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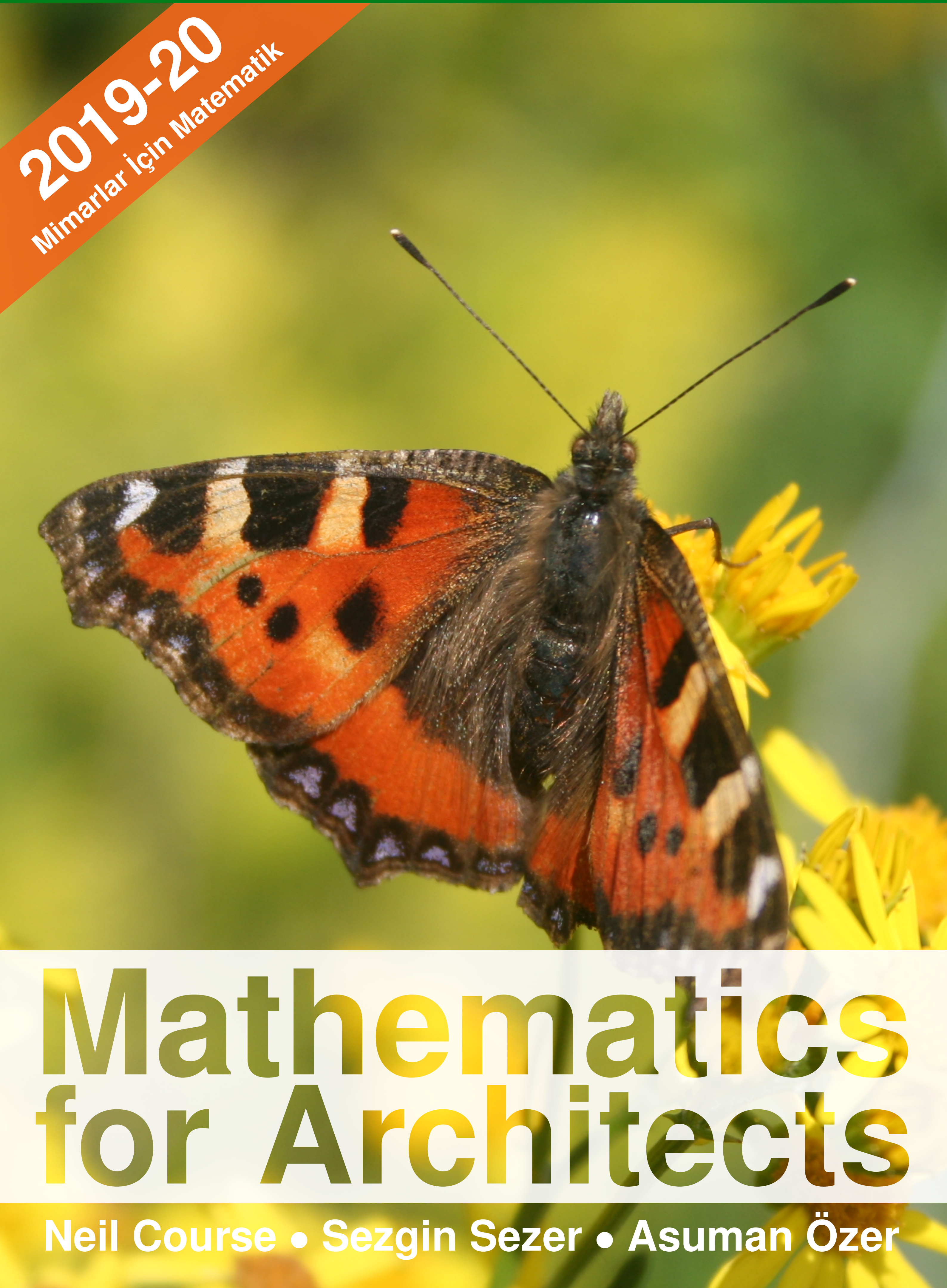

2019-20

Mimarlar İçin Matematik

Mathematics for Architects

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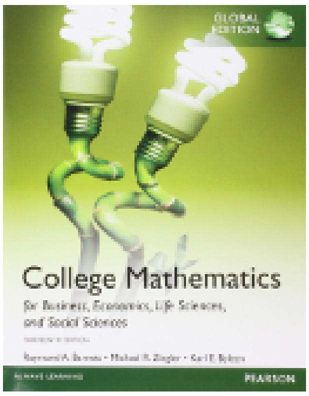
Suggested further reading:



Raymond A. Barnett, Michael R. Ziegler and Karl E. Byleen,
College Mathematics for Business, Economics, Life Sciences, and Social Sciences,
Pearson. ISBN: 978-1-292-05766-8

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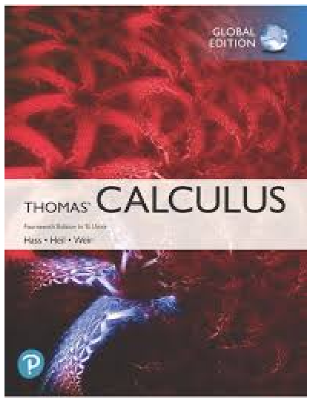
Thomas' Calculus,
Pearson. ISBN: 978-1-292-25322-0



Friday 20th December, 2019

Contents

Part I

Introduction

1

Symbolic Logic

Definition. A ***proposition*** is a statement which is either *true* or *false* (but not both).

Example 1.1.

- "Grass is green" (true)
- "2+5=5" (false)
- "My name is Neil" (true)

are propositions, but

- "Close the door"
- "Is it cold today?"
- "1"

are not propositions.

Notation. The symbol for ***or*** (veya) is $\vee$.

Example 1.2. If P is the proposition "It is snowing today" and Q is the proposition "It is raining today", then $P \vee Q$ is the proposition "It is snowing or raining today".

Example 1.3. If $M = (x \in A)$ and $N = (x \in B)$, then $M \vee N = (x \in A \cup B)$

Truth Table 1.1. (T = true, F = false)

P	Q	$P \vee Q$
T	T	T
T	F	T
F	T	T
F	F	F

Notation. The symbol for ***and*** (ve) is $\wedge$.

Example 1.4. If P = "I am hungry" and Q = "I am sleepy", then $P \wedge Q$ = "I am hungry and sleepy".

Example 1.5. If $M = (x \in A)$ and $N = (x \in B)$, then $M \wedge N = (x \in A \cap B)$

Truth Table 1.2.

P	Q	$P \wedge Q$
T	T	T
T	F	F
F	T	F
F	F	F

Sembolik Mantık

Tanım. *Doğru* yada *yanlış* bir hüküm bildiren ifadeye ***önerme*** denir.

Örnek 1.1.

- "Çim yeşildir" (doğru)
- "2+5=5" (yanlış)
- "Benim adım Asuman" (doğru)

hükümleri birer önermedir. Fakat

- "Kapıyı kapat"
- "Bugün soğuk mu?"
- "1"

cümleleri bir hüküm belirtmediğinden önerme olarak kabul edilmezler.

Tanım. Doğruluğu ispat edilemeyen veya doğruluğu ispatına gerek duyulmayan ancak doğruluğu kabul edilen önermelere ***aksiyom*** denir.

Tanım. Doğruluğu ispatlanabilen önermelere ***teorem*** denir.

Tanım. İki yada daha fazla önermeyi ***ve***, ***veya(yada)***, ***ise***, ***ancak ve ancak*** gibi bağlaçlarla birleştirerek veya bir önermenin sonuna ***değil*** ekleneren elde edilen önermelere ***bileşik önerme*** denir.

Notasyon. ***veya*** bağlacı $\vee$ ile sembolize edilir.

Örnek 1.2. Eğer "Bu gün hava karlı" önermesini P ile "Bu gün hava yağmurlu" önermesini Q ile gösterecek olursak, $P \vee Q$ bileşik önermesi "Bu gün hava karlı yada(veya) yağmurlu" olarak ifade edilir.

Örnek 1.3. Eğer $M = (x \in A)$ ve $N = (x \in B)$ ise $M \vee N = (x \in A \cup B)$ dir.

Doğruluk Tablosu 1.1. (D = doğru, Y = yanlış)

P	Q	$P \vee Q$
D	D	D
D	Y	D
Y	D	D
Y	Y	Y

Notation. The symbol for ***not*** (değil) is $\neg$.

Example 1.6. If P = "Sizin hocanız kahve seviyor", then $\neg P$ = "Sizin hocanız kahve sevmiyor".

Example 1.7. If $M = (x \geq 7)$, then $\neg M = (x < 7)$

Truth Table 1.3.

P	$\neg P$
T	F
F	T

Notation. The symbol for ***if and only if*** (iff/ancak ve ancak) is $\iff$.

Truth Table 1.4.

P	Q	$P \iff Q$
T	T	T
T	F	F
F	T	F
F	F	T

Notation. The symbol for ***implies*** (ise) is $\implies$.

Example 1.8. Let P = "I am in London" and Q = "I am in the UK." Then $P \implies Q$.

Truth Table 1.5.

P	Q	$P \implies Q$
T	T	T
T	F	F
F	T	T
F	F	T

Remark. We must only write "$P \implies Q$" if both P and Q are propositions. I don't want to see nonsense like

$$\int_0^1 3x^2\, dx = \left[x^3\right]_0^1 \implies 1$$

in your work. Yes, "$\int_0^1 3x^2\, dx = \left[x^3\right]_0^1$" is a proposition. In fact, it is a *true* proposition. But "1" is not a proposition.

If you mean "=", then write "=".

Remark. If P and Q are propositions, then $(P \vee Q)$, $(P \wedge Q)$, $(\neg P)$, $(P \implies Q)$ and $(P \iff Q)$ are also propositions.

Definition. The ***converse*** (zıt) of $(P \implies Q)$ is $(Q \implies P)$.

Definition. The ***contapositive*** (devrik) of $(P \implies Q)$ is $(\neg Q \implies \neg P)$.

Example 1.9.
P = "It is raining"
Q = "I get wet"
$(P \implies Q)$ = "If it is raining, then I get wet"
converse: $(Q \implies P)$ = "If I get wet, then it is raining"
contrapositive: $(\neg Q \implies \neg P)$ = "If I do not get wet, then it is not raining"

Notasyon. ***ve*** bağlacı $\wedge$ ile sembolize edilir.

Örnek 1.4. Eğer "Ben açım" önermesini P ile "Uykum var" önermesini Q ile gösterecek olursak, $P \wedge Q$ bileşik önermesi "Ben açım ve uykum var" olarak ifade edilir.

Örnek 1.5. Eğer $M = (x \in A)$ ve $N = (x \in B)$ ise $M \wedge N = (x \in A \cap B)$ dir.

Doğruluk Tablosu 1.2.

P	Q	$P \wedge Q$
D	D	D
D	Y	Y
Y	D	Y
Y	Y	Y

Notasyon. ***Değil*** bağlacı $\sim$ veya $\neg$ ile sembolize edilir.

Örnek 1.6. If P = "Sizin hocanız kahve seviyor", then $\sim P$ = "Sizin hocanız kahve sevmiyor".

Örnek 1.7. If $M = (x \geq 7)$, then $\sim M = (x < 7)$

Doğruluk Tablosu 1.3.

P	$\sim P$
D	Y
Y	D

Notasyon. Ancak ve ancak bağlacı $\iff$ ile sembolize edilir.

Doğruluk Tablosu 1.4.

P	Q	$P \iff Q$
D	D	D
D	Y	Y
Y	D	Y
Y	Y	D

Notasyon. ***İse*** bağlacı $\implies$ ile sembolize edilir.

Örnek 1.8. P = "Londradayım." Q = "Birleşik Krallıktayım." $P \implies Q$.

Doğruluk Tablosu 1.5.

P	Q	$P \implies Q$
D	D	D
D	Y	Y
Y	D	D
Y	Y	D

Not. Eğer P ve Q birer önerme ise "$P \implies Q$" de bir önermedir.

$$\int_0^1 3x^2\, dx = \left[x^3\right]_0^1 \implies 1$$

gösterimi mantıksal olarak yanlıştır. "$\int_0^1 3x^2\, dx = \left[x^3\right]_0^1$" ifadesi bir önermedir.Fakat "1" bir önerme değildir.

İse bağlacının sol tarafında "=" ile inşa edilen bir önerme var ise, sağ tarafında da bir önerme olmalıdır.

Not. Eğer P ve Q birer önerme ise, $(P \vee Q)$, $(P \wedge Q)$, $(\neg P)$, $(P \implies Q)$ ve $(P \iff Q)$ ifadeleri de birer önermedir.

The 22 Identities.

1. $(P \vee P) = P$
2. $(P \wedge P) = P$
3. $(P \vee Q) = (Q \vee P)$
4. $(P \wedge Q) = (Q \wedge P)$
5. $((P \vee Q) \vee R) = (P \vee (Q \vee R))$
6. $((P \wedge Q) \wedge R) = (P \wedge (Q \wedge R))$
7. $\neg(P \vee Q) = (\neg P \wedge \neg Q)$
8. $\neg(P \wedge Q) = (\neg P \vee \neg Q)$
9. $(P \wedge (Q \vee R)) = ((P \wedge Q) \vee (P \wedge R))$
10. $(P \vee (Q \wedge R)) = ((P \vee Q) \wedge (P \vee R))$
11. $(P \vee \text{true}) = \text{true}$
12. $(P \wedge \text{false}) = \text{false}$
13. $(P \vee \text{false}) = P$
14. $(P \wedge \text{true}) = P$
15. $(P \vee \neg P) = \text{true}$
16. $(P \wedge \neg P) = \text{false}$
17. $\neg(\neg P) = P$
18. $(P \implies Q) = (\neg P \vee Q)$
19. $(P \iff Q) = ((P \implies Q) \wedge (Q \implies P))$
20. $((P \wedge Q) \implies R) = (P \implies (Q \implies R))$
21. $((P \implies Q) \wedge (P \implies \neg Q)) = \neg P$
22. $(P \implies Q) = (\neg Q \implies \neg P)$

Proof of Identity 18.

P	Q	$P \implies Q$	$\neg P$	Q	$\neg P \vee Q$
T	T	T	F	T	T
T	F	F	F	F	F
F	T	T	T	T	T
F	F	T	T	F	T

Note that the 3rd and 6th columns are the same:

$$T, F, T, T.$$

Therefore $(P \implies Q) = (\neg P \vee Q)$. □

Proof of Identity 22.

P	Q	$P \implies Q$	$\neg Q$	$\neg P$	$\neg Q \implies \neg P$
T	T	T	F	F	T
T	F	F	T	F	F
F	T	T	F	T	T
F	F	T	T	T	T

Therefore $(P \implies Q) = (\neg Q \implies \neg P)$. □

Tanım. $(P \implies Q)$ önermesinin ***zıttı*** $(Q \implies P)$ dır.

Tanım. $(P \implies Q)$ önermesinin ***devriği*** $(\sim Q \implies \sim P)$ dir.

Örnek 1.9.
$P =$ "Hava yağmırlu"
$Q =$ "ben ıslandım."
$(P \implies Q) =$ "Eğer hava yağmurlu ***ise*** ben ıslandım."
zıttı: $(Q \implies P) =$ "Eğer ben ıslanmıs ***isem*** hava yağmurludur. "
devriği: $(\sim Q \implies \sim P) =$ "Eğer ben ıslanmamış ***isem*** hava yağmurlu değildir."

The 22 Identities.

1. $(P \vee P) = P$
2. $(P \wedge P) = P$
3. $(P \vee Q) = (Q \vee P)$
4. $(P \wedge Q) = (Q \wedge P)$
5. $((P \vee Q) \vee R) = (P \vee (Q \vee R))$
6. $((P \wedge Q) \wedge R) = (P \wedge (Q \wedge R))$
7. $\neg(P \vee Q) = (\neg P \wedge \neg Q)$
8. $\neg(P \wedge Q) = (\neg P \vee \neg Q)$
9. $(P \wedge (Q \vee R)) = ((P \wedge Q) \vee (P \wedge R))$
10. $(P \vee (Q \wedge R)) = ((P \vee Q) \wedge (P \vee R))$
11. $(P \vee \text{true}) = \text{true}$
12. $(P \wedge \text{false}) = \text{false}$
13. $(P \vee \text{false}) = P$
14. $(P \wedge \text{true}) = P$
15. $(P \vee \neg P) = \text{true}$
16. $(P \wedge \neg P) = \text{false}$
17. $\neg(\neg P) = P$
18. $(P \implies Q) = (\neg P \vee Q)$
19. $(P \iff Q) = ((P \implies Q) \wedge (Q \implies P))$
20. $((P \wedge Q) \implies R) = (P \implies (Q \implies R))$
21. $((P \implies Q) \wedge (P \implies \neg Q)) = \neg P$
22. $(P \implies Q) = (\neg Q \implies \neg P)$

Proof of Identity 18.

P	Q	$P \implies Q$	$\neg P$	Q	$\neg P \vee Q$
D	D	D	Y	D	D
D	Y	Y	Y	Y	Y
Y	D	D	D	D	D
Y	Y	D	D	Y	D

Note that the 3rd and 6th columns are the same:

$$T, F, T, T.$$

Therefore $(P \implies Q) = (\neg P \vee Q)$. □

Example 1.10. Prove that $\neg(P \implies Q) = (P \wedge \neg Q)$
solution:

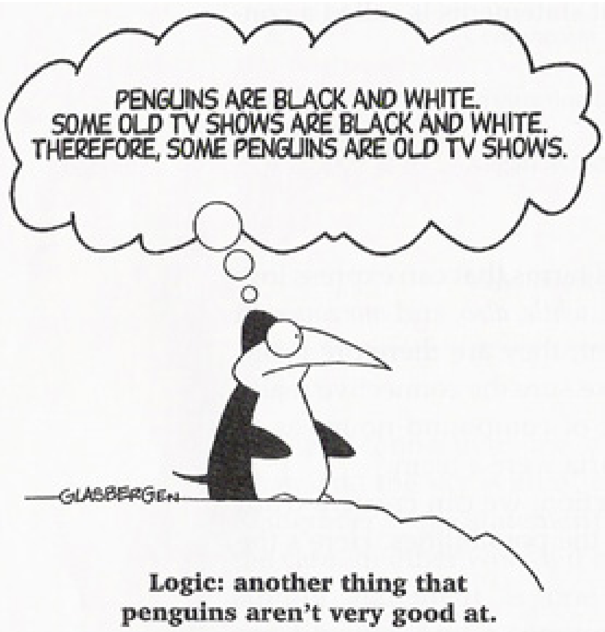
$$\neg(P \implies Q) \overset{18}{=} \neg(\neg P \vee Q) \overset{7}{=} (\neg\neg P \wedge \neg Q) \overset{17}{=} (P \wedge \neg Q).$$

Example 1.11. Prove that $(P \vee (P \wedge Q)) = P$
solution:

$$\begin{aligned}(P \vee (P \wedge Q)) &\overset{14}{=} ((P \wedge \text{true}) \vee (P \wedge Q)) \overset{9}{=} (P \wedge (\text{true} \vee Q)) \\ &\overset{3}{=} (P \wedge (Q \vee \text{true})) \overset{11}{=} (P \wedge \text{true}) \overset{14}{=} P.\end{aligned}$$

Notation. The symbol for ***for all*** (her) is $\forall$.

Notation. The symbol for ***there exists*** (vardır) is $\exists$.



Proof of Identity 22.

P	Q	$P \implies Q$	$\neg Q$	$\neg P$	$\neg Q \implies \neg P$
D	D	D	Y	Y	D
D	Y	Y	T	Y	Y
Y	D	D	Y	D	D
Y	Y	D	D	D	D

Therefore $(P \implies Q) = (\neg Q \implies \neg P)$. □

Örnek 1.10. Prove that $\neg(P \implies Q) = (P \wedge \neg Q)$
çözüm:

$$\neg(P \implies Q) \overset{18}{=} \neg(\neg P \vee Q) \overset{7}{=} (\neg\neg P \wedge \neg Q) \overset{17}{=} (P \wedge \neg Q).$$

Örnek 1.11. Prove that $(P \vee (P \wedge Q)) = P$
çözüm:

$$\begin{aligned}(P \vee (P \wedge Q)) &\overset{14}{=} ((P \wedge \text{true}) \vee (P \wedge Q)) \overset{9}{=} (P \wedge (\text{true} \vee Q)) \\ &\overset{3}{=} (P \wedge (Q \vee \text{true})) \overset{11}{=} (P \wedge \text{true}) \overset{14}{=} P.\end{aligned}$$

Notasyon. The symbol for ***for all*** (her) is $\forall$.

Notasyon. The symbol for ***there exists*** (vardır) is $\exists$.

Problems

Problem 1.1 (Truth Tables)**.** Use truth tables to prove the following

(a). Identity 5 : $((P \vee Q) \vee R) = (P \vee (Q \vee R))$

(b). Identity 6 : $((P \wedge Q) \wedge R) = (P \wedge (Q \wedge R))$

(c). Identity 7 : $\neg(P \vee Q) = (\neg P \wedge \neg Q)$

(d). Identity 8 : $\neg(P \wedge Q) = (\neg P \vee \neg Q)$

(e). Identity 9 : $(P \wedge (Q \vee R)) = ((P \wedge Q) \vee (P \wedge R))$

(f). Identity 10 : $(P \vee (Q \wedge R)) = ((P \vee Q) \wedge (P \vee R))$

(g). Identity 15 : $(P \vee \neg P) = \text{true}$

(h). Identity 16 : $(P \wedge \neg P) = \text{false}$

(i). Identity 21 : $((P \implies Q) \wedge (P \implies \neg Q)) = \neg P$

Sorular

Soru 1.1 (Truth Tables)**.** Use truth tables to prove the following

(a). Identity 5 : $((P \vee Q) \vee R) = (P \vee (Q \vee R))$

(b). Identity 6 : $((P \wedge Q) \wedge R) = (P \wedge (Q \wedge R))$

(c). Identity 7 : $\neg(P \vee Q) = (\neg P \wedge \neg Q)$

(d). Identity 8 : $\neg(P \wedge Q) = (\neg P \vee \neg Q)$

(e). Identity 9 : $(P \wedge (Q \vee R)) = ((P \wedge Q) \vee (P \wedge R))$

(f). Identity 10 : $(P \vee (Q \wedge R)) = ((P \vee Q) \wedge (P \vee R))$

(g). Identity 15 : $(P \vee \neg P) = \text{true}$

(h). Identity 16 : $(P \wedge \neg P) = \text{false}$

(i). Identity 21 : $((P \implies Q) \wedge (P \implies \neg Q)) = \neg P$

2

Numbers

The Natural Numbers

The set
$$\mathbb{N} = \{1, 2, 3, 4, 5, 6, \ldots\}$$
is called the set of ***natural numbers***. These are the first numbers that children learn. The symbol $\in$ means "*in*". For example

$$\begin{aligned} 2 \in \mathbb{N} &\quad \text{means "2 is a natural number"} \\ 7 \in \mathbb{N} &\quad \text{means "7 is a natural number"} \\ \tfrac{1}{2} \notin \mathbb{N} &\quad \text{means "}\tfrac{1}{2}\text{ is \textbf{not} a natural number"} \\ 0 \notin \mathbb{N} &\quad \text{means "0 is \textbf{not} a natural number"} \\ -5 \notin \mathbb{N} &\quad \text{means "}-5\text{ is \textbf{not} a natural number"} \end{aligned}$$

In the natural numbers, we can do "+" and "×"
$$2 + 7 = 9 \in \mathbb{N}, \qquad 2 \times 7 = 14 \in \mathbb{N}.$$
However we can not do "−" because
$$2 - 7 \notin \mathbb{N}.$$

So we invent new numbers!

The Integers

The set
$$\mathbb{Z} = \{\ldots, -4, -3, -2, -1, 0, 1, 2, 3, 4, \ldots\}$$
is called the set of ***integers***. We use a $\mathbb{Z}$ for the German word 'zahlen' (numbers). In $\mathbb{Z}$, we can do "+", "−" and "×" but we can not do "÷". For example $3 \in \mathbb{Z}$, $4 \in \mathbb{Z}$, $-5 \in \mathbb{Z}$ and
$$3 + 4 \in \mathbb{Z}, \quad 3 - 4 \in \mathbb{Z}, \quad 3 \times 4 \in \mathbb{Z}, \quad 3 \div 4 \notin \mathbb{Z},$$
$$3 + (-5) \in \mathbb{Z}, \quad 3 - (-5) \in \mathbb{Z}, \quad 3 \times (-5) \in \mathbb{Z}, \quad 3 \div (-5) \notin \mathbb{Z}.$$

So we invent new numbers!

Sayılar

Doğal sayılar

$$\mathbb{N} = \{1, 2, 3, 4, 5, 6, \ldots\}$$
kümesi ***doğal sayılar*** kümesi olarak adlandırırılır. Bunlar çocukluğumuzda ilk öğrenilen sayılardır. $\in$ sembolü "elemanıdır" anlamına gelir. Örneğin,

$$\begin{aligned} 2 \in \mathbb{N} &\quad \text{demek "2 bir doğal sayıdır"} \\ 7 \in \mathbb{N} &\quad \text{anlamı "7 bir doğal sayıdır"} \\ \tfrac{1}{2} \notin \mathbb{N} &\quad \text{anlamı "}\tfrac{1}{2}\text{ bir doğal sayı \textbf{değildir}"} \\ 0 \notin \mathbb{N} &\quad \text{anlamı "0 bir doğal sayı \textbf{değildir}"} \\ -5 \notin \mathbb{N} &\quad \text{anlamı "}-5\text{ bir doğal sayı \textbf{değildir}"} \end{aligned}$$

Doğal sayılarla "+" ve "×" işlemlerini yaparız.
$$2 + 7 = 9 \in \mathbb{N}, \qquad 2 \times 7 = 14 \in \mathbb{N}.$$
Ne yazık ki "−" işlemini yapamayı, çünkü, örneğin
$$2 - 7 \notin \mathbb{N}.$$
dir.

Bu yüzden yeni sayılar keşfederiz.!

Tam sayılar

$$\mathbb{Z} = \{\ldots, -4, -3, -2, -1, 0, 1, 2, 3, 4, \ldots\}$$
kümesine ***tam sayılar*** denir. Bunu Almanca 'zahlen' (sayılar) kelimesinden $\mathbb{Z}$ ile gösteririz. $\mathbb{Z}$ içerisinde, "+", "−" ve "×" yapabiliriz ama "÷" yapamayız. Örneğin $3 \in \mathbb{Z}$, $4 \in \mathbb{Z}$, $-5 \in \mathbb{Z}$ ve
$$3 + 4 \in \mathbb{Z}, \quad 3 - 4 \in \mathbb{Z}, \quad 3 \times 4 \in \mathbb{Z}, \quad 3 \div 4 \notin \mathbb{Z},$$
$$3 + (-5) \in \mathbb{Z}, \quad 3 - (-5) \in \mathbb{Z}, \quad 3 \times (-5) \in \mathbb{Z}, \quad 3 \div (-5) \notin \mathbb{Z}.$$

Dolayısıyla yeni sayılar keşfederiz!

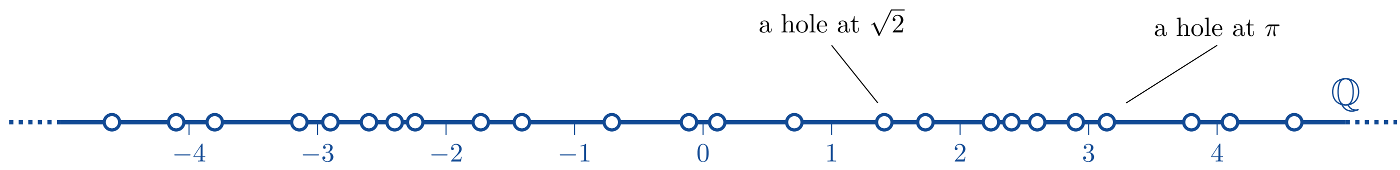


Figure 2.1: The Rational Numbers
Şekil 2.1: Rasyonel Sayılar

The Rational Numbers

The set

$$\mathbb{Q} = \{\text{all fractions}\} = \left\{ \frac{a}{b} \;\middle|\; a, b \in \mathbb{Z} \text{ and } b \neq 0 \right\}$$

is called the set of ***rational numbers***. We use a $\mathbb{Q}$ for the word 'quotient'. For example

$$\begin{aligned} 0 &= \frac{0}{1} \in \mathbb{Q} & \frac{100}{13} &\in \mathbb{Q} \\ 1 &= \frac{1}{1} \in \mathbb{Q} & \sqrt{2} &\notin \mathbb{Q} \\ \frac{3}{4} &\in \mathbb{Q} & -4 = \frac{8}{-2} &\in \mathbb{Q} \\ \pi &\notin \mathbb{Q} & 0.12345 = \frac{12345}{100000} &\in \mathbb{Q}. \end{aligned}$$

In $\mathbb{Q}$ we can do "+", "−", "×" and "÷(by a number $\neq 0$)".

Are we happy now?

No!

Why?

Because if we draw all the rational numbers in a line, then the line has lots of holes in it – see figure 2.1. In fact, $\mathbb{Q}$ has ∞ many holes in it.

So we invent new numbers!

The Real Numbers

The set

$$\mathbb{R} = \{\text{all numbers which can be written as a decimal}\}$$

is called the set of ***real numbers***. For example

$$\begin{aligned} 0 = 0.0 &\in \mathbb{R} & \frac{100}{13} &= 7.692307\ldots \in \mathbb{R} \\ \frac{23}{99} = 0.232323\ldots &\in \mathbb{R} & \sqrt{2} &= 1.414213\ldots \in \mathbb{R} \\ \frac{3}{4} = 0.75 &\in \mathbb{R} & \frac{123}{999} &= 0.123123\ldots \in \mathbb{R} \\ \pi = 3.141592\ldots &\in \mathbb{R} & \frac{12345}{100000} &= 0.12345 \in \mathbb{R}. \end{aligned}$$

The real numbers are complete – this means that if we draw all the real numbers in a line, then there are no holes in the line. See figure 2.2 on page 9.

Are we happy now?

Yes!

Rasyonel Sayılar

$$\mathbb{Q} = \{\text{tüm kesirler}\} = \left\{ \frac{a}{b} \;\middle|\; a, b \in \mathbb{Z} \text{ ve } b \neq 0 \right\}$$

kümesine ***rasyonel sayılar*** denir. Bunu $\mathbb{Q}$ ile gösteririz. Örneğin

$$\begin{aligned} 0 &= \frac{0}{1} \in \mathbb{Q} & \frac{100}{13} &\in \mathbb{Q} \\ 1 &= \frac{1}{1} \in \mathbb{Q} & \sqrt{2} &\notin \mathbb{Q} \\ \frac{3}{4} &\in \mathbb{Q} & -4 = \frac{8}{-2} &\in \mathbb{Q} \\ \pi &\notin \mathbb{Q} & 0.12345 = \frac{12345}{100000} &\in \mathbb{Q}. \end{aligned}$$

$\mathbb{Q}$ daki sayılarla "+", "−", "×" ve ($\neq 0$ sayılarla) "÷ yapabiliriz".

Şimdi oldu mu?

Hayır!

Neden?

Çünkü rasyonel sayıları bir sayı doğrusu üzerinde gösterirsek, o zaman – şekil 2.1 deki gibi bir sürü rasyonel olmayan sayının karşılık geldiği nokta buluruz . Aslında, $\mathbb{Q}$ da ∞ sayıda delik bulmak mümkündür.

Böylece hala yeni sayılara ihtiyacımız var!

Reel Sayılar

$$\mathbb{R} = \{\text{ondalık olarak yazılabilen sayılar}\}$$

kümesine ***reel sayılar*** kümesi denir. Örneğin

$$\begin{aligned} 0 = 0.0 &\in \mathbb{R} & \frac{100}{13} &= 7.692307\ldots \in \mathbb{R} \\ \frac{23}{99} = 0.232323\ldots &\in \mathbb{R} & \sqrt{2} &= 1.414213\ldots \in \mathbb{R} \\ \frac{3}{4} = 0.75 &\in \mathbb{R} & \frac{123}{999} &= 0.123123\ldots \in \mathbb{R} \\ \pi = 3.141592\ldots &\in \mathbb{R} & \frac{12345}{100000} &= 0.12345 \in \mathbb{R}. \end{aligned}$$

Reel sayılar tamdır – yani bütün reel sayıları sayı eksnınde gösterecek olursak, eksen üzerinde reel sayı karşılık gelmeyen nokta kalmadığını görürüz. Sayfa 9 şekil 2.2i inceleyiniz.

Şimdi tamam mı?

Evet!

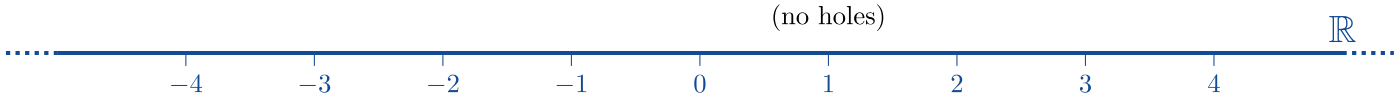


Figure 2.2: The Real Numbers
Şekil 2.2: Reel Sayılar

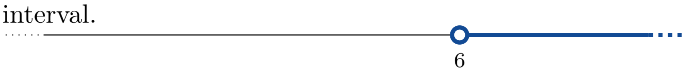
Intervals

A subset of $\mathbb{R}$ is called an ***interval*** if

(i). it contains atleast 2 numbers; and

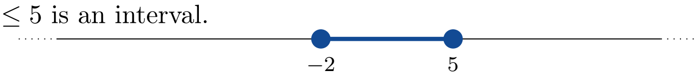
(ii). it doesn't have any holes in it.

Example 2.1. The set $\{x \mid x \text{ is a real number and } x > 6\}$ is an interval.



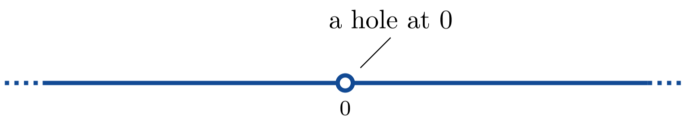
Because 6 is not in this set, we use ○ at 6.

Example 2.2. The set of all real numbers x such that $-2 \leq x \leq 5$ is an interval.



Because -2 and 5 are in this set, we use ● at -2 and 5.

Example 2.3. The set $\{x \mid x \in \mathbb{R} \text{ and } x \neq 0\}$ is not an interval.



A finite interval is

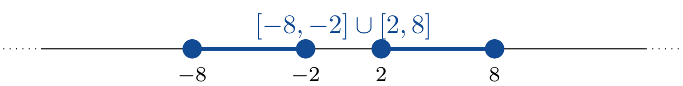
- ***closed*** if it contains both its endpoints;
- ***half-open*** if it contains one of its endpoints;
- ***open*** if it does not contain its endpoints;

as shown in table 2.1 on page 10. An infinite interval is

- ***closed*** if it contains a finite endpoint;
- ***open*** if it is not closed.

There is one exception to this rule: The whole real line is called both open and closed. See table 2.2 on page 10.

We can combine two (or more) intervals with the notation $\cup$. For example, $[-8,-2] \cup [2,8]$ is called the ***union*** of $[-8,-2]$ and $[2,8]$ and is shown below.



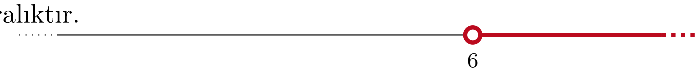
Intervals

$\mathbb{R}$ nin şu iki özelliğini sağlayan bir altkümesine ***aralık*** denir

(i). en az 2 sayı içeriyorsa; ve

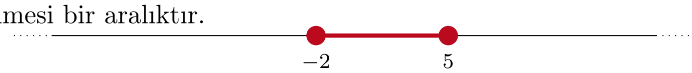
(ii). içerisinde hiç boşluk yoksa.

Örnek 2.1. The set $\{x \mid x \text{ reel sayı ve } x > 6\}$ kümesi bir aralıktır.



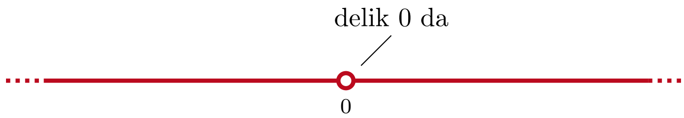
6 bu kümede olmadığından, 6 noktasında ○ olarak gösteririz.

Örnek 2.2. $-2 \leq x \leq 5$ olacak şekildeki tüm x reel sayılarının kümesi bir aralıktır.



-2 ve 5 bu kümede yer aldıklarından, -2 ve 5 noktalarında ● kullanırız.

Örnek 2.3. $\{x \mid x \in \mathbb{R} \text{ ve } x \neq 0\}$ kümesi bir aralık değildir.



Bir sonlu aralık

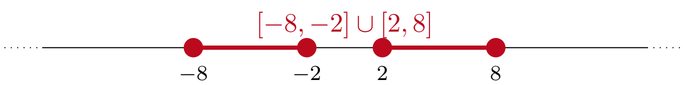
- uç noktalarının her ikisini de içeriyorsa ***kapalı***;
- uç noktalarının birisini içeriyorsa ***yarı-açık***;
- uç noktalarının hiçbirini içermiyorsa, ***açık*** olarak adlandırılır.

10 daki tablo 2.1 gösterilmektedir. Bir sonsuz aralık

- bir sonlu uç noktasını içeriyorsa ***kapalı***;
- kapalı değilse de ***açık*** adını alır.

Bu kuralın bir istisnası vardır: Tüm reel sayı doğrusu hem açık hem kapalıdır. Bakınız sayfa 10 tablo 2.2.

İki (veya daha fazla) aralığı, $\cup$ notasyonu ile birleştirebiliriz. Örneğin $[-8,-2] \cup [2,8]$ 'a $[-8,-2]$ ve $[2,8]$ in ***birleşimi*** denir ve aşağıdaki şekilde gösterilmiştir.



Notation Notasyon	Set Açıklama	Type Tip	Picture Resim
(a, b)	$\{x \mid a < x < b\}$	open / açık	
$[a, b]$	$\{x \mid a \leq x \leq b\}$	closed / kapalı	
$[a, b)$	$\{x \mid a \leq x < b\}$	half open / yarı-açık	
$(a, b]$	$\{x \mid a < x \leq b\}$	half open / yarı-açık	

Table 2.1: Types of Finite Interval
Tablo 2.1: Sonlu Aralık Çeşitleri

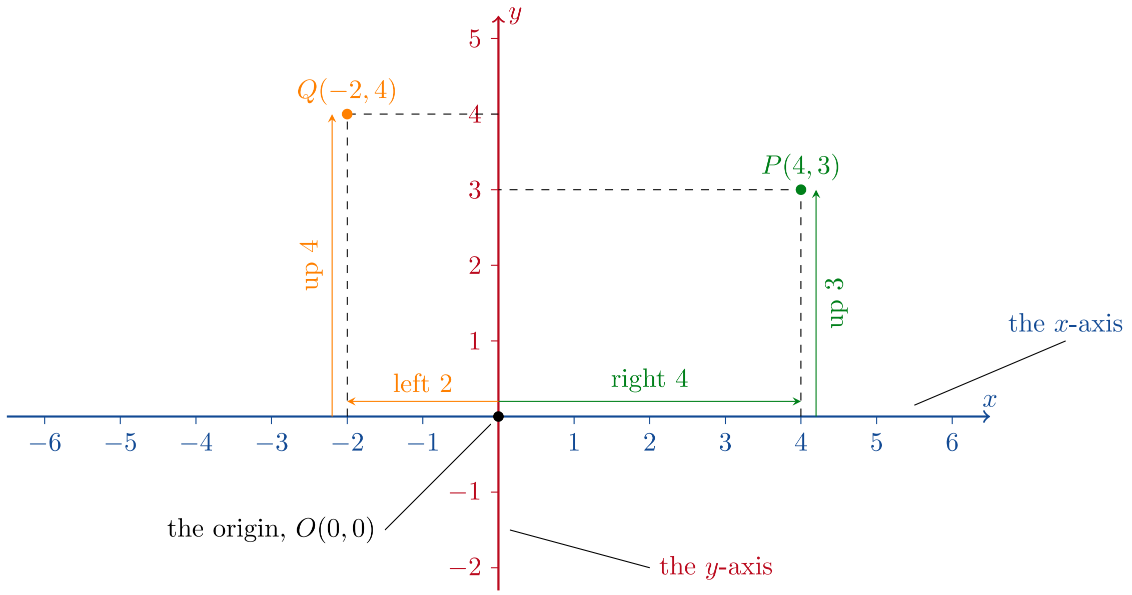
Notation Notasyon	Set Açıklama	Type Tip	Picture Resim
(a, ∞)	$\{x \mid a < x\}$	open / açık	
$[a, \infty)$	$\{x \mid a \leq x\}$	closed / kapalı	
$(-\infty, b)$	$\{x \mid x < b\}$	open / açık	
$(-\infty, b]$	$\{x \mid x \leq b\}$	closed / kapalı	
$(-\infty, \infty)$	$\mathbb{R}$	both open and closed hem açık hem kapalı	

Table 2.2: Types of Infinite Interval
Tablo 2.2: Sonsuz Aralık Çeşitleri

3

Cartesian Coordinates

Kartezyen Koordinatlar



Definition. The set

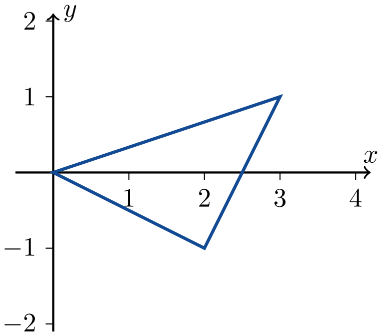
$$\{(x,y)|x,y\in\mathbb{R}\}$$

is denoted by $\mathbb{R}^2$.

Definition. The point $O(0,0)$ is called the ***origin***.

Example 3.1. Let $A(2,-1)$ and $B(3,1)$be points in $\mathbb{R}^2$. Draw the triangle OAB.

solution:



Tanım.

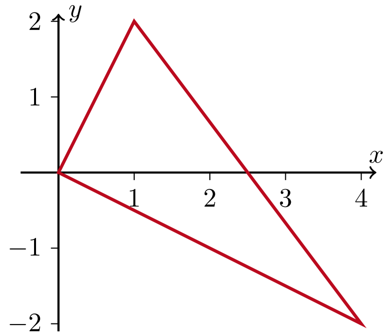
$$\{(x,y)|x,y\in\mathbb{R}\}$$

kümesini $\mathbb{R}^2$ ile gösteririz.

Tanım. $O(0,0)$ noktası ***orijin*** olarak adlandırılır.

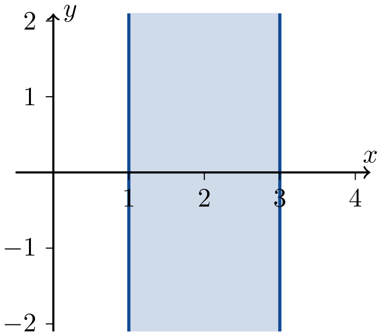
Örnek 3.2. $A(1,2)$ ve $B(4,-2)$, $\mathbb{R}^2$ de noktalar olsun. OAB üçgenini çiziniz.

çözüm:



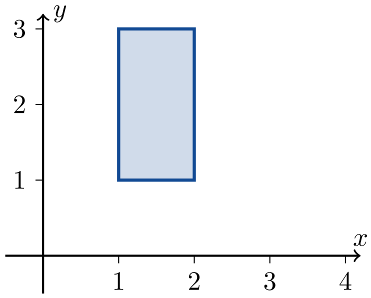
Example 3.3. Draw the region of points which satisfy $1 \leq x \leq 3$.

solution:



Example 3.5. Draw the region of points which satisfy $1 \leq x \leq 2$ and $1 \leq y \leq 3$.

solution:



Distance in $\mathbb{R}^2$.

Definition. The ***distance*** between $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$ is

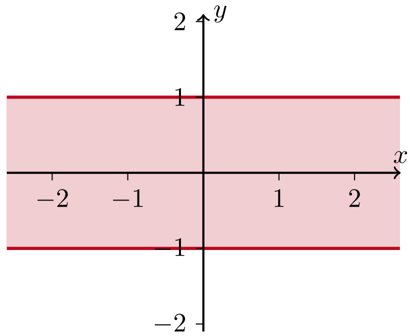
$$\|P_1P_2\| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}.$$

Example 3.7. The distance between $A(1, 3)$ and $B(4, -1)$ is

$$\|AB\| = \sqrt{(4-1)^2 + (-1-3)^2} = \sqrt{3^2 + (-4)^2} = \sqrt{25} = 5.$$

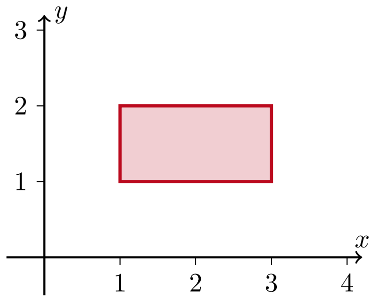
Örnek 3.4. $-1 \leq y \leq 1$ koşulunu sağlayan bölgeyi çiziniz.

çözüm:



Örnek 3.6. $1 \leq x \leq 3$ ve $1 \leq y \leq 2$ eşitsizliklerinin sağladığı bölgeyi çiziniz.

çözüm:



$\mathbb{R}^2$ 'de Uzaklık

Tanım. $P_1(x_1, y_1)$ ve $P_2(x_2, y_2)$ arasındaki ***uzaklık***

$$\|P_1P_2\| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}.$$

Örnek 3.7. $A(1, 3)$ ve $B(4, -1)$ arasındaki uzaklık

$$\|AB\| = \sqrt{(4-1)^2 + (-1-3)^2} = \sqrt{3^2 + (-4)^2} = \sqrt{25} = 5.$$

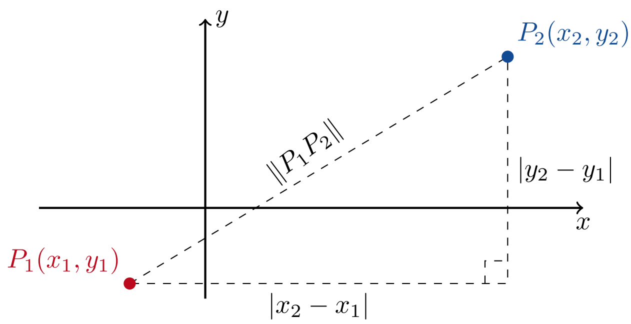


Figure 3.1: The distance between P_1 and P_2 is easy to calculate using Pythagoras.
Şekil 3.1: P_1 ve P_2 arasındaki uzaklık Pisagor bağıntısı kullanarak kolayca elde edilebilir.

Problems

Problem 3.1. Draw the regions of points in $\mathbb{R}^2$ which satisfy each of the following rules:

(a). $-1 \le x \le 2$,

(b). $-2 \le x \le 0$ and $0 \le y \le 2$,

(c). $-1 \le y \le 1$ and $-1 \le x \le 1$,

(d). $3 \le y \le 3$,

Problem 3.2. Let $A(1,1)$, $B(4,2)$ and $C(3,3)$ be points in $\mathbb{R}^2$. Which of the following three numbers is largest?

(i). $\|AB\|$,

(ii). $\|BC\|$,

(iii). $\|CA\|$.

Sorular

Soru 3.1. Draw the regions of points in $\mathbb{R}^2$ which satisfy each of the following rules:

(e). $1 \le x \le 3$ and $y = 1$,

(f). $x = 4$ and $y \ge 0$,

(g). $-2 \le x \le 1$ and $y \le 0$,

(h). $0 \le x \le 1$ and $0 \le y \le 1$.

Soru 3.2. Let $A(1,1)$, $B(4,2)$ and $C(3,3)$ be points in $\mathbb{R}^2$. Which of the following three numbers is largest?

(i). $\|AB\|$,

(ii). $\|BC\|$,

(iii). $\|CA\|$.

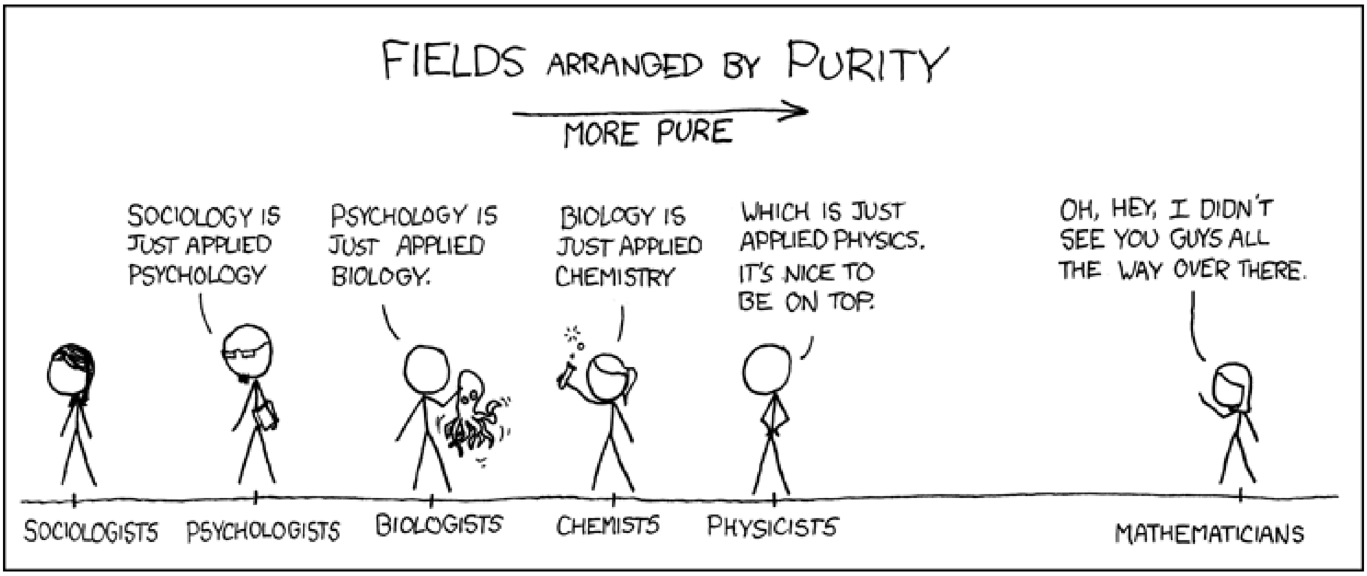


Figure 3.2: A web comic from https://xkcd.com/435/.
Şekil 3.2: https://xkcd.com/435/ adresinden alınan bir web çizgi romanı.

4

Functions

Fonksiyonlar

dependent variable
bağımlı değişken

“y is equal to f of x”

$$y = f(x)$$

“y eşittir f x”

function
fonksiyon

independent variable
bağımsız değişken

Definition. A ***function*** from a set D to a set Y is a rule that assigns a unique element of Y to each element of D.

Definition. The set D of all possible values of x is called the ***domain*** of f.

Definition. The set Y is called the ***target*** of f.

Definition. The set of all possible values of $f(x)$ is called the ***range*** of f.

If f is a function with domain D and target Y, we can write

$$f : D \to Y$$

domain target

Example 4.1. $f : \mathbb{R} \to \mathbb{R},\ f(x) = x^2.$

Example 4.2. $f : (-\infty, \infty) \to [0, \infty),\ f(x) = x^2.$

Tanım. D ve Y boş olmayan iki küme olmak üzere D nin her bir elemanını Y nin sadece bir elemanına eşleyen kurala ***fonksiyon*** denir.

Tanım. D kümesine f nin ***tanım kümesi*** denir.

Tanım. Y kümesine f nin ***değer kümesi*** denir.

Tanım. Bütün mümkün $f(x)$ değerlerinin kümesine f nin ***görüntü kümesi*** denir.

Eğer f tanım kümesi D ve değer kümesi Y olan bir fonksiyon ise, bunu şöyle gösteririz

$$f : D \to Y$$

tanım kümesi değer kümesi

Örnek 4.1. $f : \mathbb{R} \to \mathbb{R},\ f(x) = x^2.$

Örnek 4.2. $f : (-\infty, \infty) \to [0, \infty),\ f(x) = x^2.$

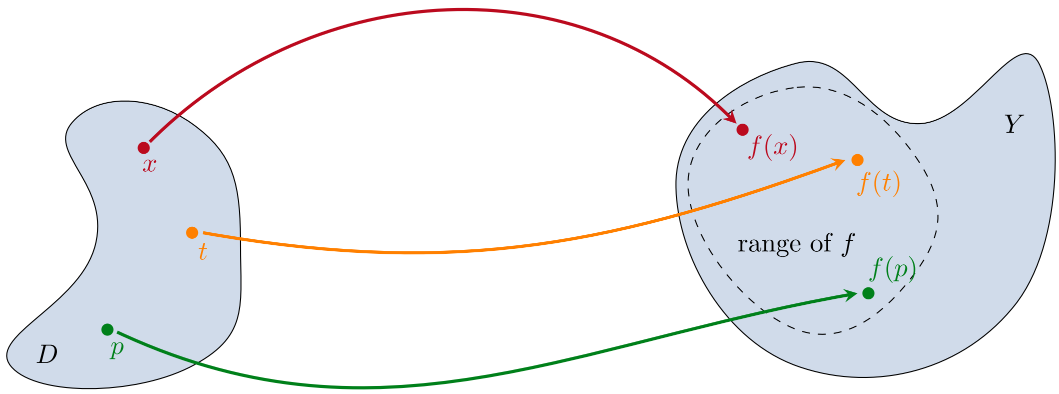


Figure 4.1: A function $f : D \to Y$.
Şekil 4.1: $f : D \to Y$ Bir Fonksiyon.

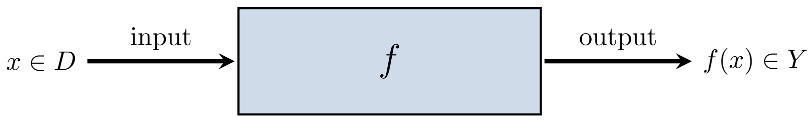


Figure 4.2: A function $f : D \to Y$.
Şekil 4.2: $f : D \to Y$ Bir Fonksiyon.

function fonksiyon	domain (x) tanım kümesi (x)	range (y) görüntü kümesi (y)
$y = x^2$	$(-\infty, \infty)$	$[0, \infty)$
$y = \frac{1}{x}$	$\{x \mid x \in \mathbb{R}, x \neq 0\}$	$\{x \mid x \in \mathbb{R}, x \neq 0\}$
$y = \sqrt{x}$	$[0, \infty)$	$[0, \infty)$
$y = \sqrt{4 - x}$	$(-\infty, 4]$	$[0, \infty)$
$y = \sqrt{1 - x^2}$	$[-1, 1]$	$[0, 1]$
$y = x^2$	$[1, 2]$	$[1, 4]$
$y = x^2$	$[2, \infty)$	$[4, \infty)$
$y = x^2$	$(-\infty, -2]$	$[4, \infty)$
$y = 1 + x^2$	$[1, 3)$	$[2, 10)$
$y = 1 - \sqrt{x}$	$[0, \infty)$	$(-\infty, 1]$

Table 4.1: Domains and ranges of some fuctions.
Tablo 4.1: Bazı fonksiyonların tanım ve görüntü kümeleri.

Graphs of Functions

Definition. The ***graph*** of f is the set containing all the points (x,y) which satisty $y=f(x)$.

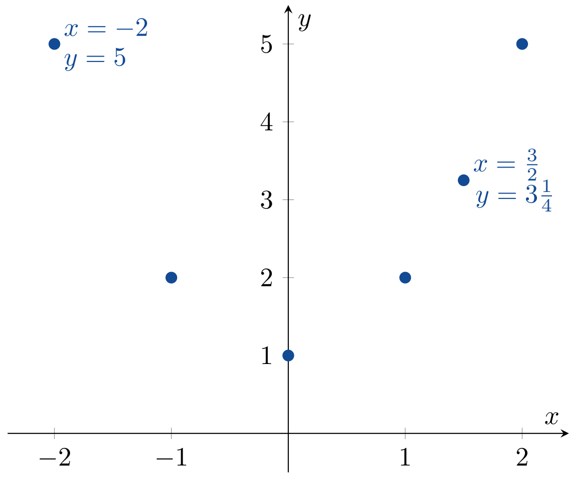
Example 4.3. Graph the function $y=1+x^2$ over the interval $[-2,2]$.

solution:

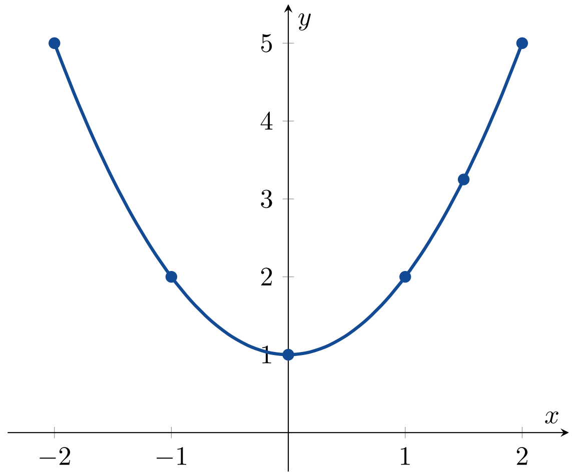
STEP 1. Make a table of (x,y) points which satisfy $y=1+x^2$.

x	y
-2	5
-1	2
0	1
1	2
$\frac{3}{2}$	$\frac{13}{4}=3\frac{1}{4}$
2	5

STEP 2. Plot these points.



STEP 3. Draw a smooth curve through these points.



Fonksiyonların Grafikleri

Tanım. $y=f(x)$ eşitliğini sağlayan (x,y) noktalarının kümesine f nin ***grafiği*** denir.

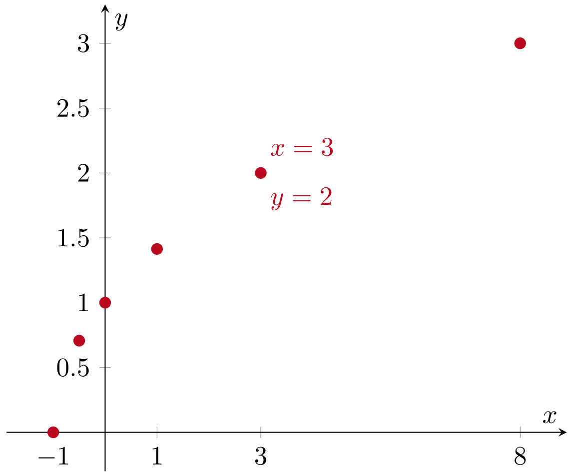
Örnek 4.4. $y=\sqrt{1+x}$ fonksiyonunun $[-1,8]$ aralığındaki grafiğini çiziniz.

çözüm:

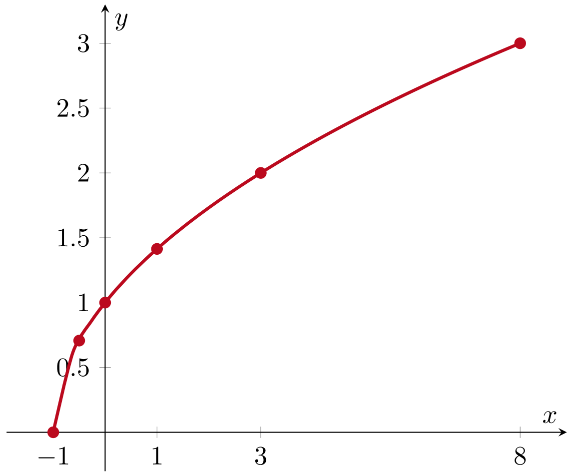
ADIM 1. $y=\sqrt{1+x}$ eşitliğini sağlayan (x,y) noktalarının bir tablosunu yapın.

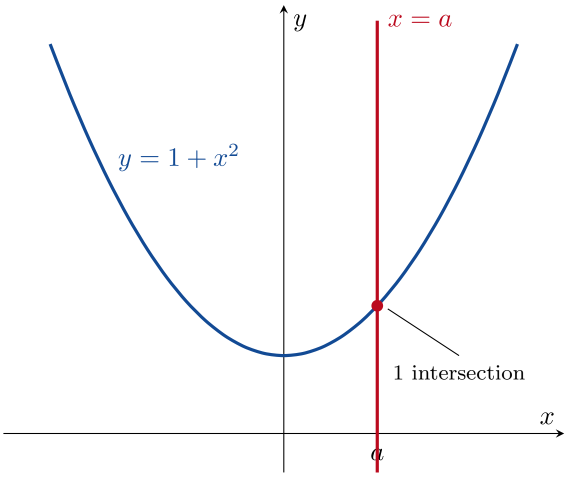
x	y
-1	0
$-\frac{1}{2}$	≈ 0.707
0	1
1	≈ 1.414
3	2
8	3

ADIM 2. Bu noktaları koordinat sisteminde gösterin.



ADIM 3. Bu noktalardan geçen pürüzsüz bir eğri çiziniz.





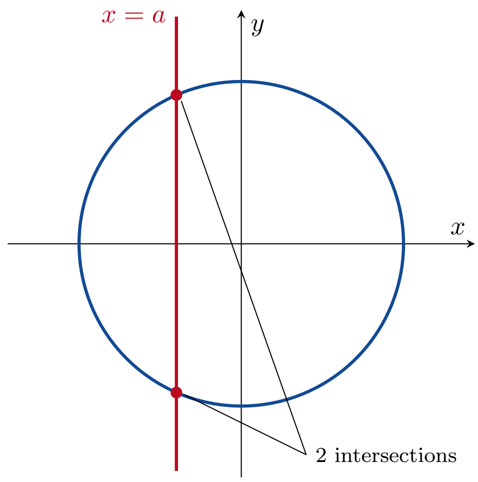


Figure 4.3: The Vertical Line Test.
Şekil 4.3: Dikey Doğru Testi

The Vertical Line Test

Not every curve that you draw is a graph of a function. A function can have only one value $f(x)$ for each $x \in D$. This means that a vertical line can intersect the graph of a function at most once.

See figure 4.3. A circle can not be the graph of a function because some vertical lines intersect the circle at two points.

If $a \in D$, then the vertical line $x = a$ will intersect the graph of $f : D \to Y$ only at the point $(a, f(a))$.

Düşey Doğru Testi

Çizdiğiniz her eğri bir fonksiyonun grafiği değildir. Bir fonksiyon her $x \in D$ için yalnızca bir tane $f(x)$ değerine sahip olabilir. Bu, düşey her doğrunun, bir fonksiyonunun grafiğini en fazla bir kez kesebileceği anlamına gelir.

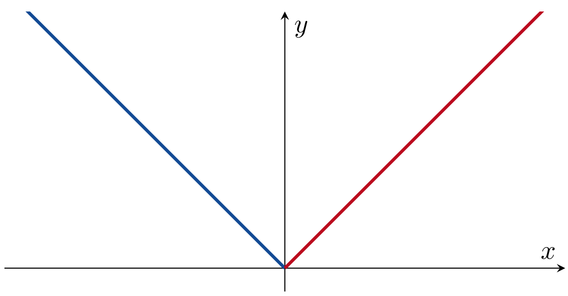
Bakınız şekil 4.3. Bir çember, bir fonksiyonun grafiği olamaz; çünkü bazı düşey doğrular çemberi iki noktada keser.

$a \in D$ ise, $x = a$ düşey doğrusu $f : D \to Y$'nin grafiğini $(a, f(a))$ noktasında kesecektir.

Piecewise-Defined Functions

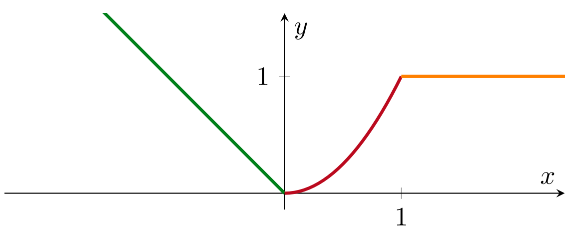
Example 4.5.

$$|x| = \begin{cases} x & x \geq 0 \\ -x & x < 0 \end{cases}$$



Example 4.6.

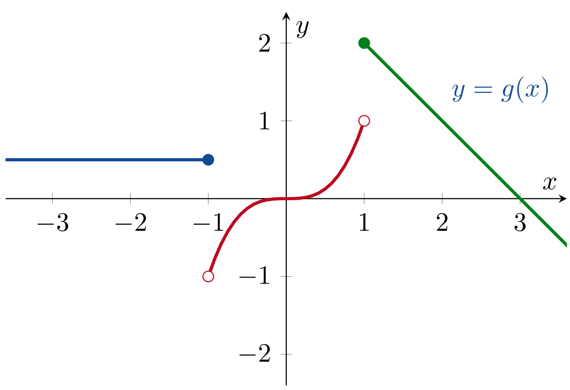
$$f(x) = \begin{cases} -x & x < 0 \\ x^2 & 0 \leq x \leq 1 \\ 1 & x > 1 \end{cases}$$



Parçalı Tanımlı Fonksiyonlar

Örnek 4.7.

$$g(x) = \begin{cases} \frac{1}{2} & x \leq -1 \\ x^3 & -1 < x < 1 \\ 3 - x & x \geq 1 \end{cases}$$



Increasing and Decreasing Functions

Definition. Let I be an interval. Let $f : I \to \mathbb{R}$ be a function.

(i). f is called ***increasing on I*** if

$$f(x_1) < f(x_2)$$

for all $x_1, x_2 \in I$ which satisfy $x_1 < x_2$;

(ii). f is called ***decreasing on I*** if

$$f(x_1) > f(x_2)$$

for all $x_1, x_2 \in I$ which satisfy $x_1 < x_2$.

Artan ve Azalan Fonksiyonlar

Tanım. I bir aralık ve $f : I \to \mathbb{R}$ bir fonksiyon olsun.

(i). her $x_1, x_2 \in I$ için $x_1 < x_2$ iken

$$f(x_1) < f(x_2)$$

oluyorsa f ye ***I da artan*** denir;

(ii). her $x_1, x_2 \in I$ için $x_1 < x_2$ iken

$$f(x_1) > f(x_2)$$

oluyorsa f ye ***I da azalan*** denir.

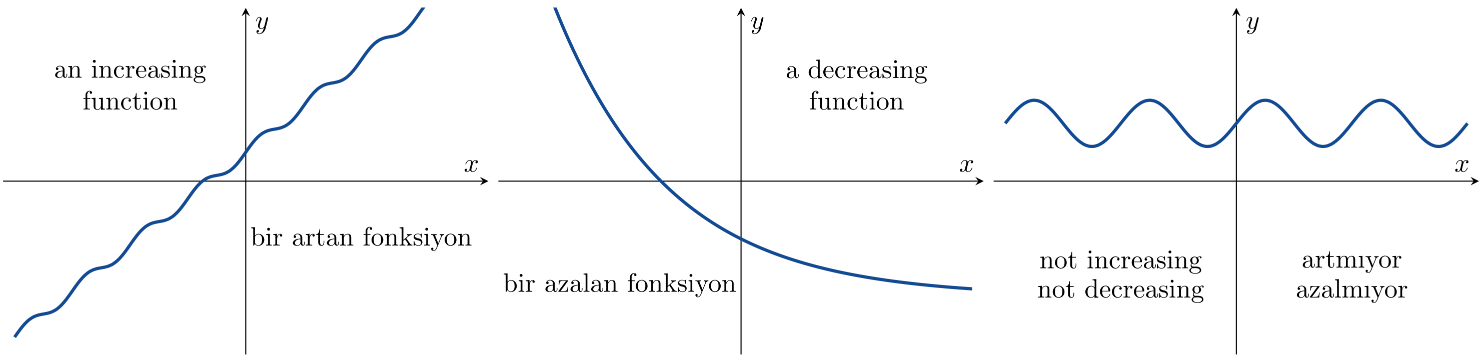


Figure 4.4: A increasing function, a decreasing function and a function which is neither increasing nor decreasing.
Şekil 4.4:

Even Functions and Odd Functions

Recall that

- $2, 4, 6, 8, 10, \dots$ are even numbers; and
- $1, 3, 5, 7, 9, \dots$ are odd numbers.

Definition.

(i). $f : D \to \mathbb{R}$ is an ***even function*** if $f(-x) = f(x)$ for all $x \in D$;

(ii). $f : D \to \mathbb{R}$ is an ***odd function*** if $f(-x) = -f(x)$ for all $x \in D$.

Example 4.8. $f(x) = x^2$ is an even function because

$$f(-x) = (-x)^2 = x^2 = f(x).$$

See figure 4.5.

Example 4.9. $f(x) = x^3$ is an odd function because

$$f(-x) = (-x)^3 = -x^3 = -f(x).$$

See figure 4.6.

Çift Fonksiyonlar ve Tek Fonsiyonlar

Hatırlayalım ki

- $2, 4, 6, 8, 10, \dots$ sayıları çift; ve
- $1, 3, 5, 7, 9, \dots$ sayıları da tek sayılardır.

Tanım.

(i). Bir $f : D \to \mathbb{R}$ fonksiyona her $x \in D$ için $f(-x) = f(x)$ oluyorsa ***çift fonksiyon*** denir ;

(ii). $f : D \to \mathbb{R}$ fonksiyonu her $x \in D$ için $f(-x) = -f(x)$ oluyorsa ***tek fonksiyon*** adını alır.

Örnek 4.8. $f(x) = x^2$ bir çift fonksiyondur çünkü

$$f(-x) = (-x)^2 = x^2 = f(x).$$

Bakınız şekil 4.5.

Örnek 4.9. $f(x) = x^3$ bir tek fonksiyondur çünkü

$$f(-x) = (-x)^3 = -x^3 = -f(x).$$

Bakınız şekil 4.6.

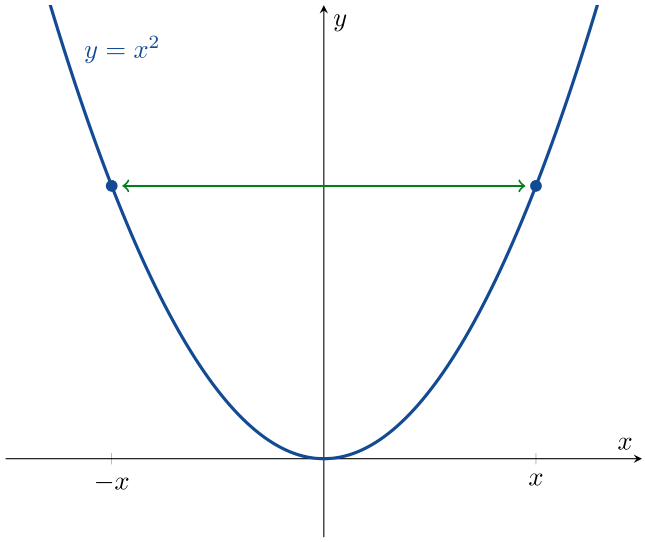


Figure 4.5: 2 is an even number and $f(x) = x^2$ is an even function.
Şekil 4.5: 2 bir çift sayıdır ve $f(x) = x^2$ bir çift fonksiyondur.

Example 4.10. Is $f(x) = x^2 + 1$ even, odd or neither?

solution: Since

$$f(-x) = (-x)^2 + 1 = x^2 + 1 = f(x),$$

f is an even function.

Example 4.11. Is $g(x) = x + 1$ even, odd or neither?

solution: Since $g(-2) = -2 + 1 = -1$ and $g(2) = 3$, we have $g(-2) \neq g(2)$ and $g(-2) \neq -g(2)$. Hence g is neither even nor odd.

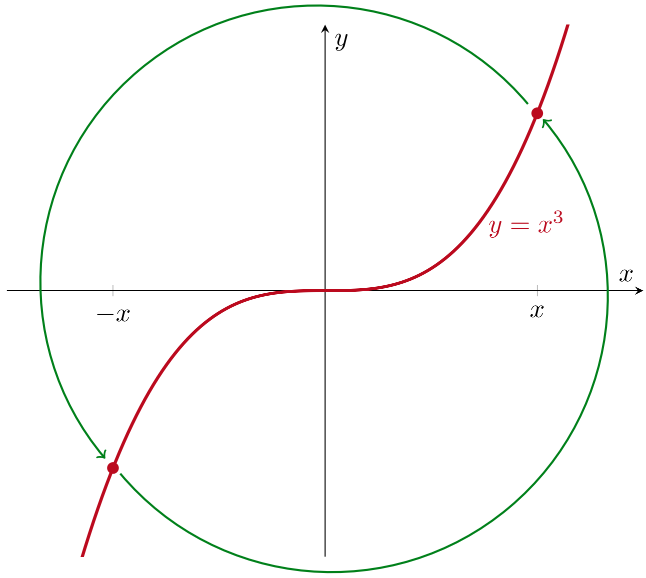


Figure 4.6: 3 is an odd number and $f(x) = x^3$ is an odd function.
Şekil 4.6: 3 bir tek sayıdır ve $f(x) = x^3$ bir tek fonksiyon.

Örnek 4.10. $f(x) = x^2 + 1$ fonksiyonu çift, tek yoksa hiçbiri mi?

çözüm:

$$f(-x) = (-x)^2 + 1 = x^2 + 1 = f(x),$$

f olduğundan bir çift fonksiyondur.

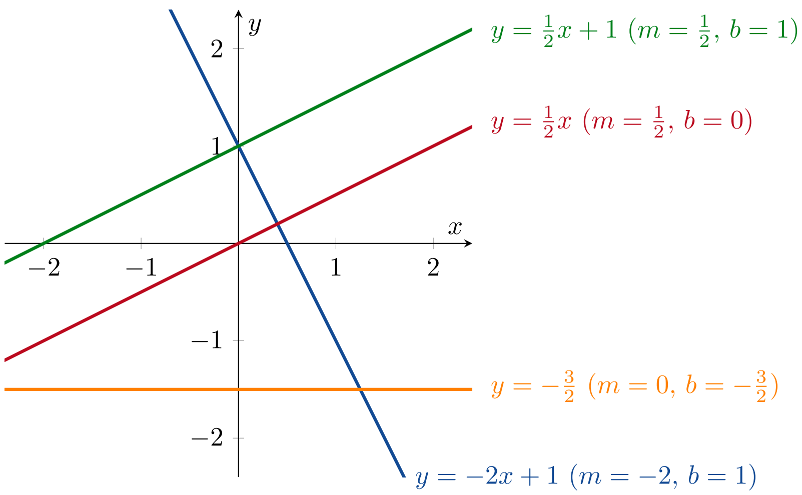
Örnek 4.11. $g(x) = x + 1$ fonksiyonu çift, tek veya hiçbirisi mi?

çözüm: $g(-2) = -2 + 1 = -1$ ve $g(2) = 3$ olduğundan, $g(-2) \neq g(2)$ ve $g(-2) \neq -g(2)$ olur. Böylece g fonksiyonu ne çift fonksiyondur ne de tek.

Linear Functions

Lineer Fonksiyonlar

$$f(x) = mx + b$$
$$(m, b \in \mathbb{R})$$

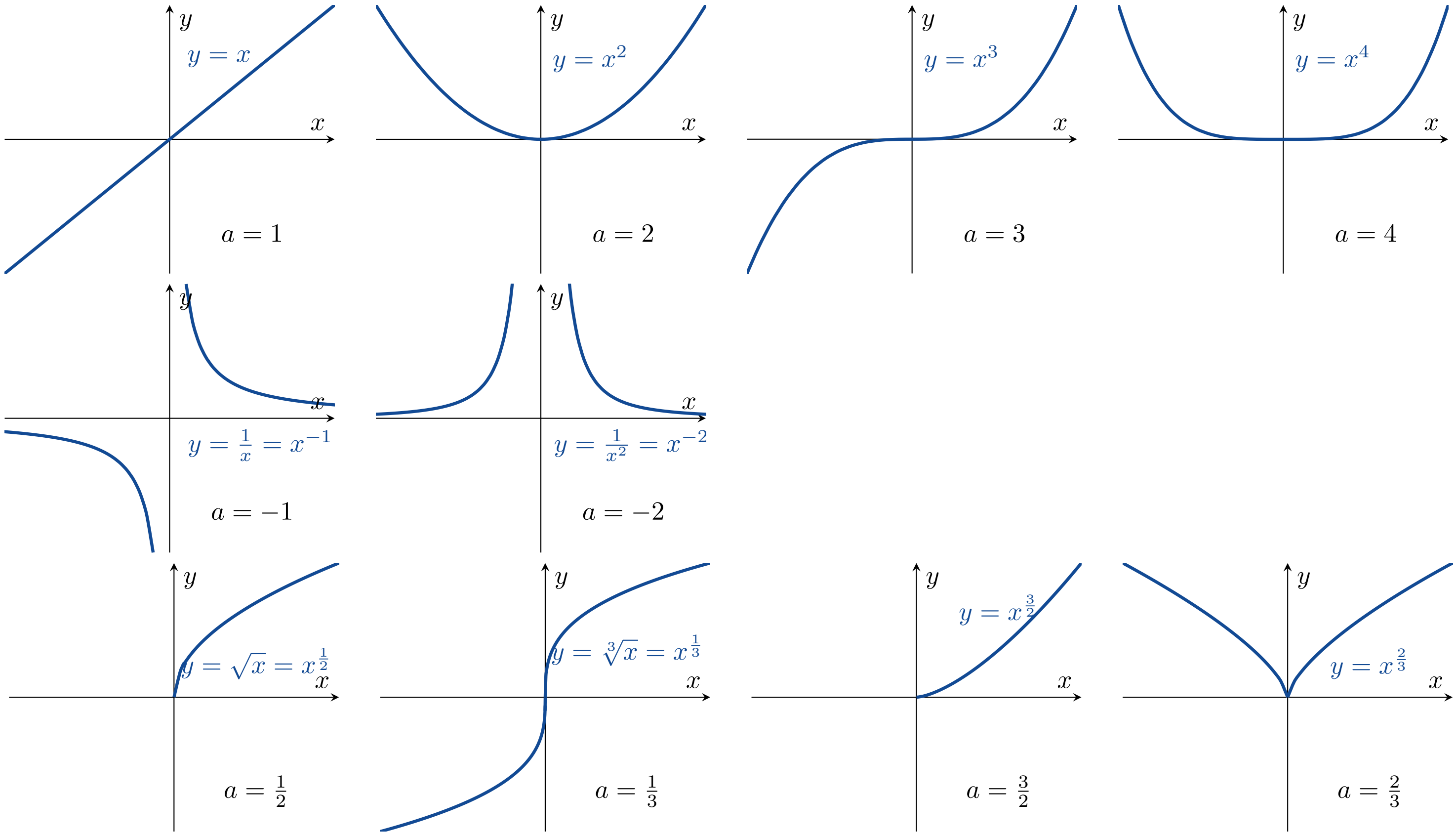


Power Functions

Kuvvet Fonksiyonları

"x to the power of a"

$$f(x) = x^a \qquad (a \in \mathbb{R})$$



Polynomials

Polinomlar

$$p(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_2 x^2 + a_1 x + a_0$$

$(n \in \mathbb{N} \cup \{0\},\ a_j \in \mathbb{R}).$

The domain of a polynomial is always $(-\infty, \infty)$. If $n > 0$ and $a_n \neq 0$, then n is called the ***degree*** of $p(x)$.

Bir polinomun tanım kümesi $(-\infty, \infty)$ dur. $n > 0$ ve $a_n \neq 0$ ise, n tamsayısına $p(x)$ in ***derecesi*** denir.

Rational Functions

Rasyonel Fonsiyonlar

rational function
rasyonel fonksiyon

$$f(x) = \frac{p(x)}{q(x)}$$

polynomial
polinom fonksiyon

Example 4.12.

$$f(x) = \frac{2x^3 - 3}{7x + 4}$$

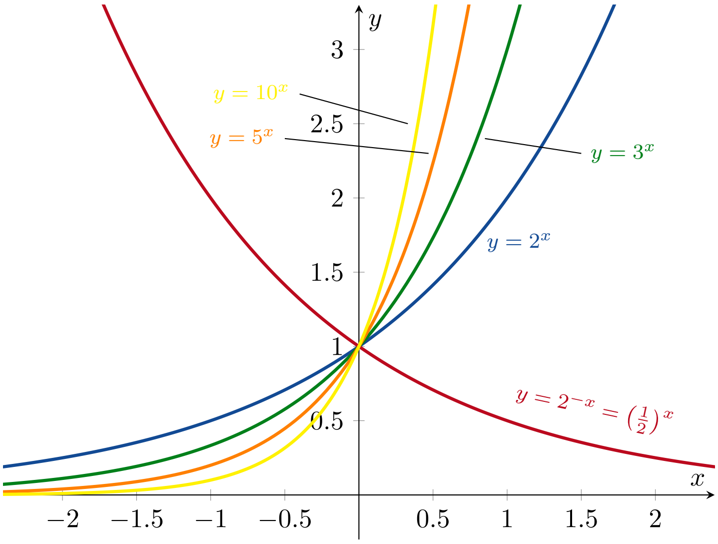
Örnek 4.13.

$$g(x) = \frac{5x^2 + 8x - 3}{3x^2 + 2}$$

Exponential Functions

Üstel Fonksiyonlar

$$f(x) = a^x$$
$$(a \in \mathbb{R},\ a > 0,\ a \neq 1)$$



The domain of an exponential function is $(-\infty, \infty)$.

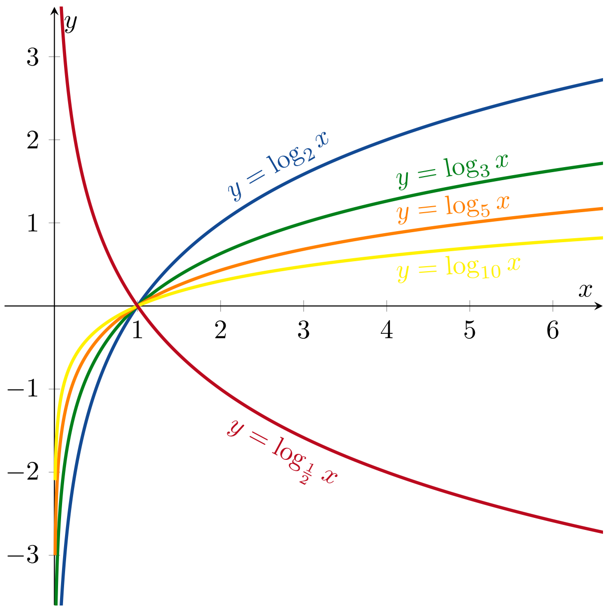
Üstel fonksiyonun tanım kümesi $(-\infty, \infty)$ dur.

Logarithmic Functions

Logaritmik Fonksiyonlar

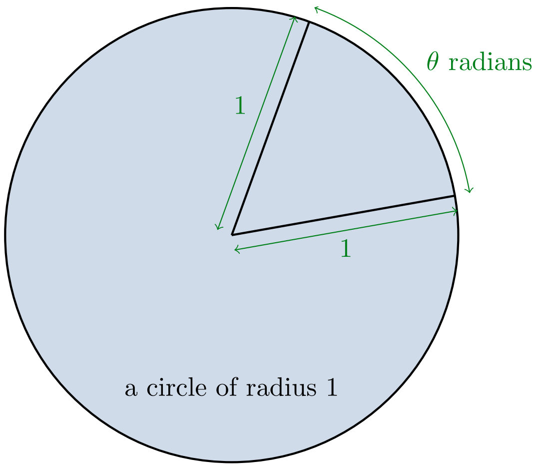
$$y = \log_a x \iff x = a^y$$
$$(a \in \mathbb{R},\ a > 0,\ a \neq 1)$$

"log base a of x"



Angles

There are two ways to measure angles. Using degrees or using radians.



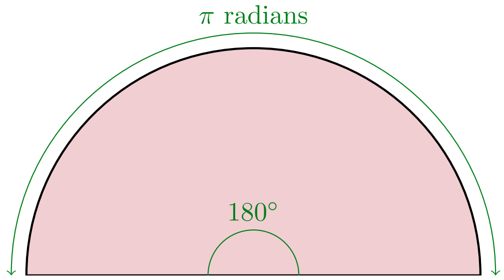
We have that

$$\pi \text{ radians} = 180 \text{ degrees}$$
$$1 \text{ radian} = \frac{180}{\pi} \text{ degrees}$$
$$1 \text{ degree} = \frac{\pi}{180} \text{ radians.}$$

Remark. In Calculus, we use radians!!!! If you see an angle in Part IV of this course, it will be in radians. Calculus doesn't work with degrees!!

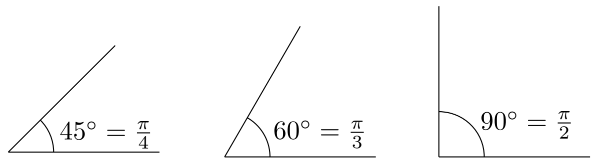
Açılar

Açı ölçmede iki yol vardır. Derece kullanarak veya radyan kullanarak.



Şu bağıntılar mevcuttur.

$$\pi \text{ radyan} = 180 \text{ derece}$$
$$1 \text{ radyan} = \frac{180}{\pi} \text{ derece}$$
$$1 \text{ derece} = \frac{\pi}{180} \text{ radyan.}$$

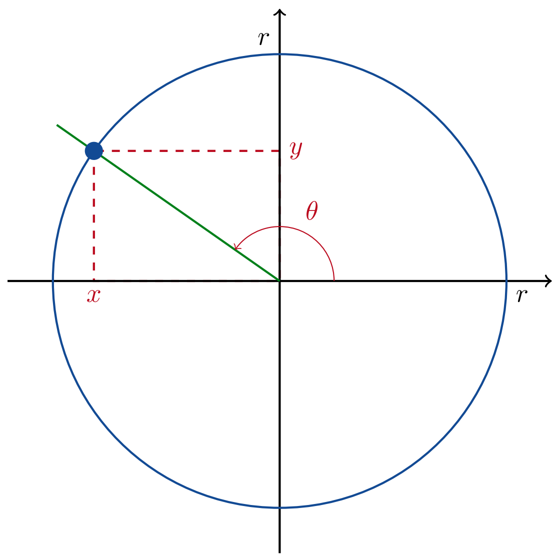


Not. Kalkülüste radyan kullanırız!!!! Bu dersin IV kısmında bir açı görürseniz, o radyan cinsinden olacaktır. Kalkülüste derece kullanmayacağız!!

Trigonometric Functions

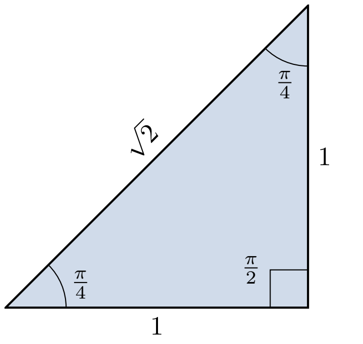
Trigonometrik Fonsiyonlar

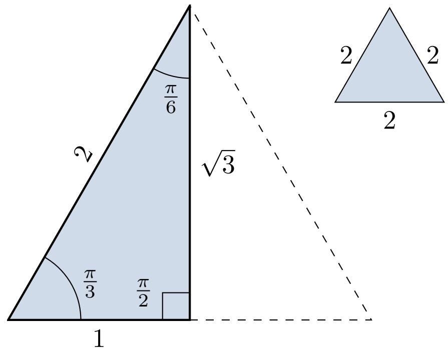
sine	$\sin\theta = \frac{y}{r}$	sinüs
cosine	$\cos\theta = \frac{x}{r}$	kosinüs
tangent	$\tan\theta = \frac{\sin\theta}{\cos\theta}$	tanjant
secant	$\sec\theta = \frac{1}{\cos\theta}$	sekant
cosecant	$\operatorname{cosec}\theta = \csc\theta = \frac{1}{\sin\theta}$	kosekant
cotangent	$\cot\theta = \frac{1}{\tan\theta}$	kotanjant



Remark. Note that $\tan\theta$ and $\sec\theta$ are only defined if $\cos\theta \neq 0$; and $\operatorname{cosec}\theta$ and $\cot\theta$ are only defined if $\sin\theta \neq 0$.

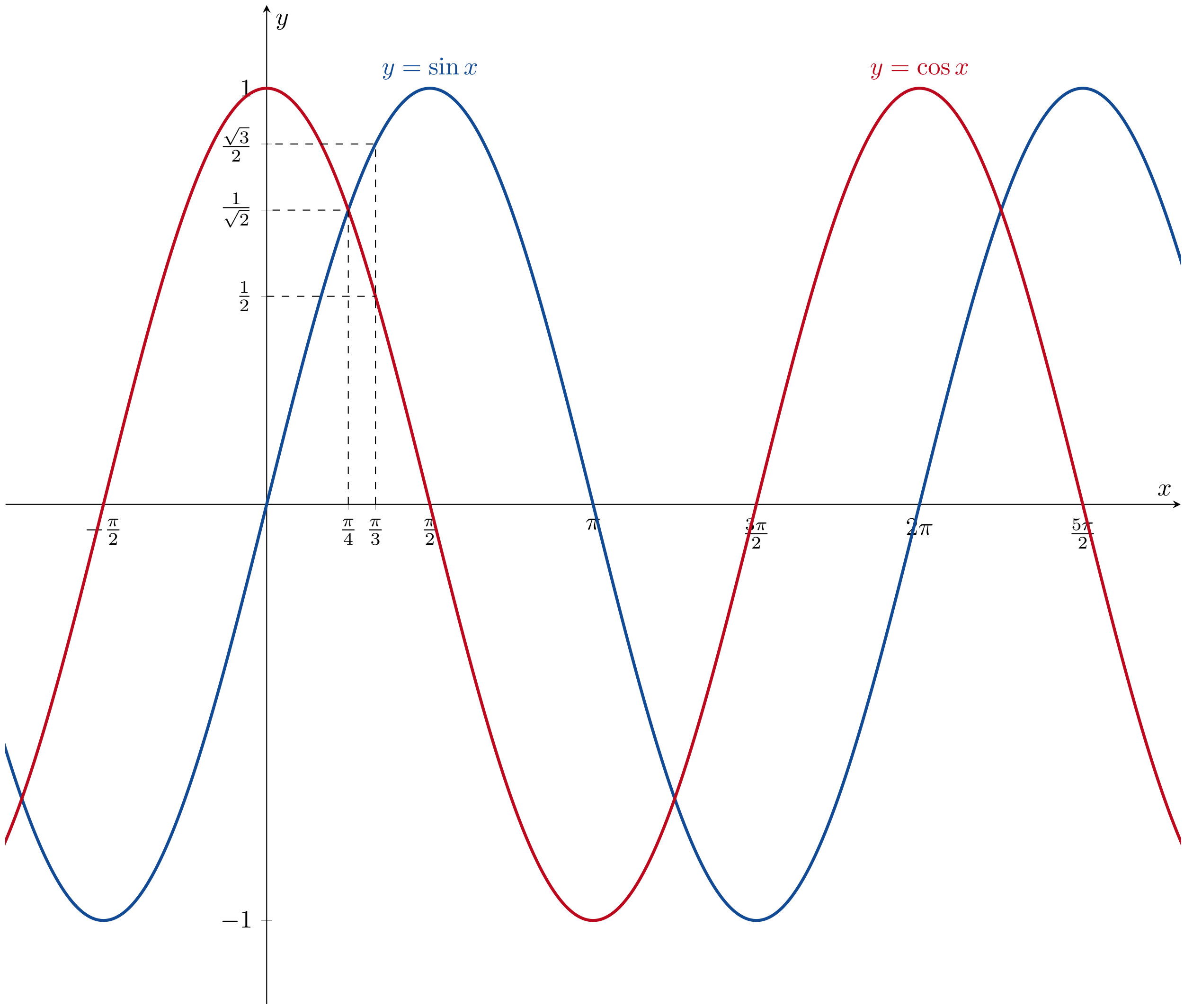
Not. $\tan\theta$ ve $\sec\theta$ nın sadece $\cos\theta \neq 0$ olduğunda; ve $\operatorname{cosec}\theta$ ve $\cot\theta$ nın da tam olarak $\sin\theta \neq 0$ ise tanımlı olduklarına dikkat edin.





$$\sin 45^\circ = \sin\frac{\pi}{4} = \frac{1}{\sqrt{2}}$$
$$\cos 45^\circ = \cos\frac{\pi}{4} = \frac{1}{\sqrt{2}}$$
$$\tan 45^\circ = \tan\frac{\pi}{4} = 1$$
$$\sec 45^\circ = \sec\frac{\pi}{4} = \sqrt{2}$$
$$\operatorname{cosec} 45^\circ = \operatorname{cosec}\frac{\pi}{4} = \sqrt{2}$$
$$\cot 45^\circ = \cot\frac{\pi}{4} = 1$$

$$\sin 60^\circ = \sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}$$
$$\cos 60^\circ = \cos\frac{\pi}{3} = \frac{1}{2}$$
$$\tan 60^\circ = \tan\frac{\pi}{3} = \sqrt{3}$$
$$\sec 60^\circ = \sec\frac{\pi}{3} = 2$$
$$\operatorname{cosec} 60^\circ = \operatorname{cosec}\frac{\pi}{3} = \frac{2}{\sqrt{3}}$$
$$\cot 60^\circ = \cot\frac{\pi}{3} = \frac{1}{\sqrt{3}}$$



Please see
Lütfen bkz.
https://tinyurl.com/ocd35mf

Problems

Problem 4.1 (Even and Odd Functions). State whether the following functions are even, odd or neither.

(a) $f(x) = 3$
(b) $f(x) = x^{77}$
(c) $f(x) = x^2 + 1$
(d) $f(x) = x^3 + x$
(e) $f(x) = x^3 + x^2$
(f) $f(x) = x^3 + 1$
(g) $f(x) = \frac{1}{x^2-1}$
(h) $f(x) = \frac{1}{x^2+1}$
(i) $f(x) = \frac{1}{x-1}$
(j) $f(x) = \sin x$
(k) $f(x) = 2x + 1$
(l) $f(x) = \cos x$

Problem 4.2 (Pointwise-Defined Functions). Graph the function $g : \mathbb{R} \to \mathbb{R}$ defined by

$$g(x) = \begin{cases} \frac{1}{x}, & x < 0 \\ x & x \geq 0. \end{cases}$$

Problem 4.3 (Rational Functions). Graph the following three functions on the same axes:

(a). $f : (0, \infty) \to \mathbb{R}$, $f(x) = x$;

(b). $g : (0, \infty) \to \mathbb{R}$, $g(x) = \frac{1}{x}$;

(c). $h : (0, \infty) \to \mathbb{R}$, $h(x) = x + \frac{1}{x}$.

Problem 4.4 (Angles). Convert the following angles into radians:

(a). $-90°$,
(b). $135°$,
(c). $120°$,
(d). $180°$,
(e). $36°$,
(f). $20°$.

Convert the following angles into degrees:

(g). $\frac{3\pi}{2}$ radians,
(h). $\frac{\pi}{10}$ radians,
(i). $\frac{\pi}{6}$ radians,
(j). $\frac{5\pi}{6}$ radians,
(k). $\frac{-\pi}{5}$ radians,
(l). 3π radians.

Problem 4.5 (Domains). Give the largest possible set of real numbers on which each of the following functions is defined:

(a). $a(x) = 1 + x^2$,
(b). $b(x) = 1 - \sqrt{x}$,
(c). $c(x) = \sqrt{5x + 10}$,
(d). $d(x) = \sqrt{x^2 - 3x}$,
(e). $e(x) = \frac{4}{3-x}$,
(f). $f(x) = \frac{2}{x^2-16}$.

Sorular

Soru 4.1 (Tek ve Çift Fonksiyonlar). Aşağıdaki fonksiyonların çift, tek veya hiçbirisi olup olmadığını bulunuz.

(a). $f(x) = 3$
(b). $f(x) = x^{77}$
(c). $f(x) = x^2 + 1$
(d). $f(x) = x^3 + x$
(e). $f(x) = x^3 + x^2$
(f). $f(x) = x^3 + 1$
(g). $f(x) = \frac{1}{x^2-1}$
(h). $f(x) = \frac{1}{x^2+1}$
(i). $f(x) = \frac{1}{x-1}$
(j). $f(x) = \sin x$
(k). $f(x) = 2x + 1$
(l). $f(x) = \cos x$

Soru 4.2 (Parçalı-Tanımlı Fonksiyonlar).

$$g(x) = \begin{cases} \frac{1}{x}, & x < 0 \\ x & x \geq 0. \end{cases}$$

ile tanımlı $g : \mathbb{R} \to \mathbb{R}$ fonksiyonunun grafiğini çiziniz.

Soru 4.3 (Rayonel Fonksiyonlar). Aşağıdaki üç fonksiyonun grafiğini aynı koordinat düzleminde çiziniz:

(a). $f : (0, \infty) \to \mathbb{R}$, $f(x) = x$;

(b). $g : (0, \infty) \to \mathbb{R}$, $g(x) = \frac{1}{x}$;

(c). $h : (0, \infty) \to \mathbb{R}$, $h(x) = x + \frac{1}{x}$.

Soru 4.4 (Açılar). Convert the following angles into radians:

(a). $-90°$,
(b). $135°$,
(c). $120°$,
(d). $180°$,
(e). $36°$,
(f). $20°$.

Convert the following angles into degrees:

(g). $\frac{3\pi}{2}$ radians,
(h). $\frac{\pi}{10}$ radians,
(i). $\frac{\pi}{6}$ radians,
(j). $\frac{5\pi}{6}$ radians,
(k). $\frac{-\pi}{5}$ radians,
(l). 3π radians.

Soru 4.5 (Tanım Kümeleri). Give the largest possible set of real numbers on which each of the following functions is defined:

(a). $a(x) = 1 + x^2$,
(b). $b(x) = 1 - \sqrt{x}$,
(c). $c(x) = \sqrt{5x + 10}$,
(d). $d(x) = \sqrt{x^2 - 3x}$,
(e). $e(x) = \frac{4}{3-x}$,
(f). $f(x) = \frac{2}{x^2-16}$.

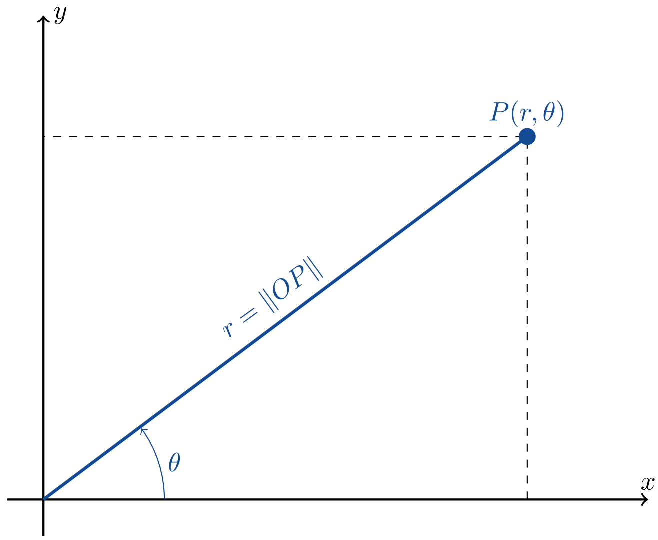
Part II

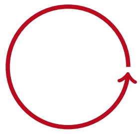
The Geometry of Space

5

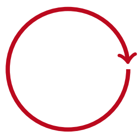
Polar Coordinates

Kutupsal Koordinatlar

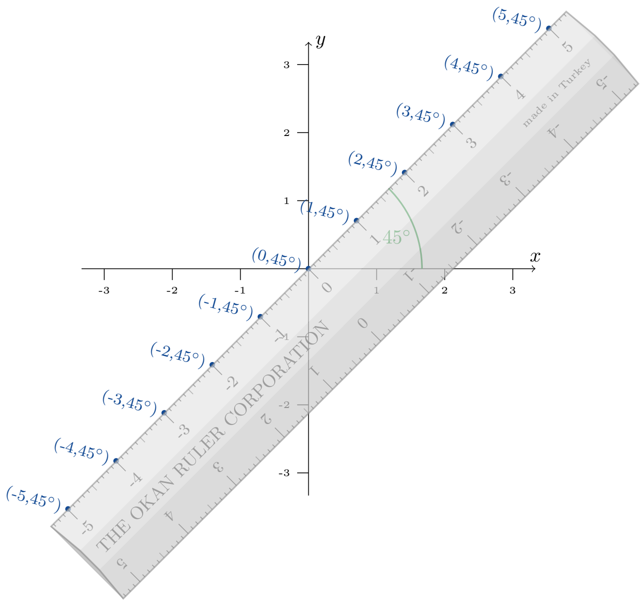


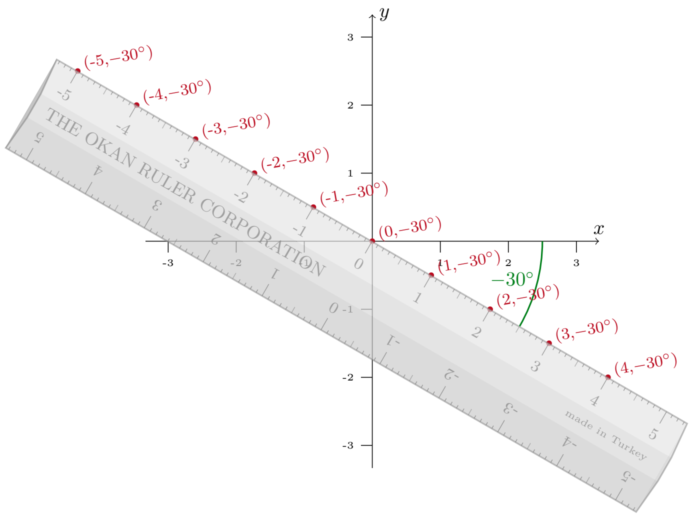


anticlockwise = positive angle
saat yönünün tersi = pozitif açı

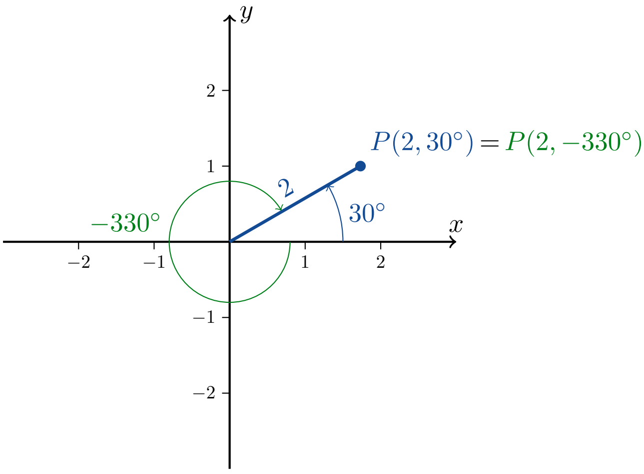


clockwise = negative angle
saat yönünde = negatif açı





Example 5.1.



Example 5.3. Find all the polar coordinates of $P(2, 30°)$.

solution: We can have either $r = 2$ or $r = -2$. For $r = 2$, we can have

$$\theta = 30°, 30° \pm 360°, 30° \pm 720°, 30° \pm 1080°, \ldots$$

For $r = -2$, we can have

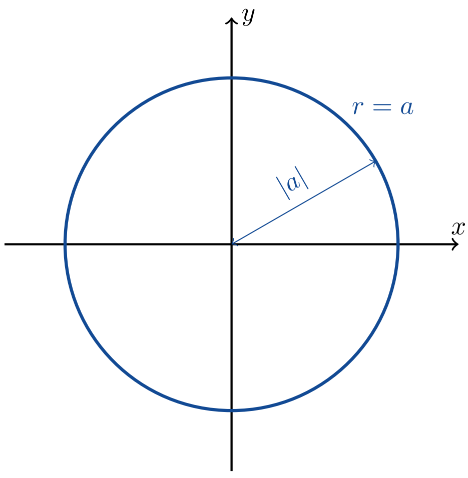
$$\theta = 210°, 210° \pm 360°, 210° \pm 720°, 210° \pm 1080°, \ldots$$

Therefore

$$P(2, 30°) = P\big(2, (30 + 360n)°\big) = P\big(-2, (210 + 360m)°\big)$$

for all $m, n \in \mathbb{Z}$.

Example 5.4.

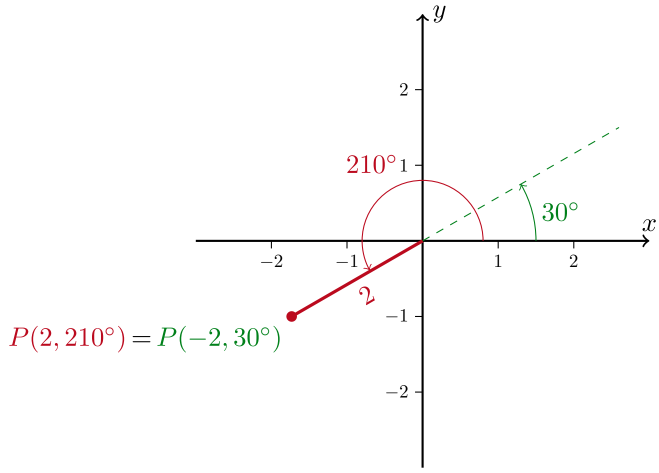


Example 5.6.

(a). $r = 1$ and $r = -1$ are both equations for a circle of radius 1 centred at the origin.

(b). $\theta = 30°$, $\theta = 210°$ and $\theta = -150°$ are all equations for the same line.

Örnek 5.2.



Örnek 5.3. $P(2, 30°)$ noktasının tüm kutupsal koordinatlarını bulunuz.

çözüm: Ya $r = 2$ ya da $r = -2$ olmalıdır. $r = 2$ ise,

$$\theta = 30°, 30° \pm 360°, 30° \pm 720°, 30° \pm 1080°, \ldots$$

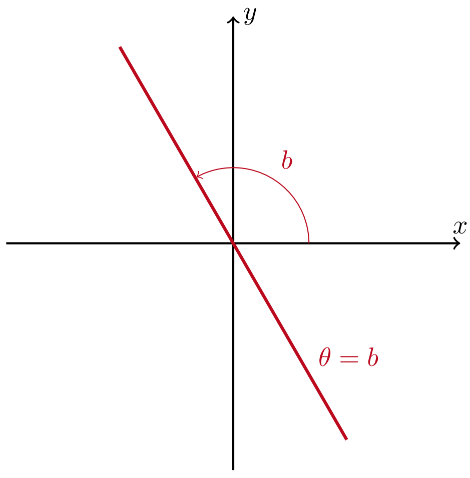
olmalıdır. $r = -2$ olduğunda ise,

$$\theta = 210°, 210° \pm 360°, 210° \pm 720°, 210° \pm 1080°, \ldots$$

olmalıdır. Böylece her $m, n \in \mathbb{Z}$ için,

$$P(2, 30°) = P\big(2, (30 + 360n)°\big) = P\big(-2, (210 + 360m)°\big).$$

Örnek 5.5.



Örnek 5.6.

(a). $r = 1$ ve $r = -1$ her ikisi merkezi orijin yarıçapı 1 olan çemberin denklemleridir.

(b). $\theta = 30°$, $\theta = 210°$ ve $\theta = -150°$ herbiri aynı doğruya ait denklemlerdir.

Example 5.7. Draw the sets of points whose polar coordinates satisfy the following:

(a). $1 \leq r \leq 2$ and $0 \leq \theta \leq 90°$.

(b). $-3 \leq r \leq 2$ and $\theta = 45°$.

(c). $r \leq 0$ and $\theta = 60°$.

(d). $120° \leq \theta \leq 150°$.

solution:

Örnek 5.7. Polar koordinatları aşağıdakileri sağlayan noktaları çiziniz:

(a). $1 \leq r \leq 2$ ve $0 \leq \theta \leq 90°$.

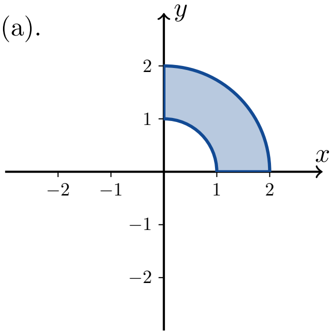
(b). $-3 \leq r \leq 2$ ve $\theta = 45°$.

(c). $r \leq 0$ ve $\theta = 60°$.

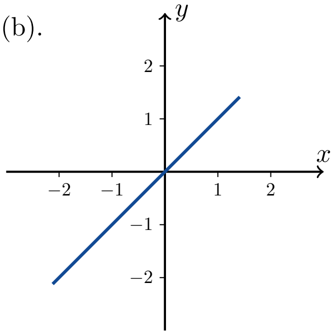
(d). $120° \leq \theta \leq 150°$.

çözüm:

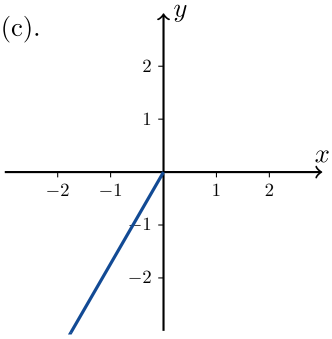
(a).

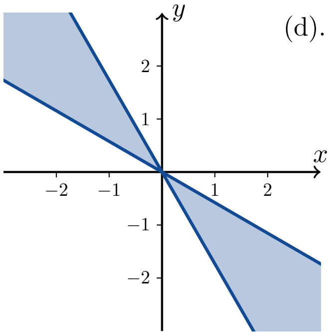


(b).



(c).





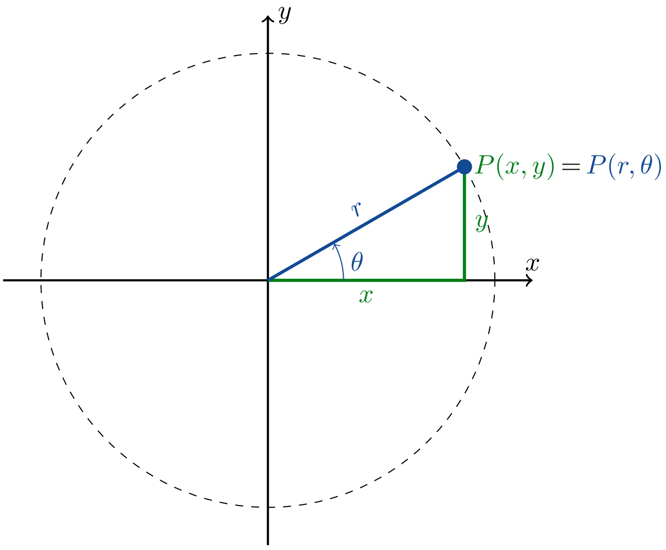
(d).

Relating Polar and Cartesian Coordinates

Kutupsal ve Kartezyen Koordinatlar Arasındaki İlişki

$$x = r\cos\theta \qquad x^2 + y^2 = r^2$$

$$y = r\sin\theta \qquad \tan\theta = \frac{y}{x}$$



Example 5.8. Convert the polar coordinates $(r, \theta) = (-3, 90°)$ into Cartesian coordinates.

solution:

$$(x, y) = (r\cos\theta, r\sin\theta) = (-3\cos 90°, -3\sin 90°) = (0, -3).$$

Örnek 5.8. $(r, \theta) = (-3, 90°)$ kutupsal koordinatlarını kartezyen koordinatlarına dönüştürünüz.

çözüm:

$$(x, y) = (r\cos\theta, r\sin\theta) = (-3\cos 90°, -3\sin 90°) = (0, -3).$$

Example 5.9. Find polar coordinates for the Cartesian coordinates $(x, y) = (5, -12)$.

solution: Choosing $r > 0$, we calculate that

$$r = \sqrt{x^2 + y^2} = \sqrt{25 + 144} = \sqrt{169} = 13.$$

To find θ we use the equation $y = r \sin\theta$ to calculate that

$$\theta = \sin^{-1} \frac{y}{r} = \sin^{-1} \frac{-12}{13} \approx -67.38°.$$

Therefore

$$(r, \theta) = (13, -67.38°).$$

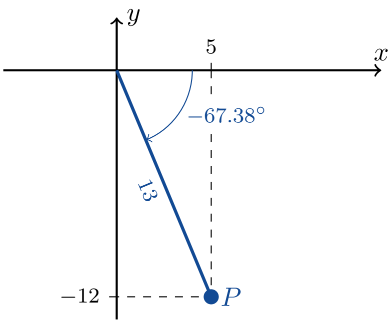


Figure 5.1: The point P has Cartesian coordinates $(x, y) = (5, -12)$ and polar coordinates $(r, \theta) = (13, -67.38°)$
Şekil 5.1:

Örnek 5.9. $(x, y) = (5, -12)$ noktasının kutupsal koordinatlarını bulunuz.

çözüm: $r > 0$ alarak,

$$r = \sqrt{x^2 + y^2} = \sqrt{25 + 144} = \sqrt{169} = 13$$

buluruz. Şimdi θ'yı bulmak için $y = r \sin\theta$ denklemi kullanılır ve

$$\theta = \sin^{-1} \frac{y}{r} = \sin^{-1} \frac{-12}{13} \approx -67.38°$$

elde edilir. Dolayısıyla

$$(r, \theta) = (13, -67.38°).$$

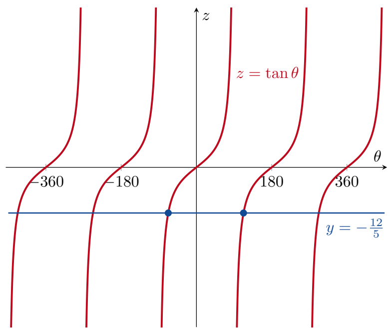


Figure 5.2: The graph of $z = \tan\theta$. Note that $\tan -67.38° = -\frac{12}{5} = \tan 112.62°$.
Şekil 5.2: $z = \tan\theta$ gragiği gösterilmektedir. Dikkat edilirse, $\tan -67.38° = -\frac{12}{5} = \tan 112.62°$.

Problems

Problem 5.1. Convert the following polar coordinates to Cartesian coordinates.

(a). $(3, 0)$
(b). $(-3, 0)$
(c). $(2, 120°)$
(d). $(2, 420°)$
(e). $(2, 60°)$
(f). $(-3, 360°)$
(g). $(-2, -60°)$
(h). $(1, 180°)$
(i). $(2\sqrt{2}, 45°)$.

Problem 5.2. Find polar coordinates for each of the following sets of Cartesian coordinates.

(a). $(1, 1)$
(b). $(-3, 0)$
(c). $(\sqrt{3}, -1)$
(d). $(-3, 4)$
(e). $(-2, -2)$
(f). $(-\sqrt{3}, 1)$.

Problem 5.3. Draw the sets of points whose polar coordinates satisfy the following:

(a). $r = 2$
(b). $0 \le r \le 2$
(c). $r \ge 2$
(d). $0 \le \theta \le 30°$ & $r \ge 0$
(e). $\theta = 120°$ & $r \le -2$
(f). $0 \le \theta \le 90°$ & $1 \le |r| \le 2$
(g). $45° \le \theta \le 315°$ & $1 \le r \le 2$
(h). $-45° \le \theta \le 45°$ & $1 \le r \le 2$
(i). $-45° \le \theta \le 45°$ & $-2 \le r \le 1$

Sorular

Soru 5.1. Aşağıdaki kutupsal koordinatları Kartezyen koordinatlara dönüştürünüz.

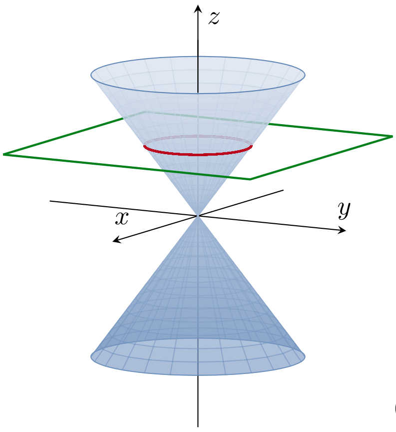
Soru 5.2. Aşağıdaki Kartezyen koordinatların herbiri için bir kutupsal koordinat bulunuz.

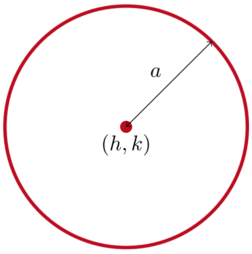
Soru 5.3. Kutupsal koordinatları aşağıdakilari sağlayan noktaların kümesini çiziniz:

6

Conic Sections

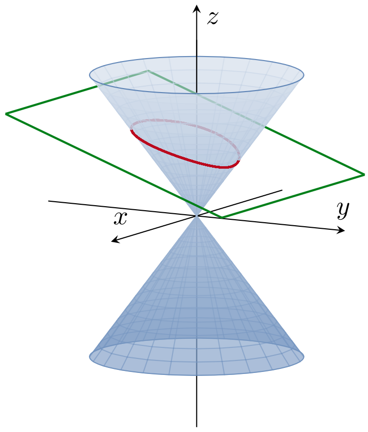
Konik Kesitler

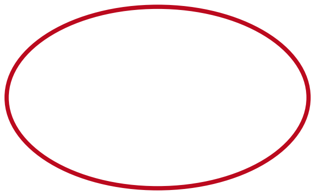




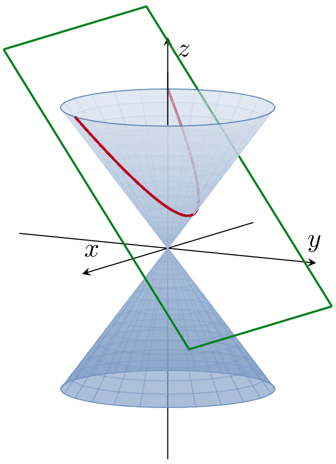
a circle
çember

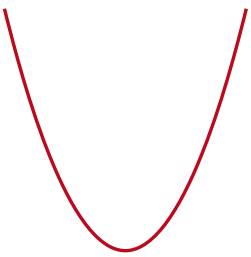
$$(x-h)^2 + (y-k)^2 = a^2$$



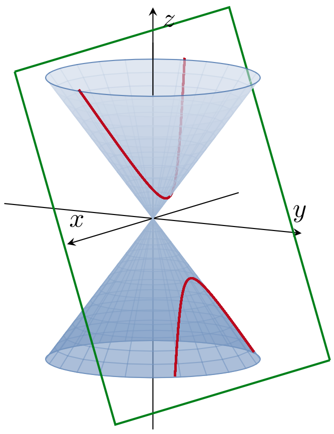


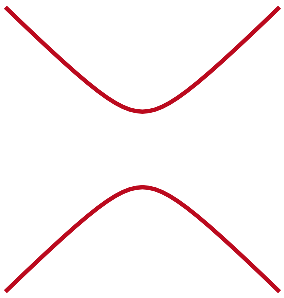
an ellipse
elips





a parabola
parabol





a hyperbola
hiperbol

Parabolas

Figure 6.1: Clifton suspension bridge, Bristol, UK. The cables of a suspension bridges hang in a shape which is almost (but not exactly) a parabola.
Şekil 6.1: Clifton süspansiyon köprüsü, Bristol, Birleşik Krallık. Asma köprülerin halatları, neredeyse (ama tam olarak değil) bir parabol biçiminde asılı durmakta.

To describe a parabola, we need a point called a ***focus*** and a line called a ***directrix***. See figure 6.2.

Definition. A point $P(x, y)$ lies on the ***parabola*** if and only if

$$\|PF\| = \|PQ\|.$$

Now

$$\begin{aligned}\|PF\| &= \text{distance between } P(x,y) \text{ and } F(0,p)\\ &= \sqrt{(x-0)^2+(y-p)^2} = \sqrt{x^2+(y-p)^2}\end{aligned}$$

and

$$\begin{aligned}\|PQ\| &= \text{distance between } P(x,y) \text{ and } Q(x,-p)\\ &= \sqrt{(x-x)^2+(y+p)^2} = \sqrt{(y+p)^2} = y+p.\end{aligned}$$

Therefore

$$\begin{aligned}\|PF\| &= \|PQ\|\\ \sqrt{x^2+(y-p)^2} &= y+p\\ x^2+(y-p)^2 &= (y+p)^2\\ x^2+y^2-2py+p^2 &= y^2+2py+p^2\\ x^2-2py &= 2py\\ &\boxed{x^2 = 4py}\end{aligned}$$

Paraboller

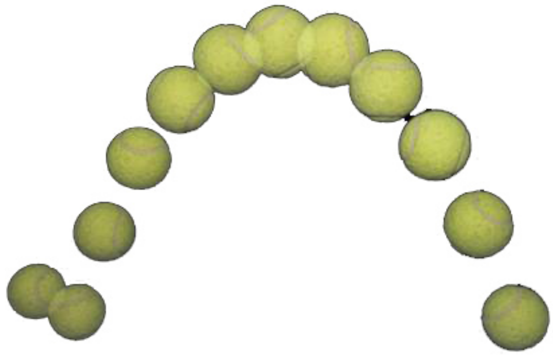


Figure 6.3: The motion of a tennis ball.
Şekil 6.3: Bir tenis topunun hareketi.

Figure 6.4: Satellite dishes.
Şekil 6.4: Uydu antenleri.

Bir parabolü tanımlamak için, ***odak*** adı verilen bir noktaya ve ***doğrultman*** adı verilen bir doğruya ihtiyaç var. Bkz şekil 6.2.

Tanım. Bir $P(x, y)$ noktası bir ***parabol*** üzerindedir ancak ve ancak

$$\|PF\| = \|PQ\|.$$

Şimdi

$$\begin{aligned}\|PF\| &= P(x,y) \text{ ile } F(0,p) \text{ arasındaki uzaklık}\\ &= \sqrt{(x-0)^2+(y-p)^2} = \sqrt{x^2+(y-p)^2}\end{aligned}$$

ve

$$\begin{aligned}\|PQ\| &= P(x,y) \text{ ile } Q(x,-p) \text{ arasındaki uzaklık}\\ &= \sqrt{(x-x)^2+(y+p)^2} = \sqrt{(y+p)^2} = y+p.\end{aligned}$$

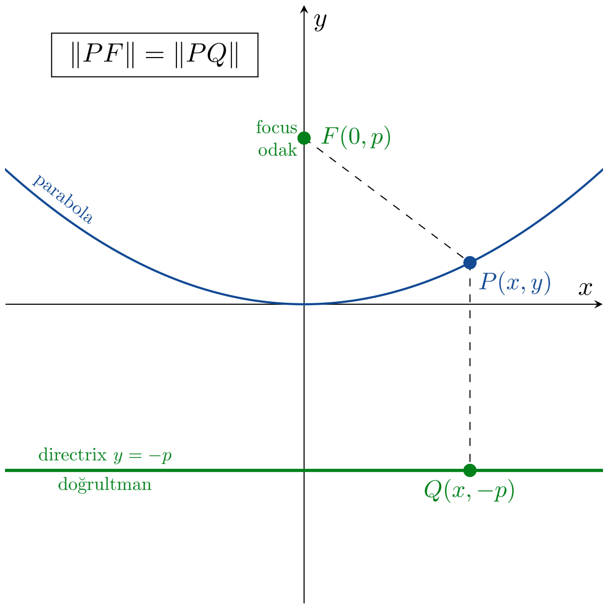
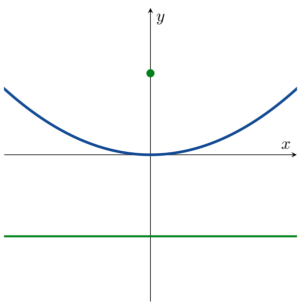
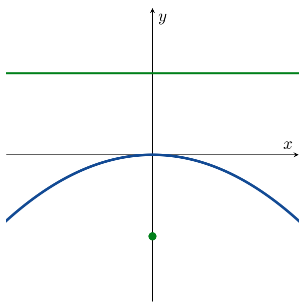
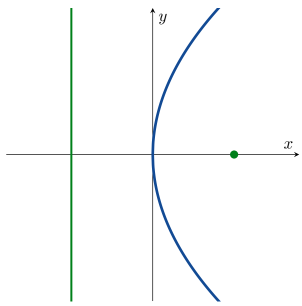
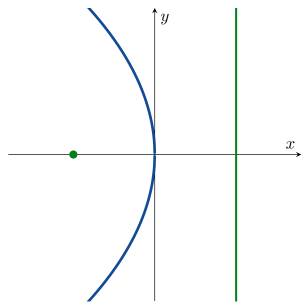


Figure 6.2: A parabola with focus at $F(0,p)$ and directrix $y = -p$.
Şekil 6.2: Odak noktası $F(0,p)$ ve doğrultmanı $y = -p$ olan parabol.

Bu nedenle

$$\begin{aligned}
\|PF\| &= \|PQ\| \\
\sqrt{x^2+(y-p)^2} &= y+p \\
x^2+(y-p)^2 &= (y+p)^2 \\
x^2+y^2-2py+p^2 &= y^2+2py+p^2 \\
x^2-2py &= 2py \\
\boxed{x^2} &\boxed{= 4py}
\end{aligned}$$

graph graf				
equation denklem	$x^2 = 4py$	$x^2 = -4py$	$y^2 = 4px$	$y^2 = -4px$
focus odak	$F(0,p)$	$F(0,-p)$	$F(p,0)$	$F(-p,0)$
directrix doğrultman	$y = -p$	$y = p$	$x = -p$	$x = p$

Example 6.1. Find the focus and directrix of the parabola $y^2 = 10x$.

solution: Our equation $y^2 = 10x$ looks like $y^2 = 4px$ with $p = \frac{10}{4} = 2.5$. Therefore the focus is at the point $F(2.5,0)$ and the directrix is the line $x = -2.5$.

Example 6.2. Find the equation for the parabola which has focus $F(0,-10)$ and directrix $y = 10$.

solution: Clearly $p = 10$ and $x^2 = -4py$. Therefore the answer is $x^2 = -40y$.

Örnek 6.1. $y^2 = 10x$ parabolünün odak noktasını ve doğrultmanını bulunuz.

çözüm: $y^2 = 10x$ denklemimiz olmak üzere $y^2 = 4px$ biçimindedir. Yani odak noktası $F(2.5,0)$ ve doğrultmanı da $x = -2.5$ olur.

Örnek 6.2. Odağı $F(0,-10)$ noktası ve doğrultmanı $y = 10$ doğrusu olan parabolün denklemini yazınız.

çözüm: Şurası açık ki $p = 10$ ve $x^2 = -4py$ dir. Bu nedenle yanıt $x^2 = -40y$ olur.

Ellipses

Elipsler

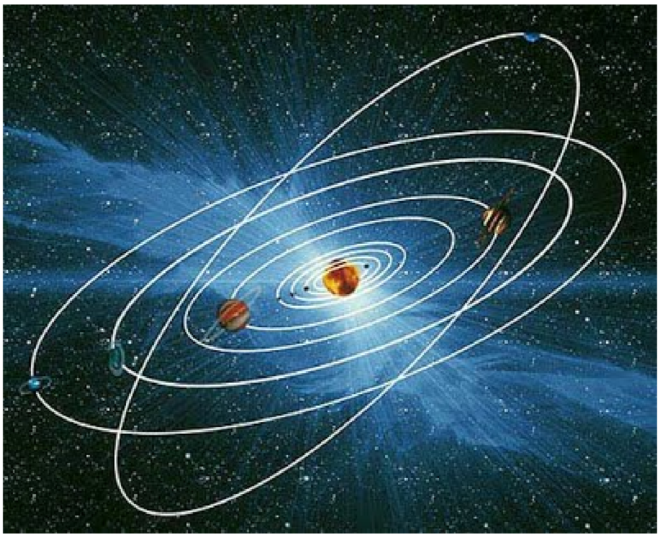


Figure 6.5: Our solar system.
Şekil 6.5: Güneş sistemimiz.

Figure 6.6: Tycho Brahe Planetarium, Copenhagen, Denmark.
Şekil 6.6: Tycho Brahe Planetaryumu, Kopenhag, Danimarka.

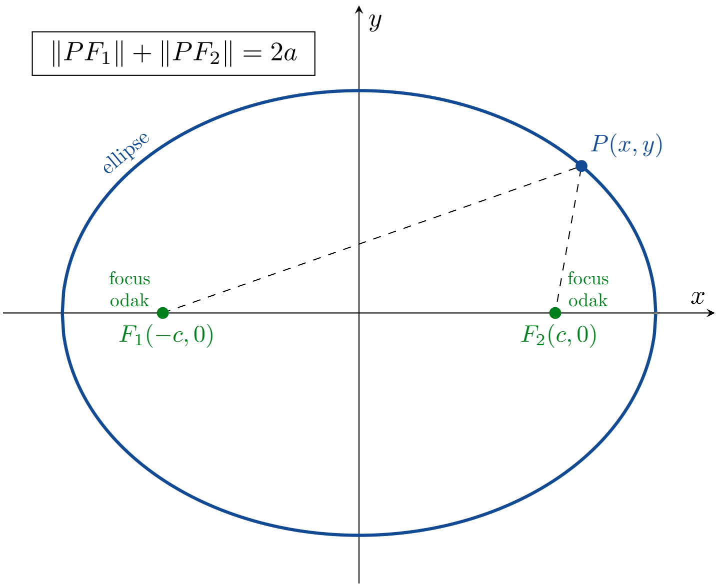


Figure 6.7: An ellipse with foci at $F_1(-c, 0)$ and $F_2(c, 0)$.
Şekil 6.7: Odakları $F_1(-c, 0)$ ve $F_2(c, 0)$ olan elips.

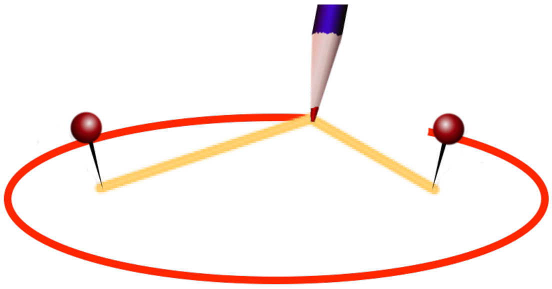


Figure 6.8: Drawing an ellipse with a pencil, two pins and a piece of string.
Şekil 6.8: İki toplu iğne, bir kalem ve biraz ip kullanarak elips çizmek.

To describe an ellipse, we need two ***foci***. See figure 6.7.

Definition. A point $P(x, y)$ is on the ***ellipse*** if and only if

$$\|PF_1\| + \|PF_2\| = 2a.$$

So

$$\sqrt{(x+c)^2 + y^2} + \sqrt{(x-c)^2 + y^2} = 2a.$$

This rearranges to

$$\frac{x^2}{a^2} + \frac{y^2}{a^2 - c^2} = 1.$$

Elipsi tanımlamak için, we need two foci. Bkz. şekil 6.7.

Tanım. Bir $P(x, y)$ noktası ***ellips*** üzerindedir ancak ve ancak

$$\|PF_1\| + \|PF_2\| = 2a.$$

Buradan hareketle

$$\sqrt{(x+c)^2 + y^2} + \sqrt{(x-c)^2 + y^2} = 2a.$$

Bunu da düzenlersek

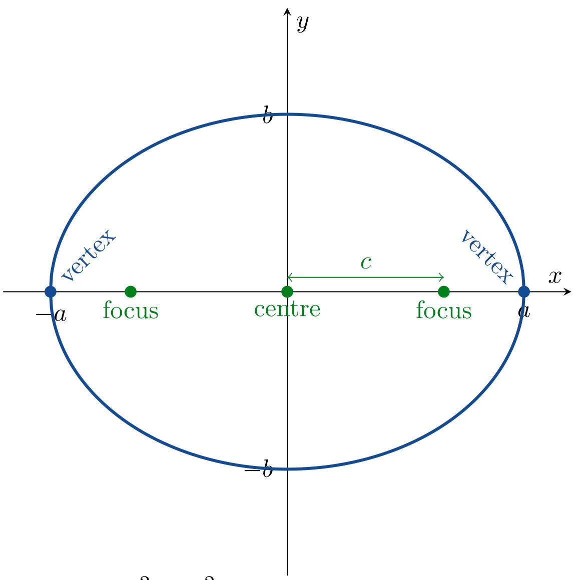
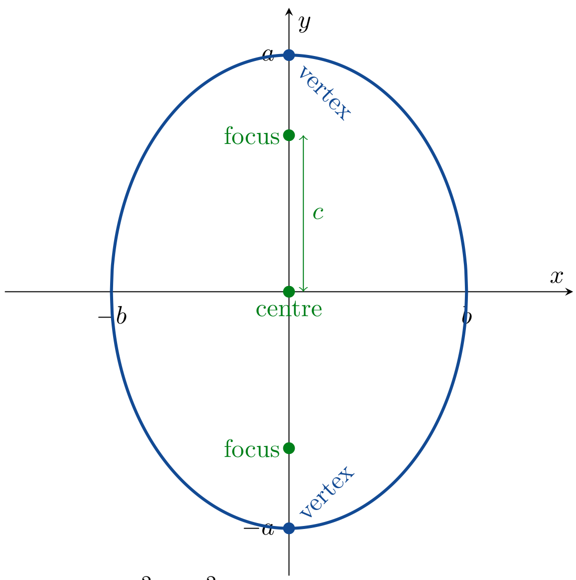
$$\frac{x^2}{a^2} + \frac{y^2}{a^2 - c^2} = 1$$

If we set $b = \sqrt{a^2 - c^2}$, then we have

$$\boxed{\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1} \qquad (0 < b < a).$$

buluruz. $b = \sqrt{a^2 - c^2}$ dersek, o zaman

$$\boxed{\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1} \qquad (0 < b < a).$$

graph			graf
equation	$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \qquad (0 < b < a)$	$\frac{x^2}{b^2} + \frac{y^2}{a^2} = 1 \qquad (0 < b < a)$	denklem
centre-to-focus distance	$c = \sqrt{a^2 - b^2}$	$c = \sqrt{a^2 - b^2}$	merkez-odak uzaklığı
foci	$F_1(-c, 0)$ & $F_2(c, 0)$	$F_1(0, -c)$ & $F_2(0, c)$	odaklar
vertices	$(-a, 0)$ & $(a, 0)$	$(0, -a)$ & $(0, a)$	tepe noktaları

Example 6.3. The ellipse $\frac{x^2}{16} + \frac{y^2}{9} = 1$ has

- $a = 4$ and $b = 3$;
- centre-to-focus distance $c = \sqrt{a^2 - b^2} = \sqrt{16 - 9} = \sqrt{7}$;
- centre at $(0, 0)$;
- foci at $(-\sqrt{7}, 0)$ and $(\sqrt{7}, 0)$; and
- vertices at $(-4, 0)$ and $(4, 0)$.

Example 6.4. The ellipse $\frac{x^2}{16} + \frac{y^2}{25} = 1$ has

- $a = 5$ and $b = 4$;
- centre-to-focus distance $c = \sqrt{a^2 - b^2} = \sqrt{25 - 16} = \sqrt{9} = 3$;
- centre at $(0, 0)$;
- foci at $(0, -3)$ and $(0, 3)$; and
- vertices at $(0, -5)$ and $(0, 5)$.

Örnek 6.3. $\frac{x^2}{16} + \frac{y^2}{9} = 1$ elipsinin

- $a = 4$ ve $b = 3$;
- merkez-odak uzaklığı $c = \sqrt{a^2 - b^2} = \sqrt{16 - 9} = \sqrt{7}$;
- merkezi $(0, 0)$;
- odakları $(-\sqrt{7}, 0)$ ve $(\sqrt{7}, 0)$; and
- tepe noktaları $(-4, 0)$ ve $(4, 0)$.

Örnek 6.4. $\frac{x^2}{16} + \frac{y^2}{25} = 1$ elipsi

- $a = 5$ ve $b = 4$;
- merkez-odak uzaklığı $c = \sqrt{a^2 - b^2} = \sqrt{25 - 16} = \sqrt{9} = 3$;
- merkezi $(0, 0)$;
- odakları $(0, -3)$ ve $(0, 3)$; ve
- tepe noktaları da $(0, -5)$ ve $(0, 5)$.

Hyperbolas

Figure 6.9: Cooling towers.
Şekil 6.9:

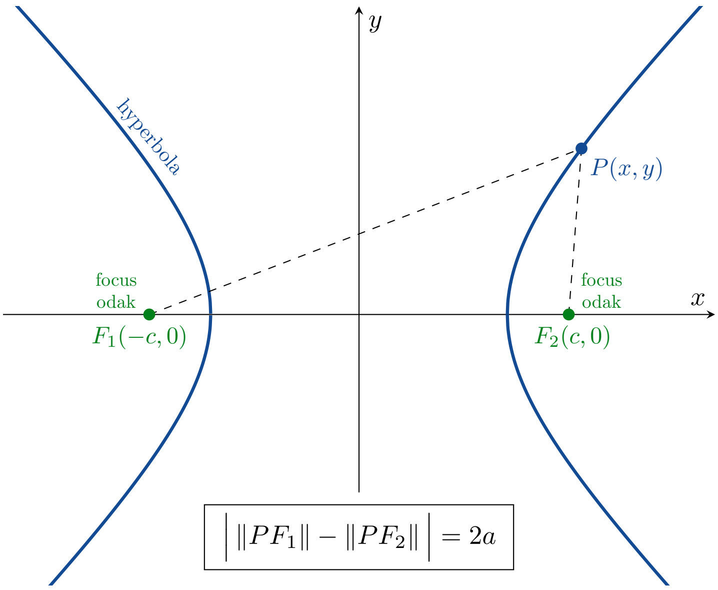


Figure 6.11: A hyperbola with foci at $F_1(-c,0)$ and $F_2(c,0)$.
Şekil 6.11:

To describe a hyperbola, we again need two foci. See figure 6.11.

Definition. A point $P(x,y)$ is on the ***hyperbola*** if and only if

$$\Big| \|PF_1\| - \|PF_2\| \Big| = 2a.$$

So

$$\sqrt{(x+c)^2+y^2} - \sqrt{(x-c)^2+y^2} = \pm 2a.$$

This rearranges to

$$\frac{x^2}{a^2} + \frac{y^2}{a^2-c^2} = 1$$

where $c > a > 0$. If we set $b = \sqrt{c^2-a^2}$, then

$$\boxed{\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1.}$$

Hiperboller

Figure 6.10: Twin Arch 138, Ichinomiya City, Japan.
Şekil 6.10:

Hiperbolü tanımlamak için, yine iki odak noktasına ihtiyaç var. Bkz. şekil 6.11.

Tanım. Bir $P(x,y)$ noktası bir ***hiperbol*** üzerindedir ancak ve ancak

$$\Big| \|PF_1\| - \|PF_2\| \Big| = 2a.$$

Bundan hareketle,

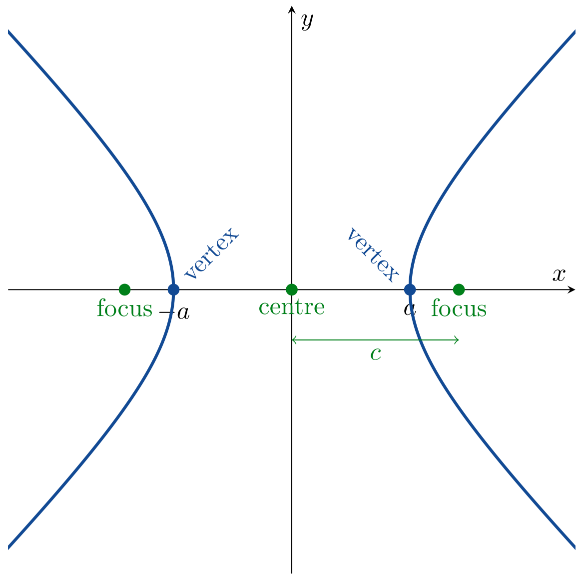
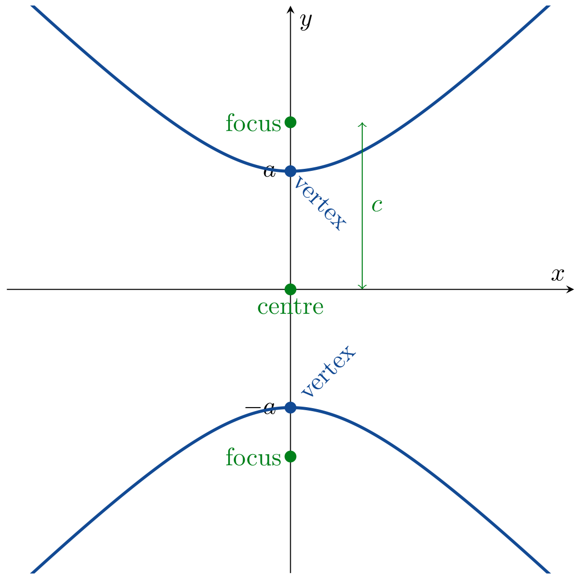
$$\sqrt{(x+c)^2+y^2} - \sqrt{(x-c)^2+y^2} = \pm 2a.$$

Düzenlersek,

$$\frac{x^2}{a^2} + \frac{y^2}{a^2-c^2} = 1$$

buluruz ki burada $c > a > 0$. Şimdi $b = \sqrt{c^2-a^2}$ dersek, o zaman

$$\boxed{\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1.}$$

graph			graf
equation	$\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$	$\dfrac{y^2}{a^2} - \dfrac{x^2}{b^2} = 1$	denklem
centre-to-focus distance	$c = \sqrt{a^2 + b^2}$	$c = \sqrt{a^2 + b^2}$	merkez-odak uzaklığı
foci	$F_1(-c, 0)$ & $F_2(c, 0)$	$F_1(0, -c)$ & $F_2(0, c)$	odaklar
vertices	$(-a, 0)$ & $(a, 0)$	$(0, -a)$ & $(0, a)$	tepe noktaları

Example 6.5. The hyperbola $\dfrac{x^2}{4} - \dfrac{y^2}{5} = 1$ has

- $a = 2$ and $b = \sqrt{5}$;
- centre at $(0, 0)$;
- centre-to-focus distance $c = \sqrt{a^2 + b^2} = \sqrt{4 + 5} = 3$;
- foci at $(-3, 0)$ and $(3, 0)$; and
- vertices at $(-2, 0)$ and $(2, 0)$.

Örnek 6.5. Hiperbol olarak $\dfrac{x^2}{4} - \dfrac{y^2}{5} = 1$ alırsak,

- $a = 2$ ve $b = \sqrt{5}$;
- merkezi $(0, 0)$;
- merkez-odak uzaklığı $c = \sqrt{a^2 + b^2} = \sqrt{4 + 5} = 3$;
- odakları $(-3, 0)$ ve $(3, 0)$; ve
- tepe noktaları da $(-2, 0)$ ve $(2, 0)$.

Example 6.6. The hyperbola $\dfrac{y^2}{9} - \dfrac{x^2}{16} = 1$ has

- $a = 3$ and $b = 4$;
- centre at $(0, 0)$;
- centre-to-focus distance $c = \sqrt{a^2 + b^2} = \sqrt{9 + 16} = 5$;
- foci at $(0, -5)$ and $(0, 5)$; and
- vertices at $(0, -3)$ and $(0, 3)$.

Örnek 6.6. $\dfrac{y^2}{9} - \dfrac{x^2}{16} = 1$ hiperbolü için

- $a = 3$ ve $b = 4$;
- merkez $(0, 0)$;
- merkez-odak uzaklığı $c = \sqrt{a^2 + b^2} = \sqrt{9 + 16} = 5$;
- odaklar $(0, -5)$ ve $(0, 5)$; ve
- tepe noktaları $(0, -3)$ ve $(0, 3)$.

Reflective Properties

Parabolas, ellipses and hyperbolas are useful in architecture and engineering because of their reflective properties.

Yansıma Özellikleri

Paraboller, elipsler ve hiperboller, yansıma özellikleri nedeniyle mimaride ve mühendislikte kullanışlıdırlar.

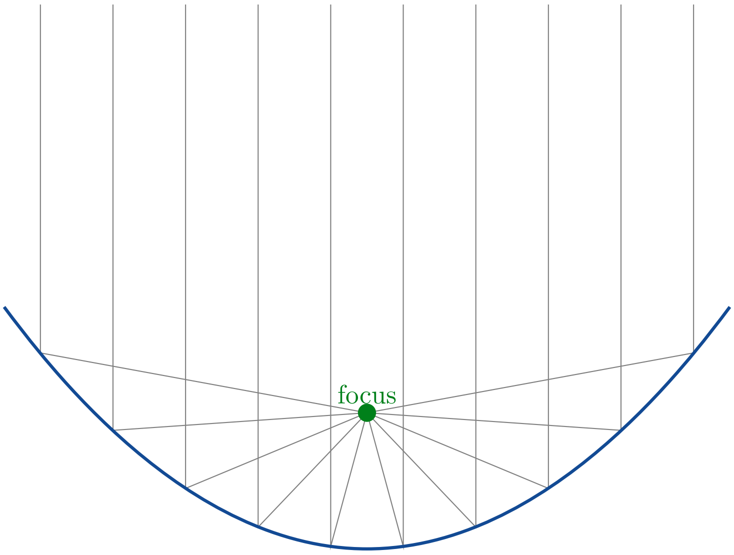


Figure 6.12: Rays originating at the focus of a parabola are reflected out of the parabola as parallel lines.
Şekil 6.12: Parabolün odağından çıkan ışınlar parabolün dışında paralel doğrular olarak yoluna devam ederler

Figure 6.13: One of a pair of whispering dishes in San Francisco, USA.
Şekil 6.13: A.B.D. San Fransisko'daki bir çift akustik çanak.

Figure 6.14: A car headlight
Şekil 6.14: Bir araba farı.

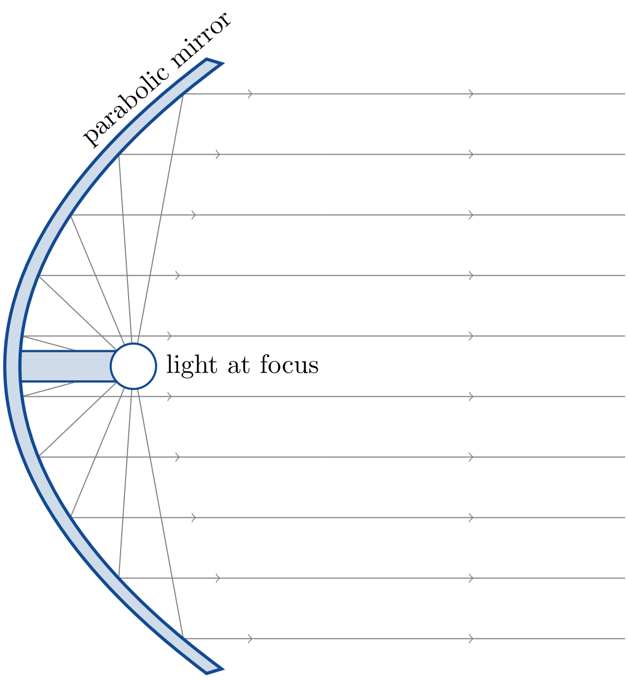


Figure 6.15: A car headlight
Şekil 6.15: Bir araba farı.

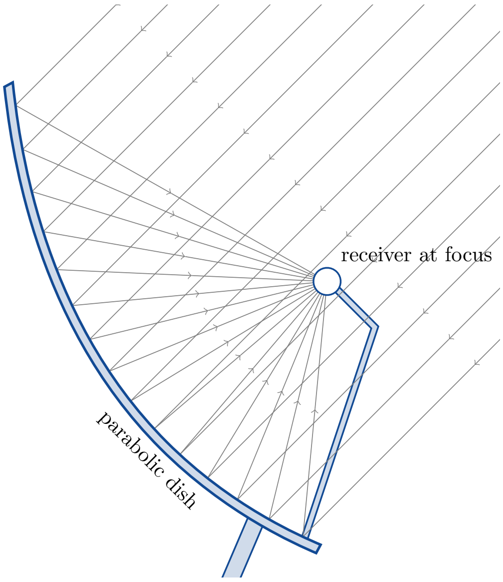


Figure 6.16: A satellite dish.
Şekil 6.16: Bir çanak anten

Figure 6.17: A satellite dish.
Şekil 6.17: Bir çanak anten

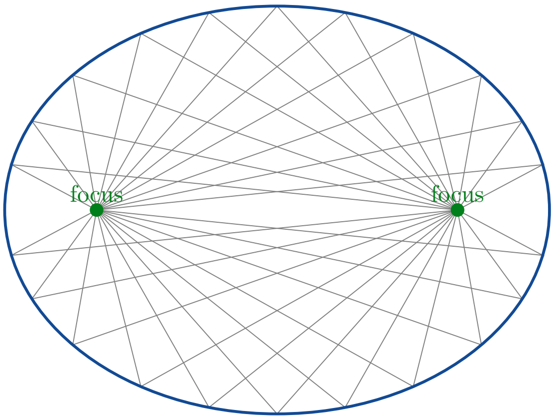


Figure 6.18: Rays originating from one focus of an ellipse are reflected toward the other focus.
Şekil 6.18: Elipsin bir odağından çıkan ışınlar diğer odağa yansıyorlar.

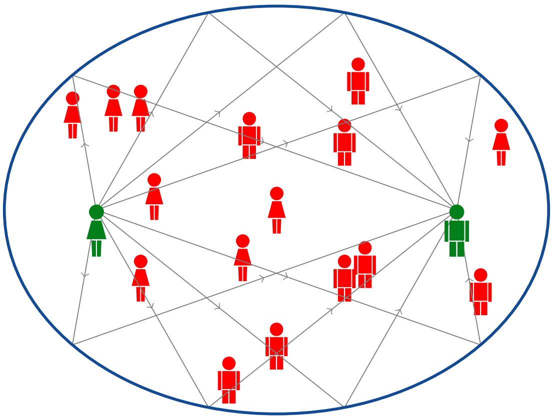


Figure 6.19: A whispering gallery.
Şekil 6.19:

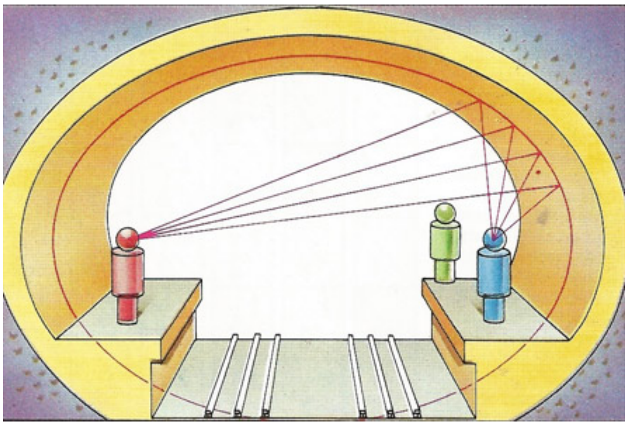


Figure 6.20: A whispering gallery.
Şekil 6.20:

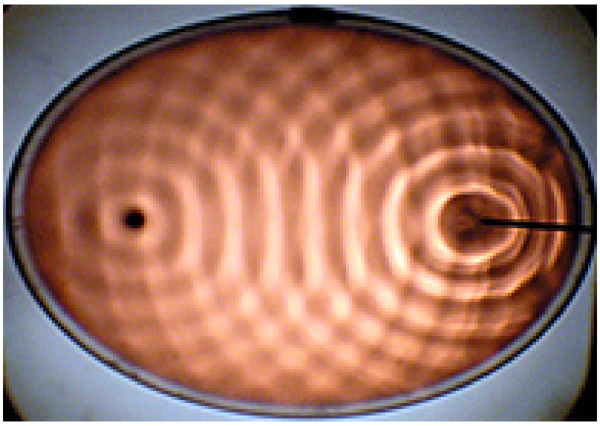


Figure 6.21: A whispering gallery.
Şekil 6.21:

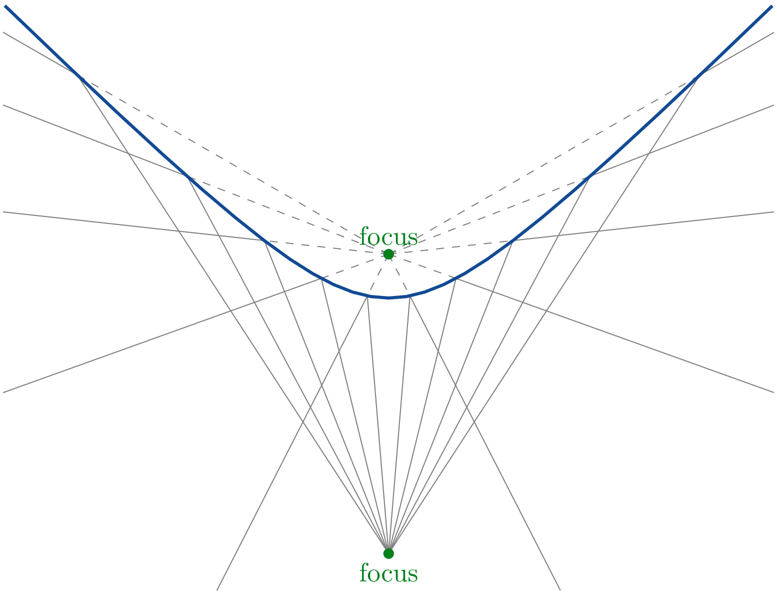


Figure 6.22: One half of a hyperbola. Rays aimed at one focus are reflected to the second focus.
Şekil 6.22: Hiperbolün bir yarısı. Odaklardan birine gelen ışınlar ikinci odağa yansıyorlar.

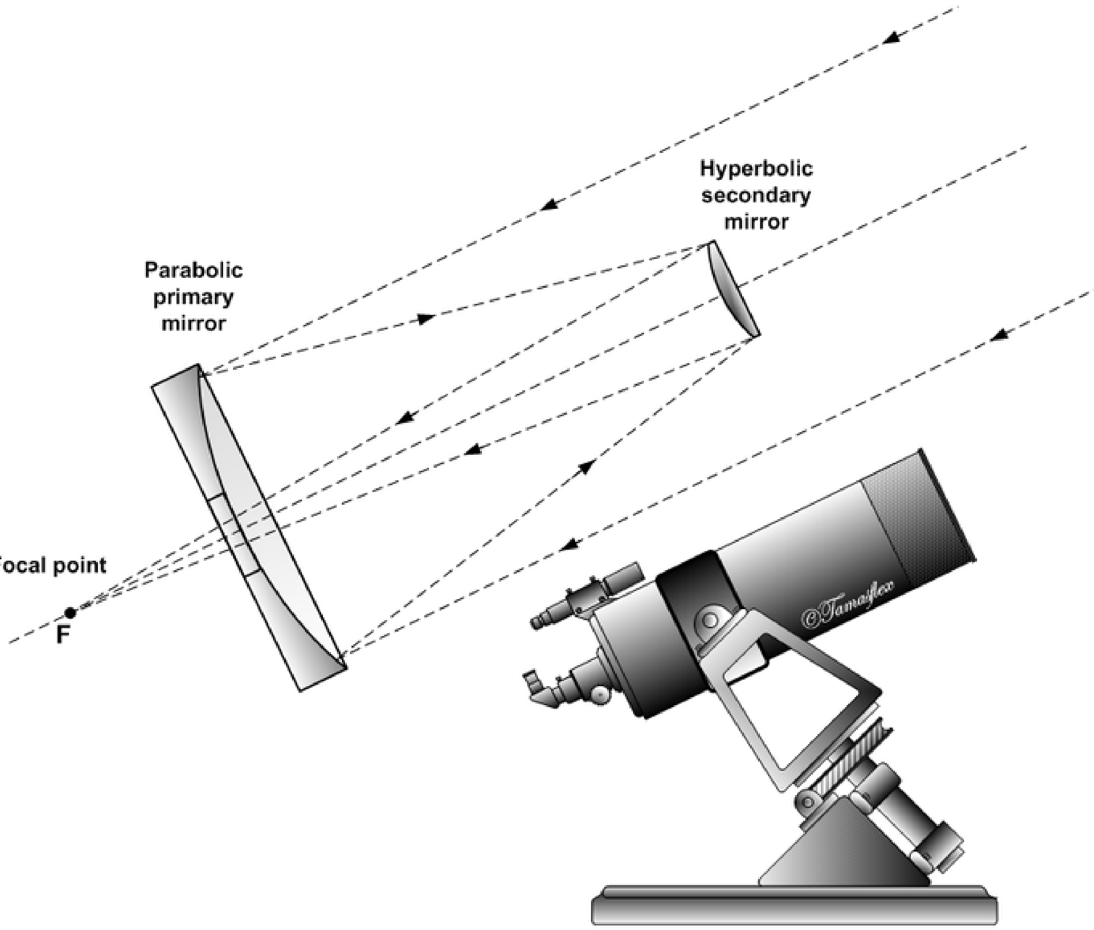


Figure 6.23: A telescope using a parabola and a hyperbola.
Şekil 6.23: Bir parabol ve bir hiperbol kullanılan teleskop

Problems

Problem 6.1 (Identifying Graphs)**.** Match the the following equations with the conic sections shown in figure 6.24.

(a). $y^2 = -4x$

(b). $x^2 = 2y$

(c). $\frac{x^2}{4} + \frac{y^2}{9} = 1$

(d). $\frac{x^2}{16} + \frac{y^2}{9} = 1$

(e). $\frac{y^2}{4} - x^2 = 1$

(f). $\frac{x^2}{4} - \frac{y^2}{9} = 1$

Problem 6.2 (Parabolas)**.**

(a). Find the focus of the parabola $y^2 = 12x$.

(b). Find the focus of the parabola $x^2 = -8y$.

(c). Find the focus of the parabola $y = 4x^2$.

Problem 6.3 (Ellipses)**.**

(a). Find the foci of the ellipse $7x^2 + 16y^2 = 112$.

(b). Find the foci of the ellipse $16x^2 + 25y^2 = 400$.

(c). Find the foci of the ellipse $2x^2 + y^2 = 2$.

(d). An ellipse has foci $(\pm\sqrt{2}, 0)$ and vertices $(\pm 2, 0)$. Find an equation for the ellipse.

Problem 6.4 (Hyperbolas)**.**

(a). Find the foci of the hyperbola $x^2 - y^2 = 1$.

(b). Find the foci of the hyperbola $y^2 - x^2 = 8$.

(c). Find the foci of the hyperbola $8x^2 - 2y^2 = 16$.

(d). A hyperbola has foci $(\pm 10, 0)$ and vertices $(\pm 6, 0)$. Find an equation for the hyperbola.

Sorular

Soru 6.1 (Grafikleri Belirlemek)**.** Match the the following equations with the conic sections shown in figure 6.24.

Soru 6.2 (Paraboller)**.**

(a). $y^2 = 12x$ parabolünün odağını bulunuz.

(b). $x^2 = -8y$ parabolünün odağını bulunuz.

(c). $y = 4x^2$ parabolünün odağını bulunuz.

Soru 6.3 (Elipsler)**.**

(a). $7x^2 + 16y^2 = 112$ elipsinin odaklarını bulunuz.

(b). $16x^2 + 25y^2 = 400$ elipsinin odaklarını bulunuz.

(c). $2x^2 + y^2 = 2$ elipsinin odaklarını bulunuz.

(d). An ellipse has foci $(\pm\sqrt{2}, 0)$ and vertices $(\pm 2, 0)$. Find an equation for the ellipse.

Soru 6.4 (Hiperboller)**.**

(a). $x^2 - y^2 = 1$ hyperbolünün odaklarını bulunuz.

(b). $y^2 - x^2 = 8$ hyperbolünün odaklarını bulunuz.

(c). $8x^2 - 2y^2 = 16$ hyperbolünün odaklarını bulunuz.

(d). A hyperbola has foci $(\pm 10, 0)$ and vertices $(\pm 6, 0)$. Find an equation for the hyperbola.

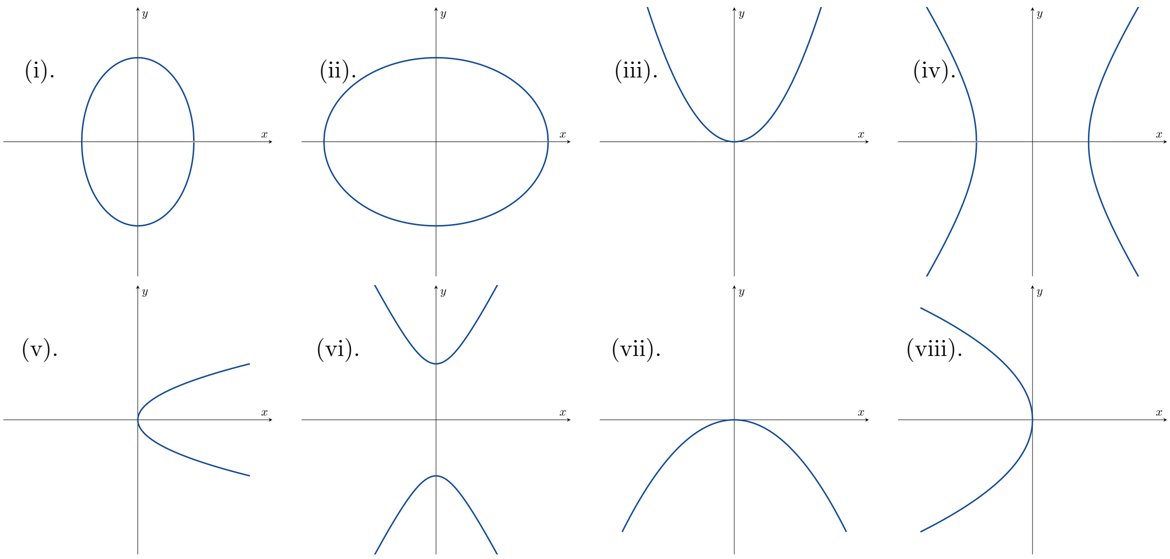
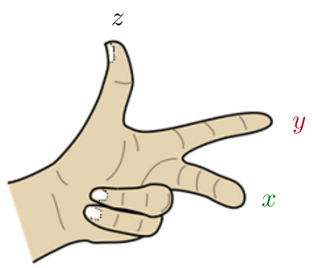


Figure 6.24: Eight conic sections.
Şekil 6.24:

7 Three Dimensional Cartesian Coordinates

Üç Boyutlu Kartezyen Koordinatlar



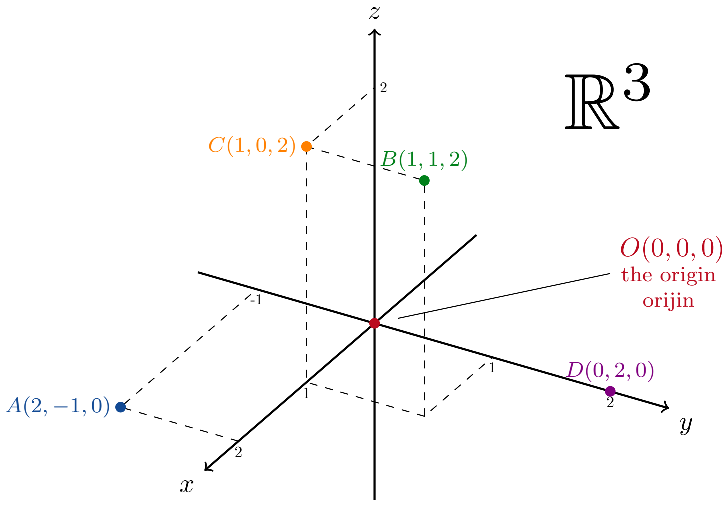


Figure 7.1: The Left-Handed Coordinate System
Şekil 7.1:

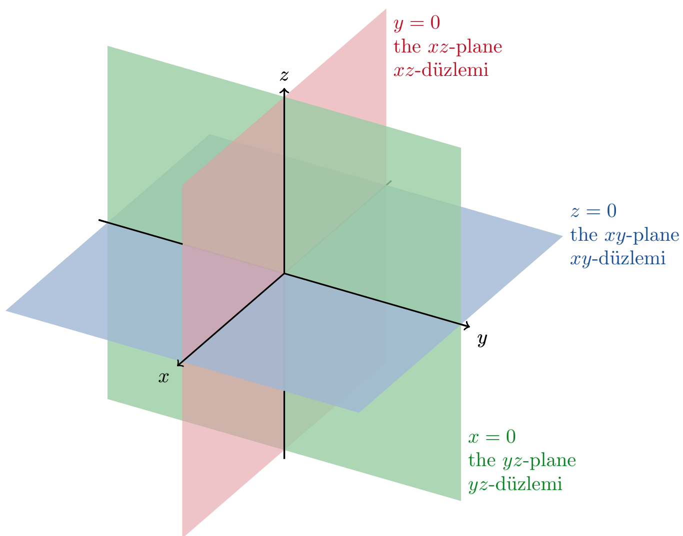


Figure 7.2: The planes $x = 0$, $y = 0$ and $z = 0$.
Şekil 7.2: $x = 0$, $y = 0$ ve $z = 0$ düzlemleri.

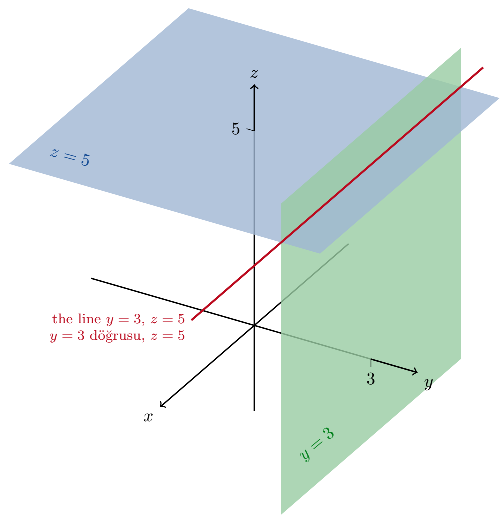


Figure 7.3: The planes $y = 3$ and $z = 5$, and the line $y = 3$, $z = 5$.
Şekil 7.3:

Example 7.1. Which points $P(x,y,z)$ satisfy $x^2+y^2=4$ and $z=3$?

solution: We know that $z=3$ is a horizontal plane and we recognise that $x^2+y^2=4$ is the equation of a circle of radius 2. Putting these together, we obtain figure 7.4.

Örnek 7.1. Hangi $P(x,y,z)$ noktaları $x^2+y^2=4$ ve $z=3$'ü sağlar?

çözüm: Biliyoruz ki $z=3$ yatay bir düzlem ve $x^2+y^2=4$ denklemi 2 yarıçaplı bir çemberdir. Bunları bir araya getirirsek, şekil 7.4'yi elde ederiz.

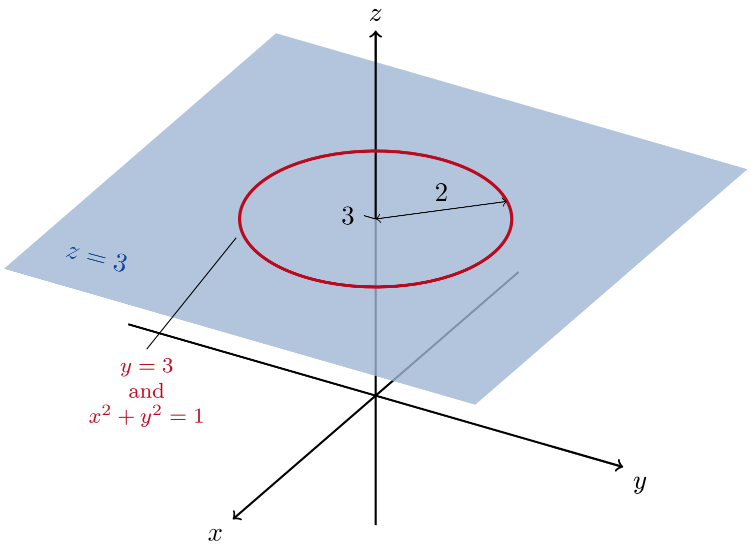


Figure 7.4: The circle $x^2+y^2=4$ in the plane $z=3$.
Şekil 7.4: $z=3$ düzlemindeki $x^2+y^2=4$ çemberi.

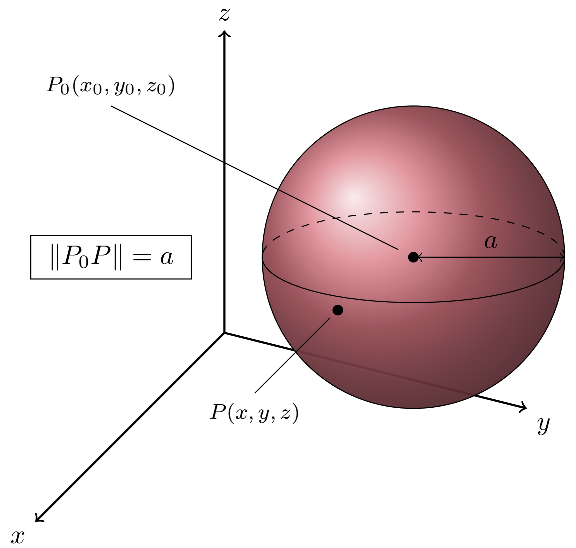


Figure 7.5: The sphere of radius a centred at $P_0(x_0,y_0,z_0)$.
Şekil 7.5: Yarıçapı a ver merkezı $P_0(x_0,y_0,z_0)$ noktası olan küre.

Distance in $\mathbb{R}^3$

Definition. The set

$$\{(x,y,z) \mid x,y,z \in \mathbb{R}\}$$

is denoted by $\mathbb{R}^3$.

Definition. The ***distance*** between $P_1(x_1,y_1,z_1)$ and $P_2(x_2,y_2,z_2)$ is

$$\|P_1P_2\| = \sqrt{(x_2-x_1)^2+(y_2-y_1)^2+(z_2-z_1)^2}.$$

Example 7.2. The distance between $A(2,1,5)$ and $B(-2,3,0)$ is

$$\begin{aligned}\|AB\| &= \sqrt{((-2)-2)^2+(3-1)^2+(0-5)^2} \\ &= \sqrt{16+4+25} = \sqrt{45} \\ &= 3\sqrt{5} \approx 6.7.\end{aligned}$$

$\mathbb{R}^3$ de Uzaklık

Tanım.

$$\{(x,y,z) \mid x,y,z \in \mathbb{R}\}$$

kümesini $\mathbb{R}^3$ ile gösteririz.

Tanım. $P_1(x_1,y_1,z_1)$ ve $P_2(x_2,y_2,z_2)$ noktaları arasındaki ***uzaklık***

$$\|P_1P_2\| = \sqrt{(x_2-x_1)^2+(y_2-y_1)^2+(z_2-z_1)^2}.$$

Örnek 7.3. $C(1,2,3)$ ve $D(3,2,1)$ noktaları arasındaki uzaklık aşağıdakı gibidir;

$$\begin{aligned}\|AB\| &= \sqrt{(3-1)^2+(2-2)^2+(1-3)^2} \\ &= \sqrt{4+0+4} = \sqrt{8} \\ &= 2\sqrt{2} \approx 2.8.\end{aligned}$$

Spheres

See figure 7.5.

Definition. The ***standard equation for a sphere*** of radius a centred at $P_0(x_0, y_0, z_0)$ is

$$(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2 = a^2.$$

Example 7.4. Find the centre and radius of the sphere

$$x^2 + y^2 + z^2 + 3x - 4z + 1 = 0.$$

solution: We need to put this equation into the standard form. Since $(x - b)^2 = x^2 - 2b + b^2$ we have that

$$\begin{aligned}
x^2 + y^2 + z^2 + 3x - 4z + 1 &= 0\\
(x^2 + 3x) + y^2 + (z^2 - 4z) &= -1\\
\left(x^2 + 3x + \frac{9}{4}\right) - \frac{9}{4} + y^2 + (z^2 - 4z + 4) - 4 &= -1\\
\left(x^2 + 3x + \frac{9}{4}\right) + y^2 + (z^2 - 4z + 4) &= -1 + \frac{9}{4} + 4\\
\left(x + \frac{3}{2}\right)^2 + y^2 + (z - 2)^2 &= \frac{21}{4}.
\end{aligned}$$

The centre is at $P_0(x_0, y_0, z_0) = P_0(-\frac{3}{2}, 0, 2)$ and the radius is $a = \sqrt{\frac{21}{4}} = \frac{\sqrt{3}\sqrt{7}}{2}$.

Spheres

Bkz. şekil 7.5.

Tanım. Yarıçapı a ve merkezi $P_0(x_0, y_0, z_0)$ olan ***Bir kürenin standart denklemi***

$$(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2 = a^2.$$

Örnek 7.5. Verilen kürenin merkez ve yarıçapını bulunuz:

$$x^2 + y^2 + z^2 + 6x - 6y + 6z = 22.$$

çözüm: Bu denklemi standart forma getirmemiz gerek. Şimdi $(x - b)^2 = x^2 - 2b + b^2$ olduğundan

$$\begin{aligned}
x^2 + y^2 + z^2 + 6x - 6y + 6z &= 22\\
(x^2 + 6x) + (y^2 - 6y) + (z^2 + 6z) &= 22\\
(x^2 + 6x + 9) - 9 + (y^2 - 6y + 9) - 9 + (z^2 + 6z + 9) - 9 &= 22\\
(x^2 + 6x + 9) + (y^2 - 6y + 9) + (z^2 + 6z + 9) &= 49\\
(x + 3)^2 + (y - 3)^2 + (z + 3)^2 &= 49
\end{aligned}$$

Merkezi $P_0(x_0, y_0, z_0) = P_0(-3, 3, -3)$ olup yarıçapı $a = \sqrt{49} = 7$.

Problems

Problem 7.1. Find the distance between the following pairs of points.

(a). $P_1(-1, 1, 5)$ and $P_2(2, 5, 0)$.

(b). $A(1, 0, 0)$ and $B(0, 0, 1)$.

(c). $C(10, 5, -8)$ and $D(10, -25, 32)$.

(d). $E(8, 9, 7)$ and $F(2, 2, 3)$.

(e). $G(-4, 2, -4)$ and $O(0, 0, 0)$.

Problem 7.2. Find the centre and the radius of the sphere

$$x^2 + y^2 + z^2 - 6y + 8z = 0.$$

Problem 7.3. Find the centre and the radius of the sphere

$$x^2 + y^2 + z^2 - 2\sqrt{2}x - 2\sqrt{2}y + 2\sqrt{2}z + 4 = 0.$$

Sorular

Soru 7.1. Aşağısaki nokta çiftleri arasındaki uzaklığı bulunuz.

(a). $P_1(-1, 1, 5)$ ve $P_2(2, 5, 0)$.

(b). $A(1, 0, 0)$ ve $B(0, 0, 1)$.

(c). $C(10, 5, -8)$ ve $D(10, -25, 32)$.

(d). $E(8, 9, 7)$ ve $F(2, 2, 3)$.

(e). $G(-4, 2, -4)$ ve $O(0, 0, 0)$.

Soru 7.2. Verilen denklemdeki kürenin merkezini ve yarıçapını bulunuz

$$x^2 + y^2 + z^2 - 6y + 8z = 0.$$

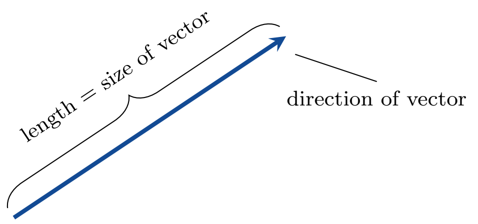
Soru 7.3. Verilen denklemdeki kürenin merkezini ve yarıçapını bulunuz

$$x^2 + y^2 + z^2 - 2\sqrt{2}x - 2\sqrt{2}y + 2\sqrt{2}z + 4 = 0.$$

Vectors

8 Vektörler

For some quantities (mass, time, distance, ...) we only need a number. For some quantities (velocity, force, ...) we need a number and a direction.



A ***vector*** is an object which has a size (length) and a direction.

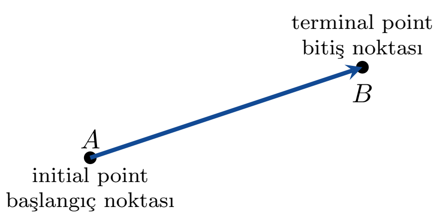


Figure 8.1: The initial point and terminal point of a vector.
Şekil 8.1:

Definition. The vector $\overrightarrow{AB}$ has ***initial point*** A and ***terminal point*** B.

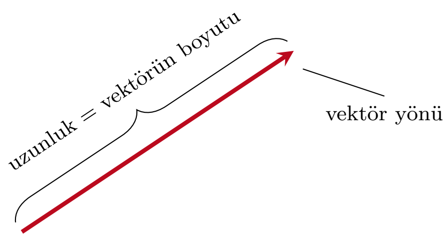
The ***length*** of $\overrightarrow{AB}$ is written $\left\|\overrightarrow{AB}\right\|$.

Two vectors are equal if they have the same length and the same direction. In figure 8.2, we can say that

$$\overrightarrow{AB} = \overrightarrow{CD} = \overrightarrow{EF} = \overrightarrow{OP}.$$

Note that $\overrightarrow{AB} \neq \overrightarrow{GH}$ because the lengths are different, and $\overrightarrow{AB} \neq \overrightarrow{IJ}$ because the directions are different.

Bazı büyüklükler (kütle, zaman, mesafe, ldots) sadece bir sayı yeterli oluyor. Ancak bazı büyüklükler için (hız, kuvvet, ldots) bir sayıyla bir de yöne ihtiyacımız var.



Vektör bir büyüklüğü (uzunluğu) ve bir yönü olan nesnedir.

Tanım. $\overrightarrow{AB}$ vektörünün ***başlangıç noktası*** A ve ***bitiş noktası*** B dir.

$\overrightarrow{AB}$'nin ***uzunluğu*** $\left\|\overrightarrow{AB}\right\|$ ile gösterilir.

İki vektörün eşit olmaları için gerek ve yeter şart uzunlukları ve boylarının aynı olmasıdır. Şekil 8.2 de, şunu söylemek mümkün

$$\overrightarrow{AB} = \overrightarrow{CD} = \overrightarrow{EF} = \overrightarrow{OP}.$$

Unutmayınız ki $\overrightarrow{AB} \neq \overrightarrow{GH}$ çünkü uzunluklar farklı ve $\overrightarrow{AB} \neq \overrightarrow{IJ}$ çünkü yönleri farklı.

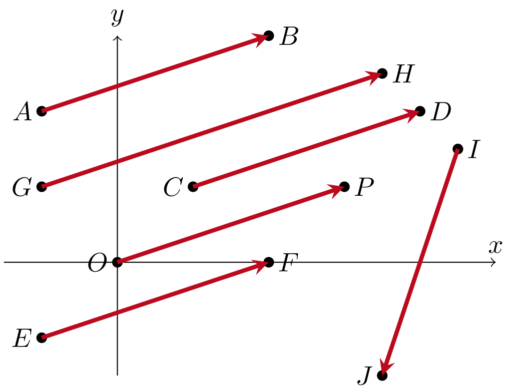


Figure 8.2: Six vectors.
Şekil 8.2: Altı vektör.

Notation

When we use a computer, we use bold letters for vectors: $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$, ... When we use a pen, we use underlined letters for vectors: $\underline{u}$, $\underline{v}$, $\underline{w}$, ...

If we type $a\mathbf{u} + b\mathbf{v}$ or write $a\underline{u} + b\underline{v}$, then

- a and b are numbers; and
- $\mathbf{u}$, $\mathbf{v}$, $\underline{u}$ and $\underline{v}$ are vectors.

Definition. In $\mathbb{R}^2$: If $\mathbf{v}$ has initial point $(0,0)$ and terminal point (v_1, v_2), then the ***component form*** of $\mathbf{v}$ is $\mathbf{v} = (v_1, v_2)$.

In $\mathbb{R}^3$: If $\mathbf{v}$ has initial point $(0,0,0)$ and terminal point (v_1, v_2, v_3), then the ***component form*** of $\mathbf{v}$ is $\mathbf{v} = (v_1, v_2, v_3)$.

Notasyon

Bilgisayar kullanırken, vektör için kalın harfler kullanırız: $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$, ... Kalemle yazarken, vektör için altı çizili harfler kullanırız: $\underline{u}$, $\underline{v}$, $\underline{w}$, ...

$a\mathbf{u} + b\mathbf{v}$ olarak yazarsak veya $a\underline{u} + b\underline{v}$ yazarsak,

- a and b are numbers; and
- $\mathbf{u}$, $\mathbf{v}$, $\underline{u}$ and $\underline{v}$ are vectors.

Tanım. In $\mathbb{R}^2$: If $\mathbf{v}$ has initial point $(0,0)$ and terminal point (v_1, v_2), then the ***component form*** of $\mathbf{v}$ is $\mathbf{v} = (v_1, v_2)$.

In $\mathbb{R}^3$: If $\mathbf{v}$ has initial point $(0,0,0)$ and terminal point (v_1, v_2, v_3), then the ***component form*** of $\mathbf{v}$ is $\mathbf{v} = (v_1, v_2, v_3)$.

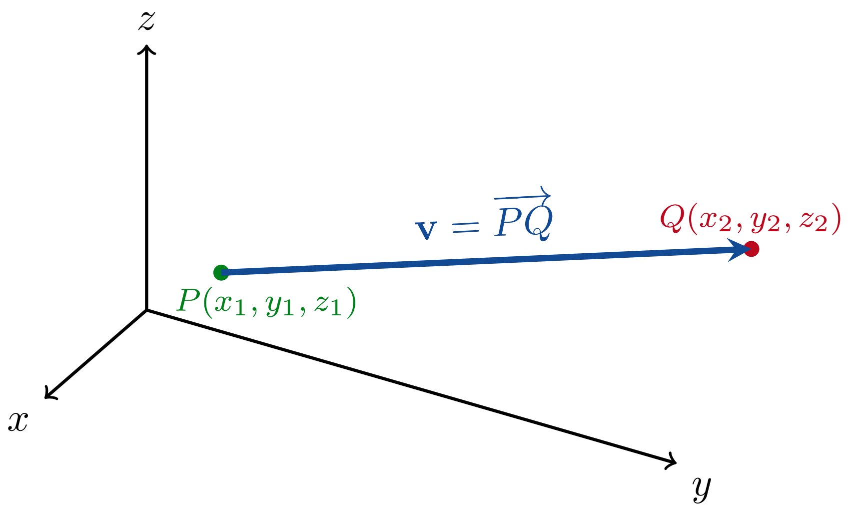


Figure 8.3: The vector $(v_1, v_2, v_3) = \mathbf{v} = (x_2 - x_1, y_2 - y_1, z_2 - z_1)$.
Şekil 8.3:

Definition. In $\mathbb{R}^2$: The ***norm*** (or ***length***) of $\mathbf{v} = (v_1, v_2)$ is

$$\|\mathbf{v}\| = \sqrt{v_1^2 + v_2^2}$$

In $\mathbb{R}^3$: The ***norm*** of $\mathbf{v} = \overrightarrow{PQ}$ is

$$\begin{aligned}\|\mathbf{v}\| &= \sqrt{v_1^2 + v_2^2 + v_3^2} \\ &= \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}.\end{aligned}$$

The vectors $\mathbf{0} = (0,0)$ and $\mathbf{0} = (0,0,0)$ have norm $\|\mathbf{0}\| = 0$. If $\mathbf{v} \neq \mathbf{0}$, then $\|\mathbf{v}\| > 0$.

Tanım. In $\mathbb{R}^2$: The ***norm*** (or ***length***) of $\mathbf{v} = (v_1, v_2)$ is

$$\|\mathbf{v}\| = \sqrt{v_1^2 + v_2^2}$$

In $\mathbb{R}^3$: The ***norm*** of $\mathbf{v} = \overrightarrow{PQ}$ is

$$\begin{aligned}\|\mathbf{v}\| &= \sqrt{v_1^2 + v_2^2 + v_3^2} \\ &= \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}.\end{aligned}$$

The vectors $\mathbf{0} = (0,0)$ and $\mathbf{0} = (0,0,0)$ have norm $\|\mathbf{0}\| = 0$. If $\mathbf{v} \neq \mathbf{0}$, then $\|\mathbf{v}\| > 0$.

Example 8.1. Find (a) the component form; and (b) the norm of the vector with initial point $P(-3,4,1)$ and terminal point $Q(-5,2,2)$.

solution:

(a). $\mathbf{v} = (v_1, v_2, v_3) = Q - P = (-5,2,2) - (-3,4,1)$
$= (-2,-2,1)$.

(b). $\|\mathbf{v}\| = \sqrt{v_1^2 + v_2^2 + v_3^2} = \sqrt{(-2)^2 + (-2)^2 + 1^2} = \sqrt{9} = 3$.

Örnek 8.1. Find (a) the component form; and (b) the norm of the vector with initial point $P(-3,4,1)$ and terminal point $Q(-5,2,2)$.

çözüm:

(a). $\mathbf{v} = (v_1, v_2, v_3) = Q - P = (-5,2,2) - (-3,4,1)$
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Vector Algebra

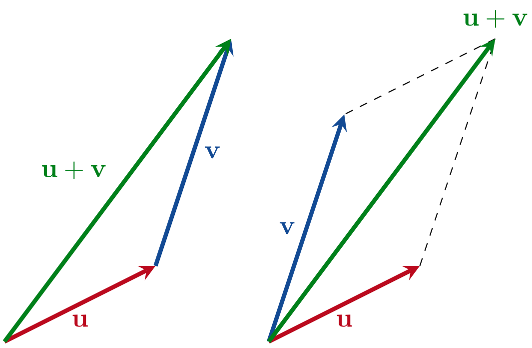

Figure 8.4: $\mathbf{u}+\mathbf{v}$ considered in two ways.
Şekil 8.4:

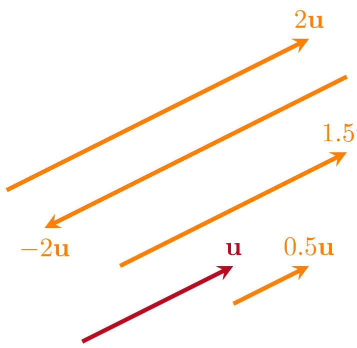

Figure 8.5: Constant multiples of $\mathbf{u}$.
Şekil 8.5:

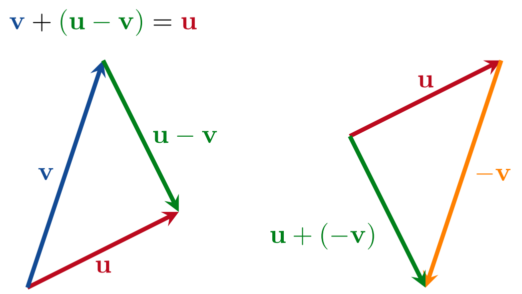

Figure 8.6: $\mathbf{u}-\mathbf{v}$ considered in two ways.
Şekil 8.6:

Let $\mathbf{u} = (u_1, u_2, u_3)$ and $\mathbf{v} = (v_1, v_2, v_3)$ be vectors. Let k be a number. Then

$$\mathbf{u} + \mathbf{v} = (u_1 + v_1, u_2 + v_2, u_3 + v_3)$$

and

$$k\mathbf{u} = (ku_1, ku_2, ku_3).$$

Note that

$$\begin{aligned} \|k\mathbf{u}\| &= \|(ku_1, ku_2, ku_3)\| = \sqrt{(ku_1)^2 + (ku_2)^2 + (ku_3)^2} \\ &= \sqrt{k^2u_1^2 + k^2u_2^2 + k^2u_3^2} = \sqrt{k^2(u_1^2 + u_2^2 + u_3^2)} \\ &= \sqrt{k^2}\sqrt{u_1^2 + u_2^2 + u_3^2} = |k| \, \|\mathbf{u}\| \,. \end{aligned}$$

The vector $-\mathbf{u} = (-1)\mathbf{u}$ has the same length as $\mathbf{u}$, but points in the opposite direction.

Example 8.2. Let $\mathbf{u} = (-1, 3, 1)$ and $\mathbf{v} = (4, 7, 0)$. Find (a) $2\mathbf{u} + 3\mathbf{v}$, (b) $\mathbf{u} - \mathbf{v}$, and (c) $\left\|\frac{1}{2}\mathbf{u}\right\|$.

solution:
(a) $2\mathbf{u}+3\mathbf{v} = 2(-1,3,1)+3(4,7,0) = (-2,6,2)+(12,21,0) = (10,27,2)$;

(b) $\mathbf{u} - \mathbf{v} = (-1, 3, 1) - (4, 7, 0) = (-5, -4, 1)$;

(c) $\left\|\frac{1}{2}\mathbf{u}\right\| = \frac{1}{2}\|\mathbf{u}\| = \frac{1}{2}\sqrt{(-1)^2 + 3^2 + 1^2} = \frac{1}{2}\sqrt{11}$.

Properties of Vector Operations

Let $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ be vectors. Let a and b be numbers. Then

(i). $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$;

(ii). $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$;

(iii). $\mathbf{u} + \mathbf{0} = \mathbf{u}$;

(iv). $\mathbf{u} + (-\mathbf{u}) = \mathbf{0}$;

(v). $0\mathbf{u} = \mathbf{0}$;

(vi). $1\mathbf{u} = \mathbf{u}$;

???????

Let $\mathbf{u} = (u_1, u_2, u_3)$ and $\mathbf{v} = (v_1, v_2, v_3)$ be vectors. Let k be a number. Then

$$\mathbf{u} + \mathbf{v} = (u_1 + v_1, u_2 + v_2, u_3 + v_3)$$

and

$$k\mathbf{u} = (ku_1, ku_2, ku_3).$$

Note that

$$\begin{aligned} \|k\mathbf{u}\| &= \|(ku_1, ku_2, ku_3)\| = \sqrt{(ku_1)^2 + (ku_2)^2 + (ku_3)^2} \\ &= \sqrt{k^2u_1^2 + k^2u_2^2 + k^2u_3^2} = \sqrt{k^2(u_1^2 + u_2^2 + u_3^2)} \\ &= \sqrt{k^2}\sqrt{u_1^2 + u_2^2 + u_3^2} = |k| \, \|\mathbf{u}\| \,. \end{aligned}$$

The vector $-\mathbf{u} = (-1)\mathbf{u}$ has the same length as $\mathbf{u}$, but points in the opposite direction.

Örnek 8.2. Let $\mathbf{u} = (-1, 3, 1)$ and $\mathbf{v} = (4, 7, 0)$. Find (a) $2\mathbf{u} + 3\mathbf{v}$, (b) $\mathbf{u} - \mathbf{v}$, and (c) $\left\|\frac{1}{2}\mathbf{u}\right\|$.

çözüm:
(a) $2\mathbf{u}+3\mathbf{v} = 2(-1,3,1)+3(4,7,0) = (-2,6,2)+(12,21,0) = (10,27,2)$;

(b) $\mathbf{u} - \mathbf{v} = (-1, 3, 1) - (4, 7, 0) = (-5, -4, 1)$;

(c) $\left\|\frac{1}{2}\mathbf{u}\right\| = \frac{1}{2}\|\mathbf{u}\| = \frac{1}{2}\sqrt{(-1)^2 + 3^2 + 1^2} = \frac{1}{2}\sqrt{11}$.

Properties of Vector Operations

Let $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ be vectors. Let a and b be numbers. Then

(i). $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$;

(ii). $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$;

(iii). $\mathbf{u} + \mathbf{0} = \mathbf{u}$;

(iv). $\mathbf{u} + (-\mathbf{u}) = \mathbf{0}$;

(v). $0\mathbf{u} = \mathbf{0}$;

(vi). $1\mathbf{u} = \mathbf{u}$;

(vii). $a(b\mathbf{u}) = (ab)\mathbf{u}$;

(viii). $a(\mathbf{u}+\mathbf{v}) = a\mathbf{u} + a\mathbf{v}$;

(ix). $(a+b)\mathbf{u} = a\mathbf{u} + b\mathbf{u}$.

Remark. We **can not** multiply vectors. Never never never never write "**uv**".

Unit Vectors

Definition. $\mathbf{u}$ is called a ***unit vector*** $\iff$ $\|\mathbf{u}\| = 1$.

Example 8.3. $\mathbf{u} = (2^{-\frac{1}{2}}, \frac{1}{2}, -\frac{1}{2})$ is a unit vector because

$$\|\mathbf{u}\| = \sqrt{\left(\frac{1}{\sqrt{2}}\right)^2 + \left(\frac{1}{2}\right)^2 + \left(-\frac{1}{2}\right)^2} = \sqrt{\frac{1}{2} + \frac{1}{4} + \frac{1}{4}} = 1.$$

In $\mathbb{R}^2$: The ***standard unit vectors*** are $\mathbf{i} = (1,0)$ and $\mathbf{j} = (0,1)$.

In $\mathbb{R}^3$: The ***standard unit vectors*** are $\mathbf{i} = (1,0,0)$, $\mathbf{j} = (0,1,0)$ and $\mathbf{k} = (0,0,1)$. Any vector $\mathbf{v} \in \mathbb{R}^3$ can be written

$$\begin{aligned}\mathbf{v} &= (v_1, v_2, v_3) = (v_1,0,0) + (0,v_2,0) + (0,0,v_3) \\ &= v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}.\end{aligned}$$

If $\|\mathbf{v}\| \neq 0$, then $\frac{\mathbf{v}}{\|\mathbf{v}\|}$ is a unit vector because

$$\left\|\frac{\mathbf{v}}{\|\mathbf{v}\|}\right\| = \left\|\frac{1}{\|\mathbf{v}\|}\mathbf{v}\right\| = \frac{1}{\|\mathbf{v}\|}\|\mathbf{v}\| = 1.$$

Clearly $\frac{\mathbf{v}}{\|\mathbf{v}\|}$ and $\mathbf{v}$ point in the same direction.

Example 8.4. Find a unit vector $\mathbf{u}$ which points in the same direction as $\overrightarrow{P_1P_2}$, where $P_1(1,0,1)$ and $P_2(3,2,0)$.

solution:
We calculate that $\overrightarrow{P_1P_2} = P_2 - P_1 = (3,2,0) - (1,0,1) = (2,2,-1) = 2\mathbf{i} + 2\mathbf{j} - \mathbf{k}$ and that $\left\|\overrightarrow{P_1P_2}\right\| = \sqrt{2^2 + 2^2 + (-1)^2} = 3$. The required unit vector is

$$\mathbf{u} = \frac{\overrightarrow{P_1P_2}}{\left\|\overrightarrow{P_1P_2}\right\|} = \frac{2\mathbf{i} + 2\mathbf{j} - \mathbf{k}}{3} = \frac{2}{3}\mathbf{i} + \frac{2}{3}\mathbf{j} - \frac{1}{3}\mathbf{k}.$$

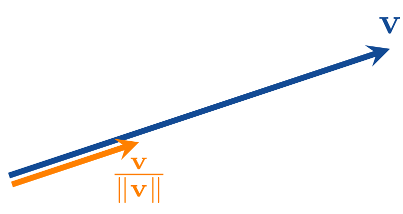

Figure 8.7: $\frac{\mathbf{v}}{\|\mathbf{v}\|}$ is a unit vector which points in the same direction as $\mathbf{v}$.
Şekil 8.7:

(vii). $a(b\mathbf{u}) = (ab)\mathbf{u}$;

(viii). $a(\mathbf{u}+\mathbf{v}) = a\mathbf{u} + a\mathbf{v}$;

(ix). $(a+b)\mathbf{u} = a\mathbf{u} + b\mathbf{u}$.

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In $\mathbb{R}^2$: The ***standard unit vectors*** are $\mathbf{i} = (1,0)$ and $\mathbf{j} = (0,1)$.

In $\mathbb{R}^3$: The ***standard unit vectors*** are $\mathbf{i} = (1,0,0)$, $\mathbf{j} = (0,1,0)$ and $\mathbf{k} = (0,0,1)$. Any vector $\mathbf{v} \in \mathbb{R}^3$ can be written

$$\begin{aligned}\mathbf{v} &= (v_1, v_2, v_3) = (v_1,0,0) + (0,v_2,0) + (0,0,v_3) \\ &= v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}.\end{aligned}$$

If $\|\mathbf{v}\| \neq 0$, then $\frac{\mathbf{v}}{\|\mathbf{v}\|}$ is a unit vector because

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Clearly $\frac{\mathbf{v}}{\|\mathbf{v}\|}$ and $\mathbf{v}$ point in the same direction.

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çözüm:
We calculate that $\overrightarrow{P_1P_2} = P_2 - P_1 = (3,2,0) - (1,0,1) = (2,2,-1) = 2\mathbf{i} + 2\mathbf{j} - \mathbf{k}$ and that $\left\|\overrightarrow{P_1P_2}\right\| = \sqrt{2^2 + 2^2 + (-1)^2} = 3$. The required unit vector is

$$\mathbf{u} = \frac{\overrightarrow{P_1P_2}}{\left\|\overrightarrow{P_1P_2}\right\|} = \frac{2\mathbf{i} + 2\mathbf{j} - \mathbf{k}}{3} = \frac{2}{3}\mathbf{i} + \frac{2}{3}\mathbf{j} - \frac{1}{3}\mathbf{k}.$$

Problems

Problem 8.1. Let $\mathbf{u} = (3, -2)$ and $\mathbf{v} = (-2, 5)$. Find the following:

(a). $\|\mathbf{u}\|$

(b). $\|\mathbf{v}\|$

(c). $3\,\|\mathbf{u}\|$

(d). $3\mathbf{u}$

(e). $\|3\mathbf{u}\|$

(f). $-3\mathbf{u}$

(g). $\|-3\mathbf{u}\|$

(h). $\mathbf{u} + \mathbf{v}$

(i). $\|\mathbf{u} + \mathbf{v}\|$

(j). $\|\mathbf{u}\| + \|\mathbf{v}\|$

(k). $2\mathbf{u} - 3\mathbf{v}$

(l). $\|2\mathbf{u} - 3\mathbf{v}\|$

(m). $\frac{3}{5}\mathbf{u} + \frac{4}{5}\mathbf{v}$

(n). $\left\|\frac{3}{5}\mathbf{u} + \frac{4}{5}\mathbf{v}\right\|$

(o). $\left\|-\frac{5}{13}\mathbf{u} + \frac{12}{13}\mathbf{v}\right\|$

Problem 8.2.

(a). Find $(5\mathbf{a} - 3\mathbf{b})$ if $\mathbf{a} = \mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$ and $\mathbf{b} = 2\mathbf{i} + 5\mathbf{k}$.

(b). Find $\overrightarrow{AB} + \overrightarrow{CD}$, where $A(1, -1, 1)$, $B(2, 0, 0)$, $C(-1, 3, 0)$ and $D(-2, 2, 1)$.

Problem 8.3 (Unit Vectors)**.**

(a). Find a unit vector which points in the same direction as $\mathbf{v} = 6\mathbf{i} + 2\mathbf{j} - 3\mathbf{k}$.

(b). Find a unit vector which points in the same direction as $\mathbf{v} = 2\mathbf{i} + \mathbf{j} - 2\mathbf{k}$.

(c). Find a vector $\mathbf{w}$ which points in the same direction as $\mathbf{v} = 12\mathbf{i} - 5\mathbf{k}$ and which satisfies $\|\mathbf{w}\| = 7$.

Sorular

Soru 8.1. Let $\mathbf{u} = (3, -2)$ and $\mathbf{v} = (-2, 5)$. Find the following:

Soru 8.2.

(a). Find $(5\mathbf{a} - 3\mathbf{b})$ if $\mathbf{a} = \mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$ and $\mathbf{b} = 2\mathbf{i} + 5\mathbf{k}$.

(b). Find $\overrightarrow{AB} + \overrightarrow{CD}$, where $A(1, -1, 1)$, $B(2, 0, 0)$, $C(-1, 3, 0)$ and $D(-2, 2, 1)$.

Soru 8.3 (Unit Vectors)**.**

(a). Find a unit vector which points in the same direction as $\mathbf{v} = 6\mathbf{i} + 2\mathbf{j} - 3\mathbf{k}$.

(b). Find a unit vector which points in the same direction as $\mathbf{v} = 2\mathbf{i} + \mathbf{j} - 2\mathbf{k}$.

(c). Find a vector $\mathbf{w}$ which points in the same direction as $\mathbf{v} = 12\mathbf{i} - 5\mathbf{k}$ and which satisfies $\|\mathbf{w}\| = 7$.

Figure 8.8: A web comic.
Şekil 8.8: Bir web çizgi romanı.

9

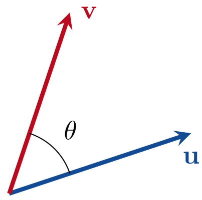
The Dot Product

Definition. In $\mathbb{R}^2$, the ***dot product*** of $\mathbf{u} = (u_1, u_2) = u_1\mathbf{i} + u_2\mathbf{j}$ and $\mathbf{v} = (v_1, v_2) = v_1\mathbf{i} + v_2\mathbf{j}$ is

$$\mathbf{u} \cdot \mathbf{v} = u_1 v_1 + u_2 v_2.$$

Definition. In $\mathbb{R}^3$, the ***dot product*** of $\mathbf{u} = (u_1, u_2, u_3) = u_1\mathbf{i} + u_2\mathbf{j} + u_3\mathbf{k}$ and $\mathbf{v} = (v_1, v_2, v_3) = v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}$ is

$$\mathbf{u} \cdot \mathbf{v} = u_1 v_1 + u_2 v_2 + u_3 v_3.$$



Theorem 9.1. *The angle between* $\mathbf{u}$ *and* $\mathbf{v}$ *is*

$$\theta = \cos^{-1}\left(\frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{u}\|\,\|\mathbf{v}\|}\right).$$

Example 9.1.

$$\begin{aligned}(1,-2,-1) \cdot (-6,2,-3) &= (1 \times -6) + (-2 \times 2) + (-1 \times -3) \\ &= -6 - 4 + 3 = -7.\end{aligned}$$

Example 9.2.

$$\begin{aligned}(\tfrac{1}{2}\mathbf{i} + 3\mathbf{j} + \mathbf{k}) \cdot (4\mathbf{i} - \mathbf{j} + 2\mathbf{k}) &= (\tfrac{1}{2} \times 4) + (3 \times -1) + (1 \times 2) \\ &= 2 - 3 + 2 = 1.\end{aligned}$$

Example 9.3. Find the angle between $\mathbf{u} = \mathbf{i} - 2\mathbf{j} - 2\mathbf{k}$ and $\mathbf{v} = 6\mathbf{i} + 3\mathbf{j} + 2\mathbf{k}$.

solution: Since $\mathbf{u} \cdot \mathbf{v} = (1,-2,-2) \cdot (6,3,2) = (1 \times 6) + (-2 \times 3) + (-2 \times 2) = 6 - 6 - 4 = -4$, $\|\mathbf{u}\| = \sqrt{1^2 + (-2)^2 + (-2)^2} = \sqrt{9} = 3$ and $\|\mathbf{v}\| = \sqrt{6^2 + 3^2 + 2^2} = \sqrt{49} = 7$, we have that

$$\theta = \cos^{-1}\left(\frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{u}\|\,\|\mathbf{v}\|}\right) = \cos^{-1}\left(-\frac{4}{21}\right) \approx 1.76 \text{ radians} \approx 98.5^\circ.$$

Example 9.4. If $A(0,0)$, $B(3,5)$ and $C(5,2)$, find $\theta = \measuredangle ACB$.

solution: θ is the angle between $\overrightarrow{CA}$ and $\overrightarrow{CB}$. We calcu-

Nokta Çarpım

Tanım. In $\mathbb{R}^2$, the ***dot product*** of $\mathbf{u} = (u_1, u_2) = u_1\mathbf{i} + u_2\mathbf{j}$ and $\mathbf{v} = (v_1, v_2) = v_1\mathbf{i} + v_2\mathbf{j}$ is

$$\mathbf{u} \cdot \mathbf{v} = u_1 v_1 + u_2 v_2.$$

Tanım. In $\mathbb{R}^3$, the ***dot product*** of $\mathbf{u} = (u_1, u_2, u_3) = u_1\mathbf{i} + u_2\mathbf{j} + u_3\mathbf{k}$ and $\mathbf{v} = (v_1, v_2, v_3) = v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}$ is

$$\mathbf{u} \cdot \mathbf{v} = u_1 v_1 + u_2 v_2 + u_3 v_3.$$

Teorem 9.1. *The angle between* $\mathbf{u}$ *and* $\mathbf{v}$ *is*

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Örnek 9.1.

$$\begin{aligned}(1,-2,-1) \cdot (-6,2,-3) &= (1 \times -6) + (-2 \times 2) + (-1 \times -3) \\ &= -6 - 4 + 3 = -7.\end{aligned}$$

Örnek 9.2.

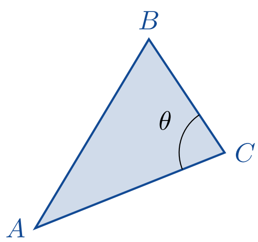
$$\begin{aligned}(\tfrac{1}{2}\mathbf{i} + 3\mathbf{j} + \mathbf{k}) \cdot (4\mathbf{i} - \mathbf{j} + 2\mathbf{k}) &= (\tfrac{1}{2} \times 4) + (3 \times -1) + (1 \times 2) \\ &= 2 - 3 + 2 = 1.\end{aligned}$$

Örnek 9.3. Find the angle between $\mathbf{u} = \mathbf{i} - 2\mathbf{j} - 2\mathbf{k}$ and $\mathbf{v} = 6\mathbf{i} + 3\mathbf{j} + 2\mathbf{k}$.

çözüm: Since $\mathbf{u} \cdot \mathbf{v} = (1,-2,-2) \cdot (6,3,2) = (1 \times 6) + (-2 \times 3) + (-2 \times 2) = 6 - 6 - 4 = -4$, $\|\mathbf{u}\| = \sqrt{1^2 + (-2)^2 + (-2)^2} = \sqrt{9} = 3$ and $\|\mathbf{v}\| = \sqrt{6^2 + 3^2 + 2^2} = \sqrt{49} = 7$, we have that

$$\theta = \cos^{-1}\left(\frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{u}\|\,\|\mathbf{v}\|}\right) = \cos^{-1}\left(-\frac{4}{21}\right) \approx 1.76 \text{ radians} \approx 98.5^\circ.$$

Örnek 9.4. If $A(0,0)$, $B(3,5)$ and $C(5,2)$, find $\theta = \measuredangle ACB$.



late that $\overrightarrow{CA} = A - C = (0,0) - (5,2) = (-5,-2)$, $\overrightarrow{CB} = B - C = (3,5) - (5,2) = (-2,3)$, $\overrightarrow{CA} \cdot \overrightarrow{CB} = (-5,-2) \cdot (-2,3) = 4$, $\left\|\overrightarrow{CA}\right\| = \sqrt{(-5)^2 + (-2)^2} = \sqrt{29}$ and $\left\|\overrightarrow{CB}\right\| = \sqrt{(-2)^2 + 3^2} = \sqrt{13}$. Therefore

$$\theta = \cos^{-1}\left(\frac{\overrightarrow{CA} \cdot \overrightarrow{CB}}{\left\|\overrightarrow{CA}\right\| \left\|\overrightarrow{CB}\right\|}\right) = \cos^{-1}\left(\frac{4}{\sqrt{29}\sqrt{13}}\right)$$
$$\approx 78.1^\circ \approx 1.36 \text{ radians}.$$

Definition. $\mathbf{u}$ and $\mathbf{v}$ are ***orthogonal*** $\iff \mathbf{u} \cdot \mathbf{v} = 0$.

Remark. Note that

$$\mathbf{u} \cdot \mathbf{v} = \|\mathbf{u}\| \|\mathbf{v}\| \cos\theta$$

by Theorem 9.1. Therefore

$$\mathbf{u} \text{ and } \mathbf{v} \text{ are orthogonal} \iff \begin{pmatrix} \mathbf{u} = \mathbf{0} \\ \text{or} \\ \mathbf{v} = \mathbf{0} \\ \text{or} \\ \theta = 90^\circ. \end{pmatrix}$$

Example 9.5. $\mathbf{u} = (3,-2)$ and $\mathbf{v} = (4,6)$ are orthogonal because $\mathbf{u} \cdot \mathbf{v} = (3,-2) \cdot (4,6) = (3 \times 4) + (-2 \times 6) = 12 - 12 = 0$.

Example 9.6. $\mathbf{u} = 3\mathbf{i} - 2\mathbf{j} + \mathbf{k}$ and $\mathbf{v} = 2\mathbf{j} + 4\mathbf{k}$ are orthogonal because $\mathbf{u} \cdot \mathbf{v} = (3 \times 0) + (-2 \times 2) + (1 \times 4) = 0 - 4 + 4 = 0$.

Example 9.7. $\mathbf{0}$ is orthogonal to every vector $\mathbf{u}$ because $\mathbf{0} \cdot \mathbf{u} = (0,0,0) \cdot (u_1, u_2, u_3) = 0u_1 + 0u_2 + 0u_3 = 0$.

Properties of the Dot Product

Let $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ be vectors. Let k be a number. Then

(i). $\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u}$;

(ii). $(k\mathbf{u}) \cdot \mathbf{v} = \mathbf{u} \cdot (k\mathbf{v}) = k(\mathbf{u} \cdot \mathbf{v})$;

(iii). $\mathbf{u} \cdot (\mathbf{v} + \mathbf{w}) = (\mathbf{u} \cdot \mathbf{v}) + (\mathbf{u} \cdot \mathbf{w})$;

(iv). $\mathbf{u} \cdot \mathbf{u} = \|\mathbf{u}\|^2$; and

(v). $\mathbf{0} \cdot \mathbf{u} = 0$.

çözüm: θ is the angle between $\overrightarrow{CA}$ and $\overrightarrow{CB}$. We calculate that $\overrightarrow{CA} = A - C = (0,0) - (5,2) = (-5,-2)$, $\overrightarrow{CB} = B - C = (3,5) - (5,2) = (-2,3)$, $\overrightarrow{CA} \cdot \overrightarrow{CB} = (-5,-2) \cdot (-2,3) = 4$, $\left\|\overrightarrow{CA}\right\| = \sqrt{(-5)^2 + (-2)^2} = \sqrt{29}$ and $\left\|\overrightarrow{CB}\right\| = \sqrt{(-2)^2 + 3^2} = \sqrt{13}$. Therefore

$$\theta = \cos^{-1}\left(\frac{\overrightarrow{CA} \cdot \overrightarrow{CB}}{\left\|\overrightarrow{CA}\right\| \left\|\overrightarrow{CB}\right\|}\right) = \cos^{-1}\left(\frac{4}{\sqrt{29}\sqrt{13}}\right)$$
$$\approx 78.1^\circ \approx 1.36 \text{ radians}.$$

Tanım. $\mathbf{u}$ and $\mathbf{v}$ are ***orthogonal*** $\iff \mathbf{u} \cdot \mathbf{v} = 0$.

Not. Note that

$$\mathbf{u} \cdot \mathbf{v} = \|\mathbf{u}\| \|\mathbf{v}\| \cos\theta$$

by Theorem 9.1. Therefore

$$\mathbf{u} \text{ and } \mathbf{v} \text{ are orthogonal} \iff \begin{pmatrix} \mathbf{u} = \mathbf{0} \\ \text{or} \\ \mathbf{v} = \mathbf{0} \\ \text{or} \\ \theta = 90^\circ. \end{pmatrix}$$

Örnek 9.5. $\mathbf{u} = (3,-2)$ and $\mathbf{v} = (4,6)$ are orthogonal because $\mathbf{u} \cdot \mathbf{v} = (3,-2) \cdot (4,6) = (3 \times 4) + (-2 \times 6) = 12 - 12 = 0$.

Örnek 9.6. $\mathbf{u} = 3\mathbf{i} - 2\mathbf{j} + \mathbf{k}$ and $\mathbf{v} = 2\mathbf{j} + 4\mathbf{k}$ are orthogonal because $\mathbf{u} \cdot \mathbf{v} = (3 \times 0) + (-2 \times 2) + (1 \times 4) = 0 - 4 + 4 = 0$.

Örnek 9.7. $\mathbf{0}$ is orthogonal to every vector $\mathbf{u}$ because $\mathbf{0} \cdot \mathbf{u} = (0,0,0) \cdot (u_1, u_2, u_3) = 0u_1 + 0u_2 + 0u_3 = 0$.

Properties of the Dot Product

Let $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ be vectors. Let k be a number. Then

(i). $\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u}$;

(ii). $(k\mathbf{u}) \cdot \mathbf{v} = \mathbf{u} \cdot (k\mathbf{v}) = k(\mathbf{u} \cdot \mathbf{v})$;

(iii). $\mathbf{u} \cdot (\mathbf{v} + \mathbf{w}) = (\mathbf{u} \cdot \mathbf{v}) + (\mathbf{u} \cdot \mathbf{w})$;

(iv). $\mathbf{u} \cdot \mathbf{u} = \|\mathbf{u}\|^2$; and

(v). $\mathbf{0} \cdot \mathbf{u} = 0$.

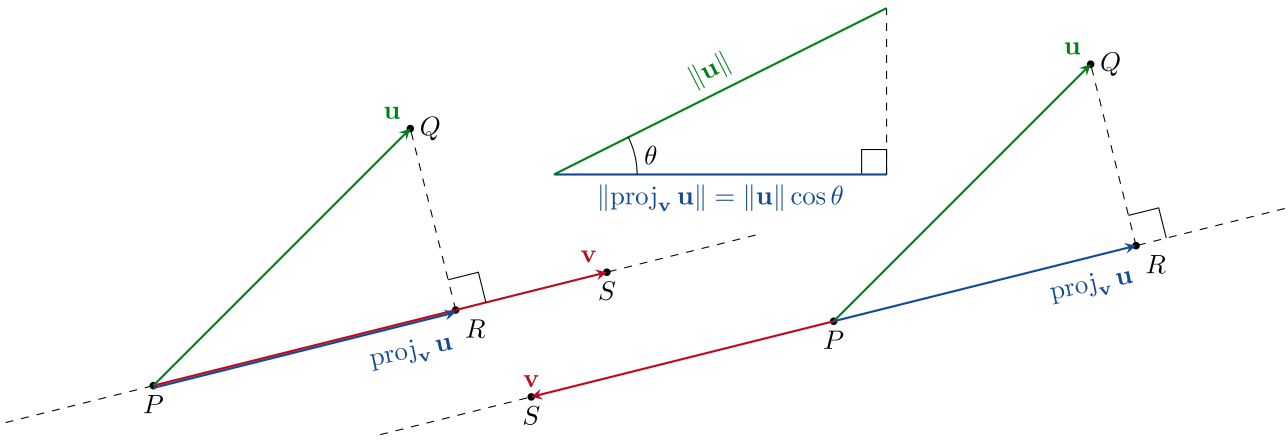


Figure 9.1: Vector Projections
Şekil 9.1: Vektör İzdüşümleri

Vector Projections

See figure 9.1.

Definition. The ***vector projection*** of $\mathbf{u}$ onto $\mathbf{v}$ is the vector

$$\text{proj}_{\mathbf{v}}\,\mathbf{u} = \overrightarrow{PR}.$$

Now

$$\begin{aligned}\text{proj}_{\mathbf{v}}\,\mathbf{u} &= \left(\text{length of } \text{proj}_{\mathbf{v}}\,\mathbf{u}\right)\begin{pmatrix}\text{a unit vector in the}\\ \text{same direction as } \mathbf{v}\end{pmatrix}\\ &= \|\text{proj}_{\mathbf{v}}\,\mathbf{u}\|\left(\frac{\mathbf{v}}{\|\mathbf{v}\|}\right)\\ &= \|\mathbf{u}\|\,(\cos\theta)\left(\frac{\mathbf{v}}{\|\mathbf{v}\|}\right)\\ &= \left(\frac{\|\mathbf{u}\|\,\|\mathbf{v}\|\cos\theta}{\|\mathbf{v}\|^2}\right)\mathbf{v}\\ &= \left(\frac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{v}\|^2}\right)\mathbf{v}.\end{aligned}$$

Since this is an important formula, we write it as a theorem.

Theorem 9.2. *The* ***vector projection*** *of* $\mathbf{u}$ *onto* $\mathbf{v}$ *is*

$$\text{proj}_{\mathbf{v}}\,\mathbf{u} = \left(\frac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{v}\|^2}\right)\mathbf{v}.$$

Example 9.8. Find the vector projection of $\mathbf{u} = 6\mathbf{i}+3\mathbf{j}+2\mathbf{k}$ onto $\mathbf{v} = \mathbf{i}-2\mathbf{j}-2\mathbf{k}$.

solution:

$$\begin{aligned}\text{proj}_{\mathbf{v}}\,\mathbf{u} &= \left(\frac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{v}\|^2}\right)\mathbf{v} = \left(\frac{6-6-4}{1+4+4}\right)(\mathbf{i}-2\mathbf{j}-2\mathbf{k})\\ &= -\frac{4}{9}\mathbf{i}+\frac{8}{9}\mathbf{j}+\frac{8}{9}\mathbf{k}.\end{aligned}$$

Example 9.9. Find the vector projection of $\mathbf{F} = 5\mathbf{i}+2\mathbf{j}$ onto $\mathbf{v} = \mathbf{i}-3\mathbf{j}$.
solution:

$$\begin{aligned}\text{proj}_{\mathbf{v}}\,\mathbf{F} &= \left(\frac{\mathbf{F}\cdot\mathbf{v}}{\|\mathbf{v}\|^2}\right)\mathbf{v} = \left(\frac{5-6}{1+9}\right)(\mathbf{i}-3\mathbf{j})\\ &= -\frac{1}{10}\mathbf{i}+\frac{3}{10}\mathbf{j}.\end{aligned}$$

Vector Projections

See figure 9.1.

Tanım. The ***vector projection*** of $\mathbf{u}$ onto $\mathbf{v}$ is the vector

$$\text{proj}_{\mathbf{v}}\,\mathbf{u} = \overrightarrow{PR}.$$

Now

$$\begin{aligned}\text{proj}_{\mathbf{v}}\,\mathbf{u} &= \left(\text{length of } \text{proj}_{\mathbf{v}}\,\mathbf{u}\right)\begin{pmatrix}\text{a unit vector in the}\\ \text{same direction as } \mathbf{v}\end{pmatrix}\\ &= \|\text{proj}_{\mathbf{v}}\,\mathbf{u}\|\left(\frac{\mathbf{v}}{\|\mathbf{v}\|}\right)\\ &= \|\mathbf{u}\|\,(\cos\theta)\left(\frac{\mathbf{v}}{\|\mathbf{v}\|}\right)\\ &= \left(\frac{\|\mathbf{u}\|\,\|\mathbf{v}\|\cos\theta}{\|\mathbf{v}\|^2}\right)\mathbf{v}\\ &= \left(\frac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{v}\|^2}\right)\mathbf{v}.\end{aligned}$$

Since this is an important formula, we write it as a theorem.

Teorem 9.2. *The* ***vector projection*** *of* $\mathbf{u}$ *onto* $\mathbf{v}$ *is*

$$\text{proj}_{\mathbf{v}}\,\mathbf{u} = \left(\frac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{v}\|^2}\right)\mathbf{v}.$$

Örnek 9.8. Find the vector projection of $\mathbf{u} = 6\mathbf{i}+3\mathbf{j}+2\mathbf{k}$ onto $\mathbf{v} = \mathbf{i}-2\mathbf{j}-2\mathbf{k}$.

çözüm:

$$\begin{aligned}\text{proj}_{\mathbf{v}}\,\mathbf{u} &= \left(\frac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{v}\|^2}\right)\mathbf{v} = \left(\frac{6-6-4}{1+4+4}\right)(\mathbf{i}-2\mathbf{j}-2\mathbf{k})\\ &= -\frac{4}{9}\mathbf{i}+\frac{8}{9}\mathbf{j}+\frac{8}{9}\mathbf{k}.\end{aligned}$$

Örnek 9.9. Find the vector projection of $\mathbf{F} = 5\mathbf{i}+2\mathbf{j}$ onto $\mathbf{v} = \mathbf{i}-3\mathbf{j}$.
çözüm:

$$\begin{aligned}\text{proj}_{\mathbf{v}}\,\mathbf{F} &= \left(\frac{\mathbf{F}\cdot\mathbf{v}}{\|\mathbf{v}\|^2}\right)\mathbf{v} = \left(\frac{5-6}{1+9}\right)(\mathbf{i}-3\mathbf{j})\\ &= -\frac{1}{10}\mathbf{i}+\frac{3}{10}\mathbf{j}.\end{aligned}$$

Problems

Problem 9.1. For each pair of vectors below, find

(i). $\mathbf{u} \cdot \mathbf{v}$;

(ii). $\|\mathbf{u}\|$ and $\|\mathbf{v}\|$;

(iii). $\cos\theta$ (where θ is the angle between $\mathbf{u}$ and $\mathbf{v}$); and

(iv). $\operatorname{proj}_{\mathbf{v}} \mathbf{u}$.

(a). $\mathbf{u} = -2\mathbf{i} + 4\mathbf{j} - \sqrt{5}\mathbf{k}$
$\mathbf{v} = 2\mathbf{i} - 4\mathbf{j} + \sqrt{5}\mathbf{k}$

(b). $\mathbf{u} = 3\mathbf{j} + 4\mathbf{k}$
$\mathbf{v} = 10\mathbf{i} + 11\mathbf{j} - 2\mathbf{k}$

(c). $\mathbf{u} = 2\mathbf{i} + 2\mathbf{j} + \mathbf{k}$
$\mathbf{v} = 2\mathbf{i} + 10\mathbf{j} - 11\mathbf{k}$

Problem 9.2. A triangle has vertices at $A(-1, 0)$, $B(2, 1)$ and $C(1, -2)$. Find the internal angles of the triangle.

Problem 9.3. Let $A(1, 1, 1)$, $B(2, 3, 2)$, $C(1, 4, 4)$ and $D(0, 2, 3)$ be four points in $\mathbb{R}^3$. Are the vectors $\overrightarrow{AC}$ and $\overrightarrow{BD}$ orthogonal?

Problem 9.4. Let $\mathbf{u}$ and $\mathbf{v}$ be vectors. Let θ denote the angle between $\mathbf{u}$ and $\mathbf{u}+\mathbf{v}$; and let ϕ denote the angle between $\mathbf{u}+\mathbf{v}$ and $\mathbf{v}$. See figure 9.2.

(a). Show that if $\|\mathbf{u}\| = \|\mathbf{v}\|$, then $\mathbf{u} \cdot (\mathbf{u}+\mathbf{v}) = (\mathbf{u}+\mathbf{v}) \cdot \mathbf{v}$

(b). Show that if $\|\mathbf{u}\| = \|\mathbf{v}\|$, then $\theta = \phi$.

Problem 9.5. A water pipe runs due north then due east. The northwards part slopes upwards with a slope of 20%. The eastwards part slopes upwards with a slope of 10%. See figure 9.3. Find the angle θ required at the turn from north to east.

Sorular

Soru 9.1. For each pair of vectors below, find

(i). $\mathbf{u} \cdot \mathbf{v}$;

(ii). $\|\mathbf{u}\|$ and $\|\mathbf{v}\|$;

(iii). $\cos\theta$ (where θ is the angle between $\mathbf{u}$ and $\mathbf{v}$); and

(iv). $\operatorname{proj}_{\mathbf{v}} \mathbf{u}$.

Soru 9.2. A triangle has vertices at $A(-1, 0)$, $B(2, 1)$ and $C(1, -2)$. Find the internal angles of the triangle.

Soru 9.3. Let $A(1, 1, 1)$, $B(2, 3, 2)$, $C(1, 4, 4)$ and $D(0, 2, 3)$ be four points in $\mathbb{R}^3$. Are the vectors $\overrightarrow{AC}$ and $\overrightarrow{BD}$ orthogonal?

Soru 9.4. Let $\mathbf{u}$ and $\mathbf{v}$ be vectors. Let θ denote the angle between $\mathbf{u}$ and $\mathbf{u}+\mathbf{v}$; and let ϕ denote the angle between $\mathbf{u}+\mathbf{v}$ and $\mathbf{v}$. See figure 9.2.

(a). Show that if $\|\mathbf{u}\| = \|\mathbf{v}\|$, then $\mathbf{u} \cdot (\mathbf{u}+\mathbf{v}) = (\mathbf{u}+\mathbf{v}) \cdot \mathbf{v}$

(b). Show that if $\|\mathbf{u}\| = \|\mathbf{v}\|$, then $\theta = \phi$.

Soru 9.5. A water pipe runs due north then due east. The northwards part slopes upwards with a slope of 20%. The eastwards part slopes upwards with a slope of 10%. See figure 9.3. Find the angle θ required at the turn from north to east.

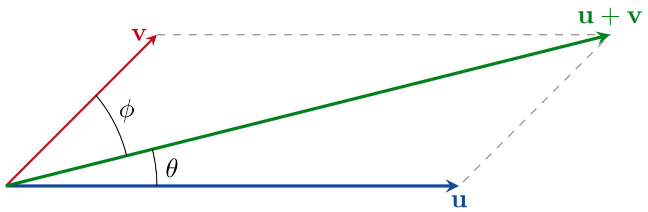


Figure 9.2: The vectors considered in Exercise 9.4.
Şekil 9.2:

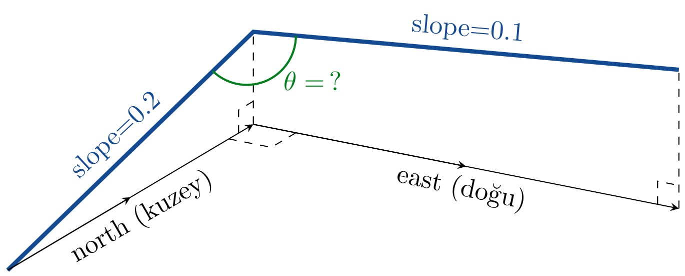
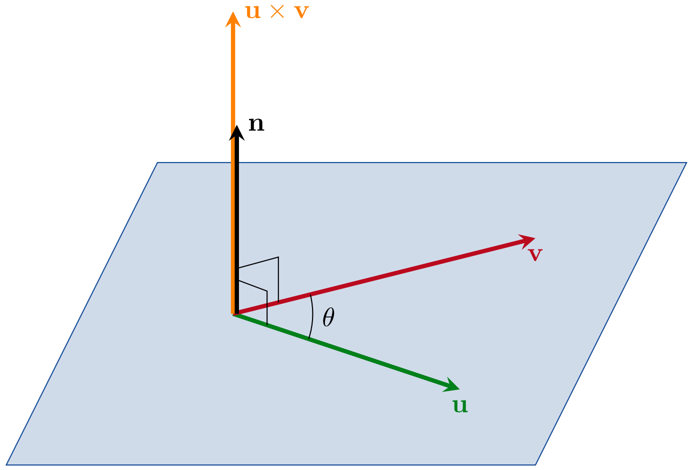


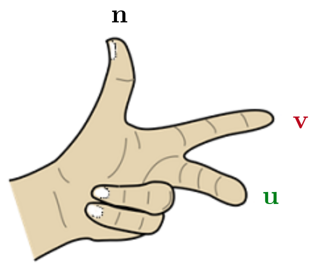
Figure 9.3: A water pipe.
Şekil 9.3:

10

The Cross Product

Vektörel Çarpım





Let $\mathbf{n}$ be a unit vector which satisfies

(i). $\mathbf{n}$ is orthogonal to $\mathbf{u}$ $\left(\begin{smallmatrix}\uparrow^{\mathbf{n}} & \\ \llcorner\!\longrightarrow & \mathbf{u}\end{smallmatrix}\right)$;

(ii). $\mathbf{n}$ is orthogonal to $\mathbf{v}$ $\left(\begin{smallmatrix}\uparrow^{\mathbf{n}} & \\ \llcorner\!\longrightarrow & \mathbf{v}\end{smallmatrix}\right)$; and

(iii). the direction of $\mathbf{n}$ is chosen using the left-hand rule.

Definition. The ***cross product*** of $\mathbf{u}$ and $\mathbf{v}$ is

$$\mathbf{u}\times\mathbf{v} = \|\mathbf{u}\|\,\|\mathbf{v}\|\,(\sin\theta)\mathbf{n}.$$

Remark.

- $\mathbf{u}\cdot\mathbf{v}$ is a number.
- $\mathbf{u}\times\mathbf{v}$ is a vector.

Remark.

$$\left(\begin{smallmatrix}\mathbf{u}\text{ and }\mathbf{v}\\ \text{are parallel}\end{smallmatrix}\right) \iff \theta = 0^\circ \text{ or } 180^\circ$$
$$\implies \sin\theta = 0 \implies \mathbf{u}\times\mathbf{v} = \mathbf{0}.$$

Let $\mathbf{n}$ be a unit vector which satisfies

(i). $\mathbf{n}$ is orthogonal to $\mathbf{u}$ $\left(\begin{smallmatrix}\uparrow^{\mathbf{n}} & \\ \llcorner\!\longrightarrow & \mathbf{u}\end{smallmatrix}\right)$;

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Tanım. The ***cross product*** of $\mathbf{u}$ and $\mathbf{v}$ is

$$\mathbf{u}\times\mathbf{v} = \|\mathbf{u}\|\,\|\mathbf{v}\|\,(\sin\theta)\mathbf{n}.$$

Not.

- $\mathbf{u}\cdot\mathbf{v}$ is a number.
- $\mathbf{u}\times\mathbf{v}$ is a vector.

Not.

$$\left(\begin{smallmatrix}\mathbf{u}\text{ and }\mathbf{v}\\ \text{are parallel}\end{smallmatrix}\right) \iff \theta = 0^\circ \text{ or } 180^\circ$$
$$\implies \sin\theta = 0 \implies \mathbf{u}\times\mathbf{v} = \mathbf{0}.$$

Properties of the Cross Product

Let $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ be vectors. Let r and s be numbers. Then

(i). $(r\mathbf{u}) \times (s\mathbf{v}) = (rs)(\mathbf{u} \times \mathbf{v})$;

(ii). $\mathbf{u} \times (\mathbf{v} + \mathbf{w}) = (\mathbf{u} \times \mathbf{v}) + (\mathbf{u} \times \mathbf{w})$;

(iii). $\mathbf{v} \times \mathbf{u} = -\mathbf{u} \times \mathbf{v}$;

(iv). $(\mathbf{v} + \mathbf{w}) \times \mathbf{u} = (\mathbf{v} \times \mathbf{u}) + (\mathbf{w} \times \mathbf{u})$;

(v). $\mathbf{0} \times \mathbf{u} = \mathbf{0}$; and

(vi). $\mathbf{u} \times (\mathbf{v} \times \mathbf{w}) = (\mathbf{u} \cdot \mathbf{w})\mathbf{v} - (\mathbf{u} \cdot \mathbf{v})\mathbf{w}$.

Properties of the Cross Product

Let $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ be vectors. Let r and s be numbers. Then

(i). $(r\mathbf{u}) \times (s\mathbf{v}) = (rs)(\mathbf{u} \times \mathbf{v})$;

(ii). $\mathbf{u} \times (\mathbf{v} + \mathbf{w}) = (\mathbf{u} \times \mathbf{v}) + (\mathbf{u} \times \mathbf{w})$;

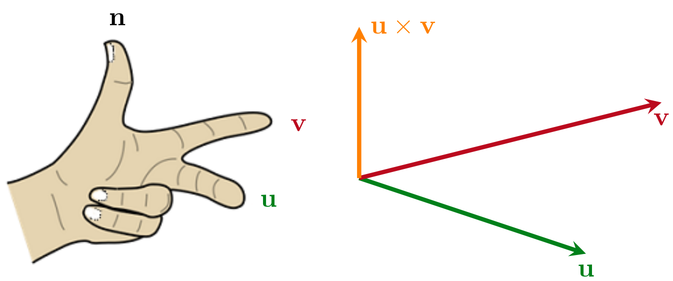
(iii). $\mathbf{v} \times \mathbf{u} = -\mathbf{u} \times \mathbf{v}$;

(iv). $(\mathbf{v} + \mathbf{w}) \times \mathbf{u} = (\mathbf{v} \times \mathbf{u}) + (\mathbf{w} \times \mathbf{u})$;

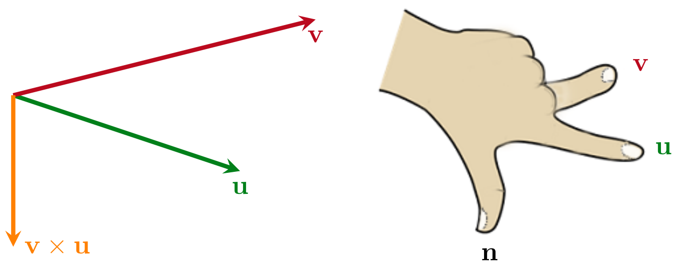
(v). $\mathbf{0} \times \mathbf{u} = \mathbf{0}$; and

(vi). $\mathbf{u} \times (\mathbf{v} \times \mathbf{w}) = (\mathbf{u} \cdot \mathbf{w})\mathbf{v} - (\mathbf{u} \cdot \mathbf{v})\mathbf{w}$.

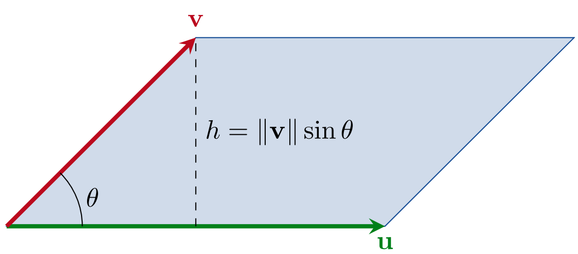
Property (iii)



Özellik (iii)



Area of a Parallelogram

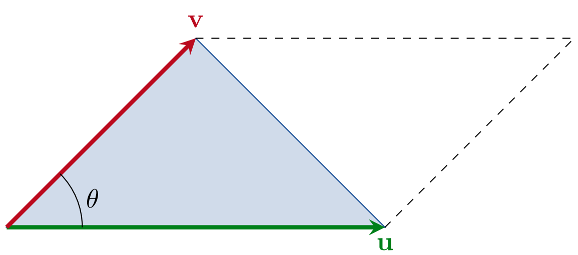


Paralelkenarın Alanı

$$\text{area} = (\text{base})\,(\text{height}) = \|\mathbf{u}\|\,\|\mathbf{v}\| \sin\theta = \|\mathbf{u} \times \mathbf{v}\|\,.$$

$$\text{alan} = (\text{taban})\,(\text{yükseklik}) = \|\mathbf{u}\|\,\|\mathbf{v}\| \sin\theta = \|\mathbf{u} \times \mathbf{v}\|\,.$$

Area of a Triangle



Üçgenin Alanı

$$\begin{aligned}\text{area of triangle} &= \tfrac{1}{2}\,(\text{area of parallelogram})\\ &= \tfrac{1}{2}\,\|\mathbf{u} \times \mathbf{v}\|\,.\end{aligned}$$

$$\begin{aligned}\text{üçgenin alanı} &= \tfrac{1}{2}\,(\text{paralelkenarın alanı})\\ &= \tfrac{1}{2}\,\|\mathbf{u} \times \mathbf{v}\|\,.\end{aligned}$$

A Formula for $\mathbf{u} \times \mathbf{v}$

???

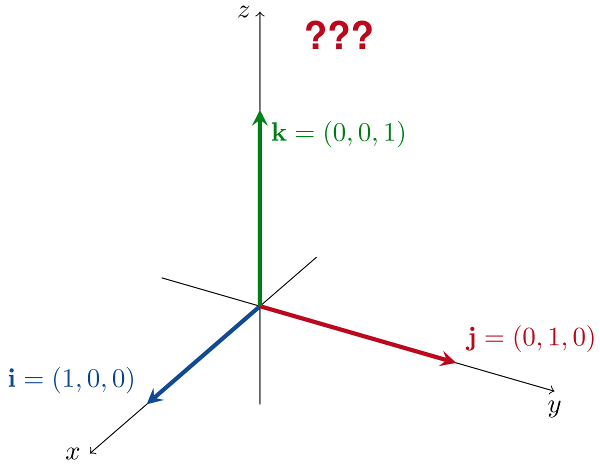


Figure 10.1: The standard unit vectors in $\mathbb{R}^3$.
Şekil 10.1:

Note first that

$$\mathbf{i} \times \mathbf{i} = \|\mathbf{i}\|\,\|\mathbf{i}\| \sin 0^\circ \mathbf{n} = \mathbf{0}.$$

Similarly $\mathbf{j} \times \mathbf{j} = \mathbf{0}$ and $\mathbf{k} \times \mathbf{k} = \mathbf{0}$ also.

Next note that $\mathbf{i} \times \mathbf{j}$ must point in the same direction at $\mathbf{k}$ by the left-hand rule. Thus

$$\mathbf{i} \times \mathbf{j} = \|\mathbf{i}\|\,\|\mathbf{j}\| \sin 90^\circ \mathbf{k} = \mathbf{k}.$$

We then immediately also have

$$\mathbf{j} \times \mathbf{i} = -\left(\mathbf{i} \times \mathbf{j}\right) = -\mathbf{k}.$$

It is left for you to check that

$$\mathbf{j} \times \mathbf{k} = \mathbf{i}, \qquad \mathbf{k} \times \mathbf{j} = -\mathbf{i}, \qquad \mathbf{i} \times \mathbf{k} = -\mathbf{j} \quad \text{and} \quad \mathbf{k} \times \mathbf{i} = \mathbf{j}.$$

Now suppose that $\mathbf{u} = u_1\mathbf{i}+u_2\mathbf{j}+u_3\mathbf{k}$ and $\mathbf{v} = v_1\mathbf{i}+v_2\mathbf{j}+v_3\mathbf{k}$. Then we can calculate that

Note first that

$$\mathbf{i} \times \mathbf{i} = \|\mathbf{i}\|\,\|\mathbf{i}\| \sin 0^\circ \mathbf{n} = \mathbf{0}.$$

Similarly $\mathbf{j} \times \mathbf{j} = \mathbf{0}$ and $\mathbf{k} \times \mathbf{k} = \mathbf{0}$ also.

Next note that $\mathbf{i} \times \mathbf{j}$ must point in the same direction at $\mathbf{k}$ by the left-hand rule. Thus

$$\mathbf{i} \times \mathbf{j} = \|\mathbf{i}\|\,\|\mathbf{j}\| \sin 90^\circ \mathbf{k} = \mathbf{k}.$$

We then immediately also have

$$\mathbf{j} \times \mathbf{i} = -\left(\mathbf{i} \times \mathbf{j}\right) = -\mathbf{k}.$$

It is left for you to check that

$$\mathbf{j} \times \mathbf{k} = \mathbf{i}, \qquad \mathbf{k} \times \mathbf{j} = -\mathbf{i}, \qquad \mathbf{i} \times \mathbf{k} = -\mathbf{j} \quad \text{and} \quad \mathbf{k} \times \mathbf{i} = \mathbf{j}.$$

Now suppose that $\mathbf{u} = u_1\mathbf{i}+u_2\mathbf{j}+u_3\mathbf{k}$ and $\mathbf{v} = v_1\mathbf{i}+v_2\mathbf{j}+v_3\mathbf{k}$. Then we can calculate that

$$\begin{aligned}
\mathbf{u} \times \mathbf{v} &= (u_1\mathbf{i} + u_2\mathbf{j} + u_3\mathbf{k}) \times (v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}) \\
&= u_1v_1\mathbf{i} \times \mathbf{i} + u_1v_2\mathbf{i} \times \mathbf{j} + u_1v_3\mathbf{i} \times \mathbf{k} + u_2v_1\mathbf{j} \times \mathbf{i} + u_2v_2\mathbf{j} \times \mathbf{j} + u_2v_3\mathbf{j} \times \mathbf{k} + u_3v_1\mathbf{k} \times \mathbf{i} + u_3v_2\mathbf{k} \times \mathbf{j} + u_3v_3\mathbf{k} \times \mathbf{k} \\
&= \mathbf{0} + u_1v_2\mathbf{k} - u_1v_3\mathbf{j} - u_2v_1\mathbf{k} + \mathbf{0} + u_2v_3\mathbf{i} + u_3v_1\mathbf{j} - u_3v_2\mathbf{i} + \mathbf{0} \\
&= (u_2v_3 - u_3v_2)\mathbf{i} - (u_1v_3 - u_3v_1)\mathbf{j} + (u_1v_2 - u_2v_1)\mathbf{k}.
\end{aligned}$$

Theorem 10.1. *If* $\mathbf{u} = u_1\mathbf{i} + u_2\mathbf{j} + u_3\mathbf{k}$ *and* $\mathbf{v} = v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}$*, then*

$$\boxed{\mathbf{u} \times \mathbf{v} = (u_2v_3 - u_3v_2)\mathbf{i} - (u_1v_3 - u_3v_1)\mathbf{j} + (u_1v_2 - u_2v_1)\mathbf{k}}$$

Teorem 10.1. *If* $\mathbf{u} = u_1\mathbf{i} + u_2\mathbf{j} + u_3\mathbf{k}$ *and* $\mathbf{v} = v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}$*, then*

$$\boxed{\mathbf{u} \times \mathbf{v} = (u_2v_3 - u_3v_2)\mathbf{i} - (u_1v_3 - u_3v_1)\mathbf{j} + (u_1v_2 - u_2v_1)\mathbf{k}}$$

If you studied matrices and determinants at high school, then you may prefer to use the following symbolic determinant formula instead.

$$\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}.$$

If you studied matrices and determinants at high school, then you may prefer to use the following symbolic determinant formula instead.

$$\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}.$$

Example 10.1. Find $\mathbf{u}\times\mathbf{v}$ and $\mathbf{v}\times\mathbf{u}$ if $\mathbf{u}=2\mathbf{i}+\mathbf{j}+\mathbf{k}$ and $\mathbf{v}=-4\mathbf{i}+3\mathbf{j}+\mathbf{k}$.

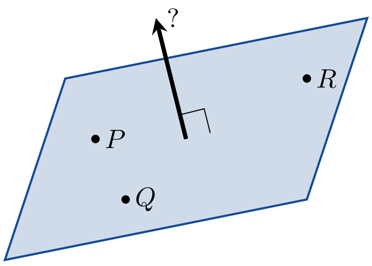
solution:

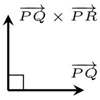
$$\mathbf{u}\times\mathbf{v}=(1-3)\mathbf{i}-(2--4)\mathbf{j}+(6--4)\mathbf{k}=-2\mathbf{i}-6\mathbf{j}+10\mathbf{k}$$

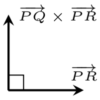
and

$$\mathbf{v}\times\mathbf{u}=-\mathbf{u}\times\mathbf{v}=2\mathbf{i}+6\mathbf{j}-10\mathbf{k}.$$

Example 10.2. Find a vector perpendicular to the plane containing the three points $P(1,-1,0)$, $Q(2,1,-1)$ and $R(-1,1,2)$.



solution: The vector $\overrightarrow{PQ}\times\overrightarrow{PR}$ is perpendicular to the plane because 

and 

. We calculate that

$$\begin{aligned}\overrightarrow{PQ}&=Q-P=(2,1,-1)-(1,-1,0)\\&=(2-1,1+1,-1-0)=\mathbf{i}+2\mathbf{j}-\mathbf{k}\\\overrightarrow{PR}&=R-P=(-1,1,2)-(1,-1,0)\\&=(-1-1,1+1,2-0)=-2\mathbf{i}+2\mathbf{j}+2\mathbf{k}\\\overrightarrow{PQ}\times\overrightarrow{PR}&=(4+2)\mathbf{i}-(2-2)\mathbf{j}+(2+4)\mathbf{k}=6\mathbf{i}+6\mathbf{k}.\end{aligned}$$

Example 10.3. Find the area of triangle PQR.

solution: The area of the triangle is

$$\begin{aligned}\text{area}&=\frac{1}{2}\left\|\overrightarrow{PQ}\times\overrightarrow{PR}\right\|=\frac{1}{2}\left\|6\mathbf{i}+6\mathbf{k}\right\|\\&=\frac{1}{2}\sqrt{6^2+0^2+6^2}=3\sqrt{2}.\end{aligned}$$

Example 10.4. Find a unit vector perpendicular to the plane containing P, Q and R.

solution: We know that $\overrightarrow{PQ}\times\overrightarrow{PR}$ is perpendicular to the plane. We just need to normalise this vector to find a unit vector.

$$\mathbf{n}=\frac{\overrightarrow{PQ}\times\overrightarrow{PR}}{\left\|\overrightarrow{PQ}\times\overrightarrow{PR}\right\|}=\frac{6\mathbf{i}+6\mathbf{k}}{6\sqrt{2}}=\frac{1}{\sqrt{2}}\mathbf{i}+\frac{1}{\sqrt{2}}\mathbf{k}.$$

Example 10.5. A triangle is inscribed inside a cube of side 2 as shown in figure 10.2. Use the cross product to find the area of the triangle.

solution: First we draw coordinate axes and assign coordinates to the vertices of the triangle. See figure 10.3. Then we can calculate

$$\overrightarrow{AB}=B-A=(2,2,0)-(2,0,0)=(0,2,0)=2\mathbf{j}$$

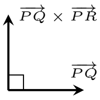
Örnek 10.1. Find $\mathbf{u}\times\mathbf{v}$ and $\mathbf{v}\times\mathbf{u}$ if $\mathbf{u}=2\mathbf{i}+\mathbf{j}+\mathbf{k}$ and $\mathbf{v}=-4\mathbf{i}+3\mathbf{j}+\mathbf{k}$.

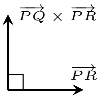
çözüm:

$$\mathbf{u}\times\mathbf{v}=(1-3)\mathbf{i}-(2--4)\mathbf{j}+(6--4)\mathbf{k}=-2\mathbf{i}-6\mathbf{j}+10\mathbf{k}$$

and

$$\mathbf{v}\times\mathbf{u}=-\mathbf{u}\times\mathbf{v}=2\mathbf{i}+6\mathbf{j}-10\mathbf{k}.$$

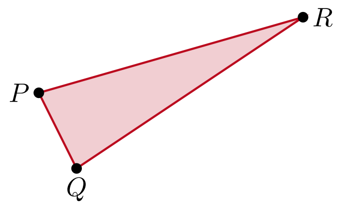
Örnek 10.2. Find a vector perpendicular to the plane containing the three points $P(1,-1,0)$, $Q(2,1,-1)$ and $R(-1,1,2)$.
çözüm: The vector $\overrightarrow{PQ}\times\overrightarrow{PR}$ is perpendicular to the plane because 

and 

. We calculate that

$$\begin{aligned}\overrightarrow{PQ}&=Q-P=(2,1,-1)-(1,-1,0)\\&=(2-1,1+1,-1-0)=\mathbf{i}+2\mathbf{j}-\mathbf{k}\\\overrightarrow{PR}&=R-P=(-1,1,2)-(1,-1,0)\\&=(-1-1,1+1,2-0)=-2\mathbf{i}+2\mathbf{j}+2\mathbf{k}\\\overrightarrow{PQ}\times\overrightarrow{PR}&=(4+2)\mathbf{i}-(2-2)\mathbf{j}+(2+4)\mathbf{k}=6\mathbf{i}+6\mathbf{k}.\end{aligned}$$

Örnek 10.3. Find the area of triangle PQR.



çözüm: The area of the triangle is

$$\begin{aligned}\text{area}&=\frac{1}{2}\left\|\overrightarrow{PQ}\times\overrightarrow{PR}\right\|=\frac{1}{2}\left\|6\mathbf{i}+6\mathbf{k}\right\|\\&=\frac{1}{2}\sqrt{6^2+0^2+6^2}=3\sqrt{2}.\end{aligned}$$

Örnek 10.4. Find a unit vector perpendicular to the plane containing P, Q and R.

çözüm: We know that $\overrightarrow{PQ}\times\overrightarrow{PR}$ is perpendicular to the plane. We just need to normalise this vector to find a unit vector.

$$\mathbf{n}=\frac{\overrightarrow{PQ}\times\overrightarrow{PR}}{\left\|\overrightarrow{PQ}\times\overrightarrow{PR}\right\|}=\frac{6\mathbf{i}+6\mathbf{k}}{6\sqrt{2}}=\frac{1}{\sqrt{2}}\mathbf{i}+\frac{1}{\sqrt{2}}\mathbf{k}.$$

Örnek 10.5. A triangle is inscribed inside a cube of side 2 as shown in figure 10.2. Use the cross product to find the area of the triangle.

çözüm: First we draw coordinate axes and assign coordinates to the vertices of the triangle. See figure 10.3. Then we can calculate

$$\overrightarrow{AB}=B-A=(2,2,0)-(2,0,0)=(0,2,0)=2\mathbf{j}$$

and

$$\overrightarrow{AC}=C-A=(0,0,2)-(2,0,0)=(-2,0,2)=-2\mathbf{i}+2\mathbf{k}.$$

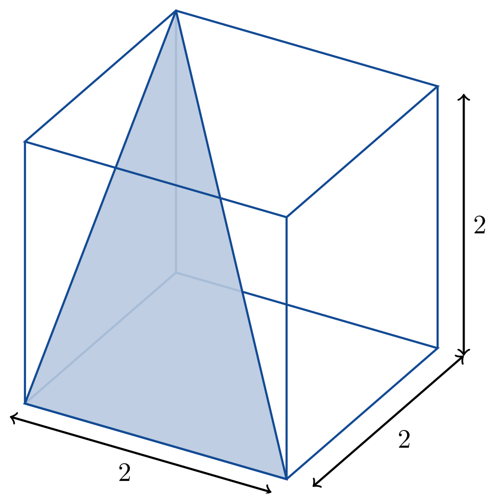


Figure 10.2: A triangle inscribed inside a cube of side 2.
Şekil 10.2:

and

$$\overrightarrow{AC} = C - A = (0,0,2) - (2,0,0) = (-2,0,2) = -2\mathbf{i} + 2\mathbf{k}.$$

It follows that

$$\overrightarrow{AB} \times \overrightarrow{AC} = (2\mathbf{j}) \times (-2I \times 2\mathbf{k}) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 2 & 0 \\ -2 & 0 & 2 \end{vmatrix}$$
$$= \mathbf{i}(4-0) - \mathbf{j}(0-0) + \mathbf{k}(0 - -4) = 4\mathbf{i} + 4\mathbf{k}.$$

Therefore

$$\text{area of triangle} = \frac{1}{2}\left\|\overrightarrow{AB} \times \overrightarrow{AC}\right\| = \frac{1}{2}\sqrt{4^2 + 0^2 + 4^2}$$
$$= \frac{1}{2}\sqrt{32} = \frac{1}{2}\sqrt{4}\sqrt{8} = \sqrt{8} = 2\sqrt{2}.$$

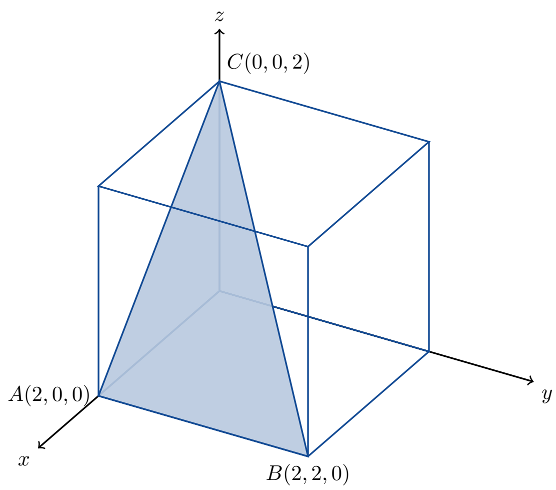


Figure 10.3: A triangle inscribed inside a cube of side 2.
Şekil 10.3:

It follows that

$$\overrightarrow{AB} \times \overrightarrow{AC} = (2\mathbf{j}) \times (-2I \times 2\mathbf{k}) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 2 & 0 \\ -2 & 0 & 2 \end{vmatrix}$$
$$= \mathbf{i}(4-0) - \mathbf{j}(0-0) + \mathbf{k}(0 - -4) = 4\mathbf{i} + 4\mathbf{k}.$$

Therefore

$$\text{area of triangle} = \frac{1}{2}\left\|\overrightarrow{AB} \times \overrightarrow{AC}\right\| = \frac{1}{2}\sqrt{4^2 + 0^2 + 4^2}$$
$$= \frac{1}{2}\sqrt{32} = \frac{1}{2}\sqrt{4}\sqrt{8} = \sqrt{8} = 2\sqrt{2}.$$

The Triple Scalar Product

Definition. The ***triple scalar product*** of $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ is

$$(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w}.$$

The Triple Scalar Product

Tanım. The ***triple scalar product*** of $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ is

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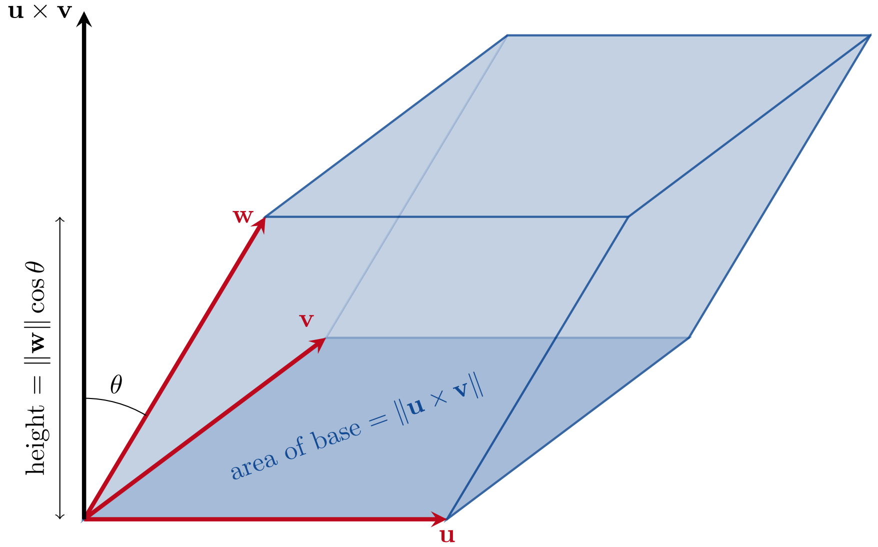
The Volume of a Parallelepiped

Paralelyüzlünün Hacmi



$$\begin{aligned} \text{volume} &= \big(\text{area of base}\big)\big(\text{height}\big) \\ &= \|\mathbf{u} \times \mathbf{v}\|\,\|\mathbf{w}\| \cos\theta \\ &= |(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w}| \end{aligned}$$

$$\begin{aligned} \text{hacim} &= \big(\text{taban alanı}\big)\big(\text{yükseklik}\big) \\ &= \|\mathbf{u} \times \mathbf{v}\|\,\|\mathbf{w}\| \cos\theta \\ &= |(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w}| \end{aligned}$$

One Final Comment

We can do the dot product in both $\mathbb{R}^2$ and $\mathbb{R}^3$. But we can only do the cross product in $\mathbb{R}^3$. There is no cross product in $\mathbb{R}^2$.

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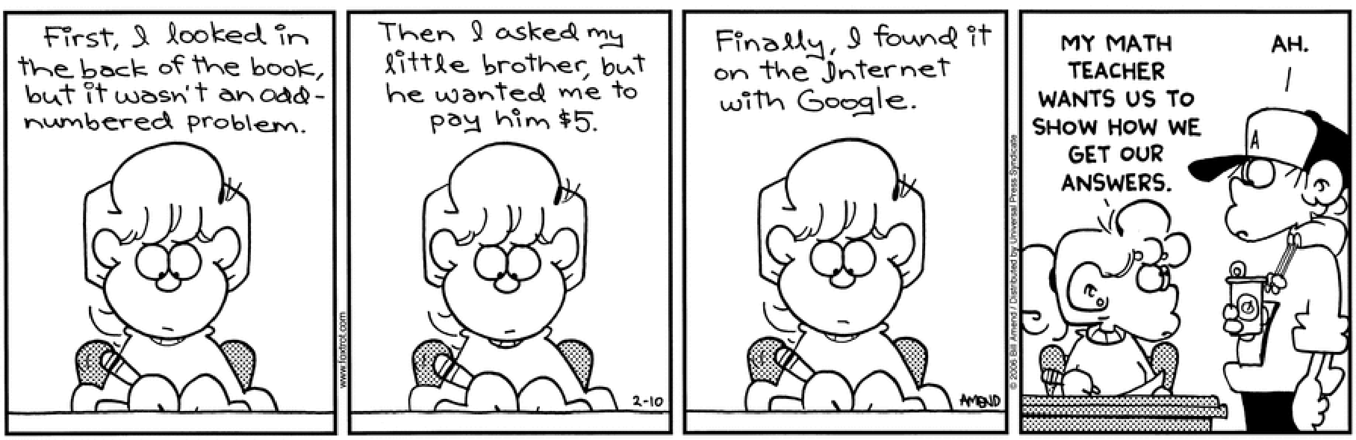

Figure 10.4: A web comic taken from https://www.gocomics.com/foxtrot/2006/02/10 .
Şekil 10.4: https://www.gocomics.com/foxtrot/2006/02/10 adresinden alınan bir web çizgi romanı.

Problems

Problem 10.1. For each pair of vectors below, find $\mathbf{u} \times \mathbf{v}$.

(a). $\mathbf{u} = 2\mathbf{i} - 2\mathbf{j} - \mathbf{k}$
$\mathbf{v} = \mathbf{i} - \mathbf{k}$

(b). $\mathbf{u} = 2\mathbf{i} - 2\mathbf{j} + 4\mathbf{k}$
$\mathbf{v} = -\mathbf{i} + \mathbf{j} - 2\mathbf{k}$

(c). $\mathbf{u} = 2\mathbf{i}$
$\mathbf{v} = -3\mathbf{j}$

(d). $\mathbf{u} = -8\mathbf{i} - 2\mathbf{j} - 4\mathbf{k}$
$\mathbf{v} = 2\mathbf{i} + 2\mathbf{j} + \mathbf{k}$

(e). $\mathbf{u} = \mathbf{i} - \mathbf{k}$
$\mathbf{v} = \mathbf{j} + \mathbf{k}$

(f). $\mathbf{u} = \mathbf{i} + \mathbf{j}$
$\mathbf{v} = \mathbf{i} - \mathbf{j}$

(g). $\mathbf{u} = \mathbf{i} + \mathbf{j} - \mathbf{k}$
$\mathbf{v} = \mathbf{0}$

(h). $\mathbf{u} = \mathbf{i} + \mathbf{j} - \mathbf{k}$
$\mathbf{v} = \mathbf{i}$

(i). $\mathbf{u} = \frac{3}{2}\mathbf{i} - \frac{1}{2}\mathbf{j} + \mathbf{k}$
$\mathbf{v} = \mathbf{i} + \mathbf{j} - 2\mathbf{k}$

Problem 10.2.

(a). Find the area of the triangle with vertices at $A(0, 0, 0)$, $B(-1, 1, -1)$ and $C(3, 0, 3)$.

(b). Find a unit vector which is perpendicular to the plane containing A, B and C.

Problem 10.3. Let $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ be vectors. Which of the following make sense? Give reasons for your answers. b

(a). $1 \cdot \mathbf{u}$.

(b). $(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w}$

(c). $\mathbf{u} \times (\mathbf{v} \cdot \mathbf{w})$

(d). $\mathbf{u} \times (\mathbf{v} \times \mathbf{w})$

(e). $\mathbf{u} \cdot (\mathbf{v} \cdot \mathbf{w})$

Problem 10.4. Use the cross product to calculate the area of the triangles shown in figures 10.5 and 10.6.

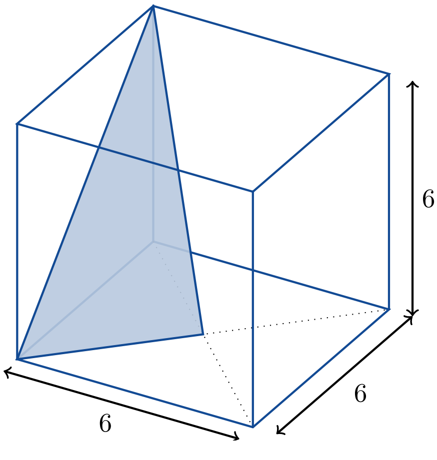

Figure 10.5: Another triangle inscribed inside a cube.
Şekil 10.5:

Problem 10.5. Calculate the triple scalar product of $\mathbf{u} = 2\mathbf{i} + \mathbf{j}$, $\mathbf{v} = 2\mathbf{i} - \mathbf{j} + \mathbf{k}$ and $\mathbf{w} = \mathbf{i} + 2\mathbf{k}$.

Sorular

Soru 10.1. For each pair of vectors below, find $\mathbf{u} \times \mathbf{v}$.

Soru 10.2.

(a). Find the area of the triangle with vertices at $A(0, 0, 0)$, $B(-1, 1, -1)$ and $C(3, 0, 3)$.

(b). Find a unit vector which is perpendicular to the plane containing A, B and C.

Soru 10.3. Let $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ be vectors. Which of the following make sense? Give reasons for your answers.

(a). $1 \cdot \mathbf{u}$.

(b). $(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w}$

(c). $\mathbf{u} \times (\mathbf{v} \cdot \mathbf{w})$

(d). $\mathbf{u} \times (\mathbf{v} \times \mathbf{w})$

(e). $\mathbf{u} \cdot (\mathbf{v} \cdot \mathbf{w})$

Soru 10.4. Use the cross product to calculate the area of the triangles shown in figures 10.5 and 10.6.

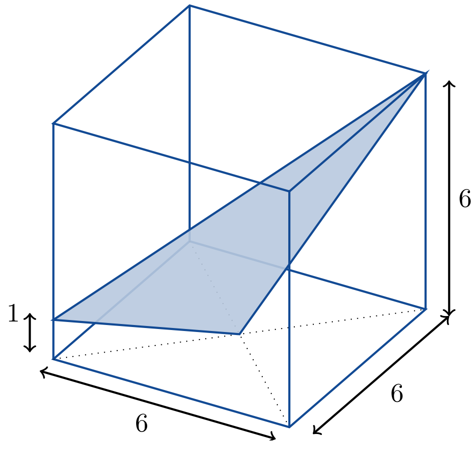

Figure 10.6: Yet another triangle inscribed inside a cube.
Şekil 10.6:

Soru 10.5. Calculate the triple scalar product of $\mathbf{u} = 2\mathbf{i} + \mathbf{j}$, $\mathbf{v} = 2\mathbf{i} - \mathbf{j} + \mathbf{k}$ and $\mathbf{w} = \mathbf{i} + 2\mathbf{k}$.

11 Lines

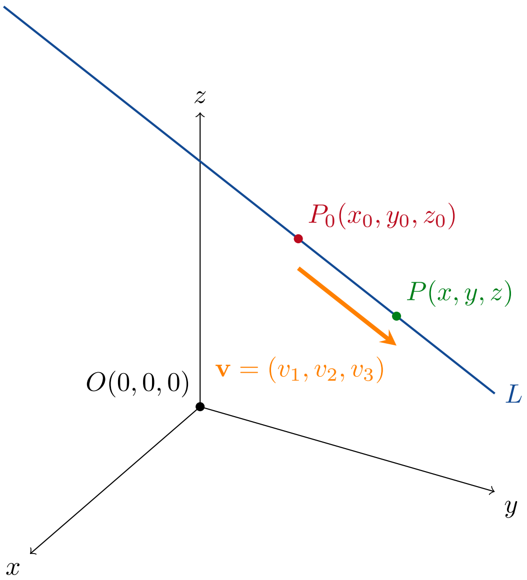

Figure 11.1: A line in $\mathbb{R}^3$ passing through the point P_0 parallel to $\mathbf{v}$.
Şekil 11.1:

Lines

To describe a line in $\mathbb{R}^3$, we need

- a point $P_0(x_0, y_0, z_0)$ which the line passes through; and
- a vector $\mathbf{v}$ which gives the direction of the line.

Let $\mathbf{r}_0 = \overrightarrow{OP_0}$ and $\mathbf{r} = \overrightarrow{OP}$.

Definition. The ***line*** L ***passing through*** $P_0(x_0, y_0, z_0)$ ***parallel to*** $\mathbf{v} = (v_1, v_2, v_3)$ has the vector equation

$$\mathbf{r} = \mathbf{r}_0 + t\mathbf{v}, \qquad -\infty < t < \infty.$$

This equation is equivalent to

$$(x, y, z) = (x_0, y_0, z_0) + t(v_1, v_2, v_3)$$

or to the set of three equations

$$x = x_0 + tv_1, \qquad y = y_0 + tv_2, \qquad z = z_0 + tv_3.$$

Doğrular

Doğrular

To describe a line in $\mathbb{R}^3$, we need

- a point $P_0(x_0, y_0, z_0)$ which the line passes through; and
- a vector $\mathbf{v}$ which gives the direction of the line.

Let $\mathbf{r}_0 = \overrightarrow{OP_0}$ and $\mathbf{r} = \overrightarrow{OP}$.

Tanım. The ***line*** L ***passing through*** $P_0(x_0, y_0, z_0)$ ***parallel to*** $\mathbf{v} = (v_1, v_2, v_3)$ has the vector equation

$$\mathbf{r} = \mathbf{r}_0 + t\mathbf{v}, \qquad -\infty < t < \infty.$$

This equation is equivalent to

$$(x, y, z) = (x_0, y_0, z_0) + t(v_1, v_2, v_3)$$

or to the set of three equations

$$x = x_0 + tv_1, \qquad y = y_0 + tv_2, \qquad z = z_0 + tv_3.$$

Tanım. The ***parametric equations*** for the line L passing through $P_0(x_0, y_0, z_0)$ parallel to $\mathbf{v} = (v_1, v_2, v_3)$ are

$$x = x_0 + tv_1, \qquad y = y_0 + tv_2, \qquad z = z_0 + tv_3.$$

Örnek 11.1. Find parametric equations for the line passing through $P_0(-2, 0, 4)$ parallel to $\mathbf{v} = 2\mathbf{i} + 4\mathbf{j} - 2\mathbf{k}$.

çözüm: We can write

$$x = -2 + 2t, \qquad y = 4t, \qquad z = 4 - 2t.$$

See figure 11.3

Örnek 11.2. Find parametric equations for the line passing through $P(-3, 2, -3)$ and $Q(1, -1, 4)$.

çözüm: Choose $P_0 = P$ and $\mathbf{v} = \overrightarrow{PQ} = (4, -3, 7) = 4\mathbf{i} - 3\mathbf{j} + 7\mathbf{k}$. Then we can write

$$x = -3 + 4t, \qquad y = 2 - 3t, \qquad z = -3 + 7t.$$

Tanım. The vector equation

$$\mathbf{r} = \mathbf{r}_0 + t\mathbf{v}, \qquad a \le t \le b$$

denotes a ***line segment***.

Definition. The ***parametric equations*** for the line L passing through $P_0(x_0, y_0, z_0)$ parallel to $\mathbf{v} = (v_1, v_2, v_3)$ are

$$x = x_0 + tv_1, \qquad y = y_0 + tv_2, \qquad z = z_0 + tv_3.$$

Example 11.1. Find parametric equations for the line passing through $P_0(-2, 0, 4)$ parallel to $\mathbf{v} = 2\mathbf{i} + 4\mathbf{j} - 2\mathbf{k}$.

solution: We can write

$$x = -2 + 2t, \qquad y = 4t, \qquad z = 4 - 2t.$$

See figure 11.3

Example 11.2. Find parametric equations for the line passing through $P(-3, 2, -3)$ and $Q(1, -1, 4)$.

solution: Choose $P_0 = P$ and $\mathbf{v} = \overrightarrow{PQ} = (4, -3, 7) = 4\mathbf{i} - 3\mathbf{j} + 7\mathbf{k}$. Then we can write

$$x = -3 + 4t, \qquad y = 2 - 3t, \qquad z = -3 + 7t.$$

Definition. The vector equation

$$\mathbf{r} = \mathbf{r}_0 + t\mathbf{v}, \qquad a \leq t \leq b$$

denotes a ***line segment***.

Example 11.3. Parametrise the line segment joining $P(-3, 2, -3)$ and $Q(1, -1, 4)$.

solution: We know that $x = -3+4t$, $y = 2-3t$ and $z = -3+7t$. The line passes through P then $t = 0$ and passed through Q when $t = 1$. Therefore

$$x = -3 + 4t, \qquad y = 2 - 3t, \qquad z = -3 + 7t, \qquad 0 \leq t \leq 1$$

denotes the line segment from P to Q. See figure 11.2.

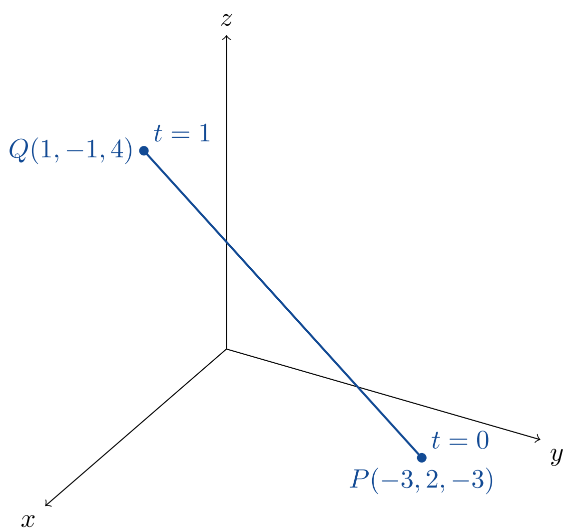

Figure 11.2: The line segment $\mathbb{R}^3$ joining $P(-3, 2, -3)$ and $Q(1, -1, 4)$.
Şekil 11.2:

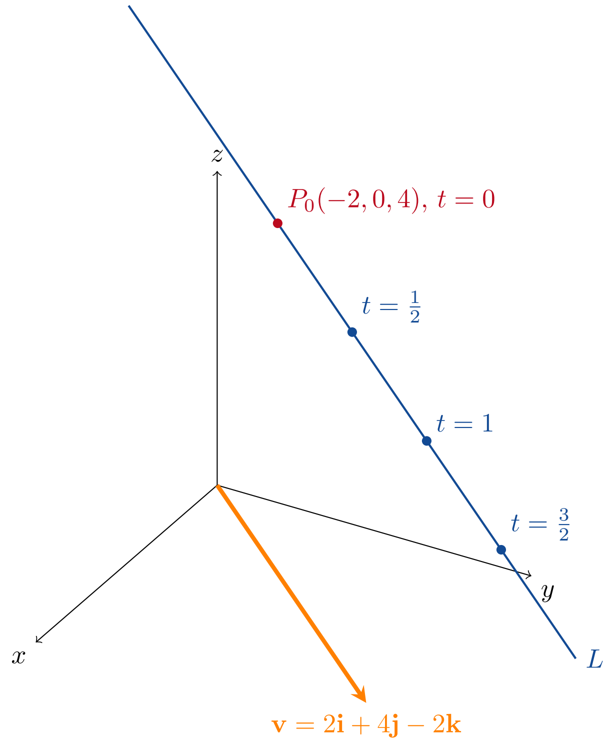

Figure 11.3: A line in $\mathbb{R}^3$ passing through the point P_0 parallel to $\mathbf{v}$.
Şekil 11.3:

Örnek 11.3. Parametrise the line segment joining $P(-3, 2, -3)$ and $Q(1, -1, 4)$.

çözüm: We know that $x = -3+4t$, $y = 2-3t$ and $z = -3+7t$. The line passes through P then $t = 0$ and passed through Q when $t = 1$. Therefore

$$x = -3 + 4t, \qquad y = 2 - 3t, \qquad z = -3 + 7t, \qquad 0 \leq t \leq 1$$

denotes the line segment from P to Q. See figure 11.2.

The Distance from a Point to a Line

Let d be the shortest distance from the point S to the line L as shown in figure 11.4. We can see from this figure that

$$d = \left\| \overrightarrow{PS} \right\| \sin\theta.$$

But remember that $\overrightarrow{PS} \times \mathbf{v} = \left\| \overrightarrow{PS} \right\| \|\mathbf{v}\| \sin\theta \; \mathbf{n}$. Therefore

$$\boxed{d = \frac{\left\| \overrightarrow{PS} \times \mathbf{v} \right\|}{\|\mathbf{v}\|}.}$$

Example 11.4. Find the distance from the point $S(1, 1, 5)$ to the line

$$x = 1 + t, \qquad y = 3 - t, \qquad z = 2t.$$

solution: The line passes through the point $P(1, 3, 0)$ in the direction $\mathbf{v} = \mathbf{i} - \mathbf{j} + 2\mathbf{k}$. Thus

$$\overrightarrow{PS} = S - P = (1, 1, 5) - (1, 3, 0) = (0, -2, 5) = -2\mathbf{j} + 5\mathbf{k}$$

and

$$\overrightarrow{PS} \times \mathbf{v} = (-4 + 5)\mathbf{i} - (0 - 5)\mathbf{j} + (0 + 2)\mathbf{k} = \mathbf{i} + 5\mathbf{j} + 2\mathbf{k}.$$

Therefore

$$d = \frac{\left\| \overrightarrow{PS} \times \mathbf{v} \right\|}{\|\mathbf{v}\|} = \frac{\sqrt{1^2 + 5^2 + 2^2}}{\sqrt{1^2 + 1^2 + 2^2}} = \frac{\sqrt{30}}{\sqrt{6}} = \sqrt{5}.$$

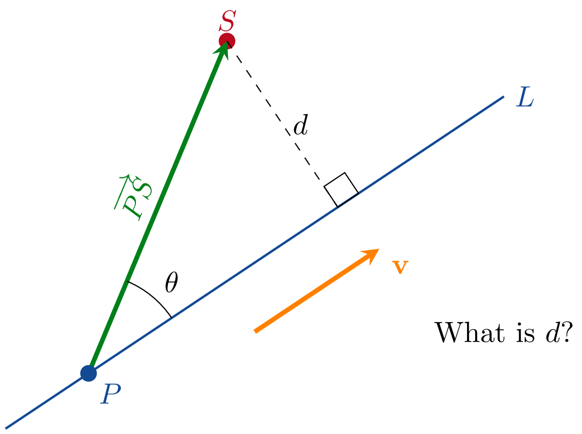

Figure 11.4: The distance from a point S to a line L.
Şekil 11.4:

The Distance from a Point to a Line

Let d be the shortest distance from the point S to the line L as shown in figure 11.4. We can see from this figure that

$$d = \left\| \overrightarrow{PS} \right\| \sin\theta.$$

But remember that $\overrightarrow{PS} \times \mathbf{v} = \left\| \overrightarrow{PS} \right\| \|\mathbf{v}\| \sin\theta \; \mathbf{n}$. Therefore

$$\boxed{d = \frac{\left\| \overrightarrow{PS} \times \mathbf{v} \right\|}{\|\mathbf{v}\|}.}$$

Örnek 11.4. Find the distance from the point $S(1, 1, 5)$ to the line

$$x = 1 + t, \qquad y = 3 - t, \qquad z = 2t.$$

çözüm: The line passes through the point $P(1, 3, 0)$ in the direction $\mathbf{v} = \mathbf{i} - \mathbf{j} + 2\mathbf{k}$. Thus

$$\overrightarrow{PS} = S - P = (1, 1, 5) - (1, 3, 0) = (0, -2, 5) = -2\mathbf{j} + 5\mathbf{k}$$

and

$$\overrightarrow{PS} \times \mathbf{v} = (-4 + 5)\mathbf{i} - (0 - 5)\mathbf{j} + (0 + 2)\mathbf{k} = \mathbf{i} + 5\mathbf{j} + 2\mathbf{k}.$$

Therefore

$$d = \frac{\left\| \overrightarrow{PS} \times \mathbf{v} \right\|}{\|\mathbf{v}\|} = \frac{\sqrt{1^2 + 5^2 + 2^2}}{\sqrt{1^2 + 1^2 + 2^2}} = \frac{\sqrt{30}}{\sqrt{6}} = \sqrt{5}.$$

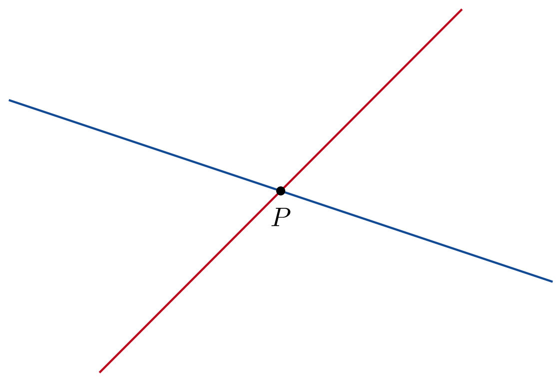

Figure 11.5: Intersecting lines.
Şekil 11.5:

Points of Intersection

Definition. Two lines intersect at a point P if and only if P lies on both lines.

Example 11.5. Do the following two lines intersect? Is yes, where?

Line 1: $x = 7 - t$, $y = 3 + 3t$, $z = 2t$.

Line 2: $x = -1 + 2s$, $y = 3s$, $z = 1 + s$.

solution: The two lines intersect if and only if there exist $s, t \in \mathbb{R}$ such that

$$\begin{aligned} 7 - t = x = -1 + 2s \qquad &\implies t = 8 - 2s \\ 3 + 3t = y = 3s \qquad &\implies s = t + 1 \\ 2t = z = 1 + s \qquad & \end{aligned}$$

The first equation tells us that $t = 8 - 2s$. Putting this into the second equation gives $s = t + 1 = (8 - 2s) + 1 = 9 - 2s$ which implies that $s = 3$ and $t = 2$. We must check the third equation: $2t = 2 \times 2 = 4 = 1 + 3 = 1 + s$. Because the third equation is also true, we know that they two lines intersect at $P(5, 9, 4)$.

Example 11.6. Do the following two lines intersect? If yes, where?

Line 1: $x = 1 + t$, $y = 3t$, $z = 3 + 3t$.

Line 2: $x = -1 + 2s$, $y = 3s$, $z = 1 + s$.

solution: Can we find $s, t \in \mathbb{R}$ such that

$$\begin{aligned} 1 + t = x = -1 + 2s \qquad & \\ 3t = y = 3s \qquad &\implies s = t \\ 3 + 3t = z = 1 + s \qquad & \end{aligned}$$

are all true?

The second equation gives $s = t$. Thus $1 + t = -1 + 2t \implies 2 + t = 2t \implies t = 2$. However $3 + 3t = 1 + t \implies 2 + 2t = 0 \implies t = -2 \neq 2$. Therefore it is not possible to find an s and a t. Hence the lines do not intersect.

Points of Intersection

Tanım. Two lines intersect at a point P if and only if P lies on both lines.

Örnek 11.5. Do the following two lines intersect? Is yes, where?

Doğru 1: $x = 7 - t$, $y = 3 + 3t$, $z = 2t$.

Doğru 2: $x = -1 + 2s$, $y = 3s$, $z = 1 + s$.

çözüm: The two lines intersect if and only if there exist $s, t \in \mathbb{R}$ such that

$$\begin{aligned} 7 - t = x = -1 + 2s \qquad &\implies t = 8 - 2s \\ 3 + 3t = y = 3s \qquad &\implies s = t + 1 \\ 2t = z = 1 + s \qquad & \end{aligned}$$

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Örnek 11.6. Do the following two lines intersect? If yes, where?

Doğru 1: $x = 1 + t$, $y = 3t$, $z = 3 + 3t$.

Doğru 2: $x = -1 + 2s$, $y = 3s$, $z = 1 + s$.

çözüm: Can we find $s, t \in \mathbb{R}$ such that

$$\begin{aligned} 1 + t = x = -1 + 2s \qquad & \\ 3t = y = 3s \qquad &\implies s = t \\ 3 + 3t = z = 1 + s \qquad & \end{aligned}$$

are all true?

The second equation gives $s = t$. Thus $1 + t = -1 + 2t \implies 2 + t = 2t \implies t = 2$. However $3 + 3t = 1 + t \implies 2 + 2t = 0 \implies t = -2 \neq 2$. Therefore it is not possible to find an s and a t. Hence the lines do not intersect.

The Distance Between Two Lines

There are three cases to consider:

- the lines intersect;
- the lines do not intersect and are parallel ($\mathbf{v}_1 = k\mathbf{v}_2$ for some $k \in \mathbb{R}$); or
- the lines do not intersect and are skew ($\mathbf{v}_1 \neq k\mathbf{v}_2$ for all $k \in \mathbb{R}$).

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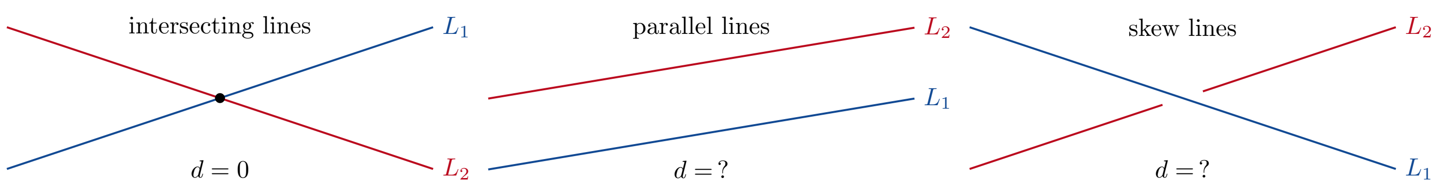

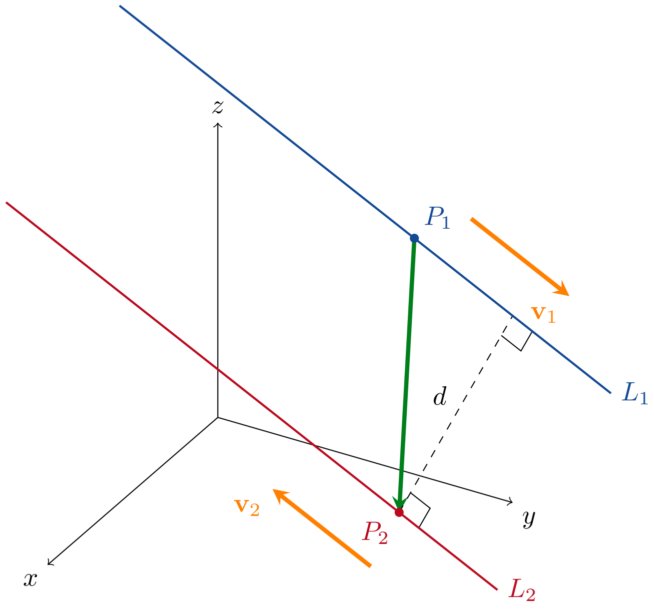

Figure 11.6: The distance between parallel lines.
Şekil 11.6:

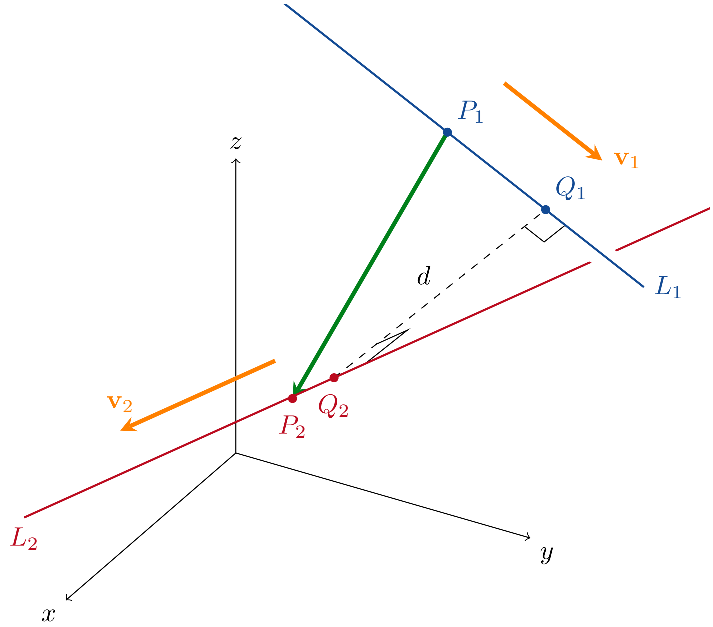

Figure 11.7: The distance between skew lines.
Şekil 11.7:

Intersecting Lines

Clearly the distance between intersecting lines is zero.

$$\boxed{d = 0.}$$

Parallel Lines ($\mathbf{v}_1 \times \mathbf{v}_2 = \mathbf{0}$)

Next we consider parallel lines. We can see from figure 11.6 that the distance between the two parallel lines is the same as the distance between P_2 and the line L_1. Hence

$$\boxed{d = \frac{\left\|\overrightarrow{P_1P_2} \times \mathbf{v}_1\right\|}{\|\mathbf{v}_1\|}.}$$

Skew Lines ($\mathbf{v}_1 \times \mathbf{v}_2 \neq \mathbf{0}$)

Finally we consider skew lines. See figure 11.7. Let $\mathbf{n} = \mathbf{v}_1 \times \mathbf{v}_2$. Then $\mathbf{n}$ is orthogonal to both $\mathbf{v}_1$ and $\mathbf{v}_2$. So

$$d = \left\|\overrightarrow{Q_1Q_2}\right\| = \left\|\operatorname{proj}_{\mathbf{n}} \overrightarrow{P_1P_2}\right\| = \frac{\left|\overrightarrow{P_1P_2} \bullet \mathbf{n}\right|}{\|\mathbf{n}\|}.$$

Thus

$$\boxed{d = \frac{\left|\overrightarrow{P_1P_2} \bullet (\mathbf{v}_1 \times \mathbf{v}_2)\right|}{\|\mathbf{v}_1 \times \mathbf{v}_2\|}.}$$

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Skew Lines ($\mathbf{v}_1 \times \mathbf{v}_2 \neq \mathbf{0}$)

Finally we consider skew lines. See figure 11.7. Let $\mathbf{n} = \mathbf{v}_1 \times \mathbf{v}_2$. Then $\mathbf{n}$ is orthogonal to both $\mathbf{v}_1$ and $\mathbf{v}_2$. So

$$d = \left\|\overrightarrow{Q_1Q_2}\right\| = \left\|\operatorname{proj}_{\mathbf{n}} \overrightarrow{P_1P_2}\right\| = \frac{\left|\overrightarrow{P_1P_2} \bullet \mathbf{n}\right|}{\|\mathbf{n}\|}.$$

Thus

$$\boxed{d = \frac{\left|\overrightarrow{P_1P_2} \bullet (\mathbf{v}_1 \times \mathbf{v}_2)\right|}{\|\mathbf{v}_1 \times \mathbf{v}_2\|}.}$$

Example 11.7. Find the distance between the following two lines.
line 1: $x = 0$, $y = -t$, $z = t$,
line 2: $x = 1 + 2s$, $y = s$, $z = -3s$.

solution: We have that $P_1(0,0,0)$, $\mathbf{v}_1 = -\mathbf{j}+\mathbf{k}$, $P_2(1,0,0)$ and $\mathbf{v}_2 = 2\mathbf{i}+\mathbf{j}-3\mathbf{k}$. Since

$$\mathbf{v}_1 \times \mathbf{v}_2 = 2\mathbf{i}+2\mathbf{j}+2\mathbf{k} \neq \mathbf{0},$$

the lines are skew. (Recall that we have $\mathbf{v}_1 \times \mathbf{v}_2 = \mathbf{0}$ for parallel vectors.) Moreover note that $\overrightarrow{P_1P_2} = \mathbf{i}$. Then we calculate that

$$d = \frac{\left|\overrightarrow{P_1P_2} \cdot (\mathbf{v}_1 \times \mathbf{v}_2)\right|}{\|\mathbf{v}_1 \times \mathbf{v}_2\|} = \frac{|(\mathbf{i}) \cdot (2\mathbf{i}+2\mathbf{j}+2\mathbf{k})|}{\|2\mathbf{i}+2\mathbf{j}+2\mathbf{k}\|}$$
$$= \frac{|2+0+0|}{\sqrt{2^2+2^2+2^2}} = \frac{1}{\sqrt{3}}.$$

Örnek 11.7. Find the distance between the following two lines.
doğru 1: $x = 0$, $y = -t$, $z = t$,
doğru 2: $x = 1 + 2s$, $y = s$, $z = -3s$.

çözüm: We have that $P_1(0,0,0)$, $\mathbf{v}_1 = -\mathbf{j}+\mathbf{k}$, $P_2(1,0,0)$ and $\mathbf{v}_2 = 2\mathbf{i}+\mathbf{j}-3\mathbf{k}$. Since

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$$= \frac{|2+0+0|}{\sqrt{2^2+2^2+2^2}} = \frac{1}{\sqrt{3}}.$$

Problems

Problem 11.1. Find parametric equations for the line through $P(3,-4,-1)$ which is parallel to the vector $\mathbf{v} = \mathbf{i}+\mathbf{j}-\mathbf{k}$.

Problem 11.2. Find parametric equations for the line through the points $P(1,2,-1)$ and $Q(-1,0,1)$.

Problem 11.3. Find parametric equations for the line through the point $P(2,3,0)$ which is perpendicular to the vectors $\mathbf{u} = \mathbf{i}+2\mathbf{j}+3\mathbf{k}$ and $\mathbf{v} = 3\mathbf{i}+4\mathbf{j}+5\mathbf{k}$.

Problem 11.4. Find the distance from the point $S(-1,4,3)$ to the line $x = 10+4t$, $y = -3$, $z = 4t$.

Problem 11.5. Find the distance from the point $S(2,1,3)$ to the line $x = 2+2t$, $y = 1+6t$, $z = 3$.

Problem 11.6. Consider the following two lines:
line 1: $x = 7+t$, $y = 8+t$, $z = 9-t$,
line 2: $x = 15-3s$, $y = 16-3s$, $z = 7$.

(a). Do these lines intersect? If yes, where?

(b). Find the distance between these two lines.

Problem 11.7. The following two lines do not intersect. Find the distance between them.
line 1: $x = 10+4t$, $y = -3$, $z = 4t$,
line 2: $x = 10-4s$, $y = 0$, $z = 2-4s$.

Problem 11.8. The following two lines do not intersect. Find the distance between them.
line 1: $x = 10+4t$, $y = -t$, $z = 4t$,
line 2: $x = 10-4s$, $y = 1$, $z = 2-4s$.

Sorular

Soru 11.1. Find parametric equations for the line through $P(3,-4,-1)$ which is parallel to the vector $\mathbf{v} = \mathbf{i}+\mathbf{j}-\mathbf{k}$.

Soru 11.2. Find parametric equations for the line through the points $P(1,2,-1)$ and $Q(-1,0,1)$.

Soru 11.3. Find parametric equations for the line through the point $P(2,3,0)$ which is perpendicular to the vectors $\mathbf{u} = \mathbf{i}+2\mathbf{j}+3\mathbf{k}$ and $\mathbf{v} = 3\mathbf{i}+4\mathbf{j}+5\mathbf{k}$.

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doğru 1: $x = 10+4t$, $y = -3$, $z = 4t$,
doğru 2: $x = 10-4s$, $y = 0$, $z = 2-4s$.

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doğru 1: $x = 10+4t$, $y = -t$, $z = 4t$,
doğru 2: $x = 10-4s$, $y = 1$, $z = 2-4s$.

12

Planes

To describe a plane, we need

- a point $P_0(x_0, y_0, z_0)$ which the plane passes through; and
- a vector $\mathbf{n} = A\mathbf{i} + B\mathbf{j} + C\mathbf{k}$ which is perpendicular to the plane.

The vector $\mathbf{n}$ is said to be ***normal*** to the plane.

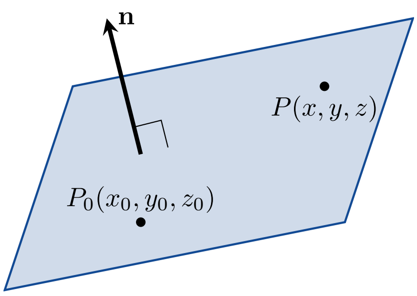

Figure 12.1: A plane passing through the point $P_0(x_0, y_0, z_0)$ with normal vector $\mathbf{n} = A\mathbf{i} + B\mathbf{j} + C\mathbf{k}$.
Şekil 12.1:

Definition. The plane passing through the point $P_0(x_0, y_0, z_0)$ with normal vector $\mathbf{n} = A\mathbf{i} + B\mathbf{j} + C\mathbf{k}$ has the vector equation

$$\mathbf{n} \cdot \overrightarrow{P_0P} = 0.$$

Writing this equation in coordinates, we have

$$A(x - x_0) + B(y - y_0) + C(z - z_0) = 0$$

or

$$\boxed{Ax + By + Cz = D}$$

where $D = Ax_0 + By_0 + Cz_0$ is a constant.

Example 12.1. Find an equation for the plane passing through $P_0(-3, 0, 7)$ normal to $\mathbf{n} = 5\mathbf{i} + 2\mathbf{j} - \mathbf{k}$.

solution:

$$\begin{aligned} 5(x - (-3)) + 2(y - 0) + (-1)(z - 7) &= 0 \\ 5x - 15 + 2y - z + 7 &= 0 \\ 5x + 2y - z &= -22. \end{aligned}$$

Düzlemler

To describe a plane, we need

- a point $P_0(x_0, y_0, z_0)$ which the plane passes through; and
- a vector $\mathbf{n} = A\mathbf{i} + B\mathbf{j} + C\mathbf{k}$ which is perpendicular to the plane.

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Tanım. The plane passing through the point $P_0(x_0, y_0, z_0)$ with normal vector $\mathbf{n} = A\mathbf{i} + B\mathbf{j} + C\mathbf{k}$ has the vector equation

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where $D = Ax_0 + By_0 + Cz_0$ is a constant.

Örnek 12.1. Find an equation for the plane passing through $P_0(-3, 0, 7)$ normal to $\mathbf{n} = 5\mathbf{i} + 2\mathbf{j} - \mathbf{k}$.

çözüm:

$$\begin{aligned} 5(x - (-3)) + 2(y - 0) + (-1)(z - 7) &= 0 \\ 5x - 15 + 2y - z + 7 &= 0 \\ 5x + 2y - z &= -22. \end{aligned}$$

Not. The vector $\mathbf{n} = A\mathbf{i} + B\mathbf{j} + C\mathbf{k}$ is normal to the plane $Ax + By + Cz = D$.

Örnek 12.2. Find a vector normal to the plane $x+2y+3z = 4$.

çözüm: We can immediately write down $\mathbf{n} = \mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$.

Örnek 12.3. Find an equation for the plane containing the points $E(0, 0, 1)$, $F(2, 0, 0)$ and $G(0, 3, 0)$.

çözüm: First we need to find a vector normal to the plane. Since $\overrightarrow{EF} = 2\mathbf{i} - \mathbf{k}$ and $\overrightarrow{EG} = 3\mathbf{j} - \mathbf{k}$, we have that

$$\begin{aligned} \mathbf{n} = \overrightarrow{EF} \times \overrightarrow{EG} &= (0 - -3)\mathbf{i} - (-2 - 0)\mathbf{j} + (6 - 0)\mathbf{k} \\ &= 3\mathbf{i} + 2\mathbf{j} + 6\mathbf{k} \end{aligned}$$

is normal to the plane. See figure 12.2. Using $P_0 = E(0, 0, 1)$,

Remark. The vector $\mathbf{n} = A\mathbf{i} + B\mathbf{j} + C\mathbf{k}$ is normal to the plane $Ax + By + Cz = D$.

Example 12.2. Find a vector normal to the plane $x+2y+3z = 4$.

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Example 12.3. Find an equation for the plane containing the points $E(0, 0, 1)$, $F(2, 0, 0)$ and $G(0, 3, 0)$.

solution: First we need to find a vector normal to the plane. Since $\overrightarrow{EF} = 2\mathbf{i} - \mathbf{k}$ and $\overrightarrow{EG} = 3\mathbf{j} - \mathbf{k}$, we have that

$$\begin{aligned}\mathbf{n} &= \overrightarrow{EF} \times \overrightarrow{EG} = (0 - -3)\mathbf{i} - (-2 - 0)\mathbf{j} + (6 - 0)\mathbf{k} \\ &= 3\mathbf{i} + 2\mathbf{j} + 6\mathbf{k}\end{aligned}$$

is normal to the plane. See figure 12.2. Using $P_0 = E(0, 0, 1)$, the equation for the plane is

$$\begin{aligned}3(x - 0) + 2(y - 0) + 6(z - 1) &= 0 \\ 3x + 2y + 6z &= 6.\end{aligned}$$

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$$\begin{aligned}3(x - 0) + 2(y - 0) + 6(z - 1) &= 0 \\ 3x + 2y + 6z &= 6.\end{aligned}$$

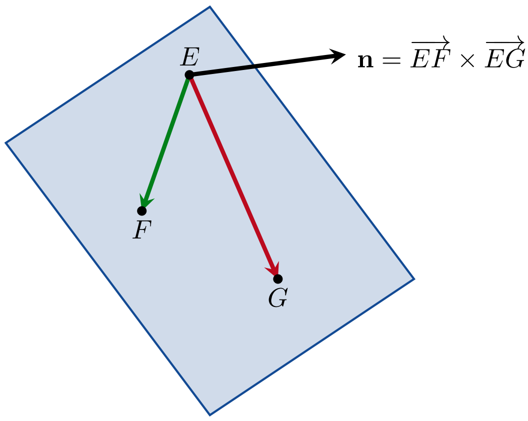


Figure 12.2: The vector $\mathbf{n}$ is perpendicular to the plane containing E, F and G.
Şekil 12.2:

Lines of Intersection

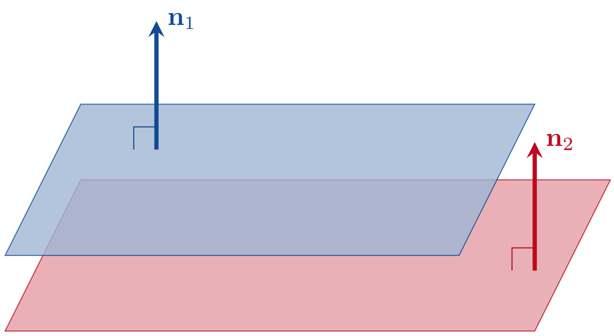


Figure 12.3: Two planes are parallel $\iff$ $\mathbf{n}_1 = k\mathbf{n}_2$ for some $k \in \mathbb{R}$.
Şekil 12.3:

Example 12.4. Find a vector parallel of the line of intersection of the planes $3x - 6y - 2z = 15$ and $2x + y - 2z = 5$.

solution: We can immediately write down $\mathbf{n}_1 = 3\mathbf{i} - 6\mathbf{j} - 2\mathbf{k}$ and $\mathbf{n}_2 = 2\mathbf{i}+\mathbf{j}-2\mathbf{k}$. A vector parallel to the line of intersection is

$$\mathbf{n}_1 \times \mathbf{n}_2 = (12 + 2)\mathbf{i} - (-6 + 4)\mathbf{j} + (3 + 12)\mathbf{k} = 14\mathbf{i} + 2\mathbf{j} + 15\mathbf{k}.$$

Example 12.5. Find the point where the line $x = \frac{8}{3} + 2t$, $y = -2t$, $z = 1 + t$ intersects the plane $3x + 2y + 6z = 6$.

solution: We calculate that

Kesişim Doğruları

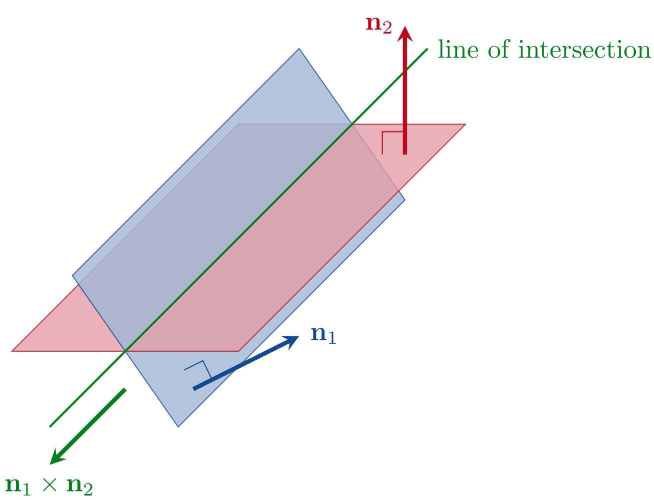


Figure 12.4: Two different planes intersect in a line $\iff$ $\mathbf{n}_1 \neq k\mathbf{n}_2$ for all $k \in \mathbb{R}$.
Şekil 12.4:

Örnek 12.4. Find a vector parallel of the line of intersection of the planes $3x - 6y - 2z = 15$ and $2x + y - 2z = 5$.

çözüm: We can immediately write down $\mathbf{n}_1 = 3\mathbf{i} - 6\mathbf{j} - 2\mathbf{k}$ and $\mathbf{n}_2 = 2\mathbf{i} + \mathbf{j} - 2\mathbf{k}$. A vector parallel to the line of intersection is

$$\mathbf{n}_1 \times \mathbf{n}_2 = (12 + 2)\mathbf{i} - (-6 + 4)\mathbf{j} + (3 + 12)\mathbf{k} = 14\mathbf{i} + 2\mathbf{j} + 15\mathbf{k}.$$

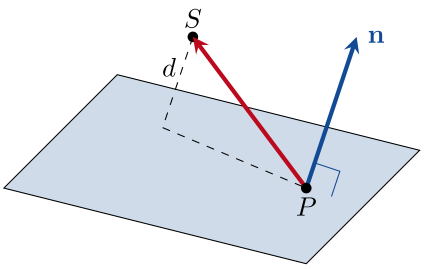
Örnek 12.5. Find the point where the line $x = \frac{8}{3}+2t$, $y = -2t$, $z = 1 + t$ intersects the plane $3x + 2y + 6z = 6$.

$$\begin{aligned} 3x + 2y + 6z &= 6 \\ 3\left(\tfrac{8}{3} + 2t\right) + 2(-2t) + 6(1+t) &= 6 \\ 8 + 6t - 4t + 6 + 6t &= 6 \\ 8t &= -8 \\ t &= -1. \end{aligned}$$

The point of intersection is

$$P(x,y,z)|_{t=-1} = P\left(\frac{8}{3} + 2t, -2t, 1+t\right)\bigg|_{t=-1} = P\left(\frac{2}{3}, 2, 0\right).$$

The Distance from a Point to a Plane



We can see from figure 12.5 that $d = \left\|\text{proj}_{\mathbf{n}} \overrightarrow{PS}\right\|$. Therefore the distance from a point S to a plane containing the point P is

$$\boxed{d = \frac{\left|\overrightarrow{PS} \cdot \mathbf{n}\right|}{\|\mathbf{n}\|}.}$$

Example 12.6. Find the distance from the point $S(1,2,3)$ to the plane $x + 2y + 3z = 4$.

solution: First we need a point in the plane. Setting $y = 0$ and $z = 0$ we must have $x = 4 - 2y - 3z = 4$. Therefore $P(4,0,0)$ is in the plane. Clearly $\mathbf{n} = \mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$.

Therefore the required distance is

$$\begin{aligned} d = \frac{\left|\overrightarrow{PS} \cdot \mathbf{n}\right|}{\|\mathbf{n}\|} &= \frac{|(-3\mathbf{i} + 2\mathbf{j} + 3\mathbf{k}) \cdot (\mathbf{i} + 2\mathbf{j} + 3\mathbf{k})|}{\|\mathbf{i} + 2\mathbf{j} + 3\mathbf{k}\|} \\ &= \frac{|-3 + 4 + 9|}{\sqrt{1^2 + 2^2 + 3^2}} = \frac{10}{\sqrt{14}}. \end{aligned}$$

çözüm: We calculate that

$$\begin{aligned} 3x + 2y + 6z &= 6 \\ 3\left(\tfrac{8}{3} + 2t\right) + 2(-2t) + 6(1+t) &= 6 \\ 8 + 6t - 4t + 6 + 6t &= 6 \\ 8t &= -8 \\ t &= -1. \end{aligned}$$

The point of intersection is

$$P(x,y,z)|_{t=-1} = P\left(\frac{8}{3} + 2t, -2t, 1+t\right)\bigg|_{t=-1} = P\left(\frac{2}{3}, 2, 0\right).$$

Bir Noktadan Bir Düzleme Olan Uzaklık

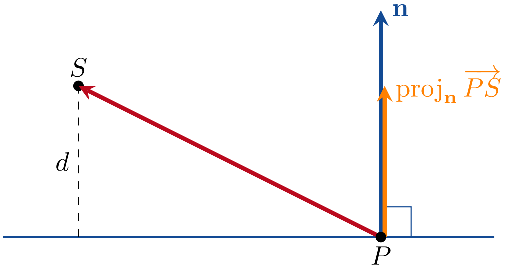


Figure 12.5: The distance from a Point to a Place.
Şekil 12.5: Bir Noktadan Bir Düzleme Olan Uzaklık.

We can see from figure 12.5 that $d = \left\|\text{proj}_{\mathbf{n}} \overrightarrow{PS}\right\|$. Therefore the distance from a point S to a plane containing the point P is

$$\boxed{d = \frac{\left|\overrightarrow{PS} \cdot \mathbf{n}\right|}{\|\mathbf{n}\|}.}$$

Örnek 12.6. Find the distance from the point $S(1,2,3)$ to the plane $x + 2y + 3z = 4$.

çözüm: First we need a point in the plane. Setting $y = 0$ and $z = 0$ we must have $x = 4 - 2y - 3z = 4$. Therefore $P(4,0,0)$ is in the plane. Clearly $\mathbf{n} = \mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$.

Therefore the required distance is

$$\begin{aligned} d = \frac{\left|\overrightarrow{PS} \cdot \mathbf{n}\right|}{\|\mathbf{n}\|} &= \frac{|(-3\mathbf{i} + 2\mathbf{j} + 3\mathbf{k}) \cdot (\mathbf{i} + 2\mathbf{j} + 3\mathbf{k})|}{\|\mathbf{i} + 2\mathbf{j} + 3\mathbf{k}\|} \\ &= \frac{|-3 + 4 + 9|}{\sqrt{1^2 + 2^2 + 3^2}} = \frac{10}{\sqrt{14}}. \end{aligned}$$

Angles Between Planes

There are two possible angles that can be measured between planes. We are interested in the smaller angle. See figure 12.6.

Definition. The angle between two planes is defined to be equal to whichever of the following angles is smaller

- the angle between $\mathbf{n}_1$ and $\mathbf{n}_2$;
- 180° minus the angle between $\mathbf{n}_1$ and $\mathbf{n}_2$.

The angle between two planes will always be between 0° and 90°.

Example 12.7. Find the angle between the planes $3x - 6y - 2z = 15$ and $-2x - y + 2z = 5$.

solution: We have normal vectors $\mathbf{n}_1 = 3\mathbf{i} - 6\mathbf{j} - 2\mathbf{k}$ and $\mathbf{n}_2 = -2\mathbf{i} - \mathbf{j} + 2\mathbf{k}$. The angle between $\mathbf{n}_1$ and $\mathbf{n}_2$ is

$$\theta = \cos^{-1}\left(\frac{\mathbf{n}_1 \cdot \mathbf{n}_2}{\|\mathbf{n}_1\|\,\|\mathbf{n}_2\|}\right) = \cos^{-1}\left(\frac{-4}{21}\right) \approx 101°.$$

Because $101° > 90°$, the angle between the two planes is approximately $180 - 101° = 79°$.

Düzlemler Arasındaki Açı

There are two possible angles that can be measured between planes. We are interested in the smaller angle. See figure 12.6.

Tanım. The angle between two planes is defined to be equal to whichever of the following angles is smaller

- the angle between $\mathbf{n}_1$ and $\mathbf{n}_2$;
- 180° minus the angle between $\mathbf{n}_1$ and $\mathbf{n}_2$.

The angle between two planes will always be between 0° and 90°.

Örnek 12.7. Find the angle between the planes $3x - 6y - 2z = 15$ and $-2x - y + 2z = 5$.

çözüm: We have normal vectors $\mathbf{n}_1 = 3\mathbf{i} - 6\mathbf{j} - 2\mathbf{k}$ and $\mathbf{n}_2 = -2\mathbf{i} - \mathbf{j} + 2\mathbf{k}$. The angle between $\mathbf{n}_1$ and $\mathbf{n}_2$ is

$$\theta = \cos^{-1}\left(\frac{\mathbf{n}_1 \cdot \mathbf{n}_2}{\|\mathbf{n}_1\|\,\|\mathbf{n}_2\|}\right) = \cos^{-1}\left(\frac{-4}{21}\right) \approx 101°.$$

Because $101° > 90°$, the angle between the two planes is approximately $180 - 101° = 79°$.

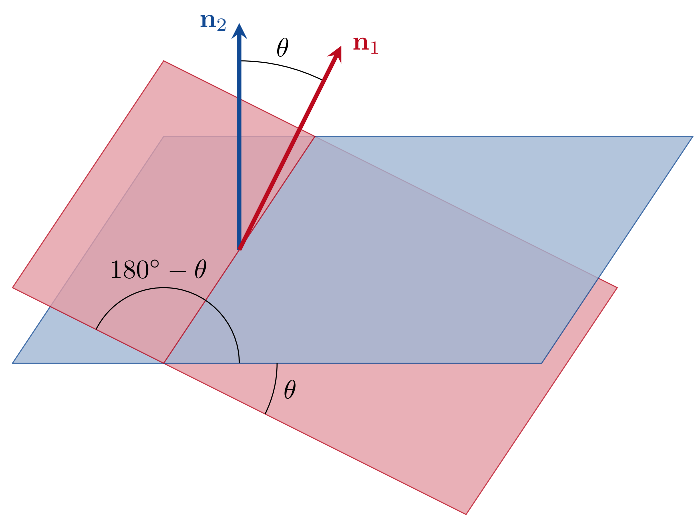


Figure 12.6: The angle between two planes is either θ or $(180° - \theta)$, whichever is smaller.
Şekil 12.6:

Problems

Problem 12.1. Find an equation for the plane passing through the points $E(2,4,5)$, $F(1,5,7)$ and $G(-1,6,8)$.

Problem 12.2. Let $O(0,0,0)$ be the origin. Find an equation for the plane through the point $A(1,-2,1)$ which is perpendicular to the vector $\overrightarrow{OA}$.

Problem 12.3. Find the point where the line intersects the plane.

(a). Line: $x = 1 - t$, $y = 3t$, $z = 1 + t$,
Plane: $2x - y + 3z = 6$.

(b). Line: $x = 2$, $y = 3 + 2t$, $z = -2 - 2t$,
Plane: $6x + 3y - 4z = -12$.

Problem 12.4. Find parametric equations for the lines in which the following pairs of planes intersect.

(a). Plane 1: $x + y + z = 1$,
Plane 2: $x + y = 2$.

(b). Plane 1: $3x - 6y - 2z = 3$,
Plane 2: $2x + y - 2z = 2$.

Problem 12.5.

(a). Find the distance from the point $S(2,-3,4)$ to the plane $x + 2y + 2y = 13$.

(b). Find the distance from the point $S(1,0,-1)$ to the plane $-4x + y + z = 4$.

Problem 12.6. Find the angle between the plane $x + y = 1$ and the plane $2x + y - 2x = 2$.

Problem 12.7. Find a formula for the distance between two planes.

Sorular

Soru 12.1. Find an equation for the plane passing through the points $E(2,4,5)$, $F(1,5,7)$ and $G(-1,6,8)$.

Soru 12.2. Let $O(0,0,0)$ be the origin. Find an equation for the plane through the point $A(1,-2,1)$ which is perpendicular to the vector $\overrightarrow{OA}$.

Soru 12.3. Find the point where the line intersects the plane.

(a). Line: $x = 1 - t$, $y = 3t$, $z = 1 + t$,
Plane: $2x - y + 3z = 6$.

(b). Line: $x = 2$, $y = 3 + 2t$, $z = -2 - 2t$,
Plane: $6x + 3y - 4z = -12$.

Soru 12.4. Find parametric equations for the lines in which the following pairs of planes intersect.

(a). Plane 1: $x + y + z = 1$,
Plane 2: $x + y = 2$.

(b). Plane 1: $3x - 6y - 2z = 3$,
Plane 2: $2x + y - 2z = 2$.

Soru 12.5.

(a). Find the distance from the point $S(2,-3,4)$ to the plane $x + 2y + 2y = 13$.

(b). Find the distance from the point $S(1,0,-1)$ to the plane $-4x + y + z = 4$.

Soru 12.6. Find the angle between the plane $x + y = 1$ and the plane $2x + y - 2x = 2$.

Soru 12.7. Find a formula for the distance between two planes.

13 Projections

Recall that in chapter 9 we defined the projection of a vector **u** onto a vector **v** to be

$$\operatorname{proj}_{\mathbf{v}} \mathbf{u} = \left(\frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{v}\|^2}\right) \mathbf{v}.$$

Projection of a Vector onto a Line

Definition. Let L be the line passing through the point P in the direction $\mathbf{v}$. The projection of a vector $\mathbf{u}$ onto the line L is

$$\operatorname{proj}_L \mathbf{u} = \operatorname{proj}_{\mathbf{v}} \mathbf{u}.$$

Example 13.1. Find the projection of the vector $\mathbf{u} = 2\mathbf{i} - \mathbf{j} + 3\mathbf{k}$ onto the line $x = 1 + 2t$, $y = 2 - t$, $z = 4 - 4t$.

solution: . Clearly $\mathbf{v} = 2\mathbf{i} - \mathbf{j} - 4\mathbf{k}$ is parallel to the line. Thus

$$\begin{aligned}
\operatorname{proj}_L \mathbf{u} = \operatorname{proj}_{\mathbf{v}} \mathbf{u} &= \left(\frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{v}\|^2}\right) \mathbf{v} \\
&= \left(\frac{4+1-12}{2^2+(-1)^2+(-4)^2}\right)(2\mathbf{i} - \mathbf{j} - 4\mathbf{k}) \\
&= \left(\frac{-7}{21}\right)(2\mathbf{i} - \mathbf{j} - 4\mathbf{k}) \\
&= -\frac{1}{3}(2\mathbf{i} - \mathbf{j} - 4\mathbf{k}) \\
&= -\frac{2}{3}\mathbf{i} + \frac{1}{3}\mathbf{j} + \frac{4}{3}\mathbf{k}.
\end{aligned}$$

İzdüşümler

Recall that in chapter 9 we defined the projection of a vector **u** onto a vector **v** to be

$$\operatorname{proj}_{\mathbf{v}} \mathbf{u} = \left(\frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{v}\|^2}\right) \mathbf{v}.$$

Projection of a Vector onto a Line

Tanım. Let L be the line passing through the point P in the direction $\mathbf{v}$. The projection of a vector $\mathbf{u}$ onto the line L is

$$\operatorname{proj}_L \mathbf{u} = \operatorname{proj}_{\mathbf{v}} \mathbf{u}.$$

Örnek 13.1. Find the projection of the vector $\mathbf{u} = 2\mathbf{i} - \mathbf{j} + 3\mathbf{k}$ onto the line $x = 1 + 2t$, $y = 2 - t$, $z = 4 - 4t$.

çözüm: . Clearly $\mathbf{v} = 2\mathbf{i} - \mathbf{j} - 4\mathbf{k}$ is parallel to the line. Thus

$$\begin{aligned}
\operatorname{proj}_L \mathbf{u} = \operatorname{proj}_{\mathbf{v}} \mathbf{u} &= \left(\frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{v}\|^2}\right) \mathbf{v} \\
&= \left(\frac{4+1-12}{2^2+(-1)^2+(-4)^2}\right)(2\mathbf{i} - \mathbf{j} - 4\mathbf{k}) \\
&= \left(\frac{-7}{21}\right)(2\mathbf{i} - \mathbf{j} - 4\mathbf{k}) \\
&= -\frac{1}{3}(2\mathbf{i} - \mathbf{j} - 4\mathbf{k}) \\
&= -\frac{2}{3}\mathbf{i} + \frac{1}{3}\mathbf{j} + \frac{4}{3}\mathbf{k}.
\end{aligned}$$

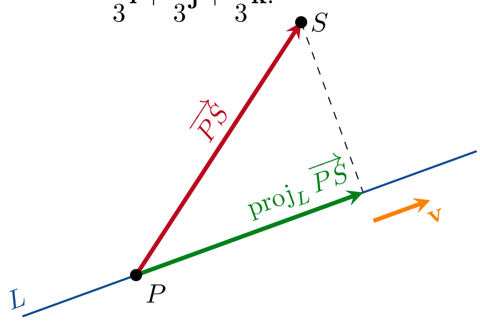


Figure 13.1: Projection of a vector onto a line.
Şekil 13.1:

Projection of a Vector onto a Plane

Definition. The ***projection*** of a vector $\mathbf{u}$ onto a plane with normal vector $\mathbf{n}$ is

$$\operatorname{proj}_{\text{plane}} \mathbf{u} = \mathbf{u} - \operatorname{proj}_{\mathbf{n}} \mathbf{u} = \mathbf{u} - \left(\frac{\mathbf{u} \cdot \mathbf{n}}{\|\mathbf{n}\|^2}\right) \mathbf{n}.$$

See figure 13.2.

Example 13.2. Find the projection of the vector $\mathbf{u} = \mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$ onto the plane $3x - y + 2z = 7$.

solution: Clearly $\mathbf{n} = 3\mathbf{i} - \mathbf{j} + 2\mathbf{k}$ and

$$\begin{aligned}\operatorname{proj}_{\mathbf{n}} \mathbf{u} &= \left(\frac{\mathbf{u} \cdot \mathbf{n}}{\|\mathbf{n}\|^2}\right) \mathbf{n} = \left(\frac{3-2+6}{3^2+(-1)^2+2^2}\right)(3\mathbf{i} - \mathbf{j} + 2\mathbf{k}) \\ &= \frac{1}{2}(3\mathbf{i} - \mathbf{j} + 2\mathbf{k}) = \frac{3}{2}\mathbf{i} - \frac{1}{2}\mathbf{j} + \mathbf{k}.\end{aligned}$$

Therefore

$$\begin{aligned}\operatorname{proj}_{\text{plane}} \mathbf{u} &= \mathbf{u} - \operatorname{proj}_{\mathbf{n}} \mathbf{u} \\ &= (\mathbf{i} + 2\mathbf{j} + 3\mathbf{k}) - \left(\frac{3}{2}\mathbf{i} - \frac{1}{2}\mathbf{j} + \mathbf{k}\right) \\ &= -\frac{1}{2}\mathbf{i} + \frac{5}{2}\mathbf{j} + 2\mathbf{k}.\end{aligned}$$

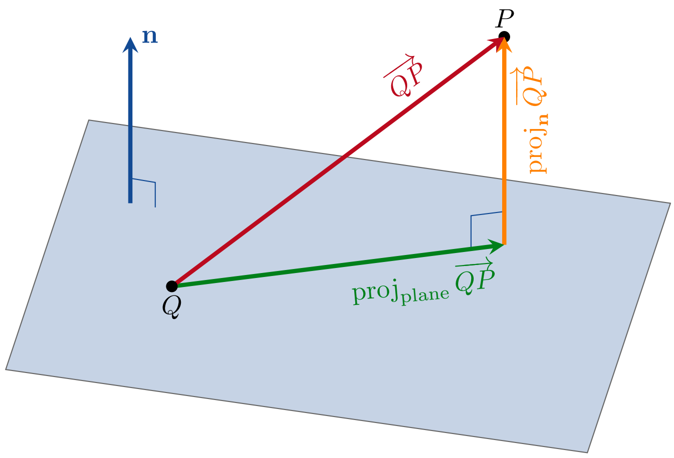


Figure 13.2: The projection of a vector onto a plane.
Şekil 13.2:

Projection of a Vector onto a Plane

Tanım. The ***projection*** of a vector $\mathbf{u}$ onto a plane with normal vector $\mathbf{n}$ is

$$\operatorname{proj}_{\text{düzlem}} \mathbf{u} = \mathbf{u} - \operatorname{proj}_{\mathbf{n}} \mathbf{u} = \mathbf{u} - \left(\frac{\mathbf{u} \cdot \mathbf{n}}{\|\mathbf{n}\|^2}\right) \mathbf{n}.$$

See figure 13.2.

Örnek 13.2. Find the projection of the vector $\mathbf{u} = \mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$ onto the plane $3x - y + 2z = 7$.

çözüm: Clearly $\mathbf{n} = 3\mathbf{i} - \mathbf{j} + 2\mathbf{k}$ and

$$\begin{aligned}\operatorname{proj}_{\mathbf{n}} \mathbf{u} &= \left(\frac{\mathbf{u} \cdot \mathbf{n}}{\|\mathbf{n}\|^2}\right) \mathbf{n} = \left(\frac{3-2+6}{3^2+(-1)^2+2^2}\right)(3\mathbf{i} - \mathbf{j} + 2\mathbf{k}) \\ &= \frac{1}{2}(3\mathbf{i} - \mathbf{j} + 2\mathbf{k}) = \frac{3}{2}\mathbf{i} - \frac{1}{2}\mathbf{j} + \mathbf{k}.\end{aligned}$$

Therefore

$$\begin{aligned}\operatorname{proj}_{\text{düzlem}} \mathbf{u} &= \mathbf{u} - \operatorname{proj}_{\mathbf{n}} \mathbf{u} \\ &= (\mathbf{i} + 2\mathbf{j} + 3\mathbf{k}) - \left(\frac{3}{2}\mathbf{i} - \frac{1}{2}\mathbf{j} + \mathbf{k}\right) \\ &= -\frac{1}{2}\mathbf{i} + \frac{5}{2}\mathbf{j} + 2\mathbf{k}.\end{aligned}$$

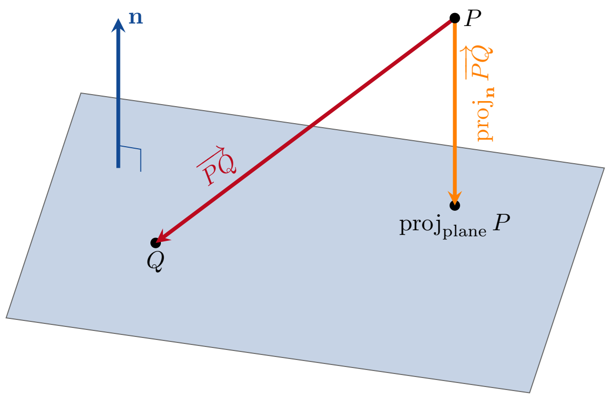


Figure 13.3: The projection of a point onto a plane.
Şekil 13.3:

Projection of a Point onto a Plane

Definition. Let P be a point and let $Ax+By+Cz=D$ be a plane. Let Q be a point on the plane and let $\mathbf{n}=A\mathbf{i}+B\mathbf{j}+C\mathbf{k}$ denote a vector normal to the plane.

The projection of the point P onto this plane is

$$\operatorname{proj}_{\text{plane}} P = P + \operatorname{proj}_{\mathbf{n}} \overrightarrow{PQ}$$

as shown in figure 13.3

Example 13.3. Find the projection of the point $P(1,2,-4)$ on the plane $2x+y+4z=2$.

solution: Note first that $\mathbf{n}=2\mathbf{i}+\mathbf{j}+4\mathbf{k}$ and that the point $Q(1,0,0)$ lies on the plane. Since

$$\overrightarrow{PQ} = Q-P = (1,0,0)-(1,2,-4) = (0,-2,4) = -2\mathbf{j}+4\mathbf{k},$$

we have

$$\begin{aligned}\operatorname{proj}_{\mathbf{n}} \overrightarrow{PQ} &= \left(\frac{\overrightarrow{PQ}\cdot\mathbf{n}}{\|\mathbf{n}\|^2}\right)\mathbf{n}\\ &= \left(\frac{0-2+16}{2^2+1^2+4^2}\right)(2\mathbf{i}+\mathbf{j}+4\mathbf{k})\\ &= \left(\frac{14}{21}\right)(2\mathbf{i}+\mathbf{j}+4\mathbf{k})\\ &= \frac{2}{3}(2\mathbf{i}+\mathbf{j}+4\mathbf{k})\\ &= \frac{4}{3}\mathbf{i}+\frac{2}{3}\mathbf{j}+\frac{8}{3}\mathbf{k}.\end{aligned}$$

Therefore

$$\begin{aligned}\operatorname{proj}_{\text{plane}} P &= P + \operatorname{proj}_{\mathbf{n}} \overrightarrow{PQ}\\ &= (1,2,-4)+\left(\frac{4}{3},\frac{2}{3},\frac{8}{3}\right)\\ &= \left(\frac{7}{3},\frac{8}{3},-\frac{4}{3}\right).\end{aligned}$$

We should double check that this point is on the plane.

$$2x+y+4z = 2\left(\frac{7}{3}\right)+\left(\frac{8}{3}\right)+4\left(-\frac{4}{3}\right) = \frac{14}{3}+\frac{8}{3}-\frac{16}{3} = \frac{6}{3} = 2 \checkmark$$

Projection of a Point on a Plane

Tanım. Let P be a point and let $Ax+By+Cz=D$ be a plane. Let Q be a point on the plane and let $\mathbf{n}=A\mathbf{i}+B\mathbf{j}+C\mathbf{k}$ denote a vector normal to the plane.

The projection of the point P onto this plane is

$$\operatorname{proj}_{\text{düzlem}} P = P + \operatorname{proj}_{\mathbf{n}} \overrightarrow{PQ}$$

as shown in figure 13.3

Örnek 13.3. Find the projection of the point $P(1,2,-4)$ on the plane $2x+y+4z=2$.

çözüm: Note first that $\mathbf{n}=2\mathbf{i}+\mathbf{j}+4\mathbf{k}$ and that the point $Q(1,0,0)$ lies on the plane. Since

$$\overrightarrow{PQ} = Q-P = (1,0,0)-(1,2,-4) = (0,-2,4) = -2\mathbf{j}+4\mathbf{k},$$

we have

$$\begin{aligned}\operatorname{proj}_{\mathbf{n}} \overrightarrow{PQ} &= \left(\frac{\overrightarrow{PQ}\cdot\mathbf{n}}{\|\mathbf{n}\|^2}\right)\mathbf{n}\\ &= \left(\frac{0-2+16}{2^2+1^2+4^2}\right)(2\mathbf{i}+\mathbf{j}+4\mathbf{k})\\ &= \left(\frac{14}{21}\right)(2\mathbf{i}+\mathbf{j}+4\mathbf{k})\\ &= \frac{2}{3}(2\mathbf{i}+\mathbf{j}+4\mathbf{k})\\ &= \frac{4}{3}\mathbf{i}+\frac{2}{3}\mathbf{j}+\frac{8}{3}\mathbf{k}.\end{aligned}$$

Therefore

$$\begin{aligned}\operatorname{proj}_{\text{düzlem}} P &= P + \operatorname{proj}_{\mathbf{n}} \overrightarrow{PQ}\\ &= (1,2,-4)+\left(\frac{4}{3},\frac{2}{3},\frac{8}{3}\right)\\ &= \left(\frac{7}{3},\frac{8}{3},-\frac{4}{3}\right).\end{aligned}$$

We should double check that this point is on the plane.

$$2x+y+4z = 2\left(\frac{7}{3}\right)+\left(\frac{8}{3}\right)+4\left(-\frac{4}{3}\right) = \frac{14}{3}+\frac{8}{3}-\frac{16}{3} = \frac{6}{3} = 2 \checkmark$$

Projection of a Line onto a Plane

Let L be a line passing through the point P in the direction $\mathbf{v}$. Let $Ax + By + Cz = D$ be a plane with normal vector $\mathbf{n} = A\mathbf{i} + B\mathbf{j} + C\mathbf{k}$.

There are three cases to consider:

(i). The line is orthogonal to the plane ($\mathbf{v} \times \mathbf{n} = \mathbf{0}$);

(ii). The line is parallel to the plane ($\mathbf{v} \cdot \mathbf{n} = 0$); and

(iii). The line is not parallel to the plane and is not orthogonal to the plane ($\mathbf{v} \cdot \mathbf{n} \neq 0$ and $\mathbf{v} \times \mathbf{n} \neq \mathbf{0}$).

Projection of a Line onto a Plane

Let L be a line passing through the point P in the direction $\mathbf{v}$. Let $Ax + By + Cz = D$ be a plane with normal vector $\mathbf{n} = A\mathbf{i} + B\mathbf{j} + C\mathbf{k}$.

There are three cases to consider:

(i). The line is orthogonal to the plane ($\mathbf{v} \times \mathbf{n} = \mathbf{0}$);

(ii). The line is parallel to the plane ($\mathbf{v} \cdot \mathbf{n} = 0$); and

(iii). The line is not parallel to the plane and is not orthogonal to the plane ($\mathbf{v} \cdot \mathbf{n} \neq 0$ and $\mathbf{v} \times \mathbf{n} \neq \mathbf{0}$).

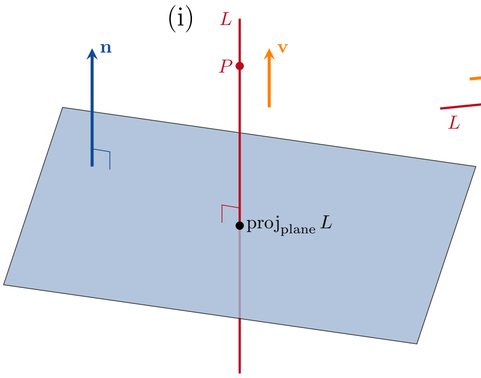


Figure 13.4: The projection of a line onto an orthogonal plane.
Şekil 13.4:

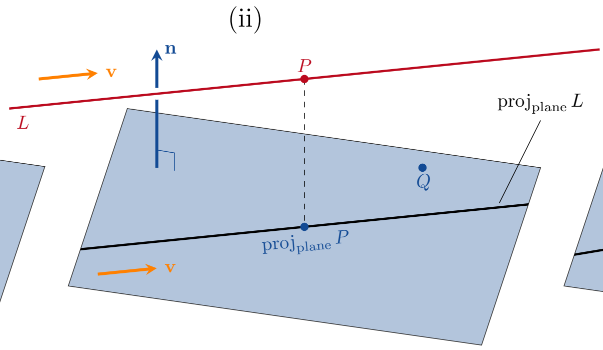


Figure 13.5: The projection of a line onto a parallel plane.
Şekil 13.5:

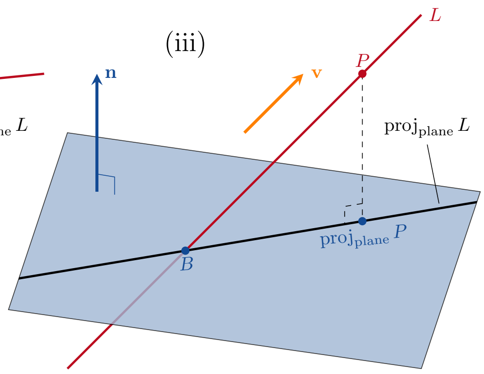


Figure 13.6: The projection of a line onto a plane which is neither orthogonal to, nor parallel to, the line.
Şekil 13.6:

A Line Orthogonal to a Plane ($\mathbf{v} \times \mathbf{n} = \mathbf{0}$)

This is the easiest case: The projection of the line onto the plane is just the point where they intersect. See figure 13.4. Therefore

$$\operatorname{proj}_{\text{plane}} L = \operatorname{proj}_{\text{plane}} P.$$

A Line Parallel to a Plane ($\mathbf{v} \cdot \mathbf{n} = 0$)

From figure 13.5, we can see that

$$\operatorname{proj}_{\text{plane}} L = \begin{pmatrix} \text{the line passing through the point} \\ \operatorname{proj}_{\text{plane}} P \text{ in the direction } \mathbf{v}. \end{pmatrix}$$

A Line which is Neither Parallel to nor Orthogonal to the Plane

See figure 13.6. If $\mathbf{v} \cdot \mathbf{n} \neq 0$, then the line must intersect the plane at some point B. Assuming $B \neq P$, we have

$$\operatorname{proj}_{\text{plane}} L = \begin{pmatrix} \text{the line passing through the} \\ \text{points } B \text{ and } \operatorname{proj}_{\text{plane}} P. \end{pmatrix}$$

A Line Orthogonal to a Plane ($\mathbf{v} \times \mathbf{n} = \mathbf{0}$)

This is the easiest case: The projection of the line onto the plane is just the point where they intersect. See figure 13.4. Therefore

$$\operatorname{proj}_{\text{düzlem}} L = \operatorname{proj}_{\text{düzlem}} P.$$

A Line Parallel to a Plane ($\mathbf{v} \cdot \mathbf{n} = 0$)

From figure 13.5, we can see that

$$\operatorname{proj}_{\text{düzlem}} L = \begin{pmatrix} \text{the line passing through the point} \\ \operatorname{proj}_{\text{düzlem}} P \text{ in the direction } \mathbf{v}. \end{pmatrix}$$

A Line which is Neither Parallel to nor Orthogonal to the Plane

See figure 13.6. If $\mathbf{v} \cdot \mathbf{n} \neq 0$, then the line must intersect the plane at some point B. Assuming $B \neq P$, we have

$$\operatorname{proj}_{\text{düzlem}} L = \begin{pmatrix} \text{the line passing through the} \\ \text{points } B \text{ and } \operatorname{proj}_{\text{düzlem}} P. \end{pmatrix}$$

Example 13.4. Find the projection of the line $x = 7 + 6t$, $y = -3 + 15t$, $z = 10 - 12t$ onto the plane $2x + 5y - 4z = 13$.

solution:

Step 1. Find $\mathbf{v}$ and $\mathbf{n}$.

$$\begin{aligned}\mathbf{v} &= 6\mathbf{i} + 15\mathbf{j} - 12\mathbf{k}\\ \mathbf{n} &= 2\mathbf{i} + 5\mathbf{j} - 4\mathbf{k}\end{aligned}$$

Step 2. Does the line intersect the plane?

Since
$$\mathbf{v} \cdot \mathbf{n} = 12 + 75 + 48 = 135 \neq 0,$$
the answer is yes, the line does intersect the plane.

Step 3. Find the point of intersection.

We calculate that

$$\begin{aligned}13 &= 2x + 5y - 4z\\ &= 2(7 + 6t) + 5(-3 + 15t) - 4(10 - 12t)\\ &= 14 + 12t - 15 + 75t - 40 + 48t\\ &= -41 + 135t\\ 54 &= 135t\\ 2 &= 5t\\ \frac{2}{5} &= t.\end{aligned}$$

Hence the point of intersection is

$$\begin{aligned}B(x, y, z)|_{t=\frac{2}{5}} &= B(7 + 6t, -3 + 15t, 10 - 12t)|_{t=\frac{2}{5}}\\ &= B(9.4, 3, 5.2)\end{aligned}$$

Step 4. Is the line orthogonal to the plane?

Since
$$\mathbf{v} \times \mathbf{n} = \begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ 6 & 15 & -12\\ 2 & 5 & -4\end{vmatrix} = 0\mathbf{i} + 0\mathbf{j} + 0\mathbf{k} = \mathbf{0},$$
the answer is yes, the line is orthogonal to the plane.

Step 5. Find $\text{proj}_{\text{plane}}\, L$.

The projection of the line on the plane is the point
$$\text{proj}_{\text{plane}}\, L = B(9.4, 3, 5.2).$$

Örnek 13.4. Find the projection of the line $x = 7 + 6t$, $y = -3 + 15t$, $z = 10 - 12t$ onto the plane $2x + 5y - 4z = 13$.

çözüm:

Adım 1. Find $\mathbf{v}$ and $\mathbf{n}$.

$$\begin{aligned}\mathbf{v} &= 6\mathbf{i} + 15\mathbf{j} - 12\mathbf{k}\\ \mathbf{n} &= 2\mathbf{i} + 5\mathbf{j} - 4\mathbf{k}\end{aligned}$$

Adım 2. Does the line intersect the plane?

Since
$$\mathbf{v} \cdot \mathbf{n} = 12 + 75 + 48 = 135 \neq 0,$$
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We calculate that

$$\begin{aligned}13 &= 2x + 5y - 4z\\ &= 2(7 + 6t) + 5(-3 + 15t) - 4(10 - 12t)\\ &= 14 + 12t - 15 + 75t - 40 + 48t\\ &= -41 + 135t\\ 54 &= 135t\\ 2 &= 5t\\ \frac{2}{5} &= t.\end{aligned}$$

Hence the point of intersection is

$$\begin{aligned}B(x, y, z)|_{t=\frac{2}{5}} &= B(7 + 6t, -3 + 15t, 10 - 12t)|_{t=\frac{2}{5}}\\ &= B(9.4, 3, 5.2)\end{aligned}$$

Adım 4. Is the line orthogonal to the plane?

Since
$$\mathbf{v} \times \mathbf{n} = \begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ 6 & 15 & -12\\ 2 & 5 & -4\end{vmatrix} = 0\mathbf{i} + 0\mathbf{j} + 0\mathbf{k} = \mathbf{0},$$
the answer is yes, the line is orthogonal to the plane.

Adım 5. Find $\text{proj}_{\text{düzlem}}\, L$.

The projection of the line on the plane is the point
$$\text{proj}_{\text{düzlem}}\, L = B(9.4, 3, 5.2).$$

Example 13.5. Find the projection of the line $x = 1 + 4t$, $y = 2 + 4t$, $z = 3 + 4t$ onto the plane $3x + 4y - 7z = 27$.

solution:

Step 1. Find $\mathbf{v}$ and $\mathbf{n}$.

$$\begin{aligned}\mathbf{v} &= 4\mathbf{i} + 4\mathbf{j} + 4\mathbf{k}\\ \mathbf{n} &= 3\mathbf{i} + 4\mathbf{j} - 7\mathbf{k}\end{aligned}$$

Step 2. Does the line intersect the plane?

Since
$$\mathbf{v} \bullet \mathbf{n} = 12 + 16 - 28 = 0,$$
the line does not intersect the plane. Therefore the line is parallel to the plane.

Step 3. Find a point on $\operatorname{proj}_{\text{plane}} L$.

$P(1, 2, 3)$ lies on the original line and $Q(9, 0, 0)$ lies on the plane. So

$$\begin{aligned}\overrightarrow{PQ} &= Q - P = (9, 0, 0) - (1, 2, 3) = (8, -2, -3)\\ &= 8\mathbf{i} - 2\mathbf{j} - 3\mathbf{k}\end{aligned}$$

and

$$\begin{aligned}\operatorname{proj}_{\mathbf{n}} \overrightarrow{PQ} &= \left(\frac{\overrightarrow{PQ} \bullet \mathbf{n}}{\|\mathbf{n}\|^2}\right)\mathbf{n} = \left(\frac{24 - 8 + 21}{9 + 16 + 49}\right)\mathbf{n}\\ &= \left(\frac{37}{74}\right)\mathbf{n} = \frac{1}{2}\mathbf{n}.\end{aligned}$$

Therefore

$$\begin{aligned}\operatorname{proj}_{\text{plane}} P &= P + \operatorname{proj}_{\mathbf{n}} \overrightarrow{PQ}\\ &= (1, 2, 3) + \left(\frac{3}{2}, 2, -\frac{7}{2}\right)\\ &= \left(\frac{5}{2}, 4, -\frac{1}{2}\right).\end{aligned}$$

We should quickly double check that our $\operatorname{proj}_{\text{plane}} P$ really is on the plane:

$$\begin{aligned}3x + 4y - 7z &= 3\left(\frac{5}{2}\right) + 4(4) - 7\left(-\frac{1}{2}\right)\\ &= \frac{15}{2} + 16 + \frac{7}{2} = 27. \checkmark\end{aligned}$$

Step 4. Find $\operatorname{proj}_{\text{plane}} L$.

The projection of the original line on the plane is the line passing through the point $\operatorname{proj}_{\text{plane}} P = \left(\frac{5}{2}, 4, -\frac{1}{2}\right)$ in the direction $\mathbf{v} = 4\mathbf{i}+4\mathbf{j}+4\mathbf{k}$, which has parametrised equations

$$x = \frac{5}{2} + 4t, \qquad y = 4 + 4t, \qquad z = -\frac{1}{2} + 4t.$$

Örnek 13.5. Find the projection of the line $x = 1 + 4t$, $y = 2 + 4t$, $z = 3 + 4t$ onto the plane $3x + 4y - 7z = 27$.

çözüm:

Adım 1. Find $\mathbf{v}$ and $\mathbf{n}$.

$$\begin{aligned}\mathbf{v} &= 4\mathbf{i} + 4\mathbf{j} + 4\mathbf{k}\\ \mathbf{n} &= 3\mathbf{i} + 4\mathbf{j} - 7\mathbf{k}\end{aligned}$$

Adım 2. Does the line intersect the plane?

Since
$$\mathbf{v} \bullet \mathbf{n} = 12 + 16 - 28 = 0,$$
the line does not intersect the plane. Therefore the line is parallel to the plane.

Adım 3. Find a point on $\operatorname{proj}_{\text{düzlem}} L$.

$P(1, 2, 3)$ lies on the original line and $Q(9, 0, 0)$ lies on the plane. So

$$\begin{aligned}\overrightarrow{PQ} &= Q - P = (9, 0, 0) - (1, 2, 3) = (8, -2, -3)\\ &= 8\mathbf{i} - 2\mathbf{j} - 3\mathbf{k}\end{aligned}$$

and

$$\begin{aligned}\operatorname{proj}_{\mathbf{n}} \overrightarrow{PQ} &= \left(\frac{\overrightarrow{PQ} \bullet \mathbf{n}}{\|\mathbf{n}\|^2}\right)\mathbf{n} = \left(\frac{24 - 8 + 21}{9 + 16 + 49}\right)\mathbf{n}\\ &= \left(\frac{37}{74}\right)\mathbf{n} = \frac{1}{2}\mathbf{n}.\end{aligned}$$

Therefore

$$\begin{aligned}\operatorname{proj}_{\text{düzlem}} P &= P + \operatorname{proj}_{\mathbf{n}} \overrightarrow{PQ}\\ &= (1, 2, 3) + \left(\frac{3}{2}, 2, -\frac{7}{2}\right)\\ &= \left(\frac{5}{2}, 4, -\frac{1}{2}\right).\end{aligned}$$

We should quickly double check that our $\operatorname{proj}_{\text{düzlem}} P$ really is on the plane:

$$\begin{aligned}3x + 4y - 7z &= 3\left(\frac{5}{2}\right) + 4(4) - 7\left(-\frac{1}{2}\right)\\ &= \frac{15}{2} + 16 + \frac{7}{2} = 27. \checkmark\end{aligned}$$

Adım 4. Find $\operatorname{proj}_{\text{düzlem}} L$.

The projection of the original line on the plane is the line passing through the point $\operatorname{proj}_{\text{düzlem}} P = \left(\frac{5}{2}, 4, -\frac{1}{2}\right)$ in the direction $\mathbf{v} = 4\mathbf{i} + 4\mathbf{j} + 4\mathbf{k}$, which has parametrised equations

$$x = \frac{5}{2} + 4t, \qquad y = 4 + 4t, \qquad z = -\frac{1}{2} + 4t.$$

Example 13.6. Find the projection of the line $x = 15 + 15t$, $y = -12 - 15t$, $z = 17 + 11t$ on the plane $13x - 9y + 16z = 69$.

solution: .

Step 1. Find $\mathbf{v}$ and $\mathbf{n}$.

$$\mathbf{v} = 15\mathbf{i} - 15\mathbf{j} + 11\mathbf{k}$$
$$\mathbf{n} = 13\mathbf{i} - 9\mathbf{j} + 16\mathbf{k}$$

Step 2. Does the line intersect the plane?

Since
$$\mathbf{v} \cdot \mathbf{n} = 506 \neq 0,$$
the line intersects the plane.

Step 3. Find the point of intersection.

We calculate that
$$\begin{aligned} 69 &= 13x - 9y + 16z \\ &= 13(15 + 15t) - 9(-12 - 15) + 16(17 + 11t) \\ &= 195 + 195t + 108 + 135t + 272 + 176t \\ &= 575 + 506t \\ -506 &= 506t \\ -1 &= t. \end{aligned}$$

Thus the line intersects the plane at
$$\begin{aligned} B(x, y, z)|_{t=-1} &= B(15 + 15t, -12 - 15t, 17 + 11t)|_{t=-1} \\ &= B(0, 3, 6). \end{aligned}$$

Step 4. Is the line orthogonal to the plane?

Since
$$\mathbf{v} \times \mathbf{n} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 15 & -15 & 11 \\ 13 & -9 & 16 \end{vmatrix} = -141\mathbf{i} - 97\mathbf{j} + 60\mathbf{k} \neq \mathbf{0},$$
the line is not orthogonal to the plane.

Step 5. Find another point on $\text{proj}_{\text{plane}}\, L$.

The point $P(15, -12, 17)$ lies on the original line. Since $\overrightarrow{PB} = (-15, 15, -11)$ and
$$\text{proj}_{\mathbf{n}}\, \overrightarrow{PB} = \left(\frac{\overrightarrow{PB} \cdot \mathbf{n}}{\|\mathbf{n}\|^2}\right)\mathbf{n} = \left(\frac{-506}{506}\right)\mathbf{n} = -\mathbf{n}$$
we have that
$$\begin{aligned} \text{proj}_{\text{plane}}\, P &= P + \text{proj}_{\mathbf{n}}\, \overrightarrow{PB} \\ &= (15, -12, 17) + (-13, 9, -16) = (2, -3, 1). \end{aligned}$$

Step 6. Find $\text{proj}_{\text{plane}}\, L$.

Let
$$\mathbf{v}_2 = \text{the vector from } B \text{ to } \text{proj}_{\text{plane}}\, P = 2\mathbf{i} - 6\mathbf{j} - 5\mathbf{k}.$$

Then $\text{proj}_{\text{plane}}\, L$ is the line passing through $B(0, 3, 6)$ in the direction $\mathbf{v}_2 = 2\mathbf{i} - 6\mathbf{j} - 5\mathbf{k}$ which has parametrised equations
$$x = 2t, \qquad y = 3 - 6t, \qquad z = 6 - 5t.$$

Örnek 13.6. Find the projection of the line $x = 15 + 15t$, $y = -12 - 15t$, $z = 17 + 11t$ on the plane $13x - 9y + 16z = 69$.

çözüm: .

Adım 1. Find $\mathbf{v}$ and $\mathbf{n}$.

$$\mathbf{v} = 15\mathbf{i} - 15\mathbf{j} + 11\mathbf{k}$$
$$\mathbf{n} = 13\mathbf{i} - 9\mathbf{j} + 16\mathbf{k}$$

Adım 2. Does the line intersect the plane?

Since
$$\mathbf{v} \cdot \mathbf{n} = 506 \neq 0,$$
the line intersects the plane.

Adım 3. Find the point of intersection.

We calculate that
$$\begin{aligned} 69 &= 13x - 9y + 16z \\ &= 13(15 + 15t) - 9(-12 - 15) + 16(17 + 11t) \\ &= 195 + 195t + 108 + 135t + 272 + 176t \\ &= 575 + 506t \\ -506 &= 506t \\ -1 &= t. \end{aligned}$$

Thus the line intersects the plane at
$$\begin{aligned} B(x, y, z)|_{t=-1} &= B(15 + 15t, -12 - 15t, 17 + 11t)|_{t=-1} \\ &= B(0, 3, 6). \end{aligned}$$

Adım 4. Is the line orthogonal to the plane?

Since
$$\mathbf{v} \times \mathbf{n} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 15 & -15 & 11 \\ 13 & -9 & 16 \end{vmatrix} = -141\mathbf{i} - 97\mathbf{j} + 60\mathbf{k} \neq \mathbf{0},$$
the line is not orthogonal to the plane.

Adım 5. Find another point on $\text{proj}_{\text{düzlem}}\, L$.

The point $P(15, -12, 17)$ lies on the original line. Since $\overrightarrow{PB} = (-15, 15, -11)$ and
$$\text{proj}_{\mathbf{n}}\, \overrightarrow{PB} = \left(\frac{\overrightarrow{PB} \cdot \mathbf{n}}{\|\mathbf{n}\|^2}\right)\mathbf{n} = \left(\frac{-506}{506}\right)\mathbf{n} = -\mathbf{n}$$
we have that
$$\begin{aligned} \text{proj}_{\text{düzlem}}\, P &= P + \text{proj}_{\mathbf{n}}\, \overrightarrow{PB} \\ &= (15, -12, 17) + (-13, 9, -16) = (2, -3, 1). \end{aligned}$$

Adım 6. Find $\text{proj}_{\text{düzlem}}\, L$.

Let
$$\mathbf{v}_2 = \text{the vector from } B \text{ to } \text{proj}_{\text{düzlem}}\, P = 2\mathbf{i} - 6\mathbf{j} - 5\mathbf{k}.$$

Then $\text{proj}_{\text{düzlem}}\, L$ is the line passing through $B(0, 3, 6)$ in the direction $\mathbf{v}_2 = 2\mathbf{i} - 6\mathbf{j} - 5\mathbf{k}$ which has parametrised equations
$$x = 2t, \qquad y = 3 - 6t, \qquad z = 6 - 5t.$$

Problems

Problem 13.1.

(a). Find the projection of the vector $\mathbf{u} = -2\mathbf{i} + 5\mathbf{j} - 3\mathbf{k}$ onto the line $x = 2 + t$, $y = 1 - 2t$, $z = 3 + 2t$.

(b). Find the projection of the vector $\mathbf{u} = \mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$ onto the line $x = 20 + 3t$, $y = 1 + 4t$, $z = -10 + 5t$.

(c). Find the projection of the vector $\mathbf{u} = \mathbf{i} - \mathbf{k}$ onto the line $x = 1 - t$, $y = 1 + t$, $z = 1 + t$.

Problem 13.2.

(a). Find the projection of the vector $\mathbf{u} = \mathbf{i} + 7\mathbf{j} + 5\mathbf{k}$ onto the plane $6x + 4z = 100$.

(b). Find the projection of the vector $\mathbf{u} = \mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$ onto the plane $3x + 2y + z = -7$.

(c). Find the projection of the vector $\mathbf{u} = \mathbf{i}$ onto the plane $7y + 4z = 13$.

Problem 13.3.

(a). Find the projection of the point $P(38, -59, 4)$ onto the plane $10x - 20y + z = 61$.

(b). Find the projection of the point $P(13, 13, 13)$ onto the plane $2x - 3y + 5z = 5$.

(c). Find the projection of the point $P(65, 70, -4)$ onto the plane $9x + 10y - z = 15$.

Problem 13.4.

(a). Find the projection of the line $x = -48 - t$, $y = 6 + t$, $z = -13 + 4t$ onto the plane $7x - y + 2z = 10$.

(b). Find the projection of the line $x = 2 + 30t$, $y = 29 - 130t$, $z = \frac{104}{5} - 114t$ onto the plane $7y + 5z = 11$.

(c). Find the projection of the line $x = -t$, $y = 14 + t$, $z = -\frac{23}{4} - t$ onto the plane $8x - 8y + 8z = 10$.

Problem 13.5. Find a formula for the projection of a point P onto a line L.

Sorular

Soru 13.1.

(a). Find the projection of the vector $\mathbf{u} = -2\mathbf{i} + 5\mathbf{j} - 3\mathbf{k}$ onto the line $x = 2 + t$, $y = 1 - 2t$, $z = 3 + 2t$.

(b). Find the projection of the vector $\mathbf{u} = \mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$ onto the line $x = 20 + 3t$, $y = 1 + 4t$, $z = -10 + 5t$.

(c). Find the projection of the vector $\mathbf{u} = \mathbf{i} - \mathbf{k}$ onto the line $x = 1 - t$, $y = 1 + t$, $z = 1 + t$.

Soru 13.2.

(a). Find the projection of the vector $\mathbf{u} = \mathbf{i} + 7\mathbf{j} + 5\mathbf{k}$ onto the plane $6x + 4z = 100$.

(b). Find the projection of the vector $\mathbf{u} = \mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$ onto the plane $3x + 2y + z = -7$.

(c). Find the projection of the vector $\mathbf{u} = \mathbf{i}$ onto the plane $7y + 4z = 13$.

Soru 13.3.

(a). Find the projection of the point $P(38, -59, 4)$ onto the plane $10x - 20y + z = 61$.

(b). Find the projection of the point $P(13, 13, 13)$ onto the plane $2x - 3y + 5z = 5$.

(c). Find the projection of the point $P(65, 70, -4)$ onto the plane $9x + 10y - z = 15$.

Soru 13.4.

(a). Find the projection of the line $x = -48 - t$, $y = 6 + t$, $z = -13 + 4t$ onto the plane $7x - y + 2z = 10$.

(b). Find the projection of the line $x = 2 + 30t$, $y = 29 - 130t$, $z = \frac{104}{5} - 114t$ onto the plane $7y + 5z = 11$.

(c). Find the projection of the line $x = -t$, $y = 14 + t$, $z = -\frac{23}{4} - t$ onto the plane $8x - 8y + 8z = 10$.

Soru 13.5. Find a formula for the projection of a point P onto a line L.

Part III

Combinatorics, Probability and Graph Theory

14

Combinatorics : Basic Counting Principles

The Addition Principle

Theorem 14.1. *For any two sets A and B,*

$$n(A \cup B) = n(A) + n(B) - n(A \cap B)$$

where $n(A)$ denotes the number of elements in set A.

Example 14.1. Ali has three yellow shirts and two blue shirts. How many shirts does Ali have?

solution: Ali has $3 + 2 = 5$ shirts. If we let S denote the set of Ali's yellow shirts, and M denote the set of Ali's blue shirts, then we have

$$n(S \cup M) = n(S) + n(M) = 3 + 2 = 5.$$

Please note that S and M are discrete sets.

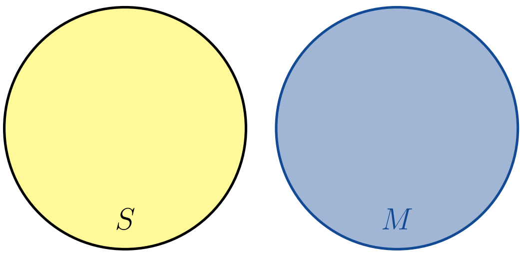

Figure 14.1: Ali's yellow and blue shirts.
Şekil 14.1: Ali'nin sarı ve mavi gömlekleri.

Example 14.2. In a college, there are 8 students studying Mathematics, 12 students studying Physics, and 5 students enrolled in a joint Chemistry-Physics-Mathematics program. How many students are studying either Mathematics or Physics?

solution: Let M and F denote the sets of students studying Mathematics and Physics respectively. Then $M \cap F$ will be

Kombinatorik : Temel Sayma Prensipleri

Toplam Prensibi

Teorem 14.1. *A ve B herhagi iki küme olsun. $n(A)$ A elemanını sayısını belirtmek üzere*

$$n(A \cup B) = n(A) + n(B) - n(A \cap B)$$

dir.

Örnek 14.1. Ali'nin 3 sarı, iki de mavi gömleği vardır. Ali gömleklerini kaç farklı şekilde giyinir?

çözüm: Ali, gömleklerini $3 + 2 = 5$ farklı şekilde giyinir. Sarı gömlekleri S, mavi gömlekleri M kümesi ile gösterirsek;

$$n(S \cup M) = n(S) + n(M) = 3 + 2 = 5$$

elde edilir.

Burada S ve M kümelerinin ayrık olduğuna dikkatinizi çekmek isterim.

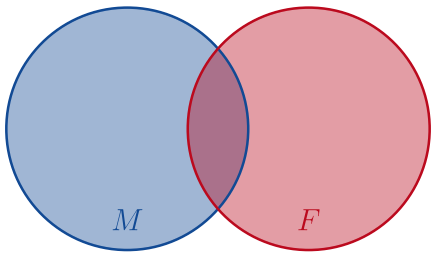

Figure 14.2: Mathematics and Physics students.
Şekil 14.2: Matematik ve Fizik eğitimi alan öğrenciler.

Örnek 14.2. Bir okulda matematik dersine kayıtlı 8, fizik dersine kayıtlı 12, kimya ve her iki derse de kayıtlı 5 öğrenci bulunmaktadır. Bu okulda matematik veya fizik dersine kayıtlı öğrenci sayısı kaçtır?

the set of students who are studying both Mathematics and Physics. The answer is

$$n(M \cup F) = n(M) + n(F) - n(M \cap F) = 8 + 12 - 5 = 15.$$

çözüm: Matematik dersini alan öğrencilerin kümesini M, fizik dersini alanları ise F ile gösterirsek hem matematik hem fizik dersini alan öğrenciler $M \cap F$ ile gösterilir. Sorunun yanıtı ise;

$$n(M \cup F) = n(M) + n(F) - n(M \cap F) = 8 + 12 - 5 = 15$$

dir.

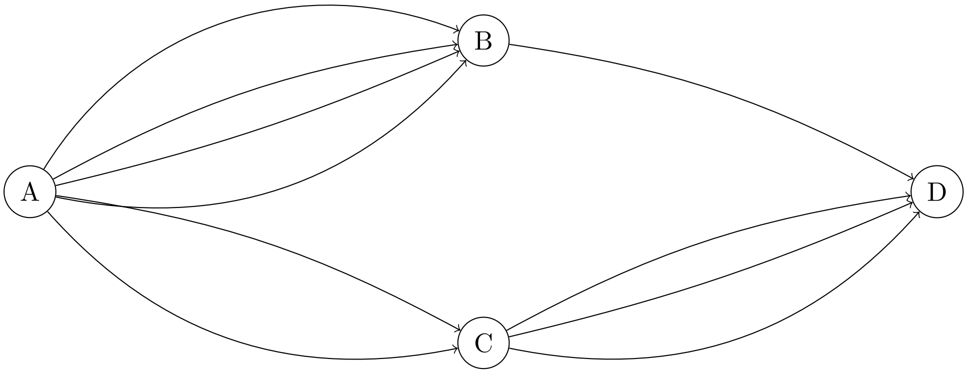

Figure 14.3: Roads linking four cities.
Şekil 14.3:

The Multiplication Principle

Theorem 14.2. *If operation O_1 can be done n ways, and operation O_2 can be done m ways, then there are*

$$n \cdot m$$

possible ways to do O_1 followed by O_2.

Example 14.3. A man has five different shirts, three different ties and two different pairs of trousers. How many different ways can this person wear a shirt, a a tie and a pair of trousers combination?

solution: Once the person has chosen a pair of trousers, he has a choice of 5 different shirts. For each shirt there are 3 different ties. Therefore he has $2 \cdot 5 \cdot 3 = 30$ choices of outfit.

Example 14.4. Four cities, called Aberdeen (A), Birmingham (B), Coventry (C) and Derby (D), are joined by roads as shown in figure 14.3. By how many different routes can a vehicle travel from A to D, without going back on itself.

solution: A vehicle that wants to travel from A to D must pass through either B or C.

If it passes through B, it can travel from A to B in 4 different ways, then from B to D in only one way – thus it can arrive at D along $4 \cdot 1 = 4$ different routes.

If the vehicle passes through C, it can travel from A to C in 2 different ways, then from C to D in 3 different ways – thus it can arrive at D along $2 \cdot 3 = 6$ different routes.

Hence the total number of different routes from A to D is $4 + 6 = 10$.

Çarpma Prensibi

Teorem 14.2. *O_1 işlemi n yoldan, O_2 işlemi m yoldan yapılabiliyorsa, O_1O_2 işlemlerinin ardışık olarak yapılması ile $n \cdot m$ sonuç ortaya çıkar.*

Örnek 14.3. Bir adamın 5 farklı gömleği, 3 farklı kravatı, 2 farklı pantolonu olsun. Bu kişi gömlek-kravat-pantolon kombinini kaç farklı şekilde giyinebilir?

çözüm: Kişinin bir pantolon seçtikten sonra o pantolon için 5 farklı gömlek seçimi vardır. Her gömlek için ise 3 farklı kravat seçimi olur. Bunu matematiksek olarak ifade edecek olursak, $2 \cdot 5 \cdot 3 = 30$ farklı şekilde giyinir.

Örnek 14.4. Antalya (A), Bursa (B), Ceyhan (C) ve Denizli (D) kentlerini baağlayan yollar yukarıdaki şekildeki gibi olsun: A noktasından çıkıp D noktasına gitmek isteyen bir araç gittiği yolu geri dönmemek üzere kaç farklı yoldan gidebilir?

çözüm: A dan D ye gitmek isteyen biri B yada C den geçmek zorundadır.

B üzerinden giderse A dan B ye 4 farklı şekilde B den D ye 1 tek yoldan gidebileceğilden en nihayetinde A dan D ye $4 \cdot 1 = 4$ farklı yoldan gidebilir.

C üzer,nden giderse A dan C ye 2, C den D ye 3 farklı yol olduğundan A dan D ye $2 \cdot 3 = 6$ farklı yoldan gidebilir.

Sonuç olarak A dan D ye $4 + 6 = 10$ farklı yoldan gidebilir.

Örnek 14.5. 2800 sayısının kaç böleni vardır?

çözüm: $2800 = 2^4 \cdot 5^2 \cdot 7$ olduğundan 2,2,2,2,5,5,7 çarpanlarından en az bir tane olmak üzere çarpanlar seçilip bunların

Example 14.5. How many divisors does 2800 have?

solution: Note first that

$$2800 = 2^4 \cdot 5^2 \cdot 7 = 1 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 5 \cdot 5 \cdot 7.$$

To create a divisor of 2800, we can use between zero and four of the 2s (so there are 4+1 choices of how many 2s to use). We can also use zero, one or both of the 5s (2+1 choices) and we can either use or not use the 7 (1+1 choices). So there are

$$(4+1)(2+1)(1+1) = 30$$

divisors of 2800. In fact, the divisors of 2800 are 1, 2, 4, 5, 7, 8, 10, 14, 16, 20, 25, 28, 35, 40, 50, 56, 70, 80, 100, 112, 140, 175, 200, 280, 350, 400, 560, 700, 1400 and 2800.

In general, suppose that $p_1, p_2, \dots, p_n$ are prime numbers and suppose that $x = p_1^{k_1}, p_2^{k_2}, \dots, p_n^{k_n}$. Then x has

$$(k_1+1)(k_2+2)\dots(k_n+1)$$

divisors.

çarpılması ile 2800 in bir böleni elde edilir. 1 sayısı her sayıyı bölebileceğinden 1 dışındaki farklı bölenlerin sayısı:
$(4+1)(2+1)(1+1) - 1 = 29$
dur.
Genel olarak bir $p_1, p_2, \dots, p_n$ asal sayılar olmak üzere bir sayının çarpanlara ayrılışı $p_1^{k_1}, p_2^{k_2}, \dots, p_n^{k_n}$ ise bu sayının "1" dahil bölenlerinin sayısı:

$$(k_1+1)(k_2+2)\dots(k_n+1)$$

ile bulunur.

Problems

Problem 14.1 (A coin and a die)**.** A coin is tossed with possible outcomes of heads (H) or tails (T). Then a die is rolled with possible outcomes of 1, 2, 3, 4, 5 or 6. How many different outcomes in total are there?

Problem 14.2 (Code words)**.**

(a). How many 3-letter code words can be formed from the letters A,B,C,D,E if no letter is repeated?

(b). How many many 3-letter code words can be formed from the letters A,B,C,D,E if letters can be repeated?

(c). How many 3-letter code words can be formed from the letters A,B,C,D,E if adjacent letters must be different?

Problem 14.3 (Postcodes)**.** The postcode for Okan University is 34959. Suppose that the first two digits of a postcode are between 01 (Adana) and 81 (Düzce), and the final three digits can be any number. How many different postcodes are possible? How many have no repeated digits?

Sorular

Soru 14.1 (A coin and a die)**.** A coin is tossed with possible outcomes of heads (H) or tails (T). Then a die is rolled with possible outcomes of 1, 2, 3, 4, 5 or 6. How many different outcomes in total are there?

Soru 14.2 (Code words)**.**

(a). How many 3-letter code words can be formed from the letters A,B,C,D,E if no letter is repeated?

(b). How many many 3-letter code words can be formed from the letters A,B,C,D,E if letters can be repeated?

(c). How many 3-letter code words can be formed from the letters A,B,C,D,E if adjacent letters must be different?

Soru 14.3 (Postcodes)**.** The postcode for Okan University is 34959. Suppose that the first two digits of a postcode are between 01 (Adana) and 81 (Düzce), and the final three digits can be any number. How many different postcodes are possible? How many have no repeated digits?

Combinatorics : Permutations and Combinations

Kombinatorik : Permütasyon ve Kombinasyonlar

Factorials

Definition. The product of the first n natural numbers is called n ***factorial*** and denoted by $n!$.

We also define the ***zero factorial***, $0!$ to be equal to 1.

$$n! = n(n-1)(n-2)\cdot \ldots \cdot 2 \cdot 1$$
$$0! = 1$$
$$n! = n \cdot (n-1)!$$

Example 15.1.

(a) $4! = 4 \cdot 3 \cdot 2 \cdot 1 = 24$

(b) $\dfrac{7!}{5!} = \dfrac{7 \cdot 6 \cdot \cancel{5!}}{\cancel{5!}} = 42$

(c) $\dfrac{52!}{5!47!} = \dfrac{52 \cdot 51 \cdot 50 \cdot 49 \cdot 48 \cdot \cancel{47!}}{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 \cdot \cancel{47!}} = 2\,598\,960$

Remark. Note that $n!$ grows very rapidly:

$$5! = 120$$
$$10! = 3\,628\,800$$
$$15! = 1\,307\,674\,368\,000$$

Faktöriyel

Tanım. n bir doğal sayı olmak üzere 1 den n e kadar olan doğal sayıların çarpımı $n!$ ile gösterilir.

Sıfır faktöriyel 1 kabul edilir.

Tanımdan aşağıdaki eşitlikler kolaylıkla elde edilebilir:

$$n! = n(n-1)(n-2)\cdot \ldots \cdot 2 \cdot 1$$
$$0! = 1$$
$$n! = n \cdot (n-1)!$$

Örnek 15.1.

(a) $4! = 4 \cdot 3 \cdot 2 \cdot 1 = 24$

(b) $\dfrac{7!}{5!} = \dfrac{7 \cdot 6 \cdot \cancel{5!}}{\cancel{5!}} = 42$

(c) $\dfrac{52!}{5!47!} = \dfrac{52 \cdot 51 \cdot 50 \cdot 49 \cdot 48 \cdot \cancel{47!}}{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 \cdot \cancel{47!}} = 2\,598\,960$

Not. Note that $n!$ grows very rapidly:

$$5! = 120$$
$$10! = 3\,628\,800$$
$$15! = 1\,307\,674\,368\,000$$

Permutations

Example 15.2. Imagine that you have four pictures to arrange on a wall. See figure 15.1 on page 87. How many different ways are there to arrange them?

solution: There are four ways to select the first picture. After we choose the first picture, we are left with three pictures to choose from. Then after we choose the second picture, we are

Permütasyon

Örnek 15.2. Imagine that you have four pictures to arrange on a wall. See figure 15.1 on page 87. How many different ways are there to arrange them?

çözüm: There are four ways to select the first picture. After we choose the first picture, we are left with three pictures to choose from. Then after we choose the second picture, we are

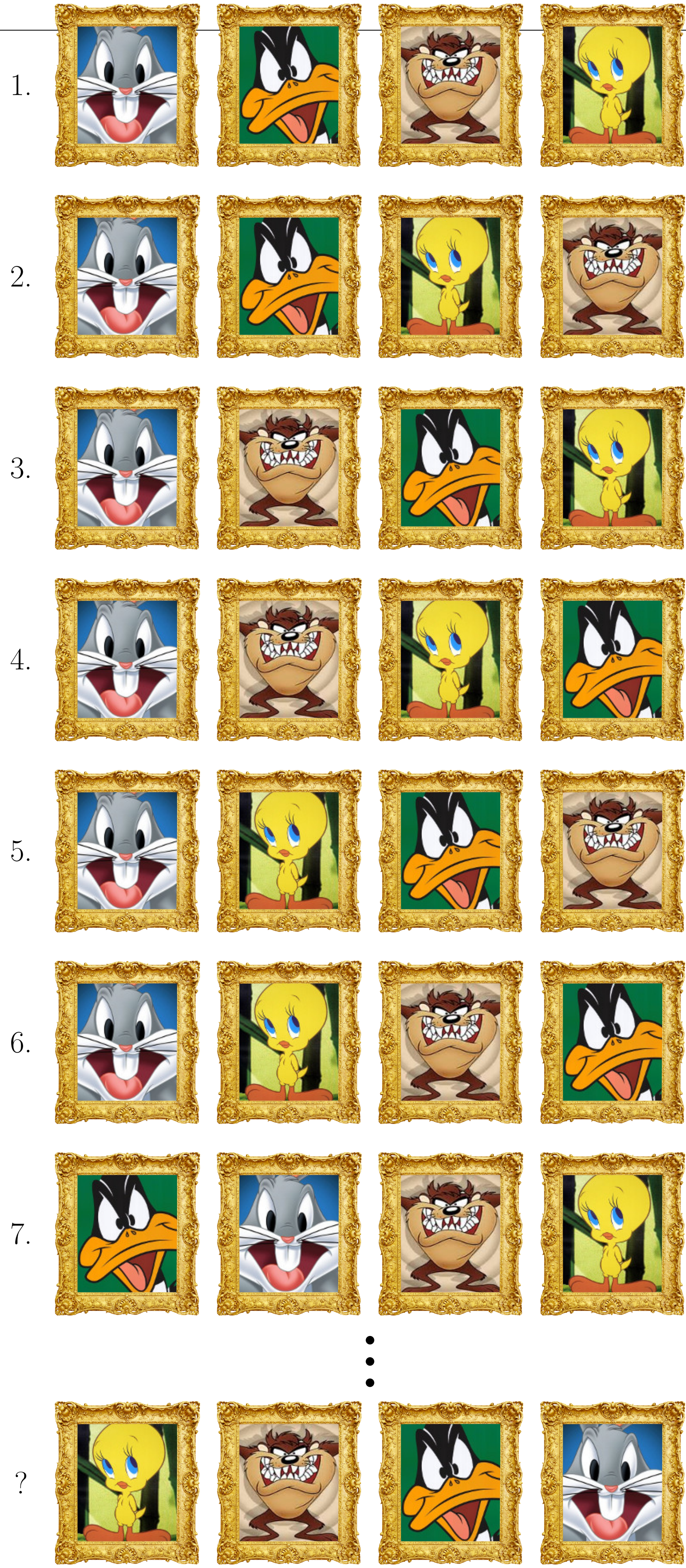

Figure 15.1: How many ways are there to arrange four (distinct) pictures on a wall?
Şekil 15.1:

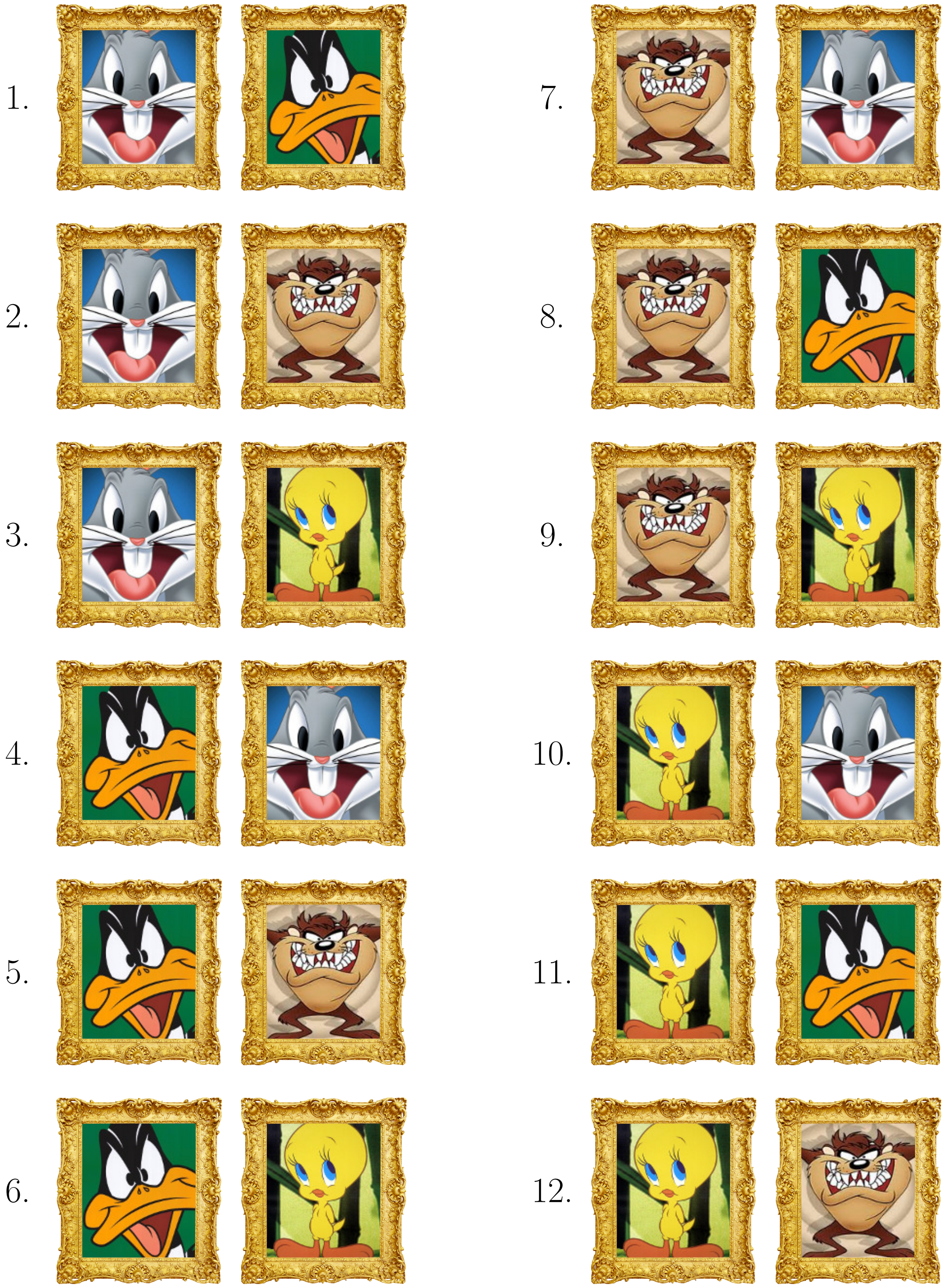

Figure 15.2: How many ways are there to select and arrange two pictures taken from a total of four pictures?
Şekil 15.2:

left with two pictures. And so on. So there are

$$4 \cdot 3 \cdot 2 \cdot 1 = 4! = 24$$

different ways to hang our pictures.

Definition. A ***permutation*** of a set of distinct objects is an arrangement of the objects in a specific order without repetition.

The number of permuations of n distinct objects (without repetition) is denoted by ${}_nP_n$ or by $P(n,n)$.

Theorem 15.1.

$${}_nP_n = n!$$

Example 15.3. Now suppose that from your four pictures, you decide to only hang two on your wall. How many ways are there to hang two out of four pictures?
solution: There are four choices for the first picture, then three for the second picture. Please see figure 15.3 on page 90. So there are

$$4 \cdot 3 = 12$$

ways to hang two out of the four pictures.

How can we write this in terms of $n!$? Note that

$$12 = 4 \cdot 3 = \frac{4 \cdot 3 \cdot 2 \cdot 1}{2 \cdot 1} = \frac{4!}{2!}.$$

Definition. A permutation of a set of n distinct objects taken r at a time (without repetition) is an arrangement of r of the n objects in a specific order. The numbers of such permutations is denoted by ${}_nP_r$ or by $P(n,r)$.

For example, suppose that we have three objects (labeled a, b and c) and suppose that we take r of the objects. The possible permuations are shown below:

left with two pictures. And so on. So there are

$$4 \cdot 3 \cdot 2 \cdot 1 = 4! = 24$$

different ways to hang our pictures.

Tanım. A ***permutation*** of a set of distinct objects is an arrangement of the objects in a specific order without repetition.

The number of permuations of n distinct objects (without repetition) is denoted by ${}_nP_n$ or by $P(n,n)$.

Teorem 15.1.

$$P(n,n) = n!$$

Örnek 15.3. Now suppose that from your four pictures, you decide to only hang two on your wall. How many ways are there to hang two out of four pictures?
çözüm: There are four choices for the first picture, then three for the second picture. Please see figure 15.3 on page 90. So there are

$$4 \cdot 3 = 12$$

ways to hang two out of the four pictures.

How can we write this in terms of $n!$? Note that

$$12 = 4 \cdot 3 = \frac{4 \cdot 3 \cdot 2 \cdot 1}{2 \cdot 1} = \frac{4!}{2!}.$$

Tanım. n tane farklı nesnenin r tanesinin tüm dizilişlerinin sayısına n nin r li permütasyonu denir ve $P(n,r)$ ile gösterilir.

Örneğin; $n = 3$ alırsak ve nesneler a, b, c olursa dizilimler r ye bağlı olarak aşağıdaki gibidir:

$n=3$		
$r=1$	$r=2$	$r=3$
a	ab	abc
b	ac	acb
c	ba	bac
	bc	bca
	ca	cab
	cb	cba
${}_3P_1 = P(3,1) = 3$	${}_3P_2 = P(3,2) = 6$	${}_3P_3 = P(3,3) = 6$

Theorem 15.2. *The number of permuations of n distinct objects taken r at a time (without repetition) is*

$$ {}_nP_r = \frac{n!}{(n-r)!} \qquad (1 \le r \le n) $$

Example 15.4. Find the number of permutations of 16 objects taken 3 at a time.

solution: Using the formula, we calculate that:

$$ {}_{16}P_3 = \frac{16!}{(16-3)!} = \frac{16 \cdot 15 \cdot 14 \cdot \cancel{13!}}{\cancel{13!}} = 16 \cdot 15 \cdot 14 = 3360. $$

Example 15.5. Please see Örnek 15.5.

Combinations

Example 15.6. To enter the Turkish national lottery (Sayısal Loto 6/49) you must select six numbers from a choice of 49 numbers.

Figure 15.3: Entering the Turkish national lottery, Sayısal Loto 6/49.
Şekil 15.3:

How many different ways are there of choosing 6 objects from 49 objects? The answer is not ${}_{49}P_6$ because the order of the numbers does not matter: For example

28 16 9 7 35 47

is the same as

7 9 16 28 35 47.

Definition. A ***combination*** of a set of n distinct objects taken r at a time (without repetition) is an r-element subset of the set of n objects. The arrangement of the elements in the subset does not matter. We denote the number of combinations by

$$ {}_nC_r \quad \text{or} \quad \binom{n}{r} \quad \text{or} \quad C(n,r). $$

Teorem 15.2. *n elemanlı bir kümenin r elemanlı permütasyonları*

$$ P(n,r) = \frac{n!}{(n-r)!} \qquad (1 \le r \le n) $$

formülü ile bulunur.

Örnek 15.4. Lütfen Example 15.4'ye bakınız.

Örnek 15.5. 17 elemanlı bir kümenin 15 elemanlı alt kümelerinin sayısı kaçtır?

çözüm: Formülü kullanırsak:

$$ P(17,15) = \frac{17!}{(17-15)!} = \frac{17!}{2!} = 177\,843\,714\,048\,000. $$

Kombinasyon

Figure 15.4: A draw of the Turkish national lottery, Sayisal Loto 6/49.
Şekil 15.4:

Örnek 15.6.

28 16 9 7 35 47

is the same as

7 9 16 28 35 47.

Tanım. n adet farklı nesnenin r tanesinin seçiminin sayısına n nin r li ***kombinasyonu*** denir ve $C(n,r)$ ile gösterilir. Kombinasyon hesabında sıralamanın bir önemi yoktur.

Önceki a, b, c örneğinde sıralamadan bağımsız olarak bu kümenin alt kümelerinin kaç farklı şekilde seçileceği sorulursa cevap aşağıdaki gibi olur:

For example, suppose again that we have three objects labeled a, b and c and suppose that we take r of these objects. The possible combinations are shown below:

$n=3$		
$r=1$	$r=2$	$r=3$
a b c	ab ac bc	abc
${}_3C_1=\binom{3}{1}=C(3,1)=3$	${}_3C_2=\binom{3}{2}=C(3,2)=3$	${}_3C_3=\binom{3}{3}=C(3,3)=1$

Theorem 15.3. *The number of combinations of n distinct objects taken r at a time (without repetition) is*

$$ {}_nC_r=\frac{n!}{(n-r)!\cdot r!} \qquad (1\leq r\leq n)$$

Example 15.7. A collector has 20 different coins. How many different ways can 6 coins be selected?

solution:

$$\begin{aligned} {}_{20}C_6 &= \frac{20!}{(20-6)!\cdot 6!}=\frac{20\cdot 19\cdot 18\cdot 17\cdot 16\cdot 15\cdot \cancel{14!}}{\cancel{14!}\cdot 6\cdot 5\cdot 4\cdot 3\cdot 2}\\ &=\frac{\cancel{20}\cdot 19\cdot \cancel{18}\cdot 17\cdot 16\cdot 15}{\cancel{6}\cdot \cancel{5}\cdot \cancel{4}\cdot \cancel{3}\cdot 2}=\frac{19\cdot 17\cdot 16\cdot 15}{2}\\ &=38\,760. \end{aligned}$$

Example 15.8. We can now answer Example 15.6. There are

$$ {}_{49}C_6=\frac{49!}{(49-6)!\cdot 6!}=\frac{49!}{43!\cdot 6!}=13\,983\,816$$

different ways to choose 6 numbers from a choice of 49 numbers.

Example 15.9. From a group of 9 people,

(a). In how many ways can a chairperson, a vice-chairperson and a secretary be elected, assuming that one person can not hold more than one position?

(b). In how many ways can we choose a subcommittee of three people?

solution:

(a). The order of election is important in the election of a chairperson, a vice-chairperson and a secretary. There is a meaning to who is the chairperson and who is the secretary. If we start with the election of the chairperson, there are 9 different candidates for him. Since one person is missing from the group of people, 8 different candidates can stand for the vice-chairperson. Finally, there are 7 different candidates for the position of secretary. Therefore, this tripartite committee can be formed in $9\ cdot8\ cdot7=504$ different ways. Or we can answer this problem with permutations:

$$ {}_9P_3=\frac{9!}{(9-3)!}=504.$$

Teorem 15.3. *n elemanlı bir kümenin r elemanlı kombinasyonları*

$$C(n,r)=\frac{n!}{(n-r)!\cdot r!} \qquad (1\leq r\leq n)$$

formülü ile bulunur.

Örnek 15.7. Bir koleksiyoncunun 20 farklı madeni parası var. Bunlar arasından 6 para kaç farklı şekilde seçilebilir?

çözüm:

$$\begin{aligned} C(20,6) &= \frac{20!}{(20-6)!\cdot 6!}=\frac{20\cdot 19\cdot 18\cdot 17\cdot 16\cdot 15\cdot \cancel{14!}}{\cancel{14!}\cdot 6\cdot 5\cdot 4\cdot 3\cdot 2}\\ &=\frac{\cancel{20}\cdot 19\cdot \cancel{18}\cdot 17\cdot 16\cdot 15}{\cancel{6}\cdot \cancel{5}\cdot \cancel{4}\cdot \cancel{3}\cdot 2}=\frac{19\cdot 17\cdot 16\cdot 15}{2}\\ &=38\,760. \end{aligned}$$

Örnek 15.8. We can now answer Example 15.6. There are

$$C(49,6)=\frac{49!}{(49-6)!\cdot 6!}=\frac{49!}{43!\cdot 6!}=13\,983\,816$$

different ways to choose 6 numbers from a choice of 49 numbers.

Örnek 15.9. 9 kişilik bir topluluktan

(a). Bir başkan bir başkan yardımcısı ve bir sekreter kaç farklı şekilde seçilebilir?

(b). 3 kişilik bir alt topluluk kaç farklı şekilde seçilir?

çözüm:

(a). Bir başkan bir başkan yardımcısı ve bir sekreter seçiminde seçim sırası önemlidir. Kimin başkan, kimin sekreter olduğunun bir anlamı vardır. Önce başkan seçiminden başlarsak onun için 9 farklı aday vardır. Topluluktan bir kişi eksildiğinden başkan yardımcısı için 8 farklı aday seçimi yapılabilir. Son olarak da sekreter için 7 farklı aday kalır. Dolayısıyla bu üçlü komite $9\cdot 8\cdot 7=504$ farklı şekilde oluşturulabilir. Dolayısıyla bu hesabı permütasyon ile yaparız:

$$P(9,3)=\frac{9!}{(9-3)!}=504.$$

(b). In order to form a sub-community of 3 people, the order does not matter! So we use combinations:

$$ {}_9C_3 = \frac{9!}{(9-3)!3!} = 84. $$

Remark. Permutations and Combinations are similar in that repetition in selections are not permitted. However, there is one important difference between them:

- In a permutation, the order is important;
- In a combination, the order is irrelevent.

You need to understand when to use ${}_nP_r$ and when to use ${}_nC_r$.

Example 15.10. From a standard deck of 52 cards,

(a). How many 5-card hands have two kings and three aces?

(b). How many 5-card hands have two clubs and three hearts?

(c). How many 3-card hands have all cards from the same suit?

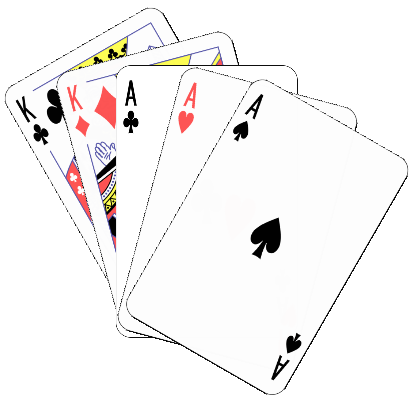

Figure 15.5: Two kings and three aces.
Şekil 15.5:

solution:

(a). We need to use the multiplication princple and combinations. We must select two kings from a total of four (that is ${}_4C_2$) and we must select three aces from a total of four (${}_4C_3$). Therefore, by the multiplication principle we have that

$$ \begin{aligned} \text{number of hands} &= {}_4C_2 \cdot {}_4C_3 \\ &= \frac{4!}{(4-2)!\cdot 2!} \cdot \frac{4!}{(4-3)!\cdot 3!} \\ &= 6 \cdot 4 = 24. \end{aligned} $$

(b). There are ${}_{13}C_2 \cdot {}_{13}C_3 = \frac{13!}{11!\cdot 2!} \cdot \frac{13!}{10!\cdot 3!} = 78 \cdot 208 = 22\,308$ hands with two clubs and three hearts.

(c). There are 13 cards in each suit, so the number of 3-card hands having all hearts, say, is

$$ {}_{13}C_3 = \frac{13!}{(13-3)!\cdot 3!} = \frac{13!}{10!\cdot 3!} = 286. $$

(b). 3 kişilik bir alt topluluğun oluşturulmasında ise sıranın bir önemi yoktur! Dolayısıyla bu hesabı kombinasyon ile yaparız:

$$ C(9,3) = \frac{9!}{(9-3)!3!} = 84. $$

Not. Permutations and Combinations are similar in that repetition in selections are not permitted. However, there is one important difference between them:

- In a permutation, the order is important;
- In a combination, the order is irrelevent.

You need to understand when to use $P(n,r)$ and when to use $C(n,r)$.

Örnek 15.10. From a standard deck of 52 cards,

(a). How many 5-card hands have two kings and three aces?

(b). How many 5-card hands have two clubs and three hearts?

(c). How many 3-card hands have all cards from the same suit?

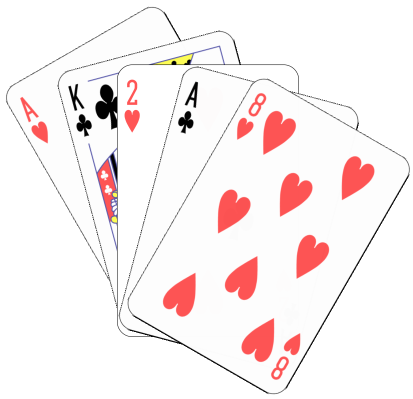

Figure 15.7: Two clubs and three hearts.
Şekil 15.7:

çözüm:

(a). We need to use the multiplication princple and combinations. We must select two kings from a total of four (that is $C(4,2)$) and we must select three aces from a total of four ($C(4,3)$). Therefore, by the multiplication principle we have that

$$ \begin{aligned} \text{number of hands} &= C(4,2) \cdot C(4,3) \\ &= \frac{4!}{(4-2)!\cdot 2!} \cdot \frac{4!}{(4-3)!\cdot 3!} \\ &= 6 \cdot 4 = 24. \end{aligned} $$

(b). There are $C(13,2) \cdot C(13,3) = \frac{13!}{11!\cdot 2!} \cdot \frac{13!}{10!\cdot 3!} = 78 \cdot 208 = 22\,308$ hands with two clubs and three hearts.

(c). There are 13 cards in each suit, so the number of 3-card hands having all hearts, say, is

$$ C(13,3) = \frac{13!}{(13-3)!\cdot 3!} = \frac{13!}{10!\cdot 3!} = 286. $$

Similarly, there are 286 hand having all clubs, 286 hands having all diamonds and 286 hands having all spades. Thus the number of hands having all cards from the same suit is

$$4 \cdot {}_{13}C_3 = 4 \cdot 286 = 1144.$$

Example 15.11. A bag contains 2 white and 3 red balls. In how many ways can 3 balls be chosen if at least one ball must be white?

solution: We can have either 1 white ball and 2 red balls, or 2 white balls and 1 red ball.

There are ${}_2C_1 \cdot {}_3C_2$ ways to get 1 white and 2 red. There are ${}_2C_2 \cdot {}_3C_1$ ways to get 2 white and 1 red. So the total number of ways is

$${}_2C_1 \cdot {}_3C_2 + {}_2C_2 \cdot {}_3C_1 = (2)(3) + (1)(3) = 9.$$

Example 15.12. You have 1 red ball, 1 blue ball, 1 green ball and 3 orange balls. The three orange balls are identical. How many visually different ways are there to arrange the balls in a line?

solution: If you had balls of six different colours, then this is easy: ${}_6P_6 = 6!$. However because you have 3 balls of the same colour, the correct answer will be less than this. For example, if we label the balls r, b, g, o_1, o_2 and o_3 then $r\,g\,o_1\,o_2\,b\,o_3$ and $r\,g\,o_2\,o_3\,b\,o_1$ will look the same, but be counted twice.

But how many different ways are there to rearrange the orange balls, $o_1\,o_2\,o_3$? There are ${}_3P_3 = 6$ different ways. This means that when we calculate ${}_6P_6$, we are counting each arrangement 6 times.

Therefore the answer to this problem is

$$\frac{{}_6P_6}{{}_3P_3} = 120.$$

Example 15.13. In how many different ways can the letters of the word 'MATHEMATICS' be arranged such that the vowels are consecutive?

solution: The word 'MATHEMATICS' has 11 letters including 4 vowels: 'A','E','A','I'. These 4 vowels must always be consecutive. Hence these 4 vowels can be grouped together and then thought of as a single object. In other words, we may assume that we have only 8 objects, MTHMTCS(AEAI).

Of these 8 objects, we have two 'M's, two 'T's, one 'H', one 'C', one 'S' and one '(AEAİ)'. The number of ways to arrange these 8 objects is

$$\frac{{}_8P_8}{{}_2P_2 \cdot {}_2P_2} = \frac{8!}{2! \cdot 2!} = 10080.$$

Next we must ask how many ways the vowels 'AEAI' can be rearranged. The letter 'A' occurs twice and the other letters occur once. Hence there are

$$\frac{{}_4P_4}{{}_2P_2} = 12$$

ways to arrange the vowels.

Similarly, there are 286 hand having all clubs, 286 hands having all diamonds and 286 hands having all spades. Thus the number of hands having all cards from the same suit is

$$4 \cdot C(13,3) = 4 \cdot 286 = 1144.$$

Örnek 15.11. A bag contains 2 white and 3 red balls. In how many ways can 3 balls be chosen if at least one ball must be white?

çözüm: We can have either 1 white ball and 2 red balls, or 2 white balls and 1 red ball.

There are $C(2,1) \cdot C(3,2)$ ways to get 1 white and 2 red. There are $C(2,2) \cdot C(3,1)$ ways to get 2 white and 1 red. So the total number of ways is

$$C(2,1) \cdot C(3,2) + C(2,2) \cdot C(3,1) = (2)(3) + (1)(3) = 9.$$

Örnek 15.12. You have 1 red ball, 1 blue ball, 1 green ball and 3 orange balls. The three orange balls are identical. How many visually different ways are there to arrange the balls in a line?

çözüm: If you had balls of six different colours, then this is easy: $P(6,6) = 6!$. However because you have 3 balls of the same colour, the correct answer will be less than this. For example, if we label the balls k, m, y, t_1, t_2 and t_3 then $k\,y\,t_1\,t_2\,m\,t_3$ and $k\,y\,t_2\,t_3\,m\,t_1$ will look the same, but be counted twice.

But how many different ways are there to rearrange the orange balls, $t_1\,t_2\,t_3$? There are $P(3,3) = 6$ different ways. This means that when we calculate $P(6,6)$, we each counting one arrangement 6 times.

Therefore the answer to this problem is

$$\frac{P(6,6)}{P(3,3)} = 120.$$

Örnek 15.13. Lütfen Example 15.13'ye bakınız.

Örnek 15.14. In how many different ways can the letters of the word 'MATEMATİK' be arranged such that the vowels are consecutive?

çözüm: The word 'MATEMATİK' has 9 letters including 4 vowels: 'A','E','A','İ'. These 4 vowels must always be consecutive. Hence these 4 vowels can be grouped together and then thought of as a single object. In other words, we may assume that we have only 6 objects, MTMTK(AEAİ).

Of these 6 objects, we have two 'M's, two 'T's, one 'K' and one '(AEAİ)'. The number of ways to arrange these 6 objects is

$$\frac{P(6,6)}{P(2,2) \cdot P(2,2)} = \frac{6!}{2! \cdot 2!} = 180.$$

Next we must ask how many ways the vowels 'AEAİ' can be rearranged. The letter 'A' occurs twice and the other letters occur once. Hence there are

$$\frac{P(4,4)}{P(2,2)} = 12$$

ways to arrange the vowels.

Multiplying these together, we get our answer: There are

$$10080 \cdot 12 = 120\,960$$

ways.

Example 15.14. Please see Örnek 15.14.

Figure 15.6: Does your calculator have nPr and nCr?
Şekil 15.6:

Multiplying these together, we get our answer: There are

$$180 \cdot 12 = 2160$$

ways.

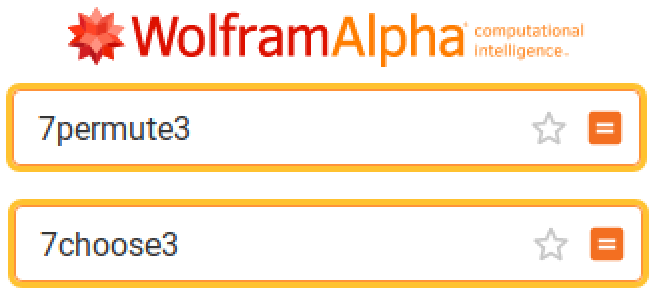

Figure 15.8: Try typing "7permute3" or "7choose3" into `wolframalpha.com` .
Şekil 15.8:

Problems

Problem 15.1 (Triangles in a Circle)**.** Five distinct points are selected on the circumference of a circle. How many different triangles can be drawn using these points as vertices?

Problem 15.2 (Playing Cards)**.** From a standard 52 card deck:

(a). How many 6-card hands consist entirely of red cards?

(b). How many 6-card hands consist entirely of hearts?

(c). How many 5-card hands consist entirely of face cards (kings, queens and jacks)?

(d). How many 5-card hands consist entirely of queens?

(e). How many 7-card hands contain four kings?

(f). How many 4-card hands contain a card from each suit?

(g). How many 3-card hands do not contain any hearts?

(h). How many 3-card hands contain at least one heart?

Problem 15.3 (Mobile Phone Shop)**.** A mobile phone shop receives a delivery of 24 smartphones, but 5 of these phones are broken. Three of these smartphones will be selected for display in the shop window.

(a). How many selections can be made?

(b). How many of these selections will contain three working phones?

Problem 15.4 (Architecture and Mimarlık)**.** In how many different ways can the letters of the word 'ARCHITECTURE' be arranged such that the vowels are consecutive? What about the word 'MİMARLIK'?

Problem 15.5 (Divisible by 5)**.** How many 3 digit numbers can be formed from the digits 2, 3, 5, 6, 7 and 9 which are divisible by 5 and none of the digits is repeated?

Problem 15.6 (Serial Numbers)**.** Serial numbers for a product are made using 2 letters followed by 3 numbers. If the letters are taken from $\{A, B, C, D, E, F, G, H\}$ with no repeats, and the numbers are taken from $\{0, 1, 2, \ldots, 9\}$ with no repeats, how many serial numbers are possible?

Sorular

Soru 15.1 (Triangles in a Circle)**.** Five distinct points are selected on the circumference of a circle. How many different triangles can be drawn using these points as vertices?

Soru 15.2 (Playing Cards)**.** From a standard 52 card deck:

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16 Introduction to Probability

Olasılığa Giriş

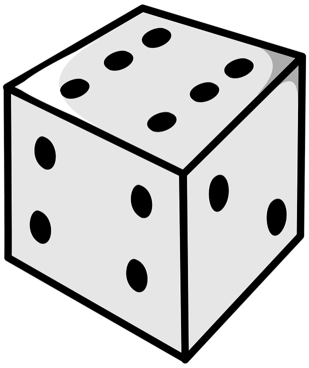

Figure 16.1: A die.
Şekil 16.1: Bir zar.

If you roll a single (standard six-sided) die, what are the possible outcomes?

$$S = \{1, 2, 3, 4, 5, 6\}$$

This is called a ***sample space***. Each of the numbers in S are called ***simple events***. Now suppose that we want to roll an even number: The subset

$$E = \{2, 4, 6\} \subseteq S$$

is called a ***compound event***.

Definition. The set

$$S = \{e_1, e_2, \ldots, e_n\}$$

is called a ***sample space*** for some experiment iff,

- S contains all possible outcomes;
- one and only one of the outcomes in S must occur.

Definition. An ***event*** E is any subset of S (including the empty set $\varnothing$ and S itself). e_i is a ***simple event***. $E = \{e_i\}$ is a ***simple event*** if E contains only one element. E is a ***compound event*** if E contains more than one element.

If you roll a single (standard six-sided) die, what are the possible outcomes?

$$S = \{1, 2, 3, 4, 5, 6\}$$

This is called a ***sample space***. Each of the numbers in S are called ***simple events***. Now suppose that we want to roll an even number: The subset

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Tanım. The set

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- S contains all possible outcomes;
- one and only one of the outcomes in S must occur.

Tanım. An ***event*** E is any subset of S (including the empty set $\varnothing$ and S itself). e_i is a ***simple event***. $E = \{e_i\}$ is a ***simple event*** if E contains only one element. E is a ***compound event*** if E contains more than one element.

1 horse	2 horses
1 woman	2 women
1 vertex	2 vertices
1 focus	2 foci
1 die	2 dice

first die / birinci zar	second die ikinci zar					
	(1,1)	(1,2)	(1,3)	(1,4)	(1,5)	(1,6)
	(2,1)	(2,2)	(2,3)	(2,4)	(2,5)	(2,6)
	(3,1)	(3,2)	(3,3)	(3,4)	(3,5)	(3,6)
	(4,1)	(4,2)	(4,3)	(4,4)	(4,5)	(4,6)
	(5,1)	(5,2)	(5,3)	(5,4)	(5,5)	(5,6)
	(6,1)	(6,2)	(6,3)	(6,4)	(6,5)	(6,6)

Table 16.1: Possible outcomes from rolling two dice.
Şekil 16.1: k

Example 16.1 (Two Dice)**.** Next suppose that you are rolling two standard six-sided dice. Please see Table 16.1. The sample space is

$$S = \{(a, b) \mid a, b \in \{1, 2, \dots, 6\}\}.$$

What is the event which corresponds to:

(a). A total score of 7.

(b). A total score of 3.

(c). A total score greater than 10.

(d). A total score of 2.

solution:

(a). $E = \{(1, 6), (2, 5), (3, 4), (4, 3), (5, 2), (6, 1)\}$.

(b). $E = \{(1, 2), (2, 1)\}$.

(c). $E = \{(5, 6), (6, 5), (6, 6)\}$.

(d). $E = \{(1, 1)\}$.

Örnek 16.1 (Two Dice)**.** Next suppose that you are rolling two standard six-sided dice. Please see Table 16.1. The sample space is

$$S = \{(a, b) \mid a, b \in \{1, 2, \dots, 6\}\}.$$

What is the event which corresponds to:

(a). A total score of 7.

(b). A total score of 3.

(c). A total score greater than 10.

(d). A total score of 2.

çözüm:

(a). $E = \{(1, 6), (2, 5), (3, 4), (4, 3), (5, 2), (6, 1)\}$.

(b). $E = \{(1, 2), (2, 1)\}$.

(c). $E = \{(5, 6), (6, 5), (6, 6)\}$.

(d). $E = \{(1, 1)\}$.

The Probability of an Event

Definition. Let

$$S = \{e_1, e_2, e_e, \ldots, e_n\}$$

be a sample space with n simple events. The ***probability of event*** e_i is a real number denoted by $P(e_i)$. We must have

(i). $P(e_i) \in [0, 1]$ for all i; and

(ii). $P(e_1) + P(e_2) + P(e_3) + \ldots + P(e_n) = 1$.

Example 16.2 (A Single Coin). Suppose that we are flipping a single coin. Then $S = \{H, T\}$. We can assume that

$$P(H) = \frac{1}{2} \qquad \text{and} \qquad P(T) = \frac{1}{2}.$$

Note that

(i). $0 \leq P(H) \leq 1$ and $0 \leq P(T) \leq 1$; and

(ii). $P(H) + P(T) = \frac{1}{2} + \frac{1}{2} = 1$.

If we flip this coin 1000 times, we would expect to get H roughly (but not exactly) 500 times.

Definition. The ***probability of an event*** E, denoted $P(E)$ must satisfy:

(i). If $E = \varnothing$ is the empty set, then $P(E) = 0$;

(ii). If $E = S$, then $P(E) = 1$.

(iii). If $E = \{e_i\}$ is a simple event, then $P(E) = P(e_i)$;

(iv). If E is a compound event, then $P(E)$ must be equal to the sum of the probabilities of all the simple events in E. E.g. if $E = \{a, b, c\}$ then $P(E) = P(a) + P(b) + P(c)$.

Remark. $P(E) = 1$ means that E is certain to occur. $P(E) = 0$ means that E will never occur.

Example 16.3 (Two coins). Now suppose that you are flipping two different coins. The sample space is

$$S = \{HH, HT, TH, TT\}.$$

We can assume that

$$P(HH) = P(HT) = P(TH) = P(TT) = \frac{1}{4}.$$

(a). What is the probability of getting one head and one tail?

(b). What is the probability of getting atleast one tail?

(c). What is the probability of getting atleast one head or one tail?

(d). What is the probability of getting two tails?

(e). What is the probability of getting three tails?

solution:

(a). We have $E = \{HT, TH\}$ and

$$P(E) = P(HT) + P(TH) = \frac{1}{4} + \frac{1}{4} = \frac{1}{2}.$$

The Probability of an Event

Tanım. Let

$$S = \{e_1, e_2, e_e, \ldots, e_n\}$$

be a sample space with n simple events. The ***probability of event*** e_i is a real number denoted by $P(e_i)$. We must have

(i). $P(e_i) \in [0, 1]$ for all i; and

(ii). $P(e_1) + P(e_2) + P(e_3) + \ldots + P(e_n) = 1$.

Örnek 16.2 (A Single Coin). Suppose that we are flipping a single coin. Then $S = \{H, T\}$. We can assume that

$$P(H) = \frac{1}{2} \qquad \text{and} \qquad P(T) = \frac{1}{2}.$$

Note that

(i). $0 \leq P(H) \leq 1$ and $0 \leq P(T) \leq 1$; and

(ii). $P(H) + P(T) = \frac{1}{2} + \frac{1}{2} = 1$.

If we flip this coin 1000 times, we would expect to get H roughly (but not exactly) 500 times.

Tanım. The ***probability of an event*** E, denoted $P(E)$ must satisfy:

(i). If $E = \varnothing$ is the empty set, then $P(E) = 0$;

(ii). If $E = S$, then $P(E) = 1$.

(iii). If $E = \{e_i\}$ is a simple event, then $P(E) = P(e_i)$;

(iv). If E is a compound event, then $P(E)$ must be equal to the sum of the probabilities of all the simple events in E. E.g. if $E = \{a, b, c\}$ then $P(E) = P(a) + P(b) + P(c)$.

Not. $P(E) = 1$ means that E is certain to occur. $P(E) = 0$ means that E will never occur.

Örnek 16.3 (Two coins). Now suppose that you are flipping two different coins. The sample space is

$$S = \{HH, HT, TH, TT\}.$$

We can assume that

$$P(HH) = P(HT) = P(TH) = P(TT) = \frac{1}{4}.$$

(a). What is the probability of getting one head and one tail?

(b). What is the probability of getting atleast one tail?

(c). What is the probability of getting atleast one head or one tail?

(d). What is the probability of getting two tails?

(e). What is the probability of getting three tails?

çözüm:

(a). We have $E = \{HT, TH\}$ and

$$P(E) = P(HT) + P(TH) = \frac{1}{4} + \frac{1}{4} = \frac{1}{2}.$$

(b). We have $E = \{HT, TH, TT\}$ and

$$P(E) = P(HT) + P(TH) + P(TT) = \frac{1}{4} + \frac{1}{4} + \frac{1}{4} = \frac{3}{4}.$$

(c). We have $E = \{HH, HT, TH, TT\} = S$ and

$$P(E) = P(S) = 1.$$

(d). We have $E = \{TT\}$ and $P(E) = \frac{1}{4}$.

(e). It is not possible to get three tails, so $E = \varnothing$ and $P(E) = P(\varnothing) = 0$.

Theorem 16.1. *If we assume that each simple event in S is equally likely, then the probability of an event E is*

$$P(E) = \frac{\textit{number of elements in } E}{\textit{number of elements in } S} = \frac{n(E)}{n(S)}.$$

Example 16.4. Suppose that we are rolling two dice and suppose that each simple event is equally likely. Find the probabilities of the following:

(a). A total score of 7.

(b). A total score of 3.

(c). A total score greater than 10.

(d). A total score of 2.

solution: Please refer to Example 16.1 and Table 16.1 again.

(a). Since $E = \{(1, 6), (2, 5), (3, 4), (4, 3), (5, 2), (6, 1)\}$, we have that

$$P(E) = \frac{n(E)}{n(S)} = \frac{6}{36} = \frac{1}{6}.$$

(b). We have that $E = \{(1, 2), (2, 1)\}$ and that

$$P(E) = \frac{n(E)}{n(S)} = \frac{2}{36} = \frac{1}{18}.$$

(c). Here $E = \{(5, 6), (6, 5), (6, 6)\}$ and

$$P(E) = \frac{n(E)}{n(S)} = \frac{3}{36} = \frac{1}{12}.$$

(d). Since $E = \{(1, 1)\}$, it follows that

$$P(E) = \frac{n(E)}{n(S)} = \frac{1}{36}.$$

Example 16.5. You randomly draw five cards from a standard deck of 52 cards. What is the probability of getting two clubs and three hearts?

solution: There are $n(S) = {}_{52}C_5$ possible 5-card hands. As we covered in Example 15.10, there are $n(E) = {}_{13}C_2 \cdot {}_{13}C_3$ 5-card hands which have two clubs and three hearts. So the probability is

$$P(E) = \frac{n(E)}{n(S)} = \frac{{}_{13}C_2 \cdot {}_{13}C_3}{{}_{52}C_5} = \frac{78 \cdot 208}{2\,598\,960} \approx 0.0062$$

(b). We have $E = \{HT, TH, TT\}$ and

$$P(E) = P(HT) + P(TH) + P(TT) = \frac{1}{4} + \frac{1}{4} + \frac{1}{4} = \frac{3}{4}.$$

(c). We have $E = \{HH, HT, TH, TT\} = S$ and

$$P(E) = P(S) = 1.$$

(d). We have $E = \{TT\}$ and $P(E) = \frac{1}{4}$.

(e). It is not possible to get three tails, so $E = \varnothing$ and $P(E) = P(\varnothing) = 0$.

Teorem 16.1. *If we assume that each simple event in S is equally likely, then the probability of an event E is*

$$P(E) = \frac{\textit{number of elements in } E}{\textit{number of elements in } S} = \frac{n(E)}{n(S)}.$$

Örnek 16.4. Suppose that we are rolling two dice and suppose that each simple event is equally likely. Find the probabilities of the following:

(a). A total score of 7.

(b). A total score of 3.

(c). A total score greater than 10.

(d). A total score of 2.

çözüm: Please refer to Example 16.1 and Table 16.1 again.

(a). Since $E = \{(1, 6), (2, 5), (3, 4), (4, 3), (5, 2), (6, 1)\}$, we have that

$$P(E) = \frac{n(E)}{n(S)} = \frac{6}{36} = \frac{1}{6}.$$

(b). We have that $E = \{(1, 2), (2, 1)\}$ and that

$$P(E) = \frac{n(E)}{n(S)} = \frac{2}{36} = \frac{1}{18}.$$

(c). Here $E = \{(5, 6), (6, 5), (6, 6)\}$ and

$$P(E) = \frac{n(E)}{n(S)} = \frac{3}{36} = \frac{1}{12}.$$

(d). Since $E = \{(1, 1)\}$, it follows that

$$P(E) = \frac{n(E)}{n(S)} = \frac{1}{36}.$$

Örnek 16.5. You randomly draw five cards from a standard deck of 52 cards. What is the probability of getting two clubs and three hearts?

çözüm: There are $n(S) = C(52, 5)$ possible 5-card hands. As we covered in Example 15.10, there are $n(E) = C(13, 2) \cdot C(13, 3)$ 5-card hands which have two clubs and three hearts. So the probability is

$$P(E) = \frac{n(E)}{n(S)} = \frac{C(13, 2) \cdot C(13, 3)}{C(52, 5)} = \frac{78 \cdot 208}{2\,598\,960} \approx 0.0062$$

Problems

Problem 16.1 (Three coins). You are flipping 3 coins. Two of the coins have a head (H) on one side and a tail (T) on the other side. The third coin has two heads. What is the probability of getting

(a). 0 tails?

(b). 1 tail?

(c). 2 tails?

(d). 3 tails?

(e). less than 2 tails?

(f). more than 1 tail?

Problem 16.2 (Playing cards). A standard deck of 52 cards has 13 hearts, 13 diamonds, 13 clubs and 13 spades. Hearts and diamonds are *red*. Clubs and spades are *black*. Kings, queens and jacks are called *face cards*.

What is the probability that:

(a). a 5-card hand consists of only red cards?

(b). a 5-card hand consists of only face cards?

(c). a 4-card hand does not have any aces?

(d). a 13-card hand does not have any clubs?

(e). a 13-card hand has only black cards?

(f). a 13-card hand has only aces and face cards?

(g). a 13-card hand contains all four aces?

Sorular

Soru 16.1 (Three coins). You are flipping 3 coins. Two of the coins have a head (H) on one side and a tail (T) on the other side. The third coin has two heads. What is the probability of getting

(a). 0 tails?

(b). 1 tail?

(c). 2 tails?

(d). 3 tails?

(e). less than 2 tails?

(f). more than 1 tail?

Soru 16.2 (Playing cards). A standard deck of 52 cards has 13 hearts, 13 diamonds, 13 clubs and 13 spades. Hearts and diamonds are *red*. Clubs and spades are *black*. Kings, queens and jacks are called *face cards*.

What is the probability that:

(a). a 5-card hand consists of only red cards?

(b). a 5-card hand consists of only face cards?

(c). a 4-card hand does not have any aces?

(d). a 13-card hand does not have any clubs?

(e). a 13-card hand has only black cards?

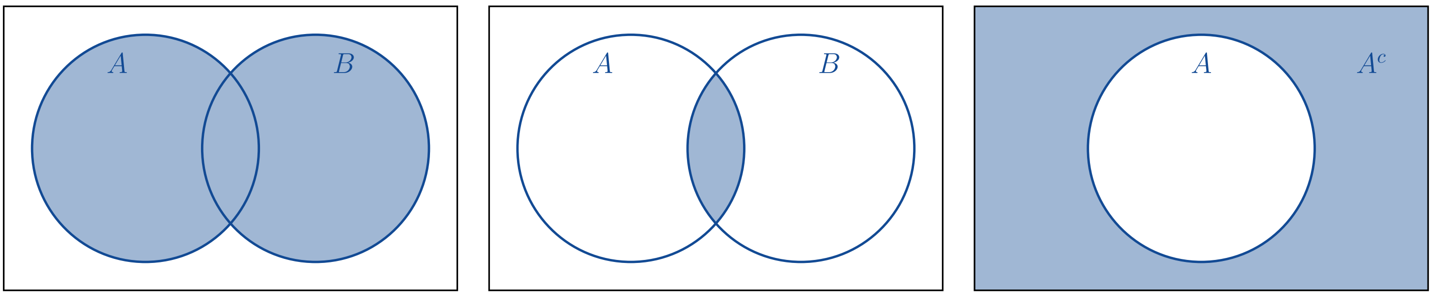
(f). a 13-card hand has only aces and face cards?

(g). a 13-card hand contains all four aces?

17 Concepts of Probability Olasılık Kavramları

Union and Intersection

Birleşim Kümesi ve Kesişim Kümesi



$A \cup B$ $\quad$ $A \cap B$ $\quad$ A^c

Definition. Suppose that A and B are events in a sample space S.

(i). The ***union*** of A and B is
$$A \cup B = \{e \in S \mid e \in A \text{ or } e \in B\}.$$

(ii). The ***intersection*** of A and B is
$$A \cap B = \{e \in S \mid e \in A \text{ and } e \in B\}.$$

Example 17.1 (One die)**.** The sample space for rolling a single die is
$$S = \{1, 2, 3, 4, 5, 6\}.$$
Assume that each of these simple events are equally likely.

(a). What is the probability of rolling a number which is even and greater than 3?

(b). What is the probability of rolling a number which is even or greater than 3?

solution: Let
$$A = \text{even numbers} = \{2, 4, 6\}$$
and
$$B = \text{numbers greater than } 3 = \{4, 5, 6\}.$$

(a). Since $A \cap B = \{4, 6\}$, we have that $P(A \cap B) = \frac{2}{6} = \frac{1}{3}$.

(b). Since $A \cup B = \{2, 4, 5, 6\}$, we have that $P(A \cup B) = \frac{4}{6} = \frac{2}{3}$.

Remark. Please recall the Addition Principle (Theorem 14.1) which stated that $n(A \cup B) = n(A) + n(B) - n(A \cap B)$.

Tanım. Suppose that A and B are events in a sample space S.

(i). The ***union*** of A and B is
$$A \cup B = \{e \in S \mid e \in A \text{ or } e \in B\}.$$

(ii). The ***intersection*** of A and B is
$$A \cap B = \{e \in S \mid e \in A \text{ and } e \in B\}.$$

Örnek 17.1 (One die)**.** The sample space for rolling a single die is
$$S = \{1, 2, 3, 4, 5, 6\}.$$
Assume that each of these simple events are equally likely.

(a). What is the probability of rolling a number which is even and greater than 3?

(b). What is the probability of rolling a number which is even or greater than 3?

çözüm: Let
$$A = \text{even numbers} = \{2, 4, 6\}$$
and
$$B = \text{numbers greater than } 3 = \{4, 5, 6\}.$$

(a). Since $A \cap B = \{4, 6\}$, we have that $P(A \cap B) = \frac{2}{6} = \frac{1}{3}$.

(b). Since $A \cup B = \{2, 4, 5, 6\}$, we have that $P(A \cup B) = \frac{4}{6} = \frac{2}{3}$.

Not. Please recall the Addition Principle (Theorem 14.1) which stated that $n(A \cup B) = n(A) + n(B) - n(A \cap B)$.

Theorem 17.1.

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

Example 17.2 (Two dice)**.** You roll two dice. What is the probability that:

(a). the sum is either 5 or 10?

(b). either the sum is greater than 9, or both dice show the same number?

solution:

(a). Let

$$\begin{aligned} A &= \text{the sum is 5} \\ &= \{(1,4),(2,3),(3,2),(4,1)\} \end{aligned}$$

and

$$\begin{aligned} B &= \text{the sum is 10} \\ &= \{(4,6),(5,5),(6,4)\}. \end{aligned}$$

Since $A \cap B = \varnothing$, we have that

$$P(A \cup B) = P(A) + P(B) = \frac{4}{36} + \frac{3}{36} = \frac{7}{36}.$$

(b). Let

$$\begin{aligned} C &= \text{the sum is greater than 9} \\ &= \{(4,6),(5,5),(5,6),(6,4),(6,5),(6,6)\} \end{aligned}$$

and

$$\begin{aligned} D &= \text{both dice show the same number} \\ &= \{(1,1),(2,2),(3,3),(4,4),(5,5),(6,6)\}. \end{aligned}$$

Note that $C \cap D = \{(5,5),(6,6)\}$. Therefore

$$\begin{aligned} P(C \cup D) &= P(C) + P(D) - P(C \cap D) \\ &= \frac{6}{36} + \frac{6}{36} - \frac{2}{36} = \frac{10}{36} = \frac{5}{18}. \end{aligned}$$

Teorem 17.1.

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

Örnek 17.2 (Two dice)**.** You roll two dice. What is the probability that:

(a). the sum is either 5 or 10?

(b). either the sum is greater than 9, or both dice show the same number?

çözüm:

(a). Let

$$\begin{aligned} A &= \text{the sum is 5} \\ &= \{(1,4),(2,3),(3,2),(4,1)\} \end{aligned}$$

and

$$\begin{aligned} B &= \text{the sum is 10} \\ &= \{(4,6),(5,5),(6,4)\}. \end{aligned}$$

Since $A \cap B = \varnothing$, we have that

$$P(A \cup B) = P(A) + P(B) = \frac{4}{36} + \frac{3}{36} = \frac{7}{36}.$$

(b). Let

$$\begin{aligned} C &= \text{the sum is greater than 9} \\ &= \{(4,6),(5,5),(5,6),(6,4),(6,5),(6,6)\} \end{aligned}$$

and

$$\begin{aligned} D &= \text{both dice show the same number} \\ &= \{(1,1),(2,2),(3,3),(4,4),(5,5),(6,6)\}. \end{aligned}$$

Note that $C \cap D = \{(5,5),(6,6)\}$. Therefore

$$\begin{aligned} P(C \cup D) &= P(C) + P(D) - P(C \cap D) \\ &= \frac{6}{36} + \frac{6}{36} - \frac{2}{36} = \frac{10}{36} = \frac{5}{18}. \end{aligned}$$

Complements

Definition. The ***complement*** of an event A in a sample space S is

$$A^c = \{e \in S \mid e \notin A\}.$$

Theorem 17.2.

$$P(E) = 1 - P(E^c).$$

Sometimes it is easier to calculate $1 - P(E^c)$, than to calculate $P(E)$ directly.

Example 17.3 (Whiteboard Markers)**.** A box containing 45 whiteboard markers is delivered to Okan University. Nine of the markers are red. The remaining markers are black. Your teacher is given 10 markers at random. He will be happy if one or more of his markers is red. What is the probability that your teacher will be happy?

solution: Let

$$E = \text{one or more of the markers is red.}$$

Then

$$E^c = \text{all 10 markers are black.}$$

Since

$$P(E^c) = \frac{n(E^c)}{n(S)} = \frac{{}_{36}C_{10}}{{}_{45}C_{10}},$$

we have that

$$P(E) = 1 - P(E^c) = 1 - \frac{{}_{36}C_{10}}{{}_{45}C_{10}} \approx 0.92.$$

Example 17.4. In a class of 30 students, what is the probability that at least two students have the same birthday? (Same month and day. Ignore 29 February)

solution: We assume that there are 365 days in a year and that each day is equally likely. We have

$$n(S) = 365^{30}$$

by the Multiplication Principle. Let

$$E = \text{2 or more people have the same birthday.}$$

Then

$$E^c = \text{all 30 students have different birthdays.}$$

We calculate that

$$n(E^c) = 365 \cdot 364 \cdot 363 \cdot \ldots \cdot 336 = \frac{365!}{335!} = {}_{365}P_{30},$$

$$P(E^c) = \frac{n(E^c)}{n(S)} = \frac{365!}{335! \cdot 365^{30}}$$

and

$$P(E) = 1 - P(E) = 1 - \frac{365!}{335! \cdot 365^{30}} \approx 0.706$$

Complements

Tanım. The ***complement*** of an event A in a sample space S is

$$A^c = \{e \in S \mid e \notin A\}.$$

Teorem 17.2.

$$P(E) = 1 - P(E^c).$$

Sometimes it is easier to calculate $1 - P(E^c)$, than to calculate $P(E)$ directly.

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çözüm: Let

$$E = \text{one or more of the markers is red.}$$

Then

$$E^c = \text{all 10 markers are black.}$$

Since

$$P(E^c) = \frac{n(E^c)}{n(S)} = \frac{C(36,10)}{C(45,10)},$$

we have that

$$P(E) = 1 - P(E^c) = 1 - \frac{C(36,10)}{C(45,10)} \approx 0.92.$$

Örnek 17.4. In a class of 30 students, what is the probability that at least two students have the same birthday? (Same month and day. Ignore 29 February)

çözüm: We assume that there are 365 days in a year and that each day is equally likely. We have

$$n(S) = 365^{30}$$

by the Multiplication Principle. Let

$$E = \text{2 or more people have the same birthday.}$$

Then

$$E^c = \text{all 30 students have different birthdays.}$$

We calculate that

$$n(E^c) = 365 \cdot 364 \cdot 363 \cdot \ldots \cdot 336 = \frac{365!}{335!} = P(365,30),$$

$$P(E^c) = \frac{n(E^c)}{n(S)} = \frac{365!}{335! \cdot 365^{30}}$$

and

$$P(E) = 1 - P(E) = 1 - \frac{365!}{335! \cdot 365^{30}} \approx 0.706$$

Problems

Problem 17.1 (Lottery). A bag contains 20 tokens numbered $1 - 20$. One ball is drawn at random. What is the probability that the number drawn is:

(a). odd or a multiple of 3?

(b). even or odd?

(c). prime or greater than 10?

(d). a multiple of 5 or a multiple of 7?

(e). less than 14 or greater than 10?

(f). even or less than 4?

Problem 17.2 (Complements).

(a). Two cards are chosen at random from a standard deck of playing cards. What is the probability that at least one of them is a face card (Jack, Queen or King)?

(b). A fair die is thrown. What is the probability that the score is not a factor of 6?

(c). The letters a,b,c,d,...,x,y,z are written on 26 cards. Two cards are chosen at random (without replacement). What is the probability that at least one of them is a consonant?

(d). Two fair dice are thrown. What is the probability that the two scores do not add to 7?

(e). A bag contains 20 balls numbered from 1 to 20. Two balls are selected at the same time from the bag. What is the probability that the two numbers selected do NOT differ by 12?

Sorular

Soru 17.1 (Lottery). A bag contains 20 tokens numbered $1 - 20$. One ball is drawn at random. What is the probability that the number drawn is:

(a). odd or a multiple of 3?

(b). even or odd?

(c). prime or greater than 10?

(d). a multiple of 5 or a multiple of 7?

(e). less than 14 or greater than 10?

(f). even or less than 4?

Soru 17.2 (Complements).

(a). Two cards are chosen at random from a standard deck of playing cards. What is the probability that at least one of them is a face card (Jack, Queen or King)?

(b). A fair die is thrown. What is the probability that the score is not a factor of 6?

(c). The letters a,b,c,d,...,x,y,z are written on 26 cards. Two cards are chosen at random (without replacement). What is the probability that at least one of them is a consonant?

(d). Two fair dice are thrown. What is the probability that the two scores do not add to 7?

(e). A bag contains 20 balls numbered from 1 to 20. Two balls are selected at the same time from the bag. What is the probability that the two numbers selected do NOT differ by 12?

18 Conditional Probability

Koşullu Olasılık

Sometimes the probability of an event will depend on another event. For example, suppose that

$$A = \text{Ali has cancer}$$

and

$$B = \text{Ali is a smoker.}$$

Clearly the probability that A occurs depends on B.

Definition. The ***conditional probability*** of A given B is

$$P(A|B) = \begin{pmatrix}\text{the probability of A, if we} \\ \text{already know that B occurs}\end{pmatrix}.$$

Theorem 18.1.

$$P(A|B) = \frac{P(A \cap B)}{P(B)}.$$

Example 18.1 (Marbles)**.** A bag contains red and blue marbles. Two marbles are drawn without replacement. The probability of selecting a red marble and then a blue marble is 0.28. The probability of selecting a red marble on the first draw is 0.5. What is the probability of selecting a blue marble on the second draw, given that the first marble drawn was red?

solution: Let

$$R = \text{the first marble is red}$$

and

$$B = \text{the second marble is blue.}$$

The required probability is

$$P(B|R) = \frac{P(B \cap R)}{P(R)} = \frac{0.28}{0.5} = 0.56$$

Example 18.2 (One die)**.** Your friend says that when she rolled a die, she rolled an odd number. What is the probability that your friend rolled a 3?

solution: Let

$$A = \text{your friend rolled a 3}$$

Sometimes the probability of an event will depend on another event. For example, suppose that

$$A = \text{Ali has cancer}$$

and

$$B = \text{Ali is a smoker.}$$

Clearly the probability that A occurs depends on B.

Tanım. The ***conditional probability*** of A given B is

$$P(A|B) = \begin{pmatrix}\text{the probability of A, if we} \\ \text{already know that B occurs}\end{pmatrix}.$$

Teorem 18.1.

$$P(A|B) = \frac{P(A \cap B)}{P(B)}.$$

Örnek 18.1 (Marbles)**.** A bag contains red and blue marbles. Two marbles are drawn without replacement. The probability of selecting a red marble and then a blue marble is 0.28. The probability of selecting a red marble on the first draw is 0.5. What is the probability of selecting a blue marble on the second draw, given that the first marble drawn was red?

çözüm: Let

$$R = \text{the first marble is red}$$

and

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The required probability is

$$P(B|R) = \frac{P(B \cap R)}{P(R)} = \frac{0.28}{0.5} = 0.56$$

Örnek 18.2 (One die)**.** Your friend says that when she rolled a die, she rolled an odd number. What is the probability that your friend rolled a 3?

çözüm: Let

$$A = \text{your friend rolled a 3}$$

and

$$B = \text{your friend rolled an odd number.}$$

and

$$B = \text{your friend rolled an odd number.}$$

Then $P(A) = \frac{1}{6}$, $P(A \cap B) = P(A) = \frac{1}{6}$ and $P(B) = \frac{1}{2}$. Hence

$$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{\frac{1}{6}}{\frac{1}{2}} = \frac{1}{3}.$$

Remark. Given that $P(A|B) = \frac{P(A\cap B)}{P(B)}$, we have that

$$P(A \cap B) = P(B)P(A|B).$$

Similarly,

$$P(A \cap B) = P(B \cap A) = P(A)P(B|A).$$

Therefore:

Theorem 18.2 (Product Rule)**.**

$$P(A \cap B) = P(A)P(B|A) = P(B)P(A|B)$$

and

$$P(A|B) = \frac{P(A)P(B|A)}{P(B)}.$$

Then $P(A) = \frac{1}{6}$, $P(A \cap B) = P(A) = \frac{1}{6}$ and $P(B) = \frac{1}{2}$. Hence

$$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{\frac{1}{6}}{\frac{1}{2}} = \frac{1}{3}.$$

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Teorem 18.2 (Product Rule)**.**

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and

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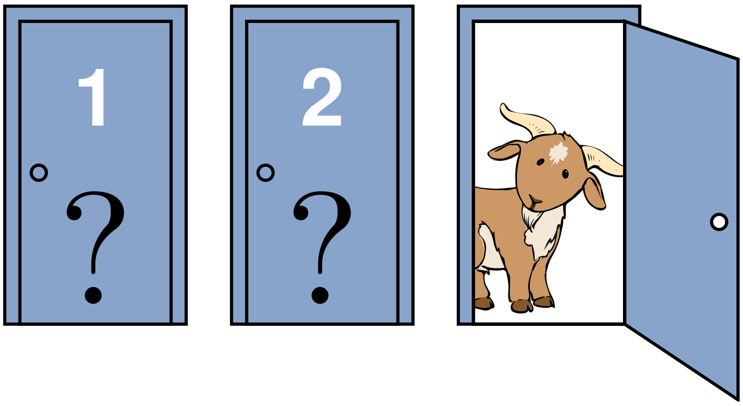

Figure 18.1: The Monty Hall Problem.
Şekil 18.1:

The Monty Hall Problem

> Suppose you're on a TV game show, and you're given the choice of three doors: Behind one door is a car; behind the others, goats. You pick a door, say No. 1, and the host, who knows what's behind the doors, opens another door, say No. 3, which has a goat. He then says to you, "Do you want to pick door No. 2?" Is it to your advantage to switch your choice?

The ***Monty Hall Problem*** is one that confuses a lot of people that haven't studied Probability.

1. You choose a door. What is the probability that the car is behind that door? Easy $P(\text{car}) = \frac{1}{3}$ right?

CORRECT

2. Then the host opens another door and shows you a goat. Now there are two closed doors: Behind one is a car and behind the other is a goat.
3. What is the probability that the car is behind the door that you chose? Two closed doors. One Car. So clearly now $P(\text{car}) = \frac{1}{2}$ right?

WRONG!!!

What? Why? To explain, let us look at all the possible outcomes if you choose door number 1 first.

behind door 1	behind door 2	behind door 3	outcome if you don't switch	outcome if you switch
car	goat	goat	win a car	win a goat
goat	car	goat	win a goat	win a car
goat	goat	car	win a goat	win a car

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car	goat	goat	win a car	win a goat
goat	car	goat	win a goat	win a car
goat	goat	car	win a goat	win a car

We can see from the table that if you don't switch your choice, then you have a $\frac{1}{3}$ chance of winning the car, but if you do switch then you have $\frac{2}{3}$ chance of winning it. These are the probabilities if you choose door number 1, but of course we would get the same results if you choose door 2 or 3 first.

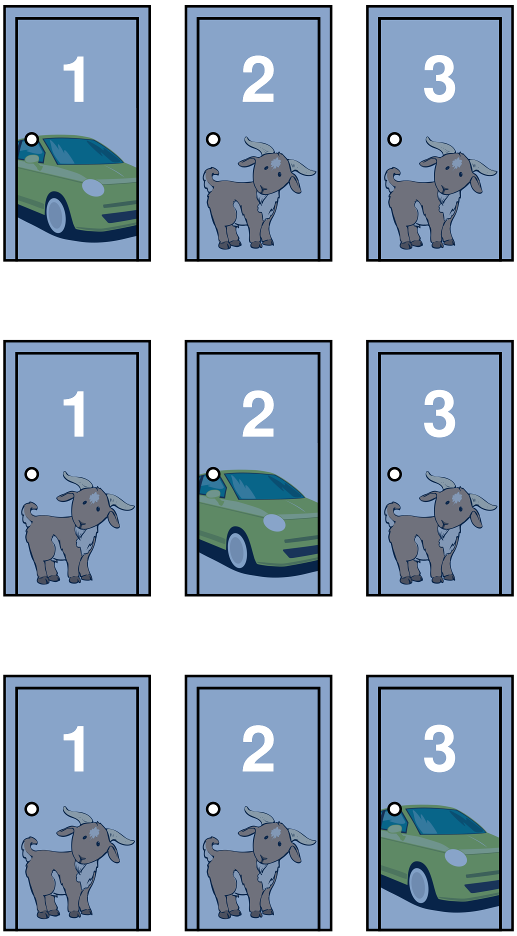

Figure 18.2: Where is the car?
Şekil 18.2:

Another way

Let's think of this another way: Please see figure 18.3. You initially choose door number 1. At the start, the chance of the car being behind door one is $\frac{1}{3}$ and the chance of the car being behind doors 2 or 3 is $\frac{2}{3}$.

Let's imagine that the host doesn't open a door, he just says you can change your choice for *both* of the other two doors. Would you switch then?

We know that atleast one of doors 2 and 3 hides a goat. Remember that the host knows where the car is: He doesn't open a door at random, he always opens a door with a goat. So he isn't really giving you any extra information. The probabilities don't change to $\frac{1}{2}$, $\frac{1}{2}$, they are still $\frac{1}{3}$, $\frac{2}{3}$.

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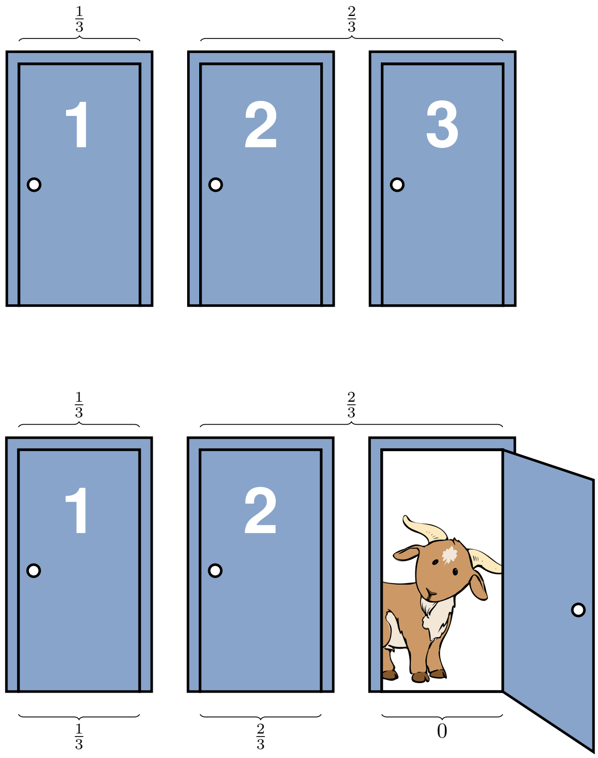

Figure 18.3: car?
Şekil 18.3:

Another way

Let's think of this another way: Please see figure 18.3. You initially choose door number 1. At the start, the chance of the car being behind door one is $\frac{1}{3}$ and the chance of the car being behind doors 2 or 3 is $\frac{2}{3}$.

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Using Conditional Probabilities

Let

$$\begin{aligned} C &= \text{door number 1 has a car behind it} \\ C^c &= \text{door number 1 does not have a car behind it} \\ &= \text{door number 1 has a goat behind it} \end{aligned}$$

and

$$E = \text{the host has opened a door with a goat behind it.}$$

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&= \text{door number 1 has a goat behind it}
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and

$$E = \text{the host has opened a door with a goat behind it.}$$

We have that $P(C) = \frac{1}{3}$ and $P(C^c) = 1 - P(C) = \frac{2}{3}$. Moreover, $P(E|C) = 1$ and $P(E|C^c) = 1$ because the host always opens a door with a goat.

Then we can calculate that

$$
\begin{aligned}
P(C|E) &= \frac{P(C)P(E|C)}{P(E)} \\
&= \frac{P(C)P(E|C)}{P(E \cap C) + P(E \cap C^c)} \\
&= \frac{P(C)P(E|C)}{P(C)P(E|C) + P(C^c)P(E|C^c)} \\
&= \frac{\frac{1}{3} \cdot 1}{\frac{1}{3} \cdot 1 + \frac{2}{3} \cdot 1} = \frac{1}{3}.
\end{aligned}
$$

This means that it doesn't matter if the host opens a door or not, the probability that the car is behind door number 1 is always $\frac{1}{3}$.

Conclusion

You should always switch door if you want to win the car.

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Conclusion

You should always switch door if you want to win the car.

Problems

Problem 18.1 (Conditional Probability)**.**

(a). Two cards are chosen at random without replacement from a pack of 52 playing cards.

If the first card chosen is an Ace, what is the probability the second card chosen is a King?

(b). Two cards are chosen at random without replacement from a pack of 52 playing cards.

If the first card chosen is an Ace, what is the probability the second card chosen is also an Ace?

(c). In Eton School, 60% of the boys play rugby, and 24% of the boys play rugby and football.

What percent of those that play rugby also play football?

Problem 18.2 (Conditional Probability)**.**

(a). Your roll two dice. Given that you get a sum is 10, what is the probability that the first die shows a 5?

(b). Harry Potter turns up at class either late or on time. He is then either shouted at or not. The probability that he turns up late is $\frac{2}{5}$. If he turns up late, the probability that he is shouted at is $\frac{7}{10}$. If he turns up on time, the probability that he is still shouted at for no particular reason is $\frac{1}{5}$.

You hear Harry being shouted at. What is the probability that he was late?

(c). The probability of raining on Sunday is 0.07. If today is Sunday then find the probability of rain today.

(d). Susan took two tests. The probability of her passing both tests is 0.6. The probability of her passing the first test is 0.8. What is the probability of her passing the second test given that she has passed the first test?

(e). A student has studied only 15 of the 25 topics covered in MATH117. When writing the final exam, the teacher randomly selects four topics to ask questions on. What is the probability that the student has studied these four topics?

Sorular

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19 Probability Trees

Example 19.1 (5 balls in a box)**.** A box contains 3 red and 2 yellow balls. Two balls are randomly drawn without replacement. What is the probability that the second ball is yellow?

solution:

A ***probability tree*** for this problem is shown in figure 19.1. From this probability tree, we can see that

$$P(YY) + P(RY) = \frac{1}{10} + \frac{3}{10} = \frac{4}{10} = \frac{2}{5}.$$

The probability that the second ball is yellow is $\frac{2}{5}$.

Olasılık Ağaçları

Örnek 19.1 (5 balls in a box)**.** A box contains 3 red and 2 yellow balls. Two balls are randomly drawn without replacement. What is the probability that the second ball is yellow?

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The probability that the second ball is yellow is $\frac{2}{5}$.

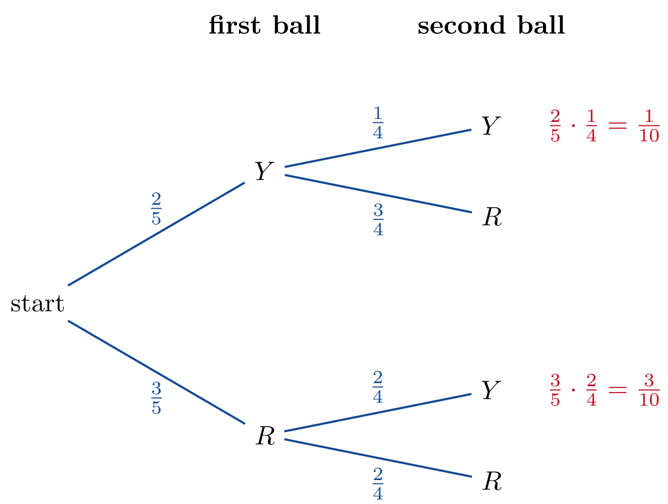

Figure 19.1: A probability tree for the 5 balls in a box problem.
Şekil 19.1:

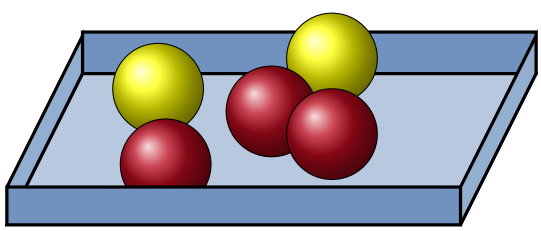

Figure 19.2: 3 red balls and 2 yellow balls in a box.
Şekil 19.2:

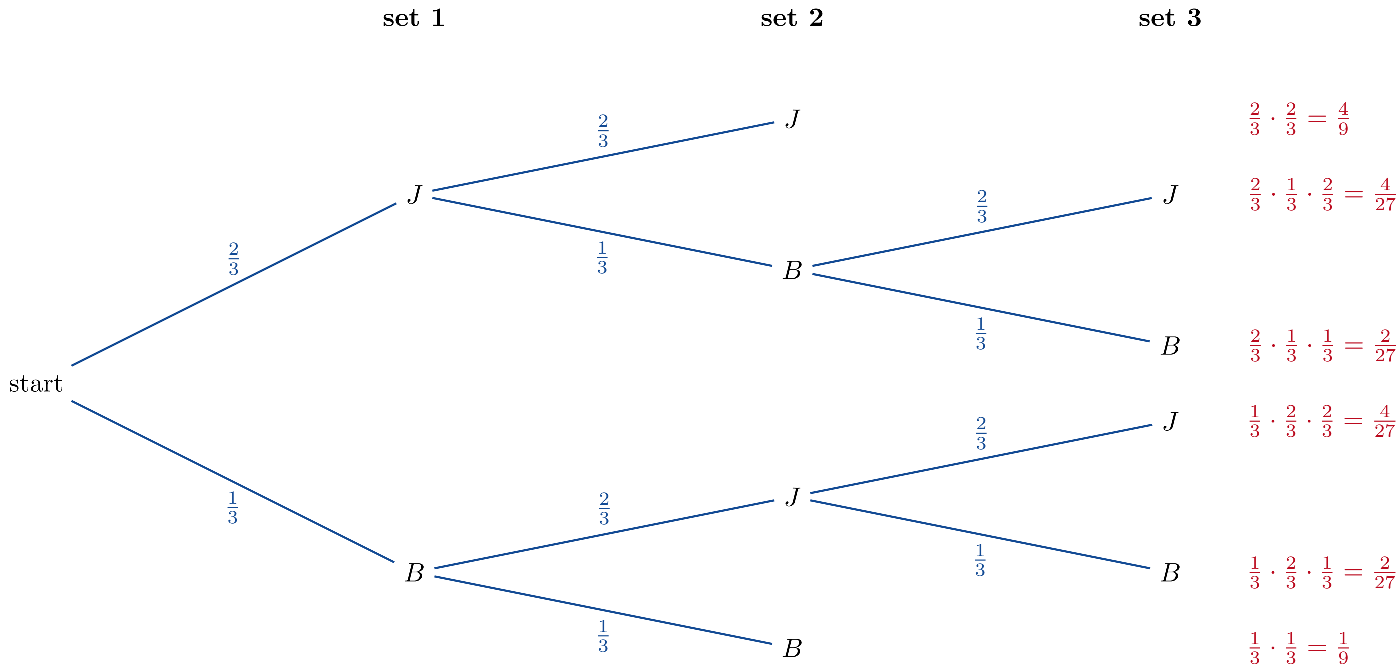

Figure 19.3: A probability tree for the tennis match between Jeremy and Boris.
Şekil 19.3:

Example 19.2 (Tennis)**.** Jeremy and Boris are playing tennis. The first player to win 2 sets wins the match. In each set, the probability that Jeremy wins that set is $\frac{2}{3}$. Find the probability that:

(a). Boris wins the match.

(b). 3 sets are played.

(c). The player who wins the first set, wins the match.

solution: A probability tree for this problem is show in figure 19.3.

(a). We calculate that

$$P(JBB)+P(BJB)+P(BB)=\frac{2}{27}+\frac{2}{27}+\frac{1}{9}=\frac{7}{27}.$$

(b). Now

$$\begin{aligned}&P(JBJ)+P(JBB)+P(BJJ)+P(BJB)\\&=\frac{4}{27}+\frac{2}{27}+\frac{4}{27}+\frac{2}{27}\\&=\frac{4}{9}.\end{aligned}$$

(c). Finally

$$\begin{aligned}&P(JJ)+P(JBJ)+P(BJB)+P(BB)\\&=\frac{4}{9}+\frac{4}{27}+\frac{2}{27}+\frac{1}{9}\\&=\frac{7}{9}.\end{aligned}$$

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Problems

Use probability trees to answer the following problems:

Problem 19.1 (Tennis)**.** Recep and Kemal are playing tennis. The first player to win 2 sets wins the match. The probability that Recep will win the first set is $\frac{4}{5}$. The probability that Recep will win the second set is $\frac{2}{5}$. The probability that Recep will win the third set (if it is played) is $\frac{1}{5}$.

(a). What is the probability that Recep will win the match?

(b). What is the probability that Kemal will win the match?

(c). What is the probability that the third set will be played?

(d). What is the probability that Kemal will win atleast one set?

(e). What is the probability that the last two sets played are won by the same person?

(f). If the match finishes after two sets, who is more likely to win?

(g). If the match finishes after three sets, who is more likely to win?

Problem 19.2 (Money Money Money)**.** A vase contains 2 ten-lira banknotes, 1 fifty-lira banknote, and 1 one-hundred-lira banknote. You randomly take the banknotes out of the vase (without replacement) until you take the one-hundred-lira note. Then you stop. You keep all the money that you have taken.

(a). What is the probability that you have taken exactly 160 liras?

(b). What is the probability that you take all of the money out of the vase?

(c). What is the probability that you take two banknotes?

Sorular

Use probability trees to answer the following problems:

Soru 19.1 (Tennis)**.** Recep and Kemal are playing tennis. The first player to win 2 sets wins the match. The probability that Recep will win the first set is $\frac{4}{5}$. The probability that Recep will win the second set is $\frac{2}{5}$. The probability that Recep will win the third set (if it is played) is $\frac{1}{5}$.

(a). What is the probability that Recep will win the match?

(b). What is the probability that Kemal will win the match?

(c). What is the probability that the third set will be played?

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20

Graph Theory

Çizge Kuramı

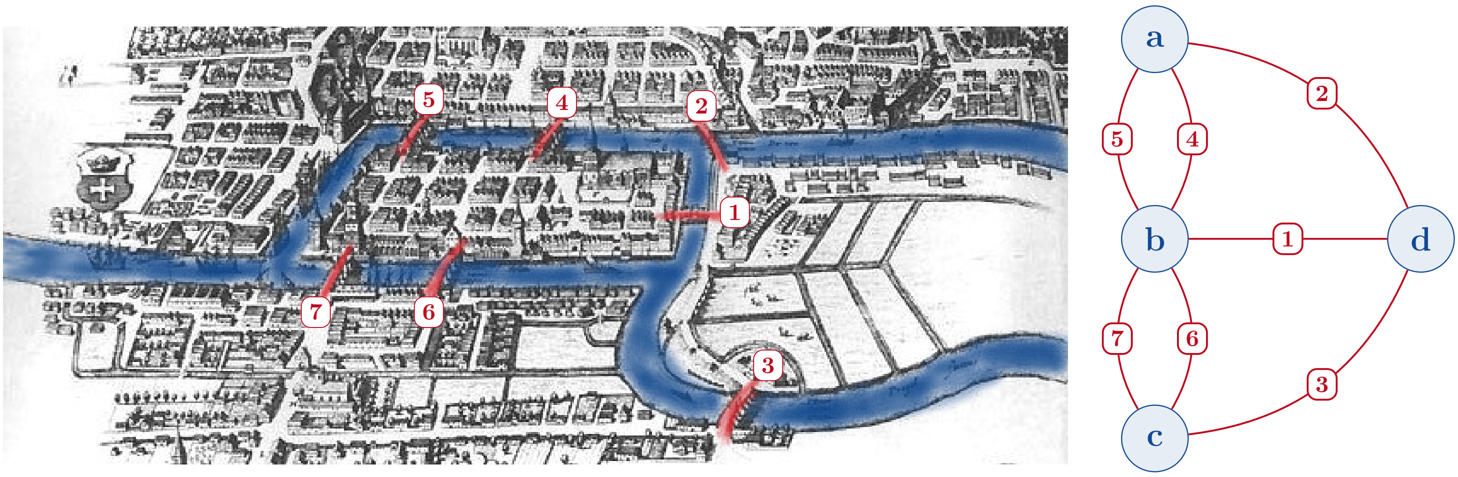


Figure 20.1: The seven bridges of Königsberg in Prussia (now Kaliningrad, Russia) in 1652.
Şekil 20.1:

Often when analysing theoretical problems, it is useful to transform the problem into a collection of vertices joined by lines. When visiting the city of Königsberg in 1736, the Swiss Mathematician Leonhard Euler (1707-1782) was set a problem by the inhabitants. To solve this problem, he invented a type of Mathematics called Graph Theory.

The town of Königsberg in Prussia (now Kaliningrad, Russia) was divided into four landmasses by the Pregel river. There were 7 separate bridges between these landmasses as shown in figure 20.1. Visitors were often asked the following problem by the locals:

> Can a person walk around the town and cross each bridge *once and only once*?

Euler was the first person to solve this problem.

Graph Theory, which has been used by mathematicians for many years to solve interesting riddles and puzzles, is nowadays in computing (algorithm design, telecommunication and GPS), physics (atomic structures), neurology (brain-like structures), chemistry (molecular structures), and many other disciplines.

Definition. A graph is formed by points called ***vertices*** (or ***nodes***), and lines called ***edges***.

Kuramsal sorunların çözümlenmesindor e belli büyüklükler yada kavramlar arasındaki bağlantıları incelerken ara bağlantıları bir çizgeye dönüştürmek ya kaçınılmaz yada en akılcı yöntemdir. Euler (1707-1782), Königsberg kentindeki köprülerle ilgili problemin çözümünü araştırırken, sorunu bir takım düğüm ve ayrıtların arabağlantılarına indirgeyerek, çizge kuramının da temellerini atmış oldu.

Prusyadaki Königsberg kasabası Pregel nehri ile ikiye ayrılıyordu ve nehrin içinde iki adacık bulunuyordu. Bu adacıklarla kasaba arasında şekildeki gibi 7 ayrı köprü bulunmakta idi. Problem, kasabanın bir yakasından gezinti yapmaya çıkan biri tüm köprüleri *sadece bir kez* geçerek başladığı noktaya dönebilirmi? sorusu idi. Bu sorunun matematiksel olarak yanıtı yıllar sonra keşfedildi.

Resim-0

Uzun yıllar, yalnız matematikçilerin uğraştığı ve ilginç bilmece/b çözmekten öteye gidemeyen çizge kuramı bu gün bilişimde (algoritma tasarlamada,telekominikasyon ve GPS'lerde), fizikte(atomi yapıların incelenmesinde), nörolojide(beyne benzer yapıların incelenmesinde), kimyada(moleküler yapıların incelenmesinde) ve daha birçok bilim dalının içinde her gün kendine yeni bir yer ediniyor.

TANIMLAR

Düğüm(Köşe): Bir çizginin (hattın) her uç noktasına düğüm

Notation. Vertices are denoted by lowercase letters: a, b, c, ... The edge from vertex u to vertex v is denoted by $e = (u, v)$. u and v are called ***endpoints*** of e.

Definition. A non-empty set of vertices V together with a set of edges E is called a ***graph*** and is denoted by $G(V, E)$

Please note that we can draw a picture to represent a graph.

Definition. An edge which starts and finishes at the same vertex is called a ***loop***. See figure 20.2.

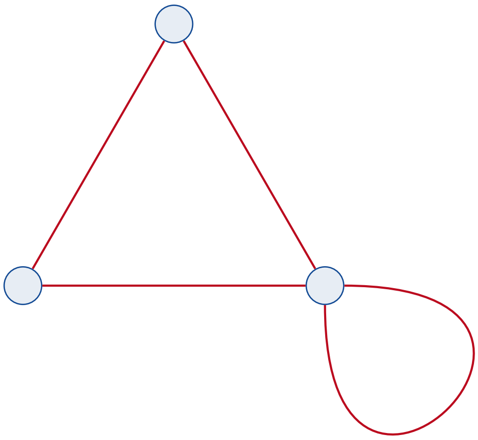


Figure 20.2: A graph with 3 vertices and 4 edges. One of the edges is a loop.
Şekil 20.2:

Example 20.1. Let $V = \{a, b, c, d\}$ and $E = \{e_1, e_2, e_3, e_4, e_5\}$ where $e_1 = (a, b)$, $e_2 = (b, c)$, $e_3 = (c, d)$, $e_4 = (a, c)$ and $e_5 = (b, d)$. Draw the graph $G = (V, E)$.

solution: Please see figure 20.3.

Definition. Two edges with the same endpoints are called ***parallel edges***.

denir.
Düğümler küçük harflerle gösterilir: a,b,...v.b.

Çizgi(Kenar): Bir çift düğüm ile etiketlenmiş hatlardır. İki uç noktasının belirtildiği $e = (v, w)$ sembolü ile gösterilir.

Çizge(Graf): **V**, köşelerin(vertex) ve **E**, kenarların(edges) boş olmayan bir kümesi olmak üzere **E** kümesindedeki elemanların **V** kümesinin farklı elemanlarına belli bir kurala bağlı oladan bağlandığı kümeye çizge(graph) denir ve **G(V,E)** ile gösterilir.

Tanımlanan her soyut çizgeye ilişkin, somut bir çizimsel gösterimin varolacağı unutulmamalıdır.

Hattın başlangıç ve bitişi aynı düğüm ise bu tür hatlara ***çevrim(cycle)*** denir.

Şekil 20.2.

Örnek 20.1. $V = a, b, c, d$ ve $E = e_1, e_2, e_3, e_4, e_5, e_1 = a, b, e_2 = b, c, e_3 = c, d, e_4 = a, c$ ve $e_5 = b, d$ olsun. $G = (V, E)$ grafını çiziniz.

çözüm: Şekil 20.3.

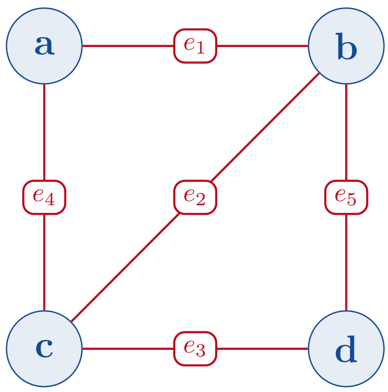


Figure 20.3: The solution to Example 20.1. This graph is a simple graph.
Şekil 20.3:

Types of Graph

Definition. A ***simple graph*** is a graph without parallel edges or loops.

Definition. If a graph contains parallel edges, it is called a ***multigraph***. See figure 20.4.

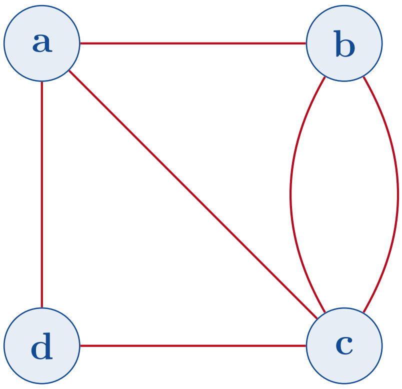


Figure 20.4: A multigraph.
Şekil 20.4:

Definition. A ***pseudograph*** is a non-simple graph in which both loops and parallel edges are permitted. See figure 20.5.

Definition. If $e = (u, v)$ is an edge of a undirected graph G, then the vertices u and v are called ***neighbours*** in G.

Definition. The ***degree*** of a vertex v in a(n undirected) graph is

$$\deg(v) = \text{the number of edges connected to } v.$$

When we calculate the degree of a vertex, a loop is counted as 2.

Example 20.2. In figure 20.6, give the degrees of all the vertices in graphs G and H.

solution:

G	H
$\deg(a) = 2$	$\deg(a) = 3$
$\deg(b) = 4$	$\deg(b) = 6$
$\deg(c) = 4$	$\deg(c) = 1$
$\deg(d) = 1$	$\deg(d) = 5$
$\deg(e) = 3$	$\deg(e) = 5$
$\deg(f) = 4$	
$\deg(g) = 0$	

Definition. A vertex without an edge (degree=0) is called an ***isolated vertex***. E.g. In figure 20.6, vertex g of graph G is an isolated vertex.

Definition. A vertex of degree 1 is called a ***pendant***. A pendant has only one edge. E.g. In figure 20.6, vertex d of graph G is a pendant.

ÇİZGE(GRAF) ÇEŞİTLERİ

1.Basit(Simple) Graflar: Ayrı iki düğümün sadece bir hatla bağlandığı, herhangi bir düğümü yine kendisine bağlayan bir hattın (çevrimin) olmadığı, hatların bir değer almadığı ve yönünün tanımlanmadığı, düğüm ve hatların sınıflandırılmadığı graflara *basit graf* denir. Bu açıdan yukarıdaki graf bir basit graftır.

2.Çoklu(multi) Graflar: İki yada daha fazla düğüm arasında birden fazla hat (paralel hatlar) varsa bu tür graflara çoklu (multi) graf denir. Çoklu graflar da yönsüz ve çevrimsizdir.

Şekil 20.4

3.Pseudo Graflar: Çevirim içeren çoklu graflardır. Yönlendirilmemiş grafların en genel halidir.

Şekil 20.5.

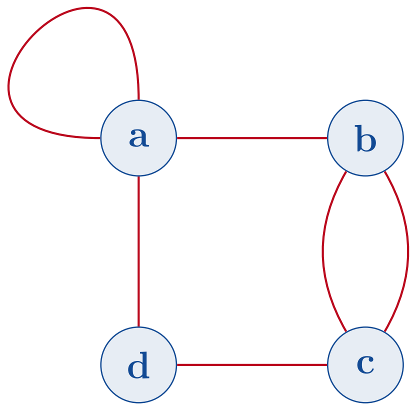


Figure 20.5: A pseudograph.
Şekil 20.5:

Eğer $e = (v, w)$ yönlendirilmemiş bir G grafının bir kenarı ise u ve v köşelerine G'de ***komşu*** denir. Eğer $e = (v, w)$, G'de bir kenar ise, e kenarına u ve v köşeleri ile ***bağlı*** denir. u ve v köşelerine $e = (v, w)$ kenarının ***uç noktaları*** denir.

Yönlendirilmemiş bir graftati bir köşenin derecesi, o köşeye bağlı olan kenarların sayısı kadardır. Ayrıca bie köşede bulunan bir çevrimin (halkanın), o köşenin derecesine katkısı 2 dir. Bir v köşesinin derecesi $\deg(v)$ ile gösterilir.

Örnek 20.2. Şekil 20.6. H ve G graflarındaki köşelerin derecelerini belirtiniz.

çözüm:

G'de;

$$\deg(a) = 2, \quad \deg(b) = \deg(c) = \deg(f) = 4$$

$$\deg(d) = 1, \qquad \deg(e) = 3, \qquad \deg(g) = 0$$

H'de;

$$\deg(a) = 4, \quad \deg(b) = \deg(e) = 6$$

$$\deg(c) = 1, \qquad \deg(d) = 5$$

Hat bağlantısı olmayan düğümlere ***izole yada ayrık düğüm (isolated vertex)*** denir. Ayrık düğümler hiçbir köşeye komşu değillerdir ve ***dereceleri "0" dır***. G grafındaki g köşesi bir ayrık düğümdür.

Derecesi "1" olan köşeye ***pendant*** denir. Bir pendant sadece <u>tek</u> bir köşeye komşudur. G grafındaki d köşesi bir pendanttır. O halde,

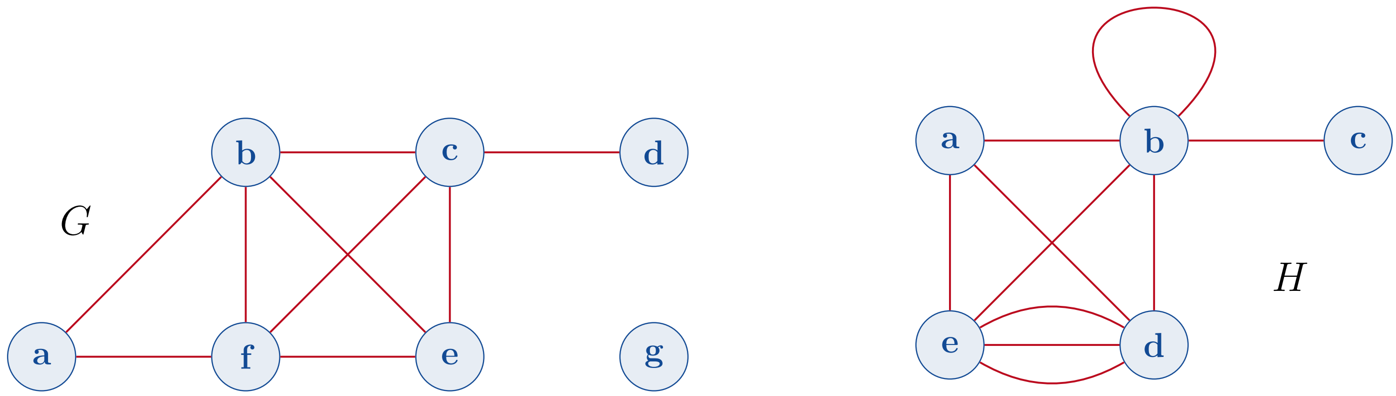


Figure 20.6: Two graphs refered to in example 20.2.
Şekil 20.6:

Theorem 20.1. *Let $G = (V, E)$ be a pseudograph and let $n(E)$ denote the number of edges in G. Then*

$$n(E) = \frac{1}{2}\sum_{v \in V} \deg(v).$$

Example 20.3. A graph has 10 vertices, each of degree 4. How many edges does this graph have?
solution:

Suppose that $V = \{v_1, \ldots, v_{10}\}$. Then by Theorem 20.1, we have that

$$\begin{aligned} n(E) &= \frac{1}{2}\sum_{i=1}^{10} \deg(v_i) \\ &= \frac{1}{2}\big(\deg(v_1) + \deg(v_2) + \ldots + \deg(v_{10})\big) \\ &= \frac{1}{2}(4 + 4 + \ldots + 4) = \frac{1}{2}(40) = 20. \end{aligned}$$

Therefore this graph has 20 edges.

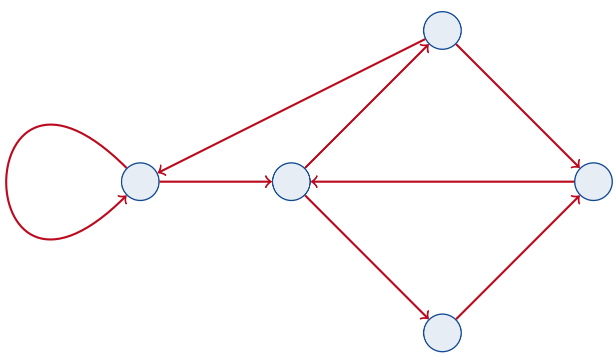


Figure 20.7: A directed graph.
Şekil 20.7:

Definition. If the edges in a graph have directions, the graph is called a ***directed graph*** (or ***digraph***). This direction indicates where the connection starts and ends. See figure 20.7.

Teorem 20.1. *$G = (V, E)$ bir pseudo graf ve $s(E)$ de bu grafın kenar sayısı olsun. O halde,*

$$s(E) = \frac{1}{2}\sum_{v \in V} \deg(v).$$

Örnek 20.3. Her köşesinin derecesi 4 olan 10 köşeli bir grafın kaç kenarı vardır?
çözüm:

$s(E)$ kenar sayısını $v_i, i = 1, 2, ..., 10$ köşeleri belirtmek üzere bu grafın toplam köşe sayısı: $\sum_{i=1}^{10} \deg(v_i) = \sum_{i=1}^{10} 4 = 40$ dir. Teorem 20.1 den $s(E) = \frac{40}{2} = 20$ bulunur.

4.Yönlü(Directed) Graflar: Eğer bir graftaki hatlar yön bilgisine sahipse bu tür graflara *yönlü graf (Directed graph / Digraph)* denir. Bu yön bilgisi bağlantının nereden başlayıp nereden bittiğini belirtir. Yön bilgisi olan graflarda düğümler arasındaki bağlantının yönü vardır.

Şekil 20.7.

5.Çoklu Yönlü Graflar: Eğer yönlü bir grafta iki yönde de bağlantı varsa ters yönde iki ayrı hat kullanılır ve bu tür graflara *çoklu yönlü graf* denir.

Şekil 20.11.

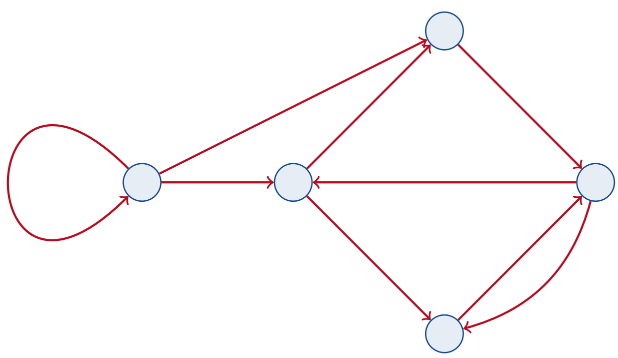


Figure 20.11: A directed multigraph
Şekil 20.11:

NOT: Graf yapısında bütün kenarlar aynı çeşittir. Yani ya

Definition. If a directed graph has parallel edges in both directions, it is called a ***directed multigraph*** (or ***multidigraph***). See figure 20.11.

Remark. In a graph, all edges are the same type. So either all edges are directed or all edges are undirected. A graph representing the road network is an example of a directed graphs, where traffic is one-way or bi-directional. See figure 20.15 on 121.

Notation. Let G be a directed graph. Now when we write $e = (u, v)$, the order of u and v is important. The edge $e = (u, v)$ starts at u and finishes at v.

Definition. Let G be a directed graph. The ***indegree*** of a number vertex v is

$$\deg^-(v) = \text{the number of edges coming } \textit{\textbf{in}}\text{to } v$$

and the ***outdegree*** of v is

$$\deg^+(v) = \text{the number of edges coming } \textit{\textbf{out}} \text{ of } v.$$

A loop is counted as 1 for both $deg^-(v)$ and $deg^+(v)$.

Example 20.4. Please see figure 20.12. Find the indegree and outdegree of each vertex in this graph.

solution:
The indegrees are:

$$\deg^-(a) = 2, \quad \deg^-(b) = 2, \quad \deg^-(c) = 3$$
$$\deg^-(d) = 2, \quad \deg^-(e) = 3, \quad \deg^-(f) = 0$$

The outdegrees are:

$$\deg^+(a) = 4, \quad \deg^+(b) = 1, \quad \deg^+(c) = 2$$
$$\deg^+(d) = 2, \quad \deg^+(e) = 3, \quad \deg^+(f) = 0$$

Theorem 20.2. *Let $G = (V, E)$ be a directed graph and let $n(E)$ denote the number of edges in G. Then*

$$n(E) = \sum_{v \in V} \deg^-(v) = \sum_{v \in V} \deg^+(v).$$

Definition. A simple graph which includes all possible edges is called a ***complete graph***.

Remark. In a complete graph, every vertex has an edge with every other vertex. A complete graph with n vertices is denoted by K_n. The degree of every vertex in K_n is $(n-1)$. Thus K_n has a total of $\frac{n(n-1)}{2}$ edges. Please see figure 20.17 on page 122.

Definition. Suppose that the edges in a graph are assigned values. See e.g. figure 20.8. This type of graph is called a ***weighted graph***.

hepsi yönlüdür ya da değildir. Yol ağını temsil eden bir grafta trafiğin tek yada çift yönlü oluşu yönlü graflar için bir örnektir. Şekil 20.15.

Bir graftaki herhangi bir ***düğümün derecesi***, kendisini diğer düğümlere birleştiren hatların sayısı kadardır. Bu düğümlerden derecesi en büyük olanı ise, aynı zamanda ***grafın derecesini*** belirler. Düğüm noktalarındaki *çevrim* düğüm derecesine iki kere katılır.

G, yönlendirilmiş bir graf olsun. Bu grafın kenarları $e(u, v)$ şeklinde ***sıralı ikililerle*** gösterilir. Burada u'ya ***v'nin komşusu*** ve $e(u, v)$'nin ***başlangıç köşesi***; v'ye ise ***u'nun komşusu*** ve $e(u, v)$'nin ***bitiş köşesi*** denir. Çevrimlerin başlangıç ve bitiş köşesi aynıdır.

Yönlendirilmiş kenarlı bir grafta v köşesine gelen kenar sayısı (in-degree) $\deg^-(v)$, v köşesinden çıkan kenar sayısı ise (out-degree) $\deg^+(v)$ ile gösterilir. Bir çevrimin hem $\deg^-(v)$'ye hem de $\deg^+(v)$'ye katkısı "1" dir.

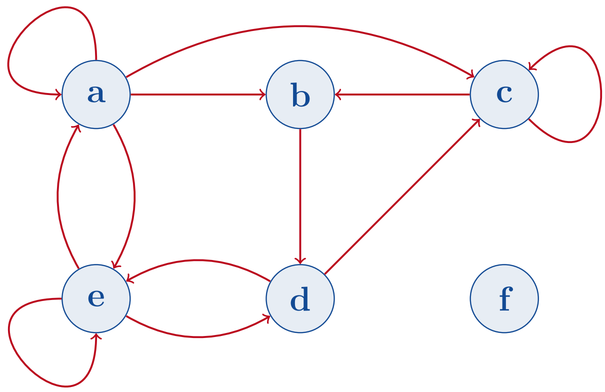


Figure 20.12: The graph refered to in example 20.4.
Şekil 20.12: 8

Örnek 20.4. Şekil 20.12. Şekildeki yönlendirilmiş G grafının $\deg^-(v)$ ve $\deg^+(v)$ değerlerini bulunuz.
çözüm:
G'nin $\deg^-(v)$ değerleri;

$$\deg^-(a) = 2, \quad \deg^-(b) = 2, \quad \deg^-(c) = 3$$
$$\deg^-(d) = 2, \quad \deg^-(e) = 3, \quad \deg^-(f) = 0$$

G' $\deg^+(v)$ değerleri;

$$\deg^+(a) = 4, \quad \deg^+(b) = 1, \quad \deg^+(c) = 2$$
$$\deg^+(d) = 2, \quad \deg^+(e) = 3, \quad \deg^+(f) = 0$$

Teorem 20.2. *$G = (V, E)$ yönlendirilmiş bir graf ve $s(E)$ bu grafın kenar sayısı olsun. O halde,*

$$s(E) = \sum_{v \in V} \deg^-(v) = \sum_{v \in V} \deg^+(v).$$

6.Tam(Complete) Graf: Graftaki her bir düğümün diğer

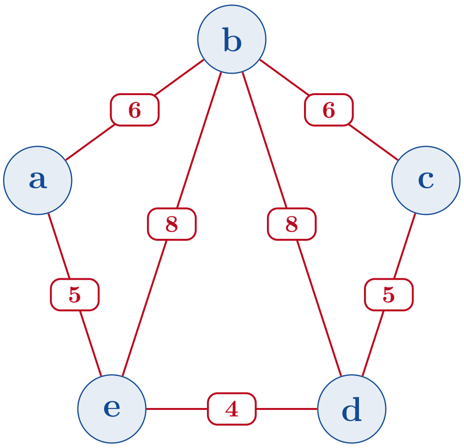


Figure 20.8: A weighted graph.
Şekil 20.8: Ağırlıklı Graf.

Definition. A ***planar graph*** is a graph that can be drawn on the plane in such a way that its edges intersect only at their endpoints. Please see figure 20.16 for an example.

Definition. A graph that is not a planar graph, and can only be drawn without intersecting edges in three-dimensional space is called a ***three-dimensional graph***. See figure 20.13.

Definition. Suppose that $n \geq 3$, that $V = \{v_1, v_2, \ldots, v_n\}$ and that $E = \{(v_1, v_2), (v_2, v_3), \ldots, (v_{n-1}, v_n), (v_n, v_1)\}$. Then the graph $C_n = (V, E)$ is called a ***cycle graph***. Please see figure 20.18 on page 122.

Definition. If we take C_n and add a new vertex which is attached to all the other vertices, we get the ***wheel graph*** W_n. Please see 20.19 on page 122.

Definition. A ***cube graph*** is a graph that is obtained by taking all vertices denoted as binary words and joining the vertices with an edge whenever the binary words differ by 1. Cube graphs are denoted by Q_n where n refers to the length of the binary number. Please see figure 20.20.

Definition. A ***bipartite graph*** is a graph where the set V of vertices can be divided into two distinct subsets V_1 and V_2 such that every edge connects a vertex in V_1 with a vertex in V_2.

Remark. In a bipartite graph, there are no edges going from V_1 to V_1, or from V_2 to V_2.

Example 20.5. Please see figure 20.9. Note that the graph C_6 is a bipartite graph because if we let $V_1 = \{v_1, v_2, v_3\}$ and $V_2 = \{v_4, v_5, v_6\}$, then every edge goes from V_1 to V_2.

Example 20.6. Please see figures 20.10 and 20.14. Is G a bipartite graph? Is H a bipartite graph?

solution: G is a bipartite graph because we can set $V_1 = \{a, b, d\}$ and $V_2 = \{c, e, f, g\}$. Note that each edge connects V_1 to V_2. The lack of a edge here between some vertices does not affect it being bipartite. For example, vertices b and g are not neighbours.

Graph H is not bipartite because the set of vertices cannot be divided into two subsets without an edge between the two corners within the same set.

tüm düğümlerle arasında bir hat mevcutsa, yani olabilecek tüm hatlara sahipse, bu tür graflara tam(tamamlanmış) graf (completed graph) denir.
Bu tür bir grafta bütün düğümlerin dereceleri birbirine eşit ve toplam düğüm sayısının bir eksiği kadardır. n düğümlü bir tamamlanmış graf K_n ile gösterilir ve grafın hat sayısı $\frac{n(n-1)}{2}$ ile hesaplanır.

Şekil 20.17.

7.Ağırlıklı(Weighted) Graf: Graf yapısındaki hatlar değer alabilir ve bu değerler grafın yapısına katılabilir. Aşağıdaki örnekte olduğu gibi, bir grafın üzerindeki hatların değerleri eşit değilse ve her biri farklı bir değer alabiliyorsa bu tip graflara maliyetli yada *ağırlıklı graf (weighted graph)* denir.

Şekil 20.8.

8.Düzlemsel Graf: Birbirini kesmeyen hatlardan oluşacak şekilde çizilebilen graflara *düzlemsel graf* denir.

Şekil 20.16.

9.Üç Boyutlu Graf: Birbirini kesmeyen hatlardan oluşacak şekilde çizilemeyen,sadece üç boyutlu uzayda ele alındığında hatlarının birbirini kesmeyecek şekilde çizilmesi mümkün olan graflara *üç boyutlu graf* denir.

Şekil 20.13.

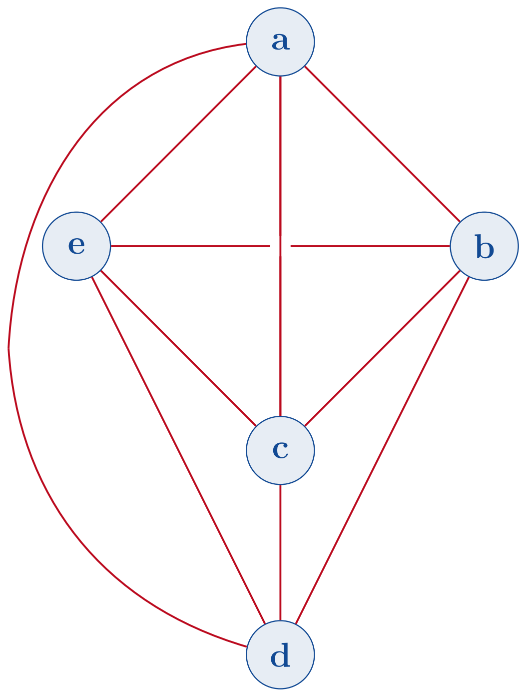


Figure 20.13: A three-dimensional graph.
Şekil 20.13: Üç Boyutlu Graf

10.Çember Graf (Cycle Graph): $n \neq 3$ olmak üzere n tane $v_1, v_2, ..., v_n$ köşelerinden ve $v_1, v_2, v_2, v_3, ..., v_{n-1}, v_n, v_n, v_1$ kenarlarından oluşan graflar çember olarak adlandırılır ve C_n ile gösterilir.

Şekil 20.18.

11.Tekerlek Graf (Wheel Graph): $n \neq 3$ olmak üzere C_n çemberindeki tüm köşelere bağlı yeni bir köşe daha eklenerek oluşturulan graf tekerlek (wheel) olarak adlandırılır ve W_n ile gösterilir.

Şekil 20.19.

12.n-Küp Graf (n-Cube Graph): Grafın köşeleri 2^n bit

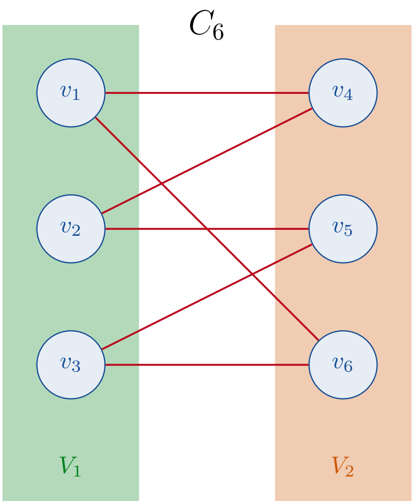


Figure 20.9: A bipartite graph.
Şekil 20.9:

dizisi (bit string) ile gösterilir. Bir köşeden diğerine geçerken düğümlerin string değerinde sadece bir bit değişmektedir. Q_n ile sembolize edilirler.

Şekil 20.20.

13.İki Parçalı (Bipartite) Graflar: G basit grafının köşelerinin kümesi olan V kümesi, boş olmayan V_1 ve V_2 gibi iki köşeye ayrılabiliyorsa ve grafın her kenarı V_1'in bir köşesini V_2'nin bir köşesine bağlıyorsa G basit grafına iki parçalı(bipartit) graf denir. Bipartit graf, V_1'in eleman sayısı m, V_2'nin eleman sayısı n olak üzere $G_{m,n}$ ile gösterilir.

Dikkat: G grafındaki hiçbir kenar V_1 kümesindeki köşeleri veya V_2 kümesindekiköşeleri kendi aralarında bağlamaz.

Örnek 20.5. Şekil 20.9. 'deki C_6 grafı bipartittir, çünkü köşelerinin kümesi kendi içinde $V_1 = v_1, v_2, v_3$ ve $V_2 = v_4, v_5, v_6$ olarak iki kümeye parçalanabiliyor. C_6'nın her kenarı V_1'deki bir köşeyi V_2'deki bir köşeye bağlıyor.

Örnek 20.6. Şekil 20.10 ve 20.14. Şekildekiş G ve H grafları bipartit midir?

çözüm: G grafı bipartittir, çünkü köşelerinin kümesi ayrı iki a, b, d ve c, e, f, g kümelerinin birleşimi ile oluşuyor ve her kenar bu kümelerin birindeki en az bir köşeyi diğer kümedeki en az bir köşeye bağlıyor. Burada bazı köşeler arasında bağ olmaması bipartitliği etkilemez. Örneğin, b ve g göşeleri komşu değildir.

H grafı bipartit değildir, çünkü aynı küme içindeki iki köşe arasında bir kenar olmaksızın köşelerin kümesi iki alt kümeye ayrılamaz.

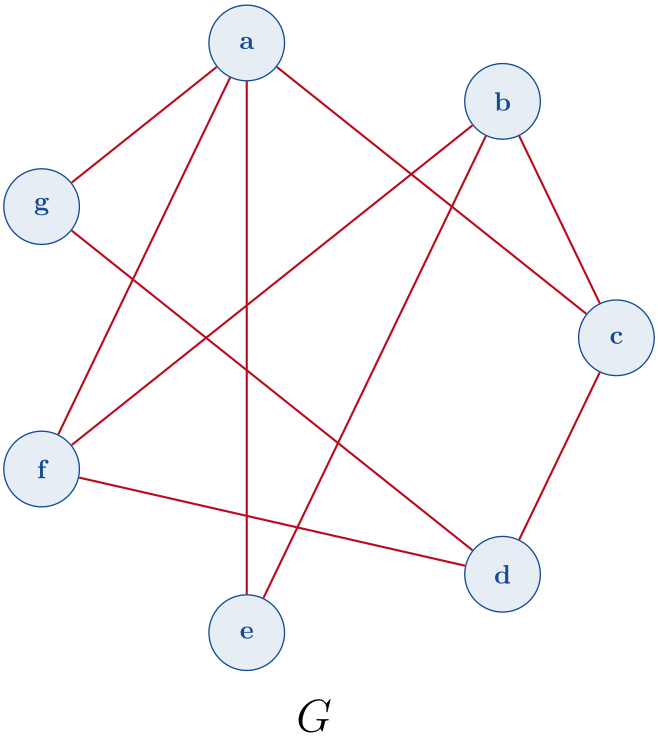


Figure 20.10: A graph refered to in example 20.6.
Şekil 20.10:

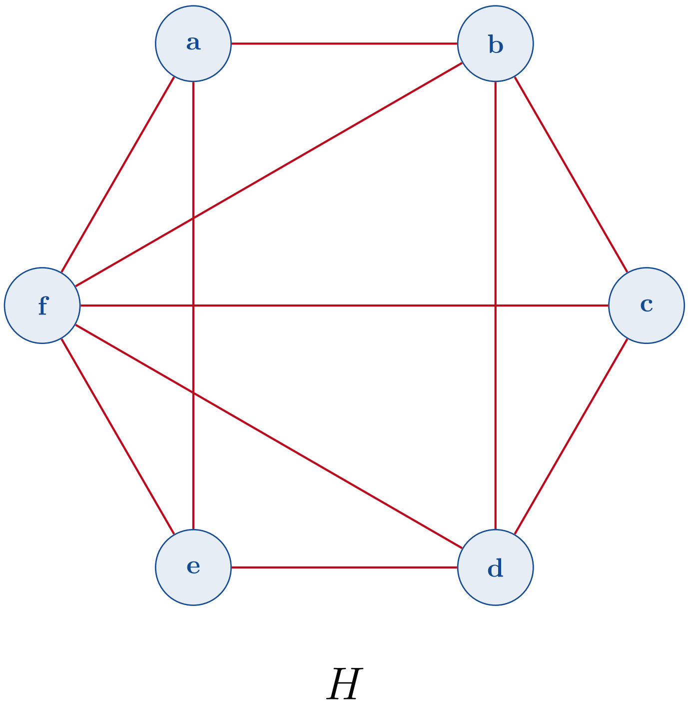


Figure 20.14: A graph refered to in example 20.6.
Şekil 20.14:

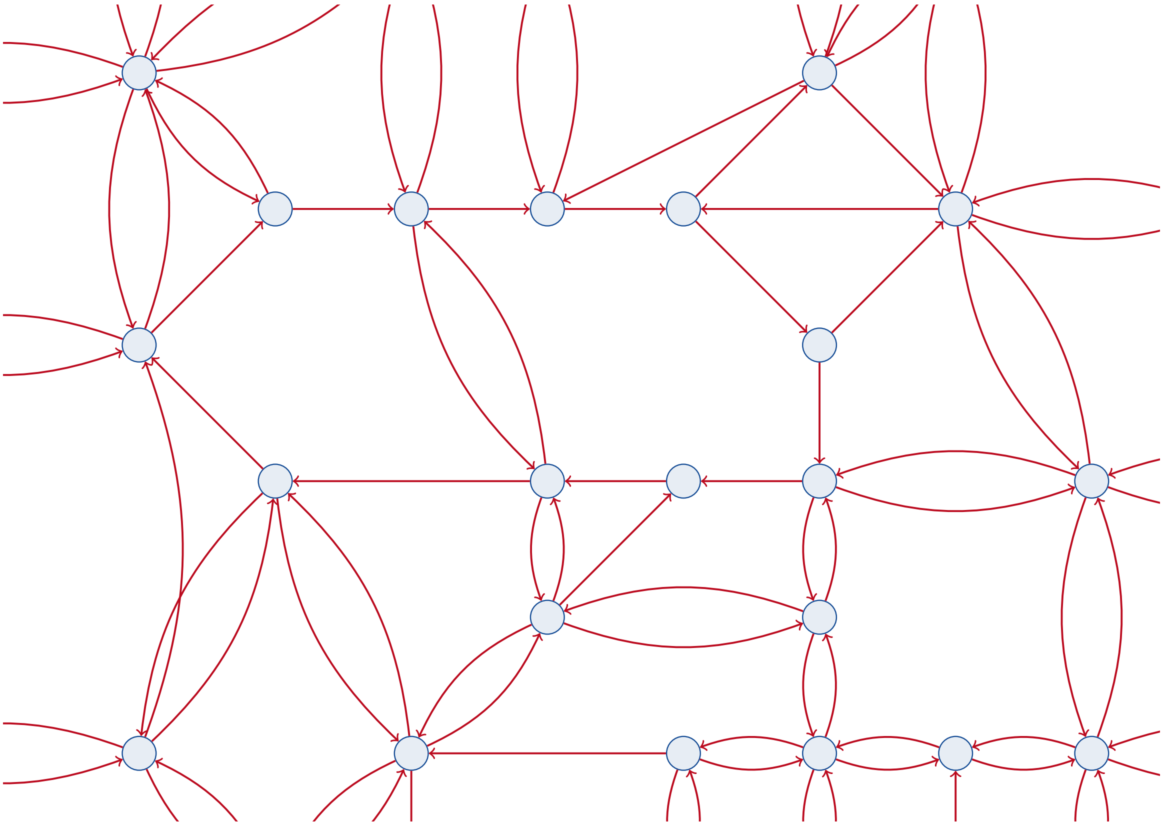


Figure 20.15: A small part of a road network.
Şekil 20.15: Yol ağını.

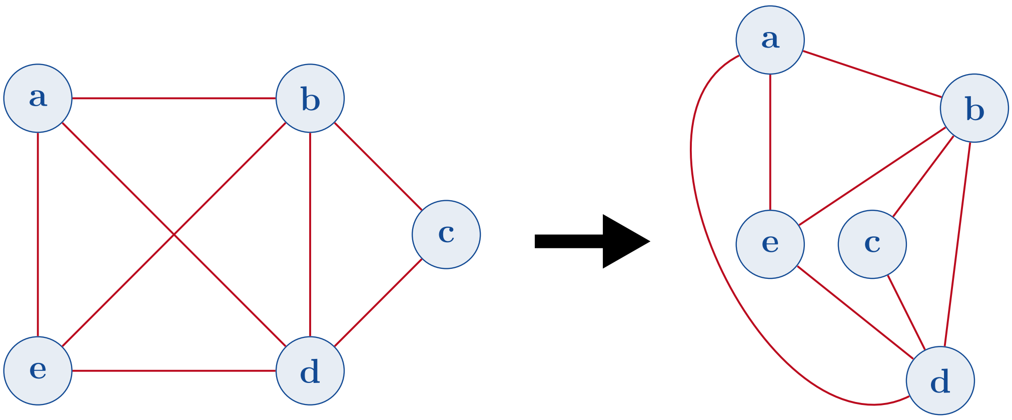


Figure 20.16: The graph on the left can also be drawn as shown on the right. Thus this is a planar graph.
Şekil 20.16: Soldaki graf, kesilmeyen kenerlerden (het) oluşacak sehilde sağdaki gibi de cizilebilir.

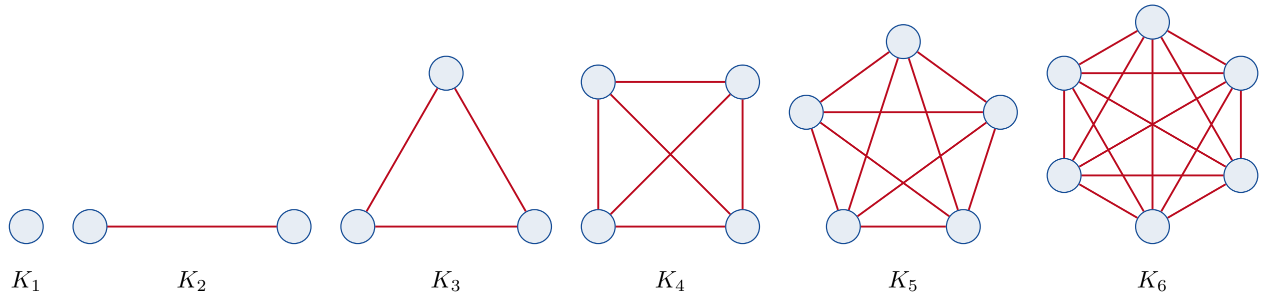


Figure 20.17: The first six complete graphs.
Şekil 20.17:

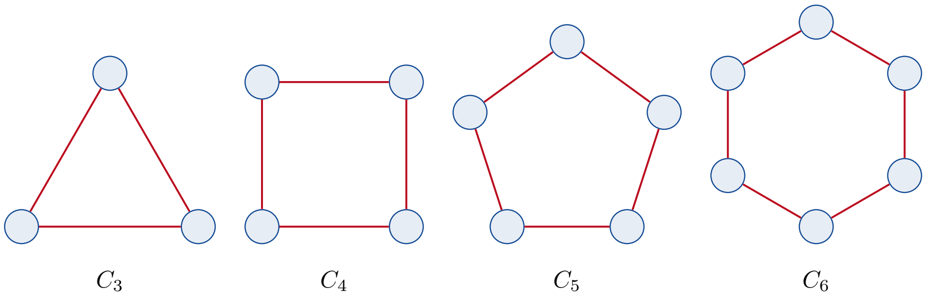


Figure 20.18: The first 4 cycle graphs.
Şekil 20.18:

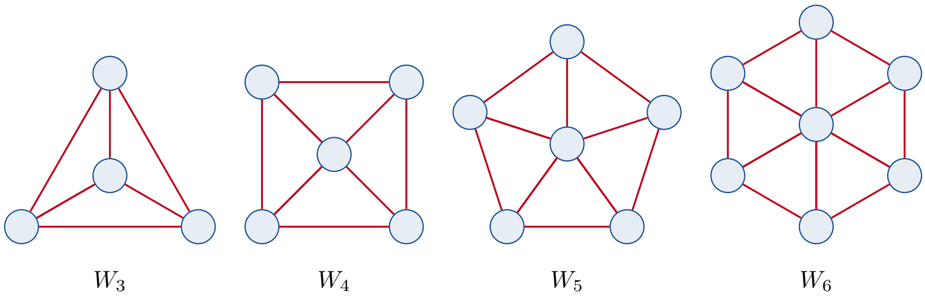


Figure 20.19: The first 4 wheel graphs.
Şekil 20.19:

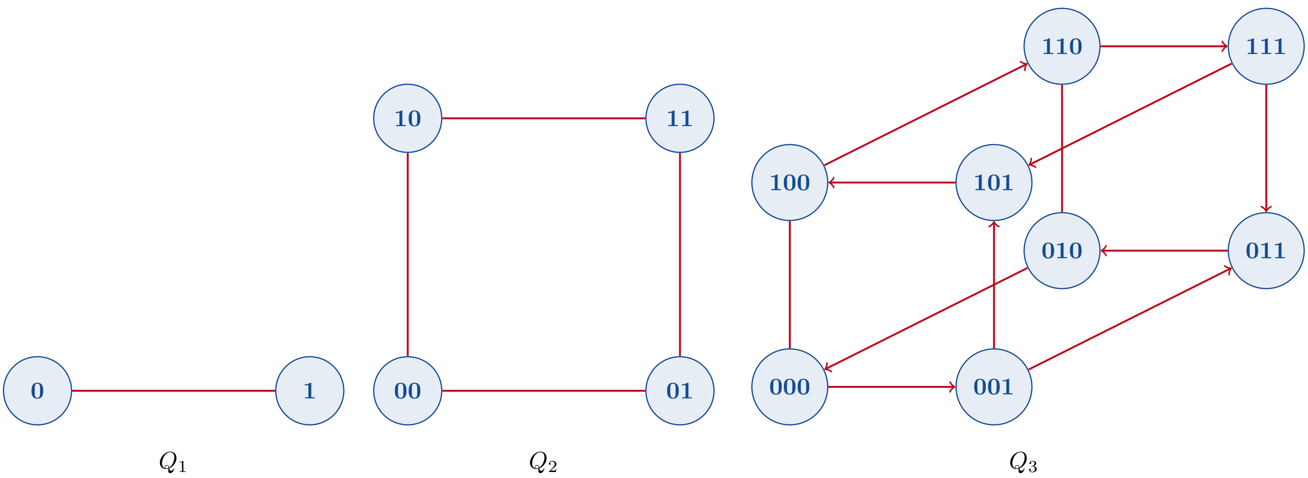


Figure 20.20: The first three cube graphs.
Şekil 20.20:

Graph Isomorphisms[1]

Definition. Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be two graphs. Let $f : V_1 \to V_2$ be a function. Suppose that

(i). f is injective (one-to-one);

(ii). f is surjective (onto); and

(iii). $a, b \in V_1$ are neighbours $\implies$ $f(a), f(b) \in V_2$ are neighbours.

Then the function f is called an ***isomorphism***. In this case, we say that the graphs G_1 and G_2 are ***isomorphic***.

This means that if two graphs are isomorphic, they have the 'same vertices' and the 'same edges between those vertices'.

Example 20.7. Please consider figure 20.24. Show that the graphs $G = (U, E)$ and $H = (V, F)$ are isomorphic.

solution: Define $f : U \to V$ by

$$f(u_1) = v_1, \qquad f(u_2) = v_2, \qquad f(u_3) = v_3, \qquad f(u_4) = v_4$$

Note that the function f provides a one-to-one correspondence between U and V. Note also that neighbourhoods are preserved, as shown below:

- u_1 and u_2 are neighbours in G. $v_1 = f(u_1)$ and $v_2 = f(u_2)$ are neighbours in H.
- u_1 and u_3 are neighbours in G. $v_1 = f(u_1)$ and $v_3 = f(u_3)$ are neighbours in H.
- u_2 and u_4 are neighbours in G. $v_2 = f(u_2)$ and $v_4 = f(u_4)$ are neighbours in H.
- u_3 and u_4 are neighbours in G. $v_3 = f(u_3)$ and $v_4 = f(u_4)$ are neighbours in H.

Often it is not easy to determine whether two simple graphs are isomorphic. Because the n clause has a one-to-one match between the corners of a simple graph, and it is very difficult to determine if each of these neighbourhoods maintains n.

However, it may be easier to show that two simple graphs are not isomorphic. This is done by showing that the two graphs have an important difference. The following properties are shared by isomorphic graphs:

- number of vertices;
- number of edges; and
- degrees of vertices.

If we find that two graphs differ in at least one of these properties, then we know that these two graphs are not isomorphic. However, even if these properties are equal for the two graphs, this does not mean that there is an isomorphism. We must look further.

Example 20.8. Please see figure 20.25. Show that G and H are not isomorphic.

solution: Note that both G and H have 5 vertices and 6 edges. However graph G does not have any vertices of degree 1, whereas in graph H we have $\deg(e) = 1$. Hence G and H are not isomorphic.

[1]This section will not be covered in 2019. Exam questions will not be asked on this topic.

GRAF İZOMORFİZMİ

$G_1 = (V_1, E_1)$ ve $G_2 = (V_2, E_2)$ iki farklı basit graflar ve f de V_1'den V_2'ye tanımlı bir fonksiyon olsun. Eğer f fonksiyonu $1-1$ (bire bir) ve örten ise G_1 ve G_2 grafları izomorfiktir denir ve f fonksiyonuna da izomorfizm adı verilir.

Tanım gereği her $a, b \in V_1$ için a ve b köşeleri G'de komşudur ancak ve ancak $f(a)$ ve $f(b)$ G_2'de komşu ise. Yani, graftaki köşeler arasındaki bire bir eşleme bu köşeler arasındaki komşuluk ilişkilerini de korur.

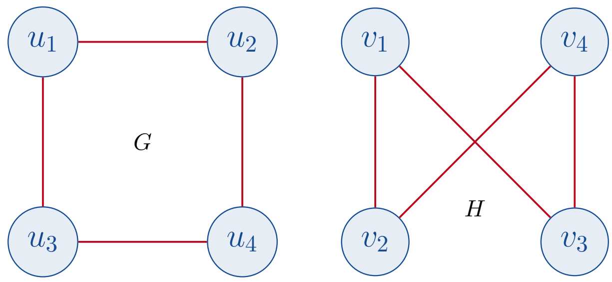


Figure 20.24: Two graphs referred to in example 20.7.
Şekil 20.24:

Örnek 20.7. Şekil 20.24
Şekildekiş $G = (U, E)$ ve $H = (V, F)$ graflarının izomorfik olduğunu gösteriniz.
çözüm:
$f(u_1) = v_1$
$f(u_2) = v_2$
$f(u_3) = v_3$
$f(u_4) = v_4$

olduğundan, f fonksiyonunun U ile V arasında bire birlik sağlayan bir fonksiyon olduğu kolaylıkla görülür.Ayrıca, aşağıda da görüldüğü gibi komşuluklar da korunmuştur.

u_1 ve u_2 G'de komşu iken $f(u_1) = v_1$ ve $f(u_2) = v_2$ de H'de komşudur.
u_1 ve u_3 G'de komşu iken $f(u_1) = v_1$ ve $f(u_3) = v_3$ de H'de komşudur.
u_2 ve u_4 G'de komşu iken $f(u_2) = v_2$ ve $f(u_4) = v_4$ de H'de komşudur.
u_3 ve u_4 G'de komşu iken $f(u_3) = v_3$ ve $f(u_4) = v_4$ de H'de komşudur.

Çoğu zaman iki basit grafın izomorfik olup olmadığını belirlemek kolay değildir. Çünkü, n koşeli bir basit grafın köşeleri arasında $n!$ tane bire bir eşleşme vardır ve bunların her birinin komşuluk ilişkilerinin koruyup korumadığının belirlenmesi n büyük olduğunda oldukça zordur.

Fakat, iki basit grafın izomorfik olmadığını göstermek daha kolay olabilir. Bu, izomorfik graflarda olması gereken özelliklerin olmadığı gösterilerek yapılır. Bu özelliklere, basit grafların izomorfizm altındaki invaryantları(değişmezleri) denir. Bunlar:
Köşe sayısı,
Kenar sayısı
Köşelerin dereceleridir.
Bu özelliklerden en az birinde eşitlik sağlanmıyorsa izomorfizma yoktur. Fakat hepsi sağlanıyorsa izomorfizma kesin vardır diyemeyiz. Bundan sonra başka şeylere bakmak gerekir.

Representations of Graphs

Definition. Let $G = (V, E)$ be a graph with n vertices

$$V = \{v_1, v_2, v_3, \ldots, v_n\}.$$

The ***adjacency matrix*** A_G is a square matrix used to represent G. The elements of the matrix are either 1 or 0 to indicate whether pairs of vertices are adjacent or not. Precisely, we define

$$a_{ij} = \begin{cases} 1 & \text{if there is an edge from } v_i \text{ to } v_j, \\ 0 & \text{otherwise.} \end{cases}$$

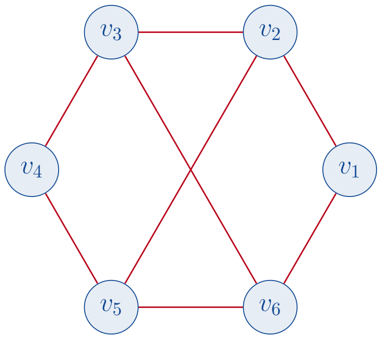


Figure 20.21: The graph referred to in example 20.9.
Şekil 20.21:

Example 20.9. Write an adjacency matrix for the graph in figure 20.21.

solution:

$$\begin{array}{c c} & \begin{array}{cccccc} v_1 & v_2 & v_3 & v_4 & v_5 & v_6 \end{array} \\ \begin{array}{c} v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \\ v_6 \end{array} & \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 \end{pmatrix} \end{array}$$

Remark. Adjacency matrices can also be used in undirected graphs with loops. Note that adjacency matrices of undirected simple graphs, multigraphs and pseudographs are symmetrical.

Example 20.10. Write an adjacency matrix for the graph in figure 20.22.

solution:

$$\begin{array}{c c} & \begin{array}{cccc} v_1 & v_2 & v_3 & v_4 \end{array} \\ \begin{array}{c} v_1 \\ v_2 \\ v_3 \\ v_4 \end{array} & \begin{pmatrix} 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \end{pmatrix} \end{array}$$

Definition. Let $G = (V, E)$ be a simple undirected graph with vertices

$$V = \{v_1, v_2, v_3, \ldots, v_n\}$$

and edges

$$E = \{e_1, e_2, e_3, \ldots, e_m\}.$$

The ***incidence matrix*** M_G is another type of matrix that is

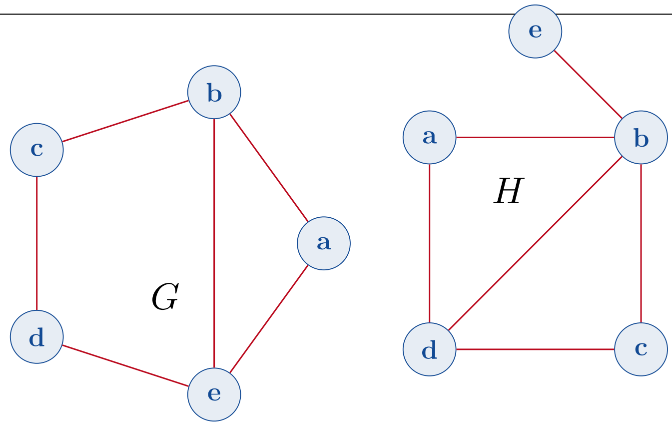


Figure 20.25: Two graphs referred to in example 20.8.
Şekil 20.25:

Örnek 20.8. Şekil 20.25 Şekildeki $G = (U, E)$ ve $H = (V, F)$ graflarının izomorfik olmadığını gösteriniz.
çözüm: $G = (U, E)$ ve $H = (V, F)$ graflarının her ikisi de 5 köşeli ve 6 kenarlıdır. Fakat, H'de derecesi 1 olan e köşesine karşılık G'de derecesi 1 olan bir köşe bulunmamaktadır. Bu nedenle H ve G izomorfik değildirler.

Grafların Gösterilişleri

Tanım (Komşuluk Matrisi). G basit grafının köşe sayısı $|V| = n$ ve G'nin köşeleri $v_1, v_2, ..., v_n$ şeklinde listelenmiş olsun. A_G ile gösterilen G'nin komşuluk matrisi 0 ve 1'lerden oluşan nxn'lik bir matristir. A_G bir komşuluk matrisi ise matrisin elemanları için aşağıdaki kural geçerlidir:

$$a_{ij} = \begin{cases} 1, & \text{eğer } \; v_i, v_j, \; \text{ G'nin bir kenarı ise} \\ 0, & \text{değilse} \end{cases}.$$

Örnek 20.9. Şekil 20.21. Şekildeki grafı komşuluk matrisi yardımıyla gösterelim.
çözüm:

$$\begin{array}{c c} & \begin{array}{cccccc} v_1 & v_2 & v_3 & v_4 & v_5 & v_6 \end{array} \\ \begin{array}{c} v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \\ v_6 \end{array} & \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 \end{pmatrix} \end{array}$$

Not. Komşuluk matrisleri, aynı zamanda halkalara ve katlı kenarlara sahip yönlendirilmemiş graflarda da kullanılabilir. Tüm yönlendirilmemiş basit grafların, multigrafların ve pisedografların komşuluk matrisleri simetriktir.

Örnek 20.10. Şekil 20.22. Şekildeki pisedografı komşuluk matrisi yardımıyla gösterelim.
çözüm:

$$\begin{array}{c c} & \begin{array}{cccc} v_1 & v_2 & v_3 & v_4 \end{array} \\ \begin{array}{c} v_1 \\ v_2 \\ v_3 \\ v_4 \end{array} & \begin{pmatrix} 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \end{pmatrix} \end{array}$$

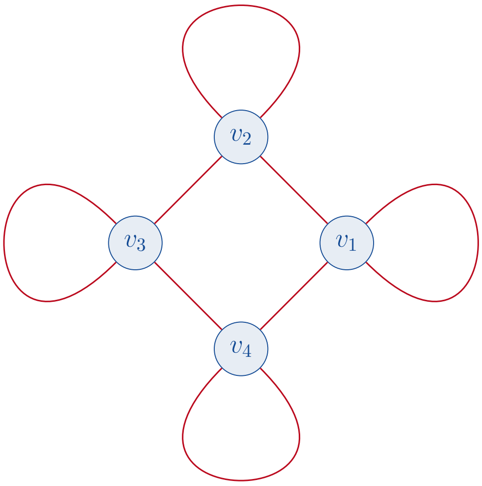


Figure 20.22: The graph referred to in example 20.10.
Şekil 20.22:

used to represent G. The elements of this matrix are either 1 or 0 to indicate if a particular edge is connected to a particular vertex. Precisely, we define

$$m_{ij} = \begin{cases} 1 & \text{if edge } e_j \text{ is connected to vertex } v_1, \\ 0 & \text{otherwise.} \end{cases}$$

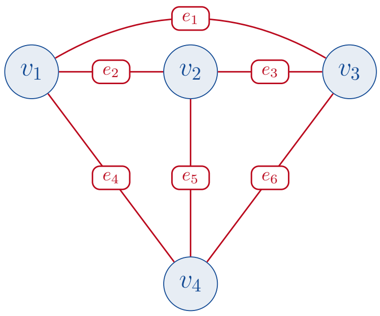


Figure 20.23: The graph referred to in example 20.11.
Şekil 20.23:

Example 20.11. Write an incidence matrix for the graph in figure 20.23.

solution:

$$\begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \end{array} \begin{array}{c} \begin{array}{cccccc} e_1 & e_2 & e_3 & e_4 & e_5 & e_6 \end{array} \\ \begin{pmatrix} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 \end{pmatrix} \end{array}$$

Tanım (İlişki "insıdıns"(incidence) Matrisi)**.** $G = (V, E)$ yönlendirilmemiş bir basit graf olsun. Köşeleri G $v_1, v_2, ..., v_n$, kenarları $e_1, e_2, ..., e_m$ şeklinde listelenmiş olan nxm'lik insıdıns matrisinin elemanlanları $M = [m_{ij}]$ olmak üzere aşağıdaki gibidir:

$$m_{ij} = \begin{cases} 1, & e_j \text{ kenarı } v_j'\text{yi bağlıyorsa} \\ 0, & \text{bağlamıyorsa} \end{cases}.$$

Örnek 20.11. Şekil 20.23. Şekildeki grafı insıdıns matris yardımıyla gösterelim.

çözüm:

$$\begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \end{array} \begin{array}{c} \begin{array}{cccccc} e_1 & e_2 & e_3 & e_4 & e_5 & e_6 \end{array} \\ \begin{pmatrix} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 \end{pmatrix} \end{array}$$

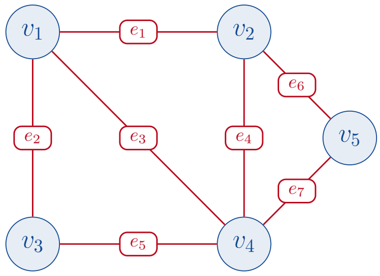


Figure 20.26: The graph referred to in example 20.12.
Şekil 20.26:

Örnek 20.12. Şekil 20.26. Şekildeki grafı insıdıns matris yardımıyla gösterelim.

çözüm:

$$\begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \end{array} \begin{array}{c} \begin{array}{ccccccc} e_1 & e_2 & e_3 & e_4 & e_5 & e_6 & e_7 \end{array} \\ \begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 \end{pmatrix} \end{array}$$

İnsıdıns matrisler, aynı zamanda halkalara ve katlı kenarlara sahip yönlendirilmemiş graflarda da kullanılabilir.Katlı kenarlar, insıdıns matriste sütunlarda aynı elemankullanılarak gösterilirler

Example 20.12. Write an incidence matrix for the graph in figure 20.26.

solution:

$$\begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \end{array}\begin{array}{c} \begin{array}{ccccccc} e_1 & e_2 & e_3 & e_4 & e_5 & e_6 & e_7 \end{array} \\ \begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 \end{pmatrix} \end{array}$$

Incidence matrices can also be used in undirected graphs with loops and folded edges. Fold edges are shown using the same element in columns in the incidence matrix.

Example 20.13. Write an incidence matrix for the graph in figure 20.26.

solution:

$$\begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \end{array}\begin{array}{c} \begin{array}{cccc} e_1 & e_2 & e_3 & e_4 \end{array} \\ \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 2 \end{pmatrix} \end{array}$$

Theorem 20.3. *If the adjacency matrices of two graphs are the same, then these two graphs are isomorphic.*

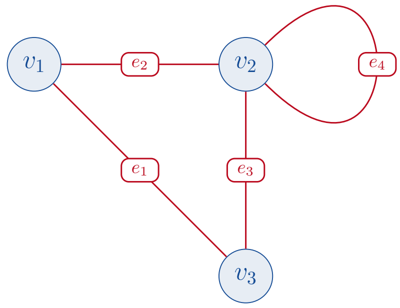


Figure 20.27: The graph referred to in example 20.13.
Şekil 20.27:

Örnek 20.13. Şekil 20.27. Yukarıdaki pisedografı insıdıns matris yardımıyla gösterelim.
çözüm:

$$\begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \end{array}\begin{array}{c} \begin{array}{cccc} e_1 & e_2 & e_3 & e_4 \end{array} \\ \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 2 \end{pmatrix} \end{array}$$

Teorem 20.3. *İki grafın komşuluk matrisleri aynı ise graflar izomorfiktir.*

Walks

Definition. A ***walk*** is a list $v_0, e_1, v_1, \ldots, e_k, v_k$ of vertices and edges such that for $1 \le i \le k$, the edge e_i has endpoints v_{i-1} and v_i.

Example 20.14. Consider figure 20.30.

$$a, e_1, b, e_5, e, e_4, d, e_3, c, e_6, f, e_7, g, e_7, f$$

is a walk in this graph.

Definition. An ***Eulerian trail*** is a walk such that

(i). no edge is repeated; and

(ii). every edge is included.

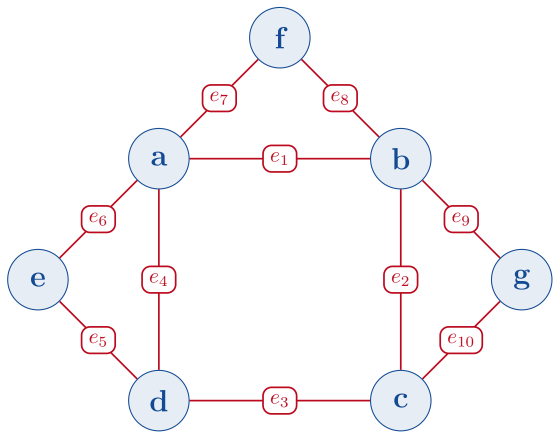


Figure 20.28: The graph refered to in example 20.15.
Şekil 20.28:

Example 20.15. Consider figure 20.28. The walk

$$d, e_5, e, e_6, b, e_2, c, e_3, e, e_4, a, e_1, b, e_8, c, e_9, d, e_{10}, a, e_7, b$$

is an Eulerian trail. Each of the ten edges appears once and only once in this list.

Remark. The Königsberg bridge problem can be rephrased as:

Does there exist an Eulerian trail in Königsberg?

Definition. A graph is ***connected*** if there exists a walk between every pair of vertices.

Example 20.16. In figure 20.6 on page 117, graph H is connected, but graph G is not connected.

Yol

Tanım. Bir ***yol***, köşe ve kenarların $v_0, e_1, v_1, \ldots, e_k, v_k$ şeklindeki bir listesidir öyle ki her $1 \le i \le k$ için e_i kenarını v_{i-1} ve v_i kçşeleri ile bağlıdır..

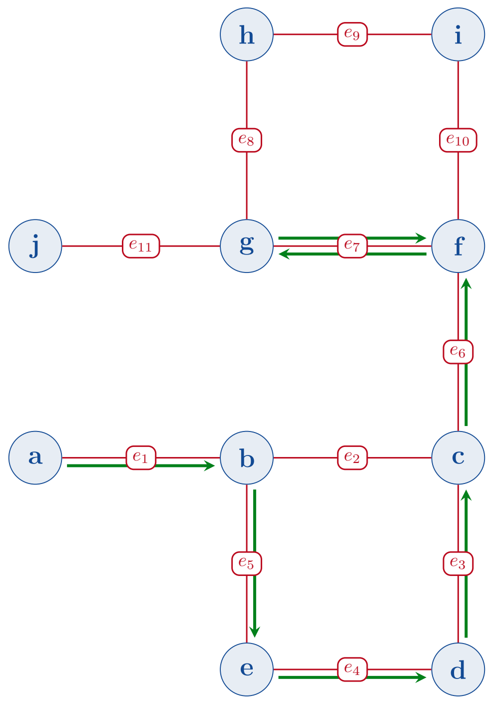


Figure 20.30: 20.14 örneğinde ifade edilen grafik.
Şekil 20.30:

Örnek 20.14. 20.30 şeklini ele alalım.

$$a, e_1, b, e_5, e, e_4, d, e_3, c, e_6, f, e_7, g, e_7, f$$

bu graf için bir yoldur.

Tanım. ***Euler yolu***,

(i). hiçhir köşenin tekrarlanmadığı, ve

(ii). tüm kenarları içeren bir yoldur.

Örnek 20.15. 20.28 şeklini ele alalım.

$$d, e_5, e, e_6, b, e_2, c, e_3, e, e_4, a, e_1, b, e_8, c, e_9, d, e_{10}, a, e_7, b$$

yolu bir Euler yoludur. On kenarın her biri bu listede bi ve sadece bir kez görünür.

Not. Königsberg köprüsü problemi şu şekilde de sorulabilirdi: b

Königsberg köprüsü bir Euler yolu oluşturur mu??

Tanım. Farklı köşelerin her çifti arasında bir yol varsa bu grafa ***bağlantılı*** graf denir.

Örnek 20.16. 20.6 117 sayfasındaki 20.6 graflarından, H bağlantılıdır, fakat G bağlantılı değildir.

Theorem 20.4. *Let G be a connected graph. Then there exists an Eulerian trail if and only if the number of vertices of odd degree is either 0 or 2.*

Furthermore, if G has 2 vertices of odd degree, then the Eulerian trail must start and finish at these two vertices.

Example 20.17. Please consider figure 20.17 on page 122. Note that in K_3 and K_5, every vertex has even degree. So there is an Eulerian trail in K_3 and in K_5.

Notice further that all four vertices in K_4 are of odd degree. This means that K_4 does not contain an Eulerian trail.

Example 20.18. Please consider figure 20.28 again. Note that

$$\deg(a) = 4,\ \deg(b) = 5,\ \deg(c) = 4,\ \deg(d) = 3,\ \deg(e) = 4.$$

Two of the vertices have odd degree, b and d. So there must exist an Eulerian trail and it must start/end at b and d. In example 20.15 we have already found this trail.

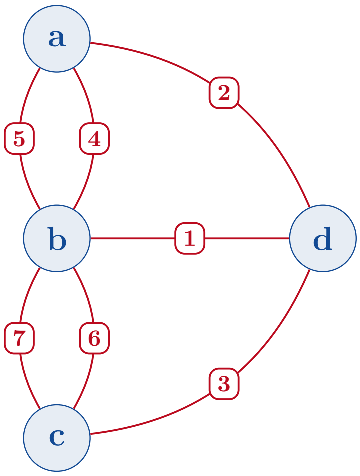


Figure 20.29: A graph of the seven bridges of Königsberg.
Şekil 20.29:

Example 20.19 (The Königsberg Bridge Problem)**.** Now consider Königsberg as shown in figure 20.29. Note that

$$\deg(a) = 3, \qquad \deg(b) = 5, \qquad \deg(c) = 3, \qquad \deg(d) = 3.$$

Since all four vertices have odd degree, there does not exist an Eulerian trail in Königsberg.

Therefore it was not possible to walk around the city of Königsberg and cross each bridge once.

Teorem 20.4. *G bağlı bir graf olsun. Bir Eulerian yolunun var olması için gerek ve yeter koşul tek dereceli köşelerin sayısının 0 ya da 2 olmasıdır.*

Dahası, Eğer G tek dereceli iki köşeye sahipse, Euler yolu Furthermore, if G has 2 vertices of odd degree, then the Eulerian trail must start and finish at these two vertices.

Örnek 20.17. Please consider figure 20.17 on page 122. Note that in K_3 and K_5, every vertex has even degree. So there is an Eulerian trail in K_3 and in K_5.

Notice further that all four vertices in K_4 are of odd degree. This means that K_4 does not contain an Eulerian trail.

Örnek 20.18. Please consider figure 20.28 again. Note that

$$\deg(a) = 4,\ \deg(b) = 5,\ \deg(c) = 4,\ \deg(d) = 3,\ \deg(e) = 4.$$

Two of the vertices have odd degree, b and d. So there must exist an Eulerian trail and it must start/end at b and d. In example 20.15 we have already found this trail.

Örnek 20.19 (The Königsberg Bridge Problem)**.** Now consider Königsberg as shown in figure 20.29. Note that

$$\deg(a) = 3, \qquad \deg(b) = 5, \qquad \deg(c) = 3, \qquad \deg(d) = 3.$$

Since all four vertices have odd degree, there does not exist an Eulerian trail in Königsberg.

Therefore it was not possible to walk around the city of Königsberg and cross each bridge once.

Euler's Formula for Polyhedra

Euler's formula is

$$n(V) - n(E) + n(F)$$

where

$$\begin{aligned} n(V) &= \text{number of vertices} \\ n(E) &= \text{number of edges} \\ n(F) &= \text{number of faces.} \end{aligned}$$

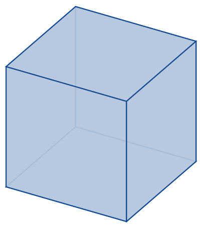
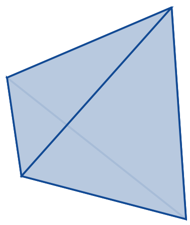
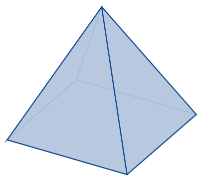
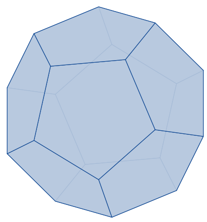
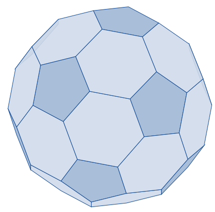
Euler'in Çok Yüzlü Cisim Formülü

Euler's formula is

$$n(V) - n(E) + n(F)$$

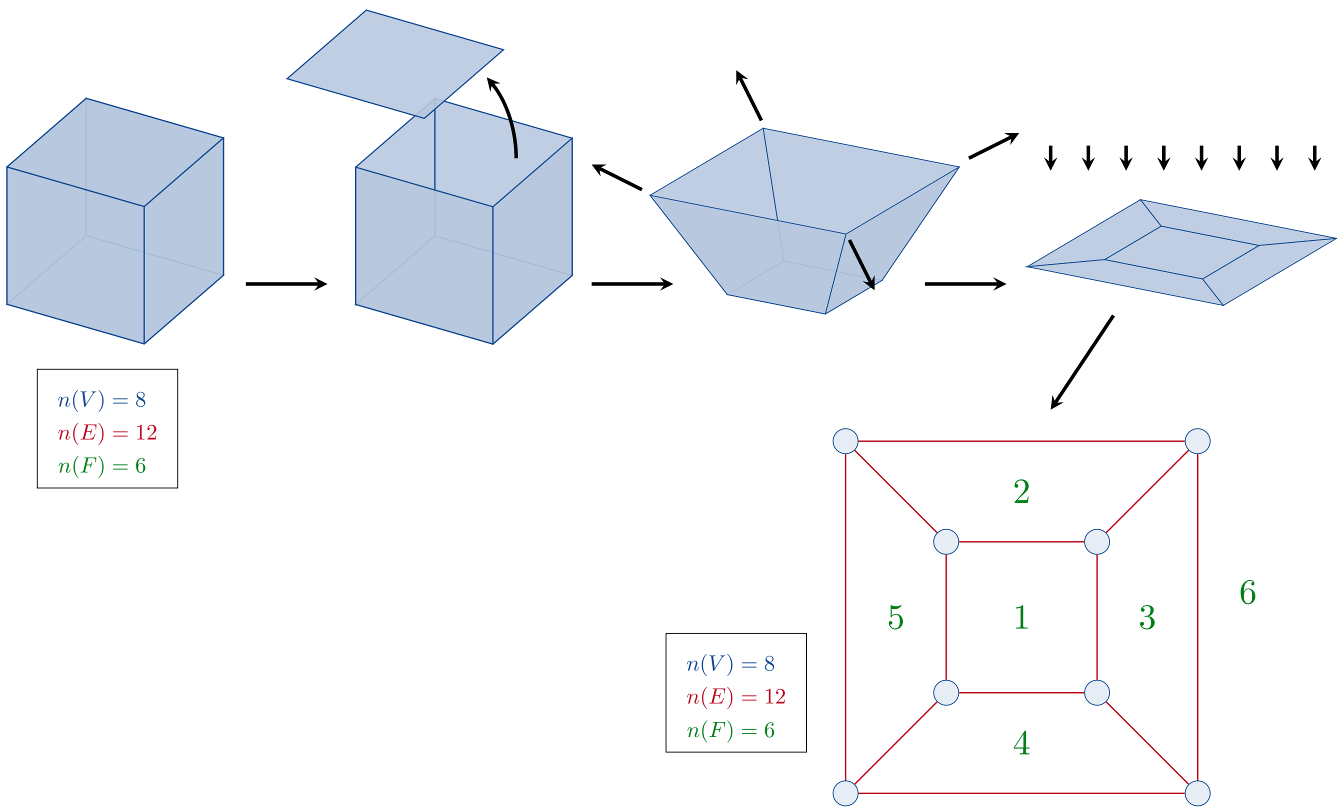
where

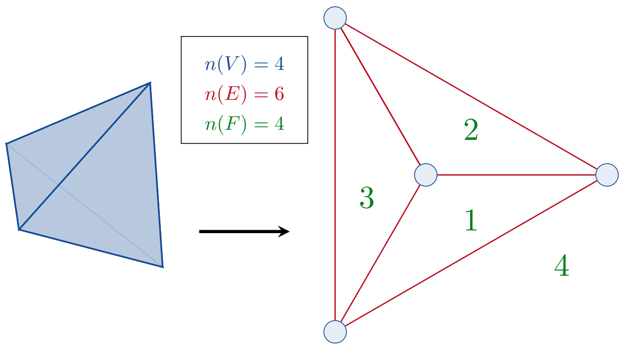
$$\begin{aligned} n(V) &= \text{number of vertices} \\ n(E) &= \text{number of edges} \\ n(F) &= \text{number of faces.} \end{aligned}$$

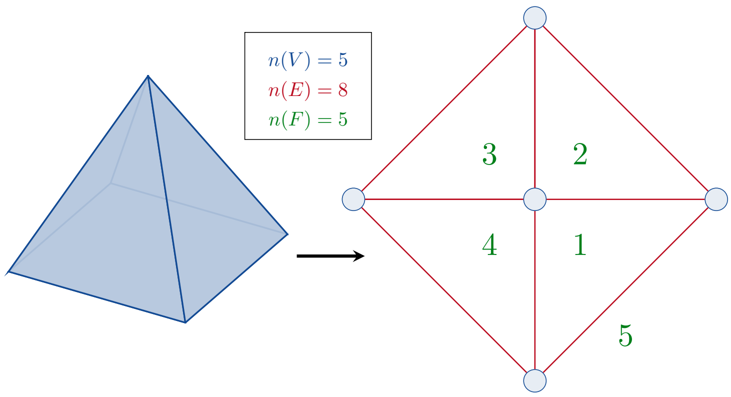
	cube	tetrahedron	pyramid	dodecahedron	football
	küp	dörtyüzlü	piramit	onikiyüzlü	futbol topu
				(12 pentagons)	(12 pentagons & 20 hexagons)
					
$n(V)$	8	4	5	20	60
$n(E)$	12	6	8	30	90
$n(F)$	6	4	5	12	32
$n(V) - n(E) + n(F)$	$8 - 12 + 6 = 2$	$4 - 6 + 4 = 2$	$5 - 8 + 5 = 2$	$20 - 30 + 12 = 2$	$60 - 90 + 32 = 2$

Remark. If we have a polyhedron without any holes in it, is Euler's formula always equal to 2? And if so, how can we prove it?

Not. If we have a polyhedron without any holes in it, is Euler's formula always equal to 2? And if so, how can we prove it?







Every three dimensional polyhedron is equivalent to a connected, planar, simple graph. So if we know something about these graphs, then we also know it about polyhedra. What do we know about such graphs?

Let us start with the first complete graph:

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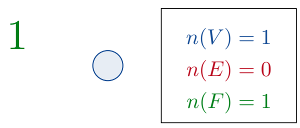

This graph, K_1, is called the ***trivial graph***. It has one vertex, zero edges and one face. So

$$n(V) - n(E) + n(F) = 1 - 0 + 1 = 2.$$

Now let us take any connected, planar, simple graph. How can we simplify this graph?

We can remove a pendant vertex (a vertex v with $\deg(v) = 1$) and its edge.

This graph, K_1, is called the ***trivial graph***. It has one vertex, zero edges and one face. So

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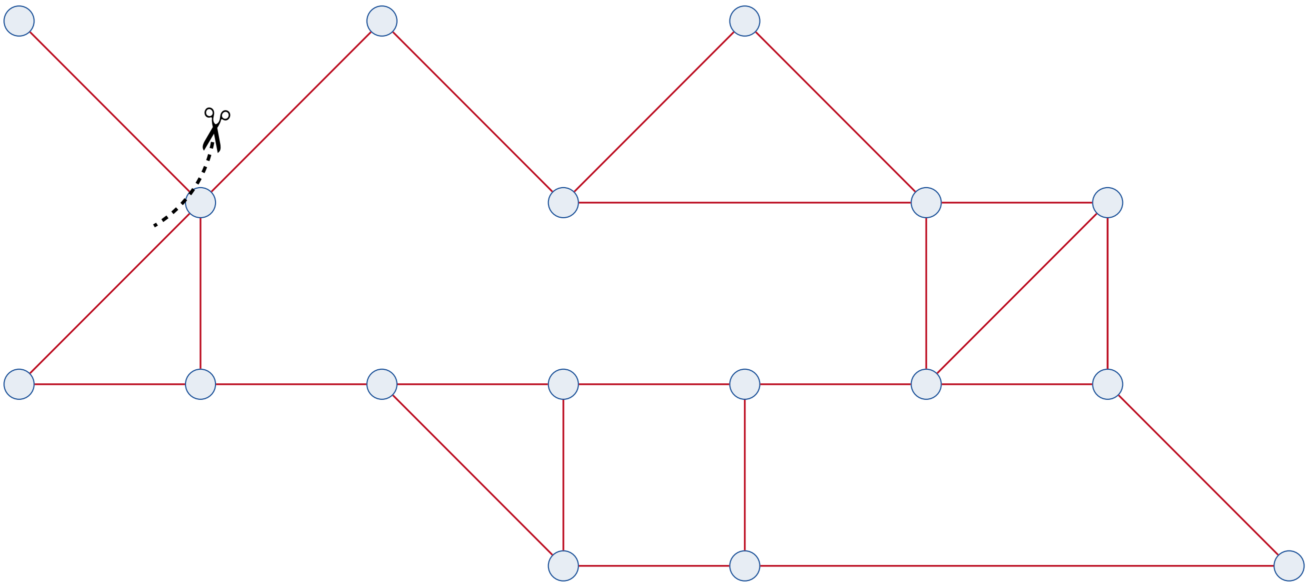

Then we still have a connected, planar, simple graph and we have decreased $n(V)$ by 1 *and* we have decreased $n(E)$ by 1. So $n(V) - n(E) + n(F)$ stays the same.

We can also remove an edge which separates two faces. E.g.

Then we still have a connected, planar, simple graph and we have decreased $n(V)$ by 1 *and* we have decreased $n(E)$ by 1. So $n(V) - n(E) + n(F)$ stays the same.

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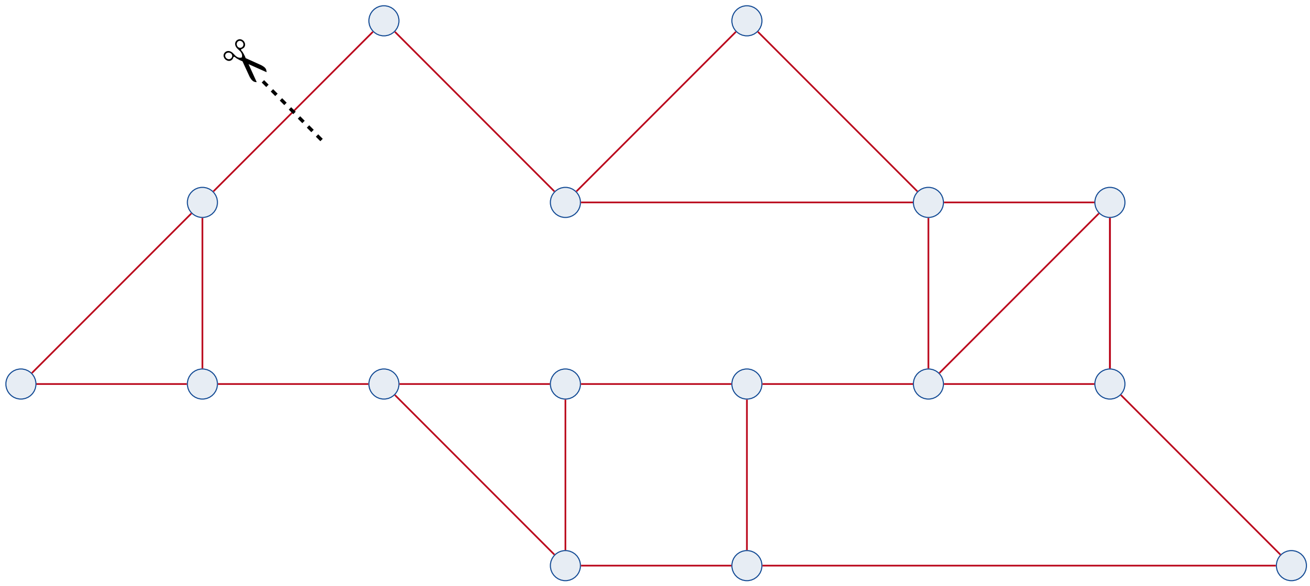

As before, we still have a connected, planar, simple graph. Moreover, we have decreased $n(E)$ by 1 *and* we have decreased $n(F)$ by 1. So again, $n(V) - n(E) + n(F)$ stays the same.

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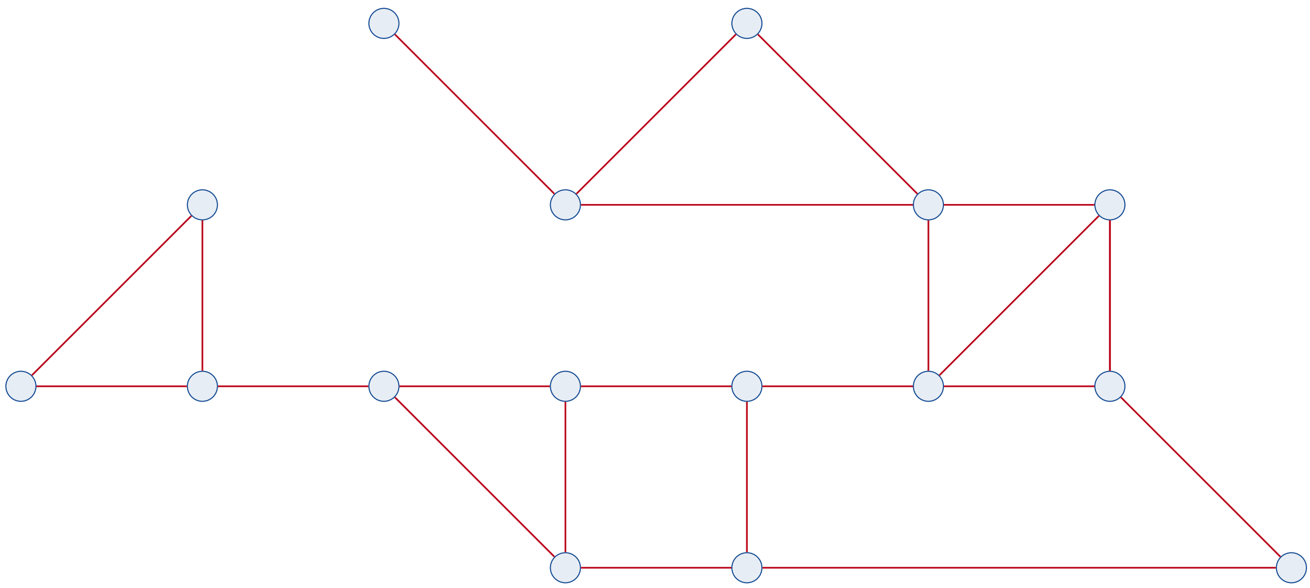

Because we can reduce every connected, planar, simple graph to the trivial graph, every such graph must have the same $n(V) - n(E) + n(F)$ number as the trivial graph. Therefore:

Because we can reduce every connected, planar, simple graph to the trivial graph, every such graph must have the same $n(V) - n(E) + n(F)$ number as the trivial graph. Therefore:

Theorem 20.5. *If G is a connected, planar, simple graph, then*

$$n(V) - n(E) + n(F) = 2.$$

Teorem 20.5. *If G is a connected, planar, simple graph, then*

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. . . and the same is true for polyhedra without holes.

. . . and the same is true for polyhedra without holes.

Problems

Problem 20.1 (Drawing Graphs). Draw the graph $G = (V, E)$ where $V = \{a, b, c, d, e\}$ and

(a). $E = \{e_1 = (a, b), e_2 = (b, c), e_3 = (c, d), e_4 = (d, e)\}$.

(b). $E = \{e_1 = (a, b), e_2 = (b, c), e_3 = (c, d), e_4 = (d, e), e_5 = (a, e), e_6 = (b, d), e_7 = (b, e), e_8 = (a, d)\}$.

(c). $E = \{e_1 = (b, c), e_2 = (d, e), e_3 = (c, d), e_4 = (a, c), e_5 = (b, d), e_6 = (c, e), e_7 = (a, d), e_8 = (b, e)\}$.

Problem 20.2 (Counting Edges). How many edges do the following graphs have?

(a). K_n

(b). K_{nm}

(c). C_n

(d). W_n

(e). Q_n

(f). the trivial graph

Problem 20.3 (Planar Graphs). Please see figures 20.10 and 20.14 on page 120. Are these planar graphs? Prove your answer.

Problem 20.4 (Bipartite Graphs). Are the following graphs bipartite graphs?

(a). Q_2

(b). Figure 20.13

(c). Figure 20.28

(d). Figure 20.30

(e). Figure 20.31

(f). Figure 20.32

Problem 20.5 (Euler's Formula). Draw a graph which does not satisfy $n(V) - n(E) + n(F) = 2$.

Problem 20.6 (Eulerian trails). Consider the two graphs below. For each graph, answer the question: Does this graph contain an Eulerian trail? If "yes", give an example of an Eulerian trail in that graph. If "no", explain how we know that it does not contain an Eulerian trail.

(a). Figure 20.31.

(b). Figure 20.32.

(c). Figure 20.21

(d). Figure 20.25

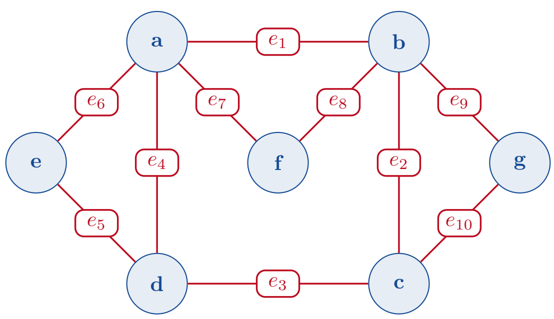

Figure 20.31: A graph refered to in exercise 20.6.
Şekil 20.31:

Sorular

Soru 20.1 (Drawing Graphs). Draw the graph $G = (V, E)$ where $V = \{a, b, c, d, e\}$ and

(a). $E = \{e_1 = (a, b), e_2 = (b, c), e_3 = (c, d), e_4 = (d, e)\}$.

(b). $E = \{e_1 = (a, b), e_2 = (b, c), e_3 = (c, d), e_4 = (d, e), e_5 = (a, e), e_6 = (b, d), e_7 = (b, e), e_8 = (a, d)\}$.

(c). $E = \{e_1 = (b, c), e_2 = (d, e), e_3 = (c, d), e_4 = (a, c), e_5 = (b, d), e_6 = (c, e), e_7 = (a, d), e_8 = (b, e)\}$.

Soru 20.2 (Counting Edges). Asagidaki graflarin kener sayilerini bulunuz.

(a). K_n

(b). K_{nm}

(c). C_n

(d). W_n

(e). Q_n

(f). the trivial graph

Soru 20.3 (Planar Graphs). Please see figures 20.10 and 20.14 on page 120. Are these planar graphs? Prove your answer.

Soru 20.4 (Bipartite Graphs). Are the following graphs bipartite graphs?

(a). Q_2

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Soru 20.5 (Euler's Formula). Draw a graph which does not satisfy $n(V) - n(E) + n(F) = 2$.

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(a). Figure 20.31.

(b). Figure 20.32.

(c). Figure 20.21

(d). Figure 20.25

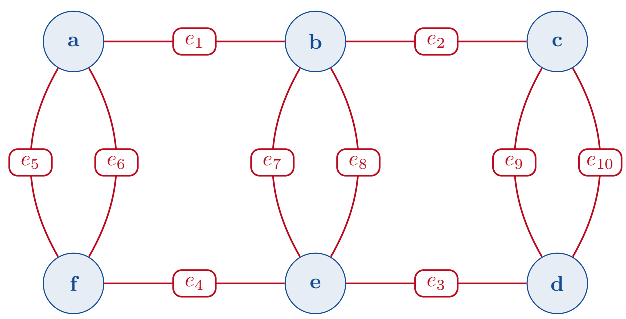


Figure 20.32: A graph refered to in exercise 20.6.
Şekil 20.32:

Part IV

Calculus

21 Limits

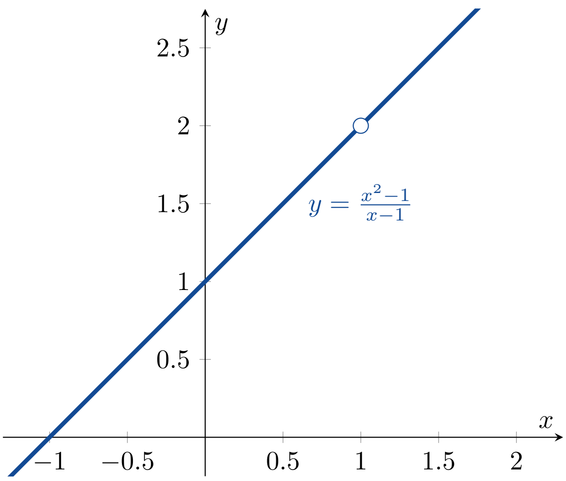


Figure 21.1: The function $f(x) = \frac{x^2-1}{x-1}$.

Consider the function $f : (-\infty, 1) \cup (1, \infty) \to \mathbb{R}$, $f(x) = \frac{x^2-1}{x-1}$ as shown in figure 21.1.

Question: How does f behave when x is close to 1?

We can see from table 21.1 that:

> "If x is close to 1, then $f(x)$ is close to 2."

Mathematically, we write this as

$$\lim_{x \to 1} f(x) = 2$$

and read it as "the limit, as x tends to 1, of $f(x)$ is equal to 2".

Limit

Şekil 21.1: $f(x) = \frac{x^2-1}{x-1}$ fonksiyonu.

x	$f(x)$
0.9	1.9
1.1	2.1
0.99	1.99
1.01	2.01
0.999	1.999
1.001	2.001

Table 21.1: Some values of $f(x) = \frac{x^2-1}{x-1}$.
Tablo 21.1: $f(x) = \frac{x^2-1}{x-1}$'nin bazı değerleri.

$f(x) = \frac{x^2-1}{x-1}$ ile tanımlı $f : (-\infty, 1) \cup (1, \infty) \to \mathbb{R}$ nın bazı değerleri şekil 21.1 de veriliyor.

Soru: x, 1'e yakın olduğunda f nasıl davranıyor?

Tablo 21.1 den şu gözlemi yapabiliriz:

> "x, 1'e yakınsa, $f(x)$ de, 2'ye yakın olur."

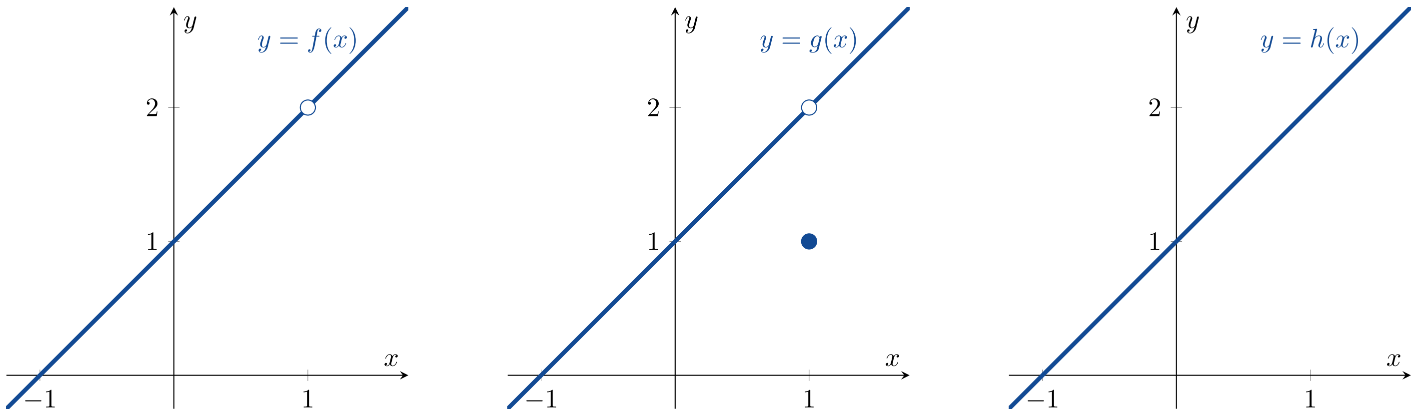
Matematiksel olarak, bunu

$$\lim_{x \to 1} f(x) = 2$$

olarak yazarız ve x, 1 e yakaşırken, $f(x)$ in limiti 2'ye eşittir olarak okuruz.

Example 21.1. Consider the following three functions:

Örnek 21.1. Aşağıdaki üç fonksiyonu inceleyecek olursak:



$$f(x) = \frac{x^2 - 1}{x - 1}$$

$$g(x) = \begin{cases} \frac{x^2-1}{x-1} & x \neq 1 \\ 1 & x = 1 \end{cases}$$

$$h(x) = x + 1$$

Note that

- $\lim_{x \to 1} f(x) = 2$, but f is not defined at $x = 1$;
- $\lim_{x \to 1} g(x) = 2$, but $g(1) \neq 2$; and
- $\lim_{x \to 1} h(x) = 2$ and $h(1) = 2$.

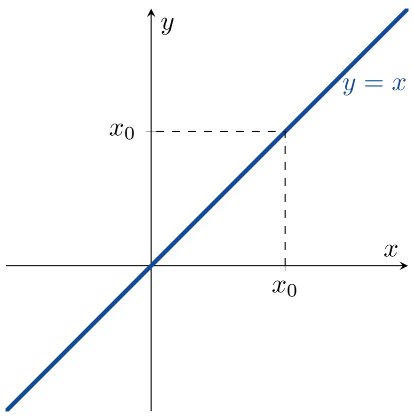


Figure 21.2: The Identity Function
Şekil 21.2: Özdeş fonksiyon.

Example 21.2 (The Identity Function)**.** $f(x) = x$

$$\lim_{x \to x_0} f(x) = \lim_{x \to x_0} x = x_0$$

Example 21.3 (A Constant Function)**.** $f(x) = 13$

$$\lim_{x \to x_0} f(x) = \lim_{x \to x_0} 13 = 13$$

- $\lim_{x \to 1} f(x) = 2$, fakat f, $x = 1$ de tanımlı değildir;
- $\lim_{x \to 1} g(x) = 2$, fakat $g(1) \neq 2$; ve
- $\lim_{x \to 1} h(x) = 2$ ve $h(1) = 2$.

olduğuna dikkat edelim.

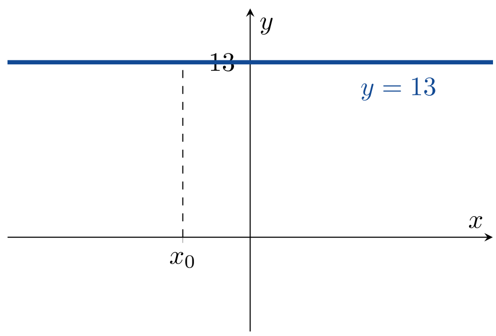


Figure 21.3: A Constant Function
Şekil 21.3: Sabit fonksiyon.

Örnek 21.2 (Birim Fonksiyon)**.** $f(x) = x$

$$\lim_{x \to x_0} f(x) = \lim_{x \to x_0} x = x_0$$

Örnek 21.3 (Sabit Fonksiyon)**.** $f(x) = 13$

$$\lim_{x \to x_0} f(x) = \lim_{x \to x_0} 13 = 13$$

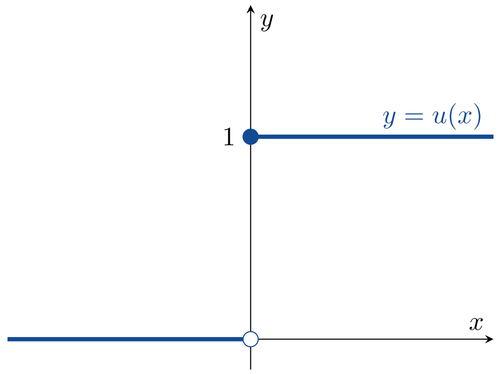


Figure 21.4: A graph of the function $u(x)$.
Şekil 21.4: $u(x)$ fonksiyonunun bir grafiği.

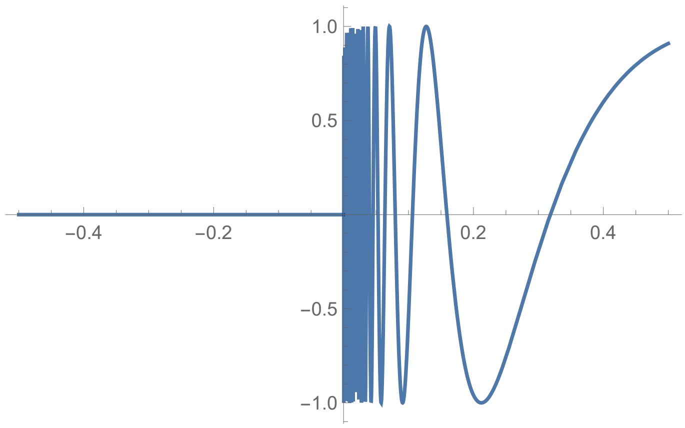


Figure 21.5: A graph of the function $v(x)$.
Şekil 21.5: $v(x)$ fonksiyonunun bir grafiği.

Example 21.4 (Sometimes Limits Do Not Exist)**.** Consider the functions

$$u(x) = \begin{cases} 0 & x < 0 \\ 1 & x \geq 0 \end{cases} \quad \text{and} \quad v(x) = \begin{cases} 0 & x \leq 0 \\ \sin \frac{1}{x} & x > 0 \end{cases}$$

as shown in figures 21.4 and 21.5.

Note that $\lim_{x \to 0} u(x)$ does not exist. To understand why, we consider x close to 0:

- If x is close to 0 and $x < 0$, then $u(x) = 0$.
- If x is close to 0 and $x > 0$, then $u(x) = 1$.

Because 0 is not close to 1, the limit as $x \to 0$ can not exist.

Moreover $\lim_{x \to 0} v(x)$ does not exist because $v(x)$ oscillates up and down too quickly if $x > 0$ and $x \to 0$.

Örnek 21.4 (Limit Her Zaman Mevcut Olmayabilir)**.** Şu fonksiyonları inceleyelim

$$u(x) = \begin{cases} 0 & x < 0 \\ 1 & x \geq 0 \end{cases} \quad \text{ve} \quad v(x) = \begin{cases} 0 & x \leq 0 \\ \sin \frac{1}{x} & x > 0 \end{cases}$$

bakınız şekil 21.4 ve 21.5.

$\lim_{x \to 0} u(x)$ limitinin mevcut olmadığına dikkat ediniz. Bunun neden mevcut olmadığını anlamak için, x 'in 0'a çok yakın olduğunu düşünelim:

- x, 0'a çok yakın ve $x < 0$ iken, $u(x) = 0$ dır.
- x, 0'a çok yakın ve $x > 0$ iken, $u(x) = 1$ olur.

0, 1'e çok yakın olmadığı için , $x \to 0$ iken limit mevcut değildir.

Ayrıca, $x > 0$ ve $x \to 0$ iken, $v(x)$, çok hızlı bir şekilde yukarıya ve aşağıya doğru salınır, çünkü $\lim_{x \to 0} v(x)$ mevcut değildir.

Theorem 21.1 (The Limit Laws). *Suppose that*

- $L, M, c, k \in \mathbb{R}$;
- f *and* g *are functions;*
- $\lim_{x\to c} f(x) = L$; *and*
- $\lim_{x\to c} g(x) = M$.

Then

(*i*). ***Sum Rule:***

$$\lim_{x\to c}\big(f(x)+g(x)\big) = L+M;$$

(*ii*). ***Difference Rule:***

$$\lim_{x\to c}\big(f(x)-g(x)\big) = L-M;$$

(*iii*). ***Constant Multiple Rule:***

$$\lim_{x\to c}\big(kf(x)\big) = kL;$$

(*iv*). ***Product Rule:***

$$\lim_{x\to c}\big(f(x)g(x)\big) = LM;$$

(*v*). ***Quotient Rule:*** *if* $M \neq 0$, *then*

$$\lim_{x\to c}\left(\frac{f(x)}{g(x)}\right) = \frac{L}{M};$$

(*vi*). ***Power Rule:*** *if* $n \in \mathbb{N}$, *then*

$$\lim_{x\to c}\big(f(x)\big)^n = L^n;$$

(*vii*). ***Root Rule:*** *if* $n \in \mathbb{N}$ *and* $\sqrt[n]{L}$ *exists, then*

$$\lim_{x\to c}\sqrt[n]{f(x)} = \sqrt[n]{L} = L^{\frac{1}{n}}.$$

Teorem 21.1 (Limit Kuralları). *Varsayalaım ki*

- $L, M, c, k \in \mathbb{R}$;
- f *ve* g *iki fonksiyon;*
- $\lim_{x\to c} f(x) = L$; *ve*
- $\lim_{x\to c} g(x) = M$ *olsun.*

O halde

(*i*). ***Toplam Kuralı:***

$$\lim_{x\to c}\big(f(x)+g(x)\big) = L+M;$$

(*ii*). ***Fark Kuralı:***

$$\lim_{x\to c}\big(f(x)-g(x)\big) = L-M;$$

(*iii*). ***Sabitle Çarpım Kuralı:***

$$\lim_{x\to c}\big(kf(x)\big) = kL;$$

(*iv*). ***Çarpım Kuralı:***

$$\lim_{x\to c}\big(f(x)g(x)\big) = LM;$$

(*v*). ***Bölüm Kuralı:*** $M \neq 0$, *ise*

$$\lim_{x\to c}\left(\frac{f(x)}{g(x)}\right) = \frac{L}{M};$$

(*vi*). ***Kuvvet Kuralı:*** $n \in \mathbb{N}$, *ise*

$$\lim_{x\to c}\big(f(x)\big)^n = L^n;$$

(*vii*). ***Kök Kuralı:*** *if* $n \in \mathbb{N}$ *ve* $\sqrt[n]{L}$ *mevcutsa, then*

$$\lim_{x\to c}\sqrt[n]{f(x)} = \sqrt[n]{L} = L^{\frac{1}{n}}.$$

Example 21.5. Find $\lim_{x\to 2}(x^3+4x^2-3)$.

solution:

$$\begin{aligned}\lim_{x\to 2}(x^3+4x^2-3) &= \big(\lim_{x\to 2} x^3\big) + \big(\lim_{x\to 2} 4x^2\big) - \big(\lim_{x\to 2} 3\big)\\ &\quad \text{(sum and difference rules)}\\ &= \big(\lim_{x\to 2} x\big)^3 + 4\big(\lim_{x\to 2} x\big)^2 - \big(\lim_{x\to 2} 3\big)\\ &\quad \text{(power and constant multiple rules)}\\ &= 2^3 + 4(2^2) - 3 = 21.\end{aligned}$$

Örnek 21.6. $\lim_{x\to 6} 8(x-5)(x-7)$ limitini bulunuz.

çözüm:

$$\begin{aligned}\lim_{x\to 6} 8(x-5)(x-7) &= 8\lim_{x\to 6}(x-5)(x-7)\\ &\quad \text{(sabitle çarpım kuralı)}\\ &= 8\big(\lim_{x\to 6}(x-5)\big)\big(\lim_{x\to 6}(x-7)\big)\\ &\quad \text{(çarpım kuralı)}\\ &= 8(1)(-1) = -8.\end{aligned}$$

Example 21.7. Find $\lim_{x\to 5}\dfrac{x^4+x^2-1}{x^2+5}$.

solution:

Örnek 21.8. $\lim_{x\to -5}\dfrac{x^2+3x-11}{x+6}$ limitini bulunuz.

çözüm:

$$\lim_{x\to 5}\frac{x^4+x^2-1}{x^2+5} = \frac{\lim_{x\to 5}(x^4+x^2-1)}{\lim_{x\to 5}(x^2+5)}$$
(quotient rule)
$$= \frac{\lim_{x\to 5}x^4+\lim_{x\to 5}x^2-\lim_{x\to 5}1}{\lim_{x\to 5}x^2+\lim_{x\to 5}5}$$
(sum and difference rules)
$$= \frac{5^4+5^2-1}{5^2+5}$$
(power rule)
$$= \frac{649}{30}.$$

$$\lim_{x\to -5}\frac{x^2+3x-11}{x+6} = \frac{\lim_{x\to -5}(x^2+3x-11)}{\lim_{x\to -5}(x+6)}$$
(bölüm kuralı)
$$= \frac{\lim_{x\to -5}x^2+\lim_{x\to -5}3x-\lim_{x\to -5}11}{\lim_{x\to -5}x+\lim_{x\to -5}6}$$
(toplam ve fark kuralı)
$$= \frac{(-5)^2-15-11}{-5+6}$$
(kuvvet kuralı)
$$= \frac{-1}{1} = -1.$$

Theorem 21.2 (Limits of Polynomial Functions). *If $P(x) = a_nx^n+a_{n-1}x^{n-1}+\ldots+a_1x+a_0$ is a polynomial function, then*
$$\lim_{x\to c}P(x) = P(c).$$

Teorem 21.2 (Polinomların Limitleri). *$P(x) = a_nx^n+a_{n-1}x^{n-1}+\ldots+a_1x+a_0$ bir polinom fonksiyonsa,*
$$\lim_{x\to c}P(x) = P(c).$$

Theorem 21.3 (Limits of Rational Functions). *If $P(x)$ and $Q(x)$ are polynomial functions and if $Q(c)\neq 0$, then*
$$\lim_{x\to c}\frac{P(x)}{Q(x)} = \frac{P(c)}{Q(c)}.$$

Teorem 21.3 (Rasyonel Fonksiyonların Limitlari). *$P(x)$ ve $Q(x)$ polinomlar ve $Q(c)\neq 0$ ise, o halde*
$$\lim_{x\to c}\frac{P(x)}{Q(x)} = \frac{P(c)}{Q(c)}.$$

Example 21.9.
$$\lim_{x\to -1}\frac{x^3+4x^2-3}{x^2+5} = \frac{(-1)^3+4(-1)^2-3}{(-1)^2+5} = \frac{0}{6} = 0.$$

Örnek 21.10.
$$\lim_{x\to 2}\frac{x^3+4x^2-3}{x^2+5} = \frac{(2)^3+4(2)^2-3}{(2)^2+5} = \frac{8+16-3}{4+5} = \frac{21}{9} = \frac{7}{3}.$$

Eliminating Zero Denominators Algebraically

$$\lim_{x\to c}\frac{P(x)}{Q(x)}$$

What can we do if $Q(c) = 0$?

Example 21.11.
$$\lim_{x\to 1}\frac{x^2+x-2}{x^2-x}$$

If we just put in $x = 1$, we would get "$\frac{0}{0}$" and we never never never want "$\frac{0}{0}$".

Instead, we try to factor x^2+x-2 and x^2-x. If $x\neq 1$, we have that
$$\frac{x^2+x-2}{x^2-x} = \frac{(x-1)(x+2)}{x(x-1)} = \frac{x+2}{x}.$$

So
$$\lim_{x\to 1}\frac{x^2+x-2}{x^2-x} = \lim_{x\to 1}\frac{x+2}{x} = \frac{1+2}{1} = 3.$$

Sıfır Paydaların Cebirsel Olarak Yok Edilmesi

$$\lim_{x\to c}\frac{P(x)}{Q(x)}$$

$Q(c) = 0$ ise ne yapılabilir?

Örnek 21.12.
$$\lim_{x\to -5}\frac{x^2+3x-10}{x^2+5x}$$

If we just put in $x = -5$ koyarsak, "$\frac{0}{0}$" buluruz ve unutmayın "$\frac{0}{0}$" asla ve asla istemediğimiz birşey.

Onun yerine, $x^2+3x-10$ ve x^2+5x yi çarpalarına ayırırız. $x\neq -5$ ise, şunu buluruz
$$\frac{x^2+3x-10}{x^2+5x} = \frac{(x+5)(x-2)}{x(x+5)} = \frac{x-2}{x}.$$

Yani
$$\lim_{x\to -5}\frac{x^2+3x-10}{x^2+5x} = \lim_{x\to -5}\frac{x-2}{x} = \frac{-5-2}{-5} = \frac{7}{5}.$$

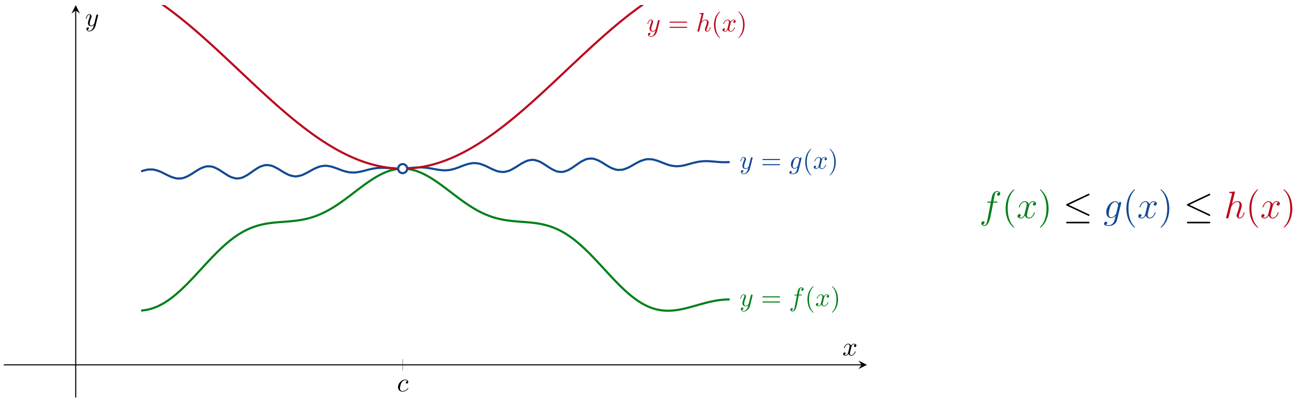


Figure 21.6: The Sandwich Theorem
Şekil 21.6: Sandöviç Teoremi

The Sandwich Theorem

See figure 21.6.

Theorem 21.4 (The Sandwich Theorem)**.** *Suppose that*

- $f(x) \leq g(x) \leq h(x)$ *for all* x *"close" to* c $(x \neq c)$*; and*
- $\lim_{x \to c} f(x) = \lim_{x \to c} h(x) = L$.

Then

$$\lim_{x \to c} g(x) = L$$

also.

Example 21.13. The inequality

$$1 - \frac{x^2}{6} < \frac{x \sin x}{2 - 2\cos x} < 1$$

holds for all x close to 0 $(x \neq 0)$. Calculate $\lim_{x \to 0} \frac{x \sin x}{2 - 2\cos x}$.

solution: Since $\lim_{x \to 0} 1 - \frac{x^2}{6} = 1 - \frac{0}{6} = 1$ and $\lim_{x \to 0} 1 = 1$, it follows by the Sandwich Theorem that $\lim_{x \to 0} \frac{x \sin x}{2 - 2\cos x} = 1$.

Theorem 21.5. *If*

- $f(x) \leq g(x)$ *for all* x *close to* c $(x \neq c)$*;*
- $\lim_{x \to c} f(x)$ *exists; and*
- $\lim_{x \to c} g(x)$ *exists,*

then

$$\lim_{x \to c} f(x) \leq \lim_{x \to c} g(x).$$

Sandöviç Teoremi

Bkz. şekil 21.6.

Teorem 21.4 (Sandöviç Teoremi)**.** *Varsayalım ki*

- c $(x \neq c)$ *ye "çok yakın" bütün* x *ler için* $f(x) \leq g(x) \leq h(x)$ *ve*
- $\lim_{x \to c} f(x) = \lim_{x \to c} h(x) = L$ *olsun.*

O zaman

$$\lim_{x \to c} g(x) = L$$

ifadesi doğrudur.

Örnek 21.13.

$$1 - \frac{x^2}{6} < \frac{x \sin x}{2 - 2\cos x} < 1$$

eşitsizliği 0 a çok yakın bütün x ler $(x \neq 0)$ için doğrudur. $\lim_{x \to 0} \frac{x \sin x}{2 - 2\cos x}$ limitini bulunuz.

çözüm: $\lim_{x \to 0} 1 - \frac{x^2}{6} = 1 - \frac{0}{6} = 1$ ve $\lim_{x \to 0} 1 = 1$ olduğundan, Sandöviç Teoremi gereğince $\lim_{x \to 0} \frac{x \sin x}{2 - 2\cos x} = 1$ olarak bulunur.

Teorem 21.5. *Eğer*

- *Her* c *ye çok yakın (ama* $x \neq c$*) bütün* x *ler için* $f(x) \leq g(x)$ *ise ;*
- $\lim_{x \to c} f(x)$ *mevcutsa ve*
- $\lim_{x \to c} g(x)$ *mevcutsa,*

o vakit

$$\lim_{x \to c} f(x) \leq \lim_{x \to c} g(x)$$

doğru olur.

Problems

Problem 21.1. Consider the function shown in figure 21.7. Decide if each of the following statements is true or false.

(a). $\lim_{x\to 0} f(x)$ exists,

(b). $\lim_{x\to 0} f(x) = 0$,

(c). $\lim_{x\to 0} f(x) = 1$,

(d). $\lim_{x\to -2} f(x)$ exists,

(e). $\lim_{x\to -1} f(x) = -1$,

(f). $\lim_{x\to 1} f(x) = 1$,

(g). $\lim_{x\to -\frac{1}{2}} f(x)$ does not exist,

(h). $\lim_{x\to -1.5} f(x) = -0.5$.

Sorular

Soru 21.1. Consider the function shown in figure 21.7. Decide if each of the following statements is true or false.

(a). $\lim_{x\to 0} f(x)$ exists,

(b). $\lim_{x\to 0} f(x) = 0$,

(c). $\lim_{x\to 0} f(x) = 1$,

(d). $\lim_{x\to -2} f(x)$ exists,

(e). $\lim_{x\to -1} f(x) = -1$,

(f). $\lim_{x\to 1} f(x) = 1$,

(g). $\lim_{x\to -\frac{1}{2}} f(x)$ does not exist,

(h). $\lim_{x\to -1.5} f(x) = -0.5$.

Problem 21.2. Find the following limits. For each one, state which limit laws or other theorems you are using.

Soru 21.2. Aşağıdaki limitleri bulunuz. her birinde, kullandığınız kural ve teoremleri yazınız.

(a). $\lim_{x\to -7}(2x+5)$

(b). $\lim_{x\to 2} \dfrac{x+3}{x+6}$

(c). $\lim_{y\to -5} \dfrac{y^2}{y-5}$

(d). $\lim_{x\to \frac{2}{3}} 3x(2x-1)$

(e). $\lim_{t\to 5} \dfrac{t-5}{t^2-25}$

(f). $\lim_{x\to 1} \dfrac{\frac{1}{x}-1}{x-1}$

(g). $\lim_{h\to 0} \dfrac{3}{\sqrt{3h+1}+1}$

(h). $\lim_{y\to 2} \dfrac{y+2}{y^2+5y+6}$

(i). $\lim_{v\to 2} \dfrac{v^3-8}{v^4-16}$

Problem 21.3. If $2 - x^2 \le g(x) \le 2\cos x$ for all x, find $\lim_{x\to 0} g(x)$. State which limit laws or other theorems you are using.

Problem 21.4. Suppose that $\lim_{x\to 4} f(x) = 0$ and $\lim_{x\to 4} g(x) = -3$. Find the following limits.

Soru 21.3. Her x içın, $2 - x^2 \le g(x) \le 2\cos x$ ise, $\lim_{x\to 0} g(x)$ limitini bulunuz. Kullandığınız kural ve teoremleri belirtiniz.

Soru 21.4. $\lim_{x\to 4} f(x) = 0$ ve $\lim_{x\to 4} g(x) = -3$ olsun. Aşağıdaki limitleri bulunuz.

(a). $\lim_{x\to 4} \left(g(x)^2\right)$

(b). $\lim_{x\to 4}(g(x)+3)$

(c). $\lim_{x\to 4} xf(x)$

(d). $\lim_{x\to 4} \dfrac{g(x)}{f(x)-1}$

(e). $\lim_{x\to 4} 4f(x) - 2g(x)$

(f). $\lim_{x\to 4} \dfrac{7f(x)+6}{2g(x)}$

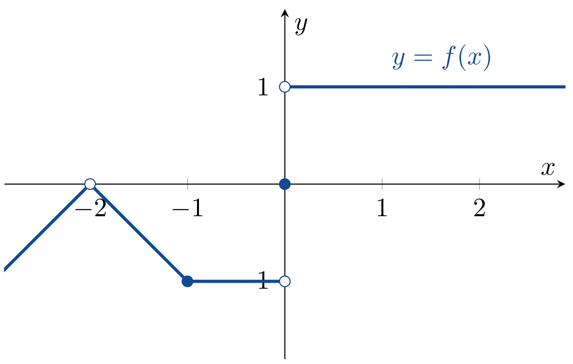
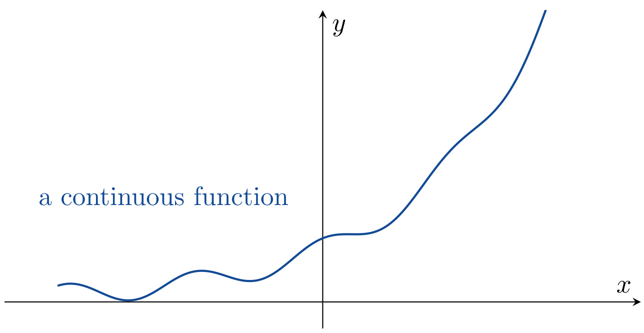


Figure 21.7: The function considered in Exercise 21.1.
Şekil 21.7:

22 Continuity

Süreklilik



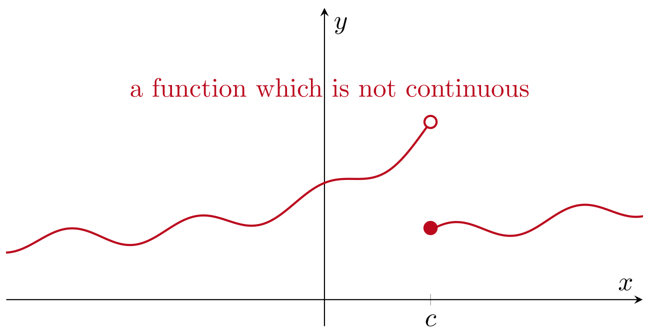


Figure 22.1: A continuous function and a function which is not continuous.
Şekil 22.1: Bir sürekli fonksiyon ve sürekli olmayan bir fonksiyon.

Definition. The function $f : D \to \mathbb{R}$ is ***continuous at*** $c \in D$ if

- $f(c)$ exists;
- $\lim_{x \to c} f(x)$ exists; and
- $\lim_{x \to c} f(x) = f(c)$.

Definition. If f is not continuous at c, we say that f is ***discontinuous*** at c – we say that c is a ***point of discontinuity*** of f.

Tanım. Şu üç koşulun hepsi sağlanırsa $f : D \to \mathbb{R}$ fonksiyonu bir $c \in D$ ***noktasında süreklidir*** denir.

- $f(c)$ tanımlı olacak;
- $\lim_{x \to c} f(x)$ mevcut olacak; ve
- $\lim_{x \to c} f(x) = f(c)$.

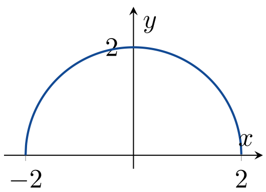
Tanım. Eğer f fonksiyonu c de sürekli değilse, f, c de ***süreksizdir*** denir – ve c'ye f'nin bir ***süreksizlik noktası*** denir.

Example 22.1. Consider the function $f : [0,4] \to \mathbb{R}$ which has its graph shown in figure 22.2. Where is f continuous? Where is f discontinuous?

solution:

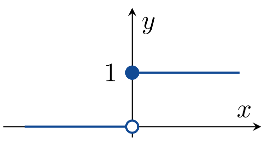
c	Is f continuous at c?	Why?
0	Yes	because $\lim\limits_{x\to 0} f(x) = 1 = f(0)$
$(0,1)$	Yes	because $\lim\limits_{x\to c} f(x) = f(c)$
1	No	because $\lim\limits_{x\to 1} f(x)$ does not exist
$(1,2)$	Yes	because $\lim\limits_{x\to c} f(x) = f(c)$
2	No	because $\lim\limits_{x\to 2} f(x) = 1 \neq 2 = f(2)$
$(2,4)$	Yes	because $\lim\limits_{x\to c} f(x) = f(c)$
4	No	because $\lim\limits_{x\to 4} f(x) = 1 \neq \frac{1}{2} = f(4)$

Example 22.2. $f : [-2,2] \to \mathbb{R}$, $f(x) = \sqrt{4-x^2}$



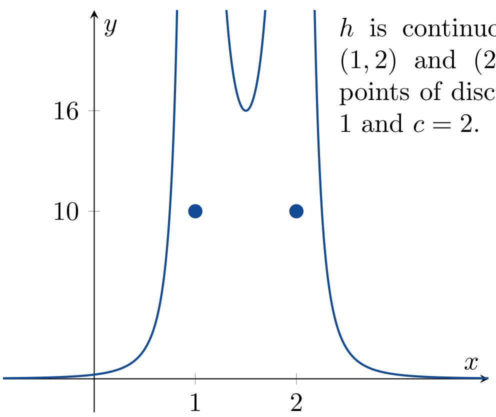
f is continuous at every $c \in [-2,2]$.

Example 22.3. $g : \mathbb{R} \to \mathbb{R}$, $g(x) = \begin{cases} 0 & x < 0 \\ 1 & x \geq 0 \end{cases}$



g has a point of discontinuity at $c = 0$. g is continuous at every point $c \neq 0$.

Example 22.4. $h : \mathbb{R} \to \mathbb{R}$, $h(x) = \begin{cases} \frac{1}{(x-1)^2(x-2)^2} & x \neq 1 \text{ or } 2 \\ 10 & x = 1 \text{ or } 2 \end{cases}$



h is continuous on $(-\infty,1)$, $(1,2)$ and $(2,\infty)$. h has a points of discontinuity at $c = 1$ and $c = 2$.

Örnek 22.1. Grafiği şekil 22.2 deki $f : [0,4] \to \mathbb{R}$ fonksiyonunu ele alalım. Bu f nerede süreklidir? Bu f nerede süreksizdir?

çözüm:

c	f fonksiyonu c de sürekli midir?	Neden?
0	Evet	çünkü $\lim\limits_{x\to 0} f(x) = 1 = f(0)$
$(0,1)$	Evet	çünkü $\lim\limits_{x\to c} f(x) = f(c)$
1	Hayır	çünkü $\lim\limits_{x\to 1} f(x)$ does not exist
$(1,2)$	Evet	çünkü $\lim\limits_{x\to c} f(x) = f(c)$
2	Hayır	çünkü $\lim\limits_{x\to 2} f(x) = 1 \neq 2 = f(2)$
$(2,4)$	Evet	çünkü $\lim\limits_{x\to c} f(x) = f(c)$
4	Hayır	çünkü $\lim\limits_{x\to 4} f(x) = 1 \neq \frac{1}{2} = f(4)$

Örnek 22.2. $f : [-2,2] \to \mathbb{R}$, $f(x) = \sqrt{4-x^2}$
f fonksiyonu her $c \in [-2,2]$ noktasında süreklidir.

Örnek 22.3. $g : \mathbb{R} \to \mathbb{R}$, $g(x) = \begin{cases} 0 & x < 0 \\ 1 & x \geq 0 \end{cases}$
g nin $c = 0$. g da bir süreksizlik noktası var ve fonksiyon her $c \neq 0$ için süreklidir.

Örnek 22.4. $h : \mathbb{R} \to \mathbb{R}$, $h(x) = \begin{cases} \frac{1}{(x-1)^2(x-2)^2} & x \neq 1 \text{ or } 2 \\ 10 & x = 1 \text{ or } 2 \end{cases}$
h fonksiyonu $(-\infty,1)$, $(1,2)$ ve $(2,\infty)$ aralıklarında süreklidir. h nin $c = 1$ ve $c = 2$ de süreksizlikleri mevcuttur.

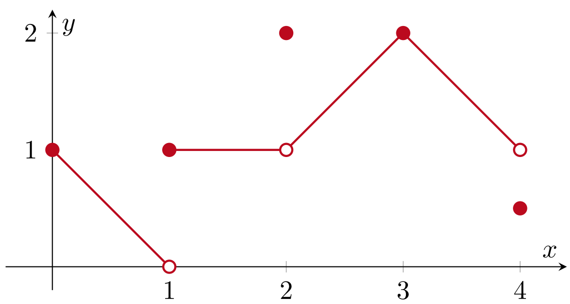


Figure 22.2: The function considered in example 22.1.
Şekil 22.2: Örnek 22.1 deki ele alınan fonksiyon.

Continuous Functions

Definition. $f : D \to \mathbb{R}$ is a ***continuous function*** if it is continuous at every $c \in D$.

Theorem 22.1. *If f and g are continuous at c, then $f+g$, $f-g$, kf $(k \in \mathbb{R})$, fg, $\frac{f}{g}$ (if $g(c) \neq 0$) and f^n $(n \in \mathbb{N})$ are all continuous at c. If $\sqrt[n]{f}$ is defined on $(c-\delta, c+\delta)$, then $\sqrt[n]{f}$ is also continuous at c $(n \in \mathbb{N})$.*

Example 22.5. Every polynominal

$$P(x) = a_n x^n + a_{n-1}x^{n-1} + \ldots + a_2 x^2 + a_1 x + a_0$$

is continuous.

Example 22.6. If

- P and Q are polynomials; and
- $Q(c) \neq 0$,

then $\frac{P(x)}{Q(x)}$ is continuous at c.

Example 22.7. $\sin x$ and $\cos x$ are continuous.

Sürekli Fonksiyonlar

Tanım. Her $c \in D$ noktasında sürekli olan bir $f : D \to \mathbb{R}$ fonksiyonuna ***sürekli fonksiyon*** denir .

Teorem 22.1. *Eğer f ve g fonksiyonları c'de sürekli iseler, o zaman $f+g$, $f-g$, kf $(k \in \mathbb{R})$, fg, $\frac{f}{g}$ $(g(c) \neq 0$ iken) ve f^n $(n \in \mathbb{N})$ fonksiyonlarının hepsi c'de süreklidir. Eğer $\sqrt[n]{f}$ fonksiyonu $(c-\delta, c+\delta)$ aralığında tanımlı ise, $\sqrt[n]{f}$ fonksiyonu da c'de süreklidir $(n \in \mathbb{N})$.*

Örnek 22.5. Her

$$P(x) = a_n x^n + a_{n-1}x^{n-1} + \ldots + a_2 x^2 + a_1 x + a_0$$

polinomu da süreklidir.

Örnek 22.6. If

- P ve Q polinomlar ve
- $Q(c) \neq 0$ ise,

o vakit $\frac{P(x)}{Q(x)}$ rasyonel fonksiyonu c'de süreklidir.

Örnek 22.7. $\sin x$ ve $\cos x$ sürekli fonksiyonlardır.

Composites

Bileşkeler

$$g \circ f(x)$$

$g \circ f(x)$ means $g(f(x))$.

Theorem 22.2. *If*

- *f is continuous at c; and*
- *g is continuous at $f(c)$,*

then $g \circ f$ is continuous at c.

Example 22.8. Show that $h(x) = \sqrt{x^2 - 2x - 5}$ is continuous on its domain.

solution: The function $g(t) = \sqrt{t}$ is continuous by Theorem 22.1. The function $f(x) = x^2 - 2x - 5$ is continuous because all polynomials are continuous. Therefore $h(x) = g \circ f(x)$ is continuous.

Example 22.9. Show that $\frac{x^{\frac{2}{3}}}{1+x^4}$ is continuous.

solution: $x^{\frac{2}{3}}$ and $1+x^4$ are continuous. Because $1+x^4 \neq 0$ for all x, we have that $\frac{x^{\frac{2}{3}}}{1+x^4}$ is continuous.

$g \circ f(x)$ demek $g(f(x))$ anlamındadır.

Teorem 22.2. *Eğer*

- *f fonksiyonu c'de sürekli ve*
- *g fonksiyonu da $f(c)$'de sürekli ise,*

bu durumda $g \circ f$ fonksiyonu da c'de süreklidir.

Örnek 22.8. $h(x) = \sqrt{x^2 - 2x - 5}$ fonksiyonunun tanım kümesind sürekli olduğunu gösteriniz.

çözüm: Teorem 22.1 den $g(t) = \sqrt{t}$ fonksiyonu süreklidir. $f(x) = x^2 - 2x - 5$ fonksiyonu da süreklidir çünkü bütün polinomlar süreklidir. Bundan ötürü $h(x) = g \circ f(x)$ sürekli olur.

Örnek 22.9. Gösteriniz ki $\frac{x^{\frac{2}{3}}}{1+x^4}$ süreklidir.

çözüm: $x^{\frac{2}{3}}$ ve $1+x^4$ süreklidir. Her x için, $1+x^4 \neq 0$ olduğundan , $\frac{x^{\frac{2}{3}}}{1+x^4}$ süreklidir.

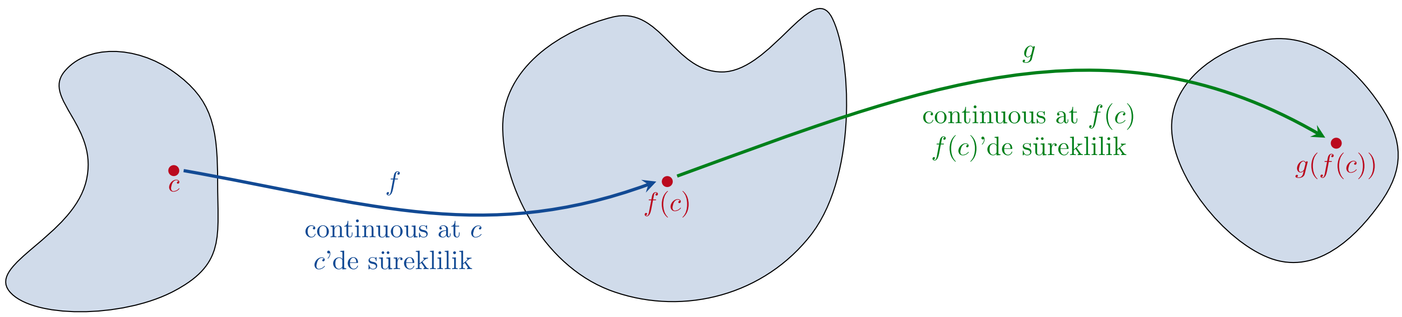


Figure 22.3: Composites of continuous functions are continuous.
Şekil 22.3: Sürekli fonsiyonların bileşkesi de süreklidir.

Theorem 22.3. *If*

- $g(x)$ *is continuous at* $x = b$*; and*
- $\lim_{x\to c} f(x) = b$,

then

$$\lim_{x\to c} g(f(x)) = g\Big(\lim_{x\to c} f(x)\Big).$$

Teorem 22.3. *Eğer*

- $g(x)$ *fonksiyonu* $x = b$ *de sürekli ve*
- $\lim_{x\to c} f(x) = b$ *ise,*

o halde

$$\lim_{x\to c} g(f(x)) = g\Big(\lim_{x\to c} f(x)\Big).$$

Example 22.10. By Theorem 22.3,

$$\begin{aligned}
&\lim_{x\to\frac{\pi}{2}} \cos\left[2x + \sin\left(\frac{3\pi}{2} + x\right)\right] \\
&\quad= \cos\left[\lim_{x\to\frac{\pi}{2}}\left(2x + \sin\left(\frac{3\pi}{2} + x\right)\right)\right] \\
&\quad= \cos\left[\lim_{x\to\frac{\pi}{2}}(2x) + \lim_{x\to\frac{\pi}{2}}\left(\sin\left(\frac{3\pi}{2} + x\right)\right)\right] \\
&\quad= \cos\left[\pi + \sin\left(\lim_{x\to\frac{\pi}{2}}\left(\frac{3\pi}{2} + x\right)\right)\right] \\
&\quad= \cos\left[\pi + \sin 2\pi\right] = \cos\left[\pi + 0\right] = -1.
\end{aligned}$$

Örnek 22.11. Teorem 22.3'den,

$$\begin{aligned}
&\lim_{x\to\frac{\pi}{2}} \tan\left[\frac{5x}{2} - \pi\cos\left(\frac{\pi}{2} - x\right)\right] \\
&\quad= \tan\left[\lim_{x\to\frac{\pi}{2}}\left(\frac{5x}{2} - \pi\cos\left(\frac{\pi}{2} - x\right)\right)\right] \\
&\quad= \tan\left[\lim_{x\to\frac{\pi}{2}}\left(\frac{5x}{2}\right) - \pi\lim_{x\to\frac{\pi}{2}}\left(\cos\left(\frac{\pi}{2} - x\right)\right)\right] \\
&\quad= \tan\left[\frac{5\pi}{4} - \pi\cos\left(\lim_{x\to\frac{\pi}{2}}\left(\frac{\pi}{2} - x\right)\right)\right] \\
&\quad= \tan\left[\frac{5\pi}{4} - \pi\cos 0\right] = \tan\left[\frac{5\pi}{4} - \pi\right] = \tan\frac{\pi}{4} = 1.
\end{aligned}$$

Problems

Problem 22.1. For what value(s) of b is

$$f(x) = \begin{cases} x & x < -2 \\ bx^2 & x \geq -2. \end{cases}$$

continuous at every x? Why?

Problem 22.2. Let

$$f(x) = \begin{cases} \frac{x^3-8}{x^2-4} & x \neq 2, x \neq -2 \\ 3 & x = 2 \\ 4 & x = -2. \end{cases}$$

(a). Show that f is continuous on $(-\infty, -2)$, on $(-2, 2)$ and on $(2, \infty)$.

(b). Show that f is continuous at $x = 2$.

(c). Show that f is discontinuous at $x = -2$.

Problem 22.3. Calculate $\lim_{t \to 0} \tan\left(\frac{\pi}{4} \cos\left(\sin t^{\frac{1}{3}}\right)\right)$.

Sorular

Soru 22.1. b'nin hangi değer(ler)i için,

$$f(x) = \begin{cases} x & x < -2 \\ bx^2 & x \geq -2. \end{cases}$$

her x noktasında süreklidir? Neden?

Soru 22.2. Farzedelim ki

$$f(x) = \begin{cases} \frac{x^3-8}{x^2-4} & x \neq 2, x \neq -2 \\ 3 & x = 2 \\ 4 & x = -2. \end{cases}$$

(a). f'nin $(-\infty, -2)$ de, $(-2, 2)$ de ve $(2, \infty)$ da sürekli olduğunu gösteriniz.

(b). f'nin $x = 2$'de sürekli olduğunu gösteriniz.

(c). f'nin $x = -2$'de süreksiz olduğunu gösteriniz.

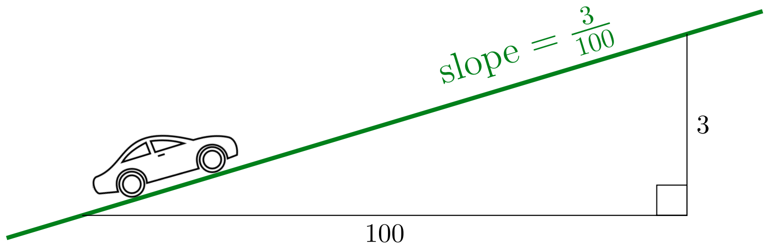
Soru 22.3. $\lim_{t \to 0} \tan\left(\frac{\pi}{4} \cos\left(\sin t^{\frac{1}{3}}\right)\right)$ limitini bulunuz.

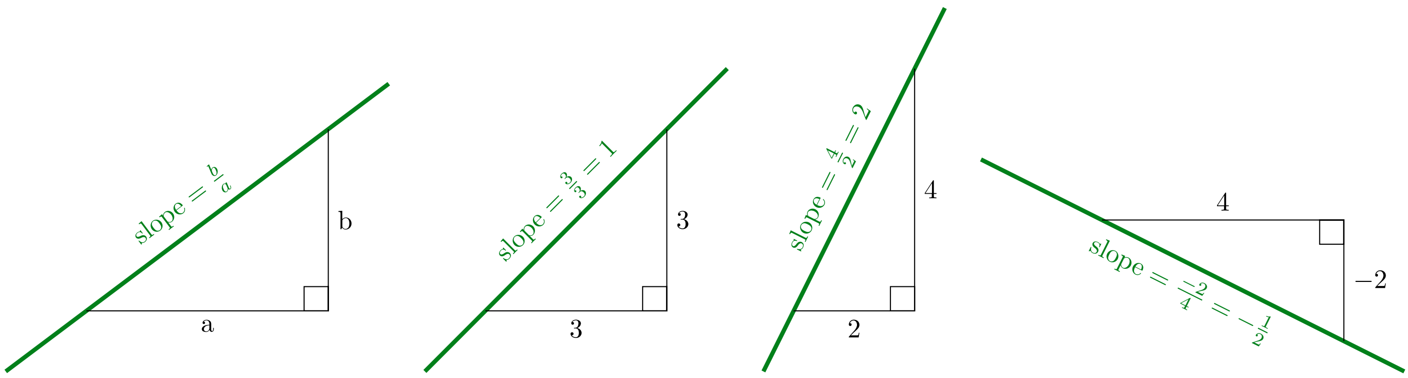
23

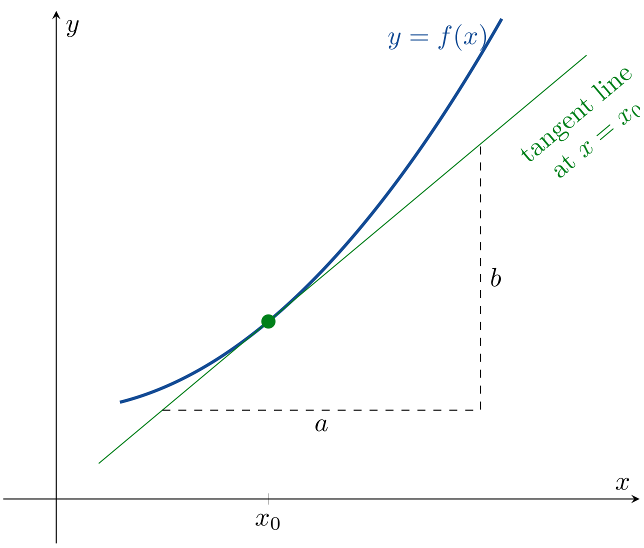
Differentiation

Türev

means





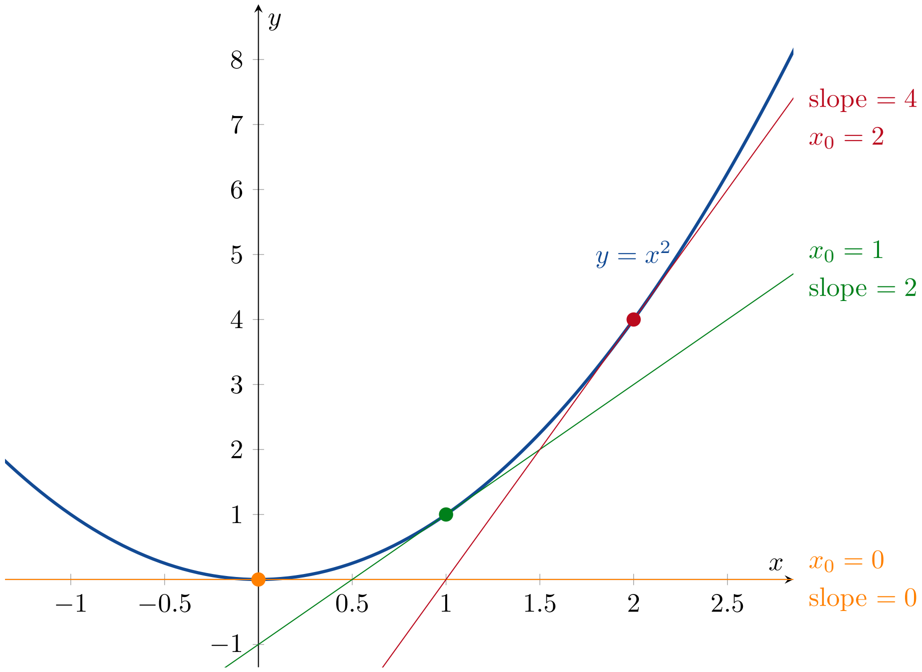


We can say that

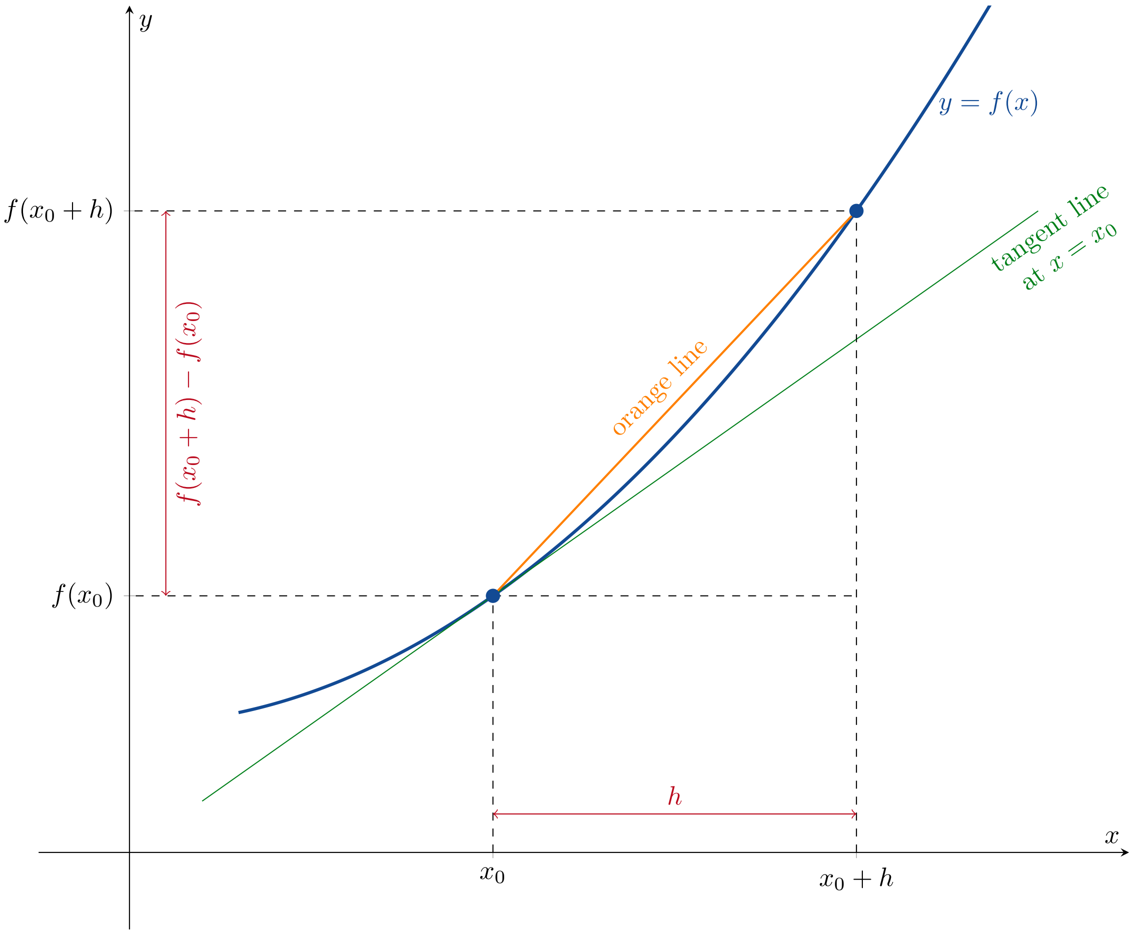
$$\begin{pmatrix}\text{slope of } y = f(x)\\ \text{at } x = x_0\end{pmatrix} = \begin{pmatrix}\text{slope of the tangent}\\ \text{line at } x = x_0\end{pmatrix}$$

Example 23.1.

Örnek 23.1.



The slope of $y = x^2$ at $x_0 = 0$ is 0.
The slope of $y = x^2$ at $x_0 = 1$ is 2.
The slope of $y = x^2$ at $x_0 = 2$ is 4.
How do we know this?



If h is very very small, then

$$\begin{pmatrix}\text{slope of the}\\ \text{tangent line}\end{pmatrix} \approx \begin{pmatrix}\text{slope of the}\\ \text{orange line}\end{pmatrix} = \frac{f(x_0+h)-f(x_0)}{h}$$

h çok ama çok küçükse, o zaman

$$\begin{pmatrix}\text{slope of the}\\ \text{tangent line}\end{pmatrix} \approx \begin{pmatrix}\text{slope of the}\\ \text{orange line}\end{pmatrix} = \frac{f(x_0+h)-f(x_0)}{h}$$

The Derivative of f

Definition. The ***derivative of a function f at a point x_0*** is

$$f'(x_0) = \lim_{h\to 0} \frac{f(x_0+h) - f(x_0)}{h}$$

<u>**if**</u> the limit exists.

(f' is pronounced "f prime")

Example 23.2. Consider the function $g(x) = \frac{1}{x}$, $x \neq 0$.

If $x_0 \neq 0$, then

$$\begin{aligned} g'(x_0) &= \lim_{h\to 0} \frac{g(x_0+h) - g(x_0)}{h} \\ &= \lim_{h\to 0} \frac{\frac{1}{x_0+h} - \frac{1}{x_0}}{h} \\ &= \lim_{h\to 0} \frac{\left(\frac{x_0}{x_0(x_0+h)} - \frac{x_0+h}{x_0(x_0+h)}\right)}{h} \\ &= \lim_{h\to 0} \frac{x_0 - x_0 - h}{hx_0(x_0+h)} \\ &= \lim_{h\to 0} \frac{-1}{x_0(x_0+h)} \\ &= -\frac{1}{x_0^2}. \end{aligned}$$

See figure 23.1.

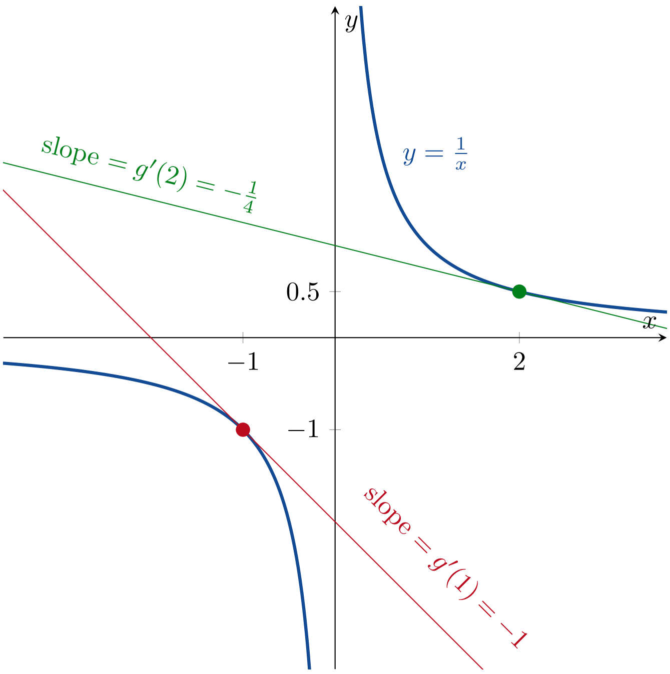


Figure 23.1: The graph of $g(x) = \frac{1}{x}$, $x \neq 0$ and two tangents to this graph.
Şekil 23.1: $g(x) = \frac{1}{x}$, $x \neq 0$ grafiği ve buna teğet iki doğru.

Definition. If $f'(x_0)$ exists, we say that ***f is differentiable at x_0***.

Definition. Let $f : D \to \mathbb{R}$ be a function. If f is differentiable at every $x_0 \in D$, we say that f is ***differentiable***.

f Türevi

Tanım. Bir ***f fonksiyonunun x_0 noktasındaki türevi*** limitin mevcut olması koşuyla

$$f'(x_0) = \lim_{h\to 0} \frac{f(x_0+h) - f(x_0)}{h}$$

olarak tanımlanır.

(f' sembolü "f üssü" olarak okunur)

Örnek 23.3. $g(x) = \frac{1}{x}$, $x \neq 0$ fonksiyonunu ele alalım.

$x_0 \neq 0$ ise,

$$\begin{aligned} g'(x_0) &= \lim_{h\to 0} \frac{g(x_0+h) - g(x_0)}{h} \\ &= \lim_{h\to 0} \frac{\frac{1}{x_0+h} - \frac{1}{x_0}}{h} \\ &= \lim_{h\to 0} \frac{\left(\frac{x_0}{x_0(x_0+h)} - \frac{x_0+h}{x_0(x_0+h)}\right)}{h} \\ &= \lim_{h\to 0} \frac{x_0 - x_0 - h}{hx_0(x_0+h)} \\ &= \lim_{h\to 0} \frac{-1}{x_0(x_0+h)} \\ &= -\frac{1}{x_0^2}. \end{aligned}$$

Bkz. şekil 23.1.

Tanım. $f'(x_0)$ mevcutsa, ***f fonksiyonu x_0'da türevlenebilirdir*** deriz.

Tanım. $f : D \to \mathbb{R}$ bir fonksiyon olsun. f her $x_0 \in D$ noktasında türevlenebilir ise, f bir ***türevlenebilir*** fonksiyondur deriz.

$f : D \to \mathbb{R}$ türevlenebilir ise, elimizde yeni bir $f' : D \to \mathbb{R}$ fonksiyonu olur.

Tanım. f' fonksiyonuna f'nin ***türevi*** denir.

Örnek 23.4. $f(x) = \frac{x}{x-1}$'nin türevini bulunuz.

çözüm: İlk olarak $f(x+h) = \frac{x+h}{x+h-1}$. Buradan

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h\to 0} \frac{\frac{x+h}{x+h-1} - \frac{x}{x-1}}{h} \\ &= \lim_{h\to 0} \frac{1}{h}\left(\frac{(x+h)(x-1) - x(x+h-1)}{(x-1)(x+h-1)}\right) \\ &= \lim_{h\to 0} \frac{1}{h}\left(\frac{-h}{(x-1)(x+h-1)}\right) \\ &= \lim_{h\to 0} \frac{-1}{(x-1)(x+h-1)} \\ &= \frac{-1}{(x-1)(x+0-1)} \\ &= \frac{-1}{(x-1)^2} \end{aligned}$$

buluruz.

If $f : D \to \mathbb{R}$ is differentiable, then we have a new function $f' : D \to \mathbb{R}$.

Definition. f' is called the ***derivative*** of f.

Example 23.3. Differentiate $f(x) = \frac{x}{x-1}$.

solution: First note that $f(x+h) = \frac{x+h}{x+h-1}$. Therefore

$$\begin{aligned}
f'(x) &= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} \\
&= \lim_{h\to 0} \frac{\frac{x+h}{x+h-1} - \frac{x}{x-1}}{h} \\
&= \lim_{h\to 0} \frac{1}{h}\left(\frac{(x+h)(x-1)-x(x+h-1)}{(x-1)(x+h-1)}\right) \\
&= \lim_{h\to 0} \frac{1}{h}\left(\frac{-h}{(x-1)(x+h-1)}\right) \\
&= \lim_{h\to 0} \frac{-1}{(x-1)(x+h-1)} \\
&= \frac{-1}{(x-1)(x+0-1)} \\
&= \frac{-1}{(x-1)^2}.
\end{aligned}$$

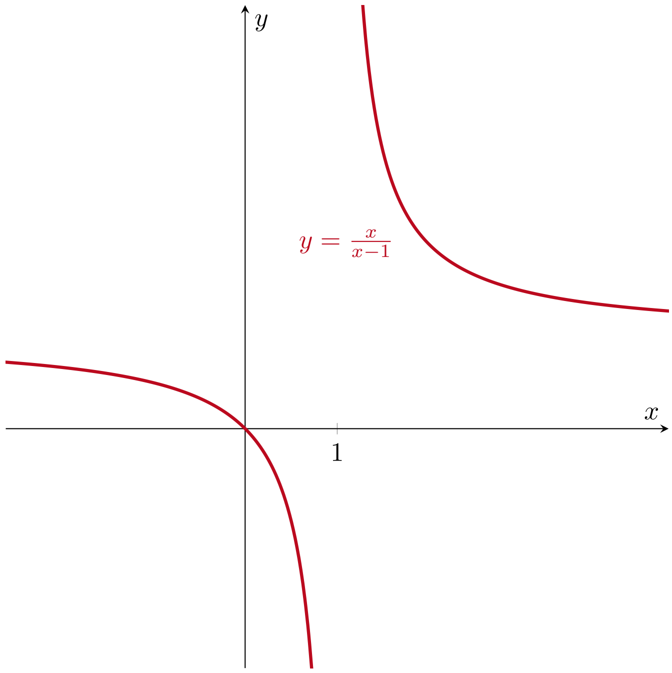


Figure 23.2: The graph of $y = \frac{x}{x-1}$.
Şekil 23.2: $y = \frac{x}{x-1}$'in grafiği

Notations

There are many ways to write the derivative of $y = f(x)$.

$$f'(x) = y' = \frac{dy}{dx} = \frac{df}{dx} = \frac{d}{dx}f(x) = \dot{y} = \dot{f}(x)$$

"the derivative of y with respect to x"

Calculus was started by two men who hated each other: Sir Isaac Newton (UK, 1642-1726) used $\dot{f}$ and $\dot{y}$. Gottfried Leibniz (GER, 1646-1716) used $\frac{df}{dx}$ and $\frac{dy}{dx}$. The f' and y' notation came later from Joseph-Louis Lagrange (ITA, 1736-1813).

If we want the derivative of $y = f(x)$ at the point $x = x_0$, we can write

$$f'(x_0) = \left.\frac{dy}{dx}\right|_{x=x_0} = \left.\frac{df}{dx}\right|_{x=x_0} = \left.\frac{d}{dx}f(x)\right|_{x=x_0}$$

"the derivative of y with respect to x at $x = x_0$"

For example, if $u(x) = \frac{1}{x}$, then

$$u'(4) = \left.\frac{d}{dx}\left(\frac{1}{x}\right)\right|_{x=4} = \left.\frac{-1}{x^2}\right|_{x=4} = \frac{-1}{4^2} = \frac{-1}{16}.$$

Notasyon

$y = f(x)$'nin türevini yazmanın birçok yolu vardır.

$$f'(x) = y' = \frac{dy}{dx} = \frac{df}{dx} = \frac{d}{dx}f(x) = \dot{y} = \dot{f}(x)$$

"y nin x'e göre türevi"

Calculus birbirinden nefret eden iki kişi tarafından başladı: Sir Isaac Newton (İngiltere, 1642-1726) $\dot{f}$ ve $\dot{y}$ kullandı. Gottfried Leibniz (Almanya, 1646-1716) $\frac{df}{dx}$ ve $\frac{dy}{dx}$ sembollerini kullandı. F' ve y' gösterimi daha sonra Joseph-Louis Lagrange'den (İtalya, 1736-1813) tarafından ilk kullanıldı.

$y = f(x)$'nin $x = x_0$'daki türevini bulmak için, şöyle yazarız

$$f'(x_0) = \left.\frac{dy}{dx}\right|_{x=x_0} = \left.\frac{df}{dx}\right|_{x=x_0} = \left.\frac{d}{dx}f(x)\right|_{x=x_0}$$

"y'nin x'e göre $x = x_0$'daki türevi"

Örneğin, $u(x) = \frac{1}{x}$ ise, o zaman

$$u'(4) = \left.\frac{d}{dx}\left(\frac{1}{x}\right)\right|_{x=4} = \left.\frac{-1}{x^2}\right|_{x=4} = \frac{-1}{4^2} = \frac{-1}{16}.$$

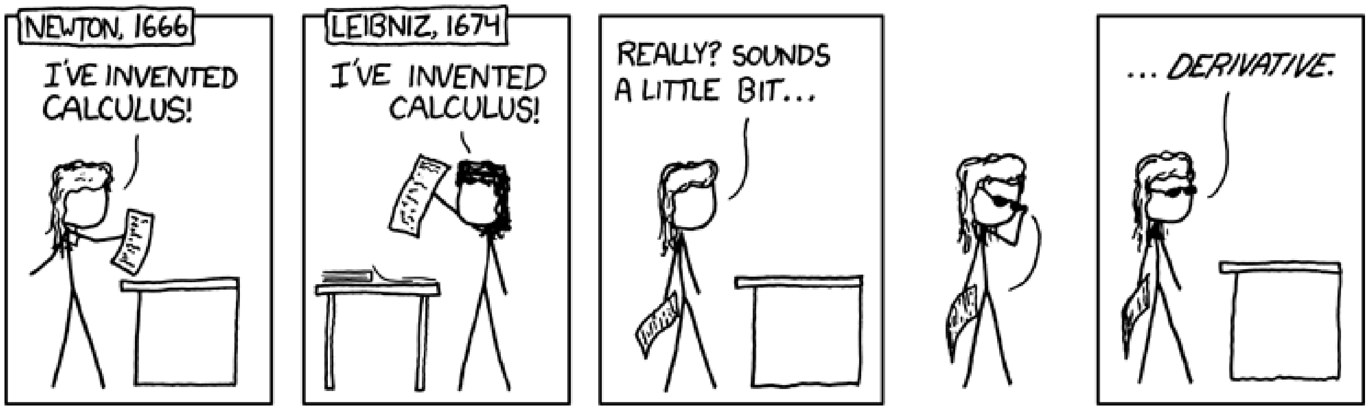


Figure 23.3: A web comic taken from https://xkcd.com/626/ .
Şekil 23.3: https://xkcd.com/626/ adresinden alınan bir web çizgi romanı.

Example 23.4. Show that $f(x) = |x|$ is differentiable on $(-\infty, 0)$ and on $(0, \infty)$, but is not differentiable at $x = 0$.

solution: If $x > 0$ then

$$\frac{df}{dx} = \frac{d}{dx}(|x|) = \frac{d}{dx}(x) = \lim_{h\to 0} \frac{(x+h)-x}{h} = \lim_{h\to 0} 1 = 1.$$

Similarly, if $x < 0$ then

$$\begin{aligned}\frac{df}{dx} &= \frac{d}{dx}(|x|) = \frac{d}{dx}(-x) = \lim_{h\to 0} \frac{(-x-h)-(-x)}{h} \\ &= \lim_{h\to 0} -1 = -1.\end{aligned}$$

Therefore f is differentiable on $(-\infty, 0)$ and on $(0, \infty)$.

Since $\lim\limits_{h\to 0} \frac{|0+h|-|0|}{h} = \lim\limits_{h\to 0} \frac{|h|}{h} = \lim\limits_{h\to 0}(\pm 1)$ does not exist, f is not differentiable at 0.

See figure 23.4.

Örnek 23.5. $f(x) = |x|$'nin $(-\infty, 0)$ ve $(0, \infty)$ aralıklarında türevlenebilir ama $x = 0$'da türevlenebilir olmadığını gösteriniz.

çözüm: $x > 0$ ise o vakit

$$\frac{df}{dx} = \frac{d}{dx}(|x|) = \frac{d}{dx}(x) = \lim_{h\to 0} \frac{(x+h)-x}{h} = \lim_{h\to 0} 1 = 1.$$

Benzer olarak, $x < 0$ ise o halde

$$\begin{aligned}\frac{df}{dx} &= \frac{d}{dx}(|x|) = \frac{d}{dx}(-x) = \lim_{h\to 0} \frac{(-x-h)-(-x)}{h} \\ &= \lim_{h\to 0} -1 = -1.\end{aligned}$$

Yani f foksiyonu $(-\infty, 0)$ ve $(0, \infty)$'da türevlenebilirdir.

$\lim\limits_{h\to 0} \frac{|0+h|-|0|}{h} = \lim\limits_{h\to 0} \frac{|h|}{h} = \lim\limits_{h\to 0}(\pm 1)$ mevcut olmadığından, f 0'da türevlenenemez.

Bkz. şekil 23.4.

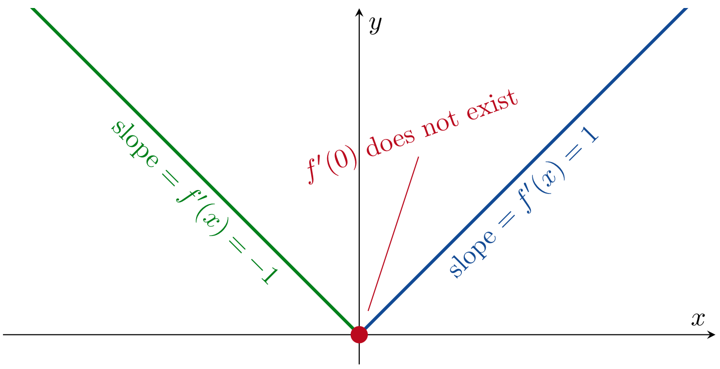
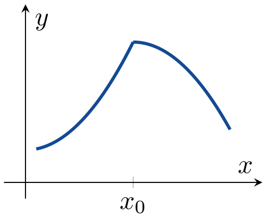


Figure 23.4: The graph of $y = |x|$.
Şekil 23.4: $y = |x|$'in grafiği.

When Does a Function Not Have a Derivative at a Point?

Hangi Durumlarda Bir Fonksiyonun Türevi Yoktur?

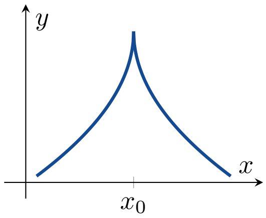


a corner

$f'(x_0)$ does not exist

köşe durumu

$f'(x_0)$ mevcut değil

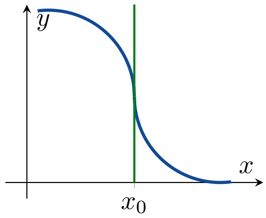


a cusp

$f'(x_0)$ does not exist

içten bükülme

$f'(x_0)$ mevcut değil

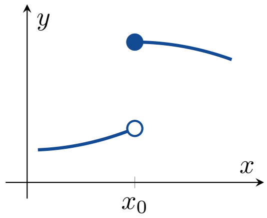


a vertical tangent

$f'(x_0)$ does not exist

dikey teğet

$f'(x_0)$ mevcut değil

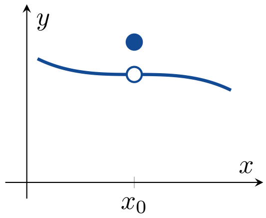


a discontinuity

$f'(x_0)$ does not exist

süreksizlik

$f'(x_0)$ mevcut değil



a discontinuity

$f'(x_0)$ does not exist

süreksizlik

$f'(x_0)$ mevcut değil

Theorem 23.1.

$$\left(\begin{array}{c} f \textit{ has a derivative} \\ \textit{at } x = x_0 \end{array}\right) \implies \left(\begin{array}{c} f \textit{ is continuous} \\ \textit{at } x = x_0 \end{array}\right)$$

Teorem 23.1.

$$\left(\begin{array}{c} f\textit{'nin at } x = x_0 \textit{ da} \\ \textit{türevi mevcut} \end{array}\right) \implies \left(\begin{array}{c} f,\ x = x_0\textit{'da} \\ \textit{sürekli} \end{array}\right)$$

24 Differentiation Rules

Constant Function

If $k \in \mathbb{R}$, then

$$\boxed{\frac{d}{dx}(k) = 0.}$$

Power Function

If $n \in \mathbb{R}$, then

$$\boxed{\frac{d}{dx}\left(x^n\right) = nx^{n-1}.}$$

Example 24.1.

$$\frac{d}{dx}\left(x^3\right) = 3x^{3-1} = 3x^2$$

Example 24.2.

$$\frac{d}{dx}\left(\sqrt{x}\right) = \frac{d}{dx}\left(x^{\frac{1}{2}}\right) = \frac{1}{2}x^{\frac{1}{2}-1} = \frac{1}{2}x^{-\frac{1}{2}} = \frac{1}{2\sqrt{x}}$$

Example 24.3.

$$\frac{d}{dx}\left(\frac{1}{x^4}\right) = \frac{d}{dx}\left(x^{-4}\right) = -4x^{-4-1} = -4x^{-5} = -\frac{4}{x^5}$$

The Constant Multiple Rule

If $u(x)$ is differentiable and $k \in \mathbb{R}$, then

$$\boxed{\frac{d}{dx}(ku) = k\frac{du}{dx}.}$$

Proof.

$$\begin{aligned}\frac{d}{dx}(ku) &= \lim_{h\to 0}\frac{ku(x+h)-ku(x)}{h}\\ &= k\lim_{h\to 0}\frac{u(x+h)-u(x)}{h} = k\frac{du}{dx}\end{aligned}$$

□

Example 24.4.

$$\frac{d}{dx}\left(3x^2\right) = 3\frac{d}{dx}\left(x^2\right) = 3\times 2x = 6x$$

Example 24.5.

$$\frac{d}{dx}(-u) = \frac{d}{dx}(-1\times u) = -1\times\frac{du}{dx} = -\frac{du}{dx}$$

Türev Kuralları

Sabit Fonksiyon

$k \in \mathbb{R}$ ise, o halde

$$\boxed{\frac{d}{dx}(k) = 0.}$$

Kuvvet Fonksiyonu

$n \in \mathbb{R}$ ise, bu durumda

$$\boxed{\frac{d}{dx}\left(x^n\right) = nx^{n-1}.}$$

Örnek 24.1.

$$\frac{d}{dx}\left(x^3\right) = 3x^{3-1} = 3x^2$$

Örnek 24.2.

$$\frac{d}{dx}\left(\sqrt{x}\right) = \frac{d}{dx}\left(x^{\frac{1}{2}}\right) = \frac{1}{2}x^{\frac{1}{2}-1} = \frac{1}{2}x^{-\frac{1}{2}} = \frac{1}{2\sqrt{x}}$$

Örnek 24.3.

$$\frac{d}{dx}\left(\frac{1}{x^4}\right) = \frac{d}{dx}\left(x^{-4}\right) = -4x^{-4-1} = -4x^{-5} = -\frac{4}{x^5}$$

Sabitle Çarpım Kuralı

$u(x)$ türevlenebilir ve $k \in \mathbb{R}$ ise,

$$\boxed{\frac{d}{dx}(ku) = k\frac{du}{dx}.}$$

Kanıt.

$$\begin{aligned}\frac{d}{dx}(ku) &= \lim_{h\to 0}\frac{ku(x+h)-ku(x)}{h}\\ &= k\lim_{h\to 0}\frac{u(x+h)-u(x)}{h} = k\frac{du}{dx}\end{aligned}$$

□

Örnek 24.4.

$$\frac{d}{dx}\left(3x^2\right) = 3\frac{d}{dx}\left(x^2\right) = 3\times 2x = 6x$$

Örnek 24.5.

$$\frac{d}{dx}(-u) = \frac{d}{dx}(-1\times u) = -1\times\frac{du}{dx} = -\frac{du}{dx}$$

The Sum Rule

If $u(x)$ and $v(x)$ are differentiable at x_0, then $u+v$ is also differentiable at x_0 and

$$\boxed{\frac{d}{dx}(u+v) = \frac{du}{dx} + \frac{dv}{dx}.}$$

Example 24.6. Differentiate $y = x^3 + \frac{4}{3}x^2 - 5x + 1$.

solution:

$$\begin{aligned}\frac{dy}{dx} &= \frac{d}{dx}\left(x^3 + \frac{4}{3}x^2 - 5x + 1\right)\\ &= \frac{d}{dx}\left(x^3\right) + \frac{d}{dx}\left(\frac{4}{3}x^2\right) - \frac{d}{dx}(5x) + \frac{d}{dx}(1)\\ &= 3x^2 + \frac{8}{3}x - 5 + 0\end{aligned}$$

Example 24.7. Does the curve $y = x^4 - 2x^2 + 2$ have any points where $\frac{dy}{dx} = 0$? If so, where?

solution: Since

$$\frac{dy}{dx} = 4x^3 - 4x = 4x(x^2-1) = 4x(x-1)(x+1),$$

we can see that $\frac{dy}{dx} = 0$ if and only if $x = -1$, 0 or 1. See figure 24.1.

Toplam Kuralı

$u(x)$ ve $v(x)$ fonksiyonları x_0'da türevlenebilirlerse, $u+v$'de x_0 türevlenebilirdir ve

$$\boxed{\frac{d}{dx}(u+v) = \frac{du}{dx} + \frac{dv}{dx}.}$$

Örnek 24.6. $y = x^3 + \frac{4}{3}x^2 - 5x + 1$ fonksiyonunun türevini bulunuz.

çözüm:

$$\begin{aligned}\frac{dy}{dx} &= \frac{d}{dx}\left(x^3 + \frac{4}{3}x^2 - 5x + 1\right)\\ &= \frac{d}{dx}\left(x^3\right) + \frac{d}{dx}\left(\frac{4}{3}x^2\right) - \frac{d}{dx}(5x) + \frac{d}{dx}(1)\\ &= 3x^2 + \frac{8}{3}x - 5 + 0\end{aligned}$$

Örnek 24.7. $y = x^4 - 2x^2 + 2$ eğrisi üzerinde $\frac{dy}{dx} = 0$ olan nokta(lar) var mıdır? Varsa, nelerdir?

çözüm:

$$\frac{dy}{dx} = 4x^3 - 4x = 4x(x^2-1) = 4x(x-1)(x+1),$$

olduğundan şunu gözlemleyebiliriz $\frac{dy}{dx} = 0$ ancak ve ancak $x = -1$, 0 veya 1 olur. Bkz. şekil 24.1.

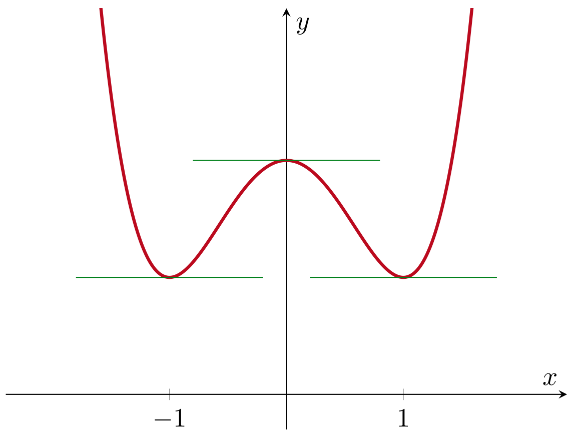


Figure 24.1: The graph of $y = x^4 - 2x^2 + 2$.
Şekil 24.1: $y = x^4 - 2x^2 + 2$'nin grafiği.

The Product Rule

If $u(x)$ and $v(x)$ are differentiable at x_0, then $u(x)v(x)$ is also differentiable at x_0 and

$$\boxed{\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}.}$$

Using prime notation, the product rule is

$$\boxed{(uv)' = u'v + uv'.}$$

Example 24.8. Differentiate $y = (x^2+1)(x^3+3)$.

solution 1: We have $y = uv$ with $u = x^2+1$ and $v = x^3+3$. So

$$\begin{aligned}\frac{dy}{dx} &= (x^2+1)'(x^3+3) + (x^2+1)(x^3+3)' \\ &= (2x+0)(x^3+3) + (x^2+1)(3x^2+0) \\ &= 2x^4 + 6x + 3x^4 + 3x^2 \\ &= 5x^4 + 3x^2 + 6x.\end{aligned}$$

solution 2: Since

$$y = (x^2+1)(x^3+3) = x^5 + x^3 + 3x^2 + 3,$$

we have that

$$\frac{dy}{dx} = 5x^4 + 3x^2 + 6x + 0.$$

Çarpım Kuralı

$u(x)$ ve $v(x)$ fonksiyonlarla x_0'da türevlenebilirlerse, $u(x)v(x)$ fonksiyonu da x_0 türevlenebilirdir ve

$$\boxed{\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}.}$$

Üs notasyonu kullanarak, çarpım kuralı da

$$\boxed{(uv)' = u'v + uv'.}$$

Örnek 24.8. $y = (x^2+1)(x^3+3)$ fonksiyonunun türevini bulunuz.

çözüm 1: Elimizde şunlar var: $y = uv$ ile $u = x^2+1$ ve $v = x^3+3$. Yani

$$\begin{aligned}\frac{dy}{dx} &= (x^2+1)'(x^3+3) + (x^2+1)(x^3+3)' \\ &= (2x+0)(x^3+3) + (x^2+1)(3x^2+0) \\ &= 2x^4 + 6x + 3x^4 + 3x^2 \\ &= 5x^4 + 3x^2 + 6x.\end{aligned}$$

çözüm 2:

$$y = (x^2+1)(x^3+3) = x^5 + x^3 + 3x^2 + 3$$

olduğundan,

$$\frac{dy}{dx} = 5x^4 + 3x^2 + 6x + 0$$

buluruz.

The Quotient Rule

If $u(x)$ and $v(x)$ are differentiable at x_0 and if $v(x_0) \neq 0$, then $\frac{u}{v}$ is also differentiable at x_0 and

$$\boxed{\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{u'v - uv'}{v^2}.}$$

Example 24.9. Differentiate $y = \frac{t^2-1}{t^3+1}$.

solution: We have $y = \frac{u}{v}$ with $u = t^2-1$ and $v = t^3+1$. Therefore

$$\begin{aligned}\frac{dy}{dt} &= \frac{u'v - uv'}{v^2} \\ &= \frac{(t^2-1)'(t^3+1) - (t^2-1)(t^3+1)'}{(t^3+1)^2} \\ &= \frac{(2t)(t^3+1) - (t^2-1)(3t^2)}{(t^3+1)^2} \\ &= \frac{2t^4 + 2t - 3t^4 + 3t^2}{(t^3+1)^2} \\ &= \frac{-t^4 + 3t^2 + 2t}{(t^3+1)^2}.\end{aligned}$$

Bölüm Kuralı

Eğer $u(x)$ ve $v(x)$ fonksiyonları x_0'da türevlenebilirlerse ve $v(x_0) \neq 0$ ise, o zaman $\frac{u}{v}$ fonksiyonu da x_0'da türevlenebilirdir ve türevi de şyledir:

$$\boxed{\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{u'v - uv'}{v^2}.}$$

Örnek 24.9. $y = \frac{t^2-1}{t^3+1}$ fonksiyonunun türevini alınız.

çözüm: $u = t^2-1$ ve $v = t^3+1$ olmak üzere $y = \frac{u}{v}$ olsun. Buradan

$$\begin{aligned}\frac{dy}{dt} &= \frac{u'v - uv'}{v^2} \\ &= \frac{(t^2-1)'(t^3+1) - (t^2-1)(t^3+1)'}{(t^3+1)^2} \\ &= \frac{(2t)(t^3+1) - (t^2-1)(3t^2)}{(t^3+1)^2} \\ &= \frac{2t^4 + 2t - 3t^4 + 3t^2}{(t^3+1)^2} \\ &= \frac{-t^4 + 3t^2 + 2t}{(t^3+1)^2}\end{aligned}$$

buluruz.

Example 24.10. Differentiate $f(s) = \frac{\sqrt{s}-1}{\sqrt{s}+1}$.

solution: We have $f(s) = \frac{u}{v}$ with $u = \sqrt{s} - 1$ and $v = \sqrt{s} + 1$. Remember that $\frac{d}{ds}\left(\sqrt{s}\right) = \frac{1}{2\sqrt{s}}$. Therefore

$$\begin{aligned} \frac{df}{ds} &= \frac{u'v - uv'}{v^2} \\ &= \frac{(\sqrt{s}-1)'(\sqrt{s}+1) - (\sqrt{s}-1)(\sqrt{s}+1)'}{\left(\sqrt{s}+1\right)^2} \\ &= \frac{\left(\frac{1}{2\sqrt{s}}\right)(\sqrt{s}+1) - (\sqrt{s}-1)\left(\frac{1}{2\sqrt{s}}\right)}{\left(\sqrt{s}+1\right)^2} \\ &= \frac{\frac{1}{2} + \frac{1}{2\sqrt{s}} - \frac{1}{2} + \frac{1}{2\sqrt{s}}}{\left(\sqrt{s}+1\right)^2} \\ &= \frac{1}{\sqrt{s}\left(\sqrt{s}+1\right)^2}. \end{aligned}$$

Örnek 24.10. $f(s) = \frac{\sqrt{s}-1}{\sqrt{s}+1}$ fonksiyonunun türevini bulunuz.

çözüm: $f(s) = \frac{u}{v}$ olsun burada $u = \sqrt{s} - 1$ ve $v = \sqrt{s} + 1$. Unutmayınız ki $\frac{d}{ds}\left(\sqrt{s}\right) = \frac{1}{2\sqrt{s}}$. Dolayısıyla

$$\begin{aligned} \frac{df}{ds} &= \frac{u'v - uv'}{v^2} \\ &= \frac{(\sqrt{s}-1)'(\sqrt{s}+1) - (\sqrt{s}-1)(\sqrt{s}+1)'}{\left(\sqrt{s}+1\right)^2} \\ &= \frac{\left(\frac{1}{2\sqrt{s}}\right)(\sqrt{s}+1) - (\sqrt{s}-1)\left(\frac{1}{2\sqrt{s}}\right)}{\left(\sqrt{s}+1\right)^2} \\ &= \frac{\frac{1}{2} + \frac{1}{2\sqrt{s}} - \frac{1}{2} + \frac{1}{2\sqrt{s}}}{\left(\sqrt{s}+1\right)^2} \\ &= \frac{1}{\sqrt{s}\left(\sqrt{s}+1\right)^2} \end{aligned}$$

buluruz.

Second Order Derivatives

If $y = f(x)$ is a differentiable function, then $f'(x)$ is also a function. If $f'(x)$ is also differentiable, then we can differentiate to find a new function called f'' ("f double prime"). f'' is called the ***second derivative*** of f. We can write

$$f''(x) = \frac{d}{dx}f'(x) = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \underbrace{\frac{d^2y}{dx^2}}_{\text{“}d\text{ squared }y,\ dx\text{ squared”}} = \frac{dy'}{dx} = y''$$

Example 24.11. If $y = x^6$, then $y' = \frac{d}{dx}\left(x^6\right) = 6x^5$ and $y'' = \frac{d}{dx}\left(y'\right) = \frac{d}{dx}\left(6x^5\right) = 30x^4$. Equivalently, we can write

$$\frac{d^2}{dx^2}\left(x^6\right) = \frac{d}{dx}\left(\frac{d}{dx}\left(x^6\right)\right) = \frac{d}{dx}\left(6x^5\right) = 30x^4.$$

İkinci Mertebeden Türevler

$y = f(x)$ türevlenebilir bir fonksiyon ise, o zaman $f'(x)$ de bir fonksiyondur. $f'(x)$ de türevlenebilir ise, bu durumda yine türev alır ve yeni bir f'' ("f iki üssü") fonksiyonu buluruz. f'' fonksiyonuna f'nin ***ikinci türevi*** denir. Şöyle dösteririz

$$f''(x) = \frac{d}{dx}f'(x) = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \underbrace{\frac{d^2y}{dx^2}}_{\text{“}d\text{ kare }y\text{ bölü }dx\text{ kare”}} = \frac{dy'}{dx} = y''$$

Örnek 24.11. $y = x^6$ ise, $y' = \frac{d}{dx}\left(x^6\right) = 6x^5$ ve $y'' = \frac{d}{dx}\left(y'\right) = \frac{d}{dx}\left(6x^5\right) = 30x^4$. Buna eşdeğer olarak,

$$\frac{d^2}{dx^2}\left(x^6\right) = \frac{d}{dx}\left(\frac{d}{dx}\left(x^6\right)\right) = \frac{d}{dx}\left(6x^5\right) = 30x^4$$

yazabiliriz.

Higher Order Derivatives

If f'' is differentiable, then its derivative $f''' = \frac{d^3f}{dx^3}$ is the ***third derivative*** of f.
If f''' is differentiable, then its derivative $f^{(4)} = \frac{d^4f}{dx^4}$ is the ***fourth derivative*** of f.
If $f^{(4)}$ is differentiable, then its derivative $f^{(5)} = \frac{d^5f}{dx^5}$ is the ***fifth derivative*** of f.

$$\vdots$$

If $f^{(n-1)}$ is differentiable, then its derivative $f^{(n)} = \frac{d^nf}{dx^n}$ is the ***nth derivative*** of f.

Example 24.12. Find the first four derivatives of $y = x^3 - 3x^2 + 2$.

solution:
First derivative: $y' = 3x^2 - 6x$
Second derivative: $y'' = 6x - 6$
Third derivative: $y''' = 6$
Fourth derivative: $y^{(4)} = 0$.

(Note that since $\frac{d}{dx}(0) = 0$, if $n \geq 4$ then $y^{(n)} = 0$ also.)

Yüksek Mertebeden Türevler

f'' türevlenebilir ise, türevi olan $f''' = \frac{d^3f}{dx^3}$ fonksiyona f'nin ***üçüncü türevi*** denir.
f''' türevlenebilir ise, türevi olan $f^{(4)} = \frac{d^4f}{dx^4}$ fonksiyonuna f'nin ***dördüncü türevi*** denir.
$f^{(4)}$ türevlenebilir ise, türevi olan $f^{(5)} = \frac{d^5f}{dx^5}$ fonksiyonuna f'nin ***beşinci türevi***.

$$\vdots$$

$f^{(n-1)}$ türevlenebilir ise, türevi olan $f^{(n)} = \frac{d^nf}{dx^n}$ fonksiyonuna f'nin ***n inci türevi*** denir.

Örnek 24.12. $y = x^3 - 3x^2 + 2$ ise, ilk dört mertebeden türevlerini bulunuz.

çözüm:
Birinci mertebeden türev: $y' = 3x^2 - 6x$
İkinci mertebeden türev: $y'' = 6x - 6$
Üçüncü mertebeden türev: $y''' = 6$
Dördüncü inci mertebeden türev: $y^{(4)} = 0$.

($\frac{d}{dx}(0) = 0$ olsuğundan, $n \geq 4$ ise $y^{(n)} = 0$ olduğunu unutmayınız.)

Problems

Problem 24.1.

(a). Find $\dfrac{ds}{dt}$ if $s = -2t^{-1} + \dfrac{4}{t^2}$.

(b). Find w'' if $w = (z+1)(z-1)(z^2+1)$.

(c). Find $\dfrac{dy}{dx}$ if $y = (2x+3)(x^4 + \frac{1}{3}x^3 + 11)$.

Problem 24.2. Find $\dfrac{db}{dx}$ if $b = \dfrac{x^2-1}{x^2+x-2}$.

Problem 24.3. Find the derivatives of the functions below:

Sorular

Soru 24.1.

(a). $s = -2t^{-1} + \dfrac{4}{t^2}$ ise $\dfrac{ds}{dt}$ yi bulunuz.

(b). $w = (z+1)(z-1)(z^2+1)$ ise w'''yü bulunuz.

(c). $y = (2x+3)(x^4 + \frac{1}{3}x^3 + 11)$ ise $\dfrac{dy}{dx}$'i bulunuz.

Soru 24.2. $b = \dfrac{x^2-1}{x^2+x-2}$ ise $\dfrac{db}{dx}$'i bulunuz.

Soru 24.3. Aşağıdaki fonksiyonların türevlerini bulunuz:

(a). $y = \dfrac{x^4}{2} - \dfrac{3}{2}x^2 - x$

(b). $y = (x-1)(x^2+3x-5)$

(c). $r = \dfrac{1}{3s^2} - \dfrac{5}{2s}$

(d). $y = \dfrac{2x+5}{3x-2}$

(e). $g(x) = \dfrac{x^2-4}{x+0.5}$

(f). $v = (1-t)(1+t^2)^{-1}$

(g). $f(s) = \dfrac{\sqrt{s}-1}{\sqrt{s}+1}$

(h). $v = \dfrac{1+x-4\sqrt{x}}{x}$

(i). $r = \dfrac{(\theta-1)(\theta^2+\theta+1)}{\theta^3}$

(j). $w = \left(\dfrac{1+3z}{3z}\right)(3-z)$

(k). $s = 5t^3 - 3t^5$

(l). $w = 3z^{-2} - \dfrac{1}{z}$

25

Derivatives of Trigonometric Functions

Sine and Cosine

$$\boxed{\frac{d}{dx}(\sin x) = \cos x}$$

Example 25.1. Differentiate $y = x^2 - \sin x$.

solution:

$$\frac{dy}{dx} = \frac{d}{dx}\left(x^2\right) - \frac{d}{dx}(\sin x) = 2x - \cos x.$$

Example 25.2. Differentiate $y = x^2 \sin x$.

solution: We will use the product rule ($(uv)' = u'v + uv'$) with $u = x^2$ and $v = \sin x$.

$$y' = (x^2)'(\sin x) + (x^2)(\sin x)' = 2x \sin x + x^2 \cos x.$$

Example 25.3. Differentiate $y = \frac{\sin x}{x}$.

solution: This time we use the quotient rule ($\left(\frac{u}{v}\right)' = \frac{u'v - uv'}{v^2}$) with $u = \sin x$ and $v = x$.

$$y' = \frac{(\sin x)'x - (\sin x)(x)'}{x^2} = \frac{x\cos x - \sin x}{x^2}.$$

Example 25.4. Differentiate $y = 5x + \cos x$.

solution:

$$\frac{dy}{dx} = \frac{d}{dx}(5x) + \frac{d}{dx}(\cos x) = 5 - \sin x.$$

Example 25.5. Differentiate $y = \sin x \cos x$.

solution: By the product rule, we have that

$$\frac{dy}{dx} = \frac{d}{dx}(\sin x)\cos x + \sin x \frac{d}{dx}(\cos x) = \cos^2 x - \sin^2 x.$$

Example 25.6. Differentiate $y = \frac{\cos x}{1 - \sin x}$.

solution: By the quotient rule, we have that

Trigonometrik Fonksiyonların Türevleri

Sinüs ve Kosinüs

$$\boxed{\frac{d}{dx}(\cos x) = -\sin x}$$

Örnek 25.1. $y = x^2 - \sin x$ fonksiyonunun türevini alınız.

çözüm:

$$\frac{dy}{dx} = \frac{d}{dx}\left(x^2\right) - \frac{d}{dx}(\sin x) = 2x - \cos x.$$

Örnek 25.2. $y = x^2 \sin x$ fonksiyonunun türevini alınız.

çözüm: Çarpım kuralı kullanırsak ($(uv)' = u'v + uv'$) burada $u = x^2$ ve $v = \sin x$ oluyor.

$$y' = (x^2)'(\sin x) + (x^2)(\sin x)' = 2x \sin x + x^2 \cos x.$$

Örnek 25.3. $y = \frac{\sin x}{x}$ fonksiyonunun türevini alınız.

çözüm: Bu sefer de bölüm kuralı kullanırsak ($\left(\frac{u}{v}\right)' = \frac{u'v - uv'}{v^2}$) burada $u = \sin x$ ve $v = x$ oluyor.

$$y' = \frac{(\sin x)'x - (\sin x)(x)'}{x^2} = \frac{x\cos x - \sin x}{x^2}.$$

Örnek 25.4. $y = 5x + \cos x$ fonksiyonunun türevini alınız.

çözüm:

$$\frac{dy}{dx} = \frac{d}{dx}(5x) + \frac{d}{dx}(\cos x) = 5 - \sin x.$$

Örnek 25.5. $y = \sin x \cos x$ fonksiyonunun türevini alınız.

çözüm: Çarpım kuralı gereğince,

$$\frac{dy}{dx} = \frac{d}{dx}(\sin x)\cos x + \sin x \frac{d}{dx}(\cos x) = \cos^2 x - \sin^2 x.$$

Örnek 25.6. $y = \frac{\cos x}{1 - \sin x}$ fonksiyonunun türevini alınız.

çözüm: Bölüm kuralından,

$$\begin{aligned}\frac{dy}{dx} &= \frac{\frac{d}{dx}(\cos x)(1-\sin x) - (\cos x)\frac{d}{dx}(1-\sin x)}{(1-\sin x)^2} \\ &= \frac{-\sin x(1-\sin x) - \cos x(0-\cos x)}{(1-\sin x)^2} \\ &= \frac{-\sin x + \sin^2 x + \cos^2 x}{(1-\sin x)^2} \\ &= \frac{-\sin x + 1}{(1-\sin x)^2} = \frac{1-\sin x}{(1-\sin x)^2} \\ &= \frac{1}{1-\sin x}.\end{aligned}$$

$$\begin{aligned}\frac{dy}{dx} &= \frac{\frac{d}{dx}(\cos x)(1-\sin x) - (\cos x)\frac{d}{dx}(1-\sin x)}{(1-\sin x)^2} \\ &= \frac{-\sin x(1-\sin x) - \cos x(0-\cos x)}{(1-\sin x)^2} \\ &= \frac{-\sin x + \sin^2 x + \cos^2 x}{(1-\sin x)^2} \\ &= \frac{-\sin x + 1}{(1-\sin x)^2} = \frac{1-\sin x}{(1-\sin x)^2} \\ &= \frac{1}{1-\sin x}.\end{aligned}$$

The Tangent Function

Tanjant Fonksiyonu

$$\boxed{\frac{d}{dx}(\tan x) = \sec^2 x}$$

Proof. Using the quotient rule, we can calculate that

$$\begin{aligned}\frac{d}{dx}(\tan x) &= \frac{d}{dx}\left(\frac{\sin x}{\cos x}\right) \\ &= \frac{\frac{d}{dx}(\sin x)(\cos x) - (\sin x)\frac{d}{dx}(\cos x)}{\cos^2 x} \\ &= \frac{(\cos x)(\cos x) - (\sin x)(-\sin x)}{\cos^2 x} \\ &= \frac{\cos^2 x + \sin^2 x}{\cos^2 x} \\ &= \frac{1}{\cos^2 x} \\ &= \sec^2 x.\end{aligned}$$

□

Kanıt. Bölüm türevinden,

$$\begin{aligned}\frac{d}{dx}(\tan x) &= \frac{d}{dx}\left(\frac{\sin x}{\cos x}\right) \\ &= \frac{\frac{d}{dx}(\sin x)(\cos x) - (\sin x)\frac{d}{dx}(\cos x)}{\cos^2 x} \\ &= \frac{(\cos x)(\cos x) - (\sin x)(-\sin x)}{\cos^2 x} \\ &= \frac{\cos^2 x + \sin^2 x}{\cos^2 x} \\ &= \frac{1}{\cos^2 x} \\ &= \sec^2 x.\end{aligned}$$

□

The Other Three

Diğer Üç Fonksiyon

$$\boxed{\frac{d}{dx}(\sec x) = \sec x \tan x} \qquad \boxed{\frac{d}{dx}(\cot x) = -\operatorname{cosec}^2 x} \qquad \boxed{\frac{d}{dx}(\operatorname{cosec} x) = -\operatorname{cosec} x \cot x}$$

You can use the quotient rule to prove these three rules. We may ask you to prove one of them in an exam.

Example 25.7. Find y'' if $y = \sec x$.

solution: Since $y' = \sec x \tan x$, we have that

$$\begin{aligned}y'' &= \frac{d}{dx}(y') = \frac{d}{dx}(\sec x \tan x) \\ &= \frac{d}{dx}(\sec x)\tan x + \sec x \frac{d}{dx}(\tan x) \\ &= (\sec x \tan x)(\tan x) + (\sec x)(\sec^2 x) \\ &= \sec x \tan^2 x + \sec^3 x.\end{aligned}$$

Bu üç kuralın kanıtlanması için bölüm kuralını kullanabilirsiniz. Bunlardan birisini sınavda kanıtlamanızı isteyebiliriz.

Örnek 25.7. $y = \sec x$ ise y''nü bulunuz.

çözüm: $y' = \sec x \tan x$ olduğundan,

$$\begin{aligned}y'' &= \frac{d}{dx}(y') = \frac{d}{dx}(\sec x \tan x) \\ &= \frac{d}{dx}(\sec x)\tan x + \sec x \frac{d}{dx}(\tan x) \\ &= (\sec x \tan x)(\tan x) + (\sec x)(\sec^2 x) \\ &= \sec x \tan^2 x + \sec^3 x\end{aligned}$$

buluruz.

Problems

Problem 25.1.

(a). Find $\frac{ds}{dx}$ if $s = (\sin x + \cos x)\sec x$.

(b). Find $\frac{dr}{d\theta}$ if $r = \theta \sin\theta + \cos\theta$.

Problem 25.2. Use the quotient rule to prove that the following are true:

(a). $\frac{d}{dx}(\sec x) = \sec x \tan x$.

(b). $\frac{d}{dx}(\cot x) = -\operatorname{cosec}^2 x$.

(c). $\frac{d}{dx}(\operatorname{cosec} x) = -\operatorname{cosec} x \cot x$.

Problem 25.3. Find the derivatives of the functions below:

Sorular

Soru 25.1.

(a). $s = (\sin x + \cos x)\sec x$ ise $\frac{ds}{dx}$'i bulunuz.

(b). $r = \theta \sin\theta + \cos\theta$ ise $\frac{dr}{d\theta}$'yı bulunuz.

Soru 25.2. Bölüm kuralı kullanarak, aşağıdakileri kanıtlayınız:

(a). $\frac{d}{dx}(\sec x) = \sec x \tan x$.

(b). $\frac{d}{dx}(\cot x) = -\operatorname{cosec}^2 x$.

(c). $\frac{d}{dx}(\operatorname{cosec} x) = -\operatorname{cosec} x \cot x$.

Soru 25.3. Aşağıdaki fonksiyonların türevlerini bulunuz:

(a). $y = -10x + 3\cos x$

(b). $y = x^2 \cos x$

(c). $y = \operatorname{cosec} x - 4\sqrt{x} + 7$

(d). $f(x) = \sin x \tan x$

(e). $g(x) = \cos x \tan x$

(f). $w = \frac{\cot z}{1 + \cot z}$

(g). $h(x) = x^3 \sin x \cos x$

(h). $p = 5 + \frac{1}{\cot t}$

(i). $r = \frac{\sin t + \cos t}{\cos t}$

(j). $y = (\sec x + \tan x)(\sec x - \tan x)$

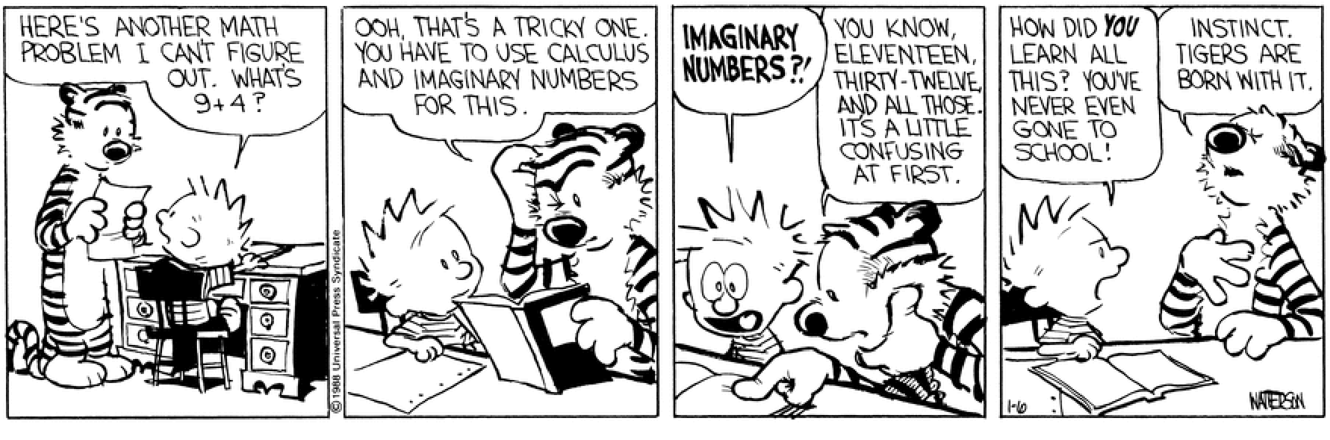


Figure 25.1: A web comic taken from https://www.gocomics.com/calvinandhobbes/1988/01/06 .
Şekil 25.1: https://www.gocomics.com/calvinandhobbes/1988/01/06 adresinden alınan bir web çizgi romanı.

26 The Chain Rule

How do we differentiate $F(x) = \sin(x^2 - 4)$?

Theorem 26.1 (The Chain Rule). *Suppose that*

- $y = f(u)$ *is differentiable at the point* $u = g(x)$*; and*
- $g(x)$ *is differentiable at* x.

Then $f \circ g$ *is differentiable at* x *and*

$$\boxed{(f \circ g)'(x) = f'\big(g(x)\big)g'(x).}$$

The Chain Rule is easier to remember if we use Leibniz's notation:

$$\boxed{\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}}$$

Example 26.1. Differentiate $y = \sin(x^2 - 4)$.

solution: We have $y = \sin u$ with $u = x^2 - 4$. Now $\frac{dy}{du} = \cos u$ and $\frac{du}{dx} = 2x$. Therefore

$$\begin{aligned} \frac{dy}{dx} &= \frac{dy}{du}\frac{du}{dx} = (\cos u)(2x) \\ &= 2x\cos u = 2x\cos(x^2 - 4) \end{aligned}$$

by the Chain Rule.

Example 26.2. Differentiate $\sin(x^2 + x)$.

solution: Let $u = x^2 + x$. Then

$$\begin{aligned} \frac{d}{dx}\big(\sin(x^2 + x)\big) &= \frac{d}{du}(\sin u)\frac{du}{dx} \\ &= (\cos u)(2x + 1) \\ &= (2x + 1)\cos(x^2 + x) \end{aligned}$$

by the Chain Rule.

Example 26.3 (Using the Chain Rule Two Times). Differentiate $g(t) = \tan(5 - \sin 2t)$.

solution: Let $u = 5 - \sin 2t$. Then $g(t) = \tan u$. Hence

$$\frac{dg}{dt} = \frac{dg}{du}\frac{du}{dt} = (\sec^2 u)\frac{d}{dt}(5 - \sin 2t).$$

Zincir Kuralı

$F(x) = \sin(x^2 - 4)$ fonksiyonunun türevini nasıl alırız?

Teorem 26.1 (Zincir Kuralı). *Varsayalım ki*

- $y = f(u)$ *fonksiyonu* $u = g(x)$ *notasında türevlenebilir ve*
- $g(x)$ *fonksiyonu da* x*'de türevlenebilir olsun.*

Bu durumda $f \circ g$ *fonksiyonu da* x *noktasında türevlenebilirdir ve türevi de*

$$\boxed{(f \circ g)'(x) = f'\big(g(x)\big)g'(x).}$$

Zincir Kuralı'nı Leibniz notasyonu kullanarak kolayca hatırlanabilir:

$$\boxed{\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}}$$

Örnek 26.1. $y = \sin(x^2 - 4)$ fonksiyonunun türevini alınız.

çözüm: $u = x^2 - 4$ olsun ve $y = \sin u$ olur. Böylece $\frac{dy}{du} = \cos u$ ve $\frac{du}{dx} = 2x$ olur. Yani

$$\begin{aligned} \frac{dy}{dx} &= \frac{dy}{du}\frac{du}{dx} = (\cos u)(2x) \\ &= 2x\cos u = 2x\cos(x^2 - 4) \end{aligned}$$

Zincir Kuralı kullanarak buluruz.

Örnek 26.2. $\sin(x^2 + x)$ fonksiyonunun türevini alınız.

çözüm: $u = x^2 + x$ diyelim. Buradan Zincir Kuralı yardımıyla,

$$\begin{aligned} \frac{d}{dx}\big(\sin(x^2 + x)\big) &= \frac{d}{du}(\sin u)\frac{du}{dx} \\ &= (\cos u)(2x + 1) \\ &= (2x + 1)\cos(x^2 + x) \end{aligned}$$

bulunur.

Örnek 26.3 (İki kez Zincir Kuralı). $g(t) = \tan(5 - \sin 2t)$ fonksiyonunun türevini alınız.

çözüm: Let $u = 5 - \sin 2t$. Then $g(t) = \tan u$. Hence

$$\frac{dg}{dt} = \frac{dg}{du}\frac{du}{dt} = (\sec^2 u)\frac{d}{dt}(5 - \sin 2t).$$

We need to use the Chain Rule a second time: Let $w = 2t$. Then

$$\begin{aligned}\frac{dg}{dt} &= (\sec^2 u)\frac{d}{dt}(5-\sin 2t)\\ &= (\sec^2 u)\frac{d}{dw}(5-\sin w)\frac{d2}{dt}\\ &= (\sec^2 u)(-\cos w)(2)\\ &= -2\cos 2t\sec^2(5-\sin 2t).\end{aligned}$$

(Note: Your final answer should not have u or w in it.)

We need to use the Chain Rule a second time: Let $w = 2t$. Then

$$\begin{aligned}\frac{dg}{dt} &= (\sec^2 u)\frac{d}{dt}(5-\sin 2t)\\ &= (\sec^2 u)\frac{d}{dw}(5-\sin w)\frac{d2}{dt}\\ &= (\sec^2 u)(-\cos w)(2)\\ &= -2\cos 2t\sec^2(5-\sin 2t).\end{aligned}$$

(Not: Cevabınız u or w içermemelidir.)

Powers of a Function

If

- f is a differentiable function of u;
- u is a differentiable function of x; and
- $y = f(u)$,

then the Chain Rule $\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$ is the same as

$$\frac{d}{dx}f(u) = f'(u)\frac{du}{dx}.$$

Now suppose that $n \in \mathbb{R}$ and $f(u) = u^n$. Then $f'(u) = nu^{n-1}$. So

$$\boxed{\frac{d}{dx}\left(u^n\right) = nu^{n-1}\frac{du}{dx}.}$$

Example 26.4.

$$\begin{aligned}\frac{d}{dx}\left(5x^3-x^4\right)^7 &= 7\left(5x^3-x^4\right)^6\frac{d}{dx}\left(5x^3-x^4\right)\\ &= 7\left(5x^3-x^4\right)^6\left(15x^2-4x^3\right).\end{aligned}$$

Example 26.5.

$$\begin{aligned}\frac{d}{dx}\left(\frac{1}{3x-2}\right) &= \frac{d}{dx}(3x-2)^{-1} = -1(3x-2)^{-2}\frac{d}{dx}(3x-2)\\ &= -\left(\frac{1}{(3x-2)^2}\right)(2) = \frac{-3}{(3x-2)^2}.\end{aligned}$$

Example 26.6.

$$\frac{d}{dx}\left(\sin^5 x\right) = 5\sin^4 x\frac{d}{dx}(\sin x) = 5\sin^4 x\cos x.$$

Example 26.7. Differentiate $|x|$.

solution: Since $|x| = \sqrt{x^2}$, we can calculate that if $x \neq 0$ then

$$\begin{aligned}\frac{d}{dx}|x| &= \frac{d}{dx}\left(\sqrt{x^2}\right) = \frac{d}{du}\left(\sqrt{u}\right)\frac{d}{dx}\left(x^2\right)\\ &= \frac{1}{2\sqrt{u}}2x = \frac{x}{\sqrt{x^2}} = \frac{x}{|x|}.\end{aligned}$$

Kuvvet Fonksiyonları

Eğer

- f, u'ya bağlı türevlenebilir fonksiyon;
- u, x'e bağlı türevlenebilir fonksiyon ve
- $y = f(u)$ ise,

Zincir Kuralı gereğince $\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$ ile

$$\frac{d}{dx}f(u) = f'(u)\frac{du}{dx}$$

ifadesi aynıdır.

Şimdi $n \in \mathbb{R}$ ve $f(u) = u^n$ olsun. O halde $f'(u) = nu^{n-1}$ olur. Böylece

$$\boxed{\frac{d}{dx}\left(u^n\right) = nu^{n-1}\frac{du}{dx}.}$$

Örnek 26.4.

$$\begin{aligned}\frac{d}{dx}\left(5x^3-x^4\right)^7 &= 7\left(5x^3-x^4\right)^6\frac{d}{dx}\left(5x^3-x^4\right)\\ &= 7\left(5x^3-x^4\right)^6\left(15x^2-4x^3\right).\end{aligned}$$

Örnek 26.5.

$$\begin{aligned}\frac{d}{dx}\left(\frac{1}{3x-2}\right) &= \frac{d}{dx}(3x-2)^{-1} = -1(3x-2)^{-2}\frac{d}{dx}(3x-2)\\ &= -\left(\frac{1}{(3x-2)^2}\right)(2) = \frac{-3}{(3x-2)^2}.\end{aligned}$$

Örnek 26.6.

$$\frac{d}{dx}\left(\sin^5 x\right) = 5\sin^4 x\frac{d}{dx}(\sin x) = 5\sin^4 x\cos x.$$

Örnek 26.7. $|x|$ fonksiyonunun türevini alınız.

çözüm: $|x| = \sqrt{x^2}$ olduğundan, $x \neq 0$ ise

$$\begin{aligned}\frac{d}{dx}|x| &= \frac{d}{dx}\left(\sqrt{x^2}\right) = \frac{d}{du}\left(\sqrt{u}\right)\frac{d}{dx}\left(x^2\right)\\ &= \frac{1}{2\sqrt{u}}2x = \frac{x}{\sqrt{x^2}} = \frac{x}{|x|}\end{aligned}$$

buluruz.

Example 26.8. Let $y = \frac{1}{(1-2x)^3}$ for $x \neq \frac{1}{2}$. Show that $\frac{dy}{dx} > 0$.

solution: First we calculate that

$$\begin{aligned}\frac{dy}{dx} &= \frac{d}{dx}(1-2x)^{-3} = -3(1-2x)^{-4}\frac{d}{dx}(1-2x)\\ &= -3(1-2x)^{-4}(-2) = \frac{6}{(1-2x)^4}\end{aligned}$$

if $x \neq \frac{1}{2}$. Since $(1-2x)^4 > 0$ if $x \neq \frac{1}{2}$ and $6 > 0$, we have that $\frac{dy}{dx} > 0$ if $x \neq \frac{1}{2}$.

Örnek 26.8. $y = \frac{1}{(1-2x)^3}$ for $x \neq \frac{1}{2}$ olsun. $\frac{dy}{dx} > 0$ olduğunu gösteriniz.

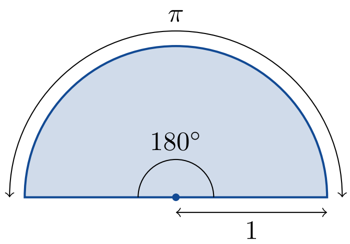
çözüm: Öncelikle, $x \neq \frac{1}{2}$ ise

$$\begin{aligned}\frac{dy}{dx} &= \frac{d}{dx}(1-2x)^{-3} = -3(1-2x)^{-4}\frac{d}{dx}(1-2x)\\ &= -3(1-2x)^{-4}(-2) = \frac{6}{(1-2x)^4}\end{aligned}$$

buluruz. Eğer $x \neq \frac{1}{2}$ ise $(1-2x)^4 > 0$ olur ve $6 > 0$ bulunur, buradan $\frac{dy}{dx} > 0$ if $x \neq \frac{1}{2}$ elde edilir.

Example 26.9 (Why Do We Use Radians in Calculus?)**.** Remember that $\frac{d}{dx}\sin x = \cos x$ is true *only if we use radians.* What happens if we use degrees?

Örnek 26.9 (Kalkülüste Neden Radyan Kullanırız?)**.** Unutmayınız ki $\frac{d}{dx}\sin x = \cos x$ doğrudur *tabii radyan kullanırsak.* Derece kullansaydık ne olurdu?



Remember that

$$\begin{aligned}180 \text{ degrees} &= \pi \text{ radians}\\ 180^\circ &= \pi\\ 1^\circ &= \frac{\pi}{180}\\ x^\circ &= \frac{\pi x}{180}.\end{aligned}$$

So

$$\frac{d}{dx}\sin x^\circ = \frac{d}{dx}\sin\left(\frac{\pi x}{180}\right) = \frac{\pi}{180}\cos\left(\frac{\pi x}{180}\right) = \frac{\pi}{180}\cos x^\circ.$$

Therefore we have

$$\boxed{\frac{d}{dx}\sin x = \cos x} \quad \text{and} \quad \boxed{\frac{d}{dx}\sin x^\circ = \frac{\pi}{180}\cos x^\circ}$$

a nice formula — not nice

This is why we use radians in Calculus.

Hatırlayacak olursak,

$$\begin{aligned}180 \text{ derece} &= \pi \text{ radyan}\\ 180^\circ &= \pi\\ 1^\circ &= \frac{\pi}{180}\\ x^\circ &= \frac{\pi x}{180}.\end{aligned}$$

Yani

$$\frac{d}{dx}\sin x^\circ = \frac{d}{dx}\sin\left(\frac{\pi x}{180}\right) = \frac{\pi}{180}\cos\left(\frac{\pi x}{180}\right) = \frac{\pi}{180}\cos x^\circ.$$

Elimize geçen

$$\boxed{\frac{d}{dx}\sin x = \cos x} \quad \text{ve} \quad \boxed{\frac{d}{dx}\sin x^\circ = \frac{\pi}{180}\cos x^\circ}$$

güzel bir formül — hiç güzel olmayan formül

Bu yüzden Kalkülüste radyan kullanıyoruz.

Problems

Problem 26.1. Find $\dfrac{ds}{dt}$ if $s = \left(\dfrac{t}{2} - 1\right)^{-10}$.

Problem 26.2. Find $\dfrac{dy}{dt}$ if $y = \cos\left(5\sin\left(\dfrac{t}{3}\right)\right)$.

Problem 26.3. Find $\dfrac{dy}{dx}$ if $y = \sqrt{3x^2 - 4x + 6}$.

Problem 26.4. Find $\dfrac{dy}{dx}$ if $y = \sin^3 x$.

Problem 26.5. Find $\dfrac{dy}{dx}$ if $y = \sec(\tan x)$.

Problem 26.6. Find $\dfrac{dy}{dx}$ if $y = \sin(x^2)\cos(2x)$.

Problem 26.7. Find $\dfrac{dy}{dt}$ if $y = \left(\dfrac{t^2}{t^3 - 4t}\right)^3$.

Problem 26.8. Find y'' if $y = \left(1 + \dfrac{1}{x}\right)^3$.

Problem 26.9. Find $(f \circ g)'(1)$ if $f(u) = u^5 + 1$ and $g(x) = \sqrt{x}$.

Problem 26.10. Find $(f \circ g)'(0)$ if $f(u) = \frac{2u}{u^2+1}$ and $g(x) = 10x^2 + x + 1$.

Sorular

Soru 26.1. $s = \left(\dfrac{t}{2} - 1\right)^{-10}$ ise $\dfrac{ds}{dt}$'yi bulunnuz.

Soru 26.2. $y = \cos\left(5\sin\left(\dfrac{t}{3}\right)\right)$ ise $\dfrac{dy}{dt}$'yi bulunuz.

Soru 26.3. $y = \sqrt{3x^2 - 4x + 6}$ ise $\dfrac{dy}{dx}$'i bulunuz.

Soru 26.4. $y = \sin^3 x$ ise $\dfrac{dy}{dx}$'i bulunuz.

Soru 26.5. $y = \sec(\tan x)$ ise $\dfrac{dy}{dx}$'i bulunuz.

Soru 26.6. $y = \sin(x^2)\cos(2x)$ ise $\dfrac{dy}{dx}$'i bulunuz.

Soru 26.7. $y = \left(\dfrac{t^2}{t^3 - 4t}\right)^3$ ise $\dfrac{dy}{dt}$'yi bulunuz.

Soru 26.8. $y = \left(1 + \dfrac{1}{x}\right)^3$ ise y'''yü bulunuz.

Soru 26.9. $f(u) = u^5 + 1$ ve $g(x) = \sqrt{x}$ ise $(f \circ g)'(1)$'i bulunuz.

Soru 26.10. $f(u) = \frac{2u}{u^2+1}$ ve $g(x) = 10x^2+x+1$ ise $(f \circ g)'(0)$'ı bulunuz.

27

Antiderivatives

Definition. F is an ***antiderivative*** of f on an interval I if $F'(x) = f(x)$ for all $x \in I$.

Example 27.1.
$2x$ is the derivative of x^2.
x^2 is an antiderivative of $2x$.

Example 27.2. If $g(x) = \cos x$, then an antiderivative of g is

$$G(x) = \sin x$$

because

$$G'(x) = \frac{d}{dx}(\sin x) = \cos x = g(x).$$

Example 27.3. If $h(x) = 2x + \cos x$, then $H(x) = x^2 + \sin x$ is an antiderivative of $h(x)$.

Remark. $F(x) = x^2$ is not the only antiderivative of $f(x) = 2x$.
$x^2 + 1$ is an antiderivative of $2x$ because $\frac{d}{dx}\left(x^2+1\right) = 2x$.
$x^2 + 5$ is an antiderivative of $2x$ because $\frac{d}{dx}\left(x^2+5\right) = 2x$.
$x^2 - 1234$ is an antiderivative of $2x$ because $\frac{d}{dx}\left(x^2-1234\right) = 2x$.

Theorem 27.1. *If F is an antiderivative of f on I, then the general antiderivative of f is*

$$F(x) + C$$

where C is a constant.

Example 27.4. Find an antiderivative of $f(x) = 3x^2$ that satisfies $F(1) = -1$.

solution: x^3 is an antiderivative of f because $\frac{d}{dx}\left(x^3\right) = 3x^2$. So the general antiderivative of f is

$$F(x) = x^3 + C.$$

Then we calculate that

$$-1 = F(1) = 1^3 + C = 1 + C \implies C = -2.$$

Therefore $F(x) = x^3 - 2$.

Ters Türevler

Tanım. Bir I aralığındaki her $x \in I$ için $F'(x) = f(x)$ olacak şekildeki F fonksiyonuna f fonksiyonunun bir ***ters türevi*** denir.

Örnek 27.1.
x^2 nin türevi $2x$ tir.
x^2 de $2x$ in bir ters türevidir.

Örnek 27.2. $g(x) = \cos x$ ise, g nin bir ters türevi

$$G(x) = \sin x$$

olur, çünkü

$$G'(x) = \frac{d}{dx}(\sin x) = \cos x = g(x).$$

Örnek 27.3. $h(x) = 2x + \cos x$ ise, $H(x) = x^2 + \sin x$ fonksiyonu $h(x)$ in bir ters türevidir.

Not. $F(x) = x^2$ fonksiyonu $f(x) = 2x$ in tek ters türevi değildir.
$x^2 + 1$ de $2x$ için bir ters türevdir çünkü $\frac{d}{dx}\left(x^2+1\right) = 2x$.
$x^2 + 5$ de $2x$ için bir ters türevdir çünkü $\frac{d}{dx}\left(x^2+5\right) = 2x$.
$x^2 - 1234$ de $2x$ için bir ters türevdir çünkü $\frac{d}{dx}\left(x^2-1234\right) = 2x$.

Teorem 27.1. *Eğer F fonksiyonu f nin I üzerindeki ters türevi ise, f nin genel ters türevi*

$$F(x) + C$$

burada C bir sabit oluyor.

Örnek 27.4. $F(1) = -1$ sağlayan $f(x) = 3x^2$ nin bir ters türevini bulunuz .

çözüm: x^3 fonksiyonu f nin bir ters türevidir çünkü $\frac{d}{dx}\left(x^3\right) = 3x^2$. Bu nedenle f nin genel ters türevi

$$F(x) = x^3 + C.$$

Şunları buluruz:

$$-1 = F(1) = 1^3 + C = 1 + C \implies C = -2.$$

Bu nedenle $F(x) = x^3 - 2$.

function, $f(x)$ fonksiyon, $f(x)$	derivative, $f'(x)$ türev, $f'(x)$
x^n	nx^{n-1}
$\sin kx$	$k\cos kx$
$\cos kx$	$-k\sin kx$
e^{kx}	ke^{kx}
$\ln\lvert x\rvert$	$\frac{1}{x}$

function, $f(x)$ fonksiyon, $f(x)$	general antiderivative, $F(x)$ genel ters türev, $F(x)$
$x^n\ (n \neq -1)$	$\frac{x^{n+1}}{n+1} + C$
$\sin kx$	$-\frac{1}{k}\cos kx + C$
$\cos kx$	$\frac{1}{k}\sin kx + C$
e^{kx}	$\frac{1}{k}e^{kx} + C$
$\frac{1}{x}$	$\ln\lvert x\rvert + C$

Table 27.1: Elementary derivatives and antiderivatives
Tablo 27.1:

The Sum Rule and the Constant Multiple Rule

Suppose that

- F is an antiderivative of f;
- G is an antiderivative of g;
- $k \in \mathbb{R}$.

The Sum Rule: The general antiderivative of $f+g$ is

$$F(x) + G(x) + C.$$

The Constant Multiple Rule: The general antiderivative of kf is

$$kF(x) + C.$$

Example 27.5. Find the general antiderivative of $f(x) = \frac{3}{\sqrt{x}} + \sin 2x$.

solution: We have $f = 3g + h$ where $g(x) = x^{-\frac{1}{2}}$ and $h(x) = \sin 2x$. An antiderivative of g is

$$G(x) = \frac{x^{-\frac{1}{2}+1}}{-\frac{1}{2}+1} = \frac{x^{\frac{1}{2}}}{\frac{1}{2}} = 2\sqrt{x}.$$

An antiderivative of h is

$$H(x) = -\frac{1}{2}\cos 2x.$$

Therefore the general antiderivative of f is

$$F(x) = 6\sqrt{x} - \frac{1}{2}\cos 2x + C.$$

Definition. The general antiderivative of f is also called the ***indefinite integral*** of f with respect to x, and is denoted by

$$\int f(x)\ dx.$$

Toplam ve Sabitle çarpım Kuralı

Varsayalım ki

- F fonksiyonu f nin bir ters türevi;
- G fonksiyonu da g nin bir ters türevi;
- $k \in \mathbb{R}$.

Toplam Kuralı: $f+g$'nin ilkeli (ters türevi)

$$F(x) + G(x) + C.$$

Sabitle Çarpım Kuralı: kf'nin ilkeli

$$kF(x) + C.$$

Örnek 27.5. $f(x) = \frac{3}{\sqrt{x}} + \sin 2x$ nin ilkelini bulunuz.

çözüm: $g(x) = x^{-\frac{1}{2}}$ olmak üzere elimizde $f = 3g+h$ ve $h(x) = \sin 2x$ var. g'nin bir ilkeli

$$G(x) = \frac{x^{-\frac{1}{2}+1}}{-\frac{1}{2}+1} = \frac{x^{\frac{1}{2}}}{\frac{1}{2}} = 2\sqrt{x}.$$

Ayrıca h'nin bir ilkeli

$$H(x) = -\frac{1}{2}\cos 2x.$$

Dıloayısıyla f fonksiyonunun bir ilkeli

$$F(x) = 6\sqrt{x} - \frac{1}{2}\cos 2x + C.$$

Tanım. f nin genel ters türev veya ilkeline aynı zamanda f nin x'e göre ***belirsiz integrali*** denir ve şöyle gösterilir:

$$\int f(x)\ dx.$$

$$\int \underbrace{f(x)}_{\substack{\text{the integrand}\\ \text{integralin integrandı}}} \, dx$$

the integral sign
integral işareti

x is the variable of integration
x ise integral değişkeni olarak tanımlanır

Example 27.6.

$$\int 2x \; dx = x^2 + C$$
$$\int \cos x = \sin x + C$$
$$\int (2x + \cos x) \; dx = x^2 + \sin x + C$$

Example 27.7. Calculate $\int (x^2 - 2x + 5) \; dx$.

solution 1. Since $\frac{d}{dx}\left(\frac{x^3}{3} - x^2 + 5x\right) = x^2 - 2x + 5$ we have that
$$\int (x^2 - 2x + 5) \; dx = \frac{x^3}{3} - x^2 + 5x + C.$$

solution 2.

$$\begin{aligned}\int (x^2 - 2x + 5) \; dx &= \int x^2 \; dx - \int 2x \; dx + \int 5 \; dx \\ &= \left(\frac{x^3}{3} + C_1\right) - \left(x^2 + C_2\right) + (5x + C_3) \\ &= \left(\frac{x^3}{3} - x^2 + 5x\right) + (C_1 - C_2 + C_3)\,.\end{aligned}$$

Because we only need one constant, we can define $C := C_1 - C_2 + C_3$. Therefore
$$\int (x^2 - 2x + 5) \; dx = \frac{x^3}{3} - x^2 + 5x + C.$$

Örnek 27.6.

$$\int 2x \; dx = x^2 + C$$
$$\int \cos x = \sin x + C$$
$$\int (2x + \cos x) \; dx = x^2 + \sin x + C$$

Örnek 27.7. $\int (x^2 - 2x + 5) \; dx$ integralini bulunuz.

çözüm 1. $\frac{d}{dx}\left(\frac{x^3}{3} - x^2 + 5x\right) = x^2 - 2x + 5$ olduğundan
$$\int (x^2 - 2x + 5) \; dx = \frac{x^3}{3} - x^2 + 5x + C$$
buluruz.

çözüm 2.

$$\begin{aligned}\int (x^2 - 2x + 5) \; dx &= \int x^2 \; dx - \int 2x \; dx + \int 5 \; dx \\ &= \left(\frac{x^3}{3} + C_1\right) - \left(x^2 + C_2\right) + (5x + C_3) \\ &= \left(\frac{x^3}{3} - x^2 + 5x\right) + (C_1 - C_2 + C_3)\,.\end{aligned}$$

Yalnızca bir sabite ihtiyacımız olduğundan, $C := C_1 - C_2 + C_3$ olarak tanımlarız. Yani
$$\int (x^2 - 2x + 5) \; dx = \frac{x^3}{3} - x^2 + 5x + C.$$

Example 27.8. You drop a box off the top of a tall building. The acceleration due to gravity is $9.8\,\mathrm{ms}^{-2}$. You can ignore air resistance. How far does the box fall in 5 seconds?

solution: The acceleration is

$$a(t) = 9.8\mathrm{ms}^{-2}$$

downwards. Since

$$\text{acceleration} = \frac{d}{dt}(\text{velocity}),$$

the velocity is an antiderivative of the acceleration. Therefore the velocity is

$$v(t) = 9.8t + C\ \mathrm{ms}^{-1}.$$

You let go of the box at time $t = 0$. So $v(0) = 0$. Thus $C = 0$. Hence

$$v(t) = 9.8t\ \mathrm{ms}^{-1}.$$

Now velocity $= \frac{d}{dt}$(position). So the distance fallen is an antiderivative of velocity. Hence

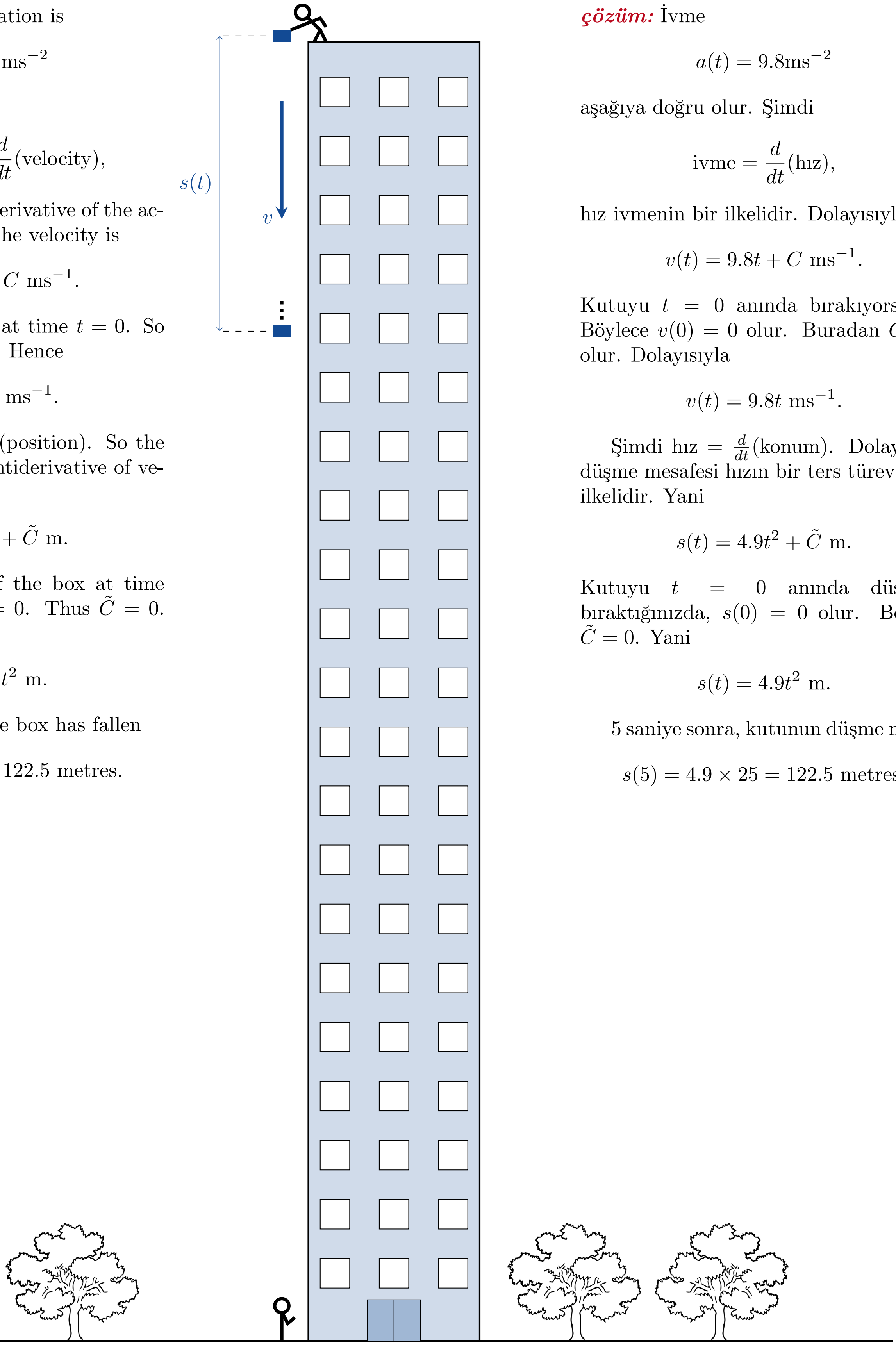
$$s(t) = 4.9t^2 + \tilde{C}\ \mathrm{m}.$$

Because you let go of the box at time $t = 0$, we have $s(0) = 0$. Thus $\tilde{C} = 0$. Therefore

$$s(t) = 4.9t^2\ \mathrm{m}.$$

After 5 seconds, the box has fallen

$$s(5) = 4.9 \times 25 = 122.5 \text{ metres}.$$



Örnek 27.8. Bir binanın üstünden bir kutu bırakılıyor. Yerçekimi ivmesi $9.8\,\mathrm{ms}^{-2}$ dir. Havadaki sürtünme ihmal edilebilir. Kutu 5 saniyede ne kadar yol alır?

çözüm: İvme

$$a(t) = 9.8\mathrm{ms}^{-2}$$

aşağıya doğru olur. Şimdi

$$\text{ivme} = \frac{d}{dt}(\text{hız}),$$

hız ivmenin bir ilkelidir. Dolayısıyla hız

$$v(t) = 9.8t + C\ \mathrm{ms}^{-1}.$$

Kutuyu $t = 0$ anında bırakıyorsunuz. Böylece $v(0) = 0$ olur. Buradan $C = 0$ olur. Dolayısıyla

$$v(t) = 9.8t\ \mathrm{ms}^{-1}.$$

Şimdi hız $= \frac{d}{dt}$(konum). Dolayısıyla düşme mesafesi hızın bir ters türevi veya ilkelidir. Yani

$$s(t) = 4.9t^2 + \tilde{C}\ \mathrm{m}.$$

Kutuyu $t = 0$ anında düşmeye bıraktığınızda, $s(0) = 0$ olur. Böylece $\tilde{C} = 0$. Yani

$$s(t) = 4.9t^2\ \mathrm{m}.$$

5 saniye sonra, kutunun düşme mesafesi

$$s(5) = 4.9 \times 25 = 122.5 \text{ metres}.$$

Problems

Problem 27.1. Find an antiderivative for each function, then check your answer by differentiating it.

(a). $f(x) = 200x$.

(b). $g(x) = x^3 - \dfrac{1}{x^3}$.

(c). $h(x) = \sin(\pi x) - 3\sin(3x)$.

(d). $l(x) = x^7 - 6x + 8$.

(e). $m(x) = \dfrac{2}{3}x^{-\frac{1}{3}}$.

(f). $p(x) = \dfrac{1}{3}x^{-\frac{2}{3}}$.

(g). $r(x) = \dfrac{2}{3}\sec^2\dfrac{x}{3}$.

(h). $s(x) = -\sec^2\dfrac{3x}{2}$.

Problem 27.2 (Right or Wrong?). Consider

$$\int \left((2x+1)^2 + \cos x\right)\ dx = \frac{(2x+1)^3}{3} + \sin x + C.$$

Is this correct or incorrect? Why?

Problem 27.3 (Right or Wrong?). Consider

$$\int \left(e^x \cos e^x\right)\ dx = \sin e^x + C.$$

Is this correct or incorrect? Why?

Problem 27.4 (Right or Wrong?). Consider

$$\int \left(3x^2 + 2x + 7\right)\ dx = x^3 + x^2 + 7x.$$

Is this correct or incorrect? Why?

Problem 27.5. Find the following indefinite integrals.

(a). $\displaystyle\int 2x\ dx$

(b). $\displaystyle\int \left(1 - x^2 - 3x^5\right)\ dx$

(c). $\displaystyle\int \frac{4+\sqrt{t}}{t^3}\ dt$

(d). $\displaystyle\int (2\cos 2\theta - 3\sin 3\theta)\ d\theta$

(e). $\displaystyle\int 2e^{3x}\ dx$

(f). $\displaystyle\int \frac{1}{x}\ dx$

Sorular

Soru 27.1. Aşağıdaki fonksiyonların birer ters türevini veya ilkelini bulup, sonra cevabınızı türev alarak bulup kontrol edin.

Soru 27.2 (Doğru mu yoksa Yanlış mı?).

$$\int \left((2x+1)^2 + \cos x\right)\ dx = \frac{(2x+1)^3}{3} + \sin x + C$$

yazalım. Bu doğru mu yoksa yanlış mı? Neden?

Soru 27.3 (Doğru mu yoksa Yanlış mı?).

$$\int \left(e^x \cos e^x\right)\ dx = \sin e^x + C$$

yazalım. Bu doğru mu yoksa yanlış mı? Neden?

Soru 27.4 (Doğru mu yoksa Yanlış mı?).

$$\int \left(3x^2 + 2x + 7\right)\ dx = x^3 + x^2 + 7x$$

yazalım. Bu doğru mu yoksa yanlış mı? Neden?

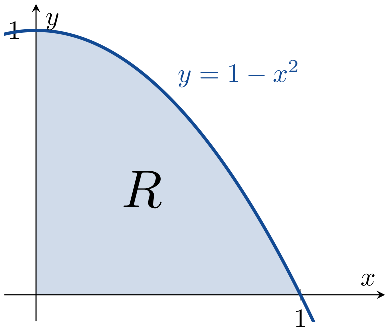
Soru 27.5. Aşağıdaki belirsiz integralleri bulunuz.

28

Integration

İntegral

Question: What is the area of R?

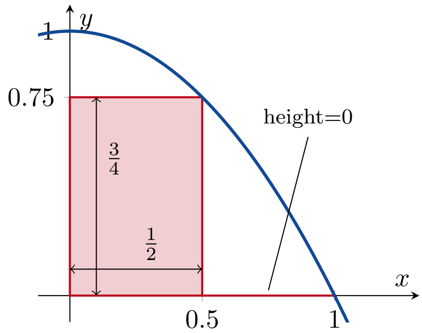


Soru: R bölgesinin alanı kaçtır?

We can use two rectangles to approximate the area of R. Then we have

$$\begin{aligned}\text{area of } R &\approx \text{area of 2 rectangles}\\ &= \left(\frac{3}{4}\times\frac{1}{2}\right)+\left(0\times\frac{1}{2}\right)\\ &= \frac{3}{8} = 0.375.\end{aligned}$$

Can we do better than this? Yes! We could use more rectangles.



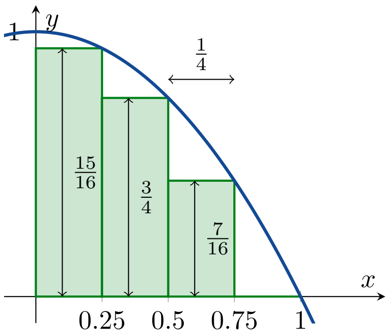
R nin alanını yaklaşık olarak hesaplamada iki dikdörtgen kullanırsak, Bu durumda

$$\begin{aligned}R\text{'nin alanı} &\approx \text{2 dikdörtgenin toplam alanı}\\ &= \left(\frac{3}{4}\times\frac{1}{2}\right)+\left(0\times\frac{1}{2}\right)\\ &= \frac{3}{8} = 0.375.\end{aligned}$$

Bundan daha iyisini yapabilir miyiz? Evet! Daha fazla dikdörtgen kullanabiliriz.

We can say that

$$\begin{aligned}\text{area of } R &\approx \text{area of 4 rectangles}\\ &= \left(\frac{15}{16}\times\frac{1}{4}\right)+\left(\frac{3}{4}\times\frac{1}{4}\right)\\ &\quad+\left(\frac{7}{16}\times\frac{1}{4}\right)+\left(0\times\frac{1}{4}\right)\\ &= \frac{17}{32} = 0.53125.\end{aligned}$$

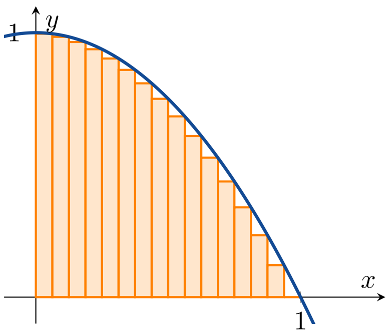


We can say that

$$\begin{aligned}\text{area of } R &\approx \text{area of 4 rectangles}\\ &= \left(\frac{15}{16}\times\frac{1}{4}\right)+\left(\frac{3}{4}\times\frac{1}{4}\right)\\ &\quad+\left(\frac{7}{16}\times\frac{1}{4}\right)+\left(0\times\frac{1}{4}\right)\\ &= \frac{17}{32} = 0.53125.\end{aligned}$$

Every time we increase the number of rectangles, the total area of the rectangles gets closer and closer to the area of R.

$$\begin{aligned}\text{area of } R &\approx \text{area of 16 rectangles}\\ &= 0.63476.\end{aligned}$$



Dikdörtgenlerin sayısını her arttırdığımızda, dikdörtgenlerin toplam alanı, R alanına daha da yakınlaşıyor.

$$\begin{aligned}R\text{'nin alanı} &\approx \text{16 dikdörtgenin toplam alanı}\\ &= 0.63476.\end{aligned}$$

Sigma Notation

Sigma Notasyonu

$$\sum_{k=1}^{n} a_k = a_1 + a_2 + a_3 + a_4 + a_5 + a_6 + a_7 + a_8 + \ldots + a_{n-1} + a_n$$

the Greek letter Sigma

$$\sum_{k=1}^{n} a_k$$

the sum finishes at $k = n$
indis k, $k = n$'de son bulur

the sum starts at $k = 1$
indis k, $k = 1$'de başlar

Example 28.1.

$$1^2 + 2^2 + 3^2 + 4^2 + 5^2 + 6^2 + 7^2 + 8^2 + 9^2 + 10^2 + 11^2 = \sum_{k=1}^{11} k^2$$

$$f(1) + f(2) + f(3) + \ldots + f(99) + f(100) = \sum_{k=1}^{100} f(k)$$

$$\sum_{k=1}^{5} k = 1 + 2 + 3 + 4 + 5 = 15$$

Örnek 28.2.

$$\sum_{k=1}^{3} (-1)^k k = (-1)(1) + (-1)^2(2) + (-1)^3(3) = -1 + 2 - 3 = -2$$

$$\sum_{k=1}^{2} \frac{k}{k+1} = \frac{1}{1+1} + \frac{2}{2+1} = \frac{1}{2} + \frac{2}{3} = \frac{7}{6}$$

$$\sum_{k=4}^{5} \frac{k^2}{k-1} = \frac{4^2}{4-1} + \frac{5^2}{5-1} = \frac{16}{3} + \frac{25}{4} = \frac{139}{12}$$

Example 28.3. I want to find a formula for $1+2+3+\ldots+n$. Note that

Örnek 28.3. $1+2+3+\ldots+n$ için bir formül bulmak istiyoruz. Dikkat edilirse

$$\begin{aligned} &2\big(1 + 2 + 3 + 4 + 5 + \ldots + (n-1) + n\big) \\ &\begin{array}{ccccccccccccccccc} = & 1 & + & 2 & + & 3 & + & 4 & + & 5 & + & \ldots & + & (n-1) & + & n \\ + & n & + & (n-1) & + & (n-2) & + & (n-3) & + & (n-4) & + & \ldots & + & 2 & + & 1 \\ = & (n+1) & + & (n+1) & + & (n+1) & + & (n+1) & + & (n+1) & + & \ldots & + & (1+n) & + & (1+n) \end{array} \\ &= n(n+1). \end{aligned}$$

Therefore

$$\boxed{\sum_{k=1}^{n} k = \frac{n(n+1)}{2}.}$$

Similarly (but more difficult) we can find that

$$\boxed{\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}}$$

and

$$\boxed{\sum_{k=1}^{n} k^3 = \left(\frac{n(n+1)}{2}\right)^2.}$$

Dolayısıyla

$$\boxed{\sum_{k=1}^{n} k = \frac{n(n+1)}{2}.}$$

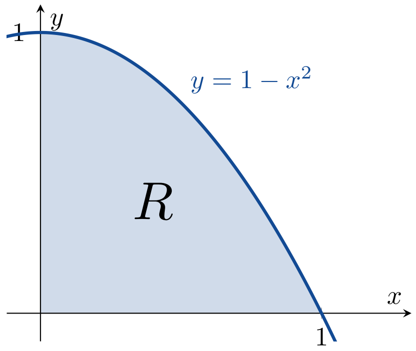
Benzer olarak (ama daha zor) şunu buluruz

$$\boxed{\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}}$$

ve

$$\boxed{\sum_{k=1}^{n} k^3 = \left(\frac{n(n+1)}{2}\right)^2.}$$

Limits of Finite Sums



Here's the plan:

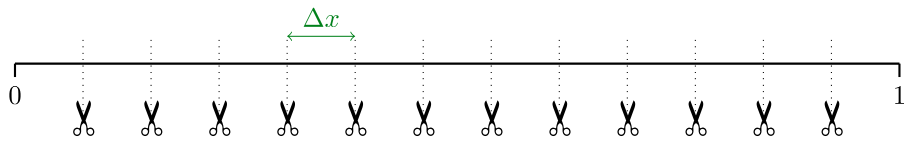
STEP 1. We will cut $[0, 1]$ in to n pieces of width

$$\Delta x = \frac{1-0}{n} = \frac{1}{n}.$$

STEP 2. We will use n rectangles to approximate the area of R. See figure 28.1.

STEP 3. Then we will take the limit as $n \to \infty$.

Sonlu Toplamların Limitleri



İşte izleyeceğimiz yol:

ADIM 1. $[0, 1]$'i n parçaya bölersek

$$\Delta x = \frac{1-0}{n} = \frac{1}{n}.$$

ADIM 2. n tane dikdörtgenle R'nin alanını yaklaşık olarak buluruz. Bkz. şekil 28.1.

ADIM 3. Daha sonra $n \to \infty$ iken limit alırız.

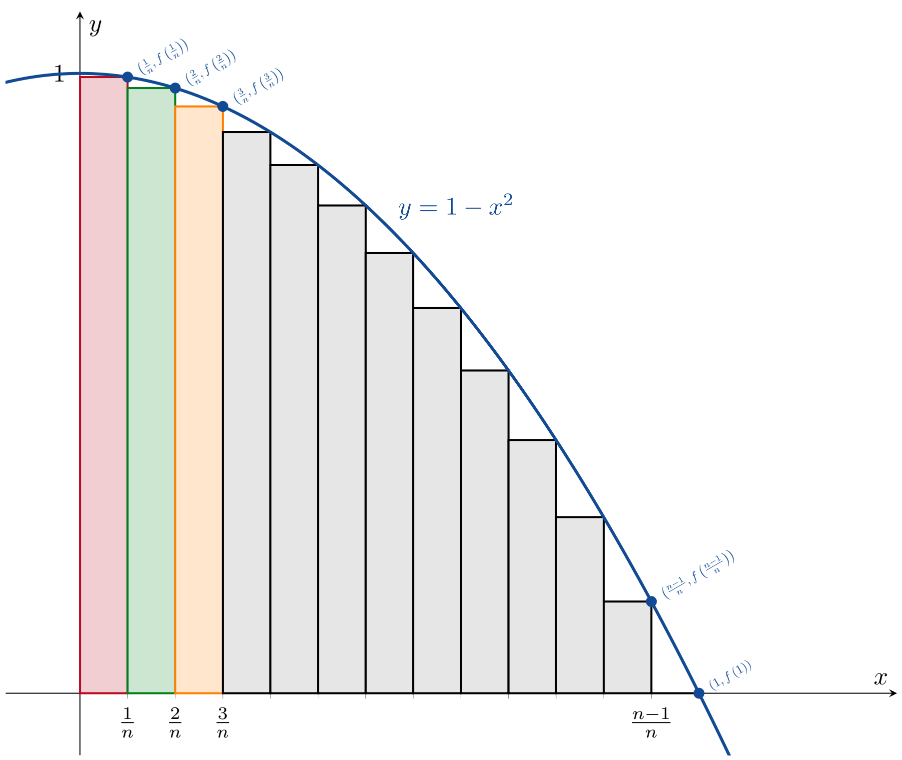


Figure 28.1: We can use n rectangles to approximate the area of R.
Şekil 28.1: n tane dikdörtgeni R'nin alanını yaklaşık hesaplamakta kullanabiliriz.

Let $f(x) = 1 - x^2$. Then

- the first rectangle has area $\frac{1}{n} f\left(\frac{1}{n}\right)$;
- the second rectangle has area $\frac{1}{n} f\left(\frac{2}{n}\right)$;
- the third rectangle has area $\frac{1}{n} f\left(\frac{3}{n}\right)$;

and so on.

Let $f(x) = 1 - x^2$. Then

- ilk dikdörtgen alanı $\frac{1}{n} f\left(\frac{1}{n}\right)$;
- ikinci dikdörtgen alanı $\frac{1}{n} f\left(\frac{2}{n}\right)$;
- üçüncü dikdörtgen alanı $\frac{1}{n} f\left(\frac{3}{n}\right)$;

ve saire.

The area of all n rectangles is

$$
\begin{aligned}
\text{area} &= \sum_{k=1}^{n} (\text{area of the } k\text{th rectangle}) \\
&= \sum_{k=1}^{n} \frac{1}{n} f\left(\frac{k}{n}\right) \\
&= \sum_{k=1}^{n} \frac{1}{n} \left(1 - \left(\frac{k}{n}\right)^2\right) \\
&= \sum_{k=1}^{n} \left(\frac{1}{n} - \frac{k^2}{n^3}\right) \\
&= \sum_{k=1}^{n} \frac{1}{n} - \sum_{k=1}^{n} \frac{k^2}{n^3} \\
&= n\left(\frac{1}{n}\right) - \frac{1}{n^3} \sum_{k=1}^{n} k^2 \\
&= 1 - \frac{1}{n^3} \left(\frac{n(n+1)(2n+1)}{6}\right) \\
&= 1 - \frac{2n^2+3n+1}{6n^2}.
\end{aligned}
$$

Taking the limit gives

$$
\begin{aligned}
\lim_{n\to\infty} \left(\sum_{k=1}^{n} \frac{1}{n} f\left(\frac{k}{n}\right)\right) &= \lim_{n\to\infty} \left(1 - \frac{2n^2+3n+1}{6n^2}\right) \\
&= 1 - \frac{2}{6} = \frac{2}{3}.
\end{aligned}
$$

Therefore the area of R is $\frac{2}{3}$.

Riemann Sums

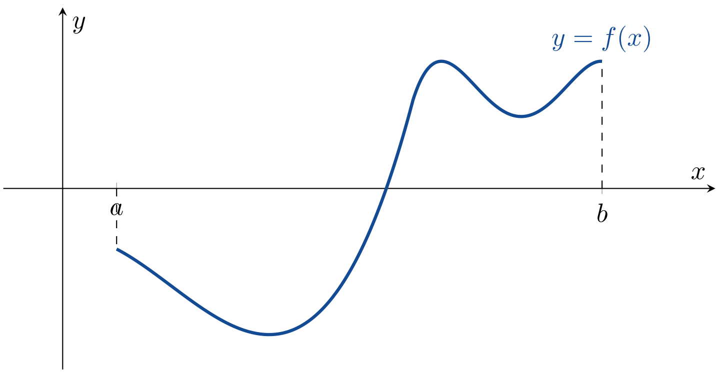


Figure 28.2: A function $f : [a, b] \to \mathbb{R}$.
Şekil 28.2: Bir fonksiyon $f : [a, b] \to \mathbb{R}$.

Now let $f[a, b] \to \mathbb{R}$ be a function. We will cut $[a, b]$ into n subintervals (the pieces don't have to all be the same size). In each subinterval we will choose one point $c_k \in [x_{k-1}, x_k]$, as shown in figure 28.3. The width of each subinterval is $\Delta x_k = x_k - x_{k-1}$.

On each subinterval $[x_{k-1}, x_k]$, we draw a rectangle of width Δx_k and height $f(c_k)$. See figure 28.4

n dikdörtgenin toplam alanı

$$
\begin{aligned}
\text{area} &= \sum_{k=1}^{n} (k \text{ inci dikdörtgen}) \\
&= \sum_{k=1}^{n} \frac{1}{n} f\left(\frac{k}{n}\right) \\
&= \sum_{k=1}^{n} \frac{1}{n} \left(1 - \left(\frac{k}{n}\right)^2\right) \\
&= \sum_{k=1}^{n} \left(\frac{1}{n} - \frac{k^2}{n^3}\right) \\
&= \sum_{k=1}^{n} \frac{1}{n} - \sum_{k=1}^{n} \frac{k^2}{n^3} \\
&= n\left(\frac{1}{n}\right) - \frac{1}{n^3} \sum_{k=1}^{n} k^2 \\
&= 1 - \frac{1}{n^3} \left(\frac{n(n+1)(2n+1)}{6}\right) \\
&= 1 - \frac{2n^2+3n+1}{6n^2}.
\end{aligned}
$$

Limit alınırsa

$$
\begin{aligned}
\lim_{n\to\infty} \left(\sum_{k=1}^{n} \frac{1}{n} f\left(\frac{k}{n}\right)\right) &= \lim_{n\to\infty} \left(1 - \frac{2n^2+3n+1}{6n^2}\right) \\
&= 1 - \frac{2}{6} = \frac{2}{3}.
\end{aligned}
$$

Buradan R'nin alanı $\frac{2}{3}$ olur.

Riemann Toplamı

Şimdi $f : [a, b] \to \mathbb{R}$ bir fonksiyon olsun. $[a, b]$'yi n aralığa böleriz (parçaların hepsinin aynı genişlikte olması gerekmez). Her alt-aralıkta, Şekil 28.3'de gösterildiği gibi $[x_{k-1}, x_k]$ cinsinden bir nokta c_k seçeriz. Her alt aralığın genişliği $\Delta x_k = x_k - x_{k-1}$'dır.

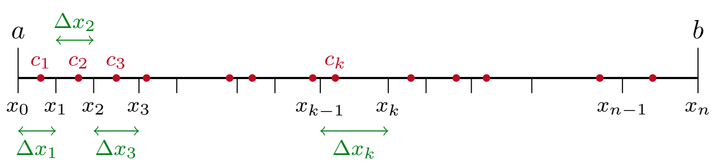


Figure 28.3: We split the interval $[a, b]$ into n subintervals. Note that $a = x_0 < x_1 < x_2 < x_3 < \ldots < x_{n-1} < x_n = b$.
Şekil 28.3: $[a, b]$ aralığını n alt-aralığa bölünür. Dikkat edilirse, $a = x_0 < x_1 < x_2 < x_3 < \ldots < x_{n-1} < x_n = b$ dir.

Her bir $[x_{k-1}, x_k]$ alt-aralığında, genişliği Δx_k ve yüksekliği $f(c_k)$ olan dikdörtgenler çizilir. Bkz. şekil 28.4

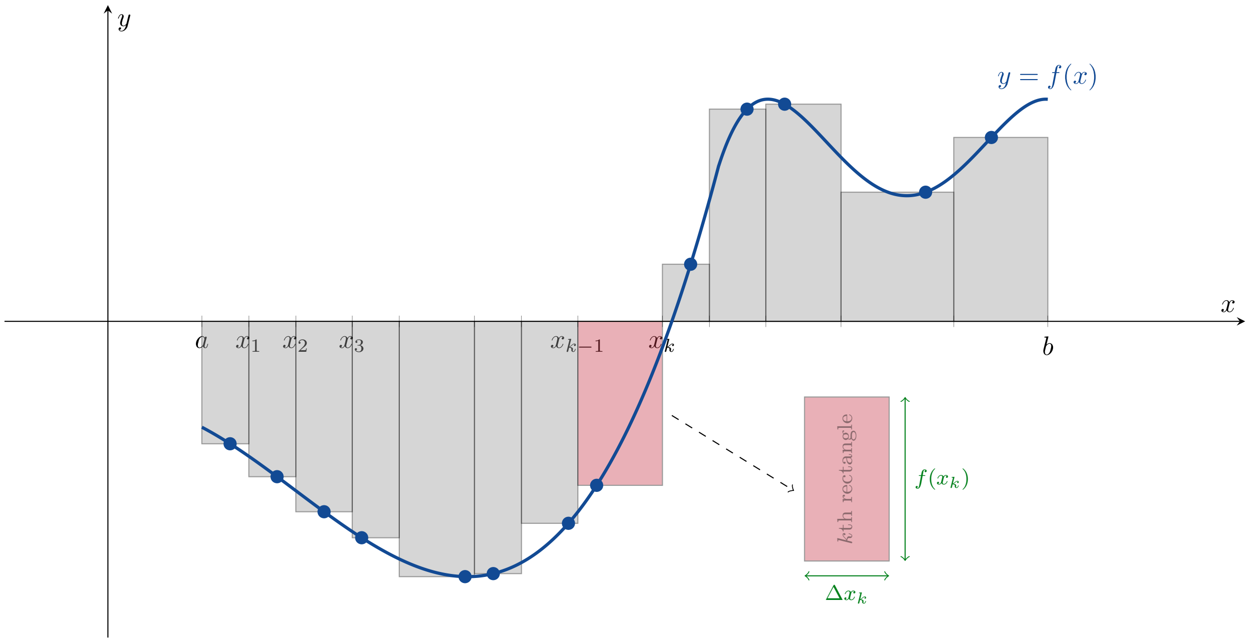


Figure 28.4: n rectangles.
Şekil 28.4: n tane dikdörtgeni.

Note that if $f(c_k) < 0$, then the rectangle on $[x_{k-1}, x_k]$ will have 'negative area' – this is ok.

The total of the n rectangles is

$$\sum_{k=1}^{n} f(c_k)\Delta x_k.$$

This is called a ***Riemann Sum for f on $[a, b]$***. Then we want to take the limit as $n \to \infty$ (or more precisely, we want to take the limit as $\max\{\Delta x_1, \Delta x_2, \ldots, \Delta x_n\} \to 0$). Sometimes this limit exists, sometimes this limit does not exist.

$f(c_k) < 0$ olduğuna dikkat edersek, tabanı $[x_{k-1}, x_k]$ olan dikdörtgen 'negatif aalanlı' – olur.

n dikdörtgenin toplam alanı

$$\sum_{k=1}^{n} f(c_k)\Delta x_k.$$

Bu toplama bir ***f nin $[a, b]$ üzerindeki bir Riemann Toplamı*** denir. Sonra $n \to \infty$ iken limit alınır (veya daha doğrusu, maks$\{\Delta x_1, \Delta x_2, \ldots, \Delta x_n\} \to 0$ iken limit alınır). Bu limit bazen mevcuttur, bazen mevcut değil.

29 The Definite Integral

Definition. **If** the limit

$$\lim_{n\to\infty}\sum_{k=1}^{n} f(c_k)\Delta x_k$$

exists, then it is called the ***definite integral of f over*** $[a,b]$. We write

$$\int_a^b f(x)\ dx = \lim_{n\to\infty}\sum_{k=1}^{n} f(c_k)\Delta x_k$$

if the limit exists.

Belirli İntegral

Tanım. **Eğer**

$$\lim_{n\to\infty}\sum_{k=1}^{n} f(c_k)\Delta x_k$$

limiti mevcutsa, bu limite ***f'nin*** $[a,b]$ ***üzerindeki belirli integrali*** adı verilir. Şöyle gösteririz

$$\int_a^b f(x)\ dx = \lim_{n\to\infty}\sum_{k=1}^{n} f(c_k)\Delta x_k$$

tabi eğer limit mevcutsa.

upper limit of integration
integralin üst sınırı

x is the variable of integration
x, integral değişkenidir

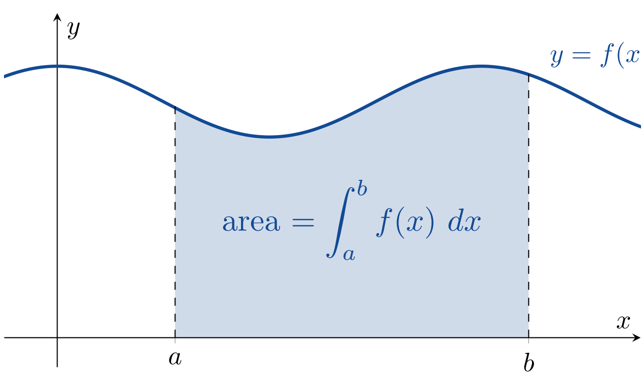
integral sign
integral işareti

$$\int_a^b \underbrace{f(x)}_{\substack{\text{the integrand}\\ \text{integralin integrandı}}}\ dx$$

"the integral of f from a to b"

"a'dan b'ye f'nin integrali"

lower limit of integration
integralin alt sınırı



Definition. If $\int_a^b f(x)\ dx$ exists, then we say that ***f is integrable on $[a,b]$***.

Example 29.1. $f(x) = 1 - x^2$ is integrable on $[0,1]$ and $\int_0^1 (1-x^2)\ dx = \frac{2}{3}$.

Remark.

$$\int_a^b f(x)\ dx = \int_a^b f(u)\ du = \int_a^b f(t)\ dt$$

It doesn't matter which letter we use for the ***dummy variable***.

Theorem 29.1. *If f is continuous on $[a,b]$, then f is integrable on $[a,b]$.*

If f has finitely many jump discontinuities but is otherwise continuous on $[a,b]$, then f is integrable on $[a,b]$.

Example 29.2. Define a function $g : [0,1] \to \mathbb{R}$ by

$$g(x) = \begin{cases} 1 & x \in \mathbb{Q} \\ 0 & x \notin \mathbb{Q}. \end{cases}$$

See figure 29.1. This function is not integrable on $[0,1]$.

Tanım. Eğer $\int_a^b f(x)\ dx$ mevcutsa, ***f fonksiyonu $[a,b]$ üzerinde integrallenebilir*** denir.

Örnek 29.1. $f(x) = 1 - x^2$ fonksiyonu $[0,1]$ üzerinde integrallenebilir ve $\int_0^1 (1-x^2)\ dx = \frac{2}{3}$.

Not.

$$\int_a^b f(x)\ dx = \int_a^b f(u)\ du = \int_a^b f(t)\ dt$$

takma değişken için hangi sembol kullandığımızın bir önemi yok.

Teorem 29.1. *Eğer f fonksiyonu $[a,b]$'de sürekli ise, $[a,b]$'de f integrallenebilirdir.*

Eğer f sonlu sayıda sıçramalı süreksizliği varsa veya $[a,b]$'de sürekli ise, then $[a,b]$ üzerinde f integrallenebilirdir.

Örnek 29.2. Şu fonksiyonu tanımlarsak $g : [0,1] \to \mathbb{R}$ öyle ki

$$g(x) = \begin{cases} 1 & x \in \mathbb{Q} \\ 0 & x \notin \mathbb{Q}. \end{cases}$$

Bkz. şekil 29.1. Bu fonksiyon $[0,1]$'de integrallenemez.

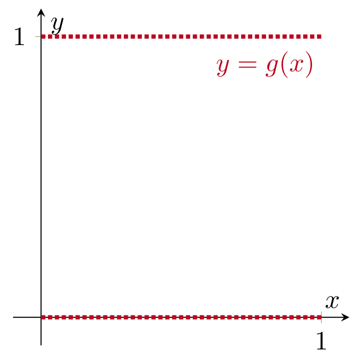


Figure 29.1: The graph of $g(x)$ defined in Example 29.2.
Şekil 29.1: Örnekteki $g(x)$ grafiği

Properties of Definite Integrals

Theorem 29.2. *Suppose that f and g are integrable. Let k be a number. Then*

(i). $\int_b^a f(x)\ dx = -\int_a^b f(x)\ dx;$

(ii). $\int_a^b kf(x)\ dx = k\int_a^b f(x)\ dx;$

(iii). $\int_a^c f(x)\ dx + \int_c^b f(x)\ dx = \int_a^b f(x)\ dx$

(iv). $\int_a^a f(x)\ dx = 0;$

(v). $\int_a^b \big(f(x)+g(x)\big)\ dx = \int_a^b f(x)\ dx + \int_a^b g(x)\ dx;$

(vi). $(b-a)\min f \leq \int_a^b f(x)\ dx \leq (b-a)\max f;$

(vii). *if* $f(x) \leq g(x)$ *on* $[a,b]$, *then*

$$\int_a^b f(x)\ dx \leq \int_a^b g(x)\ dx;$$

$(viii)$. *if* $g(x) \geq 0$ *on* $[a,b]$, *then*

$$\int_a^b g(x)\ dx \geq 0;$$

(ix). *if f is an even function, then*

$$\int_{-a}^a f(x)\ dx = 2\int_0^a f(x)\ dx;$$

and

(x). *if f is an odd function, then*

$$\int_{-a}^a f(x)\ dx = 0.$$

Belirli İntegralin Özellikleri

Teorem 29.2. *f ve g integrallenebilir olsunlar. k bir sabit sayı olsun. Bu durumda*

(i). $\int_b^a f(x)\ dx = -\int_a^b f(x)\ dx;$

(ii). $\int_a^b kf(x)\ dx = k\int_a^b f(x)\ dx;$

(iii). $\int_a^c f(x)\ dx + \int_c^b f(x)\ dx = \int_a^b f(x)\ dx$

(iv). $\int_a^a f(x)\ dx = 0;$

(v). $\int_a^b \big(f(x)+g(x)\big)\ dx = \int_a^b f(x)\ dx + \int_a^b g(x)\ dx;$

(vi). $(b-a)\min f \leq \int_a^b f(x)\ dx \leq (b-a)\,\text{maks}\, f;$

(vii). $f(x) \leq g(x)$ *on* $[a,b]$ *ise,*

$$\int_a^b f(x)\ dx \leq \int_a^b g(x)\ dx;$$

$(viii)$. $[a,b]$ *üzerinde* $g(x) \geq 0$ *ise,*

$$\int_a^b g(x)\ dx \geq 0;$$

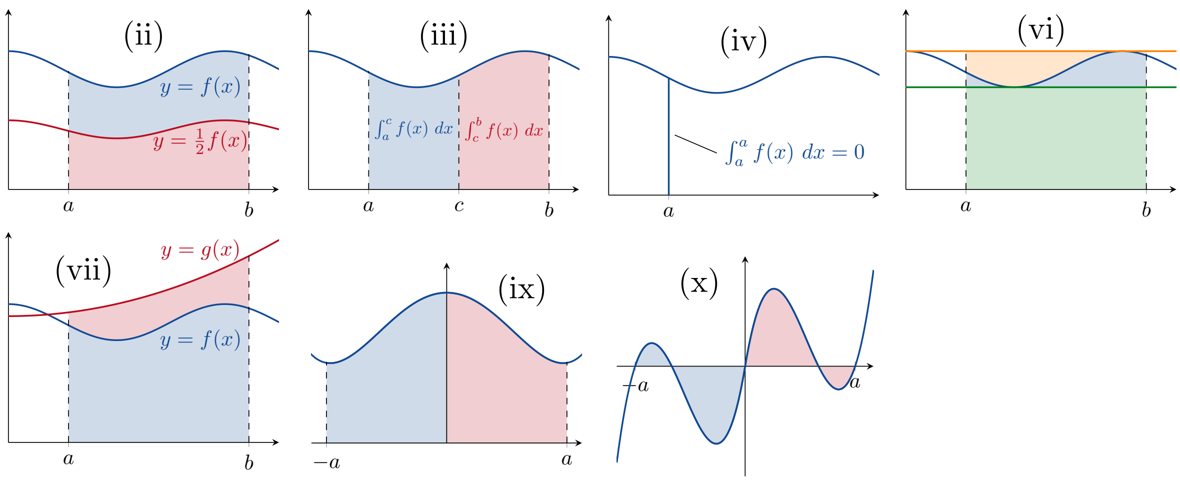
(ix). *f çift fonksiyon ise,*

$$\int_{-a}^a f(x)\ dx = 2\int_0^a f(x)\ dx;$$

ve

(x). *if f tek fonksiyon ise,*

$$\int_{-a}^a f(x)\ dx = 0.$$



Example 29.3. Suppose that $\int_{-1}^{1} f(x)\ dx = 5$, $\int_{1}^{4} f(x)\ dx = -2$ and $\int_{-1}^{1} h(x)\ dx = 7$. Then

$$\int_{4}^{1} f(x)\ dx = -\int_{1}^{4} f(x)\ dx = 2,$$

$$\begin{aligned}\int_{-1}^{1} \big(2f(x)+3h(x)\big)\ dx &= 2\int_{-1}^{1} f(x)\ dx + 3\int_{-1}^{1} h(x)\ dx\\ &= 2(5)+3(7)=31\end{aligned}$$

and

$$\begin{aligned}\int_{-1}^{4} f(x)\ dx &= \int_{-1}^{1} f(x)\ dx + \int_{1}^{4} f(x)\ dx\\ &= 5+(-2)=3.\end{aligned}$$

Example 29.4. Show that $\int_{0}^{1} \sqrt{1+\cos x}\ dx \le \sqrt{2}$.

solution: The maximum value of $\sqrt{1+\cos x}$ on $[0,1]$ is $\sqrt{1+1}=\sqrt{2}$. Therefore

$$\int_{0}^{1} \sqrt{1+\cos x}\ dx \le (1-0)\max\sqrt{1+\cos x} = 1\times\sqrt{2}.$$

Example 29.5. Calculate $\int_{-2}^{2} (x^3+x)\ dx$.

solution: Because (x^3+x) is an odd function, we have that

$$\int_{-2}^{2} (x^3+x)\ dx = 0.$$

Example 29.6. Calculate $\int_{-1}^{1} (1-x^2)\ dx$.

solution: Because $(1-x^2)$ is an even function, we have that

$$\int_{-1}^{1} (1-x^2)\ dx = 2\int_{0}^{1} (1-x^2)\ dx = 2\times\frac{2}{3}=\frac{4}{3}.$$

Örnek 29.3. Varsayalım ki $\int_{-1}^{1} f(x)\ dx = 5$, $\int_{1}^{4} f(x)\ dx = -2$ ve $\int_{-1}^{1} h(x)\ dx = 7$. O zaman

$$\int_{4}^{1} f(x)\ dx = -\int_{1}^{4} f(x)\ dx = 2,$$

$$\begin{aligned}\int_{-1}^{1} \big(2f(x)+3h(x)\big)\ dx &= 2\int_{-1}^{1} f(x)\ dx + 3\int_{-1}^{1} h(x)\ dx\\ &= 2(5)+3(7)=31\end{aligned}$$

ve

$$\begin{aligned}\int_{-1}^{4} f(x)\ dx &= \int_{-1}^{1} f(x)\ dx + \int_{1}^{4} f(x)\ dx\\ &= 5+(-2)=3.\end{aligned}$$

Örnek 29.4. Gösteriniz ki $\int_{0}^{1} \sqrt{1+\cos x}\ dx \le \sqrt{2}$.

çözüm: $[0,1]$ üzerindeki $\sqrt{1+\cos x}$'nin maksimum değeri $\sqrt{1+1}=\sqrt{2}$. Buradan

$$\int_{0}^{1} \sqrt{1+\cos x}\ dx \le (1-0)\,\text{maks}\sqrt{1+\cos x} = 1\times\sqrt{2}.$$

Örnek 29.5. $\int_{-2}^{2} (x^3+x)\ dx$ hesaplayınız.

çözüm: (x^3+x) tek fonksiyon olduğundan, şunu elde ederiz:

$$\int_{-2}^{2} (x^3+x)\ dx = 0.$$

Örnek 29.6. $\int_{-1}^{1} (1-x^2)\ dx$ hesaplayınız.

çözüm: $(1-x^2)$ çift fonksiyon olduğu için,

$$\int_{-1}^{1} (1-x^2)\ dx = 2\int_{0}^{1} (1-x^2)\ dx = 2\times\frac{2}{3}=\frac{4}{3}.$$

Example 29.7. Calculate $\int_0^b x\,dx$ for $b>0$.

solution 1: We will use a Riemann Sum. First we cut $[0,b]$ in to n pieces using

$$0 < \frac{b}{n} < \frac{2b}{n} < \frac{3b}{n} < \ldots < \frac{(n-1)b}{n} < b$$

and $c_k = \frac{kb}{n}$. Note that $\Delta x_k = \frac{b}{n}$ for all k. See figure 29.2. Then

$$\begin{aligned}\sum_{k=1}^{n} f(c_k)\Delta x_k &= \sum_{k=1}^{n} \frac{kb}{n}\frac{b}{n} = \frac{b^2}{n^2}\sum_{k=1}^{n} k\\ &= \frac{b^2}{n^2}\left(\frac{n(n+1)}{2}\right) = \frac{b^2}{2}\left(1+\frac{1}{n}\right).\end{aligned}$$

Then

$$\begin{aligned}\int_0^b x\,dx &= \lim_{n\to\infty}\sum_{k=1}^{n} f(c_k)\Delta x_k\\ &= \lim_{n\to\infty}\frac{b^2}{2}\left(1+\frac{1}{n}\right) = \frac{b^2}{2}.\end{aligned}$$

Örnek 29.7. $b>0$ ise $\int_0^b x\,dx$ integralini bulunuz.

çözüm 1: Riemann Toplamı kullanacağız. Önce $[0,b]$'yi n parçaya

$$0 < \frac{b}{n} < \frac{2b}{n} < \frac{3b}{n} < \ldots < \frac{(n-1)b}{n} < b$$

ve $c_k = \frac{kb}{n}$ kullanarak böleriz. Dikkat edilirse her k için $\Delta x_k = \frac{b}{n}$ olur. Bkz. şekil 29.2. Bu durumda

$$\begin{aligned}\sum_{k=1}^{n} f(c_k)\Delta x_k &= \sum_{k=1}^{n} \frac{kb}{n}\frac{b}{n} = \frac{b^2}{n^2}\sum_{k=1}^{n} k\\ &= \frac{b^2}{n^2}\left(\frac{n(n+1)}{2}\right) = \frac{b^2}{2}\left(1+\frac{1}{n}\right).\end{aligned}$$

O halde

$$\begin{aligned}\int_0^b x\,dx &= \lim_{n\to\infty}\sum_{k=1}^{n} f(c_k)\Delta x_k\\ &= \lim_{n\to\infty}\frac{b^2}{2}\left(1+\frac{1}{n}\right) = \frac{b^2}{2}.\end{aligned}$$

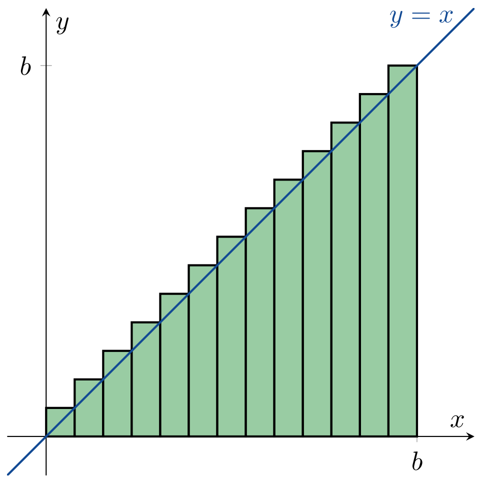


Figure 29.2: Approximating $\int_0^b x\,dx$ by n rectangles.
Şekil 29.2: n dikdörtgenle $\int_0^b x\,dx$ ye yaklaşık bulmak.

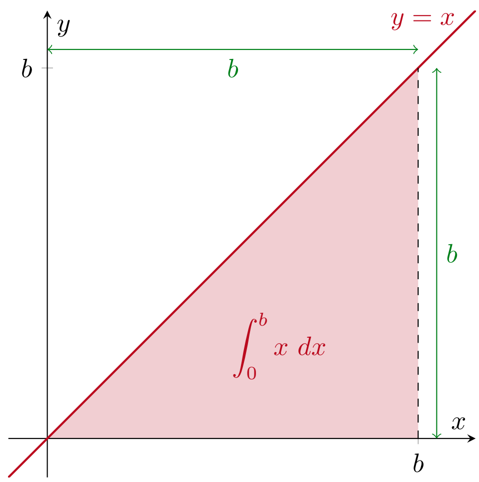


Figure 29.3: The integral of x from 0 to b.
Şekil 29.3: 0 dan b ye x in integrali.

solution 2: Alternately, we can look at figure 29.3 and say that

$$\int_0^b x\,dx = \text{area of a triangle} = \frac{1}{2}\times b\times b = \frac{b^2}{2}.$$

çözüm 2: Alternatif olarak, şekil 29.3 e bakarak

$$\int_0^b x\,dx = \text{area of a triangle} = \frac{1}{2}\times b\times b = \frac{b^2}{2}.$$

Example 29.8.

$$\begin{aligned}\int_a^b x\,dx &= \int_a^0 x\,dx + \int_0^b x\,dx\\ &= -\int_0^a x\,dx + \int_0^b x\,dx\\ &= -\frac{a^2}{2} + \frac{b^2}{2}\\ &= \frac{b^2}{2} - \frac{a^2}{2}.\end{aligned}$$

Örnek 29.8.

$$\begin{aligned}\int_a^b x\,dx &= \int_a^0 x\,dx + \int_0^b x\,dx\\ &= -\int_0^a x\,dx + \int_0^b x\,dx\\ &= -\frac{a^2}{2} + \frac{b^2}{2}\\ &= \frac{b^2}{2} - \frac{a^2}{2}.\end{aligned}$$

30 The Fundamental Theorem of Calculus

We don't want to have to use Riemann sums every time we need to calculate a definite integral – we want a better way. The following theorem is the most important theorem in Calculus. If you can only memorise one theorem for the exams, it should be this one.

Theorem 30.1 (The Fundamental Theorem of Calculus)**.** *Suppose that $f : [a, b] \to \mathbb{R}$ is a continuous function.*

(*i*). *Then the function $F : [a, b] \to \mathbb{R}$ defined by*

$$F(x) = \int_a^x f(t)\ dt$$

is continuous on $[a, b]$; differentiable on (a, b); and its derivative is

$$\boxed{F'(x) = \frac{d}{dx}\int_a^x f(t)\ dt = f(x).}$$

(*ii*). *If F is any antiderivative of f on $[a, b]$, then*

$$\boxed{\int_a^b f(x)\ dx = F(b) - F(a).}$$

Remark. Part (i) of the theorem tells how to differentiate $\int_a^x f(t)\ dt$.

Example 30.1. Find $\frac{dy}{dx}$ if $y = \int_a^x (t^3 + 1)\ dt$.

solution:

$$\frac{dy}{dx} = \frac{d}{dx}\int_a^x (t^3 + 1)\ dt = x^3 + 1.$$

Example 30.2. Find $\frac{dy}{dx}$ if $y = \int_1^x \sin t\ dt$.

solution:

$$\frac{dy}{dx} = \frac{d}{dx}\int_1^x \sin t\ dt = \sin x.$$

Example 30.3. Find $\frac{dy}{dx}$ if $y = \int_0^x \sin\ln\tan e^{t^2}\ dt$.

Kalkülüsün Temel Teoremi

Bir belirli integrali hesaplamanız gerektiğinde her defasında Riemann toplamlarını kullanmamız gerekmiyor – daha iyi bir yol istiyoruz. Aşağıdaki teorem Kalkülüsün en önemli teoremidir. Sınavlar için bir teorem ezberleyeceğim diyorsanız, işte bu o teoremdir.

Teorem 30.1 (Kalkülüsün Temel Teoremi)**.** *$f : [a, b] \to \mathbb{R}$'nin sürekli bir fonksiyon olduğunu varsayalım.*

(*i*). *Bu durumda $F : [a, b] \to \mathbb{R}$,*

$$F(x) = \int_a^x f(t)\ dt$$

de $[a, b]$ üzerinde süreklidir; (a, b) üzerinde türevlenebilir; ve türevi $f(x)$'tir

$$\boxed{F'(x) = \frac{d}{dx}\int_a^x f(t)\ dt = f(x).}$$

(*ii*). *Eğer F de f'nin $[a, b]$ üzerindeki herhangi bir ters türevi ise, bu durumda*

$$\boxed{\int_a^b f(x)\ dx = F(b) - F(a).}$$

Not. Teoremin (i) kısmı $\int_a^x f(t)\ dt$'in türevini nasıl alacağımızı söyler.

Örnek 30.1. $y = \int_a^x (t^3 + 1)\ dt$ ise, $\frac{dy}{dx}$'i bulunuz.

çözüm:

$$\frac{dy}{dx} = \frac{d}{dx}\int_a^x (t^3 + 1)\ dt = x^3 + 1.$$

Örnek 30.2. $y = \int_1^x \sin t\ dt$ ise, $\frac{dy}{dx}$'i bulunuz.

çözüm:

$$\frac{dy}{dx} = \frac{d}{dx}\int_1^x \sin t\ dt = \sin x.$$

Örnek 30.3. $y = \int_0^x \sin\ln\tan e^{t^2}\ dt$ ise, $\frac{dy}{dx}$'i bulunuz.

solution:

$$\frac{dy}{dx} = \frac{d}{dx}\int_0^x \sin\ln\tan e^{t^2}\, dt = \sin\ln\tan e^{x^2}.$$

Example 30.4. Find $\frac{dy}{dx}$ if $y = \int_x^5 3t\sin t\, dt$.

solution:

$$\begin{aligned}\frac{dy}{dx} &= \frac{d}{dx}\int_x^5 3t\sin t\, dt\\ &= \frac{d}{dx}\left(-\int_5^x 3t\sin t\, dt\right)\\ &= -3x\sin x.\end{aligned}$$

Example 30.5. Find $\frac{dy}{dx}$ if $y = \int_1^{x^2}\cos t\, dt$.

solution: This time we will need to use the Chain rule. Let $u = x^2$. Then

$$\begin{aligned}\frac{dy}{dx} &= \frac{dy}{du}\frac{du}{dx}\\ &= \left(\frac{d}{du}\int_1^u \cos t\, dt\right)\left(\frac{d}{dx}x^2\right)\\ &= (\cos u)(2x) = 2x\cos x^2.\end{aligned}$$

Remark. Part (ii) of the theorem tells us how to calculate the definite integral of f over $[a,b]$:

Step 1. Find an antiderivative F of f.

Step 2. Calculate $F(b) - F(a)$.

Notation. We will write

$$\Big[F(x)\Big]_a^b = F(b) - F(a).$$

Example 30.6.

$$\begin{aligned}\int_0^\pi \cos x\, dx &= \Big[\sin x\Big]_0^\pi\\ &\qquad \left(\text{because } \tfrac{d}{dx}\sin x = \cos x\right)\\ &= \sin\pi - \sin 0\\ &= 0 - 0\\ &= 0\end{aligned}$$

Example 30.7.

$$\begin{aligned}\int_{-\frac{\pi}{4}}^0 \sec x\tan x &= \Big[\sec x\Big]_{-\frac{\pi}{4}}^0\\ &\qquad \left(\text{because } \tfrac{d}{dx}\sec x = \sec x\tan x\right)\\ &= \sec 0 - \sec -\frac{\pi}{4}\\ &= 1 - \sqrt{2}.\end{aligned}$$

Example 30.8.

$$\begin{aligned}\int_1^4\left(\frac{3}{2}\sqrt{x} - \frac{4}{x^2}\right)dx &= \left[x^{\frac{3}{2}} + \frac{4}{x}\right]_1^4\\ &\qquad \left(\text{because } \tfrac{d}{dx}\left(x^{\frac{3}{2}} + \frac{4}{x}\right) = \frac{3}{2}\sqrt{x} - \frac{4}{x^2}\right)\\ &= \left(4^{\frac{3}{2}} + \frac{4}{4}\right) - \left(1^{\frac{3}{2}} + \frac{4}{1}\right)\\ &= (8+1) - (1+4)\\ &= 4.\end{aligned}$$

çözüm:

$$\frac{dy}{dx} = \frac{d}{dx}\int_0^x \sin\ln\tan e^{t^2}\, dt = \sin\ln\tan e^{x^2}.$$

Örnek 30.4. $y = \int_x^5 3t\sin t\, dt$ ise, $\frac{dy}{dx}$'i bulunuz.

çözüm:

$$\begin{aligned}\frac{dy}{dx} &= \frac{d}{dx}\int_x^5 3t\sin t\, dt\\ &= \frac{d}{dx}\left(-\int_5^x 3t\sin t\, dt\right)\\ &= -3x\sin x.\end{aligned}$$

Örnek 30.5. $y = \int_1^{x^2}\cos t\, dt$ ise, $\frac{dy}{dx}$'i bulunuz.

çözüm: Bu sefer Zincir kuralı kullanmamız gerekecek. $u = x^2$ diyelim. O zaman

$$\begin{aligned}\frac{dy}{dx} &= \frac{dy}{du}\frac{du}{dx}\\ &= \left(\frac{d}{du}\int_1^u \cos t\, dt\right)\left(\frac{d}{dx}x^2\right)\\ &= (\cos u)(2x) = 2x\cos x^2.\end{aligned}$$

Not. Teoremin (ii) kısmı f'nin $[a,b]$ üzerindeki belirli integrali nasıl hesaplayacağımızı söyler :

Adım 1. f'nin bir ters türevi olan F'yi bulunuz.

Adım 2. $F(b) - F(a)$ sayısını hesaplayınız.

Notasyon. We will write

$$\Big[F(x)\Big]_a^b = F(b) - F(a).$$

Örnek 30.6.

$$\begin{aligned}\int_0^\pi \cos x\, dx &= \Big[\sin x\Big]_0^\pi\\ &\qquad \left(\text{çünkü } \tfrac{d}{dx}\sin x = \cos x\right)\\ &= \sin\pi - \sin 0\\ &= 0 - 0\\ &= 0\end{aligned}$$

Örnek 30.7.

$$\begin{aligned}\int_{-\frac{\pi}{4}}^0 \sec x\tan x &= \Big[\sec x\Big]_{-\frac{\pi}{4}}^0\\ &\qquad \left(\text{çünkü } \tfrac{d}{dx}\sec x = \sec x\tan x\right)\\ &= \sec 0 - \sec -\frac{\pi}{4}\\ &= 1 - \sqrt{2}.\end{aligned}$$

Örnek 30.8.

$$\begin{aligned}\int_1^4\left(\frac{3}{2}\sqrt{x} - \frac{4}{x^2}\right)dx &= \left[x^{\frac{3}{2}} + \frac{4}{x}\right]_1^4\\ &\qquad \left(\text{çünkü } \tfrac{d}{dx}\left(x^{\frac{3}{2}} + \frac{4}{x}\right) = \frac{3}{2}\sqrt{x} - \frac{4}{x^2}\right)\\ &= \left(4^{\frac{3}{2}} + \frac{4}{4}\right) - \left(1^{\frac{3}{2}} + \frac{4}{1}\right)\\ &= (8+1) - (1+4)\\ &= 4.\end{aligned}$$

Total Area

Example 30.9. Let $f(x) = x^2 - 4$ and $g(x) = 4 - x^2$. See figure 30.1. We have that

$$\begin{aligned}\int_{-2}^{2} f(x)\ dx &= \int_{-2}^{2} (x^2-4)\ dx = \left[\frac{x^3}{3} - 4x\right]_{-2}^{2}\\ &= \left(\frac{8}{3}-8\right) - \left(\frac{-8}{3}+8\right) = -\frac{32}{3}\end{aligned}$$

and

$$\begin{aligned}\int_{-2}^{2} g(x)\ dx &= \int_{-2}^{2} (4-x^2)\ dx = \left[4x - \frac{x^3}{3}\right]_{-2}^{2}\\ &= \left(8-\frac{8}{3}\right) - \left(8+\frac{-8}{3}\right) = \frac{32}{3}.\end{aligned}$$

The total area between the graph of $y = f(x)$ and the x-axis, over $[-2, 2]$, is $\left|-\frac{32}{3}\right| = \frac{32}{3}$. The total area between the graph of $y = g(x)$ and the x-axis, over $[-2, 2]$, is $\left|\frac{32}{3}\right| = \frac{32}{3}$.

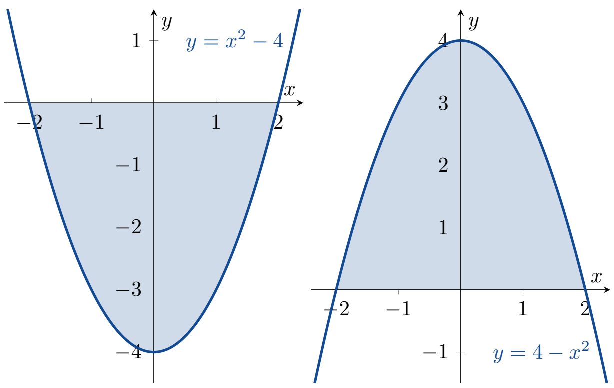


Figure 30.1: Graphs showing $\int_{-2}^{2}(x^2-4)\ dx$ and $\int_{-2}^{2}(4-x^2)\ dx$.
Şekil 30.1: $\int_{-2}^{2}(x^2-4)\ dx$ ve $\int_{-2}^{2}(4-x^2)\ dx$ integrallerini gösteren grafikler.

Example 30.10. Let $f(x) = \sin x$. Calculate

(a). the definite integral of f over $[0, 2\pi]$; and

(b). the total area between the graph of $y = f(x)$ and the x-axis over $[0, 2\pi]$.

solution:

(a).

$$\begin{aligned}\int_{0}^{2\pi} \sin x\ dx &= \Big[-\cos x\Big]_{0}^{2\pi} = -\cos 2\pi + \cos 0\\ &= -1+1 = 0.\end{aligned}$$

(b).

$$\begin{aligned}\text{total area} &= \int_{0}^{\pi} \sin x\ dx + \left|\int_{\pi}^{2\pi} \sin x\ dx\right|\\ &= \Big[-\cos x\Big]_{0}^{\pi} + \left|\Big[-\cos x\Big]_{\pi}^{2\pi}\right|\\ &= -\cos\pi + \cos 0 + |-\cos 2\pi + \cos\pi|\\ &= -(-1) + 1 + |-1+(-1)| = 4.\end{aligned}$$

Toplam Alan

Örnek 30.9. $f(x) = x^2 - 4$ ve $g(x) = 4 - x^2$ olsun. Bkz. şekil 30.1. Burada

$$\begin{aligned}\int_{-2}^{2} f(x)\ dx &= \int_{-2}^{2} (x^2-4)\ dx = \left[\frac{x^3}{3} - 4x\right]_{-2}^{2}\\ &= \left(\frac{8}{3}-8\right) - \left(\frac{-8}{3}+8\right) = -\frac{32}{3}\end{aligned}$$

ve

$$\begin{aligned}\int_{-2}^{2} g(x)\ dx &= \int_{-2}^{2} (4-x^2)\ dx = \left[4x - \frac{x^3}{3}\right]_{-2}^{2}\\ &= \left(8-\frac{8}{3}\right) - \left(8+\frac{-8}{3}\right) = \frac{32}{3}.\end{aligned}$$

$y = f(x)$ grafiği ve x-ekseni arasında kalan, $[-2, 2]$ üzerindeki toplam alan, is $\left|-\frac{32}{3}\right| = \frac{32}{3}$. $y = g(x)$ ve x-ekseni arasında kalan, $[-2, 2]$ üzerindeki toplam alan, ise $\left|\frac{32}{3}\right| = \frac{32}{3}$ olur.

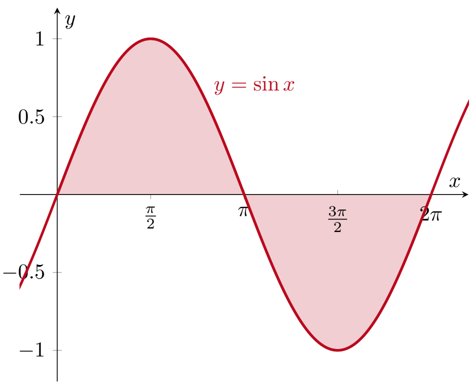


Figure 30.3: The total area between the graph $y = \sin x$ and the x-axis over $[0, 2\pi]$.
Şekil 30.3: $[0, 2\pi]$ üzerinde $y = \sin x$ grafiğiyle x-ekseni arasında kalan toplam alan.

Örnek 30.10. $f(x) = \sin x$ olsun.

(a). f'nin $[0, 2\pi]$ üzerindeki belirli integralini; ve

(b). $y = f(x)$ grafiği ile x-ekseni arasında $[0, 2\pi]$ üzerinde kalan alanı bulunuz.

çözüm:

(a).

$$\begin{aligned}\int_{0}^{2\pi} \sin x\ dx &= \Big[-\cos x\Big]_{0}^{2\pi} = -\cos 2\pi + \cos 0\\ &= -1+1 = 0.\end{aligned}$$

(b).

$$\begin{aligned}\text{toplam alan} &= \int_{0}^{\pi} \sin x\ dx + \left|\int_{\pi}^{2\pi} \sin x\ dx\right|\\ &= \Big[-\cos x\Big]_{0}^{\pi} + \left|\Big[-\cos x\Big]_{\pi}^{2\pi}\right|\\ &= -\cos\pi + \cos 0 + |-\cos 2\pi + \cos\pi|\\ &= -(-1) + 1 + |-1+(-1)| = 4.\end{aligned}$$

Summary

To find the ***total area*** between the graph of $y = f(x)$ and the x-axis over $[a, b]$:

STEP 1. Divide $[a, b]$ at the zeroes of f.

STEP 2. Integrate f over each subinterval.

STEP 3. Add the absolute values of the integrals.

Example 30.11. Find the total area between the graph of $y = x^3 - x^2 - 2x$ and the x-axis for $-1 \leq x \leq 2$.

solution:

1. Let $f(x) = x^3 - x^2 - 2x$. Since $0 = f(x) = x^3 - x^2 - 2x = x(x+1)(x-2)$ implies that $x = 0$ or $x = 1$ or $x = 2$, we divide $[-1, 2]$ into $[-1, 0]$ and $[0, 2]$.

2. We calculate that

$$\begin{aligned}\int_{-1}^{0}(x^3 - x^2 - 2x)\,dx &= \left[\frac{x^4}{4} - \frac{x^3}{3} - x^2\right]_{-1}^{0} \\ &= (0 - 0 - 0) - \left(\frac{1}{4} + \frac{1}{3} - 1\right) \\ &= \frac{5}{12}\end{aligned}$$

and

$$\begin{aligned}\int_{0}^{2}(x^3 - x^2 - 2x)\,dx &= \left[\frac{x^4}{4} - \frac{x^3}{3} - x^2\right]_{0}^{2} \\ &= \left(\frac{16}{4} - \frac{8}{3} - 4\right) - (0 - 0 - 0) \\ &= -\frac{8}{3}.\end{aligned}$$

3. Therefore

$$\text{total area} = \left|\frac{5}{12}\right| + \left|-\frac{8}{3}\right| = \frac{37}{12}.$$

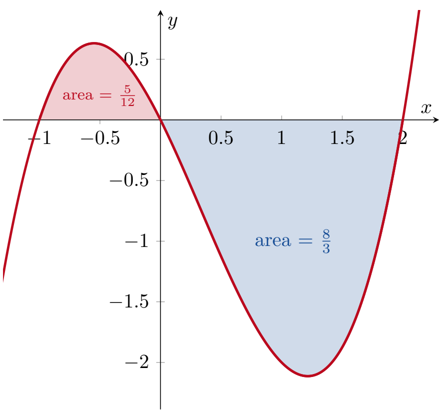


Figure 30.2: The total area between the graph $y = x^3 - x^2 - 2x$ and the x-axis over $[-1, 2]$.
Şekil 30.2: $[-1, 2]$ üzerinde olan, $y = x^3 - x^2 - 2x$ ve x-ekseni arasındaki toplam alan.

Summary

$[a, b]$ üzerindeki $y = f(x)$ grafiği ve x-ekseni arasında kalan ***toplam alanı*** bulmak için:

ADIM 1. f'nin köklerinin olduğu yerlerde $[a, b]$ bölünür .

ADIM 2. Her bir alt-aralık üzerinde f integre edilir.

ADIM 3. Her bir integralin mutlak değerleri toplanır.

Örnek 30.11. $-1 \leq x \leq 2$ ise $y = x^3 - x^2 - 2x$ grafiği ve x-ekseni arasında kalan alanı bulunuz.

çözüm:

1. $f(x) = x^3 - x^2 - 2x$ olsun. $0 = f(x) = x^3 - x^2 - 2x = x(x+1)(x-2)$ olduğundan $x = 0$ veya $x = 1$ veya $x = 2$ olduğundan, $[-1, 2]$'yi $[-1, 0]$ ve $[0, 2]$'ye ayırırız.

2. Kolayca hesaplanacağı üzere

$$\begin{aligned}\int_{-1}^{0}(x^3 - x^2 - 2x)\,dx &= \left[\frac{x^4}{4} - \frac{x^3}{3} - x^2\right]_{-1}^{0} \\ &= (0 - 0 - 0) - \left(\frac{1}{4} + \frac{1}{3} - 1\right) \\ &= \frac{5}{12}\end{aligned}$$

ve

$$\begin{aligned}\int_{0}^{2}(x^3 - x^2 - 2x)\,dx &= \left[\frac{x^4}{4} - \frac{x^3}{3} - x^2\right]_{0}^{2} \\ &= \left(\frac{16}{4} - \frac{8}{3} - 4\right) - (0 - 0 - 0) \\ &= -\frac{8}{3}.\end{aligned}$$

3. Dolayısıyla

$$\text{toplam alan} = \left|\frac{5}{12}\right| + \left|-\frac{8}{3}\right| = \frac{37}{12}.$$

olur.

The Average Value of a Continuous Function

The average of $\{1, 2, 2, 6, 9\}$ is $\frac{1+2+2+6+9}{5} = \frac{20}{5} = 4$. We can also calculate the average value of a continuous function.

Definition. If f is integrable on $[a, b]$, then the ***average value of f on $[a, b]$*** is

$$\boxed{\text{av}(f) = \frac{1}{b-a}\int_a^b f(x)\ dx.}$$

Example 30.12. Find the average value of $f(x) = \sqrt{4-x^2}$ on $[-2, 2]$.

solution: Since

$$\begin{aligned}\int_{-2}^{2} f(x)\ dx &= \frac{1}{2} \times \text{the area of a circle of radius 2}\\ &= \frac{1}{2}\pi 2^2 = 2\pi,\end{aligned}$$

we have that

$$\text{av}(f) = \frac{1}{2-(-2)}\int_{-2}^{2} f(x)\ dx = \frac{2\pi}{4} = \frac{\pi}{2}.$$

Example 30.13. Find the average value of $g(x) = x^3 - x$ on $[0, 1]$.

solution:

$$\begin{aligned}\text{av}(g) &= \frac{1}{1-0}\int_0^1 g(x)\ dx = \int_0^1 (x^3 - x)\ dx\\ &= \left[\frac{x^4}{4} - \frac{x^2}{2}\right]_0^1 = \frac{1}{4} - \frac{1}{2} = -\frac{1}{4}.\end{aligned}$$

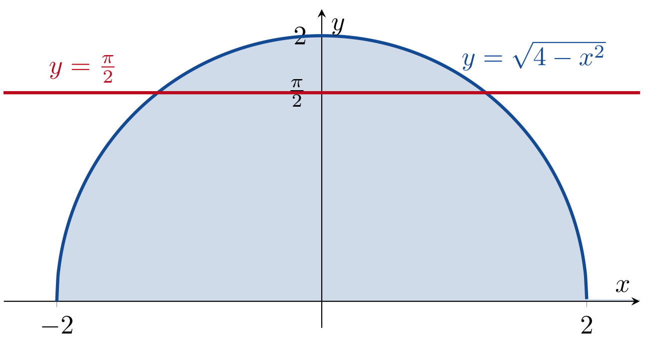


Figure 30.4: The average value of $f(x) = \sqrt{4-x^2}$ on $[-2, 2]$ is $\text{av}(f) = \frac{\pi}{2}$.
Şekil 30.4: $[-2, 2]$ üzerinde $f(x) = \sqrt{4-x^2}$'nin ortalama değeri $\text{ort}(f) = \frac{\pi}{2}$.

Sürekli Bir Fonksiyonun Ortalama Değeri

$\{1, 2, 2, 6, 9\}$ kümesinin ortalaması $\frac{1+2+2+6+9}{5} = \frac{20}{5} = 4$ tür. Sürekli bir fonksiyonun ortalama değerini de hesaplayabiliriz..

Tanım. $[a, b]$ üzerinde f integrallenebilir ise, ***f'nin $[a, b]$ üzerinde ortalama değeri***

$$\boxed{\text{ort}(f) = \frac{1}{b-a}\int_a^b f(x)\ dx.}$$

Örnek 30.12. $f(x) = \sqrt{4-x^2}$ 'nin $[-2, 2]$ üzerindeki ortalama değerini bulunuz.

çözüm:

$$\begin{aligned}\int_{-2}^{2} f(x)\ dx &= \frac{1}{2} \times \text{2 yarıçaplı çemberin alanı}\\ &= \frac{1}{2}\pi 2^2 = 2\pi,\end{aligned}$$

olduğundan,

$$\text{ort}(f) = \frac{1}{2-(-2)}\int_{-2}^{2} f(x)\ dx = \frac{2\pi}{4} = \frac{\pi}{2}.$$

Örnek 30.13. $g(x) = x^3 - x$ 'in $[0, 1]$ üzerindeki ortalama değerini bulunuz.

çözüm:

$$\begin{aligned}\text{ort}(g) &= \frac{1}{1-0}\int_0^1 g(x)\ dx = \int_0^1 (x^3 - x)\ dx\\ &= \left[\frac{x^4}{4} - \frac{x^2}{2}\right]_0^1 = \frac{1}{4} - \frac{1}{2} = -\frac{1}{4}.\end{aligned}$$

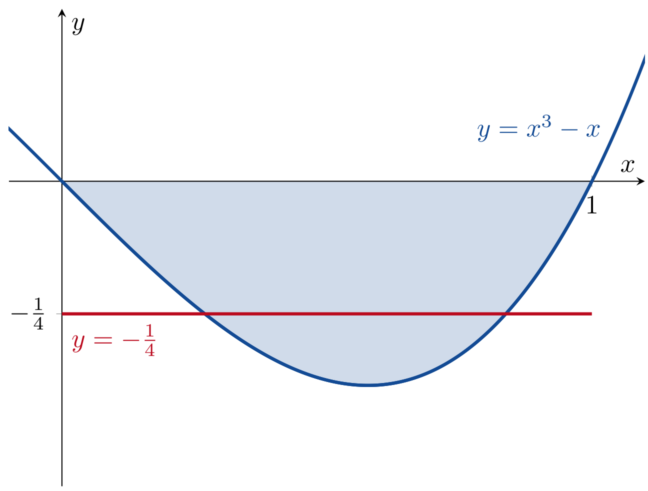


Figure 30.5: The average value of $g(x) = x^3 - x$ on $[0, 1]$ is $\text{av}(g) = -\frac{1}{4}$.
Şekil 30.5: $[0, 1]$ üzerinde $g(x) = x^3 - x$'in ortalama değeri $\text{ort}(g) = -\frac{1}{4}$.

Indefinite Integrals & Definite Integrals

Remember that

$\int f(x)\ dx$ is a function.

For example

$$\int x\ dx = \frac{x^2}{2} + C$$

and

$$\int \cos x\ dx = \sin x + C.$$

Remember that

$\int_a^b f(x)\ dx$ is a number.

For example

$$\int_0^1 x\ dx = \frac{1}{2}$$

and

$$\int_0^{\frac{\pi}{2}} \cos x\ dx = 1.$$

Belirsiz Integral ve Belirli Integral

Bilinmesi gereken

$\int f(x)\ dx$ bir fonksiyon.

Örneğin

$$\int x\ dx = \frac{x^2}{2} + C$$

ve

$$\int \cos x\ dx = \sin x + C.$$

Bilinmesi gereken

$\int_a^b f(x)\ dx$ bir sayı.

Örneğin

$$\int_0^1 x\ dx = \frac{1}{2}$$

ve

$$\int_0^{\frac{\pi}{2}} \cos x\ dx = 1.$$

Problems

Problem 30.1 (The Fundamental Theorem of Calculus)**.**

(a). Find $\frac{dy}{dx}$ if $y = \int_0^x \sqrt{1+t^2}\ dt$.

(b). Find $\frac{db}{dt}$ if $b = \int_0^{t^4} \sqrt{u}\ du$.

(c). Find $\frac{dp}{dx}$ if $p = \int_2^{x^2} \sin(t^3)\ dt$.

(d). Find $\frac{dz}{dx}$ if $z = \int_{\sqrt{x}}^{10} \sin(t^2)\ dt$.

Problem 30.2 (Definite Integrals)**.** Find the following definite integrals.

Sorular

Soru 30.1 (Kalkülüsün Temel Teorem)**.**

(a). $y = \int_0^x \sqrt{1+t^2}\ dt$ ise $\frac{dy}{dx}$'i bulunuz.

(b). $b = \int_0^{t^4} \sqrt{u}\ du$ ise $\frac{db}{dt}$'yi bulunuz.

(c). $p = \int_2^{x^2} \sin(t^3)\ dt$ ise $\frac{dp}{dx}$'i bulunuz.

(d). $z = \int_{\sqrt{x}}^{10} \sin(t^2)\ dt$ ise $\frac{dz}{dx}$'i bulunuz.

Soru 30.2 (Belirli İntegraller)**.** Aşağıdaki belirli integralleri bulunuz.

(a). $\int_{-2}^{0} (2x+5)\ dx$,

(b). $\int_0^1 x^3\ dx$,

(c). $\int_{-2}^{2} \sin\theta\ d\theta$,

(d). $\int_{-3}^{4} \left(5 - \frac{x}{2}\right)\ dx$,

(e). $\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{1-\cos 2t}{2}\ dt$,

(f). $\int_1^{32} t^{-\frac{6}{5}}\ dt$,

(g). $\int_{\frac{\pi}{2}}^{0} \frac{1+\cos 2x}{2}\ dx$,

(h). $\int_1^{-1} (r+1)^2\ dr$,

(i). $\int_1^{\sqrt{2}} \frac{s^2+\sqrt{s}}{s^2}\ ds$,

(j). $\int_{-4}^{4} |x|\ dx$,

(k). $\int_{\frac{\pi}{2}}^{\pi} \frac{\sin 2x}{2\sin x}\ dx.$,

(l). $\int_0^{\frac{\pi}{3}} 2\sec^2 x\ dx$.

31

The Substitution Method

Yerine Koyma Yöntemi

The Substitution Method for Indefinite Integrals

By the Chain rule,

$$\frac{d}{dx}\left(\frac{u^{n+1}}{n+1}\right) = u^n \frac{du}{dx}.$$

So

$$\int u^n \frac{du}{dx}\, dx = \frac{u^{n+1}}{n+1} + C.$$

But we know that

$$\int u^n\, du = \frac{u^{n+1}}{n+1} + C$$

also. So it looks like

$$\boxed{du = \frac{du}{dx}\, dx.}$$

Example 31.1. Find $\int (x^3+x)^5(3x^2+1)\, dx$.

solution: Let $u = x^3 + x$. Then $du = \frac{du}{dx}\, dx = (3x^2+1)\, dx$. By substitution, we have that

$$\begin{aligned}\int (x^3+x)^5(3x^2+1)\, dx &= \int u^5\, du \\ &= \frac{u^6}{6} + C = \frac{1}{6}(x^3+x)^6 + C.\end{aligned}$$

Example 31.2. Find $\int \sqrt{2x+1}\, dx$.

solution: Let $u = 2x+1$. Then $du = \frac{du}{dx}\, dx = 2dx$. So $dx = \frac{1}{2}\, du$. Therefore

$$\begin{aligned}\int \sqrt{2x+1}\, dx &= \int u^{\frac{1}{2}}\left(\tfrac{1}{2}du\right) = \frac{1}{2}\int u^{\frac{1}{2}}\, du \\ &= \frac{1}{2}\left(\frac{u^{\frac{3}{2}}}{\frac{3}{2}}\right) + C = \frac{1}{3}(2x+1)^{\frac{3}{2}} + C.\end{aligned}$$

Belirsiz İntegralde Yerine Koyma Yöntemi

Zincir Kuralı gereğince,

$$\frac{d}{dx}\left(\frac{u^{n+1}}{n+1}\right) = u^n \frac{du}{dx}.$$

Bu yüzden

$$\int u^n \frac{du}{dx}\, dx = \frac{u^{n+1}}{n+1} + C.$$

Biliyoruz ki

$$\int u^n\, du = \frac{u^{n+1}}{n+1} + C$$

doğrudur. Yani şuna benziyor.

$$\boxed{du = \frac{du}{dx}\, dx.}$$

Örnek 31.1. $\int (x^3+x)^5(3x^2+1)\, dx$'i bulunuz.

çözüm: $u = x^3+x$. olsun. Öyleyse $du = \frac{du}{dx}\, dx = (3x^2+1)\, dx$. Değişken değiştirerek, şunu bulmak mümkün

$$\begin{aligned}\int (x^3+x)^5(3x^2+1)\, dx &= \int u^5\, du \\ &= \frac{u^6}{6} + C = \frac{1}{6}(x^3+x)^6 + C.\end{aligned}$$

Örnek 31.2. $\int \sqrt{2x+1}\, dx$'i bulunuz.

çözüm: Diyelim ki $u = 2x+1$. O zaman $du = \frac{du}{dx}\, dx = 2dx$ olur. Yani $dx = \frac{1}{2}\, du$. Böyle olunca

$$\begin{aligned}\int \sqrt{2x+1}\, dx &= \int u^{\frac{1}{2}}\left(\tfrac{1}{2}du\right) = \frac{1}{2}\int u^{\frac{1}{2}}\, du \\ &= \frac{1}{2}\left(\frac{u^{\frac{3}{2}}}{\frac{3}{2}}\right) + C = \frac{1}{3}(2x+1)^{\frac{3}{2}} + C.\end{aligned}$$

Theorem 31.1 (The Substitution Method). *If*

- $u = g(x)$ *is differentiable;*
- $g : \mathbb{R} \to I$*; and*
- $f : I \to \mathbb{R}$ *is continuous,*

then
$$\int f(g(x))g'(x)\ dx = \int f(u)\ du.$$

Example 31.3. Find $\int 5\sec^2(5t+1)\ dt$.

solution: Let $u = 5t+1$. Then $du = \frac{du}{dt}\ dt = 5dt$. So

$$\begin{aligned}\int 5\sec^2(5t+1)\ dt &= \int \sec^2 u\ du \\ &= \tan u + C \\ &\quad \left(\text{because } \frac{d}{du}\tan u = \sec^2 u\right) \\ &= \tan(5t+1) + C.\end{aligned}$$

Example 31.4. Find $\int \cos(7\theta+3)\ d\theta$.

solution: Let $u = 7\theta+3$. Then $du = \frac{du}{d\theta}\ d\theta = 7d\theta$. So $d\theta = \frac{1}{7}du$ and

$$\begin{aligned}\int \cos(7\theta+3)\ d\theta &= \frac{1}{7}\int \cos u\ du \\ &= \frac{1}{7}\sin u + C = \frac{1}{7}\sin(7\theta+3) + C.\end{aligned}$$

Example 31.5. Find $\int x^2\sin(x^3)\ dx$.

solution: Let $u = x^3$. Then $du = \frac{du}{dx}\ dx = 3x^2\ dx$. So $\frac{1}{3}du = x^2\ dx$ and

$$\begin{aligned}\int x^2\sin(x^3)\ dx &= \int \frac{1}{3}\sin u\ du = -\frac{1}{3}\cos u + C \\ &= -\frac{1}{3}\cos(x^3) + C.\end{aligned}$$

Example 31.6. Find $\int x\sqrt{2x+1}\ dx$.

solution: Let $u = 2x+1$. Then $du = \frac{du}{dx}\ dx = 2dx$. So $dx = \frac{1}{2}du$ and

$$\int x\sqrt{2x+1}\ dx = \int x\sqrt{u}\ \frac{1}{2}du.$$

But we still have an x here. We can't integrate until we change all the x terms to u terms. Note that

$$u = 2x+1 \implies u-1 = 2x \implies \frac{1}{2}(u-1) = x.$$

Teorem 31.1 (Yerine Koyma Yöntemi).

- $u = g(x)$ *türevlenebilir;*
- $g : \mathbb{R} \to I$*; ve*
- $f : I \to \mathbb{R}$ *sürekli,*

bunun üzerine
$$\int f(g(x))g'(x)\ dx = \int f(u)\ du.$$

Örnek 31.3. $\int 5\sec^2(5t+1)\ dt$'yi bulunuz.

çözüm: $u = 5t+1$ diyelim. Buradan $du = \frac{du}{dt}\ dt = 5dt$ olur. Yani

$$\begin{aligned}\int 5\sec^2(5t+1)\ dt &= \int \sec^2 u\ du \\ &= \tan u + C \\ &\quad \left(\frac{d}{du}\tan u = \sec^2 u \text{ olduğundan}\right) \\ &= \tan(5t+1) + C.\end{aligned}$$

Örnek 31.4. $\int \cos(7\theta+3)\ d\theta$'yı bulunuz.

çözüm: $u = 7\theta+3$ olsun. Buradan $du = \frac{du}{d\theta}\ d\theta = 7d\theta$. Böylece $d\theta = \frac{1}{7}du$ ve

$$\begin{aligned}\int \cos(7\theta+3)\ d\theta &= \frac{1}{7}\int \cos u\ du \\ &= \frac{1}{7}\sin u + C = \frac{1}{7}\sin(7\theta+3) + C.\end{aligned}$$

bulunur.

Örnek 31.5. $\int x^2\sin(x^3)\ dx$'i bulunuz.

çözüm: $u = x^3$ olsun. Yani $du = \frac{du}{dx}\ dx = 3x^2\ dx$. Böylece $\frac{1}{3}du = x^2\ dx$ ve

$$\begin{aligned}\int x^2\sin(x^3)\ dx &= \int \frac{1}{3}\sin u\ du = -\frac{1}{3}\cos u + C \\ &= -\frac{1}{3}\cos(x^3) + C.\end{aligned}$$

bulunur.

Örnek 31.6. $\int x\sqrt{2x+1}\ dx$'i bulunuz.

çözüm: $u = 2x+1$ diyelim. Bu durumda $du = \frac{du}{dx}\ dx = 2dx$ olur. Yani $dx = \frac{1}{2}du$ ve

$$\int x\sqrt{2x+1}\ dx = \int x\sqrt{u}\ \frac{1}{2}du$$

buluruz. Elimizde hala x var. Bütün x'li terimleri u'lu terimlere dönüştürmedikçe integre edemiyoruz. Şunu akılda tutarak,

$$u = 2x+1 \implies u-1 = 2x \implies \frac{1}{2}(u-1) = x.$$

Therefore

$$\begin{aligned}\int x\sqrt{2x+1}\,dx &= \int \frac{1}{2}(u-1)\sqrt{u}\,\frac{1}{2}du\\ &= \frac{1}{4}\int u^{\frac{3}{2}} - u^{\frac{1}{2}}\,du\\ &= \frac{1}{4}\left(\frac{2}{5}u^{\frac{5}{2}} - \frac{2}{3}u^{\frac{3}{2}}\right) + C\\ &= \frac{1}{10}u^{\frac{5}{2}} - \frac{1}{6}u^{\frac{3}{2}} + C\\ &= \frac{1}{10}(2x+1)^{\frac{5}{2}} - \frac{1}{6}(2x+1)^{\frac{3}{2}} + C.\end{aligned}$$

Example 31.7. Find $\int \frac{2z}{\sqrt[3]{z^2+1}}\,dz$.

solution: Let $u = z^2+1$. Then $du = \frac{du}{dx}\,dx = 2z\,dz$ and

$$\begin{aligned}\int \frac{2z}{\sqrt[3]{z^2+1}}\,dz &= \int \frac{du}{u^{\frac{1}{3}}}\\ &= \int u^{-\frac{1}{3}}\,du\\ &= \frac{u^{\frac{2}{3}}}{\frac{2}{3}} + C\\ &= \frac{3}{2}u^{\frac{2}{3}} + C\\ &= \frac{3}{2}(z^2+1)^{\frac{2}{3}} + C.\end{aligned}$$

Example 31.8. Find $\int \sin^2 x\,dx$.

solution: We use the identity

$$\sin^2 x = \frac{1-\cos 2x}{2}$$

to calculate that

$$\begin{aligned}\int \sin^2 x\,dx &= \int \frac{1-\cos 2x}{2}\,dx\\ &= \frac{1}{2}\int (1-\cos 2x)\,dx\\ &= \frac{1}{2}\left(x - \frac{1}{2}\sin 2x\right) + C\\ &= \frac{1}{2}x - \frac{1}{4}\sin 2x + C.\end{aligned}$$

Example 31.9. Similarly

$$\int \cos^2 x\,dx = \int \frac{1+\cos 2x}{2}\,dx = \frac{1}{2}x + \frac{1}{4}\sin 2x + C.$$

Bu yüzden

$$\begin{aligned}\int x\sqrt{2x+1}\,dx &= \int \frac{1}{2}(u-1)\sqrt{u}\,\frac{1}{2}du\\ &= \frac{1}{4}\int u^{\frac{3}{2}} - u^{\frac{1}{2}}\,du\\ &= \frac{1}{4}\left(\frac{2}{5}u^{\frac{5}{2}} - \frac{2}{3}u^{\frac{3}{2}}\right) + C\\ &= \frac{1}{10}u^{\frac{5}{2}} - \frac{1}{6}u^{\frac{3}{2}} + C\\ &= \frac{1}{10}(2x+1)^{\frac{5}{2}} - \frac{1}{6}(2x+1)^{\frac{3}{2}} + C.\end{aligned}$$

bulunmuş olur.

Örnek 31.7. $\int \frac{2z}{\sqrt[3]{z^2+1}}\,dz$ integralini bulunuz.

çözüm: $u = z^2+1$ diyelim. Buradan $du = \frac{du}{dx}\,dx = 2z\,dz$ ve oradan da

$$\begin{aligned}\int \frac{2z}{\sqrt[3]{z^2+1}}\,dz &= \int \frac{du}{u^{\frac{1}{3}}}\\ &= \int u^{-\frac{1}{3}}\,du\\ &= \frac{u^{\frac{2}{3}}}{\frac{2}{3}} + C\\ &= \frac{3}{2}u^{\frac{2}{3}} + C\\ &= \frac{3}{2}(z^2+1)^{\frac{2}{3}} + C.\end{aligned}$$

elde edilir

Örnek 31.8. $\int \sin^2 x\,dx$ integralini bulunuz.

çözüm: Burada kullanacağımız özdeşlik

$$\sin^2 x = \frac{1-\cos 2x}{2}$$

ve buradan da

$$\begin{aligned}\int \sin^2 x\,dx &= \int \frac{1-\cos 2x}{2}\,dx\\ &= \frac{1}{2}\int (1-\cos 2x)\,dx\\ &= \frac{1}{2}\left(x - \frac{1}{2}\sin 2x\right) + C\\ &= \frac{1}{2}x - \frac{1}{4}\sin 2x + C\end{aligned}$$

bulunur.

Örnek 31.9. Benzer şekilde

$$\int \cos^2 x\,dx = \int \frac{1+\cos 2x}{2}\,dx = \frac{1}{2}x + \frac{1}{4}\sin 2x + C$$

bulunur.

The Substitution Method for Definite Integrals

Theorem 31.2 (The Substitution Method)**.** *If*

- $u = g(x)$ *is differentiable on* $[a, b]$;
- g' *is continuous on* $[a, b]$; *and*
- f *is continous on the range of* g,

then
$$\int_a^b f(g(x))g'(x)\;dx = \int_{g(a)}^{g(b)} f(u)\;du.$$

Example 31.10. Calculate $\displaystyle\int_{-1}^{1} 3x^2\sqrt{x^3+1}\;dx$.

solution 1. We can use the previous theorem to solve this example. Let $u = x^3 + 1$. Then $du = 3x^2\;dx$. Moreover $x = -1 \implies u = 0$ and $x = 1 \implies u = 2$. So
$$\int_{x=-1}^{x=1} 3x^2\sqrt{x^3+1}\;dx = \int_{u=0}^{u=2}\sqrt{u}\;du = \left[\frac{2}{3}u^{\frac{3}{2}}\right]_0^2$$
$$= \frac{2}{3}\left(2^{\frac{3}{2}} - 0^{\frac{3}{2}}\right) = \frac{2}{3}2\sqrt{2} = \frac{4\sqrt{2}}{3}.$$

solution 2. Alternately, we can first find the indefinite integral, then find the required definite integral.

Let $u = x^3 + 1$. Then $du = 3x^2\;dx$. So
$$\int 3x^2\sqrt{x^3+1}\;dx = \int\sqrt{u}\;du = \frac{2}{3}u^{\frac{3}{2}} + C = \frac{2}{3}(x^3+1)^{\frac{3}{2}} + C.$$
Therefore
$$\int_{-1}^{1} 3x^2\sqrt{x^3+1}\;dx = \left[\frac{2}{3}(x^3+1)^{\frac{3}{2}}\right]_{-1}^{1}$$
$$= \left(\frac{2}{3}(1+1)^{\frac{3}{2}}\right) - \left(\frac{2}{3}(-1+1)^{\frac{3}{2}}\right)$$
$$= \frac{2}{3}\times 2^{\frac{3}{2}} = \frac{4\sqrt{2}}{3}.$$

Example 31.11. Calculate $\displaystyle\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \cot\theta\,\mathrm{cosec}^2\,\theta\;d\theta$.

solution: Let $u = \cot\theta$. Then $du = \frac{du}{d\theta}\;d\theta = -\,\mathrm{cosec}^2\,\theta\;d\theta$. So $-du = \mathrm{cosec}^2\,\theta\;d\theta$. Moreover $\theta = \frac{\pi}{4} \implies u = \cot\frac{\pi}{4} = 1$ and $\theta = \frac{\pi}{2} \implies u = \cot\frac{\pi}{2} = 0$. Hence
$$\int_{\theta=\frac{\pi}{4}}^{\theta=\frac{\pi}{2}} \cot\theta\,\mathrm{cosec}^2\,\theta\;d\theta = \int_{u=1}^{u=0} u\;(-du) = -\int_1^0 u\;du$$
$$= -\left[\frac{u^2}{2}\right]_1^0 = -\left(\frac{0^2}{2} - \frac{1^2}{2}\right) = \frac{1}{2}.$$

Belirli İntegralde Değişken Değiştirme

Teorem 31.2 (Değişken Değiştirme Yöntemi)**.** *Eğer*

- $u = g(x)$ *fonksiyonu* $[a, b]$*'de türevliyse;*
- g' *fonksiyonu* $[a, b]$*'de sürekliyse; ve*
- f *fonksiyonu da* g*'nin görüntü kümesinde sürekliyse,*

bu durumda
$$\int_a^b f(g(x))g'(x)\;dx = \int_{g(a)}^{g(b)} f(u)\;du$$
olur.

Örnek 31.10. $\displaystyle\int_{-1}^{1} 3x^2\sqrt{x^3+1}\;dx$ integralini bulunuz.

çözüm 1. Bu soruyu yapmak için önceki teoremi kullanabiliriz. Diyelim ki, $u = x^3 + 1$ olsun. Bu durumda $du = 3x^2\;dx$ olur. Ayrıca $x = -1 \implies u = 0$ ve $x = 1 \implies u = 2$ olur. Buradan
$$\int_{x=-1}^{x=1} 3x^2\sqrt{x^3+1}\;dx = \int_{u=0}^{u=2}\sqrt{u}\;du = \left[\frac{2}{3}u^{\frac{3}{2}}\right]_0^2$$
$$= \frac{2}{3}\left(2^{\frac{3}{2}} - 0^{\frac{3}{2}}\right) = \frac{2}{3}2\sqrt{2} = \frac{4\sqrt{2}}{3}$$
bulunmuş olur.

çözüm 2. Değişimli olarak, önce belirsiz integrali bulur, daha sonra da belirli integrali bulabiliriz.

Şimdi $u = x^3 + 1$ olsun. Buradan $du = 3x^2\;dx$ olur. Bu sebeple
$$\int 3x^2\sqrt{x^3+1}\;dx = \int\sqrt{u}\;du = \frac{2}{3}u^{\frac{3}{2}} + C = \frac{2}{3}(x^3+1)^{\frac{3}{2}} + C.$$
Böylece
$$\int_{-1}^{1} 3x^2\sqrt{x^3+1}\;dx = \left[\frac{2}{3}(x^3+1)^{\frac{3}{2}}\right]_{-1}^{1}$$
$$= \left(\frac{2}{3}(1+1)^{\frac{3}{2}}\right) - \left(\frac{2}{3}(-1+1)^{\frac{3}{2}}\right)$$
$$= \frac{2}{3}\times 2^{\frac{3}{2}} = \frac{4\sqrt{2}}{3}.$$

Örnek 31.11. $\displaystyle\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \cot\theta\,\mathrm{cosec}^2\,\theta\;d\theta$'yı bulunuz.

çözüm: $u = \cot\theta$ olsun. Buradan $du = \frac{du}{d\theta}\;d\theta = -\,\mathrm{cosec}^2\,\theta\;d\theta$ olur. Böylece $-du = \mathrm{cosec}^2\,\theta\;d\theta$ bulunur. Ayrıca $\theta = \frac{\pi}{4} \implies u = \cot\frac{\pi}{4} = 1$ ve $\theta = \frac{\pi}{2} \implies u = \cot\frac{\pi}{2} = 0$ bulunur. Bunun sonucu olarak da
$$\int_{\theta=\frac{\pi}{4}}^{\theta=\frac{\pi}{2}} \cot\theta\,\mathrm{cosec}^2\,\theta\;d\theta = \int_{u=1}^{u=0} u\;(-du) = -\int_1^0 u\;du$$
$$= -\left[\frac{u^2}{2}\right]_1^0 = -\left(\frac{0^2}{2} - \frac{1^2}{2}\right) = \frac{1}{2}$$
bulunur

Problems

Problem 31.1. Use a substitution to evaluate the following indefinite integrals. You must show your working.

Sorular

Soru 31.1. Yerine koyma (değişken değiştirme) yöntemi kullanarak, aşağıdaki belirsiz integralleri bulunuz. İşlemlerini açıklamalısınız.

(a). $\int 2(2x+4)^5 \; dx.$

(b). $\int 2x(x^2+5)^{-4} \; dx.$

(c). $\int (3x+2)(3x^2+4x)^4 \; dx.$

(d). $\int \sec 2t \tan 2t \; dt.$

(e). $\int \dfrac{9r^2 \; dr}{\sqrt{1-r^3}}.$

(f). $\int \sqrt{x}\sin^2(x^{\frac{3}{2}}-1) \; dx.$

(g). $\int \dfrac{1}{\sqrt{x}(1+\sqrt{x})^2} \; dx$

(h). $\int r^4\left(7-\dfrac{r^5}{10}\right)^3 \; dr.$

(i). $\int \dfrac{1}{\theta^2}\sin\dfrac{1}{\theta}\cos\dfrac{1}{\theta} \; d\theta.$

(j). $\int x(x-1)^{10} \; dx.$

(k). $\int x^3\sqrt{x^2+1} \; dx.$

(l). $\int z^2 e^{(z^3)} \; dz.$

Problem 31.2. Use a substitution to evaluate the following definite integrals. You must show your working.

Soru 31.2. Yerine koyma (değişken değiştirme) yöntemi kullanarak, aşağıdaki belirli integralleri bulunuz. İşlemlerini açıklamalısınız.

(a). $\int_0^3 \sqrt{y+1} \; dy.$

(b). $\int_{-1}^0 \sqrt{y+1} \; dy.$

(c). $\int_0^{\frac{\pi}{4}} \tan x \sec^2 x \; dx$

(d). $\int_{-1}^1 t^3(1+t^4)^3 \; dt.$

(e). $\int_{-\sqrt{3}}^{\sqrt{3}} \dfrac{4x}{\sqrt{x^2+1}} \; dx.$

(f). $\int_0^{\frac{\pi}{6}} (1-\cos 3t)\sin 3t \; dt.$

(g). $\int_0^1 (4y-y^2+4y^3+1)^{-\frac{2}{3}}(12y^2-2y+4) \; dy$

(h). $\int_{-\frac{\pi}{2}}^0 \dfrac{\sin x}{(3+2\cos x)^2} \; dx.$

(i). $\int_0^{\frac{\pi}{2}} \dfrac{\sin x}{(3+2\cos x)^2} \; dx.$

32

Area Between Curves

Eğriler Arasındaki Alanlar

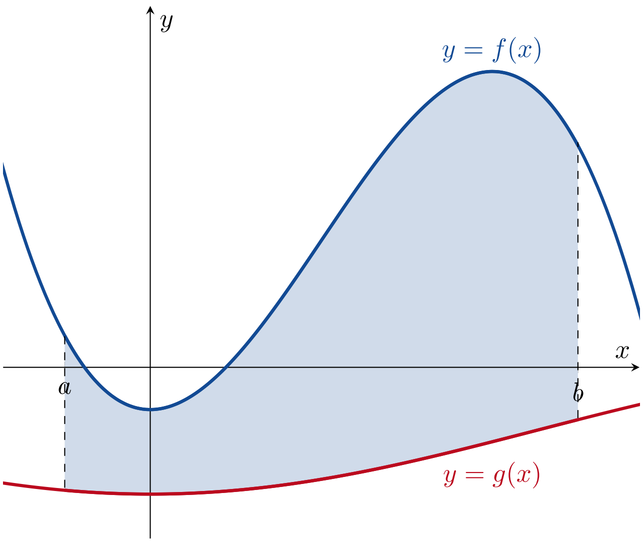


Figure 32.1: The region between the curves $y = f(x)$ and $y = g(x)$ for $a \leq x \leq b$.
Şekil 32.1: $a \leq x \leq b$ iken $y = f(x)$ ve $y = g(x)$ eğrileri arasındaki alan.

Definition. If

- f is continous;
- g is continous; and
- $f(x) \geq g(x)$ on $[a, b]$,

then the ***area of the region between the curves*** $y = f(x)$ ***and*** $y = g(x)$ ***for*** $a \leq x \leq b$ is

$$\boxed{\text{area} = \int_a^b \Big(f(x) - g(x)\Big)\, dx.}$$

Example 32.1. Find the area between $y = 2 - x^2$ and $y = -x$.

solution: First we need to find the limits of integration:

Tanım. Eğer

- f sürekli;
- g sürekli; ve
- $[a, b]$ üzerinde $f(x) \geq g(x)$ 'se,

o zaman $a \leq x \leq b$ ***oldukça*** $y = f(x)$ ***ve*** $y = g(x)$ ***eğrileri arasındaki alan***

$$\boxed{\text{alan} = \int_a^b \Big(f(x) - g(x)\Big)\, dx.}$$

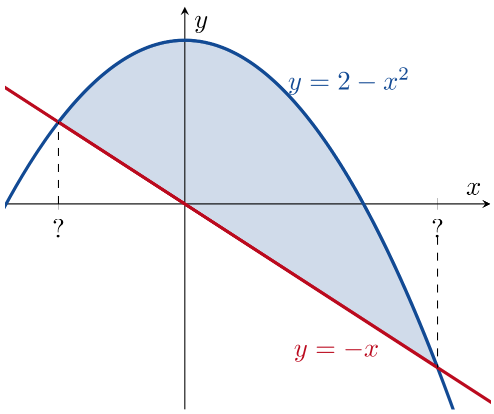


Figure 32.2: The region between the curves $y = 2 - x^2$ and $y = -x$.
Şekil 32.2: $y = 2 - x^2$ ve $y = -x$ arasındaki bölge

Örnek 32.1. $y = 2 - x^2$ ve $y = -x$ arasındaki alanı bulunuz.

çözüm: İlk olarak integrasyon sınırlarını buluruz:

$$\begin{aligned} 2 - x^2 &= -x \\ 0 &= x^2 - x - 2 \\ 0 &= (x+1)(x-2) \implies x = -1 \text{ veya } 2. \end{aligned}$$

$$\begin{aligned}2-x^2&=-x\\0&=x^2-x-2\\0&=(x+1)(x-2)\implies x=-1 \text{ or } 2.\end{aligned}$$

We need to integrate from $x=-1$ to $x=2$. Therefore

$$\begin{aligned}\text{area}&=\int_{-1}^{2}\Big((2-x^2)-(-x)\Big)\,dx\\&=\int_{-1}^{2}(2+x-x^2)\,dx\\&=\left[2x+\tfrac{1}{2}x^2-\tfrac{1}{3}x^3\right]_{-1}^{2}\\&=\left(4+\frac{4}{2}-\frac{8}{3}\right)-\left(-2+\frac{1}{2}+\frac{1}{3}\right)\\&=\frac{9}{2}.\end{aligned}$$

Example 32.2. Find the area bounded by $y=\sqrt{x}$, $y=x-2$ and the x-axis, for $x\geq 0$ and $y\geq 0$.

solution: First we calculate that

$$\begin{aligned}\sqrt{x}&=x-2\\x&=(x-2)^2=x^2-4x+4\\0&=x^2-5x+4=(x-1)(x-4)\implies x=1 \text{ or } 4.\end{aligned}$$

Since $\sqrt{1}\neq 1-2$, we must have $x=4$. See figure 32.3. Therefore

$$\begin{aligned}\text{area}&=\text{blue area}+\text{red area}\\&=\int_0^2\sqrt{x}\,dx+\int_2^4\Big(\sqrt{x}-(x-2)\Big)\,dx\\&=\int_0^2 x^{\frac{1}{2}}\,dx+\int_2^4(x^{\frac{1}{2}}-x+2)\,dx\\&=\left[\frac{2}{3}x^{\frac{3}{2}}\right]_0^2+\left[\frac{2}{3}x^{\frac{3}{2}}-\frac{1}{2}x^2+2x\right]_2^4\\&=\left(\frac{2}{3}(2)^{\frac{3}{2}}-0\right)+\left(\frac{2}{3}(4)^{\frac{3}{2}}-\frac{1}{2}(16)+2(4)\right)\\&\quad-\left(\frac{2}{3}(2)^{\frac{3}{2}}-\frac{1}{2}(4)+2(2)\right)\\&=\frac{4\sqrt{2}}{3}+\frac{16}{3}-8+8-\frac{4\sqrt{2}}{3}+2-4\\&=\frac{10}{3}.\end{aligned}$$

$x=-1$ den $x=2$'ye integre ederiz. Böylece

$$\begin{aligned}\text{area}&=\int_{-1}^{2}\Big((2-x^2)-(-x)\Big)\,dx\\&=\int_{-1}^{2}(2+x-x^2)\,dx\\&=\left[2x+\tfrac{1}{2}x^2-\tfrac{1}{3}x^3\right]_{-1}^{2}\\&=\left(4+\frac{4}{2}-\frac{8}{3}\right)-\left(-2+\frac{1}{2}+\frac{1}{3}\right)\\&=\frac{9}{2}.\end{aligned}$$

Örnek 32.2. $x\geq 0$ ve $y\geq 0$ olmak üzere $y=\sqrt{x}$, $y=x-2$ ve x-ekseni ile sınırlı alanı bulunuz.

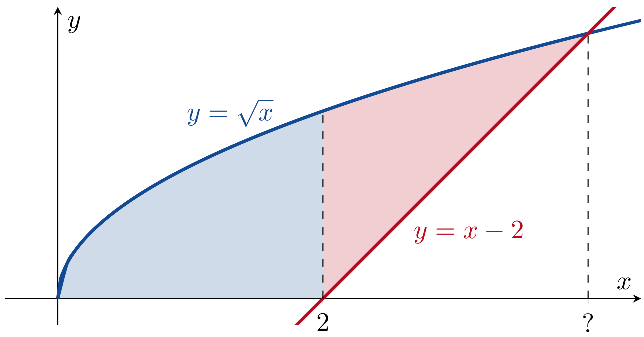


Figure 32.3: The region between the curves $y=\sqrt{x}$, $y=x-2$ and the x-axis for $x\geq 0$ and $y\geq 0$.
Şekil 32.3: $x\geq 0$ and $y\geq 0$ olduğunda $y=\sqrt{x}$, $y=x-2$ ve x-ekseni ile sınırlı bölge.

çözüm: İlk olarak

$$\begin{aligned}\sqrt{x}&=x-2\\x&=(x-2)^2=x^2-4x+4\\0&=x^2-5x+4=(x-1)(x-4)\implies x=1 \text{ veya } 4.\end{aligned}$$

$\sqrt{1}\neq 1-2$ olduğundan, $x=4$ buluruz. Bkz. şekil 32.3. Buradan

$$\begin{aligned}\text{area}&=\text{mavi alan}+\text{kırmızı alan}\\&=\int_0^2\sqrt{x}\,dx+\int_2^4\Big(\sqrt{x}-(x-2)\Big)\,dx\\&=\int_0^2 x^{\frac{1}{2}}\,dx+\int_2^4(x^{\frac{1}{2}}-x+2)\,dx\\&=\left[\frac{2}{3}x^{\frac{3}{2}}\right]_0^2+\left[\frac{2}{3}x^{\frac{3}{2}}-\frac{1}{2}x^2+2x\right]_2^4\\&=\left(\frac{2}{3}(2)^{\frac{3}{2}}-0\right)+\left(\frac{2}{3}(4)^{\frac{3}{2}}-\frac{1}{2}(16)+2(4)\right)\\&\quad-\left(\frac{2}{3}(2)^{\frac{3}{2}}-\frac{1}{2}(4)+2(2)\right)\\&=\frac{4\sqrt{2}}{3}+\frac{16}{3}-8+8-\frac{4\sqrt{2}}{3}+2-4\\&=\frac{10}{3}\end{aligned}$$

elde edilir.

Problems

Problem 32.1 (Total Area). Calculate the total area between the curve $y = 2x^2$ and the curve $y = x^4 - 2x^2$ for $-2 \le x \le 2$.

Problem 32.2 (Total Area). Find the total areas of the regions shown in the following figures:

(a). Figure 32.5. (b). Figure 32.6. (c). Figure 32.7.

Sorular

Soru 32.1 (Toplam Alan). $y = 2x^2$ eğrisiyle $y = x^4 - 2x^2$ eğrisi arasındaki alanı $-2 \le x \le 2$ ise bulunuz.

Soru 32.2 (Toplam Alan). Find the total areas of the regions shown in the following figures:

(a). Şekil 32.5. (b). Şekil 32.6. (c). Şekil 32.7.

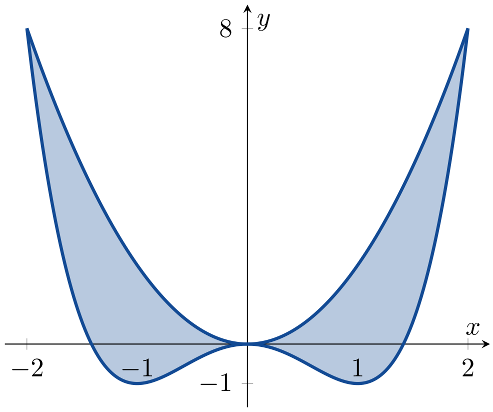


Figure 32.4: The total area between the curve $y = 2x^2$ and the curve $y = x^4 - 2x^2$, for $-2 \le x \le 2$.
Şekil 32.4: $-2 \le x \le 2$ olduğunda $y = 2x^2$ ve $y = x^4 - 2x^2$ arasındaki alan.

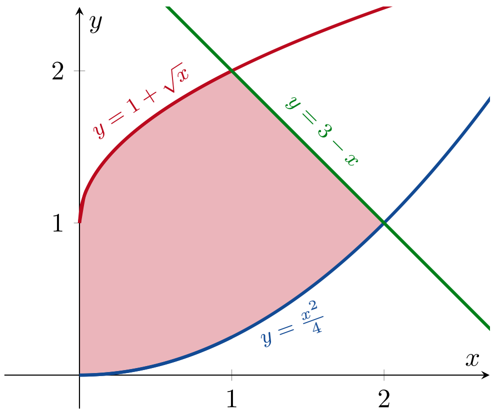


Figure 32.6: The region bounded by $y = 1+\sqrt{x}$, $y = 3-x$ and $y = \frac{x^2}{4}$ in the first quadrant.
Şekil 32.6:

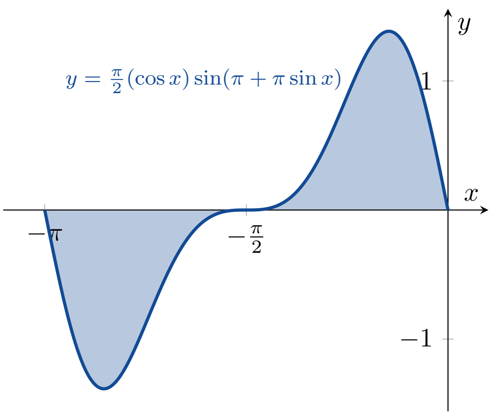


Figure 32.5: The total area between the curve $y = \frac{\pi}{2}(\cos x)\sin(\pi + \pi\sin x)$ and x-axis, for $-\pi \le x \le 0$.
Şekil 32.5:

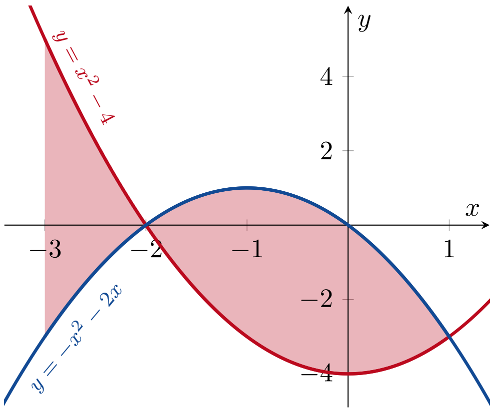


Figure 32.7: The total area between the curve $y = x^2 - 4$ and the curve $y = -x^2 - 2x$, for $-3 \le x \le 1$.
Şekil 32.7: $-3 \le x \le 1$ olduğunda $y = x^2 - 4$ ve $y = -x^2 - 2x$ arasındaki alan.

Solutions to Selected Problems

1.1.

(a).

P	Q	R	$P \vee Q$	$(P \vee Q) \vee R$	$Q \vee R$	$P \vee (Q \vee R)$
T	T	T	T	T	T	T
T	T	F	T	T	T	T
T	F	T	T	T	T	T
T	F	F	T	T	F	T
F	T	T	T	T	T	T
F	T	F	T	T	T	T
F	F	T	F	T	T	T
F	F	F	F	F	F	F

(c).

P	Q	$P \vee Q$	$\neg(P \vee Q)$	$\neg P$	$\neg Q$	$\neg P \wedge \neg Q$
T	T	T	F	F	F	F
T	F	T	F	F	T	F
F	T	T	F	T	F	F
F	F	F	T	T	T	T

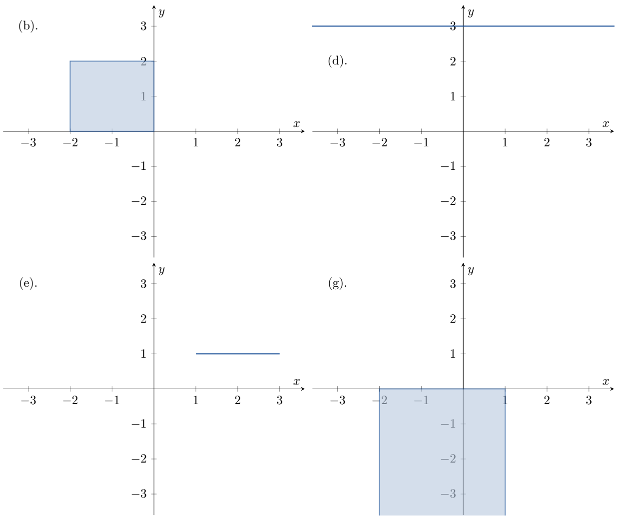
(d).

P	Q	$P \wedge Q$	$\neg(P \wedge Q)$	$\neg P$	$\neg Q$	$\neg P \vee \neg Q$
T	T	T	F	F	F	F
T	F	F	T	F	T	T
F	T	F	T	T	F	T
F	F	F	T	T	T	T

(g).

P	$\neg P$	$P \vee \neg P$	true
T	F	T	T
F	T	T	T

3.1.



3.2. We calculate that

$$\|AB\| = \sqrt{(4-1)^2 + (2-1)^2} = \sqrt{3^2 + 1^2} = \sqrt{10} \approx 3.16$$
$$\|BC\| = \sqrt{(3-4)^2 + (3-2)^2} = \sqrt{(-1)^2 + 1^2} = \sqrt{2} \approx 1.41$$
$$\|CA\| = \sqrt{(1-3)^2 + (1-3)^2} = \sqrt{(-2)^2 + (-2)^2} = \sqrt{8} \approx 2.83.$$

Hence $\|AB\|$ is the largest number.

4.1.

(a). $f(x) = 3$ is even.

(b). $f(x) = x^{77}$ is odd.

(c). $f(x) = x^2 + 1$ is even.

(d). $f(x) = x^3 + x$ is odd.

(e). $f(x) = x^3 + x^2$ is neither even nor odd.

(f). $f(x) = x^3 + 1$ is neither even nor odd.

(g). $f(x) = \frac{1}{x^2-1}$ is even.

(h). $f(x) = \frac{1}{x^2+1}$ is even.

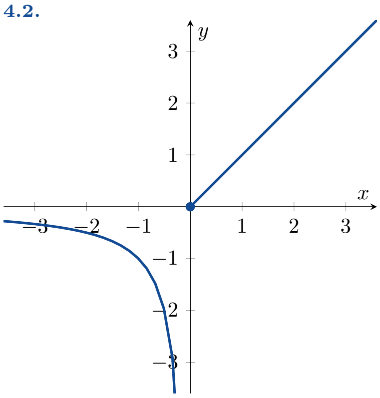
(i). $f(x) = \frac{1}{x-1}$ is neither even nor odd.

(j). $f(x) = \sin x$ is odd.

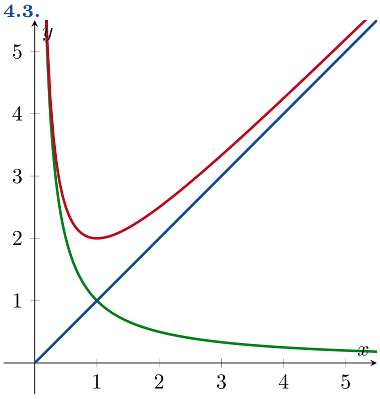
(k). $f(x) = 2x+1$ is neither even nor odd.

(l). $f(x) = \cos x$ is even.

4.2.



4.3.



4.4.

(a). $-\frac{\pi}{2}$

(b). $\frac{3\pi}{4}$

(c). $\frac{2\pi}{3}$

(d). π

(e). $\frac{\pi}{5}$

(f). $\frac{\pi}{9}$

(g). $270°$

(h). $18°$

(i). $30°$

(j). $150°$

(k). $-36°$

(l). $540° = 180°$

4.5.

(a). $\mathbb{R}$

(b). $[0, \infty)$

(c). $[-2, \infty)$

(d). $(-\infty, 0] \cup [3, \infty)$

(e). $(-\infty, 3) \cup (3, \infty)$

(f). $(-\infty, -4) \cup (-4, 4) \cup (4, \infty)$

5.1.

(a). $(3, 0)$

(b). $(-3, 0)$

(c). $(-1, \sqrt{3})$

(d). $(1, \sqrt{3})$

(e). $(1, \sqrt{3})$

(f). $(-3, 0)$

(g). $(-1, \sqrt{3})$

(h). $(-1, 0)$

(i). $(2, 2)$.

5.2.

(a). $(\sqrt{2}, 45°)$

(b). $(3, 180°)$

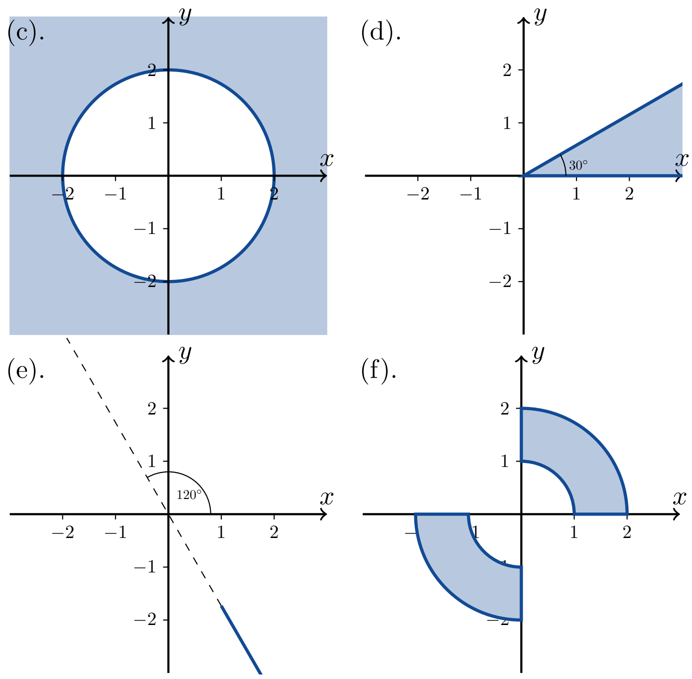
(c). $(2, -30°)$

(d). $(5, \pi - \tan^{-1}\frac{4}{3}) \approx (5, 126.87°)$

(e). $(2\sqrt{2}, -135°)$

(f). $(2, 150°)$.

5.3.



6.1.
(a). viii (b). iii (c). i (d). ii (e). vi (f). iv

6.2.
(a). $(3, 0)$ (b). $(0, -2)$ (c). $(0, \frac{1}{16})$

6.3.
(a). $(\pm 3, 0)$ (b). $(\pm 3, 0)$ (c). $(0, \pm 1)$ (d). $\frac{x^2}{4} + \frac{y^2}{2} = 1$

6.4.
(a). $(\pm\sqrt{2}, 0)$ (c). $(\pm\sqrt{10}, 0)$

(b). $(0, \pm 4)$ (d). $64x^2 - 36y^2 = 2304$

7.1.
(a). The distance is

$$\begin{aligned} d &= \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2} \\ &= \sqrt{(2 - -1)^2 + (5 - 1)^2 + (0 - 5)^2} = \sqrt{3^2 + 4^2 + 5^2} \\ &= \sqrt{9 + 16 + 25} = \sqrt{50} \end{aligned}$$

(b). $\sqrt{2}$ (c). 50 (d). $\sqrt{101}$ (e). 6

7.2. First we rearrange the equation into standard form

$$\begin{aligned} x^2 + y^2 + z^2 - 6y + 8z &= 0 \\ x^2 + (y^2 - 6y + 9) - 9 + (z^2 + 8z + 16) - 16 &= 0 \\ x^2 + (y - 3)^2 - 9 + (z + 4)^2 - 16 &= 0 \\ x^2 + (y - 3)^2 + (z + 4)^2 &= 25 \\ (x - 0)^2 + (y - 3)^2 + (z + 4)^2 &= 5^2. \end{aligned}$$

Then it is easy to see that the centre of the sphere is $(0, 3, -4)$ and the radius is 5.

7.3. Centre $= (\sqrt{2}, \sqrt{2}, -\sqrt{2})$. Radius $= \sqrt{2}$.

8.1.
(a). $3\sqrt{13}$ (e). $3\sqrt{13}$ (i). $\sqrt{10}$ (m). $(\frac{1}{5}, \frac{14}{5})$

(b). $\sqrt{29}$ (f). $(-9, 6)$ (j). $\sqrt{13} + \sqrt{29}$ (n). $\frac{\sqrt{197}}{5}$

(c). $3\sqrt{13}$ (g). $3\sqrt{13}$ (k). $(12, -19)$

(d). $(9, -6)$ (h). $(1, 3)$ (l). $\sqrt{505}$

8.2.

(a). We have that

$$\begin{aligned} 5\mathbf{a} - 3\mathbf{b} &= 5(\mathbf{i} + 2\mathbf{j} + 3\mathbf{k}) - 3(2\mathbf{i} + 5\mathbf{k}) = 5\mathbf{i} + 10\mathbf{j} + 15\mathbf{k} - 6\mathbf{i} - 15\mathbf{k} \\ &= -\mathbf{i} + 10\mathbf{j}. \end{aligned}$$

(b). $(0, 0, 0)$

8.3.

(a). We require the vector

$$\mathbf{u} = \frac{\mathbf{v}}{\|\mathbf{v}\|} = \frac{6\mathbf{i} + 2\mathbf{j} - 3\mathbf{k}}{\sqrt{6^2 + 2^2 + (-3)^2}} = \frac{6\mathbf{i} + 2\mathbf{j} - 3\mathbf{k}}{\sqrt{49}} = \frac{6}{7}\mathbf{i} + \frac{2}{7}\mathbf{j} - \frac{3}{7}\mathbf{k}.$$

(b). $\mathbf{u} = \frac{2}{3}\mathbf{i} + \frac{1}{3}\mathbf{j} - \frac{2}{3}\mathbf{k}$

(c). $\mathbf{w} = \frac{84}{13}\mathbf{i} - \frac{35}{13}\mathbf{k}$

9.1.
(a). (i) -25 (ii) 5 and 5 (iii) -1 (iv) $-2\mathbf{i} + 4\mathbf{j} - \sqrt{5}\mathbf{k}$

(b). (i) 25 (ii) 5 and 15 (iii) $\frac{1}{3}$ (iv) $\frac{10}{9}\mathbf{i} + \frac{11}{9}\mathbf{j} - \frac{2}{9}\mathbf{k}$

(c). (i) 13 (ii) 3 and 15 (iii) $\frac{1}{15}$ (iv) $\frac{26}{225}\mathbf{i} + \frac{26}{45}\mathbf{j} - \frac{143}{225}\mathbf{k}$

9.2. $\measuredangle BAC = \cos^{-1}\frac{1}{\sqrt{5}} \approx 63.435°$, $\measuredangle ABC = \cos^{-1}\frac{3}{5} \approx 53.130°$ and $\measuredangle BCA = \cos^{-1}\frac{1}{\sqrt{5}} \approx 63.435°$.

9.3. Yes.

9.4.
(a). If $\|\mathbf{u}\| = \|\mathbf{v}\|$, then we have that $\mathbf{u}\cdot(\mathbf{u}+\mathbf{v}) = \mathbf{u}\cdot\mathbf{u} + \mathbf{u}\cdot\mathbf{v} = \|\mathbf{u}\|^2 + \mathbf{u}\cdot\mathbf{v} = \mathbf{u}\cdot\mathbf{v} + \|\mathbf{v}\|^2 = \mathbf{u}\cdot\mathbf{v} + \mathbf{v}\cdot\mathbf{v} = (\mathbf{u}+\mathbf{v})\cdot\mathbf{v}$ as required.

(b). Since $\mathbf{u}\cdot(\mathbf{u}+\mathbf{v}) = \|\mathbf{u}\|\,\|\mathbf{u}+\mathbf{v}\|\cos\theta$ and $(\mathbf{u}+\mathbf{v})\cdot\mathbf{v} = \|\mathbf{u}+\mathbf{v}\|\,\|\mathbf{v}\|\cos\phi = \|\mathbf{u}+\mathbf{v}\|\,\|\mathbf{u}\|\cos\phi$, we can see from part (a) that $\cos\theta = \cos\phi$. Therefore $\theta = \phi$.

9.5. Suppose that the x-axis points to the east, that the y-axis points to the north and that the z-axis points upwards. The vector $\mathbf{u} = (0, -5, -1)$ is parallel to the northwards part of the pipe (pointing to the south). The vector $\mathbf{v} = (10, 0, 1)$ is parallel to the eastwards part of the pipe. Thus

$$\theta = \cos^{-1}\left(\frac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{u}\|\,\|\mathbf{v}\|}\right) = \cos^{-1}\left(\frac{-1}{\sqrt{26}\sqrt{101}}\right) = 91.12°.$$

The question could be interpreted in different ways, so I would accept both $91.12°$ and $180° - 91.12° = 88.88°$ as correct answers.

10.1.
(a). $2\mathbf{i} + \mathbf{j} + 2\mathbf{k}$ (d). $6\mathbf{i} - 12\mathbf{k}$ (g). $\mathbf{0}$

(b). $\mathbf{0}$ (e). $\mathbf{i} - \mathbf{j} + \mathbf{k}$ (h). $-\mathbf{j} - \mathbf{k}$

(c). $-6\mathbf{k}$ (f). $-2\mathbf{k}$ (i). $4\mathbf{j} + 2\mathbf{k}$

10.2.
(a). $\frac{3}{\sqrt{2}}$

(b). $(\frac{1}{\sqrt{2}}, 0, -\frac{1}{\sqrt{2}})$

10.3.
(a). no (b). yes (c). no (d). yes (e). no

10.4. $9\sqrt{3}$ and $\frac{21}{\sqrt{2}}$

10.5. -7

11.1. $x = 3 + t$, $y = -4 + t$, $z = -1 - t$.

11.2. $x = 1 - 2t$, $y = 2 - 2t$, $z = -1 + 2t$.

11.3. $x = 2 - 2t$, $y = 3 + 4t$, $z = -2t$.

11.4. $d = 7\sqrt{3}$

11.5. $d = 0$ (the point is on the line)

11.6. (a). Yes, at the point $P(9, 10, 7)$ $(t = 2 = s)$. (b). $d = 0$.

11.7. Immediately we can see that we have $P_1(10, -3, 0)$, $\mathbf{v}_1 = 4\mathbf{i} + 4\mathbf{k}$, $P_2(10, 0, 2)$ and $\mathbf{v}_2 = -4\mathbf{i} - 4\mathbf{k}$. Since $\mathbf{v}_1 \times \mathbf{v}_2 = \mathbf{0}$, the lines are parallel. We calculate that

$$\overrightarrow{P_1P_2} = P_2 - P_1 = (10, 0, 2) - (10, -3, 0) = (0, 3, 2)$$
$$\overrightarrow{P_1P_2} \times \mathbf{v}_1 = (0, 3, 2) \times (4, 0, 4) = (12, 8, -12)$$
$$d = \frac{\left\|\overrightarrow{P_1P_2} \times \mathbf{v}_1\right\|}{\|\mathbf{v}_1\|} = \frac{\|(12, 8, -12)\|}{\|(4, 0, 4)\|} = \frac{4\sqrt{22}}{4\sqrt{2}} = \sqrt{11}$$

11.8. First we have $P_1(10, 0, 0)$, $\mathbf{v}_1 = 4\mathbf{i} - \mathbf{j} + 4\mathbf{k}$, $P_2(10, 1, 2)$ and $\mathbf{v}_2 = -4\mathbf{i} - 4\mathbf{k}$. Note that the lines are skew because $\mathbf{v}_1 \times \mathbf{v}_2 = 4\mathbf{i} - 4\mathbf{k} \neq \mathbf{0}$. Thus

$$\overrightarrow{P_1P_2} = P_2 - P_1 = (10, 1, 2) - (10, 0, 0) = (0, 1, 2)$$
$$\overrightarrow{P_1P_2} \bullet (\mathbf{v}_1 \times \mathbf{v}_2) = (0, 1, 2) \bullet (4, 0, -4) = -8$$
$$d = \frac{\left|\overrightarrow{P_1P_2} \bullet (\mathbf{v}_1 \times \mathbf{v}_2)\right|}{\|\mathbf{v}_1 \times \mathbf{v}_2\|} = \frac{|-8|}{4\sqrt{2}} = \frac{8}{4\sqrt{2}} = \sqrt{2}.$$

12.1. $x + 3y - z = 9$

12.2. $x - 2y + z = 6$

12.3.
(a). $P(\frac{3}{2}, -\frac{3}{2}, \frac{1}{2})$

(b). $P(2, -\frac{20}{7}, \frac{27}{7})$

12.4.
(a). $x = 1 - t,\ y = 1 + t,\ z = -1$

(b). $x = 1 + 14t,\ y = 2t,\ z = 15t$

12.5.
(a). 3

(b). $\frac{3\sqrt{2}}{2}$

12.6. Let $\mathbf{n}_1 = \mathbf{i} + \mathbf{j}$ and $\mathbf{n}_2 = 2\mathbf{i} + \mathbf{j} - 2\mathbf{k}$. Then we have

$$\theta = \cos^{-1}\left(\frac{\mathbf{n}_1 \bullet \mathbf{n}_2}{\|\mathbf{n}_1\|\,\|\mathbf{n}_2\|}\right) = \cos^{-1}\left(\frac{2+1}{\sqrt{2}\sqrt{9}}\right) = \cos^{-1}\left(\frac{1}{\sqrt{2}}\right) = 45°$$

13.1. (a). $-2\mathbf{i} + 4\mathbf{j} - 4\mathbf{k}$ (b). $\frac{39}{25}\mathbf{i} + \frac{52}{25}\mathbf{j} + \frac{13}{5}\mathbf{k}$ (c). $\frac{2}{3}\mathbf{i} - \frac{2}{3}\mathbf{j} - \frac{2}{3}\mathbf{k}$

13.2. (a). $-2\mathbf{i} + 7\mathbf{j} + 3\mathbf{k}$ (b). $-\frac{8}{7}\mathbf{i} + \frac{4}{7}\mathbf{j} + \frac{10}{7}\mathbf{k}$ (c). $\mathbf{i}$

13.3. (a). $(8, 1, 1)$ (b). $(\frac{213}{20}, \frac{401}{40}, \frac{57}{8})$ (c). $(2, 0, 3)$

13.4.
(a). $x = 1 - t,\ y = -1 + t,\ z = 1 + 4t$

(b). $x = 8 - 6t,\ y = 3 - 2t,\ z = -2 + \frac{14}{5}t$.

(c). $B(7, 7, \frac{5}{4})$

14.1. 12

14.2.

(a). $5 \cdot 4 \cdot 3 = 60$ (b). $5 \cdot 5 \cdot 5 = 125$ (c). $5 \cdot 4 \cdot 4 = 80$

14.3. There are $81 \cdot 10 \cdot 10 \cdot 10 = 81000$ different postcodes.

If we insist that no digits are repeated, then the first two digits can not be 11 (Bilecik), 22 (Edirne), 33 (Mersin), 44 (Malatya), 55 (Samsun), 66 (Yozgat) or 77 (Yalova). That leaves $81 - 7 = 74$ choices for the first two digits. There are then 8 choices for the 3rd digit, 7 choices for the 4th digit and 6 choices for the final digit. Hence there are $74 \cdot 8 \cdot 7 \cdot 6 = 24864$ possible postcodes.

15.1. ${}_5C_3 = 10$

15.3.

(a). ${}_{24}C_3 = 2024$ (b). ${}_{19}C_3 = 969$

15.4.
(a). $\left(\frac{{}_8P_8}{{}_2P_2 \cdot {}_2P_2 \cdot {}_2P_2}\right)\left(\frac{{}_5P_5}{{}_2P_2}\right) = 302\,400$

(b). $\left(\frac{{}_6P_6}{{}_2P_2}\right)({}_3P_3) = 2160$

15.5. Note that a number is divisible by 5 if the its last digit is 0 or 5. Since we don't have a 0, we must find the number of 3 digit numbers which end in a 5 and such that none of the digits are repeated. The answer is ${}_5P_2 = 20$.

15.6. ${}_8P_2 \cdot {}_{10}P_3 = 40\,320$.

16.1. The sample space is $S = \{HHH, HTH, THH, TTH\}$ since the third coin is always H.

(a). $\frac{1}{4}$ (b). $\frac{1}{2}$ (c). $\frac{1}{4}$ (d). 0 (e). $\frac{3}{4}$ (f). $\frac{1}{4}$

16.2.

(a). $\frac{{}_{26}C_5}{{}_{52}C_5} \approx 0.0253$ (d). $\frac{{}_{39}C_{13}}{{}_{52}C_{13}} \approx 0.0128$

(c). $\frac{{}_{48}C_4}{{}_{52}C_4} \approx 0.7187$ (f). $\frac{{}_{16}C_{13}}{{}_{52}C_{13}} \approx 8.8187 \times 10^{-10}$

17.1.

(a). $\frac{13}{20}$ (c). $\frac{7}{10}$ (d). $\frac{1}{4}$ (f). $\frac{3}{5}$.

17.2.

(a). $\frac{7}{17}$ (b). $\frac{1}{3}$ (c). $\frac{63}{65}$ (d). $\frac{5}{6}$ (e). $\frac{91}{95}$

18.1.

(a). $\frac{4}{51}$ (b). $\frac{1}{17}$ (c). 40%

18.2.

(a). $\frac{1}{3}$ (c). 0, 49 (d). $\frac{3}{4}$

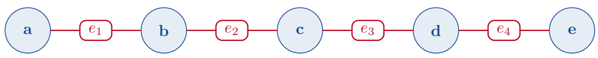
19.1.

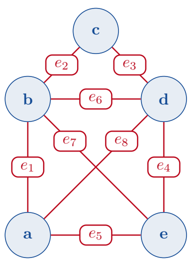
(a). $\frac{54}{125}$ (b). $\frac{71}{125}$ (c). $\frac{14}{25}$ (d). $\frac{17}{25}$ (e). $\frac{21}{25}$

19.2.

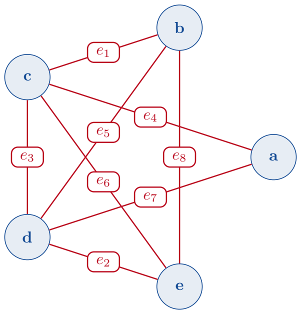
(a). $\frac{1}{6}$. (b). $\frac{1}{4}$. (c). $\frac{1}{4}$.

20.1. One way to draw each graph is shown below, but of course there are ∞ different ways to draw them.

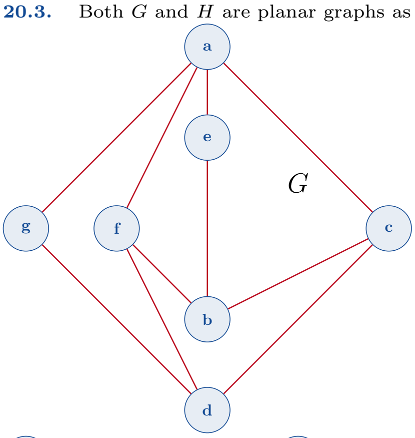
(a). 

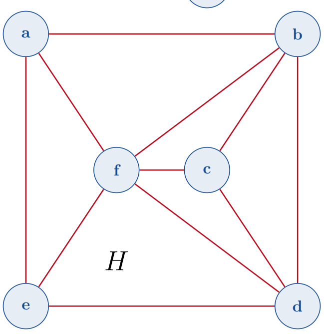
(b). 

(c).



20.3. Both G and H are planar graphs as shown below.





20.4.

(a). yes (b). no (c). no (d). yes

20.5. One such example is

This graph has 2 vertices, 0 edges and 1 face. So $n(V) - n(E) + n(F) = 3$. This example does not disprove theorem 20.4 because it is not connected.

21.1.

(a). false	(c). false	(e). true	(g). false
(b). false	(d). true	(f). true	(h). true

21.2.

(a). -9 (c). $-\frac{5}{2}$ (d). $\frac{2}{3}$ (e). $\frac{1}{10}$ (f). -1 (h). $\frac{1}{5}$

(b). $\frac{5}{8}$ (g). $\frac{3}{2}$

(i). $\lim\limits_{v\to 2} \dfrac{v^3-8}{v^4-16} = \lim\limits_{v\to 2} \dfrac{(v-2)(v^2+2v+4)}{(v-2)(v+2)(v^2+4)} = \lim\limits_{v\to 2} \dfrac{v^2+2v+4}{(v+2)(v^2+4)} = \frac{12}{32} = \frac{3}{8}$

21.3. Clearly $\lim\limits_{x\to 0} 2 - x^2 = 2 - 0^2 = 2$ and $\lim\limits_{x\to 0} 2\cos x = 2\cos 0 = 2$. It follows by the Sandwich Theorem that $\lim\limits_{x\to 0} g(x) = 2$ also.

21.4.

(a). $\lim\limits_{x\to 4} \left(g(x)^2\right) = 9$

(b). $\lim\limits_{x\to 4} (g(x)+3) = 0$

(c). $\lim\limits_{x\to 4} xf(x) = 0$

(d). $\lim\limits_{x\to 4} \dfrac{g(x)}{f(x)-1} = 3$

22.1. First note that f is clearly continuous for all $x \neq -2$. Since $f(-2) = 4b$, we require that $\lim\limits_{x\to -2} f(x) = 4b$ also. But $\lim\limits_{x\to -2} x = -2$. So we must have $b = -\frac{1}{2}$.

22.2.

(a). Since $x^2 - 4 \neq 0$ if $x \neq \pm 2$, it follows that the rational function $\frac{x^3-8}{x^2-4}$ is continuous on $(-\infty, -2)$, on $(-2, 2)$ and on $(2, \infty)$. Hence $f(x)$ is also continuous on these open intervals.

(b). Since

$$\begin{aligned} \lim_{x\to 2} f(x) &= \lim_{x\to 2} \frac{x^3-8}{x^2-4} = \lim_{x\to 2} \frac{(x-2)(x^2+2x+4)}{(x-2)(x+2)} \\ &= \lim_{x\to 2} \frac{x^2+2x+4}{x+2} = \frac{12}{4} = 3 = f(2), \end{aligned}$$

it follows that f is continuous at $x = 2$.

(c). Since $\lim\limits_{x\to -2} f(x) = \lim\limits_{x\to -2} \dfrac{x^3-8}{x^2-4}$ does not exist, it follows that f is discontinuous at $x = -2$.

22.3. Since tan, cos and sin are continuous functions, we use theorem 22.3 to calculate that

$$\begin{aligned} \lim_{t\to 0} \tan\left(\frac{\pi}{4}\cos\left(\sin t^{\frac{1}{3}}\right)\right) &= \tan \lim_{t\to 0}\left(\frac{\pi}{4}\cos\left(\sin t^{\frac{1}{3}}\right)\right) \\ &= \tan\left(\frac{\pi}{4}\lim_{t\to 0}\cos\left(\sin t^{\frac{1}{3}}\right)\right) \\ &= \tan\left(\frac{\pi}{4}\cos\left(\lim_{t\to 0}\sin t^{\frac{1}{3}}\right)\right) \\ &= \tan\left(\frac{\pi}{4}\cos\left(\sin \lim_{t\to 0} t^{\frac{1}{3}}\right)\right) \\ &= \tan\left(\frac{\pi}{4}\cos\left(\sin 0\right)\right) \\ &= \tan\left(\frac{\pi}{4}\cos 0\right) = \tan\frac{\pi}{4} = 1. \end{aligned}$$

24.1.

(a). $\dfrac{ds}{dt} = 2t^{-2} - 8t^{-3} = \dfrac{2}{t^2} - \dfrac{8}{t^3}$.

(b). We calculate that

$$\begin{aligned} w' &= \frac{d}{dz}(z+1)(z-1)(z^2+1) = \frac{d}{dz}(z^2-1)(z^2+1) \\ &= \frac{d}{dz}(z^4-1) = 4z^3 \end{aligned}$$

and

$$w'' = \frac{d}{dz}4z^3 = 12z^2.$$

(c). We calculate that

$$\begin{aligned} \frac{dy}{dx} &= (2x+3)'(x^4+\tfrac{1}{3}x^3+11) + (2x+3)(x^4+\tfrac{1}{3}x^3+11)' \\ &= 2(x^4+\tfrac{1}{3}x^3+11) + (2x+3)(4x^3+x^2) \\ &= 2x^4+\tfrac{2}{3}x^3+22+8x^4+2x^3+12x^3+3x^2 \\ &= 10x^4+\tfrac{44}{3}x^3+3x^2+22 \end{aligned}$$

by the product rule.

24.2. First note that

$$b = \frac{x^2-1}{x^2+x-2} = \frac{(x-1)(x+1)}{(x+2)(x-1)} = \frac{x+1}{x+2}.$$

Using the quotient rule, we calculate that

$$\begin{aligned} \frac{db}{dx} &= \left(\frac{u}{v}\right)' = \frac{u'v-uv'}{v^2} = \frac{(x+1)'(x+2)-(x+1)(x+2)'}{(x+2)^2} \\ &= \frac{(x+2)-(x+1)}{(x+2)^2} = \frac{1}{(x+2)^2}. \end{aligned}$$

24.3.

(a). $y' = 2x^3 - 3x - 1$

(b). $y' = 3x^2 + 4x - 8$

(c). $r' = -\frac{2}{3s^3} + \frac{5}{2s^2}$

(d). $y = \frac{-19}{(3x-2)^2}$

(e). $g'(x) = \frac{x^2+x+4}{(x+0.5)^2}$

(f). $v' = \frac{t^2-2t-1}{(1+t^2)^2}$

(g). $f'(s) = \frac{1}{\sqrt{s}(\sqrt{s}+1)^2}$

(h). $v' = -\frac{1}{x^2} + 2x^{-\frac{3}{2}}$

(i). $r' = 3\theta^{-4}$

(j). $w' = -z^{-2} - 1$

(k). $s' = 15t^2 - 15t^4$

(l). $w' = -\frac{6}{z^3} + \frac{1}{z^2}$

25.1.

(a). $\dfrac{ds}{dx} = \sec^2 x.$ (b). $\dfrac{dr}{d\theta} = \theta\cos\theta.$

25.2.

(b). By the quotient rule, we have that

$$\begin{aligned}\frac{d}{dx}(\cot x) &= \frac{d}{dx}\left(\frac{\cos x}{\sin x}\right)\\ &= \frac{(\cos x)'\sin x - (\cos x)(\sin x)'}{\sin^2 x}\\ &= \frac{-\sin x\sin x - \cos x\cos x}{\sin^2 x}\\ &= \frac{-1}{\sin^2 x} = -\operatorname{cosec}^2 x\end{aligned}$$

as required.

25.3.

(a). $y' = -10 - 3\sin x$

(b). $y' = 2x\cos x - x^2\sin x$

(c). $y' = -\operatorname{cosec} x\cot x - \frac{2}{\sqrt{x}}$

(d). $f'(x) = \sin x\sec^2 x + \sin x$

(e). $g'(x) = \cos x$

(f). $w' = \frac{-\operatorname{cosec}^2 z}{(1+\cot z)^2}$

(g). $h'(x) = 3x^2\sin x\cos x + x^3\cos^2 x - x^3\sin^2 x$

(h). $p' = \sec^2 t$

(i). $r' = \sec^2 t$

(j). 0

26.1. $\dfrac{ds}{dt} = -5\left(\dfrac{t}{2} - 1\right)^{-11}.$

26.2. $\dfrac{dy}{dt} = -\dfrac{5}{3}\sin\left(5\sin\left(\dfrac{t}{3}\right)\right)\cos\left(\dfrac{t}{3}\right).$

26.3. $\dfrac{dy}{dx} = \dfrac{3x-2}{\sqrt{3x^2-4x+6}}.$

26.4. $\dfrac{dy}{dx} = 3\sin^2 x\cos x.$

26.5. $\dfrac{dy}{dx} = \tan(\tan x)\sec^2 x\sec(\tan x).$

26.6. $\dfrac{dy}{dx} = 2x\cos(2x)\cos(x^2) - 2\sin(2x)\sin(x^2).$

26.7. Let $u = \frac{t}{t^2-4}$. Then

$$\begin{aligned}\frac{dy}{dx} &= \frac{d}{dx}\left(\frac{t^2}{t^3-4t}\right)^3 = \frac{d}{dx}\left(\frac{t}{t^2-4}\right)^3\\ &= \frac{d}{dx}u^3 = \left(\frac{d}{du}u^3\right)\frac{du}{dx} = 3u^2\frac{d}{dx}\left(\frac{t}{t^2-4}\right)\\ &= \left(\frac{3t^2}{(t^2-4)^2}\right)\left(\frac{(t)'(t^2-4)-(t)(t^2-4)'}{(t^2-4)^2}\right)\\ &= \left(\frac{3t^2}{(t^2-4)^2}\right)\left(\frac{(t^2-4)-(t)(2t)}{(t^2-4)^2}\right)\\ &= -\frac{3t^2(t^2+4)}{(t^2-4)^4}.\end{aligned}$$

26.8. $y'' = \dfrac{6}{x^3}\left(1+\dfrac{1}{x}\right)\left(1+\dfrac{2}{x}\right).$

26.9. $\frac{5}{2}$.

26.10. 0.

27.1.

(a) $F(x) = 100x^2$.

(b) $G(x) = \frac{x^4}{4} + \frac{1}{2x^2}$.

(c) $H(x) = -\frac{1}{\pi}\cos(\pi x) + \cos(3x)$.

(d) $L(x) = \frac{x^8}{8} - 3x^2 + 8x$.

(e) $M(x) = x^{\frac{2}{3}}$ is an antiderivative of $m(x) = \frac{2}{3}x^{-\frac{1}{3}}$ because

$$M'(x) = \frac{d}{dx}x^{\frac{2}{3}} = \frac{2}{3}x^{\frac{2}{3}-1} = \frac{2}{3}x^{-\frac{1}{3}} = m(x).$$

(f) $P(x) = x^{\frac{1}{3}}$.

(g) $R(x) = 2\tan\frac{x}{3}$.

(h) $S(x) = -\frac{2}{3}\tan\frac{3x}{2}$.

27.2. It is incorrect because $\frac{d}{dx}\left(\frac{(2x+1)^3}{3} + \sin x\right) \neq (2x+1)^2 + \cos x$.

27.3. It is correct. By the Chain Rule (with $u = e^x$),

$$\frac{d}{dx}\sin e^x = \left(\frac{d}{du}\sin u\right)\left(\frac{d}{dx}e^x\right) = (\cos u)(e^x) = e^x\cos e^x.$$

Thus $\sin e^x$ is an antiderivative of $e^x\cos e^x$.

27.4. It is incorrect because the "$+C$" is missing.

27.5.

(a). $\int 2x\ dx = x^2 + C$

(b). $\int (1 - x^2 - 3x^5)\ dx = x - \frac{x^3}{3} - \frac{x^6}{2} + C$

(c). $\int \frac{4+\sqrt{t}}{t^3}\ dt = \int 4t^{-3} + t^{-\frac{5}{2}}\ dt = -2t^{-2} - \frac{2}{3}t^{-\frac{3}{2}} + C$

(d). $\int (2\cos 2\theta - 3\sin 3\theta)\ d\theta = \sin 2\theta + \cos 3\theta + C$

(e). $\int 2e^{3x}\ dx = \frac{2}{3}e^{3x} + C$

(f). $\int \frac{1}{x}\ dx = \ln|x| + C$

30.1.

(a). $\frac{dy}{dx} = \sqrt{1+x^2}$.

(b). $\frac{db}{dt} = 4t^5$.

(c). We calculate that

$$\begin{aligned}\frac{dp}{dx} &= \frac{d}{dx}\int_2^{x^2}\sin(t^3)\ dt = \left(\frac{d}{du}\int_2^u \sin(t^3)\ dt\right)\left(\frac{d}{dx}x^2\right)\\ &= \sin(u^3)\ 2x = 2x\sin(x^6).\end{aligned}$$

(d). $\frac{dz}{dx} = -\frac{1}{2}x^{-\frac{1}{2}}\sin x = -\frac{\sin x}{2\sqrt{x}}$.

30.2.

(a). 6

(d). $\frac{133}{4}$

(e). $\frac{\pi}{4} - \frac{1}{2}$

(f). $\frac{5}{2}$

(g). 0

(h). $-\frac{8}{3}$

(i). $\sqrt{2} - \sqrt[4]{8} + 1$

(j). 16

(k). -1

(l). $2\sqrt{3}$

31.1.

(a). $\frac{1}{6}(2x+4)^6+C.$

(b). $-\frac{(x^2+5)^{-3}}{3}+C.$

(c). $\frac{(3x^2+4x)^5}{10}+C.$

(d). $\frac{1}{2}\sec 2t+C.$

(e). $-6\sqrt{1-r^3}+C.$

(f). $-\frac{1}{3}(x^{\frac{3}{2}}-1)-\frac{1}{6}\sin(x^{\frac{3}{2}}-1)+C.$

(g). $-\frac{2}{1+\sqrt{x}}+C$

(h). $-\frac{1}{2}\left(7-\frac{r^5}{10}\right)^4+C.$

(i). $-\frac{1}{2}\sin^2\frac{1}{\theta}+C.$

(j). $\frac{1}{12}(x-1)^{12}+\frac{1}{11}(x-1)^{11}+C$

(k). $\frac{1}{5}(x^2+1)^{\frac{5}{2}}-\frac{1}{3}(x^2+1)^{\frac{3}{2}}+C.$

(l). $\frac{1}{3}e^{z^3}+C.$

31.2.

(a). $\frac{14}{3}$ (b). $\frac{2}{3}$ (c). $\frac{1}{2}$ (d). 0 (e). 0 (f). $\frac{1}{6}$ (g). 3 (h). $-\frac{1}{15}$ (i). $\frac{1}{15}$

32.1. We calculate that

$$\begin{aligned}\text{total area} &= \int_{-2}^{2} 2x^2-(x^4-2x^2)\ dx = 2\int_0^2 4x^2-x^4\ dx\\ &= 2\left[\frac{4}{3}x^3-\frac{x^5}{5}\right]_0^2 = 2\left(\left(\frac{4}{3}(8)-\frac{32}{5}\right)-(0-0)\right)\\ &= \frac{128}{15}.\end{aligned}$$

32.2.

(a). 2 (b). $\frac{5}{2}$ (c). $\frac{38}{3}$

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