

# Lecture 4

- 11. Vectors
- 12. The Dot Product
- 13. The Cross Product

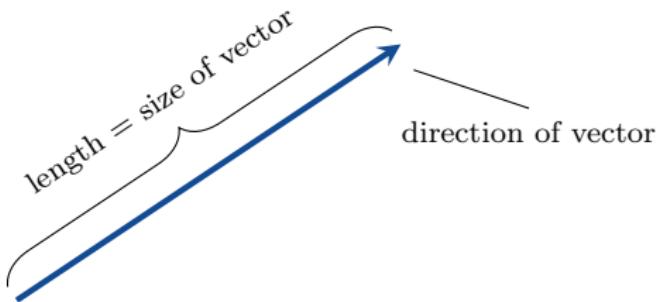


# Vectors

## 11. Vectors

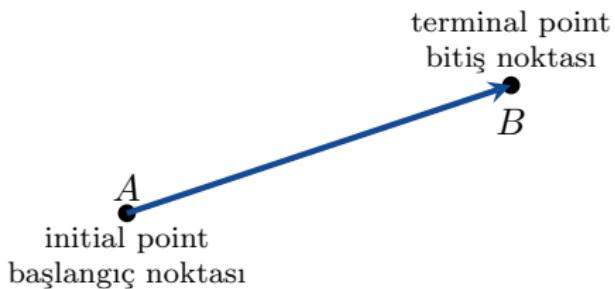


For some quantities (mass, time, distance, ...) we only need a number. For some quantities (velocity, force, ...) we need a number and a direction.



A *vector* is an object which has a size (length) and a direction.

## 11. Vectors

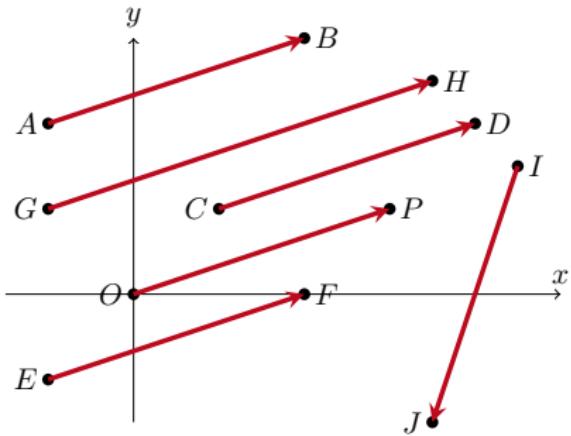


### Definition

The vector  $\overrightarrow{AB}$  has *initial point*  $A$  and *terminal point*  $B$ .

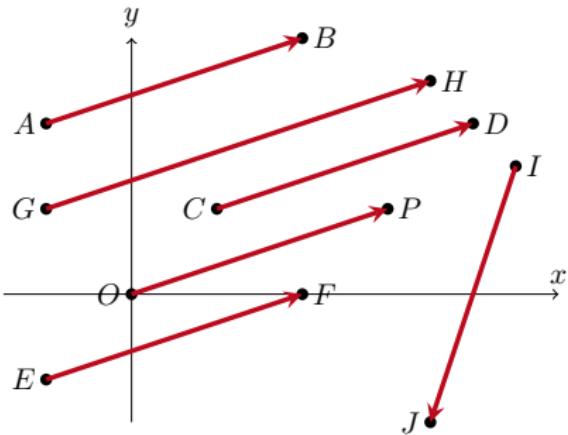
The *length* of  $\overrightarrow{AB}$  is written  $\|\overrightarrow{AB}\|$  (or  $|\overrightarrow{AB}|$ ).

# 11. Vectors



Two vectors are equal if they have the same length and the same direction.

# 11. Vectors

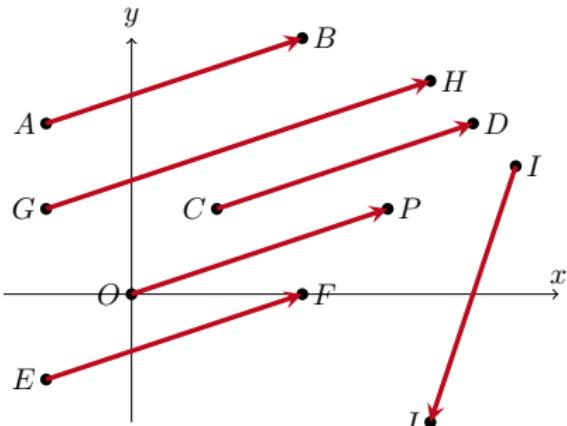


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$$\overrightarrow{AB} = \overrightarrow{CD} = \overrightarrow{EF} = \overrightarrow{OP}.$$

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We can say that

$$\overrightarrow{AB} = \overrightarrow{CD} = \overrightarrow{EF} = \overrightarrow{OP}.$$

Note that  $\overrightarrow{AB} \neq \overrightarrow{GH}$  because the lengths are different, and  $\overrightarrow{AB} \neq \overrightarrow{IJ}$  because the directions are different.



## Notation

When we use a computer, we use bold letters for vectors: **u**, **v**, **w**, ...



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When we use a computer, we use bold letters for vectors:  $\mathbf{u}$ ,  $\mathbf{v}$ ,  $\mathbf{w}$ , . . . When we use a pen, we use underlined letters for vectors:  $\underline{u}$ ,  $\underline{v}$ ,  $\underline{w}$ , . . .

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If we type  $a\mathbf{u} + b\mathbf{v}$  or write  $a\underline{u} + b\underline{v}$ , then

- $a$  and  $b$  are numbers; and
- $\mathbf{u}$ ,  $\mathbf{v}$ ,  $\underline{u}$  and  $\underline{v}$  are vectors.

# 11. Vectors



## Definition

In  $\mathbb{R}^2$ : If  $\mathbf{v}$  has initial point  $(0, 0)$  and terminal point  $(v_1, v_2)$ , then the *component form* of  $\mathbf{v}$  is  $\mathbf{v} = (v_1, v_2)$ .

# 11. Vectors

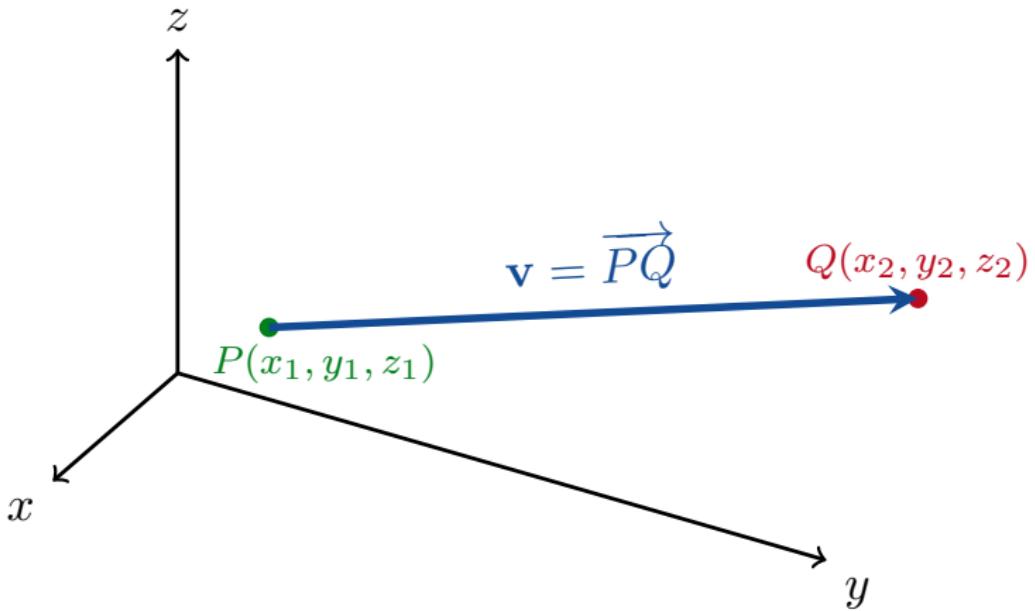


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In  $\mathbb{R}^3$ : If  $\mathbf{v}$  has initial point  $(0, 0, 0)$  and terminal point  $(v_1, v_2, v_3)$ , then the *component form* of  $\mathbf{v}$  is  $\mathbf{v} = (v_1, v_2, v_3)$ .

## 11. Vectors



$$(v_1, v_2, v_3) = \mathbf{v} = (x_2 - x_1, y_2 - y_1, z_2 - z_1)$$

# 11. Vectors



## Definition

In  $\mathbb{R}^2$ : The *norm* (or *length*) of  $\mathbf{v} = (v_1, v_2)$  is

$$\|\mathbf{v}\| = \sqrt{v_1^2 + v_2^2}$$

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The vectors  $\mathbf{0} = (0, 0)$  and  $\mathbf{0} = (0, 0, 0)$  have norm  $\|\mathbf{0}\| = 0$ . If  $\mathbf{v} \neq \mathbf{0}$ , then  $\|\mathbf{v}\| > 0$ .

## 11. Vectors



### Example

Find (1) the component form; and (2) the norm of the vector with initial point  $P(-3, 4, 1)$  and terminal point  $Q(-5, 2, 2)$ .

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 $= (-2, -2, 1).$

## 11. Vectors

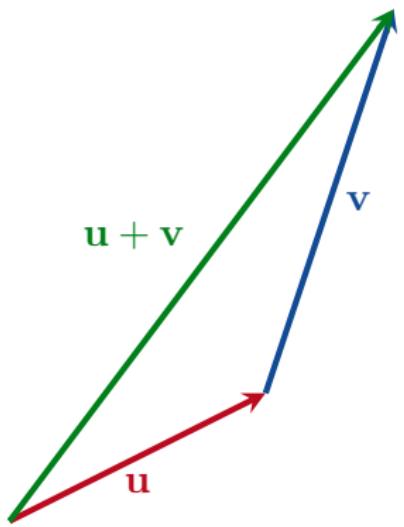
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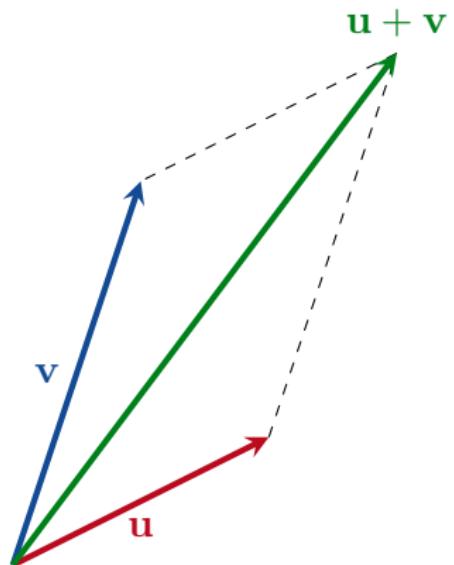
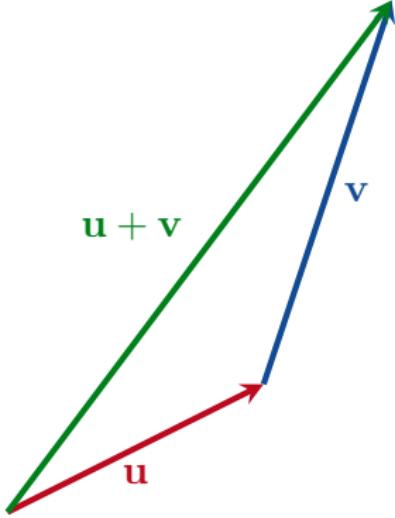
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2  $\|\mathbf{v}\| = \sqrt{v_1^2 + v_2^2 + v_3^2} = \sqrt{(-2)^2 + (-2)^2 + 1^2} = \sqrt{9} = 3.$

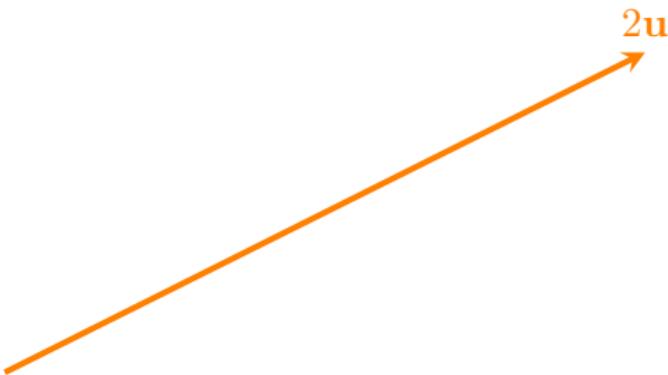
## Vector Algebra: Addition



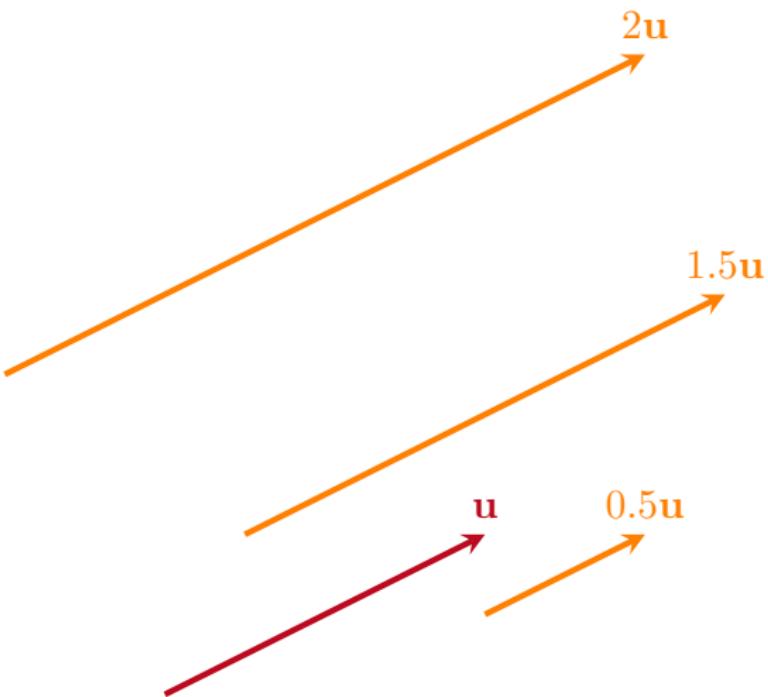
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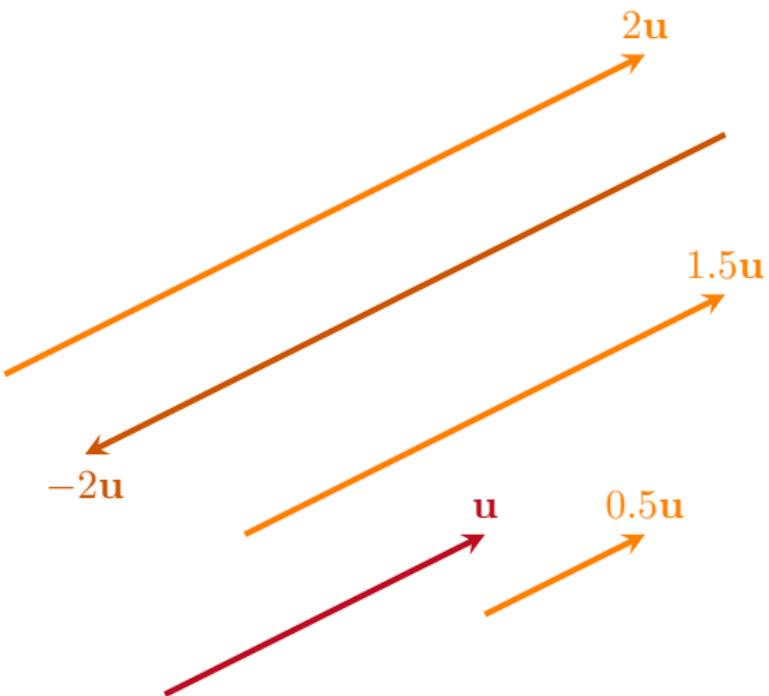
# Vector Algebra: Multiplication by a Constant



## Vector Algebra: Multiplication by a Constant

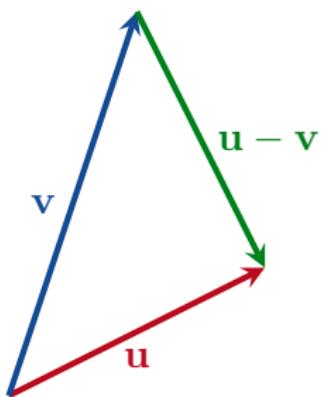


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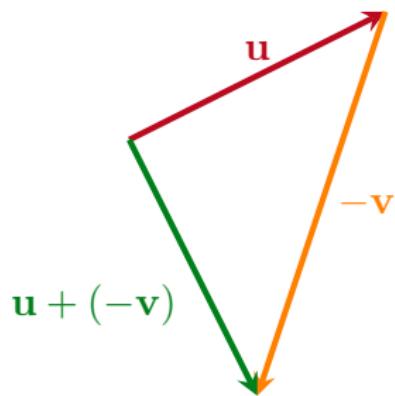
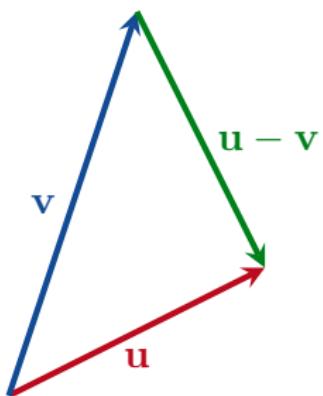
## Vector Algebra: Subtraction

$$\mathbf{v} + (\mathbf{u} - \mathbf{v}) = \mathbf{u}$$



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## 11. Vectors



Let  $\mathbf{u} = (u_1, u_2, u_3)$  and  $\mathbf{v} = (v_1, v_2, v_3)$  be vectors. Let  $k$  be a number.

## 11. Vectors



Let  $\mathbf{u} = (u_1, u_2, u_3)$  and  $\mathbf{v} = (v_1, v_2, v_3)$  be vectors. Let  $k$  be a number. Then

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and

$$k\mathbf{u} = (ku_1, ku_2, ku_3).$$

## 11. Vectors



Note that

$$\|k\mathbf{u}\| = \|(ku_1, ku_2, ku_3)\|$$

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## 11. Vectors



Note that

$$\begin{aligned}\|k\mathbf{u}\| &= \|(ku_1, ku_2, ku_3)\| \\ &= \sqrt{(ku_1)^2 + (ku_2)^2 + (ku_3)^2}\end{aligned}$$

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## 11. Vectors



The vector  $-\mathbf{u} = (-1)\mathbf{u}$  has the same length as  $\mathbf{u}$ , but points in the opposite direction.

# 11. Vectors



## Example

Let  $\mathbf{u} = (-1, 3, 1)$  and  $\mathbf{v} = (4, 7, 0)$ .

Find  $2\mathbf{u} + 3\mathbf{v}$ ,  $\mathbf{u} - \mathbf{v}$ , and  $\left\| \frac{1}{2}\mathbf{u} \right\|$ .

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# Properties of Vector Operations

Let  $\mathbf{u}$ ,  $\mathbf{v}$  and  $\mathbf{w}$  be vectors. Let  $a$  and  $b$  be numbers. Then

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- 7  $a(b\mathbf{u}) = (ab)\mathbf{u};$
- 8  $a(\mathbf{u} + \mathbf{v}) = a\mathbf{u} + a\mathbf{v};$
- 9  $(a + b)\mathbf{u} = a\mathbf{u} + b\mathbf{u}.$

## 11. Vectors



### Remark

We **can not** multiply vectors. Never never never never write "**uv**".

## Unit Vectors

### Definition

$\mathbf{u}$  is called a *unit vector*  $\iff \|\mathbf{u}\| = 1$ .

## 11. Vectors



Example

$\mathbf{u} = (2^{-\frac{1}{2}}, \frac{1}{2}, -\frac{1}{2})$  is a unit vector because

$$\|\mathbf{u}\| = \sqrt{\left(\frac{1}{\sqrt{2}}\right)^2 + \left(\frac{1}{2}\right)^2 + \left(-\frac{1}{2}\right)^2} = \sqrt{\frac{1}{2} + \frac{1}{4} + \frac{1}{4}} = 1.$$

# Standard Unit Vectors

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## Standard Unit Vectors

In  $\mathbb{R}^2$ : The *standard unit vectors* are  $\mathbf{i} = (1, 0)$  and  $\mathbf{j} = (0, 1)$ .

In  $\mathbb{R}^3$ : The *standard unit vectors* are  $\mathbf{i} = (1, 0, 0)$ ,  $\mathbf{j} = (0, 1, 0)$  and  $\mathbf{k} = (0, 0, 1)$ . Any vector  $\mathbf{v} \in \mathbb{R}^3$  can be written

$$\begin{aligned}\mathbf{v} &= (v_1, v_2, v_3) = (v_1, 0, 0) + (0, v_2, 0) + (0, 0, v_3) \\ &= v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}.\end{aligned}$$

## Normalising a Vector

If  $\|\mathbf{v}\| \neq 0$ , then  $\frac{\mathbf{v}}{\|\mathbf{v}\|}$  is a unit vector because

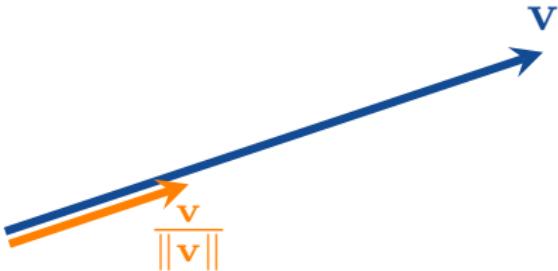
$$\left\| \frac{\mathbf{v}}{\|\mathbf{v}\|} \right\| = \left\| \frac{1}{\|\mathbf{v}\|} \mathbf{v} \right\| = \frac{1}{\|\mathbf{v}\|} \|\mathbf{v}\| = 1.$$

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$$\left\| \frac{\mathbf{v}}{\|\mathbf{v}\|} \right\| = \left\| \frac{1}{\|\mathbf{v}\|} \mathbf{v} \right\| = \frac{1}{\|\mathbf{v}\|} \|\mathbf{v}\| = 1.$$

Clearly  $\frac{\mathbf{v}}{\|\mathbf{v}\|}$  and  $\mathbf{v}$  point in the same direction.



## 11. Vectors

### Example

Find a unit vector  $\mathbf{u}$  which points in the same direction as  $\overrightarrow{P_1P_2}$ , where  $P_1(1, 0, 1)$  and  $P_2(3, 2, 0)$ .

## 11. Vectors

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Find a unit vector  $\mathbf{u}$  which points in the same direction as  $\overrightarrow{P_1P_2}$ , where  $P_1(1, 0, 1)$  and  $P_2(3, 2, 0)$ .

We calculate that

$$\overrightarrow{P_1P_2} = P_2 - P_1 = (3, 2, 0) - (1, 0, 1) = (2, 2, -1) = 2\mathbf{i} + 2\mathbf{j} - \mathbf{k}$$

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### Example

Find a unit vector  $\mathbf{u}$  which points in the same direction as  $\overrightarrow{P_1P_2}$ , where  $P_1(1, 0, 1)$  and  $P_2(3, 2, 0)$ .

We calculate that

$$\overrightarrow{P_1P_2} = P_2 - P_1 = (3, 2, 0) - (1, 0, 1) = (2, 2, -1) = 2\mathbf{i} + 2\mathbf{j} - \mathbf{k}$$

and that

$$\left\| \overrightarrow{P_1P_2} \right\| = \sqrt{2^2 + 2^2 + (-1)^2} = 3.$$

## 11. Vectors

### Example

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We calculate that

$$\overrightarrow{P_1P_2} = P_2 - P_1 = (3, 2, 0) - (1, 0, 1) = (2, 2, -1) = 2\mathbf{i} + 2\mathbf{j} - \mathbf{k}$$

and that

$$\left\| \overrightarrow{P_1P_2} \right\| = \sqrt{2^2 + 2^2 + (-1)^2} = 3.$$

The required unit vector is

$$\mathbf{u} = \frac{\overrightarrow{P_1P_2}}{\left\| \overrightarrow{P_1P_2} \right\|} = \frac{2\mathbf{i} + 2\mathbf{j} - \mathbf{k}}{3} = \frac{2}{3}\mathbf{i} + \frac{2}{3}\mathbf{j} - \frac{1}{3}\mathbf{k}.$$

**EXAMPLE 5** If  $\mathbf{v} = 3\mathbf{i} - 4\mathbf{j}$  is a velocity vector, express  $\mathbf{v}$  as a product of its speed times its direction of motion.

**Solution** Speed is the magnitude (length) of  $\mathbf{v}$ :

$$|\mathbf{v}| = \sqrt{(3)^2 + (-4)^2} = \sqrt{9 + 16} = 5.$$

The unit vector  $\mathbf{v}/|\mathbf{v}|$  is the direction of  $\mathbf{v}$ :

$$\frac{\mathbf{v}}{|\mathbf{v}|} = \frac{3\mathbf{i} - 4\mathbf{j}}{5} = \frac{3}{5}\mathbf{i} - \frac{4}{5}\mathbf{j}.$$

So

$$\mathbf{v} = 3\mathbf{i} - 4\mathbf{j} = 5 \left( \underbrace{\frac{3}{5}\mathbf{i} - \frac{4}{5}\mathbf{j}}_{\substack{\text{Length} \\ (\text{speed})}} \right).$$

■

If  $\mathbf{v} \neq \mathbf{0}$ , then

1.  $\frac{\mathbf{v}}{|\mathbf{v}|}$  is a unit vector called the direction of  $\mathbf{v}$ ;
2. the equation  $\mathbf{v} = |\mathbf{v}| \frac{\mathbf{v}}{|\mathbf{v}|}$  expresses  $\mathbf{v}$  as its length times its direction.

**EXAMPLE 6** A force of 6 newtons is applied in the direction of the vector  $\mathbf{v} = 2\mathbf{i} + 2\mathbf{j} - \mathbf{k}$ . Express the force  $\mathbf{F}$  as a product of its magnitude and direction.

**Solution** The force vector has magnitude 6 and direction  $\frac{\mathbf{v}}{|\mathbf{v}|}$ , so

$$\begin{aligned}\mathbf{F} &= 6 \frac{\mathbf{v}}{|\mathbf{v}|} = 6 \frac{2\mathbf{i} + 2\mathbf{j} - \mathbf{k}}{\sqrt{2^2 + 2^2 + (-1)^2}} = 6 \frac{2\mathbf{i} + 2\mathbf{j} - \mathbf{k}}{3} \\ &= 6 \left( \frac{2}{3} \mathbf{i} + \frac{2}{3} \mathbf{j} - \frac{1}{3} \mathbf{k} \right).\end{aligned}$$



## Midpoint of a Line Segment

Vectors are often useful in geometry. For example, the coordinates of the midpoint of a line segment are found by averaging.

The **midpoint**  $M$  of the line segment joining points  $P_1(x_1, y_1, z_1)$  and  $P_2(x_2, y_2, z_2)$  is the point

$$\left( \frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}, \frac{z_1 + z_2}{2} \right).$$

To see why, observe (Figure 12.16) that

$$\begin{aligned}\overrightarrow{OM} &= \overrightarrow{OP}_1 + \frac{1}{2}(\overrightarrow{P_1P_2}) = \overrightarrow{OP}_1 + \frac{1}{2}(\overrightarrow{OP}_2 - \overrightarrow{OP}_1) \\ &= \frac{1}{2}(\overrightarrow{OP}_1 + \overrightarrow{OP}_2) \\ &= \frac{x_1 + x_2}{2}\mathbf{i} + \frac{y_1 + y_2}{2}\mathbf{j} + \frac{z_1 + z_2}{2}\mathbf{k}.\end{aligned}$$

**EXAMPLE 7** The midpoint of the segment joining  $P_1(3, -2, 0)$  and  $P_2(7, 4, 4)$  is

$$\left( \frac{3+7}{2}, \frac{-2+4}{2}, \frac{0+4}{2} \right) = (5, 1, 2). \quad \blacksquare$$

## 11. Vectors



Please read the final two examples in this section of the textbook.



# The Dot Product

## 12. The Dot Product

### Definition

In  $\mathbb{R}^2$ , the *dot product* of  $\mathbf{u} = (u_1, u_2) = u_1\mathbf{i} + u_2\mathbf{j}$  and  $\mathbf{v} = (v_1, v_2) = v_1\mathbf{i} + v_2\mathbf{j}$  is

$$\mathbf{u} \cdot \mathbf{v} = u_1v_1 + u_2v_2.$$

## 12. The Dot Product

### Definition

In  $\mathbb{R}^2$ , the *dot product* of  $\mathbf{u} = (u_1, u_2) = u_1\mathbf{i} + u_2\mathbf{j}$  and  $\mathbf{v} = (v_1, v_2) = v_1\mathbf{i} + v_2\mathbf{j}$  is

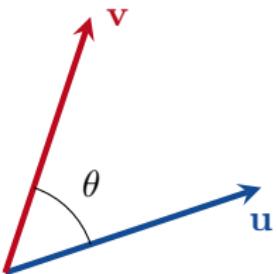
$$\mathbf{u} \cdot \mathbf{v} = u_1v_1 + u_2v_2.$$

### Definition

In  $\mathbb{R}^3$ , the *dot product* of  $\mathbf{u} = (u_1, u_2, u_3) = u_1\mathbf{i} + u_2\mathbf{j} + u_3\mathbf{k}$  and  $\mathbf{v} = (v_1, v_2, v_3) = v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}$  is

$$\mathbf{u} \cdot \mathbf{v} = u_1v_1 + u_2v_2 + u_3v_3.$$

## 12. The Dot Product

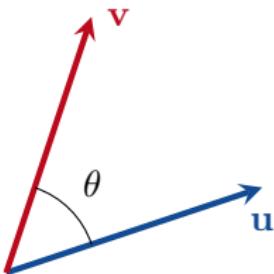


### Theorem

*The angle between  $\mathbf{u}$  and  $\mathbf{v}$  is*

$$\theta = \cos^{-1} \left( \frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{u}\| \|\mathbf{v}\|} \right).$$

## 12. The Dot Product



### Theorem

*The angle between  $\mathbf{u}$  and  $\mathbf{v}$  is*

$$\theta = \cos^{-1} \left( \frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{u}\| \|\mathbf{v}\|} \right).$$

This means that

$$\mathbf{u} \cdot \mathbf{v} = \|\mathbf{u}\| \|\mathbf{v}\| \cos \theta.$$

## 12. The Dot Product

### Example

$$\begin{aligned}(1, -2, -1) \cdot (-6, 2, -3) &= (1 \times -6) + (-2 \times 2) + (-1 \times -3) \\&= -6 - 4 + 3 = -7.\end{aligned}$$

## 12. The Dot Product

### Example

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### Example

$$\begin{aligned}\left(\frac{1}{2}\mathbf{i} + 3\mathbf{j} + \mathbf{k}\right) \cdot (4\mathbf{i} - \mathbf{j} + 2\mathbf{k}) &= \left(\frac{1}{2} \times 4\right) + (3 \times -1) + (1 \times 2) \\&= 2 - 3 + 2 = 1.\end{aligned}$$

12.

$$\theta = \cos^{-1} \left( \frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{u}\| \|\mathbf{v}\|} \right).$$



### Example

Find the angle between  $\mathbf{u} = \mathbf{i} - 2\mathbf{j} - 2\mathbf{k}$  and  $\mathbf{v} = 6\mathbf{i} + 3\mathbf{j} + 2\mathbf{k}$ .

12.

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Find the angle between  $\mathbf{u} = \mathbf{i} - 2\mathbf{j} - 2\mathbf{k}$  and  $\mathbf{v} = 6\mathbf{i} + 3\mathbf{j} + 2\mathbf{k}$ .

Since

$$\begin{aligned}\mathbf{u} \cdot \mathbf{v} &= (1, -2, -2) \cdot (6, 3, 2) = (1 \times 6) + (-2 \times 3) + (-2 \times 2) \\ &= 6 - 6 - 4 = -4,\end{aligned}$$

12.

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$$\|\mathbf{u}\| = \sqrt{1^2 + (-2)^2 + (-2)^2} = \sqrt{9} = 3$$

and

$$\|\mathbf{v}\| = \sqrt{6^2 + 3^2 + 2^2} = \sqrt{49} = 7,$$

12.

$$\theta = \cos^{-1} \left( \frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{u}\| \|\mathbf{v}\|} \right).$$

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we have that

$$\theta = \cos^{-1} \left( \frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{u}\| \|\mathbf{v}\|} \right) =$$

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Find the angle between  $\mathbf{u} = \mathbf{i} - 2\mathbf{j} - 2\mathbf{k}$  and  $\mathbf{v} = 6\mathbf{i} + 3\mathbf{j} + 2\mathbf{k}$ .

Since

$$\begin{aligned}\mathbf{u} \cdot \mathbf{v} &= (1, -2, -2) \cdot (6, 3, 2) = (1 \times 6) + (-2 \times 3) + (-2 \times 2) \\ &= 6 - 6 - 4 = -4,\end{aligned}$$

$$\|\mathbf{u}\| = \sqrt{1^2 + (-2)^2 + (-2)^2} = \sqrt{9} = 3$$

and

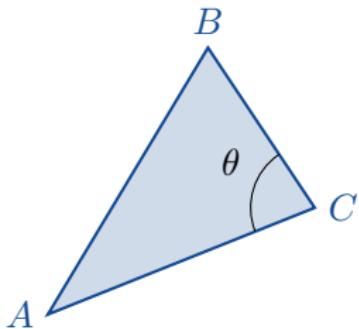
$$\|\mathbf{v}\| = \sqrt{6^2 + 3^2 + 2^2} = \sqrt{49} = 7,$$

we have that

$$\theta = \cos^{-1} \left( \frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{u}\| \|\mathbf{v}\|} \right) = \cos^{-1} \left( -\frac{4}{21} \right) \approx 1.76 \text{ radians} \approx 98.5^\circ.$$

12.

$$\theta = \cos^{-1} \left( \frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{u}\| \|\mathbf{v}\|} \right).$$



### Example

If  $A(0, 0)$ ,  $B(3, 5)$  and  $C(5, 2)$ , find  $\theta = \angle ACB$ .

## 12. The Dot Product



$\theta$  is the angle between  $\overrightarrow{CA}$  and  $\overrightarrow{CB}$ .

## 12. The Dot Product

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$$\overrightarrow{CA} = A - C = (0, 0) - (5, 2) = (-5, -2),$$

$$\overrightarrow{CB} = B - C = (3, 5) - (5, 2) = (-2, 3),$$

$$\overrightarrow{CA} \cdot \overrightarrow{CB} = (-5, -2) \cdot (-2, 3) = 4,$$

$$\|\overrightarrow{CA}\| = \sqrt{(-5)^2 + (-2)^2} = \sqrt{29}$$

and

$$\|\overrightarrow{CB}\| = \sqrt{(-2)^2 + 3^2} = \sqrt{13}.$$

## 12. The Dot Product

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$$\overrightarrow{CA} \cdot \overrightarrow{CB} = (-5, -2) \cdot (-2, 3) = 4,$$

$$\|\overrightarrow{CA}\| = \sqrt{(-5)^2 + (-2)^2} = \sqrt{29}$$

and

$$\|\overrightarrow{CB}\| = \sqrt{(-2)^2 + 3^2} = \sqrt{13}.$$

Therefore

$$\theta = \cos^{-1} \left( \frac{\overrightarrow{CA} \cdot \overrightarrow{CB}}{\|\overrightarrow{CA}\| \|\overrightarrow{CB}\|} \right) = \cos^{-1} \left( \frac{4}{\sqrt{29}\sqrt{13}} \right)$$

$$\approx 78.1^\circ \approx 1.36 \text{ radians.}$$

## 12. The Dot Product

### Definition

$\mathbf{u}$  and  $\mathbf{v}$  are *orthogonal*  $\iff \mathbf{u} \cdot \mathbf{v} = 0$ .

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### Remark

Recall that

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## 12. The Dot Product

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### Remark

Recall that

$$\mathbf{u} \cdot \mathbf{v} = \|\mathbf{u}\| \|\mathbf{v}\| \cos \theta.$$

Therefore

$$\mathbf{u} \text{ and } \mathbf{v} \text{ are orthogonal} \iff \begin{cases} \mathbf{u} = \mathbf{0} \\ \text{or} \\ \mathbf{v} = \mathbf{0} \\ \text{or} \\ \theta = 90^\circ. \end{cases}$$

## 12. The Dot Product

### Example

$\mathbf{u} = (3, -2)$  and  $\mathbf{v} = (4, 6)$  are orthogonal because

$$\mathbf{u} \cdot \mathbf{v} = (3, -2) \cdot (4, 6) = (3 \times 4) + (-2 \times 6) = 12 - 12 = 0.$$

## 12. The Dot Product

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### Example

$\mathbf{u} = 3\mathbf{i} - 2\mathbf{j} + \mathbf{k}$  and  $\mathbf{v} = 2\mathbf{j} + 4\mathbf{k}$  are orthogonal because

$$\mathbf{u} \cdot \mathbf{v} = (3 \times 0) + (-2 \times 2) + (1 \times 4) = 0 - 4 + 4 = 0.$$

## 12. The Dot Product

### Example

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### Example

$\mathbf{u} = 3\mathbf{i} - 2\mathbf{j} + \mathbf{k}$  and  $\mathbf{v} = 2\mathbf{j} + 4\mathbf{k}$  are orthogonal because

$$\mathbf{u} \cdot \mathbf{v} = (3 \times 0) + (-2 \times 2) + (1 \times 4) = 0 - 4 + 4 = 0.$$

### Example

$\mathbf{0}$  is orthogonal to every vector  $\mathbf{u}$  because

$$\mathbf{0} \cdot \mathbf{u} = (0, 0, 0) \cdot (u_1, u_2, u_3) = 0u_1 + 0u_2 + 0u_3 = 0.$$

## 12. The Dot Product



### Properties of the Dot Product

Let  $\mathbf{u}$ ,  $\mathbf{v}$  and  $\mathbf{w}$  be vectors. Let  $k$  be a number. Then

**1**  $\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u};$

## 12. The Dot Product



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- 2  $(k\mathbf{u}) \cdot \mathbf{v} = \mathbf{u} \cdot (k\mathbf{v}) = k(\mathbf{u} \cdot \mathbf{v})$ ;

## 12. The Dot Product



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- 3  $\mathbf{u} \cdot (\mathbf{v} + \mathbf{w}) = (\mathbf{u} \cdot \mathbf{v}) + (\mathbf{u} \cdot \mathbf{w})$ ;

## 12. The Dot Product



### Properties of the Dot Product

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- 3  $\mathbf{u} \cdot (\mathbf{v} + \mathbf{w}) = (\mathbf{u} \cdot \mathbf{v}) + (\mathbf{u} \cdot \mathbf{w})$ ;
- 4  $\mathbf{u} \cdot \mathbf{u} = \|\mathbf{u}\|^2$ ; and

## 12. The Dot Product



### Properties of the Dot Product

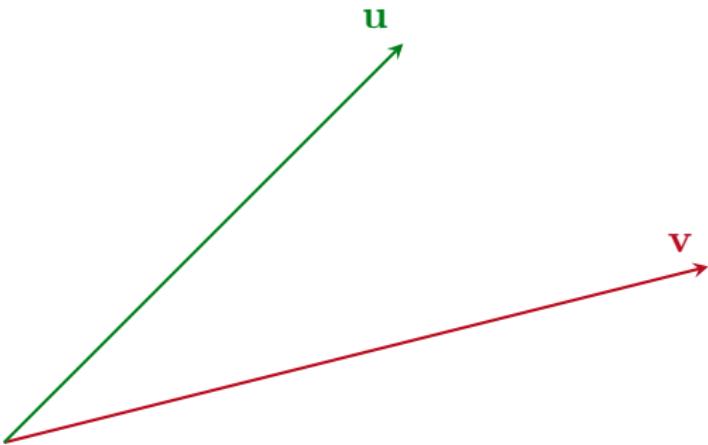
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- 1  $\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u}$ ;
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- 4  $\mathbf{u} \cdot \mathbf{u} = \|\mathbf{u}\|^2$ ; and
- 5  $\mathbf{0} \cdot \mathbf{u} = 0$ .

## 12. The Dot Product



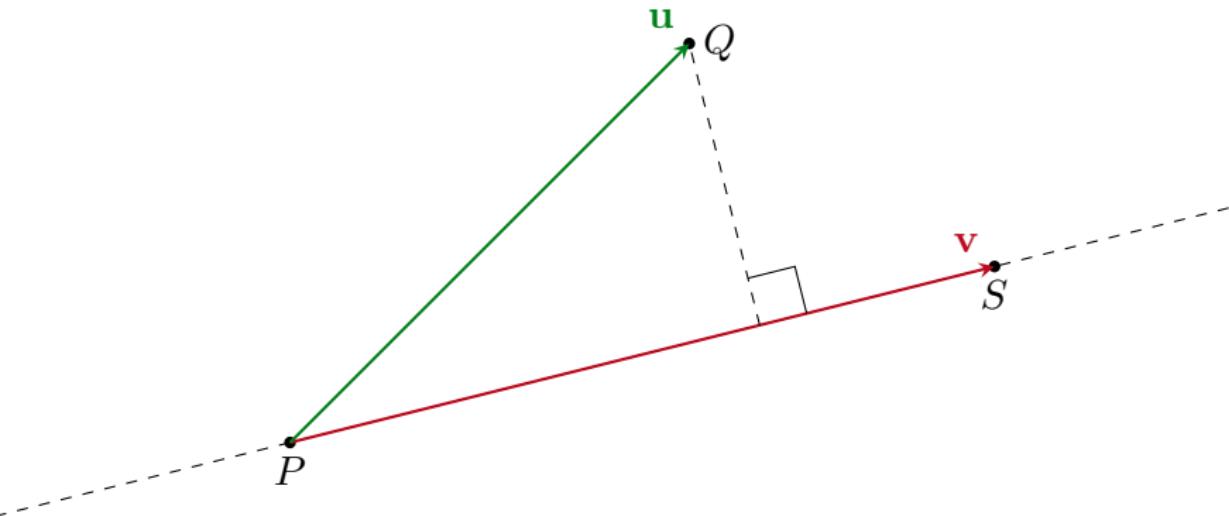
# Vector Projections



## 12. The Dot Product



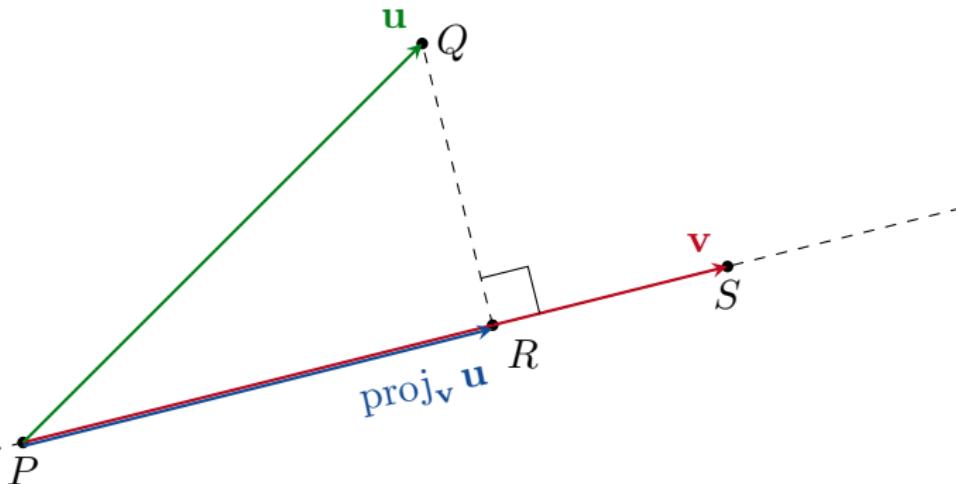
### Vector Projections



## 12. The Dot Product



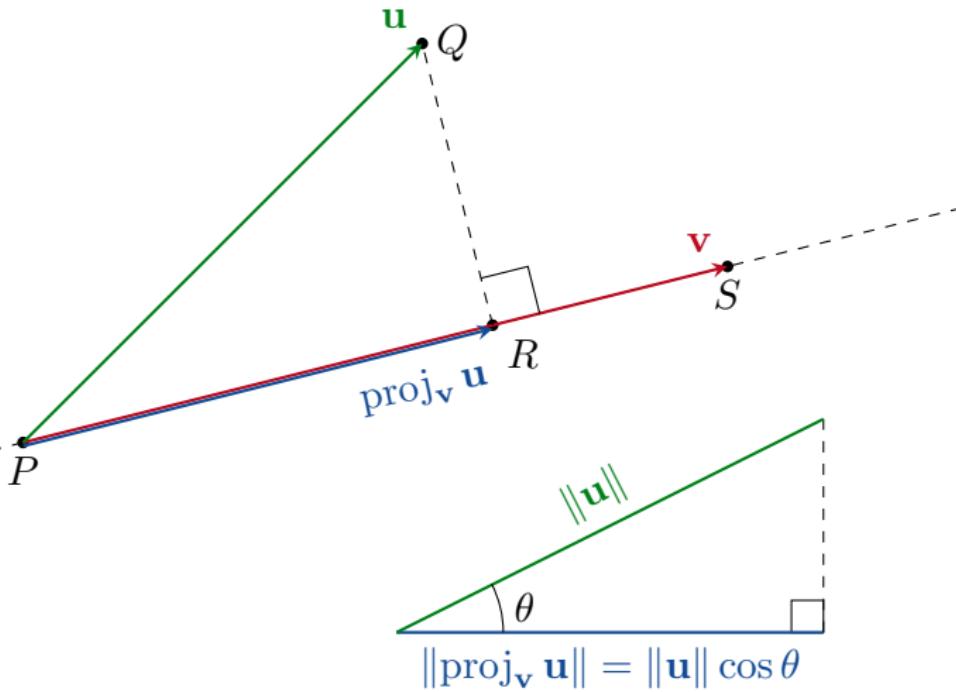
### Vector Projections



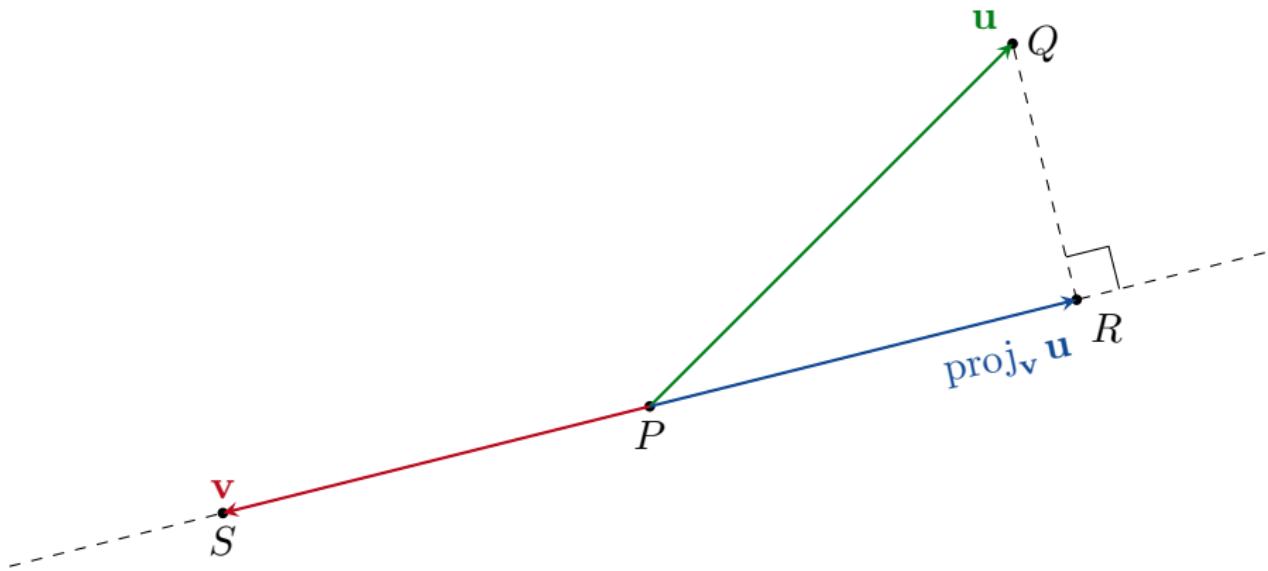
## 12. The Dot Product



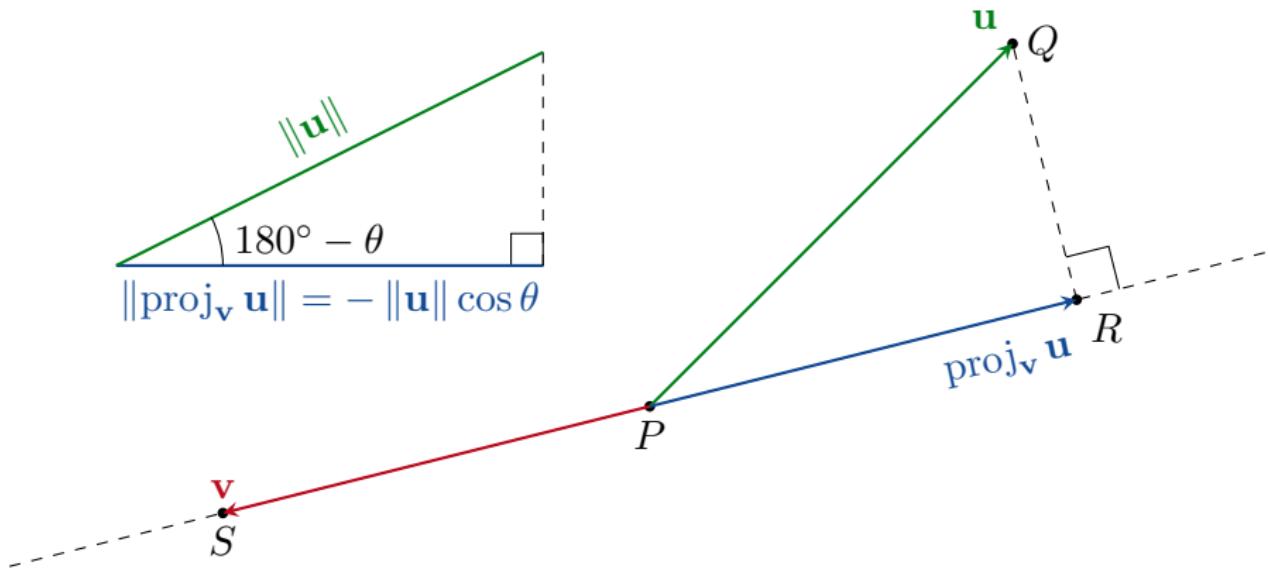
### Vector Projections



## 12. The Dot Product



## 12. The Dot Product



## 12. The Dot Product



### Definition

The *vector projection* of  $\mathbf{u}$  onto  $\mathbf{v}$  is the vector

$$\text{proj}_{\mathbf{v}} \mathbf{u} = \overrightarrow{PR}.$$

## 12. The Dot Product

Now

$$\text{proj}_{\mathbf{v}} \mathbf{u} = (\text{length of } \text{proj}_{\mathbf{v}} \mathbf{u}) \begin{pmatrix} \text{a unit vector in} \\ \text{the same} \\ \text{direction as } \mathbf{v} \end{pmatrix}$$

=

=

=

=

## 12. The Dot Product

Now

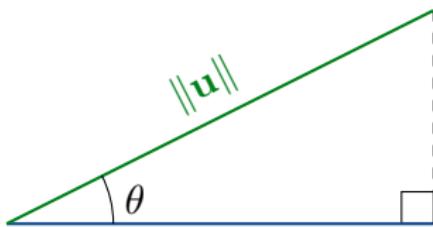
$$\text{proj}_{\mathbf{v}} \mathbf{u} = (\text{length of } \text{proj}_{\mathbf{v}} \mathbf{u}) \begin{pmatrix} \text{a unit vector in} \\ \text{the same} \\ \text{direction as } \mathbf{v} \end{pmatrix}$$

$$= \|\text{proj}_{\mathbf{v}} \mathbf{u}\| \left( \frac{\mathbf{v}}{\|\mathbf{v}\|} \right)$$

=

=

=



$$\|\text{proj}_{\mathbf{v}} \mathbf{u}\| = \|\mathbf{u}\| \cos \theta$$

## 12. The Dot Product

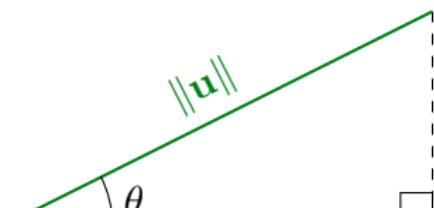
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$$= \|\text{proj}_{\mathbf{v}} \mathbf{u}\| \left( \frac{\mathbf{v}}{\|\mathbf{v}\|} \right)$$

$$= \|\mathbf{u}\| (\cos \theta) \left( \frac{\mathbf{v}}{\|\mathbf{v}\|} \right)$$

=



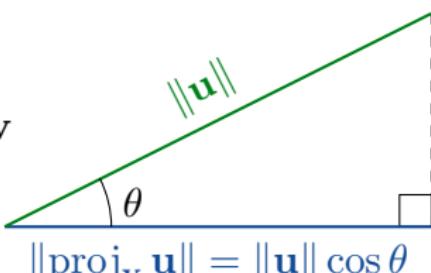
=

$$\|\text{proj}_{\mathbf{v}} \mathbf{u}\| = \|\mathbf{u}\| \cos \theta$$

## 12. The Dot Product

Now

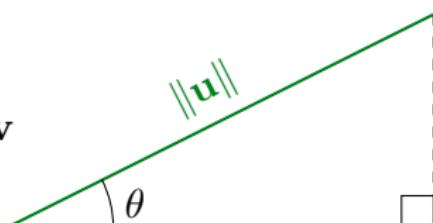
$$\begin{aligned}\text{proj}_{\mathbf{v}} \mathbf{u} &= (\text{length of } \text{proj}_{\mathbf{v}} \mathbf{u}) \left( \begin{array}{c} \text{a unit vector in} \\ \text{the same} \\ \text{direction as } \mathbf{v} \end{array} \right) \\ &= \|\text{proj}_{\mathbf{v}} \mathbf{u}\| \left( \frac{\mathbf{v}}{\|\mathbf{v}\|} \right) \\ &= \|\mathbf{u}\| (\cos \theta) \left( \frac{\mathbf{v}}{\|\mathbf{v}\|} \right) \\ &= \left( \frac{\|\mathbf{u}\| \|\mathbf{v}\| \cos \theta}{\|\mathbf{v}\|^2} \right) \mathbf{v} \\ &= \|\mathbf{proj}_{\mathbf{v}} \mathbf{u}\| = \|\mathbf{u}\| \cos \theta\end{aligned}$$



## 12. The Dot Product

Now

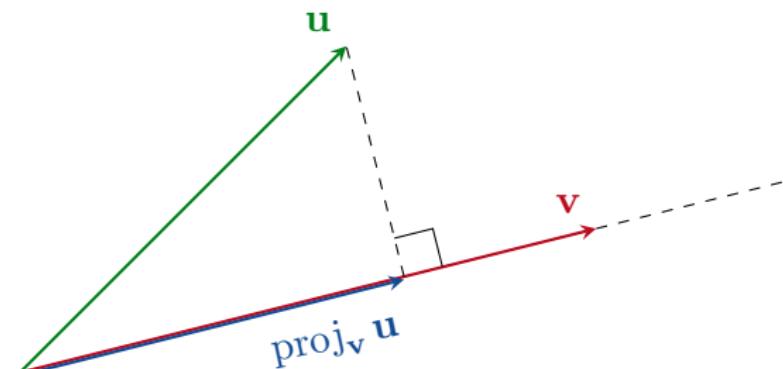
$$\begin{aligned}\text{proj}_{\mathbf{v}} \mathbf{u} &= (\text{length of } \text{proj}_{\mathbf{v}} \mathbf{u}) \begin{pmatrix} \text{a unit vector in} \\ \text{the same} \\ \text{direction as } \mathbf{v} \end{pmatrix} \\ &= \|\text{proj}_{\mathbf{v}} \mathbf{u}\| \left( \frac{\mathbf{v}}{\|\mathbf{v}\|} \right) \\ &= \|\mathbf{u}\| (\cos \theta) \left( \frac{\mathbf{v}}{\|\mathbf{v}\|} \right) \\ &= \left( \frac{\|\mathbf{u}\| \|\mathbf{v}\| \cos \theta}{\|\mathbf{v}\|^2} \right) \mathbf{v} \\ &= \left( \frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{v}\|^2} \right) \mathbf{v}.\end{aligned}$$



$$\|\text{proj}_{\mathbf{v}} \mathbf{u}\| = \|\mathbf{u}\| \cos \theta$$

Since this is an important formula, we write it as a theorem.

## 12. The Dot Product



Theorem

*The vector projection of  $\mathbf{u}$  onto  $\mathbf{v}$  is*

$$\text{proj}_{\mathbf{v}} \mathbf{u} = \left( \frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{v}\|^2} \right) \mathbf{v}.$$

12.

$$\text{proj}_{\mathbf{v}} \mathbf{u} = \left( \frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{v}\|^2} \right) \mathbf{v}$$



### Example

Find the vector projection of  $\mathbf{u} = 6\mathbf{i} + 3\mathbf{j} + 2\mathbf{k}$  onto  $\mathbf{v} = \mathbf{i} - 2\mathbf{j} - 2\mathbf{k}$ .

12.

$$\text{proj}_{\mathbf{v}} \mathbf{u} = \left( \frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{v}\|^2} \right) \mathbf{v}$$



### Example

Find the vector projection of  $\mathbf{u} = 6\mathbf{i} + 3\mathbf{j} + 2\mathbf{k}$  onto  $\mathbf{v} = \mathbf{i} - 2\mathbf{j} - 2\mathbf{k}$ .

$$\begin{aligned}\text{proj}_{\mathbf{v}} \mathbf{u} &= \left( \frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{v}\|^2} \right) \mathbf{v} = \left( \frac{6 - 6 - 4}{1 + 4 + 4} \right) (\mathbf{i} - 2\mathbf{j} - 2\mathbf{k}) \\ &= -\frac{4}{9}\mathbf{i} + \frac{8}{9}\mathbf{j} + \frac{8}{9}\mathbf{k}.\end{aligned}$$

12.

$$\text{proj}_{\mathbf{v}} \mathbf{u} = \left( \frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{v}\|^2} \right) \mathbf{v}$$



### Example

Find the vector projection of  $\mathbf{F} = 5\mathbf{i} + 2\mathbf{j}$  onto  $\mathbf{v} = \mathbf{i} - 3\mathbf{j}$ .

$$\begin{aligned}\text{proj}_{\mathbf{v}} \mathbf{F} &= \left( \frac{\mathbf{F} \cdot \mathbf{v}}{\|\mathbf{v}\|^2} \right) \mathbf{v} = \left( \frac{5 - 6}{1 + 9} \right) (\mathbf{i} - 3\mathbf{j}) \\ &= -\frac{1}{10}\mathbf{i} + \frac{3}{10}\mathbf{j}.\end{aligned}$$

12.

$$\text{proj}_{\mathbf{v}} \mathbf{u} = \left( \frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{v}\|^2} \right) \mathbf{v}$$



### Example

Verify that the vector  $\mathbf{u} - \text{proj}_{\mathbf{v}} \mathbf{u}$  is orthogonal to  $\text{proj}_{\mathbf{v}} \mathbf{u}$ .

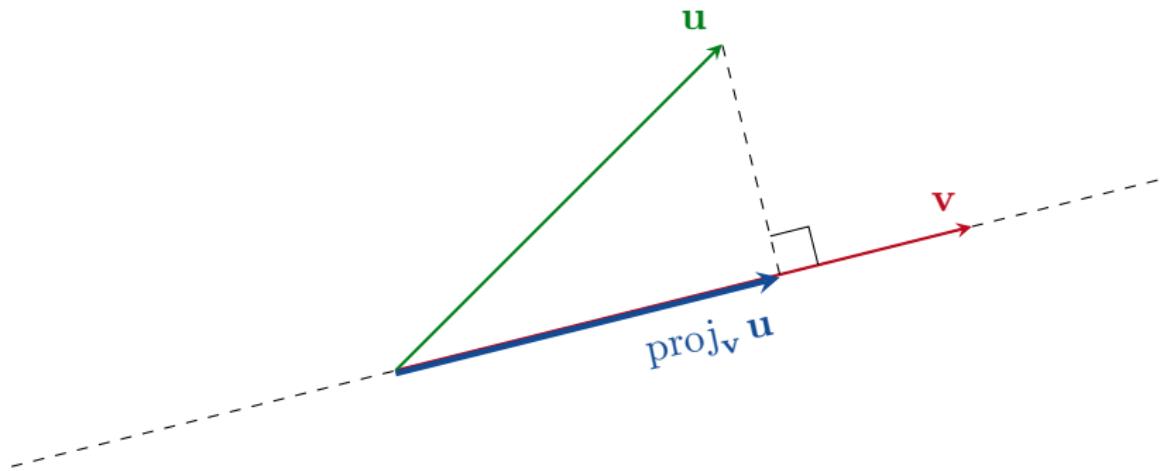
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### Example

Verify that the vector  $\mathbf{u} - \text{proj}_{\mathbf{v}} \mathbf{u}$  is orthogonal to  $\text{proj}_{\mathbf{v}} \mathbf{u}$ .



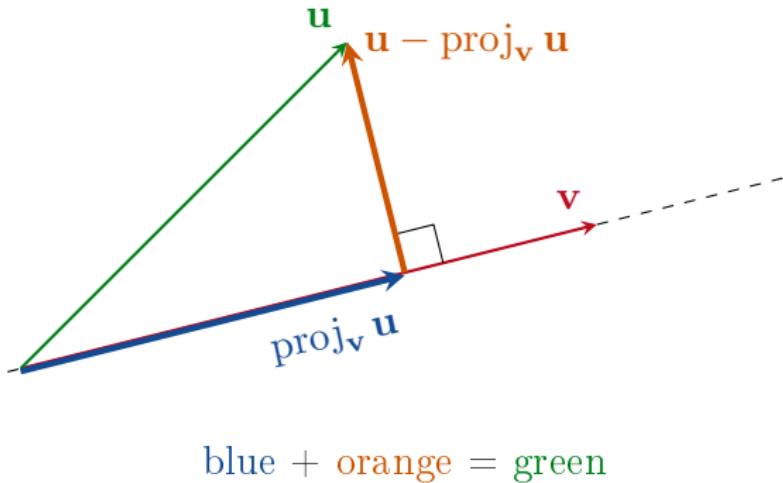
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### Example

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12.

$$\text{proj}_{\mathbf{v}} \mathbf{u} = \left( \frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{v}\|^2} \right) \mathbf{v}$$



Clearly

$$\text{proj}_{\mathbf{v}} \mathbf{u} = \left( \frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{v}\|^2} \right) \mathbf{v} = (\text{a number}) \mathbf{v}$$

is parallel to  $\mathbf{v}$ .

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is parallel to  $\mathbf{v}$ . So it is enough to show that  $\mathbf{u} - \text{proj}_{\mathbf{v}} \mathbf{u}$  is orthogonal to  $\mathbf{v}$ .

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$$\text{proj}_{\mathbf{v}} \mathbf{u} = \left( \frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{v}\|^2} \right) \mathbf{v}$$

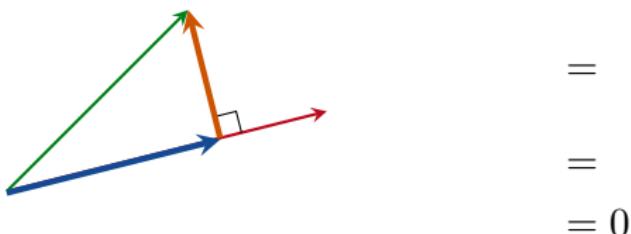
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$$\text{proj}_{\mathbf{v}} \mathbf{u} = \left( \frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{v}\|^2} \right) \mathbf{v} = (\text{a number}) \mathbf{v}$$

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Since

$$(\mathbf{u} - \text{proj}_{\mathbf{v}} \mathbf{u}) \cdot \mathbf{v} =$$



$$\begin{aligned}
 &= \\
 &= \\
 &= 0
 \end{aligned}$$

we have shown that  $\mathbf{u} - \text{proj}_{\mathbf{v}} \mathbf{u}$  is orthogonal to  $\mathbf{v}$ .

12.

$$\text{proj}_{\mathbf{v}} \mathbf{u} = \left( \frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{v}\|^2} \right) \mathbf{v}$$

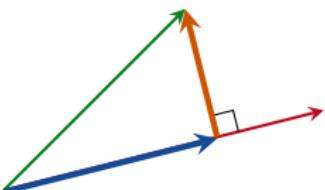
Clearly

$$\text{proj}_{\mathbf{v}} \mathbf{u} = \left( \frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{v}\|^2} \right) \mathbf{v} = (\text{a number}) \mathbf{v}$$

is parallel to  $\mathbf{v}$ . So it is enough to show that  $\mathbf{u} - \text{proj}_{\mathbf{v}} \mathbf{u}$  is orthogonal to  $\mathbf{v}$ .

Since

$$\begin{aligned}
 (\mathbf{u} - \text{proj}_{\mathbf{v}} \mathbf{u}) \cdot \mathbf{v} &= \mathbf{u} \cdot \mathbf{v} - \left( \frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{v}\|^2} \right) \mathbf{v} \cdot \mathbf{v} \\
 &= \mathbf{u} \cdot \mathbf{v} - \frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{v}\|^2} \|\mathbf{v}\|^2 \\
 &= \mathbf{u} \cdot \mathbf{v} - \mathbf{u} \cdot \mathbf{v} \\
 &= 0
 \end{aligned}$$

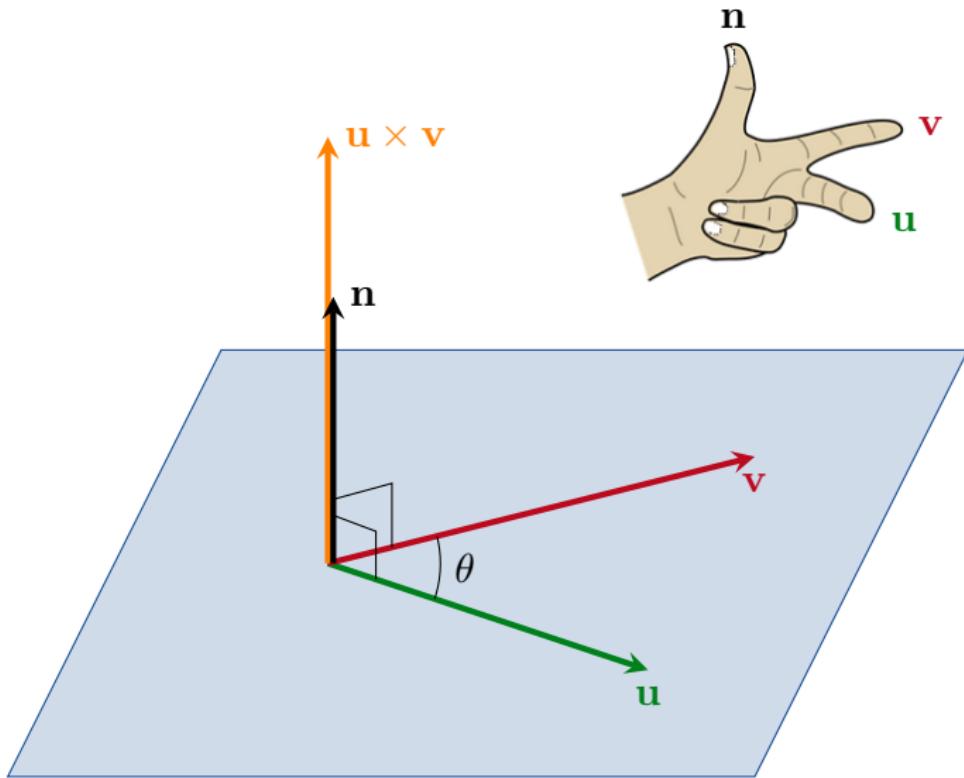


we have shown that  $\mathbf{u} - \text{proj}_{\mathbf{v}} \mathbf{u}$  is orthogonal to  $\mathbf{v}$ .



# The Cross Product

## 13. The Cross Product

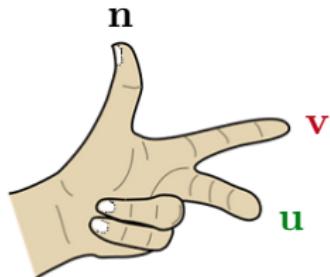


## 13. The Cross Product



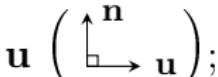
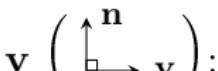
Let  $\mathbf{n}$  be a unit vector which satisfies

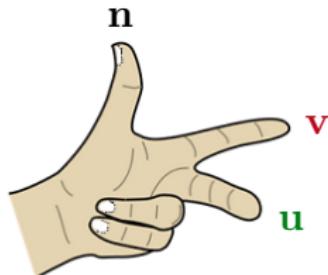
- 1  $\mathbf{n}$  is orthogonal to  $\mathbf{u}$  ( $\begin{smallmatrix} \mathbf{n} \\ \perp \\ \mathbf{u} \end{smallmatrix}$ );
- 2  $\mathbf{n}$  is orthogonal to  $\mathbf{v}$  ( $\begin{smallmatrix} \mathbf{n} \\ \perp \\ \mathbf{v} \end{smallmatrix}$ ); and
- 3 the direction of  $\mathbf{n}$  is chosen using the left-hand rule.



## 13. The Cross Product

Let  $\mathbf{n}$  be a unit vector which satisfies

- 1  $\mathbf{n}$  is orthogonal to  $\mathbf{u}$  () ;
- 2  $\mathbf{n}$  is orthogonal to  $\mathbf{v}$  () ; and
- 3 the direction of  $\mathbf{n}$  is chosen using the left-hand rule.



### Definition

The *cross product* of  $\mathbf{u}$  and  $\mathbf{v}$  is

$$\mathbf{u} \times \mathbf{v} = \|\mathbf{u}\| \|\mathbf{v}\| (\sin \theta) \mathbf{n}.$$

## 13. The Cross Product



### Remark

- $\mathbf{u} \cdot \mathbf{v}$  is a number.
- $\mathbf{u} \times \mathbf{v}$  is a vector.

13.

$$\mathbf{u} \times \mathbf{v} = \|\mathbf{u}\| \|\mathbf{v}\| (\sin \theta) \mathbf{n}$$



### Remark

$$\begin{pmatrix} \mathbf{u} \text{ and } \mathbf{v} \\ \text{are} \\ \text{parallel} \end{pmatrix} \iff \theta = 0^\circ \text{ or } 180^\circ$$
$$\implies \sin \theta = 0 \implies \mathbf{u} \times \mathbf{v} = \mathbf{0}.$$

## 13. The Cross Product



### Properties of the Cross Product

Let  $\mathbf{u}$ ,  $\mathbf{v}$  and  $\mathbf{w}$  be vectors. Let  $r$  and  $s$  be numbers. Then

1  $(r\mathbf{u}) \times (s\mathbf{v}) = (rs)(\mathbf{u} \times \mathbf{v});$

## 13. The Cross Product



### Properties of the Cross Product

Let  $\mathbf{u}$ ,  $\mathbf{v}$  and  $\mathbf{w}$  be vectors. Let  $r$  and  $s$  be numbers. Then

- 1  $(r\mathbf{u}) \times (s\mathbf{v}) = (rs)(\mathbf{u} \times \mathbf{v});$
- 2  $\mathbf{u} \times (\mathbf{v} + \mathbf{w}) = (\mathbf{u} \times \mathbf{v}) + (\mathbf{u} \times \mathbf{w});$

# 13. The Cross Product



## Properties of the Cross Product

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- 3  $\mathbf{v} \times \mathbf{u} = -\mathbf{u} \times \mathbf{v};$
- 4  $(\mathbf{v} + \mathbf{w}) \times \mathbf{u} = (\mathbf{v} \times \mathbf{u}) + (\mathbf{w} \times \mathbf{u});$

## 13. The Cross Product



### Properties of the Cross Product

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- 2  $\mathbf{u} \times (\mathbf{v} + \mathbf{w}) = (\mathbf{u} \times \mathbf{v}) + (\mathbf{u} \times \mathbf{w});$
- 3  $\mathbf{v} \times \mathbf{u} = -\mathbf{u} \times \mathbf{v};$
- 4  $(\mathbf{v} + \mathbf{w}) \times \mathbf{u} = (\mathbf{v} \times \mathbf{u}) + (\mathbf{w} \times \mathbf{u});$
- 5  $\mathbf{0} \times \mathbf{u} = \mathbf{0};$  and

# 13. The Cross Product



## Properties of the Cross Product

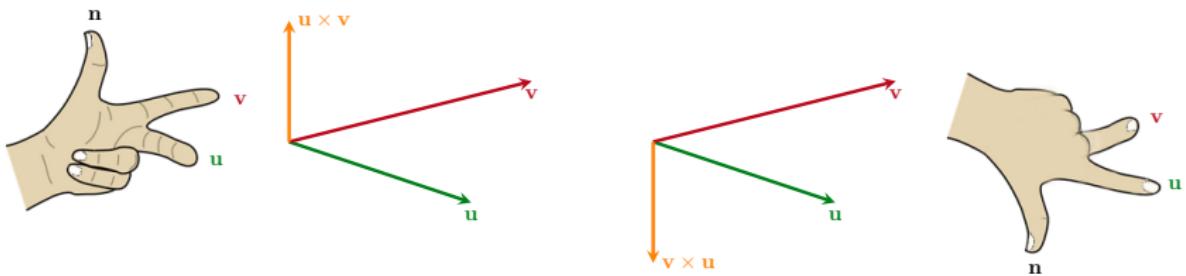
Let  $\mathbf{u}$ ,  $\mathbf{v}$  and  $\mathbf{w}$  be vectors. Let  $r$  and  $s$  be numbers. Then

- 1  $(r\mathbf{u}) \times (s\mathbf{v}) = (rs)(\mathbf{u} \times \mathbf{v});$
- 2  $\mathbf{u} \times (\mathbf{v} + \mathbf{w}) = (\mathbf{u} \times \mathbf{v}) + (\mathbf{u} \times \mathbf{w});$
- 3  $\mathbf{v} \times \mathbf{u} = -\mathbf{u} \times \mathbf{v};$
- 4  $(\mathbf{v} + \mathbf{w}) \times \mathbf{u} = (\mathbf{v} \times \mathbf{u}) + (\mathbf{w} \times \mathbf{u});$
- 5  $\mathbf{0} \times \mathbf{u} = \mathbf{0};$  and
- 6  $\mathbf{u} \times (\mathbf{v} \times \mathbf{w}) = (\mathbf{u} \cdot \mathbf{w})\mathbf{v} - (\mathbf{u} \cdot \mathbf{v})\mathbf{w}.$

# 13. The Cross Product



## Property (iii)



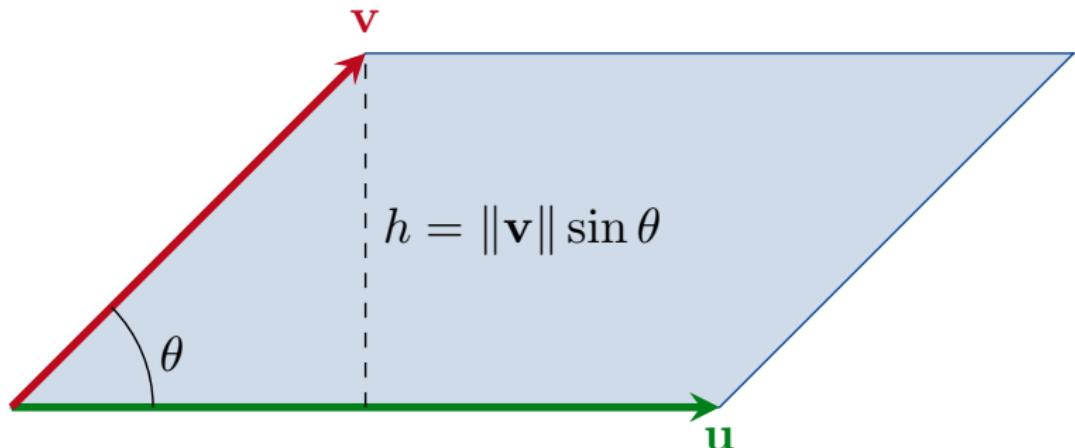
$$\mathbf{v} \times \mathbf{u} = -\mathbf{u} \times \mathbf{v}$$

13.

$$\mathbf{u} \times \mathbf{v} = \|\mathbf{u}\| \|\mathbf{v}\| (\sin \theta) \mathbf{n}$$



## Area of a Parallelogram

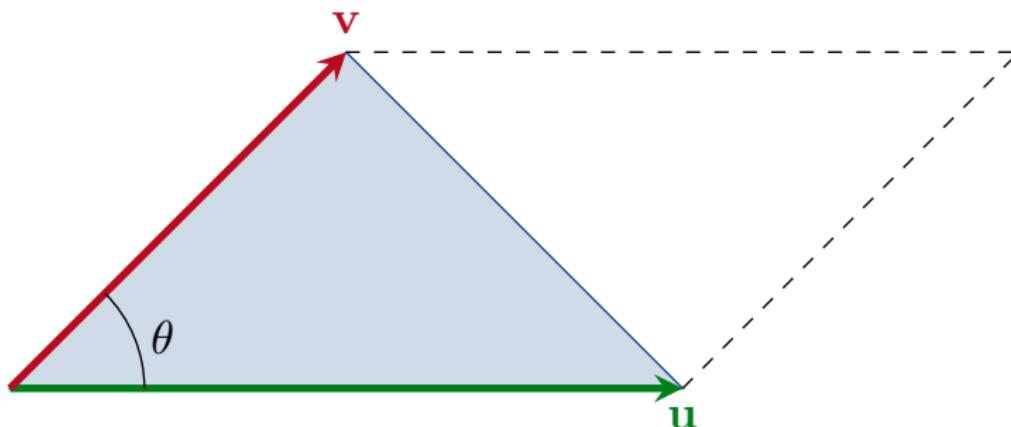


$$\text{area} = (\text{base}) (\text{height}) = \|\mathbf{u}\| \|\mathbf{v}\| \sin \theta = \|\mathbf{u} \times \mathbf{v}\| .$$

# 13. The Cross Product



## Area of a Triangle

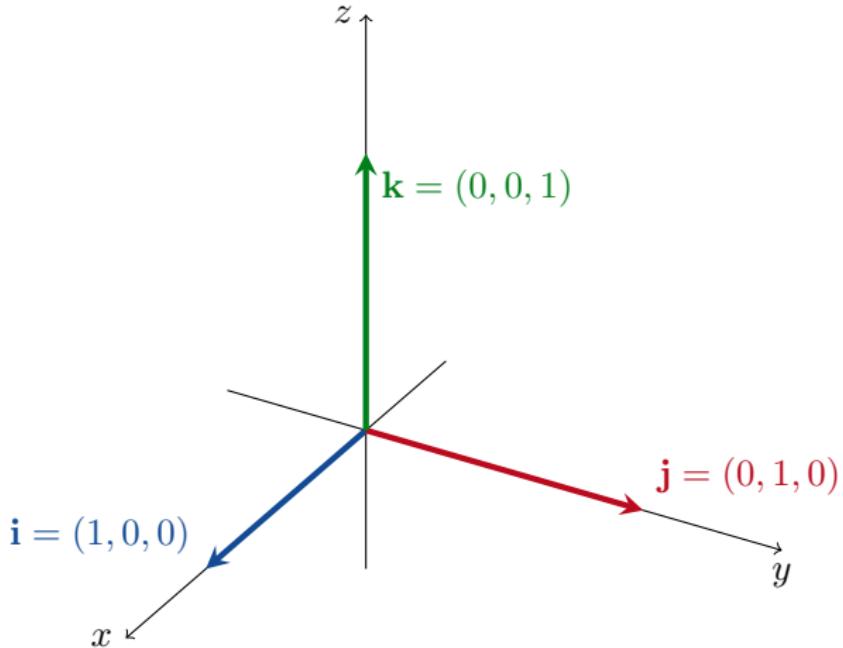


$$\begin{aligned}\text{area of triangle} &= \frac{1}{2} (\text{area of parallelogram}) \\ &= \frac{1}{2} \|\mathbf{u} \times \mathbf{v}\|.\end{aligned}$$

# 13. The Cross Product



## A Formula for $\mathbf{u} \times \mathbf{v}$



13.

$$\mathbf{u} \times \mathbf{v} = \|\mathbf{u}\| \|\mathbf{v}\| (\sin \theta) \mathbf{n}$$



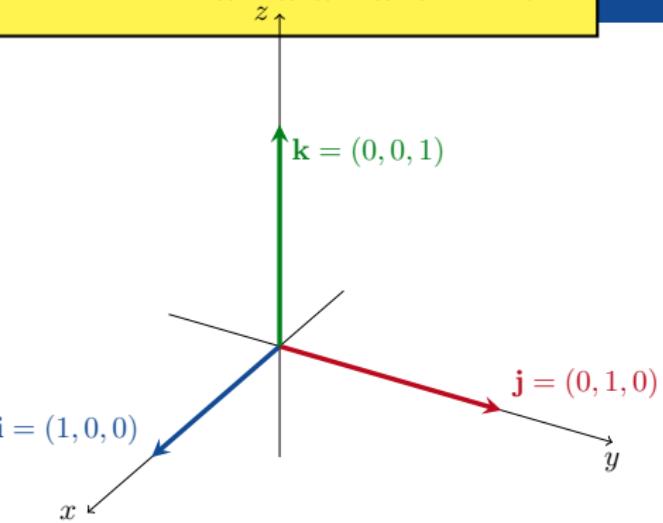
Note first that

$$\mathbf{i} \times \mathbf{i} = \|\mathbf{i}\| \|\mathbf{i}\| \sin 0^\circ \mathbf{n} = \mathbf{0}.$$

Similarly  $\mathbf{j} \times \mathbf{j} = \mathbf{0}$  and  $\mathbf{k} \times \mathbf{k} = \mathbf{0}$  also.

13.

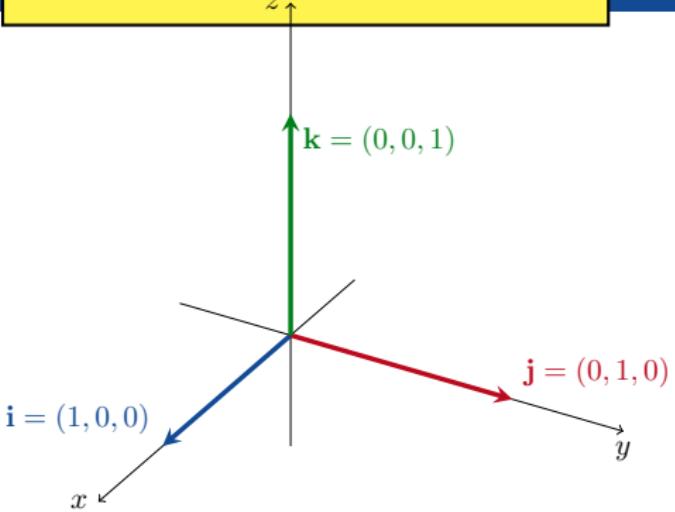
$$\mathbf{u} \times \mathbf{v} = \|\mathbf{u}\| \|\mathbf{v}\| (\sin \theta) \mathbf{n}$$



Next note that  $\mathbf{i} \times \mathbf{j}$  must point in the same direction as  $\mathbf{k}$  by the left-hand rule.

13.

$$\mathbf{u} \times \mathbf{v} = \|\mathbf{u}\| \|\mathbf{v}\| (\sin \theta) \mathbf{n}$$

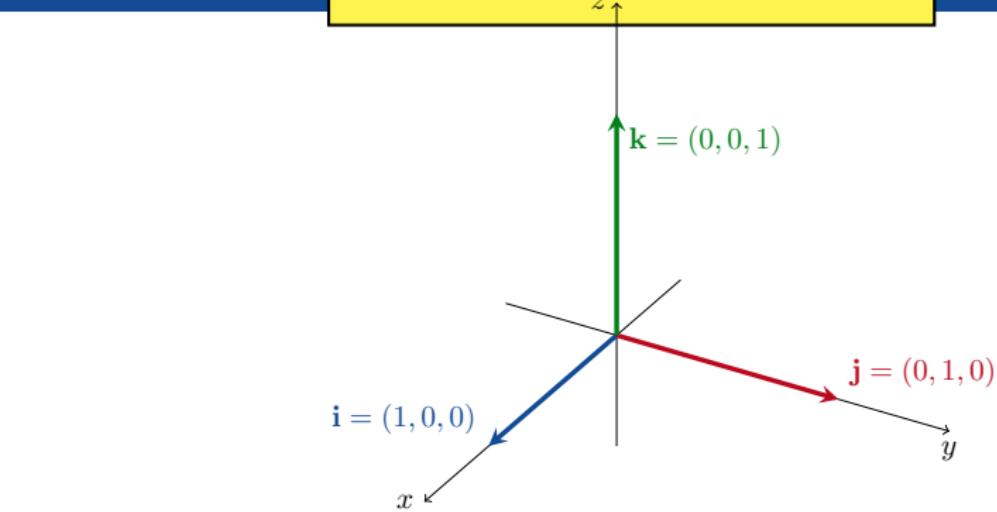


Next note that  $\mathbf{i} \times \mathbf{j}$  must point in the same direction as  $\mathbf{k}$  by the left-hand rule. Thus

$$\mathbf{i} \times \mathbf{j} = \|\mathbf{i}\| \|\mathbf{j}\| \sin 90^\circ \mathbf{k} = \mathbf{k}.$$

13.

$$\mathbf{u} \times \mathbf{v} = \|\mathbf{u}\| \|\mathbf{v}\| (\sin \theta) \mathbf{n}$$



Next note that  $\mathbf{i} \times \mathbf{j}$  must point in the same direction as  $\mathbf{k}$  by the left-hand rule. Thus

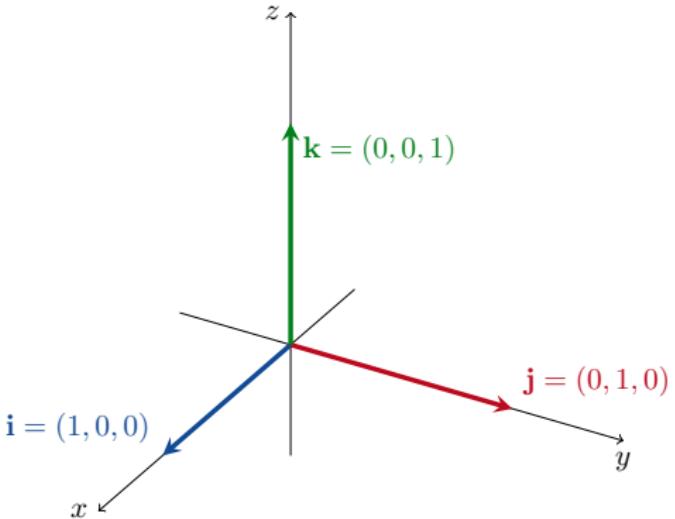
$$\mathbf{i} \times \mathbf{j} = \|\mathbf{i}\| \|\mathbf{j}\| \sin 90^\circ \mathbf{k} = \mathbf{k}.$$

We then immediately also have

$$\mathbf{j} \times \mathbf{i} = -(\mathbf{i} \times \mathbf{j}) = -\mathbf{k}.$$

13.

$$\mathbf{u} \times \mathbf{v} = \|\mathbf{u}\| \|\mathbf{v}\| (\sin \theta) \mathbf{n}$$



I leave it for you to check that

$$\mathbf{j} \times \mathbf{k} = \mathbf{i}, \quad \mathbf{k} \times \mathbf{j} = -\mathbf{i}, \quad \mathbf{i} \times \mathbf{k} = -\mathbf{j} \quad \text{and} \quad \mathbf{k} \times \mathbf{i} = \mathbf{j}.$$

## 13. The Cross Product



Now suppose that  $\mathbf{u} = u_1\mathbf{i} + u_2\mathbf{j} + u_3\mathbf{k}$   
and  $\mathbf{v} = v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}$ .

## 13. The Cross Product



Now suppose that  $\mathbf{u} = u_1\mathbf{i} + u_2\mathbf{j} + u_3\mathbf{k}$   
and  $\mathbf{v} = v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}$ . Then we can calculate  
that

$$\mathbf{u} \times \mathbf{v} = (u_1\mathbf{i} + u_2\mathbf{j} + u_3\mathbf{k}) \times (v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k})$$

=

=

=

## 13. The Cross Product

Now suppose that  $\mathbf{u} = u_1\mathbf{i} + u_2\mathbf{j} + u_3\mathbf{k}$   
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that

$$\begin{aligned}\mathbf{u} \times \mathbf{v} &= (u_1\mathbf{i} + u_2\mathbf{j} + u_3\mathbf{k}) \times (v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}) \\&= u_1v_1\mathbf{i} \times \mathbf{i} + u_1v_2\mathbf{i} \times \mathbf{j} + u_1v_3\mathbf{i} \times \mathbf{k} + u_2v_1\mathbf{j} \times \mathbf{i} + u_2v_2\mathbf{j} \times \mathbf{j} \\&\quad + u_2v_3\mathbf{j} \times \mathbf{k} + u_3v_1\mathbf{k} \times \mathbf{i} + u_3v_2\mathbf{k} \times \mathbf{j} + u_3v_3\mathbf{k} \times \mathbf{k} \\&= \\&= \end{aligned}$$

## 13. The Cross Product

$$\begin{aligned}\mathbf{i} \times \mathbf{j} &= \mathbf{k} \\ \mathbf{j} \times \mathbf{i} &= -\mathbf{k} \\ \mathbf{j} \times \mathbf{k} &= \mathbf{i} \\ \mathbf{k} \times \mathbf{j} &= -\mathbf{i} \\ \mathbf{i} \times \mathbf{k} &= -\mathbf{j} \\ \mathbf{k} \times \mathbf{i} &= \mathbf{j}\end{aligned}$$

Now suppose that  $\mathbf{u} = u_1\mathbf{i} + u_2\mathbf{j} + u_3\mathbf{k}$   
and  $\mathbf{v} = v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}$ . Then we can calculate  
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$$\begin{aligned}\mathbf{u} \times \mathbf{v} &= (u_1\mathbf{i} + u_2\mathbf{j} + u_3\mathbf{k}) \times (v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}) \\ &= u_1v_1\mathbf{i} \times \mathbf{i} + u_1v_2\mathbf{i} \times \mathbf{j} + u_1v_3\mathbf{i} \times \mathbf{k} + u_2v_1\mathbf{j} \times \mathbf{i} + u_2v_2\mathbf{j} \times \mathbf{j} \\ &\quad + u_2v_3\mathbf{j} \times \mathbf{k} + u_3v_1\mathbf{k} \times \mathbf{i} + u_3v_2\mathbf{k} \times \mathbf{j} + u_3v_3\mathbf{k} \times \mathbf{k} \\ &= \\ &= \end{aligned}$$

## 13. The Cross Product

$$\begin{aligned}\mathbf{i} \times \mathbf{j} &= \mathbf{k} \\ \mathbf{j} \times \mathbf{i} &= -\mathbf{k} \\ \mathbf{j} \times \mathbf{k} &= \mathbf{i} \\ \mathbf{k} \times \mathbf{j} &= -\mathbf{i} \\ \mathbf{i} \times \mathbf{k} &= -\mathbf{j} \\ \mathbf{k} \times \mathbf{i} &= \mathbf{j}\end{aligned}$$

Now suppose that  $\mathbf{u} = u_1\mathbf{i} + u_2\mathbf{j} + u_3\mathbf{k}$   
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that

$$\begin{aligned}\mathbf{u} \times \mathbf{v} &= (u_1\mathbf{i} + u_2\mathbf{j} + u_3\mathbf{k}) \times (v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}) \\ &= u_1v_1\mathbf{i} \times \mathbf{i} + u_1v_2\mathbf{i} \times \mathbf{j} + u_1v_3\mathbf{i} \times \mathbf{k} + u_2v_1\mathbf{j} \times \mathbf{i} + u_2v_2\mathbf{j} \times \mathbf{j} \\ &\quad + u_2v_3\mathbf{j} \times \mathbf{k} + u_3v_1\mathbf{k} \times \mathbf{i} + u_3v_2\mathbf{k} \times \mathbf{j} + u_3v_3\mathbf{k} \times \mathbf{k} \\ &= \mathbf{0} + u_1v_2\mathbf{k} - u_1v_3\mathbf{j} - u_2v_1\mathbf{k} + \mathbf{0} + u_2v_3\mathbf{i} + u_3v_1\mathbf{j} - u_3v_2\mathbf{i} + \mathbf{0} \\ &= \end{aligned}$$

## 13. The Cross Product

$$\begin{aligned}\mathbf{i} \times \mathbf{j} &= \mathbf{k} \\ \mathbf{j} \times \mathbf{i} &= -\mathbf{k} \\ \mathbf{j} \times \mathbf{k} &= \mathbf{i} \\ \mathbf{k} \times \mathbf{j} &= -\mathbf{i} \\ \mathbf{i} \times \mathbf{k} &= -\mathbf{j} \\ \mathbf{k} \times \mathbf{i} &= \mathbf{j}\end{aligned}$$

Now suppose that  $\mathbf{u} = u_1\mathbf{i} + u_2\mathbf{j} + u_3\mathbf{k}$   
and  $\mathbf{v} = v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}$ . Then we can calculate  
that

$$\begin{aligned}\mathbf{u} \times \mathbf{v} &= (u_1\mathbf{i} + u_2\mathbf{j} + u_3\mathbf{k}) \times (v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}) \\ &= u_1v_1\mathbf{i} \times \mathbf{i} + u_1v_2\mathbf{i} \times \mathbf{j} + u_1v_3\mathbf{i} \times \mathbf{k} + u_2v_1\mathbf{j} \times \mathbf{i} + u_2v_2\mathbf{j} \times \mathbf{j} \\ &\quad + u_2v_3\mathbf{j} \times \mathbf{k} + u_3v_1\mathbf{k} \times \mathbf{i} + u_3v_2\mathbf{k} \times \mathbf{j} + u_3v_3\mathbf{k} \times \mathbf{k} \\ &= \mathbf{0} + u_1v_2\mathbf{k} - u_1v_3\mathbf{j} - u_2v_1\mathbf{k} + \mathbf{0} + u_2v_3\mathbf{i} + u_3v_1\mathbf{j} - u_3v_2\mathbf{i} + \mathbf{0} \\ &= (u_2v_3 - u_3v_2)\mathbf{i} - (u_1v_3 - u_3v_1)\mathbf{j} + (u_1v_2 - u_2v_1)\mathbf{k}.\end{aligned}$$

## 13. The Cross Product



### Theorem

If  $\mathbf{u} = u_1\mathbf{i} + u_2\mathbf{j} + u_3\mathbf{k}$  and  $\mathbf{v} = v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}$ , then

$$\mathbf{u} \times \mathbf{v} = (u_2v_3 - u_3v_2)\mathbf{i} - (u_1v_3 - u_3v_1)\mathbf{j} + (u_1v_2 - u_2v_1)\mathbf{k}.$$

## 13. The Cross Product



If you studied matrices and determinants at high school, then you may prefer to use the following symbolic determinant formula instead.

$$\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}.$$

$$\mathbf{u} \times \mathbf{v} = (u_2 v_3 - u_3 v_2) \mathbf{i} - (u_1 v_3 - u_3 v_1) \mathbf{j} + (u_1 v_2 - u_2 v_1) \mathbf{k}$$

## Example

Find  $\mathbf{u} \times \mathbf{v}$  and  $\mathbf{v} \times \mathbf{u}$  if  $\mathbf{u} = 2\mathbf{i} + \mathbf{j} + \mathbf{k}$  and  $\mathbf{v} = -4\mathbf{i} + 3\mathbf{j} + \mathbf{k}$ .

$$\mathbf{u} \times \mathbf{v} = (u_2 v_3 - u_3 v_2) \mathbf{i} - (u_1 v_3 - u_3 v_1) \mathbf{j} + (u_1 v_2 - u_2 v_1) \mathbf{k}$$

## Example

Find  $\mathbf{u} \times \mathbf{v}$  and  $\mathbf{v} \times \mathbf{u}$  if  $\mathbf{u} = 2\mathbf{i} + \mathbf{j} + \mathbf{k}$  and  $\mathbf{v} = -4\mathbf{i} + 3\mathbf{j} + \mathbf{k}$ .

$$\mathbf{u} \times \mathbf{v} = (1 - 3)\mathbf{i} - (2 - -4)\mathbf{j} + (6 - -4)\mathbf{k} = -2\mathbf{i} - 6\mathbf{j} + 10\mathbf{k}$$

$$\mathbf{u} \times \mathbf{v} = (u_2 v_3 - u_3 v_2) \mathbf{i} - (u_1 v_3 - u_3 v_1) \mathbf{j} + (u_1 v_2 - u_2 v_1) \mathbf{k}$$

## Example

Find  $\mathbf{u} \times \mathbf{v}$  and  $\mathbf{v} \times \mathbf{u}$  if  $\mathbf{u} = 2\mathbf{i} + \mathbf{j} + \mathbf{k}$  and  $\mathbf{v} = -4\mathbf{i} + 3\mathbf{j} + \mathbf{k}$ .

$$\mathbf{u} \times \mathbf{v} = (1 - 3)\mathbf{i} - (2 - -4)\mathbf{j} + (6 - -4)\mathbf{k} = -2\mathbf{i} - 6\mathbf{j} + 10\mathbf{k}$$

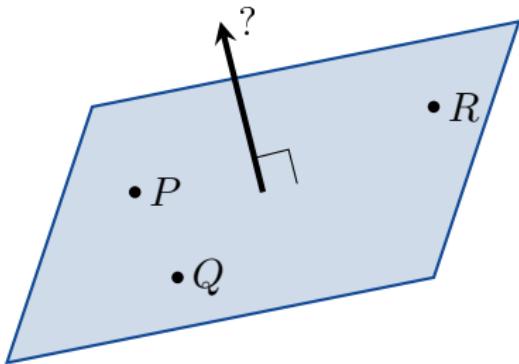
and

$$\mathbf{v} \times \mathbf{u} = -\mathbf{u} \times \mathbf{v} = 2\mathbf{i} + 6\mathbf{j} - 10\mathbf{k}.$$

$$\mathbf{u} \times \mathbf{v} = (u_2 v_3 - u_3 v_2) \mathbf{i} - (u_1 v_3 - u_3 v_1) \mathbf{j} + (u_1 v_2 - u_2 v_1) \mathbf{k}$$

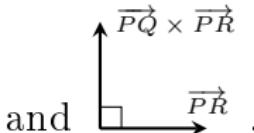
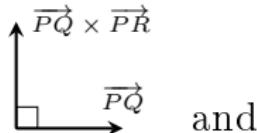
## Example

Find a vector perpendicular to the plane containing the three points  $P(1, -1, 0)$ ,  $Q(2, 1, -1)$  and  $R(-1, 1, 2)$ .



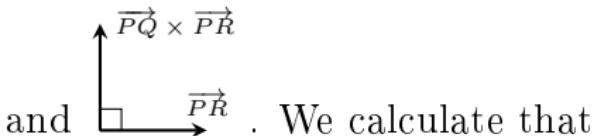
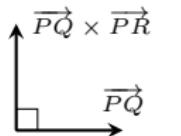
$$\mathbf{u} \times \mathbf{v} = (u_2 v_3 - u_3 v_2) \mathbf{i} - (u_1 v_3 - u_3 v_1) \mathbf{j} + (u_1 v_2 - u_2 v_1) \mathbf{k}$$

The vector  $\overrightarrow{PQ} \times \overrightarrow{PR}$  is perpendicular to the plane because



$$\mathbf{u} \times \mathbf{v} = (u_2 v_3 - u_3 v_2) \mathbf{i} - (u_1 v_3 - u_3 v_1) \mathbf{j} + (u_1 v_2 - u_2 v_1) \mathbf{k}$$

The vector  $\overrightarrow{PQ} \times \overrightarrow{PR}$  is perpendicular to the plane because



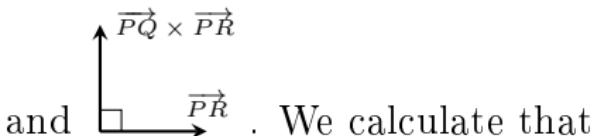
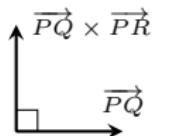
and . We calculate that

$$\begin{aligned}\overrightarrow{PQ} &= Q - P = (2, 1, -1) - (1, -1, 0) \\ &= (2 - 1, 1 + 1, -1 - 0) = \mathbf{i} + 2\mathbf{j} - \mathbf{k}\end{aligned}$$

$$\begin{aligned}\overrightarrow{PR} &= R - P = (-1, 1, 2) - (1, -1, 0) \\ &= (-1 - 1, 1 + 1, 2 - 0) = -2\mathbf{i} + 2\mathbf{j} + 2\mathbf{k}\end{aligned}$$

$$\mathbf{u} \times \mathbf{v} = (u_2 v_3 - u_3 v_2) \mathbf{i} - (u_1 v_3 - u_3 v_1) \mathbf{j} + (u_1 v_2 - u_2 v_1) \mathbf{k}$$

The vector  $\overrightarrow{PQ} \times \overrightarrow{PR}$  is perpendicular to the plane because



and . We calculate that

$$\begin{aligned}\overrightarrow{PQ} &= Q - P = (2, 1, -1) - (1, -1, 0) \\ &= (2 - 1, 1 + 1, -1 - 0) = \mathbf{i} + 2\mathbf{j} - \mathbf{k}\end{aligned}$$

$$\begin{aligned}\overrightarrow{PR} &= R - P = (-1, 1, 2) - (1, -1, 0) \\ &= (-1 - 1, 1 + 1, 2 - 0) = -2\mathbf{i} + 2\mathbf{j} + 2\mathbf{k}\end{aligned}$$

and

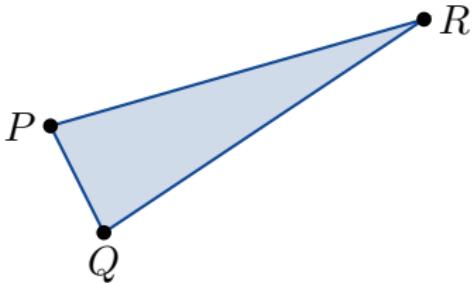
$$\overrightarrow{PQ} \times \overrightarrow{PR} = (4 + 2)\mathbf{i} - (2 - 2)\mathbf{j} + (2 + 4)\mathbf{k} = 6\mathbf{i} + 6\mathbf{k}.$$

## 13. The Cross Product

### Example

Find the area of triangle  $PQR$ .

$P(1, -1, 0)$ ,  $Q(2, 1, -1)$  and  $R(-1, 1, 2)$

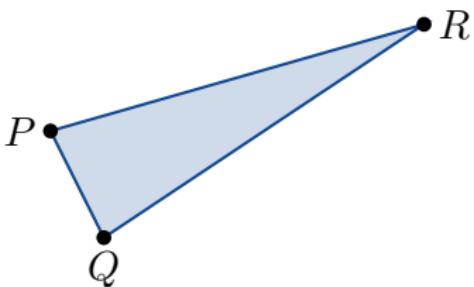


## 13. The Cross Product

### Example

Find the area of triangle  $PQR$ .

$P(1, -1, 0)$ ,  $Q(2, 1, -1)$  and  $R(-1, 1, 2)$



The area of the triangle is

$$\begin{aligned}\text{area} &= \frac{1}{2} \left\| \overrightarrow{PQ} \times \overrightarrow{PR} \right\| = \frac{1}{2} \|6\mathbf{i} + 6\mathbf{k}\| \\ &= \frac{1}{2} \sqrt{6^2 + 0^2 + 6^2} = 3\sqrt{2}.\end{aligned}$$

## 13. The Cross Product



### Example

Find a unit vector perpendicular to the plane containing  $P$ ,  $Q$  and  $R$ .

$P(1, -1, 0)$ ,  $Q(2, 1, -1)$  and  $R(-1, 1, 2)$

## 13. The Cross Product



### Example

Find a unit vector perpendicular to the plane containing  $P$ ,  $Q$  and  $R$ .

$P(1, -1, 0)$ ,  $Q(2, 1, -1)$  and  $R(-1, 1, 2)$

We know that  $\overrightarrow{PQ} \times \overrightarrow{PR}$  is perpendicular to the plane. We just need to normalise this vector to find a unit vector.

## 13. The Cross Product



### Example

Find a unit vector perpendicular to the plane containing  $P$ ,  $Q$  and  $R$ .

$P(1, -1, 0)$ ,  $Q(2, 1, -1)$  and  $R(-1, 1, 2)$

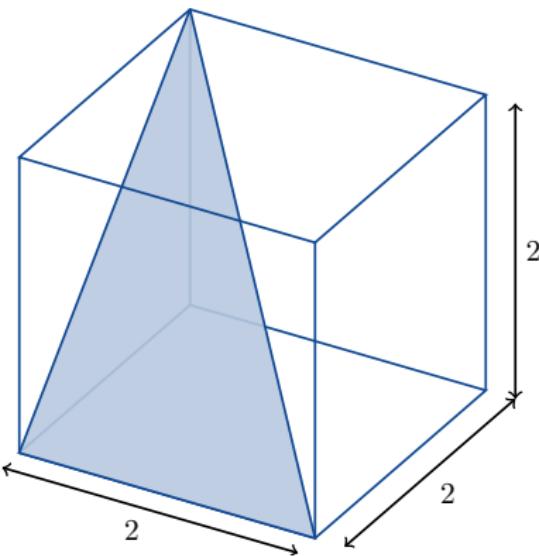
We know that  $\overrightarrow{PQ} \times \overrightarrow{PR}$  is perpendicular to the plane. We just need to normalise this vector to find a unit vector.

$$\mathbf{n} = \frac{\overrightarrow{PQ} \times \overrightarrow{PR}}{\|\overrightarrow{PQ} \times \overrightarrow{PR}\|} = \frac{6\mathbf{i} + 6\mathbf{k}}{6\sqrt{2}} = \frac{1}{\sqrt{2}}\mathbf{i} + \frac{1}{\sqrt{2}}\mathbf{k}.$$

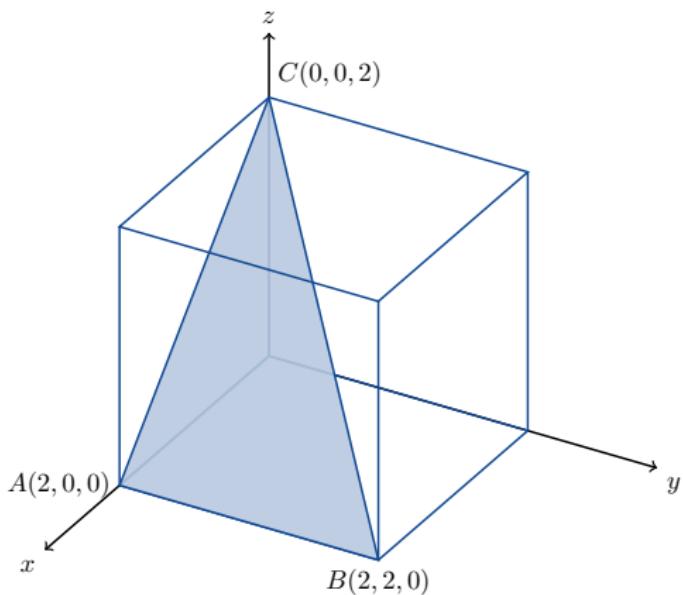
## 13. The Cross Product

### Example

A triangle is inscribed inside a cube of side 2 as shown below.  
Use the cross product to find the area of the triangle.



## 13. The Cross Product



First we draw coordinate axes and assign coordinates to the vertices of the triangle.

## 13. The Cross Product

Then we can calculate

$$\overrightarrow{AB} = B - A = (2, 2, 0) - (2, 0, 0) = (0, 2, 0) = 2\mathbf{j}$$

and

$$\overrightarrow{AC} = C - A = (0, 0, 2) - (2, 0, 0) = (-2, 0, 2) = -2\mathbf{i} + 2\mathbf{k}.$$

## 13. The Cross Product

Then we can calculate

$$\overrightarrow{AB} = B - A = (2, 2, 0) - (2, 0, 0) = (0, 2, 0) = 2\mathbf{j}$$

and

$$\overrightarrow{AC} = C - A = (0, 0, 2) - (2, 0, 0) = (-2, 0, 2) = -2\mathbf{i} + 2\mathbf{k}.$$

It follows that

$$\begin{aligned}\overrightarrow{AB} \times \overrightarrow{AC} &= (2\mathbf{j}) \times (-2\mathbf{i} + 2\mathbf{k}) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 2 & 0 \\ -2 & 0 & 2 \end{vmatrix} \\ &= \mathbf{i}(4 - 0) - \mathbf{j}(0 - 0) + \mathbf{k}(0 - -4) = 4\mathbf{i} + 4\mathbf{k}.\end{aligned}$$

## 13. The Cross Product

Then we can calculate

$$\overrightarrow{AB} = B - A = (2, 2, 0) - (2, 0, 0) = (0, 2, 0) = 2\mathbf{j}$$

and

$$\overrightarrow{AC} = C - A = (0, 0, 2) - (2, 0, 0) = (-2, 0, 2) = -2\mathbf{i} + 2\mathbf{k}.$$

It follows that

$$\begin{aligned}\overrightarrow{AB} \times \overrightarrow{AC} &= (2\mathbf{j}) \times (-2\mathbf{i} + 2\mathbf{k}) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 2 & 0 \\ -2 & 0 & 2 \end{vmatrix} \\ &= \mathbf{i}(4 - 0) - \mathbf{j}(0 - 0) + \mathbf{k}(0 - -4) = 4\mathbf{i} + 4\mathbf{k}.\end{aligned}$$

Therefore

$$\begin{aligned}\text{area of triangle} &= \frac{1}{2} \left\| \overrightarrow{AB} \times \overrightarrow{AC} \right\| = \frac{1}{2} \sqrt{4^2 + 0^2 + 4^2} \\ &= \frac{1}{2} \sqrt{32} = \frac{1}{2} \sqrt{4} \sqrt{8} = \sqrt{8} = 2\sqrt{2}.\end{aligned}$$

## 13. The Cross Product



### The Triple Scalar Product

#### Definition

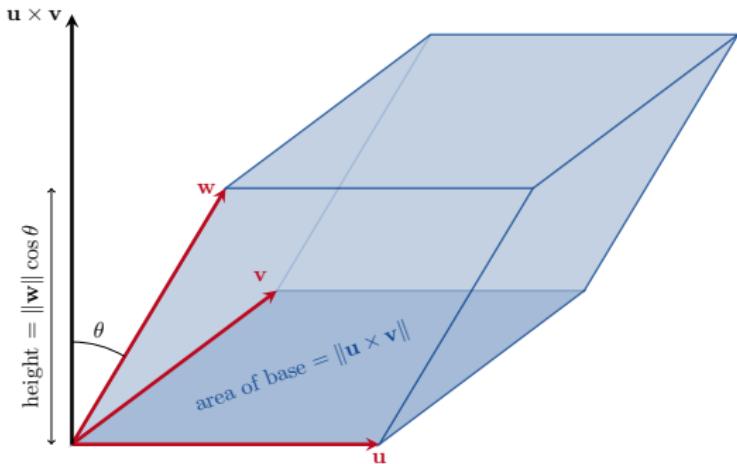
The *triple scalar product* of  $\mathbf{u}$ ,  $\mathbf{v}$  and  $\mathbf{w}$  is

$$(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w}.$$

# 13. The Cross Product



## The Volume of a Parallelepiped



$$\text{volume} = (\text{area of base})(\text{height}) = \|\mathbf{u} \times \mathbf{v}\| \|\mathbf{w}\| \cos \theta = |(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w}|$$

## 13. The Cross Product



### One Final Comment

We can do the dot product in both  $\mathbb{R}^2$  and  $\mathbb{R}^3$ . But we can only do the cross product in  $\mathbb{R}^3$ . There is no cross product in  $\mathbb{R}^2$ .



# Next Time

- 14. Lines
- 15. Planes
- 16. Projections