Interpretation of Linear Models

In this short handout we will consider the interpretation of linear regression model coefficients in models with different combinations of outcome and regressor variables:

1. continuous level-level
2. continuous-discrete
3. discrete-continuous
4. discrete-discrete
5. log-level
6. level-log
7. log-log

In all instances, we will work on the CLRM model assumptions 1 & 2, which tell us that the conditional expectation function is linear in parameters:

$$E[Y_i|X_i] = X_i'\beta$$

Continuous, level-level models

If Y_i and X_i are both continuously distributed random variables then,

$$\beta_j = \frac{\partial E[Y_i|X_i]}{\partial X_{ij}}$$

or, as a vector,

$$\beta = \frac{\partial E[Y_i|X_i]}{\partial X_i} = \begin{bmatrix} \frac{\partial E[Y_i|X_i]}{\partial X_{i1}} \\ \vdots \\ \frac{\partial E[Y_i|X_i]}{\partial X_{ik}} \end{bmatrix} = \begin{bmatrix} \beta_1 \\ \vdots \\ \beta_k \end{bmatrix}$$

The regression parameter has a partial derivative interpretation with respect to the CEF. As discussed in Handout 1, this is often used to motivate the experimental language of *ceteris paribus*: "holding all else fixed.

Continuous-discrete models

Consider a case where there is a single discrete regressor: $D_i \in \{0, 1\}$. For example,

$$Y_i = \beta_1 + \beta_2 D_i + \varepsilon_i$$

We cannot apply the partial derivative interpretation since D is not continuous. Instead, we will look at differences:

$$\begin{aligned} E[Y_i|D_i = 1] =& \beta_1 + \beta_2 \\ E[Y_i|D_i = 0] =& \beta_1 \\ \Rightarrow \beta_2 =& E[Y_i|D_i = 1] - E[Y_i|D_i = 0] \end{aligned}$$

We can easily extend this the case where the model includes additional (discrete or continuous) covariates, as well as case where the variable takes on multiple discrete values.

Discrete-continuous models

If the outcome is discrete ($Y_i \in \{0, 1\}$) while the regressors are continuous, the resulting linear model is referred to as a linear probability model.

$$E[Y_i|X_i] = Pr(Y_i = 1|X_i) = X_i'\beta$$

This is differentiable, since X is continuous and the same partial derivative interpretation follows.

$$\beta_j = \frac{\partial Pr(Y_i = 1|X_i)}{\partial X_{ij}}$$

Note, the unit of Y is probability-points ($\in [0, 1]$), not %-points ($\in [0, 100]$). Of course, the conversion of units can be made by $\times 100$ to measure in **%-points**.

Discrete-discrete models

If both the outcome and regressor(s) are discrete, then the parameter identifies a difference in conditional probabilities,

$$\beta_2 = Pr(Y_i|D_i = 1) - Pr(Y_i = 1|D_i = 0)$$

Note, the unit of Y is probability-points ($\in [0, 1]$), not %-points ($\in [0, 100]$).

Log-level models

Consider the model,

$$\ln(Y_i) = X_i'\beta + \varepsilon_i$$

Then,

$$X_i'\beta = E[\ln(Y_i)|X_i]$$

$$\beta_j = \frac{\partial E[\ln(Y_i)|X_i]}{\partial X_{ij}}$$

The coefficient is therefore measured in log-units of Y. The relation to a change in the (expected) level of Y is given by,

$$\%\Delta E[Y_i|X_i] = (exp(\beta) - 1) \times 100$$

For reasonably small values of β (i.e. within the range $[-0.1, 0.1]$) this can be approximated by,

$$\%\Delta E[Y_i|X_i] = \beta \times 100$$

A 1-unit change in X_{i1} is associated with a β_1 **percent** change in the expected value of Y.

This referred to as a **semi-elasticity**.

Level-log models

If the regressor(s) is measure in log-units; for example,

$$Y_i = \beta_1 + \beta_2 \ln(X_i)_i + \varepsilon_i$$

Then,

$$\beta_2 = \frac{\partial E[Y_i|X_i]}{\partial \ln(X_i)}$$

A 1 **percent** increase in X is given by $X \times 1.01$. This is equivalent to a change in $\ln(X)$ of,

$$\ln(X_i \times 1.01) - \ln(X_i) = \ln(1.01) \approx 0.01$$

Thus, a 1 **percent** increase in the level of X is associated with a $\beta_2/100$ increase in the expected value of Y. Or, more accurate

$$\Delta E[Y_i|X_i] = \beta_2 \times \ln(1.01)$$

This is also a semi-elasticity.

Log-log models

In models where both the outcome and regressor are log-transformed with an **elasticity** interpretation.

$$\ln(Y_i) = \beta_1 + \beta_2 \ln(X_i)_i + \varepsilon_i$$
$$\beta_2 = \frac{\partial E[\ln(Y_i)|X_i]}{\partial \ln(X_i)}$$

β_2 is a the % change in the expected value of Y from a 1 % change in X.