Cointegration in R: Spot Rates Market Data

- Traditionally, cointegration is tested in the very long run

- Case Study A tests for an equilibrium between T-Bill rates and Treasury yields over the horizon of 1960-2010.

  HOWEVER

- As quants we have to look for co-movement in the current, frequent market data.

We will use this opportunity to get introduced to R.

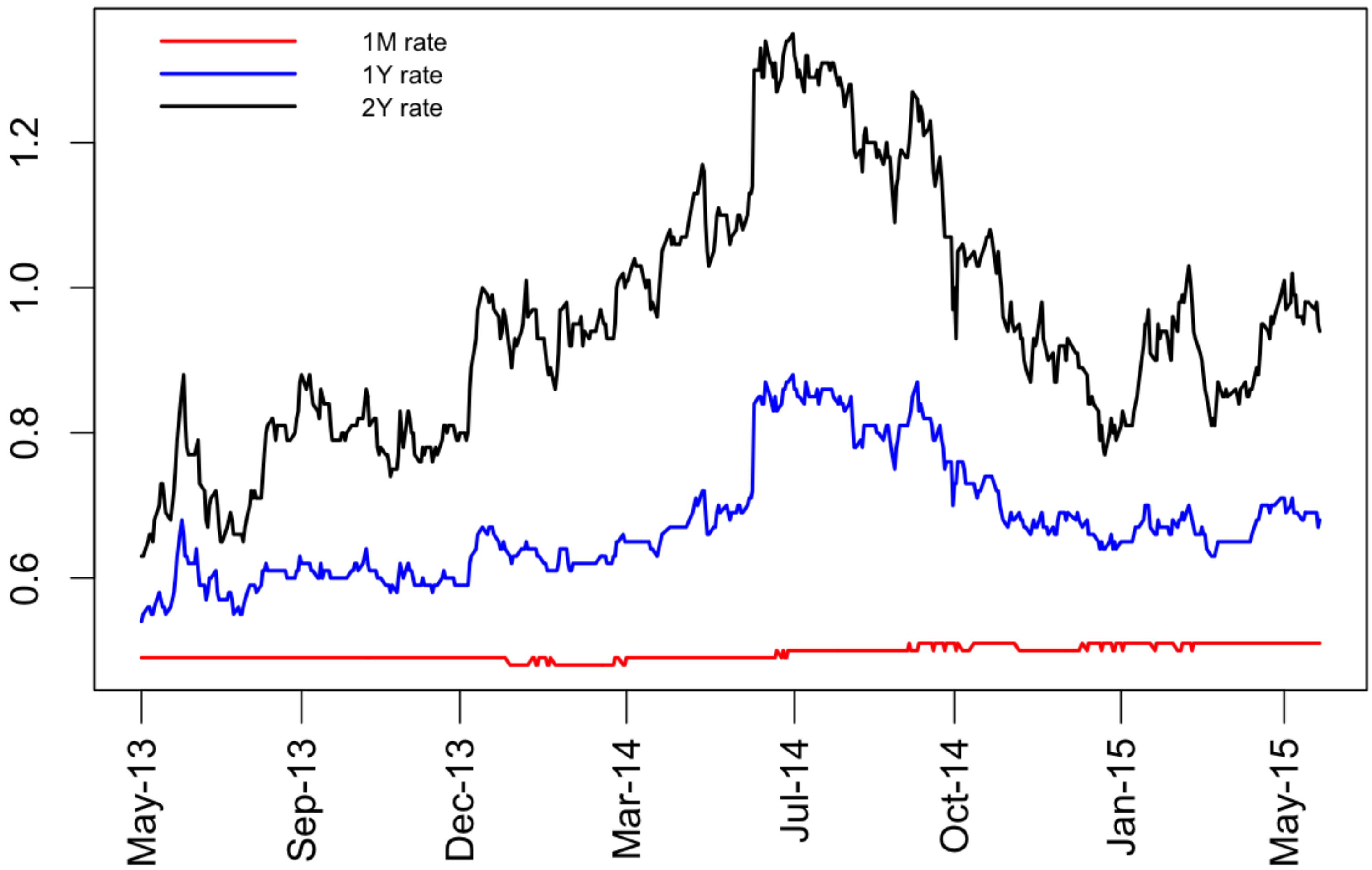
Spot Curve

The Bank of England provides the daily yield curve data. It makes sense to consider smaller windows of the long timeframe:

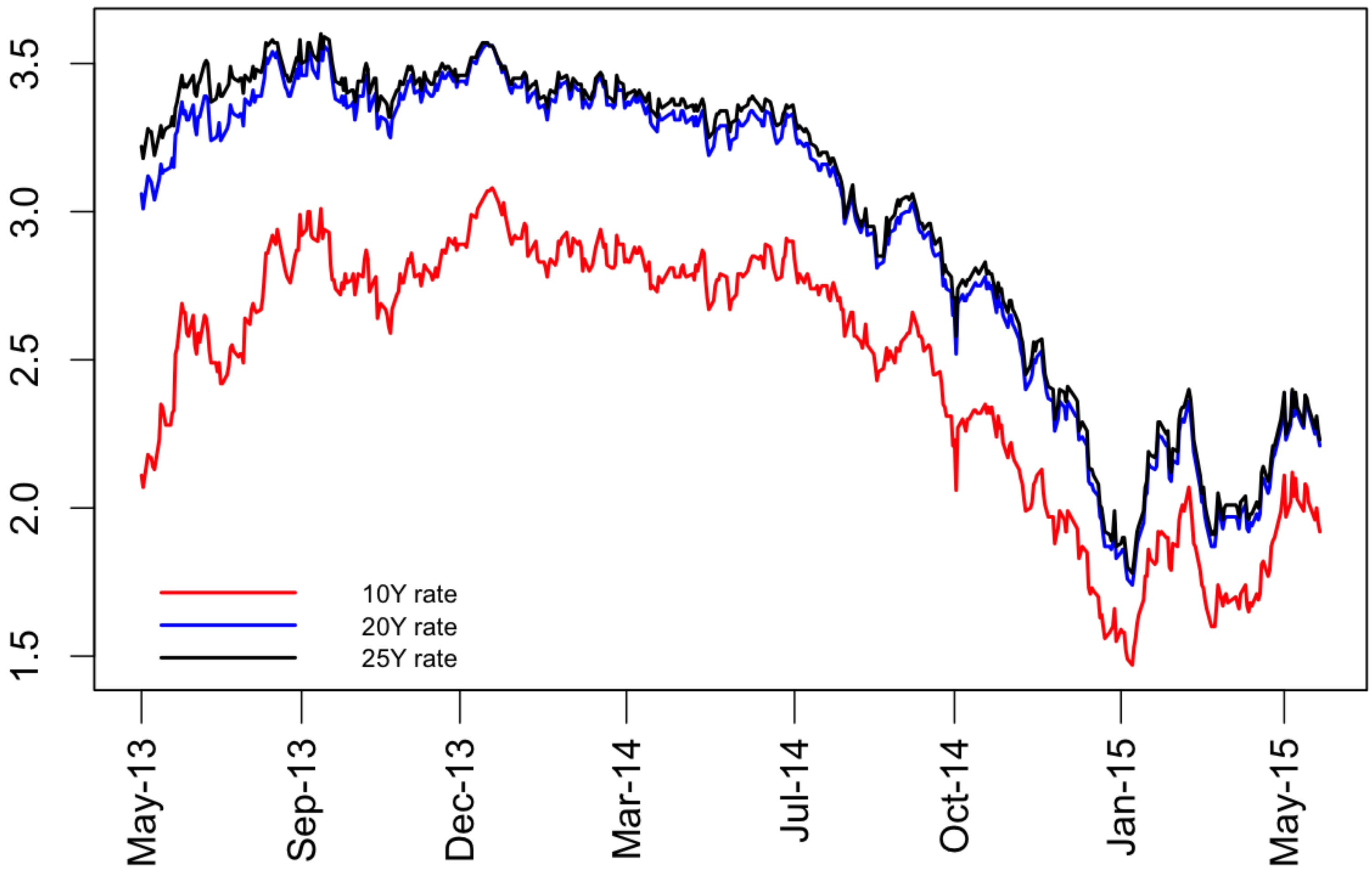
- two-year window May 2013 – May 2015 (charts below) vs.
- all data from from Jan 2005 to May 2015.

We have to learn the equilibrium-correction mechanics (**ECM**) but it's worthwhile to have a peek from the multivariate test for cointegration.

Spot Rates at Short End



Spot Rates at Long End



Problems with curve data

1. r_t at the short end (0.8Y, 1Y, 2Y) and y_t at the long end (7Y, 10Y, 20Y) **do not** come as cointegrated in samples of two-three year period.

   There is simply not enough horizon for a cointegrated relationship to transpire.

2. Let's play a game: **Which long-end rates are co-integrated?** Choose pairs among 10Y, 20Y, 25Y.

   Can't decouple that easily.

   Similar pattern comes up for the short end, if all data included in the testing.

   Parallel to that, short rates have independent co-movement.

Engle-Granger preview

Let's choose a model with **10Y and 25Y tenors** because of their importance as benchmarks.

- We set up a naive cointegrating equation

$$r_{10Y} = \beta\, r_{25Y} + e_t \qquad \Rightarrow \qquad \hat{e}_t = r_{10Y} - \beta\, r_{25Y}$$

- We test this estimated residual $\hat{e}_t$ for stationarity by CADF.

  If the residual is stationary, it means that r_{10Y} and r_{25Y} have a unit root **in common**, removable by differencing

Long-run relationship r_{10Y} on r_{25Y}

```
lm(formula = curve2.this$X10 ~ curve2.this$X25)

                Estimate Std. Error t value Pr(>|t|)
(Intercept)      0.15878    0.03132    5.07  5.6e-07 ***
curve2.this$X25  0.76980    0.01023   75.28  < 2e-16 ***

Residual standard error: 0.1231 on 504 degrees of freedom
Multiple R-squared:  0.9183,Adjusted R-squared:  0.9182

Residuals:
     Min       1Q   Median       3Q      Max
-0.53675 -0.03449  0.01926  0.07920  0.18461
```

As usual, regressing one non-stationary series on another gives *extremely* significant coefficients. Large N_{obs} makes $R^2 \to 1$.

Long-run relationship if cointegrated

$$\widehat{r}_{10Y} = 0.159 + 0.77\, r_{25Y} + \widehat{e}_t$$

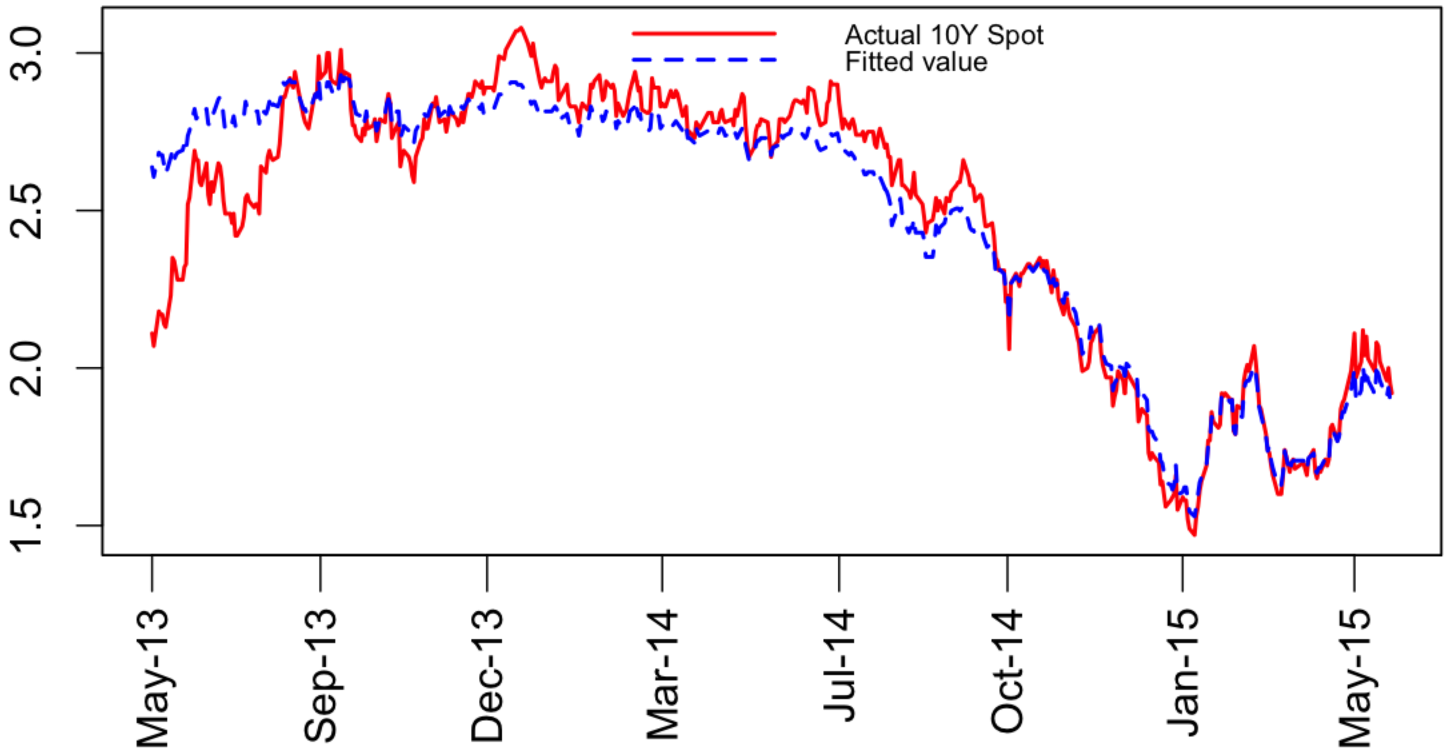
This model is valid only if it produces stationary $\widehat{e}_t$, so there is co-integration between r_{10Y} and r_{25Y}

It only works in the context of the equilibrium correction over the long-run, producing stationary and mean-reverting residual:

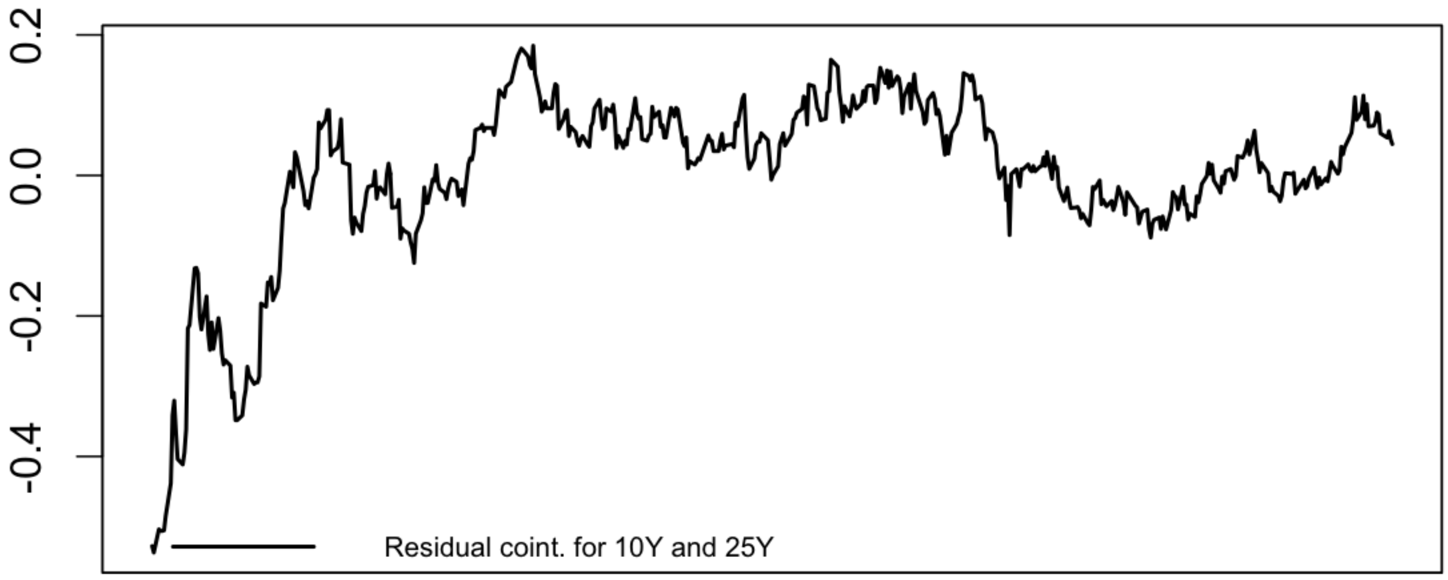
$$\widehat{e}_t = r_{10Y} - (0.159 + 0.77\, r_{25Y}).$$

Linear regression FIT for r_{10Y}

Our linear model aims to obtain $\widehat{e}_t$ so we would be differencing actual r_{10Y} with fitted $\widehat{r}_{10Y}$.



Stationary cointegrating residual $\widehat{e}_t$



We will confirm the stationarity of residual, and proceed with forming error-correction equations.

Stationarity test for $\hat{e}_t$

```
################################################
# Augmented Dickey-Fuller Test Unit Root Test #
################################################

lm(formula = z.diff ~ z.lag.1 - 1 + z.diff.lag)

            Estimate Std. Error t value Pr(>|t|)
z.lag.1    -0.038559   0.008548  -4.511 8.06e-06 ***
z.diff.lag -0.042376   0.043711  -0.969    0.333

[DF test-statistic is -4.5107, for which critical values]
      1pct  5pct 10pct
tau1 -2.58 -1.95 -1.62

Residual standard error: 0.02318 on 502 degrees of freedom
Multiple R-squared:  0.04071,Adjusted R-squared:  0.03689
```

Dickey-Fuller Test reminder

DF Test performed on cointegrated residual is referred to as CADF.

H_0 : time series has a unit root. For the coint residual e_t we want that to be rejected. Usually more stronger than the critical value -4.511 for z.lag.1

$$\Delta e_t = \text{Const} + \phi e_{t-1} + \phi_1 \Delta e_{t-1}$$

Presence of Const is not trivial, it suggests a linear increasing trend $\Delta e_t = \text{Const}$ (time-independent, constant drift).

By implication, $\phi \neq 0$ gives $\beta_{AR(1)} - 1 \neq 0$, so no unit root.

Long-run relationship (cointegrated)

ECM estimation [R code provided for your exploration] gives

- **the calbirated parameter** of interest is the speed of correction towards the equilibrium $(1-\alpha)$

  It is inevitably small but **must be** significant for cointegration to exist.

- We have quite good correlation between differences Δr_{10Y} and Δr_{25Y}. There is co-movement on the short timescale.

  For the lower frequency samples, you might find that Δr_t (for the short rate) and Δy_t (for some long-term rate) are cointegrated but correlated weakly negatively.

Equilibrium Correction Model: two-way, two residuals

$$\Delta r_{10Y} = 1.086 \Delta r_{25Y} - 0.02716\, e^{10Y}_{t-1} + \epsilon_t$$

```
              Estimate Std. Error t value Pr(>|t|)
tenorX.diff   1.085090   0.022986  47.206  < 2e-16 ***
ec_term.lag -0.027164   0.007202  -3.772 0.000181 ***

Residual standard error: 0.01981   Multiple R-squared: 0.8202
```

$$\Delta r_{25Y} = 0.752 \Delta r_{10Y} - 0.01206 e^{25Y}_{t-1} + \epsilon_t$$

```
              Estimate Std. Error t value Pr(>|t|)
tenorY.diff   0.751627   0.015910  47.243   <2e-16 ***
ec_term1.lag -0.012059   0.004851  -2.486   0.0132 *

Residual standard error: 0.01649   Multiple R-squared: 0.8175
```

Summary

Please take away the following ideas...

- this case of evolution of spot rates at different tenors is a case of a basis relationship,
- so imposing a long-run relationship and using Engle–Granger procedure has more statistical power,
- r_{10Y} and r_{25Y} series each have a unit root,
- it turns out that by differencing these time series, the unit root got cancelled and a stationary residual obtained,
- that means the time series are co-integrated.

Case Extra Slides

- Restricted VECM from Johansen Procedure
- Engle-Granger Procedure for r_{25Y} on r_{10Y} (other way)
- Linear regression on differences Δr_{25Y}, Δr_{10Y}
- Hedging ratio puzzle

Restricted VECM for Δr_{10Y} and Δr_{25Y}

```
cajorls(johansen.test)

lm(formula = substitute(form1), data = data.mat)

         X10.d     X25.d
ect1     -0.05842  -0.02647
X10.dl1  -0.13888  -0.09543
X25.dl1   0.07943   0.06495

[Cointegrating Equation (EC term)]
               ect1
X10.l2     1.0000000
X25.l2    -0.7870489
constant  -0.1435463
```

Long-run relationship r_{25Y} on r_{10Y} (other way)

The linear model $r_{25Y} = \beta\, r_{10Y} + \epsilon_t$ only aims to obtain $\widehat{e}_t$.

```
lm(formula = curve2.this$X25 ~ curve2.this$X10)

                 Estimate Std. Error t value Pr(>|t|)
(Intercept)       0.05686    0.03989   1.425    0.155
curve2.this$X10  1.19295    0.01585  75.285   <2e-16 ***

Residual standard error: 0.1532 on 504 degrees of freedom
Multiple R-squared:  0.9183,Adjusted R-squared:  0.9182
F-statistic:  5668 on 1 and 504 DF,  p-value: < 2.2e-16

Residuals:
     Min       1Q   Median       3Q      Max
-0.18591 -0.08516 -0.03819  0.02177  0.65373
```

Stationarity test for $\hat{e}_t$ (other way)

```
###############################################
# Augmented Dickey-Fuller Test Unit Root Test #
###############################################

lm(formula = z.diff ~ z.lag.1 - 1 + z.diff.lag)

            Estimate Std. Error t value Pr(>|t|)
z.lag.1    -0.033920   0.007759  -4.372  1.5e-05 ***
z.diff.lag -0.038024   0.043779  -0.869    0.386

[DF test-statistic is -4.3718, for which critical values]
      1pct  5pct 10pct
tau1 -2.58 -1.95 -1.62

Residual standard error: 0.02619 on 502 degrees of freedom
Multiple R-squared:  0.03792,Adjusted R-squared:  0.03409
```

Comparison to linear regression

OLS on simple differences Δr_{25Y} and Δr_{10Y} gives min variance relationship – cointegration plays a completely separate role.

```
lm(formula = diff(curve2.this$X25) ~ diff(curve2.this$X10) + 0)

                      Estimate Std. Error t value Pr(>|t|)
diff(curve2.this$X10)  0.74570    0.01581   47.16   <2e-16 ***

Residual standard error: 0.01657 on 504 degrees of freedom
Multiple R-squared:  0.8153, Adjusted R-squared:  0.8149

Residuals:
      Min        1Q    Median        3Q       Max
-0.081683 -0.010172 -0.002371  0.007629  0.050172

cor(diff(curve2.this$X25), diff(curve2.this$X10))
[1] 0.903719
```

Hedging ratio puzzle

What would you use as a hedging ratio for assets r_{10Y} and r_{10Y} in presence of cointegration between them?

Multiple Choice:

- 0.7698 from linear regression of r_{10Y} on r_{25Y}
- 0.7457 from linear regression on differences Δr_{25Y} on Δr_{10Y}
- 0.7516 from correction equation of Δr_{25Y}
- 0.7870 from a coint regression VECM output.