Arithmetic Progressions of length 4 and the Gowers U^3-norm

<u>Theorem</u> (Gowers, 1998): $\dfrac{r_4(N)}{N} \ll \dfrac{1}{(\log\log N)^c}$ for some fixed $c>0$.

[The constant $c>0$ can be take to be 2^{-40} (say), but we will not keep careful track it these notes (well, it's unlikely).]

<u>Proposition</u> (Dichotomy)

Let P be an arith. prog. of integers and $A \subseteq P$ with density $\delta > 0$.
If $|P| \geq e^{\delta^{-c}}$ for some large $c>0$, then either

(i) $\#$ 4AP's in $A \geq \dfrac{\delta^4}{72} |P|^2$ [In particular, at least one non-trivial 4AP.]
(inc. trivial)

<u>OR</u>

(ii) $\exists$ arith. prog. $P' \subseteq P$ with $|P'| \geq |P|^{\delta^c}$ such that

$$|A \cap P'| \geq (\delta + \delta^c)\,|P'| .$$

<u>Exercise 1</u>: Verify that this proposition implies the theorem.

- As before, in the proof of this proposition we may (via a rescaling argument) assume that $P = [1, N]$.
- We will again identify $[1,N] \simeq \mathbb{Z}_N$, but will now assume that $(N, 6) = 1$ (N neither a multiple of 3 or 2).

For $f_1, f_2, f_3, f_4 : \mathbb{Z}_N \to \mathbb{C}$ we define the operator

$$AP_4(f_1,f_2,f_3,f_4) = \frac{1}{N^2} \sum_{x,d \in \mathbb{Z}_N} f_1(x) f_2(x+d) f_3(x+2d) f_4(x+3d)$$

Note: If $A \subseteq \mathbb{Z}_N$, then

. $AP_4(1_A, 1_A, 1_A, 1_A) = \frac{1}{N^2} \times \#\ \mathbb{Z}_N$-4AP's in A (inc. trivial)

while if $B := A \cap [\frac{2}{5}N, \frac{3}{5}N]$, then

$AP_4(1_B, 1_B, 1_A, 1_A) \leq \frac{1}{N^2} \times \#$ (genuine) 4AP's in A (inc. trivial).

* In proving the proposition, we may assume that $|B| \geq \frac{\delta}{6}N$. *

(If not, then $\max\{|A \cap [1, \frac{2N}{5}]|, |A \cap [\frac{3N}{5}, N]|\} \geq \frac{5}{12}\delta N = (\delta + \frac{\delta}{24})(\frac{2N}{5})$.)

Gowers' Norms: For $f : \mathbb{Z}_N \to \mathbb{C}$ we define

$$\|f\|_{U^2}^4 = \frac{1}{N^3} \sum_{x,h_1,h_2} f(x) \overline{f(x+h_1)}\, \overline{f(x+h_2)} f(x+h_1+h_2)$$

$$\|f\|_{U^3}^8 = \frac{1}{N^4} \sum_{x,h_1,h_2,h_3} f(x) \overline{f(x+h_1)}\, \overline{f(x+h_2)}\, \overline{f(x+h_3)} f(x+h_1+h_2) f(x+h_1+h_3) f(x+h_2+h_3) \cdot \overline{f(x+h_1+h_2+h_3)}$$

$\vdots$

Exercise 2

(a) Show that $\|f\|_{U^2} \leq \|f\|_{U^3} \leq \cdots$ (Hint: Cauchy-Schwarz)

(b) Show that $\exists\ f : \mathbb{Z}_N \to \mathbb{D}$ with $\|f\|_{U^3} = 1$, but $\|f\|_{U^2} \leq N^{-1/4}$.

(Hint: Try $f(x) = e^{2\pi i x^2/N}$).

Last time we proved that the U^2-norm "controlled 3AP's", namely

<u>Lemma</u> (Generalized von-Neumann Theorem for U^2-norm)

If $f_1, f_2, f_3 : \mathbb{Z}_N \to \mathbb{D}$, then $|AP_3(f_1,f_2,f_3)| \leq \|f_j\|_{U^2}$ $j=1,2,3$.

Today we see that the U^3-norm "controls 4AP's", specifically we will prove the following

<u>Lemma</u> (Generalized von-Neumann Theorem for U^3-norm)

If $f_1, f_2, f_3, f_4 : \mathbb{Z}_N \to \mathbb{D}$, then $|AP_4(f_1,f_2,f_3,f_4)| \leq \|f_j\|_{U^3}$ $j=1,2,3,4$.

<u>Remark</u>: The analogous result for higher Gowers norms is also true. The proof of this lemma is presented at the end of these notes.

Proof of Proposition - 1^{st} steps

Let $A \subseteq \mathbb{Z}_N$ with $|A| = \delta N$ and $N \geq e^{\delta^{-C}}$ for some large $C>0$.

<u>Lemma 1</u>: Let $f = 1_A - \delta$. If $\|f\|_{U^3} \leq \varepsilon$, then

$$\left| AP_4(1_B, 1_B, 1_A, 1_A) - \left(\frac{|B|}{N}\right)^2 \delta^2 \right| \leq 2\varepsilon .$$

which we can assume

In particular, if $|B| \geq \frac{\delta}{6} N$ and $\|f\|_{U^3} \leq \delta^4/144$, then A contains at least $\frac{\delta^4}{72} N^2$ (genuine) 4AP's (inc. trivial).

Hence if <u>(i) doesn't hold</u>, then we must have <u>$\|f\|_{U^3} \geq \delta^4/144$</u>.

Proof of Lemma 1: Let $f = 1_A - \delta$,

$$\begin{aligned} AP_4(1_B, 1_B, 1_A, 1_A) &= AP_4(1_B, 1_B, 1_A, \delta) + AP_4(1_B, 1_B, 1_A, f) \\ &= \underbrace{AP_4(1_B, 1_B, \delta, \delta)}_{= \left(\frac{|B|}{N}\right)^2 \delta^2} + \underbrace{AP_4(1_B, 1_B, f, \delta) + AP_4(1_B, 1_B, 1_A, f)}_{|-''-| \leq 2\|f\|_{U^3} \text{ by Lemma.}} \end{aligned}$$

Hence

$$\left| AP_4(1_B, 1_B, 1_A, 1_A) - \left(\frac{|B|}{N}\right)^2 \delta^2 \right| \leq 2\|f\|_{U^3} \leq 2\varepsilon.$$

In particular,

$$\begin{aligned} AP_4(1_B, 1_B, 1_A, 1_A) &\geq \left(\frac{|B|}{N}\right)^2 \delta^2 - 2\|f\|_{U^3} \\ &\geq \frac{\delta^4}{72} \quad \underline{\text{if}} \quad \frac{|B|}{N} \geq \frac{\delta}{6} \ \& \ \|f\|_{U^3} \leq \delta^4/144. \ \square \end{aligned}$$

In light of this observation we see that the proof of the proposition reduces to "simply" establishing

Lemma 2: If $f : \mathbb{Z}_N \to [-1,1]$ satisfies $\sum_x f(x) = 0$ & $\|f\|_{U^3} \geq \varepsilon$, then $\exists$ genuine arith. prog. $P \subseteq \mathbb{Z}_N$ with $|P| \geq N^{\varepsilon^c}$ s.t. $\frac{1}{|P|}\sum_{x \in P} f(x) \geq \varepsilon^c$.

(in particular, if $f = 1_A - \delta$ & $\|f\|_{U^3} \geq \varepsilon$, then for the P above

$$\frac{|A \cap P|}{|P|} \geq \delta + \varepsilon^c.)$$

Remark: $\|f\|_{U^3} \geq \varepsilon$ is a weaker assumption than $\|f\|_{U^2} \geq \varepsilon$ (Exercise 1)

Indeed, the proof of Lemma 2 is a tour de force that will take a long time...

Proof of Lemma (Generalized von-Neumann for U^3-norm)

First observe that

$$\|f\|_{U^3}^8 = \frac{1}{N^4}\sum_{x,h_1,h_2,h_3} \Delta_{h_1} f(x)\,\overline{\Delta_{h_1} f(x+h_2)}\,\overline{\Delta_{h_1} f(x+h_3)}\,\Delta_{h_1} f(x+h_2+h_3)$$

$$= \frac{1}{N}\sum_{h_1} \|\Delta_{h_1} f\|_{U^2}^4 \quad , \text{ where } \Delta_h f(x) = f(x)\overline{f(x+h)} .$$

Now $\quad AP_4(f_1,f_2,f_3,f_4) = \frac{1}{N}\sum_x \frac{1}{N}\sum_d f_1(x)\left[f_2(x+d)f_3(x+2d)f_4(x+3d)\right]$.

Since $|f_1(x)|\le 1$, Cauchy-Schwarz implies that

$$|AP_4(f_1,f_2,f_3,f_4)|^2 \le \frac{1}{N}\sum_x \frac{1}{N^2}\sum_{d,d'} f_2(\underset{\substack{\parallel\\ x}}{x+d})\,\overline{f_2(\underset{\substack{\parallel\\ x+y\\ (y=d'-d)}}{x+d'})}\,f_3(x+\underset{\substack{\parallel\\ z}}{2d})\,\overline{f_3(x+2d')}\,f_4(x+3d)\,\overline{f_4(x+3}$$

$$= \frac{1}{N^3}\sum_{x,y,z} f_2(x)\overline{f_2(x+y)}\,f_3(x+z)\overline{f_3(x+z+2y)}\,f_4(x+2z)\overline{f_4(x+2z+3y}$$

$$= \frac{1}{N^3}\sum_{x,y,z} \Delta_y f_2(x)\,\Delta_{2y} f_3(x+z)\,\Delta_{3y} f_4(x+2z)$$

$$= \frac{1}{N}\sum_y AP_3(\Delta_y f_2, \Delta_{2y} f_3, \Delta_{3y} f_4)$$

GvN for U^2-norm

$$\longrightarrow \le \frac{1}{N}\sum_y \|\Delta_{3y} f_4\|_{U^2}\cdot 1$$

Hölder

$$\longrightarrow \le \left(\frac{1}{N}\sum_y \|\Delta_{3y} f_4\|_{U^2}^4\right)^{1/4}$$

$$= \left(\frac{1}{N}\sum_y \|\Delta_y f_4\|_{U^2}^4\right)^{1/4} \quad \text{since } 3\nmid N .$$

$$= \left(\|f_4\|_{U^3}^8\right)^{1/4} \quad (\text{by } 1^{st} \text{ observation})$$

$$\Rightarrow \quad |AP_4(f_1,f_2,f_3,f_4)| \le \|f_4\|_{U^3} \qquad (j=1,2,3 \text{ similar}) \qquad \square .$$