In this lecture we will bring everything together and conclude the proof of

Theorem (Freiman-Ruzsa on $\mathbb{Z}$)

If $A \subseteq \mathbb{Z}$ finite with $|A+A| \leq K|A|$, then A is contained in a GAP Q of dimension $d = O(K^c)$ and size $|Q| \leq e^{O(K^c)}|A|$.

We will show that one can take $c = 34$ (say).

Proof

Initial Steps:

- Since $|A+A| \leq K|A|$, Plünnecke implies that $|8A-8A| \leq K^{16}|A|$.

- Ruzsa's model lemma $\Rightarrow$ $\exists A' \subseteq A$ with $|A'| \geq |A|/8$ and $B \subseteq \mathbb{Z}_N$ with $K^{16}|A| \leq N \leq 2K^{16}|A|$ prime s.t. $A' \simeq_8 B$.

- Bogolyubov's lemma $\Rightarrow$ $2B-2B$ contains a Bohr set $B(\Gamma, \frac{1}{2\pi})$ with rank $d := |\Gamma| \leq 2^{10}K^{32}$.

- Geometry of Numbers $\Rightarrow$ $2B-2B$ contains a proper GAP Q of dimension d and size $|Q| \geq e^{-cd\log d}|A|$.

- Since $A' \simeq_8 B \Rightarrow 2A'-2A' \simeq_2 2B-2B$.

  Thus $2A'-2A'$, and hence also $2A-2A$ contains a proper GAP Q of dimension $d \leq 2^{10}K^{32}$ and size $|Q| \approx e^{-cK^{33}}|A|$.

Final Step

- **1st Attempt:** Note that $|A+Q| \leq |3A-2A| \leq K^5|A| \leq e^{cK^{33}}|Q|$.

Ruzsa's covering lemma $\Rightarrow$ $\exists$ set $X \subseteq A$ with $|X| \leq e^{cK^{33}}$ s.t.

$$A \subseteq X+Q-Q.$$

Since $X \subseteq$ GAP of dimension $|X|$ and size $2^{|X|}$ it follows that

dominate!

$$X+Q-Q \subseteq \text{GAP of dimension } |X|+d \leq e^{cK^{33}}$$

$$\text{and size } \leq 2^{|X|}|Q-Q| \leq 2^{|X|}2^d|Q| \leq e^{e^{cK^{33}}}.$$

* The wasteful step was allowing X to be so big. We can do better:

- **2nd Attempt: Chang's Refinement:** Set $Q_0 = Q$ and let X_0 denote maximal subset of A such that $\{x+Q_0 : x \in X_0\}$ pairwise disjoint.

If $|X_0| \leq 2K$ – STOP.

If $|X_0| > 2K$, let Y_0 be *any* subset of X_0 with $|Y_0| = 2K$ and $Q_1 = Q_0 + Y_0$.

Now choose $X_1 \subseteq A$ maximally such that $\{x+Q_1 : x \in X_1\}$ pairwise disjoint.

If $|X_1| \leq 2K$ – STOP.

If $|X_1| > 2K$ ...

* Proceed until we terminate in a set X_J of size $\leq 2K$. *

[Since $|Q_J| = (2K)^J|Q| \approx (2K)^J e^{-cK^{33}}|A|$, and $|Q_J| \leq |JA+Q| \leq |(J+2)A-2A| \leq K^{J+4}|A|$ we see that $2^J \leq e^{cK^{33}} \Rightarrow J \leq CK^{33}$.]

By maximality of each X_j we get $A \subseteq X_J + Q_J - Q_J = X_J + \sum_{j=0}^{J-1}(Y_j - Y_j) + (Q-Q)$.

Hence $A \subseteq$ GAP of dimension $\leq (J+1)2K + 2^{10}K^{32} \leq CK^{34}$ & size $\leq 3^{(J+1)2K}2^{cK^{32}}|Q| \leq e^{cK^{34}}$