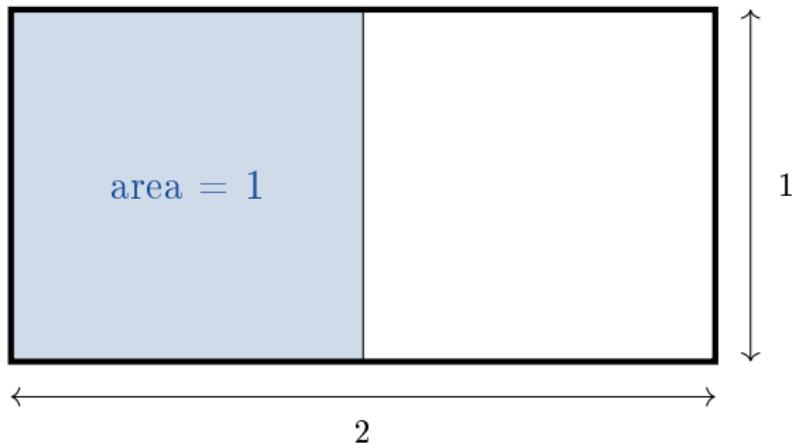
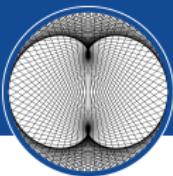


Example

Looking at the area of the rectangle above, we can see that

$$2 =$$

9.2 Infinite Series

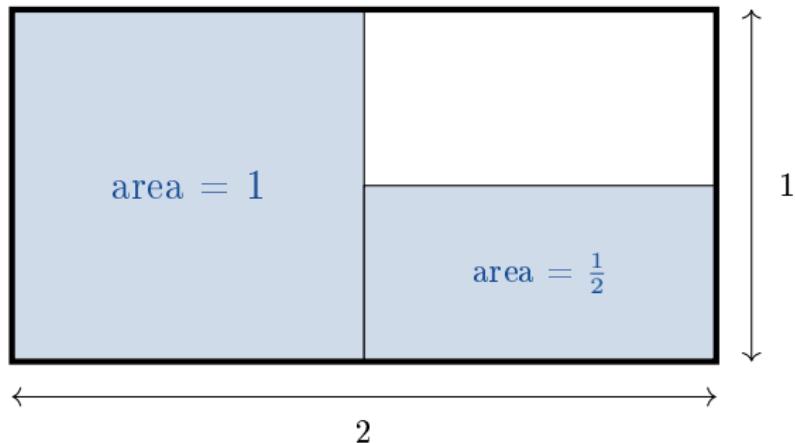
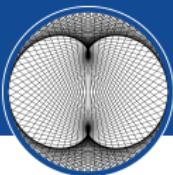


Example

Looking at the area of the rectangle above, we can see that

$$2 = 1 +$$

9.2 Infinite Series

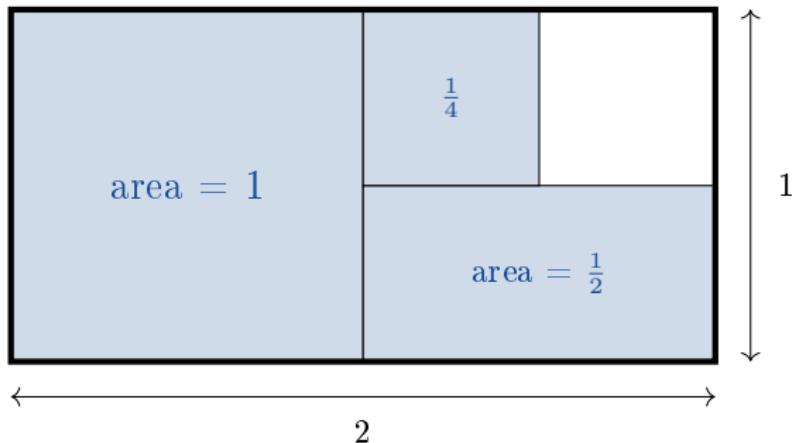
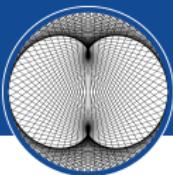


Example

Looking at the area of the rectangle above, we can see that

$$2 = 1 + \frac{1}{2} +$$

9.2 Infinite Series

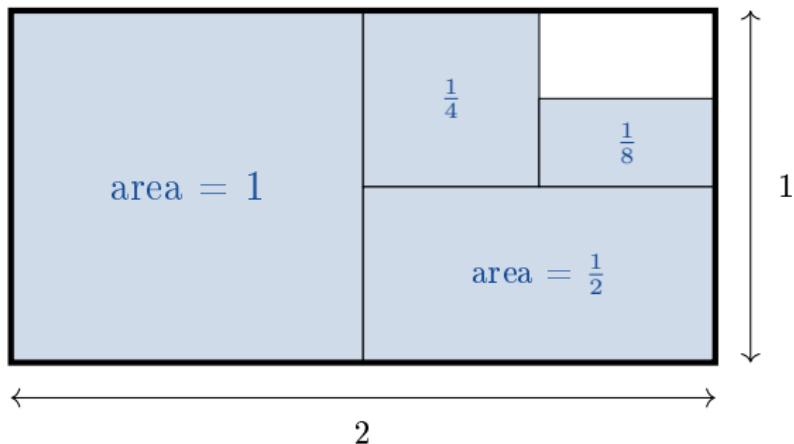
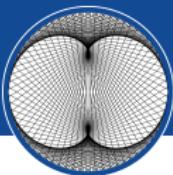


Example

Looking at the area of the rectangle above, we can see that

$$2 = 1 + \frac{1}{2} + \frac{1}{4} +$$

9.2 Infinite Series

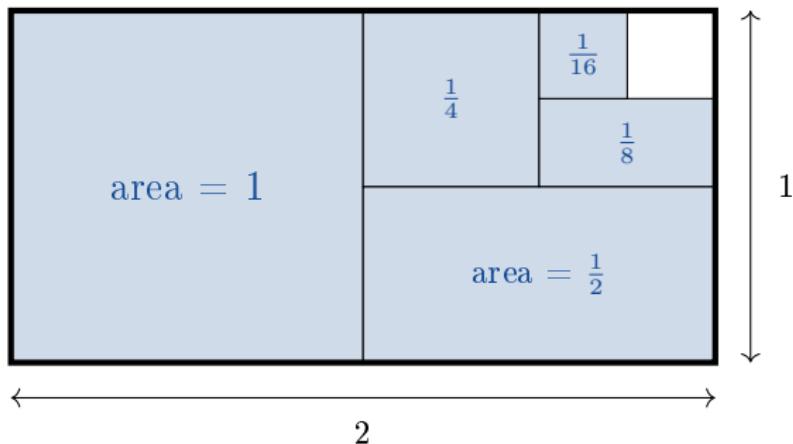
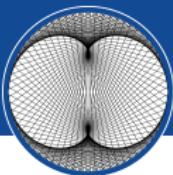


Example

Looking at the area of the rectangle above, we can see that

$$2 = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} +$$

9.2 Infinite Series

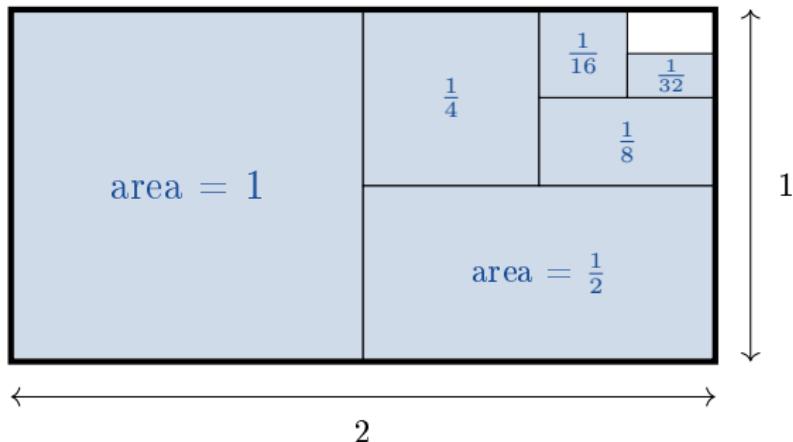
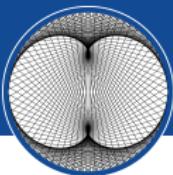


Example

Looking at the area of the rectangle above, we can see that

$$2 = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} +$$

9.2 Infinite Series

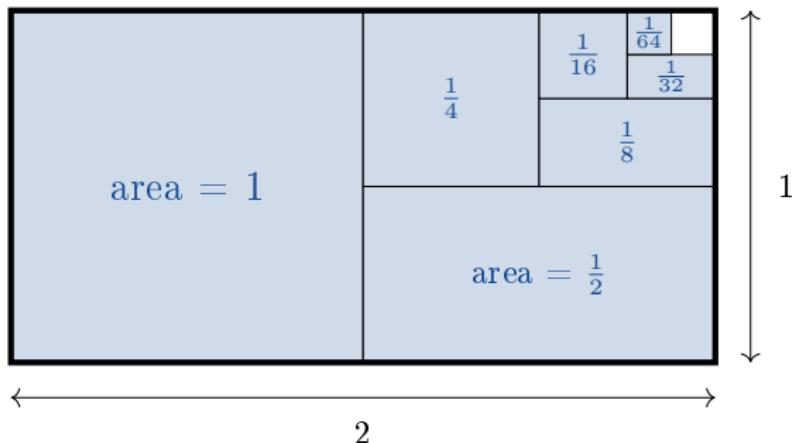
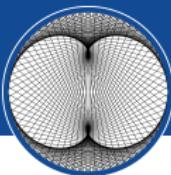


Example

Looking at the area of the rectangle above, we can see that

$$2 = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} +$$

9.2 Infinite Series

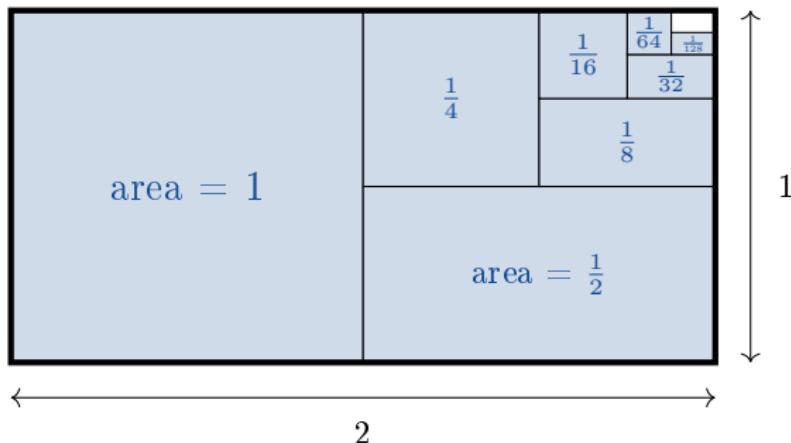
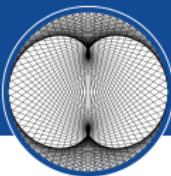


Example

Looking at the area of the rectangle above, we can see that

$$2 = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \frac{1}{64} +$$

9.2 Infinite Series

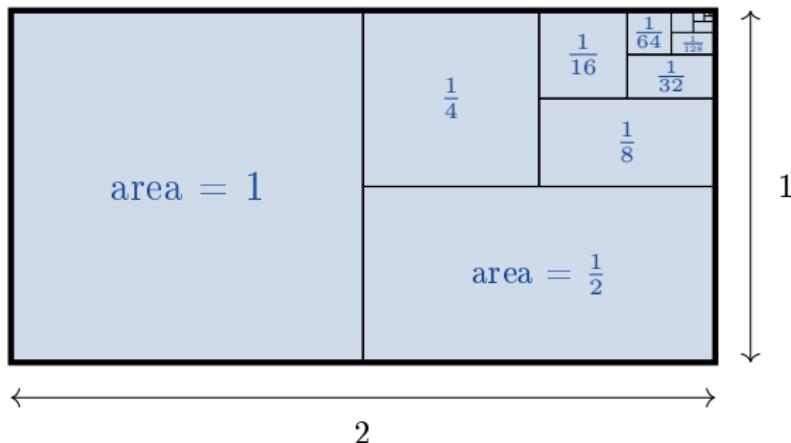
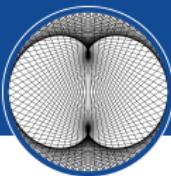


Example

Looking at the area of the rectangle above, we can see that

$$2 = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \frac{1}{64} + \frac{1}{128} +$$

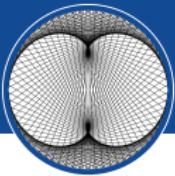
9.2 Infinite Series



Example

Looking at the area of the rectangle above, we can see that

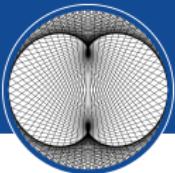
$$2 = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \frac{1}{64} + \frac{1}{128} + \frac{1}{256} + \dots$$



Geometric Series

Now consider the series $1 + x + x^2 + x^3 + x^4 + x^5 + x^6 + \dots$

for $x \in \mathbb{R}$.

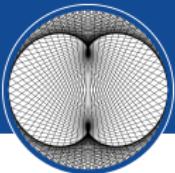


Geometric Series

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$$s_n = 1 + x + x^2 + x^3 + \dots + x^{n-1}.$$



Geometric Series

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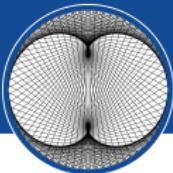
for $x \in \mathbb{R}$. The partial sums are

$$s_n = 1 + x + x^2 + x^3 + \dots + x^{n-1}.$$

Then we can see that

$$\begin{aligned}(1 - x)s_n &= s_n - xs_n \\&= (1 + x + x^2 + x^3 + x^4 + \dots + x^{n-1}) \\&\quad - (x + x^2 + x^3 + x^4 + \dots + x^{n-1} + x^n)\end{aligned}$$

9.2 Infinite Series



Geometric Series

Now consider the series $1 + x + x^2 + x^3 + x^4 + x^5 + x^6 + \dots$

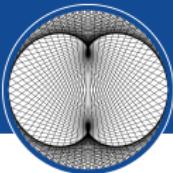
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Then we can see that

$$\begin{aligned}(1 - x)s_n &= s_n - xs_n \\&= (1 + \cancel{x} + \cancel{x^2} + \cancel{x^3} + \cancel{x^4} + \dots + \cancel{x^{n-1}}) \\&\quad - (\cancel{x} + \cancel{x^2} + \cancel{x^3} + \cancel{x^4} + \dots + \cancel{x^{n-1}} + x^n) \\&= 1 - x^n.\end{aligned}$$

9.2 Infinite Series



Geometric Series

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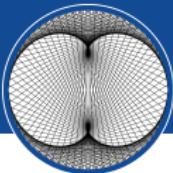
Then we can see that

$$\begin{aligned}(1 - x)s_n &= s_n - xs_n \\&= (1 + \cancel{x} + \cancel{x^2} + \cancel{x^3} + \cancel{x^4} + \dots + \cancel{x^{n-1}}) \\&\quad - (\cancel{x} + \cancel{x^2} + \cancel{x^3} + \cancel{x^4} + \dots + \cancel{x^{n-1}} + x^n) \\&= 1 - x^n.\end{aligned}$$

If $x \neq 1$, then

$$s_n = \frac{1 - x^n}{1 - x}.$$

9.2 Infinite Series

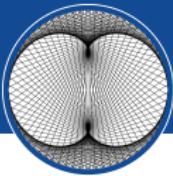


Now

- If $x = 1$, then $s_n = 1 + 1 + 1 + 1 + \dots + 1 = n \rightarrow \infty$ as $n \rightarrow \infty$. So

$$x = 1 \implies \sum_{k=0}^{\infty} x^k \text{ diverges.}$$

9.2 Infinite Series



Now

- If $x = 1$, then $s_n = 1 + 1 + 1 + 1 + \dots + 1 = n \rightarrow \infty$ as $n \rightarrow \infty$. So

$$x = 1 \implies \sum_{k=0}^{\infty} x^k \text{ diverges.}$$

- If $|x| < 1$, then $s_n = \frac{1-x^n}{1-x} \rightarrow \frac{1}{1-x}$ as $n \rightarrow \infty$. So

$$|x| < 1 \implies \sum_{k=0}^{\infty} x^k \text{ converges and } \sum_{k=0}^{\infty} x^k = \frac{1}{1-x}.$$

9.2 Infinite Series



- If $x = -1$, then $s_n = \frac{1 - (-1)^n}{2}$ and (s_n) does not have a limit as $n \rightarrow \infty$. So

$$x = -1 \implies \sum_{k=0}^{\infty} x^k \text{ diverges.}$$

9.2 Infinite Series



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$$x = -1 \implies \sum_{k=0}^{\infty} x^k \text{ diverges.}$$

- If $|x| > 1$, then $|s_n| = \frac{|x^n - 1|}{|x - 1|} \geq \frac{|x|^n - 1}{|x| + 1} \rightarrow \infty$ as $n \rightarrow \infty$. So

$$|x| > 1 \implies \sum_{k=0}^{\infty} x^k \text{ diverges.}$$



9.2 Infinite Series

- If $x = -1$, then $s_n = \frac{1 - (-1)^n}{2}$ and (s_n) does not have a limit as $n \rightarrow \infty$. So

$$x = -1 \implies \sum_{k=0}^{\infty} x^k \text{ diverges.}$$

- If $|x| > 1$, then $|s_n| = \frac{|x^n - 1|}{|x - 1|} \geq \frac{|x|^n - 1}{|x| + 1} \rightarrow \infty$ as $n \rightarrow \infty$. So

$$|x| > 1 \implies \sum_{k=0}^{\infty} x^k \text{ diverges.}$$

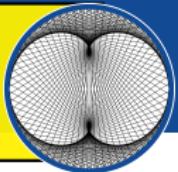
Theorem

$$\sum_{k=0}^{\infty} x^k \begin{cases} \text{converges if } |x| < 1 \\ \text{diverges if } |x| \geq 1. \end{cases}$$

Moreover, if $|x| < 1$ then $\sum_{k=0}^{\infty} x^k = \frac{1}{1-x}$.

9.2 Infinite Series

$$\sum_{k=0}^{\infty} x^k = \frac{1}{1-x}$$



Example

$$\begin{aligned} 7 + \frac{7}{3} + \frac{7}{9} + \frac{7}{27} + \frac{7}{81} + \frac{7}{243} + \dots \\ &= 7 \left(1 + \frac{1}{3} + \frac{1}{9} + \frac{1}{27} + \frac{1}{81} + \frac{1}{243} + \dots \right) \\ &= 7 \left(\frac{1}{1 - \frac{1}{3}} \right) \\ &= \frac{21}{2} = 10.5. \end{aligned}$$

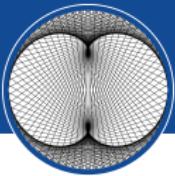
EXAMPLE 4 Express the repeating decimal $5.232323\dots$ as the ratio of two integers.

Solution From the definition of a decimal number, we get a geometric series

$$\begin{aligned} 5.232323\dots &= 5 + \frac{23}{100} + \frac{23}{(100)^2} + \frac{23}{(100)^3} + \dots \\ &= 5 + \frac{23}{100} \underbrace{\left(1 + \frac{1}{100} + \left(\frac{1}{100}\right)^2 + \dots\right)}_{1/(1 - 0.01)} \quad \begin{matrix} a = 1, \\ r = 1/100 \end{matrix} \\ &= 5 + \frac{23}{100} \left(\frac{1}{0.99}\right) = 5 + \frac{23}{99} = \frac{518}{99} \end{aligned}$$



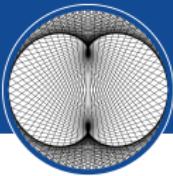
9.2 Infinite Series



Example

Does $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$ converge or diverge?

9.2 Infinite Series



Example

Does $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$ converge or diverge?

If we write $a_n = \frac{1}{n(n+1)}$ in partial fractions, we get

$$a_n = \frac{1}{n} - \frac{1}{n+1}. \text{ So}$$

$$a_1 = 1 - \frac{1}{2}$$

$$a_2 = \frac{1}{2} - \frac{1}{3}$$

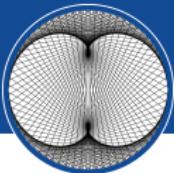
$$a_3 = \frac{1}{3} - \frac{1}{4}$$

⋮

$$a_{n-1} = \frac{1}{n-1} - \frac{1}{n}$$

$$a_n = \frac{1}{n} - \frac{1}{n+1}.$$

9.2 Infinite Series

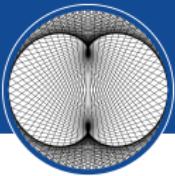


Thus

$$\begin{aligned}s_n &= a_1 + a_2 + a_3 + a_4 + \dots + a_{n-1} + a_n \\&= \left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \left(\frac{1}{4} - \frac{1}{5}\right) + \dots \\&\quad + \left(\frac{1}{n-1} - \frac{1}{n}\right) + \left(\frac{1}{n} - \frac{1}{n+1}\right)\end{aligned}$$

=

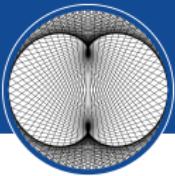
9.2 Infinite Series



Thus

$$\begin{aligned}s_n &= a_1 + a_2 + a_3 + a_4 + \dots + a_{n-1} + a_n \\&= \left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \left(\frac{1}{4} - \frac{1}{5}\right) + \dots \\&\quad + \left(\frac{1}{n-1} - \frac{1}{n}\right) + \left(\frac{1}{n} - \frac{1}{n+1}\right) \\&= 1 - \frac{1}{2} + \frac{1}{2} - \frac{1}{3} + \frac{1}{3} - \frac{1}{4} + \frac{1}{4} - \frac{1}{5} + \dots + \frac{1}{n-1} - \frac{1}{n} + \frac{1}{n} - \frac{1}{n+1} \\&= \end{aligned}$$

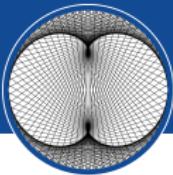
9.2 Infinite Series



Thus

$$\begin{aligned}s_n &= a_1 + a_2 + a_3 + a_4 + \dots + a_{n-1} + a_n \\&= \left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \left(\frac{1}{4} - \frac{1}{5}\right) + \dots \\&\quad + \left(\frac{1}{n-1} - \frac{1}{n}\right) + \left(\frac{1}{n} - \frac{1}{n+1}\right) \\&= 1 - \frac{1}{2} + \frac{1}{2} - \frac{1}{3} + \frac{1}{3} - \frac{1}{4} + \frac{1}{4} - \frac{1}{5} + \dots + \frac{1}{n-1} - \frac{1}{n} + \frac{1}{n} - \frac{1}{n+1} \\&= 1 - \frac{1}{n+1}\end{aligned}$$

9.2 Infinite Series

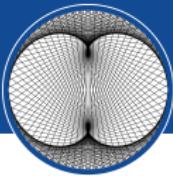


Thus

$$\begin{aligned}s_n &= a_1 + a_2 + a_3 + a_4 + \dots + a_{n-1} + a_n \\&= \left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \left(\frac{1}{4} - \frac{1}{5}\right) + \dots \\&\quad + \left(\frac{1}{n-1} - \frac{1}{n}\right) + \left(\frac{1}{n} - \frac{1}{n+1}\right) \\&= 1 - \frac{1}{2} + \frac{1}{2} - \frac{1}{3} + \frac{1}{3} - \frac{1}{4} + \frac{1}{4} - \frac{1}{5} + \dots + \frac{1}{n-1} - \frac{1}{n} + \frac{1}{n} - \frac{1}{n+1} \\&= 1 - \frac{1}{n+1} \\&\rightarrow 1\end{aligned}$$

as $n \rightarrow \infty$.

9.2 Infinite Series



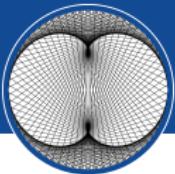
Thus

$$\begin{aligned}s_n &= a_1 + a_2 + a_3 + a_4 + \dots + a_{n-1} + a_n \\&= \left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \left(\frac{1}{4} - \frac{1}{5}\right) + \dots \\&\quad + \left(\frac{1}{n-1} - \frac{1}{n}\right) + \left(\frac{1}{n} - \frac{1}{n+1}\right) \\&= 1 - \frac{1}{2} + \frac{1}{2} - \frac{1}{3} + \frac{1}{3} - \frac{1}{4} + \frac{1}{4} - \frac{1}{5} + \dots + \frac{1}{n-1} - \frac{1}{n} + \frac{1}{n} - \frac{1}{n+1} \\&= 1 - \frac{1}{n+1} \\&\rightarrow 1\end{aligned}$$

as $n \rightarrow \infty$.

Therefore $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$ converges and $\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = 1$.

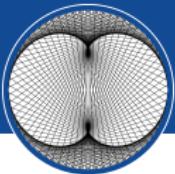
9.2 Infinite Series



Theorem (The Sum Rule)

If $\sum_{k=1}^{\infty} a_k = s$ and $\sum_{k=1}^{\infty} b_k = t$ are convergent series, then
 $\sum_{k=1}^{\infty} (a_k + b_k) = s + t$ is convergent.

9.2 Infinite Series



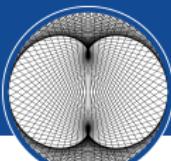
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 $\sum_{k=1}^{\infty} (a_k + b_k) = s + t$ is convergent.

Proof.

Let $s_n = a_1 + a_2 + a_3 + \dots + a_n$ and $t_n = b_1 + b_2 + b_3 + \dots + b_n$ be the partial sums of $\sum_{k=1}^{\infty} a_k$ and $\sum_{k=1}^{\infty} b_k$ respectively.
Then $s_n \rightarrow s$ and $t_n \rightarrow t$ as $n \rightarrow \infty$.

9.2 Infinite Series



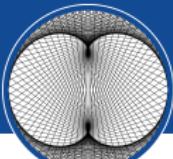
continued.

So

$$\begin{aligned}\sum_{k=1}^n (a_k + b_k) &= (a_1 + b_1) + (a_2 + b_2) + (a_3 + b_3) + \dots + (a_n + b_n) \\ &= s_n + t_n \rightarrow s + t\end{aligned}$$

as $n \rightarrow \infty$.

9.2 Infinite Series



continued.

So

$$\begin{aligned}\sum_{k=1}^n (a_k + b_k) &= (a_1 + b_1) + (a_2 + b_2) + (a_3 + b_3) + \dots + (a_n + b_n) \\ &= s_n + t_n \rightarrow s + t\end{aligned}$$

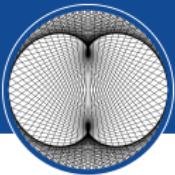
as $n \rightarrow \infty$. Therefore $\sum_{k=1}^{\infty} (a_k + b_k)$ converges and

$$\sum_{k=1}^{\infty} (a_k + b_k) = \sum_{k=1}^{\infty} a_k + \sum_{k=1}^{\infty} b_k.$$



So if two series are convergent, we can add them.

9.2 Infinite Series

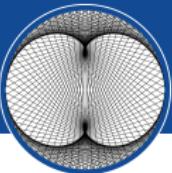


Theorem (The Constant Multiple Rule)

If $\sum_{k=1}^{\infty} a_k = s$ is a convergent series, then $\sum_{k=1}^{\infty} c a_k = cs$ is convergent for any number $c \in \mathbb{R}$.

(you prove)

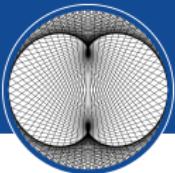
9.2 Infinite Series



Theorem (The Divergence Test / Iraksaklık Testi)

If $a_n \not\rightarrow 0$ as $n \rightarrow \infty$, then $\sum_{n=1}^{\infty} a_n$ diverges.

9.2 Infinite Series



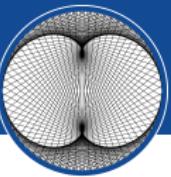
Theorem (The Divergence Test / Iraksaklık Testi)

If $a_n \not\rightarrow 0$ as $n \rightarrow \infty$, then $\sum_{n=1}^{\infty} a_n$ diverges.

Proof.

Let $s_n = a_1 + a_2 + a_3 + \dots + a_n$. We will use proof by contrapositive: Suppose that $\sum_{k=1}^{\infty} a_k$ converges.

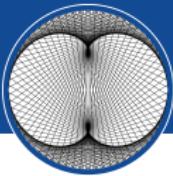
9.2 Infinite Series



Proof continued.

Then $s_n \rightarrow s$ as $n \rightarrow \infty$.

9.2 Infinite Series

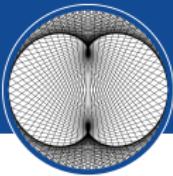


Proof continued.

Then $s_n \rightarrow s$ as $n \rightarrow \infty$. But then $s_{n-1} \rightarrow s$ as $n \rightarrow \infty$ also.
Hence

$$a_n = s_n - s_{n-1}$$

9.2 Infinite Series



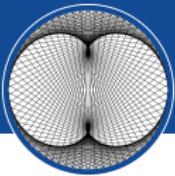
Proof continued.

Then $s_n \rightarrow s$ as $n \rightarrow \infty$. But then $s_{n-1} \rightarrow s$ as $n \rightarrow \infty$ also.
Hence

$$a_n = s_n - s_{n-1} \rightarrow s - s = 0$$

as $n \rightarrow \infty$.

9.2 Infinite Series



Proof continued.

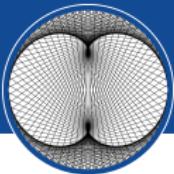
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as $n \rightarrow \infty$. So

$$\sum_{k=1}^{\infty} a_k \text{ converges} \implies a_n \rightarrow 0 \text{ as } n \rightarrow \infty.$$

9.2 Infinite Series



Proof continued.

Then $s_n \rightarrow s$ as $n \rightarrow \infty$. But then $s_{n-1} \rightarrow s$ as $n \rightarrow \infty$ also.
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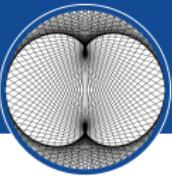
$$\sum_{k=1}^{\infty} a_k \text{ converges} \implies a_n \rightarrow 0 \text{ as } n \rightarrow \infty.$$

Therefore

$$a_n \not\rightarrow 0 \text{ as } n \rightarrow \infty \implies \sum_{k=1}^{\infty} a_k \text{ diverges.}$$



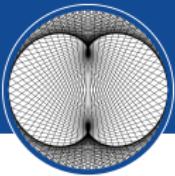
9.2 Infinite Series



Corollary

If $\sum_{k=1}^{\infty} a_k$ converges, then $a_n \rightarrow 0$ as $n \rightarrow \infty$.

9.2 Infinite Series



Corollary

If $\sum_{k=1}^{\infty} a_k$ converges, then $a_n \rightarrow 0$ as $n \rightarrow \infty$.

Remark

$$\sum_{k=1}^{\infty} a_k \text{ converges} \implies a_n \rightarrow 0 \text{ as } n \rightarrow \infty.$$

But

$$a_n \rightarrow 0 \text{ as } n \rightarrow \infty \not\implies \sum_{k=1}^{\infty} a_k \text{ converges.}$$

Be careful!!!

Example

9.2

Let $a_n = \frac{1}{n}$ for all $n \in \mathbb{N}$. Then $a_n \rightarrow 0$ as $n \rightarrow \infty$.

Does $\sum_{k=1}^{\infty} \frac{1}{n}$ converge or diverge?



Example

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Does $\sum_{k=1}^{\infty} \frac{1}{n}$ converge or diverge?

Let

$$s_n = a_1 + a_2 + a_3 + \dots + a_n = 1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{n}.$$

Clearly (s_n) is an increasing sequence.

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$$\begin{aligned}s_{2^n} &= 1 + \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4}\right) + \left(\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}\right) + \dots \\ &\quad + \left(\frac{1}{2^{n-1}+1} + \dots + \frac{1}{2^n}\right)\end{aligned}$$

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as $n \rightarrow \infty$, we have that (s_n) is **not bounded above**.

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Let $a_n = \frac{1}{n}$ for all $n \in \mathbb{N}$. Then $a_n \rightarrow 0$ as $n \rightarrow \infty$.

Does $\sum_{k=1}^{\infty} \frac{1}{n}$ converge or diverge?

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as $n \rightarrow \infty$, we have that (s_n) is **not bounded above**. Hence $s_n \rightarrow \infty$ as $n \rightarrow \infty$ by a theorem from last week.

Example

9.2

Let $a_n = \frac{1}{n}$ for all $n \in \mathbb{N}$. Then $a_n \rightarrow 0$ as $n \rightarrow \infty$.

Does $\sum_{k=1}^{\infty} \frac{1}{n}$ converge or diverge?

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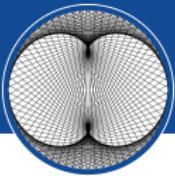
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as $n \rightarrow \infty$, we have that (s_n) is **not bounded above**. Hence $s_n \rightarrow \infty$ as $n \rightarrow \infty$ by a theorem from last week.

Therefore $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges.

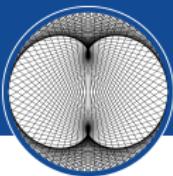
9.2 Infinite Series



Lemma

$$\sum_{n=1}^{\infty} \frac{1}{n} \text{ diverges.}$$

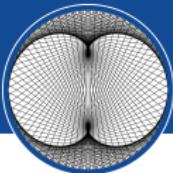
9.2 Infinite Series



Example

Does $\sum_{k=1}^{\infty} b_n = \sum_{k=1}^{\infty} \left(\frac{3n+1}{5n+1} \right)^4$ converge or diverge?

9.2 Infinite Series



Example

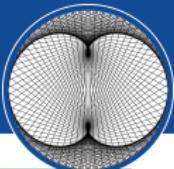
Does $\sum_{k=1}^{\infty} b_n = \sum_{k=1}^{\infty} \left(\frac{3n+1}{5n+1} \right)^4$ converge or diverge?

Since

$$b_n = \left(\frac{3n+1}{5n+1} \right)^4 \rightarrow \left(\frac{3}{5} \right)^4 \neq 0$$

as $n \rightarrow \infty$, it follows that $\sum_{n=1}^{\infty} \left(\frac{3n+1}{5n+1} \right)^4$ diverges by the Divergence Test.

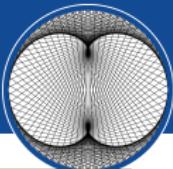
9.2 Infinite Series



Example

Does $\sum_{k=1}^{\infty} \frac{1}{\sqrt{n}}$ converge or diverge?

9.2 Infinite Series



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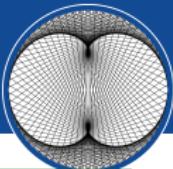
Does $\sum_{k=1}^{\infty} \frac{1}{\sqrt{n}}$ converge or diverge?

Let $a_n = \frac{1}{\sqrt{n}}$ and $b_n = \frac{1}{n}$. Then

$$a_n = \frac{1}{\sqrt{n}} \geq \frac{1}{n} = b_n$$

for all $n \in \mathbb{N}$.

9.2 Infinite Series



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Does $\sum_{k=1}^{\infty} \frac{1}{\sqrt{n}}$ converge or diverge?

Let $a_n = \frac{1}{\sqrt{n}}$ and $b_n = \frac{1}{n}$. Then

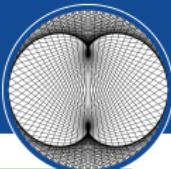
$$a_n = \frac{1}{\sqrt{n}} \geq \frac{1}{n} = b_n$$

for all $n \in \mathbb{N}$. Let $s_n = a_1 + a_2 + a_3 + \dots + a_n$ and $t_n = b_1 + b_2 + b_3 + \dots + b_n$. Then

$$s_n \geq t_n$$

for all $n \in \mathbb{N}$.

9.2 Infinite Series



Example

Does $\sum_{k=1}^{\infty} \frac{1}{\sqrt{n}}$ converge or diverge?

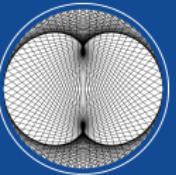
Let $a_n = \frac{1}{\sqrt{n}}$ and $b_n = \frac{1}{n}$. Then

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for all $n \in \mathbb{N}$. Let $s_n = a_1 + a_2 + a_3 + \dots + a_n$ and $t_n = b_1 + b_2 + b_3 + \dots + b_n$. Then

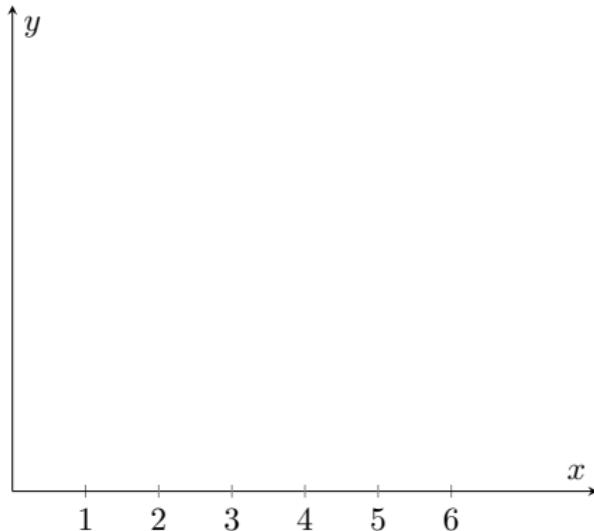
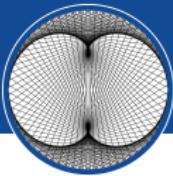
$$s_n \geq t_n$$

for all $n \in \mathbb{N}$. Since $t_n \rightarrow \infty$ as $n \rightarrow \infty$, we have that $s_n \rightarrow \infty$ as $n \rightarrow \infty$ also. Therefore $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ diverges.



The Integral Test

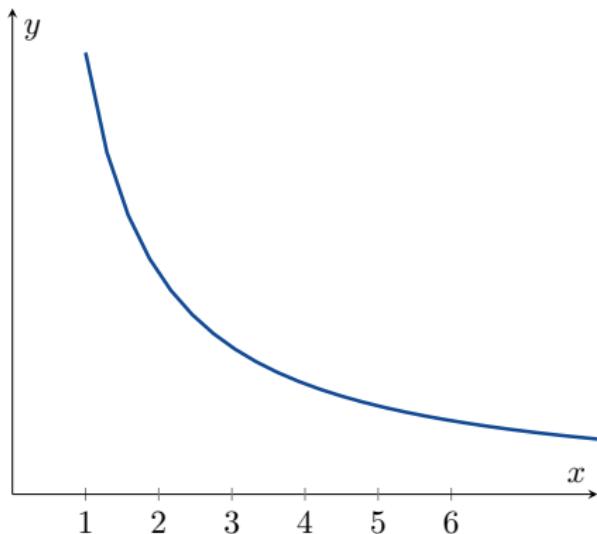
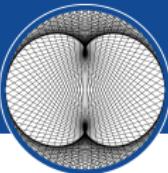
9.3 The Integral Test



Let $f : [1, \infty) \rightarrow [0, \infty)$ be a function which is

- continuous;
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- positive ($f(x) \geq 0 \ \forall x \in [1, \infty)$).

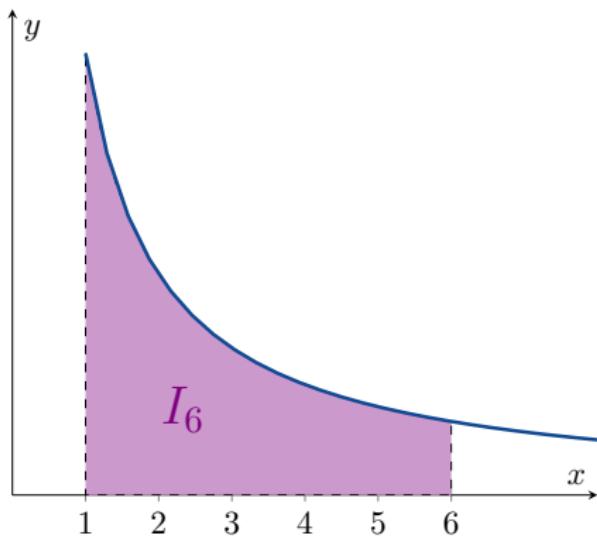
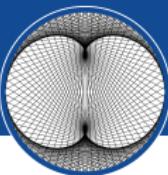
9.3 The Integral Test



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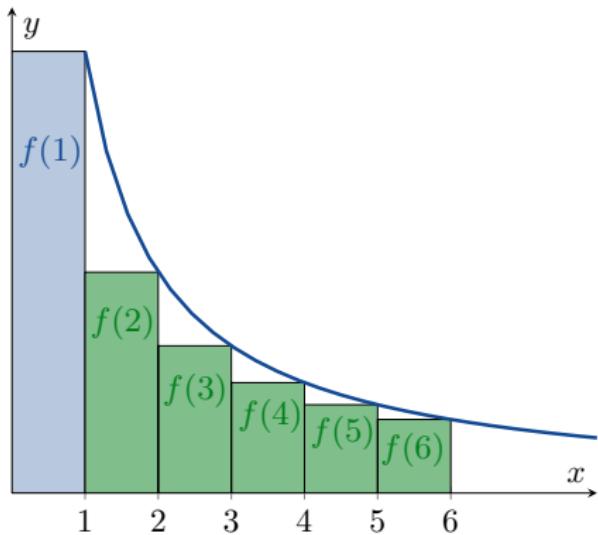
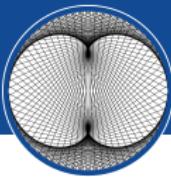
9.3 The Integral Test



Define

$$I_n := \int_1^n f(x) \, dx$$

9.3 The Integral Test



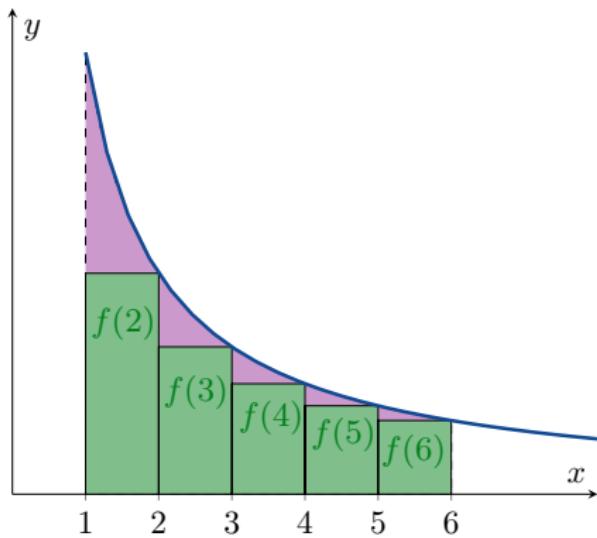
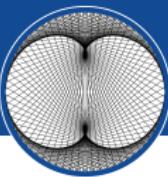
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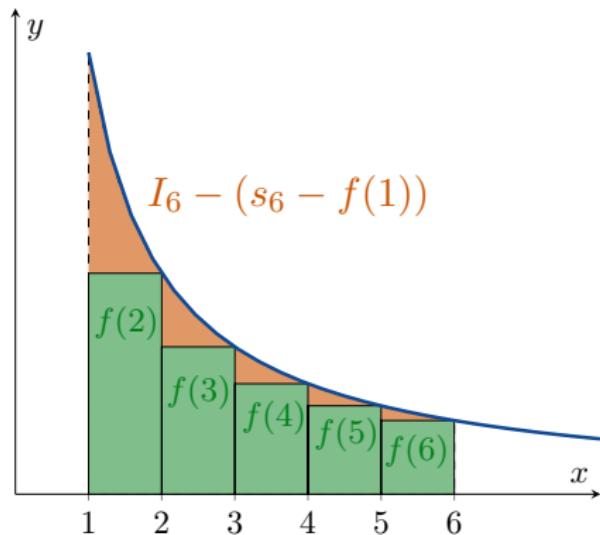
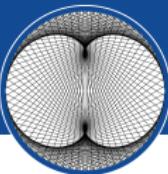
$$s_n := f(1) + f(2) + f(3) + \dots + f(n) = \sum_{k=1}^n f(k).$$

9.3 The Integral Test



Notice that $f(2) + f(3) + f(4) + \dots + f(n) \leq I_n$.

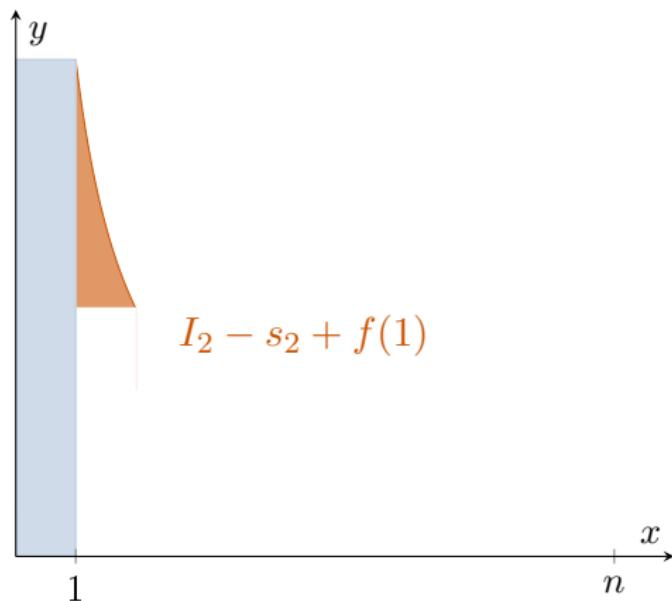
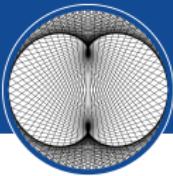
9.3 The Integral Test



Notice that $f(2) + f(3) + f(4) + \dots + f(n) \leq I_n$. The difference is

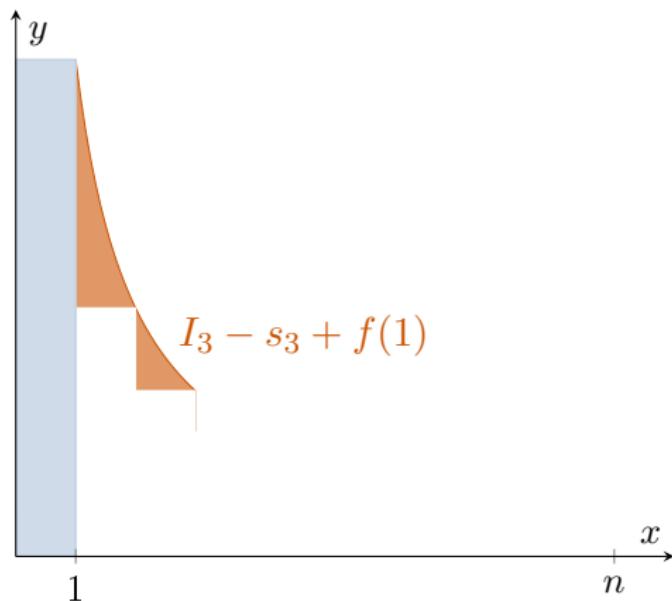
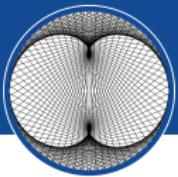
$$I_n - f(2) - f(3) - f(4) - \dots - f(n) = I_n - (s_n - f(1)) = I_n - s_n + f(1).$$

9.3 The Integral Test



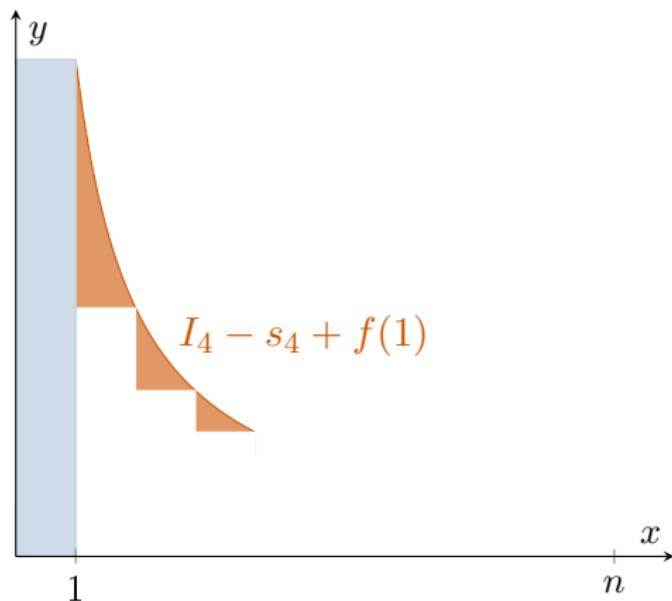
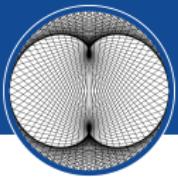
As n increases, $I_n - s_n + f(1)$ increases. Since $f(1)$ is a constant, we have that $(I_n - s_n)$ is an increasing sequence.

9.3 The Integral Test



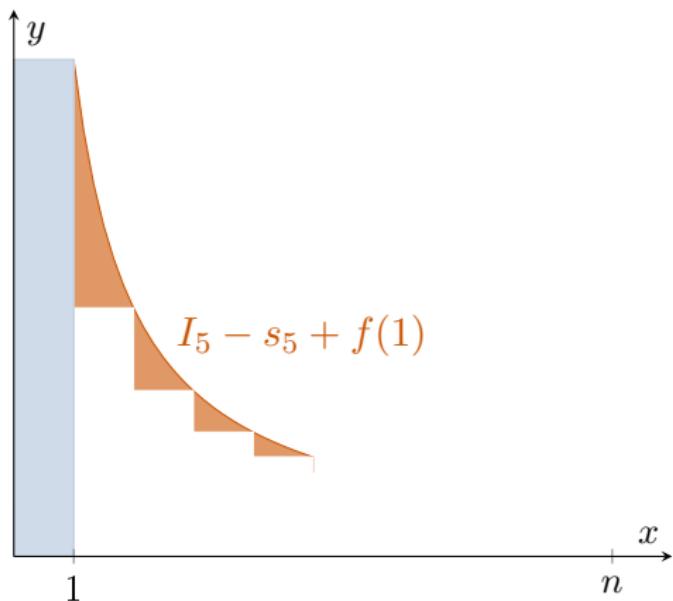
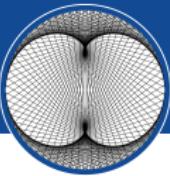
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9.3 The Integral Test



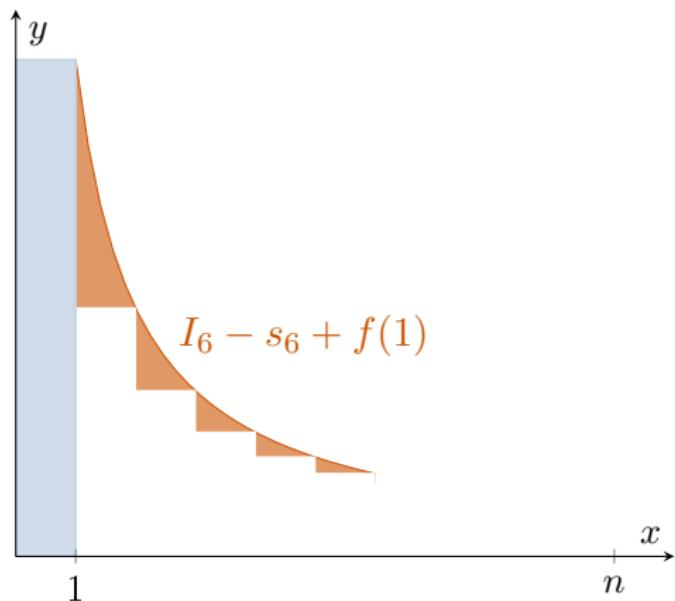
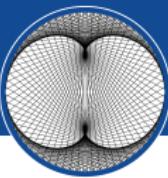
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9.3 The Integral Test



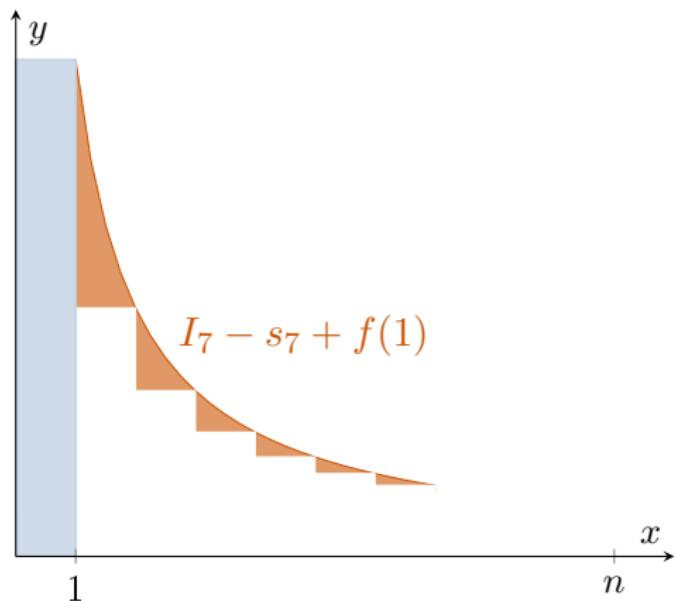
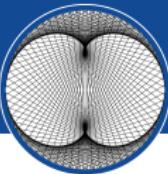
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9.3 The Integral Test



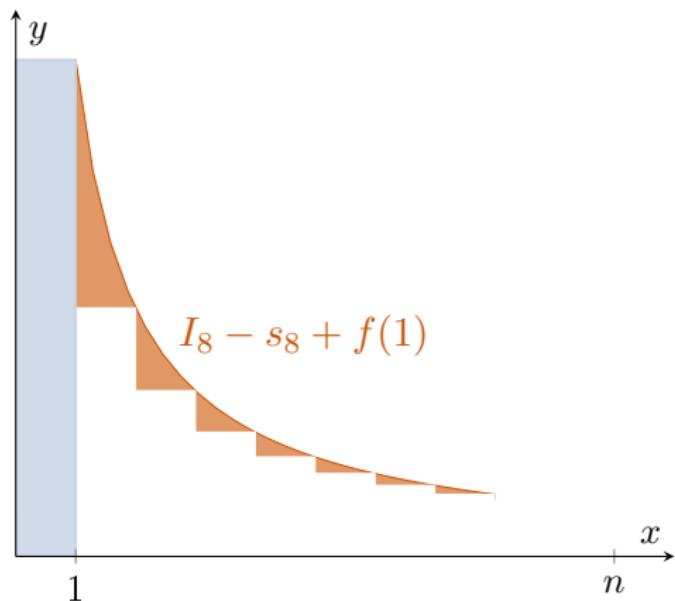
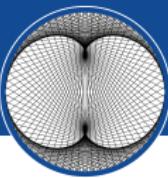
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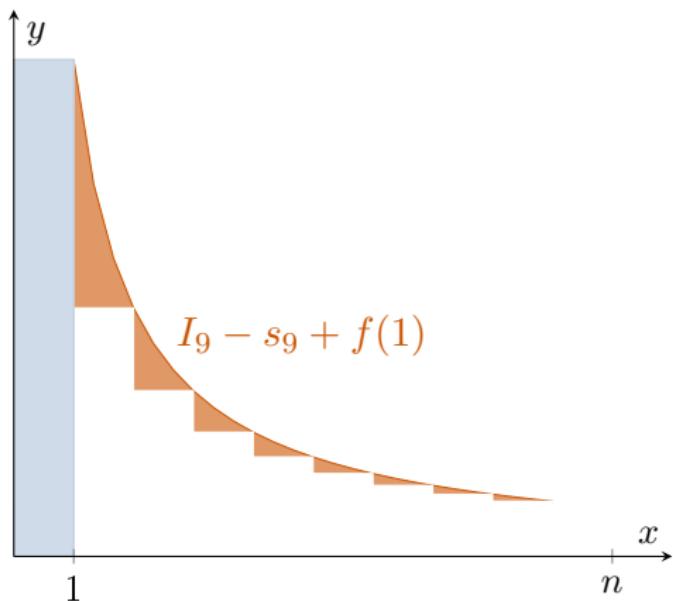
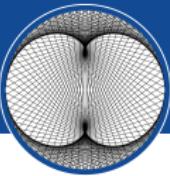
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9.3 The Integral Test



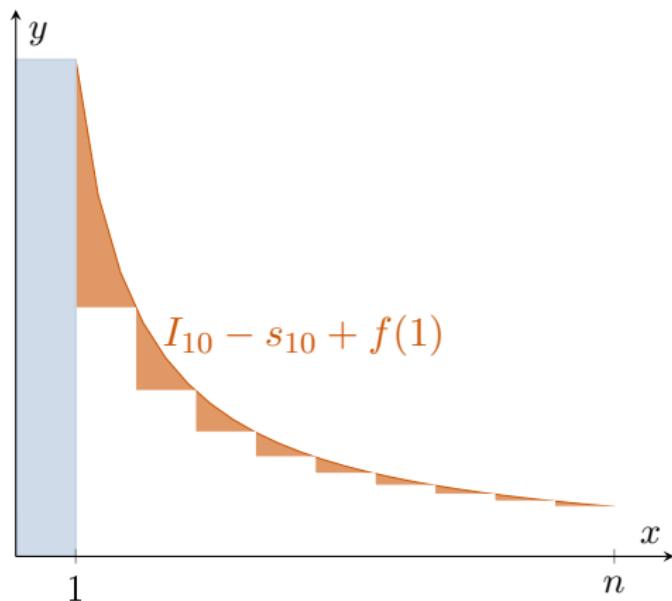
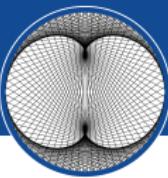
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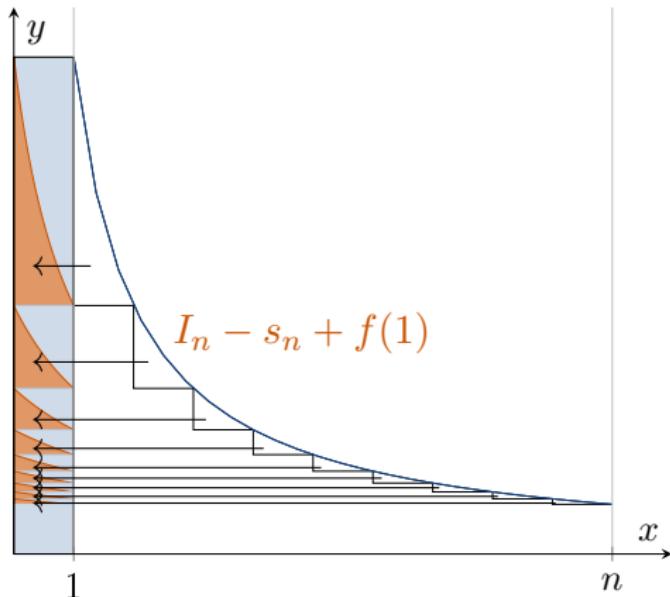
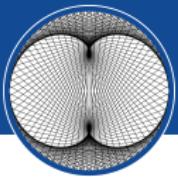
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9.3 The Integral Test



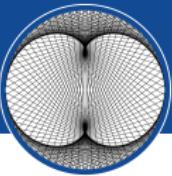
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9.3 The Integral Test



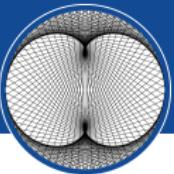
We can see from the picture above that $I_n - s_n + f(1) \leq f(1)$. Therefore $I_n - s_n \leq 0$ for all $n \in \mathbb{N}$.

9.3 The Integral Test



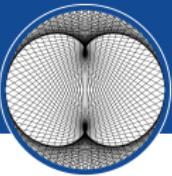
So $(I_n - s_n)$ is an increasing sequence which is bounded above.

9.3 The Integral Test



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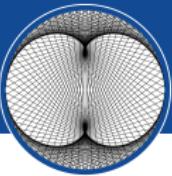
Lemma

Let $f : [1, \infty) \rightarrow [0, \infty)$ be a positive, decreasing, continuous function. Let $s_n = f(1) + f(2) + f(3) + \dots + f(n)$ and

$$I_n := \int_1^n f(x) dx \text{ for all } n \in \mathbb{N}.$$

Then $(I_n - s_n)$ is convergent.

9.3 The Integral Test



Theorem (The Integral Test / Integral Testi)

Let $f : [1, \infty) \rightarrow [0, \infty)$ be a positive, decreasing, continuous function.

- 1 If $\int_1^\infty f(x) dx < \infty$, then $\sum_{n=1}^{\infty} f(n)$ converges.
- 2 If $\int_1^\infty f(x) dx = \infty$, then $\sum_{n=1}^{\infty} f(n)$ diverges.

9.3 The Integral Test

Proof.

Let s_n and I_n be as defined above. Let $c_n = s_n - I_n$. By the previous lemma, we know that $c_n \rightarrow c$ as $n \rightarrow \infty$.



9.3 The Integral Test



Proof.

Let s_n and I_n be as defined above. Let $c_n = s_n - I_n$. By the previous lemma, we know that $c_n \rightarrow c$ as $n \rightarrow \infty$.

Since $f(x) > 0$ for all $x \in [1, \infty)$, (s_n) and (I_n) are both increasing sequences. Either

- 1 (I_n) is bounded above; or
- 2 (I_n) is not bounded above.

9.3 The Integral Test



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CASE 1: If (I_n) is increasing and bounded above, then (I_n) must converge, $I_n \rightarrow I$ as $n \rightarrow \infty$. But then

$s_n = c_n + I_n \rightarrow c + I$ as $n \rightarrow \infty$. So $\sum_{n=1}^{\infty} f(n)$ converges.

9.3 The Integral Test



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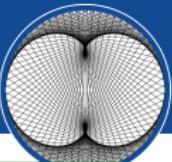
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CASE 2: If (I_n) is increasing and not bounded above, then $I_n \rightarrow \infty$ as $n \rightarrow \infty$ and we have that $s_n = c_n + I_n \rightarrow \infty$ as $n \rightarrow \infty$ also. So $\sum_{n=1}^{\infty} f(n)$ diverges. □

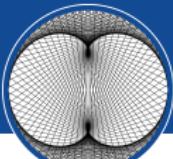
9.3 The Integral Test



Example

For which $\alpha > 0$ does $\sum_{n=1}^{\infty} \frac{1}{n^\alpha}$ converge?

9.3 The Integral Test



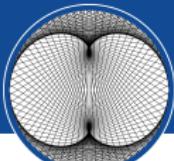
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For which $\alpha > 0$ does $\sum_{n=1}^{\infty} \frac{1}{n^\alpha}$ converge?

Let $f(x) = \frac{1}{x^\alpha}$ for some $\alpha > 0$. Then f is continuous, decreasing and positive $\forall x \geq 1$. So

$$I_n = \int_1^n f(x) \, dx = \int_1^n \frac{1}{x^\alpha} dx = \begin{cases} \frac{1}{\alpha-1} \left[-\frac{1}{x^{\alpha-1}} \right]_1^n & \text{if } \alpha \neq 1 \\ [\ln x]_1^n & \text{if } \alpha = 1 \end{cases}$$

9.3 The Integral Test



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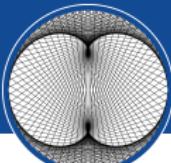
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- Suppose that $\alpha > 1$. Then

$$I_n = \frac{1}{\alpha-1} \left(1 - \frac{1}{n^{\alpha-1}} \right) \rightarrow \frac{1}{\alpha-1} < \infty$$

as $n \rightarrow \infty$. So $\sum_{n=1}^{\infty} \frac{1}{n^\alpha}$ converges by the Integral Test.

9.3 The Integral Test



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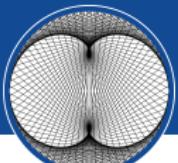
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- Suppose that $\alpha = 1$. Then

$$I_n = \ln n - \ln 1 = \ln n \rightarrow \infty$$

as $n \rightarrow \infty$. So $\sum_{n=1}^{\infty} \frac{1}{n^\alpha}$ diverges by the Integral Test.

9.3 The Integral Test



Example

For which $\alpha > 0$ does $\sum_{n=1}^{\infty} \frac{1}{n^\alpha}$ converge?

Let $f(x) = \frac{1}{x^\alpha}$ for some $\alpha > 0$. Then f is continuous, decreasing and positive $\forall x \geq 1$. So

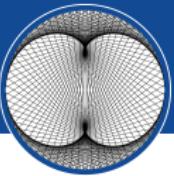
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- Suppose that $0 < \alpha < 1$. Then

$$I_n = \frac{1}{1-\alpha} (n^{1-\alpha} - 1) \rightarrow \infty$$

as $n \rightarrow \infty$. So $\sum_{n=1}^{\infty} \frac{1}{n^\alpha}$ diverges by the Integral Test.

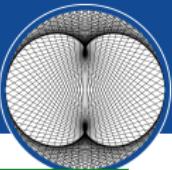
9.3 The Integral Test



Theorem

The series $\sum_{n=1}^{\infty} \frac{1}{n^\alpha}$ diverges for $0 < \alpha \leq 1$ and converges for $\alpha > 1$.

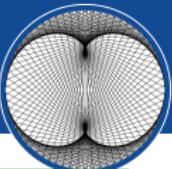
9.3 The Integral Test



Example

Consider $\sum_{n=2}^{\infty} \frac{1}{n \ln n}$. (Q: Why am I starting at $n = 2$?)

9.3 The Integral Test

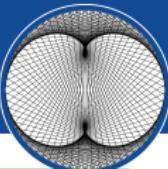


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Use the Integral Test to decide if this series converges or diverges.

9.3 The Integral Test



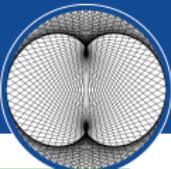
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9.3 The Integral Test



Example

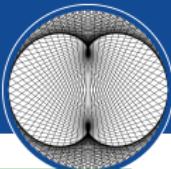
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Use the Integral Test to decide if this series converges or diverges.

Let $f(x) = \frac{1}{x \ln x}$ for $x \geq 2$. Then $f : [2, \infty) \rightarrow \mathbb{R}$ is continuous, decreasing and positive. Moreover, for $n \geq 2$,

$$\begin{aligned}\int_2^{\infty} f(x) dx &= \lim_{n \rightarrow \infty} \int_2^n f(x) dx \\&= \lim_{n \rightarrow \infty} \int_2^n \frac{1}{x \ln x} dx = \lim_{n \rightarrow \infty} \left[\ln(\ln x) \right]_2^n \\&= \lim_{n \rightarrow \infty} (\ln \ln n - \ln \ln 2) = \infty.\end{aligned}$$

9.3 The Integral Test



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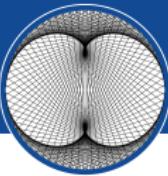
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Therefore $\sum_{n=2}^{\infty} \frac{1}{n \log n}$ diverges by the Integral Test.

9.3 The Integral Test



EXAMPLE 5 Determine the convergence or divergence of the series.

(a) $\sum_{n=1}^{\infty} ne^{-n^2}$

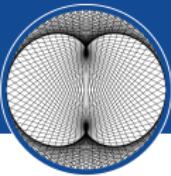
Solutions

(a) We apply the Integral Test and find that

$$\begin{aligned}\int_1^{\infty} \frac{x}{e^{x^2}} dx &= \frac{1}{2} \int_1^{\infty} \frac{du}{e^u} \quad u = x^2, du = 2x dx \\ &= \lim_{b \rightarrow \infty} \left[-\frac{1}{2} e^{-u} \right]_1^b \\ &= \lim_{b \rightarrow \infty} \left(-\frac{1}{2e^b} + \frac{1}{2e} \right) = \frac{1}{2e}.\end{aligned}$$

Since the integral converges, the series also converges.

9.3 The Integral Test



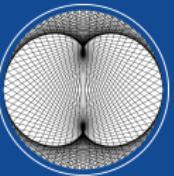
EXAMPLE 5 Determine the convergence or divergence of the series.

$$(b) \sum_{n=1}^{\infty} \frac{1}{2^{\ln n}}$$

(b) Again applying the Integral Test,

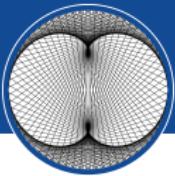
$$\begin{aligned}\int_1^{\infty} \frac{dx}{2^{\ln x}} &= \int_0^{\infty} \frac{e^u du}{2^u} \quad u = \ln x, x = e^u, dx = e^u du \\ &= \int_0^{\infty} \left(\frac{e}{2}\right)^u du \\ &= \lim_{b \rightarrow \infty} \frac{1}{\ln\left(\frac{e}{2}\right)} \left(\left(\frac{e}{2}\right)^b - 1 \right) = \infty. \quad (e/2) > 1\end{aligned}$$

The improper integral diverges, so the series diverges also.



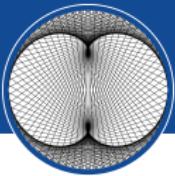
Comparison Tests

9.4 Comparison Tests



We continue with two more tests for convergence.

9.4 Comparison Tests



Theorem (The Comparison Test / Karşılaştırma Testi)

Let $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ be two series of non-negative real numbers (i.e. $a_n \geq 0$ and $b_n \geq 0$ for all n).

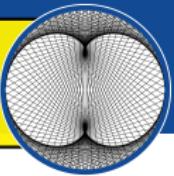
Suppose that

- 1 $0 \leq a_n \leq Kb_n$ for all $n \in \mathbb{N}$ and for some $K > 0$; and
- 2 $\sum_{n=1}^{\infty} b_n$ converges.

Then $\sum_{n=1}^{\infty} a_n$ converges.

9.4 Comparison Tests

$$0 \leq a_n \leq K b_n$$



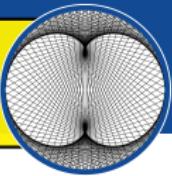
Proof.

Let $s_n = a_1 + a_2 + a_3 + \dots + a_n$ and $t_n = b_1 + b_2 + b_3 + \dots + b_n$ be the partial sums.

Since $a_k \geq 0$ and $b_k \geq 0$ for all $k \in \mathbb{N}$, (s_n) and (t_n) are increasing sequences.

9.4 Comparison Tests

$$0 \leq a_n \leq K b_n$$



Proof.

Let $s_n = a_1 + a_2 + a_3 + \dots + a_n$ and $t_n = b_1 + b_2 + b_3 + \dots + b_n$ be the partial sums.

Since $a_k \geq 0$ and $b_k \geq 0$ for all $k \in \mathbb{N}$, (s_n) and (t_n) are increasing sequences.

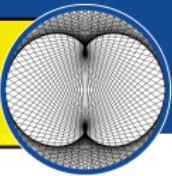
Since $\sum_{n=1}^{\infty} b_n$ converges, $\exists t \in \mathbb{R}$ such that $t_n \rightarrow t$ as $n \rightarrow \infty$. So

$$\begin{aligned}s_n &= a_1 + a_2 + a_3 + \dots + a_n \\&\leq K b_1 + K b_2 + K b_3 + \dots + K b_n = K t_n \leq K t\end{aligned}$$

for all $n \in \mathbb{N}$.

9.4 Comparison Tests

$$0 \leq a_n \leq K b_n$$



Proof.

Let $s_n = a_1 + a_2 + a_3 + \dots + a_n$ and $t_n = b_1 + b_2 + b_3 + \dots + b_n$ be the partial sums.

Since $a_k \geq 0$ and $b_k \geq 0$ for all $k \in \mathbb{N}$, (s_n) and (t_n) are increasing sequences.

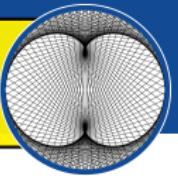
Since $\sum_{n=1}^{\infty} b_n$ converges, $\exists t \in \mathbb{R}$ such that $t_n \rightarrow t$ as $n \rightarrow \infty$. So

$$\begin{aligned}s_n &= a_1 + a_2 + a_3 + \dots + a_n \\&\leq K b_1 + K b_2 + K b_3 + \dots + K b_n = K t_n \leq K t\end{aligned}$$

for all $n \in \mathbb{N}$. So (s_n) is an increasing sequence which is bounded above.

9.4 Comparison Tests

$$0 \leq a_n \leq K b_n$$



Proof.

Let $s_n = a_1 + a_2 + a_3 + \dots + a_n$ and $t_n = b_1 + b_2 + b_3 + \dots + b_n$ be the partial sums.

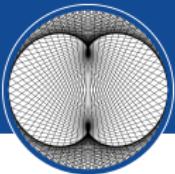
Since $a_k \geq 0$ and $b_k \geq 0$ for all $k \in \mathbb{N}$, (s_n) and (t_n) are increasing sequences.

Since $\sum_{n=1}^{\infty} b_n$ converges, $\exists t \in \mathbb{R}$ such that $t_n \rightarrow t$ as $n \rightarrow \infty$. So

$$\begin{aligned}s_n &= a_1 + a_2 + a_3 + \dots + a_n \\&\leq K b_1 + K b_2 + K b_3 + \dots + K b_n = K t_n \leq K t\end{aligned}$$

for all $n \in \mathbb{N}$. So (s_n) is an increasing sequence which is bounded above. Therefore (s_n) is convergent by a theorem from last week. □

9.4 Comparison Tests



Corollary

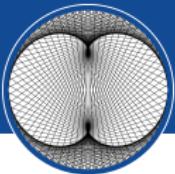
Let $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} c_n$ be two series of non-negative real numbers.

Suppose that

- 1 $a_n \geq kc_n \geq 0$ for all $n \in \mathbb{N}$ and for some $k > 0$; and
- 2 $\sum_{n=1}^{\infty} c_n$ diverges.

Then $\sum_{n=1}^{\infty} a_n$ diverges.

9.4 Comparison Tests



Proof.

Let $K = \frac{1}{k}$. Then $c_n \leq Ka_n$ for all $n \in \mathbb{N}$. By the Comparison Test we have that

$$\sum_{n=1}^{\infty} a_n \text{ converges} \implies \sum_{n=1}^{\infty} c_n \text{ converges.}$$

By proof by contrapositive, we have that

$$\sum_{n=1}^{\infty} c_n \text{ diverges} \implies \sum_{n=1}^{\infty} a_n \text{ diverges.}$$





9.4 Comparison Tests

Corollary

Let $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ be two series of non-negative real numbers.

Suppose that

- 1 $0 \leq a_n \leq K b_n$ for all $n \geq N_0$ for some $N_0 \in \mathbb{N}$ and $K > 0$;
and
- 2 $\sum_{n=1}^{\infty} b_n$ converges.

Then $\sum_{n=1}^{\infty} a_n$ also converges.

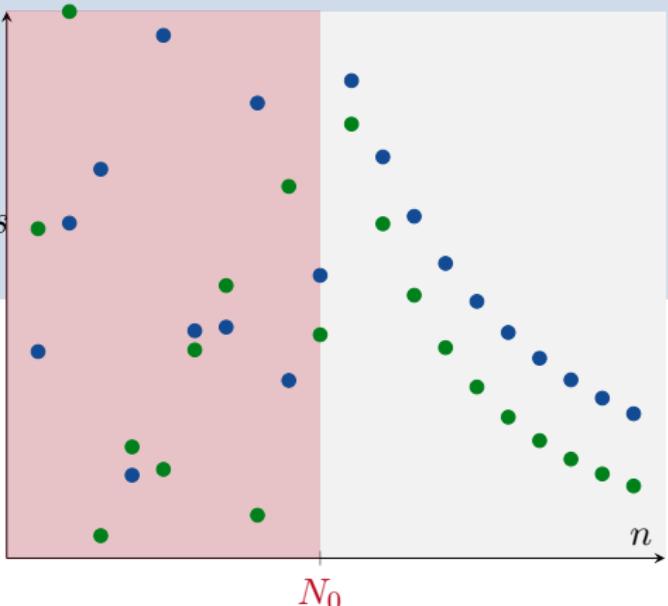
9.4 Comparison Tests



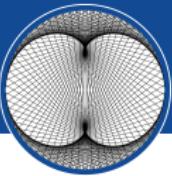
Let $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ be two series of non-negative real numbers.
Suppose that

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and
- 2 $\sum_{n=1}^{\infty} b_n$ converges.

Then $\sum_{n=1}^{\infty} a_n$ also converges



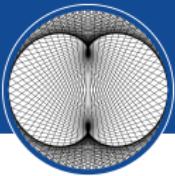
9.4 Comparison Tests



Example

Does $\sum_{n=1}^{\infty} \frac{2}{4n-1}$ converge or diverge?

9.4 Comparison Tests



Example

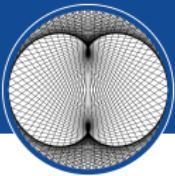
Does $\sum_{n=1}^{\infty} \frac{2}{4n-1}$ converge or diverge?

Since

$$\frac{2}{4n-1} \geq \frac{2}{4n} = \frac{1}{2n} = \left(\frac{1}{2}\right) \left(\frac{1}{n}\right) = K \left(\frac{1}{n}\right)$$

and since $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges,

9.4 Comparison Tests



Example

Does $\sum_{n=1}^{\infty} \frac{2}{4n-1}$ converge or diverge?

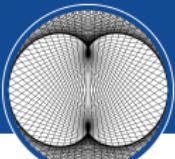
Since

$$\frac{2}{4n-1} \geq \frac{2}{4n} = \frac{1}{2n} = \left(\frac{1}{2}\right) \left(\frac{1}{n}\right) = K \left(\frac{1}{n}\right)$$

and since $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges, it follows by the Comparison Test that

$$\sum_{n=1}^{\infty} \frac{2}{4n-1} \text{ also diverges.}$$

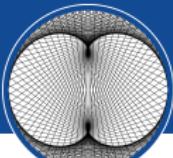
9.4 Comparison Tests



Theorem (The Limit Comparison Test / Limit Karşılaştırma Testi)

Let $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ be two series of strictly positive real numbers
(i.e. $a_n > 0$ and $b_n > 0 \forall n$).

9.4 Comparison Tests



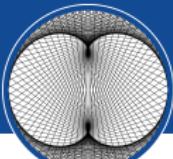
Theorem (The Limit Comparison Test / Limit Karşılaştırma Testi)

Let $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ be two series of strictly positive real numbers
(i.e. $a_n > 0$ and $b_n > 0 \forall n$).

Suppose that

- 1 $\frac{a_n}{b_n} \rightarrow l$ as $n \rightarrow \infty$;
- 2 $l \in \mathbb{R}$; and
- 3 $l \neq 0$.

9.4 Comparison Tests



Theorem (The Limit Comparison Test / Limit Karşılaştırma Testi)

Let $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ be two series of strictly positive real numbers
(i.e. $a_n > 0$ and $b_n > 0 \forall n$).

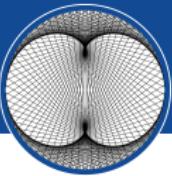
Suppose that

- 1 $\frac{a_n}{b_n} \rightarrow l$ as $n \rightarrow \infty$;
- 2 $l \in \mathbb{R}$; and
- 3 $l \neq 0$.

Then either

- $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ both converge; or
- $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ both diverge.

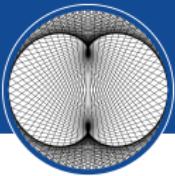
9.4 Comparison Tests



Proof.

Since $a_n > 0$, $b_n > 0$, $l \neq 0$ and $\frac{a_n}{b_n} \rightarrow l$ as $n \rightarrow \infty$, we must have that $l > 0$.

9.4 Comparison Tests



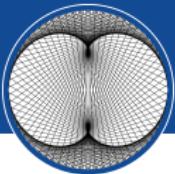
Proof.

Since $a_n > 0$, $b_n > 0$, $l \neq 0$ and $\frac{a_n}{b_n} \rightarrow l$ as $n \rightarrow \infty$, we must have that $l > 0$.

So $\exists N \in \mathbb{N}$ such that

$$\begin{aligned} n > N &\implies \left| \frac{a_n}{b_n} - l \right| < \frac{l}{2} \\ &\implies \frac{l}{2} < \frac{a_n}{b_n} < \frac{3l}{2} \\ &\implies \frac{l}{2}b_n < a_n < \frac{3l}{2}b_n. \end{aligned}$$

9.4 Comparison Tests

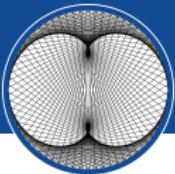


Proof continued.

Now

- $\sum_{n=1}^{\infty} b_n$ converges $\implies \sum_{n=1}^{\infty} a_n$ converges, by the corollary above, since $0 < a_n < \left(\frac{3l}{2}\right) b_n$ for all $n > N$;

9.4 Comparison Tests

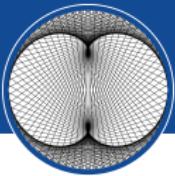


Proof continued.

Now

- $\sum_{n=1}^{\infty} b_n$ converges $\implies \sum_{n=1}^{\infty} a_n$ converges, by the corollary above, since $0 < a_n < \left(\frac{3l}{2}\right) b_n$ for all $n > N$; and
- $\sum_{n=1}^{\infty} a_n$ converges $\implies \sum_{n=1}^{\infty} b_n$ converges, by the corollary above, since $0 < b_n < \left(\frac{2}{l}\right) a_n$ for all $n > N$.

9.4 Comparison Tests

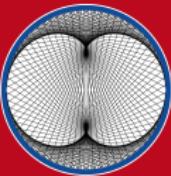


Proof continued.

Now

- $\sum_{n=1}^{\infty} b_n$ converges $\implies \sum_{n=1}^{\infty} a_n$ converges, by the corollary above, since $0 < a_n < \left(\frac{3l}{2}\right) b_n$ for all $n > N$; and
- $\sum_{n=1}^{\infty} a_n$ converges $\implies \sum_{n=1}^{\infty} b_n$ converges, by the corollary above, since $0 < b_n < \left(\frac{2}{l}\right) a_n$ for all $n > N$.

So the two series both converge, or both diverge. □



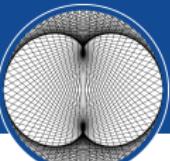
Break

We will continue at 3pm



KEEP
CALM
AND
PASS
CALCULUS

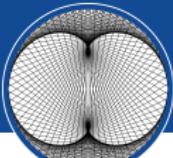
9.4 Comparison Tests



Lemma

$\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges.

9.4 Comparison Tests



Lemma

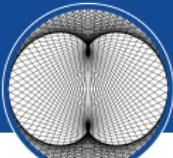
$$\sum_{n=1}^{\infty} \frac{1}{n^2} \text{ converges.}$$

Proof.

Let $a_n = \frac{1}{n^2}$ and $b_n = \frac{1}{n(n+1)}$. Earlier we showed that

$$\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \frac{1}{n(n+1)} \text{ converges.}$$

9.4 Comparison Tests



Lemma

$\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges.

Proof.

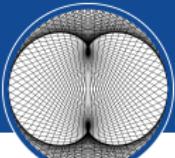
Let $a_n = \frac{1}{n^2}$ and $b_n = \frac{1}{n(n+1)}$. Earlier we showed that

$\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \frac{1}{n(n+1)}$ converges. Note that $\forall n \in \mathbb{N}$, $a_n > 0$ and $b_n > 0$. Moreover

$$\frac{a_n}{b_n} = \frac{n(n+1)}{n^2} = 1 + \frac{1}{n} \rightarrow 1$$

as $n \rightarrow \infty$.

9.4 Comparison Tests



Lemma

$$\sum_{n=1}^{\infty} \frac{1}{n^2} \text{ converges.}$$

Proof.

Let $a_n = \frac{1}{n^2}$ and $b_n = \frac{1}{n(n+1)}$. Earlier we showed that

$\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \frac{1}{n(n+1)}$ converges. Note that $\forall n \in \mathbb{N}$, $a_n > 0$ and $b_n > 0$. Moreover

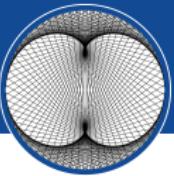
$$\frac{a_n}{b_n} = \frac{n(n+1)}{n^2} = 1 + \frac{1}{n} \rightarrow 1$$

as $n \rightarrow \infty$. It follows by the Limit Comparison Test that

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{1}{n^2} \text{ also converges.}$$



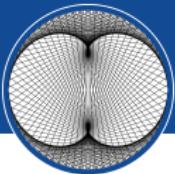
9.4 Comparison Tests



Lemma

$\sum_{n=1}^{\infty} \frac{1}{n^\alpha}$ converges for all $\alpha \geq 2$.

9.4 Comparison Tests



Lemma

$\sum_{n=1}^{\infty} \frac{1}{n^\alpha}$ converges for all $\alpha \geq 2$.

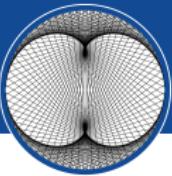
Proof.

Let $a_n = \frac{1}{n^\alpha}$ where $\alpha \geq 2$ and $b_n = \frac{1}{n^2}$. Then $\forall n \in \mathbb{N}$, $a_n > 0$, $b_n > 0$ and

$$0 < a_n = \frac{1}{n^\alpha} \leq \frac{1}{n^2} = b_n.$$

Since $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{1}{n^2}$ converges, it follows by the Comparison Test that $\sum_{n=1}^{\infty} a_n$ also converges. □

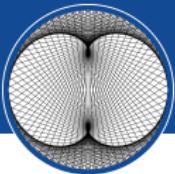
9.4 Comparison Tests



Lemma

$$\sum_{n=1}^{\infty} \frac{1}{n^\alpha} \text{ diverges for all } \alpha \leq 1.$$

9.4 Comparison Tests



Lemma

$$\sum_{n=1}^{\infty} \frac{1}{n^{\alpha}} \text{ diverges for all } \alpha \leq 1.$$

Proof.

Let $a_n = \frac{1}{n}$ and $b_n = \frac{1}{n^{\alpha}}$ where $\alpha \leq 1$. Then $\forall n \in \mathbb{N}$, $a_n > 0$, $b_n > 0$ and

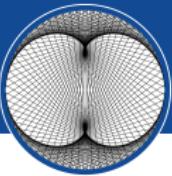
$$0 < a_n = \frac{1}{n} \leq \frac{1}{n^{\alpha}} = b_n.$$

Since $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges, it follows by the corollary above that

$\sum_{n=1}^{\infty} b_n$ also diverges.

□

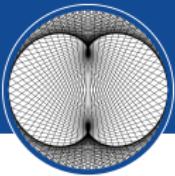
9.4 Comparison Tests



Lemma

$$\sum_{n=1}^{\infty} \sin \frac{1}{n} \text{ diverges.}$$

9.4 Comparison Tests



Lemma

$$\sum_{n=1}^{\infty} \sin \frac{1}{n} \text{ diverges.}$$

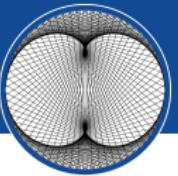
Proof.

Let $a_n = \sin \frac{1}{n}$ and $b_n = \frac{1}{n}$. Then $\forall n \in \mathbb{N}, 0 < \frac{1}{n} \leq 1 < \frac{\pi}{2}$. So $\sin \frac{1}{n} > 0$ for all $n \in \mathbb{N}$. Hence $\forall n \in \mathbb{N}, a_n > 0, b_n > 0$ and

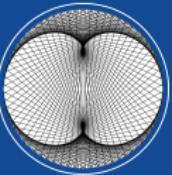
$$\frac{a_n}{b_n} = \frac{\sin \frac{1}{n}}{\frac{1}{n}} \rightarrow 1$$

as $n \rightarrow \infty$. Since $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges, it follows by the Limit Comparison Test that $\sum_{n=1}^{\infty} a_n$ also diverges. □

9.4 Comparison Tests

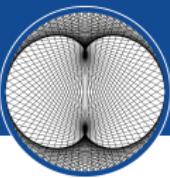


There are more examples in the textbook.



Absolute Convergence; The Ratio and Root Tests

9.5 Absolute Convergence; The Ratio and Root Tests



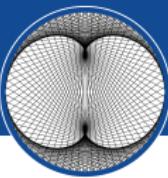
Theorem (The Ratio Test / Oran Testi)

Let $\sum_{n=1}^{\infty} a_n$ be a series of strictly positive real numbers (i.e. $a_n > 0$ for all n). Suppose that

$$\frac{a_{n+1}}{a_n} \rightarrow l \in \mathbb{R} \cup \{\infty\}$$

as $n \rightarrow \infty$.

9.5 Absolute Convergence; The Ratio and Root Tests



Theorem (The Ratio Test / Oran Testi)

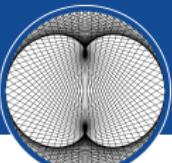
Let $\sum_{n=1}^{\infty} a_n$ be a series of strictly positive real numbers (i.e. $a_n > 0$ for all n). Suppose that

$$\frac{a_{n+1}}{a_n} \rightarrow l \in \mathbb{R} \cup \{\infty\}$$

as $n \rightarrow \infty$.

- 1 If $l < 1$, then $\sum_{n=1}^{\infty} a_n$ converges.
- 2 If $l > 1$, then $\sum_{n=1}^{\infty} a_n$ diverges.
- 3 If $l = \infty$, then $\sum_{n=1}^{\infty} a_n$ diverges.

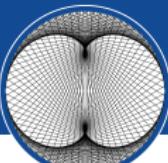
9.5 Absolute Convergence; The Ratio and Root Tests



Proof.

CASE 1 ($l < 1$): Let $k \in (l, 1)$. Then $k - l > 0$.

9.5 Absolute Convergence; The Ratio and Root Tests

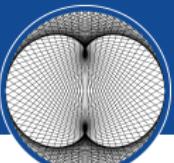


Proof.

CASE 1 ($l < 1$): Let $k \in (l, 1)$. Then $k - l > 0$. Since $\frac{a_{n+1}}{a_n} \rightarrow l$ as $n \rightarrow \infty$, $\exists N \in \mathbb{N}$ such that

$$n > N \implies \left| \frac{a_{n+1}}{a_n} - l \right| < k - l \implies \frac{a_{n+1}}{a_n} < k \implies a_{n+1} < ka_n.$$

9.5 Absolute Convergence; The Ratio and Root Tests



Proof.

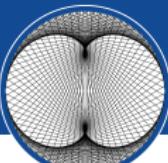
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Thus

$$\begin{aligned} n > N + 1 &\implies 0 < a_n < ka_{n-1} < k^2 a_{n-2} < \dots \\ &< k^{n-N-1} a_{N+1} = k^n \left(\frac{a_{N+1}}{k^{N+1}} \right). \end{aligned}$$

9.5 Absolute Convergence; The Ratio and Root Tests



Proof.

CASE 1 ($l < 1$): Let $k \in (l, 1)$. Then $k - l > 0$. Since $\frac{a_{n+1}}{a_n} \rightarrow l$ as $n \rightarrow \infty$, $\exists N \in \mathbb{N}$ such that

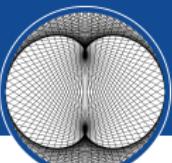
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Now $\frac{a_{N+1}}{k^{N+1}}$ is a constant, so $0 < a_n < k^n C$ for all $n > N + 1$.

9.5 Absolute Convergence; The Ratio and Root Tests



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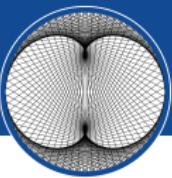
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Now $\frac{a_{N+1}}{k^{N+1}}$ is a constant, so $0 < a_n < k^n C$ for all $n > N + 1$.

We know that $\sum_{k=1}^{\infty} k^n$ converges, since $0 < k < 1$. By the Comparison Test, $\sum_{n=1}^{\infty} a_n$ also converges.

9.5 Absolute Convergence; The Ratio and Root Tests

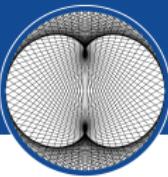


Proof continued.

CASE 2 ($l > 1$): Since $\frac{a_{n+1}}{a_n} \rightarrow l$ as $n \rightarrow \infty$, $\exists M$ such that

$$n > M \implies \frac{a_{n+1}}{a_n} > 1 \implies a_{n+1} > a_n.$$

9.5 Absolute Convergence; The Ratio and Root Tests



Proof continued.

CASE 2 ($l > 1$): Since $\frac{a_{n+1}}{a_n} \rightarrow l$ as $n \rightarrow \infty$, $\exists M$ such that

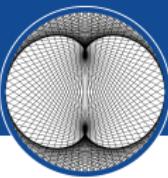
$$n > M \implies \frac{a_{n+1}}{a_n} > 1 \implies a_{n+1} > a_n.$$

So

$$n > M + 1 \implies a_n > a_{n-1} > a_{n-2} > \dots > a_{M+1}.$$

So $a_n \not\rightarrow 0$ as $n \rightarrow \infty$.

9.5 Absolute Convergence; The Ratio and Root Tests



Proof continued.

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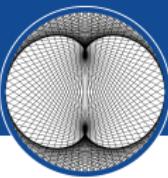
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Therefore $\sum_{n=1}^{\infty} a_n$ diverges.

9.5 Absolute Convergence; The Ratio and Root Tests



Proof continued.

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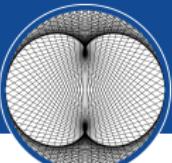
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Therefore $\sum_{n=1}^{\infty} a_n$ diverges.

CASE 3 ($l = \infty$): I leave this for you to prove.



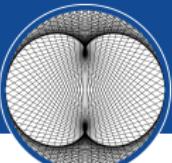
9.5 Absolute Convergence; The Ratio and Root Tests



Remark

If $l = 1$, the Ratio Test tells us nothing.

9.5 Absolute Convergence; The Ratio and Root Tests

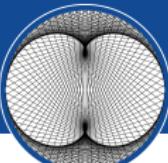


Remark

If $l = 1$, the Ratio Test tells us nothing.

For example, let $a_n = \frac{1}{n}$ and $b_n = \frac{1}{n^2}$. We know that $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges and $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges.

9.5 Absolute Convergence; The Ratio and Root Tests



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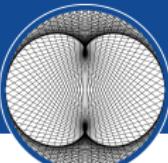
$$\frac{a_{n+1}}{a_n} = \frac{\frac{1}{n+1}}{\frac{1}{n}} = \frac{n}{n+1} = \frac{1}{1 + \frac{1}{n}} \rightarrow 1$$

as $n \rightarrow \infty$ and

$$\frac{b_{n+1}}{b_n} = \frac{\frac{1}{(n+1)^2}}{\frac{1}{n^2}} = \frac{n^2}{(n+1)^2} = \frac{1}{\left(1 + \frac{1}{n}\right)^2} \rightarrow 1$$

as $n \rightarrow \infty$.

9.5 Absolute Convergence; The Ratio and Root Tests



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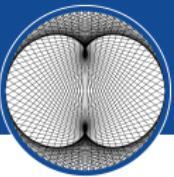
as $n \rightarrow \infty$ and

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as $n \rightarrow \infty$.

If we get $\frac{a_{n+1}}{a_n} \rightarrow 1$ as $n \rightarrow \infty$, then we cannot use the Ratio Test – we have to use a different test to see if $\sum_{n=1}^{\infty} a_n$ converges or diverges.

9.5 Absolute Convergence; The Ratio and Root Tests

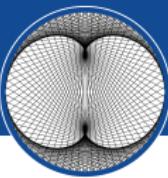


Remark

Moreover, note that if $a_n = \frac{1}{n}$, then $\frac{a_{n+1}}{a_n} < 1$ for all $n \in \mathbb{N}$.

Please remember that when we use the Ratio Test, we look at $\lim_{n \rightarrow \infty} \frac{a_{n+1}}{a_n}$, not at $\frac{a_{n+1}}{a_n}$.

9.5 Absolute Convergence; The Ratio and Root Tests

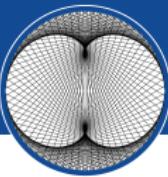


Example

Does $\sum_{n=1}^{\infty} \frac{(2n)!}{7^n(n!)^2}$ converge or diverge?

Let $z_n = \frac{(2n)!}{7^n(n!)^2}$. Then $z_n > 0$ for all $n \in \mathbb{N}$

9.5 Absolute Convergence; The Ratio and Root Tests



Example

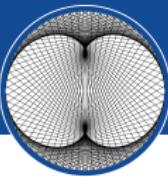
Does $\sum_{n=1}^{\infty} \frac{(2n)!}{7^n(n!)^2}$ converge or diverge?

Let $z_n = \frac{(2n)!}{7^n(n!)^2}$. Then $z_n > 0$ for all $n \in \mathbb{N}$ and

$$\begin{aligned}\frac{z_{n+1}}{z_n} &= \frac{(2n+2)!}{7^{n+1}((n+1)!)^2} \cdot \frac{7^n(n!)^2}{(2n)!} = \frac{(2n+2)(2n+1)}{7(n+1)^2} \\ &= \frac{(2 + \frac{2}{n})(2 + \frac{1}{n})}{7(1 + \frac{1}{n})(1 + \frac{1}{n})} \rightarrow \frac{4}{7} < 1\end{aligned}$$

as $n \rightarrow \infty$.

9.5 Absolute Convergence; The Ratio and Root Tests



Example

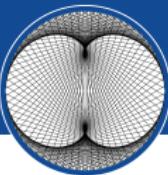
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as $n \rightarrow \infty$. By the Ratio Test, $\sum_{n=1}^{\infty} \frac{(2n)!}{7^n(n!)^2}$ converges.

9.5 Absolute Convergence; The Ratio and Root Tests

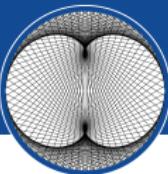


Example

Does $\sum_{n=1}^{\infty} n^2 e^{-n(n+1)}$ converge or diverge?

Let $y_n = n^2 e^{-n(n+1)}$. Then $y_n > 0$ for all $n \in \mathbb{N}$

9.5 Absolute Convergence; The Ratio and Root Tests



Example

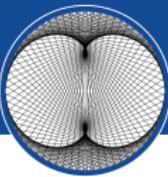
Does $\sum_{n=1}^{\infty} n^2 e^{-n(n+1)}$ converge or diverge?

Let $y_n = n^2 e^{-n(n+1)}$. Then $y_n > 0$ for all $n \in \mathbb{N}$ and

$$\frac{y_{n+1}}{y_n} = \frac{(n+1)^2 e^{-(n+1)(n+2)}}{n^2 e^{-n(n+1)}} = \left(1 + \frac{1}{n}\right)^2 e^{-2(n+1)} \rightarrow 0 < 1$$

as $n \rightarrow \infty$.

9.5 Absolute Convergence; The Ratio and Root Tests



Example

Does $\sum_{n=1}^{\infty} n^2 e^{-n(n+1)}$ converge or diverge?

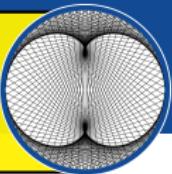
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as $n \rightarrow \infty$. By the Ratio Test, $\sum_{n=1}^{\infty} n^2 e^{-n(n+1)}$ converges.

9.5 Absolute Convergence Tests

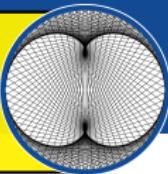
$$\sum_{k=0}^{\infty} x^k \begin{cases} \text{converges if } |x| < 1 \\ \text{diverges if } |x| \geq 1. \end{cases}$$



When some of the terms in a series are positive and some are negative then the series may or may not converge.

9.5 Absolute Convergence Tests

$$\sum_{k=0}^{\infty} x^k \begin{cases} \text{converges if } |x| < 1 \\ \text{diverges if } |x| \geq 1. \end{cases}$$



When some of the terms in a series are positive and some are negative then the series may or may not converge.

For example, the geometric series

$$5 - \frac{5}{4} + \frac{5}{16} - \frac{5}{64} + \frac{5}{256} - \dots$$

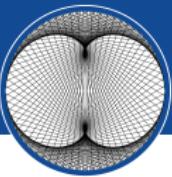
converges because $x = -\frac{1}{4}$ and $\left|-\frac{1}{4}\right| < 1$.

However, the geometric series

$$1 - \frac{5}{4} + \frac{25}{16} - \frac{125}{64} + \frac{625}{256} - \dots$$

diverges because $x = -\frac{5}{4}$ and $\left|-\frac{5}{4}\right| \geq 1$.

9.5 Absolute Convergence; The Ratio and Root Tests



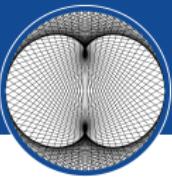
Definition

Let $\sum_{n=1}^{\infty} a_n$ be a series of real numbers. If $\sum_{n=1}^{\infty} |a_n|$ is convergent,

then we say that $\sum_{n=1}^{\infty} a_n$ is *absolutely convergent*.

(We can also say that $\sum_{k=1}^{\infty} a_k$ converges absolutely in this case.)

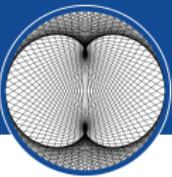
9.5 Absolute Convergence; The Ratio and Root Tests



Theorem

Every absolutely convergent series is convergent.

9.5 Absolute Convergence; The Ratio and Root Tests



Theorem

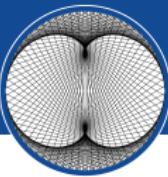
Every absolutely convergent series is convergent.

Remark

The theorem says that

$$\sum_{n=1}^{\infty} |a_n| \text{ converges} \implies \sum_{n=1}^{\infty} a_n \text{ converges.}$$

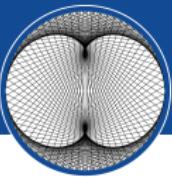
9.5 Absolute Convergence; The Ratio and Root Tests



Proof.

Let $\sum_{k=1}^{\infty} a_k$ be absolutely convergent. Then $\sum_{n=1}^{\infty} |a_n|$ converges.

9.5 Absolute Convergence; The Ratio and Root Tests



Proof.

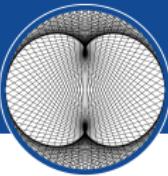
Let $\sum_{k=1}^{\infty} a_k$ be absolutely convergent. Then $\sum_{n=1}^{\infty} |a_n|$ converges.

Some of the a_n might be ≥ 0 and some might be < 0 . We want to separate these two types of a_n . Define

$$b_n := \begin{cases} a_n & \text{if } a_n \geq 0 \\ 0 & \text{if } a_n < 0 \end{cases} \quad \text{and} \quad c_n := \begin{cases} 0 & \text{if } a_n \geq 0 \\ -a_n & \text{if } a_n < 0. \end{cases}$$

Note that $b_n \geq 0 \ \forall n$, $c_n \geq 0 \ \forall n$ and $a_n = b_n - c_n$.

9.5 Absolute Convergence; The Ratio and Root Tests



Proof continued.

Let

$$s_n = |a_1| + |a_2| + |a_3| + \dots + |a_n|,$$

$$t_n = a_1 + a_2 + a_3 + \dots + a_n,$$

$$r_n = b_1 + b_2 + b_3 + \dots + b_n,$$

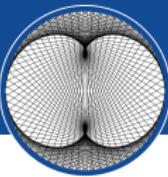
$$u_n = c_1 + c_2 + c_3 + \dots + c_n.$$

Now

$$\sum_{n=1}^{\infty} |a_n| \text{ converges} \implies s_n \rightarrow s \text{ as } n \rightarrow \infty.$$

We want to prove that (t_n) converges.

9.5 Absolute Convergence; The Ratio and Root Tests



Proof continued.

Let

$$s_n = |a_1| + |a_2| + |a_3| + \dots + |a_n|,$$

$$t_n = a_1 + a_2 + a_3 + \dots + a_n,$$

$$r_n = b_1 + b_2 + b_3 + \dots + b_n,$$

$$u_n = c_1 + c_2 + c_3 + \dots + c_n.$$

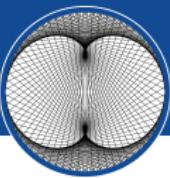
Now

$$\sum_{n=1}^{\infty} |a_n| \text{ converges} \implies s_n \rightarrow s \text{ as } n \rightarrow \infty.$$

We want to prove that (t_n) converges.

Since $|a_k| \geq 0$ for all $k \in \mathbb{N}$, (s_n) is increasing. So $s_n \leq s$ for all $n \in \mathbb{N}$.

9.5 Absolute Convergence; The Ratio and Root Tests



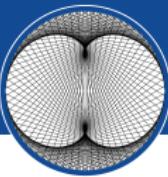
Proof continued.

Hence

$$r_n = b_1 + b_2 + b_3 + \dots + b_n \leq |a_1| + |a_2| + |a_3| + \dots + |a_n| = s_n \leq s$$

for all $n \in \mathbb{N}$. Since $b_k \geq 0$ for all $k \in \mathbb{N}$, (r_k) is an increasing sequence which is bounded above. So $r_n \rightarrow r$ as $n \rightarrow \infty$.

9.5 Absolute Convergence; The Ratio and Root Tests



Proof continued.

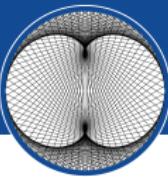
Hence

$$r_n = b_1 + b_2 + b_3 + \dots + b_n \leq |a_1| + |a_2| + |a_3| + \dots + |a_n| = s_n \leq s$$

for all $n \in \mathbb{N}$. Since $b_k \geq 0$ for all $k \in \mathbb{N}$, (r_k) is an increasing sequence which is bounded above. So $r_n \rightarrow r$ as $n \rightarrow \infty$.

Similarly $u_n \rightarrow u$ as $n \rightarrow \infty$.

9.5 Absolute Convergence; The Ratio and Root Tests



Proof continued.

Hence

$$r_n = b_1 + b_2 + b_3 + \dots + b_n \leq |a_1| + |a_2| + |a_3| + \dots + |a_n| = s_n \leq s$$

for all $n \in \mathbb{N}$. Since $b_k \geq 0$ for all $k \in \mathbb{N}$, (r_k) is an increasing sequence which is bounded above. So $r_n \rightarrow r$ as $n \rightarrow \infty$.

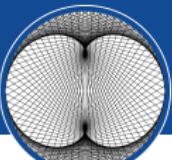
Similarly $u_n \rightarrow u$ as $n \rightarrow \infty$.

Therefore $t_n = r_n - u_n \rightarrow r - u$ as $n \rightarrow \infty$.

So $\sum_{k=1}^{\infty} a_k$ converges.

□

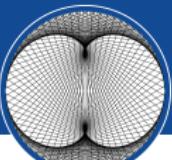
9.5 Absolute Convergence; The Ratio and Root Tests



Remark

$\sum_{n=1}^{\infty} a_n$ is absolutely convergent $\implies \sum_{n=1}^{\infty} |a_n|$ is convergent.

9.5 Absolute Convergence; The Ratio and Root Tests



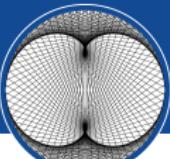
Remark

$\sum_{n=1}^{\infty} a_n$ is absolutely convergent $\implies \sum_{n=1}^{\infty} |a_n|$ is convergent.

But

$\sum_{n=1}^{\infty} a_n$ is convergent $\not\implies \sum_{n=1}^{\infty} |a_n|$ is absolutely convergent.

9.5 Absolute Convergence; The Ratio and Root Tests



Remark

$$\sum_{n=1}^{\infty} a_n \text{ is absolutely convergent} \implies \sum_{n=1}^{\infty} |a_n| \text{ is convergent.}$$

But

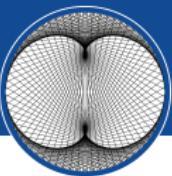
$$\sum_{n=1}^{\infty} a_n \text{ is convergent} \not\implies \sum_{n=1}^{\infty} |a_n| \text{ is absolutely convergent.}$$

For example, consider $a_n = \frac{(-1)^{n+1}}{n}$. The series

$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} - \frac{1}{8} + \dots$ is convergent, but the series

$\sum_{n=1}^{\infty} \left| \frac{(-1)^{n+1}}{n} \right| = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \dots$ is divergent.

9.5 Absolute Convergence; The Ratio and Root Tests



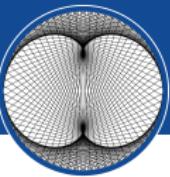
Corollary (The Triangle Inequality)

If $\sum_{k=1}^{\infty} a_k$ is absolutely convergent, then

$$\left| \sum_{n=1}^{\infty} a_n \right| \leq \sum_{n=1}^{\infty} |a_n|.$$

(you prove)

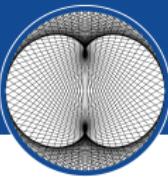
9.5 Absolute Convergence; The Ratio and Root Tests



Example

Is $\sum_{n=1}^{\infty} (-1)^{n+1} \sin^2 \frac{1}{n}$ absolutely convergent?

9.5 Absolute Convergence; The Ratio and Root Tests

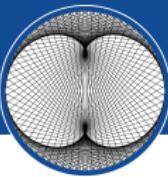


Example

Is $\sum_{n=1}^{\infty} (-1)^{n+1} \sin^2 \frac{1}{n}$ absolutely convergent?

First note that $|(-1)^{n+1} \sin^2 \frac{1}{n}| = \sin^2 \frac{1}{n}$.

9.5 Absolute Convergence; The Ratio and Root Tests



Example

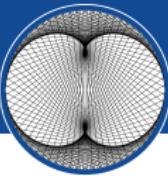
Is $\sum_{n=1}^{\infty} (-1)^{n+1} \sin^2 \frac{1}{n}$ absolutely convergent?

First note that $|(-1)^{n+1} \sin^2 \frac{1}{n}| = \sin^2 \frac{1}{n}$. Let $a_n = \sin^2 \frac{1}{n}$ and $b_n = \frac{1}{n^2}$ for all $n \in \mathbb{N}$. Then $a_n > 0$ and $b_n > 0$ for all n , and

$$\frac{a_n}{b_n} = \left(\frac{\sin \frac{1}{n}}{\frac{1}{n}} \right) \left(\frac{\sin \frac{1}{n}}{\frac{1}{n}} \right) \rightarrow 1 \cdot 1 = 1$$

as $n \rightarrow \infty$.

9.5 Absolute Convergence; The Ratio and Root Tests



Example

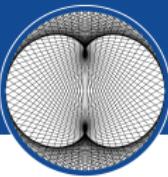
Is $\sum_{n=1}^{\infty} (-1)^{n+1} \sin^2 \frac{1}{n}$ absolutely convergent?

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as $n \rightarrow \infty$. We know that $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges. By the Limit Comparison Test, $\sum_{n=1}^{\infty} \sin^2 \frac{1}{n}$ also converges.

9.5 Absolute Convergence; The Ratio and Root Tests



Example

Is $\sum_{n=1}^{\infty} (-1)^{n+1} \sin^2 \frac{1}{n}$ absolutely convergent?

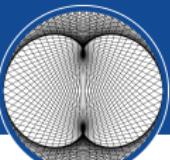
First note that $|(-1)^{n+1} \sin^2 \frac{1}{n}| = \sin^2 \frac{1}{n}$. Let $a_n = \sin^2 \frac{1}{n}$ and $b_n = \frac{1}{n^2}$ for all $n \in \mathbb{N}$. Then $a_n > 0$ and $b_n > 0$ for all n , and

$$\frac{a_n}{b_n} = \left(\frac{\sin \frac{1}{n}}{\frac{1}{n}} \right) \left(\frac{\sin \frac{1}{n}}{\frac{1}{n}} \right) \rightarrow 1 \cdot 1 = 1$$

as $n \rightarrow \infty$. We know that $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges. By the Limit Comparison Test, $\sum_{n=1}^{\infty} \sin^2 \frac{1}{n}$ also converges.

Hence $\sum_{n=1}^{\infty} |(-1)^{n+1} \sin^2 \frac{1}{n}|$ converges and therefore $\sum_{n=1}^{\infty} (-1)^{n+1} \sin^2 \frac{1}{n}$ converges absolutely.

9.5 Absolute Convergence; The Ratio and Root Tests



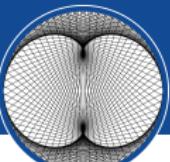
Theorem (The Ratio Test v2)

Let $\sum_{k=1}^{\infty} a_k$ be a series of non-zero real numbers (i.e. $a_n \neq 0 \ \forall n$). Suppose that

$$\left| \frac{a_{n+1}}{a_n} \right| \rightarrow l \in \mathbb{R} \cup \{\infty\}$$

as $n \rightarrow \infty$.

9.5 Absolute Convergence; The Ratio and Root Tests



Theorem (The Ratio Test v2)

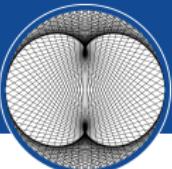
Let $\sum_{k=1}^{\infty} a_k$ be a series of non-zero real numbers (i.e. $a_n \neq 0 \ \forall n$). Suppose that

$$\left| \frac{a_{n+1}}{a_n} \right| \rightarrow l \in \mathbb{R} \cup \{\infty\}$$

as $n \rightarrow \infty$.

- 1 If $l < 1$, then $\sum_{k=1}^{\infty} a_k$ is absolutely convergent.
- 2 If $l > 1$, then $\sum_{k=1}^{\infty} a_k$ is divergent.
- 3 If $l = \infty$, then $\sum_{k=1}^{\infty} a_k$ is divergent.

9.5 Absolute Convergence; The Ratio and Root Tests



Proof.

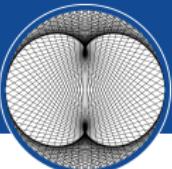
Let $b_n = |a_n|$. Then $b_n > 0 \forall n$ and

$$\frac{b_{n+1}}{b_n} = \frac{|a_{n+1}|}{|a_n|} = \left| \frac{a_{n+1}}{a_n} \right| \rightarrow l$$

as $n \rightarrow \infty$. Then we can use the Ratio Test to see that

$$l < 1 \implies \sum_{n=1}^{\infty} b_n \text{ converges} \implies \sum_{n=1}^{\infty} a_n \text{ converges absolutely.}$$

9.5 Absolute Convergence; The Ratio and Root Tests



Proof.

Let $b_n = |a_n|$. Then $b_n > 0 \forall n$ and

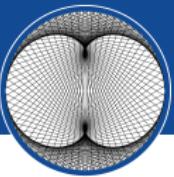
$$\frac{b_{n+1}}{b_n} = \frac{|a_{n+1}|}{|a_n|} = \left| \frac{a_{n+1}}{a_n} \right| \rightarrow l$$

as $n \rightarrow \infty$. Then we can use the Ratio Test to see that

$$l < 1 \implies \sum_{n=1}^{\infty} b_n \text{ converges} \implies \sum_{n=1}^{\infty} a_n \text{ converges absolutely.}$$

If $l > 1$ or $l = \infty$, then $\exists N$ such that $\frac{b_{n+1}}{b_n} > 1 \forall n > N$. Hence $b_n > b_{N+1} \forall n > N + 1$. Therefore $b_n \not\rightarrow 0$ as $n \rightarrow \infty$. So $a_n \not\rightarrow 0$ as $n \rightarrow \infty$ and thus $\sum_{k=1}^{\infty} a_k$ diverges by the Divergence Test. □

9.5 Absolute Convergence; The Ratio and Root Tests



Remark

If $l = 1$, then the Ratio Test v2 tells us nothing.

EXAMPLE 2

Investigate the convergence of the following series.

$$(a) \sum_{n=0}^{\infty} \frac{2^n + 5}{3^n}$$

$$(b) \sum_{n=1}^{\infty} \frac{(2n)!}{n!n!}$$

Solution We apply the Ratio Test to each series.

(a) For the series $\sum_{n=0}^{\infty} (2^n + 5)/3^n$,

$$\left| \frac{a_{n+1}}{a_n} \right| = \frac{(2^{n+1} + 5)/3^{n+1}}{(2^n + 5)/3^n} = \frac{1}{3} \cdot \frac{2^{n+1} + 5}{2^n + 5} = \frac{1}{3} \cdot \left(\frac{2 + 5 \cdot 2^{-n}}{1 + 5 \cdot 2^{-n}} \right) \rightarrow \frac{1}{3} \cdot \frac{2}{1} = \frac{2}{3}.$$

The series converges absolutely (and thus converges) because $\rho = 2/3$ is less than 1. This does *not* mean that $2/3$ is the sum of the series. In fact,

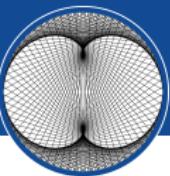
$$\sum_{n=0}^{\infty} \frac{2^n + 5}{3^n} = \sum_{n=0}^{\infty} \left(\frac{2}{3} \right)^n + \sum_{n=0}^{\infty} \frac{5}{3^n} = \frac{1}{1 - (2/3)} + \frac{5}{1 - (1/3)} = \frac{21}{2}.$$

(b) If $a_n = \frac{(2n)!}{n!n!}$, then $a_{n+1} = \frac{(2n+2)!}{(n+1)!(n+1)!}$ and

$$\begin{aligned} \left| \frac{a_{n+1}}{a_n} \right| &= \frac{n!n!(2n+2)(2n+1)(2n)!}{(n+1)!(n+1)!(2n)!} \\ &= \frac{(2n+2)(2n+1)}{(n+1)(n+1)} = \frac{4n+2}{n+1} \rightarrow 4. \end{aligned}$$

The series diverges because $\rho = 4$ is greater than 1.

9.5 Absolute Convergence; The Ratio and Root Tests



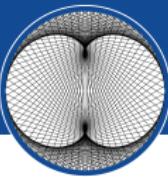
Theorem (The Root Test / Kök Testi)

Suppose that

$$\sqrt[n]{|a_n|} \rightarrow l$$

as $n \rightarrow \infty$.

9.5 Absolute Convergence; The Ratio and Root Tests



Theorem (The Root Test / Kök Testi)

Suppose that

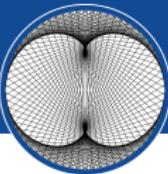
$$\sqrt[n]{|a_n|} \rightarrow l$$

as $n \rightarrow \infty$.

1 $l < 1 \implies \sum_{k=1}^{\infty} a_k$ converges absolutely;

2 $l > 1$ (or $l = \infty$) $\implies \sum_{k=1}^{\infty} a_k$ diverges.

9.5 Absolute Convergence; The Ratio and Root Tests



Theorem (The Root Test / Kök Testi)

Suppose that

$$\sqrt[n]{|a_n|} \rightarrow l$$

as $n \rightarrow \infty$.

1 $l < 1 \implies \sum_{k=1}^{\infty} a_k$ converges absolutely;

2 $l > 1$ (or $l = \infty$) $\implies \sum_{k=1}^{\infty} a_k$ diverges.

(proof in textbook)

Remark

If $l = 1$, then the Root Test tells us nothing.

EXAMPLE 3 Consider again the series with terms $a_n = \begin{cases} n/2^n, & n \text{ odd} \\ 1/2^n, & n \text{ even.} \end{cases}$

Does $\sum a_n$ converge?

Solution We apply the Root Test, finding that

$$\sqrt[n]{|a_n|} = \begin{cases} \sqrt[n]{n}/2, & n \text{ odd} \\ 1/2, & n \text{ even.} \end{cases}$$

Therefore,

$$\frac{1}{2} \leq \sqrt[n]{|a_n|} \leq \frac{\sqrt[n]{n}}{2}.$$

Since $\sqrt[n]{n} \rightarrow 1$ (Section 10.1, Theorem 5), we have $\lim_{n \rightarrow \infty} \sqrt[n]{|a_n|} = 1/2$ by the Sandwich Theorem. The limit is less than 1, so the series converges absolutely by the Root Test. ■

EXAMPLE 4 Which of the following series converge, and which diverge?

(a) $\sum_{n=1}^{\infty} \frac{n^2}{2^n}$

(b) $\sum_{n=1}^{\infty} \frac{2^n}{n^3}$

(c) $\sum_{n=1}^{\infty} \left(\frac{1}{1+n} \right)^n$

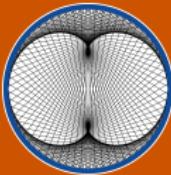
Solution We apply the Root Test to each series, noting that each series has positive terms.

(a) $\sum_{n=1}^{\infty} \frac{n^2}{2^n}$ converges because $\sqrt[n]{\frac{n^2}{2^n}} = \frac{\sqrt[n]{n^2}}{\sqrt[n]{2^n}} = \frac{(\sqrt[n]{n})^2}{2} \rightarrow \frac{1^2}{2} < 1$.

(b) $\sum_{n=1}^{\infty} \frac{2^n}{n^3}$ diverges because $\sqrt[n]{\frac{2^n}{n^3}} = \frac{2}{(\sqrt[n]{n})^3} \rightarrow \frac{2}{1^3} > 1$.

(c) $\sum_{n=1}^{\infty} \left(\frac{1}{1+n} \right)^n$ converges because $\sqrt[n]{\left(\frac{1}{1+n} \right)^n} = \frac{1}{1+n} \rightarrow 0 < 1$.





Next Time

- 9.6 Alternating Series and Conditional Convergence
- 9.7 Power Series
- 9.8 Taylor and Maclaurin Series