Dual Spaces

Linear Functionals

- A linear functional on a vector space X over $\mathbb{C}$ is a mapping

$$L: X \to \mathbb{C}$$

which satisfies $L(ax+by) = aL(x) + bL(y)$ $\forall a,b \in \mathbb{C}$ and $x,y \in X$.

Theorem 1: Let L be a linear functional on a normed vector space $(X, \|$
The following are all equivalent:

(i) L continuous

(ii) L continuous at 0

"L bounded"

(iii) $\exists\, C \geq 0$ s.t. $|L(x)| \leq C\|x\|$ $\forall x \in X$.

Proof:

(i) $\Rightarrow$ (ii): Immediate.

(ii) $\Rightarrow$ (iii): $\exists\, \delta > 0$ s.t. $\|x\| < \delta \Rightarrow |L(x)| \leq 1$

Hence $|L(x)| = \left|L\left(\frac{\|x\|}{\delta} \cdot \frac{\delta}{\|x\|} x\right)\right|$

$$= \delta^{-1}\|x\| \left|L\left(\tfrac{\delta}{\|x\|} x\right)\right| \leq \underbrace{\delta^{-1}}_{\text{"}C\text{"}} \|x\|$$

(iii) $\Rightarrow$ (i): Since $|L(x-y)| \leq C\|x-y\|$

$$\Rightarrow |L(x) - L(y)| < \varepsilon \text{ whenever } \|x-y\| < C^{-1}\varepsilon.$$

- If X is a normed vector space over $\mathbb{C}$, then the space of all continuous linear functionals on X is called the dual space of X and is denoted by X^*.

Easy Observations:

① X^* is itself a vector space over $\mathbb{C}$.

② the function $L \longmapsto \|L\|_{X^*}$ defined by

$$\|L\|_{X^*} := \sup_{\substack{x \in X \\ \|x\|=1}} |L(x)|$$

is a norm on X^* (called the operator norm).

Theorem 2: X^* is in fact a Banach space.

Proof: We only need to check that X^* is complete.

Let $\{L_n\}$ be a Cauchy sequence in X^*. It follows that for each $x \in X$, $\{L_n(x)\}$ is Cauchy in $\mathbb{C}$ and hence converges to a limit which we denote by $L(x)$. It is clear that L is a linear functional, but less clear that L is continuous.

So it suffice to establish that L is continuous and that

$$\lim_{n\to\infty} \| L_n - L \|_{X^*} = 0 .$$

Let $\varepsilon > 0$. We know (since $\{L_n\}$ Cauchy in X^*) that $\exists\, N$ such that

$$n, m \geq N \Rightarrow \| L_n - L_m \| < \varepsilon$$

$$\Rightarrow | L_n(x) - L_m(x) | < \varepsilon \quad \forall\, x \in X \text{ with } \|x\| = 1 .$$

Letting $m \to \infty$ we see that

$$n \geq N \Rightarrow | L_n(x) - L(x) | < \varepsilon \quad \forall\, x \in X \text{ with } \|x\| = 1 .$$

$$\Rightarrow \| L_n - L \|_{X^*} < \varepsilon . \quad \text{as required.}$$

Finally we note that for any $x \in X$,

$$|L(x)| \leq \underbrace{|L_N(x)|}_{\leq C\|x\|} + \underbrace{|L_N(x) - L(x)|}_{< \varepsilon \|x\|} \leq (C + \varepsilon) \|x\| .$$

(since L_N continuous, by Theorem 1)

It follows from Theorem 1 that L is continuous. $\square$

In general, given a Banach space X it is interesting/useful to be able to describe its dual X^*.