Mathematics IV

Dr. Neil Course

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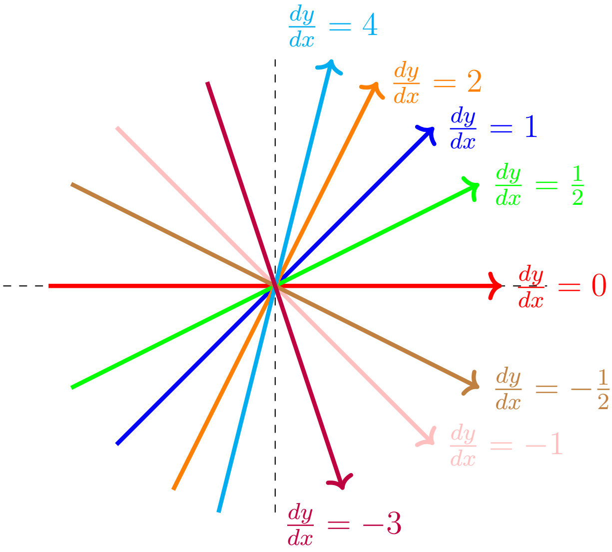
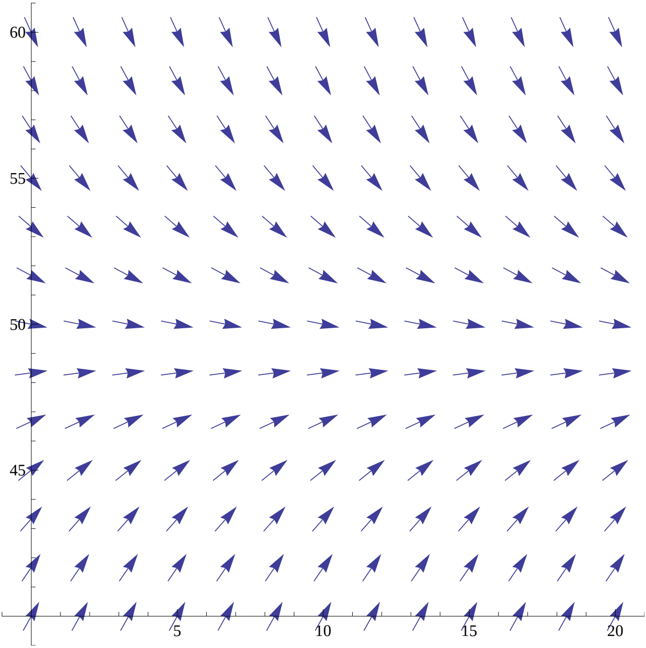
1 Introduction

Direction Fields



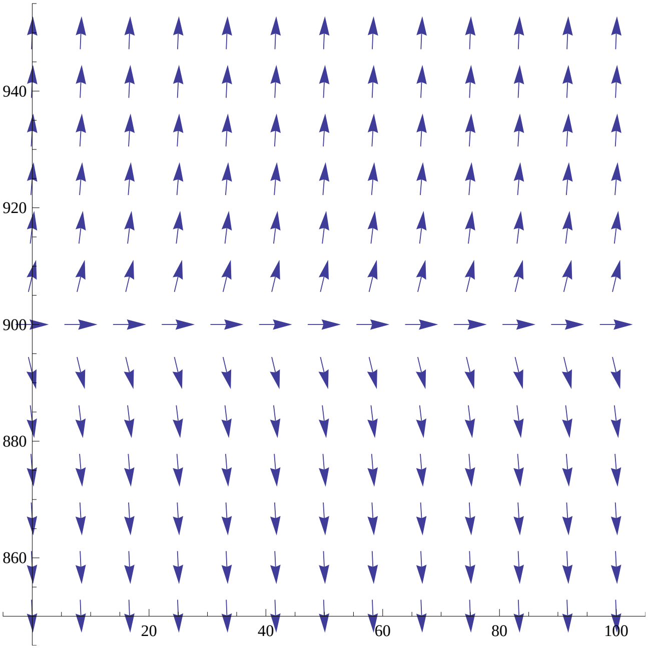
A Falling Object

$$\frac{dv}{dt} = 9.8 - \frac{v}{5}$$



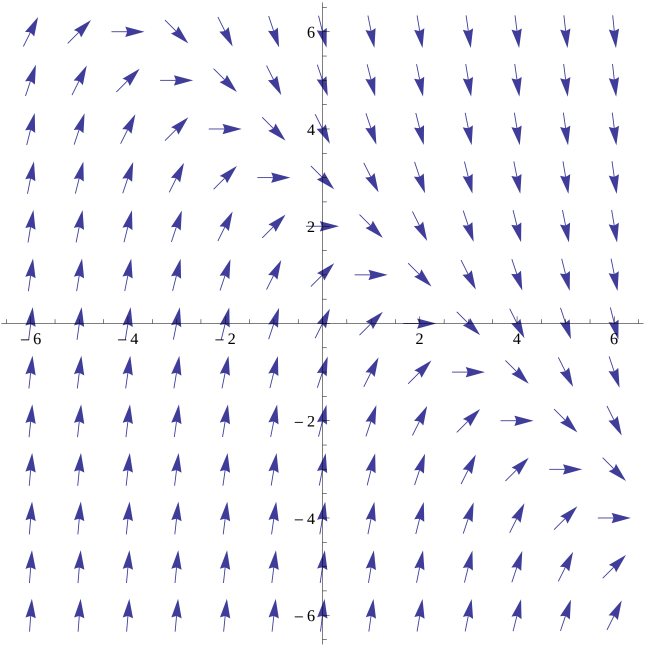
Mice and Owls

$$\frac{dp}{dt} = \frac{p}{2} - 450$$



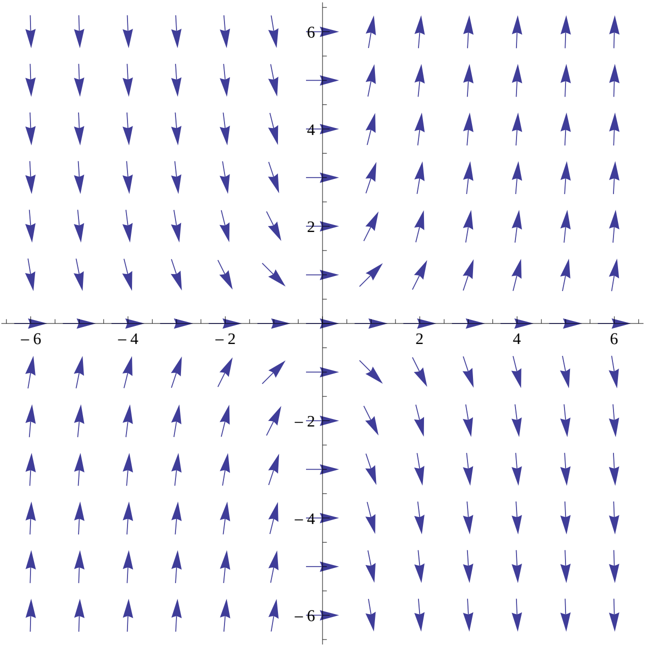
Example

$$\frac{dy}{dx} = 2 - x - y$$



Example

$$\frac{dy}{dx} = xy$$



Classification

- ODE = ordinary differential equation (adi diferansiyel denklemler)
- PDE = partial differential equation (kısmi türevli diferansiyel denklemler)
- IVP = initial value problem
- BVP = boundary value problem

Greek Letters

alpha	A	α
beta	B	β
gamma	Γ	γ
delta	Δ	δ
epsilon	E	ϵ ε
zeta	Z	ζ
eta	N	η
theta	Θ	θ
iota	I	ι
kappa	K	κ
lambda	Λ	λ
mu	M	μ
nu	N	ν
xi	Ξ	ξ
omnicron	O	o
pi	Π	π
rho	P	ρ
sigma	Σ	σ
tau	T	τ
upsilon	Y	υ
phi	Φ	ϕ φ
chi	X	χ
psi	Ψ	ψ
omega	Ω	ω

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First Order ODEs

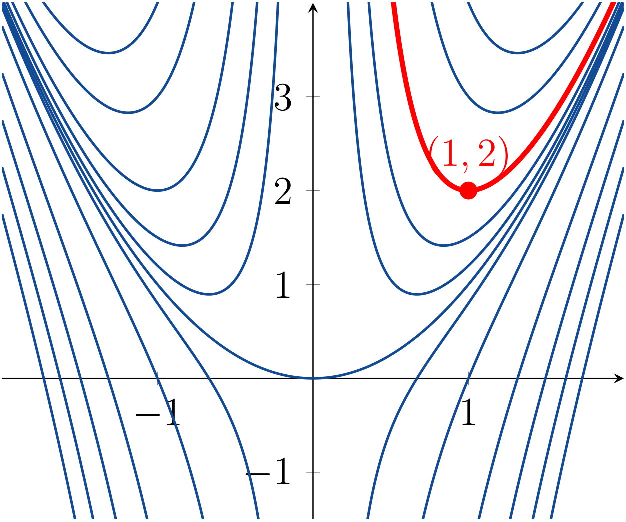
A Linear Equation

The solution to

$$ty' + 2y = 4t$$

is

$$y(t) = t^2 + \frac{c}{t^2}.$$



Note that the solution satisfying $y(1) = 2$ is a differentiable function $y : (0, \infty) \to \mathbb{R}$.

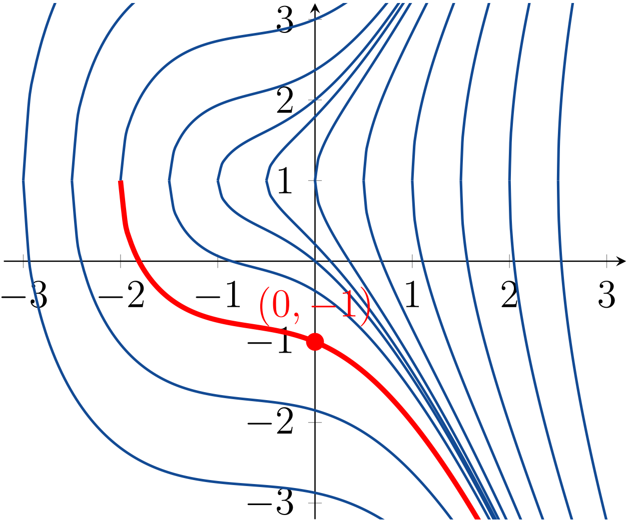
A Separable Equation

The solution to

$$\frac{dy}{dx} = \frac{3x^2 + 4x + 2}{2(y-1)}$$

is

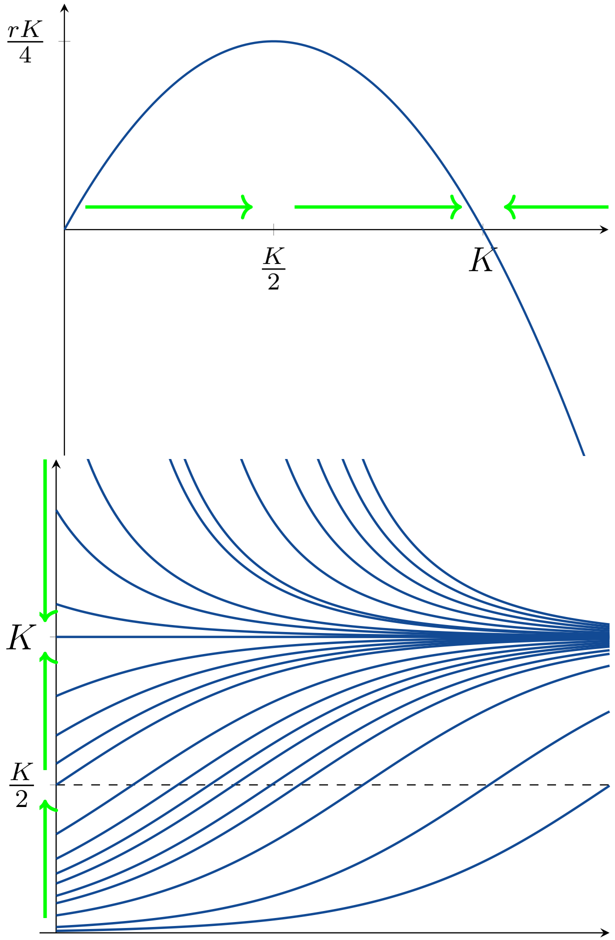
$$y(x) = 1 \pm \sqrt{x^3 + 2x^2 + 2x + c}.$$



Note that the solution satisfying $y(0) = -1$ is a differentiable function $y : (-2, \infty) \to \mathbb{R}$.

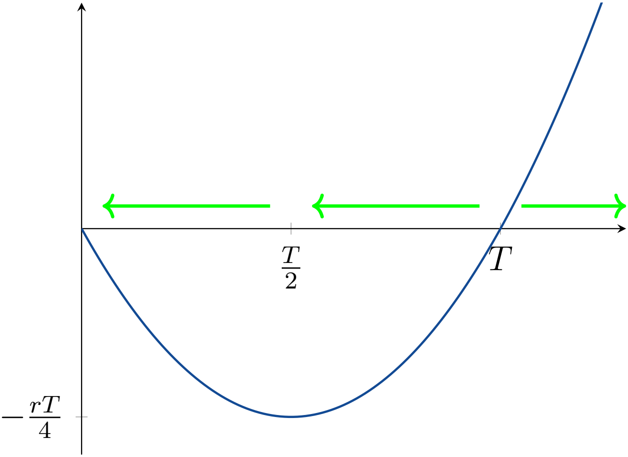
Logistic Growth

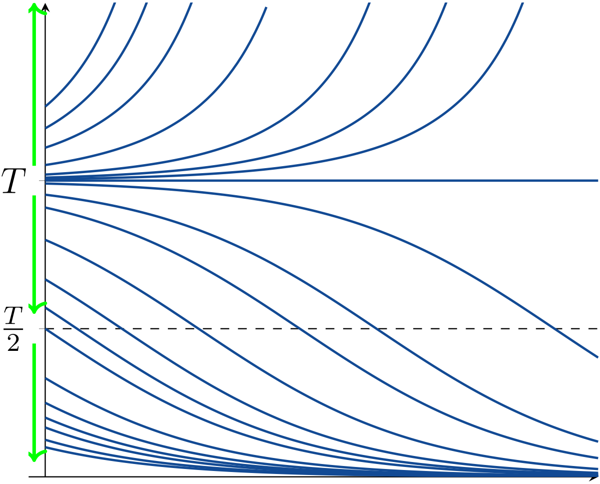
$$\frac{dy}{dt} = r\left(1 - \frac{y}{K}\right)y$$



A Critical Threshold

$$\frac{dy}{dt} = -r\left(1 - \frac{y}{T}\right)y$$

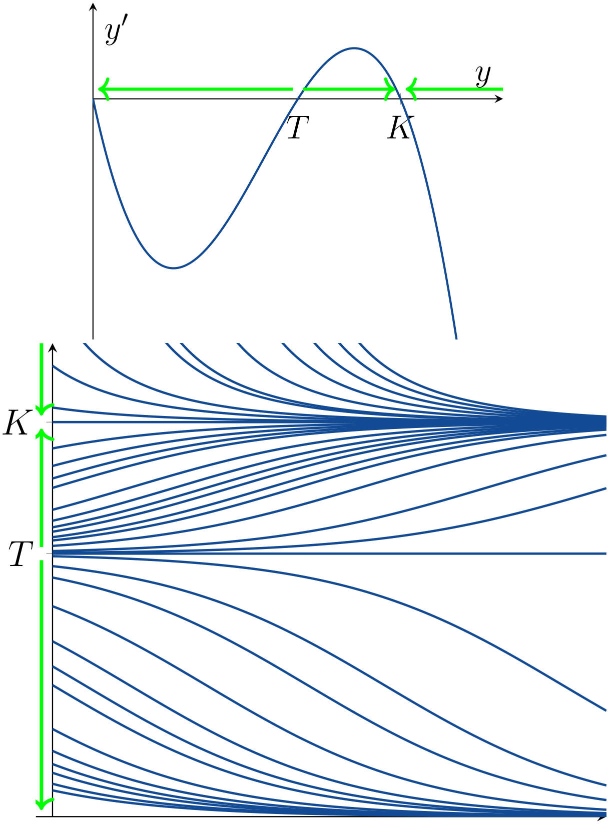




Logistic Growth with a Threshold

$0 < T < K,\ r > 0$

$$\frac{dy}{dt} = -r\left(1 - \frac{y}{T}\right)\left(1 - \frac{y}{K}\right)y$$



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Second Order Linear ODEs

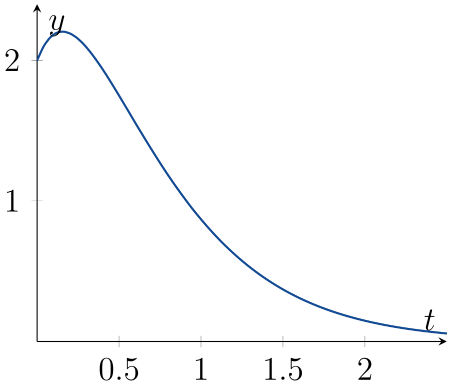
Example

The initial value problem

$$\begin{cases} y'' + 5y' + 6y = 0 \\ y(0) = 2 \\ y'(0) = 3 \end{cases}$$

has solution

$$y = 9e^{-2t} - 7e^{-3t}.$$



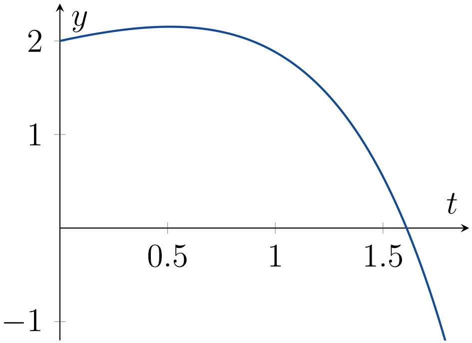
Example

The initial value problem

$$\begin{cases} 4y'' - 8y' + 3y = 0 \\ y(0) = 2 \\ y'(0) = \frac{1}{2} \end{cases}$$

has solution

$$y = -\frac{1}{2}e^{\frac{3t}{2}} + \frac{5}{2}e^{\frac{t}{2}}.$$



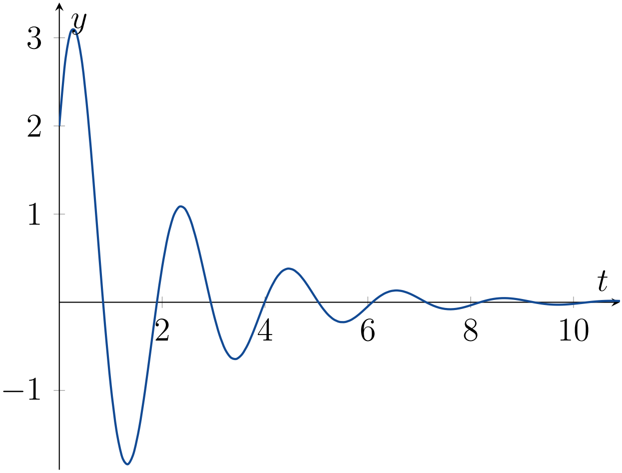
Example

The initial value problem

$$\begin{cases} y'' + y' + 9.25y = 0 \\ y(0) = 2 \\ y'(0) = 8 \end{cases}$$

has solution

$$y = e^{-\frac{t}{2}} \left(2\cos 3t + 3\sin 3t\right).$$



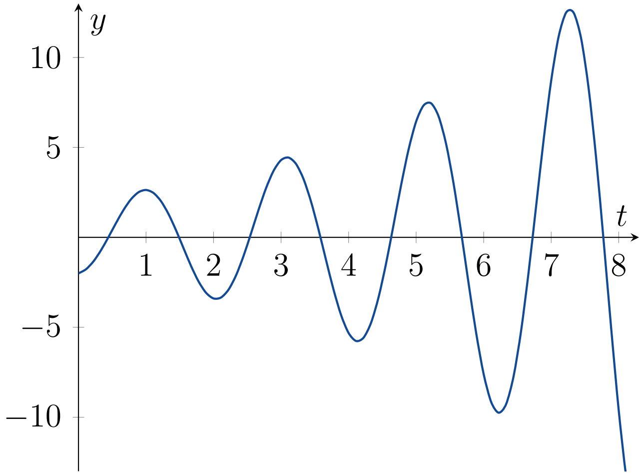
Example

The initial value problem

$$\begin{cases} 16y'' - 8y' + 145y = 0 \\ y(0) = -2 \\ y'(0) = 1 \end{cases}$$

has solution

$$y = -2e^{\frac{t}{4}} \cos 3t + \frac{1}{2} e^{\frac{t}{4}} \sin 3t.$$



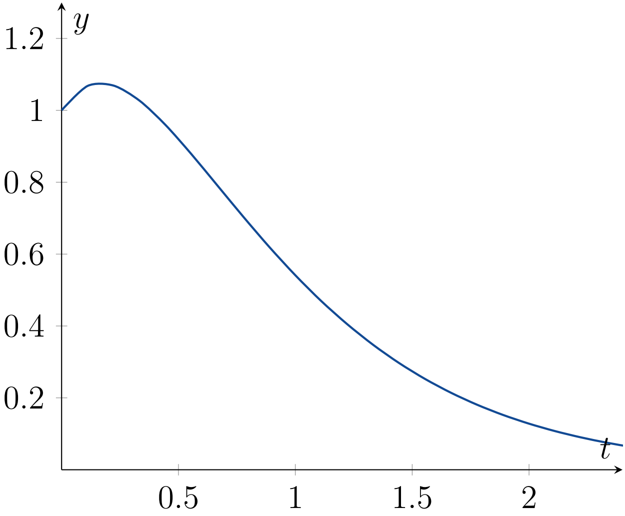
Example

The initial value problem

$$\begin{cases} y'' + 4y' + 4y = 0 \\ y(0) = 1 \\ y'(0) = 1 \end{cases}$$

has solution

$$y = e^{-2t} + 3te^{-2t}.$$



Example

The initial value problems

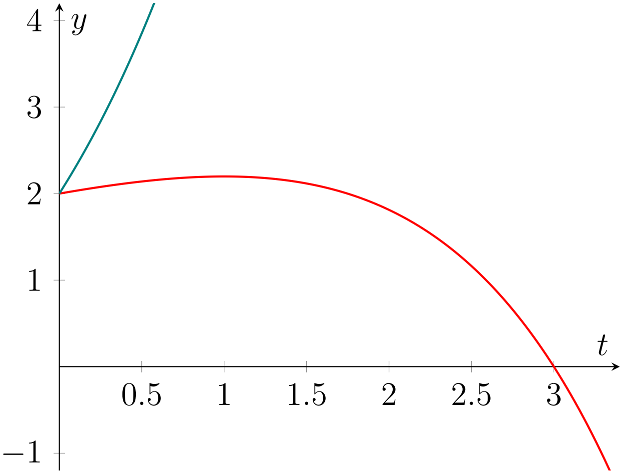
$$\begin{cases} y'' - y' + \frac{1}{4}y = 0 \\ y(0) = 2 \\ y'(0) = \frac{1}{3} \end{cases} \quad \text{and} \quad \begin{cases} y'' - y' + \frac{1}{4}y = 0 \\ y(0) = 2 \\ y'(0) = 2 \end{cases}$$

have solutions

$$y = 2e^{\frac{t}{2}} - \frac{2}{3}te^{\frac{t}{2}}$$

and

$$y = 2e^{\frac{t}{2}} + te^{\frac{t}{2}}.$$



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Higher Order Linear ODEs

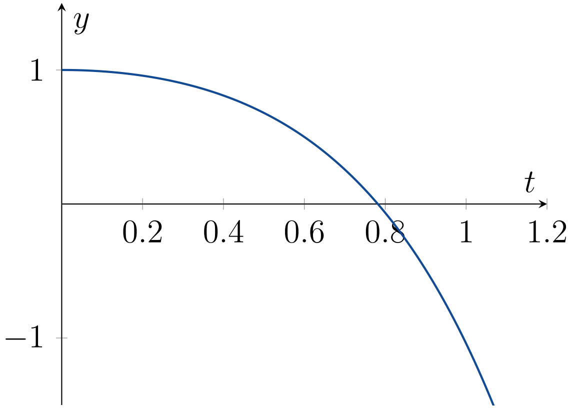
Example

The initial value problem

$$\begin{cases} y^{(4)} + y''' - 7y'' - y' + 6y = 0 \\ y(0) = 1 \\ y'(0) = 0 \\ y''(0) = -2 \\ y'''(0) = -1 \end{cases}$$

has solution

$$y = \frac{11}{8}e^{t} + \frac{5}{12}e^{-t} - \frac{2}{3}e^{2t} - \frac{1}{8}e^{-3t}.$$



Example

The initial value problems

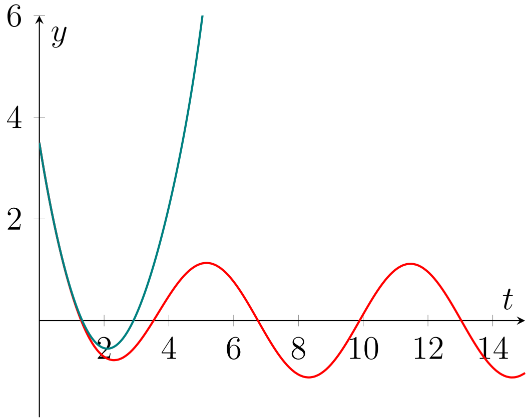
$$\begin{cases} y^{(4)} - y = 0 \\ y(0) = \frac{7}{2} \\ y'(0) = -4 \\ y''(0) = \frac{5}{2} \\ y'''(0) = -2 \end{cases} \quad \text{and} \quad \begin{cases} y^{(4)} - y = 0 \\ y(0) = \frac{7}{2} \\ y'(0) = -4 \\ y''(0) = \frac{5}{2} \\ y'''(0) = -\frac{15}{8} \end{cases}$$

have solutions

$$y = 3e^{-t} + \tfrac{1}{2}\cos t - \sin t$$

and

$$y = \tfrac{1}{32}e^{t} + \tfrac{95}{32}e^{-t} + \tfrac{1}{2}\cos t - \tfrac{17}{16}\sin t.$$



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The Laplace Transform

Definition: Suppose that

(i) $A > 0,\ K > 0,\ M > 0,\ a \in \mathbb{R}$;

(ii) f is piecewise continuous on $[0, A]$; and

(iii) $|f(t)| \leq Ke^{at}$ for all $t \geq M$.

The ***Laplace Transform*** of f is

$$F(s) = \mathcal{L}[f](s) = \int_0^\infty e^{-st} f(t)\, dt.$$

Note that

$$\begin{aligned}
\mathcal{L}[f'](s) &= sF(s) - f(0) \\
\mathcal{L}[f''](s) &= s^2F(s) - sf(0) - f'(0) \\
\mathcal{L}[f^{(n)}](s) &= s^nF(s) - s^{n-1}f(0) - s^{(n-2)}f'(0) \\
&\quad - \ldots - sf^{(n-2)}(0) - f^{(n-1)}(0).
\end{aligned}$$

Elementary Laplace Transforms

$f(t)$	$F(s)=\mathcal{L}[f](s)$	
1	$\frac{1}{s}$	$s>0$
e^{at}	$\frac{1}{s-a}$	$s>a$
$t^n \quad (n\in\mathbb{N})$	$\frac{n!}{s^{n+1}}$	$s>0$
$\sin at$	$\frac{a}{s^2+a^2}$	$s>0$
$\cos at$	$\frac{s}{s^2+a^2}$	$s>0$
$\sinh at$	$\frac{a}{s^2-a^2}$	$s>\lvert a\rvert$
$\cosh at$	$\frac{s}{s^2-a^2}$	$s>\lvert a\rvert$
$e^{at}\sin bt$	$\frac{b}{(s-a)^2+b^2}$	$s>a$
$e^{at}\cos bt$	$\frac{s-a}{(s-a)^2+b^2}$	$s>a$
$t^n e^{at} \quad (n\in\mathbb{N})$	$\frac{n!}{(s-a)^{n+1}}$	$s>a$
$u_c(t)$	$\frac{e^{-cs}}{s}$	$s>0$
$u_c(t)f(t-c)$	$e^{-cs}F(s)$	
$e^{ct}f(t)$	$F(s-c)$	
$f(ct) \quad (c>0)$	$\frac{1}{c}F\left(\frac{s}{c}\right)$	
$\int_0^t f(t-\tau)g(\tau)d\tau$	$F(s)G(s)$	
$(-1)^n f(t)$	$F^{(n)}(s)$	
$t^n f(t)$	$(-1)^n\frac{d^nF}{ds^n}$	

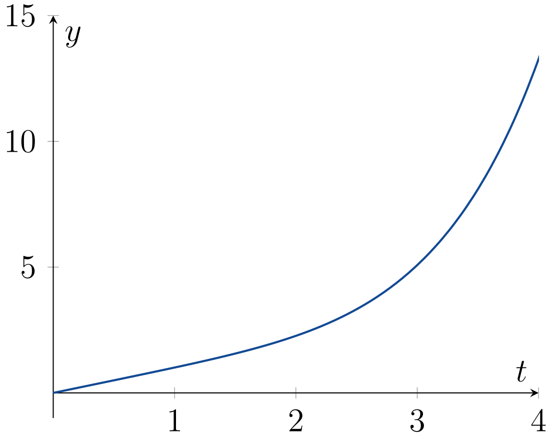
Example

The initial value problem

$$\begin{cases} y^{(4)} - y = 0 \\ y(0) = 0 \\ y'(0) = 1 \\ y''(0) = 0 \\ y'''(0) = 0 \end{cases}$$

has solution

$$y = \tfrac{1}{2}\sinh t + \tfrac{1}{2}\sin t.$$



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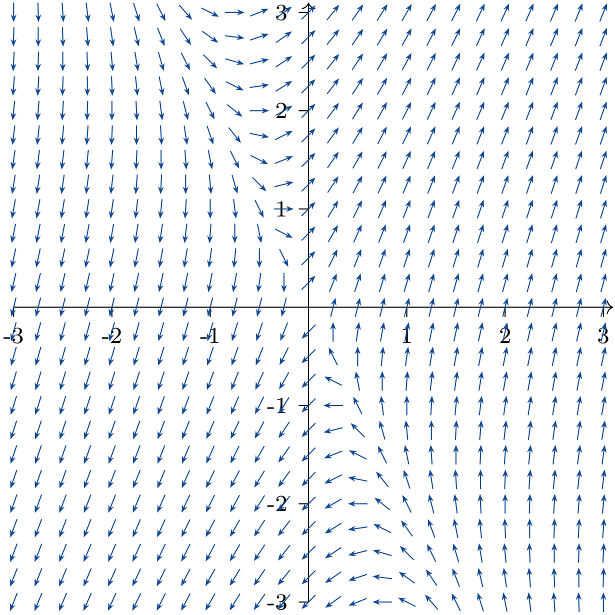
Systems of First Order Linear ODEs

Example

A direction field for the linear system

$$\mathbf{x}' = \begin{pmatrix} 1 & 1 \\ 4 & 1 \end{pmatrix} \mathbf{x}$$

is



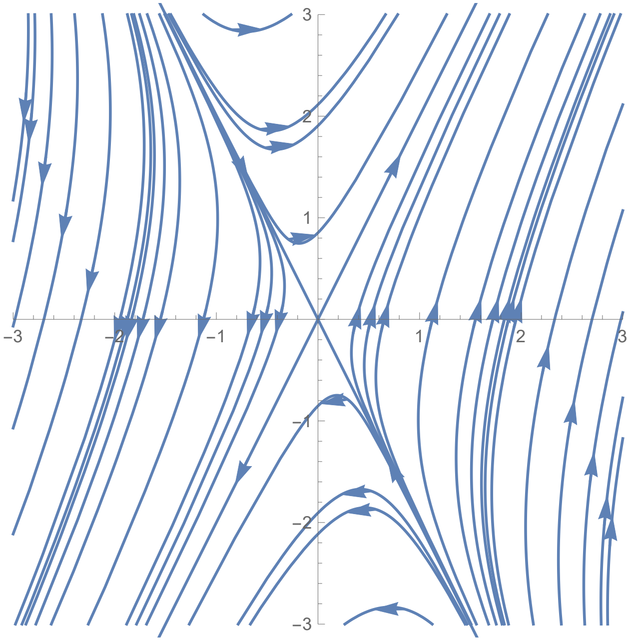
Example

The linear system

$$\mathbf{x}' = \begin{pmatrix} 1 & 1 \\ 4 & 1 \end{pmatrix} \mathbf{x}$$

has general solution

$$\mathbf{x}(t) = c_1 \begin{pmatrix} 1 \\ 2 \end{pmatrix} e^{3t} + c_2 \begin{pmatrix} 1 \\ -2 \end{pmatrix} e^{-t}.$$

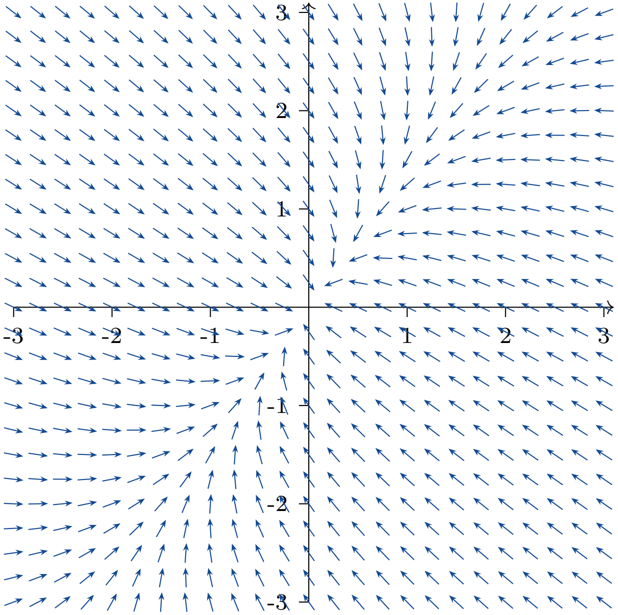


Example

A direction field for the linear system

$$\mathbf{x}' = \begin{pmatrix} -3 & \sqrt{2} \\ \sqrt{2} & -2 \end{pmatrix} \mathbf{x}$$

is



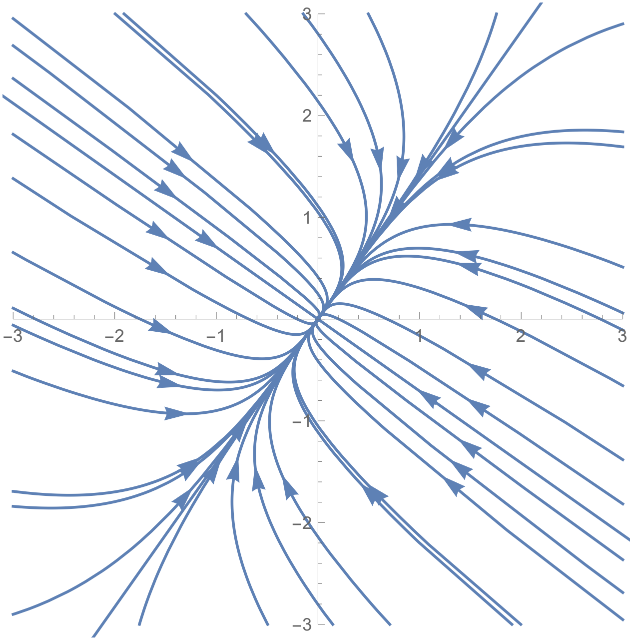
Example

The linear system

$$\mathbf{x}' = \begin{pmatrix} -3 & \sqrt{2} \\ \sqrt{2} & -2 \end{pmatrix} \mathbf{x}$$

has general solution

$$\mathbf{x}(t) = c_1 \begin{pmatrix} 1 \\ \sqrt{2} \end{pmatrix} e^{-t} + c_2 \begin{pmatrix} -\sqrt{2} \\ 1 \end{pmatrix} e^{-4t}.$$

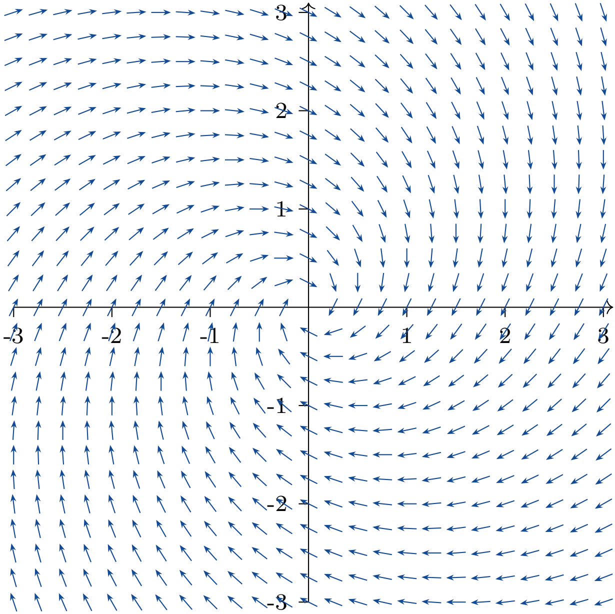


Example

A direction field for the linear system

$$\mathbf{x}' = \begin{pmatrix} -\frac{1}{2} & 1 \\ -1 & -\frac{1}{2} \end{pmatrix} \mathbf{x}$$

is



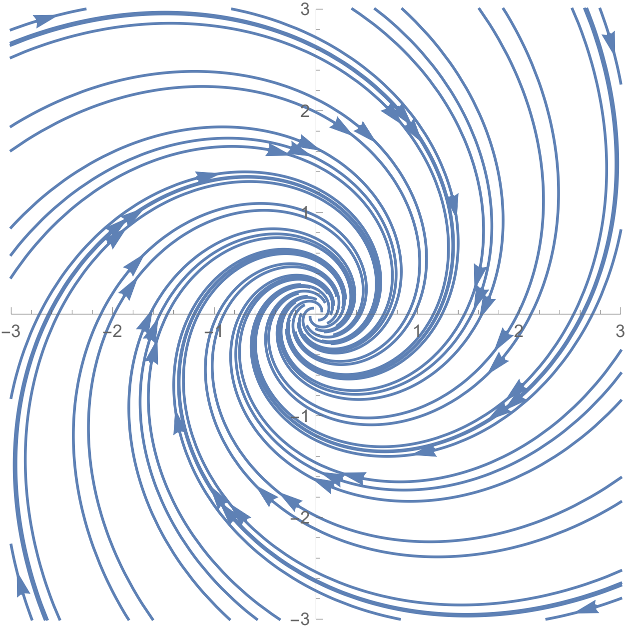
Example

The linear system

$$\mathbf{x}' = \begin{pmatrix} -\frac{1}{2} & 1 \\ -1 & -\frac{1}{2} \end{pmatrix} \mathbf{x}$$

has general solution

$$\mathbf{x}(t) = c_1 \begin{pmatrix} \cos t \\ -\sin t \end{pmatrix} e^{-\frac{t}{2}} + c_2 \begin{pmatrix} -\sin t \\ \cos t \end{pmatrix} e^{-\frac{t}{2}}.$$



Example

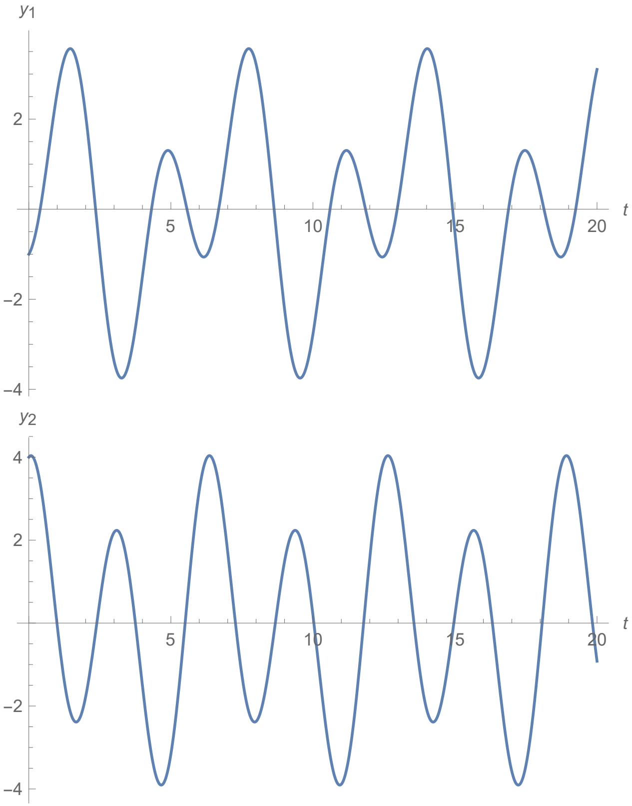
The initial value problem

$$\mathbf{y}' = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -2 & \frac{3}{2} & 0 & 0 \\ \frac{4}{3} & -3 & 0 & 0 \end{pmatrix} \mathbf{y}, \qquad \mathbf{y}(0) = \begin{pmatrix} -1 \\ 4 \\ 1 \\ 1 \end{pmatrix}$$

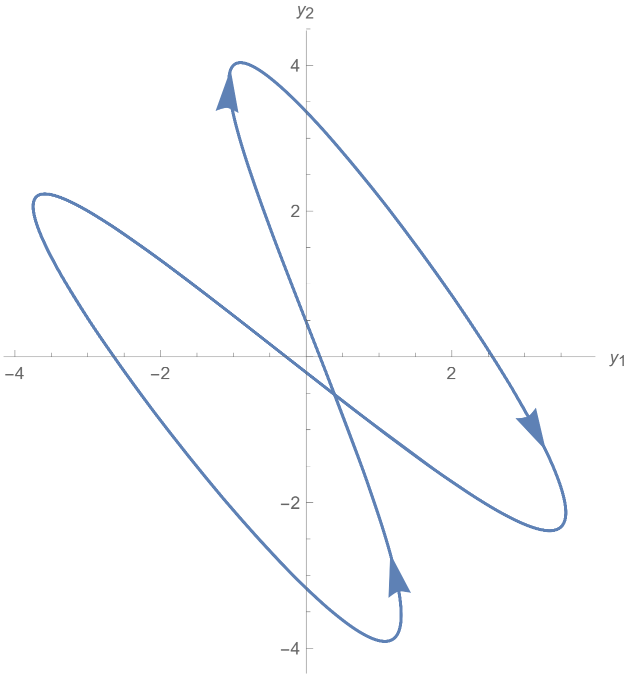
has solution

$$\begin{aligned} y(t) = \frac{4}{9} \begin{pmatrix} 3\cos t \\ 2\cos t \\ -3\sin t \\ -2\sin t \end{pmatrix} + \frac{7}{18} \begin{pmatrix} 3\sin t \\ 2\sin t \\ 3\cos t \\ 2\cos t \end{pmatrix} \\ - \frac{7}{9} \begin{pmatrix} 3\cos 2t \\ -4\cos 2t \\ -6\sin 2t \\ 8\sin 2t \end{pmatrix} - \frac{1}{36} \begin{pmatrix} 3\sin 2t \\ -4\sin 2t \\ 6\cos 2t \\ -8\cos 2t \end{pmatrix} \end{aligned}$$

Example (continued)



Example (continued)

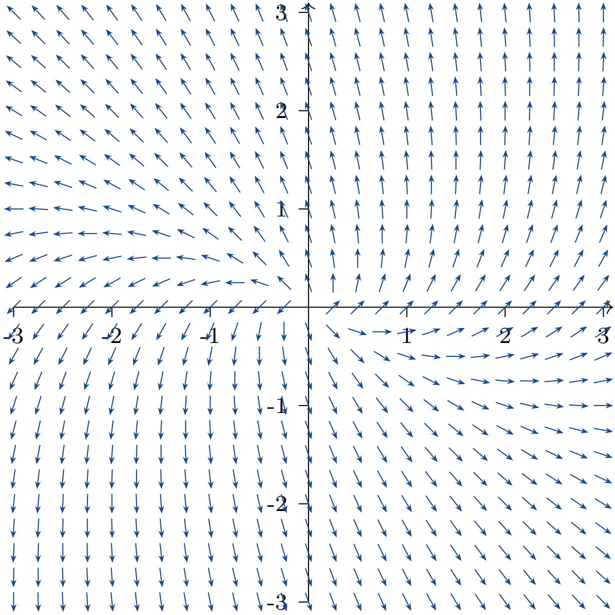


Example

A direction field for the linear system

$$\mathbf{x}' = \begin{pmatrix} 1 & -1 \\ 1 & 3 \end{pmatrix} \mathbf{x}$$

is



Example

The linear system

$$\mathbf{x}' = \begin{pmatrix} 1 & -1 \\ 1 & 3 \end{pmatrix} \mathbf{x}$$

has general solution

$$\mathbf{x}(t) = c_1 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{2t} + c_2 \left[\begin{pmatrix} 1 \\ -1 \end{pmatrix} t + \begin{pmatrix} 0 \\ -1 \end{pmatrix} \right] e^{2t}.$$