Conditional Expectation Function

Consider the random variable $Y_i \in \mathbb{R}$ and the random vector $X_i \in \mathbb{R}^k$, $k \geq 1$.[1]

Definition

The Conditional Expecation Function (CEF) - denoted $E[Y_i|X_i]$ - is a **random function**. It is a function that returns the expected value of Y_i for each realized value of X_i. Since X_i is a random vector the resulting function is random itself.

If we fix $X_i = x$, then the value at which we are evaluating the function is no longer random. The result is a constant: the expected value of Y_i at the given x.

$$E[Y_i|X_i = x] = \int y \cdot f_Y(y|X_i = x)dy = \int ydF_Y(y|X_i = x)$$

This follows the same logic that the expectation of a random variable is, $E[Y_i]$, is not random.

Discrete case. The book devotes a lot time to the discussion of cases were X_i is a discrete random *variable*; using the notation $W_i \in \{0, 1\}$ or $D_i \in \{0, 1\}$. In this unique case, we can write the CEF as,

$$E[Y_i|D_i] = E[Y_i|D_i = 0] + D_i \cdot \left(E[Y_i|D_i = 1] - E[Y_i|D_i = 0]\right)$$

The above function returns $E[Y_i|D_i = 0]$ when $D_i = 0$ and $E[Y_i|D_i = 1]$ when $D_i = 1$. This expression for the CEF will be useful in latter chapters of the book.

Law of Iterated Expectations

The Law of Iterated Expectations says that given two random variables[2] $[Y_i, X_i]$, we can express the unconditional expected value of Y_i as the expected value of the conditional expectation of Y_i on X_i.

$$E[Y_i] = E\big[E[Y_i|X_i]\big]$$

[1] The subscript i is not necessary here. However, this notation is consistent with the rest of the book. In this book, Y_i denotes a random variable, $\in \mathbb{R}$, and Y a random vector, $\in \mathbb{R}^n$. Likewise, X_i is a random vector, $\in \mathbb{R}^k$, while X will represent a random matrix, $\in \mathbb{R}^n \times \mathbb{R}^k$.

[2] This can be extended to random vectors.

Where the outside expectation is with respect to X_i,[3] since the CEF is a random function of X_i. We can expand this as follows,

$$E[Y_i] = \int t \cdot f_{Y_i}(t)dt = \int \int y \cdot f_{Y_i|X}(y|x)dyf_X(x)dx = E[E[Y_i|X_i]]$$

Example 0.1. Suppose Y_i and X_i are both discrete, $Y_i \in \{1, 2\}$ and $X_i \in \{3, 4\}$, with the joint distribution:

Table 1: $f_{Y,X}$

	$X_i = 3$	$X_i = 2$
$Y_i = 1$	**1/10**	**3/10**
$Y_i = 2$	**2/10**	**4/10**

We can then define the two marginal distributions,

Table 2: f_Y

$Y_i = 1$	$Y_i = 2$
4/10	**6/10**

and,

Table 3: f_X

$X_i = 3$	$X_i = 4$
3/10	**7/10**

Likewise, we know the conditional distribution $f_{Y|X}$; which we get by dividing the joint distribution by the marginal distribution of X_i. Each column of the conditional distribution should add up to 1.

Table 4: $f_{Y|X}$

	$X_i = 3$	$X_i = 4$
$Y_i = 1$	**1/3**	**3/7**
$Y_i = 2$	**2/3**	**4/7**

Now we can calculate the following objects:

[3]Some texts use the notation $E_X[E[Y_i|X_i]]$ to demonstrate that the outside expectation is with respect to X_i.

1. $E[Y_i]$

$$
\begin{aligned}
E[Y_i] =& 1 \cdot Pr(Y_i = 1) + 2 \cdot Pr(Y_i = 2) \\
=& 1 \cdot 4/10 + 2 \cdot 6/10 \\
=& 16/10
\end{aligned}
$$

2. $E[Y_i|X_i = 3]$

$$
\begin{aligned}
E[Y_i|X_i = 3] =& 1 \cdot Pr(Y_i = 1|X_i = 3) + 2 \cdot Pr(Y_i = 2|X_i = 3) \\
=& 1 \cdot 1/3 + 2 \cdot 2/3 \\
=& 5/3
\end{aligned}
$$

3. $E[Y_i|X_i = 4]$

$$
\begin{aligned}
E[Y_i|X_i = 4] =& 1 \cdot Pr(Y_i = 1|X_i = 4) + 2 \cdot Pr(Y_i = 2|X_i = 4) \\
=& 1 \cdot 3/7 + 2 \cdot 4/7 \\
=& 11/7
\end{aligned}
$$

4. $E\big[E[Y_i|X_i]\big]$

$$
\begin{aligned}
E\big[E[Y_i|X_i]\big] =& E[Y_i|X_i = 3] \cdot Pr(X_i = 3) + E[Y_i|X_i = 4] \cdot Pr(X_i = 4) \\
=& 5/3 \cdot 3/10 + 11/7 \cdot 7/10 \\
=& 16/10
\end{aligned}
$$

We have therefore demonstrated the law of iterated expectations.

We can extend this principle to conditional expectations. Suppose you have three random variables/vectors $\{Y_i, X_i, Z_i\}$, we can express the conditional expected value of Y_i on X_i as the (conditional) expected value of the conditional expectation of Y_i on X_i *and* Z_i.

$$
E[Y_i|X_i] = E\big[E[Y_i|X_i, Z_i]|X_i\big]
$$

Here the outside expectation is with respect Z_i *conditional on* X_i. It utilizes the conditional distribution $f_{Z|X}$ to form the outside expectation,

$$
E[Y_i|X_i] = \int y \cdot f_{Y|X}(y|X_i)dt = \int \int y \cdot f_{Y|X,Z}(y|X_i, z)dy f_{Z|X}(z|X_i)dz = E\big[E[Y_i|X_i, Z_i]|X_i\big]
$$

Properties of the CEF

The following three theorems can be found in a range of Econometrics textbooks and Microeconometrics texts, including MM & MHE

Theorem 0.1. *We can express the* observed *outcome* Y_i *as a sum of* $E[Y_i|X_i] + \varepsilon_i$ *where* $E[\varepsilon_i|X_i] = 0$ *(i.e., mean independent).*

Proof.

1. $E[\varepsilon_i|X_i] = E[Y_i - E[Y_i|X_i]|X_i] = E[Y_i|X_i] - E[Y_i|X_i] = 0$
2. $E[h(X_i)\varepsilon_i] = E[h(X_i)E[\varepsilon_i|X_i]] = E[h(X_i) \times 0] = 0$

□

Theorem 0.2. $E[Y_i|X_i]$ *is the best predictor of* Y_i*.*

Proof.

$$\begin{aligned}
(Y_i - m(X_i))^2 =& \left((Y_i - E[Y_i|X_i]) + (E[Y_i|X_i] - m(X_i))\right)^2 \\
=& (Y_i - E[Y_i\|X_i])^2 + (E[Y_i|X_i] - m(X_i))^2 \\
&+ 2(Y_i - E[Y_i|X_i]) \times (E[Y_i|X_i] - m(X_i))
\end{aligned}$$

The last term (cross product) is mean zero. Thus, the function is minimized by setting $m(X_i) = E[Y_i|X_i]$. □

Theorem 0.3. *[ANOVA Theorem] The variance of* Y_i *can be decomposed as* $V(E[Y_i|X_i]) + E(V(Y_i|X_i))$

Proof.

$$\begin{aligned}
V(Y_i) =& V(E[Y_i|X_i] + \varepsilon_i) \\
=& V(E[Y_i|X_i]) + V(\varepsilon_i) \\
=& V(E[Y_i|X_i]) + E[\varepsilon_i^2]
\end{aligned}$$

The second line follows from Theorem 1.1 (independence) and

$$E[\varepsilon_i^2] = E\left[E[\varepsilon_i^2|X_i]\right] = E\left[V(Y_i|X_i)\right]$$

□