



Welcome to

Mathematics II

with Dr Neil Course

Lecture 1

- Information about this course
- 8.1 Using Basic Integration Formulae
- 8.2 Integration by Parts
- 8.3 Trigonometric Integrals

Information about this course

- \approx 12 classes. Friday afternoons 1pm-3:30pm.



13:00

14:00

15:00

Information about this course

- \approx 12 classes. Friday afternoons 1pm-3:30pm.
- 2 lectures with a break between.

lecture

lecture

13:00

14:00

15:00

Information about this course

- \approx 12 classes. Friday afternoons 1pm-3:30pm.
- 2 lectures with a break between.
- Then I will answers your questions.

lecture

lecture

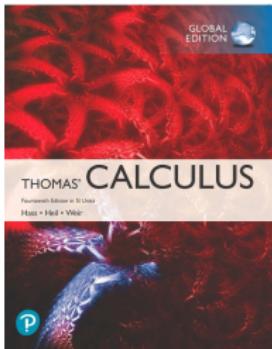
questions

13:00

14:00

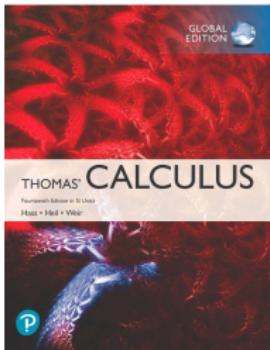
15:00

The Book



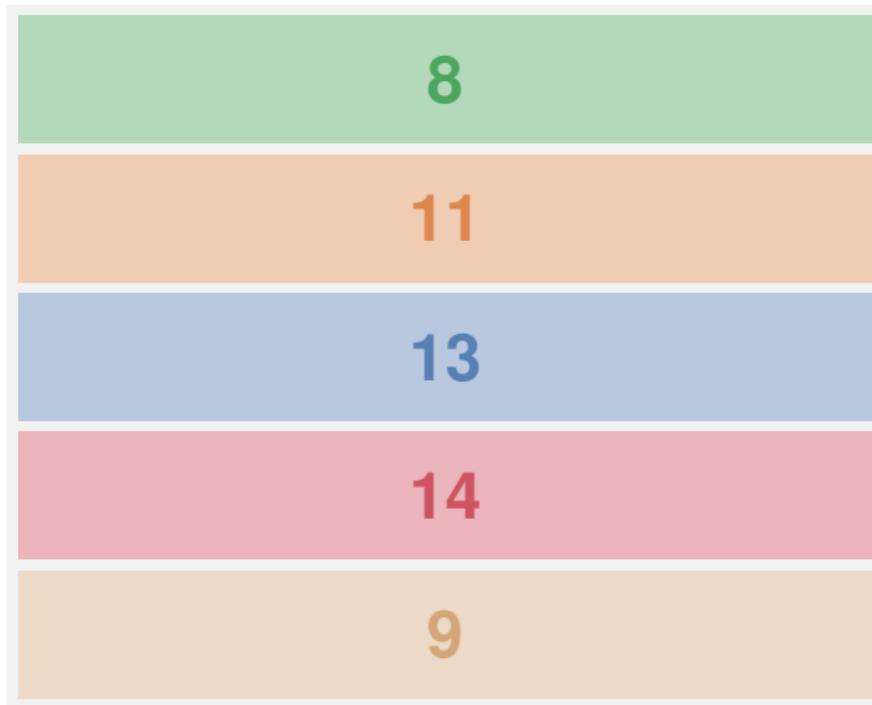
Joel R. Hass, Christopher E. Heil and Maurice D. Weir,
Thomas' Calculus,
14th Edition in SI Units, Pearson.

The Book



Joel R. Hass, Christopher E. Heil and Maurice D. Weir,
Thomas' Calculus,
14th Edition in SI Units, Pearson.

This is a required purchase.
You need to have this book to be
able to do the homework.



8. Techniques of Integration

11

13

14

9

8. Techniques of Integration

11. Vectors and the Geometry of Space

13

14

9

8. Techniques of Integration

11. Vectors and the Geometry of Space

13. Partial Derivatives

14

9

8. Techniques of Integration

11. Vectors and the Geometry of Space

13. Partial Derivatives

14. Multiple Integrals

8. Techniques of Integration

11. Vectors and the Geometry of Space

13. Partial Derivatives

14. Multiple Integrals

9. Infinite Sequences and Series

8. Techniques of Integration

2 weeks

11. Vectors and the Geometry of Space

2 weeks

13. Partial Derivatives

2 weeks

14. Multiple Integrals

3 weeks

9. Infinite Sequences and Series

3 weeks



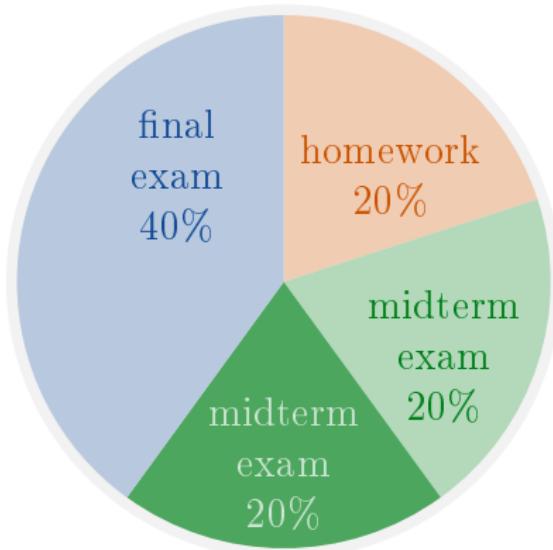
Exams and homework

(This information may change based on the University's decisions)



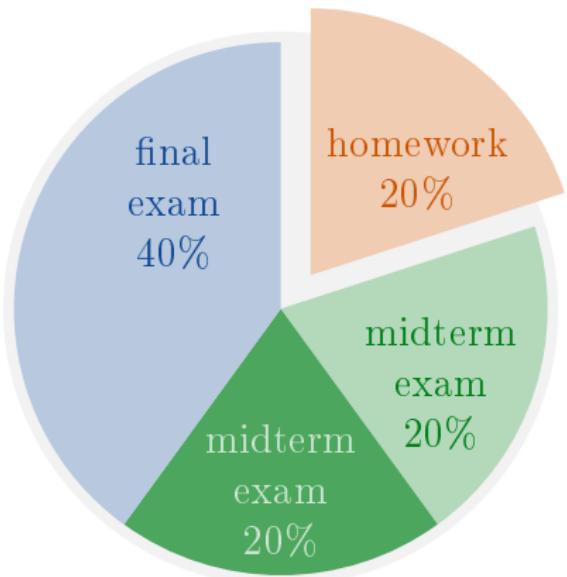
Exams and homework

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Exams and homework

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using Pearson
MyLab Math

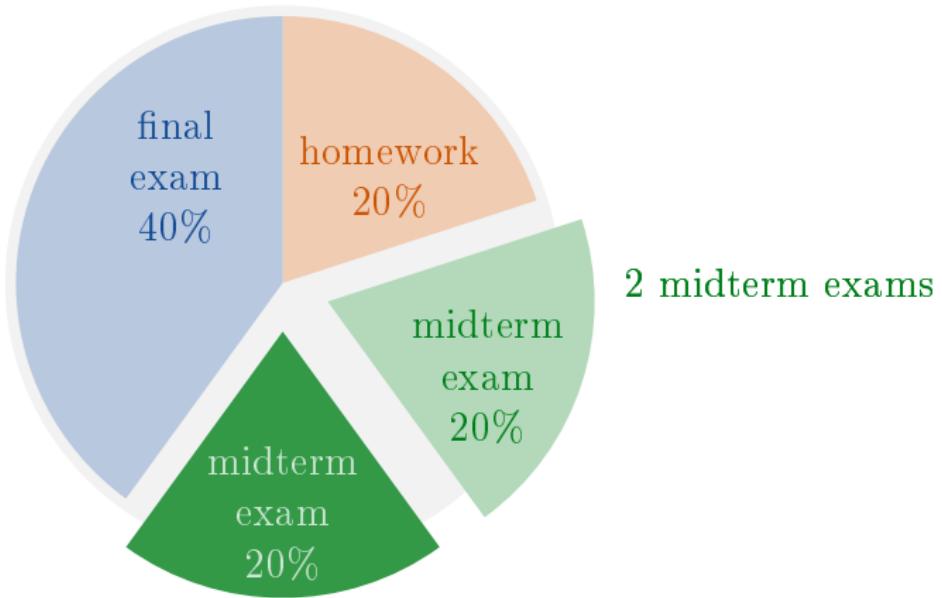
one piece of
homework for
each lesson

deadline = end of
term

more details in
O'Learn

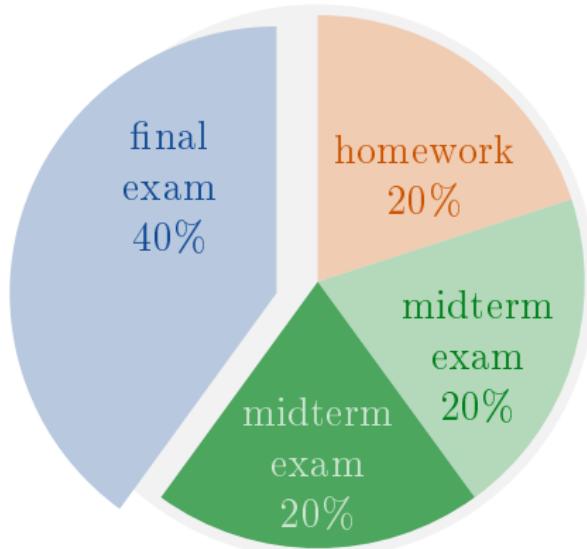
Exams and homework

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Exams and homework

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Expectations

If this was a classroom course, the expectation is that for each hour of lectures, you would study an extra 1-2 hours outside of class.

classroom
course

lectures (5 hours)

other study (5-10 hours)

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classroom
course

lectures (5 hours)

other study (5-10 hours)

For an online course, you are still expected to study a total of 10-15 hours each week.

online
course

class
(2.5 hours)

other study (7.5-12.5 hours)

This may include:

- Do the online homework on MyLab;

⋮

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- Use the O'Learn Discussion Board;

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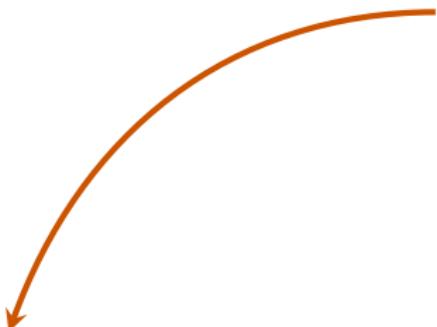
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- Solve the exercises in the textbook;
- Use the O'Learn Discussion Board;
- Read other books?;
- Watch online videos;

:

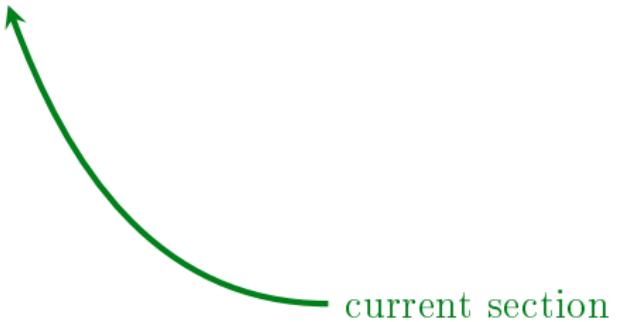
99.9 Section Title



slide number



99.9 Section Title





Using Basic Integration Formulae

Table 8.1

Basic integration formulas

1. $\int k \, dx = kx + C$ (any number k)	12. $\int \tan x \, dx = \ln \sec x + C$
2. $\int x^n \, dx = \frac{x^{n+1}}{n+1} + C$ ($n \neq -1$)	13. $\int \cot x \, dx = \ln \sin x + C$
3. $\int \frac{dx}{x} = \ln x + C$	14. $\int \sec x \, dx = \ln \sec x + \tan x + C$
4. $\int e^x \, dx = e^x + C$	15. $\int \csc x \, dx = -\ln \csc x + \cot x + C$
5. $\int a^x \, dx = \frac{a^x}{\ln a} + C$ ($a > 0, a \neq 1$)	16. $\int \sinh x \, dx = \cosh x + C$
6. $\int \sin x \, dx = -\cos x + C$	17. $\int \cosh x \, dx = \sinh x + C$
7. $\int \cos x \, dx = \sin x + C$	18. $\int \frac{dx}{\sqrt{a^2 - x^2}} = \sin^{-1} \left(\frac{x}{a} \right) + C$
8. $\int \sec^2 x \, dx = \tan x + C$	19. $\int \frac{dx}{a^2 + x^2} = \frac{1}{a} \tan^{-1} \left(\frac{x}{a} \right) + C$
9. $\int \csc^2 x \, dx = -\cot x + C$	20. $\int \frac{dx}{x\sqrt{x^2 - a^2}} = \frac{1}{a} \sec^{-1} \left \frac{x}{a} \right + C$
10. $\int \sec x \tan x \, dx = \sec x + C$	21. $\int \frac{dx}{\sqrt{a^2 + x^2}} = \sinh^{-1} \left(\frac{x}{a} \right) + C$ ($a > 0$)
11. $\int \csc x \cot x \, dx = -\csc x + C$	22. $\int \frac{dx}{\sqrt{x^2 - a^2}} = \cosh^{-1} \left(\frac{x}{a} \right) + C$ ($x > a > 0$)

8.1 Using Basic Integration Formulae

Example

Calculate $\int_3^5 \frac{2x - 3}{\sqrt{x^2 - 3x + 1}} dx.$

8.1 Using Basic Integration Formulae

Example

Calculate $\int_3^5 \frac{2x - 3}{\sqrt{x^2 - 3x + 1}} dx.$

We use the substitution $u = x^2 - 3x + 1$.

8.1 Using Basic Integration Formulae

Example

Calculate $\int_3^5 \frac{2x - 3}{\sqrt{x^2 - 3x + 1}} dx$.

We use the substitution $u = x^2 - 3x + 1$. Then $du = (2x - 3) dx$ and

$$x = 3 \implies u = 9 - 9 + 1 = 1$$

$$x = 5 \implies u = 25 - 15 + 1 = 11.$$

8.1 Using Basic Integration Formulae

Example

Calculate $\int_3^5 \frac{2x - 3}{\sqrt{x^2 - 3x + 1}} dx.$

We use the substitution $u = x^2 - 3x + 1$. Then $du = (2x - 3) dx$ and

$$x = 3 \implies u = 9 - 9 + 1 = 1$$

$$x = 5 \implies u = 25 - 15 + 1 = 11.$$

Hence

$$\int_3^5 \frac{2x - 3}{\sqrt{x^2 - 3x + 1}} dx = \int_1^{11} u^{-\frac{1}{2}} du = [2\sqrt{u}]_1^{11} = 2(\sqrt{11} - 1).$$

8.1

$$\int \frac{dx}{\sqrt{a^2 - x^2}} = \sin^{-1} \left(\frac{x}{a} \right) + C$$



Example

Find $\int \frac{dx}{\sqrt{8x - x^2}}$.

8.1

$$\int \frac{dx}{\sqrt{a^2 - x^2}} = \sin^{-1} \left(\frac{x}{a} \right) + C$$



Example

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This time we will complete the square of $x^2 - 8x$ and use that to simplify the integral:

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Example

Find $\int \frac{dx}{\sqrt{8x - x^2}}$.

This time we will complete the square of $x^2 - 8x$ and use that to simplify the integral:

$$x^2 - 8x = x^2 - 8x + 16 - 16$$

8.1

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$$x^2 - 8x = x^2 - 8x + 16 - 16 = (x^2 - 8x + 16) - 16 = (x - 4)^2 - 16.$$

8.1

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So

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=

=

=

8.1

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$$= \int \frac{du}{\sqrt{16 - u^2}}$$

=

=

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$$\int \frac{dx}{\sqrt{a^2 - x^2}} = \sin^{-1} \left(\frac{x}{a} \right) + C$$



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=

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8.1 Using Basic Integration Formulae



Example

Find $\int \cos x \sin 2x + \sin x \cos 2x \, dx$.

8.1 Using Basic Integration Formulae



Example

Find $\int \cos x \sin 2x + \sin x \cos 2x \, dx$.

$$\begin{aligned}\int \cos x \sin 2x + \sin x \cos 2x \, dx &= \int \sin(x + 2x) \, dx \\ &= \int \sin 3x \, dx \\ &= \dots\end{aligned}$$

8.1 Using Basic Integration Formulae



Example

Find $\int_0^{\frac{\pi}{4}} \frac{dx}{1 - \sin x}$.

8.1 Using Basic Integration Formulae



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Here is a trick for dealing with $\frac{1}{A-B}$:

8.1 Using Basic Integration Formulae

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Here is a trick for dealing with $\frac{1}{A-B}$: Multiply by $\frac{A+B}{A+B}$.

8.1 Using Basic Integration Formulae



Example

Find $\int_0^{\frac{\pi}{4}} \frac{dx}{1 - \sin x}.$

Here is a trick for dealing with $\frac{1}{A-B}$: Multiply by $\frac{A+B}{A+B}$.
Then we get

$$\frac{1}{A-B} = \left(\frac{1}{A-B} \right) \left(\frac{A+B}{A+B} \right) = \frac{A+B}{A^2 - B^2}$$

which is sometimes easier to deal with.

$$\begin{aligned}
 \int_0^{\pi/4} \frac{dx}{1 - \sin x} &= \int_0^{\pi/4} \frac{1}{1 - \sin x} \cdot \frac{1 + \sin x}{1 + \sin x} dx && \text{Multiply and divide by conjugate.} \\
 &= \int_0^{\pi/4} \frac{1 + \sin x}{1 - \sin^2 x} dx && \text{Simplify.} \\
 &= \int_0^{\pi/4} \frac{1 + \sin x}{\cos^2 x} dx && 1 - \sin^2 x = \cos^2 x \\
 &= \int_0^{\pi/4} (\sec^2 x + \sec x \tan x) dx && \text{Use Table 8.1, Formulas 8 and 10} \\
 &= \left[\tan x + \sec x \right]_0^{\pi/4} = (1 + \sqrt{2} - (0 + 1)) = \sqrt{2}.
 \end{aligned}$$



8.1 Using Basic Integration Formulae



Example

Find $\int \frac{3x^2 - 7x}{3x + 2} dx.$

8.1 Using Basic Integration Formulae



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Find $\int \frac{3x^2 - 7x}{3x + 2} dx$.

Solution The integrand is an improper fraction since the degree of the numerator is greater than the degree of the denominator. To integrate it, we perform long division to obtain a quotient plus a remainder that is a proper fraction:

$$\frac{3x^2 - 7x}{3x + 2} = x - 3 + \frac{6}{3x + 2}.$$

8.1 Using Basic Integration Formulae



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$$\frac{3x^2 - 7x}{3x + 2} = x - 3 + \frac{6}{3x + 2}.$$

Therefore,

$$\int \frac{3x^2 - 7x}{3x + 2} dx = \int \left(x - 3 + \frac{6}{3x + 2} \right) dx = \frac{x^2}{2} - 3x + 2 \ln |3x + 2| + C. \quad \blacksquare$$

8.1

$$\int \frac{dx}{\sqrt{a^2 - x^2}} = \sin^{-1} \left(\frac{x}{a} \right) + C$$



Example

Find $\int \frac{3x + 2}{\sqrt{1 - x^2}}.$

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$$\int \frac{dx}{\sqrt{a^2 - x^2}} = \sin^{-1} \left(\frac{x}{a} \right) + C$$



Example

Find $\int \frac{3x + 2}{\sqrt{1 - x^2}}$.

First note that

$$\int \frac{3x + 2}{\sqrt{1 - x^2}} = 3 \int \frac{x \, dx}{\sqrt{1 - x^2}} + 2 \int \frac{dx}{\sqrt{1 - x^2}}$$

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So we just need to calculate $\int \frac{x \, dx}{\sqrt{1 - x^2}}$.

8.1 Using Basic Integration Formulae



$$\int \frac{x \, dx}{\sqrt{1 - x^2}}$$

Let $u = 1 - x^2$.

8.1 Using Basic Integration Formulae



$$\int \frac{x \, dx}{\sqrt{1 - x^2}}$$

Let $u = 1 - x^2$. Then $du = -2x \, dx$ and $-\frac{1}{2} du = x \, dx$.

8.1 Using Basic Integration Formulae



$$\int \frac{x \, dx}{\sqrt{1 - x^2}}$$

Let $u = 1 - x^2$. Then $du = -2x \, dx$ and $-\frac{1}{2} du = x \, dx$. It follows that

$$\int \frac{x \, dx}{\sqrt{1 - x^2}} = \int \frac{-\frac{1}{2} du}{\sqrt{u}} = -\frac{1}{2} \int u^{-\frac{1}{2}} \, du = \dots = -\sqrt{1 - x^2} + C.$$

8.1 Using Basic Integration Formulae



$$\int \frac{x \, dx}{\sqrt{1 - x^2}}$$

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Therefore

$$\begin{aligned}\int \frac{3x + 2}{\sqrt{1 - x^2}} &= 3 \int \frac{x \, dx}{\sqrt{1 - x^2}} + 2 \int \frac{dx}{\sqrt{1 - x^2}} \\ &= -3\sqrt{1 - x^2} + 2 \sin^{-1} x + C.\end{aligned}$$

8.1 Using Basic Integration Formulae

Example

Find $\int \frac{dx}{(1 + \sqrt{x})^3}$.

8.1 Using Basic Integration Formulae

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$$\text{Find } \int \frac{dx}{(1 + \sqrt{x})^3}.$$

I want to make a substitution to make this integral easier, but what u should I choose?

8.1 Using Basic Integration Formulae

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Find $\int \frac{dx}{(1 + \sqrt{x})^3}$.

I want to make a substitution to make this integral easier, but what u should I choose?

First guess: $u = \sqrt{x}$.

8.1 Using Basic Integration Formulae

Example

Find $\int \frac{dx}{(1 + \sqrt{x})^3}$.

I want to make a substitution to make this integral easier, but what u should I choose?

First guess: $u = \sqrt{x}$. But then $du = \frac{1}{2\sqrt{x}} dx$ and we would have to deal with this extra $\sqrt{x} = u$ term.

8.1 Using Basic Integration Formulae

Example

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I want to make a substitution to make this integral easier, but what u should I choose?

First guess: $u = \sqrt{x}$. But then $du = \frac{1}{2\sqrt{x}} dx$ and we would have to deal with this extra $\sqrt{x} = u$ term.

Second guess: Instead let us try $u = 1 + \sqrt{x}$.

8.1 Using Basic Integration Formulae

Example

$$\text{Find } \int \frac{dx}{(1 + \sqrt{x})^3}.$$

I want to make a substitution to make this integral easier, but what u should I choose?

First guess: $u = \sqrt{x}$. But then $du = \frac{1}{2\sqrt{x}} dx$ and we would have to deal with this extra $\sqrt{x} = u$ term.

Second guess: Instead let us try $u = 1 + \sqrt{x}$. Then again we have $du = \frac{1}{2\sqrt{x}} dx$ and $dx = 2\sqrt{x} du = 2(u - 1) du$. Hence

$$\int \frac{dx}{(1 + \sqrt{x})^3} = \int \frac{2(u - 1) du}{u^3} = \int \frac{2}{u^2} - \frac{2}{u^3} du = \dots$$

8.1 Using Basic Integration Formulae

Example

Calculate $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} x^3 \cos x dx$.

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This is actually easy: The integrand is an odd function

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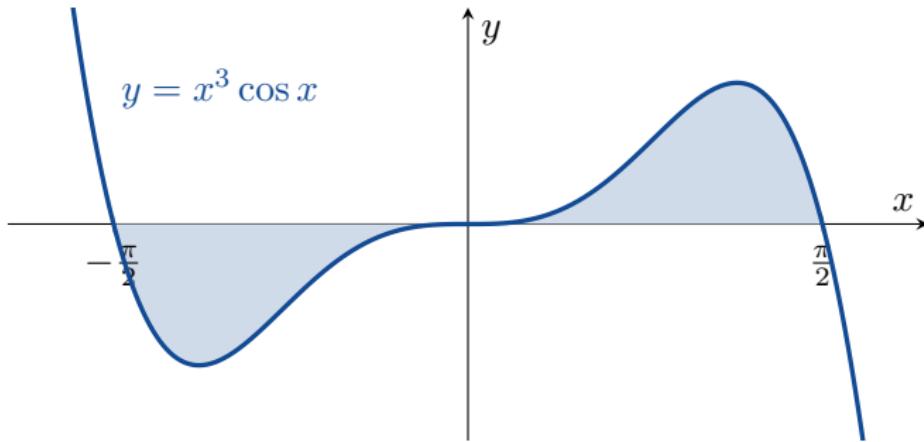
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Integration by Parts

8.2 Integration by Parts

How can we calculate

$$\int x \cos x \, dx$$

or

$$\int x^2 e^x \, dx ?$$

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$$\int \text{function} \times \text{function} \, dx$$

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Theorem (Integration by Parts)

$$\int u(x)v'(x) \, dx =$$

8.2 Integration by Parts

How can we calculate

$$\int x \cos x \, dx$$

or

$$\int x^2 e^x \, dx ?$$

$$\int \text{function} \times \text{function} \, dx$$

Theorem (Integration by Parts)

$$\int u(x)v'(x) \, dx = u(x)v(x) - \int u'(x)v(x) \, dx$$

$$\int \color{red}{uv'}\,dx = \color{red}{uv} - \int \color{red}{u'}v\,dx$$

Example

Find $\int x \cos x\,dx$.

$$\int \textcolor{red}{uv}' dx = \textcolor{red}{uv} - \int \textcolor{red}{u}'\textcolor{green}{v} dx$$

Example

Find $\int x \cos x dx$.

We need to choose a $\textcolor{red}{u}(x)$ and a $v'(x)$.

8.2

$$\int \textcolor{red}{u} \textcolor{green}{v}' dx = \textcolor{red}{u} \textcolor{green}{v} - \int \textcolor{red}{u}' \textcolor{green}{v} dx$$



Example

Find $\int x \cos x dx$.

We need to choose a $\textcolor{red}{u}(x)$ and a $v'(x)$.

Let

$$\textcolor{red}{u} = x \qquad \qquad v' = \cos x$$

Then

$$\int \textcolor{red}{x} \cos x dx = \qquad \qquad - \int \qquad \qquad dx = \qquad \qquad .$$

$$\int \textcolor{brown}{u} \textcolor{green}{v}' dx = \textcolor{brown}{u} \textcolor{green}{v} - \int \textcolor{brown}{u}' \textcolor{green}{v} dx$$

Example

Find $\int x \cos x dx$.

We need to choose a $\textcolor{brown}{u}(x)$ and a $v'(x)$.

Let

$$\begin{aligned} u &= x & v' &= \cos x \\ u' &= 1 \end{aligned}$$

Then

$$\int \textcolor{brown}{x} \cos x dx = \quad - \int \quad dx = \quad .$$

$$\int \textcolor{red}{u} \textcolor{green}{v}' dx = \textcolor{red}{u} \textcolor{green}{v} - \int \textcolor{red}{u}' \textcolor{green}{v} dx$$

Example

Find $\int x \cos x dx$.

We need to choose a $\textcolor{red}{u}(x)$ and a $v'(x)$.

Let

$$\begin{array}{ll} u = x & v' = \cos x \\ u' = 1 & v = \sin x. \end{array}$$

Then

$$\int \textcolor{brown}{x} \cos x dx = \quad - \int \quad dx = \quad .$$

$$\int \textcolor{brown}{u} \textcolor{green}{v}' dx = \textcolor{brown}{u} \textcolor{green}{v} - \int \textcolor{brown}{u}' \textcolor{green}{v} dx$$

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Find $\int x \cos x dx$.

We need to choose a $\textcolor{brown}{u}(x)$ and a $v'(x)$.

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$$\begin{array}{ll} u = x & v' = \cos x \\ u' = 1 & v = \sin x. \end{array}$$

Then

$$\int \textcolor{brown}{x} \cos x dx = \textcolor{brown}{x} \sin x - \int \quad dx = \quad .$$

$$\int \textcolor{red}{uv}' dx = \textcolor{red}{uv} - \int \textcolor{red}{u'}v dx$$

Example

Find $\int x \cos x dx$.

We need to choose a $\textcolor{red}{u}(x)$ and a $v'(x)$.

Let

$$\begin{array}{ll} u = x & v' = \cos x \\ u' = 1 & v = \sin x. \end{array}$$

Then

$$\int \textcolor{red}{x} \cos x dx = \textcolor{red}{x} \sin x - \int \textcolor{red}{1} \sin x dx = \quad .$$

$$\int \textcolor{brown}{u} \textcolor{green}{v}' dx = \textcolor{brown}{u} \textcolor{green}{v} - \int \textcolor{brown}{u}' \textcolor{green}{v} dx$$

Example

Find $\int x \cos x dx$.

We need to choose a $\textcolor{brown}{u}(x)$ and a $v'(x)$.

Let

$$\begin{array}{ll} u = x & v' = \cos x \\ u' = 1 & v = \sin x. \end{array}$$

Then

$$\int \textcolor{brown}{x} \cos x dx = \textcolor{brown}{x} \sin x - \int 1 \sin x dx = x \sin x + \cos x + C.$$

$$\int \textcolor{red}{uv}' dx = \textcolor{red}{uv} - \int \textcolor{red}{u}'\textcolor{green}{v} dx$$

Example

Find $\int \ln x dx$.

$$\int \textcolor{red}{uv}' dx = \textcolor{red}{uv} - \int \textcolor{red}{u}'\textcolor{green}{v} dx$$

Example

Find $\int \ln x dx$.

We will consider $\int \ln x \cdot 1 dx$.

$$\int \textcolor{red}{uv}' dx = \textcolor{red}{uv} - \int \textcolor{red}{u}'\textcolor{green}{v} dx$$

Example

Find $\int \ln x dx$.

We will consider $\int \ln x \cdot 1 dx$.

Let

$$u = \ln x \quad v' = 1$$

$$\int \textcolor{red}{uv}' dx = \textcolor{red}{uv} - \int \textcolor{red}{u}'\textcolor{green}{v} dx$$

Example

Find $\int \ln x dx$.

We will consider $\int \ln x \cdot 1 dx$.

Let

$$u = \ln x \qquad \qquad v' = 1$$

$$u' = \frac{1}{x}$$

$$\int \textcolor{red}{uv}' dx = \textcolor{red}{uv} - \int \textcolor{red}{u}'\textcolor{green}{v} dx$$

Example

Find $\int \ln x dx$.

We will consider $\int \ln x \cdot 1 dx$.

Let

$$u = \ln x \quad v' = 1$$

$$u' = \frac{1}{x} \quad v = x.$$

8.2

$$\int \textcolor{red}{uv}' dx = \textcolor{red}{uv} - \int \textcolor{red}{u}'\textcolor{green}{v} dx$$



Example

Find $\int \ln x dx.$

We will consider $\int \ln x \cdot 1 dx.$

Let

$$u = \ln x \quad v' = 1$$

$$u' = \frac{1}{x} \quad v = x.$$

Then

$$\int \ln x \cdot 1 dx = \ln x \cdot x - \int \frac{1}{x} \cdot x dx$$

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=

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$$\int \textcolor{red}{uv}' dx = \textcolor{red}{uv} - \int \textcolor{red}{u}'\textcolor{green}{v} dx$$

Example

Find $\int \ln x dx.$

We will consider $\int \ln x \cdot 1 dx.$

Let

$$\begin{array}{ll} u = \ln x & v' = 1 \\ u' = \frac{1}{x} & v = x. \end{array}$$

Then

$$\begin{aligned} \int \ln x \cdot 1 dx &= \ln x \cdot x - \int \frac{1}{x} \cdot x dx \\ &= x \ln x - \int 1 dx \\ &= . \end{aligned}$$

$$\int \textcolor{red}{uv}' dx = \textcolor{red}{uv} - \int \textcolor{red}{u}'\textcolor{green}{v} dx$$

Example

Find $\int \ln x dx.$

We will consider $\int \ln x \cdot 1 dx.$

Let

$$\begin{array}{ll} u = \ln x & v' = 1 \\ u' = \frac{1}{x} & v = x. \end{array}$$

Then

$$\begin{aligned} \int \ln x \cdot 1 dx &= \ln x \cdot x - \int \frac{1}{x} \cdot x dx \\ &= x \ln x - \int 1 dx \\ &= x \ln x - x + C. \end{aligned}$$

8.2

$$\int \color{red}{uv'}\,dx = \color{red}{uv} - \int \color{red}{u'}v\,dx$$



Sometimes we have to use integration by parts more than once.

$$\int \textcolor{red}{u} \textcolor{green}{v}' dx = \textcolor{red}{u} \textcolor{green}{v} - \int \textcolor{red}{u}' \textcolor{green}{v} dx$$

Example

Find $\int \textcolor{brown}{x}^2 e^x dx$.

$$\int uv' dx = uv - \int u'v dx$$

Example

Find $\int x^2 e^x dx$.

We calculate that

$$\int x^2 e^x dx = x^2 e^x - 2 \int x e^x dx.$$

$$\int \textcolor{red}{uv}' dx = \textcolor{red}{uv} - \int \textcolor{red}{u}'\textcolor{green}{v} dx$$

Example

Find $\int \textcolor{brown}{x}^2 e^x dx$.

We calculate that

$$\int \textcolor{brown}{x}^2 e^x dx = \textcolor{brown}{x}^2 e^x - 2 \int \textcolor{brown}{x} e^x dx.$$

But what do we do with $\int \textcolor{brown}{x} e^x dx$?

$$\int \textcolor{red}{uv}' dx = \textcolor{red}{uv} - \int \textcolor{red}{u}'v dx$$

Example

Find $\int \textcolor{brown}{x}^2 e^x dx$.

We calculate that

$$\int \textcolor{brown}{x}^2 e^x dx = \textcolor{brown}{x}^2 e^x - 2 \int \textcolor{brown}{x} e^x dx.$$

But what do we do with $\int \textcolor{brown}{x} e^x dx$?

$$\int \textcolor{brown}{x} e^x dx = \textcolor{brown}{x} e^x - \int \textcolor{brown}{1} e^x dx = x e^x - e^x + C_1.$$

8.2

$$\int uv' dx = uv - \int u'v dx$$



Example

Find $\int x^2 e^x dx$.

We calculate that

$$\int x^2 e^x dx = x^2 e^x - 2 \int x e^x dx.$$

But what do we do with $\int x e^x dx$?

$$\int x e^x dx = x e^x - \int 1 e^x dx = x e^x - e^x + C_1.$$

Putting it all together, we have

$$\int x^2 e^x dx = x^2 e^x - 2 \int x e^x dx = x^2 e^x - 2x e^x + 2e^x + C.$$

$$\int \textcolor{red}{uv}' dx = \textcolor{red}{uv} - \int \textcolor{red}{u}'\textcolor{green}{v} dx$$

Remark

We can use the same technique to calculate $\int x^{\textcolor{red}{n}} e^x dx$.

We would have to do integration by parts $\textcolor{red}{n}$ times.

$$\int \textcolor{red}{u} \textcolor{green}{v}' dx = \textcolor{red}{u} \textcolor{green}{v} - \int \textcolor{red}{u}' \textcolor{green}{v} dx$$

Theorem

$$\int u dv = \textcolor{red}{u} \textcolor{green}{v} - \int v du$$

EXAMPLE 4 Evaluate

$$\int e^x \cos x \, dx.$$

Solution Let $u = e^x$ and $dv = \cos x \, dx$. Then $du = e^x \, dx$, $v = \sin x$, and

$$\int e^x \cos x \, dx = e^x \sin x - \int e^x \sin x \, dx.$$

The second integral is like the first except that it has $\sin x$ in place of $\cos x$. To evaluate it, we use integration by parts with

$$u = e^x, \quad dv = \sin x \, dx, \quad v = -\cos x, \quad du = e^x \, dx.$$

Then

$$\begin{aligned}\int e^x \cos x \, dx &= e^x \sin x - \left(-e^x \cos x - \int (-\cos x)(e^x \, dx) \right) \\ &= e^x \sin x + e^x \cos x - \int e^x \cos x \, dx.\end{aligned}$$

The unknown integral now appears on both sides of the equation. Adding the integral to both sides and adding the constant of integration give

$$2 \int e^x \cos x \, dx = e^x \sin x + e^x \cos x + C_1.$$

Dividing by 2 and renaming the constant of integration give

$$\int e^x \cos x \, dx = \frac{e^x \sin x + e^x \cos x}{2} + C.$$



EXAMPLE 5 Obtain a formula that expresses the integral

$$\int \cos^n x \, dx$$

in terms of an integral of a lower power of $\cos x$.

Solution We may think of $\cos^n x$ as $\cos^{n-1} x \cdot \cos x$. Then we let

$$u = \cos^{n-1} x \quad \text{and} \quad dv = \cos x \, dx,$$

so that

$$du = (n - 1) \cos^{n-2} x (-\sin x \, dx) \quad \text{and} \quad v = \sin x.$$

Integration by parts then gives

$$\begin{aligned}\int \cos^n x \, dx &= \cos^{n-1} x \sin x + (n - 1) \int \sin^2 x \cos^{n-2} x \, dx \\&= \cos^{n-1} x \sin x + (n - 1) \int (1 - \cos^2 x) \cos^{n-2} x \, dx \\&= \cos^{n-1} x \sin x + (n - 1) \int \cos^{n-2} x \, dx - (n - 1) \int \cos^n x \, dx.\end{aligned}$$

If we add

$$(n - 1) \int \cos^n x dx$$

to both sides of this equation, we obtain

$$n \int \cos^n x dx = \cos^{n-1} x \sin x + (n - 1) \int \cos^{n-2} x dx.$$

We then divide through by n , and the final result is

$$\int \cos^n x dx = \frac{\cos^{n-1} x \sin x}{n} + \frac{n-1}{n} \int \cos^{n-2} x dx.$$

The formula found in Example 5 is called a **reduction formula** because it replaces an integral containing some power of a function with an integral of the same form having the power reduced. When n is a positive integer, we may apply the formula repeatedly until the remaining integral is easy to evaluate. For example, the result in Example 5 tells us that

$$\begin{aligned}\int \cos^3 x dx &= \frac{\cos^2 x \sin x}{3} + \frac{2}{3} \int \cos x dx \\ &= \frac{1}{3} \cos^2 x \sin x + \frac{2}{3} \sin x + C.\end{aligned}$$



$$\int \textcolor{red}{u} \textcolor{green}{v}' dx = \textcolor{red}{u} \textcolor{green}{v} - \int \textcolor{red}{u}' \textcolor{green}{v} dx$$

Theorem

$$\int_{\textcolor{red}{a}}^{\textcolor{brown}{b}} \textcolor{red}{u} \textcolor{green}{v}' dx =$$

$$\int uv' dx = uv - \int u'v dx$$

Theorem

$$\int_a^b uv' dx = [uv]_a^b - \int_a^b u'v dx$$

8.2

$$\int_a^b uv' dx = [uv]_a^b - \int_a^b u'v dx$$



Example

Calculate the area of the region bounded by the curve $y = xe^{-x}$ and the x -axis from $x = 0$ to $x = 4$.

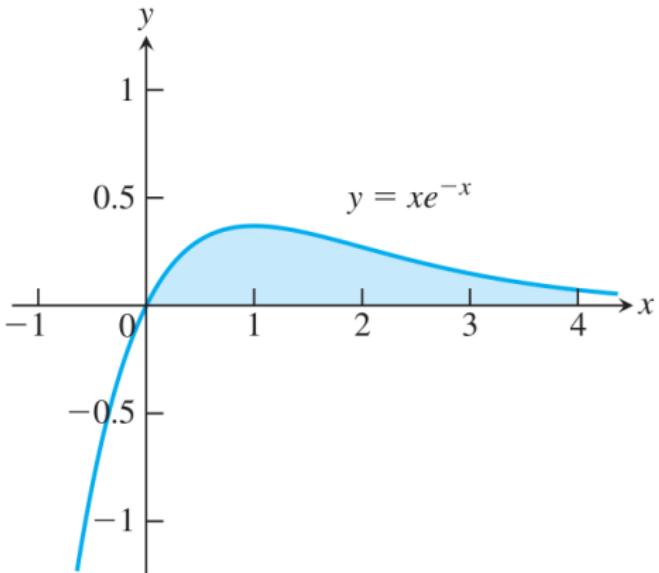
8.2

$$\int_a^b uv' dx = [uv]_a^b - \int_a^b u'v dx$$



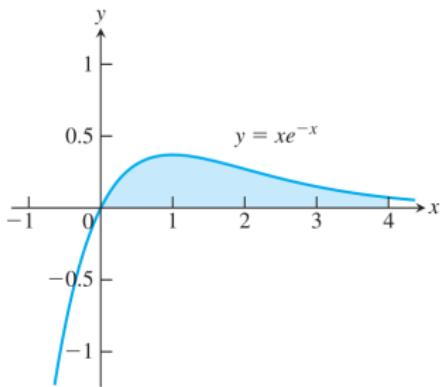
Example

Calculate the area of the region bounded by the curve $y = xe^{-x}$ and the x -axis from $x = 0$ to $x = 4$.



8.2

$$\int_a^b uv' dx = [uv]_a^b - \int_a^b u'v dx$$



We calculate that

$$\int_0^4 xe^{-x} dx =$$

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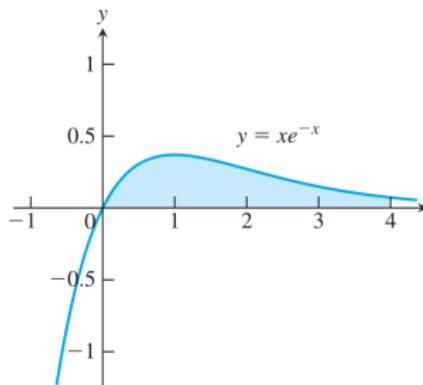
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8.2



$$\int_a^b uv' dx = [uv]_a^b - \int_a^b u'v dx$$



$$u = x$$

$$u' = 1$$

$$v' = e^{-x}$$

$$v = -e^{-x}$$

We calculate that

$$\int_0^4 xe^{-x} dx =$$

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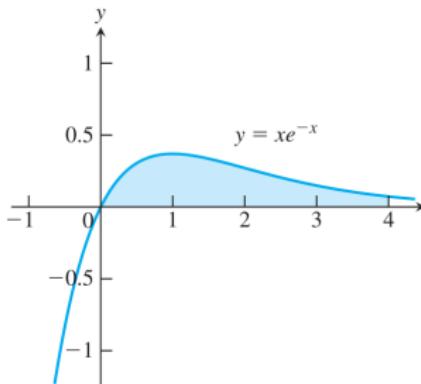
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8.2



$$\int_a^b uv' dx = [uv]_a^b - \int_a^b u'v dx$$



$$u = x$$

$$u' = 1$$

$$v' = e^{-x}$$

$$v = -e^{-x}$$

We calculate that

$$\int_0^4 xe^{-x} dx = [-xe^{-x}]_0^4 - \int_0^4 1(-e^{-x}) dx$$

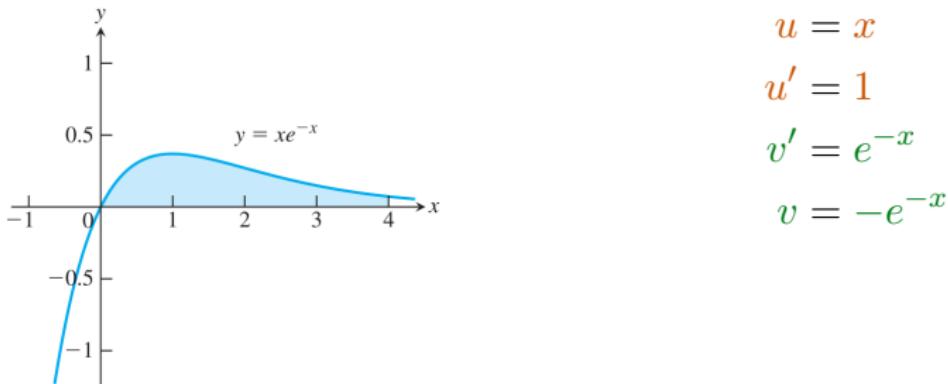
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8.2

$$\int_a^b uv' dx = [uv]_a^b - \int_a^b u'v dx$$



$$u = x$$

$$u' = 1$$

$$v' = e^{-x}$$

$$v = -e^{-x}$$

We calculate that

$$\begin{aligned}\int_0^4 xe^{-x} dx &= \left[-xe^{-x} \right]_0^4 - \int_0^4 1(-e^{-x}) dx \\ &= (-4e^{-4} + 0) + [-e^{-x}]_0^4 \\ &= -4e^{-4} + (-e^{-4} + 1) = 1 - 5e^{-4}.\end{aligned}$$

8.2

$$\int_a^b uv' dx = [uv]_a^b - \int_a^b u'v dx$$



Example

Find $\int_0^1 \sin^{-1} x dx$.

8.2

$$\int_a^b uv' dx = [uv]_a^b - \int_a^b u'v dx$$



Example

Find $\int_0^1 \sin^{-1} x dx$.

Recall that $\frac{d}{dx} \sin^{-1} x = \frac{1}{\sqrt{1-x^2}}$.

8.2

$$\int_a^b uv' dx = [uv]_a^b - \int_a^b u'v dx$$



Example

Find $\int_0^1 \sin^{-1} x dx$.

Recall that $\frac{d}{dx} \sin^{-1} x = \frac{1}{\sqrt{1-x^2}}$.

Let $u = \sin^{-1} x$ and $v' = 1$.

8.2

$$\int_a^b uv' dx = [uv]_a^b - \int_a^b u'v dx$$



Example

Find $\int_0^1 \sin^{-1} x dx$.

Recall that $\frac{d}{dx} \sin^{-1} x = \frac{1}{\sqrt{1-x^2}}$.

Let $u = \sin^{-1} x$ and $v' = 1$. Then $u' = \frac{1}{\sqrt{1-x^2}}$ and $v = x$.

8.2

$$\int_a^b uv' dx = [uv]_a^b - \int_a^b u'v dx$$



Example

Find $\int_0^1 \sin^{-1} x dx$.

Recall that $\frac{d}{dx} \sin^{-1} x = \frac{1}{\sqrt{1-x^2}}$.

Let $u = \sin^{-1} x$ and $v' = 1$. Then $u' = \frac{1}{\sqrt{1-x^2}}$ and $v = x$. It follows that

$$\int_0^1 \sin^{-1} x \cdot 1 dx = [\cancel{x} \sin^{-1} x]_0^1 - \int_0^1 \frac{\cancel{x}}{\sqrt{1-x^2}} dx$$

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=

=

.

8.2

$$\int_a^b uv' dx = [uv]_a^b - \int_a^b u'v dx$$



Example

Find $\int_0^1 \sin^{-1} x dx$.

Recall that $\frac{d}{dx} \sin^{-1} x = \frac{1}{\sqrt{1-x^2}}$.

Let $u = \sin^{-1} x$ and $v' = 1$. Then $u' = \frac{1}{\sqrt{1-x^2}}$ and $v = x$. It follows that

$$\begin{aligned}\int_0^1 \sin^{-1} x \cdot 1 dx &= [\cancel{x} \sin^{-1} x]_0^1 - \int_0^1 \frac{\cancel{x}}{\sqrt{1-\cancel{x}^2}} dx \\&= [\cancel{x} \sin^{-1} x]_0^1 - \left[-\sqrt{1-\cancel{x}^2} \right]_0^1 \\&= \left(\frac{\pi}{2} - 0 \right) - (-0 + 1) \\&= \frac{\pi}{2} - 1.\end{aligned}$$

Break

We will continue at 2pm





Trigonometric Integrals

8.3 Trigonometric Integrals



$$\cos \theta = \sin \left(\frac{\pi}{2} - \theta \right)$$

$$\cos^2 \theta + \sin^2 \theta = 1$$

$$1 + \tan^2 \theta = \sec^2 \theta$$

$$1 + \cot^2 \theta = \operatorname{cosec}^2 \theta$$

$$\cos(A + B) = \cos A \cos B - \sin A \sin B$$

$$\sin(A + B) = \sin A \cos B + \cos A \sin B$$

$$\cos 2\theta = \cos^2 \theta - \sin^2 \theta$$

$$\sin 2\theta = 2 \sin \theta \cos \theta$$

$$\cos^2 \theta = \frac{1}{2}(1 + \cos 2\theta)$$

$$\sin^2 \theta = \frac{1}{2}(1 - \cos 2\theta)$$

8.3 Trigonometric Integrals

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8.3 Trigonometric Integrals



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8.3 Trigonometric Integrals

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8.3 Trigonometric Integrals



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8.3 Trigonometric Integrals

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8.3 Trigonometric Integrals



How can we find

$$\int \sin^m x \cos^n x dx$$

if $m, n \in \{0, 1, 2, 3, 4, 5, \dots\}$?

8.3 Trigonometric Integrals



$$\int \sin^m x \cos^n x dx$$

We need to look at 3 different cases:

8.3 Trigonometric Integrals



$$\int \sin^m x \cos^n x dx$$

We need to look at 3 different cases:

- 1 **m is odd:**

8.3 Trigonometric Integrals



$$\int \sin^m x \cos^n x dx$$

We need to look at 3 different cases:

1 m is odd:

2 m is even and n is odd:

8.3 Trigonometric Integrals



$$\int \sin^m x \cos^n x dx$$

We need to look at 3 different cases:

1 m is odd:

2 m is even and n is odd:

3 both m and n are even:

8.3 Trigonometric Integrals



$$\int \sin^m x \cos^n x dx \quad \cos^2 x + \sin^2 x = 1$$

We need to look at 3 different cases:

- 1 **m is odd:** Write $m = 2k + 1$ and use

$$\sin^m x = \sin^{2k+1} x = (\sin^2 x)^k \sin x = (1 - \cos^2 x)^k \sin x$$

and the substitution $u = \cos x$.

- 2 **m is even and n is odd:**

- 3 **both m and n are even:**

8.3 Trigonometric Integrals

$$\int \sin^m x \cos^n x dx \quad \cos^2 x + \sin^2 x = 1$$

We need to look at 3 different cases:

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and the substitution $u = \sin x$.

- 3** **both m and n are even:**

8.3 Trigonometric Integrals

$$\int \sin^m x \cos^n x dx \quad \cos^2 x + \sin^2 x = 1$$

We need to look at 3 different cases:

- 1** **m is odd:** Write $m = 2k + 1$ and use

$$\sin^m x = \sin^{2k+1} x = (\sin^2 x)^k \sin x = (1 - \cos^2 x)^k \sin x$$

and the substitution $u = \cos x$.

- 2** **m is even and n is odd:** Write $n = 2k + 1$ use

$$\cos^n x = \cos^{2k+1} x = (\cos^2 x)^k \cos x = (1 - \sin^2 x)^k \cos x$$

and the substitution $u = \sin x$.

- 3** **both m and n are even:** Use

$$\sin^2 x = \frac{1 - \cos 2x}{2} \quad \text{and} \quad \cos^2 x = \frac{1 + \cos 2x}{2}.$$

8.3 Trigonometric Integrals



Example

Find $\int \sin^3 x \cos^2 x dx.$

8.3 Trigonometric Integrals

Example

Find $\int \sin^3 x \cos^2 x dx$.

Solution This is an example of Case 1.

$$\begin{aligned}\int \sin^3 x \cos^2 x dx &= \int \sin^2 x \cos^2 x \sin x dx && m \text{ is odd.} \\ &= \int (1 - \cos^2 x)(\cos^2 x)(-d(\cos x)) && \sin x dx = -d(\cos x) \\ &= \int (1 - u^2)(u^2)(-du) && u = \cos x \\ &= \int (u^4 - u^2) du && \text{Multiply terms.} \\ &= \frac{u^5}{5} - \frac{u^3}{3} + C = \frac{\cos^5 x}{5} - \frac{\cos^3 x}{3} + C\end{aligned}$$



8.3 Trigonometric Integrals



Example

Find $\int \cos^5 x dx.$

8.3 Trigonometric Integrals

Example

Find $\int \cos^5 x \, dx.$

Solution This is an example of Case 2, where $m = 0$ is even and $n = 5$ is odd.

$$\begin{aligned}
 \int \cos^5 x \, dx &= \int \cos^4 x \cos x \, dx = \int (1 - \sin^2 x)^2 d(\sin x) && \cos x \, dx = d(\sin x) \\
 &= \int (1 - u^2)^2 \, du && u = \sin x \\
 &= \int (1 - 2u^2 + u^4) \, du && \text{Square } 1 - u^2. \\
 &= u - \frac{2}{3}u^3 + \frac{1}{5}u^5 + C = \sin x - \frac{2}{3}\sin^3 x + \frac{1}{5}\sin^5 x + C
 \end{aligned}$$



8.3 Trigonometric Integrals



Example

Find $\int \sin^2 x \cos^4 x dx$.

8.3 Trigonometric Integrals

Example

Find $\int \sin^2 x \cos^4 x dx$.

Solution This is an example of Case 3.

$$\begin{aligned}
 \int \sin^2 x \cos^4 x dx &= \int \left(\frac{1 - \cos 2x}{2} \right) \left(\frac{1 + \cos 2x}{2} \right)^2 dx && m \text{ and } n \text{ both even} \\
 &= \frac{1}{8} \int (1 - \cos 2x)(1 + 2\cos 2x + \cos^2 2x) dx \\
 &= \frac{1}{8} \int (1 + \cos 2x - \cos^2 2x - \cos^3 2x) dx \\
 &= \frac{1}{8} \left[x + \frac{1}{2} \sin 2x - \int (\cos^2 2x + \cos^3 2x) dx \right]
 \end{aligned}$$

For the term involving $\cos^2 2x$, we use

$$\begin{aligned}\int \cos^2 2x \, dx &= \frac{1}{2} \int (1 + \cos 4x) \, dx \\ &= \frac{1}{2} \left(x + \frac{1}{4} \sin 4x \right).\end{aligned}$$

Omit constant of
integration until final result.

For the $\cos^3 2x$ term, we have

$$\begin{aligned}\int \cos^3 2x \, dx &= \int (1 - \sin^2 2x) \cos 2x \, dx && u = \sin 2x, \, du = 2 \cos 2x \, dx \\ &= \frac{1}{2} \int (1 - u^2) \, du = \frac{1}{2} \left(\sin 2x - \frac{1}{3} \sin^3 2x \right). && \text{Again omit } C.\end{aligned}$$

Combining everything and simplifying, we get

$$\int \sin^2 x \cos^4 x \, dx = \frac{1}{16} \left(x - \frac{1}{4} \sin 4x + \frac{1}{3} \sin^3 2x \right) + C.$$



8.3 Trigonometric Integrals

Example

Find $\int_0^{\frac{\pi}{4}} \sqrt{1 + \cos 4x} dx.$

8.3 Trigonometric Integrals

Example

Find $\int_0^{\frac{\pi}{4}} \sqrt{1 + \cos 4x} dx.$

Solution To eliminate the square root, we use the identity

$$\cos^2 \theta = \frac{1 + \cos 2\theta}{2} \quad \text{or} \quad 1 + \cos 2\theta = 2 \cos^2 \theta.$$

With $\theta = 2x$, this becomes

$$1 + \cos 4x = 2 \cos^2 2x.$$

Therefore,

$$\begin{aligned} \int_0^{\pi/4} \sqrt{1 + \cos 4x} dx &= \int_0^{\pi/4} \sqrt{2 \cos^2 2x} dx = \int_0^{\pi/4} \sqrt{2} \sqrt{\cos^2 2x} dx \\ &= \sqrt{2} \int_0^{\pi/4} |\cos 2x| dx = \sqrt{2} \int_0^{\pi/4} \cos 2x dx \quad \begin{matrix} \cos 2x \geq 0 \text{ on} \\ [0, \pi/4] \end{matrix} \\ &= \sqrt{2} \left[\frac{\sin 2x}{2} \right]_0^{\pi/4} = \frac{\sqrt{2}}{2} [1 - 0] = \frac{\sqrt{2}}{2}. \end{aligned}$$



8.3

$$\sec^2 x = 1 + \tan^2 x \quad \frac{d}{dx} \tan x = \sec^2 x$$



Example

Find $\int \tan^4 x dx$.

8.3

$$\sec^2 x = 1 + \tan^2 x \quad \frac{d}{dx} \tan x = \sec^2 x$$



Example

Find $\int \tan^4 x \, dx$.

Solution

$$\begin{aligned}\int \tan^4 x \, dx &= \int \tan^2 x \cdot \tan^2 x \, dx = \int \tan^2 x \cdot (\sec^2 x - 1) \, dx \\&= \int \tan^2 x \sec^2 x \, dx - \int \tan^2 x \, dx \\&= \int \tan^2 x \sec^2 x \, dx - \int (\sec^2 x - 1) \, dx \\&= \int \tan^2 x \sec^2 x \, dx - \int \sec^2 x \, dx + \int dx\end{aligned}$$

In the first integral, we let

$$u = \tan x, \quad du = \sec^2 x \, dx$$

and have

$$\int u^2 \, du = \frac{1}{3} u^3 + C_1.$$

The remaining integrals are standard forms, so

$$\int \tan^4 x \, dx = \frac{1}{3} \tan^3 x - \tan x + x + C.$$



8.3 Trigonometric Integrals



Example

Find $\int \sec^3 x \, dx$.

Solution We integrate by parts using

$$u = \sec x, \quad dv = \sec^2 x dx, \quad v = \tan x, \quad du = \sec x \tan x dx.$$

Then

$$\begin{aligned}\int \sec^3 x dx &= \sec x \tan x - \int (\tan x)(\sec x \tan x dx) \\&= \sec x \tan x - \int (\sec^2 x - 1) \sec x dx && \tan^2 x = \sec^2 x - 1 \\&= \sec x \tan x + \int \sec x dx - \int \sec^3 x dx.\end{aligned}$$

Combining the two secant-cubed integrals gives

$$2 \int \sec^3 x dx = \sec x \tan x + \int \sec x dx$$

and

$$\int \sec^3 x dx = \frac{1}{2} \sec x \tan x + \frac{1}{2} \ln |\sec x + \tan x| + C.$$



8.3

$$\sec^2 x = 1 + \tan^2 x \quad \frac{d}{dx} \tan x = \sec^2 x$$



Example

Find $\int \tan^4 x \sec^4 x dx.$

8.3

$$\sec^2 x = 1 + \tan^2 x \quad \frac{d}{dx} \tan x = \sec^2 x$$



Example

Find $\int \tan^4 x \sec^4 x \, dx.$

Solution

$$\begin{aligned}
 \int (\tan^4 x)(\sec^4 x) \, dx &= \int (\tan^4 x)(1 + \tan^2 x)(\sec^2 x) \, dx && \sec^2 x = 1 + \tan^2 x \\
 &= \int (\tan^4 x + \tan^6 x)(\sec^2 x) \, dx \\
 &= \int (\tan^4 x)(\sec^2 x) \, dx + \int (\tan^6 x)(\sec^2 x) \, dx \\
 &= \int u^4 \, du + \int u^6 \, du = \frac{u^5}{5} + \frac{u^7}{7} + C && u = \tan x, \\
 && du = \sec^2 x \, dx \\
 &= \frac{\tan^5 x}{5} + \frac{\tan^7 x}{7} + C
 \end{aligned}$$



8.3 Trigonometric Integrals

How do we calculate

$$\int \sin mx \sin nx \, dx$$

or

$$\int \sin mx \cos nx \, dx$$

or

$$\int \cos mx \cos nx \, dx$$

?

8.3 Trigonometric Integrals

How do we calculate

$$\int \sin mx \sin nx \, dx$$

or

$$\int \sin mx \cos nx \, dx$$

or

$$\int \cos mx \cos nx \, dx$$

?

It is possible to use integration by parts (twice), but there is an easier way.

8.3

$$\cos(A + B) = \cos A \cos B - \sin A \sin B$$



$$\cos(mx - nx) - \cos(mx + nx) =$$

8.3

$$\cos(A + B) = \cos A \cos B - \sin A \sin B$$



$$\begin{aligned}\cos(mx - nx) - \cos(mx + nx) &= \cos mx \cos nx + \sin mx \sin nx \\ &\quad - \cos mx \cos nx + \sin mx \sin nx\end{aligned}$$

$$\cos(A + B) = \cos A \cos B - \sin A \sin B$$

$$\begin{aligned}\cos(mx - nx) - \cos(mx + nx) &= \cos mx \cos nx + \sin mx \sin nx \\ &\quad - \cos mx \cos nx + \sin mx \sin nx \\ &= 2 \sin mx \sin nx\end{aligned}$$

8.3

$$\cos(A + B) = \cos A \cos B - \sin A \sin B$$



$$\begin{aligned}\cos(mx - nx) - \cos(mx + nx) &= \cos mx \cos nx + \sin mx \sin nx \\ &\quad - \cos mx \cos nx + \sin mx \sin nx \\ &= 2 \sin mx \sin nx\end{aligned}$$

Therefore

$$\sin mx \sin nx = \frac{1}{2}(\cos(m - n)x - \cos(m + n)x).$$

$$\cos(A + B) = \cos A \cos B - \sin A \sin B$$

$$\begin{aligned}\cos(mx - nx) - \cos(mx + nx) &= \cos mx \cos nx + \sin mx \sin nx \\ &\quad - \cos mx \cos nx + \sin mx \sin nx \\ &= 2 \sin mx \sin nx\end{aligned}$$

Therefore

$$\sin mx \sin nx = \frac{1}{2}(\cos(m-n)x - \cos(m+n)x).$$

Similarly

$$\sin mx \cos nx = \frac{1}{2}(\sin(m-n)x + \sin(m+n)x)$$

and

$$\cos mx \cos nx = \frac{1}{2}(\cos(m-n)x + \cos(m+n)x).$$

$$\sin mx \cos nx = \frac{1}{2}(\sin(m-n)x + \sin(m+n)x)$$

Example

Find $\int \sin 3x \cos 5x \, dx$.

$$\sin mx \cos nx = \frac{1}{2} (\sin(m-n)x + \sin(m+n)x)$$

Example

Find $\int \sin 3x \cos 5x \, dx$.

Solution From Equation (4) with $m = 3$ and $n = 5$, we get

$$\begin{aligned}\int \sin 3x \cos 5x \, dx &= \frac{1}{2} \int [\sin(-2x) + \sin 8x] \, dx \\ &= \frac{1}{2} \int (\sin 8x - \sin 2x) \, dx \\ &= -\frac{\cos 8x}{16} + \frac{\cos 2x}{4} + C.\end{aligned}$$



$$\cos mx \cos nx = \frac{1}{2}(\cos(m-n)x + \cos(m+n)x)$$

Example

Find $\int \cos 3x \cos 2x dx$.

$$\cos mx \cos nx = \frac{1}{2}(\cos(m-n)x + \cos(m+n)x)$$

Example

Find $\int \cos 3x \cos 2x dx$.

We have $m = 3$ and $n = 2$. It follows that

$$\begin{aligned}\int \cos 3x \cos 2x dx &= \frac{1}{2} \int \cos(3-2)x dx + \frac{1}{2} \int \cos(3+2)x dx \\ &= \dots\end{aligned}$$



Next Time

- 8.4 Trigonometric Substitutions
- 8.5 Integration of Rational Functions by Partial Fractions
- 8.8 Improper Integrals