Welcome to

# Mathematics II

with Dr Neil Course

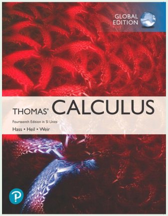
# Lecture 1

- Information about this course
- 8.1 Using Basic Integration Formulae
- 8.2 Integration by Parts
- 8.3 Trigonometric Integrals


neilcourse@gmail.com

MATH113 MATH114 MATH215 MATH216

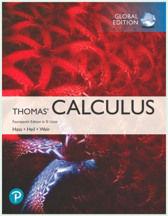
MATH113 MATH114 MATH215 MATH216




Calculus

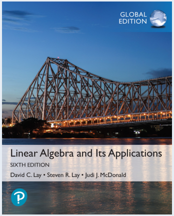
# MATH114 Mathematics II

MATH113 MATH114




Calculus

MATH215

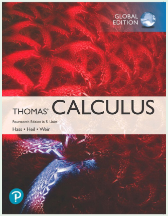



Linear Algebra

MATH216

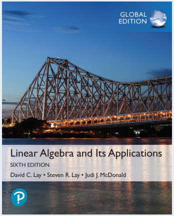
# MATH114 Mathematics II

MATH113 MATH114



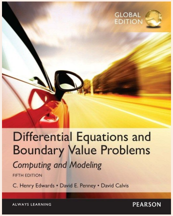

Calculus

MATH215




Linear Algebra

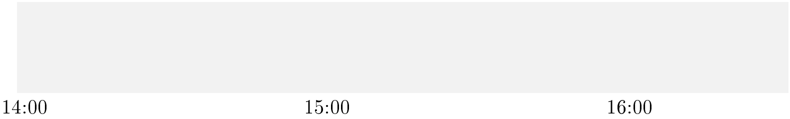
MATH216




Differential Equations

# Information about this course

- $\approx 12$ classes. Friday afternoons 2pm-4:30pm.



# Information about this course

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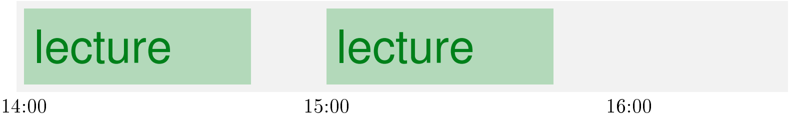
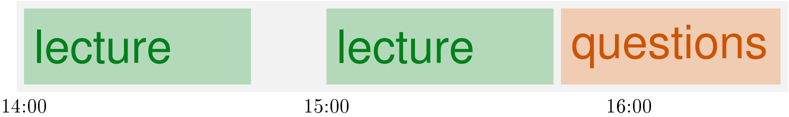
- 2 lectures with a break between.

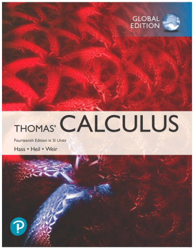


# Information about this course

- ≈ 12 classes. Friday afternoons 2pm-4:30pm.
- 2 lectures with a break between.
- Then I will answers your questions.

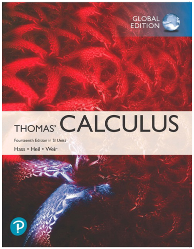


# The Book




Joel R. Hass, Christopher E. Heil and Maurice D. Weir,
*Thomas' Calculus*,
14th Edition in SI Units, Pearson.

# The Book




Joel R. Hass, Christopher E. Heil and Maurice D. Weir,
*Thomas' Calculus*,
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This is a required purchase.
You need to have this book to be able to do the homework.

| |
|---|
| 8 |
| 11 |
| 13 |
| 14 |
| 9 |

| 8. Techniques of Integration |
| --- |
| 11 |
| 13 |
| 14 |
| 9 |

8. Techniques of Integration

11. Vectors and the Geometry of Space

13

14

9

8. Techniques of Integration

11. Vectors and the Geometry of Space

13. Partial Derivatives

14

9

**8. Techniques of Integration**

**11. Vectors and the Geometry of Space**

**13. Partial Derivatives**

**14. Multiple Integrals**

**9**

**8. Techniques of Integration**

**11. Vectors and the Geometry of Space**

**13. Partial Derivatives**

**14. Multiple Integrals**

**9. Infinite Sequences and Series**

| | |
|---|---|
| **8. Techniques of Integration** | 2 weeks |
| **11. Vectors and the Geometry of Space** | 2 weeks |
| **13. Partial Derivatives** | 2 weeks |
| **14. Multiple Integrals** | 3 weeks |
| **9. Infinite Sequences and Series** | 3 weeks |

# Exams and homework

(This information may change based on the University's decisions)

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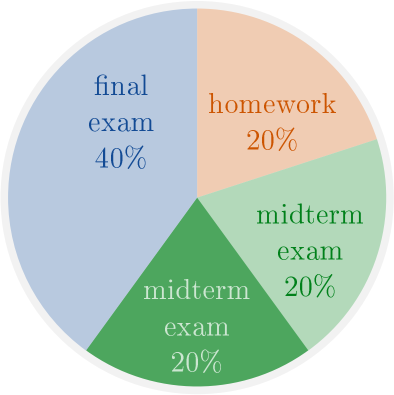

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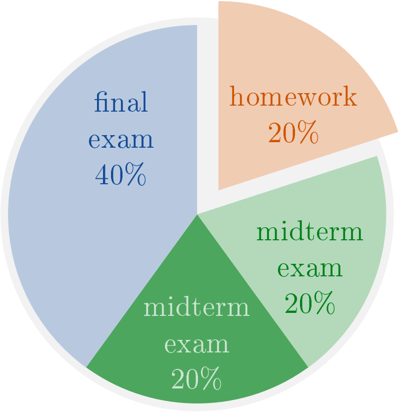


using Pearson's MyLab Math

12 pieces of homework

deadline = one day before the final exam

more details in O'Learn

# Exams and homework

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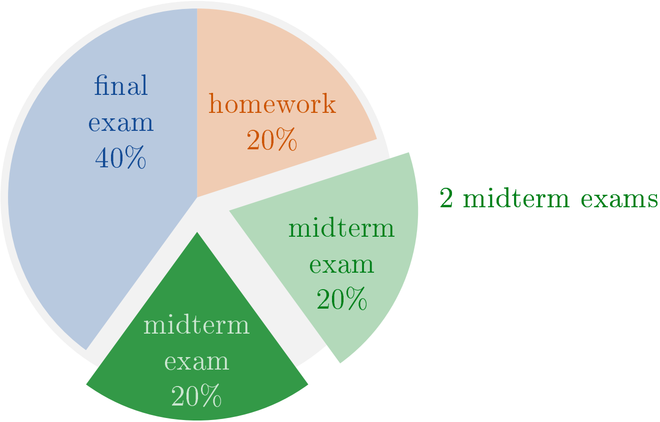

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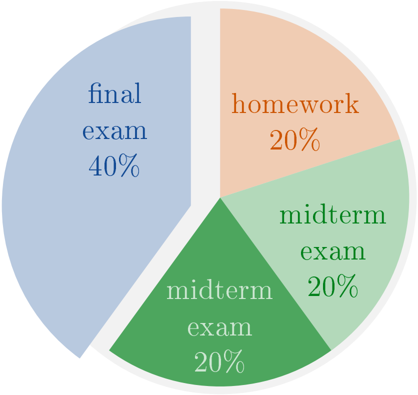

# Expectations

If this was a classroom course, the expectation is that for each hour of lectures, you would study an extra 1-2 hours outside of class.

| | | |
|---|---|---|
| classroom course | lectures (5 hours) | other study (5-10 hours) |

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| classroom course | lectures (5 hours) | other study (5-10 hours) |

For an online course, you are still expected to study a total of 10-15 hours each week.

| | | |
|---|---|---|
| online course | class (2.5 hours) | other study (7.5-12.5 hours) |

This may include:

- Do the online homework on MyLab;

$\vdots$

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- Use the O'Learn Discussion Board;
- Read other books?;
- Watch online videos;

⋮

slide number

10 minute break

current section

$$1 + 2 = 3$$

something important

# Using Basic Integration Formulae

# 22 Basic Integration Formulae

Let's start with a little bit of revision

## 22 Basic Integration Formulae

1. $\int k\,dx = kx + C$ $\quad$ ($k$ is a number)

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2. $\displaystyle\int x^n\,dx = \frac{x^{n+1}}{n+1} + C$

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4. $\displaystyle\int e^x\,dx = e^x + C$

# 22 Basic Integration Formulae

1. $\displaystyle\int k\,dx = kx + C$ $\qquad$ ($k$ is a number)
2. $\displaystyle\int x^n\,dx = \frac{x^{n+1}}{n+1} + C$
3. $\displaystyle\int \frac{1}{x}\,dx = \ln|x| + C$
4. $\displaystyle\int e^x\,dx = e^x + C$
5. $\displaystyle\int a^x\,dx = \frac{a^x}{\ln a} + C$ $\qquad$ $(a > 0,\ a \neq 1)$

# 22 Basic Integration Formulae

6. $$\int \sin x \, dx = -\cos x + C$$
7. $$\int \cos x \, dx = \sin x + C$$

## 22 Basic Integration Formulae

6 $\displaystyle\int \sin x\,dx = -\cos x + C$

7 $\displaystyle\int \cos x\,dx = \sin x + C$

8 $\displaystyle\int \sec^2 x\,dx = \tan x + C$

9 $\displaystyle\int \operatorname{cosec}^2 x\,dx = -\cot x + C$

10 $\displaystyle\int \sec x \tan x\,dx = \sec x + C$

11 $\displaystyle\int \operatorname{cosec} x \cot x\,dx = -\operatorname{cosec} x + C$

## 22 Basic Integration Formulae

12 $\displaystyle\int \tan x\,dx = \ln|\sec x| + C$

13 $\displaystyle\int \cot x\,dx = \ln|\sin x| + C$

14 $\displaystyle\int \sec x\,dx = \ln|\sec x + \tan x| + C$

15 $\displaystyle\int \operatorname{cosec} x\,dx = -\ln|\operatorname{cosec} x + \cot x| + C$

## 22 Basic Integration Formulae

16 $\displaystyle\int \sinh x \, dx = \cosh x + C$

17 $\displaystyle\int \cosh x \, dx = \sinh x + C$

## 22 Basic Integration Formulae

18 $$\int \frac{dx}{\sqrt{a^2 - x^2}} = \sin^{-1}\left(\frac{x}{a}\right) + C$$

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## 22 Basic Integration Formulae

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22 $$\int \frac{dx}{\sqrt{x^2 - a^2}} = \cosh^{-1}\left(\frac{x}{a}\right) + C \qquad (x > a > 0)$$

**Example**

Calculate $\displaystyle\int_{3}^{5} \frac{2x-3}{\sqrt{x^2-3x+1}}\,dx.$

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We use the substitution $u = x^2 - 3x + 1$.

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**Example**

Calculate $\displaystyle\int_3^5 \frac{2x-3}{\sqrt{x^2-3x+1}}\,dx$.

We use the substitution $u = x^2 - 3x + 1$. Then $du = (2x-3)\,dx$ and

$$x = 5 \implies u = x^2 - 3x + 1 = 25 - 15 + 1 = 11$$
$$x = 3 \implies u = x^2 - 3x + 1 = 9 - 9 + 1 = 1.$$

**Example**

Calculate $\displaystyle\int_3^5 \frac{2x-3}{\sqrt{x^2-3x+1}}\,dx$.

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$$x = 5 \implies u = x^2 - 3x + 1 = 25 - 15 + 1 = 11$$
$$x = 3 \implies u = x^2 - 3x + 1 = 9 - 9 + 1 = 1.$$

Hence

$$\int_3^5 \frac{2x-3}{\sqrt{x^2-3x+1}}\,dx = \int_1^{11} u^{-\frac{1}{2}}\,du = \left[2\sqrt{u}\right]_1^{11} = 2\left(\sqrt{11}-1\right).$$

# 0.1 Using Basic Integration Formulae

$$\int \frac{dx}{\sqrt{a^2 - x^2}} = \sin^{-1}\left(\frac{x}{a}\right) + C$$

**Example**

Find $\displaystyle\int \frac{dx}{\sqrt{8x - x^2}}$.

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**Example**

Find $\displaystyle\int \frac{dx}{\sqrt{8x - x^2}}$.

This time we will complete the square of $x^2 - 8x$ and use that to simplify the integral:

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**Example**

Find $\displaystyle\int \frac{dx}{\sqrt{8x - x^2}}$.

This time we will complete the square of $x^2 - 8x$ and use that to simplify the integral:

$x^2 - 8x = x^2 - 8x + 16 - 16$

$$\int \frac{dx}{\sqrt{a^2 - x^2}} = \sin^{-1}\left(\frac{x}{a}\right) + C$$

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Find $\displaystyle\int \frac{dx}{\sqrt{8x - x^2}}$.

This time we will complete the square of $x^2 - 8x$ and use that to simplify the integral:

$$x^2 - 8x = x^2 - 8x + 16 - 16 = (x^2 - 8x + 16) - 16 = (x - 4)^2 - 16.$$

$$\int \frac{dx}{\sqrt{a^2 - x^2}} = \sin^{-1}\left(\frac{x}{a}\right) + C$$

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So

$$\int \frac{dx}{\sqrt{8x - x^2}} = \int \frac{dx}{\sqrt{16 - (x-4)^2}}$$

$$=$$

$$=$$

$$=$$

$$\int \frac{dx}{\sqrt{a^2 - x^2}} = \sin^{-1}\left(\frac{x}{a}\right) + C$$

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So

$$\begin{aligned} \int \frac{dx}{\sqrt{8x - x^2}} &= \int \frac{dx}{\sqrt{16 - (x - 4)^2}} \\ &= \int \frac{du}{\sqrt{16 - u^2}} \\ &= \\ &= \end{aligned}$$

## 0.1 Using Basic Integration Formulae

$$\int \frac{dx}{\sqrt{a^2 - x^2}} = \sin^{-1}\left(\frac{x}{a}\right) + C$$

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\int \frac{dx}{\sqrt{8x - x^2}} &= \int \frac{dx}{\sqrt{16 - (x-4)^2}} \\
&= \int \frac{du}{\sqrt{16 - u^2}} \\
&= \sin^{-1}\left(\frac{u}{4}\right) + C \\
&=
\end{aligned}$$

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This time we will complete the square of $x^2 - 8x$ and use that to simplify the integral:

$$x^2 - 8x = x^2 - 8x + 16 - 16 = (x^2 - 8x + 16) - 16 = (x-4)^2 - 16.$$

So

$$\begin{aligned}
\int \frac{dx}{\sqrt{8x - x^2}} &= \int \frac{dx}{\sqrt{16 - (x-4)^2}} \\
&= \int \frac{du}{\sqrt{16 - u^2}} \\
&= \sin^{-1}\left(\frac{u}{4}\right) + C \\
&= \sin^{-1}\left(\frac{x-4}{4}\right) + C.
\end{aligned}$$

**Example**

Find $\displaystyle\int \cos x \sin 2x + \sin x \cos 2x \, dx$.

**Example**

Find $\int \cos x \sin 2x + \sin x \cos 2x \, dx$.

$$\begin{aligned}\int \cos x \sin 2x + \sin x \cos 2x \, dx &= \int \sin(x + 2x) \, dx \\ &= \int \sin 3x \, dx \\ &= \dots\end{aligned}$$

**Example**

Find $\displaystyle\int_0^{\frac{\pi}{4}} \frac{dx}{1-\sin x}$.

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Here is a trick for dealing with $\frac{1}{A-B}$:

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**Example**

Find $\displaystyle\int_0^{\frac{\pi}{4}} \frac{dx}{1-\sin x}$.

Here is a trick for dealing with $\frac{1}{A-B}$: Multiply by $\frac{A+B}{A+B}$. Then we get

$$\frac{1}{A-B} = \left(\frac{1}{A-B}\right)\left(\frac{A+B}{A+B}\right) = \frac{A+B}{A^2-B^2}$$

which is sometimes easier to deal with.

# 0.1 Using Basic Integration Formulae

$$\int_0^{\frac{\pi}{4}} \frac{dx}{1-\sin x} = \int_0^{\frac{\pi}{4}} \frac{1}{1-\sin x} \cdot \qquad dx =$$

$$= \qquad\qquad =$$

$$=$$

$$= \qquad\qquad = \qquad\qquad = \qquad .$$

$$\int_0^{\frac{\pi}{4}} \frac{dx}{1-\sin x} = \int_0^{\frac{\pi}{4}} \frac{1}{1-\sin x} \cdot \frac{1+\sin x}{1+\sin x}\, dx =$$

$$= \qquad\qquad =$$

$$=$$

$$= \qquad\qquad = \qquad\qquad = \quad .$$

$$\int_0^{\frac{\pi}{4}} \frac{dx}{1-\sin x} = \int_0^{\frac{\pi}{4}} \frac{1}{1-\sin x}\cdot\frac{1+\sin x}{1+\sin x}\,dx = \int_0^{\frac{\pi}{4}} \frac{1+\sin x}{1-\sin^2 x}\,dx$$

$$=\qquad\qquad =$$

$$=$$

$$=\qquad\qquad =\qquad\qquad = \quad .$$

$$\int_0^{\frac{\pi}{4}} \frac{dx}{1-\sin x} = \int_0^{\frac{\pi}{4}} \frac{1}{1-\sin x} \cdot \frac{1+\sin x}{1+\sin x}\, dx = \int_0^{\frac{\pi}{4}} \frac{1+\sin x}{1-\sin^2 x}\, dx$$

$$= \int_0^{\frac{\pi}{4}} \frac{1+\sin x}{\cos^2 x}\, dx =$$

$$=$$

$$= \qquad = \qquad = \qquad .$$

$$
\begin{aligned}
\int_0^{\frac{\pi}{4}} \frac{dx}{1-\sin x} &= \int_0^{\frac{\pi}{4}} \frac{1}{1-\sin x} \cdot \frac{1+\sin x}{1+\sin x}\, dx = \int_0^{\frac{\pi}{4}} \frac{1+\sin x}{1-\sin^2 x}\, dx \\
&= \int_0^{\frac{\pi}{4}} \frac{1+\sin x}{\cos^2 x}\, dx = \int_0^{\frac{\pi}{4}} \frac{1}{\cos^2 x} + \frac{\sin x}{\cos^2 x}\, dx \\
&= \int_0^{\frac{\pi}{4}} \sec^2 x + \sec x \tan x\, dx \\
&= \qquad\qquad = \qquad\qquad = \quad .
\end{aligned}
$$

# 0.1 Using Basic Integration Formulae

$$
\begin{aligned}
\int_0^{\frac{\pi}{4}} \frac{dx}{1-\sin x} &= \int_0^{\frac{\pi}{4}} \frac{1}{1-\sin x}\cdot\frac{1+\sin x}{1+\sin x}\,dx = \int_0^{\frac{\pi}{4}} \frac{1+\sin x}{1-\sin^2 x}\,dx \\
&= \int_0^{\frac{\pi}{4}} \frac{1+\sin x}{\cos^2 x}\,dx = \int_0^{\frac{\pi}{4}} \frac{1}{\cos^2 x}+\frac{\sin x}{\cos^2 x}\,dx \\
&= \int_0^{\frac{\pi}{4}} \sec^2 x + \sec x \tan x\,dx \\
&= \Big[\tan x + \sec x\Big]_0^{\frac{\pi}{4}} = \qquad = \quad .
\end{aligned}
$$

$$\int_0^{\frac{\pi}{4}} \frac{dx}{1-\sin x} = \int_0^{\frac{\pi}{4}} \frac{1}{1-\sin x} \cdot \frac{1+\sin x}{1+\sin x}\, dx = \int_0^{\frac{\pi}{4}} \frac{1+\sin x}{1-\sin^2 x}\, dx$$

$$= \int_0^{\frac{\pi}{4}} \frac{1+\sin x}{\cos^2 x}\, dx = \int_0^{\frac{\pi}{4}} \frac{1}{\cos^2 x} + \frac{\sin x}{\cos^2 x}\, dx$$

$$= \int_0^{\frac{\pi}{4}} \sec^2 x + \sec x \tan x \, dx$$

$$= \Big[\tan x + \sec x\Big]_0^{\frac{\pi}{4}} = \left(1+\sqrt{2}\right) - (0+1) = \sqrt{2}.$$

## Example

Find $\displaystyle\int \frac{3x^2 - 7x}{3x + 2}\, dx.$

**Example**

Find $\int \frac{3x^2 - 7x}{3x + 2}\, dx$.

The integrand is an improper fraction because the degree of $3x^2 - 7x$ (2nd degree) is greater than the degree of $3x + 2$ (1st degree).

**Example**

Find $\displaystyle\int \frac{3x^2 - 7x}{3x + 2}\, dx.$

The integrand is an improper fraction because the degree of $3x^2 - 7x$ (2nd degree) is greater than the degree of $3x + 2$ (1st degree). We want to write our integrand as

$$\frac{3x^2 - 7x}{3x + 2} = \text{(linear function)} + \frac{\text{something}}{3x + 2}$$

[Do you remember how we studied *oblique asymptotes* in MATH113?]

**Example**

Find $\displaystyle\int \frac{3x^2 - 7x}{3x + 2}\, dx.$

The integrand is an improper fraction because the degree of $3x^2 - 7x$ (2nd degree) is greater than the degree of $3x + 2$ (1st degree). We want to write our integrand as

$$\frac{3x^2 - 7x}{3x + 2} = (ax + b) + \frac{\text{something}}{3x + 2}$$

[Do you remember how we studied *oblique asymptotes* in MATH113?]

**Example**

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$$\frac{3x^2-7x}{3x+2}\cdot(3x+2) = (ax+b)(3x+2) + \frac{c}{3x+2}\cdot(3x+2)$$

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The integrand is an improper fraction because the degree of $3x^2 - 7x$ (2nd degree) is greater than the degree of $3x + 2$ (1st degree). We want to write our integrand as

$$3x^2 - 7x = (ax + b)(3x + 2) + c$$

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The integrand is an improper fraction because the degree of $3x^2-7x$ (2nd degree) is greater than the degree of $3x+2$ (1st degree). We want to write our integrand as

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$$3x^2 - 7x = 3ax^2 + (2a + 3b)x + (2b + c)$$

$a = 1$, $b = -3$, $c = 6$.

**Example**

Find $\int \frac{3x^2 - 7x}{3x+2}\, dx$.

The integrand is an improper fraction because the degree of $3x^2 - 7x$ (2nd degree) is greater than the degree of $3x + 2$ (1st degree). We want to write our integrand as

$$\frac{3x^2 - 7x}{3x+2} = 2x - 3 + \frac{6}{3x+2}.$$

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$$\frac{3x^2 - 7x}{3x + 2} = 2x - 3 + \frac{6}{3x + 2}.$$

$$\int \frac{3x^2 - 7x}{3x + 2}\,dx = \int 2x - 3 + \frac{6}{3x + 2}\,dx = \frac{x^2}{2} - 3x + 2\ln|3x + 2| + C.$$

$$\int \frac{dx}{\sqrt{a^2 - x^2}} = \sin^{-1}\left(\frac{x}{a}\right) + C$$

**Example**

Find $\displaystyle\int \frac{3x+2}{\sqrt{1-x^2}}$.

$$\int \frac{dx}{\sqrt{a^2 - x^2}} = \sin^{-1}\left(\frac{x}{a}\right) + C$$

**Example**

Find $\displaystyle\int \frac{3x+2}{\sqrt{1-x^2}}$.

First note that

$$\int \frac{3x+2}{\sqrt{1-x^2}} = 3\int \frac{x\,dx}{\sqrt{1-x^2}} + 2\int \frac{dx}{\sqrt{1-x^2}}$$

$$\int \frac{dx}{\sqrt{a^2 - x^2}} = \sin^{-1}\left(\frac{x}{a}\right) + C$$

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So we just need to calculate $\displaystyle\int \frac{x\,dx}{\sqrt{1-x^2}}$.

$$\int \frac{x\,dx}{\sqrt{1-x^2}}$$

Let $u = 1 - x^2$.

$$\int \frac{x\,dx}{\sqrt{1-x^2}}$$

Let $u = 1 - x^2$. Then $du = -2x\,dx$ and $-\frac{1}{2}\,du = x\,dx$.

$$\int \frac{x\,dx}{\sqrt{1-x^2}}$$

Let $u = 1 - x^2$. Then $du = -2x\,dx$ and $-\frac{1}{2}\,du = x\,dx$. It follows that

$$\int \frac{x\,dx}{\sqrt{1-x^2}} = \int \frac{-\frac{1}{2}\,du}{\sqrt{u}} = -\frac{1}{2}\int u^{-\frac{1}{2}}\,du = \ldots = -\sqrt{1-x^2} + C.$$

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Let $u = 1 - x^2$. Then $du = -2x\,dx$ and $-\frac{1}{2}\,du = x\,dx$. It follows that

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Therefore

$$\begin{aligned}\int \frac{3x+2}{\sqrt{1-x^2}} &= 3\int \frac{x\,dx}{\sqrt{1-x^2}} + 2\int \frac{dx}{\sqrt{1-x^2}} \\ &= -3\sqrt{1-x^2} + 2\sin^{-1} x + C.\end{aligned}$$

**Example**

Find $\displaystyle\int \frac{dx}{\left(1+\sqrt{x}\right)^3}$.

**Example**

Find $\displaystyle\int \frac{dx}{(1+\sqrt{x})^3}$.

I want to make a substitution to make this integral easier, but what $u$ should I choose?

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First guess: $u = \sqrt{x}$.

**Example**

Find $\int \frac{dx}{(1+\sqrt{x})^3}$.

I want to make a substitution to make this integral easier, but what $u$ should I choose?

First guess: $u = \sqrt{x}$. But then $du = \frac{1}{2\sqrt{x}}\,dx$ and we would have to deal with this extra $\sqrt{x} = u$ term.

**Example**

Find $\displaystyle\int \frac{dx}{(1+\sqrt{x})^3}$.

I want to make a substitution to make this integral easier, but what $u$ should I choose?

First guess: $u = \sqrt{x}$. But then $du = \frac{1}{2\sqrt{x}}\, dx$ and we would have to deal with this extra $\sqrt{x} = u$ term.

Second guess: Instead let us try $u = 1 + \sqrt{x}$.

**Example**

Find $\displaystyle\int \frac{dx}{(1+\sqrt{x})^3}$.

I want to make a substitution to make this integral easier, but what $u$ should I choose?

First guess: $u = \sqrt{x}$. But then $du = \frac{1}{2\sqrt{x}}\,dx$ and we would have to deal with this extra $\sqrt{x} = u$ term.

Second guess: Instead let us try $u = 1 + \sqrt{x}$. Then again we have $du = \frac{1}{2\sqrt{x}}\,dx$ and $dx = 2\sqrt{x}\,du = 2(u-1)\,du$. Hence

$$\int \frac{dx}{(1+\sqrt{x})^3} = \int \frac{2(u-1)\,du}{u^3} = \int \frac{2}{u^2} - \frac{2}{u^3}\,du = \ldots$$

**Example**

Calculate $\displaystyle\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} x^3 \cos x \, dx.$

**Example**

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This is actually easy: The integrand is an odd function

# 0.1 Using Basic Integration Formulae

**Example**

Calculate $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} x^3 \cos x \, dx$.

This is actually easy: The integrand is an odd function so

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} x^3 \cos x \, dx = 0.$$

**Example**

Calculate $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} x^3 \cos x \, dx$.

This is actually easy: The integrand is an odd function so

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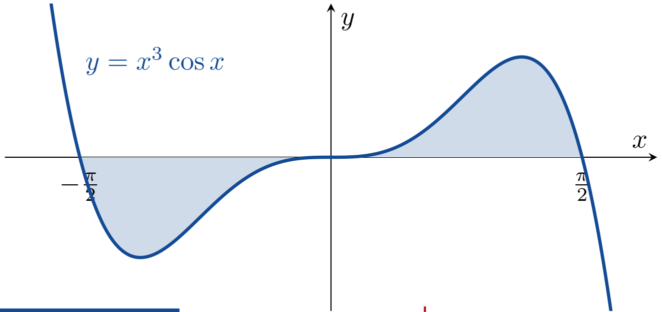

# 0.2 Integration by Parts

How can we calculate

$$\int x \cos x \, dx$$

or

$$\int x^2 e^x \, dx \ ?$$

How can we calculate

$$\int x\cos x\,dx$$

or

$$\int x^2 e^x\,dx\ ?$$

How do we integrate function $\times$ function?

We know how to differentiate function $\times$ function. By the product rule we have

$$(uv)' = u'v + uv' .$$

We know how to differentiate function $\times$ function. By the product rule we have

$$\int (uv)'\,dx = \int u'v\,dx + \int uv'\,dx.$$

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$$uv \qquad = \int u'v\,dx + \int uv'\,dx.$$

# 0.2 Integration by Parts

We know how to differentiate function $\times$ function. By the product rule we have

$$uv = \int u'v\,dx + \int uv'\,dx.$$

This rearranges to

**Theorem (Integration by Parts)**

$$\int u(x)v'(x)\,dx = u(x)v(x) - \int u'(x)v(x)\,dx$$

**Example**

Find $\displaystyle\int x\cos x\,dx$.

**Example**

Find $\displaystyle\int x\cos x\,dx$.

We need to choose a $u(x)$ and a $v'(x)$.

$$\int uv'\,dx = uv - \int u'v\,dx$$

**Example**

Find $\displaystyle\int x\cos x\,dx$.

We need to choose a $u(x)$ and a $v'(x)$.
Let

$$u = x \qquad\qquad v' = \cos x$$

Then

$$\int x\cos x\,dx = \qquad - \int \qquad dx = \qquad .$$

$$\int uv'\,dx = uv - \int u'v\,dx$$

**Example**

Find $\displaystyle\int x\cos x\,dx$.

We need to choose a $u(x)$ and a $v'(x)$.
Let

$$u = x \qquad\qquad v' = \cos x$$
$$u' = 1$$

Then

$$\int x\cos x\,dx = \qquad - \int \qquad dx = \qquad .$$

$$\int uv'\,dx = uv - \int u'v\,dx$$

**Example**

Find $\displaystyle\int x\cos x\,dx$.

We need to choose a $u(x)$ and a $v'(x)$.
Let

$$\begin{aligned} u &= x & v' &= \cos x \\ u' &= 1 & v &= \sin x. \end{aligned}$$

Then

$$\int x\cos x\,dx = \qquad - \int \qquad dx = \qquad .$$

**Example**

Find $\int x \cos x \, dx$.

We need to choose a $u(x)$ and a $v'(x)$.
Let

$$\begin{aligned} u &= x & \qquad v' &= \cos x \\ u' &= 1 & \qquad v &= \sin x. \end{aligned}$$

Then

$$\int x \cos x \, dx = x \sin x - \int \qquad dx = \qquad .$$

$$\int uv'\,dx = uv - \int u'v\,dx$$

**Example**

Find $\int x \cos x \, dx$.

We need to choose a $u(x)$ and a $v'(x)$.
Let

$$\begin{aligned} u &= x & v' &= \cos x \\ u' &= 1 & v &= \sin x. \end{aligned}$$

Then

$$\int x \cos x \, dx = x \sin x - \int 1 \sin x \, dx = \qquad .$$

$$\int uv'\,dx = uv - \int u'v\,dx$$

**Example**

Find $\displaystyle\int x\cos x\,dx$.

We need to choose a $u(x)$ and a $v'(x)$.
Let

$$\begin{aligned} u &= x & v' &= \cos x \\ u' &= 1 & v &= \sin x. \end{aligned}$$

Then

$$\int x\cos x\,dx = x\sin x - \int 1\sin x\,dx = x\sin x + \cos x + C.$$

# 0.2 Integration by Parts

$$\int uv'\,dx = uv - \int u'v\,dx$$

**Example**

Find $\displaystyle\int \ln x\,dx$.

$$\int uv' \, dx = uv - \int u'v \, dx$$

**Example**

Find $\displaystyle\int \ln x \, dx$.

We will consider $\displaystyle\int \ln x \cdot 1 \, dx$.

$$\int uv'\,dx = uv - \int u'v\,dx$$

**Example**

Find $\displaystyle\int \ln x\,dx$.

We will consider $\displaystyle\int \ln x \cdot 1\,dx$.

Let

$$u = \ln x \qquad\qquad v' = 1$$

$$\int uv'\,dx = uv - \int u'v\,dx$$

**Example**

Find $\displaystyle\int \ln x\,dx$.

We will consider $\displaystyle\int \ln x \cdot 1\,dx$.

Let

$$u = \ln x \qquad\qquad v' = 1$$

$$u' = \frac{1}{x}$$

$$\int uv'\,dx = uv - \int u'v\,dx$$

**Example**

Find $\displaystyle\int \ln x\,dx$.

We will consider $\displaystyle\int \ln x \cdot 1\,dx$.

Let

$$\begin{aligned} u &= \ln x & v' &= 1 \\ u' &= \frac{1}{x} & v &= x. \end{aligned}$$

**Example**

Find $\displaystyle\int \ln x\,dx$.

We will consider $\displaystyle\int \ln x \cdot 1\,dx$.

Let

$$u = \ln x \qquad\qquad v' = 1$$

$$u' = \frac{1}{x} \qquad\qquad v = x.$$

Then

$$\int \ln x \cdot 1\,dx = \ln x \cdot x - \int \frac{1}{x} \cdot x\,dx$$

$$=$$

$$= \qquad .$$

$$\int uv'\,dx = uv - \int u'v\,dx$$

**Example**

Find $\displaystyle\int \ln x\,dx$.

We will consider $\displaystyle\int \ln x \cdot 1\,dx$.

Let

$$u = \ln x \qquad v' = 1$$
$$u' = \frac{1}{x} \qquad v = x.$$

Then

$$\begin{aligned}\int \ln x \cdot 1\,dx &= \ln x \cdot x - \int \frac{1}{x} \cdot x\,dx \\ &= x \ln x - \int 1\,dx \\ &= \qquad\qquad .\end{aligned}$$

$$\int uv'\,dx = uv - \int u'v\,dx$$

**Example**

Find $\displaystyle\int \ln x\,dx$.

We will consider $\displaystyle\int \ln x \cdot 1\,dx$.

Let

$$\begin{aligned} u &= \ln x & v' &= 1 \\ u' &= \frac{1}{x} & v &= x. \end{aligned}$$

Then

$$\begin{aligned} \int \ln x \cdot 1\,dx &= \ln x \cdot x - \int \frac{1}{x} \cdot x\,dx \\ &= x\ln x - \int 1\,dx \\ &= x\ln x - x + C. \end{aligned}$$

$$\int uv' \, dx = uv - \int u'v \, dx$$

Sometimes we have to use integration by parts more than once.

$$\int uv'\,dx = uv - \int u'v\,dx$$

**Example**

Find $\int x^2 e^x \, dx$.

$$\int uv'\,dx = uv - \int u'v\,dx$$

**Example**

Find $\int x^2 e^x\,dx$.

We calculate that

$$\int x^2 e^x\,dx = x^2 e^x - 2\int x e^x\,dx.$$

$$\int uv'\,dx = uv - \int u'v\,dx$$

**Example**

Find $\int x^2 e^x\,dx$.

We calculate that

$$\int x^2 e^x\,dx = x^2 e^x - 2\int x e^x\,dx.$$

But what do we do with $\int x e^x\,dx$?

$$\int uv'\,dx = uv - \int u'v\,dx$$

**Example**

Find $\displaystyle\int x^2 e^x\,dx$.

We calculate that

$$\int x^2 e^x\,dx = x^2 e^x - 2\int x e^x\,dx.$$

But what do we do with $\displaystyle\int x e^x\,dx$?

$$\int x e^x\,dx = x e^x - \int 1 e^x\,dx = x e^x - e^x + C_1.$$

$$\int uv'\,dx = uv - \int u'v\,dx$$

## Example

Find $\int x^2 e^x \, dx$.

We calculate that

$$\int x^2 e^x \, dx = x^2 e^x - 2 \int x e^x \, dx.$$

But what do we do with $\int x e^x \, dx$?

$$\int x e^x \, dx = x e^x - \int 1 e^x \, dx = x e^x - e^x + C_1.$$

Putting it all together, we have

$$\int x^2 e^x \, dx = x^2 e^x - 2 \int x e^x \, dx = x^2 e^x - 2x e^x + 2e^x + C.$$

$$\int uv' \, dx = uv - \int u'v \, dx$$

**Remark**

We can use the same technique to calculate $\int x^n e^x \, dx$.

We would have to do integration by parts $n$ times.

# 0.2 Integration by Parts

$$\int uv'\,dx = uv - \int u'v\,dx$$

**Theorem**

$$\int u\,dv = uv - \int v\,du$$

$$\int u\,dv = uv - \int v\,du$$

**Example (Using Integration by Parts Twice)**

Calculate $\displaystyle\int e^x \cos x\,dx$.

$$\int u\,dv = uv - \int v\,du$$

**Example (Using Integration by Parts Twice)**

Calculate $\int e^x \cos x\,dx$.

Let $u = e^x$ and $dv = \cos x\,dx$.

$$\int u\,dv = uv - \int v\,du$$

**Example (Using Integration by Parts Twice)**

Calculate $\int e^x \cos x\,dx$.

Let $u = e^x$ and $dv = \cos x\,dx$. Then $du = e^x\,dx$ and $v = \sin x$.

$$\int u\,dv = uv - \int v\,du$$

**Example (Using Integration by Parts Twice)**

Calculate $\int e^x \cos x\,dx$.

Let $u = e^x$ and $dv = \cos x\,dx$. Then $du = e^x\,dx$ and $v = \sin x$. Hence

$$\int e^x \cos x\,dx = e^x \sin x - \int \sin x \cdot e^x\,dx.$$

$$\int u\,dv = uv - \int v\,du$$

**Example (Using Integration by Parts Twice)**

Calculate $\int e^x \cos x\,dx$.

Let $u = e^x$ and $dv = \cos x\,dx$. Then $du = e^x\,dx$ and $v = \sin x$. Hence

$$\int e^x \cos x\,dx = e^x \sin x - \int \sin x \cdot e^x\,dx.$$

But we still have $\int (\text{function}) \times (\text{function})\,dx$. So we need to use Integration by Parts a second time.

$$\int u\,dv = uv - \int v\,du$$

So far we have

$$\int e^x \cos x\,dx = e^x \sin x - \int e^x \sin x\,dx.$$

$$\int u\,dv = uv - \int v\,du$$

So far we have

$$\int e^x \cos x\,dx = e^x \sin x - \int e^x \sin x\,dx.$$

Let $u = e^x$ and $dv = \sin x\,dx$.

So far we have

$$\int e^x \cos x\,dx = e^x \sin x - \int e^x \sin x\,dx.$$

Let $u = e^x$ and $dv = \sin x\,dx$. Then $du = e^x\,dx$ and $v = -\cos x$.

# 0.2 Integration by Parts

$$\int u\,dv = uv - \int v\,du$$

So far we have

$$\int e^x \cos x\,dx = e^x \sin x - \int e^x \sin x\,dx.$$

Let $u = e^x$ and $dv = \sin x\,dx$. Then $du = e^x\,dx$ and $v = -\cos x$. Hence

$$\begin{aligned}
\int e^x \cos x\,dx &= e^x \sin x - \int e^x \sin x\,dx \\
&= e^x \sin x - \left(-e^x \cos x - \int (-\cos x) \cdot e^x\,dx\right) \\
&=
\end{aligned}$$

So far we have

$$\int e^x \cos x\,dx = e^x \sin x - \int e^x \sin x\,dx.$$

Let $u = e^x$ and $dv = \sin x\,dx$. Then $du = e^x\,dx$ and $v = -\cos x$. Hence

$$\begin{aligned}
\int e^x \cos x\,dx &= e^x \sin x - \int e^x \sin x\,dx \\
&= e^x \sin x - \left(-e^x \cos x - \int (-\cos x)\cdot e^x\,dx\right) \\
&= e^x \sin x + e^x \cos x - \int e^x \cos x\,dx.
\end{aligned}$$

$$\int u\,dv = uv - \int v\,du$$

So far we have

$$\int e^x \cos x\,dx = e^x \sin x - \int e^x \sin x\,dx.$$

Let $u = e^x$ and $dv = \sin x\,dx$. Then $du = e^x\,dx$ and $v = -\cos x$. Hence

$$\begin{aligned}\int e^x \cos x\,dx &= e^x \sin x - \int e^x \sin x\,dx \\ &= e^x \sin x - \left(-e^x \cos x - \int (-\cos x)\cdot e^x\,dx\right) \\ &= e^x \sin x + e^x \cos x - \int e^x \cos x\,dx.\end{aligned}$$

$$\int u\,dv = uv - \int v\,du$$

$$\int e^x \cos x \, dx = e^x \sin x + e^x \cos x - \int e^x \cos x \, dx.$$

$$\int u\,dv = uv - \int v\,du$$

$$\int e^x \cos x\,dx = e^x \sin x + e^x \cos x - \int e^x \cos x\,dx.$$

It follows that

$$2\int e^x \cos x\,dx = e^x \sin x + e^x \cos x + C_1$$

$$\int u\,dv = uv - \int v\,du$$

$$\int e^x \cos x\,dx = e^x \sin x + e^x \cos x - \int e^x \cos x\,dx.$$

It follows that

$$2\int e^x \cos x\,dx = e^x \sin x + e^x \cos x + C_1$$

and hence that

$$\int e^x \cos x\,dx = \frac{e^x \sin x + e^x \cos x}{2} + C.$$

**Example**

Obtain a formula that expresses the integral $\displaystyle\int \cos^n x\,dx$ in terms of an integral of a lower power of $\cos x$.

### Example

Obtain a formula that expresses the integral $\int \cos^n x\,dx$ in terms of an integral of a lower power of $\cos x$.

To use integration by parts, we need (function) $\times$ (function). So think of the integral like this:

$$\int \cos^n x\,dx = \int \cos^{n-1} x \cdot \cos x\,dx.$$

$$\int u\,dv = uv - \int v\,du$$

$$\int \cos^n x\,dx = \int \cos^{n-1} x \cdot \cos x\,dx.$$

If $u = \cos^{n-1} x$ and $dv = \cos x\,dx$,

$$\int u\,dv = uv - \int v\,du$$

$$\int \cos^n x\,dx = \int \cos^{n-1} x \cdot \cos x\,dx.$$

If $u = \cos^{n-1} x$ and $dv = \cos x\,dx$, then we have that $du = -(n-1)\cos^{n-2} x \sin x\,dx$ and $v = \sin x$.

$$\int u\,dv = uv - \int v\,du$$

$$\int \cos^n x\,dx = \int \cos^{n-1} x \cdot \cos x\,dx.$$

If $u = \cos^{n-1} x$ and $dv = \cos x\,dx$, then we have that $du = -(n-1)\cos^{n-2} x \sin x\,dx$ and $v = \sin x$. Hence

$$\begin{aligned}
\int \cos^n x\,dx &= \int u\,dv = uv - \int v\,du \\
&= \\
&= \\
&= \\
&=
\end{aligned}$$

$$\int u\,dv = uv - \int v\,du$$

$$\int \cos^n x\,dx = \int \cos^{n-1} x \cdot \cos x\,dx.$$

If $u = \cos^{n-1} x$ and $dv = \cos x\,dx$, then we have that $du = -(n-1)\cos^{n-2} x \sin x\,dx$ and $v = \sin x$. Hence

$$\begin{aligned}\int \cos^n x\,dx &= \int u\,dv = uv - \int v\,du \\ &= \cos^{n-1} x \sin x + (n-1) \int \sin x \cos^{n-2} x \sin x\,dx \\ &= \\ &= \\ &= \end{aligned}$$

$$\int u\,dv = uv - \int v\,du$$

$$\int \cos^n x\,dx = \int \cos^{n-1} x \cdot \cos x\,dx.$$

If $u = \cos^{n-1} x$ and $dv = \cos x\,dx$, then we have that $du = -(n-1)\cos^{n-2} x \sin x\,dx$ and $v = \sin x$. Hence

$$\begin{aligned}
\int \cos^n x\,dx &= \int u\,dv = uv - \int v\,du \\
&= \cos^{n-1} x \sin x + (n-1) \int \sin x \cos^{n-2} x \sin x\,dx \\
&= \cos^{n-1} x \sin x + (n-1) \int \sin^2 x \cos^{n-2} x\,dx \\
&= \\
&=
\end{aligned}$$

# 0.2 Integration by Parts

$$\int u\,dv = uv - \int v\,du$$

$$\int \cos^n x\,dx = \int \cos^{n-1} x \cdot \cos x\,dx.$$

If $u = \cos^{n-1} x$ and $dv = \cos x\,dx$, then we have that $du = -(n-1)\cos^{n-2} x \sin x\,dx$ and $v = \sin x$. Hence

$$\begin{aligned}
\int \cos^n x\,dx &= \int u\,dv = uv - \int v\,du \\
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&= \cos^{n-1} x \sin x + (n-1)\int \sin^2 x \cos^{n-2} x\,dx \\
&= \cos^{n-1} x \sin x + (n-1)\int (1-\cos^2 x)\cos^{n-2} x\,dx \\
&=
\end{aligned}$$

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If we add $(n-1) \int \cos^n x\,dx$ to both sides then we obtain

$$n \int \cos^n x\,dx = \cos^{n-1} x \sin x + (n-1) \int \cos^{n-2} x\,dx.$$

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Finally we divide by $n$ to obtain our answer

$$\int \cos^n x\,dx = \frac{\cos^{n-1} x \sin x}{n} + \frac{n-1}{n}\int \cos^{n-2} x\,dx.$$

### Remark

$$\int \cos^n x\,dx = \frac{\cos^{n-1} x \sin x}{n} + \frac{n-1}{n}\int \cos^{n-2} x\,dx.$$

The formula above is called a *reduction formula* because it replaces it replaces an integral containing $\cos^n x$ with an integral containing a smaller power of $\cos x$.

$$\int u\,dv = uv - \int v\,du$$

### Remark

$$\int \cos^n x\,dx = \frac{\cos^{n-1} x \sin x}{n} + \frac{n-1}{n} \int \cos^{n-2} x\,dx.$$

The formula above is called a *reduction formula* because it replaces it replaces an integral containing $\cos^n x$ with an integral containing a smaller power of $\cos x$.

### Example ($n = 3$)

$$\begin{aligned}\int \cos^3 x\,dx &= \frac{1}{3}\cos^2 x \sin x + \frac{2}{3}\int \cos x\,dx \\ &= \frac{1}{3}\cos^2 x \sin x + \frac{2}{3}\sin x + C.\end{aligned}$$

# Break

We will continue at 3pm

# Integration by Parts for Definite Integrals

**Theorem**

$$\int_a^b uv'\,dx =$$

# Integration by Parts for Definite Integrals

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### Example

Calculate the area of the region bounded by the curve $y = xe^{-x}$ and the $x$-axis from $x = 0$ to $x = 4$.

$$\int_a^b uv'\,dx = \Big[uv\Big]_a^b - \int_a^b u'v\,dx$$

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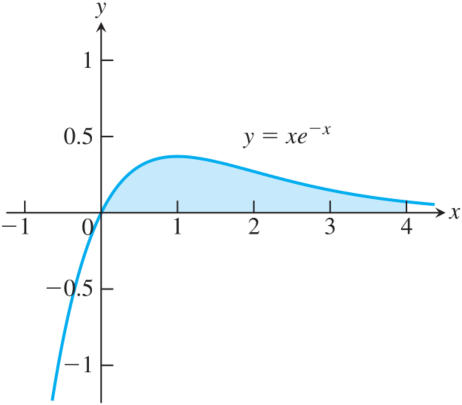

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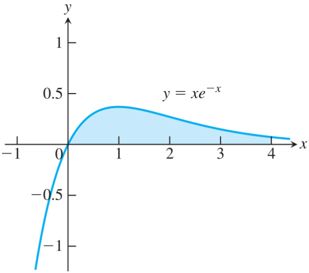


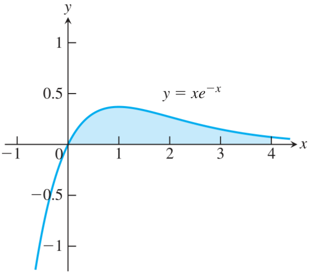

We calculate that

$$\int_0^4 xe^{-x}\,dx =$$

$$=$$

$$= \qquad\qquad\qquad .$$

$$\int_a^b uv'\,dx = \Big[uv\Big]_a^b - \int_a^b u'v\,dx$$




$$u = x$$
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$$v = -e^{-x}$$

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$$= \qquad .$$

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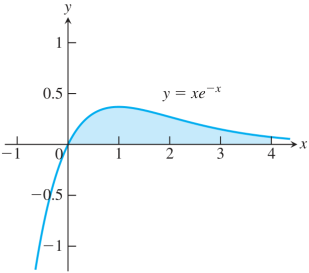


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$$=$$
$$= \qquad .$$

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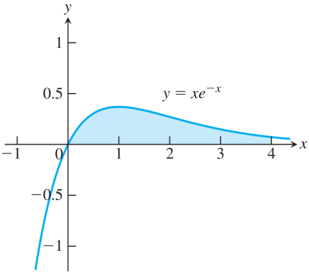


$$u = x$$
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We calculate that

$$\begin{aligned}\int_0^4 xe^{-x}\,dx &= \Big[-xe^{-x}\Big]_0^4 - \int_0^4 1(-e^{-x})\,dx \\ &= \left(-4e^{-4}+0\right) + \left[-e^{-x}\right]_0^4 \\ &= -4e^{-4} + \left(-e^{-4}+1\right) = 1-5e^{-4}.\end{aligned}$$

$$\int_a^b uv'\,dx = \Big[uv\Big]_a^b - \int_a^b u'v\,dx$$

**Example**

Find $\displaystyle\int_0^1 \sin^{-1} x\,dx$.

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Recall that $\frac{d}{dx}\sin^{-1} x = \frac{1}{\sqrt{1-x^2}}$.

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**Example**

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**Example**

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$$\begin{aligned}
\int_0^1 \sin^{-1} x \cdot 1\,dx &= \left[x\sin^{-1} x\right]_0^1 - \int_0^1 \frac{x}{\sqrt{1-x^2}}dx \\
&= \\
&= \\
&= \qquad .
\end{aligned}$$

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&= \left[x\sin^{-1} x\right]_0^1 - \Big[-\sqrt{1-x^2}\Big]_0^1 \\
&= \left(\frac{\pi}{2} - 0\right) - (-0+1) \\
&= \frac{\pi}{2} - 1.
\end{aligned}$$

# 0.3 Trigonometric Integrals

# 0.3 Trigonometric Integrals

$$
\begin{aligned}
\cos\theta &= \sin\left(\frac{\pi}{2}-\theta\right)\\
\cos^2\theta+\sin^2\theta &= 1\\
1+\tan^2\theta &= \sec^2\theta\\
1+\cot^2\theta &= \mathrm{cosec}^2\,\theta\\
\cos(A+B) &= \cos A\cos B-\sin A\sin B\\
\sin(A+B) &= \sin A\cos B+\cos A\sin B\\
\cos 2\theta &= \cos^2\theta-\sin^2\theta\\
\sin 2\theta &= 2\sin\theta\cos\theta\\
\cos^2\theta &= \frac{1}{2}(1+\cos 2\theta)\\
\sin^2\theta &= \frac{1}{2}(1-\cos 2\theta)
\end{aligned}
$$

$$\cos\theta = \sin\left(\frac{\pi}{2} - \theta\right)$$

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\end{aligned}$$

How can we find

$$\int \sin^m x \, \cos^n x \, dx$$

if $m, n \in \{0, 1, 2, 3, 4, 5, \ldots\}$?

$$\int \sin^m x \, \cos^n x \, dx$$

We need to look at 3 different cases:

1. $m$ is an odd number;
2. $n$ is an odd number;
3. both $m$ and $n$ are even.

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[If both $m$ and $n$ are odd, then you can choose to use case 1 or case 2.]

# 0.3 Trigonometric Integrals

$$\int \sin^{m} x \, \cos^{n} x \, dx$$

1. $m$ is an odd number;

$$\int \sin^m x \cos^n x \, dx$$

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The idea is:

a. Use $\sin^2 x = 1 - \cos^2 x$ to change all but one of the sin terms into cos terms;
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$$\int \sin^m x \cos^n x \, dx = \int \sin^{2k+1} x \cos^n x \, dx$$

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$$\int \sin^m x \, \cos^n x \, dx = \int (1 - u^2)^k u^n \, (-du)$$

# 0.3 Trigonometric Integrals

$$\int \sin^m x \, \cos^{n} x \, dx$$

2. $n$ is odd;

Same idea, but with cos and sin swapped:

$$\int \sin^{m} x \, \cos^{n} x \, dx$$

2 $n$ is odd;

Same idea, but with cos and sin swapped:

a Use $\cos^2 x = 1 - \sin^2 x$ to change all but one of the cos terms into sin terms;

b Use the substitution $u = \sin x$. The final $\cos x$ term will disappear.

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$$\int \sin^m x \, \cos^n x \, dx = \int \sin^m x \cos x (\cos^2 x)^k \, dx$$

$$=$$

$$=$$

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$$= \int \sin^m x \cos x (1 - \sin^2 x)^k \, dx = \int u^m (1 - u^2)^k \, du.$$

$$\int \sin^{m} x \, \cos^{n} x \, dx$$

3. both $m$ and $n$ are even.

$$\int \sin^{m} x \cos^{n} x \, dx$$

3 both $m$ and $n$ are even.

This time the idea is to use

$$\sin^2 x = \frac{1 - \cos 2x}{2} \qquad \text{and} \qquad \cos^2 x = \frac{1 + \cos 2x}{2}$$

to reduce the powers of sin and cos that we need to deal with.

**Example**

Find $\int \sin^3 x \cos^2 x \, dx$.

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Find $\int \sin^{3} x \cos^{2} x \, dx$.

This is an example of case 1 since 3 is an odd number.

**Example**

Find $\int \sin^3 x \cos^2 x \, dx$.

This is an example of case $\boxed{1}$ since 3 is an odd number. We are going to be using the substitution $u = \cos x$ and $-du = \sin x \, dx$.

**Example**

Find $\int \sin^3 x \cos^2 x \, dx$.

This is an example of case 1 since 3 is an odd number. We are going to be using the substitution $u = \cos x$ and $-du = \sin x \, dx$.

$$\begin{aligned}\int \sin^3 x \cos^2 x \, dx &= \int \sin^2 x \cos^2 x \sin x \, dx \\ &= \\ &= \\ &= \\ &= \end{aligned}$$

**Example**

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This is an example of case 1 since 3 is an odd number. We are going to be using the substitution $u = \cos x$ and $-du = \sin x \, dx$.

$$\begin{aligned}\int \sin^3 x \cos^2 x \, dx &= \int \sin^2 x \cos^2 x \sin x \, dx \\ &= \int (1 - \cos^2 x) \cos^2 x \sin x \, dx \\ &= \\ &= \\ &= \end{aligned}$$

**Example**

Find $\int \sin^3 x \cos^2 x \, dx$.

This is an example of case $\boxed{1}$ since $3$ is an odd number. We are going to be using the substitution $u = \cos x$ and $-du = \sin x \, dx$.

$$\begin{aligned}
\int \sin^3 x \cos^2 x \, dx &= \int \sin^2 x \cos^2 x \sin x \, dx \\
&= \int (1 - \cos^2 x) \cos^2 x \sin x \, dx \\
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&= \\
&=
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**Example**

Find $\int \sin^3 x \cos^2 x \, dx$.

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\int \sin^3 x \cos^2 x \, dx &= \int \sin^2 x \cos^2 x \sin x \, dx \\
&= \int (1 - \cos^2 x) \cos^2 x \sin x \, dx \\
&= \int (1 - u^2) \, u^2 \, (-du) \\
&= \int u^4 - u^2 \, du \\
&= \frac{u^5}{5} - \frac{u^3}{3} + C = \frac{\cos^5 x}{5} - \frac{\cos^3 x}{3} + C.
\end{aligned}$$

**Example**

Find $\int \cos^5 x \, dx$.

**Example**

Find $\int \cos^5 x \, dx$.

This is an example of case $\boxed{2}$ since 5 is an odd number. We are going to be using the substitution $u = \sin x$ and $du = \cos x \, dx$.

**Example**

Find $\int \cos^5 x\, dx$.

This is an example of case 2 since 5 is an odd number. We are going to be using the substitution $u = \sin x$ and $du = \cos x\, dx$.

$$\int \cos^5 x\, dx = \int \cos^4 x \cos x\, dx$$
$$=$$
$$=$$
$$=$$

**Example**

Find $\int \cos^5 x\, dx$.

This is an example of case $\boxed{2}$ since 5 is an odd number. We are going to be using the substitution $u = \sin x$ and $du = \cos x\, dx$.

$$\begin{aligned}\int \cos^5 x\, dx &= \int \cos^4 x \cos x\, dx \\ &= \int (\cos^2 x)^2 \cos x\, dx \\ &= \\ &= \end{aligned}$$

**Example**

Find $\int \cos^5 x \, dx$.

This is an example of case $\boxed{2}$ since 5 is an odd number. We are going to be using the substitution $u = \sin x$ and $du = \cos x \, dx$.

$$\begin{aligned}
\int \cos^5 x \, dx &= \int \cos^4 x \cos x \, dx \\
&= \int (\cos^2 x)^2 \cos x \, dx \\
&= \int (1 - \sin^2 x)^2 \cos x \, dx \\
&=
\end{aligned}$$

**Example**

Find $\int \cos^5 x\, dx$.

This is an example of case $\boxed{2}$ since 5 is an odd number. We are going to be using the substitution $u = \sin x$ and $du = \cos x\, dx$.

$$\begin{aligned} \int \cos^5 x\, dx &= \int \cos^4 x \cos x\, dx \\ &= \int (\cos^2 x)^2 \cos x\, dx \\ &= \int (1 - \sin^2 x)^2 \cos x\, dx \\ &= \int (1 - u^2)^2\, du = \ldots \end{aligned}$$

$$\sin^2 x = \frac{1-\cos 2x}{2} \qquad \cos^2 x = \frac{1+\cos 2x}{2}$$

**Example**

Find $\int \sin^2 x \cos^4 x \, dx$.

Both 2 and 4 are even so this is an example of case 3.

$$\sin^2 x = \frac{1-\cos 2x}{2} \qquad \cos^2 x = \frac{1+\cos 2x}{2}$$

**Example**

Find $\int \sin^2 x \cos^4 x \, dx$.

Both 2 and 4 are even so this is an example of case 3.

$$\begin{aligned} \int \sin^2 x \cos^4 x \, dx &= \int \left(\frac{1-\cos 2x}{2}\right)\left(\frac{1+\cos 2x}{2}\right)^2 dx \\ &= \\ &= \\ &= \end{aligned}$$

$$\sin^2 x = \frac{1-\cos 2x}{2} \qquad \cos^2 x = \frac{1+\cos 2x}{2}$$

**Example**

Find $\displaystyle\int \sin^{2} x \cos^{4} x \, dx$.

Both 2 and 4 are even so this is an example of case 3.

$$\begin{aligned}
\int \sin^{2} x \cos^{4} x \, dx &= \int \left(\frac{1-\cos 2x}{2}\right)\left(\frac{1+\cos 2x}{2}\right)^2 dx \\
&= \frac{1}{8}\int (1-\cos 2x)(1+2\cos 2x+\cos^2 2x)\, dx \\
&= \frac{1}{8}\int 1+\cos 2x-\cos^2 2x-\cos^3 2x \;\, dx \\
&=
\end{aligned}$$

$$\sin^2 x = \frac{1-\cos 2x}{2} \qquad \cos^2 x = \frac{1+\cos 2x}{2}$$

**Example**

Find $\int \sin^2 x \cos^4 x \, dx$.

Both 2 and 4 are even so this is an example of case 3.

$$\begin{aligned}
\int \sin^2 x \cos^4 x \, dx &= \int \left(\frac{1-\cos 2x}{2}\right)\left(\frac{1+\cos 2x}{2}\right)^2 dx \\
&= \frac{1}{8}\int (1-\cos 2x)(1+2\cos 2x+\cos^2 2x)\, dx \\
&= \frac{1}{8}\int \underbrace{1+\cos 2x}_{\text{easy}} - \cos^2 2x - \cos^3 2x \; dx \\
&=
\end{aligned}$$

$$\sin^2 x = \frac{1-\cos 2x}{2} \qquad \cos^2 x = \frac{1+\cos 2x}{2}$$

**Example**

Find $\displaystyle\int \sin^2 x \cos^4 x \, dx$.

Both 2 and 4 are even so this is an example of case $\boxed{3}$.

$$\begin{aligned}
\int \sin^2 x \cos^4 x \, dx &= \int \left(\frac{1-\cos 2x}{2}\right)\left(\frac{1+\cos 2x}{2}\right)^2 dx \\
&= \frac{1}{8}\int (1-\cos 2x)(1+2\cos 2x+\cos^2 2x)\, dx \\
&= \frac{1}{8}\int 1+\cos 2x-\underbrace{\cos^2 2x}_{\text{case } \boxed{3}}-\cos^3 2x \;\, dx \\
&=
\end{aligned}$$

$$\sin^2 x = \frac{1-\cos 2x}{2} \qquad \cos^2 x = \frac{1+\cos 2x}{2}$$

**Example**

Find $\int \sin^2 x \cos^4 x \, dx$.

Both 2 and 4 are even so this is an example of case $\boxed{3}$.

$$\begin{aligned}\int \sin^2 x \cos^4 x \, dx &= \int \left(\frac{1-\cos 2x}{2}\right)\left(\frac{1+\cos 2x}{2}\right)^2 dx \\ &= \frac{1}{8}\int (1-\cos 2x)(1+2\cos 2x+\cos^2 2x)\,dx \\ &= \frac{1}{8}\int 1+\cos 2x-\cos^2 2x-\underbrace{\cos^3 2x}_{\text{case } \boxed{2}}\,dx \\ &= \end{aligned}$$

$$\sin^2 x = \frac{1-\cos 2x}{2} \qquad \cos^2 x = \frac{1+\cos 2x}{2}$$

**Example**

Find $\int \sin^2 x \cos^4 x \, dx$.

Both 2 and 4 are even so this is an example of case 3.

$$\begin{aligned}
\int \sin^2 x \cos^4 x \, dx &= \int \left(\frac{1-\cos 2x}{2}\right)\left(\frac{1+\cos 2x}{2}\right)^2 dx \\
&= \frac{1}{8}\int (1-\cos 2x)(1+2\cos 2x+\cos^2 2x)\, dx \\
&= \frac{1}{8}\int 1+\cos 2x-\cos^2 2x-\cos^3 2x \;\, dx \\
&= \frac{1}{8}\left(x+\frac{1}{2}\sin 2x-\int \cos^2 2x\, dx-\int \cos^3 2x\, dx\right)
\end{aligned}$$

$$\sin^2 x = \frac{1-\cos 2x}{2} \qquad \cos^2 x = \frac{1+\cos 2x}{2}$$

For the orange integral we calculate that

$$\int \cos^2 2x \, dx = $$

$$= $$

$$= $$

For the green integral we calculate that

$$\int \cos^3 2x \, dx = $$

$$= $$

$$= $$

$$\sin^2 x = \frac{1 - \cos 2x}{2} \qquad \cos^2 x = \frac{1 + \cos 2x}{2}$$

For the orange integral we calculate that

$$\int \cos^2 2x \, dx = \frac{1}{2} \int 1 + \cos 4x \, dx$$

$$=$$

$$=$$

For the green integral we calculate that

$$\int \cos^3 2x \, dx =$$

$$=$$

$$=$$

$$\sin^2 x = \frac{1 - \cos 2x}{2} \qquad \cos^2 x = \frac{1 + \cos 2x}{2}$$

For the orange integral we calculate that

$$\begin{aligned}\int \cos^2 2x \, dx &= \frac{1}{2} \int 1 + \cos 4x \, dx \\ &= \frac{1}{2}\left(x + \frac{1}{4}\sin 4x\right) + C_1 \\ &= \frac{x}{2} + \frac{1}{8}\sin 4x + C_1.\end{aligned}$$

For the green integral we calculate that

$$\begin{aligned}\int \cos^3 2x \, dx &= \\ &= \\ &= \end{aligned}$$

$$\sin^2 x = \frac{1-\cos 2x}{2} \qquad \cos^2 x = \frac{1+\cos 2x}{2}$$

For the orange integral we calculate that

$$\begin{aligned}\int \cos^2 2x\,dx &= \frac{1}{2}\int 1+\cos 4x\,dx\\ &= \frac{1}{2}\left(x+\frac{1}{4}\sin 4x\right)+C_1\\ &= \frac{x}{2}+\frac{1}{8}\sin 4x + C_1.\end{aligned}$$

For the green integral we calculate that

$$\begin{aligned}\int \cos^3 2x\,dx &= \int (1-\sin^2 2x)\cos 2x\,dx\\ &= \\ &= \end{aligned}$$

$$\sin^2 x = \frac{1-\cos 2x}{2} \qquad \cos^2 x = \frac{1+\cos 2x}{2}$$

For the orange integral we calculate that

$$\begin{aligned}\int \cos^2 2x\,dx &= \frac{1}{2}\int 1+\cos 4x\,dx\\ &= \frac{1}{2}\left(x+\frac{1}{4}\sin 4x\right)+C_1\\ &= \frac{x}{2}+\frac{1}{8}\sin 4x + C_1.\end{aligned}$$

For the green integral we calculate that

$$\begin{aligned}\int \cos^3 2x\,dx &= \int (1-\sin^2 2x)\cos 2x\,dx\\ &= \frac{1}{2}\int (1-u^2)\,du\\ &=\end{aligned}$$

$$\sin^2 x = \frac{1 - \cos 2x}{2} \qquad \cos^2 x = \frac{1 + \cos 2x}{2}$$

For the orange integral we calculate that

$$\begin{aligned}
\int \cos^2 2x \, dx &= \frac{1}{2} \int 1 + \cos 4x \, dx \\
&= \frac{1}{2} \left( x + \frac{1}{4} \sin 4x \right) + C_1 \\
&= \frac{x}{2} + \frac{1}{8} \sin 4x + C_1.
\end{aligned}$$

For the green integral we calculate that

$$\begin{aligned}
\int \cos^3 2x \, dx &= \int (1 - \sin^2 2x) \cos 2x \, dx \\
&= \frac{1}{2} \int (1 - u^2) \, du \\
&= \frac{1}{2} \left( \sin 2x - \frac{1}{3} \sin^3 2x \right) + C_2
\end{aligned}$$

$$\sin^2 x = \frac{1 - \cos 2x}{2} \qquad \cos^2 x = \frac{1 + \cos 2x}{2}$$

Now let's put it all together.

$$\int \sin^2 x \cos^4 x \, dx$$
$$= \frac{1}{8}\left(x + \frac{1}{2}\sin 2x - \int \cos^2 2x \, dx - \int \cos^3 2x \, dx\right)$$
$$=$$
$$=$$

$$\sin^2 x = \frac{1 - \cos 2x}{2} \qquad \cos^2 x = \frac{1 + \cos 2x}{2}$$

Now let's put it all together.

$$\begin{aligned}
&\int \sin^2 x \cos^4 x \, dx \\
&\qquad = \frac{1}{8}\left( x + \frac{1}{2}\sin 2x - \int \cos^2 2x \, dx - \int \cos^3 2x \, dx \right) \\
&\qquad = \frac{1}{8}\left( x + \frac{1}{2}\sin 2x - \left( \frac{x}{2} + \frac{1}{8}\sin 4x \right) - \frac{1}{2}\left( \sin 2x - \frac{1}{3}\sin^3 2x \right) \right) + C \\
&\qquad =
\end{aligned}$$

$$\sin^2 x = \frac{1 - \cos 2x}{2} \qquad \cos^2 x = \frac{1 + \cos 2x}{2}$$

Now let's put it all together.

$$\begin{aligned}
&\int \sin^2 x \cos^4 x \, dx \\
&\quad = \frac{1}{8}\left(x + \frac{1}{2}\sin 2x - \int \cos^2 2x \, dx - \int \cos^3 2x \, dx\right) \\
&\quad = \frac{1}{8}\left(x + \frac{1}{2}\sin 2x - \left(\frac{x}{2} + \frac{1}{8}\sin 4x\right) - \frac{1}{2}\left(\sin 2x - \frac{1}{3}\sin^3 2x\right)\right) + C \\
&\quad = \frac{x}{16} - \frac{1}{64}\sin 4x + \frac{1}{48}\sin^3 2x + C.
\end{aligned}$$

$$\sin^2 x = \frac{1-\cos 2x}{2} \qquad \cos^2 x = \frac{1+\cos 2x}{2}$$

**Example**

Find $\displaystyle\int_0^{\frac{\pi}{4}} \sqrt{1+\cos 4x}\, dx$.

$$\sin^2 x = \frac{1 - \cos 2x}{2} \qquad \cos^2 x = \frac{1 + \cos 2x}{2}$$

**Example**

Find $\int_0^{\frac{\pi}{4}} \sqrt{1 + \cos 4x}\, dx$.

To get rid of the $\sqrt{}$, we are going to use this .

$$\sin^2 x = \frac{1 - \cos 2x}{2} \qquad \cos^2 x = \frac{1 + \cos 2x}{2}$$

**Example**

Find $\displaystyle\int_0^{\frac{\pi}{4}} \sqrt{1 + \cos 4x}\, dx$.

To get rid of the $\sqrt{\phantom{x}}$, we are going to use this . Replacing $x$ by $2x$ and rearranging, this is

$$1 + \cos 4x = 2\cos^2 2x.$$

$$\sin^2 x = \frac{1 - \cos 2x}{2} \qquad \cos^2 x = \frac{1 + \cos 2x}{2}$$

**Example**

Find $\int_0^{\frac{\pi}{4}} \sqrt{1 + \cos 4x}\, dx$.

To get rid of the $\sqrt{\ }$, we are going to use this. Replacing $x$ by $2x$ and rearranging, this is

$$1 + \cos 4x = 2\cos^2 2x.$$

Hence

$$\int_0^{\frac{\pi}{4}} \sqrt{1 + \cos 4x}\, dx = \int_0^{\frac{\pi}{4}} \sqrt{2\cos^2 2x}\, dx =$$

$$=$$

$$=$$

$$\sin^2 x = \frac{1-\cos 2x}{2} \qquad \cos^2 x = \frac{1+\cos 2x}{2}$$

**Example**

Find $\displaystyle\int_0^{\frac{\pi}{4}} \sqrt{1+\cos 4x}\, dx$.

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Hence

$$\int_0^{\frac{\pi}{4}} \sqrt{1+\cos 4x}\, dx = \int_0^{\frac{\pi}{4}} \sqrt{2\cos^2 2x}\, dx = \int_0^{\frac{\pi}{4}} \sqrt{2}\sqrt{\cos^2 2x}\, dx$$

$$=$$

$$=$$

$$\sin^2 x = \frac{1-\cos 2x}{2} \qquad \cos^2 x = \frac{1+\cos 2x}{2}$$

**Example**

Find $\displaystyle\int_0^{\frac{\pi}{4}} \sqrt{1+\cos 4x}\,dx$.

To get rid of the $\sqrt{\phantom{x}}$, we are going to use this . Replacing $x$ by $2x$ and rearranging, this is

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Hence

$$\int_0^{\frac{\pi}{4}} \sqrt{1+\cos 4x}\,dx = \int_0^{\frac{\pi}{4}} \sqrt{2\cos^2 2x}\,dx = \int_0^{\frac{\pi}{4}} \sqrt{2}\sqrt{\cos^2 2x}\,dx$$

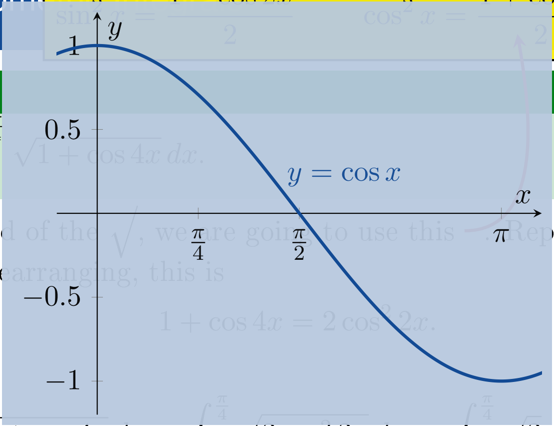
$$= \sqrt{2}\int_0^{\frac{\pi}{4}} |\cos 2x|\ dx$$

$$=$$

$$\sin^2 x = \frac{1-\cos 2x}{2} \qquad \cos^2 x = \frac{1+\cos 2x}{2}$$

**Example**

Find $\displaystyle\int_0^{\frac{\pi}{4}} \sqrt{1+\cos 4x}\, dx$.




To get rid of the $\sqrt{\ }$, we are going to use this . Replacing $x$ by $2x$ and rearranging, this is

$$1+\cos 4x = 2\cos^2 2x.$$

Hence

$$\int_0^{\frac{\pi}{4}} \sqrt{1+\cos 4x}\, dx = \int_0^{\frac{\pi}{4}} \sqrt{2\cos^2 2x}\, dx = \int_0^{\frac{\pi}{4}} \sqrt{2}\sqrt{\cos^2 2x}\, dx$$

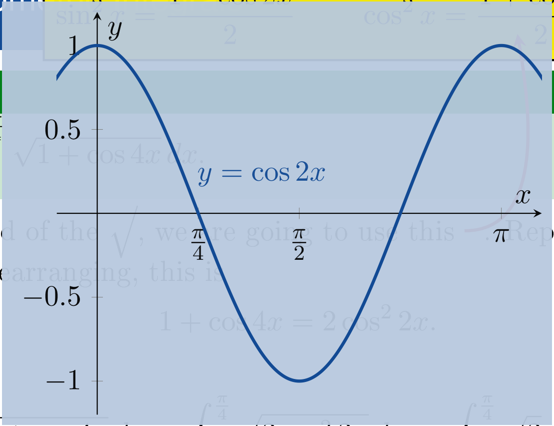
$$= \sqrt{2}\int_0^{\frac{\pi}{4}} |\cos 2x|\, dx$$

$$=$$

$$\sin^2 x = \frac{1 - \cos 2x}{2} \qquad \cos^2 x = \frac{1 + \cos 2x}{2}$$

**Example**

Find $\displaystyle\int_0^{\frac{\pi}{4}} \sqrt{1 + \cos 4x}\, dx$.




To get rid of the $\sqrt{\ }$, we are going to use this . Replacing $x$ by $2x$ and rearranging, this is

$$1 + \cos 4x = 2\cos^2 2x.$$

Hence

$$\int_0^{\frac{\pi}{4}} \sqrt{1 + \cos 4x}\, dx = \int_0^{\frac{\pi}{4}} \sqrt{2\cos^2 2x}\, dx = \int_0^{\frac{\pi}{4}} \sqrt{2}\sqrt{\cos^2 2x}\, dx$$

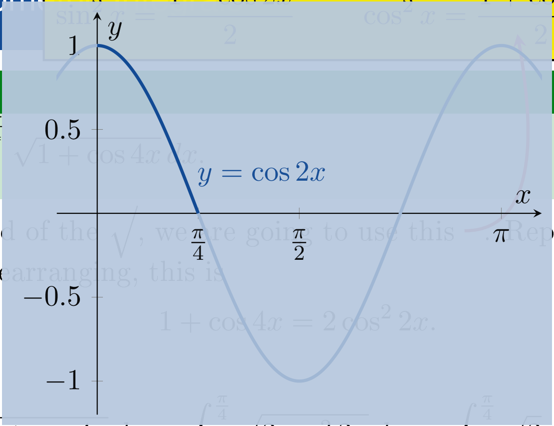
$$= \sqrt{2} \int_0^{\frac{\pi}{4}} |\cos 2x|\ dx$$

$$=$$

$$\sin^2 x = \frac{1-\cos 2x}{2} \qquad \cos^2 x = \frac{1+\cos 2x}{2}$$

**Example**

Find $\int_0^{\frac{\pi}{4}} \sqrt{1+\cos 4x}\,dx$.




To get rid of the $\sqrt{\ }$, we are going to use this . Replacing $x$ by $2x$ and rearranging, this is

$$1+\cos 4x = 2\cos^2 2x.$$

Hence

$$\begin{aligned}\int_0^{\frac{\pi}{4}} \sqrt{1+\cos 4x}\,dx &= \int_0^{\frac{\pi}{4}} \sqrt{2\cos^2 2x}\,dx = \int_0^{\frac{\pi}{4}} \sqrt{2}\sqrt{\cos^2 2x}\,dx \\ &= \sqrt{2}\int_0^{\frac{\pi}{4}} \left|\cos 2x\right|\,dx \\ &= \end{aligned}$$

$$\sin^2 x = \frac{1-\cos 2x}{2} \qquad \cos^2 x = \frac{1+\cos 2x}{2}$$

**Example**

Find $\displaystyle\int_0^{\frac{\pi}{4}} \sqrt{1+\cos 4x}\,dx.$

To get rid of the $\sqrt{\phantom{x}}$, we are going to use this . Replacing $x$ by $2x$ and rearranging, this is

$$1+\cos 4x = 2\cos^2 2x.$$

Hence

$$\int_0^{\frac{\pi}{4}} \sqrt{1+\cos 4x}\,dx = \int_0^{\frac{\pi}{4}} \sqrt{2\cos^2 2x}\,dx = \int_0^{\frac{\pi}{4}} \sqrt{2}\sqrt{\cos^2 2x}\,dx$$

$$= \sqrt{2}\int_0^{\frac{\pi}{4}} \cos 2x \quad dx$$

$$=$$

$$\sin^2 x = \frac{1-\cos 2x}{2} \qquad \cos^2 x = \frac{1+\cos 2x}{2}$$

**Example**

Find $\displaystyle\int_0^{\frac{\pi}{4}} \sqrt{1+\cos 4x}\,dx$.

To get rid of the $\sqrt{\ }$, we are going to use this . Replacing $x$ by $2x$ and rearranging, this is

$$1 + \cos 4x = 2\cos^2 2x.$$

Hence

$$\begin{aligned}
\int_0^{\frac{\pi}{4}} \sqrt{1+\cos 4x}\,dx &= \int_0^{\frac{\pi}{4}} \sqrt{2\cos^2 2x}\,dx = \int_0^{\frac{\pi}{4}} \sqrt{2}\sqrt{\cos^2 2x}\,dx \\
&= \sqrt{2}\int_0^{\frac{\pi}{4}} \cos 2x \quad dx \\
&= \sqrt{2}\left[\frac{\sin 2x}{2}\right]_0^{\frac{\pi}{4}} = \sqrt{2}\left(\frac{1}{2} - 0\right) = \frac{\sqrt{2}}{2}.
\end{aligned}$$

$$\sec^2 x = 1 + \tan^2 x \qquad \frac{d}{dx}\tan x = \sec^2 x$$

**Example**

Find $\displaystyle\int \tan^4 x\,dx$.

$$\sec^2 x = 1 + \tan^2 x \qquad \frac{d}{dx} \tan x = \sec^2 x$$

**Example**

Find $\displaystyle\int \tan^4 x \, dx$.

**Solution**

$$\int \tan^4 x \, dx = \int \tan^2 x \cdot \tan^2 x \, dx = \int \tan^2 x \cdot (\sec^2 x - 1) \, dx$$

$$= \int \tan^2 x \sec^2 x \, dx - \int \tan^2 x \, dx$$

$$= \int \tan^2 x \sec^2 x \, dx - \int (\sec^2 x - 1) \, dx$$

$$= \int \tan^2 x \sec^2 x \, dx - \int \sec^2 x \, dx + \int dx$$

In the first integral, we let

$$u = \tan x, \qquad du = \sec^2 x \, dx$$

and have

$$\int u^2 \, du = \frac{1}{3} u^3 + C_1.$$

The remaining integrals are standard forms, so

$$\int \tan^4 x \, dx = \frac{1}{3} \tan^3 x - \tan x + x + C. \qquad \blacksquare$$

**Example**

Find $\displaystyle\int \sec^3 x\,dx$.

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Find $\int \sec^3 x\,dx$.

Think of this as $\int \sec x \sec^2 x\,dx$. We are going to use Integration by Parts.

**Example**

Find $\displaystyle\int \sec^3 x\,dx$.

Think of this as $\displaystyle\int \sec x \sec^2 x\,dx$. We are going to use Integration by Parts. Let $u = \sec x$ and $dv = \sec^2 x\,dx$. Then $du = \sec x \tan x\,dx$ and $v = \tan x$.

$$\int u\,dv = uv - \int v\,du$$

**Example**

Find $\displaystyle\int \sec^3 x\,dx$.

Think of this as $\displaystyle\int \sec x \sec^2 x\,dx$. We are going to use Integration by Parts. Let $u = \sec x$ and $dv = \sec^2 x\,dx$. Then $du = \sec x \tan x\,dx$ and $v = \tan x$. Hence

$$\int \sec x \sec^2 x\,dx = \sec x \tan x - \int \tan x \sec x \tan x\,dx$$

$$=$$

$$=$$

$$=$$

$$\int u\,dv = uv - \int v\,du$$

**Example**

Find $\displaystyle\int \sec^3 x\,dx$.

Think of this as $\displaystyle\int \sec x \sec^2 x\,dx$. We are going to use Integration by Parts. Let $u = \sec x$ and $dv = \sec^2 x\,dx$. Then $du = \sec x \tan x\,dx$ and $v = \tan x$. Hence

$$\begin{aligned}\int \sec x \sec^2 x\,dx &= \sec x \tan x - \int \tan x \sec x \tan x\,dx \\ &= \sec x \tan x - \int \tan^2 x\, \sec x\,dx \\ &= \\ &= \end{aligned}$$

$$\int u\,dv = uv - \int v\,du$$

**Example**

Find $\displaystyle\int \sec^3 x\,dx$.

Think of this as $\displaystyle\int \sec x \sec^2 x\,dx$. We are going to use Integration by Parts. Let $u = \sec x$ and $dv = \sec^2 x\,dx$. Then $du = \sec x \tan x\,dx$ and $v = \tan x$. Hence

$$\begin{aligned}\int \sec x \sec^2 x\,dx &= \sec x \tan x - \int \tan x \sec x \tan x\,dx \\ &= \sec x \tan x - \int \tan^2 x \sec x\,dx \\ &= \sec x \tan x - \int (\sec^2 x - 1) \sec x\,dx \\ &= \end{aligned}$$

$$\int u\,dv = uv - \int v\,du$$

**Example**

Find $\displaystyle\int \sec^3 x\,dx$.

Think of this as $\displaystyle\int \sec x \sec^2 x\,dx$. We are going to use Integration by Parts. Let $u = \sec x$ and $dv = \sec^2 x\,dx$. Then $du = \sec x \tan x\,dx$ and $v = \tan x$. Hence

$$\begin{aligned}
\int \sec x \sec^2 x\,dx &= \sec x \tan x - \int \tan x \sec x \tan x\,dx \\
&= \sec x \tan x - \int \tan^2 x \sec x\,dx \\
&= \sec x \tan x - \int (\sec^2 x - 1) \sec x\,dx \\
&= \sec x \tan x + \int \sec x\,dx - \int \sec^3 x\,dx.
\end{aligned}$$

So we have

$$\int \sec^3 x\,dx = \sec x \tan x + \int \sec x\,dx - \int \sec^3 x\,dx.$$

So we have

$$\int \sec^3 x \, dx = \sec x \tan x + \int \sec x \, dx - \int \sec^3 x \, dx.$$

Thus

$$2 \int \sec^3 x \, dx = \sec x \tan x + \int \sec x \, dx$$

So we have

$$\int \sec^3 x\,dx = \sec x \tan x + \int \sec x\,dx - \int \sec^3 x\,dx.$$

Thus

$$2\int \sec^3 x\,dx = \sec x \tan x + \int \sec x\,dx$$

and

$$\begin{aligned}\int \sec^3 x\,dx &= \frac{1}{2}\sec x \tan x + \frac{1}{2}\int \sec x\,dx \\ &= \frac{1}{2}\sec x \tan x + \frac{1}{2}\ln|\sec x + \tan x| + C.\end{aligned}$$

$$\sec^2 x = 1 + \tan^2 x \qquad \frac{d}{dx}\tan x = \sec^2 x$$

**Example**

Find $\displaystyle\int \tan^4 x \sec^4 x \, dx$.

$$\sec^2 x = 1 + \tan^2 x \qquad \frac{d}{dx}\tan x = \sec^2 x$$

**Example**

Find $\displaystyle\int \tan^4 x \sec^4 x \, dx$.

**Solution**

$$\begin{aligned}
\int (\tan^4 x)(\sec^4 x)\,dx &= \int (\tan^4 x)(1 + \tan^2 x)(\sec^2 x)\,dx && \sec^2 x = 1 + \tan^2 x \\
&= \int (\tan^4 x + \tan^6 x)(\sec^2 x)\,dx \\
&= \int (\tan^4 x)(\sec^2 x)\,dx + \int (\tan^6 x)(\sec^2 x)\,dx \\
&= \int u^4\,du + \int u^6\,du = \frac{u^5}{5} + \frac{u^7}{7} + C && u = \tan x,\ du = \sec^2 x\,dx \\
&= \frac{\tan^5 x}{5} + \frac{\tan^7 x}{7} + C
\end{aligned}$$

How do we calculate

$$\int \sin mx \sin nx \, dx$$

or

$$\int \sin mx \cos nx \, dx$$

or

$$\int \cos mx \cos nx \, dx$$

?

How do we calculate

$$\int \sin mx \sin nx \, dx$$

or

$$\int \sin mx \cos nx \, dx$$

or

$$\int \cos mx \cos nx \, dx$$

?

It is possible to use integration by parts (twice), but there is an easier way.

$$\cos(A + B) = \cos A \cos B - \sin A \sin B$$

$$\cos(mx - nx) - \cos(mx + nx) =$$

$$\cos(A+B) = \cos A \cos B - \sin A \sin B$$

$$\cos(mx - nx) - \cos(mx + nx) = \cos mx \cos nx + \sin mx \sin nx$$
$$- \cos mx \cos nx + \sin mx \sin nx$$

$$\cos(A+B) = \cos A \cos B - \sin A \sin B$$

$$\begin{aligned}
\cos(mx - nx) - \cos(mx + nx) = {} & \cos mx \cos nx + \sin mx \sin nx \\
& - \cos mx \cos nx + \sin mx \sin nx \\
= {} & 2 \sin mx \sin nx
\end{aligned}$$

$$\cos A + B) = \cos A \cos B - \sin A \sin B$$

$$\begin{aligned}\cos(mx - nx) - \cos(mx + nx) &= \cos mx \cos nx + \sin mx \sin nx \\ &\quad - \cos mx \cos nx + \sin mx \sin nx \\ &= 2 \sin mx \sin nx\end{aligned}$$

Therefore

$$\sin mx \sin nx = \frac{1}{2}\big(\cos(m-n)x - \cos(m+n)x\big).$$

$$\cos(A+B) = \cos A \cos B - \sin A \sin B$$

$$\begin{aligned}\cos(mx - nx) - \cos(mx + nx) &= \cos mx \cos nx + \sin mx \sin nx \\ &\quad - \cos mx \cos nx + \sin mx \sin nx \\ &= 2 \sin mx \sin nx\end{aligned}$$

Therefore

$$\sin mx \sin nx = \frac{1}{2}\big(\cos(m-n)x - \cos(m+n)x\big).$$

Similarly

$$\sin mx \cos nx = \frac{1}{2}\big(\sin(m-n)x + \sin(m+n)x\big)$$

and

$$\cos mx \cos nx = \frac{1}{2}\big(\cos(m-n)x + \cos(m+n)x\big).$$

$$\sin mx \cos nx = \frac{1}{2}\big(\sin(m-n)x + \sin(m+n)x\big)$$

**Example**

Find $\displaystyle\int \sin 3x \cos 5x \, dx$.

$$\sin mx \cos nx = \frac{1}{2}\big(\sin(m-n)x + \sin(m+n)x\big)$$

**Example**

Find $\displaystyle\int \sin 3x \cos 5x \, dx$.

**Solution** From Equation (4) with $m = 3$ and $n = 5$, we get

$$\begin{aligned}\int \sin 3x \cos 5x \, dx &= \frac{1}{2}\int [\sin(-2x) + \sin 8x]\, dx \\ &= \frac{1}{2}\int (\sin 8x - \sin 2x)\, dx \\ &= -\frac{\cos 8x}{16} + \frac{\cos 2x}{4} + C.\end{aligned}$$

■

$$\cos mx \cos nx = \frac{1}{2}\big(\cos(m-n)x + \cos(m+n)x\big)$$

**Example**

Find $\displaystyle\int \cos 3x \cos 2x \, dx$.

$$\cos mx \cos nx = \tfrac{1}{2}\big(\cos(m-n)x + \cos(m+n)x\big)$$

**Example**

Find $\displaystyle\int \cos 3x \cos 2x \, dx$.

We have $m = 3$ and $n = 2$. It follows that

$$\begin{aligned}\int \cos 3x \cos 2x \, dx &= \frac{1}{2}\int \cos(3-2)x \, dx + \frac{1}{2}\int \cos(3+2)x \, dx \\ &= \dots\end{aligned}$$

# Next Time

- 8.4 Trigonometric Substitutions
- 8.5 Integration of Rational Functions by Partial Fractions
- 8.8 Improper Integrals