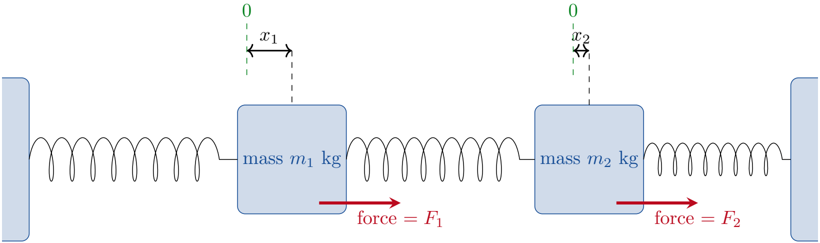
Week 13

- 5.1 Introduction
- 5.2 Basic Theory of Systems of First Order Linear Equations
- 5.3 Homogeneous Linear Systems with Constant Coefficients
- 5.4 Complex Eigenvalues

5.1 Introduction



Consider the dynamical system shown above. There are two blocks and three springs. Forces F_1 and F_2 act on the blocks as shown.

See `https://tinyurl.com/wm2ogdh`

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So we expect that:

$$\begin{cases} \dfrac{d^2x_1}{dt^2} = f_1(x_1, x_2, F_1, m_1) \\[2ex] \dfrac{d^2x_2}{dt^2} = f_2(x_1, x_2, F_2, m_2). \end{cases}$$

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This is a system of two ODEs. To find $x_1(t)$ and $x_2(t)$, we would need to solve these equations at the same time.

The most famous system of ODEs is the system of *Predator-Prey* equations:

$$\begin{cases} \dfrac{dx}{dt} = \alpha x - \beta xy \\[2ex] \dfrac{dy}{dt} = \delta xy - \gamma y \end{cases}$$

where

$$\begin{aligned} x(t) &= \text{number of prey (e.g. mice)} \\ y(t) &= \text{number of predators (e.g. owls)}, \end{aligned}$$

which originate circa 1925.

It is possible to convert an nth order linear ODE into a system of n first order linear ODEs. Or vice versa.

$$\begin{aligned} a_n y^{(n)} + a_{n-1} y^{(n-1)} + \dots \\ + a_1 y' + a_0 y = g(t) \end{aligned} \quad \longleftrightarrow \quad \begin{cases} x_1' = b_{11}x_1 + \ldots + b_{1n}x_n + h_1(t) \\ x_2' = b_{21}x_1 + \ldots + b_{2n}x_n + h_2(t) \\ \vdots \\ x_n' = b_{n1}x_1 + \ldots + b_{nn}x_n + h_n(t) \end{cases}$$

Example

Write

$$u'' + 0.25u' + u = 0$$

as a system of two first order ODEs.

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$$x_2' = u'' = -0.25u' - u = -0.25x_2 - x_1.$$

Therefore

$$\begin{cases} x_1' = x_2 \\ x_2' = -x_1 - 0.25x_2. \end{cases}$$

Remark

We will need

- matrices,
- eigenvalues,
- eigenvectors,
- the Wronskian,
- linear independence,
- and more

from MATH215 – please either revise your Linear Algebra lecture notes or read your Linear Algebra book or read §7.2-7.3 in the textbook by Boyce and DiPrima.

Basic Theory of Systems of First Order Linear Equations

5.2

$$\begin{cases} x_1' = p_{11}(t)x_1 + p_{12}(t)x_2 + \ldots + p_{1n}(t)x_n + g_1(t) \\ x_2' = p_{21}(t)x_1 + p_{22}(t)x_2 + \ldots + p_{2n}(t)x_n + g_2(t) \\ \qquad \vdots \\ x_n' = p_{n1}(t)x_1 + p_{n2}(t)x_2 + \ldots + p_{nn}(t)x_n + g_n(t) \end{cases}$$

is a system of n linear ODEs and n variables: x_1, x_2, $\ldots$, x_n.

If we write

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, \ \mathbf{x}' = \begin{bmatrix} x_1' \\ x_2' \\ \vdots \\ x_n' \end{bmatrix}, \ P = \begin{bmatrix} p_{11} & p_{12} & \dots & p_{1n} \\ p_{21} & p_{22} & \dots & p_{2n} \\ \vdots & \vdots & & \vdots \\ p_{n1} & p_{n2} & \dots & p_{nn} \end{bmatrix}, \text{ and } \mathbf{g} = \begin{bmatrix} g_1 \\ g_2 \\ \vdots \\ g_n \end{bmatrix}$$

then we can write this system as

$$\mathbf{x}' = P(t)\mathbf{x} + \mathbf{g}(t).$$

First we will consider the homogeneous system

$$\mathbf{x}' = P(t)\mathbf{x}.$$

In Chapters 3 and 4 when we had multiple solutions, we wrote them as $y_1(t)$, $y_2(t)$, $\ldots$. But we are already using x_1, x_2, $\ldots$ to denote coordinates. So we need a new type of notation.

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Notation

We use $\mathbf{x}^{(1)}(t)$, $\mathbf{x}^{(2)}(t)$, ... to denote different vector solutions.

Recall from Chapter 3 that if $y_1(t)$ and $y_2(t)$ are both solutions to

$$ay'' + by' + cy = 0,$$

then

$$c_1 y_1 + c_2 y_2$$

is also a solution.

Theorem

If $\mathbf{x}^{(1)}(t)$ *and* $\mathbf{x}^{(2)}(t)$ *are solutions to* $\mathbf{x}' = P(t)\mathbf{x}$*, then*

$$c_1\mathbf{x}^{(1)} + c_2\mathbf{x}^{(2)}$$

is also a solution for any $c_1, c_2 \in \mathbb{R}$*.*

Example

$\mathbf{x}^{(1)} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} e^{3t}$ and $\mathbf{x}^{(2)} = \begin{bmatrix} 1 \\ -2 \end{bmatrix} e^{-t}$ are both solutions to $\mathbf{x}' = \begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix} \mathbf{x}$ (we will see this later).

Example

$\mathbf{x}^{(1)} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} e^{3t}$ and $\mathbf{x}^{(2)} = \begin{bmatrix} 1 \\ -2 \end{bmatrix} e^{-t}$ are both solutions to $\mathbf{x}' = \begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix} \mathbf{x}$ (we will see this later).Therefore

$$\mathbf{x}(t) = c_1\mathbf{x}^{(1)} + c_2\mathbf{x}^{(2)} = c_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} e^{3t} + c_2 \begin{bmatrix} 1 \\ -2 \end{bmatrix} e^{-t}$$

is also a solution to this system.

(Suppose that $P(t)$ is an $n \times n$ matrix.)

Theorem

If $\mathbf{x}^{(1)}(t)$, $\mathbf{x}^{(2)}(t)$, $\ldots$, $\mathbf{x}^{(n)}(t)$ *are linearly independent solutions to* $\mathbf{x}' = P(t)\mathbf{x}$, *then every solution to this system can be written as*

$$\mathbf{x}(t) = c_1\mathbf{x}^{(1)} + c_2\mathbf{x}^{(2)} + \ldots + c_n\mathbf{x}^{(n)}$$

in exactly one way.

Definition

In this case, we say that $\mathbf{x}^{(1)}(t)$, $\mathbf{x}^{(2)}(t)$, $\ldots$, $\mathbf{x}^{(n)}(t)$ form a *fundamental set of solutions* to $\mathbf{x}' = P(t)\mathbf{x}$.

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Definition

In this case,

$$\mathbf{x}(t) = c_1\mathbf{x}^{(1)} + c_2\mathbf{x}^{(2)} + \ldots + c_n\mathbf{x}^{(n)}$$

is called the *general solution* to $\mathbf{x}' = P(t)\mathbf{x}$.

Homogeneous Linear Systems with Constant Coefficients

5.3

5.3 Homogeneous Linear Systems with Constant Coefficients

Consider

$$\mathbf{x}' = A\mathbf{x}$$

where $A \in \mathbb{R}^{n\times n}$.

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If $n = 1$, then we just have

$$\frac{dx}{dt} = ax$$

which has general solution $x(t) = ce^{at}$.

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For $n > 1$, we guess that

$$\mathbf{x}(t) = \boldsymbol{\xi} e^{rt}$$

is a solution to $\mathbf{x}' = A\mathbf{x}$, for some number $r \in \mathbb{C}$ and some vector $\boldsymbol{\xi} \in \mathbb{C}^n$.

But if $\mathbf{x}(t) = \boldsymbol{\xi} e^{rt}$, then

$$\mathbf{x}' = A\mathbf{x}$$

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$$\begin{aligned} r\boldsymbol{\xi} e^{rt} = \mathbf{x}' = A\mathbf{x} &= A\boldsymbol{\xi} e^{rt} \\ r\boldsymbol{\xi} &= A\boldsymbol{\xi} \\ (A - rI)\boldsymbol{\xi} &= \mathbf{0} \end{aligned}$$

where I is the identity matrix.

But if $\mathbf{x}(t) = \boldsymbol{\xi} e^{rt}$, then

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where I is the identity matrix. Hence r must be an eigenvalue of A and $\boldsymbol{\xi}$ must be a corresponding eigenvector of A.

Remark

So the idea is:

1. Find the eigenvalues;
2. Find the eigenvectors; then
3. Write $\mathbf{x}^{(j)}(t) = \boldsymbol{\xi}^{(j)} e^{r_j t}$.

Example

Solve

$$\mathbf{x}' = \begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix} \mathbf{x}.$$

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First we find the eigenvalues. Since

$$\begin{aligned} 0 = \det(A - rI) &= \begin{vmatrix} 1-r & 1 \\ 4 & 1-r \end{vmatrix} = (1-r)^2 - 4 \\ &= r^2 - 2r - 3 = (r+1)(r-3), \end{aligned}$$

the eigenvalues are $r_1 = 3$ and $r_2 = -1$.

Using the first eigenvalue $r_1 = 3$, we calculate that

$$\mathbf{0} = (A - r_1 I)\boldsymbol{\xi} = \begin{bmatrix} -2 & 1 \\ 4 & -2 \end{bmatrix} \begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix}$$

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Hence we can choose $\boldsymbol{\xi}^{(1)} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$. Then using the second eigenvalue $r_2 = -1$, we calculate that

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Hence we can choose $\boldsymbol{\xi}^{(2)} = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$. This gives us two solutions:

$$\mathbf{x}^{(1)}(t) = \begin{bmatrix} 1 \\ 2 \end{bmatrix} e^{3t} \qquad \text{and} \qquad \mathbf{x}^{(2)}(t) = \begin{bmatrix} 1 \\ -2 \end{bmatrix} e^{-t}.$$

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$$W\left(\mathbf{x}^{(1)}, \mathbf{x}^{(2)}\right)(t) = \begin{vmatrix} e^{3t} & e^{-t} \\ 2e^{3t} & -2e^{-t} \end{vmatrix} = -4e^{2t} \neq 0.$$

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Since $W \neq 0$, we have that $\mathbf{x}^{(1)}(t)$ and $\mathbf{x}^{(2)}(t)$ are linearly independent. So $\mathbf{x}^{(1)}(t)$ and $\mathbf{x}^{(2)}(t)$ form a fundamental set of solutions. Therefore the general solution is

$$\boxed{\mathbf{x}(t) = c_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} e^{3t} + c_2 \begin{bmatrix} 1 \\ -2 \end{bmatrix} e^{-t}.}$$

Example

Solve

$$\begin{cases} \mathbf{x}' = \begin{bmatrix} 8 & -1 \\ 6 & 1 \end{bmatrix} \mathbf{x} \\ \mathbf{x}(0) = \begin{bmatrix} 1 \\ -2 \end{bmatrix}. \end{cases}$$

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The eigenvalues are $r_1 = 7$ and $r_2 = 2$.

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$$\mathbf{x}(t) = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{7t} + c_2 \begin{bmatrix} 1 \\ 6 \end{bmatrix} e^{2t}.$$

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Setting $t = 0$, we have

$$\begin{bmatrix} 1 \\ -2 \end{bmatrix} = \mathbf{x}(0)$$

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$$\implies \begin{cases} c_1 = \frac{8}{5} \\ c_2 = -\frac{3}{5}. \end{cases}$$

$$\mathbf{x}(t) = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{7t} + c_2 \begin{bmatrix} 1 \\ 6 \end{bmatrix} e^{2t}.$$

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$$\implies \begin{cases} c_1 = \frac{8}{5} \\ c_2 = -\frac{3}{5}. \end{cases}$$

Therefore the solution to the IVP is

$$\boxed{\mathbf{x}(t) = \frac{8}{5} \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{7t} - \frac{3}{5} \begin{bmatrix} 1 \\ 6 \end{bmatrix} e^{2t}.}$$

Example

Solve

$$\mathbf{x}' = \begin{bmatrix} -3 & \sqrt{2} \\ \sqrt{2} & -2 \end{bmatrix} \mathbf{x}.$$

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The eigenvalues are $r_1 = -1$ and $r_2 = -4$.

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Solve

$$\mathbf{x}' = \begin{bmatrix} -3 & \sqrt{2} \\ \sqrt{2} & -2 \end{bmatrix} \mathbf{x}.$$

The eigenvalues are $r_1 = -1$ and $r_2 = -4$. The corresponding eigenvectors are $\boldsymbol{\xi}^{(1)} = \begin{bmatrix} 1 \\ \sqrt{2} \end{bmatrix}$ and $\boldsymbol{\xi}^{(2)} = \begin{bmatrix} -\sqrt{2} \\ 1 \end{bmatrix}$.

Example

Solve

$$\mathbf{x}' = \begin{bmatrix} -3 & \sqrt{2} \\ \sqrt{2} & -2 \end{bmatrix} \mathbf{x}.$$

The eigenvalues are $r_1 = -1$ and $r_2 = -4$. The corresponding eigenvectors are $\boldsymbol{\xi}^{(1)} = \begin{bmatrix} 1 \\ \sqrt{2} \end{bmatrix}$ and $\boldsymbol{\xi}^{(2)} = \begin{bmatrix} -\sqrt{2} \\ 1 \end{bmatrix}$. Hence the general solution is

$$\boxed{\mathbf{x}(t) = c_1 \begin{bmatrix} 1 \\ \sqrt{2} \end{bmatrix} e^{-t} + c_2 \begin{bmatrix} -\sqrt{2} \\ 1 \end{bmatrix} e^{-4t}.}$$

Remark

$$\det(A - rI) = 0$$

There are three possibilities for the eigenvalues of A.

Remark

$$\det(A - rI) = 0$$

There are three possibilities for the eigenvalues of A.

1. All the eigenvalues are real and different;
2. Some eigenvalues occur in complex conjugate pairs;
3. Some eigenvalues are repeated.

If all the eigenvalues are real and different, then the eigenvectors are linearly independent:

If all the eigenvalues are real and different, then the eigenvectors are linearly independent:

So $W\left(\mathbf{x}^{(1)}, \ldots, \mathbf{x}^{(n)}\right)(t) \neq 0$ and $\mathbf{x}^{(1)}(t)$, $\ldots$, $\mathbf{x}^{(n)}(t)$ form a fundamental set of solutions.

If some eigenvalues are repeated, *but there are n linearly independent eigenvectors*, then this is also true: $\mathbf{x}^{(1)}(t), \ldots,$ $\mathbf{x}^{(n)}(t)$ form a fundamental set of solutions.

Example

Solve

$$\mathbf{x}' = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} \mathbf{x}.$$

The eigenvalues and eigenvectors are

$$r_1 = 2 \qquad r_2 = -1 \qquad r_3 = -1$$

$$\boldsymbol{\xi}^{(1)} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \qquad \boldsymbol{\xi}^{(2)} = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \qquad \boldsymbol{\xi}^{(3)} = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$$

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which gives us the following three solutions

$$\mathbf{x}^{(1)}(t) = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} e^{2t} \qquad \mathbf{x}^{(2)}(t) = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} e^{-t} \qquad \mathbf{x}^{(3)}(t) = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} e^{-t}.$$

$$\mathbf{x}^{(1)}(t) = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} e^{2t} \qquad \mathbf{x}^{(2)}(t) = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} e^{-t} \qquad \mathbf{x}^{(3)}(t) = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} e^{-t}.$$

You can check that the Wronskian of $\mathbf{x}^{(1)}$, $\mathbf{x}^{(2)}$ and $\mathbf{x}^{(3)}$ is non-zero.

$$\mathbf{x}^{(1)}(t) = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} e^{2t} \qquad \mathbf{x}^{(2)}(t) = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} e^{-t} \qquad \mathbf{x}^{(3)}(t) = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} e^{-t}.$$

You can check that the Wronskian of $\mathbf{x}^{(1)}$, $\mathbf{x}^{(2)}$ and $\mathbf{x}^{(3)}$ is non-zero. Therefore $\mathbf{x}^{(1)}$, $\mathbf{x}^{(2)}$ and $\mathbf{x}^{(3)}$ form a fundamental set of solutions.

$$\mathbf{x}^{(1)}(t) = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} e^{2t} \qquad \mathbf{x}^{(2)}(t) = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} e^{-t} \qquad \mathbf{x}^{(3)}(t) = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} e^{-t}.$$

You can check that the Wronskian of $\mathbf{x}^{(1)}$, $\mathbf{x}^{(2)}$ and $\mathbf{x}^{(3)}$ is non-zero. Therefore $\mathbf{x}^{(1)}$, $\mathbf{x}^{(2)}$ and $\mathbf{x}^{(3)}$ form a fundamental set of solutions. The general solution to the ODE is

$$\boxed{\mathbf{x}(t) = c_1 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} e^{2t} + c_2 \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} e^{-t} + c_3 \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} e^{-t}.}$$

Remark

Next we will study systems with complex eigenvalues.

5.4

Complex Eigenvalues

Consider

$$\mathbf{x}' = A\mathbf{x}$$

where $A \in \mathbb{R}^{n\times n}$.

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$$\begin{bmatrix} x_1' \\ x_2' \\ \vdots \\ x_n' \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

Consider

$$\mathbf{x}' = A\mathbf{x}$$

where $A \in \mathbb{R}^{n\times n}$.

$$\begin{cases} x_1' = a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n \\ x_2' = a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n \\ \qquad \vdots \\ x_n' = a_{n1}x_1 + a_{n2}x_2 + \ldots + a_{nn}x_n \end{cases}$$

Any complex eigenvalues of A must occur in complex conjugate pairs: If $r_1 = \lambda + i\mu$ is an eigenvalue of A, then $r_2 = \overline{r}_1 = \lambda - i\mu$ is also an eigenvalue of A.

Moreover, if $\boldsymbol{\xi}^{(1)}$ is an eigenvector of A corresponding to r_1, then $\boldsymbol{\xi}^{(2)} = \overline{\boldsymbol{\xi}^{(1)}}$ is an eigenvector of A corresponding to $r_2 = \bar{r}_1$.

Two solutions of $\mathbf{x}' = A\mathbf{x}$ are

$$\mathbf{x}^{(1)}(t) = \boldsymbol{\xi}^{(1)} e^{r_1 t} \qquad \text{and} \qquad \mathbf{x}^{(2)}(t) = \overline{\boldsymbol{\xi}^{(1)}} e^{\overline{r}_1 t}.$$

Two solutions of $\mathbf{x}' = A\mathbf{x}$ are

$$\mathbf{x}^{(1)}(t) = \boldsymbol{\xi}^{(1)} e^{r_1 t} \qquad \text{and} \qquad \mathbf{x}^{(2)}(t) = \overline{\boldsymbol{\xi}^{(1)}} e^{\bar{r}_1 t}.$$

But $\mathbf{x}^{(1)}, \mathbf{x}^{(2)} : \mathbb{R} \to \mathbb{C}^n$ and we want solutions $: \mathbb{R} \to \mathbb{R}^n$.

If $r_1 = \lambda + i\mu$, and $\boldsymbol{\xi}^{(1)} = \mathbf{a} + i\mathbf{b}$ (λ, $\mu \in \mathbb{R}$, $\mathbf{a}$, $\mathbf{b} \in \mathbb{R}^n$), then

$$\mathbf{x}^{(1)}(t) = (\mathbf{a} + i\mathbf{b})e^{(\lambda+i\mu)t}$$

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$$\begin{aligned} \mathbf{x}^{(1)}(t) &= (\mathbf{a} + i\mathbf{b})e^{(\lambda+i\mu)t} \\ &= (\mathbf{a} + i\mathbf{b})e^{\lambda t}\left(\cos\mu t + i\sin\mu t\right) \end{aligned}$$

If $r_1 = \lambda + i\mu$, and $\boldsymbol{\xi}^{(1)} = \mathbf{a} + i\mathbf{b}$ $(\lambda,\ \mu \in \mathbb{R},\ \mathbf{a},\ \mathbf{b} \in \mathbb{R}^n)$, then

$$\begin{aligned}
\mathbf{x}^{(1)}(t) &= (\mathbf{a} + i\mathbf{b})e^{(\lambda+i\mu)t} \\
&= (\mathbf{a} + i\mathbf{b})e^{\lambda t}(\cos\mu t + i\sin\mu t) \\
&= e^{\lambda t}(\mathbf{a}\cos\mu t - \mathbf{b}\sin\mu t) + ie^{\lambda t}(\mathbf{a}\sin\mu t + \mathbf{b}\cos\mu t)
\end{aligned}$$

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$$\begin{aligned}\mathbf{x}^{(1)}(t) &= (\mathbf{a} + i\mathbf{b})e^{(\lambda+i\mu)t} \\ &= (\mathbf{a} + i\mathbf{b})e^{\lambda t}(\cos\mu t + i\sin\mu t) \\ &= e^{\lambda t}(\mathbf{a}\cos\mu t - \mathbf{b}\sin\mu t) + ie^{\lambda t}(\mathbf{a}\sin\mu t + \mathbf{b}\cos\mu t) \\ &= \mathbf{u}(t) + i\mathbf{v}(t).\end{aligned}$$

The functions $\mathbf{u}(t)$ and $\mathbf{v}(t)$ will be linearly independent.

The functions $\mathbf{u}(t)$ and $\mathbf{v}(t)$ will be linearly independent. Furthermore

$$\operatorname{span}\{\mathbf{u}(t), \mathbf{v}(t)\} = \operatorname{span}\{\mathbf{x}^{(1)}, \mathbf{x}^{(2)}\}.$$

The functions $\mathbf{u}(t)$ and $\mathbf{v}(t)$ will be linearly independent. Furthermore

$$\operatorname{span}\{\mathbf{u}(t), \mathbf{v}(t)\} = \operatorname{span}\{\mathbf{x}^{(1)}, \mathbf{x}^{(2)}\}.$$

So we can include $\mathbf{u}(t)$ and $\mathbf{v}(t)$ in our fundamental set of solutions instead of $\mathbf{x}^{(1)}(t)$ and $\mathbf{x}^{(2)}(t)$.

Example

Solve

$$\mathbf{x}' = \begin{bmatrix} -\frac{1}{2} & 1 \\ -1 & -\frac{1}{2} \end{bmatrix} \mathbf{x}.$$

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$$\mathbf{x}' = \begin{bmatrix} -\frac{1}{2} & 1 \\ -1 & -\frac{1}{2} \end{bmatrix} \mathbf{x}.$$

We calculate that

$$0 = \det(A - rI) = \begin{vmatrix} -\frac{1}{2} - r & 1 \\ -1 & -\frac{1}{2} - r \end{vmatrix} = r^2 + r + \frac{5}{4}$$

and

$$r_{1,2} = \frac{-1 \pm \sqrt{1-5}}{2} = \frac{-1 \pm 2i}{2} = -\frac{1}{2} \pm i.$$

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So we have $r_1 = -\frac{1}{2} + i$ and $r_2 = -\frac{1}{2} - i$.

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and

$$r_{1,2} = \frac{-1 \pm \sqrt{1-5}}{2} = \frac{-1 \pm 2i}{2} = -\frac{1}{2} \pm i.$$

So we have $r_1 = -\frac{1}{2} + i$ and $r_2 = -\frac{1}{2} - i$. We will use r_1. We do not need r_2.

Since

$$0 = (A - r_1 I)\boldsymbol{\xi}^{(1)} = \begin{bmatrix} -i & 1 \\ -1 & -i \end{bmatrix} \begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix}$$

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we can choose

$$\boldsymbol{\xi}^{(1)} = \begin{bmatrix} 1 \\ i \end{bmatrix}.$$

Note that we also have

$$\boldsymbol{\xi}^{(2)} = \overline{\boldsymbol{\xi}^{(1)}} = \overline{\begin{bmatrix} 1 \\ i \end{bmatrix}} = \begin{bmatrix} 1 \\ -i \end{bmatrix},$$

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$$\boldsymbol{\xi}^{(2)} = \overline{\boldsymbol{\xi}^{(1)}} = \overline{\begin{bmatrix} 1 \\ i \end{bmatrix}} = \begin{bmatrix} 1 \\ -i \end{bmatrix},$$

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Next we look at $\mathbf{x}^{(1)}(t)$:

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$$\mathbf{x}^{(1)}(t) = \boldsymbol{\xi}^{(1)} e^{r_1 t} = \begin{bmatrix} 1 \\ i \end{bmatrix} e^{-\frac{t}{2}} \left(\cos t + i \sin t\right)$$

$$=$$

$$=$$

Next we look at $\mathbf{x}^{(1)}(t)$:

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= e^{-\frac{t}{2}} \begin{bmatrix} \cos t + i \sin t \\ i \cos t - \sin t \end{bmatrix} \\
= e^{-\frac{t}{2}} \begin{bmatrix} \cos t \\ - \sin t \end{bmatrix} + i e^{-\frac{t}{2}} \begin{bmatrix} \sin t \\ \cos t \end{bmatrix}
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&= \mathbf{u}(t) + i \mathbf{v}(t).
\end{aligned}$$

Hence we choose

$$\mathbf{u}(t) = e^{-\frac{t}{2}} \begin{bmatrix} \cos t \\ -\sin t \end{bmatrix} \quad \text{and} \quad \mathbf{v}(t) = e^{-\frac{t}{2}} \begin{bmatrix} \sin t \\ \cos t \end{bmatrix}.$$

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But are $\mathbf{u}(t)$ and $\mathbf{v}(t)$ linearly independent?

$$\mathbf{u}(t) = e^{-\frac{t}{2}} \begin{bmatrix} \cos t \\ -\sin t \end{bmatrix} \quad \text{and} \quad \mathbf{v}(t) = e^{-\frac{t}{2}} \begin{bmatrix} \sin t \\ \cos t \end{bmatrix}.$$

But are $\mathbf{u}(t)$ and $\mathbf{v}(t)$ linearly independent? Since

$$\begin{aligned} W\big(\mathbf{u}(t), \mathbf{v}(t)\big)(t) &= \begin{vmatrix} u_1 & v_1 \\ u_2 & v_2 \end{vmatrix} = \begin{vmatrix} e^{-\frac{t}{2}} \cos t & e^{-\frac{t}{2}} \sin t \\ -e^{-\frac{t}{2}} \sin t & e^{-\frac{t}{2}} \cos t \end{vmatrix} \\ &= e^{-t} \cos^2 t + e^{-t} \sin^2 t = e^{-t} \\ &\neq 0 \end{aligned}$$

the answer is yes.

$$\mathbf{u}(t) = e^{-\frac{t}{2}} \begin{bmatrix} \cos t \\ -\sin t \end{bmatrix} \quad \text{and} \quad \mathbf{v}(t) = e^{-\frac{t}{2}} \begin{bmatrix} \sin t \\ \cos t \end{bmatrix}.$$

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the answer is yes. Therefore $\mathbf{u}(t)$ and $\mathbf{v}(t)$ form a fundamental set of solutions.

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the answer is yes. Therefore $\mathbf{u}(t)$ and $\mathbf{v}(t)$ form a fundamental set of solutions.

Therefore the general solution to $\mathbf{x}' = \begin{bmatrix} -\frac{1}{2} & 1 \\ -1 & -\frac{1}{2} \end{bmatrix} \mathbf{x}$ is

$$\boxed{\mathbf{x}(t) = c_1 e^{-\frac{t}{2}} \begin{bmatrix} \cos t \\ -\sin t \end{bmatrix} + c_2 e^{-\frac{t}{2}} \begin{bmatrix} \sin t \\ \cos t \end{bmatrix}.}$$

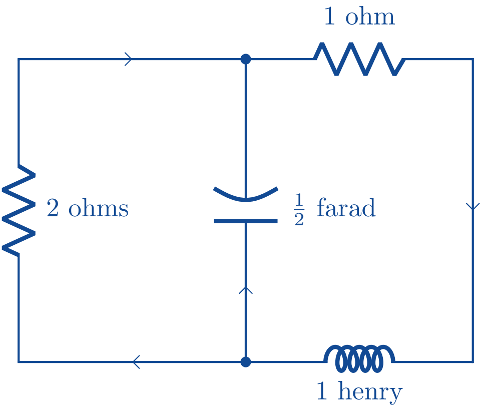
Remark

Our method is

1. Find the eigenvalues;
2. Find the eigenvectors;
3. - If r_j is real, just use the solution $\mathbf{x}^{(j)}(t) = \boldsymbol{\xi}^{(j)} e^{r_j t}$;
   - But if r_j is complex, write

     $$\mathbf{x}^{(j)}(t) = \boldsymbol{\xi}^{(j)} e^{r_j t} = \begin{pmatrix} \text{real valued} \\ \text{function} \end{pmatrix} + i \begin{pmatrix} \text{real valued} \\ \text{function} \end{pmatrix}$$

     and use these two functions.


1 ohm
2 ohms
$\frac{1}{2}$ farad
1 henry

Example

The electric circuit shown above is described by

$$\begin{cases} I' = -I - V \\ V' = 2I - V \end{cases}$$

where

$$\begin{aligned} I &= \text{the current through the inductor} \\ V &= \text{the voltage drop across the capacitor.} \end{aligned}$$

(Ask an Electrical Engineer.)

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(Ask an Electrical Engineer.)

Suppose that at time $t = 0$ the current is 2 amperes and the voltage drop is 2 volts. Find $I(t)$ and $V(t)$.

$$\begin{cases} I' = -I - V \\ V' = 2I - V \end{cases}$$

Suppose that at time $t = 0$ the current is 2 amperes and the voltage drop is 2 volts. Find $I(t)$ and $V(t)$.

We must solve the IVP

$$\begin{cases} \dfrac{d}{dt}\begin{bmatrix} I \\ V \end{bmatrix} = \begin{bmatrix} -1 & -1 \\ 2 & -1 \end{bmatrix}\begin{bmatrix} I \\ V \end{bmatrix} \\ \\ \begin{bmatrix} I \\ V \end{bmatrix}(0) = \begin{bmatrix} 2 \\ 2 \end{bmatrix}. \end{cases}$$

The eigenvalues of $\begin{bmatrix} -1 & -1 \\ 2 & -1 \end{bmatrix}$ are $r_1 = -1 + i\sqrt{2}$ and $r_2 = -1 - i\sqrt{2}$ (please check). The corresponding eigenvectors are

$$\boldsymbol{\xi}^{(1)} = \begin{bmatrix} 1 \\ -i\sqrt{2} \end{bmatrix} \qquad \text{and} \qquad \boldsymbol{\xi}^{(2)} = \begin{bmatrix} 1 \\ i\sqrt{2} \end{bmatrix}.$$

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$$\boldsymbol{\xi}^{(1)} = \begin{bmatrix} 1 \\ -i\sqrt{2} \end{bmatrix} \qquad \text{and} \qquad \boldsymbol{\xi}^{(2)} = \begin{bmatrix} 1 \\ i\sqrt{2} \end{bmatrix}.$$

Then we calculate that

$$\begin{aligned} \mathbf{x}^{(1)}(t) = \boldsymbol{\xi}^{(1)} e^{r_1 t} &= \begin{bmatrix} 1 \\ -i\sqrt{2} \end{bmatrix} e^{(-1+i\sqrt{2})t} \\ &= \begin{bmatrix} 1 \\ -i\sqrt{2} \end{bmatrix} e^{-t} \big(\cos\sqrt{2}t + i \sin\sqrt{2}t \big) \\ &= e^{-t} \begin{bmatrix} \cos\sqrt{2}t + i\sin\sqrt{2}t \\ -i\sqrt{2}\cos\sqrt{2}t + \sqrt{2}\sin\sqrt{2}t \end{bmatrix} \\ &= e^{-t} \begin{bmatrix} \cos\sqrt{2}t \\ \sqrt{2}\sin\sqrt{2}t \end{bmatrix} + ie^{-t} \begin{bmatrix} \sin\sqrt{2}t \\ -\sqrt{2}\cos\sqrt{2}t \end{bmatrix}. \end{aligned}$$

Hence the general solution to the ODE is

$$\begin{bmatrix} I(t) \\ V(t) \end{bmatrix} = c_1 e^{-t} \begin{bmatrix} \cos\sqrt{2}t \\ \sqrt{2}\sin\sqrt{2}t \end{bmatrix} + c_2 e^{-t} \begin{bmatrix} \sin\sqrt{2}t \\ -\sqrt{2}\cos\sqrt{2}t \end{bmatrix}.$$

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$$\begin{bmatrix} I(t) \\ V(t) \end{bmatrix} = c_1 e^{-t} \begin{bmatrix} \cos\sqrt{2}t \\ \sqrt{2}\sin\sqrt{2}t \end{bmatrix} + c_2 e^{-t} \begin{bmatrix} \sin\sqrt{2}t \\ -\sqrt{2}\cos\sqrt{2}t \end{bmatrix}.$$

Using the initial condition, we calculate that

$$\begin{bmatrix} 2 \\ 2 \end{bmatrix} = \begin{bmatrix} I(0) \\ V(0) \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ -\sqrt{2} \end{bmatrix} \qquad \Longrightarrow \qquad \begin{cases} c_1 = 2 \\ c_2 = -\sqrt{2}. \end{cases}$$

Hence the general solution to the ODE is

$$\begin{bmatrix} I(t) \\ V(t) \end{bmatrix} = c_1 e^{-t} \begin{bmatrix} \cos\sqrt{2}t \\ \sqrt{2}\sin\sqrt{2}t \end{bmatrix} + c_2 e^{-t} \begin{bmatrix} \sin\sqrt{2}t \\ -\sqrt{2}\cos\sqrt{2}t \end{bmatrix}.$$

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Thus

$$\begin{bmatrix} I(t) \\ V(t) \end{bmatrix} = 2\, e^{-t} \begin{bmatrix} \cos\sqrt{2}t \\ \sqrt{2}\sin\sqrt{2}t \end{bmatrix} - \sqrt{2}\, e^{-t} \begin{bmatrix} \sin\sqrt{2}t \\ -\sqrt{2}\cos\sqrt{2}t \end{bmatrix}.$$

$$\begin{bmatrix} I(t) \\ V(t) \end{bmatrix} = 2\,e^{-t} \begin{bmatrix} \cos\sqrt{2}t \\ \sqrt{2}\sin\sqrt{2}t \end{bmatrix} - \sqrt{2}\,e^{-t} \begin{bmatrix} \sin\sqrt{2}t \\ -\sqrt{2}\cos\sqrt{2}t \end{bmatrix}$$

So the answers to this problem are

$$\boxed{I(t) = \qquad\qquad\qquad\qquad}$$

and

$$\boxed{V(t) = \qquad\qquad\qquad\qquad}$$

$$\begin{bmatrix} I(t) \\ V(t) \end{bmatrix} = 2\,e^{-t} \begin{bmatrix} \cos\sqrt{2}t \\ \sqrt{2}\sin\sqrt{2}t \end{bmatrix} - \sqrt{2}\,e^{-t} \begin{bmatrix} \sin\sqrt{2}t \\ -\sqrt{2}\cos\sqrt{2}t \end{bmatrix}$$

So the answers to this problem are

$$\boxed{I(t) = 2e^{-t}\cos\sqrt{2}t - \sqrt{2}e^{-t}\sin\sqrt{2}t}$$

and

$$\boxed{V(t) = \qquad\qquad\qquad\qquad\qquad\qquad}$$

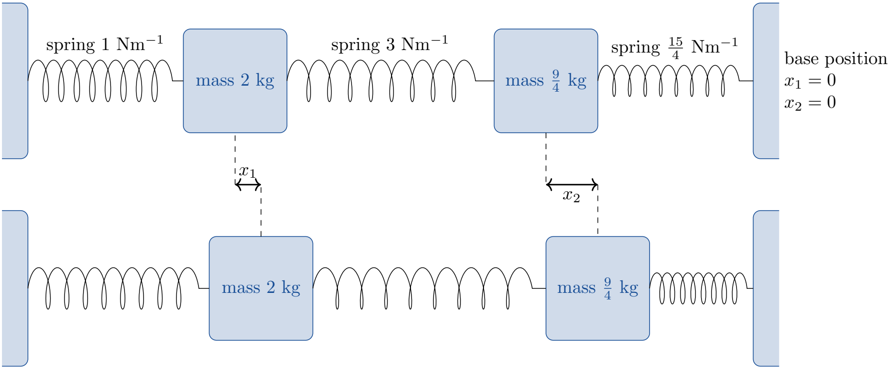
$$\begin{bmatrix} I(t) \\ V(t) \end{bmatrix} = 2\,e^{-t} \begin{bmatrix} \cos\sqrt{2}t \\ \sqrt{2}\sin\sqrt{2}t \end{bmatrix} - \sqrt{2}\,e^{-t} \begin{bmatrix} \sin\sqrt{2}t \\ -\sqrt{2}\cos\sqrt{2}t \end{bmatrix}$$

So the answers to this problem are

$$\boxed{I(t) = 2e^{-t}\cos\sqrt{2}t - \sqrt{2}e^{-t}\sin\sqrt{2}t}$$

and

$$\boxed{V(t) = 2\sqrt{2}e^{-t}\sin\sqrt{2}t + 2e^{-t}\cos\sqrt{2}t.}$$

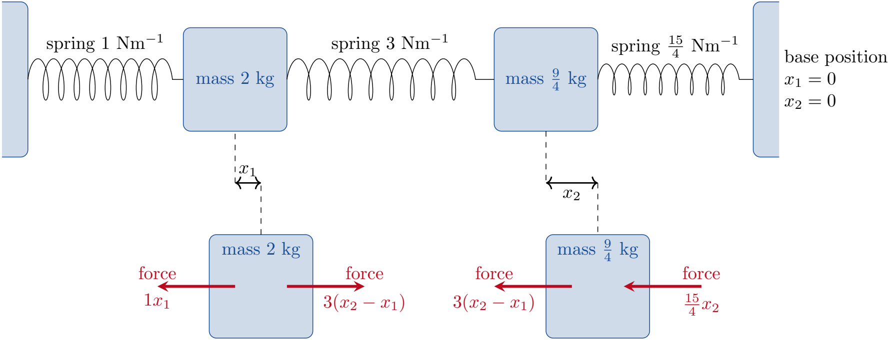


See `https://tinyurl.com/wm2ogdh` for an animated figure.

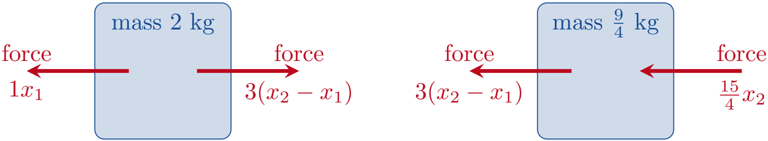
Example

For the dynamical system shown above, find $x_1(t)$ and $x_2(t)$.

As the springs are stretched and compressed, they apply forces on the blocks as shown below (Hooke's Law).



5.4 Complex Eigenvalues



We calculate that

$$\text{mass} \times \text{acceleration} = \text{force}$$

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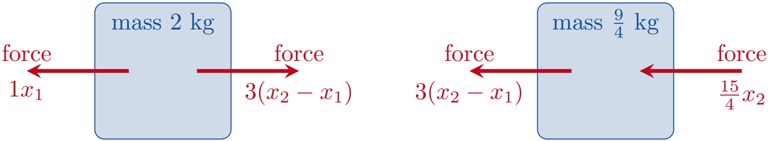
5.4 Complex Eigenvalues

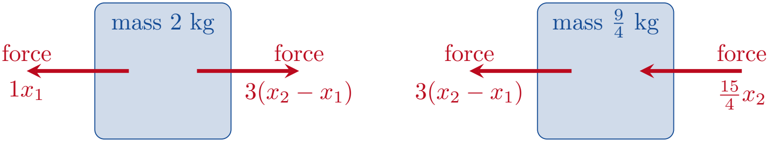


We calculate that

$$2\frac{d^2x_1}{dt^2} = \text{mass} \times \text{acceleration} = \text{force} = -x_1 + 3(x_2 - x_1)$$

$$\text{mass} \times \text{acceleration} = \text{force}$$

5.4 Complex Eigenvalues



We calculate that

$$2\frac{d^2x_1}{dt^2} = \text{mass} \times \text{acceleration} = \text{force} = -x_1 + 3(x_2 - x_1)$$

$$\frac{9}{4}\frac{d^2x_2}{dt^2} = \text{mass} \times \text{acceleration} = \text{force} = -3(x_2 - x_1) - \frac{15}{4}x_2.$$

$$2\frac{d^2x_1}{dt^2} = -x_1 + 3(x_2 - x_1)$$
$$\frac{9}{4}\frac{d^2x_2}{dt^2} = -3(x_2 - x_1) - \frac{15}{4}x_2.$$

This is a system of 2 second order ODEs.

$$
\begin{aligned}
2\frac{d^2x_1}{dt^2} &= -x_1 + 3(x_2 - x_1) \\
\frac{9}{4}\frac{d^2x_2}{dt^2} &= -3(x_2 - x_1) - \frac{15}{4}x_2.
\end{aligned}
$$

This is a system of 2 second order ODEs. We want a system of first order ODEs.

$$2\frac{d^2x_1}{dt^2} = -x_1 + 3(x_2 - x_1)$$
$$\frac{9}{4}\frac{d^2x_2}{dt^2} = -3(x_2 - x_1) - \frac{15}{4}x_2.$$

Now let $y_1 = x_1$, $y_2 = x_2$, $y_3 = x_1'$ and $y_4 = x_2'$.

$$
\begin{aligned}
2\frac{d^2x_1}{dt^2} &= -x_1 + 3(x_2 - x_1) \\
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Now let $y_1 = x_1$, $y_2 = x_2$, $y_3 = x_1'$ and $y_4 = x_2'$. Then

$$
\begin{aligned}
y_1' &= x_1' = y_3 \\
y_2' &= \\
y_3' &= \\
y_4' &=
\end{aligned}
$$

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$$\begin{aligned} y_1' &= x_1' = y_3 \\ y_2' &= x_2' = y_4 \\ y_3' &= \\ y_4' &= \end{aligned}$$

$$
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$$
\begin{aligned}
y_1' &= x_1' = y_3 \\
y_2' &= x_2' = y_4 \\
y_3' &= x_1'' = \frac{1}{2}\Big(- x_1 + 3x_2 - 3x_1 \Big) = -2y_1 + \frac{3}{2}y_2 \\
y_4' &=
\end{aligned}
$$

$$\begin{aligned} 2\frac{d^2x_1}{dt^2} &= -x_1 + 3(x_2 - x_1) \\ \frac{9}{4}\frac{d^2x_2}{dt^2} &= -3(x_2 - x_1) - \frac{15}{4}x_2. \end{aligned}$$

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$$\begin{aligned} y_1' &= x_1' = y_3 \\ y_2' &= x_2' = y_4 \\ y_3' &= x_1'' = \frac{1}{2}\Big(- x_1 + 3x_2 - 3x_1\Big) = -2y_1 + \frac{3}{2}y_2 \\ y_4' &= x_2'' = \frac{4}{9}\left(-3x_2 + 3x_1 - \frac{15}{4}x_2\right) = \frac{4}{3}y_1 - 3y_2. \end{aligned}$$

So

$$\mathbf{y}' = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -2 & \frac{3}{2} & 0 & 0 \\ \frac{4}{3} & -3 & 0 & 0 \end{bmatrix} \mathbf{y}.$$

The characteristic polynomial of this matrix is

$$0 = r^4 + 5r^2 + 4 = (r^2 + 1)(r^2 + 4).$$

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So $r_1 = i$, $r_2 = -i$, $r_3 = 2i$ and $r_4 = -2i$. We will use r_1 and r_3 (we do not need r_2 and r_4).

The corresponding eigenvectors (please check) are

$$\boldsymbol{\xi}^{(1)} = \begin{bmatrix} 3 \\ 2 \\ 3i \\ 2i \end{bmatrix} \qquad \text{and} \qquad \boldsymbol{\xi}^{(3)} = \begin{bmatrix} 3 \\ -4 \\ 6i \\ -8i \end{bmatrix}.$$

It follows that

$$\boldsymbol{\xi}^{(1)} e^{r_1 t} = \begin{bmatrix} 3 \\ 2 \\ 3i \\ 2i \end{bmatrix} (\cos t + i \sin t) = \begin{bmatrix} 3\cos t \\ 2\cos t \\ -3\sin t \\ -2\sin t \end{bmatrix} + i \begin{bmatrix} 3\sin t \\ 2\sin t \\ 3\cos t \\ 2\cos t \end{bmatrix}$$

$$= \mathbf{u}(t) + i\mathbf{v}(t)$$

and

It follows that

$$\boldsymbol{\xi}^{(1)}e^{r_1 t} = \begin{bmatrix} 3 \\ 2 \\ 3i \\ 2i \end{bmatrix} (\cos t + i \sin t) = \begin{bmatrix} 3\cos t \\ 2\cos t \\ -3\sin t \\ -2\sin t \end{bmatrix} + i \begin{bmatrix} 3\sin t \\ 2\sin t \\ 3\cos t \\ 2\cos t \end{bmatrix}$$

$$= \mathbf{u}(t) + i\mathbf{v}(t)$$

and

$$\boldsymbol{\xi}^{(3)}e^{r_3 t} = \begin{bmatrix} 3 \\ -4 \\ 6i \\ -8i \end{bmatrix} (\cos 2t + i \sin 2t) = \begin{bmatrix} 3\cos 2t \\ -4\cos 2t \\ -6\sin 2t \\ +8\sin 2t \end{bmatrix} + i \begin{bmatrix} 3\sin 2t \\ -4\sin 2t \\ 6\cos 2t \\ -8\cos 2t \end{bmatrix}$$

$$= \mathbf{w}(t) + i\mathbf{z}(t)$$

Therefore the general solution is

$$\mathbf{y}(t) = c_1\mathbf{u}(t) + c_2\mathbf{v}(t) + c_3\mathbf{w}(t) + c_4\mathbf{z}(t)$$

Therefore the general solution is

$$\begin{aligned}\mathbf{y}(t) &= c_1\mathbf{u}(t) + c_2\mathbf{v}(t) + c_3\mathbf{w}(t) + c_4\mathbf{z}(t)\\ &= c_1\begin{bmatrix}3\cos t\\ 2\cos t\\ -3\sin t\\ -2\sin t\end{bmatrix} + c_2\begin{bmatrix}3\sin t\\ 2\sin t\\ 3\cos t\\ 2\cos t\end{bmatrix} + c_3\begin{bmatrix}3\cos 2t\\ -4\cos 2t\\ -6\sin 2t\\ 8\sin 2t\end{bmatrix} + c_4\begin{bmatrix}3\sin 2t\\ -4\sin 2t\\ 6\cos 2t\\ -8\cos 2t\end{bmatrix}.\end{aligned}$$

Example

Suppose that the above system has initial condition

$$\mathbf{y}(0) = \begin{bmatrix} -1 \\ 4 \\ 1 \\ 1 \end{bmatrix}.$$

Sketch graphs of $y_1(t)$ and $y_2(t)$.

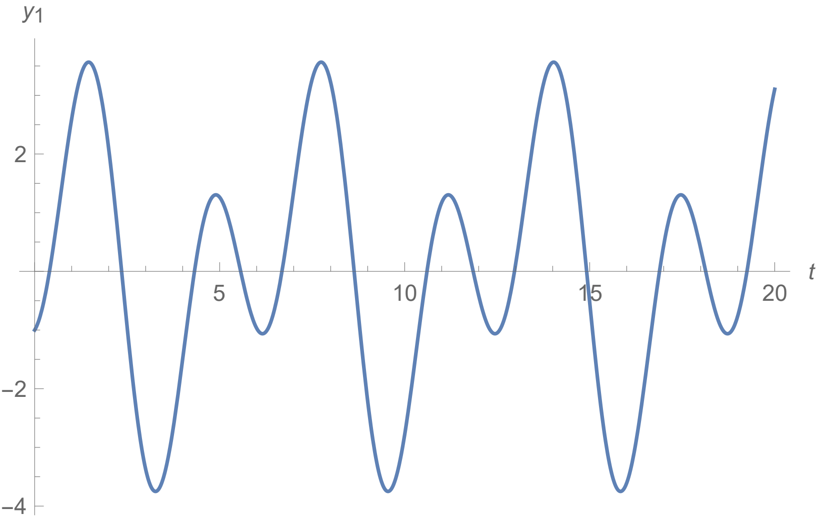
The initial value problem

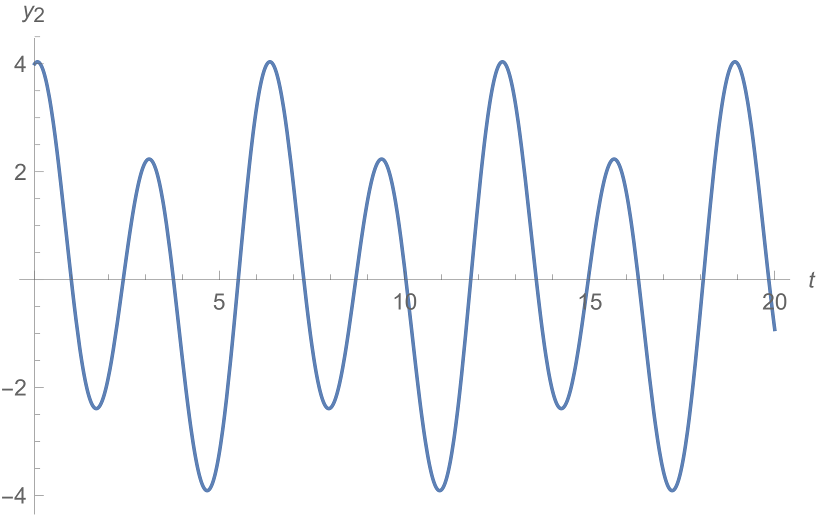
$$\mathbf{y}' = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -2 & \frac{3}{2} & 0 & 0 \\ \frac{4}{3} & -3 & 0 & 0 \end{bmatrix} \mathbf{y}, \qquad \mathbf{y}(0) = \begin{bmatrix} -1 \\ 4 \\ 1 \\ 1 \end{bmatrix}$$

has solution

$$\mathbf{y}(t) = \frac{4}{9} \begin{bmatrix} 3\cos t \\ 2\cos t \\ -3\sin t \\ -2\sin t \end{bmatrix} + \frac{7}{18} \begin{bmatrix} 3\sin t \\ 2\sin t \\ 3\cos t \\ 2\cos t \end{bmatrix} - \frac{7}{9} \begin{bmatrix} 3\cos 2t \\ -4\cos 2t \\ -6\sin 2t \\ 8\sin 2t \end{bmatrix} - \frac{1}{36} \begin{bmatrix} 3\sin 2t \\ -4\sin 2t \\ 6\cos 2t \\ -8\cos 2t \end{bmatrix}.$$

Then we can draw the graphs of y_1 and y_2:


y_1
2
−2
−4
5
10
15
20
t



y_2
4
2
−2
−4
5
10
15
20
t

Please see `https://tinyurl.com/s7uww7m`

Next Week

- 5.5 Fundamental Matrices
- 5.6 Repeated Eigenvalues