# Van der Waerden's Theorem

One of the earliest results in Ramsey theory is due to van der Waerden. This coloring theorem says that if the natural numbers are partitioned into finitely many classes, then one of the classes must contain an arbitrarily long arithmetic progression. In this note, we will sketch a combinatorial proof of van der Waerden.

**Definition 1.** *Let $k \in \mathbb{N}$. A k-term* arithmetic progression *is a set of the form*

$$\{x, x+h, \ldots, x+(k-1)h\},$$

*where $x, h \in \mathbb{N}$. We will write $x + [0, k-1]h$ to denote this arithmetic progression.*

**Definition 2.** *Let $k \geq 1$, $d \geq 0$, $x \in \mathbb{N}$ and $h_1, \ldots, h_d \in \mathbb{N}$. A* fan of radius $k$, dimension $d$, and base point $x$ *is a $d$-tuple*

$$(x + [0, k-1]h_1, \ldots, x + [0, k-1]h_d).$$

*We call the progressions $x + [1, k-1]h_i$ the* spokes *of the fan. A fan is said to be* polychromatic *if the base point $x$ and the spokes, $x + [1, k-1]h_i$, $(1 \leq i \leq d)$ are all monochromatic with distinct colors.*

**Theorem 1** (van der Waerden, 1927)**.** *Let $k, r \in \mathbb{N}$. If $N$ is sufficiently large and $[1, N]$ is $r$-colored, then there is a monochromatic arithmetic progression of length $k$. We denote the smallest $N$ for which the theorem is true by $W_r(k)$.*

Our proof will consist of two inductive steps. We will induct on the length of the progression, $k$. To complete this induction, we will show that a coloring must either contain a monochromatic $k$-term arithmetic progression, or a polychromatic fan of radius $k$ and dimension $d$ by inducting on $d$. The key observation is that a polychromatic fan consisting of $r$ colors, can only have $r-1$ spokes.

*Proof.* The base case, when $k = 1$, is trivial. Assume that $k \geq 2$ and that there exists an $N$ so that if $[1, N]$ is $r$-colored, then there is a monochromatic $(k-1)$-term arithmetic progression.

**Claim 1.** *For any $d \geq 1$ there exists $M$ so that if a block of $M$ consecutive colors is $r$-colored, then either there is a monochromatic $k$-term arithmetic progression, or there exists a polychromatic fan of radius $k$ and dimension $d$.*

The base case $d = 1$ follows from the existence of $W_r(k-1)$ claimed in the inductive hypothesis on progression length (check this!). Let $d \geq 2$ and assume the claim is true for $d-1$. Let $M_1$ and $M_2$ be large parameters. Consider $M_2$ consecutive blocks of $M_1$ consecutive integers. By the induction hypothesis, assuming $M_1$ is sufficiently large, either some block contains a $k$-term monochromatic progression (in which case we have proved Theorem 1) or each block must contain a polychromatic fan of radius $k$ and dimension $d-1$. We will assume the latter.

Since there are $r^{M_1}$ possible colorings of each block, as long as $M_2$ is sufficiently large, there must be an arithmetic progression of $k$ blocks,

$$B, B+a, \ldots, B+(k-1)a,$$

the last $k-1$ of which are identically colored (here we are using our first inductive hypothesis and the fact that we have assumed that there ar no monochromatic $k$-term arithmetic progressions). We note that it is sufficient to take $M_2 = 2W_{r^{M_1}}(k-1)$. Furthermore, the inductive hypothesis implies that the block $B+a$ must contain the elements of a polychromatic fan,

$$F+a = (x+a+[0,k-1]h_1, \ldots, x+a+[0,k-1]h_{d-1})$$

of radius $k$, dimension $d-1$ and base point $x+a$. Since the last $k-1$ blocks are identically colored, we now have a $k-1$ term progression of identically colored polychromatic fans

$$F+a, F+2a, \ldots, F+(k-1)a.$$

Then, the set

$$F \cup F+a \cup \cdots \cup F+(k-1)a$$

contains a polychromatic fan of radius $k$ and dimension $d$. In particular, the fan

$$\big(x+[0,k-1]a, x+[0,k-1](a+h_1), \ldots, x+[0,k-1](a+h_{d-1})\big)$$

is polychromatic. This completes the proof of our claim.

Let $d = r$. By the claim, as long as $N \geq M_1M_2$ any $r$-coloring of $[1,N]$ must contain a $k$-term monochromatic arithmetic progression, or a polychromatic fan of radius $k$ and dimension $d$. The latter case is impossible, which means that the coloring must contain a $k$-term monochromatic progression. $\square$

## References


[1] T. Tao, "The ergodic and combinatorial approaches to Szemerédi's Theorem," (2006) [arXiv:math/0604456v1].