EXTENSIONS OF VARNAVIDES' PROOF ON 3-TERM ARITHMETIC PROGRESSIONS

LUCIA PETITO

Abstract

Varnavides proved, as a result of Roth's theorem, a lower bound on the number of three term arithmetic progressions in a set $[1, N]$. We will extend it to find a lower bound on the number of K-term arithmetic progressions and the number of solutions to the equation $c_1x_1 + c_2x_2 + c_3x_3 = 0$ such that $c_1 + c_2 + c_3 = 0$ in a subset of $[1, N]$.

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1. Introduction

In 1927, dutch mathematician Baartel Leendert van der Waerden (1903-1996) proved the following theorem about arithmetic progressions:

Given r *and* K*, there exists a positive integer* $N = N(r, K)$ *such that if the integers in* $[1, N]$ *are* r*-colored, there will exist at least one* K*-term arithmetic progression in* $[1, N]$

However, the bound he got on the size of N needed was not very strong. Hungarian mathematicians Paul Erdös (1913-1996) and Paul Turán (1910-1976) posed the following conjecture in 1936 with hopes to improve on the bounds:

Given δ *and* K*, there exists a positive integer* $N = N(\delta, K)$ *such that any subset* A *of* $[1, N]$ *with* $|A| \geq \delta N$ *contains a non-trivial* K*-term arithmetic progression.*

No progress was made on this conjecture for more than 20 years. Finally, in 1953, Klaus Roth (b. 1925) proved it for the case $K = 3$. Szemerédi (b. 1940) proved the $K = 4$ case in 1969, with Roth presenting an alternative proof in 1972, but it was not until 1975 that Szemerédi was able to finally prove the Erdös/Turán conjecture.

During the course of this program we were able to prove the following theorem using the method used in Roth's proof of the $K = 3$ case.

Given an equation $c_1x_1 + \cdots + c_kx_k = 0$ *such that* $\sum_{i=1}^{k} c_i = 0$ *in* $A \subseteq [1, N]$*,* $\delta > 0$*, and* $|A| \geq \delta N$ *for* $N \geq N(\delta, K)$*, a solution* $(x_1, x_2, \ldots, x_k)$ *to the given equation exists in* A*.*

The constant δ, as in all the theorems above, is also known as the density of A in $[1, N]$. While Roth and Szemerédi were thinking about proving the existence of one K-term arithmetic progression, Varnavides was proving the existence of multiple K-term arithmetic progressions in $[1, N]$. In 1955, Varnavides was able to prove the existence of multiple 3-term arithmetic progressions in [1, N], assuming that one existed. In 1958, he published a second paper with a better bound on the number of 3-term arithmetic progressions in $[1, N]$:

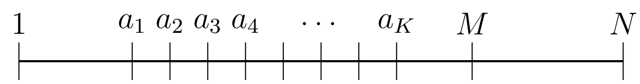
Given $\delta > 0$*, there exists* $N \geq N(\delta)$ *such that any subset* A *of* $[1, N]$ *with* $|A| \geq \delta N$ *contains at least* CN^2 *3-term arithmetic progressions, where* C *is a constant depending on* δ*.*

In this paper we will extend the argument Varnavides presented in 1958 to K-term arithmetic progressions, as well as solutions to linear polynomials in 3 variables, as in $c_1x_1 + c_2x_2 + c_3x_3 = 0$.

2. Varnavides' Bound for K-term Arithmetic Progressions

Theorem: *Given $\delta > 0$, $K \geq 3$ and a set $A \subseteq [1, N]$ of distinct integers with $|A| \geq \delta N$ and $N > N(\delta)$, there exist at least CN^2 distinct K-term arithmetic progressions in A.*

Proof: First, we must create a situation to which we can apply Szemerédi's theorem. Given $\frac{1}{2}\delta > 0$, look at the set of integers $[1, M]$, where $M > N(\frac{1}{2}\delta)$. In a subset $A \subseteq [1, M]$, with $|A| \geq \frac{\delta M}{2}$ there exists at least one K-term arithmetic progression according to Szemerédi's theorem.



Now consider all M-term arithmetic progressions in $[1, N]$ of the form:

$$1 \leq u \leq u + d \leq \ldots \leq u + (M-1)d \leq N$$

We will refer to these progressions as $P_{u,d}$, where u is the start point of the progression, and d is its step size. We are interested in the size of the intersection of $P_{u,d}$ and A, i.e. $|A \cap P_{u,d}|$. We define a *good progression* to be a $P_{u,d}$ such that $|A \cap P_{u,d}| > \frac{\delta M}{2}$. We assume $d < \frac{\delta}{M^2}N$.

Claim: The number of good progressions in $[1, N]$ with a fixed step size d is at least $\frac{1}{4}\delta N$, that is

$$|\{P_{u,d} : |A \cap P_{u,d}| > \tfrac{1}{2}\delta M\}| > \tfrac{1}{4}\delta N$$

Let GP be the total number of good progressions in A for any step size. Let GP_d be the total number of good progressions in A with a fixed step size d.

Look at all $a \in A$ such that $Md \leq a \leq N - Md$. We claim each of those a is contained in exactly M M-term arithmetic progressions with a fixed step size d.

Each a is of the form $a = u + ld$ where $l = 0, 1, 2, \ldots, M-1$. We know $Md \leq a \leq N - Md$. We must consider the end points, that is $a = Md$ and $b = N - Md$. It is clear that if a M-term arithmetic progression begins at $a = Md$, the M-term progression will fit in the interval [1, N]; the same is clear for a M-term arithmetic progression that ends at $a = N - Md$. It suffices to check that a M-term arithmetic progression ending at $a = Md$ fits in $[1, N]$ and a M-term arithmetic progression starting at $b = N - Md$ fits in $[1, N]$. Let $a = Md = d + (M-1)d$.

$$\begin{aligned} a - (M-1)d &= a - (d + (M-2)d) \\ &= d + (M-1)d - d - (M-2)d \\ &= d > 1 \end{aligned}$$

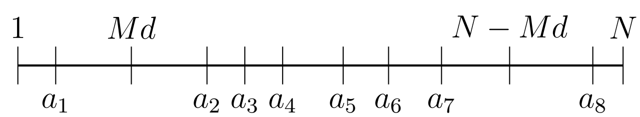
Now let $b = N - Md$.

$$\begin{aligned} b + (M-1)d &= N - Md + (M-1)d \\ &= N - d < N \end{aligned}$$

Since each of the endpoints of our interval can be the beginning and the ending of an M-term arithmetic progression, they can also occur at any point in the M-term arithmetic progression. Therefore, they can be contained in$M$ M-term arithmetic progressions. Since the endpoints can be contained in M M-term arithmetic progressions, each a inside our interval can be contained in M M-term arithmetic progressions with fixed step size d.

Since we know $Md < \frac{\delta N}{M}$ (from $d < \frac{\delta N}{M^2}$), the number of a's that fall in the interval $[Md, N - Md]$ is at least

$$\delta N - 2Md < \delta N - 2\tfrac{\delta N}{M} = \delta(1 - \tfrac{2}{M})N$$



Keep d fixed. Now find a lower bound on the sum of the sizes of the intersections of A and $P_{u,d}$

$$\sum_{u=1}^{N-(M-1)d} |A \cap P_{u,d}| \geq M\delta(1 - \tfrac{2}{M})N > \tfrac{3}{4}\delta MN$$

Now find an upper bound on the sum of the sizes of the intersections of A and $P_{u,d}$. The number of M-term arithmetic progressions with fixed d such that $1 \leq u \leq \cdots \leq u + (M-1)d \leq N$ is clearly less than N (this is a very rough estimate), so

$$\sum_{u=1}^{N-(M-1)d} |A \cap P_{u,d}| < \tfrac{1}{2}\delta MN + (GP_d)M$$

$$\tfrac{1}{4}\delta MN < (GP_d)M \Rightarrow (GP_d) > \tfrac{1}{4}\delta N$$

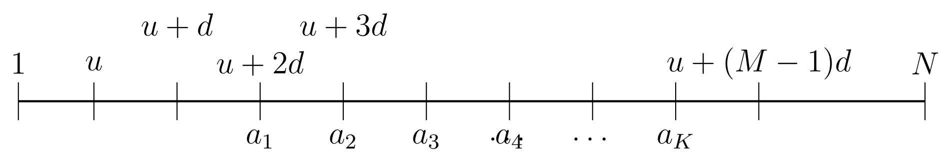
Since we have a lower bound on the number of good progressions for a given step size d, we can find a lower bound on the number of good progressions in A.

$$GP > \tfrac{1}{4}\delta N \tfrac{\delta N}{M^2} = \tfrac{\delta^2 N^2}{4M^2} = CN^2$$

Each $P_{u,d}$ in the set of good progressions contains a K-term arithmetic progression, by Szemerédi's theorem applied to the set:

$$\{1 + \tfrac{a_i - u}{d} : a_i \in A \cap P_{u,d}\}$$

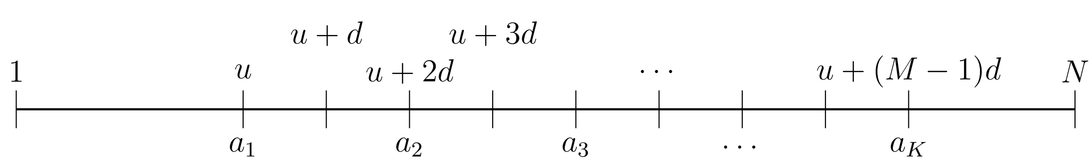
However, this approximation double counts some K-term arithmetic progressions.



Given $P_{u,d}$, an M-term arithmetic progression, our solution with $d = d'$ can be found in at most $M - K + 1$ progressions.

a_1	a_K
$u + 0d$	$u + (K-1)d$
$u + d$	$u + Kd$
$\vdots$	$\vdots$
$u + (M-K)d$	$u + (M-1)d$

The K-term arithmetic progression does not fit in $P_{u,d}$ if a_1 is any of the last $M-K+1$ terms in the M-term arithmetic progression; ie if $l = \{M-K, M-K+1, \ldots, M-1\}$. Thus, we have $M - K + 1$ options for the beginning placement of the K-term arithmetic progression, $l = \{0, 1, 2, \ldots, M-K\}$



Now consider our K-term arithmetic progression iwth step size $\frac{d'}{t} = d$. We have a progression $P_{u,d}$ with step size d. We need $Md > (K-1)d'$ to guarantee that the

K-term arithmetic progression fits inside a M-term arithmetic progression with step size d. This implies

$$M\frac{d'}{t} > (K-1)d' \rightarrow t < \frac{M}{K-1} \quad (d' < \frac{dM}{K-1})$$

We have $\frac{M}{(K-1)}$ options for step sizes, and a very rough bound on the number of M-term arithmetic progressions our K-term arithmetic progression fits in, $M-K+1$. Thus, our K-term arithmetic progression can be found in at most $\frac{M}{K-1}(M-K+1)$ M-term arithmetic progressions.

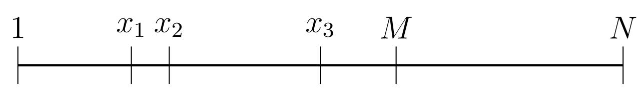
Thus, the same K-term arithmetic progression cannot occur more than C_2 times (where C_2 is a constant depending on M, which in turn depends on δ and K). We can say the number of distinct K-term arithmetic progressions is at least $\frac{C_1}{C_2}N^2 = CN^2$.

3. Varnavides' Bound for Solutions to Linear Polynomials in 3 Variables

As stated in the introduction, it is known that given $\delta > 0$, a set $A \subseteq [1, N]$ of distinct integers with $|A| \geq \delta N$ with $N > N(\delta)$, and an equation $c_1x_1 + \cdots + c_kx_k = 0$ such that $\sum_{i=1}^{k} c_i = 0$, we are guaranteed to have at least one solution in A.

Theorem: *Given $\delta > 0$, a set $A \subseteq [1, N]$ of distinct integers with $|A| \geq \delta N$ with $N > N(\delta)$, and an equation $c_1x_1 + c_2x_2 + c_3x_3 = 0$ with $c_1 + c_2 + c_3 = 0$, there exist at least CN^2 solutions to this equation in A.*

Proof: First, we must create a situation that fits our theorem. Look at $[1, M]$, where $M \geq N(\frac{1}{2}\delta)$. Given $c_1x_1 + c_2x_2 + c_3x_3 = 0$ with $c_1 + c_2 + c_3 = 0$, in a subset $A \subseteq [1, M]$, with $|A| \geq \frac{1}{2}\delta M$, there exists at least one solution to this equation, according to our theorem.



Now consider all M-term arithmetic progressions that satisfy

$$1 \leq u \leq u + d \leq \ldots \leq u + (M-1)d \leq N$$

We assume $d < \frac{\delta}{M^2}N$.

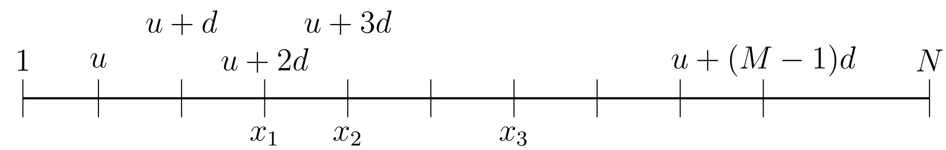
Fix d. The number of good progressions in $[1, N]$ is at least $\frac{1}{4}\delta N$, as shown in section 2. As before, let GP stand for the total number of good progressions for all step sizes.

$$\text{GP} > \frac{1}{4}\delta N\frac{\delta N}{M^2} = \frac{\delta^2 N^2}{4M^2} = CN^2$$

Each $P_{u,d}$ in the set of good progressions contains a solution to our monomial in 3 variables, by the theorem stated earlier applied to the set:

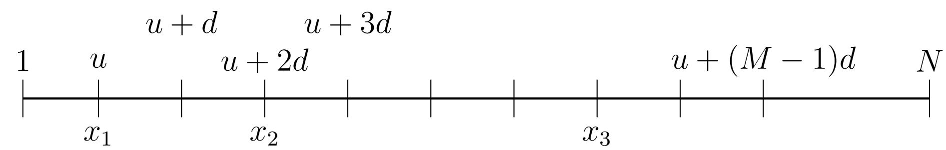
$$\{1 + \tfrac{a_i - u}{d} : a_i \in A \cap P_{u,d}\}$$

However, this approximation double counts a some solutions.



Given $P_{u,d}$, an M-term arithmetic progression, our solution with $d = gcd|x_i - x_j|$ for $i, j \in \{1, 2, 3\}$ can be found in at most $M - 2$ M-term arithmetic progressions.

Worst case scenario: Assuming without loss of generality $x_1 < x_2 < x_3$, if $d = 1 = |x_2 - x_1| = |x_3 - x_2|$, then we have a 3-term arithmetic progression, so we can place this solution in $M - 2$ M-term progressions. The solution does not fit in $P_{u,d} = \{u, u + d, \ldots, u + ld, \ldots, u + (M-1)d\}$ if x_1 is any of the last two terms in the M-term arithmetic progression; ie if $l = \{M - 2, M - 1\}$. Thus, we have M-2 options for the placement of $x_1 = u + ld$, $l = \{0, 1, 2, \ldots, M - 3\}$



Now consider our solution with $\frac{gcd|x_i - x_j|}{t} = d$ for $i, j \in [1, 3]$. We have a progression $P_{u,d}$ with step size d. We know $Md > \max|x_i - x_j|(i, j \in \{1, 3\})$ to ensure that the solution fits in the M-term arithmetic progression. By definition, $\max|x_i - x_j| = wgcd|x_i - x_j|(i, j \in \{1, 2, 3\})$, where $w \in [1, N]$ This means

$$M\frac{gcd|x_i - x_j|}{t} > \max|x_i - x_j| = wgcd|x_i - x_j|$$
$$t < \frac{M}{w} \quad (d' < \frac{dM}{2})$$

We have $\frac{M}{2}$ options for step sizes, and a very rough bound on the number of M-term arithmetic progressions our solution fits in, $M - 2$. Thus, our solution can be found in at most $\frac{M}{2}(M - 2)$ M-term arithmetic progressions.

Thus, the same solution cannot occur more than C_2 times (where C_2 is a constant depending on M, which in turn depends on δ). Therefore, we can say the number of distinct solutions is at least $\frac{C_1}{C_2}N^2 = CN^2$.

References

[1] Roth, Klaus. *On Certain Sets of Integers* Journal London Math. Soc., 28 (1953), 104-109.

[2] Varnavides, P. *On Certain Sets of Positive Density* Journal London Math. Soc. 34 (1959) 358-360.

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