Bogolyubov's Lemma

Lemma (Bogolyubov) Suppose $A \subseteq \mathbb{Z}_N$ with $|A| = \delta N$.

Then $2A - 2A$ contains a Bohr set of rank at most $4/\delta^2$ and radius at least $\frac{1}{2\pi}$.

Recall: • Given $\Gamma \subseteq \mathbb{Z}_N$ and $r > 0$, we define a Bohr set $B(\Gamma, r)$ with spectrum (of frequencies) Γ and radius r as follows:

$$B(\Gamma, r) := \{ x \in \mathbb{Z}_N : \|\tfrac{x\xi}{N}\| \leq r \ \ \forall\, \xi \in \Gamma \}$$

$\uparrow$ distance to nearest integer.

[• Note: $\|\alpha\| \leq r \Rightarrow |e^{2\pi i \alpha} - 1| \leq 2\pi r \Rightarrow \cos 2\pi\alpha \geq 1 - 2\pi r$.]

• $d := |\Gamma|$ is called the rank (or sometimes dimension) of B.

Proof of Lemma: Let $\varepsilon > 0$ be a parameter to be determined.

Let $\Gamma = \mathrm{spec}_\varepsilon(A) := \{ \xi \in \mathbb{Z}_N : |\hat{1}_A(\xi)| \geq \varepsilon\delta \}$.

Planchere($\Rightarrow |\Gamma| \leq \delta^{-1}\varepsilon^{-2}$

[Pf: $\delta = \sum_{\xi \in \mathbb{Z}_N} |\hat{1}_A(\xi)|^2 \geq \sum_{\xi \in \Gamma}' |\hat{1}_A(\xi)|^2 \geq |\Gamma|\, \delta^2 \varepsilon^2$.]

* Even more can be said about the structure of $\mathrm{spec}_\varepsilon(A)$ — See Chang's Structure Theorem.

Consider

$$f(x) = 1_A * 1_A * 1_{-A} * 1_{-A}(x)$$

Note that

(i) $\mathrm{supp}(f) \subseteq 2A - 2A$

& (ii) $\hat{f}(\xi) = |\hat{1}_A(\xi)|^4$ (since $\hat{1}_{-A}(\xi) = \overline{\hat{1}_A(\xi)}$)

Fourier inversion (and the fact that f is real) implies

$$f(x) = \sum_{\xi} |\hat{1}_A(\xi)|^4 e^{2\pi i x \xi / N} = \sum_{\xi} |\hat{1}_A(\xi)|^4 \cos(2\pi x \xi / N)$$

$$= \underbrace{\delta^4}_{\xi = 0 \text{ term}} + \underbrace{\sum_{\xi \in \Gamma \setminus \{0\}} |\hat{1}_A(\xi)|^4 \cos(2\pi x \xi / N)}_{?} + \underbrace{\sum_{\xi \notin \Gamma} |\hat{1}_A(\xi)|^4 \cos(2\pi x \xi / N)}_{|\text{—"—}| \leq (\varepsilon^2 \delta^2) \cdot \delta}$$

Plancherel

$\leq \delta^4 / 4$ ($\forall x$)

if $\varepsilon = \frac{\delta^{1/2}}{2}$

Since when $x \in B(\Gamma, \frac{1}{2\pi})$, we have $\cos(2\pi x \xi / N) \geq 0$

it follows that

$$f(x) \geq \delta^4 - \frac{\delta^4}{4} = \frac{3\delta^4}{4} > 0 \quad \forall x \in B(\Gamma, \tfrac{1}{2\pi})$$

and hence that $B(\Gamma, \frac{1}{2\pi}) \subseteq 2A - 2A$, since $\mathrm{supp}(f) \subseteq 2A - 2A$.

$\square$