(Local) Quadratic Bias $\Rightarrow$ Density Increment

Theorem

Let $\varepsilon > 0$ and $f : \mathbb{Z}_N \to [-1,1]$ (with $N \geq e^{\varepsilon^{-c}}$ for some $c>0$) satisfy $\sum_{x \in \mathbb{Z}_N} f(x) = 0$. If $\exists$ $\mathbb{Z}_N$-prog Q with $|Q| \geq N^{\varepsilon^c}$ and quadratics $\psi_1, \ldots, \psi_N$ such that

$$\frac{1}{N} \sum_{x \in \mathbb{Z}_N} \left| \frac{1}{|Q|} \sum_{h \in Q+x} f(h) e^{2\pi i \psi_x(h)/N} \right| \geq \varepsilon$$

then $\exists$ <u>genuine</u> progression $P \subseteq [1,N]$ with $|P| \geq \frac{\varepsilon}{20} |Q|^{1/400}$ such that

$$\frac{1}{|P|} \sum_{x \in P} f(x) \geq \varepsilon/20.$$

Lemma

Let Q be a $\mathbb{Z}_N$-prog of length L and $\psi(h) = ah^2 + bh$, then $\exists$ partition $\{Q_j\}_{j=1}^{J}$ of Q into $\mathbb{Z}_N$-progs such that for all $1 \leq j \leq J$:

(i) $L^{1/200} \leq |Q_j| \leq 2L^{1/200}$ (& hence $J \leq L^{199/200}$)

and (ii) $\left| e^{+2\pi i \frac{\psi(h)}{N}} - e^{2\pi i \frac{\psi(h')}{N}} \right| \leq \frac{4}{L^{1/200}}$ $\forall\, h, h' \in Q_j$.

Note: $\frac{4}{L^{1/200}} \leq \varepsilon/2$ provided $L \geq (8\varepsilon^{-1})^{200}$ Which is true for us!

Lemma $\Rightarrow$ Theorem

Fix $x \in \mathbb{Z}_N$. The lemma gives us a partition $\{Q_{x,j}\}_{j=1}^{J_x}$ of $Q+x$ into $\mathbb{Z}_N$-prog with the property that $\forall\ 1 \le j \le J_x$

$$\text{(i)} \quad |Q|^{1/200} \le |Q_{x,j}| \le 2|Q|^{1/200} \qquad \left(\& \text{ hence } \tfrac{1}{2}|Q|^{\frac{199}{200}} \le J_x \le |Q|^{\frac{199}{200}}\right)$$

$$\& \quad \text{(ii)} \quad \left| e^{2\pi i \psi_x(h)/N} - e^{2\pi i \psi_x(h')/N} \right| \le \varepsilon/2 \quad \forall\ h, h' \in Q_{j,x}.$$

<u>Note</u>: $$\sum_x \sum_{h \in Q+x} f(h) = \sum_x \sum_{j=1}^{J_x} \sum_{h \in Q_{x,j}} f(h) = |Q| \sum_n f(n) = 0.$$

Now

$$\varepsilon N |Q| \le \sum_x \left| \sum_{h \in Q+x} f(h) e^{2\pi i \psi_x(h)/N} \right|$$

$$\le \sum_x \sum_{j=1}^{J_x} \left| \sum_{h \in Q_{x,j}} f(h) \right| + \frac{\varepsilon}{2} N |Q|$$

$$\Rightarrow \sum_x \sum_{j=1}^{J_x} \left\{ \left| \sum_{h \in Q_{x,j}} f(h) \right| + \sum_{h \in Q_{x,j}} f(h) \right\} \ge \frac{\varepsilon}{2} N |Q|$$

$\Rightarrow \exists\ x, j$ such that

$$\left| \sum_{h \in Q_{x,j}} f(h) \right| + \sum_{h \in Q_{x,j}} f(h) \ge \frac{\varepsilon}{2} \frac{|Q|}{J_x} \ge \frac{\varepsilon}{2} |Q|^{1/200} \ge \frac{\varepsilon}{4} |Q_{x,j}|.$$

$$\Rightarrow \quad \frac{1}{|Q_{x,j}|} \sum_{h \in Q_{x,j}} f(h) \ge \frac{\varepsilon}{8}.$$

That is great, but how do we get genuine ?

Exercise 1: Any $\mathbb{Z}_\infty$-prog of length L can be partitioned into fewer that $3\sqrt{L}$ genuine arithmetic progressions. all of length $\leq \sqrt{L}$.

Assuming this, we proceed as follows:

Go back to the very beginning of the proof and start with the observation that the lemma & the exercises gives us a partition, for each fixed x, of $Q+x$ into genuine progressions $\{Q_{x,j}\}_{j=1}^{J_x}$ where now $J_x \leq 5|Q|^{399/400}$. Arguing as before we see that

$$\sum_x \sum_{j=1}^{J_x} \left\{ \left| \sum_{n\in Q_{x,j}} f(n) \right| + \sum_{n\in Q_{x,j}} f(n) \right\} \geq \frac{\varepsilon}{2} N|Q|.$$

* The contribution from the terms with $|Q_{x,j}| \leq \frac{\varepsilon}{20}|Q|^{1/400}$ above is less that $\frac{\varepsilon}{4}N|Q|$. Thus $\exists\, x, j$ with $\underbrace{|Q_{x,j}| \geq \frac{\varepsilon}{20}|Q|^{1/400}}_{|Q|^{1/400} \geq}$ such that

$$\left| \sum_{n\in Q_{x,j}} f(n) \right| + \sum_{n\in Q_{x,j}} f(n) \geq \frac{\varepsilon}{2}\frac{|Q|}{J_x} \geq \frac{\varepsilon}{10}|Q|^{1/400} \geq \frac{\varepsilon}{10}|Q_{x,j}|$$

~~If necessary, further divide this $Q_{x,j}$ into subprog of length $\approx |Q|^{1/400}$.~~

$$\Longrightarrow \frac{1}{|Q_{x,j}|}\sum_{n\in Q_{x,j}} f(n) \geq \frac{\varepsilon}{20} \qquad \square$$

Proof of Lemma

We will need to following (whose proof we will discuss next time)

> **Theorem** (Heilbronn Property)
>
> Given any $\alpha \in \mathbb{R}$ & $Q \in \mathbb{N}$, $\exists\ 1 \leq q \leq Q$ such that $\|\alpha q^2\| \leq \frac{1}{Q^{1/10}}$.

We will assume that our $\mathbb{Z}_N$-prog Q has length L. It follows immediately that

$$\left| e^{2\pi i \psi(h)/N} - e^{2\pi i \psi(h')/N} \right| \leq 2\pi \left\| \frac{\psi(h) - \psi(h')}{N} \right\|.$$

Let $L_1 = L^{1/200}$. By the Heilbronn property we know

$$\exists\ 1 \leq q \leq L_1^{100} \text{ s.t. } \left\|\frac{a}{N} q^2\right\| \leq L_1^{-10}.$$

We now partition Q into congruence classes mod q. & further divide each of these into sub progressions of length $\approx L_1^3$.

This gives us a partition $\{Q_i\}_{i=1}^{I}$ of Q s.t. $\forall\ 1 \leq i \leq I$

$$Q_i = \{x_i + lq : 1 \leq l \leq L_i\} \text{ where } L_1^3 \leq L_i \leq 2L_1^3.$$

Note that if $h, h' \in Q_i$, ie. $h = x_i + lq$ & $h = x_i + l'q$, then

$$\left\| \frac{\psi(h) - \psi(h')}{N} \right\| = \left\| \frac{aq^2}{N}(l^2 - l'^2) - \frac{(2ax_i + b)}{N} q\,(l - l') \right\|$$

$$\leq \underbrace{\left\|\frac{aq^2}{N}\right\| |l^2 - l'^2|}_{\substack{\leq L_1^{-10} \times 4L_1^6 \\ = 4L_1^{-4}.}} + \underbrace{\left\| \frac{(2ax_i + b)q}{N}(l - l') \right\|}_{(*)}$$

¿What to do about $(*)$?

By Dirichlet, $\exists\ 1 \le q_i \le L_1^2$ s.t. $\left\| \frac{(2ax_i+b)q}{N} q_i \right\| \le \frac{1}{L_1^2}$.

Now partition Q_i further into subprog of difference qq_i, namely

$$\{Q_{i,k}\}_{k=1}^{K} \text{ with } L_1 \le |Q_{i,k}| \le 2L_1 \text{ for all } 1 \le k \le K.$$

Hence, for any $h, h' \in Q_{i,k}$ we have

$$\left\| \frac{\psi(h) - \psi(h')}{N} \right\| \le 4L_1^{-4} + \underbrace{\underbrace{\left\| \frac{(2ax_i+b)qq_i}{N} \right\|}_{\le \frac{1}{L_1^2}} \underbrace{|m-m'|}_{\le 2L_1}}_{\le 2L_1^{-1}}$$

$$\le 4L_1^{-4} + 2L_1^{-1}$$

$$\le \underline{\underline{6L_1^{-1}}}$$

as required. $\square$