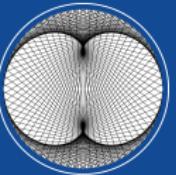


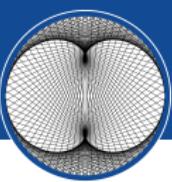
Lecture 7

- 14.1 Double and Iterated Integrals over Rectangles
- 14.2 Double Integrals over General Regions
- 14.3 Area by Double Integration
- 10.3 Polar Coordinates

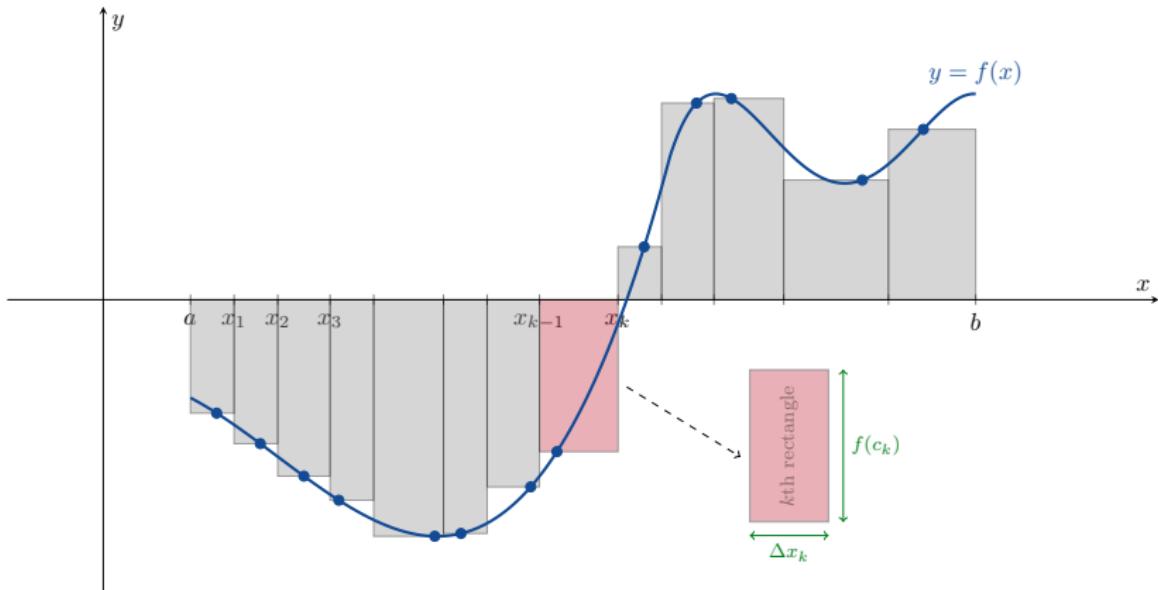


Double and Iterated Integrals over Rectangles

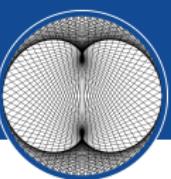
14.1 Double and Iterated Integrals over Rectangles



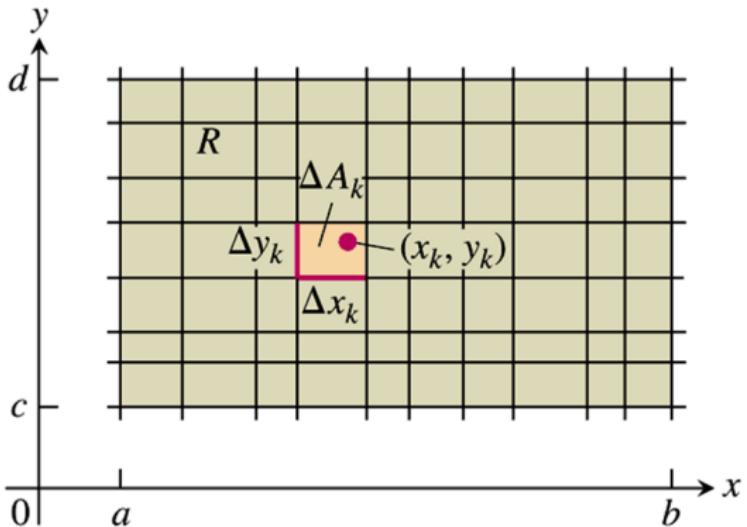
Riemann Sums



$$\int_a^b f(x) dx = \lim_{\|P\| \rightarrow 0} \sum_{k=1}^n f(c_k) \Delta x_k.$$

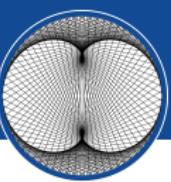


Riemann Sums for Double Integrals

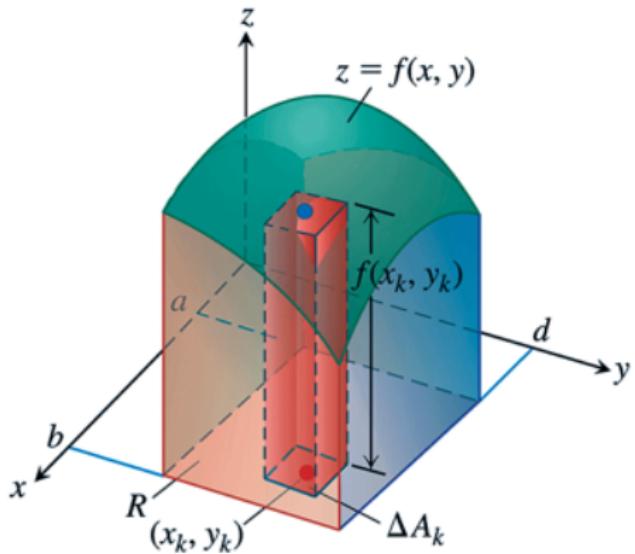


Partition a rectangle R into small rectangles of area

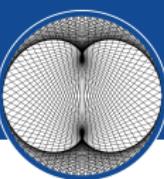
$$\Delta A_k = \Delta x_k \Delta y_k.$$



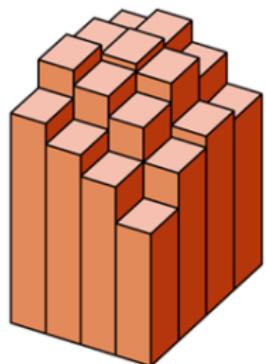
Riemann Sums for Double Integrals



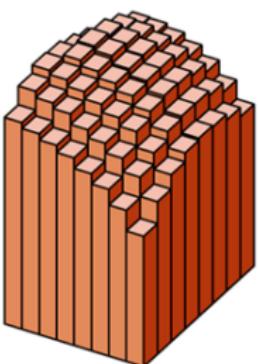
Then approximate the graph of $z = f(x, y)$ by drawing a cuboid on each small rectangle.



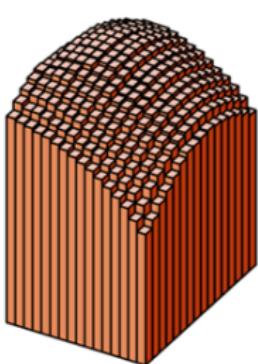
Riemann Sums for Double Integrals



(a) $n = 16$



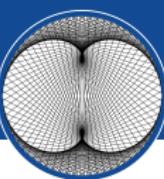
(b) $n = 64$



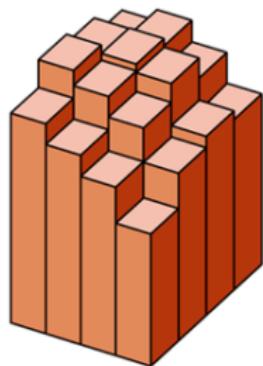
(c) $n = 256$

$$\sum_k f(x_k, y_k) \Delta A_k.$$

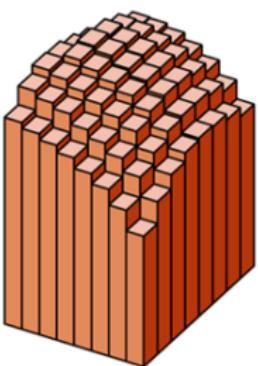
14.1 Double and Iterated Integrals over Rectangles



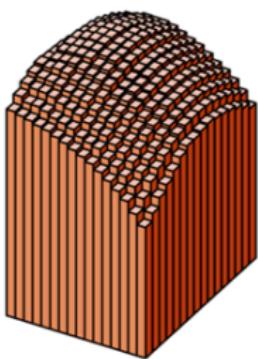
Riemann Sums for Double Integrals



(a) $n = 16$



(b) $n = 64$



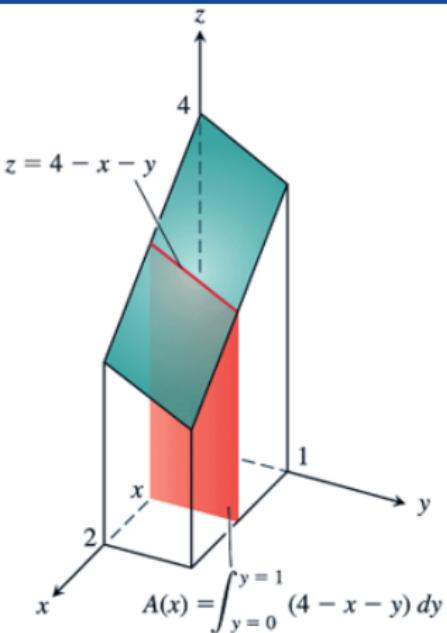
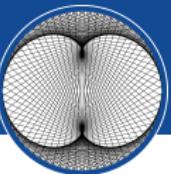
(c) $n = 256$

Then take the limit,

$$\iint_R f \, dA = \lim_{\|P\| \rightarrow 0} \sum_k f(x_k, y_k) \Delta A_k.$$

14.1 Double and Iterated

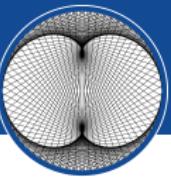
Rectangles



Example

Calculate the volume under the plane $z = 4 - x - y$ over the rectangle $R = \{(x, y) | 0 \leq x \leq 2, 0 \leq y \leq 1\}$.

14.1 Double and Iterated Integrals over Rectangles



Since $R = \{(x, y) \mid 0 \leq x \leq 2, 0 \leq y \leq 1\}$, we calculate that

$$\iint_R (4 - x - y) dA = \int_{x=0}^{x=2} \int_{y=0}^{y=1} (4 - x - y) dy dx$$

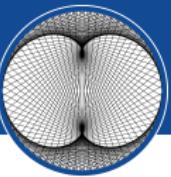
=

=

=

=

14.1 Double and Iterated Integrals over Rectangles



Since $R = \{(x, y) \mid 0 \leq x \leq 2, 0 \leq y \leq 1\}$, we calculate that

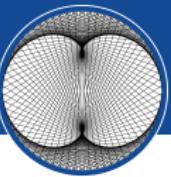
$$\begin{aligned}\iint_R (4 - x - y) dA &= \int_{x=0}^{x=2} \int_{y=0}^{y=1} (4 - x - y) dy dx \\ &= \int_{x=0}^{x=2} \left(\int_{y=0}^{y=1} (4 - x - y) dy \right) dx\end{aligned}$$

=

=

=

14.1 Double and Iterated Integrals over Rectangles



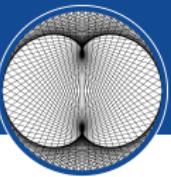
Since $R = \{(x, y) \mid 0 \leq x \leq 2, 0 \leq y \leq 1\}$, we calculate that

$$\begin{aligned}\iint_R (4 - x - y) dA &= \int_{x=0}^{x=2} \int_{y=0}^{y=1} (4 - x - y) dy dx \\&= \int_{x=0}^{x=2} \left(\int_{y=0}^{y=1} (4 - x - y) dy \right) dx \\&= \int_{x=0}^{x=2} \left[4y - xy - \frac{y^2}{2} \right]_{y=0}^{y=1} dx \\&= \end{aligned}$$

=

=

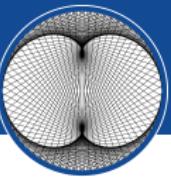
14.1 Double and Iterated Integrals over Rectangles



Since $R = \{(x, y) \mid 0 \leq x \leq 2, 0 \leq y \leq 1\}$, we calculate that

$$\begin{aligned}\iint_R (4 - x - y) dA &= \int_{x=0}^{x=2} \int_{y=0}^{y=1} (4 - x - y) dy dx \\&= \int_{x=0}^{x=2} \left(\int_{y=0}^{y=1} (4 - x - y) dy \right) dx \\&= \int_{x=0}^{x=2} \left[4y - xy - \frac{y^2}{2} \right]_{y=0}^{y=1} dx \\&= \int_{x=0}^{x=2} \left(\frac{7}{2} - x \right) dx \\&= \end{aligned}$$

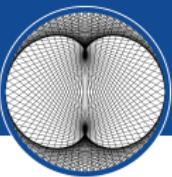
14.1 Double and Iterated Integrals over Rectangles



Since $R = \{(x, y) \mid 0 \leq x \leq 2, 0 \leq y \leq 1\}$, we calculate that

$$\begin{aligned}\iint_R (4 - x - y) dA &= \int_{x=0}^{x=2} \int_{y=0}^{y=1} (4 - x - y) dy dx \\&= \int_{x=0}^{x=2} \left(\int_{y=0}^{y=1} (4 - x - y) dy \right) dx \\&= \int_{x=0}^{x=2} \left[4y - xy - \frac{y^2}{2} \right]_{y=0}^{y=1} dx \\&= \int_{x=0}^{x=2} \left(\frac{7}{2} - x \right) dx \\&= \left[\frac{7}{2}x - \frac{x^2}{2} \right]_{x=0}^{x=2} = 5.\end{aligned}$$

14.1 Double and Iterated Integrals over Rectangles

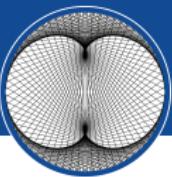


Remark

You don't need to write " $x =$ " and " $y =$ " if understand which integral is which:

$$\int_{x=0}^{x=2} \int_{y=0}^{y=1} (4 - x - y) \, dy \, dx = \int_0^2 \int_0^1 (4 - x - y) \, dy \, dx.$$

14.1 Double and Iterated Integrals over Rectangles

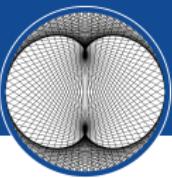


Remark

You don't need to write " $x =$ " and " $y =$ " if understand which integral is which:

$$\int_{x=0}^{x=2} \int_{y=0}^{y=1} (4 - x - y) \, dy \, dx = \int_0^2 \int_0^1 (4 - x - y) \, dy \, dx.$$

14.1 Double and Iterated Integrals over Rectangles



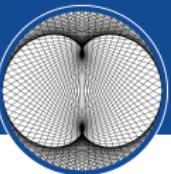
Remark

You don't need to write " $x =$ " and " $y =$ " if understand which integral is which:

$$\int_{x=0}^{x=2} \int_{y=0}^{y=1} (4 - x - y) \, dy \, dx = \int_0^2 \int_0^1 (4 - x - y) \, dy \, dx.$$



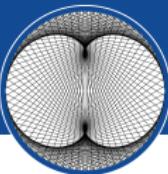
14.1 Double and Iterated Integrals over Rectangles



Previously we did

$$\int_0^2 \int_0^1 (4 - x - y) \, dy \, dx = 5.$$

14.1 Double and Iterated Integrals over Rectangles



Previously we did

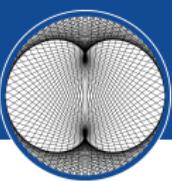
$$\int_0^2 \int_0^1 (4 - x - y) \, dy \, dx = 5.$$

But note that

$$\begin{aligned}\int_0^1 \int_0^2 (4 - x - y) \, dx \, dy &= \int_0^1 \left(\int_0^2 (4 - x - y) \, dx \right) \, dy \\&= \int_0^1 \left[4x - \frac{x^2}{2} - xy \right]_0^2 \, dy \\&= \int_0^1 (6 - 2y) \, dy \\&= [6y - y^2]_0^1 = 5\end{aligned}$$

also.

14.1 Double and Iterated Integrals over Rectangles



Guido Fubini

BORN

19 January 1879

DECEASED

6 June 1943

NATIONALITY

Italian

Theorem (Fubini's Theorem)

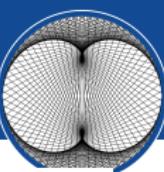
If $f(x, y)$ is continuous on the rectangle

$$R = \{(x, y) \mid a \leq x \leq b, c \leq y \leq d\} = [a, b] \times [c, d],$$

then

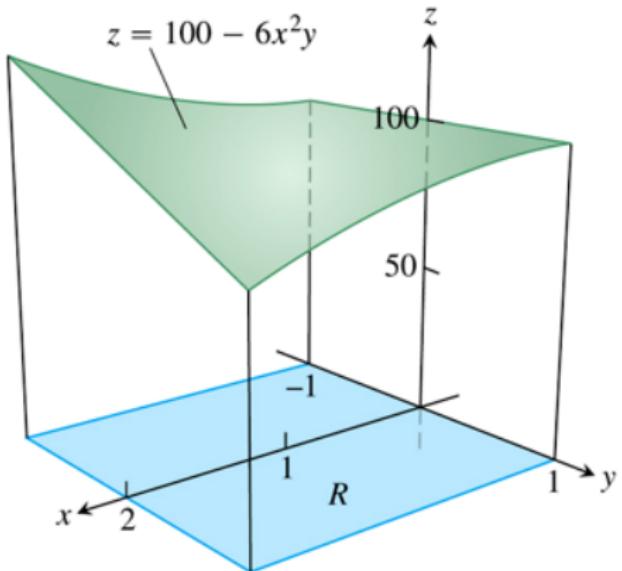
$$\iint_R f(x, y) dA = \int_a^b \int_c^d f(x, y) dy dx = \int_c^d \int_a^b f(x, y) dx dy.$$

14.1 Double and Iterated Integrals over Rectangles



EXAMPLE 1 Calculate $\iint_R f(x, y) dA$ for

$$f(x, y) = 100 - 6x^2y \quad \text{and} \quad R: 0 \leq x \leq 2, -1 \leq y \leq 1.$$



Solution Figure 15.6 displays the volume beneath the surface. By Fubini's Theorem,

$$\begin{aligned}\iint_R f(x, y) dA &= \int_{-1}^1 \int_0^2 (100 - 6x^2y) dx dy = \int_{-1}^1 \left[100x - 2x^3y \right]_{x=0}^{x=2} dy \\ &= \int_{-1}^1 (200 - 16y) dy = \left[200y - 8y^2 \right]_{-1}^1 = 400.\end{aligned}$$

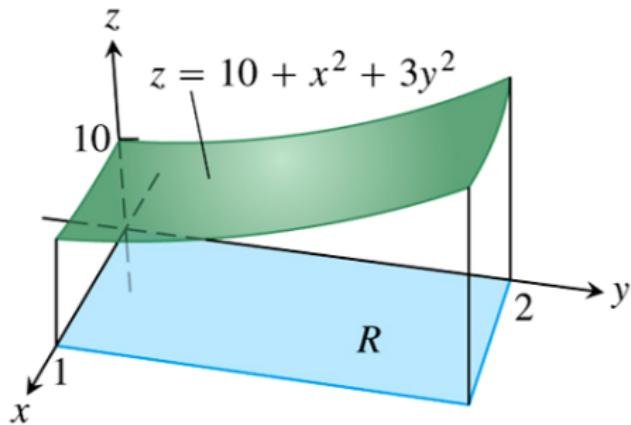
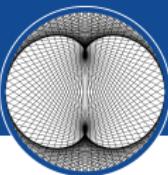
Reversing the order of integration gives the same answer:

$$\begin{aligned}\int_0^2 \int_{-1}^1 (100 - 6x^2y) dy dx &= \int_0^2 \left[100y - 3x^2y^2 \right]_{y=-1}^{y=1} dx \\ &= \int_0^2 [(100 - 3x^2) - (-100 - 3x^2)] dx \\ &= \int_0^2 200 dx = 400.\end{aligned}$$



14.1 Double an

gles

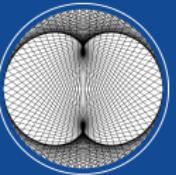


EXAMPLE 2 Find the volume of the region bounded above by the elliptical paraboloid $z = 10 + x^2 + 3y^2$ and below by the rectangle R : $0 \leq x \leq 1$, $0 \leq y \leq 2$.

Solution The surface and volume are shown in Figure 15.7. The volume is given by the double integral

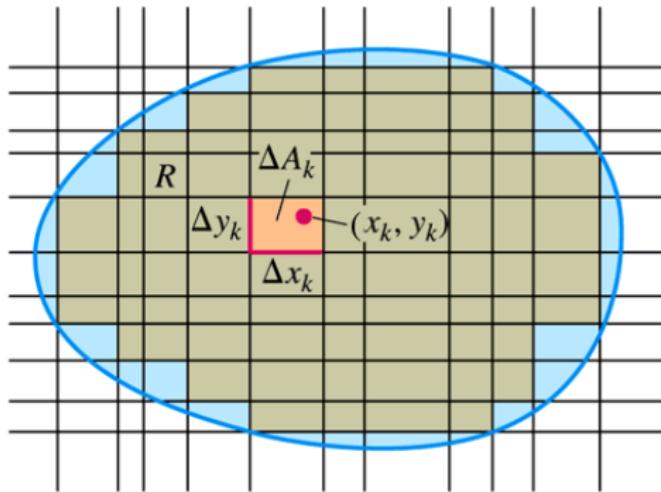
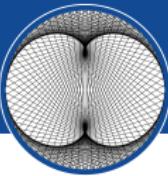
$$\begin{aligned}
 V &= \iint_R (10 + x^2 + 3y^2) dA = \int_0^1 \int_0^2 (10 + x^2 + 3y^2) dy dx \\
 &= \int_0^1 \left[10y + x^2y + y^3 \right]_{y=0}^{y=2} dx \\
 &= \int_0^1 (20 + 2x^2 + 8) dx = \left[20x + \frac{2}{3}x^3 + 8x \right]_0^1 = \frac{86}{3}.
 \end{aligned}$$

■



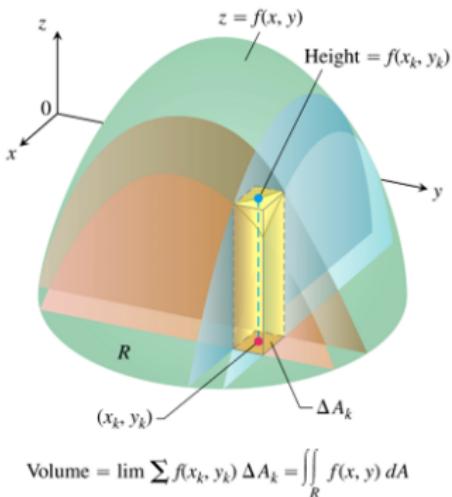
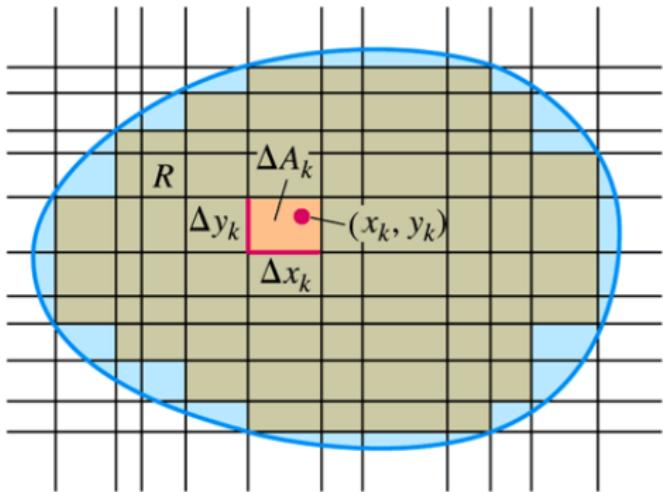
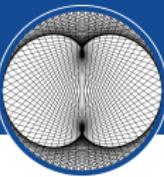
Double Integrals over General Regions

14.2 Double Integrals over General Regions



We only use rectangles that are completely inside R .

14.2 Double Integrals over General Regions

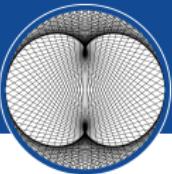


$$\text{Volume} = \lim \sum f(x_k, y_k) \Delta A_k = \iint_R f(x, y) dA$$

We only use rectangles that are completely inside R .

$$\iint_R f dA = \lim_{\|P\| \rightarrow 0} \sum_k f(x_k, y_k) \Delta A_k.$$

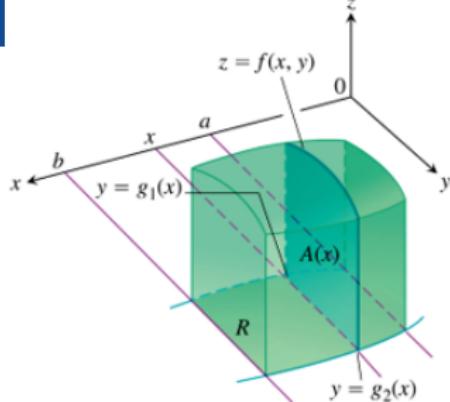
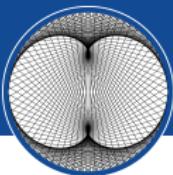
14.2 Double Integrals over General Regions



Theorem (Fubini's Theorem)

Let $f(x, y)$ be continuous on R . Let g_1, g_2, h_1 and h_2 be continuous.

14.2 Double Integrals over General Regions



Theorem (Fubini's Theorem)

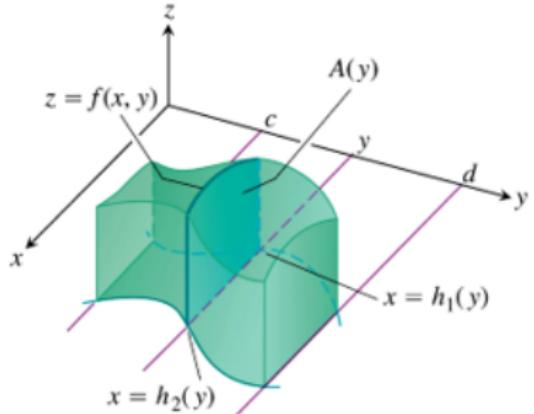
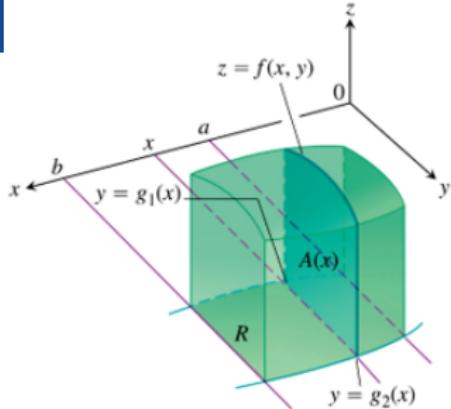
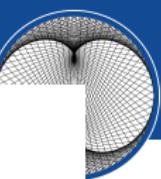
- 1 If R is defined by $a \leq x \leq b$ and $g_1(x) \leq y \leq g_2(x)$, then

$$\iint_R f \, dA = \int_{x=a}^{x=b} \int_{y=g_1(x)}^{y=g_2(x)} f(x, y) \, dy \, dx$$

only numbers

you can have functions here

14.2 Double Integrals over General Regions



Theorem (Fubini's Theorem)

2 If R is defined by $c \leq y \leq d$ and $h_1(x) \leq x \leq h_2(x)$, then

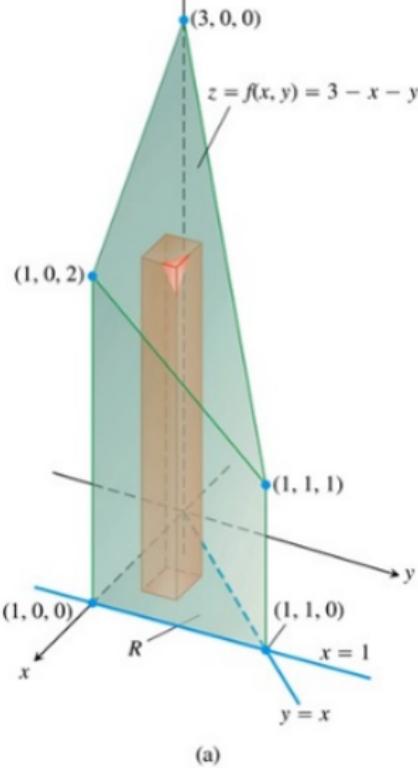
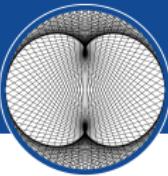
$$\iint_R f \, dA = \int_{y=c}^{y=d} \int_{x=h_1(x)}^{x=h_2(x)} f(x, y) \, dx \, dy$$

only numbers

you can have functions here

14.2 Double Integral

ons

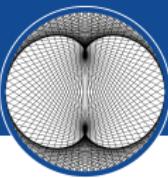
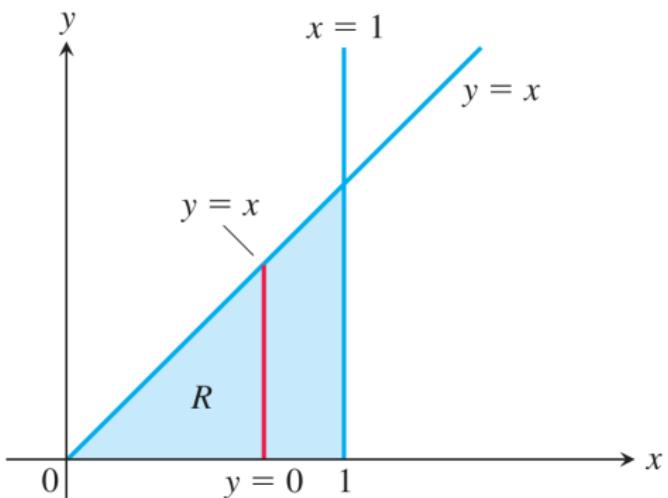


(a)

EXAMPLE 1 Find the volume of the prism whose base is the triangle in the xy -plane bounded by the x -axis and the lines $y = x$ and $x = 1$ and whose top lies in the plane

$$z = f(x, y) = 3 - x - y.$$

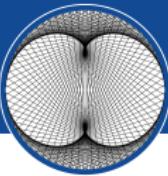
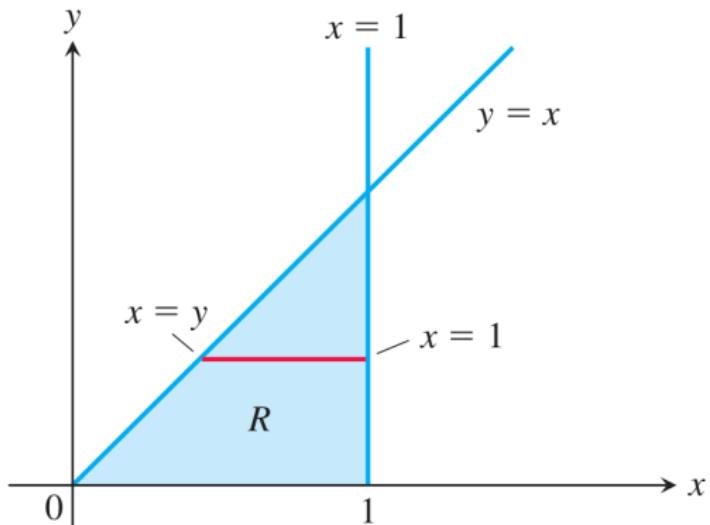
14.2 Double Integrals



Solution See Figure 15.12. For any x between 0 and 1, y may vary from $y = 0$ to $y = x$ (Figure 15.12b). Hence,

$$\begin{aligned}
 V &= \int_0^1 \int_0^x (3 - x - y) \, dy \, dx = \int_0^1 \left[3y - xy - \frac{y^2}{2} \right]_{y=0}^{y=x} \, dx \\
 &= \int_0^1 \left(3x - \frac{3x^2}{2} \right) \, dx = \left[\frac{3x^2}{2} - \frac{x^3}{2} \right]_{x=0}^{x=1} = 1.
 \end{aligned}$$

14.2 Double Integrals

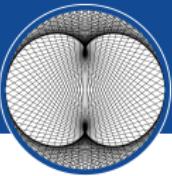


When the order of integration is reversed (Figure 15.12c), the integral for the volume is

$$\begin{aligned}
 V &= \int_0^1 \int_y^1 (3 - x - y) \, dx \, dy = \int_0^1 \left[3x - \frac{x^2}{2} - xy \right]_{x=y}^{x=1} dy \\
 &= \int_0^1 \left(3 - \frac{1}{2} - y - 3y + \frac{y^2}{2} + y^2 \right) dy \\
 &= \int_0^1 \left(\frac{5}{2} - 4y + \frac{3}{2}y^2 \right) dy = \left[\frac{5}{2}y - 2y^2 + \frac{y^3}{2} \right]_{y=0}^{y=1} = 1.
 \end{aligned}$$

The two integrals are equal, as they should be.

14.2 Double Integrals over General Regions



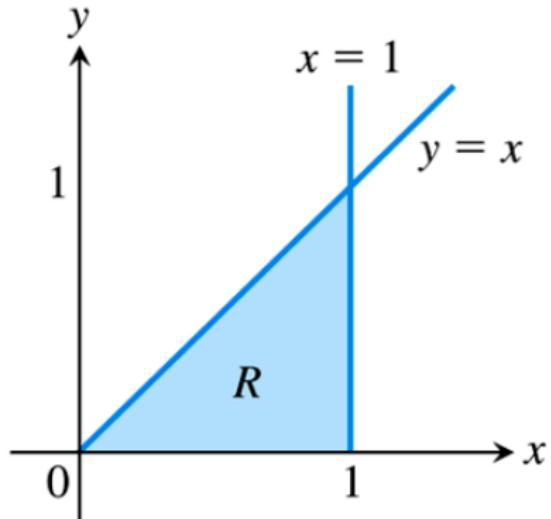
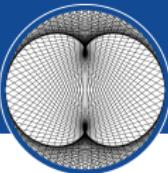
Remark

Fubini's Theorem tells us that

$$\iint f \, dx \, dy = \iint f \, dy \, dx.$$

But sometimes one is easier to calculate than the other.

14.2 Double Integrals over General Regions



EXAMPLE 2 Calculate

$$\iint_R \frac{\sin x}{x} dA,$$

where R is the triangle in the xy -plane bounded by the x -axis, the line $y = x$, and the line $x = 1$.

Solution The region of integration is shown in Figure 15.13. If we integrate first with respect to y and next with respect to x , then because x is held fixed in the first integration, we find

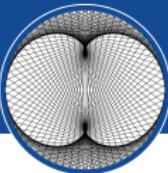
$$\int_0^1 \left(\int_0^x \frac{\sin x}{x} dy \right) dx = \int_0^1 \left[y \frac{\sin x}{x} \right]_{y=0}^{y=x} dx = \int_0^1 \sin x dx = -\cos(1) + 1 \approx 0.46.$$

If we reverse the order of integration and attempt to calculate

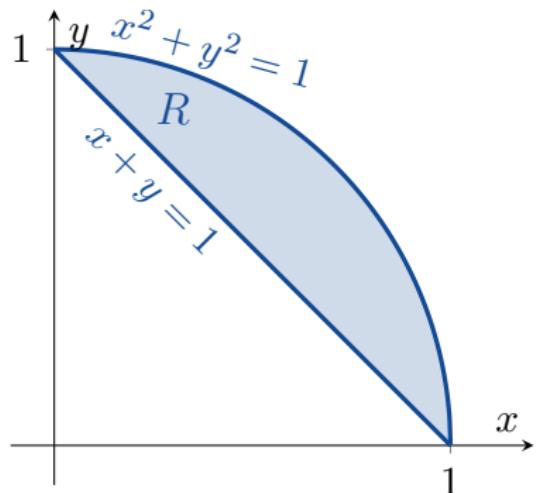
$$\int_0^1 \int_y^1 \frac{\sin x}{x} dx dy,$$

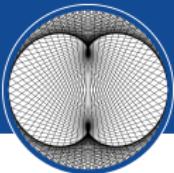
we run into a problem because $\int ((\sin x)/x) dx$ cannot be expressed in terms of elementary functions (there is no simple antiderivative).

There is no general rule for predicting which order of integration will be the good one in circumstances like these. If the order you first choose doesn't work, try the other. Sometimes neither order will work, and then we may need to use numerical approximations. ■

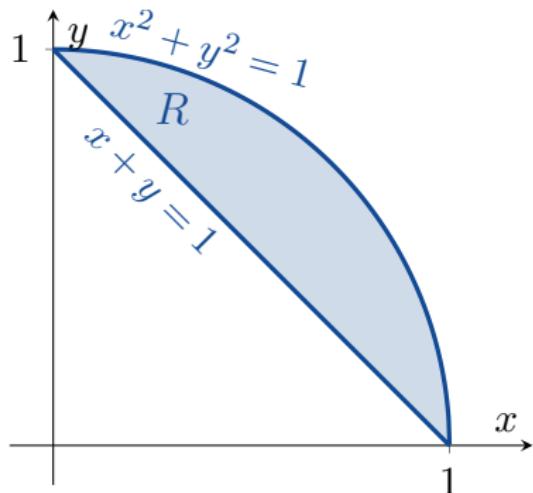


Finding Limits of Integration



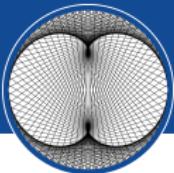


Finding Limits of Integration

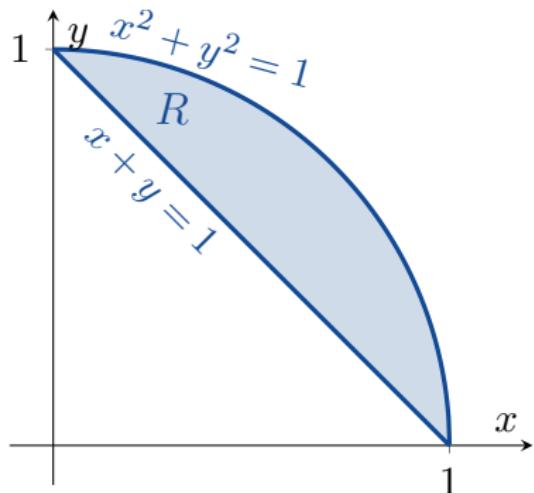


Question 1: Shall we do $dxdy$ or $dydx$?

$$\int \int f(x, y)$$



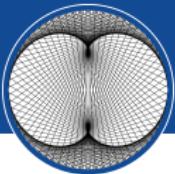
Finding Limits of Integration



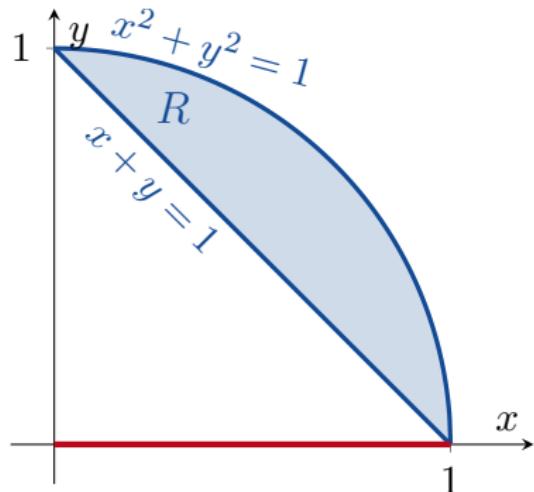
Question 1: Shall we do $dxdy$ or $dydx$?

$dydx$

$$\int \int f(x, y) dydx$$

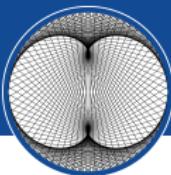


Finding Limits of Integration

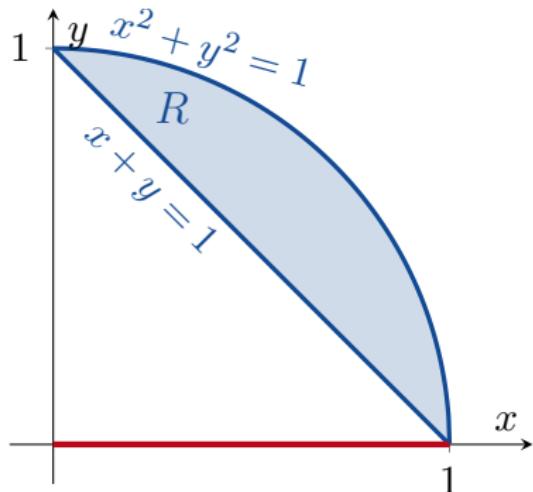


Question 2: What are the smallest and biggest possible values of x in R ?

$$\int \int f(x, y) dy dx$$



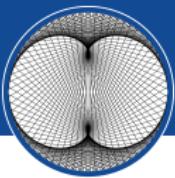
Finding Limits of Integration



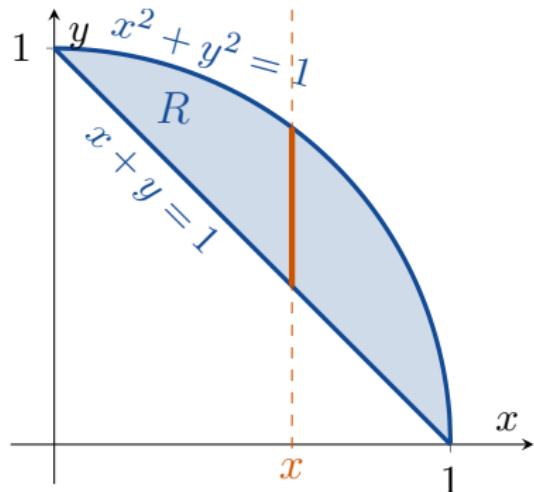
Question 2: What are the smallest and biggest possible values of x in R ?

$$0 \leq x \leq 1$$

$$\int_0^1 \int f(x, y) dy dx$$

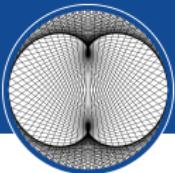


Finding Limits of Integration

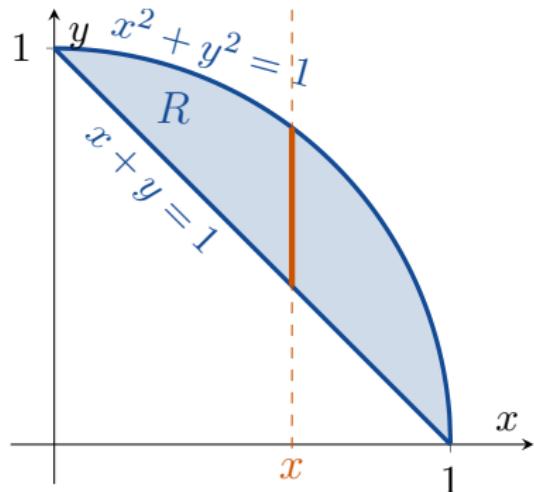


Now choose an x between 0 and 1 and draw the **vertical** line through x .

$$\int_0^1 \int f(x, y) dy dx$$



Finding Limits of Integration

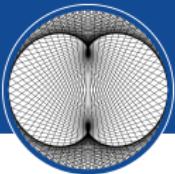


Now choose an x between 0 and 1 and draw the **vertical line** through x .

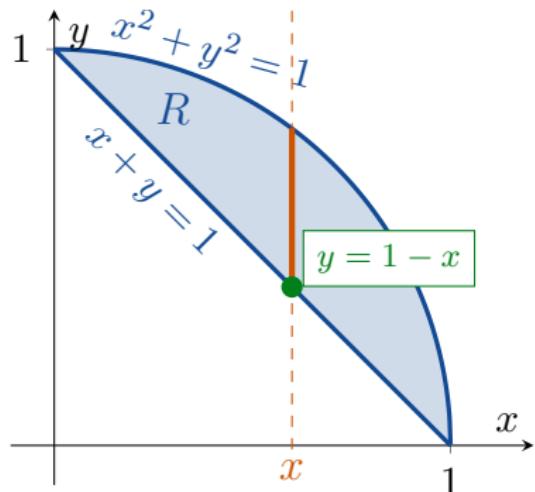
Question 3: What are the smallest and biggest possible values of y (on this **line**) in R ?

$$\leq y \leq$$

$$\int_0^1 \int f(x, y) dy dx$$



Finding Limits of Integration

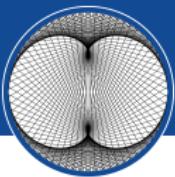


Now choose an x between 0 and 1 and draw the **vertical line** through x .

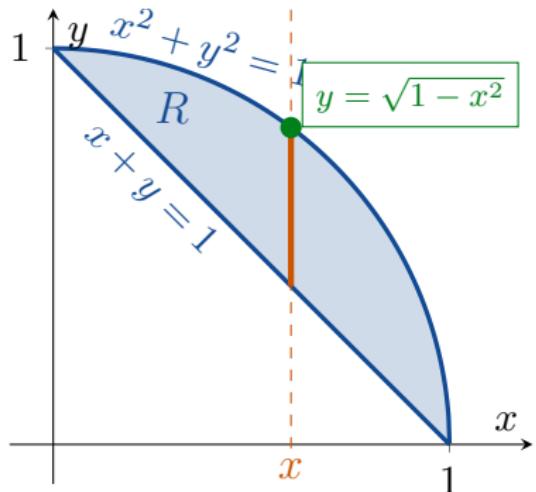
Question 3: What are the smallest and biggest possible values of y (on this **line**) in R ?

$$1 - x \leq y \leq$$

$$\int_0^1 \int_{1-x}^{1-\sqrt{1-x^2}} f(x, y) dy dx$$



Finding Limits of Integration

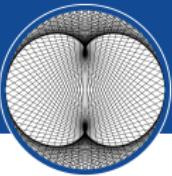


Now choose an x between 0 and 1 and draw the **vertical** line through x .

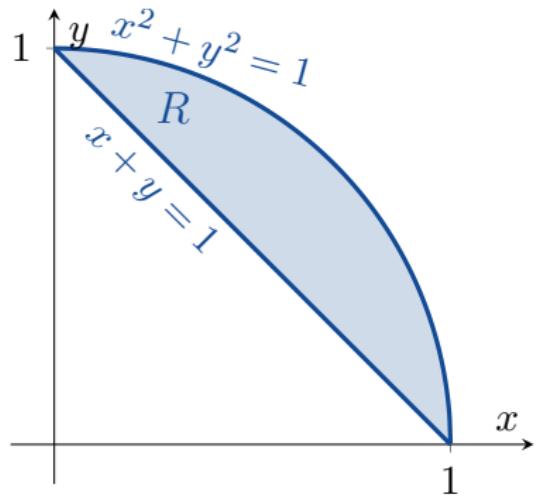
Question 3: What are the smallest and biggest possible values of y (on this **line**) in R ?

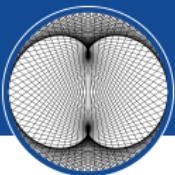
$$1 - x \leq y \leq \sqrt{1 - x^2}$$

$$\int_0^1 \int_{1-x}^{\sqrt{1-x^2}} f(x, y) \, dy \, dx$$

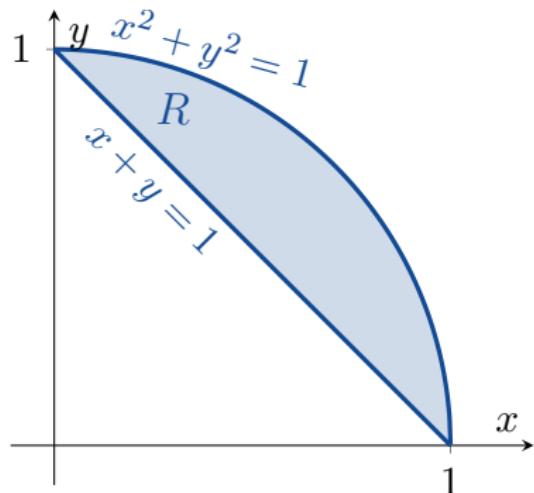


Finding Limits of Integration (again)





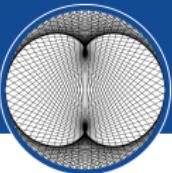
Finding Limits of Integration (again)



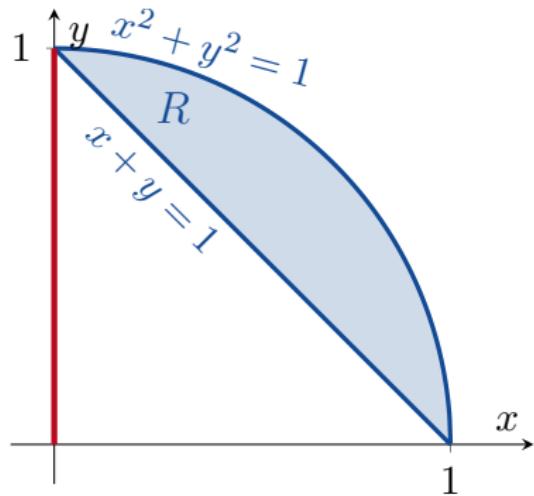
Question 1: Shall we do $dxdy$ or $dydx$?

$dxdy$

$$\int \int f(x, y) \, dxdy$$



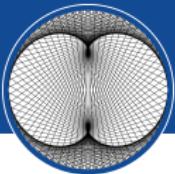
Finding Limits of Integration (again)



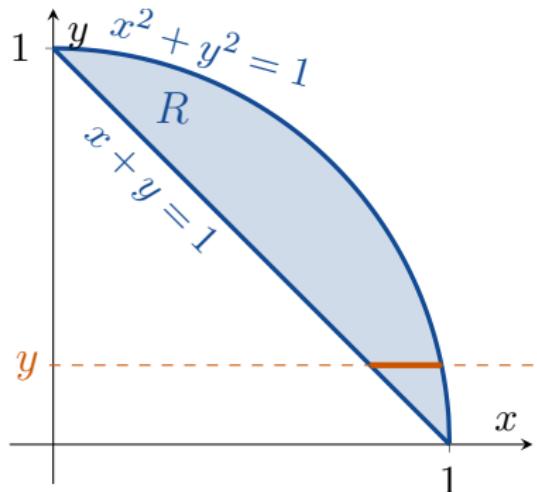
Question 2: What are the smallest and biggest possible values of y in R ?

$$0 \leq y \leq 1$$

$$\int_0^1 \int f(x, y) \, dx \, dy$$



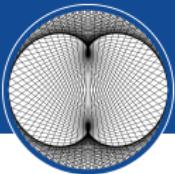
Finding Limits of Integration (again)



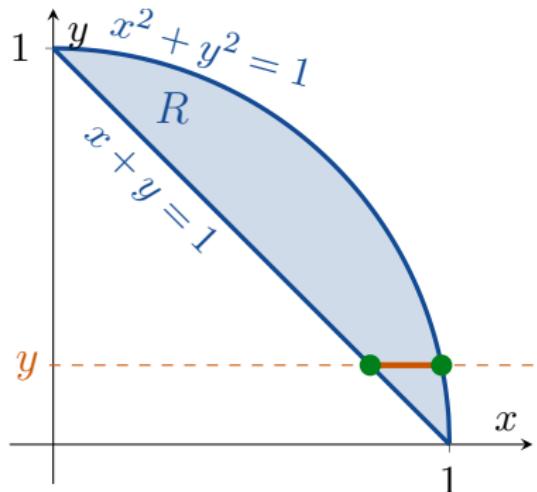
Now choose an y between 0 and 1 and draw the **horizontal line** through y .

Question 3: What are the smallest and biggest possible values of x (on this **line**) in R ?

$$\int_0^1 \int f(x, y) \, dx \, dy$$



Finding Limits of Integration (again)



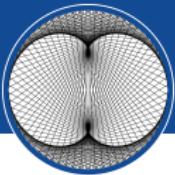
Now choose an y between 0 and 1 and draw the **horizontal** line through y .

Question 3: What are the smallest and biggest possible values of x (on this **line**) in R ?

$$1 - y \leq x \leq \sqrt{1 - y^2}$$

$$\int_0^1 \int_{1-y}^{\sqrt{1-y^2}} f(x, y) \, dx \, dy$$

14.2 Double Integrals over General Regions



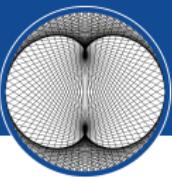
Example

Sketch the region of integration for the integral

$$\int_0^2 \int_{x^2}^{2x} (4x + 2) \, dy \, dx.$$

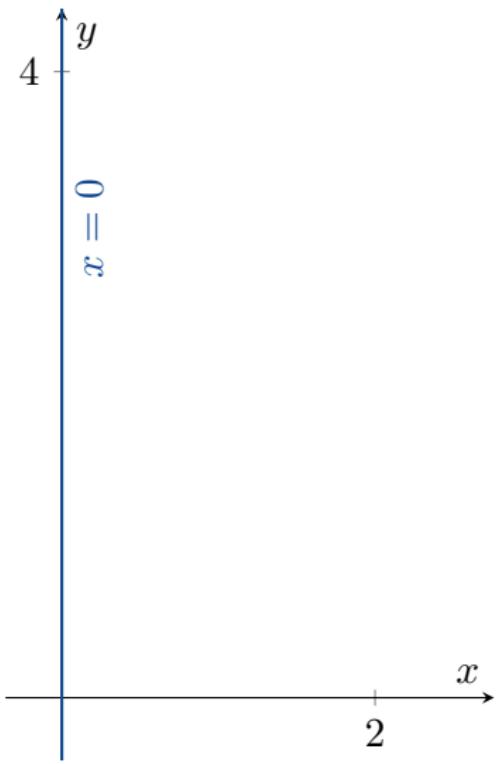
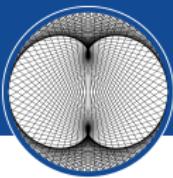
Then write an equivalent integral with $dxdy$.

14.2 Double Integrals over General Regions



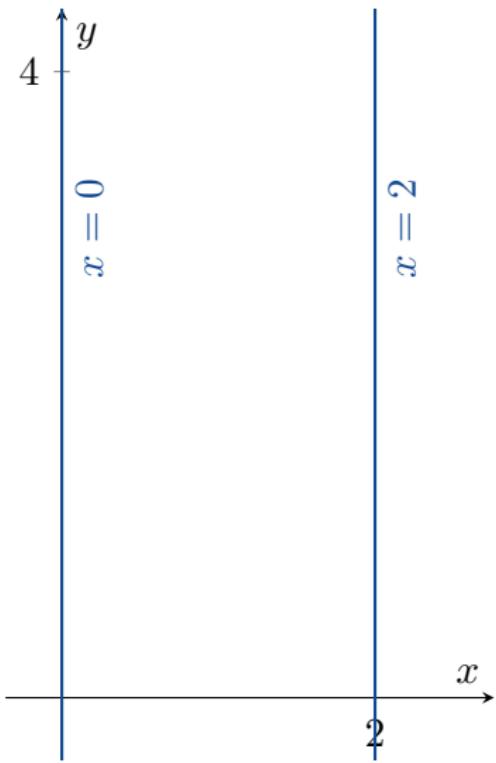
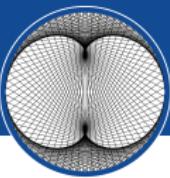
$$\int_0^2 \int_{x^2}^{2x} (4x + 2) \, dy \, dx$$

14.2 Double Integrals over General Regions



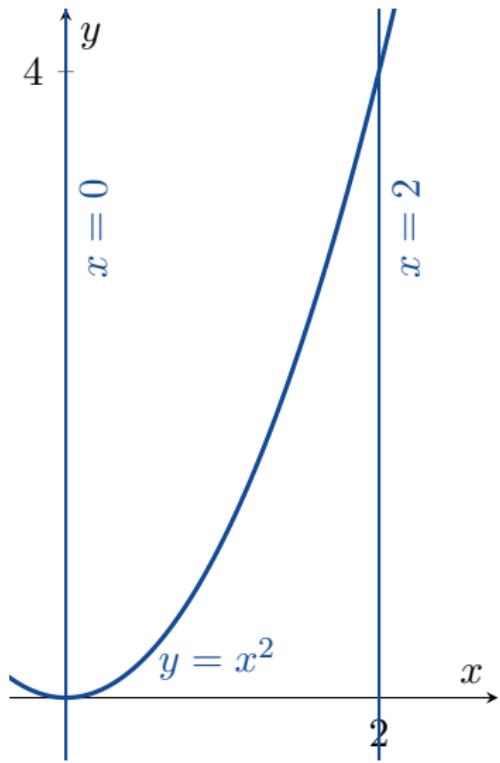
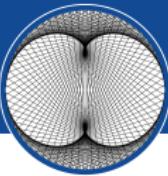
$$\int_0^2 \int_{x^2}^{2x} (4x + 2) dy dx$$

14.2 Double Integrals over General Regions



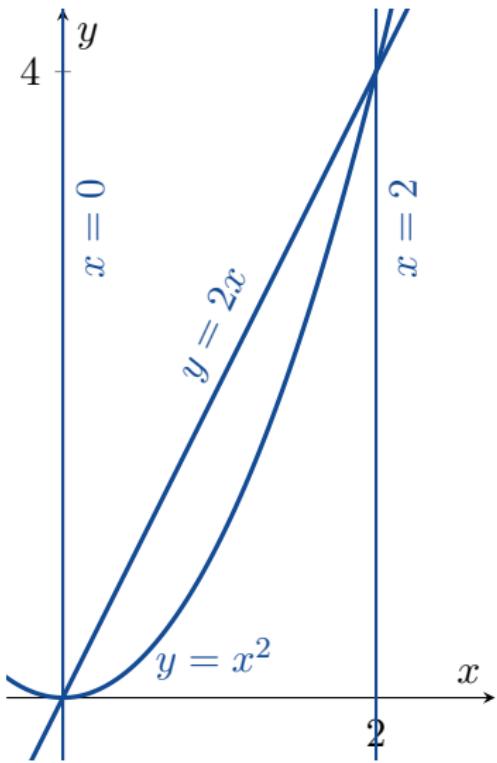
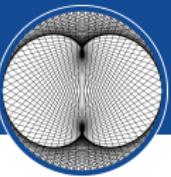
$$\int_0^2 \int_{x^2}^{2x} (4x + 2) dy dx$$

14.2 Double Integrals over General Regions



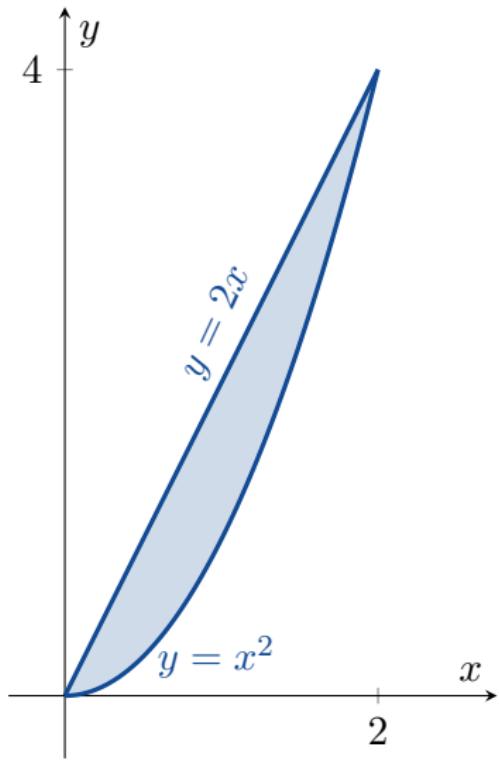
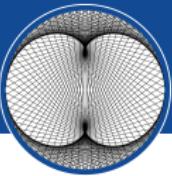
$$\int_0^2 \int_{x^2}^{2x} (4x + 2) dy dx$$

14.2 Double Integrals over General Regions



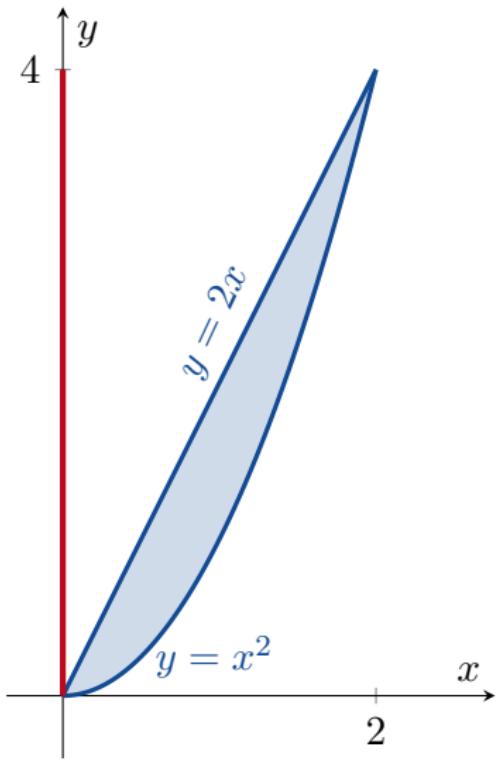
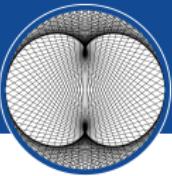
$$\int_0^2 \int_{x^2}^{2x} (4x + 2) dy dx$$

14.2 Double Integrals over General Regions



$$\int_0^2 \int_{x^2}^{2x} (4x + 2) \, dy \, dx \\ = \int \int (4x + 2) \, dx \, dy.$$

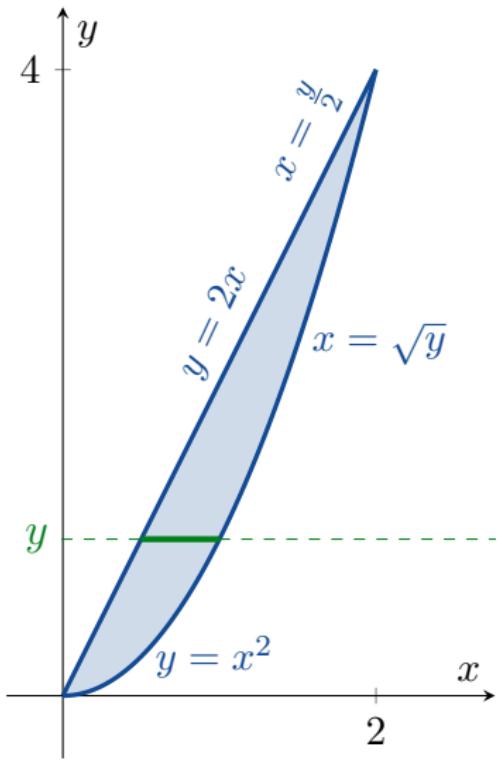
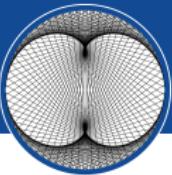
14.2 Double Integrals over General Regions



$$\int_0^2 \int_{x^2}^{2x} (4x + 2) \, dy \, dx$$

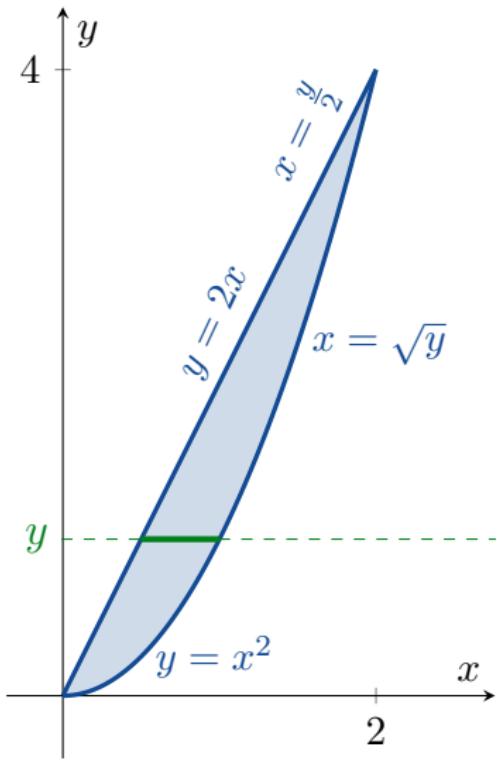
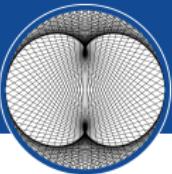
$$= \int_0^4 \int \quad (4x + 2) \, dx \, dy.$$

14.2 Double Integrals over General Regions



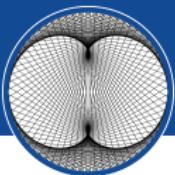
$$\int_0^2 \int_{x^2}^{2x} (4x + 2) \, dy \, dx \\ = \int_0^4 \int \quad (4x + 2) \, d\textcolor{green}{x} \, dy.$$

14.2 Double Integrals over General Regions



$$\int_0^2 \int_{x^2}^{2x} (4x + 2) \, dy \, dx \\ = \int_0^4 \int_{\frac{y}{2}}^{\sqrt{y}} (4x + 2) \, d\text{x} \, dy.$$

14.2 Double Integrals over General Regions

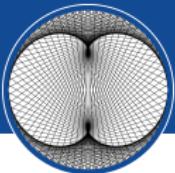


Properties of Double Integrals

Suppose that $f(x, y)$ and $g(x, y)$ are continuous on the bounded region R .

Theorem

14.2 Double Integrals over General Regions



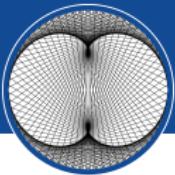
Properties of Double Integrals

Suppose that $f(x, y)$ and $g(x, y)$ are continuous on the bounded region R .

Theorem

$$\iint_R cf \, dA = c \iint_R f \, dA$$

for any number c .

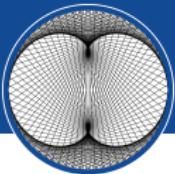


Properties of Double Integrals

Suppose that $f(x, y)$ and $g(x, y)$ are continuous on the bounded region R .

Theorem

$$\iint_R (f + g) \, dA = \iint_R f \, dA + \iint_R g \, dA.$$



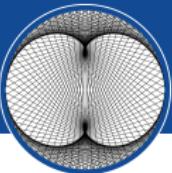
Properties of Double Integrals

Suppose that $f(x, y)$ and $g(x, y)$ are continuous on the bounded region R .

Theorem

$$f \geq 0 \text{ on } R \implies \iint_R f \, dA \geq 0.$$

14.2 Double Integrals over General Regions

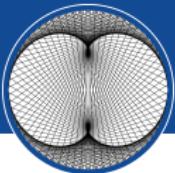


Properties of Double Integrals

Suppose that $f(x, y)$ and $g(x, y)$ are continuous on the bounded region R .

Theorem

$$f \geq g \text{ on } R \implies \iint_R f \, dA \geq \iint_R g \, dA.$$



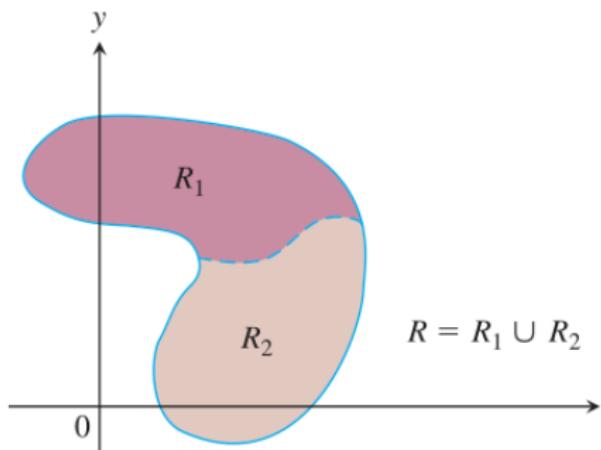
Properties of Double Integrals

Suppose that $f(x, y)$ and $g(x, y)$ are continuous on the bounded region R .

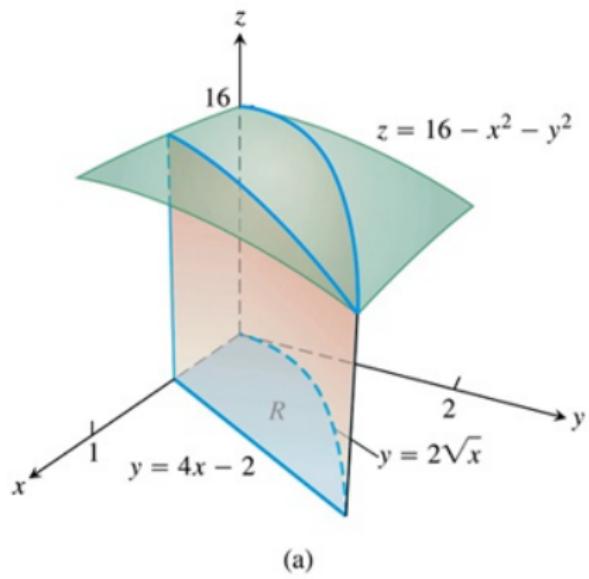
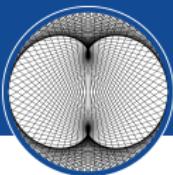
Theorem

If $R = R_1 \cup R_2$ where R_1 and R_2 don't overlay, then

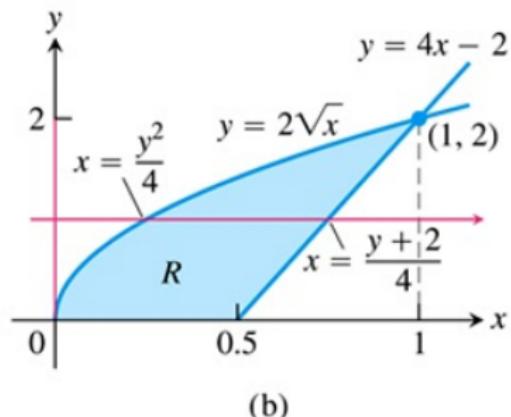
$$\iint_R f \, dA = \iint_{R_1} f \, dA + \iint_{R_2} f \, dA.$$



14.2 Double Integrals over General Regions



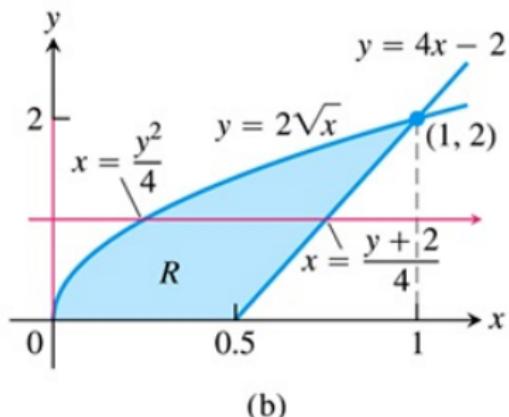
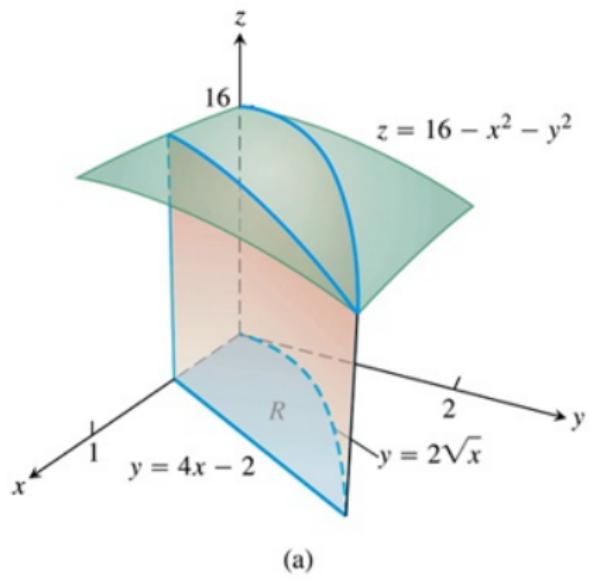
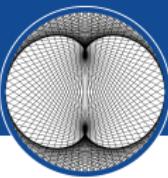
(a)



(b)

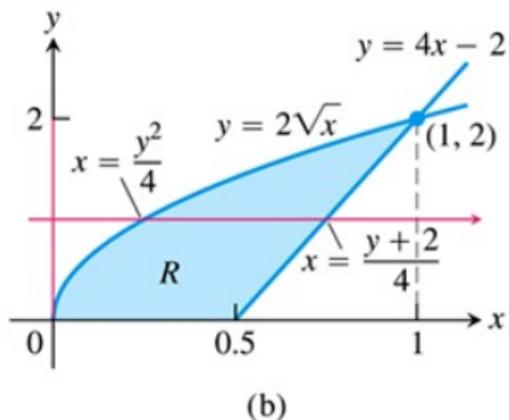
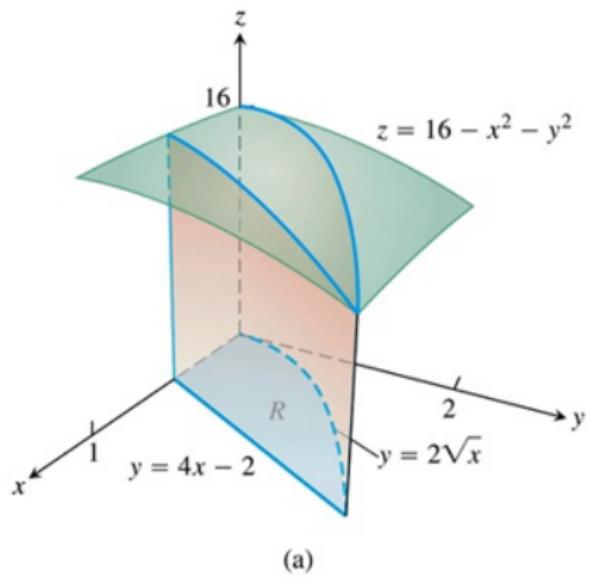
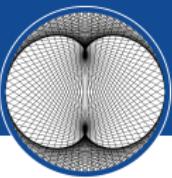
EXAMPLE 4 Find the volume of the wedgelike solid that lies beneath the surface $z = 16 - x^2 - y^2$ and above the region R bounded by the curve $y = 2\sqrt{x}$, the line $y = 4x - 2$, and the x -axis.

14.2 Double Integrals over General Regions



$$0 \leq y \leq 2$$

14.2 Double Integrals over General Regions



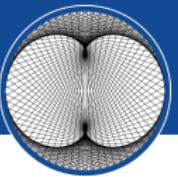
$$0 \leq y \leq 2$$

$$\frac{y^2}{4} \leq x \leq \frac{y+2}{4}$$

Solution Figure 15.18a shows the surface and the “wedgelike” solid whose volume we want to calculate. Figure 15.18b shows the region of integration in the xy -plane. If we integrate in the order $dy\ dx$ (first with respect to y and then with respect to x), two integrations will be required because y varies from $y = 0$ to $y = 2\sqrt{x}$ for $0 \leq x \leq 0.5$, and then varies from $y = 4x - 2$ to $y = 2\sqrt{x}$ for $0.5 \leq x \leq 1$. So we choose to integrate in the order $dx\ dy$, which requires only one double integral whose limits of integration are indicated in Figure 15.18b. The volume is then calculated as the iterated integral:

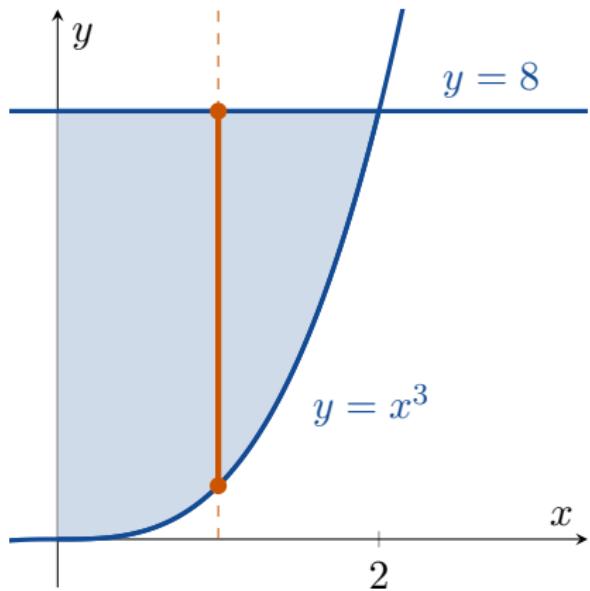
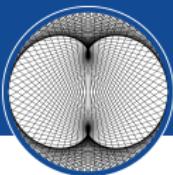
$$\begin{aligned}
 & \iint_R (16 - x^2 - y^2) \, dA \\
 &= \int_0^2 \int_{y^2/4}^{(y+2)/4} (16 - x^2 - y^2) \, dx \, dy \\
 &= \int_0^2 \left[16x - \frac{x^3}{3} - xy^2 \right]_{x=y^2/4}^{x=(y+2)/4} \, dx \\
 &= \int_0^2 \left[4(y+2) - \frac{(y+2)^3}{3 \cdot 64} - \frac{(y+2)y^2}{4} - 4y^2 + \frac{y^6}{3 \cdot 64} + \frac{y^4}{4} \right] \, dy \\
 &= \left[\frac{191y}{24} + \frac{63y^2}{32} - \frac{145y^3}{96} - \frac{49y^4}{768} + \frac{y^5}{20} + \frac{y^7}{1344} \right]_0^2 = \frac{20803}{1680} \approx 12.4.
 \end{aligned}$$

14.2 Double Integrals over General Regions



Now some problems for you.

14.2 Double Integrals over General Regions



Which is correct?

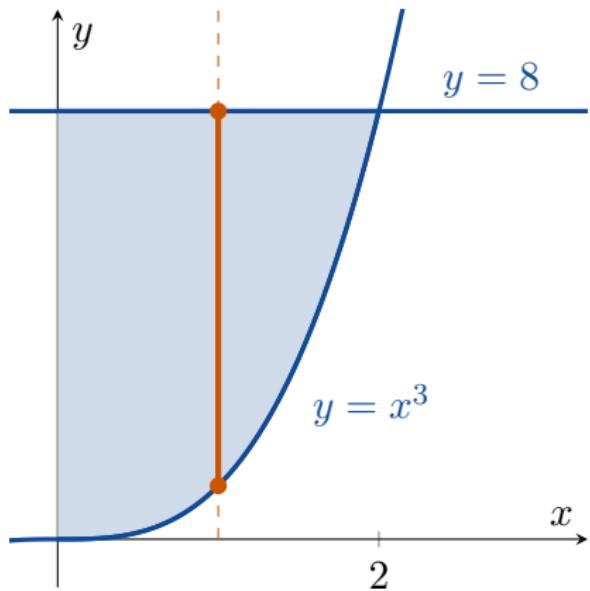
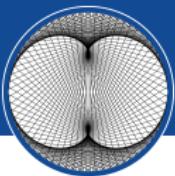
A $\int_0^2 \int_{x^3}^8 f(x, y) dy dx$

B $\int_0^2 \int_{x^3}^8 f(x, y) dy dx$

C $\int_{x^3}^8 \int_0^2 f(x, y) dy dx$

D $\int_0^1 \int_0^{x^3} f(x, y) dy dx$

14.2 Double Integrals over General Regions



Which is correct?

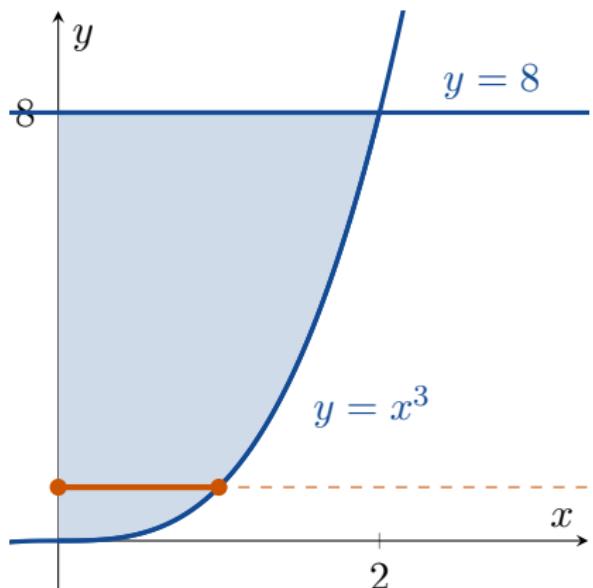
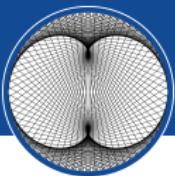
A $\int_0^2 \int_{x^3}^8 f(x, y) dy dx$

B $\int_0^2 \int_{x^3}^8 f(x, y) dy dx$

C $\int_{x^3}^8 \int_0^2 f(x, y) dy dx$

D $\int_0^1 \int_0^{x^3} f(x, y) dy dx$

14.2 Double Integrals over General Regions



Which is correct?

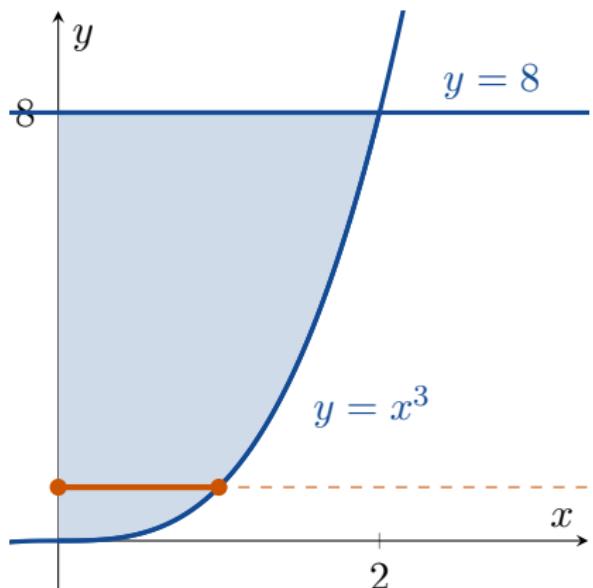
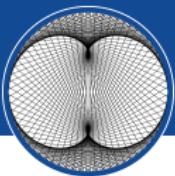
A $\int_0^8 \int_0^{y^{\frac{1}{3}}} f(x, y) \, dxdy$

B $\int_0^8 \int_0^{y^{\frac{1}{3}}} f(x, y) \, dydx$

C $\int_0^8 \int_0^{x^3} f(x, y) \, dxdy$

D $\int_0^8 \int_0^3 f(x, y) \, dydx$

14.2 Double Integrals over General Regions



Which is correct?

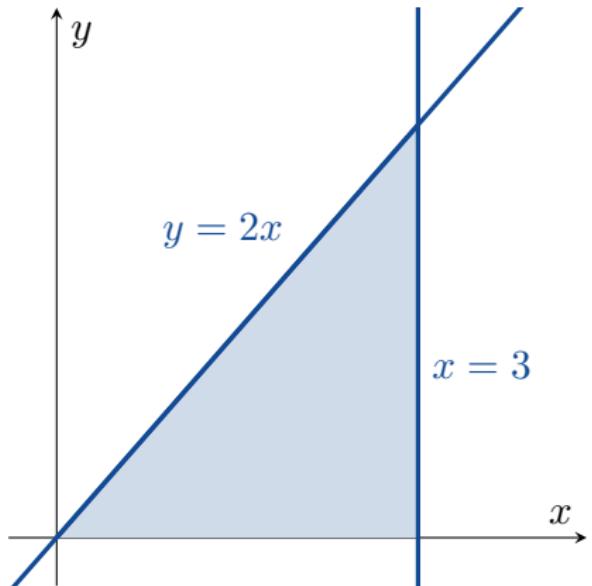
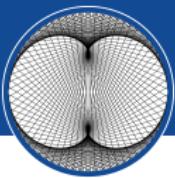
A $\int_0^8 \int_0^{y^{\frac{1}{3}}} f(x, y) dx dy$

B $\int_0^8 \int_0^{y^{\frac{1}{3}}} f(x, y) dy dx$

C $\int_0^8 \int_0^{x^3} f(x, y) dx dy$

D $\int_0^8 \int_0^3 f(x, y) dy dx$

14.2 Double Integrals over General Regions



Which is correct?

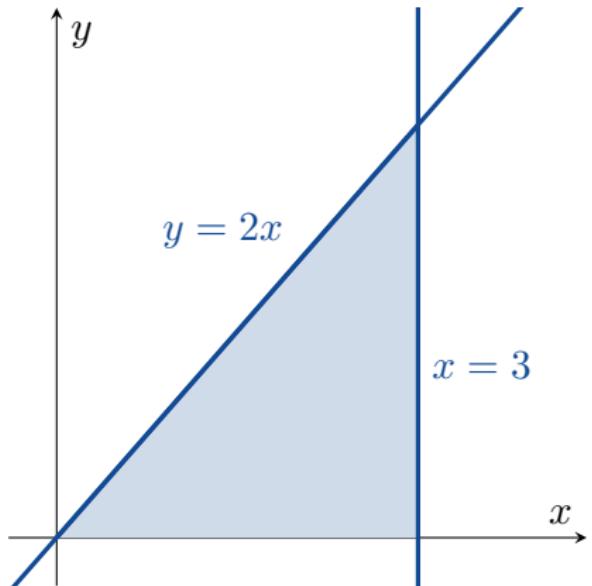
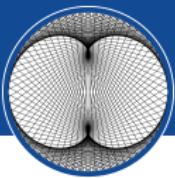
A $\int_0^6 \int_0^{\frac{y}{2}} f(x, y) \, dxdy$

B $\int_0^3 \int_0^{2x} f(x, y) \, dxdy$

C $\int_0^3 \int_0^{2x} f(x, y) \, dydx$

D $\int_0^6 \int_{\frac{y}{2}}^3 f(x, y) \, dxdy$

14.2 Double Integrals over General Regions



Which is correct?

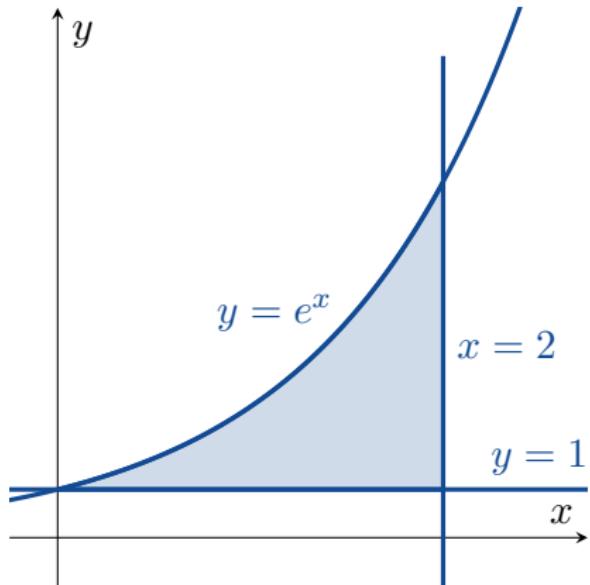
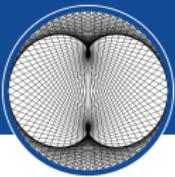
A $\int_0^6 \int_0^{\frac{y}{2}} f(x, y) \, dxdy$

B $\int_0^3 \int_0^{2x} f(x, y) \, dxdy$

C $\int_0^3 \int_0^{2x} f(x, y) \, dydx$

D $\int_0^6 \int_{\frac{y}{2}}^3 f(x, y) \, dxdy$

14.2 Double Integrals over General Regions



Which is correct?

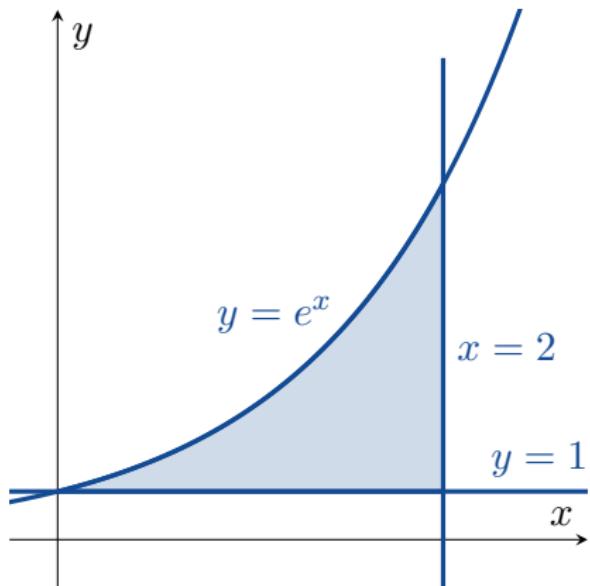
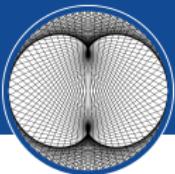
A $\int_1^{e^x} \int_0^2 f(x, y) \, dxdy$

B $\int_0^2 \int_0^{e^x} f(x, y) \, dxdy$

C $\int_0^2 \int_0^{e^x} f(x, y) \, dydx$

D $\int_0^2 \int_1^{e^x} f(x, y) \, dydx$

14.2 Double Integrals over General Regions



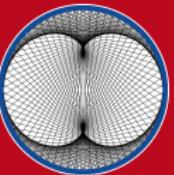
Which is correct?

A $\int_1^{e^x} \int_0^2 f(x, y) \, dxdy$

B $\int_0^2 \int_0^{e^x} f(x, y) \, dxdy$

C $\int_0^2 \int_0^{e^x} f(x, y) \, dydx$

D $\int_0^2 \int_1^{e^x} f(x, y) \, dydx$

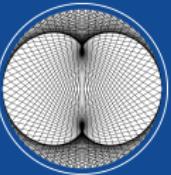


Break

We will continue at 3pm

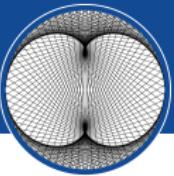


"The next part of this recipe will involve some calculus."



11 Area by Double Integration 3

14.3 Area by Double Integration

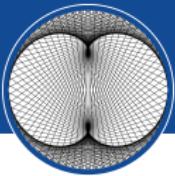


Definition

The *area* of a closed, bounded region R is

$$A = \iint_R dA.$$

14.3 Area by Double Integration

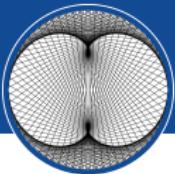


Definition

The *area* of a closed, bounded region R is

$$A = \iint_R 1 dA.$$

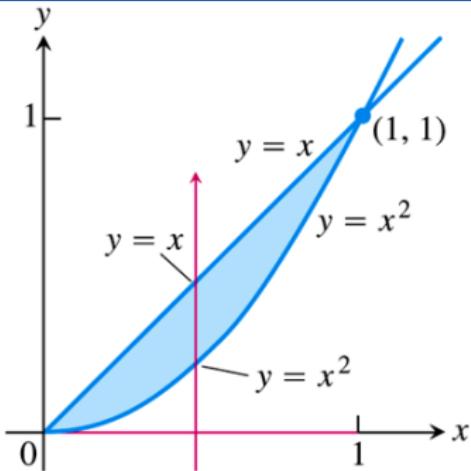
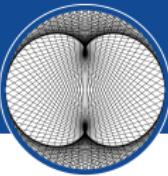
14.3 Area by Double Integration



Example

Find the area of the region R bounded by $y = x$ and $y = x^2$ in the first quadrant.

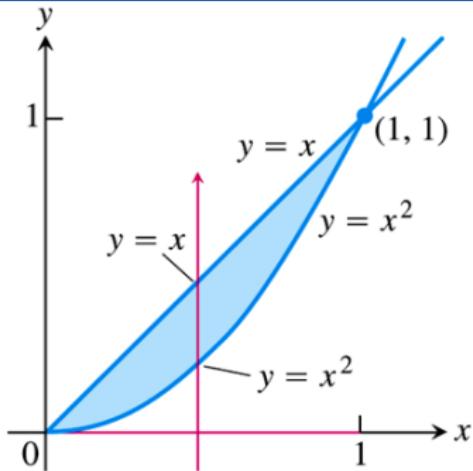
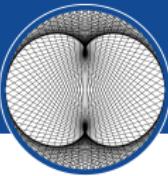
14.3 Area by Double Integration



Example

Find the area of the region R bounded by $y = x$ and $y = x^2$ in the first quadrant.

14.3 Area by Double Integration

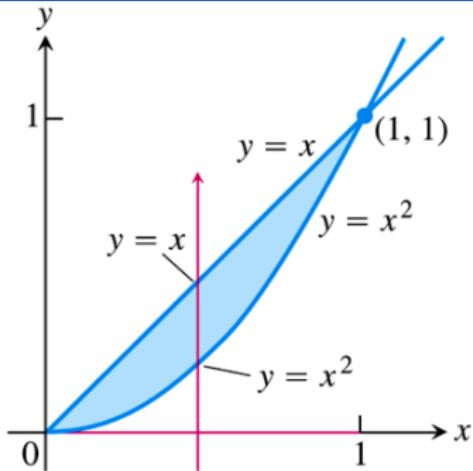
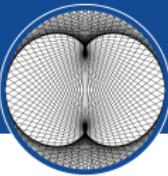


Example

Find the area of the region R bounded by $y = x$ and $y = x^2$ in the first quadrant.

$$A = \int_0^1 \int_{x^2}^x dy dx$$

14.3 Area by Double Integration

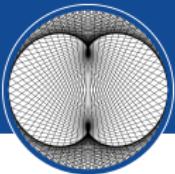


Example

Find the area of the region R bounded by $y = x$ and $y = x^2$ in the first quadrant.

$$\begin{aligned} A &= \int_0^1 \int_{x^2}^x dy dx = \int_0^1 [y]_{x^2}^x dx = \int_0^1 (x - x^2) dx \\ &= \left[\frac{x^2}{2} - \frac{x^3}{3} \right]_0^1 = \frac{1}{6}. \end{aligned}$$

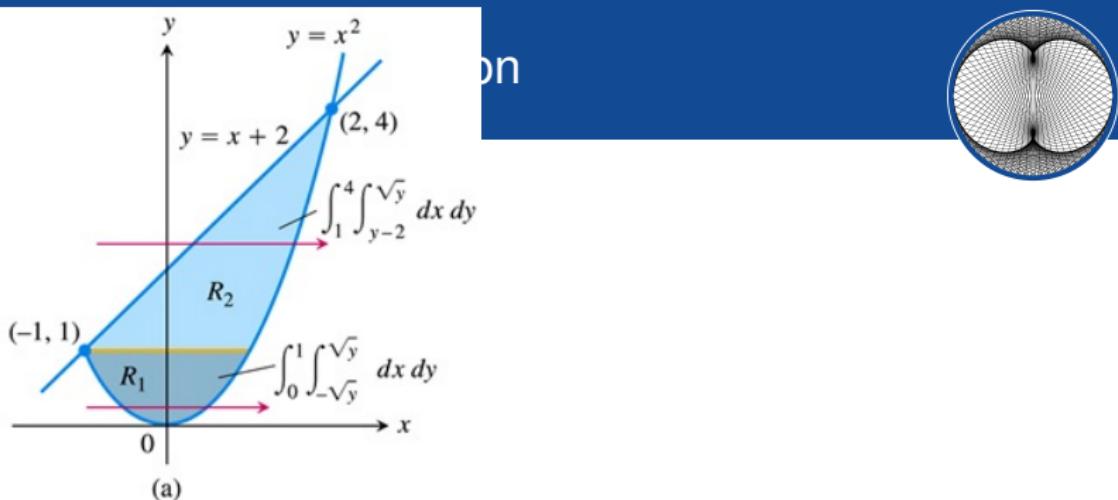
14.3 Area by Double Integration



Example

Find the area of the region R bounded by $y = x + 2$ and $y = x^2$ in the first quadrant.

14.3 Arc

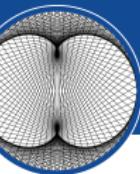
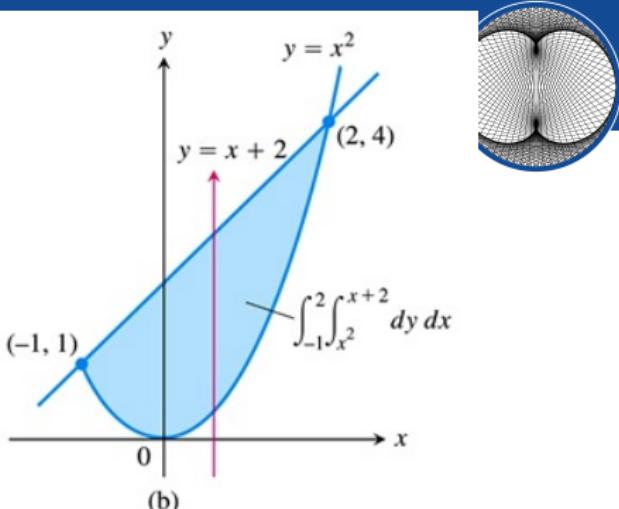
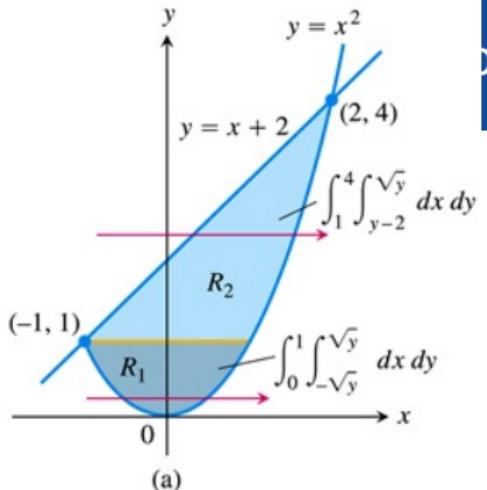


Example

Find the area of the region R bounded by $y = x + 2$ and $y = x^2$ in the first quadrant.

$$A = \iint_{R_1} dA + \iint_{R_2} dA = \int_0^1 \int_{-\sqrt{y}}^{\sqrt{y}} dx dy + \int_1^4 \int_{y-2}^{\sqrt{y}} dx dy = \dots$$

14.3 Area



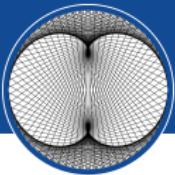
Example

Find the area of the region R bounded by $y = x + 2$ and $y = x^2$ in the first quadrant.

$$A = \iint_{R_1} dA + \iint_{R_2} dA = \int_0^1 \int_{-\sqrt{y}}^{\sqrt{y}} dx dy + \int_1^4 \int_{y-2}^{\sqrt{y}} dx dy = \dots$$

$$A = \int_{-1}^2 \int_{x^2}^{x+2} dy dx = \int_{-1}^2 [y]_{x^2}^{x+2} = \int_{-1}^2 (x+2-x^2) dx = \dots = \frac{9}{2}.$$

14.3 Area by Double Integration

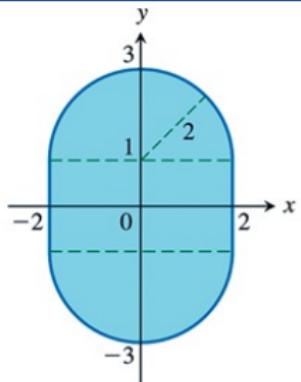
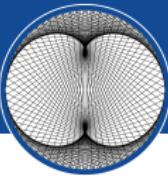


Example

Find the area of the region R described by $-2 \leq x \leq 2$ and $-1 - \sqrt{4 - x^2} \leq y \leq 1 + \sqrt{4 - x^2}$.

14.3 Area by

tion



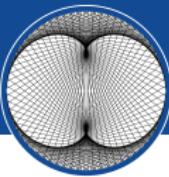
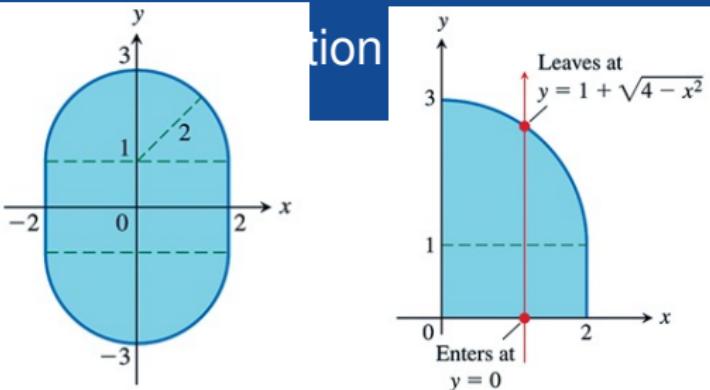
Example

Find the area of the region R described by $-2 \leq x \leq 2$ and $-1 - \sqrt{4 - x^2} \leq y \leq 1 + \sqrt{4 - x^2}$.

$$A = \iint_R dA$$

14.3 Area by

tion



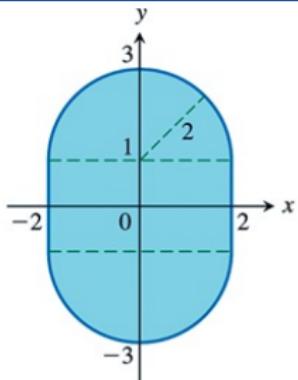
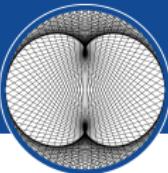
Example

Find the area of the region R described by $-2 \leq x \leq 2$ and $-1 - \sqrt{4 - x^2} \leq y \leq 1 + \sqrt{4 - x^2}$.

$$A = \iint_R dA = 4 \int_0^2 \int_{0}^{1+\sqrt{4-x^2}} dy dx = \dots = 8 + 4\pi.$$

14.3 Area by

tion



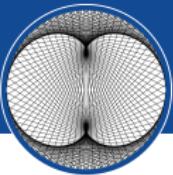
Example

Find the area of the region R described by $-2 \leq x \leq 2$ and $-1 - \sqrt{4 - x^2} \leq y \leq 1 + \sqrt{4 - x^2}$.

$$A = \iint_R dA = 4 \int_0^2 \int_{0}^{1+\sqrt{4-x^2}} dy dx = \dots = 8 + 4\pi.$$

or

$$A = \left(\begin{array}{l} \text{area of a} \\ \text{circle of} \\ \text{radius 2} \end{array} \right) + \left(\begin{array}{l} \text{area of a } 4 \times 2 \\ \text{rectangle} \end{array} \right) = 4\pi + 8.$$

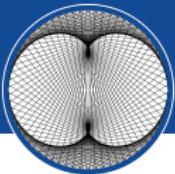


Average Value of a Function

Definition

The *average value* of f over R is

$$\text{av}(f) = \frac{1}{\text{area of } R} \iint_R f \, dA$$



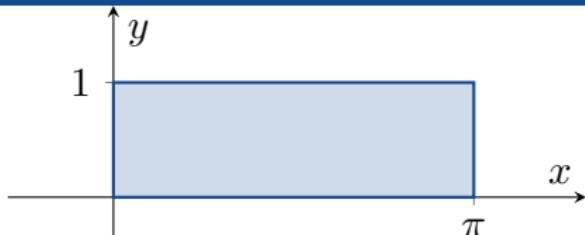
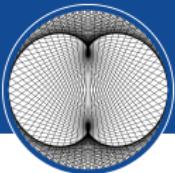
Average Value of a Function

Definition

The *average value* of f over R is

$$\text{av}(f) = \frac{1}{\text{area of } R} \iint_R f \, dA = \frac{\iint_R f \, dA}{\iint_R dA}.$$

14.3 Area by Double Integration

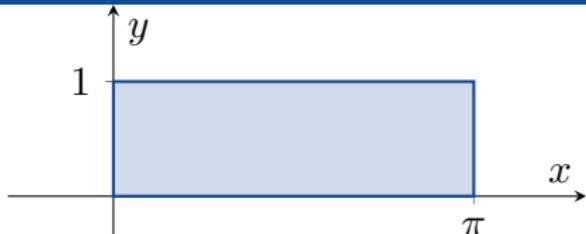
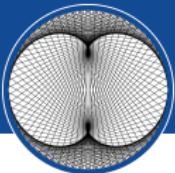


Example

Find the average value of $f(x, y) = x \cos xy$ over the rectangle $R = [0, \pi] \times [0, 1]$.

$$\text{av}(f) = \frac{1}{\text{area of } R} \iint_R f \, dA$$

14.3 Area by Double Integration

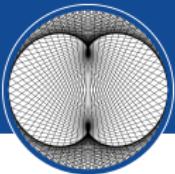


Example

Find the average value of $f(x, y) = x \cos xy$ over the rectangle $R = [0, \pi] \times [0, 1]$.

$$\text{av}(f) = \frac{1}{\text{area of } R} \iint_R f \, dA = \frac{1}{\pi} \int_0^\pi \int_0^1 x \cos xy \, dy \, dx$$

14.3 Area by Double Integration

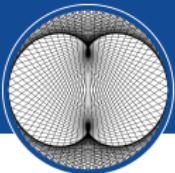


Example

Find the average value of $f(x, y) = x \cos xy$ over the rectangle $R = [0, \pi] \times [0, 1]$.

$$\begin{aligned}\text{av}(f) &= \frac{1}{\text{area of } R} \iint_R f \, dA = \frac{1}{\pi} \int_0^\pi \int_0^1 x \cos xy \, dy \, dx \\ &= \frac{1}{\pi} \int_0^\pi \left[\sin xy \right]_{y=0}^{y=1} dx = \frac{1}{\pi} \int_0^\pi (\sin x - 0) \, dx\end{aligned}$$

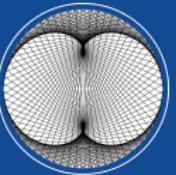
14.3 Area by Double Integration



Example

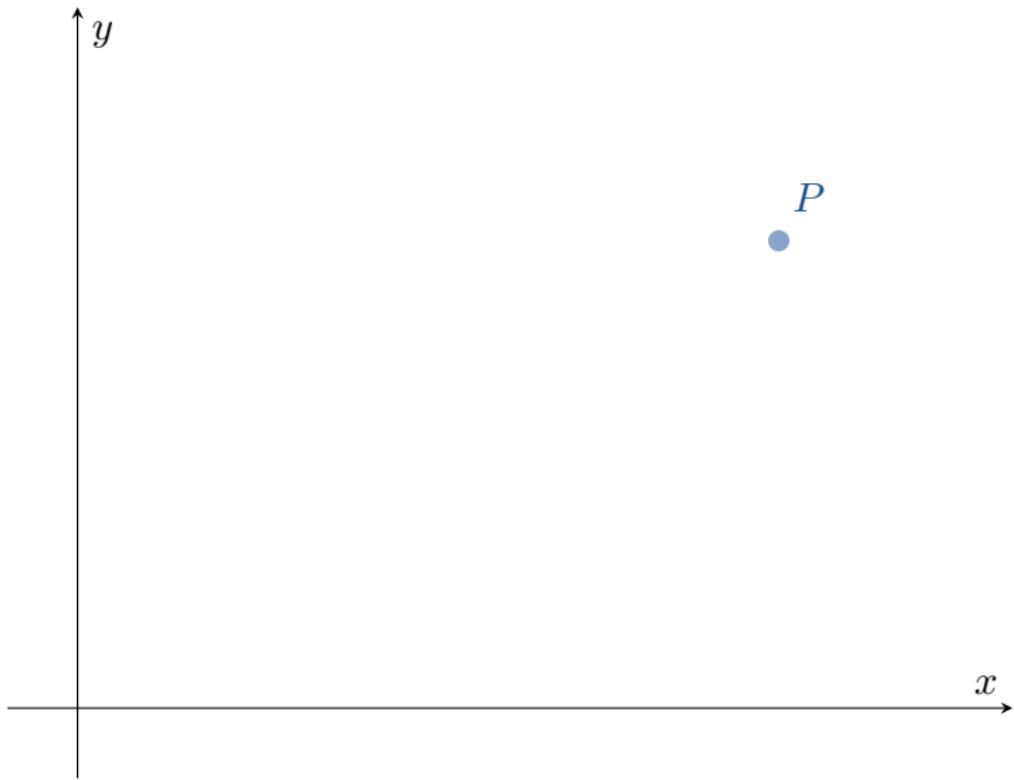
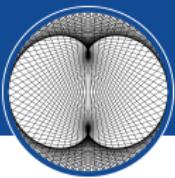
Find the average value of $f(x, y) = x \cos xy$ over the rectangle $R = [0, \pi] \times [0, 1]$.

$$\begin{aligned}\text{av}(f) &= \frac{1}{\text{area of } R} \iint_R f \, dA = \frac{1}{\pi} \int_0^\pi \int_0^1 x \cos xy \, dy \, dx \\ &= \frac{1}{\pi} \int_0^\pi \left[\sin xy \right]_{y=0}^{y=1} dx = \frac{1}{\pi} \int_0^\pi (\sin x - 0) \, dx \\ &= \frac{1}{\pi} \left[-\cos x \right]_0^\pi = \frac{1}{\pi} (1 + 1) = \frac{2}{\pi}.\end{aligned}$$

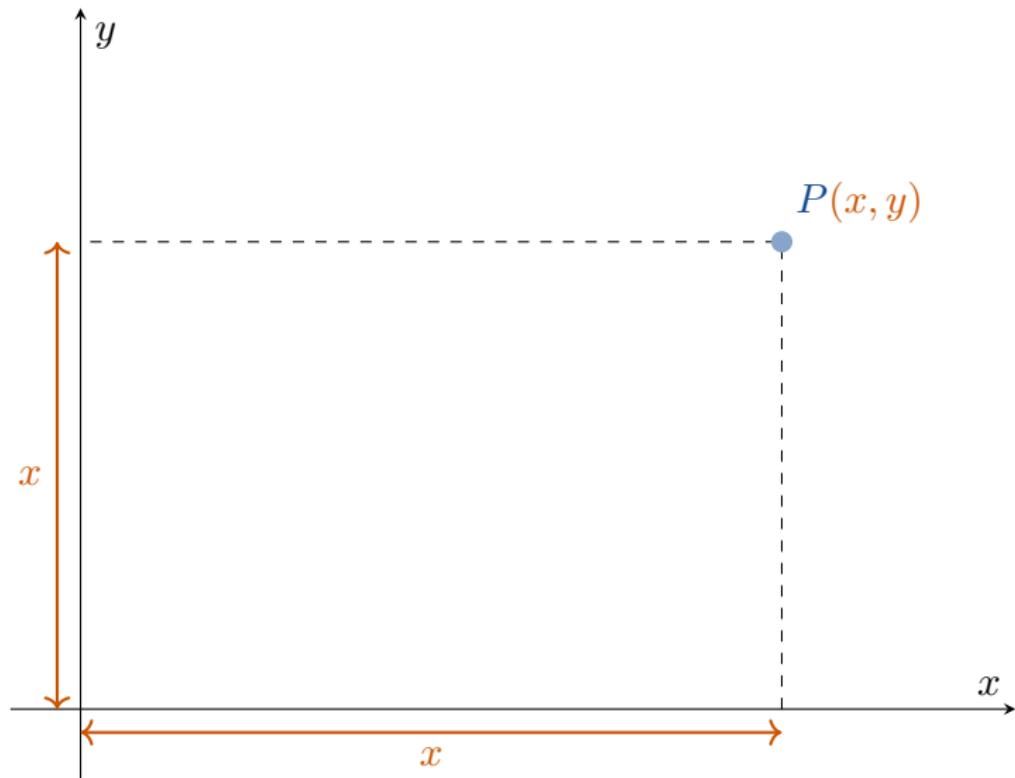
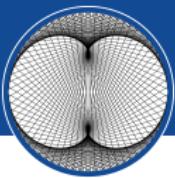


10.3 Polar Coordinates

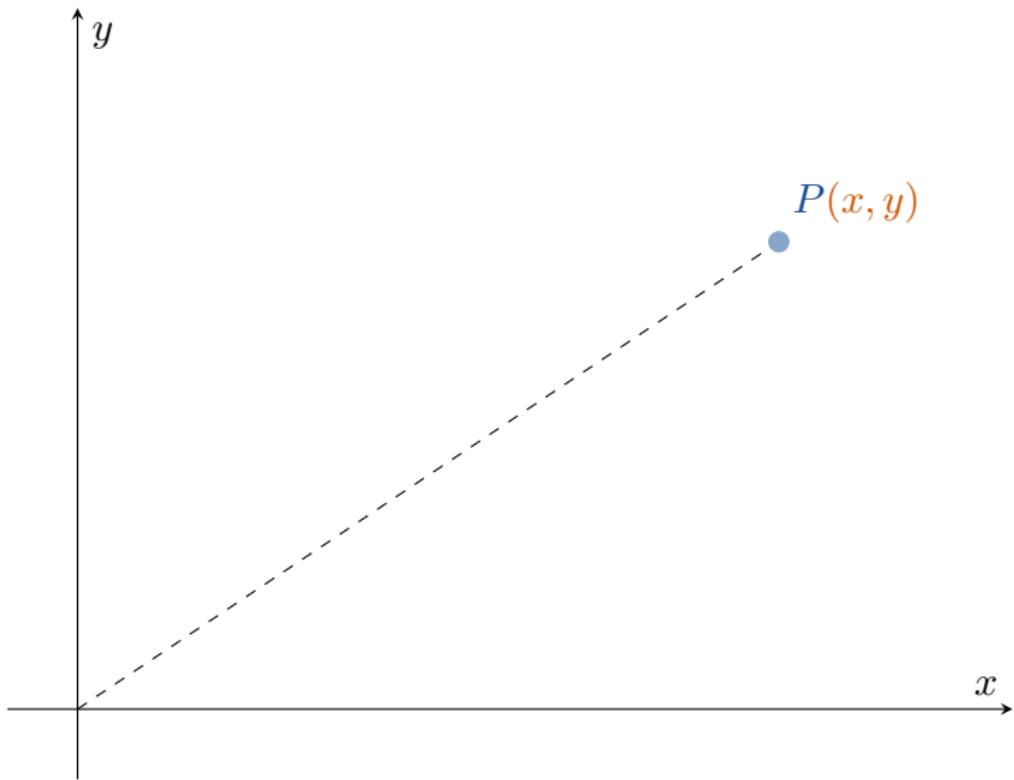
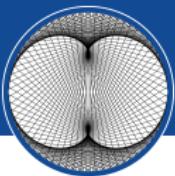
10.3 Polar Coordinates



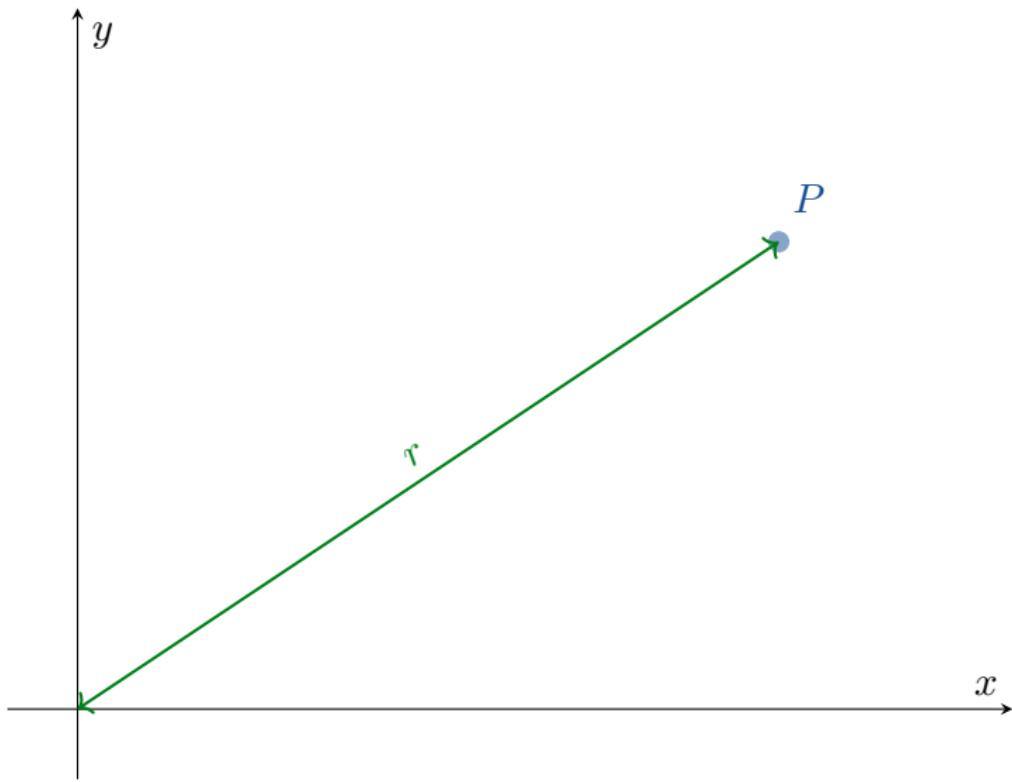
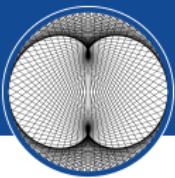
10.3 Polar Coordinates



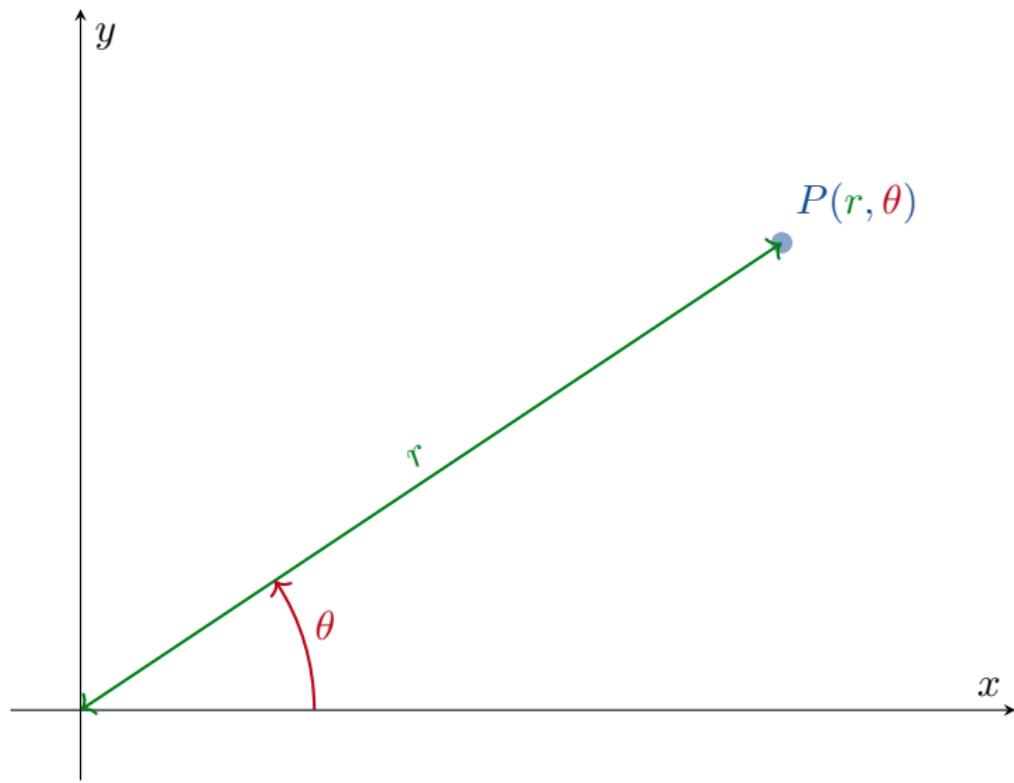
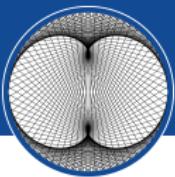
10.3 Polar Coordinates



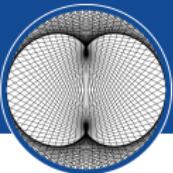
10.3 Polar Coordinates



10.3 Polar Coordinates



10.3 Polar Coordinates

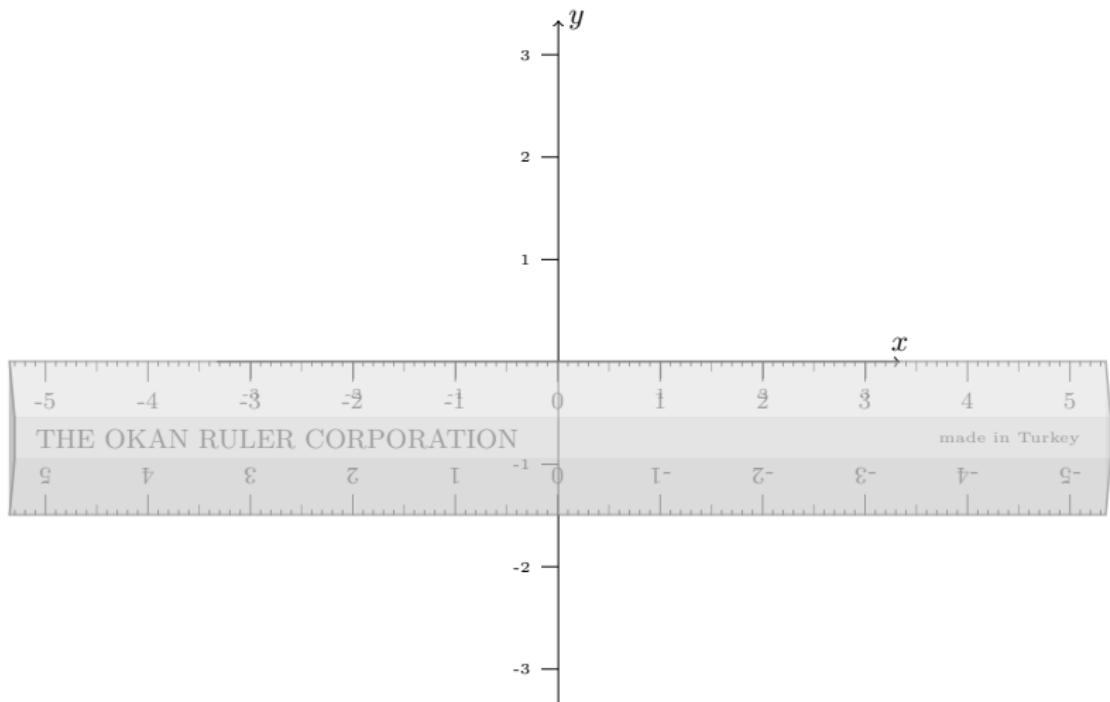
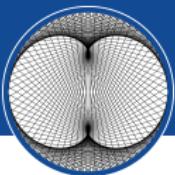


anticlockwise = positive angle
saat yönünün tersi = pozitif açı

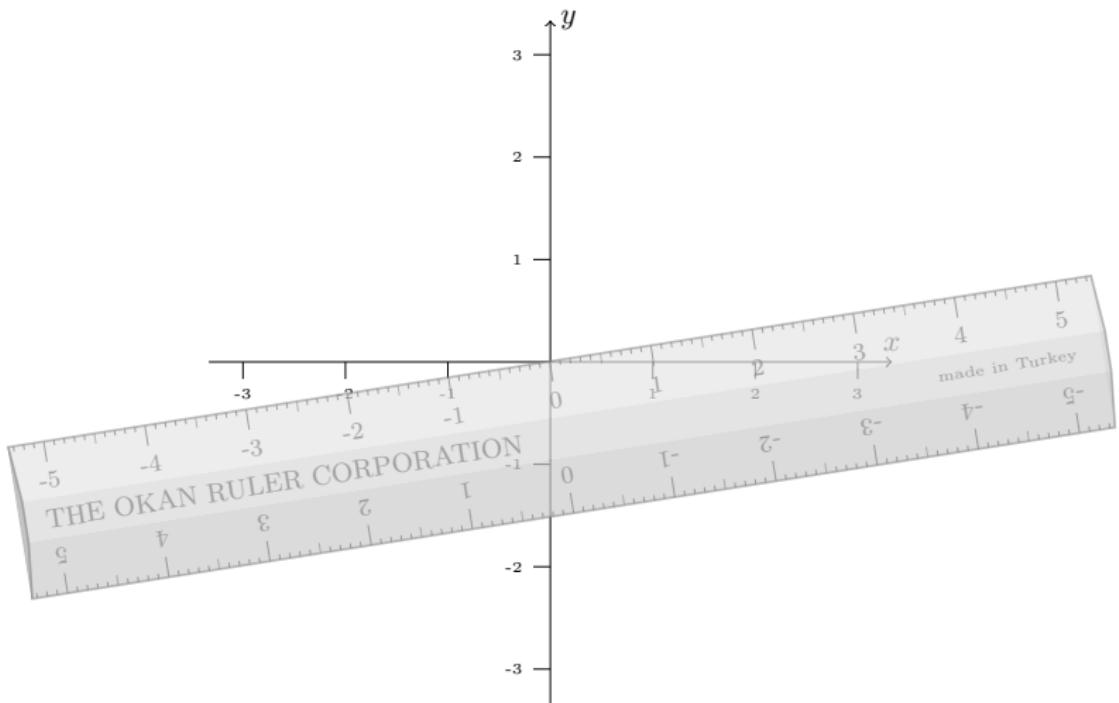
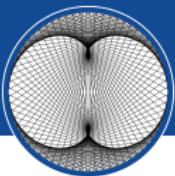


clockwise = negative angle
saat yönünde = negatif açı

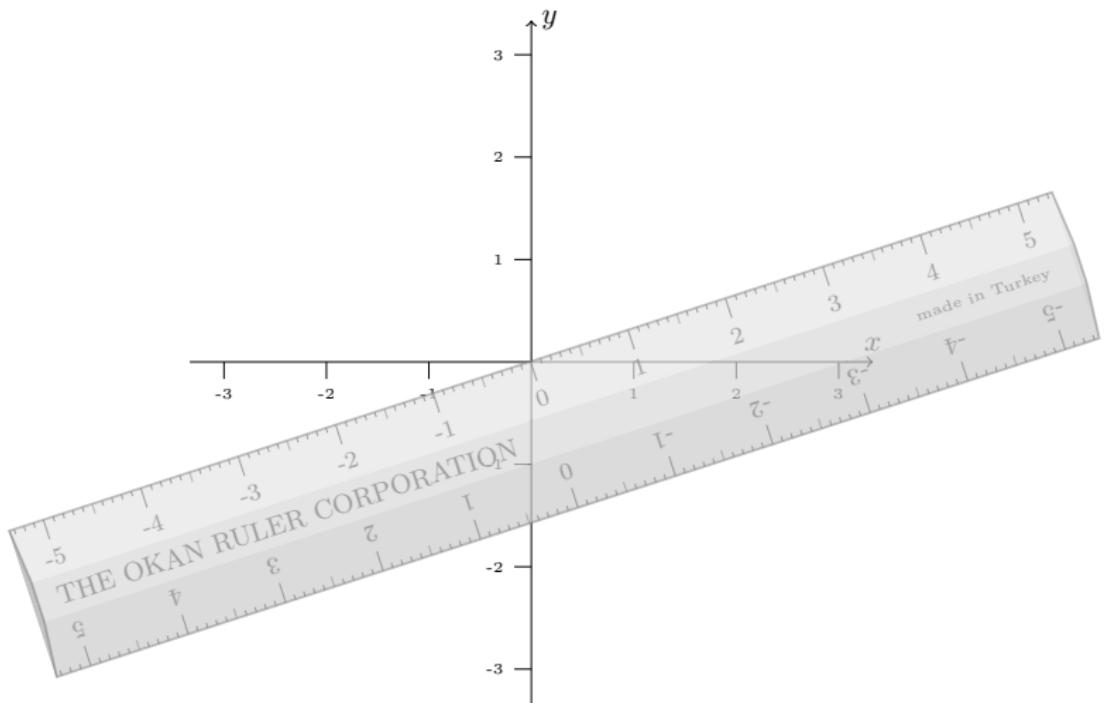
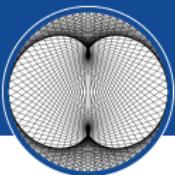
10.3 Polar Coordinates



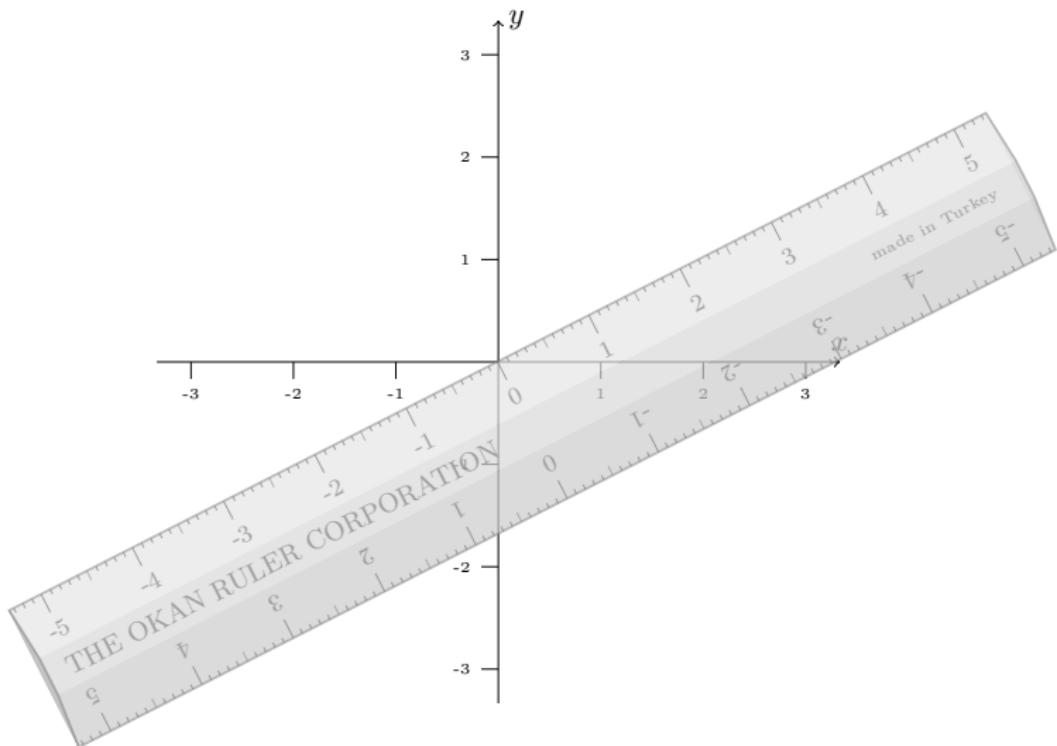
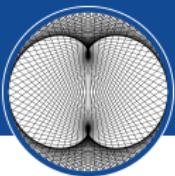
10.3 Polar Coordinates



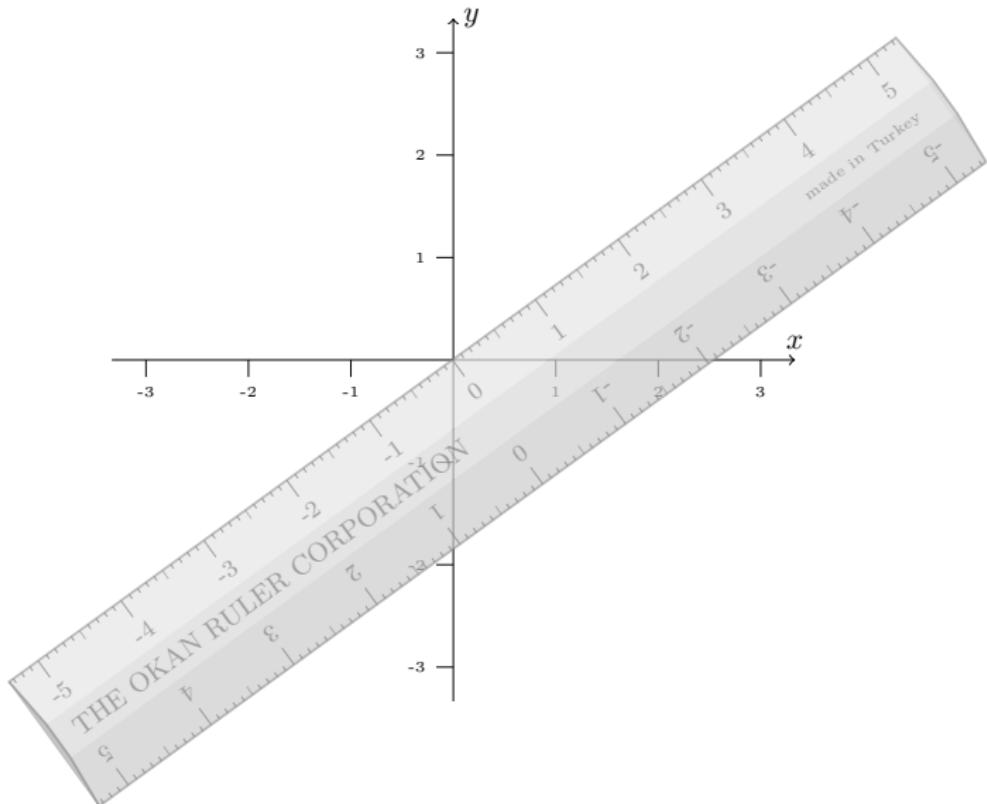
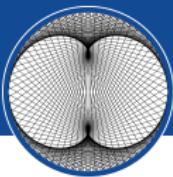
10.3 Polar Coordinates



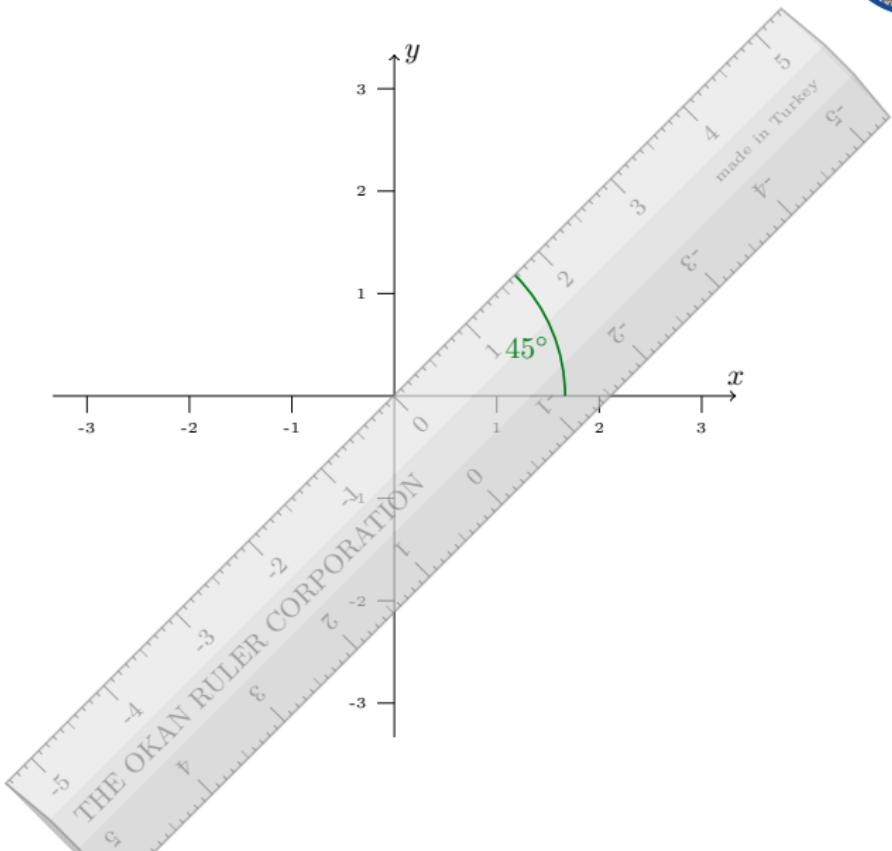
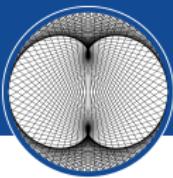
10.3 Polar Coordinates



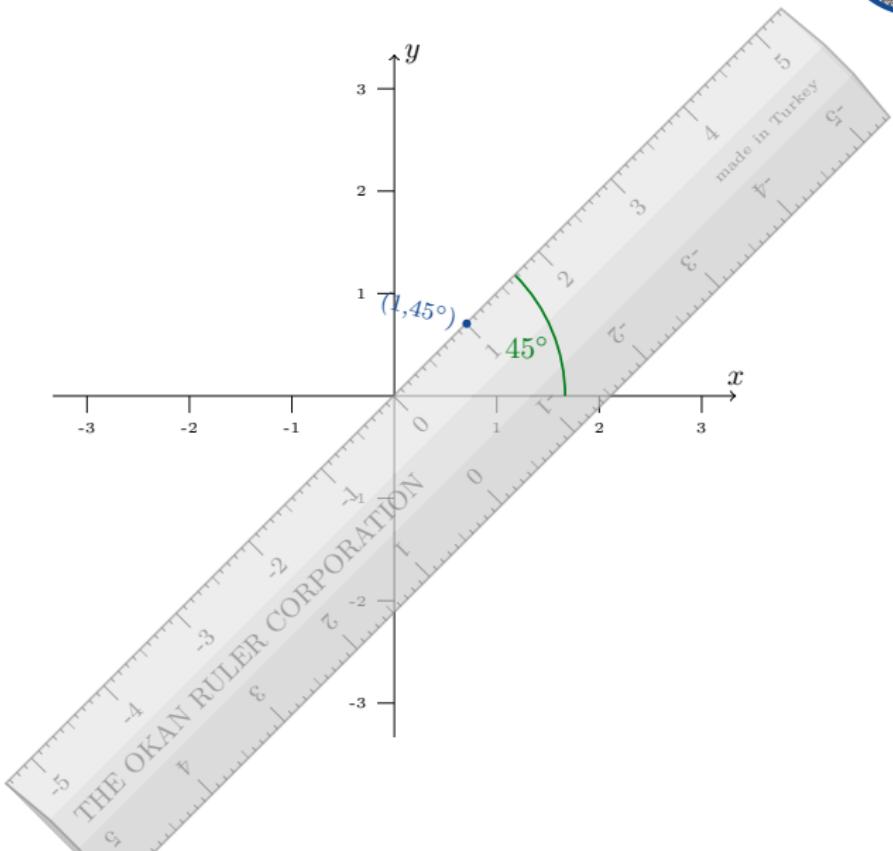
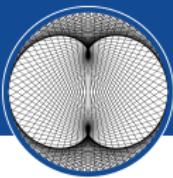
10.3 Polar Coordinates



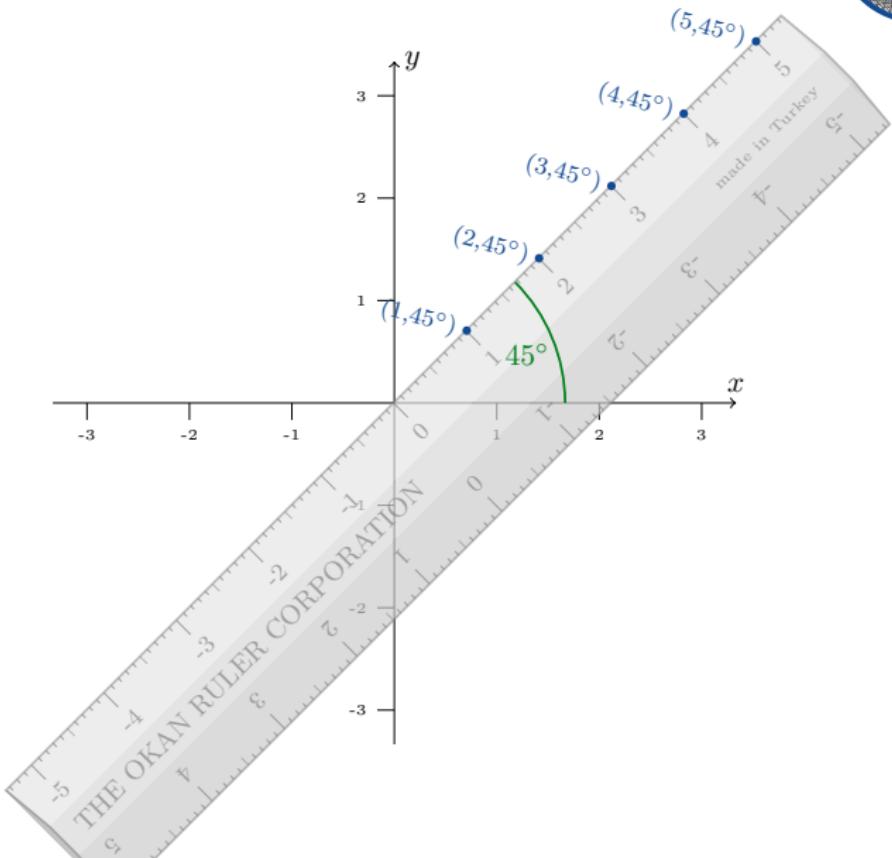
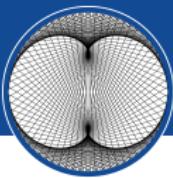
10.3 Polar Coordinates



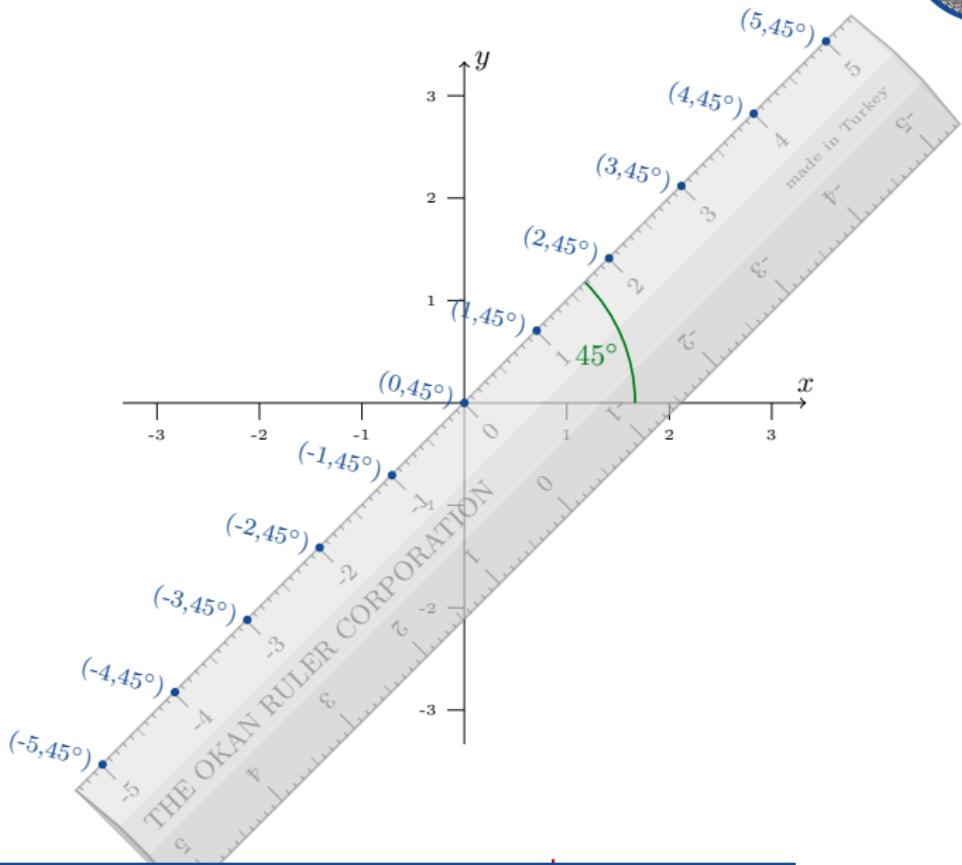
10.3 Polar Coordinates



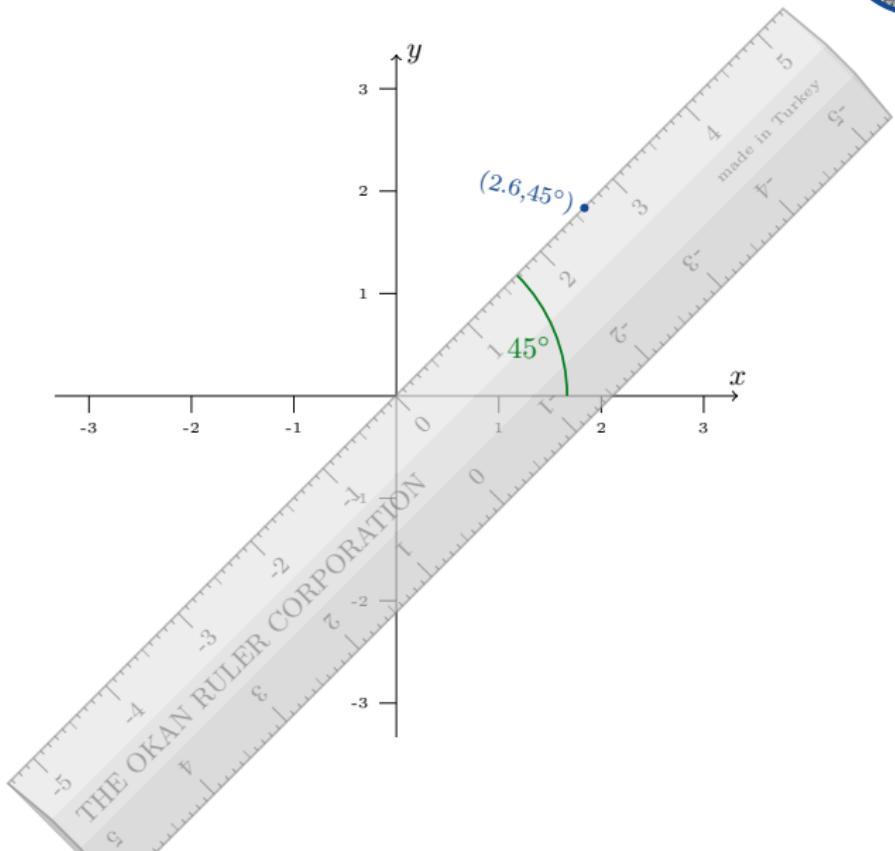
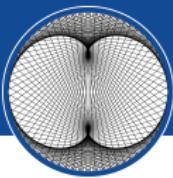
10.3 Polar Coordinates



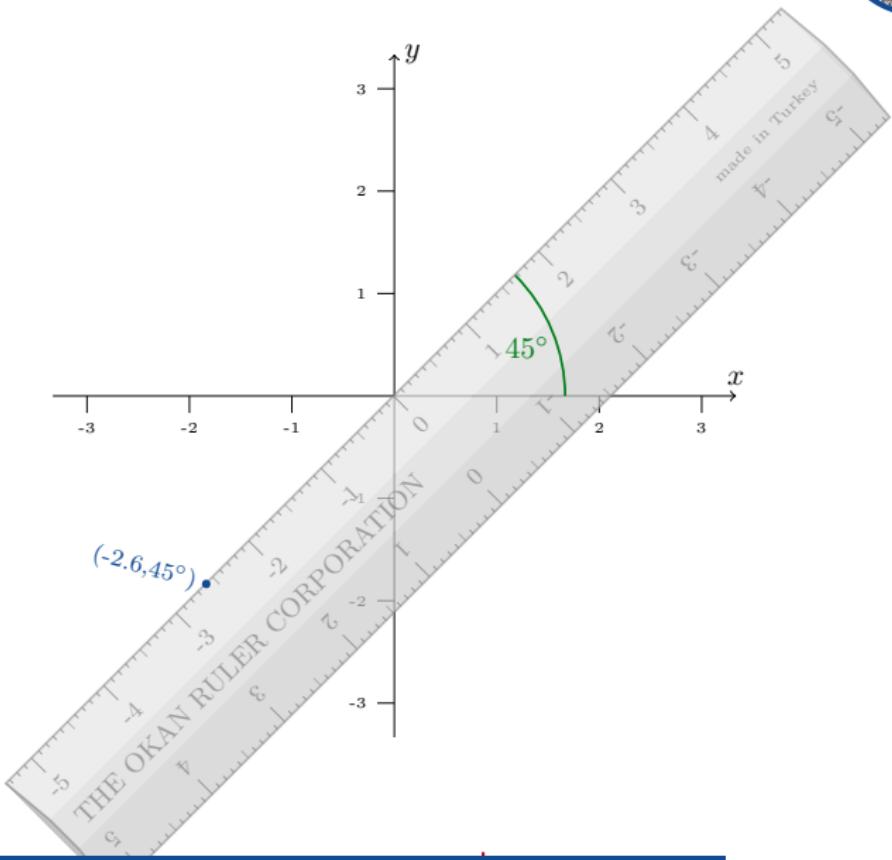
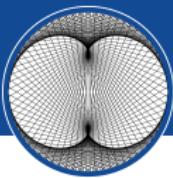
10.3 Polar Coordinates



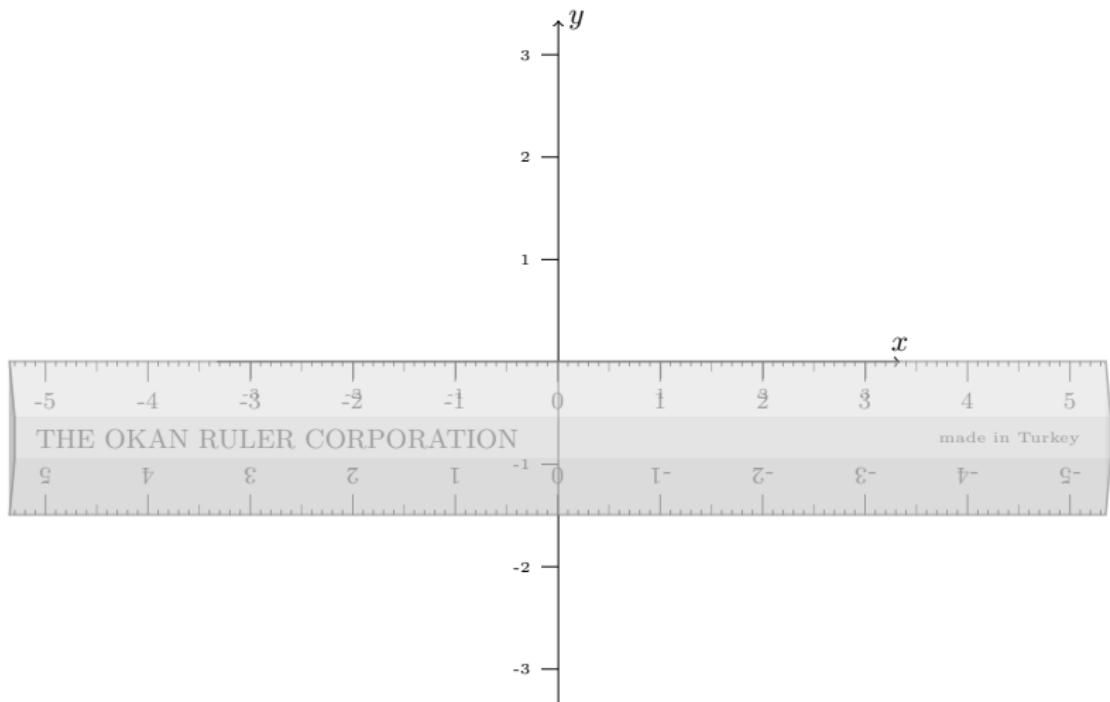
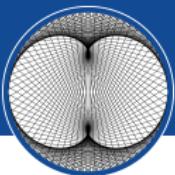
10.3 Polar Coordinates



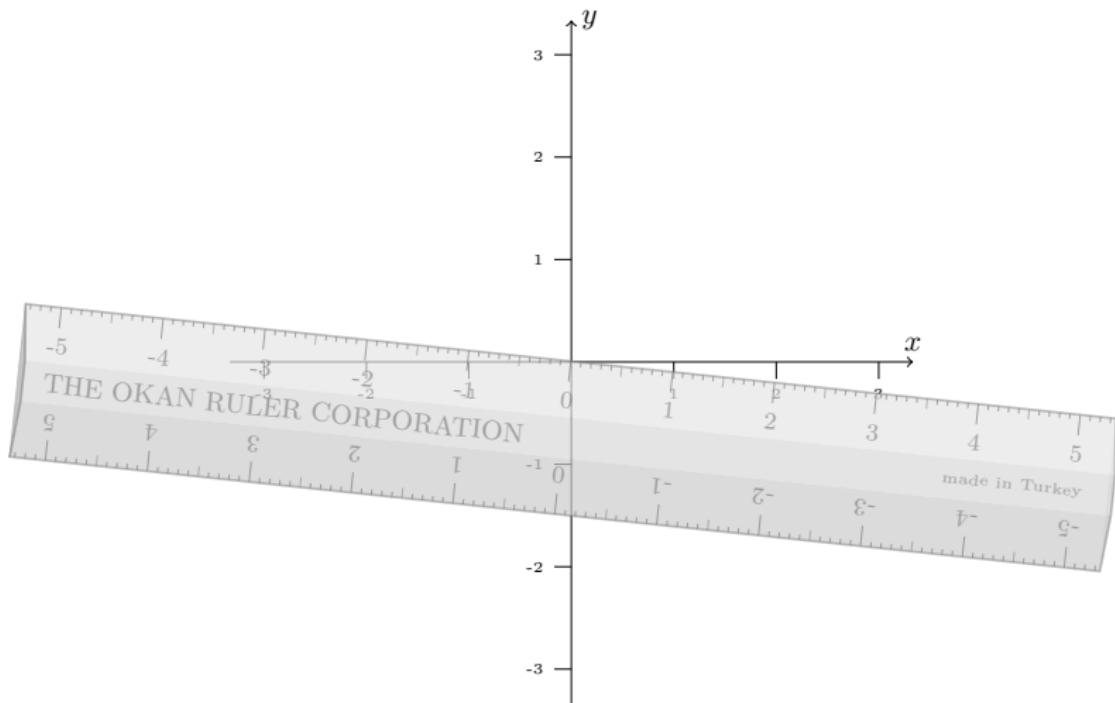
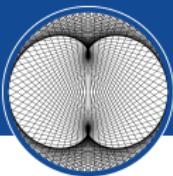
10.3 Polar Coordinates



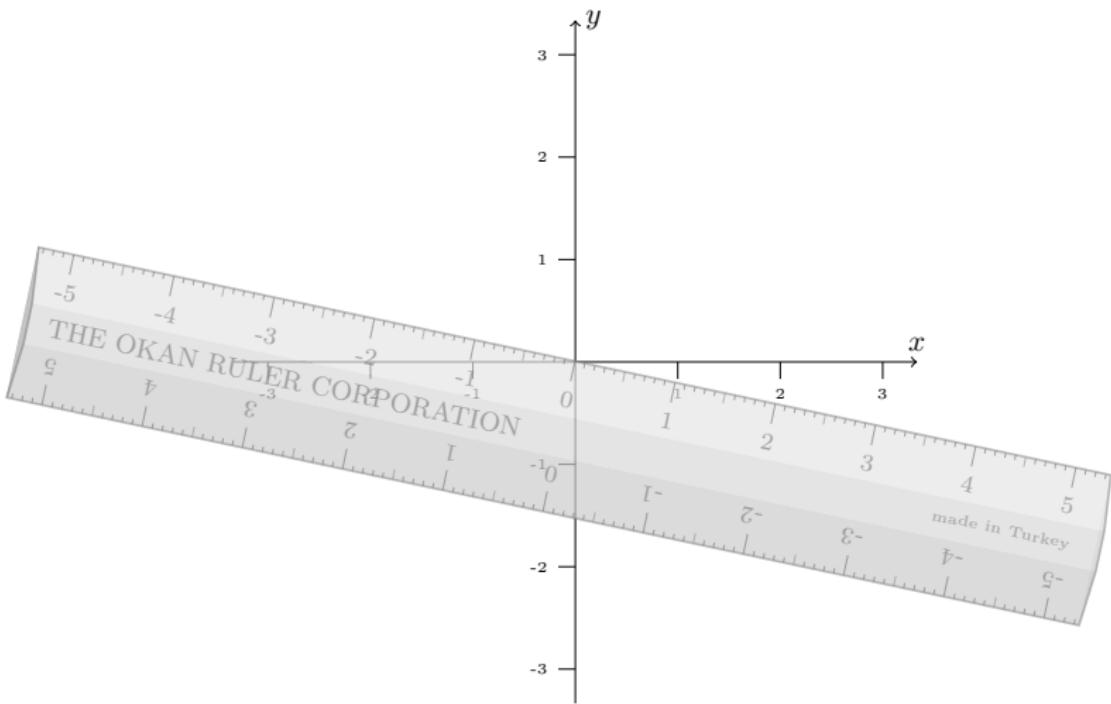
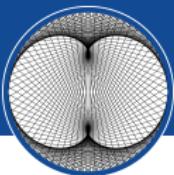
10.3 Polar Coordinates



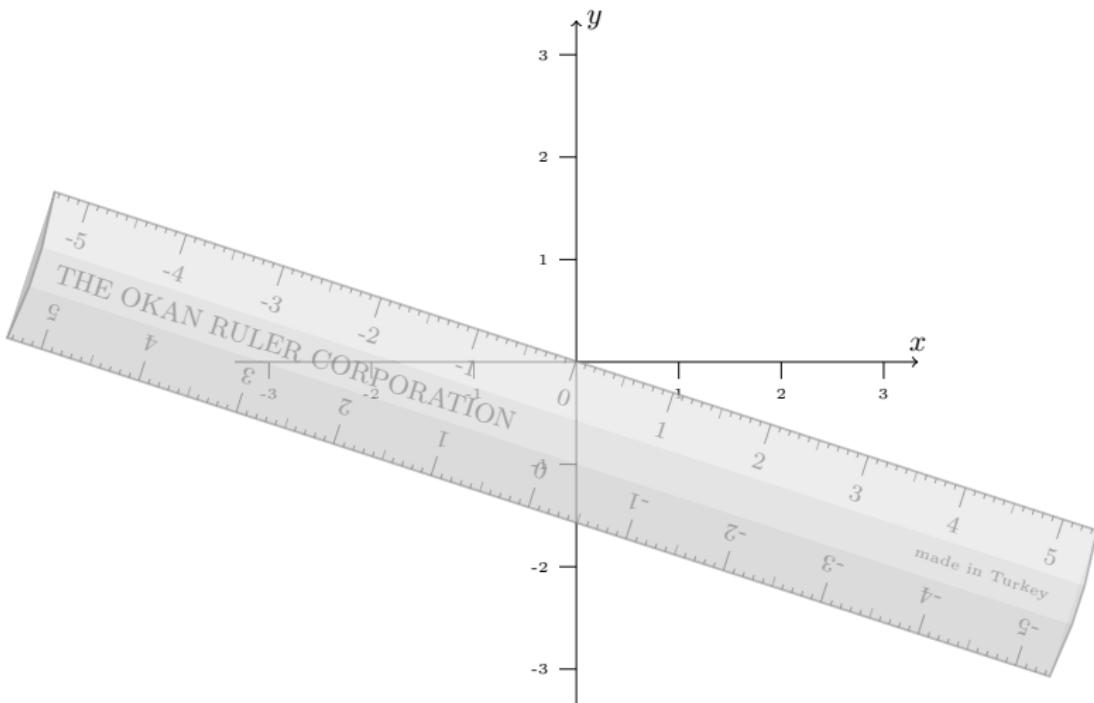
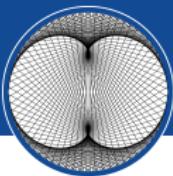
10.3 Polar Coordinates



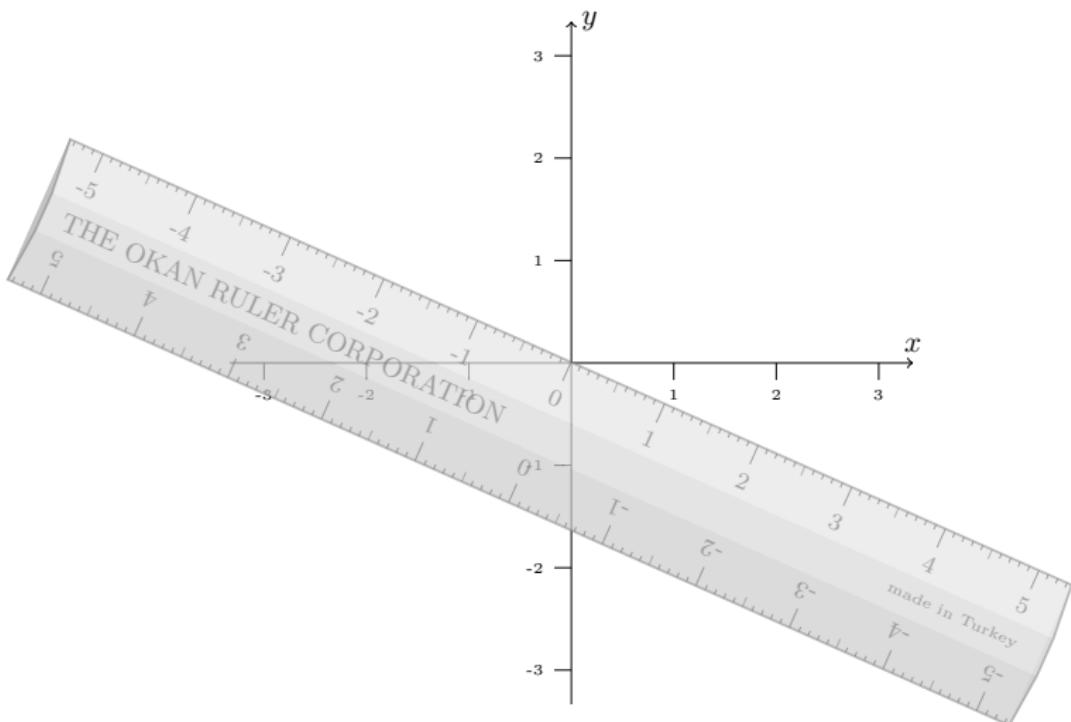
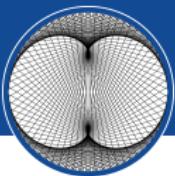
10.3 Polar Coordinates



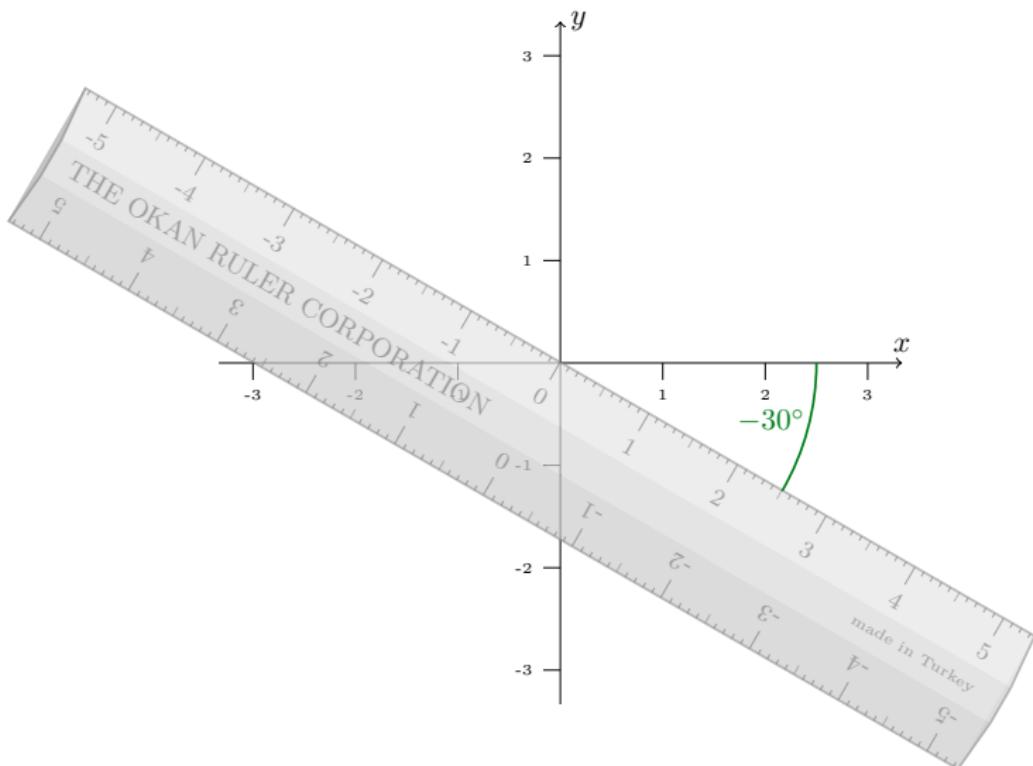
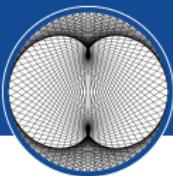
10.3 Polar Coordinates



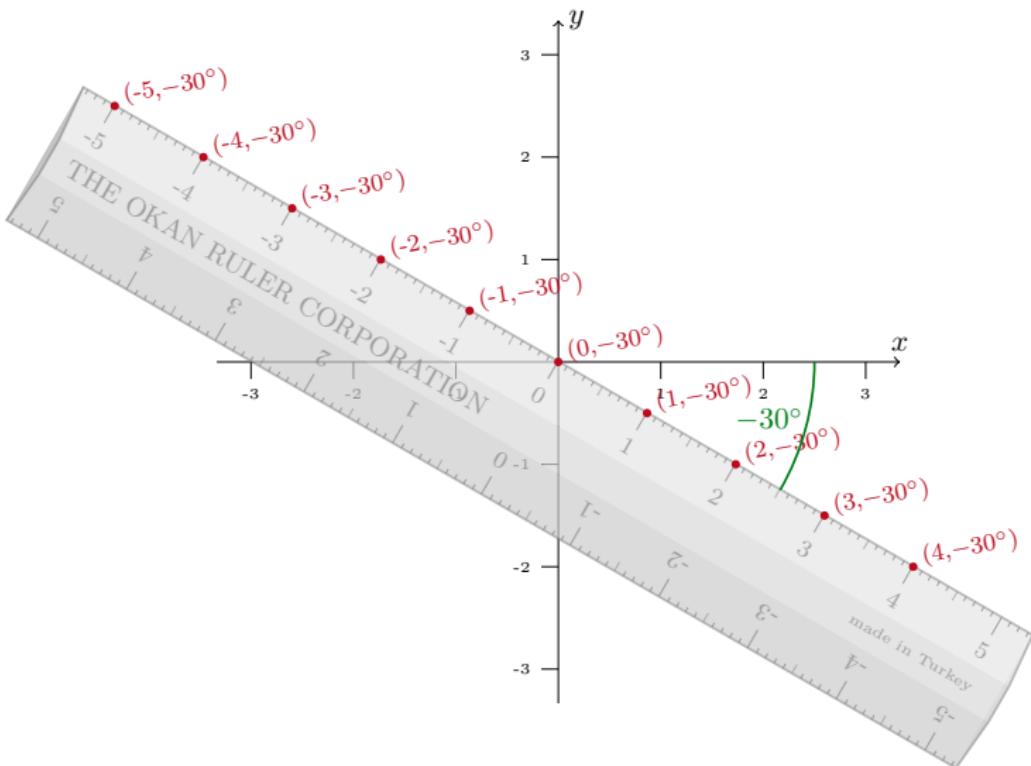
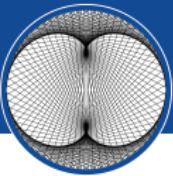
10.3 Polar Coordinates



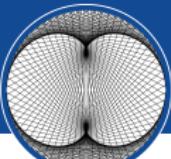
10.3 Polar Coordinates



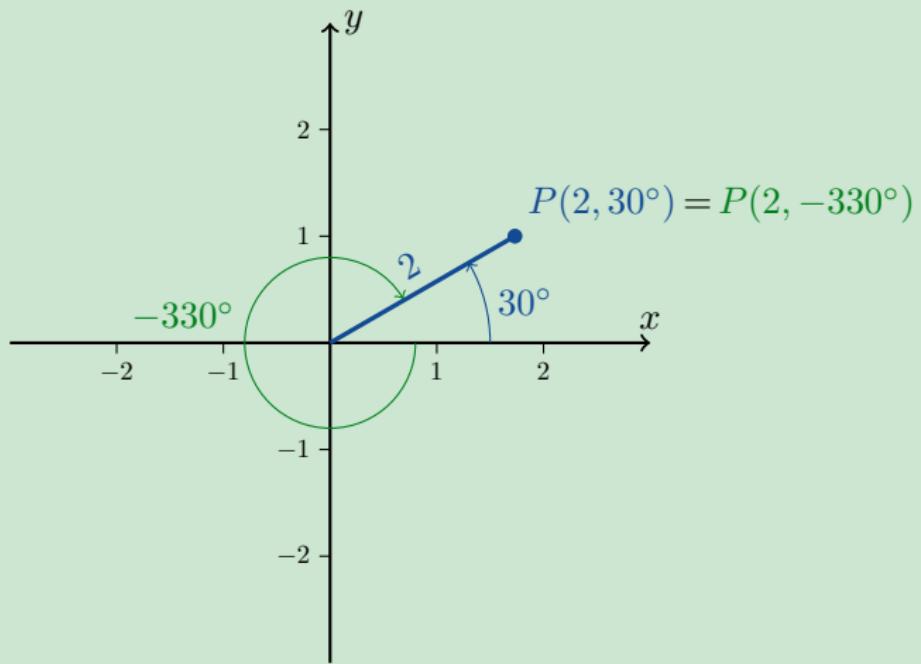
10.3 Polar Coordinates



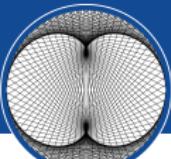
10.3 Polar Coordinates



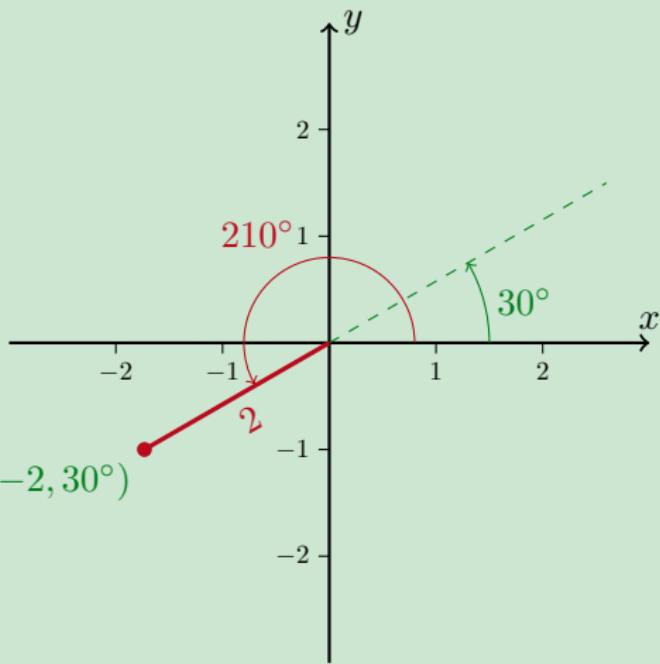
Example



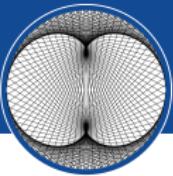
10.3 Polar Coordinates



Example



10.3 Polar Coordinates

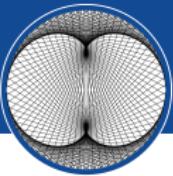


Example

Find all the polar coordinates of $P(2, 30^\circ)$.

We can have either $r = 2$ or $r = -2$.

10.3 Polar Coordinates



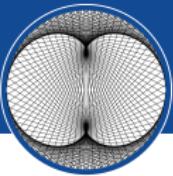
Example

Find all the polar coordinates of $P(2, 30^\circ)$.

We can have either $r = 2$ or $r = -2$. For $r = 2$, we can have

$$\theta = 30^\circ, 30^\circ \pm 360^\circ, 30^\circ \pm 720^\circ, 30^\circ \pm 1080^\circ, \dots$$

10.3 Polar Coordinates



Example

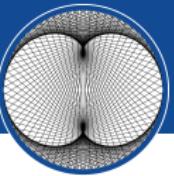
Find all the polar coordinates of $P(2, 30^\circ)$.

We can have either $r = 2$ or $r = -2$. For $r = 2$, we can have

$$\theta = 30^\circ, 30^\circ \pm 360^\circ, 30^\circ \pm 720^\circ, 30^\circ \pm 1080^\circ, \dots$$

For $r = -2$, we can have

$$\theta = 210^\circ, 210^\circ \pm 360^\circ, 210^\circ \pm 720^\circ, 210^\circ \pm 1080^\circ, \dots$$



10.3 Polar Coordinates

Example

Find all the polar coordinates of $P(2, 30^\circ)$.

We can have either $r = 2$ or $r = -2$. For $r = 2$, we can have

$$\theta = 30^\circ, 30^\circ \pm 360^\circ, 30^\circ \pm 720^\circ, 30^\circ \pm 1080^\circ, \dots$$

For $r = -2$, we can have

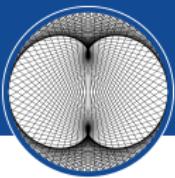
$$\theta = 210^\circ, 210^\circ \pm 360^\circ, 210^\circ \pm 720^\circ, 210^\circ \pm 1080^\circ, \dots$$

Therefore

$$P(2, 30^\circ) = P\left(2, (30 + 360n)^\circ\right) = P\left(-2, (210 + 360m)^\circ\right)$$

for all $m, n \in \mathbb{Z}$.

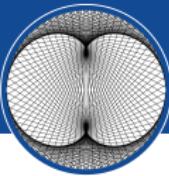
10.3 Polar Coordinates



Example

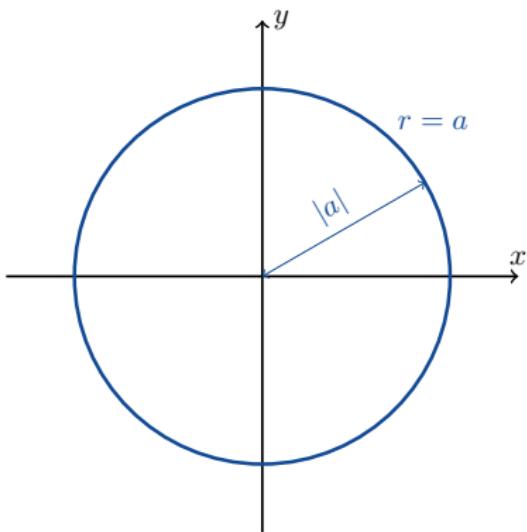
Draw the graph of $r = a$.

10.3 Polar Coordinates

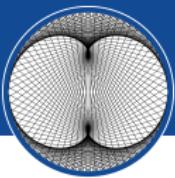


Example

Draw the graph of $r = a$.



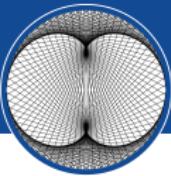
10.3 Polar Coordinates



Example

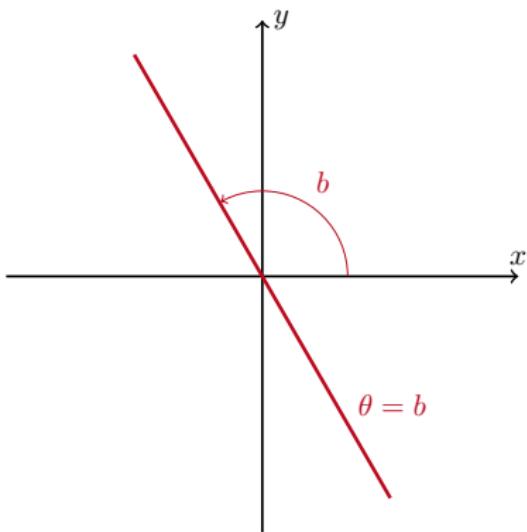
Draw the graph of $\theta = b$.

10.3 Polar Coordinates

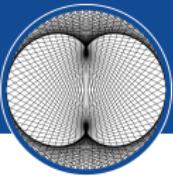


Example

Draw the graph of $\theta = b$.



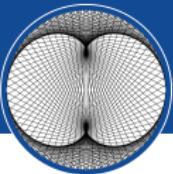
10.3 Polar Coordinates



Remark

$r = 1$ and $r = -1$ are both equations for a circle of radius 1 centred at the origin.

10.3 Polar Coordinates



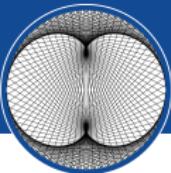
Remark

$r = 1$ and $r = -1$ are both equations for a circle of radius 1 centred at the origin.

Remark

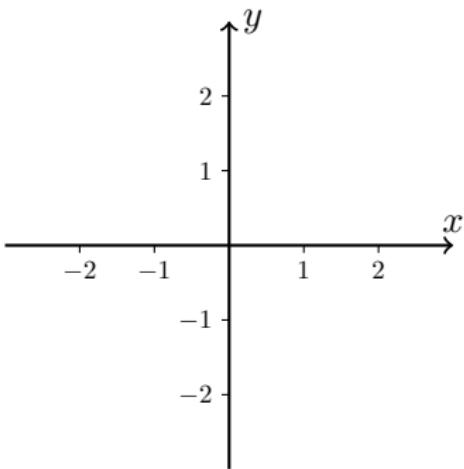
$\theta = 30^\circ$, $\theta = 210^\circ$ and $\theta = -150^\circ$ are all equations for the same line.

10.3 Polar Coordinates

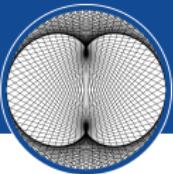


Example

Draw the sets of points whose polar coordinates satisfy the following: $1 \leq r \leq 2$ and $0^\circ \leq \theta \leq 90^\circ$.

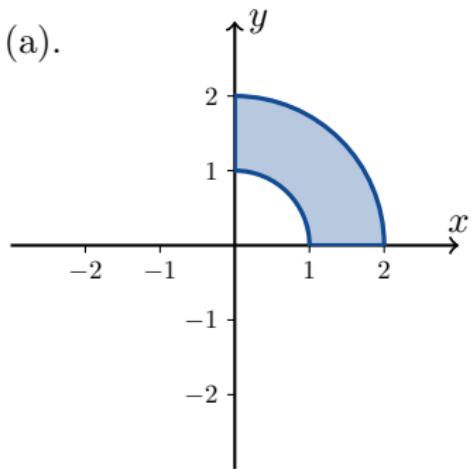


10.3 Polar Coordinates

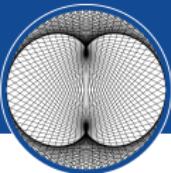


Example

Draw the sets of points whose polar coordinates satisfy the following: $1 \leq r \leq 2$ and $0^\circ \leq \theta \leq 90^\circ$.

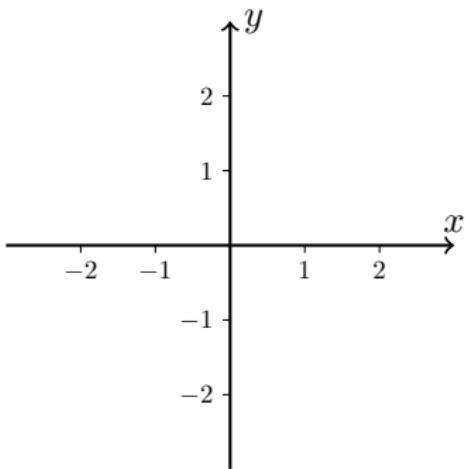


10.3 Polar Coordinates

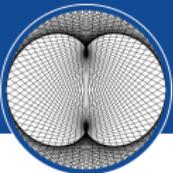


Example

Draw the sets of points whose polar coordinates satisfy the following: $-3 \leq r \leq 2$ and $\theta = 45^\circ$.

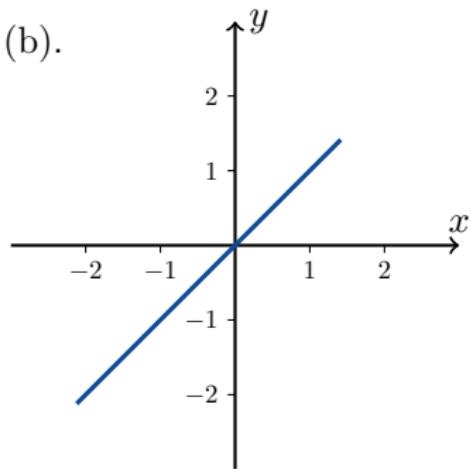


10.3 Polar Coordinates

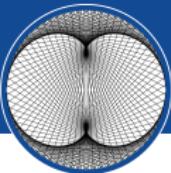


Example

Draw the sets of points whose polar coordinates satisfy the following: $-3 \leq r \leq 2$ and $\theta = 45^\circ$.

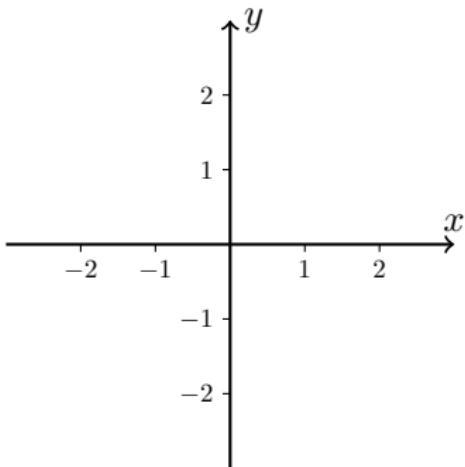


10.3 Polar Coordinates

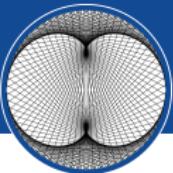


Example

Draw the sets of points whose polar coordinates satisfy the following: $r \leq 0$ and $\theta = 60^\circ$.

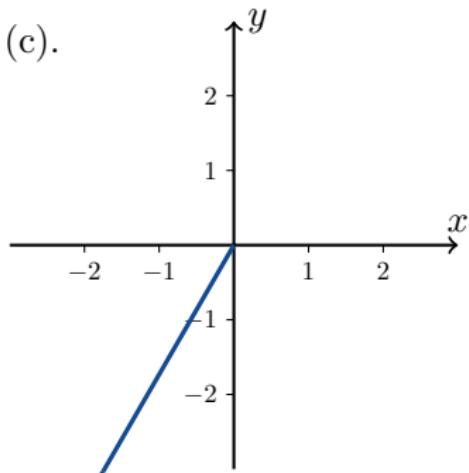


10.3 Polar Coordinates

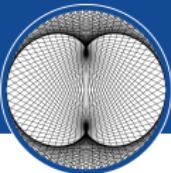


Example

Draw the sets of points whose polar coordinates satisfy the following: $r \leq 0$ and $\theta = 60^\circ$.

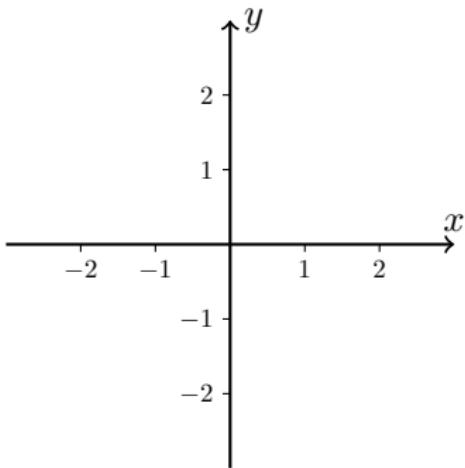


10.3 Polar Coordinates

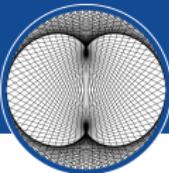


Example

Draw the sets of points whose polar coordinates satisfy the following: $120^\circ \leq \theta \leq 150^\circ$.

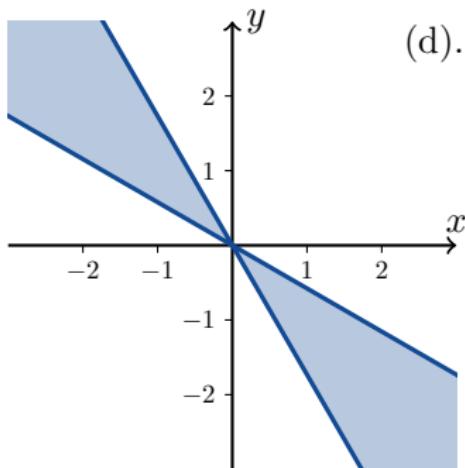


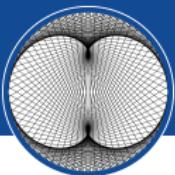
10.3 Polar Coordinates



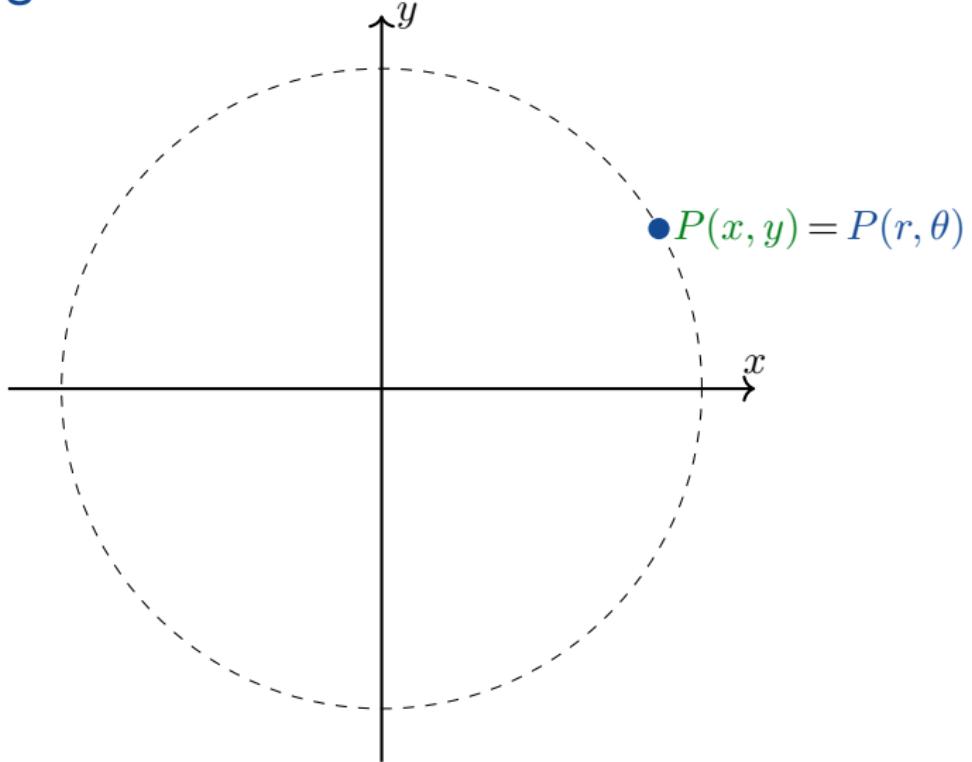
Example

Draw the sets of points whose polar coordinates satisfy the following: $120^\circ \leq \theta \leq 150^\circ$.

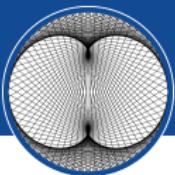




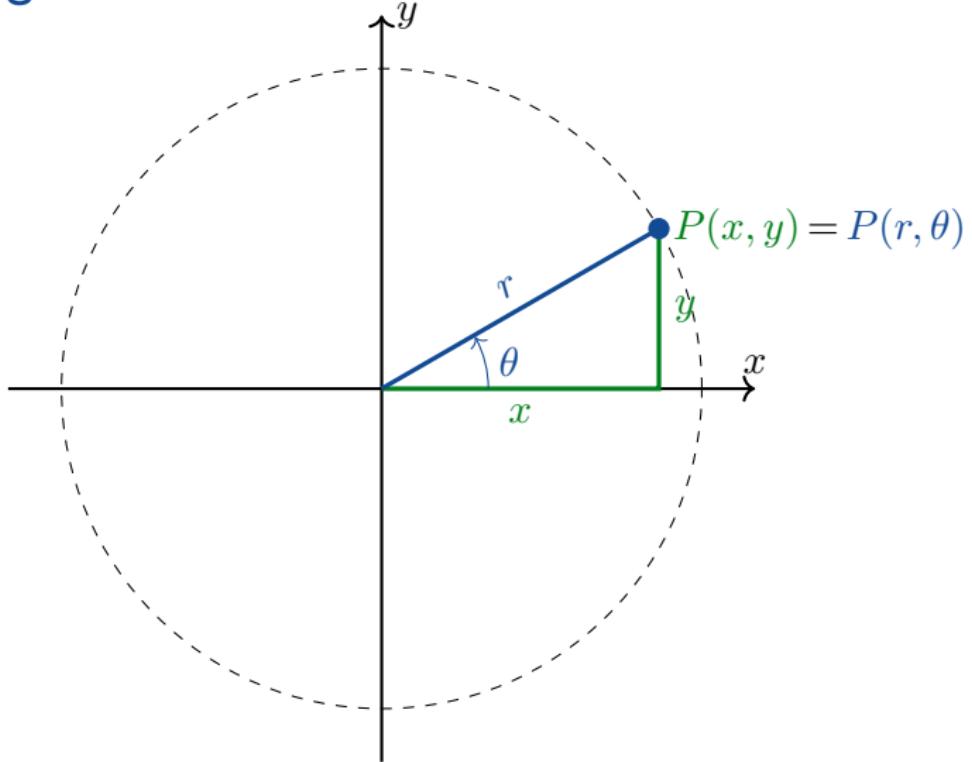
Relating Polar and Cartesian Coordinates

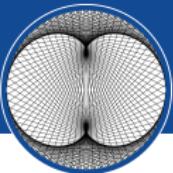


10.3 Polar Coordinates

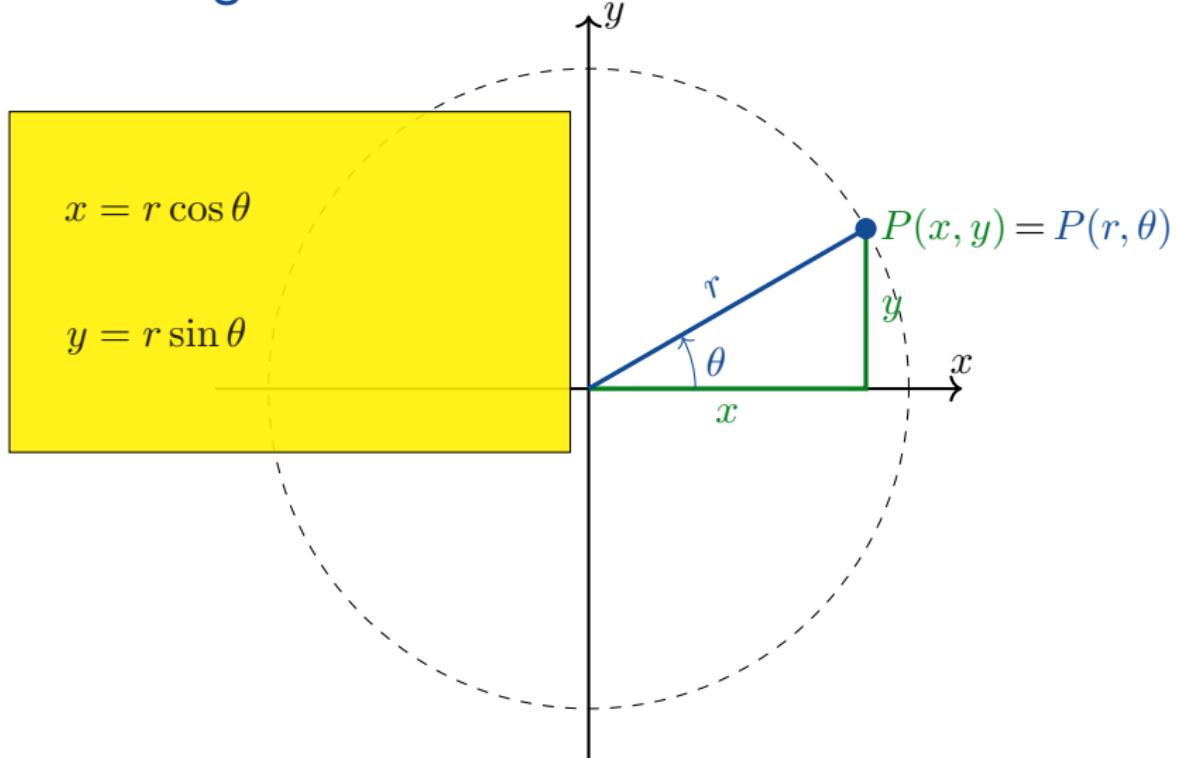


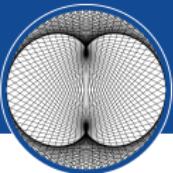
Relating Polar and Cartesian Coordinates





Relating Polar and Cartesian Coordinates

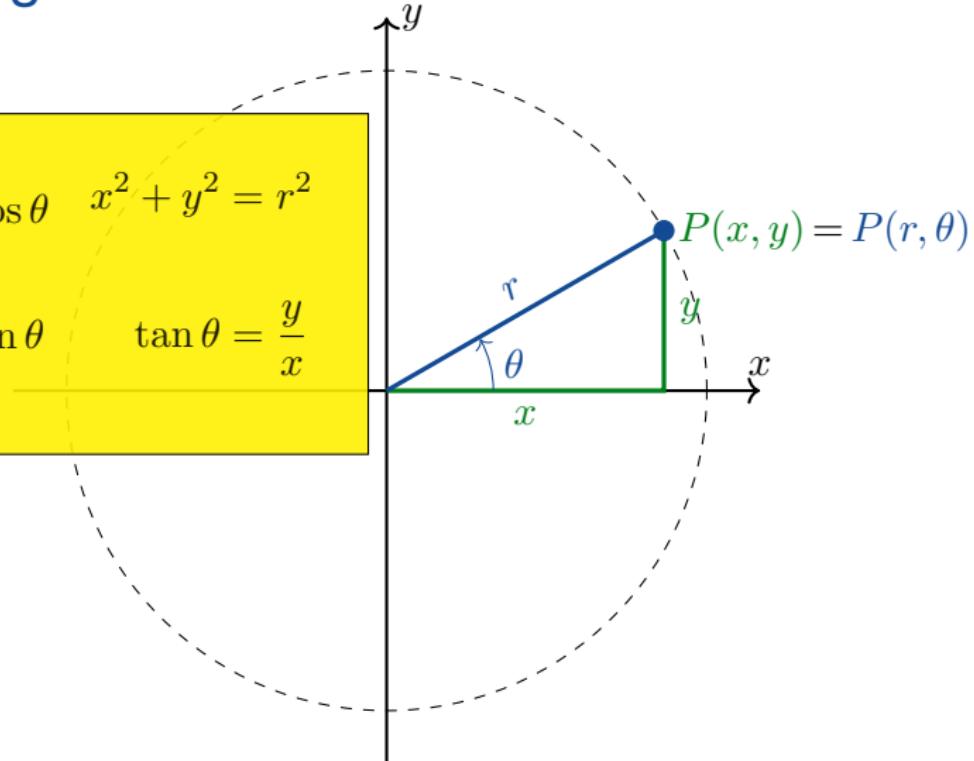




Relating Polar and Cartesian Coordinates

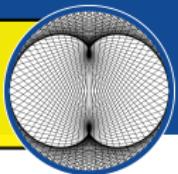
$$x = r \cos \theta \quad x^2 + y^2 = r^2$$

$$y = r \sin \theta \quad \tan \theta = \frac{y}{x}$$



10.3 Polar

$$x = r \cos \theta \quad y = r \sin \theta \quad x^2 + y^2 = r^2 \quad \tan \theta = \frac{y}{x}$$

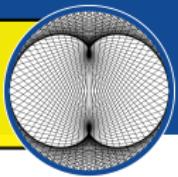


Example

Convert the polar coordinates $(r, \theta) = (-3, 90^\circ)$ into Cartesian coordinates.

10.3 Polar

$$x = r \cos \theta \quad y = r \sin \theta \quad x^2 + y^2 = r^2 \quad \tan \theta = \frac{y}{x}$$



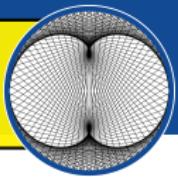
Example

Convert the polar coordinates $(r, \theta) = (-3, 90^\circ)$ into Cartesian coordinates.

$$(x, y) = (r \cos \theta, r \sin \theta) = (-3 \cos 90^\circ, -3 \sin 90^\circ) = (0, -3).$$

10.3 Polar

$$x = r \cos \theta \quad y = r \sin \theta \quad x^2 + y^2 = r^2 \quad \tan \theta = \frac{y}{x}$$

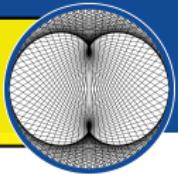


Example

Find polar coordinates for the Cartesian coordinates $(x, y) = (5, -12)$.

10.3 Polar

$$x = r \cos \theta \quad y = r \sin \theta \quad x^2 + y^2 = r^2 \quad \tan \theta = \frac{y}{x}$$



Example

Find polar coordinates for the Cartesian coordinates $(x, y) = (5, -12)$.

Choosing $r > 0$, we calculate that

$$r = \sqrt{x^2 + y^2} = \sqrt{25 + 144} = \sqrt{169} = 13.$$

To find θ we use the equation $y = r \sin \theta$ to calculate that

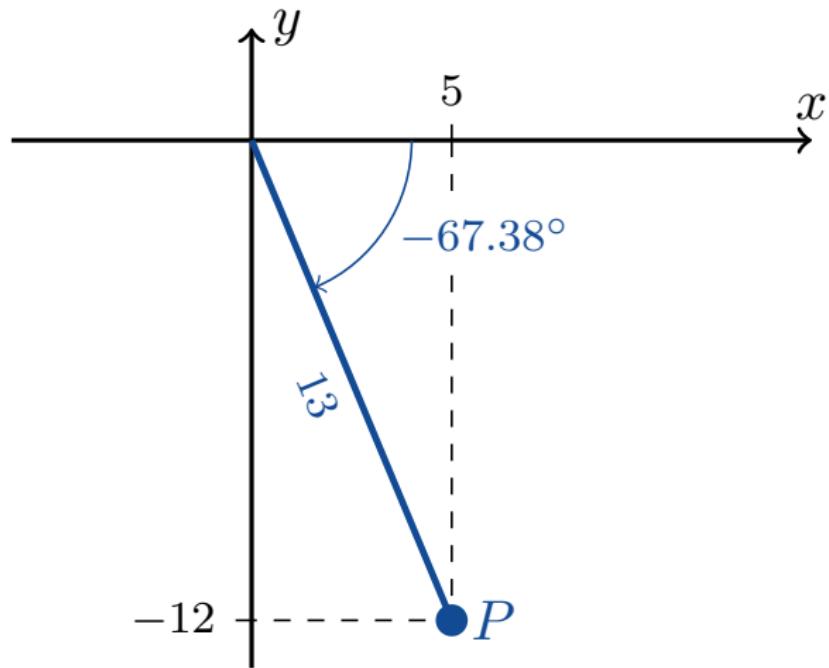
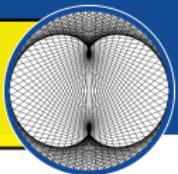
$$\theta = \sin^{-1} \frac{y}{r} = \sin^{-1} \frac{-12}{13} \approx -67.38^\circ.$$

Therefore

$$(r, \theta) = (13, -67.38^\circ).$$

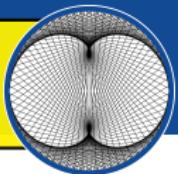
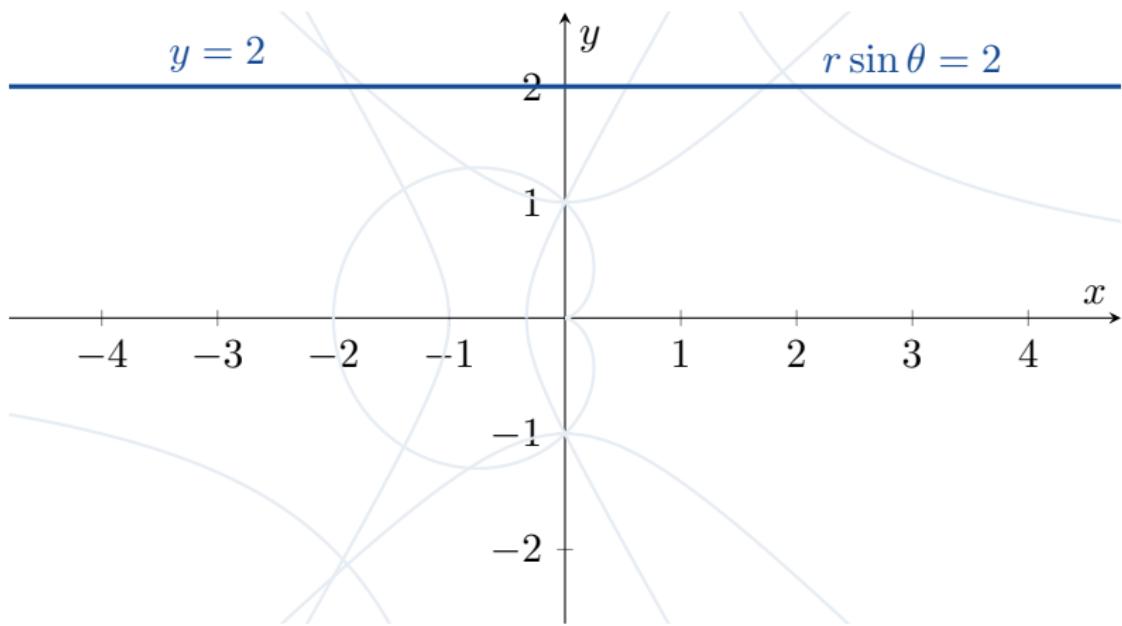
10.3 Polar

$$x = r \cos \theta \quad y = r \sin \theta \quad x^2 + y^2 = r^2 \quad \tan \theta = \frac{y}{x}$$



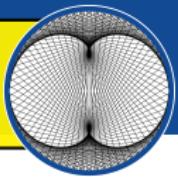
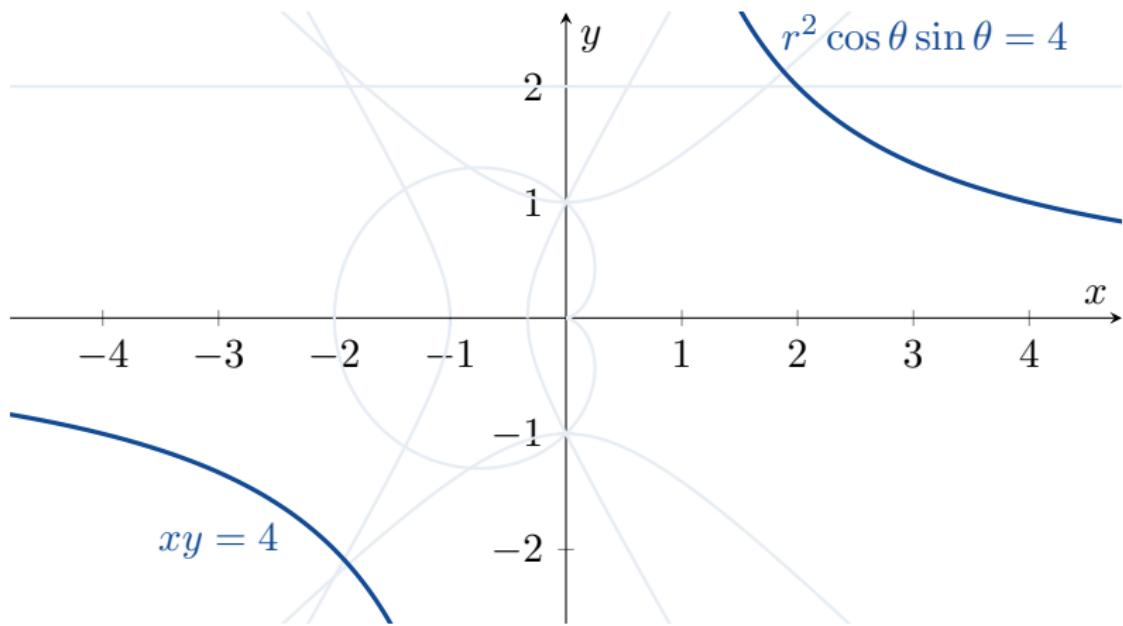
10.3 Polar

$$x = r \cos \theta \quad y = r \sin \theta \quad x^2 + y^2 = r^2 \quad \tan \theta = \frac{y}{x}$$

Cartesian Equation \longleftrightarrow Polar Equation

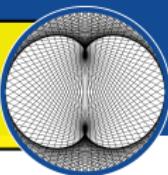
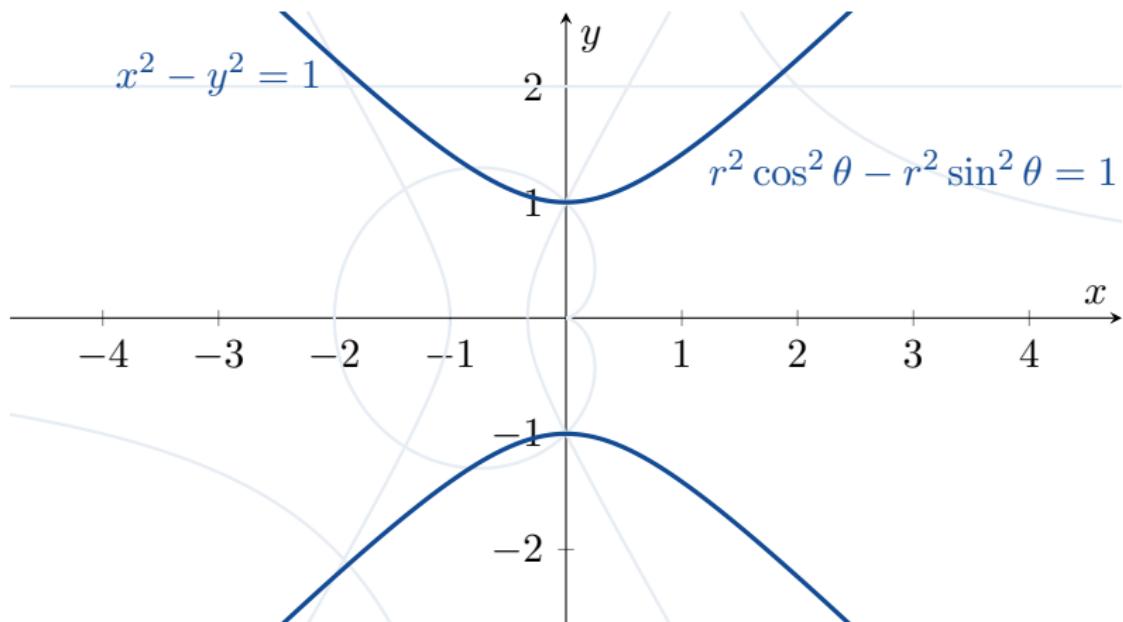
10.3 Polar

$$x = r \cos \theta \quad y = r \sin \theta \quad x^2 + y^2 = r^2 \quad \tan \theta = \frac{y}{x}$$

Cartesian Equation \longleftrightarrow Polar Equation

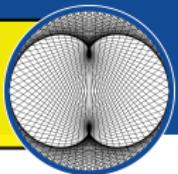
10.3 Polar

$$x = r \cos \theta \quad y = r \sin \theta \quad x^2 + y^2 = r^2 \quad \tan \theta = \frac{y}{x}$$

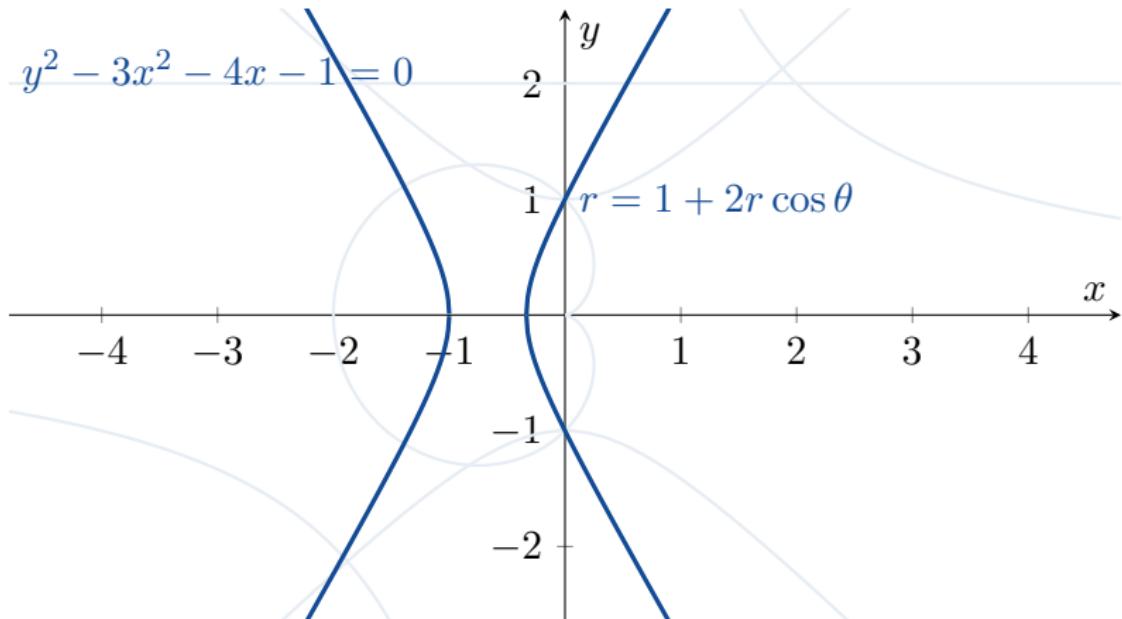
Cartesian Equation \longleftrightarrow Polar Equation

10.3 Polar

$$x = r \cos \theta \quad y = r \sin \theta \quad x^2 + y^2 = r^2 \quad \tan \theta = \frac{y}{x}$$

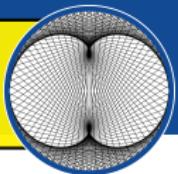
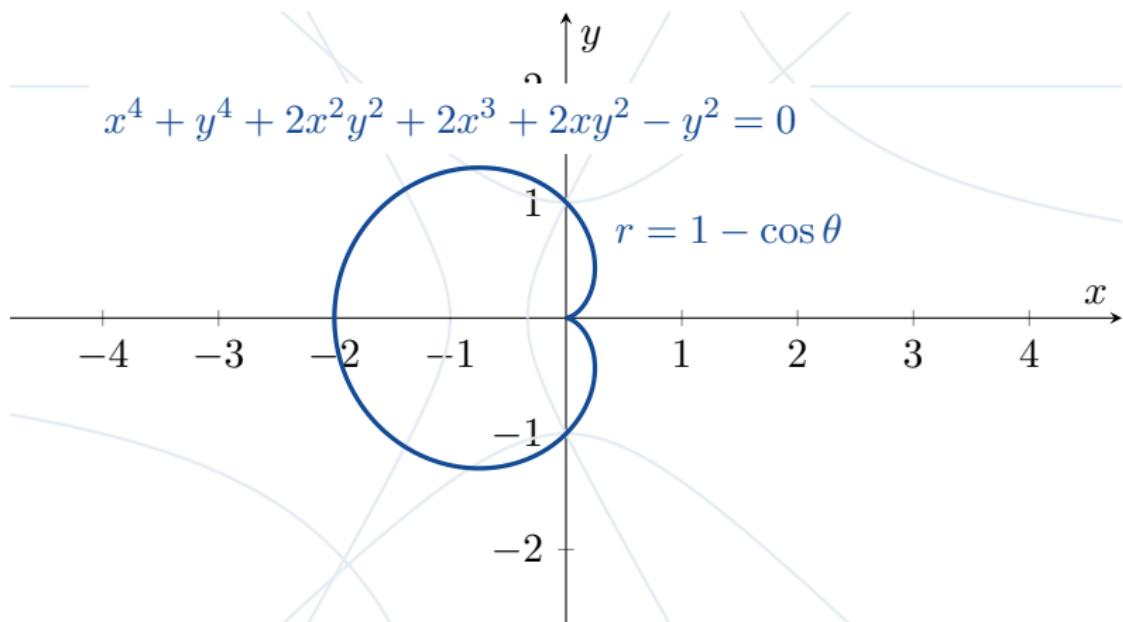


Cartesian Equation \longleftrightarrow Polar Equation



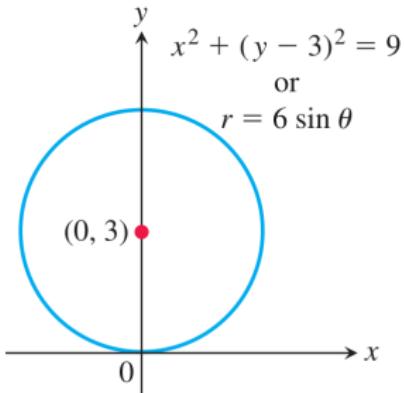
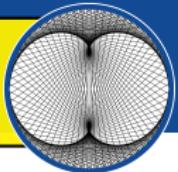
10.3 Polar

$$x = r \cos \theta \quad y = r \sin \theta \quad x^2 + y^2 = r^2 \quad \tan \theta = \frac{y}{x}$$

Cartesian Equation \longleftrightarrow Polar Equation

10.3 Polar

$$x = r \cos \theta \quad y = r \sin \theta \quad x^2 + y^2 = r^2 \quad \tan \theta = \frac{y}{x}$$



EXAMPLE 5 Find a polar equation for the circle $x^2 + (y - 3)^2 = 9$

Solution We apply the equations relating polar and Cartesian coordinates:

$$x^2 + (y - 3)^2 = 9$$

$$x^2 + y^2 - 6y + 9 = 9 \qquad \text{Expand } (y - 3)^2.$$

$$x^2 + y^2 - 6y = 0 \qquad \text{Cancelation}$$

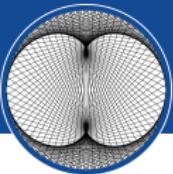
$$r^2 - 6r \sin \theta = 0 \qquad x^2 + y^2 = r^2, y = r \sin \theta$$

$$r = 0 \quad \text{or} \quad r - 6 \sin \theta = 0$$

$$r = 6 \sin \theta \qquad \text{Includes both possibilities}$$

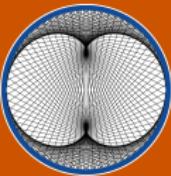


10.3 Polar Coordinates



Please quickly look through **10.4 Graphing in Polar Coordinates** (only 3 pages) in your textbook.

We will not be testing you on section 10.4, but it can help you to understand section 14.4 which we will study next.



Next Time

- 14.4 Double Integrals in Polar Form
- 14.5 Triple Integrals in Rectangular Coordinates
- 14.7 Triple Integrals in Cylindrical and Spherical Coordinates
- 14.8 Substitutions in Multiple Integrals