Math 3100
Sample Exam 3 – Version 1

No calculators. Show your work. Give full explanations. Good luck!

1. (4 points) Explain why there exist no examples of the following:
   (a) A continuous function on $[0, 1]$ with range equal to $(0, 1)$.
   (b) A continuous function on $[0, 1]$ with range equal to $[0, 1] \cap \mathbb{Q}$

2. (8 points) Evaluate the following infinite series

$$\text{(a)} \quad \sum_{n=1}^{\infty} \frac{n}{4^n} \qquad\qquad \text{(b)} \quad \sum_{n=1}^{\infty} \frac{(-1)^n}{n 4^n}$$

3. (14 points)
   (a) i. Find the sixth order Maclaurin polynomial for the function

$$f(x) = \frac{x^2}{2 + x^2}$$

   ii. Without differentiating find the value of $f^{(6)}(0)$.
   (b) Let $P_3(x)$ denote the third order Taylor polynomial centered at $x_0 = 1$ of $f(x) = \log x$.
   i. Find $P_3(x)$.
   ii. Give an estimate for how well $P_3(1.5)$ approximates $\log(1.5)$.
   (c) i. Carefully state the *Lagrangian Remainder Estimate* for Maclaurin series.
   ii. Find a polynomial that approximates e^x to within 10^{-3} for all $|x| \leq 1/2$.

4. (14 points)
   (a) Carefully state what it mean to say that a function $f : \mathbb{R} \to \mathbb{R}$ is differentiable at x_0 and prove that if f is differentiable at x_0, then f is continuous at x_0.
   (b) Let $h(x) = \begin{cases} x^2, & x \in \mathbb{Q} \\ 0, & x \notin \mathbb{Q} \end{cases}$.
   i. Prove that h is discontinuous at all $x \neq 0$.
   ii. Prove that h is differentiable at $x = 0$.
   iii. What can you say about the continuity of h at $x = 0$ and the differentiability of h at $x \neq 0$?
   (c) Let $f : [a, b] \to \mathbb{R}$.
   Prove that if f has a minimum at a point $c \in (a, b)$, and if $f'(c)$ exists, then $f'(c) = 0$.

5. (10 points) Let $h_n(x) = \dfrac{x}{(1 + x)^{n+1}}$.
   (a) Prove that h_n converges uniformly to 0 on $[0, \infty)$.
   (b) i. Verify that

$$\sum_{n=0}^{\infty} h_n(x) = \begin{cases} 1 \text{ if } x > 0 \\ 0 \text{ if } x = 0 \end{cases}$$

   ii. Does $\sum_{n=0}^{\infty} h_n$ converge uniformly on $[0, \infty)$?
   (c) Prove that $\sum_{n=0}^{\infty} h_n$ converges uniformly on $[a, \infty)$ for any $a > 0$.

   Hint: Recall that the Binomial Theorem implies $(1 + x)^{n+1} \geq \dfrac{n(n + 1)}{2} x^2$ *for all* $x \geq 0$.

Math 3100 - Sample Exam 3 (Version 1) - SOLUTIONS

1. (a) The Extreme Value Theorem implies that there cannot exist a continuous function on $[0,1]$ with range equal to $(0,1)$ since any continuous function on $[0,1]$ must attain both a maximum and minimum value & $(0,1)$ does not contain a max or min element.

(b) The Intermediate Value Theorem implies that there cannot exist a continuous function on $[0,1]$ with range equal to $\mathbb{Q} \cap [0,1]$ since between any two rationals in this range the must exist an irrational (but $\mathbb{Q} \cap [0,1]$ clearly contain no irrationals.)

2.- (a) Since $\sum_{n=1}^{\infty} nx^n = \frac{x}{(1-x)^2} \ \forall |x|<1 \Rightarrow \sum_{n=1}^{\infty} n4^{-n} = \underline{\underline{\frac{4}{9}}}$

differentiate term-by-term

(b) Since $\sum_{n=1}^{\infty} \frac{(-1)^n x^n}{n} = -\log(1+x) \ \forall |x|<1 \Rightarrow \sum_{n=1}^{\infty} \frac{(-1)^n}{n4^n} = \underline{\underline{\log\left(\frac{4}{5}\right)}}$

integration term-by-term

3. (a) (i) $\frac{x^2}{2+x^2} = \frac{x^2}{2}\,\frac{1}{1+\frac{x^2}{2}} = \left(\frac{x^2}{2}\right)\left(1-\frac{x^2}{2}+\left(\frac{x^2}{2}\right)^2-\cdots\right)$ if $|x|<\sqrt{2}$

$$= \underbrace{\frac{x^2}{2}-\frac{x^4}{4}+\frac{x^6}{8}}_{=P_6(x)} - \cdots$$

"the 6th order Maclaurin polynomial for $\frac{x^2}{2+x^2}$".

(ii) $f^{(6)}(0) = 6!\left(\frac{1}{8}\right) = \underline{\underline{90}}$

coefficient in front of x^6 above

(b)(i) Since $\log(1+x) = x - \frac{x^2}{2}+\frac{x^3}{3}-\cdots$ if $|x|<1$

$$\Rightarrow \log x = \log(1+(x-1)) = \underbrace{(x-1)-\frac{(x-1)^2}{2}+\frac{(x-1)^3}{3}}_{=P_3(x)} - \cdots \quad \text{if } |x-1|<1$$

"the 3rd order Taylor poly for $\log x$ centered at 1"

(ii) Since the Taylor series above is alternating when $x>1$ and the terms are decreasing when $|x-1|<1$ it follows from the Alternating Series Remainder Estimate that

1st omitted term

$$|\log(1.5)-P(1.5)| \leq \frac{|1.5-1|^4}{4} = \frac{1}{4}\left(\frac{1}{2}\right)^4 = \underline{\underline{\frac{1}{64}}}$$

(c) (i) Lagrangian Remainder Estimate for Maclaurin Series

If f is $(n+1)$-times differentiable on $(-R,R)$, then for any $x\in(-R,R)\setminus\{0\}$ $\exists$ c between 0 & x such that

$$f(x)-\left[f(0)+f'(0)x+\cdots+\frac{f^{(n)}(0)}{n!}x^n\right] = \frac{f^{(n+1)}(c)}{(n+1)!}x^n.$$

(ii) It follows from the Lagrangian Remainder Estimate that

$$\left| e^x - \left(1 + x + \cdots + \frac{x^n}{n!}\right)\right| = \frac{e^c}{(n+1)!}|x|^{n+1} \text{ for some } c \text{ between } 0 \text{ \& } x,$$

$$\leq \frac{e^{1/2}}{(n+1)!}\left(\frac{1}{2}\right)^{n+1} \quad \text{if } |x| \leq \frac{1}{2}$$

$$\leq \frac{1}{(n+1)!}\left(\frac{1}{2}\right)^n \quad \text{since } e^{1/2} \leq 2.$$

Since $\frac{1}{(n+1)!}\left(\frac{1}{2}\right)^n < \frac{1}{1000}$ if $n = 4$

$\Rightarrow 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24}$ approximate e^x to within 10^{-3} for all $|x| \leq \frac{1}{2}$.

4. (a) We say that f is <u>differentiable at x_0</u> if

$$\lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} \text{ exists.}$$

<u>Claim</u>

If f is differentiable at x_0, then f is continuous at x_0

<u>Proof</u> Since

$$\lim_{x \to x_0} f(x) - f(x_0) = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} \cdot (x - x_0)$$

as $x \neq x_0$

$$= \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} \cdot \lim_{x \to x_0} (x - x_0)$$

limit laws (since both limits exist)

$$= f'(x_0) \cdot 0 = 0$$

it follows that $\lim_{x \to x_0} f(x) = f(x_0)$ and hence that f is conts at x_0. $\square$

(b) Let $h(x) = \begin{cases} x^2 & \text{if } x \in \mathbb{Q} \\ 0 & \text{if } x \notin \mathbb{Q}. \end{cases}$

(i) <u>Claim</u> h is discontinuous at all $x_0 \neq 0$.

<u>Proof</u>

Let $x_0 \neq 0$. Since the rationals and irrationals are both dense in $\mathbb{R}$ we know there exists

* A seq $\{x_n\}$ of rationals with $x_n \to x_0$ as $n \to \infty$.

& * A seq $\{y_n\}$ of irrationals with $y_n \to x_0$ as $n \to \infty$

Since $h(x_n) = x_n^2 \to x_0^2 \neq 0$ as $n \to \infty$

& $h(y_n) = 0 \ \forall n \in \mathbb{N}$ so $h(y_n) \to 0$ as $n \to \infty$

it follows that h is discontinuous at x_0. $\square$

(ii) <u>Claim</u> h is differentiable (and hence conts) at $x_0 = 0$.

<u>Proof</u> Since $\left|\frac{h(x)-h(0)}{x-0}\right| = \left|\frac{h(x)}{x}\right| \leq |x| \ \forall x \neq 0$

it follow from the "squeeze theorem" that $\lim_{x \to 0} \frac{h(x)-h(0)}{x-0}$ exists and equals 0. $\square$

(iii) * h is conts at $x_0 = 0$ since h is diff'ble at $x_0 = 0$ (Q5(a))

* h is <u>not</u> diff'ble at any $x_0 \neq 0$ since h is not conts at any $x_0 \neq 0$ (Q5(a) again)

5. Let $h_n(x) = \frac{x}{(1+x)^{n+1}}$.

(a) Since $(1+x)^{n+1} \geq (n+1)x \quad \forall x \geq 0$ (Binomial Thm)

$\Rightarrow |h_n(x)| \leq \frac{1}{n+1} \quad \forall x \geq 0$

$\Rightarrow \sup_{x\in[0,\infty)} |h_n(x) - 0| \to 0$ as $n \to \infty$. $\square$

(*) One could also use "Calculus" to solve this (*)

(b) (i) Clearly $\sum_{n=0}^{\infty} h_n(0) = 0$. If $x>0$, then

$$\sum_{n=0}^{\infty} h_n(x) = \frac{x}{1+x} \sum_{n=0}^{\infty} \left(\frac{1}{1+x}\right)^n = \frac{x}{1+x} \frac{1}{1-\frac{1}{1+x}} \quad \text{since } \left|\frac{1}{1+x}\right|<1$$
$$= 1.$$

(ii) $\sum_{n=0}^{\infty} h_n$ cannot converge uniformly on $[0,\infty)$ since it does not converge to a continuous function on $[0,\infty)$

(c) Let $a>0$ be fixed. Since $|h_n(x)| \leq \frac{2x}{n(n+1)x^2} \leq \frac{2}{n(n+1)a} \quad \forall x \in [a,\infty)$

& $\sum_{n=1}^{\infty} \frac{2}{n(n+1)a}$ converges it follows from the M-test that $\sum_{n=0}^{\infty} h_n(x)$ conv uniformly on $[a,\infty)$. $\square$