Math 3100 Assignment 10

Uniform Convergence

Homework due date: 5:00 pm on Monday the 22nd of April 2019

1. Consider the sequence of functions
$$f_n(x) = \frac{x+n}{n}$$
   (a) Find the pointwise limit of $\{f_n\}$ on $\mathbb{R}$.
   (b) Show that $\{f_n\}$ does not converge uniformly on $\mathbb{R}$.
   (c) Show that $\{f_n\}$ does converge uniformly on $[-M, M]$ for any $M > 0$.

2. Consider the sequence of functions
$$g_n(x) = \frac{x}{1+x^n}.$$
   (a) Find the pointwise limit of $\{g_n\}$ on $[0, \infty)$.
   (b) Explain how we know that the convergence cannot be uniform on $[0, \infty)$.
   (c) Write down a smaller set over which the convergence is uniform, no proofs required.

3. (a) Consider the sequence of functions
$$F_n(x) = \frac{x}{1+nx^2}.$$
   Find the points on $\mathbb{R}$ where each $F_n(x)$ attains it maximum and minimum value. Use this to prove that $\{F_n\}$ converges uniformly on $\mathbb{R}$.
   (b) Prove that $G_n(x) = x^n(1-x)$ converges uniformly to 0 on $[0, 1]$.

4. (a) Prove that if $\sum_{n=0}^{\infty} h_n(x)$ converges uniformly on a set A, then the sequence of functions $\{h_n\}$ must converge uniformly to 0 on A.
   (b) Let
$$h(x) = \sum_{n=0}^{\infty} \frac{1}{1+n^2x}.$$
      i. Prove that the series defining h does not converge uniformly on $(0, \infty)$.
      ii. Prove that h is however a continuous function on $(0, \infty)$.

5. Let $g_n(x) = \dfrac{nx^2}{n^3 + x^3}$.
   (a) Prove that g_n converge uniformly to 0 on $[0, M]$ for any $M > 0$, but does <u>not</u> converge uniformly to 0 on $[0, \infty)$.
   (b) i. Prove that $\sum_{n=1}^{\infty} g_n$ converges uniformly on $[0, M]$ for any $M > 0$.
      ii. Does $\sum_{n=1}^{\infty} g_n$ converge uniformly on $[0, \infty)$?
      iii. Does $\sum_{n=1}^{\infty} g_n$ define a continuous function on $[0, \infty)$?

Math 3100 – Homework 10 – SOLUTIONS

1. (a) $\lim_{n\to\infty} \frac{x+n}{n} = 1 \quad \forall\, x \in \mathbb{R}$

take $x=n$

(b) $\sup_{x\in\mathbb{R}} \left| \frac{x+n}{n} - 1 \right| = \sup_{x\in\mathbb{R}} \frac{|x|}{n} \geq 1$ so $\frac{x+n}{n} \not\to 1$ uniformly on $\mathbb{R}$

(c) Let $M>0$.

$$\sup_{x\in[-M,M]} \left| \frac{x+n}{n} - 1 \right| = \sup_{x\in[-M,M]} \frac{|x|}{n} \leq \frac{M}{n} \to 0 \text{ as } n\to\infty$$

$\Rightarrow \frac{x+n}{n} \to 1$ uniformly on $[-M,M]$ for any $M>0$.

2. (a) $\lim_{n\to\infty} \frac{x}{1+x^n} = \begin{cases} x & \text{if } 0\leq x<1 \\ \frac{1}{2} & \text{if } x=1 \\ 0 & \text{if } x>1. \end{cases}$

(b) Since the limit function is <u>not</u> cont's on $[0,\infty)$, but each function $\frac{x}{1+x^n}$ <u>is</u>, the convergence <u>cannot</u> be uniform on $[0,\infty)$.

(c) $[0,1-\varepsilon] \cup [1+\varepsilon,\infty)$ for any $\varepsilon>0$.

3. (a) It follows from "calculus" that $F_n(x) = \frac{x}{1+x^2 n}$ attains its max & min at $\pm\frac{1}{\sqrt{n}}$, so $|F_n(x)| \leq \frac{1}{2\sqrt{n}}$ $\forall x\in\mathbb{R}$

Since $\frac{1}{2\sqrt{n}} \to 0$ it follows that $\sup_{x\in\mathbb{R}} |F_n(x)| \to 0$ as $n\to\infty$.

(b) "Calculus" $\Rightarrow G_n(x) = x^n(1-x)$ attains max at $x = \frac{n}{n+1}$.

$\Rightarrow \sup_{x\in[0,1]} |G_n(x)| \leq \left(\frac{n}{n+1}\right)^n \left(\frac{1}{n+1}\right) \to 0$ (since $\left(\frac{n}{n+1}\right)^n \leq 1$ & $\frac{1}{n+1}\to 0$)

4. (a) <u>Claim</u>

If $\sum_{n=0}^{\infty} h_n$ conv unif on A, then $h_n \to 0$ unif on A.

<u>Proof</u> Let $\varepsilon > 0$.

Since $\sum_{n=0}^{\infty} h_n$ satisfies the "Cauchy Condition" we know

$\exists N$ s.t. $n > m \geq N \Rightarrow \left| \sum_{k=m+1}^{n} h_k(x) \right| < \varepsilon \quad \forall x \in A$

In particular, with $n = m+1$, we have if

$$n > N \Rightarrow |h_n(x)| < \varepsilon \quad \forall x \in A$$

$$\Rightarrow \sup_{x \in A} |h_n(x)| \leq \varepsilon .$$

D.

(b) (i) Let $h_n(x) = \frac{1}{1+n^2 x}$. Since

$$\sup_{x \in (0,\infty)} |h_n(x)| \geq h_n\left(\frac{1}{n^2}\right) = \frac{1}{2} \quad \forall n$$

$\Rightarrow h_n \not\to 0$ unif on $(0,\infty)$ & hence that

$\sum h_n$ does <u>not</u> conv. unif on $(0,\infty)$. [Q4(a)]

(ii) It suffices to show that $\sum h_n$ conv. uniformly on $[a,\infty)$ for any $a > 0$. Since $\frac{1}{1+n^2 x} \leq \frac{1}{n^2 a} \; \forall x \in [a,\infty)$

& $\sum \frac{1}{n^2 a}$ converges $\underset{\text{M-test}}{\Longrightarrow}$ $\sum \frac{1}{1+n^2 x}$ conv. unif on $[a,\infty)$ for any $a > 0$.

5. (a) Let $g_n(x) = \frac{nx^2}{n^3+x^3}$.

<u>Claim 1</u> $g_n \to 0$ unif on $[0, M]$ for any $M>0$.

<u>Pf</u> Fix $M>0$. Since $\left|\frac{nx^2}{n^3+x^3}\right| \leq \frac{x^2}{n^2} \leq \frac{M^2}{n^2}$ $\forall x \in [0,M]$

$$\Rightarrow \sup_{x\in[0,M]} |g_n(x) - 0| \leq \frac{M^2}{n^2} \to 0 \text{ as } n \to \infty. \quad \square$$

<u>Claim 2</u> $g_n \not\to 0$ unif on $[0,\infty)$

<u>Pf</u> Since $\sup_{x\in[0,\infty)} \left|\frac{nx^2}{n^3+x^3}\right| \geq \frac{n^3}{n^3+n^3} = \frac{1}{2}$

take $x=n$

$$\Rightarrow \sup_{x\in[0,\infty)} |g_n(x) - 0| \not\to 0 \text{ as } n\to\infty. \quad \square$$

(b) (i) Since $\left|\frac{nx^2}{n^3+x^3}\right| \leq \frac{M^2}{n^2}$ $\forall x \in [0,M]$ & $\sum \frac{M^2}{n^2}$ conv.

$$\overset{\text{M-test}}{\Longrightarrow} \sum_{n=1}^{\infty} \frac{nx^2}{n^3+x^3} \text{ conv. unif on } [0,M] \ \forall M>0.$$

(ii) $\sum_{n=1}^{\infty} g_n$ cannot conv. unif on $[0,\infty)$ since

$g_n \not\to 0$ unif on $[0,\infty)$ [Q5(a)(ii) & Q4(a)]

(iii) Since $\sum_{n=1}^{\infty} g_n$ conv. unif on $[0,M]$ for any $M>0$

$\Rightarrow \sum_{n=1}^{\infty} g_n(x)$ defines a cont. function on $[0,M]$ for any $M>0$

$\Rightarrow \sum_{n=1}^{\infty} g_n(x)$ is cts at every $x\in [0,\infty)$.

(Since every $x\in [0,\infty)$ is contained in $[0,M]$ for some large enough M.)