Strongly Singular Integrals on Homogeneous Groups

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joint work with:
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These are convolution operators, formally given by

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Proposition (Wainger). *If $\beta > 0$, then as $|\xi| \to \infty$*

$$\widehat{K_{\alpha,\beta}}(\xi) = c\,|\xi|^{-a} e^{i|\xi|^b/b} + O(|\xi|^{-a-1})$$

where $$\frac{1}{-\beta} + \frac{1}{b} = 1 \quad \textit{and} \quad \frac{\alpha}{\beta} + \frac{a}{b} = \frac{d}{2}$$

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By "joining the dots" we then obtain

Theorem. *If* $1 < p < \infty$, *then*

$$T : L^p \to L^p \iff \left| \frac{1}{p} - \frac{1}{2} \right| \leq \frac{d\beta - 2\alpha}{2d\beta}$$

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Objective: Generalize these results on $\mathbb{R}^d$ and consider analogous operators on homogeneous groups

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We let

$$h = a_1 + \cdots + a_d$$

denote the homogeneous dimension of $\mathbb{H}$.

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$$\left[\text{If } n = 1, \text{ then } \; (x_1, x_2) \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = x_1 y_2 - x_2 y_1 \right]$$

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The group law is inherited from matrix multiplication, for example if $m = 3$, then

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and the mapping

$$x \mapsto (\delta x_1, \delta x_2, \delta^2 x_3)$$

is an automorphism of this group.

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Theorem 1. *On any $\mathbb{H}$, $\exists$ a quasi-norm $\rho = \rho_{\mathbb{H}}$ so that*

$$\|Tf\|_{L^2(\mathbb{H})} \leq C\|f\|_{L^2(\mathbb{H})} \quad \textit{whenever} \quad \alpha \leq d\beta/2$$

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More precisely

$$|\widehat{K_{\alpha,\beta}}(\xi)| \leq C(1 + \rho_\beta(\xi))^{\alpha - \beta}$$

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- $|\widehat{K_{\alpha,\beta}}(\xi)| \leq C \Rightarrow \alpha \leq \frac{a_2+1}{2a_2}\beta$ (Laghi and NL)

We now consider the family of quasi-norms ρ_1, where

$$\{x \in \mathbb{R}^d | \rho_1(x) = 1\}$$

is a smooth, *convex* hypersurface with *everywhere non-vanishing Gaussian curvature*.

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Theorem 2. *On (nonisotropic)* $\mathbb{R}^d$ *with* $\rho = \rho_1$ *we have*

(i) If $\alpha = 0$, *then*

$$T : H^1_\beta(\mathbb{R}^d) \to L^1(\mathbb{R}^d)$$

(ii) If $1 < p < \infty$, *then*

$$T : L^p(\mathbb{R}^d) \to L^p(\mathbb{R}^d) \Longleftrightarrow \left| \frac{1}{p} - \frac{1}{2} \right| \leq \frac{d\beta - 2\alpha}{2d\beta}$$

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$\Big[$*In particular if* $\alpha \leq d\beta/2$, *then* $T : L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)\Big]$

Strategy for L^2 estimates (Oscillatory integrals)

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It suffices to establish the estimate

$$\|T_j f\|_{L^2(\mathbb{H})} \leq C 2^{j(\alpha - d\beta/2)} \|f\|_{L^2(\mathbb{H})}$$

for the (rescaled) dyadic operators

$$T_j f(x) = 2^{j\alpha} \int_{\rho(y^{-1}\cdot x)\approx 1} \Psi(x,y) e^{i2^{j\beta}\rho(y^{-1}\cdot x)^{-\beta}} f(y)\, dy$$

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The desired dyadic estimate follows immediately if

$$\det \partial_{x_j} \partial_{y_k} \left[\rho(y^{-1} \cdot x)^{-\beta}\right] \neq 0$$

for all (x, y) in the support of Ψ.

On (nonisotropic) $\mathbb{R}^d$ matter therefore reduce to

Proposition 1. *If* $x \neq y$ *and* $\beta > 0$, *then*

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<u>Proof in this special case:</u> Let $\rho = \rho_1$. Recall that

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is defined to be a smooth, *convex* hypersurface with *everywhere non-vanishing Gaussian curvature*.

Consequently,
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with $\langle\nabla\rho, u\rangle \neq 0$ (curvature).

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This is impossible as $\langle H\rho\, v, x\rangle = \langle H\rho\, x, v\rangle = 0$.

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Corollary. *Given any* $\mathbb{H}$ *there exists* $\varepsilon > 0$ *so that if* $\rho(x) = \rho_1(\varepsilon^{-1}x)$, $x \neq y$ *and* $\beta > 0$, *then*

$$\det \partial_{x_j}\partial_{y_k}\left[\rho(y^{-1}\cdot x)^{-\beta}\right] \neq 0$$

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- Define, for any $\varepsilon > 0$, the group $\mathbb{H}_\varepsilon$ with

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- Proposition 1 $\Rightarrow$ $\exists\, \varepsilon > 0$ so that

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Corollary. *Given any $\mathbb{H}$ there exists $\varepsilon > 0$ so that if $\rho(x) = \rho_1(\varepsilon^{-1}x)$, $x \neq y$ and $\beta > 0$, then*

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Proof: Given a group $\mathbb{H}$ with

$$x \cdot y = x + y + Q(x,y)$$

- Define, for any $\varepsilon > 0$, the group $\mathbb{H}_\varepsilon$ with

$$x \cdot_\varepsilon y = x + y + \varepsilon^{-1}Q(\varepsilon x, \varepsilon y)$$

- Proposition 1 $\Rightarrow$ $\exists\, \varepsilon > 0$ so that

$$\det \partial_{x_j}\partial_{y_k} \left[\rho_1(y^{-1}\cdot_\varepsilon x)^{-\beta}\right] \neq 0$$

- $x \mapsto \varepsilon^{-1}x$ is an isomorphism from $\mathbb{H}$ to $\mathbb{H}_\varepsilon$

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$$\implies T : L^2 \to L^2 \text{ whenever } \alpha \leq (n + \tfrac{1}{2})\beta$$

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$$\left[\text{Here essentially } \rho(x) = (|x'|^4 + b^2 x_{2n+1}^2)^{1/4}\right]$$