Structure of Bohr Sets

Recall that given $\Gamma \subseteq \mathbb{Z}_N$ and $\varepsilon > 0$ we defined the <u>Bohr set</u> $B(\Gamma, \varepsilon)$ by

$$B(\Gamma, \varepsilon) = \{x \in \mathbb{Z}_N : \|\tfrac{x\xi}{N}\| \leq \varepsilon \ \forall \xi \in \Gamma\}$$

and refered to $|\Gamma|$ as the rank of $B(\Gamma, \varepsilon)$ and ε as its radius.

<u>Exercise 1</u>: Use the pigeonhole principle to show that $|B(\Gamma, \varepsilon)| \geq \varepsilon^{|\Gamma|} N$ and conclude from this that $B(\Gamma, \varepsilon)$ always contains an arithmetic progression of length at least $\varepsilon N^{1/|\Gamma|}$.

The main result of this note is the following:

Theorem

Any Bohr set of rank d and radius $0 < \varepsilon < 1$ contains a <u>proper</u> symm generalized arith. prog. of dimension d and size at least $\left(\frac{\varepsilon}{d}\right)^d N$.

The proof of this theorem relies on some geometry of numbers, specifically

<u>Minkowski's Second Theorem</u>

Let $K \subseteq \mathbb{R}^d$ be a centrally symmetric convex body and Λ a non-deg lattice, then

$$\lambda_1 \lambda_2 \cdots \lambda_d \operatorname{vol}(K) \leq 2^d \det(\Lambda) \qquad (*)$$

where λ_j denotes the <u>jth successive minima of K wrt Λ</u>.

$\hookrightarrow \inf\{\lambda > 0 : \lambda K \text{ contains } j \text{ lin. indep. elements from } \Lambda\}$.

<u>Note</u>: Since $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_d$, $(*) \Rightarrow \lambda_1^d \operatorname{vol}(K) \leq 2^d \det(\Lambda)$ and hence that K must contain a non-zero lattice point whenever $\operatorname{vol}(K) > 2^d \det(\Lambda)$

- Recall that if $\Lambda \subseteq \mathbb{R}^d$ is a (non-deg) lattice, that is a discrete subgp of $\mathbb{R}^d$, generated by the linearly independent vectors $v_1, \ldots, v_d$, then
$$\det(\Lambda) := \det(v_1, v_2, \ldots, v_d).$$

- For a proof of Minkowski's 2nd Theorem, see Ben Green's notes.

<u>Proof of Theorem</u>: Suppose that our Bohr set $B(\Gamma, \varepsilon) \subseteq \mathbb{Z}_N$ with spectrum $\Gamma = \{\xi_1, \ldots, \xi_d\}$ with property that $(\xi_j, N) = 1$ for some $1 \leq j \leq d$.

Let $\Lambda := N\mathbb{Z}^d + (\xi_1, \ldots, \xi_d)\mathbb{Z}$ and $K = \{x \in \mathbb{R}^d : \underbrace{\|x\|_\infty}_{= \max_{1\leq j\leq d} |x_j|} \leq 1\}$.

<u>Exercise 2</u>: Show that $\mathrm{vol}(K) = 2^d$ and $\det(\Lambda) = N^{d-1}$.

Minkowski $\Rightarrow \exists$ lin. indep $v_1, \ldots, v_d \in \Lambda$ with $v_j \in \lambda_j K \;\forall\, 1 \leq j \leq d$ <u>&</u> $\lambda_1 \cdots \lambda_d \leq N^{d-1}$.

Since $v_j \in \Lambda$ we know $\exists\, x_j \in \mathbb{Z}$ s.t. $v_j \equiv (x_j\xi_1, \ldots, x_j\xi_d) \bmod N$, combining this with fact that $v_j \in \lambda_j K$ we conclude that $\|\frac{x_j \xi}{N}\| \leq \frac{\lambda_j}{N} \;\forall\, \xi \in \Gamma$, in other words $x_j \in B(\Gamma, \lambda_j/N)$ for each $1 \leq j \leq d$.

It follows that
$$P := \{l_1 x_1 + \cdots + l_d x_d : |l_j| \leq \frac{\varepsilon N}{2d\lambda_j}\} \subseteq B(\Gamma, \varepsilon).$$

Note that <u>if</u> P is <u>proper</u>, then it follows that $|P| \geq \left(\frac{\varepsilon}{d}\right)^d \frac{N^d}{\lambda_1 \cdots \lambda_d} \geq \underline{\underline{\left(\frac{\varepsilon}{d}\right)^d N}}$.

<u>Verification that P is proper</u>: Suppose $l_1 x_1 + \cdots + l_d x_d = l_1' x_1 + \cdots + l_d' x_d$.

$\Rightarrow l_1 v_1 + \cdots + l_d v_d \equiv l_1' v_1 + \cdots + l_d' v_d \bmod N$ where $0 \leq l_j, l_j' \leq \varepsilon N / d\lambda_j$.

$\Rightarrow w = (l_1 - l_1')v_1 + \cdots + (l_d - l_d')v_d \in N\mathbb{Z}^d$ with $\|w\|_\infty \leq \sum_{j=1}^d \frac{\varepsilon N}{d\lambda_j} \|v_j\|_\infty < \varepsilon N$.

Since $0 < \varepsilon < 1$ it follows that $w = 0$ and hence $l_j = l_j' \;\forall\, 1 \leq j \leq d$. $\square$