A non-constant f-harmonic map $S^2 \to S^2$ of degree zero

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June 13, 2005

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1 Longitudinally symmetric f-harmonic maps $S^2 \to S^2$

Consider a map

$$u : \begin{cases} S^2 \to S^2 \\ (\phi, \theta) \mapsto \big(\phi, \alpha(\theta)\big) \end{cases} \tag{1.1}$$

1.1 Gradient and Laplacian in polar coordinates

We will use: $u_1 = \cos\phi \sin\alpha$, $u_2 = \sin\phi\sin\alpha$, $u_3 = \cos\alpha$.

$$\nabla v = \frac{1}{\sin\theta}\frac{\partial v}{\partial \phi}\hat{\boldsymbol{\phi}} + \frac{\partial v}{\partial \theta}\hat{\boldsymbol{\theta}}$$

$$\Delta v = \frac{1}{\sin^2\theta} v_{\phi\phi} + \frac{1}{\sin\theta}\big(\sin\theta v_\theta\big)_\theta$$

1.2 Differentiation

$$\begin{aligned}
\nabla u_1 &= \frac{-1}{\sin\theta}\sin\phi\sin\alpha\hat{\boldsymbol{\phi}} + \alpha_\theta\cos\phi\cos\alpha\hat{\boldsymbol{\theta}} \\
\nabla u_2 &= \frac{1}{\sin\theta}\cos\phi\sin\alpha\hat{\boldsymbol{\phi}} + \alpha_\theta\sin\phi\cos\alpha\hat{\boldsymbol{\theta}} \\
\nabla u_3 &= -\alpha_\theta\sin\alpha\hat{\boldsymbol{\theta}}
\end{aligned}$$

So

$$|\nabla u|^2 = \frac{\sin^2\alpha}{\sin^2\theta} + \alpha_\theta^2$$

and

$$\begin{aligned}
\nabla f \cdot \nabla u_1 &= f_\theta \alpha_\theta \cos\phi\cos\alpha \\
\nabla f \cdot \nabla u_2 &= f_\theta \alpha_\theta \sin\phi\cos\alpha \\
\nabla f \cdot \nabla u_3 &= f_\theta \alpha_\theta \sin\alpha
\end{aligned}$$

since $\nabla f = f_\theta\hat{\boldsymbol{\theta}}$.

$$\begin{aligned}
\Delta u_1 &= \frac{-\cos\phi\sin\alpha}{\sin^2\theta} + \alpha_\theta\frac{\cos\theta}{\sin\theta}\cos\phi\cos\alpha - \alpha_\theta\alpha_\theta\cos\phi\sin\alpha + \alpha_{\theta\theta}\cos\phi\cos\alpha, \\
\Delta u_2 &= \frac{-\sin\phi\sin\alpha}{\sin^2\theta} + \alpha_\theta\frac{\cos\theta}{\sin\theta}\sin\phi\cos\alpha - \alpha_\theta\alpha_\theta\sin\phi\sin\alpha + \alpha_{\theta\theta}\sin\phi\cos\alpha, \\
\Delta u_3 &= -\alpha_\theta\frac{\cos\theta}{\sin\theta}\sin\alpha - \alpha_\theta\alpha_\theta\cos\alpha + \alpha_{\theta\theta}\sin\alpha.
\end{aligned}$$

1.3 The Euler-Lagrange equation

Therefore

$$
\begin{aligned}
0 &= \Delta u_1 + u_1|\nabla u|^2 + \frac{1}{f}\nabla f \cdot \nabla u_1 \\
&= \frac{\cos\phi}{\sin^2\theta}\Big[-\sin\alpha + \alpha_\theta \cos\alpha \sin\theta\cos\theta + \alpha_{\theta\theta}\cos\alpha\sin^2\theta \\
&\qquad + \sin^3\alpha + \frac{1}{f} f_\theta \alpha_\theta \cos\alpha \sin^2\theta \Big] \\
&= \frac{\cos\alpha\cos\phi}{\sin^2\theta}\left[\alpha_{\theta\theta}\sin^2\theta + \alpha_\theta\left(\sin\theta\cos\theta + \frac{f_\theta}{f}\sin^2\theta\right) - \sin\alpha\cos\alpha\right],
\end{aligned}
$$

and hence

$$\alpha_{\theta\theta}\sin^2\theta + \alpha_\theta\left(\sin\theta\cos\theta + \frac{f_\theta}{f}\sin^2\theta\right) = \sin\alpha\cos\alpha. \tag{1.2}$$

Rearranging, we see we must have

$$F(\theta) := \frac{f_\theta(\theta)}{f(\theta)} = \frac{\sin\alpha\cos\alpha}{\alpha_\theta \sin^2\theta} - \frac{\alpha_{\theta\theta}}{\alpha_\theta} - \frac{\cos\theta}{\sin\theta}. \tag{1.3}$$

2 Example

Let $\psi \in C_c^\infty(\mathbb{R};[0,1])$ satisfy $\psi(x) = 1$ for $|x| < \frac{\pi}{2} - 0.2$ and $\psi(x) = 0$ for $|x| > \frac{\pi}{2} - 0.1$. Define

$$
\begin{aligned}
A &= \tfrac{1}{4}, \\
B &= \tfrac{4}{\pi^2}\log[\tfrac{\pi}{12}], \\
C &= \tfrac{3\pi}{4},
\end{aligned}
$$

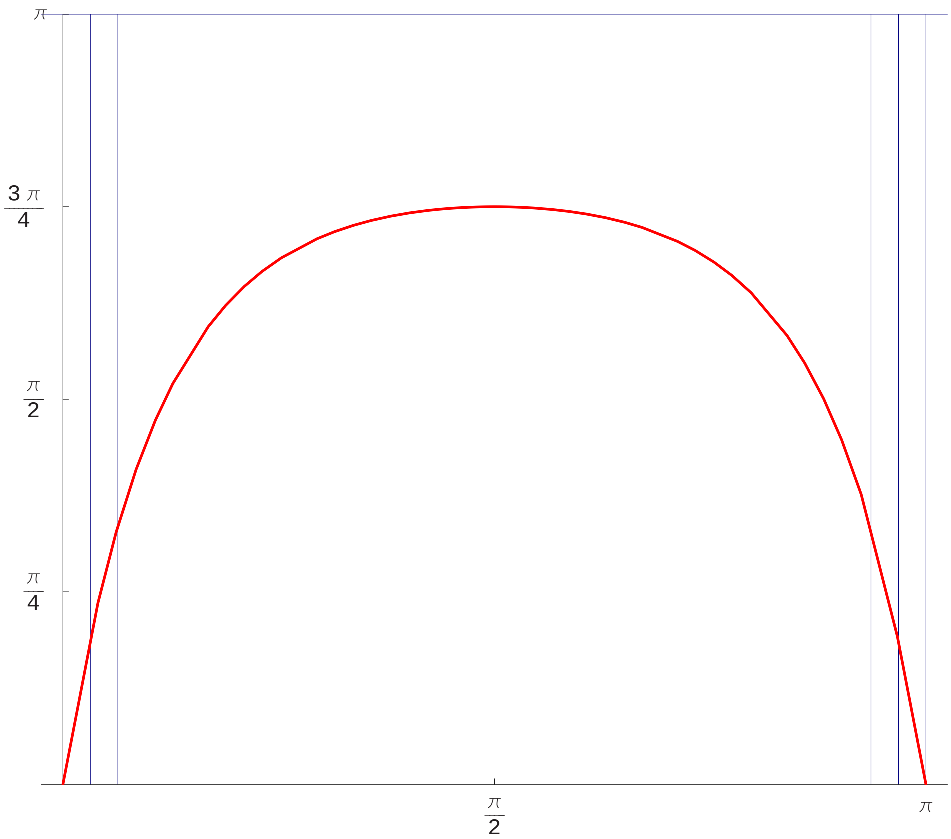
and

$$\gamma(x) = \begin{cases} \frac{C - Dx}{A(x-\frac{\pi}{2})^2} & \text{for } x \le \frac{\pi}{2} - \frac{1}{2} \\ \frac{C - D(\pi - x)}{A(x-\frac{\pi}{2})^2} & \text{for } x \ge \frac{\pi}{2} + \frac{1}{2} \\ 1 & \text{otherwise} \end{cases}$$

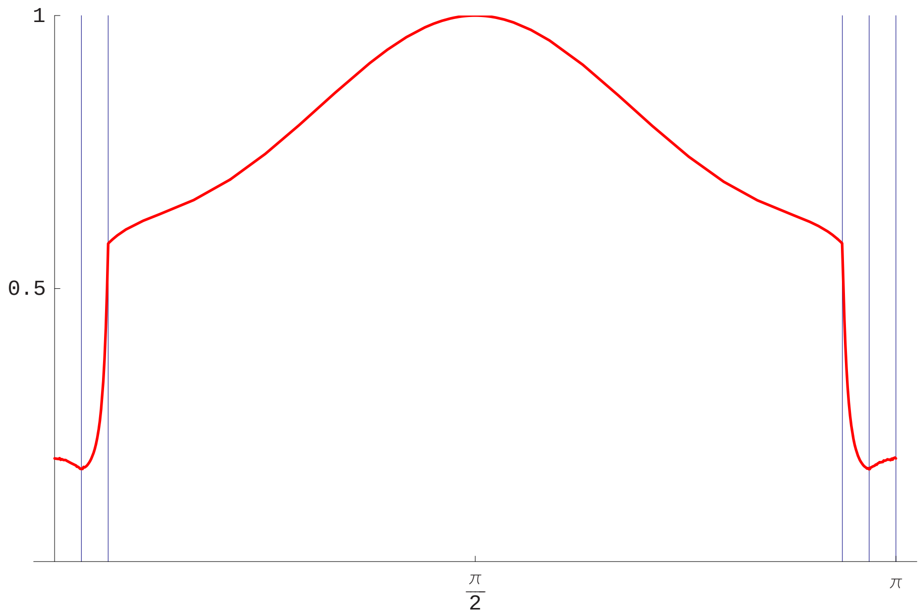
for some suitable constant D (e.g. $D = 6$ perhaps). Notice that γ has no zeros in $(0,\pi)$.

Now set

$$\alpha(\theta) := C - A\left(\theta - \tfrac{\pi}{2}\right)^2 \exp\left[-B\left(\theta - \tfrac{\pi}{2}\right)^2 \psi\left(\theta - \tfrac{\pi}{2}\right) + \left(1 - \psi\left(\theta - \tfrac{\pi}{2}\right)\right)\log\gamma(\theta)\right],$$



(a) $\alpha : [0, \pi] \longrightarrow [0, \pi]$.



(b) $f : [0, \pi] \to (0, \infty)$.

Figure 1: The maps α and f.

(see figure 1(a) on page 4) and then define F as in (1.3). Since α is smooth on $[0,\pi]$, it follows that $F(\theta)$ must also be smooth, except possibly at points where either $\alpha_\theta(\theta)=0$ or $\sin\theta=0$ – that is, for $\theta=0$, $\frac{\pi}{2}$ or π.

Notice that for θ close to zero, we have $\alpha(\theta)=D\theta$, so $F(\theta)\to 0$ as $\theta\searrow 0$ (similarly as $\theta\nearrow\pi$). Moreover, when θ is close to $\frac{\pi}{2}$, we have

$$\alpha(\theta)=C-A\left(\theta-\tfrac{\pi}{2}\right)^2 e^{-B\left(\theta-\frac{\pi}{2}\right)^2}.$$

One may check that near $\frac{\pi}{2}$ we have that

$$F(\theta)=\frac{\frac{\sin 2\alpha}{4A\sin^2\theta}+\left[1-5B\left(\theta-\frac{\pi}{2}\right)^2+2B^2\left(\theta-\frac{\pi}{2}\right)^4\right]e^{-B\left(\theta-\frac{\pi}{2}\right)^2}}{\left(\theta-\frac{\pi}{2}\right)\left[B\left(\theta-\frac{\pi}{2}\right)^2-1\right]e^{-B\left(\theta-\frac{\pi}{2}\right)^2}}-\frac{\cos\theta}{\sin\theta}.$$

Recall that $A=\frac{1}{4}$ and $\alpha(\frac{\pi}{2})=C=\frac{3\pi}{4}$. Therefore, near $\theta=\frac{\pi}{2}$, we have that

$$\begin{aligned}\sin 2\alpha &= -\cos\left[\tfrac{1}{2}(\theta-\tfrac{\pi}{2})^2 e^{-B(\theta-\frac{\pi}{2})^2}\right]=-1+o[\theta-\tfrac{\pi}{2}],\\ 4A\sin^2\theta &= 1+o[\theta-\tfrac{\pi}{2}]\end{aligned}$$

for o as defined in [Wei]. Thus, near $\theta=\frac{\pi}{2}$,

$$F(\theta)=\frac{\frac{-1+o[\theta-\frac{\pi}{2}]}{1+o[\theta-\frac{\pi}{2}]}+1+o[\theta-\frac{\pi}{2}]}{(\theta-\frac{\pi}{2})\big(1+o[\theta-\frac{\pi}{2}]\big)}-\frac{\cos\theta}{\sin\theta}.$$

So $F(\theta)\to 0$ as $\theta\to\frac{\pi}{2}$.

Finally, we can define

$$f(\theta):=e^{\int_{\frac{\pi}{2}}^{\theta}F(\omega)\ d\omega}.$$

Then the map $u:(\phi,\theta)\mapsto(\phi,\alpha(\theta))$ is a non-constant f-harmonic of degree zero. Depending on choice of ψ, f may look like figure 1(b) on page 4.

References

[Wei] Eric W. Weisstein, *Landau Symbols*, from MathWorld–A Wolfram Web Resource. http://mathworld.wolfram.com/LandauSymbols.html.