Fourier Analysis on $\mathbb{R}^n$

Given $f \in L^1(\mathbb{R}^n)$ we define its Fourier transform $\hat{f}: \mathbb{R}^n \to \mathbb{C}$ by

$$\hat{f}(\xi) = \int_{\mathbb{R}^n} f(x) e^{-2\pi i x\cdot\xi}\, dx \qquad [x\cdot\xi = x_1\xi_1 + \cdots + x_n\xi_n].$$

Basic Properties (Exercise, from 8100?) Suppose $f, g \in L^1(\mathbb{R}^n)$

(a) $(\tau_y f)^\wedge(\xi) = e^{-2\pi i \xi\cdot y}\hat{f}(\xi)$ & $\tau_y(\hat{f}) = \hat{h}$, $h(x) = e^{2\pi i y\cdot x} f(x)$

(b) If T invertible linear trans. on $\mathbb{R}^n$ & $S = (T^*)^{-1}$ is its inv. transpose,

$$(f\circ T)^\wedge = |\det T|^{-1}\, \hat{f}\circ S$$

* In particular $\hat{f_t}(\xi) = \hat{f}(t\xi)$ where $f_t(x) = \frac{1}{t^n} f\left(\frac{x}{t}\right)$ *

(c) $\widehat{f*g} = \hat{f}\hat{g}$ $\quad \left[f*g(x) = \int_{\mathbb{R}^n} f(y)g(x-y)\,dy\right]$

$\left\{\begin{array}{l} \text{(d) If } x_j f \in L^1, \text{ then } \frac{\partial}{\partial \xi_j}\hat{f} = \hat{h},\ \ h(x) = -2\pi i x_j f(x) \\ \text{(e) If } \frac{\partial}{\partial x_j} f \in L^1 \ \&\ f \in C_0, \text{ then } \widehat{\frac{\partial}{\partial x_j} f} = 2\pi i \xi_j \hat{f}(\xi) \end{array}\right.$

"smoothness of f $\leftrightsquigarrow$ decay of $\hat{f}$ at infinity" (& vice versa)

(f) [Riemann-Lebesgue lemma] $\hat{f} \in C_0(\mathbb{R}^n)$ (of course $\|\hat{f}\|_\infty \le \|f\|_1$

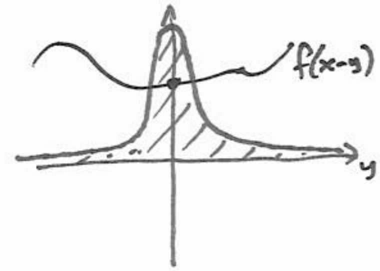
Important Example

If $g(x) = e^{-\pi|x|^2}$, then $\hat{g}(\xi) = e^{-\pi|\xi|^2}$.

(follows by complex analysis OR properties (d) & (e) & $\int_{\mathbb{R}^n} e^{-\pi x^2} dx = 1$.)

Remark (on g_t)

1. $g_t(x) = t^{-n} e^{-\pi |x|^2 / t^2}$ & $\int g_t = 1 \quad \forall t > 0$



2. Property (b) $\Rightarrow \hat{g}_t(\xi) = \hat{g}(t\xi) = e^{-\pi t^2 |\xi|^2}$

(Uncertainty Principle!)

Recall (from Math 8100?)

Theorem 1: Let $1 \le p \le \infty$. If $f \in L^p(\mathbb{R}^n)$, or bounded & unif conts if $p = \infty$, then

$$\lim_{t \to 0} \| f * g_t - f \|_p = 0 .$$

Proof:

$$f * g_t(x) - f(x) = \int [f(x-y) - f(x)] g_t(y) \, dy . \quad (\text{let } y = tz)$$

$$= \int [f(x - tz) - f(x)] g(z) \, dz$$

Hence

$$\| f * g_t - f \|_p \underset{\substack{\uparrow \\ \text{Minkowski}}}{\le} \int \underbrace{\| \tau_{tz} f - f \|_p}_{} |g(z)| \, dz$$

$\hookrightarrow$ bounded by $2\|f\|_p$ & $\to 0$ as $t \to 0$ for all fixed z.

Result follows by the dominated convergence theorem. $\square$

Theorem 2 (Fourier Inversion Formula)

If $f \in L^1(\mathbb{R}^n)$ & $\hat{f} \in L^1(\mathbb{R}^n)$, then for a.e. $x \in \mathbb{R}^n$,

$$f(x) = \int \hat{f}(\xi)\, e^{2\pi i x\cdot\xi}\, d\xi .$$

Corollary 1: If $f \in L^1$ & $\hat{f} \in L^1$, then f agrees almost everywhere with a continuous function! (In general, $\hat{f} \notin L^1$ if $f \in L^1$).

Corollary 2: If $f \in L^1$ & $\hat{f} \equiv 0$, then $f = 0$ a.e.

Remark: Simply appealling to Fubini/Tonelli fails !! (integrand not in $L^1(\mathbb{R}^n \times \mathbb{R}^n)$.)

Trick: Introduce a "convergence factor" $\widehat{g_\varepsilon}(\xi) = e^{-\pi\varepsilon^2|\xi|^2}$ in.

Proof of Theorem 2

An appeal to Fubini does give:

Lemma (Multiplication Formula) If $f, g \in L^1(\mathbb{R}^n)$, then

$$\int \hat{f} g = \int f \hat{g} .$$

Proof: (Easy exercise).

Given $t > 0$ and $x \in \mathbb{R}^n$, we set

$$\varphi(\xi) = e^{2\pi i x \cdot \xi}\, \hat{g}_t(\xi).$$

It follows that

$$\hat{\varphi}(y) = \hat{\hat{g}}_t(y - x) = g_t(x - y) \quad \longleftarrow \underline{\text{Check!}}$$

Therefore,

$$\int \hat{f}(\xi)\varphi(\xi)\, d\xi \overset{\text{Mult. Formula}}{=} \int f(y)\hat{\varphi}(y)\, dy = \int f(y)\, g_t(x-y)\, dy = f * g_t(x)$$

$$\int \hat{f}(\xi)\varphi(\xi)\, d\xi \xrightarrow[\text{(since } \hat{f} \in L^1)]{t \to 0,\ \text{DCT}} \underline{\int \hat{f}(\xi)\, e^{2\pi i x \cdot \xi}\, d\xi}$$

Since $f * g_t \to f$ in L^1 (by Theorem 1)

it follows that

$$f(x) = \int \hat{f}(\xi)\, e^{2\pi i x \cdot \xi}\, d\xi \quad \text{a.e.}$$

as required $\square$