DISTANCES IN DENSE SUBSETS OF $\mathbb{Z}^d$

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ABSTRACT. In [2] Katznelson and Weiss establish that all sufficiently large distances can always be attained between pairs of points from any given measurable subset of $\mathbb{R}^2$ of positive upper (Banach) density. A second proof of this result, as well as a stronger "pinned variant", was given by Bourgain in [1] using Fourier analytic methods. In [5] the second author adapted Bourgain's Fourier analytic approach to established a result analogous to that of Katznelson and Weiss for subsets $\mathbb{Z}^d$ provided $d \geq 5$. We present a new direct proof of this discrete distance set result and, using appropriate discrete spherical maximal function theorems, ultimately establish the natural "pinned variant".

1. INTRODUCTION

1.1. Existing results. A result of Katznelson and Weiss [2] states that all sufficiently large distances can always be attained between pairs of points from any given measurable subset of $\mathbb{R}^2$ of positive upper (Banach) density. Specifically, if A is a measurable subset of $\mathbb{R}^2$ of positive upper Banach density, they established the existence of a threshold $\lambda_0 = \lambda_0(A)$ such that the distance set

$$\operatorname{dist}(A) = \{|x - y| \,:\, x, y \in A\} \supseteq [\lambda_0, \infty).$$

Recall that the *upper Banach density* $\delta^*(A)$ of a set $A \subseteq \mathbb{R}^d$ is defined by

$$\delta^*(A) := \lim_{N\to\infty} \sup_{t\in\mathbb{R}^d} \frac{|A \cap (t + Q_N)|}{|Q_N|},$$

where $|\cdot|$ denotes Lebesgue measure on $\mathbb{R}^d$ and Q_N denotes the cube $[-N/2, N/2]^d$.

This result was later established using Fourier analytic methods by Bourgain in [1]. Bourgain also established a "pinned variant", namely that for any $\lambda_1 \geq \lambda_0$ there is a fixed $x \in A$ such that

$$\operatorname{dist}(A; x) = \{|x - y| \,:\, y \in A\} \supseteq [\lambda_0, \lambda_1].$$

In [5] the second author adapted Bourgain's Fourier analytic approach to established a result analogous to that of Katznelson and Weiss for subsets $\mathbb{Z}^d$, namely that if $A \subseteq \mathbb{Z}^d$ of positive upper Banach density and $d \geq 5$, then there exists $\lambda_0 = \lambda_0(A)$ and an integer q, depending on d and the density of A, such that

$$\operatorname{dist}^2(A) = \{|x - y|^2 \,:\, x, y \in A\} \supseteq [\lambda_0, \infty) \cap q^2\mathbb{Z}.$$

Recall that the *upper Banach density* $\delta^*(A)$ of a set $A \subseteq \mathbb{Z}^d$ is analogously defined by

$$\delta^*(A) := \lim_{N\to\infty} \sup_{t\in\mathbb{Z}^d} \frac{|A \cap (t + Q_N)|}{|Q_N|},$$

where, $|\cdot|$ now denotes counting measure on $\mathbb{Z}^d$ and Q_N the discrete cube $[-N/2, N/2]^d \cap \mathbb{Z}^d$.

Note that since A could fall entirely into a fixed congruence class of some integer $1 \leq r \leq \delta^*(A)^{-1/d}$ the value of q in the result above must be divisible by the least common multiple of all integers $1 \leq r \leq \delta^*(A)^{-1/d}$.

1.2. New results. We will denote, for any integer λ, the discrete sphere of radius $\sqrt{\lambda}$ by S_λ, namely

$$S_\lambda := \{x \in \mathbb{R}^d \,:\, |x|^2 = \lambda\} \cap \mathbb{Z}^d.$$

In this paper we will present a new direct proof of the following discrete distance set result from [5].

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Theorem 1 (Unpinned Distances). *Let $A \subseteq \mathbb{Z}^d$ with $d \geq 5$ and $\delta^*(A) > 0$.*

There exist $q = q(\delta^*(A))$ and $\lambda_0 = \lambda_0(A)$ such that for any integer $\lambda \geq \Lambda_0$ there exist a pair of points

$$\{x, x + x_1\} \subseteq A \quad \text{with} \quad |x_1|^2 = q^2\lambda.$$

In fact, for any $\varepsilon > 0$ there exist $q = q(\varepsilon, d)$ and $\Lambda_0 = \Lambda_0(A, \varepsilon)$ such that for any integer $\lambda \geq \Lambda_0$ one has

$$\frac{|A \cap (x + qS_\lambda)|}{|S_\lambda|} > \delta^*(A) - \varepsilon \quad \text{for some } x \in A.$$

By considering sets A of the form $\bigcup_{s\in\{1,\dots,q\}^d} A_s$ with each set A_s a "random" subset of the congruence class $s + (q\mathbb{Z})^d$ one can easily see that the second conclusion above is best possible or "ε-optimal".

The main new result of this paper is the following "pinned variant" of Theorem 1 above, in other words a discrete analogue of Bourgain's pinned distances theorem in [1].

Theorem 2 (Pinned Distances). *Let $\varepsilon > 0$ and $A \subseteq \mathbb{Z}^d$ with $d \geq 5$.*

There exist $q = q(\varepsilon, d)$ and $\Lambda_0 = \Lambda_0(A, \varepsilon)$ such that for any $\Lambda_1 \geq \Lambda_0$ there exists a fixed $x \in A$ such that

$$\frac{|A \cap (x + qS_\lambda)|}{|S_\lambda|} > \delta^*(A) - \varepsilon \quad \text{for all integers} \quad \Lambda_0 \leq \lambda \leq \Lambda_1.$$

1.3. Outline of paper.

In Section 2 we state analogues of Theorems 1 and 2 for uniformly distributed subsets of $\mathbb{Z}^d$ and reduce their proofs to that of analogous results for uniformly distributed compact subsets of $\mathbb{Z}^d$.

In Section 3 we complete the proof of Theorems 1 by proving the analogous result for uniformly distributed compact subsets of $\mathbb{Z}^d$, namely Proposition 1. To do this we introduce a norm which measures the uniformity of distribution within residue classes modulo q with respect to a scale L. We then prove that this norm controls the frequency with which certain distances appear in compact subset of $\mathbb{Z}^d$, this is analogous to the so-called von-Neumann type inequalities in additive combinatorics.

In Sections 4 and 5 we complete the proof of Theorem 2 by proving the analogous result for uniformly distributed compact subsets of $\mathbb{Z}^d$, namely Proposition 2. In Section 4 we reduce matters to the *Discrete Spherical Maximal Function Theorem* of Magyar, Stein and Wainger [6] and a closely related "mollified variant" thereof, namely Proposition 5, whose statement and proof we presented in Section 5.

2. Reduction to Uniformly Distributed Compact Subsets of $\mathbb{Z}^d$

2.1. Distances in Uniformly Distributed Subsets of $\mathbb{Z}^d$.

Definition 1 (Definition of q_η and η-uniform distribution). For any $\eta > 0$ we define

$$q_\eta := \text{lcm}\{1 \leq q \leq C\eta^{-2}\}$$

with $C > 0$ a (sufficiently) large absolute constant and $A \subseteq \mathbb{Z}^d$ to be *η-uniformly distributed (modulo q_η)* if its *relative* upper Banach density on any residue class modulo q_η never exceeds $(1 + \eta^2)$ times its density on $\mathbb{Z}^d$, namely if for all $s \in \{1, \dots, q_\eta\}^d$ one has

$$\delta^*(A \,|\, s + (q_\eta\mathbb{Z})^d) \leq (1 + \eta^2)\, \delta^*(A).$$

Theorems 1 and 2 are immediate consequences, via an easy density increment argument, of the following analogous results for uniformly distributed sets.

Theorem 3 (Theorem 1 for Uniformly Distributed Sets). *Let $\varepsilon > 0$, $0 < \eta \ll \varepsilon^2$, and $A \subseteq \mathbb{Z}^d$ with $d \geq 5$.*

If A is η-uniformly distributed, then there exist $\Lambda_0 = \Lambda_0(A, \eta)$ such that for any integer $\lambda \geq \Lambda_0$ one has

$$\frac{|A \cap (x + S_\lambda)|}{|S_\lambda|} > \delta^*(A) - \varepsilon \quad \text{for some } x \in A$$

In Theorem 3 above, and throughout the paper, we use the notation $\alpha \ll \beta$ to denote that $\alpha \leq c\beta$ for some suitably small constant $c > 0$.

Theorem 4 (Theorems 2 for Uniformly Distributed Sets)**.** *Let $\varepsilon > 0$, $0 < \eta \ll \varepsilon^3$, and $A \subseteq \mathbb{Z}^d$ with $d \geq 5$.*

If A is η-uniformly distributed, then there exist $\Lambda_0 = \Lambda_0(A, \eta)$ such that for any $\Lambda_1 \geq \Lambda_0$ there exists a fixed $x \in A$ such that

$$\frac{|A \cap (x + S_\lambda)|}{|S_\lambda|} > \delta^*(A) - \varepsilon \quad \textit{for all integers} \quad \Lambda_0 \leq \lambda \leq \Lambda_1.$$

2.2. **Compact variants of Theorems 3 and 4.** We shall now show that Theorems 3 and 4 can in turn can be directly deduced from analogous *compact* variants, namely Corollary 1 and Proposition 2 below.

In what follows we shall use 1_B to denote the characteristic function of any $B \subseteq \mathbb{Z}^d$ and define

$$\sigma_\lambda = |S_\lambda|^{-1} 1_{S_\lambda}.$$

First we introduce a second related notion of uniformity.

Definition 2 (Definition of (η, L)-uniform distribution)**.** Let $\eta > 0$ and $q_\eta \ll \eta^2 L \ll \eta^4 N$.

We define $A \subseteq Q_N$ to be (η, L)-*uniformly distributed* if

$$\frac{1}{|Q_N|} \sum_{t \in Q_N} \left| \frac{|A \cap (t + Q_{q_\eta, L})|}{|Q_{q_\eta, L}|} - \frac{|A|}{|Q_N|} \right|^2 \leq \eta^2,$$

where as before Q_N denotes the discrete cube $[-N/2, N/2]^d \cap \mathbb{Z}^d$ and now $Q_{q,L} := Q_L \cap (q\mathbb{Z})^d$.

Proposition 1 (Average count of distances in uniformly distributed subsets of Q_N)**.**

If $\eta > 0$ and $A \subseteq Q_N \subseteq \mathbb{Z}^d$ with $d \geq 5$ is (η, L)-uniformly distributed, then

$$\frac{1}{|Q_N|} \sum_{x \in \mathbb{Z}^d} 1_A(x) \sum_{x_1 \in \mathbb{Z}^d} 1_A(x - x_1)\, \sigma_\lambda(x_1) = \left(\frac{|A|}{|Q_N|} \right)^2 + O(\eta)$$

for all integers λ that satisfy $\eta^{-4} L^2 \leq \lambda \leq \eta^4 N^2$.

It is easy to see that Proposition 1 immediately implies the following

Corollary 1 (Unpinned distances in uniformly distributed subsets of Q_N)**.** *Let $\varepsilon > 0$ and $0 < \eta \ll \varepsilon^2$.*

If $A \subseteq Q_N \subseteq \mathbb{Z}^d$ with $d \geq 5$ is (η, L)-uniformly distributed, then for all integers λ that satisfies

$$\eta^{-4} L^2 \leq \lambda \leq \eta^4 N^2$$

there exists $x \in A$ such that

$$\frac{|A \cap (x + S_\lambda)|}{|S_\lambda|} = \sum_{x_1 \in \mathbb{Z}^d} 1_A(x - x_1)\, \sigma_\lambda(x_1) > \frac{|A|}{|Q_N|} - \varepsilon.$$

We will ultimately also establish the following "pinned" variant of Corollary 1.

Proposition 2 (Pinned distances in uniformly distributed subsets of Q_N)**.** *Let $\varepsilon > 0$ and $0 < \eta \ll \varepsilon^3$.*

If $A \subseteq Q_N \subseteq \mathbb{Z}^d$ with $d \geq 5$ is (η, L)-uniformly distributed, then there exists $x \in A$ such that

$$\frac{|A \cap (x + S_\lambda)|}{|S_\lambda|} > \frac{|A|}{|Q_N|} - \varepsilon$$

for all integers λ that satisfy $\eta^{-4} L^2 \leq \lambda \leq \eta^4 N^2$.

2.3. **Reduction of Theorems 3 and 4 to Corollary 1 and Proposition 2.**

The task of deducing Theorems 3 and 4 from Corollary 1 and Proposition 2 respectively simply amounts to establishing the following precise relationship between our two notions of uniform distribution.

Lemma 1. *Let $\eta > 0$. If $A \subseteq \mathbb{Z}^d$ with $\delta^*(A) > 0$ is η-uniformly distributed, then there exists a positive integer $L = L(A, \eta)$ and an arbitrarily large integer N with $N \geq \eta^{-4} L$ such that the set $(A - t_0) \cap Q_N$ satisfies*

$$\frac{|(A - t_0) \cap Q_N|}{|Q_N|} > (1 - \eta^4/3)\, \delta^*(A)$$

for some $t_0 \in \mathbb{Z}^d$ and simultaneously has the property that it is $(C\eta, L)$-uniformly distributed for some $C > 0$.

Proof. Since $A \subseteq \mathbb{Z}^d$ is η-uniformly distributed we know there exists a positive integer $L = L(A, \eta)$ such that

$$\frac{|A \cap (t + Q_{q_\eta,L})|}{|Q_{q_\eta,L}|} \leq (1 + \eta^4/3)\, \delta^*(A) \tag{1}$$

for all $t \in \mathbb{Z}^d$. Since $\delta^*(A) > 0$ we further know that there exist arbitrarily large $N \in \mathbb{N}$ such that

$$\frac{|A \cap (t_0 + Q_N)|}{|Q_N|} \geq (1 - \eta^4/3)\, \delta^*(A) \tag{2}$$

for some $t_0 \in \mathbb{Z}^d$. Combining (1) and (2) we see there exist $N \in \mathbb{N}$ with $N \geq \eta^{-4}L$ and $t_0 \in \mathbb{Z}^d$ such that

$$\frac{|A \cap (t + Q_{q_\eta,L})|}{|Q_{q_\eta,L}|} \leq (1 + \eta^4)\frac{|A \cap (t_0 + Q_N)|}{|Q_N|}$$

for all $t \in \mathbb{Z}^d$. Setting $A' := (A - t_0) \cap Q_N$ we further note that since $A' \cap (t + Q_{q_\eta,L})$ is only supported in $Q_N + Q_L$ it follows that

$$\frac{|A'|}{|Q_N|} = \frac{1}{|Q_N|} \sum_{t\in\mathbb{Z}^d} \frac{|A' \cap (t + Q_{q_\eta,L})|}{|Q_{q_\eta,L}|} = \frac{1}{|Q_N|} \sum_{t\in Q_N} \frac{|A' \cap (t + Q_{q_\eta,L})|}{|Q_{q_\eta,L}|} + O(L/N),$$

from which one can easily deduce that

$$\frac{1}{|Q_N|}\Big|\Big\{t \in Q_N \;:\; \frac{|A' \cap (t + Q_{q_\eta,L})|}{|Q_{q_\eta,L}|} \leq (1 - \eta^2)\, \frac{|A'|}{|Q_N|}\Big\}\Big| = O(\eta^2)$$

provided $L/N \ll \eta^2$ and hence that A' is $(C\eta, L)$-uniformly distributed for some $C > 0$. □

We are thus left with proving Propositions 1 and 2. These proofs are presented in Sections 3 and 4 below.

3. Proof of Proposition 1

3.1. Reduction to a Generalized von-Neumann Inequality.

Let Q_N denote the discrete cube $[-N/2, N/2]^d \cap \mathbb{Z}^d$ with $d \geq 5$.

Definition 3 (Counting Function for Distances)**.** For $1 \ll \lambda \ll N^2$ and functions $f_0, f_1 : Q_N \to [-1, 1]$ we define

$$T(f_0, f_1)(\lambda) = \frac{1}{|Q_N|} \sum_{x\in\mathbb{Z}^d} f_0(x) \sum_{x_1\in\mathbb{Z}^d} f_1(x - x_1)\, \sigma_\lambda(x_1).$$

Definition 4 ($U^1(q, L)$-norm)**.** For $1 \ll q \ll L \ll N$ and functions $f : Q_N \to \mathbb{R}$ we define

$$\|f\|_{U^1(q,L)} = \Big(\frac{1}{|Q_N|} \sum_{t\in\mathbb{Z}^d} |f * \chi_{q,L}(t)|^2\Big)^{1/2} \tag{3}$$

where $\chi_{q,L}$ denotes the normalized characteristic function of the cubes $Q_{q,L} := Q_L \cap (q\mathbb{Z})^d$, namely

$$\chi_{q,L}(x) = \begin{cases} \left(\frac{q}{L}\right)^d & \text{if } \; x \in (q\mathbb{Z})^d \cap [-\frac{L}{2}, \frac{L}{2}]^d \\ 0 & \text{otherwise} \end{cases}. \tag{4}$$

In (3) above and in the sequel we denote the convolution $f * g$ of two functions f and g by

$$f * g(x) := \sum_{y\in\mathbb{Z}^d} f(x - y)g(y).$$

We note that the $U^1(q, L)$-norm measures the mean square oscillation of a function with respect to cubic grids of size L and gap q. It is a simple observation, that we record precisely below, that sets $A \subseteq Q_N$ that are (η, L)-uniformly distributed have the property that their "balance functions" have small $U^1(q_\eta, L)$-norm.

Lemma 2. *Let $\eta > 0$ and $1 \ll L \ll \eta^2 N$.*

If $A \subseteq Q_N$ is (η, L)-uniformly distributed, then $\|f_A\|_{U^1(q_\eta,L)} \leq 2\eta$ where $f_A = 1_A - \frac{|A|}{|Q_N|}1_{Q_N}$.

In light of Lemma 2 we see that the engine that drives our proof of Proposition 1, and thus our short proof of Theorem 1, via Corollary 1, is the following "generalized von-Neumann inequality".

Lemma 3 (Generalized von-Neumann)**.** *Let $\eta > 0$, and λ, L, and N be integers with $\eta^{-4}L^2 \leq \lambda \leq \eta^4 N^2$.*

Given any functions $f_0, f_1 : Q_N \to [-1,1]$ on $Q_N \subseteq \mathbb{Z}^d$ with $d \geq 5$ we have

$$|T(f_0, f_1)(\lambda)| \leq \|f_1\|_{U^1(q_\eta, L)} + O(\eta).$$

Proof of Proposition 1. Let $\eta > 0$ and $A \subseteq Q_N \subseteq \mathbb{Z}^d$ with $d \geq 5$ be (η, L)-uniformly distributed.

We let $\alpha = |A|/|Q_N|$ and note that Lemma 2 ensures that $\|f_A\|_{U^1(q_\eta, L)} \leq 2\eta$ where $f_A = 1_A - \alpha 1_{Q_N}$.

Proposition 1 follows immediately from Lemma 3 since

$$T(1_A, 1_A)(\lambda) = \alpha\, T(1_A, 1_{Q_N}) + T(1_A, f_A)(\lambda) = \alpha^2 + \|f_A\|_{U^1(q_\eta, L)} + O(\eta)$$

for all integers λ that satisfy $\eta^{-4}L^2 \leq \lambda \leq \eta^4 N^2$. $\square$

In order to prove Proposition 1 we thus left with the final task of establishing Lemma 3.

3.2. **Proof of Lemma 3.** For any $f : Q_N \to [-1,1]$ we define its *Fourier transform* $\widehat{f} : \mathbb{T}^d \to \mathbb{C}$ by

$$\widehat{f}(\xi) = \sum_{x \in \mathbb{Z}^d} f(x) e^{-2\pi i x \cdot \xi}$$

noting that the support assumption on f ensures that the series defining $\widehat{f}$ converges uniformly to a continuous function on the torus $\mathbb{T}^d$, which we will freely identify with the unit cube $[0,1)^d$ in $\mathbb{R}^d$.

It is easy to verify, using Cauchy-Schwarz and basic properties of the Fourier transform, that

$$|T(f_0, f_1)(\lambda)|^2 \leq \frac{1}{|Q_N|} \int |\widehat{f_1}(\xi)|^2 |\widehat{\sigma_\lambda}(\xi)|^2 \, d\xi$$

where

$$\widehat{\sigma_\lambda}(\xi) := \frac{1}{|S_\lambda|} \sum_{x \in S_\lambda} e^{-2\pi i x \cdot \xi}. \tag{5}$$

It is clear that whenever $|\xi|^2 \ll \lambda^{-1}$ there can be no cancellation in the exponential sum (5), in fact it is easy to verify that the same is also true whenever ξ is *close* to a rational point with *small* denominator. The following proposition is a precise formulation of the fact that this is the only obstruction to cancellation.

Proposition 3 (Key exponential sum estimates, Proposition 1 in [5])**.** *Let $\eta > 0$. If $\lambda \geq C\eta^{-4}$ and*

$$\xi \notin \left(q_\eta^{-1}\mathbb{Z}\right)^d + \{\xi \in \mathbb{R}^d \,:\, |\xi|^2 \leq \eta^{-1}\lambda^{-1}\},$$

then

$$\Big|\frac{1}{|S_\lambda|} \sum_{x \in S_\lambda} e^{-2\pi i x \cdot \xi}\Big| \leq \eta.$$

We now define $\psi_{q_\eta, L}$ indirectly via the identity

$$\widehat{\psi_{q_\eta, L}}(\xi) := \widehat{\chi_{q_\eta, L}}(\xi)^2.$$

Since the definition of $\chi_{q_\eta, L}$ in (4) above clearly implies that

$$\text{(i)}\ \ 0 \leq \widehat{\psi_{q_\eta, L}}(\xi) \leq 1 \text{ for all } \xi \in \mathbb{T}^d \quad \text{ and } \quad \text{(ii)}\ \ \widehat{\psi_{q_\eta, L}}(\ell/q_\eta) = 1 \text{ for all } \ell \in \mathbb{Z}^d$$

it follows that

$$0 \leq 1 - \widehat{\psi_{q_\eta, L}}(\xi) \ll L|\xi - \ell/q_\eta|$$

for all $\xi \in \mathbb{T}^d$ and $\ell \in \mathbb{Z}^d$. In particular we note that

$$|1 - \widehat{\psi_{q_\eta, L}}(\xi)| \ll \eta \quad \text{if} \ \ |\xi - \ell/q_\eta| \leq \eta^{-1/2}\lambda^{-1/2} \ \text{ for some } \ell \in \mathbb{Z}^d. \tag{6}$$

while Proposition 3 ensures that

$$|\widehat{\sigma_\lambda}(\xi)| \leq \eta \quad \text{if} \ \ |\xi - \ell/q_\eta| > \eta^{-1/2}\lambda^{-1/2} \ \text{ for all } \ell \in \mathbb{Z}^d. \tag{7}$$

Hence, if we write

$$|\widehat{\sigma_\lambda}(\xi)|^2 = |\widehat{\sigma_\lambda}(\xi)|^2 \widehat{\psi}_{q_\eta, L}(\xi) + |\widehat{\sigma_\lambda}(\xi)|^2 (1 - \widehat{\psi}_{q_\eta, L}(\xi))$$

use the fact that $|\widehat{\sigma_\lambda}(\xi)|\leq 1$ for all $\xi\in\mathbb{T}^d$ and appeal to Plancherel we can deduce that

$$|T(f_0,f_1)(\lambda)|^2\leq\frac{1}{|Q_N|}\int|\widehat{f_1}(\xi)|^2\,\widehat{\chi}_{q_\eta,L}(\xi)^2\,d\xi+O(\eta^2)=\|f_1\|^2_{U^1(q_\eta,L)}+O(\eta^2)$$

which completes the proof of Lemma 3. □

4. Proof of Proposition 2

Let Q_N denote the discrete cube $[-N/2,N/2]^d\cap\mathbb{Z}^d$.

Definition 5 (Discrete Spherical Averages)**.** Let $f:Q_N\to\mathbb{R}$ be any function.

For any integer λ with $1\ll\lambda\ll N^2$ we define the *discrete spherical average*

$$\mathcal{A}_\lambda(f)(x):=f*\sigma_\lambda(x)=\frac{1}{|S_\lambda|}\sum_{y\in S_\lambda}f(x-y).$$

In Section 4.1 below we reduce Proposition 2 to the *Discrete Spherical Maximal Function Theorem* of Magyar, Stein and Wainger [6], see Proposition 4, and a new "mollified variant" thereof, namely Proposition 5. The statement and proof of Proposition 5 is presented in Section 5.

4.1. **Proof of Proposition 2.** Let $\varepsilon>0$ and $0<\eta\ll\varepsilon^3$. Suppose, contrary to Proposition 2, that there exists a set $A\subseteq Q_N\subseteq\mathbb{Z}^d$ with $d\geq 5$ and $\alpha=|A|/|Q_N|>0$ that it is (η,L)-uniformly distributed, but has the property that for every $x\in A$ there exists an integer λ with $\eta^{-4}L^2\leq\lambda\leq\eta^4N^2$ such that

$$\mathcal{A}_\lambda(1_A)(x)=\frac{|A\cap(x+S_\lambda)|}{|S_\lambda|}\leq\alpha-\varepsilon.$$

It easily follows that for every $x\in A$ there exists an integer λ with $\eta^{-4}L^2\leq\lambda\leq\eta^4N^2$ such that

$$\mathcal{A}_\lambda(f_A)(x)=-\varepsilon+O(\sqrt{\lambda}/N)$$

where $f_A=1_A-\alpha 1_{Q_N}$. Hence for every $x\in A$ we may conclude that

$$\mathcal{A}_*(f_A)(x)\geq\varepsilon/2 \tag{8}$$

where for any function $f:\mathbb{Z}^d\to\mathbb{R}$, $\mathcal{A}_*(f)$ denotes the *discrete spherical maximal function* defined by

$$\mathcal{A}_*(f)(x):=\sup_{\eta^{-4}L^2\leq\lambda\leq\eta^4N^2}|\mathcal{A}_\lambda(f)(x)|\,.$$

Proposition 4 (ℓ^2-Boundedness of the Discrete Spherical Maximal Function [6])**.** *If $d\geq 5$, then*

$$\sum_{x\in\mathbb{Z}^d}|\mathcal{A}_*(f)(x)|^2\leq C\sum_{x\in\mathbb{Z}^d}|f(x)|^2.$$

Since (8) implies, after an application of Cauchy-Schwarz, the inequality

$$\frac{\alpha\,\varepsilon}{2}\leq\frac{1}{|Q_N|}\sum_{x\in\mathbb{Z}^d}1_A(x)\mathcal{A}_*(f_A)(x)\leq\alpha^{1/2}\Big(\frac{1}{|Q_N|}\sum_{x\in\mathbb{Z}^d}|\mathcal{A}_*(f_A)(x)|^2\Big)^{1/2} \tag{9}$$

it follows that

$$\frac{\alpha^{1/2}\,\varepsilon}{2}\leq\Big(\frac{1}{|Q_N|}\sum_{x\in\mathbb{Z}^d}|\mathcal{A}_*(f_A*\chi_{q_\eta,L})(x)|^2\Big)^{1/2}+\Big(\frac{1}{|Q_N|}\sum_{x\in\mathbb{Z}^d}|\mathcal{A}_*(f_A-f_A*\chi_{q_\eta,L})(x)|^2\Big)^{1/2}. \tag{10}$$

In light of Proposition 4 it follows that the first sum above satisfies

$$\Big(\frac{1}{|Q_N|}\sum_{x\in\mathbb{Z}^d}|\mathcal{A}_*(f_A*\chi_{q_\eta,L})(x)|^2\Big)^{1/2}\leq C\Big(\frac{1}{|Q_N|}\sum_{t\in\mathbb{Z}^d}|f*\chi_{q,L}(t)|^2\Big)^{1/2}=C\,\|f_A\|_{U^1(q_\eta,L)}\leq 2C\eta.$$

Estimate (10) will therefore lead to a contradiction, if η is chosen sufficiently small with respect to ε^3, and hence complete the proof of Proposition 2 if we establish that the second sum in (10) satisfies

$$\Big(\frac{1}{|Q_N|}\sum_{x\in\mathbb{Z}^d}|\mathcal{A}_*(f_A-f_A*\chi_{q_\eta,L})(x)|^2\Big)^{1/2}\leq C\eta^{1/3}\alpha^{1/2} \tag{11}$$

for some absolute constant $C>0$. Estimate (11) follows immediately from Proposition 5 in Section 5 below.

5. A "Mollified" Discrete Spherical Maximal Function Theorem

Let $\eta > 0$ and λ, L, and N be integers that satisfy $\eta^{-4}L^2 \leq \lambda \leq \eta^4 N^2$. For functions $f : Q_N \to [-1,1]$ we now define

$$\mathcal{A}_{\lambda,\eta}(f)(x) := \mathcal{A}_\lambda(f - f * \chi_{q_\eta,L})(x) = f * (\sigma_\lambda - \sigma_\lambda * \chi_{q_\eta,L})(x)$$

where $\sigma_\lambda = \dfrac{1}{|S_\lambda|}1_{S_\lambda}$, and introduce the corresponding *"mollified" discrete spherical maximal function*

$$\mathcal{A}_{*,\eta}(f)(x) := \sup_{\eta^{-4}L^2 \leq \lambda \leq \eta^4 N^2} |\mathcal{A}_{\lambda,\eta}(f)(x)|\,. \tag{12}$$

We note that the convolution operator $\mathcal{A}_{\lambda,\eta}$ corresponds to the Fourier multiplier $\widehat{\sigma_{\lambda,\eta}} := \widehat{\sigma_\lambda}(1 - \widehat{\chi_{q_\eta,L}})$.

Proposition 5 (ℓ^2-Decay of the "Mollified" Discrete Spherical Maximal Function)**.** *If $d \geq 5$, then for any $\eta > 0$ we have*

$$\sum_{x \in \mathbb{Z}^d} |\mathcal{A}_{*,\eta}(f)(x)|^2 \leq C\eta^{2/3} \sum_{x \in \mathbb{Z}^d} |f(x)|^2. \tag{13}$$

Proof of Proposition 5. We follow the proof of Proposition 4 as given in [6]. For each $x \in \mathbb{Z}^d$ we now define

$$\mathcal{B}_\lambda(f)(x) = \mathcal{A}_{\lambda^2}(f)(x)$$

noting that when considering $\mathcal{B}_\lambda$ we are now allowing all values of λ for which λ^2 is an integer, and that

$$\mathcal{B}_*(f)(x) := \sup_{\eta^{-2}L \leq \lambda \leq \eta^2 N} \mathcal{B}_\lambda(f)(x) = \mathcal{A}_*(f)(x) \quad \text{and} \quad \mathcal{B}_{*,\eta}(f)(x) = \mathcal{B}_*(f - f * \chi_{q_\eta,L})(x).$$

We now recall the approximation to $\mathcal{B}_\lambda$ given in Section 3 of [6] as a convolution operator $\mathcal{M}_\lambda$ acting on functions on $\mathbb{Z}^d$ of the form

$$\mathcal{M}_\lambda = c_d \sum_{q=1}^{\infty} \sum_{\substack{1 \leq a \leq q \\ (a,q)=1}} e^{-2\pi i \lambda a/q} \mathcal{M}_\lambda^{a/q}$$

where for each reduced fraction a/q the corresponding convolution operator $\mathcal{M}_\lambda^{a/q}$ has Fourier multiplier

$$m_\lambda^{a/q}(\xi) := \sum_{\ell \in \mathbb{Z}^k} G(a/q,\ell)\varphi_q(\xi - \ell/q)\widetilde{\sigma}_\lambda(\xi - \ell/q)$$

with $\varphi_q(\xi) = \varphi(q\xi)$ a standard smooth cut-off function, $G(a/q,l)$ a normalized Gauss sum, and $\widetilde{\sigma}_\lambda(\xi) = \widetilde{\sigma}(\lambda\xi)$ where $\widetilde{\sigma}(\xi)$ is the Fourier transform (on $\mathbb{R}^d$) of the measure on the unit sphere in $\mathbb{R}^d$ induced by Lebesgue measure and normalized to have total mass 1. By Proposition 4.1 in [6] we have

$$\Big\| \sup_{\Lambda \leq \lambda \leq 2\Lambda} |\mathcal{B}_\lambda(f) - \mathcal{M}_\lambda(f)| \Big\|_{\ell^2(\mathbb{Z}^d)} \leq C\Lambda^{-1/2}\|f\|_{\ell^2(\mathbb{Z}^d)}$$

provided $d \geq 5$. Writing

$$\mathcal{M}_*(f) := \sup_{\eta^{-2}L \leq \lambda \leq \eta^2 N} |\mathcal{M}_\lambda(f)| \quad \text{and} \quad \mathcal{M}_{*,\eta}(f) := \mathcal{M}_*(f - f * \chi_{q_\eta,L})$$

this implies

$$\|\mathcal{B}_{*,\eta}(f) - \mathcal{M}_{*,\eta}(f)\|_{\ell^2} \leq C\,\eta L^{-1/2}\,\|f - f * \chi_{q_\eta,L}\|_{\ell^2} \leq C\,\eta L^{-1/2}\,\|f\|_{\ell^2}$$

thus matters reduce to showing (13) for the operator $\mathcal{M}_{*,\eta}$.

For a given reduced fraction a/q we now define the maximal operator

$$\mathcal{M}_*^{a/q}(f) := \sup_{\eta^{-2}L \leq \lambda \leq \eta^2 N} |\mathcal{M}_\lambda^{a/q}(f)|$$

where $\mathcal{M}_\lambda^{a/q}$ is the convolution operator with multiplier $m_\lambda^{a/q}(\xi)$. It is proved in Lemma 3.1 of [6] that

$$\|\mathcal{M}_*^{a/q}(f)\|_{\ell^2} \leq Cq^{-d/2}\|f\|_{\ell^2}. \tag{14}$$

We will show here that if $q \leq C\eta^{-2/3}$, then

$$\|\mathcal{M}_*^{a/q}(f - f * \chi_{q_\eta,L})\|_{\ell^2} \leq C\eta^{1/3}q^{-d/2}\|f\|_{\ell^2}. \tag{15}$$

Taking estimates (14) and (15) for granted, one obtains

$$\|\mathcal{M}_*(f - f * \chi_{q_\eta,L})\|_{\ell^2} \ll \Big(\eta^{1/3} \sum_{1\le q\le C\eta^{-2/3}} q^{-d/2+1} + \sum_{q\ge C\eta^{-2/3}} q^{-d/2+1}\Big) \|f\|_{\ell^2} \ll \eta^{1/3}\|f\|_{\ell^2}$$

as required. It thus remains to prove (15).

Writing $\varphi_q(\xi) = \varphi'_q(\xi)\varphi_q(\xi)$, with a suitable smooth cut-off function φ', we can introduce the decomposition

$$m_\lambda^{a/q}(\xi) = \Big(\sum_{\ell\in\mathbb{Z}^k} G(a/q,\ell)\varphi'_q(\xi-\ell/q)\Big)\Big(\sum_{\ell\in\mathbb{Z}^k} \varphi_q(\xi-\ell/q)\widetilde{\sigma}(\xi-\ell/q)\Big) =: g^{a/q}(\xi)\, n_\lambda^q(\xi),$$

since for each ξ at most one term in each of the above sums is non-vanishing. Accordingly

$$\mathcal{M}_*^{a/q}(f - f*\chi_{q_\eta,L}) = G^{a/q}\ \mathcal{N}_*^q(f - f*\chi_{q_\eta,L})$$

where the maximal operator $\mathcal{N}_*^q$ and the convolution operator $G_{a/q}$ correspond to the multipliers n_λ^q and $g^{a/q}$ respectively. Now by the standard Gauss sum estimate we have $|g^{a/q}(\xi)| \ll q^{-d/2}$ uniformly in ξ, hence

$$\|G^{a/q}\ \mathcal{N}_*^q(f - f*\chi_{q_\eta,L})\|_{\ell^2} \ll q^{-d/2}\|\mathcal{N}_*^q(f - f*\chi_{q_\eta,L})\|_{\ell^2}.$$

Thus by our choice $q_\eta := \mathrm{lcm}\{1 \le q \le C\eta^{-2}\}$ it remains to show that if q divides q_η then

$$\|\mathcal{N}_*^q(f - f*\chi_{q_\eta,L})\|_{\ell^2} \ll \eta^{1/3}\|f\|_{\ell^2}. \tag{16}$$

As before we write $\mathcal{N}_{*,\eta}^q(f) = \mathcal{N}_*^q(f - f*\chi_{q_\eta,L})$, and note that this is a maximal operator with multiplier

$$n_\lambda^q(\xi)(1-\widehat{\chi_{q_\eta,L}})(\xi) = \sum_{\ell\in\mathbb{Z}^d} \varphi_q(\xi-\ell/q)(1-\widehat{\chi_{q_\eta,L}})(\xi-\ell/q)\widetilde{\sigma}_\lambda(\xi-\ell/q).$$

For a fixed q, the multiplier $\varphi_q(1-\widehat{\chi_{q_\eta,L}})\widetilde{\sigma}_\lambda$ is supported on the cube $[-\frac{1}{2q},\frac{1}{2q}]^d$ thus by Corollary 2.1 in [6]

$$\|\mathcal{N}_{*,\eta}^q\|_{\ell^2\to\ell^2} \le C\, \|\widetilde{\mathcal{N}}_{*,\eta}^q\|_{L^2\to L^2}$$

where $\widetilde{\mathcal{N}}_{*,\eta}^q$ is the maximal operator corresponding to the multipliers $\varphi_q(1-\widehat{\chi_{q_\eta,L}})\widetilde{\sigma}_\lambda$, for $\eta^{-2}L \le \lambda \le \eta^2 N$, acting on $L^2(\mathbb{R}^d)$. By the definition of the function $\chi_{q_\eta,L}$

$$|1-\widehat{\chi_{q_\eta,L}}(\xi)| \ll \min\{1, L|\xi|\},$$

thus from Theorem 6.1 (with $j=1$) in [3] we obtain

$$\|\widetilde{\mathcal{N}}_{*,\eta}^q\|_{L^2\to L^2} \ll \left(\frac{L}{\eta^{-2}L}\right)^{1/6} = \eta^{1/3}$$

which establishes (16) and completes the proof. □

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