A Topological Hierarchy for Functions on Triangulated Surfaces - Summary

Peer-Timo Bremer, Member, IEEE, Herbert Edelsbrunner,
Bernd Hamann, Member, IEEE, and Valerio Pascucci, Member, IEEE

Danesh Toshniwal - IMT2021094
Neil Handa - IMT202137

ABSTRACT:
The authors of the research paper utilize a combination of topological and geometric methods to develop a multi-resolution representation for a function over a two-dimensional domain. Initially, they generate the Morse-Smale complex of the function and gradually simplify its topology by eliminating pairs of critical points. Using a straightforward concept of dependence among these eliminations, they establish a hierarchical data structure that allows for traversal and reconstruction operations akin to conventional geometry-based representations.

MAIN MOTIVE OF THIS PAPER:
This paper aims to efficiently create simplified models with simplified topology and geometry. It presents a hierarchical data structure that represents the topology of a continuous function on a triangulated surface. The structure computes and encodes the complete function's topology, allowing for quick and consistent access to adaptive topological simplifications.Furthermore, the hierarchy incorporates geometrically accurate approximations of the function that correspond to any topological refinement.

STEPS FOLLOWED FOR THE EXECUTION OF THE PAPER:

1. A decomposition of the domain into monotonic quadrangular regions is constructed by connecting critical points with lines of steepest descent.

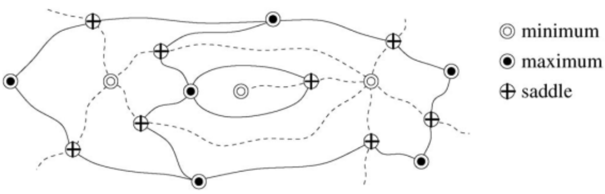

Figure 1: Morse Smale Complex
Source: A TOPOLOGICAL HIERARCHY FOR FUNCTIONS ON TRIANGULATED SURFACES - Fig. 1

   (a) From each saddle, two lines of steepest ascent and two lines of steepest descent are constructed, following the actual lines of maximal slope instead of edges of the triangulation K.
   (b) Triangles are split to create new edges in the direction of the gradient, with modifications made to avoid regions with disconnected interior or regions whose interior does not touch both saddles.
   (c) Two paths can still merge if they are both ascending or both descending. If not, an edge of the triangulation is split and a new sample is introduced with function value that preserves the structure of the MS complex.
   (d) After computing all paths, K is partitioned into quadrangular regions forming the cells of the MS complex. Here, each quadrangle is grown from a triangle incident to a saddle without crossing a path.

2. The decomposition is simplified through a sequence of cancellations ordered by persistence.

   (a) We use only one atomic operation to simplify the MS complex of a function, namely, a cancellation that eliminates two critical points.

(b) In order to cancel two critical points, they must be adjacent in the MS complex.
Only two possible combinations arise: a minimum and a saddle or a saddle and a maximum.
As both the configurations are symmetric, without loss of generality, we will limit the discussion to the second case: a saddle and a maximum

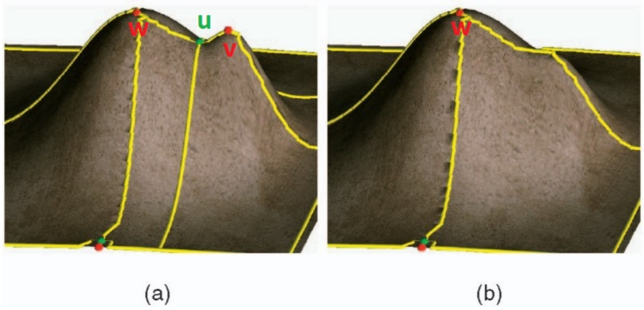

Figure 2: Cancellation

(c) Let u be the saddle and v the maximum of the cancelled pair and let w be the other maximum connected to u.
We require $w \neq v$ and $f(w) > f(v)$; otherwise, we prohibit the cancellation of u and v.

(d) We view the cancellation as merging three critical points into one, namely, u, v, w into w. All paths ending at either u, v, or w are removed and we adapt the local geometry to the new topology. Subsequently, all paths that were connected to either maximum are recomputed.

(e) There are several reasons for requiring $f(w) > f(v)$: It implies that all recomputed paths remain monotonic and ensures that we do not eliminate any level sets, except that the ones between $f(u)$ and $f(v)$ are simplified.

3. The simplified decomposition is turned into a hierarchical multiresolution data structure whose levels correspond to simplified versions of the function.

   (a) Identify all paths that are impacted by a cancellation.

   (b) Apply gradient smoothing to geometrically eliminate the canceled critical points.

   (c) Smooth the existing regions until they become monotonic (without any critical points).

   (d) Remove the paths that were canceled and recalculate new paths based on the updated geometry.

   (e) Utilize one-dimensional gradient smoothing to ensure that the new paths adhere to the given constraints.

   (f) Continuously smooth the new regions until all points within them become regular and meet the desired criteria.

CONCLUSION:
The algorithm's most fundamental application is the elimination of topological noise without smoothing. This feature is independent of the hierarchical structure and is achieved by iteratively canceling critical points with the lowest persistence. By removing the noise, the algorithm effectively reduces the overall number of critical points. This noise removal capability is particularly valuable as it addresses a significant challenge in topological data analysis: the presence of numerous spurious topological features. Thus, this cleanup process serves as a valuable preprocessing step for various recently developed techniques in the field.

NOTE: We have implemented the main algorithm, that is, the construction of Morse Smale Complex, as instructed by Sir.