University of Georgia REU - Additive Combinatorics - Summer 2010

Probability Sheet

In this exercise sheet, we will show that random sets satisfy the Fourier analytic and size estimates necessary to prove Roth's theorem in 'random' sets. The problems here are adapted from [1].

Throughout, we will assume that W is a random subset of $\mathbb{Z}_N$ such that each $x \in \mathbb{Z}_N$ belongs to W independently with probability $p \in (0, 1]$.

Notation: We will assume that X is a random variable with discrete support. We denote the *expectation* of X by

$$\mathbf{E}(X) := \sum_x x \Pr(X = x)$$

and the variance of X by

$$\operatorname{var}(X) := \mathbf{E}\left(|X - \mathbf{E}(X)|^2\right).$$

1. **Basic properties**

   (a) Suppose that $X_1, \ldots, X_n$ are random variables with discrete support. Prove that

   $$\mathbf{E}(X_1 + \cdots + \ldots X_n) = \mathbf{E}(X_1) + \cdots + \mathbf{E}(X_n).$$

   (b) Show that $\mathbf{E}(|W|) = pN$.

   (c) Show that $\operatorname{var}(X) = \mathbf{E}(|X|^2) - \mathbf{E}(|X|)^2$.

   (d) Suppose that $X_1, \ldots, X_n$ are pairwise independent random variables with discrete support. Prove that

   $$\operatorname{var}(X_1 + \cdots + \ldots X_n) = \operatorname{var}(X_1) + \cdots + \operatorname{var}(X_n).$$

   (e) Let X be a non-negative random variable. Prove that for any $\lambda > 0$

   $$\Pr(X \geq \lambda) \leq \frac{\mathbf{E}(X)}{\lambda}.$$

   Let $t > 0$. Notice that this implies

   $$\Pr(e^{tX} \geq e^{t\lambda}) \leq \frac{\mathbf{E}(e^{tX})}{e^{t\lambda}}.$$

2. **Properties of random sets**

   Theorem 1 (Chernoff's inequality I). *Assume $X_1, \ldots X_n$ are jointly independent real valued random variables so that $|X_i - \mathbf{E}(X_i)| \leq 1$ for all $1 \leq i \leq n$. Set $X := X_1 + \ldots X_n$ and let $\sigma := \sqrt{\operatorname{var}(X)}$. Then for any $\lambda > 0$*

   $$\Pr(|X - \mathbf{E}(X)| \geq \lambda\sigma) \leq 2\max\{e^{-\lambda^2/4}, e^{-\lambda\sigma/2}\}.$$

(a) Prove that the probability that

$$\left|\frac{|W|}{N} - p\right| \le C\sqrt{\log N}\sqrt{p(1-p)/N}$$

tends to 1 as $N \to \infty$.
Hint: Take $\lambda = C\sqrt{\log N}$ *for some large constant* C.

Theorem 2 (Chernoff's inequality II)**.** *Assume* $X_1, \dots X_n$ *are jointly independent complex valued random variables so that* $|X_i - \mathbf{E}(X_i)| \le 1$ *for all* $1 \le i \le n$. *Set* $X := X_1 + \dots X_n$ *and let* $\sigma := \sqrt{\operatorname{var}(X)}$. *Then for any* $\lambda > 0$

$$\Pr(|X - \mathbf{E}(X)| \ge \lambda\sigma) \le 4\max\{e^{-\lambda^2/8}, e^{-\lambda\sigma/2\sqrt{2}}\}.$$

(b) Let $p \in (0,1)$. Suppose W is a random subset of $\mathbb{Z}_N$ such that each $x \in \mathbb{Z}_N$ belongs to W independently with probability $p \in (0,1]$.

(i) Show that for any $\lambda > 0$

$$\Pr\left(|\widehat{W}(\xi)| \ge \lambda\sigma\right) \le 4\max\{e^{-\lambda^2/8}, e^{-\lambda\sigma/2\sqrt{2}}\}$$

and hence

$$\Pr\left(|\widehat{W}(\xi)| < \lambda\sigma \text{ for all } \xi \ne 0\right) \ge 1 - 4N\max\{e^{-\lambda^2/8}, e^{-\lambda\sigma/2\sqrt{2}}\}.$$

(ii) Prove that the probability that

$$|\widehat{W}(\xi)| \le C\sqrt{\log N}\sqrt{p(1-p)/N}$$

tends to 1 as $N \to \infty$.
Hint: Take $\lambda = C\sqrt{\log N}$ *for some large constant* C *as before.*

References

[1] Terence Tao and Van Vu. *Additive combinatorics*, volume 105 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 2006.