Rotated Corners in the Integer Lattice

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Abstract

In this paper we prove the existence of rotated corners within positive density subsets of the lattice $[1, N]^2$ for sufficiently large N. We further show that the number of these corners is some positive proportion of the total possible number of corners.

Preliminaries

Roth in [1] proved that for N sufficiently large, a subset of $[1, N]$ with positive density will necessarily contain a non-trivial 3-term arithmetic progression. In the integer lattice, we are interested in finding three points that are vertices of an isosceles triangle. To do this, we look for rotated corners.

Definition. *A* rotated corner *is a triple* $x, y, z \in [1, N]^2$ *such that* $y - x = R(z - x)$*, where* $R = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$.

We will eventually show that either N is bounded above in terms of the density of our subset, or our subset has increased density in some lattice.

Definition. *A* lattice *is a set* $P \subseteq [1, N]^2$ *along with some* $x \in [1, N]^2$*,* $q, k \in [1, N]$ *such that*

$$P = \{(x_1 + iq, x_2 + jq) : 1 \leq i, j \leq k\}.$$

The approach will be similar to Roth's Fourier analytic proof for 3-term arithmetic progressions. For this we will require some notation and definitions. For a in some 2-dimensional set A, we will denote its coordinates as (a_1, a_2).

Definition. *For* $x, y \in \mathbb{Z}_N^2$*, we define the* dot product $x \cdot y := x_1 y_1 + x_2 y_2$.

Definition. *Let* $f : \mathbb{Z}_N^2 \to \mathbb{C}$ *and define* $e(\alpha) := \exp(2\pi i \alpha)$*. We define the* Fourier transform *of* f *to be*

$$\widehat{f}(\xi) := \frac{1}{N^2} \sum_{x \in \mathbb{Z}_N^2} f(x) e(-x \cdot \xi / N).$$

For convenience, we will rely on context and hereafter denote the dot product by juxtaposition.

Definition. *Let* $f, g : \mathbb{Z}_N^2 \to \mathbb{C}$*. We define the* convolution *of* f *with* g *to be*

$$f * g(x) = \frac{1}{N^2} \sum_{y \in \mathbb{Z}_N^2} f(y) g(x - y).$$

Definition. *Given a set* $A \subseteq \mathbb{Z}_N^2$*, we denote its characteristic function*

$$A(x) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{otherwise} \end{cases}$$

Finally, we state without proof some fairly standard but useful identities.

Proposition. *Let* $f, g : \mathbb{Z}_N^2 \to \mathbb{C}$ *and let* $A \subseteq \mathbb{Z}_N^2$*. Then the following are true.*

(i) $\displaystyle \frac{1}{N^2} \sum_{x \in \mathbb{Z}_N^2} |f(x)|^2 = \sum_{\xi \in \mathbb{Z}_N^2} |\widehat{f}(\xi)|^2$

(ii) $\widehat{f * g}(\xi) = \widehat{f}(\xi) \widehat{g}(\xi)$

(iii) $\displaystyle \max_{\xi \in \mathbb{Z}_N^2} |\widehat{f}(\xi)| \leq \frac{1}{N^2} \sum_{x \in \mathbb{Z}_N^2} |f(x)|$

(iv) $\displaystyle \widehat{A}(0) = \frac{|A|}{N}$

Identity (i) is frequently referred to as Plancherel's identity. For our proof of existence of rotated corners, we need to consider two cases. The first is when our set is sufficiently "random" and the second is when it can be shown to be somewhat "structured".

The "Random" Case

A sufficient notion of randomness for our purposes is that of ε-uniformity.

Definition. *We say that a set $A \subseteq \mathbb{Z}_N^2$ is* ε-uniform *if $|\widehat{A}(\xi)| \leq \varepsilon$ for all $\xi \neq (0,0)$.*

Let $A \subseteq [1,N]^2$ with $|A| = \delta N^2$. If we consider A as a subset of $\mathbb{Z}_N^2$ and let $B := A \cap (N/3, 2N/3]^2$, it's easily verified that every corner counted by

$$\Delta(A) := |\{(x,y,z) \in B^2 \times A : y - x \equiv R(z-x) \ (\mathrm{mod} N)\}|.$$

is a genuine corner in $[1,N]^2$ as well. That is, $\Delta(A)$ provides a lower bound on the number of corners contained in A. We can count $\Delta(A)$ by

$$\begin{aligned}
\Delta(A) &= \frac{1}{N^2} \sum_{x \in B} \sum_{y \in B} \sum_{z \in A} \sum_{\xi \in \mathbb{Z}_N^2} e((y - x - R(z-x))\xi/N) \\
&= \frac{1}{N^2} \sum_{\xi \in \mathbb{Z}_N^2} \sum_{y \in B} e(y\xi/N) \sum_{z \in A} e(-R(z)\xi/N) \sum_{x \in B} e(-(x - R(x))\xi/N) \\
&= \frac{1}{N^2} \sum_{\xi \in \mathbb{Z}_N^2} \sum_{y \in B} e(y\xi/N) \sum_{z \in A} e(-zR^{-1}(\xi)/N) \sum_{x \in B} e(-x(\xi - R^{-1}(\xi))/N) \\
&= N^4 \sum_{\xi \in \mathbb{Z}_N^2} \widehat{B}(-\xi)\widehat{B}(\xi - R^{-1}(\xi))\widehat{A}(R^{-1}(\xi)) \qquad (1)
\end{aligned}$$

With this count for $\Delta(A)$, we can make the following observation.

Lemma 1. *If A is ε-uniform, then $|\Delta(A) - \delta|B|^2| \leq \varepsilon N^2 |B|$.*

Proof. By pulling out the terms when $\xi = 0$, the summation in (1) provides

$$|\Delta(A) - \delta|B|^2| = N^4 \left| \sum_{\xi \in \mathbb{Z}_N^2 \setminus \{0\}} \widehat{B}(-\xi)\widehat{B}(\xi - R^{-1}(\xi))\widehat{A}(R^{-1}(\xi)) \right|$$

Utilizing ε-uniformity of A and the Cauchy-Schwarz inequality, we have

$$\begin{aligned}
|\Delta(A) - \delta|B|^2| &\leq \varepsilon N^4 \left| \sum_{\xi \in \mathbb{Z}_N^2} \widehat{B}(-\xi)\widehat{B}(\xi - R^{-1}(\xi)) \right| \\
&\leq \varepsilon N^4 \left(\sum_{\xi \in \mathbb{Z}_N^2} |\widehat{B}(-\xi)|^2 \right)^{1/2} \left(\sum_{\xi \in \mathbb{Z}_N^2} |\widehat{B}(\xi - R^{-1}(\xi))|^2 \right)^{1/2}
\end{aligned}$$

Finally, noting that both $-\xi$ and $\xi - R^{-1}(\xi)$ vary completely over $\mathbb{Z}_N^2$, Plancherel's identity yields

$$|\Delta(A) - \delta|B|^2| \leq \varepsilon N^4 \left(\frac{|B|^{1/2}}{N} \right) \left(\frac{|B|^{1/2}}{N} \right) = \varepsilon N^2 |B|$$

□

In particular, Lemma 1 implies that if A is ε-uniform with $\varepsilon \leq \delta|B|/(2N^2)$, then A contains at least $\delta|B|^2/2$ corners. This is useful to prove the main result of this section.

Lemma 2. *Let $A \subseteq [1,N]^2$ with $|A| = \delta N^2$ be ε-uniform for some $\varepsilon \leq \delta|B|/(2N^2)$. Suppose that A contains no nontrivial rotated corners. Then either $N < 15/\delta$ or there exists some lattice $P \subseteq [1,N]^2$ such that*

$$\frac{|A \cap P|}{|P|} \geq \delta + \frac{\delta^2}{80}$$

Proof. First suppose $|B| \geq \delta N^2/10$. Then, by our lower bound from Lemma 1, it must be that

$$1 > \frac{\delta|B|^2}{2} \geq \frac{\delta^3 N^4}{200} \Rightarrow N < 15/\delta.$$

Suppose instead $|B| < \delta N^2/10$. Then the other 8 $N/3$ by $N/3$ lattices comparable to $(N/3, 2N/3]$ must contain at least $9\delta N^2/10$ elements of A. By the pigeonhole principle, we can conclude that one of them, call it P, has $|A \cap P| \geq 9\delta N^2/80$. But then we have

$$\frac{|A \cap P|}{|P|} = \frac{9|A \cap P|}{N^2} \geq \frac{81\delta}{80} \geq \delta + \frac{\delta^2}{80}.$$

□

The "Structured" Case

For $x \in \mathbb{R}$, let $\|x\|$ denote the distance to the nearest integer from x.

Lemma 3. *Fix $\xi \in \mathbb{Z}_N^2 \backslash \{0\}$ and let $Q \in [1, N^{1/2}]$. Then there exists some $q \in [1, Q^2]$ such that*

$$\|q\xi_1/N\| \leq 1/Q, \quad \|q\xi_2/N\| \leq 1/Q$$

Proof. Consider the lattice $[1, N]^2$ modulo 1. Split this into Q^2 lattices of the form

$$[i/Q, (i+1)/Q) \times [j/Q, (j+1)/Q)$$

for $0 \leq i, j \leq Q-1$. If there existed a q such that $q\xi/N$ was in one of the four "corner" lattices, we would be done. Suppose instead that there was no such q. Then there are $Q^2 - 4$ lattices with Q^2 choices of q, so there must exist $1 \leq q_1 < q_2 \leq Q$ such that $q_1\xi/N$ and $q_2\xi/N$ lie within the same lattice. Then we know

$$(q_2 - q_1)\xi_1/N \leq 1/Q \ (\mathrm{mod} 1), \quad (q_2 - q_1)\xi_2/N \leq 1/Q \ (\mathrm{mod} 1). \tag{2}$$

But $q_2 - q_1 \in [1, Q]$, and (2) implies that $(q_2 - q_1)\xi/N$ lies within the bottom left corner lattice. This contradicts our assumption that there were no such q, and therefore one must exist. □

Let $1 \leq L \leq N^{1/2}/8\pi$. By Lemma 3, we can now fix $1 \leq q \leq 64\pi^2 L^2$ such that $\|q\xi_1/N\| \leq 1/8\pi L$ and $\|q\xi_2/N\| \leq 1/8\pi L$ for all nonzero ξ. Define $P_q := \{(-iq, -jq) : 1 \leq i, j, \leq L\}$ and set $\eta := L^2/N^2$. We can now show that P_q has relatively large Fourier coefficients. Fixing $\xi \in \mathbb{Z}_N^2 \backslash \{0\}$,

$$\begin{aligned}
|\widehat{P_q}(\xi)| &= \frac{1}{N^2}\left|\sum_{j=1}^{L}\sum_{k=1}^{L} e((jq\xi_1 + kq\xi_2)/N)\right| \\
&= \frac{1}{N^2}\left|\sum_{j=1}^{L} e(jq\xi_1/N)\right|\left|\sum_{k=1}^{L} e(kq\xi_2/N)\right| \\
&\geq \frac{1}{N^2}\sum_{j=1}^{L}\mathrm{Re}(e(jq\xi_1/N))\sum_{k=1}^{L}\mathrm{Re}(e(kq\xi_2/N)) \\
&\geq \frac{1}{N^2}\sum_{j=1}^{L}(1 - \|2\pi jq\xi_1/N\|)\sum_{k=1}^{L}(1 - \|2\pi kq\xi_2/N\|) \\
&= \frac{1}{N^2}\left(L - 2\pi\|q\xi_1/N\|\sum_{j=1}^{L} j\right)\left(L - 2\pi\|q\xi_2/N\|\sum_{k=1}^{L} k\right) \\
&\geq \frac{1}{N^2}\left(L - \frac{2\pi L^2}{8\pi L}\right)^2 \\
&\geq \eta - \frac{4\pi L^3}{8\pi L N^2} \\
&= \eta/2
\end{aligned}$$

Define the balanced function of a set $A \subseteq \mathbb{Z}_N^2$ as $f_A := A(x) - \delta$. A simple computation shows

$$f_A * P_q(x) = \frac{|A \cap (x - P_q)|}{N^2} - \delta\eta \tag{3}$$

Suppose that A is not ε-uniform. Then we have

$$\frac{1}{N^2} \sum_{x \in \mathbb{Z}_N^2} |f_A * P_q(x)| \geq \max_{\xi \in \mathbb{Z}_N^2} \left|\widehat{f_A}(\xi)\right| \left|\widehat{P_q}(\xi)\right| = \max_{\xi \in \mathbb{Z}_N^2 \setminus \{0\}} \left|\widehat{A}(\xi)\right| \left|\widehat{P_q}(\xi)\right| \geq \frac{\varepsilon\eta}{2}$$

If we let $(f_A * P_q)_+ := \max\{f_A * P_q, 0\}$, then we see

$$\frac{1}{N^2} \sum_{x \in \mathbb{Z}_N^2} (f_A * P_q)_+(x) = \frac{1}{2N^2} \left(\sum_{x \in \mathbb{Z}_N^2} |f_A * P_q(x)| + \sum_{x \in \mathbb{Z}_N^2} f_A * P_q(x) \right)$$

But noting $\sum_{x \in \mathbb{Z}_N^2} f_A * P_q(x) = 0$, we now have

$$\frac{1}{N^2} \sum_{x \in \mathbb{Z}_N^2} (f_A * P_q)_+(x) \geq \frac{\varepsilon\eta}{4} \tag{4}$$

In particular, together with (3) and the pigeonhole principle, we have that there exists some $x \in \mathbb{Z}_N^2$ such that

$$\frac{|A \cap (x - P_q)|}{N^2} - \delta\eta \geq \frac{\varepsilon\eta}{4} \tag{5}$$

This allows us to prove the main result of this section.

Lemma 4. *Suppose $A \subseteq [1, N]^2$ with $|A| \geq \delta N^2$ is not ε-uniform. Then there exists a genuine lattice $P \subseteq [1, N]^2$ with $|P| \geq \frac{1}{2}\sqrt{\varepsilon N / 512\pi^2}$ such that*

$$\frac{|A \cap P|}{|P|} \geq \delta + \frac{\varepsilon}{8}$$

Proof. To begin, we count $x \in \mathbb{Z}_N^2$ where $x - P_q$ would be an adequate lattice in $\mathbb{Z}_N^2$. Let

$$k := \left| \left\{ x \in \mathbb{Z}_N^2 : \frac{|A \cap (x - P_q)|}{|P_q|} \geq \delta + \frac{\varepsilon}{8} \right\} \right|$$

Equivalently, we know

$$k = \left| \left\{ x \in \mathbb{Z}_N^2 : \frac{|A \cap (x - P_q)|}{N^2} - \delta\eta \geq \frac{\varepsilon\eta}{8} \right\} \right|$$

and hence by (5), we know $k \geq 1$. Note that $f_A * P_q(x) \leq \eta$ always. By (4), we have

$$\frac{\varepsilon\eta N^2}{4} \leq \sum_{x \in \mathbb{Z}_N^2} f_A * P_q(x) \leq (N^2 - k)\left(\frac{\varepsilon\eta}{8}\right) + k\eta \Rightarrow \frac{\varepsilon N^2}{8} \leq k\left(1 - \frac{\varepsilon}{8}\right) \leq k$$

In order to show that one of these k starting points is for a genuine solution, it would suffice to show that the total size of the lattice is less than k. That is, we need $q^2L^2 \leq \varepsilon N^2/8$. But we know $q^2L^2 \leq 64\pi^2 N L^4$, so it is sufficient to have

$$64\pi^2 N L^4 \leq \frac{\varepsilon N^2}{8} \Leftrightarrow L^2 \leq \sqrt{\frac{\varepsilon N}{512\pi^2}}$$

Thus we can choose our L such that $|P| \geq \frac{1}{2}\sqrt{\varepsilon N / 512\pi^2}$. □

Existence of a Rotated Corner

We now have sufficient machinery to prove the existence of a rotated corner in a positive density subset of the integer lattice.

Theorem (Roth for Rotated Corners)**.** *Let $\delta > 0$. Then there exists some absolute constant C such that, provided $N \geq \exp\exp(C/\delta)$, any subset $A \subseteq [1, N]^2$ with $|A| \geq \delta N^2$ will necessarily contain a non-trivial rotated corner.*

Proof. Let $A \subseteq [1,N]^2$ with $|A| \geq \delta N$ such that A contains no non-trivial rotated corners. Set $\varepsilon := \delta^2/10$. By Lemmas 2 and 4, since we are assuming N is sufficiently large, we have that there exists some lattice $P \subseteq [1,N]^2$ such that $|P| \geq c\delta\sqrt{N}$ and

$$\frac{|A \cap P|}{|P|} \geq \delta + \frac{\delta^2}{80}$$

Setting $N_0 := N$, $A_0 := A$, $\delta_0 := \delta$, we can iterate this argument as follows. Suppose for A_j we had a lattice $P_j = \{(x_1 + iq, x_2 + kq) : 1 \leq i, k \leq \sqrt{|P_j|}\}$ that satisfied the density increment above. Then we could set $N_{j+1} \geq \sqrt{|P_j|}$ and

$$A_{j+1} = \left\{ \left(\frac{a_1 - x_1}{q}, \frac{a_2 - x_2}{q} \right) : a_1, a_2 \in A_j \cap P_j \right\} \subseteq [1, N_{j+1}]^2$$

First notice that if A_{j+1} contains some rotated corner, then we have $a, b, c \in A_j \cap P_j$ such that

$$\frac{a-x}{q} - \frac{b-x}{q} = R\left(\frac{c-x}{q} - \frac{b-x}{q}\right) \Rightarrow a - b = R(c-b)$$

Since we have assumed A_0 contains no rotated corners, this implies no A_j contains a rotated corner. Now, note $|A_{j+1}| \geq \delta_{j+1} N_{j+1}^2$ where $\delta_{j+1} \geq \delta_j + \delta_j^2/80$. This iteration cannot continue indefinitely. Indeed, a simple induction argument demonstrates that any δ_j will double after $80/\delta_j$ iterations. A finite amount of doubling will cause some $\delta_{k+1} > 1$, where certainly

$$k \leq \sum_{i=0}^{\infty} \frac{80}{2^i \delta_0} = \frac{160}{\delta_0}.$$

But $\delta_{k+1} > 1$ is absurd, so we must have $N_k < 15/\delta_k$. Letting c vary to denote the appropriate absolute constant, we have

$$\frac{15}{\delta_0} \geq \frac{15}{\delta_k} > N_j \geq c\delta_{k-1} N_{k-1}^{1/2} \geq c\delta_{k-1}\left(\delta_{k-2} N_{k-2}^{1/2}\right)^{1/2} \geq \cdots \geq c\delta_0^2 N_0^{2^{-j}} \Rightarrow N_0 < c\delta_0^{-3\cdot 2^j}.$$

An easy calculation provides that $\log\log N_0 < \log\log(c\delta_0^{-3\cdot 2^j}) \leq C/\delta$ for some absolute constant C, contradicting our choice of N_0. Therefore, A must contain a non-trivial rotated corner. $\square$

Existence of Many Rotated Corners

Any two points chosen in $[1,N]^2$ give rise to at least one rotated corner; that is, there are approximately N^4 rotated corners in $[1,N]^2$. We can apply an argument of Varnavides from [2], originally applied to Roth's theorem on 3-term arithmetic progressions, to guarantee that a positive proportion of the total number of rotated corners are in any set with positive density.

Theorem (Varnavides for Rotated Corners)**.** *Let $\delta > 0$ and let N be sufficiently large. Then any subset $A \subseteq [1,N]^2$ with $|A| \geq \delta N^2$ will necessarily contain $c(\delta)N^4$ rotated corners, where $c(\delta)$ is a constant depending only on δ.*

Proof. We first choose an integer $k := k(\delta)$ sufficiently large to apply Roth for Rotated Corners on a subset of $[1,k]^2$ with density $\delta/2$. For $u \in [1,N]^2$ and $d \in \mathbb{N}$, we consider lattices

$$P_{u,d} = \{(u_1 + id, u_2 + jd) : 1 \leq i, j \leq k\} \subseteq [1,N]^2.$$

We call $P_{u,d}$ *good* if $|A \cap P_{u,d}| \geq \delta k^2/2$. Fix $d < \delta N/k^2$ and let g_d denote the number of good lattices with step size d. For every $a \in A$ with $kd < a_1, a_2 < N - kd$, we can choose $u \in \{(a_1 - id, a_2 - jd) : 1 \leq i, j \leq k\}$ so that a is contained in exactly k^2 lattices. The number of these a is bounded below by $|A| - 4Nkd \geq \delta N^2 - 4\delta N^2/k = \delta(1 - 4/k)N^2$. Provided $k > 16$, we have

$$\sum_{u,d : P_{u,d} \subseteq [1,N]^2} |A \cap P_{u,d}| \geq k^2 \delta \left(1 - \frac{4}{k}\right) N^2 \geq \frac{3}{4}\delta k^2 N^2 \tag{6}$$

The most A could intersect with any good lattice is precisely the size of the lattice, k^2. The most A could intersect with any other lattice is $\delta k^2/2$, and certainly the total number of such lattices is less than N^2. Thus, we have

$$\sum_{u,d:P_{u,d}\subseteq[1,N]^2} |A \cap P_{u,d}| \leq g_d k^2 + \frac{1}{2}\delta k^2 N^2 \tag{7}$$

Then by (6) and (7), we have $g_d \geq (1/4)\delta N^2$. But then we have at least $\delta N^2/4 \cdot \delta N^2/k^2 = c_1(\delta)N^4$ good lattices, each of which gives rise to a rotated corner by applying Roth for Rotated Corners.

The only concern now is that each corner is being counted multiple times. To account for this, fix some solution in A. Note in the three points in the solution, there must be at least two distinct horizontal coordinates and two distinct vertical coordinates. For the solution to lie in a lattice $P_{u,d}$, we have at most $(k-1)$ choices for u_1 and u_2, hence at most $(k-1)^2$ choices for u. Further observe that the difference between the two distinct horizontal coordinates, call it d', must be divisible by d. That is, $d = d'/t$ for some $t \geq 1$. But we must have $kd > d'$ or the lattice is not large enough to span our corner. Thus we have at most k choices for t, and so at most k choices for d. Our $P_{u,d}$ is completely determined by our choices of u and d, hence our solution can be contained in at most $k(k-1)^2$ lattices. Therefore we have at least $(1/k(k-1)^2)c_1(\delta)N^4 = c(\delta)N^4$ rotated corners in A. □

References

[1] K. Roth, *On certain sets of integers*, J. Lond. Math. Soc., 28 (1953), pp. 104–109.

[2] P. Varnavides, *On certain sets of positive density*, J. Lond. Math. Soc., 34 (1959), pp. 358–360.