

# Math 8100 Assignment 1

## Preliminaries

Due date: Tuesday the 30th of August 2022

1. The **Cantor set**  $\mathcal{C}$  is the set of all  $x \in [0, 1]$  that have a ternary expansion  $x = \sum_{k=1}^{\infty} a_k 3^{-k}$  with  $a_k \neq 1$  for all  $k$ . Thus  $\mathcal{C}$  is obtained from  $[0, 1]$  by removing the open middle third  $(\frac{1}{3}, \frac{2}{3})$ , then removing the open middle thirds  $(\frac{1}{9}, \frac{2}{9})$  and  $(\frac{7}{9}, \frac{8}{9})$  of the two remaining intervals, and so forth.
  - (a) Find a real number  $x$  belonging to the Cantor set which is not the endpoint of one of the intervals used in its construction.
  - (b) Prove that  $\mathcal{C}$  is both nowhere dense (and hence meager) and has measure zero.
  - (c) Prove that  $\mathcal{C}$  is uncountable by showing that the function  $f(x) = \sum_{k=1}^{\infty} b_k 2^{-k}$  where  $b_k = a_k/2$ , maps  $\mathcal{C}$  onto  $[0, 1]$ .
2. A set  $A \subseteq \mathbb{R}^n$  is called an  $F_\sigma$  set if it can be written as the countable union of closed subsets of  $\mathbb{R}^n$ . A set  $B \subseteq \mathbb{R}^n$  is called a  $G_\delta$  set if it can be written as the countable intersection of open subsets of  $\mathbb{R}^n$ .
  - (a) Argue that a set is a  $G_\delta$  set if and only if its complement is an  $F_\sigma$  set.
  - (b) Show that every closed set is a  $G_\delta$  set and every open set is an  $F_\sigma$  set.

*Hint: One approach is to prove that every open subset of  $\mathbb{R}^n$  can be written as a countable union of closed cubes with disjoint interiors. This approach is however very specific to open sets in  $\mathbb{R}^n$ .*
  - (c) Give an example of an  $F_\sigma$  set which is not a  $G_\delta$  set and a set which is neither an  $F_\sigma$  nor a  $G_\delta$  set.
3. (a) Let  $\{r_n\}_{n=1}^{\infty}$  be any enumeration of all the rationals in  $[0, 1]$  and define  $f : [0, 1] \rightarrow \mathbb{R}$  by setting

$$f(x) = \begin{cases} \frac{1}{n} & \text{if } x = r_n \\ 0 & \text{if } x \in [0, 1] \setminus \mathbb{Q} \end{cases}.$$

Prove that  $\lim_{x \rightarrow c} f(x) = 0$  for every  $c \in [0, 1]$  and conclude that set of all points at which  $f$  is discontinuous is precisely  $[0, 1] \cap \mathbb{Q}$ .

- (b) Let  $f : \mathbb{R} \rightarrow \mathbb{R}$  be bounded.

i. Recall that we defined the *oscillation of  $f$  at  $x$*  to be

$$\omega_f(x) := \lim_{\delta \rightarrow 0^+} \sup_{y, z \in B_\delta(x)} |f(y) - f(z)|.$$

Briefly explain why this is a well defined notion and prove that

$$f \text{ is continuous at } x \iff \omega_f(x) = 0.$$

- ii. Prove that for every  $\varepsilon > 0$  the set  $A_\varepsilon = \{x \in \mathbb{R} : \omega_f(x) \geq \varepsilon\}$  is closed and deduce from this that the set of all points at which  $f$  is discontinuous is an  $F_\sigma$  set.

4. Let  $\{x_n\}_{n=1}^{\infty}$  be any enumeration of a given countable set  $X \subseteq \mathbb{R}$ . For each  $n \in \mathbb{N}$  define

$$f_n(x) = \begin{cases} 1 & \text{if } x > x_n \\ 0 & \text{if } x \leq x_n \end{cases}.$$

Prove that

$$f(x) = \sum_{n=1}^{\infty} \frac{1}{n^2} f_n(x)$$

defines an increasing function  $f$  on  $\mathbb{R}$  that is continuous on  $\mathbb{R} \setminus X$ .

5. Let  $C([0, 1])$  denote the collection of all real-valued continuous functions with domain  $[0, 1]$ .
- Show that  $d_\infty(f, g) = \sup_{x \in [0, 1]} |f(x) - g(x)|$  defines a metric on  $C([0, 1])$  and that with the “uniform” metric  $C([0, 1])$  is in fact a *complete* metric space.
  - Prove that the unit ball  $\{f \in C([0, 1]) : d_\infty(f, 0) \leq 1\}$  is closed and bounded, but *not* compact.
  - \*\* Challenge: Can you show that  $C([0, 1])$  with the metric  $d_\infty$  is not *totally bounded*.  
A set is *totally bounded* if, for every  $\varepsilon > 0$ , it can be covered by finitely many balls of radius  $\varepsilon$ .
6. Let

$$g(x) = \sum_{n=0}^{\infty} \frac{1}{1 + n^2 x}.$$

- Show that the series defining  $g$  does not converge uniformly on  $(0, \infty)$ , but none the less still defines a continuous function on  $(0, \infty)$ .  
*Hint for the first part: Show that if  $\sum_{n=0}^{\infty} g_n(x)$  converges uniformly on a set  $X$ , then the sequence of functions  $\{g_n\}$  must converge uniformly to 0 on  $X$ .*
  - Is  $g$  differentiable on  $(0, \infty)$ ? If so, is the derivative function  $g'$  continuous on  $(0, \infty)$ ?
7. Let  $h_n(x) = \frac{x}{(1+x)^{n+1}}$ .

- Prove that  $h_n$  converges uniformly to 0 on  $[0, \infty)$ .
- i. Verify that

$$\sum_{n=0}^{\infty} h_n(x) = \begin{cases} 1 & \text{if } x > 0 \\ 0 & \text{if } x = 0 \end{cases}$$

- ii. Does  $\sum_{n=0}^{\infty} h_n$  converge uniformly on  $[0, \infty)$ ?
- Prove that  $\sum_{n=0}^{\infty} h_n$  converges uniformly on  $[a, \infty)$  for any  $a > 0$ .

### Extra Challenge Problems

*Not to be handed in with the assignment*

- Given an arbitrary  $F_\sigma$  set  $V$ , can you produce a function whose discontinuities lie precisely in  $V$ ?  
*Hint: First try to do this for an arbitrary closed set.*
- (Baire Category Theorem) Prove that if  $X$  is a non-empty *complete* metric space, then  $X$  cannot be written as a countable union of nowhere dense sets.  
*Hint: Modify the proof given in class of the special case  $X = \mathbb{R}$  replacing the use of the nested interval property with the following fact (which you should prove):*  
*If  $F_1 \supseteq F_2 \supseteq \dots$  is a nested sequence of closed non-empty and bounded sets in a complete metric space  $X$  with  $\lim_{n \rightarrow \infty} \text{diam } F_n = 0$ , then  $\bigcap_{n=1}^{\infty} F_n$  contains exactly one point.*
- Complete the proof, sketched in class, of the so-called Lebesgue Criterion: *A bounded function on an interval  $[a, b]$  is Riemann integrable if and only if its set of discontinuities has measure zero.*
  - Prove that if the set of discontinuities of  $f$  has measure zero, then  $f$  is Riemann integrable.  
*[Hint: Let  $\varepsilon > 0$ . Cover the compact set  $A_\varepsilon$  (defined in Q3(b)ii. above) by a finite number of open intervals whose total length is  $\leq \varepsilon$ . Select an appropriate partition of  $[a, b]$  and estimate the difference between the upper and lower sums of  $f$  over this partition.]*
  - Prove that if  $f$  is Riemann integrable on  $[a, b]$ , then its set of discontinuities has measure zero.  
*[Hint: The set of discontinuities of  $f$  is contained in  $\bigcup_n A_{1/n}$ . Given  $\varepsilon > 0$ , choose a partition  $P$  such that  $U(f, P) - L(f, P) < \varepsilon/n$ . Show that the total length of the intervals in  $P$  whose interiors intersect  $A_{1/n}$  is  $\leq \varepsilon$ .]*