Riesz-Thorin Interpolation & Two Standard Applications

Theorem (Riesz-Thorin)

Let $1 \le p_0, p_1, q_0, q_1 \le \infty$ & for $0 < \theta < 1$ define p & q by

$$\frac{1}{p} = \frac{1}{p_0}(1-\theta) + \frac{1}{p_1}\theta \quad \& \quad \frac{1}{q} = \frac{1}{q_0}(1-\theta) + \frac{1}{q_1}\theta$$

If T is a linear operator from $L^{p_0} + L^{p_1}$ to $L^{q_0} + L^{q_1}$ such that

$$\|Tf\|_{q_0} \le M_0 \|f\|_{p_0} \ \forall f \in L^{p_0} \quad \& \quad \|Tf\|_{q_1} \le M_1 \|f\|_{p_1} \ \forall f \in L^{p_1}$$

then

$$\|Tf\|_q \le M_0^{1-\theta} M_1^{\theta} \|f\|_p \quad \forall f \in L^p.$$

(Proof: See Folland p 200.)

Application 1: Hausdorff-Young Inequality.

If $f \in L^p$ with $1 < p < 2$, then we can write $f = f_1 + f_2$ with $f_1 \in L^1$ & $f_2 \in L^2$, therefore $\hat{f} = \hat{f}_1 + \hat{f}_2 \in L^\infty + L^2$ (a space which contains $L^{p'}$). By applying the above interpolation theorem we can see that $\hat{f}$ is in fact in $L^{p'}$.

Corollary 1 (Hausdorff-Young)

If $f \in L^p$ with $1 \le p \le 2$, then $\hat{f} \in L^{p'}$ with $\|\hat{f}\|_{p'} \le \|f\|_p$.

Proof: Apply theorem using (i) $\|\hat{f}\|_\infty \le \|f\|_1$ & (ii) $\|\hat{f}\|_2 = \|f\|_2$. □

Application 2: Young's Inequality.

Corollary 2 (Young's Inequality) If $f \in L^p$ and $g \in L^q$, then $f * g \in L^r$ with $\frac{1}{p} + \frac{1}{q} = 1 + \frac{1}{r}$ & $\|f*g\|_r \le \|f\|_p \|g\|_q$.

Proof: Fix $f \in L^p$ and apply theorem (with $M_0 = M_1 = \|f\|_p$) using

(i) $\|f*g\|_p \le \|f\|_p \|g\|_1$ (Minkowski) & (ii) $\|f*g\|_\infty \le \|f\|_p \|g\|_{p'}$ □

The Marcinkiewicz Interpolation Theorem

An operator T is said to be bounded from $L^p(\mathbb{R}^n)$ to $L^q(\mathbb{R}^n)$, or of strong-type (p,q), if $\exists$ constant $C_{p,q}>0$ such that

$$\|Tf\|_q \leq C_{p,q}\|f\|_p \quad \text{for all } f\in L^p(\mathbb{R}^n).$$

An operator T is said to be of weak-type (p,q) if $\exists\, C_{p,q}>0$ s.t.

$$\left|\{x\in\mathbb{R}^n : |Tf(x)|>\alpha\}\right| \leq \left(\frac{C_{p,q}\|f\|_p}{\alpha}\right)^q \quad \text{for all } f\in L^p(\mathbb{R}^n).$$

Note: (Chebyshev): T strong-type $(p,q) \Rightarrow T$ weak-type (p,q).

An operator T from a vector space of measurable functions to measurable functions is sublinear if

(i) $|T(f_0+f_1)(x)| \leq |Tf_0(x)| + |Tf_1(x)|$

(ii) $|T(cf)(x)| \leq |c|\,|Tf(x)|$, $c\in\mathbb{C}$.

Theorem (Marcinkiewicz Interpolation Theorem)
If a sublinear operator T is both of weak-type (p_0,p_0) and of weak-type (p_1,p_1) for some $1\leq p_0<p_1\leq\infty$, then T is bounded on $L^p(\mathbb{R}^n)$ for $p_0<p<p_1$.

Proof (See Folland p 203 for a more general result)
In this special case the proof is on the level of an exercise using the fact that if f is a measurable function on $\mathbb{R}^n$, then

$$\int_{\mathbb{R}^n} |f(x)|^p\,dx = p\int_0^\infty \alpha^{p-1}\left|\{x\in\mathbb{R}^n : |f(x)|>\alpha\}\right|\,d\alpha .$$