Problem Set 3 (SOLUTIONS)

The purpose of this problem set is for you to see how the ordinary least squares (OLS) estimator behaves under various assumptions in a linear regression model where you know what the model is – since you are going to be generating the data from a known data generating process (DGP).

The models estimated are simple bivariate regressions but the properties of the OLS estimator with vary with each case. This is demonstrated by changing the (a) distributional properties of the error term (variance-covariance structure), and (b) inducing correlation between the regressor and the error term. Any resulting bias and/or inconsistency will depend on the DGP.

To achieve certain results we will have to use a serially-correlated error structure, which is only appropriate in a time-series setting. For this reason, the models will be written with subscript t and not i.

The code has been provided for model 1. You can then modify the code for models 2-4.

Preamble

```
<IPython.core.display.HTML object>
```

You do not need to load data for this problem set.

```
clear
//or, to remove all stored values (including macros, matrices, scalars, etc.)
*clear all

* Replace $rootdir with the relevant path to on your local harddrive.
cd "$rootdir/problem-sets/ps-3"

cap log close
log using problem-set-3-log.txt, replace
```

However, since we are going to generate random variables, we should set a seed. This ensures replicability of the exercise. The number you choose is arbitrary, it simply ensures that any algorithms used to generate (pseudo) random variables start at the same place.

```
set seed 981836
```

Model 1: CLRM

This is your classical linear regression model. OLS estimator is unbiased and consistent.

$$Y_t = \beta_1 + \beta_2 X_t + \upsilon_t \qquad \text{with} \quad \upsilon_t \sim N(0, \sigma^2)$$

We know that the OLS estimator for β_2 is given by,

$$\begin{aligned}
\hat{\beta}_2 =& \frac{\sum_t \left[(X_t - \bar{X})(Y_t - \bar{Y})\right]}{\sum_t (X_t - \bar{X})^2} \\
=& \beta_2 + \frac{\sum_t \left[(X_t - \bar{X})(\upsilon_t - \bar{\upsilon})\right]}{\sum_t (X_t - \bar{X})^2} \\
=& \beta_2 + \frac{\sum_t \tilde{X}_t \tilde{\upsilon}_t}{\sum_t \tilde{X}_t^2}
\end{aligned}$$

where $\tilde{X}_t$ and $\tilde{\upsilon}_t$ represent the demeaned counterparts of these variables. Alternatively, using linear algebra notation:

$$\begin{aligned}
\hat{\beta}_2 =& \frac{X' M_\ell Y}{X' M_\ell X} \\
=& \beta_2 + \frac{X' M_\ell \upsilon}{X' M_\ell X} \\
=& \beta_2 + \frac{\tilde{X}' \tilde{\upsilon}}{\tilde{X}' \tilde{X}}
\end{aligned}$$

where $\tilde{X} = M_\ell X$, $\tilde{\upsilon} = M_\ell \upsilon$, and $M_\ell = I_n - \ell(\ell'\ell)^{-1}\ell'$ (the orthogonal projection of the constant regressor).

We know from Handouts 2 & 3,

1. $E[\hat{\beta}_2] = \beta_2$ (i.e., unbiased)
2. $p\lim \hat{\beta}_2 = \beta_2$ (i.e., consistent)

Can you demonstrate these results?

Simulation

Begin by designing a programme that takes the parameters of the model as arguments, generates the data, estimates the model, and then returns the stored values.

```
cap prog drop mc1
program define mc1, rclass
    syntax [, obs(integer 1) s(real 1) b1(real 0) b2(real 0)  sigma(real 1)]
    drop _all
    set obs `obs'
    gen u = rnormal(0,`sigma')            // sigma is the std deviation of the error distribu
    gen x=uniform()*`s'                   // s is the std devation of the x distribution
    gen y=`b1'+`b2'*x + u                   // this generates the dep variable y
    reg y x
    return scalar b1=_b[_cons]            // intercept estimate
    return scalar b2=_b[x]                  // coeff on the x variable
    return scalar se2 = _se[x]            // std error
    return scalar t2 = _b[x]/_se[x]     // t ratio
end
```

Use the the `simulate` command in Stata to estimate the model 100 times:

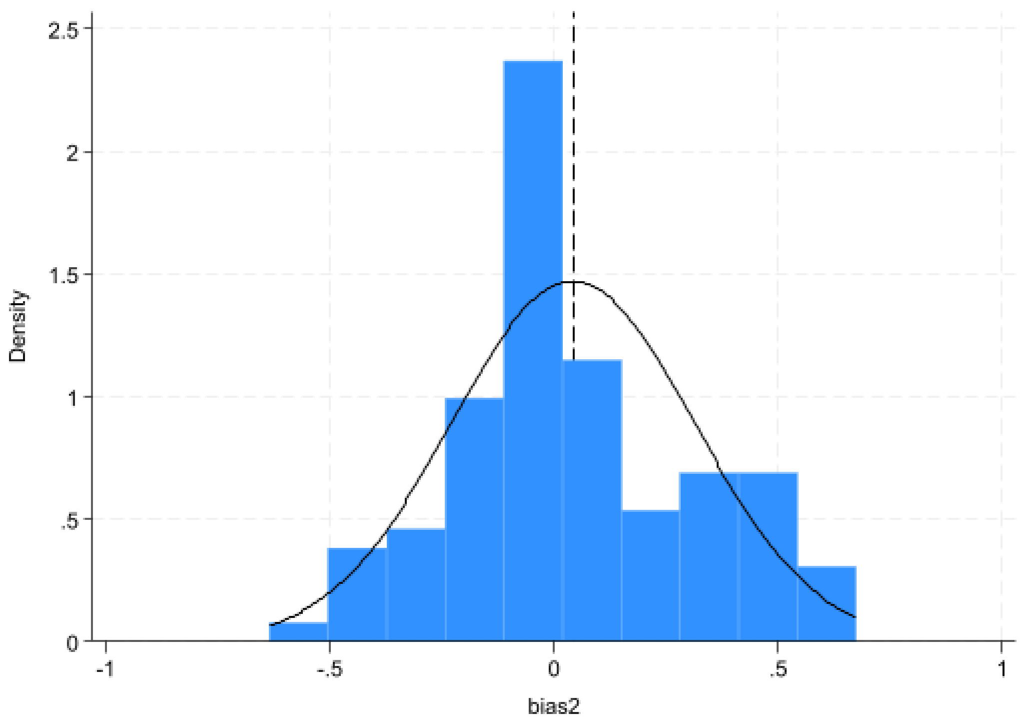
```
simulate b1=r(b1) b2=r(b2) se2=r(se2) t2=r(t2), reps(100): mc1, obs(50) s(6) b1(4) b2(2) sig
```

```
      Command: mc1, obs(50) s(6) b1(4) b2(2) sigma(3)
           b1: r(b1)
           b2: r(b2)
          se2: r(se2)
           t2: r(t2)

Simulations (100): .........10.........20.........30.........40.........50.....
> ....60.........70.........80.........90.........100 done
```

Calculate the bias and plot the distribution of the bias.

```
gen bias2=b2-2
su b1 b2 se2 t2
su bias2
histogram bias2, normal xline(`r(mean)')
```

```
    Variable |        Obs        Mean    Std. dev.       Min        Max
-------------+---------------------------------------------------------
          b1 |        100    3.880226    .9977415   1.080851   6.090028
          b2 |        100    2.041985     .271704   1.365155   2.673303
         se2 |        100    .2520885    .0291596   .1814694   .3255497
          t2 |        100    8.216185      1.4968   5.484826   12.60699

    Variable |        Obs        Mean    Std. dev.       Min        Max
-------------+---------------------------------------------------------
       bias2 |        100    .0419852     .271704  -.6348448   .6733029
(bin=10, start=-.63484478, width=.13081477)
```



Model 2: Serial Correlation

Relax the assumption of an iid error term and allow for serial correlation. The OLS estimator is unbiased and consistent. However, the std errors are wrong since the software does not know that you have serially correlated errors and you are not taking this into account in the estimation.

$$Y_t = \beta_1 + \beta_2 X_t + \upsilon_t \qquad \text{where} \quad \upsilon_t = \rho\upsilon_{t-1} + \varepsilon_t \quad \text{and} \quad \varepsilon_t \sim N(0, \sigma^2)$$

We say that U_t follows an AR(1) process. You can show that $\hat{\beta}_2$ remains unbiased and consistant. However, the standard homoskedastic-variance estimator is incorrect:

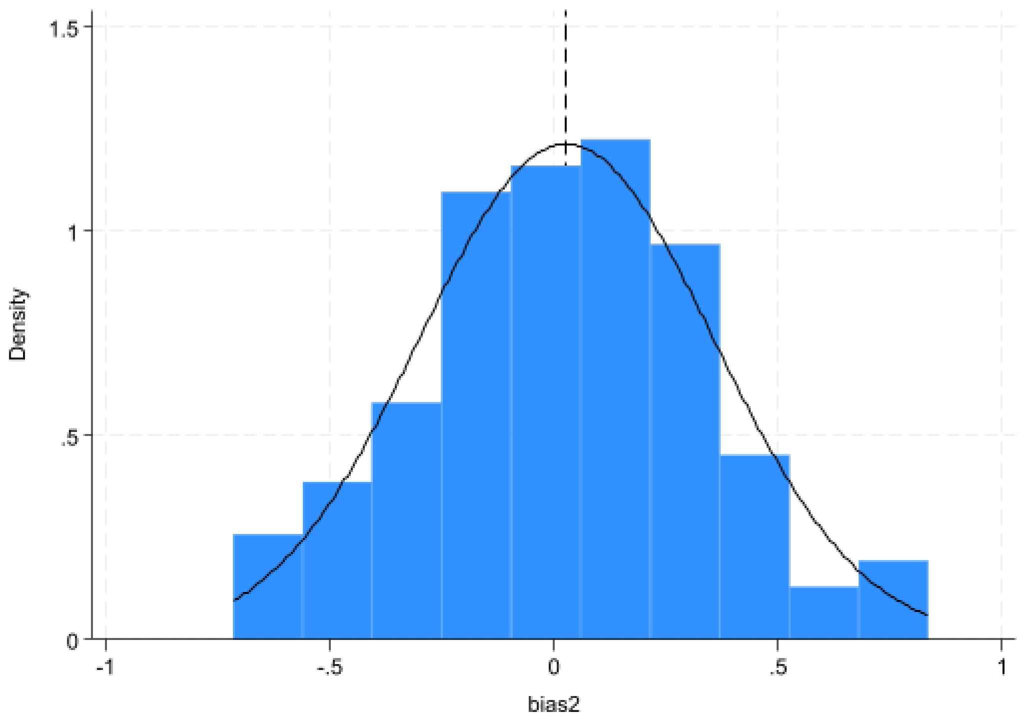
$$Var(\hat{\beta}_2) \neq \frac{\sigma^2}{Var(X_i)}$$

Simulation

```
cap prog drop mc2
program define mc2, rclass
    syntax [, obs(integer 1) s(real 1) b1(real 0) b2(real 0) bias2(real 0) sigma(real 1) rho
    drop _all
    set obs `obs'
    gen u=0
    gen time=_n
    tsset time
    gen e = rnormal(0,`sigma')
    forvalues i=2/`obs'  {
    replace u=`rho'*u[_n-1] + e if _n==`i'
    }
    gen x=uniform()*`s'
    gen y=`b1'+`b2'*x + u
    reg y x
    return scalar b1=_b[_cons]
    return scalar b2=_b[x]
    return scalar se2 = _se[x]
    return scalar t2 = _b[x]/_se[x]
end

simulate b2=r(b2) se2=r(se2) t2=r(t2), reps(100): mc2, obs(30) s(6) b1(4) b2(2) sigma(3) rho
gen bias2=b2-2
su b2 t2 se2
su bias2
histogram bias2, normal xline(`r(mean)')
```

```
Command: mc2, obs(30) s(6) b1(4) b2(2) sigma(3) rho(0.2)
```

```
         b2: r(b2)
        se2: r(se2)
         t2: r(t2)

Simulations (100): .........10.........20.........30.........40.........50.....
> ....60.........70.........80.........90.........100 done

    Variable |        Obs        Mean    Std. dev.       Min        Max
-------------+---------------------------------------------------------
          b2 |        100    2.027336    .3287378   1.284326    2.83436
          t2 |        100    6.309408    1.478218   3.236439   10.17626
         se2 |        100    .3301718    .0526629   .2165523   .4499786

    Variable |        Obs        Mean    Std. dev.       Min        Max
-------------+---------------------------------------------------------
       bias2 |        100    .0273363    .3287378  -.7156742   .8343601
(bin=10, start=-.71567416, width=.15500343)
```



Model 3: Dynamic model without serial correlation

Consider a version of Model 1, where the regressor is the lag of the dependent variable.

$$Y_t = \beta_1 + \beta_2 Y_{t-1} + \upsilon_t \qquad \text{with} \quad \upsilon_t \sim N(0, \sigma^2)$$

The OLS estimator is now,

$$\hat{\beta}_2 = \beta_2 + \frac{\sum_t \tilde{Y}_{t-1}\tilde{\upsilon}_t}{\sum_t \tilde{Y}_{t-1}^2}$$

This model is biased, since

$$E\left[\frac{\sum_t \tilde{Y}_{t-1}\tilde{\upsilon}_t}{\sum_t \tilde{Y}_{t-1}^2}\right] \neq \frac{E\left[\sum_t \tilde{Y}_{t-1}\tilde{\upsilon}_t\right]}{E\left[\sum_t \tilde{Y}_{t-1}^2\right]}$$

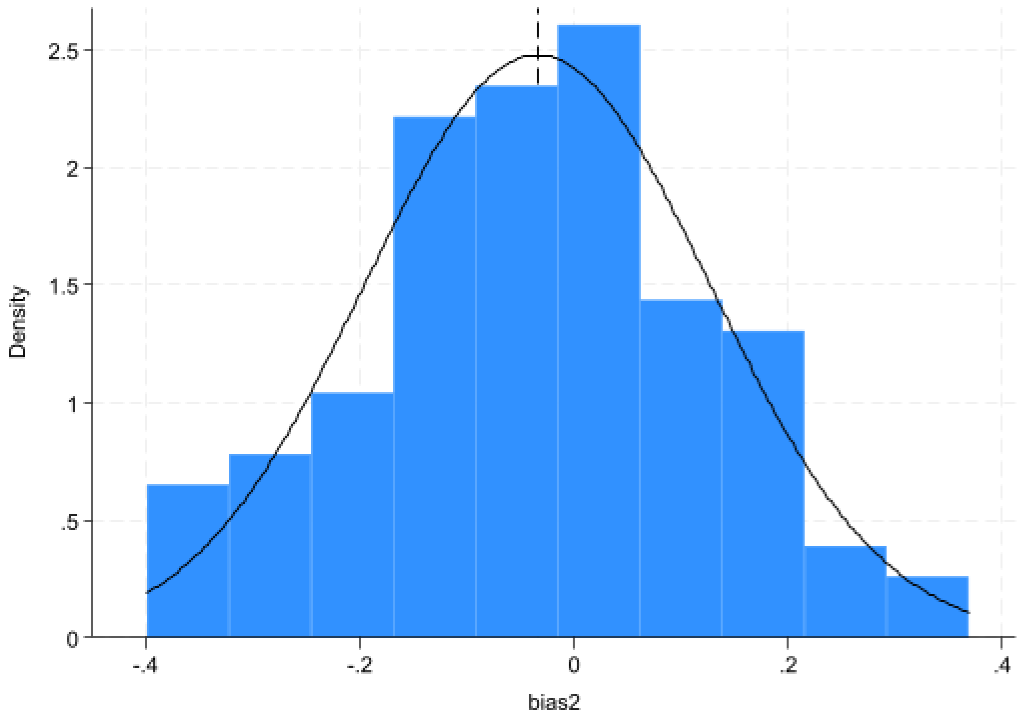
When the regressor was X_t, the above statement was true given the Law of Iterated Expectations. However, you can use Slutsky's theorem and the WLLN to show that $\hat{\beta}_2 \to_p \beta_2$. This result relies on the fact that Y_{t-1} is realized before υ_t which is iid. Thus, the bias goes to 0 as $n \to \infty$.

Simulation

```
cap prog drop mc3
program define mc3, rclass
    syntax [, obs(integer 1)  g1(real 0)  g2(real 0)  sigma(real 1)]
    drop _all
    set obs `obs'
    gen y=0
    gen u = rnormal(0,`sigma')
    gen time=_n
    tsset time
    forvalues i=2/`obs'  {
    replace y=`g1'+ `g2'* y[_n-1] + u  if _n==`i'
    }
    reg y  L.y
    return scalar g1=_b[_cons]
    return scalar g2=_b[L.y]
    return scalar se2 = _se[L.y]
  return scalar t2 = _b[L.y]/_se[L.y]
end
```

```
simulate  g2=r(g2) se2=r(se2) t2=r(t2), reps(100): mc3, obs(30) g1(4)  g2(0.20) sigma(3)   /
g bias2=g2-0.20
su g2 t2 se2
su bias2
histogram bias2, normal xline(`r(mean)')
```

```
      Command: mc3, obs(30) g1(4) g2(0.20) sigma(3)
           g2: r(g2)
          se2: r(se2)
           t2: r(t2)

Simulations (100): .........10.........20.........30.........40.........50.....
> ....60.........70.........80.........90.........100 done

    Variable |        Obs        Mean    Std. dev.       Min        Max
-------------+---------------------------------------------------------
          g2 |        100    .1656964    .1610075  -.1987682   .5684798
          t2 |        100     .931305    .9285551  -1.082788   3.572202
         se2 |        100    .1824637    .0075315   .1591399   .2018857

    Variable |        Obs        Mean    Std. dev.       Min        Max
-------------+---------------------------------------------------------
       bias2 |        100   -.0343036    .1610075  -.3987682   .3684798
(bin=10, start=-.39876819, width=.0767248)
```



Model 4: Dynamic model with serial correlation

Consider a version of Model 2, where the regressor is the lag of the dependent variable.

$$Y_t = \beta_1 + \beta_2 Y_{t-1} + \upsilon_t \text{where} \quad \upsilon_t = \rho\upsilon_{t-1} + \varepsilon_t \quad \text{and} \quad \varepsilon_t \sim N(0, \sigma^2)$$

As with model 3, the OLS estimator will be biased. In addition, since $Cov(\upsilon_t, \upsilon_{t-1}) \neq 0$ and $Cov(Y_t, \upsilon_t) \neq 0$ (for any t),

$$\Rightarrow Cov(Y_{t-1}, \upsilon_t) \neq 0$$

As a result $\hat{\beta}_2$ is inconsistent.

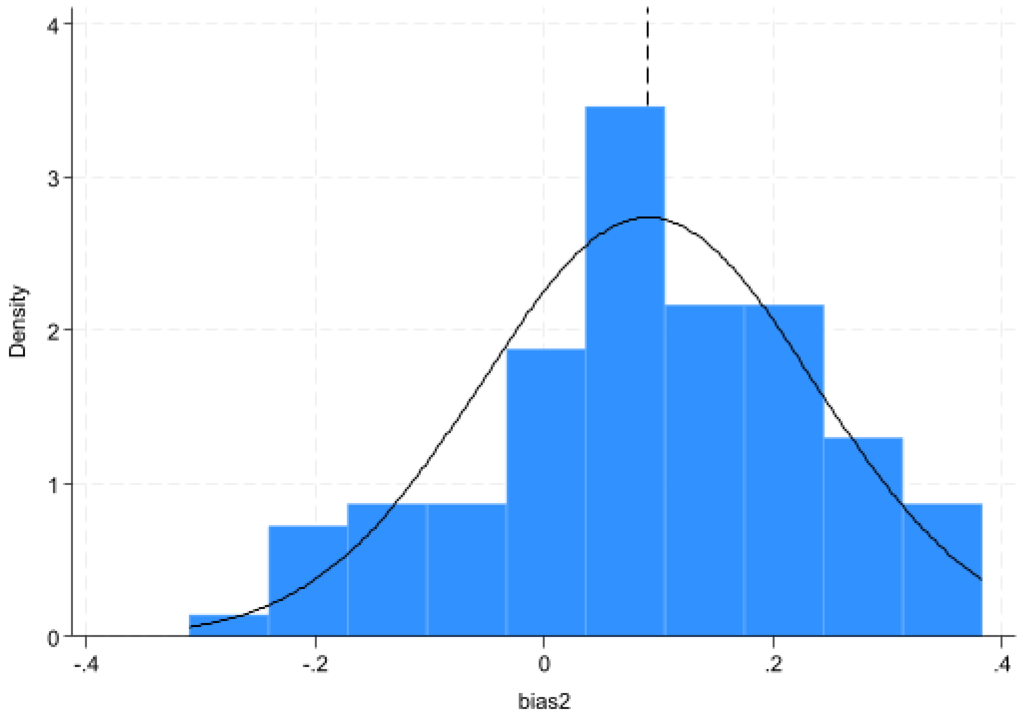
Simulation

```
cap prog drop mc4
program define mc4, rclass
    syntax [, obs(integer 1)  g1(real 0)  g2(real 0)  sigma(real 1) rho(real 0) ]
    drop _all
    set obs `obs'
```

```
    gen y=0
    gen u=0
    gen e = rnormal(0,`sigma')
    gen time=_n
    tsset time

    forvalues i=2/`obs'  {
    replace u=`rho'*u[_n-1] + e if _n==`i'
    replace y=`g1'+ `g2'* y[_n-1] + u  if _n==`i'
    }
    reg y  L.y
    return scalar g1=_b[_cons]
    return scalar g2=_b[L.y]
    return scalar se2 = _se[L.y]
  return scalar t2 = _b[L.y]/_se[L.y]
end

simulate  g2=r(g2) se2=r(se2) t2=r(t2), reps(100): mc4, obs(30) g1(4) g2(0.20) sigma(3) rho(0
g bias2=g2-0.20
su g2 t2 se2
su bias2
histogram bias2, normal xline(`r(mean)')
```

```
      Command: mc4, obs(30) g1(4) g2(0.20) sigma(3) rho(0.2)
           g2: r(g2)
          se2: r(se2)
           t2: r(t2)

Simulations (100): .........10.........20.........30.........40.........50.....
> ....60.........70.........80.........90.........100 done

    Variable |        Obs        Mean    Std. dev.       Min        Max
-------------+---------------------------------------------------------
          g2 |        100    .2905161    .1457775   -.110407   .5824866
          t2 |        100    1.682558    .9099479   -.630446   3.896481
         se2 |        100    .1767967    .0106219   .1494904    .219561

    Variable |        Obs        Mean    Std. dev.       Min        Max
-------------+---------------------------------------------------------
       bias2 |        100    .0905161    .1457775   -.310407   .3824866
```

```
(bin=10, start=-.31040695, width=.06928935)
```



Postamble

```
log close
```