Almost-Periodicity of Convolutions & Applications I

Background Material: Fourier analysis on $\mathbb{Z}_N$.

Given a function $f: \mathbb{Z}_N \to \mathbb{C}$ we define its Fourier transform

$$\hat{f}(\xi) := \mathbb{E}_x f(x) e^{-2\pi i x\xi/N}$$

Facts:

- $|\hat{f}(\xi)| \leq \mathbb{E}_x |f(x)| \quad \forall \xi \in \mathbb{Z}_N$
- $\sum_{\xi \in \mathbb{Z}_N} |\hat{f}(\xi)|^2 = \mathbb{E}_x |f(x)|^2$ (Plancherel)
- If $f(x) = g(x+t)$, then $\hat{f}(\xi) = e^{2\pi i t\xi/N} \hat{g}(\xi)$
- $f * g(x) = \mathbb{E}_y f(x-y) g(y)$ & $\widehat{f*g} = \hat{f}\,\hat{g}$.

Exercise 1: Prove these fact using the well known orthogonality relation

$$\mathbb{E}_x e^{2\pi i x\xi/N} = \begin{cases} 1 & \text{if } \xi = 0 \\ 0 & \text{if } \xi \neq 0. \end{cases}$$

Important Special Case: If $A \subseteq \mathbb{Z}_N$ with $|A| = \delta N$, then

(i) $|\hat{1}_A(\xi)| \leq \mathbb{E}_x |1_A(x)| = \delta$ for all $\xi \in \mathbb{Z}_N$

(ii) $\sum_{\xi \in \mathbb{Z}_N} |\hat{1}_A(\xi)|^2 = \mathbb{E}_x |1_A(x)|^2 = \delta$

Proposition 1 For any $\varepsilon > 0$ & $A \subseteq \mathbb{Z}_N$ with $|A| = \delta N$, there exists a symmetric arith. prog P with $|P| \geq \varepsilon N^{\varepsilon^2\delta}$ such that

$$\mathbb{E}_x |1_A * 1_A(x+t) - 1_A * 1_A(x)|^2 \leq 5\varepsilon^2\delta^3 \quad \text{for all } t \in P.$$

Proof of Proposition (Bogolyubov ?)

It follows from the properties of the Fourier transform, that

$$\mathbb{E}_x |1_A * 1_A(x+t) - 1_A * 1_A|^2 = \sum_{\xi \in \mathbb{Z}_N} |\hat{1}_A(\xi)|^4 |e^{2\pi i t\xi/N} - 1|^2$$

Let

$$\Gamma = \{\xi \in \mathbb{Z}_N : |\hat{1}_A(\xi)| \geq \delta\varepsilon\}$$

$$\delta = \sum_{\xi \in \mathbb{Z}_N} |\hat{1}_A(\xi)|^2 \geq \sum_{\xi \in \Gamma} |\hat{1}_A(\xi)|^2 \geq (\delta\varepsilon)^2 |\Gamma|$$

& note that by Plancherel $|\Gamma| \leq 1/\delta\varepsilon^2$.

Define the Bohr set $B(\Gamma, \varepsilon) := \{x \in \mathbb{Z}_N : |e^{2\pi i x\xi/N} - 1| \leq \varepsilon \ \forall \xi \in \Gamma\}$

Exercise 2: Use the pigeonhole principle to show that

- $|B(\Gamma, \varepsilon)| \geq \varepsilon^{|\Gamma|} N$

and deduce from this that $B(\Gamma, \varepsilon)$ contains a symm AP of length $\varepsilon N^{1/|\Gamma|}$.

In light of exercise 2 it suffices to show that

$$\sum_{\xi \in \mathbb{Z}_N} |\hat{1}_A(\xi)|^4 |e^{2\pi i t\xi/N} - 1|^2 \leq 5\varepsilon^2\delta^3 \quad \forall t \in B(\Gamma, \varepsilon).$$

Write

$$\sum_{\xi \in \mathbb{Z}_N} |\hat{1}_A(\xi)|^4 |e^{2\pi i t\xi/N} - 1|^2 = \underbrace{\sum_{\xi \notin \Gamma} |\hat{1}_A(\xi)|^4 |e^{2\pi i t\xi/N} - 1|^2}_{(*)} + \underbrace{\sum_{\xi \in \Gamma} |\hat{1}_A(\xi)|^4 |e^{2\pi i t\xi/N} - 1|}_{(**)}$$

$$(*) \leq 4 \sum_{\xi \notin \Gamma} |\hat{1}_A(\xi)|^4 \leq 4(\varepsilon\delta)^2 \sum_{\xi \in \mathbb{Z}_N} |\hat{1}_A(\xi)|^2 = 4\varepsilon^2\delta^3$$

$$(**) \leq \varepsilon^2 \sum_{\xi \in \Gamma} |\hat{1}_A(\xi)|^4 \leq \varepsilon^2\delta^2 \sum_{\xi \in \mathbb{Z}_N} |\hat{1}_A(\xi)|^2 = \varepsilon^2\delta^3$$

□

As an application we will sketch the proof of the following

Theorem 1: If $r_3(N) := \max_{A \subseteq \{1,\dots,N\}} \{|A| : A \text{ contains } \underline{\text{no}} \text{ nontrivial 3APs}\}$, then

$$\frac{r_3(N)}{N} \ll \exp\left(-c\sqrt{\log\log N}\right).$$

Remark: In 1936, Erdős & Turán conjectured that $\frac{r_3(N)}{N} \to 0$ as $N \to \infty$.

In 1953, Roth proved that in fact $\frac{r_3(N)}{N} \ll \frac{1}{\log\log N}$.

Erdős later conjecture that if $A \subseteq \mathbb{N}$ and contains no non-trivial 3APs then $\sum_{n\in A} \frac{1}{n} < \infty$. Note that this is equivalent to the statement that

$$\sum_{N=1}^{\infty} \frac{r_3(N)}{N^2} < \infty \qquad \text{(Exercise)}$$

and hence the Erdős conjecture would follow if one could show

$$\frac{r_3(N)}{N} \ll \left(\frac{1}{\log N}\right)^{1+\varepsilon} \text{ for some } \varepsilon > 0.$$

In 2011, Sanders proved that $\frac{r_3(N)}{N} \ll \frac{(\log\log N)^5}{\log N}$.

Sketch proof of Theorem 1: We will cheat and assume that $A \subseteq \mathbb{Z}_N$, what one should actually do is identify $\{1,\dots,N\}$ with a subset of $\mathbb{Z}_p$ with $2N < p < 4N$. I will leave it to you to fix the proof & fill in the other gaps...

We start by assuming that $A \subseteq \mathbb{Z}_N$ with $|A| = \delta N$ that contains no non-trivial 3APs. Our goal is to show that $\delta \ll \exp\left(-c\sqrt{\log\log N}\right)$.

For a given $f: \mathbb{Z}_N \to \mathbb{C}$ we define the operator

$$\underline{AP_3(f) := \mathbb{E}_x f(x)\, f*f(2x)} = \mathbb{E}_x \mathbb{E}_y f(y) f(x) f(2x-y)$$

$$y \overset{x-y}{\frown} x \overset{x-y}{\frown} 2x-y$$

Note that $\underline{AP_3(1_A) = \frac{\delta}{N}}$ (by assumption)

Proposition 1 (with $\varepsilon = \delta/1000$) gives the existence of a symmetric AP, P, with $|P| \geq \varepsilon N^{\delta\varepsilon^2}$ such that

$$\mathbb{E}_x |1_A * 1_A(x+t) - 1_A * 1_A(x)|^2 \leq 5\varepsilon^2\delta^3 .$$

Using this progression P we define a new progression Q which is also symmetric, has the same step size as P, but is $1/4$ the length.

Note: $Q+Q+Q+Q \subseteq P$ (this will be important later).

Claim: If $N \gg \delta^{-c}$, then $\exists\, x$ s.t. $\frac{|A \cap (x+Q)|}{|Q|} \geq 2\delta$

Exercise 3: Show that if this Claim is true, then Theorem 1 follows!.

Sketch Proof of Claim: Suppose $\frac{|A \cap (x+Q)|}{|Q|} < 2\delta$ for all $x \in \mathbb{Z}_N$.

Let $\mu_Q(x) := \frac{N}{|Q|} 1_Q(x)$ & note that $1_A * \mu_Q(x) = \frac{|A \cap (x+Q)|}{|Q|}$.

Now if we define $f(x) := \frac{1}{2\delta} 1_A * \mu_Q(x)$, then it follow that

$$\left.\begin{array}{l} \text{(i) } 0 \leq f \leq 1 \\ \text{(ii) } \mathbb{E}_x f(x) = \frac{1}{2} \end{array}\right\} \Longrightarrow AP_3(f) \geq \underset{\text{(say)}}{\frac{1}{800}} \Rightarrow \underline{AP_3(1_A * \mu_Q) \geq \frac{\delta^3}{100}}$$

this is the sketchy part!
(Varnavides)

"large"

But $AP_3(1_A * \mu_Q)$ should be close to $AP_3(1_A)$ and hence "small".

Indeed it is true that

$$|AP_3(1_A * \mu_Q) - AP_3(1_A)| \leq 3\varepsilon\delta^2 \qquad (*)$$

and hence

$$\underline{AP_3(1_A * \mu_Q)} \leq 3\varepsilon\delta^2 + \frac{\delta}{N} < \underline{4\varepsilon\delta^2}$$

for large enough N

a contradiction if $4\varepsilon\delta^2 \leq \frac{\delta^3}{100} \Leftrightarrow \varepsilon \leq \delta/400$. □

Proof of $(*)$:

$$AP_3(1_A * \mu_Q) = \mathbb{E}_x\, 1_A * 1_A * \mu_Q * \mu_Q(2x)\, \underbrace{1_A * \mu_Q(x)}_{= \mathbb{E}_{y\in Q} 1_A(x-y)}$$

$$= \mathbb{E}_x 1_A(x)\, \mathbb{E}_{y\in Q}\left[\mathbb{E}_{z,w\in Q} 1_A * 1_A(2x+2y-z-w)\right]$$

$$\Rightarrow |AP_3(1_A * \mu_Q) - AP_3(1_A)|$$

$$\leq \mathbb{E}_x \left| 1_A * 1_A(2x+2y-z-w) - 1_A * 1_A(2x) \right| 1_A(x)$$

Cauchy-Schwarz

$$\leq \left(\mathbb{E}_x |\text{---}''\text{---}|^2\right)^{1/2} \Big(\underbrace{\mathbb{E}_x |1_A(x)|^2}_{=\delta}\Big)^{1/2}$$

$Q+Q+2Q \subseteq P$ & Propn 1.

$$\leq (5\varepsilon^2\delta^3)^{1/2}\, \delta^{1/2}$$

$$\leq \underline{3\varepsilon\delta^2} \quad \text{as required.}$$

□