Ruzsa's Model Lemma

Theorem (Ruzsa Model Lemma)

Let $A \subseteq \mathbb{Z}$ be finite and $k \geq 2$ be an integer. Then for any integer $N \geq |kA - kA|$ there is a subset $A' \subseteq A$ with $|A'| \geq |A|/k$ which is (Freiman) k-isomorphic to some subset of $\mathbb{Z}_N$.

Definition

Two subsets A and B of two abelian groups are said to be Freiman k-isomorphic if there is a bijection $\varphi : A \to B$ such that

$$a_1 + \cdots + a_k = a_1' + \cdots + a_k' \iff \varphi(a_1) + \cdots + \varphi(a_k) = \varphi(a_1') + \cdots + \varphi(a_k').$$

whenever $a_j, a_j' \in A$; such a map is called a k-isomorphism.

Exercise 1

(a) Suppose P is a GAP and $\varphi : P \to \varphi(P)$ is a 2-isomorphism, then $\varphi(P)$ is a GAP of the same dimension.

(b) If $A \simeq_8 B$, then $2A - 2A \simeq_2 2B - 2B$.

Proof of Theorem: Let $\xi \in [0, N]$, where $[0,N]$ is an interval of reals, that we will choose later. We define $\varphi : \mathbb{Z} \to \mathbb{Z}_N$ by

$$\varphi(a) = \lfloor \xi a \rfloor \mod N$$

where $\lfloor x \rfloor$ denote the usual "floor function" (integer part).

We shall show that if ξ is chosen appropriately, then φ is a well-defined k-isomorphism on some subset $A' \subseteq A$. The set A' will be one of the sets

$$A_j = \left\{ a \in A : \frac{j-1}{k} \leq \{\xi a\} < \frac{j}{k} \right\}, \quad j = 1, \dots, k.$$

$\{x\} := x - \lfloor x \rfloor$ (fractional part).

In fact, we claim that for a suitable choice of ξ the restriction of φ is an isomorphism on each A_j; clearly at least one will have size $\geq |A|/k$.

Our goal is thus to show that $\exists\ \xi$ such that for arb. $a_1, \dots, a_k, a_1', \dots, a_k' \in A_j$

$$\lfloor \xi a_1 \rfloor + \dots + \lfloor \xi a_k \rfloor \equiv \lfloor \xi a_1' \rfloor + \dots + \lfloor \xi a_k' \rfloor \bmod N \underset{(*)}{\Longleftrightarrow} a_1 + \dots + a_k = a_1' + \dots + a_k'.$$

To see that such a ξ exist consider,

$$\sum_{i=1}^{k} \left(\lfloor \xi a_i \rfloor - \lfloor \xi a_i' \rfloor \right) \underset{(**)}{=} \xi \sum_{i=1}^{k} (a_i - a_i') - \sum_{i=1}^{k} \left(\{\xi a_i\} - \{\xi a_i'\} \right).$$

Suppose RHS of $(*)$ holds with all $a_i, a_i' \in A_j$, then the absolute value of the last sum above is < 1. It follows that the sum on the left above must $= 0$, since it is an integer < 1. Thus the LHS of $(*)$ holds, even with equality as integers, regardless of the choice of ξ.

Suppose LHS of $(*)$ holds. The LHS of $(**)$ is thus a multiple of N and the RHS is of the form $\xi t + \delta$, where $t \in kA - kA$ and $|\delta| < 1$. We want our choice of ξ to force $t = 0$. It thus suffices to pick $\xi \in [0, N]$ s.t.

$$\xi \notin \frac{1}{t} \cdot \left(dN + (-1, 1) \right)$$

for any $d \in \mathbb{Z}$ and non-zero integer $t \in kA - kA$.

Now for a fixed t, this condition excludes at most $|t|+1$ intervals from $[0,N]$ (corresponding to $d \in \{0,\dots,t\}$), the lengths of which sum to 2.

The number of t's that need be considered is $\frac{|kA-kA|-1}{2}$, since we are omitting $t=0$ and t and $-t$ give rise to the same set of excluded intervals.

The total length of the excluded intervals is thus $|kA-kA|-1$, and since $N \geq |kA-kA|$ this means that there is a $\xi \in [0,N]$ outside of these intervals, as required. $\square$