CHAPTER 6

Tensor Products, Superposition, and Quantum Entanglement

Quantum computing utilizes two major features of quantum mechanics: *superposition* and *entanglement*. Superposition, as indicated earlier, refers to the quantum phenomenon where a quantum system can exist in multiple states or places at the same time. Entanglement is an extremely strong correlation that exists between two or more quantum particles. These particles are so inextricably linked that even if separated by great distances, they change their states instantaneously in perfect unison. This might seem almost impossible, but is fundamental to the quantum world. Some knowledge of tensor products is needed to understand the concepts of superposition and entanglement.

6.1 Tensor Products

The state of a single quantum bit can be represented as a unit (column) vector in a two-dimensional vector space C^2. However, quantum information processing systems in general use multiple qubits. The joint state of such a system can only be described using a new vector space that takes into account the interaction among the qubits. This vector space is generated by using a special operation known as a *tensor product*. The tensor product, denoted by $\otimes$, combines the smaller vector spaces of the individual qubits and forms a bigger space; the elements of the bigger vector space are identified as *tensors*. A tensor product is also called *Kronecker product* or *direct product* [1, 2].

For example, the tensor product of two two-dimensional vectors $\mathbf{U} = \begin{pmatrix} x_1 \\ y_1 \end{pmatrix}$ and $\mathbf{V} = \begin{pmatrix} x_2 \\ y_2 \end{pmatrix}$ is illustrated below:

$$\mathbf{U} \otimes \mathbf{V} = \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} \otimes \begin{pmatrix} x_2 \\ y_2 \end{pmatrix}$$

$$= \begin{bmatrix} x_1 \begin{bmatrix} x_2 \\ y_2 \end{bmatrix} \\ y_1 \begin{bmatrix} x_2 \\ y_2 \end{bmatrix} \end{bmatrix}$$

$$= \begin{bmatrix} x_1 x_2 \\ x_1 y_2 \\ y_1 x_2 \\ y_1 y_2 \end{bmatrix}$$

It has been shown in Chap. 1 that the vector representation of single qubit states $|0>$ and $|1>$ are $|0> = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $|1> = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$. Using these vector representations and the definition of the tensor product the two-qubit basis states can be represented as

$$|00> = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad |01> = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} \quad |10> = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \quad |11> = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

In general, if **U** is an m-dimensional vector space with bases $\{g_0 \cdots \cdots g_{m-1}\}$ and **V** is an n-dimensional vector space with bases $\{h_0 \cdots \cdots h_{n-1}\}$, the tensor product $\mathbf{U} \otimes \mathbf{V}$ is a mn-dimensional vector space that is spanned by elements of the form $g \otimes h$ and the coefficients are given by $u_i v_j$ for each basis vector [2]:

$$\mathbf{U} \otimes \mathbf{V} = \sum_{i=0}^{m-1} u_i g_i \otimes \sum_{j=0}^{n-1} v_j h_j$$

$$= \sum_{i=0}^{m-1} u_i \sum_{j=0}^{n-1} v_j (g_i \otimes h_j)$$

Assuming both **U** and **V** are two-dimensional, that is, $m = n = 2$,

$$\mathbf{U} \otimes \mathbf{V} = \sum_{i=0}^{1} u_i g_i \otimes \sum_{j=0}^{1} v_j h_j$$

$$= (u_0 g_0 + u_1 g_1) \otimes (v_0 h_0 + v_1 h_1)$$

$$= u_0 v_0 (g_0 \otimes h_0) + u_0 v_1 (g_0 \otimes h_1) + u_1 v_0 (g_1 \otimes h_0) + u_1 v_1 (g_1 \otimes h_1)$$

The new vector space generated by the tensor product of two two-dimensional vectors **U** and **V** is four-dimensional and the new basis vectors are

$$(u_0 v_0,\ u_0 v_1,\ u_1 v_0,\ u_1 v_1)$$

This vector is shown below as a four-dimensional column vector

$$\mathbf{U} \otimes \mathbf{V} = \begin{pmatrix} u_0 v_0 \\ u_0 v_1 \\ u_1 v_0 \\ u_1 v_1 \end{pmatrix}$$

For example, if **V** is a vector space with two basis vectors $|\mathbf{u}>$ and $|\mathbf{v}>$ that correspond to two quantum bits, then the joint state of the quantum bits is

$$u> \otimes |v>$$

and it is an element of **U** ⊗ **V**.

Some of the basic properties of tensor products are:

i. If A and B are operators on m and n dimensional vectors respectively, then A ⊗ B is an operator on $n \times m$ dimensional vector.

ii. The product of any scalar s with tensor product A ⊗ B is equal to

$$s(A \otimes B) = (sA) \otimes B = A \otimes (sB)$$

iii. $(A \otimes B)^* = (A^* \otimes B^*)$ (and similarly for inverse and transpose)

iv. If A is an $m \times n$ matrix and B an $p \times q$ matrix, then their tensor product is an $mp \times nq$ matrix.

v. If A, B, C, and D are matrices then

   a. $A \otimes (B + C) = (A \otimes B) + (A \otimes C)$

   b. $A \otimes (B \otimes C) = (A \otimes B) \otimes C$

that is, derivation of a tensor product is independent of the order of evaluation; thus it is an associative operation.

c. $(A \otimes B)(C \otimes D) = (AC) \otimes (BD)$

that is, the distribution law holds.

vi. If A, B are matrices, and **U**, **V**, and **W** are vectors, then

$(A \otimes B)(\mathbf{U} \otimes \mathbf{W}) = A\mathbf{U} \otimes B\mathbf{W}$

$(\mathbf{U} + \mathbf{V}) \otimes \mathbf{W} = \mathbf{U} \otimes \mathbf{W} + \mathbf{V} \otimes \mathbf{W}$

$\mathbf{U} \otimes (\mathbf{V} + \mathbf{W}) = \mathbf{U} \otimes \mathbf{V} + \mathbf{U} \otimes \mathbf{W}$

vii. If s and t are scalars, **U** and **V** are vectors, then

$s\mathbf{U} \otimes t\mathbf{V} = s\, t\, (\mathbf{U} \otimes \mathbf{V})$

6.2 Multi-Qubit Systems

A multi-qubit quantum system can be formed by putting together a number of known quantum subsystems. A subsystem based on a single qubit is generally described by its state that is defined to be a unit vector in some complex Hilbert space. The mathematical framework for describing a multi-qubit system utilizes the concept of a *tensor product* of vector spaces. The state space of the *i*th constituent of such a system is given by a separable Hilbert space H_i. Each Hilbert space H_i has an orthonormal basis given by

$$\{|i, k_i > k_i = 1, 2, \ldots\}, i = 1, 2, \ldots, n$$

The state space of the multi-bit system is then the tensor product space H defined by

$$H = H_1 \otimes H_2 \otimes \ldots \otimes H_n$$

The basis states of a multi-qubit system are constructed from the basis-states of a single qubit using *tensor product* of vectors. The tensor product of two vectors **u** and **v**, assuming **u** and **v** are *m*-dimensional and *n*-dimensional, respectively, has the dimension $m \times n$. Since in a single qubit system $m = 2 = n$, a two-qubit system has four basis states. These basis states are constructed from the single-qubit basis $|0>$, $|1>$ using the following rule

$$|\mathbf{u}> \otimes\ |\mathbf{v}> = |\mathbf{u}>|\mathbf{v}> = |\mathbf{uv}>$$

where **u**,**v** $\in \{0, 1\}$.

For instance, if the states of the two qubits are given by

$$|\psi_1> = \alpha_0|0> + \alpha_1|1> \text{ and } |\psi_2> = \beta_0|0> + \beta_1|1>$$

then the state of the composite system is

$$|\psi\rangle = |\psi_1\rangle \otimes |\psi_2\rangle$$
$$= (\alpha_0|0\rangle + \alpha_1|1\rangle) \otimes (\beta_0|0\rangle + \beta_1|1\rangle)$$

Using the following rule

$$|\mathbf{u}\rangle \otimes |\mathbf{v}\rangle = |\mathbf{u}\rangle|\mathbf{v}\rangle = |\mathbf{uv}\rangle \qquad \text{where } \mathbf{u}, \mathbf{v} = 0, 1$$

$|\psi\rangle$ can be written as

$$|\psi\rangle = \alpha_0\beta_0|0\rangle\ |0\rangle + \alpha_0\beta_1|0\rangle|1\rangle + \alpha_1\beta_0|1\rangle\ |0\rangle$$
$$+ \alpha_1\beta_1|1\rangle\ |1\rangle \ldots\ldots\ldots\ldots \quad (6.1)$$

Thus a two-qubit system has four basis states:

$$|0\rangle \otimes |0\rangle,\ |0\rangle \otimes |1\rangle, |1\rangle \otimes |0\rangle, |1\rangle \otimes\ |1\rangle$$

The first state in the two-qubit system, for instance,

$$|0\rangle \otimes |0\rangle$$

indicates that the first qubit is in state $|0\rangle$ and the second qubit is also in state $|0\rangle$. Similarly, for the state $|0\rangle \otimes |1\rangle$ the first qubit is in state $|0\rangle$ and the second qubit is in state $|1\rangle$.

Using the following rule

$$|\mathbf{u}\rangle \otimes |\mathbf{v}\rangle = |\mathbf{u}\rangle|\mathbf{v}\rangle = |\mathbf{uv}\rangle \text{ where } (\mathbf{u},\mathbf{v})\ \varepsilon\ (0, 1)$$

Expression (6.1) can be written as

$$|\psi\rangle = \alpha_0\beta_0|0\rangle|0\rangle + \alpha_0\beta_1|0\rangle|1\rangle + \alpha_1\beta_0|1\rangle\ |0\rangle + \alpha_1\beta_1|1\rangle\ |1\rangle$$

and also as

$$|\psi\rangle = \alpha_0\beta_0\ |00\rangle + \alpha_0\beta_1|01\rangle + \alpha_1\beta_0\ |10\rangle + \alpha_1\beta_1|11\rangle$$

It should be noted, however, that every two-qubit state cannot be separated into two single-qubit states. In general, the state of a two-qubit system has the form

$$|\psi\rangle = c_0|00\rangle + c_1|01\rangle + c_2\ |10\rangle + c_3|11\rangle$$

where

$$|c_0|^2 + |c_1|^2 + |c_2|^2 + |c_3|^2 = 1$$

As indicated previously a collection of n qubits is referred to as a *quantum register* of size n. An n-qubit register has 2^n basis states, each of the form $|c_0 \otimes |c_1 \otimes \dots \dots |c_{n-1}$, with $c_i \in \{0, 1\}$. A *basis state* can be represented by a number 0 to 2^{n-1}. For example, the state 1001 in a quantum register of size 4 is denoted by $|9>_4$.

A quantum register of n qubits can be in any superposition of 2^n states

$$c_0\,|0> + c_1\,|1> + c_2\,|2> + \dots\dots\dots\dots + c_2^{\,n-1}\,|2^{n-1}>$$

However, it is not possible to retrieve the states within a superposition because this process leads to the collapse of the superposition and produce just one of the original states $|j>$ with probability $|c_j|^2$.

6.3 Superposition

The principle of superposition was introduced in Chap. 4. A quantum system with k distinguishable states can exist partly in two or more mutually exclusive states, and can also be placed in a linear superposition of these states with complex coefficients. Conversely, two or more states can be superimposed to give a new state. After measurement the system falls to one of the basis states that form the superposition, thus destroying the original configuration.

To illustrate, assume a system of two hydrogen atoms. An electron in a hydrogen atom can be regarded as a two-state quantum system since each electron can either be in the ground or in the excited state. Thus, the electrons in a system of two hydrogen atoms constitute a system of four classical states and the system can be in one of four states: 00, 01, 10, or 11.

However, unlike a classical bit which can only be in a single state at any time, a *qubit* it can not only be in one of the two discrete classical states but may also exist in more than one state at the same time due to the wave-like characteristics of subatomic particles. This is basically the essence of the *principle of superposition*; it states that if s_1 and s_2 are two *distinct* physical states, then the *complex linear superposition* of s_1 and s_2

$$\frac{1}{\sqrt{2}}\,|s_1> + \frac{1}{\sqrt{2}}\,|s_2>$$

also is a *quantum state* of the system, and

$$\frac{1}{\sqrt{2}}\,|s_1> - \frac{1}{\sqrt{2}}\,|s_2>$$

also represents a legitimate quantum state.

Thus, by the superposition principle, the quantum state of the two electron system can be in any linear combination of the four classical states:

$$|\varphi> = \alpha_{00}\,|0>|0> + \alpha_{01}\,|0>|0> + \alpha_{10}\,|1>|0> + \alpha_{11}\,|1>|1>$$

where $\sum_{i,j} |\alpha_{ij}|^2 = 1$. Note that in this case $\alpha_{ij} = \frac{1}{2}$.

The overall state can be written as a product of the individual states of the two electrons:

$$\frac{1}{2}\,|00> + \frac{1}{2}\,|01> + \frac{1}{2}\,|10> + \frac{1}{2}\,|11>$$

$$= (\frac{1}{\sqrt{2}}\,|0> + \frac{1}{\sqrt{2}}\,|1>) \otimes (\frac{1}{\sqrt{2}}\,|0> + \frac{1}{\sqrt{2}}\,|1>)$$

The fact that the overall state is *factorizable*, that is, the product of the states of the two electrons indicates that the electrons are independent of each other. This means any operation on one electron has no impact on the other. Moreover, since the overall state is an *equal* superposition of the four states, the measurement outcome will be one of these randomly and with equal probability.

As an example, consider a system with two states A and B. Suppose the observation of the system in state A gives an output *a*, and the observation in state B gives *b*. Then the observation of the system when equal percentage of A and B are in superposition, always gives an output *a* or *b* with equal probability, nothing else.

In the world of subatomic particles there may potentially be an unlimited number of different states in which the particles may be located at once. In reality, whether a particle is in an indeterminate state or in two places at the same time, can never actually be observed, only a measurement can verify this.

The working of a quantum computer relies on processing all the particles in superposition simultaneously [3]. This gives parallel processing capability to quantum computers. For example, a traditional computer can process only *one* combination of n bits (currently 64 bits). A quantum computer on the other side of the spectrum can process all 2^n combinations of two states at the same time. This corresponds to a conventional computer with 2^n processors. For example, the capability to process 64 *qubits* simultaneously instead of 64 classical *bits* increases the speed of the computation by a factor of 2^{64} ($= 2 \times 10^{19}$)!

However, a major problem with the superposition state is that it collapses into a random state once it is measured. For example, assume an electron as a qubit with spin-up orientation representing state $|0>$ and spin-down state $|1>$. If the particle enters a superposition of states, it behaves as if it were in both $|0>$ and $|1>$ states simultaneously.

Thus an operation on a single qubit affects both values of the qubit at the same time. Similarly, any operation on a system with two qubits will allow simultaneous operation on four values, and on eight values in a three-qubit system. For example, in a four-qubit system at any particular time the 4 qubits can be in any one of 16 possible configurations:

$$(0000, 0001, 0010, \ldots \ldots \ldots \ldots 1111)$$

Thus, a 4-qubit register can be represented in a superposition of the above 16 states:

$$| \Psi> = c_0 |0000> + c_1 |0001> + c_2 |0010> \cdots \cdots \cdots + c_{14} |1110> + c_{15} |1111>$$

where the numbers c_0, c_1, c_2, … …, c_{15} are complex coefficients such that

$$|c_0|^2 + |c_1|^2 + |c_2|^2 + \cdots \cdots \cdots \cdots \cdots + |c_{15}|^2 = 1$$

The states of the qubit register can be represented as tensor products resulting in

$$\begin{aligned} | \Psi> &= C_{0000} |0> \otimes |0> \otimes |0> \otimes |0> + C_{0001} |0> \otimes |0> \otimes |0> \otimes |1> \\ &\quad + C_{0010} |0> \otimes |0> \otimes |1> \otimes |0>) + \cdots \cdots \cdots \cdots \cdots + C_{1110} |1> \otimes | \\ &\quad 1> \otimes |1> \otimes |0>) + C_{1111} |1> \otimes | 1> \otimes |1> \otimes |1>) \\ &= C_{0000} |0000> + C_{0001} |0001> + C_{0010} |0010> + \cdots \cdots \cdots \cdots \cdots \\ &\quad + C_{1110} | 1110> + C_{1111} |1111> \end{aligned}$$

In a similar way, the state of an n-qubit register can be written as a normalized vector in 2^n-dimensional complex Hilbert space; this allows the superposition of 2^n different base states as indicated earlier.

A unique feature of quantum computing is that with a linear increase in the number of qubits in a register, the dimension of quantum register state space grows exponentially. The additional qubits allow parallel processing capability to quantum computers and thus help in solving certain problems in a time that is many times faster than that is possible in classical computers.

6.4 Entanglement

Entanglement is a unique kind of correlation that exists only in quantum systems. It has no analogue in classical physics and refers to the strange behavior of quantum particles such as electrons, photons that have interacted in specific ways in the past and then moved apart. It is the essential ingredient in such phenomena as *superdense coding*, which allows any of four possible messages to be transmitted via a single quantum particle, and *teleportation*, in which a quantum state is transmitted from one location to another without going through the intervening space.

EPR (Einstein, Podolsky, Rosen) showed that two particles in a quantum system may be correlated such that any measurement on one particle identifies the outcome of the same measurement on its partner particle *instantaneously*, irrespective of how wide is the physical distance between them. In other words, if two particles are entangled, any measurement on one of them can impact the behavior of its partner particle instantaneously, no matter how far away this second particle moved. This phenomenon is so strange that Einstein called it "spooky action at a distance."

For example, if the particles are correlated in spins and one of them is in the spin-up orientation then the other one must be in spin-down orientation. Thus, the state of the two particle system can be written as [4]:

$$|\text{spin-up}\rangle_1 \otimes |\text{spin-down}\rangle_2$$

It is also possible to have another state of the two particle system in which the first particle has spin-down orientation and the second one is in spin-up orientation:

$$|\text{spin-down}\rangle_1 \otimes |\text{spin-up}\rangle_2$$

Alternatively, the first particle could be in a superposition state of spin-up and spin-down orientation. If the second particle is in a spin-up orientation, the combined state of the particles is

$$(\alpha|\text{spin-up}\rangle_1 + \beta|\text{spin-down}\rangle_1) \otimes |\text{spin-up}\rangle_2$$

If the spin of the first particle is measured, the probability of getting a spin-up orientation is α^2 and the probability of getting a spin-down orientation is β^2. In either case the second particle is not affected since it has not been measured. Hence, if the outcome of the measurement of the first particle is spin-up, then the state of the system after the measurement will be

$$|\text{spin-up}\rangle_1 \otimes |\text{spin-up}\rangle_2$$

Thus in the quantum states of the kind seen above, a measurement of one particle has no impact on the state of the other.

Based on the fundamental principle of superposition one could create a new quantum state from the two-particle states

$$|\text{spin-up}\rangle_1 \otimes |\text{spin-down}\rangle_2$$

and

$$|\text{spin-down}\rangle_1 \otimes |\text{spin-up}\rangle_2$$

as follows

$$\frac{1}{\sqrt{2}}(|\text{spin-up}\rangle_1 \otimes |\text{spin-down}\rangle_2 + |\text{spin-down}\rangle_1 \otimes |\text{spin-up}\rangle_2)$$

where $\frac{1}{\sqrt{2}}$ is the normalization constant.

Notice that in the superposition state above, the second particle is in spin-up as well as in spin-down orientation. Both orientations of the second particle, however, is associated with a particular orientation of the first particle. For example, in the first term spin-up of the first particle is attached to the spin-down orientation of the second, whereas in the second term the spin-down of the first is attached to the spin-up of the second. So if the measurement of second particle results in a spin-up orientation then the final state of the two-particle system will be

$$|\text{spin-down}\rangle_1 \otimes |\text{spin-up}\rangle_2$$

that is, the second term of the expression. Alternatively, if the orientation of the second particle were spin-down then the final state will be the first term of the expression, that is

$$|\text{spin-up}\rangle_1 \otimes |\text{spin-down}\rangle_2$$

Next a measurement is made on the first particle. In the first term the measurement of the second particle had indicated spin-down orientation; therefore, the first particle can only be spin-up since this is the only orientation of the particle present in the combined state. In the second case, the second particle have a spin-up orientation thus the first particle will be in spin-down orientation. Thus, the measurement result of the first particle is determined by the outcome of the earlier measurement on the second particle.

It should now be clear that the measurement of spin on one particle of the pair will correctly predict the subsequent measurement result of the other particle in the pair. The reason for this is that the original state of the two-particle system

$$\frac{1}{\sqrt{2}}(|\text{spin-up}\rangle_1 \otimes |\text{spin-down}\rangle_2 + |\text{spin-down}\rangle_1 \otimes |\text{spin-up}\rangle_2)$$

cannot be factored into a simple tensor product of one state involving only the first particle and another state involving only the second particle; such states are called *entangled*.

It seems from the above discussion that two entangled particles have some kind of invisible link between them. The existence of this linkage can be avoided only if it is possible to write the entangled qubits as a sum of two independent qubits. To illustrate, suppose an entangled state of the two qubits could be created from two two-qubit registers, $p = |00\rangle$ and $q = |11\rangle$, by combining the two states where each has equal weight (ω):

$$\Psi_{00} = \omega|00\rangle + \omega|11\rangle \quad \ldots\ldots\ \ldots\ \ldots\ \ldots\ldots \text{(6.2)}$$

Assume that Ψ could be created by taking the tensor product of the individual states of qubits u and v, that is

$$\Psi_{00} = (u_0|0\rangle + u_1|1\rangle) \otimes (v_0|0\rangle + v_1|1\rangle)$$

If the derived values of u_0, u_1, v_0, and v_1 are found to satisfy the expression Ψ, then it can be written as a separable state. By expanding out the tensor product in Ψ,

$$\Psi_{00} = (u_0 v_0 |00\rangle + u_0 v_1 |01\rangle + u_1 v_0 |10\rangle + u_1 v_1 |11\rangle)$$

Since the expression (6.2) does not contain state $|10\rangle$ nor $|01\rangle$, both coefficients $u_1 v_0$ and $u_0 v_1$ will be 0. This means for the first coefficient either $u_1 = 0$ or $v_0 = 0$. However, u_1 is not allowed to be 0 since that will result in $u_1 v_1 = 0$, eliminating state $|11\rangle$. Similarly, if $v_0 = 0$ then $u_0 v_0 = 0$ resulting in the elimination of state $|00\rangle$. In short, neither u_1 nor v_0 can be 0. Hence, state $|10\rangle$ will have a nonzero weight that is in contradiction with expression (6.2); hence, quantum state Ψ needs to be modified as

$$\Psi_{00} = u_0 v_0 |00\rangle + u_1 v_1 \, |11\rangle + u_1 v_0 |10\rangle$$

Alternatively, if coefficient $u_0 v_1 = 0$ instead of $u_1 v_0$, then state $|01\rangle$ will have a nonzero weight. Thus, the modified quantum state Ψ will be

$$|\Psi_{00}\rangle = u_0 v_0 |00\rangle + u_0 v_1 |01\rangle + u_1 v_1 \, |11\rangle$$

A simple circuit for generating entanglement is shown in Fig. 6.1.

The first qubit is passed through a Hadamard gate and then both qubits are entangled by a CNOT gate. If the input to the circuit is $|0\rangle \otimes 0\rangle|$, then the Hadamard gate changes the state to

$$\frac{|0\rangle + |1\rangle}{\sqrt{2}} \otimes |0\rangle$$
$$= \frac{1}{\sqrt{2}} (|0\rangle + |1\rangle) \otimes |0\rangle$$
$$= \frac{1}{\sqrt{2}} (|00\rangle + |10\rangle)$$

and after passing through the CNOT gate the state changes to

$$= \frac{1}{\sqrt{2}} (|00\rangle + |11\rangle)$$

which is one of the four Bell states that are maximally entangled; this state is known as the Bell state $|\Phi^+\rangle$.

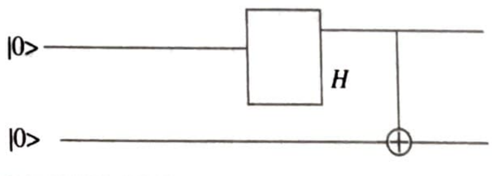

FIGURE 6.1 Entangled state generation with a Hadamard gate and a CNOT gate (From Ref. 3).

Next assume each output of the circuit is passed through two H gates, that is,

$$H \otimes H \left(\frac{|00> + |11>}{\sqrt{2}}\right)$$

$$= \frac{1}{\sqrt{2}} (H|0> \otimes H|0> + \frac{1}{\sqrt{2}} (H|1> \otimes H|1>)$$

$$= \frac{1}{\sqrt{2}} (\frac{1}{\sqrt{2}} (|0> + \frac{1}{\sqrt{2}} (|1>) \otimes \frac{1}{\sqrt{2}} (\frac{1}{\sqrt{2}} (|0> + \frac{1}{\sqrt{2}} (|1>)$$

$$+ \frac{1}{\sqrt{2}} (\frac{1}{\sqrt{2}} (|0> - \frac{1}{\sqrt{2}} (|1>) \otimes \frac{1}{\sqrt{2}} (\frac{1}{\sqrt{2}} (|0> - \frac{1}{\sqrt{2}} (|1>)$$

$$= \frac{|00> + |11>}{\sqrt{2}}$$

that is, the Bell state remains the same as the original.

It can be shown that each input combination to the circuit of Fig. 6.1, results in a Bell state; they are all listed below and are also known as the *EPR states*

$$|\psi_{00}> = \frac{|00> + |11>}{\sqrt{2}} = |\Phi^{+}>$$

$$|\psi_{01}> = \frac{|01> + |10>}{\sqrt{2}} = |\psi^{+}>$$

$$|\psi_{10}> = \frac{|00> - |11>}{\sqrt{2}} = |\Phi^{-}>$$

$$|\psi_{11}> = \frac{|01> - |10>}{\sqrt{2}} = |\psi^{-}>$$

As another example of entanglement consider a two-qubit system with standard basis chosen for both qubits. Then the basis for the system is

$$(|0>, |1>) \otimes (|0>, |1>)$$

$$= (|00>, |01>, |10>, |11>)$$

On the other hand, if the Hadamard basis is chosen for the first qubit and the standard basis for the second qubit, the basis for the two-qubit system is

$$(|+>, |->) \otimes (|0>, |1>)$$

$$= (|+0>, |+1>, |-0>, |-1>)$$

where

$$(|+0> = \frac{1}{\sqrt{2}}(|0> + |1> \otimes |0>$$

$$= \frac{1}{\sqrt{2}}(|00> + |10>$$

$$(|+1> = \frac{1}{\sqrt{2}}(|0> + |1> \otimes |1>$$

$$= \frac{1}{\sqrt{2}}(|01> + |11>$$

$$(|-0> = \frac{1}{\sqrt{2}}(|0> - |1> \otimes |0>$$

$$= \frac{1}{\sqrt{2}}(|00> - \frac{1}{\sqrt{2}}(|10>$$

$$(|-1> = \frac{1}{\sqrt{2}}(|0> - |1> \otimes |1>$$

$$= \frac{1}{\sqrt{2}}(|01> - \frac{1}{\sqrt{2}}(|11>$$

Therefore, the Hadamard-Standard base is

$$\left(\frac{1}{\sqrt{2}}(|00> + |10>\right), \left(\frac{1}{\sqrt{2}}(|00> + |11>\right), \left(\frac{1}{\sqrt{2}}(|00> - |10>\right),$$

$$\left(\frac{1}{\sqrt{2}}(|01> - |11>\right)$$

6.5 Decoherence

The superposition principle states that *any* two states $|A>$, $|B>$ of a quantum system may be superimposed, yielding a new state. Thus, a quantum bit can yield a new state from the arbitrary superposition of its two states. However, a superposed state is very fragile and therefore difficult to control [5]. As a consequence, any kind of interaction with their environment can cause superposed states to eradicate certain *coherences*, therefore preventing the associated states from interfering with each other. This effectively destroys the superposition and the system *collapses* randomly into one of the states that constitute the superposition state; this process is known as *decoherence*.

Decoherence is an undesirable effect in quantum systems. It destroys many possible advantages of quantum systems over classical systems. For example, entanglement which has potential applications in quantum computation, quantum cryptography may be lost due to decoherence. As another example, the superposition of states that allow parallel processing of quantum information, are the ones that are most susceptible to decoherence. Hence qubits need to be designed such that the effects of environmental interactions that make it difficult to maintain the quantum superposition feature for prolonged periods of time are eliminated. This is a necessary requirement in quantum computing systems. Currently decoherence is a major barrier to the development of quantum information processing systems. It is generally accepted now that reliable computation in the presence of decoherence is possible only by incorporating some form of quantum error correction.

References

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