User Guide for Johansen's Method

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1 Theoretical Background

1.1 The Model

Consider VAR(p) model

$$\mathbf{y}_t = \boldsymbol{\mu} + \boldsymbol{\Phi}_1 \mathbf{y}_{t-1} + \cdots + \boldsymbol{\Phi}_p \mathbf{y}_{t-p} + \boldsymbol{\epsilon}_t, \tag{1.1}$$

where $\mathbf{y}_t$ is $n \times 1$ vector of variables and assumed to be $I(1)$. If x_t are cointegrated, then there exists a following VECM representation by Engle and Granger (1987):

$$\Delta\mathbf{y}_t = \boldsymbol{\mu} + \boldsymbol{\Pi}\mathbf{y}_{t-1} + \mathbf{A}_1 \Delta\mathbf{y}_{t-1} + \cdots + \mathbf{A}_{p-1}\Delta\mathbf{y}_{t-p+1} + \boldsymbol{\epsilon}_t, \tag{1.2}$$

where $\boldsymbol{\Pi} = \boldsymbol{\alpha}\boldsymbol{\beta}'$ has a reduced rank r that is the number of cointegrating vectors, and $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ are $n \times r$ matrices.

As our interest is on $\boldsymbol{\Pi}$, it is more convenient to $\mathbf{A}_1, \cdots, \mathbf{A}_{p-1}$ by a partial regression.

(1.3) Regress $\Delta\mathbf{y}_t$ on $\mathbf{1} + \mathbf{A}_1\Delta\mathbf{y}_{t-1} + \cdots + \mathbf{A}_{p-1}\Delta\mathbf{y}_{t-p+1} \rightarrow$ Get residuals : $\mathbf{R}_{0t}$

(1.4) Regress $\mathbf{y}_{t-1}$ on $\mathbf{1} + \mathbf{A}_1\Delta\mathbf{y}_{t-1} + \cdots + \mathbf{A}_{p-1}\Delta\mathbf{y}_{t-p+1} \rightarrow$ Get residuals : $\mathbf{R}_{kt}$

Then, we have a concentrated regression:

$$\mathbf{R}_{0t} = \boldsymbol{\alpha}\boldsymbol{\beta}'\mathbf{R}_{kt} + \boldsymbol{\epsilon}_t \tag{1.5}$$

For notational convenience, let

$$\mathbf{S}_{ij} = \frac{1}{T}\sum_{t=1}^{T} \mathbf{R}_{it}\mathbf{R}_{jt}' \qquad i, j = 0, k \tag{1.6}$$

Note that $\boldsymbol{\alpha}$ can be easily estimated from (1.5) provided that $\boldsymbol{\beta}$ is known:

$$\begin{aligned} \hat{\boldsymbol{\alpha}}' &= (\boldsymbol{\beta}'\mathbf{R}_k'\mathbf{R}_k\boldsymbol{\beta})^{-1}\boldsymbol{\beta}'\mathbf{R}_k'\mathbf{R}_0 \\ &= (\boldsymbol{\beta}'\mathbf{S}_{kk}\boldsymbol{\beta})^{-1}\boldsymbol{\beta}'\mathbf{S}_{k0} \end{aligned} \tag{1.7}$$

Johansen (1988) estimate $\boldsymbol{\beta}$ using MLE. Consider MLE for

$$\mathbf{Y} = XB + \mathbf{U}, \qquad u_t \sim N(\mathbf{0}, \boldsymbol{\Sigma}). \tag{1.8}$$

Then, the log likelihood of (1.8) is

$$\log L = -\frac{T}{2}\log 2\pi - \frac{T}{2}\log|\boldsymbol{\Sigma}| - \frac{1}{2}(\mathbf{Y} - \mathbf{XB})'\boldsymbol{\Sigma}^{-1}(\mathbf{Y} - \mathbf{XB}) \tag{1.9}$$

The FOC of (1.9) for $\boldsymbol{\Sigma}$ is:

$$\hat{\boldsymbol{\Sigma}} = \frac{1}{T}(\mathbf{Y} - \mathbf{XB})'(\mathbf{Y} - \mathbf{XB}) \tag{1.10}$$

Plug (1.10) in (1.9), then we get a concentrated likelihood:

$$\log L = \text{constant} - \frac{T}{2}\log|\hat{\boldsymbol{\Sigma}}|, \tag{1.11}$$

which is proportional to

$$L_{max} = |\hat{\boldsymbol{\Sigma}}|^{-\frac{T}{2}}. \tag{1.12}$$

Let $L(\boldsymbol{\beta}) = |\hat{\boldsymbol{\Sigma}}|^{-\frac{T}{2}}$. Then,

$$\begin{aligned} |L(\boldsymbol{\beta})|^{-\frac{2}{T}} &= |\hat{\boldsymbol{\Sigma}}| \\ &= |\frac{1}{T}(\mathbf{R}_0 - \mathbf{R}_k\boldsymbol{\beta}\boldsymbol{\alpha}')'(\mathbf{R}_0 - \mathbf{R}_k\boldsymbol{\beta}\boldsymbol{\alpha}')| \\ &= |\frac{1}{T}(\mathbf{R}_0\mathbf{R}_0 - \boldsymbol{\alpha}\boldsymbol{\beta}'\mathbf{R}_k'\mathbf{R}_k\boldsymbol{\beta}\boldsymbol{\alpha}')| \\ &= |\mathbf{S}_{00} - \mathbf{S}_{0k}\boldsymbol{\beta}(\boldsymbol{\beta}'\mathbf{S}_{kk}\boldsymbol{\beta})^{-1}\boldsymbol{\beta}'\mathbf{S}_{k0}| \end{aligned} \tag{1.13}$$

Therefore,

$$\begin{aligned} \max_{\boldsymbol{\beta}} L(\boldsymbol{\beta}) &\Leftrightarrow \min_{\boldsymbol{\beta}} |\mathbf{S}_{00} - \mathbf{S}_{0k}\boldsymbol{\beta}(\boldsymbol{\beta}'\mathbf{S}_{kk}\boldsymbol{\beta})^{-1}\boldsymbol{\beta}'\mathbf{S}_{k0}| \\ &\Leftrightarrow \min_{\boldsymbol{\beta}} |\boldsymbol{\beta}'\mathbf{S}_{kk}\boldsymbol{\beta} - \boldsymbol{\beta}'\mathbf{S}_{k0}\mathbf{S}_{00}^{-1}\mathbf{S}_{0k}\boldsymbol{\beta}|\frac{|\mathbf{S}_{00}|}{|\boldsymbol{\beta}'\mathbf{S}_{kk}\boldsymbol{\beta}|} \\ &\Leftrightarrow \max_{\boldsymbol{\beta}} \frac{|\boldsymbol{\beta}'\mathbf{S}_{kk}\boldsymbol{\beta}|}{|\boldsymbol{\beta}'(\mathbf{S}_{kk} - \mathbf{S}_{k0}\mathbf{S}_{00}^{-1}\mathbf{S}_{0k})\boldsymbol{\beta}|}\frac{1}{|\mathbf{S}_{00}|} \end{aligned} \tag{1.14}$$

At the second line, I use the formula:

$$\left| \begin{array}{cc} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{array} \right| = |\mathbf{A}|\ |\mathbf{D} - \mathbf{C}\mathbf{A}^{-1}\mathbf{B}| = |\mathbf{D}|\ |\mathbf{A} - \mathbf{B}\mathbf{D}^{-1}\mathbf{C}| \tag{1.15}$$

Thus,

$$|\mathbf{A} - \mathbf{B}\mathbf{D}^{-1}\mathbf{C}| = |\mathbf{D} - \mathbf{C}\mathbf{A}^{-1}\mathbf{B}|\frac{|\mathbf{A}|}{|\mathbf{D}|}, \tag{1.16}$$

where $\mathbf{A} = \mathbf{S}_{00}, \mathbf{B} = \mathbf{S}_{0k}\boldsymbol{\beta}, \mathbf{C} = \boldsymbol{\beta}'\mathbf{S}_{k0}$, and $\mathbf{D} = \boldsymbol{\beta}'\mathbf{S}_{kk}\boldsymbol{\beta}$.

Note also that FOC for

$$\max_{\mathbf{x}} \frac{\mathbf{x}'\mathbf{A}\mathbf{x}}{\mathbf{x}'\mathbf{B}\mathbf{x}} \quad (\equiv \lambda) \tag{1.17}$$

is

$$(\mathbf{A} - \lambda\mathbf{B})\mathbf{x} = \mathbf{0}, \tag{1.18}$$

where λ is an eigenvalue, and $\mathbf{x}$ is an eigenvector.

Therefore, (1.14) becomes an eigenvalue problem. Let

$$\lambda_0 = \max_{\boldsymbol{\beta}} \frac{|\boldsymbol{\beta}'\mathbf{S}_{kk}\boldsymbol{\beta}|}{|\boldsymbol{\beta}'(\mathbf{S}_{kk} - \mathbf{S}_{k0}\mathbf{S}_{00}^{-1}\mathbf{S}_{0k})\boldsymbol{\beta}|}. \tag{1.19}$$

Then, the FOC is

$$\begin{aligned} & (\mathbf{S}_{kk} - \lambda_0(\mathbf{S}_{kk} - \mathbf{S}_{k0}\mathbf{S}_{00}^{-1}\mathbf{S}_{0k}))\boldsymbol{\beta} = \mathbf{0} \\ \Leftrightarrow\ & ((1-\lambda_0)\mathbf{S}_{kk} + \lambda_0(\mathbf{S}_{k0}\mathbf{S}_{00}^{-1}\mathbf{S}_{0k}))\boldsymbol{\beta} = \mathbf{0} \\ \Leftrightarrow\ & (\lambda_0(\mathbf{S}_{k0}\mathbf{S}_{00}^{-1}\mathbf{S}_{0k}) - (\lambda_0 - 1)\mathbf{S}_{kk})\boldsymbol{\beta} = \mathbf{0} \\ \Leftrightarrow\ & (\mathbf{S}_{k0}\mathbf{S}_{00}^{-1}\mathbf{S}_{0k} - (1 - \frac{1}{\lambda_0})\mathbf{S}_{kk})\boldsymbol{\beta} = \mathbf{0} \\ \Leftrightarrow\ & (\mathbf{S}_{k0}\mathbf{S}_{00}^{-1}\mathbf{S}_{0k} - \lambda\mathbf{S}_{kk})\boldsymbol{\beta} = \mathbf{0}, \end{aligned} \tag{1.20}$$

where $\lambda = 1 - \frac{1}{\lambda_0}$. Note that λ and $\boldsymbol{\beta}$ are an eigenvalue and an eigenvector of $\mathbf{S}_{kk}^{-1}\mathbf{S}_{k0}\mathbf{S}_{00}^{-1}\mathbf{S}_{0k}$, respectively.

Therefore, our maximization problem is reduced to find an eigenvalue and eigenvector of $\mathbf{S}_{kk}^{-1}\mathbf{S}_{k0}\mathbf{S}_{00}^{-1}\mathbf{S}_{0k}$.

1.2 A Rank Test

From (1.12), (1.14) and (1.19), we get

$$|L_{max}(\boldsymbol{\beta})|^{-\frac{2}{T}} = |\mathbf{S}_{00}| \prod_{i=1}^{r} \frac{1}{\lambda_{0i}} \tag{1.21}$$

$$L_{max}(\boldsymbol{\beta}) = -\frac{T}{2}|\mathbf{S}_{00}| \prod_{i=1}^{r}(1-\lambda_i) \tag{1.22}$$

Therefore, we get the LR test (or Trace test) as:

$$\begin{aligned} LR &= -2\log\frac{L_{max}(H_0=r)}{L_{max}(H_1=n)} \\ &= -T\sum_{i=r+1}^{n}\log(1-\lambda_i) \end{aligned} \tag{1.23}$$

and the maximum eigenvalue test (or λ_{max} test) as:

$$\begin{aligned} \lambda_{max} &= -2\log\frac{L_{max}(H_0=r)}{L_{max}(H_1=r+1)} \\ &= -T\log(1-\lambda_{r+1}). \end{aligned} \tag{1.24}$$

Note that alternative hypothesis is different in each test. For large values of test statistics, we reject the null hypothesis that there exist r cointegrating vectors, $H_0 = r$. The critical values are available in Johansen (1995).

1.3 Model Selection

Johansen (1995) considers five models with respect to data properties as well as cointegrating relations as follows: i) a model with a quadratic trend in $\mathbf{y}_t$ (hflag=1):

$$\Delta\mathbf{y}_t = \boldsymbol{\mu} + \boldsymbol{\rho}_0 t + \boldsymbol{\alpha}\boldsymbol{\beta}'\mathbf{y}_{t-1} + \mathbf{A}_1\Delta\mathbf{y}_{t-1} + \cdots + \mathbf{A}_{p-1}\Delta\mathbf{y}_{t-p+1} + \boldsymbol{\epsilon}_t, \tag{1.25}$$

ii) a model with a linear trend in $\mathbf{y}_t$ (hflag=2), in which deterministic cointegration is not satisfied:

$$\Delta \mathbf{y}_t = \boldsymbol{\mu} + \boldsymbol{\rho}_0 t + \boldsymbol{\alpha}\boldsymbol{\beta}' \mathbf{y}_{t-1} + \mathbf{A}_1 \Delta \mathbf{y}_{t-1} + \cdots + \mathbf{A}_{p-1} \Delta \mathbf{y}_{t-p+1} + \boldsymbol{\epsilon}_t, \tag{1.26}$$

iii) a model with a linear trend in $\mathbf{y}_t$ (hflag=3), in which deterministic cointegration is satisfied (cotrended):

$$\Delta \mathbf{y}_t = \boldsymbol{\mu} + \boldsymbol{\alpha}(\boldsymbol{\beta}' \mathbf{y}_{t-1} + \boldsymbol{\rho}_1 t) + \mathbf{A}_1 \Delta \mathbf{y}_{t-1} + \cdots + \mathbf{A}_{p-1} \Delta \mathbf{y}_{t-p+1} + \boldsymbol{\epsilon}_t, \tag{1.27}$$

iv) a model with no trend in $\mathbf{y}_t$ (hflag=4):

$$\Delta \mathbf{y}_t = \boldsymbol{\alpha}(\boldsymbol{\beta}' \mathbf{y}_{t-1} + \boldsymbol{\rho}_0) + \mathbf{A}_1 \Delta \mathbf{y}_{t-1} + \cdots + \mathbf{A}_{p-1} \Delta \mathbf{y}_{t-p+1} + \boldsymbol{\epsilon}_t, \tag{1.28}$$

and v) a model with no trend in $\mathbf{y}_t$ (hflag=5):

$$\Delta \mathbf{y}_t = \boldsymbol{\alpha}\boldsymbol{\beta}' \mathbf{y}_{t-1} + \mathbf{A}_1 \Delta \mathbf{y}_{t-1} + \cdots + \mathbf{A}_{p-1} \Delta \mathbf{y}_{t-p+1} + \boldsymbol{\epsilon}_t. \tag{1.29}$$

1.4 Estimation of Restricted Cointegrating Vectors

Johansen (1995) illustrates how to estimate restricted cointegrating vectors. Consider a trivariate model with two cointegrating vectors. Let $\mathbf{y}_t = (\mathbf{y}_{1t}, \mathbf{y}_{2t}, \mathbf{y}_{3t})'$ and $\boldsymbol{\beta} = [\boldsymbol{\beta}_1 | \boldsymbol{\beta}_2]$. One may impose a restriction of $\boldsymbol{\beta}_{11} = \boldsymbol{\beta}_{13}$ using $\mathbf{H}_1 \boldsymbol{\varphi}_1 = \boldsymbol{\beta}_1$ and $\mathbf{H}_2 \boldsymbol{\varphi}_2 = \boldsymbol{\beta}_2$, where $\mathbf{H}_i$ is an $n \times (n - q_i)$ matrix, $\boldsymbol{\varphi}_i$ is an $(n - q_i) \times 1$ matrix, and q_i is the number of restrictions on each cointegrating vector. In this particular example, letting

$$\mathbf{H}_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ -1 & 0 \end{bmatrix}, \quad \mathbf{H}_2 = \mathbf{I}_3 \tag{1.30}$$

gives the following restrictions:

$$\mathbf{H}_1 \boldsymbol{\varphi}_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} \varphi_{11} \\ \varphi_{12} \end{bmatrix} = \begin{bmatrix} \varphi_{11} \\ \varphi_{12} \\ -\varphi_{11} \end{bmatrix} = \begin{bmatrix} \boldsymbol{\beta}_{11} \\ \boldsymbol{\beta}_{12} \\ \boldsymbol{\beta}_{13} \end{bmatrix}. \tag{1.31}$$

2 Program and Output

2.1 Program: JOHANSEN.EXP

It is incomplete from here.

```
@ JOHANSE.EXP for Johansen's Rank Test @
@ Written by Kyungho Jang (Last Reviseion by Jang: 11/12/2001) @

new;


@**************** Start of Modification ****************@

@ *************************************************************** @
@ ---------------- Step 1: Specifiy Output File ----------------- @
@ *************************************************************** @

outfile = "johansen.out";

@ *************************************************************** @
@ ---------------- Step 2: Prepare the Data --------------------- @
@ *************************************************************** @

@ 1. Reading in data @
load dat[176,3] = sample2.dat;     @ nondurables, duralble goods, price @

@ 2. Specify the series for test @
nondurab = ln(dat[1:176,1]); @ Define the series for CCR @
durable = ln(dat[1:176,2]); @ Define the series for CCR @
price = ln(dat[1:176,3]); @ Define the series for CCR @

@ *************************************************************** @
@ ---------------- Step 3: Define variables and parameters ------ @
@ *************************************************************** @


@ Define variables @
yi1 = price~nondurab~durable; @ Specify endogenous I(1) varibales @
yi0 = 0;   @ Specify stationary variables, otherwise set yi0=0 @
xi1nc = 0; @ Specify I(1) exogenous variables, otherwise set xi1nc=0 @
xi1ci = 0; @ Specify I(1) exogenous variables, otherwise set xi1ci=0 @
```

```
xi0 = 0;   @ Specify I(0) exogenous variables, otherwise set xi0=0 @
dum = 0;   @ Specify dummy variables, otherwise set dum=0 @
sflag = 0; @ 1 = Seasonal Dummy (Centered), 0 = No Seasonal @

@ Set parameters for estimation @
varlag = 6;  @ The lag length of VAR: DGP - 6, 9, 12, 13 @
freq = 4;    @ Frequency: 4 = Quarterly, 12 = Monthly @
n = 3;      @ # of variables @
r = 1;       @ # of cointegrating vectors @
k = n - r;   @ # of common trends @
let bn[1,1] = 1;     @ r x 1, Normalization of cointegrating Vector @
let names = P2 C1 C2; @ Labels of variables @
conf_lev = 95; @ Confidence intervals for the rank test @
              @ Availables for 50%, 75%, 80%, 85%, 90%, 95%, 97.5%, 99% @

hflag = 3;   @ Johansen Model @
     @ 1 = H(r)  : Quadratic Trend @
     @ 2 = H*(r) : Linear Trend @
     @ 3 = H1(r) : Linear Trend, Deterministic Cointegration @
     @ 4 = H1*(r): No Trend, Det'c Cointegration @
     @ 5 = H2(r) : No Trend, No Constant, Det'c Cointegration @

@ Set control parameters for data @
mtag = 1; @ Cointegrating Vectors: 1 = Unrestricted, 2 = Restricted @
ptag = 0;    @ 1 = print detailed output, 0 = No print @

if mtag == 2;
    @ Restrictions on Beta : H*phi = Beta @
        let h1[3,1] = 1
                     -1
                      1; @ n x (n-q), q: # of restrictions @
        h = h1;
        let ngv = 1;     @ # of vector on each group @
        let rgv = 2;     @ # of restrictions on each group @
        sgv = n - rgv;   @ # of column on each H matrics @
endif;

@ ************************************************************** @
@ ----------------        End of Modification        ----------- @
@ ************************************************************** @
```

```
output file = ^outfile reset;
outfile;
datestr(0);
timestr(0);

#include johansen.set;

@ Define the Table for Trace Test @
if hflag == 2;
   names = "trend"|names;
elseif hflag == 4;
   names = "const"|names;
endif;

{z0,zk,z1,ctag,ttag} = trans(yi1,yi0,xi1nc,xi1ci,xi0,dum,varlag,hflag,sflag,freq);

n = cols(z0);
t = rows(z0);
{s00,s0k,skk} = rfu(z0,zk,z1);
format /rdn 14,4;
"*********** The rank tests ***********";
{va,ve} = evnorm(n,s00,s0k,skk,symm);
ranktst(va,ve,t,names,hflag,conf_lev);

"*********** Estimating alpha and pi ***********";
{va,ve} = evnorm(r,s00,s0k,skk,symm);
"* Unrestricted beta (normalized)";
{va,ve} = betanorm(va,ve,bn,names);
betaun = ve;
beta = betaun;

if mtag == 1;
   beta = betaun;
elseif mtag == 2;
   if rows(ngv) == 1;
{s00,s0k_h,h_skk_h,h} = rfhhflag(z0,zk,z1,h,hflag);
{va,ve} = evnorm(r,s00,s0k_h,h_skk_h,symm);
beta = h*ve;
   else;
{beta} = rfghflag(z0,zk,z1,h,betaun,ngv,rgv,hflag);
   endif;
```

```
endif;

"";
"* Estimation of parameters using given beta";
{beta,alpha,pi0,talpha,tpi0} = esthflag(va,beta,bn,s00,s0k,skk,t,names,hflag);

end;
```

2.2 Output: JOHANSEN.OUT

```
johansen.out
11/13/01
 4:06:18
Model: H1(r) - hflag = 3, Linear Trend, Deterministic Cointegration
*********** The rank tests ***********

    I(1) ANALYSIS

   EIGENV.     L-MAX     TRACE     H0:R=      N-R=   CV_L95%  CV_TR95%
    0.1374   25.1248   36.1254    0.0000    3.0000   14.0360   29.3760
    0.0473    8.2430   11.0006    1.0000    2.0000   11.4990   15.3400
    0.0161    2.7576    2.7576    2.0000    1.0000    3.8410    3.8410

    BETA(transposed)
        P2        C1        C2

   23.8057  -29.2183   18.5292
    4.0964  -26.8310    8.2816
   13.8506   10.3330   -0.4327

*********** Estimating alpha and pi ***********
* Unrestricted beta (normalized)
BETA (transposed)
        P2        C1        C2
    1.0000   -1.2274    0.7784

* Estimation of parameters using given beta
BETA (transposed)
        P2        C1        C2
    1.0000   -1.2274    0.7784
ALPHA
```

```
        P2   -0.0906
        C1    0.0310
        C2   -0.1144
t-values for alpha
        P2   -4.5147
        C1    1.0416
        C2   -1.5813

PI
        P2         C1         C2
   -0.0906    0.1112   -0.0705
    0.0310   -0.0381    0.0241
   -0.1144    0.1404   -0.0890
t-values for pi
        P2         C1         C2
   -4.5147    4.5147   -4.5147
    1.0416   -1.0416    1.0416
   -1.5813    1.5813   -1.5813
```

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