A SHORT PROOF OF ROTH'S THEOREM

NEIL LYALL ÁKOS MAGYAR

Roth's Theorem. *Let $\delta > 0$ and $N \geq \exp\exp(C\delta^{-1})$ for some absolute constant C. Then any $A \subset [1, N]$ of size $|A| = \delta N$ necessarily contains a (non-trivial) arithmetic progression of length three.*

With the aid of the Fourier transform (on $\mathbf{Z}$) we can count the number of 3AP in A, that is the number of solutions to the equation $m_1 + m_2 = 2m_3$ with $m_1, m_2, m_3 \in A$, namely

$$\mathcal{N} = \sum_{m_1 \in A} \sum_{m_2 \in A} \sum_{m_3 \in A} \int_0^1 e^{2\pi i(m_1+m_2-2m_3)\alpha} d\alpha = \int_0^1 \widehat{1_A}(\alpha)^2 \widehat{1_A}(-2\alpha)\, d\alpha.$$

Definition 1. We say that A is (ε, L) *regular* if every progression P, with $|P| \geq L$, satisfies

$$|A \cap P| \leq (\delta + \varepsilon)|P|.$$

Definition 2. We define the *balanced* function of A to be

$$f_A(m) = 1_A(m) - \delta 1_{[1,N]}(m).$$

We note that $\sum f_A(m) = 0$ and now <u>fix</u> $\varepsilon = \delta^2/10$ and $L = \varepsilon N^{1/2}$.

Proposition. *If A is (ε, L)* regular*, then $|\widehat{f_A}(\alpha)| \leq 8\varepsilon N$ uniformly in α.*

Corollary. *If A is (ε, L)* regular*, then A contains at least $\delta^3 N^2/10$ non-trivial 3AP.*

Proof of Corollary. Writing $1_A = \delta 1_{[1,N]} + f_A$ we obtain

$$\mathcal{N} = \delta \int_0^1 \widehat{1_A}(\alpha)^2 \widehat{1_{[1,N]}}(-2\alpha)\, d\alpha + \int_0^1 \widehat{1_A}(\alpha)^2 \widehat{f_A}(-2\alpha)\, d\alpha = \text{Main} + \text{Error}.$$

Now it is easy to see that

$$\text{Main} = \delta \sum_{m_1 \in A} \sum_{m_2 \in A} \sum_{m_3 \in [1,N]} \int_0^1 e^{2\pi i(m_1+m_2-2m_3)\alpha} d\alpha = \delta |A|^2 = \delta^3 N^2,$$

while

$$|\text{Error}| \leq \sup_\alpha |\widehat{f_A}(\alpha)| \int_0^1 |\widehat{1_A}(\alpha)|^2\, d\alpha \leq 8\varepsilon\delta N^2.$$

This completes the proof of the corollary as there are only δN trivial 3AP in A. □

Proof of Proposition. Let $\alpha \in [0, 1]$. By Dirichlet, there exists $q \leq 2L$ such that $\langle q\alpha \rangle \leq 1/2L$. Defining

$$P_0 = \{\ell q : |\ell| \leq L/2\}$$

it then follows that

$$\widehat{1_{P_0}}(\alpha) = \sum_{|\ell| \leq L/2} e^{2\pi i \ell q \alpha} \geq L/2$$

and hence that

$$\sum_m |f_A * 1_{P_0}(m)| \geq |\widehat{f_A}(\alpha)|\, |\widehat{1_{P_0}}(\alpha)| \geq \frac{L}{2} |\widehat{f_A}(\alpha)|.$$

But,

$$f_A * 1_{P_0}(m) = \sum_{|\ell| \leq L/2} f_A(m - \ell q) = |A \cap P_m| - \delta |P_m \cap [1, N]| \leq \varepsilon L$$

where $P_m = m + P_0 \subseteq [1, N]$. Thus, after accounting for the "tails", we can conclude that

$$\sum_m (f_A * 1_{P_0})_+(m) \leq \varepsilon LN + 2L^3 \leq 2\varepsilon LN$$

and consequently, since $\sum_m f_A * 1_{P_0}(m) = 0$ and $|g| = 2g_+ - g$, that

$$\sum_m |f_A * 1_{P_0}(m)| \leq 4\varepsilon LN.$$ □