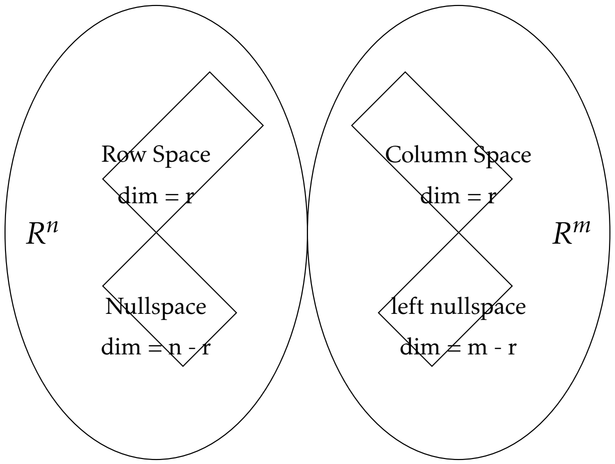
1 Four Fundamental Subspaces



2 Eigenvalues, Eigenvectors

- $Ax = \lambda x$
- $Tr(A) = \sum_{i=1}^{n} \lambda_i$
- For an upper or lower triangular matrix, the eigenvalues sit on the diagonal

2.1 Special Case: 2x2

$p(\lambda) = \lambda^2 - \text{Tr}(A) + \det(A)$

2.2 Finding Eigenspaces

To find the eigenspace given an eigenvalue, $lambda_i$:

1. Use rref to solve for the nullspace of the matrix $A - \lambda_i I$ rref($\left[A - \lambda_i I \mid 0\right]$)
2. Write down a vector equation from the rref result
3. Read off the vectors multiplied by free variables on the RHS of the equation (these are the basis for the eigenspace)

3 Diagonalization

$$\begin{aligned}
A\vec{x_1} &= \lambda_1 \vec{x_1} \\
A\vec{x_2} &= \lambda_2 \vec{x_2} \\
A\begin{bmatrix} \vec{x_1} & \vec{x_2} \end{bmatrix} &= \begin{bmatrix} \lambda_1 \vec{x_1} & \lambda_2 \vec{x_2} \end{bmatrix} \\
&= \begin{bmatrix} \vec{x_1} & \vec{x_2} \end{bmatrix} \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}
\end{aligned}$$

Let the eigenvector matrix, $\begin{bmatrix} \vec{x_1} & \vec{x_2} \end{bmatrix}$, be called P.

Let the eigenvalue matrix, $\begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$, be called D.

We have shown that we can write $AP = PD$. We can rewrite this to see other forms, like $P^{-1}AP = D$ and $A = PDP^{-1}$.

3.1 Powers of Matrices

The above diagonalization leads to an easy analysis of powers of matrices. For instance:

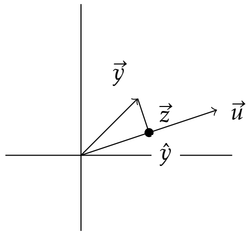
$$\begin{aligned}
A^2 &= PDP^{-1}PDP^{-1} \\
&= PDIDP^{-1} \\
&= PD^2P^{-1}
\end{aligned}$$

In general, $A^n = PD^nP^{-1}$.

3.2 Conditions for Diagonalization

Having n distinct eigenvalues is sufficient to be diagonalizable. Having n linearly independent eigenvectors is necessary and sufficient.

4 Orthogonal Projection



Let $\vec{z} = \vec{y} - \hat{y}$ and $\hat{y} = \alpha\vec{u}$. We know that $\vec{z} \perp \vec{u}$, therefore $\vec{z} \cdot \vec{u} = 0$.

$$\begin{aligned}
\vec{u} \cdot \vec{z} &= 0 \\
\vec{u} \cdot (\vec{y} - \hat{y}) &= 0 \\
\vec{u} \cdot (\vec{y} - \alpha\vec{u}) &= 0 \\
\vec{u} \cdot \vec{y} - \alpha\vec{u} \cdot \vec{u} &= 0 \\
\therefore \alpha &= \frac{\vec{u} \cdot \vec{y}}{\vec{u} \cdot \vec{u}}
\end{aligned}$$

4.1 Gram Schmidt Process

Given a set of vectors, we can produce an orthogonal basis for the same space using the gram schmidt process.

Take the first vector as is, then for each following vector subtract off the amount projected in each preceeding basis vector.

Once we have a set of orthogonal vectors, then we can normalize them to form an orthonormal set.

An m by n matrix U is orthonormal if and only if $U^T U = I$.

4.1.1 QR Factorization

Given an orthonormal basis Q, we often want to find how to decompose A such that $A = QR$. To find R, we use the fact that $Q^T = Q^{-1}$, thus $Q^T A = Q^T QR = Q^{-1} QR = R$.

5 Symmetric Matrices

A symmetric matrix is one such that $S^T = S$. These matrices will have real eigenvalues and orthogonal eigenvectors.

An antisymmetric matrix is one such that $A^T = -A$. These matrcies will have pure imaginary eigenvalues and orthogonal complex eigenvectors.

An orthogonal matrix $Q^T Q = I$ will have $|\lambda| = 1$ for each eigenvalue, and orthogonal complex eigenvectors.

Important note! When we say orthogonal eigenvectors, we mean for different eigenspaces. If you get 2 or more eigenvectors for a particular λ, these will likely not be orthogonal and you will need to use the Gram Schmidt process to make them orthogonal.

5.1 Note on Orthogonal Diagonalization

An n by n matrix, A, is orthogonally diagonalizable if and only if A is symmetric.

6 Useful Notes

If A is an n by n matrix and A is full rank, then the following are true and equivalent.

- A is invertible
- $A\vec{x} = \vec{0}$ has only the trivial solution, since the determinant cannot be 0
- rref$(A) = I_n$
- $A\vec{x} = \vec{b}$ is consistent $\forall \vec{b} \subset$ n x 1 matrices.
- $A\vec{x} = \vec{b}$ has exactly one solution $\forall \vec{b} \subset$ n x 1 matrices.
- $\det(A) \neq 0$
- Column vectors of A are linearly independent
- Row vectors of A are linearly independent
- $C(A)$ span R^n
- $C(A^T)$ span R^n
- $C(A)$ form a basis for R^n
- $C(A^T)$ form a basis for R^n
- A has rank n
- A has nullity 0
- The orthogonal complement of the nullspace of A is R^n
- The orthogonal complement of the row space of A is {0}
- $A^T A$ is invertible
- $\lambda = 0$ is not an eigenvalue of A

7 True / False Gotchas

- **FALSE**: 'If $A\vec{x} = \lambda\vec{x}$ for some $\vec{x}$ then λ is an eigenvalue of A' (not just some vector, a **non zero** vector)
- **TRUE**: 'A matrix A is not invertible iff 0 is an eigenvalue of A.' ($A\vec{x} = 0\vec{x} \to \det(A) = 0$ for a non-trivial solution)
- **FALSE**: 'An elementary row operation does not change the determinant' (it does not change the magnitude, but a row swap flips the sign)
- **FALSE**: 'If $\{\vec{v_1}, \vec{v_2}, \vec{v_3}\}$ is an orthogonal basis for W, then multiplying $\vec{v_3}$ by a scalar c gives a new orthogonal basis' (consider the scalar $c = 0$, it forms a new basis of lower dimension)