

Lecture 6

- 3.8 Solving Initial Value Problems
- 3.9 The Method of Variation of Parameters
- 3.10 Higher Order Linear ODEs

Solving Initial Value Problems

3.8 Solving Initial Value Problems



Remark

$$\begin{cases} ay'' + by' + cy = g(t) \\ y(t_0) = y_0 \\ y'(t_0) = y_1. \end{cases}$$

To solve this IVP, the method is:

3.8 Solving Initial Value Problems



Remark

$$\begin{cases} ay'' + by' + cy = g(t) \\ y(t_0) = y_0 \\ y'(t_0) = y_1. \end{cases}$$

To solve this IVP, the method is:

- 1 Find the general solution to $ay'' + by' + cy = 0$;

3.8 Solving Initial Value Problems



Remark

$$\begin{cases} ay'' + by' + cy = g(t) \\ y(t_0) = y_0 \\ y'(t_0) = y_1. \end{cases}$$

To solve this IVP, the method is:

- 1 Find the general solution to $ay'' + by' + cy = 0$;
- 2 Find a particular solution to $ay'' + by' + cy = g(t)$:
 - 1 if $g(t)$ **does not** solve the homogeneous equation, then your ansatz should look like $g(t)$;
 - 2 if $g(t)$ **does** solve the homogeneous equation, then “multiply by t ” (repeat as necessary);

Remark

$$\begin{cases} ay'' + by' + cy = g(t) \\ y(t_0) = y_0 \\ y'(t_0) = y_1. \end{cases}$$

To solve this IVP, the method is:

- 1 Find the general solution to $ay'' + by' + cy = 0$;
- 2 Find a particular solution to $ay'' + by' + cy = g(t)$:
 - 1 if $g(t)$ **does not** solve the homogeneous equation, then your ansatz should look like $g(t)$;
 - 2 if $g(t)$ **does** solve the homogeneous equation, then “multiply by t ” (repeat as necessary);
- 3 1+2;

3.8 Solving Initial Value Problems



Remark

$$\begin{cases} ay'' + by' + cy = g(t) \\ y(t_0) = y_0 \\ y'(t_0) = y_1. \end{cases}$$

To solve this IVP, the method is:

- 1 Find the general solution to $ay'' + by' + cy = 0$;
- 2 Find a particular solution to $ay'' + by' + cy = g(t)$:
 - 1 if $g(t)$ **does not** solve the homogeneous equation, then your ansatz should look like $g(t)$;
 - 2 if $g(t)$ **does** solve the homogeneous equation, then “multiply by t ” (repeat as necessary);
- 3 1+2;
- 4 Find c_1 and c_2 .

3.8 Solving Initial Value Problems



Remark

You must do step 4 last. If you try to find c_1 and c_2 before doing the other steps, you may get the wrong answer.

3.8 Solving Initial Value Problems



Remark

You must do step 4 last. If you try to find c_1 and c_2 before doing the other steps, you may get the wrong answer.

Example

Solve

$$\begin{cases} y'' - y = 2e^t \\ y(0) = 1 \\ y'(0) = 2. \end{cases}$$



Correct Solution:

- 1 First we consider $y'' - y = 0$. The characteristic equation $r^2 - 1 = 0$ has roots $r_1 = 1$ and $r_2 = -1$. Hence the general solution is $y(t) = c_1 e^t + c_2 e^{-t}$.

3.8 Solving Initial Value Problems



- 2 Next we need to find a particular solution. Since Ae^t solves the homogeneous equation, we must “multiply by t ”. We try the ansatz $Y(t) = Ate^t$ and we calculate that

$$\begin{aligned}Y' &= Ae^t + Ate^t, \\Y'' &= 2Ae^t + Ate^t\end{aligned}$$

and

$$\begin{aligned}2e^t &= Y'' - Y \\&= 2Ae^t + Ate^t - Ate^t \\&= 2Ae^t.\end{aligned}$$

We must have $A = 1$. Therefore $Y(t) = te^t$ is a particular solution.

3.8 Solving Initial Value Problems



3 Thus

$$y(t) = c_1 e^t + c_2 e^{-t} + t e^t$$

is the general solution to the ODE.

3.8 Solving Initial Value Problems



3 Thus

$$y(t) = c_1 e^t + c_2 e^{-t} + t e^t$$

is the general solution to the ODE.

4 Finally we must satisfy the initial conditions. Since

$$y'(t) = c_1 e^t - c_2 e^{-t} + e^t + t e^t$$

we have

$$1 = y(0) = c_1 + c_2 + 0$$

$$2 = y'(0) = c_1 - c_2 + 1 + 0$$

which implies that $c_1 = 1$ and $c_2 = 0$. Therefore the solution to the IVP is

$$y(t) = e^t + t e^t.$$

$$y(t) = e^t + te^t$$



3 Thus

$$y(t) = c_1 e^t + c_2 e^{-t} + te^t$$

is the general solution to the ODE.

4 Finally we must satisfy the initial conditions. Since

$$y'(t) = c_1 e^t - c_2 e^{-t} + e^t + te^t$$

we have

$$1 = y(0) = c_1 + c_2 + 0$$

$$2 = y'(0) = c_1 - c_2 + 1 + 0$$

which implies that $c_1 = 1$ and $c_2 = 0$. Therefore the solution to the IVP is

$$y(t) = e^t + te^t.$$

$$y(t) = e^t + te^t$$



Incorrect Solution:

- 1 First we consider $y'' - y = 0$. The characteristic equation $r^2 - 1 = 0$ has roots $r_1 = 1$ and $r_2 = -1$. Hence the general solution is $y(t) = c_1 e^t + c_2 e^{-t}$.

$$y(t) = e^t + te^t$$



4 Next we find c_1 and c_2 . Since

$$y'(t) = c_1 e^t - c_2 e^{-t}$$

we have

$$1 = y(0) = c_1 + c_2$$

$$2 = y'(0) = c_1 - c_2$$

which implies that $c_1 = \frac{3}{2}$ and $c_2 = -\frac{1}{2}$. Thus

$$y(t) = \frac{3}{2}e^t - \frac{1}{2}e^{-t}.$$

$$y(t) = e^t + te^t$$



- 2 Next we need to find a particular solution. Since Ae^t solves the homogeneous equation, we must “multiply by t ”. We try the ansatz $Y(t) = Ate^t$ and we calculate that

$$\begin{aligned}Y' &= Ae^t + Ate^t, \\Y'' &= 2Ae^t + Ate^t\end{aligned}$$

and

$$\begin{aligned}2e^t &= Y'' - Y \\&= 2Ae^t + Ate^t - Ate^t \\&= 2Ae^t.\end{aligned}$$

We must have $A = 1$. Therefore $Y(t) = te^t$ is a particular solution.

$$y(t) = e^t + te^t$$



3 Finally we add our solutions together to get

$$y(t) = \frac{3}{2}e^t - \frac{1}{2}e^{-t} + te^t$$

$$y(t) = e^t + te^t$$



3 Finally we add our solutions together to get

$$y(t) = \frac{3}{2}e^t - \frac{1}{2}e^{-t} + te^t$$

which **is WRONG!!!** This function does not satisfy the initial conditions.

3.8 Solving Initial Value Problems



Example

Solve

$$\begin{cases} -y'' + 6y' - 16y = 1 + 6e^{3t} \sin(2t) \\ y(0) = \frac{15}{16} \\ y'(0) = -1. \end{cases} \quad (1)$$

(This is an exam question from 2013: Students had 30 minutes to solve this.)

3.8 Solving Initial Value Problems



To solve

$$-y'' + 6y' - 16y = 1 + 6e^{3t} \sin(2t)$$

we will consider 3 ODEs:

3.8 Solving Initial Value Problems



To solve

$$-y'' + 6y' - 16y = 1 + 6e^{3t} \sin(2t)$$

we will consider 3 ODEs:

- $-y'' + 6y' - 16y = 0$
- $-y'' + 6y' - 16y = 1$
- $-y'' + 6y' - 16y = 6e^{3t} \sin(2t).$

3.8 Solving Initial Value Problems



First consider the homogeneous equation $-y'' + 6y' - 16y = 0$.

3.8 Solving Initial Value Problems



First consider the homogeneous equation $-y'' + 6y' - 16y = 0$.
The characteristic equation is $-r^2 + 6r - 16 = 0$ which has roots
 $r = 3 \pm i\sqrt{7}$.

3.8 Solving Initial Value Problems



First consider the homogeneous equation $-y'' + 6y' - 16y = 0$.
The characteristic equation is $-r^2 + 6r - 16 = 0$ which has roots $r = 3 \pm i\sqrt{7}$. Therefore the general solution to $-y'' + 6y' - 16y = 0$ is

$$y(t) = c_1 e^{3t} \sin(\sqrt{7}t) + c_2 e^{3t} \cos(\sqrt{7}t).$$

3.8 Solving Initial Value Problems



Next consider $-y'' + 6y' - 16y = 1$.

3.8 Solving Initial Value Problems



Next consider $-y'' + 6y' - 16y = 1$. Trying the ansatz $Y(t) = C$, we see that

$$1 = -Y'' + 6Y' - 16Y = -16C.$$

We must choose $C = -\frac{1}{16}$. Hence $Y(t) = -\frac{1}{16}$.

3.8 Solving Initial Value Problems



Now consider $-y'' + 6y' - 16y = 6e^{3t} \sin(2t)$.

3.8 Solving Initial Value Problems



Now consider $-y'' + 6y' - 16y = 6e^{3t} \sin(2t)$. We try the ansatz $Y(t) = Ae^{3t} \cos 2t + Be^{3t} \sin 2t$ and find that

$$\begin{aligned} 6e^{3t} \sin 2t &= -Y'' + 6Y' - 16Y \\ &= -e^{3t} \left((5A + 12B) \cos 2t + (5B - 12A) \sin 2t \right) \\ &\quad + 6e^{3t} \left((3A + 2B) \cos 2t + (3B - 2A) \sin 2t \right) \\ &\quad - 16e^{3t} (A \cos 2t + B \sin 2t) \\ &= e^{3t} \cos 2t (-5A - 12B + 16A + 12B - 16A) \\ &\quad + e^{3t} \sin 2t (-5B + 12A + 18B - 12A - 16B) \\ &= e^{3t} \cos 2t (-5A) + e^{3t} \sin 2t (-3B). \end{aligned}$$

3.8 Solving Initial Value Problems



Now consider $-y'' + 6y' - 16y = 6e^{3t} \sin(2t)$. We try the ansatz $Y(t) = Ae^{3t} \cos 2t + Be^{3t} \sin 2t$ and find that

$$\begin{aligned} 6e^{3t} \sin 2t &= -Y'' + 6Y' - 16Y \\ &= -e^{3t} \left((5A + 12B) \cos 2t + (5B - 12A) \sin 2t \right) \\ &\quad + 6e^{3t} \left((3A + 2B) \cos 2t + (3B - 2A) \sin 2t \right) \\ &\quad - 16e^{3t} (A \cos 2t + B \sin 2t) \\ &= e^{3t} \cos 2t (-5A - 12B + 16A + 12B - 16A) \\ &\quad + e^{3t} \sin 2t (-5B + 12A + 18B - 12A - 16B) \\ &= e^{3t} \cos 2t (-5A) + e^{3t} \sin 2t (-3B). \end{aligned}$$

Thus, we need $A = 0$ and $B = -2$. Hence

$$Y(t) = -2e^{3t} \sin 2t.$$

3.8 Solving Initial Value Problems



Next we add these 3 solutions together. Therefore, the general solution to the ODE is

$$y(t) = c_1 e^{3t} \sin(\sqrt{7}t) + c_2 e^{3t} \cos(\sqrt{7}t) - \frac{1}{16} - 2e^{3t} \sin(2t).$$

3.8 Solving Initial Value Problems



$$y(t) = c_1 e^{3t} \sin(\sqrt{7}t) + c_2 e^{3t} \cos(\sqrt{7}t) - \frac{1}{16} - 2e^{3t} \sin(2t)$$

The final step is to choose c_1 and c_2 to satisfy the initial conditions.

3.8 Solving Initial Value Problems



$$y(t) = c_1 e^{3t} \sin(\sqrt{7}t) + c_2 e^{3t} \cos(\sqrt{7}t) - \frac{1}{16} - 2e^{3t} \sin(2t)$$

The final step is to choose c_1 and c_2 to satisfy the initial conditions.

$$\frac{15}{16} = y(0) = 0 + c_2 - \frac{1}{16} - 0 \quad \implies \quad c_2 = 1.$$

$$\begin{aligned} -1 &= y'(0) \\ &= 3c_1 e^{3t} \sin(\sqrt{7}t) + \sqrt{7}c_1 e^{3t} \cos(\sqrt{7}t) + 3e^{3t} \cos(\sqrt{7}t) \\ &\quad - \sqrt{7}e^{3t} \sin(\sqrt{7}t) - 6e^{3t} \sin(2t) - 4e^{3t} \cos(2t) \Big|_{t=0} \\ &= 0 + \sqrt{7}c_1 + 3 - 0 - 0 - 4 \quad \implies \quad c_1 = 0. \end{aligned}$$

3.8 Solving Initial Value Problems



Therefore, the solution to the IVP is

$$y(t) = e^{3t} \cos(\sqrt{7}t) - \frac{1}{16} - 2e^{3t} \sin(2t).$$

3.8 Solving Initial Value Problems



Remark

$$ay'' + by' + cy = g(t)$$

The method of undetermined coefficients works well if $g(t)$ is a nice function: e^{kt} , $\sin kt$, $t^3 + 2t^2 + 3t + 4$, $e^{at} \cosh kt$, \dots

However if $g(t)$ is a less nice function, then we may need a different method to find a particular solution.

The Method of Variation of Parameters

3.9 The Method of Variation of Parameters



Example

Find a particular solution to

$$y'' + 4y = 3 \operatorname{cosec} t \quad (2)$$

3.9 The Method of Variation of Parameters



Example

Find a particular solution to

$$y'' + 4y = 3 \operatorname{cosec} t \quad (2)$$

The homogeneous equation $y'' + 4y = 0$ has general solution $y = c_1 \cos 2t + c_2 \sin 2t$. The idea is:

3.9 The Method of Variation of Parameters



Example

Find a particular solution to

$$y'' + 4y = 3 \operatorname{cosec} t \quad (2)$$

The homogeneous equation $y'' + 4y = 0$ has general solution $y = c_1 \cos 2t + c_2 \sin 2t$. The idea is:

- 1 Replace the constants c_1 and c_2 by functions $u_1(t)$ and $u_2(t)$:

$$Y(t) = u_1(t) \cos 2t + u_2(t) \sin 2t.$$

3.9 The Method of Variation of Parameters



Example

Find a particular solution to

$$y'' + 4y = 3 \operatorname{cosec} t \quad (2)$$

The homogeneous equation $y'' + 4y = 0$ has general solution $y = c_1 \cos 2t + c_2 \sin 2t$. The idea is:

- 1 Replace the constants c_1 and c_2 by functions $u_1(t)$ and $u_2(t)$:

$$Y(t) = u_1(t) \cos 2t + u_2(t) \sin 2t.$$

- 2 Try to find u_1 and u_2 so that Y solves (2). There will be lots of u_1 and u_2 that we can use, so we will be free to add an extra condition.

3.9 The Method of Variation of Parameters



So suppose that

$$Y = u_1 \cos 2t + u_2 \sin 2t.$$

3.9 The Method of Variation of Parameters



So suppose that

$$Y = u_1 \cos 2t + u_2 \sin 2t.$$

Then

$$Y' = u_1' \cos 2t - 2u_1 \sin 2t + u_2' \sin 2t + 2u_2 \cos 2t$$

3.9 The Method of Variation of Parameters



So suppose that

$$Y = u_1 \cos 2t + u_2 \sin 2t.$$

Then

$$Y' = u_1' \cos 2t - 2u_1 \sin 2t + u_2' \sin 2t + 2u_2 \cos 2t$$

At this point, it is getting complicated so we will use our chance to add a condition: Suppose that

$$u_1' \cos 2t + u_2' \sin 2t = 0 \quad (3)$$

3.9 The Method of Variation of Parameters



So suppose that

$$Y = u_1 \cos 2t + u_2 \sin 2t.$$

Then

$$Y' = u_1' \cos 2t - 2u_1 \sin 2t + u_2' \sin 2t + 2u_2 \cos 2t$$

At this point, it is getting complicated so we will use our chance to add a condition: Suppose that

$$u_1' \cos 2t + u_2' \sin 2t = 0 \quad (3)$$

So

$$Y' = -2u_1 \sin 2t + 2u_2 \cos 2t$$

and

$$Y'' = -2u_1' \sin 2t - 4u_1 \cos 2t + 2u_2' \cos 2t - 4u_2 \sin 2t.$$

3.9 The Method of Variation of Parameters



Then

$$\begin{aligned} 3 \operatorname{cosec} t &= Y'' + 4Y \\ &= (-2u_1' \sin 2t - 4u_1 \cos 2t + 2u_2' \cos 2t - 4u_2 \sin 2t) \\ &\quad + 4(u_1 \cos 2t + u_2 \sin 2t) \\ &= -2u_1' \sin 2t + 2u_2' \cos 2t \end{aligned}$$

3.9 The Method of Variation of Parameters



Then

$$\begin{aligned}3 \operatorname{cosec} t &= Y'' + 4Y \\&= (-2u_1' \sin 2t - 4u_1 \cos 2t + 2u_2' \cos 2t - 4u_2 \sin 2t) \\&\quad + 4(u_1 \cos 2t + u_2 \sin 2t) \\&= -2u_1' \sin 2t + 2u_2' \cos 2t\end{aligned}$$

We want to find $u_1(t)$ and $u_2(t)$ which satisfy

$$\begin{cases} 3 \operatorname{cosec} t = -2u_1' \sin 2t + 2u_2' \cos 2t \\ u_1' \cos 2t + u_2' \sin 2t = 0 \end{cases}$$

3.9 The Method of Variation of Parameters



$$\begin{cases} 3 \operatorname{cosec} t = -2u'_1 \sin 2t + 2u'_2 \cos 2t \\ u'_1 \cos 2t + u'_2 \sin 2t = 0 \end{cases}$$

From the latter condition, we have $u'_2 = -u'_1 \frac{\cos 2t}{\sin 2t}$.

3.9 The Method of Variation of Parameters



$$\begin{cases} 3 \operatorname{cosec} t = -2u'_1 \sin 2t + 2u'_2 \cos 2t \\ u'_1 \cos 2t + u'_2 \sin 2t = 0 \end{cases}$$

From the latter condition, we have $u'_2 = -u'_1 \frac{\cos 2t}{\sin 2t}$. Putting this into the first condition, we calculate that

$$3 \operatorname{cosec} t = -2u'_1 \sin 2t + 2 \left(-u'_1 \frac{\cos 2t}{\sin 2t} \right) \cos 2t$$

$$3 \operatorname{cosec} t \sin 2t = -2u'_1 \sin^2 2t - 2u'_1 \cos^2 2t = -2u'_1$$

$$u'_1 = \frac{-3 \operatorname{cosec} t \sin 2t}{2} = \frac{-3 \sin 2t}{2 \sin t} = -3 \cos t$$

and

$$u'_2 = \frac{3 \cos t \cos 2t}{\sin 2t} = \frac{3 \cos t (1 - \sin^2 t)}{2 \sin t \cos t} = \frac{3}{2} \operatorname{cosec} t - 3 \sin t.$$

3.9 The Method of Variation of Parameters



Integrating gives

$$u_1(t) = \int u_1'(t) dt = \int -3 \cos t dt = -3 \sin t$$

$$\begin{aligned} u_2(t) &= \int u_2'(t) dt = \int \frac{3}{2} \operatorname{cosec} t - 3 \sin t dt \\ &= \frac{3}{2} \ln |\operatorname{cosec} t - \cot t| + 3 \cos t \end{aligned}$$

3.9 The Method of Variation of Parameters



Integrating gives

$$\begin{aligned}u_1(t) &= \int u_1'(t) dt = \int -3 \cos t dt = -3 \sin t \\u_2(t) &= \int u_2'(t) dt = \int \frac{3}{2} \operatorname{cosec} t - 3 \sin t dt \\&= \frac{3}{2} \ln |\operatorname{cosec} t - \cot t| + 3 \cos t\end{aligned}$$

Therefore a particular solution is

$$\begin{aligned}Y(t) &= u_1(t) \cos 2t + u_2(t) \sin 2t \\&= -3 \sin t \cos 2t + \frac{3}{2} \ln |\operatorname{cosec} t - \cot t| \sin 2t + 3 \cos t \sin 2t \\&= 3 \sin t + \frac{3}{2} \ln |\operatorname{cosec} t - \cot t| \sin 2t.\end{aligned}$$

3.9 The Method of Variation of Parameters



Summary

Suppose that $c_1y_1 + c_2y_2$ is the general solution of $L[y] = 0$.

3.9 The Method of Variation of Parameters



Summary

Suppose that $c_1y_1 + c_2y_2$ is the general solution of $L[y] = 0$.

- 1 Guess $Y = u_1(t)y_1 + u_2(t)y_2$;

3.9 The Method of Variation of Parameters



Summary

Suppose that $c_1y_1 + c_2y_2$ is the general solution of $L[y] = 0$.

- 1 Guess $Y = u_1(t)y_1 + u_2(t)y_2$;
- 2 Make the extra condition $u_1'y_1 + u_2'y_2 = 0$;

3.9 The Method of Variation of Parameters



Summary

Suppose that $c_1y_1 + c_2y_2$ is the general solution of $L[y] = 0$.

- 1 Guess $Y = u_1(t)y_1 + u_2(t)y_2$;
- 2 Make the extra condition $u_1'y_1 + u_2'y_2 = 0$;
- 3 Put Y into $L[y] = g(t)$;

3.9 The Method of Variation of Parameters



Summary

Suppose that $c_1y_1 + c_2y_2$ is the general solution of $L[y] = 0$.

- 1 Guess $Y = u_1(t)y_1 + u_2(t)y_2$;
- 2 Make the extra condition $u_1'y_1 + u_2'y_2 = 0$;
- 3 Put Y into $L[y] = g(t)$;
- 4 Find u_1' and u_2' ;

3.9 The Method of Variation of Parameters



Summary

Suppose that $c_1y_1 + c_2y_2$ is the general solution of $L[y] = 0$.

- 1 Guess $Y = u_1(t)y_1 + u_2(t)y_2$;
- 2 Make the extra condition $u_1'y_1 + u_2'y_2 = 0$;
- 3 Put Y into $L[y] = g(t)$;
- 4 Find u_1' and u_2' ;
- 5 Integrate to get u_1 and u_2 ;

3.9 The Method of Variation of Parameters



Summary

Suppose that $c_1y_1 + c_2y_2$ is the general solution of $L[y] = 0$.

- 1 Guess $Y = u_1(t)y_1 + u_2(t)y_2$;
- 2 Make the extra condition $u_1'y_1 + u_2'y_2 = 0$;
- 3 Put Y into $L[y] = g(t)$;
- 4 Find u_1' and u_2' ;
- 5 Integrate to get u_1 and u_2 ;

Then Y is a particular solution to $L[y] = g(t)$.

3.9 The Method of Variation of Parameters



Example

Find a particular solution to $y'' - 2y' + y = e^t \ln t$.

3.9 The Method of Variation of Parameters



Example

Find a particular solution to $y'' - 2y' + y = e^t \ln t$.

The characteristic equation, $0 = r^2 - 2r + 1 = (r - 1)^2$ has roots $r_1 = r_2 = 1$. Hence the general solution of the homogeneous equation is $y(t) = c_1 e^t + c_2 t e^t$.

3.9 The Method of Variation of Parameters



Example

Find a particular solution to $y'' - 2y' + y = e^t \ln t$.

The characteristic equation, $0 = r^2 - 2r + 1 = (r - 1)^2$ has roots $r_1 = r_2 = 1$. Hence the general solution of the homogeneous equation is $y(t) = c_1 e^t + c_2 t e^t$.

Therefore we guess that $Y = u_1(t)e^t + u_2(t)te^t$.

3.9 The Method of Variation of Parameters



Example

Find a particular solution to $y'' - 2y' + y = e^t \ln t$.

The characteristic equation, $0 = r^2 - 2r + 1 = (r - 1)^2$ has roots $r_1 = r_2 = 1$. Hence the general solution of the homogeneous equation is $y(t) = c_1 e^t + c_2 t e^t$.

Therefore we guess that $Y = u_1(t)e^t + u_2(t)te^t$.

We make the extra condition that

$$u_1' y_1 + u_2' y_2 = 0$$

$$u_1' e^t + u_2' t e^t = 0$$

$$u_1' + u_2' t = 0.$$

3.9 The Method of Variation of Parameters



Then we calculate that

$$Y' = u_1' e^t + u_1 e^t + u_2' t e^t + u_2 e^t + u_2 t e^t$$

$$=$$

$$Y'' =$$

$$=$$

and

$$e^t \ln t = Y'' - 2Y' + Y$$

$$=$$
$$=$$

3.9 The Method of Variation of Parameters



Then we calculate that

$$Y' = \cancel{u_1' e^t} + u_1 e^t + \cancel{u_2' t e^t} + u_2 e^t + u_2 t e^t$$
$$=$$

$$Y'' =$$
$$=$$

and

$$e^t \ln t = Y'' - 2Y' + Y$$
$$=$$
$$=$$

3.9 The Method of Variation of Parameters



Then we calculate that

$$\begin{aligned} Y' &= \cancel{u_1' e^t} + u_1 e^t + \cancel{u_2' t e^t} + u_2 e^t + u_2 t e^t \\ &= u_1 e^t + u_2 e^t + u_2 t e^t, \end{aligned}$$

$$\begin{aligned} Y'' &= \\ &= \end{aligned}$$

and

$$\begin{aligned} e^t \ln t &= Y'' - 2Y' + Y \\ &= \\ &= \end{aligned}$$

3.9 The Method of Variation of Parameters



Then we calculate that

$$\begin{aligned} Y' &= \cancel{u_1' e^t} + u_1 e^t + \cancel{u_2' t e^t} + u_2 e^t + u_2 t e^t \\ &= u_1 e^t + u_2 e^t + u_2 t e^t, \end{aligned}$$

$$\begin{aligned} Y'' &= u_1' e^t + u_1 e^t + u_2' e^t + u_2 e^t + u_2' t e^t + u_2 e^t + u_2 t e^t \\ &= \end{aligned}$$

and

$$\begin{aligned} e^t \ln t &= Y'' - 2Y' + Y \\ &= \\ &= \end{aligned}$$

3.9 The Method of Variation of Parameters



Then we calculate that

$$\begin{aligned} Y' &= \cancel{u_1' e^t} + u_1 e^t + \cancel{u_2' t e^t} + u_2 e^t + u_2 t e^t \\ &= u_1 e^t + u_2 e^t + u_2 t e^t, \end{aligned}$$

$$\begin{aligned} Y'' &= \cancel{u_1' e^t} + u_1 e^t + u_2' e^t + u_2 e^t + \cancel{u_2' t e^t} + u_2 e^t + u_2 t e^t \\ &= \end{aligned}$$

and

$$\begin{aligned} e^t \ln t &= Y'' - 2Y' + Y \\ &= \\ &= \end{aligned}$$

3.9 The Method of Variation of Parameters



Then we calculate that

$$\begin{aligned} Y' &= \cancel{u_1' e^t} + u_1 e^t + \cancel{u_2' t e^t} + u_2 e^t + u_2 t e^t \\ &= u_1 e^t + u_2 e^t + u_2 t e^t, \end{aligned}$$

$$\begin{aligned} Y'' &= \cancel{u_1' e^t} + u_1 e^t + u_2' e^t + u_2 e^t + \cancel{u_2' t e^t} + u_2 e^t + u_2 t e^t \\ &= u_1 e^t + u_2' e^t + 2u_2 e^t + u_2 t e^t \end{aligned}$$

and

$$\begin{aligned} e^t \ln t &= Y'' - 2Y' + Y \\ &= \\ &= \end{aligned}$$

3.9 The Method of Variation of Parameters



Then we calculate that

$$\begin{aligned} Y' &= \cancel{u_1' e^t} + u_1 e^t + \cancel{u_2' t e^t} + u_2 e^t + u_2 t e^t \\ &= u_1 e^t + u_2 e^t + u_2 t e^t, \end{aligned}$$

$$\begin{aligned} Y'' &= \cancel{u_1' e^t} + u_1 e^t + u_2' e^t + u_2 e^t + \cancel{u_2' t e^t} + u_2 e^t + u_2 t e^t \\ &= u_1 e^t + u_2' e^t + 2u_2 e^t + u_2 t e^t \end{aligned}$$

and

$$\begin{aligned} e^t \ln t &= Y'' - 2Y' + Y \\ &= (u_1 e^t + u_2' e^t + 2u_2 e^t + u_2 t e^t) - 2(u_1 e^t + u_2 e^t + u_2 t e^t) \\ &\quad + (u_1 e^t + u_2 t e^t) \\ &= \end{aligned}$$

3.9 The Method of Variation of Parameters



Then we calculate that

$$\begin{aligned} Y' &= \cancel{u_1' e^t} + u_1 e^t + \cancel{u_2' t e^t} + u_2 e^t + u_2 t e^t \\ &= u_1 e^t + u_2 e^t + u_2 t e^t, \end{aligned}$$

$$\begin{aligned} Y'' &= \cancel{u_1' e^t} + u_1 e^t + u_2' e^t + u_2 e^t + \cancel{u_2' t e^t} + u_2 e^t + u_2 t e^t \\ &= u_1 e^t + u_2' e^t + 2u_2 e^t + u_2 t e^t \end{aligned}$$

and

$$\begin{aligned} e^t \ln t &= Y'' - 2Y' + Y \\ &= (u_1 e^t + u_2' e^t + 2u_2 e^t + u_2 t e^t) - 2(u_1 e^t + u_2 e^t + u_2 t e^t) \\ &\quad + (u_1 e^t + u_2 t e^t) \\ &= u_2' e^t. \end{aligned}$$

3.9 The Method of Variation of Parameters



Then we calculate that

$$\begin{aligned} Y' &= \cancel{u_1' e^t} + u_1 e^t + \cancel{u_2' t e^t} + u_2 e^t + u_2 t e^t \\ &= u_1 e^t + u_2 e^t + u_2 t e^t, \end{aligned}$$

$$\begin{aligned} Y'' &= \cancel{u_1' e^t} + u_1 e^t + u_2' e^t + u_2 e^t + \cancel{u_2' t e^t} + u_2 e^t + u_2 t e^t \\ &= u_1 e^t + u_2' e^t + 2u_2 e^t + u_2 t e^t \end{aligned}$$

and

$$\begin{aligned} e^t \ln t &= Y'' - 2Y' + Y \\ &= (u_1 e^t + u_2' e^t + 2u_2 e^t + u_2 t e^t) - 2(u_1 e^t + u_2 e^t + u_2 t e^t) \\ &\quad + (u_1 e^t + u_2 t e^t) \\ &= u_2' e^t. \end{aligned}$$

It follows that $u_2' = \ln t$ and thus $u_1' = -u_2' t = -t \ln t$.

3.9 The Method of Variation of Parameters



Next we integrate to find

$$u_1(t) = \int u_1'(t) dt = \int -t \ln t dt = -\frac{1}{2}t^2 \ln t + \frac{1}{4}t^2$$

and

$$u_2(t) = \int u_2'(t) dt = \int \ln t dt = t \ln t - t.$$

3.9 The Method of Variation of Parameters



Next we integrate to find

$$u_1(t) = \int u_1'(t) dt = \int -t \ln t dt = -\frac{1}{2}t^2 \ln t + \frac{1}{4}t^2$$

and

$$u_2(t) = \int u_2'(t) dt = \int \ln t dt = t \ln t - t.$$

Therefore a particular solution is

$$\begin{aligned} Y(t) &= u_1(t)e^t + u_2(t)te^t \\ &= \left(-\frac{1}{2}t^2 \ln t + \frac{1}{4}t^2\right) e^t + (t \ln t - t) te^t \\ &= \left(\frac{1}{2} \ln t - \frac{3}{4}\right) t^2 e^t. \end{aligned}$$

3.9 The Method of Variation of Parameters



Isn't there an easier way?

3.9 The Method of Variation of Parameters



Isn't there an easier way?

Theorem

Suppose that $y_1(t)$ and $y_2(t)$ form a fundamental set of solutions of $y'' + p(t)y' + q(t)y = 0$. Then a particular solution of $y'' + p(t)y' + q(t)y = g(t)$ is given by

$$Y(t) =$$

3.9 The Method of Variation of Parameters



Isn't there an easier way?

Theorem

Suppose that $y_1(t)$ and $y_2(t)$ form a fundamental set of solutions of $y'' + p(t)y' + q(t)y = 0$. Then a particular solution of $y'' + p(t)y' + q(t)y = g(t)$ is given by

$$Y(t) = -y_1 \int \frac{y_2 g}{W} + y_2 \int \frac{y_1 g}{W} \quad (4)$$

where $W = W(y_1, y_2)$ is the Wronskian.

3.9 The Method of Variation of Parameters



Example

Find a particular solution to $y'' - 2y' + y = e^t \ln t$.

3.9 The Method of Variation of Parameters



Example

Find a particular solution to $y'' - 2y' + y = e^t \ln t$.

The characteristic equation $0 = r^2 - 2r + 1 = (r - 1)^2$ has roots $r_1 = r_2 = 1$. Hence

$$y_1 = e^t \quad \text{and} \quad y_2 = te^t$$

form a fundamental set of solutions to the homogeneous equation $y'' - 2y' + y = 0$.

3.9 The Method of Variation of Parameters



Example

Find a particular solution to $y'' - 2y' + y = e^t \ln t$.

The characteristic equation $0 = r^2 - 2r + 1 = (r - 1)^2$ has roots $r_1 = r_2 = 1$. Hence

$$y_1 = e^t \quad \text{and} \quad y_2 = te^t$$

form a fundamental set of solutions to the homogeneous equation $y'' - 2y' + y = 0$.

We calculate that

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} e^t & te^t \\ e^t & e^t + te^t \end{vmatrix} = e^{2t} + te^{2t} - te^{2t} = e^{2t}.$$

3.9 The Method of Variation of Parameters



$$y_1 = e^t \quad y_2 = te^t \quad g = e^t \ln t \quad W = e^{2t}$$

It follows that

$$\begin{aligned} Y(t) &= -y_1 \int \frac{y_2 g}{W} + y_2 \int \frac{y_1 g}{W} \\ &= \\ &= \\ &= \\ &= \end{aligned}$$

is a particular solution to the ODE.

3.9 The Method of Variation of Parameters



$$y_1 = e^t \quad y_2 = te^t \quad g = e^t \ln t \quad W = e^{2t}$$

It follows that

$$\begin{aligned} Y(t) &= -y_1 \int \frac{y_2 g}{W} + y_2 \int \frac{y_1 g}{W} \\ &= -e^t \int \frac{te^t e^t \ln t}{e^{2t}} dt + te^t \int \frac{e^t e^t \ln t}{e^{2t}} dt \\ &= \\ &= \\ &= \end{aligned}$$

is a particular solution to the ODE.

3.9 The Method of Variation of Parameters



$$y_1 = e^t \quad y_2 = te^t \quad g = e^t \ln t \quad W = e^{2t}$$

It follows that

$$\begin{aligned} Y(t) &= -y_1 \int \frac{y_2 g}{W} + y_2 \int \frac{y_1 g}{W} \\ &= -e^t \int \frac{te^t e^t \ln t}{e^{2t}} dt + te^t \int \frac{e^t e^t \ln t}{e^{2t}} dt \\ &= -e^t \int t \ln t \, dt + te^t \int \ln t \, dt \\ &= \\ &= \end{aligned}$$

is a particular solution to the ODE.

3.9 The Method of Variation of Parameters



$$y_1 = e^t \quad y_2 = te^t \quad g = e^t \ln t \quad W = e^{2t}$$

It follows that

$$\begin{aligned} Y(t) &= -y_1 \int \frac{y_2 g}{W} + y_2 \int \frac{y_1 g}{W} \\ &= -e^t \int \frac{te^t e^t \ln t}{e^{2t}} dt + te^t \int \frac{e^t e^t \ln t}{e^{2t}} dt \\ &= -e^t \int t \ln t \, dt + te^t \int \ln t \, dt \\ &= -e^t \left(\frac{1}{2} t^2 \ln t - \frac{1}{4} t^2 \right) + te^t (t \ln t - t) \\ &= \end{aligned}$$

is a particular solution to the ODE.

3.9 The Method of Variation of Parameters



$$y_1 = e^t \quad y_2 = te^t \quad g = e^t \ln t \quad W = e^{2t}$$

It follows that

$$\begin{aligned} Y(t) &= -y_1 \int \frac{y_2 g}{W} + y_2 \int \frac{y_1 g}{W} \\ &= -e^t \int \frac{te^t e^t \ln t}{e^{2t}} dt + te^t \int \frac{e^t e^t \ln t}{e^{2t}} dt \\ &= -e^t \int t \ln t \, dt + te^t \int \ln t \, dt \\ &= -e^t \left(\frac{1}{2} t^2 \ln t - \frac{1}{4} t^2 \right) + te^t (t \ln t - t) \\ &= \left(\frac{1}{2} \ln t - \frac{3}{4} \right) t^2 e^t \end{aligned}$$

is a particular solution to the ODE.



Higher Order Linear ODEs

3.10 Higher Order Linear ODEs



We can use the same ideas to solve higher order linear ODEs.

3.10 Higher Order Linear ODEs



Example

Solve

$$\begin{cases} y^{(4)} + y''' - 7y'' - y' + 6y = 0 \\ y(0) = 1 \\ y'(0) = 0 \\ y''(0) = -2 \\ y'''(0) = -1. \end{cases}$$

3.10 Higher Order Linear ODEs



Example

Solve

$$\begin{cases} y^{(4)} + y''' - 7y'' - y' + 6y = 0 \\ y(0) = 1 \\ y'(0) = 0 \\ y''(0) = -2 \\ y'''(0) = -1. \end{cases}$$

The characteristic equation is

$$r^4 + r^3 - 7r^2 - r + 6 = 0$$

which has roots $r_1 = 1$, $r_2 = -1$, $r_3 = 2$ and $r_4 = -3$.

Example

Solve

$$\begin{cases} y^{(4)} + y''' - 7y'' - y' + 6y = 0 \\ y(0) = 1 \\ y'(0) = 0 \\ y''(0) = -2 \\ y'''(0) = -1. \end{cases}$$

The characteristic equation is

$$r^4 + r^3 - 7r^2 - r + 6 = 0$$

which has roots $r_1 = 1$, $r_2 = -1$, $r_3 = 2$ and $r_4 = -3$. So the general solution to the ODE is

$$y(t) = c_1 e^t + c_2 e^{-t} + c_3 e^{2t} + c_4 e^{-3t}.$$

3.10 Higher Order Linear ODEs



Then

$$1 = y(0) = c_1 + c_2 + c_3 + c_4 + 4$$

$$0 = y'(0) = c_1 - c_2 + 2c_3 - 3c_4$$

$$-2 = y''(0) = c_1 + c_2 + 4c_3 + 9c_4$$

$$-1 = y'''(0) = c_1 - c_2 + 8c_3 - 27c_4$$

3.10 Higher Order Linear ODEs



Then

$$\begin{aligned} 1 &= y(0) = c_1 + c_2 + c_3 + c_4 + 4 \\ 0 &= y'(0) = c_1 - c_2 + 2c_3 - 3c_4 \\ -2 &= y''(0) = c_1 + c_2 + 4c_3 + 9c_4 \\ -1 &= y'''(0) = c_1 - c_2 + 8c_3 - 27c_4 \end{aligned} \quad \Rightarrow \quad \begin{aligned} c_1 &= \frac{11}{8} \\ c_2 &= \frac{5}{12} \\ c_3 &= -\frac{2}{3} \\ c_4 &= -\frac{1}{8} \end{aligned}$$

3.10 Higher Order Linear ODEs



Then

$$\begin{aligned} 1 &= y(0) = c_1 + c_2 + c_3 + c_4 + 4 \\ 0 &= y'(0) = c_1 - c_2 + 2c_3 - 3c_4 \\ -2 &= y''(0) = c_1 + c_2 + 4c_3 + 9c_4 \\ -1 &= y'''(0) = c_1 - c_2 + 8c_3 - 27c_4 \end{aligned} \quad \Rightarrow \quad \begin{aligned} c_1 &= \frac{11}{8} \\ c_2 &= \frac{5}{12} \\ c_3 &= -\frac{2}{3} \\ c_4 &= -\frac{1}{8} \end{aligned}$$

Therefore the solution to the IVP is

$$y = \frac{11}{8}e^t + \frac{5}{12}e^{-t} - \frac{2}{3}e^{2t} - \frac{1}{8}e^{-3t}.$$

3.10 Higher Order Linear ODEs



Example

Solve

$$y^{(4)} - y = e^t$$

Example

Solve

$$y^{(4)} - y = e^t$$

The characteristic equation

$$0 = r^4 - 1 = (r^2 - 1)(r^2 + 1)$$

has roots $r_1 = 1$, $r_2 = -1$, $r_3 = i$ and $r_4 = -i$.

Example

Solve

$$y^{(4)} - y = e^t$$

The characteristic equation

$$0 = r^4 - 1 = (r^2 - 1)(r^2 + 1)$$

has roots $r_1 = 1$, $r_2 = -1$, $r_3 = i$ and $r_4 = -i$. Therefore

$$y = c_1 e^t + c_2 e^{-t} + c_3 \cos t + c_4 \sin t$$

is the general solution of the homogenous equation $y^{(4)} - y = 0$.

3.10 Higher Order Linear ODEs



$$y^{(4)} - y = e^t$$

Next we need to find a particular solution.

3.10 Higher Order Linear ODEs



$$y^{(4)} - y = e^t$$

Next we need to find a particular solution. Since e^t solves the homogeneous equation, we try the ansatz $Y = Ate^t$.

3.10 Higher Order Linear ODEs



$$y^{(4)} - y = e^t$$

Next we need to find a particular solution. Since e^t solves the homogeneous equation, we try the ansatz $Y = Ate^t$. Then

$$Y' = Ae^t + Ate^t$$

$$Y'' = Ae^t + Ae^t + Ate^t = 2Ae^t + Ate^t$$

$$Y''' = 2Ae^t + Ae^t + Ate^t = 3Ae^t + Ate^t$$

$$Y^{(4)} = 3Ae^t + Ae^t + Ate^t = 4Ae^t + Ate^t$$

$$y^{(4)} - y = e^t$$

Next we need to find a particular solution. Since e^t solves the homogeneous equation, we try the ansatz $Y = Ate^t$. Then

$$Y' = Ae^t + Ate^t$$

$$Y'' = Ae^t + Ae^t + Ate^t = 2Ae^t + Ate^t$$

$$Y''' = 2Ae^t + Ae^t + Ate^t = 3Ae^t + Ate^t$$

$$Y^{(4)} = 3Ae^t + Ae^t + Ate^t = 4Ae^t + Ate^t$$

and

$$e^t = Y^{(4)} - Y = 4Ae^t + Ate^t - Ate^t = 4Ae^t \quad \implies \quad A = \frac{1}{4}.$$

$$y^{(4)} - y = e^t$$

Next we need to find a particular solution. Since e^t solves the homogeneous equation, we try the ansatz $Y = Ate^t$. Then

$$Y' = Ae^t + Ate^t$$

$$Y'' = Ae^t + Ae^t + Ate^t = 2Ae^t + Ate^t$$

$$Y''' = 2Ae^t + Ae^t + Ate^t = 3Ae^t + Ate^t$$

$$Y^{(4)} = 3Ae^t + Ae^t + Ate^t = 4Ae^t + Ate^t$$

and

$$e^t = Y^{(4)} - Y = 4Ae^t + Ate^t - Ate^t = 4Ae^t \quad \implies \quad A = \frac{1}{4}.$$

Therefore $Y(t) = \frac{1}{4}te^t$ is a particular solution to the ODE.

3.10 Higher Order Linear ODEs



The general solution to the ODE is therefore

$$y(t) = c_1 e^t + c_2 e^{-t} + c_3 \cos t + c_4 \sin t + \frac{1}{4} t e^t.$$

3.10 Higher Order Linear ODEs



Remark

Any time the characteristic equation has a repeated root, just multiply by t .

3.10 Higher Order Linear ODEs



Remark

Any time the characteristic equation has a repeated root, just multiply by t . E.g. if the roots are $r_1 = 7$, $r_2 = 7$, $r_3 = 7$, $r_4 = 7$, $r_5 = 7$ and $r_6 = 8$, then the general solution is

$$y(t) = c_1 e^{7t} + c_2 t e^{7t} + c_3 t^2 e^{7t} + c_4 t^3 e^{7t} + c_5 t^4 e^{7t} + c_6 e^{8t}.$$

3.10 Higher Order Linear ODEs



Example (Going backwards)

Find a linear, homogeneous ODEs with constant coefficients, which has general solution

$$y(t) = c_1 e^t + c_2 t e^t + c_3 e^{2t} \sin t + c_4 e^{2t} \cos t + c_5 e^{2t} t \sin t + c_6 e^{2t} t \cos t.$$

3.10 Higher Order Linear ODEs



Example (Going backwards)

Find a linear, homogeneous ODEs with constant coefficients, which has general solution

$$y(t) = c_1 e^t + c_2 t e^t + c_3 e^{2t} \sin t + c_4 e^{2t} \cos t + c_5 e^{2t} t \sin t + c_6 e^{2t} t \cos t.$$

The first two terms correspond to a double root $r = 1$.

3.10 Higher Order Linear ODEs



Example (Going backwards)

Find a linear, homogeneous ODEs with constant coefficients, which has general solution

$$y(t) = c_1 e^t + c_2 t e^t + c_3 e^{2t} \sin t + c_4 e^{2t} \cos t + c_5 e^{2t} t \sin t + c_6 e^{2t} t \cos t.$$

The first two terms correspond to a double root $r = 1$. The last four terms correspond to a double complex root $r = 2 \pm i$.

3.10 Higher Order Linear ODEs



Example (Going backwards)

Find a linear, homogeneous ODEs with constant coefficients, which has general solution

$$y(t) = c_1 e^t + c_2 t e^t + c_3 e^{2t} \sin t + c_4 e^{2t} \cos t + c_5 e^{2t} t \sin t + c_6 e^{2t} t \cos t.$$

The first two terms correspond to a double root $r = 1$. The last four terms correspond to a double complex root $r = 2 \pm i$.

Consequently, the characteristic equation is

$$\begin{aligned} 0 &= (r - 1)^2 (r - 2 - i)^2 (r - 2 + i)^2 \\ &= (r - 1)^2 (r^2 - 4r + 5)^2 \\ &= r^6 - 10r^5 + 43r^4 - 100r^3 + 131r^2 - 90r + 25. \end{aligned}$$

3.10 Higher Order Linear ODEs



Example (Going backwards)

Find a linear, homogeneous ODEs with constant coefficients, which has general solution

$$y(t) = c_1 e^t + c_2 t e^t + c_3 e^{2t} \sin t + c_4 e^{2t} \cos t + c_5 e^{2t} t \sin t + c_6 e^{2t} t \cos t.$$

The first two terms correspond to a double root $r = 1$. The last four terms correspond to a double complex root $r = 2 \pm i$.

Consequently, the characteristic equation is

$$\begin{aligned} 0 &= (r - 1)^2 (r - 2 - i)^2 (r - 2 + i)^2 \\ &= (r - 1)^2 (r^2 - 4r + 5)^2 \\ &= r^6 - 10r^5 + 43r^4 - 100r^3 + 131r^2 - 90r + 25. \end{aligned}$$

Then, a differential equation is

$$\frac{d^6 y}{dt^6} - 10 \frac{d^5 y}{dt^5} + 43 \frac{d^4 y}{dt^4} - 100 \frac{d^3 y}{dt^3} + 131 \frac{d^2 y}{dt^2} - 90 \frac{dy}{dt} + 25y = 0.$$

Next Time

- 4.1 Definition of the Laplace Transform
- 4.2 Solving Initial Value Problems