A ROUGH (INFORMAL) RESEARCH SUMMARY

For Andreas, Steve, and Jim

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Abstract. Apologies for the Theorem–Theorem–Theorem nature of this. I haven't had time to write a nice short research statement yet.

1. Additive Combinatorics

General Problem: Finding patterns in dense subsets of $\mathbf{Z}$.

We say that $A \subseteq \mathbf{Z}$ has positive (upper) Banach density, if there exists $\varepsilon > 0$ and a sequence of intervals I_j with $|I_j| \to \infty$ such that

$$\delta(A \,|\, I_j) = \frac{|A \cap I_j|}{|I_j|} \geq \varepsilon \quad \text{for all } j.$$

The following result is one of the highlights of all combinatorics.

Theorem 1 (Szemerédi [31])**.** *If $A \subseteq \mathbf{Z}$ has positive Banach density, then A contains arbitrarily long arithmetic progressions.*

> Quantitative Version: Let $\varepsilon > 0$ and $k \in \mathbf{N}$, then there exists $N_0 = N_0(\varepsilon, k)$ such that if $N \geq N_0$ and $A \subseteq \{1, \ldots, N\} = [1, N]$ with $|A| = \varepsilon N$, then A contains an arithmetic progression of length k.

Szemerédi's original proof is both a long and combinatorial and gives only weak (tower-type) bounds for N_0. In [13] Furstenberg provided a second proof, showing that Szemerédi's theorem in fact follows from an ergodic theorem, now know as the multiple recurrence theorem. Furstenberg's approach, as with all such applications of ergodic theory, gives no bound for N_0. The first effective bounds for Szemerédi's theorem were obtained in the celebrated third proof by Gowers [14].

It is natural to ask if other configurations must occur in subsets of the integers (or integer lattice) of positive density. Sárközy [30] and Furstenberg [13] independently proved that if p is a polynomial such that $p(\mathbf{N}) \subseteq \mathbf{N}$ and $p(0) = 0$, then there exists $d \in \mathbf{N}$ and distinct elements $a, a' \in A$ such that $p(d) = a - a'$, in other words there exists $d \neq 0$ such that $p(d) \in A - A$.

Sárközy's approach was similar in spirit to that of Roth's Fourier analytic (circle method) proof of Szemerédi's theorem for arithmetic progressions of length three (it is this approach that Gowers was successfully able to "generalize"). Note that no combinatorial proof of this result is currently known.

Bergelson and Leibman (extending on the ideas of Furstenberg) established a far reaching generalization of Sárközy and Furstenberg's result, the so called Polynomial Szemerédi Theorem.

Theorem 2 (Bergelson-Leibman [1])**.** *If $A \subseteq \mathbf{Z}$ has positive density and $p_1, \ldots, p_k$ are polynomials such that $p_j(\mathbf{N}) \subseteq \mathbf{N}$ and $p_j(0) = 0$, then $\exists\ a \in \mathbf{Z}$, $d \in \mathbf{N}$ such that*

$$\{a + p_1(d), \ldots, a + p_k(d)\} \subseteq A.$$

We note that there is no known Fourier analytic proof of this result, in fact no effective bound is even known (in the quantative version) for the case $p_1(d) = d^2$, $p_2(d) = 2d^2$, and $p_3(d) = 3d^2$; Roth's theorem where the common difference as a perfect (non-zero) square.

One can rephrasing Sárközy's result as follows; if $A \subseteq \mathbb{Z}$ with positive Banach density and p is a polynomial such that $p(\mathbf{N}) \subseteq \mathbf{N}$ and $p(0) = 0$, then the set of polynomial return times $\{d \,|\, A \cap (A + p(d)) \neq \emptyset\}$ is non-empty. Using Fourier analysis Á. Magyar and myself have been able to show that the set of all *monomial* return times is *syndetic* and obtain uniform lower bounds for the density of these return times. Recall that a set Δ is *syndetic* if there exists $M \in \mathbf{N}$ such that every interval of length greater than M intersects Δ nontrivially.

Theorem 3 (Lyall and Magyar [23])**.** *If $A \subseteq \mathbf{Z}$ has positive Banach density and $n \in \mathbf{N}$, then $\{d \,|\, A \cap (A + d^n) \neq \emptyset\}$ is syndetic. Moreover, there exists $M = M(A)$ such that for all $\lambda \in \mathbf{Z}$ there exists $a \in A$ such that*

$$\#\{d \in [\lambda, \lambda + M] \,|\, a + d^n \in A\} \geq \delta(\epsilon)M.$$

We note that the dependence of M on A is necessary. By Szemerédi we have the following

Corollary 4. *For all $\lambda \in \mathbf{Z}$ there exists $a \in A$ and an arithmetic progression $\{d_1, \ldots, d_k\} \subseteq [\lambda, \lambda + M]$ such that*

$$\{a + d_1^n, \ldots, a + d_k^n\} \subseteq A.$$

While our arguments currently restrict us to homogeneous polynomials we have good reason to believe that we can, by lifting the problem to $\mathbf{Z}^n$, establish the results above for any polynomial that satisfies $p(\mathbf{N}) \subseteq \mathbf{N}$ and $p(0) = 0$.

The following related (very related!) result has come to our attention only very recently.

Theorem 5 (Frantzikinakis and Kra [12])**.** *Let $A \subseteq \mathbf{Z}$ and $p_1, \ldots, p_k$ be* rationally independent *polynomials such that $p_j(\mathbf{N}) \subseteq \mathbf{N}$ and $p_j(0) = 0$, then for every $\varepsilon > 0$, the set*

$$\{d \,|\, \delta^* \left(A \cap (A + p_1(d)) \cap \cdots \cap (A + p_k(d))\right) \geq \delta^*(A)^{k+1} - \varepsilon\}$$

is syndetic.

Here $\delta^*(A) = \lim\limits_{N\to\infty} \sup\limits_{\lambda \in \mathbf{Z}} |A \cap [\lambda, \lambda + N)|/N$ denotes the explicit upper Banach density of A.

Their results do not however lead to uniform lower bounds for the density of return times.

Rough outline of our argument

Fix $\varepsilon > 0$, $N > 0$ and let $A \subseteq I_N$ with $|A| \geq \varepsilon N$. We introduce the following notion of regularity; we say that A is (L, q) *regular* if for all $r \in \mathbf{Z}_q$ and for all $I_L \subseteq I_N$

$$\delta(A \,|\, (r + q\mathbf{Z}) \cap I_L) \leq \frac{10\epsilon}{9}.$$

We show (inspired by Bourgain [2] and using the circle method) that if, for a given $0 < \mu < \lambda \ll N$,

$$(A - A) \cap \{d^n \,|\, d \in [\lambda, \lambda + \mu]\} \ll \mu N \tag{1}$$

and A is $(\lambda^n, q_\varepsilon)$ *regular*, then there exists $\delta = \delta(\varepsilon)$ such that

$$\sum_{a=1}^{q_\epsilon} \int\limits_{\alpha \in \mathcal{M}} |\widehat{\chi_A}(\alpha)|^2 \, d\alpha \geq \varepsilon^2 N \quad \text{where} \quad \mathcal{M} = \left\{\alpha \,:\, \frac{1}{\lambda^n} \leq \left|\alpha - \frac{a}{q_\varepsilon}\right| \leq \frac{1}{\delta\mu^n}\right\}.$$

Hence, if we assume that our theorem is false, and further assume that A is *regular*, then we can find an arbitrarily long lacunary sequence $\{\lambda_j\}$ and a slowly increasing sequence $\{\mu_j\}$ so that (1) holds with $\lambda = \lambda_j$ and $\mu = \mu_j$ for each j, from which we can conclude that

$$\int_0^1 |\widehat{\chi_A}(\alpha)|^2 \, d\alpha > N$$

a contradiction. While if A is not *regular*, then we perform a density increment argument in the same spirit as that in the (standard) proof of Roth's theorem.

2. Strongly singular integrals

2.1. **Strongly singular integrals on $\mathbf{R}^d$.** In the model case strongly singular integrals are convolution operators on $\mathbf{R}^d$ whose distributional kernels are too singular at the origin to fall under the theory of Calderón and Zygmund. This strong singularity is compensated for by the introduction of a suitably large oscillating term. It is natural to then ask the question: *How does the respective size of the singularity and oscillation affect the regularity of the operator?*

To be more precise we consider convolution operators, formally given by

$$Tf(x) = f * K_{\alpha,\beta}(x)$$

where $K_{\alpha,\beta}$ is a distribution[1] on $\mathbf{R}^d$ that away from the origin agrees with the function

$$K_{\alpha,\beta}(x) = |x|^{-d-\alpha} e^{i|x|^{-\beta}/\beta} \chi(|x|) \tag{2}$$

with χ a smooth cutoff about the origin and $\beta > 0$.

In this setting the answer to the question phrased above is completely understood.

Proposition 6 (Wainger [32])**.** *If $\beta > 0$, then as $|\xi| \to \infty$*

$$\widehat{K_{\alpha,\beta}}(\xi) = c\ |\xi|^{-a} e^{i|\xi|^b/b} + O(|\xi|^{-a-1})$$

where

$$\frac{1}{-\beta} + \frac{1}{b} = 1 \quad \text{and} \quad \frac{\alpha}{\beta} + \frac{a}{b} = \frac{d}{2}$$

The decay at infinity of $|\widehat{K_{\alpha,\beta}}(\xi)|$ reflects the oscillatory nature of $K_{\alpha,\beta}$ near the origin. In particular

$$|\widehat{K_{\alpha,\beta}}(\xi)| \leq C(1 + |\xi|)^{\frac{\alpha - d\beta/2}{\beta+1}}$$

and hence that T extends to a bounded operator on $L^2(\mathbf{R}^d)$ whenever $\alpha \leq \frac{d\beta}{2}$.

Theorem 7 (C. Fefferman and Stein [11])**.** *Let L be a compactly supported distribution on $\mathbf{R}^d$ which coincides with a locally integrable function away from the origin. If there exists, $0 < b < 1$, such that*

$$\int_{|x|>2|y|^{1-b}} |L(x-y) - L(x)|\, dx \leq C$$

for $|y| \leq 1$, and

$$|\widehat{L}(\xi)| \leq \frac{C}{(1 + |\xi|)^{db/2}}$$

for $\xi \in \mathbf{R}^d$, then $Tf = f * L$ is bounded from $H^1(\mathbf{R}^d)$ to $L^1(\mathbf{R}^d)$.

The kernel $K_{\alpha,\beta}$ clearly satisfies the hypothesis of Theorem 7, combining this with Theorem 6 and a complex interpolation argument on Hardy spaces therefore gives us the following comprehensive result.

Theorem 8. *If $1 < p < \infty$, then*

$$T : L^p \to L^p \iff \left|\frac{1}{p} - \frac{1}{2}\right| \leq \frac{d\beta - 2\alpha}{2d\beta} \quad \left(= \frac{a}{db}\right)$$

The sharpness of this result follows from a counterexample of Wainger [32]. In addition to Theorem 7 above, C. Fefferman [9] also showed that if $\alpha = 0$, then T is of weak type (1,1).

For extensions of the results above to non-translation invariant operators, see Hörmander [15], C. Fefferman [10], and Lyall [21]; see also Stein [26] and Sjölin [25] for different generalizations.

I have been interested in generalizing these classical results to the setting of homogeneous groups. Almost no previous work had been done in this direction, with the one notable exception of B. Shayya [24] where

[1] The distribution-valued function $\alpha \mapsto K_{\alpha,\beta}$, initially defined for $\operatorname{Re}\alpha < 0$, continues analytically to $\mathbf{C}$.

sharp L^p results were obtained for a class of nonisotropic operators in the plane (of course the underlying group here is still commutative). I have also studied strongly singular analogues of singular Radon transforms, both in R^d and on Heisenberg group.

2.2. **Strongly singular integrals on homogeneous groups.** Recall that a *homogeneous group* $\mathbf{H}$ consists of $\mathbf{R}^d$ equipped with a Lie group structure and a family of dilations

$$x \mapsto \delta \circ x = (\delta^{a_1} x_1, \dots, \delta^{a_d} x_d).$$

which are group *automorphisms.* Consequently,

$$x \cdot y = x + y + Q(x, y)$$

where $Q = (Q_1, \dots, Q_d)$ is a polynomial of *isotropic* degree at least 2. Moreover, each Q_k is homogeneous of degree a_k. We let $h = a_1 + \cdots + a_d$ denote the homogeneous dimension of $\mathbf{H}$.

Examples

1. $\mathbb{R}^d$ with its usual additive structure (but with possibly nonisotropic dilations) [abelian]

2. Identifying $\mathbb{R}^{2n+1}$ with the *Heisenberg group* H^n, with dilations

$$x = (x', x_{2n+1}) \mapsto (\delta x', \delta^2 x_{2n+1})$$

and

$$x \cdot y = x + y + (0, 2\, x'^{\,\mathrm{t}}\; J\; y')$$

where J is the standard symplectic matrix on $\mathbb{R}^{2n}$

$$\left[\text{If } n = 1, \text{ then } \;(x_1, x_2) \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = x_1 y_2 - x_2 y_1\right]$$

3. The set of upper-triangular $m \times m$ matrices with ones along the diagonal identified with $\mathbb{R}^{m(m-1)/2}$. The group law is inherited from matrix multiplication, for example if $m = 3$, then

$$\begin{pmatrix} 1 & x_1 & x_3 \\ 0 & 1 & x_2 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & y_1 & y_3 \\ 0 & 1 & y_2 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & x_1 + y_1 & x_3 + y_3 + x_1 y_2 \\ 0 & 1 & x_2 + y_2 \\ 0 & 0 & 1 \end{pmatrix}$$

and the mapping $x \mapsto (\delta x_1, \delta x_2, \delta^2 x_3)$ is an automorphism of this group.

It is then natural to consider, for suitable *quasi-norms* ρ on $\mathbf{H}$, the (group) convolution operators

$$Tf(x) = \int_{\mathbf{H}} K_{\alpha,\beta}(y^{-1} \cdot x) f(y)\, dy$$

where $K_{\alpha,\beta}$ is again a distribution that for $x \neq 0$ now agrees with the function

$$K_{\alpha,\beta}(x) = \rho(x)^{-h-\alpha} e^{i\rho(x)^{-\beta}} \chi(\rho(x))$$

where χ is a smooth cutoff about the origin and $\beta > 0$.

For these operators it is satisfying to note that we have the following general L^2 (existence) result.

Theorem 9 (Laghi and Lyall [20])**.** *On any homogeneous group $\mathbf{H}$, there exists a quasi-norm $\rho = \rho_{\mathbf{H}}$ so that*

$$T : L^2(\mathbf{H}) \to L^2(\mathbf{H}) \quad \textit{whenever} \quad \alpha \leq d\beta/2$$

2.2.1. *Nonisotropic* $\mathbf{R}^d$. If ρ_1 is a quasi-norm for which the level hypersurface

$$\Sigma = \{x \in \mathbb{R}^d | \rho_1(x) = 1\}$$

is smooth and *convex* with *everywhere non-vanishing Gaussian curvature*, then we have the following

Theorem 10 (Laghi and Lyall [20], Shayya [24]). *On (nonisotropic)* $\mathbf{R}^d$ *with* $\rho = \rho_1$ *we have*

(i) *If* $\alpha = 0$*, then*

$$T : H^1_\beta(\mathbf{R}^d) \to L^1(\mathbf{R}^d)$$

(ii) *If* $1 < p < \infty$*, then*

$$T : L^p(\mathbf{R}^d) \to L^p(\mathbf{R}^d) \iff \left|\frac{1}{p} - \frac{1}{2}\right| \leq \frac{d\beta - 2\alpha}{2d\beta}$$

$$\big[\textit{In particular if } \alpha \leq d\beta/2, \textit{ then } T : L^2(\mathbf{R}^d) \to L^2(\mathbf{R}^d)\big]$$

Here H^1_β denotes the Hardy space defined via atomic decompostion with atoms supported on the non-isotropic balls associated to quasi-norms corresponding to the dilations $x \mapsto \delta^\beta(\delta \circ x)$. It is crucial in the proof of the Hardy space estimate that these new dilations are still automorphisms of nonisotrpic $\mathbf{R}^d$.

This result was initially obtained by B. Shayya [24] in $\mathbf{R}^2$ with $\Sigma = S^1$ using Fourier transform techniques.

Observation (Laghi and Lyall [20]). *Equivalent norms give rise to different* L^2 *behavior.*

More precisely, if $\rho(x) = (x_1^{2a_2} + x_2^{2a_1})^{1/2a_1a_2}$ with $a_2 \geq a_1$, then

$$|\widehat{K_{\alpha,\beta}}(\xi)| \leq C \Rightarrow \alpha \leq \frac{a_2+1}{2a_2}\beta.$$

In fact, if $\rho(x) = \left(\sum_{i=1}^d x_i^{2p/a_i}\right)^{1/2p}$ where $p = \prod_{i=1}^d a_i$, then

$$T : L^2(\mathbf{R}^d) \nrightarrow L^2(\mathbf{R}^d) \text{ for } \alpha > \Big(\frac{1}{2} + \frac{h - h_d}{2p}\Big)\beta.$$

The L^2 estimate (a *concrete* special case of Theorem 9) is of course the key to proving Theorem 10. In contrast to Shayya (and the other authors mentioned above) we take an oscillatory integral approach to establishing the desired L^2 estimates and as such we are not restricted to translation invariant settings.

Strategy for establishing L^2 estimates. By almost orthogonality it suffices to establish the estimate

$$\|T_j f\|_{L^2(\mathbf{H})} \leq C 2^{j(\alpha - d\beta/2)} \|f\|_{L^2(\mathbf{H})} \tag{3}$$

for the (rescaled) dyadic operators

$$T_j f(x) = 2^{j\alpha} \int_{\rho(y^{-1}\cdot x)\approx 1} \Psi(x,y) e^{i2^{j\beta}\rho(y^{-1}\cdot x)^{-\beta}} f(y)\, dy$$

with Ψ smooth and compactly supported in x and y. The desired dyadic estimate follows immediately if

$$\det \partial_{x_j}\partial_{y_k}\left[\rho(y^{-1}\cdot x)^{-\beta}\right] \neq 0$$

for all (x,y) in the support of Ψ.

To establish (3) on (nonisotropic) $\mathbf{R}^d$ therefore reduce to showing that if $x \neq y$ and $\beta > 0$, then

$$\det \partial_{x_j}\partial_{y_k}\left[\rho_1(x-y)^{-\beta}\right] \neq 0.^2 \tag{4}$$

Theorem 9 follows from this by continuity since for any $\mathbf{H}$ there exists $\varepsilon > 0$ so that if $\rho(x) = \rho_1(\varepsilon^{-1}x)$,

$$\det \partial_{x_j}\partial_{y_k}\left[\rho(y^{-1}\cdot x)^{-\beta}\right] \neq 0$$

whenever $x \neq y$ and $\beta > 0$.

[2] In the case of *isotropic* dilations this actually holds for all $\beta \neq -1$. See proof of (4) in appendix.

2.2.2. *Concrete results for different quasi-norms on the Heisenberg group.* We denote by H_a^n the Heisenberg group given by the multiplication law

$$x \cdot y = (x' + y', x_{2n+1} + y_{2n+1} + 2a\; x'^{\,\mathrm{t}}\; J\; y')$$

<u>Quasi-norm 1:</u> Consider the special case $\rho_1(x) = 1 \Leftrightarrow |x| = 1$ (i.e. $\Sigma = S^{2n}$). For this norm we have

Theorem 11 (Laghi and Lyall [18]). *If $\rho(x) = \rho_1(b^{-1}x)$ and $|ab| \leq 1$, then*

$$T : L^2(H_a^n) \to L^2(H_a^n) \quad \textit{whenever} \quad \alpha \leq \Big(n + \frac{1}{2}\Big)\beta.$$

Note that this corresponds to a positive result when $\rho(x) = 1 \iff |x| = b$ with $|ab| \leq 1$.

<u>Quasi-norm 2 (Koranyi norm):</u> Consider $\rho_2(x) = (|x'|^4 + x_{2n+1}^2)^{1/4}$.

In my thesis, using the *group Fourier transform*, I obtained the following (in some sense partial) result.

Theorem 12 (Lyall [22]). *If $\rho = \rho_2$ and $a \neq 0$, then*

$$\alpha \leq n\beta \Longrightarrow T : L^2(H_a^n) \to L^2(H_a^n) \Longrightarrow \alpha \leq \Big(n + \frac{1}{2}\Big)\beta$$

This result is <u>uniform</u> in $a \neq 0$. It relies on the radial nature of the Koranyi norm and follows from uniform asymptotic expansions for Laguerre functions due to Erdélyi [7] along with some careful analysis.

Sharp L^2 estimates follow from our *oscillatory integral approach.* Fix a constant C_β by the equation

$$2C_\beta^2 = (\beta + 2)\big(2\beta + 5 + \sqrt{(2\beta + 5)^2 - 9}\big)$$

and note that $C_\beta \geq 3$ for all $\beta > 0$.

Theorem 13 (Laghi and Lyall [18]). *If $\rho(x) = \rho_2(b^{-1}x)$ and $0 < |ab| < C_\beta$, then*

$$T : L^2(H_a^n) \to L^2(H_a^n) \quad \textit{whenever} \quad \alpha \leq \Big(n + \frac{1}{2}\Big)\beta.$$

Note that this essentially corresponds to a positive result if $\rho(x) = (|x'|^4 + b^2 x_{2n+1}^2)^{1/4}$ and $0 < |ab| < C_\beta$.

Theorems 11 and 13 are obtained by showing that for the respective quasi-norms ρ we have

$$\det \partial_{x_j} \partial_{y_k} \left[\rho(y^{-1} \cdot x)^{-\beta}\right] \neq 0$$

whenever $x \neq y$, $\beta > 0$. We note that when $\rho = \rho_1$ the corresponding mixed Hessian actually remains non-singular if $a = 0$, while if $\rho = \rho_2$ the corresponding mixed Hessian can drop rank when $a = 0$.

2.3. **Strongly singular Radon transforms.**

2.3.1. *Strongly singular integrals along curves in $\mathbf{R}^d$.* It is standard and well known that the Hilbert transform along curves:

$$\mathcal{H}_\gamma f(x) = \text{p.v.} \int_{-1}^{1} f(x - \gamma(t)) \frac{dt}{t},$$

is bounded on $L^p(\mathbf{R}^d)$, for $1 < p < \infty$, where $\gamma(t)$ is an appropriate curve in $\mathbf{R}^d$. In particular, it is known to map $L^2(\mathbf{R}^2)$ boundedly into $L^2(\mathbf{R}^2)$, for $1 < p < \infty$, where

$$\gamma(t) = (t, t|t|^k) \text{ or } (t, |t|^{k+1}) \tag{5}$$

with $k \geq 1$, is a curve in $\mathbf{R}^2$. This work was initiated by Fabes and Rivière [8]. The specific result stated above is due to Nagel, Rivière, and Wainger [16]. In [27], Stein and Wainger extended these results to *well-curved* γ in $\mathbf{R}^d$; smooth mappings $\gamma(t)$ such that $\gamma(0) = 0$ and

$$\frac{d^k \gamma(t)}{dt^k}\Big|_{t=0}, \quad k = 1, 2, \ldots,$$

span $\mathbf{R}^d$ (in other words smooth mappings of finite type in a small neighborhood of the origin).

It is worth pointing out, however, that $\mathcal{H}_\gamma$ displays "bad" behavior near L^1; in [4] Christ showed that $\mathcal{H}_\gamma$ maps the (parabolic) Hardy space H^1 into weak L^1 for the plane curves $\gamma(t)=(t,t^2)$, and furthermore pointed out that $H^1\to L^1$ boundedness cannot hold, while a previous result of Christ and Stein [6] established that $\mathcal{H}_\gamma$ maps $L\log L(\mathbf{R}^d)$ into $L^{1,\infty}(\mathbf{R}^d)$ for a large class of curves γ in $\mathbf{R}^d$. Seeger and Tao [28] have shown that $\mathcal{H}_\gamma$ maps the product Hardy space $H^1_{\text{prod}}(\mathbf{R}^2)$ into the Lorentz space $L^{1,2}(\mathbf{R}^2)$; the results obtained are sharp, as $\mathcal{H}_\gamma$ does not map the product Hardy space into any smaller Lorentz space. Finally, the same authors, along with Wright, have shown in [29] that $\mathcal{H}_\gamma : L\log\log L(\mathbf{R}^2)\to L^{1,\infty}(\mathbf{R}^2)$.

A natural strongly singular analogue of the singular integrals along curves in $\mathbf{R}^d$ discussed above would be operators of the form

$$T_\gamma f(x)=\text{p.v.}\int_{-1}^{1}H_{\alpha,\beta}(t)f(x-\gamma(t))dt, \tag{6}$$

with $\gamma(t)=(\gamma_1(t),\dots,\gamma_d(t))$ where $H_{\alpha,\beta}(t)=t^{-1}|t|^{-\alpha}e^{i|t|^{-\beta}}$ is now a strongly singular (convolution) kernel in $\mathbf{R}$ which enjoys some additional cancellation (note that $H_{\alpha,\beta}$ is an odd function for $t\neq 0$).

Theorem 14 (Laghi and Lyall [19])**.** *Let $\gamma(t)$ be* well-curved.

(i) *If $\alpha=0$ and $\beta>0$, then*
$$T_\gamma : L\log L(\mathbf{R}^d)\to L^{1,\infty}(\mathbf{R}^d)$$
(ii) *If $1<p\le 2$, then*
$$T_\gamma : L^p(\log L)^{2(1/p-1/2)}(\mathbf{R}^d)\to L^{p,p'}(\mathbf{R}^d)\quad \textit{whenever}\quad \frac{1}{p}-\frac{1}{2}\le\frac{\beta-(d+1)\alpha}{2\beta}$$
$$\big[\textit{In particular } T_\gamma : L^2(\mathbf{R}^d)\to L^2(\mathbf{R}^d)\iff \alpha\le\beta/(d+1)\big]$$

**** Andreas and Steve: Do I get the natural thing for $2\le p<\infty$ by duality? ****

Continuing on the work of Zielinski [33], Chandarana [3] obtained the L^2 result above for operators of the form (6) in $\mathbf{R}^2$ along the model homogeneous curves (5). Although Chandarana obtains some partial L^p results, no endpoint result near L^1 have previously been obtained for the critical value $\alpha=0$. The near L^1 result above has been obtained by applying results of Christ and Stein [6].

Refined estimates in $\mathbf{R}^2$

By apply the results of Seeger, Tao, and Wright in [29] to the operator T_γ in (6) gives the following.

Theorem 15 (Laghi and Lyall [19])**.** *Let $d=2$, $\alpha=0$ and $\gamma(t)=(t,t|t|^b)$, $b>0$.*

(i) *If $\alpha=0$ and $\beta>0$, then*
$$T_\gamma : L\log\log L(\mathbf{R}^2)\to L^{1,\infty}(\mathbf{R}^2)$$
(ii) *If $1<p\le 2$, then*
$$T_\gamma : L^p(\log\log L)^{2(1/p-1/2)}(\mathbf{R}^2)\to L^{p,p'}(\mathbf{R}^2)\quad \textit{whenever}\quad \frac{1}{p}-\frac{1}{2}\le\frac{\beta-3\alpha}{2\beta}$$

**** Andreas and Steve: Do I get the natural thing for $2\le p<\infty$ by duality? ****

2.3.2. *Strongly singular Radon transforms on the Heisenberg group.* In [21] I considered operators R on the Heisenberg group H^n obtained by taking group convolution with the distribution

$$M(x,t)=K_{\alpha,\beta}(x)\delta(t-\phi(x)), \tag{7}$$

where $K_{\alpha,\beta}$ is a model strongly singular kernel in $\mathbf{R}^{2n}$ of type (2). Using group Fourier transform techniques it was shown that if $\phi\equiv 0$, or $\phi(x)=|x|^\kappa$ with $\kappa\ge 2$, then

$$R : L^2(H^n)\to L^2(H^n)\iff \alpha\le(n-1/6)\beta.$$

In [17] we studied the following natural generalizations (to the non-translation invariant setting) of the operators R discussed above by means of oscillatory integrals;

$$Tf(x,t) = \int_{\mathbf{R}^{2n+1}} K_{\alpha,\beta}(x,y) \left(\int_{\mathbf{R}} e^{i\tau\left[t-s+2\,x^{\mathrm{t}}Jy-\phi(x,y)\right]} d\tau \right) f(y,s)dy\,ds, \tag{8}$$

where $K_{\alpha,\beta}$ is a marginally more general *strongly singular integral kernel* on $\mathbf{R}^{2n} \times \mathbf{R}^{2n}$. We showed that the problem in question can be effectively treated by establishing uniform estimates for certain oscillatory integrals whose canonical relations project with two-sided fold singularities. Our main result is the following.

Theorem 16 (Lyall [21], Laghi and Lyall [17])**.** *Consider the operator (8) with phase function ϕ satisfying either of the following conditions:*

(i) *$\phi \in C^\infty(U \setminus \Delta)$, where U is a neighbourhood of the diagonal $\Delta \subset \mathbf{R}^{2n} \times \mathbf{R}^{2n}$ with $U \supset supp(a)$, and for some $\kappa > 2$ satisfies the differential inequalities*
$$\left| D^\mu_{x,y}\phi(x,y) \right| \le C_\mu \left| x-y \right|^{\kappa-|\mu|}$$
for all $x \neq y$ and every multiindex μ.

(ii) *$\phi(x,y) = \varphi(x-y)$, where φ is smooth and supported in a small neighbourhood of the origin, with*
$$\nabla^2_x \varphi(0) = 4B$$
where $B = (b_i\delta_{i,j})$ with $b_i = b_{i+n}$ a real constant for $i = 1, \ldots, n$.

Then $T : L^2(\mathbf{H}^n) \to L^2(\mathbf{H}^n)$ if and only if $\alpha \le (n-1/6)\beta$.

Note that our second result only concerns operators associated with translation-invariant phase functions. The model example of such a phase is $\varphi(x) = |x|^2$, more generally we can also consider phases of the form $\varphi(x) = \sigma(|x|^2)$, where σ is a smooth function supported in a neighbourhood of the origin.

3. Appendix for Andreas and Steve

Suppose ρ is a quasi-norm for which the level hypersurface
$$\Sigma = \{x \in \mathbb{R}^d | \rho(x) = 1\}$$
is smooth and *convex* with *everywhere non-vanishing Gaussian curvature*. Noting that this implies
$$\langle H\rho(x)v, v\rangle > 0$$
for all $v \neq 0$ satisfying $\langle \nabla\rho(x), v\rangle = 0$.

Proposition 17. *Under the conditions above $H[\rho^{-\beta}](x)$ as full rank for every $\beta > 0$ and $x \neq 0$.*

Proof. It is easy to verify that
$$H[\rho^{-\beta}](x) = -\beta\rho(x)^{-(\beta+2)} \left[\rho(x)H\rho(x) - (\beta+1)\nabla\rho(x)\nabla\rho(x)^{\mathrm{t}} \right].$$
Thus if, for a fixed x, we suppose the existence of a *non-zero* $u \in \ker H[\rho^{-\beta}](x)$, we see that
$$\rho(x)H\rho(x)u = (\beta+1)\langle\nabla\rho(x), u\rangle\nabla\rho(x), \tag{9}$$
and note, as a consequence of the curvature condition, that necessarily
$$\langle\nabla\rho(x), u\rangle \neq 0. \tag{10}$$

The proof breaks into two cases.

Case (i) $\det H\rho(x) = 0$; if this is the case then there exists $w \neq 0$ such that
$$H\rho(x)w = 0 \quad \text{and} \quad \langle\nabla\rho(x), w\rangle \neq 0,$$
where the second observation follows from the curvature assumption and allows us to write
$$u = \lambda w + v$$

where $\langle \nabla\rho(x), v\rangle = 0$ and $\lambda \neq 0$. It then follows from (9) that

$$\rho(x)H\rho(x)v = \lambda(\beta+1)\langle \nabla\rho(x), w\rangle \nabla\rho(x).$$

This is clearly an impossibility as, by the symmetry of $H\rho(x)$, the vector on the left of this identity is orthogonal to w, whereas the vector on the right is patently not.

Case (ii) $\det H\rho(x) \neq 0$; in this case it is straightforward to see that (9) is equivalent to the identity

$$\rho(x) = (\beta+1)\langle H\rho(x)^{-1}\nabla\rho(x), \nabla\rho(x)\rangle. \tag{11}$$

Lemma 18. *If for some $x \neq 0$ we have that $\det H\rho^{-\beta}(x) = 0$ for some $\beta \neq 0$ while $\det H\rho(x) \neq 0$, then $H\rho(x)$ is positive definite.*

Proof. It follows from (11) that

$$\langle H\rho(x)^{-1}\nabla\rho(x), \nabla\rho(x)\rangle > 0.$$

If we now set

$$w = H\rho(x)^{-1}\nabla\rho(x)$$

it follows that

$$\langle H\rho(x)w, w\rangle = \langle \nabla\rho(x), w\rangle > 0.$$

We may therefore express any $z \in \mathbf{R}^d$ in the form

$$z = \lambda w + v$$

where $\langle \nabla\rho(x), v\rangle = 0$, from which it follows that

$$\langle H\rho(x)z, z\rangle = \lambda^2\langle H\rho(x)w, w\rangle + \langle H\rho(x)v, v\rangle > 0$$

since $\langle H\rho(x)w, v\rangle = 0$. □

We now wish to study (11) under linear changes of variables. It follows from Lemma 18 that there exists an orthogonal matrix B such that

$$D(x) = BH\rho(x)B^{\mathrm{t}}$$

is diagonal at $x = 0$ with *positive* eigenvalues d_{jj} for $j = 1, \dots, d$. Using a translation we may clearly assume that $x = 0$.

Now

$$\begin{aligned} H[\rho^{-\beta}(Bx)] &= BH[\rho^{-\beta}](Bx)B^{\mathrm{t}} \\ &= -\beta\rho(Bx)^{-(\beta+2)}B\left[\rho(Bx)H\rho(Bx) - (\beta+1)\nabla\rho(Bx)\nabla\rho(Bx)^{\mathrm{t}}\right]B^{\mathrm{t}}. \end{aligned}$$

Suppose $u \in \ker H[\rho^{-\beta}(Bx)]$, it would then follow that

$$\rho(Bx)D(Bx)u = (\beta+1)\langle B\nabla\rho(Bx), u\rangle B\nabla\rho(Bx) \tag{12}$$

and hence

$$\rho(Bx) = (\beta+1)\langle D(Bx)^{-1}B\nabla\rho(Bx), B\nabla\rho(Bx)\rangle. \tag{13}$$

Lemma 19. *If we let $A = \{a_{ij}\}$ with $a_{ij} = a_i\delta_{ij}$, then*

$$\rho(Bx) = \langle D(Bx)^{-1}B\nabla\rho(Bx), B\nabla\rho(Bx)\rangle - \langle D(Bx)^{-1}AB\nabla\rho(Bx), B\nabla\rho(Bx)\rangle. \tag{14}$$

Proof. This result follows from Euler's relations. By homogeneity we know that $\rho(\delta \circ Bx) = \delta\rho(Bx)$, thus differentiating with respect to δ (and then setting $\delta = 1$) we obtain the identity

$$\langle \nabla\rho(Bx), ABx\rangle = \rho(Bx).$$

If we now differentiate this with respect to x we in turn obtain

$$BH\rho(Bx)ABx + AB\nabla\rho(Bx) = B\nabla\rho(Bx)$$

and hence

$$ABx = B^{-1}D(Bx)^{-1}B\nabla\rho(Bx) - B^{-1}D(Bx)^{-1}AB\nabla\rho(Bx).$$

The result now follows by taking the dot product of both sides with respect to $\nabla\rho(Bx)$ and the fact that B is an orthogonal matrix (as in the derivation of identities (11) and (13)). □

Combining identities (13) and (14) we obtain the identity

$$\langle(\beta ID(Bx)^{-1} + D(Bx)^{-1}A)B\nabla\rho(Bx), B\nabla\rho(Bx)\rangle = 0. \tag{15}$$

Now at $x = 0$, $D(Bx)$ is diagonal and (15) reduces to

$$\langle(A + \beta I)D(Bx)^{-1}B\nabla\rho(Bx), B\nabla\rho(Bx)\rangle = \sum_{j=1}^{d}(a_j + \beta)d_{jj}^{-1}y_j^2 = 0$$

where $y = B\nabla\rho(Bx)$, an impossibility since $a_j + \beta > 0$ and $d_{jj} > 0$ for all $j = 1, \ldots, d$. □

References

[1] V. Bergelson and A. Leibman, *Polynomial extensions of van der Waerden's and Szemerédi's theorems*, J. Amer. Math. Soc., 9, No. 2 (1996), pp. 725–753.
[2] J. Bourgain, *A Szemerédi type theorem for sets of positive density in* $\mathbf{R}^k$, Israel J. Math., 54, (1986), pp. 309–316.
[3] S. Chandarana, *L^p-bounds for hypersingular integral operators along curves*, Pacific J. Math., 175, No. 2 (1996), pp. 389–415.
[4] M. Christ, *Weak type* $(1, 1)$ *bounds for rough operators*, Ann. of Math. **128**, (1988), pp. 19–42.
[5] M. Christ, A. Nagel, E. Stein, and S. Wainger, *Singular and maximal radon transforms: analysis and geometry*, Ann. of Math., 150 (1999), pp. 489–577.
[6] M. Christ and E. M. Stein, *A remark on singular Calderón-Zygmund theory*, Proc. Amer. Math. Soc. **99**, (1987), pp. 71–75.
[7] A. Erdélyi, *Asymptotic forms for Laguerre polynomials*, J. Indian Math. Soc., 24 (1960), pp. 235–250.
[8] E. B. Fabes and N. M. Rivière, *Singular integrals with mixed homogeniety*, Stud. Math., 27 (1966), pp. 19–38.
[9] C. Fefferman, *Inequalities for strongly singular convolution operators*, Acta Math., 124 (1970), pp. 9–36.
[10] ———, *L^p bounds for pseudo-differential operators*, Israel J. Math., 14 (1973), pp. 413–417.
[11] C. Fefferman and E. M. Stein, *H^p spaces of several variables*, Acta Math., 129 (1972), pp. 137–193.
[12] N. Franzikinakis and B. Kra, *Ergodic averages for independent polynomials and applications*, to appear, J. Lond. Math. Soc.
[13] H. Furstenberg, *Ergodic behavior of diagonal measures and a theorem of Szemerédi on arithmetic progressions*, J. d'Analyse Math, 71 (1977), pp. 204–256.
[14] W. T. Gowers, *A new proof of Szemerédi's theorem*, GAFA, 11 (2001), pp. 465–588.
[15] L. Hörmander, *Pseudo-differential operators and hyperbolic equations*, Proc. Symposia Pure Appl. Math., 10 (1967), pp. 138–183.
[16] A. Nagel, N. M. Rivière, and S. Wainger, *On Hilbert transform along curves*, Bull. Amer. Math. Soc., 80 (1974), pp. 106–108.
[17] N. Laghi and N. Lyall, *Strongly singular Radon transforms on the Heisenberg group and folding singularities.* Preprint, 2005.
[18] N. Laghi and N. Lyall, *Strongly singular integral operators associated to different quasi-norms on the Heisenberg group.* Preprint, 2006.
[19] N. Laghi and N. Lyall, *Strongly singular integrals along curves.* Preprint, 2006.
[20] N. Laghi and N. Lyall, *Strongly singular integral operators on homogeneous groups.* in preparation, 2006.
[21] N. Lyall, *A class of strongly singular Radon transforms on the Heisenberg group.* to appear in Proc. of Edin. Math. Soc.
[22] ———, *Strongly Singular Convolution Operators on the Heisenberg group.* to appear in Trans. Amer. Math. Soc.
[23] N. Lyall and Á. Magyar, *On a theorem of Sárközy and Furstenberg.* in preparation, 2006.
[24] B. Shayya, *Nonisotropic strongly singular integral operators*, Trans. Amer. Math. Soc., 354 (2002), pp. 4893–4907.
[25] P. Sjölin, *L^p estimates for strongly singular convolution operators in* $\mathbf{R}^n$, Ark. Mat, 14 (1976), pp. 157–179.
[26] E. M. Stein, *Singular integrals, harmonic functions, and differentiability properties of functions of several variables*, Proc. Symposia Pure Appl. Math., 10 (1967), pp. 316–335.
[27] E. M. Stein and S. Wainger, *Problems in harmonic analysis related to curvature*, Bull. Amer. Math. Soc., 84 (1978), pp. 1239–1295.
[28] A. Seeger and T. Tao, *Sharp Lorentz space estimates for rough operators*, Math. Ann., 320 (2001), pp. 381–415.
[29] A. Seeger, T. Tao and J. R. Wright, Singular maximal functions and Radon transforms near L^1, Amer. J. Math., **126**, (2004), pp. 607–647.
[30] A. Sárzözy, *On difference sets of integers III*, Acta Arith., 27 (1975), pp. 299–345.
[31] E. Szemerédi, *On sets of integers containing no k elements in arithmetic progression*, Acta Math. Acad. Sci. Hungar., 31 (1978), pp. 125–149.
[32] S. Wainger, *Special Trigonometric Series in k Dimensions,* Memoirs of the AMS ***59***, American Math. Soc., 1965.
[33] M. Zielinski, *Highly Oscillatory Integrals along Curves.* Ph.D. Thesis, University of Wisconsin–Madison, 1985.