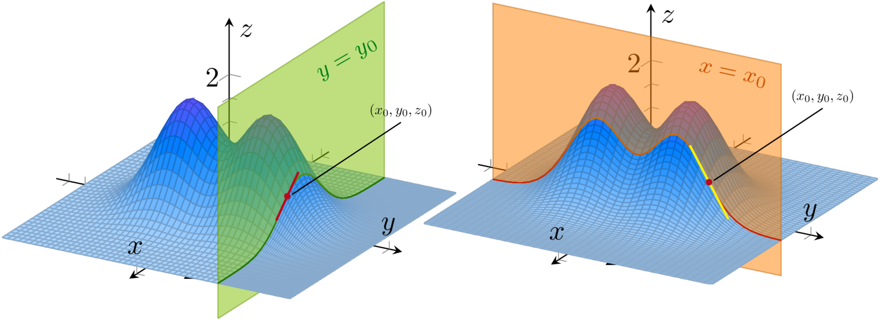
Lecture 6

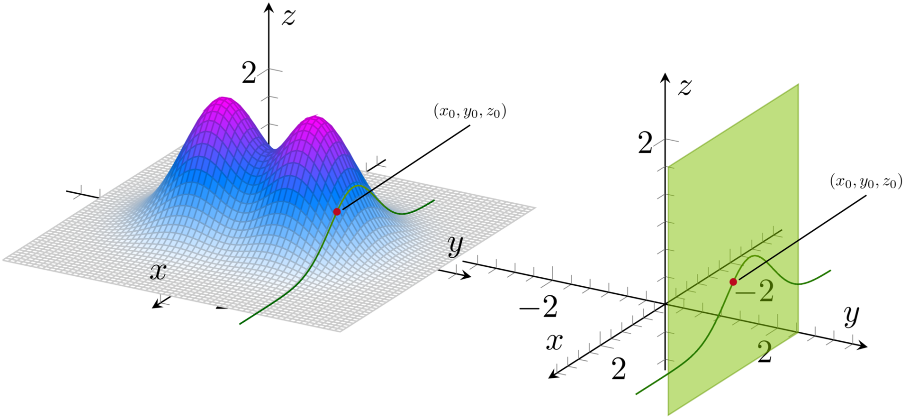
- 13.5 Directional Derivatives and Gradient Vectors
- 13.6 Tangent Planes and Differentials
- 13.7 Extreme Values and Saddle Points
- 13.8 Lagrange Multipliers

Directional Derivatives and Gradient Vectors

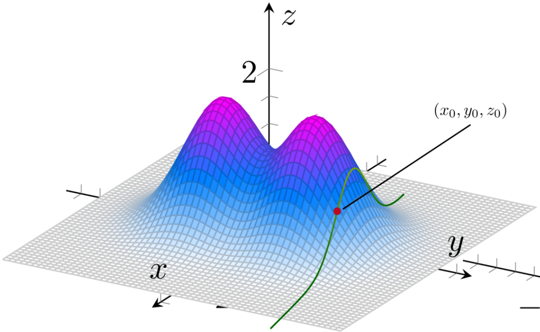
Partial Derivatives (revision)

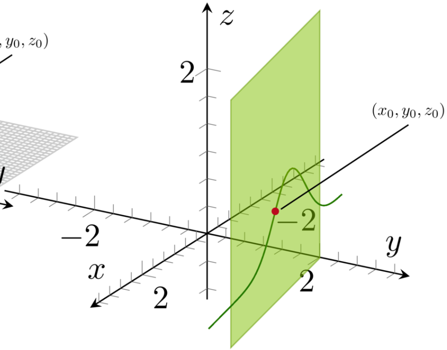


Directional Derivatives



Directional Derivatives





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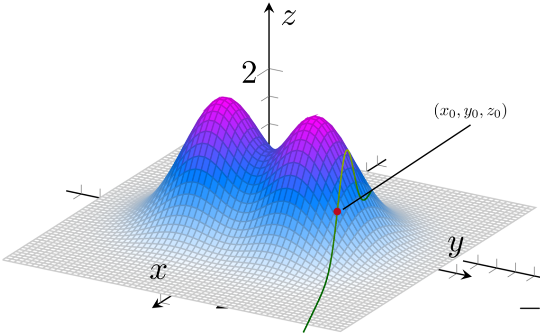

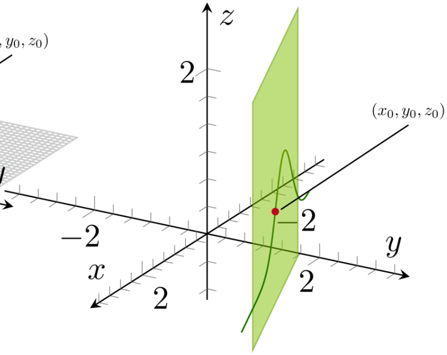

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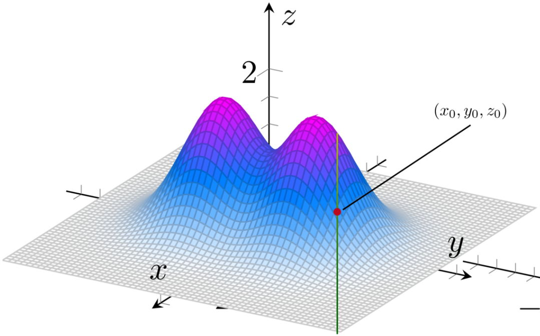
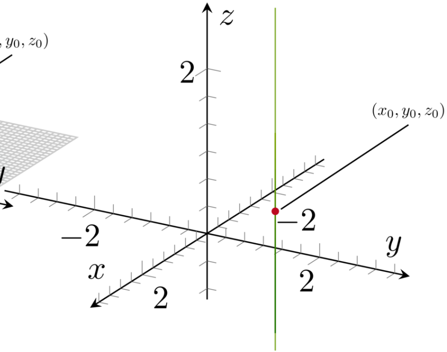
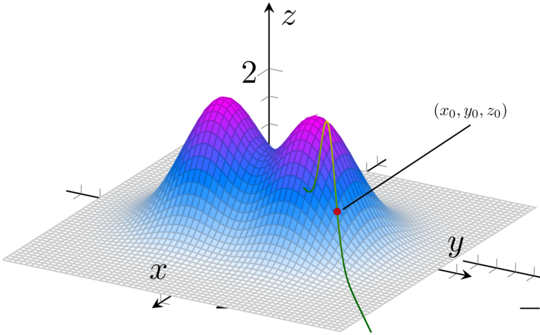
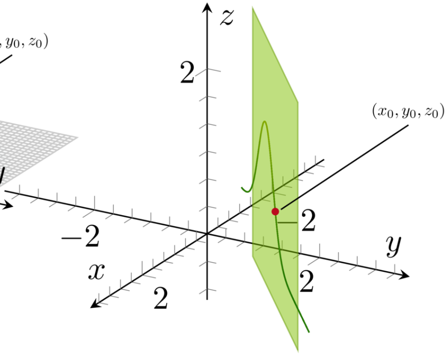




Directional Derivatives





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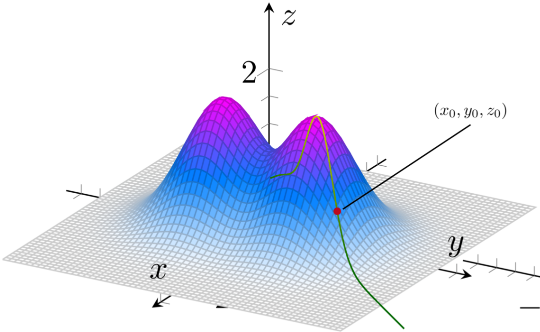

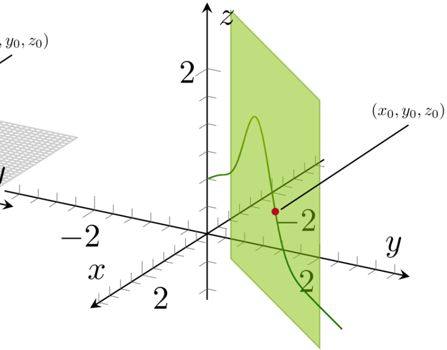

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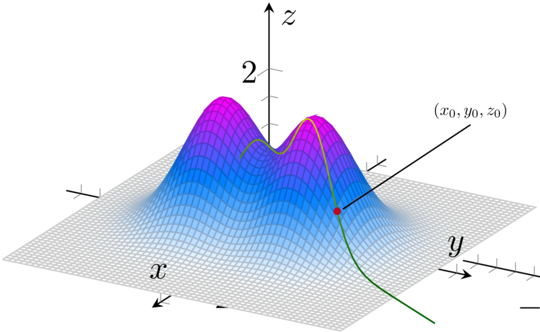

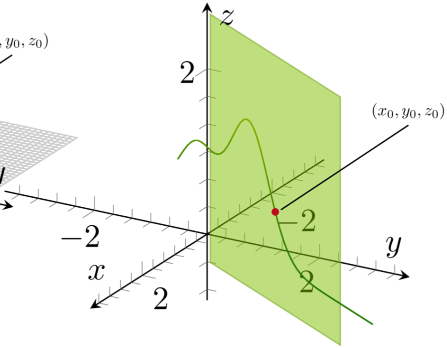

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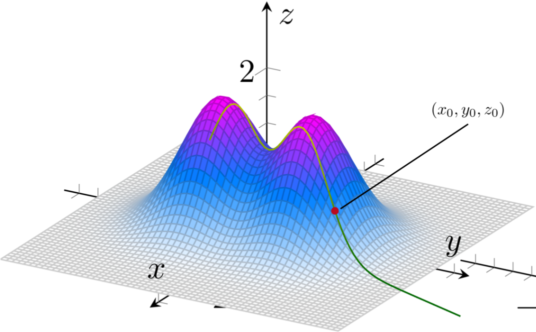

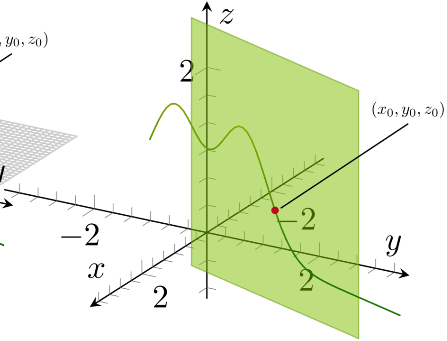

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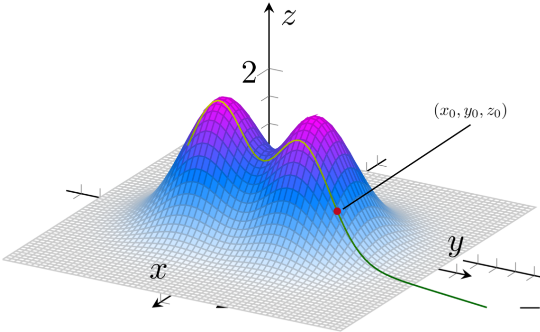

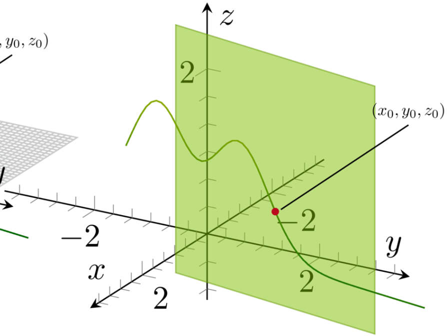

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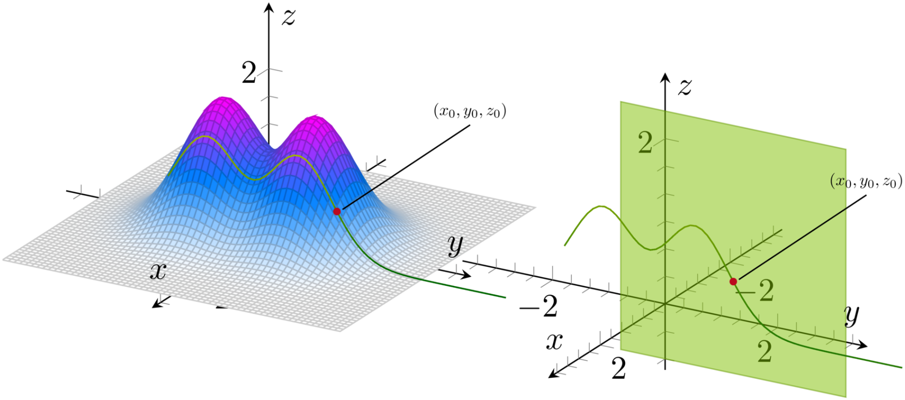

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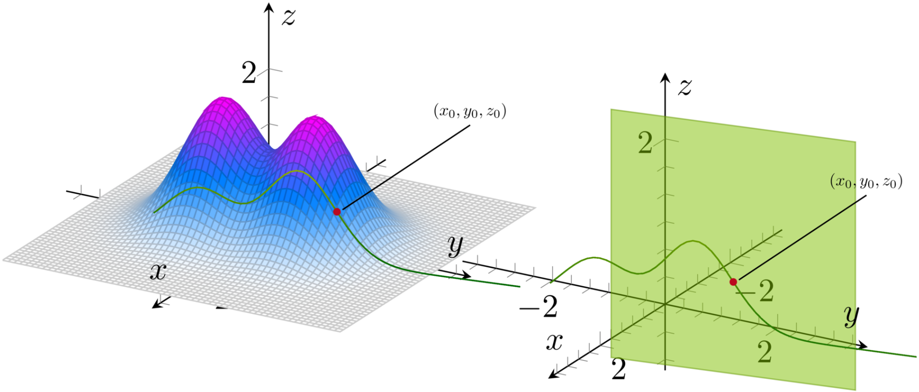

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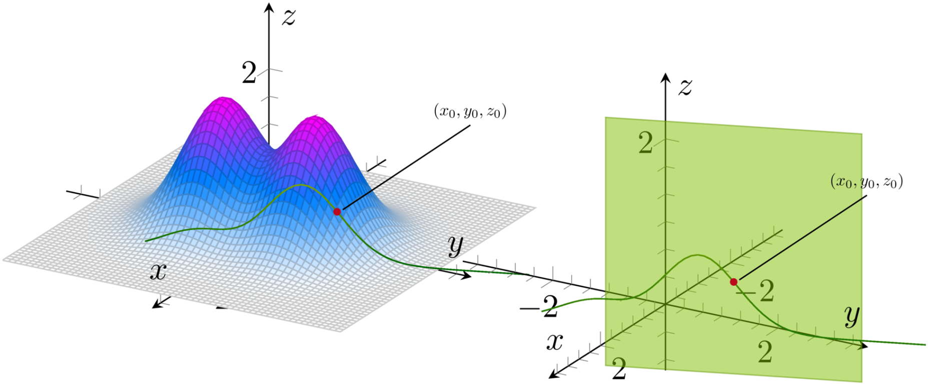

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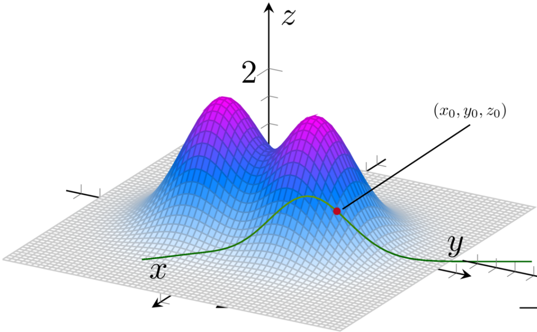

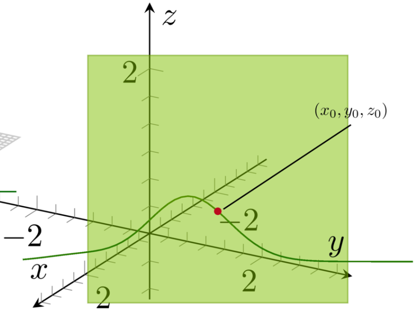

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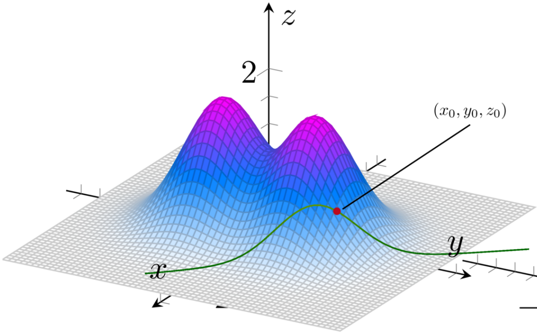

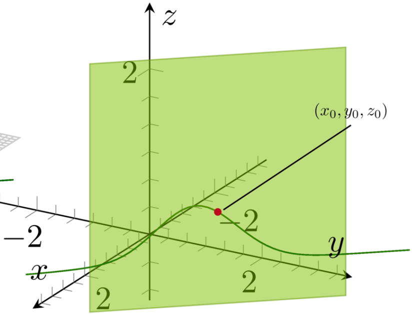

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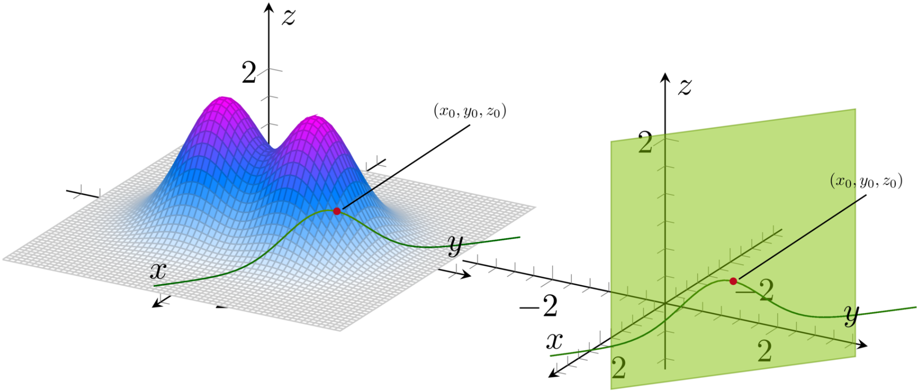

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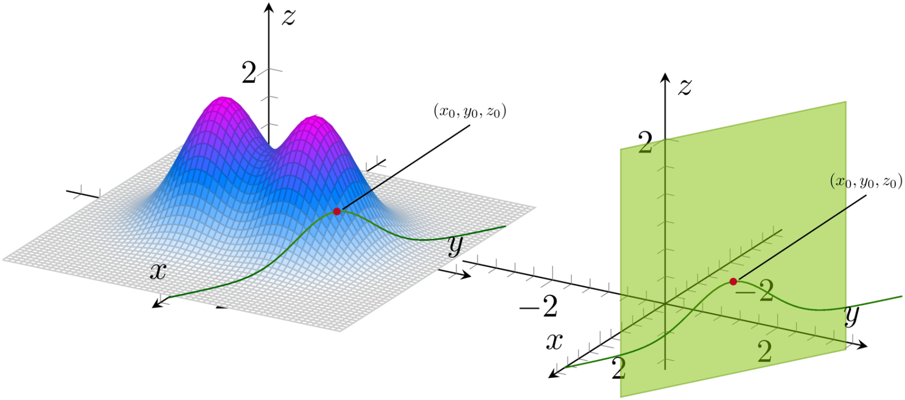
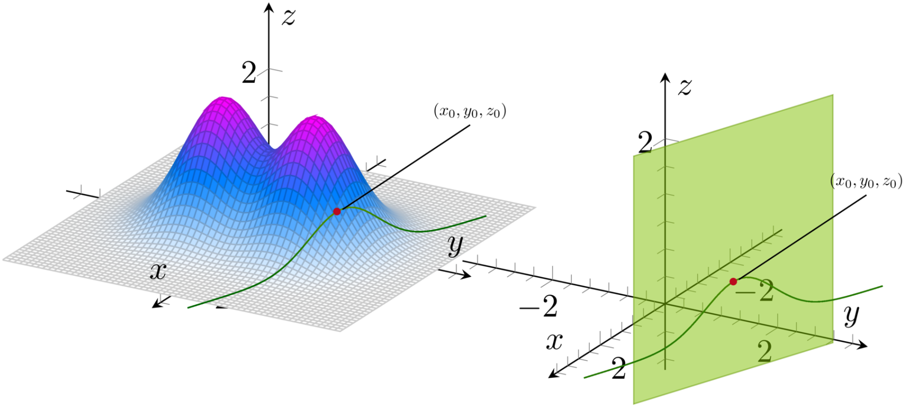
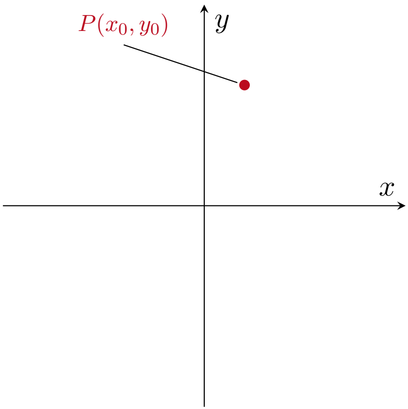


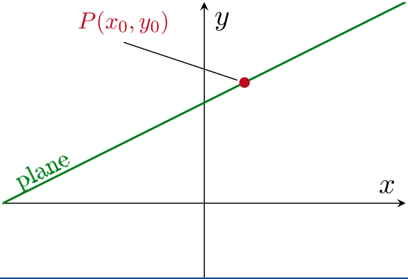
Directional Derivatives



top view

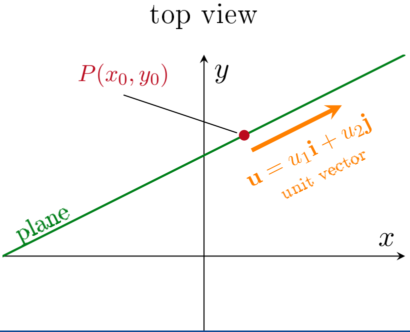


top view



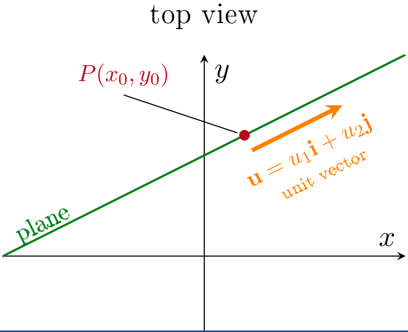
Definition

The *derivative of* f *at* $P(x_0, y_0)$



Definition

The *derivative of f at $P(x_0, y_0)$ in the direction of the unit vector* $\mathbf{u} = u_1\mathbf{i} + u_2\mathbf{j}$



Definition

The *derivative of f at $P(x_0, y_0)$ in the direction of the unit vector* $\mathbf{u} = u_1\mathbf{i} + u_2\mathbf{j}$ is

$$D_{\mathbf{u}} f(P_0) = \lim_{s \to 0} \frac{f(x_0 + su_1, y_0 + su_2) - f(x_0, y_0)}{s}$$

provided the limit exists.

$$D_{\mathbf{u}} f(P_0) = \left(\frac{df}{ds}\right)_{\mathbf{u},\ P_0}$$

EXAMPLE 1 Using the definition, find the derivative of

$$f(x, y) = x^2 + xy$$

at $P_0(1, 2)$ in the direction of the unit vector $\mathbf{u} = \left(1/\sqrt{2}\right)\mathbf{i} + \left(1/\sqrt{2}\right)\mathbf{j}$.

Solution Applying the definition in Equation (1), we obtain

$$\left(\frac{df}{ds}\right)_{\mathbf{u},P_0} = \lim_{s\to 0} \frac{f(x_0 + su_1, y_0 + su_2) - f(x_0, y_0)}{s} \qquad \text{Eq. (1)}$$

$$= \lim_{s\to 0} \frac{f\left(1 + s\cdot\frac{1}{\sqrt{2}}, 2 + s\cdot\frac{1}{\sqrt{2}}\right) - f(1, 2)}{s}$$

$$= \lim_{s\to 0} \frac{\left(1 + \frac{s}{\sqrt{2}}\right)^2 + \left(1 + \frac{s}{\sqrt{2}}\right)\left(2 + \frac{s}{\sqrt{2}}\right) - (1^2 + 1\cdot 2)}{s}$$

$$= \lim_{s\to 0} \frac{\left(1 + \frac{2s}{\sqrt{2}} + \frac{s^2}{2}\right) + \left(2 + \frac{3s}{\sqrt{2}} + \frac{s^2}{2}\right) - 3}{s}$$

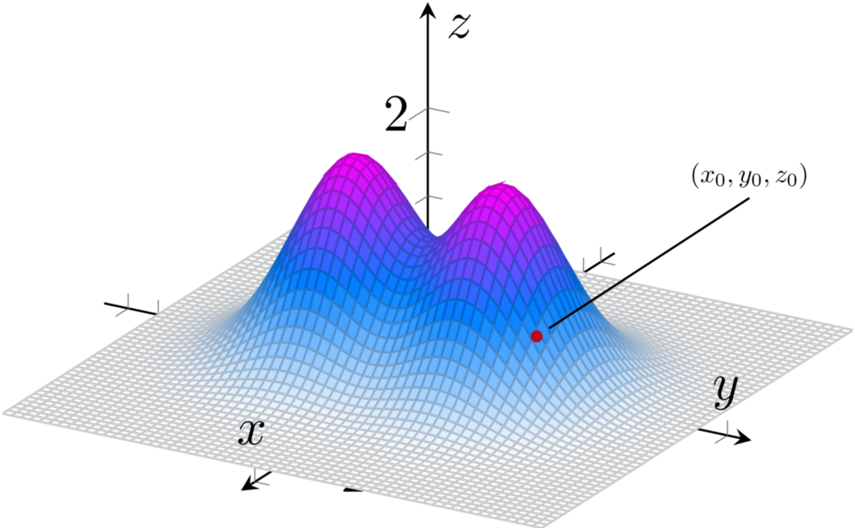
$$= \lim_{s\to 0} \frac{\frac{5s}{\sqrt{2}} + s^2}{s} = \lim_{s\to 0}\left(\frac{5}{\sqrt{2}} + s\right) = \frac{5}{\sqrt{2}}.$$

The rate of change of $f(x, y) = x^2 + xy$ at $P_0(1, 2)$ in the direction $\mathbf{u}$ is $5/\sqrt{2}$.

Remark

But it is easier to calculate directional derivatives if we use gradient vectors.

What is a Gradient Vector?



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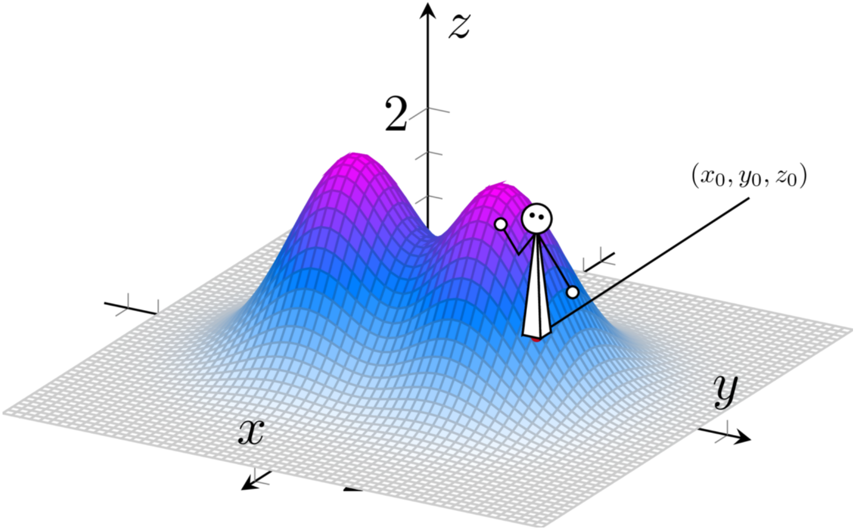

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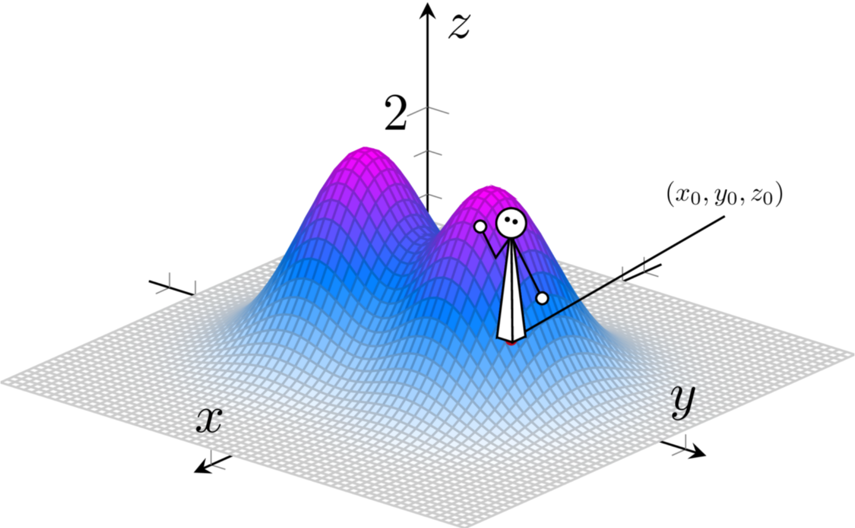

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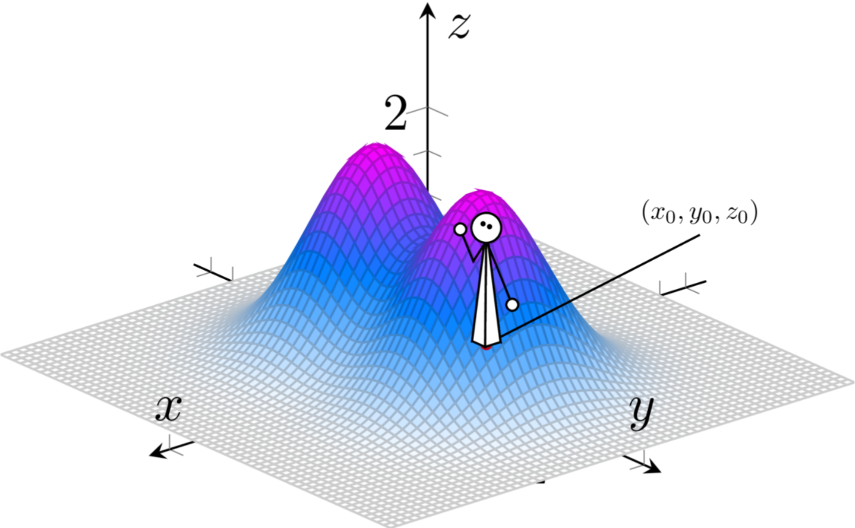

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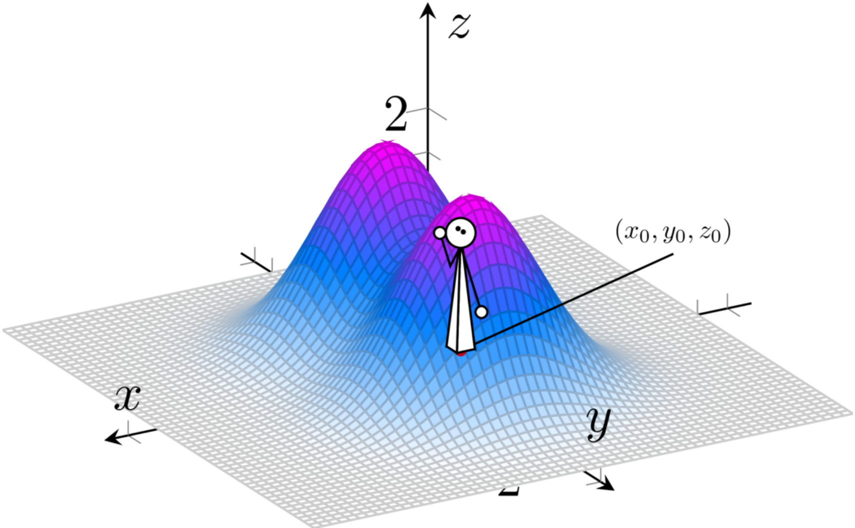

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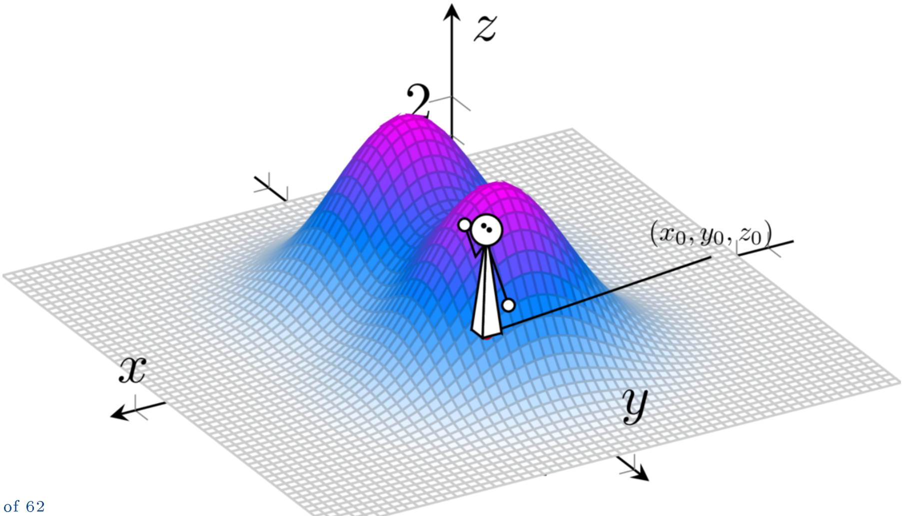

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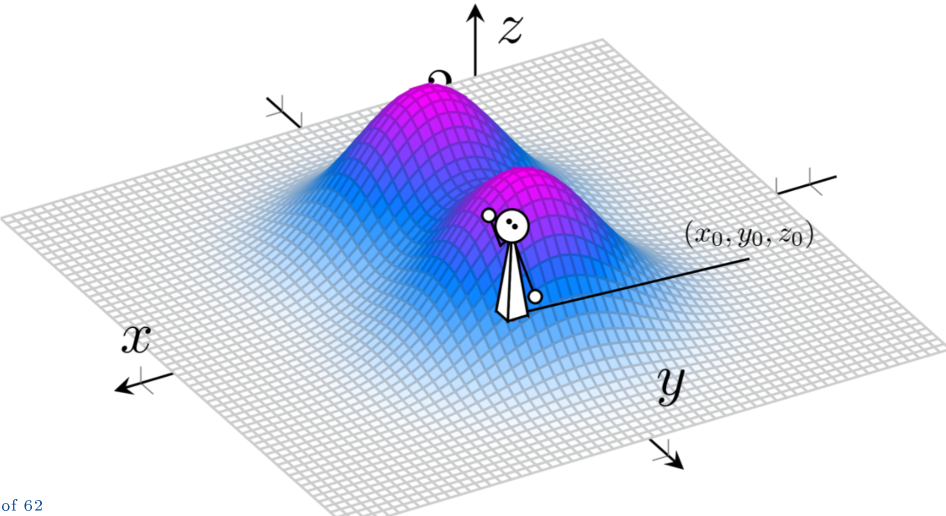

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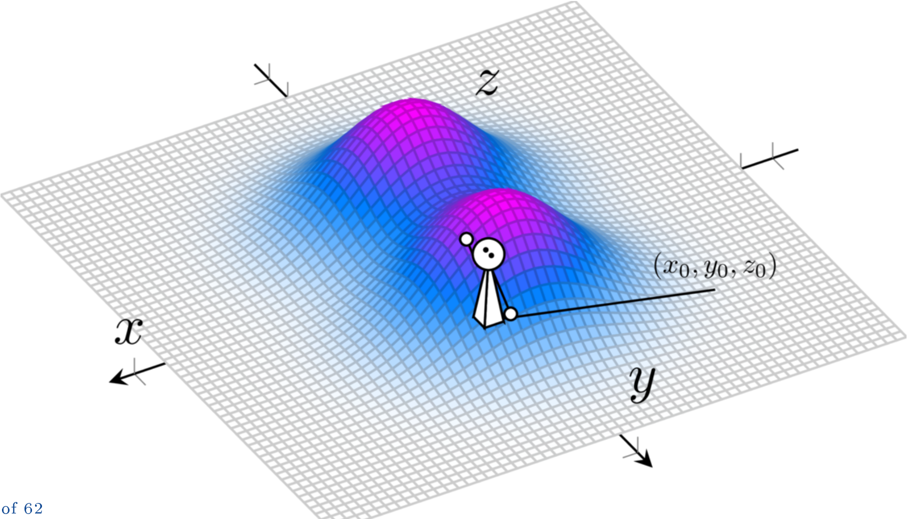

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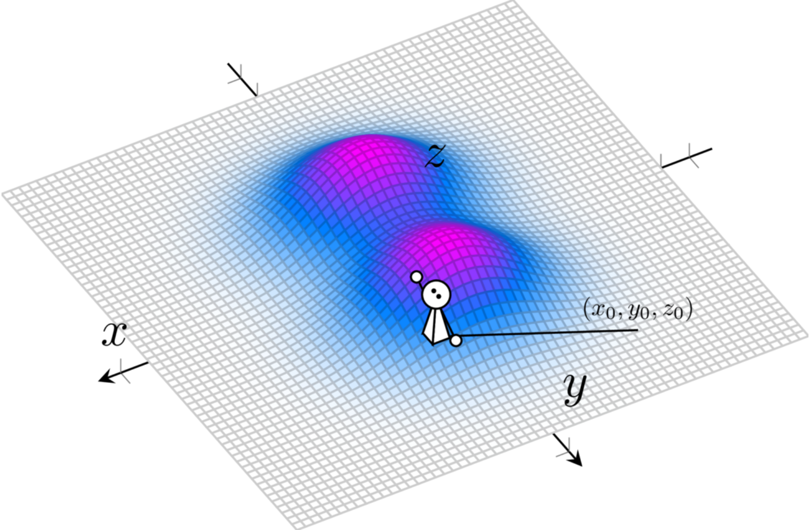

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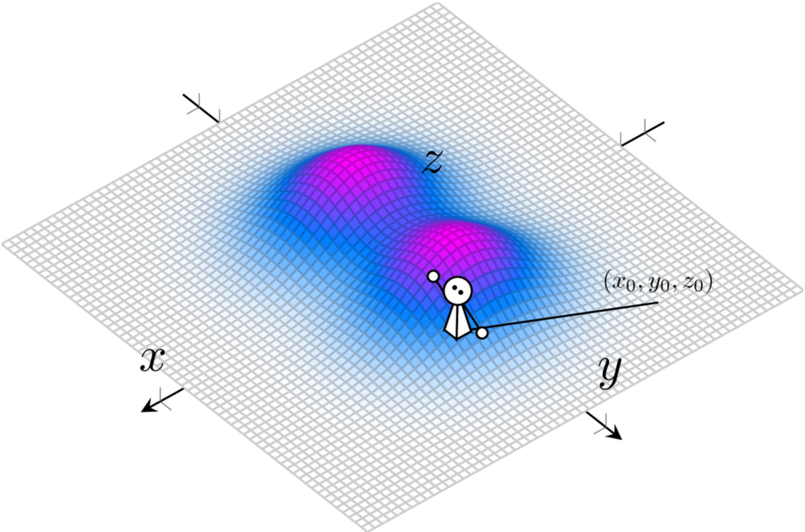

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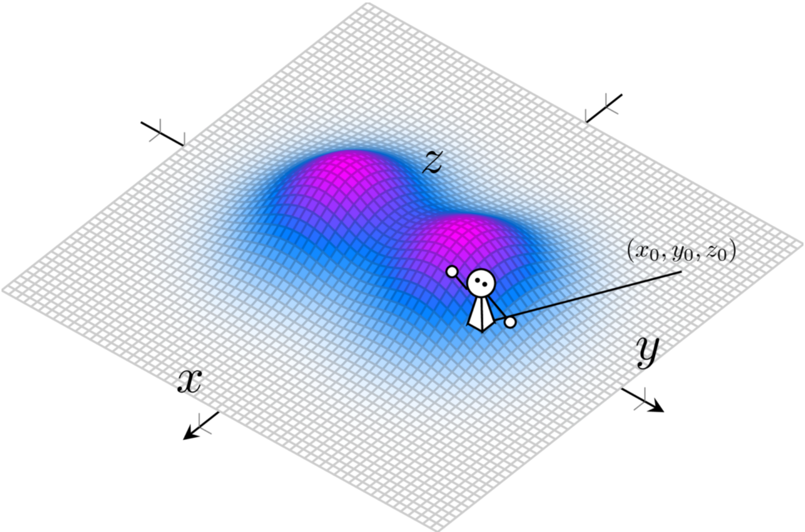

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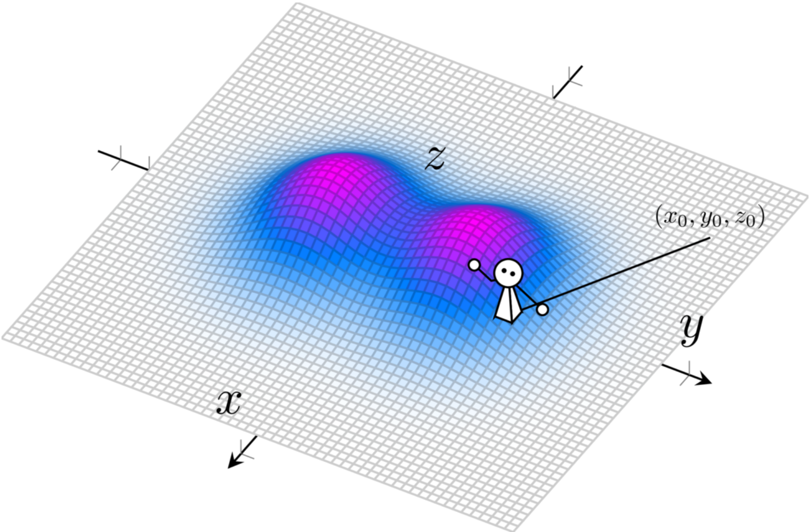

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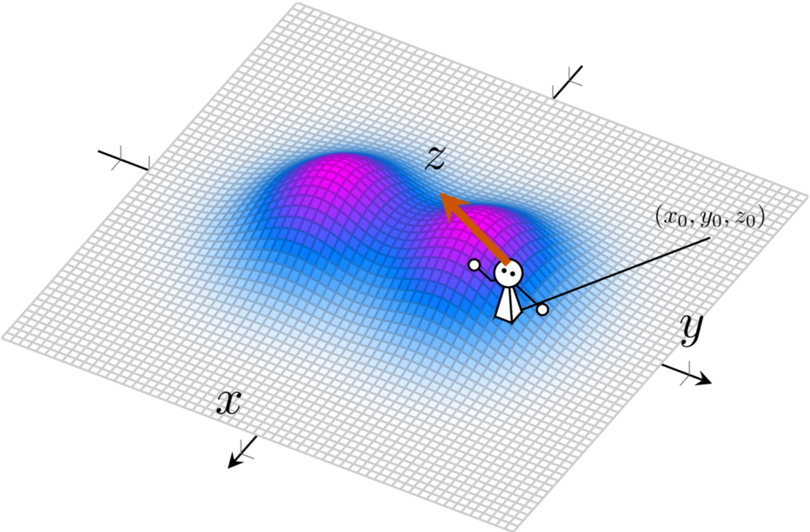

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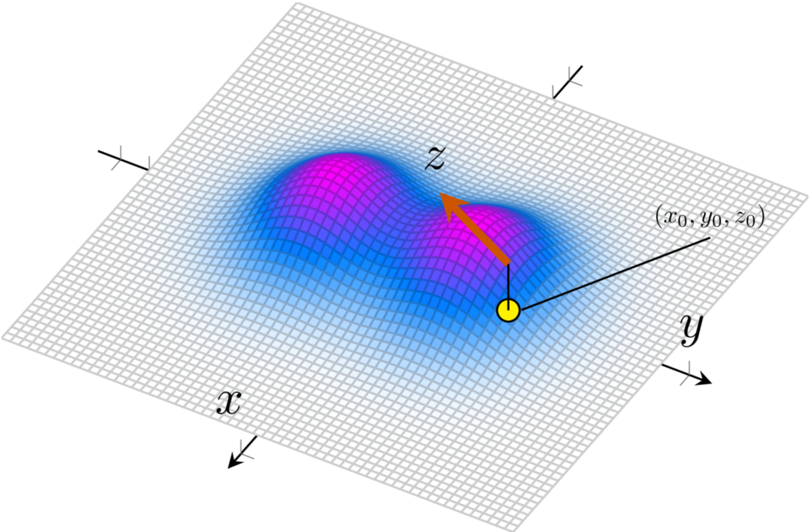

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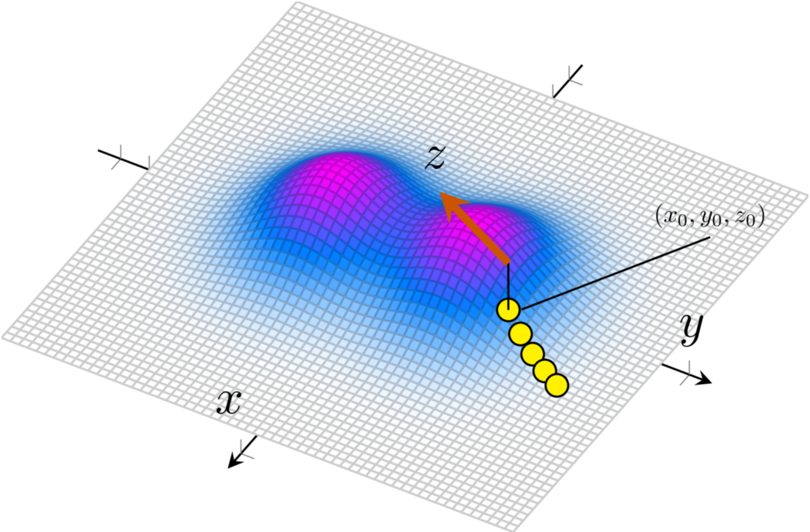

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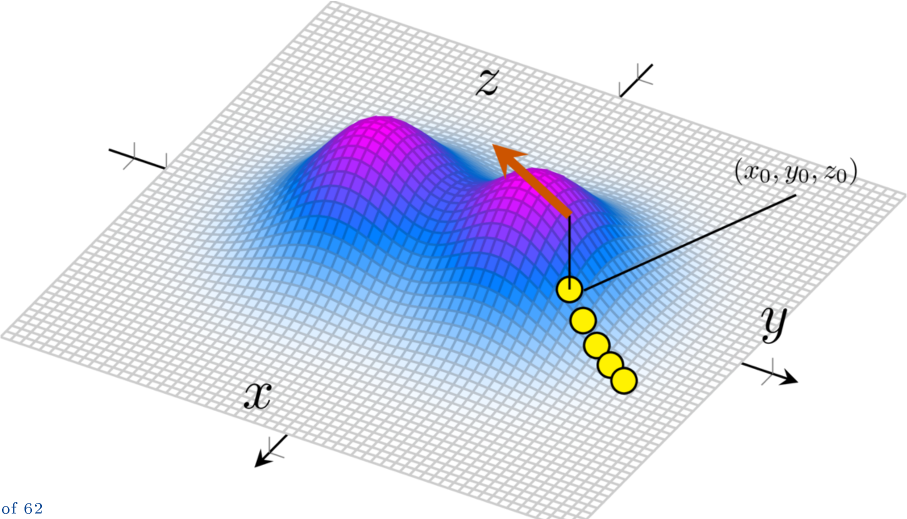

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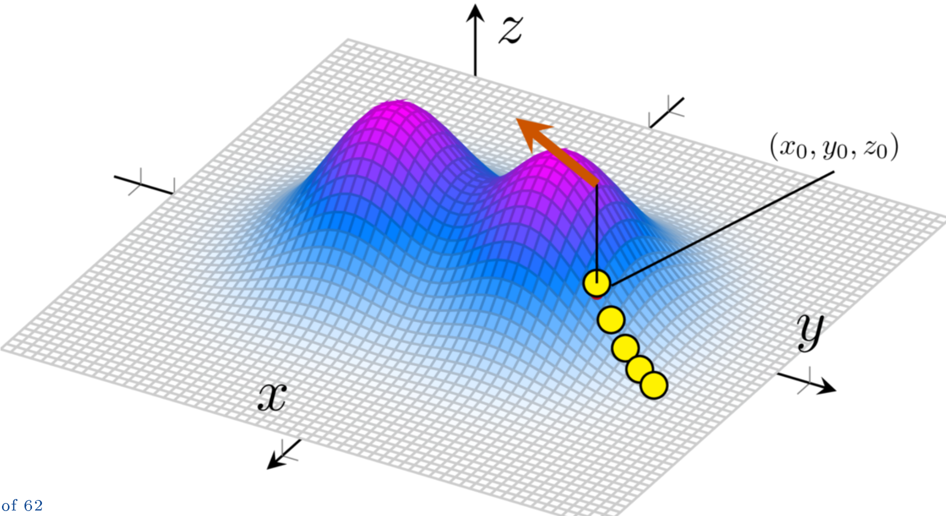

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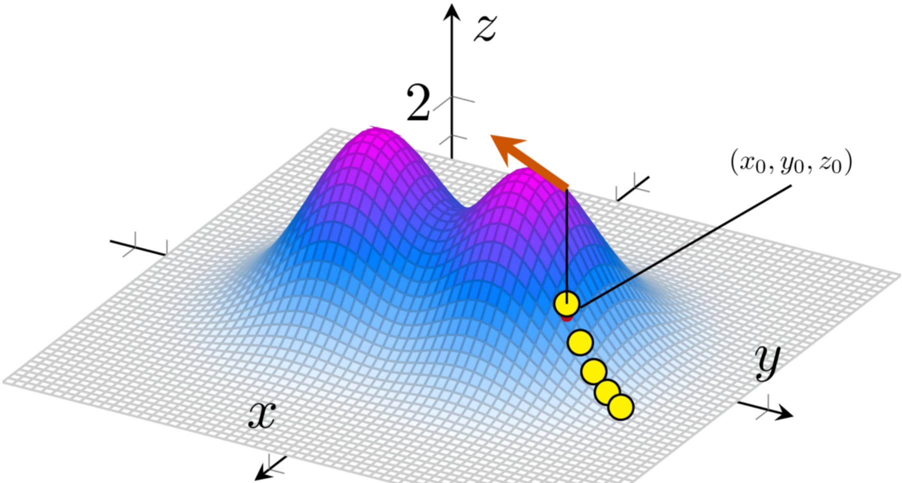

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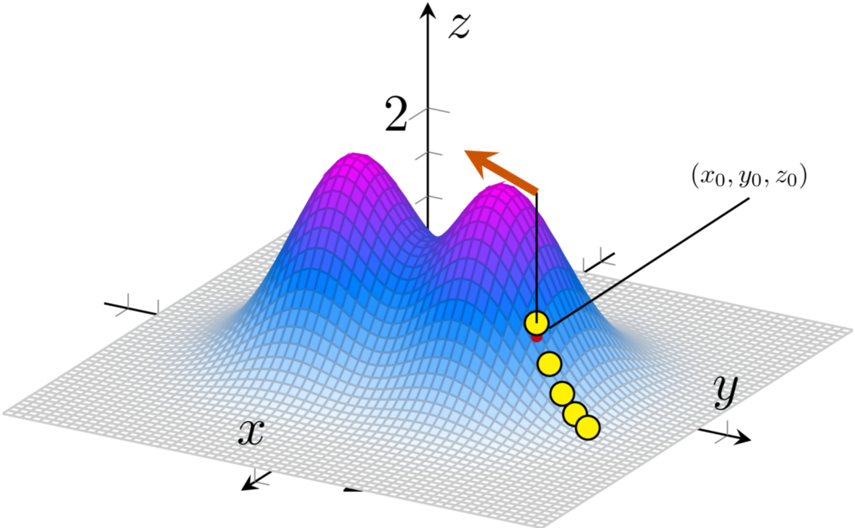

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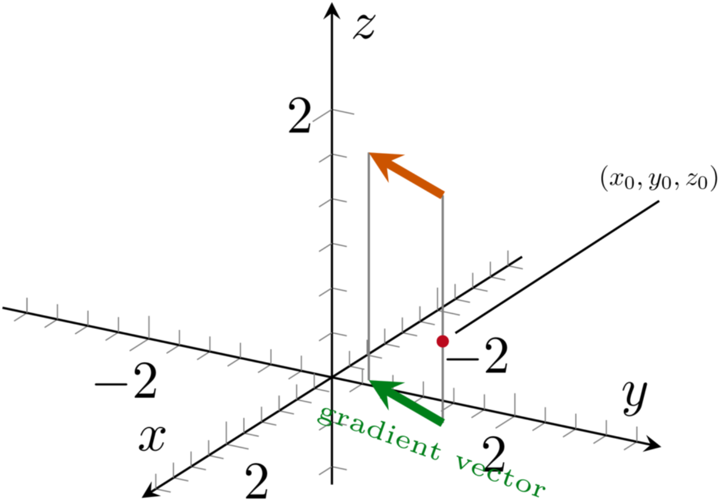

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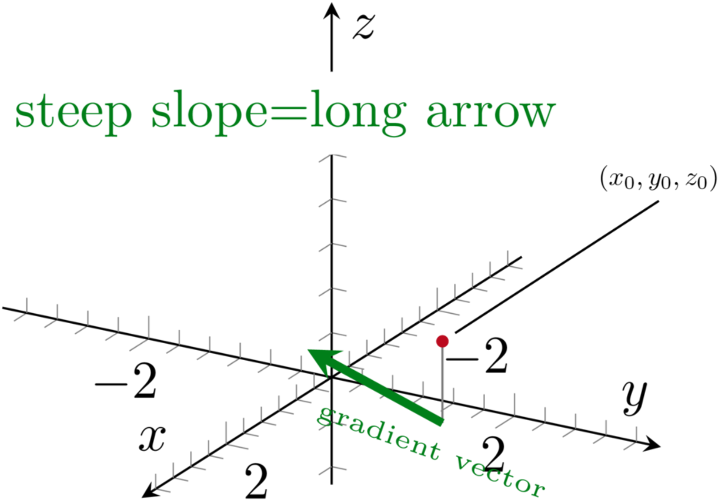

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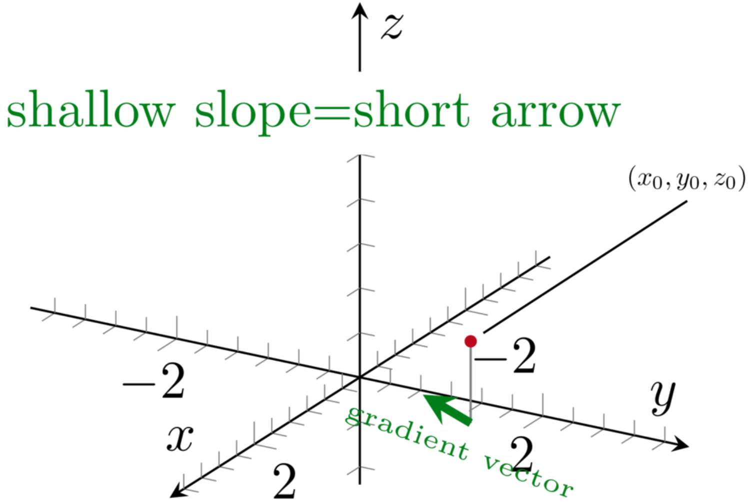

Definition

The *gradient vector* of $f(x,y)$ is the vector

$$\nabla f = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j}.$$

∇ is pronounced “nabla” or “del”.

Harps, p. 984.

$$\nabla f = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j}$$

Example

Find the gradient vector of $f(x, y) = xe^y + \cos(xy)$ at the point $(2, 0)$.

$$\nabla f = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j}$$

Example

Find the gradient vector of $f(x,y) = xe^y + \cos(xy)$ at the point $(2,0)$.

We calculate that

$$f_x(2,0) =$$

$$f_y(2,0) =$$

and

$$\nabla f\big|_{(2,0)} = f_x(2,0)\mathbf{i} + f_y(2,0)\mathbf{j} = \qquad .$$

$$\nabla f = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j}$$

Example

Find the gradient vector of $f(x, y) = xe^y + \cos(xy)$ at the point $(2, 0)$.

We calculate that

$$f_x(2,0) = \left. e^y - y\sin(xy) \right|_{(2,0)} = e^0 - 0 = 1,$$
$$f_y(2,0) =$$

and

$$\left.\nabla f\right|_{(2,0)} = f_x(2,0)\mathbf{i} + f_y(2,0)\mathbf{j} = \qquad .$$

$$\nabla f = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j}$$

Example

Find the gradient vector of $f(x, y) = xe^y + \cos(xy)$ at the point $(2, 0)$.

We calculate that

$$f_x(2,0) = e^y - y\sin(xy)\big|_{(2,0)} = e^0 - 0 = 1,$$
$$f_y(2,0) = xe^y - x\sin(xy)\big|_{(2,0)} = 2e^0 - 2\cdot 0 = 2$$

and

$$\nabla f\big|_{(2,0)} = f_x(2,0)\mathbf{i} + f_y(2,0)\mathbf{j} = \qquad .$$

$$\nabla f = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j}$$

Example

Find the gradient vector of $f(x,y) = xe^y + \cos(xy)$ at the point $(2,0)$.

We calculate that

$$\begin{aligned} f_x(2,0) &= \left. e^y - y\sin(xy)\right|_{(2,0)} = e^0 - 0 = 1, \\ f_y(2,0) &= \left. xe^y - x\sin(xy)\right|_{(2,0)} = 2e^0 - 2\cdot 0 = 2 \end{aligned}$$

and

$$\left.\nabla f\right|_{(2,0)} = f_x(2,0)\mathbf{i} + f_y(2,0)\mathbf{j} = \mathbf{i} + 2\mathbf{j}.$$

Theorem

$$D_{\mathbf{u}}f = \nabla f \bullet \mathbf{u}.$$

$$D_{\mathbf{u}}f = \nabla f \cdot \mathbf{u} \qquad \nabla f = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j}$$

Example

Find the derivative of $f(x, y) = xe^y + \cos(xy)$ at the point $(2, 0)$ in the direction of $\mathbf{v} = 3\mathbf{i} - 4\mathbf{j}$.

$$D_{\mathbf{u}}f = \nabla f \bullet \mathbf{u} \qquad \nabla f = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j}$$

Example

Find the derivative of $f(x,y) = xe^y + \cos(xy)$ at the point $(2,0)$ in the direction of $\mathbf{v} = 3\mathbf{i} - 4\mathbf{j}$.

Recall that $\nabla f|_{(2,0)} = \mathbf{i} + 2\mathbf{j}$.

$$D_{\mathbf{u}}f = \nabla f \cdot \mathbf{u} \qquad \nabla f = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j}$$

Example

Find the derivative of $f(x, y) = xe^y + \cos(xy)$ at the point $(2, 0)$ in the direction of $\mathbf{v} = 3\mathbf{i} - 4\mathbf{j}$.

Recall that $\nabla f|_{(2,0)} = \mathbf{i} + 2\mathbf{j}$. We need to find a unit vector $\mathbf{u}$ which points in the same direction as $\mathbf{v}$,

$$D_{\mathbf{u}}f = \nabla f \bullet \mathbf{u} \qquad \nabla f = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j}$$

Example

Find the derivative of $f(x, y) = xe^y + \cos(xy)$ at the point $(2, 0)$ in the direction of $\mathbf{v} = 3\mathbf{i} - 4\mathbf{j}$.

Recall that $\nabla f|_{(2,0)} = \mathbf{i} + 2\mathbf{j}$. We need to find a unit vector $\mathbf{u}$ which points in the same direction as $\mathbf{v}$, so we calculate that

$$\mathbf{u} = \frac{\mathbf{v}}{\|\mathbf{v}\|} = \frac{3\mathbf{i} - 4\mathbf{j}}{\sqrt{3^2 + (-4)^2}} = \frac{3}{5}\mathbf{i} - \frac{4}{5}\mathbf{j}.$$

$$D_{\mathbf{u}}f = \nabla f \cdot \mathbf{u} \qquad \nabla f = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j}$$

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Therefore

$$D_{\mathbf{u}}f(2,0) = \nabla f|_{(2,0)} \cdot \mathbf{u} =$$

$$D_{\mathbf{u}}f = \nabla f \cdot \mathbf{u} \qquad \nabla f = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j}$$

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Therefore

$$D_{\mathbf{u}}f(2,0) = \nabla f|_{(2,0)} \cdot \mathbf{u} = (\mathbf{i} + 2\mathbf{j}) \cdot \left(\frac{3}{5}\mathbf{i} - \frac{4}{5}\mathbf{j}\right) = \frac{3}{5} - \frac{8}{5} = -1.$$

$$\boxed{\mathbf{a} \bullet \mathbf{b} = \|\mathbf{a}\| \, \|\mathbf{b}\| \cos\theta}$$

Note that

$$D_{\mathbf{u}} f = \nabla f \bullet \mathbf{u} = \|\nabla f\| \, \|\mathbf{u}\| \cos\theta = \|\nabla f\| \cos\theta$$

since $\|\mathbf{u}\| = 1$.

$$\mathbf{a} \bullet \mathbf{b} = \|\mathbf{a}\| \, \|\mathbf{b}\| \cos\theta$$

Note that

$$D_{\mathbf{u}}f = \nabla f \bullet \mathbf{u} = \|\nabla f\| \, \|\mathbf{u}\| \cos\theta = \|\nabla f\| \cos\theta$$

since $\|\mathbf{u}\| = 1$.

So we must always have

$$-\|\nabla f\| \leq D_{\mathbf{u}}f \leq \|\nabla f\| .$$

$$D_{\mathbf{u}} f = \|\nabla f\| \cos\theta$$

Remark

f increases mostly rapidly $\implies \cos\theta = 1 \implies \theta = 0 \implies$ **u** points in the same direction as ∇f

∇f points 'uphill'

$$\boxed{D_{\mathbf{u}} f = \|\nabla f\| \cos\theta}$$

Remark

f increases mostly rapidly $\implies \cos\theta = 1 \implies \theta = 0 \implies$ $\mathbf{u}$ points in the same direction as ∇f

∇f points 'uphill'

Remark

f decreases mostly rapidly $\implies \cos\theta = -1 \implies \theta = 180^\circ \implies$ $\mathbf{u}$ points in the opposite direction from ∇f

a ball on a hill rolls in the direction $-\nabla f$

$$D_{\mathbf{u}} f = \|\nabla f\| \cos\theta$$

Remark

f increases mostly rapidly $\implies \cos\theta = 1 \implies \theta = 0 \implies$ $\mathbf{u}$ points in the same direction as ∇f

∇f points 'uphill'

Remark

f decreases mostly rapidly $\implies \cos\theta = -1 \implies \theta = 180^\circ \implies$ $\mathbf{u}$ points in the opposite direction from ∇f

a ball on a hill rolls in the direction $-\nabla f$

Remark

$$\theta = 90^\circ \implies D_{\mathbf{u}} f = 0.$$

EXAMPLE 3 Find the directions in which $f(x, y) = (x^2/2) + (y^2/2)$

(a) increases most rapidly at the point (1, 1).

(b) decreases most rapidly at (1, 1).

(c) What are the directions of zero change in f at (1, 1)?

Solution

(a) The function increases most rapidly in the direction of ∇f at (1, 1). The gradient there is

$$(\nabla f)_{(1,1)} = (x\mathbf{i} + y\mathbf{j})_{(1,1)} = \mathbf{i} + \mathbf{j}.$$

Its direction is

$$\mathbf{u} = \frac{\mathbf{i} + \mathbf{j}}{|\mathbf{i} + \mathbf{j}|} = \frac{\mathbf{i} + \mathbf{j}}{\sqrt{(1)^2 + (1)^2}} = \frac{1}{\sqrt{2}}\mathbf{i} + \frac{1}{\sqrt{2}}\mathbf{j}.$$

(b) The function decreases most rapidly in the direction of $-\nabla f$ at $(1, 1)$, which is

$$-\mathbf{u} = -\frac{1}{\sqrt{2}}\mathbf{i} - \frac{1}{\sqrt{2}}\mathbf{j}.$$

(c) The directions of zero change at $(1, 1)$ are the directions orthogonal to ∇f:

$$\mathbf{n} = -\frac{1}{\sqrt{2}}\mathbf{i} + \frac{1}{\sqrt{2}}\mathbf{j} \qquad \text{and} \qquad -\mathbf{n} = \frac{1}{\sqrt{2}}\mathbf{i} - \frac{1}{\sqrt{2}}\mathbf{j}.$$

$$D_{\mathbf{u}} f = \|\nabla f\| \cos\theta$$

Algebra Rules for ∇

Theorem

1. *Sum Rule:* $\nabla(f+g) = \nabla f + \nabla g$
2. *Difference Rule:* $\nabla(f+g) = \nabla f + \nabla g$
3. *Constant Multiple Rule:* $\nabla(kf) = k\nabla f$ *(for $k \in \mathbb{R}$)*
4. *Product Rule:* $\nabla(fg) = g\nabla f + f\nabla g$
5. *Quotient Rule:* $\nabla\left(\frac{f}{g}\right) = \frac{g\nabla f - f\nabla g}{g^2}$.

EXAMPLE 5 We illustrate two of the rules with

$$f(x, y) = x - y \qquad g(x, y) = 3y$$
$$\nabla f = \mathbf{i} - \mathbf{j} \qquad \nabla g = 3\mathbf{j}.$$

We have

1. $\nabla(f - g) = \nabla(x - 4y) = \mathbf{i} - 4\mathbf{j} = \nabla f - \nabla g$ Rule 2
2. $\nabla(fg) = \nabla(3xy - 3y^2) = 3y\mathbf{i} + (3x - 6y)\mathbf{j}$

   and

   $$f\nabla g + g\nabla f = (x - y)3\mathbf{j} + 3y(\mathbf{i} - \mathbf{j}) \quad \text{Substitute.}$$
   $$= 3y\mathbf{i} + (3x - 6y)\mathbf{j}. \quad \text{Simplify.}$$

   We have therefore verified for this example that $\nabla(fg) = f\nabla g + g\nabla f$.

Functions of Three Variables

$$\nabla f = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} + \frac{\partial f}{\partial z}\mathbf{k}$$

$$D_{\mathbf{u}}f = \nabla f \cdot \mathbf{u}.$$

EXAMPLE 6

(a) Find the derivative of $f(x, y, z) = x^3 - xy^2 - z$ at $P_0(1, 1, 0)$ in the direction of $\mathbf{v} = 2\mathbf{i} - 3\mathbf{j} + 6\mathbf{k}$.

(b) In what directions does f change most rapidly at P_0, and what are the rates of change in these directions?

Solution

(a) The direction of $\mathbf{v}$ is obtained by dividing $\mathbf{v}$ by its length:

$$|\mathbf{v}| = \sqrt{(2)^2 + (-3)^2 + (6)^2} = \sqrt{49} = 7$$

$$\mathbf{u} = \frac{\mathbf{v}}{|\mathbf{v}|} = \frac{2}{7}\mathbf{i} - \frac{3}{7}\mathbf{j} + \frac{6}{7}\mathbf{k}.$$

The partial derivatives of f at P_0 are

$$f_x = (3x^2 - y^2)\Big|_{(1,1,0)} = 2, \qquad f_y = -2xy\Big|_{(1,1,0)} = -2, \qquad f_z = -1\Big|_{(1,1,0)} = -1.$$

The gradient of f at P_0 is

$$\nabla f\big|_{(1,1,0)} = 2\mathbf{i} - 2\mathbf{j} - \mathbf{k}.$$

The derivative of f at P_0 in the direction of $\mathbf{v}$ is therefore

$$D_{\mathbf{u}}f\big|_{(1,1,0)} = \nabla f\big|_{(1,1,0)} \cdot \mathbf{u} = (2\mathbf{i} - 2\mathbf{j} - \mathbf{k}) \cdot \left(\frac{2}{7}\mathbf{i} - \frac{3}{7}\mathbf{j} + \frac{6}{7}\mathbf{k}\right)$$

$$= \frac{4}{7} + \frac{6}{7} - \frac{6}{7} = \frac{4}{7}.$$

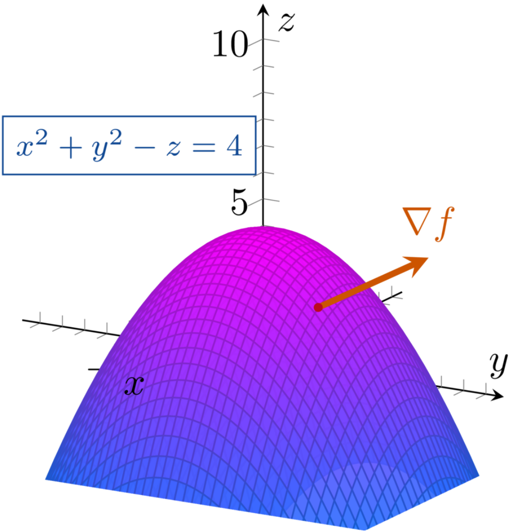
(b) The function increases most rapidly in the direction of $\nabla f = 2\mathbf{i} - 2\mathbf{j} - \mathbf{k}$ and decreases most rapidly in the direction of $-\nabla f$. The rates of change in the directions are, respectively,

$$|\nabla f| = \sqrt{(2)^2 + (-2)^2 + (-1)^2} = \sqrt{9} = 3 \qquad \text{and} \qquad -|\nabla f| = -3.$$

13.6 Tangent Planes and Differentials

$$D_{\mathbf{u}}f = \|\nabla f\| \cos\theta$$

Tangent Planes and Normal Lines



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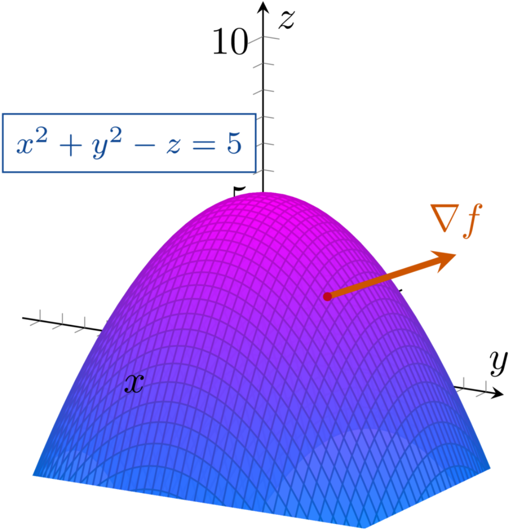
Tangent Planes and Normal Lines



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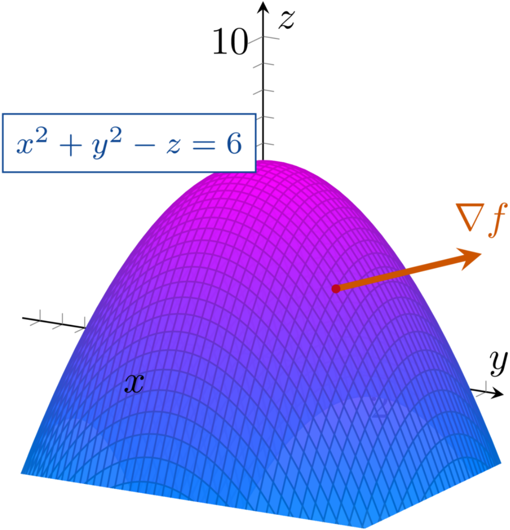
Tangent Planes and Normal Lines



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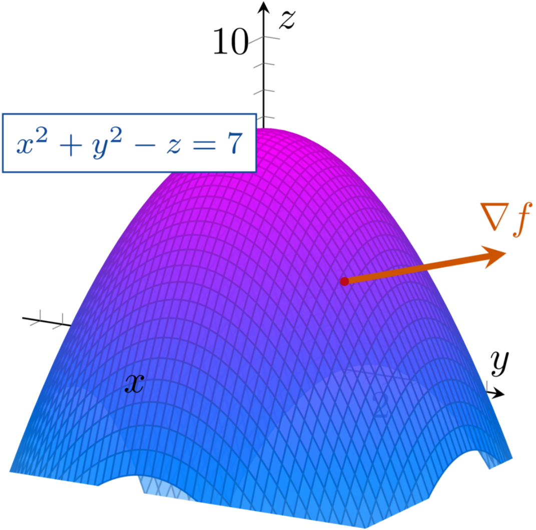
Tangent Planes and Normal Lines



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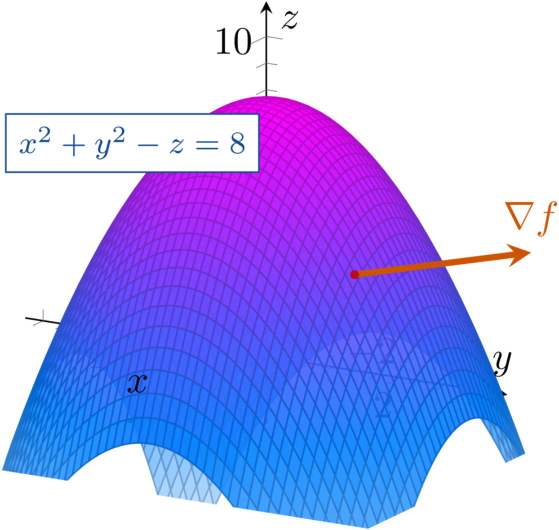

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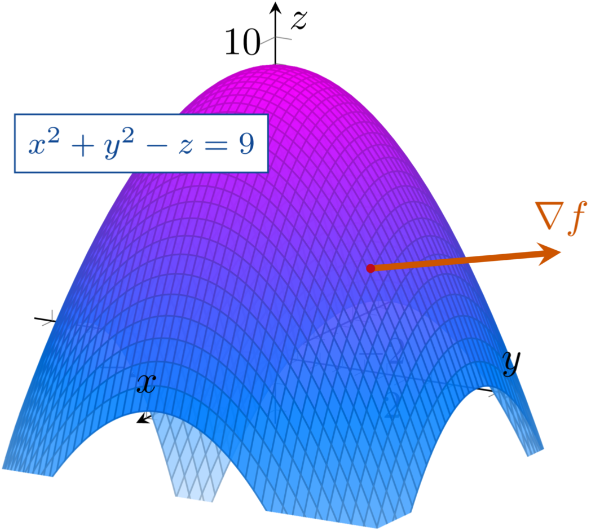

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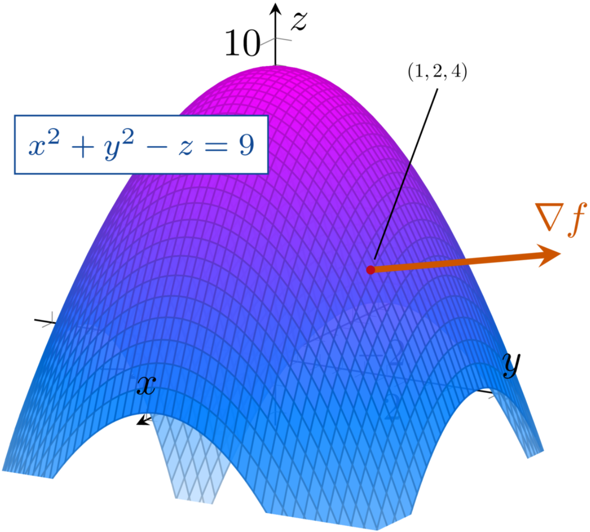

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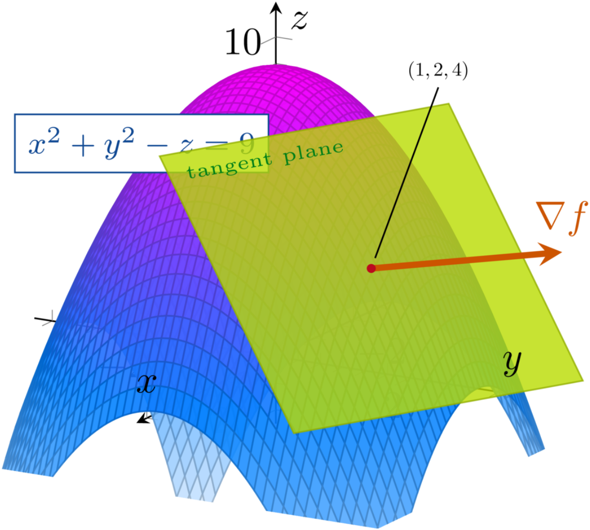

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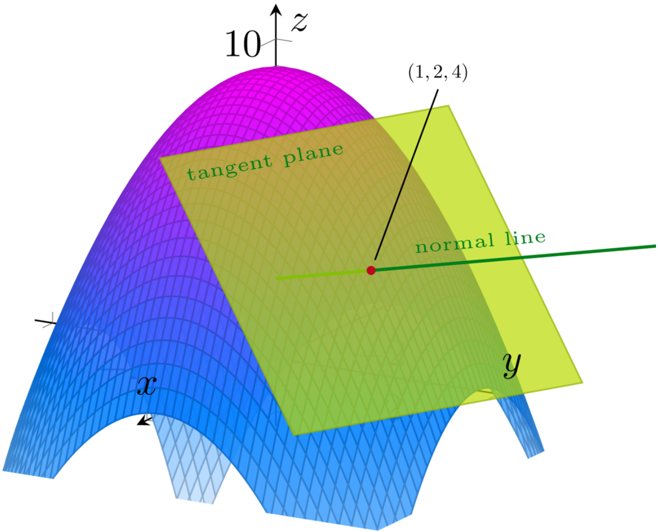

$$D_{\mathbf{u}}f = \|\nabla f\| \cos\theta$$

Definition

The *tangent plane* to the surface $f(x, y, z) = c$ at the point $P(x_0, y_0, z_0)$ (where the gradient is not zero) is the plane through P with normal vector $\nabla f|_P$.

$$D_{\mathbf{u}}f = \|\nabla f\| \cos\theta$$

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$$f_x(P)(x - x_0) + f_y(P)(y - y_0) + f_z(P)(z - z_0) = 0.$$

$$D_{\mathbf{u}}f = \|\nabla f\| \cos\theta$$

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Definition

The *normal line* to the surface $f(x, y, z) = c$ at the point P is the line through P parallel to $\nabla f|_P$.

$$D_{\mathbf{u}} f = \|\nabla f\| \cos\theta$$

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Definition

The *normal line* to the surface $f(x,y,z) = c$ at the point P is the line through P parallel to $\nabla f|_P$.

$$x = x_0 + f_x(P)t \qquad y = y_0 + f_y(P)t \qquad z = z_0 + f_z(P)t.$$

EXAMPLE 1 Find the tangent plane and normal line of the level surface

$$f(x, y, z) = x^2 + y^2 + z - 9 = 0 \qquad \text{A circular paraboloid}$$

at the point $P_0(1, 2, 4)$.

Solution The surface is shown in Figure 14.34.

The tangent plane is the plane through P_0 perpendicular to the gradient of f at P_0. The gradient is

$$\nabla f\big|_{P_0} = (2x\mathbf{i} + 2y\mathbf{j} + \mathbf{k})\Big|_{(1,2,4)} = 2\mathbf{i} + 4\mathbf{j} + \mathbf{k}.$$

The tangent plane is therefore the plane

$$2(x - 1) + 4(y - 2) + (z - 4) = 0, \qquad \text{or} \qquad 2x + 4y + z = 14.$$

The line normal to the surface at P_0 is

$$x = 1 + 2t, \qquad y = 2 + 4t, \qquad z = 4 + t.$$

$$D_{\mathbf{u}}f = \|\nabla f\| \cos\theta$$

Now consider

$$z = f(x, y).$$

$$D_{\mathbf{u}}f = \|\nabla f\| \cos\theta$$

Now consider

$$z = f(x, y).$$

This is equivalent to

$$F(x, y, z) = f(x, y) - z = 0.$$

$$\boxed{D_{\mathbf{u}}f = \|\nabla f\| \cos\theta}$$

Now consider

$$z = f(x, y).$$

This is equivalent to

$$F(x, y, z) = f(x, y) - z = 0.$$

Definition

The *tangent plane* to the surface $z = f(x, y)$ at the point $P(x_0, y_0, z_0) = (x_0, y_0, f(x_0, y_0))$ is

$$f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0) - (z - z_0) = 0.$$

EXAMPLE 2 Find the plane tangent to the surface $z = x \cos y - ye^x$ at $(0, 0, 0)$.

Solution We calculate the partial derivatives of $f(x, y) = x \cos y - ye^x$ and use Equation (3):

$$f_x(0, 0) = (\cos y - ye^x)\Big|_{(0,0)} = 1 - 0 \cdot 1 = 1$$

$$f_y(0, 0) = (-x \sin y - e^x)\Big|_{(0,0)} = 0 - 1 = -1.$$

The tangent plane is therefore

$$1 \cdot (x - 0) - 1 \cdot (y - 0) - (z - 0) = 0, \qquad \text{Eq. (3)}$$

or

$$x - y - z = 0.$$

$$D_{\mathbf{u}}f = \|\nabla f\| \cos\theta$$

EXAMPLE 3 The surfaces

$$f(x, y, z) = x^2 + y^2 - 2 = 0 \qquad \text{A cylinder}$$

and

$$g(x, y, z) = x + z - 4 = 0 \qquad \text{A plane}$$

meet in an ellipse E (Figure 14.35). Find parametric equations for the line tangent to E at the point $P_0(1, 1, 3)$.

$$D_{\mathbf{u}}f = \|\nabla f\| \cos\theta$$

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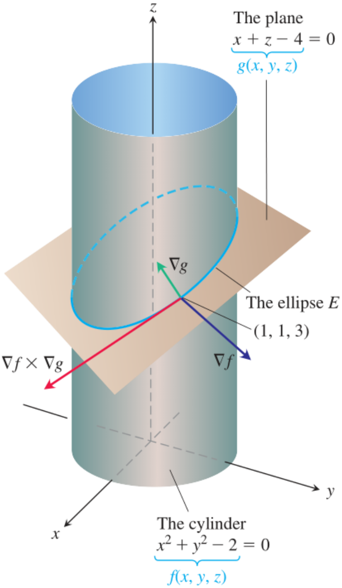
the point $P_0(1, 1, 3)$.



$$D_{\mathbf{u}} f = \|\nabla f\| \cos\theta$$

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$$g(x, y, z) = x + z - 4 = 0 \qquad \text{A plane}$$

meet in an ellipse E (Figure 14.35). Find parametric equations for the line tangent to E at the point $P_0(1, 1, 3)$.

Solution The tangent line is orthogonal to both ∇f and ∇g at P_0, and therefore parallel to $\mathbf{v} = \nabla f \times \nabla g$. The components of $\mathbf{v}$ and the coordinates of P_0 give us equations for the line. We have

$$\nabla f\big|_{(1,1,3)} = (2x\mathbf{i} + 2y\mathbf{j})\Big|_{(1,1,3)} = 2\mathbf{i} + 2\mathbf{j}$$

$$\nabla g\big|_{(1,1,3)} = (\mathbf{i} + \mathbf{k})\Big|_{(1,1,3)} = \mathbf{i} + \mathbf{k}$$

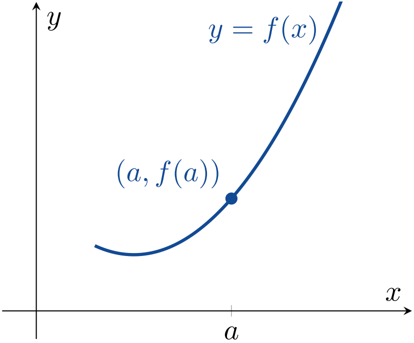
$$\mathbf{v} = (2\mathbf{i} + 2\mathbf{j}) \times (\mathbf{i} + \mathbf{k}) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 2 & 0 \\ 1 & 0 & 1 \end{vmatrix} = 2\mathbf{i} - 2\mathbf{j} - 2\mathbf{k}.$$

The tangent line to the ellipse of intersection is

$$x = 1 + 2t, \qquad y = 1 - 2t, \qquad z = 3 - 2t.$$

$$D_{\mathbf{u}}f = \|\nabla f\| \cos\theta$$

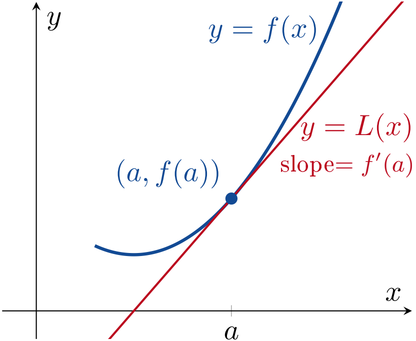
Linearisation of a Function of One Variable



The *linearisation* of f at a is

$L(x) =$

Linearisation of a Function of One Variable



The *linearisation* of f at a is

$$L(x) = f(a) + f'(a)(x - a).$$

$$L(x) = f(a) + f'(a)(x-a)$$

Linearisation of a Function of Two Variable

Definition

The *linearisation* of a function $f(x,y)$ at a point (x_0, y_0) is

$$L(x,y) = f(x_0,y_0) + f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0).$$

EXAMPLE 5 Find the linearization of

$$f(x, y) = x^2 - xy + \frac{1}{2}y^2 + 3$$

at the point (3, 2).

Solution We first evaluate f, f_x, and f_y at the point $(x_0, y_0) = (3, 2)$:

$$f(3, 2) = \left(x^2 - xy + \frac{1}{2}y^2 + 3\right)\Big|_{(3,2)} = 8$$

$$f_x(3, 2) = \frac{\partial}{\partial x}\left(x^2 - xy + \frac{1}{2}y^2 + 3\right)\Big|_{(3,2)} = (2x - y)\Big|_{(3,2)} = 4$$

$$f_y(3, 2) = \frac{\partial}{\partial y}\left(x^2 - xy + \frac{1}{2}y^2 + 3\right)\Big|_{(3,2)} = (-x + y)\Big|_{(3,2)} = -1,$$

giving

$$\begin{aligned} L(x, y) &= f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0) \\ &= 8 + (4)(x - 3) + (-1)(y - 2) = 4x - y - 2. \end{aligned}$$

The linearization of f at (3, 2) is $L(x, y) = 4x - y - 2$

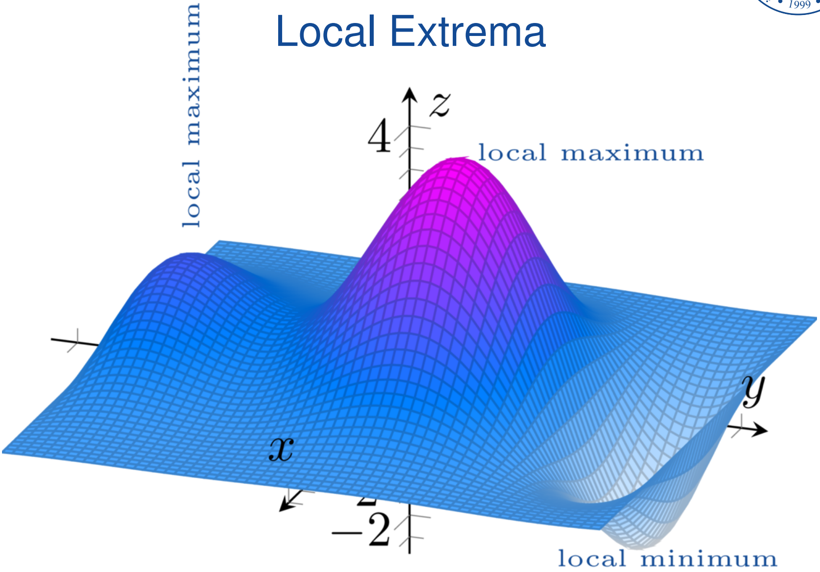
Break

We will continue at 2pm

13.7

Extreme Values and Saddle Points

Local Extrema



Definition

1. $f(a,b)$ is a local maximum value of $f(x,y)$ iff

$$f(a,b) \geq f(x,y)$$

for all (x,y) close to (a,b).

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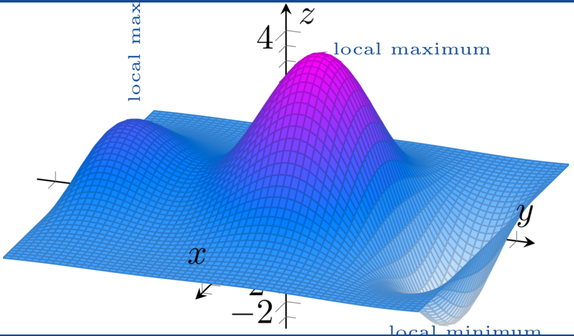
for all (x,y) close to (a,b).

2. $f(a,b)$ is a local minimum value of $f(x,y)$ iff

$$f(a,b) \leq f(x,y)$$

for all (x,y) close to (a,b).



Theorem (First Derivative Test)

$$\left(\begin{array}{c} f(x,y) \textit{ has a local} \\ \textit{extrema at an interior} \\ \textit{point (a,b) of its} \\ \textit{domain} \end{array}\right) \implies \begin{array}{c} f_x(a,b) = 0 \\ \textit{and} \\ f_y(a,b) = 0 \end{array}$$

if $f_x(a,b)$ *and* $f_y(a,b)$ *both exist.*

Definition

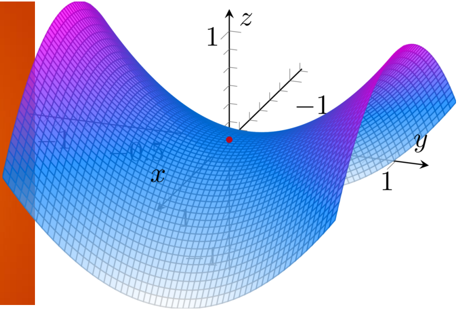
An interior point of the domain of $f(x, y)$ where either

1. $f_x = f_y = 0$;
2. f_x does not exist; or
3. f_y does not exist

is called a *critical point* of f.

Saddle Points

Saddle Points



The point $(0, 0)$ is a *saddle point* of $z = y^2 - x^2$.

EXAMPLE 1 Find the local extreme values of $f(x, y) = x^2 + y^2 - 4y + 9$.

Solution The domain of f is the entire plane (so there are no boundary points) and the partial derivatives $f_x = 2x$ and $f_y = 2y - 4$ exist everywhere. Therefore, local extreme values can occur only where

$$f_x = 2x = 0 \qquad \text{and} \qquad f_y = 2y - 4 = 0.$$

The only possibility is the point $(0, 2)$, where the value of f is 5. Since $f(x, y) = x^2 + (y - 2)^2 + 5$ is never less than 5, we see that the critical point $(0, 2)$ gives a local minimum (Figure 14.46). ■

EXAMPLE 2 Find the local extreme values (if any) of $f(x, y) = y^2 - x^2$.

Solution The domain of f is the entire plane (so there are no boundary points) and the partial derivatives $f_x = -2x$ and $f_y = 2y$ exist everywhere. Therefore, local extrema can occur only at the origin $(0, 0)$ where $f_x = 0$ and $f_y = 0$. Along the positive x-axis, however, f has the value $f(x, 0) = -x^2 < 0$; along the positive y-axis, f has the value $f(0, y) = y^2 > 0$. Therefore, every open disk in the xy-plane centered at $(0, 0)$ contains points where the function is positive and points where it is negative. The function has a saddle point at the origin and no local extreme values (Figure 14.47a). Figure 14.47b displays the level curves (they are hyperbolas) of f, and shows the function decreasing and increasing in an alternating fashion among the four groupings of hyperbolas. ■

Theorem (Second Derivative Test)

Suppose that

- $f(x, y)$, f_x, f_y, f_{xx}, f_{yy} *and* f_{xy} *are all continuous on an open disk centred at* (a, b); *and*
- $f_x(a, b) = 0 = f_y(a, b)$.

Then

Theorem (Second Derivative Test)

Suppose that

- $f(x,y)$, f_x, f_y, f_{xx}, f_{yy} *and* f_{xy} *are all continuous on an open disk centred at* (a,b)*; and*
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Then

$f_{xx} < 0$	$f_{xx}f_{yy} - f_{xy}^2 > 0$	f *has a local maximum at* (a,b)

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$f_{xx} < 0$	$f_{xx}f_{yy} - f_{xy}^2 > 0$	f *has a local maximum at* (a,b)
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	$f_{xx}f_{yy} - f_{xy}^2 < 0$	f *has a saddle point at* (a,b)

Theorem (Second Derivative Test)

Suppose that

- $f(x,y)$, f_x, f_y, f_{xx}, f_{yy} *and* f_{xy} *are all continuous on an open disk centred at* (a,b); *and*
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Then

$f_{xx} < 0$	$f_{xx}f_{yy} - f_{xy}^2 > 0$	f *has a local maximum at* (a,b)
$f_{xx} > 0$	$f_{xx}f_{yy} - f_{xy}^2 > 0$	f *has a local minimum at* (a,b)
	$f_{xx}f_{yy} - f_{xy}^2 < 0$	f *has a saddle point at* (a,b)
	$f_{xx}f_{yy} - f_{xy}^2 = 0$	*we don't know*

Otto Hesse

BORN
22 April 1811

DECEICED
4 August 1874

NATIONALITY
German

Definition

$f_{xx}f_{yy} - f_{xy}^2$ is called the *Hessian* of f.

EXAMPLE 3 Find the local extreme values of the function

$$f(x, y) = xy - x^2 - y^2 - 2x - 2y + 4.$$

Solution The function is defined and differentiable for all x and y, and its domain has no boundary points. The function therefore has extreme values only at the points where f_x and f_y are simultaneously zero. This leads to

$$f_x = y - 2x - 2 = 0, \qquad f_y = x - 2y - 2 = 0,$$

or

$$x = y = -2.$$

Therefore, the point $(-2, -2)$ is the only point where f may take on an extreme value. To see if it does so, we calculate

$$f_{xx} = -2, \qquad f_{yy} = -2, \qquad f_{xy} = 1.$$

The discriminant of f at $(a, b) = (-2, -2)$ is

$$f_{xx}f_{yy} - f_{xy}{}^2 = (-2)(-2) - (1)^2 = 4 - 1 = 3.$$

The combination

$$f_{xx} < 0 \qquad \text{and} \qquad f_{xx}f_{yy} - f_{xy}{}^2 > 0$$

tells us that f has a local maximum at $(-2, -2)$. The value of f at this point is $f(-2, -2) = 8$. ■

EXAMPLE 4 Find the local extreme values of $f(x, y) = 3y^2 - 2y^3 - 3x^2 + 6xy$.

Solution Since f is differentiable everywhere, it can assume extreme values only where

$$f_x = 6y - 6x = 0 \qquad \text{and} \qquad f_y = 6y - 6y^2 + 6x = 0.$$

From the first of these equations we find $x = y$, and substitution for y into the second equation then gives

$$6x - 6x^2 + 6x = 0 \qquad \text{or} \qquad 6x(2 - x) = 0.$$

The two critical points are therefore $(0, 0)$ and $(2, 2)$.

To classify the critical points, we calculate the second derivatives:

$$f_{xx} = -6, \qquad f_{yy} = 6 - 12y, \qquad f_{xy} = 6.$$

The discriminant is given by

$$f_{xx}f_{yy} - f_{xy}{}^2 = (-36 + 72y) - 36 = 72(y - 1).$$

At the critical point $(0, 0)$ we see that the value of the discriminant is the negative number -72, so the function has a saddle point at the origin. At the critical point $(2, 2)$ we see that the discriminant has the positive value 72. Combining this result with the negative value of the second partial $f_{xx} = -6$, Theorem 11 says that the critical point $(2, 2)$ gives a local maximum value of $f(2, 2) = 12 - 16 - 12 + 24 = 8$. A graph of the surface is shown in Figure 14.48. ■

EXAMPLE 5 Find the critical points of the function $f(x, y) = 10xye^{-(x^2+y^2)}$ and use the Second Derivative Test to classify each point as one where a saddle, local minimum, or local maximum occurs.

Solution First we find the partial derivatives f_x and f_y and set them simultaneously to zero in seeking the critical points:

$$f_x = 10ye^{-(x^2+y^2)} - 20x^2ye^{-(x^2+y^2)} = 10y(1 - 2x^2)e^{-(x^2+y^2)} = 0 \Rightarrow y = 0 \text{ or } 1 - 2x^2 = 0,$$
$$f_y = 10xe^{-(x^2+y^2)} - 20xy^2e^{-(x^2+y^2)} = 10x(1 - 2y^2)e^{-(x^2+y^2)} = 0 \Rightarrow x = 0 \text{ or } 1 - 2y^2 = 0.$$

Since both partial derivatives are continuous everywhere, the only critical points are

$$(0, 0), \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right), \left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right), \left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right), \text{ and } \left(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right).$$

Next we calculate the second partial derivatives in order to evaluate the discriminant at each critical point:

$$f_{xx} = -20xy(1 - 2x^2)e^{-(x^2+y^2)} - 40xye^{-(x^2+y^2)} = -20xy(3 - 2x^2)e^{-(x^2+y^2)},$$
$$f_{xy} = f_{yx} = 10(1 - 2x^2)e^{-(x^2+y^2)} - 20y^2(1 - 2x^2)e^{-(x^2+y^2)} = 10(1 - 2x^2)(1 - 2y^2)e^{-(x^2+y^2)},$$
$$f_{yy} = -20xy(1 - 2y^2)e^{-(x^2+y^2)} - 40xye^{-(x^2+y^2)} = -20xy(3 - 2y^2)e^{-(x^2+y^2)}.$$

The following table summarizes the values needed by the Second Derivative Test.

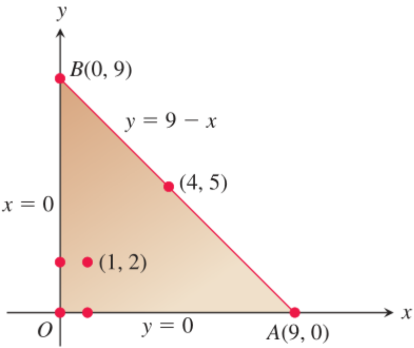
Critical Point	f_{xx}	f_{xy}	f_{yy}	**Discriminant D**
$(0, 0)$	0	10	0	-100
$\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$	$-\frac{20}{e}$	0	$-\frac{20}{e}$	$\frac{400}{e^2}$
$\left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$	$\frac{20}{e}$	0	$\frac{20}{e}$	$\frac{400}{e^2}$
$\left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)$	$\frac{20}{e}$	0	$\frac{20}{e}$	$\frac{400}{e^2}$
$\left(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)$	$-\frac{20}{e}$	0	$-\frac{20}{e}$	$\frac{400}{e^2}$

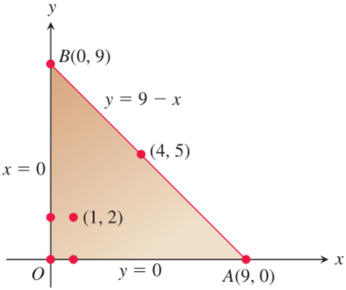
From the table we find that $D < 0$ at the critical point $(0, 0)$, giving a saddle; $D > 0$ and $f_{xx} < 0$ at the critical points $\left(1/\sqrt{2}, 1/\sqrt{2}\right)$ and $\left(-1/\sqrt{2}, -1/\sqrt{2}\right)$, giving local maximum values there; and $D > 0$ and $f_{xx} > 0$ at the critical points $\left(-1/\sqrt{2}, 1/\sqrt{2}\right)$ and $\left(1/\sqrt{2}, -1/\sqrt{2}\right)$, each giving local minimum values. A graph of the surface is shown in Figure 14.49. ■

EXAMPLE 6 Find the absolute maximum and minimum values of

$$f(x, y) = 2 + 2x + 4y - x^2 - y^2$$

on the triangular region in the first quadrant bounded by the lines $x = 0$, $y = 0$, and $y = 9 - x$.





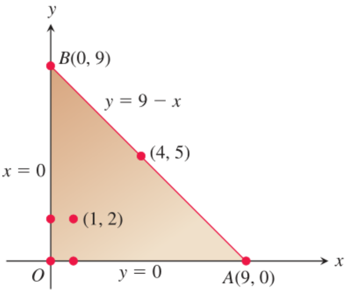
Solution Since f is differentiable, the only places where f can assume these values are points inside the triangle where $f_x = f_y = 0$ and points on the boundary (Figure 14.50a).

(a) Interior points. For these we have

$$f_x = 2 - 2x = 0, \qquad f_y = 4 - 2y = 0,$$

yielding the single point $(x, y) = (1, 2)$. The value of f there is

$$f(1, 2) = 7.$$



(b) Boundary points. We take the triangle one side at a time:

i) On the segment OA, $y = 0$. The function

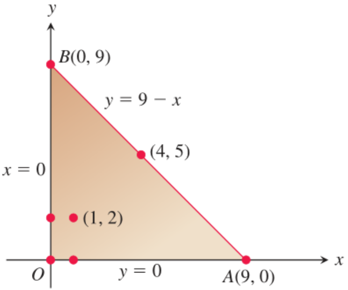
$$f(x, y) = f(x, 0) = 2 + 2x - x^2$$

may now be regarded as a function of x defined on the closed interval $0 \leq x \leq 9$. Its extreme values (as we know from Chapter 4) may occur at the endpoints

$$\begin{aligned} x &= 0 \quad \text{where} \quad f(0, 0) = 2 \\ x &= 9 \quad \text{where} \quad f(9, 0) = 2 + 18 - 81 = -61 \end{aligned}$$

or at the interior points where $f'(x, 0) = 2 - 2x = 0$. The only interior point where $f'(x, 0) = 0$ is $x = 1$, where

$$f(x, 0) = f(1, 0) = 3.$$

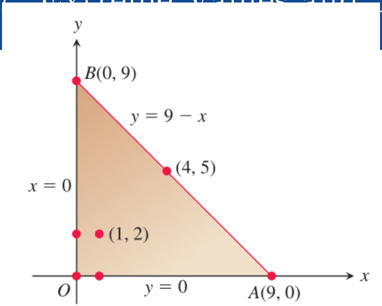


ii) On the segment OB, $x = 0$ and

$$f(x, y) = f(0, y) = 2 + 4y - y^2.$$

As in part i), we consider $f(0, y)$ as a function of y defined on the closed interval $[0, 9]$. Its extreme values can occur at the endpoints or at interior points where $f'(0, y) = 0$. Since $f'(0, y) = 4 - 2y$, the only interior point where $f'(0, y) = 0$ occurs at $(0, 2)$, with $f(0, 2) = 6$. So the candidates for this segment are

$$f(0, 0) = 2, \qquad f(0, 9) = -43, \qquad f(0, 2) = 6.$$



iii) We have already accounted for the values of f at the endpoints of AB, so we need only look at the interior points of the line segment AB. With $y = 9 - x$, we have

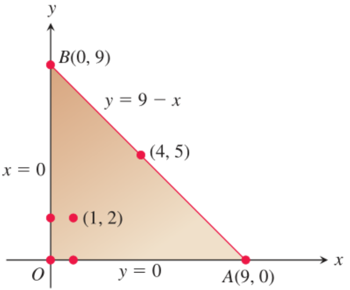
$$f(x, y) = 2 + 2x + 4(9 - x) - x^2 - (9 - x)^2 = -43 + 16x - 2x^2.$$

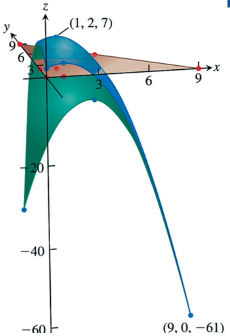
Setting $f'(x, 9 - x) = 16 - 4x = 0$ gives

$$x = 4.$$

At this value of x,

$$y = 9 - 4 = 5 \qquad \text{and} \qquad f(x, y) = f(4, 5) = -11.$$





At this value of x,

$$y = 9 - 4 = 5 \qquad \text{and} \qquad f(x, y) = f(4, 5) = -11.$$

Summary We list all the function value candidates: 7, 2, −61, 3, −43, 6, −11. The maximum is 7, which f assumes at (1, 2). The minimum is −61, which f assumes at (9, 0). See Figure 14.50b.

Please read Example 7 in the textbook.

13.8

Lagrange Multipliers

Example

Find the point $P(x, y, z)$ on the plane $2x + y - z - 5 = 0$ that is closest to the origin.

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We need to find the minimum of

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subject to the constraint that

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Let $f(x, y) = x^2 + y^2 + z^2$. We will study f instead of $\left\|\overrightarrow{OP}\right\|$.

We will let x and y be the independent variables and write

$$z = 2x + y - 5.$$

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$$z = 2x + y - 5.$$

So we want to find the minimum of

$$h(x, y) = f(x, y, 2x + y - 5) = x^2 + y^2 + (2x + y - 5)^2.$$

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So we want to find the minimum of

$$h(x, y) = f(x, y, 2x + y - 5) = x^2 + y^2 + (2x + y - 5)^2.$$

We will use the First Derivative Test:

$0 = h_x$

$0 = h_y$

We will let x and y be the independent variables and write

$$z = 2x + y - 5.$$

So we want to find the minimum of

$$h(x, y) = f(x, y, 2x + y - 5) = x^2 + y^2 + (2x + y - 5)^2.$$

We will use the First Derivative Test:

$0 = h_x = 2x + 2(2x + y - 5)(2)$

$0 = h_y = 2y + 2(2x + y - 5)$

We will let x and y be the independent variables and write

$$z = 2x + y - 5.$$

So we want to find the minimum of

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$$\begin{aligned} 0 = h_x &= 2x + 2(2x + y - 5)(2) \\ 0 = h_y &= 2y + 2(2x + y - 5) \end{aligned} \implies \begin{aligned} 10x + 4y &= 20 \\ 4x + 4y &= 10 \end{aligned}$$

We will let x and y be the independent variables and write

$$z = 2x + y - 5.$$

So we want to find the minimum of

$$h(x, y) = f(x, y, 2x + y - 5) = x^2 + y^2 + (2x + y - 5)^2.$$

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$$\begin{aligned} 0 = h_x &= 2x + 2(2x + y - 5)(2) \\ 0 = h_y &= 2y + 2(2x + y - 5) \end{aligned} \implies \begin{aligned} 10x + 4y &= 20 \\ 4x + 4y &= 10 \end{aligned} \implies \begin{aligned} x &= \tfrac{5}{3} \\ y &= \tfrac{5}{6}. \end{aligned}$$

Then we have

$$z = 2x + y - 5 = \frac{10}{3} + \frac{5}{6} - 5 = -\frac{5}{6}.$$

We will let x and y be the independent variables and write

$$z = 2x + y - 5.$$

So we want to find the minimum of

$$h(x,y) = f(x,y,2x+y-5) = x^2 + y^2 + (2x+y-5)^2.$$

We will use the First Derivative Test:

$$\begin{aligned} 0 = h_x &= 2x + 2(2x+y-5)(2) \\ 0 = h_y &= 2y + 2(2x+y-5) \end{aligned} \implies \begin{aligned} 10x + 4y &= 20 \\ 4x + 4y &= 10 \end{aligned} \implies \begin{aligned} x &= \tfrac{5}{3} \\ y &= \tfrac{5}{6}. \end{aligned}$$

Then we have

$$z = 2x + y - 5 = \frac{10}{3} + \frac{5}{6} - 5 = -\frac{5}{6}.$$

The point on this plane which is closest to the origin is

$$P\left(\frac{5}{3}, \frac{5}{6}, -\frac{5}{6}\right).$$

The Method of Lagrange Multipliers

Suppose that we want to find the maximum/minimum of

$$f(x, y, z)$$

subject to the constraint that

$$g(x, y, z) = 0.$$

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Theorem (The Method of Lagrange Multipliers)

We only need to find x, y, z and λ which satisfy

$$\nabla f = \lambda \nabla g \qquad \textit{and} \qquad g(x, y, z) = 0.$$

The Method of Lagrange Multipliers

Suppose that we want to find the maximum/minimum of

$$f(x, y, \cancel{z})$$

subject to the constraint that

$$g(x, y, \cancel{z}) = 0.$$

Theorem (The Method of Lagrange Multipliers)

We only need to find x, y, $\cancel{z}$ and λ which satisfy

$$\nabla f = \lambda \nabla g \qquad \textit{and} \qquad g(x, y, \cancel{z}) = 0.$$

$$\nabla f = \lambda \nabla g \qquad g(x, y, z) = 0.$$

Example (repeat)

Find the point $P(x, y, z)$ on the plane $2x + y - z - 5 = 0$ that is closest to the origin.

$$\nabla f = \lambda \nabla g \qquad g(x,y,z) = 0.$$

Example (repeat)

Find the point $P(x,y,z)$ on the plane $2x + y - z - 5 = 0$ that is closest to the origin.

Let $f(x,y,z) = x^2 + y^2 + z^2$ and $g(x,y,z) = 2x + y - z - 5$.

$$\nabla f = \lambda \nabla g \qquad g(x, y, z) = 0.$$

Example (repeat)

Find the point $P(x, y, z)$ on the plane $2x + y - z - 5 = 0$ that is closest to the origin.

Let $f(x, y, z) = x^2 + y^2 + z^2$ and $g(x, y, z) = 2x + y - z - 5$. Then

$$= \nabla f = \lambda \nabla g =$$

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$$2x\mathbf{i} + 2y\mathbf{j} + 2z\mathbf{k} = \nabla f = \lambda \nabla g =$$

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$$2x\mathbf{i} + 2y\mathbf{j} + 2z\mathbf{k} = \nabla f = \lambda \nabla g = 2\lambda\mathbf{i} + \lambda\mathbf{j} - \lambda\mathbf{k}$$

$$\nabla f = \lambda \nabla g \qquad g(x,y,z) = 0.$$

Example (repeat)

Find the point $P(x,y,z)$ on the plane $2x + y - z - 5 = 0$ that is closest to the origin.

Let $f(x,y,z) = x^2 + y^2 + z^2$ and $g(x,y,z) = 2x + y - z - 5$. Then

$$2x\mathbf{i} + 2y\mathbf{j} + 2z\mathbf{k} = \nabla f = \lambda \nabla g = 2\lambda\mathbf{i} + \lambda\mathbf{j} - \lambda\mathbf{k} \implies \begin{cases} x = \lambda \\ y = \frac{\lambda}{2} \\ z = -\frac{\lambda}{2}. \end{cases}$$

$$\nabla f = \lambda \nabla g \qquad g(x, y, z) = 0.$$

Example (repeat)

Find the point $P(x, y, z)$ on the plane $2x + y - z - 5 = 0$ that is closest to the origin.

Let $f(x, y, z) = x^2 + y^2 + z^2$ and $g(x, y, z) = 2x + y - z - 5$. Then

$$2x\mathbf{i} + 2y\mathbf{j} + 2z\mathbf{k} = \nabla f = \lambda \nabla g = 2\lambda\mathbf{i} + \lambda\mathbf{j} - \lambda\mathbf{k} \implies \begin{cases} x = \lambda \\ y = \frac{\lambda}{2} \\ z = -\frac{\lambda}{2}. \end{cases}$$

Hence

$$0 = 2x + y - z - 5 =$$

$$\nabla f = \lambda \nabla g \qquad g(x, y, z) = 0.$$

Example (repeat)

Find the point $P(x, y, z)$ on the plane $2x + y - z - 5 = 0$ that is closest to the origin.

Let $f(x, y, z) = x^2 + y^2 + z^2$ and $g(x, y, z) = 2x + y - z - 5$. Then

$$2x\mathbf{i} + 2y\mathbf{j} + 2z\mathbf{k} = \nabla f = \lambda \nabla g = 2\lambda\mathbf{i} + \lambda\mathbf{j} - \lambda\mathbf{k} \implies \begin{cases} x = \lambda \\ y = \frac{\lambda}{2} \\ z = -\frac{\lambda}{2}. \end{cases}$$

Hence

$$0 = 2x + y - z - 5 = 2\lambda + \frac{\lambda}{2} - \left(-\frac{\lambda}{2}\right) - 5 = 3\lambda - 5$$

$$\nabla f = \lambda \nabla g \qquad g(x,y,z) = 0.$$

Example (repeat)

Find the point $P(x,y,z)$ on the plane $2x + y - z - 5 = 0$ that is closest to the origin.

Let $f(x,y,z) = x^2 + y^2 + z^2$ and $g(x,y,z) = 2x + y - z - 5$. Then

$$2x\mathbf{i} + 2y\mathbf{j} + 2z\mathbf{k} = \nabla f = \lambda \nabla g = 2\lambda\mathbf{i} + \lambda\mathbf{j} - \lambda\mathbf{k} \implies \begin{cases} x = \lambda \\ y = \frac{\lambda}{2} \\ z = -\frac{\lambda}{2}. \end{cases}$$

Hence

$$0 = 2x + y - z - 5 = 2\lambda + \frac{\lambda}{2} - \left(-\frac{\lambda}{2}\right) - 5 = 3\lambda - 5 \implies \lambda = \frac{5}{3}.$$

$$\nabla f = \lambda \nabla g \qquad g(x,y,z) = 0.$$

Example (repeat)

Find the point $P(x,y,z)$ on the plane $2x + y - z - 5 = 0$ that is closest to the origin.

Let $f(x,y,z) = x^2 + y^2 + z^2$ and $g(x,y,z) = 2x + y - z - 5$. Then

$$2x\mathbf{i} + 2y\mathbf{j} + 2z\mathbf{k} = \nabla f = \lambda \nabla g = 2\lambda\mathbf{i} + \lambda\mathbf{j} - \lambda\mathbf{k} \implies \begin{cases} x = \lambda \\ y = \frac{\lambda}{2} \\ z = -\frac{\lambda}{2}. \end{cases}$$

Hence

$$0 = 2x + y - z - 5 = 2\lambda + \frac{\lambda}{2} - \left(-\frac{\lambda}{2}\right) - 5 = 3\lambda - 5 \implies \lambda = \frac{5}{3}.$$

Therefore

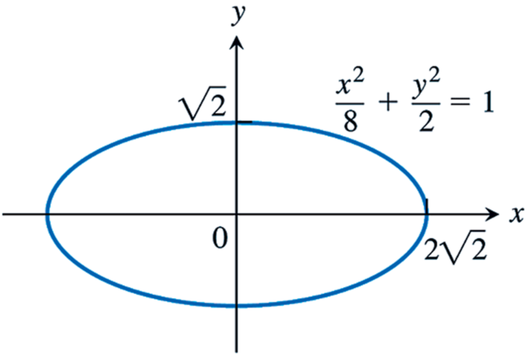
$$P(x,y,z) = \left(\lambda, \frac{\lambda}{2}, -\frac{\lambda}{2}\right) = \left(\frac{5}{3}, \frac{5}{6}, -\frac{5}{6}\right).$$

EXAMPLE 3 Find the greatest and smallest values that the function

$$f(x, y) = xy$$

takes on the ellipse (Figure 14.55)

$$\frac{x^2}{8} + \frac{y^2}{2} = 1.$$



Solution We want to find the extreme values of $f(x, y) = xy$ subject to the constraint

$$g(x, y) = \frac{x^2}{8} + \frac{y^2}{2} - 1 = 0.$$

To do so, we first find the values of x, y, and λ for which

$$\nabla f = \lambda \nabla g \qquad \text{and} \qquad g(x, y) = 0.$$

The gradient equation in Equations (1) gives

$$y\mathbf{i} + x\mathbf{j} = \frac{\lambda}{4} x\mathbf{i} + \lambda y\mathbf{j},$$

from which we find

$$y = \frac{\lambda}{4} x, \qquad x = \lambda y, \qquad \text{and} \qquad y = \frac{\lambda}{4}(\lambda y) = \frac{\lambda^2}{4} y,$$

so that $y = 0$ or $\lambda = \pm 2$. We now consider these two cases.

Case 1: If $y = 0$, then $x = y = 0$. But $(0, 0)$ is not on the ellipse. Hence, $y \neq 0$.

Case 2: If $y \neq 0$, then $\lambda = \pm 2$ and $x = \pm 2y$. Substituting this in the equation $g(x, y) = 0$ gives

$$\frac{(\pm 2y)^2}{8} + \frac{y^2}{2} = 1, \qquad 4y^2 + 4y^2 = 8 \qquad \text{and} \qquad y = \pm 1.$$

The function $f(x, y) = xy$ therefore takes on its extreme values on the ellipse at the four points $(\pm 2, 1)$, $(\pm 2, -1)$. The extreme values are $xy = 2$ and $xy = -2$.

EXAMPLE 4 Find the maximum and minimum values of the function $f(x, y) = 3x + 4y$ on the circle $x^2 + y^2 = 1$.

Solution We model this as a Lagrange multiplier problem with

$$f(x, y) = 3x + 4y, \qquad g(x, y) = x^2 + y^2 - 1$$

and look for the values of x, y, and λ that satisfy the equations

$$\begin{aligned} \nabla f = \lambda \nabla g: &\quad 3\mathbf{i} + 4\mathbf{j} = 2x\lambda\mathbf{i} + 2y\lambda\mathbf{j} \\ g(x, y) = 0: &\quad x^2 + y^2 - 1 = 0. \end{aligned}$$

The gradient equation in Equations (1) implies that $\lambda \neq 0$ and gives

$$x = \frac{3}{2\lambda}, \qquad y = \frac{2}{\lambda}.$$

These equations tell us, among other things, that x and y have the same sign. With these values for x and y, the equation $g(x, y) = 0$ gives

$$\left(\frac{3}{2\lambda}\right)^2 + \left(\frac{2}{\lambda}\right)^2 - 1 = 0,$$

so

$$\frac{9}{4\lambda^2} + \frac{4}{\lambda^2} = 1, \qquad 9 + 16 = 4\lambda^2, \qquad 4\lambda^2 = 25, \qquad \text{and} \qquad \lambda = \pm\frac{5}{2}.$$

Thus,

$$x = \frac{3}{2\lambda} = \pm\frac{3}{5}, \qquad y = \frac{2}{\lambda} = \pm\frac{4}{5},$$

and $f(x, y) = 3x + 4y$ has extreme values at $(x, y) = \pm(3/5, 4/5)$.

By calculating the value of $3x + 4y$ at the points $\pm(3/5, 4/5)$, we see that its maximum and minimum values on the circle $x^2 + y^2 = 1$ are

$$3\left(\frac{3}{5}\right) + 4\left(\frac{4}{5}\right) = \frac{25}{5} = 5 \qquad \text{and} \qquad 3\left(-\frac{3}{5}\right) + 4\left(-\frac{4}{5}\right) = -\frac{25}{5} = -5.$$

Next Time

- 14.1 Double and Iterated Integrals over Rectangles
- 14.2 Double Integrals over General Regions