The "Classical" Fourier Analytic Proof of Roth's Theorem

Theorem (Roth, 1953): $\frac{r_3(N)}{N} \ll \frac{1}{\log\log N}$

Proposition (Dichotomy) Let P be an arith. prog. of integers and $A \subseteq P$ with density $\delta > 0$. If $|P| \geq 1000\,\delta^{-10}$ (say), then either

(i) $\#$ 3AP's in A (inc. trivial) $\geq \frac{\delta^3 |P|^2}{32}$ [In part. at least one non-trivial 3AP]

close to "expected" number.

OR

(ii) $\exists$ arith. prog. $P' \subseteq P$ with $|P'| \geq |P|^{1/3}$ such that
$$|A \cap P'| \geq (\delta + \delta^2/64)\,|P'|.$$

Proposition $\Rightarrow$ Theorem:

Suppose $A \subseteq [1, N]$ with $|A| = \delta N$ and no 3AP's. Provided $N \geq 1000\,\delta^{-10}$ it follows that (ii) holds (as (i) doesn't). We define $A' := A \cap P' \subseteq P'$.

It follows that $|A'| = \delta' |P'|$ with $|P'| \geq N^{1/3}$ and $\delta' \geq \delta + \delta^2/64$.

* Since $A' \subseteq A$, it also has no 3AP's * Repeat...

This process can be repeated at most $m = \frac{128}{\delta}$ times (as in this many steps the density will necessarily exceed 1, which is rediculous!).

Thus we must have
$$N^{1/3^m} \ll \delta^{-10} \iff \delta \ll \frac{1}{\log\log N}. \qquad \square$$

Proof of Proposition

To prove the propn we can, by a simple rescaling argument, assume that $P=[1,N]$. Furthermore we shall assume that N is odd and identify

$$[1,N] \simeq \mathbb{Z}_N \quad (= \mathbb{Z}/N\mathbb{Z}).$$

Fourier Analysis on $\mathbb{Z}_N$

For $f:\mathbb{Z}_N \to \mathbb{C}$ we define $\hat{f}(\xi) = \frac{1}{N}\sum_{x\in\mathbb{Z}_N} f(x)e^{-2\pi i x\xi/N}$.

The following are simple consequences of the orthogonality relation

$$\frac{1}{N}\sum_{x\in\mathbb{Z}_N} e^{2\pi i x\xi/N} = \begin{cases} 1 & \text{if } \xi=0 \\ 0 & \text{if } \xi\neq 0 .\end{cases}$$

Exercise 1: Verify this.

- If $f\equiv 1$, then $\hat{f}(\xi) = \begin{cases} 1 & \text{if } \xi=0 \\ 0 & \text{if } \xi\neq 0 .\end{cases}$
- Fourier Inversion: $f(x) = \sum_{\xi\in\mathbb{Z}_N} \hat{f}(\xi)e^{2\pi i x\xi/N}$
- Plancherel Identity: $\frac{1}{N}\sum_{x\in\mathbb{Z}_N} |f(x)|^2 = \sum_{\xi\in\mathbb{Z}_N} |\hat{f}(\xi)|^2$.

If $A\subseteq\mathbb{Z}_N$ with $|A|=\delta N$, then

- $|\hat{1}_A(\xi)| \leq \hat{1}_A(0) = \delta$
- $\sum_{\xi\in\mathbb{Z}_N} |\hat{1}_A(\xi)|^2 = \delta$

and if we define $f := 1_A - \delta$ (the balanced function of A), then

$$\hat{f}(0) = \frac{1}{N}\sum_{x\in\mathbb{Z}_N} f(x) = 0 \quad\text{while}\quad \hat{f}(\xi) = \hat{1}_A(\xi) \text{ for all } \xi\neq 0.$$

For $f_1, f_2, f_3 : \mathbb{Z}_N \to \mathbb{C}$ we define the operator

$$AP_3(f_1, f_2, f_3) = \frac{1}{N^2} \sum_{x \in \mathbb{Z}_N} \sum_{d \in \mathbb{Z}_N} f_1(x) f_2(x+d) f_3(x+2d).$$

<u>Exercise 2</u>: Verify that $$AP_3(f_1, f_2, f_3) = \sum_{\xi \in \mathbb{Z}_N} \hat{f}_1(\xi) \hat{f}_2(-2\xi) \hat{f}_3(\xi).$$

<u>Note</u>: If $A \subseteq \mathbb{Z}_N$, then

$$AP_3(1_A, 1_A, 1_A) = \frac{1}{N^2} \times \# \ \mathbb{Z}_N\text{-3AP's in } A \quad (\text{counting trivial})$$

while if $B := A \cap [\frac{N}{3}, \frac{2N}{3}]$, then

$$AP_3(1_B, 1_B, 1_A) \leq \frac{1}{N^2} \times \# \ (\text{genuine}) \text{ 3AP's in } A \quad (\text{counting trivial}).$$

* <u>Important Observation</u>: In proving propn, we may assume that $|B| \geq \frac{\delta}{4} N$.

(if not, then $\max\{|A \cap [1, N/3]|, |A \cap [\frac{2N}{3}, N]|\} \geq \frac{3}{8} \delta N = (\delta + \frac{\delta}{8}) \frac{N}{3}$)

<u>Two Key Lemmas</u>: Let $A \subseteq \mathbb{Z}_N$ with $|A| = \delta N$ and $N \geq 1000 \delta^{-10}$.

<u>Lemma 1</u>

If $|AP_3(1_B, 1_B, 1_A) - (\frac{|B|}{N})^2 \delta| \overset{(*)}{\geq} (\frac{|B|}{N})^2 \frac{\delta}{2}$, then $\exists \ \xi \neq 0$ s.t. $|\hat{1}_A(\xi)| \geq \frac{|B|}{N} \frac{\delta}{2}$.

<u>Lemma 2</u>: Let $\varepsilon > 0$.

If $\exists \ \xi \neq 0$ s.t. $|\hat{1}_A(\xi)| \geq \varepsilon$, then $\exists$ <u>genuine</u> arith prog $P \subseteq \mathbb{Z}_N$ with

$|P| \geq (\frac{\varepsilon N}{16\pi})^{1/2}$ such that $|A \cap P| \geq (\delta + \varepsilon/8)|P|$.

<u>Proof of Proposition</u>:

$$\text{If (i) doesn't hold} \underset{\text{Exercise 3}}{\Longrightarrow} (*) \text{ holds} \underset{\text{Lemma 1}}{\Longrightarrow} \exists \ \xi \neq 0 \text{ s.t. } |\hat{1}_A(\xi)| \geq \frac{\delta^2}{8} \underset{\text{Lemma 2}}{\Longrightarrow} \text{(ii) holds.}$$

$\square$

Proof of Lemma 1

As a consequence of Exercise 2 we know that

$$AP_3(1_B, 1_B, 1_A) = \left(\frac{|B|}{N}\right)^2 \delta + \sum_{\xi \neq 0} \hat{1}_B(\xi)\hat{1}_B(-2\xi)\hat{1}_A(\xi)$$

$$\Rightarrow \left(\frac{|B|}{N}\right)^2 \frac{\delta}{2} \leq \left| AP_3(1_B, 1_B, 1_A) - \left(\frac{|B|}{N}\right)^2 \delta \right| \leq \max_{\xi \neq 0} |\hat{1}_A(\xi)| \sum_{\xi \in \mathbb{Z}_N} |\hat{1}_B(\xi)| \, |\hat{1}_B(-2\xi)|$$

An application of Cauchy-Schwarz followed by Planchel gives that

$$\sum_{\xi \in \mathbb{Z}_N} |\hat{1}_B(\xi)| \, |\hat{1}_B(-2\xi)| \leq \sum_{\xi \in \mathbb{Z}_N} |\hat{1}_B(\xi)|^2 = \frac{|B|}{N}$$

and the result follows. $\square$

Proof of Lemma 2

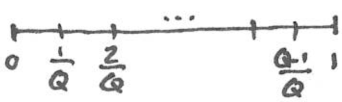
Let $\xi \neq 0$ be s.t. $|\hat{1}_A(\xi)| \geq \varepsilon$.

distance to nearest integer.

Recall, Dirichlet's Principle

Given any $\alpha \in \mathbb{R}$, $Q \in \mathbb{N}$, $\exists\ 1 \leq q \leq Q$ such that $\|q\alpha\| \leq \frac{1}{Q}$.

[Proof: "Q holes"



and "Q+1 pigeons" $\alpha, 2\alpha, \ldots, (Q+1)\alpha \mod 1$

There must exist $1 \leq k < \ell \leq Q+1$ s.t. $|(k-\ell)\alpha| \leq \frac{1}{Q} \mod 1$. Set $q = \ell - k$.]

Let Q and L be positive integer parameters to be determined.

By Dirichlet, $\exists\ 1 \leq q \leq Q$ such that $\|q\,\xi/N\| \leq \frac{1}{Q}$ and hence for any $\ell \in \mathbb{N}$

$$\left|1 - e^{2\pi i \ell q \xi/N}\right| \leq 2\pi\ell \, \|q\,\xi/N\| \leq \frac{2\pi\ell}{Q}. \quad (**)$$

Now partition $\mathbb{Z}_N$ into genuine arith. progs $\{P_j\}_{j=1}^{J}$ of step size q and length L_j where $L \leq L_j \leq 2L$.

Note that

x_j is first element in the prog. P_j

- $\frac{N}{J} \geq L \geq \frac{L_j}{2} \quad \forall\ 1 \leq j \leq J$
- If the P_j are to be genuine, then we must have $2QL \leq N$.

We now fix $Q = \frac{N}{2L}$ and seek to show that we can take L to be at least $(\varepsilon N/16\pi)^{1/2}$. Notice that if we set $f := 1_A - \delta$, then

$$\varepsilon \leq \left| \frac{1}{N} \sum_{x \in \mathbb{Z}_N} f(x) e^{-2\pi i \frac{x\xi}{N}} \right| \leq \frac{1}{N} \sum_{j=1}^{J} \left| \sum_{\ell=1}^{L_j} f(x_j + \ell q) e^{-2\pi i \ell q \xi / N} \right|$$

Assumption

by (**)

$$\leq \frac{1}{N} \sum_{j=1}^{J} \left| \sum_{\ell=1}^{L_j} f(x_j + \ell q) \right| + \frac{4\pi L}{Q}$$

Thus if we set $L = (\varepsilon N/16\pi)^{1/2}$ it follows that

$$\frac{1}{N} \sum_{j=1}^{J} \left| \sum_{x \in P_j} f(x) \right| \geq \frac{\varepsilon}{2}$$

$$\Rightarrow \frac{1}{N} \sum_{j=1}^{J} \left(\left| \sum_{x \in P_j} f(x) \right| + \sum_{x \in P_j} f(x) \right) \geq \frac{\varepsilon}{2} \qquad \left(\text{since } \sum_{x \in \mathbb{Z}_N} f(x) = 0 \right)$$

$\Rightarrow \exists\ 1 \leq j \leq J$ such that

$$\left| \sum_{x \in P_j} f(x) \right| + \sum_{x \in P_j} f(x) \geq \frac{\varepsilon}{2} \frac{N}{J} \geq \frac{\varepsilon}{4} |P_j|$$

$$\Rightarrow |A \cap P_j| - \delta |P_j| = \sum_{x \in P_j} f(x) \geq \frac{\varepsilon}{8} |P_j|$$

Since $|X| + X \geq 1 \Rightarrow X \geq \frac{1}{2}$

$$\Rightarrow \frac{|A \cap P_j|}{|P_j|} \geq \delta + \varepsilon/8 \quad \text{as required.}$$

$\square$