Fourier Transform of Spherical Measures

Let $d\sigma$ denote the measure on S^{n-1} induced from Lebesgue measure on $\mathbb{R}^n$

$$\widehat{d\sigma}(\xi) := \int_{S^{n-1}} e^{-2\pi i x\cdot\xi}\, d\sigma(x).$$

<u>Theorem</u>

$$\widehat{d\sigma}(\xi) = \frac{e^{-i\pi\frac{(n-1)}{4}} e^{2\pi i|\xi|}}{|\xi|^{\frac{n-1}{2}}} + \frac{e^{i\pi\frac{(n-1)}{4}} e^{-2\pi i|\xi|}}{|\xi|^{\frac{n-1}{2}}} + O\left(|\xi|^{\frac{n+1}{2}}\right)$$

as $|\xi|\to\infty$

In particular,

$$|\widehat{d\sigma}(\xi)| \le C(1+|\xi|)^{-\frac{n-1}{2}}.$$

<u>Proof</u>: Since $\widehat{d\sigma}$ is a radial function (that is, one which depends only on distance from the origin), it suffice to consider $\widehat{d\sigma}$ at $\xi = \lambda e_n$, for $\lambda = |\xi| > 0$.

<u>Exercise</u>: We can express S^{n-1} as a (non-disjoint) union

$$U_1 \cup U_2 \cup \bigcup_{k=3}^{m} U_k$$

of open sets, where U_1 & U_2 are small nbds of the north & south poles (i.e. $(0,\dots,0,\pm1)$) respectively & for each $3\le k\le m$ there is $1\le j\le n-1$ s.t. the proj map

$$\pi_j : \mathbb{R}^n \longrightarrow \mathbb{R}^{n-1}$$
$$x \longmapsto (x_1,\dots,x_{j-1},x_{j+1},\dots,x_n)$$

is a diffeomorphism when restricted to U_k.

Let $\{\psi_k\}$ be a partition of unity relative to the cover $\bigcup_{k=1}^{m} U_k$ such that

$$\psi_1(0,\dots,0,1) = \psi_2(0,\dots,0,-1) = 1.$$

It follows that

$$\widehat{d\sigma}(\lambda e_n) = \int_{U_1} e^{-2\pi i \lambda x_n} \psi_1(x)\,d\sigma + \int_{U_2} e^{-2\pi i \lambda x_n} \psi_2(x)\,d\sigma + \sum_{k=3}^{m} \int_{U_k} e^{-2\pi i \lambda x_n} \psi_k(x)\,d\sigma$$

For each $3 \le k \le m$ we know $\exists\, 1 \le j \le n-1$ & local coordinates $x' = (x_1, \ldots, x_{j-1}, x_{j+1}, \ldots, x_n)$ in terms of which $x_j = \pm(1-|x'|^2)^{1/2}$ & $d\sigma \doteq (1-|x'|^2)^{-1/2}\,dx'$.

and hence

$$\int_{U_k} e^{-2\pi i \lambda x_n} \psi_k(x)\,d\sigma = \int_{\mathbb{R}^{n-1}} e^{i\pi\lambda(-2x_n)} \tilde{\psi}_k(x')\,dx' = O(\lambda^{-N})$$

by Principle of Non-Stationary Phase

- The interesting integrals are those involving ψ_1 & ψ_2:

The local coordinate on U_1 are $x' = (x_1, \ldots, x_{n-1})$ and in terms of these one has $x_n = (1-|x'|^2)^{1/2}$ & $d\sigma = (1-|x'|^2)^{-1/2}\,dx'$

and hence

$$\int_{U'} e^{-2\pi i \lambda x_n} \psi_1(x)\,d\sigma = \int_{\mathbb{R}^{n-1}} e^{i\pi\lambda\phi(x')} \tilde{\psi}_1(x')\,dx'$$

where $\phi(x') = -2(1-|x'|^2)^{1/2}$. Easy to see that 0 is a non-deg critical pt for ϕ & $H\phi(0) = 2I$. Principle of Stationary Phase $\Rightarrow$

$$\int_{U_1} e^{-2\pi i \lambda x_n} \psi_1(x)\,d\sigma = \lambda^{-\frac{n-1}{2}} e^{-2\pi i \lambda} e^{i\pi\frac{(n-1)}{4}} + O(\lambda^{-\frac{n+1}{2}}).$$

Similarly,

$$\int_{U_2} e^{-2\pi i \lambda x_n} \psi_2(x)\,d\sigma = \lambda^{-\frac{n-1}{2}} e^{2\pi i \lambda} e^{-i\pi\frac{(n-1)}{4}} + O(\lambda^{-\frac{n+1}{2}}).$$

Add all of this together gives the result. $\square$

* Recall $\int_{-1}^{1} e^{-2\pi i x \xi}\,dx = \frac{\sin 2\pi\xi}{\pi\xi}$.

<u>Corollary</u>: Let B denote the unit ball in $\mathbb{R}^n$, then $|\hat{\chi}_B(\xi)| \leq C(1+|\xi|)^{-\frac{n+1}{2}}$.

<u>Proof</u>

$$\hat{\chi}_B(\xi) = \int_{|x|\leq 1} e^{-2\pi i x\cdot\xi}\,dx = \int_0^1 \underbrace{\left(\int_{S^{n-1}} e^{-2\pi i r x\cdot\xi}\,d\sigma(x)\right)}_{(*)} r^{n-1}\,dr$$

We know that

$$(*) = \frac{c_1 e^{2\pi i r|\xi|} + \bar{c}_1 e^{-2\pi i r|\xi|}}{(r|\xi|)^{\frac{n-1}{2}}} + O(r|\xi|)^{-\frac{n+1}{2}} \quad \text{where } c_1 = e^{-i\pi\frac{n-1}{4}}.$$

It therefore suffices to show that

$$\left|\int_0^1 e^{\pm 2\pi i r|\xi|}\, r^{\frac{n-1}{2}}\,dr\right| \leq C|\xi|^{-1} \quad (\underline{\text{Ex}}: \text{IBP}) \; \square$$

Using this and the Poisson Summation Formula one can prove:

<u>Thm</u> (Hlawka) Let $n \geq 2$, then

$$\#(\lambda B \cap \mathbb{Z}^n) - \text{volume}(B)\lambda^n = O\left(\lambda^{n-2+\frac{2}{n+1}}\right).$$

Conjectured $= \begin{cases} \frac{1}{2}+\varepsilon & n=2 \\ \varepsilon & n=3,4 \\ 0 & n\geq 5 \end{cases}$

<u>Proof</u> (Exercise – Hints available at request)

<u>Further Exercise</u> (Tricky?) Let $0<\delta\leq 1$.

Show that

$$\int_{|x|\leq 1} (1-|x|^2)^\delta e^{-2\pi i x\cdot\xi}\,dx = a_1 \frac{e^{2\pi i|\xi|}}{|\xi|^{\frac{n+1}{2}+\delta}} + a_2\frac{e^{-2\pi i|\xi|}}{|\xi|^{\frac{n+1}{2}+\delta}} + O\left(|\xi|^{-\frac{n+3}{2}}\right)$$

(say).

Conclude that if $\|S_R^\delta f\|_p \leq C_p\|f\|_p$, then $p \in \left(\frac{2n}{n+1+2\delta}, \frac{2n}{n-1-2\delta}\right)$.