Math 3100 Assignment 4

Subsequences and Completeness

Due at 5:00 pm on Friday the 8th of February 2019

1. Evaluate following limits or explain why they do not exist. Be sure to justify your answer.

$$\text{(a)}\quad \lim_{n\to\infty}\left(\frac{2n+1}{3-n}\right)^3 \qquad \text{(b)}\quad \lim_{n\to\infty}\left((-1)^n+\frac{1}{n}\right)$$

$$\text{(c)}\quad \lim_{n\to\infty}\frac{\cos(n)}{n^2} \qquad \text{(d)}\quad \lim_{n\to\infty}\frac{n!+n}{2^n+3n!} \qquad \text{(e)}\quad \lim_{n\to\infty}\frac{n+\log(n)}{n+1}$$

2. (a) Let $x_1 = 0$ and $x_{n+1} = \dfrac{2x_n+1}{x_n+2}$ for all $n \in \mathbb{N}$.
   i. Find x_2, x_3, and x_4.
   ii. Prove that $\{x_n\}$ converges and find the value of its limit.

   (b) Let $a_1 = \sqrt{2}$, and define
   $$a_{n+1} = \sqrt{2+a_n}$$
   for all $n \geq 1$. Prove that $\lim_{n\to\infty} a_n$ exists and equals 2.

   Hint: For both parts try to apply the Monotone Convergence Theorem

3. (a) Prove that if $\{a_n\}$ is increasing, then every subsequence of $\{a_n\}$ is also increasing.

   (b) Let $\{x_n\}$ be a sequence of real numbers.
   Prove that $\{x_n\}$ contains a subsequence converging to x <u>if and only if</u> for all $\varepsilon > 0$ there exist infinitely many terms from $\{x_n\}$ that satisfy $|x_n - x| < \varepsilon$.

4. Let $A, B \subseteq \mathbb{R}$ which are non-empty, bounded above.

   (a) Show that if $A \subseteq B$, then $\inf(B) \leq \inf(A) \leq \sup(A) \leq \sup(B)$.

   (b) Show that if $\sup A < \sup B$, then there must exist $b \in B$ that is an upper bound for A.

   (c) Prove that if $\sup(A) \notin A$, then there exists a sequence $\{a_n\}$ of points in A such that
   $$\lim_{n\to\infty} a_n = \sup(A).$$

5. Let $\{x_n\}$ be a bounded sequence of real numbers and
$$S = \{x \in \mathbb{R} : \text{there exists a subsequence of } \{x_n\} \text{ that converges to } x\}.$$

   (a) Carefully explain why both $\sup S$ and $\inf S$ exist.
   The value of $\sup S$ is called the *limit superior* of $\{x_n\}$ is usually denoted by $\limsup_{n\to\infty} x_n$, while the value of $\inf S$ is called the *limit inferior* of $\{x_n\}$ is usually denoted by $\liminf_{n\to\infty} x_n$

   (b) Argue why $\lim_{n\to\infty} x_n$ exists <u>if and only if</u> $\liminf_{n\to\infty} x_n = \limsup_{n\to\infty} x_n$.
   In this case all three share the same value.

   (c) Prove that if $\beta > \limsup_{n\to\infty} x_n$, then there exists an N such that $x_n < \beta$ whenever $n > N$.

   (d) * Let $\alpha := \limsup_{n\to\infty} x_n$. Prove that there exists a subsequence of $\{x_n\}$ that converges α.