The Bochner–Riesz Conjecture

Convergence (in norm) of Partial Fourier Integrals

Recall that if $f \in S(\mathbb{R}^n)$, then from the Fourier transform

$$\hat{f}(\xi) = \int_{\mathbb{R}^n} f(x) e^{-2\pi i x \cdot \xi}\, dx$$

we can, via the Fourier Inversion Formula, recover the original function

$$f(x) = \int_{\mathbb{R}^n} \hat{f}(\xi) e^{2\pi i x \cdot \xi}\, d\xi .$$

¿ What can we say if f <u>not</u> smooth, say merely in L^p ?

We define the so-called "disc multipliers":

$$S_R f(x) = \int_{|\xi| \le R} \hat{f}(\xi) e^{2\pi i x \cdot \xi}\, d\xi$$

and consider the natural question: Does $S_R f \to f$ in L^p for all $f \in L^p$?

(It would perhaps be more "natural" to ask about pointwise convergence, but this is <u>much</u> harder...)

The answer to this question is completely known:

(i) In <u>one</u> dimension, it follows from M. Riesz's theorem for the Hilbert transform, that $S_R f \to f$ in L^p if $1 < p < \infty$.

(ii) While if $n \geq 2$, C. Fefferman showed that

$$S_R f \to f \text{ in } L^p(\mathbb{R}^n) \iff p = 2 . \; !$$

* We will discuss the proof of both these results next week *

In order to get around this lack of convergence, we define, for $\delta \geq 0$, the <u>Bochner–Riesz means</u> of f by

$$S_R^\delta f(x) = \int_{|\xi| \leq R} \left(1 - \frac{|\xi|^2}{R^2}\right)^\delta \hat{f}(\xi) e^{2\pi i x\cdot\xi} d\xi .$$

and again consider the question: Does $S_R^\delta f \to f$ in $L^p(\mathbb{R}^n)$?

<u>Exercise 1</u>: Show that $S_R^\delta g \to g$ in $L^p(\mathbb{R}^n)$, $1 \leq p < \infty$ if $g \in S(\mathbb{R}^n)$.

<u>Exercise 2</u>: Consequently, in order to show $S_R^\delta f \to f$ in $L^p(\mathbb{R}^n)$, $1 \leq p < \infty$ it suffices to show that $\|S^\delta f\|_p \leq C\|f\|_p$, $1 \leq p < \infty$. where

$$S^\delta f(x) = \int_{|\xi| \leq 1} (1 - |\xi|^2)^\delta \hat{f}(\xi) e^{2\pi i x\cdot\xi} d\xi .$$

<u>Note</u>: If $f \in S(\mathbb{R}^n)$, then

- $S^\delta f = f * K^\delta$ where $K^\delta(x) = \int_{|\xi|\leq 1} (1-|\xi|^2)^\delta e^{2\pi i x\cdot\xi} d\xi$.
- $\widehat{S^\delta f} = \hat{f} m^\delta$ where $m^\delta(\xi) = (1-|\xi|^2)_+^\delta$.

Recall a previous exercise:

$$K^\delta(x) = \frac{c_1 e^{2\pi i |x|}}{|x|^{\frac{n+1}{2}+\delta}} + \frac{c_2 e^{-2\pi i |x|}}{|x|^{\frac{n+1}{2}+\delta}} + O\left(|x|^{-\frac{n+3}{2}-\delta}\right) \text{ as } |x| \to \infty$$

* In particular, $K^\delta \in L^1(\mathbb{R}^n) \iff \delta > \frac{n-1}{2}$ (In which case S^δ will be bounded on all L^p, $1 \leq p \leq \infty$).

* Suppose $\|S^\delta f\|_p \leq C\|f\|_p$ holds for all $f \in L^p$ with $1 \leq p \leq 2$.
  Choose $f \in S(\mathbb{R}^n)$ s.t. $\hat{f} = 1$ on unit ball $\Rightarrow$ $S^\delta f(x) = K^\delta(x) \in L^p(\mathbb{R}^n) \iff \underline{\underline{p > \frac{2n}{n+1+2\delta}}}$

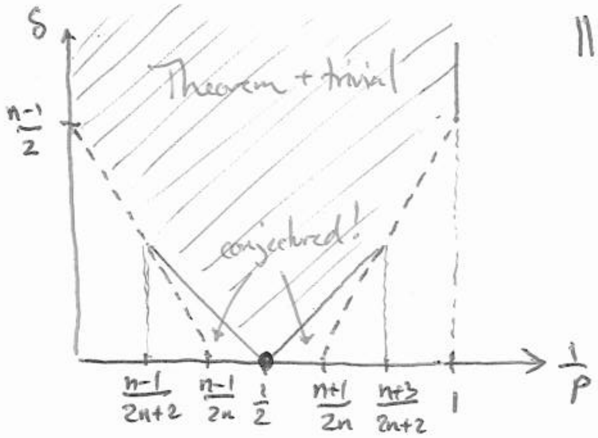
The Bochner–Riesz Conjecture Let $1 \le p \le \infty$.

If $\underbrace{\delta > \max\{n|\tfrac{1}{p}-\tfrac{1}{2}|-\tfrac{1}{2}, 0\}}$, then $\|S^\delta f\|_p \le C\|f\|_p$ for all $f \in L^p(\mathbb{R}^n)$

$\Leftrightarrow \delta > 0 \ \& \ \frac{2n}{n+1+2\delta} < p < \frac{2n}{n-1-2\delta}$ Note: These are dual exponents.

- When $p=2$ the conjecture clearly holds.

Theorem (C. Fefferman) If $1 \le p \le \frac{2n+2}{n+3}$ & $\delta > \max\{n|\frac{1}{p}-\frac{1}{2}|-\frac{1}{2}, 0\}$, then $\|S^\delta f\|_p \le C\|f\|_p$ for all $f \in L^p(\mathbb{R}^n)$.

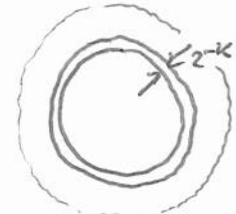


Boundedness of S^δ

Proof Let $\varphi \in C_c^\infty(\mathbb{R})$, $\operatorname{supp}\varphi \subseteq [\frac{1}{2}, 2]$ s.t. $\sum_{k=0}^{\infty} \varphi(2^k \xi) = 1$ if $|\xi| \le 1$.

Write $S^\delta f = \sum_{k=0}^{\infty} T_k^\delta f$, where $T_k^\delta f = f * K_k^\delta$

with $K_k^\delta(x) = \int_{|\xi|\le 1} \varphi(2^k(1-|\xi|^2))(1-|\xi|^2)^\delta e^{2\pi i x\cdot\xi}\, d\xi$.



Note: $K_k^\delta \in L^1$ for all small values of k.

It is straightforward to see that

$$|K_k^\delta(x)| \le C \min\left\{2^{-k(\delta+1)}, 2^{-k(\delta+1)}\left(\frac{2^k}{|x|}\right)^N\right\} \text{ for all } N.$$

size & support IBP

In particular, we have good estimates if $|x| \gtrsim 2^{k(1+\varepsilon)}$.

We now cover $\mathbb{R}^n$ with cubes of sidelength $2^{k(1+\varepsilon)}$ & let $\{\psi_j\}$. be a partition of unity relative to this cover. Define $f_j = f\psi_j$.

It follow that each individual operator

$$T_k^\delta f(x) = \sum_{j=1}^{\infty} T_k^\delta f_j(x) = \underbrace{\sum_{j=1}^{\infty} T_k^\delta f_j(x)\chi_{Q_j^*}(x)}_{\text{\underline{local} by construction}} + \underbrace{\sum_{j=1}^{\infty} T_k^\delta f_j(x)\chi_{\mathbb{R}^n\setminus Q_j^*}(x)}_{\text{\underline{Error}}}$$

(If $\operatorname{supp} f \subseteq Q$, then $\operatorname{supp} Tf \subseteq Q^*$)

* Here Q_j^* denotes the 10-fold dilate of the cube corresponding to f_j.

Let us first see that the "error operator" is bounded on L^p, $1 \le p \le \infty$:

$$\int \Big(\sum_{j=1}^{\infty} \int |K_k^\delta(x-y) f_j(y)|\, dy\, \chi_{\mathbb{R}^n\setminus Q_j^*}(x) \Big)^p dx$$

$$\le 2^{-k(\delta+1)} \int \Big(\sum_{j=1}^{\infty} \int H_k(x-y)|f_j(y)|\, dy \Big)^p dx$$

where $H_k(x) = \begin{cases} (2^k/|x|)^N & \text{if } |x| \ge 2^{k(1+\varepsilon)} \\ 0 & \text{o/w} \end{cases}$

Thus

$$\|\text{error}\|_p \le c\, 2^{-k(\delta+1)} \| H_k * |f| \|_p \le 2^{-k(\delta+1)} \|H_k\|_1 \|f\|_p$$
$$\le_c 2^{k(n-\delta-1-\varepsilon(N-n))} \|f\|_p$$

since $\|H_k\|_1 \le 2^{kn} \int_{|x|\ge 2^{k\varepsilon}} |x|^{-N} dx \le_c 2^{k(n-\varepsilon(N-n))}$

Picking N sufficiently large gives boundedness of error on L^p, $1 \le p \le \infty$.

<u>Main Term</u>: Since this operator is local

$$\Big\| \sum_{j=1}^{\infty} T_k^\delta f_j \chi_{Q_j^*} \Big\|_p \overset{(Ex)}{\leq} C \left(\sum_{j=1}^{\infty} \| T_k^\delta f_j \|^p_{L^p(Q_j^*)} \right)^{1/p} \leq C \left(\sum_{j=1}^{\infty} \|f_j\|_p^p \right)^{1/p} = C\|f\|$$

Thus, if we could show that for a <u>fixed</u> j:

$$(*) \quad \| T_k^\delta f_j \|_{L^p(Q_j^*)} \leq C \|f_j\|_{L^p}$$

we'd be done.

<u>Proof of $(*)$</u>:

$$\| T_k^\delta f_j \|_{L^p(Q_j^*)} \leq \underbrace{|Q^*|^{\frac{1}{p}-\frac{1}{2}}} \| T_k^\delta f_j \|_{L^2(Q_j^*)}.$$

$$\leq 2^{kn(1+\varepsilon)(\frac{1}{p}-\frac{1}{2})} \| T_k^\delta f_j \|_{L^2}$$

& $\| T_k^\delta f_j \|^2_{L^2} = \| \widehat{T_k^\delta f_j} \|^2_{L^2}$

$$= \int_{|\xi|\leq 1} | \varphi(2^k(1-|\xi|^2))(1-|\xi|^2)^\delta |^2 |\hat{f}_j(\xi)|^2 d\xi$$

$$= \int_0^1 | \varphi(2^k(1-r^2))(1-r^2)^\delta |^2 r^{n-1} \underbrace{\left(\int_{S^{n-1}} |\hat{f}_j(r\xi)|^2 d\sigma(\xi) \right)}_{\leq \left(\int_{\mathbb{R}^n} \left| \frac{1}{r^n} f_j(\frac{x}{r}) \right|^p dx \right)^{2/p}} dr$$

$$\leq C \left[\int_0^1 | \varphi(2^k(1-r^2))(1-r^2)^\delta |^2 dr \right] \|f_j\|_p^2 \quad \text{by Tomas-Stein}$$

$$\leq C (2^{-k\delta})^2 \underset{\uparrow \text{ support}}{2^{-k}} \|f_j\|_p^2$$

$$\Rightarrow \| T_k^\delta f_j \|_{L^p(Q_j^*)} \leq c\, 2^{kn(1+\varepsilon)(\frac{1}{p}-\frac{1}{2})} 2^{-k(\delta+\frac{1}{2})} \|f_j\|_p$$

$$\leq C 2^{-k\varepsilon'} \|f_j\|_p$$

for some $\varepsilon' > 0$ <u>if</u> $\delta > n(1+\varepsilon)(\frac{1}{p}-\frac{1}{2}) - \frac{1}{2}$ $\square$