Math 3100
Sample Exam 3 – Version 1

No calculators. Show your work. Give full explanations. Good luck!

1. (4 points) Explain why there exist no examples of the following:
   (a) A continuous function on $[0, 1]$ with range equal to $(0, 1)$.
   (b) A continuous function on $[0, 1]$ with range equal to $[0, 1] \cap \mathbb{Q}$

2. (8 points) Evaluate the following infinite series

$$\text{(a)} \quad \sum_{n=1}^{\infty} \frac{n}{4^n} \qquad\qquad \text{(b)} \quad \sum_{n=1}^{\infty} \frac{(-1)^n}{n4^n}$$

3. (14 points)
   (a) i. Find the sixth order Maclaurin polynomial for the function

$$f(x) = \frac{x^2}{2 + x^2}$$

   ii. Without differentiating find the value of $f^{(6)}(0)$.

   (b) Let $P_3(x)$ denote the third order Taylor polynomial centered at $x_0 = 1$ of $f(x) = \log x$.
   i. Find $P_3(x)$.
   ii. Give an estimate for how well $P_3(1.5)$ approximates $\log(1.5)$.

   (c) i. Carefully state the *Lagrangian Remainder Estimate* for Maclaurin series.
   ii. Find a polynomial that approximates e^x to within 10^{-3} for all $|x| \leq 1/2$.

4. (14 points)
   (a) Carefully state what it mean to say that a function $f : \mathbb{R} \to \mathbb{R}$ is differentiable at x_0 and prove that if f is differentiable at x_0, then f is continuous at x_0.

   (b) Let $h(x) = \begin{cases} x^2, & x \in \mathbb{Q} \\ 0, & x \notin \mathbb{Q} \end{cases}$.
   i. Prove that h is discontinuous at all $x \neq 0$.
   ii. Prove that h is differentiable at $x = 0$.
   iii. What can you say about the continuity of h at $x = 0$ and the differentiability of h at $x \neq 0$?

   (c) Let $f : [a, b] \to \mathbb{R}$.
   Prove that if f has a minimum at a point $c \in (a, b)$, and if $f'(c)$ exists, then $f'(c) = 0$.

5. (10 points) Let $h_n(x) = \dfrac{x}{(1 + x)^{n+1}}$.
   (a) Prove that h_n converges uniformly to 0 on $[0, \infty)$.
   (b) i. Verify that

$$\sum_{n=0}^{\infty} h_n(x) = \begin{cases} 1 \text{ if } x > 0 \\ 0 \text{ if } x = 0 \end{cases}$$

   ii. Does $\sum_{n=0}^{\infty} h_n$ converge uniformly on $[0, \infty)$?

   (c) Prove that $\sum_{n=0}^{\infty} h_n$ converges uniformly on $[a, \infty)$ for any $a > 0$.

   Hint: Recall that the Binomial Theorem implies $(1 + x)^{n+1} \geq \dfrac{n(n + 1)}{2} x^2$ *for all* $x \geq 0$.