Entanglement

So far, we have only discussed the manipulation and measurement of a single qubit. However, **quantum entanglement** is a physical phenomenon that occurs when multiple qubits are correlated with each other. Entanglement can have strange and useful consequences that could make quantum computers faster than classical computers. Qubits can be "entangled," providing hidden quantum information that does not exist in the classical world. It is this entanglement that is one of the main advantages of the quantum world!

To provide one example of the strange behavior of entanglement, suppose we have two fair coins. Classically, if you flipped two fair coins, you would measure the outcomes HH, HT, TH, or TT, each occuring with a 25% probability. However, by quantum entangling these two fair coins, it is possible to create a state $(1/\sqrt{2})(|HH\rangle + |TT\rangle)$ as illustrated in Figure 7.1. Many other types of entangled coins are possible, but this is one famous example. If you flipped this "entangled" pair of coins, they are entangled in such a way that only two measurement outcomes are possible: 1) both coins land on heads; or 2) both coins land on tails; each outcome occurring with 50% probability. Isn't that weird!

Furthermore, if the two entangled coins are separated by thousands of miles,

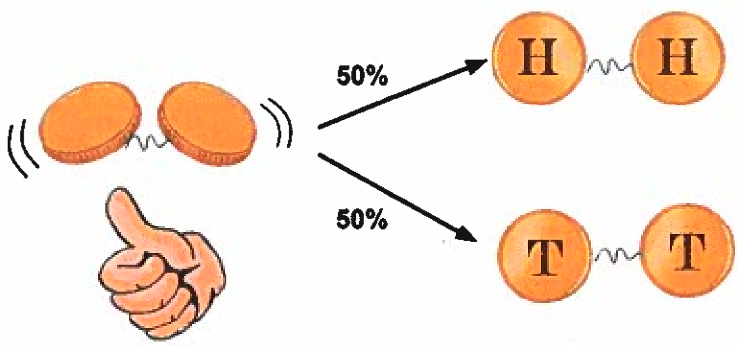


Figure 7.1: Two coins that are entangled in such a way that they either both land on HH or both land on TT.

one coin can be flipped and measured. In this case, if the measured coin produced the outcome heads, then we automatically know that the other coin must also land on heads. If the measured coin produced the outcome tails, then we automatically know that the other coin must also land of tails! If this isn't strange enough, the two coins could be separated by a distance greater than what light (which travels at the fastest speed in the universe) could travel as shown in Figure 7.2. If the two coins are flipped at the exact same time, somehow the two coins know to land on the same side as the other even though there can be no classical communication between them.[1]

How does the other coin instantaneously "know" what was measured on the other? Is information somehow being transmitted faster than the speed of light? Einstein called this behavior a "spooky action at a distance." It has since been shown that no information is being transmitted from one place to the other, and so no information is being transmitted faster than the speed of light. Rather, the particles share non-classical information at the time of entanglement, which is then observed in the measurement process. The correlation between entangled qubits is the key that allows quantum computers to perform certain computations much faster than classical computers.

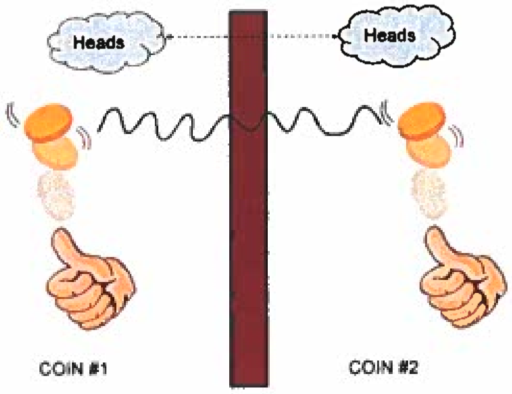


Figure 7.2: Two coins are separated with no means of communication between each other. Classically, the flip of the second coin would be unrelated to the first flip. However, entangled coins would still produce correlated results.

[1] "Bounding the speed of spooky action at a distance." *Physical Review Letters*. 110: 260407. 2013. arXiv:1303.0614.

7.1 ● Hidden Variable Theory

It is tempting to think that there may be some classical explanation for entanglement. For example, maybe when causing the coins to interact and entangling them, the same interaction might have changed the coins? Did the entanglement change the fair coins by adding extra mass to the heads side or the tails side, thereby making them unfair? To give a more realistic classical example, if one particle decays into two smaller particles, the momenta of the two particles are related according to the conservation of momentum by $\vec{p}_i = \vec{p}_{f1} + \vec{p}_{f2}$. Given a known total initial momentum, then by measuring the momentum of one of the smaller particles, we can determine the momentum of the other. By measuring one particle's momentum, we know the other. Momentum is the hidden classical variable that is encoded when the two particles are created. This is shown in Figure 7.3.

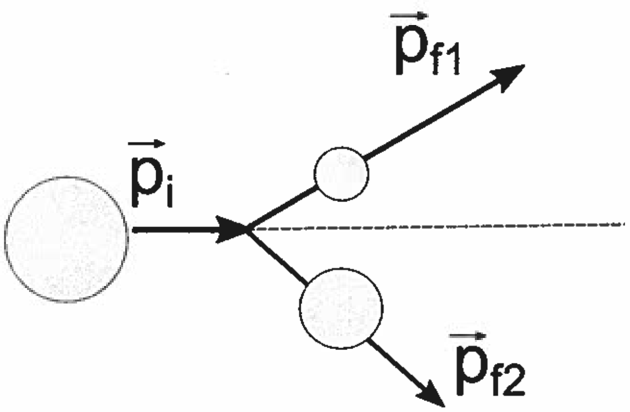


Figure 7.3: When a particle decays into two smaller particles, the decay products are "classically entangled" according to the conservation of momentum.

However, Bell's theorem[2] showed that the correlation between entangled quantum particles is more than what is possible classically, disproving the idea of a hidden variable. All other potential loopholes have been resolved as of 2016.[3] As such, entanglement is a purely quantum phenomenon with no classical explanation.

[2] https://brilliant.org/wiki/bells-theorem/

[3] The BIG Bell Test Collaboration (9 May 2018). "Challenging local realism with human choices." *Nature*. 557: 212–216. doi:10.1038/s41586-018-0085-3.

7.2 ▪ Multi-Qubit States

Given multiple qubits, the total state of a system can be written together in a single ket. For example, if coin #1 is heads and coin #2 is tails, the two-coin state is expressed as $|HT\rangle$. In general, a system of two qubits which is in a superposition of four classical states may be written as

$$|\psi\rangle = \alpha_{00}|00\rangle + \alpha_{01}|01\rangle + \alpha_{10}|10\rangle + \alpha_{11}|11\rangle.$$

As we saw for the single qubit states, the coefficients α_{ij} are called the amplitudes and are generally complex numbers. Measuring the two qubits will collapse the system into one of the four basis states with probability given by α_{ij}^2. This is shown in Figure 7.4.

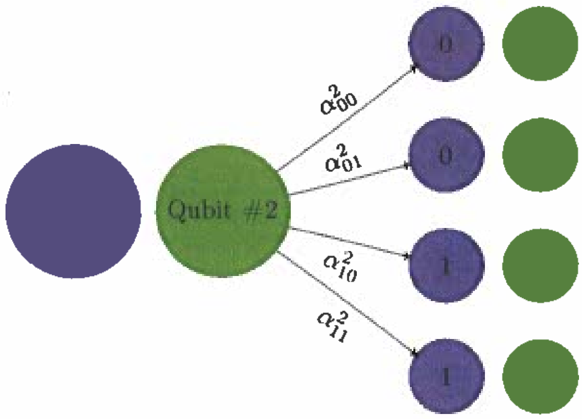


Figure 7.4: A two-qubit system can collapse into one of four states with probability α_{ij}^2.

Example

A system of two qubits is in a superposition state given by $|\psi\rangle = \frac{1}{\sqrt{2}}|00\rangle + \frac{1}{2}|10\rangle - \frac{1}{2}|11\rangle$.

a) What is the probability of measuring both qubits as 1?
   $\text{Prob}(|11\rangle) = \left(\frac{-1}{2}\right)^2 = \frac{1}{4}$.

b) If we only measure the first qubit and get a value of 1, what is the new state of the system?

   Since $|00\rangle$ is the only basis state of $|\psi\rangle$ that doesn't have a 1 in the first qubit, we eliminate the state $|00\rangle$ from the possibilities. This results in

$|\psi'\rangle = \frac{1}{2}|10\rangle - \frac{1}{2}|11\rangle.$

Finally, we renormalize the state so that the probabilities add up to 1. Therefore, the new state is $|\psi'\rangle = \frac{1}{\sqrt{2}}|10\rangle - \frac{1}{\sqrt{2}}|11\rangle$.

7.3 ● Non-Entangled Systems

It is possible to have a system of particles that are not entangled with each other. In this case, changing one particle will not cause any change in the other particle. For example, in a classical system, flipping two coins and measuring one coin as heads does not tell you any information about whether or not the other coin will land on heads or tails. These events are said to be independent. If you wanted to calculate the probability of $|HT\rangle$, you would simply multiply the probability of getting H on coin #1 by the probability of getting T on coin #2. This is given by

$$\text{Prob}(|HT\rangle) = \left(\frac{1}{2}\right)\left(\frac{1}{2}\right) = \frac{1}{4}.$$

Non-entangled states are also called product states or separable states because they can be factored into a product of single-qubit states.[4] The single-qubit probabilities multiply to produce the two-qubit probabilities.

Example:

One qubit is in a $\alpha_0|0\rangle + \alpha_1|1\rangle$ state, while another is in a $\beta_0|0\rangle + \beta_1|1\rangle$ state. What is the state of the non-interacting two-qubit system?

$$(\alpha_0|0\rangle + \alpha_1|1\rangle)(\beta_0|0\rangle + \beta_1|1\rangle) = \alpha_0\beta_0|00\rangle + \alpha_0\beta_1|01\rangle + \alpha_1\beta_0|10\rangle + \alpha_1\beta_1|11\rangle.$$

7.4 ● Entangled Systems

In an entangled system, measuring the value of one qubit changes the probability distribution of the second qubit.

Example Is $|\psi\rangle = \frac{1}{\sqrt{2}}|00\rangle + \frac{1}{\sqrt{2}}|11\rangle$ an entangled state?

Yes! To see this, examine qubit #2. The probabilities for measuring qubit #2 in the $|0\rangle$ or $|1\rangle$ states are originally 50/50 respectively. However, if we measured

[4] More recently, it has been shown that there can exist quantum correlations in separable states that are not due to entanglement. These are called quantum discord: https://en.wikipedia.org/wiki/Quantum_discord.

qubit #1, then the probability for measuring qubit #2 becomes 100%. The same argument holds if qubit #2 is measured first. As such, measuring one of the qubits affects the probability of measuring the other qubit in a certain state, and so they are entangled. Mathematically, an entangled state is a special multi-qubit superposition state that cannot be factored into a product of the individual qubits.

Example: Show that $|\psi\rangle = \frac{1}{\sqrt{2}}|00\rangle + \frac{1}{\sqrt{2}}|11\rangle$ cannot be written as a product of two single qubits.

Assume that the state can be written as the product of two states.

$$\frac{1}{\sqrt{2}}|00\rangle + \frac{1}{\sqrt{2}}|11\rangle \stackrel{?}{=} (\alpha_0|0\rangle + \alpha_1|1\rangle)(\beta_0|0\rangle + \beta_1|1\rangle), \tag{7.1}$$

$$\frac{1}{\sqrt{2}}|00\rangle + \frac{1}{\sqrt{2}}|11\rangle \stackrel{?}{=} \alpha_0\beta_0|00\rangle + \alpha_0\beta_1|01\rangle + \alpha_1\beta_0|10\rangle + \alpha_1\beta_1|11\rangle. \tag{7.2}$$

Comparing the amplitudes on the left vs. the right, the α_i's and β_j's must satisfy:

$$\alpha_0\beta_0 = \frac{1}{\sqrt{2}}, \quad \alpha_0\beta_1 = 0, \quad \alpha_1\beta_0 = 0, \quad \alpha_1\beta_1 = \frac{1}{\sqrt{2}}. \tag{7.3}$$

However, this is not possible. For example, take $\alpha_0\beta_1 = 0$. This means that either $\alpha_0 = 0$ or $\beta_1 = 0$. If $\alpha_0 = 0$, then $\alpha_0\beta_0 = 0$, but $\alpha_0\beta_0 = \frac{1}{\sqrt{2}}$ in the above equation. A similar contradiction occurs with $\beta_1 = 0$. So the initial assumption must be incorrect and this entangled state cannot be written as the product of two separate states.

7.5 ▪ Entangling Particles

As there are many different ways of building a quantum computer, there are many different ways of entangling particles. One method called "spontaneous parametric down-conversion" shines a laser at a special nonlinear crystal. The crystal splits the incoming photon into two photons with correlated polarizations. For example, one could produce a pair of photons that always have perpendicular polarizations (see Figure 7.5).

7.6 ▪ CNOT Gate

You have already learned about the X, Hadamard, and Z gates. These act on a single qubit. There are also quantum gates that perform a logic operation on two or more qubits. The most important multi-qubit gate is the controlled NOT

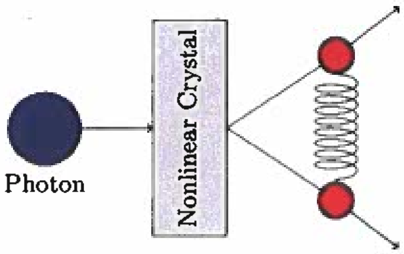


Figure 7.5: A nonlinear crystal creates two photons with entangled polarizations.

(CNOT) gate. The CNOT is used to entangle two qubits together and is essential in quantum computing/algorithms. The CNOT takes in two qubits, a control qubit and a target qubit, and outputs two qubits. The control qubit stays the same, while the target obeys the following rule.

- If the control qubit is $|0\rangle$, then leave the target qubit alone.
- If the control qubit is $|1\rangle$, then on the target qubit flip $|0\rangle \rightarrow |1\rangle$ and $|1\rangle \rightarrow |0\rangle$.

Figure 7.6 is the circuit for the CNOT gate.

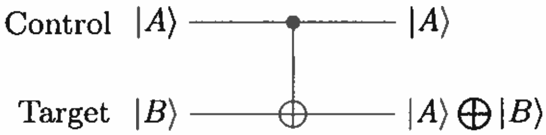


Figure 7.6: The CNOT gate performs an X gate on the target qubit if the control qubit is $|1\rangle$.

Examples

1. Figure 7.7 shows the IBM Q quantum circuit sending $|10\rangle$ through a CNOT gate. What is the output?

   The figure shows that the control qubit is q[1] and the target is q[0]. Since the control is in the $|1\rangle$ state, the target qubit is flipped to $|1\rangle$. So measurement will always result in $|11\rangle$.

2. Examine Figure 7.8. The control qubit is in a superposition of $|0\rangle$ and $|1\rangle$. What is the effect of a CNOT gate?

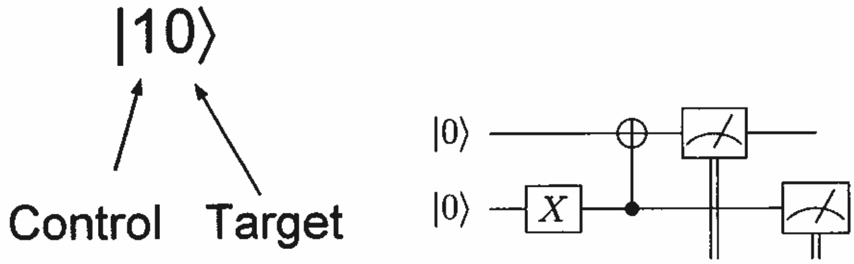


Figure 7.7: The IBM Q quantum circuit that sends a control qubit in the $|10\rangle$ state through a CNOT gate.

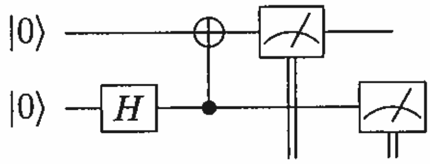


Figure 7.8: The IBM Q quantum circuit that sends a control qubit in a superposition state through a CNOT gate.

Before the CNOT operation, in ket notation, the control qubit is in the $\frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle$ state, while the target qubit is in the $|0\rangle$ state. The two-qubit input state is therefore $\frac{1}{\sqrt{2}}|00\rangle + \frac{1}{\sqrt{2}}|10\rangle$. Applying the rules for the CNOT, the first state $|00\rangle$ does not change as the control qubit is $|0\rangle$. However, for the second state $|01\rangle$, the control qubit is $|1\rangle$ and so the target qubit is flipped from $|0\rangle$ to $|1\rangle$. The result of the CNOT gate is the state $\frac{1}{\sqrt{2}}|00\rangle + \frac{1}{\sqrt{2}}|11\rangle$. The histogram from measuring this state is shown in Figure 7.9. This is a special state called the Bell state.

The two qubits are entangled after the CNOT! As illustrated in the previous example, this state cannot be written as the product of two separate qubits. As with the single-qubit gates, the CNOT gate operates on ALL states in the superposition. Quantum algorithms leverages this parallelism to ensure speed improvements over classical computers. In addition, as with all quantum gates, the CNOT is reversible, meaning the operation can be undone (which can be used to figure out the original qubit states).

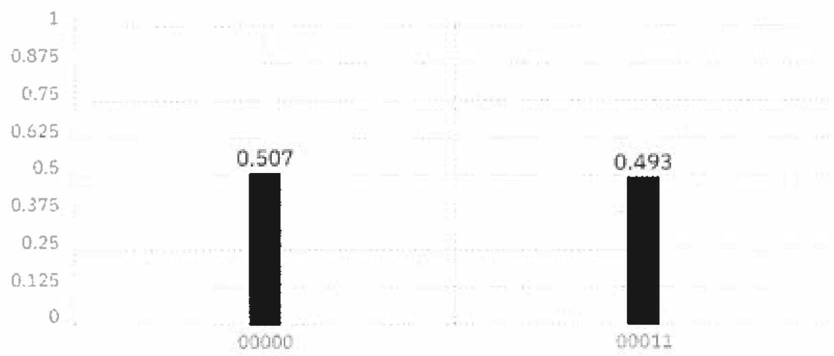


Figure 7.9: The measurement histogram produced by running the circuit in Figure 7.8. Reprint Courtesy of International Business Machines Corporation, ©International Business Machines Corporation.

7.7 Check Your Understanding

1. • For each of the questions below, assume that two-qubits start in the state

$$|\psi\rangle = \frac{1}{\sqrt{2}}|00\rangle + \frac{1}{2}|10\rangle - \frac{1}{2}|11\rangle. \tag{7.4}$$

   a) What is the probability of measuring both qubits as 0?
   b) What is the probability of measuring the first qubit as 1?
   c) What is the probability of measuring the second qubit as 0?
   d) What is the new state of the system after measuring the first qubit as 0?
   e) What is the new state of the system after measuring the first qubit as 1?

2. • Two fair coins are flipped. What is the state of the two-coin system while the coins are in the air?

3. • Two six-sided dice are rolled. What is the total probability of rolling an even number on one die and an old number on the other die?

4. ■ Is $\frac{1}{\sqrt{2}}|00\rangle + \frac{1}{\sqrt{2}}|01\rangle$ an entangled state? If so, show that it cannot be written as a product. If not, what is the individual state of the two qubits?

5. ■ Are the following two-qubit states entangled?

   a) $\frac{1}{\sqrt{2}}|01\rangle + \frac{1}{\sqrt{2}}|10\rangle$
   b) $\frac{1}{\sqrt{2}}|01\rangle - \frac{1}{\sqrt{2}}|10\rangle$
   c) $\frac{\sqrt{3}}{2}|00\rangle + \frac{1}{\sqrt{2}}|11\rangle$

d) $\frac{1}{\sqrt{2}}|10\rangle + \frac{1}{\sqrt{2}}|11\rangle$
e) $\frac{1}{2}|00\rangle + \frac{1}{2}|01\rangle + \frac{1}{2}|10\rangle - \frac{1}{2}|11\rangle$
f) $\frac{1}{\sqrt{2}}|00\rangle + \frac{1}{2}|10\rangle - \frac{1}{2}|11\rangle$

6. • Two qubits are passed through a CNOT. The first qubit is the control qubit. What is the output for the following initial states?

   a) $|00\rangle$
   b) $|01\rangle$
   c) $|11\rangle$
   d) $\frac{1}{\sqrt{2}}|01\rangle + \frac{1}{\sqrt{2}}|10\rangle$
   e) $\frac{1}{\sqrt{2}}|00\rangle + \frac{1}{2}|10\rangle - \frac{1}{2}|11\rangle$

7. • The output of a CNOT gate is shown in the figure below. What were the inputs?

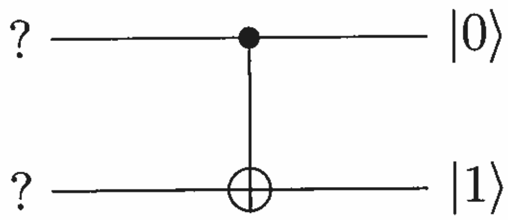
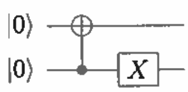
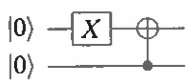
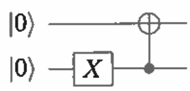


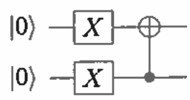
Figure 7.10: CNOT gate.

8. • Can you predict the state produced by these quantum circuits?

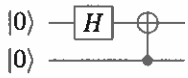
a) 

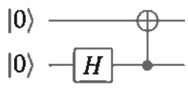
b) 

c) 

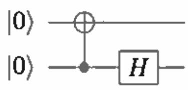
d) 

9. ■ Can you predict which states will be produced by these quantum circuits?

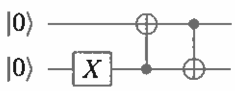
a) 

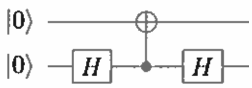
b) 

c) 

d) 

10. ◆ Can you predict the state produced by these quantum circuits?

a) 

b) 

11. ■ Use the IBM Q simulator to create the entangled state $\frac{1}{\sqrt{2}}|01\rangle + \frac{1}{\sqrt{2}}|10\rangle$.

12. ■ Suppose Alice has one half of an entangled pair and Bob has the other half. When Alice makes a measurement on her qubit, Bob's qubit instantaneously changes its state. Can Alice and Bob use entanglement to transmit information faster than the speed of light? Why or why not?