Taken from

Harmonic Analysis by Elias M. Stein

A Besicovitch set is a set of measure zero that contains line segments in all possible directions.

1. *The Besicovitch set*

Shortly after its initial construction, it was noted that the Besicovitch set could be used to give a solution to the Kakeya "needle problem". The question was to find a plane set of least area in which a segment of unit length could be moved so that it pointed in all possible directions. While this historical aspect has remained something of a curiosity, the Besicovitch set has come to play an increasingly significant role in real-variable theory and Fourier analysis. Indeed, our accumulated experience allows us to regard the structure of this set as, in many ways, representative of the complexities of two-dimensional sets, in the same sense that Cantor-like sets already display some of the typical features that arise in the one-dimensional case.

Our presentation of the Besicovitch set will be in terms of a union of a large number of congruent thin rectangles in the plane, which have a

high degree of overlap. Each rectangle will have side lengths 1 and 2^{-N}, where N is a large fixed integer. Given such a rectangle R, we define its *reach* $\tilde{R}$, to be the rectangle obtained by translating R two units, in the positive direction, along the longer side of R. The relation between R and $\tilde{R}$ is displayed in Figure 1.

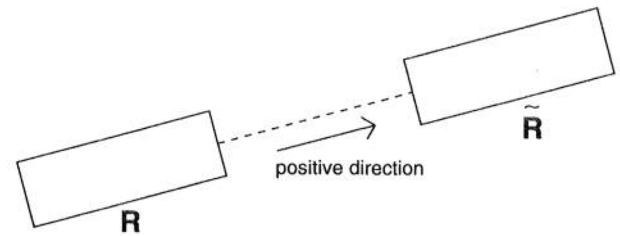

Figure 1. The reach $\tilde{R}$ of the rectangle R.

THEOREM 1. *Given any $\varepsilon > 0$, there exists an integer $N = N_\varepsilon$, and 2^N rectangles $R_1, \ldots, R_{2^N}$, each having side lengths 1 and 2^{-N}, so that*

(i) $\left|\bigcup_{j=1}^{2^N} R_j\right| < \varepsilon$, *and*

(ii) *The $\tilde{R}_j$ are mutually disjoint, $j = 1, \ldots, 2^N$, and so*

$$\left|\bigcup_{j=1}^{2^N} \tilde{R}_j\right| = 1.$$

Here $\tilde{R}_j$ denotes the "reach" of R_j, as defined above.

1.1 While the properties of the Besicovitch set used below are most easily stated in terms of rectangles, for the actual construction it seems necessary to work with triangles. More precisely, one proceeds as follows. An initial triangle is partitioned into a large number of smaller triangles, obtained by equally dividing the base of the original triangle. The main point then is that these subtriangles can be translated so that their union has small measure.

The translation procedure used arises as a series of iterations of a basic procedure, which we will now describe.

1.1.2 We start with a triangle T and construct from it a figure $\Phi(T)$, which is the union of translates of two subtriangles of T. To specify the construction, we fix a constant of proportionality α with $1/2 < \alpha < 1$. Suppose that T is the triangle ABC, whose base AB lies along the x-axis. We bisect the base AB at M, obtaining thereby two subtriangles: the "left" triangle AMC and the "right" triangle MBC (Figure 2).

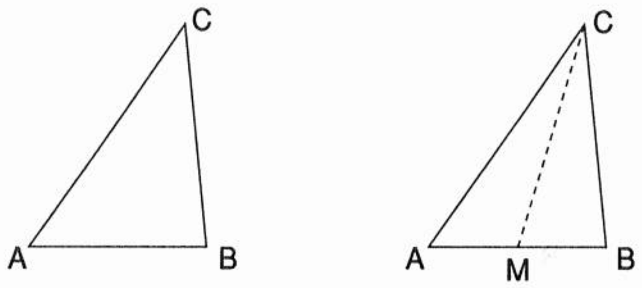

Figure 2. Bisecting the triangle T.

Now we translate the "right" triangle leftwards to obtain the overlapping figure $\Phi(T)$ (Figure 3).

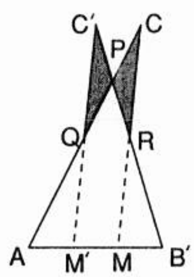

Figure 3. The overlapping figure $\Phi(T)$.

Here $M'B'C'$ is the translate of MBC; the left triangle AMC has remained fixed. The figure $\Phi(T)$ is the union of two parts: the smaller triangle $\Phi_h(T) = AB'P$ (which is similar to the original triangle ABC), and the union $\Phi_a(T)$ of the two small shaded triangles.

We shall call $\Phi_h(T)$ the "heart" of the figure $\Phi(T)$ and $\Phi_a(T)$ the "arms" of $\Phi(T)$. The constant α is the side-length ratio between the (similar) heart triangle $\Phi_h(T)$ and original triangle T. It determines the figure $\Phi(T)$, once the triangle T is given. We observe first that

$$|\Phi_h(T)| = \alpha^2 \cdot |T|. \tag{1}$$

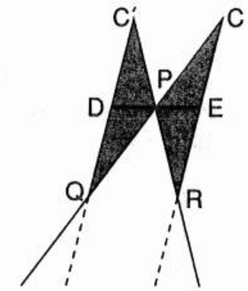

Figure 4. The "arms" $\Phi_a(T)$ of $\Phi(T)$.

Next, we fix our attention on $\Phi_a(T)$ (Figure 4). We draw the line DE parallel to the base AB' that passes through the intersection point P. It divides $\Phi_a(T)$ into four triangles. A moment's reflection shows that the triangle $C'DP$ is similar to the "right" triangle $C'M'B'$, with ratio $1-\alpha$. Also, PER is congruent to the triangle $C'DP$. Moreover, the triangle PEC is similar to the "left" triangle AMC, with ratio $1-\alpha$; also the triangle PEC is congruent to PDQ. Altogether then:

$$|\Phi_a(T)| = 2(1-\alpha)^2 \cdot |T|. \tag{2}$$

Therefore

$$|\Phi(T)| = [\alpha^2 + 2(1-\alpha)^2] \cdot |T|. \tag{3}$$

We will use this process to generate our monster, which will have a tiny heart and many arms.

1.1.2 We now describe the result of an "n-fold iteration" of this construction. Starting from the fixed triangle ABC, we subdivide the base AB into 2^n equal subintervals, with division points

$$A = A_0, A_1, \ldots, A_{2^n} = B.$$

In this way, our original triangle is divided into 2^{n-1} disjoint smaller triangles, namely

$$A_{2j}A_{2j+2}\tilde{C}, \qquad 0 \le j < 2^{n-1};$$

the base of such a triangle has midpoint A_{2j+1}.

Now with α fixed throughout, we construct the figure $\Phi(A_{2j}A_{2j+2}C)$ as above, for each of the 2^{n-1} triangles. In so doing, we obtain 2^{n-1} "hearts" and also 2^{n-1} pairs of "arms", in the above terminology. By our construction, the right side of $\Phi_h(A_{2j}A_{2j+2}C)$ is parallel to the line CA_{2j+2}, as is the left side of $\Phi_h(A_{2j+2}A_{2j+4}C)$ (here $0 \le j < 2^{n-1} - 1$). Thus the triangle $A_{2j+2}A_{2j+4}C$ can be moved leftwards so that the left side of $\Phi_h(A_{2j+2}A_{2j+4}C)$ coincides with the right side of $\Phi_h(A_{2j}A_{2j+2}C)$. Carrying out such a translation for all the triangles $A_{2j}A_{2j+2}C$, $0 < j < 2^{n-1}$, we can incorporate each of these 2^{n-1} hearts into one composite heart, which is similar to the original triangle ABC.

To summarize, we have translated the 2^n subtriangles of ABC, forming a figure that we call $\Psi_1(ABC)$. As stated above, it contains a "heart", namely the disjoint union of the translates of the

$$\Phi_h(A_{2j}A_{2j+2}C).$$

The rest of $\Psi_1(ABC)$ consists of the union of the translated

$$\Phi_a(A_{2j}A_{2j+2}C),$$

which we refer to as the "arms" of $\Psi_1(ABC)$. Observe that there can be considerable overlap among these arms, although we will not take advantage of this.

Since $|A_{2j}A_{2j+2}C| = 2^{-n+1}|ABC|$, (1) gives us that

$$|\text{heart of } \Psi_1(ABC)| = \sum_{j=0}^{2^{n-1}-1} |\Phi_h(A_{2j}A_{2j+2}C)| = \alpha^2|ABC|.$$

Also

$$|\text{arms of } \Psi_1(ABC)| \le \sum |\Phi_a(A_{2j}A_{2j+2}C)| = 2(1-\alpha)^2 \cdot |ABC|,$$

by (2). Thus

$$|\Psi_1(ABC)| \le [\alpha^2 + 2(1-\alpha)^2] \cdot |ABC|. \tag{4}$$

We now iterate the construction leading to Ψ_1 as follows. The heart of $\Psi_1(ABC)$ is given to us as the union of 2^{n-1} triangles, and so we carry out the above process on the heart of $\Psi_1(ABC)$ with n replaced by $n-1$; then we re-translate all 2^n of the original triangles $A_jA_{j+1}C$, $0 \le j < 2^n$, to obtain the figure $\Psi_2(ABC)$. The area of its heart is then $\alpha^2 \cdot \alpha^2 \cdot |ABC|$; also, the area of the additional arms generated at this stage will not exceed $2(1-\alpha^2)\alpha^2 \cdot |ABC|$. We continue in this way, finally obtaining $\Psi_n(ABC)$.

It follows from (4) that

$$|\Psi_n(ABC)| \le [\alpha^{2n} + 2(1-\alpha^2) + 2(1-\alpha^2)\alpha^2 + \cdots + 2(1-\alpha^2)\alpha^{2n-2}] \cdot |ABC|.$$

However

$$2(1-\alpha^2) + \cdots + 2(1-\alpha^2)\alpha^{2n-2} \le 2(1-\alpha^2)\sum_{j=0}^{\infty} \alpha^{2j}$$
$$= \frac{2(1-\alpha^2)}{1-\alpha^2} \le 2(1-\alpha).$$

Therefore

$$|\Psi_n(ABC)| \le [\alpha^{2n} + 2(1-\alpha)] \cdot |ABC|. \tag{5}$$

The set $\Psi_n(ABC)$ is essentially the Besicovitch set we are after. Note that if we take α close to 1 and then n large, the factor $\alpha^{2n} + 2(1-\alpha)$ can be made as small as we wish.

1.1.3 The following observation gives the thrust of the preceding construction.

Let T_j, $0 \le j < 2^n$, denote the triangles $A_jA_{j+1}C$ that make up ABC, and let T'_j denote the corresponding translated triangles comprising $\Psi_n(ABC)$. Of course, the T_j have a common vertex C; let C_j be the corresponding vertices of the T'_j. Denote by T^*_j the triangles obtained by reflecting the T'_j through C_j. While we have seen that the triangles T'_j overlap to a high degree, the reflected triangles T^*_j are mutually disjoint.

In fact, if T_{j_2} was originally to the right of T_{j_1}, then by construction T_{j_2} was moved leftwards (relative to T_{j_1}), so C_{j_2} is to the left of C_{j_1}. The relative positions of T'_{j_1} and T'_{j_2} are then as in Figure 5, from which the disjointness of $T^*_{j_1}$ and $T^*_{j_2}$ is clear.

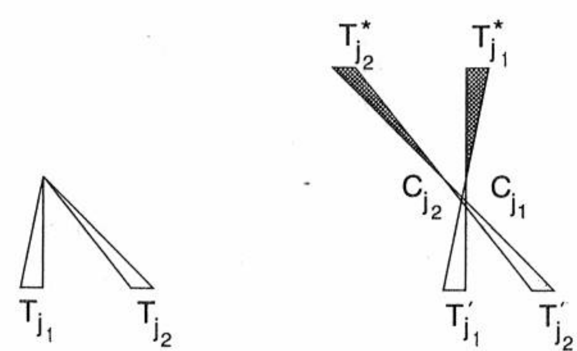

Figure 5. Reflected subtriangles are disjoint.

1.1.4 To complete our construction of the Besicovitch set, we pass from the triangles above to rectangles. We begin by fixing the original triangle ABC to be the equilateral triangle whose altitude has length 2. Next, suppose T_j' is one of the triangles making up $\Psi_n(ABC)$. We draw a line from its vertex C_j to the midpoint of its base, marking off the points P_1 and P_2 on it at distances $1/2$ and $3/2$ from the vertex. We let R_j denote the rectangle whose major axis is P_1P_2, whose side lengths are 1 and 2^{-N}; here $N = n + c_1$, where c_1 is a fixed large integer (see Figure 6). Since the angle at the vertex C_j' is larger than $c_2 \cdot 2^{-n}$, for some small positive constant c_2, we can always choose c_1 large enough so that $R_j \subset T_j'$. Now let $\tilde{R}_j \subset T_j^*$ be the reflection of R_j through C_j.

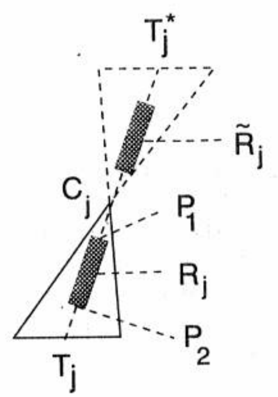

Figure 6. End of the proof of Theorem 1.

We have therefore 2^n rectangles R_j of dimension (1×2^{-N}), so that their "reaches" $\tilde{R}_j$ are disjoint. To pass to the corresponding $2^N = 2^{c_1} \cdot 2^n$ such rectangles, notice that the figure $\Psi_n(ABC)$, together with all its reflected triangles, belongs to a fixed compact set. Taking 2^{c_1} disjoint translates of such sets finally gives us 2^N rectangles, with side lengths 1 and 2^{-N}, which are contained in a set of measure at most

$$2^{c_1} \cdot [\alpha^{2n} + 2(1-\alpha)] \cdot |T|,$$

while the corresponding "reach" rectangles are all disjoint. If we take α sufficiently close to 1, and $N = n + c_1$ sufficiently large, we can make this measure arbitrarily small, completing the proof of the theorem.