SHARP L^2 ESTIMATES FOR STRONGLY SINGULAR INTEGRAL OPERATORS ON THE HEISENBERG GROUP

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Abstract. In this article we improve on the arguments in [2] and obtain sharp L^2 estimates for strongly singular integral operators on the Heisenberg group.

1. Introduction

The Heisenberg group $\mathbf{H}^n$ is a non-commutative nilpotent Lie group, with underlying manifold $\mathbf{R}^{2n+1}$ equipped the group law[1]

$$(x,t)\cdot(y,s)=(x+y,s+t-2a\ x^{\mathrm{T}}Jy) \tag{1}$$

where a is a positive real number and J denotes the standard symplectic matrix on $\mathbf{R}^{2n}$, namely

$$J=\begin{pmatrix}0 & I_n\\ -I_n & 0\end{pmatrix}$$

with inverses given by $(x,t)^{-1}=-(x,t)$. The *nonisotropic* dilations

$$(x,t)\mapsto\delta\circ(x,t)=(\delta x,\delta^2 t). \tag{2}$$

are automorphisms of $\mathbf{H}^n$ and the homogeneous distance function

$$\rho(x,t)=\rho_b(x,t)=(|x|^4+bt^2)^{1/4} \tag{3}$$

defines a quasi-norm on this group. When $a^2b=1$ it in fact defines a norm.

As in [2] we consider (group) convolution operators on $\mathbf{H}^n$ formally given by

$$Tf(x,t)=f*K_{\alpha,\beta}(x,t)$$

where $K_{\alpha,\beta}$ is a strongly singular distributional kernel on $\mathbf{H}^n$ that agrees, for $(x,t)\neq(0,0)$, with the function

$$K_{\alpha,\beta}(x,t)=\rho(x,t)^{-2n-2-\alpha}e^{i\rho(x,t)^{-\beta}}\chi(\rho(x,t)),$$

where $\beta>0$ and χ is smooth and compactly supported in a small neighborhood of the origin.

We fix the constant

$$C_\beta=(\beta+2)\left(2\beta+5+\sqrt{(2\beta+5)^2-9}\right)$$

and note that $C_\beta\geq 18$ for all $\beta>0$. Our main result is then the following.

1991 *Mathematics Subject Classification.* 42B20, 43A80.

Key words and phrases. Strongly singular integrals, Heisenberg group.

Both authors were partially supported by HARP grants from the European Commission.

[1] If we identify $\mathbf{R}^{2n+1}$ with $\mathbf{C}^n\times\mathbf{R}$ by $z_j=x_j+ix_{j+n}$, then the Heisenberg group law can be written in the complex coordinates:

$$(z,t)\cdot(w,s)=(z+w,t+s+2a\,\mathrm{Im}\,z\cdot\bar{w}).$$

Theorem 1. *Let $2a^2b < C_\beta$, then T extends to a bounded operator from $L^2(\mathbf{H}^n)$ to itself if and only if $\alpha \leq (n+\frac{1}{2})\beta$.*

2. Reductions and Remarks

The necessary condition in Theorem 1 follows from the arguments in [2]. To establish sufficiency matters reduce to considering the dyadic operator

$$T_j(x,t) = f * K_j(x,t),$$

where

$$K_j(x,t) = \vartheta(2^j\rho(x,t))K_{\alpha,\beta}(x,t),$$

where $\vartheta \in C_0^\infty(\mathbf{R})$ supported in $[\frac{1}{2},2]$ is chosen such that $\sum_{j=0}^\infty \vartheta(2^jr) = 1$ for all $0 \leq r \leq 1$.

We know from [2] that everything reduces to establishing the following key result.

Theorem 2. *Let $2a^2b < C_\beta$, then the dyadic operators T_j are bounded uniformly on $L^2(\mathbf{H}^n)$ whenever $\alpha \leq (n+\frac{1}{2})\beta$, more precisely*

$$\int_{\mathbf{H}^n} |T_jf(x,t)|^2\,dx\,dt \leq C2^{j(2\alpha-(2n+1)\beta)}\int_{\mathbf{H}^n} |f(x,t)|^2\,dx\,dt. \tag{4}$$

Theorem 1 then follows from an application of Cotlar's lemma (and a standard limiting argument), since it is easy to verify (see [2]) that the operators T_j are almost orthogonal.

Since, for $p \leq q$, the $L^p \to L^q$ operator norms of the T_j are controlled by the $L^p \to L^q$ operator norms of the rescaled operators

$$\widetilde{T}_jf(x,t) = 2^{j\alpha}2^{jd_h(1/p-1/q)}S_jf(x,t) = 2^{j\alpha}2^{jd_h(1/p-1/q)}s_j * f(x,t),$$

where

$$s_j(x,t) = \vartheta(\rho(x,t))\rho(x,t)^{-d_h-\alpha}e^{i2^{j\beta}\rho(x,t)^{-\beta}}$$

and $d_h = 2n+2$ denotes the homogeneous dimension of $\mathbf{H}^n$; establishing Theorem 2 is equivalent to showing that the L^2 operator norm of S_j is $O(2^{-jd\beta/2})$, where $d = 2n+1$, denotes the topological dimension of $\mathbf{H}^n$. Using standard interpolation techniques it follows from this that the L^p operator norm of S_j is $O(2^{-jd\beta/2}2^{jd\beta|1/p-1/2|})$, and if $p \leq 2$ this is also a bound for the $L^p \to L^{p'}$, $L^p \to L^2$, and $L^{p'} \to L^2$ operator norms and can be written more succinctly as $O(2^{-jd\beta/p'})$; from this one of course immediately obtains the corresponding results for our dyadic operator T_j. We note however that the behavior of the operator T near L^1 and the endpoint results in L^p for $p \neq 2$ remain open problems.

We've used the notation d_h and d for the homogeneous and topological dimensions of $\mathbf{H}^n$ as the arguments alluded to above also apply in the setting of homogeneous groups; of course establishing the analogue of Theorem 2 in more general group settings remains an open problem.

3. Homogeneous groups and a proposition of Hörmander

The Heisenberg group is of course one of the simplest examples of a (non-commutative) homogeneous group. Recall that a homogeneous group consists of $\mathbf{R}^d$ equipped with a Lie group structure, together with a family of dilations

$$x = (x_1,\ldots,x_d) \mapsto \delta\circ x = (\delta^{a_1}x_1,\ldots,\delta^{a_d}x_d),$$

with $a_1,\ldots,a_d$ strictly positive, that are group automorphisms, for all $\delta > 0$.

To each homogeneous group on $\mathbf{R}^d$, we can associate its Lie algebra, consisting of left-invariant vector fields on $\mathbf{R}^d$, with basis $\{X_j\}_{1\leq j\leq d}$ where each X_j is the left-invariant vector field that agrees with $\partial/\partial x_j$ at the origin.

Key to establishing Theorem 2 is the following, presumably well known, generalization of a proposition of Hörmander [1], see also [3], Chapter IX.

Proposition 3. *Let Ψ be a smooth function of compact support in x and y, and Φ be real-valued and smooth on the support of Ψ. If we assume that*

$$\det\Big(X_jY_k\Phi(x,y)\Big)\neq 0, \tag{5}$$

on the support of Ψ, then for $\lambda>0$ we have

$$\Big\|\int_{\mathbf{R}^d}\Psi(x,y)e^{i\lambda\Phi(x,y)}f(y)dy\Big\|_{L^2(\mathbf{R}^d)}\leq C\lambda^{-\frac{d}{2}}\|f\|_{L^2(\mathbf{R}^d)}. \tag{6}$$

Proof. By using a partition of unity we may assume that the amplitude Ψ has suitably small compact support in both x and y. Denoting the operator on the left hand side of inequality (6) by T_λ it is then easy to see that

$$T_\lambda^*T_\lambda f(y)=\int_{\mathbf{R}^d}K_\lambda(x,z)f(z)\,dz$$

where

$$K_\lambda(x,z)=\int_{\mathbf{R}^d}e^{i\lambda[\Phi(x,y)-\Phi(z,y)]}\Psi(x,y)\overline{\Psi(z,y)}\,dy.$$

It consequently suffices to establish the kernel estimate

$$|K_\lambda(x,z)|\leq C(1+\lambda|z^{-1}\cdot x|)^{-N}, \tag{7}$$

since it then follows that

$$\int|K_\lambda(x,z)|\,dz\approx|\{z:|z^{-1}\cdot x|\leq\lambda^{-1}\}|=C\lambda^{-d}$$

and similarly for $\int|K_\lambda(x,z)|\,dx$, and therefore by Schur's test that

$$\|T_\lambda^*T_\lambda f\|_{L^2(\mathbf{R}^d)}\leq C\lambda^{-d}\|f\|_{L^2(\mathbf{R}^d)}.$$

The kernel $K_\lambda(x,z)$ is of course always bounded, hence in order to establish (7) we need only consider the case when $|z^{-1}\cdot x|\geq\lambda^{-1}$. Now

$$\begin{aligned}
Y_k\Phi(x,y)-Y_k\Phi(z,y)&=\int_0^1\frac{d}{dt}Y_k\Phi(z\cdot t(z^{-1}\cdot x),y)\,dt\\
&=\sum_{j=1}^d(z^{-1}\cdot x)_j\int_0^1X_jY_k\Phi(z\cdot t(z^{-1}\cdot x),y)\,dt\\
&=\sum_{j=1}^d(z^{-1}\cdot x)_j\left\{X_jY_k\Phi(x,y)+\int_0^1X_jY_k[\Phi(z\cdot t(z^{-1}\cdot x),y)-\Phi(x,y)]\,dt\right\}\\
&=\sum_{j=1}^d(z^{-1}\cdot x)_jX_jY_k\Phi(x,y)+O(|z^{-1}\cdot x|^2).
\end{aligned}$$

So if we let

$$A=A(x,y)=X_jY_k\Phi(x,y)\qquad\text{and}\qquad u=u(x,y,z)=A^{-1}\frac{z^{-1}\cdot x}{|z^{-1}\cdot x|}$$

and define

$$\Delta(x,y,z) = (u_1Y_1 + \cdots + u_dY_d)[\Phi(x,y) - \Phi(z,y)]$$

it follows that

$$\Delta(x,y,z) = |z^{-1}\cdot x| + O(|z^{-1}\cdot x|^2).$$

Therefore for $|z^{-1}\cdot x|$ small enough, it is here that we use our initial suitably small support assumption, we have

$$|\Delta(x,y,z)| \geq \tfrac{1}{2}|z^{-1}\cdot x|,$$

and if we now set

$$D = \frac{1}{i\lambda\Delta(x,y,z)}(u_1Y_1 + \cdots + u_dY_d),$$

it follows that

$$\begin{aligned}\left|\int_{\mathbf{R}^d} e^{i\lambda[\Phi(x,y)-\Phi(z,y)]}\Psi(x,y)\overline{\Psi(z,y)}\,dy\right| &= \left|\int_{\mathbf{R}^d} D^N\left(e^{i\lambda[\Phi(x,y)-\Phi(z,y)]}\right)\Psi(x,y)\overline{\Psi(z,y)}\,dy\right| \\ &= \left|\int_{\mathbf{R}^d} e^{i\lambda[\Phi(x,y)-\Phi(z,y)]}(D^{\mathrm{T}})^N\left(\Psi(x,y)\overline{\Psi(z,y)}\right)\,dy\right| \\ &\leq C_N(1+\lambda|z^{-1}\cdot x|)^{-N},\end{aligned}$$

for all $N \geq 0$. □

4. Proof of Theorem 2

We have already reduced matters to establishing the estimate

$$\int_{\mathbf{H}^n} |S_jf(x,t)|^2\,dx\,dt \leq C2^{-j(2n+1)\beta)}\int_{\mathbf{H}^n}|f(x,t)|^2\,dx\,dt \tag{8}$$

for the rescaled operators S_j.

Since the S_j are local operators, in the sense that the support of S_jf is always contained in a fixed dilate of some nonisotropic ball containing the support of f, we may make the additional assumption that the integral kernels above have compact support in both (x,t) and (y,s). Estimate (8) then follows from Proposition 3 once we have verified the *non-degeneracy* condition (5) in this setting.

It is well known that

$$X_j^\ell = \frac{\partial}{\partial x_j} + 2ax_{j+n}\frac{\partial}{\partial t}, \quad X_{j+n}^\ell = \frac{\partial}{\partial x_{j+n}} - 2ax_j\frac{\partial}{\partial t} \qquad j = 1,\ldots,n,$$

and $T = \frac{\partial}{\partial t}$ form a real basis for the Lie algebra of left-invariant vector fields on $\mathbf{H}^n$, while

$$X_j^r = X_{j+n}^\ell, \quad X_{j+n}^r = X_j^\ell,$$

for $j = 1,\ldots,n$, and $T = \frac{\partial}{\partial t}$ form a real basis for the Lie algebra of right-invariant vector fields.

For convenience we shall use synonymously

$$X_{2n+1}^\ell = X_{2n+1}^r = T,$$

and furthermore denote

$$X^\ell = (X_1^\ell,\ldots,X_{2n+1}^\ell) \quad \text{and} \quad X^r = (X_1^r,\ldots,X_{2n+1}^r).$$

We note that

$$-[X_j^r\widetilde{\varphi}](x,t) = [X_j^\ell\varphi]\big((x,t)^{-1}\big),$$

where $\widetilde{\varphi}(x) = \varphi\big((x,t)^{-1}\big)$, and hence

$$X_j^\ell Y_k^\ell \left[\varphi\big((y,s)^{-1}\cdot(x,t)\big)\right] = -[X_j^\ell X_k^r \varphi]\big((y,s)^{-1}\cdot(x,t)\big).$$

The *non-degeneracy* condition (5) in this setting is therefore equivalent to the following.

Lemma 4. *Let $\Phi(x,t) = (|x|^4 + bt^2)^{-\beta/4}$, then*

$$\det\Big(X_j^\ell X_k^r \Phi(x,t)\Big) \neq 0$$

whenever $2a^2 b < C_\beta$.

Proof. Let $\varphi(x,t) = |x|^4 + bt^2$. It is straightforward to see that the 'mixed' Hessian of Φ is given by

$$X_j^\ell X_k^r \Phi(x,t) = -\tfrac{\beta}{4}\varphi^{-(\beta+8)/4}\{\varphi X_j^\ell X_k^r\varphi - \tfrac{\beta+4}{4} X_j^\ell \varphi X_k^r \varphi\}.$$

For convenience we now define

$$A := X_j^\ell X_k^r \varphi \qquad \text{and} \qquad B := X_j^\ell\varphi X_k^r\varphi.$$

Since $\operatorname{rank}(B) = 1$ it follows that

$$\det(\varphi A - \tfrac{\beta+4}{4}B) = \varphi^{2n}\left\{\varphi\det(A) - \frac{\beta+4}{4}\sum_{j=1}^{2n+1}\det\begin{pmatrix}\mathbf{a}_1\\ \vdots\\ \mathbf{b}_j\\ \vdots\\ \mathbf{a}_{2n+1}\end{pmatrix}\right\},$$

where $\mathbf{a}_j = (a_{j1},\dots,a_{j\,2n+1})$ and $\mathbf{b}_j = (b_{j1},\dots,b_{j\,2n+1})$.

It is an easy calculation to see that

$$X^\ell\varphi(x,t) = \big(4|x|^2 x + 4abt(Jx), 2bt\big),$$
$$X^r\varphi(x,t) = \big(4|x|^2 x - 4abt(Jx), 2bt\big),$$

where J is the standard symplectic matrix on $\mathbf{R}^{2n}$ coming from the group structure. Hence we have

$$A = 4\begin{pmatrix}C & 0\\ 0 & 0\end{pmatrix} + 8\begin{pmatrix}D & 0\\ 0 & 0\end{pmatrix} + 4abE, \quad \text{and} \quad B = 4|x|^2\begin{pmatrix}F & 0\end{pmatrix} + 4abtG,$$

where

$$C = |x|^2 I + abtJ \qquad D = xx^{\mathrm{T}} \qquad E = \begin{pmatrix}2a(Jx)(x^{\mathrm{T}}J) & Jx\\ x^{\mathrm{T}}J & 1/2a\end{pmatrix},$$

$$F = (X^\ell\varphi)x^{\mathrm{T}} \qquad \text{and} \qquad G = \begin{pmatrix}(X^\ell\varphi)(x^{\mathrm{T}}J) & X^\ell\varphi/2a\end{pmatrix}.$$

Now since both $\operatorname{rank}(D) = 1$ and $\operatorname{rank}(E) = 1$ it follows that

$$\begin{aligned}
\det(A) &= 2b\,4^{2n}\det(C + 2D)\\
&= 2b\,4^{2n}\left\{\big(|x|^4 + a^2b^2t^2\big)^n + \tfrac{1}{2}\big(|x|^4 + a^2b^2t^2\big)^{n-1}\sum_{j=1}^{2n} x_j X_j^\ell\varphi\right\}\\
&= 2b\,4^{2n}\big(|x|^4 + a^2b^2t^2\big)^{n-1}\big(3|x|^4 + a^2b^2t^2\big).
\end{aligned}$$

To obtain the final identity above we used the fact that

$$\sum_{j=1}^{2n} x_j X_j^\ell \varphi = 4|x|^4.$$

Using the fact that

$$\operatorname{rank}\begin{pmatrix} \mathbf{e}_1 \\ \vdots \\ \mathbf{g}_j \\ \vdots \\ \mathbf{e}_{2n} \\ \mathbf{e}_{2n+1} \end{pmatrix} = \operatorname{rank}\begin{pmatrix} \mathbf{d}_1 \\ \vdots \\ \mathbf{f}_j \\ \vdots \\ \mathbf{d}_{2n} \end{pmatrix} = 1$$

and

$$\sum_{j=1}^{2n} (X_j^\ell \varphi)^2 = 16|x|^2\big(|x|^4 + a^2b^2t^2\big)$$

we may conclude that

$$\begin{aligned}
\sum_{j=1}^{2n} \det\begin{pmatrix} \mathbf{a}_1 \\ \vdots \\ \mathbf{b}_j \\ \vdots \\ \mathbf{a}_{2n} \\ \mathbf{a}_{2n+1} \end{pmatrix} &= 2b\ 4^{2n} \sum_{j=1}^{2n} \det\left\{\begin{pmatrix} \mathbf{c}_1 \\ \vdots \\ \mathbf{0}_j \\ \vdots \\ \mathbf{c}_{2n} \end{pmatrix} + \begin{pmatrix} 2\mathbf{d}_1 \\ \vdots \\ |x|^2\mathbf{f}_j \\ \vdots \\ 2\mathbf{d}_{2n} \end{pmatrix}\right\} \\
&= 2b\ 4^{2n}|x|^2 \sum_{j=1}^{2n} X_j^\ell\varphi \det\begin{pmatrix} \mathbf{c}_1 \\ \vdots \\ \mathbf{0}_j + x^{\mathrm{T}} \\ \vdots \\ \mathbf{c}_{2n} \end{pmatrix} \\
&= 2b\ 4^{2n}\big(|x|^4 + a^2b^2t^2\big)^{n-1}|x|^2\ \frac{1}{4}\sum_{j=1}^{2n}(X_j^\ell\varphi)^2 \\
&= 2b\ 4^{2n+1}|x|^4\big(|x|^4 + a^2b^2t^2\big)^n.
\end{aligned}$$

Finally, we can combine the fact that

$$\operatorname{rank}\begin{pmatrix} \mathbf{e}_1 \\ \vdots \\ \mathbf{e}_{2n} \\ \mathbf{g}_{2n+1} \end{pmatrix} = 1,$$

that

$$\det(C + 2D) = \big(|x|^4 + a^2b^2t^2\big)^{n-1}\big(3|x|^4 + a^2b^2t^2\big),$$

and the fact that

$$\sum_{j=1}^{n} (x_j X_{j+n}^\ell \varphi - x_{j+n} X_j^\ell \varphi) = -4ab|x|^2 t$$

to obtain the identity

$$\det\begin{pmatrix}\mathbf{a}_1\\ \vdots\\ \mathbf{a}_{2n}\\ \mathbf{b}_{2n+1}\end{pmatrix} = 4^{2n+1}ab\,X^\ell_{2n+1}\varphi\det\begin{pmatrix}C+2D & Jx\\ |x|^2x^{\mathrm{T}} & t/2a\end{pmatrix}$$

$$= 4^{2n+1}b^2t\left\{2a|x|^2\sum_{j=1}^n\left\{x_j\det\begin{pmatrix}\mathbf{c}_1\\ \vdots\\ \mathbf{0}_{j+n}+x^{\mathrm{T}}\\ \vdots\\ \mathbf{c}_{2n}\end{pmatrix} - x_{j+n}\det\begin{pmatrix}\mathbf{c}_1\\ \vdots\\ \mathbf{0}_{j}+x^{\mathrm{T}}\\ \vdots\\ \mathbf{c}_{2n}\end{pmatrix}\right\} + t\det(C+2D)\right\}$$

$$= 4^{2n+1}b^2t\big(|x|^4+a^2b^2t^2\big)^{n-1}\left\{\tfrac12 a|x|^2\sum_{j=1}^n(x_jX^\ell_{j+n}\varphi - x_{j+n}X^\ell_j\varphi) + t\big(3|x|^4+a^2b^2t^2\big)\right\}$$

$$= 4^{2n+1}b^2t^2\big(|x|^4+a^2b^2t^2\big)^{n-1}\left((3-2a^2b)|x|^4+a^2b^2t^2\right).$$

Bringing this all together we see that

$$\sum_{j=1}^{2n+1}\det\begin{pmatrix}\mathbf{a}_1\\ \vdots\\ \mathbf{b}_j\\ \vdots\\ \mathbf{a}_{2n+1}\end{pmatrix} = 2b\;4^{2n}\big(|x|^4+a^2b^2t^2\big)^{n-1}\left\{4|x|^8+2bt^2\big(3|x|^4+a^2b^2t^2\big)\right\},$$

and consequently

$$\det(\varphi A - \tfrac{\beta+4}{4}B)$$
$$= -b\;(4\varphi)^{2n}\big(|x|^4+a^2b^2t^2\big)^{n-1}\left\{2(\beta+1)|x|^8+(\beta+2)bt^2\big(3|x|^4+a^2b^2t^2\big)-2|x|^4a^2b^2t^2\right\}$$
$$= -b\;(4\varphi)^{2n}\big(|x|^4+a^2b^2t^2\big)^{n-1}\left\{2(\beta+1)|x|^8+\big(3(\beta+2)-2a^2b\big)|x|^4bt^2+(\beta+2)a^2b^3t^4\right\}.$$

By analyzing the discriminant

$$\Delta = 4a^4b^2-4(\beta+2)(2\beta+5)a^2b+9(\beta+2)^2,$$

we see that our Hessian will be *non-degenerate* provided either

$$2a^2b\le 3(\beta+2)\qquad\text{or}\qquad |2a^2b-(2\beta+5)(\beta+2)|<(\beta+2)\sqrt{(2\beta+5)^2-9},$$

which reduces simply to the condition that

$$2a^2b<(\beta+2)\left(2\beta+5+\sqrt{(2\beta+5)^2-9}\right).$$

□

We conclude by remarking that when $2a^2b\ge C_\beta$ the Hessian degenerates along the paraboloids

$$|x|^4=\frac{2a^2b-3(\beta+2)\pm\sqrt{\Delta}}{4(\beta+1)}\,t^2.$$

In particular when $2a^2b=C_\beta$ we have that $\Delta=0$ and hence the Hessian degenerates along the paraboloid

$$|x|^4=\frac{C_\beta-3(\beta+2)}{4(\beta+1)}\,t^2=\frac{(\beta+1)(\beta+2)+\sqrt{(\beta+1)(\beta+4)}}{2(\beta+1)}\,t^2.$$

When $a = 0$ (not a Heisenberg type group, but still a homogeneous group) we of course have $X_j^\ell = X_j^r = \partial/\partial x_j$ and it is then straightforward to verify that in this case

$$\det(\varphi A - \tfrac{\beta+4}{4}B) = -b\,(4\varphi)^{2n}|x|^{4n}\left\{2(\beta+1)|x|^4 + 3(\beta+2)bt^2\right\}.$$

In particular the Hessian degenerates along the line $x = 0$.

We hope to further investigate these degenerate examples in the future.

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