



Welcome to

Mathematics IV (Differential Equations)

with Dr Neil Course

Lecture 1

- Information about this course
- 1.1 Introduction
- 1.2 Some Examples
- 1.3 How to Draw a Direction Field
- 1.4 Solving Our First Differential Equations

Information about this course

- ≈ 12 classes. Thursday afternoons 4pm-6pm.

16:00

17:00

18:00

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- Each lecture \approx 60 minutes.

lecture

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18:00

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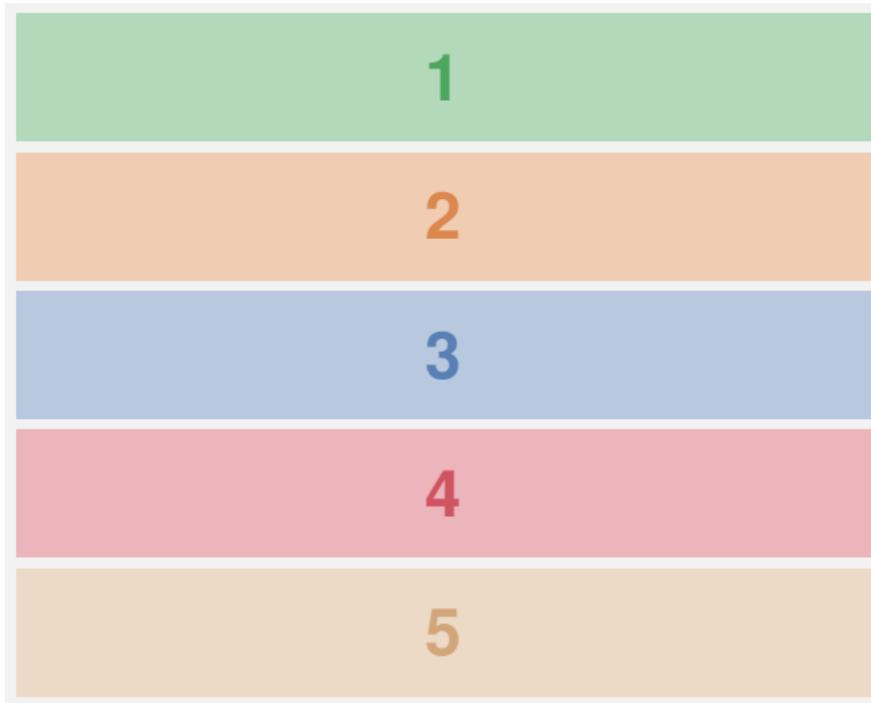
lecture

questions

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Introduction

Examples; Directions Fields; Classification.

2

3

4

5

Introduction

Examples; Directions Fields; Classification.

First Order Differential Equations

3

4

5

Introduction

Examples; Directions Fields; Classification.

First Order Differential Equations

Second and Higher Order Linear ODEs

4

5

Introduction

Examples; Directions Fields; Classification.

First Order Differential Equations

Second and Higher Order Linear ODEs

The Laplace Transform

Introduction

Examples; Directions Fields; Classification.

First Order Differential Equations

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The Laplace Transform

Systems of First Order Linear ODEs

Introduction

Examples; Directions Fields; Classification.

First Order Differential Equations

3 lectures

Second and Higher Order Linear ODEs

3 lectures

The Laplace Transform

3 lectures

Systems of First Order Linear ODEs

3 lectures



Lecture Notes

The image shows the front cover of a lecture note book. At the top, there is a blue banner with white text that reads 'First Order • Second and Higher Order • Laplace Transform • Systems'. Below this, a diagonal blue banner contains the text 'Autumn 2021-22'. The main title 'Differential Equations' is written in large, bold, blue letters on a white background. Below the title, the author's name 'Neil Course' is printed in a smaller, white font. The background of the cover features a photograph of several colored pencils (yellow, orange, purple, grey, and blue) arranged diagonally.

First Order • Second and Higher Order • Laplace Transform • Systems

Autumn
2021-22

Differential
Equations

Neil Course



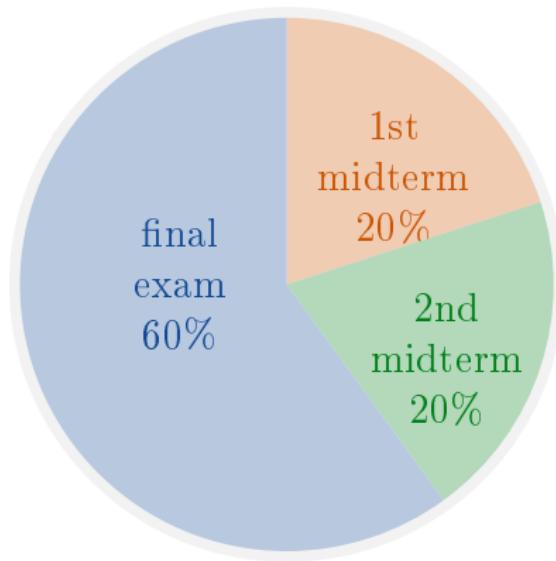
Exams

(This information may change based on the University's decisions)



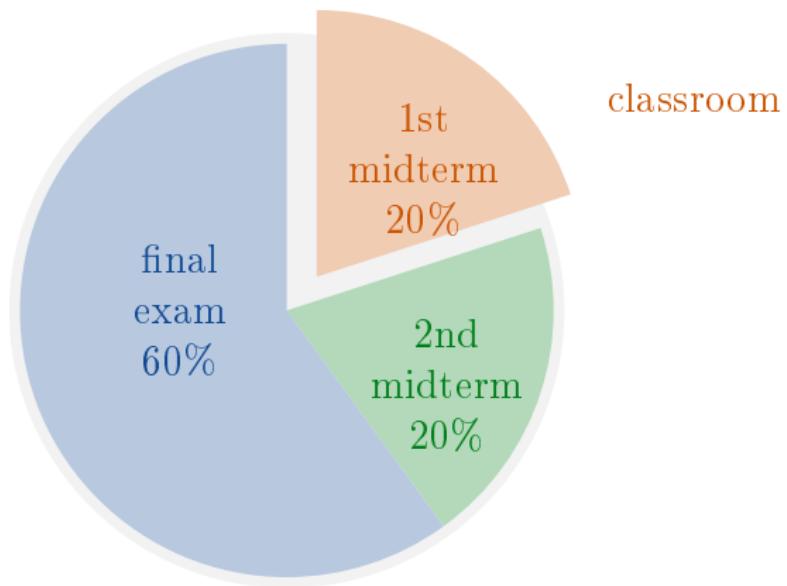
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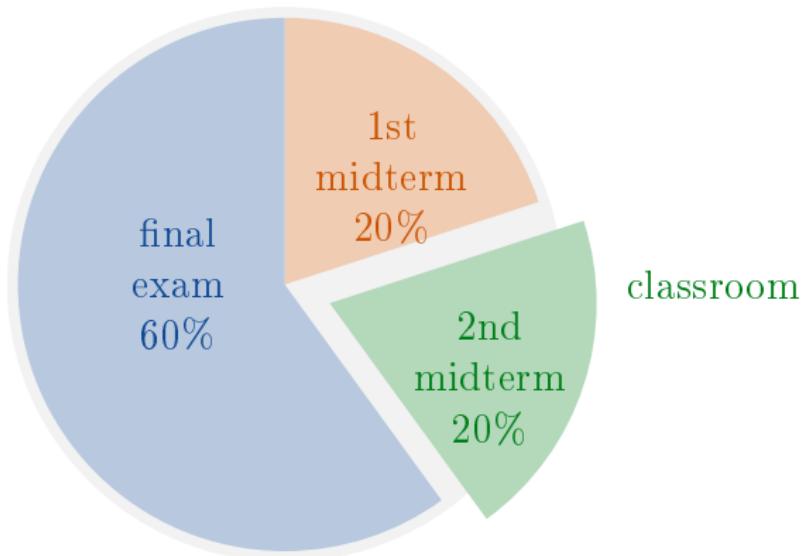
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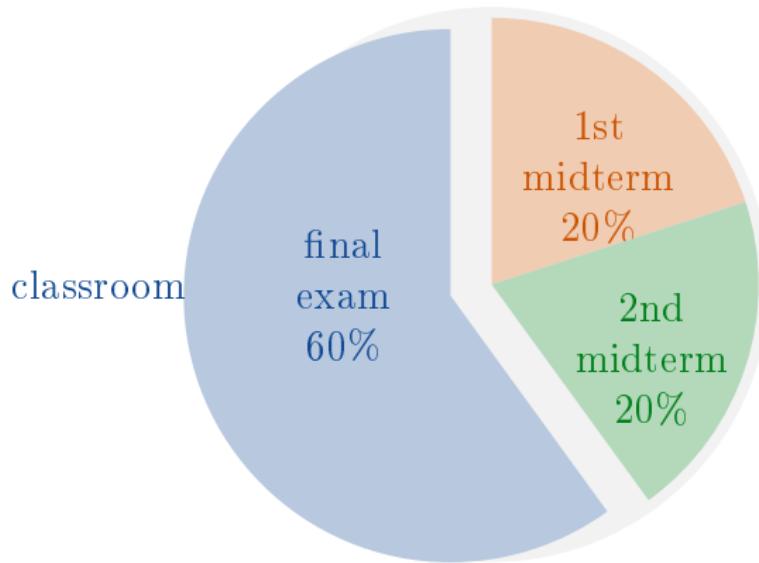
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Expectations

If this was a classroom course, the expectation is that for each hour of lectures, you would study an extra 1-2 hours outside of class.

classroom
course

lectures (4 hours)

other study (4-8 hours)

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For an online course, you are still expected to study a total of 8-12 hours each week.

online
course

class
(2 hours)

other study (6-10 hours)

This may include:

- Rewatch the recorded lectures (O'Learn & YouTube);

⋮

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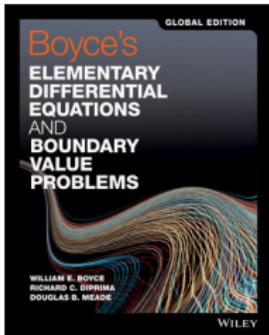
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- Watch online videos;

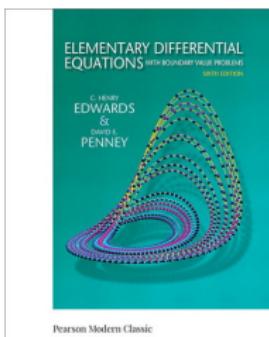
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Two good books



William E. Boyce, Richard C. DiPrima and Douglas B. Meade,

Boyce's Elementary Differential Equations and Boundary Value Problems,
Wiley.

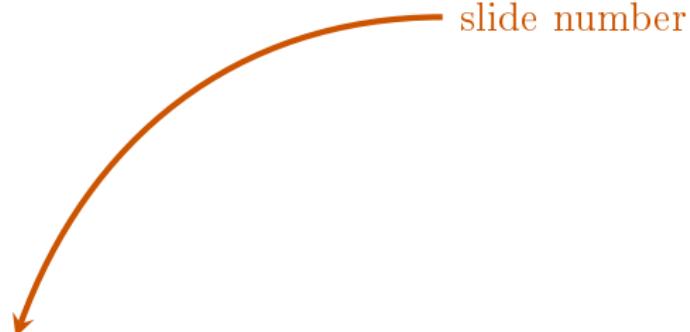


C. Henry Edwards and David E. Penney,
Elementary Differential Equations with Boundary Value Problems,
Pearson.

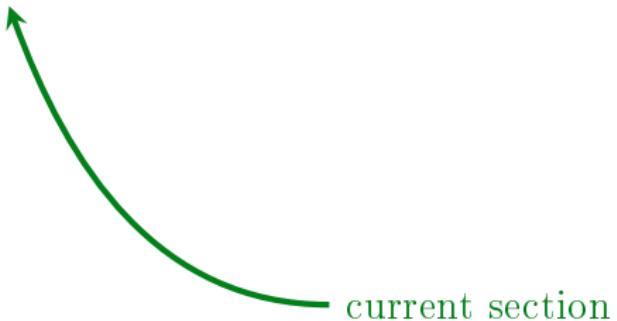
9.9 Section Title



slide number



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Introduction

1.1 Introduction



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1.1 Introduction



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But if I say “Solve $\frac{dy}{dx} = 2x$ ”, then I don’t want a number; I am looking for a function $y(x)$ which satisfies this equation.

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A *differential equation* is an equation containing a derivative.

For example, the equation $\frac{dy}{dx} = 2x$ contains a derivative – so it is a differential equation.

1.1 Introduction

Example

Solve $\frac{dy}{dx} = 2x$.

1.1 Introduction

Example

Solve $\frac{dy}{dx} = 2x$.

This differential equation is easy to solve:

$$y(x) = \int \frac{dy}{dx} dx = \int 2x dx = x^2 + c.$$

1.1 Introduction

Example

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Therefore the solution to the IVP is

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1.1 Introduction



Example

Solve $\frac{dy}{dx} = y$.

1.1 Introduction

Example

Solve $\frac{dy}{dx} = y$.

This is harder: This time we can't just integrate $\frac{dy}{dx}$ to find $y(x)$.
I will show you how to solve this later.

Some Examples

1.2 Some Examples

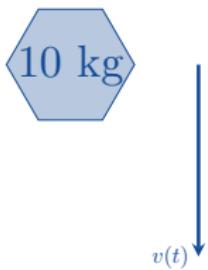


Many problems in engineering, science and the social sciences can be modelled using differential equations. We start with 3 examples.

1.2 Some Examples

Example (A Falling Object)

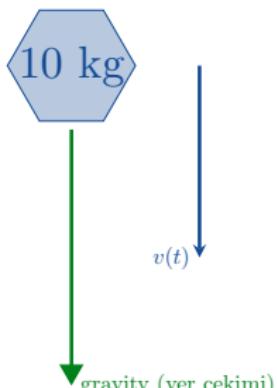
Suppose that an object of mass 10 kg is falling.



1.2 Some Examples

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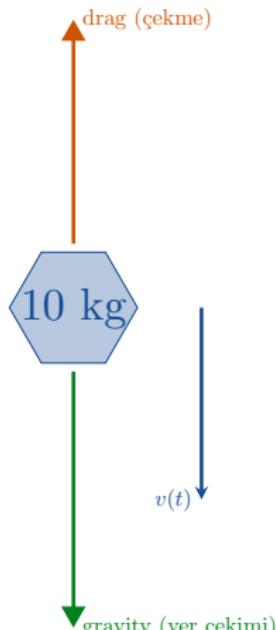
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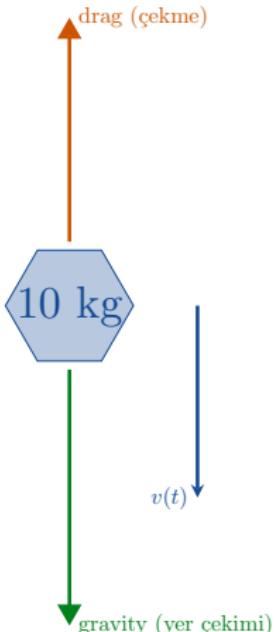
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1.2 Some Examples

Example (A Falling Object)

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Let

- $v(t)$ denote the velocity (downwards) of the object in ms^{-1} ; and
- t denote time in seconds.

1.2 Some Examples



Newton's Second Law says

$$\text{force} = \text{mass} \times \text{acceleration}$$

1.2 Some Examples



Newton's Second Law says

force = mass × acceleration

$$= 10 \times \frac{dv}{dt}.$$

1.2 Some Examples

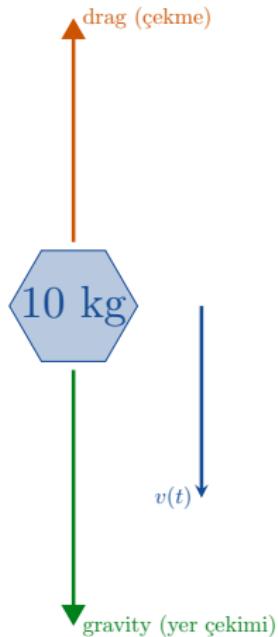


Newton's Second Law says

$$\begin{aligned} \text{force} &= \text{mass} \times \text{acceleration} \\ &= 10 \times \frac{dv}{dt}. \end{aligned}$$

Note that $\frac{dv}{dt}$ is measured in $\frac{\text{ms}^{-1}}{\text{s}} = \text{ms}^{-2}$.

1.2 Some Examples



Now

$$\text{force} = \text{gravity} - \text{drag}.$$

1.2 Some Examples



On the Earth, the gravity on an object of mass 10 kg is approximately

$$\text{gravity} = 10g$$

(where $g = 9.8 \text{ ms}^{-2}$).

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(where $g = 9.8 \text{ ms}^{-2}$). It is reasonable to assume (if the object isn't travelling too quickly) that

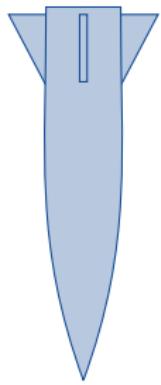
drag is proportional to velocity

drag $\propto v$

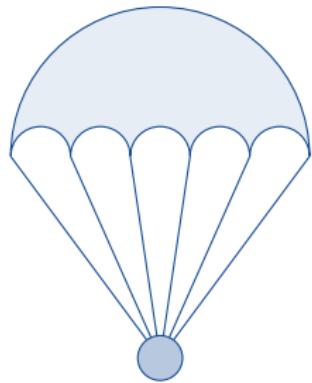
drag = γv

where $\gamma > 0$ is a constant depending on the shape of the object.

1.2 Some Examples

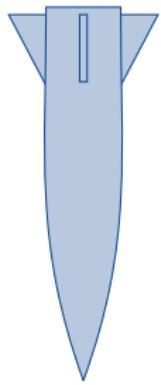


small $\gamma > 0$



big $\gamma > 0$

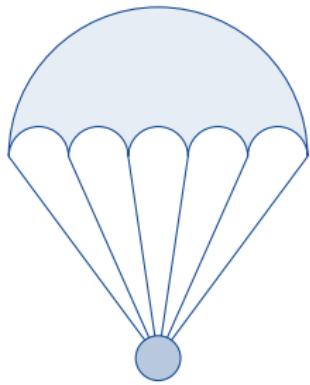
1.2 Some Examples



small $\gamma > 0$



$$\gamma = 2 \text{ kg s}^{-1}$$



big $\gamma > 0$

1.2 Some Examples

If $\gamma = 2 \text{ kg s}^{-1}$, then we have that

$$10 \frac{dv}{dt} = \text{force} = \text{gravity} - \text{drag} = 10g - \gamma v = 98 - 2v.$$

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Therefore

$$\boxed{\frac{dv}{dt} = 9.8 - \frac{v}{5}.} \quad (1)$$

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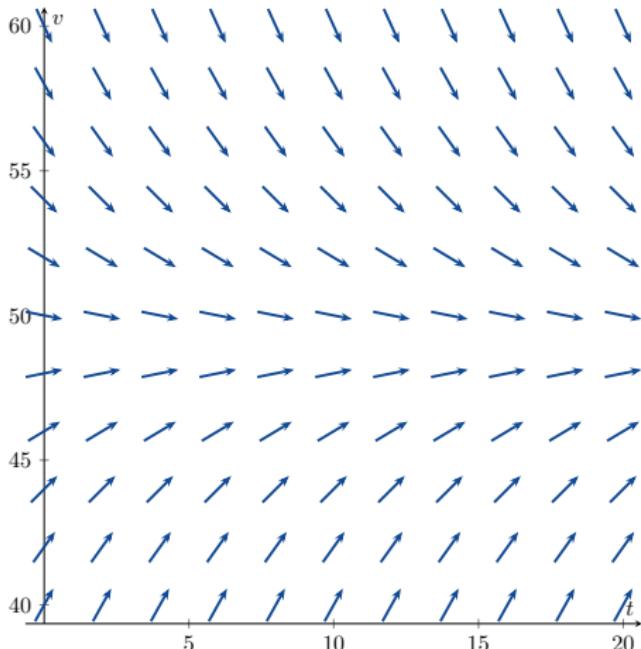
We will solve equation (1) later. First we will look at this differential equation's direction field to try to understand it.

1.2 Some Examples

A direction field is a grid of arrows in the tv -plane which show the slope of solutions to a differential equation.

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A direction field is a grid of arrows in the tv -plane which show the slope of solutions to a differential equation. A direction field for (1) looks like this:



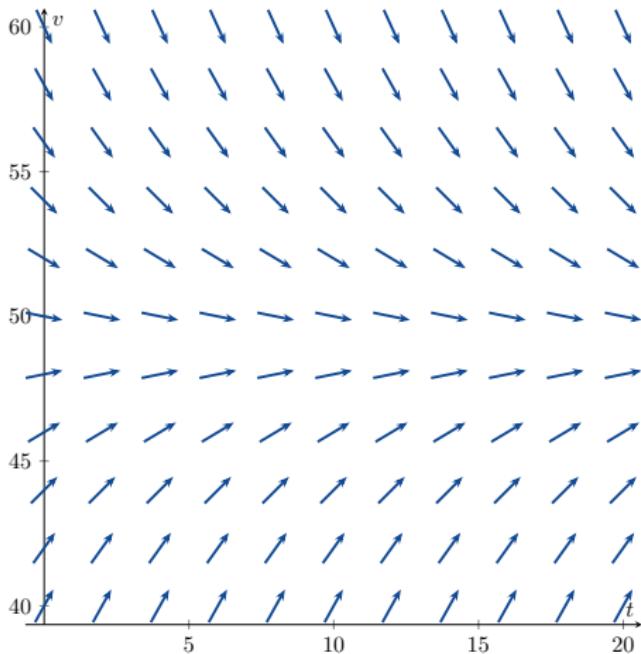
1.2 Some Examples



I will show you how to draw direction fields later. For now, I want to see what we can learn about the solutions to (1).

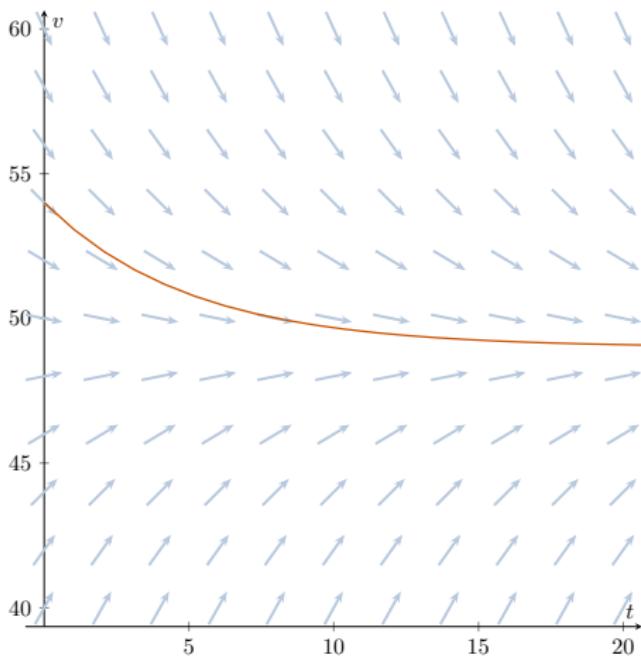
1.2 Some Examples

If we start at $v(0) = 54$ say, the arrows tell us that the solution is decreasing like this:



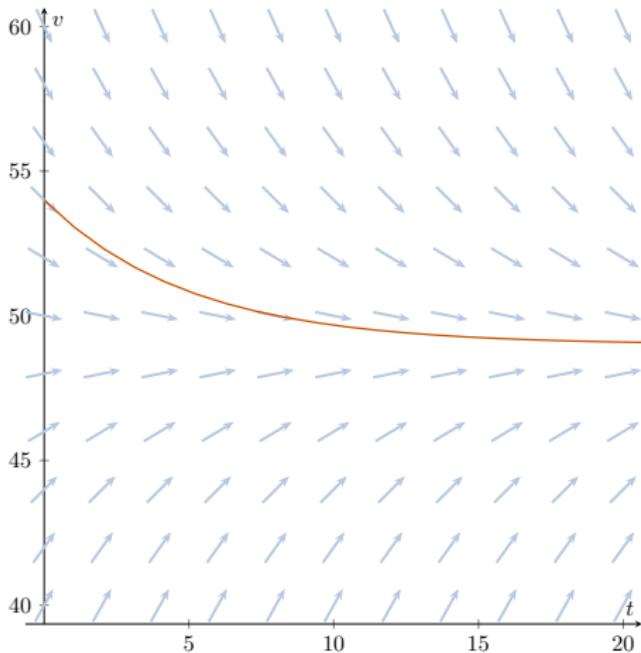
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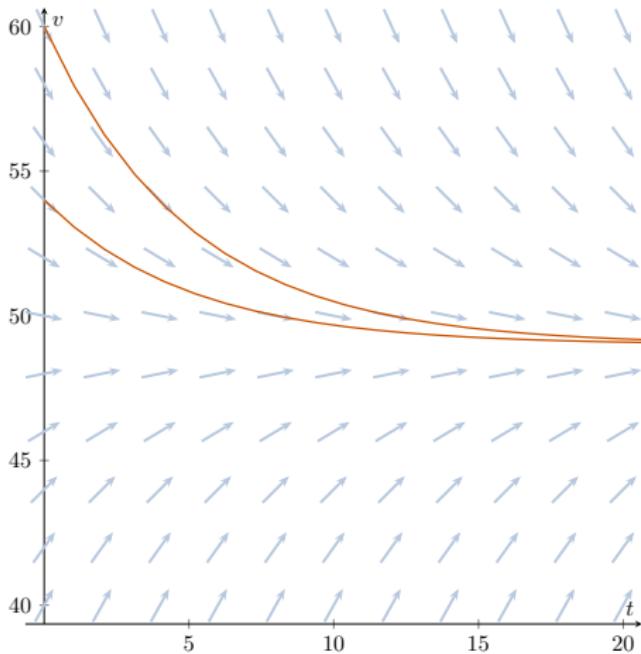
1.2 Some Examples

We can guess at some more solutions:



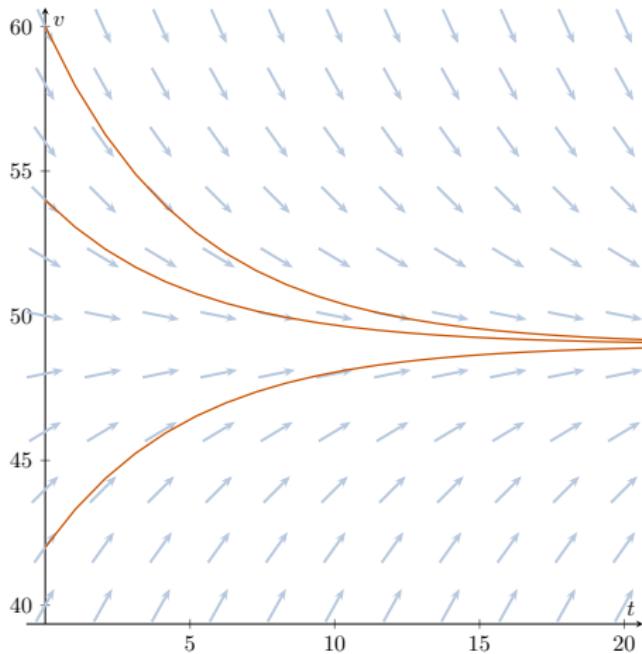
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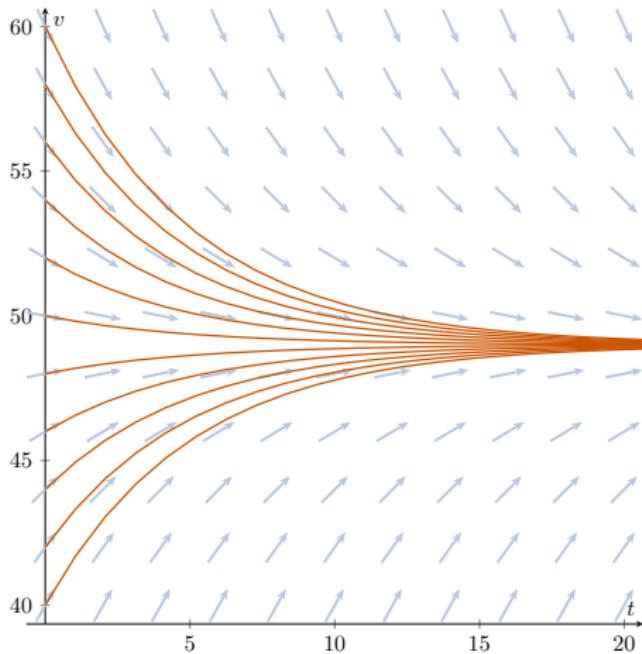
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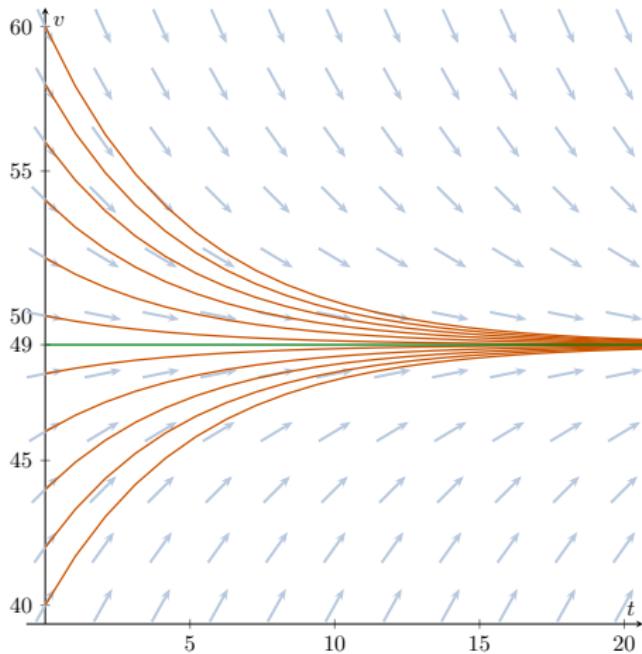
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1.2 Some Examples



Note that if $v = 49$, then we have

$$\frac{dv}{dt} = 9.8 - \frac{49}{5} = 9.8 - 9.8 = 0.$$

Hence $v(t) = 49$ is a constant solution (or *equilibrium solution*) of (1).

1.2 Some Examples

Example (Mice and Owls)

Let $p(t)$ denote the population of mice in an area, where t is measured in months.

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Let $p(t)$ denote the population of mice in an area, where t is measured in months.

We assume that there is plenty of food for the mice to eat so, if nothing eats the mice, $p(t)$ will increase at a rate proportional to $p(t)$.

$$\frac{dp}{dt} \propto p$$
$$\frac{dp}{dt} = rp$$

where $r > 0$ is a constant.

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where $r > 0$ is a constant.

Suppose that $r = 0.5$ per month. Hence

$$\frac{dp}{dt} = \frac{p}{2}.$$

1.2 Some Examples

However, suppose that 5 owls also live in this area and suppose that each owl eats 3 mice each day.

1 owl eats 3 mice per day

5 owls eat $5 \times 3 = 15$ mice per day

5 owls eat $30 \times 15 = 450$ mice per month.

1.2 Some Examples

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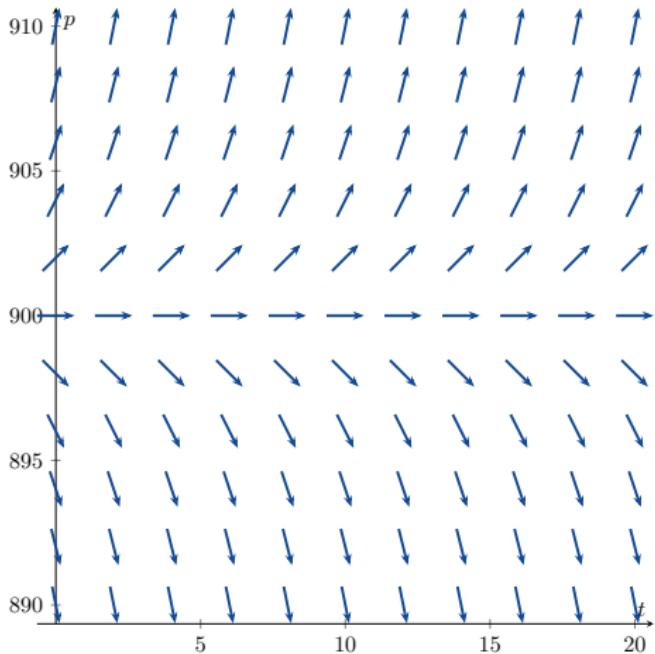
$$5 \text{ owls eat } 30 \times 15 = 450 \text{ mice per month.}$$

So we change our differential equation to

$$\boxed{\frac{dp}{dt} = \frac{p}{2} - 450.} \quad (2)$$

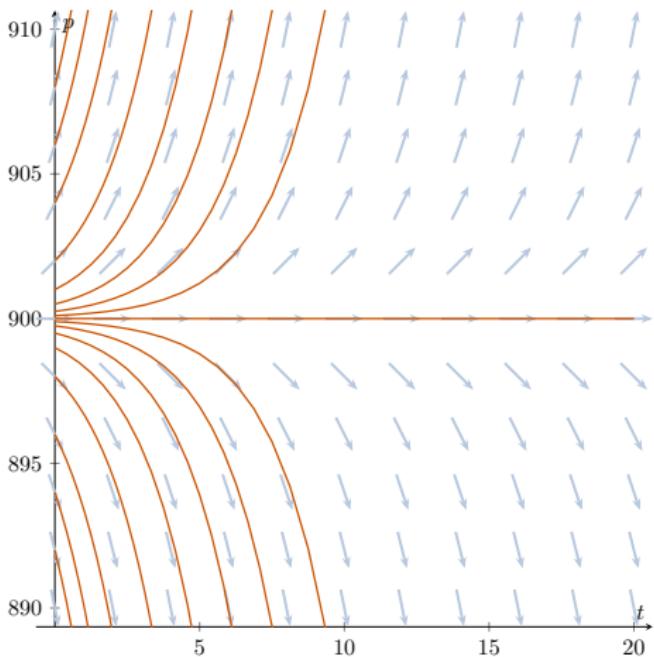
1.2 Some Examples

If we look at a direction field for (2),



1.2 Some Examples

If we look at a direction field for (2), then we can guess at some solutions:



1.2 Some Examples



Example (A cup of coffee)

Newton's law of cooling states that; the temperature of an object changes at a rate proportional to the difference between its temperature and that of its surroundings.

1.2 Some Examples

Example (A cup of coffee)

Newton's law of cooling states that; the temperature of an object changes at a rate proportional to the difference between its temperature and that of its surroundings.

Suppose that the temperature of your cup of coffee obeys Newton's law of cooling; suppose that it has a temperature of 90°C when freshly poured; and suppose that the temperature of your room is 20°C .

1.2 Some Examples



Example (A cup of coffee)

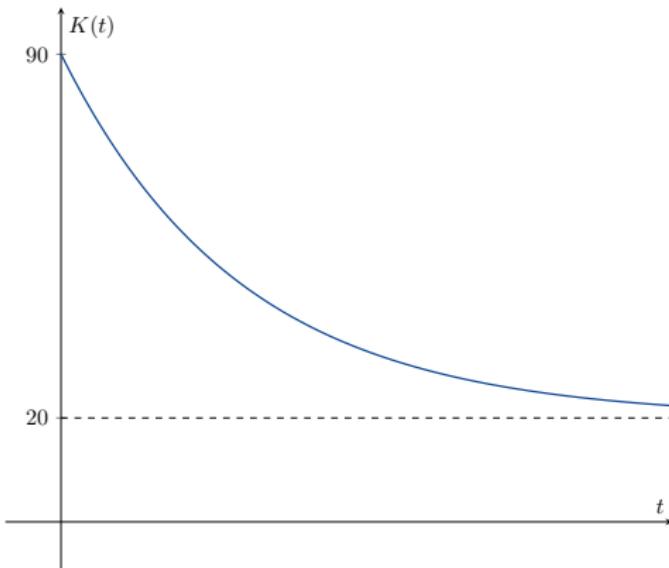
Newton's law of cooling states that; the temperature of an object changes at a rate proportional to the difference between its temperature and that of its surroundings.

Suppose that the temperature of your cup of coffee obeys Newton's law of cooling; suppose that it has a temperature of 90°C when freshly poured; and suppose that the temperature of your room is 20°C .

Write a differential equation for the temperature of your coffee.

1.2 Some Examples

We expect the cup of coffee to cool like this:



When the coffee is hot, it will cool quickly. When it is just above 20°C , it will cool slowly.

1.2 Some Examples

Let $K(t)$ denote the temperature of the coffee in $^{\circ}\text{C}$ and let t denote time measured in minutes. Then we know that

$$\frac{dK}{dt} \propto (20 - K).$$

It follows that

$$\boxed{\frac{dK}{dt} = r(20 - K)} \quad (3)$$

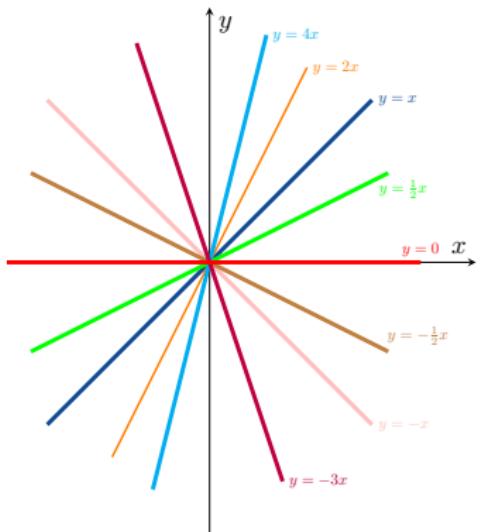
for some constant r . Since hot coffee cools down (and cold coffee warms up), we must have $r > 0$.



How to Draw a Direction Field

1.3 How to Draw a Direction Field

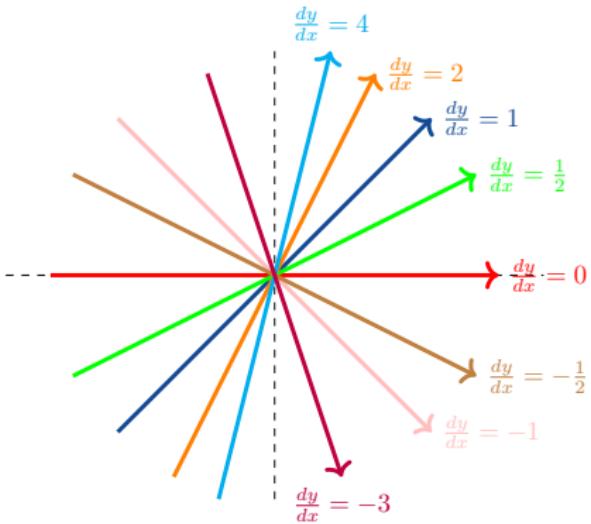
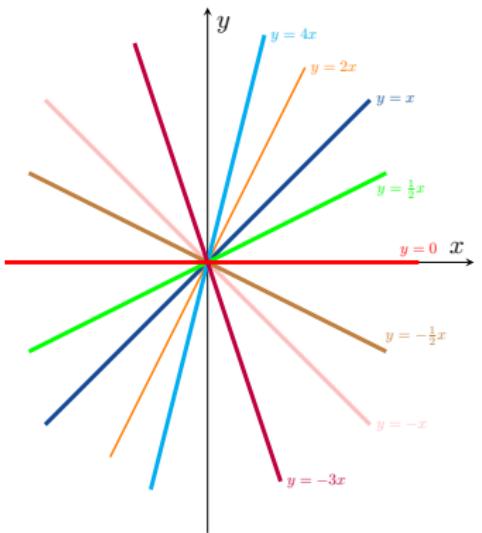
Consider the graphs of $y = mx$ for different values of $m \in \mathbb{R}$.
E.g. $y = 2x$ slopes upwards with slope 2.



1.3 How to Draw a Direction Field

Consider the graphs of $y = mx$ for different values of $m \in \mathbb{R}$.

E.g. $y = 2x$ slopes upwards with slope 2. We will use rightwards arrows to show the slope of solutions of differential equations at various points.



1.3 How to Draw a Direction Field

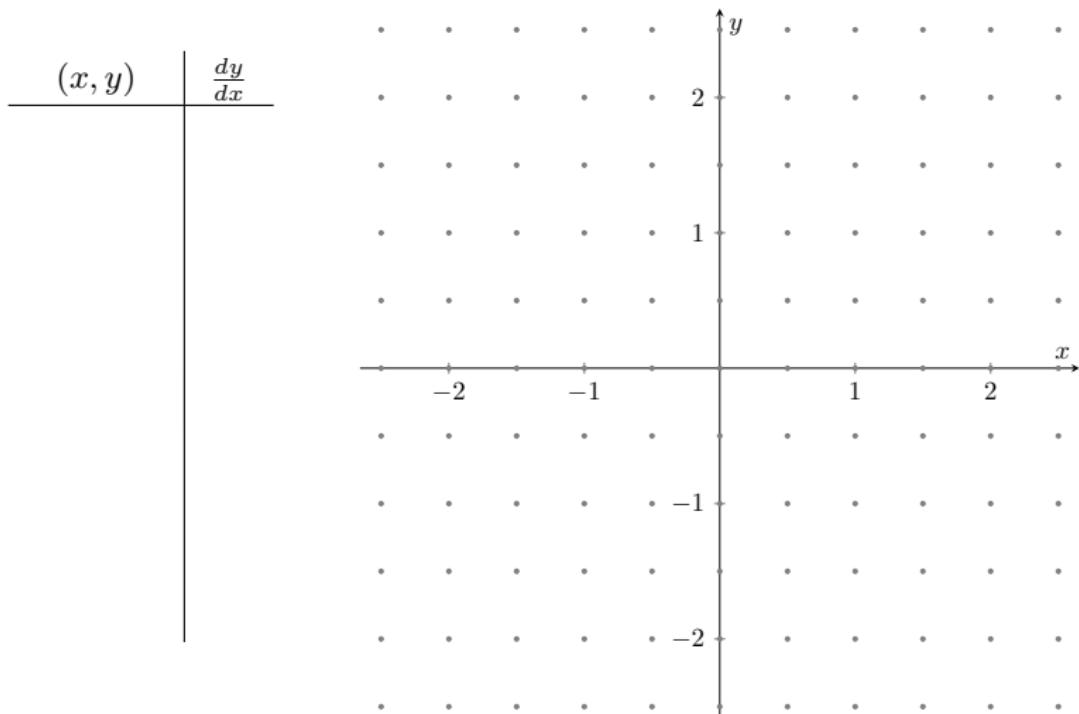
Example

Draw a direction field for $\frac{dy}{dx} = x + y$.

1.3 How to Draw a Direction Field

Example

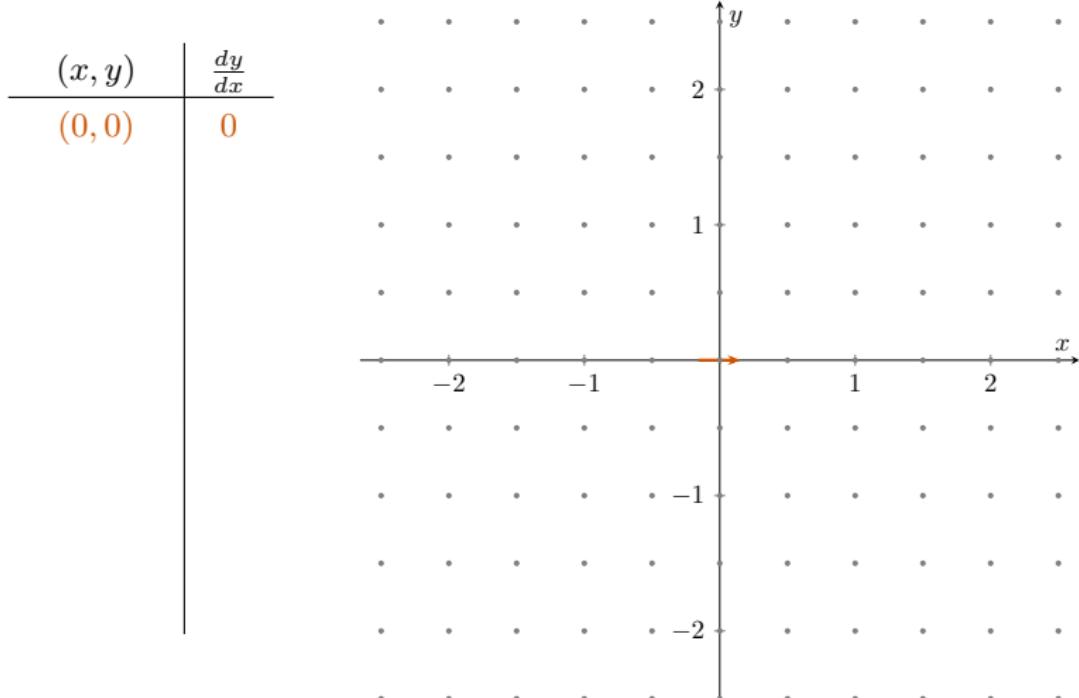
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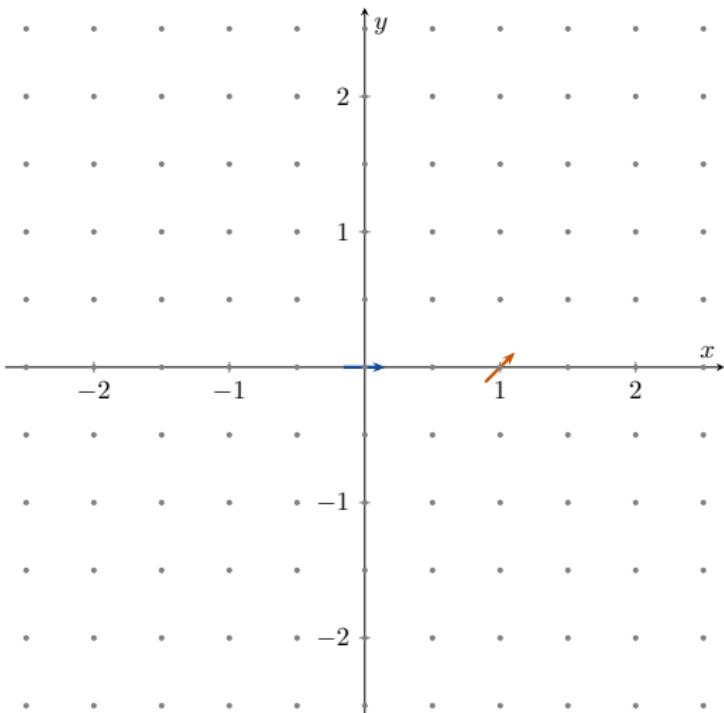


1.3 How to Draw a Direction Field

Example

Draw a direction field for $\frac{dy}{dx} = x + y$.

| (x, y) | $\frac{dy}{dx}$ |
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| $(0, 0)$ | 0 |
| $(1, 0)$ | 1 |

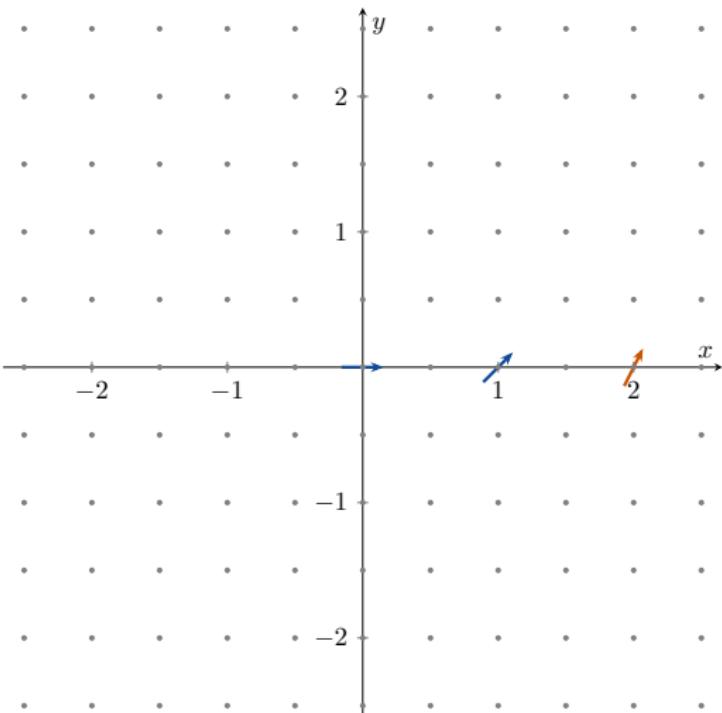


1.3 How to Draw a Direction Field

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| (x, y) | $\frac{dy}{dx}$ |
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| $(0, 0)$ | 0 |
| $(1, 0)$ | 1 |
| $(2, 0)$ | 2 |

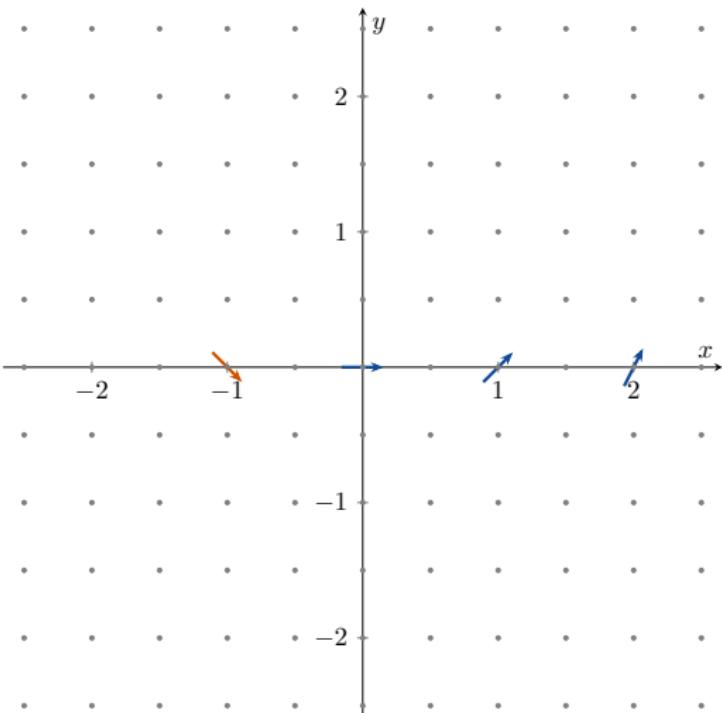


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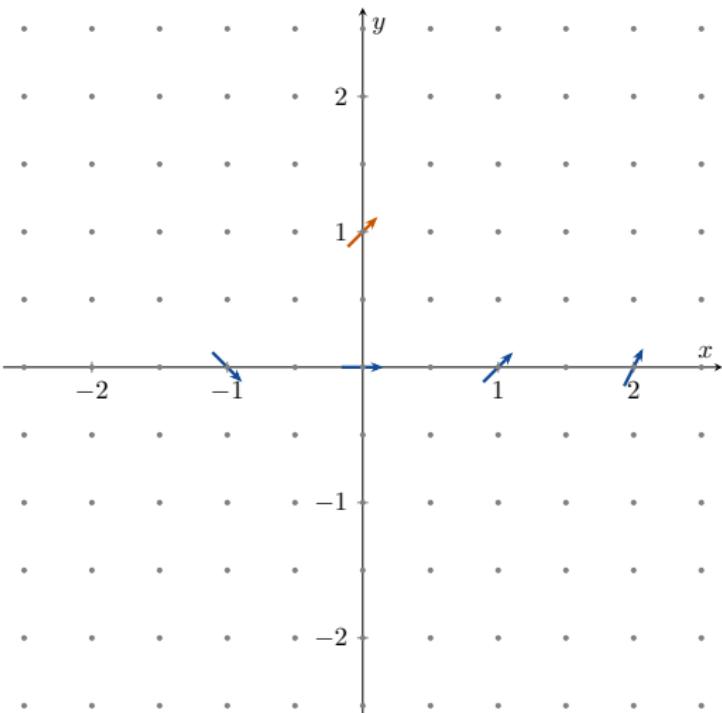


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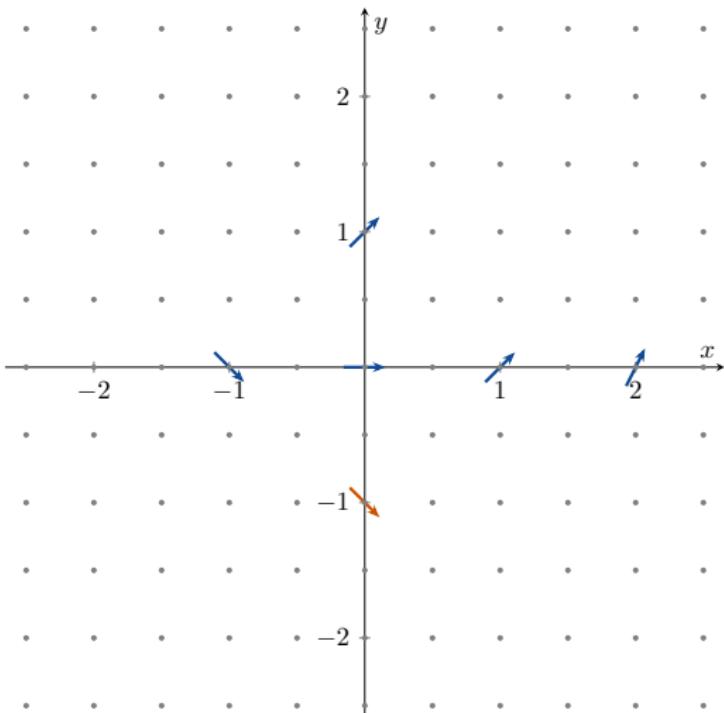


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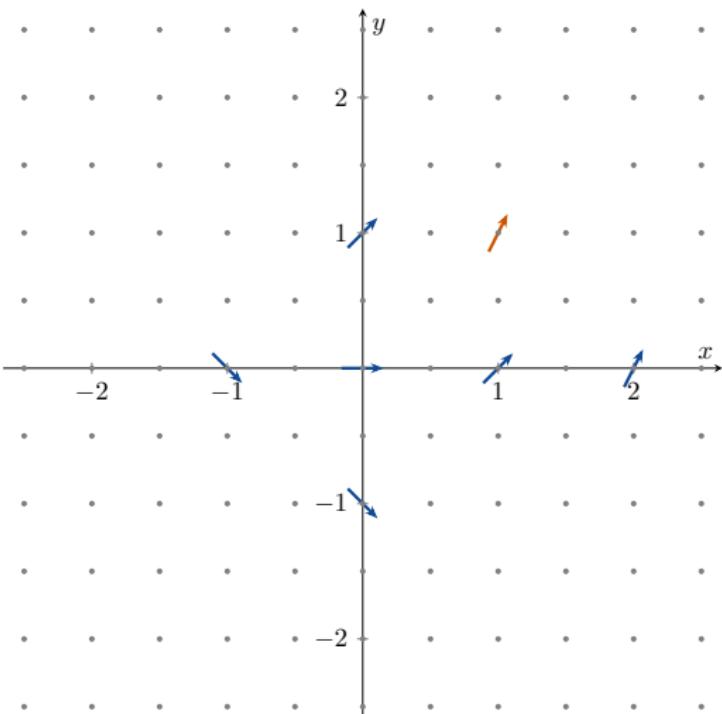


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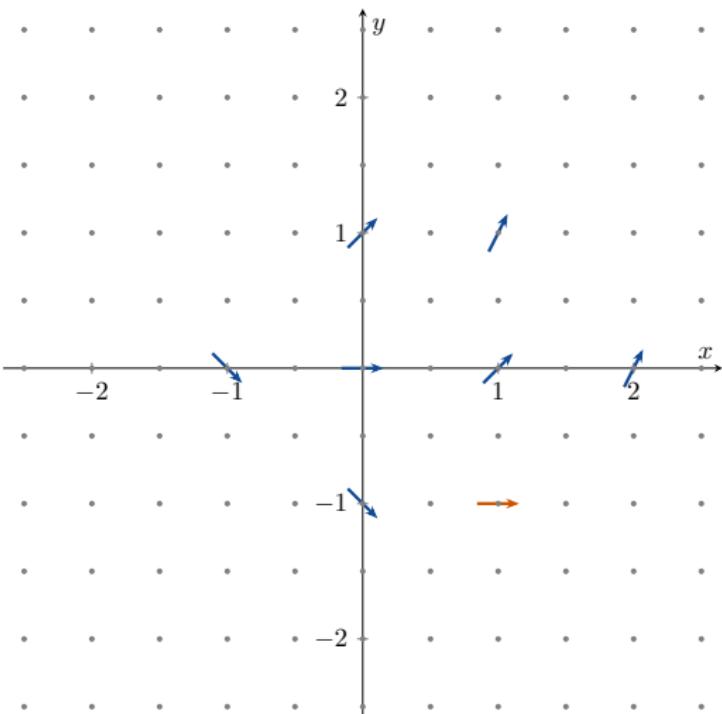


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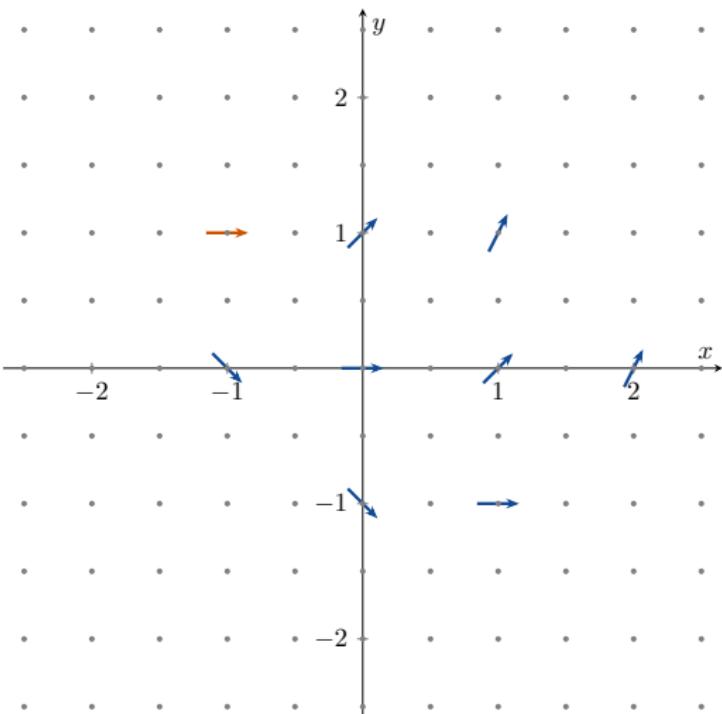


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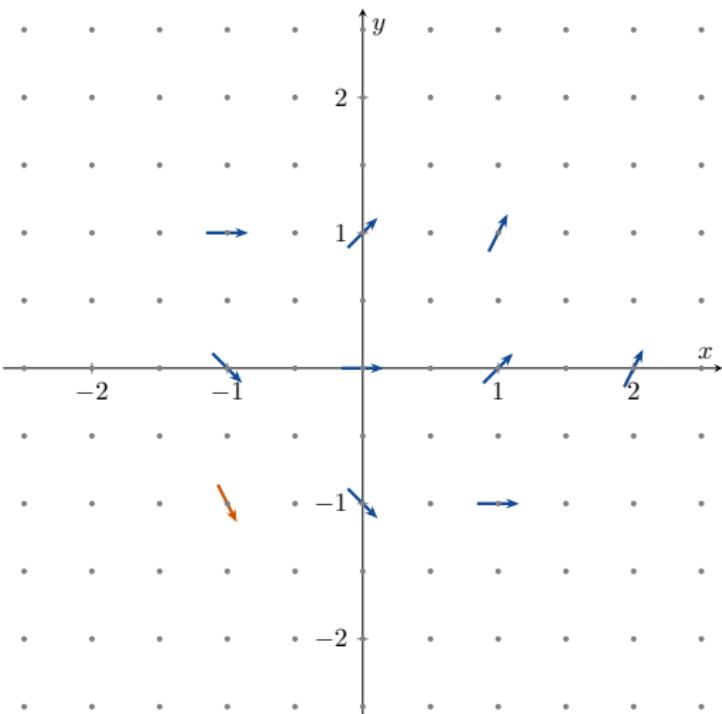


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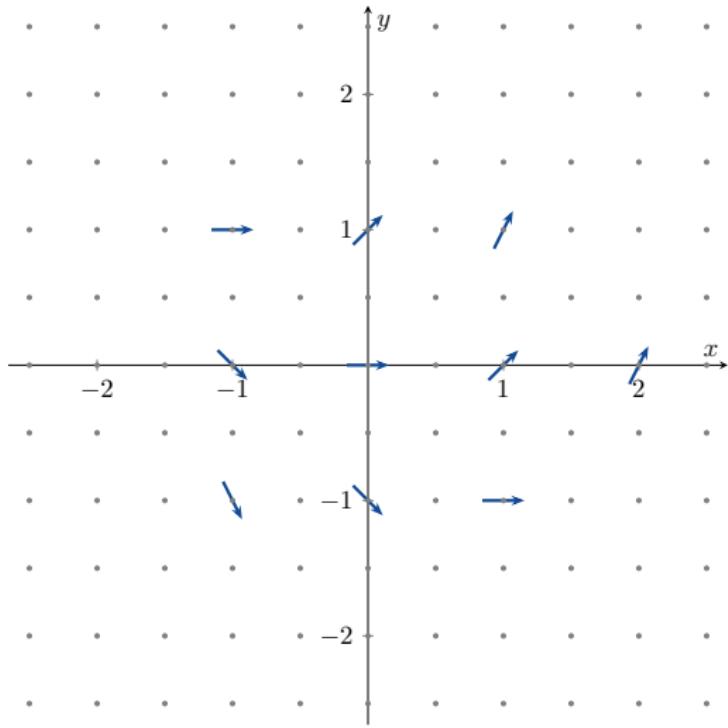
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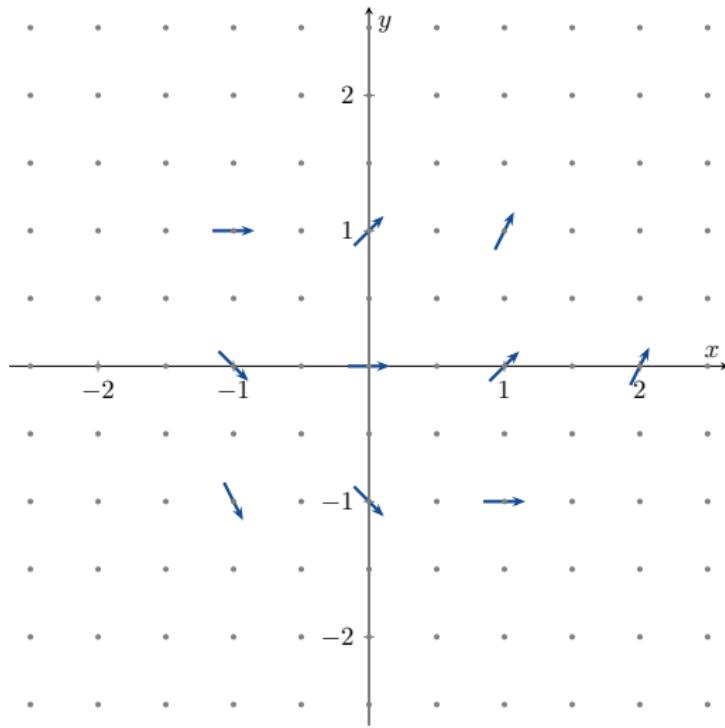
1.3 How to Draw a Direction Field



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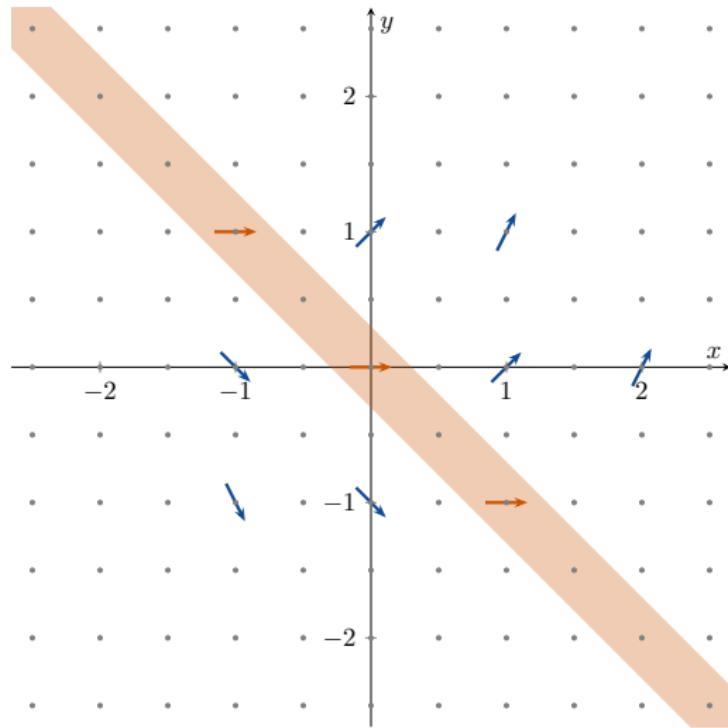
Then we look for patterns and guess!!!



1.3 How to Draw a Direction Field



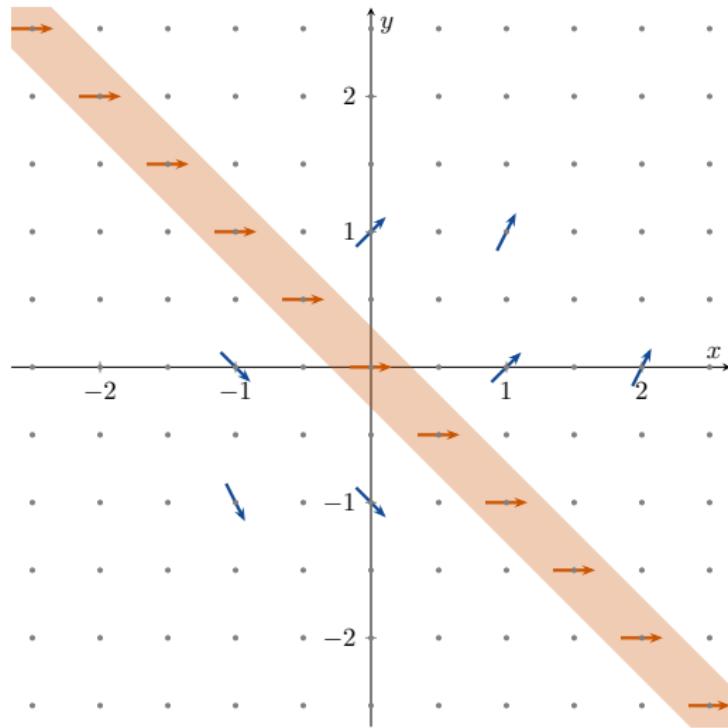
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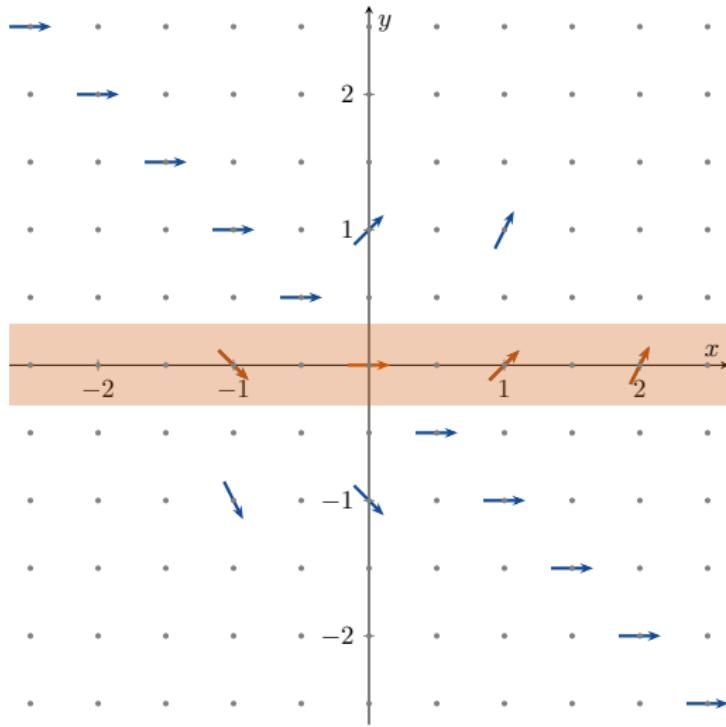
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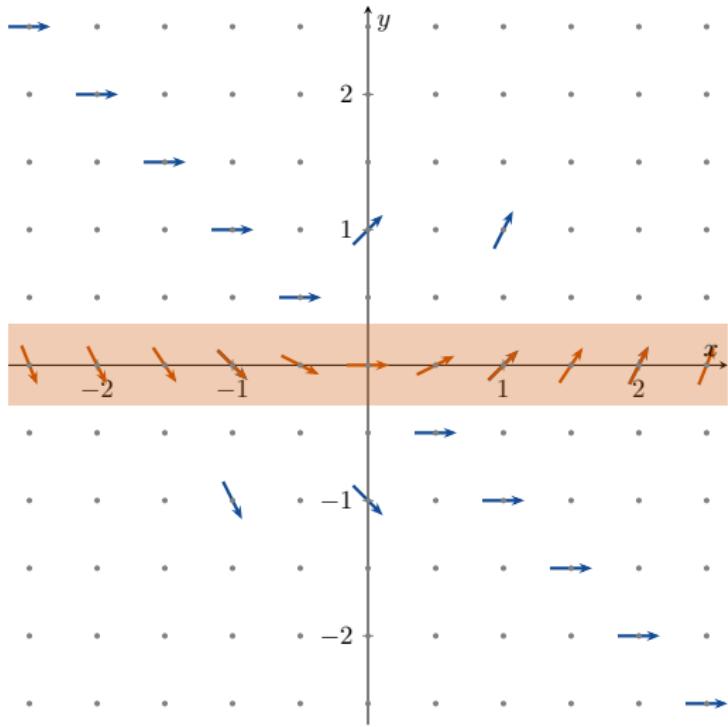
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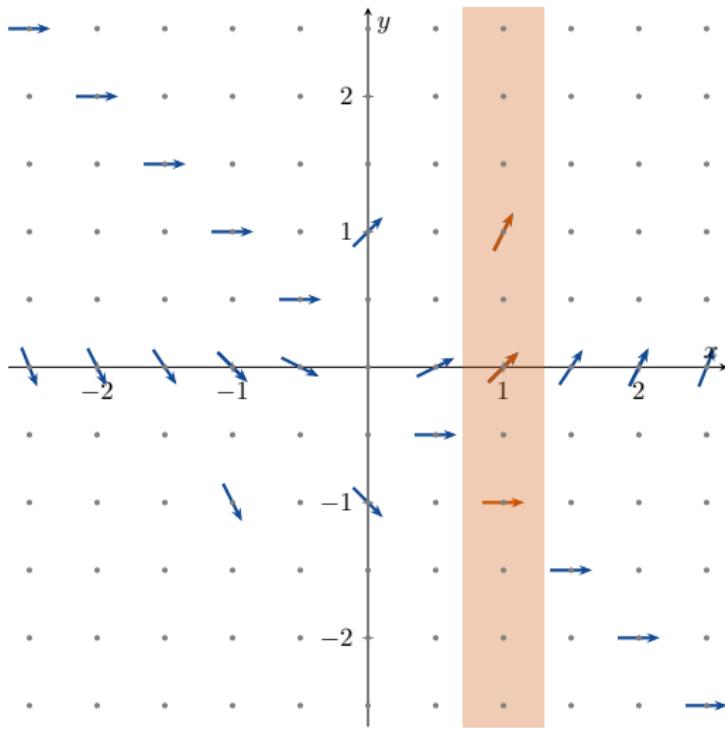
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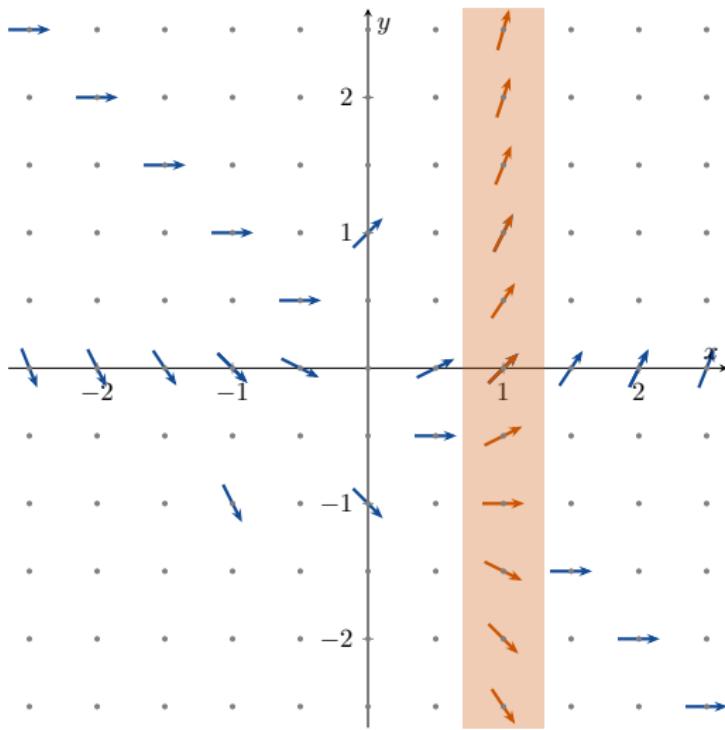
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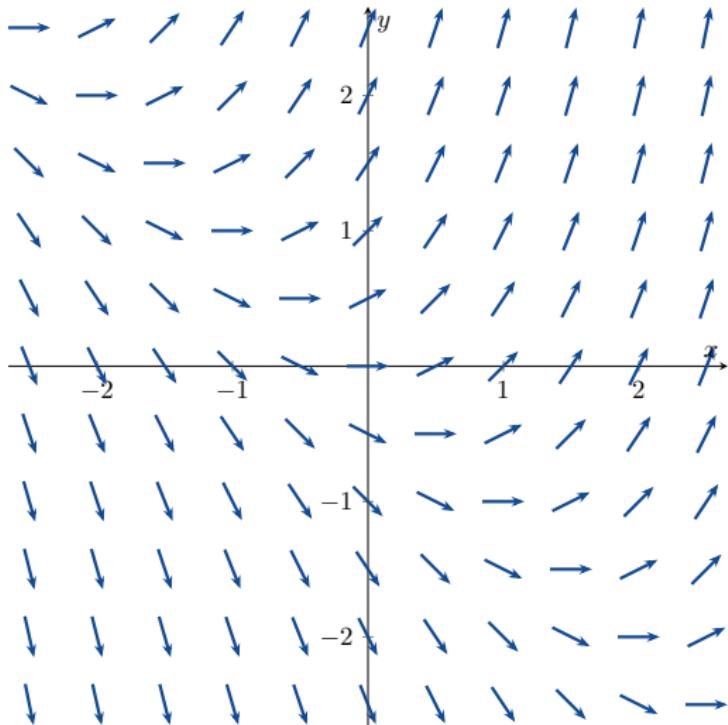
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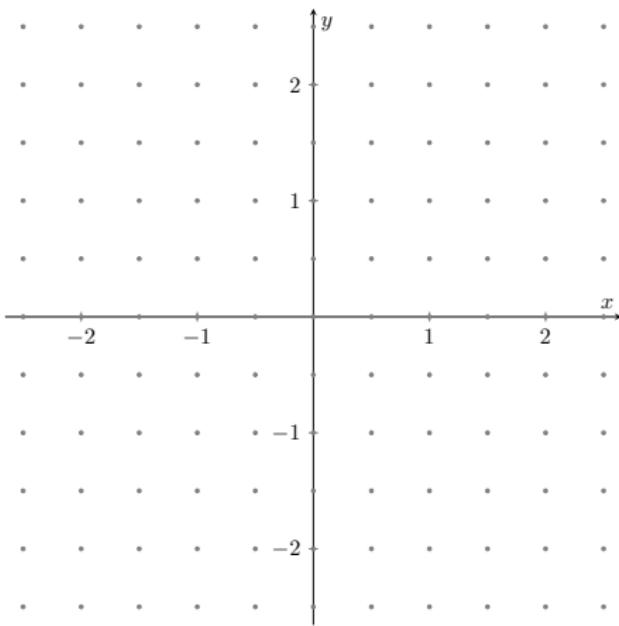
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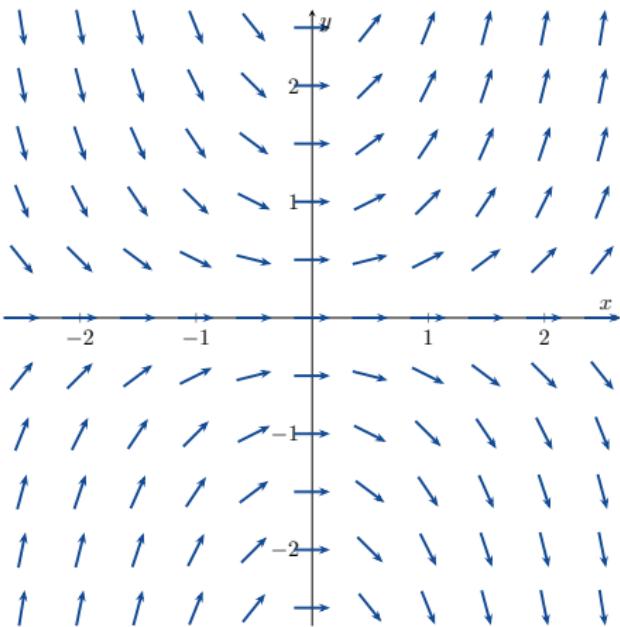
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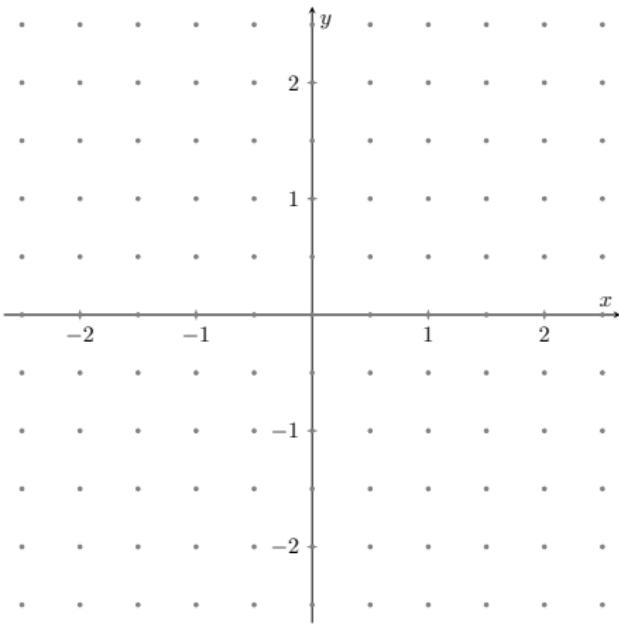
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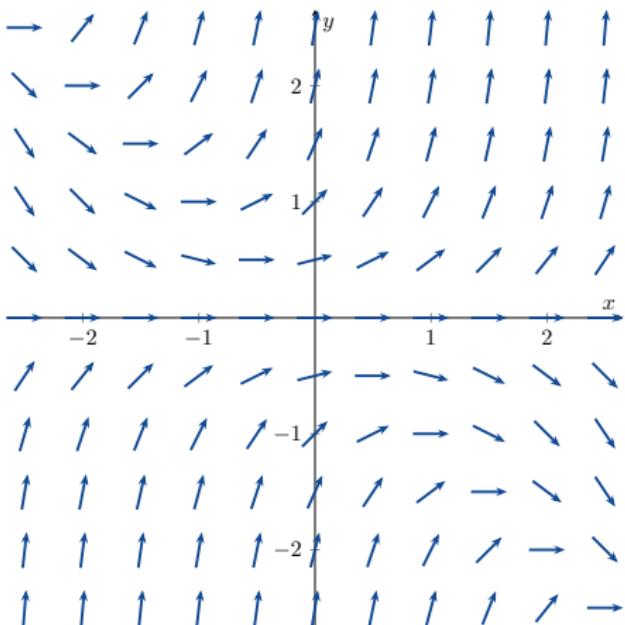
Draw a direction field for $\frac{dy}{dx} = y(x + y)$.



1.3 How to Draw a Direction Field

Example

Draw a direction field for $\frac{dy}{dx} = y(x + y)$.





Solving Our First Differential Equations

1.4 Solving Our First Differential Equations



$$\frac{dv}{dt} = 9.8 - \frac{v}{5} \quad (1)$$

$$\frac{dp}{dt} = \frac{p}{2} - 450 \quad (2)$$

Both (1) and (2) are of the form

$$\frac{dy}{dt} = ay - b \quad (4)$$

for constants a and b . We will now study how to solve equations like this.

1.4 Solving Our First Differential Equations



Example (Mice and Owls)

Solve

$$\frac{dp}{dt} = \frac{p}{2} - 450 = \frac{p - 900}{2}. \quad (2)$$

1.4 Solving Our First Differential Equations



$$\frac{dp}{dt} = \frac{p}{2} - 450 = \frac{p - 900}{2}. \quad (2)$$

If $p \neq 900$, we can rearrange (2) to

$$\frac{dp}{p - 900} = \frac{1}{2} dt.$$

1.4 Solving Our First Differential Equations



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Note that all the terms involving p are on the left, and all the terms involving t are on the right.

1.4 Solving Our First Differential Equations



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If $p \neq 900$, we can rearrange (2) to

$$\frac{dp}{p - 900} = \frac{1}{2} dt.$$

Note that all the terms involving p are on the left, and all the terms involving t are on the right. (Of course $\frac{dp}{dt}$ does not really mean $dp \div dt$, and using this method annoys “Pure Mathematicians”, but it works.)

1.4 Solving Our First Differential Equations



If we can separate the variables like this, then we are allowed to integrate:

$$\frac{dp}{p - 900} = \frac{1}{2} dt$$

1.4 Solving Our First Differential Equations



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1.4 Solving Our First Differential Equations



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$$\frac{dp}{p - 900} = \frac{1}{2} dt$$

$$\int \frac{dp}{p - 900} = \int \frac{1}{2} dt$$

$$\ln |p - 900| = \frac{t}{2} + K$$

where K is a constant.

1.4 Solving Our First Differential Equations



$$\ln |p - 900| = \frac{t}{2} + K$$

Thus

$$\begin{aligned}|p - 900| &= e^{\frac{t}{2} + K} \\ p - 900 &= \pm e^K e^{\frac{t}{2}} \\ p(t) &= 900 \pm e^K e^{\frac{t}{2}}.\end{aligned}$$

1.4 Solving Our First Differential Equations



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K is a number that we don't know.

1.4 Solving Our First Differential Equations



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1.4 Solving Our First Differential Equations



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1.4 Solving Our First Differential Equations



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1.4 Solving Our First Differential Equations



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K is a number that we don't know. So e^K is a number that we don't know. So $\pm e^K$ is a number that we don't know. We can give this unknown number a new name: Let $c = \pm e^K$. Then we have

$$p(t) = 900 + ce^{\frac{t}{2}}.$$

1.4 Solving Our First Differential Equations



Before we go on, let us just check that the function

$$p(t) = 900 + ce^{\frac{t}{2}}$$

really does solve the differential equation

$$\frac{dp}{dt} = \frac{p}{2} - 450.$$

1.4 Solving Our First Differential Equations



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We calculate that

$$\frac{dp}{dt} = \frac{d}{dt} \left(900 + ce^{\frac{t}{2}} \right) = 0 + c \left(\frac{1}{2} \right) e^{\frac{t}{2}} = \frac{c}{2} e^{\frac{t}{2}}$$

1.4 Solving Our First Differential Equations



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and

$$\frac{p}{2} - 450 = \frac{900 + ce^{\frac{t}{2}}}{2} - 450 = 450 + \frac{c}{2} e^{\frac{t}{2}} - 450 = \frac{c}{2} e^{\frac{t}{2}}.$$

1.4 Solving Our First Differential Equations



Example (A Falling Object)

Solve

$$\frac{dv}{dt} = 9.8 - \frac{v}{5}. \quad (1)$$

1.4 Solving Our First Differential Equations



We use the same method:

$$\frac{dv}{dt} = 9.8 - \frac{v}{5}$$

$$\frac{dv}{dt} = \frac{49 - v}{5}$$

$$\frac{dv}{v - 49} = -\frac{1}{5} dt$$

$$\int \frac{dv}{v - 49} = \int -\frac{1}{5} dt$$

$$\ln |v - 49| = -\frac{t}{5} + K$$

$$|v - 49| = e^{-\frac{t}{5} + K}$$

$$v - 49 = \pm e^K e^{-\frac{t}{5}}$$

$$v(t) = 49 \pm e^K e^{-\frac{t}{5}} = 49 + ce^{-\frac{t}{5}}.$$

1.4 Solving Our First Differential Equations



Example

Solve $\frac{dy}{dt} = y$.

This is $\frac{dy}{dt} = ay - b$ with $a = 1$ and $b = 0$. I leave this for you to solve.



Next Time

- 1.5 Classification
- 2.1 Linear Equations
- 2.2 Separable Equations
- 2.3 Differences Between Linear and Nonlinear Equations