STRONGLY SINGULAR CONVOLUTION OPERATORS ON $\mathbf{R}^d$

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These are convolution operators whose kernels are too singular at the origin to fall under the theory of Calderón and Zygmund, and clearly must have built-in oscillation in order to extend to bounded operators on $L^2(\mathbf{R}^d)$. What is of interest here is the precise relationship between the size of the singularity and the size of the oscillation.

Let K_α be a distribution on $\mathbf{R}^d$ that away from the origin agrees with the function

$$K_\alpha(x) = |x|^{-d-\alpha} e^{i|x|^{-\beta}} \chi(|x|),$$

where $\beta > 0$ and χ is smooth and compactly supported in a small neighborhood of the origin (say where $|x| \leq \frac{1}{10}$). The distribution-valued function $\alpha \mapsto K_\alpha$, initially defined for $\operatorname{Re}\alpha < 0$, continues analytically to the entire complex plane[1].

Theorem [7],[2]**.** *If $\alpha > 0$, then the operator $T_E f(x) = f * K_\alpha(x)$, defined initially for test functions, extends to a bounded operator on $L^p(\mathbf{R}^d)$ whenever $\left|\frac{1}{p} - \frac{1}{2}\right| \leq \frac{1}{2} - \frac{\alpha}{d\beta}$.*

The counterexample showing that this result is sharp is due to Wainger as is sufficiency up to the endpoints. The endpoint questions were settled later; for $\alpha = 0$ Fefferman in [1] showed that T_E extends to an operator of weak type (1,1) and later in [2] Fefferman and Stein showed that T_E in fact extends to a bounded operator from $H^1(\mathbf{R}^d)$ to $L^1(\mathbf{R}^d)$. The sufficient half of the Theorem then follows by interpolation with the $L^2(\mathbf{R}^d)$ result below (and duality).

Lemma 1. *If $\alpha \leq \frac{d\beta}{2}$ then T_E extends to a bounded operator from $L^2(\mathbf{R}^d)$ to itself.*

Sketch of proof. Since T_E is translation invariant it may be realized as a Fourier multiplier,

$$\widehat{T_E f}(\xi) = \widehat{f}(\xi) \cdot m(\xi),$$

where $\widehat{\ }$ denotes the Fourier transform and $m = \widehat{K_\alpha}$, the fact that K_α is a compactly supported distribution ensures that $m(\xi)$ is a function. Plancherel's theorem then implies that

$$\|T_E f\|_{L^2(\mathbf{R}^d)} \leq C\|f\|_{L^2(\mathbf{R}^d)} \quad \text{if and only if} \quad |m(\xi)| \leq C, \text{ uniformly in } \xi.$$

Since K_α is also radial we have

$$m(\xi) = (2\pi)^{\frac{d}{2}} |\xi|^{\frac{2-d}{2}} \int_0^\infty K_0(r) J_{\frac{d-2}{2}}(r|\xi|) r^{\frac{d}{2}} dr,$$

where $J_{\frac{d-2}{2}}$ is a Bessel function; see [6]. Using the well known asymptotic properties of these functions it follows that for large $|\xi|$,

$$m(\xi) = c_1 |\xi|^{\frac{\alpha - \frac{d\beta}{2}}{\beta+1}} e^{ic_2|\xi|^{\frac{\beta}{\beta+1}}} + O\big(|\xi|^{\frac{\alpha - \frac{d+1}{2}\beta}{\beta+1}}\big),$$

and in particular, since $m(\xi)$ remains bounded for small ξ, that $|m(\xi)| \leq C(1+|\xi|)^{\frac{\alpha - \frac{d\beta}{2}}{\beta+1}}$. ☐

We have in fact shown that $\|T_E f\|_{L^2(\mathbf{R}^d)} \leq C\|f\|_{L^2(\mathbf{R}^d)}$ if and only if $\alpha \leq \frac{d\beta}{2}$.

Lemma 2. *If $\alpha = 0$ then T_E extends to a bounded operator from $H^1(\mathbf{R}^d)$ to $L^1(\mathbf{R}^d)$.*

[1] Continue analytically the function $K_z^\varepsilon(x) = e^{-\varepsilon|x|^{-\beta}} K_z(x)$ via integration by parts and then let $\varepsilon \to 0$.

Sketch of proof. For any f in $H^1(\mathbf{R}^d)$ we have the atomic decomposition

$$f = \sum_Q \lambda_Q \, a_Q \qquad \text{where} \qquad \sum_Q |\lambda_Q| \sim \|f\|_{H^1(\mathbf{R}^d)},$$

and the individual atoms satisfy the following;

$$\text{(i) } \operatorname{supp} a_Q \subset Q \qquad \text{(ii) } \|a_Q\|_\infty \leq |Q|^{-1} \qquad \text{(iii) } \int a_Q(x)dx = 0.$$

As a consequence of this characterization of H^1 it suffices to check that for an individual atom a_Q

$$\int |Ta_Q(x)|dx \leq C,$$

where C is independent of a_Q. Since T is translation–invariant, we assume that $a = a_Q$ is supported in a cube centered at the origin. If $Q^* = 2Q$, then Cauchy–Schwarz and Lemma 1 imply that

$$\int_{Q^*} |Ta(x)|dx \leq C|Q^*|^{\frac{1}{2}} \|a\|_2 \leq C|Q^*|^{\frac{1}{2}} |Q|^{-\frac{1}{2}} \leq C.$$

Let $\ell = \ell(Q)$ denote the sidelength of Q. Now if $\ell \geq 1$ then it follows from the compact support of our kernel K that $\operatorname{supp} Ta \subset Q^*$, from the argument above our result follows in this case. We may now assume that $\ell < 1$.

Let us first look at x in the complement of the following, yet to be determined, exceptional set

$$\mathcal{E}_Q = \{x : |x| \leq \ell^\gamma\}.$$

Now it is straightforward to see, using the cancellation of our atom a, that

$$\int_{x \notin \mathcal{E}_Q} |Ta(x)|dx \leq \int |a(y)| \int_{x \notin \mathcal{E}_Q} |K(x-y) - K(x)|dx\,dy \leq C\ell \int_{x \notin \mathcal{E}_Q} |x|^{-d-\beta-1} dx \leq C\ell^{1-\gamma(\beta+1)}.$$

In view of the calculation above we now fix our exceptional set $\mathcal{E}_Q$ with $\gamma = \frac{1}{\beta+1}$. It remains to consider those x in $\mathcal{E}_Q$, we shall as usual use an L^2 result, namely Lemma 1.

$$\int_{\mathcal{E}_Q} |Ta(x)|dx \leq |\mathcal{E}_Q|^{\frac{1}{2}} \|Ta\|_2 \leq C|Q|^{\frac{1}{2}\frac{1}{\beta+1}} \big\|T\Lambda_{\frac{d\beta}{2(\beta+1)}} \Lambda_{-\frac{d\beta}{2(\beta+1)}} a\big\|_2 \leq C|Q|^{\frac{1}{2}\frac{1}{\beta+1}} \big\|\Lambda_{-\frac{d\beta}{2(\beta+1)}} a\big\|_2,$$

where Λ_n is the Bessel potential of order n defined on the transform side by $\widehat{\Lambda_n f}(\xi) = (1+|\xi|^2)^{\frac{n}{2}} \widehat{f}(\xi)$. Now, as $\frac{d\beta}{2(\beta+1)} < d$, we can dominate $\Lambda_{-\frac{d\beta}{2(\beta+1)}} f$ by the fractional integral $f * |x|^{-d+\frac{d\beta}{2(\beta+1)}}$. Therefore, by the Hardy–Littlewood–Sobolev inequality, we have that

$$\big\|\Lambda_{-\frac{d\beta}{2(\beta+1)}} a\big\|_2 \leq C\|a\|_{p(d)} \quad \text{where} \qquad \frac{1}{p(d)} = \frac{1}{2} + \frac{1}{2}\frac{\beta}{\beta+1},$$

and hence

$$\int_{\mathcal{E}_Q} |Ta(x)|dx \leq C|Q|^{\frac{1}{2}\frac{1}{\beta+1}} \|a\|_{p(d)} \leq C|Q|^{\frac{1}{2}\frac{1}{\beta+1}} |Q|^{-1+\frac{1}{p(d)}} \leq C. \qquad \square$$

References

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