Math 4110/6110

Problem Set 3: Measures

1. (a) Prove that the intersection of any two σ-algebras itself forms a σ-algebra.
   (b) Let X be an uncountable set and $\mathcal{A} = \{E \subseteq X : E \text{ is countable or } E^c \text{ is countable}\}$.
      i. Verify that $\mathcal{A}$ is a σ-algebra, called the *σ-algebra of countable or co-countable sets.*
      ii. Verify that the function μ on $\mathcal{M}$ defined by $\mu(E) = 0$ if E is countable and $\mu(E) = 1$ if E^c is countable is a measure on $\mathcal{A}$.
   (c) Let X be an infinite set. Defined $\mu(E) = 0$ if E is finite and $\mu(E) = \infty$ if E is infinite. Verify that μ is a finitely additive measure on $\mathcal{P}(X)$, but not a measure.
   (d) Let $(X, \mathcal{M})$ be a measurable space endowed with a measure μ and $E \in \mathcal{M}$. Define $\mu_E(F) = \mu(E \cap F)$ for all $F \in \mathcal{M}$. Prove that μ_E defines a measure on $\mathcal{M}$.

2. Prove that if μ is a measure on a σ-algebra $\mathcal{M}$, then

$$\mu(A \cup B) + \mu(A \cap B) = \mu(A) + \mu(B)$$

   for any two sets $A, B \in \mathcal{M}$.

3. Suppose that $A \subseteq E \subseteq B$, where A and B are Lebesgue measurable subsets on $\mathbb{R}^n$.
   (a) Prove that if $m(A) = m(B) < \infty$, then E is Lebesgue measurable.
   (b) Give an example showing that the same conclusion does not hold if A and B have infinite measure.

4. Suppose A and B are a pair of compact subsets of $\mathbb{R}^n$ with $A \subseteq B$, and let $a = m(A)$ and $b = m(B)$. Prove that for any c with $a < c < b$, there is a compact set E with $A \subseteq E \subseteq B$ and $m(E) = c$.

   Hint: If $n = 1$ and E is a measurable subset of $[0, 1]$, consider $m(E \cap [0, t])$ as a function of t.

5. If I is a bounded interval and $\alpha \in (0, 1)$, let us call the open interval with the same midpoint as I and length equal to α times the length of I the "open middle αth" of I. If $\{\alpha_j\}_{j=1}^{\infty}$ is any sequence of numbers in $(0, 1)$, then, we can define a decreasing sequence $\{K_j\}$ of closed sets as follows: $K_0 = [0, 1]$, and K_j is obtained by removing the the open middle α_jth from each of the intervals that make up K_{j-1}. The resulting limiting set $K = \bigcap_{j=1}^{\infty} K_j$ is called a **generalized Cantor set**.
   (a) Suppose $\{\alpha_j\}_{j=1}^{\infty}$ is any sequence of numbers in $(0, 1)$.
      i. Prove that $\prod_{j=1}^{\infty}(1 - \alpha_j) > 0$ if and only if $\sum_{j=1}^{\infty} \alpha_j < \infty$.
      ii. Given $\beta \in (0, 1)$, exhibit a sequence $\{\alpha_j\}$ such that $\prod_{j=1}^{\infty}(1 - \alpha_j) = \beta$.
   (b) Given $\beta \in (0, 1)$, construct an open set G in $[0, 1]$ whose boundary has Lebesgue measure β.

   Hint: Every closed nowhere dense set is the boundary of an open set.