Lecture 3

Mertens' Identities (and another theorem of Chebyshev)

Theorem 1: For $x \geq 2$

(a) $$\sum_{n \leq x} \frac{\Lambda(n)}{n} = \log x + O(1)$$

(b) $$\sum_{p \leq x} \frac{\log p}{p} = \log x + O(1)$$

(c) $$\int_1^x \psi(t)\, t^{-2}\, dt = \log x + O(1)$$

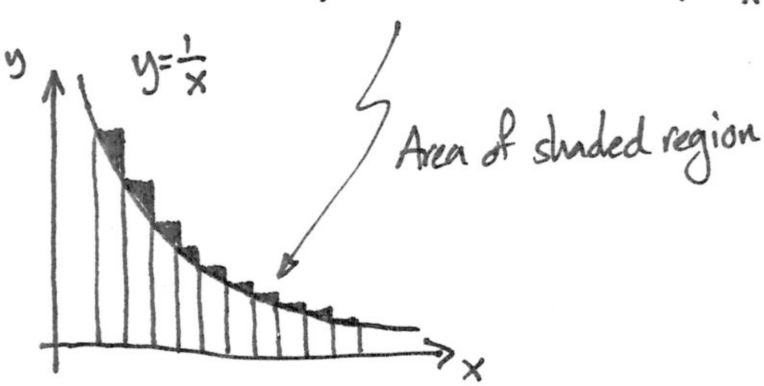
(d) $$\sum_{p \leq x} \frac{1}{p} = \log\log x + b + O\left(\frac{1}{\log x}\right)$$

(e) $$\prod_{p \leq x} \left(1 - \frac{1}{p}\right)^{-1} = e^{\gamma} \log x + O(1).$$

These two, particularly the second one, is what people usually mean when they speak of "Mertens Theorem". or "Mertens Formula".

where γ is Euler's Constant & $b = \gamma - \sum_{p} \sum_{k \geq 2} \frac{1}{k p^k}$.

Recall that $\gamma := \int_1^{\infty} \frac{1}{\lfloor t \rfloor} - \frac{1}{t}\, dt \quad \left(= \lim_{x \to \infty} \left(\sum_{n \leq x} \frac{1}{n} - \log x \right) \right)$



Proof of Theorem 1

(a): $$T(x) := \sum_{n\le x} \log n \overset{\text{Lemma 2.1}}{=} \sum_{n\le x}\sum_{d\mid n}\Lambda(d) = \sum_{d\le x}\sum_{\substack{n\le x\\ d\mid n}}\Lambda(d) = \sum_{d\le x}\Lambda(d)\left\lfloor \frac{x}{d}\right\rfloor$$

$$= x\sum_{d\le x}\frac{\Lambda(d)}{d} + O(x).$$

Since $T(x) = x\log x - x + O(\log x) = x\log x + O(x)$

$$\Rightarrow \sum_{d\le x}\frac{\Lambda(d)}{d} = \log x + O(1). \qquad \square$$

(b): Observe that

$$\sum_{d\le x}\frac{\Lambda(d)}{d} - \sum_{p\le x}\frac{\log p}{p} = \sum_{\substack{p^k\le x\\ k\ge 2}}\frac{\log p}{p^k}$$

$$\le \sum_{p\le x}\log p\sum_{k=2}^{\infty}\frac{1}{p^k}$$

$$= \sum_{p\le x}\frac{\log p}{p(p-1)} \ll 1 \qquad \square$$

(c): Here we use Summation by Parts:

$$\sum_{n\le x}\Lambda(n)\frac{1}{n} = \frac{\psi(x)}{x} + \int_2^x \psi(t)\,t^{-2}\,dt$$

Result follows since $\frac{\psi(x)}{x} = O(1)$ by Theorem 2.1. $\square$

(d): Again we use partial summation:

$$\sum_{p\le x}\frac{1}{p}=\sum_{p\le x}\frac{\log p}{p}\cdot\frac{1}{\log p}\qquad \left(\text{let } A(t)=\sum_{p\le t}\frac{\log p}{p}\ \&\ f(x)=\frac{1}{\log x}\right)$$

$$=\frac{A(x)}{\log x}+\int_2^x\frac{A(t)}{t(\log t)^2}\,dt$$

$$=\left(1+O\left(\frac{1}{\log x}\right)\right)+\left(\int_2^x\frac{dt}{t\log t}+\int_2^x\frac{A(t)-\log t}{t(\log t)^2}\,dt\right)$$

Since $\int_2^x\frac{dt}{t\log t}=\log\log x-\log\log 2$

& $\int_2^x\frac{A(t)-\log t}{t(\log t)^2}\,dt=\underbrace{\int_2^\infty\frac{A(t)-\log t}{t(\log t)^2}\,dt}_{\ll 1}-\underbrace{\int_x^\infty\frac{A(t)-\log t}{t(\log t)^2}\,dt}_{\ll\int_x^\infty\frac{dt}{t(\log t)^2}=\frac{1}{\log x}}$

$$\Rightarrow\quad \sum_{p\le x}\frac{1}{p}=\log\log x+b+O\left(\frac{1}{\log x}\right)$$

where $b=1-\log\log 2+\int_2^\infty\frac{A(t)-\log t}{t(\log t)^2}\,dt$. $\square$

(e): Since

- $\log\prod_{p\le x}\left(1-\frac{1}{p}\right)^{-1}=\sum_{p\le x}\log\left(1-\frac{1}{p}\right)^{-1}=\sum_{p\le x}\sum_{k=1}^{\infty}\frac{1}{kp^k}$ Since $\log(1-y)^{-1}=\sum_{k=1}^{\infty}\frac{y^k}{k}$ $\forall\ |y|<1$.
- $\sum_{p\le x}\frac{1}{p}=\log\log x+b+O\left(\frac{1}{\log x}\right)$
- $\sum_{p\le x}\sum_{k\ge 2}\frac{1}{kp^k}=\underbrace{\sum_{p}\sum_{k\ge 2}\frac{1}{kp^k}}_{(*)}-\underbrace{\sum_{p>x}\sum_{k\ge 2}\frac{1}{kp^k}}_{(**)}$

& we know

$$(*)\le\sum_p\frac{1}{2}\sum_{k\ge 2}\frac{1}{p^k}=\sum_p\frac{1}{2p(p-1)}<\infty\quad \&\quad (**)\ll\sum_{n\ge x}\frac{1}{n^2}\ll\frac{1}{x}.$$

it follows that

$$\log\prod_{p\le x}\left(1-\frac{1}{p}\right)^{-1}=\log\log x+\underbrace{b+\sum_p\sum_{k\ge 2}\frac{1}{kp^k}}_{=:c}+O\left(\frac{1}{\log x}\right)$$

$$\Rightarrow\quad \prod_{p\le x}\left(1-\frac{1}{p}\right)^{-1}=e^c\log x+O(1)$$ Since $e^y=1+O(y)$. $\square$

In order to complete the proof of parts (d) & (e) we must show $c = \gamma$.

Proof that $c = \gamma$

Since if $p \le x$ & $p^k > x$, then $k \geq \log x / \log p$, it follows that

$$\sum_{\substack{p \le x \\ p^k > x}} \frac{1}{kp^k} \ll \sum_{\substack{p \le x \\ p^k > x}} \frac{\log p}{\log x} \frac{1}{p^k} \ll \sum_{p} \frac{\log p}{\log x} \sum_{k \geq 2} p^{-k} \ll \frac{1}{\log x} \sum_{p} \frac{\log p}{p^2} \ll \frac{1}{\log x}.$$

From the proof of (e) we therefore now see that

$$\sum_{n \le x} \frac{\Lambda(n)}{n \log n} = \sum_{p \le x} \sum_{k=1}^{\infty} \frac{1}{kp^k} + O\left(\frac{1}{\log x}\right) = \log\log x + c + O\left(\frac{1}{\log x}\right). \quad (*)$$

Lemma 1: $\displaystyle \sum_{n \le x} \frac{1}{n} = \log x + \gamma + O\left(\frac{1}{x}\right)$

Proof:
$$\sum_{n \le x} \frac{1}{n} = \int_1^x \frac{1}{\lfloor t \rfloor} dt - \int_1^x \frac{1}{t} dt + \underbrace{\int_1^x \frac{1}{t} dt}_{= \log x}$$

$$= \log x + \underbrace{\int_1^{\infty} \frac{1}{\lfloor t \rfloor} - \frac{1}{t} dt}_{=: \gamma} - \underbrace{\int_x^{\infty} \frac{1}{\lfloor t \rfloor} - \frac{1}{t} dt}_{\ll \frac{1}{x} \ (\text{since } \lfloor t \rfloor \geq t/2)} \qquad \square$$

Combining $(*)$ & Lemma 1 gives: For all $x \geq 1$,

$$\sum_{n \le x} \frac{\Lambda(n)}{n \log n} = \sum_{n \le \log x} \frac{1}{n} + (c - \gamma) + O\left(\frac{1}{\log 2x}\right).$$

We now do something slightly surprising (at least at first), we now integrate each of these terms in x (against a cleverly chosen weight).

For $s>1$ we write

$$\zeta(s) := \sum_{n=1}^{\infty} n^{-s} .$$

Comparing the sum to an integral we can easily check that

$$\zeta(s) = \frac{1}{s-1} + O(1) \qquad (s>1) .$$

Moreover, we have

$$\sum_{n \le x} n^{-s} \le \prod_{p \le x} \left(1 - \frac{1}{p^s}\right)^{-1} \le \zeta(s)$$

as in Lecture 1, as the product over p equals the sum

$$\sum_{n=1}^{\infty} \varepsilon_n n^{-s}$$

where

$$\varepsilon_n = \begin{cases} 1 & \text{if all prime factors of } n \text{ are } \le x \\ 0 & \text{o/w} \end{cases} .$$

Letting $x \to \infty$ we obtain Euler's famous formula:

$$\zeta(s) = \prod_p \left(1 - \frac{1}{p^s}\right)^{-1} .$$

as $s \to 1^+$

Taking logarithms, we obtain for any $s>1$

$$\log \zeta(s) = \sum_p \log\left(1 - \frac{1}{p^s}\right)^{-1} = \sum_p \sum_{k=1}^{\infty} \frac{1}{k p^{ks}} = \sum_{n=1}^{\infty} \frac{\Lambda(n)}{\log n} \frac{1}{n^s}$$

Since $\log \zeta(s) = \log\left(\frac{1}{s-1}\right) + O(s-1)$ & by partial summation

$$\sum_{n=1}^{\infty} \frac{\Lambda(n)}{n \log n} \frac{1}{n^{s-1}} = \int_1^{\infty} (s-1) t^{-s} \left(\sum_{n \le t} \frac{\Lambda(n)}{n \log n} \right) dt$$

$$\Rightarrow \int_1^{\infty} (s-1) t^{-s} \left(\sum_{n \le t} \frac{\Lambda(n)}{n \log n} \right) dt = \overset{①}{\log\left(\frac{1}{s-1}\right)} + O(s-1) .$$

as $s \to 1^+$.

Recalling that for all $x \geq 1$

$$\sum_{n \leq x} \frac{\Lambda(n)}{n \log n} = \sum_{n \leq \log x} \frac{1}{n} + (c-\gamma) + O\left(\frac{1}{\log x}\right) \qquad (\dagger)$$

we now apply the "transform"

$$f \longmapsto \int_1^\infty (s-1) t^{-s} f(t)\, dt$$

to the three terms on the right of idenity $(\dagger)$.

Since
- $\displaystyle\int_1^\infty (s-1) t^{-s} \Big(\sum_{n \leq \log t} \frac{1}{n}\Big) dt = (s-1) \sum_{n=1}^\infty \frac{1}{n} \int_{e^n}^\infty t^{-s}\, dt$

$$= \sum_{n=1}^\infty \frac{(e^{1-s})^n}{n}$$

$$= \log\left(1-e^{1-s}\right)^{-1}$$

$$= \log\left((s-1)+O(s-1)^2\right)^{-1}$$

$$\underline{\underline{= \log\left(\frac{1}{s-1}\right) + O(s-1)}} \quad ②$$

- $\displaystyle\int_1^\infty (s-1) t^{-s} (c-\gamma)\, dt = \underline{\underline{c-\gamma}}$

- $\displaystyle\int_1^\infty (s-1) t^{-s} \frac{dt}{\log 2t}$

$$= \underbrace{\int_1^2 (s-1) t^{-s} \frac{dt}{\log 2t}}_{=O(s-1)} + \underbrace{\int_2^{e^{1/s-1}} (s-1) t^{-s} \frac{dt}{\log 2t}}_{\ll (s-1)\int_2^{e^{1/s-1}} \frac{dt}{t \log t}} + \underbrace{\int_{e^{1/s-1}}^\infty (s-1) t^{-s} \frac{dt}{\log 2t}}_{\ll (s-1)^2 \int_{e^{1/s-1}}^\infty t^{-s}\, dt}$$

$$\ll (s-1) \log\left(\frac{1}{s-1}\right) \qquad \ll (s-1).$$

$$\ll (s-1) \log\left(\frac{1}{s-1}\right).$$

and the main two terms ① & ② cancel, on letting $s \to 1^+$ we obtain

$$\underline{\underline{c = \gamma}} \qquad \square$$

Another Theorem of Chebyshev

Theorem 2 (Chebyshev)

If $\pi(x) \sim c \dfrac{x}{\log x}$ holds, then c must equal 1.

Proof

Recall Theorem 1 (c): $\int_1^x \psi(t)\, t^{-2}\, dt = \log x + O(1)$.

By Corollary 2.1, it suffice to show that

① $\limsup\limits_{t \to \infty} \dfrac{\psi(t)}{t} \geq 1 \qquad \left(\Rightarrow \limsup\limits_{x \to \infty} \dfrac{\pi(x)}{x/\log x} \geq 1 \right)$

② $\liminf\limits_{t \to \infty} \dfrac{\psi(t)}{t} \leq 1 \qquad \left(\Rightarrow \liminf\limits_{x \to \infty} \dfrac{\pi(x)}{x/\log x} \leq 1 \right)$

Suppose that $\limsup\limits_{t \to \infty} \dfrac{\psi(t)}{t} = a$, and suppose that $\varepsilon > 0$.

Then there has to be an x_0 such that

$$\psi(x) \leq (a+\varepsilon)x \qquad \text{for all } x \geq x_0 .$$

and hence

$$\int_1^x \psi(t)\, t^{-2}\, dt \leq \int_1^{x_0} \psi(t)\, t^{-2}\, dt + (a+\varepsilon) \int_{x_0}^x t^{-1}\, dt$$
$$\leq (a+\varepsilon)\log x + O_\varepsilon(1).$$

Since this holds for any $\varepsilon > 0$ & $\int_1^x \psi(t)\, t^{-2}\, dt = \log x + O(1)$

it follows that $a \geq 1$. Similarly $\liminf\limits_{t \to \infty} \dfrac{\psi(t)}{t} \leq 1$. $\square$