Math 3100 Assignment 1

Preliminaries

Due at the beginning of class on Wednesday the 16th of January 2019

1. (Induction)

   (a) Prove, by induction, that the following identities hold for all $n \in \mathbb{N}$:

   i. $1 + 2 + \cdots + n = \dfrac{n(n+1)}{2}$

   ii. $1^3 + 2^3 + \cdots + n^3 = \left(\dfrac{n(n+1)}{2}\right)^2$

   It seems like the two identities above must be closely related to each other.
   Challenge: Can you give a geometric proof of the second identity using only the first?

   (b) Prove, by induction, that the following inequalities hold for all $n \in \mathbb{N}$:

   i. $2n + 1 \leq 3n^2$

   ii. $2n^2 - 1 \leq n^3$

   Hint: The validity of the first inequality should help you establish the second.

2. (Absolute Value and Inequalities)

   (a) i. If $|x| < 2$, what can you say about $|x - 3|$?

   ii. If $|x - 2| < 1$, what can you say about $|x + 3|$?

   iii. If $|x + 1| < 1/2$, what can you say about $|x|^{-1}$?

   (b) Use the triangle inequality to show that

   $$\big||x| - |y|\big| \leq |x - y|$$

   for all $x, y \in \mathbb{R}$. This inequality is often refered to as the *reverse triangle inequailty.*
   Hint: Start by writing $x = (x - y) + y$.

3. (Irrational numbers are dense in the reals)

   (a) Let $q \in \mathbb{Q}$, that is let q be a rational number. Prove that $q + \sqrt{2}$ must be irrational.

   (b) Prove that given any two real numbers $x < y$, there exists an irrational number z such that $x < z < y$.
   Hint: Try to deduce this as a consequence of the denseness of rationals in the real, and part (a), after considering the real numbers $x - \sqrt{2}$ and $y - \sqrt{2}$.

Math 3100 - HOMEWORK 1 - SOLUTIONS

1 (a) (i) Claim: $1+2+\cdots+n \underset{(*)}{=} \dfrac{n(n+1)}{2} \quad \forall n \in \mathbb{N}$

Proof

Base case $(n=1)$: LHS $=1$ & RHS $= \dfrac{1(1+1)}{2} = 1$ ✓

Suppose $(*)$ holds for some $n \in \mathbb{N}$, then

$$1+2+\cdots+n+n+1 = (1+2+\cdots+n)+n+1$$
$$\underset{\text{Ind Hyp}}{=} \frac{n(n+1)}{2}+(n+1) = \frac{n(n+1)+2(n+1)}{2} = \frac{(n+1)(n+2)}{2} \qquad \square$$

(ii) Claim: $1^3+2^3+\cdots+n^3 \underset{(*)}{=} \dfrac{n^2(n+1)^2}{4} \quad \forall n \in \mathbb{N}$

Proof

Base Case $(n=1)$: LHS $= 1^3 = 1$ & RHS $= \dfrac{1^2(1+1)^2}{4} = 1$ ✓

Suppose $(*)$ holds for some $n \in \mathbb{N}$, then

$$1^3+2^3+\cdots+n^3+(n+1)^3 = (1^3+2^3+\cdots+n^3)+(n+1)^3$$
$$\underset{\text{Ind Hyp}}{=} \frac{n^2(n+1)^2}{4}+(n+1)^3$$
$$= \frac{n^2(n+1)^2+4(n+1)^3}{4}$$
$$= \frac{(n+1)^2(n^2+4(n+1))}{4} = \frac{(n+1)^2(n+2)^2}{4} \qquad \square$$

(b) (i) Claim: $2n+1 \underset{(*)}{\leq} 3n^2 \quad \forall n \in \mathbb{N}$

Proof

Base Case $(n=1)$: LHS $= 2(1)+1 = 3$ & RHS $= 3(1)^2 = 3$ ✓

Suppose $(*)$ holds for some $n \in \mathbb{N}$, then

$$2(n+1)+1 = 2n+1+2 \underset{\text{Ind Hyp}}{\leq} 3n^2+2 \leq 3n^2+6n+3 = 3(n+1)^2 \qquad \square$$

(ii) Claim: $2n^2-1 \le n^3 \quad \forall n \in \mathbb{N}$ $(*)$

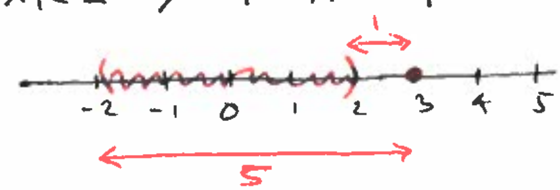
Proof

Base Case $(n=1)$: $LHS = 2(1)^2-1 = 1$ & $RHS = 1^3 = 1$ ✓

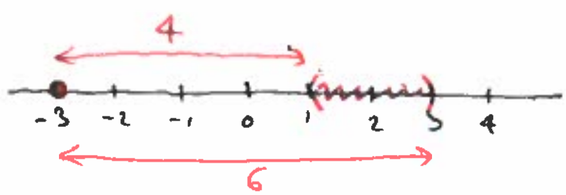
Suppose $(*)$ holds for some $n \in \mathbb{N}$, then

$$\begin{aligned} 2(n+1)^2 - 1 &= 2n^2+4n+1 \\ &= \underbrace{(2n^2-1)}_{\le\, n^3 \text{ (Ind Hyp)}} + \underbrace{(2n+1)}_{\le\, 3n^2 \text{ (Q1(b)i)}} + (2n+1) \\ &\le n^3 + 3n^2 + 2n+1 \\ &\le n^3+3n^2+3n+1 = (n+1)^3 . \qquad \square \end{aligned}$$

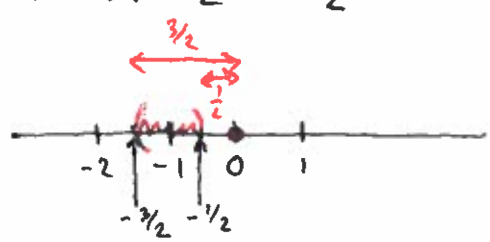
2. (a) (i) $|x|<2 \Rightarrow 1 < |x-3| < 5$



(ii) $|x-2|<1 \Rightarrow 4 < |x+3| < 6$



(iii) $|x+1| < \frac{1}{2} \Rightarrow \frac{1}{2} < |x| < \frac{3}{2} \Rightarrow \frac{2}{3} < |x|^{-1} < 2$



(b) Claim: $\big||x|-|y|\big| \le |x-y| \quad \forall x,y \in \mathbb{R}$

Proof Writing $x = (x-y)+y \overset{\Delta\text{-inequality}}{\Longrightarrow} |x| \le |x-y| + |y| \Rightarrow |x|-|y| \le |x-y|$

Interchanging role of x & y gives $\underbrace{|y|-|x|}_{=-(|x|-|y|)} \le |y-x| = |x-y|$

Since $|x|-|y|$ & $-(|x|-|y|) \le |x-y| \Longrightarrow \big||x|-|y|\big| \le |x-y|$ $\square$

3. (a)

Claim

If q is rational, then $q+\sqrt{2}$ is irrational.

Proof [Contradiction]

We will use the fact that $\sqrt{2}$ is irrational & the addition fact that the difference of two rationals is always rational.

Suppose $q+\sqrt{2}$ were rational, then

$$\sqrt{2} = (q+\sqrt{2}) - q \in \mathbb{Q} \quad \lightning \text{ Contradiction. } \square$$

(b)

i.e. z is irrational

Claim: Given any $x, y \in \mathbb{R}$ with $x<y$, $\exists\, z \in \mathbb{R} \setminus \mathbb{Q}$ such that $x < z < y$.

Proof

We will use the fact that "$\mathbb{Q}$ is dense is $\mathbb{R}$", proved in class.

Consider the real numbers $x-\sqrt{2}$ & $y-\sqrt{2}$.

Since $x-\sqrt{2} < y-\sqrt{2}$ we know, since "$\mathbb{Q}$ is dense in $\mathbb{R}$" that $\exists$ rational q such that $x-\sqrt{2} < q < y-\sqrt{2}$

but this implies $x < \underbrace{q+\sqrt{2}}_{= z} < y$.

Finally we note that from part (a) we know $q+\sqrt{2}$ is irrational

$\square$