Approximation to the Identity

The following theorem underlies many of the important applications of convolutions on $\mathbb{R}^n$. First some notation:

If φ is any m'ble function on $\mathbb{R}^n$ & $t>0$, we define

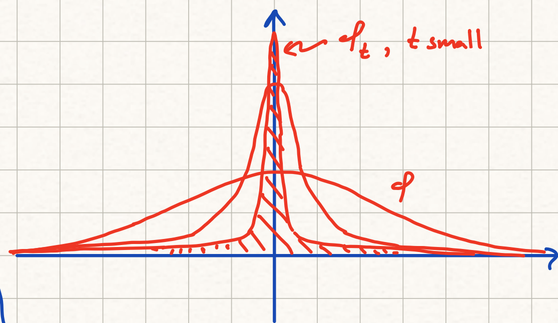
$$\varphi_t(x) = t^{-n}\varphi(t^{-1}x).$$

Note: If $\varphi \in L^1$, then $\int \varphi_t = \int \varphi \quad \forall\, t>0$

Moreover, the "mass" of φ_t becomes concentrated at the origin at $t \to 0$. In fact, we have the following:

Important Observation: For any $\eta>0$,

$$\int_{|x|\geq\eta} |\varphi_t(x)|\,dx \to 0 \quad \text{as } t \to 0$$



[This follows immediately from "small tails".]

Terminology:

If $\varphi \in L^1$ with $\int\varphi = 1$, then $\{\varphi_t\}_{t>0}$ called an approximate identity.

Theorem (Approximation to the Identity)

Suppose $\varphi \in L^1$ with $\int \varphi = 1$.

(i) If f bounded & unif conts, then $f * \varphi_t \to f$ uniformly as $t \to 0^+$.

(ii) If $f \in L^1$, then $f * \varphi_t \to f$ in L^1 as $t \to 0^+$.

<u>Remark</u>: If we further impose the additional assumption that

$$|\varphi(x)| \leq C(1+|x|)^{-n-\varepsilon} \quad \text{for some } C, \varepsilon > 0$$

then one can conclude that for every $f \in L^1$, $f * \varphi_t \to f$ a.e.
[* We do <u>not</u> prove this here.]

Proof of Theorem

<u>(i)</u>: Let $\varepsilon > 0$. We first note, using fact that $\int \varphi_t = 1$, that

$$|f * \varphi_t(x) - f(x)| \leq \int |f(x-y) - f(x)|\, |\varphi_t(y)|\, dy. \quad (*)$$

Note that $\exists\, \eta > 0$ s.t. $|f(x-y) - f(x)| < \frac{\varepsilon}{2\|\varphi\|_1}$ $\forall\, |y| < \eta$.

We now write

$$\int |f(x-y)-f(x)|\,|\varphi_t(y)|\,dy = \underbrace{\int_{|y|<\eta} |f(x-y)-f(x)|\,|\varphi_t(y)|\,dy}_{\substack{< \frac{\varepsilon}{2\|\varphi\|_1} \int |\varphi_t| \\ < \varepsilon/2.}} + \underbrace{\int_{|y|\geq\eta} |f(x-y)-f(x)|\,|\varphi_t(y)|\,dy}_{\substack{< 2M \int_{|y|\geq\eta} |\varphi_t(y)|\,dy \\ < \varepsilon/2}}$$

<u>if</u> t small enough. $\square$

<u>(ii)</u>: Let $\varepsilon > 0$. Integrating $(*)$ above gives

$$\int |f * \varphi_t(x) - f(x)|\,dx \leq \int\int |f(x-y) - f(x)|\,|\varphi_t(y)|\,dy\,dx$$

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$$= \int |\varphi_t(y)| \underbrace{\left(\int |f(x-y) - f(x)|\,dx\right)}_{(**)} dy$$

Since $\exists\ \eta > 0$ s.t. $(**) < \varepsilon / 2\|\varphi\|_1 \quad \forall\ |y| < \eta$ (by Continuity in L^1)

We can argue exactly as above to conclude that

$$\int |f * \varphi_t(x) - f(x)|\, dx < \varepsilon \quad \text{provided } t \text{ is sufficiently small.}$$

$\square$

Applications

<u>Corollary 1</u> $C_c^\infty(\mathbb{R}^n)$ is dense in $L^1(\mathbb{R}^n)$

<u>Corollary 2</u> (Weierstrass Approximation Theorem)

If $f \in C([a,b])$, then for any $\varepsilon > 0$ $\exists$ polynomial P such that

$$\sup_{x \in [a,b]} |f(x) - P(x)| < \varepsilon .$$

<u>Proof of Corollary 1</u> Let $f \in L^1$ & $\varepsilon > 0$.

We know $\exists\ g \in C_c$ s.t. $\|f - g\|_1 < \varepsilon/2$, so it suffices to show that $\exists\ h \in C_c^\infty$ s.t. $\|g - h\|_1 < \varepsilon/2$.

<u>Let $\varphi \in C_c^\infty$ with $\int \varphi = 1$.</u> We know Thm (ii) above that

$$\|g * \varphi_t - g\|_1 < \varepsilon/2 \quad \text{provided } t \text{ is suff. small.}$$

It thus suffices to establish that <u>$g * \varphi_t \in C_c^\infty \quad \forall\ t > 0$</u>.

We know, from the Corollary to Thms 1–3 in the "convolution notes" that $g * \varphi_t \in C^\infty \ \forall\ t > 0$.

To see that $g * \varphi_t$ is compactly supported note that since both g & φ_t are compactly supported $\exists\, N > 0$ s.t.

$$g(x-y) = 0 \quad \text{if} \quad |x-y| \geq N$$
$$\& \quad \varphi_t(y) = 0 \quad \text{if} \quad |y| \geq N.$$

Since $|x| \leq |x-y| + |y|$ it follows that if $|x| \geq 2N$, then either $|x-y| \geq N$ or $|y| \geq N$ $\Rightarrow$ $g * \varphi_t(x) = 0$.

$\square$

Proof of Corollary 2. Let $\varepsilon > 0$.

We may assume that $f(a) = f(b) = 0$ & set $f = 0$ on $\mathbb{R} \setminus [a,b]$.

Let $G(x) = e^{-\pi x^2}$. Since f is bounded & unif continuous, it follows from Theorem (i) that $f * G_t \to f$ uniformly on $[a,b]$.

i.e. $\exists$ fixed t such that $|f * G_t(x) - f(x)| < \varepsilon/2 \quad \forall\, x \in \mathbb{R}$.

Now $$G_t(x) = \frac{1}{t^n} e^{-\pi x^2/t^2} = \frac{1}{t^n} \sum_{k=0}^{\infty} \frac{(-1)^k \pi^k}{k!} \frac{x^{2k}}{t^{2k}}.$$

Since the convergence of this Maclaurin Series is uniform on any closed interval inside $\mathbb{R}$. Choose $M > 0$ s.t. $[-M, M] \supseteq [a,b]$, then $\exists\, K$ s.t

$$\left| G_t(x) - \underbrace{\frac{1}{t^n} \sum_{k=0}^{K} \frac{(-1)^k \pi^k}{k!} \frac{x^{2k}}{t^{2k}}}_{= Q(x)} \right| < \frac{\varepsilon}{2\|f\|_1} \quad \forall\, x \in [-2M, 2M]$$

$$\Rightarrow \quad |f * G_t(x) - f * Q(x)| \leq \int |f(x-y)|\,|G_t(y) - Q(y)|\,dy$$
$$< \varepsilon/2 \quad \forall\, x \in [-M, M].$$

This is a polynomial

$$=: P$$

& hence that $|f(x) - f * Q(x)| < \varepsilon \quad \forall\, x \in [a,b]$.

$\square$