Math 3100 **Spring 2019**

Exam 3

No calculators. Show your work. Give full explanations. Good luck!

1. (10 points) Let $f(x) = \begin{cases} x^2 \sin \dfrac{1}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$.

   (a) Compute $f'(x)$ for $x \neq 0$.

   (b) Use the definition of the derivative to find $f'(0)$.

   (c) Is f' continuous at 0? Give your reasoning.

   (d) Does $f''(0)$ exist? Give your reasoning.

2. (10 points) Evaluate the following infinite series

$$\text{(a)} \quad \sum_{n=1}^{\infty} \frac{(-1)^n n}{2^n} \qquad\qquad \text{(b)} \quad \sum_{n=1}^{\infty} \frac{1}{n 2^n}$$

3. (10 points)

   (a) Give an example of an infinitely differentiable function that is <u>not</u> equal to its Taylor series.

   (b) Let $f(x) = \dfrac{xe^{-x^2}}{x^2+1}$.

      i. Write down the 5th order Maclaurin polynomial for f.

      ii. Without differentiating find the value of $f^{(5)}(0)$.

4. (10 points)

   (a) Use the *Lagrangian Remainder Estimate* to determine how well the polynomial $1 + x/2$ approximates $\sqrt{1+x}$ on $[0, 1/10]$.

   (b) Obtain, by any means, a polynomial that approximates $\log(1+x^2)$ to within 10^{-3} for all $|x| \leq 1/2$.

5. (10 points)

   (a) Carefully state the definition of uniform convergence of a sequence of functions $\{f_n\}$ to a function f on a set A.

   (b) Consider the sequence of functions

$$f_n(x) = \frac{x}{1+x^n}.$$

      i. Find the pointwise limit of $\{f_n\}$ on $[0, \infty)$.

      ii. Explain how we know that the convergence cannot be uniform on $[0, \infty)$.

   (c) Prove that $f(x) = \sum_{n=1}^{\infty} \dfrac{x}{1+x^n}$ defines a continuous function on $(1, \infty)$.