Math 3100 Assignment 9

Taylor Series

Homework due date: 1:00 pm on Friday the 12th of April 2019

1. Find a power series representation for the function

   (a) $f(x) = \dfrac{1}{4+x^2}$ (b) $g(x) = \dfrac{1}{(1+x)^2}$ (c) $h(x) = x\log(1+x)$

2. Evaluate these sums

   (a) $\displaystyle\sum_{n=0}^{\infty} 2^{-n}$ (b) $\displaystyle\sum_{n=3}^{\infty} \frac{4^{1-n}}{2n-1}$ (c) $\displaystyle\sum_{n=1}^{\infty} n^2 3^{-n}$

3. Find the Taylor Polynomial of order n generated by f centered at x_0.

   (a) $f(x) = \log x, \quad x_0 = 1, \quad n = 3$
   (b) $f(x) = \sqrt{x+4}, \quad x_0 = 0, \quad n = 2$
   (c) $f(x) = \dfrac{xe^{-x}}{x^2+1}, \quad x_0 = 0, \quad n = 6$

4. Let $f(x) = \dfrac{1}{1+3x^2}$. Without differentiating, find $f^{(8)}(0)$. Show your work.

5. Find the Taylor Series centered at $x_0 = 0$ (the Maclaurin Series) of the following functions.

   (a) $x^2 \sin x$
   (b) $\sin^2 x$ *Hint:* $\sin^2 x = (1 - \cos 2x)/2$.

6. Find the Taylor series generated by f at x_0.

   (a) $f(x) = x^4 + x^2 + 1, \quad x_0 = -2$
   (b) $f(x) = x^{-2}, \quad x_0 = 1$

7. For what values of x do the following polynomials approximate $\sin x$ to within 0.01

   (a) $P_1(x) = x$ (b) $P_3(x) = x - x^3/6$ (c) $P_5(x) = x - x^3/6 + x^5/120$

8. How accurately does $1 + x + x^2/2$ approximate e^x for $-1 \le x \le 1$? Can you find a polynomial that approximates e^x to within 0.01 on this interval?

9. (a) How accurately does $1 - x^2 + x^4/2$ approximate e^{-x^2} for $-1 \le x \le 1$?
   (b) Can you find a polynomial that approximates e^{-x^2} to within 0.01 on this interval?

10. Find a polynomial that will approximate

$$F(x) = \int_0^x t^2 e^{-t^2}\, dt$$

for all x in the interval $[0, 1]$ with an error of magnitude less than 10^{-3}.

Math 3100 – Homework 9 – SOLUTIONS

1. (a) $$\frac{1}{4+x^2} = \frac{1}{4}\frac{1}{1+\left(\frac{x}{2}\right)^2} = \frac{1}{4}\sum_{n=0}^{\infty}(-1)^n\left(\frac{x}{2}\right)^{2n} \quad \text{if } \left|\frac{x}{2}\right|<1$$
$$= \sum_{n=0}^{\infty}\frac{(-1)^n}{4^{n+1}}x^{2n} \quad \text{if } |x|<2$$

(b) $$\frac{1}{(1+x)^2} = -\frac{d}{dx}\left(\frac{1}{1+x}\right)$$
$$= -\frac{d}{dx}\left(\sum_{n=0}^{\infty}(-1)^n x^n\right) \quad \text{if } |x|<1$$
$$= -\sum_{n=0}^{\infty}\frac{d}{dx}\left((-1)^n x^n\right) \quad \text{if } |x|<1$$

by "differentiation term-by-term".

$$= \sum_{n=0}^{\infty}(-1)^{n+1} n\, x^{n-1} \quad \text{if } |x|<1$$
$$= \sum_{n=1}^{\infty}(-1)^{n+1} n\, x^{n-1} \quad \text{if } |x|<1$$

OR

$$\sum_{n=0}^{\infty}(-1)^n(n+1)x^n \quad \text{if } |x|<1.$$

(c) $$x\log(1+x) = x\int_0^x \frac{1}{1+t}\,dt$$
$$= x\int_0^x \sum_{n=0}^{\infty}(-1)^n t^n\,dt \quad \text{if } |x|<1.$$

by "integration term by term"

$$= x\sum_{n=0}^{\infty}\int_0^x (-1)^n t^n\,dt$$
$$= x\sum_{n=0}^{\infty}(-1)^n\frac{x^{n+1}}{n+1} = \sum_{n=2}^{\infty}\frac{(-1)^n}{n-1}x^n.$$

2. (a) Since $\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$ if $|x|<1$

$$\Rightarrow \sum_{n=0}^{\infty} 2^{-n} = \frac{1}{1-\frac{1}{2}} = \underline{\underline{2}}$$

(b) Since $\sum_{n=0}^{\infty} \frac{x^{2n+1}}{2n+1} = \sum_{n=0}^{\infty} \int_0^x t^{2n}\,dt$

$$= \int_0^x \left(\sum_{n=0}^{\infty} t^{2n}\right) dt$$

"Integration term-by-term"

$$= \int_0^x \frac{1}{1-t^2}\,dt \quad \text{if } |x|<1.$$

$$= \frac{1}{2}\int_0^x \left(\frac{1}{1+t} + \frac{1}{1-t}\right) dt$$

$$= \frac{1}{2}\log\left(\frac{1+x}{1-x}\right) \quad \text{if } |x|<1$$

$$\Rightarrow \underline{\sum_{n=0}^{\infty} \frac{\left(\frac{1}{2}\right)^{2n+1}}{2n+1} = \frac{1}{2}\log 3}$$

Now $\sum_{n=1}^{\infty} \frac{4^{1-n}}{2n-1} = \sum_{n=0}^{\infty} \frac{\left(\frac{1}{2}\right)^{2n}}{2n+1} = 2\sum_{n=0}^{\infty} \frac{\left(\frac{1}{2}\right)^{2n+1}}{2n+1} = \log 3$

$$\Rightarrow \sum_{n=3}^{\infty} \frac{4^{1-n}}{2n-1} = \log 3 - (\underset{n=1}{1} + \underset{n=2}{\tfrac{1}{12}})$$

$$= \underline{\underline{\log 3 - \frac{13}{12}}}$$

(c) Since $\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$ if $|x|<1$

$$\Rightarrow \frac{d}{dx}\left(\sum_{n=0}^{\infty} x^n\right) = \frac{d}{dx}\left(\frac{1}{1-x}\right) \quad \text{if } |x|<1$$

$$\Rightarrow \sum_{n=1}^{\infty} n x^{n-1} = \frac{1}{(1-x)^2} \quad \text{if } |x|<1.$$

differentiation term-by-term

$$\Rightarrow \sum_{n=1}^{\infty} n x^{n} = \frac{x}{(1-x)^2} \quad \text{if } |x|<1$$

$$\Rightarrow \frac{d}{dx}\left(\sum_{n=1}^{\infty} n x^n\right) = \frac{d}{dx}\left(\frac{x}{(1-x)^2}\right) \quad \text{if } |x|<1$$

$$\Rightarrow \sum_{n=1}^{\infty} n^2 x^{n-1} = \frac{1+x}{(1-x)^3} \quad \text{if } |x|<1$$

$$\Rightarrow \sum_{n=1}^{\infty} n^2 x^{n} = \frac{x(1+x)}{(1-x)^3} \quad \text{if } |x|<1$$

Hence $\sum_{n=1}^{\infty} n^2 \left(\frac{1}{3}\right)^n = \frac{\left(\frac{1}{3}\right)\left(\frac{4}{3}\right)}{\left(\frac{2}{3}\right)^3} = \frac{3}{2}$

3. (a) Let $f(x) = \log x$

The 3rd order Taylor Poly of f centered at $x_0 = 1$ is:

$$f(1) + f'(1)(x-1) + \frac{f''(1)}{2}(x-1)^2 + \frac{f'''(1)}{6}(x-1)^3$$

$$= 0 + (1)(x-1) + \left(-\frac{1}{2}\right)(x-1)^2 + \left(\frac{1}{3}\right)(x-1)^3.$$

$$= (x-1) - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3.$$

OR

since we know that

$$\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots \quad \text{if } |x|<1$$

$$\Rightarrow \log x = \log(1+(x-1))$$

$$= \underbrace{(x-1) - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3}_{3^{rd}\text{ order Taylor poly of } \log x \text{ centered at } x_0=1.} + \cdots \quad \text{if } |x-1|<1$$

(b). Let $f(x) = \sqrt{x+4}$

The 2^{nd} order Maclaurin Poly for f is:

$$f(0) + f'(0)x + \frac{f''(0)}{2}x^2.$$

$$= 2 + \left(\frac{1}{4}\right)x + \left(\frac{1}{64}\right)x^2$$

(c) $$\frac{xe^{-x}}{1+x^2} = (xe^{-x})\left(\frac{1}{1+x^2}\right)$$

if $|x|<1$

$$= \left(x - x^2 + \frac{x^3}{2} - \frac{x^4}{6} + \frac{x^5}{24} - \frac{x^6}{120} + \cdots\right)\left(1 - x^2 + x^4 - x^6 + \cdots\right)$$

$$= \underbrace{x - x^2 + \underset{\frac{1}{2}-1}{\underset{\shortparallel}{\left(-\frac{1}{2}\right)}}x^3 + \underset{-\frac{1}{6}+1}{\underset{\shortparallel}{\frac{5}{6}}}x^4 + \underset{\frac{1}{24}-\frac{1}{2}+1}{\underset{\shortparallel}{\frac{13}{24}}}x^5 + \underset{-\frac{1}{120}+\frac{1}{6}-1}{\underset{\shortparallel}{\left(-\frac{101}{120}\right)}}x^6}_{6^{th}\text{ order Maclaurin Series for } \frac{xe^{-x}}{1+x^2}} + \cdots$$

4. Let $f(x) = \frac{1}{1+3x^2}$

We know that
$$f(x) = \sum_{n=0}^{\infty} (-1)^n (3x^2)^n \quad \text{if } |3x^2| < 1.$$
$$= \sum_{n=0}^{\infty} (-1)^n 3^n x^{2n} \quad \text{if } |x| < 1/\sqrt{3}.$$

* This is also the Maclaurin Series for f *

Thus the coefficient in front of x^8, namely $(-1)^4 3^4 = 81$ is equal to $\frac{f^{(8)}(0)}{8!}$.

$$\Rightarrow \quad f^{(8)}(0) = \underline{\underline{8!\,(81)}}$$

5(a) Since $\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$ for all $x \in \mathbb{R}$

$$\Rightarrow \quad x^2 \sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+3}}{(2n+1)!} \quad \text{for all } x \in \mathbb{R}$$

(b) Since $\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} \quad \forall x \in \mathbb{R}$

$$\Rightarrow \quad \cos 2x = \sum_{n=0}^{\infty} \frac{(-1)^n 4^n x^{2n}}{(2n)!} \quad \forall x \in \mathbb{R}$$

$$\Rightarrow \quad \sin^2 x = \frac{1}{2}(1 - \cos 2x)$$
$$= \frac{1}{2} \sum_{n=1}^{\infty} \frac{(-1)^n 4^n x^{2n}}{(2n)!} = \sum_{n=1}^{\infty} \frac{(-1)^n 2^{2n-1}}{(2n)!} x^{2n} \quad \forall x \in \mathbb{R}$$

6. (a) Let $f(x) = x^4 + x^2 + 1$

The 4^{th} order Taylor Poly of f centered at $x_0 = -2$ is:

$$f(-2) + f'(-2)(x+2) + \frac{f''(-2)}{2}(x+2)^2 + \frac{f'''(-2)}{6}(x+2)^3 + \frac{f^{(4)}(-2)}{24}(x+2)^4$$

$$= 21 - 36(x+2) + 25(x+2)^2 - 8(x+2)^3 + (x+2)^4.$$

$\circledast$ This is the full Taylor series of f centred at $x_0 = -2$ since $f^{(n)}(-2) = 0 \;\; \forall n \geq 5$ $\circledast$

(b)

Since $\dfrac{1}{x^2} = \dfrac{1}{(1+(x-1))^2}$

and
$$\frac{1}{(1+x)^2} = -\frac{d}{dx}\left(\frac{1}{1+x}\right)$$
$$= -\frac{d}{dx}\left(\sum_{n=0}^{\infty}(-1)^n x^n\right) \quad \text{if } |x|<1$$

differentiate term-by-term

$$= \sum_{n=0}^{\infty}(-1)^{n+1}\frac{d}{dx}x^n$$
$$= \sum_{n=1}^{\infty}(-1)^{n+1} n x^{n-1} \quad \text{if } |x|<1$$

$$\Rightarrow \frac{1}{x^2} = \sum_{n=1}^{\infty}(-1)^{n+1} n (x-1)^{n-1}$$
$$= \underbrace{\sum_{n=0}^{\infty}(-1)^n (n+1)(x-1)^n}_{\text{Taylor series for } \frac{1}{x^2} \text{ centered at } x_0 = 1.} \quad \text{if } |x-1|<1$$

7. Recall that

$$\sin x = x - \frac{x^3}{6} + \frac{x^5}{120} - \frac{x^7}{5040} + \cdots \qquad \text{for all } x \in \mathbb{R}.$$

Notice that this is alternating & the terms are decreasing provided $|x| \leq 2$ (for example).
We can therefore apply the "Alt. Series. Remainder Est"

(a) $|\sin x - x| \leq \frac{|x|^3}{6} \leq \frac{1}{100}$ if $|x| \leq \sqrt[3]{\frac{6}{100}}$

first omitted terms

(b) $\left|\sin x - \left(x - \frac{x^3}{6}\right)\right| \leq \frac{|x|^5}{120} \leq \frac{1}{100}$ if $|x| \leq \sqrt[5]{\frac{120}{100}}$.

(c) $\left|\sin x - \left(x - \frac{x^3}{6} + \frac{x^5}{120}\right)\right| \leq \frac{|x|^7}{5040} \leq \frac{1}{100}$ if $|x| \leq \sqrt[7]{\frac{5040}{100}}$

Note that this is still ≤ 2

Alternative Approach
Using Lagrange's Remainder Estimate

Let $f(x) = \sin x$.

for some c between 0 & x

(a) $|\sin x - x| = \left|\frac{f^{(3)}(c)}{6} x^3\right| \leq \frac{|x|^3}{6} \leq \frac{1}{100}$ if $|x| \leq \sqrt[3]{\frac{6}{100}}$

$|-\cos(c)| \leq 1$

actually the 2nd order Maclaurin Poly for $\sin x$!

for some c between 0 & x

(b) $\left|\sin x - \left(x - \frac{x^3}{6}\right)\right| = \left|\frac{f^{(5)}(c)}{120} x^5\right| \leq \frac{|x|^5}{120} \leq \frac{1}{100}$ if $|x| \leq \sqrt[5]{\frac{120}{100}}$

$\underbrace{x - \frac{x^3}{6}}$ actually the 4^{th} order Maclaurin Poly for $\sin x$.

for some c between 0 & x

(c) $\left|\sin x - \left(x - \frac{x^3}{6} + \frac{x^5}{120}\right)\right| = \left|\frac{f^{(7)}(c)}{7!} x^7\right| \leq \frac{|x|^7}{5040} \leq \frac{1}{100}$ if $|x| < \sqrt[7]{\frac{5040}{100}}$

$\underbrace{x - \frac{x^3}{6} + \frac{x^5}{120}}$ actually the 6^{th} order Maclaurin Poly for $\sin x$

(*) It now is unimportant that

$\sqrt[7]{\frac{5040}{100}} \leq 2$!

8. It follows from Lagrange's Remainder Estimate that

$$e^x - \left(1 + x + \frac{x^2}{2}\right) = \frac{e^c}{6} x^3 \quad \text{for some } c \text{ between } 0 \,\&\, x.$$

$$\Rightarrow \left|e^x - \left(1 + x + \frac{x^2}{2}\right)\right| \leq \frac{e}{6} \quad \text{if } |x| \leq 1.$$

Since

$$\left|e^x - \left(1 + x + \frac{x^2}{2} + \cdots + \frac{x^n}{n!}\right)\right| = \frac{e^c}{(n+1)!}|x|^{n+1} \quad \text{for some } c \text{ between } 0 \,\&\, x$$

$$\leq \frac{e}{(n+1)!} \quad \text{if } |x| \leq 1$$

& $\frac{e}{(n+1)!} \leq \frac{1}{100}$ if $n \geq 5$

$$\Rightarrow \left|e^x - \left(1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \frac{x^5}{120}\right)\right| \leq \frac{e}{6!} = \frac{e}{720} < \frac{1}{100}.$$

9. (a) Recall that

$$e^{-x^2} = 1 - x^2 + \frac{x^4}{2} - \frac{x^6}{6} + \frac{x^8}{24} - \frac{x^{10}}{120} + \cdots \quad \forall\, x \in \mathbb{R}.$$

Since this series is alternating and the terms are decreasing if $|x| \le 1$ we can use the Alternating Series Remainder Estimate & see that

$$\left| e^{-x^2} - \left(1 - x^2 + \frac{x^4}{2}\right) \right| \le \frac{|x|^6}{6} \le \frac{1}{6} \quad \text{if } |x| \le 1.$$

first omitted terms

(b) Since $\dfrac{|x|^{10}}{120} \le \dfrac{1}{120} < \dfrac{1}{100}$ if $|x| \le 1$

$$\Rightarrow \left| e^{-x^2} - \left(1 - x^2 + \frac{x^4}{2} - \frac{x^6}{6} + \frac{x^8}{24}\right) \right| \le \frac{|x|^{10}}{120} < \frac{1}{100} \text{ if } |x| \le 1.$$

Alternative Approach Using Lagrange's Rem. Estimate

(a) Recall that

$$\left| e^x - \left(1 + x + \frac{x^2}{2}\right) \right| \le \frac{e^c}{6} |x|^3 \text{ for some } c \text{ between } 0 \,\&\, x.$$

$$\Rightarrow \left| e^{-x^2} - \left(1 - x^2 + \frac{x^4}{2}\right) \right| \le \frac{e^{c'}}{6} |x|^6 \text{ for some } c' \text{ between } 0 \,\&\, -x^2$$

since $-x^2 < c' < 0$
$\Rightarrow e^{c'} \le e^0 = 1$

$$\le \frac{|x|^6}{6} \le \frac{1}{6} \text{ if } |x| \le 1$$

(b) Recall that

$$\left| e^x - \left(1 + x + \frac{x^2}{2} + \cdots + \frac{x^n}{n!}\right)\right| = \frac{e^c |x|^{n+1}}{(n+1)!} \quad \text{for some } c \text{ between } 0 \,\&\, x$$

$$\Rightarrow \left| e^{-x^2} - \left(1 - x^2 + \frac{x^4}{2} + \cdots + \frac{(-1)^n x^{2n}}{n!}\right)\right| = \frac{e^{c'} |x|^{2n+2}}{(n+1)!} \quad \text{for some } -x^2 < c < 0$$

$$\le \frac{|x|^{2n+2}}{(n+1)!} \quad \text{since } e^{c'} \le e^0 = 1$$

$$\le \frac{1}{(n+1)!} \quad \text{if } |x| \le 1$$

Since $\frac{1}{(n+1)!} \le \frac{1}{100}$ if $n \ge 4$

$$\Rightarrow \left| e^{-x^2} - \left(1 - x^2 + \frac{x^4}{2} - \frac{x^6}{6} + \frac{x^8}{24}\right)\right| \le \frac{1}{5!} = \frac{1}{120} < \frac{1}{100} \quad \forall |x| \le 1.$$

10. Recall that

$$t^2 e^{-t^2} = t^2 - t^4 + \frac{t^6}{2} - \frac{t^8}{6} + \frac{t^{10}}{24} - \frac{t^{12}}{120} + \cdots$$

$$\Rightarrow \int_0^x t^2 e^{-t^2}\,dt = \frac{x^3}{3} - \frac{x^5}{5} + \frac{x^7}{14} - \frac{x^9}{54} + \frac{x^{11}}{264} - \frac{x^{13}}{1560} + \cdots$$

Since this series is alternating & decreasing if $0 \le x \le 1$

$$\Rightarrow \left| \int_0^x t^2 e^{-t^2}\,dt - \left(\frac{x^3}{3} - \frac{x^5}{5} + \frac{x^7}{14} - \frac{x^9}{54} + \frac{x^{11}}{264}\right) \right| \le \frac{|x|^{13}}{1560} < \frac{1}{1000} \quad \text{if } 0 \le x \le 1.$$

Alt. Series Rem. Est.

first omitted term

Alternative Approach using Lagrange's Remainder Estimate

Recall that

$$e^x - \left(1+x+\frac{x^2}{2}+\cdots+\frac{x^n}{n!}\right) = \frac{e^c x^{n+1}}{(n+1)!} \quad \text{for some } c \text{ between } 0 \,\&\, x.$$

$$\Rightarrow \quad e^{-t^2} - \left(1-t^2+\frac{t^4}{2}+\cdots+(-1)^n\frac{t^{2n}}{n!}\right) = \frac{e^{c'} t^{2n+2}}{(n+1)!} \quad \text{for some } -t^2 < c' < 0$$

$$\Rightarrow \quad t^2e^{-t^2} - \left(t^2-t^4+\frac{t^6}{2}+\cdots+(-1)^n\frac{t^{2n+2}}{n!}\right) = \frac{e^{c'} t^{2n+4}}{(n+1)!}$$

$$\Rightarrow \quad \int_0^x t^2e^{-t^2}\,dt - \int_0^x \left(t^2-t^4+\frac{t^6}{2}+\cdots+(-1)^n\frac{t^{2n+2}}{n!}\right)dt = \int_0^x \frac{e^{c'} t^{2n+4}}{(n+1)!}\,dt$$

$$\Rightarrow \quad \left| \int_0^x t^2e^{-t^2}\,dt - \left(\frac{x^3}{3}-\frac{x^5}{5}+\cdots+\frac{(-1)^n x^{2n+3}}{(2n+3)\,n!}\right)\right| \le \int_0^x \frac{t^{2n+4}}{(n+1)!}\,dt$$

since $e^{c'} \le e^0 \le 1$

$$= \frac{x^{2n+5}}{(2n+5)(n+1)!}$$

$$\le \frac{1}{(2n+5)(n+1)!} \quad \text{if } 0 \le x \le 1.$$

Since $\frac{1}{(2n+5)}\frac{1}{(n+1)!} \le \frac{1}{1000}$ if $n \ge 4$

$$\Rightarrow \quad \left| \int_0^x t^2e^{-t^2}\,dt - \left(\frac{x^3}{3}-\frac{x^5}{5}+\frac{x^7}{14}-\frac{x^9}{54}-\frac{x^{11}}{264}\right)\right| \le \frac{1}{(13)(120)} < \frac{1}{1000}$$