Welcome to

Mathematics II

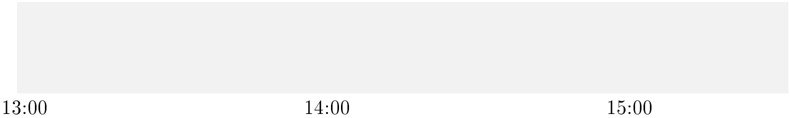
with Dr Neil Course

Lecture 1

- Information about this course
- 8.1 Using Basic Integration Formulae
- 8.2 Integration by Parts
- 8.3 Trigonometric Integrals

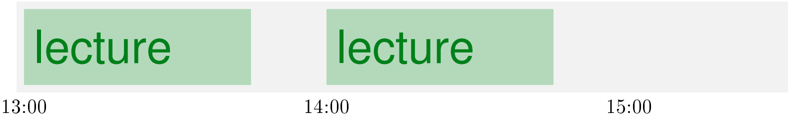
Information about this course

- ≈ 12 classes. Friday afternoons 1pm-3:30pm.



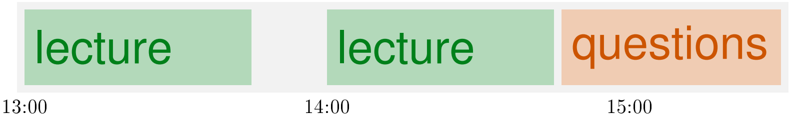
Information about this course

- ≈ 12 classes. Friday afternoons 1pm-3:30pm.
- 2 lectures with a break between.

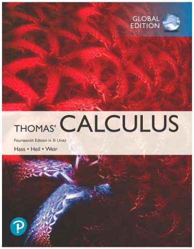


Information about this course

- $\approx$ 12 classes. Friday afternoons 1pm-3:30pm.
- 2 lectures with a break between.
- Then I will answers your questions.



The Book



Joel R. Hass, Christopher E. Heil and Maurice D. Weir,
Thomas' Calculus in SI Units,
14th Edition, Wiley.

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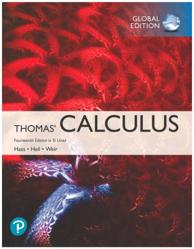

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This is a required purchase.
You need to have this book to be able to do the homework.

8
12
14
15
10

8. Techniques of Integration

12
14
15
10

8. Techniques of Integration

12. Vectors and the Geometry of Space

14

15

10

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12. Vectors and the Geometry of Space

14. Partial Derivatives

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15. Multiple Integrals

10

8. Techniques of Integration

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14. Partial Derivatives

15. Multiple Integrals

10. Infinite Sequences and Series

8. Techniques of Integration — 2 weeks

12. Vectors and the Geometry of Space — 2 weeks

14. Partial Derivatives — 2 weeks

15. Multiple Integrals — 3 weeks

10. Infinite Sequences and Series — 3 weeks

Exams and homework

(This information may change based on the University's decisions)

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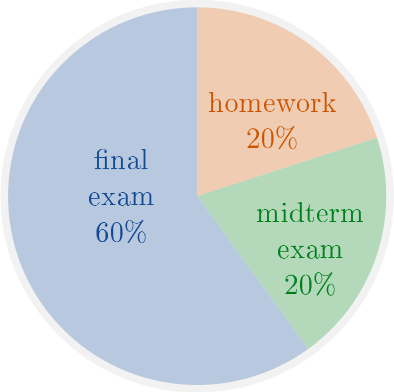

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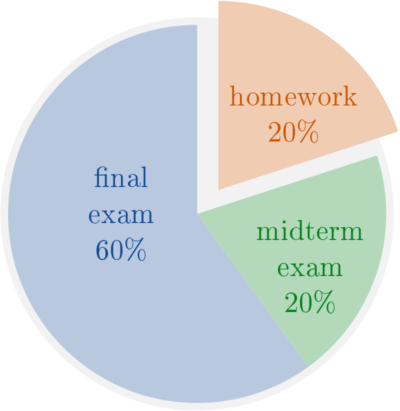

using Pearson MyLab Math

one piece of homework for each lesson

deadline = end of term

more details in O'Learn

Exams and homework

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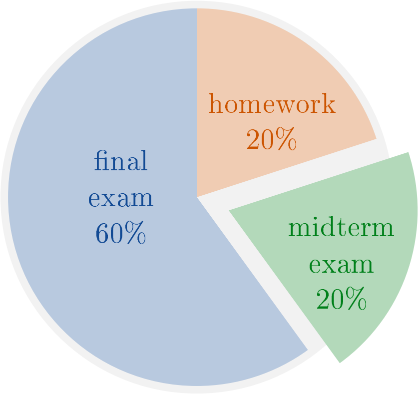

only 1 midterm exam

week 8 or week 9

Exams and homework

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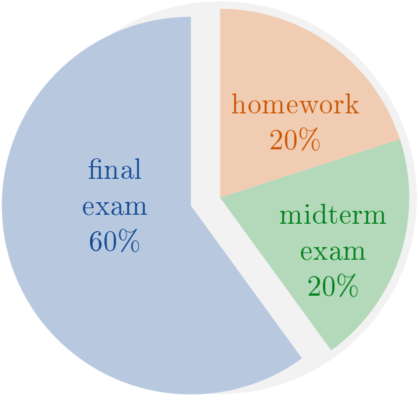

Expectations

If this was a classroom course, the expectation is that for each hour of lectures, you would study an extra 1-2 hours outside of class.

classroom course	lectures (5 hours)	other study (5-10 hours)

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classroom course	lectures (5 hours)	other study (5-10 hours)

For an online course, you are still expected to study a total of 10-15 hours each week.

online course	class (2.5 hours)	other study (7.5-12.5 hours)

This may include:

- Do the online homework on MyLab;

$\vdots$

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- Solve the exercises in the textbook;
- Use the O'Learn Discussion Board;
- Read other books?;
- Watch online videos;

$\vdots$

slide number

current section

Using Basic Integration Formulae

Table 8.1

Basic integration formulas

1. $\int k\,dx = kx + C \quad \text{(any number } k)$
2. $\int x^n\,dx = \dfrac{x^{n+1}}{n+1} + C \quad (n \neq -1)$
3. $\int \dfrac{dx}{x} = \ln|x| + C$
4. $\int e^x\,dx = e^x + C$
5. $\int a^x\,dx = \dfrac{a^x}{\ln a} + C \quad (a > 0, a \neq 1)$
6. $\int \sin x\,dx = -\cos x + C$
7. $\int \cos x\,dx = \sin x + C$
8. $\int \sec^2 x\,dx = \tan x + C$
9. $\int \csc^2 x\,dx = -\cot x + C$
10. $\int \sec x \tan x\,dx = \sec x + C$
11. $\int \csc x \cot x\,dx = -\csc x + C$
12. $\int \tan x\,dx = \ln|\sec x| + C$
13. $\int \cot x\,dx = \ln|\sin x| + C$
14. $\int \sec x\,dx = \ln|\sec x + \tan x| + C$
15. $\int \csc x\,dx = -\ln|\csc x + \cot x| + C$
16. $\int \sinh x\,dx = \cosh x + C$
17. $\int \cosh x\,dx = \sinh x + C$
18. $\int \dfrac{dx}{\sqrt{a^2 - x^2}} = \sin^{-1}\left(\dfrac{x}{a}\right) + C$
19. $\int \dfrac{dx}{a^2 + x^2} = \dfrac{1}{a}\tan^{-1}\left(\dfrac{x}{a}\right) + C$
20. $\int \dfrac{dx}{x\sqrt{x^2 - a^2}} = \dfrac{1}{a}\sec^{-1}\left|\dfrac{x}{a}\right| + C$
21. $\int \dfrac{dx}{\sqrt{a^2 + x^2}} = \sinh^{-1}\left(\dfrac{x}{a}\right) + C \quad (a > 0)$
22. $\int \dfrac{dx}{\sqrt{x^2 - a^2}} = \cosh^{-1}\left(\dfrac{x}{a}\right) + C \quad (x > a > 0)$

Example

Calculate $\displaystyle\int_3^5 \frac{2x-3}{\sqrt{x^2-3x+1}}\,dx.$

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We use the substitution $u = x^2 - 3x + 1$.

Example

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We use the substitution $u = x^2 - 3x + 1$. Then $du = (2x-3)\,dx$ and

$$\begin{aligned} x = 3 &\implies u = 9 - 9 + 1 = 1 \\ x = 5 &\implies u = 25 - 15 + 1 = 11. \end{aligned}$$

Example

Calculate $\displaystyle\int_3^5 \frac{2x-3}{\sqrt{x^2-3x+1}}\,dx$.

We use the substitution $u = x^2 - 3x + 1$. Then $du = (2x-3)\,dx$ and

$$x = 3 \implies u = 9 - 9 + 1 = 1$$
$$x = 5 \implies u = 25 - 15 + 1 = 11.$$

Hence

$$\int_3^5 \frac{2x-3}{\sqrt{x^2-3x+1}}\,dx = \int_1^{11} u^{-\frac{1}{2}}\,du = \left[2\sqrt{u}\right]_1^{11} = 2\left(\sqrt{11}-1\right).$$

$$\int \frac{dx}{\sqrt{a^2 - x^2}} = \sin^{-1}\left(\frac{u}{a}\right) + C$$

Example

Find $\displaystyle\int \frac{dx}{\sqrt{8x - x^2}}$.

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This time we will complete the square of $x^2 - 8x$ and use that to simplify the integral:

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Example

Find $\int \frac{dx}{\sqrt{8x - x^2}}$.

This time we will complete the square of $x^2 - 8x$ and use that to simplify the integral:

$x^2 - 8x = x^2 - 8x + 16 - 16$

$$\int \frac{dx}{\sqrt{a^2 - x^2}} = \sin^{-1}\left(\frac{u}{a}\right) + C$$

Example

Find $\displaystyle\int \frac{dx}{\sqrt{8x - x^2}}$.

This time we will complete the square of $x^2 - 8x$ and use that to simplify the integral:

$$x^2 - 8x = x^2 - 8x + 16 - 16 = (x^2 - 8x + 16) - 16 = (x - 4)^2 - 16.$$

Example

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$$x^2 - 8x = x^2 - 8x + 16 - 16 = (x^2 - 8x + 16) - 16 = (x - 4)^2 - 16.$$

So

$$\int \frac{dx}{\sqrt{8x - x^2}} = \int \frac{dx}{\sqrt{16 - (x - 4)^2}}$$

$$=$$

$$=$$

$$=$$

$$\int \frac{dx}{\sqrt{a^2 - x^2}} = \sin^{-1}\left(\frac{u}{a}\right) + C$$

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So

$$\begin{aligned}
\int \frac{dx}{\sqrt{8x - x^2}} &= \int \frac{dx}{\sqrt{16 - (x - 4)^2}} \\
&= \int \frac{du}{\sqrt{16 - u^2}} \\
&= \\
&=
\end{aligned}$$

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$$x^2-8x = x^2-8x+16-16 = (x^2-8x+16)-16 = (x-4)^2-16.$$

So

$$\begin{aligned}\int \frac{dx}{\sqrt{8x-x^2}} &= \int \frac{dx}{\sqrt{16-(x-4)^2}} \\ &= \int \frac{du}{\sqrt{16-u^2}} \\ &= \sin^{-1}\left(\frac{u}{4}\right)+C \\ &= \sin^{-1}\left(\frac{x-4}{4}\right)+C.\end{aligned}$$

Example

Find $\displaystyle\int \cos x \sin 2x + \sin x \cos 2x \, dx$.

Example

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$$\begin{aligned}\int \cos x \sin 2x + \sin x \cos 2x \, dx &= \int \sin(x + 2x) \, dx \\ &= \int \sin 3x \, dx \\ &= \dots\end{aligned}$$

Example

Find $\displaystyle\int_0^{\frac{\pi}{4}} \frac{dx}{1-\sin x}$.

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Example

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Here is a trick for dealing with $\frac{1}{A-B}$: Multiply by $\frac{A+B}{A+B}$. Then we get

$$\frac{1}{A-B} = \left(\frac{1}{A-B}\right)\left(\frac{A+B}{A+B}\right) = \frac{A+B}{A^2-B^2}$$

which is sometimes easier to deal with.

$$\int_0^{\pi/4} \frac{dx}{1 - \sin x} = \int_0^{\pi/4} \frac{1}{1 - \sin x} \cdot \frac{1 + \sin x}{1 + \sin x}\, dx$$

Multiply and divide by conjugate.

$$= \int_0^{\pi/4} \frac{1 + \sin x}{1 - \sin^2 x}\, dx$$

Simplify.

$$= \int_0^{\pi/4} \frac{1 + \sin x}{\cos^2 x}\, dx$$

$1 - \sin^2 x = \cos^2 x$

$$= \int_0^{\pi/4} (\sec^2 x + \sec x \tan x)\, dx$$

Use Table 8.1, Formulas 8 and 10

$$= \Big[\tan x + \sec x\Big]_0^{\pi/4} = \big(1 + \sqrt{2} - (0 + 1)\big) = \sqrt{2}. \quad \blacksquare$$

Example

Find $\displaystyle\int \frac{3x^2 - 7x}{3x + 2}\, dx$.

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Solution The integrand is an improper fraction since the degree of the numerator is greater than the degree of the denominator. To integrate it, we perform long division to obtain a quotient plus a remainder that is a proper fraction:

$$\frac{3x^2 - 7x}{3x + 2} = x - 3 + \frac{6}{3x + 2}.$$

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$$\frac{3x^2 - 7x}{3x + 2} = x - 3 + \frac{6}{3x + 2}.$$

Therefore,

$$\int \frac{3x^2 - 7x}{3x + 2}\,dx = \int \left(x - 3 + \frac{6}{3x + 2} \right) dx = \frac{x^2}{2} - 3x + 2\ln|3x + 2| + C. \qquad \blacksquare$$

$$\int \frac{dx}{\sqrt{a^2 - x^2}} = \sin^{-1}\left(\frac{u}{a}\right) + C$$

Example

Find $\displaystyle\int \frac{3x+2}{\sqrt{1-x^2}}$.

$$\int \frac{dx}{\sqrt{a^2 - x^2}} = \sin^{-1}\left(\frac{u}{a}\right) + C$$

Example

Find $\displaystyle\int \frac{3x+2}{\sqrt{1-x^2}}$.

First note that

$$\int \frac{3x+2}{\sqrt{1-x^2}} = 3\int \frac{x\,dx}{\sqrt{1-x^2}} + 2\int \frac{dx}{\sqrt{1-x^2}}$$

$$\boxed{\int \frac{dx}{\sqrt{a^2 - x^2}} = \sin^{-1}\left(\frac{u}{a}\right) + C}$$

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So we just need to calculate $\int \frac{x\,dx}{\sqrt{1-x^2}}$.

$$\int \frac{x\,dx}{\sqrt{1-x^2}}$$

Let $u = 1 - x^2$.

$$\int \frac{x\,dx}{\sqrt{1-x^2}}$$

Let $u = 1 - x^2$. Then $du = -2x\,dx$ and $-\frac{1}{2}\,du = x\,dx$.

$$\int \frac{x\,dx}{\sqrt{1-x^2}}$$

Let $u = 1 - x^2$. Then $du = -2x\,dx$ and $-\frac{1}{2}\,du = x\,dx$. It follows that

$$\int \frac{x\,dx}{\sqrt{1-x^2}} = \int \frac{-\frac{1}{2}\,du}{\sqrt{u}} = -\frac{1}{2}\int u^{-\frac{1}{2}}\,du = \ldots = -\sqrt{1-x^2} + C.$$

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Therefore

$$\begin{aligned}\int \frac{3x+2}{\sqrt{1-x^2}} &= 3\int \frac{x\,dx}{\sqrt{1-x^2}} + 2\int \frac{dx}{\sqrt{1-x^2}} \\ &= -3\sqrt{1-x^2} + 2\sin^{-1} x + C.\end{aligned}$$

Example

Find $\displaystyle\int \frac{dx}{(1+\sqrt{x})^3}$.

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First guess: $u = \sqrt{x}$.

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First guess: $u = \sqrt{x}$. But then $du = \frac{1}{2\sqrt{x}}\,dx$ and we would have to deal with this extra $\sqrt{x} = u$ term.

Example

Find $\displaystyle\int \frac{dx}{(1+\sqrt{x})^3}$.

I want to make a substitution to make this integral easier, but what u should I choose?

First guess: $u = \sqrt{x}$. But then $du = \frac{1}{2\sqrt{x}}\,dx$ and we would have to deal with this extra $\sqrt{x} = u$ term.

Second guess: Instead let us try $u = 1 + \sqrt{x}$.

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I want to make a substitution to make this integral easier, but what u should I choose?

First guess: $u = \sqrt{x}$. But then $du = \frac{1}{2\sqrt{x}}\,dx$ and we would have to deal with this extra $\sqrt{x} = u$ term.

Second guess: Instead let us try $u = 1 + \sqrt{x}$. Then again we have $du = \frac{1}{2\sqrt{x}}\,dx$ and $dx = 2\sqrt{x}\,du = 2(u-1)\,du$. Hence

$$\int \frac{dx}{(1+\sqrt{x})^3} = \int \frac{2(u-1)\,du}{u^3} = \int \frac{2}{u^2} - \frac{2}{u^3}\,du = \dots$$

Example

Calculate $\displaystyle\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} x^3 \cos x \, dx$.

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This is actually easy: The integrand is an odd function

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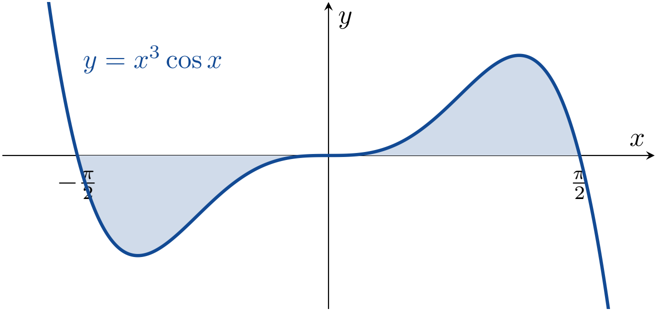

Integration by Parts

How can we calculate

$$\int x \cos x \, dx$$

or

$$\int x^2 e^x \, dx \ ?$$

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Theorem (Integration by Parts)

$$\int u(x) v'(x) \, dx =$$

8.2 Integration by Parts

How can we calculate

$$\int x \cos x \, dx$$

or

$$\int x^2 e^x \, dx \ ?$$

$$\int \text{function} \times \text{function} \, dx$$

Theorem (Integration by Parts)

$$\int u(x)v'(x) \, dx = u(x)v(x) - \int u'(x)v(x) \, dx$$

$$\int uv'\,dx = uv - \int u'v\,dx$$

Example

Find $\displaystyle\int x\cos x\,dx$.

$$\int uv'\,dx = uv - \int u'v\,dx$$

Example

Find $\displaystyle\int x\cos x\,dx$.

We need to choose a $u(x)$ and a $v'(x)$.

$$\int uv'\,dx = uv - \int u'v\,dx$$

Example

Find $\displaystyle\int x\cos x\,dx$.

We need to choose a $u(x)$ and a $v'(x)$.
Let

$$u = x \qquad\qquad v' = \cos x$$

Then

$$\int x\cos x\,dx = \qquad - \int \qquad dx = \qquad .$$

$$\boxed{\int uv'\,dx = uv - \int u'v\,dx}$$

Example

Find $\displaystyle\int x\cos x\,dx$.

We need to choose a $u(x)$ and a $v'(x)$.
Let

$$u = x \qquad\qquad v' = \cos x$$
$$u' = 1$$

Then

$$\int x\cos x\,dx = \qquad - \int \qquad dx = \qquad .$$

$$\boxed{\int uv'\,dx = uv - \int u'v\,dx}$$

Example

Find $\displaystyle\int x\cos x\,dx$.

We need to choose a $u(x)$ and a $v'(x)$.
Let

$$\begin{aligned} u &= x & v' &= \cos x \\ u' &= 1 & v &= \sin x. \end{aligned}$$

Then

$$\int x\cos x\,dx = \qquad - \int \qquad dx = \qquad .$$

$$\int uv'\,dx = uv - \int u'v\,dx$$

Example

Find $\int x\cos x\,dx$.

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Then

$$\int x\cos x\,dx = x\sin x - \int \qquad dx = \qquad .$$

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We need to choose a $u(x)$ and a $v'(x)$.
Let

$$\begin{aligned} u &= x & v' &= \cos x \\ u' &= 1 & v &= \sin x. \end{aligned}$$

Then

$$\int x\cos x\,dx = x\sin x - \int 1\sin x\,dx = \qquad\qquad .$$

$$\int uv'\,dx = uv - \int u'v\,dx$$

Example

Find $\int x \cos x \, dx$.

We need to choose a $u(x)$ and a $v'(x)$.
Let

$$\begin{aligned} u &= x & v' &= \cos x \\ u' &= 1 & v &= \sin x. \end{aligned}$$

Then

$$\int x \cos x \, dx = x \sin x - \int 1 \sin x \, dx = x \sin x + \cos x + C.$$

$$\int uv'\,dx = uv - \int u'v\,dx$$

Example

Find $\displaystyle\int \ln x\,dx$.

$$\int uv'\,dx = uv - \int u'v\,dx$$

Example

Find $\displaystyle\int \ln x\,dx$.

We will consider $\displaystyle\int \ln x \cdot 1\,dx$.

$$\int uv'\,dx = uv - \int u'v\,dx$$

Example

Find $\displaystyle\int \ln x\,dx$.

We will consider $\displaystyle\int \ln x \cdot 1\,dx$.

Let

$$u = \ln x \qquad\qquad v' = 1$$

$$\int uv'\,dx = uv - \int u'v\,dx$$

Example

Find $\int \ln x\,dx$.

We will consider $\int \ln x \cdot 1\,dx$.

Let

$$u = \ln x \qquad\qquad v' = 1$$

$$u' = \frac{1}{x}$$

$$\int uv'\,dx = uv - \int u'v\,dx$$

Example

Find $\displaystyle\int \ln x\,dx$.

We will consider $\displaystyle\int \ln x \cdot 1\,dx$.

Let

$$\begin{aligned} u &= \ln x & v' &= 1 \\ u' &= \frac{1}{x} & v &= x. \end{aligned}$$

$$\int uv'\,dx = uv - \int u'v\,dx$$

Example

Find $\displaystyle\int \ln x\,dx$.

We will consider $\displaystyle\int \ln x \cdot 1\,dx$.

Let

$$u = \ln x \qquad v' = 1$$
$$u' = \frac{1}{x} \qquad v = x.$$

Then

$$\int \ln x \cdot 1\,dx = \ln x \cdot x - \int \frac{1}{x} \cdot x\,dx$$
$$=$$
$$= \qquad .$$

$$\int uv'\,dx = uv - \int u'v\,dx$$

Example

Find $\displaystyle\int \ln x\,dx$.

We will consider $\displaystyle\int \ln x \cdot 1\,dx$.

Let

$$u = \ln x \qquad v' = 1$$
$$u' = \frac{1}{x} \qquad v = x.$$

Then

$$\begin{aligned}\int \ln x \cdot 1\,dx &= \ln x \cdot x - \int \frac{1}{x} \cdot x\,dx \\ &= x \ln x - \int 1\,dx \\ &= \qquad\qquad .\end{aligned}$$

$$\int uv'\,dx = uv - \int u'v\,dx$$

Example

Find $\displaystyle\int \ln x\,dx$.

We will consider $\displaystyle\int \ln x \cdot 1\,dx$.

Let

$$u = \ln x \qquad\qquad v' = 1$$
$$u' = \frac{1}{x} \qquad\qquad v = x.$$

Then

$$\begin{aligned}\int \ln x \cdot 1\,dx &= \ln x \cdot x - \int \frac{1}{x} \cdot x\,dx \\ &= x\ln x - \int 1\,dx \\ &= x\ln x - x + C.\end{aligned}$$

$$\int uv'\,dx = uv - \int u'v\,dx$$

Sometimes we have to use integration by parts more than once.

$$\boxed{\int uv' \, dx = uv - \int u'v \, dx}$$

Example

Find $\int x^2 e^x \, dx$.

$$\int uv'\,dx = uv - \int u'v\,dx$$

Example

Find $\int x^2 e^x \, dx$.

We calculate that

$$\int x^2 e^x \, dx = x^2 e^x - 2 \int x e^x \, dx.$$

$$\boxed{\int uv'\,dx = uv - \int u'v\,dx}$$

Example

Find $\displaystyle\int x^2 e^x\,dx$.

We calculate that

$$\int x^2 e^x\,dx = x^2 e^x - 2\int x e^x\,dx.$$

But what do we do with $\int x e^x\,dx$?

$$\int uv'\,dx = uv - \int u'v\,dx$$

Example

Find $\displaystyle\int x^2 e^x\,dx$.

We calculate that

$$\int x^2 e^x\,dx = x^2 e^x - 2\int x e^x\,dx.$$

But what do we do with $\int x e^x\,dx$?

$$\int x e^x\,dx = x e^x - \int 1 e^x\,dx = x e^x - e^x + C.$$

$$\int uv'\,dx = uv - \int u'v\,dx$$

Example

Find $\int x^2e^x\,dx$.

We calculate that

$$\int x^2e^x\,dx = x^2e^x - 2\int xe^x\,dx.$$

But what do we do with $\int xe^x\,dx$?

$$\int xe^x\,dx = xe^x - \int 1e^x\,dx = xe^x - e^x + C.$$

Putting it all together, we have

$$\int x^2e^x\,dx = x^2e^x - 2\int xe^x\,dx = x^2e^x - 2xe^x + 2e^x + C.$$

$$\int uv'\,dx = uv - \int u'v\,dx$$

Remark

We can use the same technique to calculate $\int x^n e^x\,dx$.

We would have to do integration by parts n times.

$$\boxed{\int u v' \, dx = uv - \int u' v \, dx}$$

Theorem

$$\int u \, dv = uv - \int v \, du$$

EXAMPLE 4 Evaluate

$$\int e^x \cos x \, dx.$$

Solution Let $u = e^x$ and $dv = \cos x \, dx$. Then $du = e^x \, dx$, $v = \sin x$, and

$$\int e^x \cos x \, dx = e^x \sin x - \int e^x \sin x \, dx.$$

The second integral is like the first except that it has $\sin x$ in place of $\cos x$. To evaluate it, we use integration by parts with

$$u = e^x, \qquad dv = \sin x \, dx, \qquad v = -\cos x, \qquad du = e^x \, dx.$$

Then

$$\begin{aligned} \int e^x \cos x \, dx &= e^x \sin x - \left(-e^x \cos x - \int (-\cos x)(e^x \, dx) \right) \\ &= e^x \sin x + e^x \cos x - \int e^x \cos x \, dx. \end{aligned}$$

The unknown integral now appears on both sides of the equation. Adding the integral to both sides and adding the constant of integration give

$$2\int e^x \cos x \, dx = e^x \sin x + e^x \cos x + C_1.$$

Dividing by 2 and renaming the constant of integration give

$$\int e^x \cos x \, dx = \frac{e^x \sin x + e^x \cos x}{2} + C. \qquad \blacksquare$$

EXAMPLE 5 Obtain a formula that expresses the integral

$$\int \cos^n x \, dx$$

in terms of an integral of a lower power of $\cos x$.

Solution We may think of $\cos^n x$ as $\cos^{n-1} x \cdot \cos x$. Then we let

$$u = \cos^{n-1} x \quad \text{and} \quad dv = \cos x \, dx,$$

so that

$$du = (n-1)\cos^{n-2} x\,(-\sin x \, dx) \quad \text{and} \quad v = \sin x.$$

Integration by parts then gives

$$\begin{aligned}
\int \cos^n x \, dx &= \cos^{n-1} x \sin x + (n-1)\int \sin^2 x \cos^{n-2} x \, dx \\
&= \cos^{n-1} x \sin x + (n-1)\int (1-\cos^2 x)\cos^{n-2} x \, dx \\
&= \cos^{n-1} x \sin x + (n-1)\int \cos^{n-2} x \, dx - (n-1)\int \cos^n x \, dx.
\end{aligned}$$

If we add

$$(n-1)\int \cos^n x\,dx$$

to both sides of this equation, we obtain

$$n\int \cos^n x\,dx = \cos^{n-1} x \sin x + (n-1)\int \cos^{n-2} x\,dx.$$

We then divide through by n, and the final result is

$$\int \cos^n x\,dx = \frac{\cos^{n-1} x \sin x}{n} + \frac{n-1}{n}\int \cos^{n-2} x\,dx.$$

The formula found in Example 5 is called a **reduction formula** because it replaces an integral containing some power of a function with an integral of the same form having the power reduced. When n is a positive integer, we may apply the formula repeatedly until the remaining integral is easy to evaluate. For example, the result in Example 5 tells us that

$$\begin{aligned}\int \cos^3 x\,dx &= \frac{\cos^2 x \sin x}{3} + \frac{2}{3}\int \cos x\,dx \\ &= \frac{1}{3}\cos^2 x \sin x + \frac{2}{3}\sin x + C.\end{aligned}$$

$$\int uv'\,dx = uv - \int u'v\,dx$$

Theorem

$$\int_a^b uv'\,dx =$$

$$\boxed{\int uv'\,dx = uv - \int u'v\,dx}$$

Theorem

$$\int_a^b uv'\,dx = \Big[uv\Big]_a^b - \int_a^b u'v\,dx$$

$$\int_a^b uv'\,dx = \Big[uv\Big]_a^b - \int_a^b u'v\,dx$$

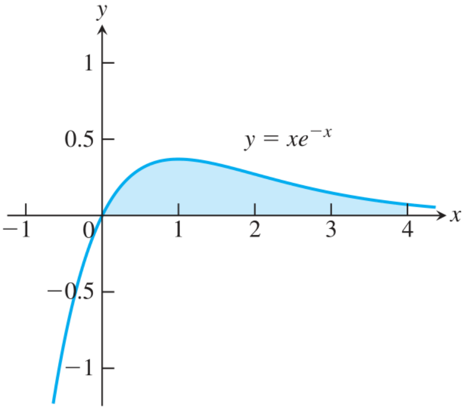
Example

Calculate the area of the region bounded by the curve $y = xe^{-x}$ and the x-axis from $x = 0$ to $x = 4$.

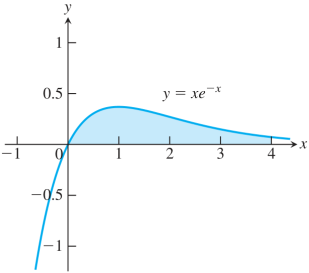
$$\int_a^b uv'\,dx = \Big[uv\Big]_a^b - \int_a^b u'v\,dx$$

Example

Calculate the area of the region bounded by the curve $y = xe^{-x}$ and the x-axis from $x = 0$ to $x = 4$.



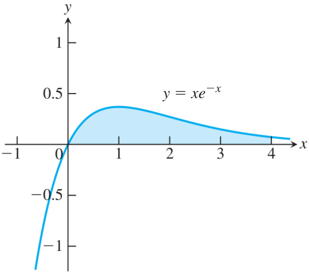
$$\int_a^b uv'\,dx = \Big[uv\Big]_a^b - \int_a^b u'v\,dx$$



We calculate that

$$\begin{aligned}\int_0^4 xe^{-x}\,dx &= \\ &= \\ &= \qquad\qquad\qquad\qquad\qquad\qquad .\end{aligned}$$

$$\int_a^b uv'\,dx = \Big[uv\Big]_a^b - \int_a^b u'v\,dx$$

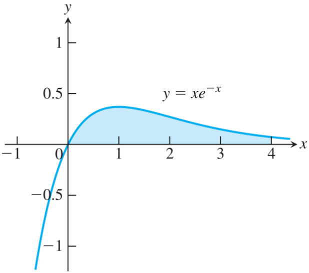


$$u = x$$
$$u' = 1$$
$$v' = e^{-x}$$
$$v = -e^{-x}$$

We calculate that

$$\int_0^4 xe^{-x}\,dx =$$
$$=$$
$$= \qquad .$$

$$\int_a^b uv'\,dx = \Big[uv\Big]_a^b - \int_a^b u'v\,dx$$

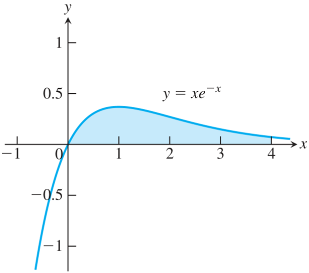


$$u = x$$
$$u' = 1$$
$$v' = e^{-x}$$
$$v = -e^{-x}$$

We calculate that

$$\int_0^4 xe^{-x}\,dx = \Big[-xe^{-x}\Big]_0^4 - \int_0^4 1(-e^{-x})\,dx$$
$$=$$
$$= \qquad\qquad .$$

$$\int_a^b uv'\,dx = \Big[uv\Big]_a^b - \int_a^b u'v\,dx$$



$$u = x$$
$$u' = 1$$
$$v' = e^{-x}$$
$$v = -e^{-x}$$

We calculate that

$$\begin{aligned}\int_0^4 xe^{-x}\,dx &= \Big[-xe^{-x}\Big]_0^4 - \int_0^4 1(-e^{-x})\,dx \\ &= \left(-4e^{-4}+0\right) + \left[-e^{-x}\right]_0^4 \\ &= -4e^{-4} + \left(-e^{-4}+1\right) = 1 - 5e^{-4}.\end{aligned}$$

$$\int_a^b uv'\,dx = \Big[uv\Big]_a^b - \int_a^b u'v\,dx$$

Example

Find $\displaystyle\int_0^1 \sin^{-1} x\,dx$.

$$\int_a^b uv'\,dx = \Big[uv\Big]_a^b - \int_a^b u'v\,dx$$

Example

Find $\displaystyle\int_0^1 \sin^{-1} x\,dx$.

Recall that $\frac{d}{dx}\sin^{-1} x = \frac{1}{\sqrt{1-x^2}}$.

$$\int_a^b uv'\,dx = \Big[uv\Big]_a^b - \int_a^b u'v\,dx$$

Example

Find $\displaystyle\int_0^1 \sin^{-1} x\,dx$.

Recall that $\frac{d}{dx}\sin^{-1} x = \frac{1}{\sqrt{1-x^2}}$.

Let $u = \sin^{-1} x$ and $v' = 1$.

$$\int_a^b uv'\,dx = \Big[uv\Big]_a^b - \int_a^b u'v\,dx$$

Example

Find $\displaystyle\int_0^1 \sin^{-1} x\,dx$.

Recall that $\frac{d}{dx}\sin^{-1} x = \frac{1}{\sqrt{1-x^2}}$.

Let $u = \sin^{-1} x$ and $v' = 1$. Then $u' = \frac{1}{\sqrt{1-x^2}}$ and $v = x$.

$$\int_a^b uv'\,dx = \Big[uv\Big]_a^b - \int_a^b u'v\,dx$$

Example

Find $\displaystyle\int_0^1 \sin^{-1} x\,dx$.

Recall that $\frac{d}{dx}\sin^{-1} x = \frac{1}{\sqrt{1-x^2}}$.

Let $u = \sin^{-1} x$ and $v' = 1$. Then $u' = \frac{1}{\sqrt{1-x^2}}$ and $v = x$. It follows that

$$\int_0^1 \sin^{-1} x \cdot 1\,dx = \left[x\sin^{-1} x\right]_0^1 - \int_0^1 \frac{x}{\sqrt{1-x^2}}dx$$

$$=$$

$$=$$

$$= \qquad .$$

$$\int_a^b uv'\,dx = \Big[uv\Big]_a^b - \int_a^b u'v\,dx$$

Example

Find $\displaystyle\int_0^1 \sin^{-1} x\,dx$.

Recall that $\frac{d}{dx}\sin^{-1} x = \frac{1}{\sqrt{1-x^2}}$.

Let $u = \sin^{-1} x$ and $v' = 1$. Then $u' = \frac{1}{\sqrt{1-x^2}}$ and $v = x$. It follows that

$$\begin{aligned}
\int_0^1 \sin^{-1} x \cdot 1\,dx &= \big[x\sin^{-1} x\big]_0^1 - \int_0^1 \frac{x}{\sqrt{1-x^2}}dx \\
&= \big[x\sin^{-1} x\big]_0^1 - \Big[-\sqrt{1-x^2}\Big]_0^1 \\
&= \left(\frac{\pi}{2} - 0\right) - (-0+1) \\
&= \frac{\pi}{2} - 1.
\end{aligned}$$

Break

We will continue at 2pm

Trigonometric Integrals

8.3 Trigonometric Integrals

$$
\begin{aligned}
\cos\theta &= \sin\left(\frac{\pi}{2}-\theta\right)\\
\cos^2\theta+\sin^2\theta &= 1\\
1+\tan^2\theta &= \sec^2\theta\\
1+\cot^2\theta &= \operatorname{cosec}^2\theta\\
\cos(A+B) &= \cos A\cos B-\sin A\sin B\\
\sin(A+B) &= \sin A\cos B+\cos A\sin B\\
\cos 2\theta &= \cos^2\theta-\sin^2\theta\\
\sin 2\theta &= 2\sin\theta\cos\theta\\
\cos^2\theta &= \frac{1}{2}(1+\cos 2\theta)\\
\sin^2\theta &= \frac{1}{2}(1-\cos 2\theta)
\end{aligned}
$$

$$
\begin{aligned}
\cos\theta &= \sin\left(\frac{\pi}{2} - \theta\right) \\
\cos^2\theta + \sin^2\theta &= 1 \\
1 + \tan^2\theta &= \sec^2\theta \\
1 + \cot^2\theta &= \operatorname{cosec}^2\theta \\
\cos(A+B) &= \cos A\cos B - \sin A\sin B \\
\sin(A+B) &= \sin A\cos B + \cos A\sin B \\
\cos 2\theta &= \cos^2\theta - \sin^2\theta \\
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$$
\begin{aligned}
\cos\theta &= \sin\left(\frac{\pi}{2} - \theta\right) \\
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\cos(A+B) &= \cos A\cos B - \sin A\sin B \\
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\cos^2\theta &= \frac{1}{2}(1+\cos 2\theta) \\
\sin^2\theta &= \frac{1}{2}(1-\cos 2\theta)
\end{aligned}
$$

How can we find

$$\int \sin^m x \, \cos^n x \, dx$$

if $m, n \in \{0, 1, 2, 3, 4, 5, \dots\}$?

$$\int \sin^m x \cos^n x \, dx$$

We need to look at 3 different cases:

$$\int \sin^m x \cos^n x \, dx$$

We need to look at 3 different cases:

1. m **is odd:**

$$\int \sin^m x \cos^n x \, dx$$

We need to look at 3 different cases:

1. m **is odd:**

2. m **is even and** n **is odd:**

$$\int \sin^m x \cos^n x \, dx$$

We need to look at 3 different cases:

1. **m is odd:**

2. **m is even and n is odd:**

3. **both m and n are even:**

$$\int \sin^m x \cos^n x \, dx \qquad\qquad \cos^2 x + \sin^2 x = 1$$

We need to look at 3 different cases:

1. m **is odd:** Write $m = 2k+1$ and use
$$\sin^m x = \sin^{2k+1} x = \left(\sin^2 x\right)^k \sin x = \left(1 - \cos^2 x\right)^k \sin x$$
and the substitution $u = \cos x$.
2. m **is even and** n **is odd:**
3. **both** m **and** n **are even:**

$$\int \sin^m x \cos^n x \, dx \qquad \cos^2 x + \sin^2 x = 1$$

We need to look at 3 different cases:

1. **m is odd:** Write $m = 2k+1$ and use
$$\sin^m x = \sin^{2k+1} x = \left(\sin^2 x\right)^k \sin x = \left(1 - \cos^2 x\right)^k \sin x$$
and the substitution $u = \cos x$.
2. **m is even and n is odd:** Write $n = 2k+1$ use
$$\cos^n x = \cos^{2k+1} x = \left(\cos^2 x\right)^k \cos x = \left(1 - \sin^2 x\right)^k \cos x$$
and the substitution $u = \sin x$.
3. **both m and n are even:**

$$\int \sin^m x \cos^n x \, dx \qquad\qquad \cos^2 x + \sin^2 x = 1$$

We need to look at 3 different cases:

1. m **is odd:** Write $m = 2k+1$ and use
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2. m **is even and** n **is odd:** Write $n = 2k+1$ use
$$\cos^n x = \cos^{2k+1} x = \left(\cos^2 x\right)^k \cos x = \left(1 - \sin^2 x\right)^k \cos x$$
and the substitution $u = \sin x$.
3. **both** m **and** n **are even:** Use
$$\sin^2 x = \frac{1 - \cos 2x}{2} \qquad \text{and} \qquad \cos^2 x = \frac{1 + \cos 2x}{2}.$$

Example

Find $\displaystyle\int \sin^3 x \cos^2 x \, dx$.

Example

Find $\int \sin^3 x \cos^2 x \, dx$.

Solution This is an example of Case 1.

$$\begin{aligned}
\int \sin^3 x \cos^2 x \, dx &= \int \sin^2 x \cos^2 x \sin x \, dx && m \text{ is odd.} \\
&= \int (1 - \cos^2 x)(\cos^2 x)(-d(\cos x)) && \sin x \, dx = -d(\cos x) \\
&= \int (1 - u^2)(u^2)(-du) && u = \cos x \\
&= \int (u^4 - u^2) \, du && \text{Multiply terms.} \\
&= \frac{u^5}{5} - \frac{u^3}{3} + C = \frac{\cos^5 x}{5} - \frac{\cos^3 x}{3} + C
\end{aligned}$$

Example

Find $\displaystyle\int \cos^5 x \, dx$.

Example

Find $\displaystyle\int \cos^5 x \, dx$.

Solution This is an example of Case 2, where $m = 0$ is even and $n = 5$ is odd.

$$\int \cos^5 x \, dx = \int \cos^4 x \cos x \, dx = \int (1 - \sin^2 x)^2 \, d(\sin x) \qquad \cos x \, dx = d(\sin x)$$

$$= \int (1 - u^2)^2 \, du \qquad u = \sin x$$

$$= \int (1 - 2u^2 + u^4) \, du \qquad \text{Square } 1 - u^2.$$

$$= u - \frac{2}{3}u^3 + \frac{1}{5}u^5 + C = \sin x - \frac{2}{3}\sin^3 x + \frac{1}{5}\sin^5 x + C \qquad \blacksquare$$

Example

Find $\int \sin^2 x \cos^4 x \, dx$.

Example

Find $\int \sin^2 x \cos^4 x \, dx$.

Solution This is an example of Case 3.

$$\int \sin^2 x \cos^4 x \, dx = \int \left(\frac{1 - \cos 2x}{2}\right)\left(\frac{1 + \cos 2x}{2}\right)^2 dx \qquad m \text{ and } n \text{ both even}$$

$$= \frac{1}{8}\int (1 - \cos 2x)(1 + 2\cos 2x + \cos^2 2x)\, dx$$

$$= \frac{1}{8}\int (1 + \cos 2x - \cos^2 2x - \cos^3 2x)\, dx$$

$$= \frac{1}{8}\left[x + \frac{1}{2}\sin 2x - \int (\cos^2 2x + \cos^3 2x)\, dx\right]$$

For the term involving $\cos^2 2x$, we use

$$\int \cos^2 2x \, dx = \frac{1}{2}\int (1 + \cos 4x)\, dx$$

$$= \frac{1}{2}\left(x + \frac{1}{4}\sin 4x\right).$$

Omit constant of integration until final result.

For the $\cos^3 2x$ term, we have

$$\int \cos^3 2x\, dx = \int (1 - \sin^2 2x) \cos 2x\, dx \qquad u = \sin 2x,\ du = 2\cos 2x\, dx$$

$$= \frac{1}{2}\int (1 - u^2)\, du = \frac{1}{2}\left(\sin 2x - \frac{1}{3}\sin^3 2x\right). \qquad \text{Again omit } C.$$

Combining everything and simplifying, we get

$$\int \sin^2 x \cos^4 x\, dx = \frac{1}{16}\left(x - \frac{1}{4}\sin 4x + \frac{1}{3}\sin^3 2x\right) + C. \qquad \blacksquare$$

Example

Find $\displaystyle\int_0^{\frac{\pi}{4}} \sqrt{1+\cos 4x}\,dx$.

Example

Find $\displaystyle\int_0^{\frac{\pi}{4}} \sqrt{1+\cos 4x}\, dx.$

Solution To eliminate the square root, we use the identity

$$\cos^2\theta = \frac{1+\cos 2\theta}{2} \qquad \text{or} \qquad 1+\cos 2\theta = 2\cos^2\theta.$$

With $\theta = 2x$, this becomes

$$1+\cos 4x = 2\cos^2 2x.$$

Therefore,

$$\int_0^{\pi/4} \sqrt{1+\cos 4x}\, dx = \int_0^{\pi/4} \sqrt{2\cos^2 2x}\, dx = \int_0^{\pi/4} \sqrt{2}\sqrt{\cos^2 2x}\, dx$$

$$= \sqrt{2}\int_0^{\pi/4} |\cos 2x|\, dx = \sqrt{2}\int_0^{\pi/4} \cos 2x\, dx \qquad \begin{array}{l}\cos 2x \geq 0 \text{ on}\\ [0, \pi/4]\end{array}$$

$$= \sqrt{2}\left[\frac{\sin 2x}{2}\right]_0^{\pi/4} = \frac{\sqrt{2}}{2}[1-0] = \frac{\sqrt{2}}{2}.$$

$$\sec^2 x = 1 + \tan^2 x \qquad \frac{d}{dx}\tan x = \sec^2 x$$

Example

Find $\displaystyle\int \tan^4 x\,dx$.

$$\sec^2 x = 1 + \tan^2 x \qquad \frac{d}{dx}\tan x = \sec^2 x$$

Example

Find $\displaystyle\int \tan^4 x\, dx$.

Solution

$$\int \tan^4 x\, dx = \int \tan^2 x \cdot \tan^2 x\, dx = \int \tan^2 x \cdot (\sec^2 x - 1)\, dx$$

$$= \int \tan^2 x \sec^2 x\, dx - \int \tan^2 x\, dx$$

$$= \int \tan^2 x \sec^2 x\, dx - \int (\sec^2 x - 1)\, dx$$

$$= \int \tan^2 x \sec^2 x\, dx - \int \sec^2 x\, dx + \int dx$$

In the first integral, we let

$$u = \tan x, \qquad du = \sec^2 x\, dx$$

and have

$$\int u^2\, du = \frac{1}{3} u^3 + C_1.$$

The remaining integrals are standard forms, so

$$\int \tan^4 x\, dx = \frac{1}{3} \tan^3 x - \tan x + x + C.$$

Example

Find $\displaystyle\int \sec^3 x\,dx$.

Solution We integrate by parts using

$$u = \sec x, \qquad dv = \sec^2 x\, dx, \qquad v = \tan x, \qquad du = \sec x \tan x\, dx.$$

Then

$$\begin{aligned}
\int \sec^3 x\, dx &= \sec x \tan x - \int (\tan x)(\sec x \tan x\, dx) \\
&= \sec x \tan x - \int (\sec^2 x - 1) \sec x\, dx \qquad \tan^2 x = \sec^2 x - 1 \\
&= \sec x \tan x + \int \sec x\, dx - \int \sec^3 x\, dx.
\end{aligned}$$

Combining the two secant-cubed integrals gives

$$2\int \sec^3 x\, dx = \sec x \tan x + \int \sec x\, dx$$

and

$$\int \sec^3 x\, dx = \frac{1}{2} \sec x \tan x + \frac{1}{2} \ln |\sec x + \tan x| + C.$$

■

$$\sec^2 x = 1 + \tan^2 x \qquad \frac{d}{dx}\tan x = \sec^2 x$$

Example

Find $\displaystyle\int \tan^4 x \sec^4 x \, dx$.

$$\sec^2 x = 1 + \tan^2 x \qquad \frac{d}{dx}\tan x = \sec^2 x$$

Example

Find $\displaystyle\int \tan^4 x \sec^4 x \, dx$.

Solution

$$\begin{aligned}
\int (\tan^4 x)(\sec^4 x)\, dx &= \int (\tan^4 x)(1 + \tan^2 x)(\sec^2 x)\, dx && \sec^2 x = 1 + \tan^2 x \\
&= \int (\tan^4 x + \tan^6 x)(\sec^2 x)\, dx \\
&= \int (\tan^4 x)(\sec^2 x)\, dx + \int (\tan^6 x)(\sec^2 x)\, dx \\
&= \int u^4\, du + \int u^6\, du = \frac{u^5}{5} + \frac{u^7}{7} + C && u = \tan x,\ du = \sec^2 x\, dx \\
&= \frac{\tan^5 x}{5} + \frac{\tan^7 x}{7} + C
\end{aligned}$$

■

How do we calculate

$$\int \sin mx \sin nx \, dx$$

or

$$\int \sin mx \cos nx \, dx$$

or

$$\int \cos mx \cos nx \, dx$$

?

How do we calculate

$$\int \sin mx \sin nx \, dx$$

or

$$\int \sin mx \cos nx \, dx$$

or

$$\int \cos mx \cos nx \, dx$$

?

It is possible to use integration by parts (twice), but there is an easier way.

$$\cos(A+B) = \cos A \cos B - \sin A \sin B$$

$$\cos(mx - nx) - \cos(mx + nx) =$$

$$\cos(A+B) = \cos A \cos B - \sin A \sin B$$

$$\begin{aligned}\cos(mx - nx) - \cos(mx + nx) = {} & \cos mx \cos nx + \sin mx \sin nx \\ & - \cos mx \cos nx + \sin mx \sin nx\end{aligned}$$

$$\cos(A+B) = \cos A \cos B - \sin A \sin B$$

$$\begin{aligned} \cos(mx - nx) - \cos(mx + nx) = {} & \cos mx \cos nx + \sin mx \sin nx \\ & - \cos mx \cos nx + \sin mx \sin nx \\ = {} & 2 \sin mx \sin nx \end{aligned}$$

$$\cos(A+B) = \cos A \cos B - \sin A \sin B$$

$$\begin{aligned} \cos(mx-nx) - \cos(mx+nx) &= \cos mx \cos nx + \sin mx \sin nx \\ &\quad - \cos mx \cos nx + \sin mx \sin nx \\ &= 2 \sin mx \sin nx \end{aligned}$$

Therefore

$$\sin mx \sin nx = \frac{1}{2}\big(\cos(m-n)x - \cos(m+n)x\big).$$

$$\cos(A+B) = \cos A \cos B - \sin A \sin B$$

$$\begin{aligned}\cos(mx-nx) - \cos(mx+nx) &= \cos mx \cos nx + \sin mx \sin nx \\ &\quad - \cos mx \cos nx + \sin mx \sin nx \\ &= 2 \sin mx \sin nx\end{aligned}$$

Therefore

$$\sin mx \sin nx = \frac{1}{2}\big(\cos(m-n)x - \cos(m+n)x\big).$$

Similarly

$$\sin mx \cos nx = \frac{1}{2}\big(\sin(m-n)x + \sin(m+n)x\big)$$

and

$$\cos mx \cos nx = \frac{1}{2}\big(\cos(m-n)x + \cos(m+n)x\big).$$

$$\sin mx \cos nx = \frac{1}{2}\big(\sin(m-n)x + \sin(m+n)x\big)$$

Example

Find $\displaystyle\int \sin 3x \cos 5x \, dx$.

$$\sin mx \cos nx = \tfrac{1}{2}\big(\sin(m-n)x + \sin(m+n)x\big)$$

Example

Find $\displaystyle\int \sin 3x \cos 5x \, dx$.

Solution From Equation (4) with $m = 3$ and $n = 5$, we get

$$\begin{aligned}\int \sin 3x \cos 5x \, dx &= \frac{1}{2}\int [\sin(-2x) + \sin 8x]\, dx \\ &= \frac{1}{2}\int (\sin 8x - \sin 2x)\, dx \\ &= -\frac{\cos 8x}{16} + \frac{\cos 2x}{4} + C.\end{aligned}$$

■

$$\cos mx \cos nx = \tfrac{1}{2}\big(\cos(m-n)x + \cos(m+n)x\big)$$

Example

Find $\displaystyle\int \cos 3x \cos 2x \, dx.$

$$\cos mx \cos nx = \tfrac{1}{2}\big(\cos(m-n)x + \cos(m+n)x\big)$$

Example

Find $\displaystyle\int \cos 3x \cos 2x \, dx$.

We have $m = 3$ and $n = 2$. It follows that

$$\begin{aligned}\int \cos 3x \cos 2x \, dx &= \frac{1}{2}\int \cos(3-2)x \, dx + \frac{1}{2}\int \cos(3+2)x \, dx \\ &= \ldots\end{aligned}$$

Next Time

- 8.4 Trigonometric Substitutions
- 8.5 Integration of Rational Functions by Partial Fractions
- 8.8 Improper Integrals