Lecture 10

- 28. Derivatives of Trigonometric Functions
- 29. The Chain Rule
- 30. Antiderivatives
- 31. Integration

Derivatives of Trigonometric Functions

Sine and Cosine

$$\boxed{\frac{d}{dx}(\sin x) = \cos x}$$

$$\boxed{\frac{d}{dx}(\cos x) = -\sin x}$$

Example

Differentiate $y = x^2 - \sin x$.

solution:

$$\frac{dy}{dx} = \frac{d}{dx}\left(x^2\right) - \frac{d}{dx}(\sin x) = 2x - \cos x.$$

Example

Differentiate $y = x^2 \sin x$.

solution: We will use the product rule ($(uv)' = u'v + uv'$) with $u = x^2$ and $v = \sin x$.

$$y' = (x^2)'(\sin x) + (x^2)(\sin x)' = 2x \sin x + x^2 \cos x.$$

Example

Differentiate $y = \frac{\sin x}{x}$.

solution: This time we use the quotient rule $\left(\left(\frac{u}{v}\right)' = \frac{u'v - uv'}{v^2}\right)$ with $u = \sin x$ and $v = x$.

$$y' = \frac{(\sin x)' x - (\sin x)(x)'}{x^2} = \frac{x \cos x - \sin x}{x^2}.$$

Example

Differentiate $y = 5x + \cos x$.

solution:

$$\frac{dy}{dx} = \frac{d}{dx}(5x) + \frac{d}{dx}(\cos x) = 5 - \sin x.$$

Example

Differentiate $y = \sin x \cos x$.

solution: By the product rule, we have that

$$\frac{dy}{dx} = \frac{d}{dx}(\sin x)\cos x + \sin x \frac{d}{dx}(\cos x) = \cos^2 x - \sin^2 x.$$

Example

Differentiate $y = \frac{\cos x}{1-\sin x}$.

solution: By the quotient rule, we have that

$$\begin{aligned}
\frac{dy}{dx} &= \frac{\frac{d}{dx}(\cos x)(1-\sin x) - (\cos x)\frac{d}{dx}(1-\sin x)}{(1-\sin x)^2} \\
&= \frac{-\sin x(1-\sin x) - \cos x(0-\cos x)}{(1-\sin x)^2} \\
&= \frac{-\sin x + \sin^2 x + \cos^2 x}{(1-\sin x)^2} \\
&= \frac{-\sin x + 1}{(1-\sin x)^2} = \frac{1-\sin x}{(1-\sin x)^2} \\
&= \frac{1}{1-\sin x}.
\end{aligned}$$

The Tangent Function

$$\boxed{\frac{d}{dx}(\tan x) = \sec^2 x}$$

Proof.

Using the quotient rule, we can calculate that

$$\begin{aligned}
\frac{d}{dx}(\tan x) &= \frac{d}{dx}\left(\frac{\sin x}{\cos x}\right) \\
&= \frac{\frac{d}{dx}(\sin x)(\cos x) - (\sin x)\frac{d}{dx}(\cos x)}{\cos^2 x} \\
&= \frac{(\cos x)(\cos x) - (\sin x)(-\sin x)}{\cos^2 x} \\
&= \frac{\cos^2 x + \sin^2 x}{\cos^2 x} \\
&= \frac{1}{\cos^2 x} = \sec^2 x.
\end{aligned}$$

$\square$

The Other Three

$$\boxed{\frac{d}{dx}(\sec x) = \sec x \tan x}$$

$$\boxed{\frac{d}{dx}(\cot x) = -\operatorname{cosec}^2 x}$$

$$\boxed{\frac{d}{dx}(\operatorname{cosec} x) = -\operatorname{cosec} x \cot x}$$

You can use the quotient rule to prove these three rules. We may ask you to prove one of them in an exam.

Example

Find y'' if $y = \sec x$.

solution: Since $y' = \sec x \tan x$, we have that

$$\begin{aligned} y'' &= \frac{d}{dx}\left(y'\right) = \frac{d}{dx}\left(\sec x \tan x\right) \\ &= \frac{d}{dx}\left(\sec x\right)\tan x + \sec x \frac{d}{dx}\left(\tan x\right) \\ &= (\sec x \tan x)(\tan x) + (\sec x)(\sec^2 x) \\ &= \sec x \tan^2 x + \sec^3 x. \end{aligned}$$

29 The Chain Rule

How do we differentiate $F(x) = \sin(x^2 - 4)$?

29. The Chain Rule

Theorem (The Chain Rule)

Suppose that

- $y = f(u)$ *is differentiable at the point* $u = g(x)$*; and*
- $g(x)$ *is differentiable at* x.

Then $f \circ g$ *is differentiable at* x *and*

$$\boxed{(f \circ g)'(x) = f'\big(g(x)\big)g'(x).}$$

The Chain Rule is easier to remember if we use Leibniz's notation:

$$\boxed{\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}}$$

Example

Differentiate $y = \sin(x^2 - 4)$.

solution: We have $y = \sin u$ with $u = x^2 - 4$. Now $\frac{dy}{du} = \cos u$ and $\frac{du}{dx} = 2x$. Therefore

$$\begin{aligned} \frac{dy}{dx} &= \frac{dy}{du}\frac{du}{dx} = (\cos u)(2x) \\ &= 2x\cos u = 2x\cos(x^2 - 4) \end{aligned}$$

by the Chain Rule.

Example

Differentiate $\sin(x^2 + x)$.

solution: Let $u = x^2 + x$. Then

$$\begin{aligned} \frac{d}{dx}\left(\sin(x^2 + x)\right) &= \frac{d}{du}\left(\sin u\right)\frac{du}{dx} \\ &= (\cos u)(2x + 1) \\ &= (2x + 1)\cos(x^2 + x) \end{aligned}$$

by the Chain Rule.

Example (Using the Chain Rule Two Times)

Differentiate $g(t) = \tan(5 - \sin 2t)$.

solution: Let $u = 5 - \sin 2t$. Then $g(t) = \tan u$. Hence

$$\frac{dg}{dt} = \frac{dg}{du}\frac{du}{dt} = (\sec^2 u)\frac{d}{dt}(5 - \sin 2t).$$

We need to use the Chain Rule a second time: Let $w = 2t$. Then

$$\begin{aligned}\frac{dg}{dt} &= (\sec^2 u)\frac{d}{dt}(5 - \sin 2t)\\ &= (\sec^2 u)\frac{d}{dw}(5 - \sin w)\frac{dw}{dt}\\ &= (\sec^2 u)(-\cos w)(2)\\ &= -2\cos 2t \sec^2(5 - \sin 2t).\end{aligned}$$

(Note: Your final answer should not have u or w in it.)

Powers of a Function

If

- f is a differentiable function of u;
- u is a differentiable function of x; and
- $y = f(u)$,

then the Chain Rule $\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$ is the same as

$$\frac{d}{dx}f(u) = f'(u)\frac{du}{dx}.$$

Now suppose that $n \in \mathbb{R}$ and $f(u) = u^n$. Then $f'(u) = nu^{n-1}$. So

$$\boxed{\frac{d}{dx}\left(u^n\right) = nu^{n-1}\frac{du}{dx}.}$$

Example

$$\begin{aligned}\frac{d}{dx}\left(5x^3 - x^4\right)^7 &= 7\left(5x^3 - x^4\right)^6 \frac{d}{dx}\left(5x^3 - x^4\right)\\ &= 7\left(5x^3 - x^4\right)^6\left(15x^2 - 4x^3\right).\end{aligned}$$

Example

$$\frac{d}{dx}\left(\frac{1}{3x-2}\right) = \frac{d}{dx}(3x-2)^{-1} = -1\,(3x-2)^{-2}\,\frac{d}{dx}(3x-2)$$
$$= -\left(\frac{1}{(3x-2)^2}\right)(3) = \frac{-3}{(3x-2)^2}.$$

Example

$$\frac{d}{dx}\left(\sin^5 x\right) = 5\sin^4 x \frac{d}{dx}(\sin x) = 5\sin^4 x \cos x.$$

Example

Differentiate $|x|$.

solution: Since $|x| = \sqrt{x^2}$, we can calculate that if $x \neq 0$ then

$$\begin{aligned} \frac{d}{dx}|x| &= \frac{d}{dx}\left(\sqrt{x^2}\right) = \frac{d}{du}\left(\sqrt{u}\right)\frac{d}{dx}\left(x^2\right) \\ &= \frac{1}{2\sqrt{u}}2x = \frac{x}{\sqrt{x^2}} = \frac{x}{|x|}. \end{aligned}$$

Example

Let $y = \frac{1}{(1-2x)^3}$ for $x \neq \frac{1}{2}$. Show that $\frac{dy}{dx} > 0$.

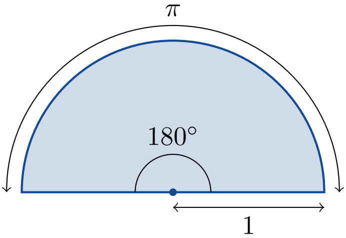
solution: First we calculate that

$$\begin{aligned}\frac{dy}{dx} &= \frac{d}{dx}(1-2x)^{-3} = -3(1-2x)^{-4}\frac{d}{dx}(1-2x) \\ &= -3(1-2x)^{-4}(-2) = \frac{6}{(1-2x)^4}\end{aligned}$$

if $x \neq \frac{1}{2}$. Since $(1-2x)^4 > 0$ if $x \neq \frac{1}{2}$ and $6 > 0$, we have that $\frac{dy}{dx} > 0$ if $x \neq \frac{1}{2}$.

Example (Why Do We Use Radians in Calculus?)

Remember that $\frac{d}{dx}\sin x = \cos x$ is true *only if we use radians.* What happens if we use degrees?



Remember that

$$\begin{aligned} 180 \text{ degrees} &= \pi \text{ radians} \\ 180^\circ &= \pi \\ 1^\circ &= \frac{\pi}{180} \\ x^\circ &= \frac{\pi x}{180}. \end{aligned}$$

So

$$\frac{d}{dx}\sin x^\circ = \frac{d}{dx}\sin\left(\frac{\pi x}{180}\right) = \frac{\pi}{180}\cos\left(\frac{\pi x}{180}\right) = \frac{\pi}{180}\cos x^\circ.$$

Therefore we have

$$\underbrace{\boxed{\frac{d}{dx}\sin x = \cos x}}_{\text{a nice formula}} \qquad \text{and} \qquad \underbrace{\boxed{\frac{d}{dx}\sin x^\circ = \frac{\pi}{180}\cos x^\circ.}}_{\text{not nice}}$$

This is why we use radians in Calculus.

30 Antiderivatives

Definition

F is an *antiderivative* of f on an interval I if $F'(x) = f(x)$ for all $x \in I$.

Example

$2x$ is the derivative of x^2.
x^2 is an antiderivative of $2x$.

Example

If $g(x) = \cos x$, then an antiderivative of g is

$$G(x) = \sin x$$

because

$$G'(x) = \frac{d}{dx}(\sin x) = \cos x = g(x).$$

Example

If $h(x) = 2x + \cos x$, then $H(x) = x^2 + \sin x$ is an antiderivative of $h(x)$.

Remark

$F(x) = x^2$ is not the only antiderivative of $f(x) = 2x$.

$x^2 + 1$ is an antiderivative of $2x$ because $\frac{d}{dx}\left(x^2 + 1\right) = 2x$.

$x^2 + 5$ is an antiderivative of $2x$ because $\frac{d}{dx}\left(x^2 + 5\right) = 2x$.

$x^2 - 1234$ is an antiderivative of $2x$ because $\frac{d}{dx}\left(x^2 - 1234\right) = 2x$.

Theorem

If F is an antiderivative of f on I, then the general antiderivative of f is

$$F(x) + C$$

where C is a constant.

Example

Find an antiderivative of $f(x) = 3x^2$ that satisfies $F(1) = -1$.

solution: x^3 is an antiderivative of f because $\frac{d}{dx}\left(x^3\right) = 3x^2$. So the general antiderivative of f is

$$F(x) = x^3 + C.$$

Then we calculate that

$$-1 = F(1) = 1^3 + C = 1 + C \implies C = -2.$$

Therefore $F(x) = x^3 - 2$.

function $f(x)$	derivative $f'(x)$
x^n	nx^{n-1}
$\sin kx$	$k\cos kx$
$\cos kx$	$-k\sin kx$
e^{kx}	ke^{kx}

30. Antiderivatives

function $f(x)$	derivative $f'(x)$
x^n	nx^{n-1}
$\sin kx$	$k\cos kx$
$\cos kx$	$-k\sin kx$
e^{kx}	ke^{kx}

function $f(x)$	general antiderivative $F(x)$
$x^n \ (n \neq -1)$	
$\sin kx$	
$\cos kx$	
e^{kx}	

30. Antiderivatives

function $f(x)$	derivative $f'(x)$
x^n	nx^{n-1}
$\sin kx$	$k \cos kx$
$\cos kx$	$-k \sin kx$
e^{kx}	ke^{kx}

function $f(x)$	general antiderivative $F(x)$
$x^n \ (n \neq -1)$	$\frac{x^{n+1}}{n+1} + C$
$\sin kx$	
$\cos kx$	
e^{kx}	

30. Antiderivatives

function $f(x)$	derivative $f'(x)$
x^n	nx^{n-1}
$\sin kx$	$k \cos kx$
$\cos kx$	$-k \sin kx$
e^{kx}	ke^{kx}

function $f(x)$	general antiderivative $F(x)$
$x^n \ (n \neq -1)$	$\frac{x^{n+1}}{n+1} + C$
$\sin kx$	$-\frac{1}{k} \cos kx + C$
$\cos kx$	
e^{kx}	

30. Antiderivatives

function $f(x)$	derivative $f'(x)$
x^n	nx^{n-1}
$\sin kx$	$k \cos kx$
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30. Antiderivatives

function $f(x)$	derivative $f'(x)$
x^n	nx^{n-1}
$\sin kx$	$k\cos kx$
$\cos kx$	$-k\sin kx$
e^{kx}	ke^{kx}

function $f(x)$	general antiderivative $F(x)$
$x^n \ (n \neq -1)$	$\frac{x^{n+1}}{n+1} + C$
$\sin kx$	$-\frac{1}{k}\cos kx + C$
$\cos kx$	$\frac{1}{k}\sin kx + C$
e^{kx}	$\frac{1}{k}e^{kx} + C$

The Sum Rule and the Constant Multiple Rule

Suppose that

- F is an antiderivative of f;
- G is an antiderivative of g;
- $k \in \mathbb{R}$.

The Sum Rule: The general antiderivative of $f + g$ is

$$F(x) + G(x) + C.$$

The Constant Multiple Rule: The general antiderivative of kf is

$$kF(x) + C.$$

Example

Find the general antiderivative of $f(x) = \frac{3}{\sqrt{x}} + \sin 2x$.

solution: We have $f = 3g + h$ where $g(x) = x^{-\frac{1}{2}}$ and $h(x) = \sin 2x$. An antiderivative of g is

$$G(x) = \frac{x^{-\frac{1}{2}+1}}{-\frac{1}{2}+1} = \frac{x^{\frac{1}{2}}}{\frac{1}{2}} = 2\sqrt{x}.$$

An antiderivative of h is

$$H(x) = -\frac{1}{2}\cos 2x.$$

Therefore the general antiderivative of f is

$$F(x) = 6\sqrt{x} - \frac{1}{2}\cos 2x + C.$$

Definition

The general antiderivative of f is also called the *indefinite integral* of f with respect to x, and is denoted by

$$\int f(x)\;dx.$$

the integral sign
integral işareti

x is the variable of integration
x ise integral değişkeni olarak tanımlanır

$$\int \underbrace{f(x)}_{\substack{\text{the integrand} \\ \text{integralin integrandı}}} \, dx$$

Example

$$\int 2x \; dx = x^2 + C$$

$$\int \cos x \; dx = \sin x + C$$

$$\int (2x + \cos x) \; dx = x^2 + \sin x + C$$

Example

Calculate $\int (x^2 - 2x + 5)\ dx$.

solution 1. Since $\frac{d}{dx}\left(\frac{x^3}{3} - x^2 + 5x\right) = x^2 - 2x + 5$ we have that

$$\int (x^2 - 2x + 5)\ dx = \frac{x^3}{3} - x^2 + 5x + C.$$

solution 2.

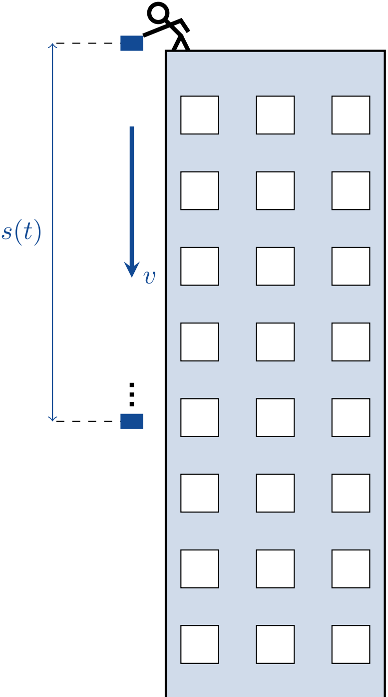
$$\begin{aligned}\int (x^2 - 2x + 5)\ dx &= \int x^2\ dx - \int 2x\ dx + \int 5\ dx \\ &= \left(\frac{x^3}{3} + C_1\right) - \left(x^2 + C_2\right) + (5x + C_3) \\ &= \left(\frac{x^3}{3} - x^2 + 5x\right) + (C_1 - C_2 + C_3)\,.\end{aligned}$$

Because we only need one constant, we can define $C := C_1 - C_2 + C_3$. Therefore

$$\int (x^2 - 2x + 5)\ dx = \frac{x^3}{3} - x^2 + 5x + C.$$

Example

You drop a box off the top of a tall building. The acceleration due to gravity is $9.8\,\mathrm{ms}^{-2}$. You can ignore air resistance. How far does the box fall in 5 seconds?


$s(t)$
v

solution: The acceleration is

$$a(t) = 9.8\text{ms}^{-2}$$

downwards. Since

$$\text{acceleration} = \frac{d}{dt}(\text{velocity}),$$

the velocity is an antiderivative of the acceleration. Therefore the velocity is

$$v(t) = 9.8t + C \text{ ms}^{-1}.$$

You let go of the box at time $t = 0$. So $v(0) = 0$. Thus $C = 0$. Hence

$$v(t) = 9.8t \text{ ms}^{-1}.$$

Now velocity $= \frac{d}{dt}(\text{position})$. So the distance fallen is an antiderivative of velocity. Hence

$$s(t) = 4.9t^2 + \tilde{C} \text{ m}.$$

Because you let go of the box at time $t = 0$, we have $s(0) = 0$. Thus $\tilde{C} = 0$. Therefore

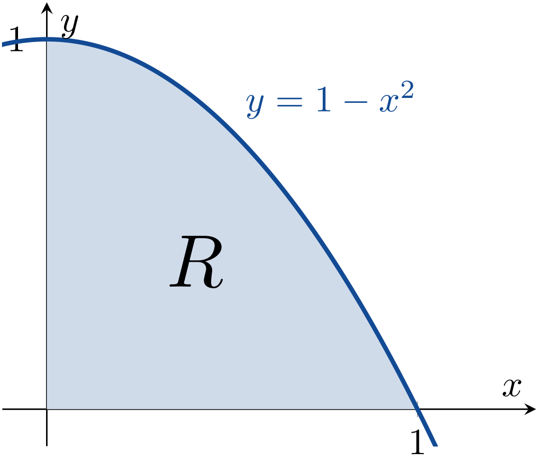
$$s(t) = 4.9t^2 \text{ m}.$$

After 5 seconds, the box has fallen

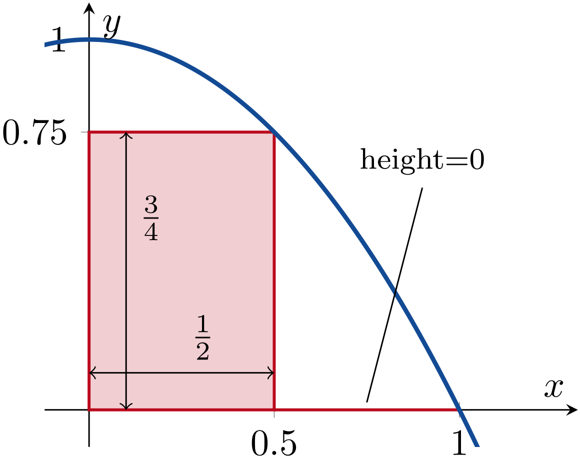
$$s(5) = 4.9 \times 25 = 122.5 \text{ metres}.$$

3.1

Integration



Question: What is the area of R?

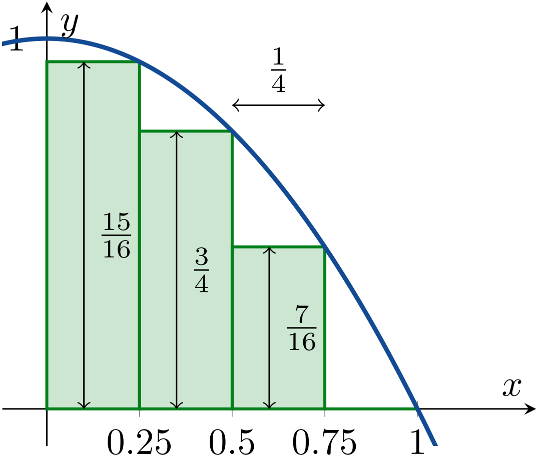


We can use two rectangles to approximate the area of R.

Then we have

$$\begin{aligned} \text{area of } R &\approx \text{area of 2 rectangles} \\ &= \left(\frac{3}{4} \times \frac{1}{2}\right) + \left(0 \times \frac{1}{2}\right) \\ &= \frac{3}{8} = 0.375. \end{aligned}$$

Can we do better than this? Yes! We could use more rectangles.

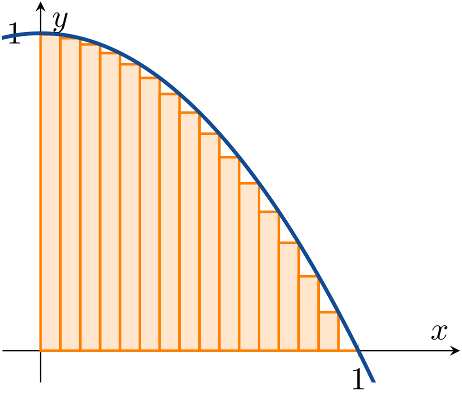


y
1
1/4
15/16
3/4
7/16
x
0.25
0.5
0.75
1

We can say that

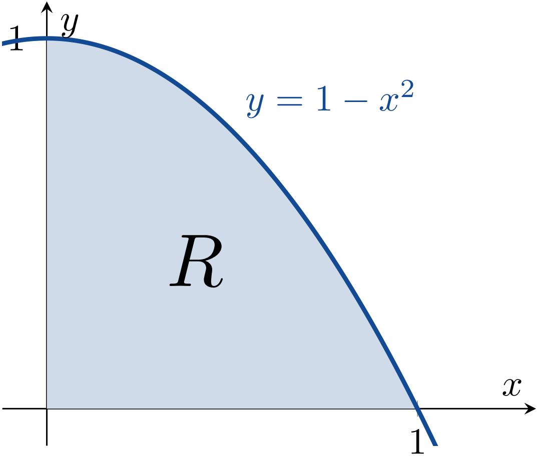
$$
\begin{aligned}
\text{area of } R &\approx \text{area of 4 rectangles} \\
&= \left(\frac{15}{16} \times \frac{1}{4}\right) + \left(\frac{3}{4} \times \frac{1}{4}\right) \\
&\qquad + \left(\frac{7}{16} \times \frac{1}{4}\right) + \left(0 \times \frac{1}{4}\right) \\
&= \frac{17}{32} = 0.53125.
\end{aligned}
$$

Every time we increase the number of rectangles, the total area of the rectangles gets closer and closer to the area of R.



$$\text{area of } R \approx \text{area of 16 rectangles}$$
$$= 0.63476.$$

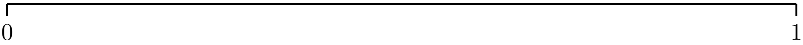
Limits of Finite Sums



31. Integration

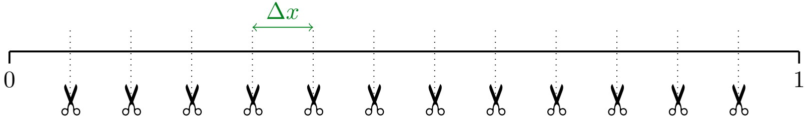
STEP 1: We will cut $[0,1]$ into n pieces of width

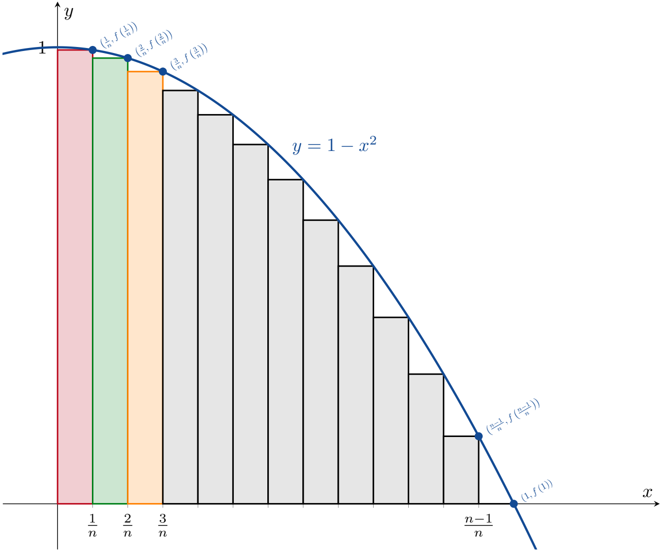
$$\Delta x = \frac{1-0}{n} = \frac{1}{n}.$$



STEP 1: We will cut $[0,1]$ into n pieces of width

$$\Delta x = \frac{1-0}{n} = \frac{1}{n}.$$





STEP 2: We will use n rectangles to approximate the area of R.

STEP 3: Then we will take the limit as $n \to \infty$.

Let $f(x) = 1 - x^2$. Then

- the first rectangle has area $\frac{1}{n} f\left(\frac{1}{n}\right)$;
- the second rectangle has area $\frac{1}{n} f\left(\frac{2}{n}\right)$;
- the third rectangle has area $\frac{1}{n} f\left(\frac{3}{n}\right)$;

and so on.

The area of all n rectangles is

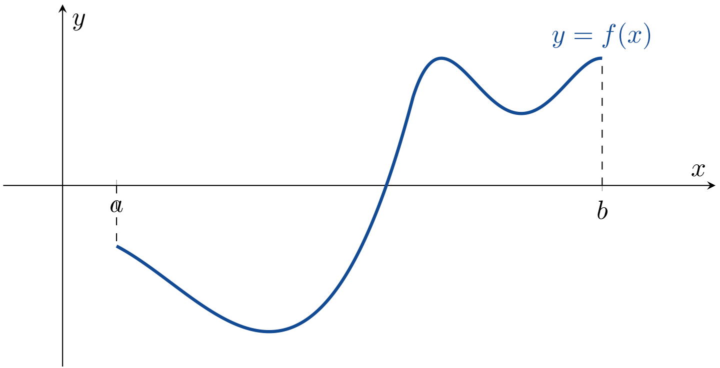
$$\begin{aligned}
\text{area} &= \sum_{k=1}^{n} (\text{area of the } k\text{th rectangle}) = \sum_{k=1}^{n} \frac{1}{n} f\left(\frac{k}{n}\right) \\
&= \sum_{k=1}^{n} \frac{1}{n}\left(1 - \left(\frac{k}{n}\right)^2\right) = \sum_{k=1}^{n}\left(\frac{1}{n} - \frac{k^2}{n^3}\right) \\
&= \sum_{k=1}^{n} \frac{1}{n} - \sum_{k=1}^{n} \frac{k^2}{n^3} \\
&= n\left(\frac{1}{n}\right) - \frac{1}{n^3}\sum_{k=1}^{n} k^2 \\
&= 1 - \frac{1}{n^3}\left(\frac{n(n+1)(2n+1)}{6}\right) \\
&= 1 - \frac{2n^2+3n+1}{6n^2}.
\end{aligned}$$

Taking the limit gives

$$\begin{aligned}\lim_{n\to\infty}\left(\sum_{k=1}^{n}\frac{1}{n}f\left(\frac{k}{n}\right)\right) &= \lim_{n\to\infty}\left(1-\frac{2n^2+3n+1}{6n^2}\right)\\ &= 1-\frac{2}{6}=\frac{2}{3}.\end{aligned}$$

Therefore the area of R is $\frac{2}{3}$.

Riemann Sums

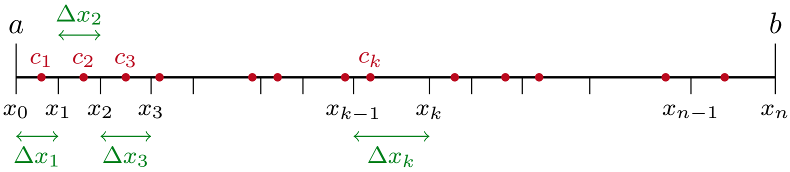


31. Integration

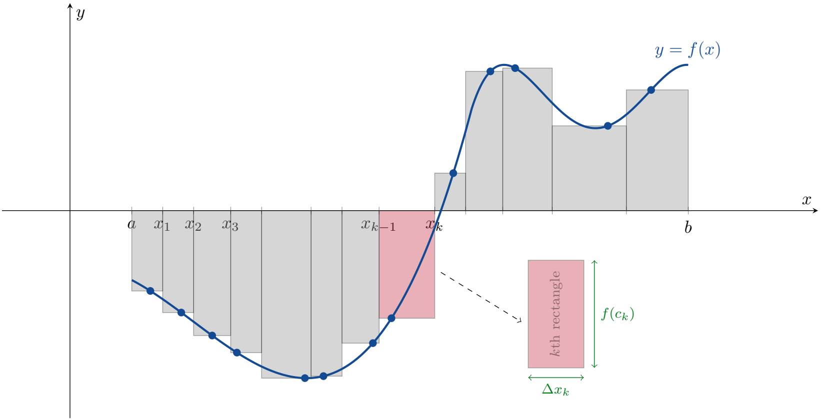
Now let $f : [a, b] \to \mathbb{R}$ be a function. We will cut $[a, b]$ into n subintervals (the pieces don't have to all be the same size).

In each subinterval we will choose one point $c_k \in [x_{k-1}, x_k]$.

The width of each subinterval is $\Delta x_k = x_k - x_{k-1}$.



On each subinterval $[x_{k-1}, x_k]$, we draw a rectangle of width Δx_k and height $f(c_k)$.



Note that if $f(c_k) < 0$, then the rectangle on $[x_{k-1}, x_k]$ will have 'negative area' – this is ok.

The total of the n rectangles is

$$\sum_{k=1}^{n} f(c_k)\Delta x_k.$$

This is called a *Riemann Sum for f on* $[a, b]$.

Then we want to take the limit as $n \to \infty$ (or more precisely, we want to take the limit as $\max\{\Delta x_1, \Delta x_2, \ldots, \Delta x_n\} \to 0$). Sometimes this limit exists, sometimes this limit does not exist.

Next Time

- 32. The Definite Integral
- 33. The Fundamental Theorem of Calculus
- 34. The Substitution Method
- 35. Area Between Curves