Introduction

A (proper) arithmetic progression of length k (kAP) is a collection of elements of the form : $a, a+q, \dots, a+(k-1)q$ with $q \neq 0$.

(This definition makes sense in any subset of an additive group)

The combinatorial study of arithmetic progressions began with Erdős & Turán in 1936, who showed:

"If $A \subseteq [1,N]$ with density at least $\frac{4}{9}$ (& N suff. large), then A contains a 3AP"

Definition

$r_k(N)$ = size of largest subset of $[1,N]$ that contains no kAP.

(Note: Erdős-Turán result $\Leftrightarrow$ $\frac{r_3(N)}{N} \leq \frac{4}{9}$.)

In the same paper they (implicitely?) made the following conjecture.

Erdős-Turán Conjecture: For all $k \geq 3$, $\frac{r_k(N)}{N} \to 0$ as $N \to \infty$.

Exercise 1 Let $k \geq 3$. Show that

Of course this is trivial!

$$\lim_{N\to\infty} \frac{r_k(N)}{N} = 0 \iff \text{If } A \subseteq \mathbb{N} \text{ with no kAPs then } \lim_{N\to\infty} \frac{|A \cap [1,N]|}{N} = 0 \iff \text{If } A \subseteq \mathbb{N} \text{ and } \limsup_{N\to\infty} \frac{|A \cap N|}{N} > 0 \text{ then A contains a kAP.}$$

Only $\Leftarrow$ is non-trivial.

History (of verification of the Erdős-Turán conjecture)

- $k=3$: Roth (1953) – Fourier Analysis
- General k:

  Szemerédi (1975) – Combinatorics

  Furstenberg (1977) – Ergodic Theory

  Gowers (1998) – "Higher Fourier Analysis"

  Gowers / Rödl et. al. (2002) – Hypergraph regularity

  $\vdots$

So the Erdős-Turán conjecture is most definitely a theorem...

Szemerédi's Theorem: $\lim_{N\to\infty} \frac{r_k(N)}{N} = 0$ for all $k \geq 3$.

However, only Roth's proof (and Gowers eventual vast generalization of it) gives effective quantitative bounds for the quantity $\frac{r_k(N)}{N}$:

- Roth established $\frac{r_3(N)}{N} \ll \frac{1}{\log\log N}$.
- Gowers established for $k \geq 4$,

$$\frac{r_k(N)}{N} \ll \frac{1}{(\log\log N)^{c_k}}, \text{ with } c_k = \frac{1}{2^{2^{k+9}}}.$$

After Szemerédi's proof of the Erdős-Turán conjecture, Erdős conjecture the following (which is really a conjecture on quantitative bounds):

Erdős Conjecture: If $A \subseteq \mathbb{N}$ contains no kAP's, then $\sum_{n \in A} \frac{1}{n} < \infty$.

This conjecture is still very much open, even for $k=3$. It has however been verified in one particular notable special case.

Theorem (Green-Tao, 2004)

The primes contain arbitrarily long arithmetic progressions.

Exercise 2: Let $k \geq 3$. Show that

Erdős Conjecture $\Leftrightarrow$ If $A \subseteq \mathbb{N}$ and $\sum_{n \in A} \frac{1}{n} = \infty$ then A contains a kAP. $\Leftrightarrow$ $\sum_{N=1}^{\infty} \frac{r_k(N)}{N^2} < \infty$

trivial, of course

Hint: Cauchy Condensation test & more...

Note: It would follow that $\sum_{N=1}^{\infty} \frac{r_k(N)}{N^2} < \infty$ if one could how that

$$\frac{r_k(N)}{N} \ll \frac{1}{(\log N)^{1+\varepsilon}} \quad \text{for some } \varepsilon > 0.$$

Quantitative Improvements (on bounds of Roth & Gowers on previous page)

- **$k=3$**:
  - Late 1980's, Heath-Brown / Szemerédi showed
  $$\frac{r_3(N)}{N} \ll \frac{1}{(\log N)^c}, \text{ with } c = \frac{1}{1000} \text{ (say)}.$$
  - Bourgain (1999) showed $\frac{r_3(N)}{N} \ll \left(\frac{\log\log N}{\log N}\right)^{1/2}$
  - Sanders (2010) showed $\frac{r_3(N)}{N} \ll \frac{(\log\log N)^5}{\log N}$

  Current Record!
- **$k=4$**:
  - Green-Tao (2005) showed $\frac{r_4(N)}{N} \ll e^{-c\sqrt{\log\log N}}$.

Lower Bounds (Behrend's Construction)

The most obvious non-trivial lower bound is $r_3(N) \gg N^{\log 2/\log 3}$ and is obtained by noting that any set of numbers whose base three expansion contains only zeros and ones will contain <u>no</u> 3AP's.

Behrend produced a significantly larger construction in 1946.

<u>Theorem</u> (Behrend, 1946). $\frac{r_3(N)}{N} \ll e^{-C\sqrt{\log N}}$

<u>Proof</u>: The key observation is that a sphere in $\mathbb{R}^n$ contains <u>no</u> 3AP's.

Consider $[1,M]^n$, with M and n to be chosen later. By the pigeonhole principle, there exists a sphere S (with radius r satisfying $n \leq r^2 \leq nM^2$) such that $|S \cap [1,M]^n| \geq M^{n-2}/n$, denote $X: S \cap [1,M]^n$.

As noted above X contains <u>no</u> 3AP's, to turn this into a subset of $[1,N]$ we consider the projection

$$\pi : [1,M]^n \to \mathbb{Z}$$
$$x \longmapsto x_1 + (2M)x_2 + \cdots + (2M)^{n-1}x_n .$$

Since every integer has a unique expansion base M, this map is injective and moreover

<u>Exercise 3</u>: $x+y \neq 2z \implies \pi(x)+\pi(y) \neq 2\pi(z)$.

It follows that $A := \pi(x) \subseteq [1,N]$ with $N = (2M)^n$ ($\Leftrightarrow M = \frac{N^{1/n}}{2}$) and <u>no</u> 3AP's.

$$\frac{|A|}{N} \geq \frac{M^{n-2}}{nN} \geq \frac{N^{-2/n}}{n2^n} = \frac{2^{-\frac{2}{n}\log_2 N}}{n2^n} = \frac{2^{-3\sqrt{\log_2 N}}}{\sqrt{\log_2 N}} \text{ choosing } n = \sqrt{\log_2 N}. \quad \square$$