Repeated Integration : Fubini & Tonelli's Theorems

Fubini's Theorem

"Finiteness of multiple int $\Rightarrow$ finiteness of all iterated ints (& all equal)".

Let $f(x,y)$ be Lebesgue int'ble on $\mathbb{R}^n = \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$, then for a.e. $x \in \mathbb{R}^{n_1}$

(i) $f_x(y) = f(x,y)$ is an int'ble function of y on $\mathbb{R}^{n_2}$

(ii) $\int_{\mathbb{R}^{n_2}} f(x,y)\,dy$ is an int'ble function of x on $\mathbb{R}^{n_1}$

Moreover,

$$\int_{\mathbb{R}^{n_1}} \left(\int_{\mathbb{R}^{n_2}} f(x,y)\,dy \right) dx = \int_{\mathbb{R}^n} f .$$

In order to fully benefit from Fubini's theorem (using it "positively") we need a viable way to check that function are integrable.

Tonelli's Theorem

"For $f \geq 0$: Finiteness of any one of Fubini's 3 ints $\Rightarrow$ Finiteness of other two!"

Let $f(x,y)$ be non-negative and measurable on $\mathbb{R}^n = \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$, then for a.e. $x \in \mathbb{R}^{n_1}$

(i) $f_x(y) = f(x,y)$ is measurable as a function of y on $\mathbb{R}^{n_2}$

(ii) $\int_{\mathbb{R}^{n_2}} f(x,y)\,dy$ is measurable as a function of x on $\mathbb{R}^{n_1}$

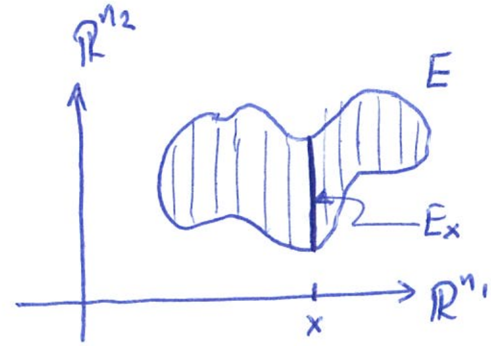
Moreover,

$$\int_{\mathbb{R}^{n_1}} \left(\int_{\mathbb{R}^{n_2}} f(x,y)\,dy \right) dx = \int_{\mathbb{R}^n} f .$$

Corollary (of Tonelli)

If E is a Lebesgue measurable subset of $\mathbb{R}^n = \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$, then for a.e. $x \in \mathbb{R}^{n_1}$ the "slice" $E_x := \{y \in \mathbb{R}^{n_2} : (x,y) \in E\}$ is a Lebesgue measurable subset of $\mathbb{R}^{n_2}$ and $m(E_x)$ is a measurable function of x in $\mathbb{R}^{n_1}$. Moreover,

$$\int_{\mathbb{R}^{n_1}} m(E_x)\,dx = m(E).$$



¿ Is it true that if for a given set $E \subset \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$ we knew that for a.e. $x \in \mathbb{R}^{n_1}$ that the slices E_x were m'ble subsets of $\mathbb{R}^{n_2}$, then E measurable in $\mathbb{R}^n = \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$

NO! Consider $E = [0,1] \times \mathcal{N} \Rightarrow E^y := \{x \in \mathbb{R}^{n_1} : (x,y) \in E\}$

$$= \begin{cases} [0,1] & \text{if } y \in \mathcal{N} \\ \phi & \text{o/w} \end{cases} \in \mathcal{M}(\mathbb{R}^{n_1}).$$

So if $E \in \mathcal{M}(\mathbb{R}^n)$, Corollary $\Rightarrow E_x \in \mathcal{M}(\mathbb{R}^{n_2})$, but $E_x = \mathcal{N}$ ↯.

Remark: In practice we often combine Fubini & Tonelli as follows:

Let $f(x,y)$ be m'ble on $\mathbb{R}^n = \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$. If either

$$\int_{\mathbb{R}^{n_1}} \left(\int_{\mathbb{R}^{n_2}} |f(x,y)|\,dy \right) dx \quad \text{or} \quad \int_{\mathbb{R}^{n_2}} \left(\int_{\mathbb{R}^{n_1}} |f(x,y)|\,dx \right) dy$$

is finite, then $f \in L^1(\mathbb{R}^n)$ (by Tonelli applied to $|f(x,y)|$), thus $\int_{\mathbb{R}^n} f < \infty$ and (by Fubini) we know that

$$\int_{\mathbb{R}^n} f = \int_{\mathbb{R}^{n_1}} \left(\int_{\mathbb{R}^{n_2}} f(x,y)\,dy \right) dx = \int_{\mathbb{R}^{n_2}} \left(\int_{\mathbb{R}^{n_1}} f(x,y)\,dx \right) dy$$

Two Examples (using Fubini to show function are non-integrable)

Example 1

Let $f(x,y) = \dfrac{x-y}{(x+y)^3}$ on $[0,1]\times[0,1]$.

Since
$$\int_0^1\left(\int_0^1 \frac{x-y}{(x+y)^3}\,dy\right)dx = \frac{1}{2} \qquad (\text{Exercise})$$
we also have that
$$\int_0^1\left(\int_0^1 \frac{x-y}{(x+y)^3}\,dx\right)dy = -\frac{1}{2}$$

and Fubini $\Rightarrow f \notin L^1([0,1]\times[0,1])$.

Example 2 (converse of Fubini false!)

Let $f(x,y) = \dfrac{xy}{(x^2+y^2)^2}$ on $[-1,1]\times[-1,1]$.

It is immediately clear that
$$\int_{-1}^1 f(x,y)\,dx = \int_{-1}^1 f(x,y)\,dy = 0$$
and hence that both iterated integrals equal 0.

However,
$$\int_{-1}^1\left(\int_{-1}^1 |f(x,y)|\,dx\right)dy \overset{\text{Exercise}}{=} 2\int_0^1\left(\frac{1}{y}-\frac{y}{1+y^2}\right)dy \quad \text{which } \underline{\text{DNE}}!$$

Fubini $\Rightarrow |f| \notin L^1([-1,1]\times[-1,1]) \Leftrightarrow f \notin L^1([-1,1]\times[-1,1])$.