The Fourier Transform on $\mathbb{R}^n$

Given $f \in L^1(\mathbb{R}^n)$ we define its Fourier transform $\hat{f} : \mathbb{R}^n \to \mathbb{C}$ by

$$\hat{f}(\xi) = \int_{\mathbb{R}^n} f(x) e^{-2\pi i x \cdot \xi} \, dx \quad , \text{where } x \cdot \xi = x_1\xi_1 + \cdots + x_n\xi_n .$$

Theorem (Fourier Inversion Formula)

If $f \in L^1(\mathbb{R}^n)$ and $\hat{f} \in L^1(\mathbb{R}^n)$, then for a.e. $x \in \mathbb{R}^n$,

$$f(x) = \int_{\mathbb{R}^n} \hat{f}(\xi) e^{2\pi i x \cdot \xi} \, d\xi .$$

Corollary 1: If $f \in L^1$ & $\hat{f} \in L^1$, then f agrees almost everywhere with a continuous function!

Certainly not true for arb. L^1 functions!!

Corollary 2: If $f \in L^1$ & $\hat{f} \equiv 0$, then $f = 0$ a.e.

Note: A simple appeal to Fubini/Tonelli fails as the integrand in:

$$\iint f(y) e^{-2\pi i \xi \cdot y} e^{2\pi i x \cdot \xi} \, dy \, d\xi$$

is not in $L^1(\mathbb{R}^{2n})$.

** Trick: We will introduce a "convergence factor" $\left(e^{-\pi t^2 |\xi|^2}\right)$ and pass to the limit. **

Before starting the proof we record the follow

<u>Basic Properties</u> (All Homework Exercises): Suppose $f, g \in L^1(\mathbb{R}^n)$

(a) (i) $(\tau_y f)^\wedge(\xi) = e^{-2\pi i y\cdot\xi}\hat{f}(\xi)$, where $\tau_y f(x) = f(x-y)$

(ii) $\tau_\eta(\hat{f}) = \hat{h}$, where $h(x) = e^{2\pi i \eta\cdot x} f(x)$.

(b) If T inv. linear. trans on $\mathbb{R}^n$ & $S = (T^*)^{-1}$ is its inv. transpose, then

$$(f\circ T)^\wedge = |\det T|^{-1}\, \hat{f}\circ S$$

* In particular,

$$\widehat{f_t}(\xi) = \hat{f}(t\xi), \text{ where } f_t(x) = t^{-n} f(t^{-1}x). \ *$$

(c) $\widehat{f * g} = \hat{f}\,\hat{g}$, where $f*g(x) = \int f(x-y)g(y)\,dy$.

(d) (i) If $x_j f \in L^1$, then $\frac{\partial}{\partial \xi_j}\hat{f}(\xi) = \hat{h}(\xi)$, where $h(x) = -2\pi i x_j f(x)$.

(ii) If $\frac{\partial}{\partial x_j} f \in L^1$ & $\lim_{|x|\to\infty} f(x) = 0$, then $\widehat{\frac{\partial}{\partial x_j} f}(\xi) = 2\pi i \xi_j \hat{f}(\xi)$.

$\rightarrow$ "smoothness of f $\Longleftrightarrow$ decay of $\hat{f}$ at infinity" (& vice versa).

(e) [Riemann-Lebesgue Lemma]

$\hat{f}$ is bounded, continuous & $\lim_{|\xi|\to\infty} \hat{f}(\xi) = 0$.

Important Example

If $g(x) = e^{-\pi|x|^2}$, then $\hat{g}(\xi) = e^{-\pi|\xi|^2}$.

(Follows by complex analysis OR $\underbrace{\text{properties (d) [(i) \& (ii)] \& } \int_{\mathbb{R}} e^{-\pi x^2}\,dx = 1}_{\text{see Homework}}$)

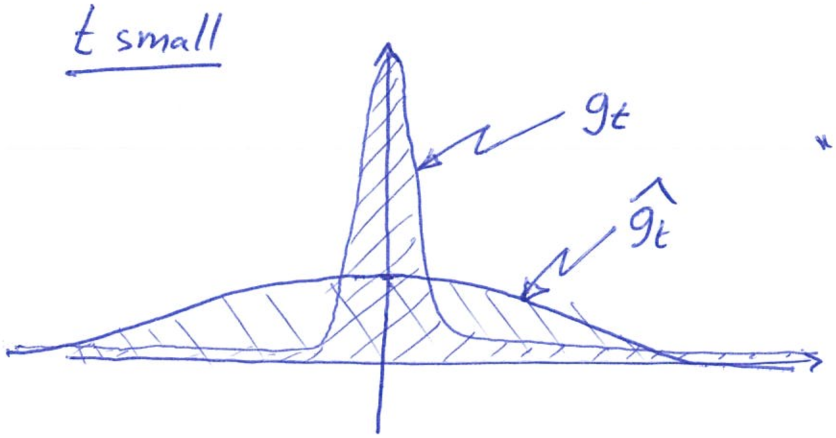
Remark (on g_t):

1. $g_t(x) = t^{-n} e^{-\pi|x|^2/t^2}$ & $\int g_t = 1 \quad \forall t > 0$

2. Property (b) [Special case]

$$\Rightarrow \quad \hat{g_t}(\xi) = \hat{g}(t\xi) = e^{-\pi t^2|\xi|^2}$$

t small



"Uncertainty Principle!!"

Key Ingredient: If $f \in L^1(\mathbb{R}^n)$, then

$$f * g_t \longrightarrow f \text{ in } L^1 \text{ as } t \to 0.$$

(This is just a special case of the "Approximate Identity Thm" from last time.)

Proof of Fourier Inversion Formula

An appeal to Fubini/~~Tonelli~~ does give:

> **Lemma** (Multiplication Formula) If $f, g \in L^1(\mathbb{R}^n)$, then
> $$\int \hat{f} g = \int f \hat{g}.$$

[Proof: Follows from Fubini since both integrals equal
$$\iint f(x) g(y) e^{-2\pi i x \cdot y} \, dx \, dy \quad \& \quad f(x) g(y) \in L^1(\mathbb{R}^n \times \mathbb{R}^n)].$$

Given $t > 0$ and $x \in \mathbb{R}^n$, we set
$$\varphi(\xi) = e^{2\pi i x \cdot \xi} \hat{g}_t(\xi). \quad \left[= e^{2\pi i x \cdot \xi} e^{-\pi t^2 |\xi|^2} \right]$$

It follows that
$$\hat{\varphi}(y) = \hat{\hat{g}}_t(y - x) = g_t(x - y)$$

Check!!

Therefore,
$$\int \hat{f}(\xi) \varphi(\xi) \, d\xi \overset{\text{Lemma}}{=} \int f(y) \hat{\varphi}(y) \, dy$$

Since $\hat{f} \in L^1$ (by DCT) $\downarrow$ as $t \to 0$

$$\int \hat{f}(\xi) e^{2\pi i x \cdot \xi} \, d\xi \qquad \qquad \int f(y) \hat{\varphi}(y)\,dy = \int f(y) g_t(x - y) \, dy = f * g_t(x).$$

Since $f * g_t \to f$ in L^1, it follows that
$$f(x) = \int \hat{f}(\xi) e^{2\pi i x \cdot \xi} \, d\xi \quad \text{for a.e. } x \in \mathbb{R}^n. \qquad \square$$