Theorem ("Converse of Hölder")

Suppose $1 \leq p, q \leq \infty$ are conjugate exponents. Given any measurable function f,

$$\|f\|_p = \sup \left| \int fg \right|$$

where the supremum is taken over all measurable functions g such that $\|g\|_q = 1$ and $\int fg$ exists. In particular:

(i) If $f \in L^p$, then $\|f\|_p = \sup\limits_{\|g\|_q = 1} \left| \int fg \right|$

(ii) Suppose f integrable on all sets of finite measure, and

$$M := \sup_{\substack{\|g\|_q = 1 \\ g \text{ simple}}} \left| \int fg \right| < \infty$$

then $f \in L^p$ and $\|f\|_p = M$.

Proof:

(i): It follows immediately from Hölder's inequality that

$$\sup_{\|g\|_q = 1} \left| \int fg \right| \leq \|f\|_p$$

It thus suffice to show that $\sup\limits_{\|g\|_q = 1} \left| \int fg \right| \geq \|f\|_p$.

- We clearly may assume that $\|f\|_p = 1$ (if not simply divide both sides by $\|f\|_p$). We will achieve our objective by exhibiting a $g \in L^q$ with $\|g\|_q = 1$ such that $\int fg = 1$.

- Write $f(x) = |f(x)| e^{i\theta(x)}$.

  - $\underline{1 < q \leq \infty}$ ($\underline{\&\ 1 \leq p < \infty}$):

    Define $g(x) := e^{-i\theta(x)} |f(x)|^{p-1}$.

    Since $q(p-1) = p$ it follows that $\|g\|_q = 1$ and

    $$\int f(x) g(x) = \int |f(x)|^p = 1 .$$

  - $\underline{q = 1}$ ($\underline{\&\ p = \infty}$): Let $\varepsilon > 0$ and E denote a set of finite positive measure where $|f(x)| \geq \|f\|_\infty - \varepsilon = 1 - \varepsilon$

    Define $g(x) := e^{-i\theta(x)} \dfrac{\chi_E(x)}{m(E)}$.

    It follows that $\|g\|_1 = 1$ and $\int fg = \frac{1}{m(E)} \int_E |f| \geq 1 - \varepsilon$.

    Since $\varepsilon > 0$ was arbitrary the result follows.

<u>(ii)</u>: Here we recall that we can find a sequence $\{\varphi_n\}$ of simple functions so that $|\varphi_n| \leq |f|$ with $\varphi_n(x) \to f(x)$ for a.e. x.

We again write $f(x) = |f(x)|\, e^{i\theta(x)}$.

- If $1 < q \leq \infty$ (and hence $1 \leq p < \infty$) we define

$$g_n(x) := e^{-i\theta(x)} \frac{|\varphi_n(x)|^{p-1}}{\|\varphi_n\|_p^{p-1}} . \quad \text{(note } g_n \text{ simple)}$$

As before $\|g_n\|_q = 1$. It follows that

$$\|f\|_p \leq \liminf_{n\to\infty} \|\varphi_n\|_p \qquad \text{Fatou's Lemma}$$

$$\boxed{\text{definition of } g_n} \qquad = \liminf_{n\to\infty} \int |\varphi_n g_n|$$

$$\leq \liminf_{n\to\infty} \int |f g_n| = \liminf_{n\to\infty} \int f g_n \leq M .$$

Since $\|f\|_p \geq M$ (by Hölder) the result follow in this case.

- If $q = 1$ (and hence $p = \infty$) and $\varepsilon > 0$ we consider any set E with finite measure for which $|f(x)| \geq M + \varepsilon$.

If $m(E) > 0$, we defie

$$g(x) := e^{-i\theta(x)} \frac{\chi_E(x)}{m(E)} . \quad \text{(note } g \text{ simple)}$$

It follows that $\|g\|_1 = 1$ and $\int fg = \frac{1}{m(E)} \int_E |f| \geq M + \varepsilon$ $\lightning$.

This contradiction implies that $m(E) = 0$ and hence that $\|f\|_\infty \leq M$.

Since $\|f\|_\infty \geq M$ clearly holds the result follows. $\square$.