Math 3100 Assignment 8
Continuity and Differentiation

Due at 5:00 pm on Friday the 5th of April 2019

1. (a) Let $f : [0,1] \to [0,1]$ be continuous. Prove that f must have a fixed point; that is, show that there must exist $x \in [0,1]$ with the property that $f(x) = x$.
   (b) Let $f : [0,1] \to \mathbb{R}$ be continuous with $f(0) = f(1)$.
   Show that there must exist $x \in [0, 1/2]$ with the property that $f(x) = f(x + 1/2)$.

2. Give an example of each of the following, or provide a short argument for why the request is impossible.
   (a) A continuous function defined on $[0,1]$ with range $(0,1)$.
   (b) A continuous function defined on $(0,1)$ with range $[0,1]$.
   (c) A continuous function defined on $(0,1]$ with range $(0,1)$.

3. Suppose f is a continuous function, $f(1) = -4$, $f(-2) = 3$, $\lim_{x \to -\infty} f(x) = 2$ and $\lim_{x \to \infty} f(x) = -1$.
   Prove that there exist $c, d \in \mathbb{R}$ so that $f(c) \leq f(x) \leq f(d)$ for all $x \in \mathbb{R}$.

4. Exactly one of the following requests is impossible. Decide which it is, and provide examples for the other three. In each case, lets assume that the functions are defined on all of $\mathbb{R}$.
   (a) Function f and g not differentiable at $x_0 = 0$, but where fg is differentiable at $x_0 = 0$.
   (b) A function f not differentiable at $x_0 = 0$ and a function g differentiable at $x_0 = 0$ where fg is differentiable at $x_0 = 0$.
   (c) A function f not differentiable at $x_0 = 0$ and a function g differentiable at $x_0 = 0$ where $f + g$ is differentiable at $x_0 = 0$.
   (d) A function f differentiable at $x_0 = 0$, but not differentiable at any other point.

5. Use the definition of the derivative to find $f'(x_0)$ for all $x_0 \in \mathbb{R}$ if:

$$\text{(a)} \quad f(x) = \sqrt{x^2 + 1} \qquad\qquad \text{(b)} \quad f(x) = \frac{1}{x^2 + 1}$$

6. (a) Let $f(x) = \begin{cases} x^2, & \text{if } x \geq 0 \\ 0, & \text{if } x < 0 \end{cases}$.
      i. Compute $f'(x)$ for $x \neq 0$.
      ii. Use the definition of the derivative to find $f'(0)$.
      iii. Is f' continuous at 0? Give your reasoning.
      iv. Does $f''(0)$ exist? Give your reasoning.
   (b) Let $g(x) = \begin{cases} x^3 \sin \dfrac{1}{x^2}, & x \neq 0 \\ 0, & x = 0 \end{cases}$.
      i. Compute $g'(x)$ for $x \neq 0$.
      ii. Use the definition of the derivative to find $g'(0)$.
      iii. Is g' continuous at 0? Give your reasoning.
      iv. Does $g''(0)$ exist? Give your reasoning.

7. Prove that if g is differentiable at x_0, and $g(x_0) \neq 0$, then $1/g$ is differentiable at x_0 and

$$\left(\frac{1}{g}\right)'(x_0) = \frac{-g'(x_0)}{g(x_0)^2}.$$

8. (a) Suppose f is continuous on $[a,b]$, twice differentiable on (a,b), and $f''(x) \neq 0$ for all $x \in (a,b)$. Prove *carefully* that f has at most 2 distinct zeros in $[a,b]$.
   Hint: Use Rolle's Theorem
   (b) Prove that the function $f(x) = x^2 - \sin x$ has *precisely* two roots.

Math 3100 – Homework 8 – SOLUTIONS

1. **Claim**

If $f:[0,1]\to\mathbb{R}$ is continuous with $f(0)=f(1)$, then $\exists\, x\in[0,\tfrac12]$ such that $f(x)=f(x+\tfrac12)$.

Proof

Since $f(0)=f(1)$

Let $g(x)=f(x+\tfrac12)-f(x)$.

Note that g is continuous on $[0,\tfrac12]$ with property that $g(0)=-g(\tfrac12)$.

If $g(0)=0$ we would be done, otherwise $g(0)$ & $g(\tfrac12)$ will have opposite signs (one must be positive & the other negative) and it follows from the Intermediate Value Theorem that $\exists\, x\in(0,\tfrac12)$ with

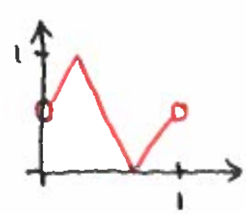
$$g(x)=0 \iff f(x)=f(x+\tfrac12).$$

$\square$

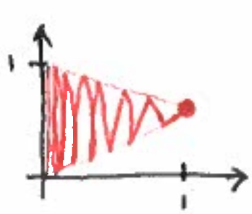
2. (a) Impossible, since the Extreme Value Thm ensures that any cont's function on $[0,1]$ must attain both a max & a min, such a function cannot have range $(0,1)$.

[$(0,1)$ has no max or min]

(b)



(c)



3. Since $\lim_{x \to \infty} f(x) = -1$ $\exists\, b \in \mathbb{R}$ such that if $x > b$, then

$$-2 < f(x) < 0. \qquad (*)$$

Since $\lim_{x \to -\infty} f(x) = 2$ $\exists\, a \in \mathbb{R}$ such that if $x < a$, then

$$1 < f(x) < 3. \qquad (**)$$

Extreme Value Thm $\Rightarrow$ $\exists\, c, d \in [a,b]$ such that

$$f(c) \leq f(x) \leq f(d) \quad \forall\, x \in [a,b].$$

Since $1, -2 \in [a,b]$ with $f(1) = -4$ & $f(-2) = 3$ we know

$$f(c) \leq -4 \quad \& \quad f(d) \geq 3$$

and hence that $f(c) \leq f(x) \leq f(d)$ $\forall\, x \in \mathbb{R}$. (by $(*)$ & $(**)$)

4. (a) <u>Example</u> $f(x) = g(x) = |x|$.

(b) <u>Example</u> $f(x) = |x|$ & $g(x) \equiv 0$

(c) Impossible! If g is diff'ble at x_0 & $f+g$ is diff'ble at x_0 then $f = (f+g) - g$ is diff'ble at x_0.

(d) <u>Claim</u>: $h(x) = \begin{cases} x^2 & \text{if } x \in \mathbb{Q} \\ 0 & \text{if } x \notin \mathbb{Q} \end{cases}$ is diff'ble at $x_0 = 0$ only!

<u>Proof</u> If $x_0 = 0$, then $\left| \frac{h(x) - h(x_0)}{x - x_0} \right| = \left| \frac{h(x)}{x} \right| \leq |x|$ $\forall\, x \neq 0$. It follows from the Squeeze Thm that $\lim_{x \to 0} \frac{h(x) - h(x_0)}{x - x_0}$ exists & equals 0.

If $x_0 \neq 0$, let $\{x_n\}$ be a seq in $\mathbb{Q}$ with $x_n \to x_0$ & $\{y_n\}$ a seq in $\mathbb{R} \setminus \mathbb{Q}$ with $y_n \to x_0$. Since $h(x_n) = x_n^2 \to x_0^2$ & $h(y_n) = 0$ $\forall n$ $\Rightarrow$ h is <u>not</u> cont's at x_0

& hence not diff'ble at

5. (a) Let $f(x) = \sqrt{x^2+1}$ & $x_0 \in \mathbb{R}$.

Since, for $x \neq x_0$

$$\frac{f(x)-f(x_0)}{x-x_0} = \frac{\sqrt{x^2+1}-\sqrt{x_0^2+1}}{x-x_0}$$

multiply by conjugate

$$= \frac{(x^2+1)-(x_0^2+1)}{(x-x_0)\left(\sqrt{x^2+1}+\sqrt{x_0^2+1}\right)}$$

$$= \frac{x+x_0}{\sqrt{x^2+1}+\sqrt{x_0^2+1}}$$

$$\longrightarrow \frac{2x_0}{2\sqrt{x_0^2+1}}$$

[Since $\lim_{x\to x_0} \sqrt{x^2+1} = \sqrt{x_0^2+1}$]

as $x \to x_0$ it follows that

$$f'(x_0) = \frac{x_0}{\sqrt{x_0^2+1}}.$$

(b) Let $f(x) = \frac{1}{x^2+1}$ & $x_0 \in \mathbb{R}$

Since, for $x \neq x_0$

$$\frac{f(x)-f(x_0)}{x-x_0} = \frac{\frac{1}{x^2+1}-\frac{1}{x_0^2+1}}{x-x_0}$$

$$= \frac{x_0^2-x^2}{(x-x_0)(x^2+1)(x_0^2+1)}$$

$$= \frac{-(x+x_0)}{(x^2+1)(x_0^2+1)}$$

$$\longrightarrow \frac{-2x_0}{(x_0^2+1)^2}$$

[Since $\lim_{x\to x_0} (x^2+1) = x_0^2+1$]

as $x \to x_0$ it follows that

$$f'(x_0) = \frac{-2x_0}{(x_0^2+1)^2}.$$

6. (a) Let $f(x) = \begin{cases} x^2 & \text{if } x \geq 0 \\ 0 & \text{if } x < 0. \end{cases}$

(i) $f'(x) = \begin{cases} 2x & \text{if } x > 0 \\ 0 & \text{if } x < 0 \end{cases}$

(ii) $\underline{f'(0) = 0}$ since if $x \neq 0$ then

$$\frac{f(x) - f(0)}{x - 0} = \frac{f(x)}{x} = \begin{cases} x & \text{if } x > 0 \\ 0 & \text{if } x < 0. \end{cases}$$

Since $\left| \frac{f(x) - f(0)}{x - 0} \right| \leq |x| \ \forall\, x \neq 0$ it follows from the Squeeze Thm that $\lim_{x \to 0} \frac{f(x) - f(0)}{x - 0}$ exists and equals 0.

(iii) $f'(x) = \begin{cases} 2x & \text{if } x \geq 0 \\ 0 & \text{if } x < 0 \end{cases}$

Let $\{x_n\}$ be any sequence with $\lim_{n \to \infty} x_n = 0$. Since

$$|f'(x_n)| \leq 2|x_n| \ \forall n \ \& \ \lim_{n \to \infty} |x_n| = 0$$

it follows from the Squeeze Thm that $\lim_{n \to \infty} f'(x_n) = 0 = f'(0)$ and hence that f' is conts at $x_0 = 0$.

(iv) $\underline{\text{Claim}}$ $f''(0)$ does $\underline{\text{not}}$ exist.

$\underline{\text{Proof}}$ If $x \neq 0$, then $\frac{f'(x) - f'(0)}{x - 0} = \begin{cases} 2 & \text{if } x > 0 \\ 0 & \text{if } x < 0. \end{cases}$

Let $\{x_n\}$ be a seq in $(0, \infty)$ with $x_n \to 0$ & $\{y_n\}$ be a seq in $(-\infty, 0)$ with $y_n \to 0$. Since $\frac{f'(x_n) - f'(0)}{x_n - 0} = 2 \ \forall n$ and $\frac{f'(y_n) - f'(0)}{y_n - 0} = 0 \ \forall n$ $\Rightarrow \lim_{x \to 0} \frac{f'(x) - f'(0)}{x - 0}$ DNE. $\square$

(b) Let $g(x) = \begin{cases} x^3 \sin \frac{1}{x^2} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$.

(i) $g'(x) = 3x^2 \sin \frac{1}{x^2} - 2 \cos \frac{1}{x^2}$ if $x \neq 0$

(ii) $g'(0) = 0$ since if $x \neq 0$ then

$$\frac{g(x) - g(0)}{x - 0} = x^2 \sin \frac{1}{x^2}$$

and hence $\left| \frac{g(x) - g(0)}{x - 0} \right| \leq |x|^2 \quad \forall x \neq 0$.

It follows from the Squeeze Thm that $\lim_{x \to 0} \frac{g(x) - g(0)}{x - 0}$ exists & $= 0$.

(iii) g' is not continuous at $x_0 = 0$ since

$\lim_{x \to 0} 3x^2 \sin \frac{1}{x^2} = 0$ but $\lim_{x \to 0} 2 \cos \frac{1}{x^2}$ DNE!

[Consider $x_n = \frac{1}{\sqrt{2\pi n}}$ & $y_n = \frac{1}{\sqrt{\frac{\pi}{2} + 2\pi n}}$]

(iv) Since g' is not continuous at $x_0 = 0$, it is not diff'ble at $x_0 = 0$,

so $g''(0)$ does not exist.

7. Claim: If g is diff'ble at x_0 & $g(x_0) \neq 0$, then $\frac{1}{g}$ is diff'ble at x_0 with $\left(\frac{1}{g}\right)'(x_0) = \frac{-g'(x_0)}{(g(x_0))^2}$.

Proof: If $x \neq x_0$, then

$$\frac{\frac{1}{g(x)} - \frac{1}{g(x_0)}}{x - x_0} = -\frac{g(x) - g(x_0)}{(x - x_0)\, g(x)\, g(x_0)} \longrightarrow \frac{-g'(x_0)}{g(x_0)^2}$$

since g is cont's at x_0. □

8. (a) Claim

If f is cnts on $[a,b]$ & twice diff'ble on (a,b) and $f''(x) \neq 0 \ \forall x \in (a,b)$, then f has at most 2 zeros in $[a,b]$.

Proof. Suppose $\exists\ x_1, x_2, x_3 \in [a,b]$, with $x_1 < x_2 < x_3$ &

$$f(x_1) = f(x_2) = f(x_3) = 0.$$

Rolle's Thm implies $\exists\ c_1 \in (x_1, x_2)$ & $c_2 \in (x_2, x_3)$ such that $f'(c_1) = 0$ and $f'(c_2) = 0$.

Rolle's Thm (this time applied to f' on $[c_1, c_2]$) implies $\exists\ c \in (c_1, c_2)$ such that $f''(c) = 0$, contradicting the hypothesis that $f''(x) \neq 0 \ \forall x \in (a,b)$. $\square$

(b) Claim $f(x) = x^2 - \sin x$ has exactly two zeros.

Proof

Since f is cnts on $[0,\pi]$ & twice diff'ble on $(0,\pi)$ with $f''(x) = 2 + \sin x \neq 0 \ \forall x \in (0,\pi)$ it follow from part (a) above that f has at most 2 zeros in $[0,\pi]$.

Note that $f(0) = 0$.

Since $f(\pi) = \pi^2 > 0$ & $f(\frac{\pi}{6}) = \frac{\pi^2}{36} - \frac{1}{2} < 0$ it follows from the Intermediate Value Thm that $\exists\ x \in (\frac{\pi}{6}, \pi)$ with $f(x) = 0$. $\square$