Math 8410 - Spring 2012

Prime Numbers

A natural number greater than 1 is called <u>prime</u> if its only positive divisors are 1 and itself, and <u>composite</u> otherwise. The sequence of prime numbers begins

$$2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, \ldots$$

Recall,

<u>The Fundamental Theorem of Arithmetic</u>

Every natural number greater than 1 can be factored uniquely into a product of primes.

Thus we see that the primes are the building blocks of the integers (if one builds multiplicatively). The following is also true:

<u>Theorem 1</u> (Schnirelmann, 1933)

There is an absolute constant S, such that every natural number greater than 1 can be written as a sum of at most S primes.

We will prove this result later in the course. Note its connection to:

<u>Goldbach's Conjecture</u> (1742)

Every <u>even</u> integer greater than 2 can be written as the sum of two primes.

It would of course follow from Goldbach's conjecture that every natural number greater than 5 can be written as the sum of 3 primes

Goldbach's conjecture is of course a famously open a difficult problem, in this course we will however be able to prove the following:

Theorem 2 (Vinogradov, 1937 (Hardy & Littlewood in 1923 subject to GRH))
Every sufficiently large odd integer can be written as the sum of three primes.

The machinary we will develop to prove this result will also give:

Theorem 3
"Almost every" even integer can be written as the sum of 2 primes, in the sense that the density

$$\delta(E) = \lim_{N\to\infty} \frac{|E \cap \{1,\dots,N\}|}{N} = 0$$

where E denotes the set of even integer that are not the sum of 2 primes.

In order to establish these results we shall need (amongst other things) to prove some deep results concerning the distribution of primes, in particular the prime number theorem & the prime number theorem for primes in arithmetic progression, and more...