Math 8100 Assignment 4
Lebesgue Integration

Due date: Tuesday the 4th of October 2022

Definition. Let E be a Lebesgue measurable subset of $\mathbb{R}^n$.

We say that a measurable function $f : E \to \mathbb{C}$ is *integrable on E* if $\displaystyle\int_E |f(x)|\,dx < \infty$.

1. (a) Give an example of a continuous integrable function f on $\mathbb{R}$ for which $f(x) \nrightarrow 0$ as $|x| \to \infty$.
   (b) Prove that if f is integrable on $\mathbb{R}$ and uniformly continuous, then $\displaystyle\lim_{|x|\to\infty} f(x) = 0$.

2. Let f be an integrable function on $\mathbb{R}^n$.
   (a) Prove that $\{x : |f(x)| = \infty\}$ has measure equal to zero.
   (b) Let $\varepsilon > 0$. Prove that there exists a measurable set E with $m(E) < \infty$ for which
$$\int_E |f| > \left(\int |f|\right) - \varepsilon.$$

3. Let f be a function in $L^+(\mathbb{R}^n)$ that is finite almost everywhere.
   Let $E_{2^k} = \{x : f(x) > 2^k\}$, $F_k = \{x : 2^k < f(x) \le 2^{k+1}\}$, and note that since f is finite almost everywhere it follows that $\bigcup_{k=-\infty}^{\infty} F_k = \{x : f(x) > 0\}$, and the sets F_k are disjoint. Prove that
$$\int f(x) < \infty \iff \sum_{k=-\infty}^{\infty} 2^k m(F_k) < \infty \iff \sum_{k=-\infty}^{\infty} 2^k m(E_{2^k}) < \infty.$$

4. Prove the following:
   (a)
$$\int_{\{x\in\mathbb{R}^n \,:\, |x|\le 1\}} |x|^{-p}\,dx < \infty \quad \text{if and only if} \quad p < n.$$
   (b)
$$\int_{\{x\in\mathbb{R}^n \,:\, |x|\ge 1\}} |x|^{-p}\,dx < \infty \quad \text{if and only if} \quad p > n.$$
   Hint: One possible approach is to use the first equivalence in Question 3 above. I suggest however that in this case you also try simply writing $\mathbb{R}^n$ as a disjoint union of the annuli $A_k = \{2^k < |x| \le 2^{k+1}\}$.

5. Given any integrable function f on $\mathbb{R}^n$, the *Fourier transform of f* is defined by
$$\widehat{f}(\xi) = \int_{\mathbb{R}^n} f(x) e^{-2\pi i x\cdot\xi}\,dx$$
   where $x \cdot \xi = x_1\xi_1 + \cdots + x_n\xi_n$. Show that $\widehat{f}$ is a bounded continuous function of ξ.

6. Let $\{f_k\}$ be a sequence of integrable functions on $\mathbb{R}^n$, f be integrable on $\mathbb{R}^n$, and $\displaystyle\lim_{k\to\infty} f_k = f$ a.e.
   (a) Suppose further that
$$\lim_{k\to\infty} \int |f_k(x)|\,dx = A < \infty \qquad \text{and} \qquad \int |f(x)|\,dx = B.$$

i. Prove that

$$\lim_{k\to\infty} \int |f_k(x) - f(x)|\, dx = A - B.$$

Hint: Use the fact that

$$|f_k(x)| - |f(x)| \le |f_k(x) - f(x)| \le |f_k(x)| + |f(x)|.$$

ii. Give an example of a sequence $\{f_k\}$ of such functions for which $A \ne B$.

(b) Deduce that

$$\int |f - f_k| \to 0 \quad \Longleftrightarrow \quad \int |f_k| \to \int |f|.$$

7. (a) Suppose that $f(x)$ and $xf(x)$ are both integrable functions on $\mathbb{R}$. Prove that the function

$$F(t) = \int_{\mathbb{R}} f(x) \cos(tx)\, dx.$$

is differentiable at every t and find a formula for $F'(t)$.

(b) Giving complete justification, evaluate

$$\lim_{t\to 0} \int_0^1 \frac{e^{t\sqrt{x}} - 1}{t}\, dx.$$

Extra Challenge Problems

Not to be handed in with the assignment

1. Assume Fatou's theorem and deduce the monotone convergence theorem from it.

2. A sequence $\{f_k\}$ of integrable functions on $\mathbb{R}^n$ is said to *converge in measure* to f if for every $\varepsilon > 0$,

$$\lim_{k\to\infty} m(\{x \in \mathbb{R}^n : |f_k(x) - f(x)| \ge \varepsilon\}) = 0.$$

(a) Prove that if $f_k \to f$ in L^1 then $f_k \to f$ in measure.

(b) Give an example to show that the converse of Question 2a is false.

(c) Prove that if we make the additional assumption that there exists an integrable function g such that $|f_k| \le g$ for all k, then $f_k \to f$ in measure implies that

i. * (Bonus points) $f \in L^1$

Hint: First show that $\{f_k\}$ contains a subsequence which converges to f almost everywhere.

ii. $f_k \to f$ in L^1.

Hint: Try using absolute continuity and "small tails property" of the Lebesgue integral.

3. Let $\Omega \subseteq \mathbb{R}^n$ be measurable with $m(\Omega) < \infty$. A set $\Phi \subseteq L^1(\Omega)$ is said to be *uniformly integrable* if, for any $\varepsilon > 0$ there exists $\delta > 0$ such that whenever $f \in \Phi$ and $E \subseteq \Omega$ is measurable with $m(E) < \delta$, then

$$\int_E |f(x)|\, dx < \varepsilon.$$

(a) Prove that if $f \in L^1(\Omega)$ and $\{f_k\}$ is a uniformly integrable sequence of functions in $L^1(\Omega)$ such that $f_k \to f$ almost everywhere on Ω, then $f_k \to f$ in $L^1(\Omega)$.

(b) Is it necessary to assume that $f \in L^1(\Omega)$?