Math 3100 **Spring 2019**

Practice Final Exam

The questions on this exam would have equal weighting

1. (a) Give the definition of $\lim_{n\to\infty} x_n = x$ and use this definition to prove that
$$\lim_{n\to\infty} \frac{n^2-1}{2n^2+n+1} = \frac{1}{2}.$$
   (b) Let $\{x_n\}$ and $\{y_n\}$ be two sequences of real numbers with
$$\lim_{n\to\infty} x_n = 0 \text{ and } \lim_{n\to\infty} y_n = y.$$
   Using only the definition of convergence prove that $\lim_{n\to\infty} x_n y_n = 0$
   (c) Prove that if $x_n < 10$ for all $n \in \mathbb{N}$ and $\lim_{n\to\infty} x_n = x$, then $x \leq 10$.

2. (a) Carefully state the definition of the *supremum* (the least upper bound) of a set of real numbers and the *Axiom of Completeness* (the least upper bound axiom).
   (b) Let $f : [0,1] \to \mathbb{R}$ be a continuous function with $f(0) = 0$ and $f(1) = 12$ and let
$$A := \{x \in [0,1] \,:\, f(x) < 10\}.$$
   i. Prove that $\alpha := \sup(A)$ exists.
   ii. Show that there exists a sequence $\{\alpha_n\}$ in A with the property that $\lim_{n\to\infty} \alpha_n = \alpha$.
   iii. Conclude that $f(\alpha) \leq 10$.

3. (a) Carefully state the *Monotone Convergence Theorem*.
   (b) Let $\{x_n\}$ be defined recursively by $x_1 = 1$ and $x_{n+1} = \dfrac{3x_n + 2}{x_n + 2}$ for each $n \in \mathbb{N}$. Prove that $\{x_n\}$ converges and find its limit.

4. (a) Carefully state the definition of a sequence of real numbers $\{a_n\}$ being a Cauchy sequence.
   (b) Prove, using the definition given in (a), that Cauchy sequences are always bounded.
   (c) Carefully state the *Bolzano-Weierstrass Theorem* and use this to show that Cauchy sequences of real numbers are always convergent.

5. (a) Show that if $\lim_{n\to\infty} \sqrt{n}a_n = 2$, then $\sum_{n=1}^{\infty} a_n$ diverges.
   (b) Determine if the following series are absolutely convergent, conditionally convergent, or divergent. Justify your answers.

$$\text{(i)} \quad \sum_{n=1}^{\infty} \frac{(-1)^n n}{n^2+1} \qquad\qquad \text{(ii)} \quad \sum_{n=1}^{\infty} \frac{\log n}{n^{3/2}}$$

(c) Find all $x \in \mathbb{R}$ for which the following two series converge and a closed form for their sum on this domain:

$$\text{i. } \sum_{n=0}^{\infty} \frac{x^n}{3^{n+1}} \qquad\qquad \text{ii. } \sum_{n=1}^{\infty} \frac{nx^n}{3^{n+1}}$$

(d) i. Find the value of $f^{(2018)}(0)$ if $f(x) = \log(1+x)$.
ii. Find a polynomial that approximates e^x to within 10^{-3} for all $|x| \leq 1/2$.

6. (a) Let $X \subseteq \mathbb{R}$, $x_0 \in X$, and $f : X \to \mathbb{R}$.
Carefully state the ε-δ definition of what it means for f to be *continuous* at x_0.
(b) Use the definition from part (a) to prove that

$$f(x) = \frac{x^2 + 5x - 2}{x + 1}$$

is continuous at 2.
(c) Prove that if $\lim_{n\to\infty} f(x_n) = f(x_0)$ for all sequences with $\lim_{n\to\infty} x_n = x_0$, then f is continuous at x_0.

7. (a) Carefully state what it mean to say that a function $f : \mathbb{R} \to \mathbb{R}$ is differentiable at x_0 and prove that if f is differentiable at x_0, then f is continuous at x_0.
(b) Let $h(x) = \begin{cases} x^2, & x \in \mathbb{Q} \\ 0, & x \notin \mathbb{Q} \end{cases}$.
i. Prove that h is discontinuous at all $x \neq 0$.
ii. Prove that h is differentiable at $x = 0$.
iii. What can you say about the continuity of h at $x = 0$ and the differentiability of h at $x \neq 0$?
(c) Let $f : [a, b] \to \mathbb{R}$.
Prove that if f has a minimum at a point $c \in (a, b)$, and $f'(c)$ exists, then $f'(c) = 0$.

8. (a) Carefully state the definition of uniform convergence of a sequence of functions $\{f_n\}$ to a function f on a set A.
(b) Prove that if $\{f_n\}$ is a sequence of continuous functions defined on an interval $[a, b]$ which converge uniformly to a limit function f on $[a, b]$, then f must also be a continuous on $[a, b]$.
(c) Conclude from part (a) that if the power series $\sum a_n x^n$ converges for all $|x| < R$, then for any $0 < c < R$ it converges uniformly on the interval $[-c, c]$. Conclude from this that $\sum a_n x^n$ in fact defines a continuous function on $(-R, R)$.

9. (a) State and prove the Weierstrass M-test.
(b) i. Show that $\sum_{n=1}^{\infty} \frac{x}{1 + x^n}$ diverges for all $x \in (0, 1]$, but convereges if $x > 1$.
ii. Let $f(x) = \sum_{n=1}^{\infty} \frac{x}{1 + x^n}$ on $(1, \infty)$.
A. Does f define a continuous function on $(1, \infty)$?
B. Does the series defining f converge uniformly on $(1, \infty)$?