On Roth's $\frac{1}{4}$-Theorem

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August 4, 2010

Abstract

We outline some Fourier-analytic properties of arithmetic progressions in $\mathbb{Z}_N$ and provide a proof of the Roth's $\frac{1}{4}$-Theorem.

1 Introduction

The goal of this paper is a proof of the Roth $\frac{1}{4}$-Theorem, which states that for any 2-coloring of the integers from 1 to N, there is an arithmetic progression contained in 1 to N with at least a constant times $N^{1/4}$ more elements of one color than the other. Our approach will be to prove the analogous result in $\mathbb{Z}_N$ but to restrict the choice of our progressions such that they easily translate into genuine progressions in the integers from 1 to N.

We begin with some basic definitions and propositions about Fourier transforms over $\mathbb{Z}_N$.

Definition 1. For $f : \mathbb{Z}_N \to \mathbb{C}$, define the Fourier transform

$$\hat{f}(\xi) = \frac{1}{N} \sum_{x \in \mathbb{Z}_N} f(x) e^{2\pi i x \xi / N}.$$

A useful basic relation between average magnitudes of a function and its transform was proven by Plancherel in the continous setting. There is a discreet analog.

Proposition 2. **(Plancherel's Identity)** *For $f : \mathbb{Z}_N \to \mathbb{C}$, we have the identity*

$$\frac{1}{N} \sum_{x \in \mathbb{Z}_N} |f(x)|^2 = \sum_{\xi \in \mathbb{Z}_N} |\hat{f}(\xi)|^2.$$

A useful way of mixing functions is that of the convolution.

Definition 3. For $f, g : \mathbb{Z}_N \to \mathbb{C}$, define the convolution of f with g by

$$f * g(x) = \frac{1}{N} \sum_{y \in \mathbb{Z}_N} f(y) g(x - y).$$

Convolutions interact nicely with tranformation.

Proposition 4. *For $f, g : \mathbb{Z}_N \to \mathbb{C}$ and ξ in $\mathbb{Z}_N$, we have the identity*

$$\widehat{f * g}(\xi) = \hat{f}(\xi)\hat{g}(\xi).$$

For simplicity in notation, we introduce the discrete interval and the indicator function.

Definition 5. For real numbers a and b, define

$$[a, b]_{\mathbb{Z}} := [a, b] \cap \mathbb{Z}$$

where $[a, b]$ is the closed real interval from a to b.

Definition 6. For a subset T of $\mathbb{Z}_N$, define its indicator function

$$T(x) = \begin{cases} 1 & \text{for} \quad x \in T \\ 0 & \text{for} \quad x \notin T \end{cases}$$

Now we begin to consider arithmetic progressions.

Definition 7. For q an element of $\mathbb{Z}_N$, define the arithemtic progression of length L

$$P_q = \{-q, -2q, \ldots, -Lq\}$$

We will assume the following proposition without proof.

Proposition 8. *For $1 \leq L \leq \frac{N}{4\pi}$, let $\eta = \frac{L}{N}$. Then for every ξ in $\mathbb{Z}_N$, there exists a q in $[1, 4\pi L]_{\mathbb{Z}}$ such that $\left|\hat{P}_q(\xi)\right| \geq \frac{\eta}{2}$.*

Thus for a fixed progression length L, for each element on the transform side, there is an arithmetic progression with a relatively large transform value.

2 Discrepency Functions

We now introduce the main consruction we wish to count. Let $f : \mathbb{Z}_N \to \{-1, 1\}$ be any coloring of $\mathbb{Z}_N$.

Definition 9. For a fixed length L, we define the quantity $\Delta_q(x)$ to be

$$\Delta_q = N|f * P_q(x)|.$$

This is a technical definition which is easy to work with using Fourier analysis; looking more closely reveals this is exactly the quantity we wish to count.

Lemma 10. *We have the identity $\Delta_q(X) = \big||f^{-1}(-1) \cap (x - P_q)| - |f^{-1}(1) \cap (x - P_q)|\big|$.*

Proof. We have

$$\Delta_q(x) = N|f * P_q(x)| = N\left|\frac{1}{N}\sum_{y\in\mathbb{Z}_N} f(y)P_q(x-y)\right|.$$

Since $P_q(x-y)$ is zero for $x-y$ not in P_q and one otherwise, we can rewrite the sum

$$\Delta_q(x) = \left|\sum_{j=1}^{L} f(x-qj)\right|.$$

But this sum contributes a negative one for every element in $f^{-1}(-1)\cap(x-P_q)$ and a positive one for every element in $f^{-1}(1)\cap(x-P_q)$. This proves our result. $\square$

Thus $\Delta_q(x)$ counts the discrepency between the number of elements colored -1 and those colored 1 in the arithemtic progression $x, x+q, \ldots, x+(L-1)q$, considered as elements of $\mathbb{Z}_N$.

We will eventually conclude that we have arithmetic progressions of large discrepency by considering the average discrepency and concluding that it is large enough. For this, it happens to be easier to consider the square of the discrepency function, but since squaring positive numbers is monotonic, there is no major loss of information.

Lemma 11. *If* $1 \le L \le \frac{N}{4\pi}$, *then we have*

$$\frac{1}{NL}\sum_{q=1}^{\lfloor 4\pi L\rfloor}\sum_{x\in\mathbb{Z}_N} \Delta_q(x)^2 \ge \frac{L}{4}.$$

Proof.

$$\begin{aligned}
\frac{1}{NL}\sum_{q=1}^{\lfloor 4\pi L\rfloor}\sum_{x\in\mathbb{Z}_N}\Delta_q(x)^2 &= \frac{1}{NL}\sum_{q=1}^{\lfloor 4\pi L\rfloor}\sum_{x\in\mathbb{Z}_N} N^2|f*P_q(x)|^2 \\
&= \frac{N^2}{L}\sum_{q=1}^{\lfloor 4\pi L\rfloor}\sum_{\xi\in\mathbb{Z}_N}|\widehat{f*P_q}(\xi)|^2 && \text{by Plancherel} \\
&= \frac{N^2}{L}\sum_{q=1}^{\lfloor 4\pi L\rfloor}\sum_{\xi\in\mathbb{Z}_N}|\hat{f}(\xi)\hat{P}_q(\xi)|^2 && \text{by Convolution} \\
&= \frac{N^2}{L}\sum_{q=1}^{\lfloor 4\pi L\rfloor}\sum_{\xi\in\mathbb{Z}_N}|\hat{f}(\xi)|^2|\hat{P}_q(\xi)|^2 \\
&= \frac{N^2}{L}\sum_{\xi\in\mathbb{Z}_N}|\hat{f}(\xi)|^2\sum_{q=1}^{\lfloor 4\pi L\rfloor}|\hat{P}_q(\xi)|^2 \\
&\geq \frac{N^2}{L}\sum_{\xi\in\mathbb{Z}_N}|\hat{f}(\xi)|^2\left(\frac{\eta}{2}\right)^2 && \text{by Corollary} \\
&= \frac{N^2}{L}\frac{L^2}{4N^2}\sum_{\xi\in\mathbb{Z}_N}|\hat{f}(\xi)|^2 \\
&= \frac{L}{4}\frac{1}{N}\sum_{x\in\mathbb{Z}_N}|f(x)|^2 && \text{by Plancherel} \\
&= \frac{L}{4}
\end{aligned}$$

□

Corollary 12. *There exists some q_0 in the interval $[1, 4\pi L]_{\mathbb{Z}}$ such that*

$$\frac{1}{N}\sum_{x\in\mathbb{Z}_N}\Delta_{q_0}(x)^2 \geq \frac{1}{4\pi}\frac{L}{4}.$$

Proof. The average of $\frac{1}{N}\sum_{x\in\mathbb{Z}_N}\Delta_q(x)^2$ over all q in $[1, 4\pi L]_{\mathbb{Z}}$ is

$$\frac{1}{\lfloor 4\pi L\rfloor}\sum_{q=1}^{\lfloor 4\pi L\rfloor}\frac{1}{N}\sum_{x\in\mathbb{Z}_N}\Delta_q(x)^2 \geq \frac{1}{4\pi}\frac{1}{NL}\sum_{q=1}^{\lfloor 4\pi L\rfloor}\sum_{x\in\mathbb{Z}_N}\Delta_q(x)^2 \geq \frac{1}{4\pi}\frac{L}{4}.$$

Since this is a bound on an average of real numbers, at least one number must satisfy the bound. □

It is useful to generalize our definition of the discrepency function to sets.

Definition 13. For any subset A of $\mathbb{Z}_N$, define

$$\Delta_A = \left| |f^{-1}(-1) \cap A| - |f^{-1}(1) \cap A| \right|$$

to be the discrepency between the number of -1 colored elements of A and the number of 1 colored elements of A.

The following lemma will be useful in translating our results from $\mathbb{Z}_N$ to the integers from 1 to N.

Lemma 14. *For any disjoint subsets A and B of $\mathbb{Z}_N$, we have $\Delta_A + \Delta_B \geq \Delta_{A\cup B}$.*

Proof. Each of A and B have more of either elements in $f^{-1}(1)$ or elements in $f^{-1}(-1)$. If both A and B have an excess of the same label, then they will add their contributions in the union, yielding $\Delta_{A\cup B} = \Delta_A + \Delta_B$. If they have an excess of the opposite label, then they will cancel their contributions in the union and $\Delta_{A\cup B} = |\Delta_A - \Delta_B| \leq \Delta_A + \Delta_B$. $\square$

3 The Proof

We now have all the pieces needed for the theorem.

Theorem 15. **(Roth's $\frac{1}{4}$-Theorem)** *In any 2-coloring of $[1, N]$, there exists an arithmetic progression that contains at least $cN^{1/4}$ more elements in one color than the other.*

Proof. Take $L = \left\lfloor \sqrt{\frac{N}{4\pi}} \right\rfloor$. Then since $q_0 \leq 4\pi L$, we have

$$q_0 L \leq 4\pi L^2 = 4\pi \left\lfloor \sqrt{\frac{N}{4\pi}} \right\rfloor^2 \leq N.$$

Thus the diameter of the entire progression is less than N, considered from the first element to the last. Since the average over all of the x in $\mathbb{Z}_N$ of $\Delta_{q_0}(x)^2$ is greater than or equal to $\frac{1}{4\pi}\frac{L}{4}$, there must be at least one x_0 in $\mathbb{Z}_N$ such that

$$\Delta_{q_0}(x_0)^2 \geq \frac{1}{16\pi} L = \frac{1}{16\pi}\sqrt{\frac{N}{4\pi}} = \frac{\sqrt{N}}{32\pi^{3/2}}.$$

Hence we have that $\Delta_{q_0}(x_0) \geq \frac{1}{4\sqrt{2}\pi^{3/4}} N^{1/4}$. If $x_0 - P_{q_0}$ is a genuine progression, we are done with $c = 1/4\sqrt{2}\pi^{3/4}$. If $x_0 - P_{q_0}$ is not a genuine progression, it splits into two genuine progressions, Q_1 and Q_2, which do not overlap since the diameter of $x_0 - P_{q_0}$ is less than N. Since $Q_1 \cup Q_2 = x_0 - P_{q_0}$ and Q_1 and Q_2 are disjoint, we have

$$\Delta_{Q_1} + \Delta_{Q_2} \geq \frac{1}{4\sqrt{2}\pi^{3/4}} N^{1/4}$$

and hence for at least one of i in $\{1,2\}$, we have $\Delta_{Q_i} \geq \frac{1}{2}\frac{1}{4\sqrt{2}\pi^{3/4}}N^{1/4}$. Therefore in all cases, we have that we can take $c = 1/8\sqrt{2}\pi^{3/4}$. $\square$

References

[1] Terence Tao and Van H. Vu, *Additive Combinatorics*, Cambridge University Press, New York, 2006.