Math 3100 Assignment 6
More Infinite Series

Due at 5:00 pm on Friday the 1st of March 2019

1. Let $\{a_n\}_{n=1}^{\infty}$ be a sequence of non-negative terms.

   Prove that $\sum\limits_{n=1}^{\infty} a_n$ converges if and only if its sequence of partial sums is bounded.

   Be sure to prove both implications

2. Let $a_n \geq 0$ for all $n \in \mathbb{N}$.

   (a) Show that if $\lim\limits_{n\to\infty} na_n$ exists and is not equal to 0, then $\displaystyle\sum_{n=1}^{\infty} a_n$ diverges.

   (b) Show that if $\lim\limits_{n\to\infty} n^2 a_n$ exists, then $\displaystyle\sum_{n=1}^{\infty} a_n$ converges.

3. Determine which of the following series converge, and which diverge. Give reasons for your answer.

   (a) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{3^n - 1}$ (b) $\displaystyle\sum_{n=1}^{\infty} \frac{\log n}{n^2}$ (c) $\displaystyle\sum_{n=3}^{\infty} \frac{1}{\sqrt{n \log n}}$ (d) $\displaystyle\sum_{n=1}^{\infty} \frac{(1+n^2)^{1/3}}{n}$ (e) $\displaystyle\sum_{n=1}^{\infty} \frac{(1+n^2)^{1/3}}{n^2}$

4. Determine which of the following series are absolutely convergent, which are conditionally convergent, and which diverge. Give reasons for your answer.

   (a) $\displaystyle\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n\sqrt{n}}$ (b) $\displaystyle\sum_{n=1}^{\infty} (-1)^n \frac{n}{n^2+1}$ (c) $\displaystyle\sum_{n=1}^{\infty} \frac{(-3)^n n}{(n+1)^5}$ (d) $\displaystyle\sum_{n=1}^{\infty} \frac{2^{n+1}}{n(-3)^n}$ (e) $\displaystyle\sum_{n=1}^{\infty} (-1)^n \frac{n!}{(2n)!}$

Math 3100 – Homework 6 – SOLUTIONS

1. <u>Claim</u> If $a_n \geq 0$ for all $n \in \mathbb{N}$, then

$$\sum_{n=1}^{\infty} a_n \text{ converges} \iff \{s_n\} \text{ bounded}$$

where $s_n = a_1 + \cdots + a_n$.

<u>Proof</u>

First we recall that (by definition)

$$\sum_{n=1}^{\infty} a_n \text{ converges} \iff \{s_n\} \text{ converges.}$$

We are thus tasked with showing that if $a_n \geq 0 \ \forall n \in \mathbb{N}$ and $s_n = a_1 + \cdots + a_n$, then

$$\{s_n\} \text{ converges} \iff \{s_n\} \text{ bounded.}$$

- <u>($\Rightarrow$)</u>: Immediate. Every convergent sequence is bounded.
- <u>($\Leftarrow$)</u>: Since $a_n \geq 0 \ \forall n \in \mathbb{N}$ it follows that $\{s_n\}$ is increasing (since $s_{n+1} - s_n = a_{n+1} \geq 0 \ \forall n \in \mathbb{N}$). It then follows from the MCT, since we are assuming $\{s_n\}$ is also bounded, that $\{s_n\}$ converges.

$\square$

2. Let $a_n \geq 0$ $\forall n \in \mathbb{N}$.

(a) Claim

If $\lim_{n\to\infty} n a_n$ exists and is not equal to 0, then $\sum_{n=1}^{\infty} a_n$ diverges.

Proof 1 (Direct Comparison)

Since $n a_n \to L > 0$ (by order limit laws we know $L \geq 0$)

we know $\exists N$ such that $n > N$ implies $n a_n \geq \frac{L}{2}$. (taking $\varepsilon = L/2$)

Since $a_n > \left(\frac{L}{2}\right)\frac{1}{n}$ $\forall n > N$

and $\sum_{n=1}^{\infty}\left(\frac{L}{2}\right)\frac{1}{n}$ diverges, it follows from direct comparison

that $\sum_{n=1}^{\infty} a_n$ diverges also. $\square$

Proof 2 (Limit Comparison)

Since $n a_n = \frac{a_n}{\left(\frac{1}{n}\right)} \longrightarrow L > 0$

and $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges, it follows from limit comparison

that $\sum_{n=1}^{\infty} a_n$ diverges also. $\square$.

This can also be done by direct comparison

(b) Claim: If $\lim_{n\to\infty} n^2 a_n$ exists, then $\sum_{n=1}^{\infty} a_n$ converges.

Proof. Since $n^2 a_n = \frac{a_n}{\left(\frac{1}{n^2}\right)} \to L \geq 0$, and $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges

it follows from the limit comparison test that $\sum_{n=1}^{\infty} a_n$ conv.

3. (a) $\sum_{n=1}^{\infty} \frac{1}{3^n - 1}$ CONVERGES

Justification using Direct Comparison

Since $\frac{1}{3^n - 1} \leq \frac{1}{3^n - 3^n/2} = \frac{2}{3^n}$ for all $n \geq 2$

and $\sum_{n=1}^{\infty} \frac{2}{3^n}$ converges (Geom Series) it follows that

$\sum_{n=1}^{\infty} \frac{1}{3^n - 1}$ converges.

Justification using Limit Comparison

Since $\frac{\frac{1}{3^n - 1}}{\frac{1}{3^n}} = \frac{3^n}{3^n - 1} = \frac{1}{1 - \frac{1}{3^n}} \longrightarrow 1$ as $n \to \infty$

and $\sum_{n=1}^{\infty} \frac{1}{3^n}$ converges (Geometric series) it follows

that $\sum_{n=1}^{\infty} \frac{1}{3^n - 1}$ converges.

(b) $\sum_{n=1}^{\infty} \frac{\log n}{n^2}$ CONVERGES.

Justification using Direct Comparison

Since $\log n \leq n^{1/2}$ eventually

$\Rightarrow \frac{\log n}{n^2} \leq \frac{1}{n^{3/2}}$ eventually.

Since $\sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$ converges (p-series) it follows that

$\sum_{n=1}^{\infty} \frac{\log n}{n^2}$ converges.

Justification using Limit Comparison

0 is O.K if series you compare to is CONV!

Since $$\frac{\frac{\log n}{n^2}}{\frac{1}{n^{3/2}}} = \frac{\log n}{n^{1/2}} \to 0 \text{ as } n \to \infty$$

and $\sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$ converges (p-series) it follows that

$\sum_{n=1}^{\infty} \frac{\log n}{n^2}$ converges.

(c) $\sum_{n=3}^{\infty} \frac{1}{\sqrt{n \log n}}$ DIVERGES.

Justification using Direct Comparison

Since $\frac{1}{\sqrt{n}\sqrt{\log n}} \geq \frac{1}{n}$ eventually (since $\log n \leq n$ eventually,

and $\sum_{n=3}^{\infty} \frac{1}{n}$ diverges (p-series) it follows that

$\sum_{n=3}^{\infty} \frac{1}{\sqrt{n}\sqrt{\log n}}$ diverges.

∞ is O.K. if series you compare to is diverging

Justification using Limit Comparison

Since $$\frac{\frac{1}{\sqrt{n}\sqrt{\log n}}}{\frac{1}{n}} = \frac{\sqrt{n}}{\sqrt{\log n}} \longrightarrow \infty \text{ as } n \to \infty$$

and $\sum_{n=3}^{\infty} \frac{1}{n}$ diverges (p-series) it follows that

$\sum_{n=3}^{\infty} \frac{1}{\sqrt{n}\sqrt{\log n}}$ diverges.

(d) $\displaystyle\sum_{n=1}^{\infty} \frac{(1+n^2)^{1/3}}{n}$ DIVERGENT

<u>Justification using Direct Comparison</u>

Since $\displaystyle\frac{(1+n^2)^{1/3}}{n} \geq \frac{n^{2/3}}{n} = \frac{1}{n^{1/3}}$, for all $n\in\mathbb{N}$ and $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^{1/3}}$ diverges.

it follows that $\displaystyle\sum_{n=1}^{\infty} \frac{(1+n^2)^{1/3}}{n}$ diverges

<u>Justification using Limit Comparison</u>

Since
$$\frac{\frac{(1+n^2)^{1/3}}{n}}{\frac{1}{n^{1/3}}} = \frac{(1+n^2)^{1/3} n^{1/3}}{n}$$
$$= \left(\frac{1+n^2}{n^2}\right)^{1/3} = \left(\frac{1}{n^2}+1\right)^{1/3} \longrightarrow 1 \text{ as } n\to\infty.$$

and $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^{1/3}}$ diverges it follows that $\displaystyle\sum_{n=1}^{\infty} \frac{(1+n^2)^{1/3}}{n}$ diverges

(e) $\displaystyle\sum_{n=1}^{\infty} \frac{(1+n^2)^{1/3}}{n^2}$ CONVERGES.

<u>Justification using Direct Comparison</u>

Since $\displaystyle\frac{(1+n^2)^{1/3}}{n^2} \leq \frac{(n^2+n^2)^{1/3}}{n^2} = 2^{1/3}\frac{1}{n^{4/3}}$ $\forall n\in\mathbb{N}$ and

$\displaystyle\sum_{n=1}^{\infty} 2^{1/3}\frac{1}{n^{4/3}}$ converges (p-series) it follows that

$\displaystyle\sum_{n=1}^{\infty} \frac{(1+n^2)^{1/3}}{n^2}$ converges.

Justification using Limit Comparison

Since
$$\frac{\frac{(1+n^2)^{1/3}}{n^2}}{\frac{1}{n^{4/3}}} = \frac{(1+n^2)^{1/3}\, n^{4/3}}{n^2}$$
$$= \frac{(1+n^2)^{1/3}(n^4)^{1/3}}{(n^6)^{1/3}}$$
$$= \left(\frac{n^4+n^6}{n^6}\right)^{1/3} = \left(\frac{1}{n^2}+1\right)^{1/3} \longrightarrow 1 \quad \text{as } n \to \infty$$

and $\sum_{n=1}^{\infty} \frac{1}{n^{4/3}}$ converges it follows that $\sum_{n=1}^{\infty} \frac{(1+n^2)^{1/3}}{n^2}$ converges.

4. (a) $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n\sqrt{n}}$ CONVERGES ABSOLUTELY

Since $\sum_{n=1}^{\infty} \left|(-1)^{n+1} \frac{1}{n\sqrt{n}}\right| = \sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$ converges (p-series).

(b) $\sum_{n=1}^{\infty} (-1)^n \frac{n}{n^2+1}$ CONVERGES CONDITIONALLY.

Since $\sum_{n=1}^{\infty} \left|(-1)^n \frac{n}{n^2+1}\right| = \sum_{n=1}^{\infty} \frac{n}{n^2+1}$ diverges

Limit Comparison

(since $\frac{n}{n^2+1} \Big/ \frac{1}{n} = \frac{n^2}{n^2+1} \longrightarrow 1$ & $\sum \frac{1}{n}$ diverges)

but $\sum_{n=1}^{\infty} (-1)^n \frac{n}{n^2+1}$ converges by Alt. Series Test

(since $\frac{n}{n^2+1} \searrow 0$).

Can you verify this?

(c) $\displaystyle\sum_{n=1}^{\infty} \frac{(-3)^n n}{(n+1)^5}$ DIVERGES.

Since $\left|\frac{(-3)^n n}{(n+1)^5}\right| = \frac{3^n n}{(n+1)^5} \not\to 0$. (actually $\to \infty$)

and hence $\frac{(-3)^n n}{(n+1)^5} \not\to 0$ as $n \to \infty$.

(d) $\displaystyle\sum_{n=1}^{\infty} \frac{2^{n+1}}{n(-3)^n}$ CONVERGES ABSOLUTELY.

Let $a_n = \frac{2^{n+1}}{n(-3)^n}$. Since

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{2^{n+2}}{(n+1)(-3)^{n+1}} \cdot \frac{n(-3)^n}{2^{n+1}}\right| = \frac{n}{n+1}\,\frac{1}{3}\,2 \to \frac{2}{3} < 1$$

it follow from the Ratio Test that $\sum_{n=1}^{\infty} |a_n|$ converges.

(e) $\displaystyle\sum_{n=1}^{\infty} (-1)^n \frac{n!}{(2n)!}$ CONVERGES ABSOLUTELY.

Let $a_n = (-1)^n \frac{n!}{(2n)!}$. Since

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{(-1)^{n+1}(n+1)!}{(2n+2)!} \cdot \frac{(2n)!}{(-1)^n n!}\right| = \frac{n+1}{(2n+2)(2n+1)} \to 0 < 1$$

it follows from the Ratio Test that $\sum_{n=1}^{\infty} |a_n|$ converges.