Convolutions

Let f and g be measurable functions on $\mathbb{R}^n$. The <u>convolution</u> of f and g is the function $f * g$ defined by

$$f * g(x) = \int f(x-y)\,g(y)\,dy$$

for all x such that the integral exists.

Remarks

- Various conditions can be imposed on f & g to ensure $f*g$ exists.
- <u>If</u>, for some ~~funct~~ x, the function $y \longmapsto f(x-y)g(y)$ is integrable then the function $y \longmapsto f(y)g(x-y)$ is also integrable and hence

$$f*g = g*f$$

[Change of variables $y \mapsto x-y$ is a translation followed by a reflection.

Theorem 1

(a) If $f \in L^1$ and g bounded, then $f*g$ is bounded & unif. continuous

(b) If f and g are both in L^1 & bounded, then $\lim_{|x|\to\infty} f*g(x) = 0$.

<u>Proof</u>: <u>Exercise</u>.

<u>Hints</u>: <u>(a)</u>: Use continuity in L^1.

<u>(b)</u>: Use $|x| \leq |x-y| + |y|$.

Theorem 2

If $f \in L^1$ and $g \in L^1$, then $f*g \in L^1$ and $\|f*g\|_1 \leq \|f\|_1 \|g\|_1$

Remark: If $f, g \geq 0$, then one in fact has equality

Proof

- $h(x,y) = f(x-y)g(y)$ measurable on $\mathbb{R}^{2n}$ (See Appendix). (& hence so is $|h(x,y)|$).

- Since
$$\int \left(\int |f(x-y)||g(y)|\,dx \right) dy = \|f\|_1 \|g\|_1 \quad \text{(See below)}$$
it follows from Fubini/Tonelli that $h \in L^1(\mathbb{R}^{2n})$ and that for a.e. $x \in \mathbb{R}^n$, $f*g(x)$ is integrable on $\mathbb{R}^n$ (and in particular exists).

- Finally we note that
$$\int |f*g(x)|\,dx \leq \int \left(\int |f(x-y)||g(y)|\,dy \right) dx$$
$$\overset{\text{Tonelli}}{=} \int \left(\int |f(x-y)||g(y)|\,dx \right) dy$$
$$\overset{(a)}{=} \int |g(y)| \left(\underbrace{\int |f(x-y)|\,dx}_{= \int |f(x)|\,dx} \right) dy = \|f\|_1 \|g\|_1$$
$\square$

Corollary (of Thms 1 & 2)

If $f \in L^1$ & $g \in L^1$ and bounded, then
$$\lim_{|x| \to \infty} f*g(x) = 0$$

Proof: Exercise.

Theorem 3

If $f \in L^1$ and g bounded & $g \in C^1$ with $\frac{\partial g}{\partial x_j}$ bounded for all $1 \le j$

then $f * g \in C^1$ and $\frac{\partial}{\partial x_j}(f*g) = f * \left(\frac{\partial}{\partial x_j} g\right)$.

Proof

Let $\{t_n\}$ be any sequence s.t. $\lim_{n\to\infty} t_n = 0$.

Since $$\left| f(y) \frac{g(x+t_n e_j - y) - g(x-y)}{t_n} \right| \le M \, |f(y)| \quad \text{(by MVT)}$$

M: bound on $\frac{\partial g}{\partial x_j}$

it follows from the DCT that

$$\frac{\partial}{\partial x_j}(f*g)(x) = \lim_{n\to\infty} \int f(y) \frac{g(x+t_n e_j - y) - g(x-y)}{t_n} \, dy$$

$$= \int f(y) \left\{ \lim_{n\to\infty} \frac{g(x+t_n e_j - y) - g(x-y)}{t_n} \right\} dy$$

$$= f * \left(\frac{\partial}{\partial x_j} g \right)(x). \qquad \square$$

Corollary

If $f \in L^1$ and $g \in C_c^\infty$, then $\underbrace{f*g \in C^\infty \text{ and } \lim_{|x|\to\infty} f*g(x) = 0}_{\text{“} f*g \in C_0^\infty \text{”}}$

Proof:
- $f*g \in C^\infty$ (Thm 3)
- $f*g(x) \to 0$ as $|x| \to \infty$ (Thm 1(b) [or Corollary to Thms 1(a) & 2]) $\square$