CHAPTER 7

Teleportation and Superdense Coding

Quantum teleportation is used to replace the state of one qubit with that of another over a long distance without the qubits directly interacting with each other. It works only at the level of individual quantum particles such as photons, electrons etc., and has not even vague resemblance with what is presented in television shows and/or science fictions stories. Superdense coding can be viewed as the process in which two classical bits of information are transmitted by sending just one quantum bit.

7.1 Quantum Teleportation

The goal of teleportation is to transfer the unknown state information of the source (first) qubit without measuring or observing, to the destination (second) qubit, thereby avoiding the disturbance of the first [1, 2, 4]. The second qubit, therefore, does not receive a *copy* of the quantum state of the first since it is impossible to produce an exact copy of an arbitrary quantum state (*no-cloning theorem*). As it turns out, the second qubit does not need a copy of the state information of the first—the *original* state of the first is *teleported* to it. Note that the process is not faster than light and a pair of entangled states has to be distributed ahead of time:

1. At the start, assume a single-qubit state

   $$|\psi> = \alpha|0> + \beta|1>$$

   at a location A, and that α and β in the state are unknown. Therefore, the necessary information to specify the state at location A are not available.
2. Generate an entangled state of a pair of qubits; assume the entangled state is a *Bell* (EPR) state and is written as

   $$|\vartheta> = \frac{1}{\sqrt{2}}(|00> + |11>)$$

The first half of the Bell state is sent to location A and the second half to location B. Thus, there are two qubits in location A (state $|\Psi>$ and half of the Bell state $|\vartheta>$), and one in location B (the second half of the Bell state).

3. To teleport the qubit at location A to location B, create a tensor product of qubit at A with ϑ

$$\omega_1 = \psi \otimes \vartheta$$
$$= (\alpha|0> + \beta|1>) \otimes \frac{1}{\sqrt{2}}(|00> + |11>)$$
$$= (\alpha|0> \otimes \frac{1}{\sqrt{2}}|0\,0> + |11>) + \beta|1> \otimes \frac{1}{\sqrt{2}}(|00> + |11>)$$
$$= \frac{1}{\sqrt{2}}(\alpha|000> + \alpha|01\,1> + \beta|10\,0> + \beta|111>)$$

Note that there are three qubits at the start:
   a. Qubit 1 is in an unknown state that is to be teleported and is located in A.
   b. Qubit 2 is the first half of the entangled pair and is located in A.
   c. Qubit 3 is the second half of the entangled pair and is located in B.

4. Next the two qubits in location A are sent through a CNOT gate. As indicated in Chap. 5 a CNOT gate inverts the state of the second qubit if the first qubit is in state 1, otherwise nothing changes. Thus, the second qubit of terms 3 and 4 in state ω_0 change giving a new state:

$$\omega_1 = \frac{1}{\sqrt{2}}(\alpha|00\,0> + \alpha|01\,1> + \beta|11\,0> + \beta\;|10\,1>)$$

5. Next Qubit 1, that is the first qubit that initially contains the state to be teleported, is sent through a Hadamard gate. There are four terms in state ω_1 with the first qubit being in state 0, 0, 1, and 1, respectively. As indicated previously a Hadamard gate transforms state $|0>$ and $|1>$ into

$$|0> = \frac{1}{\sqrt{2}}(|0> + |1>)$$
$$|1> = \frac{1}{\sqrt{2}}(|0> - |1>)$$

respectively.

By substituting $|0>$ and $|1>$ in the first qubit of the terms of ω_1 with their Hadamard transformations, another quantum state ω_2 is obtained:

$$\omega_2 = \frac{1}{\sqrt{2}}(\alpha|00\,0> + \alpha|01\,1> + \beta|110> + \beta|10\,1>)$$

$$= \frac{1}{\sqrt{2}} [\alpha (\frac{1}{\sqrt{2}} (|0> + |1>) \; |0\,0> + \alpha (\frac{1}{\sqrt{2}} (|0> + |1>) \; |1\,1>$$

$$+ \beta (\frac{1}{\sqrt{2}} (|0> - |1>) \; |1\,0> + \beta (\frac{1}{\sqrt{2}} (|0> - |1>) \; |0\,1>]$$

This indicates a superposition of eight states that can be rearranged as follows:

$$\omega_2 = \frac{1}{\sqrt{2}} [\alpha ((\frac{1}{\sqrt{2}} (|0> + |1>) \; |0\,0> + (\frac{1}{\sqrt{2}} (|0> + |1>) \; |1\,1>)$$

$$+ \beta ((\frac{1}{\sqrt{2}} (|0> - |1>) \; |1\,0> + (\frac{1}{\sqrt{2}} (|0> - |1>) \; |0\,1>)]$$

$$= \frac{1}{2} [\alpha|000> + \alpha|100> + \alpha|011> + \alpha|111> + \beta|010>$$

$$- \beta|11\,0> + \beta|001> - \beta|101>]$$

$$= \frac{1}{2} [|00> (\alpha|0> + \beta|1>) + |01> (\alpha|1> + \beta|0>)$$

$$+ |10> (\alpha|0> - \beta|1>) + |11> (\alpha|1> - \beta|0>)]$$

Note that at this stage that qubits Q1 and Q2 are in location A, and the third qubit Q3 is in location B.

Using the two-dimensional unit matrix I and the three Pauli matrices:

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$Y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

the state ω_2 can be rewritten as

$$= \frac{1}{2} [|00> \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} |\psi> + |01> \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} |\psi>$$

$$+ |10> \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} |\psi> + |11> i \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} |\psi>]$$

$$= \frac{1}{2} [|00> \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} |\psi> + |01> \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} |\psi>$$

$$+ |10> \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} |\psi> + |11> \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} |\psi>]$$

$$= \frac{1}{2} [|00> I|\psi> + |01> X|\psi> + |10> Z|\psi> + |11> XZ|\psi>]$$

Result of the Measurement of Q1 and Q2 in Location A	State of Q3 in Location B
c_0 c_1	
00	$(\alpha\|0> + \beta\|1>)$
01	$(\alpha\|1> + \beta\|0>)$
10	$(\alpha\|0> - \beta\|1>)$
11	$(\alpha\|1> - \beta\|0>)$

TABLE 7.1 One of Four Possible States of Q3 After Measurement

Note that qubits 1 and 2 are different in each term. If the qubits in location A are measured, the outcomes can be encoded using one of the following pairs of classical bits:

$$c_0c_1 = 00, 01, 10, \text{ or } 11$$

In other words, the four possible outcomes upon measuring qubits 1 and 2 result in two bits of classical information: c_0 and c_1. This measurement has an impact on Qubit 3 in location B and leaves it in one of the four distinct states as shown in Table 7.1; the classical bits c_0 and c_1 identify the state.

6. The next step is to send the classical bits c_0 and c_1 to location B via a classical channel. Depending on the values of c_0 and c_1, one of the four possible unitary operations is performed on Qubit 3 (qubit in location B) as shown in Table 7.2. This step restores state ψ, that is the original state of Qubit 1.

   As an example, suppose the qubits in location A (Qubits 1 and 2) are measured and the result is 00, then as shown previously the qubit in location B (Qubit 3) is in state $(\alpha\,|\,0> + \beta\,|\,1>)$. Note that this is the state that was initially intended for teleportation, thus the teleportation from A to B already

State of Q1 and Q2 in Location A	Unitary Operation Needed to Restore Original State of Location A
00	I
01	X
10	Z
11	ZX

TABLE 7.2 Unitary Operation Performed on Qubit 3

happened in this instance. However, this is not the only result of the measurement, there are four possible results, each of which gives a different state for the qubit in location B. If the result of the measurement in location A is 00, 01, 10, or 11 then the state of the qubit at location B becomes ($\alpha|0> + \beta|1>$), ($\alpha|1> + \beta|0>$), ($\alpha|0> - \beta|1>$), and ($\alpha|1> - \beta|0>$), respectively.

These four possible outcomes upon measuring two qubits in location A are encoded using the two classical bits c_0 and c_1. In each case a unitary transformation is applied on the state of Qubit 3 in location B so that the state of Qubit 1 and Qubit 2 in A are restored to their original value as discussed below:

i. State of the qubits 1 and 2 = 00

$$\begin{array}{cc} q_1q_2 & q_3 \\ |0\,0> & (\alpha|0>+\beta|1>) \end{array}$$

Classical bits $c_0c_1 = 00$ are sent from location A to B. Since both bits are 0s, both operators X and Z are idle, and unitary operator I is applied to Qubit 3 as indicated in Table 7.2, so it retains its state $\alpha|0> + \beta|1>$:

$$\begin{aligned} I(\alpha|0>+\beta|1>) &= \alpha I|0> + \beta I|1> \\ &= \alpha\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} + \beta\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} \\ &= \alpha\begin{pmatrix} 1 \\ 0 \end{pmatrix} + \beta\begin{pmatrix} 0 \\ 1 \end{pmatrix} \\ &= \alpha|0> + \beta|1> = \psi \end{aligned}$$

ii. State of the qubits 1 and 2 = 01

$$\begin{array}{cc} q_1q_2 & q_3 \\ |0\,1> & (\alpha|1>+\beta|0>) \end{array}$$

Classical bits $c_0c_1 = 01$ are sent from location A to B. Since $c_0 = 0$ and $c_1 = 1$, operator X is applied to the qubit in location B, that is, Qubit 3 as indicated in Table 7.2.

$$X(\alpha|1>+\beta|0>) = \alpha X\,|1> + \beta X\,|0>$$

Thus the state of the qubit changes to $\alpha|0> + \beta|1>) = \psi$

iii. State of the qubits 1 and 2 = 10

$$\begin{array}{cc} q_1q_2 & q_3 \\ |1\,0> & (\alpha|0>-\beta|1>) \end{array}$$

Classical bits $c_0c_1 = 10$ are sent from location A to B as shown in Table 7.1. Since $c_0 = 1$ and $c_1 = 0$, operator Z is applied to the qubit in location B. Thus, the state of the qubit changes to $\alpha|0\rangle + \beta|1\rangle) = \psi$:

$$Z(\alpha|0\rangle - \beta|1\rangle) = \alpha Z|0\rangle - \beta Z|1\rangle$$
$$= \alpha\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} - \beta\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix}$$
$$= \alpha\begin{pmatrix} 1 \\ 0 \end{pmatrix} + \beta\begin{pmatrix} 0 \\ 1 \end{pmatrix}$$
$$= \alpha|0\rangle + \beta|1\rangle = \psi$$

iv. State of the qubits 1 and 2 = 11

$$\begin{array}{cc} q_1q_2 & q_3 \\ |11\rangle & (\alpha|1\rangle - \beta|0\rangle) \end{array}$$

Classical bits $c_0c_1 = 11$ are sent from location A to B. Since $c_0 = 1$ and $c_1 = 1$, both X and Z are applied to the qubit in location B:

$$ZX(\alpha|1\rangle - \beta|0\rangle) = \alpha ZX|1\rangle - \beta ZX|0\rangle$$
$$= \alpha Z|0\rangle - \beta Z|1\rangle$$
$$= \alpha|0\rangle + \beta|1\rangle = \psi$$

Thus, the state of the qubit changes to $\alpha|0\rangle + \beta|1\rangle$.

In each of the above cases there is a unitary transformation that restores the state of the qubit in location B to original state ψ.

7.2 No-Cloning Theorem

In classical computing systems, it is taken for granted that digital data can be copied with perfect accuracy. The *no-cloning theorem* describes one of the most fundamental properties of quantum systems, namely, there is no unitary operation that will perfectly copy an arbitrary quantum state [3]. An arbitrary state in this context means any state of a specified Hilbert space that is being considered. This obviously limits the available resources for programming a quantum computer. However, the no-cloning feature is extremely important in quantum cryptography because the inability of copying an unknown quantum state is a contributing factor to the system security.

To illustrate the operation of cloning assume a hypothetical machine that accepts the state of a qubit as an input and produces two exact copies of the state, that is *clones* of the state. For example, a state $|\varphi\rangle$ is transformed into $|\varphi\varphi\rangle$ by the machine. Similarly, another state $|\phi>$ is converted to $|\phi\phi>$. However, if a state that is a linear combination of two states is sent through the cloning machine, the output obtained is

$$|\omega> = (a\,|\varphi\varphi> + b\,|\phi\phi>)$$

that is, a superposition of the two copies of $|\varphi>$ and two copies of $|\phi>$ because in quantum systems the linearity property is preserved. However, the output of the machine is expected to be

$$|\psi>|\psi> = (a\,|\varphi> + b\,|\phi>)(a\,|\varphi> + b\,|\phi>)$$

that is the original and a copy of $|\psi>$, and not $|\omega>$ produced by the machine! The no-cloning theorem formally states this result:

Theorem: There is *no* valid quantum operation that maps an arbitrary state $|\psi>$ to $|\psi>|\psi>$ [5,6].

Assume an initial state $|s>$ that is to be converted into any other state $|\phi>$ or $|\varphi>$. For example, if $|s>$ is to be converted to $|\varphi>$ then the initial pair of state $|s>$ and $|\varphi>$ is transformed into two copies of $|\varphi>$ by a unitary transformation U as shown in Fig. 7.1.

The copying of a state using an unitary operator U can be written as

$$U|\varphi>\otimes|s> = |\varphi>\otimes|\varphi> \tag{7.1}$$

Similarly for another state $|\phi>$,

$$U|\phi>\otimes|s> = |\phi>\otimes|\phi> \tag{7.2}$$

Take the inner product of the left-hand sides of the above equations

$$U|\varphi>\otimes|s>U|\phi>\otimes|s>$$

Replace the first half of the equation with its complex conjugate,

$$= <s|\otimes<\varphi|\;U^*\,U\;|\phi>\otimes|s>$$
$$= <s|\otimes<\varphi|\;U^*\,U\;|\phi>\otimes|s>$$

Since $U^*\,U = I$,

$$= <s|\otimes<\varphi|\phi>\otimes|s>$$
$$= <\varphi|\phi><s|s>$$

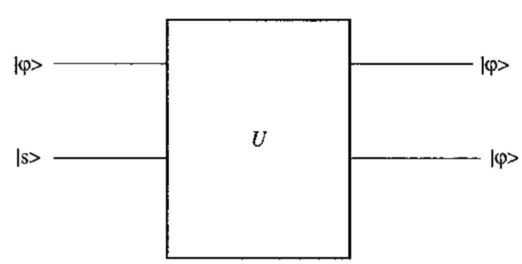

FIGURE 7.1 Unitary transformation U for copying state.

Since $<s|s> = 1$

$$= <\varphi|\phi> \tag{7.3}$$

Similarly, taking the inner product of the right-hand sides of Eqs. (7.1) and (7.2) gives

$$\begin{aligned} &<\varphi| \otimes <\varphi| \, |\phi> \otimes |\phi> \\ &= (<\varphi|\phi>)^2 \end{aligned} \tag{7.4}$$

However, Eqs. (7.3) and (7.4) must be equal; this implies

$$<\varphi|\phi> = (<\varphi|\phi>)^2$$

Thus, $<\varphi|\phi>$ is either 0 or 1, that is, $|\phi>$ and $|\varphi>$ are either orthogonal or the same state. Perfect copying is only possible for a set of states that are orthogonal, not for any arbitrary state.

An alternative proof of the no-cloning theorem is as follows. Assume a unitary operator that can copy an unknown state $|\alpha> = a|0> + b|1>$ onto a state $|s>$, then

$$\begin{aligned} U(|\alpha>|s>) = |\alpha>|\alpha> &= (a|0> + b|1>)(a|0> + b|1>) \\ &= a^2|00> + ab|01> + ab|10> + b^2|11> \end{aligned}$$

However, if the linear combination corresponding to state $|\alpha>$ is sent through the cloning machine then a different state is produced:

$$\begin{aligned} U(a|0> + b|1>)\,|s> &= (a|00> + b|11>) \\ &\neq (a|0> + b|1>)\,(a|0> + b|1>) \end{aligned}$$

This contradiction in result shows there does not exist an operator for cloning.

7.3 Superdense Coding

Superdense coding can be viewed as teleportation in reverse. The idea is to transmit two classical bits of information by sending a single qubit through the quantum channel. Suppose Alice wants to send a two-bit message to Bob [1, 4, 6]. She could send two qubits with the message encoded in them. A single qubit on its own cannot transmit two classical bits of information. However, superdense coding allows a single qubit to perform this. This option requires that the two parties initially share a pair of entangled qubits. Assume initially Alice and Bob share a Bell state:

$$\frac{1}{\sqrt{2}}(|00> + |11>)$$

The first qubit in each term is Alice's half of the state and the second qubit is used by Bob. Note that the Bell state is a fixed state; Alice and Bob do not need to prepare the state. It is assumed that some third party prepared the Bell state beforehand and sent one of entangled qubits to Alice, and the other one to Bob. Hence, Alice and Bob each possess half of the Bell state, that is they share one unit of the entangled pair.

For example, assume s_1s_0 is the two-bit string Alice wants to send to Bob. There are four possible combinations of $s_1\, s_0$(= 00, 01, 10, or 11) that Alice can send to Bob using the qubit she has. She chooses one of four unitary operations $U(= I, X, Y \text{ or } Z)$, on the entangled bit in her possession based upon which bit string she wishes to send to Bob. Applying the transformation only to her qubit means she needs to apply an identity (I) operation on the second qubit (in Bob's possession) so that it does not change. This coding process is described below, and the combined state after the application of the chosen unitary operation is also shown with Alice's qubit in bold:

i. *Classical bits to be sent: 00.* Nothing needs to be done, so Alice applies $U = I \otimes I$ on her part of the Bell state

$$I \otimes I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Hence

$$(I \otimes I)(\frac{1}{\sqrt{2}}(|00> + |11>) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} (\frac{1}{\sqrt{2}}(|00> + |11>)$$

$$= (\frac{1}{\sqrt{2}}(|0>|0> + |1>|1>)$$

ii. *Classical bits to be sent: 01.* Alice applies $U = X \otimes I$ on her part of the Bell state

$$U = X \otimes I$$

$$= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

Hence

$$(X \otimes I)\ (\frac{1}{\sqrt{2}}(|00> + |11>) = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} (\frac{1}{\sqrt{2}}(|00> + |11>)$$

$$= \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

$$= \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix}$$

$$= \frac{1}{\sqrt{2}} (|1>|0> + |0>|1>)$$

iii. *Classical bits to be sent: 10.* Alice applies $U = Z \otimes I$ on her part of the Bell state

$$U = Z \otimes I$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

Hence

$$(Z \otimes I)(\frac{1}{\sqrt{2}}(|00> + |11>)$$

$$= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} (\frac{1}{\sqrt{2}}(|00> + |11>)$$

$$= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

$$= \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ 0 \\ -1 \end{pmatrix}$$

$$= \frac{1}{\sqrt{2}} (|0>|0> - |1>1>)$$

iv. *Classical bits to be sent: 11.* Alice applies $U = XZ \otimes I$ on her part of the Bell state

Hence

$$U = iY \otimes I \text{ since } XZ = iY$$

$$= i \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Thus

$$(XZ \otimes I)(\frac{1}{\sqrt{2}}(|00> + |11>)$$

$$= \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix} (\frac{1}{\sqrt{2}}(|00> + |11>)$$

$$= \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

$$= \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \\ -1 \\ 0 \end{pmatrix} = \frac{1}{\sqrt{2}}(|0>|1> - |1>|0>)$$

Table 7.3 shows the starting state, the unitary operation applied, and the resulting final state for sending a two-bit string. Note that like the initial states, the final states are also Bell basis states.

After the application of the unitary operation Alice sends her half of the entangled qubits, that is, q_0 to Bob. Bob combines it with his qubit (q_1) and applies a controlled-NOT operation to the pair (q_0 q_1), where q_0 is assumed to be the control bit. Next a Hadamard transform on the first qubit of the pair leads to the untanglement of the Bell state and results in a unique state that corresponds to the two-bit string. This process can be explained as follows:

Classical Bits to Be Sent	Initial State	Unitary Operation	Final State
00	$(\frac{1}{\sqrt{2}}(\|00> + \|11>)$	$I \times I$	$(\frac{1}{\sqrt{2}}(\|0>\|0> + \|1>\|1>)$
01	$(\frac{1}{\sqrt{2}}(\|00> + \|11>)$	$X \times I$	$(\frac{1}{\sqrt{2}}(\|1>\|0> + \|0>\|1>)$
10	$(\frac{1}{\sqrt{2}}(\|00> + \|11>)$	$Y \times I$	$(\frac{1}{\sqrt{2}}(\|0>\|0> - \|1>\|1>)$
11	$(\frac{1}{\sqrt{2}}(\|00> + \|11>)$	$X \times Z$	$(\frac{1}{\sqrt{2}}(\|0>\|1> - \|1>\|0>)$

TABLE 7.3 Transmission of One of a Four Possible Two-Bit Strings via a Single Qubit

Case 00: The controlled-NOT operation to the pair ($q_0\, q_1$) has no effect on the $|0>|0>$ part of the Bell state

$$\frac{1}{\sqrt{2}}(|0>|0>+|1>|1>$$

but since bit q_0 is 1, it changes the $|1>|1>$ part into $|1>|0>$; thus the Bell state is transformed into

$$\frac{1}{\sqrt{2}}(|0>|0>+|1>|0> \tag{7.5}$$

Bob then applies H (Hadamard transform) on the first qubit (q_0) of the entangled pair; this changes Bell state Eq. (7.5):

$$\begin{aligned}
&\frac{1}{\sqrt{2}}[\frac{1}{\sqrt{2}}(|0>+|1>)|0>+\frac{1}{\sqrt{2}}(|0>-|1>)|0>] \\
&=\frac{1}{2}[(|0>+|1>)|0>+(|0>-|1>)|0>] \\
&=\frac{1}{2}[|00>+|10>+(|00>-|10>] \\
&=|00>
\end{aligned}$$

Bob measures both qubits and gets Alice's message 00.

Case 01: The controlled–NOT operation changes the $|1>|0>$ part of the Bell state

$$\frac{1}{\sqrt{2}}(|1>|0>+|0>|1>)$$

into $|1>|1>$, because the first that is, the control bit is 1; thus the Bell state is transformed into

$$\frac{1}{\sqrt{2}}(|1>|1>+|0>|1> \tag{7.6}$$

Bob then applies H (Hadamard transform) on the first qubit (q_0) of the entangled pair, this changes Bell state Eq. (7.6):

$$\begin{aligned}
&\frac{1}{\sqrt{2}}[\frac{1}{\sqrt{2}}(|0>-|1>)|1>+\frac{1}{\sqrt{2}}(|0>+|1>)|1>] \\
&=\frac{1}{2}[|01>-|11>+(|01>+|11>] \\
&=|01>
\end{aligned}$$

Bob measures both qubits and gets Alice's message 01.

Case 10: The controlled–NOT operation changes the $|1>|1>$ part of the Bell state

$$\frac{1}{\sqrt{2}}(|0>|0>-|1>|1>)$$

and transforms it to

$$\frac{1}{\sqrt{2}}(|0>|0>-|1>|0>) \tag{7.7}$$

The application of Hadamard transform on the q_0 bit, changes the Bell state Eq. (7.7):

$$\frac{1}{\sqrt{2}}[\frac{1}{\sqrt{2}}(|0>+|1>)|0>-\frac{1}{\sqrt{2}}(|0>-|1>)|0>]$$
$$=\frac{1}{2}[|00>+|10>-(|00>+|10>]$$
$$=|10>$$

Bob measures both qubits and gets Alice's message 10.

Case 11: The controlled–NOT operation changes the $|1>|0>$ part of the Bell state

$$\frac{1}{\sqrt{2}}(|0>|1>-|1>|0>)$$

to

$$\frac{1}{\sqrt{2}}(|0>|1>-|1>|1>) \tag{7.8}$$

The application of Hadamard transform on the q_0 bit, changes the Bell state (7.8):

$$\frac{1}{\sqrt{2}}[\frac{1}{\sqrt{2}}(|0>+|1>)|1>-\frac{1}{\sqrt{2}}(|0>-|1>)|1>]$$
$$=\frac{1}{2}[|01>+|11>-(|01>+|11>]$$
$$=|11>$$

Bob measures both qubits and gets Alice's message 11.

Thus, in each of the four cases above Bob needed two bits to decode the status of the final state. This means Alice's message to Bob was composed of two bits not one, otherwise it was not possible to decode four separate states!