The Restriction Conjecture

Fourier transform on $\mathbb{R}^n$: Given $f:\mathbb{R}^n \to \mathbb{C}$, we define

$$\hat{f}(\xi) = \int_{\mathbb{R}^n} f(x) e^{-2\pi i x\cdot\xi}\, dx .$$

The restriction problem asks when an inequality of the form

$$\| \hat{f}|_{S^{n-1}} \|_{L^q(S^{n-1})} \leq C \|f\|_{L^p(\mathbb{R}^n)} \qquad (*)$$

holds. Note that if f is an arbitrary L^p function, it is not even clear that this estimate makes sense. It follows from Plancherel that no such estimate can hold if f is an arbitrary L^2 function, since in this case $\hat{f}$ will also be an arbitrary L^2 function and as such cannot be meaningfully restricted to S^{n-1}. On the other hand, if f is a L^1 function, then $\hat{f} \in C_0$ (so $\hat{f}|_{S^{n-1}}$ makes sense) & $(*)$ holds for $q=\infty$.

¿ What happens in the intermediate values of p ?

The Restriction Conjecture Let $1 \leq p, q \leq \infty$.

$$\underset{(\text{for all } f \in S(\mathbb{R}^n))}{\| \hat{f}|_{S^{n-1}} \|_{L^q(S^{n-1})} \leq C \|f\|_{L^p(\mathbb{R}^n)}} \iff \underbrace{q \leq \tfrac{n-1}{n+1} p' \ \& \ p < \tfrac{2n}{n+1}}_{(**)}$$

We will establish the necessity of condition $(**)$ below.
It is a theorem (due to Tomas & Stein) that $(**)$ is also sufficient when $q=2$, in which case $(**)$ reduces to simply $p \leq \frac{2n+2}{n+3}$.
We will also prove this result, but only "up to the endpoint", i.e. only in the range $p < \frac{2n+2}{n+3}$.

Theorem Let $1 \le p, q \le \infty$. If $\|\hat{f}|_{S^{n-1}}\|_{L^q(S^{n-1})} \le C \|f\|_{L^p(\mathbb{R}^n)}$ $\forall f \in S(\mathbb{R}^n)$, then $q \le \frac{n-1}{n+1} p'$ & $p < \frac{2n}{n+1}$.

Proof

1. Condition $q \le \frac{n-1}{n+1} p'$ (Knapp Example)

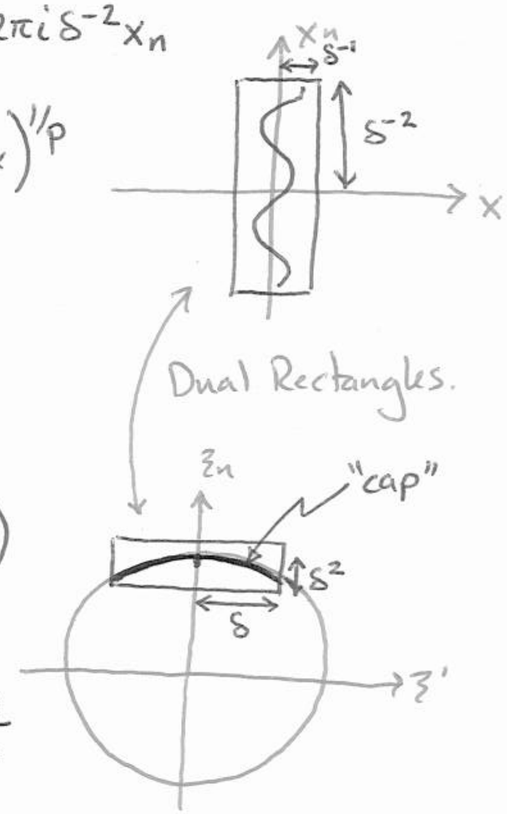
Let $\psi \in S(\mathbb{R}^n)$ such that $\hat{\psi} = 1$ on unit cube, ie. where $|x_j| \le 1$ $(1 \le j \le n)$.

Define $f(x) = \delta^{n+1} \psi(\delta x_1, \ldots, \delta x_{n-1}, \delta^2 x_n) e^{-2\pi i \delta^{-2} x_n}$

$$\Rightarrow \|f\|_{L^p(\mathbb{R}^n)} = \left(\int |\delta^{n+1} \psi(\delta x_1, \ldots, \delta x_{n-1}, \delta^2 x_n)|^p dx \right)^{1/p}$$

$$= \delta^{n+1} \delta^{-\frac{n+1}{p}} \|\psi\|_{L^p(\mathbb{R}^n)}$$

$$= c\, \delta^{\frac{n+1}{p'}}.$$



While
$$\hat{f}(\xi) = \hat{\psi}(\delta^{-1}\xi_1, \ldots, \delta^{-1}\xi_{n-1}, \delta^{-2}(\xi_n - 1))$$

$$\Rightarrow \left(\int_{S^{n-1}} |\hat{f}(\xi)|^q d\sigma(\xi) \right)^{1/q} \ge \left(\int_{\text{"cap"}} d\sigma(\xi) \right)^{1/q} = c\, \delta^{\frac{n-1}{q}}$$

provided δ small.

Thus, if $\|\hat{f}\|_{L^q(S^{n-1})} \le C \|f\|_{L^p(\mathbb{R}^n)}$ we much have

$$\delta^{\frac{n-1}{q}} \le C\, \delta^{\frac{n+1}{p'}} \quad \forall \delta > 0 \text{ small}.$$

Letting $\delta \to 0$ we see that we most have

$$\frac{n-1}{q} \ge \frac{n+1}{p'} \iff q \le \frac{n-1}{n+1} p'. \qquad \square$$

2. Condition $p < \frac{2n}{n+1}$: We use duality & the fact that

$$\|\widehat{d\sigma}\|_{L^{p'}(\mathbb{R}^n)} < \infty \iff 1 \leq p < \frac{2n}{n+1} \quad \left(\iff p' > \frac{2n}{n-1}\right)$$

(which follows immediately from $\widehat{d\sigma}(\xi) = \dfrac{c e^{2\pi i|\xi|} + \bar{c} e^{-2\pi i|\xi|}}{|\xi|^{\frac{n-1}{2}}} + O\left(|\xi|^{-\frac{n+1}{2}}\right)$ as $|\xi| \to \infty$ and the fact that $\widehat{d\sigma}$ is continuous.)

$$\|\hat{f}\|_{L^q(S^{n-1})} \leq C \|f\|_{L^p(\mathbb{R}^n)} \quad \longleftarrow \text{"Restriction Estimate"}$$

$$\iff \sup_{\|f\|_{L^p(\mathbb{R}^n)}=1} \|\hat{f}\|_{L^q(S^{n-1})} \leq C$$

$$\iff \sup_{\|f\|_{L^p(\mathbb{R}^n)}=1} \sup_{\|g\|_{L^{q'}(S^{n-1})}=1} \left| \int_{S^{n-1}} \hat{f}(\xi) g(\xi) d\sigma(\xi) \right| \leq C$$

$$\iff \sup_{\|g\|_{L^{q'}(S^{n-1})}=1} \sup_{\|f\|_{L^p(\mathbb{R}^n)}=1} \left| \int_{\mathbb{R}^n} f(x) \widehat{g d\sigma}(x) dx \right| \leq C$$

$$\iff \sup_{\|g\|_{L^{q'}(S^{n-1})}=1} \|\widehat{g d\sigma}\|_{L^{p'}(\mathbb{R}^n)} \leq C$$

$$\iff \|\widehat{g d\sigma}\|_{L^{p'}(\mathbb{R}^n)} \leq C \|g\|_{L^{q'}(S^{n-1})} \quad \longleftarrow \text{"Extension Estimate"}$$

Thus, if we choose $g \equiv 1$, then if the "restriction estimate" holds we must also have

$$\|\widehat{d\sigma}\|_{L^{p'}(\mathbb{R}^n)} \leq C$$

and hence $1 \leq p < \frac{2n}{n+1}$. $\square$

<u>Theorem</u> (Tomas–Stein) If $f \in S(\mathbb{R}^n)$ & $1 \le p \le \frac{2n+2}{n+3}$, then

$$\int_{S^{n-1}} |\hat{f}(\xi)|^2 d\sigma(\xi) \le C \left(\int_{\mathbb{R}^n} |f(x)|^p dx \right)^{2/p}.$$

* We will only prove this up to the endpoint, i.e. for $1 \le p < \frac{2n+2}{n+3}$ *

In order to prove this, it suffices to show that

$$\| f * \widehat{d\sigma} \|_{L^{p'}(\mathbb{R}^n)} \le C \|f\|_{L^p(\mathbb{R}^n)} \quad \text{for } 1 \le p \le \frac{2n+2}{n+3}.$$

[Since

$$\int_{S^{n-1}} |\hat{f}(\xi)|^2 d\sigma = \langle \hat{f}, \hat{f} d\sigma \rangle = \langle \hat{f}, \widehat{f * \widehat{d\sigma}} \rangle \underset{\substack{\uparrow \\ \text{Plancherel}}}{=} \langle f, f * \widehat{d\sigma} \rangle \underset{\substack{\uparrow \\ \text{Hölder}}}{\le} \|f\|_{L^p(\mathbb{R}^n)} \|f * \widehat{d\sigma}\|_{L^{p'}(\mathbb{R}^n)}$$
]

<u>Remark</u> (Partial Result)

Recall again that $\widehat{d\sigma} \in L^q(\mathbb{R}^n) \Leftrightarrow q > \frac{2n}{n-1}$.

Young's inequality implies that

$$\|f * \widehat{d\sigma}\|_{p'} \le \|\widehat{d\sigma}\|_q \|f\|_p \quad \underline{\underline{\text{if}}} \quad \frac{1}{p'} + 1 = \frac{1}{q} + \frac{1}{p} \Leftrightarrow q = p'/2.$$

Thus $\|f * \widehat{d\sigma}\|_{p'} \le C \|f\|_p$ <u>if</u> $\widehat{d\sigma} \in L^q(\mathbb{R}^n) \Leftrightarrow q > \frac{2n}{n-1} \Leftrightarrow \underline{\underline{p < \frac{4n}{3n+1}}}$.

* We obtain $p < \frac{2n+2}{n+3}$ by decomposing $\widehat{d\sigma}$ & using (real) interpolation *.

Let $\varphi \in C_c^\infty$ s.t. $\varphi(x) = 1$ if $|x| \le 1$ & $\varphi(x) = 0$ if $|x| \ge 2$.

Define $\psi(x) := \varphi(x) - \varphi(2x)$ & $\psi_j(x) = \psi(2^{-j}x)$.

It follows that we have the following partition of unity:

$$1 = \varphi(x) + \sum_{j=1}^{\infty} \psi_j(x) \quad \text{for all } x \in \mathbb{R}^n. \qquad \text{(Exercise)}.$$

Proof of Theorem (for $1 \le p < \frac{2n+2}{n+3}$)

Using the partition of unity above, we write

$$\widehat{d\sigma}(x) = \underbrace{\varphi(x)\widehat{d\sigma}(x)}_{=K(x)} + \sum_{j=1}^{\infty} \underbrace{\psi_j(x)\widehat{d\sigma}(x)}_{=K_j(x)}$$

"supported $|x| \le 2$" "supported $2^{j-1} \le |x| \le 2^{j+1}$"

Hence

$$f * \widehat{d\sigma} = f * K + \sum_{j=1}^{\infty} f * K_j .$$

Note: Since $\widehat{d\sigma}$ continuous, it follows that $K \in L^q$ for all q, hence by Young's inequality we have

$$\|f * K\|_{p'} \le \|K\|_q \|f\|_p \quad \text{with } q = p'/2 .$$
$$\le C\|f\|_p$$

¿What about the terms in the sum?

We want to obtain

$$\|f * K_j\|_{p'} \le C\, 2^{-\varepsilon j} \|f\|_p \quad \text{for some } \varepsilon > 0$$

since then we will be able to sum things up nicely.

* We will obtain such a (p,p') estimate by first obtaining the easier $(1,\infty)$ & $(2,2)$ estimates & then interpolating. *

Note that

$$\text{(i)} \quad \|f * K_j\|_\infty \le \|K_j\|_\infty \|f\|_1 \quad \text{(Young)}$$

$$\& \quad \text{(ii)} \quad \|f * K_j\|_2 = \|\hat{f}\, \widehat{K_j}\|_2 \le \|\widehat{K_j}\|_\infty \|f\|_2 .$$

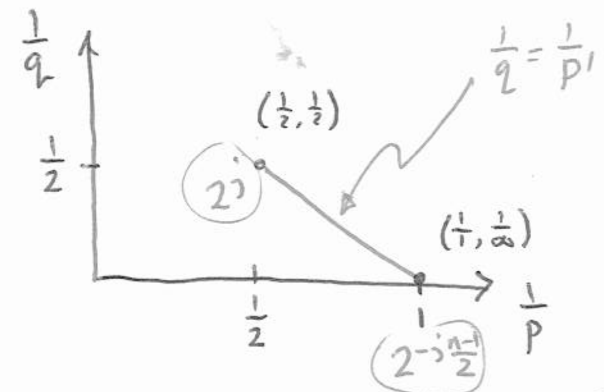
* Thus, we need simultaneous control of both K_j & $\widehat{K_j}$ *

Claim:

(i) $\|K_j\|_\infty \le C\, 2^{-\frac{n-1}{2} j}$ & (ii) $\|\hat{K}_j\|_\infty \le C 2^j$

(decay) (growth).

Assuming the Claim we finish the proof by interpolation:



$$\frac{1}{p} = \frac{\theta}{2} + \frac{1-\theta}{1}$$

$$\frac{1}{p'} = \frac{\theta}{2} + \frac{1-\theta}{\infty} = \frac{\theta}{2} \Rightarrow \underline{\underline{\theta = \frac{2}{p'}}}$$

$$\|f * K_j\|_{p'} \le C\,(2^j)^{2/p'}\left(2^{-j\frac{n-1}{2}}\right)^{1-2/p'} \|f\|_p = C\, 2^{-j\left(\frac{n-1}{2} - \frac{n+1}{p'}\right)} \|f\|_p$$

$$\& \quad \frac{n-1}{2} > \frac{n+1}{p'} \iff p < \frac{2n+2}{n+3}.$$

□

Proof of Claim:

(i): $|\widehat{d\sigma}(x)| \le C|x|^{-\frac{n-1}{2}} \Rightarrow |K_j(x)| \le C\, 2^{-j\frac{n-1}{2}}$.

since $\hat{\varphi} \in S(\mathbb{R}^n)$.

(ii): Since $\varphi_j(x) = \varphi(2^{-j}x) \Rightarrow \hat{\varphi}_j(\xi) = 2^{+jn}\hat{\varphi}(2^j\xi) \Rightarrow |\hat{\varphi}_j(\xi)| \le \frac{C 2^{jn}}{(1+2^j|\xi|)^N}$ $\forall N$.

Therefore,

$$|\hat{K}_j(\xi)| = |\hat{\varphi}_j * d\sigma(\xi)| = \left|\int \hat{\varphi}_j(\xi-\eta)\, d\sigma(\eta)\right|$$

$$\le C 2^{jn} \underbrace{\int (1+2^j|\xi-\eta|)^{-N} d\sigma(\eta)}.$$

$$= \int_{|\xi-\eta|\le 2^j} \text{—}''\text{—} + \sum_{k=0}^{\infty} \int_{|\xi-\eta|\approx 2^{-j+k}} \text{—}''\text{—}$$

$$\le C 2^{jn}\left\{2^{-j(n-1)} + \sum_{k=0}^{\infty} 2^{-kN} 2^{-(j-k)(n-1)}\right\}$$

$$\le C 2^j \quad \text{if } \underline{\underline{N \ge n}}!$$

□