

Lecture 9

- 5.4 The Fundamental Theorem of Calculus
- 5.5 Indefinite Integrals and the Substitution Method
- 5.6 Substitution and Area Between Curves



The Fundamental Theorem of Calculus

5.4 The Fundamental Theorem of Calculus



We don't want to have to use Riemann sums every time we need to calculate a definite integral – we want a better way.

5.4 The Fundamental Theorem of Calculus



We don't want to have to use Riemann sums every time we need to calculate a definite integral – we want a better way.

Remark

The following theorem is the most important theorem in Calculus. If you can only memorise one theorem for the exams, it should be this one.

5.4 The Fundamental Theorem of Calculus



Theorem (The Fundamental Theorem of Calculus)

Suppose that $f : [a, b] \rightarrow \mathbb{R}$ is a continuous function.

1

2

5.4 The Fundamental Theorem of Calculus



Theorem (The Fundamental Theorem of Calculus)

Suppose that $f : [a, b] \rightarrow \mathbb{R}$ is a continuous function.

1 Then the function $F : [a, b] \rightarrow \mathbb{R}$ defined by

$$F(x) = \int_a^x f(t) \, dt$$

is continuous on $[a, b]$; differentiable on (a, b) ; and its derivative is

$$F'(x) = \frac{d}{dx} \int_a^x f(t) \, dt = f(x).$$

2

5.4 The Fundamental Theorem of Calculus



Theorem (The Fundamental Theorem of Calculus)

Suppose that $f : [a, b] \rightarrow \mathbb{R}$ is a continuous function.

- 1 Then the function...
- 2 If F is any antiderivative of f on $[a, b]$, then

$$\int_a^b f(x) \, dx = F(b) - F(a).$$

5.4 The Fundamental Theorem of Calculus



Remark

Part 1 of the theorem tells how to differentiate $\int_a^x f(t) dt$.

5.4 The Fundamental Theorem of Calculus

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Example

Find $\frac{dy}{dx}$ if $y = \int_a^x (t^3 + 1) dt$.

$$\frac{dy}{dx} = \frac{d}{dx} \int_a^x (t^3 + 1) dt = x^3 + 1.$$

5.4 The Fundamental Theorem of Calculus

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Example

Find $\frac{dy}{dx}$ if $y = \int_1^x \sin t dt$.

$$\frac{dy}{dx} = \frac{d}{dx} \int_1^x \sin t dt = \sin x.$$

5.4 The Fundamental Theorem of Calculus



Example

Find $\frac{dy}{dx}$ if $y = \int_0^x \sin \ln \tan e^{t^2} dt$.

5.4 The Fundamental Theorem of Calculus



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$$\frac{dy}{dx} = \frac{d}{dx} \int_0^x \sin \ln \tan e^{t^2} dt = \sin \ln \tan e^{x^2}.$$

5.4 The Fundamental Theorem of Calculus



Example

$$\text{Find } \frac{dy}{dx} \text{ if } y = \int_x^5 3t \sin t \, dt.$$

5.4 The Fundamental Theorem of Calculus



Example

Find $\frac{dy}{dx}$ if $y = \int_x^5 3t \sin t \, dt$.

$$\frac{dy}{dx} = \frac{d}{dx} \int_x^5 3t \sin t \, dt$$

5.4 The Fundamental Theorem of Calculus



Example

Find $\frac{dy}{dx}$ if $y = \int_x^5 3t \sin t \, dt$.

$$\begin{aligned}\frac{dy}{dx} &= \frac{d}{dx} \int_x^5 3t \sin t \, dt \\ &= \frac{d}{dx} \left(- \int_5^x 3t \sin t \, dt \right)\end{aligned}$$

5.4 The Fundamental Theorem of Calculus



Example

Find $\frac{dy}{dx}$ if $y = \int_x^5 3t \sin t \, dt$.

$$\begin{aligned}\frac{dy}{dx} &= \frac{d}{dx} \int_x^5 3t \sin t \, dt \\ &= \frac{d}{dx} \left(- \int_5^x 3t \sin t \, dt \right) \\ &= -3x \sin x.\end{aligned}$$

5.4 The Fundamental Theorem of Calculus



Example

Find $\frac{dy}{dx}$ if $y = \int_1^{x^2} \cos t \, dt$.

This time we will need to use the Chain rule. Let $u = x^2$.

5.4 The Fundamental Theorem of Calculus



Example

Find $\frac{dy}{dx}$ if $y = \int_1^{x^2} \cos t \, dt$.

This time we will need to use the Chain rule. Let $u = x^2$. Then

$$\begin{aligned}\frac{dy}{dx} &= \frac{dy}{du} \frac{du}{dx} \\ &= \left(\frac{d}{du} \int_1^u \cos t \, dt \right) \left(\frac{d}{dx} x^2 \right)\end{aligned}$$

5.4 The Fundamental Theorem of Calculus



Example

Find $\frac{dy}{dx}$ if $y = \int_1^{x^2} \cos t \, dt$.

This time we will need to use the Chain rule. Let $u = x^2$. Then

$$\begin{aligned}\frac{dy}{dx} &= \frac{dy}{du} \frac{du}{dx} \\&= \left(\frac{d}{du} \int_1^u \cos t \, dt \right) \left(\frac{d}{dx} x^2 \right) \\&= (\cos u) (2x) = 2x \cos x^2.\end{aligned}$$

5.4 The Fundamental Theorem of Calculus



Remark

Part 2 of the theorem tells us how to calculate the definite integral of f over $[a, b]$:

- 1 Find an antiderivative F of f .
- 2 Calculate $F(b) - F(a)$.

5.4 The Fundamental Theorem of Calculus



Remark

Part 2 of the theorem tells us how to calculate the definite integral of f over $[a, b]$:

- 1 Find an antiderivative F of f .
- 2 Calculate $F(b) - F(a)$.

Notation

We will write

$$\left[F(x) \right]_a^b = F(b) - F(a).$$

5.4 The Fundamental Theorem of Calculus



Example

$$\int_0^\pi \cos x \, dx = [\sin x]_0^\pi$$

(because $\frac{d}{dx} \sin x = \cos x$)

5.4 The Fundamental Theorem of Calculus



Example

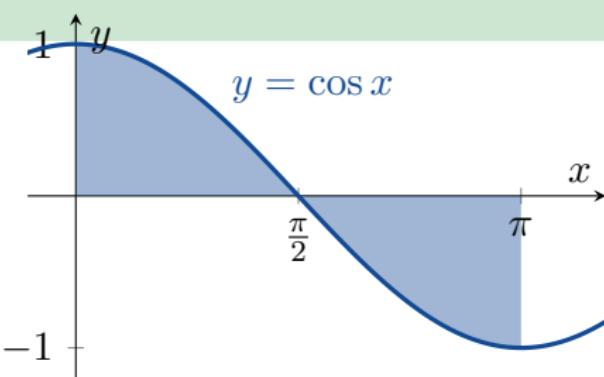
$$\begin{aligned}\int_0^\pi \cos x \, dx &= [\sin x]_0^\pi \\&\quad (\text{because } \frac{d}{dx} \sin x = \cos x) \\&= \sin \pi - \sin 0 \\&= 0 - 0 \\&= 0\end{aligned}$$

5.4 The Fundamental Theorem of Calculus

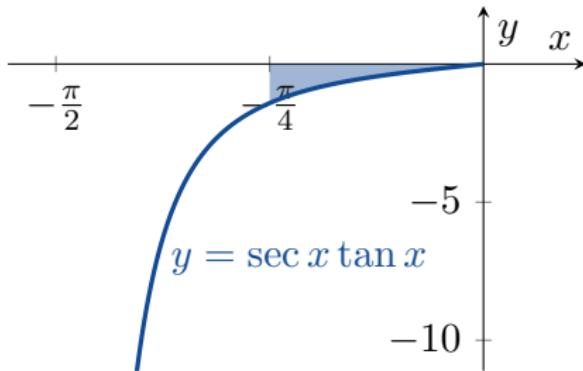


Example

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5.4 The Fundamental Theorem of Calculus

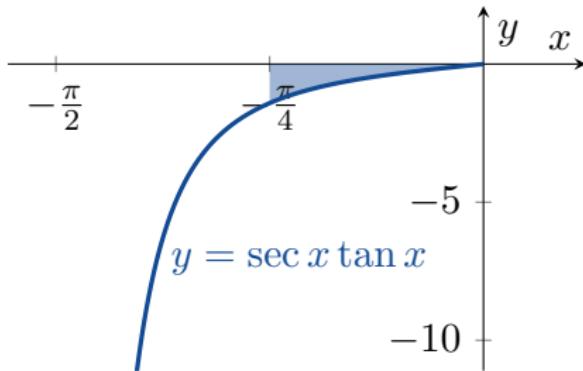


Example

$$\int_{-\frac{\pi}{4}}^0 \sec x \tan x \, dx = [\sec x]_{-\frac{\pi}{4}}^0$$

(because $\frac{d}{dx} \sec x = \sec x \tan x$)

5.4 The Fundamental Theorem of Calculus



Example

$$\begin{aligned}\int_{-\frac{\pi}{4}}^0 \sec x \tan x \, dx &= \left[\sec x \right]_{-\frac{\pi}{4}}^0 \\&\quad (\text{because } \frac{d}{dx} \sec x = \sec x \tan x) \\&= \sec 0 - \sec \left(-\frac{\pi}{4} \right) \\&= 1 - \sqrt{2}.\end{aligned}$$

5.4 The Fundamental Theorem of Calculus



Example

$$\int_1^4 \left(\frac{3}{2}\sqrt{x} - \frac{4}{x^2} \right) dx = \left[x^{\frac{3}{2}} + \frac{4}{x} \right]_1^4$$

(because $\frac{d}{dx} \left(x^{\frac{3}{2}} + \frac{4}{x} \right) = \frac{3}{2}\sqrt{x} - \frac{4}{x^2}$)

5.4 The Fundamental Theorem of Calculus



Example

$$\int_1^4 \left(\frac{3}{2}\sqrt{x} - \frac{4}{x^2} \right) dx = \left[x^{\frac{3}{2}} + \frac{4}{x} \right]_1^4$$

(because $\frac{d}{dx} \left(x^{\frac{3}{2}} + \frac{4}{x} \right) = \frac{3}{2}\sqrt{x} - \frac{4}{x^2}$)

$$= \left(4^{\frac{3}{2}} + \frac{4}{4} \right) - \left(1^{\frac{3}{2}} + \frac{4}{1} \right)$$
$$= (8 + 1) - (1 + 4)$$
$$= 4.$$

5.4 The Fundamental Theorem of Calculus



Summary

Remark

Part 1 of the *Fundamental Theorem of Calculus* says

$$\boxed{\frac{d}{dx} \int_a^x f(t) dt = f(x)}$$

Remark

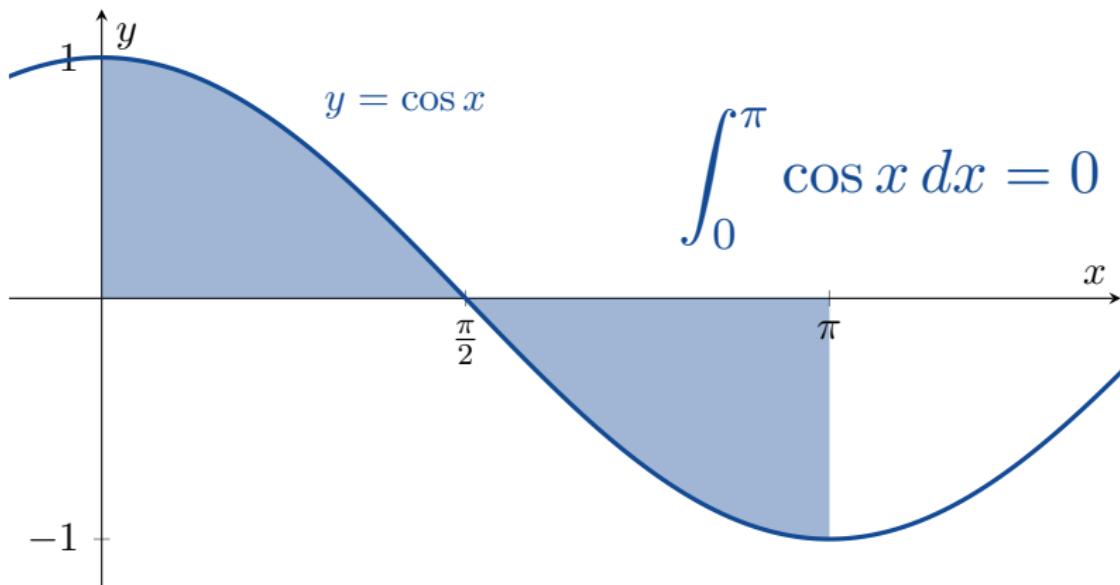
Part 2 of the *Fundamental Theorem of Calculus* says

$$\boxed{\int_a^b f(x) dx = [F(x)]_a^b}$$

5.4 The Fundamental Theorem of Calculus



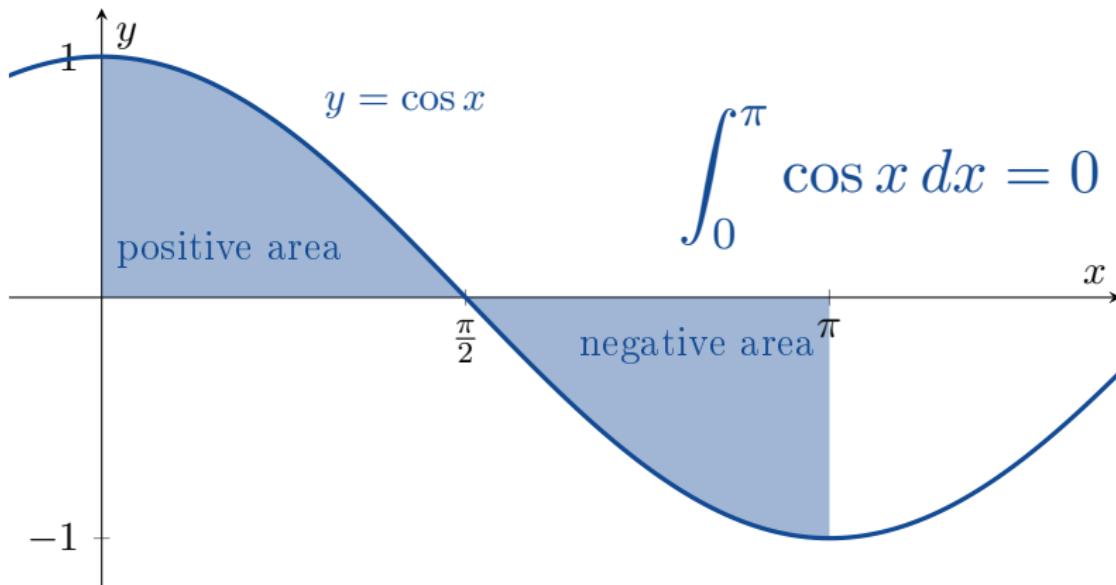
Total Area



5.4 The Fundamental Theorem of Calculus



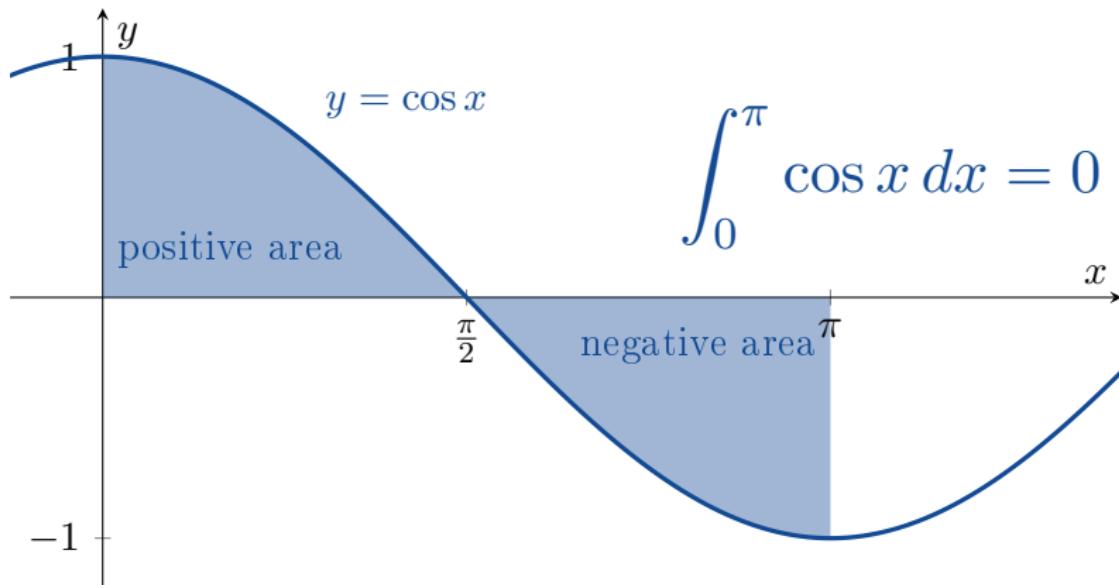
Total Area



5.4 The Fundamental Theorem of Calculus



Total Area



But what if we say that all area is “positive area”?

5.4 The Fundamental Theorem of Calculus



Example

Let $f(x) = x^2 - 4$. We have that

$$\begin{aligned}\int_{-2}^2 f(x) \, dx &= \int_{-2}^2 (x^2 - 4) \, dx = \left[\frac{x^3}{3} - 4x \right]_{-2}^2 \\ &= \left(\frac{8}{3} - 8 \right) - \left(\frac{-8}{3} + 8 \right) = -\frac{32}{3}.\end{aligned}$$

5.4 The Fundamental Theorem of Calculus



Example

Let $f(x) = x^2 - 4$. We have that

$$\begin{aligned}\int_{-2}^2 f(x) \, dx &= \int_{-2}^2 (x^2 - 4) \, dx = \left[\frac{x^3}{3} - 4x \right]_{-2}^2 \\ &= \left(\frac{8}{3} - 8 \right) - \left(\frac{-8}{3} + 8 \right) = -\frac{32}{3}.\end{aligned}$$

The total area between the graph of $y = f(x)$ and the x -axis, over $[-2, 2]$, is $\left| -\frac{32}{3} \right| = \frac{32}{3}$.

5.4 The Fundamental Theorem of Calculus



Example

Let $g(x) = 4 - x^2$. We have that

$$\begin{aligned}\int_{-2}^2 g(x) \, dx &= \int_{-2}^2 (4 - x^2) \, dx = \left[4x - \frac{x^3}{3} \right]_{-2}^2 \\ &= \left(8 - \frac{8}{3} \right) - \left(8 + \frac{-8}{3} \right) = \frac{32}{3}.\end{aligned}$$

5.4 The Fundamental Theorem of Calculus



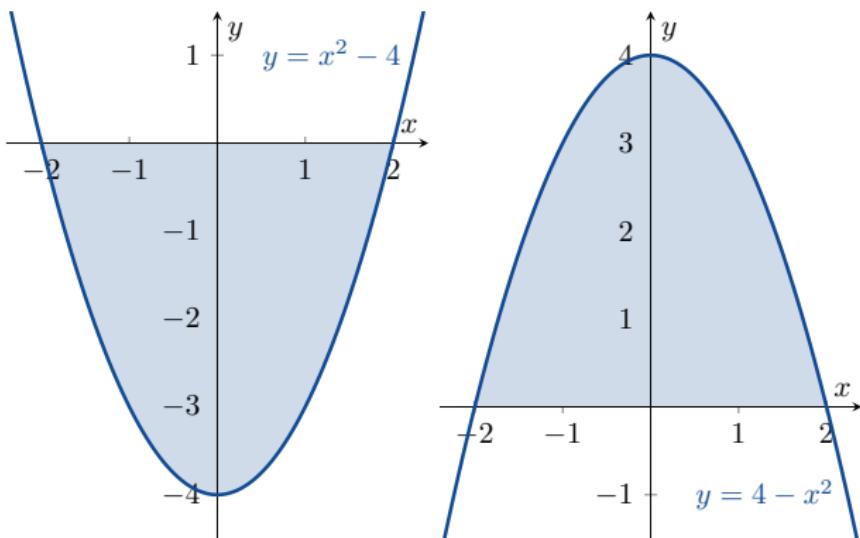
Example

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The total area between the graph of $y = g(x)$ and the x -axis, over $[-2, 2]$, is $\left| \frac{32}{3} \right| = \frac{32}{3}$.

5.4 The Fundamental Theorem of Calculus



$$\text{integral} = -\frac{32}{3}$$
$$\text{total area} = \frac{32}{3}$$

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5.4 The Fundamental Theorem of Calculus

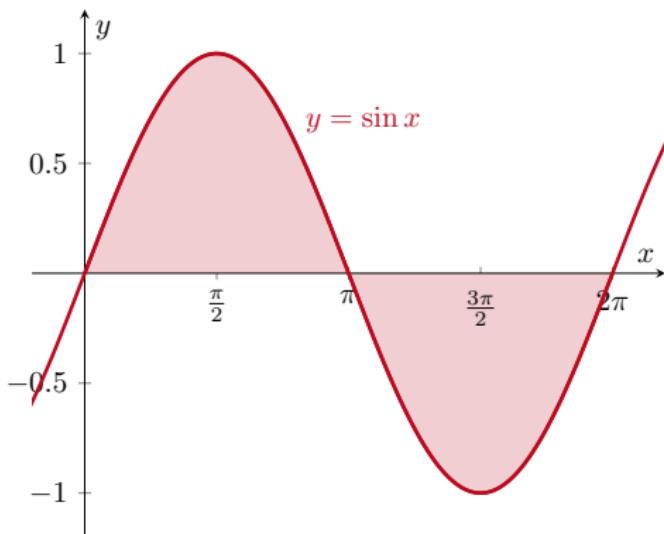


Example

Let $f(x) = \sin x$. Calculate

- 1 the definite integral of f over $[0, 2\pi]$; and
- 2 the total area between the graph of $y = f(x)$ and the x -axis over $[0, 2\pi]$.

5.4 The Fundamental Theorem of Calculus



5.4 The Fundamental Theorem of Calculus



1

$$\begin{aligned}\int_0^{2\pi} \sin x \, dx &= \left[-\cos x \right]_0^{2\pi} = -\cos 2\pi + \cos 0 \\ &= -1 + 1 = 0.\end{aligned}$$

5.4 The Fundamental Theorem of Calculus



1

$$\begin{aligned}\int_0^{2\pi} \sin x \, dx &= \left[-\cos x \right]_0^{2\pi} = -\cos 2\pi + \cos 0 \\ &= -1 + 1 = 0.\end{aligned}$$

2

$$\text{total area} = \left| \int_0^{\pi} \sin x \, dx \right| + \left| \int_{\pi}^{2\pi} \sin x \, dx \right|$$

5.4 The Fundamental Theorem of Calculus



1

$$\begin{aligned}\int_0^{2\pi} \sin x \, dx &= \left[-\cos x \right]_0^{2\pi} = -\cos 2\pi + \cos 0 \\ &= -1 + 1 = 0.\end{aligned}$$

2

$$\begin{aligned}\text{total area} &= \left| \int_0^{\pi} \sin x \, dx \right| + \left| \int_{\pi}^{2\pi} \sin x \, dx \right| \\ &= \left| \left[-\cos x \right]_0^{\pi} \right| + \left| \left[-\cos x \right]_{\pi}^{2\pi} \right| \\ &= \left| -\cos \pi + \cos 0 \right| + \left| -\cos 2\pi + \cos \pi \right| \\ &= \left| -(-1) + 1 \right| + \left| -1 + (-1) \right| = 4.\end{aligned}$$

5.4 The Fundamental Theorem of Calculus



Summary

To find the *total area* between the graph of $y = f(x)$ and the x -axis over $[a, b]$:

- 1 Divide $[a, b]$ at the zeroes of f .
- 2 Integrate f over each subinterval.
- 3 Add the absolute values of the integrals.

5.4 The Fundamental Theorem of Calculus



Example

Find the total area between the graph of $y = x^3 - x^2 - 2x$ and the x -axis for $-1 \leq x \leq 2$.

5.4 The Fundamental Theorem of Calculus



Example

Find the total area between the graph of $y = x^3 - x^2 - 2x$ and the x -axis for $-1 \leq x \leq 2$.

- 1 Let $f(x) = x^3 - x^2 - 2x$.

Since

$$\begin{aligned}0 &= f(x) \\&= x^3 - x^2 - 2x \\&= x(x + 1)(x - 2)\end{aligned}$$

5.4 The Fundamental Theorem of Calculus



Example

Find the total area between the graph of $y = x^3 - x^2 - 2x$ and the x -axis for $-1 \leq x \leq 2$.

- 1 Let $f(x) = x^3 - x^2 - 2x$.

Since

$$\begin{aligned} 0 &= f(x) && x = 0 \\ &= x^3 - x^2 - 2x &\implies & \text{or} \\ &= x(x+1)(x-2) && x = -1 \\ & && \text{or} \\ & && x = 2, \end{aligned}$$

5.4 The Fundamental Theorem of Calculus



Example

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- 1 Let $f(x) = x^3 - x^2 - 2x$.

Since

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we divide $[-1, 2]$ into $[-1, 0]$ and $[0, 2]$.

5.4 The Fundamental Theorem of Calculus



2 We calculate that

$$\begin{aligned}\int_{-1}^0 (x^3 - x^2 - 2x) \, dx &= \left[\frac{x^4}{4} - \frac{x^3}{3} - x^2 \right]_{-1}^0 \\&= (0 - 0 - 0) - \left(\frac{1}{4} + \frac{1}{3} - 1 \right) \\&= \frac{5}{12}\end{aligned}$$

5.4 The Fundamental Theorem of Calculus



2 We calculate that

$$\begin{aligned}\int_{-1}^0 (x^3 - x^2 - 2x) \, dx &= \left[\frac{x^4}{4} - \frac{x^3}{3} - x^2 \right]_{-1}^0 \\&= (0 - 0 - 0) - \left(\frac{1}{4} + \frac{1}{3} - 1 \right) \\&= \frac{5}{12}\end{aligned}$$

and

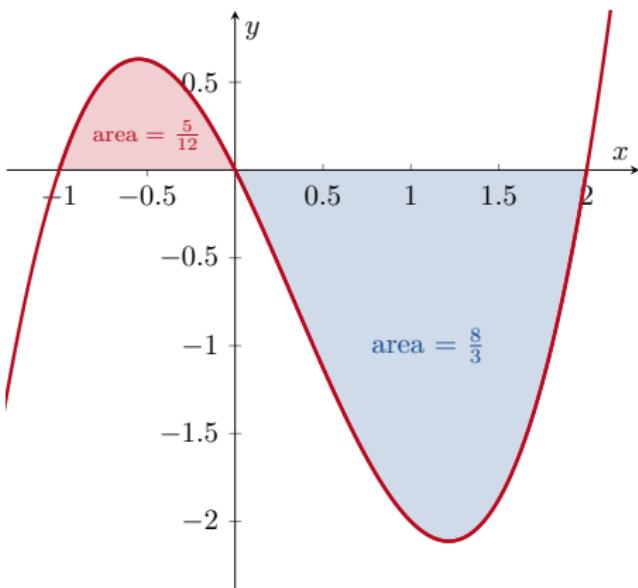
$$\begin{aligned}\int_0^2 (x^3 - x^2 - 2x) \, dx &= \left[\frac{x^4}{4} - \frac{x^3}{3} - x^2 \right]_0^2 \\&= \left(\frac{16}{4} - \frac{8}{3} - 4 \right) - (0 - 0 - 0) \\&= -\frac{8}{3}.\end{aligned}$$

5.4 The Fundamental Theorem of Calculus



3 Therefore

$$\text{total area} = \left| \frac{5}{12} \right| + \left| -\frac{8}{3} \right| = \frac{37}{12}.$$





The Average Value of a Continuous Function

The average of $\{1, 2, 2, 6, 9\}$ is $\frac{1+2+2+6+9}{5} = \frac{20}{5} = 4$. We can also calculate the average value of a continuous function.



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Definition

If f is integrable on $[a, b]$, then the *average value of f on $[a, b]$* is

$$\text{av}(f) = \frac{1}{b-a} \int_a^b f(x) \, dx.$$

5.4

$$\text{av}(f) = \frac{1}{b-a} \int_a^b f(x) \, dx$$



Example

Find the average value of $f(x) = \sqrt{4 - x^2}$ on $[-2, 2]$.

$$\text{av}(f) = \frac{1}{b-a} \int_a^b f(x) \, dx$$

Example

Find the average value of $f(x) = \sqrt{4 - x^2}$ on $[-2, 2]$.

Since

$$\begin{aligned}\int_{-2}^2 f(x) \, dx &= \frac{1}{2} \times \text{the area of a circle of radius 2} \\ &= \frac{1}{2}\pi 2^2 = 2\pi,\end{aligned}$$

$$\text{av}(f) = \frac{1}{b-a} \int_a^b f(x) \, dx$$

Example

Find the average value of $f(x) = \sqrt{4 - x^2}$ on $[-2, 2]$.

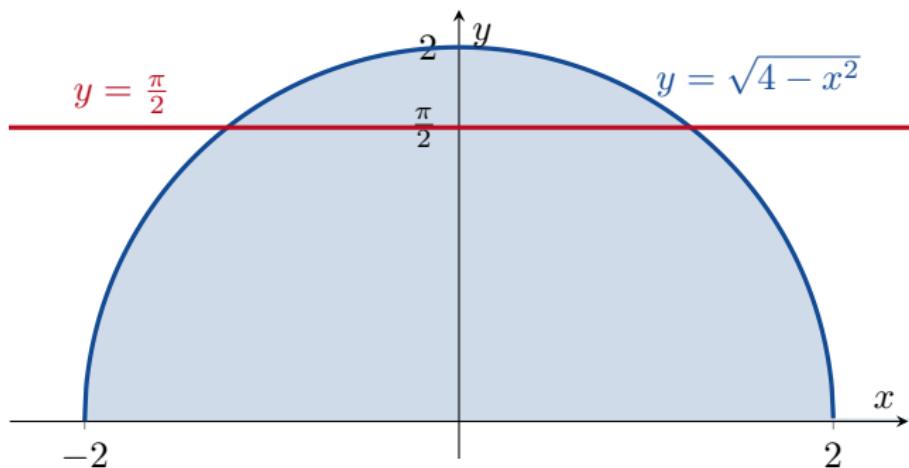
Since

$$\begin{aligned}\int_{-2}^2 f(x) \, dx &= \frac{1}{2} \times \text{the area of a circle of radius 2} \\ &= \frac{1}{2}\pi 2^2 = 2\pi,\end{aligned}$$

we have that

$$\text{av}(f) = \frac{1}{2 - (-2)} \int_{-2}^2 f(x) \, dx = \frac{2\pi}{4} = \frac{\pi}{2}.$$

5.4 The Fundamental Theorem of Calculus



5.4

$$\text{av}(f) = \frac{1}{b-a} \int_a^b f(x) \, dx$$



Example

Find the average value of $g(x) = x^3 - x$ on $[0, 1]$.

$$\text{av}(g) = \frac{1}{1-0} \int_0^1 g(x) \, dx$$

5.4

$$\text{av}(f) = \frac{1}{b-a} \int_a^b f(x) \, dx$$

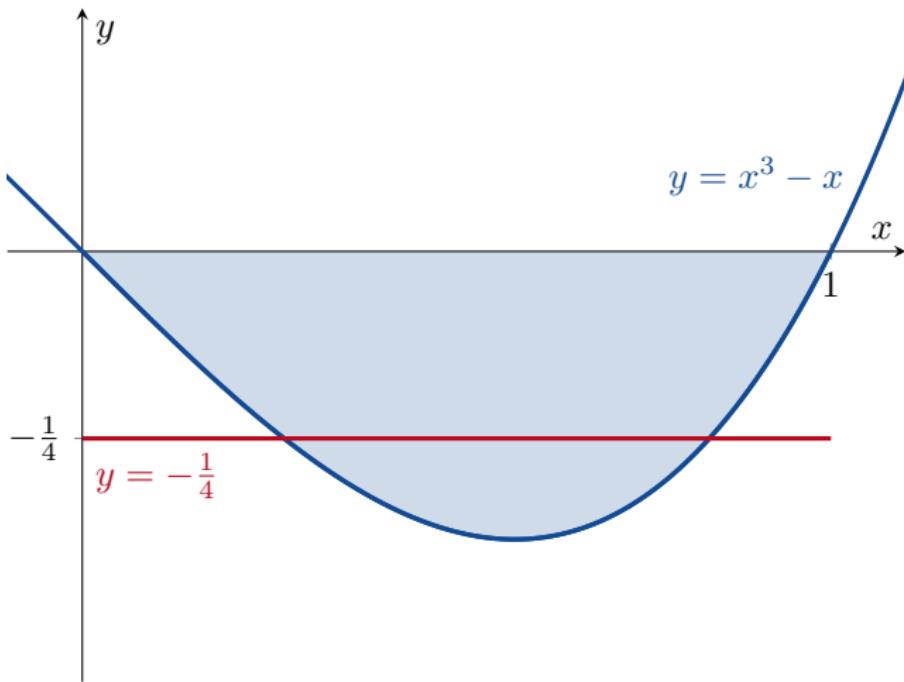


Example

Find the average value of $g(x) = x^3 - x$ on $[0, 1]$.

$$\begin{aligned}\text{av}(g) &= \frac{1}{1-0} \int_0^1 g(x) \, dx \\ &= \int_0^1 (x^3 - x) \, dx \\ &= \left[\frac{x^4}{4} - \frac{x^2}{2} \right]_0^1 \\ &= \frac{1}{4} - \frac{1}{2} = -\frac{1}{4}.\end{aligned}$$

5.4 The Fundamental Theorem of Calculus



Indefinite Integrals & Definite Integrals

Remember that

$\int f(x) dx$ is a function.

Remember that

$\int_a^b f(x) dx$ is a number.

Indefinite Integrals & Definite Integrals

Remember that

$\int f(x) dx$ is a function.

For example

$$\int x \, dx = \frac{x^2}{2} + C$$

and

$$\int \cos x \, dx = \sin x + C.$$

Remember that

$\int_a^b f(x) dx$ is a number.

Indefinite Integrals & Definite Integrals

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$$\int x \, dx = \frac{x^2}{2} + C$$

and

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Remember that

$\int_a^b f(x) dx$ is a number.

For example

$$\int_0^1 x \, dx = \frac{1}{2}$$

and

$$\int_0^{\frac{\pi}{2}} \cos x \, dx = 1.$$



Indefinite Integrals and the Substitution Method

5.5 Indefinite Integrals and the Substitution Method



By the Chain rule,

$$\frac{d}{dx} \left(\frac{u^{n+1}}{n+1} \right) = u^n \frac{du}{dx}.$$

So

$$\int u^n \frac{du}{dx} dx = \frac{u^{n+1}}{n+1} + C.$$

5.5 Indefinite Integrals and the Substitution Method



By the Chain rule,

$$\frac{d}{dx} \left(\frac{u^{n+1}}{n+1} \right) = u^n \frac{du}{dx}.$$

So

$$\int u^n \frac{du}{dx} dx = \frac{u^{n+1}}{n+1} + C.$$

But we know that

$$\int u^n du = \frac{u^{n+1}}{n+1} + C$$

also.

5.5 Indefinite Integrals and the Substitution Method



By the Chain rule,

$$\frac{d}{dx} \left(\frac{u^{n+1}}{n+1} \right) = u^n \frac{du}{dx}.$$

So

$$\int u^n \frac{du}{dx} dx = \frac{u^{n+1}}{n+1} + C.$$

But we know that

$$\int u^n du = \frac{u^{n+1}}{n+1} + C$$

also. So it looks like

$$du = \frac{du}{dx} dx.$$

5.5 Indefinite Integrals and the Substitution Method



Example

Find $\int (x^3 + x)^5 (3x^2 + 1) \, dx.$

5.5 Indefinite Integrals and the Substitution Method



Example

Find $\int (x^3 + x)^5 (3x^2 + 1) \, dx.$

Let $u = x^3 + x.$

5.5 Indefinite Integrals and the Substitution Method



Example

Find $\int (x^3 + x)^5 (3x^2 + 1) \, dx$.

Let $u = x^3 + x$. Then $du = \frac{du}{dx} \, dx = (3x^2 + 1) \, dx$.

5.5 Indefinite Integrals and the Substitution Method



Example

Find $\int (x^3 + x)^5 (3x^2 + 1) \, dx$.

Let $u = x^3 + x$. Then $du = \frac{du}{dx} \, dx = (3x^2 + 1) \, dx$. By substitution, we have that

$$\int (x^3 + x)^5 (3x^2 + 1) \, dx = \int u^5 \, du$$

5.5 Indefinite Integrals and the Substitution Method



Example

Find $\int (x^3 + x)^5 (3x^2 + 1) \, dx$.

Let $u = x^3 + x$. Then $du = \frac{du}{dx} \, dx = (3x^2 + 1) \, dx$. By substitution, we have that

$$\begin{aligned}\int (x^3 + x)^5 (3x^2 + 1) \, dx &= \int u^5 \, du \\ &= \frac{u^6}{6} + C \\ &= \frac{1}{6}(x^3 + x)^6 + C.\end{aligned}$$

5.5 Indefinite Integrals and the Substitution Method


$$\begin{pmatrix} \text{The} \\ \text{substitution} \\ \text{method} \end{pmatrix} = \begin{pmatrix} \text{doing the} \\ \text{Chain Rule} \\ \text{backwards.} \end{pmatrix}$$

5.5 Indefinite Integrals and the Substitution Method



Example

Find $\int \sqrt{2x + 1} dx$.

Let $u = 2x + 1$.

5.5 Indefinite Integrals and the Substitution Method



Example

Find $\int \sqrt{2x + 1} dx$.

Let $u = 2x + 1$. Then $du = \frac{du}{dx} dx = 2dx$. So $dx = \frac{1}{2} du$.

5.5 Indefinite Integrals and the Substitution Method



Example

Find $\int \sqrt{2x+1} dx$.

Let $u = 2x + 1$. Then $du = \frac{du}{dx} dx = 2dx$. So $dx = \frac{1}{2} du$.
Therefore

$$\begin{aligned}\int \sqrt{2x+1} dx &= \int u^{\frac{1}{2}} \left(\frac{1}{2}du\right) = \frac{1}{2} \int u^{\frac{1}{2}} du \\ &= \frac{1}{2} \left(\frac{u^{\frac{3}{2}}}{\frac{3}{2}} \right) + C = \frac{1}{3}(2x+1)^{\frac{3}{2}} + C.\end{aligned}$$

5.5 Indefinite Integrals and the Substitution Method



Theorem (The Substitution Method)

If

- $u = g(x)$ is differentiable;
- $g : \mathbb{R} \rightarrow I$; and
- $f : I \rightarrow \mathbb{R}$ is continuous,

then

$$\int f(g(x))g'(x) \, dx = \int f(u) \, du.$$

5.5 Indefinite Integrals and the Substitution Method



Example

Find $\int 5 \sec^2(5t + 1) dt.$

Let $u = 5t + 1$. Then $du = \frac{du}{dt} dt = 5dt$.

5.5 Indefinite Integrals and the Substitution Method



Example

Find $\int 5 \sec^2(5t + 1) dt$.

Let $u = 5t + 1$. Then $du = \frac{du}{dt} dt = 5dt$. So

$$\begin{aligned}\int 5 \sec^2(5t + 1) dt &= \int \sec^2 u du \\ &= \tan u + C \\ &\quad (\text{because } \frac{d}{du} \tan u = \sec^2 u) \\ &= \tan(5t + 1) + C.\end{aligned}$$

5.5 Indefinite Integrals and the Substitution Method



Example

Find $\int \cos(7\theta + 3) d\theta$.

Let $u = 7\theta + 3$. Then $du = \frac{du}{d\theta} d\theta = 7d\theta$. So $d\theta = \frac{1}{7}du$

5.5 Indefinite Integrals and the Substitution Method



Example

Find $\int \cos(7\theta + 3) d\theta$.

Let $u = 7\theta + 3$. Then $du = \frac{du}{d\theta} d\theta = 7d\theta$. So $d\theta = \frac{1}{7}du$ and

$$\begin{aligned}\int \cos(7\theta + 3) d\theta &= \frac{1}{7} \int \cos u du \\ &= \frac{1}{7} \sin u + C = \frac{1}{7} \sin(7\theta + 3) + C.\end{aligned}$$

5.5 Indefinite Integrals and the Substitution Method



Example

Find $\int x^2 \sin(x^3) dx.$

There are both x^2 and x^3 in the integrand. Which should we choose to be equal to u ?

5.5 Indefinite Integrals and the Substitution Method



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There are both x^2 and x^3 in the integrand. Which should we choose to be equal to u ?

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5.5 Indefinite Integrals and the Substitution Method



Example

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There are both x^2 and x^3 in the integrand. Which should we choose to be equal to u ?

Let $u = x^3$. Then $du = \frac{du}{dx} dx = 3x^2 dx$. So $\frac{1}{3}du = x^2 dx$ and

$$\begin{aligned}\int x^2 \sin(x^3) dx &= \int \frac{1}{3} \sin u du = -\frac{1}{3} \cos u + C \\ &= -\frac{1}{3} \cos(x^3) + C.\end{aligned}$$

5.5 Indefinite Integrals and the Substitution Method



Example

Find $\int x\sqrt{2x+1} dx.$

Let $u = 2x + 1$. Then $du = \frac{du}{dx} dx = 2 dx$. So $dx = \frac{1}{2} du$ and

$$\int x\sqrt{2x+1} dx = \int \cancel{x} \sqrt{u} \frac{1}{2} du.$$

5.5 Indefinite Integrals and the Substitution Method



Example

Find $\int x\sqrt{2x+1} dx$.

Let $u = 2x + 1$. Then $du = \frac{du}{dx} dx = 2 dx$. So $dx = \frac{1}{2} du$ and

$$\int x\sqrt{2x+1} dx = \int \cancel{x} \sqrt{u} \frac{1}{2} du.$$

But we still have an x here. We can't integrate until we change all the x terms to u terms.

5.5 Indefinite Integrals and the Substitution Method



Example

Find $\int x\sqrt{2x+1} dx$.

Let $u = 2x + 1$. Then $du = \frac{du}{dx} dx = 2 dx$. So $dx = \frac{1}{2} du$ and

$$\int x\sqrt{2x+1} dx = \int \cancel{x} \sqrt{u} \frac{1}{2} du.$$

But we still have an x here. We can't integrate until we change all the x terms to u terms. Note that

$$u = 2x + 1 \implies u - 1 = 2x \implies \frac{1}{2}(u - 1) = x.$$

5.5 Indefinite Integrals and the Substitution Method



Therefore

$$\begin{aligned}\int x\sqrt{2x+1} \, dx &= \int \cancel{x} \sqrt{u} \frac{1}{2} \, du \\ &= \int \frac{1}{2}(\cancel{u}-1)\sqrt{u} \frac{1}{2} \, du\end{aligned}$$

5.5 Indefinite Integrals and the Substitution Method



Therefore

$$\begin{aligned}\int x\sqrt{2x+1} \, dx &= \int \cancel{x} \sqrt{u} \frac{1}{2} du \\&= \int \frac{1}{2}(\cancel{u}-1)\sqrt{u} \frac{1}{2} du \\&= \frac{1}{4} \int u^{\frac{3}{2}} - u^{\frac{1}{2}} \, du \\&= \frac{1}{4} \left(\frac{2}{5}u^{\frac{5}{2}} - \frac{2}{3}u^{\frac{3}{2}} \right) + C \\&= \frac{1}{10}u^{\frac{5}{2}} - \frac{1}{6}u^{\frac{3}{2}} + C \\&= \frac{1}{10}(2x+1)^{\frac{5}{2}} - \frac{1}{6}(2x+1)^{\frac{3}{2}} + C.\end{aligned}$$

5.5 Indefinite Integrals and the Substitution Method



Example

Find $\int \sin^2 x \, dx$.

We use the identity

$$\sin^2 x = \frac{1 - \cos 2x}{2}$$

5.5 Indefinite Integrals and the Substitution Method



Example

Find $\int \sin^2 x \, dx$.

We use the identity

$$\sin^2 x = \frac{1 - \cos 2x}{2}$$

to calculate that

$$\begin{aligned}\int \sin^2 x \, dx &= \int \frac{1 - \cos 2x}{2} \, dx \\&= \frac{1}{2} \int (1 - \cos 2x) \, dx \\&= \frac{1}{2} \left(x - \frac{1}{2} \sin 2x \right) + C \\&= \frac{1}{2}x - \frac{1}{4} \sin 2x + C.\end{aligned}$$

5.5 Indefinite Integrals and the Substitution Method



Example

Similarly

$$\int \cos^2 x \, dx = \int \frac{1 + \cos 2x}{2} \, dx = \frac{1}{2}x + \frac{1}{4}\sin 2x + C.$$

(c)
$$\begin{aligned} \int (1 - 2 \sin^2 x) \sin 2x \, dx &= \int (\cos^2 x - \sin^2 x) \sin 2x \, dx \\ &= \int \cos 2x \sin 2x \, dx && \cos 2x = \cos^2 x - \sin^2 x \\ &= \int \frac{1}{2} \sin 4x \, dx = \int \frac{1}{8} \sin u \, du && u = 4x, du = 4x \, dx \\ &= -\cos 4x + C. \end{aligned}$$



5.5 Indefinite Integrals and the Substitution Method



Sometimes there are more than one choice of substitution that we can use.

EXAMPLE 8 Evaluate $\int \frac{2z dz}{\sqrt[3]{z^2 + 1}}.$

5.5 Indefinite Integrals and the Substitution Method



Sometimes there are more than one choice of substitution that we can use.

EXAMPLE 8 Evaluate $\int \frac{2z dz}{\sqrt[3]{z^2 + 1}}$.

Method 1: Substitute $u = z^2 + 1$.

$$\begin{aligned} \int \frac{2z dz}{\sqrt[3]{z^2 + 1}} &= \int \frac{du}{u^{1/3}} && \text{Let } u = z^2 + 1, \\ &= \int u^{-1/3} du && du = 2z dz. \\ &= \frac{u^{2/3}}{2/3} + C && \text{In the form } \int u^n du \\ &= \frac{3}{2}u^{2/3} + C && \text{Integrate.} \\ &= \frac{3}{2}(z^2 + 1)^{2/3} + C && \text{Replace } u \text{ by } z^2 + 1. \end{aligned}$$

5.5 Indefinite Integrals and the Substitution Method



Sometimes there are more than one choice of substitution that we can use.

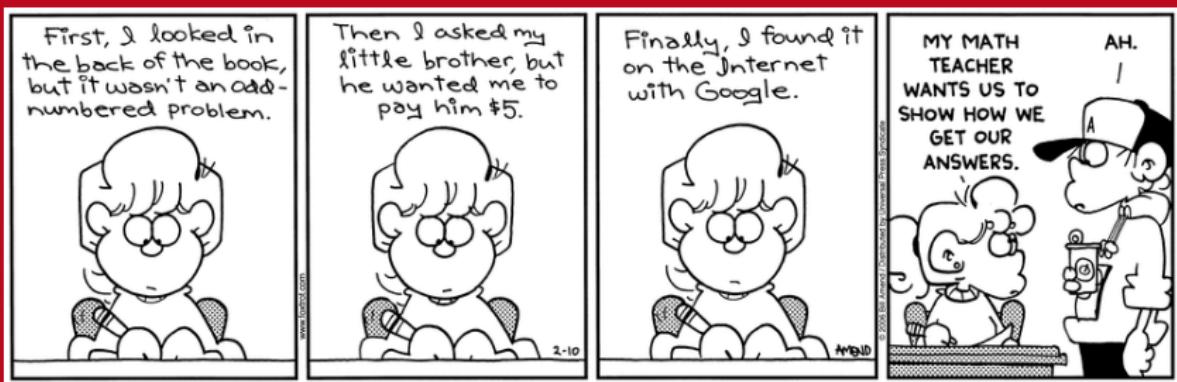
EXAMPLE 8 Evaluate $\int \frac{2z\,dz}{\sqrt[3]{z^2 + 1}}$.

Method 2: Substitute $u = \sqrt[3]{z^2 + 1}$ instead.

$$\begin{aligned}\int \frac{2z\,dz}{\sqrt[3]{z^2 + 1}} &= \int \frac{3u^2\,du}{u} && \text{Let } u = \sqrt[3]{z^2 + 1}, \\ &= 3 \int u\,du && u^3 = z^2 + 1, 3u^2\,du = 2z\,dz. \\ &= 3 \cdot \frac{u^2}{2} + C && \text{Integrate.} \\ &= \frac{3}{2}(z^2 + 1)^{2/3} + C && \text{Replace } u \text{ by } (z^2 + 1)^{1/3}. \quad \blacksquare\end{aligned}$$

Break

We will continue at 2pm





Substitution and Area Between Curves

5.6 Substitution and Area Between Curves



Theorem (The Substitution Method)

If

- $u = g(x)$ is differentiable on $[a, b]$;
- g' is continuous on $[a, b]$; and
- f is continuous on the range of g ,

then

$$\int_a^b f(g(x))g'(x) \, dx = \int_{g(a)}^{g(b)} f(u) \, du.$$

5.6 Substitution and Area Between Curves



Example

Calculate $\int_{-1}^1 3x^2 \sqrt{x^3 + 1} dx$.

solution 1. We can use the previous theorem to solve this example.

Let $u = x^3 + 1$. Then $du = 3x^2 dx$.

5.6 Substitution and Area Between Curves



Example

Calculate $\int_{-1}^1 3x^2 \sqrt{x^3 + 1} dx$.

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Let $u = x^3 + 1$. Then $du = 3x^2 dx$.

Moreover $x = -1 \implies u = 0$ and $x = 1 \implies u = 2$.

5.6 Substitution and Area Between Curves



Example

Calculate $\int_{-1}^1 3x^2 \sqrt{x^3 + 1} dx$.

solution 1. We can use the previous theorem to solve this example.

Let $u = x^3 + 1$. Then $du = 3x^2 dx$.

Moreover $x = -1 \implies u = 0$ and $x = 1 \implies u = 2$. So

$$\int_{x=-1}^{x=1} 3x^2 \sqrt{x^3 + 1} dx = \int_{u=0}^{u=2} \sqrt{u} du$$

5.6 Substitution and Area Between Curves



Example

Calculate $\int_{-1}^1 3x^2 \sqrt{x^3 + 1} dx$.

solution 1. We can use the previous theorem to solve this example.

Let $u = x^3 + 1$. Then $du = 3x^2 dx$.

Moreover $x = -1 \implies u = 0$ and $x = 1 \implies u = 2$. So

$$\begin{aligned}\int_{x=-1}^{x=1} 3x^2 \sqrt{x^3 + 1} dx &= \int_{u=0}^{u=2} \sqrt{u} du \\ &= \left[\frac{2}{3} u^{\frac{3}{2}} \right]_0^2 \\ &= \frac{2}{3} \left(2^{\frac{3}{2}} - 0^{\frac{3}{2}} \right) = \frac{2}{3} 2\sqrt{2} = \frac{4\sqrt{2}}{3}.\end{aligned}$$

5.6 Substitution and Area Between Curves



solution 2. Alternately, we can first find the indefinite integral, then find the required definite integral.

5.6 Substitution and Area Between Curves



solution 2. Alternately, we can first find the indefinite integral, then find the required definite integral.

Let $u = x^3 + 1$. Then $du = 3x^2 dx$. So

$$\int 3x^2 \sqrt{x^3 + 1} dx = \int \sqrt{u} du = \frac{2}{3}u^{\frac{3}{2}} + C = \frac{2}{3}(x^3 + 1)^{\frac{3}{2}} + C.$$

5.6 Substitution and Area Between Curves



solution 2. Alternately, we can first find the indefinite integral, then find the required definite integral.

Let $u = x^3 + 1$. Then $du = 3x^2 dx$. So

$$\int 3x^2 \sqrt{x^3 + 1} dx = \int \sqrt{u} du = \frac{2}{3}u^{\frac{3}{2}} + C = \frac{2}{3}(x^3 + 1)^{\frac{3}{2}} + C.$$

Therefore

$$\begin{aligned}\int_{-1}^1 3x^2 \sqrt{x^3 + 1} dx &= \left[\frac{2}{3}(x^3 + 1)^{\frac{3}{2}} \right]_{-1}^1 \\&= \left(\frac{2}{3}(1 + 1)^{\frac{3}{2}} \right) - \left(\frac{2}{3}(-1 + 1)^{\frac{3}{2}} \right) \\&= \frac{2}{3} \times 2^{\frac{3}{2}} = \frac{4\sqrt{2}}{3}.\end{aligned}$$

5.6 Substitution and Area Between Curves



Example

Calculate $\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \cot \theta \cosec^2 \theta \, d\theta$.

Let $u = \cot \theta$. Then $du = \frac{du}{d\theta} \, d\theta = -\cosec^2 \theta \, d\theta$. So $-du = \cosec^2 \theta \, d\theta$. Moreover $\theta = \frac{\pi}{4} \implies u = \cot \frac{\pi}{4} = 1$ and $\theta = \frac{\pi}{2} \implies u = \cot \frac{\pi}{2} = 0$.

5.6 Substitution and Area Between Curves



Example

Calculate $\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \cot \theta \cosec^2 \theta \, d\theta$.

Let $u = \cot \theta$. Then $du = \frac{du}{d\theta} \, d\theta = -\cosec^2 \theta \, d\theta$. So $-du = \cosec^2 \theta \, d\theta$. Moreover $\theta = \frac{\pi}{4} \implies u = \cot \frac{\pi}{4} = 1$ and $\theta = \frac{\pi}{2} \implies u = \cot \frac{\pi}{2} = 0$. Hence

$$\int_{\theta=\frac{\pi}{4}}^{\theta=\frac{\pi}{2}} \cot \theta \cosec^2 \theta \, d\theta = \int_{u=1}^{u=0} u (-du) = - \int_1^0 u \, du$$

5.6 Substitution and Area Between Curves



Example

Calculate $\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \cot \theta \cosec^2 \theta \, d\theta$.

Let $u = \cot \theta$. Then $du = \frac{du}{d\theta} \, d\theta = -\cosec^2 \theta \, d\theta$. So $-du = \cosec^2 \theta \, d\theta$. Moreover $\theta = \frac{\pi}{4} \implies u = \cot \frac{\pi}{4} = 1$ and $\theta = \frac{\pi}{2} \implies u = \cot \frac{\pi}{2} = 0$. Hence

$$\begin{aligned}\int_{\theta=\frac{\pi}{4}}^{\theta=\frac{\pi}{2}} \cot \theta \cosec^2 \theta \, d\theta &= \int_{u=1}^{u=0} u (-du) = - \int_1^0 u \, du \\ &= - \left[\frac{u^2}{2} \right]_1^0 = - \left(\frac{0^2}{2} - \frac{1^2}{2} \right) = \frac{1}{2}.\end{aligned}$$

5.6 Substitution and Area Between Curves



Remark

You don't need to write " $x =$ " and " $u =$ " if you understand which is which.

$$\int_{u=0}^{u=1} u \, du = \int_0^1 u \, du$$

$$\begin{aligned}
 \text{(b)} \quad & \int_0^{\pi/2} \frac{2 \sin x \cos x}{(1 + \sin^2 x)^3} dx = \int_1^2 \frac{1}{u^3} du \\
 &= -\frac{1}{2u^2} \Big|_1^2 \\
 &= -\frac{1}{8} - \left(-\frac{1}{2}\right) = \frac{3}{8}
 \end{aligned}$$

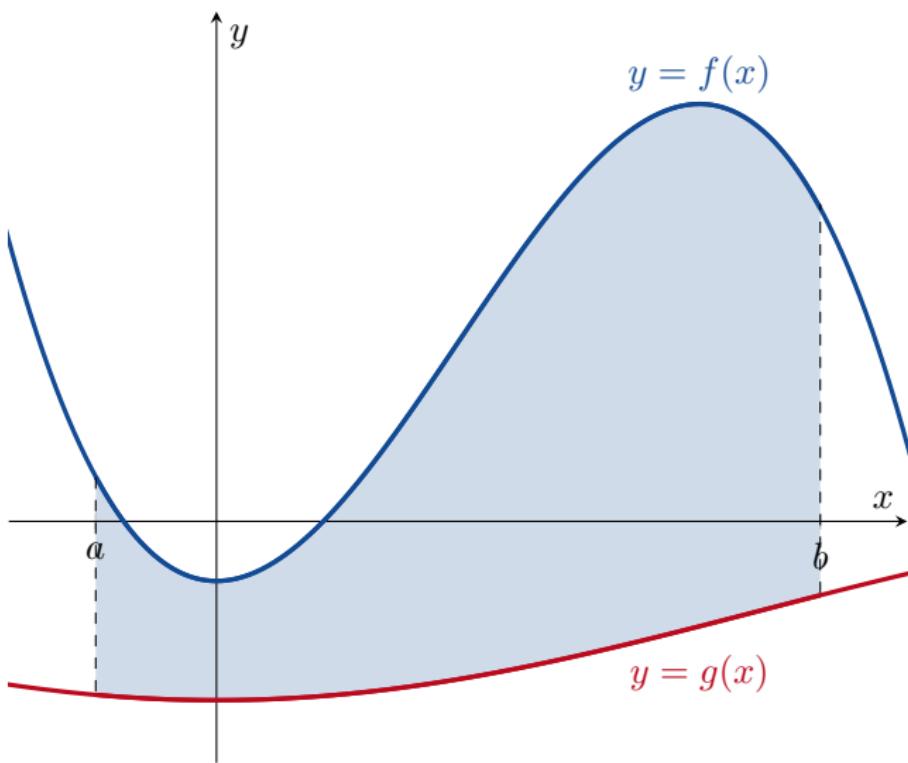
Let $u = 1 + \sin^2 x, du = 2 \sin x \cos x dx.$

When $x = 0, u = 1.$

When $x = \pi/2, u = 2.$



Area Between Curves



5.6 Substitution and Area Between Curves



Definition

If

- f is continuous;
- g is continuous; and
- $f(x) \geq g(x)$ on $[a, b]$,

then the *area of the region between the curves $y = f(x)$ and $y = g(x)$ for $a \leq x \leq b$* is

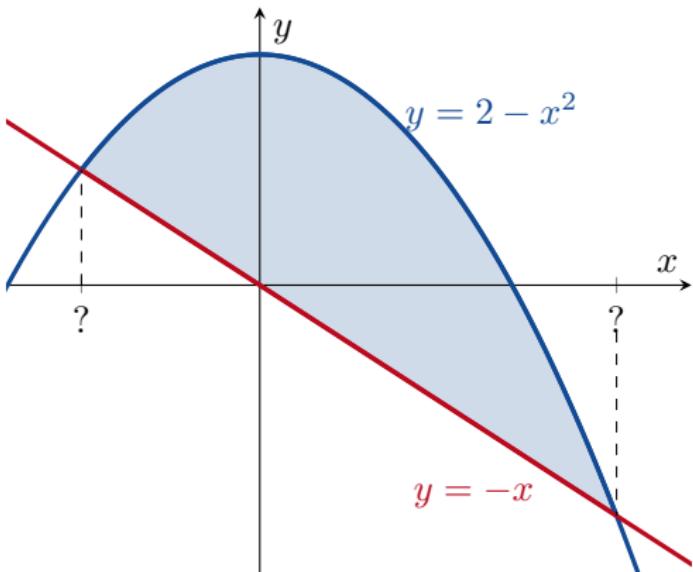
$$\text{area} = \int_a^b (f(x) - g(x)) \, dx.$$

5.6 Substitution and Area Between Curves

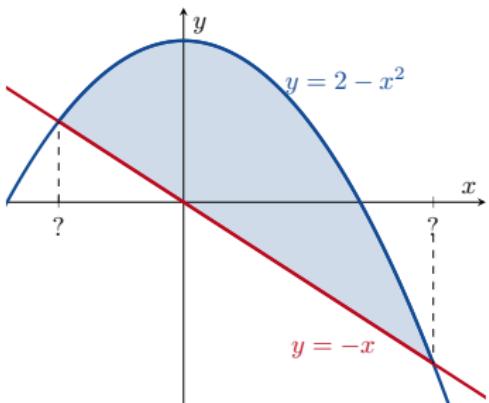


Example

Find the area between $y = 2 - x^2$ and $y = -x$.

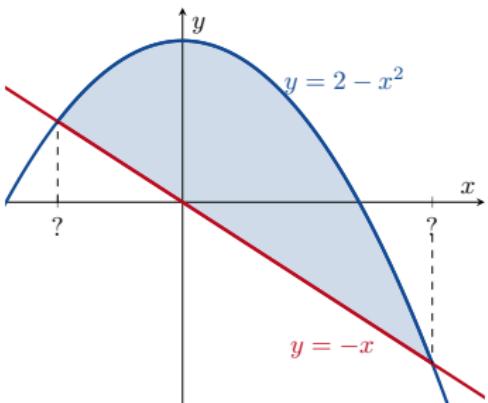


5.6 Substitution and Area Between Curves



First we need to find the limits of integration:

5.6 Substitution and Area Between Curves



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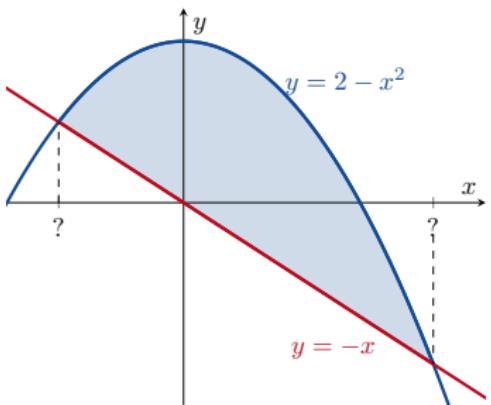
$$2 - x^2 = -x$$

$$0 = x^2 - x - 2$$

$$0 = (x + 1)(x - 2) \implies x = -1 \text{ or } 2.$$

We need to integrate from $x = -1$ to $x = 2$.

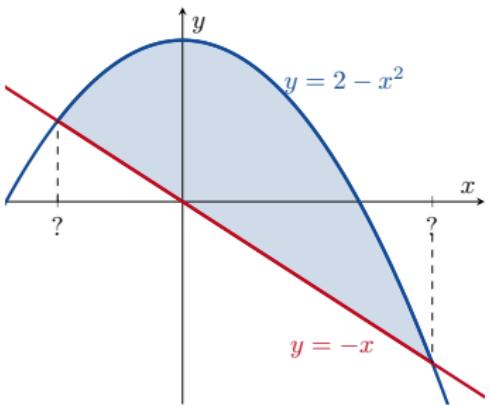
5.6 Substitution and Area Between Curves



Therefore

$$\text{area} = \int_{-1}^2 \left((2 - x^2) - (-x) \right) dx$$

5.6 Substitution and Area Between Curves



Therefore

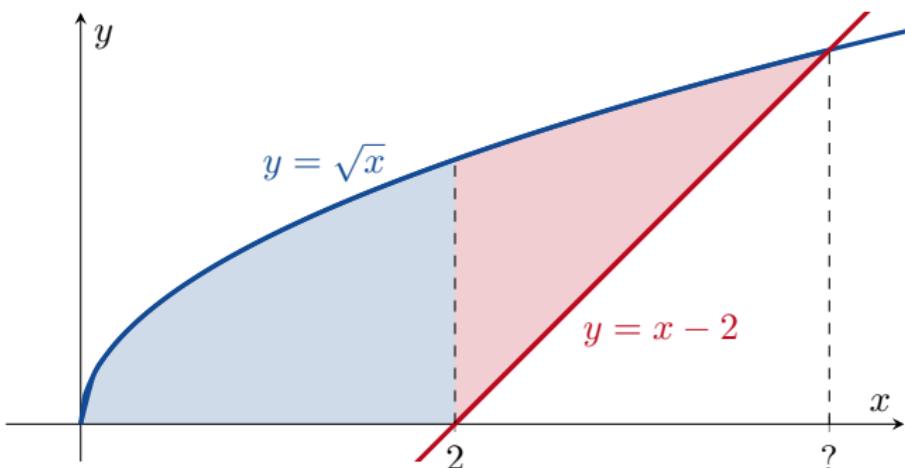
$$\begin{aligned}\text{area} &= \int_{-1}^2 \left((2 - x^2) - (-x) \right) dx \\&= \int_{-1}^2 (2 + x - x^2) dx = \left[2x + \frac{1}{2}x^2 - \frac{1}{3}x^3 \right]_{-1}^2 \\&= \left(4 + \frac{4}{2} - \frac{8}{3} \right) - \left(-2 + \frac{1}{2} + \frac{1}{3} \right) = \frac{9}{2}.\end{aligned}$$

5.6 Substitution and Area Between Curves

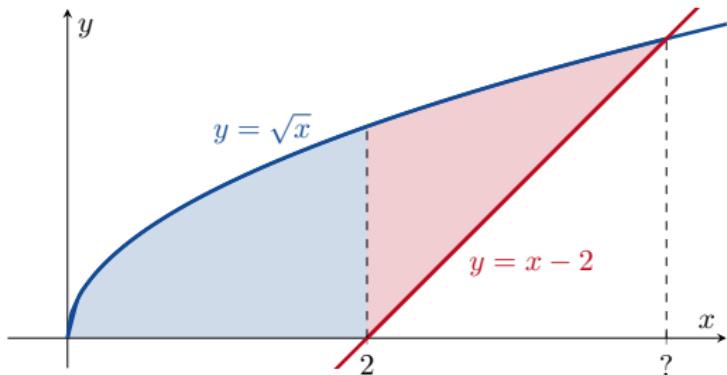


Example

Find the area bounded by $y = \sqrt{x}$, $y = x - 2$ and the x -axis, for $x \geq 0$ and $y \geq 0$.



5.6 Substitution and Area Between Curves



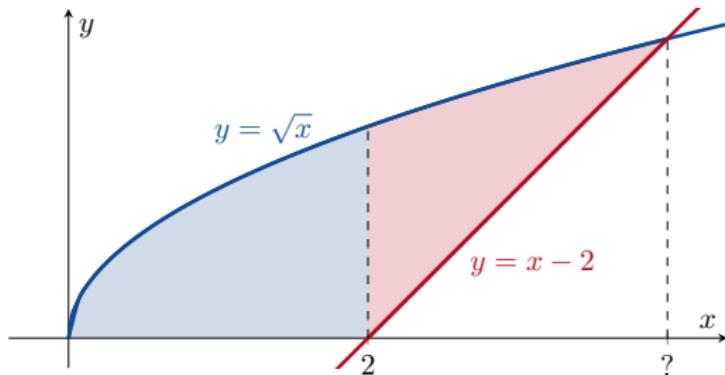
First we calculate that

$$\sqrt{x} = x - 2$$

$$x = (x - 2)^2 = x^2 - 4x + 4$$

$$0 = x^2 - 5x + 4 = (x - 1)(x - 4) \implies x = 1 \text{ or } 4.$$

5.6 Substitution and Area Between Curves



First we calculate that

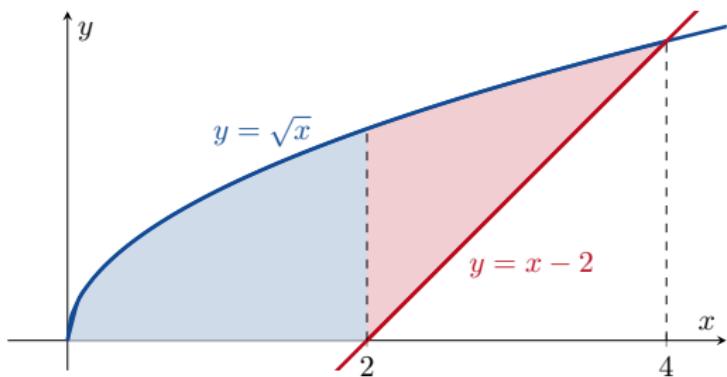
$$\sqrt{x} = x - 2$$

$$x = (x - 2)^2 = x^2 - 4x + 4$$

$$0 = x^2 - 5x + 4 = (x - 1)(x - 4) \implies x = 1 \text{ or } 4.$$

Since $\sqrt{1} \neq 1 - 2$, we must have $x = 4$.

5.6 Substitution and Area Between Curves



Therefore

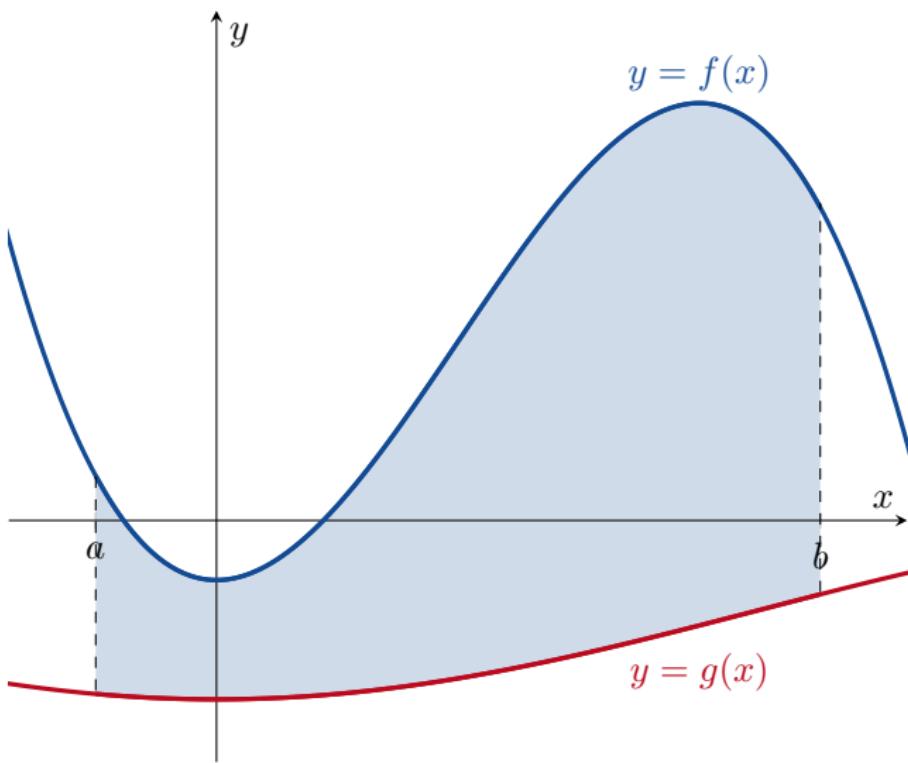
$$\text{area} = \text{blue area} + \text{red area}$$

$$= \int_0^2 (\sqrt{x} - 0) \, dx + \int_2^4 (\sqrt{x} - (x - 2)) \, dx$$

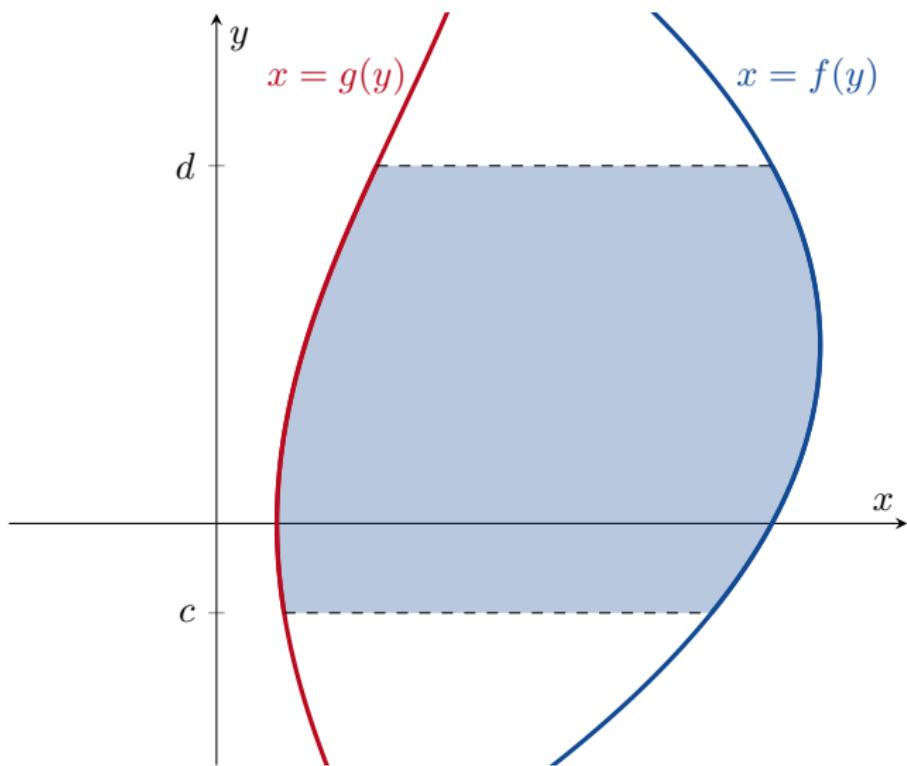
⋮

$$= \frac{10}{3}.$$

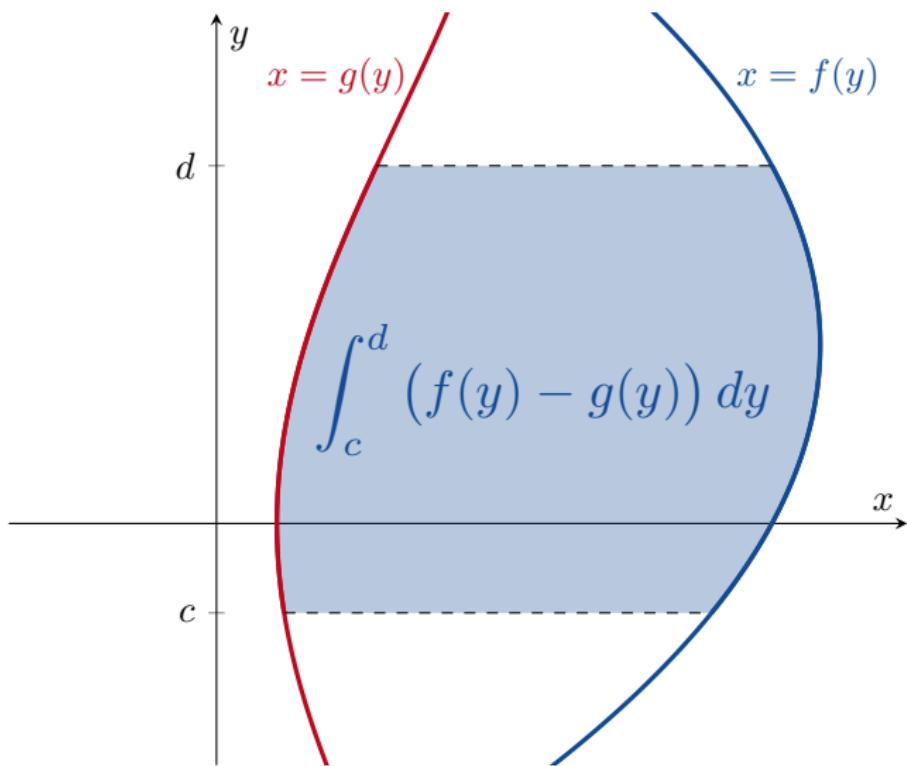
Integration With Respect To y



Integration With Respect To y



Integration With Respect To y



5.6 Substitution and Area Between Curves

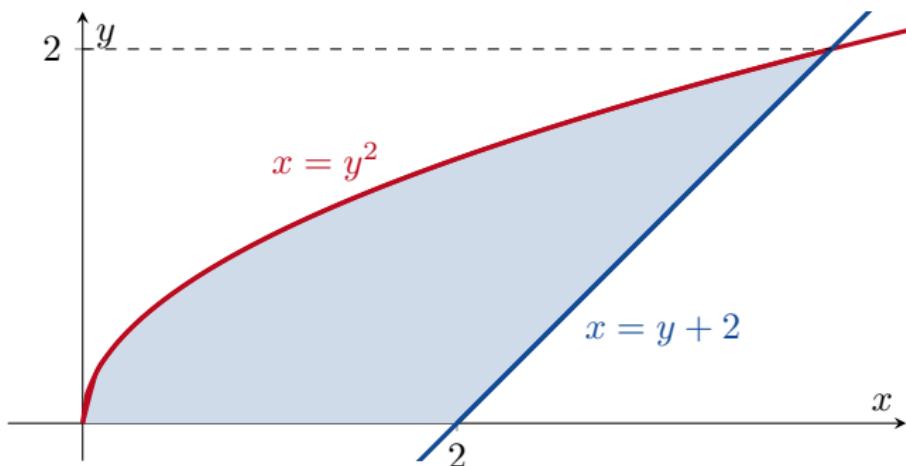


wrt = with respect to

5.6 Substitution and Area Between Curves

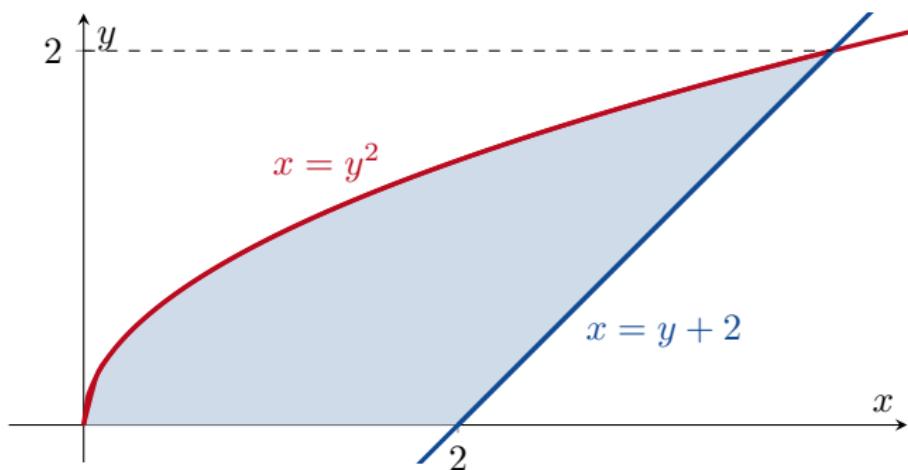
Example

Find the area bounded by $y = \sqrt{x}$, $y = x - 2$ and the x -axis, for $x \geq 0$ and $y \geq 0$, by integrating wrt y .



We will integrate between $y = 0$ and $y = 2$.

5.6 Substitution and Area Between Curves



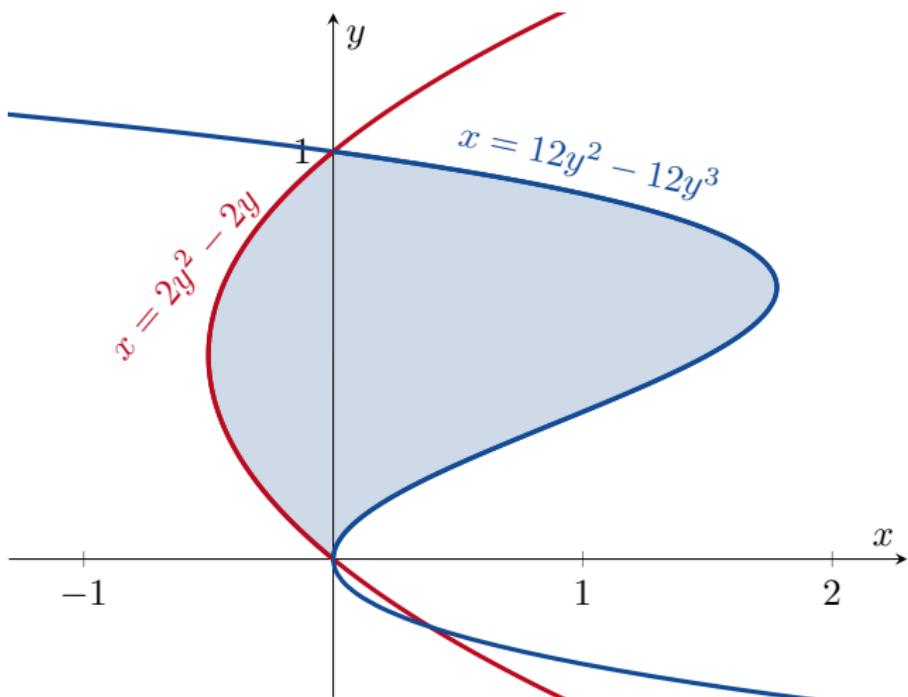
$$\text{area} = \int_c^d f(y) - g(y) dy = \int_0^2 (y+2) - (y^2) dy = \dots = \frac{10}{3}.$$

5.6 Substitution and Area Between Curves

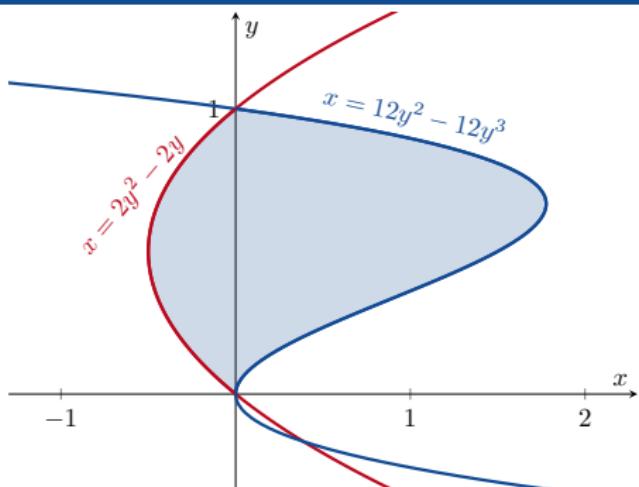


Example (page 322, exercise 33)

Find the area of the region shown below.



5.6 Substitution and Area Between Curves



$$\text{area} = \int_0^1 \left(\begin{array}{l} \text{function} \\ \text{on right} \end{array} \right) - \left(\begin{array}{l} \text{function} \\ \text{on left} \end{array} \right) dy$$

=

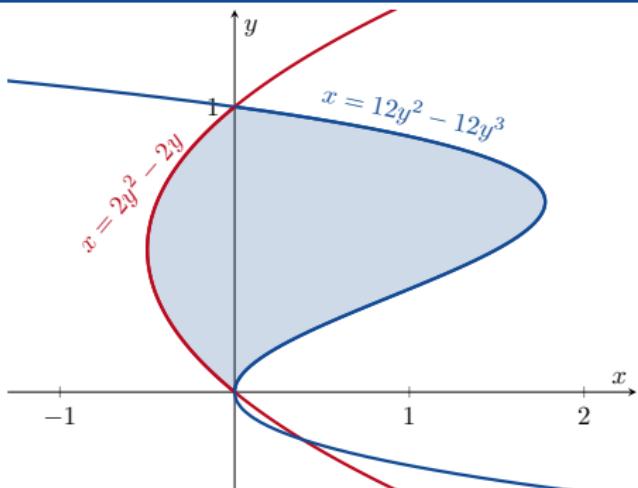
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=

= .

5.6 Substitution and Area Between Curves



$$\begin{aligned}\text{area} &= \int_0^1 \left(\begin{array}{l} \text{function} \\ \text{on right} \end{array} \right) - \left(\begin{array}{l} \text{function} \\ \text{on left} \end{array} \right) dy \\ &= \int_0^1 (12y^2 - 12y^3) - (2y^2 - 2y) dy = \int_0^1 10y^2 - 12y^3 + 2y dy \\ &= \left[\frac{10}{3}y^3 - 3y^4 + y^2 \right]_0^1 = \left(\frac{10}{3} - 3 + 1 \right) - (0 - 0 + 0) = \frac{4}{3}.\end{aligned}$$

Next Time

- 6.1 Volumes Using Cross-Sections
- 6.2 Volumes Using Cylindrical Shells
- 6.3 Arc Length
- 6.4 Areas of Surfaces of Revolution