Math 3100
Sample Exam 3 – Version 2

No calculators. Show your work. Give full explanations. Good luck!

1. (7 points)
   (a) Carefully state the *Intermediate Value Theorem.*
   (b) Let f be a continuous function on the closed interval $[0, 1]$ with range also contained in $[0, 1]$. Prove that f must have a fixed point; that is, show that $f(x) = x$ for at least one value of $x \in [0, 1]$.

2. (10 points) Let $f(x) = \begin{cases} x^4 \sin(x^{-2}), & x \neq 0 \\ 0, & x = 0 \end{cases}$.
   (a) Show that f is differentiable at 0 and compute $f'(x)$ for all $x \in \mathbb{R}$.
   (b) Is f' continuous at 0? Give your reasoning.
   (c) Is f' differentiable at 0? Give your reasoning.

3. (8 points)
   (a) Find the 4th order Maclaurin polynomial for $f(x) = \dfrac{\cos(x^2)}{1 + x}$.
   (b) Use part (a) to find the value of $f^{(4)}(0)$ without differentiating.

4. (10 points)
   (a) Carefully state the *Lagrangian Remainder Estimate* for Maclaurin series.
   (b) Use the *Lagrangian Remainder Estimate* to determine the following:
      i. An estimate for the accuracy of approximating $\sin x$ by $x - x^3/6$ when $|x| \leq 1/2$.
      ii. Values of x for which the accuracy of approximating $\sin x$ by $x - x^3/6$ is less than 10^{-3}.

   Note that you are <u>not</u> permitted to use the Alternating Series Remainder Estimate above.
   (c) Obtain, by any means, an estimate for the accuracy of approximating

$$\int_0^1 \frac{\sin x}{x}\, dx \quad \text{by} \quad 1 - \frac{1}{18}.$$

5. (15 points)
   (a) Carefully state the definition of uniform convergence of a sequence of functions $\{f_n\}$ to a function f on a set A.
   (b) Consider the sequence of functions

$$f_n(x) = \frac{x}{1 + x^n}.$$

      i. Find the pointwise limit of $\{f_n\}$ on $[0, \infty)$.
      ii. Explain how we know that the convergence cannot be uniform on $[0, \infty)$.

   (c) i. Show that $\sum_{n=1}^{\infty} \dfrac{x}{1 + x^n}$ diverges for all $x \in (0, 1]$, but converges if $x > 1$.
      ii. Let $f(x) = \sum_{n=1}^{\infty} \dfrac{x}{1 + x^n}$ on $(1, \infty)$.
         A. Prove that the series defining f does not converge uniformly on $(1, \infty)$.
         B. Prove that f is a continuous function on $(1, \infty)$.

   Hint: Show that the series defining f converges uniformly on $[a, \infty)$ for any $a > 1$.

Math 3100 - Sample Exam 3 (Version 2) - SOLUTIONS

1. (a) Intermediate Value Theorem

Let $f:[a,b]\to\mathbb{R}$ be continuous. If L is a real number between $f(a)$ & $f(b)$, then $\exists\, c\in(a,b)$ with $f(c)=L$.

(b) Claim

If f is a continuous function on $[0,1]$ with range contained in $[0,1]$, then $\exists\, x\in[0,1]$ with $f(x)=x$.

Proof

Consider $g(x)=f(x)-x$ which is also continuous on $[0,1]$.

Since $g(0)=f(0)-0=f(0)\geq 0$ and $g(1)=f(1)-1\leq 1-1=0$

IVT $\Rightarrow$ $g(x)=0$ for some $x\in[0,1]$

$\Updownarrow$

$f(x)=x$. $\square$

2. Let $f(x)=\begin{cases} x^4\sin(x^{-2}) & \text{if } x\neq 0 \\ 0 & \text{if } x=0\end{cases}$

(a) • If $x\neq 0$, then $f'(x)=4x^3\sin(x^{-2})-2x\cos(x^{-2})$.

• Claim f is diff'ble at 0 with $f'(0)=0$.

Proof

$$\lim_{x\to 0}\frac{f(x)-f(0)}{x-0}=\lim_{x\to 0}\frac{f(x)}{x}\overset{\text{as } x\neq 0}{=}\lim_{x\to 0}x^3\sin(x^{-2})=0$$ $\square$

By Squeeze Thm

(since $|x^3\sin(x^{-2})|\leq|x|^3\to 0$)

Thus $$f'(x) = \begin{cases} 4x^3\sin(x^{-2}) - 2x\cos(x^{-2}) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0. \end{cases}$$

(b) <u>Claim</u> f' is continuous at $x_0 = 0$.

<u>Proof</u>

$$\lim_{x\to 0} f'(x) \overset{\text{as } x\neq 0}{=} \lim_{x\to 0} \left[4x^3\sin(x^{-2}) - 2x\cos(x^{-2})\right] = 0 = f'(0)$$

Again by Squeeze Theorem since

$$|4x^3\sin(x^{-2}) - 2x\cos(x^{-2})| \le 4|x|^3 + 2|x| \to 0. \quad \square$$

(c) <u>Claim</u> f' is <u>not</u> differentiable at $x_0 = 0$.

<u>Proof</u>

$$\lim_{x\to 0} \frac{f'(x) - f'(0)}{x - 0} = \lim_{x\to 0} \frac{f'(x)}{x} \overset{\text{as } x\neq 0}{=} \underbrace{\lim_{x\to 0}\left(4x^2\sin(x^{-2}) - 2\cos(x^{-2})\right)}_{\text{Does not exist}} \quad \square$$

¿<u>Why</u>?

This limit does <u>not</u> exist since $\lim_{x\to 0} 4x^2\sin(x^{-2}) = 0$ by the squeeze thm (as $|4x^2\sin(x^{-2})| \le 4|x|^2 \to 0$) <u>but</u> $\lim_{x\to 0} 2\cos(x^{-2})$ does not exist (a fact that one can see readily by considering the sequence $x_n = \frac{1}{\sqrt{2\pi n}}$ & $y_n = \frac{1}{\sqrt{\frac{\pi}{2} + 2\pi n}}$)

Maclaurin Series for $\frac{1}{1+x}$ Maclaurin series for $\cos(x^2)$

3. (a)

$$f(x) = \frac{\cos(x^2)}{1+x} = \underbrace{\left(1 - x + x^2 - x^3 + x^4 - \cdots\right)}_{} \underbrace{\left(1 - \frac{x^4}{2} + \frac{x^8}{24} - \cdots\right)}_{}$$

$$= \underbrace{1 - x + x^2 - x^3 + \frac{1}{2}x^4}_{\text{"4th order Maclaurin Poly for } f\text{"}} + \cdots$$

(b) $f^{(4)}(0) = 4!\left(\frac{1}{2}\right) = \underline{\underline{12}}$

4. (a) <u>Lagrangian Remainder Estimate for Maclaurin Series</u>

If f is $(n+1)$-times differentiable on $(-R,R)$, then for any $x \in (-R,R)\setminus\{0\}$ $\exists$ c between 0 & x such that

$$f(x) - \left[f(0) + f'(0)x + \cdots + \frac{f^{(n)}(0)}{n!}x^n\right] = \frac{f^{(n+1)}(c)}{(n+1)!}x^{n+1}.$$

(b) (i) $\left|\sin x - \left(x - \frac{x^3}{6}\right)\right| \leq \frac{|x|^5}{5!} \leq \underline{\underline{\frac{1}{5!}\left(\frac{1}{2}\right)^5}}$ if $|x| \leq \frac{1}{2}$.

since $x - x^3/6$ is the $\underline{\underline{4^{th}}}$ order Maclaurin poly for $\sin x$ (& $|\sin x|$ & $|\cos x|$ always ≤ 1)

(ii) $\left|\sin x - \left(x - x^3/6\right)\right| \leq \frac{|x|^5}{5!} \leq \frac{1}{1000}$ if $\underline{\underline{|x| \leq \sqrt[5]{\frac{5!}{1000}}}}$

(c) Since $\sin x - \left(x - \frac{x^3}{6}\right) = \frac{\cos(c)}{120}x^5$ for some $0 < c < x$

$$\Rightarrow \frac{\sin x}{x} - \left(1 - \frac{x^2}{6}\right) = \frac{\cos(c)}{120}x^4 \quad \text{for some } 0 < c < x$$

$$\Rightarrow \int_0^1 \frac{\sin x}{x}dx - \underbrace{\int_0^1\left(1 - \frac{x^2}{6}\right)dx}_{= 1 - \frac{1}{18}} = \int_0^1 \frac{\cos(c)}{120}x^4 dx \leq \frac{1}{120}\int_0^1 x^4 dx = \underline{\underline{\frac{1}{600}}}$$

5. (a) $f_n \to f$ uniformly on A if $\lim_{n\to\infty} \sup_{x\in A} |f_n(x) - f(x)| = 0$.

(b) (i) $$\lim_{n\to\infty} \frac{x}{1+x^n} = \begin{cases} x & \text{if } 0 \le x < 1 \\ 1/2 & \text{if } x = 1 \\ 0 & \text{if } x > 1. \end{cases}$$

(ii) Since the limit function is <u>not</u> continuous at $x = 1$ but each f_n <u>is</u> continuous on $[0,\infty)$ the convergence <u>cannot</u> be uniform on $[0,\infty)$ (or any set A containing 1).

(c) (i) · $\sum_{n=1}^{\infty} \frac{x}{1+x^n}$ diverges if $x \in (0,1]$ since $\frac{x}{1+x^n} \not\to 0$ on $(0,1]$.

· Since $$\frac{x}{1+x^n} \leqslant \frac{x}{x^n} = \frac{1}{x^{n-1}} = \left(\frac{1}{x}\right)^{n-1}$$

& $\sum_{n=1}^{\infty} \left(\frac{1}{x}\right)^{n-1}$ converges $\forall x > 1$.

$\Rightarrow \sum_{n=1}^{\infty} \frac{x}{1+x^n}$ converges for all $x > 1$.

(ii) Let $f(x) := \sum_{n=1}^{\infty} \frac{x}{1+x^n}$ on $(1,\infty)$.

A. Since $\frac{x}{1+x^n} \not\to 0$ uniformly on $(1,\infty)$, convergence cannot be uniform.

$$\left[\sup_{x\in(1,\infty)} \left|\frac{x}{1+x^n} - 0\right| \geqslant \frac{2^{1/n}}{1+(2^{1/n})^n} \geqslant \frac{1}{2} \ \forall n \right]$$

$\uparrow$

$x = 2^{1/n}$

B. <u>Since $\sum \frac{x}{1+x^n}$ conv unif on $[a,\infty)$ $\forall a > 1$:</u>

let $a > 1$, then $\left|\frac{x}{1+x^n}\right| \le \frac{1}{x^{n-1}} \le \frac{1}{a^{n-1}}$ $\forall x \in [a,\infty)$

& $\sum \frac{1}{a^{n-1}}$ conv So M-test implies result.