

Lecture 11

- 9.6 Alternating Series and Conditional Convergence
- 9.7 Power Series
- 9.8 Taylor and Maclaurin Series



Alternating Series and Conditional Convergence



Alternating Series

Now let's talk about sequences of the form

$$\sum_{n=1}^{\infty} (-1)^{n+1} a_n = a_1 - a_2 + a_3 - a_4 + a_5 - a_6 + a_7 - a_8 + a_9 - a_{10} + \dots$$

where $a_n > 0 \ \forall n$.

9.6 Alternating Series and Conditional Convergence



$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \dots + \frac{(-1)^{n+1}}{n} + \dots$$

$$1 - 2 + 4 - 8 + 16 - 32 + \dots$$

$$4 - 2 + 1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{8} + \frac{1}{16} - \dots$$

9.6 Alternating Series and Conditional Convergence



Theorem (The Alternating Series Test / Alterne Seri Testi)

Let (a_n) be a sequence such that

- 1 $a_n > 0$ for all n ;
- 2 (a_n) is decreasing (i.e. $a_n \geq a_{n+1}$ for all n); and
- 3 $a_n \rightarrow 0$ as $n \rightarrow \infty$.

9.6 Alternating Series and Conditional Convergence



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9.6 Alternating Series and Conditional Convergence



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Then $\sum_{n=1}^{\infty} (-1)^{n+1} a_n$ converges.

Of course we can write condition 2 as “ (a_n) is decreasing eventually (i.e. $a_n \geq a_{n+1}$ for all $n > N$ for some $N \in \mathbb{N}$)” since we don’t care what happens at the start of a sequence/series.

9.6 Alternating Series and Conditional Convergence



Proof.

Let

$$s_n = a_1 - a_2 + a_3 - a_4 + a_5 - a_6 + \dots + (-1)^{n+1} a_n.$$

Then

$$s_{2n} = a_1 - a_2 + a_3 - a_4 + a_5 - a_6 + \dots - a_{2n}.$$

9.6 Alternating Series and Conditional Convergence



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So

$$s_{2n+2} - s_{2n} = a_{2n+1} - a_{2n+2} \geq 0.$$

Therefore the sequence (s_{2n}) is increasing.

9.6 Alternating Series and Conditional Convergence



Proof continued.

Moreover, since (a_n) is positive and decreasing, we have that

$$s_{2n} = a_1 - a_2 + a_3 - a_4 + a_5 - a_6 + \dots - a_{2n-2} + a_{2n-1} - a_{2n}$$

9.6 Alternating Series and Conditional Convergence



Proof continued.

Moreover, since (a_n) is positive and decreasing, we have that

$$\begin{aligned}s_{2n} &= a_1 - a_2 + a_3 - a_4 + a_5 - a_6 + \dots - a_{2n-2} + a_{2n-1} - a_{2n} \\&= a_1 - (a_2 - a_3) - (a_4 - a_5) - \dots - (a_{2n-2} - a_{2n-1}) - a_{2n}\end{aligned}$$

9.6 Alternating Series and Conditional Convergence



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Moreover, since (a_n) is positive and decreasing, we have that

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$$\begin{aligned}a_2 &\geq a_3 \\a_2 - a_3 &\geq 0 \\-(a_2 - a_3) &\leq -0\end{aligned}$$

9.6 Alternating Series and Conditional Convergence



Proof continued.

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So (s_{2n}) is bounded above.

9.6 Alternating Series and Conditional Convergence



Proof continued.

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So (s_{2n}) is bounded above.

$$\left\{ \begin{array}{l} (s_{2n}) \text{ is increasing} \\ (s_{2n}) \text{ is bounded above} \end{array} \right. \implies (s_{2n}) \text{ is convergent.}$$

9.6 Alternating Series and Conditional Convergence



Proof continued.

Let $s = \lim_{n \rightarrow \infty} s_{2n}$. Then $s_{2n} \rightarrow s$ as $n \rightarrow \infty$. Furthermore

$$\begin{aligned}s_{2n+1} &= a_1 - a_2 + a_3 - a_4 + a_5 - a_6 + \dots - a_{2n} + a_{2n+1} \\&= s_{2n} + a_{2n+1} \rightarrow s + 0 = s\end{aligned}$$

as $n \rightarrow \infty$.

9.6 Alternating Series and Conditional Convergence



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as $n \rightarrow \infty$.

It follows (you prove) that $s_n \rightarrow s$ as $n \rightarrow \infty$ also.

9.6 Alternating Series and Conditional Convergence



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as $n \rightarrow \infty$.

It follows (you prove) that $s_n \rightarrow s$ as $n \rightarrow \infty$ also.

Therefore $\sum_{n=1}^{\infty} (-1)^{n+1} a_n$ is convergent. □

9.6 Alternating Series and Conditional Convergence



Remark

If $a_n \not\rightarrow 0$ as $n \rightarrow \infty$, then $(-1)^{n+1}a_n \not\rightarrow 0$ as $n \rightarrow \infty$ and

$$\sum_{n=1}^{\infty} (-1)^{n+1}a_n$$

diverges by the Divergence Test.

9.6 Alternating Series and Conditional Convergence



Example

Let $a_n = \sin \frac{1}{n}$ for all $n \in \mathbb{N}$. Note that $0 < \frac{1}{n+1} < \frac{1}{n} \leq 1 < \frac{\pi}{2}$ for all $n \in \mathbb{N}$. Thus

$$a_n = \sin \frac{1}{n} > 0$$

and

$$a_{n+1} = \sin \frac{1}{n+1} < \sin \frac{1}{n} = a_n.$$

So (a_n) is a decreasing sequence of positive numbers. Moreover, $a_n = \sin \frac{1}{n} \rightarrow 0$ as $n \rightarrow \infty$.

9.6 Alternating Series and Conditional Convergence



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So (a_n) is a decreasing sequence of positive numbers. Moreover, $a_n = \sin \frac{1}{n} \rightarrow 0$ as $n \rightarrow \infty$. Therefore

$$\sum_{n=1}^{\infty} (-1)^{n+1} \sin \frac{1}{n}$$

converges by the Alternating Series Test.

9.6 Alternating Series and Conditional Convergence



Example

Since $a_n = \cos \frac{1}{n} \rightarrow 1 \neq 0$ as $n \rightarrow \infty$, it follows that

$$\sum_{n=1}^{\infty} (-1)^{n+1} \cos \frac{1}{n}$$

diverges by the Divergence Test.



Example

Does $\sum_{n=1}^{\infty} (-1)^{n+1} \sin^2 \frac{1}{n}$ converge or diverge?

9.6 Alternating Series and Conditional Convergence



Example

Does $\sum_{n=1}^{\infty} (-1)^{n+1} \sin^2 \frac{1}{n}$ converge or diverge?

Let $a_n = \sin^2 \frac{1}{n}$. Then $a_n > a_{n+1} > 0 \ \forall n$ and $a_n \rightarrow 0$ as $n \rightarrow \infty$.

9.6 Alternating Series and Conditional Convergence



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Does $\sum_{n=1}^{\infty} (-1)^{n+1} \sin^2 \frac{1}{n}$ converge or diverge?

Let $a_n = \sin^2 \frac{1}{n}$. Then $a_n > a_{n+1} > 0 \ \forall n$ and $a_n \rightarrow 0$ as $n \rightarrow \infty$.

Therefore

$$\sum_{n=1}^{\infty} (-1)^{n+1} \sin^2 \frac{1}{n}$$

converges by the Alternating Series Test.

9.6 Alternating Series and Conditional Convergence



Example

Does $\sum_{n=1}^{\infty} \frac{(-1)^{n+1} 10n}{n^2 + 16}$ converge or diverge?

9.6 Alternating Series and Conditional Convergence



Example

Does $\sum_{n=1}^{\infty} \frac{(-1)^{n+1} 10n}{n^2 + 16}$ converge or diverge?

Let $a_n = \frac{10n}{n^2 + 16}$ and $f(x) = \frac{10x}{x^2 + 16}$. Note that $a_n > 0$ for all n and $a_n \rightarrow 0$ as $n \rightarrow \infty$.

9.6 Alternating Series and Conditional Convergence



Example

Does $\sum_{n=1}^{\infty} \frac{(-1)^{n+1} 10n}{n^2 + 16}$ converge or diverge?

Let $a_n = \frac{10n}{n^2 + 16}$ and $f(x) = \frac{10x}{x^2 + 16}$. Note that $a_n > 0$ for all n and $a_n \rightarrow 0$ as $n \rightarrow \infty$. Moreover if $x \geq 4$, then

$$f'(x) = \frac{10(16 - x^2)}{(x^2 + 16)^2} \leq 0.$$

So (a_n) is decreasing for $n \geq 4$.

9.6 Alternating Series and Conditional Convergence



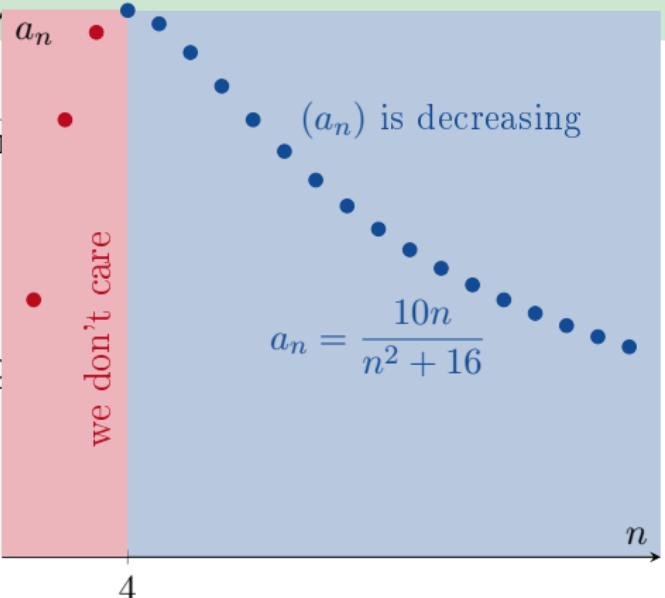
Example

Does $\sum_{n=1}^{\infty} \frac{(-1)^{n+1} 10n}{n^2 + 16}$ converge or diverge?

Let $a_n = \frac{10n}{n^2 + 16}$ and $f(x) = \frac{1}{x^2 + 16}$
and $a_n \rightarrow 0$ as $n \rightarrow \infty$. More

$$f'(x) =$$

So (a_n) is **decreasing** for $n \geq 1$



9.6 Alternating Series and Conditional Convergence



Example

Does $\sum_{n=1}^{\infty} \frac{(-1)^{n+1} 10n}{n^2 + 16}$ converge or diverge?

Let $a_n = \frac{10n}{n^2 + 16}$ and $f(x) = \frac{10x}{x^2 + 16}$. Note that $a_n > 0$ for all n and $a_n \rightarrow 0$ as $n \rightarrow \infty$. Moreover if $x \geq 4$, then

$$f'(x) = \frac{10(16 - x^2)}{(x^2 + 16)^2} \leq 0.$$

So (a_n) is decreasing for $n \geq 4$.

Therefore $\sum_{n=1}^{\infty} \frac{(-1)^{n+1} 10n}{n^2 + 16}$ converges.



Definition

If a series $\sum_{k=1}^{\infty} a_k$ is convergent, but is not absolutely convergent, then we say that it is *conditionally convergent*.

(Equivalently, we can say that the series *converges conditionally*.)

9.6 Alternating Series and Conditional Convergence



Example

$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$ is conditionally convergent.

(because $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges)

Example

$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2}$ is absolutely convergent.

9.6 Alternating Series and Conditional Convergence



Remark

We can rearrange an absolutely convergent series without changing its sum.

This is **not true** for conditionally convergent series.

9.6 Alternating Series and Conditional Convergence



Remark

We can rearrange an absolutely convergent series without changing its sum.

This is **not true** for conditionally convergent series.

For example, it is possible (see page 99 of Mary Hart's book) to show that

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \dots = \ln 2$$

and

$$\underbrace{1 + \frac{1}{3} + \frac{1}{5} + \frac{1}{7}}_{\text{4 positive terms}} \underbrace{- \frac{1}{2}}_{\text{negative term}} \underbrace{+ \frac{1}{9} + \frac{1}{11} + \frac{1}{13} + \frac{1}{15}}_{\text{4 positive terms}} \underbrace{- \frac{1}{4}}_{\text{negative term}} + \frac{1}{17} + \frac{1}{19} + \dots = \ln 4.$$



Power Series

9.7 Power Series



Let $(a_n)_{n=0}^{\infty}$ be a sequence. Then

$$\sum_{n=0}^{\infty} a_n(x - c)^n$$

is a *power series* (kuvvet serisi). This is a function of x .

9.7 Power Series



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“infinite polynomials”

9.7 Power Series

Example

The following are power series:

- $1 + 2x + 3x^2 + 4x^3 + 5x^4 + \dots = \sum_{n=1}^{\infty} nx^{n-1} = \sum_{n=0}^{\infty} (n+1)x^n;$
- $1 + x + x^2 + x^3 + x^4 + \dots = \sum_{n=0}^{\infty} x^n;$
- $1 + x + 2x^2 + 6x^3 + 24x^4 + \dots = \sum_{n=0}^{\infty} n!x^n;$
- $1 + x^2 + x^4 + x^6 + x^8 + \dots = \sum_{n=0}^{\infty} \left(\frac{1+(-1)^2}{2}\right) x^n;$
- $1 + (x-2) + (x-2)^2 + (x-2)^3 + (x-2)^4 + \dots = \sum_{n=0}^{\infty} (x-2)^n.$

9.7 Power Series



Definition

The constant c is called the *centre of expansion* of the power series $\sum_{n=0}^{\infty} a_n(x - c)^n$.

9.7 Power Series

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Remark

To make things easier, we start by looking at power series with $c = 0$. So first we will consider

$$\sum_{n=0}^{\infty} a_n x^n,$$

then we will discuss power series with $c \neq 0$ later.

9.7 Power Series



Remark

We wish to answer the following three questions about power series:

- How does a power series behave?
- Does this depend on x ?
- Is it possible for a power series to converge for some x , but diverge for other x ?

9.7 Power Series



Example

Recall that

$$\sum_{n=0}^{\infty} x^n \begin{cases} \text{converges if } |x| < 1 \\ \text{diverges if } |x| \geq 1. \end{cases}$$

9.7 Power Series



Example

Recall that

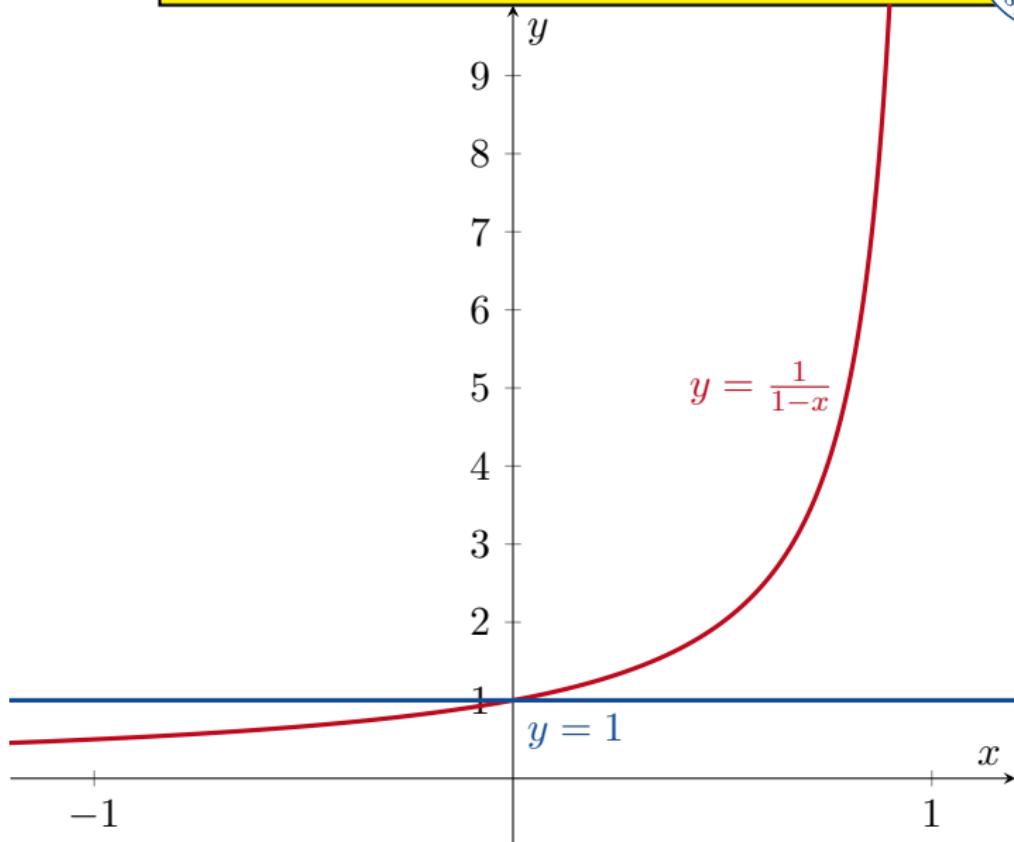
$$\sum_{n=0}^{\infty} x^n \begin{cases} \text{converges if } |x| < 1 \\ \text{diverges if } |x| \geq 1. \end{cases}$$

If $-1 < x < 1$, then

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + x^4 + x^5 + x^6 + \dots$$

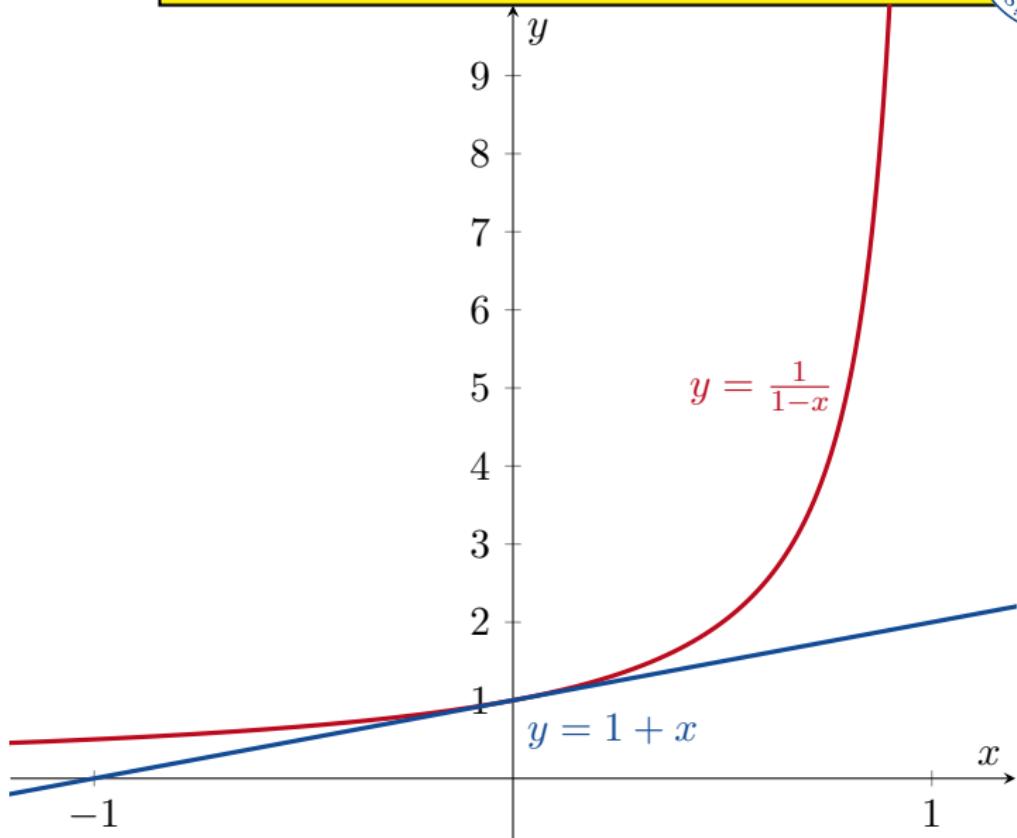
9.7 Power Series

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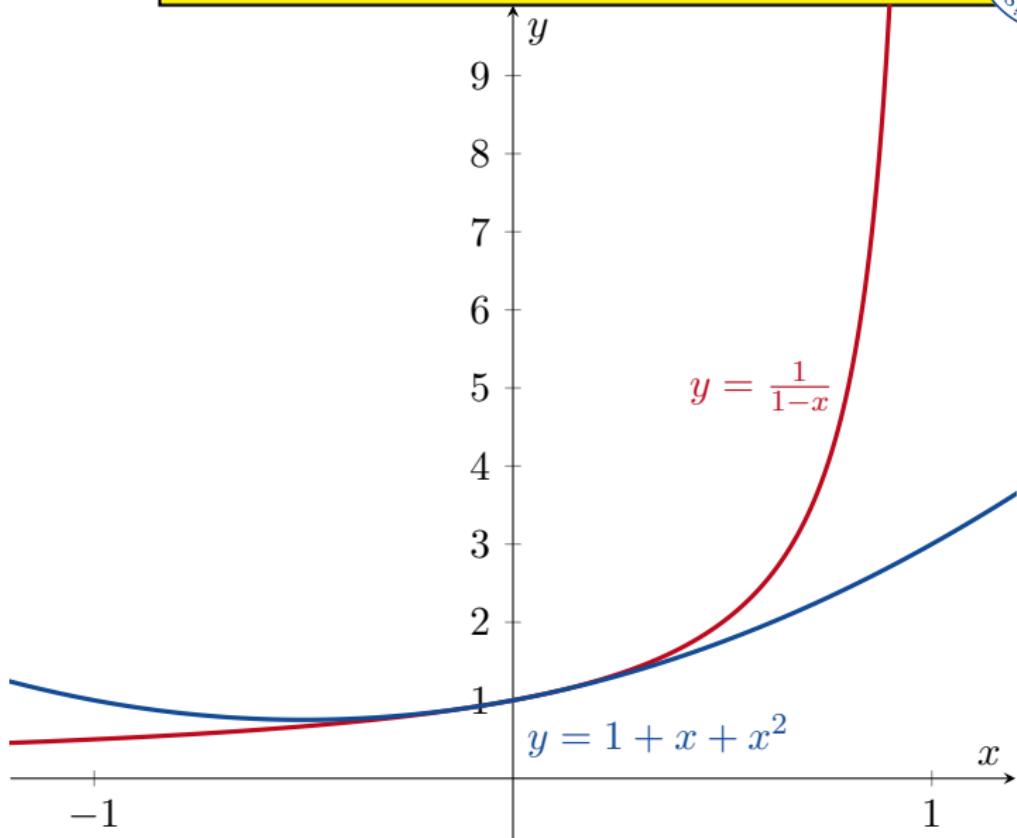
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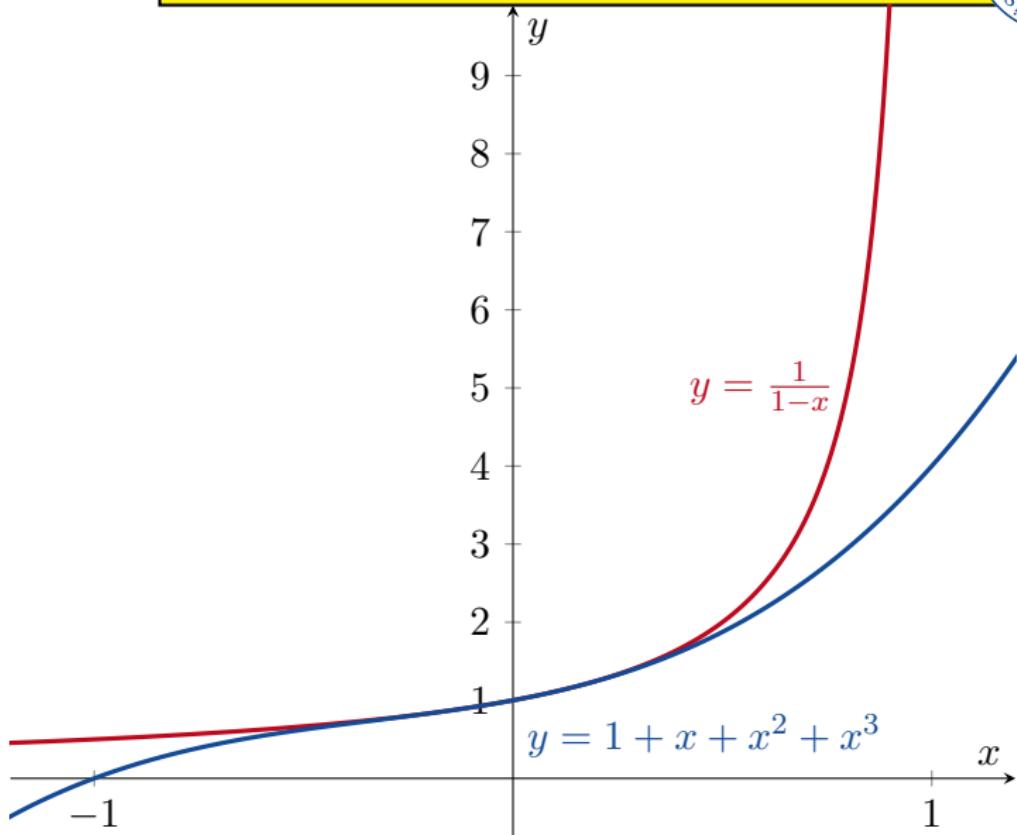
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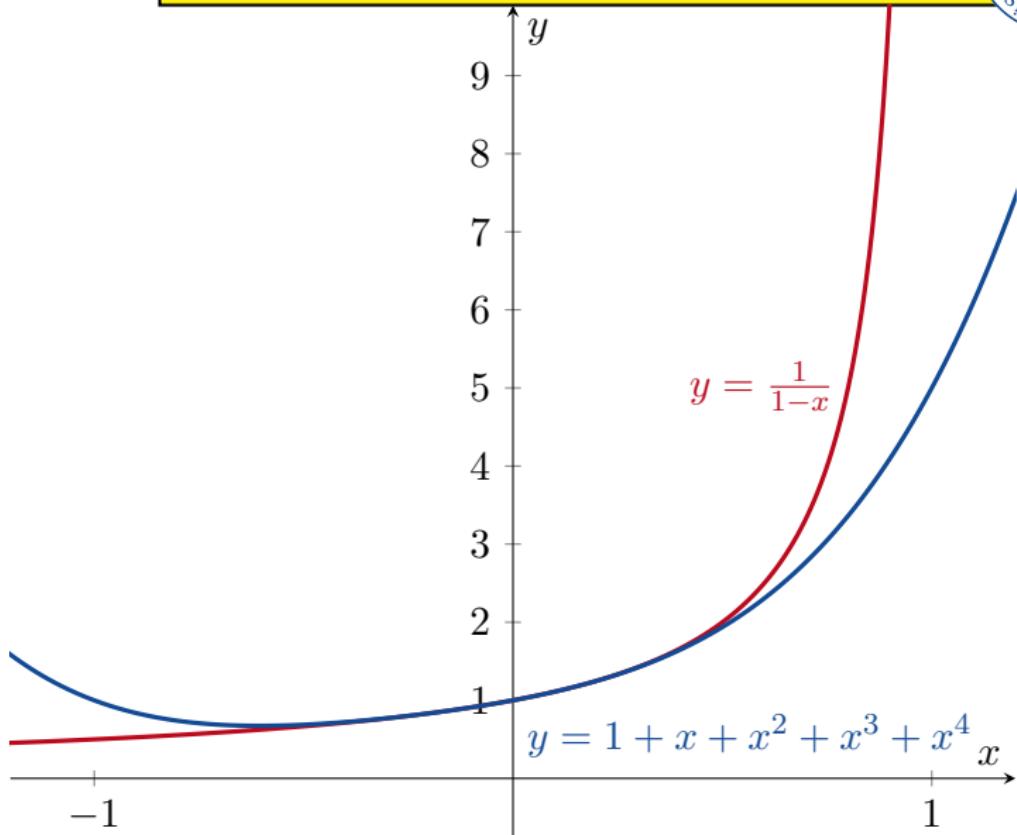
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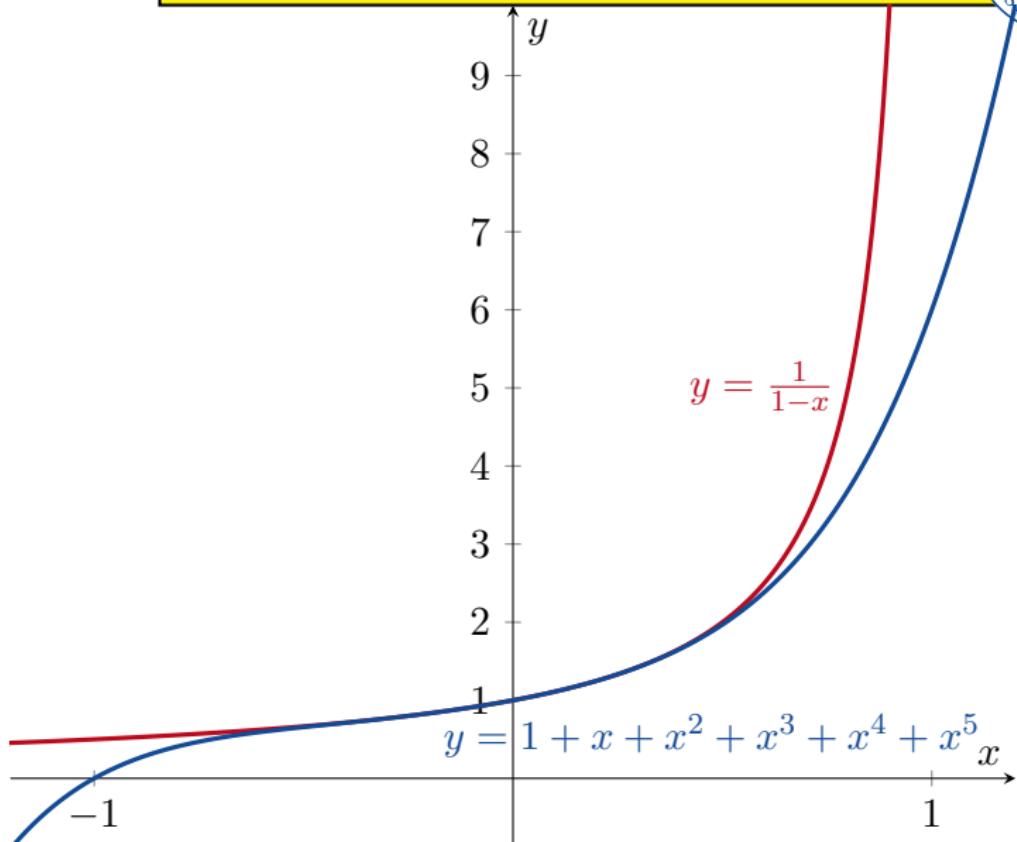
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$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + x^4 + x^5 + x^6 + \dots$$



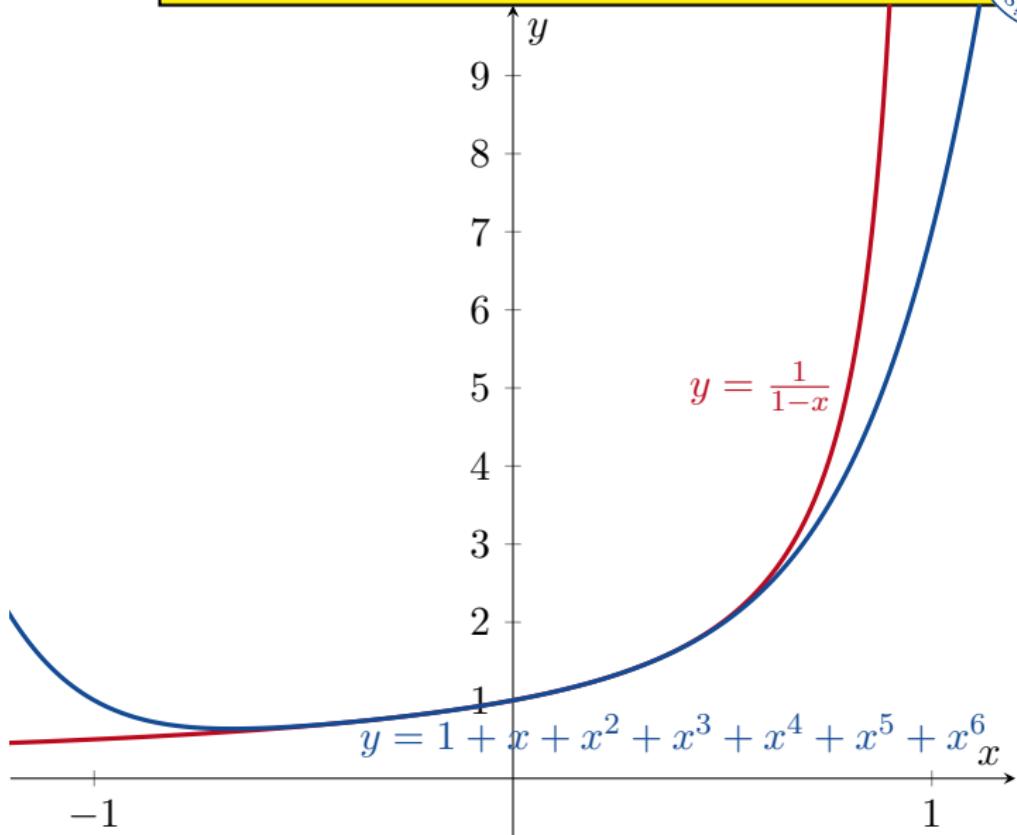
9.7 Power Series

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9.7 Power Series

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9.7 Power Series

Example

Consider $\sum_{n=0}^{\infty} \frac{x^n}{n!}$.

9.7 Power Series

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Consider $\sum_{n=0}^{\infty} \frac{x^n}{n!}$.

- If $x = 0$, then $\sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + 0 + 0 + 0 + 0 + 0 + \dots$ is convergent.

9.7 Power Series

Example

Consider $\sum_{n=0}^{\infty} \frac{x^n}{n!}$.

- If $x = 0$, then $\sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + 0 + 0 + 0 + 0 + 0 + \dots$ is convergent.
- Suppose that $x \neq 0$. Let $b_n := \frac{x^n}{n!}$. Then

$$\left| \frac{b_{n+1}}{b_n} \right| = \left| \frac{x^{n+1}}{(n+1)!} \frac{n!}{x^n} \right| = \frac{|x|}{n+1} \rightarrow 0$$

as $n \rightarrow \infty$. Hence $\sum_{n=0}^{\infty} b_n$ is absolutely convergent by the Ratio Test v2.

9.7 Power Series

Example

Consider $\sum_{n=0}^{\infty} \frac{x^n}{n!}$.

- If $x = 0$, then $\sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + 0 + 0 + 0 + 0 + 0 + \dots$ is convergent.
- Suppose that $x \neq 0$. Let $b_n := \frac{x^n}{n!}$. Then

$$\left| \frac{b_{n+1}}{b_n} \right| = \left| \frac{x^{n+1}}{(n+1)!} \frac{n!}{x^n} \right| = \frac{|x|}{n+1} \rightarrow 0$$

as $n \rightarrow \infty$. Hence $\sum_{n=0}^{\infty} b_n$ is absolutely convergent by the Ratio Test v2.

Therefore $\sum_{n=0}^{\infty} \frac{x^n}{n!}$ converges $\forall x \in \mathbb{R}$.

9.7 Power Series



Example

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9.7 Power Series



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- If $x = 0$, then $\sum_{n=0}^{\infty} n!x^n = 1 + 0 + 0 + 0 + 0 + 0 + \dots$ is convergent.
- Suppose that $x \neq 0$. Let $b_n := n!x^n$ and $t = \frac{1}{x}$. Recall that $\frac{t^n}{n!} \rightarrow 0$ as $n \rightarrow \infty \forall t \in \mathbb{R}$.

9.7 Power Series



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Consider $\sum_{n=0}^{\infty} n!x^n$.

- If $x = 0$, then $\sum_{n=0}^{\infty} n!x^n = 1 + 0 + 0 + 0 + 0 + 0 + \dots$ is convergent.
- Suppose that $x \neq 0$. Let $b_n := n!x^n$ and $t = \frac{1}{x}$. Recall that $\frac{t^n}{n!} \rightarrow 0$ as $n \rightarrow \infty \ \forall t \in \mathbb{R}$. So $|b_n| = |n!x^n| = \left| \frac{n!}{t^n} \right| \rightarrow \infty$ as $n \rightarrow \infty$. So $|b_n| \not\rightarrow 0$ as $n \rightarrow \infty$.

9.7 Power Series



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Consider $\sum_{n=0}^{\infty} n!x^n$.

- If $x = 0$, then $\sum_{n=0}^{\infty} n!x^n = 1 + 0 + 0 + 0 + 0 + 0 + \dots$ is convergent.
- Suppose that $x \neq 0$. Let $b_n := n!x^n$ and $t = \frac{1}{x}$. Recall that $\frac{t^n}{n!} \rightarrow 0$ as $n \rightarrow \infty \ \forall t \in \mathbb{R}$. So $|b_n| = |n!x^n| = \left|\frac{n!}{t^n}\right| \rightarrow \infty$ as $n \rightarrow \infty$. So $|b_n| \not\rightarrow 0$ as $n \rightarrow \infty$. By the Divergence Test, $\sum_{n=0}^{\infty} b_n$ diverges.

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Consider $\sum_{n=0}^{\infty} n!x^n$.

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- Suppose that $x \neq 0$. Let $b_n := n!x^n$ and $t = \frac{1}{|x|}$. Recall that $\frac{t^n}{n!} \rightarrow 0$ as $n \rightarrow \infty \forall t \in \mathbb{R}$. So $|b_n| = |n!x^n| = \left|\frac{n!}{t^n}\right| \rightarrow \infty$ as $n \rightarrow \infty$. So $|b_n| \not\rightarrow 0$ as $n \rightarrow \infty$. By the Divergence Test, $\sum_{n=0}^{\infty} b_n$ diverges.

Therefore $\sum_{n=0}^{\infty} \frac{x^n}{n!}$ $\begin{cases} \text{converges if } x = 0 \\ \text{diverges if } x \neq 0. \end{cases}$

9.7 Power Series



Example

Consider $\sum_{n=1}^{\infty} nx^{n-1} = 1 + 2x + 3x^2 + 4x^3 + 5x^4 + \dots$

9.7 Power Series



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9.7 Power Series

Example

Consider $\sum_{n=1}^{\infty} nx^{n-1} = 1 + 2x + 3x^2 + 4x^3 + 5x^4 + \dots$

- Suppose that $x \neq 0$. Let $b_n := nx^{n-1}$.

Then

$$\left| \frac{b_{n+1}}{b_n} \right| = \frac{(n+1)|x|^n}{n|x|^{n-1}} = \left(1 + \frac{1}{n}\right)|x| \rightarrow |x|$$

as $n \rightarrow \infty$.

9.7 Power Series

Example

Consider $\sum_{n=1}^{\infty} nx^{n-1} = 1 + 2x + 3x^2 + 4x^3 + 5x^4 + \dots$

- Suppose that $x \neq 0$. Let $b_n := nx^{n-1}$.

Then

$$\left| \frac{b_{n+1}}{b_n} \right| = \frac{(n+1)|x|^n}{n|x|^{n-1}} = \left(1 + \frac{1}{n}\right)|x| \rightarrow |x|$$

as $n \rightarrow \infty$.

By the Ratio Test v2,

$$\sum_{n=1}^{\infty} nx^{n-1} \begin{cases} \text{converges if } 0 < |x| < 1 \\ \text{diverges if } |x| > 1. \end{cases}$$

9.7 Power Series



Example

Consider $\sum_{n=1}^{\infty} nx^{n-1} = 1 + 2x + 3x^2 + 4x^3 + 5x^4 + \dots$

- Suppose that $|x| = 1$.

9.7 Power Series



Example

Consider $\sum_{n=1}^{\infty} nx^{n-1} = 1 + 2x + 3x^2 + 4x^3 + 5x^4 + \dots$

- Suppose that $|x| = 1$. Then $|nx^{n-1}| = n$ which means that $nx^{n-1} \not\rightarrow 0$ as $n \rightarrow \infty$.

9.7 Power Series



Example

Consider $\sum_{n=1}^{\infty} nx^{n-1} = 1 + 2x + 3x^2 + 4x^3 + 5x^4 + \dots$

- Suppose that $|x| = 1$. Then $|nx^{n-1}| = n$ which means that $nx^{n-1} \not\rightarrow 0$ as $n \rightarrow \infty$.

So $\sum_{n=1}^{\infty} b_n$ diverges if $|x| = 1$.

9.7 Power Series



Example

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So $\sum_{n=1}^{\infty} b_n$ diverges if $|x| = 1$.

Therefore $\sum_{n=1}^{\infty} nx^{n-1}$ $\begin{cases} \text{converges if } |x| < 1 \\ \text{diverges if } |x| \geq 1. \end{cases}$

9.7 Power Series



You can read more examples in the textbook.

9.7 Power Series



Remark

$\sum_{n=0}^{\infty} x^n$ converges for $|x| < 1$ and diverges for $|x| \geq 1$. If we differentiate each term (are we allowed to do this?), we get

$$0 + 1 + 2x + 3x^2 + 4x^3 + 5x^4 + \dots = \sum_{n=1}^{\infty} nx^{n-1}$$

which also converges for $|x| < 1$ and diverges for $|x| \geq 1$.
Interesting!

9.7 Power Series



Theorem

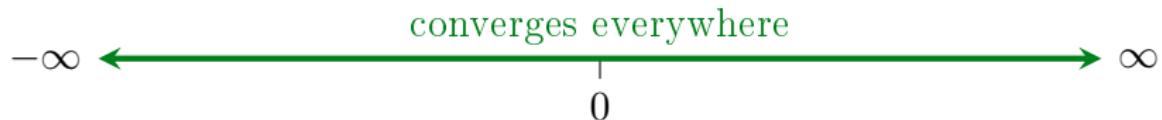
A power series $\sum_{n=0}^{\infty} a_n x^n$ satisfies one and only one of the following:

- 1 It converges absolutely $\forall x$;
- 2 It converges for $x = 0$ and diverges $\forall x \neq 0$; or
- 3 $\exists R > 0$ such that $\sum_{n=0}^{\infty} a_n x^n$ $\begin{cases} \text{converges if } |x| < R \\ \text{diverges if } |x| > R. \end{cases}$

9.7 Power Series



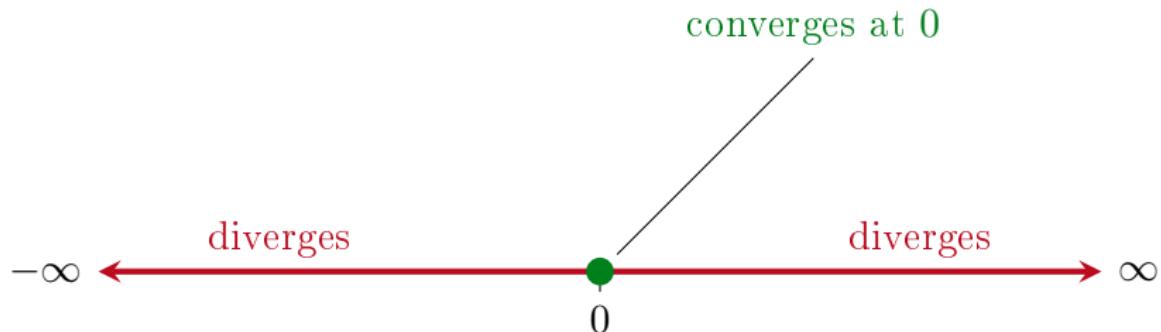
1



9.7 Power Series



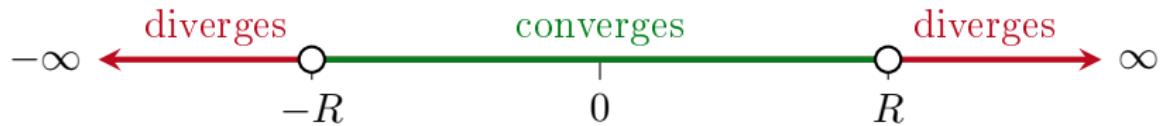
2



9.7 Power Series



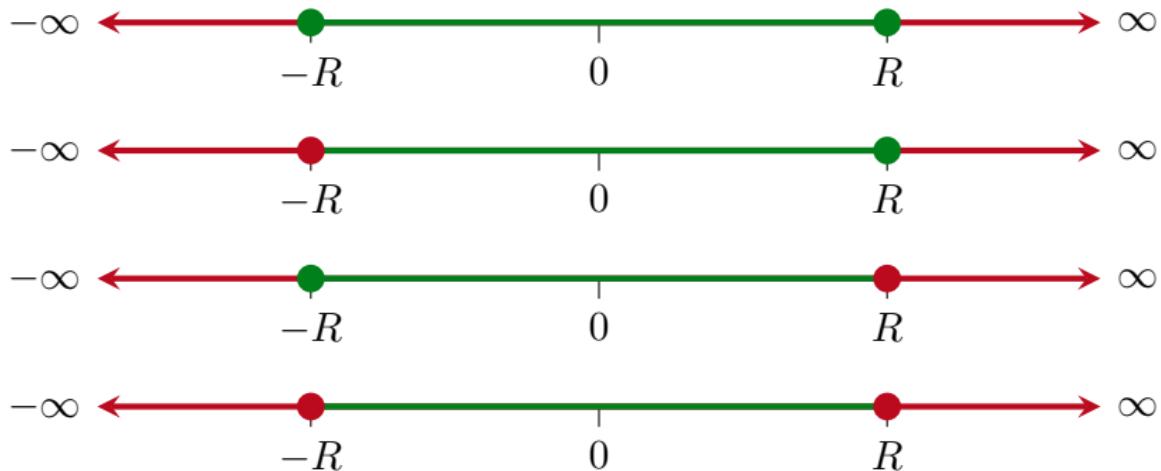
3



9.7 Power Series



3



Radius of Convergence

Definition

Let $R \in [0, \infty) \cup \{\infty\}$.

If $\sum_{n=0}^{\infty} a_n x^n$ converges $\forall |x| < R$ and diverges $\forall |x| > R$, then R is called the *radius of convergence* (yakınsaklık yarıçapı) of the power series $\sum_{n=0}^{\infty} a_n x^n$.

9.7 Power Series

Definition

If $R = \infty$, then we say that $\sum_{n=0}^{\infty} a_n x^n$ has *infinite radius of convergence*. (This means that $\sum_{n=0}^{\infty} a_n x^n$ converges $\forall x$.)

Definition

If $R = 0$, then we say that $\sum_{n=0}^{\infty} a_n x^n$ has *zero radius of convergence*. (This means that $\sum_{n=0}^{\infty} a_n x^n$ converges if $x = 0$ and diverges $\forall x \neq 0$.)

9.7 Power Series



Definition

If $R > 0$ or $R = \infty$, then the open interval $(-R, R)$ is called the *open interval of convergence* of $\sum_{n=0}^{\infty} a_n x^n$.

9.7 Power Series



Is there an easy way to find R ?

9.7 Power Series



Is there an easy way to find R ?

Theorem

Suppose that

$$\left| \frac{a_n}{a_{n+1}} \right| \rightarrow R \in \mathbb{R} \cup \{\infty\}$$

as $n \rightarrow \infty$. Then $\sum_{n=0}^{\infty} a_n x^n$ has radius of convergence R .

9.7 Power Series



Remark

A power series *always* has a radius of convergence, even if

$$\lim_{n \rightarrow \infty} \left| \frac{a_n}{a_{n+1}} \right| \text{ doesn't exist.}$$

This theorem just gives us an easy way to find R , if this limit does exist.

If the limit does not exist, then we need to use a different method to find R .

9.7 Power Series

Remark

A power series *always* has a radius of convergence, even if

$$\lim_{n \rightarrow \infty} \left| \frac{a_n}{a_{n+1}} \right| \text{ doesn't exist.}$$

This theorem just gives us an easy way to find R , if this limit does exist.

If the limit does not exist, then we need to use a different method to find R .

Remark

Never, never, never forget to use $|\cdot|$ when you use this theorem.

9.7 Power Series



Remark

The Ratio Test v2 uses $\left| \frac{a_{n+1}}{a_n} \right|$, but this theorem uses $\left| \frac{a_n}{a_{n+1}} \right|$.

Don't get these mixed up.

9.7 Power Series



We have seen that $\exists R$ such that

$$\sum_{n=0}^{\infty} a_n x^n \begin{cases} \text{converges if } |x| < R \\ \text{diverges if } |x| > R. \end{cases}$$

Suppose that $0 < R < \infty$.

9.7 Power Series



We have seen that $\exists R$ such that

$$\sum_{n=0}^{\infty} a_n x^n \begin{cases} \text{converges if } |x| < R \\ \text{diverges if } |x| > R. \end{cases}$$

Suppose that $0 < R < \infty$.

What happens when $|x| = R$?

9.7 Power Series

Example

Consider $\sum_{n=0}^{\infty} x^n$.

9.7 Power Series

Example

Consider $\sum_{n=0}^{\infty} x^n$.

This is a power series with $a_n = 1 \ \forall n$. Since

$$\left| \frac{a_n}{a_{n+1}} \right| = \frac{1}{1} = 1,$$

the radius of convergence is $R = 1$.

9.7 Power Series

Example

Consider $\sum_{n=0}^{\infty} x^n$.

This is a power series with $a_n = 1 \ \forall n$. Since

$$\left| \frac{a_n}{a_{n+1}} \right| = \frac{1}{1} = 1,$$

the radius of convergence is $R = 1$. This means that

$$\sum_{n=0}^{\infty} a_n x^n \begin{cases} \text{converges if } |x| < 1 \\ \text{diverges if } |x| > 1. \end{cases}$$

9.7 Power Series

Example

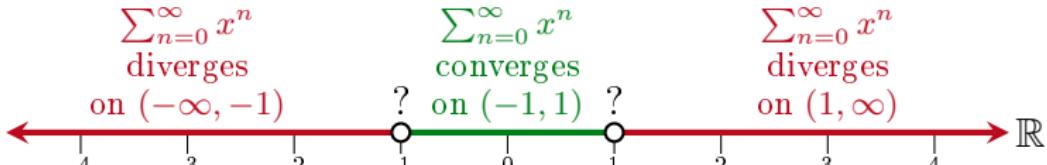
Consider $\sum_{n=0}^{\infty} x^n$.

This is a power series with $a_n = 1 \forall n$. Since

$$\left| \frac{a_n}{a_{n+1}} \right| = \frac{1}{1} = 1,$$

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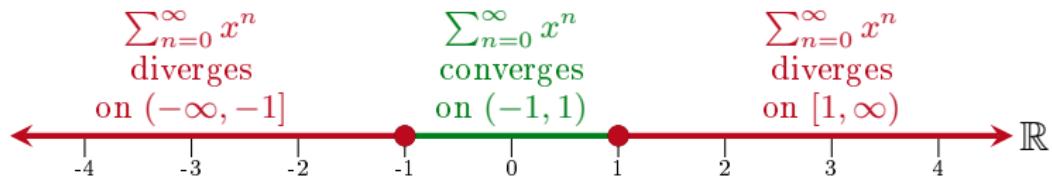
$$\sum_{n=0}^{\infty} a_n x^n \begin{cases} \text{converges if } |x| < 1 \\ \text{diverges if } |x| > 1. \end{cases}$$



9.7 Power Series



Previously we saw that $\sum_{n=0}^{\infty} x^n$ also diverges for $|x| = 1$.



For this power series, we have divergence when $x = \pm R$.

9.7 Power Series



Example

Now consider $\sum_{n=0}^{\infty} \frac{x^n}{n+1}$.

9.7 Power Series



Example

Now consider $\sum_{n=0}^{\infty} \frac{x^n}{n+1}$.

For this power series, $a_n = \frac{1}{n+1} \quad \forall n \in \mathbb{N} \cup \{0\}$ and

$$\left| \frac{a_n}{a_{n+1}} \right| = \frac{n+2}{n+1} = \frac{1 + \frac{2}{n}}{1 + \frac{1}{n}} \rightarrow 1$$

as $n \rightarrow \infty$. Thus, the radius of convergence is $R = 1$ again.

9.7 Power Series



Example

Now consider $\sum_{n=0}^{\infty} \frac{x^n}{n+1}$.

For this power series, $a_n = \frac{1}{n+1} \quad \forall n \in \mathbb{N} \cup \{0\}$ and

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as $n \rightarrow \infty$. Thus, the radius of convergence is $R = 1$ again.

This means that $\sum_{n=0}^{\infty} \frac{x^n}{n+1} \begin{cases} \text{converges if } |x| < 1 \\ \text{diverges if } |x| > 1. \end{cases}$

9.7 Power Series



When $x = 1$, the series is

$$\sum_{n=0}^{\infty} \frac{1^n}{n+1} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \dots = \sum_{n=1}^{\infty} \frac{1}{n}$$

which we know diverges.

9.7 Power Series



When $x = 1$, the series is

$$\sum_{n=0}^{\infty} \frac{1^n}{n+1} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \dots = \sum_{n=1}^{\infty} \frac{1}{n}$$

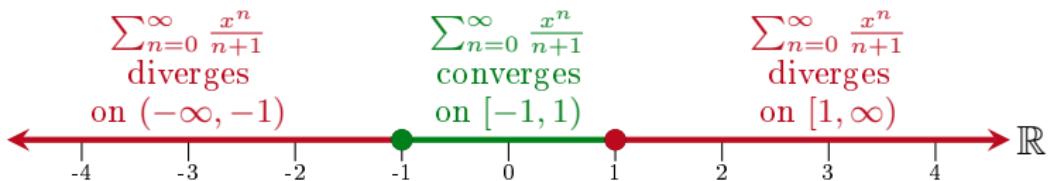
which we know diverges.

When $x = -1$, the series is

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{n+1} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \dots$$

which we know converges.

9.7 Power Series



For this power series, we have convergence when $x = -R$ and divergence when $x = R$.

9.7 Power Series



Example

Consider $\sum_{n=0}^{\infty} \frac{x^n}{(n+1)^2}$.

9.7 Power Series



Example

Consider $\sum_{n=0}^{\infty} \frac{x^n}{(n+1)^2}$.

For this power series, $a_n = \frac{1}{(n+1)^2} \quad \forall n \in \mathbb{N} \cup \{0\}$ and

$$\left| \frac{a_n}{a_{n+1}} \right| = \frac{(n+2)^2}{(n+1)^2} \rightarrow 1$$

as $n \rightarrow \infty$. Thus, the radius of convergence is $R = 1$ again.

9.7 Power Series



When $|x| = R = 1$,

$$\sum_{n=0}^{\infty} \frac{|x|^n}{(n+1)^2} = \sum_{n=0}^{\infty} \frac{1}{(n+1)^2} = \sum_{n=1}^{\infty} \frac{1}{n^2}$$

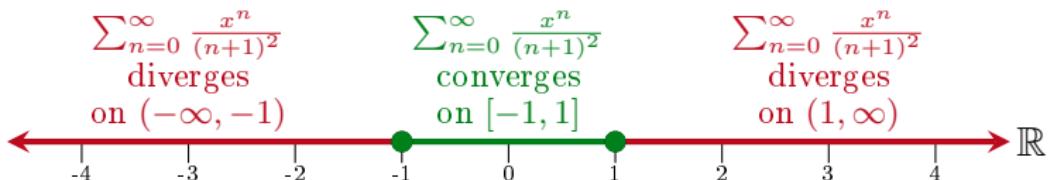
which converges.

9.7 Power Series

When $|x| = R = 1$,

$$\sum_{n=0}^{\infty} \frac{|x|^n}{(n+1)^2} = \sum_{n=0}^{\infty} \frac{1}{(n+1)^2} = \sum_{n=1}^{\infty} \frac{1}{n^2}$$

which converges.



For this power series, we have convergence when $x = \pm R$.

9.7 Power Series



Remark

The previous three examples show that when $|x| = R \in (0, \infty)$, we can have divergence, conditional convergence or absolute convergence.

Break

We will continue at 3pm

CABIN CALCULUS

- BY NANSCLARK

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$$\int \frac{1}{d} \, d\text{house} =$$



9.7 Power Series

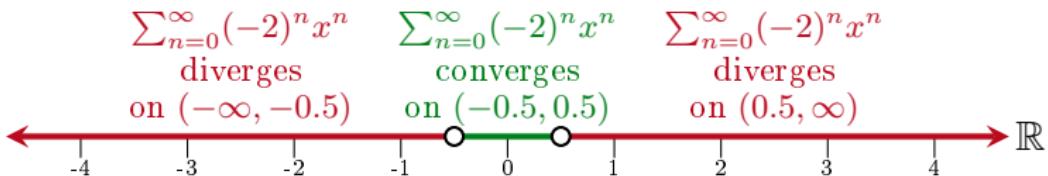
Example

Consider $\sum_{n=0}^{\infty} (-2)^n x^n$.

Since

$$\left| \frac{a_n}{a_{n+1}} \right| = \left| \frac{(-2)^n}{(-2)^{n+1}} \right| = \frac{1}{2},$$

this power series has radius of convergence $R = \frac{1}{2}$. The open interval of convergence is $(-\frac{1}{2}, \frac{1}{2})$.



9.7 Power Series

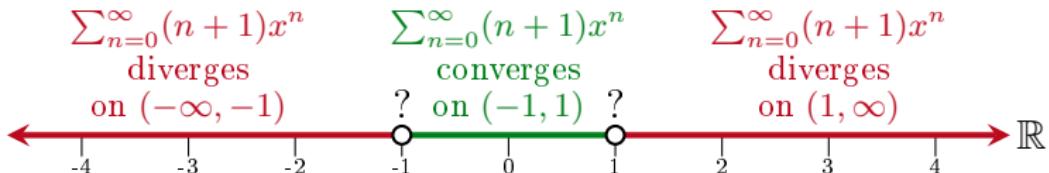
Example

Consider $\sum_{n=0}^{\infty} (n+1)x^n$.

Since

$$\left| \frac{a_n}{a_{n+1}} \right| = \frac{n+1}{n+2} \rightarrow 1$$

as $n \rightarrow \infty$, this power series has radius of convergence $R = 1$.
 The open interval of convergence is $(-1, 1)$.



9.7 Power Series

$$\cosh x = \frac{e^x + e^{-x}}{2}$$



Example

Consider $\sum_{n=0}^{\infty} (\cosh n)x^n$.

Since

$$\left| \frac{a_n}{a_{n+1}} \right| = \left| \frac{\cosh n}{\cosh(n+1)} \right| = \frac{e^n + e^{-n}}{e^{n+1} + e^{-n-1}} = \frac{1 + e^{-2n}}{e + e^{-2n-1}} \rightarrow \frac{1 + 0}{e + 1} = \frac{1}{e}$$

as $n \rightarrow \infty$, this power series has radius of convergence $R = \frac{1}{e}$.

The open interval of convergence is $(-\frac{1}{e}, \frac{1}{e})$.

$$\begin{array}{ccc} \sum_{n=0}^{\infty} (\cosh n)x^n & \sum_{n=0}^{\infty} (\cosh n)x^n & \sum_{n=0}^{\infty} (\cosh n)x^n \\ \text{diverges} & \text{converges} & \text{diverges} \\ \text{on } (-\infty, -1/e) & \text{on } (-1/e, 1/e) & \text{on } (1/e, \infty) \end{array}$$



9.7 Power Series



Example

For the power series $\sum_{n=0}^{\infty} \frac{x^n}{n!}$, we have that

$$R = \lim_{n \rightarrow \infty} \left| \frac{a_n}{a_{n+1}} \right| = \infty$$

as $n \rightarrow \infty$. The radius of convergence $R = \infty$. The open interval of convergence is $(-\infty, \infty)$.

$\sum_{n=0}^{\infty} \frac{x^n}{n!}$ converges $\forall x$





Term-by-Term Differentiation and Integration

A power series is a function. So can we differentiate it? Can we integrate it?

9.7 Power Series



Theorem

Let $\sum_{n=0}^{\infty} a_n x^n$ be a power series with radius of convergence $R > 0$.

If $|x| < R$, then

$$\frac{d}{dx} \left(\sum_{n=0}^{\infty} a_n x^n \right) = \sum_{n=0}^{\infty} \left(\frac{d}{dx} a_n x^n \right)$$

and

$$\int \left(\sum_{n=0}^{\infty} a_n x^n \right) dx = \sum_{n=0}^{\infty} \left(\int a_n x^n dx \right).$$

9.7 Power Series

EXAMPLE 4 Find series for $f'(x)$ and $f''(x)$ if

$$\begin{aligned}f(x) &= \frac{1}{1-x} = 1 + x + x^2 + x^3 + x^4 + \cdots + x^n + \cdots \\&= \sum_{n=0}^{\infty} x^n, \quad -1 < x < 1.\end{aligned}$$

9.7 Power Series

EXAMPLE 4 Find series for $f'(x)$ and $f''(x)$ if

$$\begin{aligned} f(x) &= \frac{1}{1-x} = 1 + x + x^2 + x^3 + x^4 + \cdots + x^n + \cdots \\ &= \sum_{n=0}^{\infty} x^n, \quad -1 < x < 1. \end{aligned}$$

Solution We differentiate the power series on the right term by term:

$$\begin{aligned} f'(x) &= \frac{1}{(1-x)^2} = 1 + 2x + 3x^2 + 4x^3 + \cdots + nx^{n-1} + \cdots \\ &= \sum_{n=1}^{\infty} nx^{n-1}, \quad -1 < x < 1; \\ f''(x) &= \frac{2}{(1-x)^3} = 2 + 6x + 12x^2 + \cdots + n(n-1)x^{n-2} + \cdots \\ &= \sum_{n=2}^{\infty} n(n-1)x^{n-2}, \quad -1 < x < 1. \end{aligned}$$



EXAMPLE 6 The series

$$\frac{1}{1+t} = 1 - t + t^2 - t^3 + \dots$$

converges on the open interval $-1 < t < 1$. Therefore,

$$\begin{aligned}\ln(1+x) &= \int_0^x \frac{1}{1+t} dt = t - \frac{t^2}{2} + \frac{t^3}{3} - \frac{t^4}{4} + \dots \Big|_0^x \\ &= x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots\end{aligned}$$

or

$$\ln(1+x) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1} x^n}{n}, \quad -1 < x < 1.$$

Power Series with Centre of Expansion c

The results that we have proved for the power series $\sum_{n=0}^{\infty} a_n x^n$

are also true for the power series $\sum_{n=0}^{\infty} a_n (x - c)^n$.

9.7 Power Series



Example

Recall that $\sum_{n=0}^{\infty} x^n$ has radius of convergence $R = 1$. Therefore $\sum_{n=0}^{\infty} (x - c)^n$ also has radius of convergence $R = 1$.

9.7 Power Series



Example

Recall that $\sum_{n=0}^{\infty} x^n$ has radius of convergence $R = 1$. Therefore $\sum_{n=0}^{\infty} (x - c)^n$ also has radius of convergence $R = 1$. Since

$$\sum_{n=0}^{\infty} x^n \begin{cases} \text{converges absolutely } \forall |x| < 1 \\ \text{diverges } \forall |x| > 1 \end{cases}$$

we have that

$$\sum_{n=0}^{\infty} (x - c)^n \begin{cases} \text{converges absolutely } \forall |x - c| < 1 \\ \text{diverges } \forall |x - c| > 1. \end{cases}$$

9.7 Power Series



Example

Recall that $\sum_{n=0}^{\infty} x^n$ has radius of convergence $R = 1$. Therefore $\sum_{n=0}^{\infty} (x - c)^n$ also has radius of convergence $R = 1$. Since

$$\sum_{n=0}^{\infty} x^n \begin{cases} \text{converges absolutely } \forall |x| < 1 \\ \text{diverges } \forall |x| > 1 \end{cases}$$

we have that

$$\sum_{n=0}^{\infty} (x - c)^n \begin{cases} \text{converges absolutely } \forall |x - c| < 1 \\ \text{diverges } \forall |x - c| > 1. \end{cases}$$

The open interval of convergence for $\sum_{n=0}^{\infty} (x - c)^n$ is $(c - 1, c + 1)$.

9.7 Power Series



Example

Since $\sum_{n=0}^{\infty} \frac{x^n}{n!}$ has radius of convergence $R = \infty$, it follows that

$\sum_{n=0}^{\infty} \frac{(x - c)^n}{n!}$ converges absolutely $\forall x$.

The radius of convergence of $\sum_{n=0}^{\infty} \frac{(x - c)^n}{n!}$ is $R = \infty$ and the open interval of convergence is $(-\infty, \infty)$.

9.7 Power Series

Example

Recall that $\sum_{n=0}^{\infty} \frac{x^n}{(n+1)^2}$ has radius of convergence $R = 1$. So

$$\sum_{n=0}^{\infty} \frac{(x-c)^n}{(n+1)^2} \begin{cases} \text{converges absolutely } \forall |x-c| < 1 \\ \text{diverges } \forall |x-c| > 1. \end{cases}$$

The open interval of convergence of $\sum_{n=0}^{\infty} \frac{(x-c)^n}{(n+1)^2}$ is $(c-1, c+1)$.

9.7 Power Series

Example

Recall that $\sum_{n=0}^{\infty} \frac{x^n}{(n+1)^2}$ has radius of convergence $R = 1$. So

$$\sum_{n=0}^{\infty} \frac{(x-c)^n}{(n+1)^2} \begin{cases} \text{converges absolutely } \forall |x-c| < 1 \\ \text{diverges } \forall |x-c| > 1. \end{cases}$$

The open interval of convergence of $\sum_{n=0}^{\infty} \frac{(x-c)^n}{(n+1)^2}$ is $(c-1, c+1)$.

If $x \in (c-1, c+1)$, then

$$\frac{d}{dx} \left(\sum_{n=0}^{\infty} \frac{(x-c)^n}{(n+1)^2} \right) = \sum_{n=0}^{\infty} \frac{d}{dx} \left(\frac{(x-c)^n}{(n+1)^2} \right)$$

and

$$\int \left(\sum_{n=0}^{\infty} \frac{(x-c)^n}{(n+1)^2} \right) dx = \sum_{n=0}^{\infty} \left(\int \frac{(x-c)^n}{(n+1)^2} dx \right).$$



Taylor and Maclaurin Series

9.8 Taylor and Maclaurin Series



Recall Rolle's Theorem and the Mean Value Theorem from MATH113 Mathematics I (see chapter 4 of Thomas' Calculus):

9.8 Taylor and Maclaurin Series



Michel Rolle

BORN

21 April 1652

DECEASED

8 November 1719

NATIONALITY

French

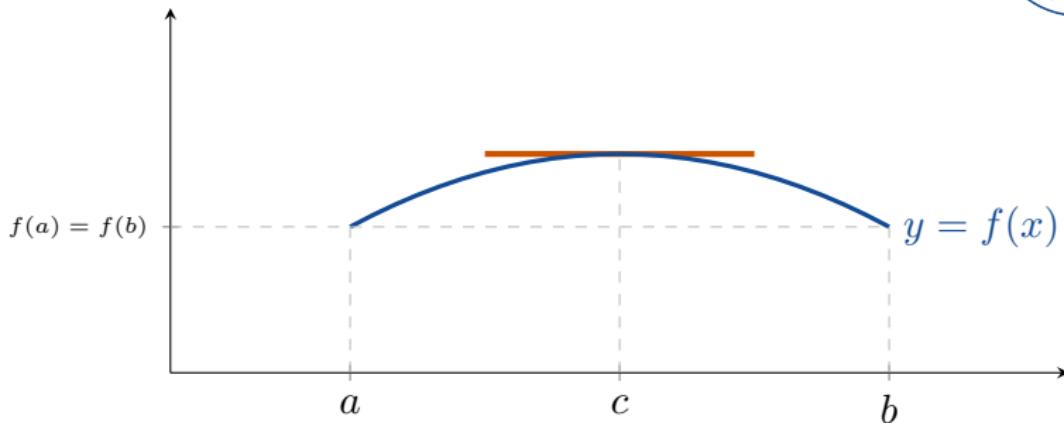
Theorem (Rolle's Theorem)

Suppose that

- 1 $f : [a, b] \rightarrow \mathbb{R}$ is continuous;
- 2 f is differentiable on (a, b) ; and
- 3 $f(a) = f(b)$.

Then $\exists c \in (a, b)$ such that $f'(c) = 0$.

9.8 Taylor and Maclaurin Series



Theorem (Rolle's Theorem)

Suppose that

- 1 $f : [a, b] \rightarrow \mathbb{R}$ is continuous;
- 2 f is differentiable on (a, b) ; and
- 3 $f(a) = f(b)$.

Then $\exists c \in (a, b)$ such that $f'(c) = 0$.



Augustin-Louis Cauchy

BORN

21 August 1789

DECEASED

23 May 1857

NATIONALITY

French

Theorem (The Mean Value Theorem)

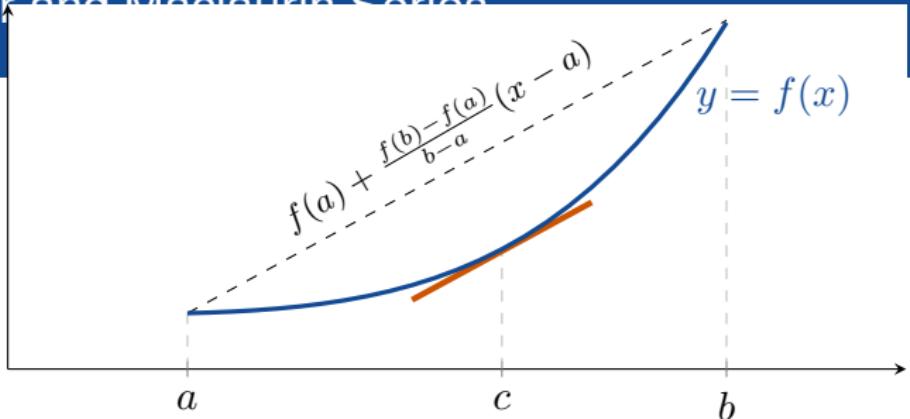
Suppose that

- 1 $f : [a, b] \rightarrow \mathbb{R}$ is continuous; and
- 2 f is differentiable on (a, b) .

Then $\exists c \in (a, b)$ such that

$$\frac{f(b) - f(a)}{b - a} = f'(c).$$

9.8 Taylor and Maclaurin Series



Theorem (The Mean Value Theorem)

Suppose that

- 1 $f : [a, b] \rightarrow \mathbb{R}$ is continuous; and
- 2 f is differentiable on (a, b) .

Then $\exists c \in (a, b)$ such that

$$\frac{f(b) - f(a)}{b - a} = f'(c).$$

9.8 Taylor and Maclaurin Series



Remark

In other words, $\exists c$ such that $a < c < b$ and

$$f(b) = f(a) + f'(c)(b - a).$$

9.8 Taylor and Maclaurin Series

Remark

In other words, $\exists c$ such that $a < c < b$ and

$$f(b) = f(a) + f'(c)(b - a).$$

Taylor's Theorem takes this formula and expands it to more terms.



Brook Taylor

BORN

18 August 1685

DECEASED

29 December 1731

NATIONALITY

British

9.8 Taylor and Maclaurin Series



Theorem (Taylor's Theorem)

Let $f : [a, b] \rightarrow \mathbb{R}$ be a function. Suppose that

- 1 $f, f', f'', f''', \dots, f^{(n-1)}$ exist and are continuous on $[a, b]$;
and
- 2 $f^{(n)}$ exists and is continuous on (a, b) .

9.8 Taylor and Maclaurin Series



Theorem (Taylor's Theorem)

Let $f : [a, b] \rightarrow \mathbb{R}$ be a function. Suppose that

- 1 $f, f', f'', f''', \dots, f^{(n-1)}$ exist and are continuous on $[a, b]$;
and
- 2 $f^{(n)}$ exists and is continuous on (a, b) .

Then $\exists c \in (a, b)$ such that

$$\begin{aligned}f(b) &= f(a) + f'(a)(b - a) + \frac{f''(a)}{2!}(b - a)^2 + \dots \\&\quad + \frac{f^{(n-1)}(a)}{(n-1)!}(b - a)^{n-1} + \frac{f^{(n)}(c)}{n!}(b - a)^n.\end{aligned}$$

9.8 Taylor and Maclaurin Series



Replacing b by x we have

$$\begin{aligned}f(x) &= f(a) + f'(a)(x - a) + \frac{f''(a)}{2!}(x - a)^2 + \dots \\&\quad + \frac{f^{(n-1)}(a)}{(n-1)!}(x - a)^{n-1} + \frac{f^{(n)}(c)}{n!}(x - a)^n.\end{aligned}$$

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This is called the *Taylor Series of $f(x)$ with centre a* .

$$a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \frac{f^{(4)}(a)}{4!}(x-a)^4 + \dots$$

Definition

The *Taylor Series of $f(x)$ with centre a* is the power series

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n.$$

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Definition

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$$\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!} (x-a)^n.$$

$$R_n(c) = \frac{f^{(n)}(c)}{n!} (x-a)^n = f(x) - \sum_{k=0}^{n-1} \frac{f^{(k)}(a)}{k!} (x-a)^k$$

is called the *remainder term*.

Remark

The Taylor Series converges to $f(x)$ $\iff R_n(c) \rightarrow 0$ as $n \rightarrow \infty$.

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Example

Find the Taylor Series for e^x centred at 0.

Let $f(x) = e^x$. Then $\frac{d^k f}{dx^k}$ exists and is continuous $\forall x$ and $\forall k$.

Let $a = 0$ and $x \neq 0$. By Taylor's Theorem,

$$\begin{aligned} f(x) &= f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \dots \\ &\quad + \frac{f^{(n-1)}(0)}{(n-1)!}x^{n-1} + \frac{f^{(n)}(c)}{n!}x^n \end{aligned}$$

for some c between 0 and x ($0 < c < x$ or $x < c < 0$).

$$a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \frac{f^{(4)}(a)}{4!}(x-a)^4 + \dots$$

Because $\frac{d}{dx}e^x = e^x$, it is easy to see that $f^{(k)}(0) = 1 \forall k$.

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$$\begin{aligned} e^x &= f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \dots \\ &\quad + \frac{f^{(n-1)}(0)}{(n-1)!}x^{n-1} + \frac{f^{(n)}(c)}{n!}x^n \\ &= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots + \frac{x^{n-1}}{(n-1)!} + \frac{e^c}{n!}x^n. \end{aligned}$$

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Since $0 < |c| < |x|$,

$$0 \leq \left| \frac{e^c}{n!}x^n \right| \leq \frac{e^{|x|}|x|^n}{n!} \rightarrow 0$$

as $n \rightarrow \infty$. Hence the remainder term $R_c(x) = \frac{e^c}{n!}x^n$ tends to zero.

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Therefore

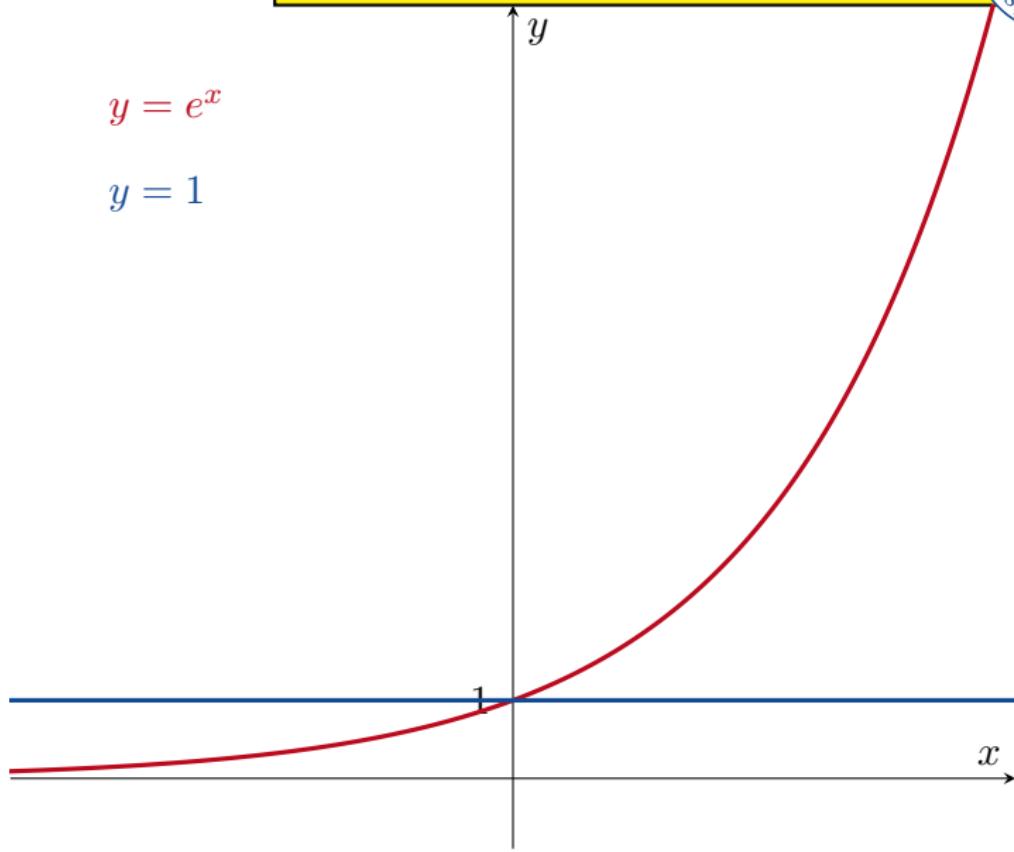
$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

is the Taylor Series of e^x with centre 0.

(Some people use this as the definition of e^x , then define $\ln x$ as the inverse of this.)

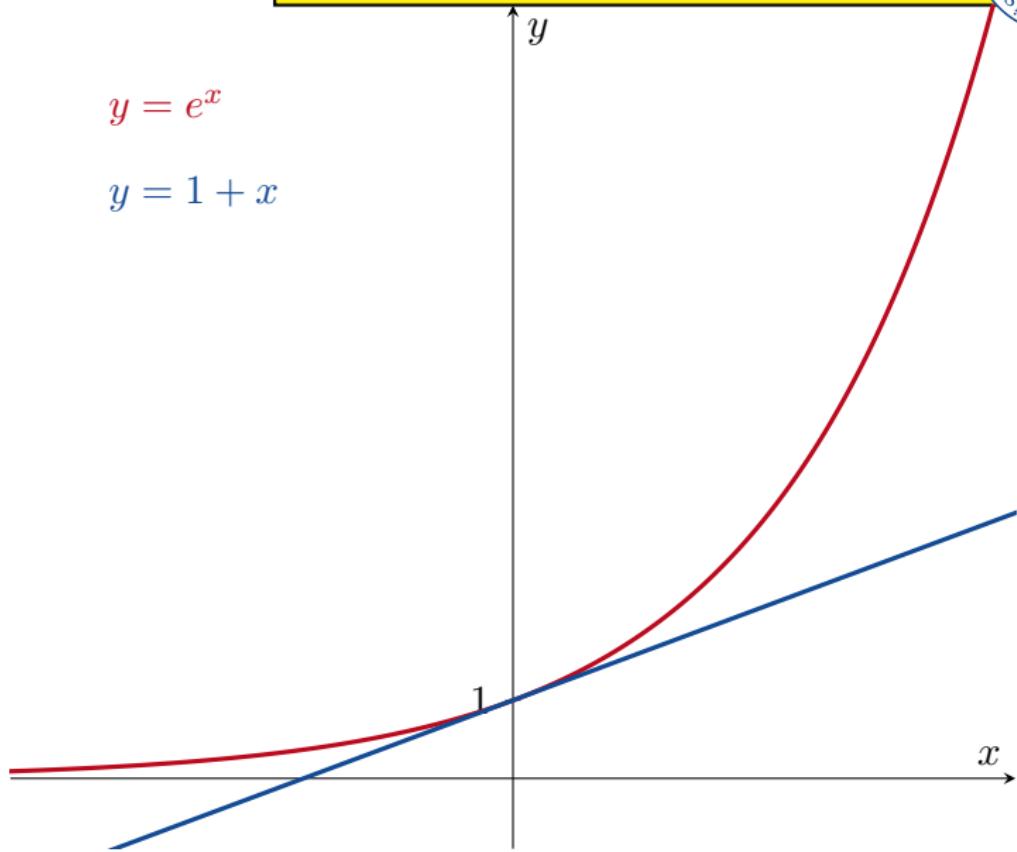
9.8 Taylor and Mac

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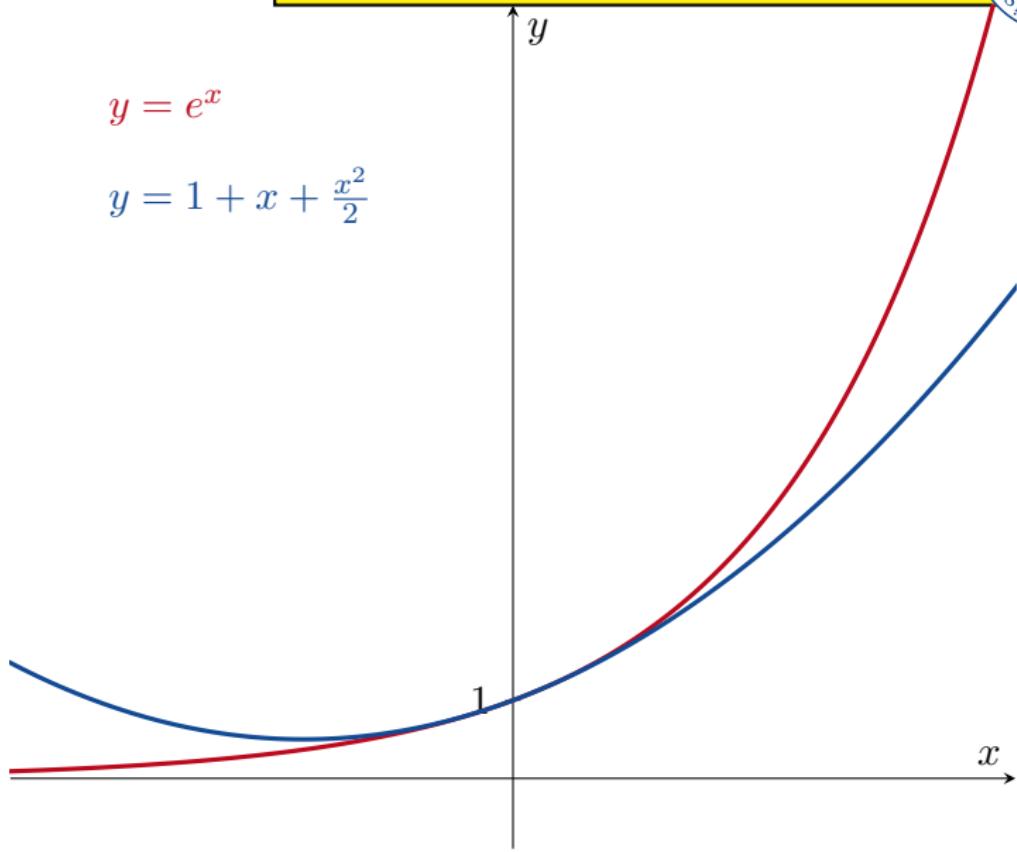
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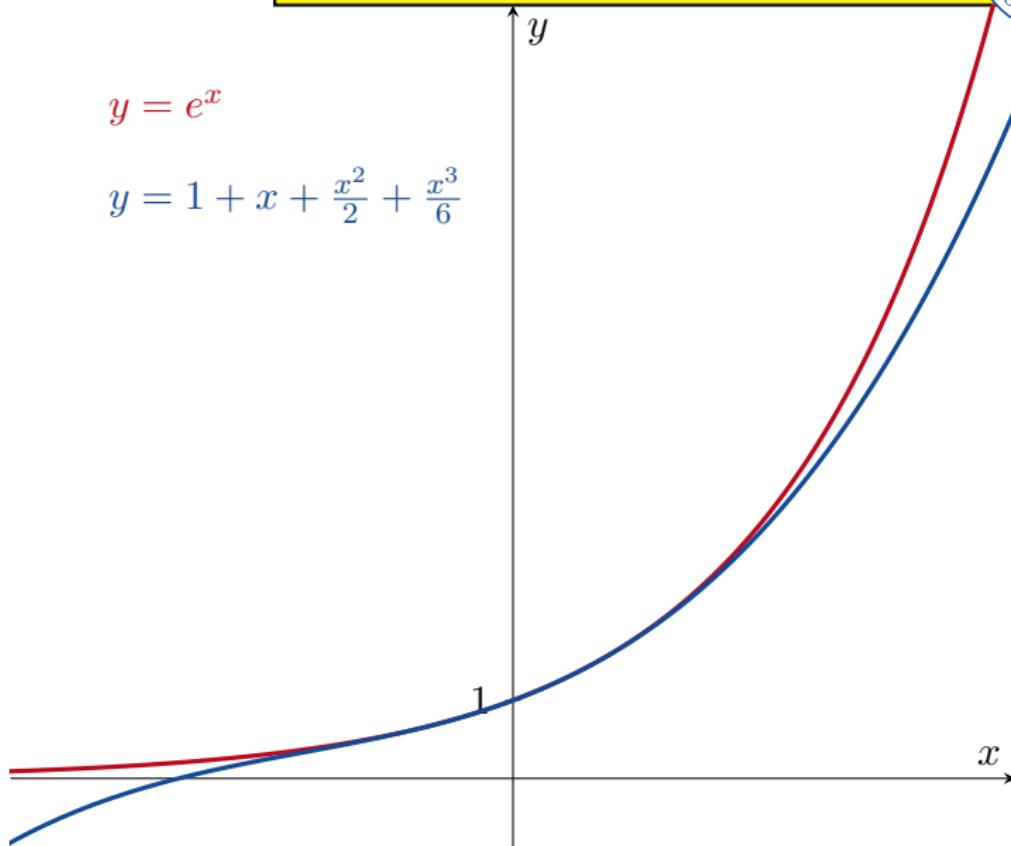
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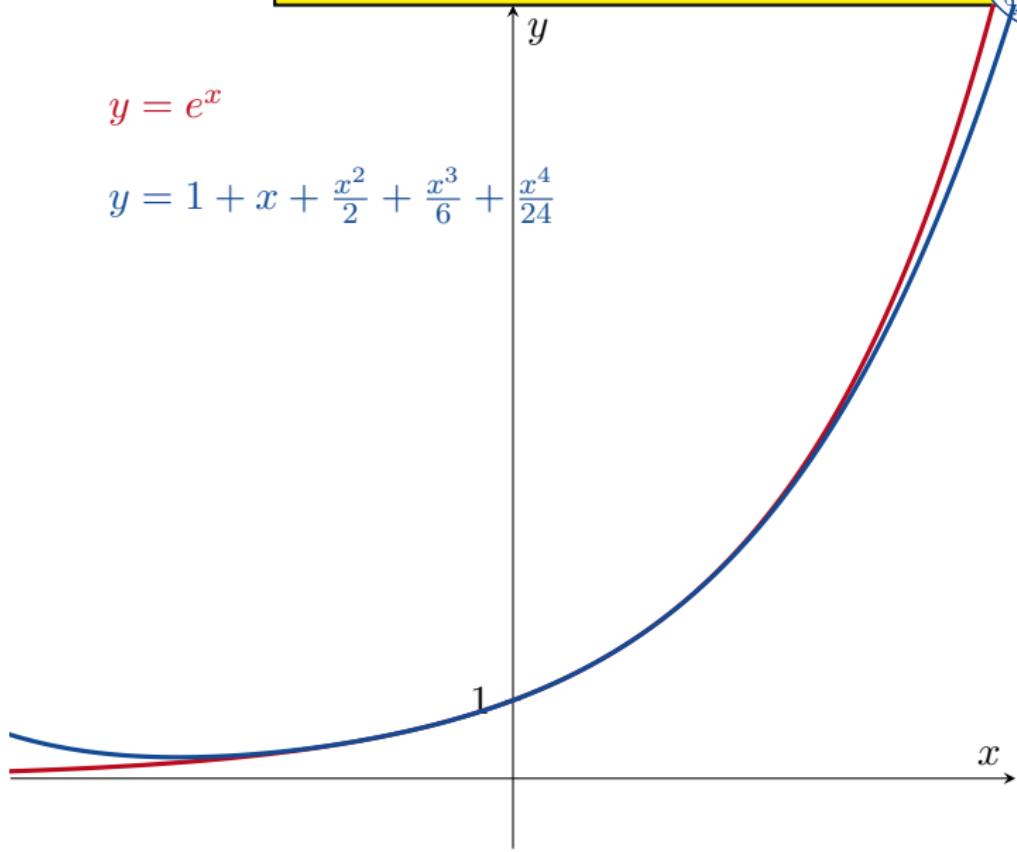
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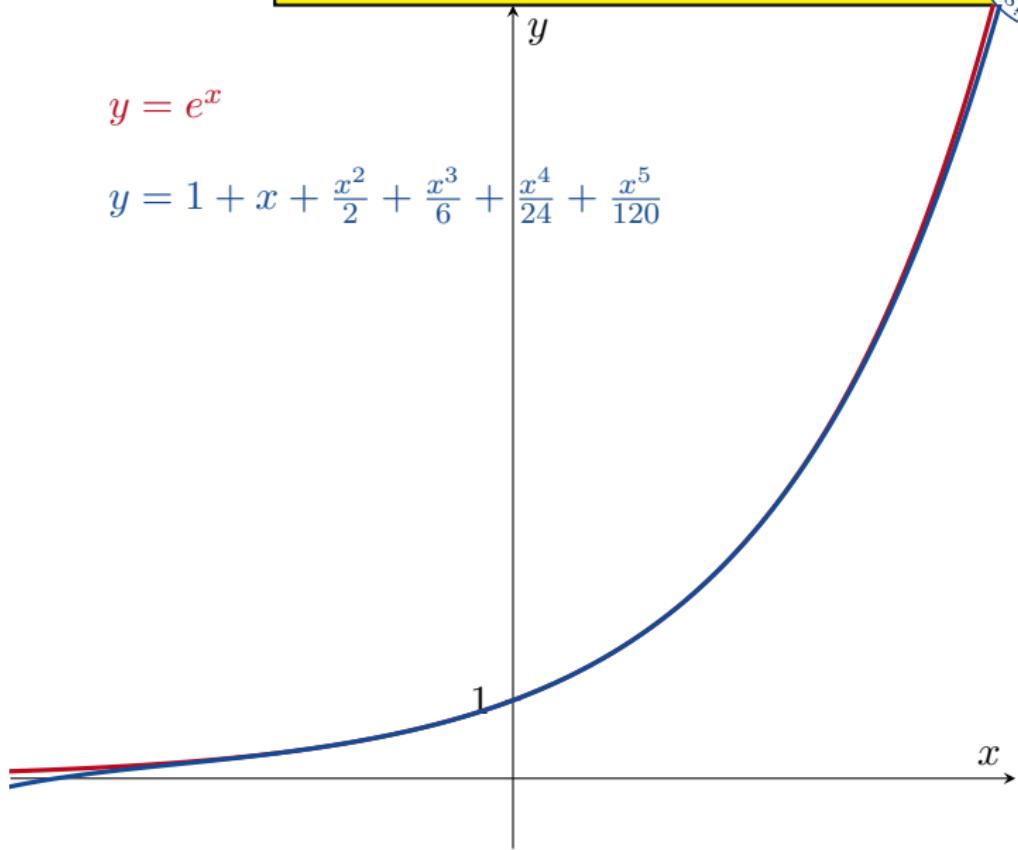
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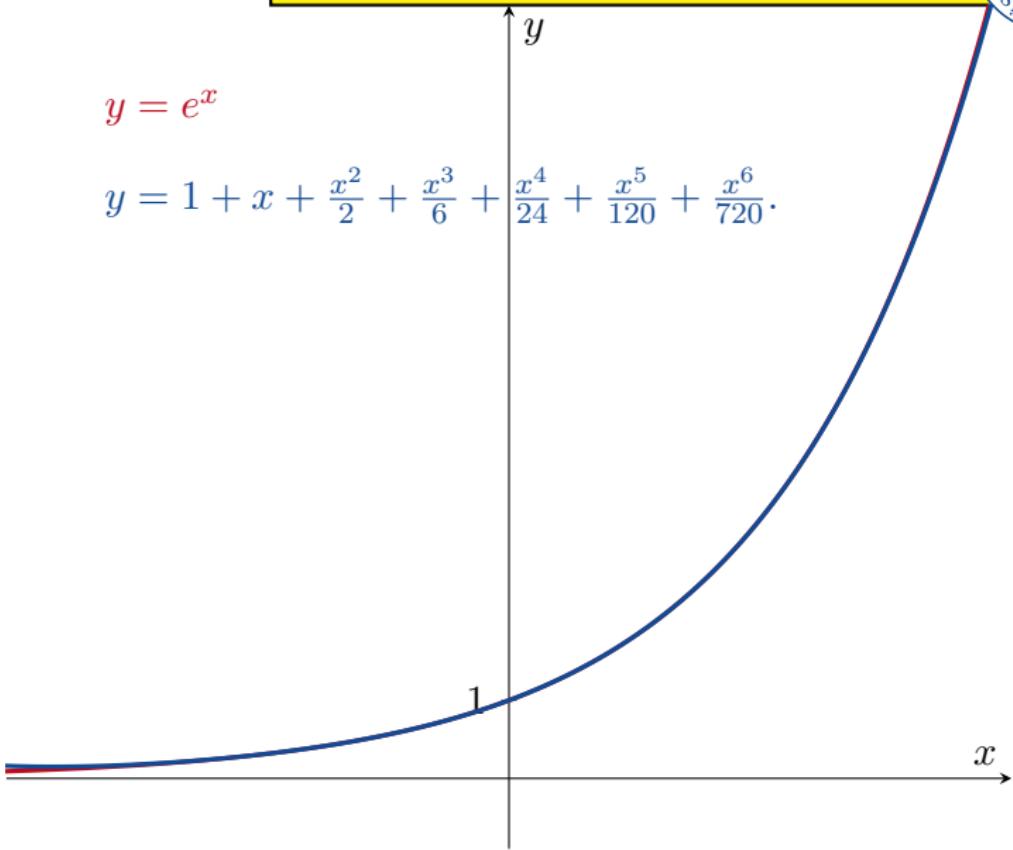
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$$y = e^x$$

$$y = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \frac{x^5}{120} + \frac{x^6}{720}.$$



$$a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \frac{f^{(4)}(a)}{4!}(x-a)^4 + \dots$$

Example

Find the Taylor Series for $\sin x$ centred at 0.

Let $f(x) = \sin x$. Then $\frac{d^k f}{dx^k}$ exists and is continuous $\forall x$ and $\forall k$.

Let $a = 0$ and $x \neq 0$.

$$a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \frac{f^{(4)}(a)}{4!}(x-a)^4 + \dots$$

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Let $a = 0$ and $x \neq 0$.

We need

- to find $\frac{d^k f}{dx^k}$ for all k ;
- to show that the **remainder term** tends to zero; and
- to calculate $\frac{d^k f}{dx^k}(0)$ for all k .

$$a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \frac{f^{(4)}(a)}{4!}(x-a)^4 + \dots$$

First note that

$$\frac{d^k}{dx^k} \sin x = \cos x \quad \text{or} \quad -\sin x \quad \text{or} \quad -\cos x \quad \text{or} \quad \sin x.$$

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Since $-1 \leq \sin c \leq 1$ and $-1 \leq \cos c \leq 1 \ \forall c$, we have that

$$0 \leq \left| \frac{f^{(n)}(c)}{n!} x^n \right| \leq \frac{|x|^n}{n!} \rightarrow 0$$

as $n \rightarrow \infty$.

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I leave it for you to check that

$$f^{(k)}(0) = \begin{cases} 1 & \text{if } k = 1, 5, 9, 13, \dots \\ 0 & \text{if } k = 0, 2, 4, 6, 8, \dots \\ -1 & \text{if } k = 3, 7, 11, 15, \dots \end{cases}$$

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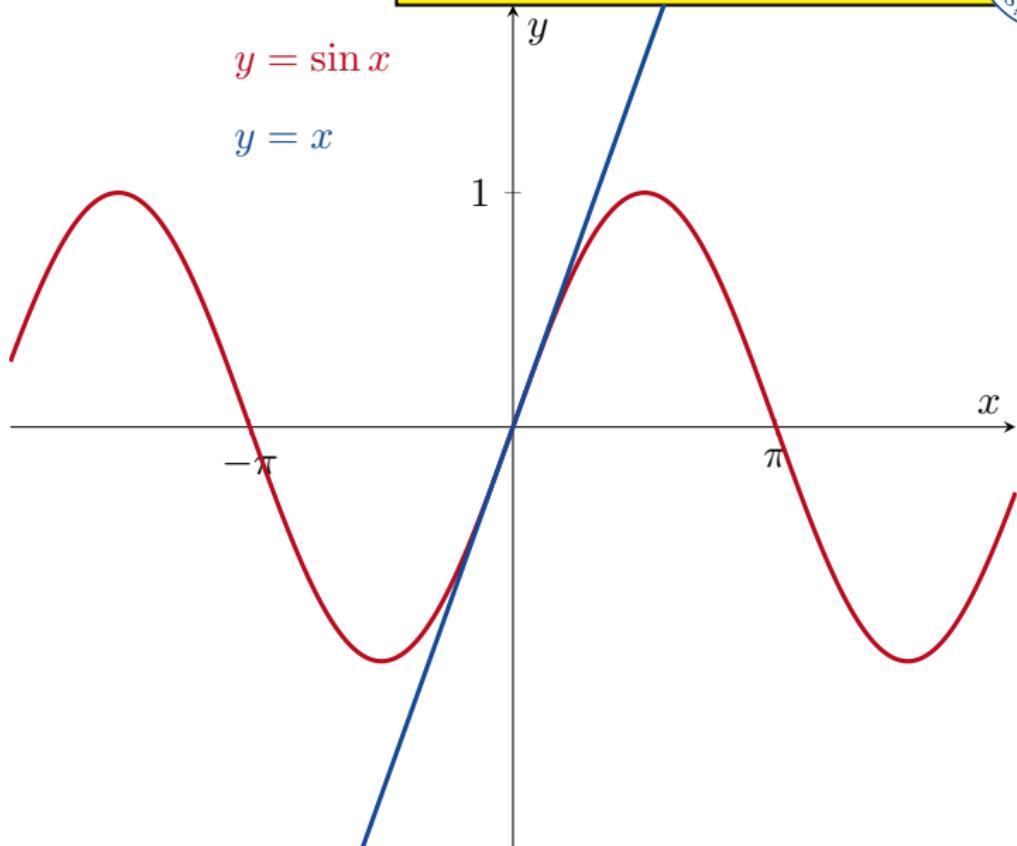
Therefore

$$\begin{aligned} \sin x &= f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \frac{f^{(4)}(0)}{4!}x^4 + \frac{f^{(5)}(0)}{5!}x^5 + \dots \\ &= 0 + x + 0 - \frac{x^3}{3!} + 0 + \frac{x^5}{5!} + \dots \\ &= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots \end{aligned}$$

is the Taylor Series of $\sin x$ with centre 0.

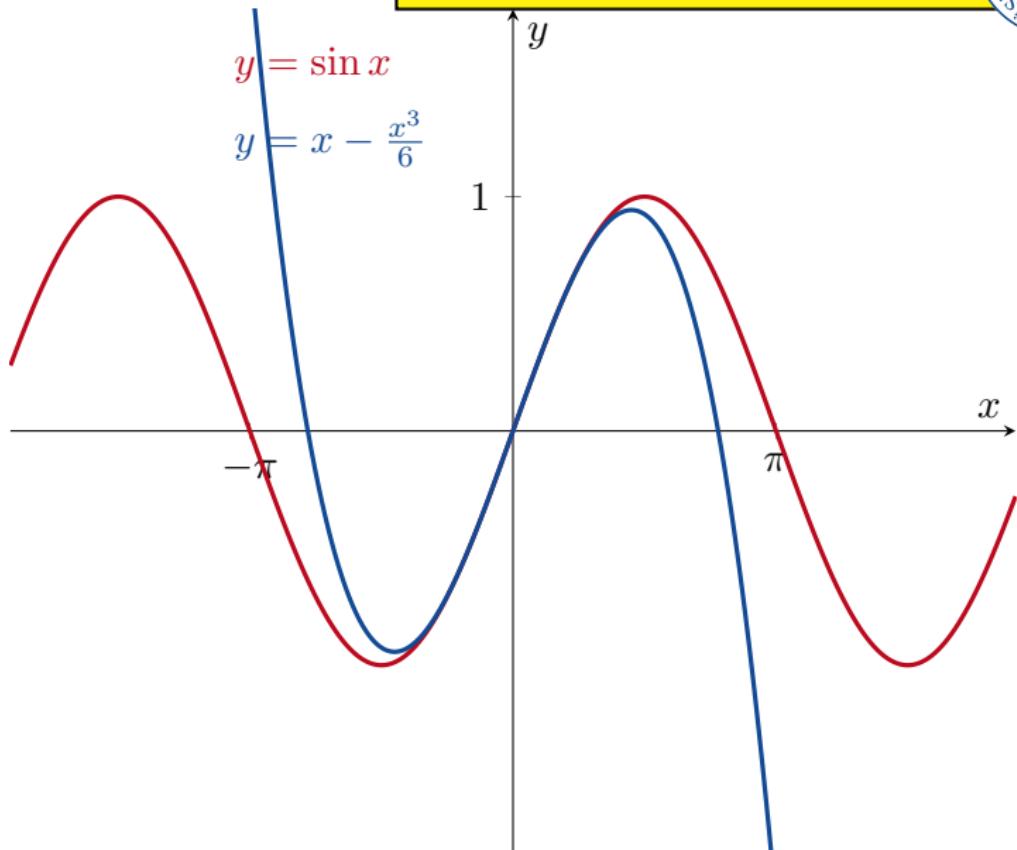
9.8 Taylor and Maclaurin

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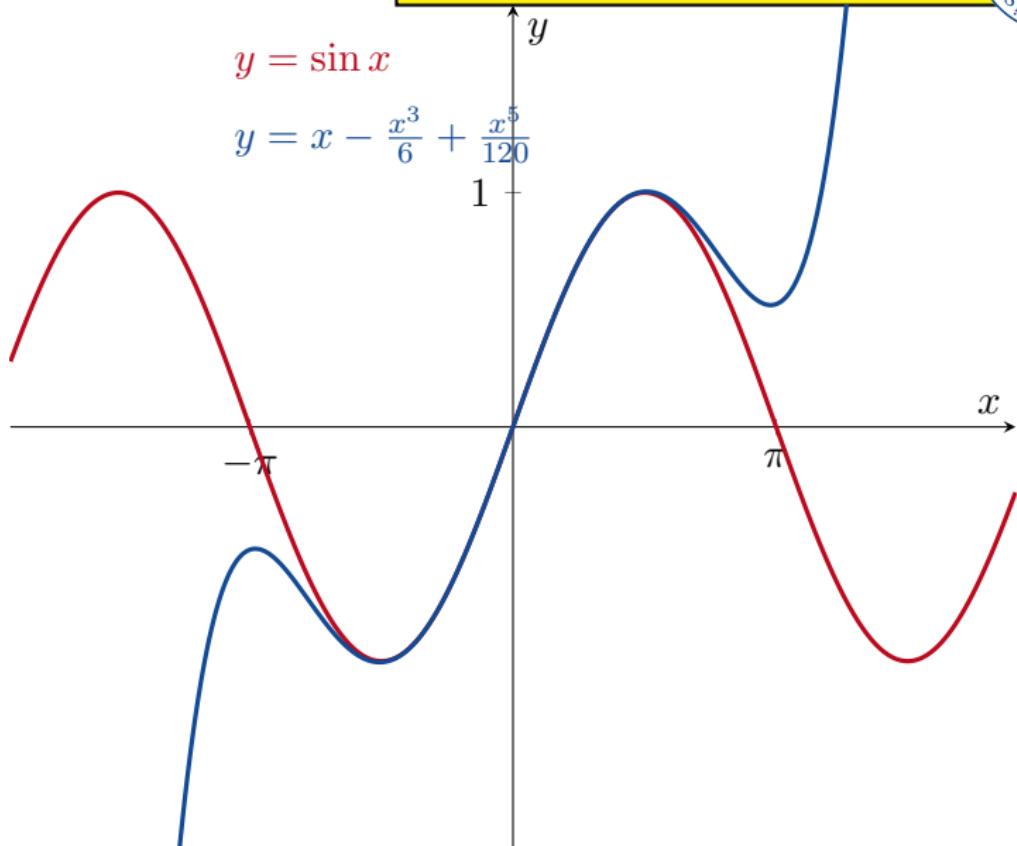
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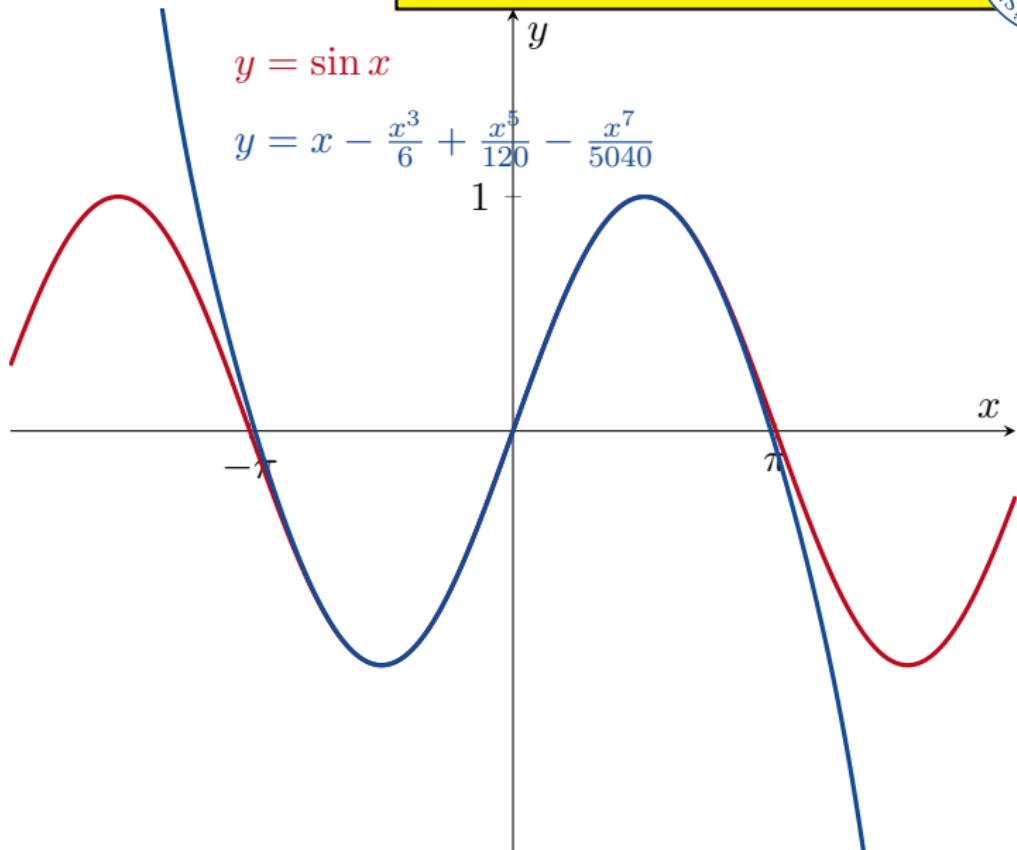
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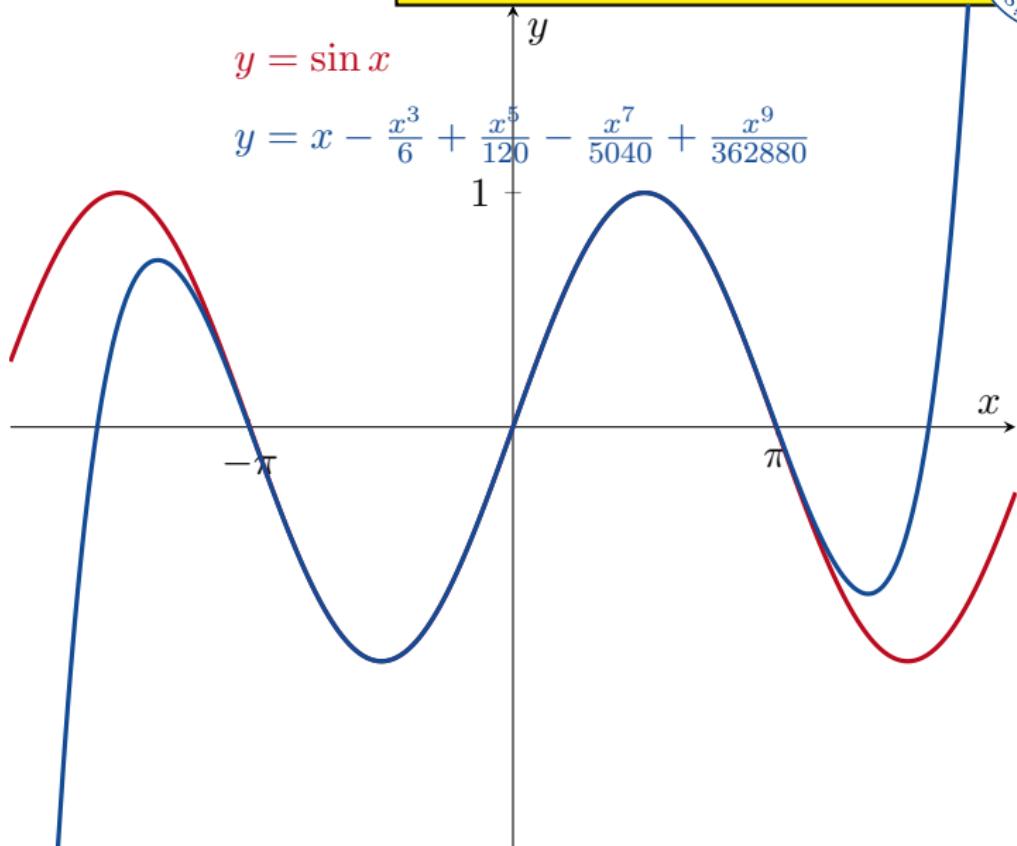
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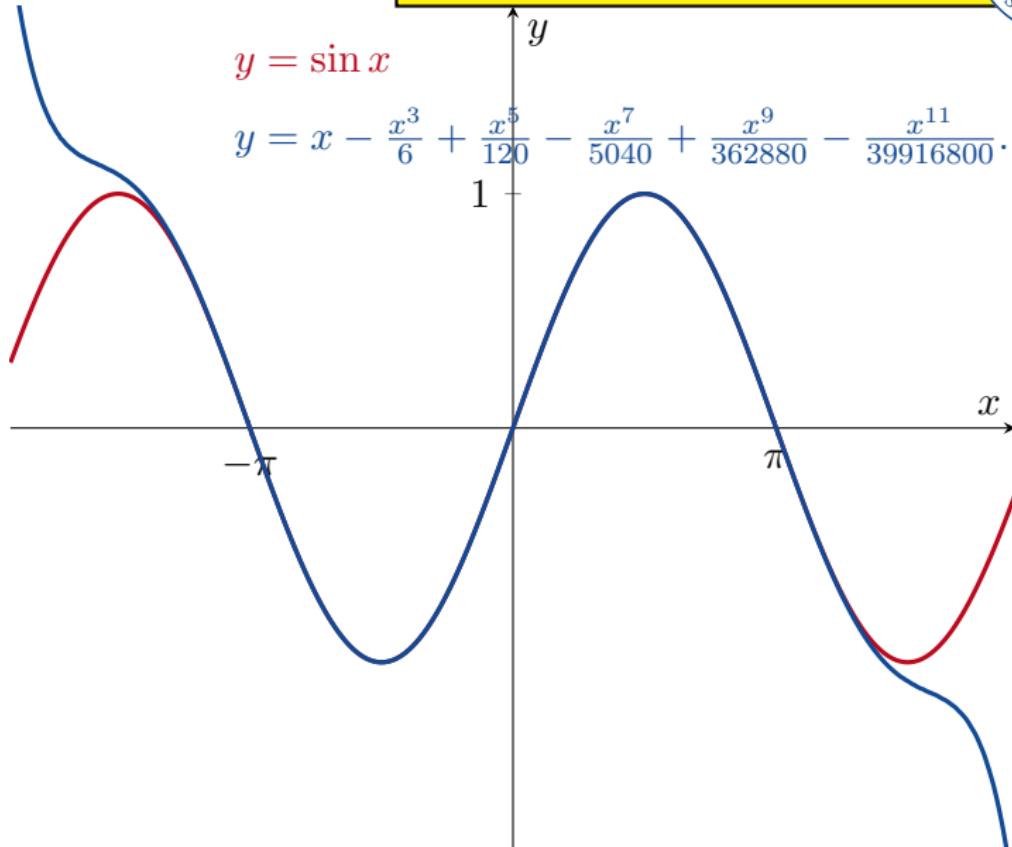
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Example

Find the Taylor Series for $\ln(1+x)$ centred at 0.

Let $f : (-1, \infty) \rightarrow \mathbb{R}$ be defined by $f(x) = \ln(1+x)$. Let $a = 0$ and let $0 < x \leq 1$. Then f and its derivatives exist and are continuous on the closed interval $[0, x]$.

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I leave it for you to check that

$$f(0) = \ln 1 = 0$$

$$f^{(k)}(x) = \frac{(-1)^{k-1}(k-1)!}{(1+x)^k}$$

$$f^{(k)}(0) = (-1)^{k-1}(k-1)!$$

for $k \in \mathbb{N}$.

$$a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \frac{f^{(4)}(a)}{4!}(x-a)^4 + \dots$$

Since $0 < c < x \leq 1$, it follows that

$$|R_n(c)| = \left| \frac{f^{(n)}(c)x^n}{n!} \right| = \left| \frac{(n-1)!x^n}{(1+c)^n n!} \right| = \frac{x^n}{(1+c)^n n!} \leq \frac{x^n}{(1+c)^n n} \leq \frac{x^n}{n} \rightarrow 0$$

as $n \rightarrow \infty$.

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Therefore, if $0 < x \leq 1$, then

$$\begin{aligned} \ln(1+x) &= f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \frac{f^{(4)}(0)}{4!}x^4 + \frac{f^{(5)}(0)}{5!}x^5 + \dots \\ &= x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{n} \end{aligned}$$

is the Taylor Series of $\ln(1+x)$ centred at 0, on the interval $[0, 1]$.

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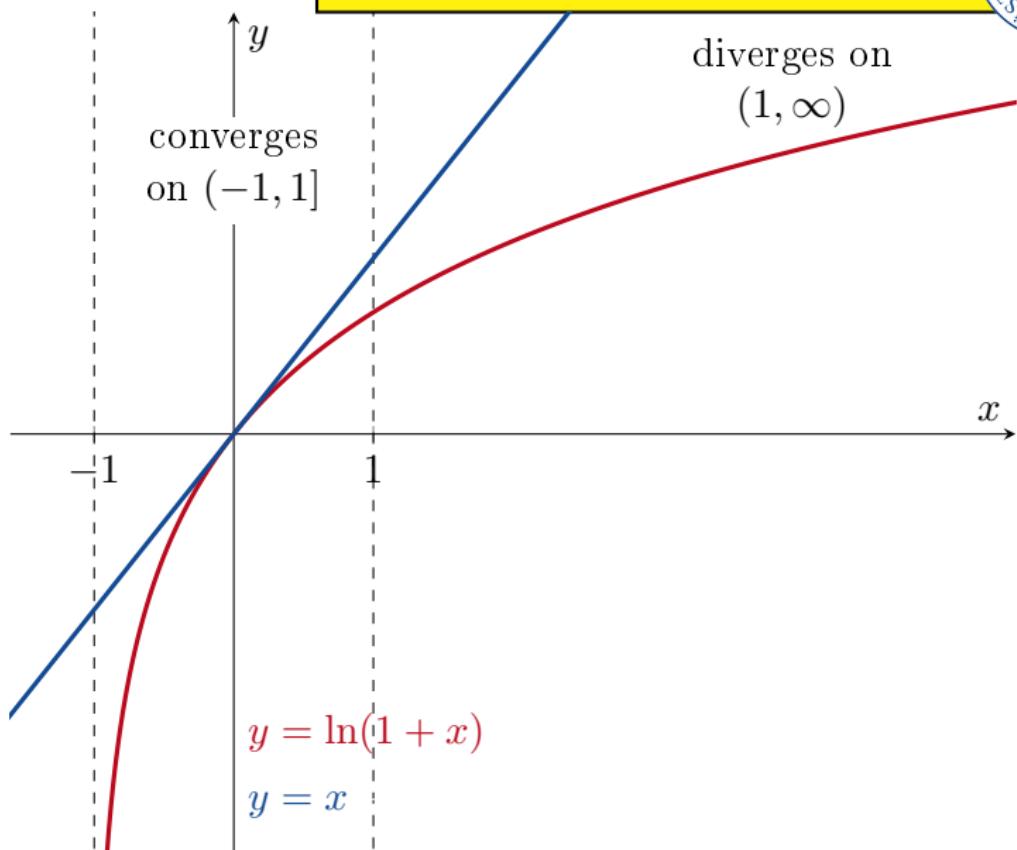
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If can be proved (more difficult) that this series also converges to $\ln(1+x) \forall x \in (-1, 0)$. If $x > 1$, then the series diverges.

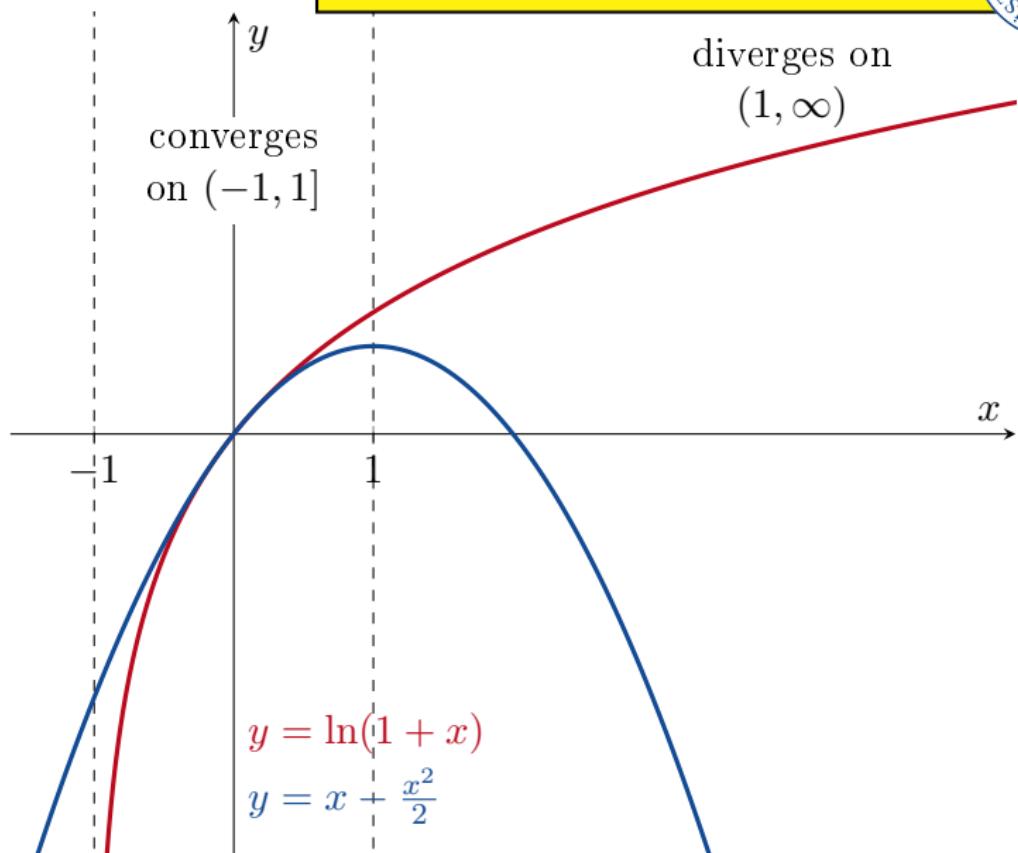
9.8 Taylor and Maclaurin Series

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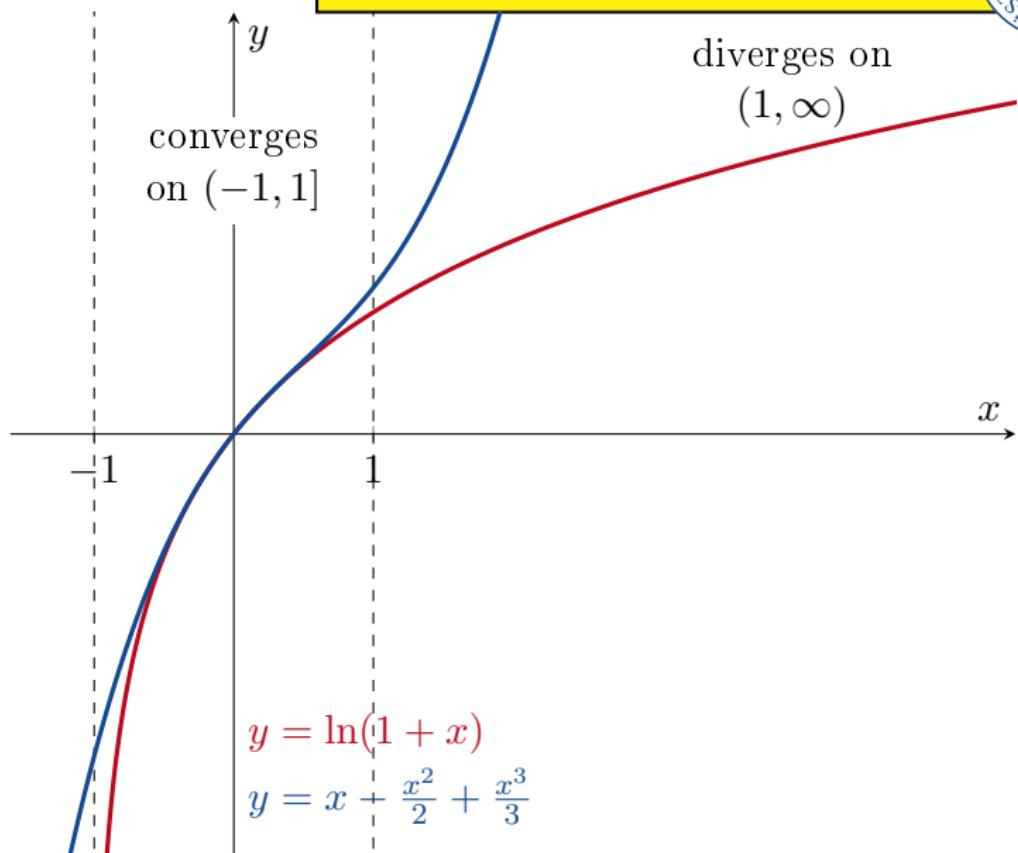
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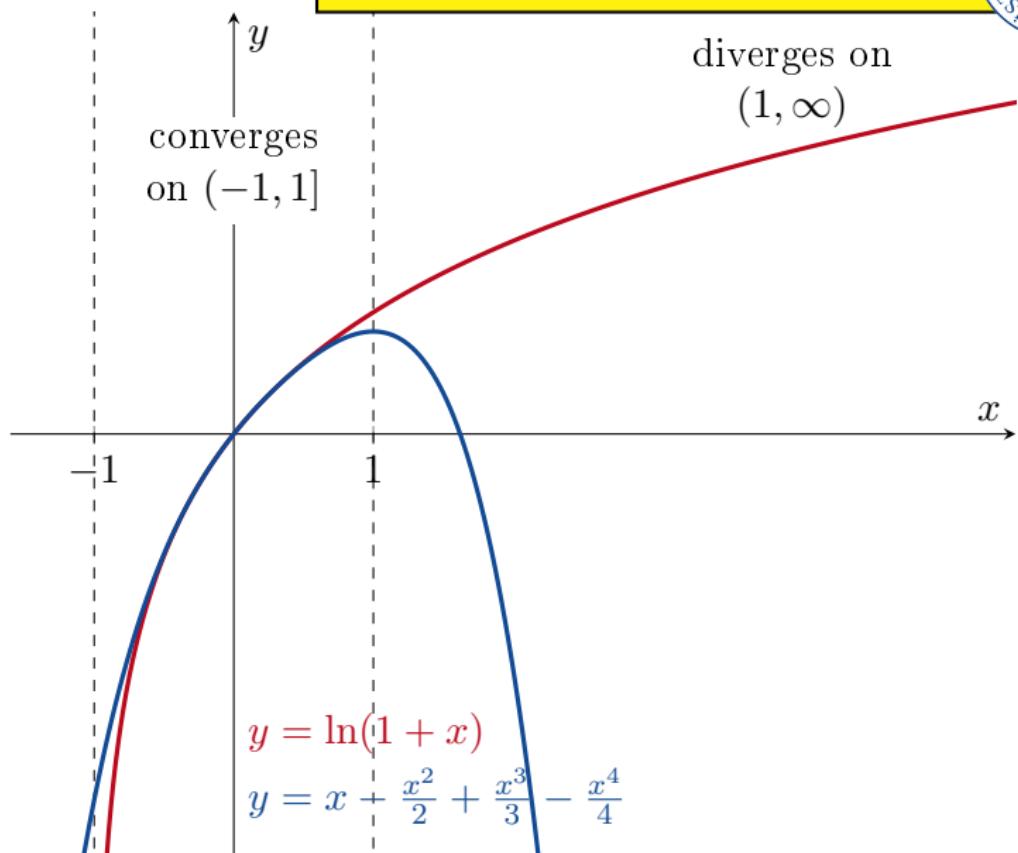
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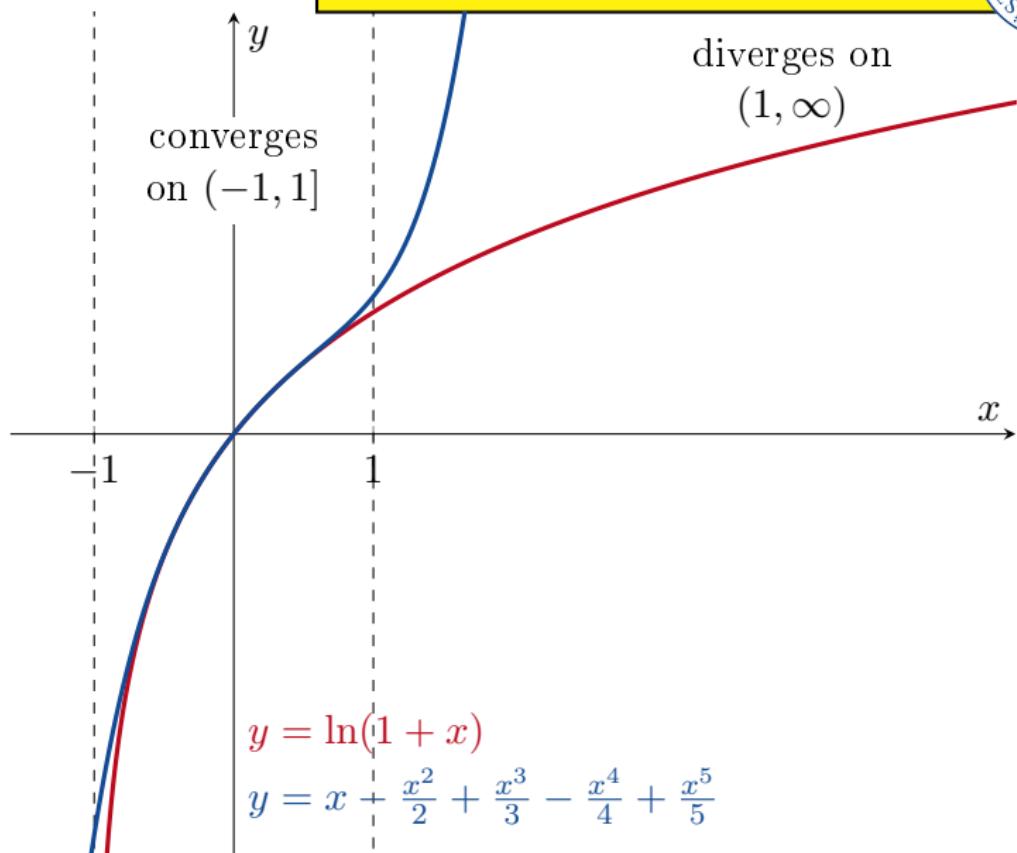
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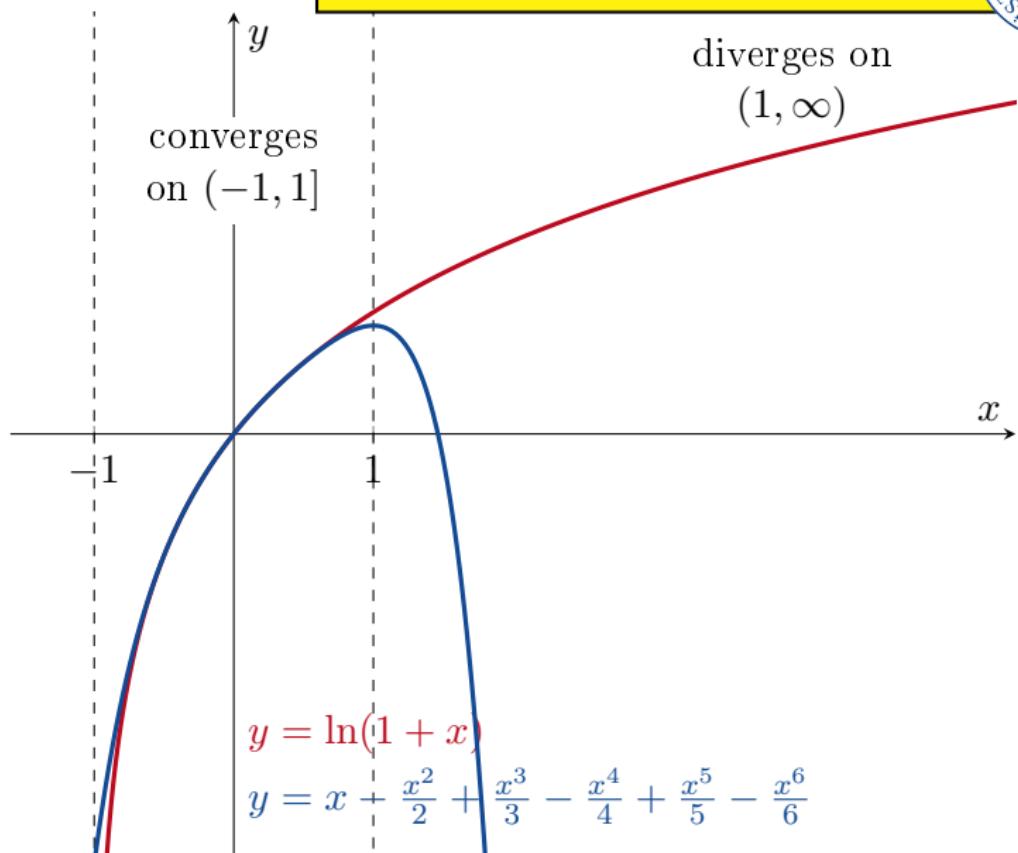
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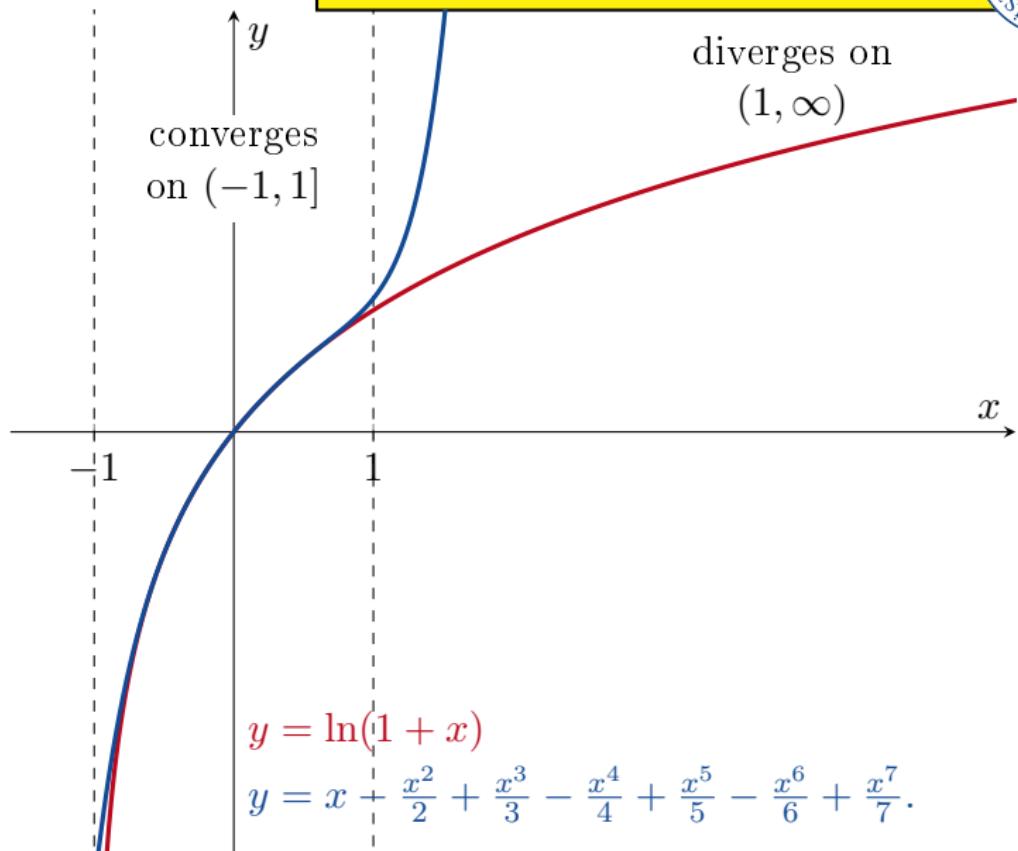
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9.8 Taylor and Maclaurin Series

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots$$



Example

Let $y = x + 1$. Then

$$\begin{aligned}\ln y &= (y - 1) - \frac{1}{2}(y - 1)^2 + \frac{1}{3}(y - 1)^3 - \frac{1}{4}(y - 1)^4 + \frac{1}{5}(y - 1)^5 \\ &\quad - \frac{1}{6}(y - 1)^6 + \dots\end{aligned}$$

is the Taylor Series of $\ln y$ with centre $a = 1$. It converges for all $y \in (0, 2]$.

9.8 Taylor and Maclaurin Series



Colin Maclaurin

BORN

February 1698

DECEASED

14 June 1746

NATIONALITY

British

Definition

A Taylor Series with centre 0 is also called a *Maclaurin Series*.

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n$$

$$f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \frac{f^{(4)}(0)}{4!}x^4 + \dots$$



Example

Calculate the Maclaurin Series for $f(x) = \sinh x$.

$$f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \frac{f^{(4)}(0)}{4!}x^4 + \dots$$



Example

Calculate the Maclaurin Series for $f(x) = \sinh x$.

This is the same as:

Example

Calculate the Taylor Series for $f(x) = \sinh x$ centred at 0.

$$f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \frac{f^{(4)}(0)}{4!}x^4 + \dots$$



Example

Calculate the Maclaurin Series for $f(x) = \sinh x$.

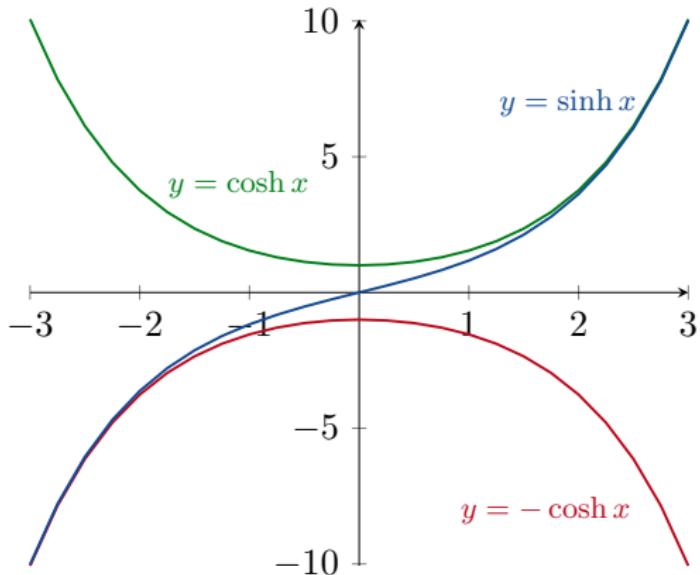
Since $\frac{d}{dx} \sinh x = \cosh x$ and $\frac{d}{dx} \cosh x = \sinh x$, we know that

$$f^{(n)}(x) = \sinh x \quad \text{or} \quad \cosh x$$

for all $n \in \mathbb{N}$.

9.8 Taylor and

$$f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \frac{f^{(4)}(0)}{4!}x^4 + \dots$$



Note that

$$-\cosh x \leq \sinh x \leq \cosh x$$

for all $x \in \mathbb{R}$.

9.8 Taylor and Maclaurin Series



$$f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \frac{f^{(4)}(0)}{4!}x^4 + \dots$$

Let $x \neq 0$ and let c be between 0 and x . (So $0 < c < x$ or $x < c < 0$.) Then

$$\left|f^{(n)}(c)\right| < \left|f^{(n)}(x)\right| \leq \cosh x.$$

So

$$0 \leq \left| \frac{f^{(n)}(c)x^n}{n!} \right| \leq \cosh x \frac{|x|^n}{n!} \rightarrow 0$$

as $n \rightarrow \infty$. By the Sandwich Rule, it follows that

$$R_c(x) = \frac{f^{(n)}(c)x^n}{n!} \rightarrow 0$$

as $n \rightarrow \infty$.

9.8 Taylor and Maclaurin Series



$$f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \frac{f^{(4)}(0)}{4!}x^4 + \dots$$

Now, since

$$f^{(n)}(x) = \begin{cases} \sinh x & \text{if } n = 0, 2, 4, 6, 8, \dots \\ \cosh x & \text{if } n = 1, 3, 5, 7, 9, \dots \end{cases}$$

we have that

$$f^{(n)}(0) = \begin{cases} 0 & \text{if } n = 0, 2, 4, 6, 8, \dots \\ 1 & \text{if } n = 1, 3, 5, 7, 9, \dots \end{cases}$$

9.8 Taylor and Maclaurin Series



$$f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \frac{f^{(4)}(0)}{4!}x^4 + \dots$$

Now, since

$$f^{(n)}(x) = \begin{cases} \sinh x & \text{if } n = 0, 2, 4, 6, 8, \dots \\ \cosh x & \text{if } n = 1, 3, 5, 7, 9, \dots \end{cases}$$

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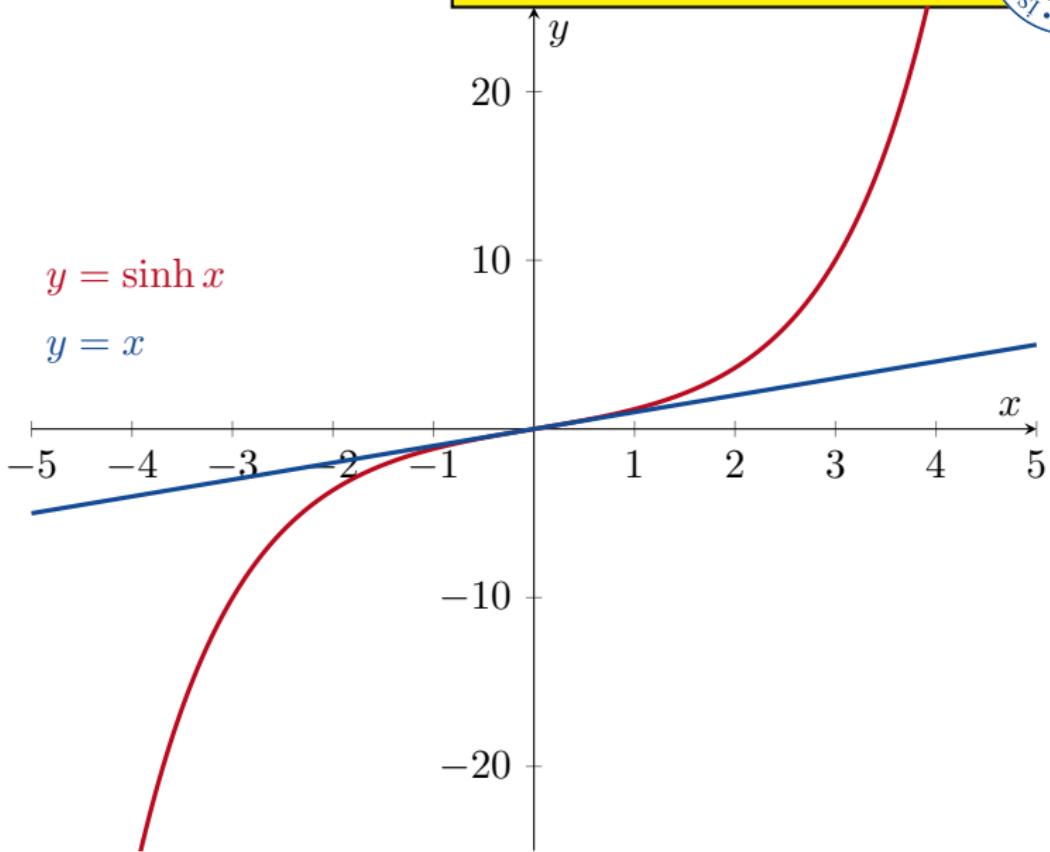
$$f^{(n)}(0) = \begin{cases} 0 & \text{if } n = 0, 2, 4, 6, 8, \dots \\ 1 & \text{if } n = 1, 3, 5, 7, 9, \dots \end{cases}$$

Therefore

$$\begin{aligned} \sinh x &= f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \frac{f^{(4)}(0)}{4!}x^4 \dots \\ &= 0 + 1x + \frac{0}{2!}x^2 + \frac{1}{3!}x^3 + \frac{0}{4!}x^4 \dots \\ &= x + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} + \frac{x^9}{9!} + \dots = \sum_{k=0}^{\infty} \frac{x^{2k+1}}{(2k+1)!}. \end{aligned}$$

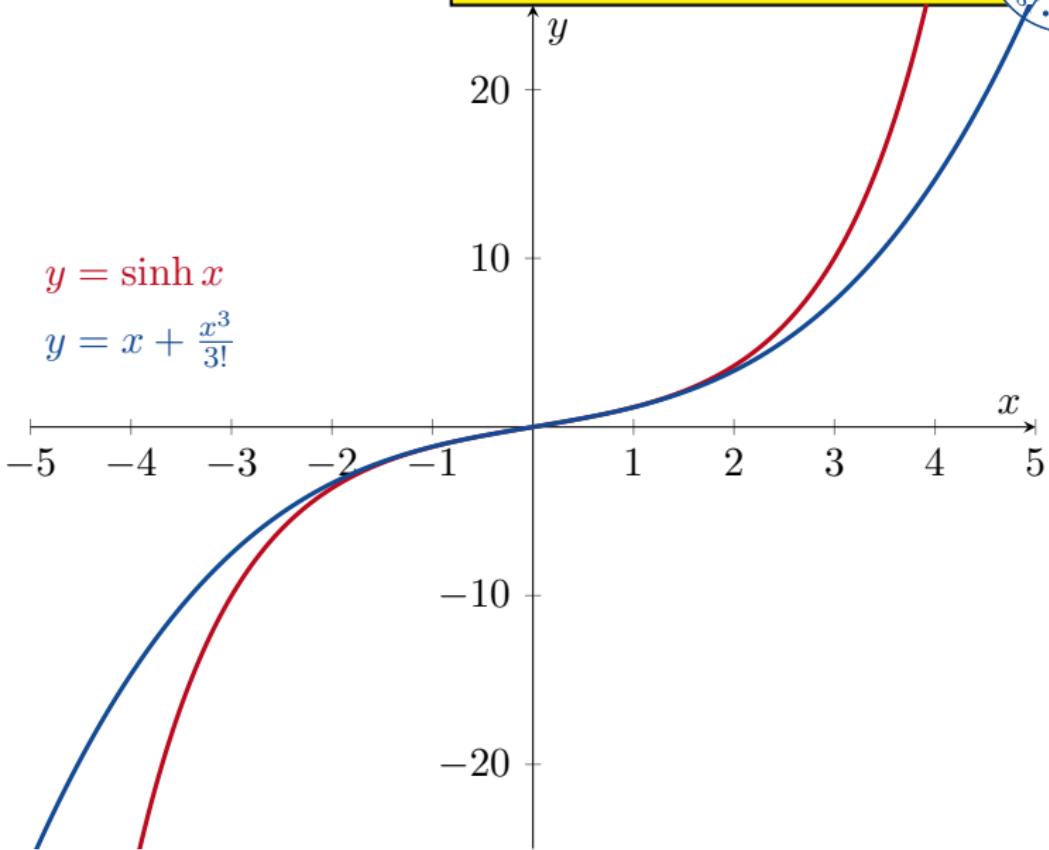
9.8 Taylor and Maclaurin S

$$x + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} + \frac{x^9}{9!} + \dots$$



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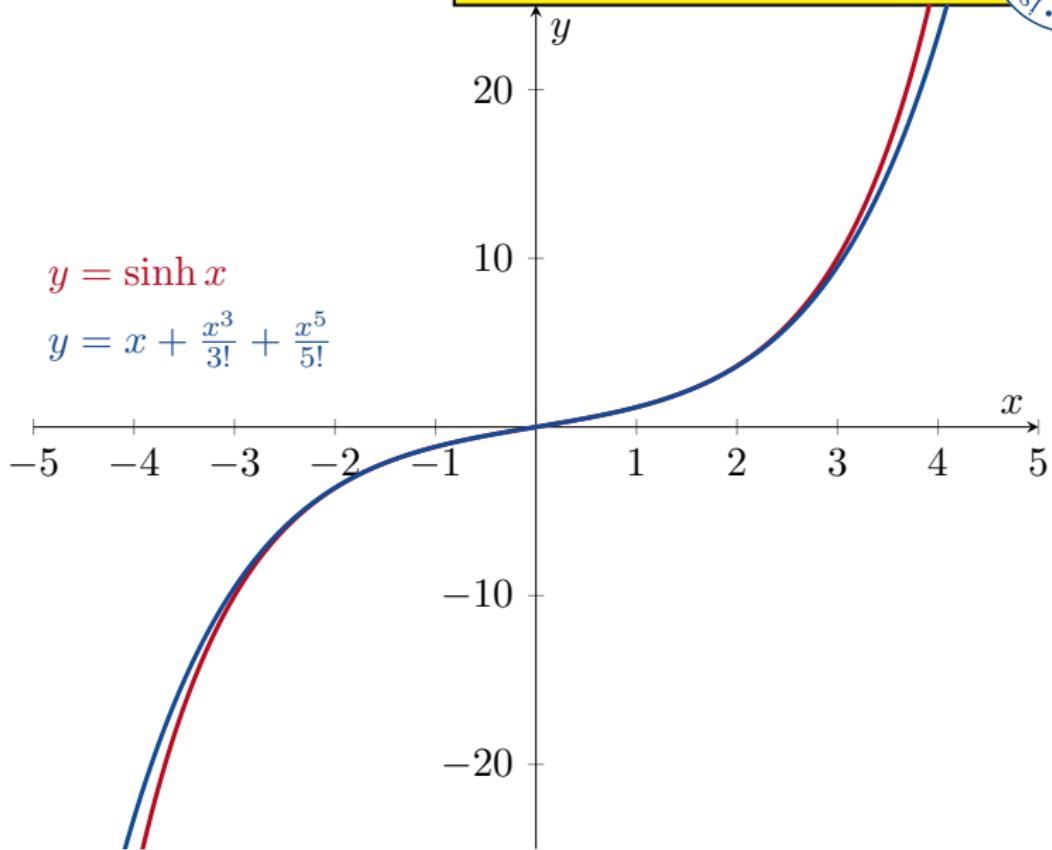


$$y = \sinh x$$

$$y = x + \frac{x^3}{3!}$$

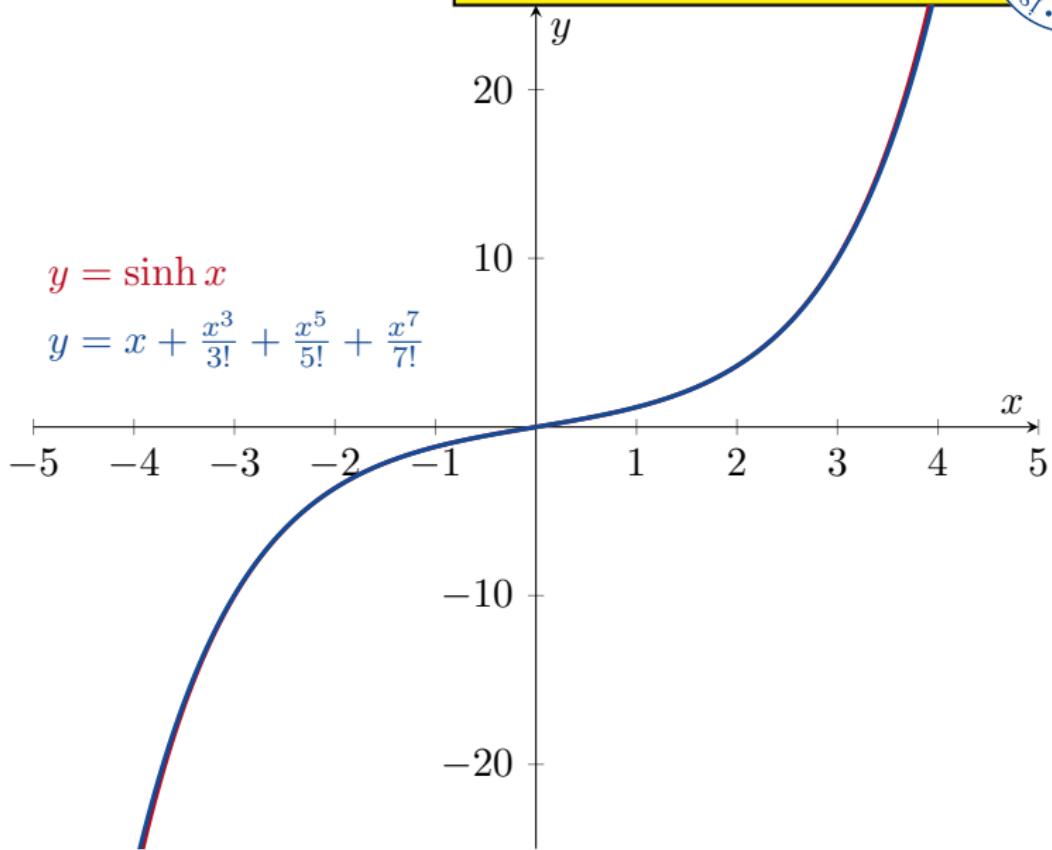
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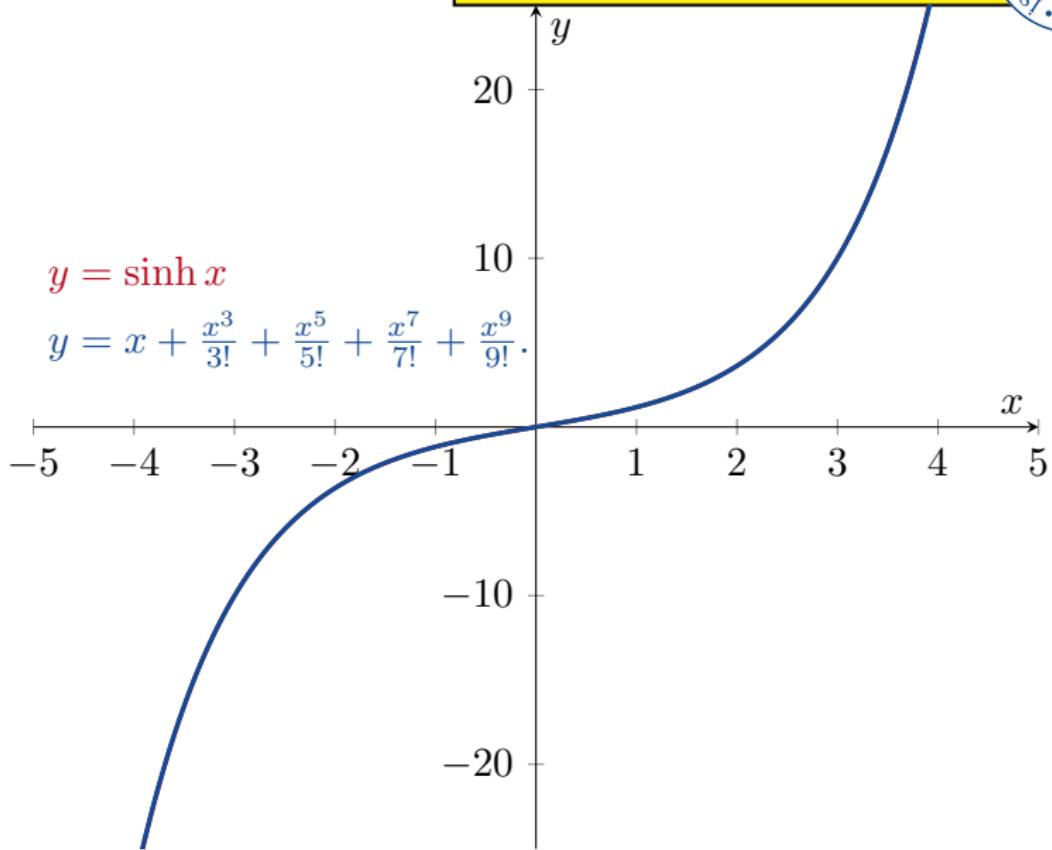
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9.8 Taylor and Maclaurin Series



Example

Calculate the Taylor Series for $f(x) = \frac{1}{x}$ with centre $a = 2$. For which $x \in \mathbb{R}$ does the series converge?

9.8 Taylor and Maclaurin Series

Example

Calculate the Taylor Series for $f(x) = \frac{1}{x}$ with centre $a = 2$. For which $x \in \mathbb{R}$ does the series converge?

Since

$$f(x) = x^{-1}$$

$$f(2) = \frac{1}{2}$$

$$f'(x) = -x^{-2}$$

$$f'(2) = -\frac{1}{4}$$

$$f''(x) = 2x^{-3}$$

$$\frac{f''(2)}{2!} = \frac{1}{8}$$

$$f'''(x) = -6x^{-4}$$

$$\frac{f'''(2)}{3!} = -\frac{1}{16}$$

⋮

⋮

$$f^{(n)}(x) = (-1)^n n! x^{-n-1}$$

$$\frac{f^{(n)}(2)}{n!} = \frac{(-1)^n}{2^{n+1}}$$

⋮

⋮

9.8 Taylor and Maclaurin Series



the Taylor Series is

$$\begin{aligned}\frac{1}{x} &= f(2) + f'(2)(x - 2) + \frac{f''(2)}{2!}(x - 2)^2 + \frac{f'''(2)}{3!}(x - 2)^3 + \frac{f^{(4)}(2)}{4!}(x - 2)^4 + \dots \\&= \frac{1}{2} - \frac{x - 2}{4} + \frac{(x - 2)^2}{8} - \frac{(x - 2)^3}{16} + \frac{(x - 2)^4}{32} - \dots \\&= \frac{1}{2} (1 + r + r^2 + r^3 + r^4 + \dots)\end{aligned}$$

where $r = -\frac{x-2}{2}$.

9.8 Taylor and Maclaurin Series



the Taylor Series is

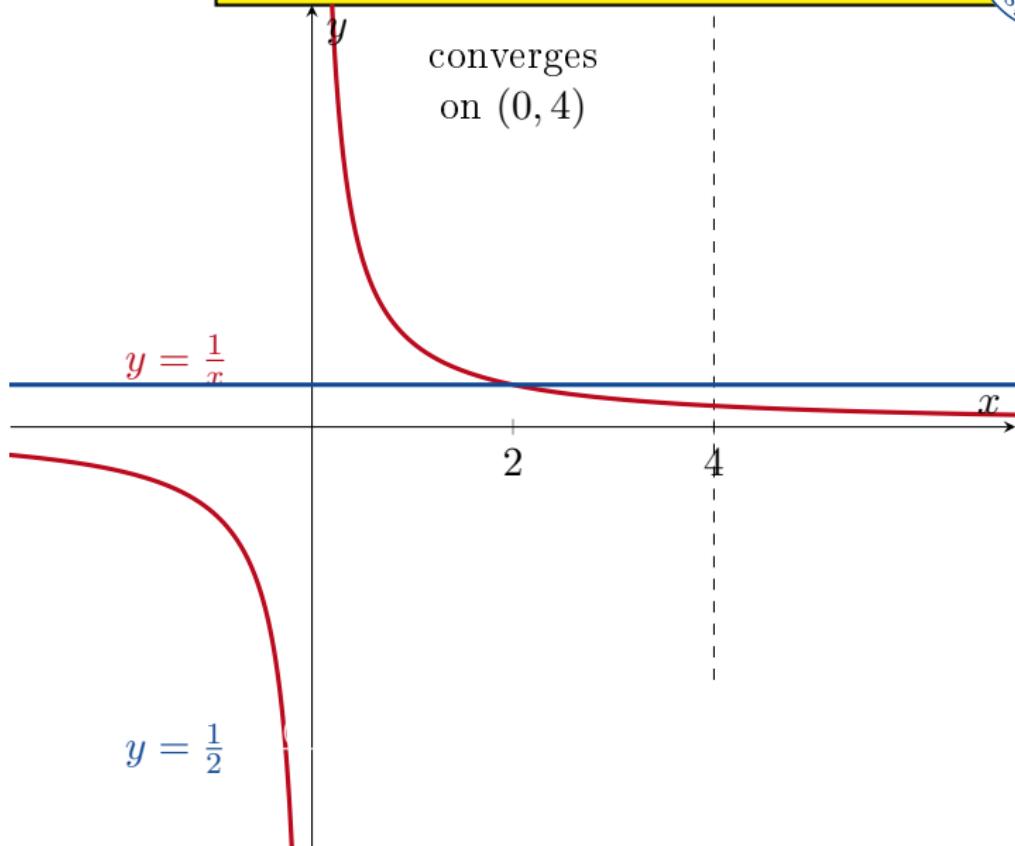
$$\begin{aligned}\frac{1}{x} &= f(2) + f'(2)(x - 2) + \frac{f''(2)}{2!}(x - 2)^2 + \frac{f'''(2)}{3!}(x - 2)^3 + \frac{f^{(4)}(2)}{4!}(x - 2)^4 + \dots \\&= \frac{1}{2} - \frac{x - 2}{4} + \frac{(x - 2)^2}{8} - \frac{(x - 2)^3}{16} + \frac{(x - 2)^4}{32} - \dots \\&= \frac{1}{2} (1 + r + r^2 + r^3 + r^4 + \dots)\end{aligned}$$

where $r = -\frac{x-2}{2}$.

This series converges absolutely for $|r| < 1$ and diverges for $|r| \geq 1$. Therefore, the Taylor Series converges for $0 < x < 4$.

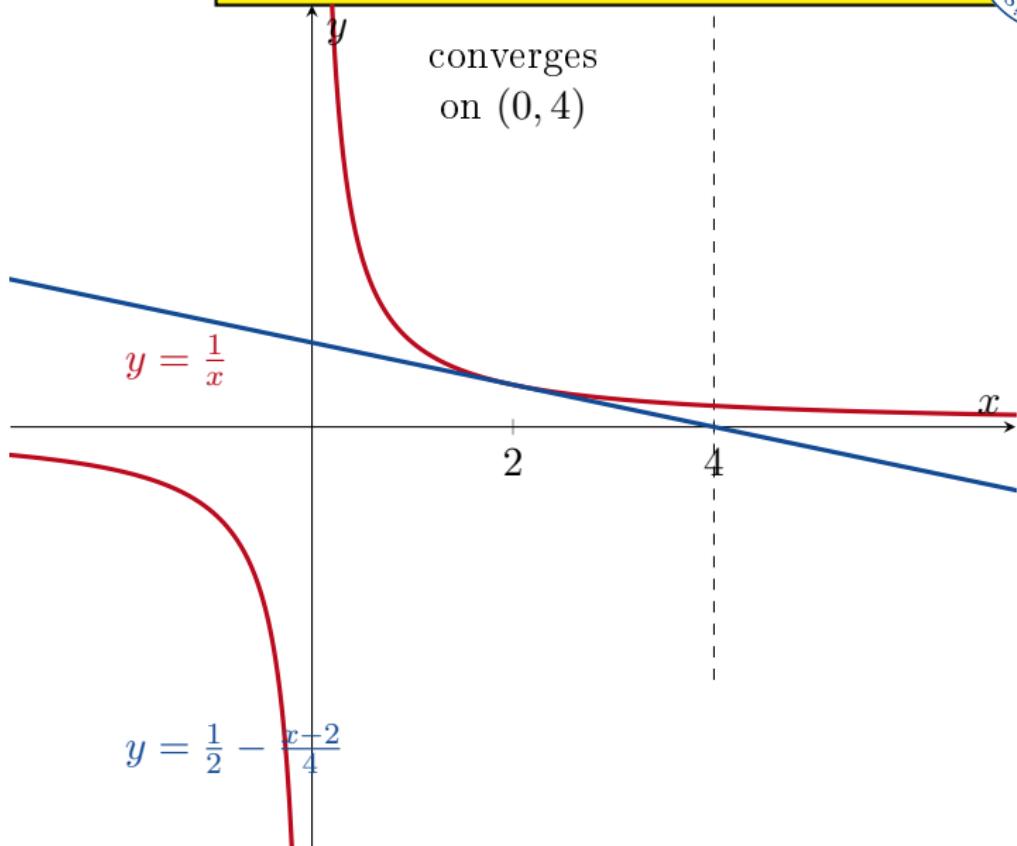
9.8 Taylor and M

$$\frac{1}{2} - \frac{x-2}{4} + \frac{(x-2)^2}{8} - \frac{(x-2)^3}{16} + \frac{(x-2)^4}{32} - \dots$$



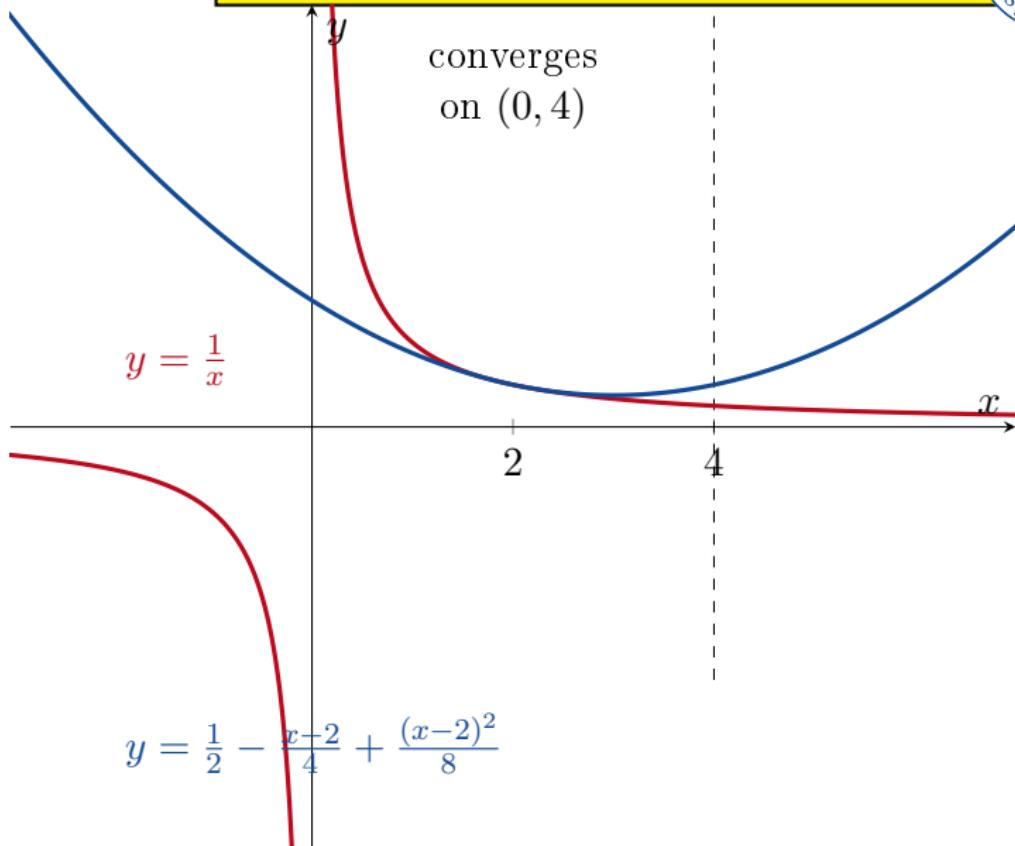
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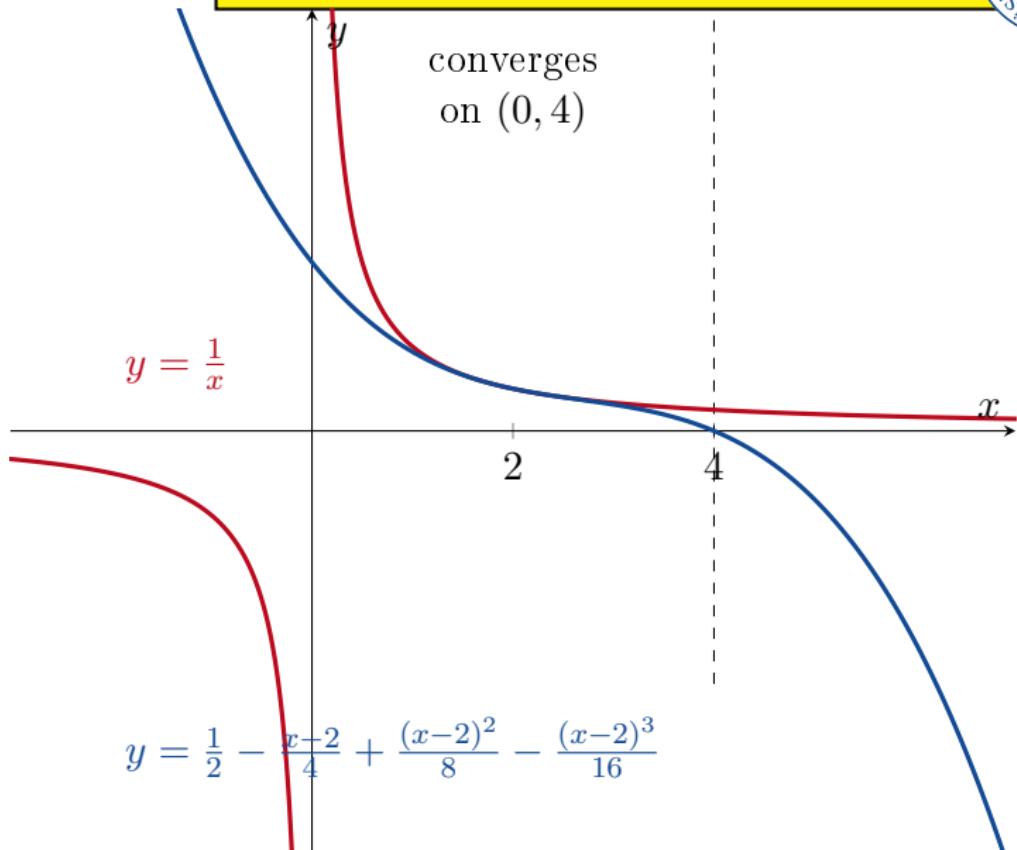
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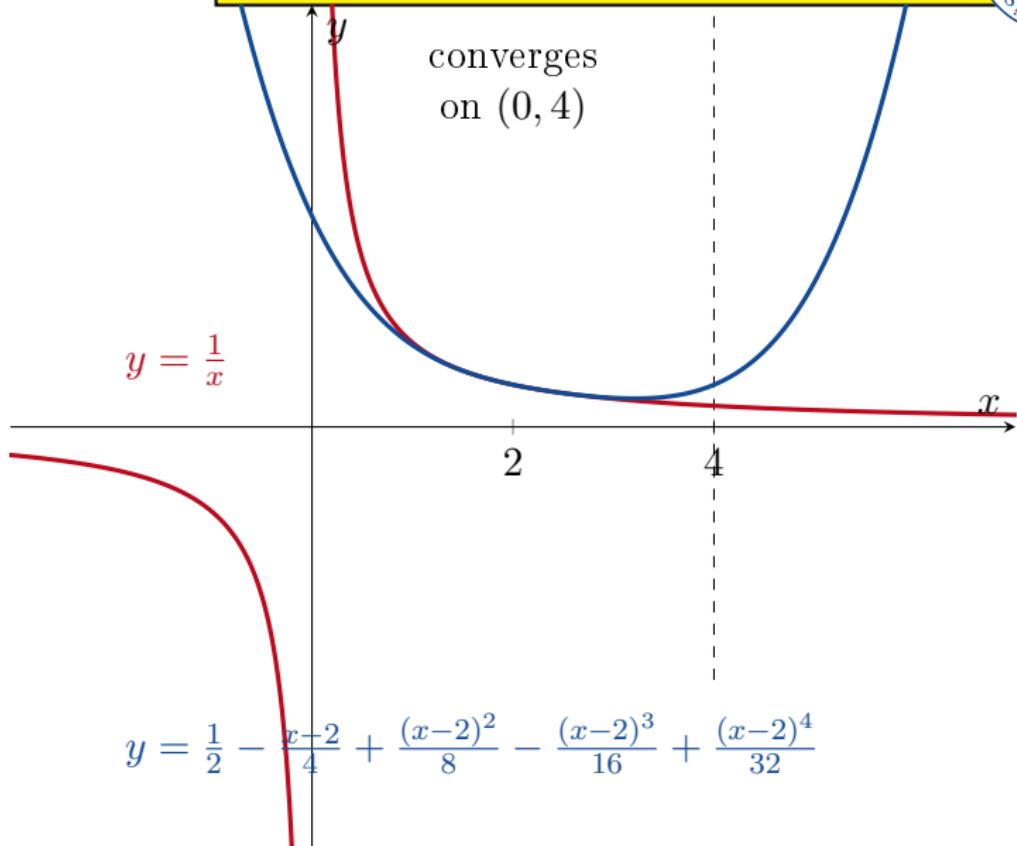
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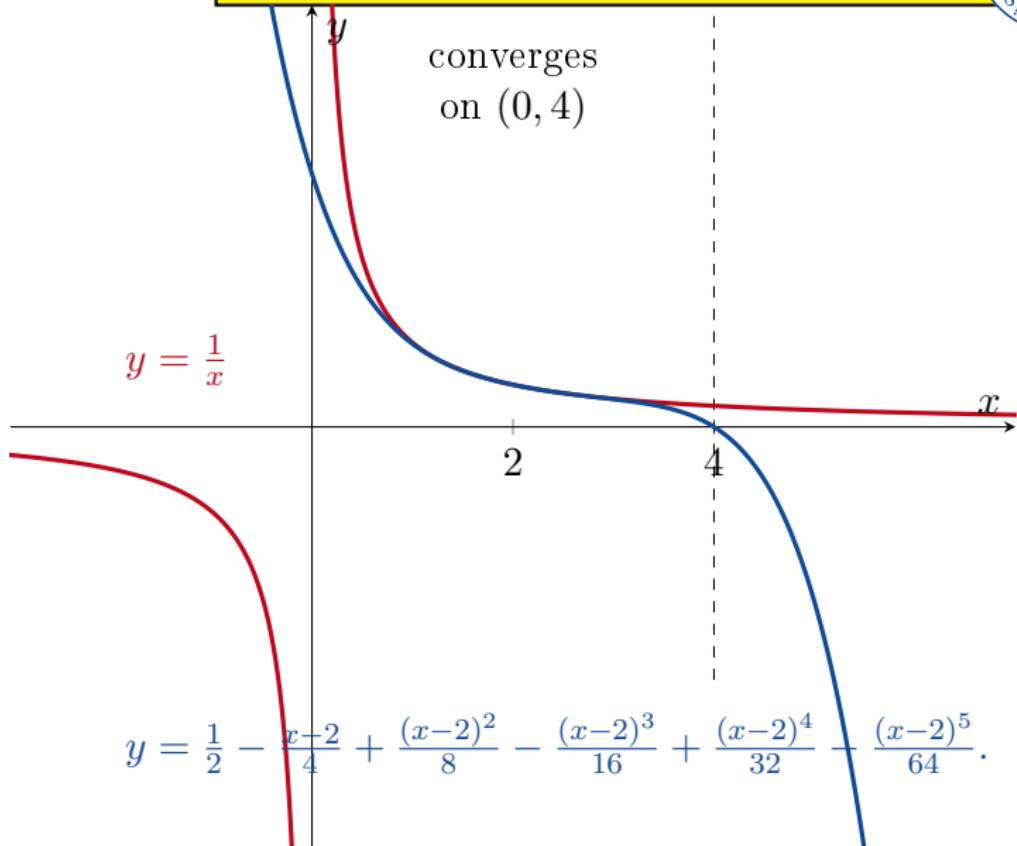
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$$\frac{1}{2} - \frac{x-2}{4} + \frac{(x-2)^2}{8} - \frac{(x-2)^3}{16} + \frac{(x-2)^4}{32} - \dots$$





The End

