Math 3100 Assignment 7
Power Series and Continuity

Due at 1:00 pm on Friday the 8th of March 2019

1. Find a power series representation for the function and determine the interval of convergence.

   (a) $f(x) = \dfrac{1}{1+x}$ (b) $g(x) = \dfrac{1}{1-4x^2}$ (c) $h(x) = \dfrac{1}{4+x^2}$ (d) $F(x) = \dfrac{x}{x-3}$

2. Find all $x \in \mathbb{R}$ for which the following power series converge:

   $$\text{(a)} \sum_{n=0}^{\infty} n^3 x^n \quad \text{(b)} \sum_{n=1}^{\infty} \frac{2^n}{n!} x^n \quad \text{(c)} \sum_{n=1}^{\infty} \frac{2^n}{n^2} x^n \quad \text{(d)} \sum_{n=1}^{\infty} \frac{n^3}{3^n} x^n \quad \text{(e)} \sum_{n=1}^{\infty} \frac{(x-1)^n}{\sqrt{n}}$$

3. Find the *radius of convergence* and *interval of convergence* of the power series.

   $$\text{(a)} \sum_{n=0}^{\infty} \frac{x^n}{n+3} \qquad \text{(b)} \sum_{n=1}^{\infty} \frac{(-1)^n x^n}{n 2^n} \qquad \text{(c)} \sum_{n=0}^{\infty} \frac{3^n x^n}{(n+1)^2} \qquad \text{(d)} \sum_{n=1}^{\infty} (-1)^n \frac{(x+2)^n}{\sqrt{n}}$$

4. Prove that each of the following functions are continuous at x_0 using the ε-δ definition of continuity.

   (a) $f(x) = 3x^2$, $x_0 = 2$

   (b) $g(x) = \dfrac{2x-3}{x-1}$, $x_0 = 2$

   (c) $h(x) = \dfrac{x^2 - x + 3}{x+1}$, $x_0 = 1$

   (d) $F(x) = x^3$, x_0 arbitrary

   (e) $G(x) = \dfrac{1}{x^2}$, $x_0 \neq 0$ arbitrary

5. Define a *modified Dirichlet's function* $h : \mathbb{R} \to \mathbb{R}$, by

   $$h(x) := \begin{cases} x & \text{if } x \in \mathbb{Q} \\ 0 & \text{if } x \notin \mathbb{Q} \end{cases}.$$

   Prove that h is continuous at $x = 0$, but discontinuous at all $x \neq 0$.

Math 3100 – Homework 7 – SOLUTIONS

1. (a) $\frac{1}{1+x} = \sum_{n=0}^{\infty} (-1)^n x^n$ if $|x|<1 \Leftrightarrow x \in (-1,1)$.

(b) $\frac{1}{1-4x^2} = \sum_{n=0}^{\infty} (4x^2)^n = \sum_{n=0}^{\infty} 4^n x^{2n}$ if $|4x^2|<1 \Leftrightarrow x \in (-\frac{1}{2}, \frac{1}{2})$.

(c) $\frac{1}{4+x^2} = \frac{1}{4}\frac{1}{1+\frac{x^2}{4}} = \frac{1}{4}\sum_{n=0}^{\infty}(-1)^n\left(\frac{x^2}{4}\right)^n$ if $\left|\frac{x^2}{4}\right|<1 \Leftrightarrow x \in (-2,2)$

$= \sum_{n=0}^{\infty} (-1)^n \frac{1}{4^{n+1}} x^{2n}$ if $x \in (-2,2)$.

(d) $\frac{x}{x-3} = -\frac{x}{3}\frac{1}{1-x/3} = -\frac{x}{3}\sum_{n=0}^{\infty}\left(\frac{x}{3}\right)^n$ if $\left|\frac{x}{3}\right|<1 \Leftrightarrow x \in (-3,3)$

$= \sum_{n=0}^{\infty} \frac{-1}{3^{n+1}} x^{n+1}$ if $x \in (-3,3)$.

2. (a) $\sum_{n=0}^{\infty} n^3 x^n$ converges $\Leftrightarrow |x|<1$.

<u>Justification</u>: Let $a_n = n^3 x^n$.

Since $\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{(n+1)^3 x^{n+1}}{n^3 x^n}\right| = \left(\frac{n+1}{n}\right)^3 |x| \longrightarrow |x|$

It follows from the <u>Ratio Test</u> that $\sum_{n=0}^{\infty} a_n$ conv. abs.

if $|x|<1$ and diverges if $|x|>1$.

Since $n^3 \not\to 0$ and $(-1)^n n^3 \not\to 0$ we can also

conclude that $\sum_{n=0}^{\infty} a_n$ diverges at $x=1$ & $x=-1$.

(b) $\sum_{n=1}^{\infty} \frac{2^n}{n!} x^n$ converges $\Leftrightarrow$ $x \in \mathbb{R}$.

Justification: Let $a_n = \frac{2^n}{n!} x^n$.

Since $\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{2^{n+1} x^{n+1}}{(n+1)!} \cdot \frac{n!}{2^n x^n}\right| = \frac{2}{n+1} |x| \to 0$

it follows from the Ratio Test that $\sum_{n=1}^{\infty} a_n$ conv. abs for all $x \in \mathbb{R}$.

(c) $\sum_{n=1}^{\infty} \frac{2^n}{n^2} x^n$ converges $\Leftrightarrow$ $|x| \leq \frac{1}{2}$.

Justification: Let $a_n = \frac{2^n}{n^2} x^n$

Since $\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{2^{n+1} x^{n+1}}{(n+1)^2} \cdot \frac{n^2}{2^n x^n}\right| = 2\left(\frac{n}{n+1}\right)^2 |x| \to 2|x|$

it follows from the Ratio Test that $\sum_{n=1}^{\infty} a_n$ conv. abs. if $|x| < \frac{1}{2}$

and diverges if $|x| > \frac{1}{2}$.

If $x = \frac{1}{2}$, then $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{1}{n^2}$ which is abs. conv.

If $x = -\frac{1}{2}$, then $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$ which is also abs. conv.

(d) $\sum_{n=1}^{\infty} \frac{n^3}{3^n} x^n$ converges $\Leftrightarrow$

Justification: Let $a_n = \frac{n^3}{3^n} x^n$

Since $\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{(n+1)^3 x^{n+1}}{3^{n+1}} \cdot \frac{3^n}{n^3 x^n}\right| = \frac{1}{3}\left(\frac{n+1}{n}\right)^3 |x| \to \frac{1}{3}|x|$

it follows from the Ratio Test that $\sum_{n=1}^{\infty} a_n$ conv. abs. if $|x| < 3$

& diverges if $|x| > 3$.

At $x=3$ we have $\sum_{n=1}^{\infty} n^3$ which diverges $(n^3 \not\to 0)$

and at $x=-3$ we have $\sum_{n=1}^{\infty} (-1)^n n^3$ which diverges $((-1)^n n^3 \not\to 0)$.

(e) $\sum_{n=1}^{\infty} \frac{(x-1)^n}{\sqrt{n}}$ converges $\Leftrightarrow -1 \le x-1 < 1 \Leftrightarrow 0 \le x < 2$

Justification: Let $a_n = \frac{(x-1)^n}{\sqrt{n}}$.

Since $\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{(x-1)^{n+1}}{\sqrt{n+1}} \cdot \frac{\sqrt{n}}{(x-1)^n}\right| = \sqrt{\frac{n}{n+1}}\,|x-1| \longrightarrow |x-1|$

it follows from the Ratio Test that $\sum_{n=1}^{\infty} a_n$ conv. abs. if $|x-1|<1$

and diverges if $|x-1|>1$.

If $x-1=1$, then we have $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ which is divergent and

if $x-1=-1$, then we have $\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}$ which is convergent by

the Alternating Series Test (since $\frac{1}{\sqrt{n}} \searrow 0$).

3. (a) $\sum_{n=0}^{\infty} \frac{x^n}{n+3}$ has Radius of Convergence 1 and converges for all $x \in [-1, 1)$

interval of convergence

Justification: Let $a_n = \frac{x^n}{n+3}$.

Since $\left|\frac{a_{n+1}}{a_n}\right| = \frac{n+3}{n+4}|x| \longrightarrow |x|$ it follows from the Ratio Test

that $\sum_{n=0}^{\infty} a_n$ conv. abs. if $|x|<1$ & diverges if $|x|>1$.

If $x=1$ we have $\sum_{n=0}^{\infty} \frac{1}{n+3}$ which is divergent & if $x=-1$

we have $\sum_{n=0}^{\infty} \frac{(-1)^n}{n+3}$ which converges by Alt. Series Test (since $\frac{1}{n+3} \searrow 0$).

(b) $\sum_{n=1}^{\infty} \frac{(-1)^n x^n}{n 2^n}$ has Radius of Convergence 2 and converges for all $x \in (-2, 2]$

interval of convergence

Justification: Let $a_n = \frac{(-1)^n x^n}{n 2^n}$.

Since $\left|\frac{a_{n+1}}{a_n}\right| = \frac{1}{2} \frac{n}{n+1} |x| \to \frac{1}{2}|x|$ it follows from the Ratio Test that $\sum_{n=1}^{\infty} a_n$ conv. abs. if $|x|<2$ & diverges if $|x|>2$.

If $x=2$ we have $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ which converges by Alt. Series Test since $\frac{1}{n} \searrow 0$. If $x=-2$ we have $\sum_{n=1}^{\infty} \frac{1}{n}$ which diverges.

(c) $\sum_{n=0}^{\infty} \frac{3^n x^n}{(n+1)^2}$ has Radius of Convergence $\frac{1}{3}$ and converges for all $x \in [-\frac{1}{3}, \frac{1}{3}]$

interval of convergence

Justification: Let $a_n = \frac{3^n x^n}{(n+1)^2}$.

Since $\left|\frac{a_{n+1}}{a_n}\right| = 3\left(\frac{n+1}{n+2}\right)^2 |x| \to 3|x|$ it follows from the Ratio Test that $\sum_{n=0}^{\infty} a_n$ conv. abs. if $|x|<\frac{1}{3}$ & diverges if $|x|>\frac{1}{3}$.

If $x=\frac{1}{3}$ we have $\sum_{n=0}^{\infty} \frac{1}{(n+1)^2}$ which converges and if $x=-\frac{1}{3}$ we have $\sum_{n=0}^{\infty} \frac{(-1)^n}{(n+1)^2}$ which conv. abs. (since $\sum_{n=0}^{\infty} \frac{1}{(n+1)^2}$ conv.).

(d) $\sum_{n=1}^{\infty}(-1)^n \frac{(x+2)^n}{\sqrt{n}}$ has Radius of Convergence 1 and converges for all $x \in (-3,-1]$.

interval of convergence

Justification: Let $a_n = \frac{(-1)^n (x+2)^n}{\sqrt{n}}$.

Since $\left|\frac{a_{n+1}}{a_n}\right| = \sqrt{\frac{n}{n+1}}\,|x+2| \longrightarrow |x+2|$ it follow from the Ratio Test that $\sum_{n=1}^{\infty} a_n$ conv. abs. if $|x+2|<1$ & diverges if $|x+2|>1$

If $x+2=1$ (i.e. if $x=-1$) then we have $\sum_{n=1}^{\infty}\frac{(-1)^n}{\sqrt{n}}$ which converges by the Alt. Series Test since $\frac{1}{\sqrt{n}} \searrow 0$.

If $x+2=-1$ (i.e. if $x=-3$) then we have $\sum_{n=1}^{\infty}\frac{1}{\sqrt{n}}$ which we know is divergent.

4.

(a) Claim: $f(x)=3x^2$ is continuous at $x_0=2$.

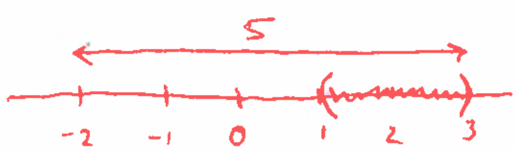
Proof Let $\varepsilon>0$ and set $\delta=\min\{1, \varepsilon/15\}$.

If $|x-2|<\delta$, then it follows that

$$|3x^2-3(2)^2| = 3|x+2||x-2| < 3(5)|x-2| < 15\left(\frac{\varepsilon}{15}\right) = \varepsilon. \qquad \square$$

Since $|x-2|<1$ we know that $|x+2|<5$

since $|x-2|<\varepsilon/15$.



(b) <u>Claim</u> $g(x) = \dfrac{2x-3}{x-1}$ is continuous at $x_0 = 2$.

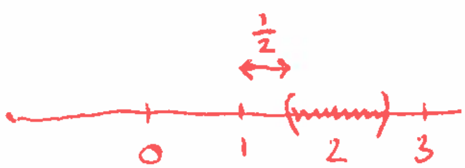
<u>Proof</u> Let $\varepsilon > 0$ and set $\delta = \min\{\frac{1}{2}, \varepsilon/2\}$.

If $|x-2| < \delta$ it follows that

$$\left| \frac{2x-3}{x-1} - \underbrace{1}_{g(2)} \right| = \frac{|x-2|}{|x-1|} < \frac{|x-2|}{(1/2)} = 2|x-2| < 2\left(\frac{\varepsilon}{2}\right) = \varepsilon \qquad \square$$

Since $|x-2| < \frac{1}{2}$ implies that $|x-1| > \frac{1}{2}$

Since $|x-2| < \varepsilon/2$



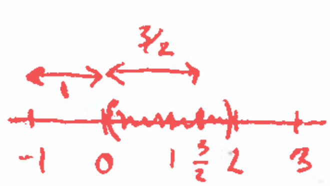
(c) <u>Claim</u> $h(x) = \dfrac{x^2 - x + 3}{x+1}$ is continuous at $x_0 = 1$

<u>Proof</u> Let $\varepsilon > 0$ and set $\delta = \min\{1, \frac{2\varepsilon}{3}\}$.

If $|x-1| < \delta$ it follows that

$$\left| \frac{x^2 - x + 3}{x+1} - \underbrace{\frac{3}{2}}_{h(1)} \right| = \frac{|2x-3||x-1|}{2|x+1|} = \frac{|x - \frac{3}{2}|}{|x+1|} |x-1|$$

$$< \frac{(3/2)}{(1)} |x-1| < \left(\frac{3}{2}\right)\left(\frac{2}{3}\varepsilon\right) = \varepsilon \qquad \square$$



Since $|x-1| < 1$ implies
(i) $|x - 3/2| < 3/2$
(ii) $|x+1| > 1$

Since $|x-1| < \frac{2\varepsilon}{3}$.

(d) <u>Claim</u> $F(x) = x^3$ is continuous at all $x_0 \in \mathbb{R}$.

<u>Proof</u> Let $\varepsilon > 0$ and set $\delta = \min\{1, \varepsilon/3(1+|x_0|)^2\}$.

If $|x - x_0| < 1$ it follows that

$$|x^3 - x_0^3| = |x^2 + xx_0 + x_0^2|\,|x - x_0|$$

$$\leq \left(|x|^2 + |x||x_0| + |x_0|^2\right) |x - x_0|$$

Since $|x-x_0|<1$ implies.
$|x| = |x - x_0 + x_0|$
$\leq |x-x_0| + |x_0|$
$< 1 + |x_0|$.

$$\leq \left((1+|x_0|)^2 + (1+|x_0|)^2 + (1+|x_0|)^2\right)|x - x_0|$$

$$< 3(1+|x_0|)^2 \left(\frac{\varepsilon}{3(1+|x_0|)^2}\right) = \varepsilon$$

since $|x - x_0| < \varepsilon/3(1+|x_0|)^2$.

$\square$

(e) <u>Claim</u> $G(x) = \frac{1}{x^2}$ is continuous at all $x_0 \neq 0$.

<u>Proof</u> Let $\varepsilon > 0$ and set $\delta = \min\left\{\frac{|x_0|}{2}, \frac{\varepsilon |x_0|^3}{10}\right\}$.

If $|x - x_0| < \delta$, it follows that

$$\left|\frac{1}{x^2} - \frac{1}{x_0^2}\right| = \frac{|x + x_0|\,|x - x_0|}{|x|^2 |x_0|^2} < \frac{\left(\frac{5}{2}|x_0|\right)}{\left(\frac{1}{2}|x_0|\right)^2 |x_0|^2} |x - x_0|$$

Since $|x-x_0| < \frac{1}{2}|x_0|$
implies

(i) $|x+x_0| = |x - x_0 + 2x_0|$
$\leq |x-x_0| + 2|x_0|$
$< \frac{5}{2}|x_0|$

$$< \frac{10}{|x_0|^3}\left(\frac{\varepsilon |x_0|^3}{10}\right) = \varepsilon$$

since $|x - x_0| < \frac{\varepsilon |x_0|^3}{10}$.

$\square$

"Reverse Δ-ineq."

& (ii) $|x| = |x_0 + x - x_0| \geq |x_0| - |x - x_0| > \frac{1}{2}|x_0|$.

5. Let $h(x) = \begin{cases} x & \text{if } x \in \mathbb{Q} \\ 0 & \text{if } x \notin \mathbb{Q}. \end{cases}$

Claim: h is continuous at $x_0 = 0$, but discontinuous at all $x_0 \neq 0$

Proof

- Let $x_0 \neq 0$: Since the rational and the irrationals are both dense in $\mathbb{R}$ we know there exists

  (i) A sequence $\{x_n\}$ with $x_n \in \mathbb{Q}$ $\forall n \in \mathbb{N}$ and $\lim_{n\to\infty} x_n = x_0$

  & (ii) A sequence $\{y_n\}$ with $y_n \notin \mathbb{Q}$ $\forall n \in \mathbb{N}$ and $\lim_{n\to\infty} y_n = x_0$.

  Since $h(x_n) = x_n$ $\forall n \in \mathbb{N} \Rightarrow \lim_{n\to\infty} h(x_n) = \lim_{n\to\infty} x_n = x_0$

  & $h(y_n) = 0$ $\forall n \in \mathbb{N} \Rightarrow \lim_{n\to\infty} h(y_n) = 0$

  and $x_0 \neq 0$ it follows that h is not continuous at x_0.

- Let $x_0 = 0$:

  Let $\{x_n\}$ be any sequence with $\lim_{n\to\infty} x_n = 0$.
  Since $|h(x_n)| \leq |x_n|$ and $|x_n| \to 0$ it follows from "Baby Squeeze" that $\lim_{n\to\infty} h(x_n) = 0 = h(0)$. Hence h is cont's at 0.