The Lebesgue Integral under Linear Transformations

We identify a linear map $T: \mathbb{R}^n \to \mathbb{R}^n$ with the matrix

$$(T_{ij}) = (e_i \cdot Te_j)$$

where $\{e_j\}$ is the standard basis for $\mathbb{R}^n$. We denote the determinant of this matrix by $\det T$ and recall that

$$\det(T \circ S) = (\det T)(\det S).$$

Furthermore, we write $GL(n,\mathbb{R})$ for the group of all invertible linear transformations of $\mathbb{R}^n$.

Basic Fact

Since every invertible matrix can be row-reduced to the identity it follows that every $T \in GL(n,\mathbb{R})$ can be written as a product of finitely many transformations of three elementary types, namely

$$T_1(x_1,\dots,x_j,\dots,x_n) = (x_1,\dots, cx_j,\dots,x_n) \quad (c \neq 0)$$

$$T_2(x_1,\dots,x_j,\dots,x_n) = (x_1,\dots,x_j + cx_k,\dots,x_n) \quad (j \neq k)$$

$$T_3(x_1,\dots,x_j,\dots,x_k,\dots,x_n) = (x_1,\dots,x_k,\dots,x_j,\dots,x_n)$$

Theorem: Let $T \in GL(n,\mathbb{R})$.

(i) If f is measurable, then so is $f \circ T$

(ii) If $f \geq 0$ or $f \in L^1$, then $\int f(x)\,dx = |\det T| \int f \circ T(x)\,dx$.

(iii) If $E \subseteq \mathbb{R}^n$ is measurable, then so is $T(E)$ & $m(T(E)) = |\det T|\, m(E)$.

Note

- (i) & (ii) $\Rightarrow$ (iii) [simply set $f = \chi_{TE}$].

- Corollary: Lebesgue measure is invariant under rotations/reflections.

Proof of Theorem

- Recall the following characterization of Lebesgue measure:

  If $E \subseteq \mathbb{R}^n$ is measurable, then

  (i) $E = V \setminus N_1$ where V is a G_δ set & $m(N_1) = 0$
  ("E is contained in a Borel set of the same measure")

  (ii) $E = H \cup N_2$ where H is a F_σ set & $m(N_2) = 0$
  ("E is the union of a Borel set & a null set")

- We first suppose that f is Borel measurable. Then $f \circ T$ is also Borel measurable since T is continuous.

  [Why?: Have to show that $T^{-1}(f^{-1}(G))$ is Borel $\forall$ open $G \subseteq \mathbb{R}^n$.
  Since f is Borel measurable we know that $f^{-1}(G)$ is Borel.
  Now consider $\{E : T^{-1}E \in \underset{\uparrow \text{Borel sets}}{B}\}$

  [Exercise: This forms a σ-algebra ~~and hence contains~~ which contains all open sets & hence all Borel sets!]]

Now if (ii) holds for T & S it is also true for $T \circ S$ since

$$\int f(x)dx = |\det T| \int f \circ T(x)\,dx = |\det T||\det S| \int (f\circ T)\circ S(x)\,dx = |\det(T\circ S)| \int f\circ(T\circ S)(x)\,dx$$

· Hence it suffices to prove (ii) when T is of the types T_1, T_2 & T_3.

· But this is a simple consequence of the Fubini – Tonelli theorem!

$$\underset{f\in L^1}{\uparrow} \qquad \underset{f\geq 0}{\uparrow}$$

- For T_3 we simply interchange the order of integration in x_j & x_k.
- For T_1 & T_2 we integrate first with respect to x_j & use the one-dimensional formulas

$$\int f(t)\,dt = |c| \int f(ct)\,dt \quad \& \quad \int f(t+h)\,dt = \int f(t)\,dt.$$

Since it is easily verified that

$$\det T_1 = c \quad , \quad \det T_2 = 1 \quad , \text{ and } \det T_3 = -1$$

the result follows, namely (ii) holds for <u>Borel measurable functions f</u>.

- It now follows that (iii) holds for all Borel sets E, since if $E\in\mathcal{B}$ then so is $T(E)$ (since T^{-1} is continuous) & we can set $f = \chi_{TE}$ in (ii).

  * In particular, the class of Borel null sets is invariant under T & T^{-1} and hence so is the class of Lebesgue null sets $\Rightarrow$ $\boxed{\text{(iii) holds for all Lebesgue measurable sets } E}$

- Now suppose that f is merely Lebesgue measurable:

$$\Rightarrow \underbrace{f^{-1}(G)}_{\text{for any open } G} = \underbrace{H}_{\text{Borel}} \cup \underbrace{N}_{\text{null.}} \quad \Rightarrow \underbrace{T^{-1}(f^{-1}(G))}_{\text{for any open } G} = T^{-1}H \cup T^{-1}N$$

But $T^{-1}H$ is also Borel & $T^{-1}N$ is also null! $\Rightarrow$ $T^{-1}(f^{-1}(G))$ measurable for any open G.

Thus $f\circ T$ is Lebesgue measurable & (i) is proved.

- <u>Finally we have to prove that (i) & (iii) $\Rightarrow$ (ii)</u>:

The hard work is done, we now follow a familiar proceedure:

- (iii) $\Rightarrow$ (ii) holds for $f = \chi_{TE}$.
- By linearity it also holds for simple functions
- By definition of the integral it thus holds all $f \in L^+$.
- Taking positive and negative parts of real and imaginary parts then yields the result for all $f \in L^1$.

$\square$