Convergence in Norm & Further Remarks

In the last lecture we introduced the Fourier transform on $L^1(\mathbb{R}^n)$, namely

$$\hat{f}(\xi) = \int_{\mathbb{R}^n} f(x) e^{-2\pi i x\cdot\xi}\,dx.$$

Let

$$X = \{ f \in L^1(\mathbb{R}^n) : \hat{f} \in L^1(\mathbb{R}^n) \}.$$

We proved

<u>Fourier Inversion Formula</u>: If $f \in X$, then

$$f(x) = \int_{\mathbb{R}^n} \hat{f}(\xi) e^{2\pi i x\cdot\xi}\,d\xi \quad \text{for a.e. } x \in \mathbb{R}^n.$$

$$\left[\Leftrightarrow \hat{\hat{f}}(x) = f(-x) \text{ for a.e. } x \in \mathbb{R}^n. \right]$$

<u>Note</u>: $X \subseteq L^2(\mathbb{R}^n)$ since if $f \in X$, then (by inversion) $f \in L^1 \cap L^\infty$ (as $\hat{f} \in L^1$) and hence $f \in L^2$ (since $\int |f|^2 \le \|f\|_\infty \int |f|$).

<u>Theorem 1</u> (Parseval / Plancherel I) If $f, g \in X$, then

$$\int f\bar{g} = \int \hat{f}\,\overline{\hat{g}} \quad \left(\text{in particular, } \|f\|_2 = \|\hat{f}\|_2\right).$$

"The Fourier transform restricted to X is an isometry on $L^2(\mathbb{R}^n)$".

<u>Proof</u>: Let $h = \overline{\hat{g}}$, then $\hat{h}(\xi) = \overline{g(\xi)}$ and hence

$$\int f\bar{g} = \int f\hat{h} \underset{\text{Mult. Formula.}}{=} \int \hat{f} h = \int \hat{f}\,\overline{\hat{g}}. \qquad \square$$

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- Assuming (we'll prove this later) that $X \subseteq L^2(\mathbb{R}^n)$ <u>dense</u>, then this
- suggests a way of extending the Fourier transform to $L^2(\mathbb{R}^n)$:

If $f \in L^2(\mathbb{R}^n) \Rightarrow \exists \{f_k\} \subseteq X$ s.t. $f_k \to f$ in $L^2(\mathbb{R}^n)$
Since $\{\hat{f}_k\}$ is Cauchy in $L^2(\mathbb{R}^n)$, this follows from PPI,
we know $\exists\, g \in L^2(\mathbb{R}^n)$ such that $\hat{f}_k \to g$ in $L^2(\mathbb{R}^n)$.

<u>Define</u> $\mathcal{F}f := g$.

<u>Note</u>: It follows immediately that if $f \in L^2(\mathbb{R}^n)$, then

$$\|\mathcal{F}f\|_2 = \lim_{k\to\infty} \|\hat{f}_k\|_2 = \lim_{k\to\infty} \|f_k\|_2 = \|f\|_2 .$$

<u>Parseval / Plancherel II</u>

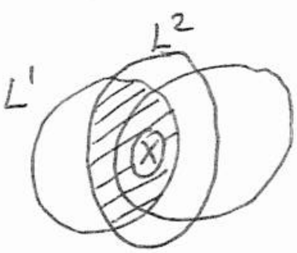
$\exists!$ bounded operator $\mathcal{F}: L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ such that

$$\mathcal{F}f = \hat{f} \text{ when } f \in X$$

and moreover

(i) $\mathcal{F}$ is a unitary operator ✓

(ii) $\mathcal{F}f = \hat{f}$ if $f \in L^1 \cap L^2$



<u>Proof</u>: Need only prove (ii). Recall that $g_t(x) = \frac{1}{t^n} e^{-\pi|x|^2/t^2}$ & $\hat{g}_t(\xi) = e^{-\pi t^2|\xi|^2}$.

Let $f \in L^1 \cap L^2$.

① <u>$f * g_t \in X$</u>:
- $\|f * g_t\|_1 \leq \|f\|_1 \|g_t\|_1$ so $f * g_t \in L^1$
- $\|\widehat{f * g_t}\|_1 \leq \|\hat{f}\|_\infty \|\hat{g}_t\|_1$ so $\widehat{f * g_t} \in L^1$

② Since $\|f * g_t - f\|_1 \to 0$ & $\|f * g_t - f\|_2 \to 0$
$\Rightarrow \|\widehat{f * g_t} - \hat{f}\|_\infty \to 0$ & $\|\underbrace{\mathcal{F}(f * g_t)}_{= \widehat{f * g_t}} - \mathcal{F}f\|_2 \to 0$ $\Bigg\} \Rightarrow \mathcal{F}f = \hat{f}$ ∎

Let $$S_R f(x) = \int_{|\xi| \le R} \hat{f}(\xi) e^{2\pi i x \cdot \xi} d\xi .$$

<u>Corollary</u> – (Strong Inversion Formula)

If $f \in L^2(\mathbb{R}^n)$, then $\|S_R f - f\|_2 \to 0$ as $R \to \infty$.

<u>Exercise</u>: This corollary implies $S_R f \to f$ a.e. if $f \in X$.

<u>Question</u>: Does $S_R f \to f$ in $L^p(\mathbb{R}^n)$ for any $p \neq 2$?

A necessary and sufficient condition for convergence in norm is

$$\|S_R f\|_p \le C_p \|f\|_p \quad (\text{with } C_p \text{ indep. of } R)$$

($\Uparrow$ follows from U.B.P. & $\Downarrow$ follow by density of $\mathcal{S}(\mathbb{R}^n)$ in $L^p(\mathbb{R}^n)$)

<u>Theorem</u>

- (M. Riesz) $\|S_R f\|_p \le C_p \|f\|_p$ if $1 < p < \infty$ <u>& $n = 1$</u>
- (C. Fefferman) $\|S_R f\|_p \le C_p \|f\|_p \Leftrightarrow p = 2$ <u>when $n \geq 2$</u>!

Equivalent to ~~(or perhaps only follows from?)~~ the L^p-boundedness of the Hilbert transform:

$$Hf(x) = \lim_{\varepsilon \to 0} \frac{1}{\pi} \int_{|y| \geq \varepsilon} \frac{f(x-y)}{y} dy .$$

In the case $n=1$:

- $S_R f(x) = f * D_R(x)$

where $$D_R(x) = \int_{-R}^{R} e^{2\pi i x\cdot\xi}\, d\xi = \frac{\sin(2\pi R x)}{\pi x}$$

This is clearly not integrable, but it is in $L^q(\mathbb{R})$ for any $q>1$, so $f*D_R$ is well defined if $f\in L^p(\mathbb{R})$, $1<p<\infty$.

- Almost everywhere convergence depends on the bound
$$\| \sup_R |S_R f| \|_p \leq C_p \|f\|_p$$
This holds if $1<p<\infty$ (Carleson-Hunt) but will not prove this here.

- For the Fourier transform, the method of Cesàro summability is
$$\sigma_R f(x) = \frac{1}{R}\int_0^R S_r f(x)\, dr \xrightarrow{?} f$$

$$\boxed{(\text{Ex}) \quad \int_{|\xi|\leq R} \left(1-\frac{|\xi|}{R}\right) \hat{f}(\xi) e^{2\pi i x\cdot\xi}\, d\xi.}$$

When $n=1$; $\sigma_R f(x) = f * F_R(x)$

where $$F_R(x) = \frac{1}{R}\int_0^R D_r(x)\, dr = \frac{1}{R}\left(\frac{\sin \pi R x}{\pi x}\right)^2.$$

* Unlike the Dirichlet kernel, the Fejér kernel $F_R \in L^1(\mathbb{R})$

It can be shown that if $1\leq p\leq \infty$, then

(i) $\sigma_R f \to f$ in L^p if $f\in L^p(\mathbb{R})$ (or f bounded & unif cont if $p=\infty$)

(ii) $\sigma_R f \to f$ a.e. ——"——

relies of L^p bounds for H-L maximal function.

¿ How to get around lack of convergence in higher dimensions ?

One way is to make the frequency cut-off more gentle. For example, as we have seen, if we take Cesàro means of S_r for $0<r\leq R$ we obtain the operator

$$\sigma_R f(x) = \int_{|\xi|\leq R} \left(1-\frac{|\xi|}{R}\right)\hat{f}(\xi)e^{2\pi i x\cdot\xi}\,d\xi .$$

The leads us to consider the family of operators

$$\tilde{S}_R^{\delta} f(x) = \int_{|\xi|\leq R} \left(1-\frac{|\xi|}{R}\right)^{\delta}\hat{f}(\xi)e^{2\pi i x\cdot\xi}\,d\xi , \quad \delta\geq 0 .$$

Note: When $n=1$ & $\delta>0$, then $\int_{|\xi|\leq R}\left(1-\frac{|\xi|}{R}\right)^{\delta} e^{2\pi i x\cdot\xi}d\xi \in L^1(\mathbb{R})$

(IBP will show $|$—"—$|\leq C(1+|x|)^{-1-\delta}$).

By Young's Inequality it follows that $\|\tilde{S}_R^{\delta} f\|_p \leq C_p\|f\|_p$, $1\leq p\leq\infty$.

* However, in higher dimensions this is no longer true !!

Rather than considering these operators directly, it is customary to replace them with the Bochner–Riesz means:

$$S_R^{\delta} f(x) := \int_{|\xi|\leq R} \left(1-\frac{|\xi|^2}{R^2}\right)^{\delta}\hat{f}(\xi)e^{2\pi i x\cdot\xi}\,d\xi , \quad \delta\geq 0 .$$

Remark: One can show that $\|S_R^{\delta}\|_{p\to p}<\infty \Leftrightarrow \|\tilde{S}_R^{\delta}\|_{p\to p}<\infty$.

Conjecture (Bochner–Riesz) Let $\delta>0$ & $1\leq p\leq\infty$, then

$$\|S_R^{\delta} f\|_p \leq C_p\|f\|_p \ \text{ if } \ \left|\frac{1}{p}-\frac{1}{2}\right| < \frac{2\delta+1}{2n} .$$