2019–20 **MATH216 Mathematics IV – Exercise Sheet 9** N. Course

Exercise 32 (Systems of First Order Linear Equations). Transform each of the following equations into a system of first order linear ODEs.

(a) $u'' + 0.5u' + 2u = 0$

(b) $u'' + 0.5u' + 2u = 3\sin t$

(c) $t^2u'' + tu' + (t^2 - 0.25)u = 0$

(d) $u^{(4)} - u = 0$

Transform each of the following systems into a single second order ODE for x_1.

(e) $\begin{cases} x_1' = 3x_1 - 2x_2 \\ x_2' = 2x_1 - 2x_2 \end{cases}$ (f) $\begin{cases} x_1' = 1.25x_1 + 0.75x_2 \\ x_2' = 0.75x_1 + 1.25x_2 \end{cases}$ (g) $\begin{cases} x_1' = x_1 - 2x_2 \\ x_2' = 3x_1 - 4x_2 \end{cases}$ (h) $\begin{cases} x_1' = 2x_2 \\ x_2' = -2x_1 \end{cases}$

Exercise 33 (Fundamental Matrices).

(a) Suppose that $\Psi(t)$ is a fundamental matrix for $\mathbf{x}' = A\mathbf{x}$, where $A \in \mathbb{R}^{n\times n}$. Show that $\Psi' = A\Psi$.

(b) Let $A = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}$. Find $e^{At} = \exp(At)$.

For each of the following;

(i) Find a fundamental matrix $\Psi(t)$ for the system; and

(ii) Find the special fundamental matrix $\Phi(t)$ which satisfies $\Phi(0) = I$.

The first one is done for you.

(ω) $\mathbf{x}' = \begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix}\mathbf{x}$

(c) $\mathbf{x}' = \begin{bmatrix} 3 & -2 \\ 2 & -2 \end{bmatrix}\mathbf{x}$

(d) $\mathbf{x}' = \begin{bmatrix} -\frac{3}{4} & \frac{1}{2} \\ \frac{1}{8} & -\frac{3}{4} \end{bmatrix}\mathbf{x}$

(e) $\mathbf{x}' = \begin{bmatrix} 2 & -5 \\ 1 & -2 \end{bmatrix}\mathbf{x}$

(f) $\mathbf{x}' = \begin{bmatrix} 1 & 1 \\ 4 & -2 \end{bmatrix}\mathbf{x}$

(g) $\mathbf{x}' = \begin{bmatrix} 2 & -1 \\ 3 & -2 \end{bmatrix}\mathbf{x}$

(h) $\mathbf{x}' = \begin{bmatrix} -1 & -4 \\ 1 & -1 \end{bmatrix}\mathbf{x}$

(i) $\mathbf{x}' = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 1 & -1 \\ -8 & -5 & -3 \end{bmatrix}\mathbf{x}$

(ω) (i) The general solution to this system is $\mathbf{x}(t) = c_1\begin{bmatrix} 1 \\ 2 \end{bmatrix}e^{3t} + c_2\begin{bmatrix} 1 \\ -2 \end{bmatrix}e^{-t}$. Therefore $\Psi(t) = \begin{bmatrix} e^{3t} & e^{-t} \\ 2e^{3t} & -2e^{-t} \end{bmatrix}$ is a fundamental matrix for this system.

(ii) We must solve $\begin{cases} \mathbf{x}' = \begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix}\mathbf{x} \\ \mathbf{x}(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \end{cases}$ and $\begin{cases} \mathbf{x}' = \begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix}\mathbf{x} \\ \mathbf{x}(0) = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \end{cases}$. Using the general solution from above, we calculate that

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \mathbf{x}(0) = c_1\begin{bmatrix} 1 \\ 2 \end{bmatrix} + c_2\begin{bmatrix} 1 \\ -2 \end{bmatrix} \implies c_1 = c_2 = \frac{1}{2} \implies \mathbf{x}^{(1)}(t) = \begin{bmatrix} \frac{1}{2}e^{3t} + \frac{1}{2}e^{-t} \\ e^{3t} - e^{-t} \end{bmatrix}$$

and

$$\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \mathbf{x}(0) = c_1\begin{bmatrix} 1 \\ 2 \end{bmatrix} + c_2\begin{bmatrix} 1 \\ -2 \end{bmatrix} \implies c_1 = \frac{1}{4},\ c_2 = -\frac{1}{4} \implies \mathbf{x}^{(2)}(t) = \begin{bmatrix} \frac{1}{4}e^{3t} - \frac{1}{4}e^{-t} \\ \frac{1}{2}e^{3t} + \frac{1}{2}e^{-t} \end{bmatrix}$$

Therefore

$$\Phi(t) = \begin{bmatrix} \frac{1}{2}e^{3t} + \frac{1}{2}e^{-t} & \frac{1}{4}e^{3t} - \frac{1}{4}e^{-t} \\ e^{3t} - e^{-t} & \frac{1}{2}e^{3t} + \frac{1}{2}e^{-t} \end{bmatrix}.$$