Math 3100 Assignment 4
Subsequences and Completeness

Due at 5:00 pm on Friday the 8th of February 2019

1. Evaluate following limits or explain why they do not exist. Be sure to justify your answer.

$$\text{(a)} \quad \lim_{n\to\infty}\left(\frac{2n+1}{3-n}\right)^3 \qquad \text{(b)} \quad \lim_{n\to\infty}\left((-1)^n + \frac{1}{n}\right)$$

$$\text{(c)} \quad \lim_{n\to\infty}\frac{\cos(n)}{n^2} \qquad \text{(d)} \quad \lim_{n\to\infty}\frac{n!+n}{2^n+3n!} \qquad \text{(e)} \quad \lim_{n\to\infty}\frac{n+\log(n)}{n+1}$$

2. (a) Let $x_1 = 0$ and $x_{n+1} = \dfrac{2x_n+1}{x_n+2}$ for all $n \in \mathbb{N}$.
   i. Find x_2, x_3, and x_4.
   ii. Prove that $\{x_n\}$ converges and find the value of its limit.

   (b) Let $a_1 = \sqrt{2}$, and define
$$a_{n+1} = \sqrt{2+a_n}$$
   for all $n \geq 1$. Prove that $\lim\limits_{n\to\infty} a_n$ exists and equals 2.

   Hint: For both parts try to apply the Monotone Convergence Theorem

3. (a) Prove that if $\{a_n\}$ is increasing, then every subsequence of $\{a_n\}$ is also increasing.

   (b) Let $\{x_n\}$ be a sequence of real numbers.
   Prove that $\{x_n\}$ contains a subsequence converging to x <u>if and only if</u> for all $\varepsilon > 0$ there exist infinitely many terms from $\{x_n\}$ that satisfy $|x_n - x| < \varepsilon$.

4. Let $A, B \subseteq \mathbb{R}$ which are non-empty, bounded above.

   (a) Show that if $A \subseteq B$, then $\inf(B) \leq \inf(A) \leq \sup(A) \leq \sup(B)$.

   (b) Show that if $\sup A < \sup B$, then there must exist $b \in B$ that is an upper bound for A.

   (c) Prove that if $\sup(A) \notin A$, then there exists a sequence $\{a_n\}$ of points in A such that
$$\lim_{n\to\infty} a_n = \sup(A).$$

5. Let $\{x_n\}$ be a bounded sequence of real numbers and
$$S = \{x \in \mathbb{R} : \text{there exists a subsequence of } \{x_n\} \text{ that converges to } x\}.$$

   (a) Carefully explain why both $\sup S$ and $\inf S$ exist.
   The value of $\sup S$ is called the *limit superior* of $\{x_n\}$ is usually denoted by $\limsup\limits_{n\to\infty} x_n$, while the value of $\inf S$ is called the *limit inferior* of $\{x_n\}$ is usually denoted by $\liminf\limits_{n\to\infty} x_n$

   (b) Argue why $\lim\limits_{n\to\infty} x_n$ exists <u>if and only if</u> $\liminf\limits_{n\to\infty} x_n = \limsup\limits_{n\to\infty} x_n$.
   In this case all three share the same value.

   (c) Prove that if $\beta > \limsup\limits_{n\to\infty} x_n$, then there exists an N such that $x_n < \beta$ whenever $n > N$.

   (d) * Let $\alpha := \limsup\limits_{n\to\infty} x_n$. Prove that there exists a subsequence of $\{x_n\}$ that converges α.

Math 3100 – Homework 4 – SOLUTIONS

1. (a) Since

$$\lim_{n\to\infty} \frac{2n+1}{3-n} = \lim_{n\to\infty} \frac{2+\frac{1}{n}}{\frac{3}{n}-1} = \frac{2+0}{0-1} = -2 \quad \text{by limit laws.}$$

$$\Rightarrow \lim_{n\to\infty} \left(\frac{2n+1}{3-n}\right)^3 = \left(\lim_{n\to\infty} \frac{2n+1}{3-n}\right)^3 = (-2)^3 = \underline{\underline{-8}}.$$

(b) Suppose $\lim_{n\to\infty} \left((-1)^n + \frac{1}{n}\right)$ exists, it would then follow from limit laws that

$$(-1)^n = \left((-1)^n + \tfrac{1}{n}\right) - \left(\tfrac{1}{n}\right)$$

converges, <u>but</u> $(-1)^n$ diverges, hence $\lim_{n\to\infty} \left((-1)^n + \frac{1}{n}\right)$ <u>DNE</u>.

(c) Since $\left|\frac{\cos(n)}{n^2}\right| \le \frac{1}{n^2}$ & $\frac{1}{n^2} \to 0$ it follows from "Baby Squeeze" that $\frac{\cos(n)}{n^2} \to 0$.

(d) Since $\frac{n}{n!} \to 0$ and $\frac{2^n}{n!} \to 0$ it follows from limit laws that

$$\lim_{n\to\infty} \frac{n!+n}{2^n+3n!} = \lim_{n\to\infty} \frac{1+\frac{n}{n!}}{\frac{2^n}{n!}+3} = \frac{1+0}{0+3} = \underline{\underline{\frac{1}{3}}}.$$

(e) Since $\frac{\log(n)}{n} \to 0$ it follows from limit laws that

$$\lim_{n\to\infty} \frac{n+\log(n)}{n+1} = \lim_{n\to\infty} \frac{1+\frac{\log n}{n}}{1+\frac{1}{n}} = \frac{1+0}{1+0} = \underline{\underline{1}}.$$

2. (a) Let $x_1 = 0$ & $x_{n+1} = \dfrac{2x_n+1}{x_n+2}$ $\forall n \in \mathbb{N}$.

(i) $x_2 = \frac{1}{2}$, $x_3 = \frac{4}{5}$, $x_4 = \frac{13}{14}$

(ii) <u>Claim 1</u> $0 \le x_n \le 1$ $\forall n \in \mathbb{N}$.

<u>Proof</u> (Induction)

Base Case ($n=1$): Since $x_1 = 0 \Rightarrow 0 \le x_1 \le 1$ ✓

If $0 \le x_n \le 1$ for some given $n \in \mathbb{N}$, then.

$$\left.\begin{array}{l} \bullet\ x_{n+1} = \dfrac{\overbrace{2x_n+1}^{\ge 0}}{\underbrace{x_n+2}_{>0}} \ge 0 \\[2ex] \bullet\ x_{n+1} - 1 = \dfrac{2x_n+1}{x_n+2} - 1 = \dfrac{\overbrace{x_n-1}^{\le 0}}{\underbrace{x_n+2}_{>0}} \le 0 \end{array}\right\} 0 \le x_{n+1} \le 1 \ ✓.$$

$\square$

<u>Claim 2</u> $\{x_n\}$ is increasing

<u>Proof</u> Since

$$x_{n+1} - x_n = \frac{2x_n+1}{x_n+2} - x_n = \frac{1-x_n^2}{x_n+2} = -\frac{\overbrace{(x_n-1)}^{\le 0}\overbrace{(x_n+1)}^{>0}}{\underbrace{x_n+2}_{>0}} \ge 0$$

$\square$

Since $\{x_n\}$ is bounded & increasing it follows from MCT & order limit laws

that $x_n \to L$ for some $0 \le L \le 1$.

Since $x_{n+1} \to L$ & $\dfrac{2x_n+1}{x_n+2} \to \dfrac{2L+1}{L+2}$

it follows by uniqueness of limits that

$$L = \frac{2L+1}{L+2} \iff L^2 = 1$$

$$\iff \underline{\underline{L = 1}} \text{ or } \cancel{L = -1}$$

2 . (b) Let $a_1 = \sqrt{2}$ & $a_{n+1} = \sqrt{2+a_n}$ $\forall n \in \mathbb{N}$.

Claim 1 $\sqrt{2} \le a_n \le 2$ $\forall n \in \mathbb{N}$

Proof (Induction)

Base case ($n=1$): $a_1 = \sqrt{2}$ ✓

Suppose $\sqrt{2} \le a_n \le 2$ for some $n \in \mathbb{N}$, then

$$2 \le a_n + 2 \le 4 \Rightarrow \sqrt{2} \le \underbrace{\sqrt{a_n+2}}_{=a_{n+1}} \le \sqrt{4} = 2 \qquad \square$$

Claim 2 $\{a_n\}$ increasing.

Proof $a_{n+1} - a_n = \sqrt{2+a_n} - a_n = \dfrac{2 + a_n - a_n^2}{\sqrt{2+a_n} + a_n}$

$$= \frac{-\underbrace{(a_n - 2)}_{\le 0}\underbrace{(a_n+1)}_{>0}}{\underbrace{\sqrt{2+a_n} + a_n}_{>0}} \ge 0 \quad \forall n \in \mathbb{N} \qquad \square$$

by Claim 1

Since $\{a_n\}$ is increasing & bounded above it follow from the MCT that $a_n \to L$ for some $\sqrt{2} \le L \le 2$.

& order limit laws

Since $a_{n+1} \to L$ & $\sqrt{a_n + 2} \to \sqrt{L+2}$ (limit laws & $x_n \to x \Rightarrow \sqrt{x_n} \to \sqrt{x}$)

it follows (by uniqueness of limits) that

$$L = \sqrt{L+2}$$

$$\Rightarrow L^2 - L - 2 = 0$$

$$\Rightarrow \underline{\underline{L = 2}} \text{ or } \cancel{L = -1}$$

2. (a) Claim

If $\{a_n\}$ is increasing, then every subsequence of $\{a_n\}$ is also increasing

Proof

Since $\{a_n\}$ is increasing we know that $\underbrace{a_m \geq a_n \ \forall m > n}_{(*)}$.
(See "Lecture Notes" on increasing/decreasing sequences).
Let $\{n_k\}_{k=1}^{\infty}$ be an arbitrary strictly increasing sequence of natural numbers, it follows that $n_{k+1} > n_k \ \forall k \in \mathbb{N}$ and that $\{a_{n_k}\}_{k=1}^{\infty}$ is an arbitrary subsequence of $\{a_n\}_{n=1}^{\infty}$.

Since $n_{k+1} > n_k \implies a_{n_{k+1}} \geq a_{n_k}$ by $(*)$ $\square$.

(b) Claim

$\{x_n\}$ contains a subsequence converging to x $\iff$ $\forall \varepsilon > 0 \ \exists$ infinitely many terms from $\{x_n\}$ for which $|x_n - x| < \varepsilon$.

Proof

$(\Rightarrow)$ Suppose $\{x_{n_k}\}$ is a subseq of $\{x_n\}$ with $\lim_{k\to\infty} x_{n_k} = x$.
Let $\varepsilon > 0$. Since $x_{n_k} \to x$ we know that all but finitely many terms from $\{x_{n_k}\}$ must lie in $(x-\varepsilon, x+\varepsilon)$, this gives us infinitely many terms from $\{x_{n_k}\}$ (which also terms in $\{x_n\}$) that satisfy $|x_{n_k} - x| < \varepsilon$.

$(\Leftarrow)$ For any $k \in \mathbb{N}$ $\exists \, n_k$ so that $|x_{n_k} - x| < \frac{1}{k}$
& $n_{k+1} > n_k \ \forall k \in \mathbb{N}$
(taking $\varepsilon = \frac{1}{k}$ and using "infinitely many term" to ensure second claim)

It follows from "Baby squeeze" that $\lim_{k\to\infty} x_{n_k} = x$. $\square$

1. (a) (i) Let $a \in A$. Since $\sup A$ is an upper bound for A and $\inf A$ is a lower bound for A it follows that

$$\inf A \le a \le \sup A \implies \underline{\inf A \le \sup A}.$$

(ii) Since $\sup B$ is an upper bound for B & $A \subseteq B$

$\implies \underline{a \le \sup B \ \forall a \in A}$

$\implies \sup A \le \sup B$ since $\sup A$ is the <u>least</u> upper bound for A

(iii) The fact that $\inf B \le \inf A$ follows in a similar manner to (ii) above <u>OR</u> one can use (ii) together with the fact that $\inf B = -\sup(-B)$ [see class notes].

(b) <u>Claim</u>

If $\sup A < \sup B$, then $\exists\, b \in B$ such that $a \le b \ \forall a \in A$.

<u>Proof</u>

Since $\varepsilon = \sup B - \sup A > 0$ we know $\exists\, b \in B$ such that $\underbrace{\sup B - \varepsilon}_{= \sup A} < b$. Since $a \le \sup A \ \forall a \in A$

$\implies a \le b \ \forall a \in A$.

("Since $\sup A$ is an upper bound for A so is b"). $\square$

(c) If $s = \sup A \notin A$, then $\exists\, a_1 \in A$ such that $s - 1 < a_1 < s$ $(\varepsilon = 1)$

More generally, for any $n \in \mathbb{N}$ $\exists\, a_n \in A$ s.t. $s - \frac{1}{n} < a_n < s$.

It follows from the "squeeze theorem" that the sequence $\{a_n\}$ satisfies $\lim_{n \to \infty} a_n = s$. $\square$

5. Let $\{x_n\}$ be bounded &

$$S = \{x \in \mathbb{R} : \{x_n\} \text{ contains a subsequence which converges to } x\}$$

"S is the collection of all subsequential limits of $\{x_n\}$"

(a) Let $M > 0$ be such that $|x_n| \le M \ \forall n \in \mathbb{N}$.

Since $\{x_n\}$ bded it follows from BW that $\{x_n\}$ contains a subsequence that converges to some $x \in \mathbb{R}$ & hence that

$S \neq \phi$ [S not equal to empty set].

Since $|x_n| \le M$ it follows that every subseq of $\{x_n\}$ is also bounded between $-M$ & M & hence (by order limit laws) that if a subsequence converges to x, then $-M \le x \le M$.

i.e. $x \in S \Rightarrow -M \le x \le M$.

In other words S is bounded.

Since S is non-empty & bounded, it follows from the AoC that $\sup S$ & $\inf S$ both exist.

(b). <u>Claim</u>! If $\{x_n\}$ is bounded, then

$$\lim_{n\to\infty} x_n = x \iff \limsup_{n\to\infty} x_n = \liminf_{n\to\infty} x_n = x.$$

In the proof of this claim below we make use of the following

Proved in Class

<u>Thm</u>

Every bounded <u>divergent</u> sequence contains at least two subsequence that converge to different limits.

Proof of Claim 1

($\Rightarrow$): If $x_n \to x$, then every subsequence of $\{x_n\}$ also converges to x & hence $S = \{x\}$.

It follows that $\sup(S) = x$ & $\inf(S) = x$.

($\Leftarrow$): If $\sup(S) = \inf(S) = x$, then $S = \{x\}$.

It follows that every convergent subsequence of $\{x_n\}$ converges to x.

It follows from the Theorem above that $x_n \to x$, since $\{x_n\}$ is bounded but does not contain two subsequences that converge to different limits. $\square$

(c) **Claim 2** Let $\alpha := \limsup_{n\to\infty} x_n$.

If $\beta > \alpha$, then $\exists N$ such that $x_n < \beta \;\forall n > N$.

"$\{x_n\}$ is eventually $< \beta$".

Proof of Claim 2

Suppose not, then $\exists$ infinitely many x_n's with $x_n \geq \beta$.

Since this subsequence of $\{x_n\}$ is also bounded above it follows from BW & order limit laws that $\{x_n\}$ admits a subsequence that converges to some x with $x \geq \beta$. [i.e. $\exists x \in S$ with $x \geq \beta$].

This contradicts the fact that α is an upper bound for S. $\square$

(d) **Claim 4** If $\alpha := \limsup_{n\to\infty} x_n$, then $\alpha \in S$.

i.e. $\exists$ subseq of $\{x_n\}$ that converges to α.

We will give two proof of Claim 4.

~~Proof of Claim 4: Take I~~

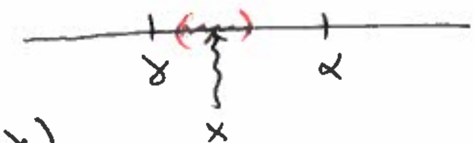
This proof use the following:

Claim 3 If $\alpha := \limsup_{n\to\infty} x_n$ & $\gamma < \alpha$, then there exists infinitely many x_n's with $x_n > \gamma$.

" $x_n > \gamma$ frequently"

Proof of Claim 3

Since $\alpha = \sup(S)$ & $\gamma < \alpha$ it follows that $\exists\, x \in S$ with $\gamma < x \leq \alpha$.



$\Updownarrow$ There is a subseq of $\{x_n\}$ that converges to x

(taking $\varepsilon = x - \gamma$)

it follows that (from Q3(b) $\Rightarrow$) that $\exists$ infinitely many x_n's with $x_n > \gamma$. $\square$

Back to proof of Claim 4: It follows from Claims 2 & 3 (with $\beta = \alpha + \varepsilon$ & $\gamma = \alpha - \varepsilon$) that for any $\varepsilon > 0$ $\exists$ infinitely many x_n's with $\alpha - \varepsilon < x_n < \alpha + \varepsilon$, which implies (by Q3(b) $\Leftarrow$) that $\exists$ subseq of $\{x_n\}$ that converges to α. $\square$.

Proof of Claim 4: Take II:

- $\exists\ \alpha_1 \in S$ with $|\alpha - \alpha_1| < \frac{1}{2}$ & x_{n_1} with $|x_{n_1} - \alpha_1| < \frac{1}{2}$ & hence $\exists\ x_{n_1}$ with $|x_{n_1} - \alpha| < 1$
- Suppose $x_{n_1}, x_{n_2}, \ldots, x_{n_{k-1}}$ have been selected with $n_1 < n_2 < \cdots < n_{k-1}$. $\exists\ x_{n_k}$ with $n_k > n_{k-1}$ such that $|x_{n_k} - \alpha| < \frac{1}{k}$.

(since $\exists\ \alpha_k \in S$ with $|\alpha - \alpha_k| < \frac{1}{2k}$ & infinitely many x_n's with $|x_n - \alpha_k| < \frac{1}{2k}$)

(by Q3(b) $\Rightarrow$)

It follows from the Squeeze Thm that $x_{n_k} \to \alpha$. $\square$