Math 4900/6900 Additional Problems 2

1. Suppose that f and g are two 2π-periodic integrable functions. We defined their **convolution** $f * g$ on $[-\pi, \pi]$ by

$$f * g\,(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(y)g(x-y)\,dy,$$

prove that

(a) $f * g$ is continuous.

(b) $\widehat{f * g}(n) = \widehat{f}(n)\widehat{g}(n)$.

(c) $f * g = g * f$.

Hint: See Proposition 2.3.1. Observe that properties (a) and (b) are easily deduced if we assume that f and g are continuous functions, then use Lemma 2.3.2. Property (c), which we have used a number of times, now follows from (a) and (b) - Why?

2. Read the proof of Theorem 2.4.1, then prove the following results.

Theorem (Addition to Theorem 2.4.1). *Let $\{K_n\}_{n=1}^{\infty}$ be a family of <u>even</u> good kernels, and f an integrable function on the circle. If f has a jump discontinuity[1] at a point x, then*

$$\lim_{n\to\infty} f * K_n\,(x) = \frac{f(x^+) + f(x^-)}{2}.$$

Corollary (Addition to Corollary 2.5.2). *If f is an integrable function on the circle with a jump discontinuity at x, then the Fourier series of f at x is Cesáro summable to $\frac{f(x^+)+f(x^-)}{2}$, that is*

$$\lim_{N\to\infty} \sigma_N(f)(x) = \frac{f(x^+) + f(x^-)}{2}.$$

3*. Prove the following.

Weyl's Criterion. *The following assertions concerning a given sequence $\{\xi_n\}$ in $[0,1)$ are equivalent:*

(i) *The sequence $\{\xi_n\}$ is equidistributed, that is for every interval $(a,b) \subset [0,1)$,*

$$\lim_{N\to\infty} \frac{\#\{1 \le n \le N \,:\, \xi_n \in (a,b)\}}{N} = b - a;$$

(ii) *For each integer $k \neq 0$,*

$$\lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} e^{2\pi i k \xi_n} = 0;$$

(iii) *For any (Riemann) integrable function f on $[0,1]$ that is periodic with period 1*

$$\lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} f(\xi_n) = \int_0^1 f(x)\,dx.$$

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[1] Recall that an integrable function is said to have a **jump discontinuity** at x if the two limits

$$f(x^+) = \lim_{h\to 0^+} f(x+h) \qquad \text{and} \qquad f(x^-) = \lim_{h\to 0^+} f(x-h)$$

both exist.