Lecture 8

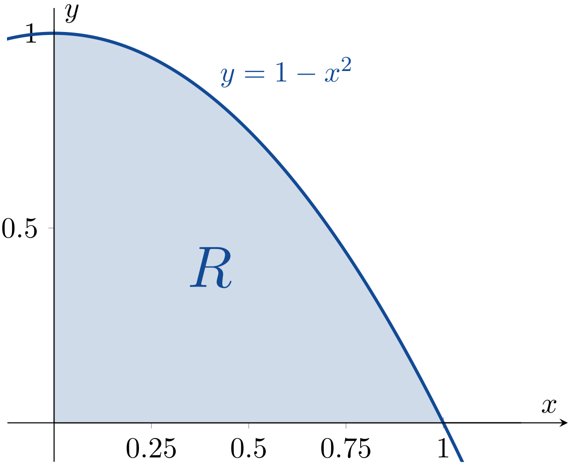
- 5.1 Area and Estimating with Finite Sums
- 5.2 Sigma Notation and Limits of Finite Sums
- 5.3 The Definite Integral

Area and Estimating with Finite Sums

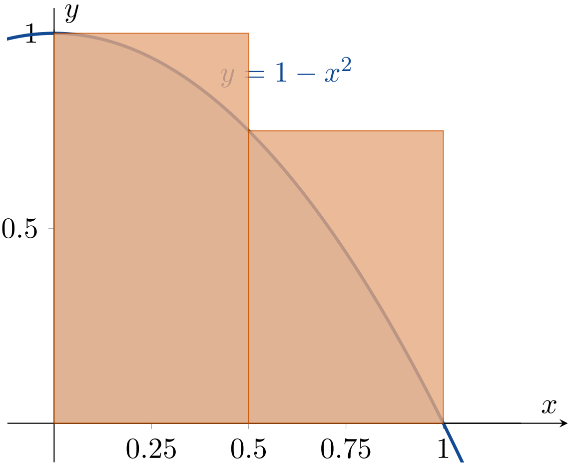
I am just going to give a quick recap of this section.

Please read more details in the textbook.

What is the area of R?

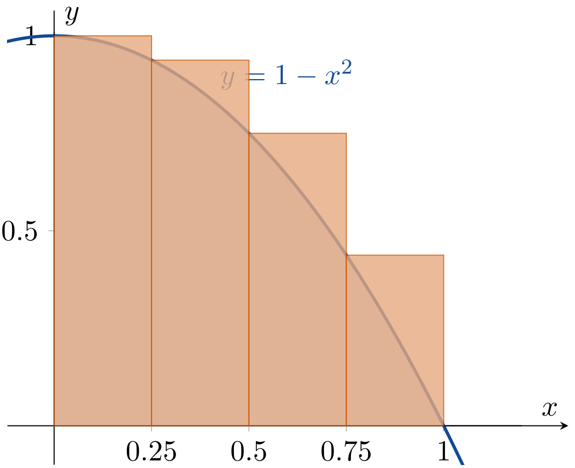


Upper Sum ($n = 2$)



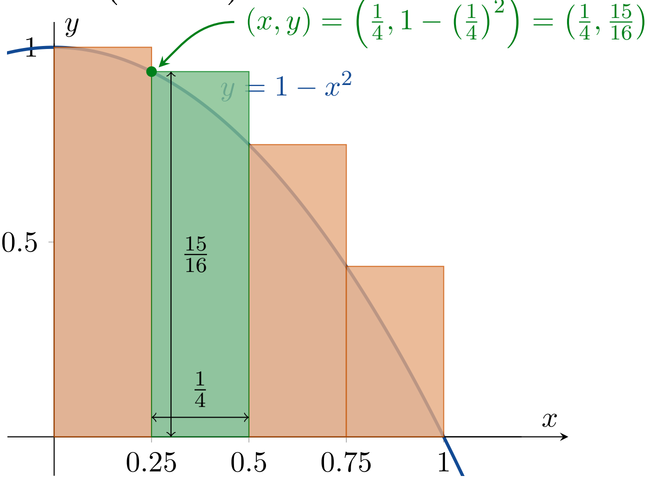
$$1 \cdot \frac{1}{2} + \frac{3}{4} \cdot \frac{1}{2} = \frac{7}{8} = 0.875$$

Upper Sum $(n = 4)$



$$1 \cdot \frac{1}{4} + \frac{15}{16} \cdot \frac{1}{4} + \frac{3}{4} \cdot \frac{1}{4} + \frac{7}{16} \cdot \frac{1}{4} = \frac{25}{32} = 0.78125$$

Upper Sum $(n = 4)$



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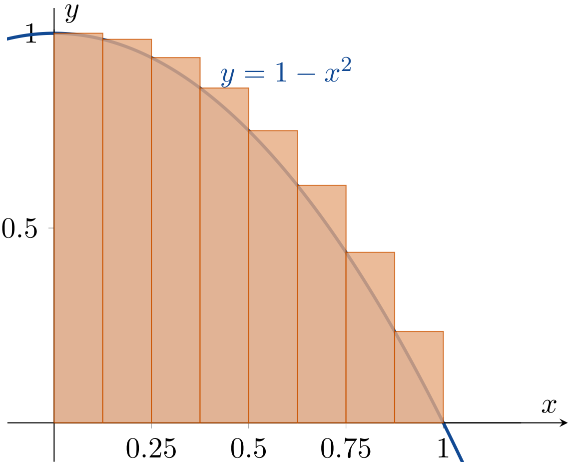
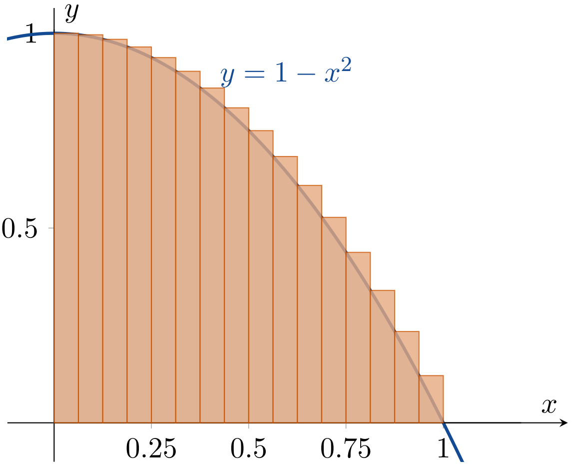
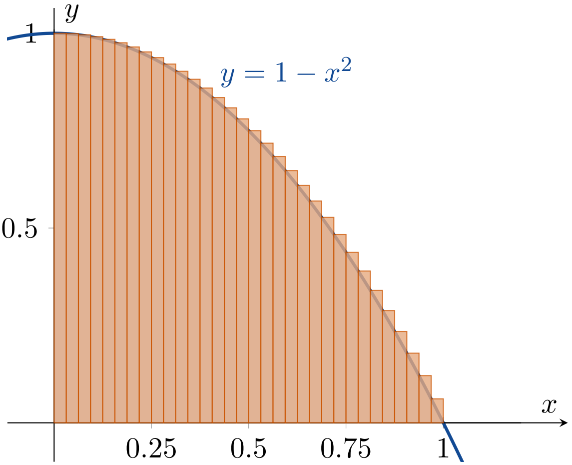
Upper Sum $(n = 8)$



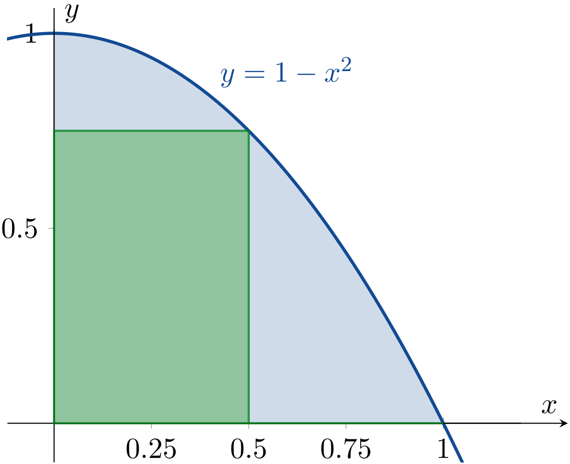
Upper Sum ($n = 16$)



Upper Sum $(n = 32)$

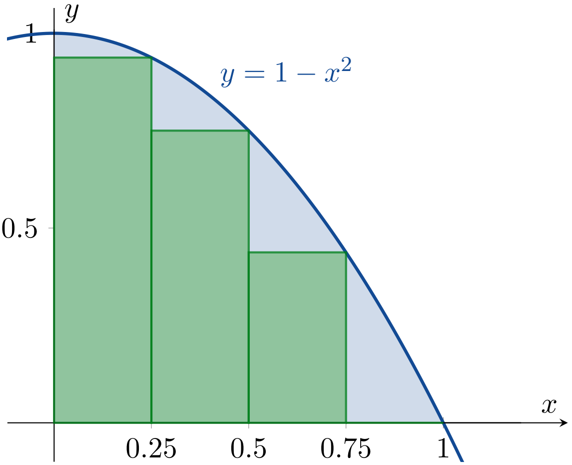


Lower Sum ($n = 2$)



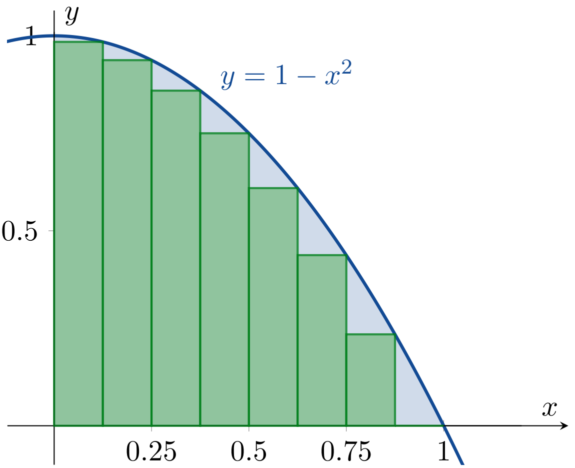
$$\frac{3}{4} \cdot \frac{1}{2} + 0 \cdot \frac{1}{2} = \frac{3}{8} = 0.375$$

Lower Sum ($n = 4$)

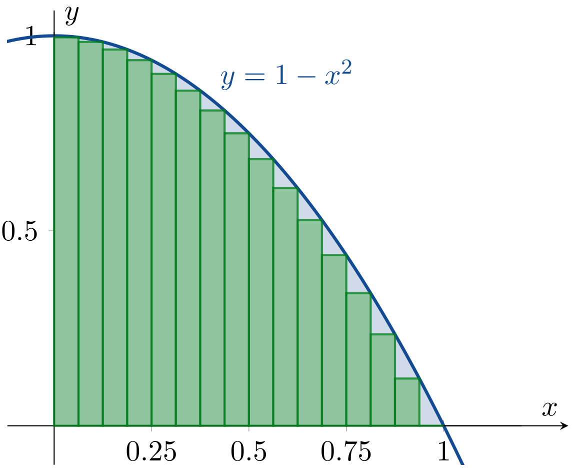


$$\frac{15}{16} \cdot \frac{1}{4} + \frac{3}{4} \cdot \frac{1}{4} + \frac{7}{16} \cdot \frac{1}{4} + 0 \cdot \frac{1}{4} = \frac{17}{32} = 0.53125$$

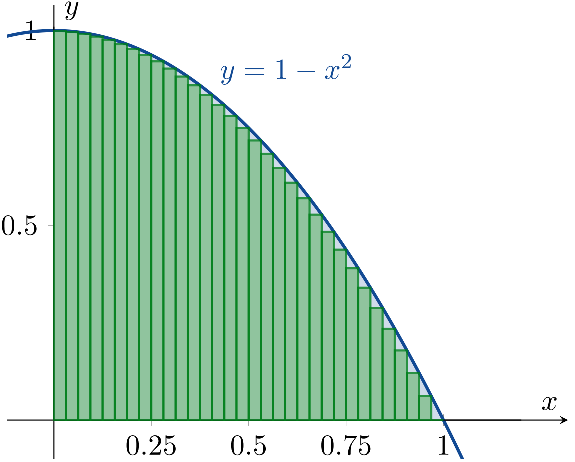
Lower Sum ($n = 8$)



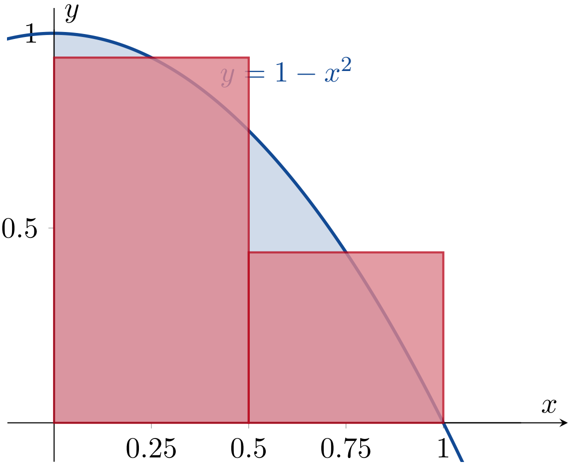
Lower Sum ($n = 16$)



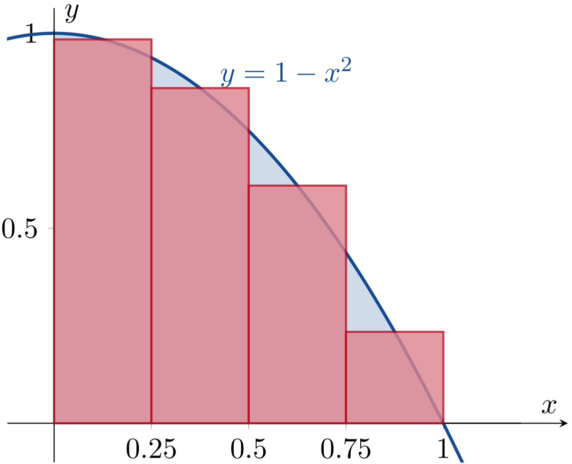
Lower Sum ($n = 32$)



Midpoint Sum ($n = 2$)

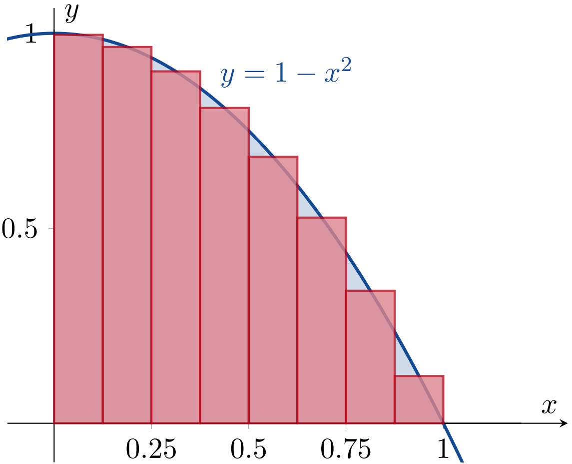


Midpoint Sum ($n = 4$)

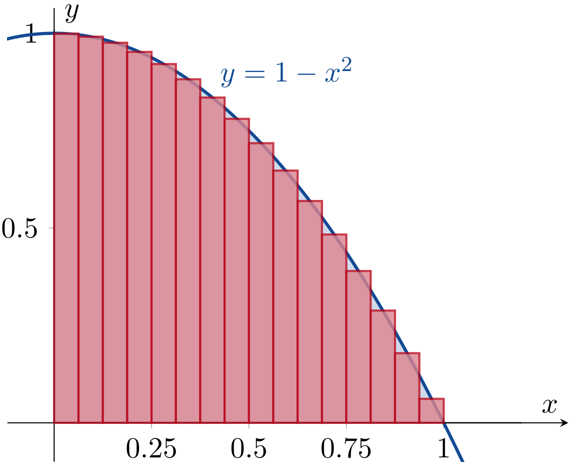


$$\frac{63}{64}\cdot\frac{1}{4}+\frac{55}{64}\cdot\frac{1}{4}+\frac{39}{64}\cdot\frac{1}{4}+\frac{15}{64}\cdot\frac{1}{4}=\frac{172}{256}=0.671875$$

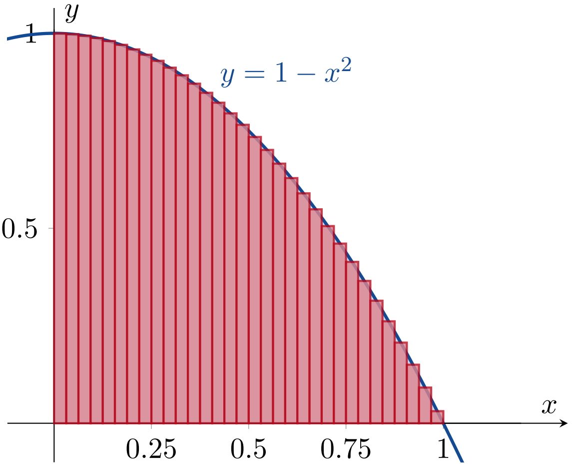
Midpoint Sum ($n = 8$)



Midpoint Sum ($n = 16$)



Midpoint Sum ($n = 32$)



Number of subintervals	Lower sum	Midpoint sum	Upper sum
2	0.375	0.6875	0.875
4	0.53125	0.671875	0.78125
16	0.634765625	0.6669921875	0.697265625
50	0.6566	0.6667	0.6766
100	0.66165	0.666675	0.67165
1000	0.6661665	0.66666675	0.6671665

Remark

As n increases, the estimates get closer and closer to the real area of R. We want to take the limit as $n \to \infty$, but we don't yet have enough notation.

Sigma Notation and Limits of Finite Sums

$$\sum_{k=1}^{n} a_k = a_1 + a_2 + a_3 + a_4 + a_5 + a_6 + a_7 + a_8 + \ldots + a_{n-1} + a_n$$

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$$\sum_{k=1}^{n} a_k$$

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the Greek letter Sigma

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$$\sum_{k=1}^{n} a_k$$

the sum starts at $k = 1$

$$\sum_{k=1}^{n} a_k = a_1 + a_2 + a_3 + a_4 + a_5 + a_6 + a_7 + a_8 + \ldots + a_{n-1} + a_n$$

the sum finishes at $k = n$

the Greek letter Sigma

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5.2 Sigma Notation and Limits of Finite Sums

$$\sum_{k=1}^{n} a_k = a_1 + a_2 + a_3 + a_4 + a_5 + a_6 + a_7 + a_8 + \ldots + a_{n-1} + a_n$$

the sum finishes at $k = n$

the Greek letter Sigma

$$\sum_{k=1}^{n} a_k$$

a_k is a formula for the k^{th} term.

the sum starts at $k = 1$

Example

$$1^2 + 2^2 + 3^2 + 4^2 + 5^2 + 6^2 + 7^2 + 8^2 + 9^2 + 10^2 + 11^2 = \sum_{k=1}^{11} k^2$$

$$f(1) + f(2) + f(3) + \ldots + f(99) + f(100) = \sum_{k=1}^{100} f(k)$$

$$\sum_{k=1}^{5} k = 1 + 2 + 3 + 4 + 5 = 15$$

Example

$$\sum_{k=1}^{3}(-1)^k k = (-1)(1) + (-1)^2(2) + (-1)^3(3) = -1 + 2 - 3 = -2$$

$$\sum_{k=1}^{2}\frac{k}{k+1} = \frac{1}{1+1} + \frac{2}{2+1} = \frac{1}{2} + \frac{2}{3} = \frac{7}{6}$$

$$\sum_{k=4}^{5}\frac{k^2}{k-1} = \frac{4^2}{4-1} + \frac{5^2}{5-1} = \frac{16}{3} + \frac{25}{4} = \frac{139}{12}$$

Algebra Rules for Finite Sums

1. *Sum Rule:* $$\sum_{k=1}^{n}(a_k + b_k) = \sum_{k=1}^{n} a_k + \sum_{k=1}^{n} b_k$$

2. *Difference Rule:* $$\sum_{k=1}^{n}(a_k - b_k) = \sum_{k=1}^{n} a_k - \sum_{k=1}^{n} b_k$$

3. *Constant Multiple Rule:* $$\sum_{k=1}^{n} ca_k = c \cdot \sum_{k=1}^{n} a_k$$ (Any number c)

4. *Constant Value Rule:* $$\sum_{k=1}^{n} c = n \cdot c$$ (Any number c)

EXAMPLE 3 We demonstrate the use of the algebra rules.

(a) $\displaystyle\sum_{k=1}^{n}(3k - k^2) = 3\sum_{k=1}^{n}k - \sum_{k=1}^{n}k^2$ — Difference Rule and Constant Multiple Rule

(b) $\displaystyle\sum_{k=1}^{n}(-a_k) = \sum_{k=1}^{n}(-1)\cdot a_k = -1\cdot\sum_{k=1}^{n}a_k = -\sum_{k=1}^{n}a_k$ — Constant Multiple Rule

(c) $\displaystyle\sum_{k=1}^{3}(k + 4) = \sum_{k=1}^{3}k + \sum_{k=1}^{3}4$ — Sum Rule

$$= (1 + 2 + 3) + (3\cdot 4)$$ Constant Value Rule

$$= 6 + 12 = 18$$

(d) $\displaystyle\sum_{k=1}^{n}\frac{1}{n} = n\cdot\frac{1}{n} = 1$ — Constant Value Rule ($1/n$ is constant)

Example

I want to find a formula for $1 + 2 + 3 + \ldots + n$.

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Note that

$$2\big(1+2+3+4+5+\ldots+(n-1)+n\big)$$

$$=$$

$$=$$

$$=$$

Example

I want to find a formula for $1+2+3+\ldots+n$.

Note that

$$
\begin{aligned}
2\big(1{+}2{+}3{+}4{+}5{+}\ldots{+}(n{-}1){+}n\big) \\
&= \begin{array}{ccccccccccccccccc} & 1 & + & 2 & + & 3 & + & 4 & + & 5 & + & \ldots & + & (n-1) & + & n \\ + & n & + & (n-1) & + & (n-2) & + & (n-3) & + & (n-4) & + & \ldots & + & 2 & + & 1 \end{array} \\
&= \\
&=
\end{aligned}
$$

Example

I want to find a formula for $1 + 2 + 3 + \ldots + n$.

Note that

$$
\begin{aligned}
&2\big(1+2+3+4+5+\ldots+(n-1)+n\big) \\
&\quad = \begin{array}{cccccccccccccccc}
& 1 & + & 2 & + & 3 & + & 4 & + & 5 & + & \ldots & + & (n-1) & + & n \\
+ & n & + & (n-1) & + & (n-2) & + & (n-3) & + & (n-4) & + & \ldots & + & 2 & + & 1
\end{array} \\
&\quad = (n+1) + (n+1) + (n+1) + (n+1) + (n+1) + \ldots + (n+1) + (n+1) \\
&\quad =
\end{aligned}
$$

Example

I want to find a formula for $1+2+3+\ldots+n$.

Note that

$$\begin{aligned} 2\big(1+2+3+4+5+\ldots+(n-1)+n\big) \\ &= \quad 1 + 2 + 3 + 4 + 5 + \ldots + (n-1) + n \\ &\quad + n + (n-1) + (n-2) + (n-3) + (n-4) + \ldots + 2 + 1 \\ &= (n+1) + (n+1) + (n+1) + (n+1) + (n+1) + \ldots + (n+1) + (n+1) \\ &= n(n+1). \end{aligned}$$

Example

I want to find a formula for $1+2+3+\ldots+n$.

Note that

$$
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&2\big(1+2+3+4+5+\ldots+(n-1)+n\big)\\
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&\quad = (n+1) + (n+1) + (n+1) + (n+1) + (n+1) + \ldots + (n+1) + (n+1)\\
&\quad = n(n+1).
\end{aligned}
$$

Therefore

$$\boxed{\sum_{k=1}^{n} k = \frac{n(n+1)}{2}.}$$

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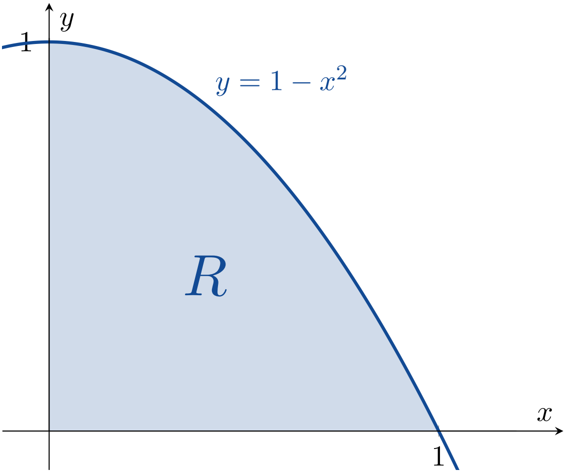
Similarly (but more difficult) we can find that

$$\boxed{\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}}$$

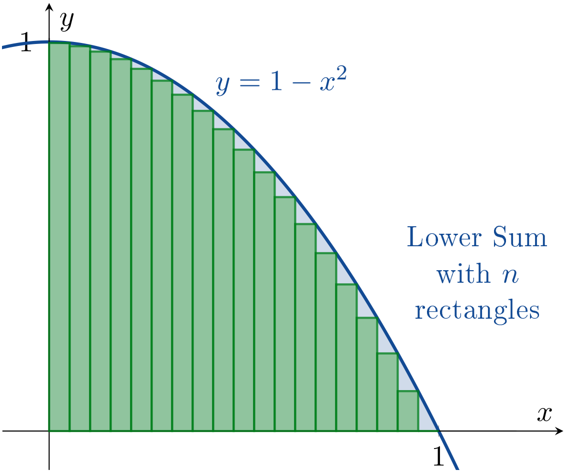
and

$$\boxed{\sum_{k=1}^{n} k^3 = \left(\frac{n(n+1)}{2}\right)^2.}$$

Limits of Finite Sums



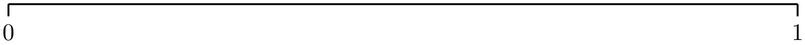
Limits of Finite Sums



5.2 Sigma Notation and Limits of Finite Sums

STEP 1: We will cut $[0, 1]$ into n pieces of width

$$\Delta x = \frac{1-0}{n} = \frac{1}{n}.$$



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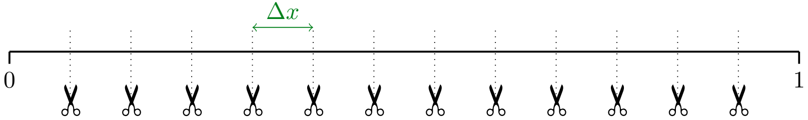
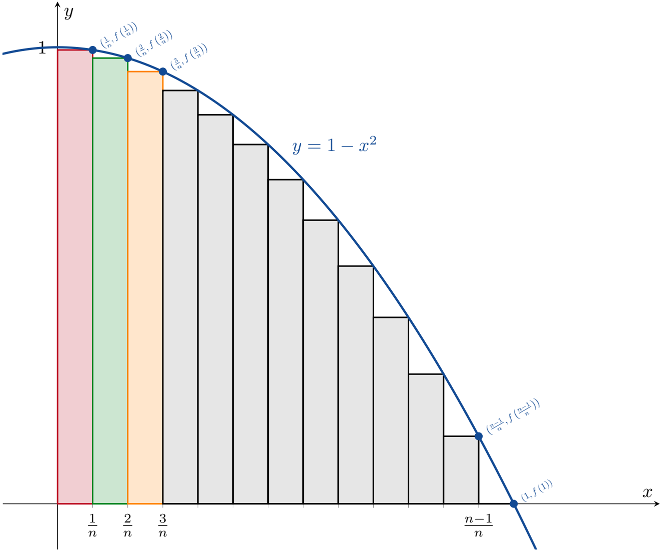




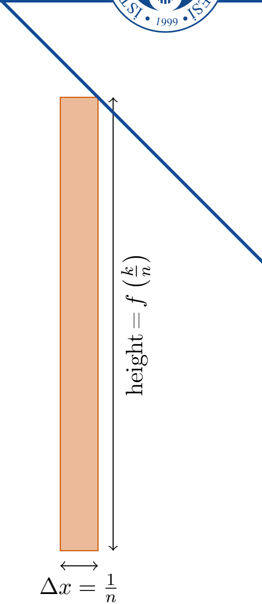
STEP 2: We will use n rectangles to approximate the area of R.

STEP 3: Then we will take the limit as $n \to \infty$.

Let $f(x) = 1 - x^2$. Then

- the first rectangle has area $\frac{1}{n} f\left(\frac{1}{n}\right)$;
- the second rectangle has area $\frac{1}{n} f\left(\frac{2}{n}\right)$;
- the third rectangle has area $\frac{1}{n} f\left(\frac{3}{n}\right)$;

and so on.



$$f(x) = 1 - x^2 \qquad \sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$$

The area of all n rectangles is

$$\text{area} = \sum_{k=1}^{n} (\text{area of the } k\text{th rectangle}) = \sum_{k=1}^{n} \frac{1}{n} f\left(\frac{k}{n}\right)$$

$$=$$

$$=$$

$$=$$

$$=$$

$$=$$

$$f(x) = 1 - x^2 \qquad \sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$$

The area of all n rectangles is

$$\begin{aligned}
\text{area} &= \sum_{k=1}^{n} (\text{area of the } k\text{th rectangle}) = \sum_{k=1}^{n} \frac{1}{n} f\left(\frac{k}{n}\right) \\
&= \sum_{k=1}^{n} \frac{1}{n}\left(1 - \left(\frac{k}{n}\right)^2\right) = \sum_{k=1}^{n} \left(\frac{1}{n} - \frac{k^2}{n^3}\right) \\
&= \\
&= \\
&= \\
&=
\end{aligned}$$

$$f(x) = 1 - x^2 \qquad \sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$$

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&= \sum_{k=1}^{n} \frac{1}{n} - \sum_{k=1}^{n} \frac{k^2}{n^3} \\
&= \\
&= \\
&=
\end{aligned}$$

$$f(x) = 1 - x^2 \qquad \sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$$

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&= \sum_{k=1}^{n} \frac{1}{n} - \sum_{k=1}^{n} \frac{k^2}{n^3} \\
&= n\left(\frac{1}{n}\right) - \frac{1}{n^3} \sum_{k=1}^{n} k^2 \\
&= \\
&=
\end{aligned}$$

$$f(x) = 1 - x^2 \qquad \sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$$

The area of all n rectangles is

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&= \sum_{k=1}^{n} \frac{1}{n} - \sum_{k=1}^{n} \frac{k^2}{n^3} \\
&= n\left(\frac{1}{n}\right) - \frac{1}{n^3}\sum_{k=1}^{n} k^2 \\
&= 1 - \frac{1}{n^3}\left(\frac{n(n+1)(2n+1)}{6}\right) \\
&=
\end{aligned}$$

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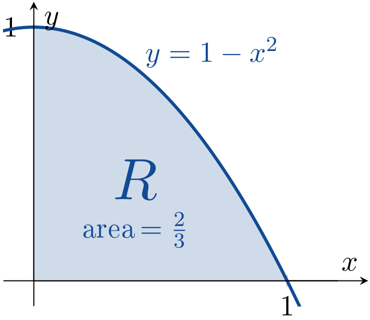
&= \sum_{k=1}^{n} \frac{1}{n} - \sum_{k=1}^{n} \frac{k^2}{n^3} \\
&= n\left(\frac{1}{n}\right) - \frac{1}{n^3}\sum_{k=1}^{n} k^2 \\
&= 1 - \frac{1}{n^3}\left(\frac{n(n+1)(2n+1)}{6}\right) \\
&= 1 - \frac{2n^2 + 3n + 1}{6n^2}.
\end{aligned}$$

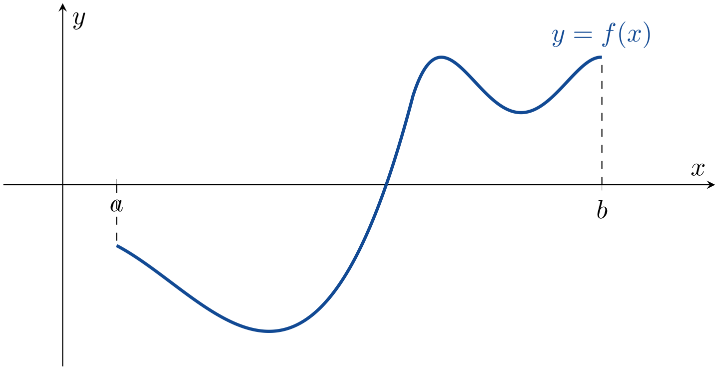
Taking the limit gives

$$\lim_{n\to\infty}\left(\sum_{k=1}^{n}\frac{1}{n}f\left(\frac{k}{n}\right)\right)=\lim_{n\to\infty}\left(1-\frac{2n^2+3n+1}{6n^2}\right)$$
$$=1-\frac{2}{6}=\frac{2}{3}.$$

Therefore the area of R is $\frac{2}{3}$.



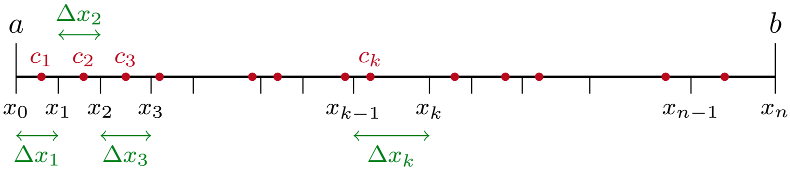
Riemann Sums



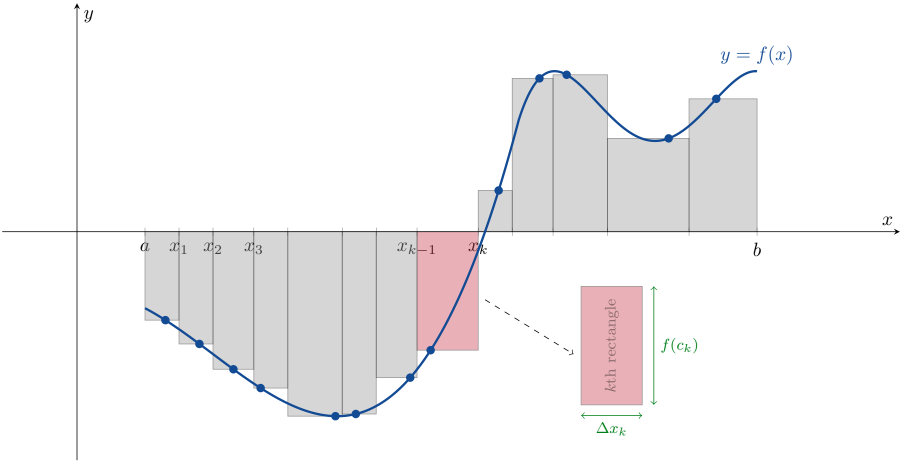
Now let $f : [a, b] \to \mathbb{R}$ be a function. We will cut $[a, b]$ into n subintervals (the pieces don't have to all be the same size).

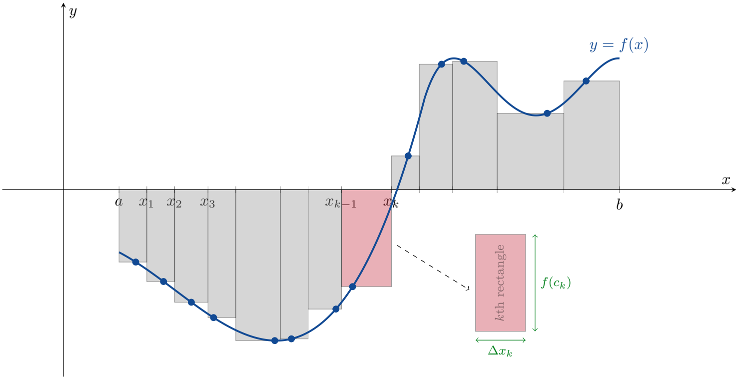
In each subinterval we will choose one point $c_k \in [x_{k-1}, x_k]$.

The width of each subinterval is $\Delta x_k = x_k - x_{k-1}$.

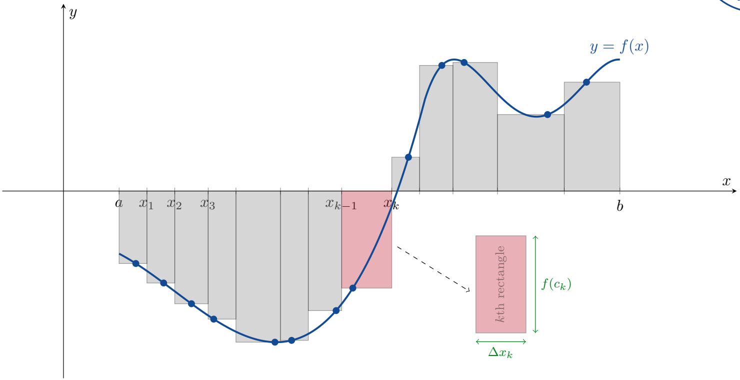


On each subinterval $[x_{k-1}, x_k]$, we draw a rectangle of width Δx_k and height $f(c_k)$.





Note that if $f(c_k) < 0$, then the rectangle on $[x_{k-1}, x_k]$ will have 'negative area' – this is ok.



The total area of the n rectangles is

$$\sum_{k=1}^{n} f(c_k)\Delta x_k.$$

This is called a *Riemann Sum for f on* $[a,b]$.

$$\sum_{k=1}^{n} f(c_k)\Delta x_k.$$

Then we want to take the limit as $n \to \infty$ (or more precisely, we want to take the limit as $\|P\| = \max\{\Delta x_1, \Delta x_2, \ldots, \Delta x_n\} \to 0$).

$$\sum_{k=1}^{n} f(c_k)\Delta x_k.$$

Then we want to take the limit as $n \to \infty$ (or more precisely, we want to take the limit as $\|P\| = \max\{\Delta x_1, \Delta x_2, \ldots, \Delta x_n\} \to 0$).

Remark

Sometimes this limit exists, sometimes this limit does not exist.

Break

We will continue at 2pm

5.3 The Definite Integral

Definition of the Definite Integral

Definition

<u>If</u> the limit

$$\lim_{n\to\infty}\sum_{k=1}^{n} f(c_k)\Delta x_k$$

exists, then it is called the *definite integral of* f *over* $[a,b]$. We write

$$\int_a^b f(x)\,dx = \lim_{n\to\infty}\sum_{k=1}^{n} f(c_k)\Delta x_k$$

if the limit exists.

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if the limit exists.

Remark

We need this limit to exist and be the same for all Riemann Sums that we can create.

DEFINITION Let $f(x)$ be a function defined on a closed interval $[a, b]$. We say that a number J is the **definite integral of f over $[a, b]$** and that J is the limit of the Riemann sums $\sum_{k=1}^{n} f(c_k)\,\Delta x_k$ if the following condition is satisfied:

Given any number $\varepsilon > 0$ there is a corresponding number $\delta > 0$ such that for every partition $P = \{x_0, x_1, \ldots, x_n\}$ of $[a, b]$ with $\|P\| < \delta$ and any choice of c_k in $[x_{k-1}, x_k]$, we have

$$\left| \sum_{k=1}^{n} f(c_k)\,\Delta x_k - J \right| < \varepsilon.$$

$$\int_a^b f(x)\,dx$$

"the integral of f from a to b" "a'dan b'ye f'nin integrali"

integral sign
integral işareti

$$\int_a^b f(x)\,dx$$

"the integral of f from a to b"

"a'dan b'ye f'nin integrali"

integral sign
integral işareti

$$\int_{\color{red}a}^{b} f(x)\,dx$$

lower limit of integration
integralin alt sınırı

"the integral of f from a to b" "a'dan b'ye f'nin integrali"

upper limit of integration
integralin üst sınırı

integral sign
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$$\int_a^{b} f(x)\,dx$$

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"the integral of f from a to b"

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$$\int_a^b \underbrace{f(x)}_{\substack{\text{the integrand} \\ \text{integralin integrandı}}} \, dx$$

lower limit of integration
integralin alt sınırı

"the integral of f from a to b" "a'dan b'ye f'nin integrali"

5.3 The Definite Integral

upper limit of integration
integralin üst sınırı

integral sign
integral işareti

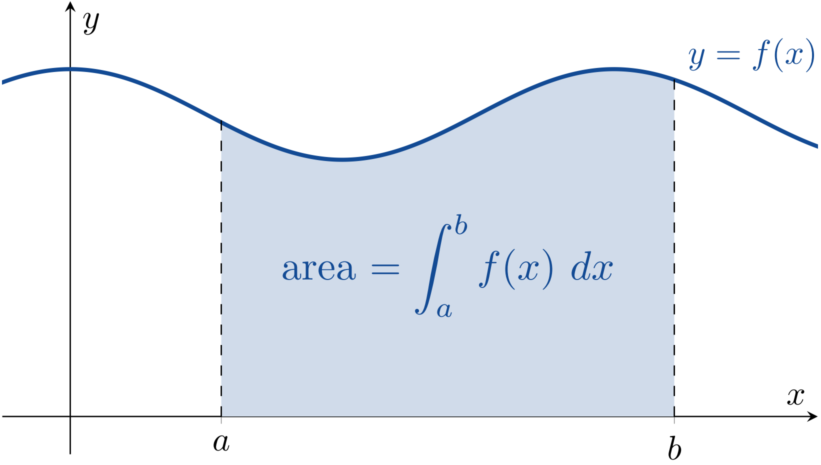
x is the variable of integration
x, integral değişkenidir

$$\int_a^b \underbrace{f(x)}_{\substack{\text{the integrand}\\ \text{integralin integrandı}}} \, dx$$

lower limit of integration
integralin alt sınırı

"the integral of f from a to b"

"a'dan b'ye f'nin integrali"



y
$y = f(x)$
$\text{area} = \int_a^b f(x)\,dx$
x
a
b

Definition

If $\int_a^b f(x)\,dx$ exists, then we say that f *is integrable on* $[a, b]$.

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Example

$f(x) = 1 - x^2$ is integrable on $[0,1]$ and $\int_0^1 (1-x^2)\,dx = \frac{2}{3}$.

Definition

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Example

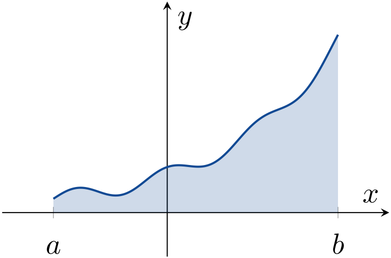
$f(x) = 1 - x^2$ is integrable on $[0, 1]$ and $\int_0^1 (1 - x^2)\ dx = \frac{2}{3}$.

Remark

$$\int_a^b f(x)\ dx = \int_a^b f(u)\ du = \int_a^b f(t)\ dt$$

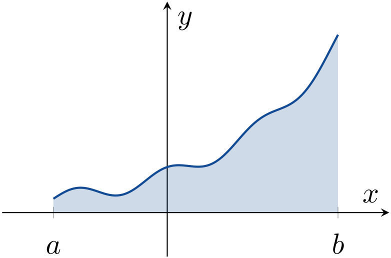
It doesn't matter which letter we use for the *dummy variable*.

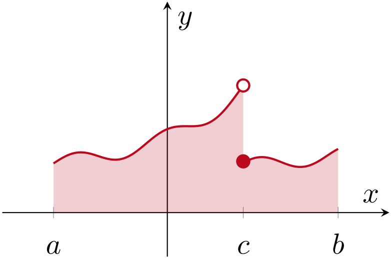
5.3 The Definite Integral



Theorem

If f is continuous on $[a, b]$, then f is integrable on $[a, b]$.

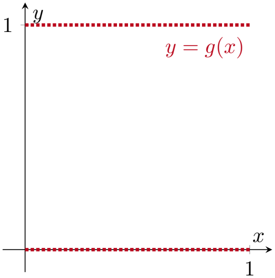




Theorem

If f is continuous on $[a, b]$, then f is integrable on $[a, b]$.

If f has finitely many jump discontinuities but is otherwise continuous on $[a, b]$, then f is integrable on $[a, b]$.

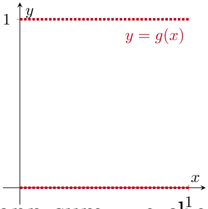


Example

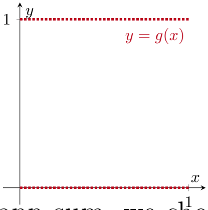
Define a function $g : [0,1] \to \mathbb{R}$ by

$$g(x) = \begin{cases} 1 & x \in \mathbb{Q} \\ 0 & x \notin \mathbb{Q}. \end{cases}$$

This function is not integrable on $[0,1]$.



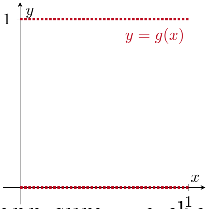
When we set up a Riemann sum, we choose the points $c_k \in [x_{k-1}, x_k]$ where we calculate the height of each rectangle.



When we set up a Riemann sum, we choose the points $c_k \in [x_{k-1}, x_k]$ where we calculate the height of each rectangle.

If we choose $c_k \in \mathbb{Q}$, then we always have $g(c_k) = 1$ and thus

$$\sum_{k=1}^{n} g(c_k)\Delta x_k = \sum_{k=1}^{n} 1 \cdot \Delta x_k = 1.$$



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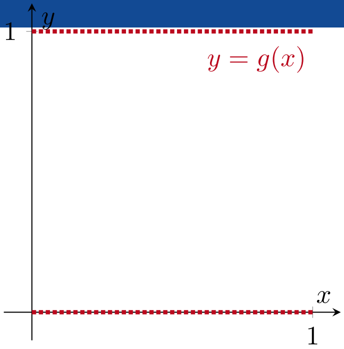
$$\sum_{k=1}^{n} g(c_k)\Delta x_k = \sum_{k=1}^{n} 1 \cdot \Delta x_k = 1.$$

However if we choose $c_k \in \mathbb{R} \setminus \mathbb{Q}$, then we always have $g(c_k) = 0$ and thus

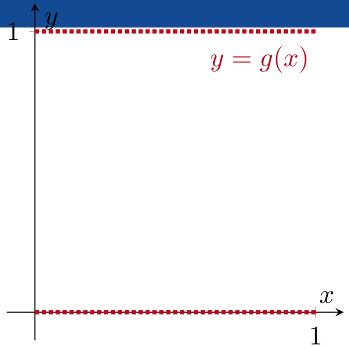
$$\sum_{k=1}^{n} g(c_k)\Delta x_k = \sum_{k=1}^{n} 0 \cdot \Delta x_k = 0.$$



Since

$$\lim_{n\to\infty} 1 \neq \lim_{n\to\infty} 0,$$

there does not exist a common limit of the Riemann sums.



Since

$$\lim_{n\to\infty} 1 \neq \lim_{n\to\infty} 0,$$

there does not exist a common limit of the Riemann sums.

Therefore $g(x) = \begin{cases} 1 & x \in \mathbb{Q} \\ 0 & x \notin \mathbb{Q}. \end{cases}$ is not integrable.

Properties of Definite Integrals

Theorem

Suppose that f and g are integrable. Let k be a number.

Properties of Definite Integrals

Theorem

Suppose that f and g are integrable. Let k be a number. Then

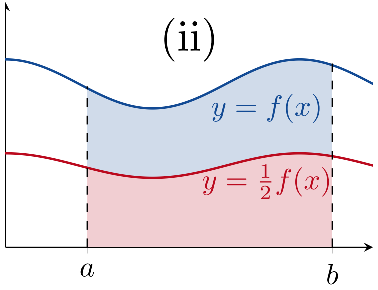
1. $\displaystyle\int_b^a f(x)\ dx = -\int_a^b f(x)\ dx;$

Properties of Definite Integrals

Theorem

Suppose that f and g are integrable. Let k be a number. Then

2. $\displaystyle\int_a^b kf(x)\,dx = k\int_a^b f(x)\,dx;$

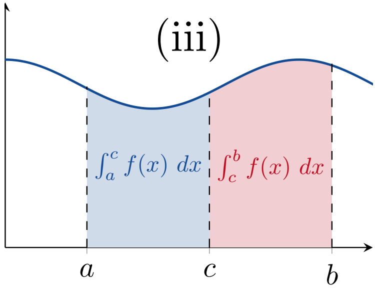


Properties of Definite Integrals

Theorem

Suppose that f and g are integrable. Let k be a number. Then

3. $$\int_a^c f(x)\ dx + \int_c^b f(x)\ dx = \int_a^b f(x)\ dx$$

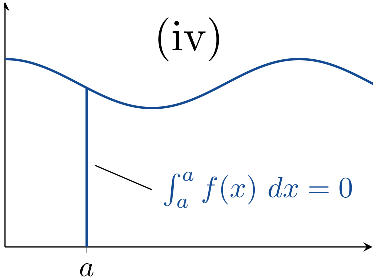


Properties of Definite Integrals

Theorem

Suppose that f and g are integrable. Let k be a number. Then

4. $\displaystyle\int_a^a f(x)\,dx = 0;$



Properties of Definite Integrals

Theorem

Suppose that f and g are integrable. Let k be a number. Then

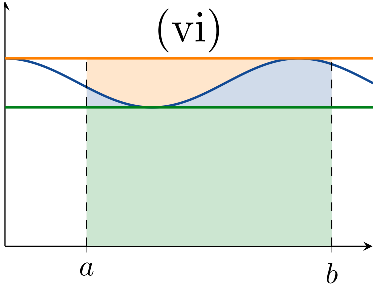
5. $$\int_a^b \big(f(x) + g(x)\big)\ dx = \int_a^b f(x)\ dx + \int_a^b g(x)\ dx;$$

Properties of Definite Integrals

Theorem

Suppose that f and g are integrable. Let k be a number. Then

6. $(b-a)\min f \leq \int_a^b f(x)\,dx \leq (b-a)\max f$;



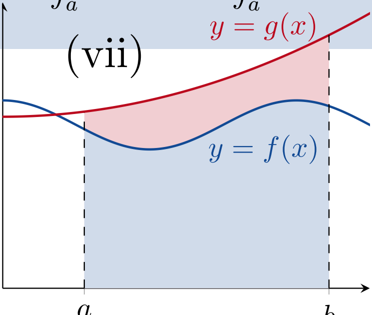
Properties of Definite Integrals

Theorem

Suppose that f and g are integrable. Let k be a number. Then

7. *if $f(x) \leq g(x)$ on $[a, b]$, then*

$$\int_a^b f(x)\ dx \leq \int_a^b g(x)\ dx;$$



Properties of Definite Integrals

Theorem

Suppose that f and g are integrable. Let k be a number. Then

8. *if $g(x) \geq 0$ on $[a, b]$, then*

$$\int_a^b g(x)\ dx \geq 0;$$

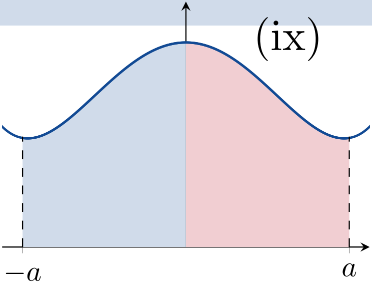
Properties of Definite Integrals

Theorem

Suppose that f and g are integrable. Let k be a number. Then

9. *if f is an even function, then*

$$\int_{-a}^{a} f(x)\ dx = 2\int_{0}^{a} f(x)\ dx;$$



Properties of Definite Integrals

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Suppose that f and g are integrable. Let k be a number. Then

10 *if f is an odd function, then*

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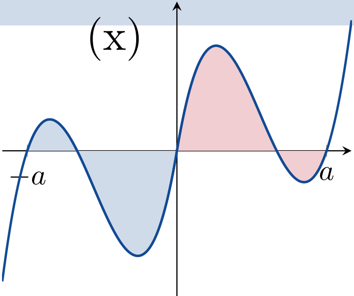

Example

Suppose that

$$\int_{-1}^{1} f(x)\ dx = 5,\ \int_{1}^{4} f(x)\ dx = -2 \text{ and } \int_{-1}^{1} h(x)\ dx = 7.$$

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$$\int_{4}^{1} f(x)\,dx = -\int_{1}^{4} f(x)\,dx = 2,$$

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$$\int_{4}^{1} f(x)\ dx = -\int_{1}^{4} f(x)\ dx = 2,$$

$$\begin{aligned}\int_{-1}^{1} \big(2f(x) + 3h(x)\big)\ dx &= 2\int_{-1}^{1} f(x)\ dx + 3\int_{-1}^{1} h(x)\ dx \\ &= 2(5) + 3(7) = 31\end{aligned}$$

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$$\int_{-1}^{1} f(x)\ dx = 5,\ \int_{1}^{4} f(x)\ dx = -2 \text{ and } \int_{-1}^{1} h(x)\ dx = 7.$$

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and

$$\begin{aligned}\int_{-1}^{4} f(x)\ dx &= \int_{-1}^{1} f(x)\ dx + \int_{1}^{4} f(x)\ dx \\ &= 5 + (-2) = 3.\end{aligned}$$

Example

Show that $\int_0^1 \sqrt{1+\cos x}\, dx \leq \sqrt{2}$.

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Show that $\int_0^1 \sqrt{1+\cos x}\, dx \leq \sqrt{2}$.

The maximum value of $\sqrt{1+\cos x}$ on $[0,1]$ is
$\sqrt{1+1} = \sqrt{2}$.

Example

Show that $\displaystyle\int_0^1 \sqrt{1+\cos x}\,dx \le \sqrt{2}$.

The maximum value of $\sqrt{1+\cos x}$ on $[0,1]$ is $\sqrt{1+1} = \sqrt{2}$. Therefore

$$\int_0^1 \sqrt{1+\cos x}\,dx \le (1-0)\max\sqrt{1+\cos x} = 1 \times \sqrt{2}.$$

Example

Calculate $\int_{-2}^{2} (x^3 + x)\ dx$.

Because $(x^3 + x)$ is an odd function, we have that

$$\int_{-2}^{2} (x^3 + x)\ dx = 0.$$

Example

Calculate $\int_{-1}^{1} (1 - x^2)\ dx$.

Because $(1 - x^2)$ is an even function, we have that

$$\int_{-1}^{1} (1 - x^2)\ dx = 2 \int_{0}^{1} (1 - x^2)\ dx = 2 \times \frac{2}{3} = \frac{4}{3}.$$

Example

Calculate $\int_0^b x\,dx$ for $b > 0$.

solution 1: We will use a Riemann Sum.

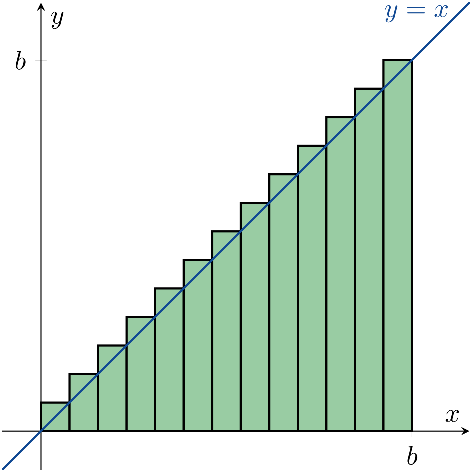
Example

Calculate $\int_0^b x\,dx$ for $b > 0$.

solution 1: We will use a Riemann Sum. First we cut $[0, b]$ in to n pieces using

$$0 < \frac{b}{n} < \frac{2b}{n} < \frac{3b}{n} < \ldots < \frac{(n-1)b}{n} < b$$

and $c_k = \frac{kb}{n}$. Note that $\Delta x_k = \frac{b}{n}$ for all k.



$y = x$
y
b
x
b

Then

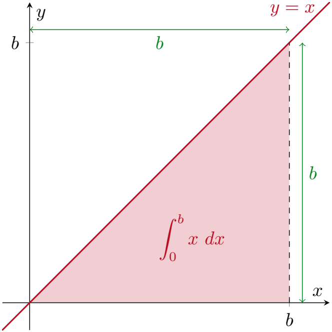
$$\sum_{k=1}^{n} f(c_k)\Delta x_k = \sum_{k=1}^{n} \frac{kb}{n}\frac{b}{n} = \frac{b^2}{n^2}\sum_{k=1}^{n} k$$
$$= \frac{b^2}{n^2}\left(\frac{n(n+1)}{2}\right) = \frac{b^2}{2}\left(1+\frac{1}{n}\right).$$

Then

$$\sum_{k=1}^{n} f(c_k)\Delta x_k = \sum_{k=1}^{n} \frac{kb}{n}\frac{b}{n} = \frac{b^2}{n^2}\sum_{k=1}^{n} k$$
$$= \frac{b^2}{n^2}\left(\frac{n(n+1)}{2}\right) = \frac{b^2}{2}\left(1+\frac{1}{n}\right).$$

Therefore

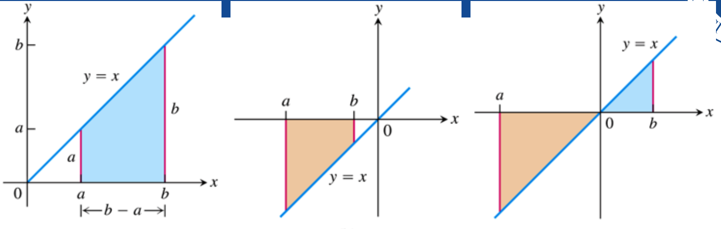
$$\int_0^b x\ dx = \lim_{n\to\infty}\sum_{k=1}^{n} f(c_k)\Delta x_k$$
$$= \lim_{n\to\infty}\frac{b^2}{2}\left(1+\frac{1}{n}\right) = \frac{b^2}{2}.$$



solution 2: Alternately, we can look at the triangle above and say that

$$\int_0^b x\ dx = \text{area of a triangle} = \frac{1}{2} \times b \times b = \frac{b^2}{2}.$$

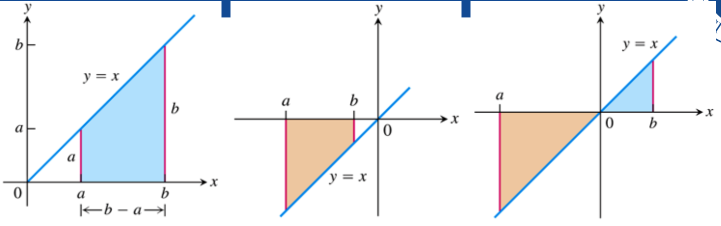
5.3 The Definite Integral



(a) (b) (c)

Example

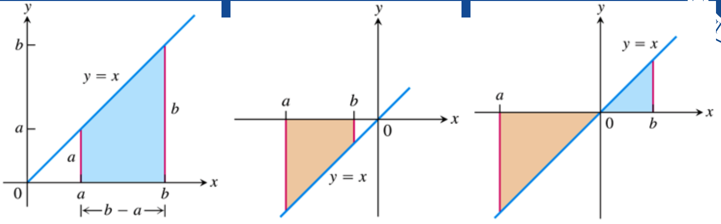
$$\int_a^b x\ dx = \int_a^0 x\ dx + \int_0^b x\ dx$$



(a) (b) (c)

Example

$$\int_a^b x\ dx = \int_a^0 x\ dx + \int_0^b x\ dx$$
$$= -\int_0^a x\ dx + \int_0^b x\ dx$$



(a) (b) (c)

Example

$$
\begin{aligned}
\int_a^b x\ dx &= \int_a^0 x\ dx + \int_0^b x\ dx \\
&= -\int_0^a x\ dx + \int_0^b x\ dx \\
&= -\frac{a^2}{2} + \frac{b^2}{2} = \frac{b^2}{2} - \frac{a^2}{2}.
\end{aligned}
$$

Next Time

- 5.4 The Fundamental Theorem of Calculus
- 5.5 Indefinite Integrals and the Substitution Method
- 5.6 Substitution and Area Between Curves