OSCILLATORY INTEGRALS IN ONE DIMENSION

MATH 8130 – SPRING 2010

In this note we give an overview of the theory of oscillatory integrals in one dimension, which gives an essentially complete description of the behavior of integrals of the form

$$I(\lambda) = \int_a^b e^{i\pi\lambda\Phi(x)}\Psi(x)\,dx, \qquad \lambda > 0,$$

as $\lambda \to \infty$, where Φ and Ψ are smooth functions. The behavior of $I(\lambda)$ is governed by three basic principles: *localization*, *scaling*, and *asymptotics*. We shall present these respective principles as three propositions: the first of these can be thought of as a principle of non–stationary phase, the second is one of van der Corput's lemmas, and the third is a formulation of the method of stationary phase.

Proposition 1 (Principle of Non-Stationary Phase)**.** *Suppose Ψ has compact support in (a,b) and Φ' is never vanishes, then for all $N \geq 0$ we have*

$$|I(\lambda)| \leq C_{N,\Phi,\Psi}\lambda^{-N}.$$

Remark. If we do not assume that Ψ vanishes near the endpoints of the interval $[a,b]$ then the best estimate we can obtain for $I(\lambda)$ is $O(\lambda^{-1})$, as the example

$$\int_a^b e^{i\pi\lambda x}\,dx = \frac{e^{i\pi\lambda b} - e^{i\pi\lambda a}}{i\pi\lambda}$$

shows. However, in the "periodic" case, i.e, if we have $\Phi^{(k)}(a) = \Phi^{(k)}(b)$ and $\Psi^{(k)}(a) = \Psi^{(k)}(b)$, we again, as in Proposition 1, obtain the rapid decrease of $I(\lambda)$.

Proposition 2 (van der Corput's Lemma)**.** *Suppose Φ is real–valued and $|\Phi^{(k)}(x)| \geq 1$ for all $x \in (a,b)$, then*

$$|I(\lambda)| \leq c_k\lambda^{-1/k}\Big[|\Psi(b)| + \int_a^b |\Psi'(x)|\,dx\Big],$$

whenever (i) $k = 1$ and $\Phi''(x)$ has at most one zero, or (ii) $k \geq 2$.

Remark. The bound c_k is independent of Φ and λ. A very useful feature of this result (in the case where $\Psi = 1$) is that the estimates are independent of the interval $[a,b]$.

Of course if Φ is completely stationary then the best one can do is $|I(\lambda)| \leq (b-a)\|\Psi\|_\infty$.

Proposition 3 (Principle of Stationary Phase)**.** *Suppose Φ is real–valued, $\Phi'(x_0) = 0$, while $\Phi''(x_0) \neq 0$. If Ψ is supported in a sufficiently small neighborhood of x_0, then*

$$I(\lambda) = a_0\lambda^{-1/2}e^{i\pi\lambda\Phi(x_0)} + O(\lambda^{-3/2})$$

as $\lambda \to \infty$, where $a_0 = e^{i\pi/4}\Big(\frac{2}{\Phi''(x_0)}\Big)^{1/2}\Psi(x_0)$, and the bounds occuring in the error term depend on upper bounds for finitely many derivatives of Φ and Ψ on the $\operatorname{supp}\Psi$*, the size of this support, and on a lower bound for $|\Phi''(x_0)|$.*