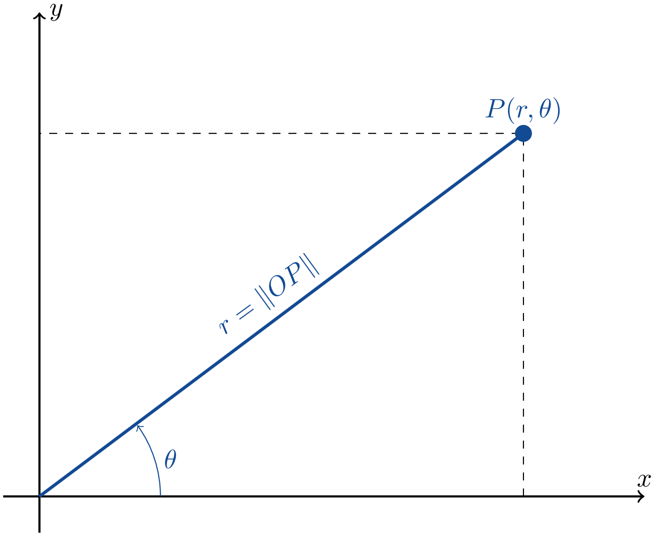
Lecture 3

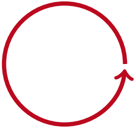
- 8. Polar Coordinates
- 9. Conic Sections
- 10. Three Dimensional Cartesian Coordinates

8 Polar Coordinates

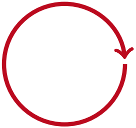


y
$P(r, \theta)$
$r = \|OP\|$
θ
x

8. Polar Coordinates

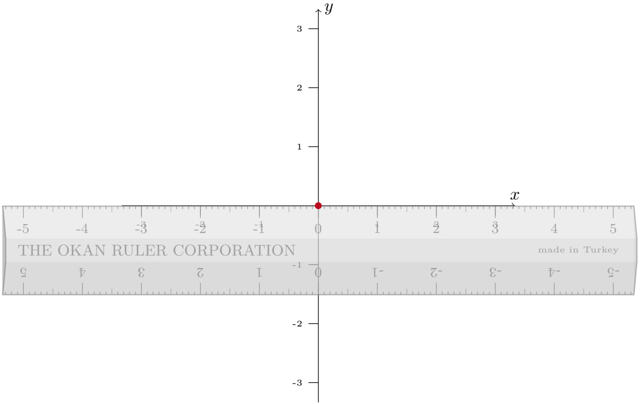


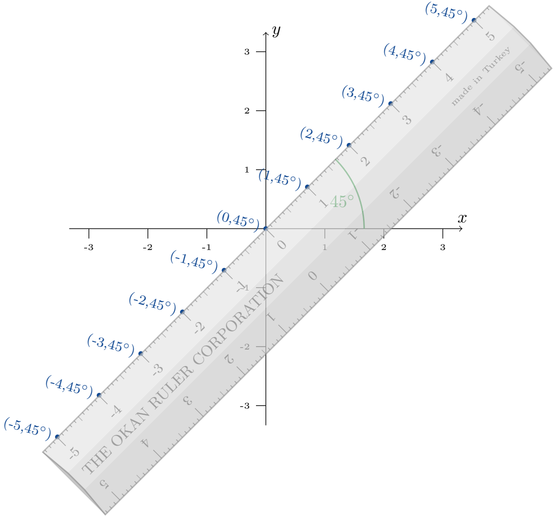
anticlockwise = positive angle
saat yönünün tersi = pozitif açı



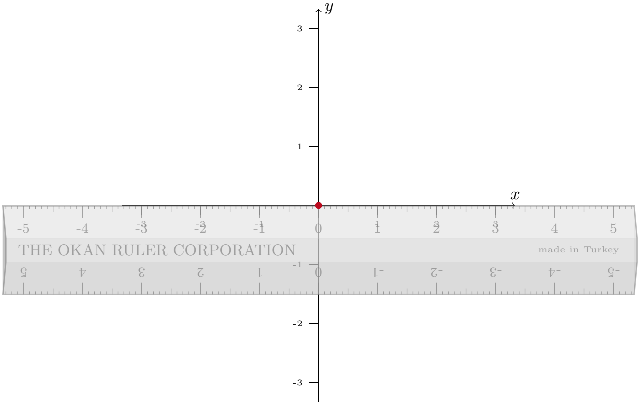
clockwise = negative angle
saat yönünde = negatif açı

8. Polar Coordinates

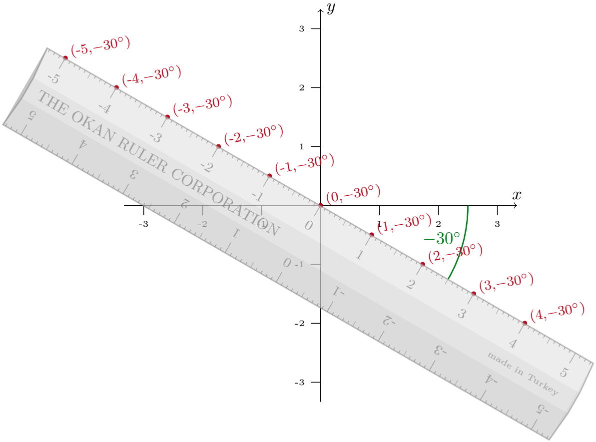




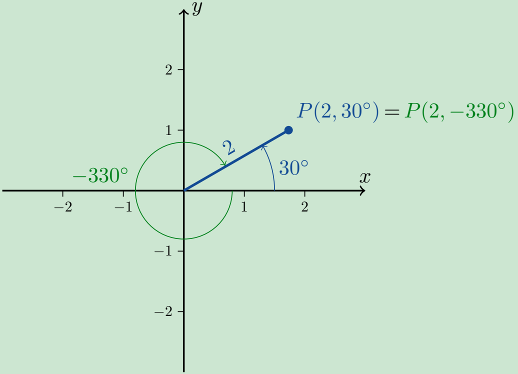
(5,45°)
(4,45°)
(3,45°)
(2,45°)
(1,45°)
(0,45°)
(-1,45°)
(-2,45°)
(-3,45°)
(-4,45°)
(-5,45°)
45°
x
y
THE OKAN RULER CORPORATION
made in Turkey


y
3
2
1
x
-1
-2
-3
-5 -4 -3 -2 -1 0 1 2 3 4 5
THE OKAN RULER CORPORATION
made in Turkey

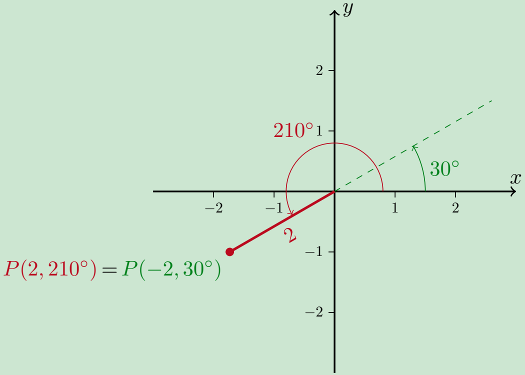
8. Polar Coordinates



Example



Example



Example

Find all the polar coordinates of $P(2, 30^\circ)$.

solution: We can have either $r = 2$ or $r = -2$. For $r = 2$, we can have

$$\theta = 30^\circ, 30^\circ \pm 360^\circ, 30^\circ \pm 720^\circ, 30^\circ \pm 1080^\circ, \ldots$$

For $r = -2$, we can have

$$\theta = 210^\circ, 210^\circ \pm 360^\circ, 210^\circ \pm 720^\circ, 210^\circ \pm 1080^\circ, \ldots$$

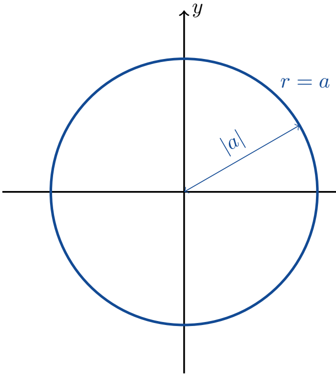
Therefore

$$P(2, 30^\circ) = P\big(2, (30 + 360n)^\circ\big) = P\big(-2, (210 + 360m)^\circ\big)$$

for all $m, n \in \mathbb{Z}$.

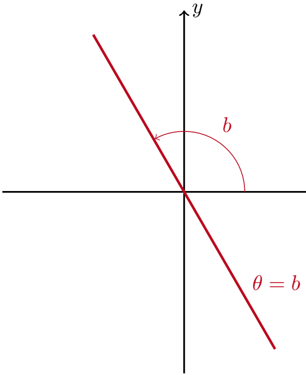
Example

Draw the graph of $r = a$.



Example

Draw the graph of $\theta = b$.



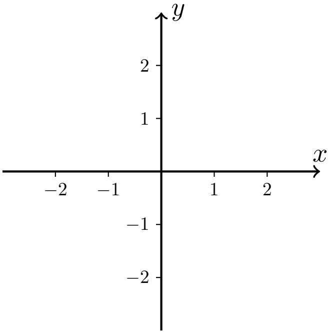
Example

1. $r = 1$ and $r = -1$ are both equations for a circle of radius 1 centred at the origin.
2. $\theta = 30°$, $\theta = 210°$ and $\theta = -150°$ are all equations for the same line.

Example

Draw the sets of points whose polar coordinates satisfy the following: $1 \leq r \leq 2$ and $0 \leq \theta \leq 90°$.

solution:

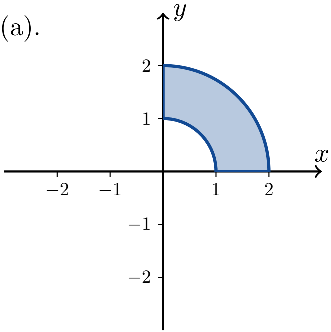


Example

Draw the sets of points whose polar coordinates satisfy the following: $1 \le r \le 2$ and $0 \le \theta \le 90°$.

solution:

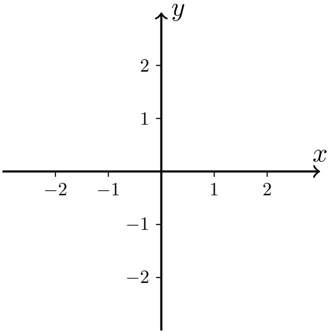
(a).



Example

Draw the sets of points whose polar coordinates satisfy the following: $-3 \leq r \leq 2$ and $\theta = 45°$.

solution:

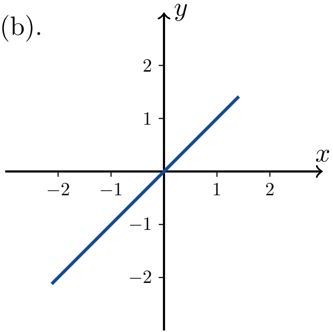


Example

Draw the sets of points whose polar coordinates satisfy the following: $-3 \le r \le 2$ and $\theta = 45°$.

solution:

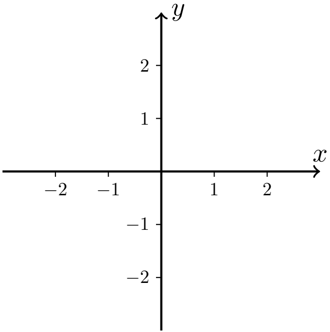
(b).



Example

Draw the sets of points whose polar coordinates satisfy the following: $r \leq 0$ and $\theta = 60°$.

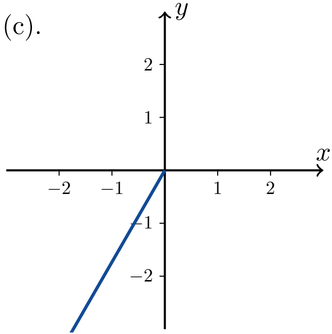
solution:



Example

Draw the sets of points whose polar coordinates satisfy the following: $r \leq 0$ and $\theta = 60°$.

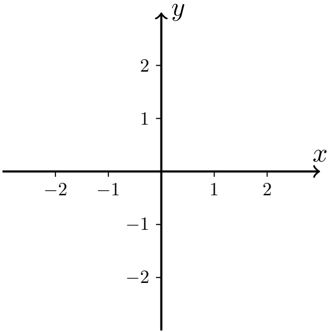
solution:



Example

Draw the sets of points whose polar coordinates satisfy the following: $120° \leq \theta \leq 150°$.

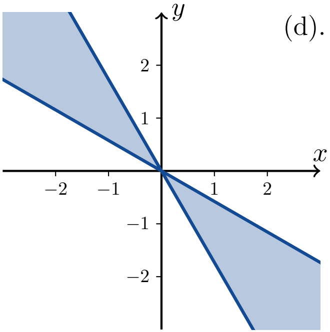
solution:



Example

Draw the sets of points whose polar coordinates satisfy the following: $120^\circ \leq \theta \leq 150^\circ$.

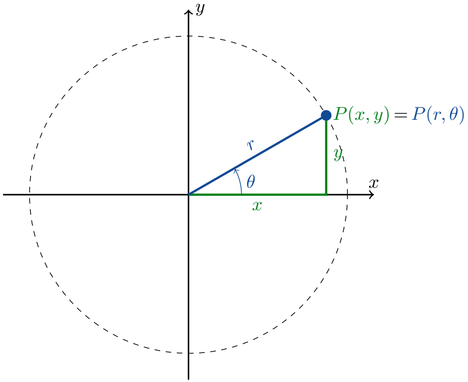
solution:



Relating Polar and Cartesian Coordinates

$$x = r\cos\theta \qquad x^2 + y^2 = r^2$$

$$y = r\sin\theta \qquad \tan\theta = \frac{y}{x}$$



$$x = r\cos\theta \qquad y = r\sin\theta \qquad x^2 + y^2 = r^2 \qquad \tan\theta = \frac{y}{x}$$

Example

Convert the polar coordinates $(r, \theta) = (-3, 90°)$ into Cartesian coordinates.

solution:

$$(x, y) = (r\cos\theta, r\sin\theta) = (-3\cos 90°, -3\sin 90°) = (0, -3).$$

Example

Find polar coordinates for the Cartesian coordinates $(x, y) = (5, -12)$.

solution: Choosing $r > 0$, we calculate that

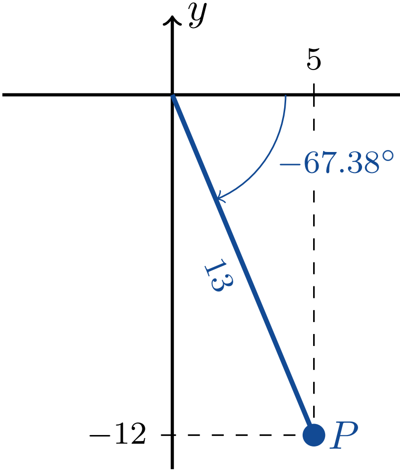
$$r = \sqrt{x^2 + y^2} = \sqrt{25 + 144} = \sqrt{169} = 13.$$

To find θ we use the equation $y = r \sin \theta$ to calculate that

$$\theta = \sin^{-1} \frac{y}{r} = \sin^{-1} \frac{-12}{13} \approx -67.38^\circ.$$

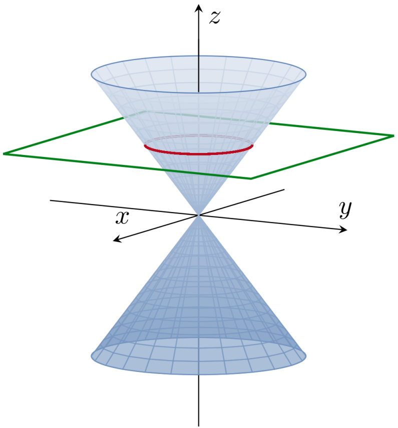
Therefore

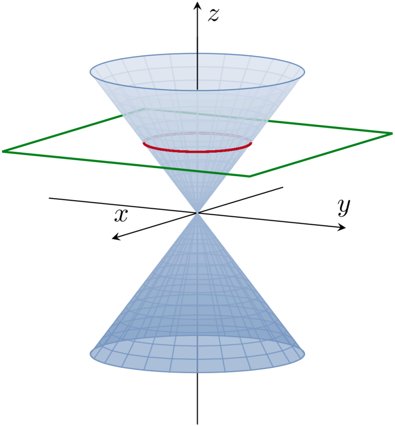
$$(r, \theta) = (13, -67.38^\circ).$$

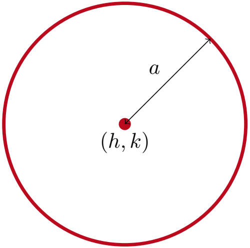


y
5
x
$-67.38°$
13
-12
P

9 Conic Sections

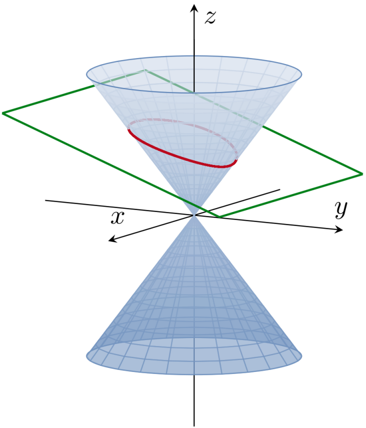

z
x
y

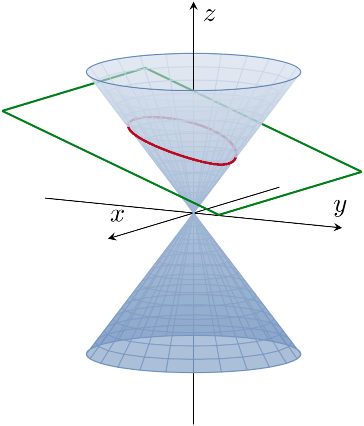


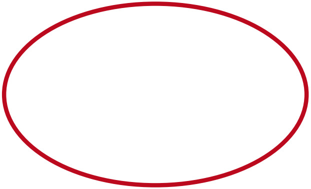


a circle

$$(x-h)^2 + (y-k)^2 = a^2$$

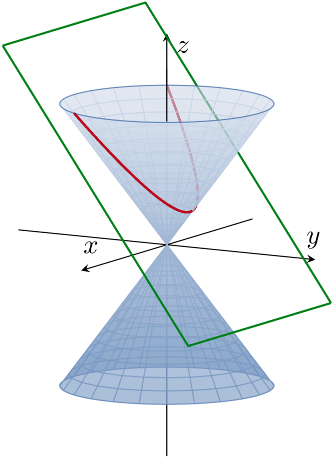

z
x
y

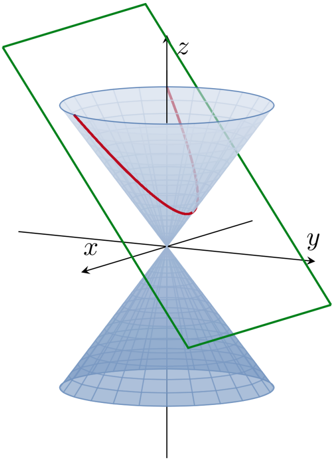


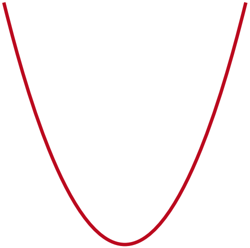


an ellipse

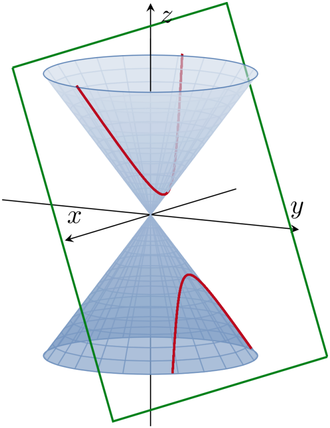
9. Conic Sections



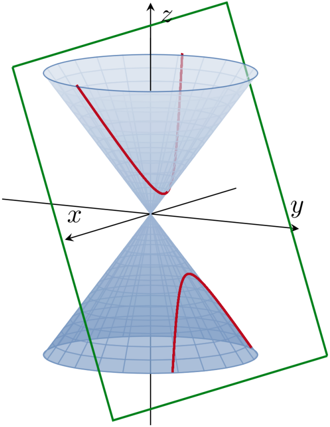


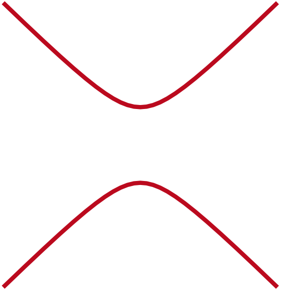


a parabola



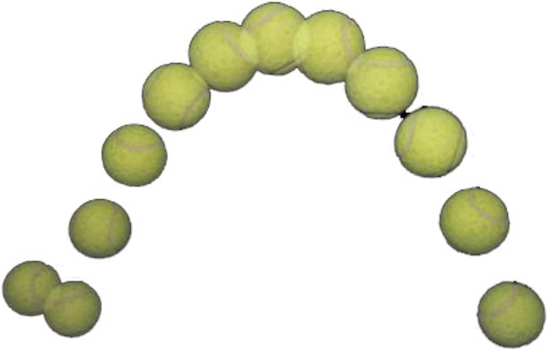
z
x
y





a hyperbola

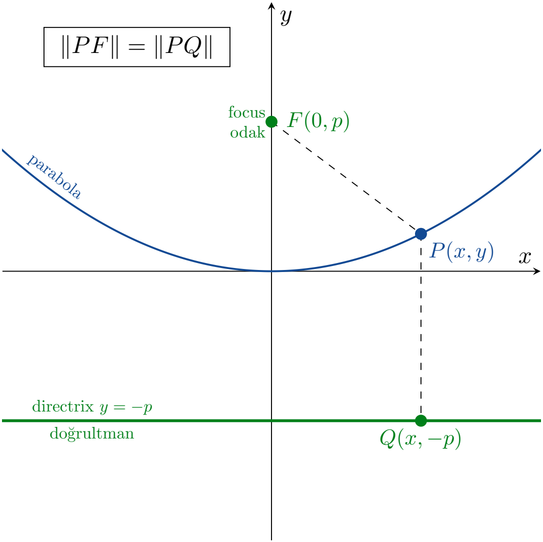
Parabolas



Clifton suspension bridge, Bristol, UK.
The cables of a suspension bridges hang in a shape which is almost (but not exactly) a parabola.


VESTEL

To describe a parabola, we need a point called a *focus* and a line called a *directrix*.


$\|PF\| = \|PQ\|$
y
focus
odak
$F(0, p)$
parabola
$P(x, y)$
x
directrix $y = -p$
doğrultman
$Q(x, -p)$

Definition

A point $P(x,y)$ lies on the *parabola* if and only if

$$\|PF\| = \|PQ\|.$$

Now

$$\begin{aligned} \|PF\| &= \text{distance between } P(x,y) \text{ and } F(0,p) \\ &= \sqrt{(x-0)^2 + (y-p)^2} = \sqrt{x^2 + (y-p)^2} \end{aligned}$$

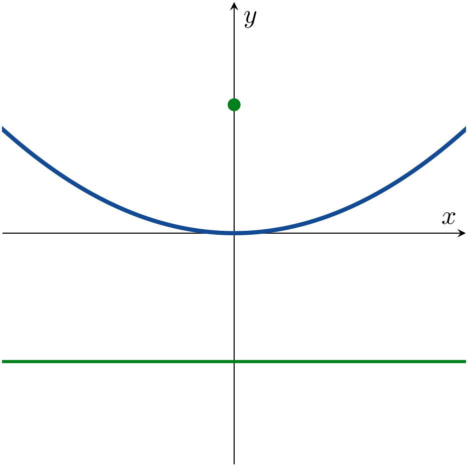
and

$$\begin{aligned} \|PQ\| &= \text{distance between } P(x,y) \text{ and } Q(x,-p) \\ &= \sqrt{(x-x)^2 + (y+p)^2} = \sqrt{(y+p)^2} = y+p. \end{aligned}$$

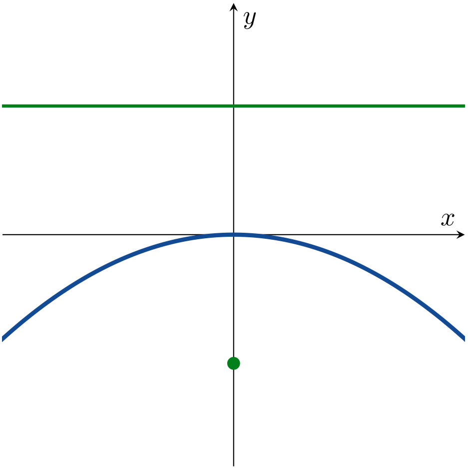
Therefore

$$\begin{aligned} \|PF\| &= \|PQ\| \\ \sqrt{x^2 + (y-p)^2} &= y+p \\ x^2 + (y-p)^2 &= (y+p)^2 \\ x^2 + y^2 - 2py + p^2 &= y^2 + 2py + p^2 \\ x^2 - 2py &= 2py \\ \boxed{x^2 = 4py} \end{aligned}$$

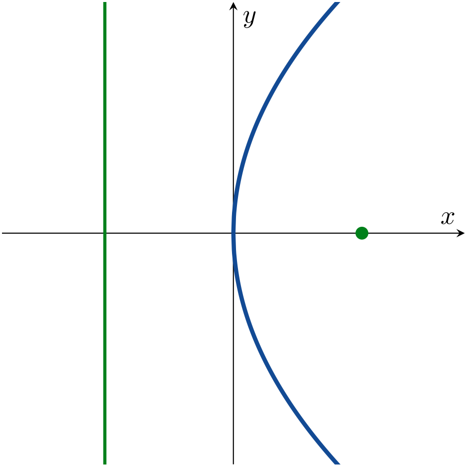
equation: $x^2 = 4py$
focus: $F(0, p)$
directrix: $y = -p$



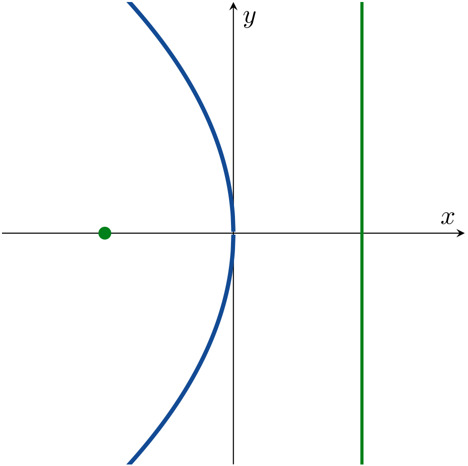
equation: $x^2 = -4py$
focus: $F(0, -p)$
directrix: $y = p$



equation: $y^2 = 4px$
focus: $F(p, 0)$
directrix: $x = -p$



equation: $y^2 = -4px$
focus: $F(-p, 0)$
directrix: $x = p$



Example

Find the focus and directrix of the parabola $y^2 = 10x$.

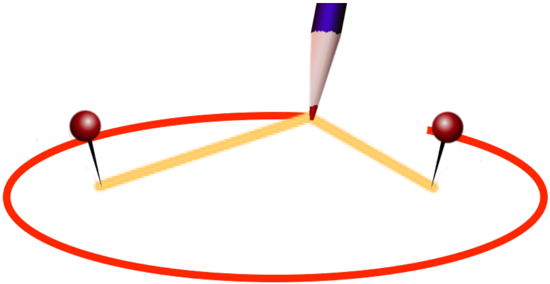
solution: Our equation $y^2 = 10x$ looks like $y^2 = 4px$ with $p = \frac{10}{4} = 2.5$. Therefore the focus is at the point $F(2.5, 0)$ and the directrix is the line $x = -2.5$.

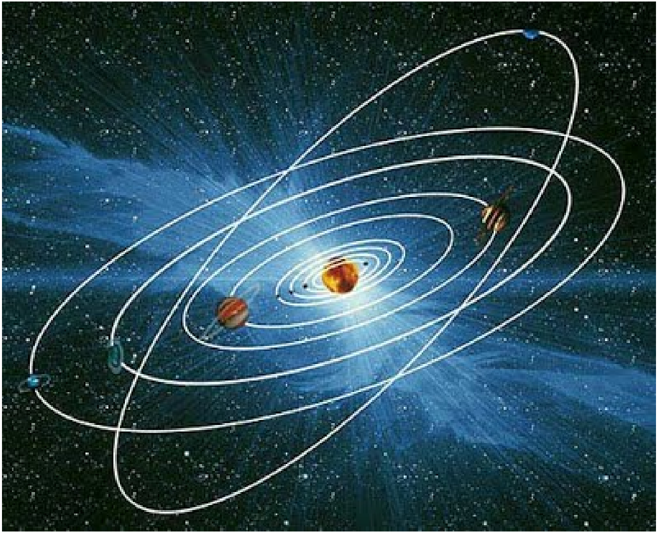
Example

Find the equation for the parabola which has focus $F(0,-10)$ and directrix $y=10$.

solution: Clearly $p=10$ and $x^2=-4py$. Therefore the answer is $x^2=-40y$.

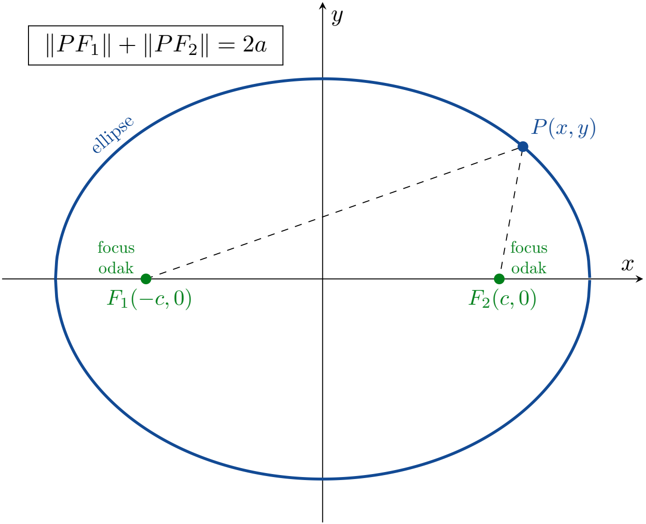
Ellipses





Tycho Brahe Planetarium, Copenhagen, Denmark.

$$\|PF_1\| + \|PF_2\| = 2a$$



To describe an ellipse, we need two *foci*.

Definition

A point $P(x, y)$ is on the *ellipse* if and only if

$$\|PF_1\| + \|PF_2\| = 2a.$$

So

$$\sqrt{(x+c)^2 + y^2} + \sqrt{(x-c)^2 + y^2} = 2a.$$

This rearranges to

$$\frac{x^2}{a^2} + \frac{y^2}{a^2 - c^2} = 1.$$

If we set $b = \sqrt{a^2 - c^2}$, then we have

$$\boxed{\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1} \qquad (0 < b < a).$$

equation:

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \quad (0 < b < a)$$

centre-to-focus distance:

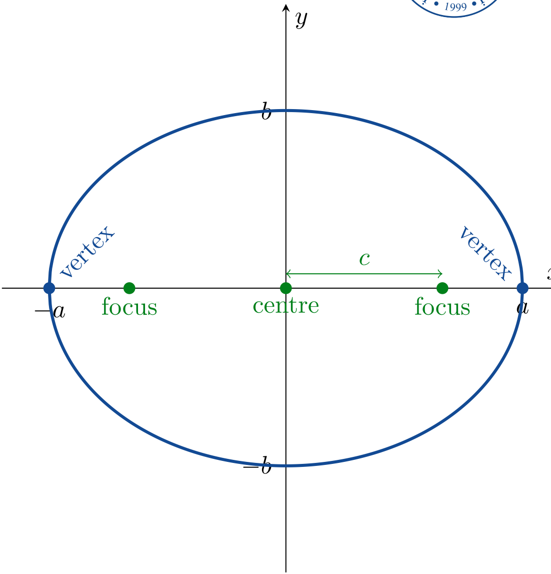
$c = \sqrt{a^2 - b^2}$

foci:

$F_1(-c, 0)$ & $F_2(c, 0)$

vertices :

$(-a, 0)$ & $(a, 0)$



equation:

$$\frac{x^2}{b^2} + \frac{y^2}{a^2} = 1 \quad (0 < b < a)$$

centre-to-focus distance:

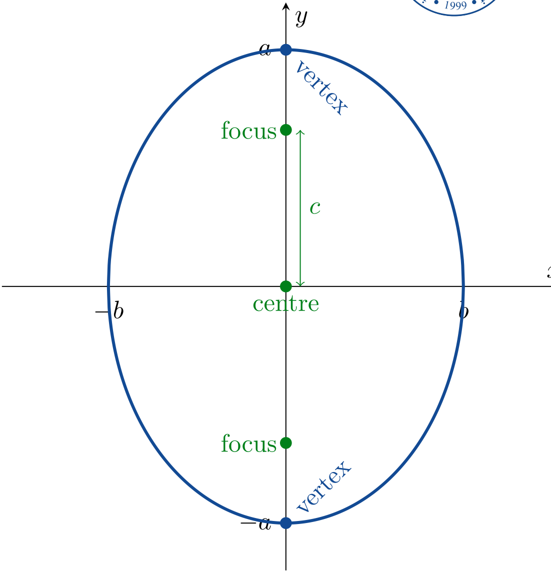
$c = \sqrt{a^2 - b^2}$

foci:

$F_1(0, -c) \;\; \& \;\; F_2(0, c)$

vertices :

$(0, -a) \;\; \& \;\; (0, a)$



Example

The ellipse $\dfrac{x^2}{16} + \dfrac{y^2}{9} = 1$ has

- $a = 4$ and $b = 3$;
- centre-to-focus distance $c = \sqrt{a^2 - b^2} = \sqrt{16 - 9} = \sqrt{7}$;
- centre at $(0, 0)$;
- foci at $(-\sqrt{7}, 0)$ and $(\sqrt{7}, 0)$; and
- vertices at $(-4, 0)$ and $(4, 0)$.

Example

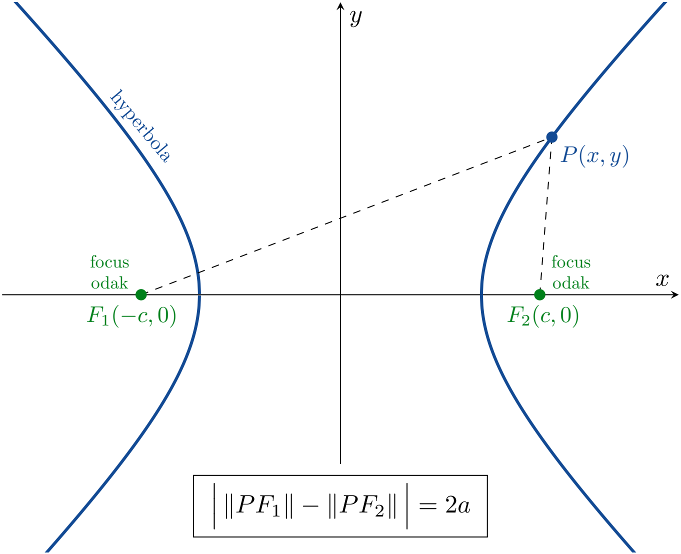
The ellipse $\dfrac{x^2}{16} + \dfrac{y^2}{25} = 1$ has

- $a = 5$ and $b = 4$;
- centre-to-focus distance $c = \sqrt{a^2 - b^2} = \sqrt{25 - 16} = \sqrt{9} = 3$;
- centre at $(0, 0)$;
- foci at $(0, -3)$ and $(0, 3)$; and
- vertices at $(0, -5)$ and $(0, 5)$.

Hyperbolas

9. Conic Sections

Twin Arch 138, Ichinomiya City, Japan.



$$\boxed{\Big| \|PF_1\| - \|PF_2\| \Big| = 2a}$$

To describe a hyperbola, we again need two foci.

Definition

A point $P(x, y)$ is on the *hyperbola* if and only if

$$\Big| \|PF_1\| - \|PF_2\| \Big| = 2a.$$

So

$$\sqrt{(x+c)^2+y^2}-\sqrt{(x-c)^2+y^2}=\pm 2a.$$

This rearranges to

$$\frac{x^2}{a^2}+\frac{y^2}{a^2-c^2}=1$$

where $c>a>0$. If we set $b=\sqrt{c^2-a^2}$, then

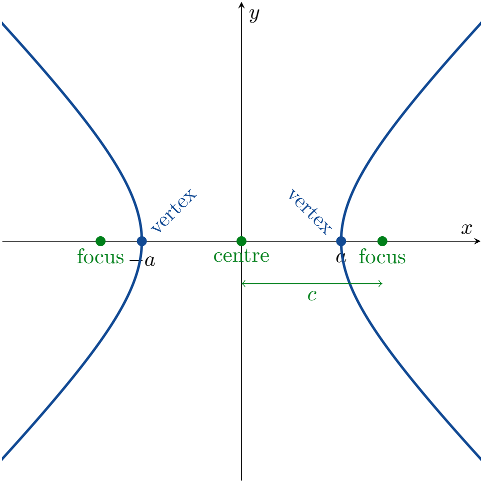
$$\boxed{\frac{x^2}{a^2}-\frac{y^2}{b^2}=1.}$$

equation:
$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

centre-to-focus distance:
$c = \sqrt{a^2 + b^2}$

foci:
$F_1(-c, 0)$ & $F_2(c, 0)$

vertices :
$(-a, 0)$ & $(a, 0)$

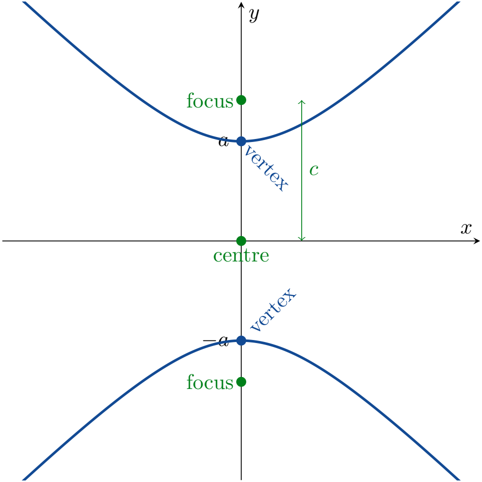


equation:
$$\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$$

centre-to-focus distance:
$$c = \sqrt{a^2 + b^2}$$

foci:
$$F_1(0,-c) \;\;\&\;\; F_2(0,c)$$

vertices :
$$(0,-a) \;\;\&\;\; (0,a)$$



Example

The hyperbola $\dfrac{x^2}{4} - \dfrac{y^2}{5} = 1$ has

- $a = 2$ and $b = \sqrt{5}$;
- centre at $(0, 0)$;
- centre-to-focus distance $c = \sqrt{a^2 + b^2} = \sqrt{4 + 5} = 3$;
- foci at $(-3, 0)$ and $(3, 0)$; and
- vertices at $(-2, 0)$ and $(2, 0)$.

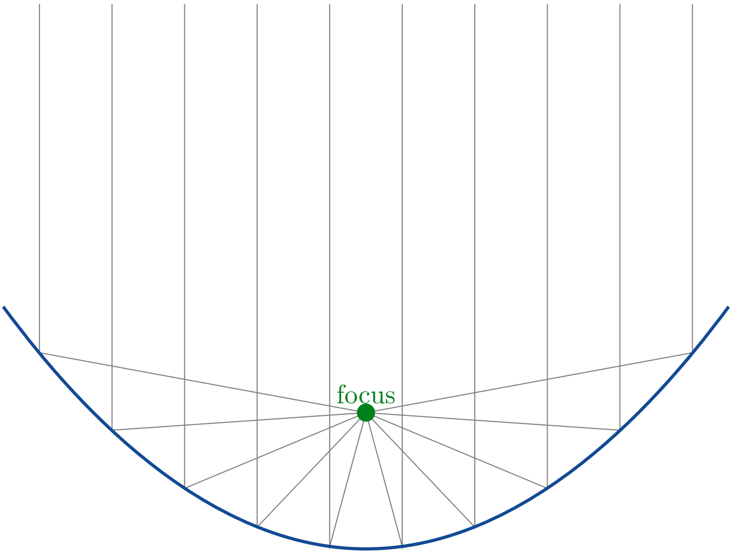
Example

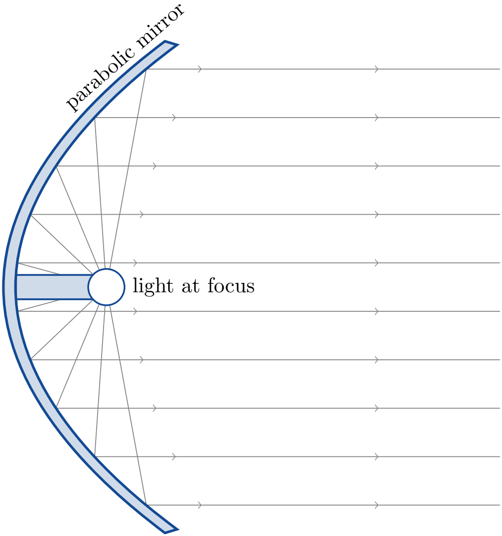
The hyperbola $\dfrac{y^2}{9} - \dfrac{x^2}{16} = 1$ has

- $a = 3$ and $b = 4$;
- centre at $(0, 0)$;
- centre-to-focus distance $c = \sqrt{a^2 + b^2} = \sqrt{9 + 16} = 5$;
- foci at $(0, -5)$ and $(0, 5)$; and
- vertices at $(0, -3)$ and $(0, 3)$.

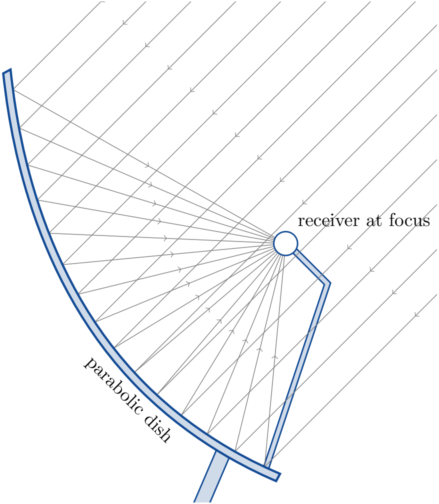
Reflective Properties

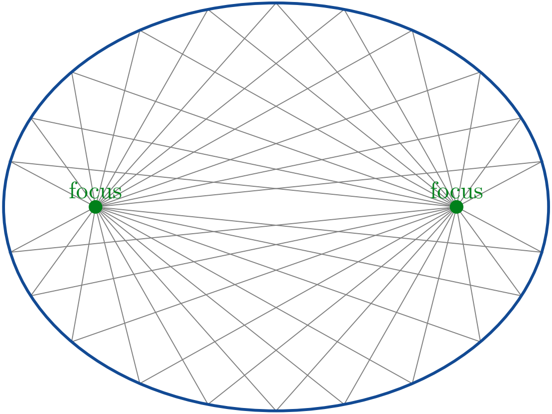
Parabolas, ellipses and hyperbolas are useful in architecture and engineering because of their reflective properties.

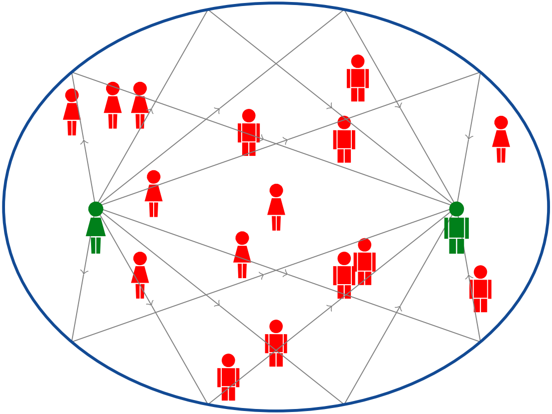

focus

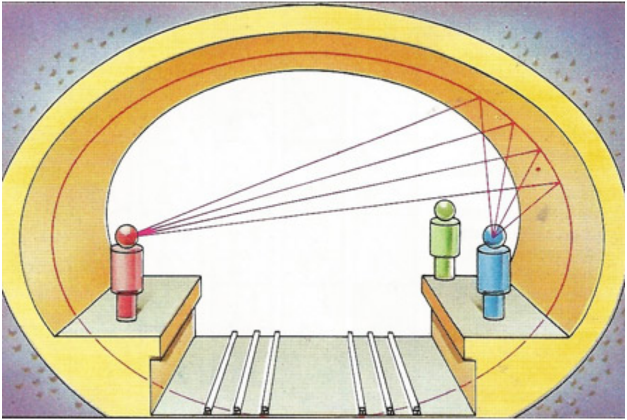

parabolic mirror
light at focus

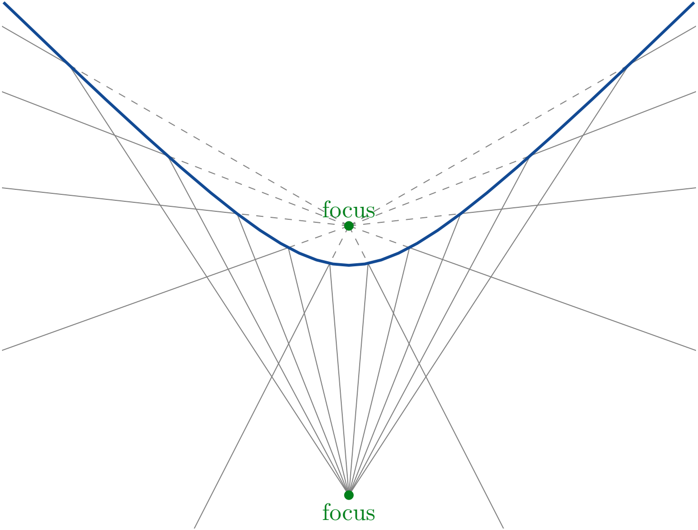

Digiturk

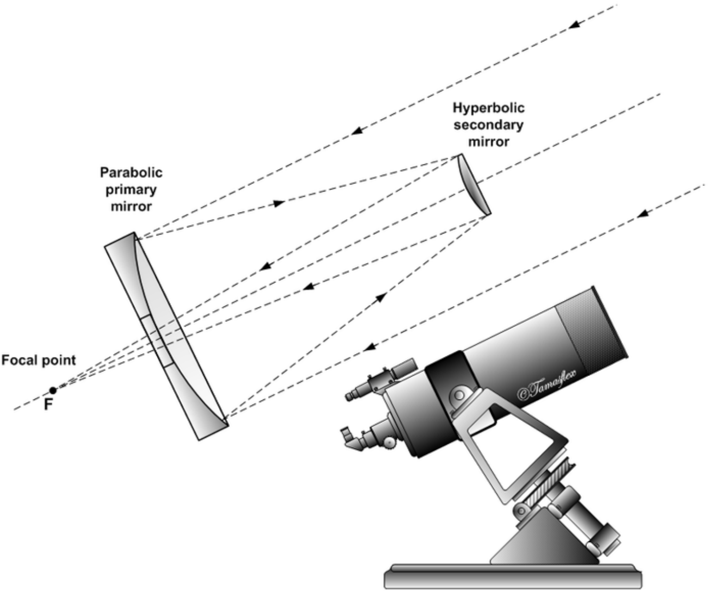

receiver at focus
parabolic dish


focus
focus

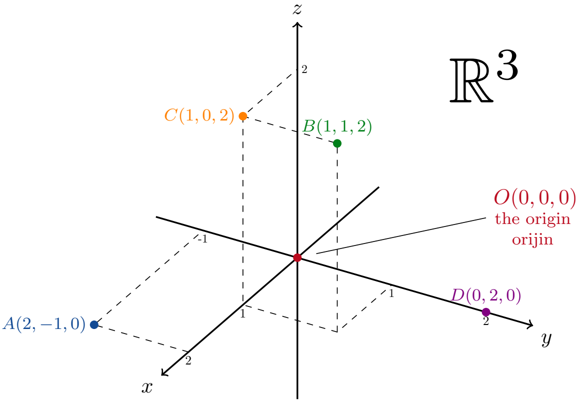


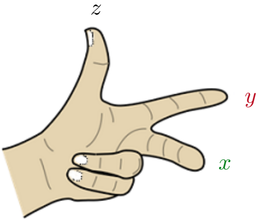


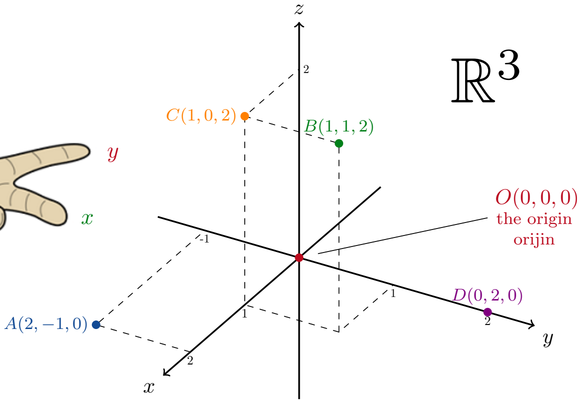

focus
focus


Hyperbolic
secondary
mirror
Parabolic
primary
mirror
Focal point
F
©Tamaiflex

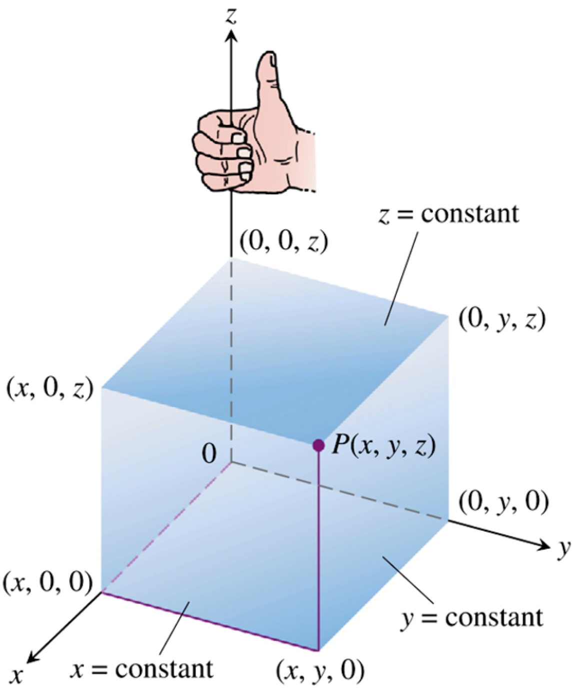
Three Dimensional Cartesian Coordinates

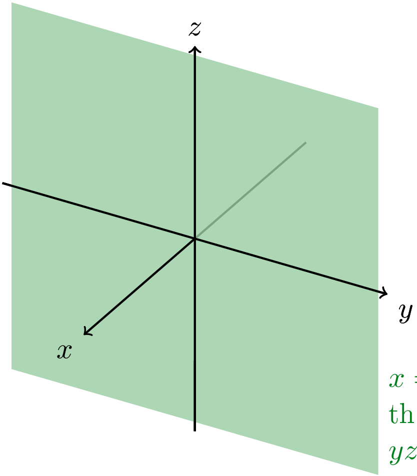

z
2
$C(1,0,2)$
$B(1,1,2)$
$\mathbb{R}^3$
$O(0,0,0)$
the origin
orijin
-1
1
$A(2,-1,0)$
$D(0,2,0)$
1
2
y
2
x





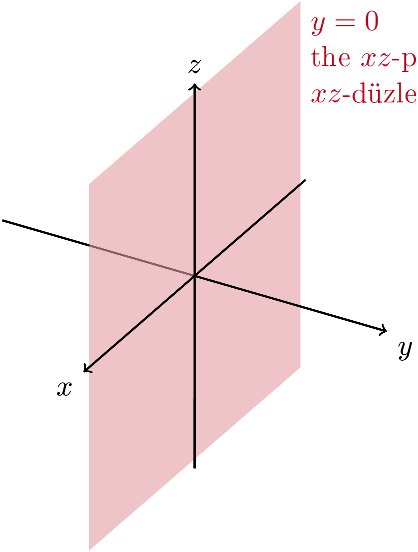
$$\mathbb{R}^3$$


z
(0, 0, z)
z = constant
(0, y, z)
(x, 0, z)
P(x, y, z)
0
(0, y, 0)
y
(x, 0, 0)
y = constant
x
x = constant
(x, y, 0)



$x = 0$
the yz-plane
yz-düzlemi

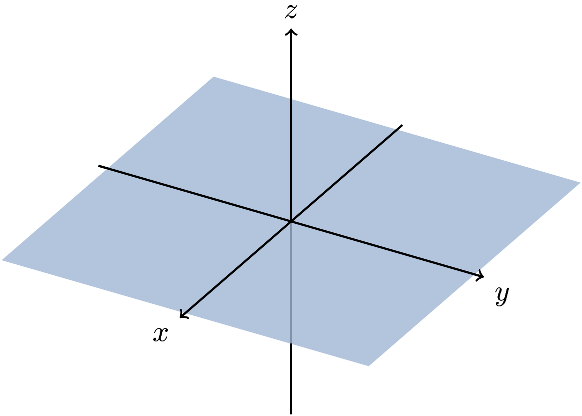
10. Three Dimensional Cartesian Coordinates



$y = 0$

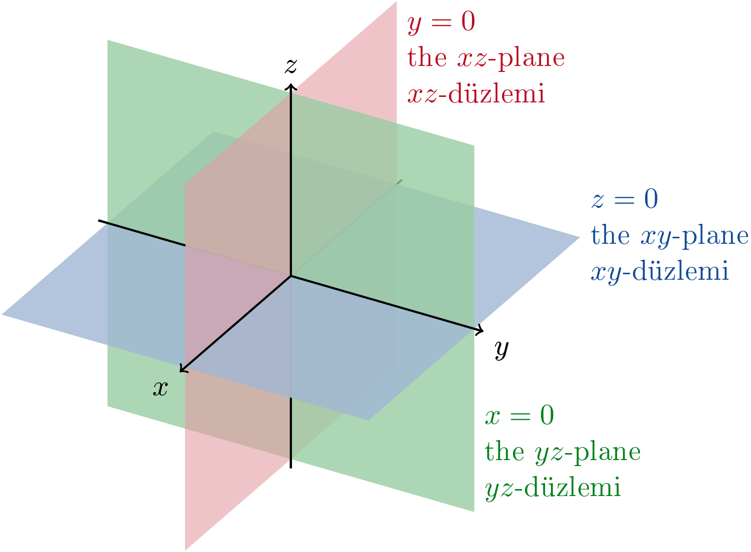
the xz-plane

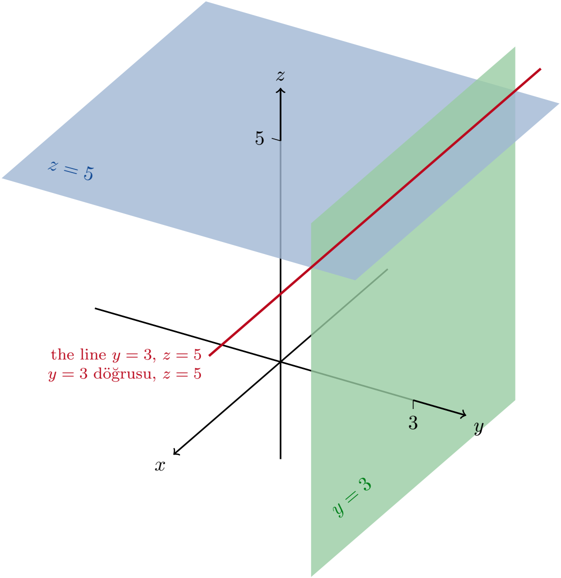
xz-düzlemi



$z = 0$
the xy-plane
xy-düzlemi

10. Three Dimensional Cartesian Coordinates




z
5
$z = 5$
the line $y = 3$, $z = 5$
$y = 3$ döğrusu, $z = 5$
3
y
x
$y = 3$

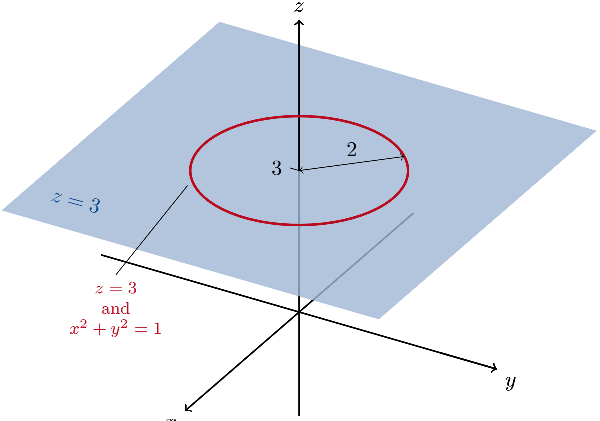
EXAMPLE 1 We interpret these equations and inequalities geometrically.

(a) $z \geq 0$	The half-space consisting of the points on and above the xy-plane.
(b) $x = -3$	The plane perpendicular to the x-axis at $x = -3$. This plane lies parallel to the yz-plane and 3 units behind it.
(c) $z = 0, x \leq 0, y \geq 0$	The second quadrant of the xy-plane.
(d) $x \geq 0, y \geq 0, z \geq 0$	The first octant.
(e) $-1 \leq y \leq 1$	The slab between the planes $y = -1$ and $y = 1$ (planes included).
(f) $y = -2, z = 2$	The line in which the planes $y = -2$ and $z = 2$ intersect. Alternatively, the line through the point $(0, -2, 2)$ parallel to the x-axis. ■

Example

Which points $P(x, y, z)$ satisfy $x^2 + y^2 = 4$ and $z = 3$?

We know that $z = 3$ is a horizontal plane and we recognise that $x^2 + y^2 = 4$ is the equation of a circle of radius 2.



Distance in $\mathbb{R}^3$

Definition

The set

$$\{(x, y, z) \mid x, y, z \in \mathbb{R}\}$$

is denoted by $\mathbb{R}^3$.

Definition

The *distance* between $P_1(x_1, y_1, z_1)$ and $P_2(x_2, y_2, z_2)$ is

$$\|P_1P_2\| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}.$$

10.

$$\|P_1P_2\| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}.$$

Example

The distance between $A(2, 1, 5)$ and $B(-2, 3, 0)$ is

10.

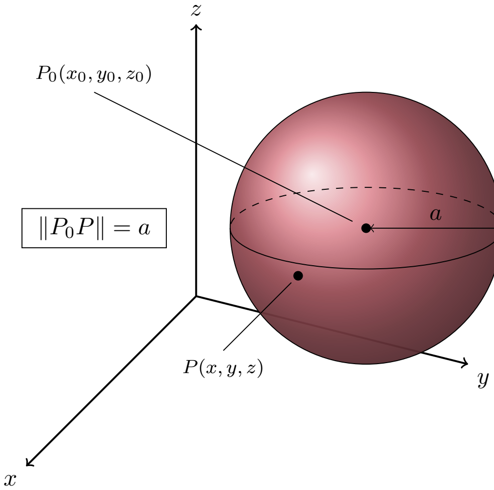
$$\|P_1P_2\| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}.$$

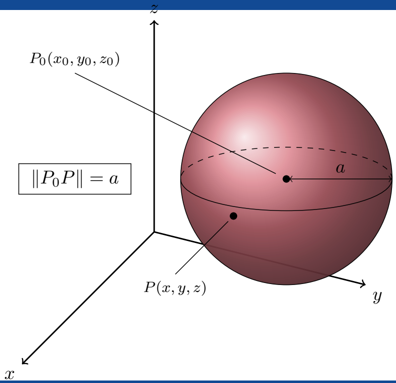
Example

The distance between $A(2, 1, 5)$ and $B(-2, 3, 0)$ is

$$\begin{aligned} \|AB\| &= \sqrt{((-2) - 2)^2 + (3 - 1)^2 + (0 - 5)^2} \\ &= \sqrt{16 + 4 + 25} = \sqrt{45} \\ &= 3\sqrt{5} \approx 6.7. \end{aligned}$$

Spheres





Definition

The *standard equation for a sphere* of radius a centred at $P_0(x_0, y_0, z_0)$ is

$$(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2 = a^2.$$

10. $$(x-x_0)^2+(y-y_0)^2+(z-z_0)^2=a^2.$$

Example

Find the centre and radius of the sphere

$$x^2+y^2+z^2+3x-4z+1=0.$$

$$(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2 = a^2.$$

Example

Find the centre and radius of the sphere

$$x^2 + y^2 + z^2 + 3x - 4z + 1 = 0.$$

First we need to put this equation into the standard form.

10.

$$(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2 = a^2.$$

Since $(x-b)^2 = x^2 - 2b + b^2$ we have that

$$x^2 + y^2 + z^2 + 3x - 4z + 1 = 0$$
$$(x^2 + 3x) + y^2 + (z^2 - 4z) = -1$$

10. $(x-x_0)^2+(y-y_0)^2+(z-z_0)^2=a^2.$

Since $(x-b)^2 = x^2 - 2b + b^2$ we have that

$$x^2+y^2+z^2+3x-4z+1=0$$
$$(x^2+3x)+y^2+(z^2-4z)=-1$$
$$\left(x^2+3x+\frac{9}{4}\right)-\frac{9}{4}+y^2+(z^2-4z+4)-4=-1$$

10.

$$(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2 = a^2.$$

Since $(x-b)^2 = x^2 - 2b + b^2$ we have that

$$\begin{aligned} x^2 + y^2 + z^2 + 3x - 4z + 1 &= 0 \\ (x^2 + 3x) + y^2 + (z^2 - 4z) &= -1 \\ \left(x^2 + 3x + \frac{9}{4}\right) - \frac{9}{4} + y^2 + (z^2 - 4z + 4) - 4 &= -1 \\ \left(x^2 + 3x + \frac{9}{4}\right) + y^2 + (z^2 - 4z + 4) &= -1 + \frac{9}{4} + 4 \end{aligned}$$

10. $(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2 = a^2.$

Since $(x-b)^2 = x^2 - 2b + b^2$ we have that

$$
\begin{aligned}
x^2 + y^2 + z^2 + 3x - 4z + 1 &= 0 \\
(x^2 + 3x) + y^2 + (z^2 - 4z) &= -1 \\
\left(x^2 + 3x + \frac{9}{4}\right) - \frac{9}{4} + y^2 + (z^2 - 4z + 4) - 4 &= -1 \\
\left(x^2 + 3x + \frac{9}{4}\right) + y^2 + (z^2 - 4z + 4) &= -1 + \frac{9}{4} + 4 \\
\left(x + \frac{3}{2}\right)^2 + y^2 + (z-2)^2 &= \frac{21}{4}.
\end{aligned}
$$

10. $(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2 = a^2.$

Since $(x-b)^2 = x^2 - 2b + b^2$ we have that

$$\begin{aligned}
x^2 + y^2 + z^2 + 3x - 4z + 1 &= 0 \\
(x^2 + 3x) + y^2 + (z^2 - 4z) &= -1 \\
\left(x^2 + 3x + \frac{9}{4}\right) - \frac{9}{4} + y^2 + (z^2 - 4z + 4) - 4 &= -1 \\
\left(x^2 + 3x + \frac{9}{4}\right) + y^2 + (z^2 - 4z + 4) &= -1 + \frac{9}{4} + 4 \\
\left(x + \frac{3}{2}\right)^2 + y^2 + (z-2)^2 &= \frac{21}{4}.
\end{aligned}$$

The centre is at $P_0(x_0, y_0, z_0) = P_0(-\frac{3}{2}, 0, 2)$ and the radius is $a = \sqrt{\frac{21}{4}} = \frac{\sqrt{3}\sqrt{7}}{2}$.

10. $(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2 = a^2.$

Example

Find the centre and radius of the sphere

$$x^2 + y^2 + z^2 + 6x - 6y + 6z = 7.$$

10. $(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2 = a^2.$

Since $(x-b)^2 = x^2 - 2b + b^2$ we have that

$$\begin{aligned}
x^2 + y^2 + z^2 + 6x - 6y + 6z &= 7 \\
(x^2 + 6x) + (y^2 - 6y) + (z^2 + 6z) &= 7 \\
(x^2 + 6x + 9) - 9 + (y^2 - 6y + 9) - 9 + (z^2 + 6z + 9) - 9 &= 7 \\
(x^2 + 6x + 9) + (y^2 - 6y + 9) + (z^2 + 6z + 9) &= 7 + 9 \\
(x+3)^2 + (y-3)^2 + (z+3)^2 &= 16
\end{aligned}$$

The centre is at $P_0(x_0, y_0, z_0) = P_0(-3, 3, -3)$ and the radius is $a = \sqrt{16} = 4$.

EXAMPLE 5 Here are some geometric interpretations of inequalities and equations involving spheres.

(a) $x^2 + y^2 + z^2 < 4$	The interior of the sphere $x^2 + y^2 + z^2 = 4$.
(b) $x^2 + y^2 + z^2 \leq 4$	The solid ball bounded by the sphere $x^2 + y^2 + z^2 = 4$. Alternatively, the sphere $x^2 + y^2 + z^2 = 4$ together with its interior.
(c) $x^2 + y^2 + z^2 > 4$	The exterior of the sphere $x^2 + y^2 + z^2 = 4$.
(d) $x^2 + y^2 + z^2 = 4, z \leq 0$	The lower hemisphere cut from the sphere $x^2 + y^2 + z^2 = 4$ by the xy-plane (the plane $z = 0$).

■

Next Time

- 11. Vectors
- 12. The Dot Product
- 13. The Cross Product