STRONGLY SINGULAR CONVOLUTION OPERATORS ON $\mathbf{R}^d$ A EXAMPLE COMING FROM CONVEX SETS

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1. Introduction

Suppose that $\mathcal{D} \subset \mathbf{R}^d$ is a centrally symmetric convex body with a smooth boundary and everywhere non-vanishing Gaussian curvature. This convex body induces the usual norm

$$\rho(x) = \sup\{t \geq 0 : x \notin t\mathcal{D}\} \tag{1}$$

on $\mathbf{R}^d$, the Minkowski functional with respect to $\mathcal{D}$.

Let $K_{\alpha,\beta}$ be a distribution[1] on $\mathbf{R}^d$ that away from the origin agrees with the function

$$K_{\alpha,\beta}(x) = \rho(x)^{-d-\alpha} e^{i\rho(x)^{-\beta}} \chi(\rho(x)), \tag{2}$$

where $\beta > 0$ and χ is smooth and compactly supported in a small neighborhood of the origin.

Theorem 1. *If $\alpha \leq \frac{d\beta}{2}$ then $Tf = f * K_{\alpha,\beta}$ extends to a bounded operator from $L^2(\mathbf{R}^d)$ to itself.*

A model case for operators of this type would be when we take $\mathcal{D}$ to be the standard Euclidean ball as in this case we would of course have $\rho(x) = |x|$. Operators of this type were first studied by Hirschman [1] in the case $d = 1$ and then in higher dimensions by Wainger [4].

In tackling the model case it is efficient to use Fourier transform methods. Since $K_{\alpha,\beta}$ is radial it is well known that its Fourier transform is given by

$$m(\xi) = (2\pi)^{\frac{d}{2}} \int_0^\infty \chi(r) r^{-1-\alpha} e^{ir^{-\beta}} J_{\frac{d-2}{2}}(r|\xi|)(r|\xi|)^{\frac{2-d}{2}}\, dr, \tag{3}$$

where $J_{\frac{d-2}{2}}$ is a Bessel function; see [3]. Using Plancherel's theorem and the asymptotics of Bessel functions it is then straightforward to establish Theorem 1 in this case.

In our proof of Theorem 1 in general however we do not appeal to Fourier transform methods.

2. Proof of Theorem 1

We now wish to decompose our operator $T = \sum_{j=0}^\infty T_j$. In order to do this we consider the following partition of unity; choose $\vartheta \in C_0^\infty(\mathbf{R})$ supported in $[\frac{1}{2}, 2]$ such that $\sum_{j=0}^\infty \vartheta(2^j r) = 1$ for all $0 \leq r \leq 1$, and write

$$T_j f(x) = f * K_j(x) \quad \text{where} \quad K_j(x) = \vartheta(2^j \rho(x)) K_{\alpha,\beta}(x).$$

Theorem 2. *The operator norms of T_j are uniformly bounded whenever $\alpha \leq \frac{d\beta}{2}$, more precisely*

$$\int_{\mathbf{R}^d} |T_j f(x)|^2 dx \leq C 2^{j(2\alpha - d\beta)} \int_{\mathbf{R}^d} |f(x)|^2 dx. \tag{4}$$

[1] The distribution-valued function $\alpha \mapsto K_{\alpha,\beta}$, initially defined for $\operatorname{Re}\alpha < 0$, continues analytically to all of $\mathbf{C}$.

We note that as the operator norms of T_j are equal to that of

$$S_j f(x) = 2^{j\alpha} \int_{\mathbf{R}^d} \vartheta(\rho(x-y))\rho(x-y)^{-d-\alpha} e^{i2^{j\beta}\rho(x-y)^{-\beta}} f(y)dy,$$

to prove Theorem 2 it suffices to establish estimate (4) for the operators S_j.

Key to establishing this result is the following proposition of Hörmander, which may be thought of as a variable coefficient version of Plancherel's theorem. See [5], Chapter 7 or [2], Chapter IX.

Proposition 3. *Let Ψ be a smooth function supported on the set $\{(x,y) \in \mathbf{R}^d \times \mathbf{R}^d : \rho(x-y) \leq C\}$ and Φ be real-valued and smooth on the support of Ψ. If we assume that all partial derivatives of Ψ and Φ are bounded and that*

$$\det\Big(\frac{\partial^2 \Phi}{\partial x_i \partial y_j}\Big) \neq 0 \tag{5}$$

on the support of Ψ, then

$$\Big\| \int_{\mathbf{R}^d} \Psi(x,y) e^{i\lambda\Phi(x,y)} f(y)dy \Big\|_{L^2(\mathbf{R}^d)} \leq C\lambda^{-\frac{d}{2}} \|f\|_{L^2(\mathbf{R}^d)}.$$

It is clear that estimate (4) for the operators S_j will be an immediate consequence of Proposition 3 once we have established that the phase of its kernel is non-degenerate (in the sense of (5) above).

Lemma 4. *Let $\Phi(x,y) = \rho(x-y)^{-\beta}$, then $\det\big(\frac{\partial^2 \Phi}{\partial x_i \partial y_j}\big) \neq 0$ whenever $\beta \neq -1$.*

Proof. We start by letting $\varphi(x) = \rho(x)^2$ and noting that as a consequence φ must be homogeneous of degree two. It clearly then suffice to verify that if $\beta \neq -1$, then

$$\det H\varphi^{-\beta/2}(x) \neq 0$$

where $H\varphi^{-\beta/2}$ denotes the (pure) Hessian matrix of the phase function $\varphi^{-\beta/2}$. Now an easy calculation shows that

$$\det H\varphi^{-\beta/2} = \frac{-\beta(\beta+2)}{4}\, \varphi^{-(\beta+4)/2}(A-B),$$

where

$$A = \frac{2\varphi}{\beta+2} H\varphi \quad \text{and} \quad B = (\nabla\varphi)(\nabla\varphi)^T.$$

Matters therefore reduce to showing that whenever $\beta \neq -1$ and $u \neq 0$

$$(A-B)u \neq 0.$$

Assuming for now that $H\varphi$ is non-singular and using the fact that $Bu = \langle \nabla\varphi, u\rangle \nabla\varphi$ it is easy to see that

$$(A-B)u = 0 \Longleftrightarrow \langle [H\varphi]^{-1}\nabla\varphi, \nabla\varphi\rangle = \frac{2\varphi}{\beta+2}.$$

Now it follows from Euler's homogeniety relations that

$$\langle H\varphi(x)x, x\rangle = \langle \nabla\varphi(x), x\rangle = 2\varphi(x), \tag{6}$$

in particular we have that $[H\varphi(x)]^{-1}\nabla\varphi(x) = x$ and hence from (6) it follow that

$$\langle [H\varphi]^{-1}\nabla\varphi, \nabla\varphi\rangle = 2\varphi,$$

establishing the result.

We still need to verify that $H\varphi$ is indeed invertible, we shall infact establish that for any $v \in \mathbf{R}^d$

$$\langle H\varphi(x)v, v\rangle > 0.$$

Since $\langle \nabla\varphi(x), x\rangle = \nabla\varphi(x) \neq 0$ we may write $v = \lambda x + y$ where y is a vector perpendicular to $\nabla\varphi(x)$. Using the fact that

$$H\varphi(x) = 2\big(\rho(x)H\rho(x) + \nabla\rho(x)\nabla\rho(x)^T\big)$$

and the identities contained in (6) we see that

$$\begin{aligned}\langle H\varphi(x)v, v\rangle &= \lambda^2\langle H\varphi(x)x, x\rangle + 2\lambda\langle H\varphi(x)x, y\rangle + \langle H\varphi(x)y, y\rangle \\ &= \lambda^2\varphi(x) + \rho(x)\langle H\rho(x)y, y\rangle.\end{aligned}$$

The result then follows from the fact that ρ was induce from a convex body with smooth boundary and everywhere non-vanishing Gaussian curvature since this is equivalent to the having

$$\langle H\rho(x)y, y\rangle > 0 \qquad \text{for all } y \perp \nabla\varphi(x). \tag{7}$$

□

Theorem 1 now follows from Theorem 2 and an application of Cotlar's lemma (plus a standard limiting argument) once we have verified that the T_j are, in the following sense, almost orthogonal.

Lemma 5. *If* $\alpha = \frac{d\beta}{2}$ *then* $\|T_i^*T_j\|_{Op} + \|T_iT_j^*\|_{Op} \leq C2^{-\frac{d\beta}{2}|i-j|}$.

Proof. This follows trivially from Theorem 2 whenever $|i-j| \leq 10$, since $\|T_i^*T_j\|_{Op} \leq \|T_i\|_{Op}\|T_j\|_{Op}$. We shall therefore, without loss of generality, assume that $j \geq i + 10$. Now $T_i^*T_j$ has a kernel

$$L_{ij}(x) = K_j * \bar{K}_i(-x),$$

and the same operator norm as the operator with kernel

$$\begin{aligned}\widetilde{L}_{ij}(x) &= 2^{-jd}L_{ij}(2^{-j}x) \\ &= 2^{-j2d}\int K_j(2^{-j}y)\bar{K}_i(2^{-j}(x-y))dy \\ &= 2^{j2\alpha}\int\limits_{\substack{\rho(y)\sim 1 \\ \rho(x-y)\sim 2^{j-i}}} \rho(y)^{-d-\alpha}\rho(x-y)^{-d-\alpha}e^{i2^{j\beta}[\rho(y)^{-\beta}-\rho(x-y)^{-\beta}]}dy.\end{aligned}$$

We trivially have the estimate $|\widetilde{L}_{ij}(x)| \leq C2^{j2\alpha}2^{(i-j)(d+\alpha)}$. It is easy to verify that

$$|\nabla_y[\rho(y)^{-\beta} - \rho(x-y)^{-\beta}]| \geq C_0,$$

thus there is always a direction in which we may integrating by parts, in doing so d times we obtain

$$|\widetilde{L}_{ij}(x)| \leq C2^{j(2\alpha-d\beta)}2^{(i-j)(d+\alpha)} = 2^{(i-j)(d+\alpha)}.$$

This of course implies that

$$\int |\widetilde{L}_{ij}(x)|\,dx \leq C2^{(i-j)\alpha}.$$

□

References

[1] I. I. Hirschman, *Multiplier Transforms I*, Duke Math. J., 26 (1956), pp. 222–242.
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