STRONGLY SINGULAR INTEGRAL OPERATORS ON $\mathbf{R}^d$

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Abstract. We generalize known results for classical strongly singular integrals on $L^p(\mathbf{R}^d)$.

1. Introduction

These are operators T, initially defined as mappings from test functions in $\mathcal{S}(\mathbf{R}^d)$ to distributions in $\mathcal{S}'(\mathbf{R}^d)$, to which are associated kernels $K_{\alpha,\beta}(x,y)$, defined when $x \neq y$, that take the form

$$K_{\alpha,\beta}(x,y) = a(x,y)e^{i\varphi(x,y)}. \tag{1}$$

We assume that the amplitude and phase satisfy the differential inequalities

$$|\partial_x^\mu \partial_y^\nu a(x,y)| \leq C_{\mu,\nu}|x-y|^{-d-\alpha-|\mu|-|\nu|} \tag{2a}$$

$$|\partial_x^\mu \partial_y^\nu \varphi(x,y)| \leq C_{\mu,\nu}|x-y|^{-\beta-|\mu|-|\nu|}, \tag{2b}$$

that φ is real-valued and furthermore that

$$|\nabla_x \varphi(x,y)|,\, |\nabla_y \varphi(x,y)| \geq C|x-y|^{-\beta-1} \tag{2c}$$

with $\alpha \geq 0$ and $\beta > 0$. In the case where $\alpha = 0$ we must make the further assumption that our amplitude a is compactly supported in a neighborhood of the diagonal $x = y$, this is of course also the only region of any interest when $\alpha > 0$.

It is clear that the estimates (2) also hold uniformly for the dilated functions

$$a_\lambda(x,y) = \lambda^{d+\alpha}a(\lambda x, \lambda y) \quad \text{and} \quad \varphi_\lambda(x,y) = \lambda^\beta \varphi(\lambda x, \lambda y),$$

and in additions to the differential inequalities (2) above we also make the following *non-degeneracy* assumption, namely that

$$\left|\det\left(\frac{\partial^2 \varphi_\lambda(x,y)}{\partial x_i \partial y_j}\right)\right| \geq C > 0 \tag{3}$$

uniformly in λ, such kernels we shall call *(non-degenerate) strongly singular integral kernels.*

Our strongly singular integral operators T are related to these kernels $K_{\alpha,\beta}$ as follows: for $f \in \mathcal{S}$ with compact support we identify the distribution Tf with the function

$$Tf(x) = \int K_{\alpha,\beta}(x,y)f(y)\,dy, \tag{4}$$

for x is outside the support of f. Our result for such operators is the following.

Theorem 1.1. *If $1 < p < \infty$, then the operator T, initially given by (4), extends to a bounded operator on $L^p(\mathbf{R}^d)$ if and only if*

$$\left|\frac{1}{p} - \frac{1}{2}\right| \leq \frac{1}{2} - \frac{\alpha}{d\beta}.$$

The counterexample showing that this result is sharp is due to Wainger [6] as is sufficiency up to the endpoints in the model case, which we describe in §2 below. The full theorem follows from the L^2 and H^1 estimates below by an interpolation theorem of Fefferman and Stein [2] and 'duality'.

Theorem 1.2. *If $\alpha \leq \frac{d\beta}{2}$ then T, initially given by (4), extends to a bounded operator on $L^2(\mathbf{R}^d)$.*

Theorem 1.3. *If $\alpha = 0$ then T, initially given by (4), extends to a bounded operator from $H^1(\mathbf{R}^d)$ to $L^1(\mathbf{R}^d)$.*

In this note we present proofs of the two key results above as well as a new argument establishing the sharp necessary condition, in each case we do not make use of Fourier transform methods.

2. Model Operator

The model case for operators of this type are those with kernels

$$K_{\alpha,\beta}(x,y) = \widetilde{K}_{\alpha,\beta}(x-y)$$

where $\widetilde{K}_{\alpha,\beta}$ is a distribution[1] on $\mathbf{R}^d$ that away from the origin agrees with the *radial* function

$$\widetilde{K}_{\alpha,\beta}(x) = |x|^{-d-\alpha} e^{i|x|^{-\beta}} \chi(|x|), \tag{5}$$

again $\beta > 0$ and here χ is smooth and compactly supported in a small neighborhood of the origin.

Operators of this type were first studied by Hirschman [3] in the case $d = 1$ and then in higher dimensions by Wainger [6].

In establishing this Theorem 1.2 in the model case it is efficient to use Fourier transform methods. Since $\widetilde{K}_{\alpha,\beta}$ is radial it is well known that its Fourier transform is given by

$$m(\xi) = (2\pi)^{\frac{d}{2}} \int_0^\infty \chi(r) r^{-1-\alpha} e^{ir^{-\beta}} J_{\frac{d-2}{2}}(r|\xi|)(r|\xi|)^{\frac{2-d}{2}}\, dr, \tag{6}$$

where $J_{\frac{d-2}{2}}$ is a Bessel function; see [5]. Using Plancherel's theorem and the asymptotics of Bessel functions it is then straightforward to establish Theorem 1.2 in this case.

Using this L^2 result the arguments of Fefferman [1] and Fefferman and Stein [2] can be employed to show that when $\alpha = 0$, T extends to an operator that is of weak type (1,1) and maps $H^1(\mathbf{R}^d)$ boundedly into $L^1(\mathbf{R}^d)$.

The simple lemma below establishes that these kernels do indeed satisfy the required non-degeneracy hypothesis.

Lemma 2.1. *Let $\varphi(x,y) = |x-y|^{-\beta}$, then $\det\big(\frac{\partial^2\varphi}{\partial x_i \partial y_j}\big) \neq 0$ whenever $\beta \neq -1$.*

Proof. Recall that $\nabla|x|^{-\beta} = -\beta|x|^{-\beta-1}\frac{x}{|x|}$, it is then easy to see that

$$\partial_{x_i}\partial_{y_j}\varphi(x) = \beta|x-y|^{-\beta-2}\left(\delta_{ij} - (\beta+2)u_i u_j\right),$$

where $u_i = \frac{(x-y)_i}{|x-y|}$. We therefore need to check that $I - (\beta+2)uu^t$ is non-singular. To do this we shall denote by R the rotation matrix such that $Ru = e_1$, of course $\det R = 1$ and it is clear that

$$\det(I - (\beta+2)uu^t) = \det\big(R(I - (\beta+2)uu^t)R^t\big) = 1 - (\beta+2) = -(\beta+1). \qquad \square$$

[1] The distribution-valued function $\alpha \mapsto \widetilde{K}_{\alpha,\beta}$, initially defined for $\operatorname{Re}\alpha < 0$, continues analytically to all of $\mathbf{C}$.

3. Proof of Theorem 1.2

We make the assumption that our kernels $K_{\alpha,\beta}(x,y)$ are supported in a small neighborhood of the diagonal, we can do this since in the complement of such a region $K_{\alpha,\beta}$ is dominated by an integrable function of $|x-y|$. We additionally choose to dyadically decompose the operator

$$T = \sum_{j=0}^{\infty} T_j.$$

In order to do this we consider the following partition of unity; choose $\vartheta \in C_0^\infty(\mathbf{R})$ supported in $[\frac{1}{2}, 2]$ such that $\sum_{j=0}^{\infty} \vartheta(2^j r) = 1$ for all $0 \le r \le 1$, and then write for $f \in \mathcal{S}$ with compact support

$$T_j f(x) = \int K_j(x,y) f(y)\, dy$$

whenever x is in the complement of the support of f, where

$$K_j(x,y) = \vartheta(2^j|x-y|) K_{\alpha,\beta}(x,y).$$

The main result here is the following.

Lemma 3.1. *The operator norms of T_j are uniformly bounded whenever $\alpha \le \frac{d\beta}{2}$, more precisely*

$$\int_{\mathbf{R}^d} |T_j f(x)|^2 dx \le C 2^{j(2\alpha - d\beta)} \int_{\mathbf{R}^d} |f(x)|^2 dx. \tag{7}$$

Key to establishing this result is the following proposition of Hörmander, which may be thought of as a variable coefficient version of Plancherel's theorem. See [7], Chapter 7 or [4], Chapter IX.

Proposition 3.2. *Let Ψ be a smooth function supported on the set $\{(x,y) \in \mathbf{R}^d \times \mathbf{R}^d : |x-y| \le C\}$ and Φ be real-valued and smooth on the support of Ψ. If we assume that all partial derivatives of Ψ and Φ are bounded and that*

$$\det\Big(\frac{\partial^2 \Phi}{\partial x_i \partial y_j}\Big) \neq 0,$$

on the support of Ψ, then

$$\Big\| \int_{\mathbf{R}^d} \Psi(x,y) e^{i\lambda\Phi(x,y)} f(y) dy \Big\|_{L^2(\mathbf{R}^d)} \le C \lambda^{-\frac{d}{2}} \|f\|_{L^2(\mathbf{R}^d)}.$$

Proof of Lemma 3.1. We note that as the operator norms of T_j are equal to that of

$$\begin{aligned} S_j f(x) &= 2^{-jd} \int K_j(2^{-j}x, 2^{-j}y) f(y)\, dy \\ &= 2^{j\alpha} \int_{\mathbf{R}^d} \vartheta(|x-y|) a_{2^{-j}}(x,y) e^{i2^{j\beta}\varphi_{2^{-j}}(x,y)} f(y) dy \end{aligned}$$

it suffices to establish estimate (7) for the operators S_j, which follows from Proposition 3.2. □

Theorem 1.2 now follows from Lemma 3.1 and an application of Cotlar's lemma (plus a standard limiting argument) once we have verified that the T_j are, in the following sense, almost orthogonal.

Lemma 3.3. *If $\alpha = \frac{d\beta}{2}$ then $\|T_i^* T_j\|_{Op} + \|T_i T_j^*\|_{Op} \le C 2^{-\frac{d\beta}{2}|i-j|}$.*

Proof. This follows trivially from Lemma 3.1 whenever $|i-j| \leq 10$, since $\|T_i^* T_j\|_{Op} \leq \|T_i\|_{Op} \|T_j\|_{Op}$. We shall therefore, without loss of generality, assume that $j \geq i+10$. Now $T_i^* T_j$ has a kernel

$$L_{ij}(x,y) = \int \bar{K}_i(z,x) K_j(z,y) dz,$$

and the same operator norm as the operator with kernel

$$\begin{aligned}
\widetilde{L}_{ij}(x,y) &= 2^{-jd} L_{ij}(2^{-j}x, 2^{-j}y) \\
&= 2^{-jd} \int \bar{K}_i(z, 2^{-j}x) K_j(z, 2^{-j}y) dz \\
&= 2^{j2\alpha} \int\limits_{\substack{|z-y|\sim 1 \\ |z-x|\sim 2^{j-i}}} a_{2^{-j}}(z,x) a_{2^{-j}}(z,y) e^{i2^{j\beta}[\varphi_{2^{-j}}(z,y) - \varphi_{2^{-j}}(z,x)]} dz.
\end{aligned}$$

Trivially we get the estimate $|\widetilde{L}_{ij}(x,y)| \leq C 2^{j2\alpha} 2^{(i-j)(d+\alpha)}$. However from (2c) it follows that

$$|\nabla_z[\varphi_{2^{-j}}(z,y) - \varphi_{2^{-j}}(z,x)]| \geq C_0,$$

thus there is always a direction in which we may integrating by parts, in doing so d times we obtain

$$|\widetilde{L}_{ij}(x,y)| \leq C 2^{j(2\alpha - d\beta)} 2^{(i-j)(d+\alpha)} = 2^{(i-j)(d+\alpha)}.$$

This of course implies that

$$\sup_x \int |\widetilde{L}_{ij}(x,y)|\, dy \leq C 2^{(i-j)\alpha}$$

and

$$\sup_y \int |\widetilde{L}_{ij}(x,y)|\, dx \leq C 2^{(i-j)\alpha},$$

and so by Schur's test we are done. □

4. Proof of Theorem 1.3

For any f in $H^1(\mathbf{R}^d)$ we have the atomic decomposition

$$f = \sum_Q \lambda_Q \, a_Q \qquad \text{where} \qquad \sum_Q |\lambda_Q| \sim \|f\|_{H^1(\mathbf{R}^d)},$$

and the individual atoms satisfy the following;

$$\text{(i)}\ \operatorname{supp} a_Q \subset Q \qquad \text{(ii)}\ \|a_Q\|_\infty \leq |Q|^{-1} \qquad \text{(iii)}\ \int a_Q(x) dx = 0.$$

As a consequence of this characterization of H^1 it suffices to check that for an individual atom a_Q

$$\int |T a_Q(x)| dx \leq C,$$

where C is independent of a_Q. Although the operators T_j are not translation invarient the 'translated' kernels

$$K^{x_0}_{\alpha,\beta}(x,y) = K_{\alpha,\beta}(x+x_0, y+x_0)$$

do satisfy assumptions (2) and (3), for the same α and β as $K_{\alpha,\beta}$, uniformly in x_0, and hence we may, without loss in generality, assume that $a = a_Q$ is supported in a cube centered at the origin. Note that if $Q^* = 2Q$, then Cauchy–Schwarz and Theorem 1.2 imply that

$$\int_{Q^*} |Ta(x)|dx \leq C|Q^*|^{\frac{1}{2}}\|a\|_2 \leq C|Q^*|^{\frac{1}{2}}|Q|^{-\frac{1}{2}} \leq C.$$

Let $\ell = \ell(Q)$ denote the sidelength of Q. Now if $\ell \geq 1$ then it follows from the compact support of our kernel K that $\operatorname{supp} Ta \subset Q^*$ and from the argument above our result follows in this case. We may now assume that $\ell < 1$. It suffices to establish the following estimate for each dyadic operator.

Lemma 4.1. *If $\alpha = 0$, then*

$$\int |T_j a(x)|dx \leq C \min\{\ell 2^{j(\beta+1)}, (\ell 2^{j(\beta+1)})^{-\frac{d\beta}{2(\beta+1)}}\}.$$

Theorem 1.3 now follows, since

$$\int |Ta(x)|dx \leq \sum_{j=1}^{\infty} \int |T_j a(x)|dx \leq C\Big(\ell \sum_{2^j \leq \ell^{-\frac{1}{\beta+1}}} 2^{j(\beta+1)} + \ell^{-\frac{d\beta}{2(\beta+1)}} \sum_{2^{-j} \leq \ell^{\frac{1}{\beta+1}}} 2^{-j\frac{d\beta}{2}}\Big) \leq C.$$

Proof of Lemma 4.1. Our estimate naturally splits into two cases.

(i) If $2^j \leq \ell^{-\frac{1}{\beta+1}}$ then it is straightforward to see, using the cancellation of our atom a, that

$$\int |T_j a(x)|dx \leq \int |a(y)| \int |K_j(x,y) - K_j(x,y')|dx\, dy \leq C\ell \int \vartheta(2^j|x|)|x|^{-d-\beta-1}dx \leq C\ell 2^{j(\beta+1)}.$$

(ii) If $2^{-j} \leq \ell^{\frac{1}{\beta+1}}$ then we shall as usual use an L^2 result, namely Lemma 3.1;

$$\int |T_j a(x)|dx \leq C\ell^{\frac{d}{2(\beta+1)}}\|T_j a\|_2 \leq C\ell^{\frac{d}{2(\beta+1)}} 2^{-j\frac{d\beta}{2}}\|a\|_2 \leq C(\ell 2^{j(\beta+1)})^{-\frac{d\beta}{2(\beta+1)}}. \qquad \square$$

5. Necessary Condition

It suffices to establish a necessary condition for the dyadic operators T_j to bounded uniformly on $L^p(\mathbf{R}^d)$ for $1 \leq p \leq 2$. Recall that

$$T_j f(x) = \int_{\mathbf{R}^d} \vartheta(2^j|x-y|)|x-y|^{-d-\alpha} e^{i|x-y|^{-\beta}} f(y)\, dy.$$

We now choose a suitable test function, namely $f_0(x) = |x|^{-\gamma}\chi(10|x|)$, where $\gamma < \frac{d}{p}$. It follows that $T_j f_0$ will be a radial function, being the convolution of two radial functions and as such we may with no loss in generality assume that $x = (|x|, 0, \ldots, 0)$. As a result we see that

$$T_j f_0(x) = |x|^{-\alpha-\gamma}\vartheta(2^j|x|) \int_{\mathbf{R}^d} e^{i|x|^{-\beta}\varphi(s)}\psi(s)\, ds,$$

where

$$\varphi(s) = (1 - 2s_1 + |s|^2)^{-\frac{\beta}{2}} \quad \text{and} \quad \psi(s) = (1 - 2s_1 + |s|^2)^{-\frac{d+\alpha}{2}}\chi(|s|)|s|^{-\gamma}.$$

We have therefore now reduced matters to the analysis of the oscillatory singular integral

$$I(\lambda) = \int_{\mathbf{R}^d} e^{i\lambda\varphi(s)}\psi(s)\, ds,$$

as $\lambda \to +\infty$.

We now write

$$I(\lambda) = M(\lambda) + E_1(\lambda) + E_2(\lambda) + E_3(\lambda),$$

where

$$M(\lambda) = e^{i\lambda} \int_{\mathbf{R}^d} \chi(\lambda^{1-\epsilon}|s|) e^{ic\lambda s_1} |s|^{-\gamma}\, ds,$$

$$E_1(\lambda) = \int_{\mathbf{R}^d} \chi(\lambda^{1-\epsilon}|s|)\big[e^{i\lambda\varphi(s)} - e^{i\lambda(1+cs_1)}\big]|s|^{-\gamma}\, ds,$$

$$E_2(\lambda) = \int_{\mathbf{R}^d} \chi(\lambda^{1-\epsilon}|s|) e^{i\lambda\varphi(s)}\big[\psi(s) - |s|^{-\gamma}\big]\, ds,$$

and

$$E_3(\lambda) = \int_{\mathbf{R}^d} \big[1 - \chi(\lambda^{1-\epsilon}|s|)\big] e^{i\lambda\varphi(s)}\psi(s)\, ds.$$

Let us first take care of the error terms. It is easy to verify that whenever $|s| \leq \lambda^{-1+\epsilon}$ we have

$$|e^{i\lambda\varphi(s)} - e^{i\lambda(1+cs_1)}| \leq C\lambda^{\epsilon}|s| \quad \text{ and } \quad |\psi(s) - |s|^{-\gamma}| \leq C\lambda^{-1+\epsilon}|s|^{-\gamma},$$

and hence that

$$|E_1(\lambda)| \leq C\lambda^{\epsilon} \int_{|s|\leq\lambda^{-1+\epsilon}} |s|^{-\gamma+1}\, ds \leq C\lambda^{\epsilon}\lambda^{-(d-\gamma+1)(1-\epsilon)} = C\lambda^{-d+\gamma}\lambda^{-1+\epsilon(2+d-\gamma)},$$

while

$$|E_2(\lambda)| \leq C\lambda^{-1+\epsilon} \int_{|s|\leq\lambda^{-1+\epsilon}} |s|^{-\gamma}\, ds \leq C\lambda^{-1+\epsilon}\lambda^{-(d-\gamma)(1-\epsilon)} = C\lambda^{-d+\gamma}\lambda^{-1+\epsilon(1+d-\gamma)}.$$

Now in the error integral $E_3(\lambda)$ it shall be advantageous to repeatedly apply integration by parts in the s_1 direction since $C\lambda^{-1+\epsilon} \leq |s| \leq \frac{1}{10}$. In fact it is clear that in this region

$$\partial_1\varphi(s) = \beta(1-s_1)(1-2s_1+|s|^2)^{-\frac{\beta+2}{2}} \geq C(\beta),$$

while

$$|\partial_1^{\ell}\varphi(s)| \leq c_{\ell} \qquad \text{and} \qquad \big|\partial_1^{\ell}[1-\chi(\lambda^{1-\epsilon}|s|)]\psi(s)\big| \leq c_{\ell}\big(\lambda^{(1-\epsilon)\ell}|s|^{-\gamma}\vartheta(10\lambda^{1-\epsilon}|s|) + |s|^{-\gamma-\ell}\big).$$

It therefore follows that after integrating by parts N times we obtain the estimate

$$|E_3(\lambda)| \leq C\lambda^{-N}\Big(\lambda^{(1-\epsilon)N} \int_{|s|\approx\lambda^{-1+\epsilon}} |s|^{-\gamma}\, ds + \int_{|s|\geq\lambda^{-1+\epsilon}} |s|^{-\gamma-N}\, ds\Big) \leq C\lambda^{-d+\gamma}\lambda^{-\epsilon(N-d+\gamma)}.$$

It remains for us to show that $|M(\lambda)| \geq C\lambda^{-d+\gamma}$. Assuming this for the moment we see that it would then follow that $|I(\lambda)| \geq \lambda^{-d+\gamma}$ for large enough λ and hence that

$$\|T_j f_0\|_p^p \geq C \int_{\mathbf{R}^d} \vartheta(2^j|x|)|x|^{-(\alpha+\gamma-\beta(d-\gamma))p}\, dx = C2^{j(\alpha+\gamma-\beta(d-\gamma))p-d}.$$

Therefore if T_j were to extend to a bounded operator on $L^p(\mathbf{R}^d)$ for $1 \leq p \leq 2$ with operator norm independent of j we see that one must have

$$\alpha - \frac{d\beta}{p'} \leq \Big(\frac{d}{p} - \gamma\Big)(\beta+1)$$

for all $\gamma < \frac{d}{p'}$. It follows immediately that one must then necessarily have the condition $\alpha \leq \frac{d\beta}{p'}$.

The lower bound estimate for the main term will be an immediate consequence of the following, slightly more general lemma.

Lemma 5.1. *If* $0 < \gamma < d$ *then*

$$\int_{\mathbf{R}^d} \chi(|s|)e^{i\xi\cdot s}|s|^{-\gamma}\, ds = C|\xi|^{\gamma-d} + O(|\xi|^{\gamma-d-1}).$$

Proof. This is merely a Fourier transform and hence

$$\int_{\mathbf{R}^d} \chi(|s|)e^{i\xi\cdot s}|s|^{-\gamma}\, ds = C\int_{\mathbf{R}^d} \widehat{\chi}(|\eta-\xi|)|\eta|^{\gamma-d}\, d\eta.$$

Now since χ is smooth and of compact support $\widehat{\chi}$ is a Schwartz function and satisfies the inequality

$$|\widehat{\chi}(|\eta-\xi|)| \le C_N(1+|\eta-\xi|)^{-N},$$

for all $N \ge 0$. Using this standard estimate it is easy to see that whenever $|\xi| \notin [\frac{1}{2}|\eta|, 2|\eta|]$ we have

$$\left|\int_{\mathbf{R}^d} \widehat{\chi}(|\eta-\xi|)|\eta|^{\gamma-d}\, d\eta\right| \le C|\xi|^{-N+\gamma}.$$

Now if $|\xi| \in [\frac{1}{2}|\eta|, 2|\eta|]$, then

$$\begin{aligned}\int_{\mathbf{R}^d} \widehat{\chi}(|\eta-\xi|)|\eta|^{\gamma-d}\, d\eta &= |\xi|^{\gamma-d}\int_{\mathbf{R}^d} \widehat{\chi}(|\eta-\xi|)\, d\eta + \int_{\mathbf{R}^d} \widehat{\chi}(|\eta-\xi|)\left[|\eta|^{\gamma-d} - |\xi|^{\gamma-d}\right] d\eta \\ &= |\xi|^{\gamma-d}\chi(0) + O\Big(\int_{|\eta|\approx|\xi|} |\eta|^{\gamma-d-1}\, d\eta\Big) \\ &= |\xi|^{\gamma-d} + O(|\xi|^{\gamma-d-1}).\end{aligned}$$

□

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