Welcome to

Mathematics IV (Differential Equations)

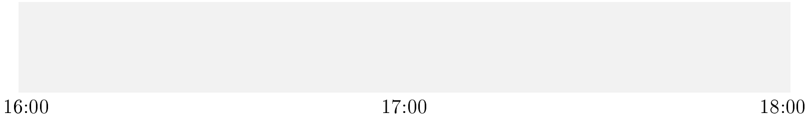
with Dr Neil Course

Lecture 1

- Information about this course
- 1.1 Introduction
- 1.2 Some Examples
- 1.3 How to Draw a Direction Field
- 1.4 Solving Our First Differential Equations

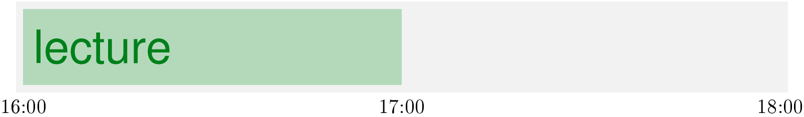
Information about this course

- ≈ 12 classes. Thursday afternoons 4pm-6pm.



Information about this course

- $\approx$ 12 classes. Thursday afternoons 4pm-6pm.
- Each lecture $\approx$ 60 minutes.



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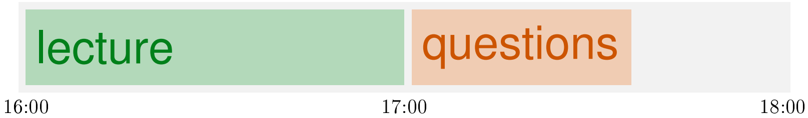

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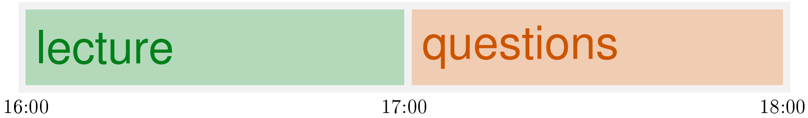

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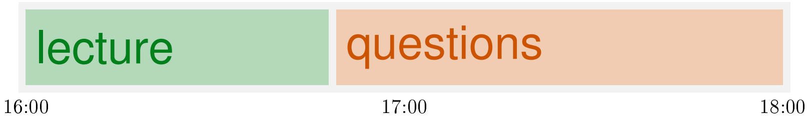

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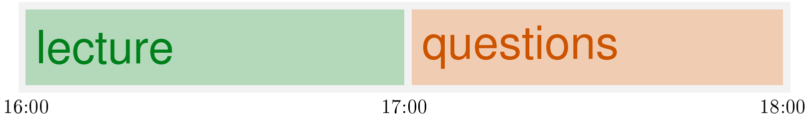

1

2

3

4

5

Introduction

Examples; Directions Fields; Classification.

2

3

4

5

Introduction
Examples; Directions Fields; Classification.

First Order Differential Equations

Second and Higher Order Linear ODEs

4

5

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The Laplace Transform

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3 lectures

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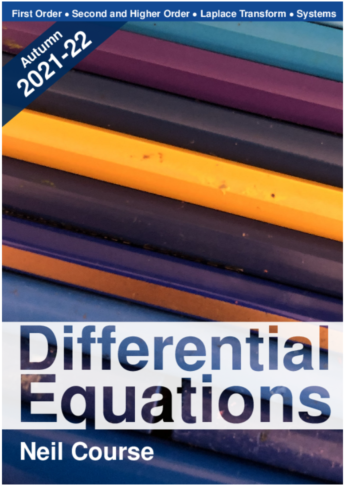
Lecture Notes



Exams

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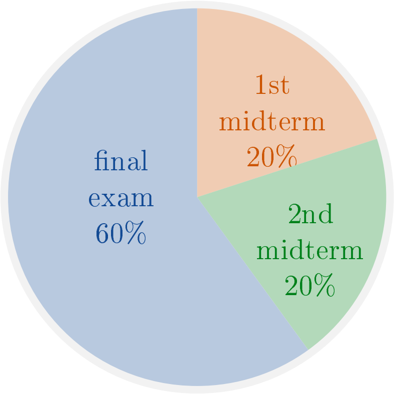

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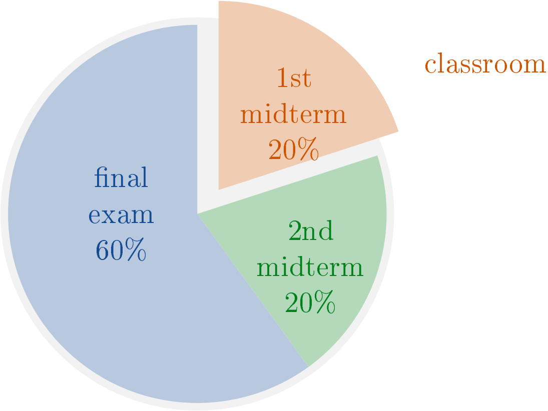

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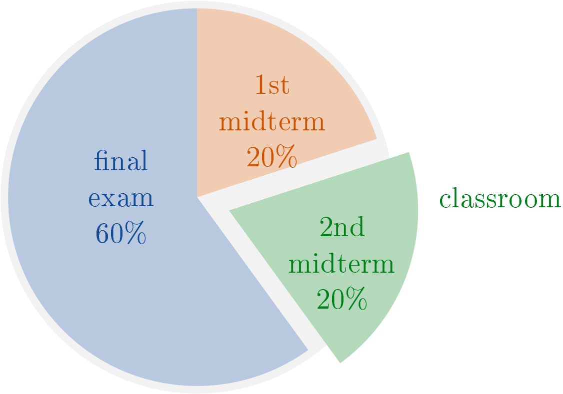

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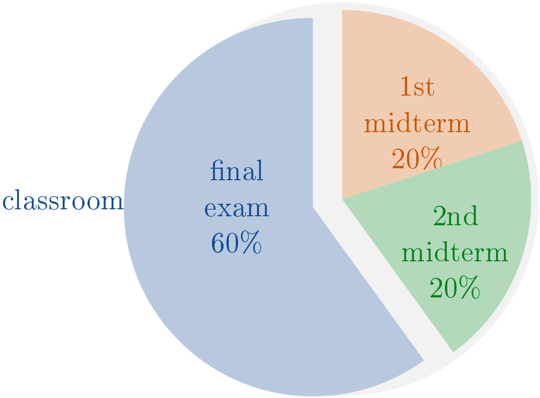

Expectations

If this was a classroom course, the expectation is that for each hour of lectures, you would study an extra 1-2 hours outside of class.

classroom course	lectures (4 hours)	other study (4-8 hours)

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For an online course, you are still expected to study a total of 8-12 hours each week.

online course	class (2 hours)	other study (6-10 hours)

This may include:

- Rewatch the recorded lectures (O'Learn & YouTube);

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- Watch online videos;

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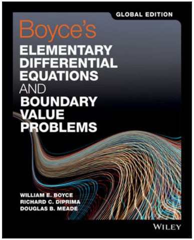
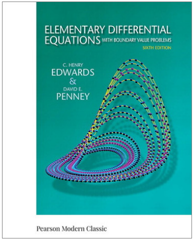
Two good books



William E. Boyce, Richard C. DiPrima and Douglas B. Meade,
Boyce's Elementary Differential Equations and Boundary Value Problems,
Wiley.



C. Henry Edwards and David E. Penney,
Elementary Differential Equations with Boundary Value Problems,
Pearson.

slide number

current section

1.1 Introduction

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A *differential equation* is an equation containing a derivative. For example, the equation $\frac{dy}{dx} = 2x$ contains a derivative – so it is a differential equation.

Example

Solve $\dfrac{dy}{dx} = 2x$.

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This differential equation is easy to solve:

$$y(x) = \int \frac{dy}{dx}\,dx = \int 2x\,dx = x^2 + c.$$

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Therefore the solution to the IVP is

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This is harder: This time we can't just integrate $\frac{dy}{dx}$ to find $y(x)$. I will show you how to solve this later.

1.2

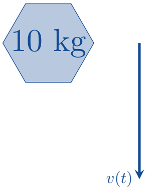
Some Examples

1.2 Some Examples

Many problems in engineering, science and the social sciences can be modelled using differential equations. We start with 3 examples.

Example (A Falling Object)

Suppose that an object of mass 10 kg is falling.



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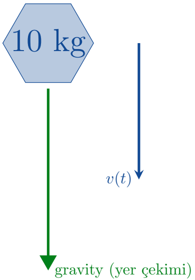

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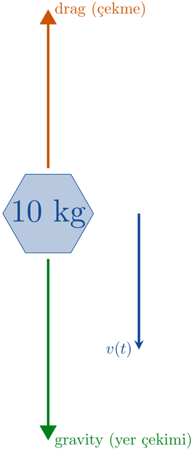

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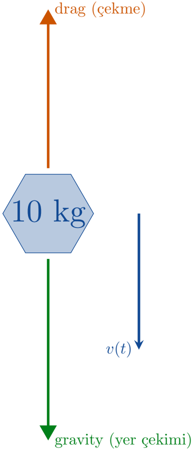

Let

- $v(t)$ denote the velocity (downwards) of the object in ms^{-1}; and
- t denote time in seconds.

1.2 Some Examples

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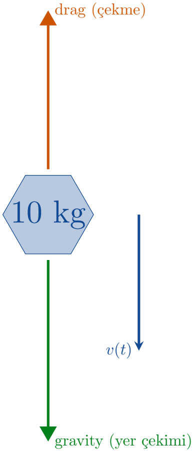
Note that $\frac{dv}{dt}$ is measured in $\frac{\mathrm{ms}^{-1}}{\mathrm{s}} = \mathrm{ms}^{-2}$.



Now

$$\text{force} = \text{gravity} - \text{drag}.$$

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$$\text{gravity} = 10g$$

(where $g = 9.8 \text{ ms}^{-2}$).

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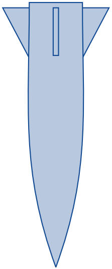
(where $g = 9.8\ \text{ms}^{-2}$). It is reasonable to assume (if the object isn't travelling too quickly) that

$$\begin{aligned} \text{drag} &\text{ is proportional to velocity} \\ \text{drag} &\propto v \\ \text{drag} &= \gamma v \end{aligned}$$

where $\gamma > 0$ is a constant depending on the shape of the object.



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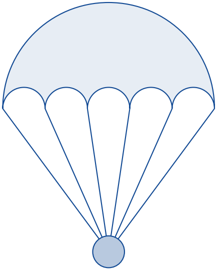

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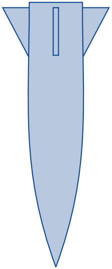
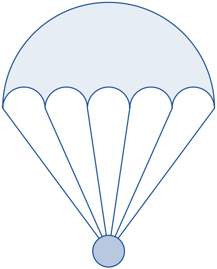


small $\gamma > 0$

$\gamma = 2\ \mathrm{kg\,s^{-1}}$



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If $\gamma = 2 \text{ kg s}^{-1}$, then we have that

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$$\boxed{\frac{dv}{dt} = 9.8 - \frac{v}{5}.} \tag{1}$$

We will solve equation (1) later. First we will look at this differential equation's direction field to try to understand it.

A direction field is a grid of arrows in the tv-plane which show the slope of solutions to a differential equation.

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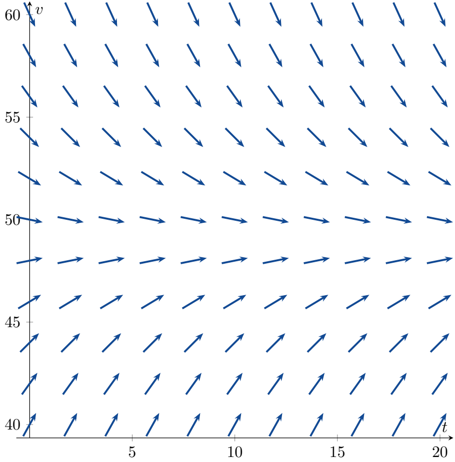
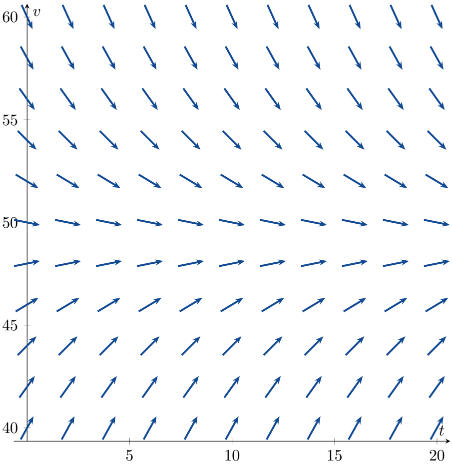


I will show you how to draw direction fields later. For now, I want to see what we can learn about the solutions to (1).

If we start at $v(0) = 54$ say, the arrows tell us that the solution is decreasing like this:



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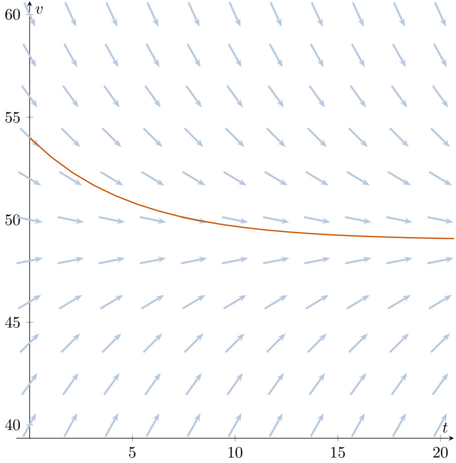
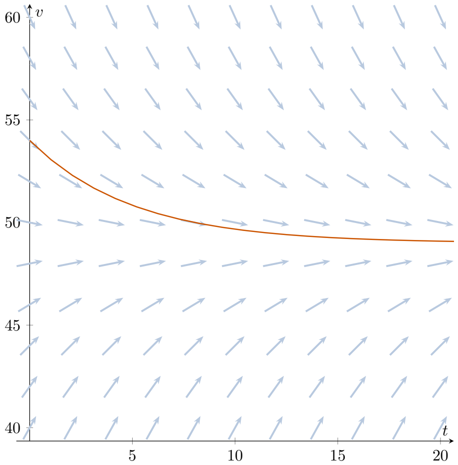


We can guess at some more solutions:



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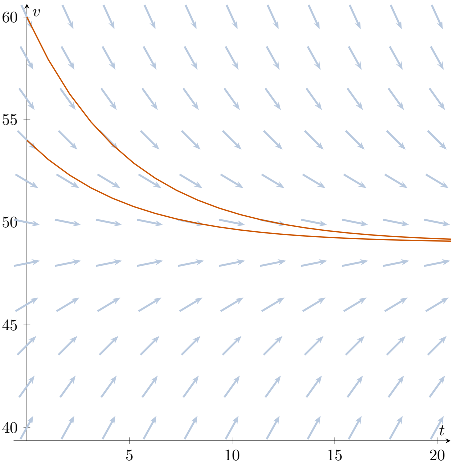

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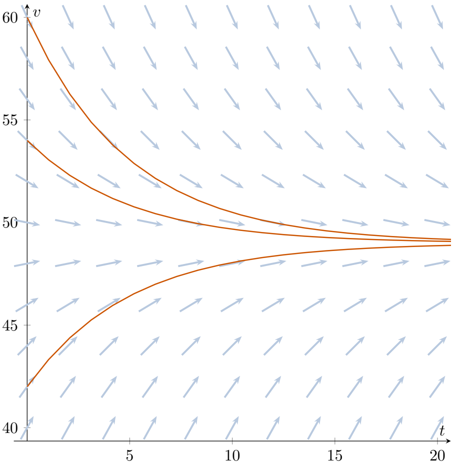

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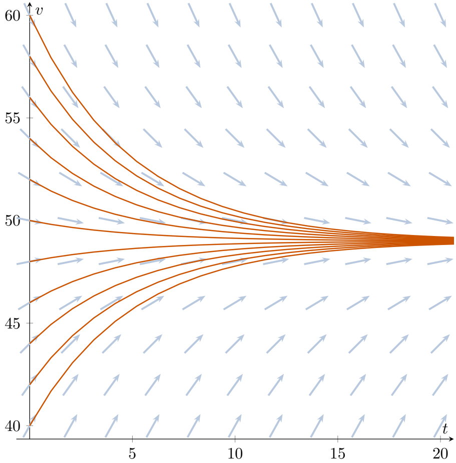

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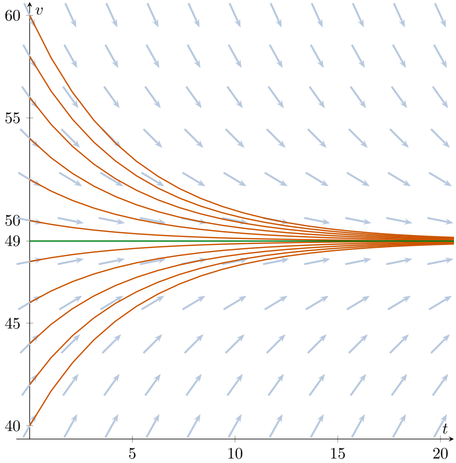

Note that if $v = 49$, then we have

$$\frac{dv}{dt} = 9.8 - \frac{49}{5} = 9.8 - 9.8 = 0.$$

Hence $v(t) = 49$ is a constant solution (or *equilibrium solution*) of (1).

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where $r > 0$ is a constant.
Suppose that $r = 0.5$ per month. Hence

$$\frac{dp}{dt} = \frac{p}{2}.$$

However, suppose that 5 owls also live in this area and suppose that each owl eats 3 mice each day.

$$\begin{aligned}1 \text{ owl eats } 3 \text{ mice per day}\\ 5 \text{ owls eat } 5 \times 3 = 15 \text{ mice per day}\\ 5 \text{ owls eat } 30 \times 15 = 450 \text{ mice per month.}\end{aligned}$$

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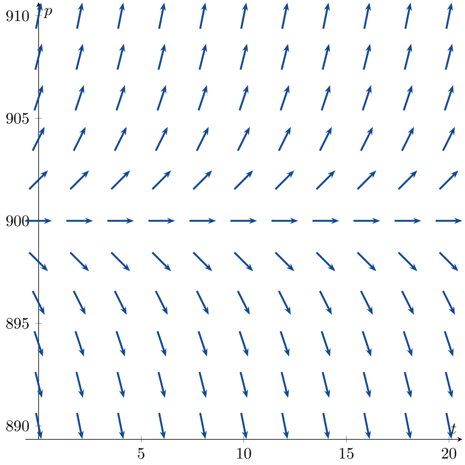
$$\begin{aligned} 1 \text{ owl eats } 3 \text{ mice per day} \\ 5 \text{ owls eat } 5 \times 3 = 15 \text{ mice per day} \\ 5 \text{ owls eat } 30 \times 15 = 450 \text{ mice per month.} \end{aligned}$$

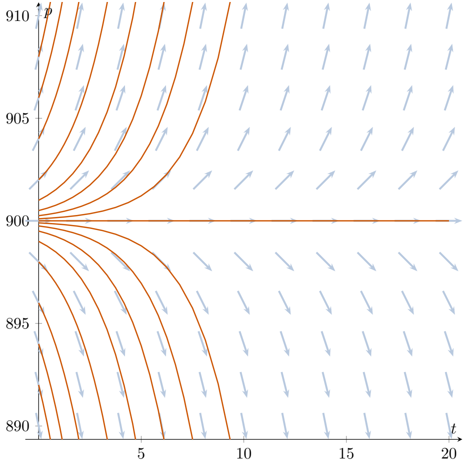
So we change our differential equation to

$$\boxed{\frac{dp}{dt} = \frac{p}{2} - 450.} \tag{2}$$

If we look at a direction field for (2),



If we look at a direction field for (2), then we can guess at some solutions:



Example (A cup of coffee)

Newton's law of cooling states that; the temperature of an object changes at a rate proportional to the difference between its temperature and that of its surroundings.

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Suppose that the temperature of your cup of coffee obeys Newton's law of cooling; suppose that it has a temperature of 90°C when freshly poured; and suppose that the temperature of your room is 20°C.

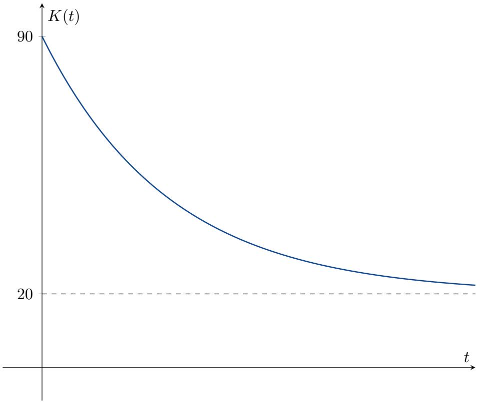
Example (A cup of coffee)

Newton's law of cooling states that; the temperature of an object changes at a rate proportional to the difference between its temperature and that of its surroundings.

Suppose that the temperature of your cup of coffee obeys Newton's law of cooling; suppose that it has a temperature of 90°C when freshly poured; and suppose that the temperature of your room is 20°C.

Write a differential equation for the temperature of your coffee.

We expect the cup of coffee to cool like this:



When the coffee is hot, it will cool quickly. When it is just above 20°C, it will cool slowly.

Let $K(t)$ denote the temperature of the coffee in °C and let t denote time measured in minutes. Then we know that

$$\frac{dK}{dt} \propto (20 - K).$$

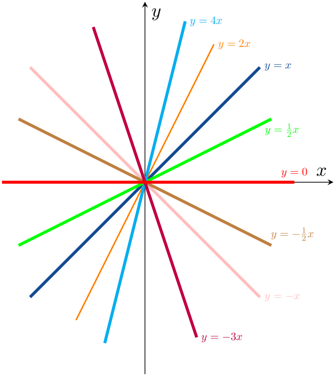
It follows that

$$\boxed{\frac{dK}{dt} = r(20 - K)} \tag{3}$$

for some constant r. Since hot coffee cools down (and cold coffee warms up), we must have $r > 0$.

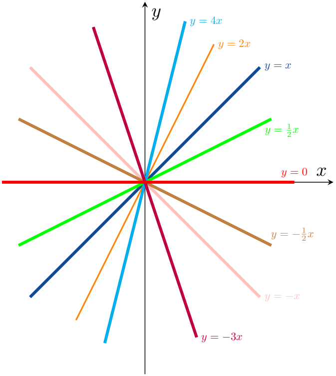
How to Draw a Direction Field

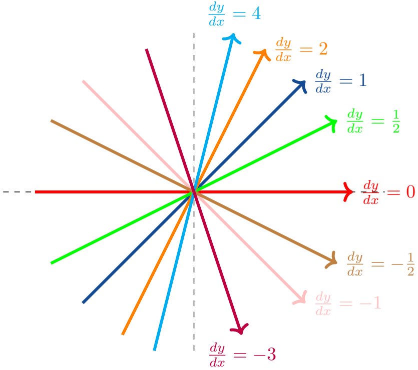
Consider the graphs of $y = mx$ for different values of $m \in \mathbb{R}$.
E.g. $y = 2x$ shopes upwards with slope 2.



1.3 How to Draw a Direction Field

Consider the graphs of $y = mx$ for different values of $m \in \mathbb{R}$. E.g. $y = 2x$ shopes upwards with slope 2. We will use rightwards arrows to show the slope of solutions of differential equations at various points.





Example

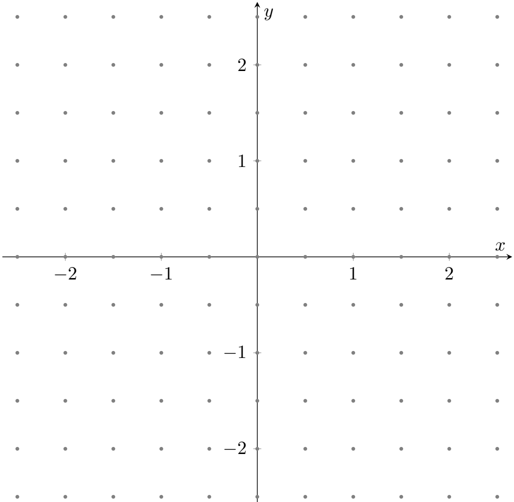
Draw a direction field for $\frac{dy}{dx} = x + y$.

1.3 How to Draw a Direction Field

Example

Draw a direction field for $\frac{dy}{dx} = x + y$.

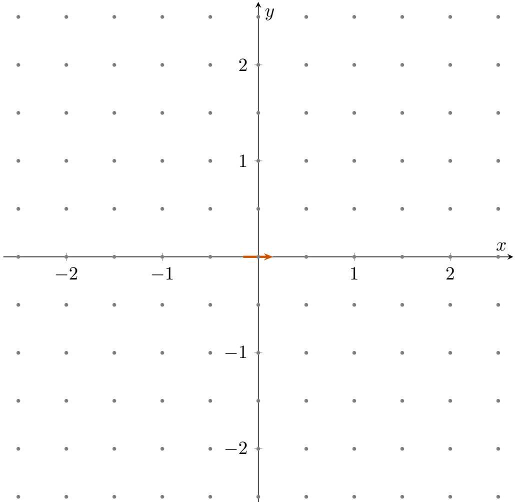
(x, y)	$\frac{dy}{dx}$



Example

Draw a direction field for $\frac{dy}{dx} = x + y$.

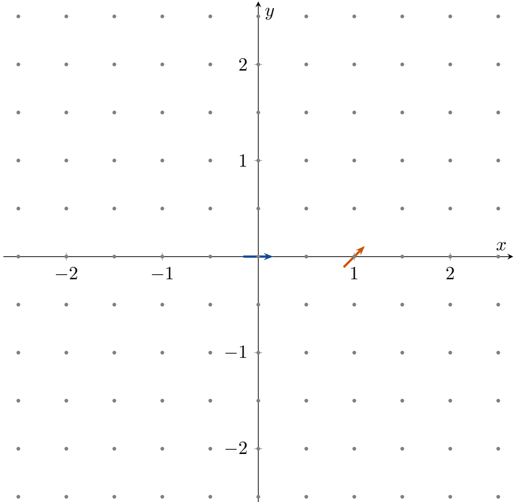
(x, y)	$\frac{dy}{dx}$
$(0, 0)$	0



Example

Draw a direction field for $\frac{dy}{dx} = x + y$.

(x,y)	$\frac{dy}{dx}$
$(0,0)$	0
$(1,0)$	1

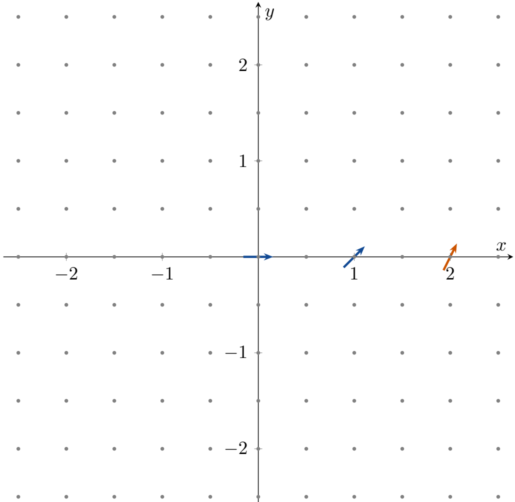


1.3 How to Draw a Direction Field

Example

Draw a direction field for $\frac{dy}{dx} = x + y$.

(x,y)	$\frac{dy}{dx}$
$(0,0)$	0
$(1,0)$	1
$(2,0)$	2



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Draw a direction field for $\frac{dy}{dx} = x + y$.

(x, y)	$\frac{dy}{dx}$
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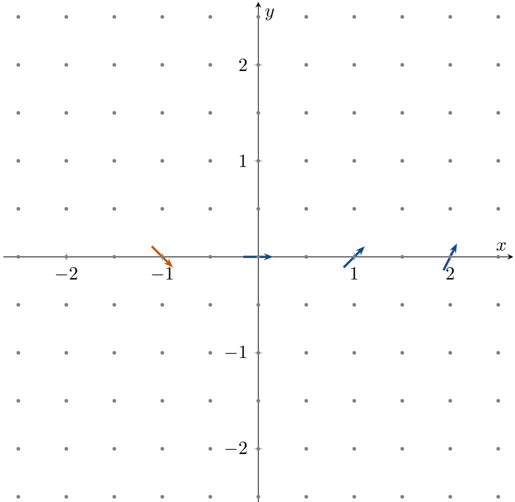
$(-1, 0)$	-1



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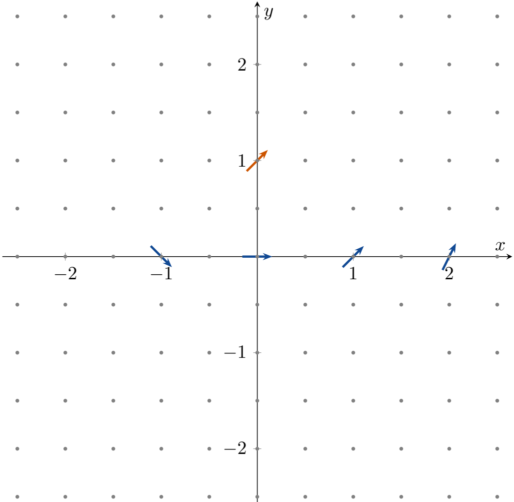

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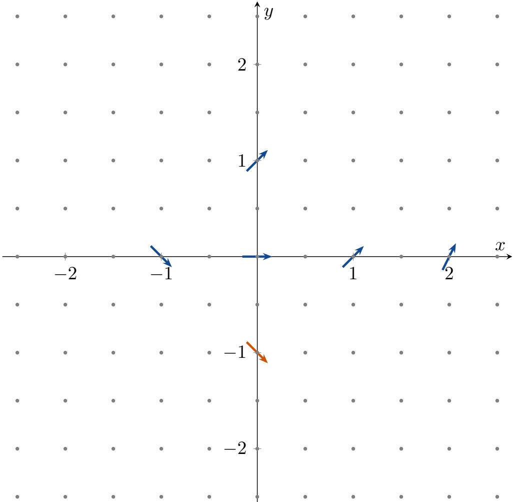

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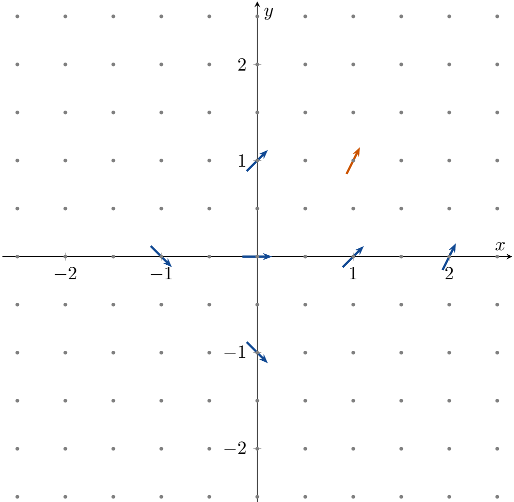

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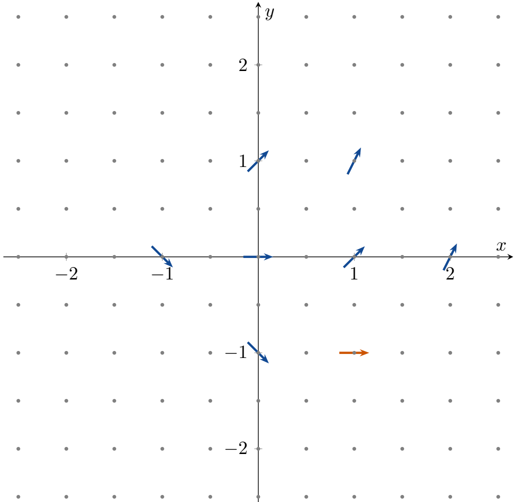

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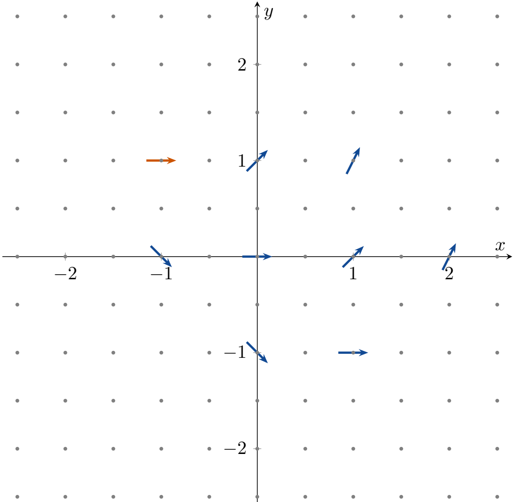

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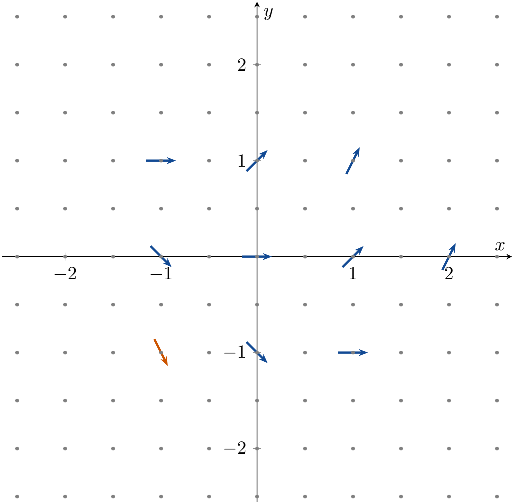
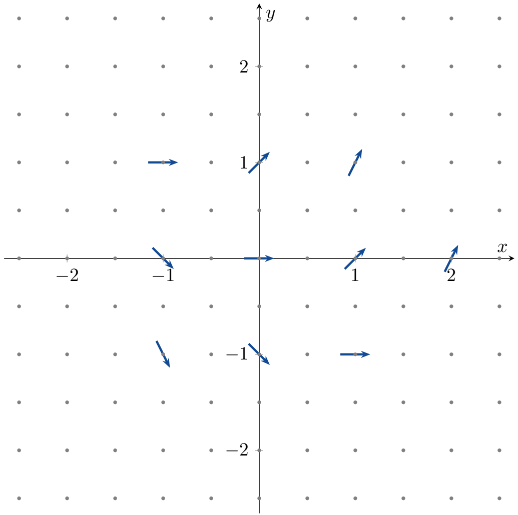
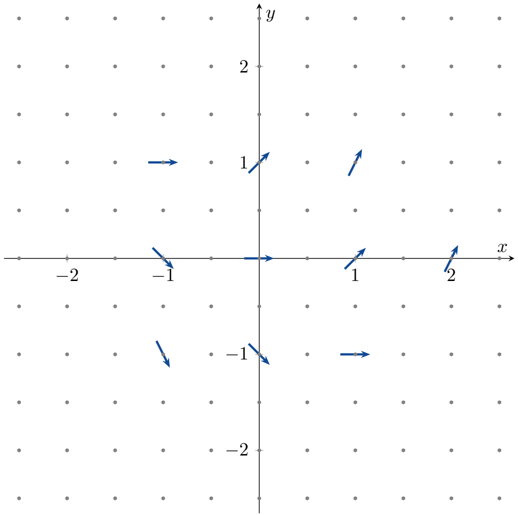
$(0,-1)$	-1
$(1,1)$	2
$(1,-1)$	0
$(-1,1)$	0
$(-1,-1)$	-2





y
x
2
1
−1
−2
−2
−1
1
2

Then we look for patterns and guess!!!



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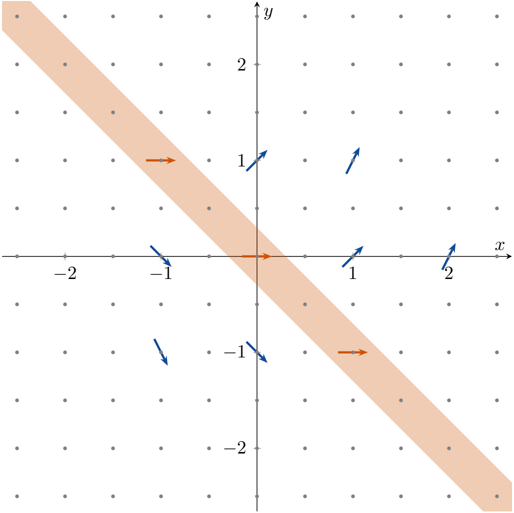

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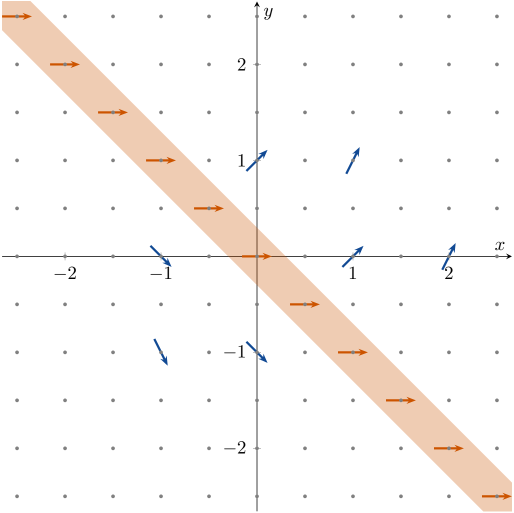

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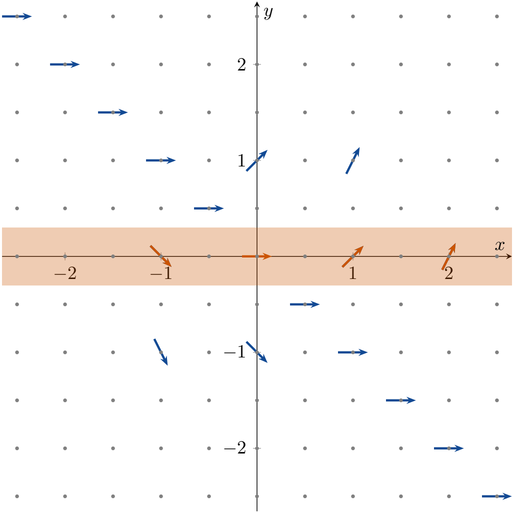

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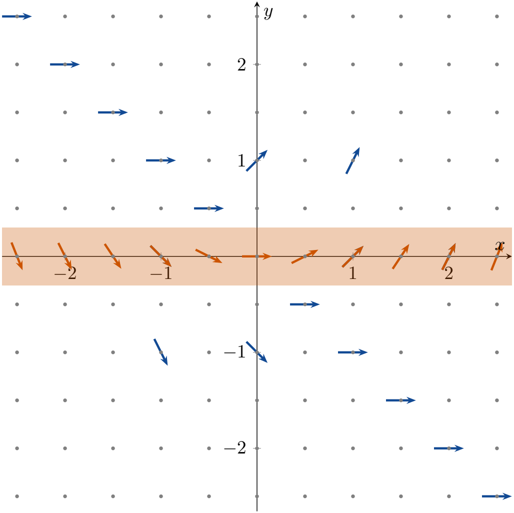

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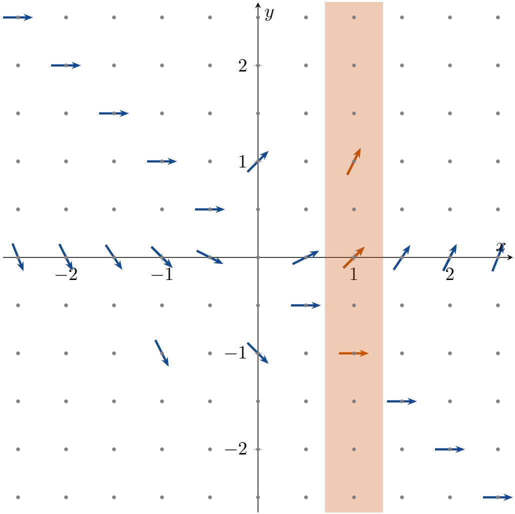

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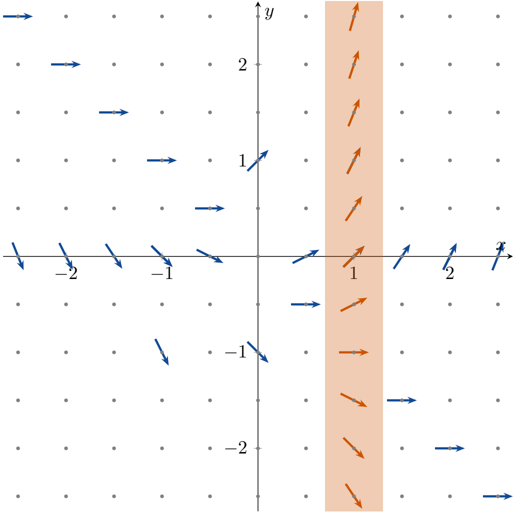

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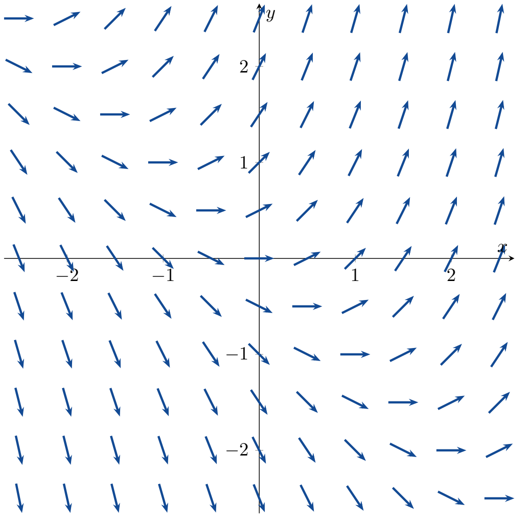
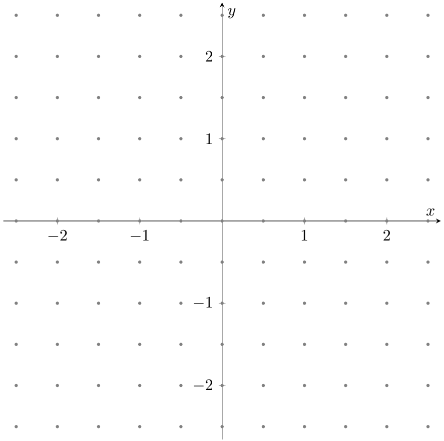


Example

Draw a direction field for $\frac{dy}{dx} = xy$.



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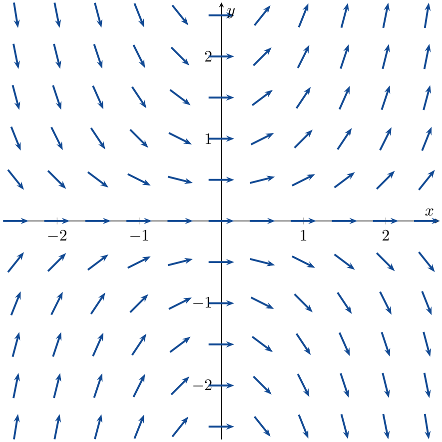
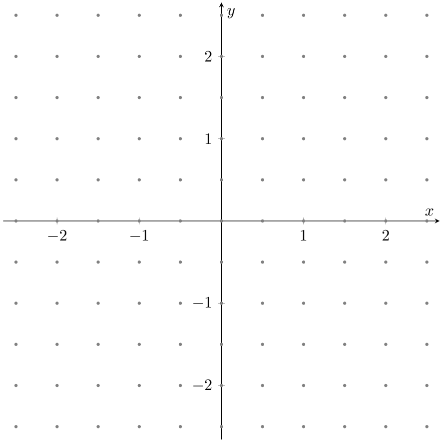


Example

Draw a direction field for $\frac{dy}{dx} = y(x+y)$.



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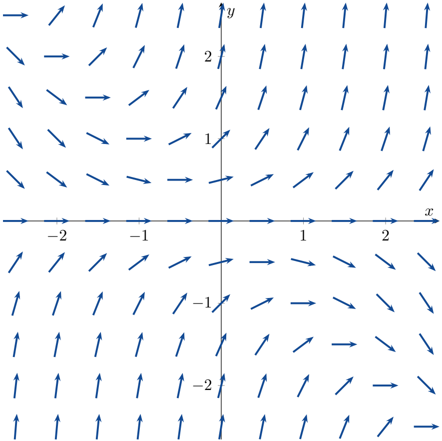

Solving Our First Differential Equations

$$\frac{dv}{dt} = 9.8 - \frac{v}{5} \tag{1}$$

$$\frac{dp}{dt} = \frac{p}{2} - 450 \tag{2}$$

Both (1) and (2) are of the form

$$\frac{dy}{dt} = ay - b \tag{4}$$

for constants a and b. We will now study how to solve equations like this.

Example (Mice and Owls)

Solve

$$\frac{dp}{dt} = \frac{p}{2} - 450 = \frac{p - 900}{2}. \tag{2}$$

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If $p \neq 900$, we can rearrange (2) to

$$\frac{dp}{p - 900} = \frac{1}{2}\, dt.$$

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Note that all the terms involving p are on the left, and all the terms involving t are on the right. (Of course $\frac{dp}{dt}$ does not really mean $dp \div dt$, and using this method annoys "Pure Mathematicians", but it works.)

If we can separate the variables like this, then we are allowed to integrate:

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$$\ln|p-900| = \frac{t}{2} + K$$

where K is a constant.

$$\ln|p - 900| = \frac{t}{2} + K$$

Thus

$$\begin{aligned} |p - 900| &= e^{\frac{t}{2}+K} \\ p - 900 &= \pm e^{K} e^{\frac{t}{2}} \\ p(t) &= 900 \pm e^{K} e^{\frac{t}{2}}. \end{aligned}$$

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Thus

$$\begin{aligned} |p - 900| &= e^{\frac{t}{2}+K} \\ p - 900 &= \pm e^{K} e^{\frac{t}{2}} \\ p(t) &= 900 \pm e^{K} e^{\frac{t}{2}}. \end{aligned}$$

K is a number that we don't know.

$$\ln|p-900| = \frac{t}{2} + K$$

Thus

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K is a number that we don't know. So e^K is a number that we don't know.

$$\ln |p - 900| = \frac{t}{2} + K$$

Thus

$$\begin{aligned} |p - 900| &= e^{\frac{t}{2} + K} \\ p - 900 &= \pm e^K e^{\frac{t}{2}} \\ p(t) &= 900 \pm e^K e^{\frac{t}{2}}. \end{aligned}$$

K is a number that we don't know. So e^K is a number that we don't know. So $\pm e^K$ is a number that we don't know.

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Thus

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K is a number that we don't know. So e^K is a number that we don't know. So $\pm e^K$ is a number that we don't know. We can give this unknown number a new name: Let $c = \pm e^K$.

$$\ln|p-900| = \frac{t}{2} + K$$

Thus

$$\begin{aligned} |p-900| &= e^{\frac{t}{2}+K} \\ p-900 &= \pm e^K e^{\frac{t}{2}} \\ p(t) &= 900 \pm e^K e^{\frac{t}{2}}. \end{aligned}$$

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$$\boxed{p(t) = 900 + ce^{\frac{t}{2}}.}$$

Before we go on, let us just check that the function

$$p(t) = 900 + ce^{\frac{t}{2}}$$

really does solve the differential equation

$$\frac{dp}{dt} = \frac{p}{2} - 450.$$

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We calculate that

$$\frac{dp}{dt} = \frac{d}{dt}\left(900 + ce^{\frac{t}{2}}\right) = 0 + c\left(\frac{1}{2}\right)e^{\frac{t}{2}} = \frac{c}{2}e^{\frac{t}{2}}$$

Before we go on, let us just check that the function

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$$\frac{dp}{dt} = \frac{d}{dt}\left(900 + ce^{\frac{t}{2}}\right) = 0 + c\left(\frac{1}{2}\right)e^{\frac{t}{2}} = \frac{c}{2}e^{\frac{t}{2}}$$

and

$$\frac{p}{2} - 450 = \frac{900 + ce^{\frac{t}{2}}}{2} - 450 = 450 + \frac{c}{2}e^{\frac{t}{2}} - 450 = \frac{c}{2}e^{\frac{t}{2}}.$$

Example (A Falling Object)

Solve

$$\frac{dv}{dt} = 9.8 - \frac{v}{5}. \tag{1}$$

We use the same method:

$$
\begin{aligned}
\frac{dv}{dt} &= 9.8 - \frac{v}{5} \\
\frac{dv}{dt} &= \frac{49 - v}{5} \\
\frac{dv}{v - 49} &= -\frac{1}{5}\,dt \\
\int \frac{dv}{v - 49} &= \int -\frac{1}{5}\,dt \\
\ln|v - 49| &= -\frac{t}{5} + K \\
|v - 49| &= e^{-\frac{t}{5} + K} \\
v - 49 &= \pm e^{K} e^{-\frac{t}{5}} \\
v(t) &= 49 \pm e^{K} e^{-\frac{t}{5}} = 49 + ce^{-\frac{t}{5}}.
\end{aligned}
$$

Example

Solve $\dfrac{dy}{dt} = y$.

This is $\dfrac{dy}{dt} = ay - b$ with $a = 1$ and $b = 0$. I leave this for you to solve.

Next Time

- 1.5 Classification
- 2.1 Linear Equations
- 2.2 Separable Equations
- 2.3 Differences Between Linear and Nonlinear Equations