Math 3100

Sample Exam 1 – Version 2

No calculators. Show your work. Give full explanations. Good luck!

1. (8 points) Give counterexamples to the following **false** statements, no proofs are required.

   Note that in each instance the converse statement is in fact true.

   (a) If $\{x_n\}$ is bounded, then $\{x_n\}$ is convergent.

   (b) If $\{x_n\}$ is convergent, then $\{x_n\}$ is both bounded and monotone.

   (c) If $\{x_n\}$ contains a convergent subsequence, then $\{x_n\}$ is bounded.

   (d) If A contains its supremum, then A has finitely many elements.

2. (4 points) Let $\{x_n\}$ and $\{y_n\}$ be two sequences of real numbers. Prove that if $\lim_{n\to\infty} x_n = x$ and $|x_n - y_n| \leq \frac{1}{n}$ for all $n \in \mathbb{N}$, then $\lim_{n\to\infty} y_n = x$.

3. (14 points)

   (a) Let $\{x_n\}$ be a sequence of real numbers. Carefully state the definition of $\lim_{n\to\infty} x_n = x$.

   (b) Use your definition to prove that $\lim_{n\to\infty} \frac{3n+4}{n+1} = 3$.

   (c) Assume that $\lim_{n\to\infty} x_n = x > 0$. Using <u>only</u> the definition of convergence prove the following two statements;

      i. there exists a number N such that if $n > N$, then $x_n > \frac{1}{2}x$.

      ii. $\lim_{n\to\infty} \frac{1}{x_n} = \frac{1}{x}$

4. (14 points)

   (a) Carefully state the *Axiom of Completeness* (the least upper bound axiom).

   (b) Let $\{x_n\}$ be a bounded increasing sequence of real numbers. Use the *Axiom of Completeness* to prove that $\lim_{n\to\infty} x_n$ exists and equals $\sup\{x_n : n \in \mathbb{N}\}$.

   (c) Prove that if $x_1 = \sqrt{2}$ and $x_{n+1} = \sqrt{2x_n}$ for all $n \in \mathbb{N}$, then the sequence $\{x_n\}$ converges and find the value of its limit.

5. (10 points) Let $\{x_n\}$ be a sequence of real numbers that satisfy the property that $|x_n| \leq 1$ for all $n \in \mathbb{N}$.

   (a) Prove that <u>if</u> $\lim_{n\to\infty} x_n$ exists and equals x, then $x \leq 1$.

   (b) Carefully explain why $\limsup_{n\to\infty} x_n$ exists and why $\limsup_{n\to\infty} x_n \leq 1$.

Math 3100 – Sample Exam 1 (Version 2) – SOLUTIONS

1. (a) $x_n = (-1)^n$

   (b) $x_n = \dfrac{(-1)^n}{n}$

   (c) $x_n = \begin{cases} \frac{1}{n} & \text{if } n \text{ odd} \\ n & \text{if } n \text{ even} \end{cases}$

   (d) $A = [0,1]$.

2. Claim

   If $\lim\limits_{n\to\infty} x_n = x$ & $|x_n - y_n| \leq \frac{1}{n}$ $\forall n \in \mathbb{N}$, then $\lim\limits_{n\to\infty} y_n = x$.

   Proof It follows from "Baby Squeeze" that $(y_n - x_n) \longrightarrow 0$. Hence

   $y_n = x_n + (y_n - x_n) \longrightarrow x + 0 = x$ by limit laws. $\square$

3. (a) $\lim\limits_{n\to\infty} x_n = x \iff \forall \varepsilon > 0\ \exists N$ such that $n > N$ implies $|x_n - x| < \varepsilon$.

   (b) Claim $\lim\limits_{n\to\infty} \dfrac{3n+4}{n+1} = 3$

   Proof Let $\varepsilon > 0$ & set $N = \varepsilon^{-1}$. It follows that if

   $n > N \Rightarrow \left| \dfrac{3n+4}{n+1} - 3 \right| = \dfrac{1}{n+1} < \dfrac{1}{n} < \dfrac{1}{N} = \varepsilon.$ $\square$

   (c)(i) Since $x_n \to x$ & $x > 0$ it follows (taking $\varepsilon = \frac{x}{2}$) that $\exists N$ such that if $n > N$ then

   $$|x_n - x| < \frac{x}{2} \iff \underline{-\frac{x}{2} < x_n - x} < \frac{x}{2}$$

   $$\Rightarrow x_n > \frac{x}{2}$$

(ii) Claim If $x_n \to x > 0$, then $\frac{1}{x_n} \to \frac{1}{x}$.

Proof Let $\varepsilon > 0$.

Since $x_n \to x$ $\exists N_1$ s.t. $x_n > \frac{x}{2}$ $\forall n > N_1$ (Part (i) above)

Since $x_n \to x$ $\exists N_2$ s.t. $|x_n - x| < \frac{x^2}{2}\varepsilon$, $\forall n > N_2$.

If $n > \max\{N_1, N_2\}$ it follows that

$$\left|\frac{1}{x_n} - \frac{1}{x}\right| = \frac{|x_n - x|}{|x_n|\,x} \underset{\text{since } n>N_1}{\leq} \frac{|x_n - x|}{(x^2/2)} \underset{\text{since } n>N_2}{<} \frac{1}{(x^2/2)}\left(\frac{x^2}{2}\varepsilon\right) = \varepsilon.$$

□

4. (a) A∘C $\Leftrightarrow$ Every non-empty set of reals that is bounded above has a least upper bound.

(b) Claim (MCT)

If $\{x_n\}$ is bounded increasing sequence of reals then

$\lim_{n\to\infty} x_n$ exists & equals $\sup\{x_n : n \in \mathbb{N}\}$.

Proof Since $\{x_n\}$ is bounded the A∘C ensures $s = \sup\{x_n : n \in \mathbb{N}\}$ exists.

Let $\varepsilon > 0$. Since $s = \sup\{x_n : n \in \mathbb{N}\}$ $\exists N$ so that

$$s - \varepsilon < x_N \leq s$$

$\{x_n\}$ increasing

$$\Rightarrow \quad s - \varepsilon < x_n \leq s < s + \varepsilon \quad \forall n > N$$

$$\Leftrightarrow \quad |x_n - s| < \varepsilon \quad \forall n > N.$$

□

(b) Let $a_1 = \sqrt{2}$ and $a_{n+1} = \sqrt{2a_n}$ $\forall n \in \mathbb{N}$

Claim 1 $\sqrt{2} \le a_n \le 2$ $\forall n \in \mathbb{N}$

Proof (Induction)

Base Case ($n=1$): $a_1 = \sqrt{2}$ ✓

Suppose $\sqrt{2} \le a_n \le 2$ for some $n \in \mathbb{N}$, it then follows that

$$2 \le 2a_n \le 4 \Rightarrow \sqrt{2} \le \underbrace{\sqrt{2a_n}}_{=a_{n+1}} \le \sqrt{4} = 2$$

$\square$.

Claim 2 $\{a_n\}$ increasing.

by Claim 1

Proof $a_{n+1} - a_n = \sqrt{2a_n} - a_n$

$$= \frac{2a_n - a_n^2}{\sqrt{2a_n} + a_n} = \frac{\overset{\geq 0}{(2-a_n)}\overset{>0}{a_n}}{\underbrace{\sqrt{2a_n} + a_n}_{>0}} \geq 0 \quad \forall n \in \mathbb{N}$$

$\square$

Since $\{a_n\}$ is increasing and bounded above it follows from the MCT (& order limit laws) that

$$a_n \to L \quad \text{for some} \quad \sqrt{2} \le L \le 2.$$

Since $a_{n+1} \to L$ and $\sqrt{2a_n} \to \sqrt{2L}$

$$\Rightarrow \quad L = \sqrt{2L}$$

$$\Rightarrow \quad L^2 = 2L \Rightarrow \cancel{L=0} \text{ or } \underline{\underline{L=2}}.$$

This is a special case of the "order limit law".

5.

(a) Claim If $|x_n| \le 1 \ \forall n \in \mathbb{N}$ & $\lim_{n\to\infty} x_n = x$, then $|x| \le 1$.

Proof 1 Suppose $|x| > 1$. Since $|x_n| \to |x|$ $\exists N$ such that

$$\big||x_n| - |x|\big| < (|x| - 1) \quad \forall n > N \quad (\varepsilon = |x| - 1)$$

$$\Updownarrow$$

$$1 - |x| < |x_n| - |x| < |x| - 1 \quad \forall n > N$$

$$\Rightarrow |x_n| > 1 \quad \forall n > N \quad \lightning .$$

$\square$

Proof 2 Let $\varepsilon > 0$. Since $x_n \to x$ $\exists N$ s.t. $|x_N - x| < \varepsilon$.

$$\Rightarrow |x| = |x - x_N + x_N| \le |x - x_N| + |x_N| \le 1 + \varepsilon .$$

Since $\varepsilon > 0$ was arbitrary it follows that $|x| \le 1$. $\square$

(b) Recall that

$$\limsup_{n\to\infty} x_n := \sup(S) \text{ where } S = \{\text{subsequential limits of } \{x_n\}\}.$$

Since $\{x_n\}$ is bounded it follows from BW that $S \neq \emptyset$ and from (a) above that every $x \in S$ must satisfy $|x| \le 1$.

$A_oC \Rightarrow \sup(S)$ exists!

Since 1 is an upper bound for S & $\sup(S)$ is the least upper bound it follows that $\sup(S) \le 1$.