2016–17 **MAT461 Fonksiyonel Analiz I – Ödev 4'in Çözümleri** N. Course

8. First, note that since U is a bijection, $\forall k \in X$, $\exists f \in X$ such that $k = Uf$. Now $Uf \in U(M^\perp) \iff f \in M^\perp \iff \langle g, f\rangle = 0\ \forall g \in M \iff \langle Ug, Uf\rangle = 0\ \forall g \in M \iff \langle h, Uf\rangle = 0\ \forall h \in UM \iff Uf \in (UM)^\perp$. Therefore $U(M^\perp) = (UM)^\perp$.

9. Clearly $\|f\|^2 = \left\|f_\parallel + f_\perp\right\|^2 = \left\|f_\parallel\right\|^2 + \|f_\perp\|^2 \geq \left\|f_\parallel\right\|^2 = \|\Pi_M f\|^2$, which implies that $\|\Pi_m\| \leq 1$. Conversely: If $g \in M$, then $\Pi_M g = g$, which implies that $\|\Pi_M\| \geq 1$. Therefore $\|\Pi_M\| = 1$.

10. (a) If $f \in M$ then $\exists\, g \in X$ such that $f = Pg$. But then $Pf = P(Pg) = P^2 g = Pg = f$.
    (b) Suppose that M is not closed. Then $M^c = X \setminus M$ is not open. So $\exists\, f \in M^c$ and a sequence $f_n \in M$ such that $f_n \to f$. So $Pf_n \to Pf$. But $Pf_n = f_n$ by (a). So $f_n \to Pf$ and $Pf = f$. But therefore we must have $f \in M$. Contradiction.
    (c) Let $g \in M^\perp$. Then $\langle f, g\rangle = 0$ for all $f \in M$. By (a) $Pf = f$, so $\langle f, Pg\rangle = \langle Pf, g\rangle = \langle f, g\rangle = 0$ for all $f \in M$. Therefore $Pg \in M^\perp$ too.
    (d) Let $g \in M^\perp \subseteq X$. Then $Pg \in M$ by definition of M. But by (c), $Pg \in M \cap M^\perp = \{0\}$, so $Pg = 0$.
    (e) For any $f \in X$ we can uniquely write $f = f_\parallel + f_\perp$ where $f_\parallel \in M$ and $f_\perp \in M^\perp$. Then $Pf = Pf_\parallel + Pf_\perp = f_\parallel + 0 = f_\parallel = \Pi_M f$. So $P = \Pi_M$.

11. (a) $\|Af\| = \|\langle u, f\rangle v\| = |\langle u, f\rangle|\,\|v\| \leq \|u\|\,\|f\|\,\|v\|$ by Cauchy-Schwarz. Therefore $\|A\| \leq \|u\|\,\|v\|$.
    (b) By Cauchy-Schwarz, $\|Af\| = \|u\|\,\|f\|\,\|v\|$ iff f is parallel to u. Therefore $\|A\| = \|u\|\,\|v\|$.
    (c) Since $\langle A^* g, f\rangle = \langle g, Af\rangle = \langle g, \langle u, f\rangle v\rangle = \langle u, f\rangle \langle g, v\rangle = \overline{\langle v, g\rangle} \langle u, f\rangle = \langle \langle v, g\rangle u, f\rangle$, it follows that $A^* g = \langle v, g\rangle u$.

For the final exam, you must study the following sections in Teschl:

- all of **Chapter 1**;
- Sections **2.1** – **2.2**;
- Sections **3.1** – **3.2.**

You should be able to answer every problem in these sections.

Make sure that learn and understand all of the definitions – I like to start questions by asking for a relevant definition. You will be able to get ≈20 points just from learning the definitions.

In addition, read and try to understand all of the proofs in Teschl/from the lessons. The proofs are examples of how to use a definition. Plus, sometimes I like to turn one or more of the shorter proofs into an exam question.

For this course, we had 3 hours of lectures per week. You should have been studying an extra 3-6 hours at home each week. Multiply that by 13 weeks and I expect that you have already studied ≈100 hours for this course. As you know, this is a difficult course, but I am confident that 100 hours of study is enough for all of you to do well in the final exam.

I will not be in my office after 21st December. If you have any questions, or would like extra help in understanding part of the course, please feel free to email me at `neil.course@okan.edu.tr` .