Math 3100 Assignment 8
Continuity and Differentiation

Due at 5:00 pm on Friday the 5th of April 2019

1. (a) Let $f : [0,1] \to [0,1]$ be continuous. Prove that f must have a fixed point; that is, show that there must exist $x \in [0,1]$ with the property that $f(x) = x$.

   (b) Let $f : [0,1] \to \mathbb{R}$ be continuous with $f(0) = f(1)$.
   Show that there must exist $x \in [0, 1/2]$ with the property that $f(x) = f(x + 1/2)$.

2. Give an example of each of the following, or provide a short argument for why the request is impossible.

   (a) A continuous function defined on $[0,1]$ with range $(0,1)$.

   (b) A continuous function defined on $(0,1)$ with range $[0,1]$.

   (c) A continuous function defined on $(0,1]$ with range $(0,1)$.

3. Suppose f is a continuous function, $f(1) = -4$, $f(-2) = 3$, $\lim_{x\to-\infty} f(x) = 2$ and $\lim_{x\to\infty} f(x) = -1$.

   Prove that there exist $c, d \in \mathbb{R}$ so that $f(c) \le f(x) \le f(d)$ for all $x \in \mathbb{R}$.

4. Exactly one of the following requests is impossible. Decide which it is, and provide examples for the other three. In each case, lets assume that the functions are defined on all of $\mathbb{R}$.

   (a) Function f and g not differentiable at $x_0 = 0$, but where fg is differentiable at $x_0 = 0$.

   (b) A function f not differentiable at $x_0 = 0$ and a function g differentiable at $x_0 = 0$ where fg is differentiable at $x_0 = 0$.

   (c) A function f not differentiable at $x_0 = 0$ and a function g differentiable at $x_0 = 0$ where $f + g$ is differentiable at $x_0 = 0$.

   (d) A function f differentiable at $x_0 = 0$, but not differentiable at any other point.

5. Use the definition of the derivative to find $f'(x_0)$ for all $x_0 \in \mathbb{R}$ if:

   $$\text{(a)}\quad f(x) = \sqrt{x^2+1} \qquad\qquad \text{(b)}\quad f(x) = \frac{1}{x^2+1}$$

6. (a) Let $f(x) = \begin{cases} x^2, & \text{if } x \ge 0 \\ 0, & \text{if } x < 0 \end{cases}$.

   i. Compute $f'(x)$ for $x \neq 0$.

   ii. Use the definition of the derivative to find $f'(0)$.

   iii. Is f' continuous at 0? Give your reasoning.

   iv. Does $f''(0)$ exist? Give your reasoning.

   (b) Let $g(x) = \begin{cases} x^3 \sin \dfrac{1}{x^2}, & x \neq 0 \\ 0, & x = 0 \end{cases}$.

   i. Compute $g'(x)$ for $x \neq 0$.

   ii. Use the definition of the derivative to find $g'(0)$.

   iii. Is g' continuous at 0? Give your reasoning.

   iv. Does $g''(0)$ exist? Give your reasoning.

7. Prove that if g is differentiable at x_0, and $g(x_0) \neq 0$, then $1/g$ is differentiable at x_0 and

   $$\left(\frac{1}{g}\right)'(x_0) = \frac{-g'(x_0)}{g(x_0)^2}.$$

8. (a) Suppose f is continuous on $[a,b]$, twice differentiable on (a,b), and $f''(x) \neq 0$ for all $x \in (a,b)$. Prove *carefully* that f has at most 2 distinct zeros in $[a,b]$.
   Hint: Use Rolle's Theorem

   (b) Prove that the function $f(x) = x^2 - \sin x$ has *precisely* two roots.