Math 3100
Sample Exam 2 – Version 2

No calculators. Show your work. Give full explanations. Good luck!

1. (15 points)

   (a) Carefully state the definition of what it means to say that $\sum_{n=1}^{\infty} a_n$ is convergent.

   (b) Use this definition to prove that if $\sum_{n=1}^{\infty} a_n$ is convergent and $\sum_{n=1}^{\infty} b_n$ is divergent (not convergent), then $\sum_{n=1}^{\infty} (a_n + b_n)$ is divergent.

   (c) Prove that if $0 \leq a_n \leq c_n$ and $\sum_{n=1}^{\infty} c_n$ is convergent, then $\sum_{n=1}^{\infty} a_n$ is also convergent.

2. (15 points)

   (a) Show that if $\lim_{n\to\infty} \sqrt{n} a_n = 2$, then $\sum_{n=1}^{\infty} a_n$ diverges.

   (b) Find all $x \in \mathbb{R}$ for which $\sum_{n=1}^{\infty} \frac{x^n}{n3^{n+1}}$ converges.

   (c) Find a sequence $\{a_n\}$ so that $\sum_{n=1}^{\infty} a_n x^n = \frac{4x}{2-x}$ for all $|x| < 2$.

3. (15 points)

   (a) Carefully state the ε-δ definition of what it means for a function $f : \mathbb{R} \to \mathbb{R}$ to be *continuous* at a point $x_0 \in \mathbb{R}$. Use this to show that $f(x) = \frac{2x+1}{x^2+1}$ is continuous at $x_0 = 2$.

   (b) Prove that if a function $f : \mathbb{R} \to \mathbb{R}$ is *continuous* at x_0, then $\lim_{n\to\infty} f(x_n) = f(x_0)$ for all sequences $\{x_n\}$ with $\lim_{n\to\infty} x_n = x_0$. Use this to show that

$$g(x) = \begin{cases} \cos(x^{-2}) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

   is not continuous at $x_0 = 0$.

4. (5 points) Give examples of the following, no proofs are required:

   (a) A function that is continuous at 0 and discontinuous on $\mathbb{R} \setminus \{0\}$.

   (b) A series with bounded partial sums that is divergent.

   (c) Bonus Points:

   A sequence $\{b_n\}$ with $0 \leq b_n \leq \frac{1}{n}$ for each $n \in \mathbb{N}$, but for which $\sum_{n=1}^{\infty} (-1)^{n+1} b_n$ diverges.

Math 3100 – Sample Exam 2 (Version 2) – SOLUTIONS

1. (a) $\sum_{n=1}^{\infty} a_n$ converges $\Leftrightarrow$ $\lim_{n\to\infty} (a_1+a_2+\cdots+a_n)$ exists.

(b) Claim

If $\sum_{n=1}^{\infty} a_n$ convs & $\sum_{n=1}^{\infty} b_n$ divs, then $\sum_{n=1}^{\infty} (a_n+b_n)$ diverges.

Proof

Since $\sum_{n=1}^{\infty} a_n$ conv we know $\lim_{n\to\infty} (a_1+\cdots+a_n)$ exists.

Since $\sum_{n=1}^{\infty} b_n$ div we know $\lim_{n\to\infty} (b_1+\cdots+b_n)$ does not exist.

If $\sum_{n=1}^{\infty} (a_n+b_n)$ converged this would mean that

$$\lim_{n\to\infty} \left((a_1+b_1)+\cdots+(a_n+b_n)\right) \text{ exists.}$$

Since $b_1+\cdots+b_n = \underbrace{\left((a_1+b_1)+\cdots+(a_n+b_n)\right)}_{\text{convergent sequence}} - \underbrace{(a_1+\cdots+a_n)}_{\text{convergent sequence}}$

this would contradict $\sum_{n=1}^{\infty} a_n$ conv & $\sum_{n=1}^{\infty} b_n$ div. □.

(c) Claim (Direct Comparison Test).

If $0 \le a_n \le c_n$ & $\sum_{n=1}^{\infty} c_n$ convergent, then $\sum_{n=1}^{\infty} a_n$ converges

Proof Let $L = \lim_{n\to\infty} (c_1+\cdots+c_n)$. Since $0\le a_n \le c_n$ $\forall n\in\mathbb{N}$ it follows that $(a_1+\cdots+a_n) \le L$ $\forall n\in\mathbb{N}$. Since the seq of partial sums $\{a_1+\cdots+a_n\}$ is also increasing it follows from the MCT that $\lim_{n\to\infty} (a_1+\cdots+a_n)$ exists $\Leftrightarrow$ $\sum_{n=1}^{\infty} a_n$ converges.

2. (a) <u>Claim</u>

If $\lim\limits_{n\to\infty} \sqrt{n}\, a_n = 2$, then $\sum\limits_{n=1}^{\infty} a_n$ diverges.

<u>Proof</u>

Since $\sqrt{n}\, a_n = \dfrac{a_n}{\frac{1}{\sqrt{n}}} \longrightarrow 2$ and $\sum\limits_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ diverges

it follows from the "limit comparison test" that $\sum\limits_{n=1}^{\infty} a_n$ diverges $\square$

(b) <u>Claim</u> $\sum\limits_{n=1}^{\infty} \dfrac{x^n}{n\,3^{n+1}}$ converges $\iff x \in [-3, 3)$

<u>Proof</u> Let $a_n = \dfrac{x^n}{n\,3^{n+1}}$.

Since $\left|\dfrac{a_{n+1}}{a_n}\right| = \dfrac{n}{n+1}\,\dfrac{1}{3}\,|x| \longrightarrow \dfrac{1}{3}|x|$ it follows from

the "Ratio Test" that $\sum\limits_{n=1}^{\infty} a_n$ conv. abs. if $|x| < 3$

and diverges if $|x| > 3$.

If $x = 3$ then $\sum\limits_{n=1}^{\infty} a_n = \sum\limits_{n=1}^{\infty} \dfrac{1}{3n}$ which diverges.

If $x = -3$ then $\sum\limits_{n=1}^{\infty} a_n = \sum\limits_{n=1}^{\infty} (-1)^n \dfrac{1}{3n}$ which converges

by the Alt. Series Test since $\frac{1}{3n} \searrow 0$.

(c) Since $\dfrac{4x}{2-x} = 2x\,\dfrac{1}{1-\frac{x}{2}}$ & $\dfrac{1}{1-\frac{x}{2}} = \sum\limits_{n=0}^{\infty} \dfrac{x^n}{2^n} = \sum\limits_{n=1}^{\infty} \dfrac{x^{n-1}}{2^{n-1}}$ if $|x| < 2$

it follows that $\dfrac{4x}{2-x} = \sum\limits_{n=1}^{\infty} \dfrac{1}{2^{n-2}}\, x^n$ if $|x| < 2$

$= a_n$.

3. (a) (i) See Sample Exam 2 (Version 1) Q3(a)(i).

(ii) <u>Claim</u>: $f(x) = \frac{2x+1}{x^2+1}$ is continuous at $x_0 = 2$.

<u>Proof</u> Let $\varepsilon > 0$ and set $\delta = \min\{1, \varepsilon/3\}$.

since $|x-2| < \frac{\varepsilon}{3}$.

If $|x - x_0| < \delta$, then

since $x^2 \geq 0$ always.

$$\left|\frac{2x+1}{x^2+1} - 1\right| = \frac{|x|}{x^2+1}|x-2| \leq |x||x-2| < 3|x-2| < 3\left(\frac{\varepsilon}{3}\right) = \varepsilon$$

$f(2)$

since $|x-2| < 1$
$\Rightarrow |x| < 3$

$\square$

(b) (i) See Sample Exam 2 (Version 1) Q3(c) direction ($\Rightarrow$).

(ii) <u>Claim</u> $g(x) = \begin{cases} \cos(x^{-2}) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$ is discontinuous at $x_0 = 0$.

<u>Proof</u> By (i) it suffices to exhibit two sequences $\{x_n\}$ & $\{y_n\}$ with $\lim_{n\to\infty} x_n = \lim_{n\to\infty} y_n = 0$, <u>but</u> $\lim_{n\to\infty} g(x_n) \neq \lim_{n\to\infty} g(y_n)$.

Take $x_n = \frac{1}{\sqrt{2\pi n}}$ and $y_n = \frac{1}{\sqrt{2\pi n + \frac{\pi}{2}}}$ for each $n \in \mathbb{N}$.

Clearly $x_n \to 0$ and $y_n \to 0$, while $g(x_n) = 1$ & $g(y_n) = 0$ $\forall n \in \mathbb{N}$

$\square$

4. (a) $f(x) = \begin{cases} x & \text{if } x \in \mathbb{Q} \\ 0 & \text{if } x \in \mathbb{R} \setminus \mathbb{Q}. \end{cases}$

(b) $\sum_{n=1}^{\infty} (-1)^n$

(c) $b_n = \begin{cases} \frac{1}{n} & \text{if } n \text{ even} \\ 0 & \text{if } n \text{ odd.} \end{cases}$