Hilbert Spaces

Among the L^p spaces, L^2 has the property that the product of any two of its elements is integrable.

- $L^2(\mathbb{R}^n)$ is naturally equipped with the <u>inner product</u>

$$\langle f, g \rangle = \int f \bar{g}$$

(<u>Note</u>: $\langle f, f \rangle^{1/2} = \|f\|_2$)

This leads to some important extra (geometric) structure in L^2.

We now discuss this in the general setting of an arbitrary <u>Hilbert space</u>.

Inner Product Space

Let V be a vector space over $\mathbb{C}$. An <u>inner product</u> on V is a map $(x,y) \longmapsto \langle x,y \rangle$ from $V \times V \mapsto \mathbb{C}$ such that

(i) $\langle ax+by, z \rangle = a\langle x,z \rangle + b\langle y,z \rangle \quad \forall\, x,y,z \in V \;\&\; a,b \in \mathbb{C}$

(ii) $\langle y,x \rangle = \overline{\langle x,y \rangle} \quad \forall\, x,y \in V$

(iii) $\langle x,x \rangle \in (0,\infty) \quad \forall\, x \in V \setminus \{0\}$

A vector space over $\mathbb{C}$ with an inner product is called a <u>inner product space</u> (or pre-Hilbert space).

Note:

(a) (i) & (ii) $\Rightarrow \langle x, ay+bz\rangle = \bar{a}\langle x,y\rangle + \bar{b}\langle x,z\rangle \;\; \forall x,y,z\in V$ & $a,b\in\mathbb{C}$.

(b) (i) $\Leftrightarrow$ "For every $z\in V$, the mapping $x\mapsto\langle x,z\rangle$ is a linear functional on V".

- We define $\|x\| := \langle x,x\rangle^{1/2}$ for all $x\in V$.

Propn: The function $x\mapsto \|x\|$ is a norm on V.

It is easy to see that $\|x\|=0 \Leftrightarrow x=0$ & $\|\lambda x\| = |\lambda|\,\|x\|$, so the only thing to check is the Δ-inequality. As with L^2 spaces, the key to verifying this is the following:

Schwarz Inequality: If V is an inner product space, then

$$|\langle x,y\rangle| \le \|x\|\,\|y\| \quad \text{for all } x,y\in V$$

with equality iff $x=\lambda y$ for some $\lambda\in\mathbb{C}$.

Proof of Δ-inequality:

$$\|x+y\|^2 := \langle x+y, x+y\rangle = \|x\|^2 + 2\,\mathrm{Re}\langle x,y\rangle + \|y\|^2$$

$$\text{Schwarz} \Rightarrow \|x+y\|^2 \le \|x\|^2 + 2\|x\|\,\|y\| + \|y\|^2 = (\|x\|+\|y\|)^2$$

$$\Rightarrow \|x+y\| \le \|x\| + \|y\|. \qquad \square$$

Proof of Schwarz Inequality

- If $x = \lambda y$ for some $\lambda \in \mathbb{C}$, then clearly both sides equal $|\lambda| \|y\|^2$.

- Suppose $x \neq \lambda y$ for any $\lambda \in \mathbb{C}$, hence $x - \lambda y \neq 0$

$$\Rightarrow \quad 0 < \langle x - \lambda y, x - \lambda y \rangle$$
$$= \|x\|^2 - 2\,\mathrm{Re}(\bar{\lambda}\langle x, y\rangle) + |\lambda|^2 \|y\|^2 \quad \text{(Check!)}$$

- Pick $u \in \mathbb{C}$ such that $\bar{u}\langle x, y\rangle = |\langle x, y\rangle|$. $\left(u = \dfrac{\langle x, y\rangle}{|\langle x, y\rangle|}\right)$

Putting $\lambda = tu$ we see that for any $t \in \mathbb{R}$

$$0 < \underbrace{\|x\|^2 - 2|\langle x, y\rangle|\, t + \|y\|^2 t^2}_{\text{a quadratic in } t\,!}$$

It follows that "$B^2 - 4AC < 0$", i.e. must have

$$4|\langle x, y\rangle|^2 - 4\|y\|^2 \|x\|^2 < 0$$

$$\Updownarrow$$

$$|\langle x, y\rangle| < \|x\| \, \|y\| .$$

$\square$.