Sums of Random Variables

Khintchine's Inequality: Let $1 < p < \infty$. If $X_j = \varepsilon_j a_j$ with $\{a_j\} \subseteq \mathbb{C}$ and the ε_j i.i.d. random signs ± 1 (with equal probabilities), then

$$\left(\mathbb{E} \left| \sum_{j=1}^{K} X_j \right|^p \right)^{1/p} \leq C p^{1/2} \left(\mathbb{E} \left| \sum_{j=1}^{K} X_j \right|^2 \right)^{1/2} \qquad (*)$$

$\Vert \leftarrow$ Independence

$$\mathbb{E} \left(\sum_{j=1}^{K} |X_j| \right)^{1/2} = \left(\sum_{j=1}^{K} |a_j|^2 \right)^{1/2}.$$

Remark: It follows from $(*)$ and Hölder's inequality that

$$\left(\mathbb{E} \left| \sum_{j=1}^{K} X_j \right|^2 \right)^{1/2} \leq C p^{1/2} \left(\mathbb{E} \left| \sum_{j=1}^{K} X_j \right|^{p'} \right)^{1/p'}.$$

where $\frac{1}{p} + \frac{1}{p'} = 1$.

Proof: We may assume that $\sum_{j=1}^{K} |a_j|^2 = 1$ & that $\{a_j\} \subseteq \mathbb{R}$.

Consider the expression

$$\mathbb{E}\left(e^{t \sum X_j}\right) = \mathbb{E}\left(\prod e^{t X_j}\right) \overset{\text{Independence}}{=} \prod \left(\mathbb{E} e^{t X_j}\right) = \prod \left(\frac{e^{t a_j} + e^{-t a_j}}{2} \right).$$

Since $\frac{1}{2}(e^x + e^{-x}) \leq e^{x^2/2}$ (compare Taylor series)

we conclude that

$$\mathbb{E}\left(e^{t \sum X_j}\right) \leq e^{\frac{t^2}{2} \sum a_j^2} = e^{t^2/2}.$$

It follows from Markov/Chebyshev that

$$\mathbb{P}(\Sigma X_j \geq t) = \mathbb{P}(e^{t\Sigma X_j} \geq e^{t^2}) \leq \frac{\mathbb{E}(e^{t\Sigma X_j})}{e^{t^2}} \leq e^{-t^2/2}.$$

By symmetry about origin we conclude that

$$\boxed{\mathbb{P}(|\Sigma_j X_j| \geq t) \leq 2e^{-t^2/2}} \qquad (**)$$

(Bernstein's large deviation inequality & holds given any indep. random variables $X_1, \dots, X_k$ with $\mathbb{E}X_j = 0$ & $\mathrm{var}(X_j) = 1$.)

Since

$$\mathbb{E}|\Sigma X_j|^p = p\int_0^\infty t^{p-1}\,\mathbb{P}(|\Sigma X_j| \geq t)\,dt$$

it follows from $(**)$ that

$$\begin{aligned}
\mathbb{E}|\Sigma X_j|^p &\leq 2p\int_0^\infty t^{p-1}e^{-t^2/2}\,dt \\
&= 4p\int_0^\infty \underbrace{\left(t^{p-2}e^{-t^2/4}\right)}_{\leq\, p^{p/2-1} \text{ on } (0,\infty)\ [\text{Calculus!}]} \frac{t}{2}e^{-t^2/4}\,dt \\
&\leq 4p^{p/2}\int_0^\infty \frac{t}{2}e^{-t^2/4}\,dt \\
&= 4p^{p/2}. \qquad \square
\end{aligned}$$

Marcinkiewicz - Zygmund Inequality : Let $1<p<\infty$. If $X_1, \dots, X_K$ are independent, mean-zero (complex-valued) random variables with $\mathbb{E}|X_j|^p < \infty$, then

$$\left(\mathbb{E}\left|\sum_{j=1}^{K} X_j\right|^p\right)^{1/p} \leq C p^{1/2} \left(\mathbb{E}\left(\sum_{j=1}^{K} |X_j|^2\right)^{p/2}\right)^{1/p}.$$

Proof: [In case where $X_1, \dots, X_K$ assume only finitely many values ?]

We can clearly assume that the X_j's are in fact real-valued.

- We will first assume that the X_j's are symmetric, that is

$$\mathbb{P}(X_j = a) = \mathbb{P}(X_j = -a) \quad \forall\, a \in \mathbb{R}.$$

We now partition our probability space Ω into "atoms" such that on each atom the X_j's are symmetric and assume at most 2 values.

It follows that

unique atom containing x.

$$\mathbb{E}_{x\in\Omega}\left|\sum_{j=1}^{K} X_j(x)\right|^p = \mathbb{E}_{x\in\Omega}\, \mathbb{E}_{y\in\Omega(x)}\left|\sum_{j=1}^{K} \underbrace{X_j(y)}_{\text{"}=\varepsilon_j a_j\text{"}}\right|^p$$

Khintchine

$$\leq (Cp)^{p/2}\, \mathbb{E}_{x\in\Omega}\, \mathbb{E}_{y\in\Omega(x)}\left(\sum_{j=1}^{K} |X_j(y)|^2\right)^{p/2}$$

$$= (Cp)^{p/2}\, \mathbb{E}_{x\in\Omega}\left(\sum_{j=1}^{K} |X_j(x)|^2\right)^{p/2}.$$

- Now we suppose that the variables $X_1, \dots, X_k$ are given and $Y_1, \dots, Y_k$ are such that $X_j \sim Y_j$ (identically distributed) and $X_1, \dots, X_k, Y_1, \dots, Y_k$ are independent.

Note that $X_j - Y_j$ is now a symmetric random variable and

$$\mathbb{E}_x \left| \sum_{j=1}^{k} X_j(x) \right|^p = \mathbb{E}_x \left| \sum_{j=1}^{k} X_j(x) - \underbrace{\mathbb{E}_y \left(\sum_{j=1}^{k} Y_j(y) \right)}_{=0} \right|^p$$

$$= \mathbb{E}_x \left| \mathbb{E}_y \left(\sum_{j=1}^{k} X_j(x) - Y_j(y) \right) \right|^p$$

Cauchy-Schwarz

$$\leq \mathbb{E}_x \mathbb{E}_y \left| \sum_{j=1}^{k} X_j(x) - Y_j(y) \right|^p$$

Symmetric Case

$$\leq (C_p)^{p/2} \, \mathbb{E}_x \mathbb{E}_y \left(\sum_{j=1}^{k} \underbrace{|X_j(x) - Y_j(y)|^2}_{\leq 2 \left(|X_j(x)|^2 + |Y_j(y)|^2 \right)} \right)^{p/2}$$

$$\leq (2C_p)^{p/2} \, \mathbb{E}_x \mathbb{E}_y \underbrace{\left(\sum_{j=1}^{k} |X_j(x)|^2 + \sum_{j=1}^{k} |Y_j(y)|^2 \right)^{p/2}}_{\leq 2^{p/2} \left(\left(\sum_{j=1}^{k} |X_j(x)|^2 \right)^{p/2} + \left(\sum_{j=1}^{k} |Y_j(y)|^2 \right)^{p/2} \right)}$$

$$\leq 2 (4C_p)^{p/2} \, \mathbb{E}_x \left(\sum_{j=1}^{k} |X_j(x)|^2 \right)^{p/2}$$

as required. $\square$