Math 4110/6110

Problem Set 4: Consequences of the Existence of Non-Measurable Sets

1. Prove that any $E \subset \mathbb{R}$ with $m_*(E) > 0$ necessarily contains a non-measurable set.

2. Let $\mathcal{N}$ denote the non-measurable subset of $[0,1]$ that was constructed in lecture.
   (a) Prove that if E is a measurable subset of $\mathcal{N}$, then $m(E) = 0$.
   (b) Show that $m_*([0,1] \setminus \mathcal{N}) = 1$
   Hint: Argue by contradiction and pick an open set G such that $[0,1] \setminus \mathcal{N} \subseteq G \subseteq [0,1]$ with $m_*(G) \leq 1 - \varepsilon$.
   (c) Conclude that there exists *disjoint* sets $E_1 \subseteq [0,1]$ and $E_2 \subseteq [0,1]$ for which
   $$m_*(E_1 \cup E_2) \neq m_*(E_1) + m_*(E_2).$$

3. Recall that the **Cantor set** $\mathcal{C}$ is the set of all $x \in [0,1]$ that have a ternary expansion $x = \sum_{k=1}^{\infty} a_k 3^{-k}$ with $a_k \neq 1$ for all k. Consider the function
   $$f(x) = \sum_{k=1}^{\infty} b_k 2^{-k} \quad \text{where} \quad b_k = a_k/2.$$
   (a) Show that f is well defined and continuous on $\mathcal{C}$, and moreover $f(0) = 0$ as well as $f(1) = 1$.
   (b) Prove that there exists a continuous function that maps a measurable set to a non-measurable set.

4. Let us examine the map f defined in Question 3 even more closely. One readily sees that if $x, y \in \mathcal{C}$ and $x < y$, then $f(x) < f(y)$ unless x and y are the two endpoints of one of the intervals removed from $[0,1]$ to obtain $\mathcal{C}$. In this case $f(x) = \ell 2^m$ for some integers ℓ and m, and $f(x)$ and $f(y)$ are the two binary expansions of this number. We can therefore extend f to a map $F : [0,1] \to [0,1]$ by declaring it to be constant on each interval missing from $\mathcal{C}$. F is called the **Cantor-Lebesgue function**.
   (a) Prove that F is non-decreasing and continuous.
   (b) Let $G(x) = F(x) + x$. Show that G is a bijection from $[0,1]$ to $[0,2]$.
   (c) i. Show that $m(G(\mathcal{C})) = 1$.
   ii. By considering rational translates of $\mathcal{N}$ (the non-measurable subset of $[0,1]$ that we constructed in class), prove that $G(\mathcal{C})$ necessarily contains a (Lebesgue) non-measurable set $\mathcal{N}'$.
   iii. Let $E = G^{-1}(\mathcal{N}')$. Show that E is Lebesgue measurable, but not Borel.