Tools for Computing Limits

Easy Exercise, right?

Recall that

$$\lim_{n\to\infty} a_n = a \iff \lim_{n\to\infty} (a_n - a) = 0 \iff \lim_{n\to\infty} |a_n - a| = 0$$

A common technique for showing that a given sequence converges to 0 is the following:

Proposition ("Baby Squeeze Theorem")

If $|x_n| \leq y_n$ for all $n \in \mathbb{N}$ & $\lim_{n\to\infty} y_n = 0$, then $\lim_{n\to\infty} x_n = 0$ also.

Proof: Let $\varepsilon > 0$. Since $y_n \to 0$ we know $\exists\, N$ such that if $n > N$ then $|y_n| < \varepsilon$ & hence $|x_n - 0| = |x_n| < \varepsilon$. □

(*) Note: "Baby Squeeze" only really requires that $|x_n| \leq y_n$ for all sufficiently large $n \in \mathbb{N}$ since in the proof above we can simply ensure that N is chosen large enough to ensure that not only $|y_n| < \varepsilon$ but also $|x_n| \leq |y_n|$.

In order to use "Baby Squeeze" we need to have a collection of examples of sequences that converge to 0.

Example 1: If $\lim_{n\to\infty} a_n = 0$ and $p > 0$, then $\lim_{n\to\infty} a_n^p = 0$.

[In particular, since $\frac{1}{n} \to 0$, it follows that $\frac{1}{n^p} \to 0$ $\forall p > 0$.]

Example 2: If $\lim_{n\to\infty} a_n = 0$ and $k \geq 0$, then $\lim_{n\to\infty} k a_n = 0$.

Verification of Example 1

Let $\varepsilon > 0$ & $p > 0$. Since $a_n \to 0$ we know $\exists N$ such that $n > N$ implies $|a_n - 0| = |a_n| < \varepsilon^{1/p}$ (since $\varepsilon^{1/p} > 0$).

This in turn implies that if $n > N$ then $|a_n^p - 0| = |a_n|^p < (\varepsilon^{1/p})^p = \varepsilon$. $\square$

Verification of Example 2

Let $\varepsilon > 0$ & $k > 0$ (note that example is obvious if $k = 0$).

Since $a_n \to 0$ we know $\exists N$ such that $n > N$ implies

$$|a_n - 0| = |a_n| < \varepsilon/k \quad (\text{since } \varepsilon/k > 0).$$

This in turn implies that if $n > N$ then $|ka_n - 0| = k|a_n| < k\left(\frac{\varepsilon}{k}\right) = \varepsilon$. $\square$

It is easy to see, using the limit laws, that "Baby Squeeze" in fact implies the following more general "Squeeze Theorem":

Proposition ("Squeeze Theorem")

If $a_n \leq b_n \leq c_n$ for all sufficiently large $n \in \mathbb{N}$ and $\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n = L$, then $\lim_{n\to\infty} b_n = L$ too.

Proof: Since $0 \leq b_n - a_n \leq c_n - a_n$ $\forall$ suff. large $n \in \mathbb{N}$ and $\lim_{n\to\infty} (a_n - c_n) = L$ (by limit law 1 & 2 (with $k = -1$)) it follows by "Baby Squeeze" that

$$\lim_{n\to\infty} (b_n - a_n) = 0 \Rightarrow \lim_{n\to\infty} b_n = \lim_{n\to\infty} ((b_n - a_n) + a_n) = 0 + L = L. \quad \square$$

limit law 1

Examples of using "Baby Squeeze"

Claim 1: $\lim_{n\to\infty} \frac{\sin(n)}{n^{1/2}} = 0$

Ex 1 above

Proof: Since $\left|\frac{\sin(n)}{n^{1/2}}\right| \leq \frac{1}{n^{1/2}}$ & $\frac{1}{n^{1/2}} \to 0$ it follows from "Baby Squeeze" that $\frac{\sin(n)}{n^{1/2}} \to 0$ also. $\square$

Claim 2: $\lim_{n\to\infty} \frac{1}{1+nx} = 0$ if $x>0$.

By Ex 2 above since $\frac{1}{x}>0$ constant.

Proof Since $\left|\frac{1}{1+nx}\right| \leq \frac{1}{x}\cdot\frac{1}{n}$ & $\frac{1}{x}\cdot\frac{1}{n} \to 0$ it follows from "Baby Squeeze" that $\frac{1}{1+nx} \to 0$ if $x>0$. $\square$

Claim 3: If $a_n \geq 0$ for all $n\in\mathbb{N}$ and $\lim_{n\to\infty} a_n = a$, then $\lim_{n\to\infty}\sqrt{a_n} = \sqrt{a}$.

⊛ In homework you have been asked to verify this claim by arguing directly from the definition, i.e. "using ε's". Here is a verification using "Baby Squeeze":

Proof: It follows from "Order Limit Laws" that $a \geq 0$.

- If $a=0$, then result follows from Example 1 with $p=1/2$.
- If $a>0$, then

$$|\sqrt{a_n}-\sqrt{a}| = \frac{|a_n-a|}{\sqrt{a_n}+\sqrt{a}} \leq \frac{|a_n-a|}{\sqrt{a}}$$

multiplying by conjugate

by Ex 2 above

Since $\frac{1}{\sqrt{a}}|a_n-a| \to 0$ it follows from "Baby Squeeze" that $\sqrt{a_n}-\sqrt{a} \to 0 \Leftrightarrow \sqrt{a_n}\to\sqrt{a}$. $\square$

We close with one more important example.

<u>Example 3</u>: $\lim_{n\to\infty} r^n = 0$ if $|r|<1$.

<u>Verification</u>: If $|r|<1$, then $\frac{1}{|r|}>1$ and hence we can write $\frac{1}{|r|}=1+x$ for some $x>0$. It follows that

$$\frac{1}{|r|^n} = (1+x)^n \geq 1+nx \text{ for all } n\in\mathbb{N} \quad \text{(by Binomial Thm)}$$

$$\Rightarrow |r^n - 0| = |r^n| = |r|^n \leq \frac{1}{1+nx} \quad \forall n\in\mathbb{N}.$$

Since $\frac{1}{1+nx}\to 0$ (Claim 2 above) it follows by "Baby Squeeze" that $r^n \to 0$ whenever $|r|<1$. $\square$