OKAN ÜNİVERSİTESİ
FEN EDEBİYAT FAKÜLTESİ
MATEMATİK BÖLÜMÜ

04.01.2012 **MAT 461 – Fonksiyonel Analiz I – Yarıyıl Sonu Sınavı** N. Course

ADI SOYADI
ÖĞRENCİ NO
İMZA

Do not open the exam until you are told that you may begin.
Sınavın başladığı yüksek sesle söylenene kadar sayfayı çevirmeyin.

1. You will have 120 minutes to answer 4 questions from a choice of 5. If you choose to answer more than 4 questions, then only your best 4 answers will be counted.
2. The points awarded for each part, of each question, are stated next to it.
3. All of the questions are in English. You may answer in English or in Turkish.
4. You should write your student number on every page.
5. If you wish to leave before the end of the exam, give your exam script to an invigilator and leave the room quietly. You may not leave in the final 10 minutes of the exam.
6. Calculators, mobile phones and any digital means of communication are forbidden. The sharing of pens, erasers or any other item between students is forbidden.
7. All bags, coats, books, notes, etc. must be placed away from your desks and away from the seats next to you. You may not access these during the exam. Take out everything that you will need before the exam starts.
8. Any student found cheating or attempting to cheat will receive a mark of zero (0), and will be investigated according to the regulations of Yükseköğretim Kurumları Öğrenci Disiplin Yönetmeliği.

1. Sınav süresi toplam 120 dakikadır. Sınavda 5 soru sorulmuştur. Bu sorulardan 4 tanesini seçerek cevaplayınız. 4'den fazla soruyu cevaplarsanız, en yüksek puanı aldığınız 4 sorunun cevapları geçerli olacaktır.
2. Soruların her bölümünün kaç puan olduğu yanlarında belirtilmiştir.
3. Tüm sorular İngilizce'dir. Cevaplarınızı İngilizce yada Türkçe verebilirsiniz.
4. Öğrenci numaranızı her sayfaya yazınız.
5. Sınav süresi sona ermeden sınavınızı teslim edip çıkmak isterseniz, sınav kağıdınızı gözetmenlerden birine veriniz ve sınav salonundan sessizce çıkınız. Sınavın son 10 dakikası içinde sınav salonundan çıkmanız yasaktır.
6. Sınav esnasında hesap makinesi, cep telefonu ve dijital bilgi alışverişi yapılan her türlü malzemelerin kullanımı ile diğer silgi, kalem, vb. alışverişlerin yapılması kesinlikle yasaktır.
7. Çanta, palto, kitap ve ders notlarınız gibi eşyalarınız sıraların üzerinden ve yanınızdaki sandalyeden kaldırılmalıdır. Sınav süresince bu tür eşyaları kullanmanız yasaktır, bu nedenle ihtiyacınız olacak herşeyi sınav başlamadan yanınıza alınız.
8. Her türlü sınav, ve diğer çalışmada, kopya çeken veya kopya çekme girişiminde bulunan bir öğrenci, o sınav ya da çalışmadan sıfır (0) not almış sayılır, ve o öğrenci hakkında Yükseköğretim Kurumları Öğrenci Disiplin Yönetmeliği hükümleri uyarınca disiplin kovuşturması yapılır.

1	2	3	4	5	TOTAL

Notation:

$$C([a,b]) = \{f : [a,b] \to \mathbb{C} : f \text{ is continuous } \}$$
$$C^1([a,b]) = \{f : [a,b] \to \mathbb{C} : f \text{ and } f' \text{ are continuous } \}$$
$$C^\infty([a,b]) = \{f : [a,b] \to \mathbb{C} : \tfrac{d^n f}{dx^n} \text{ exists and is continuous } \forall n\}$$

$$\|f\|_\infty = \max_{x\in[0,1]} |f(x)|$$
$$\|f\|_{\infty,1} = \|f\|_\infty + \|f'\|_\infty$$

$$\mathcal{L}^2_{cont}([a,b]) = \big(C([a,b]), \langle\cdot,\cdot\rangle_{L^2}\big)$$
$$\langle f,g\rangle_{L^2} = \int_a^b \overline{f(x)}g(x)\ dx$$

$$\mathcal{B}(X,Y) = \{A : X \to Y : A \text{ is linear and bounded}\}$$
$$\mathcal{B}(X) = \mathcal{B}(X,X)$$
$$\mathcal{K}(X,Y) = \{A : X \to Y : A \text{ is linear and compact}\}$$

$$\overline{x+iy} = x - iy$$
$$A^* = \text{adjoint of } A$$
$$\text{Ker}(A) = \text{kernal of } A = \{f \in X : Af = 0\}$$
$$\text{Ran}(A) = \text{range of } A = \{Af : f \in X\}$$
$$M^\perp = \text{orthogonal complement of } M$$

Question 1 (Eigenvalues and Eigenvectors)**.** Let X be a Hilbert space and let $A : X \to X$ be a linear operator.

(a) [3p] Give the definition of a *symmetrical* operator.

(b) [5p] Give an example of a Hilbert space $(X, \langle\cdot,\cdot\rangle)$ and a symmetrical operator $A : X \to X$.
[HINT: You do not need to prove that your X is a Hilbert space, but you should prove that your A is symmetrical.]

(c) [3p] Give the definitions of an *eigenvalue* and an *eigenvector*, of an operator A.

For the rest of this question, suppose that:

- A is a symmetrical operator;
- $\lambda \in \mathbb{C}$ is an eigenvalue of A;
- u is an eigenvector of A corresponding to λ;
- $\|u\| = 1$;
- $\mu \in \mathbb{C}$ is an eigenvalue of A;
- v is an eigenvector of A corresponding to μ;
- $\|v\| = 1$;
- $\lambda \neq \mu$.

(d) [7p] Show that $\lambda \in \mathbb{R}$.
[HINT: Show that $\lambda = \overline{\lambda}$.]

(e) [5p] Show that

$$(\lambda - \mu) \langle u, v \rangle = 0.$$

(f) [2p] Show that u is orthogonal to v.

Question 2 (Compact Operators)**.** Let X, Y and Z be normed spaces.

(a) [3p] Give the definition of a *compact* operator $A : X \to Y$.

(b) [2p] Give the definition of a *bounded* operator $A : X \to Y$.

Let $\mathcal{B}(X, Y) = \{A : X \to Y : A \text{ is linear and bounded}\}$ and $\mathcal{K}(X, Y) = \{A : X \to Y : A \text{ is linear and compact}\}$.

(c) [5p] Show that $\mathcal{K}(X, Y) \subseteq \mathcal{B}(X, Y)$ (In other words; show that every compact operator is bounded).

(d) [5p] Let $A, B \in \mathcal{K}(X,Y)$ and $\alpha \in \mathbb{C}$. Show that $\alpha A + B \in \mathcal{K}(X,Y)$.

(e) [5p] Let $K \in \mathcal{K}(X,Y)$ and $B \in \mathcal{B}(Y,Z)$. Show that $(BK) \in \mathcal{K}(X,Z)$.

(f) [5p] Let $B \in \mathcal{B}(X,Y)$ and $K \in \mathcal{K}(Y,Z)$. Show that $(KB) \in \mathcal{K}(X,Z)$.

Question 3 (Non-Negative Operators)**.** Let X be a Hilbert space.

(a) [3p] Give the definition of a *non-negative* operator $A : X \to X$.

For the rest of this question: Suppose that $A \geq \varepsilon \mathbb{I}$ for some $\varepsilon > 0$, where $\mathbb{I} : X \to X$ is the identity operator ($\mathbb{I}f = f \ \forall f$). In other words; suppose that $(A - \varepsilon \mathbb{I})$ is non-negative.

(b) [5p] Show that $\|Af\| \geq \varepsilon \|f\|$ for all $f \in X$.
[HINT: Consider $\langle f, Af \rangle$ and show that $\|Af\| \|f\| \geq \varepsilon \|f\|^2$.]

(c) [2p] Show that $\mathrm{Ker}(A) = \{0\}$.

Part (c) shows that $\forall g \in \operatorname{Ran}(A)$, $\exists$ a unique $f = A^{-1}g \in X$ such that $Af = g$. Therefore $A^{-1} : \operatorname{Ran}(A) \to X$ is an operator (A^{-1} is linear because A is linear).

(d) [5p] Show that A^{-1} is bounded.
[HINT: First show that $\|A^{-1}g\| \leq \frac{1}{\varepsilon}\|g\|$ for all $g \in \operatorname{Ran}(A)$.]

(e) [5p] Show that $\operatorname{Ran}(A)$ is closed.
[HINT: Suppose g_n is a sequence in $\operatorname{Ran}(A)$ and $g_n \to g \in X$. Use (d) to show that $g \in \operatorname{Ran}(A)$.]

(f) [5p] Show that $\operatorname{Ran}(A)^{\perp} = \{0\}$.
[HINT: Let $h \in \operatorname{Ran}(A)^{\perp}$. Then Ah is orthogonal to h. Use $\frac{1}{\varepsilon}A \geq \mathbb{I}$ to prove that $\|h\|^2 = 0$.]

This proves that $A : X \to X$ is a bijection.

Question 4 (Adjoints of Operators)**.** Let X be a Hilbert space and let $\mathcal{B}(X) = \{A : X \to X : A$ is linear and bounded$\}$.

(a) [3p] Let $A \in \mathcal{B}(X)$ be an operator. Give the definition of the *adjoint*, A^*, of A.

Let $A, B \in \mathcal{B}(X)$ and $\alpha \in \mathbb{C}$.

(b) [4p] Show that $(A + B)^* = A^* + B^*$.

(c) [4p] Show that $(\alpha A)^* = \overline{\alpha} A^*$.

(d) [4p] Show that $A^{**} = A$.

(e) [4p] Show that $(AB)^* = B^* A^*$.

(f) [6p] Suppose that A is a bijection and that A^{-1} is bounded. Show that

$$(A^*)^{-1} = (A^{-1})^*.$$

Question 5 (The Vector Space of Continuous Functions)**.**

(a) [3 pts] Give the definition of an inner product.

(b) [2 pts] Give the definition of a Hilbert space.

Let $I = [0, 2] \subseteq \mathbb{R}$,

$$C(I) = \{f : I \to \mathbb{C} : f \text{ is continuous}\}$$

and

$$\langle f, g \rangle = \int_0^2 \overline{f(x)} g(x)\ dx.$$

(c) [5 pts] Show that $\langle \cdot, \cdot \rangle$ is an inner product on $C(I)$.

Define

$$f_n(x) = \begin{cases} 0 & 0 \le x \le 1 - \frac{1}{n}, \\ 1 + n(x-1) & 1 - \frac{1}{n} \le x \le 1, \\ 1 & 1 \le x \le 2. \end{cases}$$

(d) [7 pts] Show that $(f_n)_{n=1}^{\infty}$ is a Cauchy sequence in $(C(I), \langle \cdot, \cdot \rangle)$.

(e) [7 pts] Show that $(f_n)_{n=1}^{\infty}$ does not have a limit in $C(I)$.

(f) [1 pts] Is $(C(I), \langle \cdot, \cdot \rangle)$ a Hilbert space)? ☐ yes ☐ no