Hausdorff Measure and Dimension.

The theory of Hausdorff measure and dimension was invented in order to provide a notion of size not captured by existing theories, such as that of Lebesgue measure.

The idea is to measure the size of a set through choosing some α-dependent measure μ which selects set of dimension α. From the perspective of μ, sets of dimension $<\alpha$ should be "small", i.e. have measure zero, and sets of dimension $>\alpha$ should be "large", i.e. have measure ∞. Lebesgue measure accomplishes this in $\mathbb{R}^n$, but only gives an integer value for dimension, and hence misses out on some structure.

Defn: Let $\alpha>0$ and $E \subseteq \mathbb{R}^n$. For any $\delta>0$ we define

$$h_\alpha^\delta(E) = \inf\left\{ \sum_{j=1}^{\infty} r_j^\alpha : E \subseteq \bigcup_{j=1}^{\infty} B_j \text{ with each } B_j \text{ a ball of radius } r_j \leq \delta \right\}.$$

Then
$$h_\alpha^*(E) = \sup_{\delta>0} h_\alpha^\delta(E) = \lim_{\delta\to 0^+} h_\alpha^\delta(E)$$

is the (outer) α-dimensional Hausdorff measure of E.

Note: 1. $h_\alpha^\delta(E)$ clearly increases as δ decreases.

2. $h_\alpha^*(E)$ can be (and usually is!) equal to 0 or ∞.

3. Covering with small balls needed to ensure basic additive properties of h_α^* and to provide accurate measure of irregular shapes.

Exercise 1: Let $h_\alpha^\infty(E) := \inf\left\{ \sum_{j=1}^{\infty} r_j^\alpha : E \subseteq \bigcup_{j=1}^{\infty} B_j \text{ with each } B_j \text{ a ball of radius } r_j \right\}$

Show that this defines a outer measure, but not necessarily a metric one.

Lemma 1: h_α^* defines a metric outer measure and hence

$$h_\alpha := h_\alpha^* \Big|_{\mathcal{B}_{\mathbb{R}^n}}$$ is a Borel measure on $\mathbb{R}^n$.

Lemma 2: Given any Borel set $E \subseteq \mathbb{R}^n$ and $\alpha \geq 0$;

(i) $h_\alpha(E+x) = h_\alpha(E)$ for all $x \in \mathbb{R}^n$

(ii) $h_\alpha(RE) = h_\alpha(E)$ for all rotations R in $\mathbb{R}^n$.

(iii) $h_\alpha(\lambda E) = \lambda^\alpha h_\alpha(E)$ for all $\lambda > 0$.

Exercise 2: (i) Show that for any $E \subseteq \mathbb{R}^n$, $h_0(E) =$ cardinality of E.

(ii) Show that h_n is a locally-finite regular Borel measure on $\mathbb{R}^n$ and hence that $\exists\, c > 0$ such that

$$h_n(E) = c\, m_n(E) \text{ for all Borel sets } E \subseteq \mathbb{R}^n$$

(where m_n denotes Lebesgue measure on $\mathbb{R}^n$).

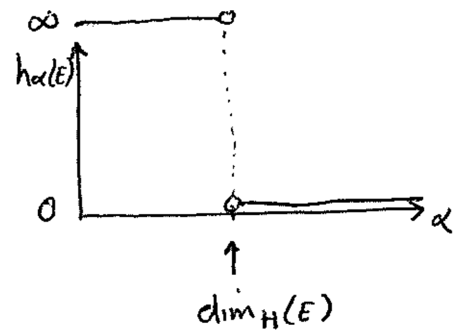
Since h_n is translation-invariant

Lemma 3: $h_n(E) = \frac{1}{\omega_n} m_n(E)$ for all Borel sets $E \subseteq \mathbb{R}^n$, where $\omega_n =$ Lebesgue measure of unit ball in $\mathbb{R}^n$.

(Note: Lemma 3 is obvious when $n=1$).

Lemma 4: For $0 \le \alpha < \beta < \infty$ and E a Borel subset of $\mathbb{R}^n$,

1. $h_\alpha(E) < \infty \Rightarrow h_\beta(E) = 0$
2. $h_\beta(E) > 0 \Rightarrow h_\alpha(E) = \infty$



Proof: Since

$$r_j^\beta = r_j^{\beta-\alpha} r_j^\alpha \le \delta^{\beta-\alpha} r_j^\alpha \quad \text{if } r_j \le \delta$$

$$\Rightarrow h_\beta^\delta(E) \le \delta^{\beta-\alpha} h_\alpha^\delta(E) \le \delta^{\beta-\alpha} h_\alpha(E) \quad \text{for all } \delta > 0$$

$$\Rightarrow h_\beta(E) = 0. \qquad \square.$$

Exercise 3: Show that if $U \subseteq \mathbb{R}^n$ is open and non-empty, then

$$h_\alpha(U) = \infty \quad \text{for all } \alpha < n.$$

Definition: The Hausdorff dimension of a Borel set $E \subseteq \mathbb{R}$ is:

$$\dim_H(E) := \sup\{\beta \ge 0 : h_\beta(E) = \infty\} = \sup\{\beta \ge 0 : h_\beta(E) > 0\}$$
$$= \inf\{\alpha \ge 0 : h_\alpha(E) = 0\} = \inf\{\alpha \ge 0 : h_\alpha(E) < \infty\}.$$

Note: Clearly, if $\alpha = \dim_H(E)$, then $0 \le h_\alpha(E) \le \infty$.
If $0 < h_\alpha(E) < \infty$, then we say that E has strict Hausdorff dimension α. (Sometimes we say that E is an α-set).

Proof of Lemma 1

- <u>Monotonicity</u>: (if $E_1 \subseteq E_2$, then $h_\alpha^*(E_1) \leq h_\alpha^*(E_2)$).

  Immediate, as any cover of E_2 is also a cover for E_1.

- <u>Subadditivity</u>: ($h_\alpha^*(\bigcup_{k=1}^\infty E_k) \leq \sum_{k=1}^\infty h_\alpha^*(E_k)$). Let $\alpha \geq 0$ & $\delta > 0$.

  For any $\varepsilon > 0$, $\exists$ cover $\{B_{k,j}\}$ of E_k with $r_j = \text{radius}(B_{k,j}) \leq \delta$ s.t.

  $$\sum_{j=1}^\infty r_j^\alpha \leq h_\alpha^\delta(E_k) + \varepsilon/2^k$$

  $$\Rightarrow \quad h_\alpha^\delta\left(\bigcup_{k=1}^\infty E_k\right) \leq \sum_{k=1}^\infty h_\alpha^\delta(E_k) + \varepsilon \leq \sum_{k=1}^\infty h_\alpha^*(E_k) + \varepsilon$$

  since $\{B_{k,j}\}_{j,k=1}^\infty$ covers $\bigcup_{k=1}^\infty E_k$.

  $$\Rightarrow \quad h_\alpha^\delta\left(\bigcup_{k\geq 1} E_k\right) \leq \sum_{k=1}^\infty h_\alpha^*(E_k) \quad \forall \delta > 0 \quad (\text{as } \varepsilon > 0 \text{ arb})$$

  $$\Rightarrow \quad h_\alpha^*\left(\bigcup_{k\geq 1} E_k\right) \leq \sum_{k\geq 1}^\infty h_\alpha^*(E_k).$$

- <u>Metric</u>: (if $d(E_1, E_2) > 0$, then $h_\alpha^*(E_1 \cup E_2) = h_\alpha^*(E_1) + h_\alpha^*(E_2)$).

  Suffices to show that $h_\alpha^*(E_1 \cup E_2) \geq h_\alpha^*(E_1) + h_\alpha^*(E_2)$.

  Fix $\varepsilon > 0$ with $\varepsilon < d(E_1, E_2)$. Given any cover of $E_1 \cup E_2$ by $\{B_j\}$ with $r_j \leq \delta$ <u>for some $\delta < \varepsilon$</u>, let

  $$B_j^1 = \{B_j : B_j \cap E_1 \neq \phi\} \quad \& \quad B_j^2 = \{B_j : B_j \cap E_2 \neq \phi\}.$$

  $\Rightarrow$ $\{B_j^1\}$ & $\{B_j^2\}$ cover E_1 & E_2 resp. are are <u>disjoint</u>.

  $$\Rightarrow \quad \sum {r_j^1}^\alpha + \sum {r_j^2}^\alpha \leq \sum r_j^\alpha$$

  Taking infimum over coverings and letting $\delta \to 0$ gives result. $\square$.

Proof of Lemma 2

Immediate from dilation, rotation and scaling properties of balls.

Exercise 6: Show that the Borel sets are closed under translations, dilations and rotations.

Proof of Lemma 3

Check!

- Suffices to show that $h_n \leq \frac{1}{\omega_n} m_n$ on $\mathcal{B}_{\mathbb{R}^n}$ as other inequality obvious.
- **Exercise 7**: Show that $h_n \ll m_n$.
- Since $h_n(K) < \infty$ $\forall$ compact $K \subseteq \mathbb{R}^n$ (Exercise 2 (ii)) and $\mathbb{R}^n$ is both locally compact & σ-compact we know that h_n is regular. Hence we need only verify that $h_n(E) \leq \frac{1}{\omega_n} m_n(E)$ $\forall$ open sets $E \subseteq \mathbb{R}^n$ with $m_n(E) < \infty$
- **Claim**: Let $E \subseteq \mathbb{R}^n$ be open with $m_n(E) < \infty$, $\exists$ $E_1 = \bigcup_{j \geq 1}^{\infty} B_j$ with the balls B_j pairwise disjoint with radius $r_j \leq \delta$ such that $E_1 \subseteq E$ & $m_n(E \setminus E_1) = 0$

It follows from this claim that $h_n^{\delta}(E_1) \leq \frac{1}{\omega_n} m_n(E)$. Repeating this process with $E = E_1$ we obtain $E_2 = \bigcup B_j$ (disjoint) with $r_j \leq \delta/2$ (now) such that

$$E_2 \subseteq E_1 \subseteq E \text{ and } m_n(E_1 \setminus E_2) = 0 \Rightarrow h_n^{\delta/2}(E_2) \leq \frac{1}{\omega_n} m_n(E_1).$$

$\vdots$

$\forall k$ $\exists$ $E_k = \bigcup B_j$ (disjoint) with $r_j \leq \delta/2^{k-1}$ s.t. $E_k \subseteq \cdots \subseteq E$ & $h_n^{\delta/2^{k-1}}(E_k) \leq \frac{1}{\omega_n} m_n(E)$

Let $V = \bigcap_{k=1}^{\infty} E_k$, then $V \subseteq E$ with $m_n(E \setminus V) = 0$ & $h_n^{\delta/2^k}(V) \leq \frac{1}{\omega_n} m_n(E)$ $\forall k$.

$$\Rightarrow h_n(E) = \underbrace{h_n(E \setminus V)}_{=0 \text{ by Ex 7}} + h_n(V) \leq \frac{1}{\omega_n} m_n(E). \qquad \square$$

- **Exercise 8**: Prove Claim!! [Write $E = \bigcup_{j \geq 1}^{\infty} B_j$'s with $r_j \leq \delta$ & apply (Vitali's) covering lemma