Gowers' Inverse Argument - (Big) Overview

- We now begin the long task of proving the following inverse theorem:

  Proposition 1 (Lemma 2 from last lecture). Let $\varepsilon > 0$ and $N \geq e^{\varepsilon^{-C}}$.

  If $f: \mathbb{Z}_N \to [-1,1]$ with $\sum_{x \in \mathbb{Z}_N} f(x) = 0$ and $\|f\|_{U^3} \geq \varepsilon$, then

  $\exists$ genuine arith. prog. P with $|P| \geq N^{\varepsilon^C}$ s.t. $\frac{1}{|P|} \sum_{x \in P} f(x) \geq \varepsilon^C$.

  In other words we are trying to extract precise structural information about f from the "pseudo-random" assumption $\|f\|_{U^3} \geq \varepsilon$.

- Recall the structure of the "inverse portion" of the Roth (revisited) argument:

  Step 1: (Easy) inverse theorem for the U^2-norm.

  $$\|f\|_{U^2} \geq \varepsilon \implies \underbrace{\exists\, \xi \in \mathbb{Z}_N \text{ s.t. } \left| \frac{1}{N} \sum_{x \in \mathbb{Z}_N} f(x) e^{-2\pi i x \xi / N} \right| \geq \varepsilon^2}_{\text{"f has linear bias"}}.$$

  Step 2: Linear Bias $\Rightarrow$ Density Increment (on long arith. prog.)

  $\exists\, \xi \in \mathbb{Z}_N$ s.t. $\left| \frac{1}{N} \sum_{x \in \mathbb{Z}_N} f(x) e^{-2\pi i x \xi / N} \right| \geq \varepsilon \implies$ $\exists$ genuine arith. prog. P with $|P| \geq N^{1/3}$ such that

  $$\frac{1}{|P|} \sum_{x \in P} f(x) \geq \varepsilon/8.$$

The proof of Proposition 1 will follow the following basic outline:

New Step 1: (Local) Inverse Theorem for the U^3-norm

$\|f\|_{U^3} \geq \varepsilon \implies$ $\exists$ $\mathbb{Z}_N$-prog Q with $|Q| \geq N^{\varepsilon^c}$ and quadratics $\psi_1, \ldots, \psi_N$ such that

$$\underbrace{\frac{1}{N}\sum_{x\in\mathbb{Z}_N}\left|\frac{1}{|Q|}\sum_{h\in Q+x} f(h)e^{2\pi i \psi_x(h)/N}\right| \geq \varepsilon^c}_{\text{"(Local) Quadratic bias"}}.$$

Using alot of stuff!!

- Note in particular that $\exists$ $x \in \mathbb{Z}_N$ such that
$$\left|\frac{1}{|Q|}\sum_{h\in Q+x} f(h)e^{2\pi i \psi_x(h)/N}\right| \geq \varepsilon^c.$$

[Motivating Example: If $f:\mathbb{Z}_N \to \mathbb{D}$ has (global) quadratic bias in the sense that $\left|\frac{1}{N}\sum_{x\in\mathbb{Z}_N} f(x)e^{2\pi i(ax^2+bx)/N}\right| \geq \varepsilon$ for some $a,b\in\mathbb{Z}_N$

then $\|f\|_{U^3} \geq \varepsilon$.

* See next page for the verification of this fact.

* New Step 1 is of course an approximate converse to this example.]

New Step 2: (Local) Quadratic Bias $\implies$ Density Increment.

$\exists$ $\mathbb{Z}_N$-prog Q with $|Q| \geq N^{\varepsilon^c}$ and quadratics $\psi_1, \ldots, \psi_N$ s.t. $\frac{1}{N}\sum_{x\in\mathbb{Z}_N}\left|\frac{1}{|Q|}\sum_{h\in Q+x} f(h)e^{2\pi i \psi_x(h)/N}\right| \geq \varepsilon$ $\implies$ $\exists$ genuine arith. prog. P with $|P| \geq \frac{\varepsilon}{20}|Q|^{1/400}$ such that $\frac{1}{|P|}\sum_{x\in P} f(x) \geq \varepsilon/20$.

(Using some Diophantine approx.)

Verification of Motivating (for New Step 1) Example.

We will, surprise surprise, use Cauchy-Schwarz, namely if $F : \mathbb{Z}_N \to \mathbb{D}$, then

$$\left(\frac{1}{N} \sum_{x \in \mathbb{Z}_N} |F(x)| \right)^2 \leqslant \frac{1}{N} \sum_{x \in \mathbb{Z}_N} |F(x)|^2$$

We are supposing that for a given $f : \mathbb{Z}_N \to \mathbb{D}$ and $\varepsilon > 0$ $\exists$ $a, b \in \mathbb{Z}_N$ s.t.

$$\left| \frac{1}{N} \sum_x f(x) e^{2\pi i (ax^2 + bx)/N} \right| \geqslant \varepsilon .$$

Squaring both sides (and squaring out the LHS) we obtain

$$\frac{1}{N^2} \sum_{x, h_1} f(x) \overline{f(x+h_1)} e^{-2\pi i (2axh_1 + ah_1^2 + bh_1)/N} \geqslant \varepsilon^2$$

$$\Rightarrow \frac{1}{N} \sum_{h_1} \left| \frac{1}{N} \sum_x f(x) \overline{f(x+h_1)} e^{-2\pi i (2axh_1)/N} \right| \geqslant \varepsilon^2$$

Cauchy-Schwarz

$$\Rightarrow \underbrace{\frac{1}{N} \sum_{h_1} \left| \frac{1}{N} \sum_x f(x) \overline{f(x+h_1)} e^{-2\pi i (2axh_1)/N} \right|^2}_{} \geqslant \varepsilon^4$$

$$\left[= \frac{1}{N} \sum_h |\widehat{\Delta_h f}(2ah)|^2 \geqslant \varepsilon^4 \right]$$

This will be an intermediate step inside New Step 1 (approximately !!)

$$\Rightarrow \frac{1}{N} \sum_{h_1} \frac{1}{N^2} \sum_{x, h_2} f(x) \overline{f(x+h_1)}\, \overline{f(x+h_2)} f(x+h_1+h_2) e^{2\pi i (2ah_1h_2)/N} \geqslant \varepsilon^4$$

$$\Rightarrow \frac{1}{N^2} \sum_{h_1, h_2} \left| \frac{1}{N} \sum_x f(x) \overline{f(x+h_1)}\, \overline{f(x+h_2)} f(x+h_1+h_2) \right| \geqslant \varepsilon^4$$

$$\Rightarrow \underbrace{\frac{1}{N^2} \sum_{h_1, h_2} \left| \frac{1}{N} \sum_x f(x) \overline{f(x+h_1)}\, \overline{f(x+h_2)} f(x+h_1+h_2) \right|^2}_{= \|f\|_{U^3}^8} \geqslant \varepsilon^8$$

Cauchy-Schwarz

Check! $\square$