Structure of the large spectrum

For $A \subseteq \mathbb{Z}_N$ with density $\delta > 0$ we define, for any $\varepsilon > 0$

$$\mathrm{Spec}_\varepsilon(A) = \{ \xi \in \mathbb{Z}_N : |\hat{1}_A(\xi)| \geq \varepsilon\delta \}.$$

Remarks: (i) $\mathrm{Spec}_\varepsilon(A)$ is symmetric (since 1_A is real)

(ii) $0 \in \mathrm{Spec}_\varepsilon(A)$ (since 1_A is non-neg)

(iii) $|\mathrm{Spec}_\varepsilon(A)| \leq \delta^{-1}\varepsilon^{-2}$

[Since $$\delta \overset{\text{Plancherel}}{=} \sum_{\xi \in \mathbb{Z}_N} |\hat{1}_A(\xi)|^2 \geq \sum_{\xi \in \mathrm{Spec}_\varepsilon(A)} |\hat{1}_A(\xi)|^2 \geq |\mathrm{Spec}_\varepsilon(A)|\,\varepsilon^2\delta^2.$$]

The purpose of this note is to prove

Chang's Structure Theorem:

Let $\varepsilon > 0$ and $A \subseteq \mathbb{Z}_N$ with density $\delta > 0$, then

$\exists \quad \Gamma = \{\eta_1, \ldots, \eta_d\} \subseteq \mathbb{Z}_N$ with $d \leq C\varepsilon^{-2}\log\delta^{-1}$

such that
$$\mathrm{Spec}_\varepsilon(A) \subseteq \left\{ \xi = \sum_{j=1}^{d} \varepsilon_j \eta_j : \varepsilon_j \in \{-1, 0, 1\} \right\}.$$

Remark: It follows immediately from this that for any $r > 0$

$$\mathrm{Bohr}(\Gamma, r/d) \subseteq \mathrm{Bohr}(\mathrm{Spec}_\varepsilon(A), r).$$

We will deduce Chang's theorem from the following Fourier analogue of Khintchine's inequality concerning functions whose Fourier spectrum has "no arithmetic structure".

- We say that a set $\Gamma = \{\xi_1, \dots, \xi_m\} \subseteq \mathbb{Z}_N$ is <u>dissociated</u> if the only solution to the equation

$$\varepsilon_1 \xi_1 + \dots + \varepsilon_m \xi_m = \varepsilon_1' \xi_1 + \dots + \varepsilon_m' \xi_m$$

with $\varepsilon_j, \varepsilon_j' \in \{-1, 0, 1\}$ is the trivial solution $\varepsilon_j = \varepsilon_j' = 0$.

<u>Rudin's Inequality</u>: Let $2 \le p < \infty$. If $f : \mathbb{Z}_N \to \mathbb{C}$ is a function with a <u>dissociated</u> spectrum $\Gamma = \{\xi \in \mathbb{Z}_N : \hat{f}(\xi) \neq 0\}$, then

$$\left(\mathbb{E}_x |f(x)|^p\right)^{1/p} \le C p^{1/2} \left(\mathbb{E}_x |f(x)|^2\right)^{1/2}$$

$$\Updownarrow$$

$$\left(\mathbb{E}_x \Big| \sum_{\xi \in \Gamma} \hat{f}(\xi) e^{2\pi i x \xi / N} \Big|^p\right)^{1/p} \le C p^{1/2} \left(\sum_{\xi \in \Gamma} |\hat{f}(\xi)|^2\right)^{1/2}.$$

<u>Remark</u>: This is equivalent to the following dual statement:

Given <u>any</u> $f : \mathbb{Z}_N \to \mathbb{C}$ and dissociated set $\Gamma \subseteq \mathbb{Z}_N$, then

$$\left(\sum_{\xi \in \Gamma} |\hat{f}(\xi)|^2\right)^{1/2} \le C p^{1/2} \left(\mathbb{E}_x |f(x)|^{p'}\right)^{1/p'}.$$

Proof (Rudin $\Rightarrow$ Dual Version)

$$\sum_{\xi\in\Gamma}|\hat{f}(\xi)|^2 = \sum_{\xi\in\mathbb{Z}_N}\hat{f}(\xi)\Big|_{\Gamma}\overline{\hat{f}(\xi)} \overset{\text{Parseval}}{=} \mathbb{E}_x\Big(\sum_{\xi\in\Gamma}\hat{f}(\xi)e^{2\pi i x\xi/N}\Big)\cdot\overline{f(x)}$$

$$\overset{\text{Hölder}}{\leq} \Big(\mathbb{E}_x\Big|\sum_{\xi\in\Gamma}\hat{f}(\xi)e^{2\pi i\frac{x\xi}{N}}\Big|^p\Big)^{1/p}\cdot\Big(\mathbb{E}_x|f(x)|^{p'}\Big)^{1/p'}$$

$$\leq C p^{1/2}\Big(\sum_{\xi\in\Gamma}|\hat{f}(\xi)|^2\Big)^{1/2}\cdot\Big(\mathbb{E}_x|f(x)|^{p'}\Big)^{1/p'} \quad \square$$

Corollary. (of dual version of Rudin)

If $A\subseteq\mathbb{Z}_N$ with density $\delta>0$ & $\Gamma\subseteq\mathbb{Z}_N$ is dissociated, then

$$\sum_{\xi\in\Gamma}|\hat{1}_A(\xi)|^2 \leq C\,\delta^2\log\delta^{-1}$$

beats trivial bound for small $\delta>0$.

Moreover, if $\Gamma\subseteq \mathrm{Spec}_\varepsilon(A)$ is dissociated, then $|\Gamma|\lesssim_c \varepsilon^{-2}\log\delta^{-1}$.

Proof: We know that

choose $p = c\log\delta^{-1}$

$$\sum_{\xi\in\Gamma}|\hat{1}_A(\xi)|^2 \leq Cp\,\delta^{2/p'} = C\delta^2(p\,\delta^{-2/p}) = C\delta^2\log\delta^{-1}.$$

If in addition $\Gamma\subseteq\mathrm{Spec}_\varepsilon(A)$, then $|\Gamma|\varepsilon^2\delta^2 \leq \sum_{\xi\in\Gamma}|\hat{1}_A(\xi)|^2\leq C\delta^2\log\delta^{-1}$. $\square$

Proof of Chang's Theorem:

Let Γ be a maximal dissociated subset of $\mathrm{Spec}_\varepsilon(A)$. We know from the Corollary above that $|\Gamma|\leq c\,\varepsilon^{-2}\log\delta^{-1}$. Now the maximality of $\Gamma=\{\eta_1,\dots,\eta_d\}$ ensures that any $\xi\in\mathrm{Spec}_\varepsilon(A)$ is involved in some equation like $\varepsilon\xi+\sum_{j=1}^{d}\varepsilon_j\eta_j=0$ since otherwise we could add ξ to Γ and obtain a larger dissociated subset. $\square$.

Proof of Rudin's Inequality:

Let $$f_\varepsilon(x) = \sum_{\xi\in\Gamma}' \hat{f}(\xi)\,\varepsilon(\xi)\,e^{2\pi i x\xi/N}$$

ε: any function $\varepsilon : \Gamma \to \{\pm 1\}$.

Note: We know from Khinchine's inequality that $\exists$ choice of ε s.t.

$$\left(\mathbb{E}_x |f_\varepsilon(x)|^p\right)^{1/p} \le c\,p^{1/2}\left(\sum_{\xi\in\Gamma}|\hat{f}(\xi)|^2\right)^{1/2}.$$

[Exercise]

Claim: $f = f_\varepsilon * p_\varepsilon$ where $p_\varepsilon(x) = 2\prod_{\xi\in\Gamma}\left(1+\varepsilon(\xi)\frac{e^{2\pi i x\xi/N}+e^{-2\pi i x\xi/N}}{2}\right)$.

Assuming this for now we see that Young's inequality gives

$$\|f\|_p \le \|f_\varepsilon\|_p\,\|p_\varepsilon\|_1$$

and since

$$\|p_\varepsilon\|_1 \underset{\text{positivity}}{=} \mathbb{E}_x p_\varepsilon(x) = 2\,\mathbb{E}_x\prod_{\xi\in\Gamma}\left(1+\frac{\varepsilon(\xi)}{2}e^{2\pi i x\xi/N}+\frac{\varepsilon(\xi)}{2}e^{-2\pi i x\xi/N}\right)$$

multiplying out product gives a weighted sum of terms like

$$\mathbb{E}_x e^{2\pi i x(\xi_1+\dots+\xi_r-\xi_1'-\dots-\xi_s')/N}$$

with $\xi_1,\dots,\xi_r,\xi_1',\dots,\xi_s'$ distinct elements of Γ.

By dissociativity, these all zero except when $r=s=0$. It is easy to then see that

$$\mathbb{E}_x p_\varepsilon(x) = \underline{\underline{2}}\,.$$

This completes the proof, modulo the claim. $\square$

Proof of Claim

Since

$$f_\varepsilon * p_\varepsilon(x) = \mathbb{E}_y \, f_\varepsilon(x-y) p_\varepsilon(y)$$

$$= 2\,\mathbb{E}_y \sum_{\eta\in\Gamma}{}' \hat{f}(\eta)\,\varepsilon(\eta)\, e^{2\pi i(x-y)\eta/N} \prod_{\xi\in\Gamma}\left(1 + \frac{\varepsilon(\xi)}{2} e^{2\pi i y\xi/N} + \frac{\varepsilon(\xi)}{2} e^{-2\pi i y\xi/2}\right).$$

Multiplying out the product and interchanging order of summation gives a weighted sum of terms like

$$\mathbb{E}_y \, e^{2\pi i y(\xi_1 + \cdots + \xi_r - \xi_1' - \cdots - \xi_s' - \eta)/N} \quad (*)$$

with $\xi_1, \ldots, \xi_r, \xi_1', \ldots, \xi_s'$ distinct elements of Γ & $\eta \in \Gamma$.

By dissociativity, the <u>only</u> non-zero term is when $r=1$, $s=0$ & $\xi_1 = \eta$.

It is easy to see that in this case $(*) = 1$ and

$$f_\varepsilon * p_\varepsilon(x) = 2 \sum_{\eta\in\Gamma} \hat{f}(\eta)\,\varepsilon(\eta)\, e^{2\pi i x\eta/N} \frac{\varepsilon(\eta)}{2}$$

$$= \sum_{\eta\in\Gamma} \hat{f}(\eta) e^{2\pi i x\eta/N}$$

$$=: f(x). \qquad \square$$