# Math 8100 Assignment 3
## Lebesgue measurable sets and functions

*Due date: Thursday the 22nd of September 2022*

1. Prove that if $E \subseteq \mathbb{R}^n$ is Lebesgue measurable with $m(E) < \infty$, then for every $\varepsilon > 0$ there exists a set $A$ that is a finite union of closed cubes such that $m(E \triangle A) < \varepsilon$.

   [*Recall that $E \triangle A$ stands for the symmetric difference, defined by $E \triangle A = (E \setminus A) \cup (A \setminus E)$*]

2. A set $E \subseteq \mathbb{R}^n$ is said to be *Carathéodory measurable* if $m_*(A) = m_*(A \cap E) + m_*(E^c \cap A)$ for every set $A \subseteq \mathbb{R}^n$, where $m_*$ denotes Lebesgue outer measure.

   Prove that a set $E \subseteq \mathbb{R}^n$ is *Lebegue measurable* if and only if $E$ is *Carathéodory measurable*.

   *Hint: If $E$ is Lebesgue measurable and $A$ is any set, choose a $G_\delta$ set $B \supseteq A$ with $m_*(A) = m(B)$. Conversely, if $E$ is Carathéodory measurable and $m_*(E) < \infty$, choose a $G_\delta$ set $B$ with $B \supseteq E$ with $m_*(E) = m(B)$ and conclude that $m_*(B \setminus E) = 0$. Be sure to also consider the case when $m_*(E) = \infty$.*

3. We say that a function $f : \mathbb{R}^n \to \mathbb{R}$ is *upper semicontinuous* at a point $x$ in $\mathbb{R}^n$ if

   $$f(x) \geq \limsup_{y \to x} f(y).$$

   Prove that if $f$ is upper semicontinuous at every point $x$ in $\mathbb{R}^n$, then $f$ is Borel measurable.

4. Let $\{f_n\}$ be a sequence of measurable functions on $\mathbb{R}^n$. Prove that $\{x \in \mathbb{R}^n \ : \ \lim_{n \to \infty} f_n(x) \text{ exists}\}$ defines a measurable set.

5. Recall that the **Cantor set** $\mathcal{C}$ is the set of all $x \in [0,1]$ that have a ternary expansion $x = \sum_{k=1}^{\infty} a_k 3^{-k}$ with $a_k \neq 1$ for all $k$. Consider the function

   $$f(x) = \sum_{k=1}^{\infty} b_k 2^{-k} \quad \text{where} \quad b_k = a_k/2.$$

   (a) Show that $f$ is well defined and continuous on $\mathcal{C}$, and moreover $f(0) = 0$ as well as $f(1) = 1$.

   (b) Prove the existence of a continuous function that maps a measurable set to a non-measurable set.

6. Let us examine the map $f$ defined in Question 5 even more closely. One readily sees that if $x, y \in \mathcal{C}$ and $x < y$, then $f(x) < f(y)$ unless $x$ and $y$ are the two endpoints of one of the intervals removed from $[0,1]$ to obtain $\mathcal{C}$. In this case $f(x) = \ell 2^m$ for some integers $\ell$ and $m$, and $f(x)$ and $f(y)$ are the two binary expansions of this number. We can therefore extend $f$ to a map $F : [0,1] \to [0,1]$ by declaring it to be constant on each interval missing from $\mathcal{C}$. $F$ is called the **Cantor-Lebesgue function**.

   (a) Prove that $F$ is non-decreasing and continuous.

   (b) Let $G(x) = F(x) + x$. Show that $G$ is a bijection from $[0,1]$ to $[0,2]$.

   (c)   i. Show that $m(G(\mathcal{C})) = 1$.

      ii. By considering rational translates of $\mathcal{N}$ (the non-measurable subset of $[0,1]$ that we constructed in class), prove that $G(\mathcal{C})$ necessarily contains a (Lebesgue) non-measurable set $\mathcal{N}'$.

      iii. Let $E = G^{-1}(\mathcal{N}')$. Show that $E$ is Lebesgue measurable, but not Borel.

   (d) Give an example of a measurable function $\varphi$ such that $\varphi \circ G^{-1}$ is not measurable.

      *Hint: Let $\varphi$ be the characteristic function of a null set whose image under $G$ is not measurable.*

**Extra Challenge Problems**

*Not to be handed in with the assignment*

1. Let $\chi_{[0,1]}$ be the characteristic function of $[0,1]$. Show that there is no function $f$ satisfying $f = \chi_{[0,1]}$ almost everywhere which is also continuous on all of $\mathbb{R}$.

2. Question 6d above supplies us with an example that if $f$ and $g$ are Lebesgue measurable, then it does not necessarily follow that $f \circ g$ will be Lebesgue measurable, even if $g$ is assumed to be continuous.

   Prove that if $f$ is Borel measurable, then $f \circ g$ will be Lebesgue or Borel measurable whenever $g$ is.

3. Let $f$ be a measurable function on $[0,1]$ with $|f(x)| < \infty$ for a.e. $x$. Prove that there exists a sequence of continuous functions $\{g_n\}$ on $[0,1]$ such that $g_n \to f$ for a.e. $x \in [0,1]$.