Lecture 7

- Rank and Nullity
- The Fundamental Matrix Spaces
- Linear Transformations
- Composition and Inverse Transformations
- Isomorphisms

Rank and Nullity

Recall, to find a basis for the row space of a matrix A, we reduce it to REF, then take the rows with a pivot.

$$A = \begin{bmatrix} 1 & -3 & 4 & -2 & 5 & 4 \\ 2 & -6 & 9 & -1 & 8 & 2 \\ 2 & -6 & 9 & -1 & 9 & 7 \\ -1 & 3 & -4 & 2 & -5 & -4 \end{bmatrix} \sim R = \begin{bmatrix} 1 & -3 & 4 & -2 & 5 & 4 \\ 0 & 0 & 1 & 3 & -2 & -6 \\ 0 & 0 & 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

For example

$$\begin{aligned} \mathbf{r}_1 &= \begin{bmatrix} 1 & -3 & 4 & -2 & 5 & 4 \end{bmatrix} \\ \mathbf{r}_2 &= \begin{bmatrix} 0 & 0 & 1 & 3 & -2 & -6 \end{bmatrix} \\ \mathbf{r}_3 &= \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 5 \end{bmatrix} \end{aligned}$$

form a basis for Row $A =$ Row R. Since R has three pivots, the dimension of Row A is 3.

And to find a basis for a column space, we take the pivot columns.

$$A = \begin{bmatrix} 1 & -3 & 4 & -2 & 5 & 4 \\ 2 & -6 & 9 & -1 & 8 & 2 \\ 2 & -6 & 9 & -1 & 9 & 7 \\ -1 & 3 & -4 & 2 & -5 & -4 \end{bmatrix} \sim R = \begin{bmatrix} 1 & -3 & 4 & -2 & 5 & 4 \\ 0 & 0 & 1 & 3 & -2 & -6 \\ 0 & 0 & 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

The the pivot columns of A are also the first, third and fifth columns. Hence

$$\left\{ \mathbf{c}_1 = \begin{bmatrix} 1 \\ 2 \\ 2 \\ -1 \end{bmatrix}, \mathbf{c}_3 = \begin{bmatrix} 4 \\ 9 \\ 9 \\ -4 \end{bmatrix}, \mathbf{c}_5 = \begin{bmatrix} 5 \\ 8 \\ 9 \\ -5 \end{bmatrix} \right\}$$

is a basis for $\operatorname{Col} A$. Since A has three pivot columns, the dimension of $\operatorname{Col} A$ is 3.

Rank and Nullity

Theorem

The row space and the column space of a matrix A have the same dimension.

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Rank and Nullity

Definition

The *rank* of a matrix A is

$$\operatorname{rank} A = \dim(\operatorname{Col} A) = \dim(\operatorname{Row} A).$$

Theorem

The row space and the column space of a matrix A have the same dimension.

Rank and Nullity

Definition

The *rank* of a matrix A is

$$\operatorname{rank} A = \dim(\operatorname{Col} A) = \dim(\operatorname{Row} A).$$

Definition

The *nullity* of A is

$$\operatorname{nullity} A = \dim(\operatorname{Nul} A).$$

Example

Find the rank and nullity of the matrix

$$A = \begin{bmatrix} -1 & 2 & 0 & 4 & 5 & -3 \\ 3 & -7 & 2 & 0 & 1 & 4 \\ 2 & -5 & 2 & 4 & 6 & 1 \\ 4 & -9 & 2 & -4 & -4 & 7 \end{bmatrix}.$$

Example

Find the rank and nullity of the matrix

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Please check that A is row equivalent to the RREF matrix

$$R = \begin{bmatrix} 1 & 0 & -4 & -28 & -37 & 13 \\ 0 & 1 & -2 & -12 & -16 & 5 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

The matrix

$$R = \begin{bmatrix} 1 & 0 & -4 & -28 & -37 & 13 \\ 0 & 1 & -2 & -12 & -16 & 5 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

has two pivots. So both $\operatorname{Row} A$ and $\operatorname{Col} A$ are two dimensional. Hence $\operatorname{rank} A = 2$.

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To find the nullity of A, we need to find the dimension of the solution space of $A\mathbf{x} = \mathbf{0}$.

Rank and Nullity

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has two pivots. So both $\mathrm{Row}\, A$ and $\mathrm{Col}\, A$ are two dimensional. Hence $\mathrm{rank}\, A = 2$.

To find the nullity of A, we need to find the dimension of the solution space of $A\mathbf{x} = \mathbf{0}$. We can see from R that the general solution of $A\mathbf{x} = \mathbf{0}$ is

$$\begin{cases} x_1 = 4x_3 + 28x_4 + 37x_5 - 13x_6 \\ x_2 = 2x_3 + 12x_4 + 16x_5 - 5x_6 \\ x_3, x_4, x_5, x_6 \text{ are free.} \end{cases}$$

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In vector form, the general solution is

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} = r \begin{bmatrix} 4 \\ 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + s \begin{bmatrix} 28 \\ 12 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} 37 \\ 16 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} + u \begin{bmatrix} -13 \\ -5 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

for any $r, s, t, u \in \mathbb{R}$.

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for any $r, s, t, u \in \mathbb{R}$. These four vectors (on the right hand side) form a basis for $\operatorname{Nul} A$. Hence nullity $A = 4$.

Note that

$$\operatorname{rank} A = \text{number of pivot positions}$$

and

$$\operatorname{nullity} A = \text{number of free variables.}$$

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Theorem

If A is an $m \times n$ matrix (i.e. with n columns), then

$$\operatorname{rank} A + \operatorname{nullity} A = n.$$

$$\text{rank}\, A + \text{nullity}\, A = \text{number of columns}$$

Example

The matrix

$$A = \begin{bmatrix} -1 & 2 & 0 & 4 & 5 & -3 \\ 3 & -7 & 2 & 0 & 1 & 4 \\ 2 & -5 & 2 & 4 & 6 & 1 \\ 4 & -9 & 2 & -4 & -4 & 7 \end{bmatrix}$$

has 6 columns. Therefore

$$\text{rank}\, A + \text{nullity}\, A = 6.$$

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has 6 columns. Therefore

$$\text{rank}\, A + \text{nullity}\, A = 6.$$

In the previous example we showed that $\text{rank}\, A = 2$ and $\text{nullity}\, A = 4$.

$$\text{rank}\, A + \text{nullity}\, A = \text{number of columns}$$

Ask the audience 1/2

Suppose that A is a 5×7 matrix of rank 3. Find the nullity of A.

1. nullity $A = 1$
2. nullity $A = 2$
3. nullity $A = 3$
4. nullity $A = 4$

rank A + nullity A = number of columns

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Ask the audience 2/2

Suppose that the solution space of $B\mathbf{x} = \mathbf{0}$ is two dimensional. If B is a 5×7, find the rank of B.

1. $\operatorname{rank} B = 4$
2. $\operatorname{rank} B = 5$
3. $\operatorname{rank} B = 6$
4. $\operatorname{rank} B = 7$

$$\text{rank } A + \text{nullity } A = \text{number of columns}$$

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4. $\text{rank } B = 7$

Rank and Nullity

Theorem

If $A\mathbf{x} = \mathbf{b}$ *is a consistent linear system of* m *equations in* n *unknowns, and if* A *has rank* r*, then the general solution of the system contains* $n - r$ *free variables (parameters).*

The Fundamental Matrix Spaces

The Fundamental Matrix Spaces

Consider the following six vector spaces:

- the row space of A
- the column space of A
- the null space of A
- the row space of A^T
- the column space of A^T
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When we take the transpose of a matrix, rows become columns and columns become rows. That means that

$$\operatorname{Row} A = \operatorname{Col} A^T$$

and

$$\operatorname{Col} A = \operatorname{Row} A^T.$$

The Fundamental Matrix Spaces

Definition

The *fundamental spaces* of a matrix A are

1. the row space of A
2. the column space of A
3. the null space of A
4. the null space of A^T.

We will talk about how these four vector spaces are related.

Theorem

If A is any matrix, then

$$\operatorname{rank} A = \operatorname{rank} A^T.$$

Theorem

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$$\operatorname{rank} A = \operatorname{rank} A^T.$$

Proof.

$$\begin{aligned}\operatorname{rank} A &= \dim(\text{row space of } A)\\ &= \dim(\text{column space of } A^T) = \operatorname{rank} A^T.\end{aligned}$$

□

If A is an $m \times n$ matrix, then A^T is an $n \times m$ matrix.

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$$\operatorname{rank} A^T + \operatorname{nullity} A^T = m.$$

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If $\operatorname{rank} A = r$, then

$$\dim(\operatorname{Row} A) = r \qquad \dim(\operatorname{Col} A) = r$$
$$\dim(\operatorname{Nul} A) = n - r \qquad \dim\left(\operatorname{Nul} A^T\right) = m - r.$$

The Fundamental Matrix Spaces

Now suppose that $\mathbf{x}$ is a solution of

$$\begin{bmatrix} 1 & 3 & -2 & 0 & 2 & 0 \\ 2 & 6 & -5 & -2 & 4 & -3 \\ 0 & 0 & 5 & 10 & 0 & 15 \\ 2 & 6 & 0 & 8 & 4 & 18 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

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Note that

$$\mathbf{r}_1 \cdot \mathbf{x} = (1, 3, -2, 0, 2, 0) \cdot (x_1, x_2, x_3, x_4, x_5, x_6)$$

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Note that

$$\begin{aligned} \mathbf{r}_1 \cdot \mathbf{x} &= (1, 3, -2, 0, 2, 0) \cdot (x_1, x_2, x_3, x_4, x_5, x_6) \\ &= \begin{bmatrix} 1 & 3 & -2 & 0 & 2 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} = 0. \end{aligned}$$

So

$$\mathbf{x} \text{ solves } A\mathbf{x} = \mathbf{0} \quad \Longrightarrow \quad \mathbf{x} \text{ is orthogonal to } \mathbf{r}_1.$$

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Similarly $\mathbf{r}_2 \cdot \mathbf{x} = 0$, $\mathbf{r}_3 \cdot \mathbf{x} = 0$ and $\mathbf{r}_4 \cdot \mathbf{x} = 0$.

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Similarly $\mathbf{r}_2 \cdot \mathbf{x} = 0$, $\mathbf{r}_3 \cdot \mathbf{x} = 0$ and $\mathbf{r}_4 \cdot \mathbf{x} = 0$.

Theorem

If A is an $m \times n$ matrix, then the null space of A (i.e. the solution space of $A\mathbf{x} = \mathbf{0}$) consists of all vectors in $\mathbb{R}^n$ that are orthogonal to every row vector of A.

Definition

If W is a subspace of $\mathbb{R}^n$, then the set of all vectors in $\mathbb{R}^n$ that are orthogonal to every vector in W is called the orthogonal complement of W and is denoted by the symbol $W^\perp$.

In other words,

$$W^\perp = \{\mathbf{v} \in \mathbb{R}^n \mid \mathbf{v} \cdot \mathbf{w} = 0 \text{ for all } \mathbf{w} \in W\}.$$

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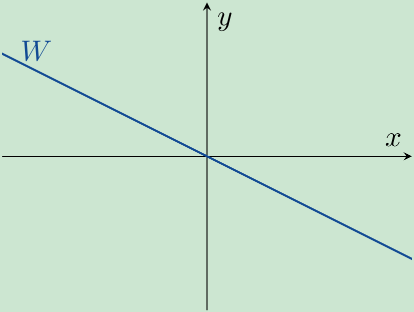
Theorem

If W is a subspace of $\mathbb{R}^n$, then:

1. $W^\perp$ *is also a subspace of* $\mathbb{R}^n$.
2. $W \cap W^\perp = \{\mathbf{0}\}$.
3. $\left(W^\perp\right)^\perp = W$.

Example

In $\mathbb{R}^2$:



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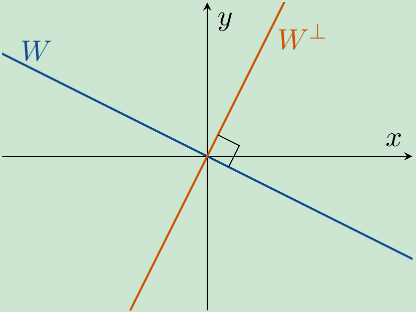
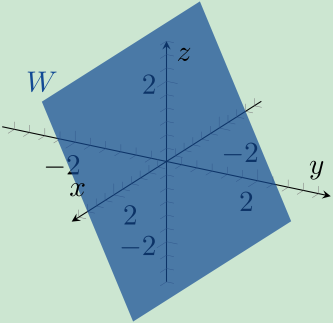


Example

In $\mathbb{R}^3$:



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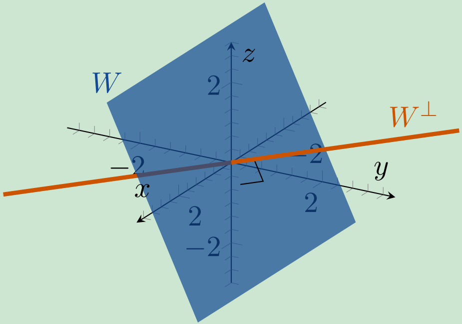

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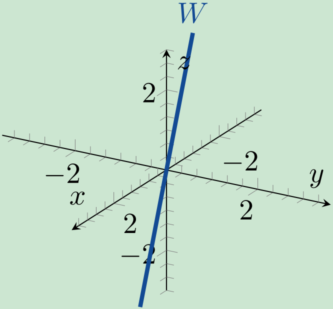

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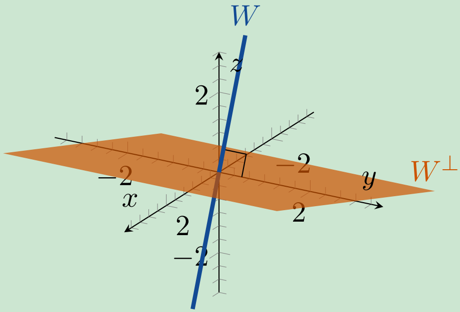

Remark

Note that

$$(\mathbb{R}^n)^\perp = \{\mathbf{0}\}$$

because only the zero vector is orthogonal to every vector in $\mathbb{R}^n$; and hence

$$\{\mathbf{0}\}^\perp = \mathbb{R}^n.$$

Theorem

Let A be an $m \times n$ matrix. Then

1. *The null space of A is the orthogonal complement of the row space of A, in $\mathbb{R}^n$.*

$$\operatorname{Nul} A = (\operatorname{Row} A)^{\perp} \qquad \textit{and} \qquad \operatorname{Row} A = (\operatorname{Nul} A)^{\perp}.$$

The Fundamental Matrix Spaces

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$$\text{Nul}\, A = (\text{Row}\, A)^\perp \qquad \textit{and} \qquad \text{Row}\, A = (\text{Nul}\, A)^\perp .$$

2. *The null space of A^T is the orthogonal complement of the column space of A, in $\mathbb{R}^m$.*

$$\text{Nul}\, A^T = (\text{Col}\, A)^\perp \qquad \textit{and} \qquad \text{Col}\, A = \left(\text{Nul}\, A^T\right)^\perp .$$

The Fundamental Matrix Spaces

Theorem

Let A be an $n \times n$ matrix. The following statements are equivalent:

1. *A is invertible.*
2. *$A\mathbf{x} = \mathbf{0}$ has only the trivial solution.*
3. *The reduced row echelon form of A is I_n.*
4. *A is expressible as a product of elementary matrices.*
5. *$A\mathbf{x} = \mathbf{b}$ is consistent for every $n \times 1$ matrix $\mathbf{b}$.*
6. *$A\mathbf{x} = \mathbf{b}$ has exactly one solution for every $n \times 1$ matrix $\mathbf{b}$.*
7. $\det(A) \neq 0$.
8.
9.
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7. $\det(A) \neq 0$.
8. *The column vectors of A are linearly independent.*
9. *The row vectors of A are linearly independent.*
10. *The column vectors of A span $\mathbb{R}^n$.*
11. *The row vectors of A span $\mathbb{R}^n$.*
12. *The column vectors of A form a basis for $\mathbb{R}^n$.*
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14. $\operatorname{rank} A = n$.
15. $\operatorname{nullity} A = 0$.
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14. $\operatorname{rank} A = n$.
15. $\operatorname{nullity} A = 0$.
16. $(\operatorname{Nul} A)^{\perp} = \mathbb{R}^n$.
17. $(\operatorname{Row} A)^{\perp} = \{\mathbf{0}\}$.

Linear Transformations

Linear Transformations

Definition

Let V and W be vector spaces. A function $T : V \to W$ is called a *linear transformation* iff

1. $T(\mathbf{u} + \mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v})$
2. $T(k\mathbf{u}) = kT(\mathbf{u})$; and

for all vectors $\mathbf{u}, \mathbf{v} \in V$ and all scalars k.

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for all vectors $\mathbf{u}, \mathbf{v} \in V$ and all scalars k.

Definition

If $V = W$, then the linear transformation $T : V \to V$ is called a *linear operator* on V.

$$T(\mathbf{u}+\mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v}) \text{ and } T(k\mathbf{u}) = kT(\mathbf{u})$$

Remark

I have tried to avoid any mention of linear transformations up to now. Your textbook introduces them in section 1.8.

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Remark

"Transformation" is just another word for "function". The important word in "linear transformation" is "linear".

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"Transformation" is just another word for "function". The important word in "linear transformation" is "linear".

Remark

We can combine rules 1 and 2 to show that

$$T(k_1\mathbf{v}_1 + k_2\mathbf{v}_2) = k_1T(\mathbf{v}_1) + k_2T(\mathbf{v}_2)$$

$T(\mathbf{u}+\mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v})$ and $T(k\mathbf{u}) = kT(\mathbf{u})$

Remark

I have tried to avoid any mention of linear transformations up to now. Your textbook introduces them in section 1.8.

Remark

"Transformation" is just another word for "function". The important word in "linear transformation" is "linear".

Remark

We can combine rules 1 and 2 to show that

$$T(k_1\mathbf{v}_1 + k_2\mathbf{v}_2) = k_1T(\mathbf{v}_1) + k_2T(\mathbf{v}_2)$$

or more generally that

$$T(k_1\mathbf{v}_1 + k_2\mathbf{v}_2 + \ldots + k_r\mathbf{v}_r) = k_1T(\mathbf{v}_1) + k_2T(\mathbf{v}_2) + \ldots + k_rT(\mathbf{v}_r).$$

$T(\mathbf{u}+\mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v})$ and $T(k\mathbf{u}) = kT(\mathbf{u})$

Theorem

If $T : V \to W$ is a linear transformation, then

$$T(\mathbf{0}) = \mathbf{0}.$$

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Proof.

Let $\mathbf{u} \in V$ be any vector. Since $0\mathbf{u} = \mathbf{0}$, it follows that

$$T(\mathbf{0}) = T(0\mathbf{u}) = 0T(\mathbf{u}) = \mathbf{0}.$$

□

$T(\mathbf{u}+\mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v})$ and $T(k\mathbf{u}) = kT(\mathbf{u})$

Example (Matrix Transformations)

Let A be an $m \times n$ matrix. We can define a function $T_A : \mathbb{R}^n \to \mathbb{R}^m$ by

$$T_A(\mathbf{x}) = A\mathbf{x}.$$

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Note that if $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$ are any vectors, and k is any number, then

$$T_A(\mathbf{u}+\mathbf{v}) = A(\mathbf{u}+\mathbf{v}) = A\mathbf{u} + A\mathbf{v} = T_A(\mathbf{u}) + T_A(\mathbf{v})$$

and

$$T_A(k\mathbf{u}) = A(k\mathbf{u}) = k(A\mathbf{u}) = kT_A(\mathbf{u}).$$

Therefore T_A is a linear transformation.

$T(\mathbf{u}+\mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v})$ and $T(k\mathbf{u}) = kT(\mathbf{u})$

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and

$$T_A(k\mathbf{u}) = A(k\mathbf{u}) = k(A\mathbf{u}) = kT_A(\mathbf{u}).$$

Therefore T_A is a linear transformation.

Functions of the form $T_A : \mathbb{R}^n \to \mathbb{R}^m$ are called *matrix transformations*.

$T(\mathbf{u}+\mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v})$ and $T(k\mathbf{u}) = kT(\mathbf{u})$

Example (The Zero Transformation)

Let V and W be any two vector spaces. Define a function $T: V \to W$ by $T(\mathbf{v}) = \mathbf{0}$ for all $\mathbf{v} \in V$.

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and

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for all $\mathbf{u}$, $\mathbf{v}$ and k. Hence T is linear.

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for all $\mathbf{u}$, $\mathbf{v}$ and k. Hence T is linear.

Example (The Identity Operator)

Let V be any vector space. The function $I : V \to V$ defined by $I(\mathbf{v}) = \mathbf{v}$ is called the *identity operator* on V. I leave it to you to show that I is linear.

$T(\mathbf{u}+\mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v})$ and $T(k\mathbf{u}) = kT(\mathbf{u})$

Example (A Linear Transformation from $\mathbb{P}^n$ to $\mathbb{P}^{n+1}$)

Let $\mathbf{p} = a_0 + a_1 x + \ldots + a_n x^n$ be a polynomial in $\mathbb{P}^n$. Define $T : \mathbb{P}^n \to \mathbb{P}^{n+1}$ by $T(\mathbf{p}) = x\mathbf{p}$. That is

$$T(\mathbf{p}) = a_0 x + a_1 x^2 + \ldots + a_n x^{n+1}.$$

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$$T(\mathbf{p}) = a_0 x + a_1 x^2 + \ldots + a_n x^{n+1}.$$

This function is a linear transformation because

$$T(\mathbf{p}_1 + \mathbf{p}_2) = x(\mathbf{p}_1 + \mathbf{p}_2) = x\mathbf{p}_1 + x\mathbf{p}_2 = T(\mathbf{p}_1) + T(\mathbf{p}_2)$$

and

$$T(k\mathbf{p}) = x(k\mathbf{p}) = k(x\mathbf{p}) = kT(\mathbf{p}).$$

$T(\mathbf{u}+\mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v})$ and $T(k\mathbf{u}) = kT(\mathbf{u})$

Example

Let $\mathbb{R}^{n\times n} = M_{nn}$ be the vector space of $n \times n$ matrices.

1. Is $T_1 : \mathbb{R}^{n\times n} \to \mathbb{R}^{n\times n}$, $T_1(A) = A^T$ linear?
2. Is $T_2 : \mathbb{R}^{n\times n} \to \mathbb{R}$, $T_2(A) = \det(A)$ linear?

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1. Since
$$T_1(A+B) = (A+B)^T = A^T + B^T = T_1(A) + T_1(B)$$
and
$$T_1(kA) = (kA)^T = kA^T = kT_1(A)$$
the answer is YES.

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Example

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1. Since
$$T_1(A+B) = (A+B)^T = A^T + B^T = T_1(A) + T_1(B)$$
and
$$T_1(kA) = (kA)^T = kA^T = kT_1(A)$$
the answer is YES.
2. Recall from Lecture 4 that
$$T_2(kA) = \det(kA) = k^n \det(A) = k^n T_2(A).$$
If $n \geq 2$, then we have $T_2(kA) \neq kT_2(A)$. So the answer is NO (if $n \geq 2$).

$T(\mathbf{u}+\mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v})$ and $T(k\mathbf{u}) = kT(\mathbf{u})$

Recall that Linear Transformations always have $T(\mathbf{0}) = \mathbf{0}$.

Example (A Nonlinear Transformation)

Let $\mathbf{b} \in \mathbb{R}^2$ such that $\mathbf{b} \neq \mathbf{0}$. Define $T : \mathbb{R}^2 \to \mathbb{R}^2$ by $T(\mathbf{x}) = \mathbf{x} + \mathbf{b}$. Note that T is not linear because

$$T(\mathbf{0}) = \mathbf{b} \neq \mathbf{0}.$$

$T(\mathbf{u}+\mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v})$ and $T(k\mathbf{u}) = kT(\mathbf{u})$

Example

$\frac{d}{dx} : C^1(-\infty,\infty) \to F(-\infty,\infty)$ is a linear transformation because

$$\frac{d}{dx}(f+g) = \frac{d}{dx}(f) + \frac{d}{dx}(g)$$

and

$$\frac{d}{dx}(kf) = k\frac{d}{dx}(f).$$

Finding Linear Transformations from Images of Basis Vectors

Theorem

Let $T : V \to W$ be a linear transformation, where V is finite-dimensional. Let $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ be a basis for V and let

$$\mathbf{v} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \ldots + c_n\mathbf{v}_n$$

be any vector in V.

Finding Linear Transformations from Images of Basis Vectors

Theorem

Let $T : V \to W$ be a linear transformation, where V is finite-dimensional. Let $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ be a basis for V and let

$$\mathbf{v} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \ldots + c_n\mathbf{v}_n$$

be any vector in V.

Then the image of $\mathbf{v} \in V$ can be expressed as

$$T(\mathbf{v}) = c_1T(\mathbf{v}_1) + c_2T(\mathbf{v}_2) + \ldots + c_nT(\mathbf{v}_n).$$

EXAMPLE 10 Computing with Images of Basis Vectors

Consider the basis $S = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ for R^3, where

$$\mathbf{v}_1 = (1, 1, 1), \quad \mathbf{v}_2 = (1, 1, 0), \quad \mathbf{v}_3 = (1, 0, 0)$$

Let $T : R^3 \to R^2$ be the linear transformation for which

$$T(\mathbf{v}_1) = (1, 0), \quad T(\mathbf{v}_2) = (2, -1), \quad T(\mathbf{v}_3) = (4, 3)$$

Find a formula for $T(x_1, x_2, x_3)$, and then use that formula to compute $T(2, -3, 5)$.

Solution We first need to express $\mathbf{x} = (x_1, x_2, x_3)$ as a linear combination of $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$. If we write

$$(x_1, x_2, x_3) = c_1(1, 1, 1) + c_2(1, 1, 0) + c_3(1, 0, 0)$$

then on equating corresponding components, we obtain

$$\begin{aligned} c_1 + c_2 + c_3 &= x_1 \\ c_1 + c_2 \phantom{{}+c_3} &= x_2 \\ c_1 \phantom{{}+c_2+c_3} &= x_3 \end{aligned}$$

which yields $c_1 = x_3$, $c_2 = x_2 - x_3$, $c_3 = x_1 - x_2$, so

$$\begin{aligned} (x_1, x_2, x_3) &= x_3(1, 1, 1) + (x_2 - x_3)(1, 1, 0) + (x_1 - x_2)(1, 0, 0) \\ &= x_3\mathbf{v}_1 + (x_2 - x_3)\mathbf{v}_2 + (x_1 - x_2)\mathbf{v}_3 \end{aligned}$$

Thus

$$\begin{aligned} T(x_1, x_2, x_3) &= x_3T(\mathbf{v}_1) + (x_2 - x_3)T(\mathbf{v}_2) + (x_1 - x_2)T(\mathbf{v}_3) \\ &= x_3(1, 0) + (x_2 - x_3)(2, -1) + (x_1 - x_2)(4, 3) \\ &= (4x_1 - 2x_2 - x_3, 3x_1 - 4x_2 + x_3) \end{aligned}$$

From this formula we obtain

$$T(2, -3, 5) = (9, 23)$$

Break

We will continue at 3pm

"The next part of this recipe will involve some calculus."

Kernal and Range

Definition

If $T : V \to W$ is a linear transformation, then the set of vectors in V that T maps to $\mathbf{0}$ is called the kernel of T and is denoted by $\ker(T)$.

$$\ker(T) = \{\mathbf{v} \in V \mid T(\mathbf{v}) = \mathbf{0}\}$$

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$$\ker(T) = \{\mathbf{v} \in V \mid T(\mathbf{v}) = \mathbf{0}\}$$

Definition

The set of all vectors in W that are images under T of at least one vector in V is called the range of T and is denoted by $R(T)$.

$$R(T) = \{\mathbf{w} \in W \mid \exists\ \mathbf{v} \in V \text{ such that } T(\mathbf{v}) = \mathbf{w}\}$$

Example (Matrix Transformations)

If $T_A : \mathbb{R}^n \to \mathbb{R}^m$ is a matrix transformation (i.e. $T_A(\mathbf{x}) = A\mathbf{x}$), then

$$\ker(T_A) = \operatorname{Nul} A \quad \text{and} \quad R(T_A) = \operatorname{Col}(A).$$

Example (Matrix Transformations)

If $T_A : \mathbb{R}^n \to \mathbb{R}^m$ is a matrix transformation (i.e. $T_A(\mathbf{x}) = A\mathbf{x}$), then

$$\ker(T_A) = \mathrm{Nul}\, A \quad \text{and} \quad R(T_A) = \mathrm{Col}(A).$$

Example (Zero Transformation)

Let $T : V \to W$ be the zero transformation. Since T maps every vector in V to $\mathbf{0}$, it follows that

$$\ker(T) = V \quad \text{and} \quad R(T) = \{\mathbf{0}\}.$$

Example (Kernel and Range of the Identity Operator)

Let $I : V \to V$ be the identity operator. Since $I(\mathbf{v}) = \mathbf{v}$ for all vectors in V, every vector in V is the image of some vector (namely, itself); thus

$$R(I) = V.$$

Since the only vector that I maps to $\mathbf{0}$ is $\mathbf{0}$ itself, it follows that

$$\ker(I) = \{\mathbf{0}\}.$$

Example

Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be the orthogonal projection onto the xy-plane (i.e. $T(x, y, z) = (x, y, 0)$).

Example

Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be the orthogonal projection onto the xy-plane (i.e. $T(x, y, z) = (x, y, 0)$). Then

$$\begin{aligned}(x, y, z) \in \ker(T) \quad &\Longrightarrow \quad (0, 0, 0) = T(x, y, z) = (x, y, 0) \\ &\Longrightarrow \quad x = y = 0.\end{aligned}$$

Example

Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be the orthogonal projection onto the xy-plane (i.e. $T(x, y, z) = (x, y, 0)$). Then

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Therefore

$$\ker(T) = \{(0, 0, z) \mid z \in \mathbb{R}\}.$$

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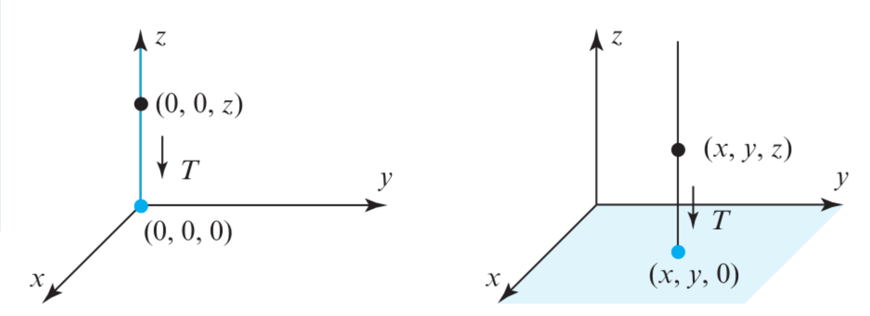
Therefore

$$\ker(T) = \{(0, 0, z) \mid z \in \mathbb{R}\}.$$

Moreover $R(T)$ is the set of points of the form $(x, y, 0)$.

Example

Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be the orthogonal projection onto the xy-plane (i.e. $T(x, y, z) = (x, y, 0)$). Then



(a) ker(T) is the z-axis. (b) R(T) is the entire xy-plane

EXAMPLE 17 Kernel and Range of a Rotation

Let $T: R^2 \rightarrow R^2$ be the linear operator that rotates each vector in the xy-plane through the angle θ (Figure 8.1.3). Since *every* vector in the xy-plane can be obtained by rotating some vector through the angle θ, it follows that $R(T) = R^2$. Moreover, the only vector that rotates into $\mathbf{0}$ is $\mathbf{0}$, so $\ker(T) = \{\mathbf{0}\}$.

Properties of Kernal and Range

Theorem

If $T : V \to W$ is a linear transformation, then:

1. *The kernel of T is a subspace of V.*
2. *The range of T is a subspace of W.*

Rank and Nullity

Let $T : V \to W$ be a linear transformation.

Definition

If the range of T is finite-dimensional, then its dimension is called the *rank of* T.

Definition

If the kernel of T is finite-dimensional, then its dimension is called the nullity of T.

Theorem

If $T : V \to W$ is a linear transformation from a finite-dimensional vector space V to a vector space W, then the range of T is finite-dimensional, and

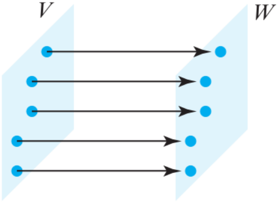
$$\operatorname{rank} T + \operatorname{nullity} T = \dim(V).$$

Composition and Inverse Transformations

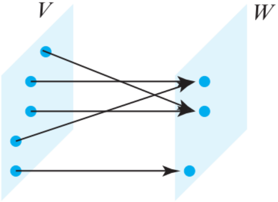
One-to-One and Onto

Definition

If $T : V \to W$ is a linear transformation from a vector space V to a vector space W, then T is said to be *one-to-one* if T maps distinct vectors in V into distinct vectors in W.



One-to-one. Distinct vectors in V have distinct images in W.

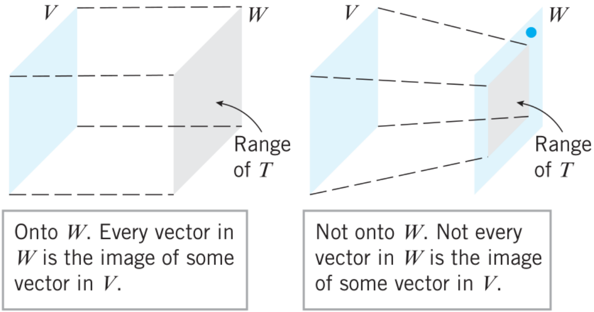


Not one-to-one. There exist distinct vectors in V with the same image.

Composition and Inverse Transformations

Definition

If $T : V \to W$ is a linear transformation from a vector space V to a vector space W, then T is said to be *onto* if every vector in W is the image of at least one vector in V.



Composition and Inverse Transformations

Theorem

If $T : V \to W$ is a linear transformation, then the following statements are equivalent:

1. *T is one-to-one.*
2. $\ker(T) = \{\mathbf{0}\}$.

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If $T : V \to W$ is a linear transformation, then the following statements are equivalent:

1. *T is one-to-one.*
2. $\ker(T) = \{\mathbf{0}\}$.

Proof.

1 $\Longrightarrow$ 2: We know that $T(\mathbf{0}) = \mathbf{0}$ because T is linear. But if T is one-to-one, then $\mathbf{0}$ is the only vector that maps to $\mathbf{0}$. Hence $\ker(T) = \{\mathbf{0}\}$.

Theorem

If $T : V \to W$ is a linear transformation, then the following statements are equivalent:

1. *T is one-to-one.*
2. $\ker(T) = \{\mathbf{0}\}$.

Proof.

1 $\implies$ 2: We know that $T(\mathbf{0}) = \mathbf{0}$ because T is linear. But if T is one-to-one, then $\mathbf{0}$ is the only vector that maps to $\mathbf{0}$. Hence $\ker(T) = \{\mathbf{0}\}$.

1 $\impliedby$ 2: Suppose that $\ker(T) = \{\mathbf{0}\}$. Let $\mathbf{u}, \mathbf{v}$ be two different vectors in V. So $\mathbf{u} - \mathbf{v} \neq \mathbf{0}$ which implies that $T(\mathbf{u} - \mathbf{v}) \neq \mathbf{0}$.

Theorem

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1 $\impliedby$ 2: Suppose that $\ker(T) = \{\mathbf{0}\}$. Let $\mathbf{u}, \mathbf{v}$ be two different vectors in V. So $\mathbf{u} - \mathbf{v} \neq \mathbf{0}$ which implies that $T(\mathbf{u} - \mathbf{v}) \neq \mathbf{0}$. But then

$$T(\mathbf{u}) - T(\mathbf{v}) = T(\mathbf{u} - \mathbf{v}) \neq \mathbf{0}$$

Composition and Inverse Transformations

Theorem

If $T : V \to W$ is a linear transformation, then the following statements are equivalent:

1. *T is one-to-one.*
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Proof.

1 $\implies$ 2: We know that $T(\mathbf{0}) = \mathbf{0}$ because T is linear. But if T is one-to-one, then $\mathbf{0}$ is the only vector that maps to $\mathbf{0}$. Hence $\ker(T) = \{\mathbf{0}\}$.

1 $\impliedby$ 2: Suppose that $\ker(T) = \{\mathbf{0}\}$. Let $\mathbf{u}, \mathbf{v}$ be two different vectors in V. So $\mathbf{u} - \mathbf{v} \neq \mathbf{0}$ which implies that $T(\mathbf{u} - \mathbf{v}) \neq \mathbf{0}$. But then

$$T(\mathbf{u}) - T(\mathbf{v}) = T(\mathbf{u} - \mathbf{v}) \neq \mathbf{0} \quad \implies \quad T(\mathbf{u}) \neq T(\mathbf{v}).$$

□

Theorem

If V and W are <u>finite-dimensional</u> vector spaces with the <u>same dimension</u>, and if $T : V \to W$ is a linear transformation, then the following statements are equivalent:

1. *T is one-to-one.*
2. $\ker(T) = \{\mathbf{0}\}$.
3. *T is onto [i.e., $R(T) = W$].*

(proof omitted.)

Example (Matrix Transformations)

Consider $T_A : \mathbb{R}^n \to \mathbb{R}^m$, $T_A(\mathbf{x}) = A\mathbf{x}$ where A is an $m \times n$ matrix.

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- If $m < n$, then T_A is not one-to-one.

Example (Matrix Transformations)

Consider $T_A : \mathbb{R}^n \to \mathbb{R}^m$, $T_A(\mathbf{x}) = A\mathbf{x}$ where A is an $m \times n$ matrix.

- If $m < n$, then T_A is not one-to-one.
- If $m > n$, then T_A is not onto.

Example (Matrix Transformations)

Consider $T_A : \mathbb{R}^n \to \mathbb{R}^m$, $T_A(\mathbf{x}) = A\mathbf{x}$ where A is an $m \times n$ matrix.

- If $m < n$, then T_A is not one-to-one.
- If $m > n$, then T_A is not onto.
- If $m = n$, then

$$T_A \text{ is both one-to-one and onto} \quad \Longleftrightarrow \quad A \text{ is invertible}$$

Example

The linear transformation $T_1 : \mathbb{P}^3 \to \mathbb{R}^4$,

$$T_1(a + bx + cx^2 + dx^3) = (a, b, c, d)$$

is both one-to-one and onto because $\ker(T_1) = \{\mathbf{0}\}$ (you prove).

Example

The linear transformation $T_1 : \mathbb{P}^3 \to \mathbb{R}^4$,

$$T_1(a + bx + cx^2 + dx^3) = (a, b, c, d)$$

is both one-to-one and onto because $\ker(T_1) = \{\mathbf{0}\}$ (you prove).

Example

The linear transformation $T_2 : \mathbb{R}^{2\times 2} \to \mathbb{R}^4$,

$$T_2\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = (a, b, c, d)$$

is both one-to-one and onto because $\ker(T_2) = \{\mathbf{0}\}$ (you prove).

Example

Let $T : \mathbb{P}^n \to \mathbb{P}^{n+1}$ be the linear transformation

$$T(\mathbf{p}) = x\mathbf{p}.$$

If $\mathbf{p} \neq \mathbf{q}$, then $x\mathbf{p} \neq x\mathbf{q}$. Therefore T is one-to-one.

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Example

$\frac{d}{dx} : C^1(-\infty, \infty) \to F(-\infty, \infty)$ is not one-to-one because, for example,

$$\frac{d}{dx}(x^2) = \frac{d}{dx}(x^2 + 1).$$

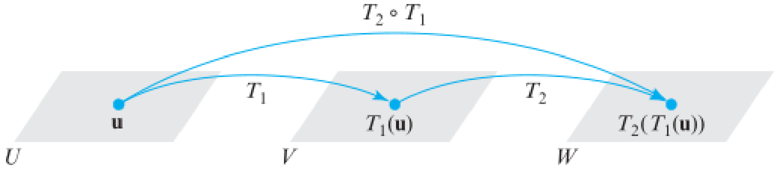
Compositions of Linear Transformations

Definition

If $T_1 : U \to V$ and $T_2 : V \to W$ are linear transformations, then the composition of T_2 with T_1, denoted by $T_2 \circ T_1$, is the function defined by the formula

$$(T_2 \circ T_1)(\mathbf{u}) = T_2\big(T_1(\mathbf{u})\big)$$

where $\mathbf{u}$ is a vector in U.



Composition and Inverse Transformations

Theorem

If $T_1 : U \to V$ *and* $T_2 : V \to W$ *are linear transformations, then* $(T_2 \circ T_1) : U \to W$ *is also a linear transformation.*

EXAMPLE 6 Composition of Linear Transformations

Let $T_1\colon P_1 \to P_2$ and $T_2\colon P_2 \to P_2$ be the linear transformations given by the formulas

$$T_1(p(x)) = xp(x) \quad \text{and} \quad T_2(p(x)) = p(2x+4)$$

Then the composition $(T_2 \circ T_1)\colon P_1 \to P_2$ is given by the formula

$$(T_2 \circ T_1)(p(x)) = T_2(T_1(p(x))) = T_2(xp(x)) = (2x+4)p(2x+4)$$

In particular, if $p(x) = c_0 + c_1 x$, then

$$\begin{aligned}(T_2 \circ T_1)(p(x)) = (T_2 \circ T_1)(c_0 + c_1 x) &= (2x+4)(c_0 + c_1(2x+4)) \\ &= c_0(2x+4) + c_1(2x+4)^2\end{aligned}$$

EXAMPLE 7 Composition with the Identity Operator

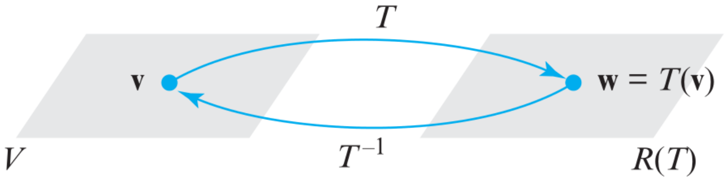
If $T: V \to V$ is any linear operator, and if $I: V \to V$ is the identity operator (Example 3 of Section 8.1), then for all vectors $\mathbf{v}$ in V, we have

$$(T \circ I)(\mathbf{v}) = T(I(\mathbf{v})) = T(\mathbf{v})$$
$$(I \circ T)(\mathbf{v}) = I(T(\mathbf{v})) = T(\mathbf{v})$$

It follows that $T \circ I$ and $I \circ T$ are the same as T; that is,

$$\boxed{T \circ I = T} \quad \text{and} \quad \boxed{I \circ T = T} \tag{2}$$

Inverse Linear Transformations



If $T : V \to W$ is a one-to-one linear transformation, then its *inverse* $T^{-1} : R(T) \to V$ is also a linear transformation.

$$T(\mathbf{v}) = \mathbf{w} \qquad \Longleftrightarrow \qquad \mathbf{v} = T^{-1}(\mathbf{w}).$$

EXAMPLE 8 An Inverse Transformation

We showed in Example 3 of this section that the linear transformation $T: P_n \to P_{n+1}$ given by

$$T(\mathbf{p}) = T(p(x)) = xp(x)$$

is one-to-one but not onto. The fact that it is not onto can be seen explicitly from the formula

$$T(c_0 + c_1x + \cdots + c_nx^n) = c_0x + c_1x^2 + \cdots + c_nx^{n+1} \tag{6}$$

which makes it clear that the range of T consists of all polynomials in P_{n+1} that have zero constant term. Since T is one-to-one it has an inverse, and from (6) this inverse is given by the formula

$$T^{-1}(c_0x + c_1x^2 + \cdots + c_nx^{n+1}) = c_0 + c_1x + \cdots + c_nx^n$$

For example, in the case where $n \geq 3$,

$$T^{-1}(2x - x^2 + 5x^3 + 3x^4) = 2 - x + 5x^2 + 3x^3$$

Theorem

If $T_1 : U \to V$ and $T_2 : V \to W$ are one-to-one linear transformations, then:

1. $T_2 \circ T_1$ *is one-to-one.*
2. $(T_2 \circ T_1)^{-1} = T_1^{-1} \circ T_2^{-1}$.

Isomorphisms

Isomorphisms

Definition

A linear transformation $T : V \to W$ that is both one-to-one and onto is called an *isomorphism*.

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Definition

If there exists an isomorphism $V \to W$, then we say that V and W are *isomorphic*.

Remark

$$\begin{array}{c} T : V \to W \\ \text{is an isomorphism} \end{array} \implies \begin{array}{c} T^{-1} : W \to V \\ \text{is also an isomorphism} \end{array}$$

(you prove)

What does isomorphic really mean?

Consider the following isomorphism $T : \mathbb{P}^2 \to \mathbb{R}^3$

$$a_0 + a_1 x + a_2 x^2 \quad \underset{T^{-1}}{\overset{T}{\rightleftarrows}} \quad (a_0, a_1, a_2)$$

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$$a_0 + a_1 x + a_2 x^2 \quad \underset{T^{-1}}{\overset{T}{\rightleftarrows}} \quad (a_0, a_1, a_2)$$

On the left we have a polynomial function. On the right we have a vector. But the numbers a_0, a_1 and a_2 are the same.

Isomorphisms

The behaviour under vector addition and scalar multiplication are also the same:

$$(a_0 + a_1x + a_2x^2) + (b_0 + b_1x + b_2x^2) \quad \underset{T^{-1}}{\overset{T}{\rightleftarrows}} \quad (a_0, a_1, a_2) + (b_0, b_1, b_2)$$

$$(a_0 + b_0) + (a_1 + b_1)x + (a_2 + b_2)x^2 \quad \underset{T^{-1}}{\overset{T}{\rightleftarrows}} \quad (a_0 + b_0, a_1 + b_1, a_2 + b_2)$$

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$$(a_0 + b_0) + (a_1 + b_1)x + (a_2 + b_2)x^2 \quad \underset{T^{-1}}{\overset{T}{\rightleftarrows}} \quad (a_0 + b_0, a_1 + b_1, a_2 + b_2)$$

and

$$k(a_0 + a_1x + a_2x^2) \quad \underset{T^{-1}}{\overset{T}{\rightleftarrows}} \quad k(a_0, a_1, a_2)$$

$$ka_0 + ka_1x + ka_2x^2 \quad \underset{T^{-1}}{\overset{T}{\rightleftarrows}} \quad (ka_0, ka_1, ka_2)$$

Remark

In the Greek language,

- ‘morphe’ means ‘form’; and
- ‘iso’ means ‘identical’.

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An Important Theorem

Theorem ($\star\ \star\ \star\ \star\ \star$)

Every real n-dimensional vector space is isomorphic to $\mathbb{R}^n$.

Isomorphisms

Theorem

If $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is a basis for a vector space V, then the coordinate map

$$T : \mathbf{u} \mapsto (\mathbf{u})_S$$

is an isomorphism between V and $\mathbb{R}^n$.

Isomorphisms

Theorem

If $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ *is a basis for a vector space* V*, then the coordinate map*

$$T : \mathbf{u} \mapsto (\mathbf{u})_S$$

is an isomorphism between V *and* $\mathbb{R}^n$*.*

Example (The Natural Isomorphism Between $\mathbb{P}^{n-1}$ and $\mathbb{R}^n$)

The linear transformation $T : \mathbb{P}^{n-1} \to \mathbb{R}^n$,

$$T : a_0 + a_1x + a_2x^2 + \ldots + a_{n-1}x^{n-1} \mapsto (a_0, a_1, a_2, \ldots, a_{n-1})$$

is an isomorphism between $\mathbb{P}^{n-1}$ and $\mathbb{R}^n$. This is called the *natural isomorphism* between these vector spaces.

Example (The Natural Isomorphism Between $\mathbb{R}^{2\times 2}$ and $\mathbb{R}^4$)

The coordinate map

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \mapsto (a, b, c, d)$$

defines an isomorphism between $\mathbb{R}^{2\times 2} = M_{22}$ and $\mathbb{R}^4$. We call this the *natural isomorphism between* $\mathbb{R}^{2\times 2}$ *and* $\mathbb{R}^4$.

Working with Isomorphisms

Example

Are the following polynomials linearly independent?

$$\mathbf{p}_1 = 1 + 2x - 3x^2 + 4x^3 + x^5$$
$$\mathbf{p}_2 = 1 + 3x - 4x^2 + 6x^3 + 5x^4 + 4x^5$$
$$\mathbf{p}_3 = 3 + 8x - 11x^2 - 16x^3 + 10x^4 + 9x^5$$

Isomorphisms

We will use the fact that $\mathbb{P}^5$ is isomorphic to $\mathbb{R}^6$.

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The following two questions are equivalent: Are the following polynomials linearly independent?

$$\begin{array}{rcrcrcrcrcrcr} \mathbf{p}_1 & = & 1 & + & 2x & - & 3x^2 & + & 4x^3 & & & + & x^5 \\ \mathbf{p}_2 & = & 1 & + & 3x & - & 4x^2 & + & 6x^3 & + & 5x^4 & + & 4x^5 \\ \mathbf{p}_3 & = & 3 & + & 8x & - & 11x^2 & - & 16x^3 & + & 10x^4 & + & 9x^5 \end{array}$$

Are the following vectors linearly independent?

$$\begin{array}{crrrrrrc} (& 1, & 2, & -3, & 4, & 0 & 1 &) \\ (& 1, & 3, & -4, & 6, & 5, & 4 &) \\ (& 3, & 8, & -11, & -16, & 10, & 9 &) \end{array}$$

Another equivalent question is: Is the row space of the following matrix 3 dimensional?

$$A = \begin{bmatrix} 1 & 2 & -3 & 4 & 0 & 1 \\ 1 & 3 & -4 & 6 & 5 & 4 \\ 3 & 8 & -11 & 16 & 10 & 9 \end{bmatrix}$$

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$$A = \begin{bmatrix} 1 & 2 & -3 & 4 & 0 & 1 \\ 1 & 3 & -4 & 6 & 5 & 4 \\ 3 & 8 & -11 & 16 & 10 & 9 \end{bmatrix}$$

I leave it to you to check that A is row equivalent to

$$R = \begin{bmatrix} 1 & 2 & -3 & 4 & 0 & 1 \\ 0 & 1 & -1 & 2 & 5 & 3 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

We can see that the answer to all three questions is **NO**.

4. (a) $\boxed{\text{10 points}}$ Let $L : \mathbb{R}^3 \to \mathbb{R}^4$ be a linear transformation defined by $L\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} 2y+3z \\ x+y-2z \\ 4x+y \\ 3x-y-z \end{bmatrix}$. Find the matrix representation of L.

(b) $\boxed{\text{15 points}}$ Define $T : \mathbb{P}_2 \to \mathbb{R}^2$ by $T(\mathbf{p}) = \begin{bmatrix} \mathbf{p}(0) \\ \mathbf{p}(1) \end{bmatrix}$. Find a polynomial $\mathbf{p} \in \mathbb{P}_2$ which is a basis for kernel of T.

Solution:

$$\ker T = \{\mathbf{p} : \mathbf{p} \in \mathbb{P}_2 \text{ and } T(\mathbf{p}) = \mathbf{0}\}$$

$$\mathbf{p}(t) = a + bt + ct^2 \Rightarrow T(\mathbf{p}(t)) = \begin{bmatrix} \mathbf{p}(0) \\ \mathbf{p}(1) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} \mathbf{p}(0) \\ \mathbf{p}(1) \end{bmatrix} = \begin{bmatrix} a \\ a+b+c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{matrix} a = 0 \\ b + c = 0 \end{matrix} \Rightarrow \mathbf{p}(t) = -ct + ct^2$$

$$\ker T = \{\mathbf{p} : \mathbf{p}(t) = (-t + t^2)\, c, c \in \mathbb{R}\} = \operatorname{Span}\{-t + t^2\}$$

$$\mathbf{p}(t) = -t + t^2$$

Next Time

- Matrices for Linear Transformations
- Similarity
- Complex Numbers