Selection of "undergraduate" Real Analysis Qual Problems

Fall 2021

(1) Let $\{x_n\}_{n=1}^{\infty}$ be a sequence of real numbers such that $x_1 > 0$ and
$$x_{n+1} = 1 - (2 + x_n)^{-1} = \frac{1 + x_n}{2 + x_n}.$$
Prove that the sequence $\{x_n\}$ converges, and find its limit.

Fall 2020

1. Show that if x_n is a decreasing sequence of positive real numbers such that $\sum_{n=1}^{\infty} x_n$ converges, then
$$\lim_{n\to\infty} n x_n = 0.$$

Spring 2020

1. Prove that if $f : [0, 1] \to \mathbb{R}$ be continuous, then
$$\lim_{k\to\infty} \int_0^1 k x^{k-1} f(x)\, dx = f(1).$$

Fall 2019

1. Let $\{a_n\}_{n=1}^{\infty}$ is a sequence of real numbers.
   (a) Prove that if $\lim\limits_{n\to\infty} a_n = 0$, then $\lim\limits_{n\to\infty} \dfrac{a_1 + \cdots + a_n}{n} = 0$.
   (b) Prove that if $\sum\limits_{n=1}^{\infty} \dfrac{a_n}{n}$ converges, then $\lim\limits_{n\to\infty} \dfrac{a_1 + \cdots + a_n}{n} = 0$.
2. Prove that $\left| \dfrac{d^n}{dx^n} \dfrac{\sin x}{x} \right| \leq \dfrac{1}{n}$ for all $x \neq 0$ and positive integers n.

   Hint: Consider $\displaystyle\int_0^1 \cos(tx)\, dt$.

Spring 2019

1. Let $C([0, 1])$ denote the space of all continuous real-valued functions on $[0, 1]$.
   (a) Prove that $C([0, 1])$ is complete under the uniform norm $\|f\|_u := \sup\limits_{x\in[0,1]} |f(x)|$.
   (b) Prove that $C([0, 1])$ is not complete under the L^1-norm $\|f\|_1 = \displaystyle\int_0^1 |f(x)|\, dx$

Fall 2018

Problem 1. *Let* $f(x) = \dfrac{1}{x}$. *Show that* $f(x)$ *is uniformly continuous on* $(1, \infty)$ *but not on* $(0, \infty)$.

Spring 2018

2. Let $f_n(x) := \frac{x}{1+x^n}$, $x \geq 0$.

   a) This sequence of functions converges pointwise. Find its limit. Is the convergence uniform on $[0, \infty)$? Justify your answer.

   b) Compute $\lim_{n\to\infty} \int_0^\infty f_n(x)\,dx$.

Fall 2017

1. Describe the intervals on which the series $\displaystyle\sum_{n=0}^{\infty} \frac{x^n}{n!}$ converges uniformly and those on which it does not converge uniformly *and* prove your assertion.

$n\to\infty$

4. Let $f_n(x) = nx(1-x)^n, n \in \mathbb{N}$.

   (i) Show f_n converges to zero pointwise, but not uniformly on $[0, 1]$.

   Hint: Consider the maximum of f_n.

   (ii) Show that $\lim_{n\to\infty} \int_0^1 n(1-x)^n \sin x\,dx = 0$.

Spring 2017

1. Let K be the set of numbers in $[0, 1]$ whose decimal expansions do not use the digit 4 (we use the convention that when a decimal number ends with 4 but all other digits are different from 4, we replace the digit 4 with 399.... For example, 0.8754=0.8753999....)

   Show that K is a compact, nowhere dense set without isolated points, and find the Lebesgue measure $m(K)$.

Fall 2016

1. Prove that the series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^x}$ converges to a differentiable function on $(1, \infty)$ and that

$$\left(\sum_{n=1}^{\infty} \frac{1}{n^x}\right)' = \sum_{n=1}^{\infty} \left(\frac{1}{n^x}\right)',$$

where $'$ means derivative with respect to x. (Recall that $(n^{-x})' = -n^{-x} \ln n$.)

Spring 2016

1. For $n \in \mathbb{N}$, let $e_n = (1 + \frac{1}{n})^n$ and $E_n = (1 + \frac{1}{n})^{n+1}$. It is obvious that $e_n < E_n$. Prove Bernoulli's inequality:

$(1 + x)^n \geq 1 + nx$ for $-1 < x < \infty$ and $n \in \mathbb{N}$.

Then use Bernoulli's inequality or any other method to show that

(a) The sequence e_n is increasing;

(b) The sequence E_n is decreasing;

(c) $2 \leq e_n < E_n \leq 4$;

(d) $\lim_{n\to\infty} e_n = \lim_{n\to\infty} E_n$.

2. Choose $0 < \lambda < 1$ and construct the Cantor set C_λ as follows: Remove from $[0, 1]$ its open middle part of length λ; we left with two intervals I_{11} and I_{12} of equal length. Remove from each of them their open middle parts of length $\lambda m(I_{11})$, etc. and keep doing this *ad infinitum*. We are left with the set C_λ. Prove that the set C_λ has Lebesgue measure zero.

Fall 2015

1. Let $f(x) = c_0 + c_1x^1 + c_2x^2 + ... + c_nx^n$ with n even and $c_n > 0$. Show that there is a number x_m such that $f(x_m) \leq f(x)$ for all $x \in \mathbb{R}$

$n\to\infty \; J_1 \quad 1 + nx^2$

4. Let $f(x)$ be real-valued, defined for $x \geq 1$, satisfying $f(1) = 1$ and

$$f'(x) = 1/(x^2 + f(x)^2)$$

Prove $\lim_{x\to\infty} f(x)$ exists and $\lim_{x\to\infty} f(x) \leq 1 + \pi/4$.

Spring 2015

1. Let (X, d) and (Y, ρ) be metric spaces, $f : X \to Y$ and $x_0 \in X$. Prove that the following two statements are equivalent:

(i) For every $\varepsilon > 0$, there exists a $\delta > 0$ such that $\rho(f(x), f(x_0)) < \varepsilon$ whenever $d(x, x_0) < \delta$.

(ii) The sequence $\{f(x_n)\}_{n=1}^\infty$ converges to $f(x_0)$ for every sequence $\{x_n\}_{n=1}^\infty$ in X which converges to x_0.

Fall 2014

1. Let $\{f_n\}_{n=1}^{\infty}$ be a sequence of continuous functions on $\mathbb{R}$ for which the series $\sum_{n=1}^{\infty} f_n$ converges uniformly. Prove that the sum function $f := \sum_{n=1}^{\infty} f_n$ is also continuous.
2. Let I be an index set and $a : I \to (0, \infty)$.
   (a) Show that if
   $$\sum_{i \in I} a(i) := \sup_{J \subset I, J \text{ finite}} \sum_{i \in J} a(i) < \infty,$$
   then I is countable.
   (b) Suppose $I = \mathbb{Q}$, and that $\sum_{q \in \mathbb{Q}} a(q) < \infty$. Show that the function f, defined for all $x \in \mathbb{R}$ by
   $$f(x) := \sum_{q \in \mathbb{Q}, q \leq x} a(q),$$
   is continuous at x if and only if $x \notin \mathbb{Q}$.

Fall 2013

1. Let $f(x)$ denote the series $\displaystyle\sum_{n=1}^{\infty} \frac{nx^2}{n^3 + x^3}$:

(a) Prove that this series does **not** converge uniformly on $[0, +\infty)$;

(b) Prove that $f(x)$ is continuous on $[0, +\infty)$.

Spring 2013

1. Suppose $\{a_n\}$ is a sequence of real numbers and define $c_n = \dfrac{a_1 + \cdots + a_n}{n}$.

   (a) Prove that if $\lim_{n\to\infty} a_n = L$, then $\lim_{n\to\infty} c_n = L$ also.
   (b) Is the converse to part (a) true? Give either a proof or counterexample.

Fall 2012

1. Suppose $f : \mathbb{R} \to \mathbb{R}$ is continuous and $\lim_{x\to\pm\infty} f(x) = 0$. Prove that f is uniformly continuous.