University of Georgia REU - Additive Combinatorics - Summer 2010

Exercise Sheet 1

Reading Assignment

1. Chapter 1 of *Ramsey Theory on the Integers* by Landman and Robertson

Problems and Exercises

1. Let $A \subseteq \{1, \dots, 2n\}$ with $|A| = n + 1$. Show that A must contain two elements that are relatively prime and two elements such that one divides the other.

2. Prove that if the numbers 1,2,...,12 are randomly positioned around a circle, then some set of three consecutively positioned numbers must have a sum of at least 19.

3. Suppose we are given n integers $a_1, \dots, a_n$, which need not be distinct. Then there is always a set of consecutive integers $a_{k+1}, \dots, a_\ell$ such that $a_{k+1} + \cdots + a_\ell$ is a multiple of n.

4. (a) Prove that within any sequence of $n^2 + 1$ integers there exists a monotone subsequence of length $n + 1$. (This is Example 1.8 in Landman and Robertson)

   (b) Show that given a sequence of only n^2 integers, there need not be a monotone subsequence of length $n + 1$.

   (c) More generally, given integers n and m, show that any sequence of length at least $nm+1$ contains either a monotonically increasing subsequence of length $n+1$, or a monotonically decreasing subsequence of length $m + 1$.

5. Let $r \geq 2$. Show that there exists a least positive integer $M = M(k; r)$ so that any r-coloring of M integers contains a monochromatic monotonic k-term subsequence. Determine $M(k; r)$. (Note that from Example 1.8 in Landman and Robertson, $M(k; 1) = k^2 + 1$)

6. Show that any 3-coloring of the xy-plane must contain two points, a unit distance apart, of the same color. Is there anything special about the distance 1? Is the result true if we only use two colors?

7. A collection of sets $A_1, \dots, A_\ell$ forms a *sunflower* if the pairwise intersections $A_i \cap A_j$ for $i \neq j$ are all the same. We will allow this pairwise intersection to be empty.

   (a) Use the pigeonhole principle to show that if $\mathcal{A}$ is a collection of sets, each of size at most k, and $|\mathcal{A}| > (\ell - 1)k$, then either $\mathcal{A}$ contains ℓ disjoint sets, or that there exist at least $|\mathcal{A}|/(\ell - 1)k$ sets which all have a common element x_0.

   (b) If $\mathcal{A}$ is a collection of sets, each of size at most k, and $|\mathcal{A}| > (\ell - 1)^k k!$, then $\mathcal{A}$ contains ℓ sets forming a sunflower.

8. (a) Prove that any r-coloring of the integer lattice contains a monochromatic rectangle. (The case $r = 2$ is Example 1.9 in Landman and Robertson)

   (b) Prove that any r-coloring of the integer lattice contains a monochromatic $k \times k$ grid.

9. (a) We showed that $R_2(3) = 6$. Hence we know that any two coloring of K_6 must contain at least one monochromatic triangle. Prove that any 2-coloring of K_6 must in fact contain at least two monochromatic triangles.

(b) Show that any 3-coloring of K_{17} contains a monochromatic triangle. (Use the fact that $R_2(3) = 6$.)

(c) Prove that

$$R_r(3) \leq \lfloor er! \rfloor + 1$$

by using induction on r (and a fine tuning of the argument given to prove Lemma 8.7 in Landman and Robertson). In fact, show that

$$R_{r+1}(3) - 1 \leq (r+1)(R_r(3) - 1) + 1$$

and use the fact that $1 + \frac{1}{2!} + \ldots + \frac{1}{r!} \leq e$ for each r.
Prove that equality holds in the case when $r = 2, 3$.

10. Prove that for $k \geq 2$,

$$2^{k/2} \leq R_2(k) \leq 2^{2k}.$$

Hint: For the lower bound, see Exercise 1.15 in Landman and Robertson. For the upper bound prove that if G is a complete graph with 2^{2k} vertices, then there exists a sequence of sets of vertices $V_{2k} \subseteq V_{2k-1} \subseteq \cdots \subseteq V_1$ with $|V_j| \geq 2^{2k-j}$ for each $1 \leq j \leq k$ and a sequence of vertices $x_1, \ldots, x_{2k}$ such that $x_j \in V_{j-1}$ for each $2 \leq j \leq k$ and each edge from x_j to V_j is the same color.

11. (a) Let $k \in \mathbb{N}$. Show that there exists a set $A \subseteq \{1, \ldots, 3^k - 1\}$ with $|A| \geq 2^k - 1$ such that no three elements of A lie in arithmetic progression.

(b) Prove that if $N \geq R_r(3)^{1/\log_3 2}$, then any r-coloring of $\{1, \ldots, N\}$ must contain a monochromatic solution to the equation $x + y = z$ with x and y distinct.

12. Prove that the Schur number $S(r)$ satisfies $S(r) \geq \frac{3^r+1}{2}$ by completing the following steps.

(a) $S(2) = 5$

(b) If c is an r coloring of $[1, N]$ such that there is no monochromatic Schur triple, then define an $r+1$ coloring of $[1, 3N+1]$ the following way. Color the two blocks $[1, N]$ and $[2N+2, 3N+1]$ the same way as original coloring, and color each number in $[N+1, 2N+1]$ by a new color. Show that there is no monochromatic Schur triple in the new coloring.

Conclude that $S(r+1) \geq 3S(r) - 1$, and do induction on r.

13. Show that there is at least one monochromatic arithmetic progression of length three in every 2-coloring of [1,9].

14. Decide whether the following assertion is true: If the set of natural numbers is r-colored, then there must be a monochromatic solution to

(a) $x + y = 3z$

(b) $x + 2y = z$

Hints: For (a) use the fact that every natural number can be expressed in the form $5^k(5\ell + j)$ for some $1 \leq j \leq 4$. For (b) use induction on r and van der Waerden's theorem.

15. Prove **Rado's theorem**: Let $k \geq 2$ and $c_i \in \mathbb{Z} \setminus \{0\}$ for $1 \leq i \leq k$. Given any r-coloring of the natural numbers there exists a monochromatic solution to the equation

$$c_1 x_1 + \cdots c_k x_k = 0$$

if and only if there exists a nonempty set $J \subseteq \{1, \ldots, k\}$ such that $\sum_{j \in J} c_j = 0$.

Remark: Some problems on this sheet were taken from Landman-Robertson [1] and Tao-Vu [2].

References

[1] Bruce M. Landman and Aaron Robertson. *Ramsey theory on the integers*, volume 24 of *Student Mathematical Library*. American Mathematical Society, Providence, RI, 2004.

[2] Terence Tao and Van Vu. *Additive combinatorics*, volume 105 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 2006.