Convolutions

Let f and g be measurable functions on $\mathbb{R}^n$. The convolution of f and g is the function $f*g$ define by

$$f*g(x) := \int f(x-y)g(y)\,dy$$

for all x such that the integral exists.

Remark: If, for some x, the function $y \mapsto f(x-y)g(y)$ is integrable, then the function $y \mapsto f(y)g(x-y)$ is also integrable and hence

$$f*g(x) = g*f(x).$$

[Change of variables $y \mapsto x-y$ is a translation followed by a reflection.]

Theorem 1

If $f, g \in L^1$, then $f*g \in L^1$ and $\|f*g\|_1 \leq \|f\|_1 \|g\|_1$

[Remark: If $f, g \geq 0$, then one in fact has equality]

Theorem 2

If $f \in L^1$ and g bounded, then $f*g$ is both bounded & unif continuous.

Theorem 3

If $f \in L^1$ and g bounded & g diff'ble with $\frac{\partial g}{\partial x_j}$ bounded $\forall\, 1 \leq j \leq n$, then $f*g \in C^1$ and $\frac{\partial}{\partial x_j}(f*g) = f*\left(\frac{\partial}{\partial x_j} g\right)$.

Corollary (of Theorems 1–3)

If $f \in L^1$ & $g \in C_c^\infty$, then $\underbrace{f * g \in C^\infty \ \& \ \lim_{|x| \to \infty} f*g(x) = 0}_{\text{" } f*g \in C_0^\infty \text{ "}}$.

Proof of Theorem 1

Let $h(x,y) = f(x-y)g(y)$. Since $F(x,y) = f(x-y)$ & $G(x,y) = g(y)$ are both measurable on $\mathbb{R}^{2n}$ (by "Appendix on Measurability"), it follows that $h(x,y)$ is m'ble on $\mathbb{R}^{2n}$ (& hence so is $|h(x,y)|$).

Since

$$\int \left(\int |f(x-y)||g(y)|\,dx \right) dy$$

$$= \int |g(y)| \Big(\underbrace{\int |f(x-y)|\,dx}_{= \int |f(x)|\,dx \text{ by translation invariance.}} \Big)\,dy$$

$$= \|f\|_1 \|g\|_1 < \infty$$

it follow from Tonelli that $h \in L^1(\mathbb{R}^{2n})$ & hence Fubini implies that for a.e. x, $h_x(y) = f(x-y)g(y)$ is an integrable function of y & $\int f(x-y)g(y)\,dy = f*g(x)$ is an int'ble fn of x.

Finally, by Tonelli we have that

$$\int |f*g(x)|\,dx \leq \iint |f(x-y)||g(y)|\,dy\,dx = \|f\|_1 \|g\|_1 \text{ as above}$$

∎

Proof of Theorem 2

Choose $M > 0$ such that $|g(y)| \le M \ \forall y$, then

$$|f * g(x+h) - f * g(x)| \le \int |f(x+h-y) - f(x-y)|\,|g(y)|\,dy$$

$$\le M \int |f(x+h-y) - f(x-y)|\,dy$$

$$= M \underbrace{\int |f(y+h) - f(y)|\,dy}_{\to 0 \text{ as } h \to 0 \text{ by "Continuity in } L^1\text{"}} \qquad \square$$

Proof of Theorem 3

Choose $M > 0$ such that $\left|\frac{\partial g}{\partial x_j}(x)\right| \le M \quad \forall x$.

Let $\{t_n\}$ be any sequence such that $t_n \to 0$ as $n \to \infty$.

Since

$$\left| f(y) \cdot \frac{g(x + t_n e_j - y) - g(x-y)}{t_n} \right| \le M\,|f(y)| \quad (\text{by MVT})$$

for all n, it follow from the DCT that

$$\frac{\partial}{\partial x_j}(f * g)(x) = \lim_{n \to \infty} \frac{f * g(x + t_n) - f * g(x)}{t_n}$$

$$= \lim_{n \to \infty} \int f(y) \frac{g(x + t_n e_j - y) - g(x-y)}{t_n}\,dy$$

DCT

$$= \int f(y) \left[\lim_{n \to \infty} \frac{g(x + t_n - y) - g(x-y)}{t_n} \right] dy$$

$$= \int f(y) \frac{\partial}{\partial x_j} g(x-y)\,dy$$

$$= f * \left(\frac{\partial g}{\partial x_j}\right)(x). \qquad \square$$