Lecture 8

- 4.3 Solving More Initial Value Problems
- 4.4 Step Functions

4.3 Solving More Initial Value Problems

Theorem

1. $\mathcal{L}[f'](s) = s\mathcal{L}[f](s) - f(0).$
2. $\mathcal{L}[f''](s) = s^2\mathcal{L}[f](s) - sf(0) - f'(0).$
3. $\mathcal{L}[f'''](s) = s^3\mathcal{L}[f](s) - s^2 f(0) - sf'(0) - f''(0).$
4. $$\mathcal{L}[f^{(n)}](s) = s^n\mathcal{L}[f](s) - s^{n-1}f(0) - s^{n-2}f'(0) - \dots - sf^{(n-2)}(0) - f^{(n-1)}(0).$$

Example

Use the Laplace Transform to solve

$$\begin{cases} y'' - 3y' + 2y = \cos t \\ y(0) = 0 \\ y'(0) = 0. \end{cases}$$

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Taking the Laplace Transform of the ODE gives

$$\begin{aligned} \mathcal{L}[y''] - 3\mathcal{L}[y'] + 2\mathcal{L}[y] &= \mathcal{L}[\cos t] \\ \big(s^2Y - sy(0) - y'(0)\big) - 3\big(sY - y(0)\big) + 2Y &= \frac{s}{s^2+1} \\ (s^2 - 3s + 2)Y &= \frac{s}{s^2+1} \end{aligned}$$

$$Y(s) = \frac{s}{(s^2+1)(s^2-3s+2)} = \frac{s}{(s^2+1)(s-2)(s-1)}$$

$$=$$

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$$
\begin{aligned}
Y(s) &= \frac{s}{(s^2+1)(s^2-3s+2)} = \frac{s}{(s^2+1)(s-2)(s-1)} \\
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&= \\
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&=
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&= \frac{1}{10}\mathcal{L}\left[\cos t\right] - \frac{3}{10}\mathcal{L}\left[\sin t\right] + \frac{2}{5}\mathcal{L}\left[e^{2t}\right] - \frac{1}{2}\mathcal{L}\left[e^{t}\right].
\end{aligned}
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$$Y(s) = \frac{1}{10}\mathcal{L}\left[\cos t\right] - \frac{3}{10}\mathcal{L}\left[\sin t\right] + \frac{2}{5}\mathcal{L}\left[e^{2t}\right] - \frac{1}{2}\mathcal{L}\left[e^{t}\right]$$

Therefore the solution to the IVP is

$$y(t) = \mathcal{L}^{-1}\left[Y\right](t) =$$

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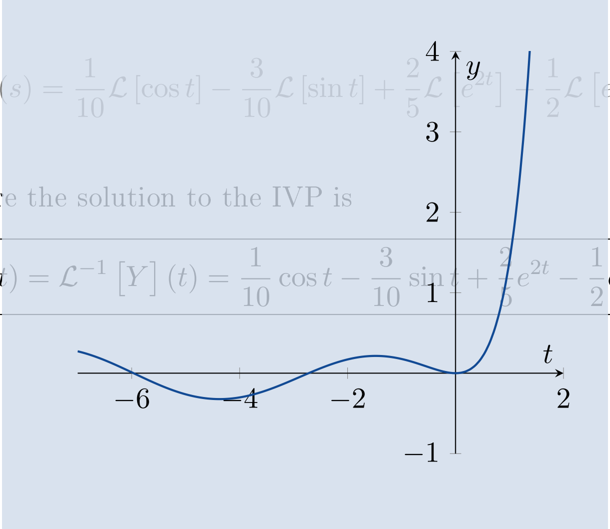

Example

Use the Laplace Transform to solve

$$\begin{cases} y'' + 2y' + y = 4e^{-t} \\ y(0) = 2 \\ y'(0) = -1. \end{cases}$$

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$$\mathcal{L}\left[y''\right] + 2\mathcal{L}\left[y'\right] + \mathcal{L}\left[y\right] = \mathcal{L}\left[4e^{-t}\right]$$

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$$(s^2Y - 2s + 1) + 2(sY - 2) + Y = \frac{4}{s+1}$$

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Use the Laplace Transform to solve

$$\begin{cases} y'' + 2y' + y = 4e^{-t} \\ y(0) = 2 \\ y'(0) = -1. \end{cases}$$

$$(s^2 + 2s + 1)Y - 2s + 1 - 4 = \frac{4}{s+1}$$

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Use the Laplace Transform to solve

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$$\begin{cases} y'' + 2y' + y = 4e^{-t} \\ y(0) = 2 \\ y'(0) = -1. \end{cases}$$

$$(s+1)^2 Y = \frac{2s^2 + 5s + 7}{s+1}$$

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Use the Laplace Transform to solve

$$\begin{cases} y'' + 2y' + y = 4e^{-t} \\ y(0) = 2 \\ y'(0) = -1. \end{cases}$$

$$y(t) = \mathcal{L}^{-1}\left[\frac{2s^2 + 5s + 7}{(s+1)^3}\right]$$

I leave it for you to check that if

$$\frac{2s^2+5s+7}{(s+1)^3} = \frac{A}{s+1} + \frac{B}{(s+1)^2} + \frac{C}{(s+1)^3}$$

then $A = 2$, $B = 1$ and $C = 4$.

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$$\frac{2s^2+5s+7}{(s+1)^3} = \frac{2}{s+1} + \frac{1}{(s+1)^2} + \frac{4}{(s+1)^3}$$

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$$\begin{aligned}\frac{2s^2+5s+7}{(s+1)^3} &= \frac{2}{s+1} + \frac{1}{(s+1)^2} + \frac{4}{(s+1)^3} \\ &= 2\left(\frac{1}{s+1}\right) + \left(\frac{1}{(s+1)^2}\right) + 2\left(\frac{2}{(s+1)^3}\right).\end{aligned}$$

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In our table of Laplace Transforms, we find that $\mathcal{L}\left[e^{-t}\right] = \frac{1}{s+1}$, $\mathcal{L}\left[te^{-t}\right] = \frac{1}{(s+1)^2}$ and $\mathcal{L}\left[t^2e^{-t}\right] = \frac{2}{(s+1)^3}$.

Therefore the solution to the IVP is

$$\begin{aligned} y(t) &= \mathcal{L}^{-1}\left[\frac{2s^2+5s+7}{(s+1)^3}\right] \\ &= 2\mathcal{L}^{-1}\left[\frac{1}{s+1}\right] + \mathcal{L}^{-1}\left[\frac{1}{(s+1)^2}\right] + 2\mathcal{L}^{-1}\left[\frac{2}{(s+1)^3}\right] \\ &= 2\left(e^{-t}\right) + \left(te^{-t}\right) + 2\left(t^2e^{-t}\right) \\ &= \boxed{(2t^2+t+2)e^{-t}.} \end{aligned}$$

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y(t) &= \mathcal{L}^{-1}\left[\frac{2s^2+5s+7}{(s+1)^3}\right] \\
&= 2\mathcal{L}^{-1}\left[\frac{1}{s+1}\right] + \mathcal{L}^{-1}\left[\frac{1}{(s+1)^2}\right] + 2\mathcal{L}^{-1}\left[\frac{2}{(s+1)^3}\right] \\
&= 2\left(e^{-t}\right) + \left(te^{-t}\right) + 2\left(t^2e^{-t}\right) \\
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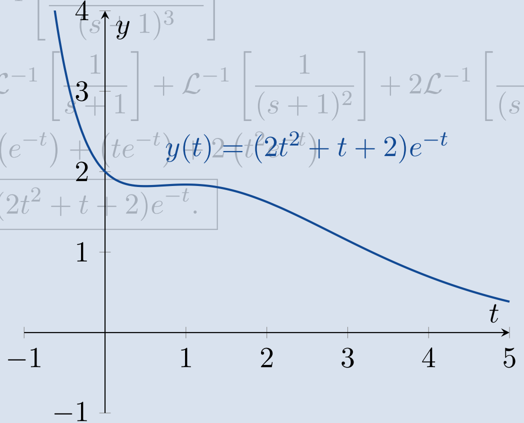
$$



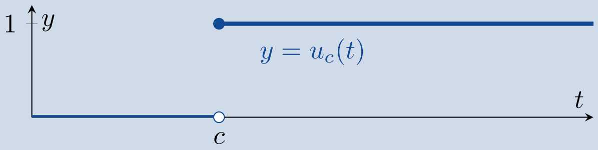
4.4 Step Functions

Definition

The *unit step function* $u_c : [0,\infty) \to \mathbb{R}$ is defined by

$$u_c(t) = \begin{cases} 0 & t < c \\ 1 & t \geq c \end{cases}$$

for $c \geq 0$.



Example

Draw the graph of $y = 1 - u_c(t)$.

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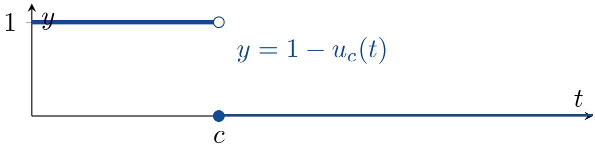

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Example

Draw the graph of $y = u_1(t) - u_2(t)$.

Clearly $t = 1$ and $t = 2$ are important points. So we consider the function on the intervals $[0, 1)$, $[1, 2)$ and $[2, \infty)$.

$$u_1(t) - u_2(t) = \begin{cases} u_1(t) - u_2(t) & 0 \leq t < 1 \\ u_1(t) - u_2(t) & 1 \leq t < 2 \\ u_1(t) - u_2(t) & 2 \leq t \end{cases}$$

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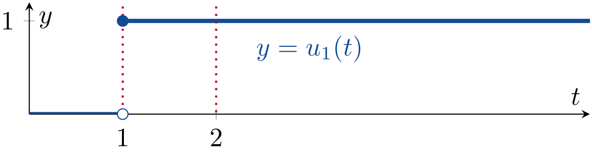

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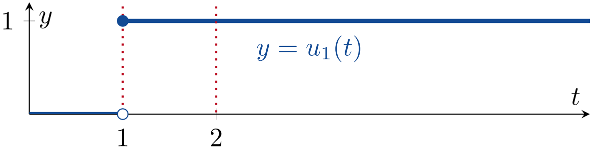
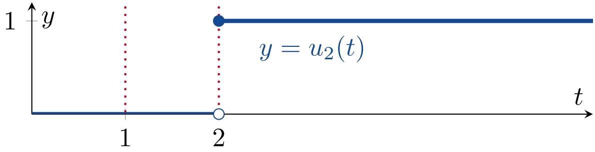


$$u_1(t) - u_2(t) = \begin{cases} 0 - u_2(t) & 0 \le t < 1 \\ 1 - u_2(t) & 1 \le t < 2 \\ 1 - u_2(t) & 2 \le t \end{cases}$$



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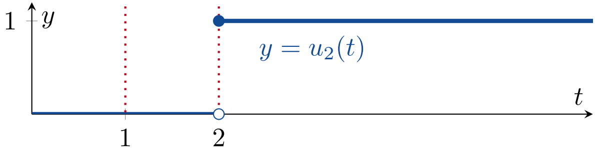

$$u_1(t) - u_2(t) = \begin{cases} 0 - 0 & 0 \leq t < 1 \\ 1 - 0 & 1 \leq t < 2 \\ 1 - 1 & 2 \leq t \end{cases}$$

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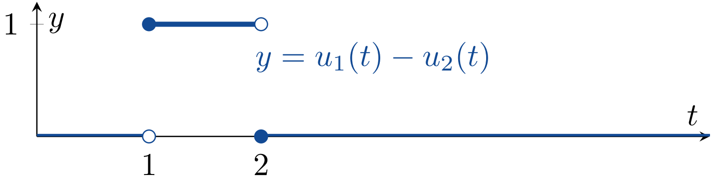
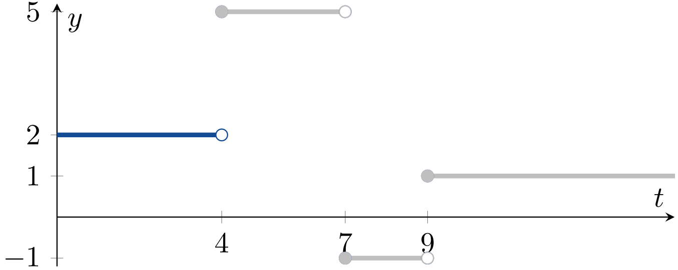


Example

Write the function

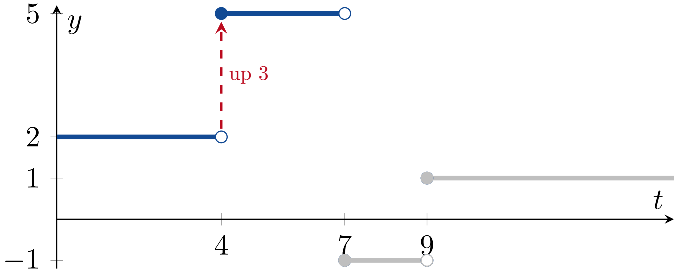
$$f(t) = \begin{cases} 2 & 0 \leq t < 4 \\ 5 & 4 \leq t < 7 \\ -1 & 7 \leq t < 9 \\ 1 & 9 \leq t \end{cases}$$

in terms of the unit step function.



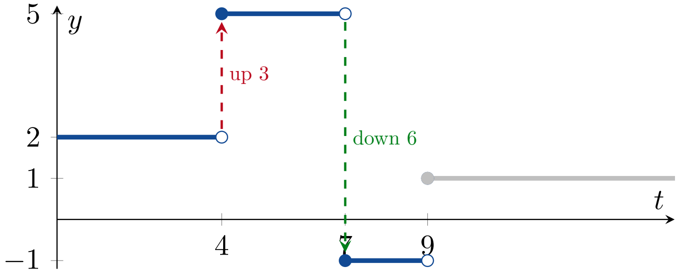
The function starts at $f(0) = 2$. So we will have

$$f(t) = 2 + (\text{something}).$$



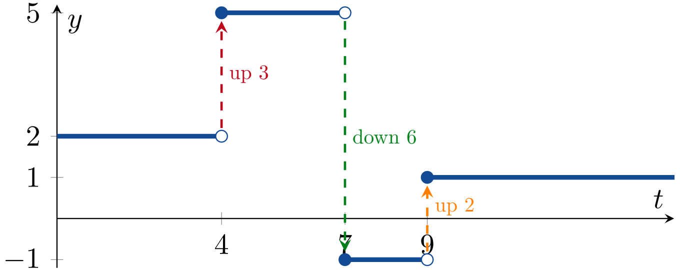
At $t = 4$, the function jumps from 2 to 5 (it goes "up 3"). So

$$f(t) = 2 + 3u_4(t) + (\text{something}).$$



Then it goes "down 6" when $t = 7$. So

$$f(t) = 2 + 3u_4(t) - 6u_7(t) + (\text{something}).$$



Finally it goes "up 2" when $t = 9$. Therefore

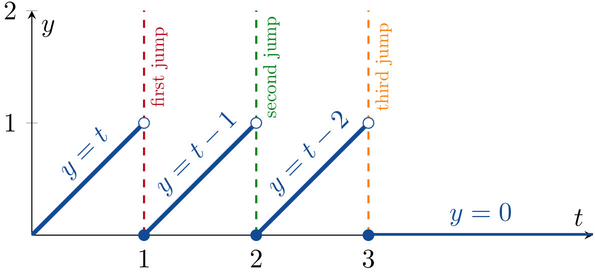
$$f(t) = 2 + 3u_4(t) - 6u_7(t) + 2u_9(t).$$

Example

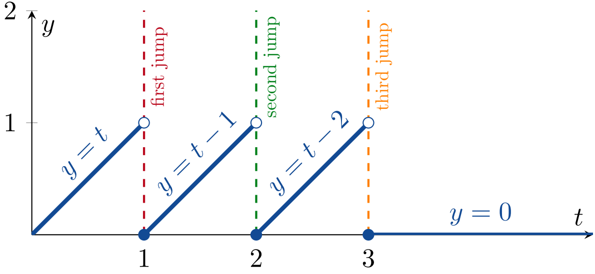
Write the function

$$f(t) = \begin{cases} t & 0 \leq t < 1 \\ t-1 & 1 \leq t < 2 \\ t-2 & 2 \leq t < 3 \\ 0 & 3 \leq t \end{cases}$$

in terms of the unit step function.

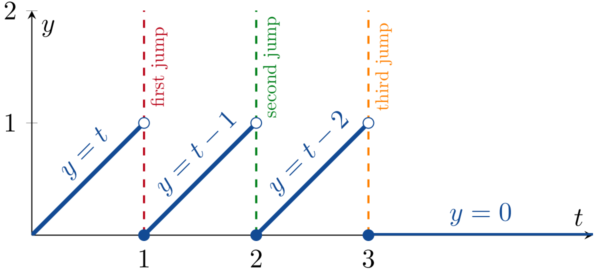


This function starts with $f(t) = t$, then changes when $t = 1$, $t = 2$ and $t = 3$:



This function starts with $f(t) = t$, then changes when $t = 1$, $t = 2$ and $t = 3$: So we must have

$$f(t) = t + \begin{pmatrix} \text{first} \\ \text{jump} \end{pmatrix} u_1(t) + \begin{pmatrix} \text{second} \\ \text{jump} \end{pmatrix} u_2(t) + \begin{pmatrix} \text{third} \\ \text{jump} \end{pmatrix} u_3(t).$$



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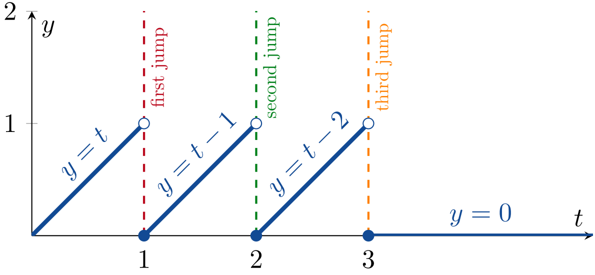
At each "jump" we calculate

$$\text{jump} = \begin{pmatrix} \text{function} \\ \text{on right} \end{pmatrix} - \begin{pmatrix} \text{function} \\ \text{on left} \end{pmatrix}.$$

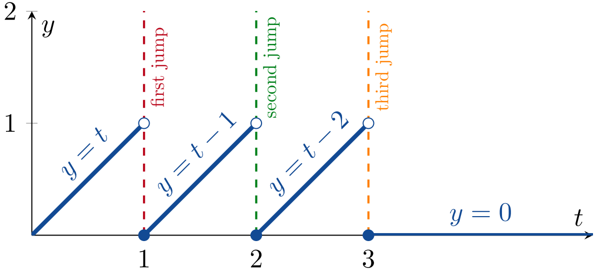


So we have

$$\begin{pmatrix}\text{first}\\ \text{jump}\end{pmatrix} =$$

$$\begin{pmatrix}\text{second}\\ \text{jump}\end{pmatrix} =$$

$$\begin{pmatrix}\text{third}\\ \text{jump}\end{pmatrix} =$$



So we have

$$\begin{pmatrix}\text{first}\\\text{jump}\end{pmatrix} = (t-1) - t = -1$$

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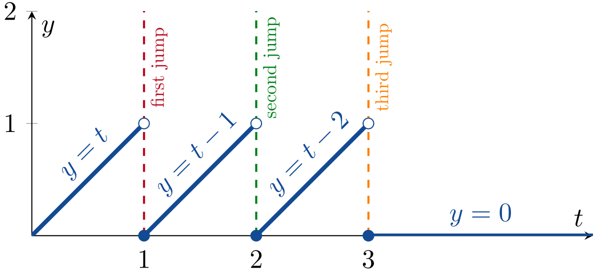

So we have

$$\begin{pmatrix}\text{first}\\ \text{jump}\end{pmatrix} = (t-1) - t = -1$$

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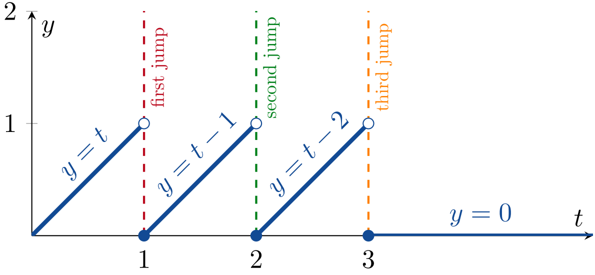

So we have

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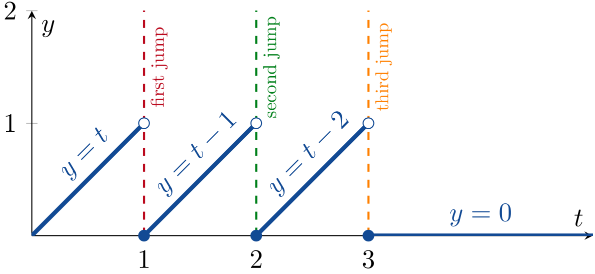
$$\begin{pmatrix}\text{third}\\ \text{jump}\end{pmatrix} = 0 - (t-2) = 2 - t$$



Hence

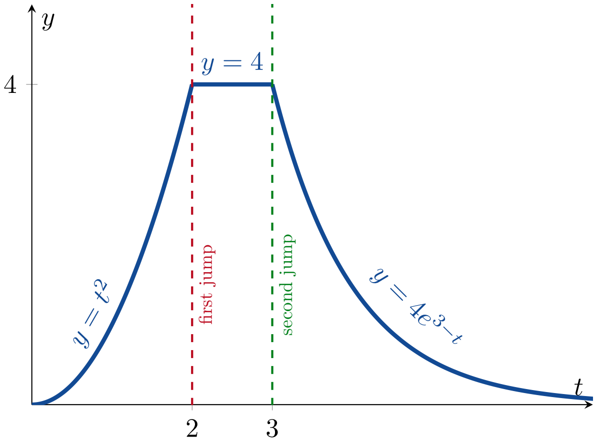
$$f(t) = t - u_1(t) - u_2(t) + (2 - t)u_3(t).$$

Example

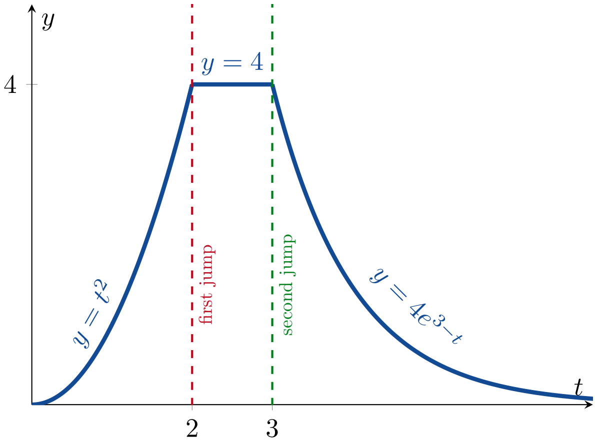
Write the function

$$f(t) = \begin{cases} t^2 & 0 \leq t < 2 \\ 4 & 2 \leq t < 3 \\ 4e^{t-3} & 3 \leq t \end{cases}$$

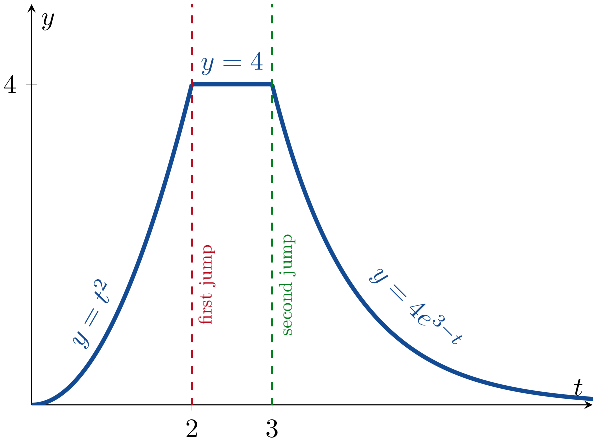
in terms of the unit step function.



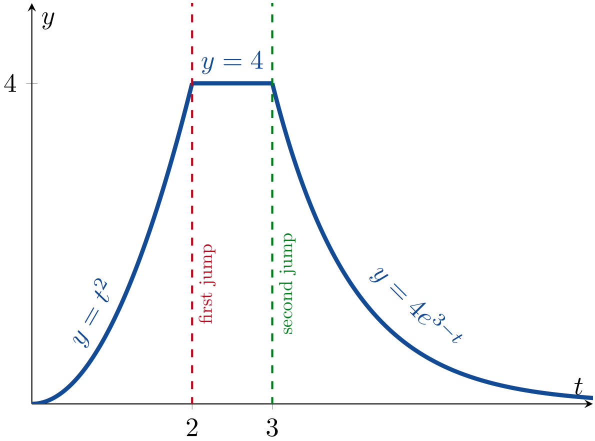
y
$y = 4$
4
$y = t^2$
first jump
second jump
$y = 4e^{3-t}$
t
2
3



$$f(t) = t^2 + \begin{pmatrix} \text{first} \\ \text{jump} \end{pmatrix} u_2(t) + \begin{pmatrix} \text{second} \\ \text{jump} \end{pmatrix} u_3(t).$$



$$f(t) = t^2 + (4 - t^2)u_2(t) + \begin{pmatrix}\text{second}\\ \text{jump}\end{pmatrix} u_3(t).$$



$$f(t) = t^2 + \left(4 - t^2\right)u_2(t) + \left(4e^{t-3} - 4\right)u_3(t).$$

What is the Laplace Transform of the unit step function?

What is the Laplace Transform of the unit step function?

We calculate that

$$\mathcal{L}\left[u_c\right](s) = \int_0^\infty e^{-st} u_c(t)\,dt =$$

$$= \qquad\qquad =$$

for $s > 0$.

What is the Laplace Transform of the unit step function?

We calculate that

$$\mathcal{L}[u_c](s) = \int_0^\infty e^{-st} u_c(t)\,dt = \int_0^c e^{-st} 0\,dt + \int_c^\infty e^{-st} 1\,dt$$

$$= \qquad\qquad =$$

for $s > 0$.

What is the Laplace Transform of the unit step function?

We calculate that

$$\begin{aligned}\mathcal{L}\left[u_c\right](s) &= \int_0^\infty e^{-st} u_c(t)\,dt = \int_0^c e^{-st} 0\,dt + \int_c^\infty e^{-st} 1\,dt \\ &= \int_c^\infty e^{-st}\,dt =\end{aligned}$$

for $s > 0$.

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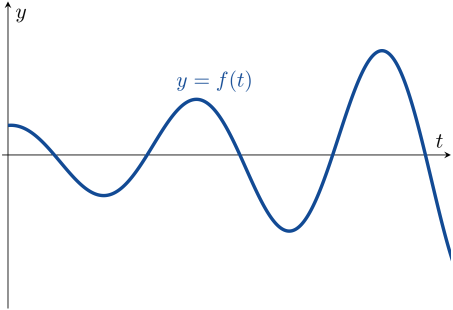
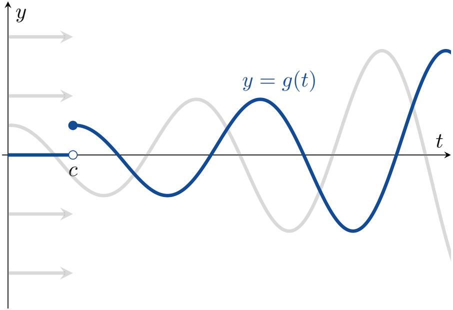
for $s > 0$.

Theorem

$$\mathcal{L}\left[u_c\right](s) = \frac{e^{-cs}}{s}$$



Now suppose that we have some function $f : [0, \infty) \to \mathbb{R}$



Now suppose that we have some function $f : [0, \infty) \to \mathbb{R}$ and we define a new function $g : [0, \infty) \to \mathbb{R}$ by

$$g(t) = \begin{cases} 0 & t < c \\ f(t-c) & t \geq c. \end{cases}$$

We can write $g(t) = u_c(t) f(t-c)$.

What is the Laplace Transform of $g(t) = u_c(t)f(t-c)$?

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$$\mathcal{L}\left[g\right] = \mathcal{L}\left[u_c(t)f(t-c)\right]$$

What is the Laplace Transform of $g(t) = u_c(t)f(t-c)$?

$$\mathcal{L}\left[g\right] = \mathcal{L}\left[u_c(t)f(t-c)\right] = \int_0^\infty e^{-st}u_c(t)f(t-c)\,dt$$

What is the Laplace Transform of $g(t) = u_c(t)f(t-c)$?

$$\mathcal{L}\left[g\right] = \mathcal{L}\left[u_c(t)f(t-c)\right] = \int_0^\infty e^{-st}u_c(t)f(t-c)\,dt$$

$$= \int_c^\infty e^{-st}f(t-c)\,dt.$$

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$$\begin{aligned}\mathcal{L}\left[g\right] = \mathcal{L}\left[u_c(t)f(t-c)\right] &= \int_0^\infty e^{-st}u_c(t)f(t-c)\,dt \\ &= \int_c^\infty e^{-st}f(t-c)\,dt.\end{aligned}$$

Let $u = t - c$.

What is the Laplace Transform of $g(t) = u_c(t)f(t-c)$?

$$\mathcal{L}\left[g\right] = \mathcal{L}\left[u_c(t)f(t-c)\right] = \int_0^\infty e^{-st}u_c(t)f(t-c)\,dt$$
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Let $u = t - c$. Then $du = dt$ and $t = c \iff u = 0$.

What is the Laplace Transform of $g(t) = u_c(t)f(t-c)$?

$$\begin{aligned}\mathcal{L}\left[g\right] = \mathcal{L}\left[u_c(t)f(t-c)\right] &= \int_0^\infty e^{-st}u_c(t)f(t-c)\,dt \\ &= \int_c^\infty e^{-st}f(t-c)\,dt.\end{aligned}$$

Let $u = t - c$. Then $du = dt$ and $t = c \iff u = 0$. Therefore

$$\mathcal{L}\left[g\right] = \int_0^\infty e^{-s(u+c)}f(u)\,du$$

What is the Laplace Transform of $g(t) = u_c(t)f(t-c)$?

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$$\mathcal{L}\left[g\right] = \int_0^\infty e^{-s(u+c)}f(u)\,du = e^{-cs}\int_0^\infty e^{-su}f(u)\,du$$

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$$\mathcal{L}\left[g\right] = \int_0^\infty e^{-s(u+c)}f(u)\,du = e^{-cs}\int_0^\infty e^{-su}f(u)\,du = e^{-cs}\mathcal{L}\left[f\right].$$

What is the Laplace Transform of $g(t) = u_c(t)f(t-c)$?

$$\mathcal{L}[g] = \mathcal{L}\left[u_c(t)f(t-c)\right] = \int_0^\infty e^{-st}u_c(t)f(t-c)\,dt$$
$$= \int_c^\infty e^{-st}f(t-c)\,dt.$$

Let $u = t - c$. Then $du = dt$ and $t = c \iff u = 0$. Therefore

$$\mathcal{L}[g] = \int_0^\infty e^{-s(u+c)}f(u)\,du = e^{-cs}\int_0^\infty e^{-su}f(u)\,du = e^{-cs}\mathcal{L}[f].$$

Theorem

$$\mathcal{L}\left[u_c(t)f(t-c)\right](s) = e^{-cs}F(s)$$

Example

Find the Laplace Transform of

$$f(t) = \begin{cases} t & 0 \le t < 1 \\ t-1 & 1 \le t < 2 \\ t-2 & 2 \le t < 3 \\ 0 & 3 \le t. \end{cases}$$

Since

$$f(t) = t - u_1(t) - u_2(t) + (2-t)u_3(t)$$

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$$\begin{aligned} f(t) &= t - u_1(t) - u_2(t) + (2-t)u_3(t) \\ &= t - u_1(t) - u_2(t) - u_3(t) - u_3(t)(t-3) \end{aligned}$$

Since

$$\begin{aligned} f(t) &= t - u_1(t) - u_2(t) + (2-t)u_3(t) \\ &= t - u_1(t) - u_2(t) - u_3(t) - u_3(t)(t-3) \end{aligned}$$

we have that

$$\begin{aligned} F(s) &= \mathcal{L}\left[t\right] - \mathcal{L}\left[u_1\right] - \mathcal{L}\left[u_2\right] - \mathcal{L}\left[u_3\right] - \mathcal{L}\left[u_3(t)(t-3)\right] \\ &= \frac{1}{s^2} - \frac{e^{-s}}{s} - \frac{e^{-2s}}{s} - \frac{e^{-3s}}{s} - \frac{e^{-3s}}{s^2}. \end{aligned}$$

Example

Find the Laplace Transform of

$$f(t) = \begin{cases} \sin t & 0 \leq t \leq \frac{\pi}{4} \\ \sin t + \cos(t - \frac{\pi}{4}) & \frac{\pi}{4} \leq t. \end{cases}$$

Note that $f(t) = \sin t + g(t)$ where

$$g(t) = \begin{cases} 0 & 0 \le t \le \frac{\pi}{4} \\ \cos(t - \frac{\pi}{4}) & \frac{\pi}{4} \le t \end{cases} = u_{\frac{\pi}{4}}(t) \cos\left(t - \frac{\pi}{4}\right).$$

$$\mathcal{L}\left[u_c(t)f(t-c)\right](s) = e^{-cs}F(s)$$

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So

$$F(s) = \mathcal{L}\left[f\right] = \mathcal{L}\left[\sin t\right] + \mathcal{L}\left[u_{\frac{\pi}{4}}(t)\cos\left(t - \frac{\pi}{4}\right)\right]$$

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So

$$\begin{aligned} F(s) = \mathcal{L}\left[f\right] &= \mathcal{L}\left[\sin t\right] + \mathcal{L}\left[u_{\frac{\pi}{4}}(t)\cos\left(t - \frac{\pi}{4}\right)\right] \\ &= \mathcal{L}\left[\sin t\right] + e^{-\frac{\pi s}{4}}\mathcal{L}\left[\cos t\right] = \frac{1}{s^2+1} + e^{-\frac{\pi s}{4}}\frac{s}{s^2+1} \end{aligned}$$

$$\mathcal{L}\left[u_c(t)f(t-c)\right](s) = e^{-cs}F(s)$$

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Example

Find the inverse Laplace Transform of $F(s) = \frac{1-e^{-2s}}{s^2}$.

Example

Find the inverse Laplace Transform of $F(s) = \frac{1-e^{-2s}}{s^2}$.

$$f(t) = \mathcal{L}^{-1}\left[F\right] = \mathcal{L}^{-1}\left[\frac{1}{s^2}\right] - \mathcal{L}^{-1}\left[\frac{e^{-2s}}{s^2}\right] = t - u_2(t)(t-2)$$

$$= \begin{cases} t & 0 \le t < 2 \\ 2 & t \ge 2. \end{cases}$$

And what is the Laplace Transform of $e^{ct}f(t)$?

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$$\mathcal{L}\left[e^{ct}f(t)\right] = \int_0^\infty e^{-st}e^{ct}f(t)\,dt = \int_0^\infty e^{-(s-c)t}f(t)\,dt = F(s-c).$$

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Theorem

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Example

Find the inverse Laplace Transform of $G(s) = \dfrac{1}{s^2 - 4s + 5}$.

$$\mathcal{L}\left[e^{ct}f(t)\right] = F(s-c)$$

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Note first that

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If $F(s) = \frac{1}{s^2+1}$, then we have $G(s) = F(s-2)$.

$$\mathcal{L}\left[e^{ct}f(t)\right] = F(s-c)$$

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Therefore

$$g(t) = \mathcal{L}^{-1}\left[G\right] = \mathcal{L}^{-1}\left[F(s-2)\right]$$

$$\mathcal{L}\left[e^{ct}f(t)\right] = F(s-c)$$

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Therefore

$$g(t) = \mathcal{L}^{-1}\left[G\right] = \mathcal{L}^{-1}\left[F(s-2)\right] = e^{2t}\mathcal{L}^{-1}\left[F\right] = e^{2t}\sin t.$$

How to find the inverse Laplace Transform of $G(s) = \dfrac{ms + n}{as^2 + bs + c}$

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$\downarrow$

Does $as^2+bs+c=0$ have roots in $\mathbb{R}$?

How to find the inverse Laplace Transform of $G(s) = \dfrac{ms+n}{as^2+bs+c}$

↓

yes ← Does $as^2 + bs + c = 0$ have roots in $\mathbb{R}$?

How to find the inverse Laplace Transform of $G(s) = \dfrac{ms+n}{as^2+bs+c}$

Does $as^2+bs+c=0$ have roots in $\mathbb{R}$?

yes

Factorise

$$G(s) = \frac{ms+n}{a(s-d)(s-e)}$$

How to find the inverse Laplace Transform of $G(s) = \dfrac{ms+n}{as^2+bs+c}$

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Factorise
$$G(s) = \frac{ms+n}{a(s-d)(s-e)}$$

Use partial fractions
$$G(s) = \frac{A}{s-d} + \frac{B}{s-e}$$

How to find the inverse Laplace Transform of $G(s) = \dfrac{ms+n}{as^2+bs+c}$

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Use partial fractions
$$G(s) = \frac{A}{s-d} + \frac{B}{s-e}$$

no

Complete the square
$$G(s) = \frac{ms+n}{(s-d)^2+e}$$

How to find the inverse Laplace Transform of $G(s) = \dfrac{ms+n}{as^2+bs+c}$

Does $as^2+bs+c=0$ have roots in $\mathbb{R}$?

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Factorise

$$G(s) = \frac{ms+n}{a(s-d)(s-e)}$$

Use partial fractions

$$G(s) = \frac{A}{s-d} + \frac{B}{s-e}$$

no

Complete the square

$$G(s) = \frac{ms+n}{(s-d)^2+e}$$

Use the formula

$$\mathcal{L}\left[e^{ct}f(t)\right] = F(s-c)$$

Example

Find the inverse Laplace Transform of $G(s) = \dfrac{30s + 440}{s^2 + 32s + 240}$.

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First note that $s^2 + 32s + 240 = 0$ has roots $s_1 = -12$ and $s_2 = -20$.

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First note that $s^2 + 32s + 240 = 0$ has roots $s_1 = -12$ and $s_2 = -20$. In fact

$$G(s) = \frac{30s + 440}{s^2 + 32s + 240} = \frac{10}{s + 12} + \frac{20}{s + 20}.$$

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I leave this example for you to finish.

Example

Find the inverse Laplace Transform of $G(s) = \dfrac{10s + 12}{s^2 + 40s + 420}$.

Example

Find the inverse Laplace Transform of $G(s) = \dfrac{10s + 12}{s^2 + 40s + 420}$.

Since the roots of $s^2 + 40s + 420 = 0$ are $s = -20 \pm 2i\sqrt{5}$, we must complete the square.

Example

Find the inverse Laplace Transform of $G(s) = \dfrac{10s+12}{s^2+40s+420}$.

Since the roots of $s^2+40s+420=0$ are $s=-20 \pm 2i\sqrt{5}$, we must complete the square. You can check that

$$G(s) = \frac{10s+12}{s^2+40s+420} = \frac{10s+12}{(s+20)^2+20}.$$

Now

$$G(s) = \frac{10s + 12}{(s+20)^2 + 20}$$
$$= 10\left(\frac{s}{(s+20)^2 + 20}\right) + \frac{12}{\sqrt{20}}\left(\frac{\sqrt{20}}{(s+20)^2 + 20}\right)$$

Now

$$
\begin{aligned}
G(s) &= \frac{10s+12}{(s+20)^2+20} \\
&= 10\left(\frac{s}{(s+20)^2+20}\right) + \frac{12}{\sqrt{20}}\left(\frac{\sqrt{20}}{(s+20)^2+20}\right) \\
&= 10F(s+20) + \frac{12}{\sqrt{20}}H(s+20)
\end{aligned}
$$

where $F(s) = \frac{s}{s^2+20}$ and $H(s) = \frac{\sqrt{20}}{s^2+20}$.

Now

$$
\begin{aligned}
G(s) &= \frac{10s+12}{(s+20)^2+20} \\
&= 10\left(\frac{s}{(s+20)^2+20}\right) + \frac{12}{\sqrt{20}}\left(\frac{\sqrt{20}}{(s+20)^2+20}\right) \\
&= 10F(s+20) + \frac{12}{\sqrt{20}}H(s+20)
\end{aligned}
$$

where $F(s) = \frac{s}{s^2+20}$ and $H(s) = \frac{\sqrt{20}}{s^2+20}$.

Note that

$$f(t) = \mathcal{L}^{-1}\left[F\right](t) = \cos\sqrt{20}t$$

and

$$h(t) = \mathcal{L}^{-1}\left[H\right](t) = \sin\sqrt{20}t.$$

$$\mathcal{L}\left[e^{ct}f(t)\right] = F(s-c) \qquad G(s) = 10F(s+20) + \frac{12}{20}H(s+20)$$

Therefore

$$\begin{aligned} g(t) &= 10\mathcal{L}^{-1}\left[F(s+20)\right] + \frac{12}{\sqrt{20}}\mathcal{L}^{-1}\left[H(s+20)\right] \\ &= \\ &= \qquad\qquad\qquad\qquad . \end{aligned}$$

$$\mathcal{L}\left[e^{ct}f(t)\right] = F(s-c) \qquad G(s) = 10F(s+20) + \frac{12}{20}H(s+20)$$

Therefore

$$\begin{aligned} g(t) &= 10\mathcal{L}^{-1}\left[F(s+20)\right] + \frac{12}{\sqrt{20}}\mathcal{L}^{-1}\left[H(s+20)\right] \\ &= 10e^{-20t}\mathcal{L}^{-1}\left[F\right] + \frac{12}{\sqrt{20}}e^{-20t}\mathcal{L}^{-1}\left[H\right] \\ &= \qquad\qquad\qquad\qquad\qquad . \end{aligned}$$

$$\mathcal{L}\left[e^{ct} f(t)\right] = F(s-c) \qquad G(s) = 10F(s+20) + \frac{12}{20}H(s+20)$$

Therefore

$$\begin{aligned} g(t) &= 10\mathcal{L}^{-1}\left[F(s+20)\right] + \frac{12}{\sqrt{20}}\mathcal{L}^{-1}\left[H(s+20)\right] \\ &= 10e^{-20t}\mathcal{L}^{-1}\left[F\right] + \frac{12}{\sqrt{20}}e^{-20t}\mathcal{L}^{-1}\left[H\right] \\ &= 10e^{-20t}\cos\sqrt{20}t + \frac{12}{\sqrt{20}}e^{-20t}\sin\sqrt{20}t. \end{aligned}$$

Next Time

- 4.5 ODEs with Discontinuous Forcing Functions
- 4.6 The Convolution Integral