

## Interpretation of Linear Models

In this short handout we will consider the interpretation of linear regression model coefficients in models with different combinations of outcome and regressor variables:

1. continuous level-level
2. continuous-discrete
3. discrete-continuous
4. discrete-discrete
5. log-level
6. level-log
7. log-log

In all instances, we will work on the CLRM model assumptions 1 & 2, which tell us that the conditional expectation function is linear in parameters:

$$E[Y_i|X_i] = X_i'\beta$$

### Continuous, level-level models

If  $Y_i$  and  $X_i$  are both continuously distributed random variables then,

$$\beta_j = \frac{\partial E[Y_i|X_i]}{\partial X_{ij}}$$

or, as a vector,

$$\beta = \frac{\partial E[Y_i|X_i]}{\partial X_i} = \begin{bmatrix} \frac{\partial E[Y_i|X_i]}{\partial X_{i1}} \\ \vdots \\ \frac{\partial E[Y_i|X_i]}{\partial X_{ik}} \end{bmatrix} = \begin{bmatrix} \beta_1 \\ \vdots \\ \beta_k \end{bmatrix}$$

The regression parameter has a partial derivative interpretation with respect to the CEF. As discussed in Handout 1, this is often used to motivate the experimental language of *ceteris paribus*: “holding all else fixed.”

## Continuous-discrete models

Consider a case where there is a single discrete regressor:  $D_i \in \{0, 1\}$ . For example,

$$Y_i = \beta_1 + \beta_2 D_i + \varepsilon_i$$

We cannot apply the partial derivative interpretation since  $D$  is not continuous. Instead, we will look at differences:

$$\begin{aligned} E[Y_i | D_i = 1] &= \beta_1 + \beta_2 \\ E[Y_i | D_i = 0] &= \beta_1 \\ \Rightarrow \beta_2 &= E[Y_i | D_i = 1] - E[Y_i | D_i = 0] \end{aligned}$$

We can easily extend this the case where the model includes additional (discrete or continuous) covariates, as well as case where the variable takes on multiple discrete values.

## Discrete-continuous models

If the outcome is discrete ( $Y_i \in \{0, 1\}$ ) while the regressors are continuous, the resulting linear model is referred to as a linear probability model.

$$E[Y_i | X_i] = Pr(Y_i = 1 | X_i) = X_i' \beta$$

This is differentiable, since  $X$  is continuous and the same partial derivative interpretation follows.

$$\beta_j = \frac{\partial Pr(Y_i = 1 | X_i)}{\partial X_{ij}}$$

Note, the unit of  $Y$  is probability-points ( $\in [0, 1]$ ), not %-points ( $\in [0, 100]$ ). Of course, the conversion of units can be made by  $\times 100$  to measure in **%-points**.

## Discrete-discrete models

If both the outcome and regressor(s) are discrete, then the parameter identifies a difference in conditional probabilities,

$$\beta_2 = Pr(Y_i | D_i = 1) - Pr(Y_i = 1 | D_i = 0)$$

Note, the unit of  $Y$  is probability-points ( $\in [0, 1]$ ), not %-points ( $\in [0, 100]$ ).

## Log-level models

Consider the model,

$$\ln(Y_i) = X_i'\beta + \varepsilon_i$$

Then,

$$X_i'\beta = E[\ln(Y_i)|X_i]$$

$$\beta_j = \frac{\partial E[\ln(Y_i)|X_i]}{\partial X_{ij}}$$

The coefficient is therefore measured in log-units of  $Y$ . The relation to a change in the (expected) level of  $Y$  is given by,

$$\% \Delta E[Y_i|X_i] = (\exp(\beta) - 1) \times 100$$

For reasonably small values of  $\beta$  (i.e. within the range  $[-0.1, 0.1]$ ) this can be approximated by,

$$\% \Delta E[Y_i|X_i] = \beta \times 100$$

A 1-unit change in  $X_{i1}$  is associated with a  $\beta_1$  **percent** change in the expected value of  $Y$ .

This referred to as a **semi-elasticity**.

## Level-log models

If the regressor(s) is measure in log-units; for example,

$$Y_i = \beta_1 + \beta_2 \ln(X_i)_i + \varepsilon_i$$

Then,

$$\beta_2 = \frac{\partial E[Y_i|X_i]}{\partial \ln(X_i)}$$

A 1 **percent** increase in  $X$  is given by  $X \times 1.01$ . This is equivalent to a change in  $\ln(X)$  of,

$$\ln(X_i \times 1.01) - \ln(X_i) = \ln(1.01) \approx 0.01$$

Thus, a 1 **percent** increase in the level of  $X$  is associated with a  $\beta_2/100$  increase in the expected value of  $Y$ . Or, more accurate

$$\Delta E[Y_i|X_i] = \beta_2 \times \ln(1.01)$$

This is also a semi-elasticity.

### **Log-log models**

In models where both the outcome and regressor are log-transformed with an **elasticity** interpretation.

$$\begin{aligned}\ln(Y_i) &= \beta_1 + \beta_2 \ln(X_i)_i + \varepsilon_i \\ \beta_2 &= \frac{\partial E[\ln(Y_i)|X_i]}{\partial \ln(X_i)}\end{aligned}$$

$\beta_2$  is a the % change in the expected value of  $Y$  from a 1 % change in  $X$ .