Math 8100 Assignment 3
Lebesgue measurable functions

Due date: Friday the 12th of September 2014

1. Let $\chi_{[0,1]}$ be the characteristic function of $[0, 1]$. Show that there is no function f satisfying $f = \chi_{[0,1]}$ almost everywhere which is also continuous on all of $\mathbb{R}$.

2. We say that a function $f : \mathbb{R}^n \to \mathbb{R}$ is *upper semicontinuous* at a point x in $\mathbb{R}^n$ if

$$f(x) \geq \limsup_{y \to x} f(y).$$

   Prove that if f is upper semicontinuous at every point x in $\mathbb{R}^n$, then f is Borel measurable.

3. Let $\{f_n\}$ be a sequence of measurable functions on $\mathbb{R}^n$. Prove that

$$\{x \in \mathbb{R}^n : \lim_{n\to\infty} f_n(x) \text{ exists}\}$$

   is a measurable set.

4. Let $\{f_n\}$ be a sequence of measurable functions on $[0, 1]$ with $|f_n(x)| < \infty$ for a.e. x. Show that there exists a sequence of positive real numbers $\{a_n\}$ such that $a_n f_n \to 0$ a.e.

   Hint: Pick a_n such that $m(\{x : a_n|f_n(x)| > 1/n\}) < 2^{-n}$, and apply the Borel-Cantelli lemma.

5. Recall that the **Cantor set** $\mathcal{C}$ is the set of all $x \in [0, 1]$ that have a ternary expansion $x = \sum_{k=1}^{\infty} a_k 3^{-k}$ with $a_k \neq 1$ for all k. Consider the function

$$f(x) = \sum_{k=1}^{\infty} b_k 2^{-k} \quad \text{where} \quad b_k = a_k/2.$$

   (a) Show that f is well defined and continuous on $\mathcal{C}$, and moreover $f(0) = 0$ as well as $f(1) = 1$.

   (b) Prove that there exists a continuous function that maps a measurable set to a non-measurable set.

6. Let us examine the map f defined in Question 5 even more closely. One readily sees that if $x, y \in \mathcal{C}$ and $x < y$, then $f(x) < f(y)$ unless x and y are the two endpoints of one of the intervals removed from $[0, 1]$ to obtain $\mathcal{C}$. In this case $f(x) = \ell 2^m$ for some integers ℓ and m, and $f(x)$ and $f(y)$ are the two binary expansions of this number. We can therefore extend f to a map $F : [0, 1] \to [0, 1]$ by declaring it to be constant on each interval missing from $\mathcal{C}$. F is called the **Cantor-Lebesgue function**.

   (a) Prove that F is non-decreasing and continuous.

   (b) Let $G(x) = F(x) + x$. Show that G is a bijection from $[0, 1]$ to $[0, 2]$.

   (c) i. Show that $m(G(\mathcal{C})) = 1$.

      ii. By considering rational translates of $\mathcal{N}$ (the non-measurable subset of $[0, 1]$ that we constructed in class), prove that $G(\mathcal{C})$ necessarily contains a (Lebesgue) non-measurable set $\mathcal{N}'$.

      iii. Let $E = G^{-1}(\mathcal{N}')$. Show that E is Lebesgue measurable, but not Borel.

   (d) Give an example of a measurable function φ such that $\varphi \circ G^{-1}$ is not measurable.

   Hint: Let φ be the characteristic function of a set of measure zero whose image under G is not measurable.

Extra Challenge Problems

Not to be handed in with the assignment

1. Question 6d above supplies us with an example that if f and g are Lebesgue measurable, then it does not necessarily follow that $f \circ g$ will be Lebesgue measurable, even if g is assumed to be continuous.

   Prove that if f is Borel measurable, then $f \circ g$ will be Lebesgue or Borel measurable whenever g is.

2. Let f be a measurable function on $[0, 1]$ with $|f(x)| < \infty$ for a.e. x. Prove that there exists a sequence of continuous functions $\{g_n\}$ on $[0, 1]$ such that $g_n \to f$ for a.e. $x \in [0, 1]$.