Further Equivalences (continuation of Lecture 9)

Theorem 3: $\frac{1}{x}\sum_{n\le x}\mu(n)\to 0 \iff \sum_{n=1}^{\infty}\frac{\mu(n)}{n}=0$.

Before giving the proof of this result we present a useful lemma.

Lemma 1: If $\sum_{n=1}^{\infty}\frac{a_n}{n}$ converges, then $\frac{1}{x}\sum_{n=1}^{x}a_n\to 0$.

Proof: Let $\varepsilon>0$ and choose y such that for all $x\ge y$

$$\Big|\sum_{y<n\le x}\frac{a_n}{n}\Big|\le\varepsilon$$

It follows by partial summation that

$$\frac{1}{x}\sum_{y<n\le x}a_n=\frac{1}{x}\sum_{y<n\le x}\frac{a_n}{n}n=\sum_{y<n\le x}\frac{a_n}{n}-\frac{1}{x}\int_y^x\Big(\sum_{y<n\le t}a_n/n\Big)dt$$

$$\le 2\varepsilon.$$

The result follows by writing

$$\frac{1}{x}\sum_{n\le x}a_n=\frac{1}{x}\sum_{n\le y}a_n+\frac{1}{x}\sum_{y<n\le x}a_n$$

and letting $x\to\infty$. $\square$

Proof of Theorem 3:

($\Leftarrow$): Follows immediately from Lemma 1.

$\underline{M(x)=o(x) \Rightarrow \sum_{n=1} \frac{\mu(n)}{n}=0}$: Since $1*\mu=\delta$ it follows that

$$\sum_{d\le x}\mu(d)\left[\frac{x}{d}\right]=\sum_{n\le x}1*\mu(n)=1 \quad \text{whenever } x\ge 1.$$

Consequently

$$\sum_{d\le x}\frac{\mu(d)}{d}=\frac{1}{x}+\frac{1}{x}\sum_{d\le x}\mu(d)\overset{\frac{x}{d}-\left[\frac{x}{d}\right]}{\overset{\|}{\left\{\frac{x}{d}\right\}}}$$

and it suffice to show that $\frac{1}{x}\sum_{d\le x}\mu(d)\left\{\frac{x}{d}\right\}\to 0$ as $x\to\infty$.

By Abel summation, for any $y\in\mathbb{N}$

$$\sum_{y<d\le x}\mu(d)\left\{\frac{x}{d}\right\}=M(x)\left\{\frac{x}{x+1}\right\}-M(y)\left\{\frac{x}{y+1}\right\}+\sum_{y<d\le x}M(d)\left(\left\{\frac{x}{d}\right\}-\left\{\frac{x}{d+1}\right\}\right).$$

Since $\frac{x}{d}$ monotonically arranged in the interval $[1, x/y]$ it follows that

$$\left|\sum_{y<d\le x}M(d)\left(\left\{\frac{x}{d}\right\}-\left\{\frac{x}{d+1}\right\}\right)\right|\le \max_{y<d\le x}|M(d)|\,\frac{x}{y}$$

and hence that

$$\left|\frac{1}{x}\sum_{y<d\le x}\mu(d)\left\{\frac{x}{d}\right\}\right|\le\frac{|M(x)|}{x}+\frac{y}{x}\frac{|M(y)|}{y}+\max_{y<d\le x}\frac{|M(d)|}{d}\,\frac{x}{y} \qquad \underline{\text{for any } \varepsilon>0}$$

for any y. This can be made arb. small by setting $y=\varepsilon x$ and letting $x\to\infty$, while trivially $\left|\frac{1}{x}\sum_{d\le\varepsilon x}\mu(d)\left\{\frac{x}{d}\right\}\right|\le\frac{1}{x}\varepsilon x\le\varepsilon$.

Since $\varepsilon>0$ can be arbitrary, this completes the proof. $\square$

Theorem 4: $\psi(x) \sim x \iff \sum_{n \le x} \frac{\Lambda(n)}{n} = \log x - \gamma + o(1)$.

Proof: We will make use of Lemma 1, and Theorems 2 & 3.

($\Leftarrow$): Follow immediately from Lemma 1, we omit the details.

($\Rightarrow$): We show that $\sum_{n=1}^{\infty} \frac{\mu(n)}{n} = 0 \Rightarrow \sum_{n \le x} \frac{\Lambda(n)}{n} = \log x - \gamma + o(1)$.

We again use the fact that

$$\Lambda - 1 = \underbrace{(\log - \tau + 2\gamma\, 1)}_{=:\, f} * \mu - 2\gamma\delta$$

$$\Rightarrow \sum_{n \le x} \frac{\Lambda(n) - 1}{n} = \sum_{dm \le x} \frac{\mu(d)}{d} \frac{f(m)}{m} - 2\gamma$$

Since $\sum_{n \le x} \frac{\Lambda(n) - 1}{n} = \sum_{n \le x} \frac{\Lambda(n)}{X} - \log x - \gamma + O\left(\frac{1}{x}\right)$ it suffices to show

$$\sum_{dm \le x} \frac{\mu(d)}{d} \frac{f(m)}{m} \longrightarrow 0 \quad \text{as } x \to \infty.$$

Applying the hyperbola method we see that for any $1 \le y \le x$,

$$\sum_{dm \le x} \frac{\mu(d)}{d} \frac{f(m)}{m} = \sum_{d \le x/y} \frac{\mu(d)}{d} \sum_{m \le x/d} \frac{f(m)}{m} + \underbrace{\sum_{m \le y} \frac{f(m)}{m} \sum_{d \le x/m} \frac{\mu(d)}{d} - \sum_{m \le y} \frac{f(m)}{m} \sum_{d \le x/y} \frac{\mu(d)}{d}}_{\text{For fixed } y \text{, this } \to 0 \text{ as } x \to \infty}.$$

Consequently, for any fixed y,

$$\limsup_{x\to\infty} \sum_{dm\le x} \frac{\mu(d)}{d}\frac{f(m)}{m} \le \limsup_{x\to\infty} \sum_{d\le x/y} \frac{\mu(d)}{d} \sum_{m\le x/d} \frac{f(m)}{m}.$$

Exercise ②: Use summation by parts to show that $\exists\ C>0$ s.t.

$$\sum_{m\le z} \frac{f(m)}{m} = C + O\left(\frac{1}{\sqrt{z}}\right), \quad z\ge 1.$$

Hint: Recall that $\Delta(x) := \sum_{m\le z} f(m) \ll \sqrt{z}$.

Using this exercise it follows that

$$\sum_{d\le x/y} \frac{\mu(d)}{d} \sum_{m\le x/d} \frac{f(m)}{m} = C\sum_{d\le x/y}\frac{\mu(d)}{d} + O\left(\frac{1}{\sqrt{y}}\right)$$

and hence that for any fixed y

$$\limsup_{x\to\infty} \left| \sum_{dm\le x} \frac{\mu(d)}{d}\frac{f(m)}{m} \right| \ll \frac{1}{\sqrt{y}}.$$

Since y can be chosen arbitrarily large it follows that

$$\sum_{n\le x} \frac{\Lambda(n)-1}{n} = -2\gamma + o(1) \qquad \square$$

First show $\lambda(n) = \sum_{d^2|n} \mu(n/d^2)$

Exercise ③: Liouville's λ function is defined as the completely multiplicative function with $\lambda(p)=-1$. Prove that $M(x)=o(x) \Leftrightarrow \sum_{n\le x}\lambda(n) = o(x)$.

Hint: Show that $\sum_{n\le x}\lambda(n) \overset{?}{=} \sum_{d\le\sqrt{x}} M(x/d^2)$ & $M(x) = \sum_{d\le\sqrt{x}}\mu(d)\left(\sum_{n\le x/d^2}\lambda(n)\right)$.