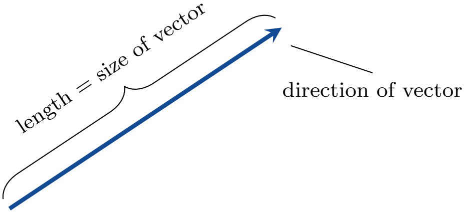
Lecture 4

- 11. Vectors
- 12. The Dot Product
- 13. The Cross Product

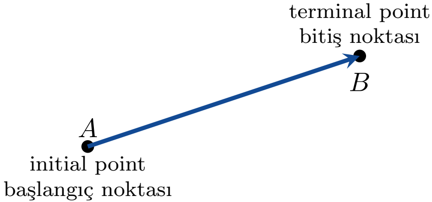
Happy Guy Fawkes Night.

11 Vectors

For some quantities (mass, time, distance, ...) we only need a number. For some quantities (velocity, force, ...) we need a number and a direction.



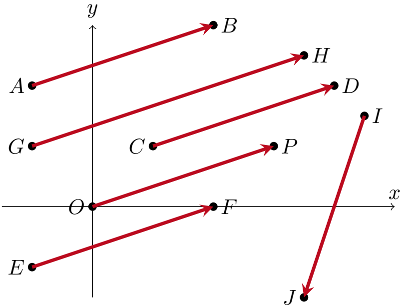
A *vector* is an object which has a size (length) and a direction.



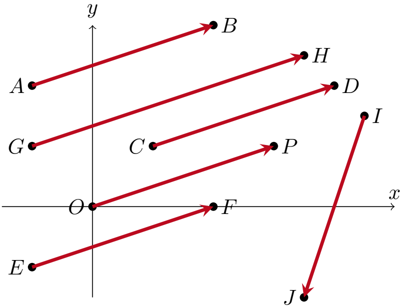
Definition

The vector $\overrightarrow{AB}$ has *initial point* A and *terminal point* B.

The *length* of $\overrightarrow{AB}$ is written $\left\|\overrightarrow{AB}\right\|$ (or $\left|\overrightarrow{AB}\right|$).



Two vectors are equal if they have the same length and the same direction.



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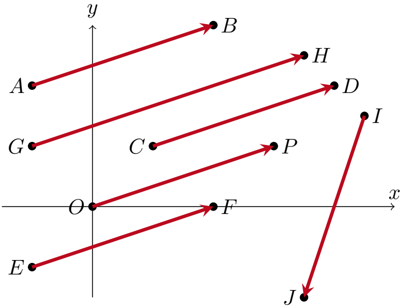
We can say that

$$\overrightarrow{AB} = \overrightarrow{CD} = \overrightarrow{EF} = \overrightarrow{OP}.$$



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We can say that

$$\overrightarrow{AB} = \overrightarrow{CD} = \overrightarrow{EF} = \overrightarrow{OP}.$$

Note that $\overrightarrow{AB} \neq \overrightarrow{GH}$ because the lengths are different, and $\overrightarrow{AB} \neq \overrightarrow{IJ}$ because the directions are different.

Notation

When we use a computer, we use bold letters for vectors: $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$, ...

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If we type $a\mathbf{u} + b\mathbf{v}$ or write $a\underline{u} + b\underline{v}$, then

- a and b are numbers; and
- $\mathbf{u}$, $\mathbf{v}$, $\underline{u}$ and $\underline{v}$ are vectors.

Definition

In $\mathbb{R}^2$: If $\mathbf{v}$ has initial point $(0, 0)$ and terminal point (v_1, v_2), then the *component form* of $\mathbf{v}$ is $\mathbf{v} = (v_1, v_2)$.

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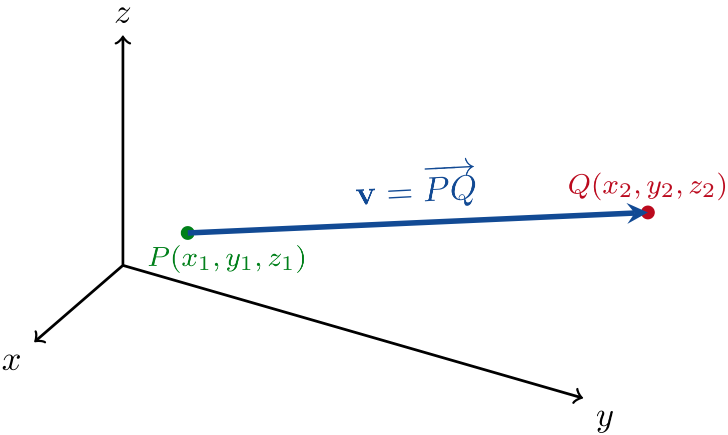
In $\mathbb{R}^3$: If $\mathbf{v}$ has initial point $(0,0,0)$ and terminal point (v_1, v_2, v_3), then the *component form* of $\mathbf{v}$ is $\mathbf{v} = (v_1, v_2, v_3)$.



$$(v_1, v_2, v_3) = \mathbf{v} = (x_2 - x_1, y_2 - y_1, z_2 - z_1)$$

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$$\begin{aligned} \|\mathbf{v}\| &= \sqrt{v_1^2 + v_2^2 + v_3^2} \\ &= \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}. \end{aligned}$$

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The vectors $\mathbf{0} = (0,0)$ and $\mathbf{0} = (0,0,0)$ have norm $\|\mathbf{0}\| = 0$. If $\mathbf{v} \neq \mathbf{0}$, then $\|\mathbf{v}\| > 0$.

Example

Find (1) the component form; and (2) the norm of the vector with initial point $P(-3, 4, 1)$ and terminal point $Q(-5, 2, 2)$.

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1. $\mathbf{v} = (v_1, v_2, v_3) = Q - P = (-5, 2, 2) - (-3, 4, 1)$
   $= (-2, -2, 1).$
2. $\|\mathbf{v}\| = \sqrt{v_1^2 + v_2^2 + v_3^2} = \sqrt{(-2)^2 + (-2)^2 + 1^2} = \sqrt{9} = 3.$

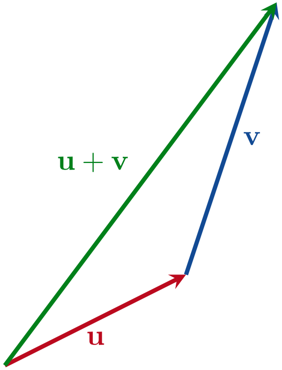
EXAMPLE 2 A small cart is being pulled along a smooth horizontal floor with a 20-lb force $\mathbf{F}$ making a 45° angle to the floor (Figure 12.11). What is the *effective* force moving the cart forward?

Solution The effective force is the horizontal component of $\mathbf{F} = \langle a, b \rangle$, given by

$$a = |\mathbf{F}| \cos 45^\circ = (20)\left(\frac{\sqrt{2}}{2}\right) \approx 14.14 \text{ lb.}$$

Notice that $\mathbf{F}$ is a two-dimensional vector. ■

Vector Algebra: Addition



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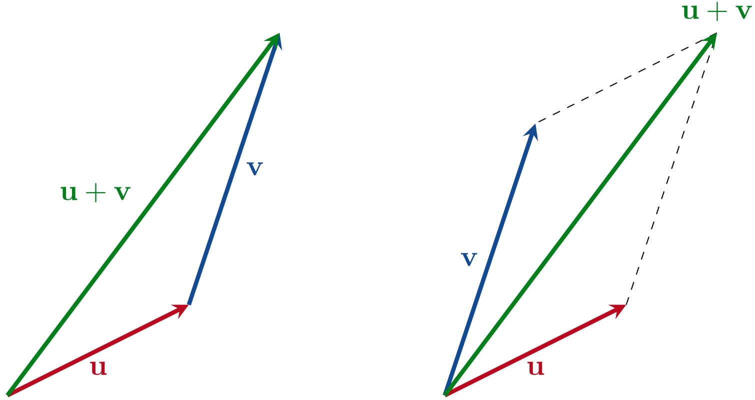
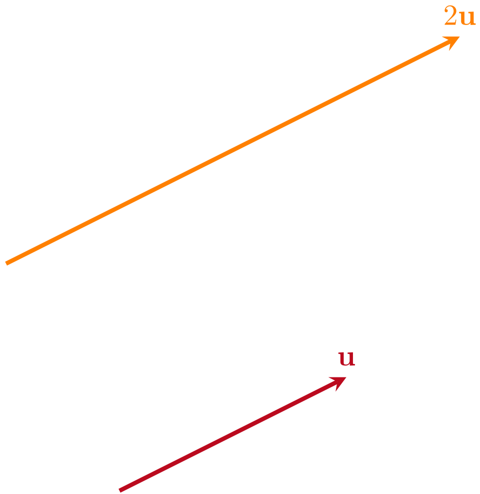


Vector Algebra: Multiplication by a Constant



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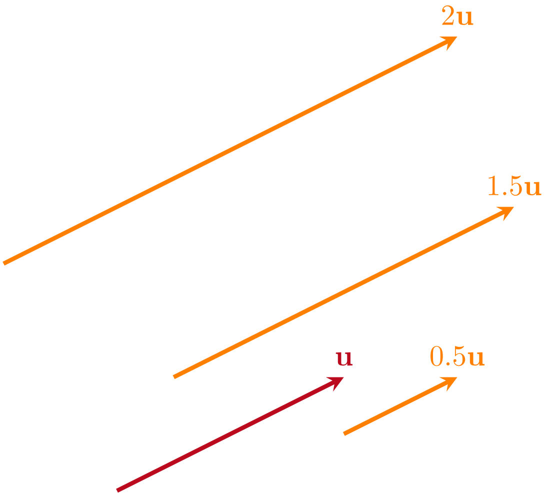

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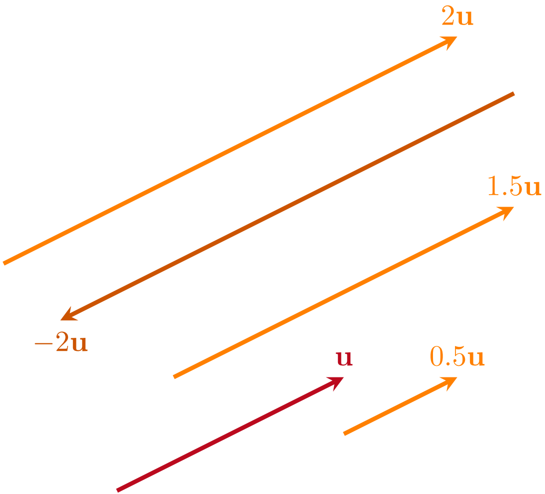
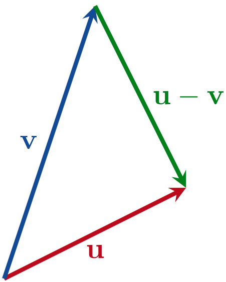


Vector Algebra: Subtraction

$$\mathbf{v} + (\mathbf{u} - \mathbf{v}) = \mathbf{u}$$



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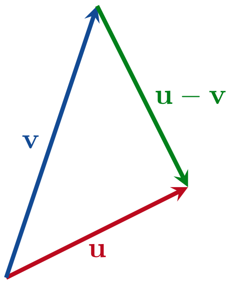

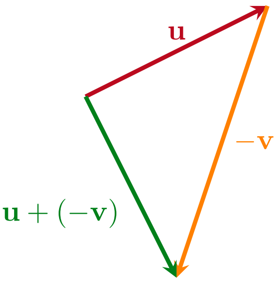

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and

$$k\mathbf{u} = (ku_1, ku_2, ku_3).$$

Note that

$$
\begin{aligned}
\|k\mathbf{u}\| &= \|(ku_1, ku_2, ku_3)\| \\
&= \\
&= \\
&= \\
&= \\
&= \qquad\qquad .
\end{aligned}
$$

Note that

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&= \sqrt{k^2}\sqrt{u_1^2 + u_2^2 + u_3^2} \\
&= |k| \, \|\mathbf{u}\| \, .
\end{aligned}
$$

The vector $-\mathbf{u} = (-1)\mathbf{u}$ has the same length as $\mathbf{u}$, but points in the opposite direction.

Example

Let $\mathbf{u} = (-1, 3, 1)$ and $\mathbf{v} = (4, 7, 0)$.

Find $2\mathbf{u} + 3\mathbf{v}$, $\mathbf{u} - \mathbf{v}$, and $\left\|\frac{1}{2}\mathbf{u}\right\|$.

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3. $\left\| \frac{1}{2}\mathbf{u} \right\| = \frac{1}{2} \|\mathbf{u}\| = \frac{1}{2}\sqrt{(-1)^2 + 3^2 + 1^2} = \frac{1}{2}\sqrt{11}$.

Properties of Vector Operations

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8. $a(\mathbf{u} + \mathbf{v}) = a\mathbf{u} + a\mathbf{v}$;
9. $(a + b)\mathbf{u} = a\mathbf{u} + b\mathbf{u}$.

Remark

We **can not** multiply vectors. Never never never never write "$\mathbf{uv}$".

Unit Vectors

Definition

$\mathbf{u}$ is called a *unit vector* $\iff$ $\|\mathbf{u}\| = 1$.

Example

$\mathbf{u} = (2^{-\frac{1}{2}}, \frac{1}{2}, -\frac{1}{2})$ is a unit vector because

$$\|\mathbf{u}\| = \sqrt{\left(\frac{1}{\sqrt{2}}\right)^2 + \left(\frac{1}{2}\right)^2 + \left(-\frac{1}{2}\right)^2} = \sqrt{\frac{1}{2} + \frac{1}{4} + \frac{1}{4}} = 1.$$

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In $\mathbb{R}^3$: The *standard unit vectors* are $\mathbf{i} = (1, 0, 0)$, $\mathbf{j} = (0, 1, 0)$ and $\mathbf{k} = (0, 0, 1)$. Any vector $\mathbf{v} \in \mathbb{R}^3$ can be written

$$\begin{aligned} \mathbf{v} &= (v_1, v_2, v_3) = (v_1, 0, 0) + (0, v_2, 0) + (0, 0, v_3) \\ &= v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}. \end{aligned}$$

Normalising a Vector

If $\|\mathbf{v}\| \neq 0$, then $\frac{\mathbf{v}}{\|\mathbf{v}\|}$ is a unit vector because

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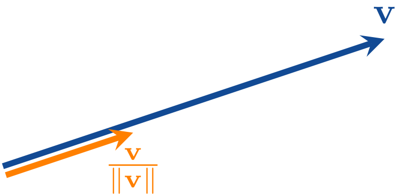
Clearly $\frac{\mathbf{v}}{\|\mathbf{v}\|}$ and $\mathbf{v}$ point in the same direction.



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We calculate that

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and that

$$\left\|\overrightarrow{P_1P_2}\right\| = \sqrt{2^2 + 2^2 + (-1)^2} = 3.$$

The required unit vector is

$$\mathbf{u} = \frac{\overrightarrow{P_1P_2}}{\left\|\overrightarrow{P_1P_2}\right\|} = \frac{2\mathbf{i} + 2\mathbf{j} - \mathbf{k}}{3} = \frac{2}{3}\mathbf{i} + \frac{2}{3}\mathbf{j} - \frac{1}{3}\mathbf{k}.$$

EXAMPLE 5 If $\mathbf{v} = 3\mathbf{i} - 4\mathbf{j}$ is a velocity vector, express $\mathbf{v}$ as a product of its speed times its direction of motion.

Solution Speed is the magnitude (length) of $\mathbf{v}$:

$$|\mathbf{v}| = \sqrt{(3)^2 + (-4)^2} = \sqrt{9 + 16} = 5.$$

The unit vector $\mathbf{v}/|\mathbf{v}|$ is the direction of $\mathbf{v}$:

$$\frac{\mathbf{v}}{|\mathbf{v}|} = \frac{3\mathbf{i} - 4\mathbf{j}}{5} = \frac{3}{5}\mathbf{i} - \frac{4}{5}\mathbf{j}.$$

So

$$\mathbf{v} = 3\mathbf{i} - 4\mathbf{j} = \underset{\substack{\text{Length}\\\text{(speed)}}}{5}\underbrace{\left(\frac{3}{5}\mathbf{i} - \frac{4}{5}\mathbf{j}\right)}_{\text{Direction of motion}}.$$

If $\mathbf{v} \neq \mathbf{0}$, then

1. $\dfrac{\mathbf{v}}{|\mathbf{v}|}$ is a unit vector called the direction of $\mathbf{v}$;

2. the equation $\mathbf{v} = |\mathbf{v}|\dfrac{\mathbf{v}}{|\mathbf{v}|}$ expresses $\mathbf{v}$ as its length times its direction.

EXAMPLE 6 A force of 6 newtons is applied in the direction of the vector $\mathbf{v} = 2\mathbf{i} + 2\mathbf{j} - \mathbf{k}$. Express the force $\mathbf{F}$ as a product of its magnitude and direction.

Solution The force vector has magnitude 6 and direction $\dfrac{\mathbf{v}}{|\mathbf{v}|}$, so

$$\begin{aligned}\mathbf{F} = 6\frac{\mathbf{v}}{|\mathbf{v}|} &= 6\frac{2\mathbf{i} + 2\mathbf{j} - \mathbf{k}}{\sqrt{2^2 + 2^2 + (-1)^2}} = 6\frac{2\mathbf{i} + 2\mathbf{j} - \mathbf{k}}{3} \\ &= 6\left(\frac{2}{3}\mathbf{i} + \frac{2}{3}\mathbf{j} - \frac{1}{3}\mathbf{k}\right).\end{aligned}$$

Midpoint of a Line Segment

Vectors are often useful in geometry. For example, the coordinates of the midpoint of a line segment are found by averaging.

The **midpoint** M of the line segment joining points $P_1(x_1, y_1, z_1)$ and $P_2(x_2, y_2, z_2)$ is the point

$$\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}, \frac{z_1 + z_2}{2}\right).$$

To see why, observe (Figure 12.16) that

$$\begin{aligned}\overrightarrow{OM} &= \overrightarrow{OP_1} + \frac{1}{2}(\overrightarrow{P_1P_2}) = \overrightarrow{OP_1} + \frac{1}{2}(\overrightarrow{OP_2} - \overrightarrow{OP_1}) \\ &= \frac{1}{2}(\overrightarrow{OP_1} + \overrightarrow{OP_2}) \\ &= \frac{x_1 + x_2}{2}\mathbf{i} + \frac{y_1 + y_2}{2}\mathbf{j} + \frac{z_1 + z_2}{2}\mathbf{k}.\end{aligned}$$

EXAMPLE 7 The midpoint of the segment joining $P_1(3, -2, 0)$ and $P_2(7, 4, 4)$ is

$$\left(\frac{3 + 7}{2}, \frac{-2 + 4}{2}, \frac{0 + 4}{2}\right) = (5, 1, 2).$$

■

Please read the final two examples in this section of the textbook.

12 The Dot Product

12. The Dot Product

Definition

In $\mathbb{R}^2$, the *dot product* of $\mathbf{u} = (u_1, u_2) = u_1\mathbf{i} + u_2\mathbf{j}$ and $\mathbf{v} = (v_1, v_2) = v_1\mathbf{i} + v_2\mathbf{j}$ is

$$\mathbf{u} \bullet \mathbf{v} = u_1v_1 + u_2v_2.$$

12. The Dot Product

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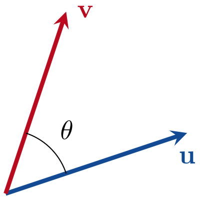
$$\mathbf{u} \bullet \mathbf{v} = u_1 v_1 + u_2 v_2.$$

Definition

In $\mathbb{R}^3$, the *dot product* of $\mathbf{u} = (u_1, u_2, u_3) = u_1\mathbf{i} + u_2\mathbf{j} + u_3\mathbf{k}$ and $\mathbf{v} = (v_1, v_2, v_3) = v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}$ is

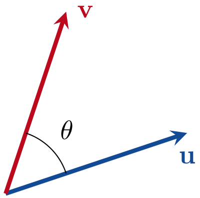
$$\mathbf{u} \bullet \mathbf{v} = u_1 v_1 + u_2 v_2 + u_3 v_3.$$



Theorem

The angle between $\mathbf{u}$ *and* $\mathbf{v}$ *is*

$$\theta = \cos^{-1}\left(\frac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{u}\|\,\|\mathbf{v}\|}\right).$$



Theorem

The angle between $\mathbf{u}$ *and* $\mathbf{v}$ *is*

$$\theta = \cos^{-1}\left(\frac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{u}\|\,\|\mathbf{v}\|}\right).$$

This means that

$$\mathbf{u}\cdot\mathbf{v} = \|\mathbf{u}\|\,\|\mathbf{v}\|\cos\theta.$$

Example

$$\begin{aligned}(1,-2,-1)\bullet(-6,2,-3) &= (1\times -6)+(-2\times 2)+(-1\times -3)\\ &= -6-4+3=-7.\end{aligned}$$

12. The Dot Product

Example

$$
\begin{aligned}
(1,-2,-1)\cdot(-6,2,-3) &= (1\times -6)+(-2\times 2)+(-1\times -3)\\
&= -6-4+3=-7.
\end{aligned}
$$

Example

$$
\begin{aligned}
(\tfrac{1}{2}\mathbf{i}+3\mathbf{j}+\mathbf{k})\cdot(4\mathbf{i}-\mathbf{j}+2\mathbf{k}) &= (\tfrac{1}{2}\times 4)+(3\times -1)+(1\times 2)\\
&= 2-3+2=1.
\end{aligned}
$$

12.

$$\theta = \cos^{-1}\left(\frac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{u}\|\,\|\mathbf{v}\|}\right).$$

Example

Find the angle between $\mathbf{u} = \mathbf{i} - 2\mathbf{j} - 2\mathbf{k}$ and $\mathbf{v} = 6\mathbf{i} + 3\mathbf{j} + 2\mathbf{k}$.

$$\theta = \cos^{-1}\left(\frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{u}\| \, \|\mathbf{v}\|}\right).$$

Example

Find the angle between $\mathbf{u} = \mathbf{i} - 2\mathbf{j} - 2\mathbf{k}$ and $\mathbf{v} = 6\mathbf{i} + 3\mathbf{j} + 2\mathbf{k}$.

Since

$$\begin{aligned} \mathbf{u} \cdot \mathbf{v} &= (1, -2, -2) \cdot (6, 3, 2) = (1 \times 6) + (-2 \times 3) + (-2 \times 2) \\ &= 6 - 6 - 4 = -4, \end{aligned}$$

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Find the angle between $\mathbf{u} = \mathbf{i} - 2\mathbf{j} - 2\mathbf{k}$ and $\mathbf{v} = 6\mathbf{i} + 3\mathbf{j} + 2\mathbf{k}$.

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$$\|\mathbf{u}\| = \sqrt{1^2 + (-2)^2 + (-2)^2} = \sqrt{9} = 3$$

and

$$\|\mathbf{v}\| = \sqrt{6^2 + 3^2 + 2^2} = \sqrt{49} = 7,$$

$$\theta = \cos^{-1}\left(\frac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{u}\|\,\|\mathbf{v}\|}\right).$$

Example

Find the angle between $\mathbf{u} = \mathbf{i} - 2\mathbf{j} - 2\mathbf{k}$ and $\mathbf{v} = 6\mathbf{i} + 3\mathbf{j} + 2\mathbf{k}$.

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$$\theta = \cos^{-1}\left(\frac{\mathbf{u} \bullet \mathbf{v}}{\|\mathbf{u}\|\,\|\mathbf{v}\|}\right).$$

Example

Find the angle between $\mathbf{u} = \mathbf{i} - 2\mathbf{j} - 2\mathbf{k}$ and $\mathbf{v} = 6\mathbf{i} + 3\mathbf{j} + 2\mathbf{k}$.

Since

$$\begin{aligned}\mathbf{u} \bullet \mathbf{v} &= (1, -2, -2) \bullet (6, 3, 2) = (1 \times 6) + (-2 \times 3) + (-2 \times 2) \\ &= 6 - 6 - 4 = -4,\end{aligned}$$

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and

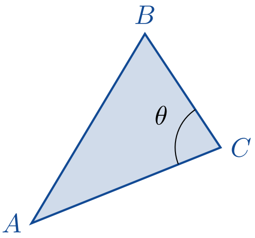
$$\|\mathbf{v}\| = \sqrt{6^2 + 3^2 + 2^2} = \sqrt{49} = 7,$$

we have that

$$\theta = \cos^{-1}\left(\frac{\mathbf{u} \bullet \mathbf{v}}{\|\mathbf{u}\|\,\|\mathbf{v}\|}\right) = \cos^{-1}\left(-\frac{4}{21}\right) \approx 1.76 \text{ radians} \approx 98.5^\circ.$$

12.

$$\theta = \cos^{-1}\left(\frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{u}\| \, \|\mathbf{v}\|}\right).$$



Example

If $A(0,0)$, $B(3,5)$ and $C(5,2)$, find $\theta = \measuredangle ACB$.

θ is the angle between $\overrightarrow{CA}$ and $\overrightarrow{CB}$.

θ is the angle between $\overrightarrow{CA}$ and $\overrightarrow{CB}$. We calculate that

$$\overrightarrow{CA} = A - C = (0,0) - (5,2) = (-5,-2),$$

$$\overrightarrow{CB} = B - C = (3,5) - (5,2) = (-2,3),$$

$$\overrightarrow{CA} \bullet \overrightarrow{CB} = (-5,-2) \bullet (-2,3) = 4,$$

$$\left\| \overrightarrow{CA} \right\| = \sqrt{(-5)^2 + (-2)^2} = \sqrt{29}$$

and

$$\left\| \overrightarrow{CB} \right\| = \sqrt{(-2)^2 + 3^2} = \sqrt{13}.$$

θ is the angle between $\overrightarrow{CA}$ and $\overrightarrow{CB}$. We calculate that

$$\begin{aligned}
\overrightarrow{CA} &= A - C = (0,0) - (5,2) = (-5,-2), \\
\overrightarrow{CB} &= B - C = (3,5) - (5,2) = (-2,3), \\
\overrightarrow{CA} \cdot \overrightarrow{CB} &= (-5,-2) \cdot (-2,3) = 4, \\
\left\| \overrightarrow{CA} \right\| &= \sqrt{(-5)^2 + (-2)^2} = \sqrt{29}
\end{aligned}$$

and

$$\left\| \overrightarrow{CB} \right\| = \sqrt{(-2)^2 + 3^2} = \sqrt{13}.$$

Therefore

$$\begin{aligned}
\theta &= \cos^{-1} \left(\frac{\overrightarrow{CA} \cdot \overrightarrow{CB}}{\left\| \overrightarrow{CA} \right\| \left\| \overrightarrow{CB} \right\|} \right) = \cos^{-1} \left(\frac{4}{\sqrt{29}\sqrt{13}} \right) \\
&\approx 78.1^\circ \approx 1.36 \text{ radians.}
\end{aligned}$$

Definition

$\mathbf{u}$ and $\mathbf{v}$ are *orthogonal* $\iff \mathbf{u} \bullet \mathbf{v} = 0$.

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Remark

Recall that

$$\mathbf{u} \cdot \mathbf{v} = \|\mathbf{u}\| \, \|\mathbf{v}\| \cos\theta.$$

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Remark

Recall that

$$\mathbf{u} \bullet \mathbf{v} = \|\mathbf{u}\| \, \|\mathbf{v}\| \cos\theta.$$

Therefore

$$\mathbf{u} \text{ and } \mathbf{v} \text{ are orthogonal} \iff \begin{pmatrix} \mathbf{u} = \mathbf{0} \\ \text{or} \\ \mathbf{v} = \mathbf{0} \\ \text{or} \\ \theta = 90^\circ. \end{pmatrix}$$

Example

$\mathbf{u} = (3, -2)$ and $\mathbf{v} = (4, 6)$ are orthogonal because
$\mathbf{u} \cdot \mathbf{v} = (3, -2) \cdot (4, 6) = (3 \times 4) + (-2 \times 6) = 12 - 12 = 0.$

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Example

$\mathbf{u} = 3\mathbf{i} - 2\mathbf{j} + \mathbf{k}$ and $\mathbf{v} = 2\mathbf{j} + 4\mathbf{k}$ are orthogonal because
$\mathbf{u} \bullet \mathbf{v} = (3 \times 0) + (-2 \times 2) + (1 \times 4) = 0 - 4 + 4 = 0.$

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Example

$\mathbf{u} = 3\mathbf{i} - 2\mathbf{j} + \mathbf{k}$ and $\mathbf{v} = 2\mathbf{j} + 4\mathbf{k}$ are orthogonal because
$\mathbf{u} \bullet \mathbf{v} = (3 \times 0) + (-2 \times 2) + (1 \times 4) = 0 - 4 + 4 = 0.$

Example

$\mathbf{0}$ is orthogonal to every vector $\mathbf{u}$ because
$\mathbf{0} \bullet \mathbf{u} = (0, 0, 0) \bullet (u_1, u_2, u_3) = 0u_1 + 0u_2 + 0u_3 = 0.$

Properties of the Dot Product

Let $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ be vectors. Let k be a number. Then

1. $\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u}$;

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4. $\mathbf{u} \cdot \mathbf{u} = \|\mathbf{u}\|^2$; and

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Let $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ be vectors. Let k be a number. Then

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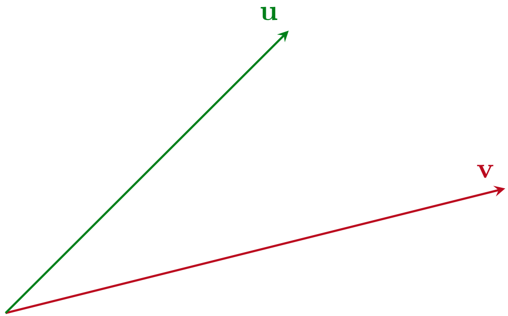
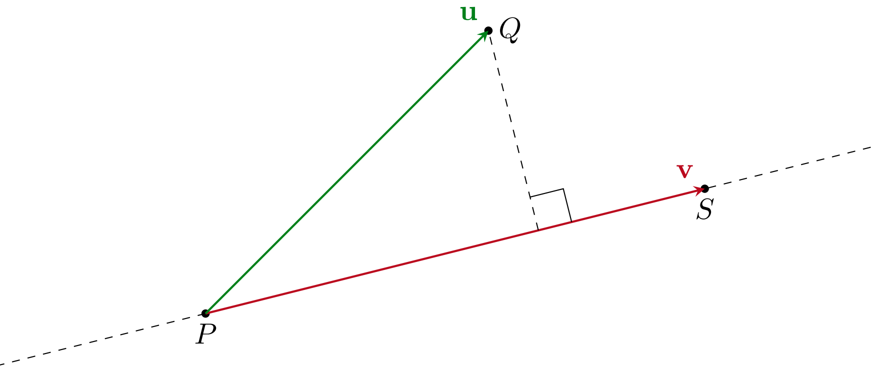
2. $(k\mathbf{u}) \cdot \mathbf{v} = \mathbf{u} \cdot (k\mathbf{v}) = k(\mathbf{u} \cdot \mathbf{v})$;
3. $\mathbf{u} \cdot (\mathbf{v} + \mathbf{w}) = (\mathbf{u} \cdot \mathbf{v}) + (\mathbf{u} \cdot \mathbf{w})$;
4. $\mathbf{u} \cdot \mathbf{u} = \|\mathbf{u}\|^2$; and
5. $\mathbf{0} \cdot \mathbf{u} = 0$.

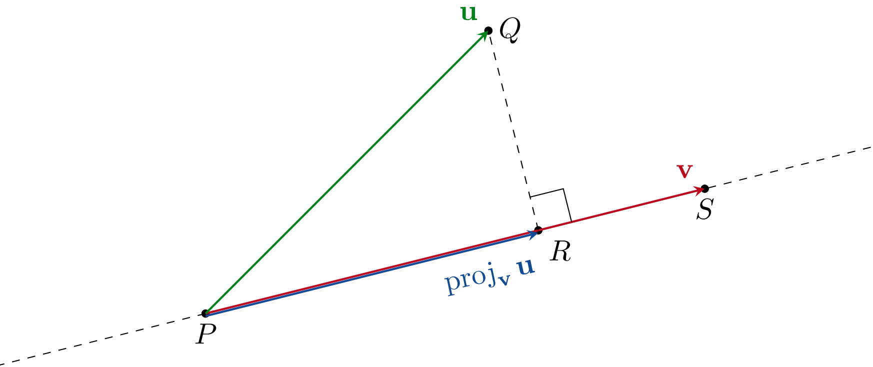
Vector Projections



Vector Projections



Vector Projections



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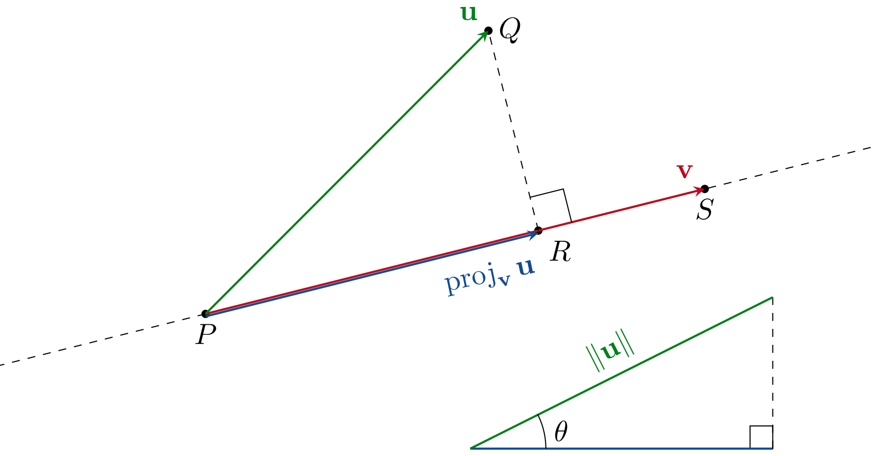
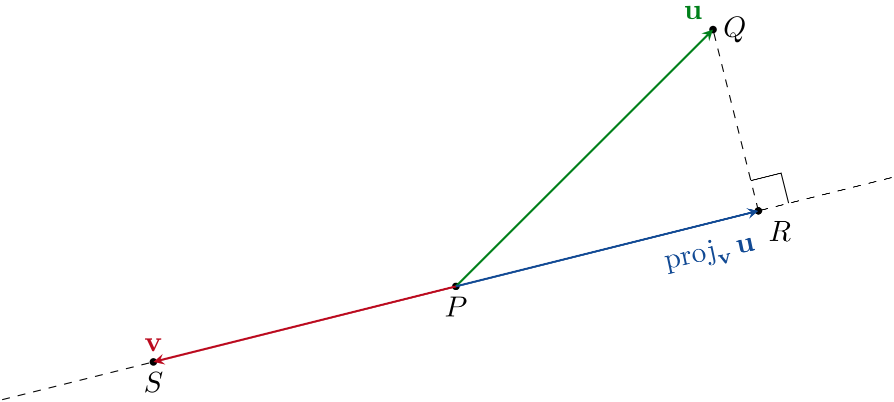
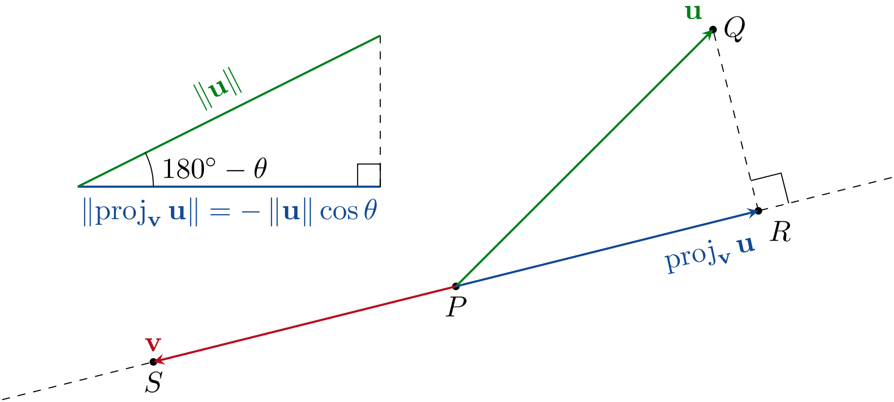


$$\|\text{proj}_{\mathbf{v}}\,\mathbf{u}\| = \|\mathbf{u}\|\cos\theta$$


u
Q
R
proj$_{\mathbf{v}}$ u
P
v
S



$\|\mathbf{u}\|$
$180° - \theta$
$\|\text{proj}_{\mathbf{v}}\,\mathbf{u}\| = -\|\mathbf{u}\|\cos\theta$
$\mathbf{u}$
Q
R
$\text{proj}_{\mathbf{v}}\,\mathbf{u}$
P
$\mathbf{v}$
S

Definition

The *vector projection* of $\mathbf{u}$ onto $\mathbf{v}$ is the vector

$$\operatorname{proj}_{\mathbf{v}} \mathbf{u} = \overrightarrow{PR}.$$

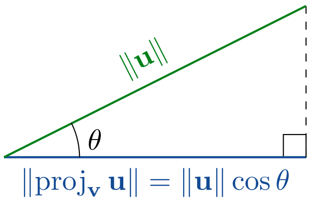
Now

$$\begin{aligned}
\operatorname{proj}_{\mathbf{v}} \mathbf{u} &= (\text{length of } \operatorname{proj}_{\mathbf{v}} \mathbf{u}) \begin{pmatrix} \text{a unit vector in} \\ \text{the same} \\ \text{direction as } \mathbf{v} \end{pmatrix} \\
&= \\
&= \\
&= \\
&=
\end{aligned}$$

12. The Dot Product

Now

$$
\begin{aligned}
\operatorname{proj}_{\mathbf{v}} \mathbf{u} &= \left(\text{length of } \operatorname{proj}_{\mathbf{v}} \mathbf{u}\right) \begin{pmatrix} \text{a unit vector in} \\ \text{the same} \\ \text{direction as } \mathbf{v} \end{pmatrix} \\
&= \|\operatorname{proj}_{\mathbf{v}} \mathbf{u}\| \left(\frac{\mathbf{v}}{\|\mathbf{v}\|} \right) \\
&= \\
&= \\
&=
\end{aligned}
$$



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$$
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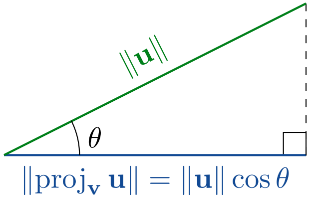
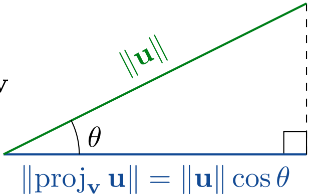
\operatorname{proj}_{\mathbf{v}} \mathbf{u} &= (\text{length of } \operatorname{proj}_{\mathbf{v}} \mathbf{u}) \begin{pmatrix} \text{a unit vector in} \\ \text{the same} \\ \text{direction as } \mathbf{v} \end{pmatrix} \\
&= \|\operatorname{proj}_{\mathbf{v}} \mathbf{u}\| \left(\frac{\mathbf{v}}{\|\mathbf{v}\|} \right) \\
&= \|\mathbf{u}\| (\cos\theta) \left(\frac{\mathbf{v}}{\|\mathbf{v}\|} \right) \\
&= \\
&=
\end{aligned}
$$



12. The Dot Product

Now

$$
\begin{aligned}
\mathrm{proj}_{\mathbf{v}}\,\mathbf{u} &= \left(\text{length of } \mathrm{proj}_{\mathbf{v}}\,\mathbf{u}\right)\begin{pmatrix}\text{a unit vector in}\\ \text{the same}\\ \text{direction as } \mathbf{v}\end{pmatrix}\\
&= \left\|\mathrm{proj}_{\mathbf{v}}\,\mathbf{u}\right\|\left(\frac{\mathbf{v}}{\|\mathbf{v}\|}\right)\\
&= \|\mathbf{u}\|\,(\cos\theta)\left(\frac{\mathbf{v}}{\|\mathbf{v}\|}\right)\\
&= \left(\frac{\|\mathbf{u}\|\,\|\mathbf{v}\|\cos\theta}{\|\mathbf{v}\|^2}\right)\mathbf{v}\\
&=
\end{aligned}
$$



Now

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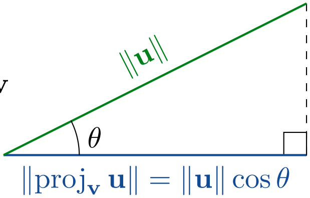
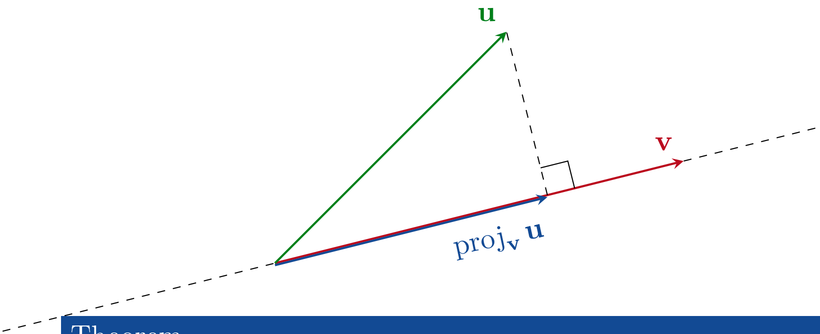
&= \|\operatorname{proj}_{\mathbf{v}} \mathbf{u}\| \left(\frac{\mathbf{v}}{\|\mathbf{v}\|} \right) \\
&= \|\mathbf{u}\| (\cos\theta) \left(\frac{\mathbf{v}}{\|\mathbf{v}\|} \right) \\
&= \left(\frac{\|\mathbf{u}\| \, \|\mathbf{v}\| \cos\theta}{\|\mathbf{v}\|^2} \right) \mathbf{v} \\
&= \left(\frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{v}\|^2} \right) \mathbf{v}.
\end{aligned}$$



Since this is an important formula, we write it as a theorem.



Theorem

The vector projection of $\mathbf{u}$ *onto* $\mathbf{v}$ *is*

$$\operatorname{proj}_{\mathbf{v}} \mathbf{u} = \left(\frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{v}\|^2} \right) \mathbf{v}.$$

$$\operatorname{proj}_{\mathbf{v}} \mathbf{u} = \left(\frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{v}\|^2} \right) \mathbf{v}$$

Example

Find the vector projection of $\mathbf{u} = 6\mathbf{i} + 3\mathbf{j} + 2\mathbf{k}$ onto $\mathbf{v} = \mathbf{i} - 2\mathbf{j} - 2\mathbf{k}$.

$$\boxed{\mathrm{proj}_{\mathbf{v}}\,\mathbf{u} = \left(\frac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{v}\|^2}\right)\mathbf{v}}$$

Example

Find the vector projection of $\mathbf{u} = 6\mathbf{i} + 3\mathbf{j} + 2\mathbf{k}$ onto $\mathbf{v} = \mathbf{i} - 2\mathbf{j} - 2\mathbf{k}$.

$$\begin{aligned}\mathrm{proj}_{\mathbf{v}}\,\mathbf{u} &= \left(\frac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{v}\|^2}\right)\mathbf{v} = \left(\frac{6-6-4}{1+4+4}\right)(\mathbf{i} - 2\mathbf{j} - 2\mathbf{k})\\ &= -\frac{4}{9}\mathbf{i} + \frac{8}{9}\mathbf{j} + \frac{8}{9}\mathbf{k}.\end{aligned}$$

$$\boxed{\operatorname{proj}_{\mathbf{v}} \mathbf{u} = \left(\frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{v}\|^2}\right) \mathbf{v}}$$

Example

Find the vector projection of $\mathbf{F} = 5\mathbf{i} + 2\mathbf{j}$ onto $\mathbf{v} = \mathbf{i} - 3\mathbf{j}$.

$$\begin{aligned} \operatorname{proj}_{\mathbf{v}} \mathbf{F} &= \left(\frac{\mathbf{F} \cdot \mathbf{v}}{\|\mathbf{v}\|^2}\right) \mathbf{v} = \left(\frac{5-6}{1+9}\right)(\mathbf{i} - 3\mathbf{j}) \\ &= -\frac{1}{10}\mathbf{i} + \frac{3}{10}\mathbf{j}. \end{aligned}$$

$$\operatorname{proj}_{\mathbf{v}} \mathbf{u} = \left(\frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{v}\|^2} \right) \mathbf{v}$$

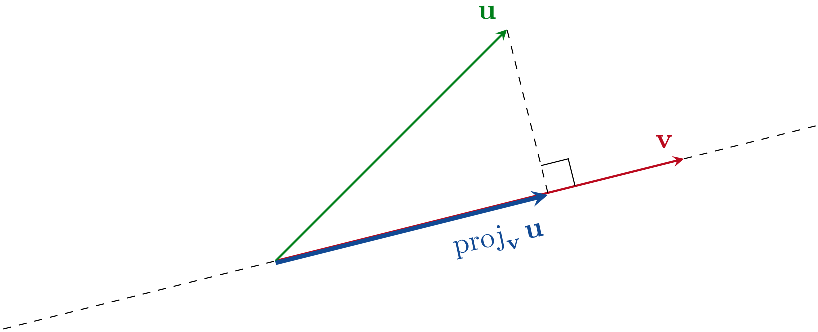
Example

Verify that the vector $\mathbf{u} - \operatorname{proj}_{\mathbf{v}} \mathbf{u}$ is orthogonal to $\operatorname{proj}_{\mathbf{v}} \mathbf{u}$.

12. $\operatorname{proj}_{\mathbf{v}} \mathbf{u} = \left(\frac{\mathbf{u} \bullet \mathbf{v}}{\|\mathbf{v}\|^2} \right) \mathbf{v}$

Example

Verify that the vector $\mathbf{u} - \operatorname{proj}_{\mathbf{v}} \mathbf{u}$ is orthogonal to $\operatorname{proj}_{\mathbf{v}} \mathbf{u}$.



$$\operatorname{proj}_{\mathbf{v}} \mathbf{u} = \left(\frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{v}\|^2} \right) \mathbf{v}$$

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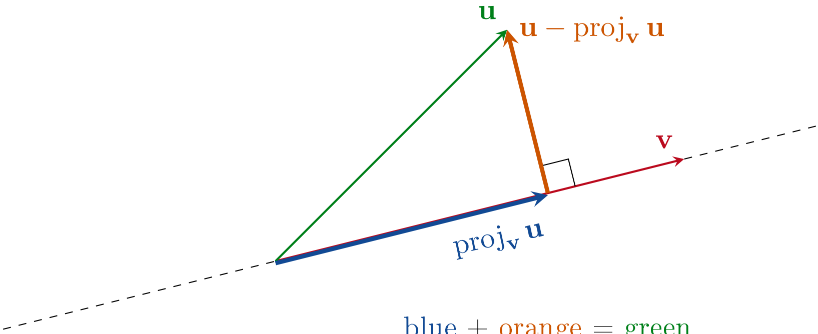
Verify that the vector $\mathbf{u} - \operatorname{proj}_{\mathbf{v}} \mathbf{u}$ is orthogonal to $\operatorname{proj}_{\mathbf{v}} \mathbf{u}$.



blue + orange = green

$$\boxed{\operatorname{proj}_{\mathbf{v}} \mathbf{u} = \left(\frac{\mathbf{u} \bullet \mathbf{v}}{\|\mathbf{v}\|^2}\right) \mathbf{v}}$$

Clearly

$$\operatorname{proj}_{\mathbf{v}} \mathbf{u} = \left(\frac{\mathbf{u} \bullet \mathbf{v}}{\|\mathbf{v}\|^2}\right) \mathbf{v} = \big(\text{a number}\big)\mathbf{v}$$

is parallel to $\mathbf{v}$.

12. $\boxed{\operatorname{proj}_{\mathbf{v}} \mathbf{u} = \left(\frac{\mathbf{u} \bullet \mathbf{v}}{\|\mathbf{v}\|^2}\right) \mathbf{v}}$

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is parallel to $\mathbf{v}$. So it is enough to show that $\mathbf{u} - \operatorname{proj}_{\mathbf{v}} \mathbf{u}$ is orthogonal to $\mathbf{v}$.

$$\operatorname{proj}_{\mathbf{v}} \mathbf{u} = \left(\frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{v}\|^2} \right) \mathbf{v}$$

Clearly

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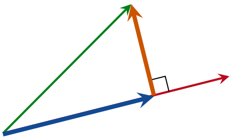
Since



$$\begin{aligned}(\mathbf{u} - \operatorname{proj}_{\mathbf{v}} \mathbf{u}) \cdot \mathbf{v} &= \\ &= \\ &= \\ &= 0\end{aligned}$$

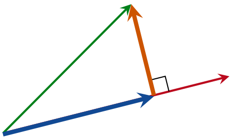
we have shown that $\mathbf{u} - \operatorname{proj}_{\mathbf{v}} \mathbf{u}$ is orthogonal to $\mathbf{v}$.

$$\boxed{\operatorname{proj}_{\mathbf{v}} \mathbf{u} = \left(\frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{v}\|^2} \right) \mathbf{v}}$$

Clearly

$$\operatorname{proj}_{\mathbf{v}} \mathbf{u} = \left(\frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{v}\|^2} \right) \mathbf{v} = (\text{a number})\mathbf{v}$$

is parallel to $\mathbf{v}$. So it is enough to show that $\mathbf{u} - \operatorname{proj}_{\mathbf{v}} \mathbf{u}$ is orthogonal to $\mathbf{v}$.
Since



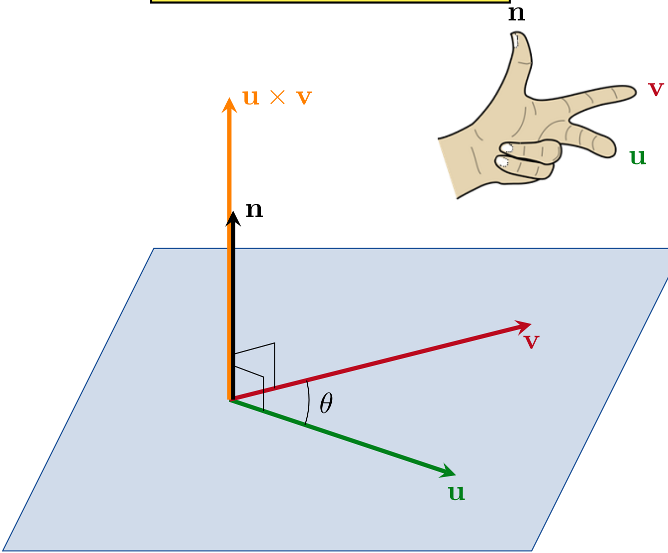
$$\begin{aligned} (\mathbf{u} - \operatorname{proj}_{\mathbf{v}} \mathbf{u}) \cdot \mathbf{v} &= \mathbf{u} \cdot \mathbf{v} - \left(\frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{v}\|^2} \right) \mathbf{v} \cdot \mathbf{v} \\ &= \mathbf{u} \cdot \mathbf{v} - \frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{v}\|^2} \|\mathbf{v}\|^2 \\ &= \mathbf{u} \cdot \mathbf{v} - \mathbf{u} \cdot \mathbf{v} \\ &= 0 \end{aligned}$$

we have shown that $\mathbf{u} - \operatorname{proj}_{\mathbf{v}} \mathbf{u}$ is orthogonal to $\mathbf{v}$.

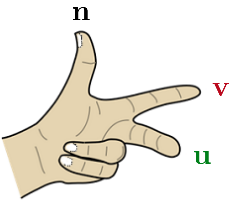
13

The Cross Product

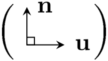
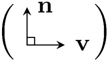
$$\text{proj}_{\mathbf{v}}\,\mathbf{u} = \left(\frac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{v}\|^2}\right)\mathbf{v}$$



$$\operatorname{proj}_{\mathbf{v}} \mathbf{u} = \left(\frac{\mathbf{u} \bullet \mathbf{v}}{\|\mathbf{v}\|^2} \right) \mathbf{v}$$



Let $\mathbf{n}$ be a unit vector which satisfies

1. $\mathbf{n}$ is orthogonal to $\mathbf{u}$ ;
2. $\mathbf{n}$ is orthogonal to $\mathbf{v}$ ; and
3. the direction of $\mathbf{n}$ is chosen using the left-hand rule.

13.

$$\operatorname{proj}_{\mathbf{v}} \mathbf{u} = \left(\frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{v}\|^2} \right) \mathbf{v}$$

Let $\mathbf{n}$ be a unit vector which satisfies

1. $\mathbf{n}$ is orthogonal to $\mathbf{u}$ ($\mathbf{n}$ $\mathbf{u}$);
2. $\mathbf{n}$ is orthogonal to $\mathbf{v}$ ($\mathbf{n}$ $\mathbf{v}$); and
3. the direction of $\mathbf{n}$ is chosen using the left-hand rule.

Definition

The *cross product* of $\mathbf{u}$ and $\mathbf{v}$ is

$$\mathbf{u} \times \mathbf{v} = \|\mathbf{u}\| \, \|\mathbf{v}\| \, (\sin\theta)\mathbf{n}.$$

13. $\boxed{\mathrm{proj}_{\mathbf{v}}\,\mathbf{u} = \left(\frac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{v}\|^2}\right)\mathbf{v}}$

Remark

- $\mathbf{u}\cdot\mathbf{v}$ is a number.
- $\mathbf{u}\times\mathbf{v}$ is a vector.

$$\mathbf{u} \times \mathbf{v} = \|\mathbf{u}\| \, \|\mathbf{v}\| \, (\sin\theta)\mathbf{n}$$

Remark

$$\begin{aligned} \begin{pmatrix} \mathbf{u} \text{ and } \mathbf{v} \\ \text{are} \\ \text{parallel} \end{pmatrix} &\iff \theta = 0^\circ \text{ or } 180^\circ \\ &\implies \sin\theta = 0 \implies \mathbf{u} \times \mathbf{v} = \mathbf{0}. \end{aligned}$$

Properties of the Cross Product

Let $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ be vectors. Let r and s be numbers. Then

1. $(r\mathbf{u}) \times (s\mathbf{v}) = (rs)(\mathbf{u} \times \mathbf{v})$;

Properties of the Cross Product

Let $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ be vectors. Let r and s be numbers. Then

1. $(r\mathbf{u}) \times (s\mathbf{v}) = (rs)(\mathbf{u} \times \mathbf{v})$;
2. $\mathbf{u} \times (\mathbf{v} + \mathbf{w}) = (\mathbf{u} \times \mathbf{v}) + (\mathbf{u} \times \mathbf{w})$;

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3. $\mathbf{v} \times \mathbf{u} = -\mathbf{u} \times \mathbf{v}$;
4. $(\mathbf{v} + \mathbf{w}) \times \mathbf{u} = (\mathbf{v} \times \mathbf{u}) + (\mathbf{w} \times \mathbf{u})$;

Properties of the Cross Product

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3. $\mathbf{v} \times \mathbf{u} = -\mathbf{u} \times \mathbf{v}$;
4. $(\mathbf{v} + \mathbf{w}) \times \mathbf{u} = (\mathbf{v} \times \mathbf{u}) + (\mathbf{w} \times \mathbf{u})$;
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Properties of the Cross Product

Let $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ be vectors. Let r and s be numbers. Then

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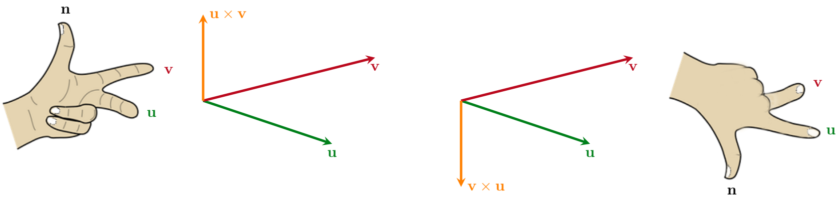
3. $\mathbf{v} \times \mathbf{u} = -\mathbf{u} \times \mathbf{v}$;
4. $(\mathbf{v} + \mathbf{w}) \times \mathbf{u} = (\mathbf{v} \times \mathbf{u}) + (\mathbf{w} \times \mathbf{u})$;
5. $\mathbf{0} \times \mathbf{u} = \mathbf{0}$; and
6. $\mathbf{u} \times (\mathbf{v} \times \mathbf{w}) = (\mathbf{u} \cdot \mathbf{w})\mathbf{v} - (\mathbf{u} \cdot \mathbf{v})\mathbf{w}$.

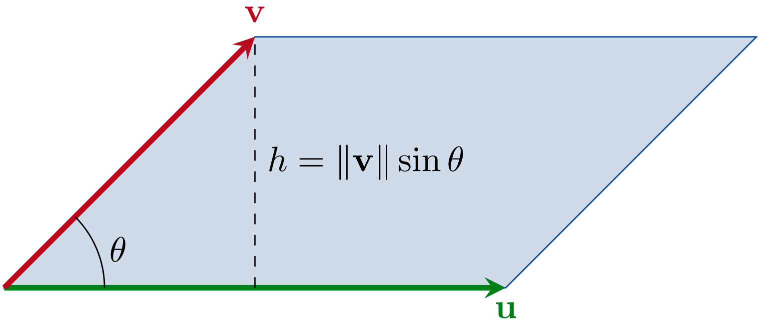
Property (iii)



$$\mathbf{v} \times \mathbf{u} = -\mathbf{u} \times \mathbf{v}$$

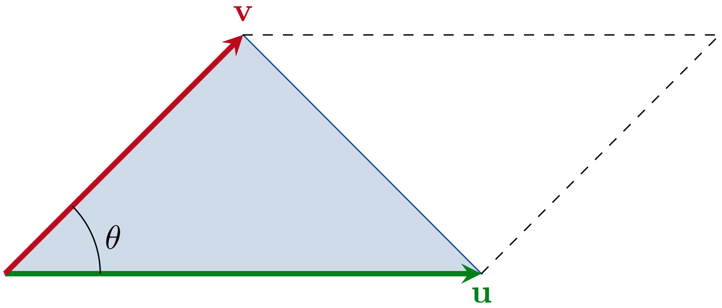
$$\mathbf{u} \times \mathbf{v} = \|\mathbf{u}\|\,\|\mathbf{v}\|\,(\sin\theta)\mathbf{n}$$

Area of a Parallelogram



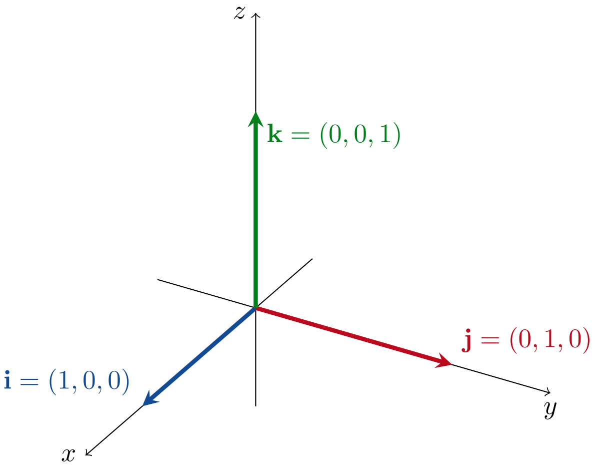
$$\text{area} = (\text{base})\,(\text{height}) = \|\mathbf{u}\|\,\|\mathbf{v}\| \sin\theta = \|\mathbf{u} \times \mathbf{v}\|\,.$$

Area of a Triangle



$$\text{area of triangle} = \frac{1}{2}\,(\text{area of parallelogram})$$
$$= \frac{1}{2}\,\|\mathbf{u} \times \mathbf{v}\|\,.$$

A Formula for $\mathbf{u} \times \mathbf{v}$



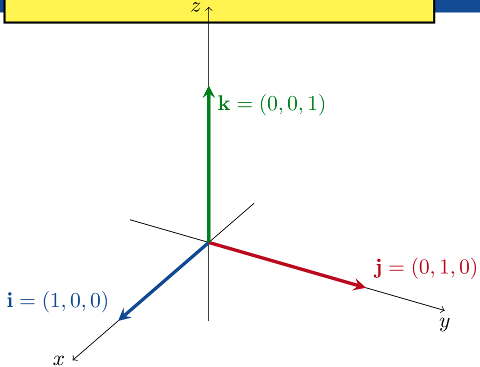
$$\mathbf{u} \times \mathbf{v} = \|\mathbf{u}\| \, \|\mathbf{v}\| \, (\sin\theta)\mathbf{n}$$

Note first that

$$\mathbf{i} \times \mathbf{i} = \|\mathbf{i}\| \, \|\mathbf{i}\| \sin 0^\circ \mathbf{n} = \mathbf{0}.$$

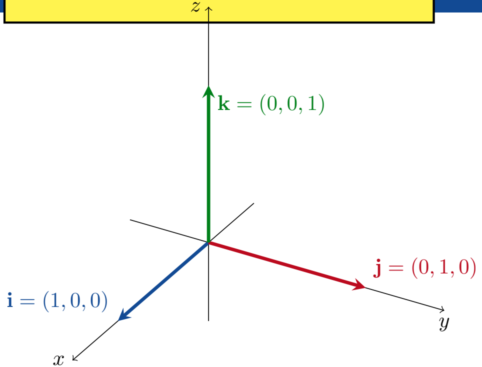
Similarly $\mathbf{j} \times \mathbf{j} = \mathbf{0}$ and $\mathbf{k} \times \mathbf{k} = \mathbf{0}$ also.

$$\mathbf{u} \times \mathbf{v} = \|\mathbf{u}\| \, \|\mathbf{v}\| \, (\sin\theta)\mathbf{n}$$



Next note that $\mathbf{i} \times \mathbf{j}$ must point in the same direction at $\mathbf{k}$ by the left-hand rule.

$$\mathbf{u} \times \mathbf{v} = \|\mathbf{u}\|\,\|\mathbf{v}\|\,(\sin\theta)\mathbf{n}$$

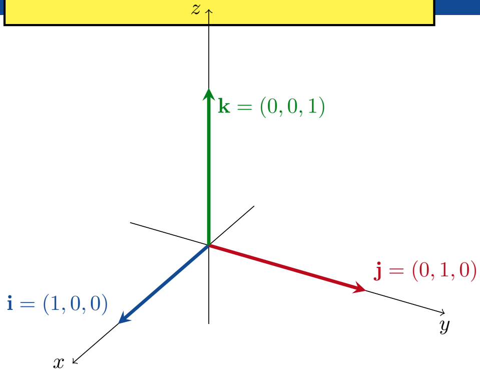


Next note that $\mathbf{i} \times \mathbf{j}$ must point in the same direction at $\mathbf{k}$ by the left-hand rule. Thus

$$\mathbf{i} \times \mathbf{j} = \|\mathbf{i}\|\,\|\mathbf{j}\|\sin 90^\circ \mathbf{k} = \mathbf{k}.$$

13.

$$\boxed{\mathbf{u} \times \mathbf{v} = \|\mathbf{u}\| \, \|\mathbf{v}\| \, (\sin\theta)\mathbf{n}}$$



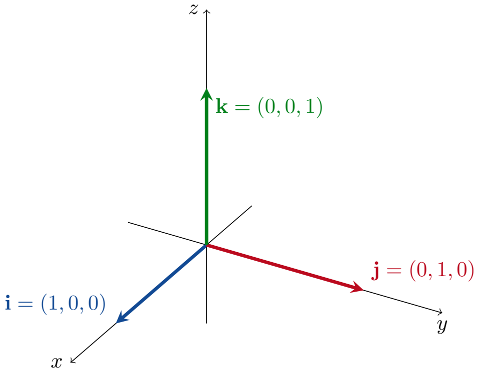
Next note that $\mathbf{i} \times \mathbf{j}$ must point in the same direction at $\mathbf{k}$ by the left-hand rule. Thus

$$\mathbf{i} \times \mathbf{j} = \|\mathbf{i}\| \, \|\mathbf{j}\| \sin 90^\circ \mathbf{k} = \mathbf{k}.$$

We then immediately also have

$$\mathbf{j} \times \mathbf{i} = -(\mathbf{i} \times \mathbf{j}) = -\mathbf{k}.$$

$$\mathbf{u} \times \mathbf{v} = \|\mathbf{u}\| \, \|\mathbf{v}\| \, (\sin\theta)\mathbf{n}$$



I leave it for you to check that

$$\mathbf{j} \times \mathbf{k} = \mathbf{i}, \qquad \mathbf{k} \times \mathbf{j} = -\mathbf{i}, \qquad \mathbf{i} \times \mathbf{k} = -\mathbf{j} \quad \text{and} \quad \mathbf{k} \times \mathbf{i} = \mathbf{j}.$$

Now suppose that $\mathbf{u} = u_1\mathbf{i} + u_2\mathbf{j} + u_3\mathbf{k}$
and $\mathbf{v} = v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}$.

Now suppose that $\mathbf{u} = u_1\mathbf{i} + u_2\mathbf{j} + u_3\mathbf{k}$ and $\mathbf{v} = v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}$. Then we can calculate that

$$\begin{aligned}
\mathbf{u} \times \mathbf{v} &= (u_1\mathbf{i} + u_2\mathbf{j} + u_3\mathbf{k}) \times (v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}) \\
&= \\
\\
&= \\
&=
\end{aligned}$$

Now suppose that $\mathbf{u} = u_1\mathbf{i} + u_2\mathbf{j} + u_3\mathbf{k}$ and $\mathbf{v} = v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}$. Then we can calculate that

$$\begin{aligned}\mathbf{u} \times \mathbf{v} &= (u_1\mathbf{i} + u_2\mathbf{j} + u_3\mathbf{k}) \times (v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}) \\ &= u_1v_1\mathbf{i} \times \mathbf{i} + u_1v_2\mathbf{i} \times \mathbf{j} + u_1v_3\mathbf{i} \times \mathbf{k} + u_2v_1\mathbf{j} \times \mathbf{i} + u_2v_2\mathbf{j} \times \mathbf{j} \\ &\quad + u_2v_3\mathbf{j} \times \mathbf{k} + u_3v_1\mathbf{k} \times \mathbf{i} + u_3v_2\mathbf{k} \times \mathbf{j} + u_3v_3\mathbf{k} \times \mathbf{k} \\ &= \\ &= \end{aligned}$$

13. The Cross Product

$$\mathbf{i} \times \mathbf{j} = \mathbf{k}$$
$$\mathbf{j} \times \mathbf{i} = -\mathbf{k}$$
$$\mathbf{j} \times \mathbf{k} = \mathbf{i}$$
$$\mathbf{k} \times \mathbf{j} = -\mathbf{i}$$
$$\mathbf{i} \times \mathbf{k} = -\mathbf{j}$$
$$\mathbf{k} \times \mathbf{i} = \mathbf{j}$$

Now suppose that $\mathbf{u} = u_1\mathbf{i} + u_2\mathbf{j} + u_3\mathbf{k}$ and $\mathbf{v} = v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}$. Then we can calculate that

$$\begin{aligned}
\mathbf{u} \times \mathbf{v} &= (u_1\mathbf{i} + u_2\mathbf{j} + u_3\mathbf{k}) \times (v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}) \\
&= u_1v_1\mathbf{i} \times \mathbf{i} + u_1v_2\mathbf{i} \times \mathbf{j} + u_1v_3\mathbf{i} \times \mathbf{k} + u_2v_1\mathbf{j} \times \mathbf{i} + u_2v_2\mathbf{j} \times \mathbf{j} \\
&\quad + u_2v_3\mathbf{j} \times \mathbf{k} + u_3v_1\mathbf{k} \times \mathbf{i} + u_3v_2\mathbf{k} \times \mathbf{j} + u_3v_3\mathbf{k} \times \mathbf{k} \\
&= \\
&=
\end{aligned}$$

13. The Cross Product

$$\mathbf{i} \times \mathbf{j} = \mathbf{k}$$
$$\mathbf{j} \times \mathbf{i} = -\mathbf{k}$$
$$\mathbf{j} \times \mathbf{k} = \mathbf{i}$$
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$$\mathbf{i} \times \mathbf{k} = -\mathbf{j}$$
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Now suppose that $\mathbf{u} = u_1\mathbf{i} + u_2\mathbf{j} + u_3\mathbf{k}$ and $\mathbf{v} = v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}$. Then we can calculate that

$$\begin{aligned}
\mathbf{u} \times \mathbf{v} &= (u_1\mathbf{i} + u_2\mathbf{j} + u_3\mathbf{k}) \times (v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}) \\
&= u_1v_1\mathbf{i} \times \mathbf{i} + u_1v_2\mathbf{i} \times \mathbf{j} + u_1v_3\mathbf{i} \times \mathbf{k} + u_2v_1\mathbf{j} \times \mathbf{i} + u_2v_2\mathbf{j} \times \mathbf{j} \\
&\quad + u_2v_3\mathbf{j} \times \mathbf{k} + u_3v_1\mathbf{k} \times \mathbf{i} + u_3v_2\mathbf{k} \times \mathbf{j} + u_3v_3\mathbf{k} \times \mathbf{k} \\
&= \mathbf{0} + u_1v_2\mathbf{k} - u_1v_3\mathbf{j} - u_2v_1\mathbf{k} + \mathbf{0} + u_2v_3\mathbf{i} + u_3v_1\mathbf{j} - u_3v_2\mathbf{i} + \mathbf{0} \\
&=
\end{aligned}$$

13. The Cross Product

$$\mathbf{i} \times \mathbf{j} = \mathbf{k}$$
$$\mathbf{j} \times \mathbf{i} = -\mathbf{k}$$
$$\mathbf{j} \times \mathbf{k} = \mathbf{i}$$
$$\mathbf{k} \times \mathbf{j} = -\mathbf{i}$$
$$\mathbf{i} \times \mathbf{k} = -\mathbf{j}$$
$$\mathbf{k} \times \mathbf{i} = \mathbf{j}$$

Now suppose that $\mathbf{u} = u_1\mathbf{i} + u_2\mathbf{j} + u_3\mathbf{k}$ and $\mathbf{v} = v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}$. Then we can calculate that

$$\begin{aligned}
\mathbf{u} \times \mathbf{v} &= (u_1\mathbf{i} + u_2\mathbf{j} + u_3\mathbf{k}) \times (v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}) \\
&= u_1v_1\mathbf{i} \times \mathbf{i} + u_1v_2\mathbf{i} \times \mathbf{j} + u_1v_3\mathbf{i} \times \mathbf{k} + u_2v_1\mathbf{j} \times \mathbf{i} + u_2v_2\mathbf{j} \times \mathbf{j} \\
&\quad + u_2v_3\mathbf{j} \times \mathbf{k} + u_3v_1\mathbf{k} \times \mathbf{i} + u_3v_2\mathbf{k} \times \mathbf{j} + u_3v_3\mathbf{k} \times \mathbf{k} \\
&= \mathbf{0} + u_1v_2\mathbf{k} - u_1v_3\mathbf{j} - u_2v_1\mathbf{k} + \mathbf{0} + u_2v_3\mathbf{i} + u_3v_1\mathbf{j} - u_3v_2\mathbf{i} + \mathbf{0} \\
&= (u_2v_3 - u_3v_2)\mathbf{i} - (u_1v_3 - u_3v_1)\mathbf{j} + (u_1v_2 - u_2v_1)\mathbf{k}.
\end{aligned}$$

Theorem

If $\mathbf{u} = u_1\mathbf{i} + u_2\mathbf{j} + u_3\mathbf{k}$ *and* $\mathbf{v} = v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}$*, then*

$$\boxed{\mathbf{u} \times \mathbf{v} = (u_2v_3 - u_3v_2)\mathbf{i} - (u_1v_3 - u_3v_1)\mathbf{j} + (u_1v_2 - u_2v_1)\mathbf{k}.}$$

If you studied matrices and determinants at high school, then you may prefer to use the following symbolic determinant formula instead.

$$\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}.$$

$$\mathbf{u} \times \mathbf{v} = (u_2v_3 - u_3v_2)\mathbf{i} - (u_1v_3 - u_3v_1)\mathbf{j} + (u_1v_2 - u_2v_1)\mathbf{k}$$

Example

Find $\mathbf{u} \times \mathbf{v}$ and $\mathbf{v} \times \mathbf{u}$ if $\mathbf{u} = 2\mathbf{i} + \mathbf{j} + \mathbf{k}$ and $\mathbf{v} = -4\mathbf{i} + 3\mathbf{j} + \mathbf{k}$.

$$\mathbf{u} \times \mathbf{v} = (u_2v_3 - u_3v_2)\mathbf{i} - (u_1v_3 - u_3v_1)\mathbf{j} + (u_1v_2 - u_2v_1)\mathbf{k}$$

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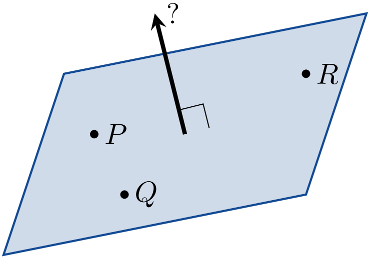
Example

Find a vector perpendicular to the plane containing the three points $P(1, -1, 0)$, $Q(2, 1, -1)$ and $R(-1, 1, 2)$.



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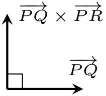
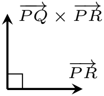
The vector $\overrightarrow{PQ} \times \overrightarrow{PR}$ is perpendicular to the plane because



and



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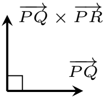
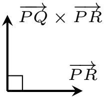


and


. We calculate that

$$\begin{aligned}
\overrightarrow{PQ} &= Q - P = (2, 1, -1) - (1, -1, 0) \\
&= (2 - 1, 1 + 1, -1 - 0) = \mathbf{i} + 2\mathbf{j} - \mathbf{k} \\
\overrightarrow{PR} &= R - P = (-1, 1, 2) - (1, -1, 0) \\
&= (-1 - 1, 1 + 1, 2 - 0) = -2\mathbf{i} + 2\mathbf{j} + 2\mathbf{k}
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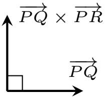

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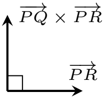

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\end{aligned}$$

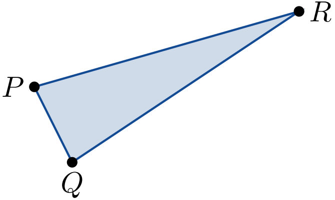
and

$$\overrightarrow{PQ} \times \overrightarrow{PR} = (4 + 2)\mathbf{i} - (2 - 2)\mathbf{j} + (2 + 4)\mathbf{k} = 6\mathbf{i} + 6\mathbf{k}.$$

Example

Find the area of triangle PQR.

$P(1, -1, 0)$, $Q(2, 1, -1)$ and $R(-1, 1, 2)$



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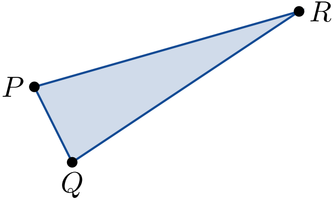
$P(1,-1,0)$, $Q(2,1,-1)$ and $R(-1,1,2)$



The area of the triangle is

$$\begin{aligned}\text{area} &= \frac{1}{2}\left\|\overrightarrow{PQ}\times\overrightarrow{PR}\right\| = \frac{1}{2}\left\|6\mathbf{i}+6\mathbf{k}\right\| \\ &= \frac{1}{2}\sqrt{6^2+0^2+6^2} = 3\sqrt{2}.\end{aligned}$$

Example

Find a unit vector perpendicular to the plane containing P, Q and R.

$P(1,-1,0)$, $Q(2,1,-1)$ and $R(-1,1,2)$

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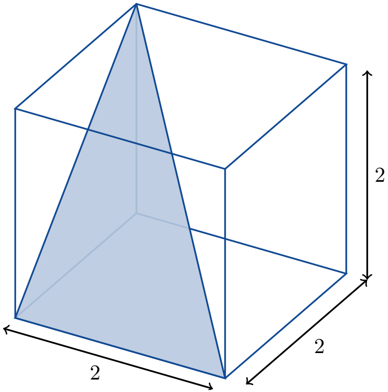
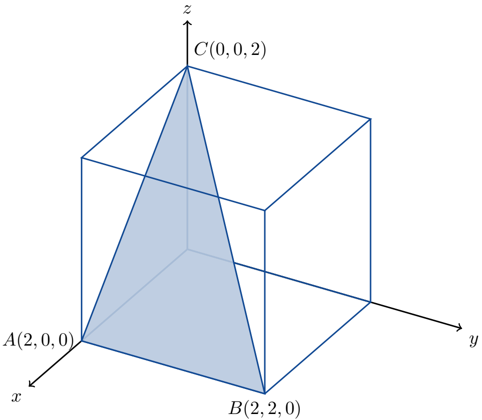
We know that $\overrightarrow{PQ} \times \overrightarrow{PR}$ is perpendicular to the plane. We just need to normalise this vector to find a unit vector.

$$\mathbf{n} = \frac{\overrightarrow{PQ} \times \overrightarrow{PR}}{\left\| \overrightarrow{PQ} \times \overrightarrow{PR} \right\|} = \frac{6\mathbf{i} + 6\mathbf{k}}{6\sqrt{2}} = \frac{1}{\sqrt{2}}\mathbf{i} + \frac{1}{\sqrt{2}}\mathbf{k}.$$

Example

A triangle is inscribed inside a cube of side 2 as shown below. Use the cross product to find the area of the triangle.





First we draw coordinate axes and assign coordinates to the vertices of the triangle.

Then we can calculate

$$\overrightarrow{AB} = B - A = (2,2,0) - (2,0,0) = (0,2,0) = 2\mathbf{j}$$

and

$$\overrightarrow{AC} = C - A = (0,0,2) - (2,0,0) = (-2,0,2) = -2\mathbf{i} + 2\mathbf{k}.$$

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It follows that

$$\begin{aligned}\overrightarrow{AB} \times \overrightarrow{AC} = (2\mathbf{j}) \times (-2I \times 2\mathbf{k}) &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 2 & 0 \\ -2 & 0 & 2 \end{vmatrix} \\ &= \mathbf{i}(4-0) - \mathbf{j}(0-0) + \mathbf{k}(0 - -4) = 4\mathbf{i} + 4\mathbf{k}.\end{aligned}$$

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Therefore

$$\begin{aligned}\text{area of triangle} &= \frac{1}{2}\left\|\overrightarrow{AB} \times \overrightarrow{AC}\right\| = \frac{1}{2}\sqrt{4^2 + 0^2 + 4^2} \\ &= \frac{1}{2}\sqrt{32} = \frac{1}{2}\sqrt{4}\sqrt{8} = \sqrt{8} = 2\sqrt{2}.\end{aligned}$$

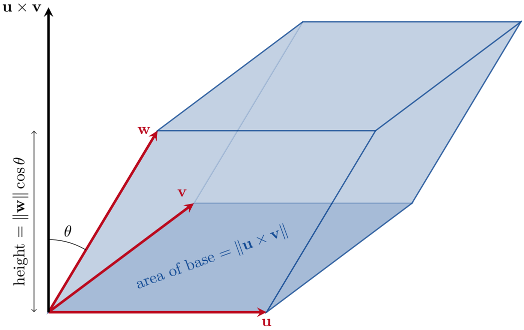
The Triple Scalar Product

Definition

The *triple scalar product* of $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ is

$$(\mathbf{u} \times \mathbf{v}) \bullet \mathbf{w}.$$

The Volume of a Parallelepiped



$$\text{volume} = \big(\text{area of base}\big)\big(\text{height}\big) = \|\mathbf{u} \times \mathbf{v}\|\, \|\mathbf{w}\| \cos\theta = |(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w}|$$

One Final Comment

We can do the dot product in both $\mathbb{R}^2$ and $\mathbb{R}^3$. But we can only do the cross product in $\mathbb{R}^3$. There is no cross product in $\mathbb{R}^2$.

Next Time

- 14. Lines
- 15. Planes
- 16. Projections