Lecture 7

- 4.1 Definition of the Laplace Transform
- 4.2 Solving Initial Value Problems

Recall that $\displaystyle\int_a^\infty f(t)\,dt$ means $\displaystyle\lim_{R\to\infty}\int_a^R f(t)\,dt.$

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Example

Let $c \neq 0$. Then

$$\int_0^\infty e^{ct}\,dt = \lim_{R\to\infty}\int_0^R e^{ct}\,dt =$$

$$= \qquad\qquad\qquad =$$

Recall that $\int_a^\infty f(t)\,dt$ means $\lim_{R\to\infty}\int_a^R f(t)\,dt$.

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Let $c \neq 0$. Then

$$\int_0^\infty e^{ct}\,dt = \lim_{R\to\infty}\int_0^R e^{ct}\,dt = \lim_{R\to\infty}\left[\frac{1}{c}e^{ct}\right]_0^R$$

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$$\int_0^\infty e^{ct}\,dt = \lim_{R\to\infty}\int_0^R e^{ct}\,dt = \lim_{R\to\infty}\left[\frac{1}{c}e^{ct}\right]_0^R$$

$$= \lim_{R\to\infty}\frac{1}{c}\left(e^{cR}-1\right) =$$

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Example

Let $c \neq 0$. Then

$$\int_0^\infty e^{ct}\,dt = \lim_{R\to\infty}\int_0^R e^{ct}\,dt = \lim_{R\to\infty}\left[\frac{1}{c}e^{ct}\right]_0^R$$
$$= \lim_{R\to\infty}\frac{1}{c}\left(e^{cR}-1\right) = \begin{cases}\infty & c>0\\ -\frac{1}{c} & c<0.\end{cases}$$

Example

$$\int_1^\infty \frac{1}{t}\,dt =$$

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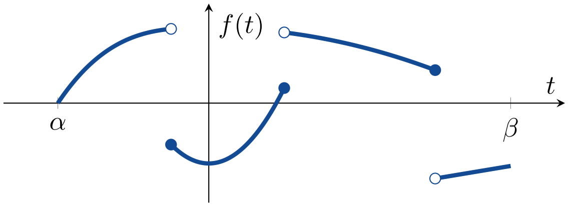
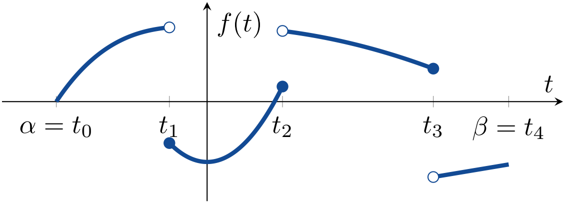


Definition

A function $f : [\alpha, \beta] \to \mathbb{R}$ is ***piecewise continuous*** on $[\alpha, \beta]$ iff



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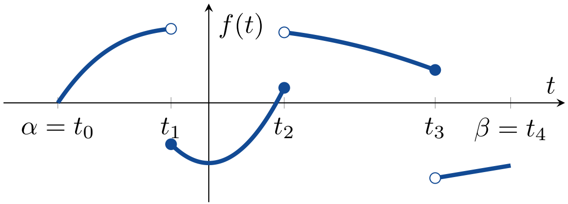
A function $f : [\alpha, \beta] \to \mathbb{R}$ is ***piecewise continuous*** on $[\alpha, \beta]$ iff there exists a finite partition $\alpha = t_0 < t_1 < t_2 < \ldots < t_n = \beta$ such that



Definition

A function $f : [\alpha, \beta] \to \mathbb{R}$ is ***piecewise continuous*** on $[\alpha, \beta]$ iff there exists a finite partition $\alpha = t_0 < t_1 < t_2 < \ldots < t_n = \beta$ such that

- f is continuous on each subinterval (t_{j-1}, t_j); and
- every one-sided limit $\lim\limits_{t \searrow t_j} f(t)$ and $\lim\limits_{t \nearrow t_j} f(t)$ is finite.

Definition of the Laplace Transform

Pierre-Simon Laplace
FRA, 1749-1827

$\frac{d}{dt}$ changes a function $f(t)$ into a new function $f'(t)$.

4.1 Definition of the Laplace Transform

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Definition

Suppose that

1. $K > 0$, $M > 0$, $a \in \mathbb{R}$;
2. f is piecewise continuous on $[0, A]$ for any $A > 0$; and
3. $|f(t)| \leq Ke^{at}$ for all $t \geq M$.

$\frac{d}{dt}$ changes a function $f(t)$ into a new function $f'(t)$.

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The ***Laplace Transform*** of $f : [0, \infty) \to \mathbb{R}$ is a new function defined by

$$F(s) = \mathcal{L}\big[f\big](s) = \int_0^\infty e^{-st} f(t)\, dt.$$

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$F(s)$ exists for $s > a$.

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Example

$$\mathcal{L}[1](s) =$$

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4.1 Definition of the Laplace Transform

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Example

$$\mathcal{L}[1](s) = \int_0^\infty e^{-st}\, dt = \lim_{R\to\infty} \left[-\frac{e^{-st}}{s} \right]_0^R = \frac{1}{s} \qquad \text{if } s > 0.$$

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$$\mathcal{L}[e^{at}](s) = \int_0^\infty e^{-st} e^{at}\, dt$$

4.1 Definition of the Laplace Transform

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Example

$$\mathcal{L}\big[e^{at}\big](s) = \int_0^\infty e^{-st} e^{at}\, dt = \int_0^\infty e^{-(s-a)t}\, dt = \frac{1}{s-a} \qquad \text{if } s > a.$$

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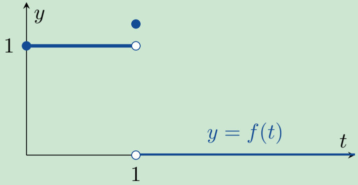
The Laplace Transform of $e^{at} : [0, \infty) \to \mathbb{R}$ is $\frac{1}{s-a} : (a, \infty) \to \mathbb{R}$.

$$F(s) = \mathcal{L}[f](s) = \int_0^\infty e^{-st} f(t)\, dt$$

Example

Let

$$f(t) = \begin{cases} 1 & 0 \leq t < 1 \\ k & t = 1 \\ 0 & t > 1. \end{cases}$$



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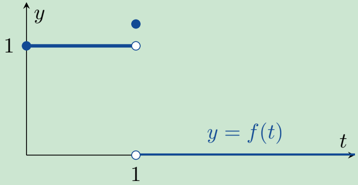

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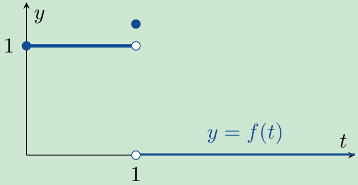

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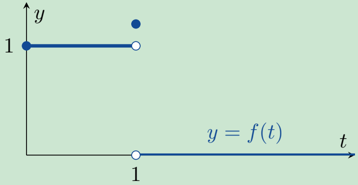

Then
$$F(s) = \mathcal{L}[f](s) = \int_0^\infty e^{-st} f(t)\, dt = \int_0^1 e^{-st}\, dt$$
$$= \left[-\frac{1}{s} e^{-st}\right]_0^1 = \frac{1 - e^{-s}}{s} \qquad \text{if } s > 1.$$

$$F(s) = \mathcal{L}[f](s) = \int_0^\infty e^{-st} f(t)\ dt$$

Example

Find the Laplace Transform of $g(t) = \sin at \ (t \geq 0)$.

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Example

Find the Laplace Transform of $g(t) = \sin at$ $(t \geq 0)$.

Using integration by parts ($\int_a^b uv' = [uv]_a^b - \int_a^b u'v$), we have

$$G(s) = \mathcal{L}[g](s) = \int_0^\infty e^{-st} \sin at\, dt = \lim_{R\to\infty} \int_0^R e^{-st} \sin at\, dt$$

$$= \lim_{R\to\infty} \Bigg(\qquad\qquad\qquad\qquad \Bigg)$$

$$=$$

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$$= \lim_{R\to\infty} \left(\left[-\frac{1}{a} e^{-st} \cos at \right]_0^R - \frac{s}{a} \int_0^R e^{-st} \cos at\,dt \right)$$

$$=$$

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$$\begin{aligned} G(s) = \mathcal{L}[g](s) = \int_0^\infty e^{-st} \sin at\, dt &= \lim_{R\to\infty} \int_0^R e^{-st} \sin at\, dt \\ &= \lim_{R\to\infty} \left(\left[-\frac{1}{a} e^{-st} \cos at \right]_0^R - \frac{s}{a} \int_0^R e^{-st} \cos at\, dt \right) \\ &= \frac{1}{a} - \frac{s}{a} \int_0^\infty e^{-st} \cos at\, dt. \end{aligned}$$

$$G(s) = \int_0^\infty e^{-st} \sin at \, dt = \frac{1}{a} - \frac{s}{a} \int_0^\infty e^{-st} \cos at \, dt$$

Using integration by parts a second time, we have

$$G(s) = \int_0^\infty e^{-st} \sin at \, dt = \frac{1}{a} - \frac{s}{a} \int_0^\infty e^{-st} \cos at \, dt$$

Using integration by parts a second time, we have

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Therefore

$$\mathcal{L}\big[\sin at\big](s) = G(s) = \frac{a}{s^2 + a^2} \qquad \text{if } s > 0.$$

$$\mathcal{L}\left[\sin at\right](s) = \frac{a}{s^2 + a^2}$$

Example

$$\mathcal{L}\left[\cos at\right](s) = \frac{s}{s^2 + a^2} \qquad \text{if } s > 0.$$

You prove.

Example

$$\mathcal{L}\left[\sinh at\right] = \frac{a}{s^2 - a^2} \qquad \text{if } s > |a|.$$

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Example

$$\mathcal{L}\left[\cosh at\right] = \frac{s}{s^2 - a^2} \qquad \text{if } s > |a|.$$

You prove.

Theorem

$$\mathcal{L}\left[c_1 f_1 + c_2 f_2\right] = c_1 \mathcal{L}\left[f_1\right] + c_2 \mathcal{L}\left[f_2\right].$$

You prove.

Example

If $h(t) = 5e^{-2t} - 3\sin 4t$ $(t \geq 0)$, then

$$\begin{aligned} H(s) &= \mathcal{L}\big[h\big](s) \\ &= \mathcal{L}\big[5e^{-2t} - 3\sin 4t\big](s) \\ &= \\ &= \\ &= \end{aligned}$$

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The Inverse Laplace Transform

We also have an ***inverse Laplace Transform***:

$$F(s) = \mathcal{L}\left[f(t)\right](s) \qquad \Longleftrightarrow \qquad f(t) = \mathcal{L}^{-1}\left[F(s)\right](t).$$

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Example

$\mathcal{L}\left[1\right] = \frac{1}{s}$ and $\mathcal{L}^{-1}\left[\frac{1}{s}\right] = 1$.

Theorem

$$\mathcal{L}^{-1}\left[c_1 f_1 + c_2 f_2\right] = c_1 \mathcal{L}^{-1}\left[f_1\right] + c_2 \mathcal{L}^{-1}\left[f_2\right].$$

You prove.

Example

Find the inverse Laplace Transform of $\dfrac{10}{s^2-25}$.

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$$\mathcal{L}^{-1}\left[\frac{10}{s^2 - 25}\right] = 2\mathcal{L}^{-1}\left[\frac{5}{s^2 - 25}\right]$$

Example

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We know that $\mathcal{L}\left[\sinh at\right] = \dfrac{a}{s^2-a^2}$. Therefore

$$\mathcal{L}^{-1}\left[\frac{10}{s^2-25}\right] = 2\mathcal{L}^{-1}\left[\frac{5}{s^2-25}\right] = 2\sinh 5t.$$

$$\mathcal{L}\left[1\right] = \frac{1}{s} \qquad \mathcal{L}\left[e^{at}\right] = \frac{1}{s-a}$$

Example

Find the inverse Laplace Transform of $\dfrac{1}{s} + \dfrac{1}{s-2}$.

$$\mathcal{L}\left[1\right] = \frac{1}{s} \qquad \mathcal{L}\left[e^{at}\right] = \frac{1}{s-a}$$

Example

Find the inverse Laplace Transform of $\dfrac{1}{s} + \dfrac{1}{s-2}$.

$$\mathcal{L}^{-1}\left[\frac{1}{s} + \frac{1}{s-2}\right] = \mathcal{L}^{-1}\left[\frac{1}{s}\right] + \mathcal{L}^{-1}\left[\frac{1}{s-2}\right] = 1 + e^{2t}.$$

4.1 Definition of the Laplace Transform

Theorem

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Proof: First we calculate that

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$$= $$

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$$\mathcal{L}\left[t^n f(t)\right] = (-1)^n \frac{d^n F}{ds^n}$$

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$$=$$

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$$\mathcal{L}\left[t^n f(t)\right] = (-1)^n \frac{d^n F}{ds^n}$$

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$$\begin{aligned}
-\frac{dF}{ds} &= -\frac{d}{ds}\int_0^\infty e^{-st} f(t)\ dt = -\int_0^\infty \frac{d}{ds} e^{-st} f(t)\ dt \\
&= -\int_0^\infty -te^{-st} f(t)\ dt \\
&= \qquad\qquad\qquad = \qquad\qquad .
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&= -\int_0^\infty -t e^{-st} f(t)\ dt \\
&= \int_0^\infty e^{-st} t f(t)\ dt = \mathcal{L}\left[t f(t)\right].
\end{aligned}$$

Therefore the formula holds for $n = 1$.

By repeatedly using

$$-\frac{dF}{ds} = \mathcal{L}\left[tf(t)\right],$$

we can also show that

$$(-1)^2 \frac{d^n F}{ds^2} = \mathcal{L}\left[t^2 f(t)\right]$$

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$$\vdots$$
$$(-1)^n \frac{d^n F}{ds^n} = \mathcal{L}\left[t^n f(t)\right].$$

$\square$

$$\mathcal{L}\left[t^n f(t)\right] = (-1)^n \frac{d^n F}{ds^n} \qquad \mathcal{L}\left[\,\cosh at\right] = \frac{s}{s^2 - a^2}$$

Example

$$\begin{aligned}\mathcal{L}\left[t^2 \cosh 2t\right] &= \\ &= \end{aligned}$$

$$\mathcal{L}\left[t^n f(t)\right] = (-1)^n \frac{d^n F}{ds^n} \qquad \mathcal{L}\left[\cosh at\right] = \frac{s}{s^2 - a^2}$$

Example

$$\begin{aligned} \mathcal{L}\left[t^2 \cosh 2t\right] &= (-1)^2 \frac{d^2}{ds^2} \mathcal{L}\left[\cosh 2t\right] \\ &= \end{aligned}$$

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Example

$$\begin{aligned}\mathcal{L}\left[t^2 \cosh 2t\right] &= (-1)^2 \frac{d^2}{ds^2} \mathcal{L}\left[\cosh 2t\right] \\ &= \frac{d^2}{ds^2}\left(\frac{s}{s^2 - 2^2}\right)\end{aligned}$$

$$\mathcal{L}\left[t^n f(t)\right] = (-1)^n \frac{d^n F}{ds^n} \qquad \mathcal{L}\left[\cosh at\right] = \frac{s}{s^2 - a^2}$$

Example

$$\begin{aligned}\mathcal{L}\left[t^2 \cosh 2t\right] &= (-1)^2 \frac{d^2}{ds^2} \mathcal{L}\left[\cosh 2t\right] \\ &= \frac{d^2}{ds^2}\left(\frac{s}{s^2 - 2^2}\right) = \ldots = \frac{2s(s^2+12)}{(s^2-4)^3}.\end{aligned}$$

$$\mathcal{L}\left[t^n f(t)\right] = (-1)^n \frac{d^n F}{ds^n}$$

Example

Find the Laplace Transform of t^n for $n \in \mathbb{N}$.

$$\mathcal{L}\left[t^n f(t)\right] = (-1)^n \frac{d^n F}{ds^n}$$

Example

Find the Laplace Transform of t^n for $n \in \mathbb{N}$.

$$\mathcal{L}\left[t^n\right] = \mathcal{L}\left[t^n \cdot 1\right]$$

$$\mathcal{L}\left[t^n f(t)\right] = (-1)^n \frac{d^n F}{ds^n} \qquad \mathcal{L}\left[1\right] = \frac{1}{s}$$

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Find the Laplace Transform of t^n for $n \in \mathbb{N}$.

$$\mathcal{L}\left[t^n\right] = \mathcal{L}\left[t^n \cdot 1\right] = (-1)^n \frac{d^n}{ds^n} \mathcal{L}\left[1\right]$$

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$$\mathcal{L}\left[t^n f(t)\right] = (-1)^n \frac{d^n F}{ds^n} \qquad \mathcal{L}\left[1\right] = \frac{1}{s}$$

Example

Find the Laplace Transform of t^n for $n \in \mathbb{N}$.

$$\mathcal{L}\left[t^n\right] = \mathcal{L}\left[t^n \cdot 1\right] = (-1)^n \frac{d^n}{ds^n} \mathcal{L}\left[1\right] = (-1)^n \frac{d^n}{ds^n}\left(\frac{1}{s}\right)$$
$$= (-1)^n \frac{(-1)^n n!}{s^{n+1}}$$

$$\mathcal{L}\left[t^n f(t)\right] = (-1)^n \frac{d^n F}{ds^n} \qquad \mathcal{L}\left[1\right] = \frac{1}{s}$$

Example

Find the Laplace Transform of t^n for $n \in \mathbb{N}$.

$$\begin{aligned}\mathcal{L}\left[t^n\right] = \mathcal{L}\left[t^n \cdot 1\right] &= (-1)^n \frac{d^n}{ds^n}\mathcal{L}\left[1\right] = (-1)^n \frac{d^n}{ds^n}\left(\frac{1}{s}\right) \\ &= (-1)^n \frac{(-1)^n n!}{s^{n+1}} = \frac{n!}{s^{n+1}}.\end{aligned}$$

$f(t)$	$F(s) = \mathcal{L}[f](s)$	
1	$\frac{1}{s}$	$s > 0$
e^{at}	$\frac{1}{s-a}$	$s > a$
$t^n \quad (n \in \mathbb{N})$	$\frac{n!}{s^{n+1}}$	$s > 0$
$\sin at$	$\frac{a}{s^2+a^2}$	$s > 0$
$\cos at$	$\frac{s}{s^2+a^2}$	$s > 0$
$\sinh at$	$\frac{a}{s^2-a^2}$	$s > \|a\|$
$\cosh at$	$\frac{s}{s^2-a^2}$	$s > \|a\|$
	$\vdots$	

$f(t)$	$F(s) = \mathcal{L}[f](s)$	
e^{at}	$\frac{1}{s-a}$	$s > a$
$t^n \qquad (n \in \mathbb{N})$	$\frac{n!}{s^{n+1}}$	$s > 0$
$\sin at$	$\frac{a}{s^2+a^2}$	$s > 0$
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$\cosh at$	$\frac{s}{s^2-a^2}$	$s > \|a\|$
$e^{at} \sin bt$	$\frac{b}{(s-a)^2+b^2}$	$s > a$
	$\vdots$	

4.1 Definition of the Laplace Transform

$f(t)$	$F(s) = \mathcal{L}[f](s)$	
$t^n \qquad (n \in \mathbb{N})$	$\frac{n!}{s^{n+1}}$	$s > 0$
$\sin at$	$\frac{a}{s^2+a^2}$	$s > 0$
$\cos at$	$\frac{s}{s^2+a^2}$	$s > 0$
$\sinh at$	$\frac{a}{s^2-a^2}$	$s > \|a\|$
$\cosh at$	$\frac{s}{s^2-a^2}$	$s > \|a\|$
$e^{at} \sin bt$	$\frac{b}{(s-a)^2+b^2}$	$s > a$
$e^{at} \cos bt$	$\frac{s-a}{(s-a)^2+b^2}$	$s > a$
	$\vdots$	

$f(t)$	$F(s)=\mathcal{L}[f](s)$	
$\sin at$	$\frac{a}{s^2+a^2}$	$s>0$
$\cos at$	$\frac{s}{s^2+a^2}$	$s>0$
$\sinh at$	$\frac{a}{s^2-a^2}$	$s>\lvert a\rvert$
$\cosh at$	$\frac{s}{s^2-a^2}$	$s>\lvert a\rvert$
$e^{at}\sin bt$	$\frac{b}{(s-a)^2+b^2}$	$s>a$
$e^{at}\cos bt$	$\frac{s-a}{(s-a)^2+b^2}$	$s>a$
$t^n e^{at} \quad (n\in\mathbb{N})$	$\frac{n!}{(s-a)^{n+1}}$	$s>a$
	$\vdots$	

$f(t)$	$F(s) = \mathcal{L}[f](s)$	
$\cos at$	$\frac{s}{s^2+a^2}$	$s > 0$
$\sinh at$	$\frac{a}{s^2-a^2}$	$s > \|a\|$
$\cosh at$	$\frac{s}{s^2-a^2}$	$s > \|a\|$
$e^{at} \sin bt$	$\frac{b}{(s-a)^2+b^2}$	$s > a$
$e^{at} \cos bt$	$\frac{s-a}{(s-a)^2+b^2}$	$s > a$
$t^n e^{at} \quad (n \in \mathbb{N})$	$\frac{n!}{(s-a)^{n+1}}$	$s > a$
$u_c(t)$	$\frac{e^{-cs}}{s}$	$s > 0$
	$\vdots$	

$f(t)$	$F(s) = \mathcal{L}[f](s)$	
$\sinh at$	$\frac{a}{s^2-a^2}$	$s > \lvert a \rvert$
$\cosh at$	$\frac{s}{s^2-a^2}$	$s > \lvert a \rvert$
$e^{at} \sin bt$	$\frac{b}{(s-a)^2+b^2}$	$s > a$
$e^{at} \cos bt$	$\frac{s-a}{(s-a)^2+b^2}$	$s > a$
$t^n e^{at} \quad (n \in \mathbb{N})$	$\frac{n!}{(s-a)^{n+1}}$	$s > a$
$u_c(t)$	$\frac{e^{-cs}}{s}$	$s > 0$
$u_c(t) f(t-c)$	$e^{-cs} F(s)$	
	$\vdots$	

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$\cosh at$	$\frac{s}{s^2-a^2}$	$s > \|a\|$
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$u_c(t)$	$\frac{e^{-cs}}{s}$	$s > 0$
$u_c(t)f(t-c)$	$e^{-cs}F(s)$	
$e^{ct}f(t)$	$F(s-c)$	
	$\vdots$	

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$u_c(t)$	$\frac{e^{-cs}}{s}$	$s > 0$
$u_c(t)f(t-c)$	$e^{-cs}F(s)$	
$e^{ct}f(t)$	$F(s-c)$	
$f(ct) \quad (c > 0)$	$\frac{1}{c}F\left(\frac{s}{c}\right)$	
	$\vdots$	

$f(t)$	$F(s) = \mathcal{L}[f](s)$	
$e^{at}\cos bt$	$\frac{s-a}{(s-a)^2+b^2}$	$s > a$
$t^n e^{at} \quad (n \in \mathbb{N})$	$\frac{n!}{(s-a)^{n+1}}$	$s > a$
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$f(ct) \quad (c > 0)$	$\frac{1}{c}F\left(\frac{s}{c}\right)$	
$\int_0^t f(t-\tau)g(\tau)d\tau$	$F(s)G(s)$	
	$\vdots$	

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$u_c(t)$	$\frac{e^{-cs}}{s}$	$s > 0$
$u_c(t) f(t-c)$	$e^{-cs} F(s)$	
$e^{ct} f(t)$	$F(s-c)$	
$f(ct) \quad (c > 0)$	$\frac{1}{c} F\left(\frac{s}{c}\right)$	
$\int_0^t f(t-\tau) g(\tau) d\tau$	$F(s)G(s)$	
$t^n f(t)$	$(-1)^n F^{(n)}(s)$	

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Find the inverse Laplace Transform of $F(s) = \ln\left(1 + \frac{1}{s^2}\right)$.

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Setting $n = 1$

$$\mathcal{L}\left[t f(t)\right] = (-1) \frac{dF}{ds}$$

Example

Find the inverse Laplace Transform of $F(s) = \ln\left(1 + \frac{1}{s^2}\right)$.

Again we will use the formula

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Setting $n = 1$

$$\mathcal{L}\left[t f(t)\right] = (-1) \frac{dF}{ds}$$

and taking $\mathcal{L}^{-1}$ of both sides gives

$$t f(t) = -\mathcal{L}^{-1}\left[\frac{dF}{ds}\right].$$

$$F(s) = \ln\left(1 + \frac{1}{s^2}\right)$$

Now

$$\frac{dF}{ds} = \frac{d}{ds}\ln\left(1 + \frac{1}{s^2}\right)$$

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Now

$$\frac{dF}{ds} = \frac{d}{ds}\ln\left(1 + \frac{1}{s^2}\right) = \frac{\frac{-2}{s^3}}{\left(1 + \frac{1}{s^2}\right)}$$

$$F(s) = \ln\left(1 + \frac{1}{s^2}\right)$$

Now

$$\frac{dF}{ds} = \frac{d}{ds}\ln\left(1 + \frac{1}{s^2}\right) = \frac{\frac{-2}{s^3}}{\left(1 + \frac{1}{s^2}\right)} = \frac{-2}{s\left(s^2+1\right)}.$$

$$F(s) = \ln\left(1 + \frac{1}{s^2}\right)$$

Now

$$\frac{dF}{ds} = \frac{d}{ds}\ln\left(1 + \frac{1}{s^2}\right) = \frac{\frac{-2}{s^3}}{\left(1 + \frac{1}{s^2}\right)} = \frac{-2}{s\left(s^2+1\right)}.$$

Therefore

$$tf(t) = -\mathcal{L}^{-1}\left[\frac{dF}{ds}\right] = \mathcal{L}^{-1}\left[\frac{2}{s\left(s^2+1\right)}\right].$$

$$F(s) = \ln\left(1 + \frac{1}{s^2}\right)$$

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$$tf(t) = -\mathcal{L}^{-1}\left[\frac{dF}{ds}\right] = \mathcal{L}^{-1}\left[\frac{2}{s\left(s^2+1\right)}\right].$$

To proceed, we need to write $\frac{2}{s(s^2+1)}$ in partial fractions.

We calculate that

$$\frac{2}{s\left(s^{2}+1\right)}=\frac{A}{s}+\frac{Bs+C}{s^{2}+1}$$

We calculate that

$$\frac{2}{s\left(s^2+1\right)} = \frac{A}{s} + \frac{Bs+C}{s^2+1}$$
$$= \frac{A(s^2+1)+Bs^2+Cs}{s\left(s^2+1\right)}$$

We calculate that

$$\begin{aligned}
\frac{2}{s\left(s^2+1\right)} &= \frac{A}{s} + \frac{Bs+C}{s^2+1} \\
&= \frac{A(s^2+1) + Bs^2 + Cs}{s\left(s^2+1\right)} \\
&= \frac{(A+B)s^2 + Cs + A}{s\left(s^2+1\right)}
\end{aligned}$$

We calculate that

$$\begin{aligned}\frac{2}{s\left(s^2+1\right)} &= \frac{A}{s}+\frac{Bs+C}{s^2+1} \\ &= \frac{A(s^2+1)+Bs^2+Cs}{s\left(s^2+1\right)} \\ &= \frac{(A+B)s^2+Cs+A}{s\left(s^2+1\right)}\end{aligned} \qquad \Longrightarrow \qquad \begin{aligned}A &= 2 \\ B &= -2 \\ C &= 0\end{aligned}$$

We calculate that

$$\begin{aligned}\frac{2}{s\left(s^2+1\right)} &= \frac{A}{s} + \frac{Bs+C}{s^2+1} \\ &= \frac{A(s^2+1)+Bs^2+Cs}{s\left(s^2+1\right)} \\ &= \frac{(A+B)s^2+Cs+A}{s\left(s^2+1\right)} \\ &= \frac{2}{s} - \frac{2s}{s^2+1}.\end{aligned} \qquad \Longrightarrow \qquad \begin{aligned} A &= 2 \\ B &= -2 \\ C &= 0 \end{aligned}$$

Thus

$$tf(t) = \mathcal{L}^{-1}\left[\frac{2}{s\left(s^2+1\right)}\right] = \mathcal{L}^{-1}\left[\frac{2}{s} - \frac{2s}{s^2+1}\right]$$

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$f(t)$	$F(s) = \mathcal{L}[f](s)$	
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$t^n \quad (n \in \mathbb{N})$	$\frac{n!}{s^{n+1}}$	$s > 0$
$\sin at$	$\frac{a}{s^2+a^2}$	$s > 0$
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$$\mathcal{L}\left[1\right] = \frac{1}{s} \qquad \mathcal{L}\left[\cos at\right] = \frac{s}{s^2+a^2}$$

Thus

$$
\begin{aligned}
tf(t) &= \mathcal{L}^{-1}\left[\frac{2}{s\left(s^2+1\right)}\right] = \mathcal{L}^{-1}\left[\frac{2}{s} - \frac{2s}{s^2+1}\right] \\
&= 2\mathcal{L}^{-1}\left[\frac{1}{s}\right] - 2\mathcal{L}^{-1}\left[\frac{s}{s^2+1}\right]
\end{aligned}
$$

$$
\mathcal{L}\left[1\right] = \frac{1}{s} \qquad\qquad \mathcal{L}\left[\cos at\right] = \frac{s}{s^2+a^2}
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tf(t) &= \mathcal{L}^{-1}\left[\frac{2}{s\left(s^2+1\right)}\right] = \mathcal{L}^{-1}\left[\frac{2}{s} - \frac{2s}{s^2+1}\right] \\
&= 2\mathcal{L}^{-1}\left[\frac{1}{s}\right] - 2\mathcal{L}^{-1}\left[\frac{s}{s^2+1}\right] \\
&= 2 - 2\cos t.
\end{aligned}
$$

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Thus

$$\begin{aligned} tf(t) &= \mathcal{L}^{-1}\left[\frac{2}{s\left(s^2+1\right)}\right] = \mathcal{L}^{-1}\left[\frac{2}{s} - \frac{2s}{s^2+1}\right] \\ &= 2\mathcal{L}^{-1}\left[\frac{1}{s}\right] - 2\mathcal{L}^{-1}\left[\frac{s}{s^2+1}\right] \\ &= 2 - 2\cos t. \end{aligned}$$

Therefore

$$\boxed{f(t) = \frac{2\left(1-\cos t\right)}{t}.}$$

$$\mathcal{L}\left[1\right] = \frac{1}{s} \qquad \mathcal{L}\left[\cos at\right] = \frac{s}{s^2+a^2}$$

Solving Initial Value Problems

Theorem

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3. $\mathcal{L}[f'''](s) = s^3\mathcal{L}[f](s) - s^2 f(0) - sf'(0) - f''(0)$.
4. $$\mathcal{L}[f^{(n)}](s) = s^n\mathcal{L}[f](s) - s^{n-1}f(0) - s^{n-2}f'(0) - \dots \\ -sf^{(n-2)}(0) - f^{(n-1)}(0).$$

Proof:

1. Using integration-by-parts ($\int uv' = uv - \int u'v$) we calculate that

$$\mathcal{L}[f'](s) = \int_0^\infty e^{-st} f'(t)\,dt = \left[e^{-st} f(t)\right]_0^\infty - \int_0^\infty \left(\frac{d}{dt} e^{-st}\right) f(t)\,dt$$

$$=$$

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as required.

$$\mathcal{L}[f'](s) = s\mathcal{L}[f](s) - f(0)$$

2 Using part 1, but replacing each f by f' we get

$$\begin{aligned}\mathcal{L}[f''](s) &= s\mathcal{L}[f'](s) - f'(0) \\ &= \\ &= \end{aligned}$$

$$\mathcal{L}[f'](s) = s\mathcal{L}[f](s) - f(0)$$

2. Using part 1, but replacing each f by f' we get

$$\begin{aligned}\mathcal{L}[f''](s) &= s\mathcal{L}[f'](s) - f'(0) \\ &= s\Big(s\mathcal{L}[f](s) - f(0)\Big) - f'(0) \\ &= \end{aligned}$$

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You prove parts 3 and 4. $\square$

Example

Solve

$$\begin{cases} y'' - y' - 2y = 0 \\ y(0) = 1 \\ y'(0) = 0. \end{cases}$$

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solution 1 (method from Chapter 3): The characteristic equation

$$0 = r^2 - r - 2 = (r-2)(r+1)$$

has roots $r_1 = -1$ and $r_2 = 2$.

Example

Solve

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$$0 = r^2 - r - 2 = (r-2)(r+1)$$

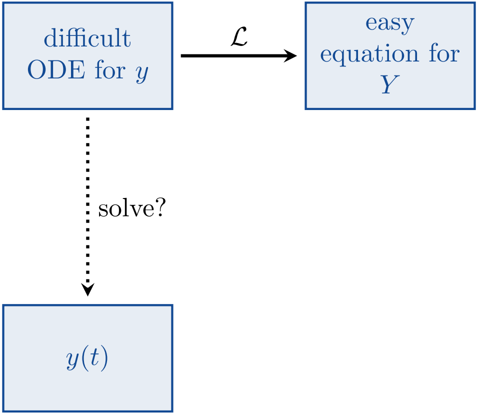
has roots $r_1 = -1$ and $r_2 = 2$. So $y = c_1 e^{-t} + c_2 e^{2t}$. Using the initial conditions we find that $c_1 = \frac{2}{3}$ and $c_2 = \frac{1}{3}$. Therefore

$$\boxed{y(t) = \frac{2}{3}e^{-t} + \frac{1}{3}e^{2t}.}$$

solution 2 (Laplace Transform):

$$\begin{array}{c}\boxed{\begin{array}{c}\text{difficult}\\ \text{ODE for } y\end{array}}\\ \Big\downarrow \text{ solve?}\\ \boxed{y(t)}\end{array}$$

solution 2 (Laplace Transform):



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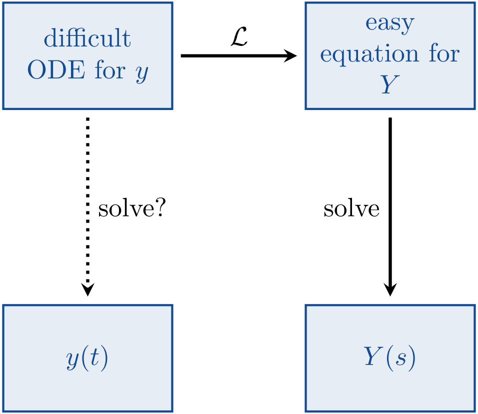
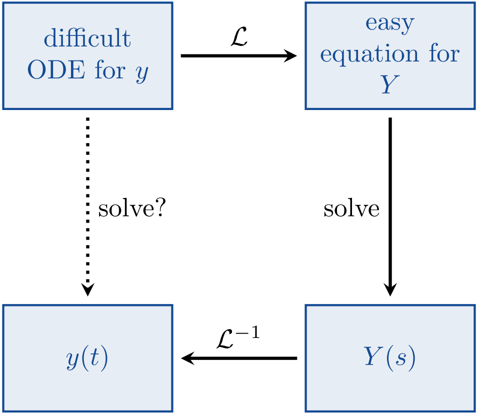


4.2 Solving Initial Value Problems

solution 2 (Laplace Transform):



$$y'' - y' - 2y = 0$$

First we take the Laplace Transform of the ODE

$$\mathcal{L}\left[y'' - y' - 2y\right] = \mathcal{L}\left[0\right]$$

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$$\mathcal{L}\left[y''\right] = s^2Y - sy(0) - y'(0) \qquad \mathcal{L}\left[y'\right] = sY - y(0)$$

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It follows that

$$\left(s^2Y - sy(0) - y'(0)\right) - \left(sY - y(0)\right) - 2Y = 0$$

$$y'' - y' - 2y = 0$$

$$\mathcal{L}\left[y''\right] = s^2Y - sy(0) - y'(0) \qquad \mathcal{L}\left[y'\right] = sY - y(0)$$

First we take the Laplace Transform of the ODE

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$$\mathcal{L}\Big[y''\Big] - \mathcal{L}\Big[y'\Big] - 2\mathcal{L}\Big[y\Big] = 0$$

It follows that

$$\begin{aligned} \big(s^2Y - sy(0) - y'(0)\big) - (sY - y(0)) - 2Y &= 0 \\ \big(s^2Y - s - 0\big) - (sY - 1) - 2Y &= 0 \\ (s^2 - s - 2)Y + (1 - s) &= 0. \end{aligned}$$

Thus

$$Y(s) = \frac{s-1}{s^2 - s - 2} = \frac{s-1}{(s-2)(s+1)}.$$

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Using partial fractions we obtain

$$\begin{aligned} Y(s) &= \frac{s-1}{(s-2)(s+1)} = \frac{A}{s-2} + \frac{B}{s+1} = \frac{As + A + Bs - 2B}{(s-2)(s+1)} \\ &= \frac{1}{3}\left(\frac{1}{s-2}\right) + \frac{2}{3}\left(\frac{1}{s+1}\right). \end{aligned}$$

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$$Y(s) = \frac{s-1}{s^2 - s - 2} = \frac{s-1}{(s-2)(s+1)}.$$

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But recall that $\mathcal{L}\left[e^{2t}\right] = \frac{1}{s-2}$ and $\mathcal{L}\left[e^{-t}\right] = \frac{1}{s+1}$.

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$$Y(s) = \frac{s-1}{(s-2)(s+1)} = \frac{A}{s-2} + \frac{B}{s+1} = \frac{As+A+Bs-2B}{(s-2)(s+1)}$$
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But recall that $\mathcal{L}\left[e^{2t}\right] = \frac{1}{s-2}$ and $\mathcal{L}\left[e^{-t}\right] = \frac{1}{s+1}$. Therefore

$$y(t) = \mathcal{L}^{-1}[Y] = \frac{1}{3}\mathcal{L}^{-1}\left[\frac{1}{s-2}\right] + \frac{2}{3}\mathcal{L}^{-1}\left[\frac{1}{s+1}\right] = \boxed{\frac{1}{3}e^{2t} + \frac{2}{3}e^{-t}.}$$

Example

Solve

$$\begin{cases} y'' + y = \sin 2t \\ y(0) = 2 \\ y'(0) = 1. \end{cases}$$

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$$\begin{cases} y'' + y = \sin 2t \\ y(0) = 2 \\ y'(0) = 1. \end{cases}$$

$$\mathcal{L}\left[y''\right] + \mathcal{L}\left[y\right] = \mathcal{L}\left[\sin 2t\right]$$

Example

Solve

$$\begin{cases} y'' + y = \sin 2t \\ y(0) = 2 \\ y'(0) = 1. \end{cases}$$

$$\mathcal{L}\left[y''\right] + \mathcal{L}\left[y\right] = \mathcal{L}\left[\sin 2t\right]$$

$$\left(s^2 Y - sy(0) - y'(0)\right) + Y = \frac{2}{s^2+4}$$

Example

Solve

$$\begin{cases} y'' + y = \sin 2t \\ y(0) = 2 \\ y'(0) = 1. \end{cases}$$

$$\mathcal{L}\left[y''\right] + \mathcal{L}\left[y\right] = \mathcal{L}\left[\sin 2t\right]$$

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Example

Solve

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$$(s^2+1)Y = 2s + 1 + \frac{2}{s^2+4}$$

Example

Solve

$$\begin{cases} y'' + y = \sin 2t \\ y(0) = 2 \\ y'(0) = 1. \end{cases}$$

$$\mathcal{L}\left[y''\right] + \mathcal{L}\left[y\right] = \mathcal{L}\left[\sin 2t\right]$$

$$Y = \frac{2s+1}{s^2+1} + \frac{2}{(s^2+1)(s^2+4)}$$

$$Y = \frac{2s+1}{s^2+1} + \frac{2}{(s^2+1)(s^2+4)} =$$

$$=$$

$$=$$

$$=$$

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$$Y = \frac{2s+1}{s^2+1} + \frac{2}{(s^2+1)(s^2+4)} = \frac{2s+1}{s^2+1} + \frac{As+B}{s^2+1} + \frac{Cs+D}{s^2+4}$$

$$=$$

$$=$$

$$=$$

$$=$$

$$
\begin{aligned}
Y &= \frac{2s+1}{s^2+1} + \frac{2}{(s^2+1)(s^2+4)} = \frac{2s+1}{s^2+1} + \frac{As+B}{s^2+1} + \frac{Cs+D}{s^2+4} \\
&= \frac{2s+1}{s^2+1} + \frac{\frac{2}{3}}{s^2+1} - \frac{\frac{2}{3}}{s^2+4} \\
&= \\
&= \\
&=
\end{aligned}
$$

$$
\begin{aligned}
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&= \frac{2s+1}{s^2+1} + \frac{\frac{2}{3}}{s^2+1} - \frac{\frac{2}{3}}{s^2+4} \\
&= \frac{2s}{s^2+1} + \frac{\frac{5}{3}}{s^2+1} - \frac{\frac{2}{3}}{s^2+4} \\
&= \\
&=
\end{aligned}
$$

$$
\begin{aligned}
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&= \frac{2s}{s^2+1} + \frac{\frac{5}{3}}{s^2+1} - \frac{\frac{2}{3}}{s^2+4} \\
&= 2\left(\frac{s}{s^2+1}\right) + \frac{5}{3}\left(\frac{1}{s^2+1}\right) - \frac{1}{3}\left(\frac{2}{s^2+4}\right) \\
&=
\end{aligned}
$$

$$
\begin{aligned}
Y &= \frac{2s+1}{s^2+1} + \frac{2}{(s^2+1)(s^2+4)} = \frac{2s+1}{s^2+1} + \frac{As+B}{s^2+1} + \frac{Cs+D}{s^2+4} \\
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&= 2\mathcal{L}\left[\cos t\right] + \frac{5}{3}\mathcal{L}\left[\sin t\right] - \frac{1}{3}\mathcal{L}\left[\sin 2t\right].
\end{aligned}
$$

$$\begin{aligned} Y &= \frac{2s+1}{s^2+1} + \frac{2}{(s^2+1)(s^2+4)} = \frac{2s+1}{s^2+1} + \frac{As+B}{s^2+1} + \frac{Cs+D}{s^2+4} \\ &= \frac{2s+1}{s^2+1} + \frac{\frac{2}{3}}{s^2+1} - \frac{\frac{2}{3}}{s^2+4} \\ &= \frac{2s}{s^2+1} + \frac{\frac{5}{3}}{s^2+1} - \frac{\frac{2}{3}}{s^2+4} \\ &= 2\left(\frac{s}{s^2+1}\right) + \frac{5}{3}\left(\frac{1}{s^2+1}\right) - \frac{1}{3}\left(\frac{2}{s^2+4}\right) \\ &= 2\mathcal{L}\left[\cos t\right] + \frac{5}{3}\mathcal{L}\left[\sin t\right] - \frac{1}{3}\mathcal{L}\left[\sin 2t\right]. \end{aligned}$$

Therefore

$$\boxed{y(t) = 2\cos t + \frac{5}{3}\sin t - \frac{1}{3}\sin 2t.}$$

Example

Solve

$$\begin{cases} y^{(4)} - y = 0 \\ y(0) = 0 \\ y'(0) = 1 \\ y''(0) = 0 \\ y'''(0) = 0. \end{cases}$$

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Example

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$$\begin{aligned} 0 &= \mathcal{L}\left[y^{(4)}\right] - \mathcal{L}\left[y\right] \\ &= \left(s^4 Y - s^3 y(0) - s^2 y'(0) - s y''(0) - y'''(0)\right) - Y \\ &= s^4 Y - s^2 - Y = (s^4 - 1)Y - s^2. \end{aligned}$$

$$0 = (s^4 - 1)Y - s^2$$

Thus

$$Y(s) = \frac{s^2}{s^4 - 1}$$

$$0 = (s^4 - 1)Y - s^2$$

Thus

$$Y(s) = \frac{s^2}{s^4 - 1} = \frac{s^2}{(s^2 - 1)(s^2 + 1)}$$

$$0 = (s^4 - 1)Y - s^2$$

Thus

$$Y(s) = \frac{s^2}{s^4 - 1} = \frac{s^2}{(s^2-1)(s^2+1)} = \frac{\frac{1}{2}}{s^2-1} + \frac{\frac{1}{2}}{s^2+1}.$$

$$0 = (s^4 - 1)Y - s^2$$

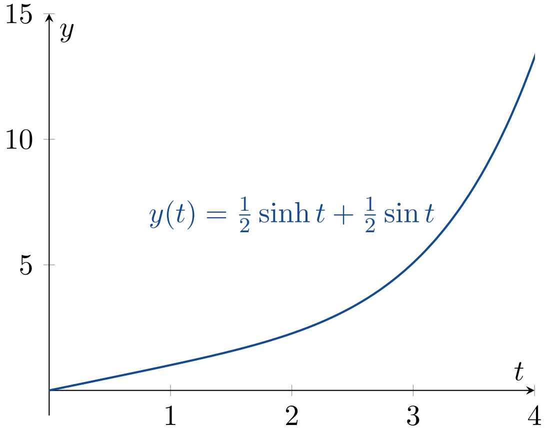
Thus

$$Y(s) = \frac{s^2}{s^4 - 1} = \frac{s^2}{(s^2 - 1)(s^2 + 1)} = \frac{\frac{1}{2}}{s^2 - 1} + \frac{\frac{1}{2}}{s^2 + 1}.$$

Therefore

$$y = \frac{1}{2}\mathcal{L}^{-1}\left[\frac{1}{s^2 - 1}\right] + \frac{1}{2}\mathcal{L}^{-1}\left[\frac{1}{s^2 + 1}\right] = \boxed{\frac{1}{2}\sinh t + \frac{1}{2}\sin t.}$$

$$\begin{cases} y^{(4)} - y = 0 \\ y(0) = 0 \\ y'(0) = 1 \\ y''(0) = 0 \\ y'''(0) = 0. \end{cases}$$



Next Time

- 4.3 Solving More Initial Value Problems
- 4.4 Step Functions