DISCRETE MULTILINEAR MAXIMAL OPERATORS ASSOCIATED TO SIMPLICES

BRIAN COOK NEIL LYALL ÁKOS MAGYAR

Abstract. We prove $\ell^{p_1} \times \cdots \times \ell^{p_k} \to \ell^r$ bounds for multilinear maximal operators associated to averages over all isometric copies of a given non-degenerate k-simplex. This provides a natural extension of $\ell^p \to \ell^p$ bounds for the discrete spherical maximal operator, which also serves as the key ingredient of our proof.

1. Introduction

The study of discrete analogues of central constructs of Euclidean harmonic analysis, initiated by Bourgain [2, 3, 4], has grown into a vast, active area of research. An important result in this development is the ℓ^p-boundedness of the so-called discrete spherical maximal operator [7]. The aim of this short note is to show that this result implies $\ell^{p_1} \times \cdots \times \ell^{p_k} \to \ell^r$ type bounds for certain, seemingly more singular, multilinear discrete maximal operators associated to averages over similar copies of a given non-degenerate simplex.

We start by recalling the discrete spherical maximal operator and the main result of [7]. Let $d \geq 5$, $\lambda^2 \in \mathbb{N}$, and $N_\lambda := |\{y \in \mathbb{Z}^d : \ |y| = \lambda\}|$. It is well-known, see for example [10], that $c_d \lambda^{d-2} \leq N_\lambda \leq C_d \lambda^{d-2}$ for some constants $0 < c_d < C_d$. For $f : \mathbb{Z}^d \to \mathbb{R}$ define the averages

$$A_\lambda f(x) = N_\lambda^{-1} \sum_{|y|=\lambda} f(x+y),$$

and the maximal operator

$$A_* f(x) = \sup_\lambda |A_\lambda f(x)|.$$

All variables x, y above and throughout this short note are always assumed to in $\mathbb{Z}^d$, unless explicitly specified otherwise, and the parameter λ is assumed be in $\sqrt{\mathbb{N}}$, i.e. $\lambda^2 \in \mathbb{N}$.

In [7] it was shown that for $p > d/(d-2)$ one has the estimate

$$\|A_* f\|_p \leq C_{p,d} \, \|f\|_p, \tag{1}$$

where $\|f\|_p$ denotes the $\ell^p(\mathbb{Z}^d)$ norm of the function f. It was further noted in [7] that the condition that $d \geq 5$ and $p > d/(d-2)$ are both sharp.

Let $k \in \mathbb{N}$ and let $\Delta = \{v_0 = 0, v_1, \ldots, v_k\} \subseteq \mathbb{Z}^d$ be a non-degenerate k-simplex, i.e. assume that the vectors $v_1, \ldots, v_k$ are linearly independent. Given $\lambda \in \sqrt{\mathbb{N}}$ we say that a simplex $\Delta' = \{y_0 = 0, y_1, \ldots, y_k\} \subseteq \mathbb{Z}^d$ is *isometric* to Δ if $|y_i - y_j| = \lambda |v_i - v_j|$ for all $0 \leq i, j \leq k$. We will write $\Delta' \simeq \lambda\Delta$ in this case and denote by $N_{\lambda\Delta}$ the number of isometric copies of $\lambda\Delta$, i.e define

$$N_{\lambda\Delta} := |\{(y_1, \ldots, y_k) \in \mathbb{Z}^{dk} : \ \Delta' = \{0, y_1, \ldots, y_k\} \simeq \lambda\Delta\}|.$$

Note that for $k = 1$ and $v_1 = (1, 0, \ldots, 0)$ we have that $N_{\lambda\Delta} = N_\lambda$.

Given a simplex $\Delta = \{v_0 = 0, v_1, \ldots, v_k\}$ we introduce the associated *inner product matrix* $T = T_\Delta = (t_{ij})_{1 \leq i,j \leq k}$ with entries $t_{ij} := v_i \cdot v_j$, where "$\cdot$" stands for the dot product in $\mathbb{R}^d$. Note that T is a positive semi-definite matrix with integer entries and T is positive definite if and only if Δ is non-degenerate. It is easy to see that $\Delta' \simeq \lambda\Delta$ if and only if

$$y_i \cdot y_j = \lambda^2 t_{ij} \quad \text{for all} \quad 1 \leq i, j \leq k. \tag{2}$$

Extending the work of Siegel [9] and Raghavan [8], Kitaoke [5] has proved that if Δ is non-degenerate, then one has the estimate

$$c_{d,k} \det(\lambda^2 T)^{(d-k-1)/2} \leq N_{\lambda\Delta} \leq C_{d,k} \det(\lambda^2 T)^{(d-k-1)/2} \tag{3}$$

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in dimensions $d \geq 2k+3$ for $\lambda \geq \lambda_{d,k,\Delta}$. It is important to note that the constants $0 < c_{d,k} < C_{d,k}$ depending only on the parameters d and k and are independent of the matrix T and hence the simplex Δ. For a self contained treatment of the upper bound in (3), see Lemma 2.2 in [6]. In particular for sufficiently large λ one has that $N_{\lambda\Delta} > 0$, in fact $N_{\lambda\Delta} \asymp \lambda^{kd-k(k+1)}$ with implicit constants may depending on Δ.

For a family of functions $f_1, \dots, f_k : \mathbb{Z}^d \to \mathbb{R}$ and $\lambda \in \sqrt{\mathbb{N}}$ such that $N_{\lambda\Delta} > 0$ we define the multi-linear averages

$$A_\lambda(f_1, \dots, f_k)(x) := N_{\lambda\Delta}^{-1} \sum_{y_1,\dots,y_k} f_1(x+y_1) \cdots f_k(x+y_k)\, S_{\lambda^2 T}(y_1, \dots, y_k) \tag{4}$$

where $S_{\lambda^2 T}(y_1, \dots, y_k) = 1$ if $y_1, \dots, y_k \in \mathbb{Z}^d$ satisfies (2) and is equal to 0 otherwise, i.e. the indicator function of the relation $\Delta' \simeq \lambda\Delta$, and the associated maximal operator

$$A_*(f_1, \dots, f_k)(x) := \sup_\lambda |A_\lambda(f_1, \dots, f_k)(x)| \tag{5}$$

where the supremum is restricted to those $\lambda \in \sqrt{\mathbb{N}}$ for which $N_{\lambda\Delta} > 0$.

We choose to present our results in an increasing order of generality, first presenting the following special case of our most general result in the special case of bilinear maximal operators associated to triangles.

Theorem 1. *Let $\Delta = \{v_0 = 0, v_1, v_2\} \subseteq \mathbb{Z}^d$ be a non-degenerate triangle.*

(i) *If $d \geq 9$, $r > 2d/(d-2)$, and $1 \leq p_1, p_2 \leq \infty$ with $1/r = 1/p_1 + 1/p_2$, then one has the estimate*

$$\|A_*(f_1, f_2)\|_r \leq C_{d,\Delta}\, \|f_1\|_{p_1} \|f_2\|_{p_2}. \tag{6}$$

(ii) *If $d \geq 11$, then for any $r > d/(d-2)$ and $p_1, p_2 > 2d/(d-2)$ that satisfies $1/r = 1/p_1 + 1/p_2$, one has*

$$\|A_*(f_1, f_2)\|_r \leq C_{d,\Delta}\, \|f_1\|_{p_1} \|f_2\|_{p_2}.$$

Note that if we know that A_* is bounded on $\ell^{p_1} \times \ell^{p_2} \to \ell^r$, then we automatically get all bounds $\ell^{q_1} \times \ell^{q_2} \to \ell^s$ for all $q_1 \leq p_1, q_2 \leq p_2$, and $s \geq r$ due to the nested properties of the discrete norms.

Furthermore, note that in Theorem 1 above, and in all subsequent theorem and propositions in this paper (except for Theorem 3), part (ii) implies part (i) for the range of dimensions in which part (ii) holds.

We remark that it was independently and simultaneously established by Anderson, Kumchev and Palsson in [1] that in dimensions $d \geq 9$, with Δ being a equilateral triangle, that estimate (6) holds in the larger range $r > \max\{32/(d+8), (d+4)/(d-2)\}$. Their result follows as a direct corollary of $\ell^p \times \ell^\infty \to \ell^p$ bounds obtained by employing very different methods than those contained in this short note.

Our proof of (i) above also follows from $\ell^p \times \ell^\infty \to \ell^p$ estimates. In Section 4 we discuss a generalization of our method that allows us to obtain better bounds in larger dimensions. In particular, we obtain $\ell^{p_1} \times \ell^{p_2} \to \ell^r$ bounds whenever $r > m/(m-1) \cdot d/(d-2)$ and $1 \leq p_1, p_2 \leq \infty$ with $1/r \leq 1/p_1 + 1/p_2$, provided $d \geq 2m+5$. This represents an improvement on the results in [1] for $d \geq 15$.

We remark that our proof of (ii) above, which we emphasize gives non-trivial estimates for a range of p_1 and p_2 for any given $r > d/(d-2)$, provided $d \geq 11$, does not follow as a corollary of $\ell^p \times \ell^\infty \to \ell^p$ estimates.

Before stating our next result, Theorem 2 below, which generalizes Theorem 1 to multilinear maximal operators associated to k-simplices, we define for each $k \in \mathbb{N}$, a symmetric convex region $\mathcal{C}_k \subseteq [0,1]^k$. We define $\mathcal{C}_k$ to be all those points $(x_1, \dots, x_k) \in [0,1]^k$ with $x_1 + \cdots + x_k < 1$ that also have the property that for any $1 \leq j \leq k-1$ one has $y_1 + \cdots + y_j < 1 - 2^{-j}$ for any choice $\{y_1, \dots, y_j\} \subset \{x_1, \dots, x_k\}$.

We note, in particular, that if $(x_1, \dots, x_k) \in \mathcal{C}_k$, then $0 \leq x_1, \dots, x_k < 1/2$, and that both the points $(1/k, \dots, 1/k)$ and $(1/2, 0, \dots, 0)$, while not in $\mathcal{C}_k$, are contained in the boundary of $\mathcal{C}_k$.

Theorem 2. *Let $k \in \mathbb{N}$ and $\Delta = \{v_0 = 0, v_1, \dots, v_k\} \subseteq \mathbb{Z}^d$ be a non-degenerate k-simplex.*

(i) *If $d \geq 4k+1$, $r > 2d/(d-2)$, and $1 \leq p_1, \dots, p_k \leq \infty$ with $1/r = 1/p_1 + \cdots + 1/p_k$, then one has*

$$\|A_*(f_1, \dots, f_k)\|_r \leq C_{d,k,\Delta}\, \|f_1\|_{p_1} \cdots \|f_k\|_{p_k}.$$

(ii) *If $d \geq 4k+3$, then for any $r > d/(d-2)$ and $p_1, \dots, p_k > 2d/(d-2)$ whose reciprocals*
$$(1/p_1, \dots, 1/p_k) \in (d-2)/d \cdot \mathcal{C}_k$$
and satisfy $1/r = 1/p_1 + \cdots + 1/p_k$, one has the estimate
$$\|A_*(f_1, \dots, f_k)\|_r \leq C_{d,k,\Delta} \, \|f_1\|_{p_1} \cdots \|f_k\|_{p_k}.$$

Note, as above, that if we know that A_* is bounded on $\ell^{p_1} \times \cdots \times \ell^{p_k} \to \ell^r$, then it is automatically bounded on $\ell^{q_1} \times \cdots \times \ell^{q_k} \to \ell^s$ for all $q_1 \leq p_1, \dots, q_k \leq p_k$, and $s \geq r$.

In Section 4 we discuss a generalization of our method that allows us to obtain better $\ell^{p_1} \times \cdots \times \ell^{p_k} \to \ell^r$ bounds provided that d is sufficiently large. In particular, we obtain $\ell^{p_1} \times \cdots \times \ell^{p_k} \to \ell^r$ bounds whenever $r > m/(m-1) \cdot d/(d-2)$ and $1 \leq p_1, \dots, p_k \leq \infty$ with $1/r \leq 1/p_1 + \cdots + 1/p_k$, provided $d \geq 2m(k-1)+5$.

We conclude matters in Section 5 by demonstrating that $\ell^p \times \ell^\infty \times \cdots \times \ell^\infty \to \ell^p$ boundedness fails for every $p \leq d/(d-2)$ in dimensions $d \geq 2k+3$.

2. Proof of Theorem 2

The crucial ingredient in our proof of Theorem 2 is pointwise estimates for $A_*(f_1, \dots f_k)$ in terms of the spherical maximal operator applied to appropriate powers of the functions f_j, specifically

Proposition 1. *Let $k \in \mathbb{N}$ and $\Delta = \{v_0 = 0, v_1, \dots, v_k\} \subseteq \mathbb{Z}^d$ be a non-degenerate k-simplex.*

(i) *If $d \geq 4k+1$, then for any $f_1, \dots, f_k : \mathbb{Z}^d \to \mathbb{R}$, one has*
$$A_*(f_1, \dots, f_k)(x) \leq C_{d,k,\Delta} \, \|f_1\|_\infty \cdots \|f_{k-1}\|_\infty \; A_*(f_k^2)(x)^{1/2} \tag{7}$$
uniformly for $x \in \mathbb{Z}^d$.

(ii) *If $d \geq 4k+3$, then for any $f_1, \dots, f_k : \mathbb{Z}^d \to \mathbb{R}$, one has*
$$A_*(f_1, \dots, f_k)(x) \leq C_{d,k,\Delta} \, A_*(f_1^2, \dots, f_{k-1}^2)(x)^{1/2} \, A_*(f_k^2)(x)^{1/2} \tag{8}$$
and hence
$$A_*(f_1, \dots, f_k)(x) \leq C_{d,k,\Delta} \, A_*(f_1^{2^{k-1}})(x)^{1/2^{k-1}} A_*(f_2^{2^{k-1}})(x)^{1/2^{k-1}} \prod_{j=3}^{k} A_*(f_j^{2^{k+1-j}})(x)^{1/2^{k+1-j}} \tag{9}$$
uniformly for $x \in \mathbb{Z}^d$.

We prove Proposition 1 in Section 3 below. It is straightforward to see that Theorem 2 (i) follows immediately from (7) and (1), indeed these estimates imply
$$\|A_*(f_1, \dots, f_k)\|_{p_k} \leq C_{d,k,\Delta} \, \|f_1\|_\infty \cdots \|f_{k-1}\|_\infty \|A_*(f_k^2)\|_{p_k/2}^{1/2} \leq C_{d,k,\Delta} \, \|f_1\|_\infty \cdots \|f_{k-1}\|_\infty \|f_k\|_{p_k}$$
provided $p_k > 2d/(d-2)$. By symmetry and interpolation we then obtain part (i) of Theorem 2.

Assuming the validity (9) for now, we can also quickly establish Theorem 2 (ii). An application of Hölder gives that
$$\|A_*(f_1, \dots, f_k)\|_r \leq C_{d,k,\Delta} \|A_*(f_1^{2^{k-1}})\|_{p_1/2^{k-1}}^{1/2^{k-1}} \|A_*(f_2^{2^{k-1}})\|_{p_2/2^{k-1}}^{1/2^{k-1}} \prod_{j=3}^{k} \|A_*(f_j^{2^{k+1-j}})\|_{p_j/2^{k+1-j}}^{1/2^{k+1-j}}$$
whenever $1/r = 1/p_1 + \cdots + 1/p_k$. Now if
$$p_1, p_2 > 2^{k-1} \frac{d}{d-2} \quad \text{and} \quad p_j > 2^{k+1-j} \frac{d}{d-2} \quad \text{for} \quad 3 \leq j \leq k$$
then by (1) we obtain
$$\|A_*(f_1, \dots, f_k)\|_r \leq C_{d,k,\Delta} \|f_1\|_{p_1} \cdots \|f_k\|_{p_k}$$
with $1/r = 1/p_1 + \cdots + 1/p_k < (d-2)/d$. Theorem 2 (ii) now follows by symmetry and interpolation. □

3. Proof of Proposition 1

The key ingredient of the proof of this proposition is an upper bound on the ℓ^1 norm of the function $S_T(y_1,\ldots,y_k)$ defined in (2) (when $\lambda=1$), proved in Lemma 2.2 in [6], namely if $T=(t_{ij})$ is a positive definite integral $k\times k$ matrix then for $d\geq 2k+3$ one has

$$\sum_{y_1,\ldots,y_k\in\mathbb{Z}^d} S_T(y_1,\ldots,y_k)\leq C_{d,k}\left(\det(T)^{(d-k-1)/2}+|T|^{(d-k)(k-1)/2}\right) \tag{10}$$

with $|T|:=(\sum_{i,j}t_{ij}^2)^{1/2}$.

Let $\Delta=\{v_0=0,v_1,\ldots,v_k\}$ be a non-degenerate k-simplex with inner product matrix $T=(t_{ij})$. Note that for $\lambda\leq\lambda_{d,k,\Delta}$ we have that $N_{\lambda\Delta}\leq C_{d,k,\Delta}$ thus by Hölder's and Minkowski's inequalities we have that $\|A_\lambda(f_1,\ldots,f_k)\|_r\leq C_{d,k,\Delta}\,\|f_1\|_{p_1}\ldots\|f_k\|_{p_k}$, whenever $1/p_1+\cdots+1/p_k=1/r$. Thus the supremum in (5) can be restricted to sufficiently large λ. Then because of $N_{\lambda\Delta}\asymp\lambda^{k(d-k-1)}$ one may replace the factor $N_{\lambda\Delta}^{-1}$ with $\lambda^{-k(d-k-1)}$ in formula (4) and assume without loss of generality that $\lambda\geq\lambda_{d,k,\Delta}$.

We choose to focus first on establishing part (ii) of Proposition 1.

Proof of Propostion 1 (ii). For a solution $y_1,\ldots,y_k$ to the system of equations (2) we will write $\underline{y}_1=(y_1,\ldots,y_{k-1})$ to group the first $k-1$ variables and T_1 for the corresponding inner product matrix, i.e. for the $k-1\times k-1$ minor of T. For given $x\in\mathbb{Z}^d$, by the Cauchy-Schwarz inequality, in dimensions $d>2k$ we have

$$\begin{aligned} A_\lambda(f_1,\ldots,f_k)(x)^2\leq\ &\lambda^{-d(k-1)+k(k-1)}\sum_{\underline{y}_1}S_{\lambda^2T_1}(\underline{y}_1)f_1^2(x+y_1)\cdots f_{k-1}^2(x+y_{k-1})\\ &\times\ \lambda^{-d(k+1)+k^2+3k}\sum_{\underline{y}_1}\Big(\sum_{y_k}f_k(x+y_k)S_{\lambda^2T}(\underline{y}_1,y_k)\Big)^2\\ \leq\ &A_*(f_1^2,\ldots,f_{k-1}^2)(x)\ B_\lambda(f_k,f_k)(x)\end{aligned}$$

where

$$B_\lambda(f_k,f_k)(x)=\lambda^{-d(k+1)+k^2+3k}\sum_{y_k,y_k'}f_k(x+y_k)f_k(x+y_k')W_{\lambda^2T}(y_k,y_k')$$

with a weight function

$$W_{\lambda^2T}(y_k,y_k')=\sum_{\underline{y}_1}S_{\lambda^2T}(\underline{y}_1,y_k)\,S_{\lambda^2T}(\underline{y}_1,y_k'). \tag{11}$$

By a slight abuse of notation let $S_\lambda(y)=1$ if $|y|^2=t_{kk}\lambda^2$ and equal to 0 otherwise. Then one may write

$$B_\lambda(f_k,f_k)(x)=\lambda^{-d(k+1)+k^2+3k}\sum_{y_k,y_k'}f_k(x+y_k)f_k(x+y_k')S_\lambda(y_k)S_\lambda(y_{k'})W_{\lambda^2T}(y_k,y_k')$$

and an application of Cauchy-Schwarz gives

$$B_\lambda(f_k,f_k)(x)^2\leq\Big(\lambda^{-d+2}\sum_y f_k^2(x+y)S_\lambda(y)\Big)^2\Big(\lambda^{-2dk+2k^2+6k-4}\sum_{y_k,y_k'}W_{\lambda^2T}(y_k,y_k')^2\Big).$$

Thus, in order to establish (8) and complete the proof of the proposition, it suffices to show that

$$\sum_{y_k,y_k'}W_{\lambda^2T}(y_k,y_k')^2\leq C\,\lambda^{2dk-2k^2-6k+4}$$

with a constant $C=C_{d,k,T}>0$. By (11), we have that

$$\sum_{y_k,y_k'}W_{\lambda^2T}(y_k,y_k')^2=\sum_{\underline{y}_1,\underline{y}_1',y_k,y_k'}S_{\lambda^2T}(\underline{y}_1,y_k)S_{\lambda^2T}(\underline{y}_1',y_k)S_{\lambda^2T}(\underline{y}_1,y_k')S_{\lambda^2T}(\underline{y}_1',y_k').$$

The above expression is the number of solutions $y_1,\dots,y_k,y'_1,\dots,y'_k \in \mathbb{Z}^d$ to the system of quadratic equations

$$\begin{aligned} & y_i\cdot y_j = y'_i\cdot y'_j = \lambda^2 t_{ij},\ \text{ for }\ 1\le i,j\le k-1\\ & y_i\cdot y_k = y'_i\cdot y_k = y_i\cdot y'_k = y'_i\cdot y'_k = \lambda^2 t_{ik},\ \text{ for }\ 1\le i\le k-1 \\ & y_k\cdot y_k = y'_k\cdot y'_k = \lambda^2 t_{kk}. \end{aligned} \tag{12}$$

For any solution $y_1,\dots,y_k,y'_1,\dots,y'_k$ of the system (12) introduce the parameters $(s_{ij})_{1\le i,j\le k-1}$ and s_{kk} such that

$$y_i\cdot y'_j = \lambda^2 s_{ij}\ \text{ for }\ 1\le i,j\le k-1 \text{ and }\ y_k\cdot y'_k = \lambda^2 s_{kk}. \tag{13}$$

We call the set of parameters $S=(s_{ij},s_{kk})_{1\le i,j\le k-1}$ *admissible* if the system (12)-(13) have a solution. For any admissible set of parameters S let $\lambda^2 T_S$ denote the $2k\times 2k$ inner product matrix of the system (12)-(13), and note that $\lambda^2 T_S$ is a positive semi-definite integral matrix with entries $O_T(\lambda^2)$.

We consider two cases.

Case 1: Assume that the matrix T_S is positive definite. Then in dimensions $d\ge 4k+3$ one may apply estimate (10) to the matrix $\lambda^2 T_S$ which shows that the number of solutions to the system (12)-(13) is bounded by $C\,\lambda^{2dk-2k(2k+1)}$. Since there at most $C\,\lambda^{2(k-1)^2+2}$ admissible sets S, such admissible sets contribute to at most $C\,\lambda^{2dk-2k^2-6k+4}$ solutions to the system (12), for some constant $C=C_{d,k,T}>0$.

Case 2: Assume $\det(T_S)=0$. Then the vectors $y_1,\dots,y_k,y'_1,\dots,y'_k$ are linearly dependent. Let $M:=\operatorname{span}\{y_1,\dots,y_k,y'_1,\dots,y'_k\}\subseteq\mathbb{R}^d$. Since $y_1,\dots,y_k$ are linearly independent one may extend these vectors with vectors $y'_{i_1},\dots y'_{i_l}$, for some $1\le l<k$, to obtain a basis of of the vector space M. Write $I=\{i_1,\dots,i_l\}$, if $j\notin I$, then $y'_j\in M$ moreover the inner products $y_j\cdot y_i$ for $1\le i\le k$, and $y_j\cdot y'_i$ for $i\in I$ are all determined by equations (12)-(13). It follows that y'_j is uniquely determined for $j\notin I$, thus the number of solutions for a fixed index set I is bounded by the number of $k+l$-tuples $y_1,\dots,y_k,y'_{i_1},\dots y'_{i_l}$ satisfying equations (12)-(13). The inner products of these vectors form a positive definite matrix, thus applying estimate (10) we obtain that number of solutions is bounded by $C\lambda^{d(k+l)-(k+l)(k+l+1)} < C\,\lambda^{2dk-2k(2k+1)}$, in dimensions $d>4k$. As the number of possible index sets I depends only on k, the total number of linearly dependent solutions to the system (12)-(13) is also bounded by $C\,\lambda^{2dk-2k^2-6k+4}$. $\square$

Proof of Propostion 1 (i). We use the same notation as above and assume that $\|f_1\|_\infty,\dots,\|f_{k-1}\|_\infty\le 1$.

For any given $x\in\mathbb{Z}^d$ we have

$$A_\lambda(f_1,\dots,f_k)(x)\le\ \lambda^{-dk+k(k+1)}\ \sum_{y_k} f_k(x-y_k)S_\lambda(y_k)\ \sum_{\underline{y}_1} S_{\lambda^2T}(\underline{y}_1,y_k)$$

and hence, after an application of Cauchy-Schwarz, we obtain

$$A_\lambda(f_1,\dots,f_k)(x)^2\le\ A_*(f_k^2)(x)\ \lambda^{-d(2k-1)+2k(k+1)-2}\sum_{y_k,\underline{y}_1,\underline{y}'_1} S_{\lambda^2T}(\underline{y}_1,y_k)\,S_{\lambda^2T}(\underline{y}'_1,y_k).$$

The sum in the expression above is the number of solutions $y_1,\dots,y_{k-1},y'_1,\dots,y'_{k-1}\in\mathbb{Z}^d$ and $y_k\in\mathbb{Z}^d$ to the system of quadratic equations

$$\begin{aligned} & y_i\cdot y_j = y'_i\cdot y'_j = \lambda^2 t_{ij},\ \text{ for }\ 1\le i,j\le k-1\\ & y_i\cdot y_k = y'_i\cdot y_k = \lambda^2 t_{ik},\ \text{ for }\ 1\le i\le k-1 \\ & y_k\cdot y_k = \lambda^2 t_{kk}. \end{aligned} \tag{14}$$

If one now argues, as in the proof of part (ii) above, it follows from estimate (10) that

$$\sum_{y_k,\underline{y}_1,\underline{y}'_1} S_{\lambda^2T}(\underline{y}_1,y_k)\,S_{\lambda^2T}(\underline{y}'_1,y_k)\le C_{d,k,T}\,\lambda^{d(2k-1)-2k(k+1)+2}.$$

We choose to omit the details of this calculation. $\square$

4. A strengthing of Theorem 2 in high dimensions

If, in the proof of Proposition 1, we apply Hölder's inequality with conjugate exponents $m/(m-1)$ and m instead of the Cauchy-Schwarz inequality, this results in $y_1, \dots, y_{k-1}$ and $y_1, \dots, y_k$ being increased m-fold as opposed to being doubled, in parts (i) and (ii) respectively.

Working through these details, which we omit, one obtains the following

Proposition 2. *Let $k \in \mathbb{N}$ and $\Delta = \{v_0 = 0, v_1, \dots, v_k\} \subseteq \mathbb{Z}^d$ be a non-degenerate k-simplex.*

Let $m \geq 2$ be an integer and set $q = m/(m-1)$.

(i) *If $d \geq 2m(k-1)+5$, then for any $f_1, \dots, f_k : \mathbb{Z}^d \to \mathbb{R}$, one has*

$$A_*(f_1, \dots, f_k)(x) \leq C_{d,k,\Delta} \, \|f_1\|_\infty \cdots \|f_{k-1}\|_\infty \; A_*(f_k^q)(x)^{1/q}$$

uniformly for $x \in \mathbb{Z}^d$.

(ii) *If $d \geq 2mk+3$, then for any $f_1, \dots, f_k : \mathbb{Z}^d \to \mathbb{R}$, one has*

$$A_*(f_1, \dots, f_k)(x) \leq C_{d,k,\Delta} \, A_*(f_1^q, \dots, f_{k-1}^q)(x)^{1/q} \, A_*(f_k^q)(x)^{1/q}$$

and hence

$$A_*(f_1, \dots, f_k)(x) \leq C_{d,k,\Delta} \, A_*(f_1^{q^{k-1}})(x)^{1/q^{k-1}} A_*(f_2^{q^{k-1}})(x)^{1/q^{k-1}} \prod_{j=3}^{k} A_*(f_j^{q^{k+1-j}})(x)^{1/q^{k+1-j}}$$

uniformly for $x \in \mathbb{Z}^d$.

This proposition allows us to establish the following strengthening of Theorem 2 in high dimensions.

Theorem 3. *Let $k \in \mathbb{N}$ and $\Delta = \{v_0 = 0, v_1, \dots, v_k\} \subseteq \mathbb{Z}^d$ be a non-degenerate k-simplex.*

(i) *If $d \geq 4k+1$, $r > q' \, d/(d-2)$, and $1 \leq p_1, \dots, p_k \leq \infty$ with $1/r \leq 1/p_1 + \cdots + 1/p_k$, one has*

$$\|A_*(f_1, \dots, f_k)\|_r \leq C_{d,k,\Delta} \, \|f_1\|_{p_1} \cdots \|f_k\|_{p_k}$$

where $q' = q'_{d,k} = \lfloor (d-5)/2(k-1) \rfloor / (\lfloor (d-5)/2(k-1) \rfloor - 1)$.

(ii) *If $d \geq 4k+3$, then for any*

$$r > (q^{-1} + q^{-2} + \cdots + q^{-(k-1)} + q^{-(k-1)})^{-1} \, d/(d-2) \text{ and } p_1, \dots, p_k > q \, d/(d-2)$$

whose reciprocals $(1/p_1, \dots, 1/p_k) \in (d-2)/d \cdot \mathcal{C}_{k,q}$ and satisfy $1/r = 1/p_1 + \cdots + 1/p_k$, one has

$$\|A_*(f_1, \dots, f_k)\|_r \leq C_{d,k,\Delta} \, \|f_1\|_{p_1} \cdots \|f_k\|_{p_k}$$

where $q = q_{d,k} = \lfloor (d-3)/2k \rfloor / (\lfloor (d-3)/2k \rfloor - 1)$ and $\mathcal{C}_{k,q}$ denotes all points $(x_1, \dots, x_k) \in [0,1]^k$ with $x_1 + \cdots + x_k < q^{-1} + q^{-2} + \cdots + q^{-(k-1)} + q^{-(k-1)}$ that also have the property that for any $1 \leq j \leq k-1$ one has $y_1 + \cdots + y_j < q^{-1} + \cdots + q^{-j}$ for any choice $\{y_1, \dots, y_j\} \subset \{x_1, \dots, x_k\}$.

Note that Theorem 3 provides us with a strengthening of Theorem 2 (i) and (ii) for all $d \geq 6k-1$ and $d \geq 6k+3$, respectively. Note that Theorem 3 is of particular interest as $d \to \infty$ for fixed k, since this corresponds to $q, q' \to 1$ through values of the form $m/(m-1)$ with $m \in \mathbb{N}$.

Proof of Theorem 3. To establish part (i) we set $m = \lfloor (d-5)/2(k-1) \rfloor$. Proposition 2 (i) then implies

$$\|A_*(f_1, \dots, f_k)\|_{p_k} \leq C_{d,k,\Delta} \, \|f_1\|_\infty \cdots \|f_{k-1}\|_\infty \|A_*(f_k^q)\|_{p_k/q'}^{1/q'} \leq C_{d,k,\Delta} \, \|f_1\|_\infty \cdots \|f_{k-1}\|_\infty \|f_k\|_{p_k}$$

provided $p_k > q' \, d/(d-2)$. By symmetry and interpolation we then obtain part (i) of Theorem 3.

To establish part (ii) we set $m = \lfloor (d-3)/2k \rfloor$. Proposition 2 (ii) then ensures that

$$A_*(f_1, \dots, f_k)(x) \leq C_{d,k,\Delta} \, A_*(f_1^{q^{k-1}})(x)^{1/q^{k-1}} A_*(f_2^{q^{k-1}})(x)^{1/q^{k-1}} \prod_{j=3}^{k} A_*(f_j^{q^{k+1-j}})(x)^{1/q^{k+1-j}}.$$

An application of Hölder, as in the proof of Theorem 2, then gives

$$\|A_*(f_1,\dots,f_k)\|_r \le C_{d,k,\Delta}\|A_*(f_1^{q^{k-1}})\|_{p_1/q^{k-1}}^{1/q^{k-1}}\|A_*(f_2^{q^{k-1}})\|_{p_2/q^{k-1}}^{1/q^{k-1}}\prod_{j=3}^{k}\|A_*(f_j^{q^{k+1-j}})\|_{p_j/q^{k+1-j}}^{1/q^{k+1-j}}$$

whenever $1/r = 1/p_1 + \cdots + 1/p_k$. Now if

$$p_1, p_2 > q^{k-1}\frac{d}{d-2} \quad \text{and} \quad p_j > q^{k+1-j}\frac{d}{d-2} \quad \text{for} \quad 3 \le j \le k$$

then by (1) we obtain

$$\|A_*(f_1,\dots,f_k)\|_r \le C_{d,k,\Delta}\|f_1\|_{p_1}\cdots\|f_k\|_{p_k}$$

with $1/r = 1/p_1 + \cdots + 1/p_k < (1/q + 1/q^2 + \cdots + 1/q^{k-1} + 1/q^{k-1})(d-2)/d$.

Part (ii) of Theorem 3 now follows by symmetry and interpolation. □

5. An example

Simple examples show that estimates of the form $\|A_*(f_1,f_2,\dots,f_k)\|_p \le C\,\|f_1\|_p\|f_2\|_\infty\cdots\|f_k\|_\infty$ are not possible for $1 \le p \le d/(d-2)$, in dimensions $d \ge 2k+3$.

Indeed, let $f_1 := \delta_0$ the point mass at the origin, and let $f_2 = \cdots = f_k = 1$. For given $x \in \mathbb{Z}^d$ and $\lambda \in \sqrt{\mathbb{N}}$, we have

$$A_\lambda(f_1,f_2,\dots,f_k)(x) \ge C\,\lambda^{-dk+k(k+1)}\sum_{y_2,\dots,y_k} S_{\lambda^2 T}(x,y_2,\dots,y_k).$$

Choosing $\lambda = |x|$, one has

$$A_*(f_1,f_2,\dots,f_k)(x) \ge C\,|x|^{-dk+k(k+1)}\sum_{y_2,\dots,y_k} S_{|x|^2 T}(x,y_2,\dots,y_k) = |x|^{-d+2}\,W_{|x|}(x),$$

where

$$W_{|x|}(x) = |x|^{-d(k-1)+k(k+1)-2}\sum_{y_2,\dots,y_k} S_{|x|^2 T}(x,y_2,\dots,y_k). \tag{15}$$

Let $p \ge 1$. Summing for $2^j \le |x| < 2^{j+1}$, one estimates by Hölder's inequality

$$\sum_{2^j \le |x| < 2^{j+1}} A_*(f_1,f_2,\dots,f_k)(x)^p \ge C\,2^{jd-jp(d-2)}\big(2^{-jd}\sum_{2^j \le |x| < 2^{j+1}} W_{|x|}(x)\big)^p. \tag{16}$$

Moreover, by (15), one has

$$2^{-jd}\sum_{2^j \le |x| < 2^{j+1}} W_{|x|}(x) \ge C\,2^{-j(dk-k(k+1)+2)}\sum_{2^j \le |x| < 2^{j+1}}\sum_{y_2,\dots,y_k} S_{|x|^2 T}(x,y_2,\dots,y_k).$$

Writing $\lambda = |x| \in \sqrt{N}$ the right side of the above expression can further estimated from below by

$$2^{-j(dk-k(k+1)+2)}\sum_{2^j \le \lambda < 2^{j+1}}\sum_{y_1,\dots,y_k} S_{\lambda^2 T}(y_1,\dots,y_k) \ge C\,2^{-2j}\sum_{2^j \le \lambda < 2^{j+1}} 1 \ \ge\ C,$$

for some constant $C = C_{d,k,T} > 0$, using estimate (3) and the fact that there are approximately 2^{2j} values of $\lambda \in \sqrt{\mathbb{N}}$ satisfying $2^j \le \lambda < 2^{j+1}$. This implies that for $1 \le p \le d/(d-2)$ the left side of (16) is bigger than a constant for every $j \in \mathbb{N}$ thus $\|A_*(f_1,f_2,\dots,f_k)\|_p = \infty$ while $\|f_1\|_p = 1$ and $\|f_j\|_\infty = 1$ for all $2 \le j \le k$.

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Department of Mathematics,Virginia Tech, Blacksburg, VA 24061, USA

Email address: briancookmath@gmail.com

Department of Mathematics, The University of Georgia, Athens, GA 30602, USA

Email address: lyall@math.uga.edu

Department of Mathematics, The University of Georgia, Athens, GA 30602, USA

Email address: magyar@math.uga.edu