# Math 3100
# Sample Exam 2 – Version 0

*No calculators. Show your work. Give full explanations. Good luck!*

1. (15 points)

   (a) Carefully state the definition of what it means to say that $\sum_{n=1}^{\infty} a_n$ is convergent.

   (b) i. Prove that if $\sum_{n=1}^{\infty} a_n$ is convergent, then $\lim_{n\to\infty} a_n = 0$.

   ii. Is it true that if $\lim_{n\to\infty} a_n = 0$, then $\sum_{n=1}^{\infty} a_n$ is convergent? Give either a proof or counterexample.

   (c) Let $b_n \geq 0$ for all $n \in \mathbb{N}$.

   i. Prove that if $\sum_{n=1}^{\infty} b_n$ converges, then $\sum_{n=1}^{\infty} b_n^2$ also converges and that in fact

   $$\sum_{n=1}^{\infty} b_n^2 \leq \left(\sum_{n=1}^{\infty} b_n\right)^2.$$

   ii. Is it true that if $\sum_{n=1}^{\infty} b_n^2$ converges, then $\sum_{n=1}^{\infty} b_n$ also converges? Give either a proof or counterexample.

2. (20 points)

   (a) Determine if the following series are absolutely convergent, conditionally convergent, or divergent. Justify your answers.

   $$\text{(i)} \quad \sum_{n=1}^{\infty} \frac{(-1)^n n}{n^3+1} \qquad \text{(ii)} \quad \sum_{n=1}^{\infty} \frac{(-1)^n n}{n+1} \qquad \text{(iii)} \quad \sum_{n=1}^{\infty} \frac{(\log n)^3}{n^2}$$

   (b) Find all $x \in \mathbb{R}$ for which $\sum_{n=1}^{\infty} \frac{(-1)^n x^n}{\sqrt{n}}$ converges.

   (c) Find a sequence $\{a_n\}$ so that $\sum_{n=2}^{\infty} a_n x^n = \frac{x^2}{2+x}$ for all $|x| < 2$.

3. (15 points)

   (a) i. Let $X \subseteq \mathbb{R}$ and $f : X \to \mathbb{R}$. Carefully state the $\varepsilon$-$\delta$ definition of what it means for $f$ to be *continuous* at a point $x_0 \in X$.

   ii. Use this $\varepsilon$-$\delta$ definition to prove that $f(x) = \frac{3-x}{x^2}$ is continuous at $x_0 = 2$.

   *Hint: Use the fact that* $\left|\frac{3-x}{x^2} - \frac{1}{4}\right| = \frac{|x+6|}{4x^2}|x-2|$

   (b) i. Let $g : \mathbb{R} \to \mathbb{R}$. Carefully state the *sequential characterization* of what it means for $g$ to be *continuous* at a point $x_0 \in \mathbb{R}$.

   ii. Prove, using the sequential characterization or otherwise, that the function

   $$g(x) = \begin{cases} x & \text{if } x \leq 1 \\ 0 & \text{if } x > 1 \end{cases}$$

   is not continuous at $x_0 = 1$.

# Math 3100 – Sample Exam 2 (Version 0) – SOLUTIONS

1. (a) We say that $\sum_{n=1}^{\infty} a_n$ converges if

$$\lim_{n\to\infty} (a_1 + a_2 + \cdots + a_n) \text{ exists.}$$

$\left[ s_n := a_1 + \cdots + a_n \text{ is called the } n^{th} \text{ partial sum of } \sum_{n=1}^{\infty} a_n \right]$

(b)(i) Claim If $\sum_{n=1}^{\infty} a_n$ converges, then $\lim_{n\to\infty} a_n = 0$.

Proof 1

Let $s_n = a_1 + \cdots + a_n$. Since $\sum_{n=1}^{\infty} a_n$ converges we know that $\{s_n\}$ converges. Let $s := \lim_{n\to\infty} s_n$.

Note that $a_n = s_n - s_{n-1}$ and $\lim_{n\to\infty} s_{n-1} = s$ also.

It follow from basic limit laws that

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} (s_n - s_{n-1}) = \lim_{n\to\infty} s_n - \lim_{n\to\infty} s_{n-1} = s - s = 0.$$

□

Proof 2

Let $\varepsilon > 0$. Since $\sum_{n=1}^{\infty} a_n$ converges it follows from the "Cauchy Criterion for Series" that $\exists N$ such that $n > m > N \Rightarrow \left| \sum_{k=m+1}^{n} a_k \right| < \varepsilon$.

In particular (with $m = n-1$) we see that if $n > N+1$, then

$$\left| \sum_{k=n}^{n} a_k \right| = |a_n| < \varepsilon$$

and hence that $\lim_{n\to\infty} a_n = 0$.

□

(b)(ii) Since $\frac{1}{n} \to 0$ but $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges we see that

$$\lim_{n\to\infty} a_n = 0 \not\Rightarrow \sum_{n=1}^{\infty} a_n \text{ convergent.}$$

(c)(i) Let $b_n \geq 0$ for all $n \in \mathbb{N}$.

<u>Claim</u>

If $\sum_{n=1}^{\infty} b_n$ converges, then $\sum_{n=1}^{\infty} b_n^2$ converges and in fact

$$\sum_{n=1}^{\infty} b_n^2 \leq \left(\sum_{n=1}^{\infty} b_n\right)^2$$

<u>Proof</u>

Since $\sum_{n=1}^{\infty} b_n$ converges we know that $\lim_{n\to\infty} (b_1 + b_2 + \cdots + b_n)$ exists.

Let $B := \lim_{n\to\infty} (b_1 + \cdots + b_n)$ and note that $\{b_1 + \cdots + b_n\}$ increases.

It further follows from the fact that $b_n \geq 0$ $\forall n \in \mathbb{N}$ that

$$\underbrace{b_1^2 + \cdots + b_n^2 \leq (b_1 + \cdots + b_n)^2}_{\text{all "cross-terms" are } \geq 0} \underbrace{\leq B^2}_{\text{Since } b_1 + \cdots + b_n \nearrow B = \sum_{n=1}^{\infty} b_n \Rightarrow b_1 + \cdots + b_n \leq B \text{ for all } n \in \mathbb{N}.}$$

Since $\{b_1^2 + \cdots + b_n^2\}$ is an increasing sequences which is bounded above (by $B^2$) it follows from MCT and "order limit laws" that

$$\sum_{n=1}^{\infty} b_n^2 := \lim_{n\to\infty} (b_1^2 + \cdots + b_n^2) \text{ exists and is } \leq B^2 = \left(\sum_{n=1}^{\infty} b_n\right)^2. \quad \square$$

(ii) Since $\sum \frac{1}{n^2}$ conv & $\sum \frac{1}{n}$ div, $\sum b_n^2$ conv $\not\Rightarrow \sum b_n$ conv.

2. (a) (i) $\sum_{n=1}^{\infty} (-1)^n \frac{n}{n^3+1}$ CONV. ABS.

since $\frac{n}{n^3+1} \le \frac{1}{n^2}$ $\forall n \in \mathbb{N}$ and $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges.

(ii) $\sum_{n=1}^{\infty} (-1)^n \frac{n}{n+1}$ DIVERGES

since $(-1)^n \frac{n}{n+1} \not\to 0$ as $n \to \infty$.

(iii) $\sum_{n=1}^{\infty} \frac{(\log n)^3}{n^2}$ CONV. ABS.

since $\log n \le n^{1/4}$ for all suff. large $n$ implies

$\frac{(\log n)^3}{n^2} \le \frac{n^{3/4}}{n^2} = \frac{1}{n^{5/4}}$ "eventually" & $\sum_{n=1}^{\infty} \frac{1}{n^{5/4}}$ converges.

(b) Claim $\sum_{n=1}^{\infty} \frac{(-1)^n x^n}{\sqrt{n}}$ converges $\Leftrightarrow$ $x \in (-1, 1]$.

Proof Let $a_n = \frac{(-1)^n x^n}{\sqrt{n}}$. Since

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{(-1)^{n+1} x^{n+1}}{\sqrt{n+1}} \frac{\sqrt{n}}{(-1)^n x^n}\right| = \sqrt{\frac{n}{n+1}}\,|x| \longrightarrow |x|$$

it follows from the "Ratio Test" that $\sum_{n=1}^{\infty} a_n$ conv. abs.
if $|x|<1$ and $\sum_{n=1}^{\infty} a_n$ diverges if $|x|>1$.

If $x=1$, then $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} (-1)^n \frac{1}{\sqrt{n}}$ which converges by the Alt. Series Test since $\frac{1}{\sqrt{n}} \searrow 0$.

If $x=-1$, then $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ which is divergent. $\square$

(c) Since

$$\frac{x^2}{2+x} = \frac{x^2}{2} \cdot \frac{1}{1+\frac{x}{2}} \quad \& \quad \frac{1}{1+\frac{x}{2}} = \sum_{n=0}^{\infty} (-1)^n \frac{x^n}{2^n} \quad \text{if } |x|<2$$

it follows that

$$\frac{x^2}{2+x} = \frac{x^2}{2} \sum_{n=0}^{\infty} (-1)^n \frac{x^n}{2^n} \quad \text{if } |x|<2$$

$$= \sum_{n=0}^{\infty} (-1)^n \frac{x^{n+2}}{2^{n+1}} \quad \text{if } |x|<2$$

$$= \sum_{n=2}^{\infty} (-1)^n \frac{x^n}{2^{n-1}} \quad \text{if } |x|<2 .$$

So $\underline{a_n = (-1)^n \frac{1}{2^{n-1}}}$.

3. (a) (i) Let $X \subseteq \mathbb{R}$ and $x_0 \in X$.

$f: X \to \mathbb{R}$ conts at $x_0 \iff$ $\forall \varepsilon>0 \ \exists \delta>0$ such that if $x \in X$ with $|x-x_0|<\delta$, then $|f(x)-f(x_0)|<\varepsilon$

(ii) __Claim__ $f(x) = \frac{3-x}{x^2}$ is conts at $x_0 = 2$.

__Proof__ Let $\varepsilon>0$ and set $\delta = \min\{1, 4\varepsilon/9\}$.

If $|x-2|<\delta$, then

since $|x-2|<1 \Rightarrow |x+6|<9$ & $x>1$

$$\left|\frac{3-x}{x^2} - \underset{\uparrow f(2)}{\frac{1}{4}}\right| = \frac{|x+6|}{4x^2}|x-2| \le \frac{9}{4}|x-2| < \frac{9}{4}\left(\frac{4\varepsilon}{9}\right) = \varepsilon .$$

since $|x-2|<\frac{4\varepsilon}{9}$. □

(b) (i) $g: \mathbb{R} \to \mathbb{R}$ conts at $x_0 \iff g(x_n) \to g(x_0)$ for __all__ seq $x_n \to x_0$.

(ii) __Claim__ $g(x) = \begin{cases} x & \text{if } x \le 1 \\ 0 & \text{if } x>1 \end{cases}$ is __not__ conts at $x_0 = 1$.

__Proof__ let $x_n = 1-\frac{1}{n}$ & $y_n = 1+\frac{1}{n}$, then $x_n \to 1$ & $y_n \to 1$

__BUT__ $g(x_n) = x_n \to 1$ and $g(y_n) = 0 \ \forall n$ & hence $g(y_n) \to 0$. □