THE WEYL INEQUALITY

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The purpose of this note is to derive the following estimate, due to Weyl, for exponential sums of the form

$$S = \sum_{n=1}^{N} e^{2\pi i P(n)} \tag{1}$$

where $P(x)$ is a polynomial with real coefficients, in the special case when $P(x) = \alpha x^2$. We closely follow the fine treatments that can be found in [1] and [2].

The Weyl Inequality (for quadratic monomials). *Let $a \in \mathbf{Z}$ and $q \in \mathbf{N}$ with $(a,q) = 1$, and $N \in \mathbf{N}$ with $N \geq 2$. If $\alpha \in \mathbf{R}$ with $|\alpha - a/q| \leq q^{-2}$, then*

$$\Big|\sum_{n=1}^{N} e^{2\pi i \alpha n^2}\Big| \leq 10 N \log N (N + q + N^2/q)^{-1/2}.$$

We remark that this gives a non-trivial estimate whenever $N^\eta \leq q \leq N^{2-\varepsilon}$ for some $0 < \eta, \varepsilon < 1$. We begin with the following elementary lemma.

Lemma 1. *Let $\alpha \in \mathbf{R}$. Then for all $N \in \mathbf{N}$,*

$$\Big|\sum_{n=1}^{N} e^{2\pi i \alpha n}\Big| \leq \min\Big\{N, \frac{1}{2\|\alpha\|}\Big\}$$

where $\|\alpha\|$ is the distance from α to the nearest integer.

Proof. If $\alpha = 0$, then the sum is N. If $\alpha \neq 0$, then

$$\Big|\sum_{n=1}^{N} e^{2\pi i \alpha n}\Big| \leq \frac{|1 - e^{2\pi i \alpha N}|}{|1 - e^{2\pi i \alpha}|} \leq \frac{|\sin \pi \alpha N|}{|\sin \pi \alpha|} \leq \frac{1}{2\|\alpha\|}. \qquad \square$$

The method of *Weyl differencing* allows us to treat higher degree polynomials, the idea is simply to *square-out* the Weyl sum (1);

$$\begin{aligned}
|S|^2 &= \sum_{n=1}^{N} \sum_{m=1}^{N} e^{2\pi i [P(m) - P(n)]} \\
&= \sum_{n=1}^{N} \sum_{h=1-n}^{N-n} e^{2\pi i [P(n+h) - P(n)]} \\
&= N + \sum_{h=1}^{N-1} \sum_{n=1}^{N-h} e^{2\pi i [P(n+h) - P(n)]} + \sum_{h=1-N}^{-1} \sum_{n=1-h}^{N} e^{2\pi i [P(n+h) - P(n)]} \\
&= N + 2\,\mathrm{Re} \sum_{h=1}^{N-1} \sum_{n=1}^{N-h} e^{2\pi i [P(n+h) - P(n)]} \\
&\leq N + 2 \sum_{h=1}^{N-1} \Big| \sum_{n=1}^{N-h} e^{2\pi i [P(n+h) - P(n)]} \Big|.
\end{aligned}$$

Since $P(x+h) - P(x)$ is a polynomial of degree one less than that of $P(x)$, the possibility of inducting on the degree of P arises.

Since we are only considering Weyl sums with $P(x) = \alpha x^2$ the difference $P(x+h)-P(x) = 2xh+h^2$, and it follows from *Weyl differencing* and Lemma 1 that

$$\begin{aligned} |S|^2 &\leq N + 2\sum_{h=1}^{N-1}\Big|\sum_{n=1}^{N-h} e^{2\pi i(2\alpha h)n}\Big| \\ &\leq N + 2\sum_{h=1}^{N-1}\min\Big\{N-h, \frac{1}{\|2\alpha h\|}\Big\} \\ &\leq N + 2\sum_{h=1}^{2N}\min\Big\{N, \frac{1}{\|\alpha h\|}\Big\}. \end{aligned}$$

The Weyl inequality therefore follows immediately from the following proposition (with $H = 2N$).

Proposition 2. *Let $a \in \mathbf{Z}$ and $q \in \mathbf{N}$ with $(a,q) = 1$, $N \in \mathbf{N}$ with $N \geq 2$, and $H \in \mathbf{N}$. If $\alpha \in \mathbf{R}$ with $|\alpha - a/q| \leq q^{-2}$, then*

$$\sum_{h=1}^{H}\min\Big\{N, \frac{1}{\|\alpha h\|}\Big\} \leq 24\log N(N+q+H+HN/q).$$

The proof of this proposition follows from the lemma below together with the key observation that if $0 < |h_2 - h_1| \leq q/2$, then $\|\alpha h_2 - \alpha h_1\| \geq 1/2q$.

Lemma 3. *Let $L, M, N \in \mathbf{N}$ with $N \geq 2$ and $L \leq M$. If $\alpha_1, \dots, \alpha_L \in \mathbf{R}$ with $\|\alpha_\ell - \alpha_{\ell'}\| \geq M^{-1}$ whenever $\ell \neq \ell'$, then*

$$\sum_{\ell=1}^{L}\min\Big\{N, \frac{1}{\|\alpha_\ell\|}\Big\} \leq 6(N+M)\log N.$$

Proof of Proposition 2. Write $\alpha = a/q + \beta$. We first note that if $0 < |h_2 - h_1| \leq q/2$, then

$$\|\alpha h_2 - \alpha h_1\| \geq \|(h_2-h_1)a/q\| - \|(h_2-h_1)\beta\| \geq 1/q - 1/2q = 1/2q$$

since $(h_2 - h_1)a \neq 0 \pmod{q}$. It then follows from Lemma 3 that

$$\sum_{h=1}^{H}\min\Big\{N, \frac{1}{\|\alpha h\|}\Big\} \leq \sum_{k=0}^{\lfloor 2H/q\rfloor}\sum_{h=k\lfloor q/2\rfloor+1}^{(k+1)\lfloor q/2\rfloor}\min\Big\{N, \frac{1}{\|\alpha h\|}\Big\} \leq 6(1+2H/q)(N+2q)\log N. \qquad \square$$

Proof of Lemma 3. Without loss of generality we may assume that each $\alpha_\ell \in [-1/2, 1/2]$ and that

$$S^+ = \sum_{\substack{1\leq \ell \leq L\\ \alpha_\ell \geq 0}}\min\Big\{N, \frac{1}{\|\alpha_\ell\|}\Big\} \geq \frac{1}{2}\sum_{\ell=1}^{L}\min\Big\{N, \frac{1}{\|\alpha_\ell\|}\Big\}.$$

Relabeling the non-negative α_ℓ as $0 \leq \alpha_1 < \alpha_2 < \cdots < \alpha_K$ and noting that $\alpha_k \geq (k-1)/M$ for $k = 1, \dots, K$, we see that

$$S^+ = \sum_{k=0}^{K-1}\min\Big\{N, \frac{M}{k}\Big\} = \sum_{k=0}^{\lfloor M/N\rfloor} N + \sum_{M/N<k<K}\frac{M}{k} \leq (N+M) + 2M\log N. \qquad \square$$

References

[1] W. T. Gowers, *Additive and Combinatorial Number Theory*, www.dpmms.cam.ac.uk/∼wtg10/addnoth.notes.dvi.
[2] H. L. Montgomery, *Ten Lectures on the Interface Between Analytic Number Theory and Harmonic Analysis*, CBMS Regional Conference Series in Mathematics, 84.