Convergence in Norm & Further Remarks

Recall that for any $f \in L^1(\mathbb{T})$ we define

$$S_N f(x) = \sum_{|n| \leq N} \hat{f}(n) e^{2\pi i n x}$$

where for each $n \in \mathbb{Z}$,

$$\hat{f}(n) = \int_0^1 f(x) e^{-2\pi i n x} \, dx \, .$$

* The development of measure theory & L^p spaces led to a new approach to the problem of convergence of $S_N f$. We can now ask:

1. Does $\lim_{N \to \infty} \| S_N f - f \|_p = 0$ for $f \in L^p(\mathbb{T})$?

2. Does $\lim_{N \to \infty} S_N f(x) = f(x)$ almost everywhere if $f \in L^p(\mathbb{T})$?

The second question is much harder than the first and we will not be able to discuss it in any detail here. But this is what is known:

- In 1926 Kolmogorov gave an example of a function in $L^1(\mathbb{T})$ whose Fourier Series diverges at every point!
- If $1 < p < \infty$, then the answer is YES:
  - $p = 2$: Carleson (1965)
  - $p > 1$: Hunt (1967)

* Before Carleson, this wasn't even known for continuous functions!!! *

Some Remarks on Question 1

We can restate the first question by means of the following:

Lemma

$S_N f \to f$ in L^p norm $(1 \le p < \infty)$ $\iff$ $\exists\, C_p < \infty$ independent of N s.t. $\|S_N f\|_p \le C_p \|f\|_p$.

Proof:

($\Rightarrow$): Follows from the Uniform Boundedness Principle.

($\Leftarrow$): Since trig polys are dense in L^p (see Corollary to Fejér's Theorem) it follows that for any $\varepsilon > 0$, there exists trig poly g with $\|f-g\|_p < \varepsilon$ and so for N suff. large

$$\|S_N f - f\|_p \le \|S_N(f-g)\|_p + \|S_N g - g\|_p + \|f-g\|_p \le (C_p+1)\varepsilon. \ \square$$

Results:

- M. Riesz showed that if $f \in L^p(\mathbb{T})$ with $1 < p < \infty$ ($p=2$ is EASY!) then $\|S_N f\|_p \le C_p \|f\|_p$.

  * This is equivalent to the L^p-boundedness of the Hilbert transform
  $$Hf(x) = \lim_{\varepsilon \to 0} \frac{1}{\pi} \int_{|y| \ge \varepsilon} \frac{f(x-y)}{y}\,dy \ \ *$$

- $\exists\, f \in C(\mathbb{T})$ with $\|f\|_\infty = 1$ s.t. $\sup_N \|S_N f\|_\infty = \infty$ ← See proof of Du Bois-Reymond
- $\exists\, g \in L^1(\mathbb{T})$ with $\|g\|_1 = 1$ s.t. $\sup_N \|S_N f\|_1 = \infty$ (Take $g =$ "Fejér kernel")

¿ Why is answer to question 1 easy when $p=2$?

– This is because the functions $\{e^{2\pi i n x}\}$ form an orthonormal basis for $L^2(\mathbb{T})$, since trig polys are dense in $L^2(\mathbb{T})$.

Recall (from Hilbert space theory): TFAE

(i) $\{e^{2\pi i n x}\}$ forms an orthonormal basis for $L^2(\mathbb{T})$

(ii) finite linear comb. of elements from $\{e^{2\pi i n x}\}$, namely trig polys, are dense in $L^2(\mathbb{T})$.

(iii) $\|\hat{f}\|_{\ell^2} = \|f\|_{L^2}$ (Parseval)

(iv) $S_N f \to f$ in L^2 norm.

This was probably shown to you in Math 8100. We will show that trig polys are dense in $L^2(\mathbb{T})$, in the section on Fejér's theorem below.

Before doing this we note the follow corollary of the above discussion:

Theorem 1: If $f \in L^1(\mathbb{T})$ & $\hat{f} \in \ell^1(\mathbb{Z})$, then $S_N f \to f$ uniformly

Proof: Since $\hat{f} \in \ell^1(\mathbb{Z})$ we know that $S_N f \to g$ uniformly, for some g.
Since $\ell^1(\mathbb{Z}) \subseteq \ell^2(\mathbb{Z}) \Rightarrow f \in L^2(\mathbb{T})$ & hence $S_N f \to f$ in $L^2(\mathbb{T})$.
Thus we must have $f = g$ a.e. $\square$

Corollaries of Theorem 1

Corollary 1: If $f \in C^1(\mathbb{T})$, then $S_N f \to f$ uniformly.

Proof: Key is the observation that $\widehat{f'}(n) = 2\pi i n \hat{f}(n)$ $\forall n \in \mathbb{Z}$.
(This follows by Integration by Parts)

Given this it follows that

Cauchy-Schwarz

$$\sum_{n \neq 0} |\hat{f}(n)| = \sum_{n\neq 0}{}' \underbrace{2\pi |n| |\hat{f}(n)|}_{\substack{\| \\ |\widehat{f'}(n)|}} \cdot \frac{1}{2\pi |n|} \leq \|\widehat{f'}\|_{\ell^2} \Big(\sum_{n \neq 0} \frac{1}{4\pi^2 |n|^2} \Big)^{1/2}$$

$$= \frac{1}{\sqrt{12}} \cdot \|f'\|_{L^2(\mathbb{T})} < \infty .$$

Now add $|\hat{f}(0)|$ to both sides and apply Theorem 1. $\square$

Recall: **The Hausdorff-Young Inequality** (for Fourier series)

Suppose $1 \leq p \leq 2$ & q is the conjugate exponent to p.
If $f \in L^p(\mathbb{T})$, then $\hat{f} \in \ell^q(\mathbb{Z})$ & $\|\hat{f}\|_{\ell^q} \leq \|f\|_{L^p}$.

[**Proof**: Follows from the Riesz-Thorin interpolation theorem, since $\|\hat{f}\|_\infty \leq \|f\|_1$ and $\|\hat{f}\|_2 = \|f\|_2$ for $f \in L^1$ or $f \in L^2$. $\square$]

Exercise 1: Show that if $f \in C(\mathbb{T})$ & $f' \in L^p(\mathbb{T})$ for some $p > 1$, then $\hat{f} \in \ell^1$ and hence $S_N f \to f$ uniformly.

Remark: Both Corollary 1 & Exercise 1 follow easily (from the harder (?) to prove) Dini's Theorem.