Math 4900/6900 **Spring 2007**

Exam 1 - Fourier Series

Instructions

4900 students: Answer any three of the following four problems.
6900 students: Answer all four of the following four problems.
Carefully state any results from class which you use in your arguments.

1. We say that a function f is *Hölder continuous* of order $\alpha > 0$ if

$$|f(x+h) - f(x)| \leq C|h|^\alpha$$

for some constant $C > 0$ and all x and h. Let f be a 2π-periodic Riemann integrable function.

(a) Show that if f is Hölder continuous of order $\alpha > 0$ as defined above, then

$$|\widehat{f}(n)| \leq \Big(\frac{C\pi^\alpha}{2}\Big)|n|^{-\alpha}$$

for all $n \neq 0$.
Hint: Show that

$$\widehat{f}(n) = -\frac{1}{2\pi}\int_{-\pi}^{\pi} f(x+\pi/n)e^{-inx}\,dx.$$

(b) Show that every continuously differentiable 2π-periodic function is Hölder continuous of order $\alpha = 1$. What can you say about a function f which is Hölder continuous of order $\alpha > 1$?

Functions which are Hölder continuous of order $\alpha = 1$ are sometimes called Lipschitz continuous. Thus all continuously differentiable functions are Lipschitz continuous.

(c) Show that if f is continuously differentiable, then in fact $\widehat{f}(n) = o(|n|^{-1})$, that is

$$n\widehat{f}(n) \to 0 \quad \text{as} \quad |n| \to \infty.$$

Hint: Use the identity in problem 3(a) below, namely that

$$\widehat{f'}(n) = in\widehat{f}(n)$$

for all $n \in \mathbb{Z}$.
(4900 students should prove this identity if not attempting problem 3).

2. A function f on the circle is said to be *Wiener* if it is continuous and its Fourier series converges absolutely.

   (a) Give examples that show that this is a stronger notion than continuity, but weaker than the notion of being continuously differentiable.

   (b) Prove that if f is a Wiener function, then the partial sums of the Fourier series of f in fact converge uniformly to f.

   You may assume the uniqueness of Fourier series: If f is continuous on the circle and $\widehat{f}(n) = 0$ for all $n \in \mathbb{Z}$, then $f = 0$.

   (c) Prove that if g and h are Riemann integrable functions on the circle, then $f = g * h$ is actually a Wiener function.

   You may assume (as you proved it in your homework) that f is continuous.

3. Let f be a continuously differentiable function which is 2π-periodic.

   (a) Prove that
   $$\widehat{f'}(n) = in\widehat{f}(n)$$
   for all $n \in \mathbb{Z}$. Be careful to treat both $n \neq 0$ and $n = 0$.

   (b) *Wirtinger's inequality:* Show that if $\int_0^{2\pi} f(x)\,dx = 0$, then
   $$\int_0^{2\pi} |f(x)|^2\,dx \leq \int_0^{2\pi} |f'(x)|^2\,dx$$
   with equality if and only if $f(x) = A\cos x + B\sin x$.

   Wirtinger's inequality is in fact equivalent to a stronger version of the isoperimetric inequality where Γ not necessarily assumed to be simple, see Exercise 4.4.

4. (a) Show that the Nth Cesàro mean of the Fourier series of f can be expressed as
   $$\sigma_N(f)(x) = \sum_{|n|\leq N} \left(1 - \frac{|n|}{N}\right)\widehat{f}(n)e^{inx}.$$

   (b) State Fejér's theorem and the periodic analogue of the Weierstrass approximation theorem (a corollary of Fejér's theorem of course).

   (c) *Hausdorff's moment problem:* Let $[a, b]$ be an interval and let $f,\ g : [a, b] \to \mathbb{C}$ be continuous. Prove that if
   $$\int_a^b x^k f(x)\,dx = \int_a^b x^k g(x)\,dx$$
   for all integers $k \geq 0$, then $f(x) = g(x)$ for all $x \in [a, b]$.

   Hint: Set $h = f - g$ and use the *Weierstrass approximation theorem to approximate $\overline{h(x)}$ by a sequence of polynomials.*