The Weyl Inequality and Heilbronn Property (for squares)

A **Weyl sum** is an exponential sum of the form

$$\sum_{m=1}^{M} e^{2\pi i P(m)}$$

where $P(m)$ is a polynomial with real coefficients. We begin this lecture by outlining the proof of Weyl's estimates for these sums in the special case when $P(m) = \alpha m^2$.

Theorem (Weyl's Inequality) Let $\alpha \in \mathbb{R}$.

If $\left|\alpha - \frac{a}{q}\right| \leq \frac{1}{q^2}$ with $(a,q)=1$, then

$$\left| \sum_{m=1}^{M} e^{2\pi i m^2 \alpha} \right| \leq C \left(\frac{M^2}{q} + M \log q + q \log q \right)^{1/2}.$$

Remark: This gives a non-trivial estimate whenever $M^{\eta} \leq q \leq M^{2-\varepsilon}$ for some $0 < \eta, \varepsilon < 1$. In particular, if $M^{1/4} \leq q \leq M^{7/4}$ the we see that

$$\left| \sum_{m=1}^{M} e^{2\pi i m^2 \alpha} \right| \leq C (\log M) M^{7/8}.$$

distance to nearest integer

We begin our proof with the following simple lemma.

Lemma 1 Let $\alpha \in \mathbb{R}$, then $\left| \sum_{m=1}^{M} e^{2\pi i m \alpha} \right| \leq \min \left\{ M, \frac{1}{2\|\alpha\|} \right\}$.

Proof: If $\alpha = 0$, then sum is M. If $\alpha \neq 0$, then

$$\left| \sum_{m=1}^{M} e^{2\pi i m \alpha} \right| \leq \frac{|1 - e^{2\pi i \alpha M}|}{|1 - e^{2\pi i \alpha}|} \leq \frac{|\sin \pi \alpha M|}{|\sin \pi \alpha|} \leq \frac{1}{2\|\alpha\|}. \qquad \square$$

Proof (of Weyl's Inequality)

The following procedure is known as Weyl differencing:

$$\left|\sum_{m=1}^{M} e^{2\pi i m^2 \alpha}\right|^2 = \sum_{n,m=1}^{M} e^{2\pi i (n^2-m^2)\alpha}$$

$n=m+h$

$$= \sum_{m=1}^{M} \sum_{h=1-m}^{M-m} e^{2\pi i ((m+h)^2-m^2)\alpha}$$

$$= \sum_{m=1}^{M} \sum_{h=1-m}^{M-m} e^{2\pi i (2mh+h^2)\alpha}$$

Exercise 1: $\rightarrow$ $\vdots$

$$= M + 2\,\mathrm{Re}\left\{ \sum_{h=1}^{M-1} \sum_{m=1}^{M-h} e^{2\pi i (2mh+h^2)\alpha} \right\}$$

$$\leq M + 2 \sum_{h=1}^{M-1} \left| \sum_{m=1}^{M-h} e^{2\pi i (2mh)\alpha} \right|$$

Lemma 1

$$\leq M + 2 \sum_{h=1}^{M-1} \min\left\{ M-h, \frac{1}{2\|2h\alpha\|} \right\}$$

$$\leq M + 2 \sum_{h=1}^{2M} \min\left\{ M, \frac{1}{\|h\alpha\|} \right\}.$$

Claim: If $|\alpha - \frac{a}{q}| \leq \frac{1}{q^2}$ with $(a,q)=1$, then for any $H \in \mathbb{N}$

$$\sum_{h=1}^{H} \min\left\{ M, \frac{1}{\|h\alpha\|} \right\} \leq C\left(\frac{H^2}{q} + H\log q + q\log q \right).$$

Weyl's Inequality clearly follows immediately from this claim. $\square$

Proof of Claim 1

Write $\alpha = \frac{a}{q} + \beta$ and note that $|\beta| \leq \frac{1}{q^2}$.

We initially assume that $\underline{\beta = 0}$ and divide H into blocks of length q:

$$\sum_{h=1}^{H} \min\left\{H, \frac{1}{\|h\alpha\|}\right\} \leq \sum_{j=1}^{\frac{H}{q}+1} \sum_{h=1}^{q} \min\left\{H, \frac{1}{\|ha/q\|}\right\}$$

$$\leq \left(\frac{H}{q}+1\right)\left(H + \sum_{h=1}^{q-1} \frac{1}{\|ha/q\|}\right)$$

since $(a,q)=1$

$$= \left(\frac{H}{q}+1\right)\left(H + \sum_{h=1}^{q-1} \frac{1}{\|h/q\|}\right)$$

$$\leq \left(\frac{H}{q}+1\right)\left(H + 2\sum_{h=1}^{q/2} \frac{q}{h}\right)$$

$$\leq \left(\frac{H}{q}+1\right)\left(H + 2q\log q\right)$$

$$\leq C\left(\frac{H^2}{q} + H\log q + q\log q\right)$$

as required.

<u>Exercise 2</u>: Deal with the case when $\beta \neq 0$. $\square$

Heilbronn Property for Squares

Recall

Dirichlet's Principle

Given any $\alpha \in \mathbb{R}$ and $M \in \mathbb{N}$, $\exists\ 1 \le q \le M$ such that $\|q\alpha\| \le \frac{1}{M}$.

In this section we shall prove

Theorem (Heilbronn Property)

Given any $\alpha \in \mathbb{R}$ and $M \in \mathbb{N}$, $\exists\ 1 \le q \le M$ such that $\|q^2\alpha\| \le \frac{1}{M^{1/10}}$.

Proof

We may assume that $\alpha \in \mathbb{Q}$, say $\alpha = \frac{n}{N}$. $\leftarrow$ Why?

Our proof will be by contradiction.

Suppose that for all $1 \le q \le M$ we have $\|q^2\alpha\| > \frac{1}{M^{1/10}}$ and consider the set

$$A = \{n, 4n, 9n, \dots, M^2 n\} \subseteq \mathbb{Z}_N \text{ "} \simeq [-\tfrac{N}{2}, \tfrac{N}{2}] \text{"}.$$

Since $\|q^2\alpha\| > \frac{1}{M^{1/10}}$ for all $1 \le q \le M$ it follows that

$$(*) \quad A \cap \left[-\frac{N}{M^{1/10}}, \frac{N}{M^{1/10}}\right] = \phi \quad \leftarrow \text{"A is non-random"}$$

$\left(|nq^2| > \frac{N}{M^{1/10}} \text{ where } |\cdot| \text{ denotes distance to nearest multiple of } N \right)$.

The "non-random" property (*) should be reflected in the non-zero Fourier coefficients of 1_A and indeed:

<u>Claim 2</u>: $\exists\ 0<|\xi|\leq M^{1/5}$ such that $\left|\sum_{m=1}^{M} e^{-2\pi i m^2\alpha\xi}\right| \geq \frac{M^{9/10}}{2}$.

LARGE!

Now it follows from the Dirichlet Principle that $\exists\ 1\leq q\leq M$ such that $\|\alpha\xi q\|\leq \frac{1}{M}$ and hence that $\exists\ a\in\mathbb{Z}$ such that

$$\left|\alpha\xi - \frac{a}{q}\right| \leq \frac{1}{qM} \quad \left(\leq \frac{1}{q^2}\right).$$

It thus follows from the remark proceeding Weyl's inequality that we must have $1\leq q\leq M^{1/4}$. [since otherwise (if $M^{1/4}\leq q\leq M$) it would follow that $\left|\sum_{m=1}^{M} e^{-2\pi i m^2\alpha\xi}\right| \leq C M^{7/8}\log M \ll M^{9/10}$]

<u>But</u>, if $1\leq q\leq M^{1/4}$, then

$$\|\alpha(\xi q)^2\| \leq \|\alpha(\xi q)\|\, M^{1/5}M^{1/4} \leq \frac{1}{M}M^{9/20} = \frac{1}{M^{11/20}}$$

a contradiction $\lightning$. $\square$

Proof of Claim 2

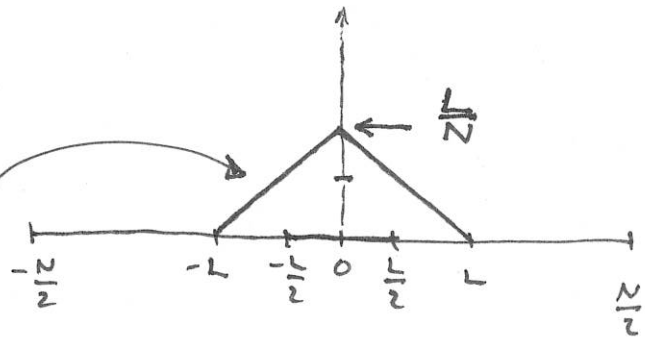
We phrase this as a more general lemma.

Lemma 2: Let $A \subseteq \mathbb{Z}_N$ with $|A| = M$ & $A \cap [-L, L] = \phi$, then $\exists\ 0 < |\xi| \leq \frac{N^2}{L^2}$ such that $|\widehat{1_A}(\xi)| \geq \frac{LM}{2N^2}$.

Proof of Lemma 2:

Let $I = [-\frac{L}{2}, \frac{L}{2}]$

& $1_I * 1_I(x) = \frac{1}{N}\sum_y 1_I(y) 1_I(x-y)$



Now

$$0 = \underbrace{\frac{1}{N}\sum_x \frac{1}{N}\sum_y 1_I(y) 1_I(x-y) 1_A(x)}$$

since $A \cap [-L, L] = \phi$

$$= \sum_{\xi \in \mathbb{Z}_N} \widehat{1_I}(\xi)^2 \overline{\widehat{1_A}(\xi)}$$

$$= \left(\frac{L}{N}\right)^2 \frac{M}{N} + \sum_{\xi \neq 0} \widehat{1_I}(\xi)^2 \widehat{1_A}(\xi)$$

$$\Rightarrow \quad \sum_{\xi \neq 0} |\widehat{1_I}(\xi)|^2 |\widehat{1_A}(\xi)| \geq \frac{L^2 M}{N^3}.$$

But

- $|\widehat{1_A}(\xi)| \leq \frac{M}{N}$ for all $\xi \neq 0$

- $|\widehat{1_I}(\xi)| \leq \min\left\{\frac{L}{N}, \frac{1}{2|\xi|}\right\}$

Lemma 1

distance to nearest multiple of N.

Using these estimates it follows that, for a parameter γ to be determined,

$$\sum_{\xi \neq 0} |\hat{1}_I(\xi)|^2 |\hat{1}_A(\xi)|$$

$$= \sum_{0<|\xi|\leq \gamma} |\hat{1}_I(\xi)|^2 |\hat{1}_A(\xi)| + \sum_{\gamma<|\xi|\leq \frac{N}{2}} |\hat{1}_I(\xi)|^2 |\hat{1}_A(\xi)|$$

Plancherel

$$\leq \max_{0<|\xi|\leq \gamma} |\hat{1}_A(\xi)| \cdot \left(\frac{L}{N}\right) + \frac{M}{N} \sum_{\gamma\leq|\xi|\leq \frac{N}{2}} \frac{1}{4|\xi|^2}.$$

Since $\displaystyle\sum_{\gamma\leq|\xi|\leq \frac{N}{2}} \frac{1}{4|\xi|^2} \leq \frac{1}{2} \sum_{n=\gamma}^{\infty} \frac{1}{n^2} \leq \frac{1}{2\gamma}$

and $\displaystyle\frac{M}{N} \cdot \frac{1}{2\gamma} \leq \frac{L^2 M}{2N^3}$ if $\gamma = \dfrac{N^2}{L^2}$

it follows that (by setting $\gamma = N^2/L^2$)

$$\max_{0<|\xi|\leq \frac{N^2}{L^2}} |\hat{1}_A(\xi)| \geq \frac{LM}{2N^2} \qquad \text{as required.} \qquad \square$$