



Welcome to

# Mathematics II

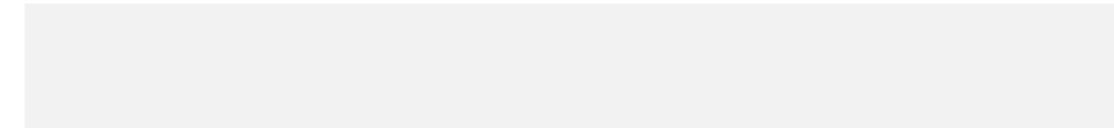
with Dr Neil Course

# Lecture 1

- Information about this course
- 8.1 Using Basic Integration Formulae
- 8.2 Integration by Parts
- 8.3 Trigonometric Integrals

## Information about this course

- $\approx$  12 classes. Friday afternoons 1pm-3:30pm.



13:00

14:00

15:00

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- $\approx$  12 classes. Friday afternoons 1pm-3:30pm.
- 2 lectures with a break between.

lecture

lecture

13:00

14:00

15:00

## Information about this course

- $\approx$  12 classes. Friday afternoons 1pm-3:30pm.
- 2 lectures with a break between.
- Then I will answers your questions.

lecture

lecture

questions

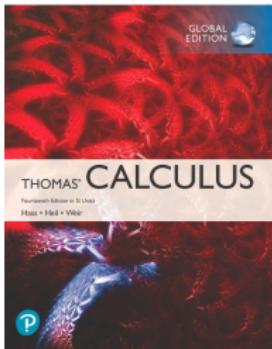
13:00

14:00

15:00

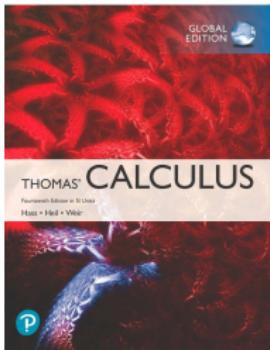


## The Book



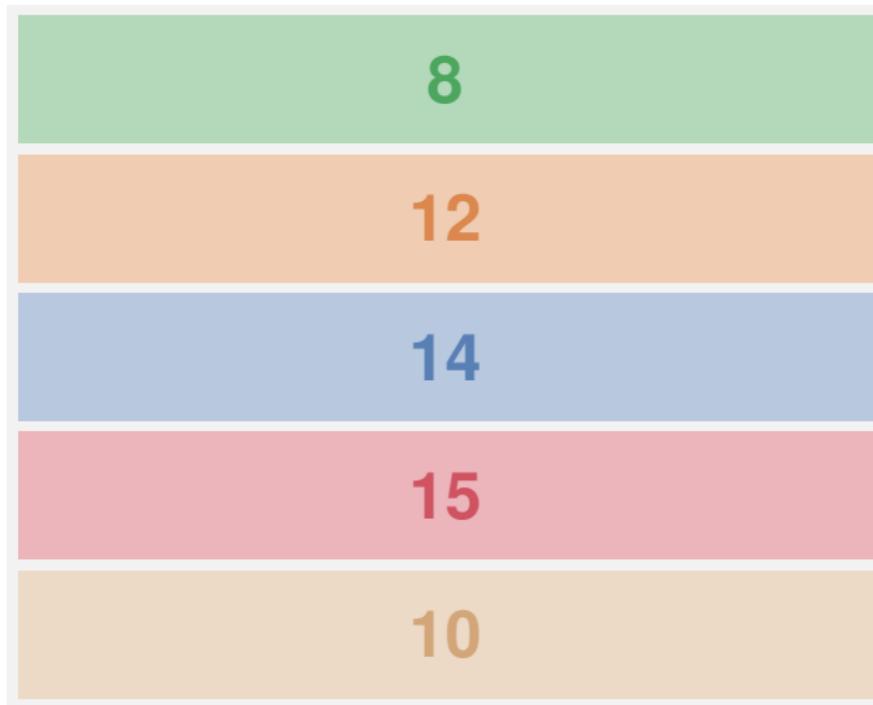
Joel R. Hass, Christopher E. Heil and Maurice D. Weir,  
*Thomas' Calculus in SI Units*,  
14th Edition, Wiley.

## The Book



Joel R. Hass, Christopher E. Heil and Maurice D. Weir,  
*Thomas' Calculus in SI Units*,  
14th Edition, Wiley.

This is a required purchase.  
You need to have this book to be  
able to do the homework.



## 8. Techniques of Integration

12

14

15

10

## 8. Techniques of Integration

## 12. Vectors and the Geometry of Space

14

15

10

## 8. Techniques of Integration

## 12. Vectors and the Geometry of Space

## 14. Partial Derivatives

15

10

**8. Techniques of Integration**

**12. Vectors and the Geometry of Space**

**14. Partial Derivatives**

**15. Multiple Integrals**

**10**

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**12. Vectors and the Geometry of Space**

**14. Partial Derivatives**

**15. Multiple Integrals**

**10. Infinite Sequences and Series**

## 8. Techniques of Integration

} 2 weeks

## 12. Vectors and the Geometry of Space

} 2 weeks

## 14. Partial Derivatives

} 2 weeks

## 15. Multiple Integrals

} 3 weeks

## 10. Infinite Sequences and Series

} 3 weeks



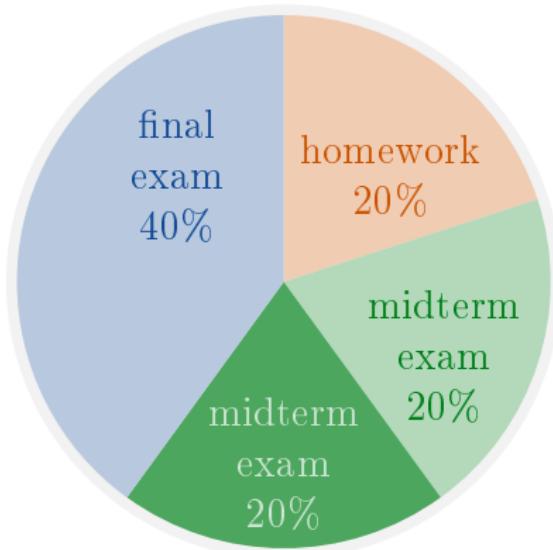
# Exams and homework

(This information may change based on the University's decisions)



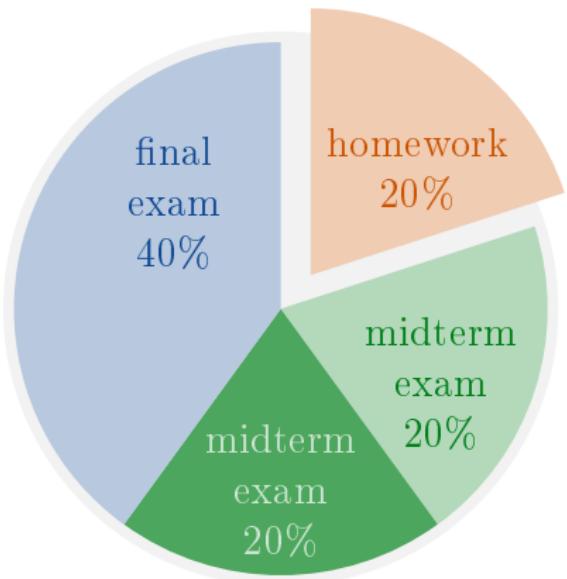
# Exams and homework

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using Pearson  
MyLab Math

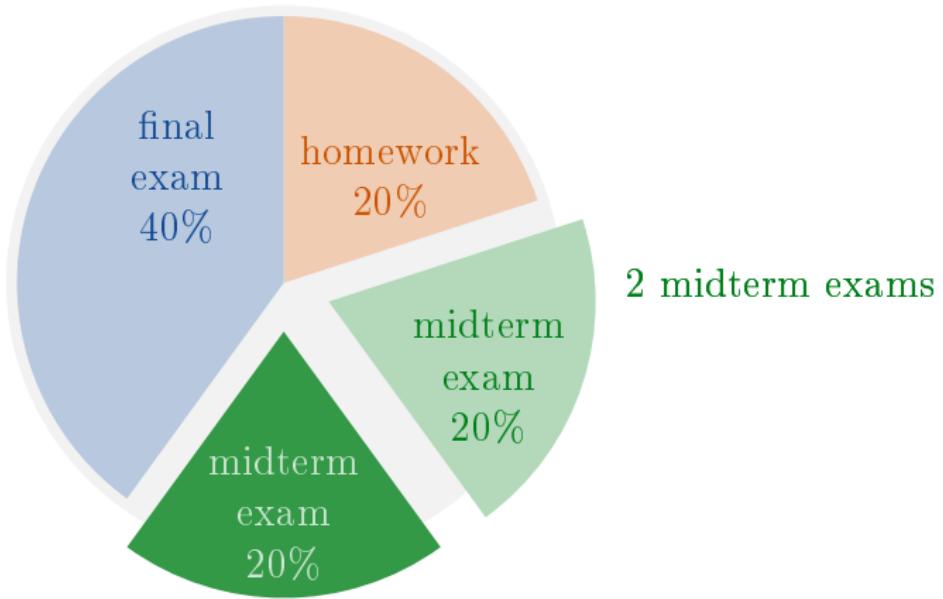
one piece of  
homework for  
each lesson

deadline = end of  
term

more details in  
O'Learn

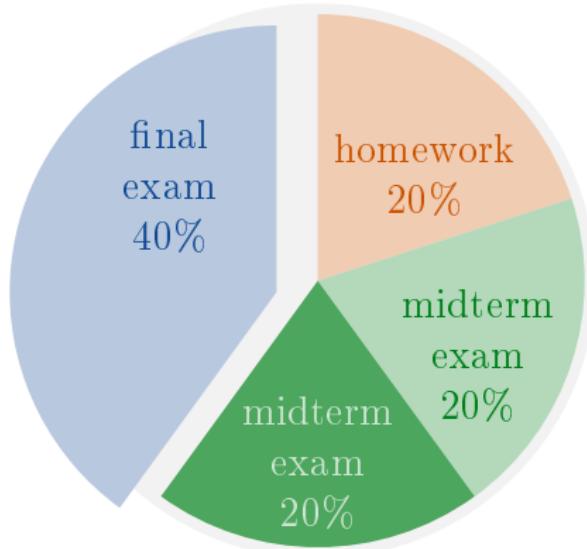
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## Expectations

If this was a classroom course, the expectation is that for each hour of lectures, you would study an extra 1-2 hours outside of class.

classroom  
course

lectures (5 hours)

other study (5-10 hours)

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classroom  
course

lectures (5 hours)

other study (5-10 hours)

For an online course, you are still expected to study a total of 10-15 hours each week.

online  
course

class  
(2.5 hours)

other study (7.5-12.5 hours)



This may include:

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⋮

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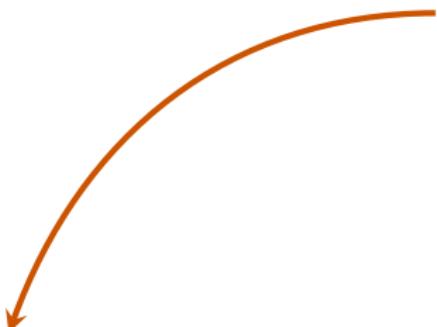
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:

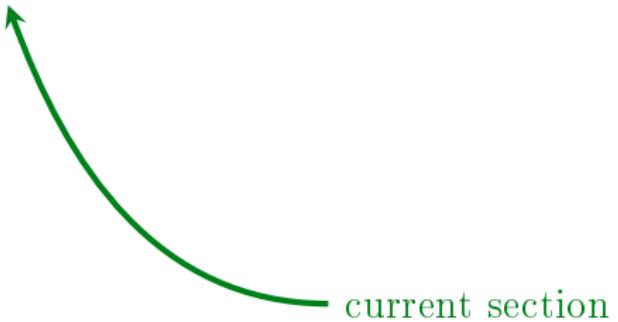
# 99.9 Section Title



slide number



# 99.9 Section Title





# Using Basic Integration Formulae

# Table 8.1

## Basic integration formulas

1. $\int k \, dx = kx + C$ (any number $k$ )	12. $\int \tan x \, dx = \ln  \sec x  + C$
2. $\int x^n \, dx = \frac{x^{n+1}}{n+1} + C$ ( $n \neq -1$ )	13. $\int \cot x \, dx = \ln  \sin x  + C$
3. $\int \frac{dx}{x} = \ln  x  + C$	14. $\int \sec x \, dx = \ln  \sec x + \tan x  + C$
4. $\int e^x \, dx = e^x + C$	15. $\int \csc x \, dx = -\ln  \csc x + \cot x  + C$
5. $\int a^x \, dx = \frac{a^x}{\ln a} + C$ ( $a > 0, a \neq 1$ )	16. $\int \sinh x \, dx = \cosh x + C$
6. $\int \sin x \, dx = -\cos x + C$	17. $\int \cosh x \, dx = \sinh x + C$
7. $\int \cos x \, dx = \sin x + C$	18. $\int \frac{dx}{\sqrt{a^2 - x^2}} = \sin^{-1} \left( \frac{x}{a} \right) + C$
8. $\int \sec^2 x \, dx = \tan x + C$	19. $\int \frac{dx}{a^2 + x^2} = \frac{1}{a} \tan^{-1} \left( \frac{x}{a} \right) + C$
9. $\int \csc^2 x \, dx = -\cot x + C$	20. $\int \frac{dx}{x\sqrt{x^2 - a^2}} = \frac{1}{a} \sec^{-1} \left  \frac{x}{a} \right  + C$
10. $\int \sec x \tan x \, dx = \sec x + C$	21. $\int \frac{dx}{\sqrt{a^2 + x^2}} = \sinh^{-1} \left( \frac{x}{a} \right) + C$ ( $a > 0$ )
11. $\int \csc x \cot x \, dx = -\csc x + C$	22. $\int \frac{dx}{\sqrt{x^2 - a^2}} = \cosh^{-1} \left( \frac{x}{a} \right) + C$ ( $x > a > 0$ )

## 8.1 Using Basic Integration Formulae

### Example

Calculate  $\int_3^5 \frac{2x - 3}{\sqrt{x^2 - 3x + 1}} dx.$

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Calculate  $\int_3^5 \frac{2x - 3}{\sqrt{x^2 - 3x + 1}} dx.$

We use the substitution  $u = x^2 - 3x + 1.$

## 8.1 Using Basic Integration Formulae

### Example

Calculate  $\int_3^5 \frac{2x - 3}{\sqrt{x^2 - 3x + 1}} dx$ .

We use the substitution  $u = x^2 - 3x + 1$ . Then  $du = (2x - 3) dx$  and

$$x = 3 \implies u = 9 - 9 + 1 = 1$$

$$x = 5 \implies u = 25 - 15 + 1 = 11.$$

## 8.1 Using Basic Integration Formulae

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$$x = 5 \implies u = 25 - 15 + 1 = 11.$$

Hence

$$\int_3^5 \frac{2x - 3}{\sqrt{x^2 - 3x + 1}} dx = \int_1^{11} u^{-\frac{1}{2}} du = [2\sqrt{u}]_1^{11} = 2(\sqrt{11} - 1).$$

8.1

$$\int \frac{dx}{\sqrt{a^2 - x^2}} = \sin^{-1} \left( \frac{u}{a} \right) + C$$



Example

Find  $\int \frac{dx}{\sqrt{8x - x^2}}$ .

8.1

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### Example

Find  $\int \frac{dx}{\sqrt{8x - x^2}}$ .

This time we will complete the square of  $x^2 - 8x$  and use that to simplify the integral:

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This time we will complete the square of  $x^2 - 8x$  and use that to simplify the integral:

$$x^2 - 8x = x^2 - 8x + 16 - 16$$

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## 8.1 Using Basic Integration Formulae



### Example

Find  $\int \cos x \sin 2x + \sin x \cos 2x \, dx$ .

## 8.1 Using Basic Integration Formulae



### Example

Find  $\int \cos x \sin 2x + \sin x \cos 2x \, dx$ .

$$\begin{aligned}\int \cos x \sin 2x + \sin x \cos 2x \, dx &= \int \sin(x + 2x) \, dx \\ &= \int \sin 3x \, dx \\ &= \dots\end{aligned}$$

## 8.1 Using Basic Integration Formulae



### Example

Find  $\int_0^{\frac{\pi}{4}} \frac{dx}{1 - \sin x}$ .

## 8.1 Using Basic Integration Formulae



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## 8.1 Using Basic Integration Formulae



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Here is a trick for dealing with  $\frac{1}{A-B}$ : Multiply by  $\frac{A+B}{A+B}$ .  
Then we get

$$\frac{1}{A-B} = \left( \frac{1}{A-B} \right) \left( \frac{A+B}{A+B} \right) = \frac{A+B}{A^2 - B^2}$$

which is sometimes easier to deal with.

$$\begin{aligned}
 \int_0^{\pi/4} \frac{dx}{1 - \sin x} &= \int_0^{\pi/4} \frac{1}{1 - \sin x} \cdot \frac{1 + \sin x}{1 + \sin x} dx && \text{Multiply and divide by conjugate.} \\
 &= \int_0^{\pi/4} \frac{1 + \sin x}{1 - \sin^2 x} dx && \text{Simplify.} \\
 &= \int_0^{\pi/4} \frac{1 + \sin x}{\cos^2 x} dx && 1 - \sin^2 x = \cos^2 x \\
 &= \int_0^{\pi/4} (\sec^2 x + \sec x \tan x) dx && \text{Use Table 8.1, Formulas 8 and 10} \\
 &= \left[ \tan x + \sec x \right]_0^{\pi/4} = (1 + \sqrt{2} - (0 + 1)) = \sqrt{2}.
 \end{aligned}$$



## 8.1 Using Basic Integration Formulae



### Example

Find  $\int \frac{3x^2 - 7x}{3x + 2} dx.$

## 8.1 Using Basic Integration Formulae



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Find  $\int \frac{3x^2 - 7x}{3x + 2} dx$ .

**Solution** The integrand is an improper fraction since the degree of the numerator is greater than the degree of the denominator. To integrate it, we perform long division to obtain a quotient plus a remainder that is a proper fraction:

$$\frac{3x^2 - 7x}{3x + 2} = x - 3 + \frac{6}{3x + 2}.$$

## 8.1 Using Basic Integration Formulae

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$$\frac{3x^2 - 7x}{3x + 2} = x - 3 + \frac{6}{3x + 2}.$$

Therefore,

$$\int \frac{3x^2 - 7x}{3x + 2} dx = \int \left( x - 3 + \frac{6}{3x + 2} \right) dx = \frac{x^2}{2} - 3x + 2 \ln |3x + 2| + C. \quad \blacksquare$$

8.1

$$\int \frac{dx}{\sqrt{a^2 - x^2}} = \sin^{-1} \left( \frac{x}{a} \right) + C$$



### Example

Find  $\int \frac{3x + 2}{\sqrt{1 - x^2}}.$

$$\int \frac{dx}{\sqrt{a^2 - x^2}} = \sin^{-1} \left( \frac{u}{a} \right) + C$$



### Example

Find  $\int \frac{3x + 2}{\sqrt{1 - x^2}}$ .

First note that

$$\int \frac{3x + 2}{\sqrt{1 - x^2}} = 3 \int \frac{x \, dx}{\sqrt{1 - x^2}} + 2 \int \frac{dx}{\sqrt{1 - x^2}}$$

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So we just need to calculate  $\int \frac{x \, dx}{\sqrt{1 - x^2}}$ .

## 8.1 Using Basic Integration Formulae



$$\int \frac{x \, dx}{\sqrt{1 - x^2}}$$

Let  $u = 1 - x^2$ .

## 8.1 Using Basic Integration Formulae



$$\int \frac{x \, dx}{\sqrt{1 - x^2}}$$

Let  $u = 1 - x^2$ . Then  $du = -2x \, dx$  and  $-\frac{1}{2} du = x \, dx$ .

## 8.1 Using Basic Integration Formulae



$$\int \frac{x \, dx}{\sqrt{1 - x^2}}$$

Let  $u = 1 - x^2$ . Then  $du = -2x \, dx$  and  $-\frac{1}{2} du = x \, dx$ . It follows that

$$\int \frac{x \, dx}{\sqrt{1 - x^2}} = \int \frac{-\frac{1}{2} du}{\sqrt{u}} = -\frac{1}{2} \int u^{-\frac{1}{2}} \, du = \dots = -\sqrt{1 - x^2} + C.$$

## 8.1 Using Basic Integration Formulae



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Therefore

$$\begin{aligned}\int \frac{3x + 2}{\sqrt{1 - x^2}} &= 3 \int \frac{x \, dx}{\sqrt{1 - x^2}} + 2 \int \frac{dx}{\sqrt{1 - x^2}} \\ &= -3\sqrt{1 - x^2} + 2 \sin^{-1} x + C.\end{aligned}$$

## 8.1 Using Basic Integration Formulae

### Example

Find  $\int \frac{dx}{(1 + \sqrt{x})^3}$ .

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I want to make a substitution to make this integral easier, but what  $u$  should I choose?

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First guess:  $u = \sqrt{x}$ .

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First guess:  $u = \sqrt{x}$ . But then  $du = \frac{1}{2\sqrt{x}} dx$  and we would have to deal with this extra  $\sqrt{x} = u$  term.

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First guess:  $u = \sqrt{x}$ . But then  $du = \frac{1}{2\sqrt{x}} dx$  and we would have to deal with this extra  $\sqrt{x} = u$  term.

Second guess: Instead let us try  $u = 1 + \sqrt{x}$ . Then again we have  $du = \frac{1}{2\sqrt{x}} dx$  and  $dx = 2\sqrt{x} du = 2(u - 1) du$ . Hence

$$\int \frac{dx}{(1 + \sqrt{x})^3} = \int \frac{2(u - 1) du}{u^3} = \int \frac{2}{u^2} - \frac{2}{u^3} du = \dots$$

## 8.1 Using Basic Integration Formulae

### Example

Calculate  $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} x^3 \cos x dx$ .

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This is actually easy: The integrand is an odd function

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$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} x^3 \cos x dx = 0.$$

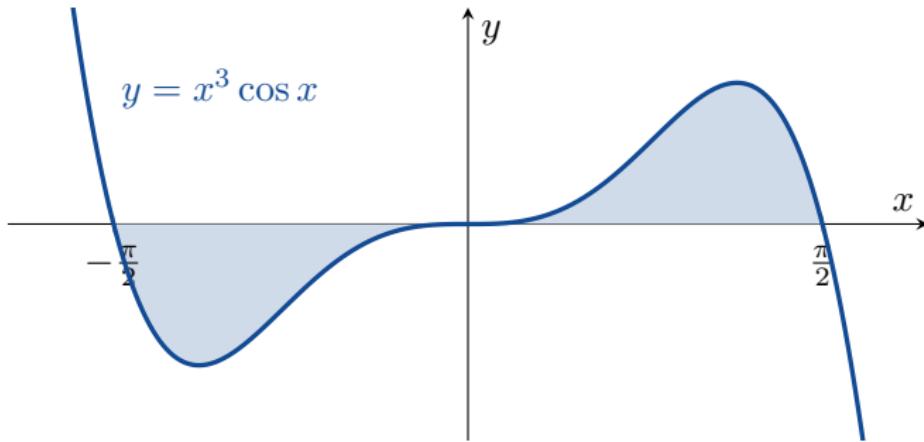
## 8.1 Using Basic Integration Formulae

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# Integration by Parts

## 8.2 Integration by Parts

How can we calculate

$$\int x \cos x \, dx$$

or

$$\int x^2 e^x \, dx ?$$

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$$\int \text{function} \times \text{function} \, dx$$

## 8.2 Integration by Parts

How can we calculate

$$\int x \cos x \, dx$$

or

$$\int x^2 e^x \, dx ?$$

$$\int \text{function} \times \text{function} \, dx$$

Theorem (Integration by Parts)

$$\int u(x)v'(x) \, dx =$$

## 8.2 Integration by Parts

How can we calculate

$$\int x \cos x \, dx$$

or

$$\int x^2 e^x \, dx ?$$

$$\int \text{function} \times \text{function} \, dx$$

Theorem (Integration by Parts)

$$\int u(x)v'(x) \, dx = u(x)v(x) - \int u'(x)v(x) \, dx$$

$$\int \color{red}{uv'}\,dx = \color{red}{uv} - \int \color{red}{u'}v\,dx$$

### Example

Find  $\int x \cos x\,dx$ .

$$\int \textcolor{red}{uv}' dx = \textcolor{red}{uv} - \int \textcolor{red}{u}'\textcolor{green}{v} dx$$

## Example

Find  $\int x \cos x dx$ .

We need to choose a  $\textcolor{red}{u}(x)$  and a  $v'(x)$ .

8.2

$$\int \textcolor{red}{u} \textcolor{green}{v}' dx = \textcolor{red}{u} \textcolor{green}{v} - \int \textcolor{red}{u}' \textcolor{green}{v} dx$$



### Example

Find  $\int x \cos x dx$ .

We need to choose a  $\textcolor{red}{u}(x)$  and a  $v'(x)$ .

Let

$$\textcolor{red}{u} = x \qquad \qquad v' = \cos x$$

Then

$$\int \textcolor{red}{x} \cos x dx = \qquad \qquad - \int \qquad \qquad dx = \qquad \qquad .$$

$$\int \textcolor{brown}{u} \textcolor{green}{v}' dx = \textcolor{brown}{u} \textcolor{green}{v} - \int \textcolor{brown}{u}' \textcolor{green}{v} dx$$

## Example

Find  $\int x \cos x dx$ .

We need to choose a  $\textcolor{brown}{u}(x)$  and a  $v'(x)$ .

Let

$$\begin{aligned} u &= x & v' &= \cos x \\ u' &= 1 \end{aligned}$$

Then

$$\int \textcolor{brown}{x} \cos x dx = \quad - \int \quad dx = \quad .$$

$$\int \textcolor{brown}{u} \textcolor{green}{v}' dx = \textcolor{brown}{u} \textcolor{green}{v} - \int \textcolor{brown}{u}' \textcolor{green}{v} dx$$

## Example

Find  $\int x \cos x dx$ .

We need to choose a  $\textcolor{brown}{u}(x)$  and a  $v'(x)$ .

Let

$$\begin{array}{ll} u = x & v' = \cos x \\ u' = 1 & v = \sin x. \end{array}$$

Then

$$\int \textcolor{brown}{x} \cos x dx = \quad - \int \quad dx = \quad .$$

$$\int \textcolor{red}{u} \textcolor{green}{v}' dx = \textcolor{red}{u} \textcolor{green}{v} - \int \textcolor{red}{u}' \textcolor{green}{v} dx$$

## Example

Find  $\int x \cos x dx$ .

We need to choose a  $\textcolor{red}{u}(x)$  and a  $v'(x)$ .

Let

$$\begin{array}{ll} u = x & v' = \cos x \\ u' = 1 & v = \sin x. \end{array}$$

Then

$$\int \textcolor{brown}{x} \cos x dx = \textcolor{brown}{x} \sin x - \int \quad dx = \quad .$$

$$\int \textcolor{red}{uv}' dx = \textcolor{red}{uv} - \int \textcolor{red}{u}'\textcolor{green}{v} dx$$

### Example

Find  $\int x \cos x dx$ .

We need to choose a  $\textcolor{red}{u}(x)$  and a  $v'(x)$ .

Let

$$\begin{array}{ll} u = x & v' = \cos x \\ u' = 1 & v = \sin x. \end{array}$$

Then

$$\int \textcolor{red}{x} \cos x dx = \textcolor{red}{x} \sin x - \int \textcolor{red}{1} \sin x dx = \quad .$$

$$\int \textcolor{brown}{u} \textcolor{green}{v}' dx = \textcolor{brown}{u} \textcolor{green}{v} - \int \textcolor{brown}{u}' \textcolor{green}{v} dx$$

## Example

Find  $\int x \cos x dx$ .

We need to choose a  $\textcolor{brown}{u}(x)$  and a  $v'(x)$ .

Let

$$\begin{array}{ll} u = x & v' = \cos x \\ u' = 1 & v = \sin x. \end{array}$$

Then

$$\int \textcolor{brown}{x} \cos x dx = \textcolor{brown}{x} \sin x - \int 1 \sin x dx = x \sin x + \cos x + C.$$

$$\int \textcolor{red}{uv}' dx = \textcolor{red}{uv} - \int \textcolor{red}{u}'\textcolor{green}{v} dx$$

### Example

Find  $\int \ln x dx$ .

$$\int \textcolor{red}{uv}' dx = \textcolor{red}{uv} - \int \textcolor{red}{u}'\textcolor{green}{v} dx$$

### Example

Find  $\int \ln x dx$ .

We will consider  $\int \ln x \cdot 1 dx$ .

$$\int \textcolor{red}{uv}' dx = \textcolor{red}{uv} - \int \textcolor{red}{u}'\textcolor{green}{v} dx$$

### Example

Find  $\int \ln x dx$ .

We will consider  $\int \ln x \cdot 1 dx$ .

Let

$$u = \ln x \quad v' = 1$$

$$\int \textcolor{red}{uv}' dx = \textcolor{red}{uv} - \int \textcolor{red}{u}'\textcolor{green}{v} dx$$

## Example

Find  $\int \ln x dx$ .

We will consider  $\int \ln x \cdot 1 dx$ .

Let

$$u = \ln x \qquad \qquad v' = 1$$

$$u' = \frac{1}{x}$$

$$\int \textcolor{red}{uv}' dx = \textcolor{red}{uv} - \int \textcolor{red}{u}'\textcolor{green}{v} dx$$

### Example

Find  $\int \ln x dx$ .

We will consider  $\int \ln x \cdot 1 dx$ .

Let

$$u = \ln x \qquad \qquad v' = 1$$

$$u' = \frac{1}{x} \qquad \qquad v = x.$$

$$\int \textcolor{red}{uv}' dx = \textcolor{red}{uv} - \int \textcolor{red}{u}'\textcolor{green}{v} dx$$

### Example

Find  $\int \ln x dx.$

We will consider  $\int \ln x \cdot 1 dx.$

Let

$$u = \ln x \quad v' = 1$$

$$u' = \frac{1}{x} \quad v = x.$$

Then

$$\int \ln x \cdot 1 dx = \ln x \cdot x - \int \frac{1}{x} \cdot x dx$$

=

=

.

$$\int \textcolor{red}{uv}' dx = \textcolor{red}{uv} - \int \textcolor{red}{u}'\textcolor{green}{v} dx$$

### Example

Find  $\int \ln x dx.$

We will consider  $\int \ln x \cdot 1 dx.$

Let

$$\begin{array}{ll} u = \ln x & v' = 1 \\ u' = \frac{1}{x} & v = x. \end{array}$$

Then

$$\begin{aligned} \int \ln x \cdot 1 dx &= \ln x \cdot x - \int \frac{1}{x} \cdot x dx \\ &= x \ln x - \int 1 dx \\ &= . \end{aligned}$$

$$\int \textcolor{red}{uv}' dx = \textcolor{red}{uv} - \int \textcolor{red}{u}'\textcolor{green}{v} dx$$

### Example

Find  $\int \ln x dx.$

We will consider  $\int \ln x \cdot 1 dx.$

Let

$$\begin{array}{ll} u = \ln x & v' = 1 \\ u' = \frac{1}{x} & v = x. \end{array}$$

Then

$$\begin{aligned} \int \ln x \cdot 1 dx &= \ln x \cdot x - \int \frac{1}{x} \cdot x dx \\ &= x \ln x - \int 1 dx \\ &= x \ln x - x + C. \end{aligned}$$

8.2

$$\int \color{red}{uv'}\,dx = \color{red}{uv} - \int \color{red}{u'}v\,dx$$



Sometimes we have to use integration by parts more than once.

$$\int \textcolor{red}{u} \textcolor{green}{v}' dx = \textcolor{red}{u} \textcolor{green}{v} - \int \textcolor{red}{u}' \textcolor{green}{v} dx$$

### Example

Find  $\int \textcolor{brown}{x}^2 e^x dx$ .

$$\int uv' dx = uv - \int u'v dx$$

### Example

Find  $\int x^2 e^x dx$ .

We calculate that

$$\int x^2 e^x dx = x^2 e^x - 2 \int x e^x dx.$$

$$\int \textcolor{red}{uv}' dx = \textcolor{red}{uv} - \int \textcolor{red}{u}'\textcolor{green}{v} dx$$

### Example

Find  $\int \textcolor{brown}{x}^2 e^x dx$ .

We calculate that

$$\int \textcolor{brown}{x}^2 e^x dx = \textcolor{brown}{x}^2 e^x - 2 \int \textcolor{brown}{x} e^x dx.$$

But what do we do with  $\int \textcolor{brown}{x} e^x dx$ ?

8.2

$$\int \textcolor{red}{uv}' dx = \textcolor{red}{uv} - \int \textcolor{red}{u}'v dx$$



### Example

Find  $\int \textcolor{brown}{x}^2 e^x dx$ .

We calculate that

$$\int \textcolor{brown}{x}^2 e^x dx = \textcolor{brown}{x}^2 e^x - 2 \int \textcolor{brown}{x} e^x dx.$$

But what do we do with  $\int \textcolor{brown}{x} e^x dx$ ?

$$\int \textcolor{brown}{x} e^x dx = \textcolor{brown}{x} e^x - \int 1 e^x dx = x e^x - e^x + C.$$

8.2

$$\int uv' dx = uv - \int u'v dx$$



### Example

Find  $\int x^2 e^x dx$ .

We calculate that

$$\int x^2 e^x dx = x^2 e^x - 2 \int x e^x dx.$$

But what do we do with  $\int x e^x dx$ ?

$$\int x e^x dx = x e^x - \int 1 e^x dx = x e^x - e^x + C.$$

Putting it all together, we have

$$\int x^2 e^x dx = x^2 e^x - 2 \int x e^x dx = x^2 e^x - 2 x e^x + 2 e^x + C.$$

$$\int \textcolor{red}{uv'} dx = \textcolor{red}{uv} - \int \textcolor{red}{u'}v dx$$

### Remark

We can use the same technique to calculate  $\int x^{\textcolor{red}{n}} e^x dx$ .

We would have to do integration by parts  $\textcolor{red}{n}$  times.

$$\int \textcolor{red}{u} \textcolor{green}{v}' dx = \textcolor{red}{u} \textcolor{green}{v} - \int \textcolor{red}{u}' \textcolor{green}{v} dx$$

## Theorem

$$\int u dv = \textcolor{red}{u} \textcolor{green}{v} - \int v du$$

**EXAMPLE 4** Evaluate

$$\int e^x \cos x \, dx.$$

**Solution** Let  $u = e^x$  and  $dv = \cos x \, dx$ . Then  $du = e^x \, dx$ ,  $v = \sin x$ , and

$$\int e^x \cos x \, dx = e^x \sin x - \int e^x \sin x \, dx.$$

The second integral is like the first except that it has  $\sin x$  in place of  $\cos x$ . To evaluate it, we use integration by parts with

$$u = e^x, \quad dv = \sin x \, dx, \quad v = -\cos x, \quad du = e^x \, dx.$$

Then

$$\begin{aligned}\int e^x \cos x \, dx &= e^x \sin x - \left( -e^x \cos x - \int (-\cos x)(e^x \, dx) \right) \\ &= e^x \sin x + e^x \cos x - \int e^x \cos x \, dx.\end{aligned}$$

The unknown integral now appears on both sides of the equation. Adding the integral to both sides and adding the constant of integration give

$$2 \int e^x \cos x \, dx = e^x \sin x + e^x \cos x + C_1.$$

Dividing by 2 and renaming the constant of integration give

$$\int e^x \cos x \, dx = \frac{e^x \sin x + e^x \cos x}{2} + C.$$



**EXAMPLE 5** Obtain a formula that expresses the integral

$$\int \cos^n x \, dx$$

in terms of an integral of a lower power of  $\cos x$ .

**Solution** We may think of  $\cos^n x$  as  $\cos^{n-1} x \cdot \cos x$ . Then we let

$$u = \cos^{n-1} x \quad \text{and} \quad dv = \cos x \, dx,$$

so that

$$du = (n - 1) \cos^{n-2} x (-\sin x \, dx) \quad \text{and} \quad v = \sin x.$$

Integration by parts then gives

$$\begin{aligned}\int \cos^n x \, dx &= \cos^{n-1} x \sin x + (n - 1) \int \sin^2 x \cos^{n-2} x \, dx \\&= \cos^{n-1} x \sin x + (n - 1) \int (1 - \cos^2 x) \cos^{n-2} x \, dx \\&= \cos^{n-1} x \sin x + (n - 1) \int \cos^{n-2} x \, dx - (n - 1) \int \cos^n x \, dx.\end{aligned}$$

If we add

$$(n - 1) \int \cos^n x dx$$

to both sides of this equation, we obtain

$$n \int \cos^n x dx = \cos^{n-1} x \sin x + (n - 1) \int \cos^{n-2} x dx.$$

We then divide through by  $n$ , and the final result is

$$\int \cos^n x dx = \frac{\cos^{n-1} x \sin x}{n} + \frac{n-1}{n} \int \cos^{n-2} x dx.$$

The formula found in Example 5 is called a **reduction formula** because it replaces an integral containing some power of a function with an integral of the same form having the power reduced. When  $n$  is a positive integer, we may apply the formula repeatedly until the remaining integral is easy to evaluate. For example, the result in Example 5 tells us that

$$\begin{aligned}\int \cos^3 x dx &= \frac{\cos^2 x \sin x}{3} + \frac{2}{3} \int \cos x dx \\ &= \frac{1}{3} \cos^2 x \sin x + \frac{2}{3} \sin x + C.\end{aligned}$$



$$\int \textcolor{red}{u} \textcolor{green}{v}' dx = \textcolor{red}{u} \textcolor{green}{v} - \int \textcolor{red}{u}' \textcolor{green}{v} dx$$

## Theorem

$$\int_{\textcolor{red}{a}}^{\textcolor{brown}{b}} \textcolor{red}{u} \textcolor{green}{v}' dx =$$

$$\int uv' dx = uv - \int u'v dx$$

## Theorem

$$\int_a^b uv' dx = [uv]_a^b - \int_a^b u'v dx$$

8.2

$$\int_a^b uv' dx = [uv]_a^b - \int_a^b u'v dx$$



### Example

Calculate the area of the region bounded by the curve  $y = xe^{-x}$  and the  $x$ -axis from  $x = 0$  to  $x = 4$ .

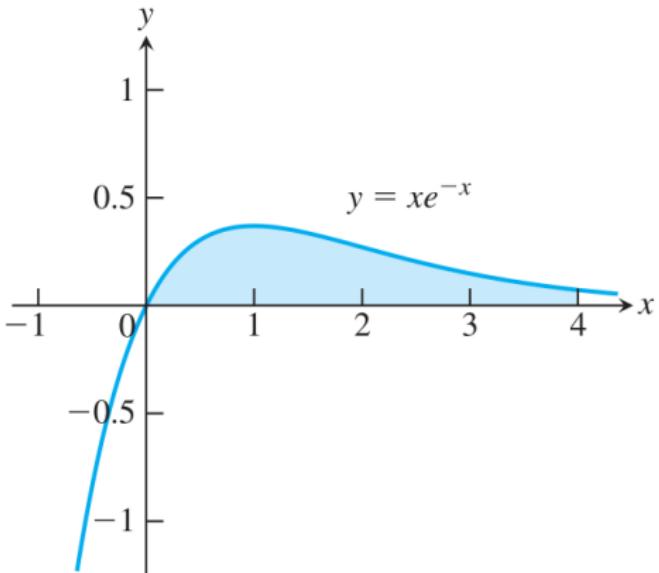
8.2

$$\int_a^b uv' dx = [uv]_a^b - \int_a^b u'v dx$$



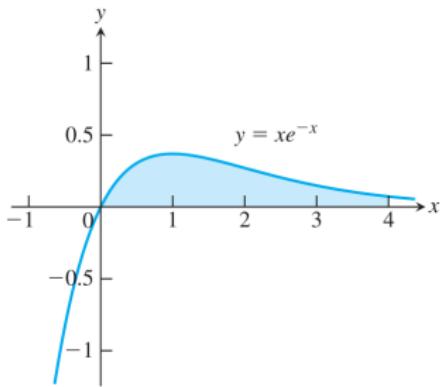
### Example

Calculate the area of the region bounded by the curve  $y = xe^{-x}$  and the  $x$ -axis from  $x = 0$  to  $x = 4$ .



8.2

$$\int_a^b uv' dx = [uv]_a^b - \int_a^b u'v dx$$



We calculate that

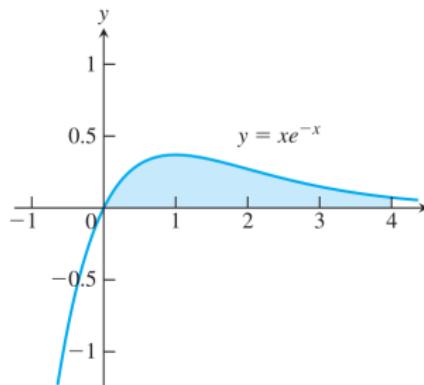
$$\int_0^4 xe^{-x} dx =$$

=

=

8.2

$$\int_a^b uv' dx = [uv]_a^b - \int_a^b u'v dx$$



$$u = x$$

$$u' = 1$$

$$v' = e^{-x}$$

$$v = -e^{-x}$$

We calculate that

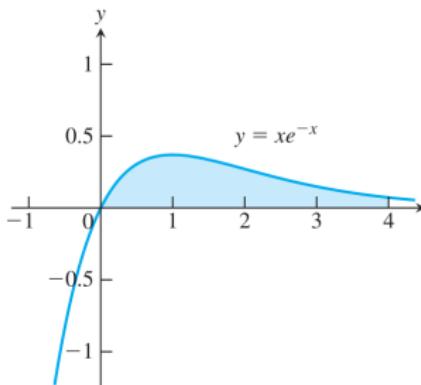
$$\int_0^4 xe^{-x} dx =$$

=

=

8.2

$$\int_a^b uv' dx = [uv]_a^b - \int_a^b u'v dx$$



$$u = x$$

$$u' = 1$$

$$v' = e^{-x}$$

$$v = -e^{-x}$$

We calculate that

$$\int_0^4 xe^{-x} dx = [-xe^{-x}]_0^4 - \int_0^4 1(-e^{-x}) dx$$

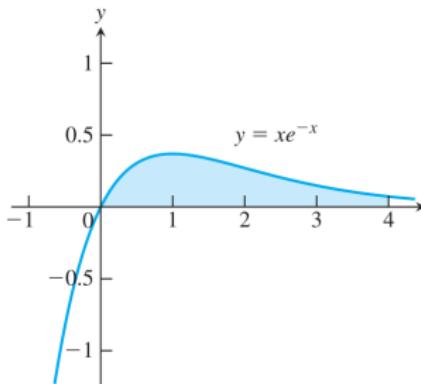
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8.2

$$\int_a^b uv' dx = [uv]_a^b - \int_a^b u'v dx$$



$$u = x$$

$$u' = 1$$

$$v' = e^{-x}$$

$$v = -e^{-x}$$

We calculate that

$$\begin{aligned}\int_0^4 xe^{-x} dx &= \left[ -xe^{-x} \right]_0^4 - \int_0^4 1(-e^{-x}) dx \\ &= (-4e^{-4} + 0) + [-e^{-x}]_0^4 \\ &= -4e^{-4} + (-e^{-4} + 1) = 1 - 5e^{-4}.\end{aligned}$$

8.2

$$\int_a^b uv' dx = [uv]_a^b - \int_a^b u'v dx$$



Example

Find  $\int_0^1 \sin^{-1} x dx$ .

8.2

$$\int_a^b uv' dx = [uv]_a^b - \int_a^b u'v dx$$



Example

Find  $\int_0^1 \sin^{-1} x dx$ .

Recall that  $\frac{d}{dx} \sin^{-1} x = \frac{1}{\sqrt{1-x^2}}$ .

8.2

$$\int_a^b uv' dx = [uv]_a^b - \int_a^b u'v dx$$



Example

Find  $\int_0^1 \sin^{-1} x dx$ .

Recall that  $\frac{d}{dx} \sin^{-1} x = \frac{1}{\sqrt{1-x^2}}$ .

Let  $u = \sin^{-1} x$  and  $v' = 1$ .

8.2

$$\int_a^b uv' dx = [uv]_a^b - \int_a^b u'v dx$$



Example

Find  $\int_0^1 \sin^{-1} x dx$ .

Recall that  $\frac{d}{dx} \sin^{-1} x = \frac{1}{\sqrt{1-x^2}}$ .

Let  $u = \sin^{-1} x$  and  $v' = 1$ . Then  $u' = \frac{1}{\sqrt{1-x^2}}$  and  $v = x$ .

8.2

$$\int_a^b uv' dx = [uv]_a^b - \int_a^b u'v dx$$



Example

Find  $\int_0^1 \sin^{-1} x dx$ .

Recall that  $\frac{d}{dx} \sin^{-1} x = \frac{1}{\sqrt{1-x^2}}$ .

Let  $u = \sin^{-1} x$  and  $v' = 1$ . Then  $u' = \frac{1}{\sqrt{1-x^2}}$  and  $v = x$ . It follows that

$$\int_0^1 \sin^{-1} x \cdot 1 dx = [\cancel{x} \sin^{-1} x]_0^1 - \int_0^1 \frac{\cancel{x}}{\sqrt{1-x^2}} dx$$

=

=

=

.

8.2

$$\int_a^b uv' dx = [uv]_a^b - \int_a^b u'v dx$$



Example

Find  $\int_0^1 \sin^{-1} x dx$ .

Recall that  $\frac{d}{dx} \sin^{-1} x = \frac{1}{\sqrt{1-x^2}}$ .

Let  $u = \sin^{-1} x$  and  $v' = 1$ . Then  $u' = \frac{1}{\sqrt{1-x^2}}$  and  $v = x$ . It follows that

$$\begin{aligned}\int_0^1 \sin^{-1} x \cdot 1 dx &= [\cancel{x} \sin^{-1} x]_0^1 - \int_0^1 \frac{\cancel{x}}{\sqrt{1-\cancel{x}^2}} dx \\ &= [\cancel{x} \sin^{-1} x]_0^1 - \left[ -\sqrt{1-\cancel{x}^2} \right]_0^1 \\ &= \left( \frac{\pi}{2} - 0 \right) - (-0 + 1) \\ &= \frac{\pi}{2} - 1.\end{aligned}$$

# Break

We will continue at 2pm





# Trigonometric Integrals

## 8.3 Trigonometric Integrals

$$\cos \theta = \sin \left( \frac{\pi}{2} - \theta \right)$$

$$\cos^2 \theta + \sin^2 \theta = 1$$

$$1 + \tan^2 \theta = \sec^2 \theta$$

$$1 + \cot^2 \theta = \operatorname{cosec}^2 \theta$$

$$\cos(A + B) = \cos A \cos B - \sin A \sin B$$

$$\sin(A + B) = \sin A \cos B + \cos A \sin B$$

$$\cos 2\theta = \cos^2 \theta - \sin^2 \theta$$

$$\sin 2\theta = 2 \sin \theta \cos \theta$$

$$\cos^2 \theta = \frac{1}{2}(1 + \cos 2\theta)$$

$$\sin^2 \theta = \frac{1}{2}(1 - \cos 2\theta)$$

## 8.3 Trigonometric Integrals



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## 8.3 Trigonometric Integrals



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## 8.3 Trigonometric Integrals



How can we find

$$\int \sin^m x \cos^n x dx$$

if  $m, n \in \{0, 1, 2, 3, 4, 5, \dots\}$ ?

## 8.3 Trigonometric Integrals



$$\int \sin^m x \cos^n x dx$$

We need to look at 3 different cases:

## 8.3 Trigonometric Integrals



$$\int \sin^m x \cos^n x dx$$

We need to look at 3 different cases:

- 1  **$m$  is odd:**

## 8.3 Trigonometric Integrals



$$\int \sin^m x \cos^n x dx$$

We need to look at 3 different cases:

**1  $m$  is odd:**

**2  $m$  is even and  $n$  is odd:**

## 8.3 Trigonometric Integrals



$$\int \sin^m x \cos^n x dx$$

We need to look at 3 different cases:

**1  $m$  is odd:**

**2  $m$  is even and  $n$  is odd:**

**3 both  $m$  and  $n$  are even:**

## 8.3 Trigonometric Integrals



$$\int \sin^m x \cos^n x dx \quad \cos^2 x + \sin^2 x = 1$$

We need to look at 3 different cases:

- 1 m is odd:** Write  $m = 2k + 1$  and use

$$\sin^m x = \sin^{2k+1} x = (\sin^2 x)^k \sin x = (1 - \cos^2 x)^k \sin x$$

and the substitution  $u = \cos x$ .

- 2 m is even and n is odd:**

- 3 both m and n are even:**

## 8.3 Trigonometric Integrals

$$\int \sin^m x \cos^n x dx \quad \cos^2 x + \sin^2 x = 1$$

We need to look at 3 different cases:

- 1**  **$m$  is odd:** Write  $m = 2k + 1$  and use

$$\sin^m x = \sin^{2k+1} x = (\sin^2 x)^k \sin x = (1 - \cos^2 x)^k \sin x$$

and the substitution  $u = \cos x$ .

- 2**  **$m$  is even and  $n$  is odd:** Write  $n = 2k + 1$  use

$$\cos^n x = \cos^{2k+1} x = (\cos^2 x)^k \cos x = (1 - \sin^2 x)^k \cos x$$

and the substitution  $u = \sin x$ .

- 3** **both  $m$  and  $n$  are even:**

## 8.3 Trigonometric Integrals

$$\int \sin^m x \cos^n x dx \quad \cos^2 x + \sin^2 x = 1$$

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and the substitution  $u = \sin x$ .

- 3** **both  $m$  and  $n$  are even:** Use

$$\sin^2 x = \frac{1 - \cos 2x}{2} \quad \text{and} \quad \cos^2 x = \frac{1 + \cos 2x}{2}.$$

## 8.3 Trigonometric Integrals



### Example

Find  $\int \sin^3 x \cos^2 x dx$ .

## 8.3 Trigonometric Integrals

### Example

Find  $\int \sin^3 x \cos^2 x dx$ .

**Solution** This is an example of Case 1.

$$\begin{aligned}\int \sin^3 x \cos^2 x dx &= \int \sin^2 x \cos^2 x \sin x dx && m \text{ is odd.} \\ &= \int (1 - \cos^2 x)(\cos^2 x)(-\sin x) dx && \sin x dx = -d(\cos x) \\ &= \int (1 - u^2)(u^2)(-du) && u = \cos x \\ &= \int (u^4 - u^2) du && \text{Multiply terms.} \\ &= \frac{u^5}{5} - \frac{u^3}{3} + C = \frac{\cos^5 x}{5} - \frac{\cos^3 x}{3} + C\end{aligned}$$



## 8.3 Trigonometric Integrals



### Example

Find  $\int \cos^5 x dx.$

## 8.3 Trigonometric Integrals

### Example

Find  $\int \cos^5 x dx.$

**Solution** This is an example of Case 2, where  $m = 0$  is even and  $n = 5$  is odd.

$$\begin{aligned}
 \int \cos^5 x dx &= \int \cos^4 x \cos x dx = \int (1 - \sin^2 x)^2 d(\sin x) && \cos x dx = d(\sin x) \\
 &= \int (1 - u^2)^2 du && u = \sin x \\
 &= \int (1 - 2u^2 + u^4) du && \text{Square } 1 - u^2. \\
 &= u - \frac{2}{3}u^3 + \frac{1}{5}u^5 + C = \sin x - \frac{2}{3}\sin^3 x + \frac{1}{5}\sin^5 x + C
 \end{aligned}$$



## 8.3 Trigonometric Integrals



### Example

Find  $\int \sin^2 x \cos^4 x dx$ .

## 8.3 Trigonometric Integrals

### Example

Find  $\int \sin^2 x \cos^4 x dx$ .

**Solution** This is an example of Case 3.

$$\begin{aligned}
 \int \sin^2 x \cos^4 x dx &= \int \left( \frac{1 - \cos 2x}{2} \right) \left( \frac{1 + \cos 2x}{2} \right)^2 dx && m \text{ and } n \text{ both even} \\
 &= \frac{1}{8} \int (1 - \cos 2x)(1 + 2\cos 2x + \cos^2 2x) dx \\
 &= \frac{1}{8} \int (1 + \cos 2x - \cos^2 2x - \cos^3 2x) dx \\
 &= \frac{1}{8} \left[ x + \frac{1}{2} \sin 2x - \int (\cos^2 2x + \cos^3 2x) dx \right]
 \end{aligned}$$

For the term involving  $\cos^2 2x$ , we use

$$\begin{aligned}\int \cos^2 2x \, dx &= \frac{1}{2} \int (1 + \cos 4x) \, dx \\ &= \frac{1}{2} \left( x + \frac{1}{4} \sin 4x \right).\end{aligned}$$

Omit constant of  
integration until final result.

For the  $\cos^3 2x$  term, we have

$$\begin{aligned}\int \cos^3 2x \, dx &= \int (1 - \sin^2 2x) \cos 2x \, dx && u = \sin 2x, \, du = 2 \cos 2x \, dx \\ &= \frac{1}{2} \int (1 - u^2) \, du = \frac{1}{2} \left( \sin 2x - \frac{1}{3} \sin^3 2x \right). && \text{Again omit } C.\end{aligned}$$

Combining everything and simplifying, we get

$$\int \sin^2 x \cos^4 x \, dx = \frac{1}{16} \left( x - \frac{1}{4} \sin 4x + \frac{1}{3} \sin^3 2x \right) + C.$$



## 8.3 Trigonometric Integrals

### Example

Find  $\int_0^{\frac{\pi}{4}} \sqrt{1 + \cos 4x} dx.$

## 8.3 Trigonometric Integrals

### Example

Find  $\int_0^{\frac{\pi}{4}} \sqrt{1 + \cos 4x} dx.$

**Solution** To eliminate the square root, we use the identity

$$\cos^2 \theta = \frac{1 + \cos 2\theta}{2} \quad \text{or} \quad 1 + \cos 2\theta = 2 \cos^2 \theta.$$

With  $\theta = 2x$ , this becomes

$$1 + \cos 4x = 2 \cos^2 2x.$$

Therefore,

$$\begin{aligned} \int_0^{\pi/4} \sqrt{1 + \cos 4x} dx &= \int_0^{\pi/4} \sqrt{2 \cos^2 2x} dx = \int_0^{\pi/4} \sqrt{2} \sqrt{\cos^2 2x} dx \\ &= \sqrt{2} \int_0^{\pi/4} |\cos 2x| dx = \sqrt{2} \int_0^{\pi/4} \cos 2x dx \quad \begin{matrix} \cos 2x \geq 0 \text{ on} \\ [0, \pi/4] \end{matrix} \\ &= \sqrt{2} \left[ \frac{\sin 2x}{2} \right]_0^{\pi/4} = \frac{\sqrt{2}}{2} [1 - 0] = \frac{\sqrt{2}}{2}. \end{aligned}$$



8.3

$$\sec^2 x = 1 + \tan^2 x \quad \frac{d}{dx} \tan x = \sec^2 x$$



Example

Find  $\int \tan^4 x dx$ .

8.3

$$\sec^2 x = 1 + \tan^2 x \quad \frac{d}{dx} \tan x = \sec^2 x$$



### Example

Find  $\int \tan^4 x \, dx$ .

### Solution

$$\begin{aligned}\int \tan^4 x \, dx &= \int \tan^2 x \cdot \tan^2 x \, dx = \int \tan^2 x \cdot (\sec^2 x - 1) \, dx \\&= \int \tan^2 x \sec^2 x \, dx - \int \tan^2 x \, dx \\&= \int \tan^2 x \sec^2 x \, dx - \int (\sec^2 x - 1) \, dx \\&= \int \tan^2 x \sec^2 x \, dx - \int \sec^2 x \, dx + \int dx\end{aligned}$$

In the first integral, we let

$$u = \tan x, \quad du = \sec^2 x \, dx$$

and have

$$\int u^2 \, du = \frac{1}{3} u^3 + C_1.$$

The remaining integrals are standard forms, so

$$\int \tan^4 x \, dx = \frac{1}{3} \tan^3 x - \tan x + x + C.$$



## 8.3 Trigonometric Integrals



### Example

Find  $\int \sec^3 x \, dx$ .

**Solution** We integrate by parts using

$$u = \sec x, \quad dv = \sec^2 x dx, \quad v = \tan x, \quad du = \sec x \tan x dx.$$

Then

$$\begin{aligned}\int \sec^3 x dx &= \sec x \tan x - \int (\tan x)(\sec x \tan x dx) \\&= \sec x \tan x - \int (\sec^2 x - 1) \sec x dx && \tan^2 x = \sec^2 x - 1 \\&= \sec x \tan x + \int \sec x dx - \int \sec^3 x dx.\end{aligned}$$

Combining the two secant-cubed integrals gives

$$2 \int \sec^3 x dx = \sec x \tan x + \int \sec x dx$$

and

$$\int \sec^3 x dx = \frac{1}{2} \sec x \tan x + \frac{1}{2} \ln |\sec x + \tan x| + C.$$



8.3

$$\sec^2 x = 1 + \tan^2 x \quad \frac{d}{dx} \tan x = \sec^2 x$$



Example

$$\text{Find } \int \tan^4 x \sec^4 x \, dx.$$

8.3

$$\sec^2 x = 1 + \tan^2 x \quad \frac{d}{dx} \tan x = \sec^2 x$$



## Example

Find  $\int \tan^4 x \sec^4 x \, dx.$

### Solution

$$\begin{aligned}
 \int (\tan^4 x)(\sec^4 x) \, dx &= \int (\tan^4 x)(1 + \tan^2 x)(\sec^2 x) \, dx && \sec^2 x = 1 + \tan^2 x \\
 &= \int (\tan^4 x + \tan^6 x)(\sec^2 x) \, dx \\
 &= \int (\tan^4 x)(\sec^2 x) \, dx + \int (\tan^6 x)(\sec^2 x) \, dx \\
 &= \int u^4 \, du + \int u^6 \, du = \frac{u^5}{5} + \frac{u^7}{7} + C && u = \tan x, \\
 &= \frac{\tan^5 x}{5} + \frac{\tan^7 x}{7} + C && du = \sec^2 x \, dx
 \end{aligned}$$



## 8.3 Trigonometric Integrals

How do we calculate

$$\int \sin mx \sin nx \, dx$$

or

$$\int \sin mx \cos nx \, dx$$

or

$$\int \cos mx \cos nx \, dx$$

?

## 8.3 Trigonometric Integrals

How do we calculate

$$\int \sin mx \sin nx \, dx$$

or

$$\int \sin mx \cos nx \, dx$$

or

$$\int \cos mx \cos nx \, dx$$

?

It is possible to use integration by parts (twice), but there is an easier way.

8.3

$$\cos(A + B) = \cos A \cos B - \sin A \sin B$$



$$\cos(mx - nx) - \cos(mx + nx) =$$

8.3

$$\cos(A + B) = \cos A \cos B - \sin A \sin B$$



$$\begin{aligned}\cos(mx - nx) - \cos(mx + nx) &= \cos mx \cos nx + \sin mx \sin nx \\ &\quad - \cos mx \cos nx + \sin mx \sin nx\end{aligned}$$

8.3

$$\cos(A + B) = \cos A \cos B - \sin A \sin B$$



$$\begin{aligned}\cos(mx - nx) - \cos(mx + nx) &= \cos mx \cos nx + \sin mx \sin nx \\ &\quad - \cos mx \cos nx + \sin mx \sin nx \\ &= 2 \sin mx \sin nx\end{aligned}$$

8.3

$$\cos(A + B) = \cos A \cos B - \sin A \sin B$$



$$\begin{aligned}\cos(mx - nx) - \cos(mx + nx) &= \cos mx \cos nx + \sin mx \sin nx \\ &\quad - \cos mx \cos nx + \sin mx \sin nx \\ &= 2 \sin mx \sin nx\end{aligned}$$

Therefore

$$\sin mx \sin nx = \frac{1}{2}(\cos(m - n)x - \cos(m + n)x).$$

$$\cos(A + B) = \cos A \cos B - \sin A \sin B$$

$$\begin{aligned}\cos(mx - nx) - \cos(mx + nx) &= \cos mx \cos nx + \sin mx \sin nx \\ &\quad - \cos mx \cos nx + \sin mx \sin nx \\ &= 2 \sin mx \sin nx\end{aligned}$$

Therefore

$$\sin mx \sin nx = \frac{1}{2}(\cos(m-n)x - \cos(m+n)x).$$

Similarly

$$\sin mx \cos nx = \frac{1}{2}(\sin(m-n)x + \sin(m+n)x)$$

and

$$\cos mx \cos nx = \frac{1}{2}(\cos(m-n)x + \cos(m+n)x).$$

$$\sin mx \cos nx = \frac{1}{2}(\sin(m-n)x + \sin(m+n)x)$$

### Example

Find  $\int \sin 3x \cos 5x \, dx$ .

$$\sin mx \cos nx = \frac{1}{2}(\sin(m-n)x + \sin(m+n)x)$$

## Example

Find  $\int \sin 3x \cos 5x \, dx$ .

**Solution** From Equation (4) with  $m = 3$  and  $n = 5$ , we get

$$\begin{aligned}\int \sin 3x \cos 5x \, dx &= \frac{1}{2} \int [\sin(-2x) + \sin 8x] \, dx \\ &= \frac{1}{2} \int (\sin 8x - \sin 2x) \, dx \\ &= -\frac{\cos 8x}{16} + \frac{\cos 2x}{4} + C.\end{aligned}$$



$$\cos mx \cos nx = \frac{1}{2}(\cos(m-n)x + \cos(m+n)x)$$



### Example

Find  $\int \cos 3x \cos 2x dx$ .

$$\cos mx \cos nx = \frac{1}{2}(\cos(m-n)x + \cos(m+n)x)$$

### Example

Find  $\int \cos 3x \cos 2x dx$ .

We have  $m = 3$  and  $n = 2$ . It follows that

$$\begin{aligned}\int \cos 3x \cos 2x dx &= \frac{1}{2} \int \cos(3-2)x dx + \frac{1}{2} \int \cos(3+2)x dx \\ &= \dots\end{aligned}$$



# Next Time

- 8.4 Trigonometric Substitutions
- 8.5 Integration of Rational Functions by Partial Fractions
- 8.8 Improper Integrals