Math 3100 **Spring 2019**

Exam 2

No calculators. Show your work. Give full explanations. Good luck!

1. (15 points)

   (a) Carefully state the definition of what it means to say that $\sum_{n=1}^{\infty} a_n$ is convergent.

   (b) i. Prove that if $\sum_{n=1}^{\infty} a_n$ is convergent, then $\lim_{n\to\infty} a_n = 0$.

   ii. Is it true that if $\lim_{n\to\infty} a_n = 0$, then $\sum_{n=1}^{\infty} a_n$ is convergent? Give either a proof or counterexample.

   (c) Suppose that both $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ are convergent series.

   i. Prove that $\sum_{n=1}^{\infty} (a_n + b_n)$ is convergent.

   ii. Prove that $\sum_{n=1}^{\infty} (a_n\, b_n)$ is convergent <u>if</u> both $a_n \geq 0$ and $b_n \geq 0$ for all $n \in \mathbb{N}$.

   iii. Is it true that $\sum_{n=1}^{\infty} (a_n\, b_n)$ is always convergent without the additional assumptions that both $a_n \geq 0$ and $b_n \geq 0$ for all $n \in \mathbb{N}$? Give either a proof or counterexample.

2. (20 points)

   (a) Determine if the following series are absolutely convergent, conditionally convergent, or divergent. Justify your answers.

   (i) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n n}{\sqrt{n^2+1}}$ (ii) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n^2+1}}$ (iii) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{n\sqrt{n^2+1}}$ (iv) $\displaystyle\sum_{n=1}^{\infty} \frac{(\log n)^{10}}{n^{5/4}}$

   (b) Find all $x \in \mathbb{R}$ for which $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n x^n}{\sqrt{n}}$ converges.

   (c) Find a sequence $\{a_n\}$ so that $\sum_{n=2}^{\infty} a_n x^n = \dfrac{x^2}{2+x}$ for all $|x| < 2$.

3. (15 points)

   (a) i. Let $X \subseteq \mathbb{R}$ and $f : X \to \mathbb{R}$. Carefully state the ε-δ definition of what it means for f to be *continuous* at a point $x_0 \in X$.

   ii. Use this ε-δ definition to prove that $f(x) = \dfrac{1}{x^2}$ is continuous at $x_0 = 1$.

   (b) i. Prove that if a function $g : \mathbb{R} \to \mathbb{R}$ is *continuous* at x_0, then $\lim_{n\to\infty} g(x_n) = g(x_0)$ for all sequences $\{x_n\}$ with $\lim_{n\to\infty} x_n = x_0$.

   ii. Prove that the function

   $$g(x) = \begin{cases} x & \text{if } x \in \mathbb{Q} \\ 0 & \text{if } x \in \mathbb{R} \setminus \mathbb{Q} \end{cases}$$

   is not continuous at $x_0 = 1$.
   List all points $x_0 \in \mathbb{R}$ where g <u>is</u> continuous, no justification is required.