Appendix (on Measurability on $\mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$).

Lemma

If f measurable on $\mathbb{R}^{n_1}$, then $F(x,y) = f(x)$ is measurable on $\mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$.

Proof: Assume that $n_2 = 1$. Need to show that for all $a \in \mathbb{R}$

$$\{(x,y) \in \mathbb{R}^n \times \mathbb{R} : F(x,y) > a\} \in \mathcal{M}(\mathbb{R}^{n+1}).$$

$$\| $$

$$\{x \in \mathbb{R}^n : f(x) > a\} \times \mathbb{R}$$

* Things thus reduce to showing that if $E \in \mathcal{M}(\mathbb{R}^n)$, then $E \times \mathbb{R} \in \mathcal{M}(\mathbb{R}^{n+1})$:

- Write $E = H \cup N$ with H a F_σ-set and $m(N) = 0$.

  $\Rightarrow \quad E \times \mathbb{R} = (H \times \mathbb{R}) \cup (N \times \mathbb{R})$.

  Since $H \times \mathbb{R}$ is clearly a F_σ-set in $\mathbb{R}^{n+1}$ we will be done if we can show that $N \times \mathbb{R}$ has <u>measure zero</u> in $\mathbb{R}^{n+1}$:

- Define $E_k = \{x \in \mathbb{R} : |x| \leq k\}$, then $E_1 \subseteq E_2 \subseteq \cdots$ & $\bigcup_k E_k = \mathbb{R}$.

  $\Rightarrow \quad N \times E_1 \subseteq N \times E_2 \subseteq \cdots$ and $\bigcup_k (N \times E_k) = N \times \mathbb{R}$.

  and hence that $m(N \times \mathbb{R}) = \lim_{k \to \infty} m(N \times E_k) = 0$ $\square$

  Claim: For each $k \in \mathbb{N}$, $m(N \times E_k) = 0$.

  Pf: Fix k & let $\varepsilon > 0$. Since N is null in $\mathbb{R}^n$ we know that

  $$N \subseteq \bigcup_j Q_j \text{ with } \sum_j |Q_j| < \varepsilon/2k. \quad (\text{with } \{Q_j\} \text{ closed cubes})$$

  $$\Rightarrow N \times E_k \subseteq \bigcup_j \underbrace{(Q_j \times E_k)}_{\text{cubes!}} \text{ with } \sum_j |Q_j \times E_k| = \sum_j 2k|Q_j| < \varepsilon \quad \square$$

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Consequence of Lemma 1

① f & g m'ble on $\mathbb{R}^{n_1}$ & $\mathbb{R}^{n_2}$ $\Rightarrow$ $H(x,y) = f(x)g(y)$ m'ble on $\mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$

$\left[H(x,y) = F(x,y)G(x,y) \text{ where } F(x,y) = f(x) \text{ \& } G(x,y) = g(y). \right]$

② f, g m'ble on $\mathbb{R}^n$ $\Rightarrow$ $h(x,y) = f(x-y)g(y)$ m'ble on $\mathbb{R}^{2n}$.

$$\left[\begin{aligned} h(x,y) &= F \circ T(x,y)\, G(x,y) \text{ where } F(x,y) = f(x),\ G(x,y) = g(y) \\ &= F(x-y, x+y)\, G(x,y) \ \underline{\text{and}}\ \underline{T = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}} \end{aligned} \right]$$

③ $f \geq 0$ & m'ble on $\mathbb{R}^n$ $\Rightarrow$ $\tilde{F}(x,y) = y - f(x)$ m'ble on $\mathbb{R}^{n+1}$ for any $y \in \mathbb{R}$.

$\left[\tilde{F}(x,y) = G(x,y) - F(x,y) \text{ where } G(x,y) = y \text{ \& } F(x,y) = f(x) \right]$

"Area under Graph"

Suppose $f(x) \geq 0$ on $\mathbb{R}^n$ & $\mathcal{A} := \{(x,y) \in \mathbb{R}^n \times \mathbb{R} : 0 \leq y \leq f(x)\}$, then

(i) f m'ble on $\mathbb{R}^n$ $\Leftrightarrow$ $\mathcal{A} \in \mathcal{M}(\mathbb{R}^{n+1})$

(ii) f m'ble on $\mathbb{R}^n$ $\Rightarrow$ $\int_{\mathbb{R}^n} f(x)\,dx = m(\mathcal{A})$.

Proof: (i): ($\Rightarrow$) follows from ③ since $\mathcal{A} = \{y \geq 0\} \cap \{\tilde{F} \leq 0\}$

($\Leftarrow$) Corollary of Tonelli $\Rightarrow$ $f(x) = m(\mathcal{A}_x)$ is m'ble.

(ii) Corollary of Tonelli $\Rightarrow$ $m(\mathcal{A}) = \int_{\mathbb{R}^n} m(\mathcal{A}_x)\,dx = \int_{\mathbb{R}^n} f(x)\,dx$. $\square$