Exercise Sheet 3

Problems and Exercises

1. Prove that $W_r(2) = r + 1$ for all $r \geq 1$.

2. Show that there exists a 2-coloring of $\mathbb{N}$ that does not contain an infinitely long arithmetic progression.

3. Let k, r, N, x and d be positive integers. Prove that every r-coloring of $[1, N]$ contains a monochromatic k-term arithmetic progression if and only if every r-coloring of

$$S = \{x, x + d, x + 2d, \ldots, x + (N - 1)d\}$$

contains a monochromatic k-term arithmetic progression.

4. Show that any two coloring of $[1, 325]$ contains at least one monochromatic arithmetic progression of length three using the following steps (this strategy will be useful when we prove Van der Waerden's Theorem):

   (a) i. Show that any 2-coloring of five consecutive natural numbers must contain a 3-term arithmetic progression whose first two elements are monochromatic.

   ii. How many consecutive numbers are needed to ensure that any r-coloring contains a 3-term arithmetic progression whose first two elements are monochromatic?

   (b) Consider a 2-coloring of $\mathbb{N}$. How many consecutive blocks of the form

$$\{x, x + 1, x + 2, x + 3, x + 4\}$$

   are needed to ensure a 3-term arithmetic progression of blocks where the first two blocks are identically colored?

   (c) Prove that any 2-coloring of $[1, 325]$ must contain a monochromatic 3-term arithmetic progression.

5. Let $r \in \mathbb{N}$ and assume $W_r(3)$ exists.

   (a) Let $m \geq 1$ be an integer and 2-color $[1, mW_{2^m}(3)]$. Prove that there exists a block, B, of m consecutive numbers such that $B, B + d$ and $B + 2d$ are identically colored.

   (b) Fix $m \geq \frac{3}{2}W_2(3)$. Notice that any block of length m necessarily contains a 4-term arithmetic progression whose first three terms are monochromatic. Use this observation, together with part (a), to show that any two coloring of $[1, \frac{2}{3}mW_{2^m}(3)]$ contain a monochromatic 4-term arithmetic progression.

6. A *fan of radius* 3*, dimension* d*, and base point* x is a d-tuple

$$(\{x, x + h_1, x + 2h_1\}, \ldots, \{x, x + h_d, x + 2h_d\}).$$

We say that a fan is *polychromatic* if the base point, x, and the *spokes*,

$$\{x + h_1, x + 2h_1\}, \ldots, \{x + h_2, x + 2h_2\},$$

are all monochromatic with distinct colors.

Suppose that N is sufficiently large, $[1, N]$ is 3-colored and contains no monochromatic 3-term arithmetic progressions.

(a) Show that the coloring contains two identically colored blocks containing a polychromatic fan of radius 3 and dimension 1.

(b) Use part (a) to find two identically colored polychromatic fans of radius 3 and dimension 2. Use this to show that the coloring must contain a 3-term arithmetic progression.

(c) Can you now extend this argument to prove the existence of $W_r(3)$ for all $r \geq 1$?

Some of these problems were taken from one of the following sources:

[1] B. Landman and A. Robertson *Ramsey Theory on the Integers.* American Math Society, 2004.

[2] T. Tao and V. Vu *Additive combinatorics.* Cambridge University Press, Cambridge, UK, 2006.