Lecture 9

- 24. Limits
- 25. Continuity
- 26. Differentiation

24 Limits

Consider the function $f(x) = \frac{x^2-1}{x-1}$, $f : (-\infty, 1) \cup (1, \infty) \to \mathbb{R}$.

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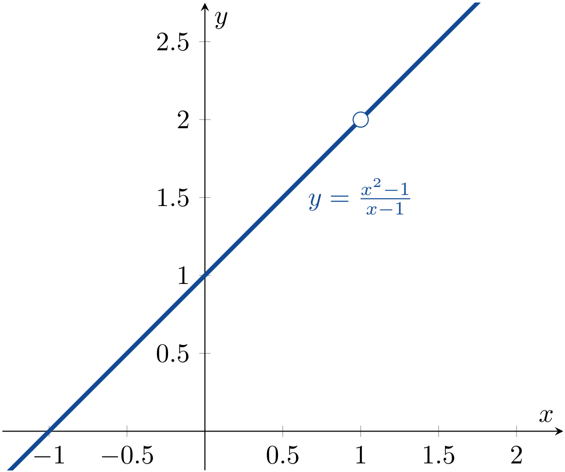

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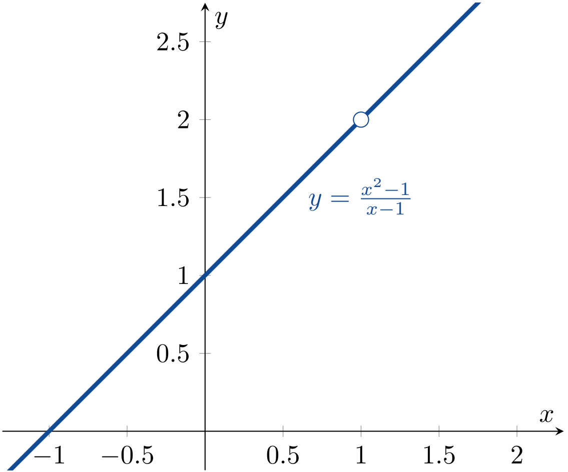
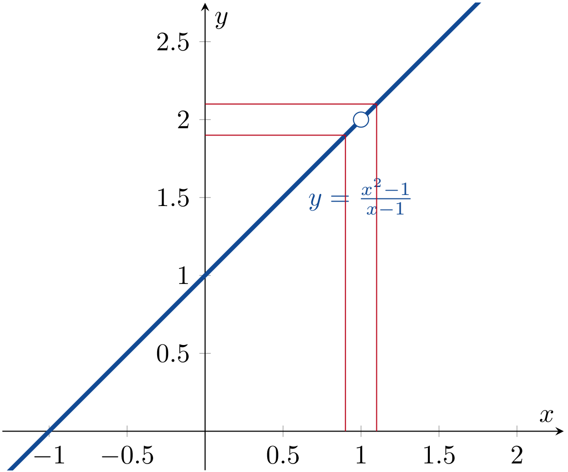


Question: How does f behave when x is close to 1?

Consider the function $f(x) = \frac{x^2-1}{x-1}$, $f : (-\infty, 1) \cup (1, \infty) \to \mathbb{R}$.



x	$f(x)$
0.9	1.9
1.1	2.1

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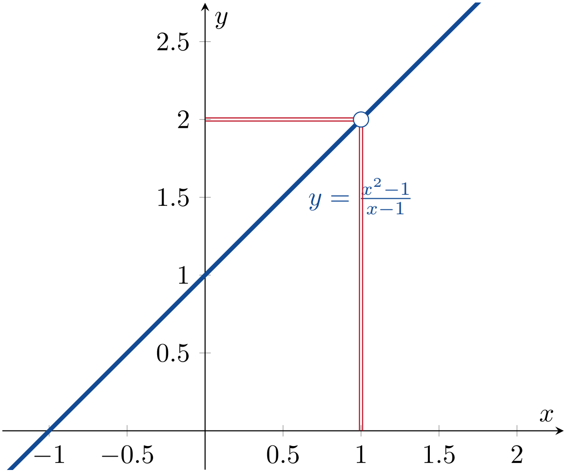

x	$f(x)$
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1.1	2.1
0.99	1.99
1.01	2.01

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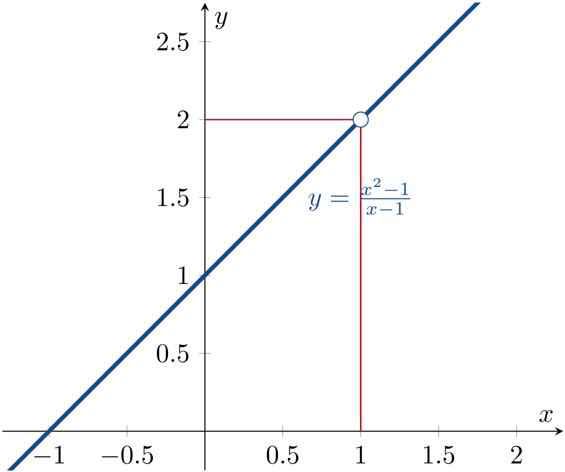

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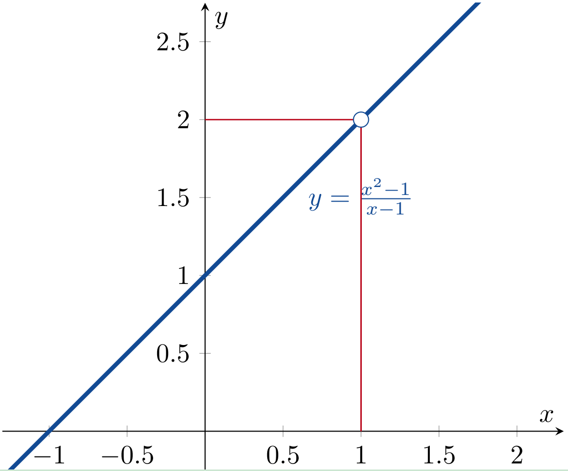

x	$f(x)$
0.9	1.9
1.1	2.1
0.99	1.99
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"If x is close to 1, then $f(x)$ is close to 2."

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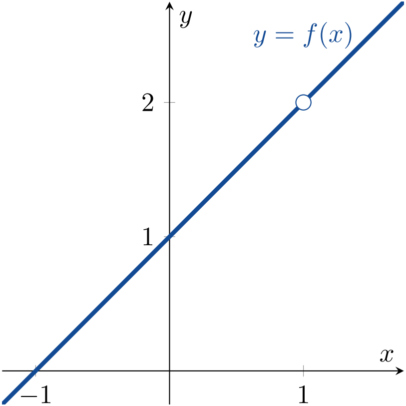
Mathematically, we write this as

$$\lim_{x \to 1} f(x) = 2$$

and read it as "the limit, as x tends to 1, of $f(x)$ is equal to 2".

Example

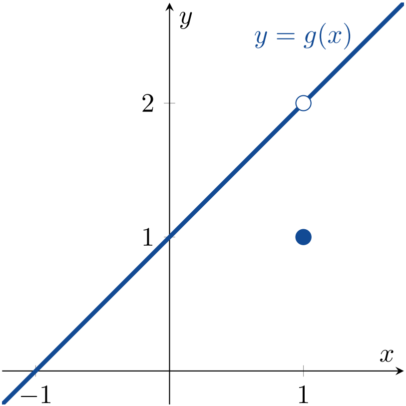
$$f(x) = \frac{x^2 - 1}{x - 1}$$



Note that $\lim_{x \to 1} f(x) = 2$,
but f is not defined at $x = 1$.

Example

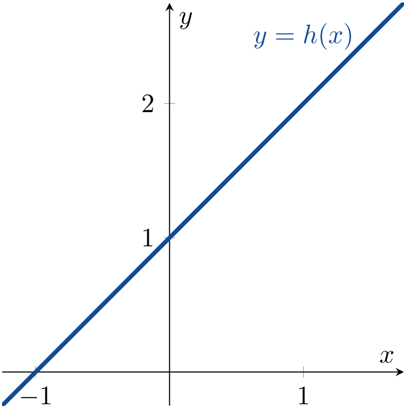
$$g(x) = \begin{cases} \frac{x^2-1}{x-1} & x \neq 1 \\ 1 & x = 1 \end{cases}$$



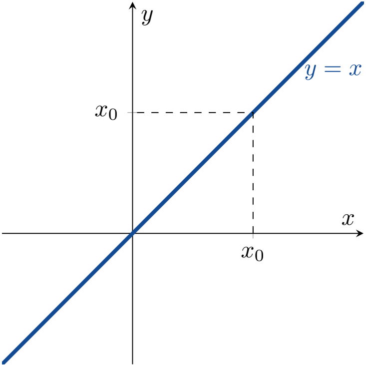
Note that $\lim_{x \to 1} g(x) = 2$, but $g(1) \neq 2$.

Example

$h(x) = x + 1$

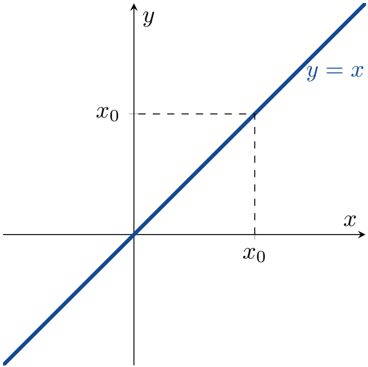


Note that $\lim_{x \to 1} h(x) = 2$ and $h(1) = 2$.



Example (The Identity Function)

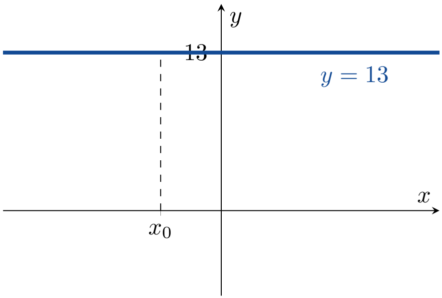
$f(x) = x$



Example (The Identity Function)

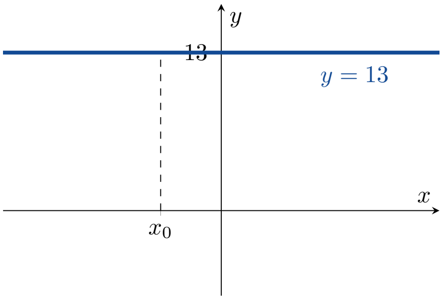
$f(x) = x$

$$\lim_{x \to x_0} f(x) = \lim_{x \to x_0} x = x_0$$



Example (A Constant Function)

$f(x) = 13$



Example (A Constant Function)

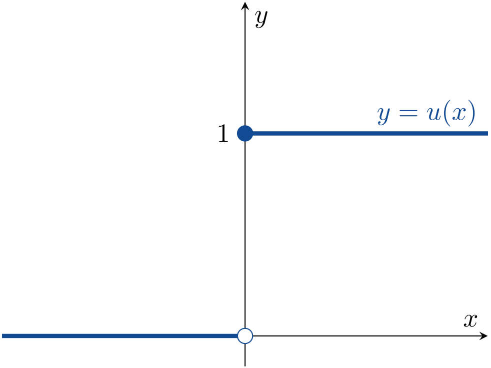
$f(x) = 13$

$$\lim_{x \to x_0} f(x) = \lim_{x \to x_0} 13 = 13$$

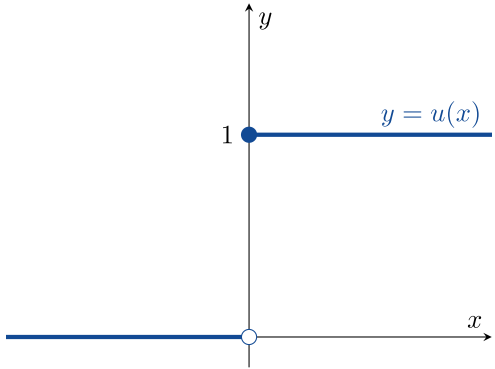
Example (Sometimes Limits Do Not Exist)

Consider the functions

$$u(x) = \begin{cases} 0 & x < 0 \\ 1 & x \geq 0 \end{cases} \qquad \text{and} \qquad v(x) = \begin{cases} 0 & x \leq 0 \\ \sin \frac{1}{x} & x > 0. \end{cases}$$

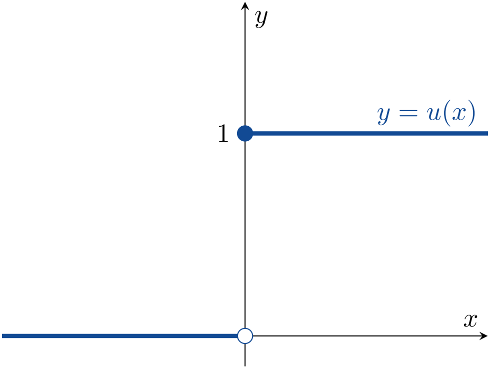


Note that $\lim_{x \to 0} u(x)$ does not exist.



Note that $\lim_{x\to 0} u(x)$ does not exist. To understand why, we consider x close to 0:

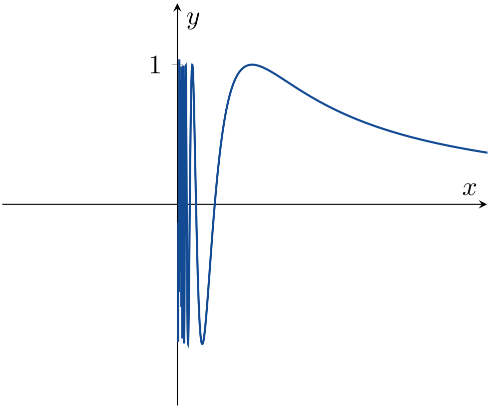
- If x is close to 0 and $x < 0$, then $u(x) = 0$.
- If x is close to 0 and $x > 0$, then $u(x) = 1$.



Note that $\lim_{x \to 0} u(x)$ does not exist. To understand why, we consider x close to 0:

- If x is close to 0 and $x < 0$, then $u(x) = 0$.
- If x is close to 0 and $x > 0$, then $u(x) = 1$.

Because 0 is not close to 1, the limit as $x \to 0$ can not exist.



Moreover $\lim_{x \to 0} v(x)$ does not exist because $v(x)$ oscillates up and down too quickly if $x > 0$ and $x \to 0$.

Theorem (The Limit Laws)

Suppose that

- $L, M, c, k \in \mathbb{R}$;
- f *and* g *are functions;*
- $\lim_{x \to c} f(x) = L$; *and*
- $\lim_{x \to c} g(x) = M$.

Then

Theorem (The Limit Laws)

Suppose that

- $L, M, c, k \in \mathbb{R}$;
- f *and* g *are functions;*
- $\lim_{x \to c} f(x) = L$; *and*
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Then

1. *Sum Rule:*
$$\lim_{x \to c} \big(f(x) + g(x) \big) = L + M;$$

Theorem (The Limit Laws)

Suppose that

- $L, M, c, k \in \mathbb{R}$;
- f *and* g *are functions;*
- $\lim_{x \to c} f(x) = L$; *and*
- $\lim_{x \to c} g(x) = M$.

Then

2. *Difference Rule:*

$$\lim_{x \to c} \big(f(x) - g(x) \big) = L - M;$$

Theorem (The Limit Laws)

Suppose that

- $L, M, c, k \in \mathbb{R}$;
- f *and* g *are functions;*
- $\lim_{x \to c} f(x) = L$; *and*
- $\lim_{x \to c} g(x) = M$.

Then

3. *Constant Multiple Rule:*

$$\lim_{x \to c} \big(kf(x)\big) = kL;$$

Theorem (The Limit Laws)

Suppose that

- $L, M, c, k \in \mathbb{R}$;
- f *and* g *are functions;*
- $\lim_{x \to c} f(x) = L$; *and*
- $\lim_{x \to c} g(x) = M$.

Then

4. *Product Rule:*
$$\lim_{x \to c} \big(f(x)g(x)\big) = LM;$$

Theorem (The Limit Laws)

Suppose that

- $L, M, c, k \in \mathbb{R}$;
- f *and* g *are functions;*
- $\lim_{x\to c} f(x) = L$; *and*
- $\lim_{x\to c} g(x) = M$.

Then

5. *Quotient Rule: if* $M \neq 0$, *then*

$$\lim_{x\to c}\left(\frac{f(x)}{g(x)}\right) = \frac{L}{M};$$

Theorem (The Limit Laws)

Suppose that

- $L, M, c, k \in \mathbb{R}$;
- f *and* g *are functions;*
- $\lim\limits_{x \to c} f(x) = L$; *and*
- $\lim\limits_{x \to c} g(x) = M$.

Then

6. *Power Rule: if* $n \in \mathbb{N}$, *then*

$$\lim_{x \to c} \big(f(x)\big)^n = L^n;$$

Theorem (The Limit Laws)

Suppose that

- $L, M, c, k \in \mathbb{R}$;
- f *and* g *are functions;*
- $\lim\limits_{x \to c} f(x) = L$; *and*
- $\lim\limits_{x \to c} g(x) = M$.

Then

7. *Root Rule: if* $n \in \mathbb{N}$ *and* $\sqrt[n]{L}$ *exists, then*

$$\lim_{x \to c} \sqrt[n]{f(x)} = \sqrt[n]{L} = L^{\frac{1}{n}}.$$

Example

Find $\lim_{x\to 2}(x^3 + 4x^2 - 3)$.

solution:

$$\lim_{x\to 2}(x^3 + 4x^2 - 3) = \Big(\lim_{x\to 2} x^3\Big) + \Big(\lim_{x\to 2} 4x^2\Big) - \Big(\lim_{x\to 2} 3\Big)$$

(sum and difference rules)

$$= \Big(\lim_{x\to 2} x\Big)^3 + 4\Big(\lim_{x\to 2} x\Big)^2 - \Big(\lim_{x\to 2} 3\Big)$$

(power and constant multiple rules)

$$= 2^3 + 4(2^2) - 3 = 21.$$

Example

Find $\lim\limits_{x\to 5} \dfrac{x^4 + x^2 - 1}{x^2 + 5}$.

solution:

$$\begin{aligned}
\lim_{x\to 5} \frac{x^4 + x^2 - 1}{x^2 + 5} &= \frac{\lim_{x\to 5}(x^4 + x^2 - 1)}{\lim_{x\to 5}(x^2 + 5)} \\
&\quad \text{(quotient rule)} \\
&= \frac{\lim_{x\to 5} x^4 + \lim_{x\to 5} x^2 - \lim_{x\to 5} 1}{\lim_{x\to 5} x^2 + \lim_{x\to 5} 5} \\
&\quad \text{(sum and difference rules)} \\
&= \frac{5^4 + 5^2 - 1}{5^2 + 5} \\
&\quad \text{(power rule)} \\
&= \frac{649}{30}.
\end{aligned}$$

Theorem (Limits of Polynomial Functions)

If $P(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_0$ *is a polynomial function, then*

$$\lim_{x \to c} P(x) = P(c).$$

Theorem (Limits of Rational Functions)

If $P(x)$ and $Q(x)$ are polynomial functions and if $Q(c) \neq 0$, then

$$\lim_{x \to c} \frac{P(x)}{Q(x)} = \frac{P(c)}{Q(c)}.$$

Example

$$\lim_{x\to -1} \frac{x^3+4x^2-3}{x^2+5} = \frac{(-1)^3+4(-1)^2-3}{(-1)^2+5} = \frac{0}{6} = 0.$$

Eliminating Zero Denominators Algebraically

$$\lim_{x \to c} \frac{P(x)}{Q(x)}$$

What can we do if $Q(c) = 0$?

Example

$$\lim_{x\to 1} \frac{x^2 + x - 2}{x^2 - x}$$

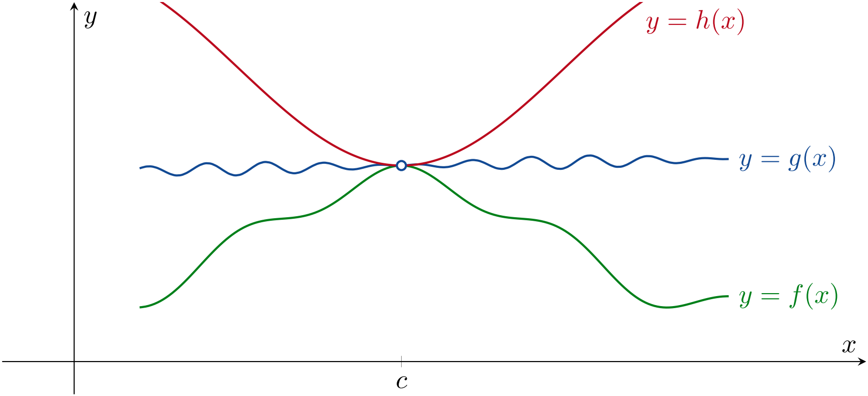
If we just put in $x = 1$, we would get "$\frac{0}{0}$" and we never never never want "$\frac{0}{0}$".
Instead, we try to factor $x^2 + x - 2$ and $x^2 - x$. If $x \neq 1$, we have that

$$\frac{x^2 + x - 2}{x^2 - x} = \frac{(x-1)(x+2)}{x(x-1)} = \frac{x+2}{x}.$$

So

$$\lim_{x\to 1} \frac{x^2 + x - 2}{x^2 - x} = \lim_{x\to 1} \frac{x+2}{x} = \frac{1+2}{1} = 3.$$

The Sandwich Theorem



Theorem (The Sandwich Theorem)

Suppose that

- $f(x) \leq g(x) \leq h(x)$ *for all* x *"close" to* c $(x \neq c)$*; and*
- $\lim\limits_{x \to c} f(x) = \lim\limits_{x \to c} h(x) = L$.

Then

$$\lim_{x \to c} g(x) = L$$

also.

Example

The inequality

$$1 - \frac{x^2}{6} < \frac{x \sin x}{2 - 2\cos x} < 1$$

holds for all x close to 0 ($x \neq 0$). Calculate $\lim\limits_{x \to 0} \dfrac{x \sin x}{2 - 2\cos x}$.

solution: Since $\lim\limits_{x \to 0} 1 - \dfrac{x^2}{6} = 1 - \dfrac{0}{6} = 1$ and $\lim\limits_{x \to 0} 1 = 1$, it follows by the Sandwich Theorem that $\lim\limits_{x \to 0} \dfrac{x \sin x}{2 - 2\cos x} = 1$.

Theorem

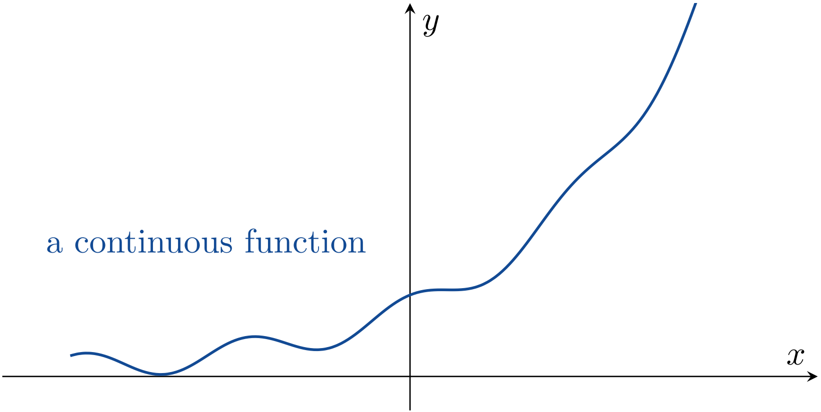
If

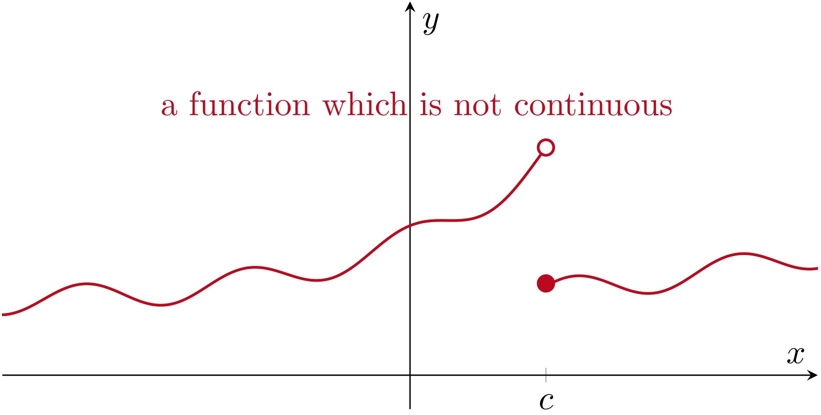
- $f(x) \le g(x)$ *for all* x *close to* c *(*$x \neq c$*);*
- $\lim\limits_{x \to c} f(x)$ *exists; and*
- $\lim\limits_{x \to c} g(x)$ *exists,*

then

$$\lim_{x \to c} f(x) \le \lim_{x \to c} g(x).$$

Continuity


y
a continuous function
x


y
a function which is not continuous
x
c

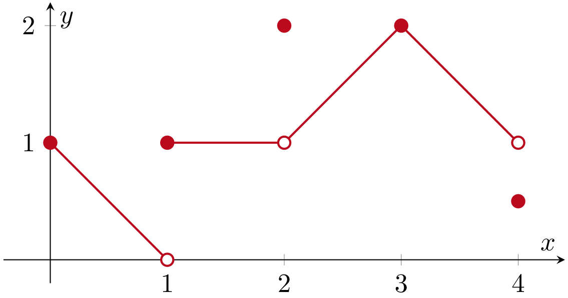
Definition

The function $f : D \to \mathbb{R}$ is *continuous at* $c \in D$ if

- $f(c)$ exists;
- $\lim_{x \to c} f(x)$ exists; and
- $\lim_{x \to c} f(x) = f(c)$.

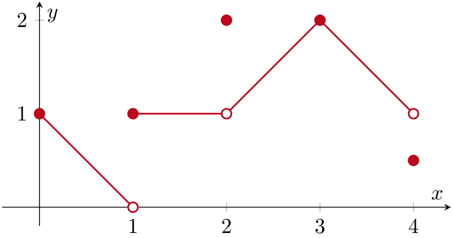
Definition

If f is not continuous at c, we say that f is *discontinuous* at c – we say that c is a *point of discontinuity* of f.



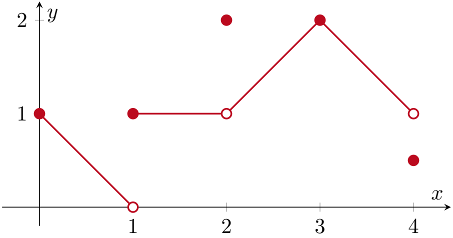
Example

Consider the function $f : [0, 4] \to \mathbb{R}$ above. Where is f continuous? Where is f discontinuous?



solution:

c	Is f continuous at c?	Why?
0	Yes	because $\lim_{x\to 0} f(x) = 1 = f(0)$
$(0,1)$	Yes	because $\lim_{x\to c} f(x) = f(c)$
1	No	because $\lim_{x\to 1} f(x)$ does not exist

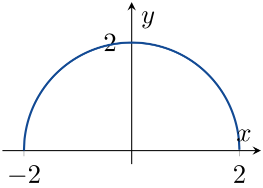


solution:

c	Is f continuous at c?	Why?
$(1, 2)$	Yes	because $\lim_{x\to c} f(x) = f(c)$
2	No	because $\lim_{x\to 2} f(x) = 1 \neq 2 = f(2)$
$(2, 4)$	Yes	because $\lim_{x\to c} f(x) = f(c)$
4	No	because $\lim_{x\to 4} f(x) = 1 \neq \frac{1}{2} = f(4)$

Example

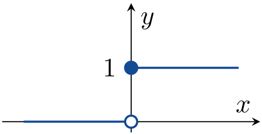
$f : [-2, 2] \to \mathbb{R},\ f(x) = \sqrt{4 - x^2}$



f is continuous at every $c \in [-2, 2]$.

Example

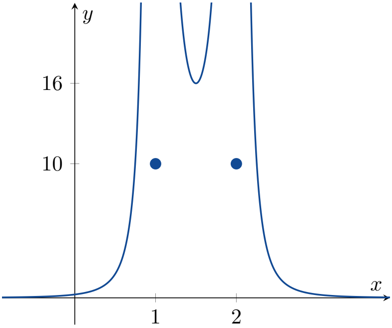
$$g : \mathbb{R} \to \mathbb{R},\ g(x) = \begin{cases} 0 & x < 0 \\ 1 & x \geq 0 \end{cases}$$



g has a point of discontinuity at $c = 0$. g is continuous at every point $c \neq 0$.

Example

$$h : \mathbb{R} \to \mathbb{R},\ h(x) = \begin{cases} \frac{1}{(x-1)^2(x-2)^2} & x \neq 1 \text{ or } 2 \\ 10 & x = 1 \text{ or } 2 \end{cases}$$



h is continuous on $(-\infty, 1)$, $(1, 2)$ and $(2, \infty)$. h has a points of discontinuity at $c = 1$ and $c = 2$.

Continuous Functions

Definition

$f : D \to \mathbb{R}$ is a *continuous function* if it is continuous at every $c \in D$.

Theorem

If f and g are continuous at c, then $f+g$, $f-g$, kf ($k \in \mathbb{R}$), fg, $\frac{f}{g}$ (if $g(c) \neq 0$) and f^n ($n \in \mathbb{N}$) are all continuous at c. If $\sqrt[n]{f}$ is defined on $(c-\delta, c+\delta)$, then $\sqrt[n]{f}$ is also continuous at c ($n \in \mathbb{N}$).

Example

Every polynominal

$$P(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_2 x^2 + a_1 x + a_0$$

is continuous.

Example

If

- P and Q are polynomials; and
- $Q(c) \neq 0$,

then $\frac{P(x)}{Q(x)}$ is continuous at c.

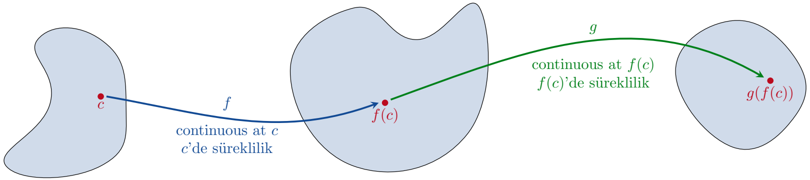
Example

$\sin x$ and $\cos x$ are continuous.

Composites

$$g \circ f(x)$$

$g \circ f(x)$ means $g(f(x))$.



Theorem

If

- f *is continuous at* c*; and*
- g *is continuous at* $f(c)$,

then $g \circ f$ *is continuous at* c.

Example

Show that $h(x) = \sqrt{x^2 - 2x - 5}$ is continuous on its domain.

solution: The function $g(t) = \sqrt{t}$ is continuous by Theorem 24. The function $f(x) = x^2 - 2x - 5$ is continuous because all polynomials are continuous. Therefore $h(x) = g \circ f(x)$ is continuous.

Example

Show that $\frac{x^{\frac{2}{3}}}{1+x^4}$ is continuous.

solution: $x^{\frac{2}{3}}$ and $1 + x^4$ are continuous. Because $1 + x^4 \neq 0$ for all x, we have that $\frac{x^{\frac{2}{3}}}{1+x^4}$ is continuous.

Theorem

If

- $g(x)$ *is continuous at* $x = b$*; and*
- $\lim_{x \to c} f(x) = b$,

then

$$\lim_{x \to c} g(f(x)) = g\left(\lim_{x \to c} f(x)\right).$$

Example

$$\begin{aligned}
&\lim_{x\to\frac{\pi}{2}} \cos\left[2x + \sin\left(\frac{3\pi}{2} + x\right)\right] \\
&\qquad = \cos\left[\lim_{x\to\frac{\pi}{2}} \left(2x + \sin\left(\frac{3\pi}{2} + x\right)\right)\right] \\
&\qquad = \cos\left[\lim_{x\to\frac{\pi}{2}} (2x) + \lim_{x\to\frac{\pi}{2}} \left(\sin\left(\frac{3\pi}{2} + x\right)\right)\right] \\
&\qquad = \cos\left[\pi + \sin\left(\lim_{x\to\frac{\pi}{2}} \left(\frac{3\pi}{2} + x\right)\right)\right] \\
&\qquad = \cos\left[\pi + \sin 2\pi\right] = \cos\left[\pi + 0\right] = -1.
\end{aligned}$$

26 Differentiation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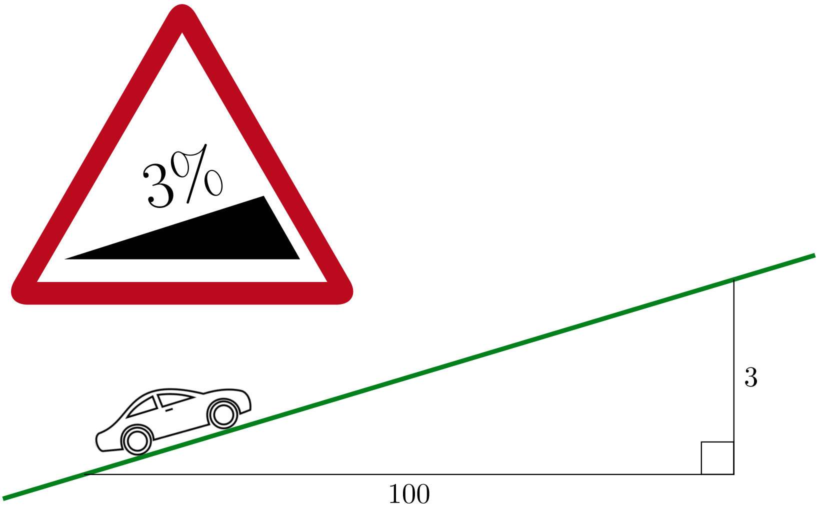
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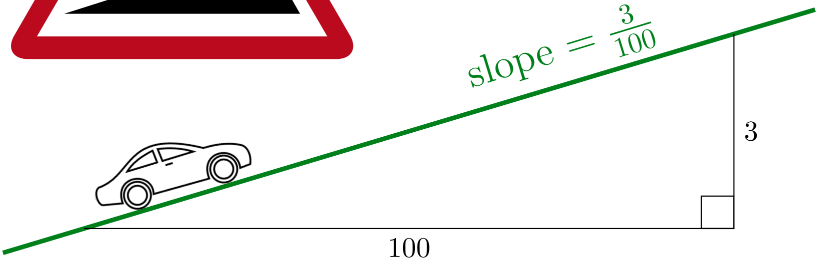
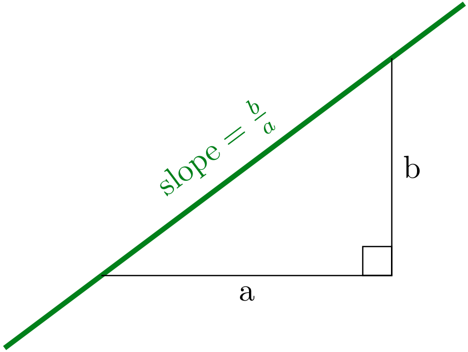
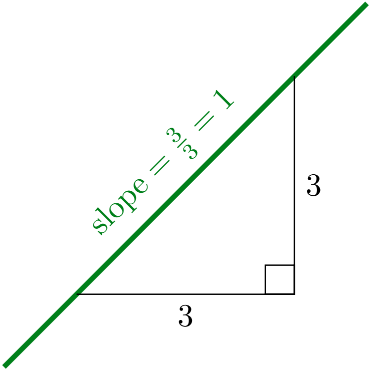
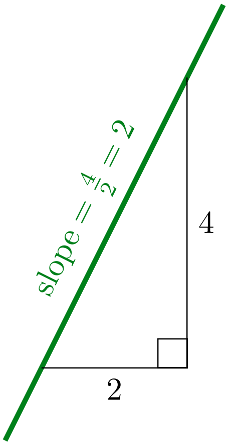
3
100

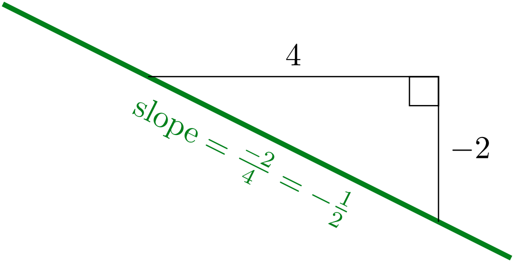

3%

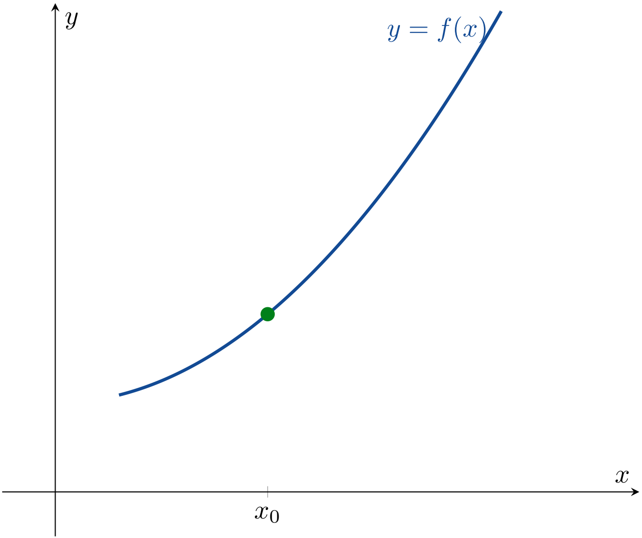

slope = $\frac{3}{100}$
3
100


slope = $^b/_a$
b
a

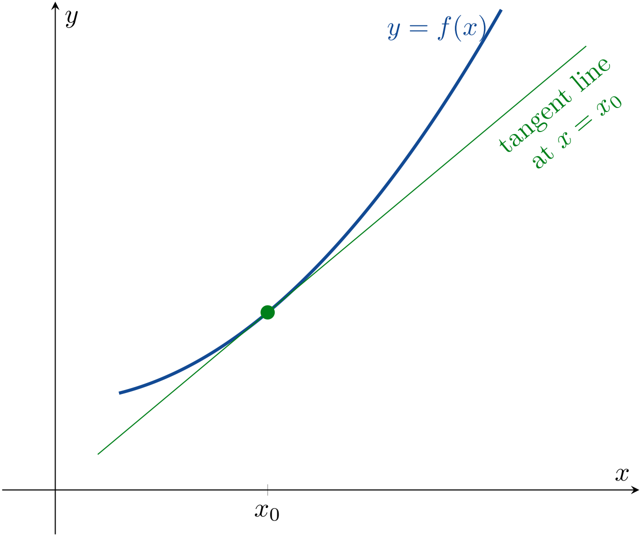

slope = $^3\!/_3$ = 1
3
3

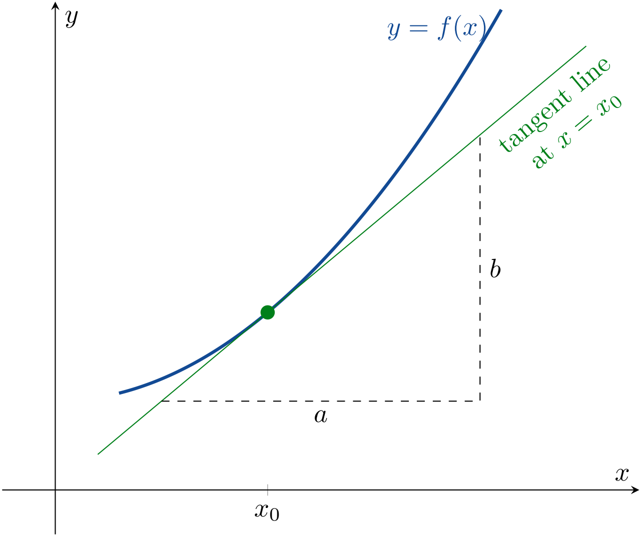

slope = $\frac{4}{2}$ = 2
4
2


4
−2
slope = $\frac{-2}{4} = -\frac{1}{2}$

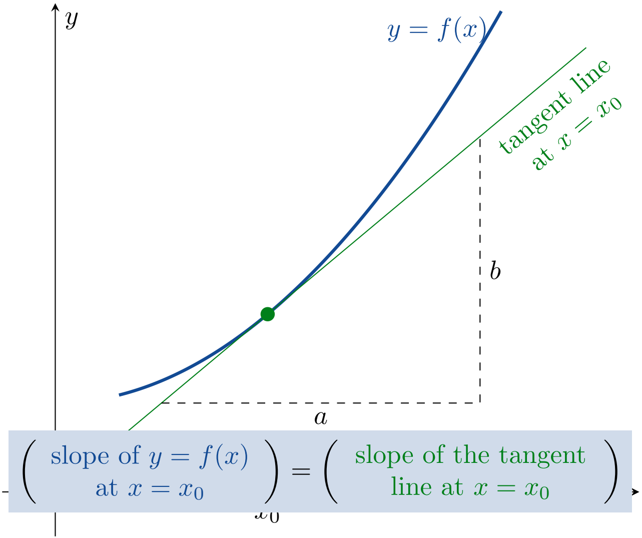


y
$y = f(x)$
x
x_0


y
$y = f(x)$
tangent line
at $x = x_0$
x
x_0



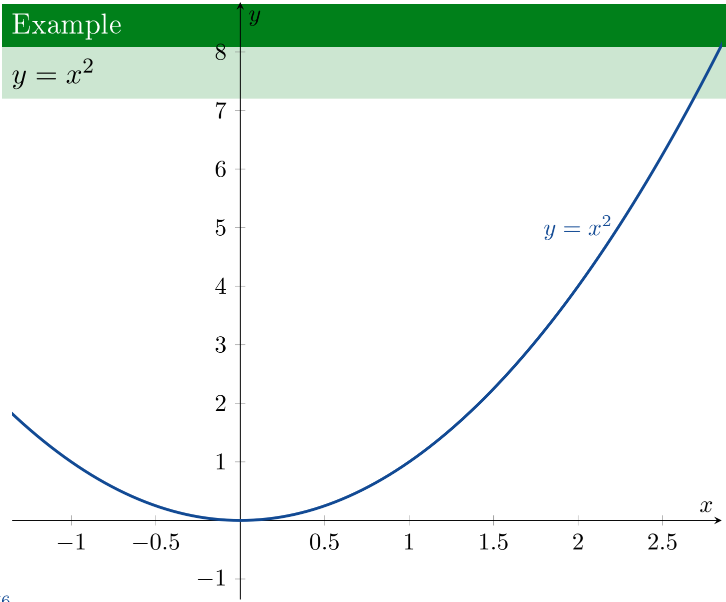
y
$y = f(x)$
tangent line
at $x = x_0$
b
a
x
x_0



$$\begin{pmatrix} \text{slope of } y = f(x) \\ \text{at } x = x_0 \end{pmatrix} = \begin{pmatrix} \text{slope of the tangent} \\ \text{line at } x = x_0 \end{pmatrix}$$

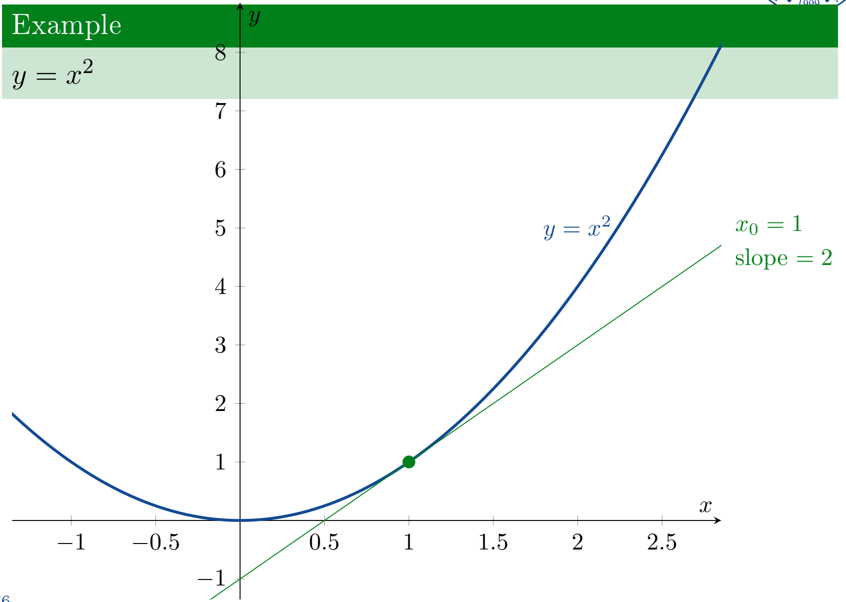
Example

$y = x^2$



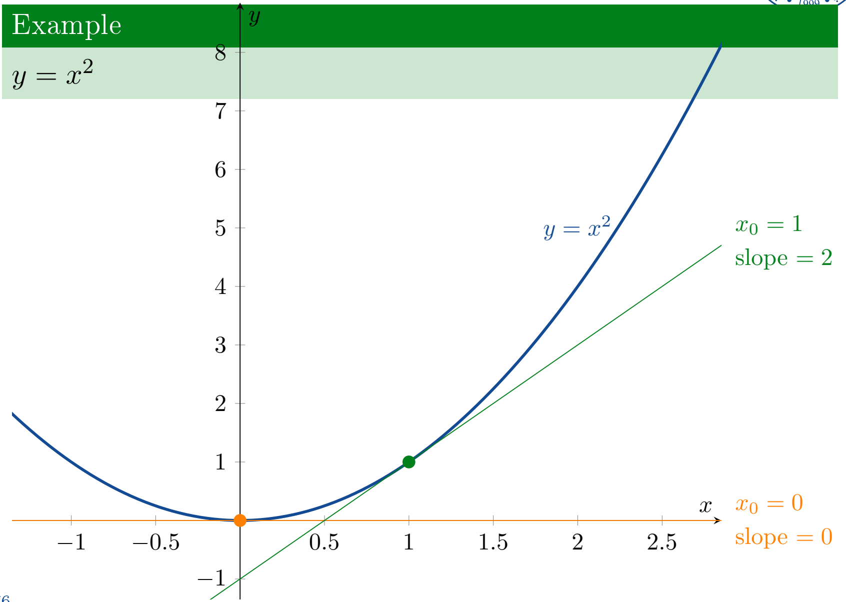
Example

$y = x^2$



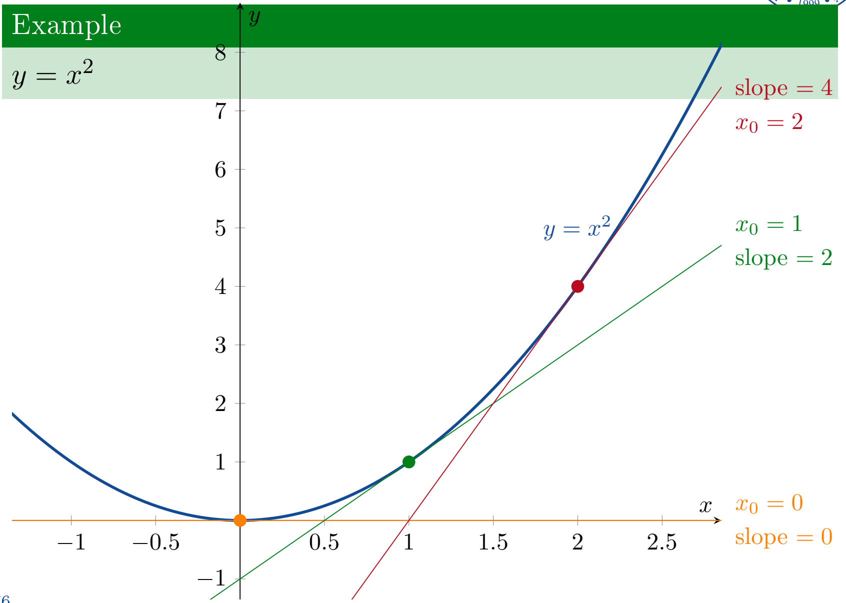
Example

$y = x^2$



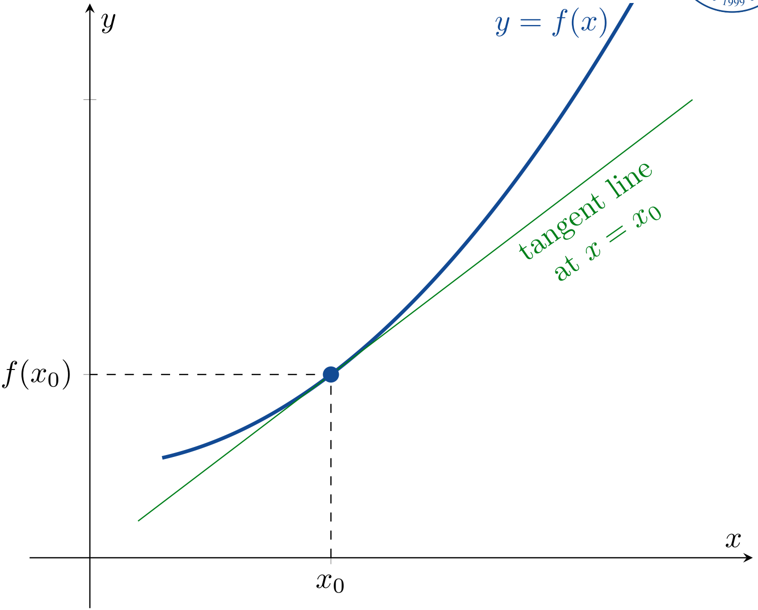
Example

$y = x^2$

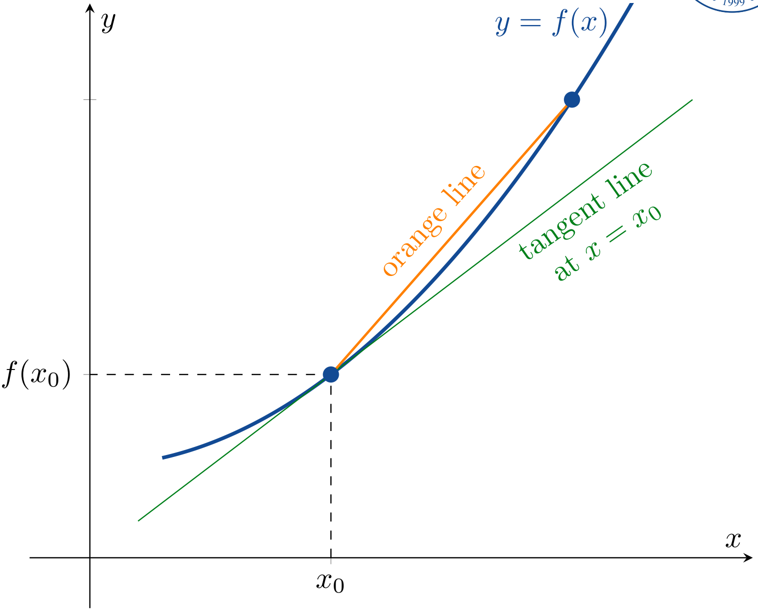


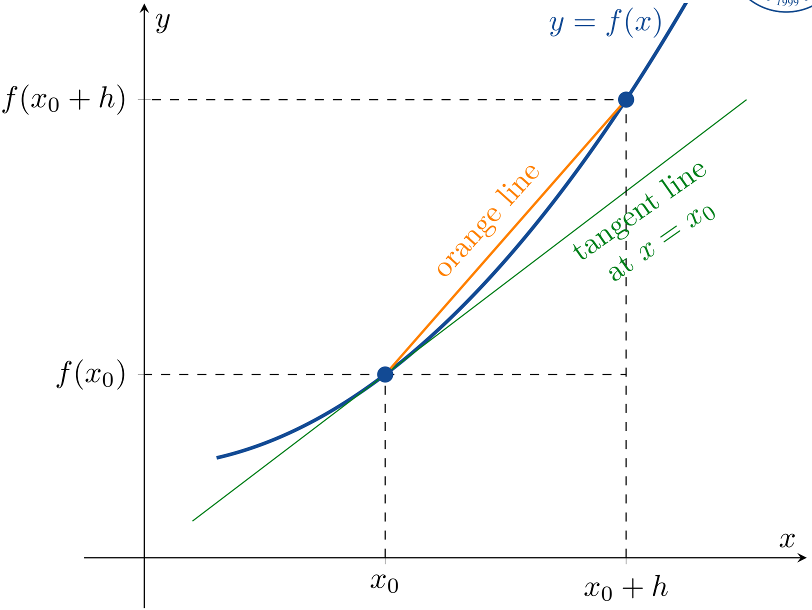
y
8
7
6
5
4
3
2
1
−1
x
−1
−0.5
0.5
1
1.5
2
2.5
$y = x^2$
slope = 4
$x_0 = 2$
$x_0 = 1$
slope = 2
$x_0 = 0$
slope = 0

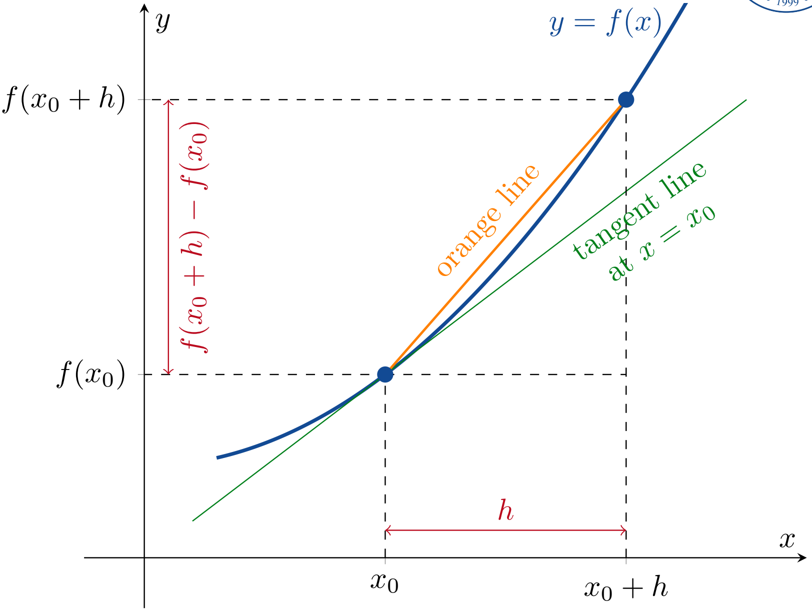
How can we calculate the slope of the tangent line?

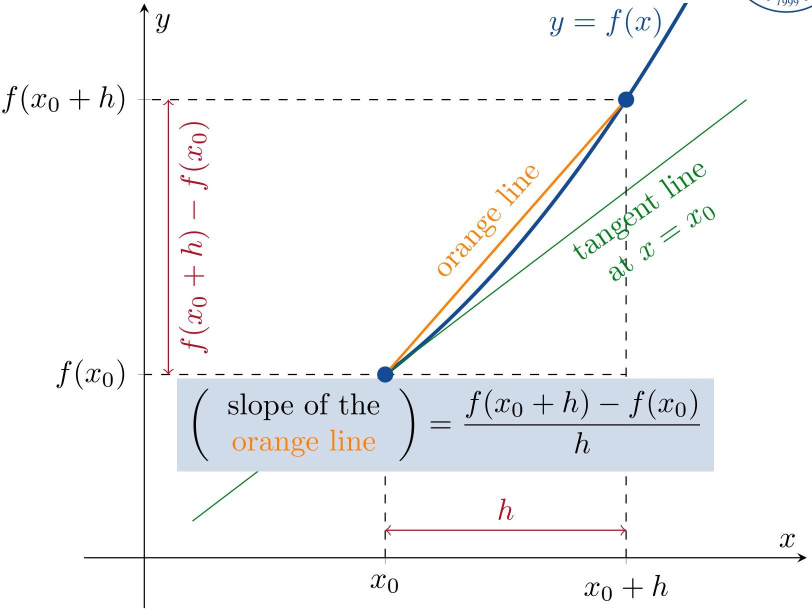


y
$y = f(x)$
tangent line
at $x = x_0$
$f(x_0)$
x
x_0

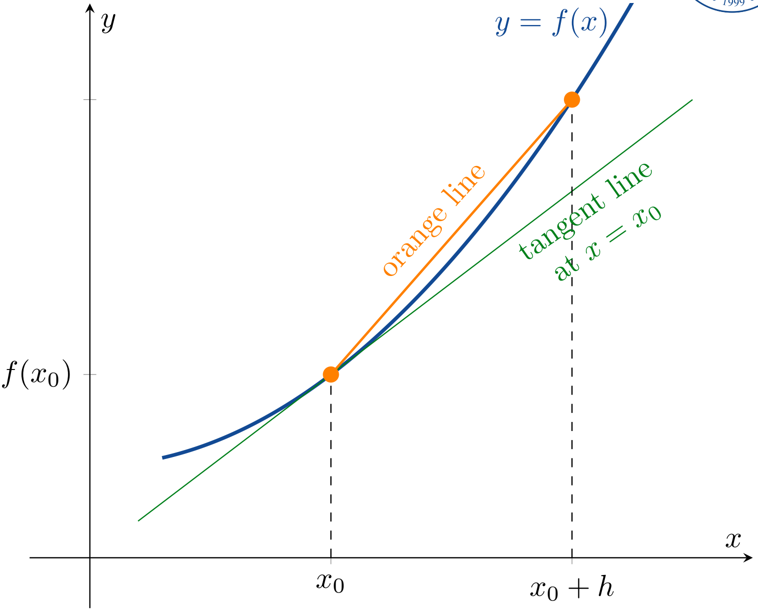

y
$y = f(x)$
orange line
tangent line
at $x = x_0$
$f(x_0)$
x
x_0


y
$y = f(x)$
$f(x_0 + h)$
$f(x_0)$
orange line
tangent line
at $x = x_0$
x
x_0
$x_0 + h$


y
$y = f(x)$
$f(x_0 + h)$
$f(x_0 + h) - f(x_0)$
orange line
tangent line
at $x = x_0$
$f(x_0)$
h
x
x_0
$x_0 + h$

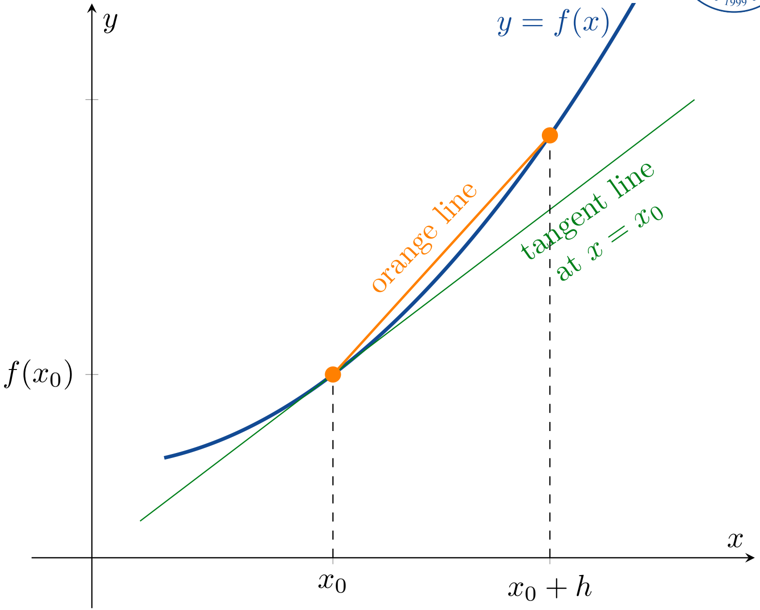


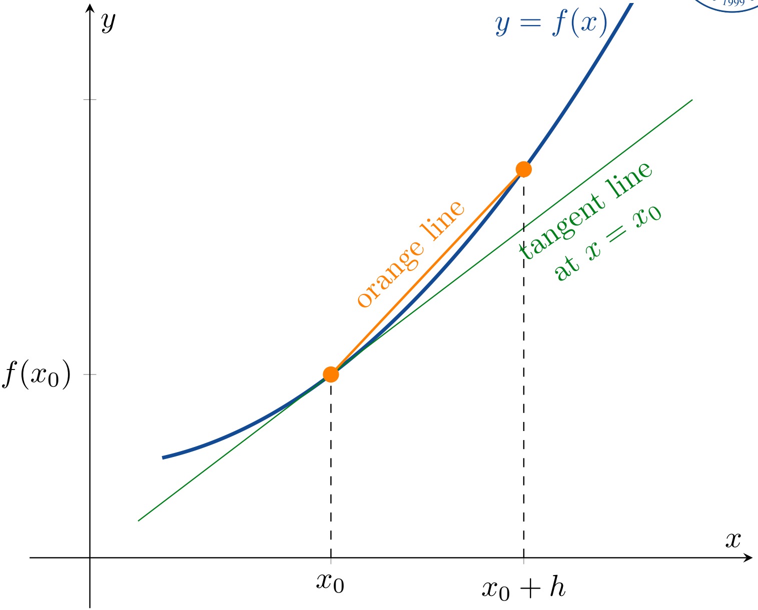
$$\begin{pmatrix} \text{slope of the} \\ \text{orange line} \end{pmatrix} = \frac{f(x_0 + h) - f(x_0)}{h}$$



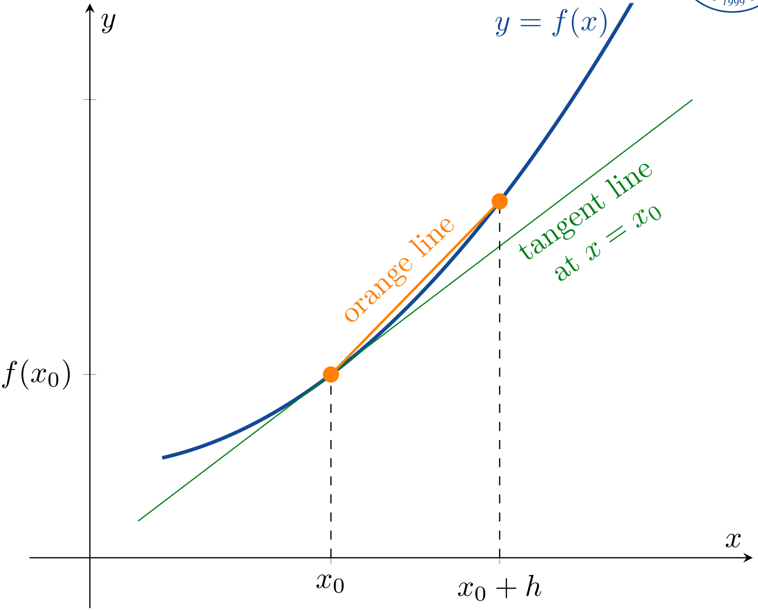
y
$y = f(x)$
orange line
tangent line
at $x = x_0$
$f(x_0)$
x
x_0
$x_0 + h$

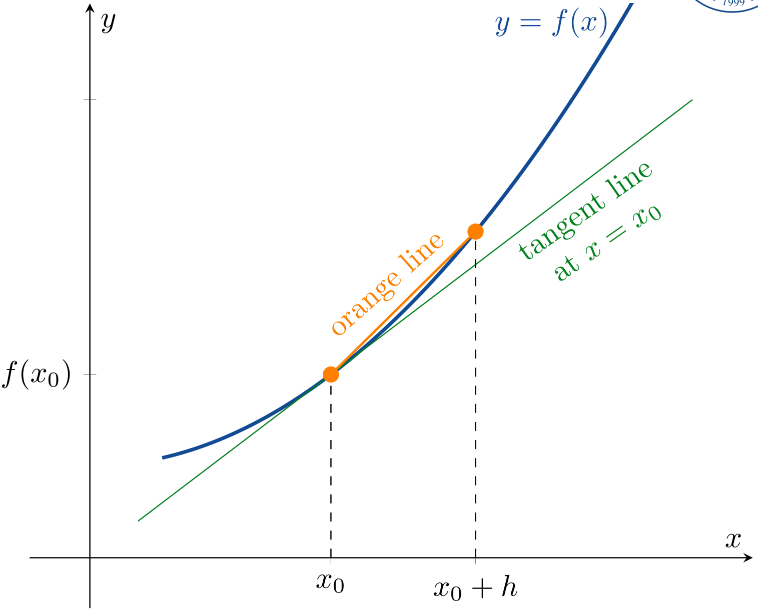
26. Differentiation



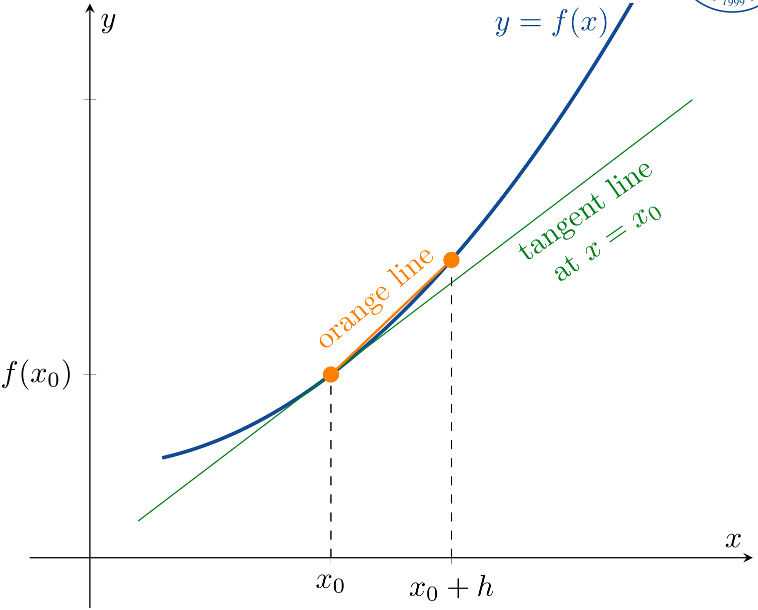


y
$y = f(x)$
orange line
tangent line
at $x = x_0$
$f(x_0)$
x
x_0
$x_0 + h$


y
$y = f(x)$
orange line
tangent line
at $x = x_0$
$f(x_0)$
x
x_0
$x_0 + h$

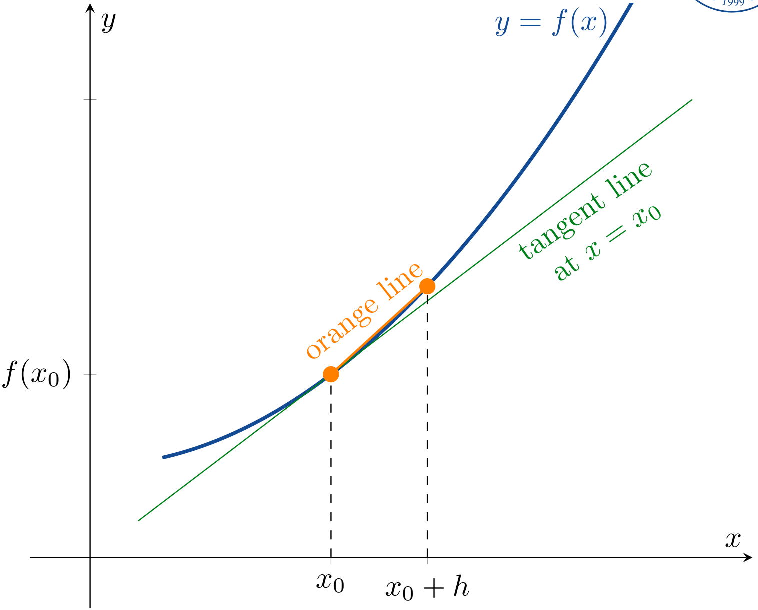


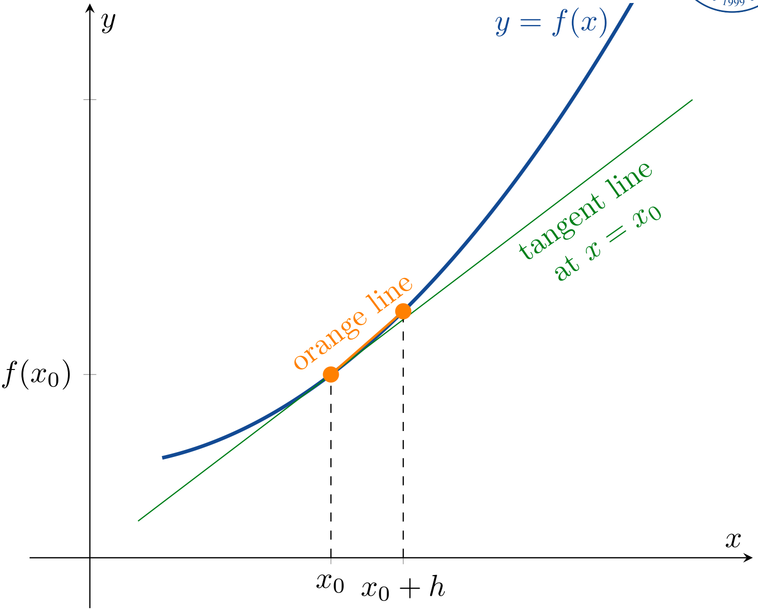
y
$y = f(x)$
tangent line
at $x = x_0$
orange line
$f(x_0)$
x
x_0
$x_0 + h$



y
$y = f(x)$
tangent line
at $x = x_0$
orange line
$f(x_0)$
x
x_0
$x_0 + h$

26. Differentiation





y
$y = f(x)$
tangent line
at $x = x_0$
orange line
$f(x_0)$
x
x_0
$x_0 + h$

If h is very very small, then

$$\begin{pmatrix} \text{slope of the} \\ \text{tangent line} \end{pmatrix} \approx \begin{pmatrix} \text{slope of the} \\ \text{orange line} \end{pmatrix} = \frac{f(x_0+h)-f(x_0)}{h}$$

The Derivative of f

Definition

The *derivative of a function* f *at a point* x_0 is

$$f'(x_0) = \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h}$$

if the limit exists.

(f' is pronounced "f prime")

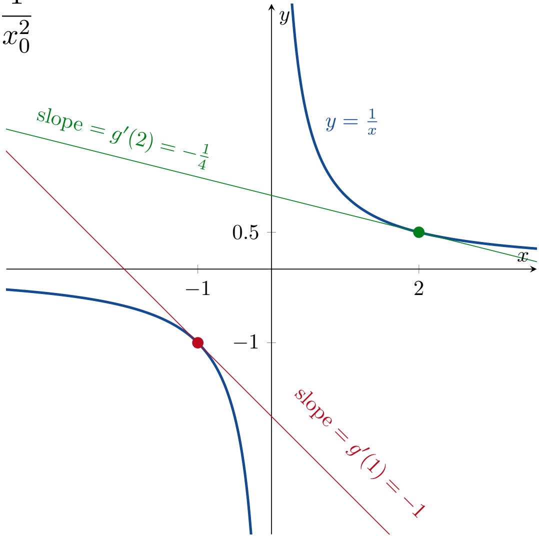
Example

Consider the function $g(x) = \frac{1}{x}$, $x \neq 0$.

If $x_0 \neq 0$, then

$$\begin{aligned}
g'(x_0) &= \lim_{h\to 0} \frac{g(x_0+h) - g(x_0)}{h} \\
&= \lim_{h\to 0} \frac{\frac{1}{x_0+h} - \frac{1}{x_0}}{h} \\
&= \lim_{h\to 0} \frac{\left(\frac{x_0}{x_0(x_0+h)} - \frac{x_0+h}{x_0(x_0+h)}\right)}{h} \\
&= \lim_{h\to 0} \frac{x_0 - x_0 - h}{hx_0(x_0+h)} \\
&= \lim_{h\to 0} \frac{-1}{x_0(x_0+h)} = -\frac{1}{x_0^2}.
\end{aligned}$$

$$g'(x_0) = -\frac{1}{x_0^2}$$



Definition

If $f'(x_0)$ exists, we say that f *is differentiable at* x_0.

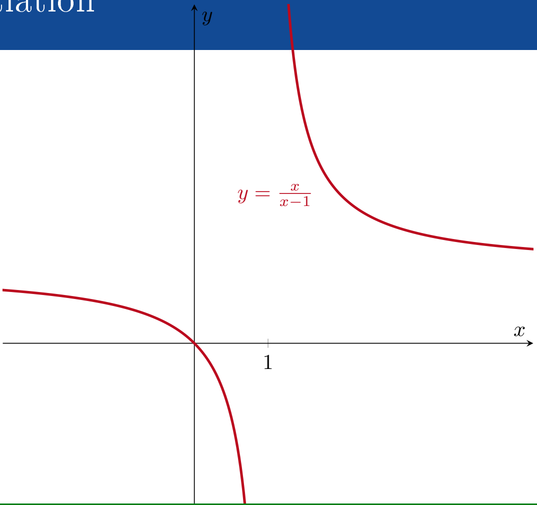
Definition

Let $f : D \to \mathbb{R}$ be a function. If f is differentiable at every $x_0 \in D$, we say that f is *differentiable*.

If $f : D \to \mathbb{R}$ is differentiable, then we have a new function $f' : D \to \mathbb{R}$.

Definition

f' is called the *derivative* of f.



Example

Differentiate $f(x) = \frac{x}{x-1}$.

solution: First note that $f(x+h) = \frac{x+h}{x+h-1}$. Therefore

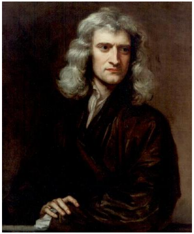
$$\begin{aligned}
f'(x) &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \\
&= \lim_{h\to 0} \frac{\frac{x+h}{x+h-1} - \frac{x}{x-1}}{h} \\
&= \lim_{h\to 0} \frac{1}{h}\left(\frac{(x+h)(x-1) - x(x+h-1)}{(x-1)(x+h-1)} \right) \\
&= \lim_{h\to 0} \frac{1}{h}\left(\frac{-h}{(x-1)(x+h-1)} \right) \\
&= \lim_{h\to 0} \frac{-1}{(x-1)(x+h-1)} \\
&= \frac{-1}{(x-1)(x+0-1)} \\
&= \frac{-1}{(x-1)^2}.
\end{aligned}$$

Notations

There are many ways to write the derivative of $y = f(x)$.

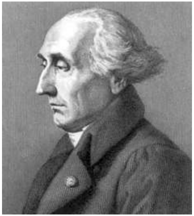
$$f'(x) = y' = \frac{dy}{dx} = \frac{df}{dx} = \frac{d}{dx} f(x) = \dot{y} = \dot{f}(x)$$

"the derivative of y with respect to x"



Sir Isaac Newton
UK, 1642-1726

Gottfried Leibniz
GER, 1646-1716



Joseph-Louis Lagrange
ITA, 1736-1813

Calculus was started by two men who hated each other: Sir Isaac Newton used $\dot{f}$ and $\dot{y}$. Gottfried Leibniz used $\frac{df}{dx}$ and $\frac{dy}{dx}$.

The f' and y' notation came later from Joseph-Louis Lagrange.

26. Differentiation

If we want the derivative of $y = f(x)$ at the point $x = x_0$, we can write

$$f'(x_0) = \left.\frac{dy}{dx}\right|_{x=x_0} = \left.\frac{df}{dx}\right|_{x=x_0} = \left.\frac{d}{dx}f(x)\right|_{x=x_0}$$

"the derivative of y with respect to x at $x = x_0$"

For example, if $u(x) = \frac{1}{x}$, then

$$u'(4) = \frac{d}{dx}\left(\frac{1}{x}\right)\bigg|_{x=4} = \frac{-1}{x^2}\bigg|_{x=4} = \frac{-1}{4^2} = \frac{-1}{16}.$$

Example

Show that $f(x) = |x|$ is differentiable on $(-\infty, 0)$ and on $(0, \infty)$, but is not differentiable at $x = 0$.

solution: If $x > 0$ then

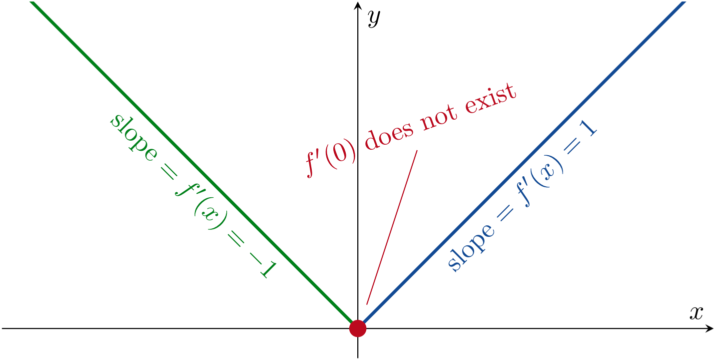
$$\frac{df}{dx} = \frac{d}{dx}\left(|x|\right) = \frac{d}{dx}(x) = \lim_{h \to 0} \frac{(x+h)-x}{h} = \lim_{h \to 0} 1 = 1.$$

Similarly, if $x < 0$ then

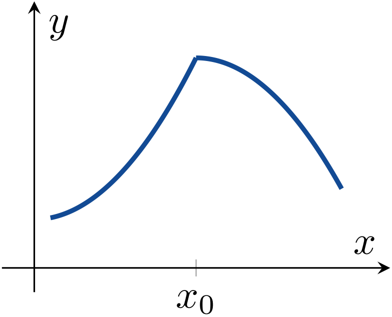
$$\begin{aligned}\frac{df}{dx} = \frac{d}{dx}\left(|x|\right) = \frac{d}{dx}(-x) &= \lim_{h\to 0}\frac{(-x-h)-(-x)}{h}\\ &= \lim_{h\to 0} -1 = -1.\end{aligned}$$

Therefore f is differentiable on $(-\infty, 0)$ and on $(0, \infty)$.

Since $\lim_{h\to 0} \frac{|0+h|-|0|}{h} = \lim_{h\to 0} \frac{|h|}{h} = \lim_{h\to 0}(\pm 1)$ does not exist, f is not differentiable at 0.



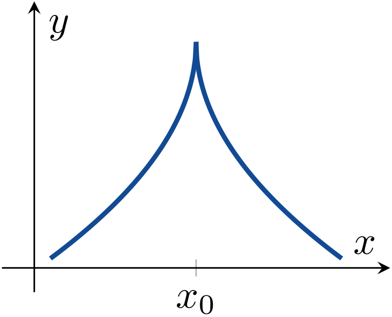
When Does a Function Not Have a Derivative at a Point?



a corner

$f'(x_0)$ does not exist

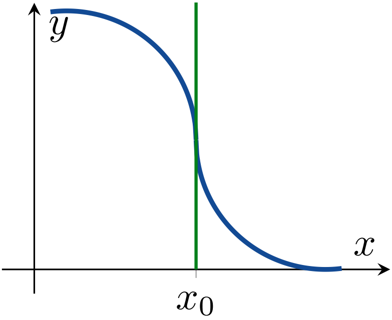
When Does a Function Not Have a Derivative at a Point?



a cusp

$f'(x_0)$ does not exist

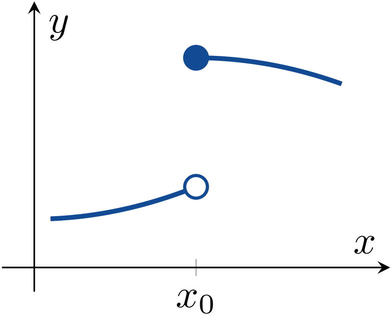
When Does a Function Not Have a Derivative at a Point?



a vertical tangent

$f'(x_0)$ does not exist

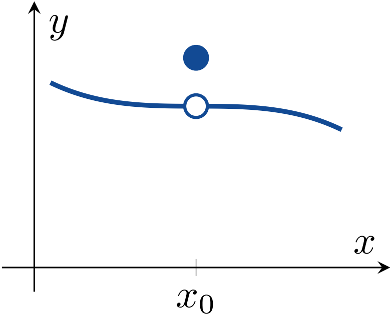
When Does a Function Not Have a Derivative at a Point?



a discontinuity

$f'(x_0)$ does not exist

When Does a Function Not Have a Derivative at a Point?



a discontinuity

$f'(x_0)$ does not exist

Theorem

$$\begin{pmatrix} f \text{ has a derivative} \\ \text{at } x = x_0 \end{pmatrix} \implies \begin{pmatrix} f \text{ is continuous} \\ \text{at } x = x_0 \end{pmatrix}$$

Next Time

- 27. Differentiation Rules
- 28. Derivatives of Trigonometric Functions
- 29. The Chain Rule