Math 3100 Assignment 3
Convergence of Sequences

Due at 5:00 pm on Friday the 1st of February 2019

1. What happens if we interchange or reverse the order of the quantifiers in the definition of convergence of a sequence?

   (a) *Definition:* A sequence $\{a_n\}$ *verconges* to a if *there exists* an $\varepsilon > 0$ such that *for all* $N \in \mathbb{N}$ it is true that $n > N$ implies $|a_n - a| < \varepsilon$.
   Give an example of a vercongent sequence. Can you give an example a vercongent sequence that is divergent? What exactly is being described in this strange definition?

   (b) *Definition:* A sequence $\{a_n\}$ *conconges* to a if *there exists* a number N such that $n > N$ implies $|a_n - a| < \varepsilon$ *for all* $\varepsilon > 0$.
   Give an example of a concongent sequence. Can you give an example a concongent sequence that is divergent? What exactly is being described in this strange definition?

2. Verify the following using the definition of convergence of a sequence:

   (a) If $a_n \to a$, then $|a_n| \to |a|$. Is the converse true?

   (b) Let $a_n \geq 0$ for all $n \in \mathbb{N}$.
      i. Show that if $a_n \to 0$, then $\sqrt{a_n} \to 0$.
      ii. Show that if $a_n \to a$, then $\sqrt{a_n} \to \sqrt{a}$.

   (c) If $\lim\limits_{n\to\infty} x_n = 3$, then $\lim\limits_{n\to\infty} \dfrac{1}{x_n} = \dfrac{1}{3}$.
   Hint: First argue that there exists a number N such that if $n > N$, then $x_n \geq 2$.

   (d) If $\{a_n\}$ is bounded (but not necessarily convergent) and $\lim\limits_{n\to\infty} b_n = 0$, then $\lim\limits_{n\to\infty} a_n b_n = 0$.

3. Let $\{a_n\}$ be a convergent sequence with $\lim\limits_{n\to\infty} a_n = a$. Prove the following two statements:

   (a) If $a_n \leq b$ for all $n \in \mathbb{N}$, then $a \leq b$.

   (b) If $\{a_n\}$ is increasing, then $a_n \leq a$ for all $n \in \mathbb{N}$.

4. We say that $\{a_n\}$ *diverges to infinity*, and write $\lim\limits_{n\to\infty} a_n = \infty$, if for every $M > 0$ there exists a number N such that $n > N$ implies that $a_n > M$.

   (a) Prove, using the definition above, that $\lim\limits_{n\to\infty} n^p = \infty$ for all $p > 0$.

   (b) Prove that if $a_n > 0$ for all $n \in \mathbb{N}$, then $\lim\limits_{n\to\infty} a_n = \infty$ if and only if $\lim\limits_{n\to\infty} a_n^{-1} = 0$.

   (c) Prove that if $\lim\limits_{n\to\infty} a_n = \infty$ and $\lim\limits_{n\to\infty} b_n = 2$, then $\lim\limits_{n\to\infty} (a_n b_n) = \infty$.

5. Let $x_1 = 3$ and $x_{n+1} = \dfrac{1}{4 - x_n}$ for all $n \in \mathbb{N}$.

   (a) Show that $\{x_n\}$ is decreasing and satisfies $2 - \sqrt{3} \leq x_n \leq 3$ for all $n \in \mathbb{N}$.

   (b) Conclude that **if** the sequence $\{x_n\}$ converges, then it must converge to $2 - \sqrt{3}$.

   We shall soon establish in class, using the "completeness of the real numbers" (the defining property that distinguishes the reals from the rationals), that bounded monotone sequences of real numbers always converge.

Math 3100 – Homework 3 – SOLUTIONS

1. (a) $\{a_n\}$ vercongs to something $\Leftrightarrow$ $\{a_n\}$ bounded

   (Since $\{a_n\}$ verconge to a $\Leftrightarrow$ $\exists\, \varepsilon>0$ such that $a-\varepsilon<a_n<a+\varepsilon \;\; \forall n\in\mathbb{N}$),

   (b) $\{a_n\}$ concongs to a $\Leftrightarrow$ $\{a_n\}$ is eventually constantly equal to a.

   (Since $|a_n-a|<\varepsilon \;\; \forall \varepsilon>0 \Leftrightarrow a_n=a$)

2. (a) <u>Claim</u>: If $a_n\to a$, then $|a_n|\to|a|$.

   <u>Proof</u> Let $\varepsilon>0$. Since $a_n\to a$ we know $\exists\, N$ such that $n>N$ implies $|a_n-a|<\varepsilon$. Since $\big||a_n|-|a|\big|\le|a_n-a| \;\; \forall n\in\mathbb{N}$ it follows that if $n>N$, then $\big||a_n|-|a|\big|<\varepsilon$. $\square$

   * Converse is <u>FALSE</u> <u>Ex</u>: $a_n=(-1)^n$.

   (b) Let $a_n\ge 0 \;\; \forall n\in\mathbb{N}$.

   (i) <u>Claim</u>: If $a_n\to 0$, then $\sqrt{a_n}\to 0$ also

   <u>Proof</u> Let $\varepsilon>0$. Since $a_n\to 0$ we know $\exists\, N$ such that if $n>N$, then $|a_n-0|= a_n<\varepsilon^2$ (since $\varepsilon^2>0$).

   and hence that $|\sqrt{a_n}-0|=\sqrt{a_n}<\sqrt{\varepsilon^2}=\varepsilon$ $\square$

   (ii) <u>Claim</u>: If $a_n\to a$, then $\sqrt{a_n}\to\sqrt{a}$.

   (*) and we may assume $a>0$

   <u>Proof</u>: Note that it follows from the "Order Limit law" that $a\ge 0$

   Let $\varepsilon>0$. Since $a_n\to a$ we know $\exists\, N$ such that if $n>N$, then $|a_n-a|<\sqrt{a}\,\varepsilon$ (since $\sqrt{a}\,\varepsilon>0$)

   Since $|\sqrt{a_n}-\sqrt{a}|=\dfrac{|a_n-a|}{\sqrt{a_n}+\sqrt{a}}\le\dfrac{|a_n-a|}{\sqrt{a}}$ it follows that

   multiplying by conjugate

   if $n>N$, then $|\sqrt{a_n}-\sqrt{a}|\le\dfrac{|a_n-a|}{\sqrt{a}}<\dfrac{1}{\sqrt{a}}(\sqrt{a}\,\varepsilon)=\varepsilon$. $\square$

(c). <u>Claim</u> If $\lim_{n\to\infty} x_n = 3$, then $\lim_{n\to\infty} \frac{1}{x_n} = \frac{1}{3}$.

<u>Proof</u> First we note that $\exists N_1$ such that

$$n > N_1 \text{ implies } |x_n - 3| < 1 \text{ \& hence that } x_n > 2.$$

(this is the definition of $x_n \to 3$ with $\varepsilon = 1$)

Let $\varepsilon > 0$. Since $x_n \to 3$ we know $\exists N_2$ such that

$$n > N_2 \text{ implies } |x_n - 3| < 6\varepsilon$$

and hence that if $n > \max\{N_1, N_2\}$, then

$$\left|\frac{1}{x_n} - \frac{1}{3}\right| = \frac{|x_n - 3|}{3|x_n|} \underset{\substack{\uparrow \\ \text{Since } n>N_1 \Rightarrow |x_n|>2}}{\leq} \frac{|x_n - 3|}{6} \overset{\text{since } n>N_2 \Rightarrow |x_n-3|<6\varepsilon}{<} \frac{1}{6}(6\varepsilon) = \varepsilon. \qquad \square$$

(d) <u>Claim</u> If $\{a_n\}$ bounded & $b_n \to 0$, then $(a_n b_n) \to 0$ also.

<u>Proof</u> Since $\{a_n\}$ bounded we know $\exists M > 0$ so that $|a_n| \leq M \ \forall n \in \mathbb{N}$.

Let $\varepsilon > 0$. Since $b_n \to 0$ we know $\exists N$ so that if $n > N$, then

$$|b_n - 0| = |b_n| < \varepsilon/M \quad (\text{since } \varepsilon/M > 0).$$

It follows that if $n > N$, then

$$|a_n b_n - 0| = |a_n b_n| = |a_n||b_n| \leq M|b_n| < M\left(\tfrac{\varepsilon}{M}\right) = \varepsilon. \qquad \square$$

3. (a) <u>Claim</u>: If $a_n \to a$ and $a_n \leq b \ \forall n \in \mathbb{N}$, then $a \leq b$.

<u>Proof</u> Suppose $a > b \Rightarrow a - b > 0$.

Since $a_n \to a$ we know $\exists N$ such that $n > N$ implies that

$$|a_n - a| < a - b \quad (\text{since } a - b > 0)$$

$$\Updownarrow$$

$$\underbrace{b - a < a_n - a}_{} < a - b$$

Adding a to the left most inequality reveals that

$$b < a_n \quad \forall n > N \quad \lightning$$

This contradicts the fact that $a_n \leq b$ for all $n \in \mathbb{N}$. $\square$

(b) Claim: If $\{a_n\}$ increasing and $a_n \to a$, then $a_n \le a \ \forall n \in \mathbb{N}$

Proof Suppose $\exists N_1$ such that $a_{N_1} > a \iff a_{N_1} - a > 0$.

Since $a_n \to a$ we know $\exists N_2$ such that $n > N_2$ implies that

$$|a_n - a| < a_{N_1} - a \quad (\text{since } a_{N_1} - a > 0)$$

& hence that $a_n < a_{N_1}$ $\lightning$ if $n > N_2$.

This contradicts the fact that $\{a_n\}$ is increasing if $n > \max\{N_1, N_2\}$.

(¿ Is this clear ?)

4. (a) Claim If $p > 0$, then $\lim_{n\to\infty} n^p = \infty$.

Proof Let $M > 0$ and set $N = M^{1/p}$.

If $n > N$, then it follows that $n > M^{1/p}$ and hence $n^p > M$. $\square$

(b) Claim If $a_n > 0 \ \forall n \in \mathbb{N}$, then $\lim_{n\to\infty} a_n = \infty \iff \lim_{n\to\infty} \frac{1}{a_n} = 0$.

Proof

($\Rightarrow$) Let $\varepsilon > 0$. Since $a_n \to \infty$ we know $\exists N$ such that $n > N$ implies $a_n > \varepsilon^{-1}$ (since we can take $M = \varepsilon^{-1} > 0$)

which in turn implies $|a_n^{-1} - 0| = \frac{1}{a_n} < \varepsilon$.

($\Leftarrow$) Let $M > 0$. Since $a_n^{-1} \to 0$ we know $\exists N$ such that $n > N$ implies $|a_n^{-1} - 0| = \frac{1}{a_n} < \frac{1}{M}$ (since can take $\varepsilon = M^{-1} > 0$)

which in turn implies that $a_n > M$. $\square$

(c) Claim If $\lim_{n\to\infty} a_n = \infty$ & $\lim_{n\to\infty} b_n = 2$, then $\lim_{n\to\infty} (a_n b_n) = \infty$

Proof let $M > 0$.

Since $b_n \to 2$ $\exists N_1$ such that $b_n \ge 1 \ \forall n \ge N_1$

Since $a_n \to \infty$ $\exists N_2$ such that $a_n > M \ \forall n \ge N_2$

Thus, if $n > \max\{N_1, N_2\} \Rightarrow a_n b_n \ge a_n > M$ $\square$

since $n > N_2$.

$n > N_1$

5. (a) Let $x_1 = 3$ & $x_{n+1} = \frac{1}{4-x_n}$ $\forall n \in \mathbb{N}$.

<u>Claim 1</u> $2-\sqrt{3} \le x_n \le 3$ for all $n \in \mathbb{N}$

<u>Proof</u> (Induction)

Base case ($n=1$): $x_1 = 3$ ✓

Suppose $2-\sqrt{3} \le x_n \le 3$ for <u>some</u> given $n \in \mathbb{N}$, it follows that

- $x_{n+1} = \frac{1}{4-x_n} \le 1 \le 3$ (since $x_n \le 3 \Rightarrow 4-x_n \ge 1 \Rightarrow \frac{1}{4-x_n} \le 1$)
- $x_{n+1} = \frac{1}{4-x_n} \ge 2-\sqrt{3}$ (since $x_n \ge 2-\sqrt{3} \Rightarrow 4-x_n \le 2+\sqrt{3}$
  $\Rightarrow \frac{1}{4-x_n} \ge \frac{1}{2+\sqrt{3}} = \frac{2-\sqrt{3}}{(2+\sqrt{3})(2-\sqrt{3})} = 2-\sqrt{3}$)

□

<u>Claim 2</u> $\{x_n\}$ is decreasing

<u>Proof</u>: $x_{n+1} - x_n = \frac{1}{4-x_n} - x_n = \frac{1-4x_n+x_n^2}{4-x_n} = \frac{(x_n-(2+\sqrt{3}))(x_n-(2-\sqrt{3}))}{4-x_n}$

≤ 0 by Claim 1 □

(b) <u>Claim</u>: <u>If</u> $\{x_n\}$ converges, then $\lim_{n\to\infty} x_n = 2-\sqrt{3}$.

<u>Proof</u> Suppose $\lim_{n\to\infty} x_n = L$. It follows from Claim 1 & the "Order limit law" that $2-\sqrt{3} \le L \le 3$.

- Since $x_n \to L$ it follows that $\lim_{n\to\infty} x_{n+1} = L$ also.
- Since $x_n \to L$ & $L \ne 4$ it follow by limit laws that
  $$\lim_{n\to\infty} \frac{1}{4-x_n} = \frac{1}{4-L}$$

Since $x_{n+1} = \frac{1}{4-x_n}$ $\forall n \in \mathbb{N}$ <u>and</u> limits are unique it follows that

that $L = \frac{1}{4-L} \Rightarrow L^2 - 4L + 1 = 0$

$\Rightarrow L = 2-\sqrt{3}$ or $\cancel{2+\sqrt{3}}$.

since $L \le 3$. □