Strongly Singular Integrals on the Heisenberg Group

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Euclidean case: Let $x \in \mathbf{R}^d$, we consider

$$Sf(x) = f * K(x)$$

where K is a distribution on $\mathbf{R}^d$ that away from the origin agrees with the function

$$K(x) = |x|^{-d-\alpha} e^{i|x|^{-\beta}} \chi(|x|),$$

with $\beta > 0$ and χ supported in nbd of origin.

Question: What relationship between α and β will ensure S extends to a bounded operator on $L^2(\mathbf{R}^d)$?

Answer:

$$\|Sf\|_{L^2(\mathbf{R}^d)} \leq A\|f\|_{L^2(\mathbf{R}^d)} \Longleftrightarrow \alpha \leq \frac{d\beta}{2}$$

Remark:

The assumption that K be *radial* may be relaxed.

For example we may consider the operators

$$\widetilde{S}f = f * \widetilde{K}$$

with distributional kernel $\widetilde{K}$ that now agree away from $x = 0$ with the function

$$\widetilde{K}(x) = a(x)e^{i\varphi(x)},$$

where the amplitude and phase satisfy

(i) $|\partial^{\varrho} a(x)| \leq C|x|^{-d-\alpha-|\varrho|}$

(ii) $\varphi > 0$, non-deg and homogeneous of deg $-\beta$,

with $\alpha, \beta > 0$.

Theorem 0

$$\|\widetilde{S}f\|_{L^2(\mathbf{R}^d)} \leq A\|f\|_{L^2(\mathbf{R}^d)} \Longleftrightarrow \alpha \leq \frac{d\beta}{2}$$

- This can be proven via a T^*T argument.

<u>Sketch of proof in radial case:</u>

It is convenient here to use the <u>Fourier transform</u>

- By Plancherel

  $\|Sf\|_2 \leq A\|f\|_2 \Longleftrightarrow |m(\xi)| \leq A$ uniformly in ξ,

  where $m(\xi) = \widehat{K}(\xi)$.

- Since K is *radial* so is m and

  $m(\xi) = (2\pi)^{\frac{d}{2}} \int_0^\infty K_0(r) J_{\frac{d-2}{2}}(r|\xi|)(r|\xi|)^{\frac{2-d}{2}} r^{d-1} dr,$

  where $J_{\frac{d-2}{2}}$ is a Bessel function.

- Using the well known asymptotic properties of these functions it follows that

  $$m(\xi) \approx (1 + |\xi|)^{\frac{\alpha - d\beta/2}{\beta+1}} e^{ic_\beta |\xi|^{\frac{\beta}{\beta+1}}},$$

  where $c_\beta = \frac{\beta+1}{\beta}\beta^{\frac{\beta}{\beta+1}}$.

Heisenberg group: $\mathbf{H}^n = \mathbf{C}^n \times \mathbf{R}$

Group law:

$$(z,t)\cdot(w,s) = (z+w, t+s+\tfrac{1}{2}\operatorname{Im} z\cdot\bar{w}),$$

$$\mathrm{Id} = (0,0) \quad \text{and} \quad (z,t)^{-1} = (-z,-t)$$

Automorphisms:

- *Nonisotropic dilations*

$$(z,t) \mapsto \delta\circ(z,t) = (\delta z, \delta^2 t), \ \ \delta > 0$$

- *Rotations*

$$(z,t) \mapsto (Uz, t), \ \text{with } U \text{ a unitary trans of } \mathbf{C}^n$$

Norm:

$$\rho(z,t) = (|z|^4 + 16t^2)^{1/4}$$

Object 1

The *group convolution* operators

$$Tf(z,t) = f * M(z,t)$$

where M is a distribution on $\mathbf{H}^n$ that agrees for $(z,t) \neq (0,0)$ with the function

$$M(z,t) = \rho(z,t)^{-d_h-\alpha} e^{i\rho(z,t)^{-\beta}} \chi(\rho(z,t)),$$

with $\beta > 0$ and $d_h = 2n+2$.

Theorem 1 If $\alpha \leq n\beta$, then T extends to a bounded operator from $L^2(\mathbf{H}^n)$ to itself.

Remarks:

- There is a "gap of $\frac{1}{2}$" with the Euclidean condition $\alpha \leq \frac{2n+1}{2}\beta$ in all dimensions.
- If T is bounded on $L^2(\mathbf{H}^n)$ then *necessarily* $\alpha \leq (n+\frac{1}{2})\beta$.

Object 2

The *group convolution* operators

$$Rf(z,t) = f * L(z,t) \quad \text{where } L = K \otimes \delta_0,$$

and as before for the distribution K agrees away from $z = 0$ with the function

$$K(z) = |z|^{-2n-\alpha} e^{i|z|^{-\beta}} \chi(|z|).$$

Theorem 2

$$\|Rf\|_{L^2(\mathbf{H}^n)} \leq A \|f\|_{L^2(\mathbf{H}^n)} \Longleftrightarrow \alpha \leq (n - \tfrac{1}{6})\beta$$

Remark: R is of course intimately connected with the *twisted convolution* operators

$$R^\lambda f(z) = \int_{\mathbf{C}^n} f(z-w) K(w) e^{-i\lambda \frac{1}{2} \operatorname{Im} z \cdot \bar{w}} dw.$$

Theorem 3

$$\|R^\lambda f\|_{L^2(\mathbf{C}^n)} \leq A_\lambda \|f\|_{L^2(\mathbf{C}^n)} \Longleftrightarrow \alpha \leq n\beta.$$

<u>Proof of Theorem 3:</u> [Case $n = 1$]

- Suffices to show

$$\int_{|z|\le 1} |R^\lambda f(z)|^2 dz \le A_\lambda \int_{|z|\le 2} |f(z)|^2 dz$$

- For simplicity let $\lambda = 4$. Now if $z, w \in \mathbf{C}$ then

$$\left| e^{-i2\,\mathrm{Im}\, z\cdot\bar{w}} - \sum_{k=0}^{N-1} \frac{\left(\overline{(z-w)}w - (z-w)\overline{w}\right)^k}{k!} \right| \le C|w|^N |z-w|^N$$

- Matters reduces to estimating the operators

$$f \mapsto \int f(z-w)K(w)\bar{w}^\ell w^m dw$$

  for $\ell + m = k = 0, \dots, N$.

- It follows from Theorem 0 that these operators are bounded in $L^2(\mathbf{C})$ whenever $\alpha - k \le \beta$

<u>Remark:</u> As we only used Theorem 0 so we can in fact also take the more general kernel $\widetilde{K}$ here.

<u>Question:</u> Can one also relax the *radial* assumptions in Theorems 1 & 2?

Method of Proof of Theorem 1: [& Theorem 2]

Group Fourier transform (GFT)

- May realize T as a "multiplier" operator

$$\widehat{Tf}(\lambda) = \hat{f}(\lambda) \cdot \widehat{M}(\lambda),$$

where $\widehat{M}(\lambda)$ is the GFT of M.

Recall that, for each $\lambda \neq 0$, $\widehat{M}(\lambda)$ is an operator on the Hilbert space $L^2(\mathbf{R}^n)$.

- By Plancherel's theorem for the GFT

$$\|Tf\|_2 \leq A\|f\|_2 \Leftrightarrow \|\widehat{M}(\lambda)\|_{Op} \leq A \text{ unif in } \lambda \neq 0.$$

- As M is *radial* on $\mathbf{H}^n$, i.e $M(z,t) = M_0(|z|,t)$, then it is a result of Geller that

$$\widehat{M}(\lambda) = C_n\Big(\delta_{\mathbf{j},\mathbf{k}}\ \mu(|\mathbf{k}|,\lambda)\Big)_{\mathbf{j},\mathbf{k}\in\mathbf{N}^n}$$

where $\mu(k,\lambda)$ are Laguerre transforms.

- It therefore follows that

$$\|Tf\|_2 \leq A\|f\|_2 \Leftrightarrow |\mu(k,\lambda)| \lesssim A \text{ unif in } k \ \&\ \lambda \neq 0.$$

Laguerre transform estimates:

Matters reduce to the study of the following;

$$\boxed{\mu(k,\lambda) = c_k^{n-1}\int_0^\infty M_0^\lambda(r)\Lambda_k^{n-1}\left(\tfrac{|\lambda|r^2}{2}\right)\left(\tfrac{|\lambda|r^2}{2}\right)^{\frac{1-n}{2}} r^{2n-1}\,dr}$$

where

$$c_k^\delta = \left(\tfrac{k!}{(k+\delta)!}\right)^{1/2}$$

$$M^\lambda(z) = \int_{\mathbf{R}} M(z,t)e^{i\lambda t}\,dt$$

and

$$\Lambda_k^\delta(x) = c_k^\delta L_k^\delta(x)e^{-\frac{1}{2}x}x^{\frac{\delta}{2}},$$

are the Laguerre functions of type δ, $\delta > -1$. These form an orthonormal basis for $L^2(\mathbf{R}^+)$.

Theorem 1'

$$|\mu_1(k,\lambda)| \leq C(1+|\lambda|k)^{\frac{\alpha-n\beta}{2(\beta+1)}}$$

Theorem 2'

$$\mu_2(k,\lambda) \sim (1+|\lambda|k)^{\frac{\alpha-(n-\frac{1}{6})\beta}{2(\beta+1)}} e^{ic(|\lambda|k)^{\frac{\beta}{2(\beta+1)}}}$$

Asymptotics of Laguerre functions: [Erdélyi]

In what follows $\nu = 4k + 2\delta + 2$ and $k \geq k_0$.

Bessel asymptotic form: If $0 \leq x \leq b\nu$, $b < 1$

$$\Lambda_k^\delta(x) = B_\delta \left(\frac{\nu}{x}\right)^{\frac{1}{2}} \left(\frac{\psi}{\psi'}\right)^{\frac{1}{2}} \{J_\delta(\nu\psi) + O[\nu^{-1}(\tfrac{x}{\nu - x})^{\frac{1}{2}} \widetilde{J}_\delta(\nu\psi)]\}$$

Airy asymptotic form: If $0 < a\nu \leq x$, $a > 0$

$$\Lambda_k^\delta(x) = A_\delta \left(\frac{\nu^{\frac{1}{3}}}{x}\right)^{\frac{1}{2}} \left(\frac{1}{-\phi'}\right)^{\frac{1}{2}} \{Ai(-\nu^{\frac{2}{3}}\phi) + O[x^{-1} \widetilde{Ai}(-\nu^{\frac{2}{3}}\phi)]\}$$

where $\psi = \psi(\frac{x}{\nu})$ and $\phi = \phi(\frac{x}{\nu})$ satisfy

$$\psi'(t) = [\phi(t)]^{\frac{1}{2}} \phi'(t) = \tfrac{1}{2}(\tfrac{1}{t} - 1)^{\frac{1}{2}}$$

Trivial Estimates: ($\gamma_1, \gamma_2 > 0$ are fixed constants)

$$|\Lambda_k^\delta(x)| \leq C \begin{cases} (x\nu)^{\frac{\delta}{2}} & \text{if } 0 \leq x \leq \frac{1}{\nu}, \\ (x\nu)^{-\frac{1}{4}} & \text{if } \frac{1}{\nu} \leq x \leq \frac{\nu}{2}, \\ \nu^{-\frac{1}{4}}(\nu - x)^{-\frac{1}{4}} & \text{if } \frac{\nu}{2} \leq x \leq \nu - \nu^{\frac{1}{3}}, \\ \nu^{-\frac{1}{3}} & \text{if } \nu - \nu^{\frac{1}{3}} \leq x \leq \nu + \nu^{\frac{1}{3}}, \\ \nu^{-\frac{1}{4}}(x - \nu)^{-\frac{1}{4}} e^{-\gamma_1 \nu^{-\frac{1}{2}}(x-\nu)^{\frac{3}{2}}} & \text{if } \nu + \nu^{\frac{1}{3}} \leq x \leq \frac{3\nu}{2}, \\ e^{-\gamma_2 x} & \text{if } x \geq \frac{3\nu}{2}. \end{cases}$$