

Lecture 7

- 4.3 Monotonic Functions and the First Derivative Test
- 4.4 Concavity and Curve Sketching
- 4.5 Applied Optimisation
- 4.7 Antiderivatives



Monotonic Functions and the First Derivative Test

4.3 Monotonic Functions and the First Derivative Test



Theorem

Suppose that

- $f : [a, b] \rightarrow \mathbb{R}$ is continuous; and
 - f is differentiable on (a, b) .
- 1 If $f'(x) > 0$ for all $x \in (a, b)$, then f is increasing on $[a, b]$.
 - 2 If $f'(x) < 0$ for all $x \in (a, b)$, then f is decreasing on $[a, b]$.

4.3 Monotonic Functions and the First Derivative Test



Example

Let $f(x) = x^3 - 12x - 5$.

- 1 Find the critical points of f .
- 2 Identify the intervals where f is increasing and the intervals where f is decreasing.

Clearly f is continuous and differentiable everywhere.

$$\begin{aligned}0 &= f'(x) = 3x^2 - 12 = 3(x^2 - 4) = 3(x - 2)(x + 2) \\&\implies x = -2 \text{ or } 2.\end{aligned}$$

The critical points are $x = -2$ and $x = 2$.

4.3 Monotonic Functions and the First Derivative Test



These critical points cut $(-\infty, \infty)$ into 3 open intervals:
 $(-\infty, -2)$, $(-2, 2)$ and $(2, \infty)$.



4.3 Monotonic Functions and the First Derivative Test



Interval	($-\infty, -2$)	($-2, 2$)	($2, \infty$)
Calculate $f'(x_0)$ at one point			
f' is			
f is			

4.3 Monotonic Functions and the First Derivative Test



Interval	$(-\infty, -2)$	$(-2, 2)$	$(2, \infty)$
Calculate $f'(x_0)$ at one point	$f'(-3) = 15$		
f' is	> 0		
f is	increasing		

4.3 Monotonic Functions and the First Derivative Test



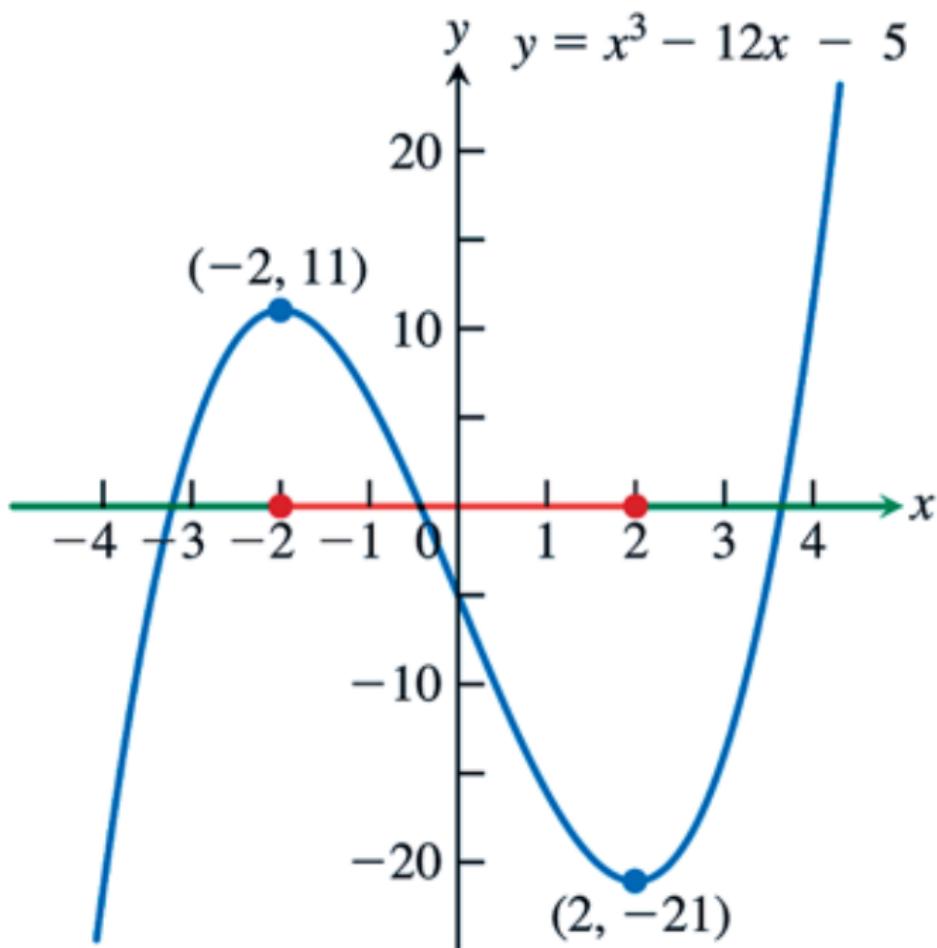
Interval	$(-\infty, -2)$	$(-2, 2)$	$(2, \infty)$
Calculate $f'(x_0)$ at one point	$f'(-3) = 15$	$f'(0) = -12$	
f' is	> 0	< 0	
f is	increasing	decreasing	

4.3 Monotonic Functions and the First Derivative Test



Interval	$(-\infty, -2)$	$(-2, 2)$	$(2, \infty)$
Calculate $f'(x_0)$ at one point	$f'(-3) = 15$	$f'(0) = -12$	$f'(3) = 15$
f' is	> 0	< 0	> 0
f is	increasing	decreasing	increasing

Therefore f is increasing on $(-\infty, -2)$ and on $(2, \infty)$, and f is decreasing on $(-2, 2)$.



4.3 Monotonic Functions and the First Derivative Test



The First Derivative Test For Local Extrema

Theorem (The First Derivative Test For Local Extrema)

Suppose that

- $f : [a, b] \rightarrow \mathbb{R}$ is continuous;
- c is a critical point of f ; and
- f is differentiable on both $(c - \delta, c)$ and $(c, c + \delta)$ for some $\delta > 0$.

<i>on the left of c</i>	<i>on the right of c</i>	<i>at c</i>

4.3 Monotonic Functions and the First Derivative Test



The First Derivative Test For Local Extrema

Theorem (The First Derivative Test For Local Extrema)

<i>on the left of c</i>	<i>on the right of c</i>	<i>at c</i>
$f' < 0$	$f' > 0$	f has a local minimum

4.3 Monotonic Functions and the First Derivative Test



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4.3 Monotonic Functions and the First Derivative Test



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$f' < 0$	$f' > 0$	f has a local minimum
$f' > 0$	$f' < 0$	f has a local maximum
$f' > 0$	$f' > 0$	f does not have a local extremem

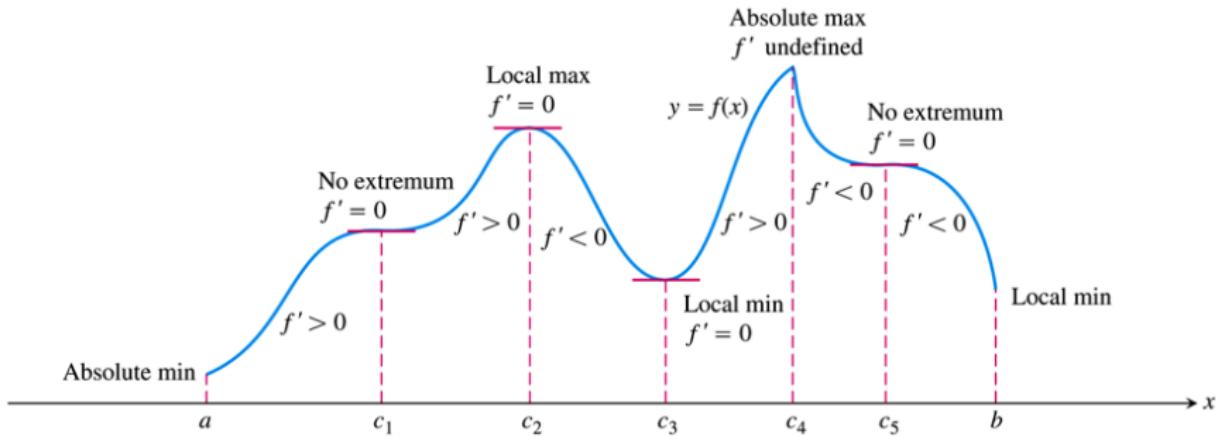
4.3 Monotonic Functions and the First Derivative Test



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$f' < 0$	$f' < 0$	f does not have a local extremem



4.3 Monotonic Functions and the First Derivative Test



Example

Let $f(x) = x^{\frac{1}{3}}(x - 4) = x^{\frac{4}{3}} - 4x^{\frac{1}{3}}$.

- 1 Find the critical points of f .
- 2 Identify the intervals on which f is increasing/decreasing.
- 3 Find the extreme values of f .

4.3 Monotonic Functions and the First Derivative Test



Example

Let $f(x) = x^{\frac{1}{3}}(x - 4) = x^{\frac{4}{3}} - 4x^{\frac{1}{3}}$.

- 1 Find the critical points of f .
- 2 Identify the intervals on which f is increasing/decreasing.
- 3 Find the extreme values of f .

f is continuous everywhere because $x^{\frac{1}{3}}$ and $(x - 4)$ are continuous functions. We can calculate that

$$\begin{aligned}f'(x) &= \frac{d}{dx} \left(x^{\frac{4}{3}} - 4x^{\frac{1}{3}} \right) = \frac{4}{3}x^{\frac{1}{3}} - \frac{4}{3}x^{-\frac{2}{3}} \\&= \frac{4}{3}x^{-\frac{2}{3}}(x - 1) = \frac{4(x - 1)}{3x^{\frac{2}{3}}}.\end{aligned}$$

$f'(x)$ does not exist if $x = 0$. $f'(x) = 0$ if and only if $x = 1$. The critical points of f are $x = 0$ and $x = 1$.

4.3 Monotonic Functions and the First Derivative Test



Using the critical points, we “cut” $(-\infty, \infty)$ into three subintervals: $(-\infty, 0)$, $(0, 1)$ and $(1, \infty)$.

4.3 Monotonic Functions and the First Derivative Test



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$(-\infty, 0)$	$(0, 1)$	$(1, \infty)$
$f' < 0$	$f' < 0$	$f' > 0$
(e.g. $f'(-1) = -\frac{8}{3}$)		
f is decreasing	f is decreasing	f is increasing

4.3 Monotonic Functions and the First Derivative Test

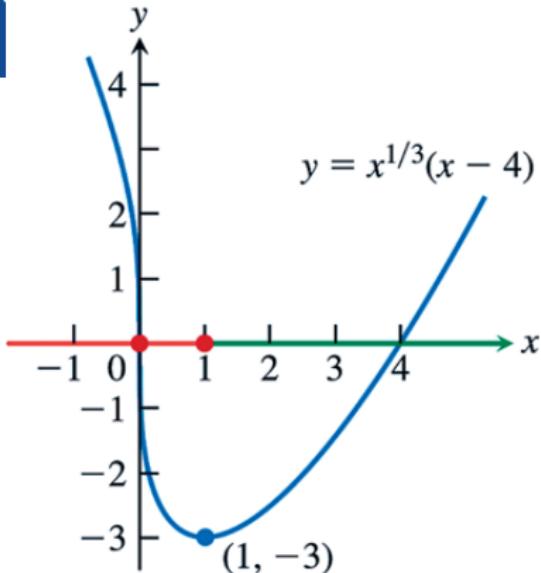


Using the critical points, we “cut” $(-\infty, \infty)$ into three subintervals: $(-\infty, 0)$, $(0, 1)$ and $(1, \infty)$.

$(-\infty, 0)$	$(0, 1)$	$(1, \infty)$
$f' < 0$	$f' < 0$	$f' > 0$
(e.g. $f'(-1) = -\frac{8}{3}$)		
f is decreasing	f is decreasing	f is increasing

We can see from this table that $x = 1$ is a local minimum and $x = 0$ is not an extremum.

4.3 Monotonic Functions and the First Derivative Test



So

$$\min_{x \in \mathbb{R}} f(x) = f(1) = 1^{\frac{1}{3}}(1 - 4) = -3.$$

Note that f does not have an absolute maximum.

Note that $\lim_{x \rightarrow 0} f'(x) = -\infty$. Therefore the graph of $y = f(x)$ has

EXAMPLE 3 Within the interval $0 \leq x \leq 2\pi$, find the critical points of

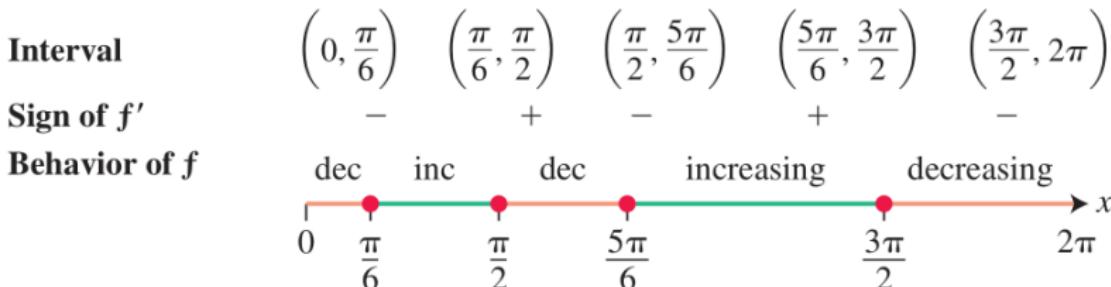
$$f(x) = \sin^2 x - \sin x - 1.$$

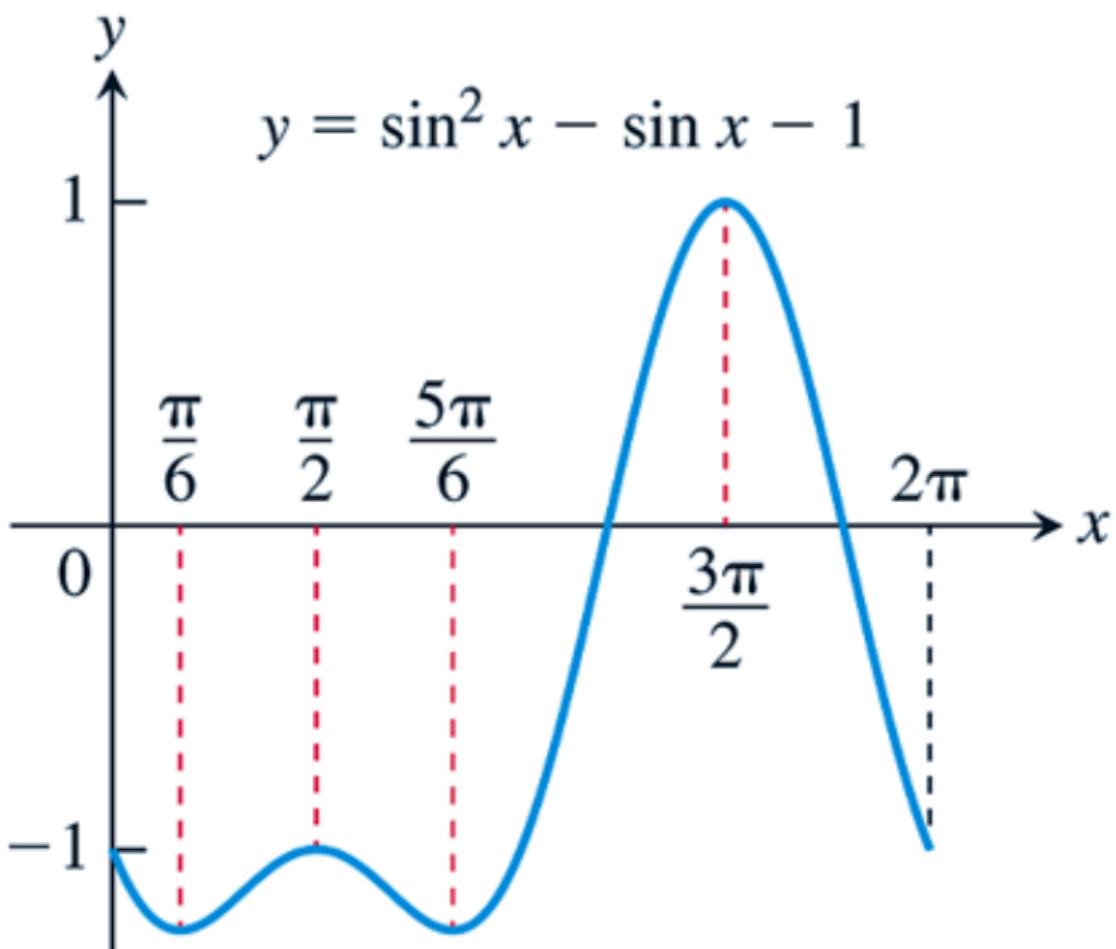
Identify the open intervals on which f is increasing and decreasing. Find the function's local and absolute extreme values.

Solution The function f is continuous over $[0, 2\pi]$ and differentiable over $(0, 2\pi)$, so the critical points occur at the zeros of f' in $(0, 2\pi)$. We find

$$f'(x) = 2 \sin x \cos x - \cos x = (2 \sin x - 1)(\cos x).$$

The first derivative is zero if and only if $\sin x = \frac{1}{2}$ or $\cos x = 0$. So the critical points of f in $(0, 2\pi)$ are $x = \pi/6$, $x = 5\pi/6$, $x = \pi/2$, and $x = 3\pi/2$. They partition $[0, 2\pi]$ into open intervals as follows.

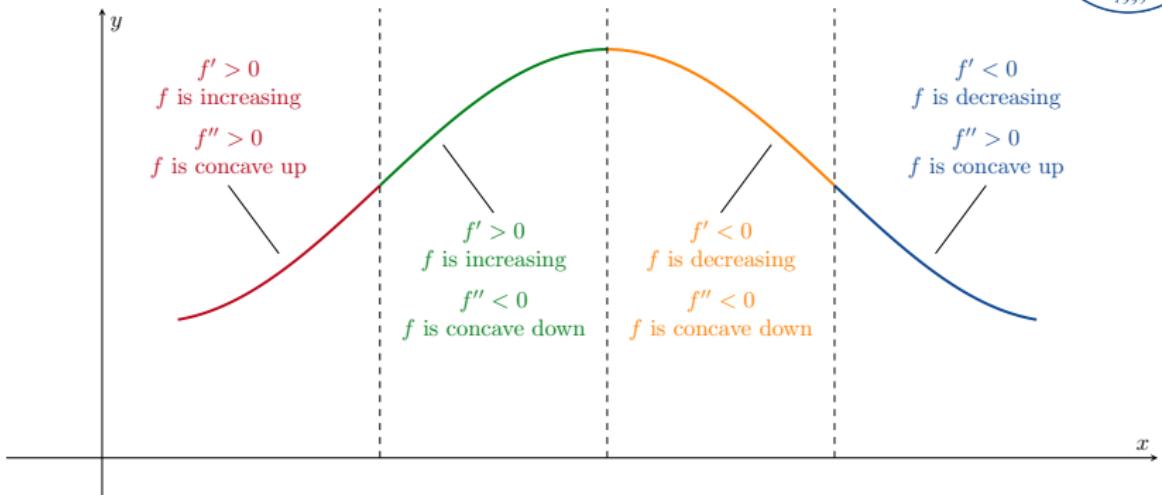




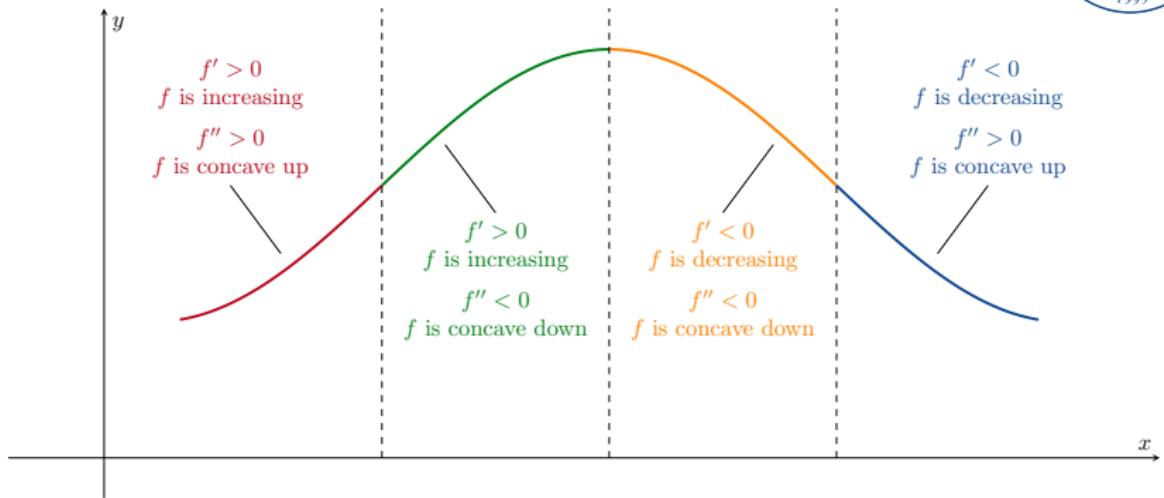


Concavity and Curve Sketching

4.4 Concavity and Curve Sketching



4.4 Concavity and Curve Sketching



Definition

$y = f(x)$ is

- 1 *concave up* if f' is increasing; and
- 2 *concave down* if f' is decreasing.

4.4 Concavity and Curve Sketching



Theorem (The Second Derivative Test for Concavity)

Suppose that $f : I \rightarrow \mathbb{R}$ is twice differentiable.

- 1 If $f'' > 0$ on I , then f is concave up on I .
- 2 If $f'' < 0$ on I , then f is concave down on I .

4.4 Concavity and Curve Sketching



Example

Consider $y = x^3$. Then $y' = 3x^2$ and $y'' = 6x$.

$(-\infty, 0)$	$(0, \infty)$
$y'' < 0$	$y'' > 0$
$y = x^3$ is concave down	$y = x^3$ is concave up

4.4 Concavity and Curve Sketching



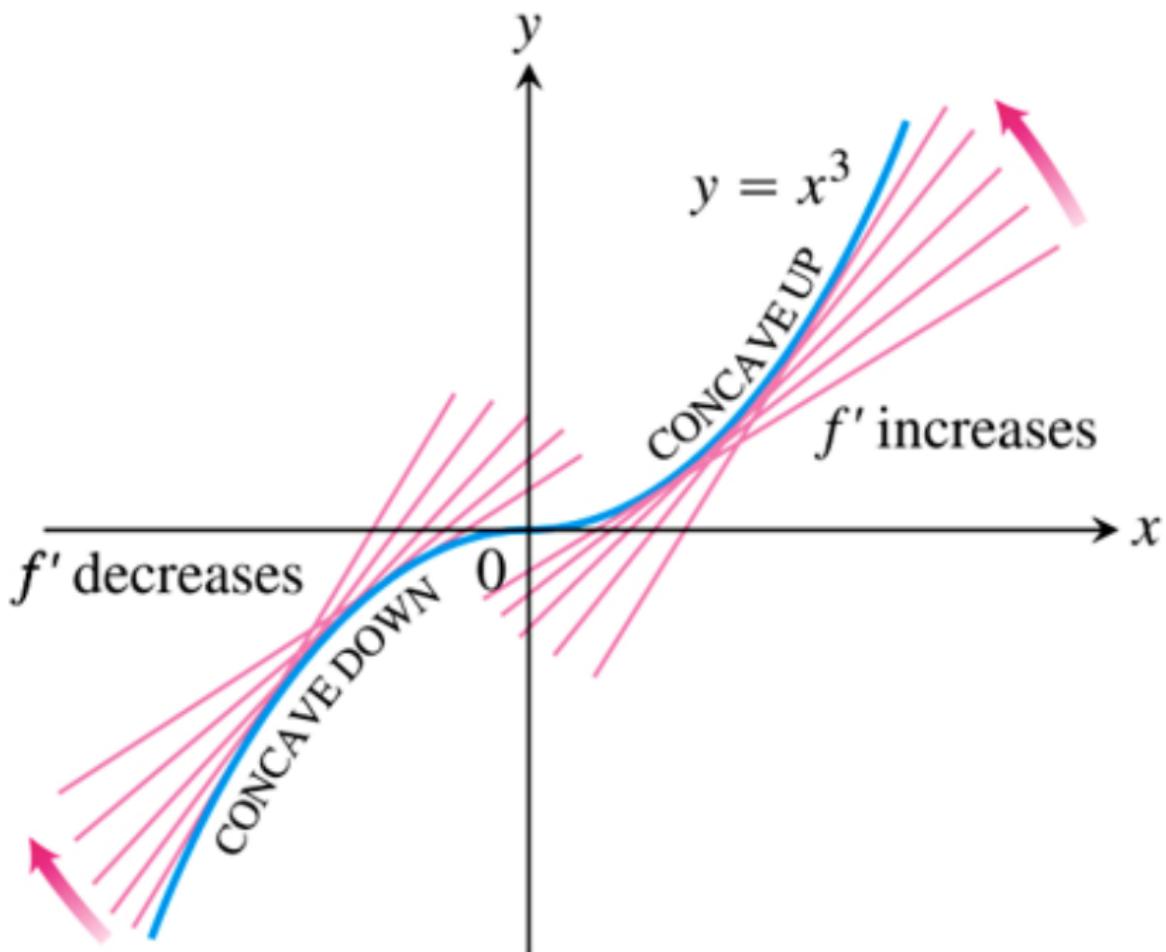
Example

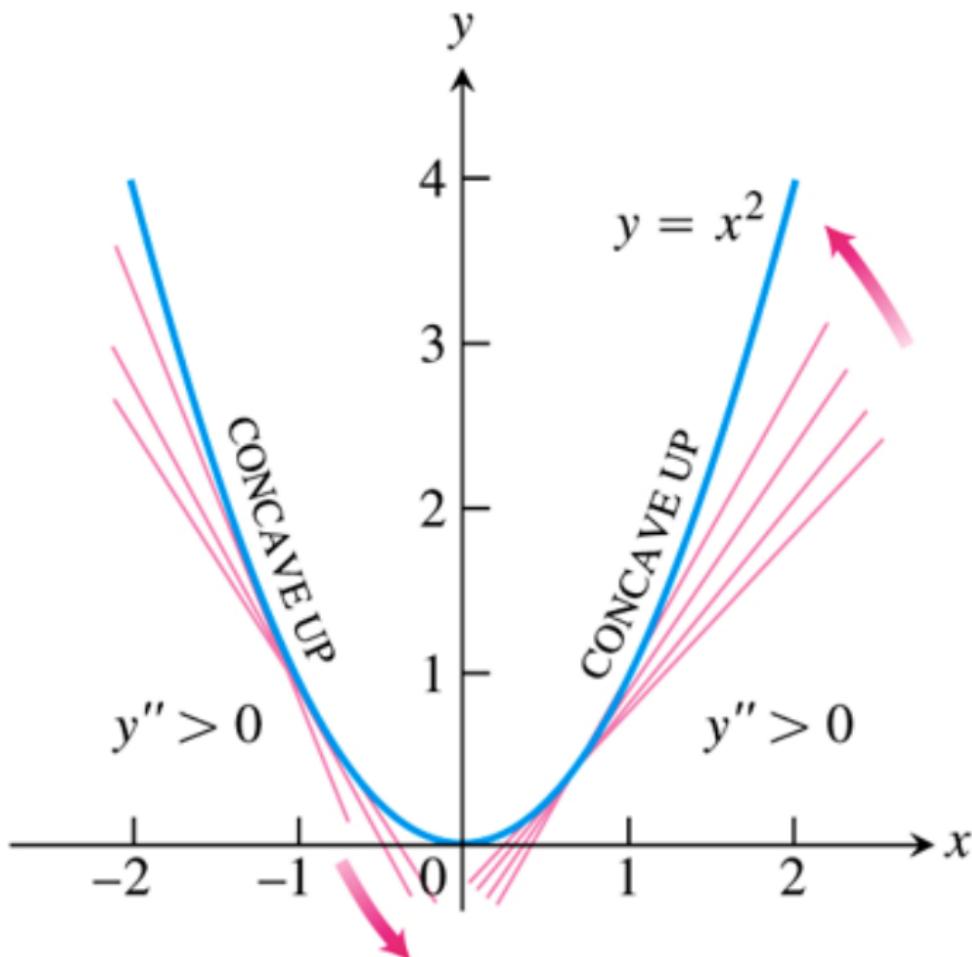
Consider $y = x^3$. Then $y' = 3x^2$ and $y'' = 6x$.

$(-\infty, 0)$	$(0, \infty)$
$y'' < 0$	$y'' > 0$
$y = x^3$ is concave down	$y = x^3$ is concave up

Example

Consider $y = x^2$. Since $y' = 2x$ and $y'' = 2$, we have that $y'' > 0$ everywhere. Therefore $y = x^2$ is concave up everywhere.





4.4 Concavity and Curve Sketching



Example

Determine the concavity of $y = 3 + \sin x$ on $[0, 2\pi]$.

4.4 Concavity and Curve Sketching

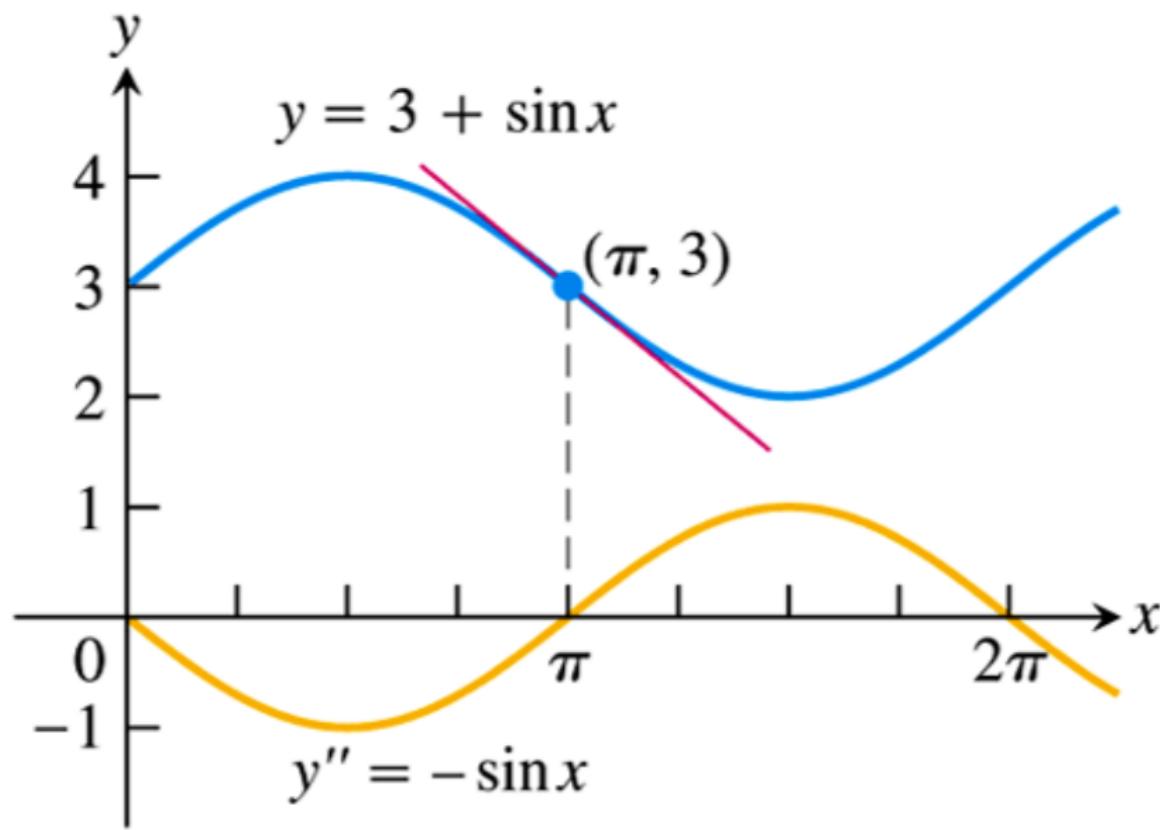


Example

Determine the concavity of $y = 3 + \sin x$ on $[0, 2\pi]$.

First we calculate that $y' = \cos x$ and $y'' = -\sin x$.

$(0, \pi)$	$(\pi, 2\pi)$
$y'' < 0$	$y'' > 0$
$y = 3 + \sin x$ is concave down	$y = 3 + \sin x$ is concave up



Points of Inflection

Definition

$(c, f(c))$ is a *point of inflection* of $y = f(x)$ if

- $y = f(x)$ has a tangent line at $x = c$; and
- the concavity of $y = f(x)$ changes at $x = c$.

4.4 Concavity and Curve Sketching



Points of Inflection

Definition

$(c, f(c))$ is a *point of inflection* of $y = f(x)$ if

- $y = f(x)$ has a tangent line at $x = c$; and
- the concavity of $y = f(x)$ changes at $x = c$.

Remark

$(c, f(c))$ is a
point of inflection

$$\implies$$

- $f''(c) = 0$;
- or
- $f''(c)$ does not exist.

EXAMPLE 3 Determine the concavity and find the inflection points of the function

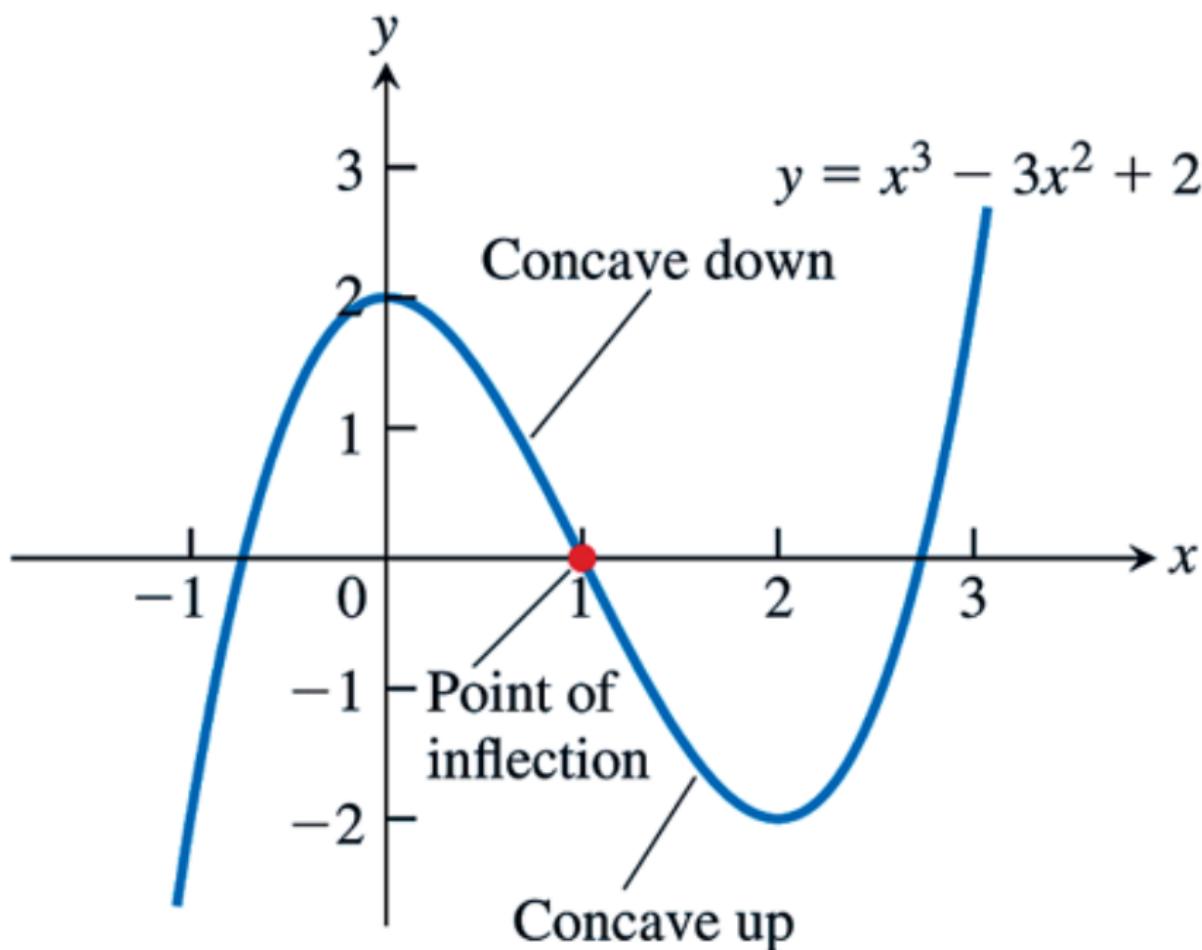
$$f(x) = x^3 - 3x^2 + 2.$$

Solution We start by computing the first and second derivatives.

$$f'(x) = 3x^2 - 6x, \quad f''(x) = 6x - 6.$$

To determine concavity, we look at the sign of the second derivative $f''(x) = 6x - 6$. The sign is negative when $x < 1$, is 0 at $x = 1$, and is positive when $x > 1$. It follows that the graph of f is concave down on $(-\infty, 1)$, is concave up on $(1, \infty)$, and has an inflection point at the point $(1, 0)$ where the concavity changes.

The graph of f is shown in Figure 4.27. Notice that we did not need to know the shape of this graph ahead of time in order to determine its concavity. ■



4.4 Concavity and Curve Sketching



Example

Find all the points of inflection of $f(x) = x^{\frac{5}{3}}$.

Since $f'(x) = \frac{5}{3}x^{\frac{2}{3}}$ and

$$f''(x) = \frac{d}{dx} \left(\frac{5}{3}x^{\frac{2}{3}} \right) = \frac{10}{9}x^{-\frac{1}{3}} = \frac{10}{9\sqrt[3]{x}},$$

4.4 Concavity and Curve Sketching



Example

Find all the points of inflection of $f(x) = x^{\frac{5}{3}}$.

Since $f'(x) = \frac{5}{3}x^{\frac{2}{3}}$ and

$$f''(x) = \frac{d}{dx} \left(\frac{5}{3}x^{\frac{2}{3}} \right) = \frac{10}{9}x^{-\frac{1}{3}} = \frac{10}{9\sqrt[3]{x}},$$

we can say that

- if $x < 0$, then $f''(x) < 0$;
- $f''(0)$ does not exist; and
- if $x > 0$, then $f''(x) > 0$.

4.4 Concavity and Curve Sketching



Example

Find all the points of inflection of $f(x) = x^{\frac{5}{3}}$.

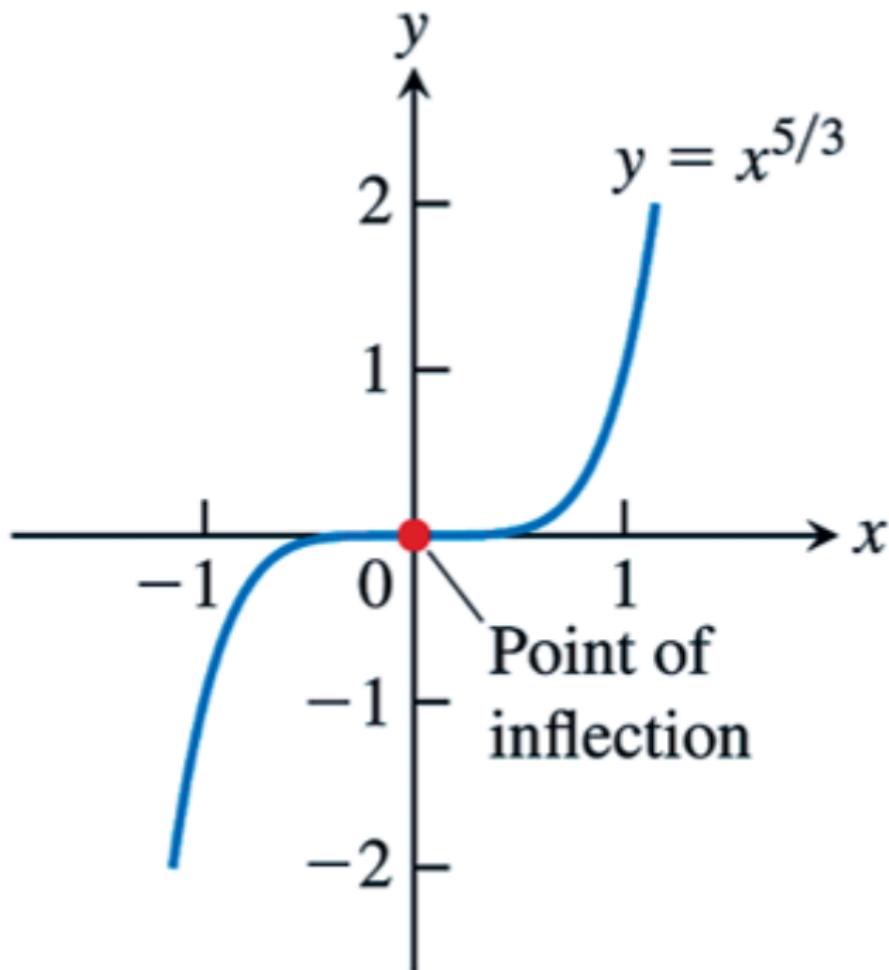
Since $f'(x) = \frac{5}{3}x^{\frac{2}{3}}$ and

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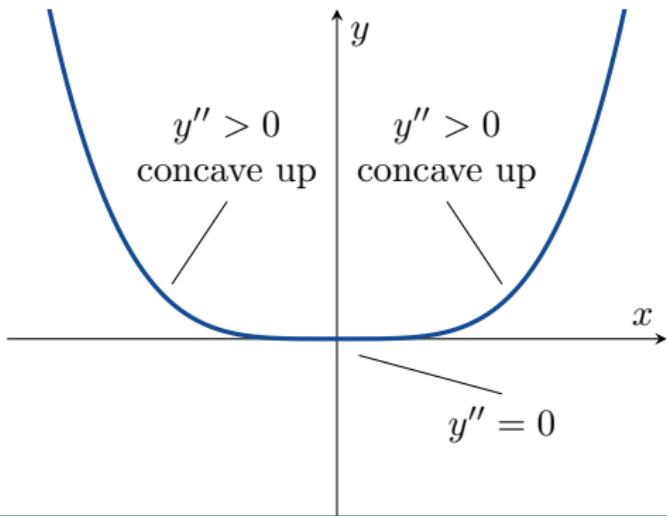
we can say that

- if $x < 0$, then $f''(x) < 0$;
- $f''(0)$ does not exist; and
- if $x > 0$, then $f''(x) > 0$.

Therefore $(0, 0)$ is an point of inflection of $y = x^{\frac{5}{3}}$.



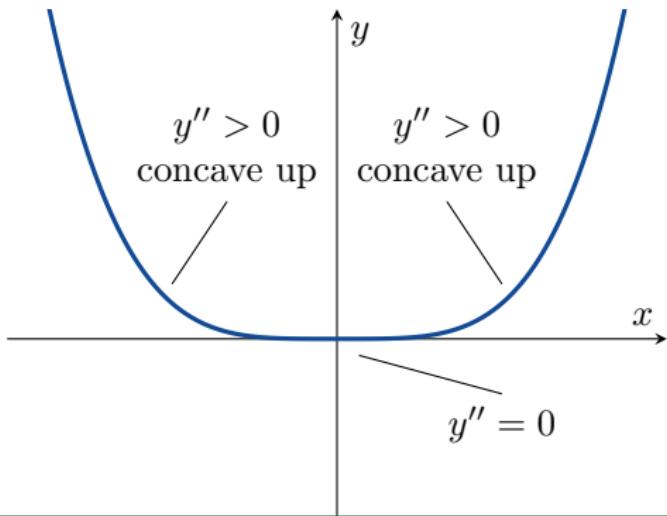
4.4 Concavity and Curve Sketching



Example

Let $y = x^4$. Then $y' = 4x^3$ and $y = 12x^2$.

4.4 Concavity and Curve Sketching



Example

Let $y = x^4$. Then $y' = 4x^3$ and $y = 12x^2$. Note that $y'' = 0$ at $x = 0$, but the concavity of the graph does not change. Hence $(0, 0)$ is not a point of inflection of $y = x^4$.

4.4 Concavity and Curve Sketching



Example

Let $y = x^{\frac{1}{3}}$. Then $y' = \frac{1}{3}x^{-\frac{2}{3}}$ and $y'' = -\frac{2}{9}x^{-\frac{5}{3}}$. Note that y'' does not exist at $x = 0$.

4.4 Concavity and Curve Sketching

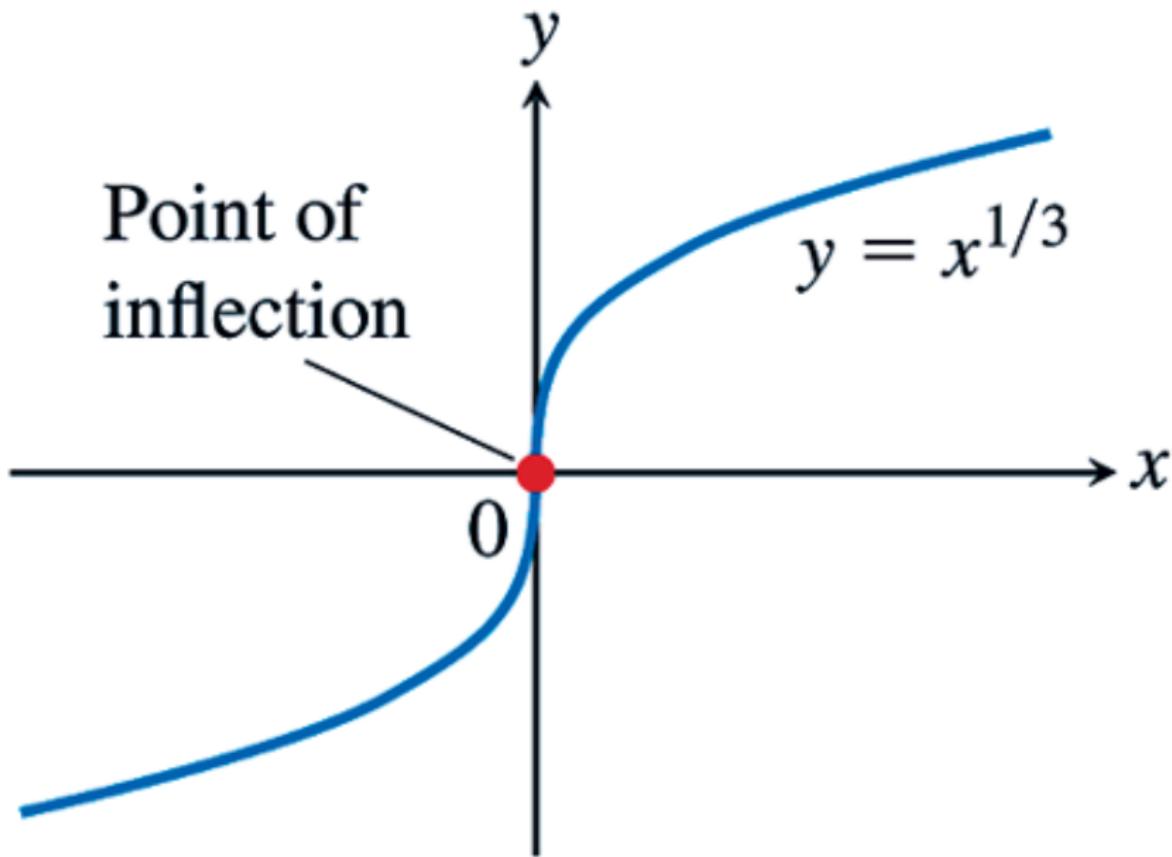


Example

Let $y = x^{\frac{1}{3}}$. Then $y' = \frac{1}{3}x^{-\frac{2}{3}}$ and $y'' = -\frac{2}{9}x^{-\frac{5}{3}}$. Note that y'' does not exist at $x = 0$.

$(-\infty, 0)$	$(0, \infty)$
$y'' > 0$	$y'' < 0$
$y = x^{\frac{1}{3}}$ is concave up	$y = x^{\frac{1}{3}}$ is concave down

$(0, 0)$ is a point of inflection of $y = x^{\frac{1}{3}}$.



The Second Derivative Test for Local Extrema

Theorem (The Second Derivative Test for Local Extrema)

Suppose that

- f'' is continuous on (a, b) ; and
- $c \in (a, b)$.

The Second Derivative Test for Local Extrema

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- 1 If $f'(c) = 0$ and $f''(c) < 0$, then f has a local maximum at $x = c$.

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- 1 If $f'(c) = 0$ and $f''(c) < 0$, then f has a local maximum at $x = c$.
 - 2 If $f'(c) = 0$ and $f''(c) > 0$, then f has a local minimum at $x = c$.
 - 3 If $f'(c) = 0$ and $f''(c) = 0$, then we don't know – we need to use a different theorem.

4.4 Concavity and Curve Sketching



Example

Let $f(x) = x^4 - 4x^3 + 10$.

- 1 Find where the local extrema are.
- 2 Find the intervals where f is increasing/decreasing.
- 3 Find the intervals where f is concave up/concave down.
- 4 Sketch the general shape of $y = f(x)$.
- 5 Plot some points which satisfy $y = f(x)$.
- 6 Graph $y = f(x)$.

4.4 Concavity and Curve Sketching



- $f(x) = x^4 - 4x^3 + 10$ is continuous because it is a polynomial.
- The domain of f is $(-\infty, \infty)$.
- Clearly $f'(x) = 4x^3 - 12x^2$.
- The domain of f' is also $(-\infty, \infty)$.
- To find the critical points, we need to solve $f'(x) = 0$.

4.4 Concavity and Curve Sketching



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- Clearly $f'(x) = 4x^3 - 12x^2$.
- The domain of f' is also $(-\infty, \infty)$.
- To find the critical points, we need to solve $f'(x) = 0$.

$$0 = f'(x) = 4x^3 - 12x^2 = 4x^2(x - 3) \implies x = 0 \text{ or } x = 3.$$

4.4 Concavity and Curve Sketching



Interval	$(-\infty, 0)$	$(0, 3)$	$(3, \infty)$
f' is	$f' < 0$	$f' < 0$	$f' > 0$
f is	decreasing	decreasing	increasing

4.4 Concavity and Curve Sketching



Interval	$(-\infty, 0)$	$(0, 3)$	$(3, \infty)$
f' is	$f' < 0$	$f' < 0$	$f' > 0$
f is	decreasing	decreasing	increasing

- 1 By the First Derivative Test, $x = 3$ is a local minimum and $x = 0$ is not an extrema.

4.4 Concavity and Curve Sketching



Interval	$(-\infty, 0)$	$(0, 3)$	$(3, \infty)$
f' is	$f' < 0$	$f' < 0$	$f' > 0$
f is	decreasing	decreasing	increasing

- 1 By the First Derivative Test, $x = 3$ is a local minimum and $x = 0$ is not an extrema.
- 2 f is decreasing on $(-\infty, 0]$ and on $[0, 3]$. f is increasing on $[3, \infty)$.

4.4 Concavity and Curve Sketching



- 3 Next we need to solve $f''(x) = 0$.

4.4 Concavity and Curve Sketching



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$$0 = f''(x) = 12x^2 - 24x \implies x = 0 \text{ or } x = 2.$$

4.4 Concavity and Curve Sketching



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Interval	$(-\infty, 0)$	$(0, 2)$	$(2, \infty)$
f'' is	$f'' > 0$	$f'' < 0$	$f'' > 0$
f is	concave up	concave down	concave up

4.4 Concavity and Curve Sketching



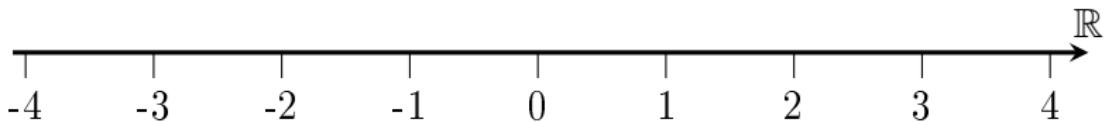
- 3 Next we need to solve $f''(x) = 0$.

$$0 = f''(x) = 12x^2 - 24x \implies x = 0 \text{ or } x = 2.$$

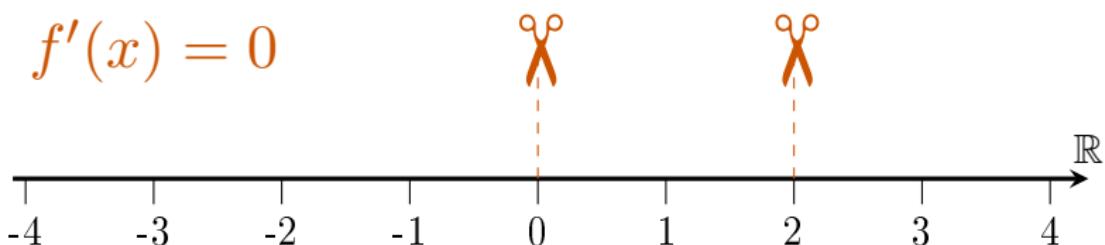
Interval	$(-\infty, 0)$	$(0, 2)$	$(2, \infty)$
f'' is	$f'' > 0$	$f'' < 0$	$f'' > 0$
f is	concave up	concave down	concave up

f is concave up on $(-\infty, 0)$ and on $(2, \infty)$. f is concave down on $(0, 2)$.

4.4 Concavity and Curve Sketching



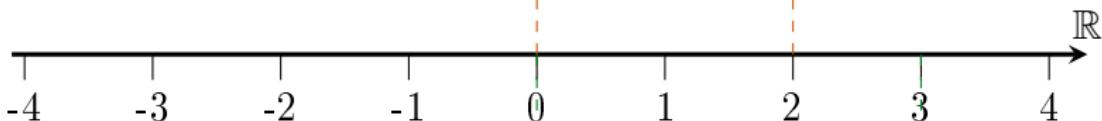
4.4 Concavity and Curve Sketching



4.4 Concavity and Curve Sketching



$$f'(x) = 0$$



$$f''(x) = 0$$



4.4 Concavity and Curve Sketching



$$f'(x) = 0$$



$$f''(x) = 0$$



We need to consider 4 intervals: $(-\infty, 0)$, $(0, 2)$, $(2, 3)$, $(3, \infty)$.

4.4 Concavity and Curve Sketching



- 4 Putting the previous two tables together, we obtain

$(-\infty, 0)$	$(0, 2)$	$(2, 3)$	$(3, \infty)$
decreasing	decreasing	decreasing	increasing
concave up	concave down	concave up	concave up
			

4.4 Concavity and Curve Sketching



- 4 Putting the previous two tables together, we obtain

$(-\infty, 0)$	$(0, 2)$	$(2, 3)$	$(3, \infty)$
decreasing	decreasing	decreasing	increasing
concave up	concave down	concave up	concave up



Therefore the general shape of f is

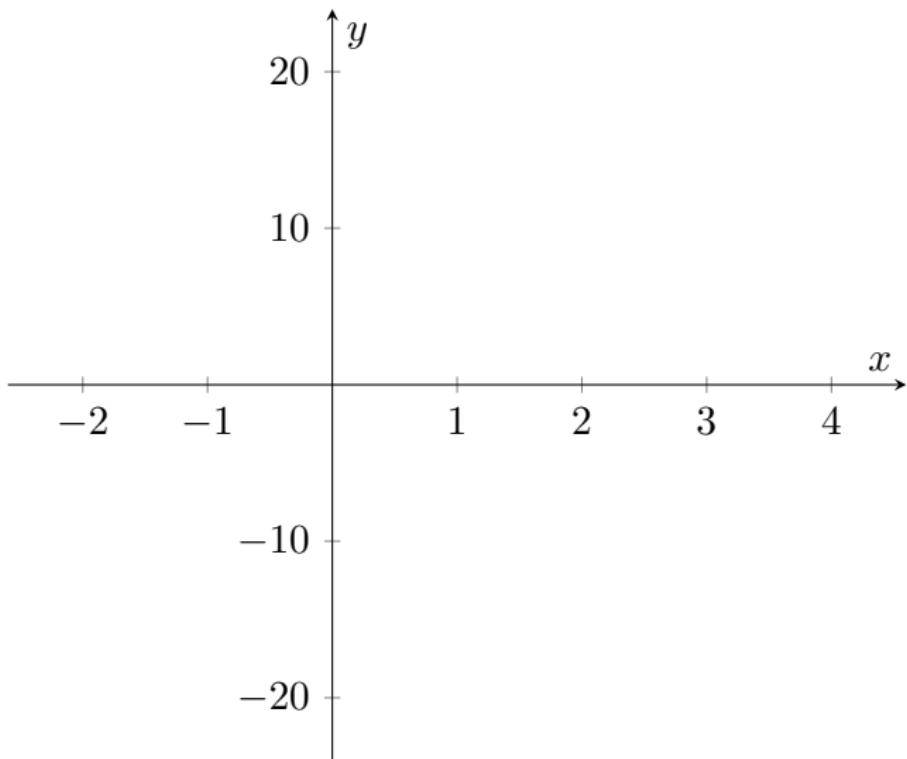


4.4 Concavity and Curve Sketching



- 5 We calculate some (x, y) points.

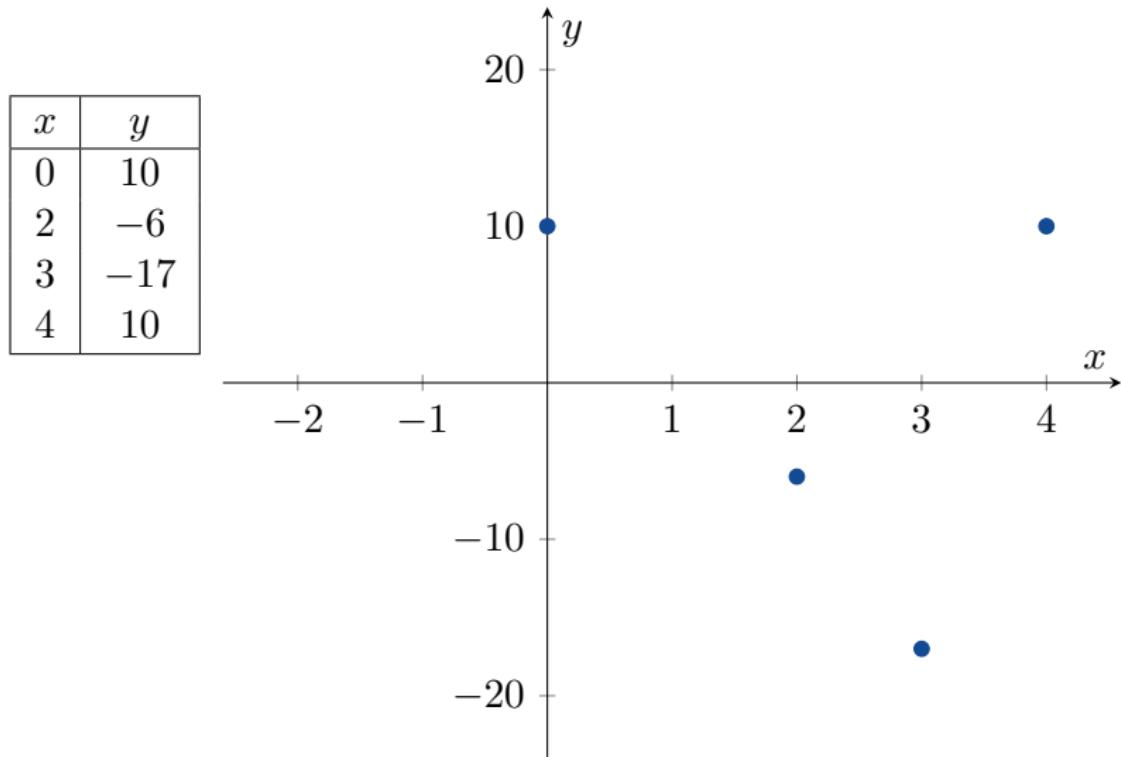
x	y
0	10
2	-6
3	-17
4	10



4.4 Concavity and Curve Sketching



- 5 We calculate some (x, y) points. Then we plot these points.

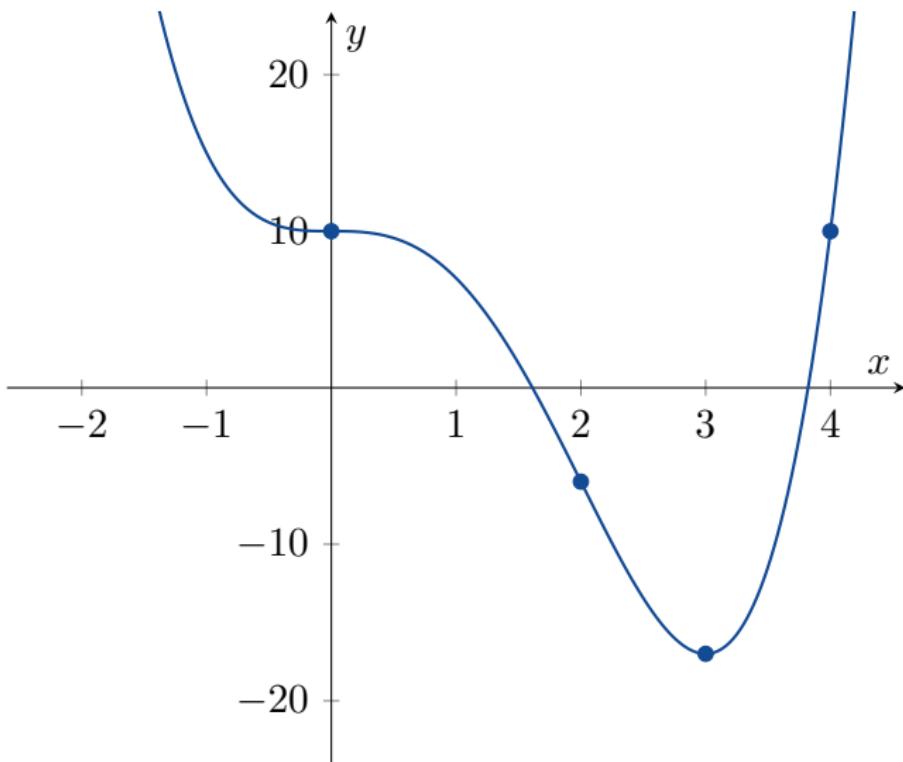


4.4 Concavity and Curve Sketching



- 6 Finally we can graph $y = x^4 - 4x^3 + 10$.

x	y
0	10
2	-6
3	-17
4	10



Procedure for Graphing $y = f(x)$

1. Identify the domain of f and any symmetries the curve may have.
2. Find the derivatives y' and y'' .
3. Find the critical points of f , if any, and identify the function's behavior at each one.
4. Find where the curve is increasing and where it is decreasing.
5. Find the points of inflection, if any occur, and determine the concavity of the curve.
6. Identify any asymptotes that may exist.
7. Plot key points, such as the intercepts and the points found in Steps 3–5, and sketch the curve together with any asymptotes that exist.

4.4 Concavity and Curve Sketching



Example

Sketch the graph of $f(x) = \frac{(x + 1)^2}{1 + x^2}$.

4.4 Concavity and Curve Sketching



Example

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The domain of f is $(-\infty, \infty)$. f is not even. f is not odd.

4.4 Concavity and Curve Sketching



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You can check that

$$f'(x) = \frac{2(1-x^2)}{(1+x^2)^2}$$

and

$$f''(x) = \frac{4x(x^2-3)}{(1+x^2)^3}.$$

4.4 Concavity and Curve Sketching

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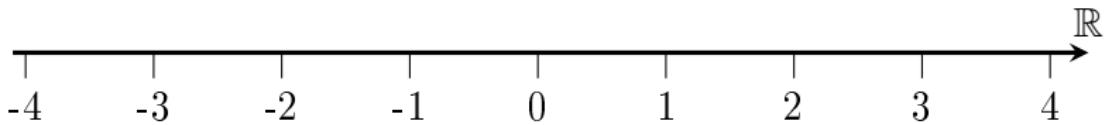
Moreover

$$f'(x) = 0 \implies x = \pm 1$$

and

$$f''(x) = 0 \implies x = 0 \text{ or } \pm \sqrt{3}.$$

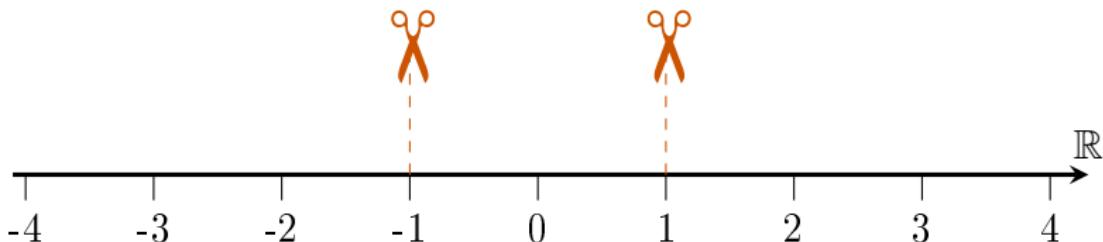
4.4 Concavity and Curve Sketching



4.4 Concavity and Curve Sketching



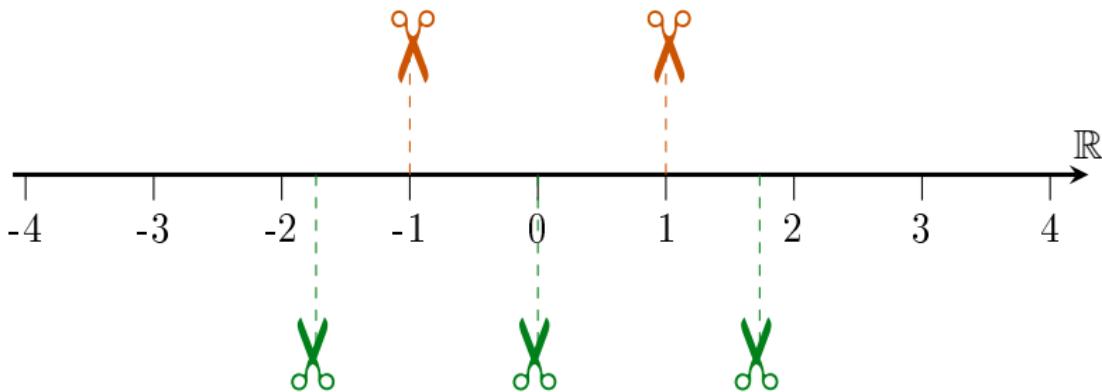
$$f'(x) = 0$$



4.4 Concavity and Curve Sketching

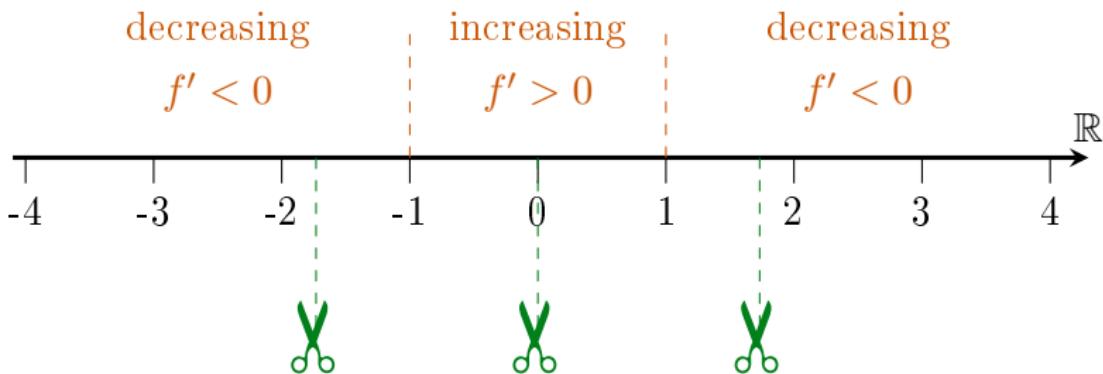


$$f'(x) = 0$$



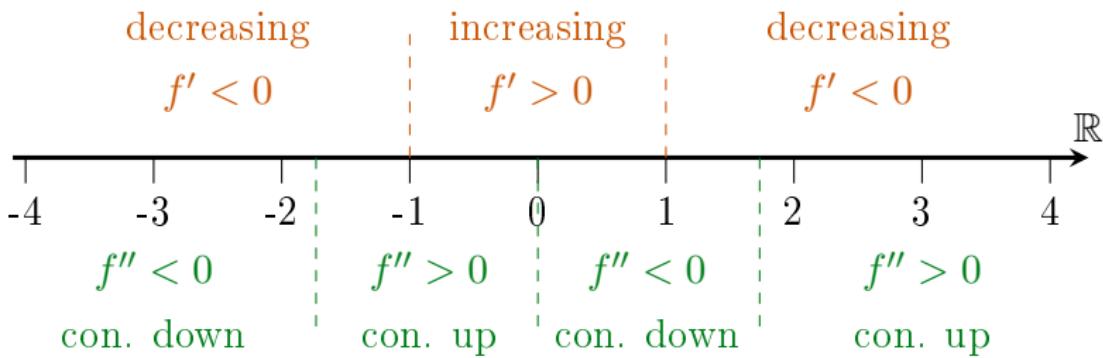
$$f''(x) = 0$$

4.4 Concavity and Curve Sketching



$$f''(x) = 0$$

4.4 Concavity and Curve Sketching



4.4 Concavity and Curve Sketching



$(-\infty, -\sqrt{3})$	$(-\sqrt{3}, -1)$	$(-1, 0)$	$(0, 1)$	$(1, \sqrt{3})$	$(\sqrt{3}, \infty)$
decrea. con. down	decrea. con. up	increa. con. up	increa. con. down	decrea. con. down	decrea. con. up

4.4 Concavity and Curve Sketching



Asymptotes:

4.4 Concavity and Curve Sketching



Asymptotes:

Since

$$\begin{aligned}\lim_{x \rightarrow \pm\infty} f(x) &= \lim_{x \rightarrow \pm\infty} \frac{(x+1)^2}{1+x^2} = \lim_{x \rightarrow \pm\infty} \frac{x^2 + 2x + 1}{1+x^2} \\ &= \lim_{x \rightarrow \pm\infty} \frac{1 + \frac{2}{x} + \frac{1}{x^2}}{\frac{1}{x^2} + 1} = 1,\end{aligned}$$

the line $y = 1$ is a horizontal asymptote.

4.4 Concavity and Curve Sketching



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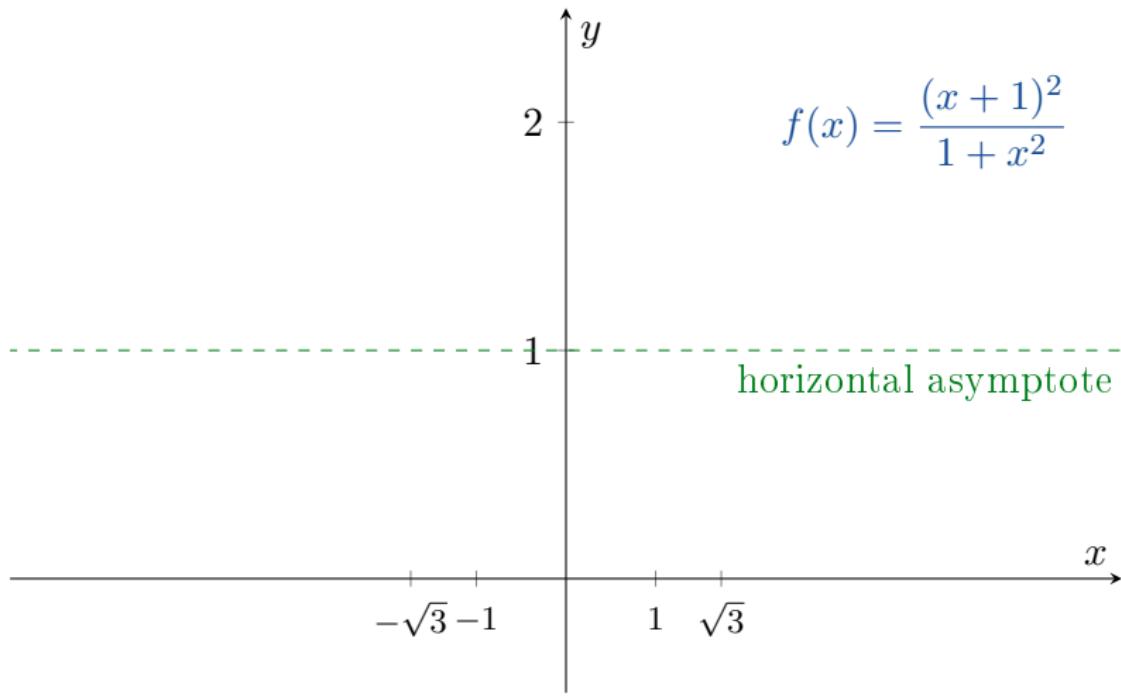
the line $y = 1$ is a horizontal asymptote.

There are no vertical asymptotes because $f(x)$ is defined everywhere.

4.4 Concavity and Curve Sketching



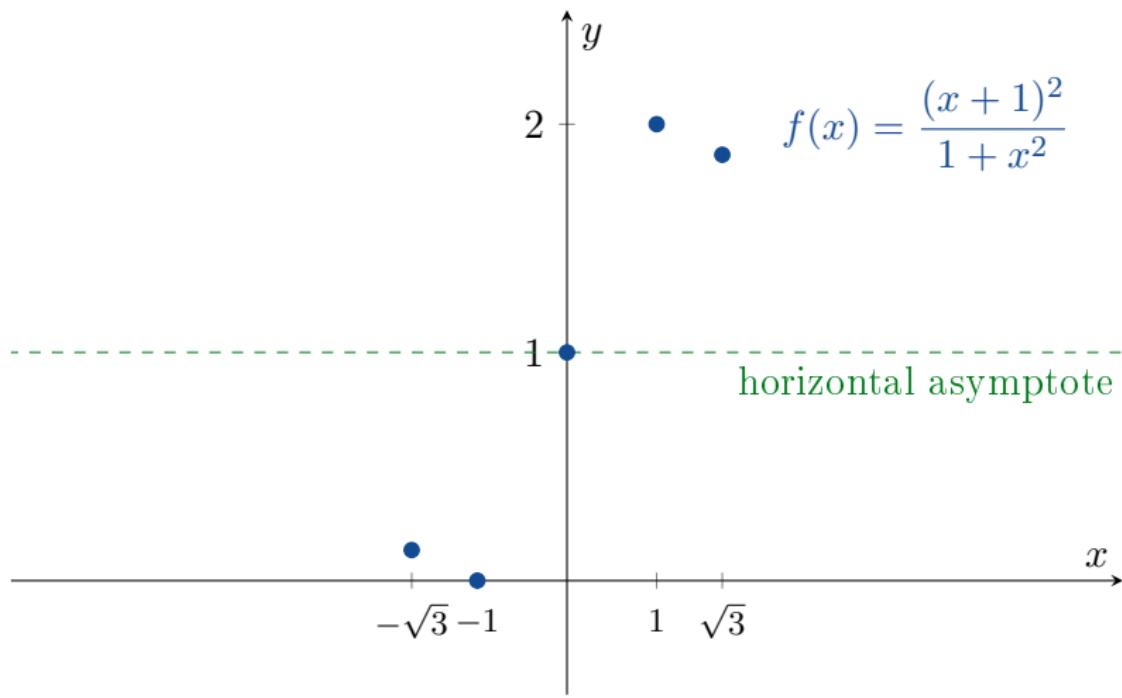
Now we have enough information to sketch the graph



4.4 Concavity and Curve Sketching



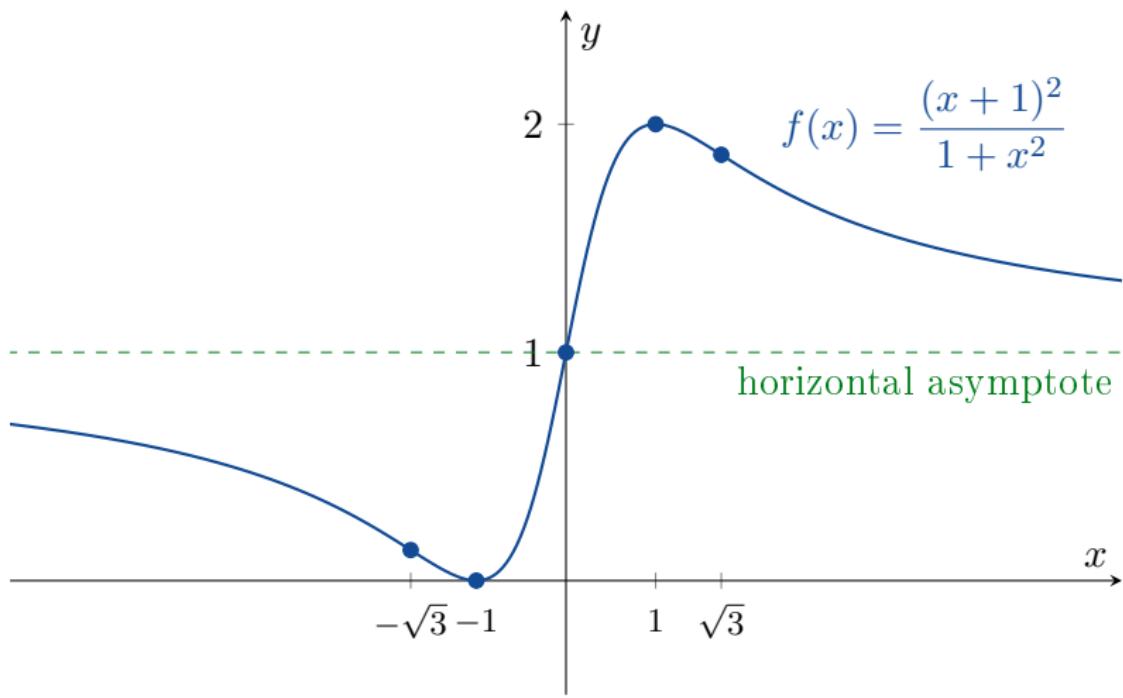
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4.4 Concavity and Curve Sketching



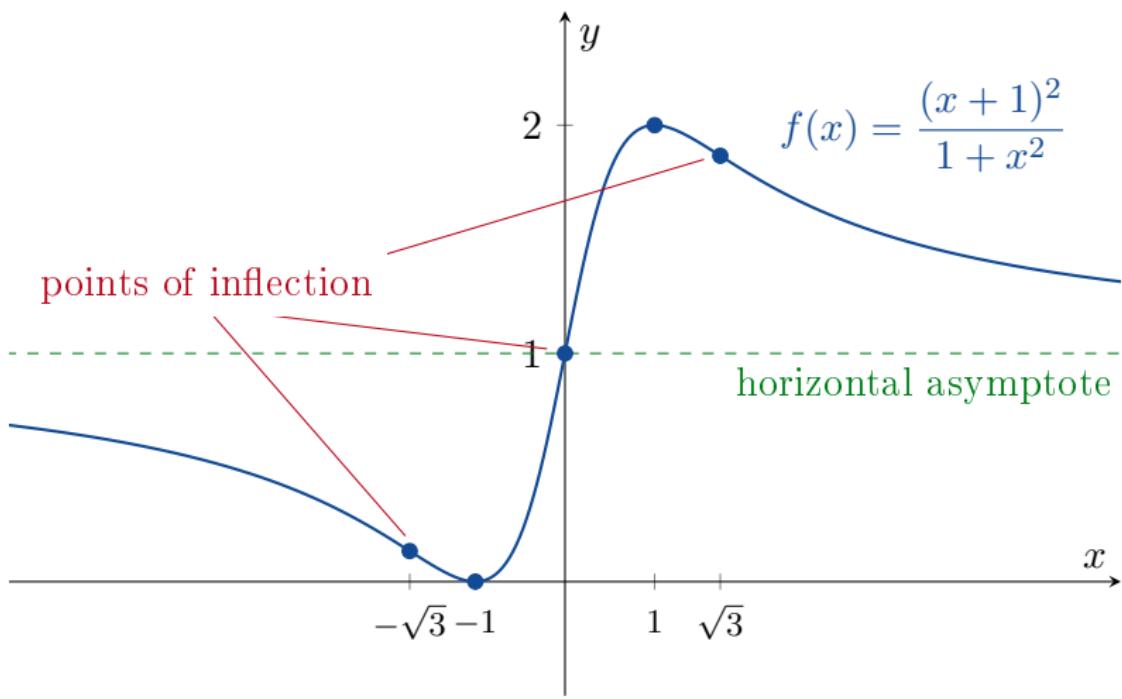
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4.4 Concavity and Curve Sketching



Now we have enough information to sketch the graph



4.4 Concavity and Curve Sketching



Example

Sketch the graph of $f(x) = \frac{x^2 + 4}{2x}$.

The domain is everything except $x = 0$. f is an odd function.

Note that

$$f(x) = \frac{x^2 + 4}{2x} = \frac{x}{2} + \frac{2}{x},$$

$$f'(x) = \frac{1}{2} - \frac{2}{x^2} = \frac{x^2 - 4}{2x^2} \quad \text{and} \quad f''(x) = \frac{4}{x^3}.$$

4.4 Concavity and Curve Sketching



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The critical points are $x = -2$ and $x = 2$.

4.4 Concavity and Curve Sketching



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4.4 Concavity and Curve Sketching



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The critical points are $x = -2$ and $x = 2$. There are no points of inflection because f'' is defined everywhere (except $x = 0$) and $f'' \neq 0$.

So we need to consider 4 intervals: $(-\infty, -2)$, $(-2, 0)$, $(0, 2)$ and $(2, \infty)$.

4.4 Concavity and Curve Sketching



$(-\infty, -2)$	$(-2, 0)$	$(0, 2)$	$(2, \infty)$
increasing	decreasing	decreasing	increasing
concave down	concave down	concave up	concave up
A blue curve segment that is concave down and decreasing from left to right.	A blue curve segment that is concave down and decreasing from left to right.	A blue curve segment that is concave up and decreasing from left to right.	A blue curve segment that is concave up and increasing from left to right.

4.4 Concavity and Curve Sketching



Asymptotes: Since

$$\lim_{x \rightarrow 0^+} f(x) = \lim_{x \rightarrow 0^+} \frac{x}{2} + \frac{2}{x} = \infty$$

and

$$\lim_{x \rightarrow 0^-} f(x) = \lim_{x \rightarrow 0^-} \frac{x}{2} + \frac{2}{x} = -\infty,$$

the line $x = 0$ is a vertical asymptote.

4.4 Concavity and Curve Sketching



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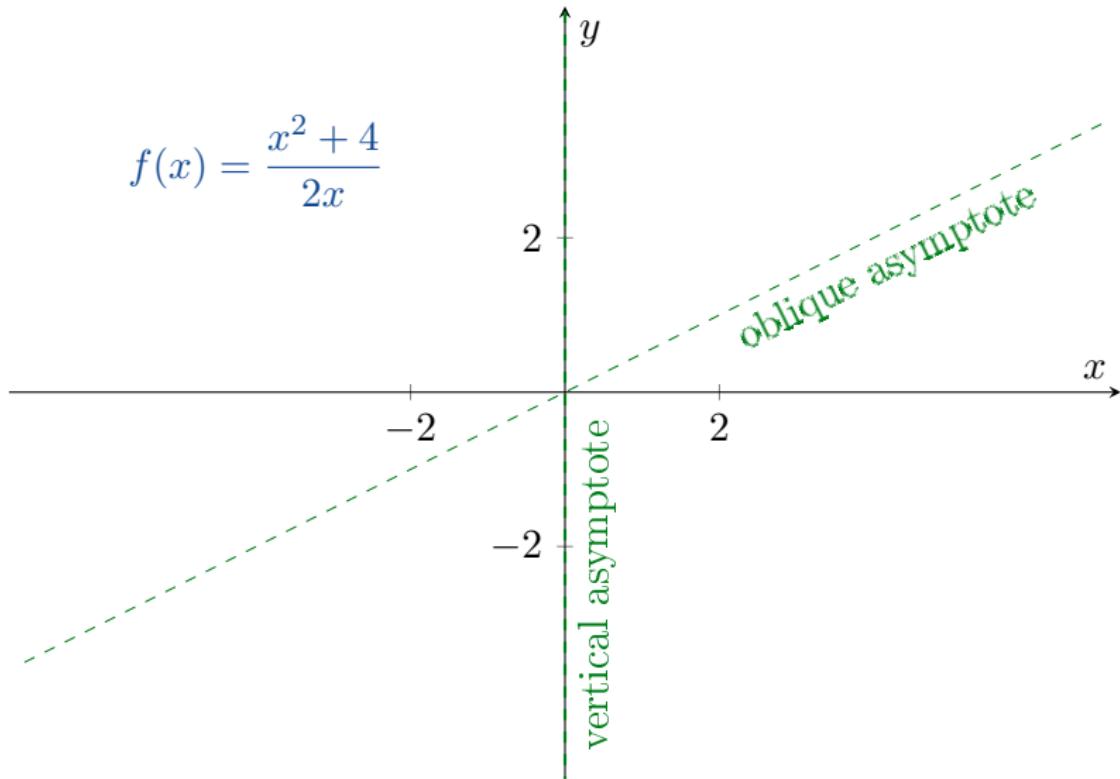
the line $x = 0$ is a vertical asymptote.

Since $f(x) = \frac{x}{2} + \frac{2}{x}$ and since $\lim_{x \rightarrow \pm\infty} \frac{2}{x} = 0$, the line $y = \frac{x}{2}$ is an oblique asymptote.

4.4 Concavity and Curve Sketching



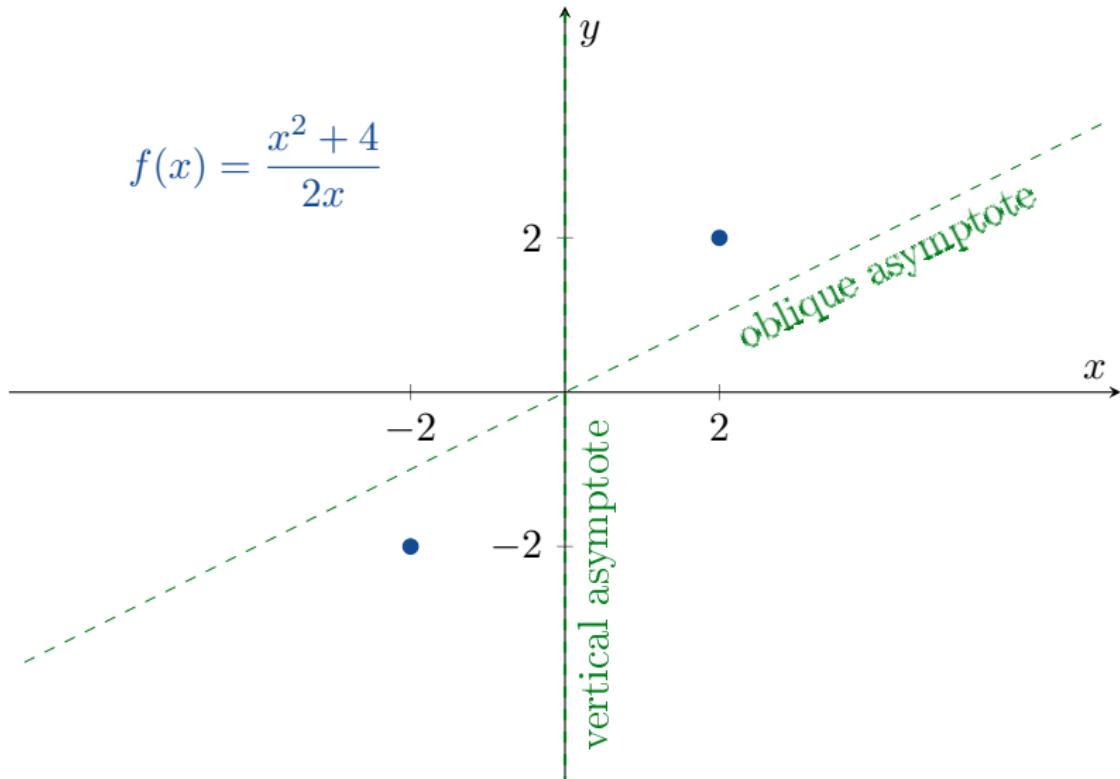
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4.4 Concavity and Curve Sketching



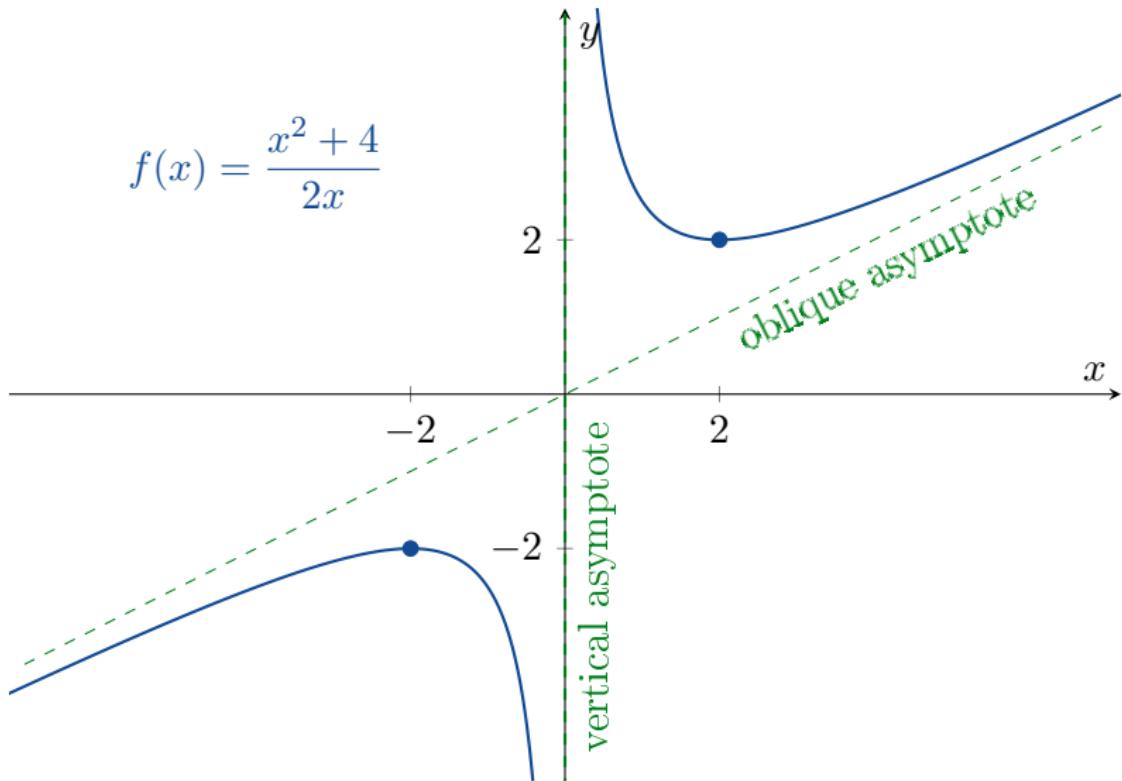
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4.4 Concavity and Curve Sketching



Now we have enough information to sketch the graph.



4.4 Concavity and Curve Sketching



Please read Example 11 in your textbook.

Break

We will continue at 2pm





Applied Optimisation

4.5 Applied Optimisation

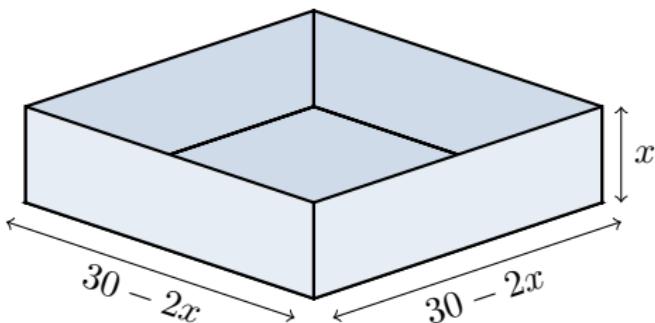
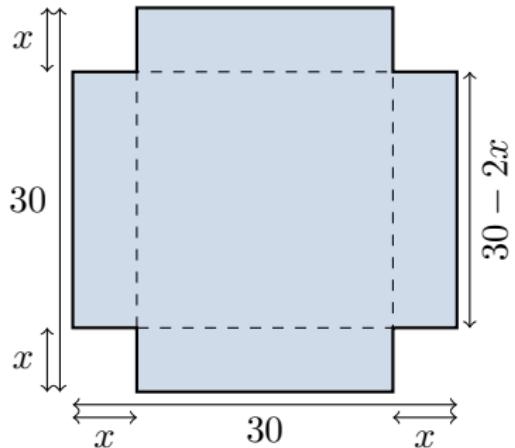


Example

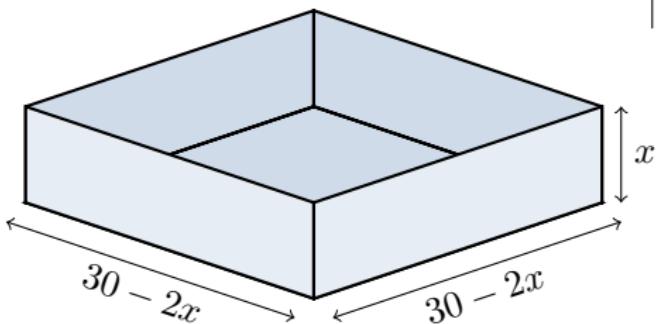
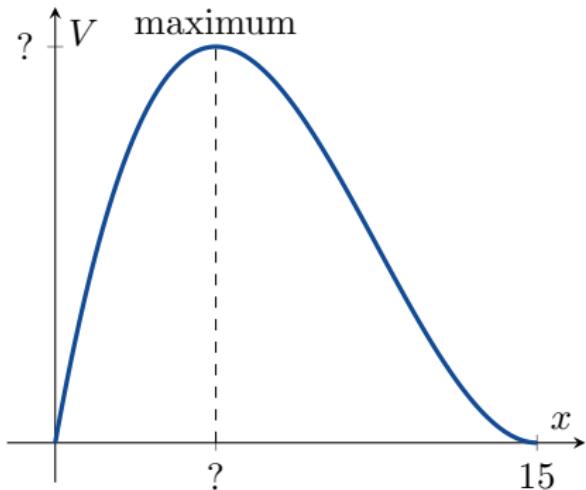
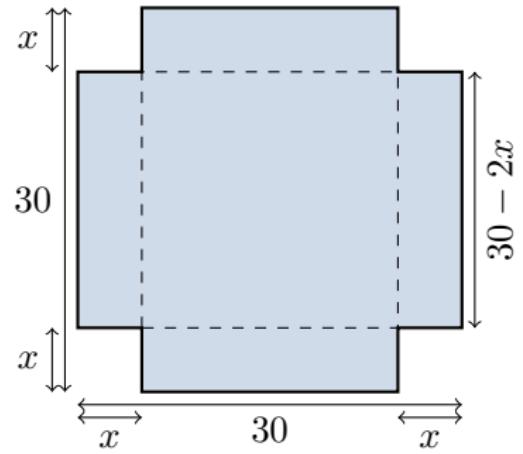
An open-top box is to be made by cutting x cm \times x cm squares from the corners of a 30 cm \times 30 cm piece of metal and bending the sides up.

How large should the squares cut from the corners be to make the box hold as much as possible?

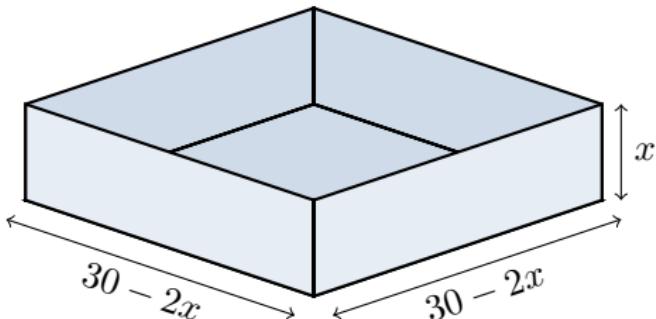
4.5 Applied Optimisation



4.5 Applied Optimisation



4.5 Applied Optimisation



The volume of the box will be

$$V(x) = x(30 - 2x)^2.$$

Note that the domain of V is $[0, 15]$.

4.5

$$V(x) = x(30 - 2x)^2$$



We calculate that

$$\begin{aligned}0 &= \frac{dV}{dx} = (x)'(30 - 2x)^2 + (x)((30 - 2x)^2)' \\&= (1)(30 - 2x)^2 + (x)2(30 - 2x)(-2) \\&= (30 - 2x)((30 - 2x) - 4x) \\&= (30 - 2x)(30 - 6x) = 12(15 - x)(5 - x).\end{aligned}$$

Therefore $x = 5$ or $x = 15$.

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Therefore $x = 5$ or $x = 15$. Since $15 \notin (0, 15)$, the only critical point of V is $x = 5$.

$$V(x) = x(30 - 2x)^2$$

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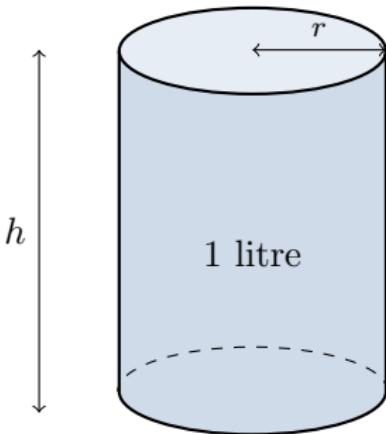
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Therefore $x = 5$ or $x = 15$. Since $15 \notin (0, 15)$, the only critical point of V is $x = 5$.

To make the largest possible box, we should choose $x = 5$. Such a box will have a volume of

$$V(5) = 5(30 - 10)^2 = 2000 \text{ cm}^3 = 2 \text{ litres.}$$

4.5 Applied Optimisation

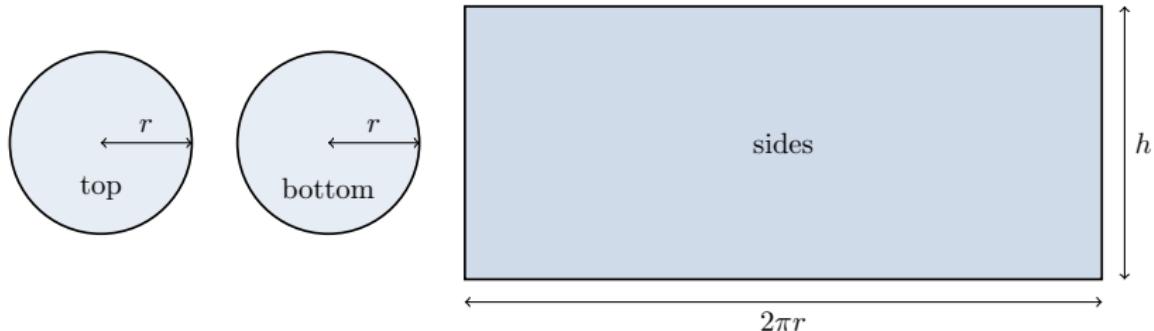


Example

You are designing a 1 litre cylindrical drinks can. You will use the same metal and the same thickness of metal for the top, bottom and sides. What dimensions will use the least metal?

We will use cm. Suppose that the radius of the can is r cm and the height of the can is h cm.

4.5 Applied Optimisation



Then the volume of the can is

$$\pi r^2 h = 1000 \text{ cm}^3$$

and the surface area of the can is

$$A = 2\pi r^2 + 2\pi r h.$$

We want to make A as small as we can.

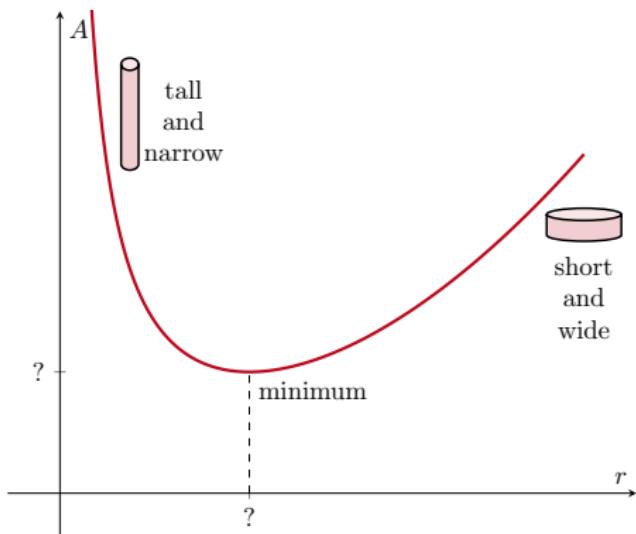
4.5 Applied Optimisation

Since

$$\pi r^2 h = 1000 \implies h = \frac{1000}{\pi r^2}$$

we have that

$$A(r) = 2\pi r^2 + 2\pi r h = 2\pi r^2 + 2\pi r \left(\frac{1000}{\pi r^2} \right) = 2\pi r^2 + \frac{2000}{r}.$$



4.5 Applied Optimisation



Then we calculate that

$$0 = \frac{dA}{dx} = 4\pi r - \frac{2000}{r^2}$$

$$4\pi r = \frac{2000}{r^2}$$

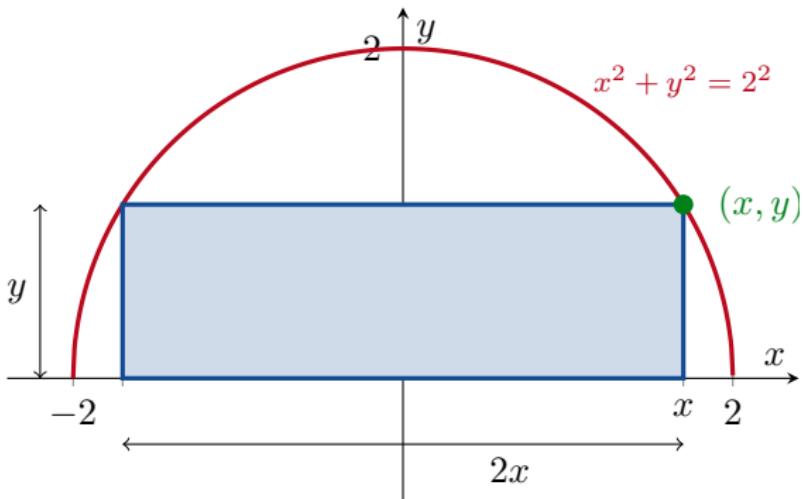
$$4\pi r^3 = 2000$$

$$r = \sqrt[3]{\frac{2000}{4\pi}} = \sqrt[3]{\frac{500}{\pi}} \approx 5.42 \text{ cm}$$

and

$$h = \frac{1000}{\pi r^2} = 2 \sqrt[3]{\frac{500}{\pi}} \approx 10.84 \text{ cm.}$$

4.5 Applied Optimisation

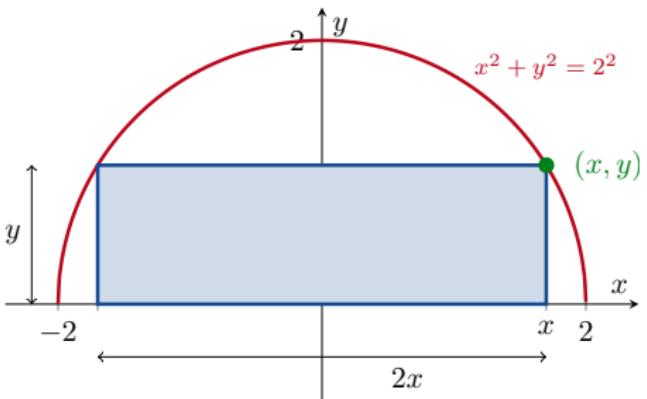


Example

A rectangle is to be inscribed in a semicircle of radius 2 as shown above.

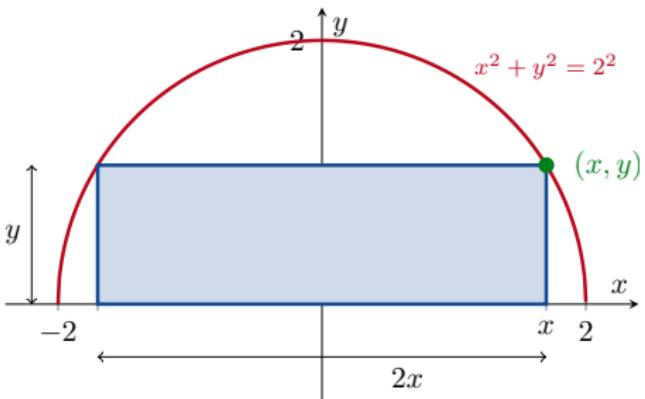
What is the largest possible area of the rectangle?

4.5 Applied Optimisation



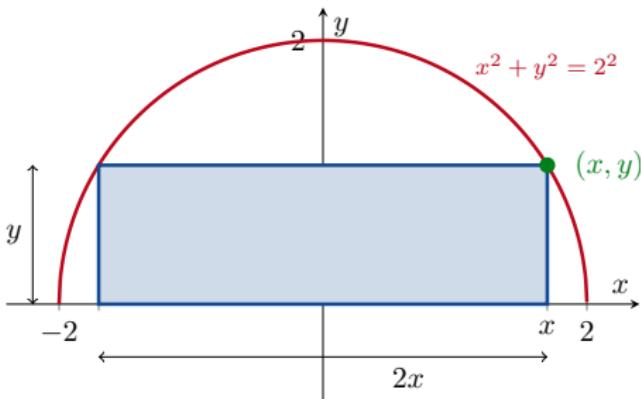
Consider a rectangle with a vertex at the point (x, y) . The area of this rectangle is clearly $A = 2xy$.

4.5 Applied Optimisation



Consider a rectangle with a vertex at the point (x, y) . The area of this rectangle is clearly $A = 2xy$. Since the point (x, y) lies on the circle $x^2 + y^2 = 2^2$, we must have $y = \sqrt{4 - x^2}$.

4.5 Applied Optimisation



Consider a rectangle with a vertex at the point (x, y) . The area of this rectangle is clearly $A = 2xy$. Since the point (x, y) lies on the circle $x^2 + y^2 = 2^2$, we must have $y = \sqrt{4 - x^2}$. Hence the area of the rectangle is

$$A(x) = 2x\sqrt{4 - x^2}.$$

We want to find $\max_{x \in [0, 2]} A(x)$.

4.5 Applied Optimisation



By differentiating A , we see that

$$\begin{aligned}0 &= \frac{dA}{dx} = \frac{d}{dx} \left(2x\sqrt{4-x^2} \right) \\&= 2\sqrt{4-x^2} + 2x \left(\frac{-2x}{2\sqrt{4-x^2}} \right) \\&= 2\sqrt{4-x^2} - \frac{2x^2}{\sqrt{4-x^2}}.\end{aligned}$$

Multiplying by $\sqrt{4-x^2}$ gives

$$0 = 2(4-x^2) - 2x^2 = 8 - 4x^2 = 4(2-x^2)$$

which implies that $x = \pm\sqrt{2}$.

4.5 Applied Optimisation

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which implies that $x = \pm\sqrt{2}$. But $-\sqrt{2} \notin [0, 2]$. So we must have $x = \sqrt{2}$. Therefore

$$\max_{x \in [0,2]} A(x) = A(\sqrt{2}) = 2\sqrt{2}\sqrt{4-2} = 2\sqrt{2}\sqrt{2} = 4.$$

4.5 Applied Optimisation



Please read three more examples in your textbook.



Antiderivatives

4.7 Antiderivatives

Definition

F is an *antiderivative* of f on an interval I if $F'(x) = f(x)$ for all $x \in I$.

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Example

$2x$ is the derivative of x^2 .

x^2 is an antiderivative of $2x$.

Example

$G(x) = \sin x$ is an antiderivative of $g(x) = \cos x$ because

$$G'(x) = \frac{d}{dx} (\sin x) = \cos x = g(x).$$

4.7 Antiderivatives

Example

$H(x) = x^2 + \sin x$ is an antiderivative of $h(x) = 2x + \cos x$
because $H'(x) = h(x)$.

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$F(x) = x^2$ is not the only antiderivative of $f(x) = 2x$.

4.7 Antiderivatives

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$x^2 + 1$ is an antiderivative of $2x$ because $\frac{d}{dx} (x^2 + 1) = 2x$.

4.7 Antiderivatives

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Remark

$F(x) = x^2$ is not the only antiderivative of $f(x) = 2x$.

$x^2 + 1$ is an antiderivative of $2x$ because $\frac{d}{dx} (x^2 + 1) = 2x$.

$x^2 + 5$ is an antiderivative of $2x$ because $\frac{d}{dx} (x^2 + 5) = 2x$.

4.7 Antiderivatives

Example

$H(x) = x^2 + \sin x$ is an antiderivative of $h(x) = 2x + \cos x$ because $H'(x) = h(x)$.

Remark

$F(x) = x^2$ is not the only antiderivative of $f(x) = 2x$.

$x^2 + 1$ is an antiderivative of $2x$ because $\frac{d}{dx}(x^2 + 1) = 2x$.

$x^2 + 5$ is an antiderivative of $2x$ because $\frac{d}{dx}(x^2 + 5) = 2x$.

$x^2 - 1234$ is an antiderivative of $2x$ because $\frac{d}{dx}(x^2 - 1234) = 2x$.

4.7 Antiderivatives



Theorem

If F is an antiderivative of f on I , then the general antiderivative of f is

$$F(x) + C$$

where C is a constant.

4.7 Antiderivatives



Example

Find an antiderivative of $f(x) = 3x^2$ that satisfies $F(1) = -1$.

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4.7 Antiderivatives

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Therefore $F(x) = x^3 - 2$.

4.7 Antiderivatives



function	derivative
$f(x)$	$f'(x)$
x^n	nx^{n-1}
$\sin kx$	$k \cos kx$
$\cos kx$	$-k \sin kx$
$\tan kx$	$k \sec^2 kx$

4.7 Antiderivatives



function	derivative	function	general antiderivative
$f(x)$	$f'(x)$	$f(x)$	$F(x)$
x^n	nx^{n-1}	x^n ($n \neq -1$)	
$\sin kx$	$k \cos kx$	$\cos kx$	
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4.7 Antiderivatives



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4.7 Antiderivatives



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x^n	nx^{n-1}	$x^n \ (n \neq -1)$	$\frac{x^{n+1}}{n+1} + C$
$\sin kx$	$k \cos kx$	$\cos kx$	
$\cos kx$	$\frac{d}{dx} \left(\frac{x^{n+1}}{n+1} + C \right) = \frac{(n+1)x^n}{n+1} = x^n$		
$\tan kx$			

4.7 Antiderivatives

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$f(x)$	$f'(x)$	$f(x)$	$F(x)$
x^n	nx^{n-1}	x^n ($n \neq -1$)	$\frac{x^{n+1}}{n+1} + C$
$\sin kx$	$k \cos kx$	$\cos kx$	$\frac{1}{k} \sin kx + C$
$\cos kx$	$-k \sin kx$	$\sin kx$	
$\tan kx$	$k \sec^2 kx$	$\sec^2 kx$	

4.7 Antiderivatives



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4.7 Antiderivatives



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EXAMPLE 3 Find the general antiderivative of each of the following functions.

(a) $f(x) = x^5$

(b) $g(x) = \frac{1}{\sqrt{x}}$

(c) $h(x) = \sin 2x$

(d) $i(x) = \cos \frac{x}{2}$

Solution In each case, we can use one of the formulas listed in Table 4.2.

(a) $F(x) = \frac{x^6}{6} + C$

Formula 1 with $n = 5$

(b) $g(x) = x^{-1/2}$, so

$$G(x) = \frac{x^{1/2}}{1/2} + C = 2\sqrt{x} + C$$

Formula 1 with $n = -1/2$

(c) $H(x) = \frac{-\cos 2x}{2} + C$

Formula 2 with $k = 2$

(d) $I(x) = \frac{\sin(x/2)}{1/2} + C = 2\sin \frac{x}{2} + C$

Formula 3 with $k = 1/2$

The Sum Rule and the Constant Multiple Rule

Suppose that

- F is an antiderivative of f ;
- G is an antiderivative of g ;
- $k \in \mathbb{R}$.

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The Sum Rule: The general antiderivative of $f + g$ is

$$F(x) + G(x) + C.$$

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The Sum Rule: The general antiderivative of $f + g$ is

$$F(x) + G(x) + C.$$

The Constant Multiple Rule: The general antiderivative of kf is

$$kF(x) + C.$$

4.7 Antiderivatives

Example

Find the general antiderivative of $f(x) = \frac{3}{\sqrt{x}} + \sin 2x$.

We have $f = 3g + h$ where $g(x) = x^{-\frac{1}{2}}$ and $h(x) = \sin 2x$.

4.7 Antiderivatives

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Find the general antiderivative of $f(x) = \frac{3}{\sqrt{x}} + \sin 2x$.

We have $f = 3g + h$ where $g(x) = x^{-\frac{1}{2}}$ and $h(x) = \sin 2x$. An antiderivative of g is

$$G(x) = \frac{x^{-\frac{1}{2}+1}}{-\frac{1}{2}+1} = \frac{x^{\frac{1}{2}}}{\frac{1}{2}} = 2\sqrt{x}.$$

4.7 Antiderivatives

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An antiderivative of h is

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An antiderivative of h is

$$H(x) = -\frac{1}{2} \cos 2x.$$

Therefore the general antiderivative of f is

$$F(x) = 6\sqrt{x} - \frac{1}{2} \cos 2x + C.$$

4.7 Antiderivatives

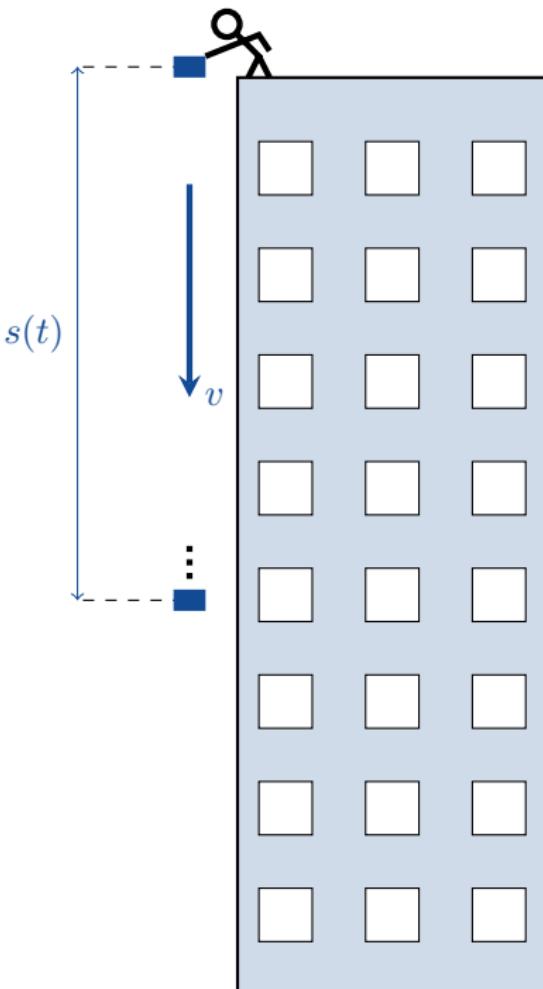


Example

You drop a box off the top of a tall building. The acceleration due to gravity is 9.8 ms^{-2} . You can ignore air resistance.

How far does the box fall in 5 seconds?

4.7 Antiderivat



4.7 Antiderivatives

The acceleration is

$$a(t) = 9.8 \text{ ms}^{-2}$$

downwards. Since

$$\text{acceleration} = \frac{d}{dt}(\text{velocity}),$$

the velocity is an antiderivative of the acceleration. Therefore the velocity is

$$v(t) = 9.8t + C \text{ ms}^{-1}.$$

4.7 Antiderivatives

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$$v(t) = 9.8t + C \text{ ms}^{-1}.$$

You let go of the box at time $t = 0$. So $v(0) = 0$. Thus $C = 0$. Hence

$$v(t) = 9.8t \text{ ms}^{-1}.$$

4.7 Antiderivatives

Now

$$\text{velocity} = \frac{d}{dt}(\text{displacement}).$$

So the distance fallen is an antiderivative of velocity. Hence

$$s(t) = 4.9t^2 + \tilde{C} \text{ m.}$$

4.7 Antiderivatives

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$$\text{velocity} = \frac{d}{dt}(\text{displacement}).$$

So the distance fallen is an antiderivative of velocity. Hence

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Because you let go of the box at time $t = 0$, we have $s(0) = 0$.

Thus $\tilde{C} = 0$. Therefore

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4.7 Antiderivatives

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$$s(t) = 4.9t^2 + \tilde{C} \text{ m.}$$

Because you let go of the box at time $t = 0$, we have $s(0) = 0$.

Thus $\tilde{C} = 0$. Therefore

$$s(t) = 4.9t^2 \text{ m.}$$

After 5 seconds, the box has fallen

$$s(5) = 4.9 \times 25 = 122.5 \text{ metres.}$$

4.7 Antiderivatives



Indefinite Integrals

Definition

The general antiderivative of f is also called the *indefinite integral* of f with respect to x , and is denoted by

$$\int f(x) \, dx.$$

4.7 Antiderivatives

the integral sign
integral işaretti

x is the variable of integration
 x ise integral değişkeni olarak tanımlanır

$$\int f(x) \, dx$$

 the integrand
integralin integrandi

4.7 Antiderivatives



Example

$$\int 2x \, dx = x^2 + C$$

$$\int \cos x \, dx = \sin x + C$$

$$\int (2x + \cos x) \, dx = x^2 + \sin x + C$$

4.7 Antiderivatives



Example

Calculate $\int (x^2 - 2x + 5) \, dx$.

4.7 Antiderivatives



Example

Calculate $\int (x^2 - 2x + 5) dx$.

solution 1. Since $\frac{d}{dx} \left(\frac{x^3}{3} - x^2 + 5x \right) = x^2 - 2x + 5$ we have that

$$\int (x^2 - 2x + 5) dx = \frac{x^3}{3} - x^2 + 5x + C.$$

4.7 Antiderivatives

solution 2.

$$\begin{aligned}\int (x^2 - 2x + 5) \, dx &= \int x^2 \, dx - \int 2x \, dx + \int 5 \, dx \\&= \left(\frac{x^3}{3} + C_1 \right) - (x^2 + C_2) + (5x + C_3) \\&= \left(\frac{x^3}{3} - x^2 + 5x \right) + (C_1 - C_2 + C_3).\end{aligned}$$

4.7 Antiderivatives

solution 2.

$$\begin{aligned}
 \int (x^2 - 2x + 5) \, dx &= \int x^2 \, dx - \int 2x \, dx + \int 5 \, dx \\
 &= \left(\frac{x^3}{3} + C_1 \right) - (x^2 + C_2) + (5x + C_3) \\
 &= \left(\frac{x^3}{3} - x^2 + 5x \right) + (C_1 - C_2 + C_3).
 \end{aligned}$$

Because we only need one constant, we can define
 $C := C_1 - C_2 + C_3$. Therefore

$$\int (x^2 - 2x + 5) \, dx = \frac{x^3}{3} - x^2 + 5x + C.$$



Next Time

- 5.1 Area and Estimating with Finite Sums
- 5.2 Sigma Notation and Limits of Finite Sums
- 5.3 The Definite Integral