The Dual Space of L^p when $1 \le p < \infty$

Suppose that $1 \le p, q \le \infty$ are conjugate exponents.

It follows from Hölder's inequality:

$$f \in L^p \text{ and } g \in L^q \Rightarrow fg \in L^1 \text{ and } \left|\int fg\right| \le \|f\|_p \|g\|_q$$

that for each $g \in L^q$ we can define $L_g \in (L^p)^*$, that is a continuous linear functional L_g on L^p, by

$$\boxed{L_g(f) = \int fg}$$

$$\left[\text{since } |L_g(f)| \le \|f\|_p \|g\|_q = C\|f\|_p\right]$$

with operator norm at most $\|g\|_q$, that is

$$\|L_g\|_{(L^p)^*} := \sup_{\|f\|_p = 1} |L_g(f)| \le \|g\|_q .$$

In fact, it follows from the "Converse to Hölder" (part (i)):

$$g \in L^q \Rightarrow \|g\|_q = \sup_{\|f\|_p = 1} \left|\int fg\right|$$

that the map $g \longmapsto L_g$ is an isometry from L^q into $(L^p)^*$.

* If $1 \le p < \infty$, then this map is infact also surjective, i.e. L^q isometrically isomorphic to $(L^p)^*$. *

Theorem (Riesz Representation Theorem for L^p functions)

Suppose $1 \le p < \infty$ and q is the conjugate exponenent to p.

Given any $L \in (L^p(\mathbb{R}^n))^*$ there exists $g \in L^q(\mathbb{R}^n)$ which represents L in the sense that

$$L(f) = \int fg \quad \text{for all } f \in L^p(\mathbb{R}^n)$$

and $\quad \|L\|_{(L^p)^*} = \|g\|_q$.

Summary:

isometrically isomorphic

(i) $(L^p(\mathbb{R}^n))^* \simeq L^q(\mathbb{R}^n)$ if $1 \le p < \infty$

but (ii) $(L^\infty(\mathbb{R}^n))^* \not\simeq L^1(\mathbb{R}^n)$

[The standard proof that $(L^\infty(\mathbb{R}^n))^*$ is a larger space than $L^1(\mathbb{R}^n)$ uses the Hahn-Banach theorem from functional analysis.]

(*) This a very important and rather deep result.

- We will see a proof of this result at the end of the semester after we have discussed abstract measures and proven the Radon-Nikodym theorem.
- We shall soon however see a proof in the special case when $p=2$, but we most first discuss some special properties of the Hilbert Space L^2.

Sketch Proof of Theorem

Let $L \in (L^p(\mathbb{R}^n))^*$. Define $\nu : \mathcal{M}(\mathbb{R}^n) \longrightarrow \mathbb{C}$, by

$$\nu(E) = L(\chi_E) \text{ for all } E \in \mathcal{M}(\mathbb{R}^n).$$

Note: (i) $\nu(\phi) = 0$

(ii) for any disjoint seq $\{E_j\}$, $\nu(\bigcup E_j) = \sum_{j=1}^{\infty} \nu(E_j)$.

not obvious (and is where we use $p \neq \infty$).

It follows that ν is a complex measure. Moreover, if $m(E) = 0$ then $\chi_E \in L^p$ and $\nu(E) = 0$, i.e. $m(E) = 0 \Rightarrow \nu(E) = 0$

"ν is absolutely conts wrt m" ($\nu \ll m$)

Radon-Nikodym Thm

$$\Longrightarrow \exists\, g \in L^1(\mathbb{R}^n) \text{ such that } \nu(E) = \int_E g(x)\,dx.$$

and hence $L(f) = \int fg$ for all simple functions f.

It follows from the "Converse of Hölder" (part (ii)) that $g \in L^q(\mathbb{R}^n)$.

Since simple functions are dense in $L^p(\mathbb{R}^n)$ this completes the proof.

$\square$