Definition

An inner product space that is complete with respect to the norm

$$\|x\| = \langle x, x\rangle^{1/2}$$

is called a Hilbert space.

(Note: Hilbert spaces are special cases of Banach spaces).

Examples

In a precise sense these are the "only two" examples!

① $\mathbb{C}^n$ with $\langle x,y\rangle = \sum_{j=1}^{n} x_j \overline{y_j}$ $\forall$ $x=(x_1,\dots,x_n), y=(y_1,\dots,y_n)\in\mathbb{C}^n$.

② $\ell^2(\mathbb{Z})$ with $\langle x,y\rangle = \sum_{j=-\infty}^{\infty} x_j\overline{y_j}$ $\forall$ $x=\{x_j\}_{j\in\mathbb{Z}}, y=\{y_j\}_{j\in\mathbb{Z}} \in \ell^2(\mathbb{Z})$.

sequences of complex numbers $\{x_j\}_{j=-\infty}^{\infty}$ such that $\sum_{j=-\infty}^{\infty} |x_j|^2 < \infty$.

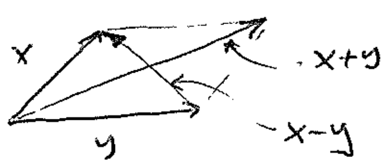
③ $L^2(\mathbb{R}^n)$ or more generally $L^2(X)$ for any $X\in\mathcal{M}(\mathbb{R}^n)$ with

$$\langle f,g\rangle = \int_X f\overline{g} \quad \forall\, f,g\in L^2(X).$$

completeness not actually needed!

Propn 2 (Parallelogram Law) Let H be a Hilbert space, then

$$\|x+y\|^2 + \|x-y\|^2 = 2\left(\|x\|^2 + \|y\|^2\right) \quad \forall\, x,y\in H.$$

[


"The sum of the □'s of the diagonals of a parallelogram is the sum of the □'s of the four sides".
]

Proof of Proposition 2

Simply sum together the two formulae:

$$\|x \pm y\|^2 = \|x\|^2 \pm 2\,\mathrm{Re}\,\langle x, y\rangle + \|y\|^2 . \qquad \square$$

- **Propn 3** (Pythagorean Theorem) again completeness not actually needed

Let H be a Hilbert space. If $x, y \in H$ and $\langle x, y\rangle = 0$, then

$$\|x+y\|^2 = \|x\|^2 + \|y\|^2 .$$

Proof:

$$\|x+y\|^2 = \langle x+y, x+y\rangle = \|x\|^2 + 2\,\mathrm{Re}\,\langle x, y\rangle + \|y\|^2$$
$$= \|x\|^2 + \|y\|^2 \text{ if } \langle x, y\rangle = 0 . \qquad \square$$

- Corollary: If $x_1, \ldots, x_n \in H$ and $\langle x_j, x_k\rangle = 0$ whenever $j \neq k$, then

$$\|x_1 + \cdots + x_n\|^2 = \|x_1\|^2 + \cdots + \|x_n\|^2 .$$

- Defn:

We say that $x, y \in H$ are orthogonal and write $x \perp y$ if $\langle x, y\rangle = 0$.

Closed subspaces and orthogonal projections: Let H be Hilbert space

- A linear subspace M of H is a subset of H that satisfies

$$ax+by \in M \quad \text{whenever} \quad x,y \in M \ \& \ a,b \in \mathbb{C}.$$

(M is also a vector space)

- M is closed if whenever $\{x_n\} \subseteq M$ converges to an element $x \in H$ then $x \in M$.

(in this case M is itself a Hilbert space)

Note: In a "finite dimensional" Hilbert space every subspace is closed.
¿ Can you think of an example of a subspace that's not closed?
($C_c \subseteq L^2(\mathbb{R}^n)$ not closed).

- Given any subset $M \subseteq H$, we define

$$M^{\perp} = \{x \in H : \langle x,y \rangle = 0 \ \forall y \in M\}.$$

Ex: $M^{\perp}$ is always a closed subspace of H.

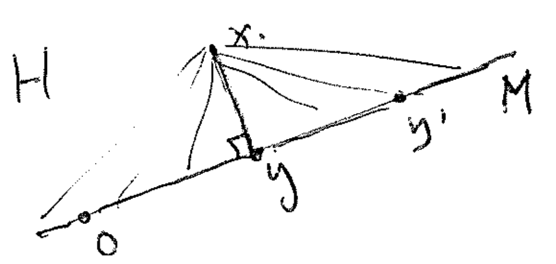
Lemma Suppose M is a closed subspace of H & $x \in H$. Then.

(i) $\exists!\, y \in M$ closest to x, in sense that

$$\|x-y\| = \inf_{y' \in M} \|x-y'\|.$$

unique

(ii) $x-y \in M^{\perp}$, i.e. $\langle x-y, y' \rangle = 0 \ \forall y' \in M$.



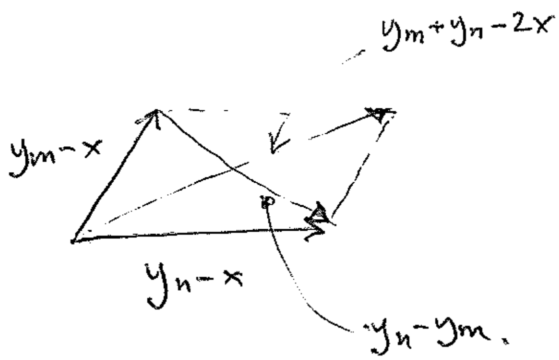
Proof

(i): Given $x \in H$, let $\delta = \inf\{\|x-y'\| : y' \in M\}$.

Let $\{y_n\} \subseteq M$ s.t. $\|x-y_n\| \longrightarrow \delta$

Parallelogram Law

$$2\left(\|y_n - x\|^2 + \|y_m - x\|^2\right) = \|y_n - y_m\|^2 + \|y_n + y_m - 2x\|^2$$



$$\Rightarrow \|y_n - y_m\|^2 = 2\|y_n - x\|^2 + 2\|y_m - x\|^2 - 4\Big\|\underbrace{\frac{y_n+y_m}{2}}_{\in M} - x\Big\|^2$$

$$\le \underbrace{2\|y_n - x\|^2 + 2\|y_m - x\|^2 - 4\delta^2}_{\longrightarrow\, 0 \text{ as } m,n \to \infty}$$

Hence $\{y_n\}$ Cauchy in H, let $y := \lim_{n\to\infty} y_n$.

Hence $y \in M$ (since M closed)

& $\|x-y\| = \delta$. (Existence ✓).

(since $\|x-y_n\| \to \|x-y\|$.)

(ii): $z = x - y \in M^\perp$:

Let $u \in M$

- Want to show that $\langle z, u \rangle = 0$
- We can assume that $\langle z, u \rangle$ is real. (Why?) [If not multiply u with scalar so that it is!].

Let $f(t) = \|\underbrace{z + tu}_{x-y+tu}\|^2 = \|z\|^2 + 2t\langle z, u\rangle + t^2\|u\|^2$, $t \in \mathbb{R}$.

We know that f has a minimum at $t=0$ (otherwise $\|x - (y - tu)\| < \|x - y\|$ ↯)

(namely δ^2).

Hence $f'(0) = 0$.

Now $f'(t) = 2\langle z, u\rangle + 2t\|u\|^2$

$\Rightarrow f'(0) = 2\langle z, u\rangle$

$\Rightarrow \langle z, u\rangle = 0$.

Uniqueness in (i): Suppose y' is another element in M s.t $\|x - y'\| = \delta$.

Pythagoras says

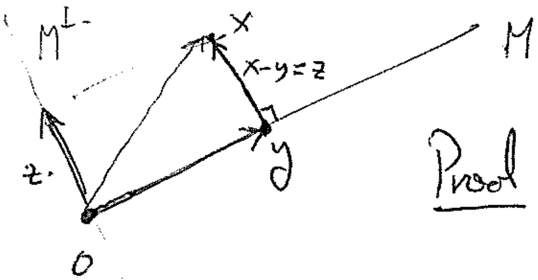
$$\|x - y'\|^2 = \|x - y\|^2 + \|y - y'\|^2$$

$$(\langle x - y, \underbrace{y - y'}_{\in M}\rangle = 0).$$

$\Rightarrow \|y - y'\| = 0 \Rightarrow y = y'$. □

Theorem

If M is a closed subspace of H, then $H = M \oplus M^{\perp}$, that is if $x \in H$, then $x = y + z$ where $y \in M$ & $z \in M^{\perp}$, moreover y & z are the unique element of M & $M^{\perp}$ whose distance to x minimal



Proof Let $x \in H$. Lemma gives $!$ $y \in M$ closest to x.

Write $x = y + \underset{\underset{z}{\|}}{(x-y)}$.

[Arguing as above we see that $\underset{\in M^{\perp}}{z}$ ~~this~~ is unique elmt closest to x.]

[Finally $x = y' + z' = y + z \Rightarrow y - y' = z' - z \in M \cap M^{\perp} = \{0\}$] . $\square$

Application:

If $y \in H$, the Schwarz inequality show that the map

$$L : x \longmapsto \langle x, y \rangle$$

is a continuous linear functional. on H. & ~~z~~ ~~$\langle x, y \rangle$~~

Defn: A linear functional on a vector space V over $\mathbb{C}$ is a mapping.

$$L : V \to \mathbb{C}.$$

s.t. $L(ax + by) = a L(x) + b L(y)$ $\forall a, b \in \mathbb{C}$ & $x, y \in V$.

[If $x_n \to x$ in H, then $\langle x_n, y \rangle \to \langle x, y \rangle$ in $\mathbb{C}$.]

<u>Thm</u> (Riesz Representation Theorem)

If L is a <u>continuous linear functional</u> on H, then $\exists!\, y \in H$ s.t.

$$L(x) = \langle x, y \rangle \quad \forall x \in H.$$

<u>Proof</u>: Uniqueness easy: If $\langle x, y' \rangle = \langle x, y \rangle \ \forall x$, then ~~in part~~ $\Rightarrow \langle x, y'-y \rangle = 0 \ \forall x$ & in particular $\langle y'-y, y'-y \rangle = 0 \Rightarrow \|y'-y\|^2 = 0 \Rightarrow y' = y$.

If $Lx = 0 \ \forall x$, take $y = 0$.

Otherwise define $M = \{x : Lx = 0\}$.

<u>Note</u>: • M is a <u>closed</u> subspace of H.

$\uparrow$ L is conts.

• $\exists\, z \in M^\perp$ with $\|z\| = 1$. (by <u>Thm</u>)

Put $u = (Lx)z - (Lz)x$, then <u>$u \in M$</u>.

$(Lu = 0)$.

~~Then $\langle z, u \rangle = 0 \Rightarrow$~~

Then

$$0 = \langle u, z \rangle = Lx \langle z, z \rangle - Lz \langle x, z \rangle$$

$$= Lx - \langle x, Lz\, z \rangle.$$

$\uparrow$ <u>y</u>

$\square$.