. Tomorrow we'll prove

Roth's Theorem : $\frac{r_3(N)}{N} \ll \frac{1}{\log\log N}$.

Recall that $r_3(N) = \max_{A \subseteq [1,N]} \{|A| : A \cap (A+d) \cap (A+2d) = \emptyset \ \forall d \neq 0\}$.

Strategy will be to identify $[1,N] \simeq \mathbb{Z}/N\mathbb{Z}$ (and assume that N odd) and use Fourier analysis on the cyclic group $\mathbb{Z}/N\mathbb{Z}$.

Model Setting. Instead of considering $[1,N]$ (or $\mathbb{Z}/N\mathbb{Z}$) we will today consider $\mathbb{F}_3^n$ and the quantity

$$r_3(\mathbb{F}_3^n) := \max_{A \subseteq \mathbb{F}_3^n} \{|A| : A \text{ contains no (non-trivial) 3APs}\}.$$

(lines!!) "A is a capset"

* This new setting is very pleasant to work with since it is more algebraic ($\mathbb{F}_3^n$ has plenty of subspaces whereas the group $\mathbb{Z}/N\mathbb{Z}$ has no non-trivial subgroups).

Theorem (Meshulam, 1995)

$$\frac{r_3(\mathbb{F}_3^n)}{N} \ll \frac{1}{\log N} \quad (N = 3^n)$$

Thm (Bateman & Katz, 2011)

$$\frac{r_3(\mathbb{F}_3^n)}{N} \ll \left(\frac{1}{\log N}\right)^{1+\varepsilon}$$

for some $\varepsilon > 0$

Propn (Dichotomy between randomness & structure)

Let $\delta > 0$ & $N \gg \delta^{-2}$. ~~Let~~ $A \subseteq \mathbb{F}_3^n$ with $|A| \geq \delta N$, then either.

If then either

(i) $\frac{1}{2}\delta^3 N^2 \leq \#\, 3AP\text{'s in } A \leq \frac{3}{2}\delta^3 N^2$

OR

(ii) $\exists$ codimension 1 affine subspace $H \leq \mathbb{F}_3^n$ on which A has increased relative density, specifically

$$|A \cap H| \geq \left(\delta + \frac{\delta^2}{4}\right)|H|.$$

Propn $\Rightarrow$ Theorem: Suppose $A \subseteq \mathbb{F}_3^n$ with $|A| \geq \delta N$ and no non-trivial 3APs.

In particular (since $N \gg \delta^{-2}$) this means that (i) does not hold, thus.

$$\exists \text{ codim } 1 \text{ affine subspace } H \text{ s.t. } \frac{|A \cap H|}{|H|} \geq \delta + \delta^2/4.$$

Identify H with $\mathbb{F}_3^{n-1}$ and conclude that we have a subset

$$A' \subseteq \mathbb{F}_3^{n-1} \text{ with } |A'| \geq \left(\delta + \frac{\delta^2}{4}\right)3^{n-1} \text{ with no 3AP's}.$$

repeat...

- We can repeat no more that n times (losing 1 dimension each time)
- Since the density increases by $\frac{\delta^2}{4}$ each time it will double in $4/\delta$ steps and infact exceed 1 in no more that $8/\delta$ steps!

$\Longrightarrow$ Must have $n \ll \frac{1}{\delta} \Leftrightarrow \delta \ll \frac{1}{n} = \frac{1}{\log_3 N}$. $\square$

Fourier Analysis on $G = \mathbb{F}_3^n$:

Given $f : G \to \mathbb{C}$ we define $\hat{f}(\xi) = \underset{x\in G}{\mathbb{E}} f(x) e^{-2\pi i x\cdot\xi/3}$

$\underset{x\in G}{\mathbb{E}} = \frac{1}{|G|}\sum_{x\in G}$.

just do $\frac{1}{N}\sum_x$.

<u>Fourier Inversion</u> : $f(x) = \sum_{\xi\in\mathbb{F}_3^n} \hat{f}(\xi) e^{2\pi i x\cdot\xi/3}$.

<u>Plancherel</u> : $\underset{x\in G}{\mathbb{E}} |f(x)|^2 = \sum_{\xi\in G} |\hat{f}(\xi)|^2$.

[<u>Proof</u>: $\sum_{\xi\in G} \hat{f}(\xi) e^{2\pi i x\cdot\xi/3} = \sum_{\xi\in G}\left(\frac{1}{N}\sum_{y\in G} f(y) e^{-2\pi i y\cdot\xi/3}\right) e^{2\pi i x\cdot\xi/3}$

$$= \sum_{y\in G} f(y) \boxed{\frac{1}{N}\sum_{\xi\in G} e^{2\pi i (x-y)\cdot\xi/3}} = f(x).\ ]$$

$$\boxed{\frac{1}{N}\sum_{\xi\in G} e^{2\pi i (x-y)\cdot\xi/3}} = \begin{cases} 1 & \text{if } x=y \\ 0 & \text{if } x\neq y \end{cases}$$

<u>Examples</u>

1. $f(x) \equiv c$ (constant).

$$\Rightarrow \hat{f}(\xi) = \frac{1}{N}\sum_{x\in\mathbb{F}_3^n} c\, e^{2\pi i x\cdot\xi/3} = \begin{cases} c & \text{if } \xi = 0 \\ 0 & \text{if } \xi\neq 0. \end{cases}$$

2. $A \subseteq \mathbb{F}_3^n$ with $|A| = \delta N$

- $|\hat{1}_A(\xi)| \leq \frac{1}{N}\sum_{x\in\mathbb{F}_3^n} 1_A(x) = \hat{1}_A(0) = \delta$.
- $\sum_{\xi} |\hat{1}_A(\xi)|^2 = \frac{1}{N}\sum_x |1_A(x)|^2 = \delta$.

3. $H \leq \mathbb{F}_3^n$ be an linear ~~affine~~ subspace. then $\exists\, \xi\in\mathbb{F}_3^n$ s.t. $H = \{x : x\cdot\xi = 0\}$.

It cosets take the form H_0, H_1, H_2 where $H_j = \{x : x\cdot\xi = j\}$.

<u>Note</u>: $\hat{1}_{H_j}(\xi) = \frac{1}{N}\sum_{x\in H_j} e^{-2\pi i x\cdot\xi/3} = e^{-2\pi i j/3}\frac{|H|}{N}$

annihilator of H

(note $|\hat{1}_{H_j}(\xi)| = \hat{1}_{H_j}(0)$.)

<u>Proof of Propn</u> Let $\delta > 0$ & $N \geq 10\delta^{-2}$. Let $A \in \mathbb{F}_3^n$ with $|A| = \delta N$.

- Suppose that (i) does not hold, i.e.

$$\left| \frac{\#\,3AP\text{'s in } A}{N^2} - \delta^3 \right| \geq \frac{\delta^3}{2}.$$

Now

$$\frac{\#\,3AP\text{'s in } A}{N^2} = \frac{1}{N^2} \sum_x \sum_d 1_A(x)\, 1_A(x+d)\, \underbrace{1_A(x+2d)}_{= \sum_\xi \hat{1}_A(\xi) e^{2\pi i (x+2d)\cdot \xi/3}}$$

$$\overset{(*)}{=} \sum_\xi \hat{1}_A(\xi) \left(\frac{1}{N} \sum_x 1_A(x) e^{-2\pi i x \cdot \xi/3} \frac{1}{N} \sum_d 1_A(x+d)\, e^{-2\pi i (x+d)(2\xi)/3} \right)$$

$$= \sum_\xi \hat{1}_A(\xi)^2 \hat{1}(-2\xi)^2$$

$$= \sum_\xi \hat{1}_A(\xi)^3$$

$$= \delta^3 + \sum_{\xi \neq 0} \hat{1}_A(\xi)^3.$$

<u>Part I</u>: If $N \geq 10\delta^{-2}$ & if (i) doesn't hold, then $\exists\, \xi \neq 0$ s.t. $|\hat{1}_A(\xi)| \geq \frac{\delta^2}{2}$

$$\Rightarrow \quad \frac{\delta^3}{2} \leq \left| \frac{\#\,3AP\text{'s in } A}{N^2} - \delta^3 \right| \leq \sum_{\xi \neq 0} |\hat{1}_A(\xi)|^3 \leq \max_{\xi \neq 0} |\hat{1}_A(\xi)| \underbrace{\sum_{\xi \in \mathbb{F}_3^n} |\hat{1}_A(\xi)|^2}_{= \delta}$$

$$\Rightarrow \quad \exists\, \xi \neq 0 \text{ s.t. } |\hat{1}_A(\xi)| \geq \frac{\delta^2}{2}.$$

♡ of argument is to now show that this $\Rightarrow \exists$ codim 1 affine subspace H s.t. $\frac{|A \cap H|}{|H|} \geq \delta + \delta^2/4$

Let $H_0 = \{x : x \cdot \xi = 0\}$ & write H_0, H_1, H_2 for the cosets

$$\frac{\delta^2}{2} \leq |\hat{1}_A(\xi)| = \left| \frac{1}{N} \sum_{x \in \mathbb{F}_3^n} 1_A(x) e^{-2\pi i x \cdot \xi / 3} \right|$$

as $\xi \neq 0$

$$= \left| \frac{1}{N} \sum_{x \in \mathbb{F}_3^n} (1_A(x) - \delta) e^{-2\pi i x \cdot \xi / 3} \right|$$

$$= \left| \frac{1}{N} \sum_{j=0}^{2} \sum_{x \in H_j} (1_A(x) - \delta) e^{-2\pi i x \cdot \xi / 3} \right|$$

$$\leq \frac{1}{3} \sum_{j=0}^{2} \left(\left| \frac{1}{|H|} \sum_{x \in H_j} (1_A(x) - \delta) e^{-2\pi i x \cdot \xi / 3} \right| + \frac{1}{|H|} \sum (1_A(x) - \delta) \right)$$

constant

$$\Rightarrow \exists\, j = 0,1,2 \text{ s.t. } \underbrace{\left| \frac{1}{|H|} \sum_{x \in H_j} (1_A(x) - \delta) \right|}_{= \frac{|A \cap H_j|}{|H|} - \delta} + \ldots \geq \frac{\delta^2}{2}$$

$$\Rightarrow \frac{|A \cap H_j|}{|H|} - \delta \geq \frac{\delta^2}{4}$$

Part II: Let $\varepsilon > 0$. If $|\hat{1}_A(\xi)| \geq \varepsilon$ for some $\xi \neq 0$
$\Rightarrow$ $\exists$ affine subspace H of codim $= 1$ s.t. $\frac{|A \cap H|}{|H|} \geq \delta + \frac{\varepsilon}{2}$.

(Note: $|x| + x \geq \delta^2/2 \Rightarrow x \geq \delta^2/4$)