

Lecture 11

- 7.1 Inverse Functions and Their Derivatives
- 7.2 Natural Logarithms
- 7.3 Exponential Functions



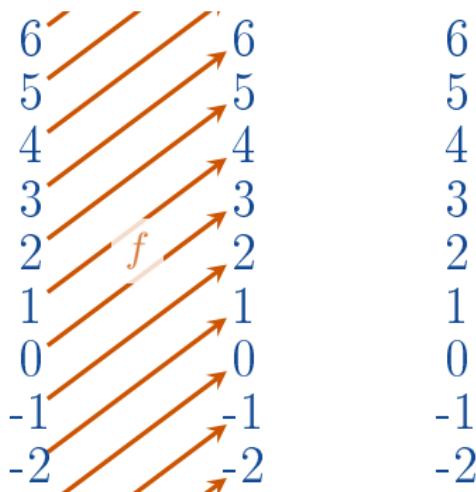
Inverse Functions and Their Derivatives

7.1 Inverse Functions and Their Derivatives



What is the opposite of adding 3 to a number?

What is the inverse of $f : \mathbb{Z} \rightarrow \mathbb{Z}$, $f(x) = x + 3$?

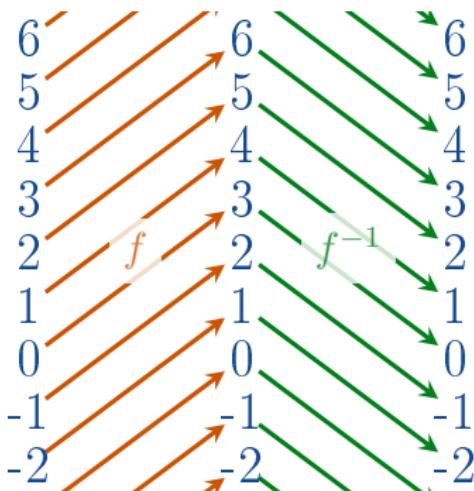


7.1 Inverse Functions and Their Derivatives



What is the opposite of adding 3 to a number?

What is the inverse of $f : \mathbb{Z} \rightarrow \mathbb{Z}$, $f(x) = x + 3$?



The inverse of f is $f^{-1} : \mathbb{Z} \rightarrow \mathbb{Z}$, $f^{-1}(x) = x - 3$.

Note that $f^{-1} \circ f(x) = x$ and $f \circ f^{-1}(x) = x$.

7.1 Inverse Functions and Their Derivatives



Some functions have an inverse.

For example $g : [0, 3] \rightarrow [0, 9]$, $g(x) = x^2$ has the inverse $g^{-1} : [0, 9] \rightarrow [0, 3]$, $g^{-1}(x) = \sqrt{x}$.

$$\sqrt{4^2} = \sqrt{16} = 4 \quad \text{and} \quad (\sqrt{9})^2 = 3^2 = 0.$$

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But some functions do not have an inverse.

For example, consider $h : [-3, 3] \rightarrow [0, 9]$, $h(x) = x^2$.

7.1 Inverse Functions and Their Derivatives



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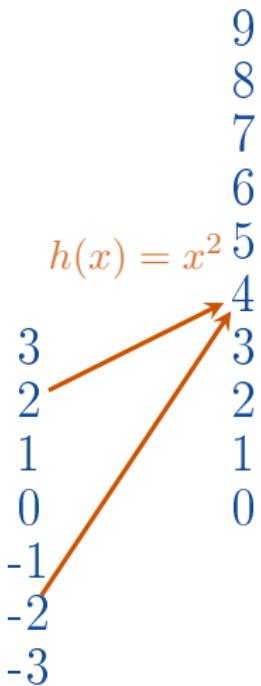
But some functions do not have an inverse.

For example, consider $h : [-3, 3] \rightarrow [0, 9]$, $h(x) = x^2$.

$$2^2 = 4 \quad \text{and} \quad (-2)^2 = 4.$$

So what do we define $h^{-1}(4)$ to be? 2 or -2?

7.1 Inverse Functions and Their Derivatives



7.1 Inverse Functions and Their Derivatives



One-to-One Functions

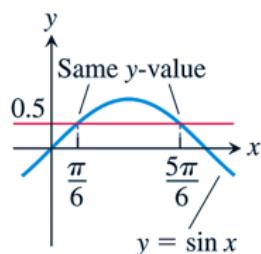
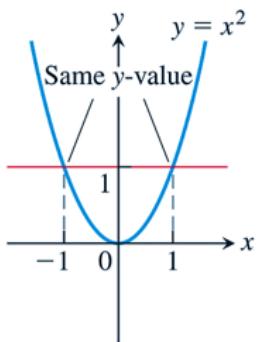
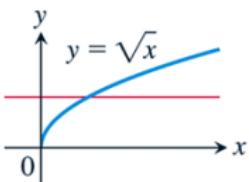
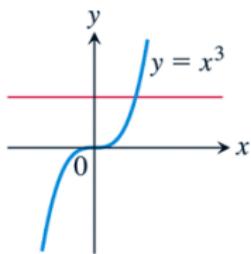
Definition

A function $f(x)$ is called *one-to-one* (or *injective*) (*bire-bir*) on a domain D if

$$x_1 \neq x_2 \implies f(x_1) \neq f(x_2)$$

for all $x_1, x_2 \in D$.

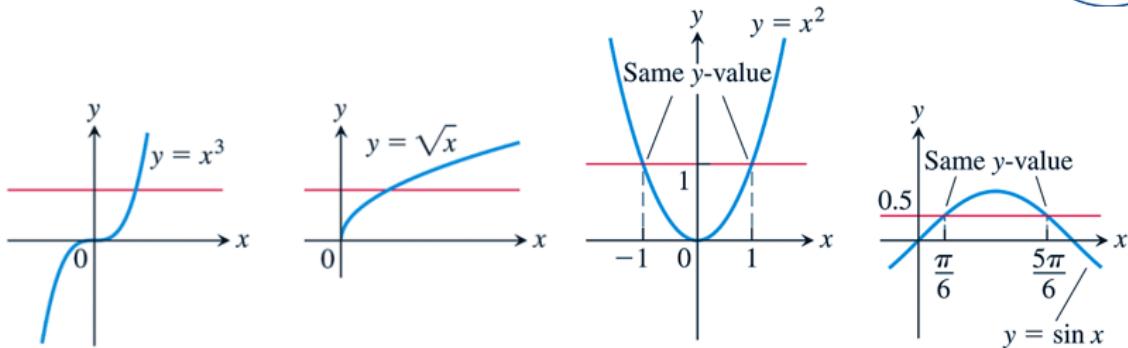
7.1 Inverse Functions and Their Derivatives



Example

- x^3 is one-to-one on any domain.

7.1 Inverse Functions and Their Derivatives

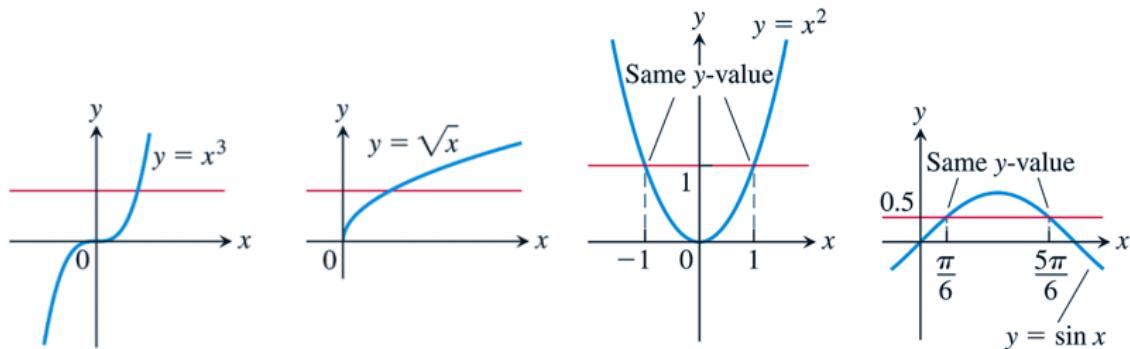


Example

- x^3 is one-to-one on any domain.
- \sqrt{x} is one-to-one on \mathbb{R} because

$$x_1 \neq x_2 \implies \sqrt{x_1} \neq \sqrt{x_2}.$$

7.1 Inverse Functions and Their Derivatives

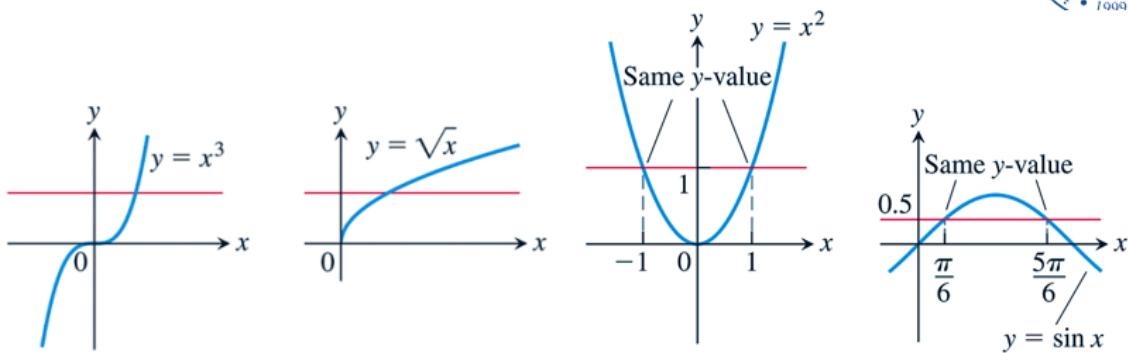


Example

- x^2 is not one-to-one on \mathbb{R} because

$$(-1)^2 = 1^2.$$

7.1 Inverse Functions and Their Derivatives

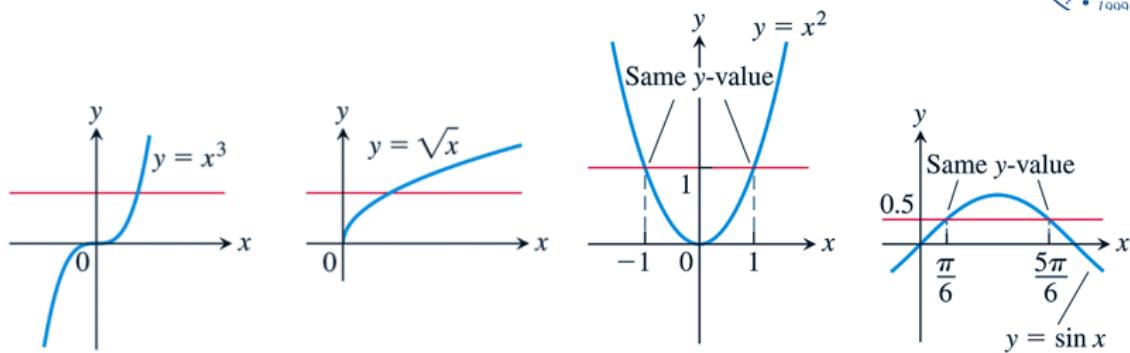


Example

- $\sin x$ is not one-to-one on $[0, \pi]$ because

$$\sin \frac{\pi}{6} = \frac{1}{2} = \sin \frac{5\pi}{6}.$$

7.1 Inverse Functions and Their Derivatives



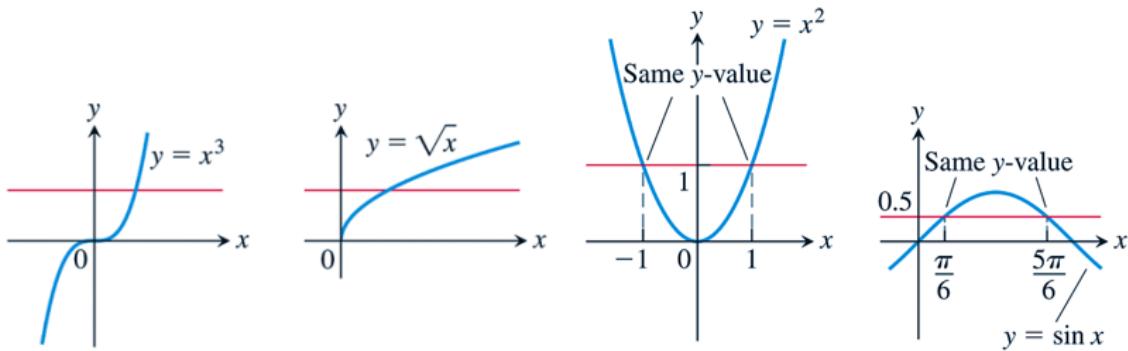
Example

- $\sin x$ is not one-to-one on $[0, \pi]$ because

$$\sin \frac{\pi}{6} = \frac{1}{2} = \sin \frac{5\pi}{6}.$$

- However, $\sin x$ is one-to-one on $[0, \frac{\pi}{2}]$.

7.1 Inverse Functions and Their Derivatives



Theorem (The Horizontal Line Test for One-to-One Functions)

A function $y = f(x)$ is one-to-one if and only if its graph intersects each horizontal line at most once.

7.1 Inverse Functions and Their Derivatives



Inverse Functions

Definition

Suppose that f is a one-to-one function on a domain D with range R . The *inverse function* f^{-1} is defined by

$$f^{-1}(b) = a \quad \text{if} \quad f(a) = b.$$

7.1 Inverse Functions and Their Derivatives



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$$f^{-1}(b) = a \quad \text{if} \quad f(a) = b.$$

The domain of f^{-1} is R and the range of f^{-1} is D .

We read f^{-1} as “ f inverse”.

Please note that f^{-1} does not mean $\frac{1}{f(x)}$.

EXAMPLE 2 Suppose a one-to-one function $y = f(x)$ is given by a table of values

x	1	2	3	4	5	6	7	8
$f(x)$	3	4.5	7	10.5	15	20.5	27	34.5

A table for the values of $x = f^{-1}(y)$ can then be obtained by simply interchanging the values in each column of the table for f :

y	3	4.5	7	10.5	15	20.5	27	34.5
$f^{-1}(y)$	1	2	3	4	5	6	7	8



7.1 Inverse Functions and Their Derivatives



Remark

If f^{-1} is the inverse of f , then

$$f^{-1} \circ f(x) = x \quad (\text{for all } x \text{ in the domain of } f)$$

and

$$f \circ f^{-1}(y) = y \quad (\text{for all } y \text{ in the range of } f).$$

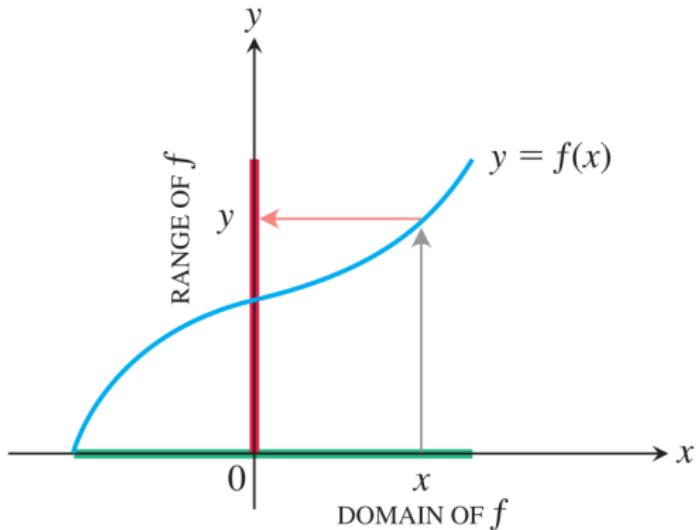
7.1 Inverse Functions and Their Derivatives



Finding Inverses

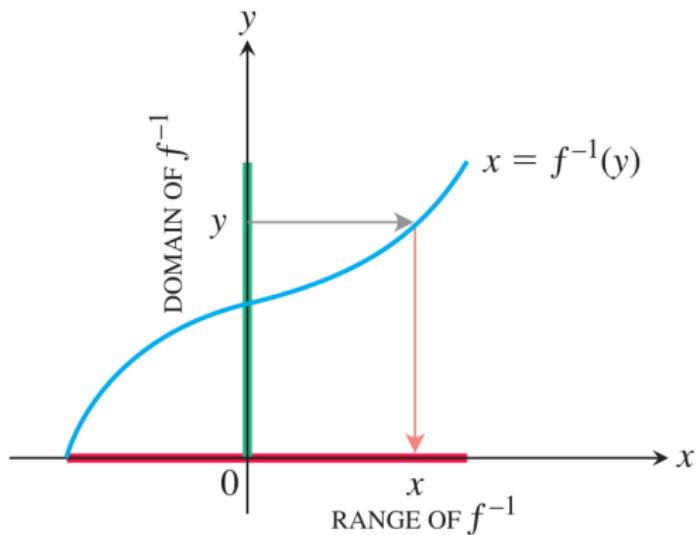
Now let's talk about how to find an inverse of a function.

7.1 Inverse Functions and Their Derivatives



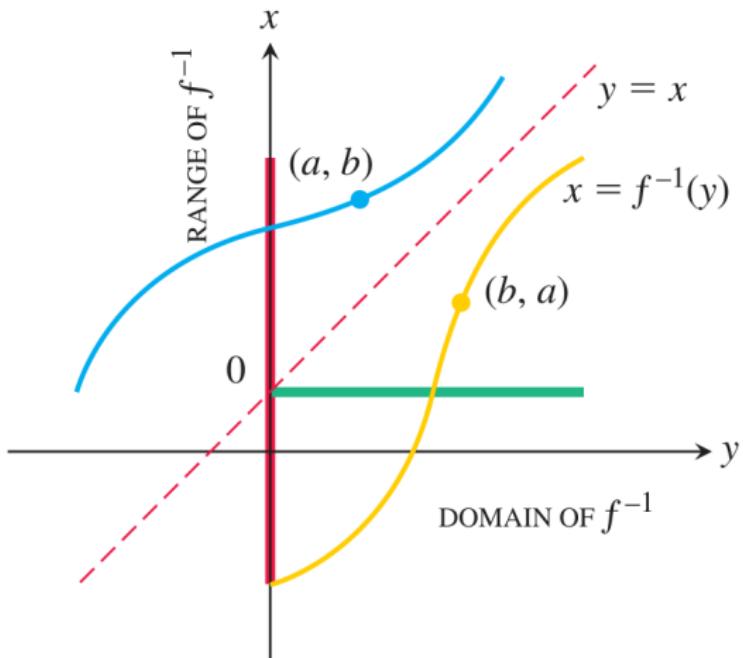
- (a) To find the value of f at x , we start at x , go up to the curve, and then over to the y -axis.

7.1 Inverse Functions and Their Derivatives



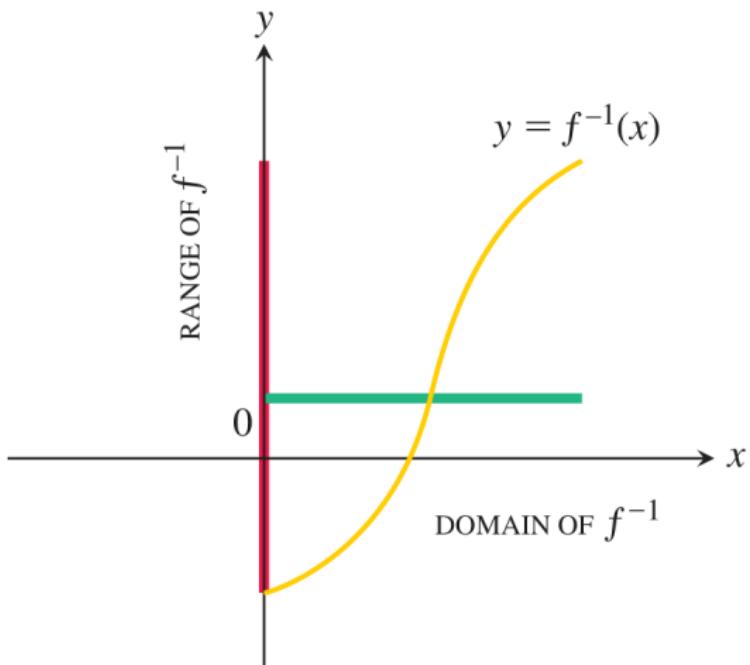
- (b) The graph of f^{-1} is the graph of f , but with x and y interchanged. To find the x that gave y , we start at y and go over to the curve and down to the x -axis. The domain of f^{-1} is the range of f . The range of f^{-1} is the domain of f .

7.1 Inverse Functions and Their Derivatives



- (c) To draw the graph of f^{-1} in the more usual way, we reflect the system across the line $y = x$.

7.1 Inverse Functions and Their Derivatives



(d) Then we interchange the letters x and y .
We now have a normal-looking graph of f^{-1}
as a function of x .

The process of passing from f to f^{-1} can be summarized as a two-step procedure.

1. Solve the equation $y = f(x)$ for x . This gives a formula $x = f^{-1}(y)$ where x is expressed as a function of y .
2. Interchange x and y , obtaining a formula $y = f^{-1}(x)$ where f^{-1} is expressed in the conventional format with x as the independent variable and y as the dependent variable.

7.1 Inverse Functions and Their Derivatives



Example

Find the inverse of $y = \frac{1}{2}x + 1$, expressed as a function of x .

- 1 Solve for x in terms of y :

- 2 Swap x and y :

7.1 Inverse Functions and Their Derivatives



Example

Find the inverse of $y = \frac{1}{2}x + 1$, expressed as a function of x .

- 1 Solve for x in terms of y :

$$y = \frac{1}{2}x + 1$$

$$2y = x + 2$$

$$2y - 2 = x$$

$$x = 2y - 2.$$

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7.1 Inverse Functions and Their Derivatives



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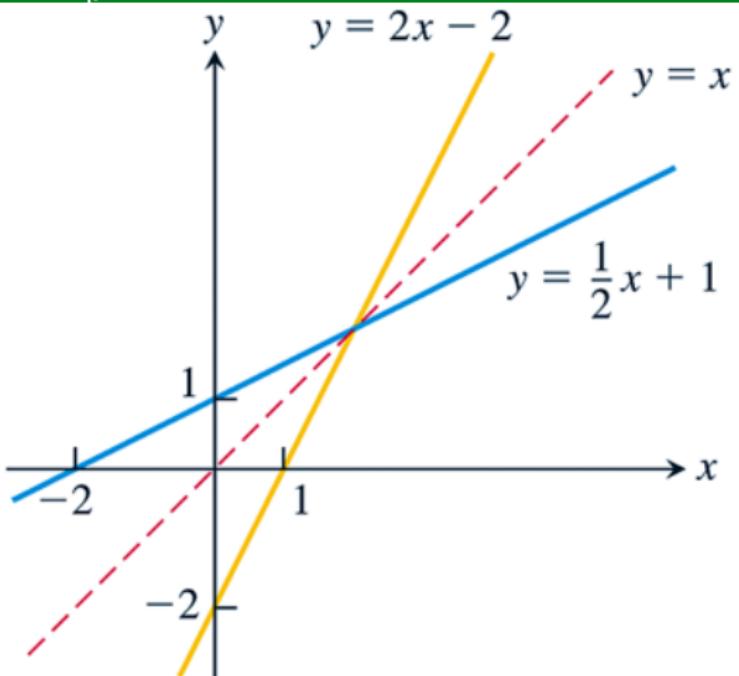
$$y = 2x - 2.$$

The inverse of $f(x) = \frac{1}{2}x + 1$ is $f^{-1}(x) = 2x - 2$.

7.1 Inverse Functions and Their Derivatives

Example

I



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7.1 Inverse Functions and Their Derivatives



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7.1 Inverse Functions and Their Derivatives



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Find the inverse of the function $y = x^2$, $x \geq 0$, expressed as a function of x .

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$$\sqrt{y} = \sqrt{x^2} = |x| = x$$

$$x = \sqrt{y}.$$

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7.1 Inverse Functions and Their Derivatives



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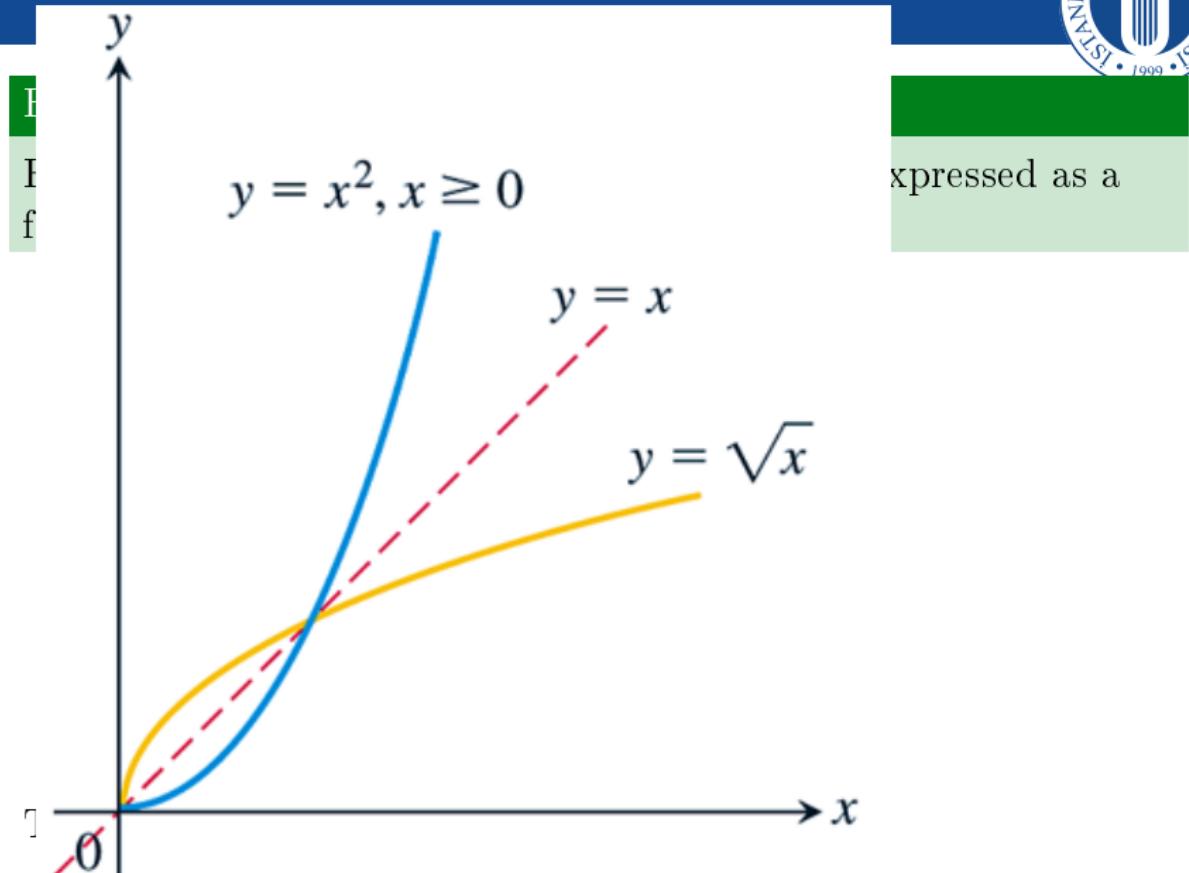
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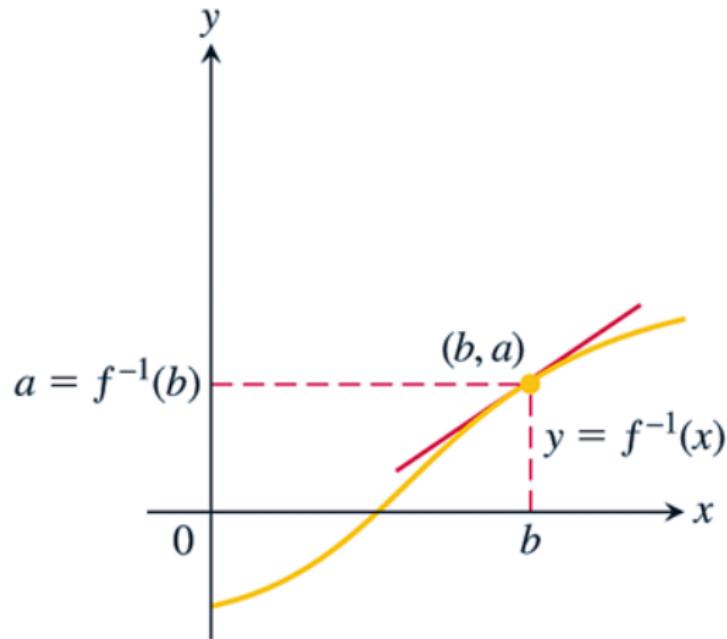
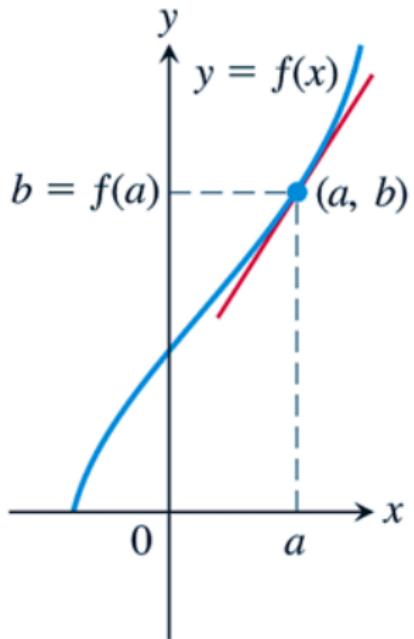
- 2 Swap x and y :

$$y = \sqrt{x}.$$

The inverse of $y = x^2$, $x \geq 0$, is $y = \sqrt{x}$.

7.1 Inverse Functions and Their Derivatives





The slopes are reciprocal: $(f^{-1})'(b) = \frac{1}{f'(a)}$ or $(f^{-1})'(b) = \frac{1}{f'(f^{-1}(b))}$

7.1 Inverse Functions and Their Derivatives



Note that

$$f(f^{-1}(x)) = x$$

7.1 Inverse Functions and Their Derivatives



Note that

$$\begin{aligned}f(f^{-1}(x)) &= x \\ \frac{d}{dx} f(f^{-1}(x)) &= \frac{d}{dx} x = 1\end{aligned}$$

7.1 Inverse Functions and Their Derivatives



Note that

$$f(f^{-1}(x)) = x$$

$$\frac{d}{dx} f(f^{-1}(x)) = \frac{d}{dx} x = 1$$

$$f'(f^{-1}(x)) \cdot \frac{d}{dx} f^{-1}(x) = 1$$

7.1 Inverse Functions and Their Derivatives



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$$\frac{d}{dx} f^{-1}(x) = \frac{1}{f'(f^{-1}(x))}.$$

7.1 Inverse Functions and Their Derivatives



Theorem

Suppose that

- $f : I \rightarrow \mathbb{R}$ for an interval I ;
- f' exists on I ; and
- $f'(x) \neq 0$ on I ,

then f^{-1} is differentiable at every point in its domain (the range of f).

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then f^{-1} is differentiable at every point in its domain (the range of f).

Moreover

$$(f^{-1})'(x) = \frac{1}{f'(f^{-1}(x))}.$$

7.1 Inverse Functions and Their Derivatives



Remark

This theorem says that

$$\frac{df^{-1}}{dx} \Big|_{x=b} = \frac{1}{\frac{df}{dx} \Big|_{x=f^{-1}(b)}}.$$

7.1 Inverse Functions and

$$(f^{-1})'(x) = \frac{1}{f'(f^{-1}(x))}$$



Example

Let $f(x) = x^2$, $x \geq 0$. Find $(f^{-1})'(4)$.

7.1 Inverse Functions and

$$(f^{-1})'(x) = \frac{1}{f'(f^{-1}(x))}$$



Example

Let $f(x) = x^2$, $x \geq 0$. Find $(f^{-1})'(4)$.

Method 1:

The inverse of $f(x) = x^2$, $x \geq 0$ is $f^{-1}(x) = \sqrt{x}$ and the derivative of \sqrt{x} is $\frac{1}{2\sqrt{x}}$.

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The inverse of $f(x) = x^2$, $x \geq 0$ is $f^{-1}(x) = \sqrt{x}$ and the derivative of \sqrt{x} is $\frac{1}{2\sqrt{x}}$.

Method 2: Using the theorem (with $f'(x) = 2x$), we calculate that

$$(f^{-1})'(x) = \frac{1}{f'(f^{-1}(x))} = \frac{1}{2(f^{-1}(x))} = \frac{1}{2(\sqrt{x})}.$$

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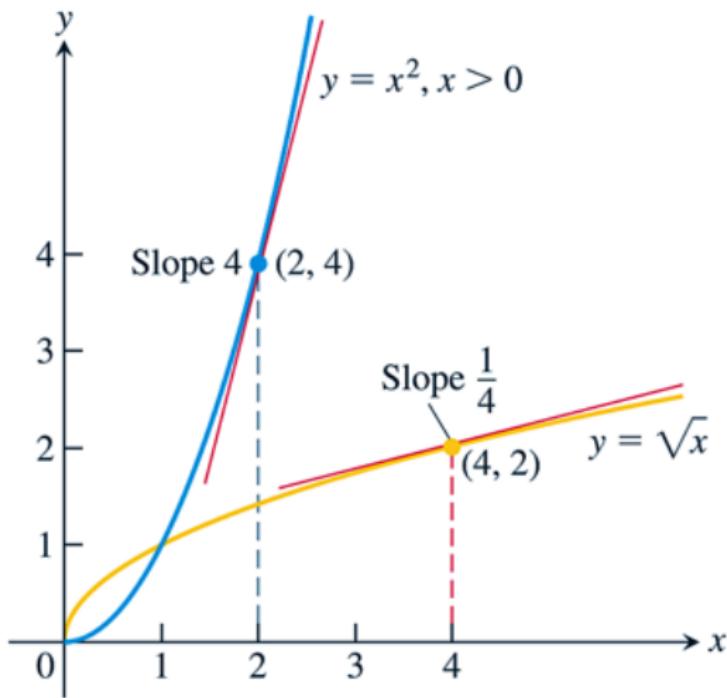
$$(f^{-1})'(x) = \frac{1}{f'(f^{-1}(x))} = \frac{1}{2(f^{-1}(x))} = \frac{1}{2(\sqrt{x})}.$$

Either way,

$$(f^{-1})'(4) = \frac{1}{2\sqrt{4}} = \frac{1}{4}.$$

7.1 Inverse Functions and

$$(f^{-1})'(x) = \frac{1}{f'(f^{-1}(x))}$$



7.1 Inverse Functions and T

$$\frac{df^{-1}}{dx} \Big|_{x=b} = \frac{1}{\frac{df}{dx} \Big|_{x=f^{-1}(b)}}$$



Example

Let $f(x) = x^3 - 2$, $x > 0$. Find the value of $\frac{df^{-1}}{dx}$ at $x = 6 = f(2)$ without finding a formula for $f^{-1}(x)$.

7.1 Inverse Functions and T

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Look at the formula in the yellow box. We want to calculate $\frac{df^{-1}}{dx}$ at $x = 6$. So we need $\frac{df}{dx}$ at $x = 2$.

7.1 Inverse Functions and T

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Since

$$\frac{df}{dx} \Big|_{x=2} = 3x^2 \Big|_{x=2} = 12,$$

7.1 Inverse Functions and T

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$$\frac{df}{dx} \Big|_{x=2} = 3x^2 \Big|_{x=2} = 12,$$

we have that

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7.1 Inverse Functions and T

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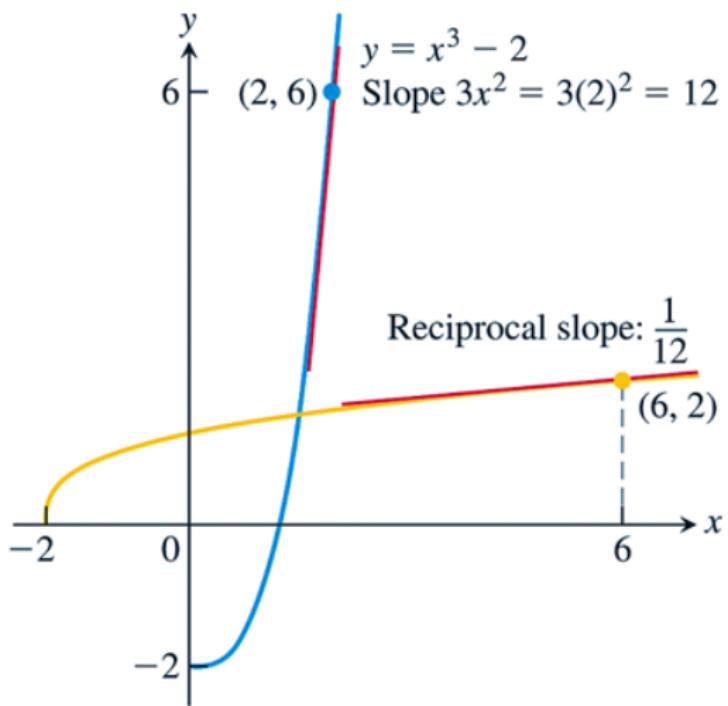
Since

$$\frac{df}{dx} \Big|_{x=2} = 3x^2 \Big|_{x=2} = 12,$$

we have that

$$\frac{df^{-1}}{dx} \Big|_{x=6} = \frac{1}{\frac{df}{dx} \Big|_{x=2}} = \frac{1}{12}.$$

7.1 Inverse Functions and Their Derivatives





Natural Logarithms

7.2 Natural Logarithms



Definition of the Natural Logarithm Function

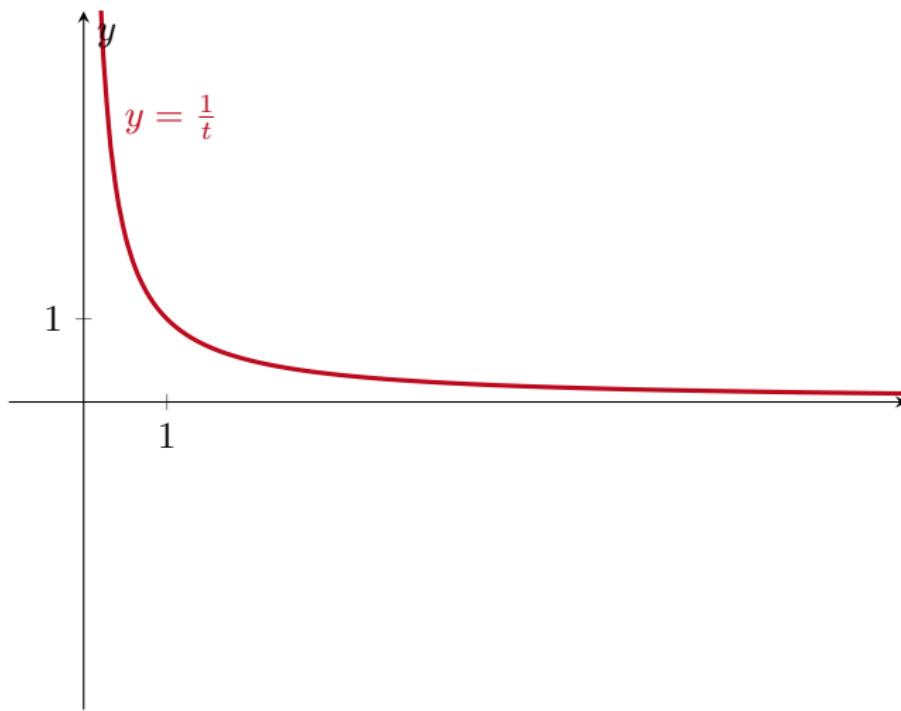
Definition

The *natural logarithm* function $\ln : (0, \infty) \rightarrow \mathbb{R}$ is defined by

$$\ln x = \int_1^x \frac{1}{t} dt.$$

7.2 Natural Logarithms

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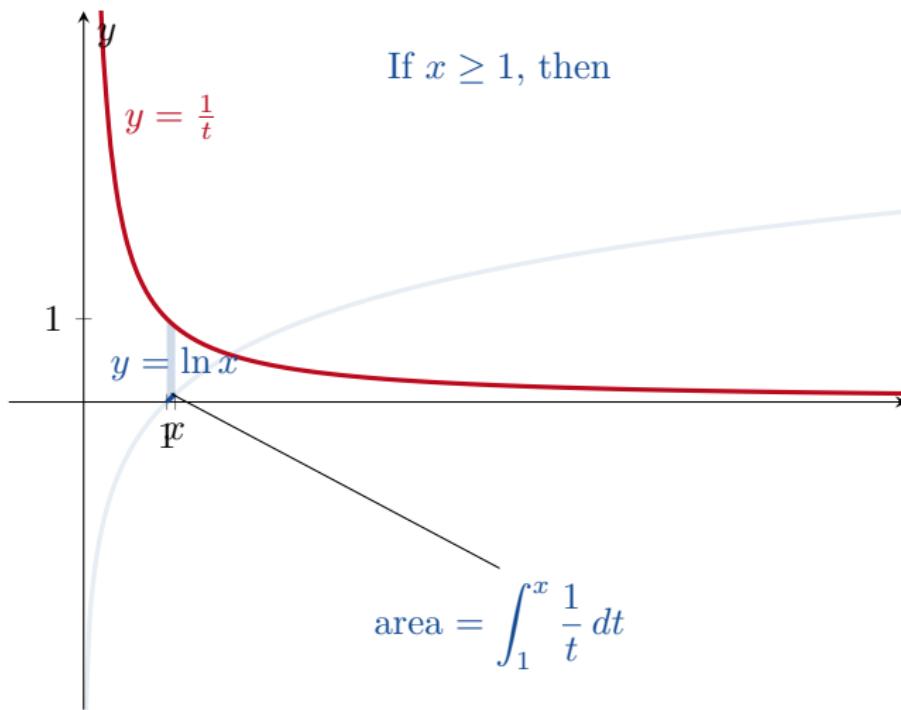


7.2 Natural Logarithms

$$\ln x = \int_1^x \frac{1}{t} dt$$



If $x \geq 1$, then

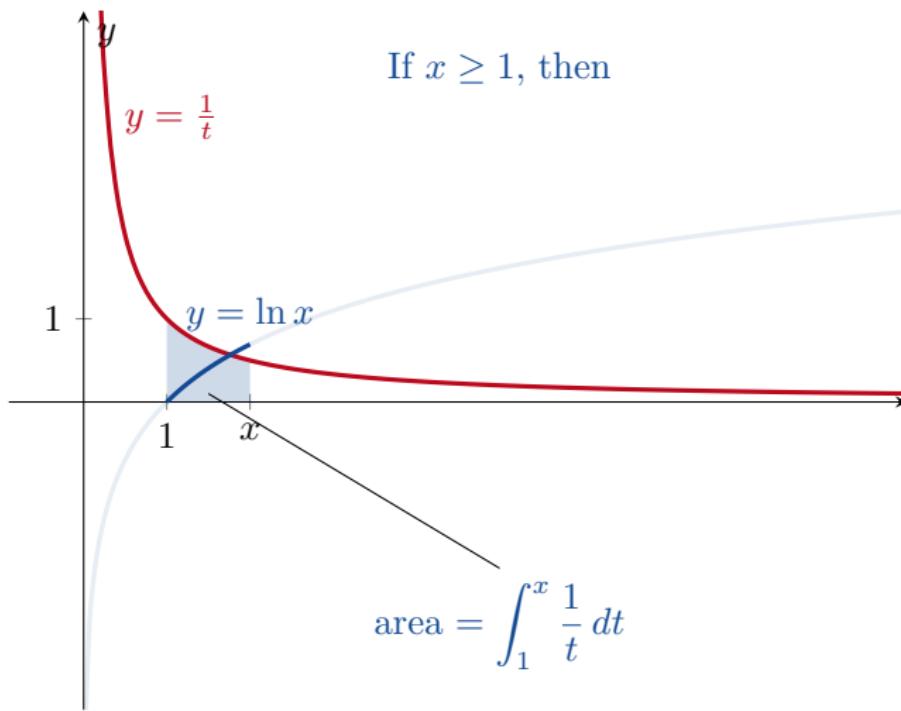


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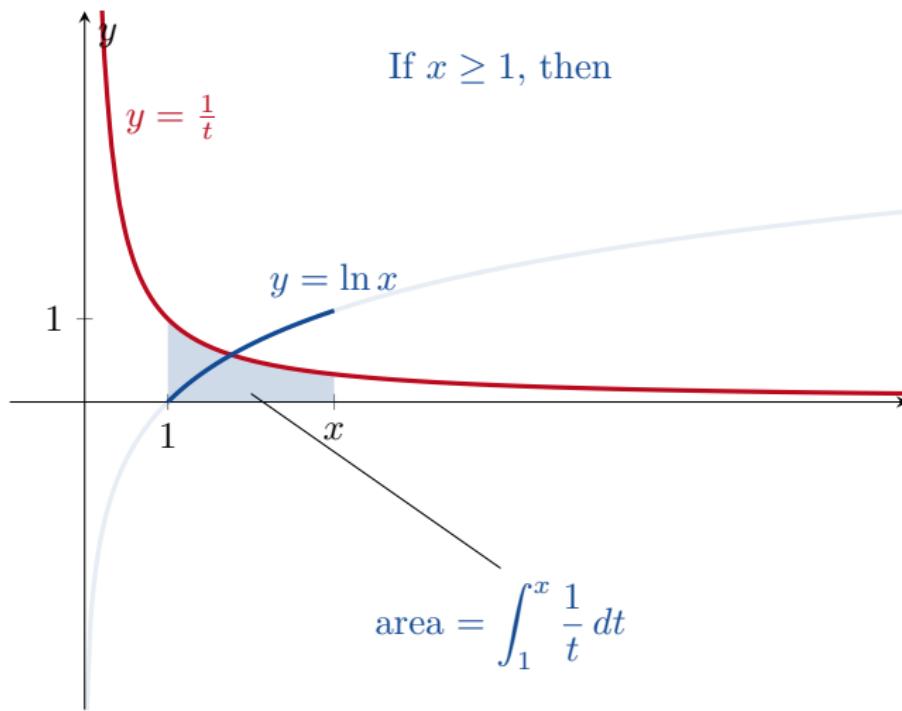


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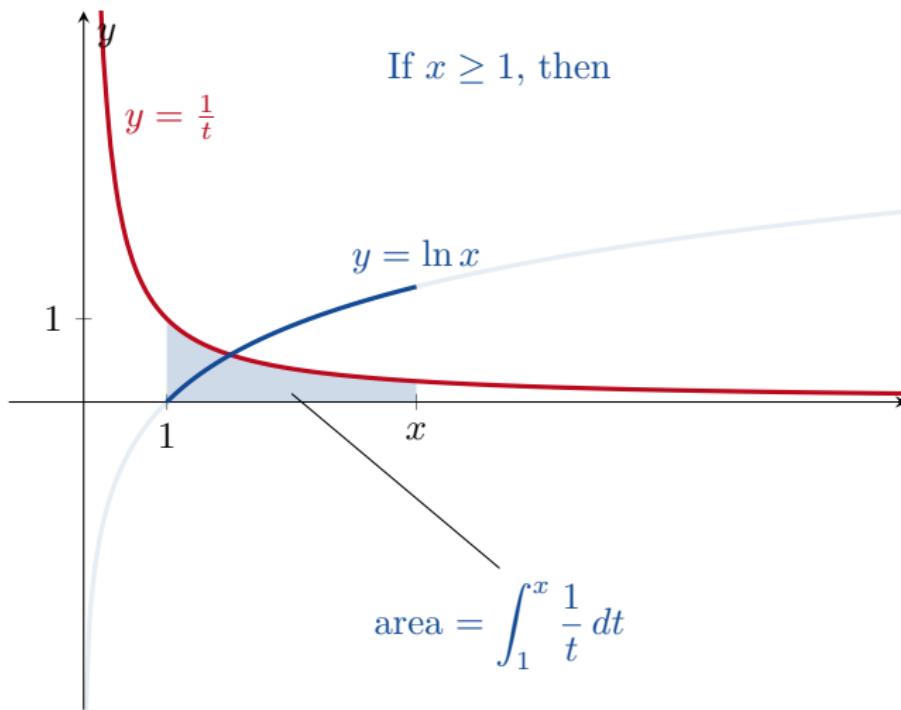


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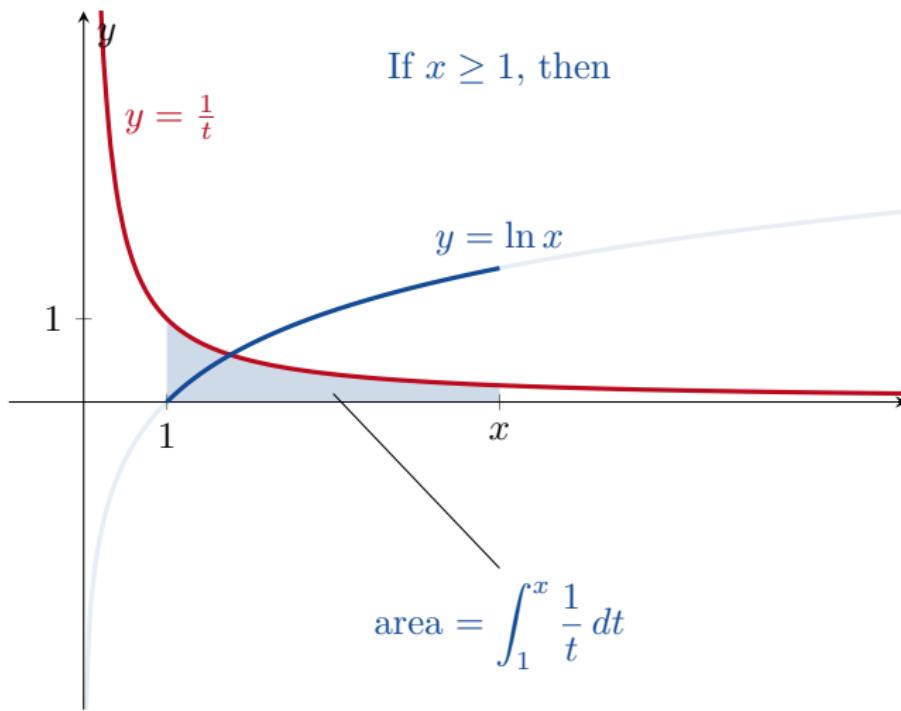


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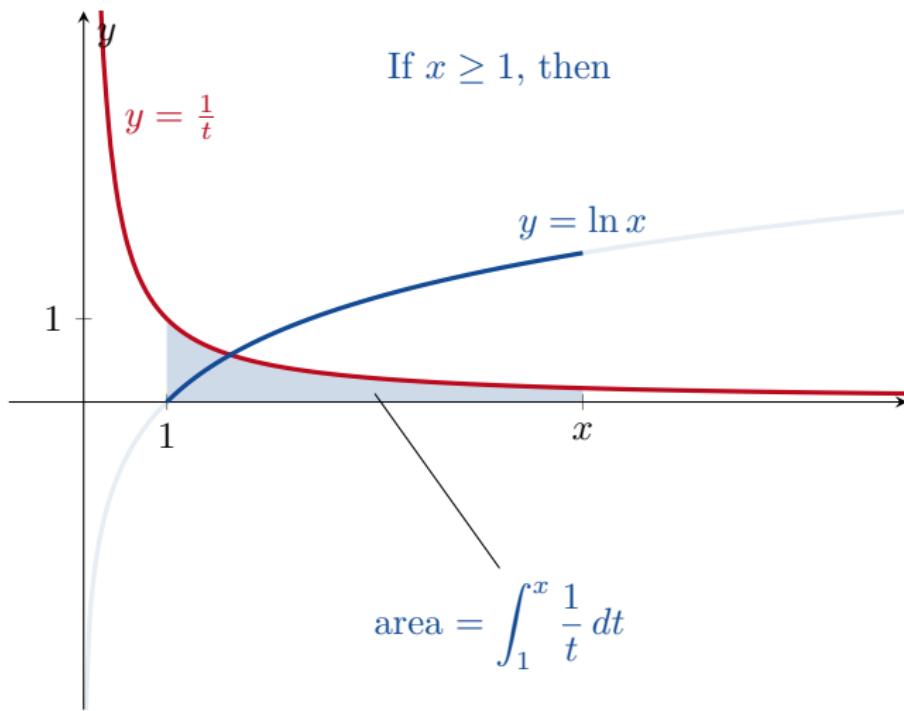


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$$\ln x = \int_1^x \frac{1}{t} dt$$



If $x \geq 1$, then

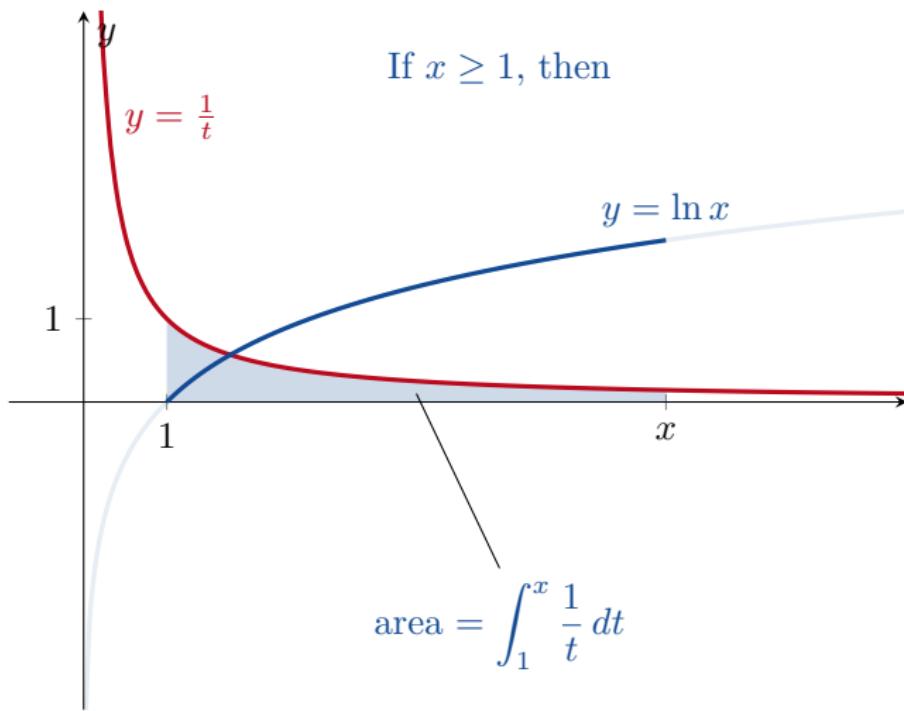


7.2 Natural Logarithms

$$\ln x = \int_1^x \frac{1}{t} dt$$



If $x \geq 1$, then



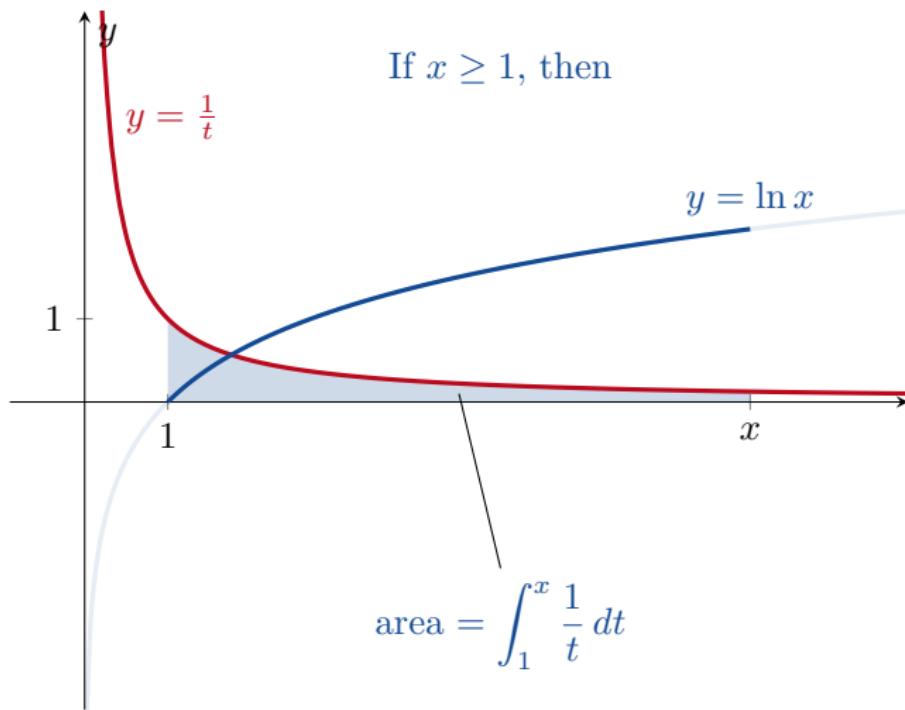
$$\text{area} = \int_1^x \frac{1}{t} dt$$

7.2 Natural Logarithms

$$\ln x = \int_1^x \frac{1}{t} dt$$



If $x \geq 1$, then

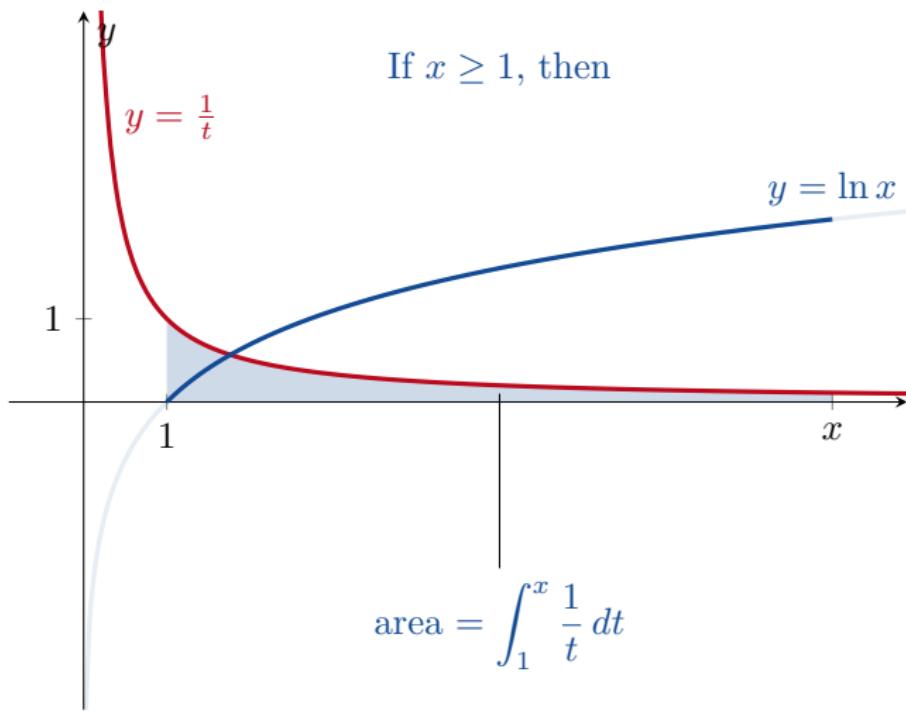


7.2 Natural Logarithms

$$\ln x = \int_1^x \frac{1}{t} dt$$

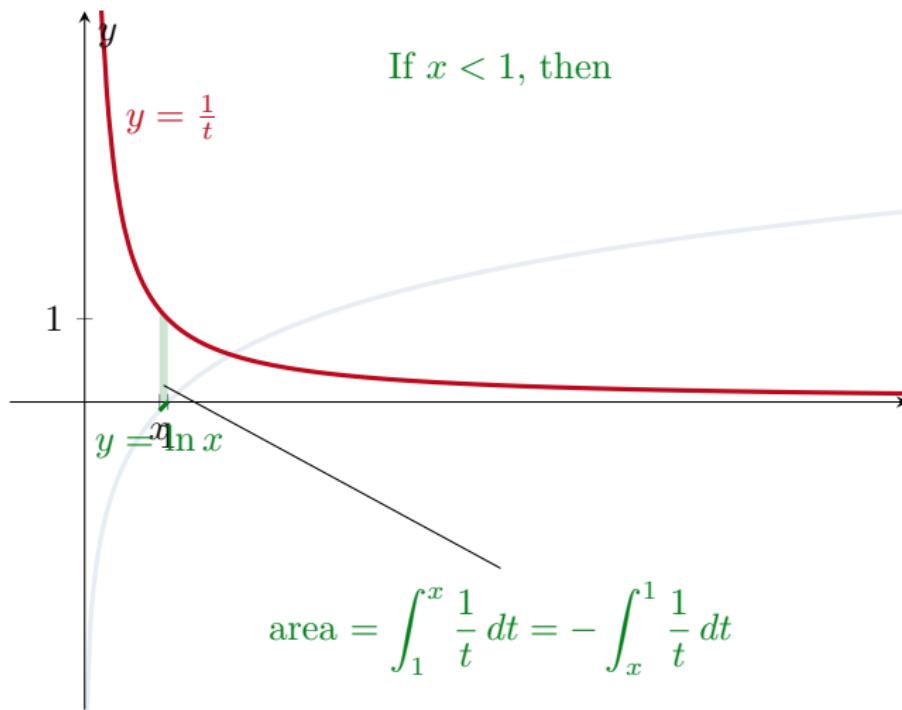


If $x \geq 1$, then



7.2 Natural Logarithms

$$\ln x = \int_1^x \frac{1}{t} dt$$

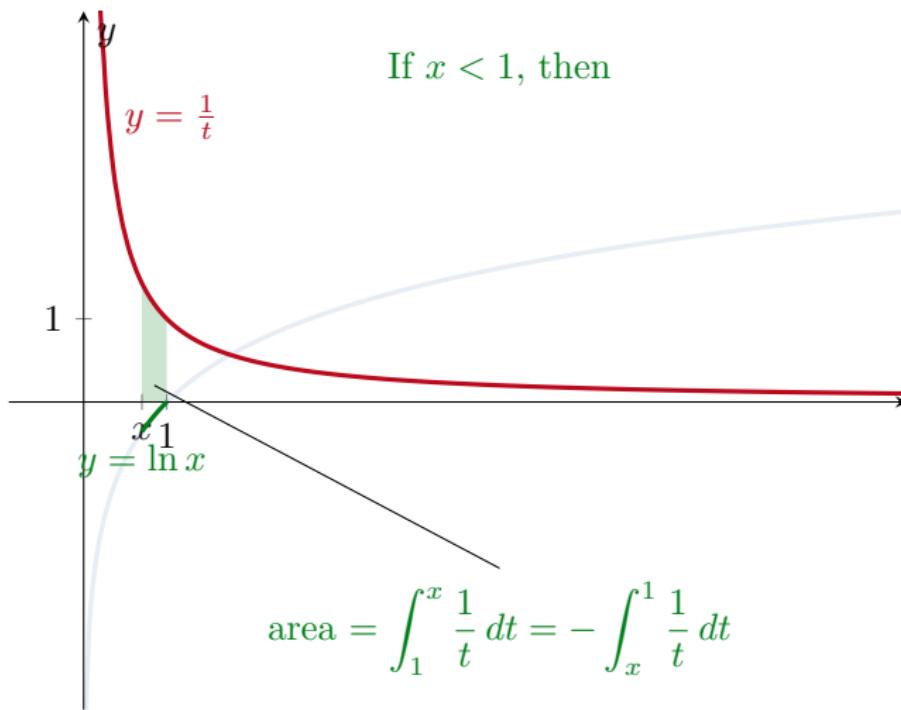


7.2 Natural Logarithms

$$\ln x = \int_1^x \frac{1}{t} dt$$



If $x < 1$, then

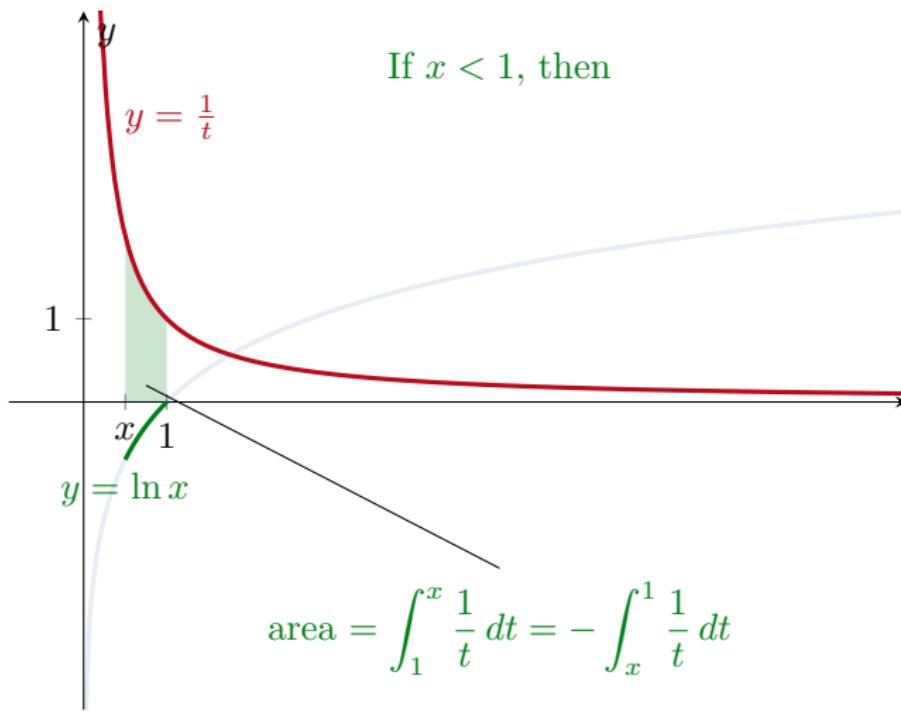


7.2 Natural Logarithms

$$\ln x = \int_1^x \frac{1}{t} dt$$

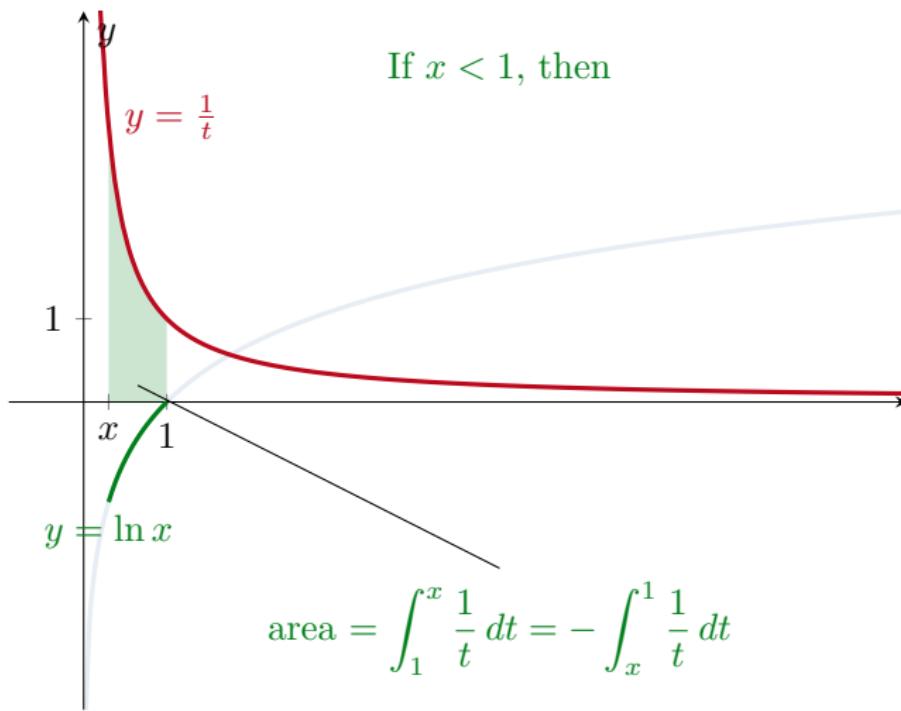


If $x < 1$, then



7.2 Natural Logarithms

$$\ln x = \int_1^x \frac{1}{t} dt$$

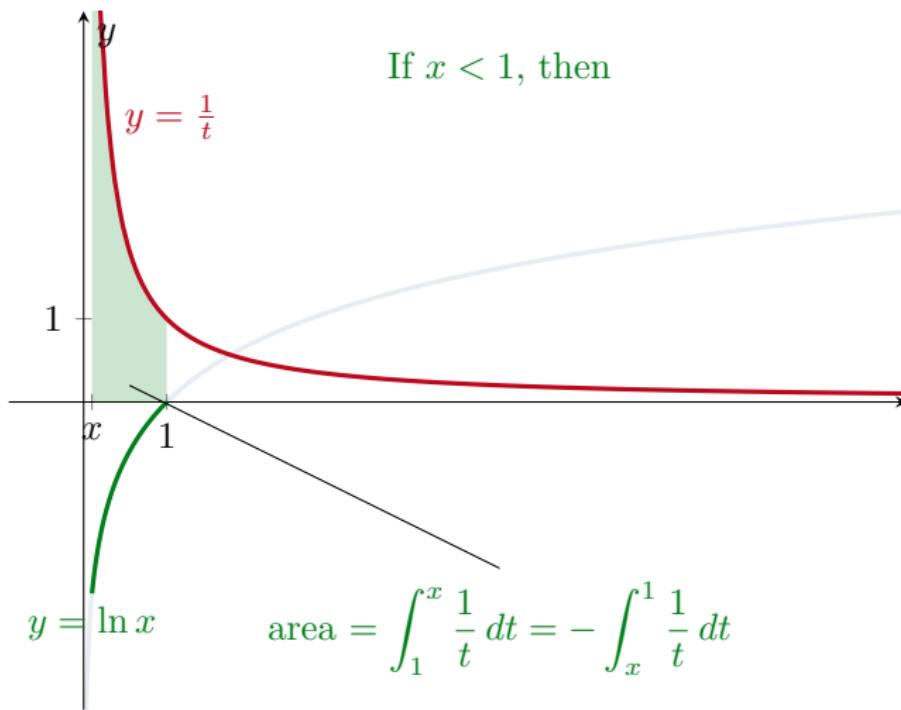


7.2 Natural Logarithms

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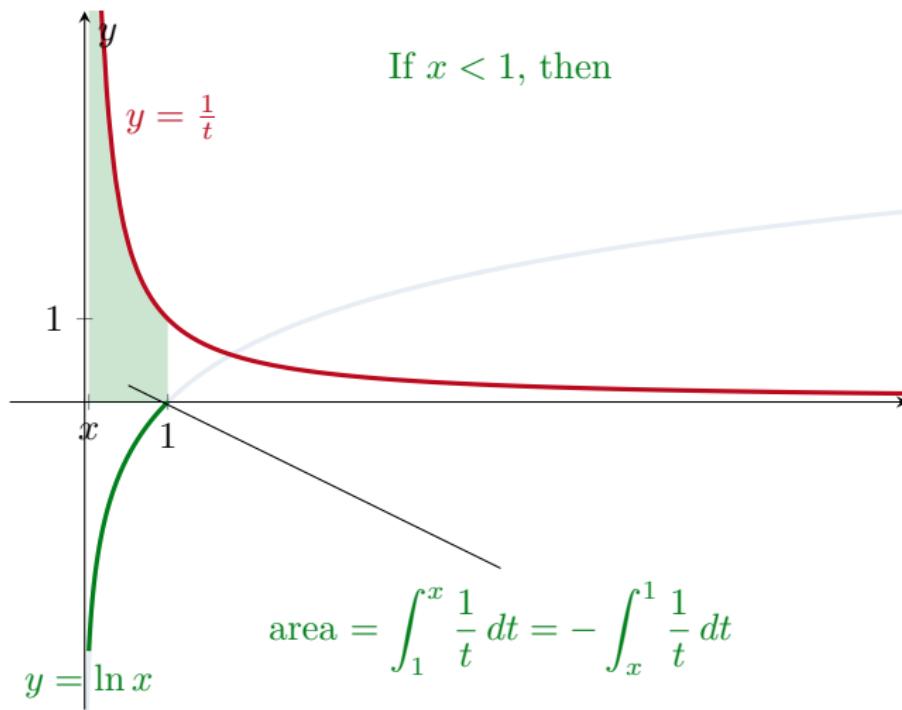


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7.2 Natural Logarithms

$$\ln x = \int_1^x \frac{1}{t} dt$$



7.2 Natural Logarithms

$$\ln x = \int_1^x \frac{1}{t} dt$$



x	ln x
0	undefined
0.05	-3.00
0.5	-0.69
1	0
2	0.69
3	1.10
4	1.39
10	2.30

7.2 Natural Logarithms

$$\ln x = \int_1^x \frac{1}{t} dt$$

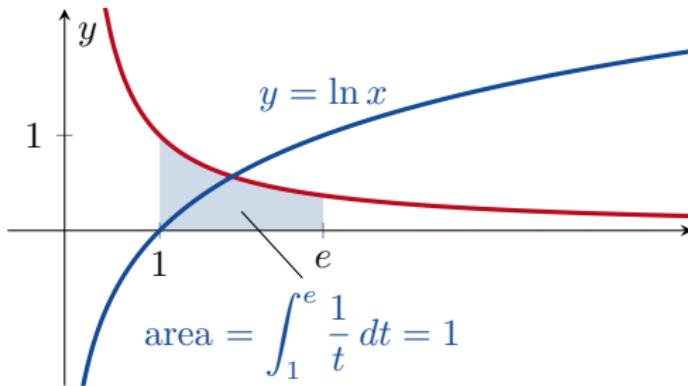


x	ln x
0	undefined
0.05	-3.00
0.5	-0.69
1	0
2	0.69
3	1.10
4	1.39
10	2.30

There is an important number between 2 and 3 where the natural logarithm is equal to 1.

7.2 Natural Logarithms

$$\ln x = \int_1^x \frac{1}{t} dt$$



Definition

The *number e* is the number which satisfies

$$\ln(e) = \int_1^e \frac{1}{t} dt = 1.$$

7.2 Natural Logarithms

$$\ln x = \int_1^x \frac{1}{t} dt$$



The Derivative of $y = \ln x$

Recall the first part of the Fundamental Theorem of Calculus:

$$\frac{d}{dx} \int_a^x f(t) dt = f(x).$$

7.2 Natural Logarithms

$$\ln x = \int_1^x \frac{1}{t} dt$$



The Derivative of $y = \ln x$

Recall the first part of the Fundamental Theorem of Calculus:

$$\frac{d}{dx} \int_a^x f(t) dt = f(x).$$

It follows that

$$\frac{d}{dx} \ln x = \frac{d}{dx} \int_1^x \frac{1}{t} dt = \frac{1}{x}.$$

7.2 Natural Logarithms



If

- $u(x)$ is differentiable; and
- $u(x) > 0$,

then it follows by the Chain Rule that



7.2 Natural Logarithms



If

- $u(x)$ is differentiable; and
- $u(x) > 0$,

then it follows by the Chain Rule that

$$\boxed{\frac{d}{dx} \ln u = \frac{1}{u} \frac{du}{dx}}.$$

EXAMPLE 1 We use Equation (2) to find derivatives.

(a) $\frac{d}{dx} \ln 2x = \frac{1}{2x} \frac{d}{dx}(2x) = \frac{1}{2x}(2) = \frac{1}{x}, \quad x > 0$

(b) Equation (2) with $u = x^2 + 3$ gives

$$\frac{d}{dx} \ln(x^2 + 3) = \frac{1}{x^2 + 3} \cdot \frac{d}{dx}(x^2 + 3) = \frac{1}{x^2 + 3} \cdot 2x = \frac{2x}{x^2 + 3}.$$

(c) Equation (2) with $u = |x|$ gives an important derivative:

$$\begin{aligned}\frac{d}{dx} \ln |x| &= \frac{d}{du} \ln u \cdot \frac{du}{dx} && u = |x|, x \neq 0 \\&= \frac{1}{u} \cdot \frac{x}{|x|} && \frac{d}{dx}(|x|) = \frac{x}{|x|} \\&= \frac{1}{|x|} \cdot \frac{x}{|x|} && \text{Substitute for } u. \\&= \frac{x}{x^2} \\&= \frac{1}{x}.\end{aligned}$$

So $1/x$ is the derivative of $\ln x$ on the domain $x > 0$, and the derivative of $\ln(-x)$ on the domain $x < 0$. ■

THEOREM 2—Algebraic Properties of the Natural Logarithm

For any numbers $b > 0$ and $x > 0$, the natural logarithm satisfies the following rules:

1. Product Rule: $\ln bx = \ln b + \ln x$

2. Quotient Rule: $\ln \frac{b}{x} = \ln b - \ln x$

3. Reciprocal Rule: $\ln \frac{1}{x} = -\ln x$ Rule 2 with $b = 1$

4. Power Rule: $\ln x^r = r \ln x$ For r rational

EXAMPLE 2 We apply the rules in Theorem 2.

- (a) $\ln 4 + \ln \sin x = \ln(4 \sin x)$ Product Rule
- (b) $\ln \frac{x+1}{2x-3} = \ln(x+1) - \ln(2x-3)$ Quotient Rule
- (c) $\ln \frac{1}{8} = -\ln 8$ Reciprocal Rule
 $= -\ln 2^3 = -3 \ln 2$ Power Rule

7.2 Natural Logarithms

Proof that $\ln bx = \ln b + \ln x$.

First note that

$$\frac{d}{dx} \ln(bx) = \frac{b}{bx} = \frac{1}{x}$$

7.2 Natural Logarithms

Proof that $\ln bx = \ln b + \ln x$.

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7.2 Natural Logarithms

Proof that $\ln bx = \ln b + \ln x$.

First note that

$$\frac{d}{dx} \ln(bx) = \frac{b}{bx} = \frac{1}{x} = \frac{d}{dx} \ln x.$$

By the second corollary of the Mean Value Theorem, this means that

$$\ln bx = \ln x + C$$

for some constant C .

7.2 Natural Logarithms

Proof that $\ln bx = \ln b + \ln x$.

First note that

$$\frac{d}{dx} \ln(bx) = \frac{b}{bx} = \frac{1}{x} = \frac{d}{dx} \ln x.$$

By the second corollary of the Mean Value Theorem, this means that

$$\ln bx = \ln x + C$$

for some constant C . Putting in $x = 1$, we have

$$\ln(b \cdot 1) = \ln 1 + C$$

7.2 Natural Logarithms

Proof that $\ln bx = \ln b + \ln x$.

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$$\ln b = \ln(b \cdot 1) = \ln 1 + C = 0 + C = C.$$

7.2 Natural Logarithms

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for some constant C . Putting in $x = 1$, we have

$$\ln b = \ln(b \cdot 1) = \ln 1 + C = 0 + C = C.$$

Therefore

$$\ln bx = \ln x + \ln b.$$



7.2 Natural Logarithms



Proof that $\ln x^r = r \ln x$ (assuming $r \in \mathbb{Q}$.)

Using the Chain Rule with $u(x) = x^r$, we find that

$$\frac{d}{dx} \ln x^r = \frac{d}{dx} \ln u = \frac{1}{u} \frac{du}{dx} = \frac{1}{x^r} rx^{r-1} = r \cdot \frac{1}{x} = \frac{d}{dx} r \ln x.$$

7.2 Natural Logarithms

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7.2 Natural Logarithms



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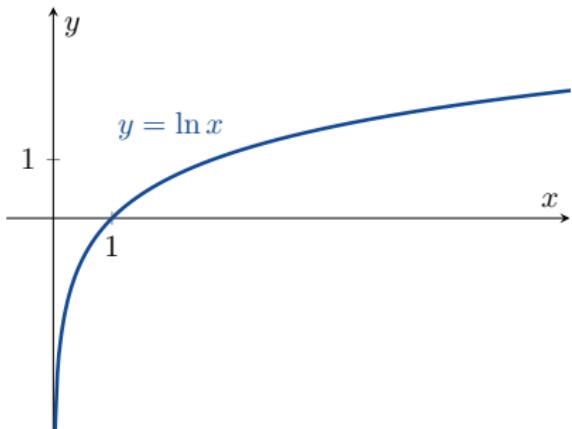
for some constant C . Putting in $x = 1$, we find $C = 0$ and we are finished.



7.2 Natural Logarithms



The Graph and Range of $\ln x$



■ Note first that

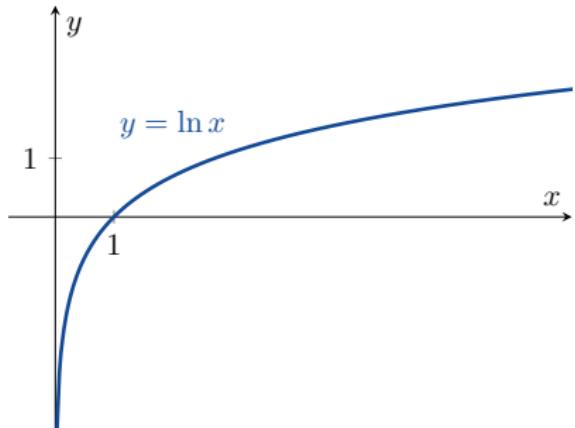
$$\frac{d}{dx} \ln x = \frac{1}{x} > 0$$

for all $x > 0$. So $\ln x$ is an increasing function.

7.2 Natural Logarithms



The Graph and Range of $\ln x$



■ Note first that

$$\frac{d}{dx} \ln x = \frac{1}{x} > 0$$

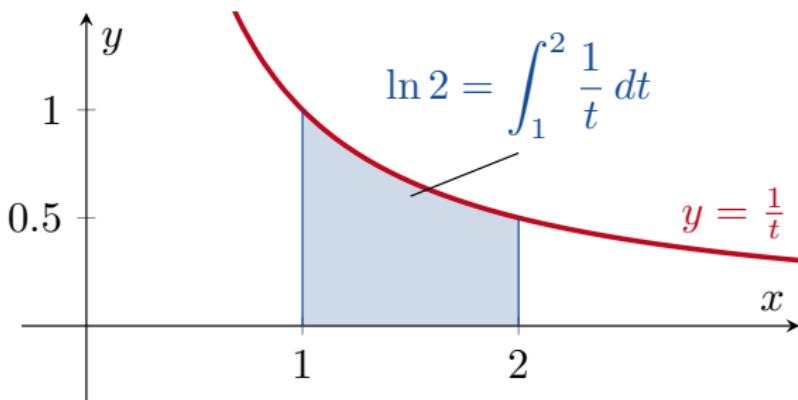
for all $x > 0$. So $\ln x$ is an increasing function.

■ Moreover

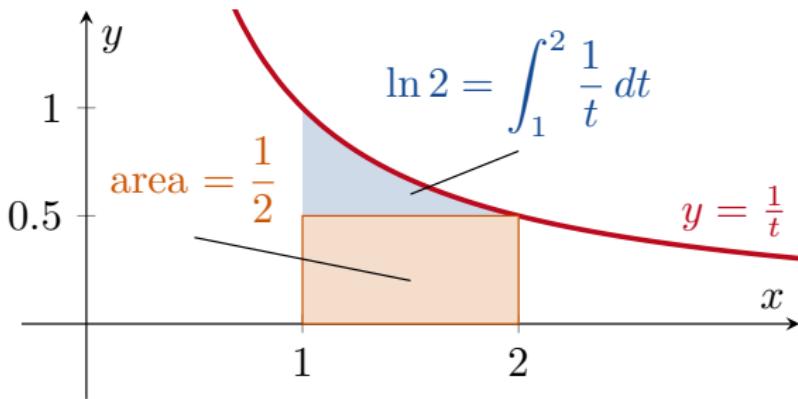
$$\frac{d^2}{dx^2} \ln x = \frac{d}{dx} \frac{1}{x} = -\frac{1}{x^2} < 0.$$

So $y = \ln x$ is concave down.

7.2 Natural Logarithms



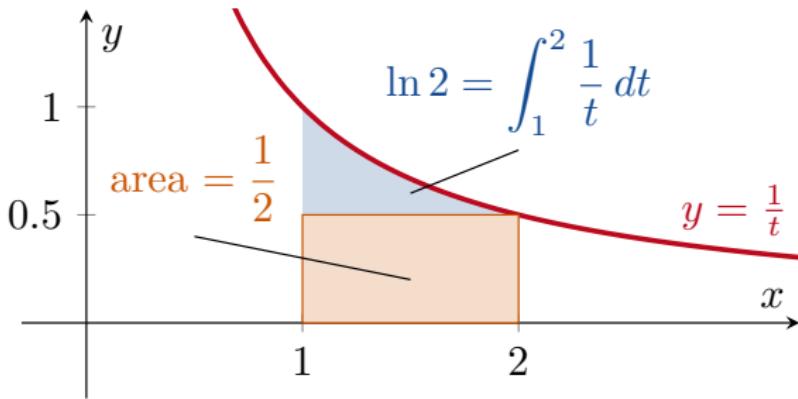
7.2 Natural Logarithms



Note that

$$\ln 2 > \frac{1}{2}.$$

7.2 Natural Logarithms



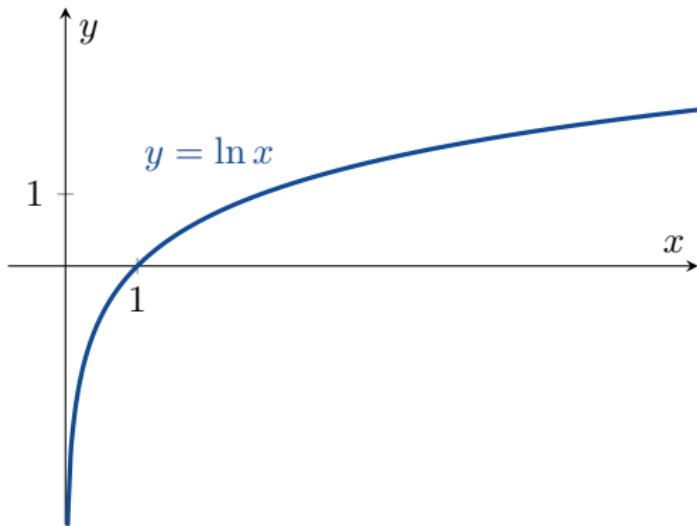
Note that

$$\ln 2 > \frac{1}{2}.$$

It follows that

$$\ln 2^n = n \ln 2 > n \left(\frac{1}{2} \right) = \frac{n}{2}.$$

7.2 Natural Logarithms



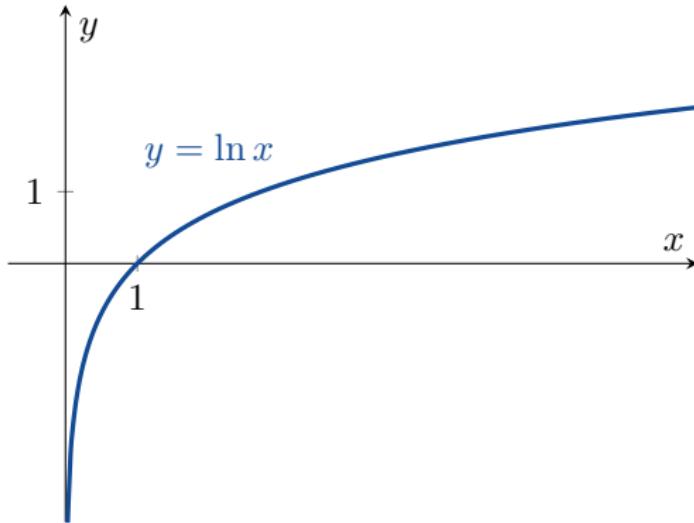
So

$$\lim_{n \rightarrow \infty} \ln 2^n \geq \lim_{n \rightarrow \infty} \frac{n}{2} = \infty.$$

Since $\ln x$ is an increasing function, we also have that

$$\lim_{x \rightarrow \infty} \ln x = \infty.$$

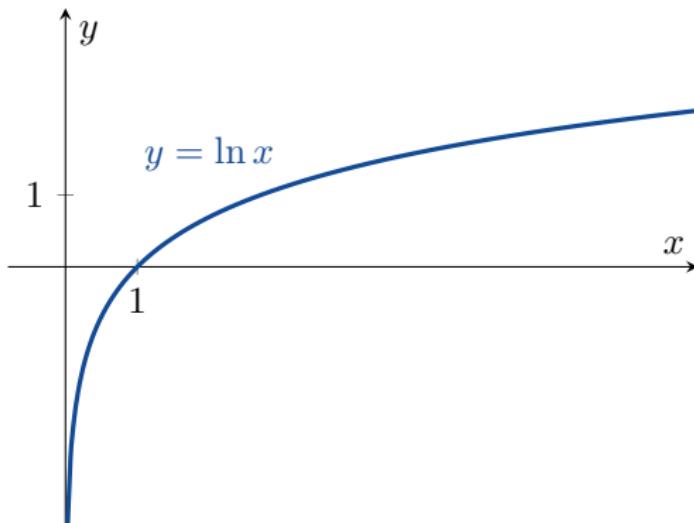
7.2 Natural Logarithms



Using $t = \frac{1}{x}$, we also have that

$$\lim_{x \rightarrow 0^+} \ln x = \lim_{t \rightarrow \infty} \ln \frac{1}{t} = \lim_{t \rightarrow \infty} \ln t^{-1} = \lim_{t \rightarrow \infty} -\ln t = -\infty.$$

7.2 Natural Logarithms



- The domain of $\ln x$ is $(0, \infty)$.
- The range of $\ln x$ is \mathbb{R} .

7.2 Natural Logarithms



The Integral $\int \frac{1}{u} du$

If

- $u(x)$ is differentiable; and
- $u(x) \neq 0$,

then

$$\int \frac{1}{u} du = \ln |u| + C.$$

7.2 Natural Logarithms

The Integral $\int \frac{1}{u} du$

If

- $u(x)$ is differentiable; and
- $u(x) \neq 0$,

then

$$\int \frac{1}{u} du = \ln |u| + C.$$

If $u = f(x)$, then $du = \frac{du}{dx} dx = f'(x) dx$. Thus

$$\int \frac{f'(x)}{f(x)} dx = \ln |f(x)| + C.$$

7.2 Natural Logarithms

$$\int \frac{du}{u} = \ln |u| + C$$



Example

Calculate $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{4 \cos \theta}{3 + 2 \sin \theta} d\theta.$

The idea is to rewrite this integral in the form $\int \frac{du}{u}.$

7.2 Natural Logarithms

$$\int \frac{du}{u} = \ln |u| + C$$



Example

Calculate $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{4 \cos \theta}{3 + 2 \sin \theta} d\theta$.

The idea is to rewrite this integral in the form $\int \frac{du}{u}$.

Let $u = 3 + 2 \cos \theta$. Then $du = -2 \sin \theta d\theta$.

7.2 Natural Logarithms

$$\int \frac{du}{u} = \ln|u| + C$$



Example

Calculate $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{4 \cos \theta}{3 + 2 \sin \theta} d\theta$.

The idea is to rewrite this integral in the form $\int \frac{du}{u}$.

Let $u = 3 + 2 \cos \theta$. Then $du = 2 \sin \theta d\theta$. Moreover

$$\theta = \frac{\pi}{2} \quad \Rightarrow \quad u = 3 + 2 \cos \frac{\pi}{2} = 3 + 2 = 5$$

$$\theta = -\frac{\pi}{2} \quad \Rightarrow \quad u = 3 + 2 \cos \left(-\frac{\pi}{2}\right) = 3 - 2 = 1.$$

7.2 Natural Logarithms

$$\int \frac{du}{u} = \ln|u| + C$$



Example

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Therefore

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{2 \cdot 2 \cos \theta}{3 + 2 \sin \theta} d\theta = \int_1^5 \frac{2}{u} du$$

7.2 Natural Logarithms

$$\int \frac{du}{u} = \ln |u| + C$$



Example

Calculate $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{4 \cos \theta}{3 + 2 \sin \theta} d\theta$.

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Therefore

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{2 \cdot 2 \cos \theta}{3 + 2 \sin \theta} d\theta = \int_1^5 \frac{2}{u} du = \left[2 \ln |u| \right]_1^5 = 2 \ln |5| - 2 \ln |1| = 2 \ln 5.$$

Break

We will continue at 2pm

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$$\int \frac{1}{d} \, d\text{house} =$$



7.2 Natural Logarithms



The Integrals of $\tan x$, $\cot x$, $\sec x$ and $\operatorname{cosec} x$

We can also use

$$\int \frac{1}{u} du = \ln |u| + C$$

to integrate $\tan x$, $\cot x$, $\sec x$ and $\operatorname{cosec} x$.

7.2 Natural Logarithms

$$\int \frac{du}{u} = \ln |u| + C$$



The Integral of $\tan x$

If $u = \cos x$, then $du = -\sin x dx$ and

$$\begin{aligned}\int \tan x \, dx &= \int \frac{\sin x}{\cos x} \, dx = \\&= \\&= \\&= \\&= .\end{aligned}$$

7.2 Natural Logarithms

$$\int \frac{du}{u} = \ln |u| + C$$



The Integral of $\tan x$

If $u = \cos x$, then $du = -\sin x dx$ and

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7.2 Natural Logarithms

$$\int \frac{du}{u} = \ln |u| + C$$



The Integral of $\tan x$

If $u = \cos x$, then $du = -\sin x dx$ and

$$\begin{aligned}\int \tan x dx &= \int \frac{\sin x}{\cos x} dx = \int \frac{-du}{u} \\&= -\ln |u| + C = -\ln |\cos x| + C \\&= \\&= \\&= .\end{aligned}$$

7.2 Natural Logarithms

$$\int \frac{du}{u} = \ln |u| + C$$



The Integral of $\tan x$

If $u = \cos x$, then $du = -\sin x dx$ and

$$\begin{aligned}\int \tan x \, dx &= \int \frac{\sin x}{\cos x} \, dx = \int \frac{-du}{u} \\&= -\ln |u| + C = -\ln |\cos x| + C \\&= \ln |\cos x|^{-1} + C \\&= \ln \frac{1}{|\cos x|} + C \\&= \ln |\sec x| + C.\end{aligned}$$

7.2 Natural Logarithms

$$\int \frac{du}{u} = \ln |u| + C$$



The Integral of $\cot x$

If $u = \sin x$, then

$$\int \cot x \, dx = \int \frac{\cos x}{\sin x} \, dx = \dots = .$$

(you fill in the details)

7.2 Natural Logarithms

$$\int \frac{du}{u} = \ln |u| + C$$



The Integral of $\cot x$

If $u = \sin x$, then

$$\int \cot x \, dx = \int \frac{\cos x}{\sin x} \, dx = \int \frac{du}{u} = \dots .$$

(you fill in the details)

7.2 Natural Logarithms

$$\int \frac{du}{u} = \ln |u| + C$$



The Integral of $\cot x$

If $u = \sin x$, then

$$\int \cot x \, dx = \int \frac{\cos x}{\sin x} \, dx = \int \frac{du}{u} = \dots = -\ln |\cosec x| + C.$$

(you fill in the details)

$$\int \frac{du}{u} = \ln|u| + C \quad (\tan x)' = \sec^2 x \quad (\sec x)' = \sec x \tan x$$



The Integral of $\sec x$

This time, we need to do an extra step. We will multiply and divide by $u = (\sec x + \tan x)$.

$$\int \frac{du}{u} = \ln|u| + C \quad (\tan x)' = \sec^2 x \quad (\sec x)' = \sec x \tan x$$



The Integral of $\sec x$

This time, we need to do an extra step. We will multiply and divide by $u = (\sec x + \tan x)$. We calculate that

$$\int \sec x \, dx = \int \sec x \left(\frac{\sec x + \tan x}{\sec x + \tan x} \right) \, dx$$

=

=

=

.

$$\int \frac{du}{u} = \ln|u| + C \quad (\tan x)' = \sec^2 x \quad (\sec x)' = \sec x \tan x$$



The Integral of $\sec x$

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$$\int \sec x \, dx = \int \sec x \left(\frac{\sec x + \tan x}{\sec x + \tan x} \right) \, dx$$

$$= \int \frac{\sec^2 x + \sec x \tan x}{\sec x + \tan x} \, dx$$

=

=

.

$$\int \frac{du}{u} = \ln|u| + C \quad (\tan x)' = \sec^2 x \quad (\sec x)' = \sec x \tan x$$



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$$\int \frac{du}{u} = \ln|u| + C \quad (\tan x)' = \sec^2 x \quad (\sec x)' = \sec x \tan x$$



The Integral of $\sec x$

This time, we need to do an extra step. We will multiply and divide by $u = (\sec x + \tan x)$. We calculate that

$$\begin{aligned}\int \sec x \, dx &= \int \sec x \left(\frac{\sec x + \tan x}{\sec x + \tan x} \right) \, dx \\ &= \int \frac{\sec^2 x + \sec x \tan x}{\sec x + \tan x} \, dx \\ &= \int \frac{du}{u} \\ &= \ln|u| + C = \ln|\sec x + \tan x| + C.\end{aligned}$$

$$\int \frac{du}{u} = \ln|u| + C \quad (\cot x)' = -\operatorname{cosec}^2 x \quad (\operatorname{cosec} x)' = -\operatorname{cosec} x \cot x$$



The Integral of $\operatorname{cosec} x$

This is similar, except that we will multiply and divide by $u = (\operatorname{cosec} x + \cot x)$ instead. We calculate that

$$\int \operatorname{cosec} x \, dx = \int \operatorname{cosec} x \left(\frac{\operatorname{cosec} x + \cot x}{\operatorname{cosec} x + \cot x} \right) \, dx$$

=

=

=

.

$$\int \frac{du}{u} = \ln|u| + C \quad (\cot x)' = -\operatorname{cosec}^2 x \quad (\operatorname{cosec} x)' = -\operatorname{cosec} x \cot x$$



The Integral of $\operatorname{cosec} x$

This is similar, except that we will multiply and divide by $u = (\operatorname{cosec} x + \cot x)$ instead. We calculate that

$$\int \operatorname{cosec} x \, dx = \int \operatorname{cosec} x \left(\frac{\operatorname{cosec} x + \cot x}{\operatorname{cosec} x + \cot x} \right) \, dx$$

$$= \int \frac{\operatorname{cosec}^2 x + \operatorname{cosec} x \cot x}{\sec x + \tan x} \, dx$$

=

=

.

$$\int \frac{du}{u} = \ln|u| + C \quad (\cot x)' = -\operatorname{cosec}^2 x \quad (\operatorname{cosec} x)' = -\operatorname{cosec} x \cot x$$



The Integral of $\operatorname{cosec} x$

This is similar, except that we will multiply and divide by $u = (\operatorname{cosec} x + \cot x)$ instead. We calculate that

$$\begin{aligned}\int \operatorname{cosec} x \, dx &= \int \operatorname{cosec} x \left(\frac{\operatorname{cosec} x + \cot x}{\operatorname{cosec} x + \cot x} \right) \, dx \\&= \int \frac{\operatorname{cosec}^2 x + \operatorname{cosec} x \cot x}{\sec x + \tan x} \, dx \\&= \int \frac{-du}{u} \\&= \end{aligned}$$

.

$$\int \frac{du}{u} = \ln |u| + C \quad (\cot x)' = -\operatorname{cosec}^2 x \quad (\operatorname{cosec} x)' = -\operatorname{cosec} x \cot x$$



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Integrals of the tangent, cotangent, secant, and cosecant functions

$$\int \tan u \, du = \ln |\sec u| + C$$

$$\int \sec u \, du = \ln |\sec u + \tan u| + C$$

$$\int \cot u \, du = \ln |\sin u| + C$$

$$\int \csc u \, du = -\ln |\csc u + \cot u| + C$$

EXAMPLE 4

$$\begin{aligned}\int_0^{\pi/6} \tan 2x \, dx &= \int_0^{\pi/3} \tan u \frac{du}{2} = \frac{1}{2} \int_0^{\pi/3} \tan u \, du \\&= \frac{1}{2} \ln |\sec u| \Big|_0^{\pi/3} = \frac{1}{2} (\ln 2 - \ln 1) = \frac{1}{2} \ln 2\end{aligned}$$

Substitute $u = 2x$,
 $dx = du/2$,
 $u(0) = 0$,
 $u(\pi/6) = \pi/3$

7.2 Natural Logarithms



Logarithmic Differentiation

Sometimes when we are trying to find a derivative, it becomes easier if we take \ln of both sides of an equation.

7.2 Natural Logarithms



Logarithmic Differentiation

Sometimes when we are trying to find a derivative, it becomes easier if we take \ln of both sides of an equation.

Example

Find $\frac{dy}{dx}$ if $y = \frac{(x^2 + 1)(x + 3)^{\frac{1}{2}}}{x - 1}$ for $x > 1$.

7.2 Natural Logarithms



We are going to take our equation and apply **ln** to both sides.

$$y = \frac{(x^2 + 1)(x + 3)^{\frac{1}{2}}}{x - 1}$$

=

=

.

7.2 Natural Logarithms



We are going to take our equation and apply \ln to both sides.

$$\ln y = \ln \frac{(x^2 + 1)(x + 3)^{\frac{1}{2}}}{x - 1}$$

=

=

.

7.2 Natural Logarithms



We are going to take our equation and apply \ln to both sides.

$$\begin{aligned}\ln y &= \ln \frac{(x^2 + 1)(x + 3)^{\frac{1}{2}}}{x - 1} \\&= \ln \left((x^2 + 1)(x + 3)^{\frac{1}{2}} \right) - \ln(x - 1) \\&= \end{aligned}$$

7.2 Natural Logarithms

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7.2 Natural Logarithms

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Now we can differentiate both sides.

$$\frac{d}{dx} \ln y = \frac{d}{dx} \left(\ln(x^2 + 1) + \frac{1}{2} \ln(x + 3) - \ln(x - 1) \right)$$

7.2 Natural Logarithms

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Now we can differentiate both sides.

$$\begin{aligned}\frac{d}{dx} \ln y &= \frac{d}{dx} \left(\ln(x^2 + 1) + \frac{1}{2} \ln(x + 3) - \ln(x - 1) \right) \\ \frac{1}{y} \frac{dy}{dx} &= \frac{2x}{x^2 + 1} + \frac{1}{2} \cdot \frac{1}{x + 3} - \frac{1}{x - 1}\end{aligned}$$

7.2 Natural Logarithms

We are going to take our equation and apply \ln to both sides.

$$\begin{aligned}\ln y &= \ln \frac{(x^2 + 1)(x + 3)^{\frac{1}{2}}}{x - 1} \\&= \ln \left((x^2 + 1)(x + 3)^{\frac{1}{2}} \right) - \ln(x - 1) \\&= \ln(x^2 + 1) + \frac{1}{2} \ln(x + 3) - \ln(x - 1).\end{aligned}$$

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$$\begin{aligned}\frac{d}{dx} \ln y &= \frac{d}{dx} \left(\ln(x^2 + 1) + \frac{1}{2} \ln(x + 3) - \ln(x - 1) \right) \\ \frac{1}{y} \frac{dy}{dx} &= \frac{2x}{x^2 + 1} + \frac{1}{2} \cdot \frac{1}{x + 3} - \frac{1}{x - 1} \\ \frac{dy}{dx} &= y \left(\frac{2x}{x^2 + 1} + \frac{1}{2x + 6} - \frac{1}{x - 1} \right).\end{aligned}$$

7.2 Natural Logarithms



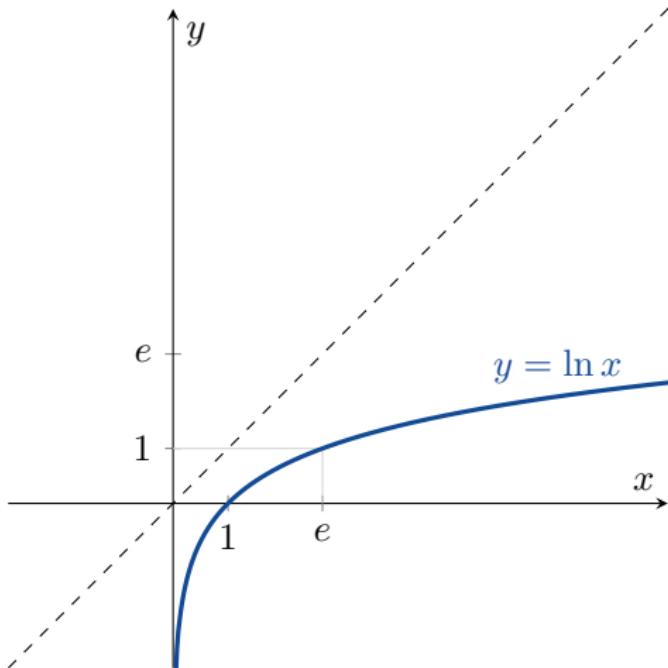
Thus

$$\begin{aligned}\frac{dy}{dx} &= y \left(\frac{2x}{x^2 + 1} + \frac{1}{2x + 6} - \frac{1}{x - 1} \right) \\ &= \left(\frac{(x^2 + 1)(x + 3)^{\frac{1}{2}}}{x - 1} \right) \left(\frac{2x}{x^2 + 1} + \frac{1}{2x + 6} - \frac{1}{x - 1} \right).\end{aligned}$$



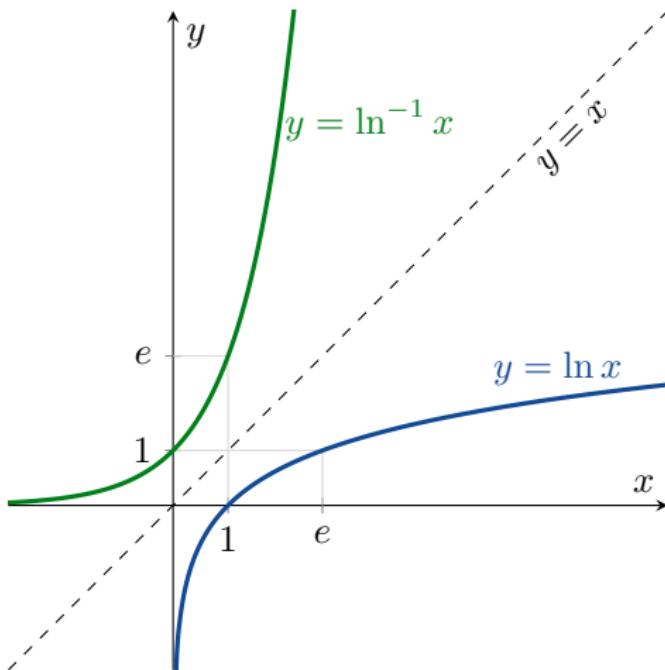
7.3 Exponential Functions

7.3 Exponential Functions



\ln is a one-to-one function with domain $(0, \infty)$ and range $(-\infty, \infty)$.

7.3 Exponential Functions



So there exists an inverse function \ln^{-1} with domain $(-\infty, \infty)$ and range $(0, \infty)$.

7.3 Exponential Functions



Definition

The inverse of the function $\ln x$ is called the *exponential function* and is written

$$\exp x = e^x.$$

7.3 Exponential Functions

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The inverse of the function $\ln x$ is called the *exponential function* and is written

$$\exp x = e^x.$$

Remark

We have

$$e^{\ln x} = x \quad (\text{for all } x > 0)$$

and

$$\ln(e^x) = x \quad (\text{for all } x).$$

EXAMPLE 1 Solve the equation $e^{2x-6} = 4$ for x .

Solution We take the natural logarithm of both sides of the equation and use the second inverse equation:

$$\ln(e^{2x-6}) = \ln 4$$

$$2x - 6 = \ln 4$$

Inverse relationship

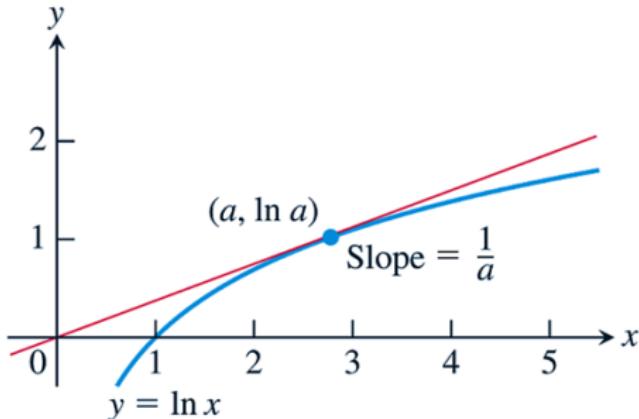
$$2x = 6 + \ln 4$$

$$x = 3 + \frac{1}{2} \ln 4 = 3 + \ln 4^{1/2}$$

$$x = 3 + \ln 2$$



7.3 Exponential Functions



EXAMPLE 2 A line with slope m passes through the origin and is tangent to the graph of $y = \ln x$. What is the value of m ?

Solution Suppose the point of tangency occurs at the unknown point $x = a > 0$. Then we know that the point $(a, \ln a)$ lies on the graph and that the tangent line at that point has slope $m = 1/a$ (Figure 7.11). Since the tangent line passes through the origin, its slope is

$$m = \frac{\ln a - 0}{a - 0} = \frac{\ln a}{a}.$$

7.3 Exponential Functions

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$$m = \frac{\ln a - 0}{a - 0} = \frac{\ln a}{a}.$$

Setting these two formulas for m equal to each other, we have

$$\frac{\ln a}{a} = \frac{1}{a}$$

$$\ln a = 1$$

$$e^{\ln a} = e^1$$

$$a = e$$

$$m = \frac{1}{e}.$$



7.3 Exponential Functions



The Derivative and Integral of e^x

$$\ln(e^x) = x$$

7.3 Exponential Functions



The Derivative and Integral of e^x

$$\frac{d}{dx} \ln(e^x) = \frac{d}{dx} x$$

7.3 Exponential Functions



The Derivative and Integral of e^x

$$\frac{d}{dx} \ln(e^x) = \frac{d}{dx} x$$

$$\frac{1}{e^x} \frac{d}{dx} e^x = 1$$

7.3 Exponential Functions



The Derivative and Integral of e^x

$$\frac{d}{dx} \ln(e^x) = \frac{d}{dx} x$$

$$\frac{1}{e^x} \frac{d}{dx} e^x = 1$$

$$\frac{d}{dx} e^x = e^x.$$

by the Chain Rule.

7.3 Exponential Functions



The Derivative and Integral of e^x

$$\frac{d}{dx} \ln(e^x) = \frac{d}{dx} x$$

$$\frac{1}{e^x} \frac{d}{dx} e^x = 1$$

$$\frac{d}{dx} e^x = e^x.$$

by the Chain Rule.

Theorem

$$\boxed{\frac{d}{dx} e^x = e^x.}$$

EXAMPLE 3 We find derivatives of the exponential using Equation (2).

(a) $\frac{d}{dx}(5e^x) = 5\frac{d}{dx}e^x = 5e^x$

(b) $\frac{d}{dx}e^{-x} = e^{-x}\frac{d}{dx}(-x) = e^{-x}(-1) = -e^{-x}$ Eq. (2) with $u = -x$

(c) $\frac{d}{dx}e^{\sin x} = e^{\sin x}\frac{d}{dx}(\sin x) = e^{\sin x} \cdot \cos x$ Eq. (2) with $u = \sin x$

(d)
$$\begin{aligned} \frac{d}{dx}\left(e^{\sqrt{3x+1}}\right) &= e^{\sqrt{3x+1}} \cdot \frac{d}{dx}\left(\sqrt{3x+1}\right) && \text{Eq. (2) with } u = \sqrt{3x+1} \\ &= e^{\sqrt{3x+1}} \cdot \frac{1}{2}(3x+1)^{-1/2} \cdot 3 &= \frac{3}{2\sqrt{3x+1}} e^{\sqrt{3x+1}} \end{aligned}$$



7.3 Exponential Functions



Since $\frac{d}{dx}e^x = e^x$, we also have

Theorem

$$\int e^x dx = e^x + C.$$

EXAMPLE 4

$$(a) \int_0^{\ln 2} e^{3x} dx = \int_0^{\ln 8} e^u \cdot \frac{1}{3} du \quad u = 3x, \quad \frac{1}{3}du = dx, \quad u(0) = 0,$$
$$u(\ln 2) = 3 \ln 2 = \ln 2^3 = \ln 8$$

$$= \frac{1}{3} \int_0^{\ln 8} e^u du$$

$$= \frac{1}{3} e^u \Big|_0^{\ln 8}$$

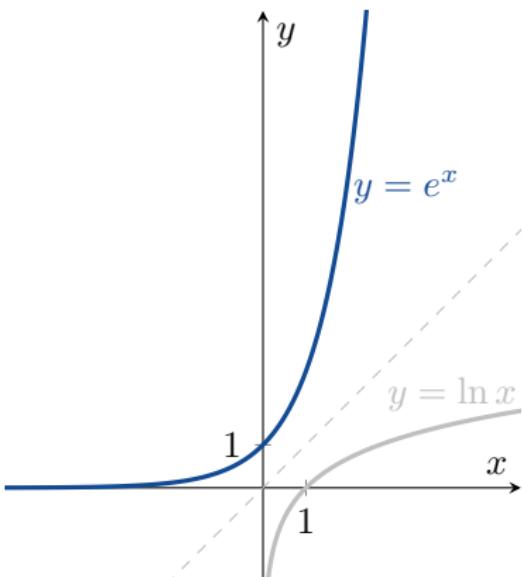
$$= \frac{1}{3}(8 - 1) = \frac{7}{3}$$

$$(b) \int_0^{\pi/2} e^{\sin x} \cos x dx = e^{\sin x} \Big|_0^{\pi/2}$$

Antiderivative from Example 2c

$$= e^1 - e^0 = e - 1$$

7.3 Exponential Functions



- e^x has domain $(-\infty, \infty)$ and range $(0, \infty)$;
- $\lim_{x \rightarrow \infty} e^x = \infty$; and
- $\lim_{x \rightarrow -\infty} e^x = 0$

7.3 Exponential Functions



Laws of Exponents

Theorem

$$1 \quad e^a e^b = e^{a+b}$$

(proof in textbook)

7.3 Exponential Functions



Laws of Exponents

Theorem

$$1 \quad e^a e^b = e^{a+b}$$

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7.3 Exponential Functions



Laws of Exponents

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$$1 \quad e^a e^b = e^{a+b}$$

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$$3 \quad \frac{e^a}{e^b} = e^{a-b}$$

(proof in textbook)

7.3 Exponential Functions



Laws of Exponents

Theorem

$$1 \quad e^a e^b = e^{a+b}$$

$$2 \quad e^{-x} = \frac{1}{e^x}$$

$$3 \quad \frac{e^a}{e^b} = e^{a-b}$$

$$4 \quad (e^a)^b = e^{ab}.$$

(proof in textbook)

7.3 Exponential Functions



The General Exponential Function a^x

Definition

For any numbers $a > 0$ and x , the *exponential function with base a* is

$$a^x = e^{x \ln a}.$$

7.3 Exponential Functions



The General Exponential Function a^x

Definition

For any numbers $a > 0$ and x , the *exponential function with base a* is

$$a^x = e^{x \ln a}.$$

Definition

For any $x > 0$ and for any $n \in \mathbb{N}$,

$$x^n = e^{n \ln x}.$$

7.3 Exponential Functions



Now we can prove

Theorem

For any $x > 0$ and any $n \in \mathbb{R}$,

$$\frac{d}{dx}x^n = nx^{n-1}.$$

7.3 Exponential Functions



Now we can prove

Theorem

For any $x > 0$ and any $n \in \mathbb{R}$,

$$\frac{d}{dx}x^n = nx^{n-1}.$$

If $x \leq 0$, then the formula holds whenever $\frac{d}{dx}x^n$, x^n and x^{n-1} all exist.

7.3 Exponential Functions



Proof.

- 1 If $x > 0$, then we calculate

$$\frac{d}{dx}x^n = \frac{d}{dx}e^{n \ln x} = e^{n \ln x} \cdot \frac{d}{dx}(n \ln x) = x^n \cdot \frac{n}{x} = nx^{n-1}.$$



7.3 Exponential Functions

Proof.

- 2 If $x < 0$, and if y' , $y = x^n$ and x^{n-1} all exists, then

$$\ln |y| = \ln |x|^n = n \ln |x| .$$



7.3 Exponential Functions

Proof.

- 2 If $x < 0$, and if y' , $y = x^n$ and x^{n-1} all exists, then

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Using Implicit Differentiation, we have

$$\ln |y| = n \ln |x|$$



7.3 Exponential Functions

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$$\ln |y| = \ln |x|^n = n \ln |x|.$$

Using Implicit Differentiation, we have

$$\frac{d}{dx} \ln |y| = \frac{d}{dx} n \ln |x|$$



7.3 Exponential Functions

Proof.

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Using Implicit Differentiation, we have

$$\begin{aligned}\frac{d}{dx} \ln |y| &= \frac{d}{dx} n \ln |x| \\ \frac{y'}{y} &= \frac{n}{x}\end{aligned}$$



7.3 Exponential Functions

Proof.

- 2 If $x < 0$, and if y' , $y = x^n$ and x^{n-1} all exists, then

$$\ln |y| = \ln |x|^n = n \ln |x|.$$

Using Implicit Differentiation, we have

$$\frac{d}{dx} \ln |y| = \frac{d}{dx} n \ln |x|$$

$$\frac{y'}{y} = \frac{n}{x}$$

$$y' = n \frac{y}{x} = n \frac{x^n}{x} = nx^{n-1}.$$



7.3 Exponential Functions

Proof.

- 1 For $x = 0$ and $n \geq 1$, we just use the definition of the derivative to show that

$$(x^n)'(0) = \lim_{h \rightarrow 0} \frac{(0+h)^n - 0^n}{h} = \lim_{h \rightarrow 0} h^{n-1} = 0.$$

(Note 0^{n-1} does not exist if $n < 1$.)



EXAMPLE 5 Differentiate $f(x) = x^x$, $x > 0$.

Solution We cannot apply the power rule here because the exponent is the *variable* x rather than being a constant value n (rational or irrational). However, from the definition of the general exponential function we note that $f(x) = x^x = e^{x \ln x}$, and differentiation gives

$$\begin{aligned}f'(x) &= \frac{d}{dx}(e^{x \ln x}) \\&= e^{x \ln x} \frac{d}{dx}(x \ln x) && \text{Eq. (2) with } u = x \ln x \\&= e^{x \ln x} \left(\ln x + x \cdot \frac{1}{x} \right) && \text{Product Rule} \\&= x^x (\ln x + 1). && x > 0\end{aligned}$$



7.3 Exponential Functions



The number e Expressed as a Limit

Theorem

$$e = \lim_{x \rightarrow 0} (1 + x)^{\frac{1}{x}}.$$

7.3 Exponential Functions



The number e Expressed as a Limit

Theorem

$$e = \lim_{x \rightarrow 0} (1 + x)^{\frac{1}{x}}.$$

Proof.

Let $f(x) = \ln x$. Then $f'(x) = \frac{1}{x}$ and $f'(1) = 1$.

7.3 Exponential Functions

The number e Expressed as a Limit

Theorem

$$e = \lim_{x \rightarrow 0} (1 + x)^{\frac{1}{x}}.$$

Proof.

Let $f(x) = \ln x$. Then $f'(x) = \frac{1}{x}$ and $f'(1) = 1$. Therefore

$$\begin{aligned} 1 &= f'(1) = \lim_{h \rightarrow 0} \frac{f(1+h) - f(1)}{h} = \\ &= \quad \quad \quad = \\ &= \quad \quad \quad = \end{aligned} .$$

7.3 Exponential Functions



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Theorem

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Proof.

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$$1 = f'(1) = \lim_{h \rightarrow 0} \frac{f(1 + h) - f(1)}{h} = \lim_{x \rightarrow 0} \frac{f(1 + x) - f(1)}{x}$$

$$= \qquad \qquad \qquad =$$

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7.3 Exponential Functions



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Proof.

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$$\begin{aligned} 1 &= f'(1) = \lim_{h \rightarrow 0} \frac{f(1+h) - f(1)}{h} = \lim_{x \rightarrow 0} \frac{f(1+x) - f(1)}{x} \\ &= \lim_{x \rightarrow 0} \frac{\ln(1+x) - 0}{x} = \lim_{x \rightarrow 0} \frac{1}{x} \ln(1+x) \\ &= \lim_{x \rightarrow 0} \ln(1+x)^{\frac{1}{x}} = \end{aligned}.$$

7.3 Exponential Functions



The number e Expressed as a Limit

Theorem

$$e = \lim_{x \rightarrow 0} (1 + x)^{\frac{1}{x}}.$$

Proof.

Let $f(x) = \ln x$. Then $f'(x) = \frac{1}{x}$ and $f'(1) = 1$. Therefore

$$\begin{aligned} 1 &= f'(1) = \lim_{h \rightarrow 0} \frac{f(1+h) - f(1)}{h} = \lim_{x \rightarrow 0} \frac{f(1+x) - f(1)}{x} \\ &= \lim_{x \rightarrow 0} \frac{\ln(1+x) - 0}{x} = \lim_{x \rightarrow 0} \frac{1}{x} \ln(1+x) \\ &= \lim_{x \rightarrow 0} \ln(1+x)^{\frac{1}{x}} = \ln \left(\lim_{x \rightarrow 0} (1+x)^{\frac{1}{x}} \right). \end{aligned}$$

7.3 Exponential Functions



Proof continued.

$$1 = \ln \left(\lim_{x \rightarrow 0} (1 + x)^{\frac{1}{x}} \right)$$

Taking the exponential of both sides gives

$$e = \lim_{x \rightarrow 0} (1 + x)^{\frac{1}{x}}$$

as required. □

7.3 Exponential Functions

Proof continued.

$$1 = \ln \left(\lim_{x \rightarrow 0} (1 + x)^{\frac{1}{x}} \right)$$

Taking the exponential of both sides gives

$$e = \lim_{x \rightarrow 0} (1 + x)^{\frac{1}{x}}$$

as required. □

Remark

By taking very small x in $(1 + x)^{\frac{1}{x}}$, we can calculate

$$e \approx 2.718281828459045$$

to 15 decimal places.

7.3 Exponential Functions



The derivative of a^x

$$\frac{d}{dx} a^x = \frac{d}{dx} e^{x \ln a} = e^{x \ln a} \cdot \frac{d}{dx} (x \ln a) = a^x \ln a.$$

7.3 Exponential Functions



The derivative of a^x

$$\frac{d}{dx} a^x = \frac{d}{dx} e^{x \ln a} = e^{x \ln a} \cdot \frac{d}{dx} (x \ln a) = a^x \ln a.$$

e is a special number because

$$\frac{d}{dx} e^x = e^x \ln e = e^x \cdot 1 = e^x.$$

7.3 Exponential Functions



By the Chain Rule, if u is differentiable then

$$\frac{d}{dx} a^u = a^u \ln a \frac{du}{dx}$$

for $a > 0$.

7.3 Exponential Functions

By the Chain Rule, if u is differentiable then

$$\frac{d}{dx} a^u = a^u \ln a \frac{du}{dx}$$

for $a > 0$.

Equivalently

$$\int a^u du = \frac{a^u}{\ln a} + C.$$

EXAMPLE 6 We find derivatives and integrals using Equations (3) and (4).

(a) $\frac{d}{dx} 3^x = 3^x \ln 3$ Eq. (3) with $a = 3, u = x$

(b) $\frac{d}{dx} 3^{-x} = 3^{-x} (\ln 3) \frac{d}{dx} (-x) = -3^{-x} \ln 3$ Eq. (3) with $a = 3, u = -x$

(c) $\frac{d}{dx} 3^{\sin x} = 3^{\sin x} (\ln 3) \frac{d}{dx} (\sin x) = 3^{\sin x} (\ln 3) \cos x$ Eq. (3) with $a = 3, u = \sin x$

(d) $\int 2^x dx = \frac{2^x}{\ln 2} + C$ Eq. (4) with $a = 2, u = x$

(e)
$$\begin{aligned} \int 2^{\sin x} \cos x dx &= \int 2^u du = \frac{2^u}{\ln 2} + C \\ &= \frac{2^{\sin x}}{\ln 2} + C \end{aligned}$$
 $u = \sin x, du = \cos x dx,$ and Eq. (4)
 u replaced by $\sin x$ ■

7.3 Exponential Functions



Logarithms with Base a

If $a > 0$ and $a \neq 1$, then a^x is a one-to-one function. Therefore it has an inverse function.

7.3 Exponential Functions



Logarithms with Base a

If $a > 0$ and $a \neq 1$, then a^x is a one-to-one function. Therefore it has an inverse function. Since $(a^x)' \neq 0$, the inverse function must be differentiable.

7.3 Exponential Functions



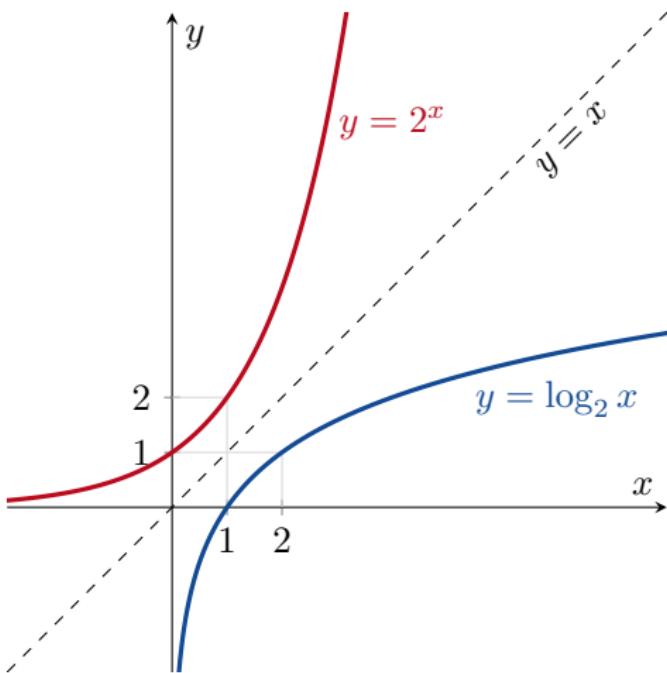
Logarithms with Base a

If $a > 0$ and $a \neq 1$, then a^x is a one-to-one function. Therefore it has an inverse function. Since $(a^x)' \neq 0$, the inverse function must be differentiable.

Definition

If $a > 0$ and $a \neq 1$, then the inverse of a^x is called the *logarithm of x with base a* and is denoted by $\log_a x$.

7.3 Exponential Functions



7.3 Exponential Functions



Remark

We have

$$a^{\log_a x} = x \quad (\text{for all } x > 0)$$

and

$$\log_a(a^x) = x \quad (\text{for all } x).$$

7.3 Exponential Functions

Remark

Note that

$$y = \log_a x$$

$$a^y = x$$

$$\ln a^y = \ln x$$

$$y \ln a = \ln x$$

$$y = \frac{\ln x}{\ln a}.$$

7.3 Exponential Functions

Remark

Note that

$$y = \log_a x$$

$$a^y = x$$

$$\ln a^y = \ln x$$

$$y \ln a = \ln x$$

$$y = \frac{\ln x}{\ln a}.$$

Therefore

$$\boxed{\log_a x = \frac{\ln x}{\ln a}.}$$

7.3 Exponential Functions



Derivatives and Integrals Involving $\log_a x$

Note that

$$\frac{d}{dx} \log_a u = \frac{d}{dx} \left(\frac{\ln u}{\ln a} \right) = \frac{1}{\ln a} \frac{d}{dx} \ln u = \frac{1}{\ln a} \cdot \frac{1}{u} \cdot \frac{du}{dx}.$$

$$\boxed{\frac{d}{dx} \log_a u = \frac{1}{\ln a} \cdot \frac{1}{u} \cdot \frac{du}{dx}}$$

EXAMPLE 7

$$(a) \frac{d}{dx} \log_{10}(3x + 1) = \frac{1}{\ln 10} \cdot \frac{1}{3x + 1} \frac{d}{dx}(3x + 1) = \frac{3}{(\ln 10)(3x + 1)}$$

$$\begin{aligned}(b) \int \frac{\log_2 x}{x} dx &= \frac{1}{\ln 2} \int \frac{\ln x}{x} dx & \log_2 x = \frac{\ln x}{\ln 2} \\ &= \frac{1}{\ln 2} \int u du & u = \ln x, \quad du = \frac{1}{x} dx \\ &= \frac{1}{\ln 2} \frac{u^2}{2} + C = \frac{1}{\ln 2} \frac{(\ln x)^2}{2} + C = \frac{(\ln x)^2}{2 \ln 2} + C\end{aligned}$$

Next Time

- 7.5 Indeterminate Forms and L'Hôpital's Rule
- 7.6 Inverse Trigonometric Functions
- 7.7 Hyperbolic Functions