A (Local) Inverse Theorem for the U^3-norm

Theorem: Let $\varepsilon>0$ and $N \geq e^{\varepsilon^{-C}}$.

if $f:\mathbb{Z}_N \to \mathbb{D}$ satisfies $\|f\|_{U^3} \geq \varepsilon$, then f has "local quadratic bias" in the sense that $\exists$ $\mathbb{Z}_N$-prog Q with $|Q| \geq N^{\varepsilon^C}$ and quadratic $\psi_1, \dots, \psi_N$ such that

$$\frac{1}{N}\sum_x \left| \frac{1}{|Q|} \sum_{h \in Q+x} f(h) e^{2\pi i \psi_x(h)/N} \right| \geq \varepsilon^C.$$

The proof of this result is very involved...

We start with a simple observation: Since

$$\|f\|_{U^3}^8 = \frac{1}{N}\sum_h \|\Delta_h f\|_{U^2}^4 \quad \text{where } \Delta_h f(x) = f(x)\overline{f(x+h)}$$

$$\left[\text{In fact } \|f\|_{U^3}^8 = \frac{1}{N^4}\sum_{x,h_1,h_2,h_3} \Delta_{h_3}(\Delta_{h_2}(\Delta_{h_1} f))(x)\right]$$

and
$$\|\Delta_h f\|_{U^2}^4 \leq \|\widehat{\Delta_h f}\|_\infty^2 \qquad (\text{Inverse Theorem for } U^2\text{-norm})$$

it is easy to see that if we set

$$H := \{h \in \mathbb{Z}_N : \|\Delta_h f\|_{U^2}^4 \geq \varepsilon^8/2\}$$

then
$$\text{(i)} \quad |H| \geq \frac{\varepsilon^8}{2} N \qquad (\text{since } \|\Delta_h f\|_{U^2} \leq 1)$$

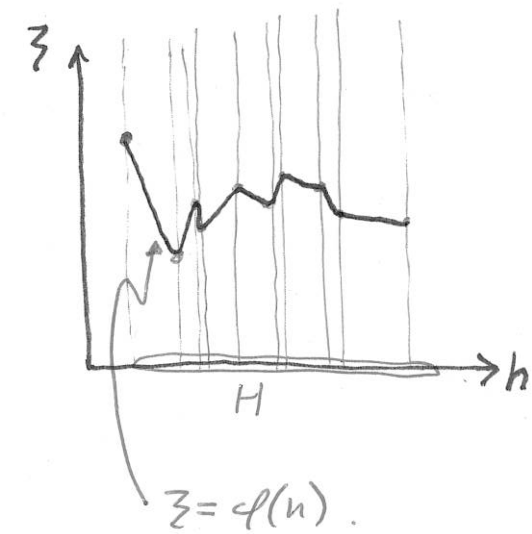
and
$$\text{(ii)} \quad \forall\, h \in H \ \exists\, \xi \in \mathbb{Z}_N \text{ s.t. } |\widehat{\Delta_h f}(\xi)| \geq \varepsilon^4/\sqrt{2}.$$

We record this observation in a lemma.

Lemma 1

If $f : \mathbb{Z}_N \to \mathbb{D}$ satisfies $\|f\|_{U^3} \geq \varepsilon$ and $H := \{h \in \mathbb{Z}_N : \|\Delta_h f\|_{U^2}^4 \geq \varepsilon^8/2\}$ then $|H| \geq \frac{\varepsilon^8}{2} N$ and $\exists\, \varphi : H \to \mathbb{Z}_N$ s.t. $|\widehat{\Delta_h f}(\varphi(h))| \geq \varepsilon^4/\sqrt{2}$.

¿ How arbitrary can this function φ be ?



Consider the special case when $f(x) = e^{2\pi i x^2/N}$.

In this case

$$\Delta_h f(x) = e^{-2\pi i h^2/N} e^{-2\pi i (2hx)/N}$$

$$\Rightarrow \quad \widehat{\Delta_h f}(\xi) = e^{-2\pi i h^2/N} \frac{1}{N} \sum_x e^{-2\pi i (2hx + \xi x)/N}$$

and hence

$$|\widehat{\Delta_h f}(\varphi(h))| = 1 \quad \text{if } \varphi(h) = -2h.$$

- Gowers argument (which we are following) hinges on the miraculous observation that the graph

$$\Gamma := \{(h, \varphi(h)) : h \in H\}$$

has considerable arithmetic structure and that one can deduce from this that φ in fact always exhibits some linear behaviour !!

Lemma 2 (Additive Structure of Large Fourier Coefficients)

Let $f: \mathbb{Z}_N \to \mathbb{D}$ and $H \subseteq \mathbb{Z}_N$ with $|H| \geq \frac{\varepsilon^8}{2} N$. If $\varphi: H \to \mathbb{Z}_N$ satisfies

$$|\widehat{\Delta_n f}(\varphi(n))| \geq \varepsilon^4/\sqrt{2}$$

the Γ contains a large number of additive quadroples, specifically

$$|\{(a,b,c,d) \in \Gamma^4 : a+b = c+d\}| \geq \frac{\varepsilon^{64}}{256} N^3 .$$

Proof: Note that

$$\frac{1}{N} \sum_{n \in H} |\widehat{\Delta_n f}(\varphi(n))|^2 \geq \left(\frac{\varepsilon^8}{2}\right)\left(\frac{\varepsilon^4}{\sqrt{2}}\right)^2 = \frac{\varepsilon^{16}}{4}$$

while

$$\frac{1}{N} \sum_{n \in H} |\widehat{\Delta_n f}(\varphi(n))|^2 = \frac{1}{N} \sum_{n \in H} \left| \frac{1}{N} \sum_x \Delta_n f(x) e^{-2\pi i x \varphi(n)/N} \right|^2$$

$$= \frac{1}{N} \sum_{n \in H} \frac{1}{N^2} \sum_{x,y} \Delta_n f(x) \overline{\Delta_n f(x+y)} e^{2\pi i y \varphi(n)/N} .$$

Now since $\Delta_n f(x) \overline{\Delta_n f(x+y)} = \Delta_y f(x) \overline{\Delta_y f(x+n)}$ [Check!]

it follows that

$$\frac{1}{N} \sum_{h \in H} |\widehat{\Delta_h f}(\varphi(h))|^2 = \frac{1}{N^3} \sum_{x,y,h} \Delta_y f(x) \overline{\Delta_y f(x+h)} \underbrace{1_H(h) e^{2\pi i y \varphi(h)/N}}_{=: G_y(h)} .$$

$$= \frac{1}{N} \sum_y \left(\sum_\xi |\widehat{\Delta_y f}(\xi)|^2 \widehat{G_y}(\xi) \right)$$

<u>Exercise 1</u> (Hint: Write $\overline{\Delta_y f(x+h)} = \sum_\xi \overline{\widehat{\Delta_y f}(\xi)} e^{-2\pi i (x+h)\xi/N}$)

Applying Hölder (to the double sum) we obtain

$$\frac{1}{N}\sum_{y}\sum_{\xi}|\widehat{\Delta_y f}(\xi)|^2\,\widehat{G_y}(\xi)$$

$$\leq \left(\frac{1}{N}\sum_{y}\sum_{\xi}|\widehat{\Delta_y f}(\xi)|^{8/3}\right)^{3/4}\left(\frac{1}{N}\sum_{y}\sum_{\xi}|\widehat{G_y}(\xi)|^4\right)^{1/4}.$$

Since

- $$\frac{1}{N}\sum_{y}\sum_{\xi}|\widehat{\Delta_y f}(\xi)|^{8/3} \leq \frac{1}{N}\sum_{y}\sum_{\xi}|\widehat{\Delta_y f}(\xi)|^2 \quad (\text{since } \|\widehat{\Delta_y f}\|_\infty \leq 1)$$
$$= \frac{1}{N}\sum_{y}\frac{1}{N}\sum_{x}|\Delta_y f(x)|^2$$
$$\leq 1 \qquad (\text{since } \|\Delta_y f\|_\infty \leq 1)$$

and

- $$\frac{1}{N}\sum_{y}\sum_{\xi}|\widehat{G_y}(\xi)|^4 = \frac{1}{N}\sum_{y}\sum_{\xi}\left|\frac{1}{N}\sum_{h\in H}e^{2\pi i[\phi(h)y-h\xi]/N}\right|^4$$
$$= \frac{1}{N}\sum_{y,\xi}\frac{1}{N^4}\sum_{h_1,h_2,h_3,h_4\in H}e^{2\pi i[\phi(h_1)+\phi(h_2)-\phi(h_3)-\phi(h_4)]\frac{y}{N}}\,e^{-2\pi i[h_1+h_2-h_3-h_4]\frac{\xi}{N}}$$
$$= \frac{1}{N^3}\sum_{h_1,h_2,h_3,h_4\in H}\left(\frac{1}{N}\sum_{y}e^{2\pi i[\text{---}''\text{---}]y/N}\right)\left(\frac{1}{N}\sum_{\xi}e^{-2\pi i[\text{---}''\text{---}]\frac{\xi}{N}}\right)$$
$$= \frac{1}{N^3}\left|\left\{(h_1,h_2,h_3,h_4)\in H^4 : \begin{array}{l} h_1+h_2=h_3+h_4 \\ \& \ \phi(h_1)+\phi(h_2)=\phi(h_3)+\phi(h_4)\end{array}\right\}\right|$$

it follows that $$|\{(a,b,c,d)\in\Gamma^4 : a+b=c+d\}| \geq \left(\frac{\varepsilon^{16}}{4}\right)^4 N^3 = \frac{\varepsilon^{64}}{256}N^3$$

as required. $\square$

Now comes the "black box" portion of this lecture:

* We will be able to deduce from the Balog-Szemerédi-Gowers Theorem (see that set of lectures) that

$$\exists \text{ subset } \Gamma' \subseteq \Gamma \text{ with } |\Gamma'| \gg \varepsilon^{128} |\Gamma| \text{ such that}$$

$$|\Gamma' + \Gamma'| \ll \varepsilon^{-24(128)} |\Gamma'| = \varepsilon^{-c} |\Gamma'|.$$

* From the fact that Γ' has "small doubling" we will then deduce the following from a (variant of) Freiman's Theorem (see the set of notes on Freiman):

$$\exists\ \mathbb{Z}_N\text{-prog } Q \text{ with } |Q| \geq N^{\varepsilon^c} \text{ s.t. } |\Gamma \cap Q| \geq \varepsilon^c |Q|.$$

Combining these observations (which we will verify later) with Lemmas 1 & 2 we obtain:

Corollary: Let $\varepsilon > 0$ and $N \geq e^{\varepsilon^{-c}}$.

If $f : \mathbb{Z}_N \to \mathbb{D}$ satisfies $\|f\|_{U^3} \geq \varepsilon$, then $\exists\ \mathbb{Z}_N$-prog Q, with $|Q| \geq N^{\varepsilon^c}$ such that

$$\frac{1}{|Q|} \sum_{h \in Q} |\widehat{\Delta_h f}(2ah+b)|^2 \geq \varepsilon^c \quad \text{for some } a, b \in \mathbb{Z}_N.$$

We have just established that if $f: \mathbb{Z}_N \to \mathbb{D}$ has large U^3-norm, then "a little piece of its derivative is linear". If we knew that it was in fact globally linear, say that $\underbrace{\frac{1}{N} \sum_{h \in \mathbb{Z}_N} |\widehat{\Delta_h f}(2h)|^2}\geq \varepsilon$, then it would be quite easy to proceed. Indeed after this out we obtain

$$\frac{1}{N^3} \sum_{x,h_1,h_2} f(x)\overline{f(x+h_1)}\,\overline{f(x+h_2)}\,f(x+h_1+h_2)\, e^{2\pi i (2h_1h_2)/N} \geq \varepsilon.$$

Using the neat observation that

$$x^2 - (x+h_1)^2 - (x+h_2)^2 + (x+h_1+h_2)^2 = 2h_1h_2$$

we seethat this can be written as

$$\|\tilde{f}\|_{U^2}^4 \geq \varepsilon \quad \text{where} \quad \tilde{f}(x) = f(x)e^{2\pi i x^2/N}$$

In light of the inverse theorem for the U^2-norm it immediately follows that $\exists\, \xi \in \mathbb{Z}_N$ such that $\left|\frac{1}{N}\sum_h f(h)e^{2\pi i (h^2 - h\xi)/N}\right| \geq \varepsilon^2$,

that is to say f correlates globally with a quadratic.

* This discussion motivates the proof that follows *

Proof of (Local) Inverse Theorem for U^3-norm

We know that $\exists$ $\mathbb{Z}_N$-prog Q with $|Q| \geq N^{\varepsilon^c}$ such that

$$\sum_{h \in Q} \left| \frac{1}{N} \sum_x \Delta_h f(x) e^{-2\pi i (2ah+b)x/N} \right| \geq \varepsilon^c |Q|.$$

Squaring out we obtain that

$$\sum_{h\in Q}\left|\frac{1}{N}\sum_{x}\Delta_h f(x)e^{-2\pi i(2ah+b)x/N}\right|^2$$

$$=\sum_{h\in Q}\frac{1}{N^2}\sum_{x}\sum_{y}f(x)\overline{f(x+h)}\,\overline{f(x+y)}f(x+h+y)e^{2\pi i(2ah+b)y/N}$$

$$=\sum_{x}f(x)\frac{1}{N^2}\sum_{h,y}\overline{f(x+h)}1_Q(h)\,\overline{f(x+y)}f(x+h+y)e^{2\pi i(2ah+b)y/N}.$$

Now write

$$(2ah+b)y=-\tilde{\varphi}_1(y)-\tilde{\varphi}_2(h)+\tilde{\varphi}_3(h+y)$$

where $\tilde{\varphi}_1(y)=ay^2-by$, $\tilde{\varphi}_2(h)=ah^2$ and $\tilde{\varphi}_3(z)=az^2$.

Setting

$$F_1(y)=f(x+y)e^{2\pi i\tilde{\varphi}_1(y)/N}$$

$$F_2(h)=f(x+h)1_Q(h)e^{2\pi i\tilde{\varphi}_2(y)/N}$$

$$F_3(z)=f(x+z)e^{2\pi i\tilde{\varphi}_3(z)/N}$$

suppressing dependence on x.

we see that

$$\sum_{x}f(x)\frac{1}{N^2}\sum_{h,y}\overline{F_1(y)}\,\overline{F_2(h)}F_3(y+h)\geq\varepsilon^c|Q|$$

$$\Rightarrow\quad\sum_{x}\underbrace{\left|\sum_{\xi}\overline{\widehat{F_1}(\xi)}\,\overline{\widehat{F_2}(\xi)}\widehat{F_3}(\xi)\right|}_{\leq\|\widehat{F_2}\|_\infty\ \text{(Cauchy-Schwarz and Plancherel)}}\geq\varepsilon^c|Q|$$

$\Rightarrow$ For each $x\in\mathbb{Z}_N$, $\exists\ \xi_x$ such that

$$\frac{1}{N}\sum_{x\in\mathbb{Z}_N}\left|\frac{1}{|Q|}\sum_{h\in Q+x}f(h)e^{-2\pi i[ah^2+(\xi_x-2ax)h]/N}\right|\geq\varepsilon^c. \qquad\square$$