Approximation to the identity

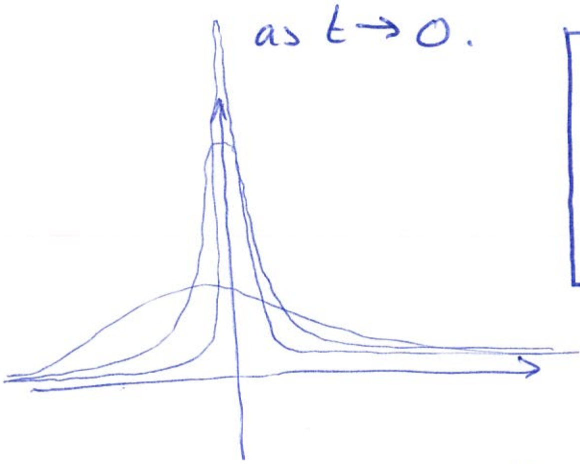
The following theorem underlies many of the important applications of convolutions on $\mathbb{R}^n$. First some notation:

If φ is any function on $\mathbb{R}^n$ and $t>0$, we define

$$\varphi_t(x) = \frac{1}{t^n}\varphi\left(\frac{x}{t}\right).$$

Note: If $\varphi \in L^1$, then $\int \varphi_t = \int \varphi\left(\frac{x}{t}\right)\frac{dx}{t^n} = \int \varphi \quad \forall t>0$.

Moreover, the "mass" of φ_t becomes concentrated at origin as $t \to 0$.



[Easy Exercise (†)
For any $\eta>0$, $\int_{|x|\geq\eta} \varphi_t(x)\,dx \to 0$ as $t\to 0$].

Terminology: If $\int\varphi = 1$, then $\{\varphi_t\}_{t>0}$ called an approximate identity.

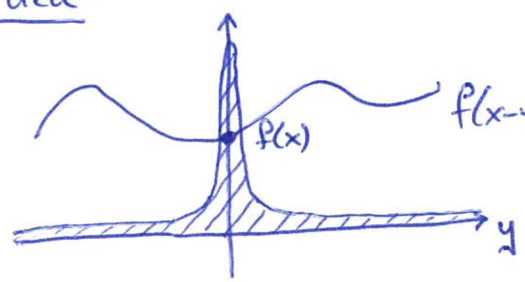
Theorem 1: Suppose $\varphi \in L^1(\mathbb{R}^n)$ and $\int \varphi = 1$. If $f \in L^1(\mathbb{R}^n)$, then

$$f * \varphi_t \to f \text{ in } L^1 \text{ as } t \to 0.$$

Remark: If we impose the additional assumption that $|\varphi(x)| \leq C(1+|x|)^{-n-\varepsilon}$ for some $C, \varepsilon > 0$, then we can conclude that for every $f \in L^1$

$f * \varphi_t(x) \to f(x)$ a.e. (We do not prove this here)

Idea:



$$\text{“}\int f(x-y)\varphi_t(y)\,dy \approx \int f(x)\varphi_t(y)\,dy\text{”}$$

for $t \to 0$.

Proof 1: $$f * \varphi_t(x) - f(x) = \int [f(x-y) - f(x)]\varphi_t(y)\,dy \quad (\text{since } \textstyle\int \varphi_t = 1)$$

$$= \int [f(x-tz) - f(x)]\varphi(z)\,dz \quad (\text{let } y = tz)$$

Fubini/Tonelli implies that

$$\|f * \varphi_t - f\|_1 \le \int |\varphi(z)| \underbrace{\left(\int |f(x-tz) - f(x)|\,dx\right)}_{\le 2\|f\|_1} dz$$

$\underline{\underline{\&}}$ $\to 0$ as $t \to 0$ for all fixed z.

Result follows by the Dominated convergence Theorem. $\square$.

Proof 2: Using that $\int \varphi_t = 1$ and Fubini/Tonelli we see that

$$\|f * \varphi_t - f\|_1 \le \int |\varphi_t(y)| \Big(\underbrace{\int |f(x-y) - f(x)|\,dx}_{(*)}\Big)\,dy$$

(same argument as above without the change of variables).

As above, we note that $(*) \le 2\|f\|_1$, and that for any $\varepsilon > 0$ $\exists\, \eta > 0$ s.t. $(*) \le \varepsilon$ provided $|y| < \eta$.

$$= \underbrace{\int_{|y|<\eta} \text{—''—}\,dy}_{\le \varepsilon \|\varphi\|_1} + \underbrace{\int_{|y|\ge\eta} \text{—''—}\,dy}_{\le \varepsilon\, 2\|f\|_1}$$

using (†)

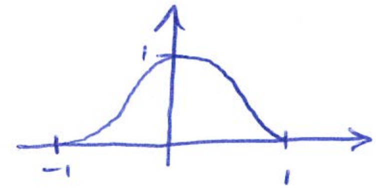
provided t small enough!

An Application

Theorem 2: $C_c^\infty(\mathbb{R}^n)$ is dense in $L^1(\mathbb{R}^n)$.

Remark: Perhap even the existence of C_c^∞ is unclear?

Ex: $\psi(x) = \begin{cases} e^{1/(|x|^2-1)} & , \text{ if } |x|<1 \\ 0 & , \text{ if } |x| \geq 1 \end{cases}$



[Recall that $\eta(t) = \begin{cases} e^{-1/t} & , t>0 \\ 0 & , t \leq 0 \end{cases}$ is C^∞ even at origin!].

Proof: Let $f \in L^1$ & $\varepsilon > 0$.

We know $\exists\, g \in C_c$ s.t. $\|f-g\|_1 < \varepsilon/2$. It thus suffices to show that $\exists\, h \in C_c^\infty$ s.t. $\|h-g\|_1 < \varepsilon/2$:

Let $\varphi \in C_c^\infty$ with $\int \varphi = 1$, ~~for example take~~

Claim: $g * \varphi_t \in C_c^\infty \ \forall\, t > 0$.

Pf: We know (from ~~Thm~~ Corollary to Thm 3 on "Convolution Notes") that $g * \varphi_t \in C^\infty \ \forall\, t>0$. To see that $g * \varphi_t$ is compactly supported, note that since both g & φ_t are compactly supported, $\exists\, N > 0$ s.t.

$$g(x-y) = 0 \text{ if } |x-y| \geq N \ \& \ \varphi_t(y) = 0 \text{ if } |y| \geq N.$$

Since $|x| \leq |x-y| + |y|$, it follow that if $|x| \geq 2N$, then *either*

$$|x-y| \geq N \text{ or } |y| \geq N \Rightarrow g * \varphi_t(x) = 0. \qquad \square$$

To conclude proof, note that $\|g * \varphi_t - g\|_1 < \varepsilon/2$ provided t suff. small (Thm 1).