Plünnecke - Ruzsa Inequalities

Let A and B be finite subsets of an abelian group satisfying $|A+B| \le K|A|$.

Theorem (Plünnecke-Ruzsa): $|kB - \ell B| \le K^{k+\ell}|A| \quad \forall\, k,\ell \in \mathbb{Z}_{\ge 0}$.

Petridis recently gave a beautifully simple new proof of this theorem, the key to his argument is the following

Proposition (Petridis)

Let $X \subseteq A$ such that $\frac{|X+B|}{|X|}$ is minimal, then given any set S

$$|S+X+B| \le K|S+X|.$$

Proof of Theorem: In addition to the above proposition we will also need the following elementary result:

Ruzsa's Δ-inequality: Given any sets U,V,W, $|U|\cdot|V-W| \le |U+V|\cdot|U+W|$

[Pf: Since $v-w = (v+u)-(w+u)$, every $v-w \in V-W$ has $|U|$ repns as $x-y$ w/ $(x,y) \in (U+V)\times(U+W)$

From proposition we know $\exists\, X \subseteq A$ s.t. for any $k \in \mathbb{Z}_{\ge 0}$

$$|X+kB| = |X+(k-1)B+B| \le K|X+(k-1)B| \le \cdots \le K^k|X|.$$

It follows that

$$|X|\cdot|kB-\ell B| \le |kB+X|\cdot|\ell B+X| \le K^{k+\ell}|X|^2$$

and hence (since $X \subseteq A$) that $|kB-\ell B| \le K^{k+\ell}|X| \le K^{k+\ell}|A|$. $\square$.

Proof of Proposition

Let $K' := \dfrac{|X+B|}{|X|}$ (note $K' \leq K$) and $S = \{s_1, \ldots, s_k\}$.

Note that

$$S + X = \bigcup_{j=1}^{k}{}^{*} (s_j + X_j) \quad \text{where} \quad s_j + X_j = (s_j + X) \setminus \bigcup_{i<j} (s_i + X)$$

defines $X_j \subseteq X$.

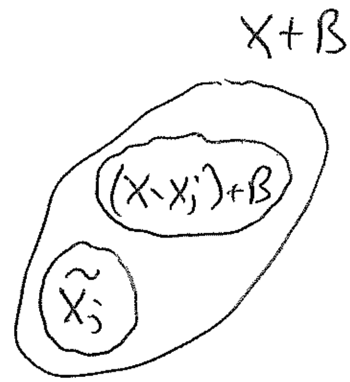
and hence $|S+X| = \sum_{j=1}^{k} |X_j|$.

Similarly

$$S + X + B = \bigcup_{j=1}^{k}{}^{*} (s_j + \widetilde{X}_j) \quad \text{where} \quad s_j + \widetilde{X}_j = (s_j + X + B) \setminus \bigcup_{i<j} (s_i + X + B)$$

defines $\widetilde{X}_j \subseteq X+B$.

and hence $|S+X+B| = \sum_{j=1}^{k} |\widetilde{X}_j|$.



We will clearly be done if we establish the following

Claim: For each $1 \leq j \leq k$, $\widetilde{X}_j \subseteq (X+B) \setminus ((X \setminus X_j) + B)$.

Since the minimality of X ensures that $|(X \setminus X_j) + B| \geq K' |X \setminus X_j|$ and hence

$$|\widetilde{X}_j| \underset{\uparrow \text{ Claim}}{\leq} |X+B| - |(X \setminus X_j) + B| \leq K'(|X| - |X \setminus X_j|) \underset{\uparrow\, K' \leq K.}{\leq} K |X_j|$$

Proof of Claim: If $x \in X \setminus X_j$, then $s_j + x \in (s_i + X)$ for some $i < j$.

$\Rightarrow$ $s_j + x + B \subseteq s_i + X + B$ and hence $x + B \subseteq (X+B) \setminus \widetilde{X}_j$.

$\Rightarrow$ $(X \setminus X_j) + B \subseteq (X+B) \setminus \widetilde{X}_j$ $\square$