Further Properties of L^p Spaces

In general $L^p \not\subseteq L^q$ for all $p \neq q$. Instructive example:

(i) $|x|^{-1}\chi_{\{|x|\leq 1\}} \in L^p(\mathbb{R}^n) \Leftrightarrow p<n$

(ii) $|x|^{-1}\chi_{\{|x|\geq 1\}} \in L^p(\mathbb{R}^n) \Leftrightarrow p>n$.

We two reasons why a function f may fail to be in L^p: either

(i) $|f|^p$ blows up too rapidly near some point

or (ii) $|f|^p$ fails to decay suff. rapidly at infinity.

Note: In the first situation the behavior of $|f|^p$ becomes worse as p increases, while in the second it becomes better. In other-words, if $p<q$, then functions in L^p can be locally more singular than functions in L^q, whereas functions in L^q can be globally more spread out than those in L^p.

Theorem 1: If $0<p<q<r\leq\infty$, then $L^q \subseteq L^p + L^r$; that is each $f\in L^q$ can be written as $f=g+h$ with $g\in L^p$ and $h\in L^q$.

Proof: Given $f\in L^q$, define $A=\{x:|f(x)|>1\}$ & $g=f\chi_A$ and $h=f\chi_{A^c}$.

Then, $|g|^p = |f|^p\chi_A \leq |f|^q\chi_A \Rightarrow g\in L^p$ (for $r=\infty$, $\|h\|_\infty \leq 1$)

& $|h|^r = |f|^r\chi_{A^c} \leq |f|^q \chi_{A^c} \Rightarrow h \in L^r$. $\square$

Arguing in a similar manner one can also obtain the follow results:

- $L^p \cap L^r \subseteq L^q$ whenever $0 < p < q < r \leq \infty$.
- $L^q(X) \subseteq L^p(X)$ whenever $0 < p < q \leq \infty$ & $m(X) < \infty$.

Using Hölder's Inequality we can obtain the following quantitative estimates:

<u>Theorem 2</u>: If $0 < p < q < r \leq \infty$, then $L^p \cap L^r \subseteq L^q$ <u>and</u>

$$\|f\|_q \leq \|f\|_p^{\lambda} \|f\|_r^{1-\lambda}$$

where $\lambda \in (0,1)$ is defined by $\frac{1}{q} = \lambda \frac{1}{p} + (1-\lambda)\frac{1}{r}$ $\left(\Leftrightarrow \lambda = \dfrac{q^{-1} - r^{-1}}{p^{-1} - r^{-1}}\right)$.

<u>Theorem 3</u>: If $m(X) < \infty$ and $0 < p < q \leq \infty$, then $L^q(X) \subseteq L^p(X)$

<u>and</u> $$\|f\|_p \leq m(X)^{\frac{1}{p} - \frac{1}{q}} \|f\|_q .$$

<u>Proof of Theorem 3</u>:

- <u>If $q = \infty$</u>: $\|f\|_p^p = \int_X |f|^p \leq \|f\|_\infty^p \int_X 1 = \|f\|_\infty^p \, m(X)$.
- <u>If $q < \infty$</u>: Use Hölder with conjugate exponents $\frac{q}{p}$ & $\frac{q}{q-p}$;

$$\|f\|_p^p = \int_X |f|^p \cdot 1 \leq \left(\int_X |f|^q\right)^{p/q} \left(\int_X 1\right)^{1-p/q} = \|f\|_q^p \, m(X)^{1-p/q}.$$

$\square$

Proof of Theorem 2

$\lambda = p/q$

- If $r = \infty$:

$$|f|^q = |f|^{\lambda q} |f|^{(1-\lambda)q} \leq \|f\|_\infty^{q-p} |f|^p$$

$$\Rightarrow \quad \|f\|_q^q \leq \|f\|_\infty^{q-p} \|f\|_p^p$$

$$\Rightarrow \quad \|f\|_q \leq \|f\|_p^\lambda \|f\|_\infty^{1-\lambda} .$$

- If $r < \infty$:

$$|f|^q = |f|^{\lambda q} |f|^{(1-\lambda)q}$$

$$\Rightarrow \quad \|f\|_q^q \leq \left(\int |f|^p \right)^{\frac{\lambda q}{p}} \cdot \left(\int |f|^r \right)^{\frac{(1-\lambda)q}{r}} = \|f\|_p^{\lambda q} \|f\|_r^{(1-\lambda)q} .$$

Hölder with conjugate exponents $\frac{p}{\lambda q}$ & $\frac{r}{(1-\lambda)q}$

[Note: $\frac{\lambda q}{p} + \frac{(1-\lambda)q}{r} = 1 \Leftrightarrow \frac{1}{q} = \lambda \frac{1}{p} + (1-\lambda)\frac{1}{r}$]

$\square$

Exercise: Let $0 < q < \infty$. Construct a function f on $(0,\infty)$ that is in $L^q((0,\infty))$, but not in $L^p((0,\infty))$ for all $p \neq q$.