that play is certain to terminate if $p > \frac{1}{3}$ and that the ultimate gain is the sum of the numbers in the initial pattern. Infinite capital is again required.

7.9. Suppose that $W_k = 1$, so that $F_k = F_0 + S_k$. Suppose that $p \ge q$ and τ is a stopping time such that $1 \le \tau \le n$ with probability 1. Show that $E[F_\tau] \le E[F_n]$, with equality in case $p = q$. Interpret this result in terms of a stock option that must be exercised by time n, where $F_0 + S_k$ represents the price of the stock at time k.

7.10. For a given policy, let A_n^* be the fortune of the gambler's adversary at time n. Consider these conditions on the policy: (i) $W_n^* \le F_{n-1}^*$; (ii) $W_n^* \le A_{n-1}^*$; (iii) $F_n^* + A_n^*$ is constant. Interpret each condition, and show that together they imply that the policy is bounded in the sense of (7.24).

7.11. Show that F_τ has infinite range if $F_0 = 1$, $W_n = 2^{-n}$, and τ is the smallest n for which $X_n = +1$.

7.12. Let u be a real function on $[0,1]$, $u(x)$ representing the *utility* of the fortune x. Consider policies bounded by 1; see (7.24). Let $Q_\pi(F_0) = E[u(F_\tau)]$; this represents the expected utility under the policy π of an initial fortune F_0. Suppose of a policy π_0 that

$$u(x) \le Q_{\pi_0}(x), \qquad 0 \le x \le 1, \tag{7.34}$$

and that

$$Q_{\pi_0}(x) \ge pQ_{\pi_0}(x+t) + qQ_{\pi_0}(x-t), \qquad 0 \le x-t \le x \le x+t \le 1. \tag{7.35}$$

Show that $Q_\pi(x) \le Q_{\pi_0}(x)$ for all x and all policies π. Such a π_0 is optimal.

Theorem 7.3 is the special case of this result for $p \le \frac{1}{2}$, bold play in the role of π_0, and $u(x) = 1$ or $u(x) = 0$ according as $x = 1$ or $x < 1$.

The condition (7.34) says that gambling with policy π_0 is at least as good as not gambling at all; (7.35) says that, although the prospects even under π_0 become on the average less sanguine as time passes, it is better to use π_0 now than to use some other policy for one step and then change to π_0.

7.13. The functional equation (7.30) and the assumption that Q is bounded suffice to determine Q completely. First, $Q(0)$ and $Q(1)$ must be 0 and 1, respectively, and so (7.31) holds. Let $T_0 x = \frac{1}{2}x$ and $T_1 x = \frac{1}{2}x + \frac{1}{2}$; let $f_0 x = px$ and $f_1 x = p + qx$. Then $Q(T_{u_1} \cdots T_{u_n} x) = f_{u_1} \cdots f_{u_n} Q(x)$. If the binary expansions of x and y both begin with the digits $u_1, \ldots, u_n$, they have the form $x = T_{u_1} \ldots T_{u_n} x'$ and $y = T_{u_1} \cdots T_{u_n} y'$. If K bounds Q and if $m = \max\{p, q\}$, it follows that $|Q(x) - Q(y)| \le Km^n$. Therefore, Q is continuous and satisfies (7.31) and (7.33).

SECTION 8. MARKOV CHAINS

As Markov chains illustrate in a clear and striking way the connection between probability and measure, their basic properties are developed here in a measure-theoretic setting.

Definitions

Let S be a finite or countable set. Suppose that to each pair i and j in S there is assigned a nonnegative number p_{ij} and that these numbers satisfy the constraint

$$\sum_{j \in S} p_{ij} = 1, \qquad i \in S. \tag{8.1}$$

Let $X_0, X_1, X_2, \ldots$ be a sequence of random variables whose ranges are contained in S. The sequence is a *Markov chain* or *Markov process* if

$$\begin{aligned} P[X_{n+1} = j \| X_0 = i_0, \ldots, X_n = i_n] \\ = P[X_{n+1} = j \| X_n = i_n] = p_{i_n j} \end{aligned} \tag{8.2}$$

for every n and every sequence $i_0, \ldots, i_n$ in S for which $P[X_0 = i_0, \ldots, X_n = i_n] > 0$. The set S is the *state space* or *phase space* of the process, and the p_{ij} are the *transition probabilities*. Part of the defining condition (8.2) is that the transition probability

$$P[X_{n+1} = j \| X_n = i] = p_{ij} \tag{8.3}$$

does not vary with n.[†]

The elements of S are thought of as the possible states of a *system*, X_n representing the state at *time* n. The sequence or process $X_0, X_1, X_2, \ldots$ then represents the history of the system, which evolves in accordance with the probability law (8.2). The conditional distribution of the *next* state X_{n+1} given the *present* state X_n must not further depend on the *past* $X_0, \ldots, X_{n-1}$. This is what (8.2) requires, and it leads to a copious theory.

The *initial probabilities* are

$$\alpha_i = P[X_0 = i]. \tag{8.4}$$

The α_i are nonnegative and add to 1, but the definition of Markov chain places no further restrictions on them.

[†] Sometimes in the definition of the Markov chain $P[X_{n+1} = j \| X_n = i]$ is allowed to depend on n. A chain satisfying (8.3) is then said to have *stationary transition probabilities*, a phrase that will be omitted here because (8.3) will always be assumed.

The following examples illustrate some of the possibilities. In each one, the state space S and the transition probabilities p_{ij} are described, but the underlying probability space $(\Omega, \mathscr{F}, P)$ and the X_n are left unspecified for now: see Theorem 8.1.[†]

Example 8.1. The Bernoulli–Laplace model of diffusion. Imagine r black balls and r white balls distributed between two boxes, with the constraint that each box contains r balls. The state of the system is specified by the number of white balls in the first box, so that the state space is $S = \{0, 1, \ldots, r\}$. The transition mechanism is this: at each stage one ball is chosen at random from each box and the two are interchanged. If the present state is i, the chance of a transition to $i-1$ is the chance i/r of drawing one of the i white balls from the first box times the chance i/r of drawing one of the i black balls from the second box. Together with similar arguments for the other possibilities, this shows that the transition probabilities are

$$p_{i,i-1} = \left(\frac{i}{r}\right)^2, \qquad p_{i,i+1} = \left(\frac{r-i}{r}\right)^2, \qquad p_{ii} = 2\frac{i(r-i)}{r^2},$$

the others being 0. This is the probablistic analogue of the model for the flow of two liquids between two containers. ■

The p_{ij} form the *transition matrix* $P = [p_{ij}]$ of the process. A *stochastic matrix* is one whose entries are nonnegative and satisfy (8.1); the transition matrix of course has this property.

Example 8.2. Random walk with absorbing barriers. Suppose that $S = \{0, 1, \ldots, r\}$ and

$$P = \begin{bmatrix} 1 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ q & 0 & p & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & q & 0 & p & \cdots & 0 & 0 & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 & \cdots & q & 0 & p & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & q & 0 & p \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 1 \end{bmatrix}$$

That is, $p_{i,i+1} = p$ and $p_{i,i-1} = q = 1-p$ for $0 < i < r$ and $p_{00} = p_{rr} = 1$. The chain represents a particle in *random walk*. The particle moves one unit to the right or left, the respective probabilities being p and q, except that each of 0 and r is an *absorbing* state—once the particle enters, it cannot leave. The state can also be viewed as a gambler's fortune; absorption in 0 represents ruin for the gambler, absorption in r ruin for his adversary (see Section 7). The gambler's initial fortune is usually regarded as nonrandom, so that (see (8.4)) $\alpha_i = 1$ for some i. ■

[†]For an excellent collection of examples from physics and biology, see FELLER, Volume 1, Chapter XV.

Example 8.3. Unrestricted random walk. Let S consist of all the integers $i = 0, \pm 1, \pm 2, \ldots$, and take $p_{i,i+1} = p$ and $p_{i,i-1} = q = 1-p$. This chain represents a random walk without barriers, the particle being free to move anywhere on the integer lattice. The walk is *symmetric* if $p = q$. ■

The state space may, as in the preceding example, be countably infinite. If so, the Markov chain consists of functions X_n on a probability space $(\Omega, \mathscr{F}, P)$, but these will have infinite range and hence will not be random variables in the sense of the preceding sections. This will cause no difficulty, however, because expected values of the X_n will not be considered. All that is required is that for each $i \in S$ the set $[\omega: X_n(\omega) = i]$ lie in $\mathscr{F}$ and hence have a probability.

Example 8.4. Symmetric random walk in space. Let S consist of the integer lattice points in k-dimensional Euclidean space R^k; $x = (x_1, \ldots, x_k)$ lies in S if the coordinates are all integers. Now x has $2k$ neighbors, points of the form $y = (x_1, \ldots, x_i \pm 1, \ldots, x_k)$; for each such y let $p_{xy} = (2k)^{-1}$. The chain represents a particle moving randomly in space; for $k = 1$ it reduces to Example 8.3 with $p = q = \frac{1}{2}$. The cases $k \le 2$ and $k \ge 3$ exhibit an interesting difference. If $k \le 2$, the particle is certain to return to its initial position, but this is not so if $k \ge 3$; see Example 8.6. ■

Since the state space in this example is not a subset of the line, the $X_0, X_1, \ldots$ do not assume real values. This is immaterial because expected values of the X_n play no role. All that is necessary is that X_n be a mapping from Ω into S (finite or countable) such that $[\omega: X_n(\omega) = i] \in \mathscr{F}$ for $i \in S$. There will be expected values $E[f(X_n)]$ for real functions f on S with finite range, but then $f(X_n(\omega))$ is a simple random variable as defined before.

Example 8.5. A selection problem. A princess must chose from among r suitors. She is definite in her preferences and if presented with all r at once could choose her favorite and could even rank the whole group. They are ushered into her presence one by one in random order, however, and she must at each stage either stop and accept the suitor or else reject him and proceed in the hope that a better one will come along. What strategy will maximize her chance of stopping with the best suitor of all?

Shorn of some details, the analysis is this. Let $S_1, S_2, \ldots, S_r$ be the suitors in order of presentation; this sequence is a random permutation of the set of suitors. Let $X_1 = 1$ and let $X_2, X_3, \ldots$ be the successive positions of suitors who dominate (are preferable to) all their predecessors. Thus $X_2 = 4$ and $X_3 = 6$ means that S_1 dominates S_2 and S_3 but S_4 dominates S_1, S_2, S_3, and that S_4 dominates S_5 but S_6 dominates $S_1, \ldots, S_5$. There can be at most r of these dominant suitors; if there are exactly m, $X_{m+1} = X_{m+2} = \cdots = r+1$ by convention.

As the suitors arrive in random order, the chance that S_i ranks highest among $S_1, \dots, S_i$ is $(i-1)!/i! = 1/i$. The chance that S_j ranks highest among $S_1, \dots, S_j$ and S_i ranks next is $(j-2)!/j! = 1/j(j-1)$. This leads to a chain with transition probabilities†

$$(8.5) \qquad P[X_{n+1} = j \mid X_n = i] = \frac{i}{j(j-1)}, \qquad 1 \le i < j \le r.$$

If $X_n = i$, then $X_{n+1} = r+1$ means that S_i dominates $S_{i+1}, \dots, S_r$ as well as $S_1, \dots, S_i$, and the conditional probability of this is

$$(8.6) \qquad P[X_{n+1} = r+1 \mid X_n = i] = \frac{i}{r}, \qquad 1 \le i \le r.$$

As downward transitions are impossible and $r+1$ is absorbing, this specifies a transition matrix for $S = \{1, 2, \dots, r+1\}$.

It is quite clear that in maximizing her chance of selecting the best suitor of all, the princess should reject those who do not dominate their predecessors. Her strategy therefore will be to stop with the suitor in position X_τ, where τ is a random variable representing her strategy. Since her decision to stop must depend only on the suitors she has seen thus far, the event $[\tau = n]$ must lie in $\sigma(X, \dots, X_n)$. If $X_\tau = i$, then by (8.6) the conditional probability of success is $f(i) = i/r$. The probability of success is therefore $E[f(X_\tau)]$, and the problem is to choose the strategy τ so as to maximize it. For the solution, see Example 8.17.‡ ■

Higher-Order Transitions

The properties of the Markov chain are entirely determined by the transition and initial probabilities. The chain rule (4.2) for conditional probabilities gives

$$\begin{aligned} P[X_0 = i_0, X_1 = i_1, X_2 = i_2] &= P[X_0 = i_0] P[X_1 = i_1 \mid X_0 = i_0] P[X_2 = i_2 \mid X_0 = i_0, X_1 = i_1] \\ &= \alpha_{i_0} p_{i_0 i_1} p_{i_1 i_2}. \end{aligned}$$

Similarly,

$$(8.7) \qquad P[X_t = i_t, 0 \le t \le m] = \alpha_{i_0} p_{i_0 i_1} \cdots p_{i_{m-1} i_m}$$

for any sequence $i_0, i_1, \dots, i_m$ of states.

Further,

$$(8.8) \quad P[X_{m+t} = j_t, 1 \le t \le n \mid X_s = i_s, 0 \le s \le m] = p_{i_m j_1} p_{j_1 j_2} \cdots p_{j_{n-1} j_n},$$

†The details can be found in DYNKIN & YUSHKEVICH, Chapter III.

‡With the princess replaced by an executive and the suitors by applicants for an office job, this is known as the *secretary problem*

as follows by expressing the conditional probability as a ratio and applying (8.7) to numerator and denominator. Adding out the intermediate states now gives the formula

$$(8.9) \qquad \begin{aligned} p_{ij}^{(n)} &= P[X_{m+n} = j \mid X_m = i] \\ &= \sum_{k_1 \dots k_{n-1}} p_{i k_1} p_{k_1 k_2} \cdots p_{k_{n-1} j} \end{aligned}$$

(the k_l range over S) for the *nth-order transition probabilities*.

Notice that $p_{ij}^{(n)}$ is the entry in position (i, j) of P^n, the nth power of the transition matrix P. If S is infinite, P is a matrix with infinitely many rows and columns; as the terms in (8.9) are nonnegative, there are no convergence problems. It is natural to put

$$p_{ij}^{(0)} = \delta_{ij} = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \ne j. \end{cases}$$

Then P^0 is the identity I, as it should be. From (8.1) and (8.9) follow

$$(8.10) \qquad p_{ij}^{(m+n)} = \sum_\nu p_{i\nu}^{(m)} p_{\nu j}^{(n)}, \qquad \sum_j p_{ij}^{(n)} = 1.$$

An Existence Theorem

Theorem 8.1. *Suppose that $P = [p_{ij}]$ is a stochastic matrix and that α_i are nonnegative numbers satisfying $\sum_{i \in S} \alpha_i = 1$. There exists on some $(\Omega, \mathcal{F}, P)$ a Markov chain $X_0, X_1, X_2, \dots$ with initial probabilities α_i and transition probabilities p_{ij}.*

PROOF. Reconsider the proof of Theorem 5.3. There the space $(\Omega, \mathcal{F}, P)$ was the unit interval, and the central part of the argument was the construction of the decompositions (5.13). Suppose for the moment that $S = \{1, 2, \dots\}$. First construct a partition $I_1^{(0)}, I_2^{(0)}, \dots$ of $(0, 1]$ into countably many† subintervals of lengths (P is again Lebesgue measure) $P(I_i^{(0)}) = \alpha_i$. Next decompose each $I_i^{(0)}$ into subintervals $I_{ij}^{(1)}$ of lengths $P(I_{ij}^{(1)}) = \alpha_i p_{ij}$. Continuing inductively gives a sequence of partitions $\{I_{i_0 \dots i_n}^{(n)}: i_0, \dots, i_n = 1, 2, \dots\}$ such that each refines the preceding and $P(I_{i_0 \dots i_n}^{(n)}) = \alpha_{i_0} p_{i_0 i_1} \cdots p_{i_{n-1} i_n}$.

Put $X_n(\omega) = i$ if $\omega \in \bigcup_{i_0 \dots i_{n-1}} I_{i_0 \dots i_{n-1} i}^{(n)}$. It follows just as in the proof of Theorem 5.3 that the set $[X_0 = i_0, \dots, X_n = i_n]$ coincides with the interval $I_{i_0 \dots i_n}^{(n)}$. Thus $P[X_0 = i_0, \dots, X_n = i_n] = \alpha_{i_0} p_{i_0 i_1} \cdots p_{i_{n-1} i_n}$. From this it follows immediately that (8.4) holds and that the first and third members of

†If $\delta_1 + \delta_2 + \cdots = b - a$ and $\delta_i \ge 0$, then $I_i = (b - \sum_{j \le i} \delta_j, b - \sum_{j < i} \delta_j]$, $i = 1, 2, \dots$, decompose $(a, b]$ into intervals of lengths δ_i.

(8.2) are the same. As for the middle member, it is $P[X_n = i_n,\ X_{n+1} = j]/P[X_n = i_n]$; the numerator is $\sum \alpha_{i_0} p_{i_0 i_1} \cdots p_{i_{n-1} i_n} p_{i_n j}$ the sum extending over all $i_0, \ldots, i_{n-1}$, and the denominator is the same thing without the factor $p_{i_n j}$, which means that the ratio is $p_{i_n j}$, as required.

That completes the construction for the case $S = \{1, 2, \ldots\}$. For the general countably infinite S, let g be a one-to-one mapping of $\{1, 2, \ldots\}$ onto S, and replace the X_n as already constructed by $g(X_n)$; the assumption $S = \{1, 2, \ldots\}$ was merely a notational convenience. The same argument obviously works if S is finite.† ■

Although strictly speaking the Markov chain *is* the sequence $X_0, X_1, \ldots$, one often speaks as though the chain were the matrix P together with the initial probabilities α_i or even P with some unspecified set of α_i. Theorem 8.1 justifies this attitude: For given P and α_i the corresponding X_n do exist, and the apparatus of probability theory—the Borel–Cantelli lemmas and so on—is available for the study of P and of systems evolving in accordance with the Markov rule.

From now on fix a chain $X_0, X_1, \ldots$ satisfying $\alpha_i > 0$ for all i. Denote by P_i probabilities conditional on $[X_0 = i]$: $P_i(A) = P[A \| X_0 = i]$. Thus

$$P_i[X_t = i_t,\ 1 \le t \le n] = p_{i i_1} p_{i_1 i_2} \cdots p_{i_{n-1} i_n} \tag{8.11}$$

by (8.8). The interest centers on these conditional probabilities, and the actual initial probabilities α_i are now largely irrelevant.

From (8.11) follows

$$\begin{aligned} P_i[X_1 = i_1, \ldots, X_m = i_m, X_{m+1} = j_1, \ldots, X_{m+n} = j_n] \\ = P_i[X_1 = i_1, \ldots, X_m = i_m] P_{i_m}[X_1 = j_1, \ldots, X_n = j_n]. \end{aligned} \tag{8.12}$$

Suppose that I is a set (finite or infinite) of m-long sequences of states, J is a set of n-long sequences of states, and every sequence in I ends in j. Adding both sides of (8.12) for $(i_1, \ldots, i_m)$ ranging over I and $(j_1, \ldots, j_n)$ ranging over J gives

$$\begin{aligned} P_i[(X_1, \ldots, X_m) \in I, (X_{m+1}, \ldots, X_{m+n}) \in J] \\ = P_i[(X_1, \ldots, X_m) \in I] P_j[(X_1, \ldots, X_n) \in J]. \end{aligned} \tag{8.13}$$

For this to hold it is essential that each sequence in I end in j. The formulas (8.12) and (8.13) are of central importance.

†For a different approach in the finite case, see Problem 8.1.

Transience and Persistence

Let

$$f_{ij}^{(n)} = P_i[X_1 \ne j, \ldots, X_{n-1} \ne j, X_n = j] \tag{8.14}$$

be the probability of a first visit to j at time n for a system that starts in i, and let

$$f_{ij} = P_i\left(\bigcup_{n=1}^{\infty} [X_n = j] \right) = \sum_{n=1}^{\infty} f_{ij}^{(n)} \tag{8.15}$$

be the probability of an eventual visit. A state i is *persistent* if a system starting at i is certain sometime to return to i: $f_{ii} = 1$. The state is *transient* in the opposite case: $f_{ii} < 1$.

Suppose that $n_1, \ldots, n_k$ are integers satisfying $1 \le n_1 < \cdots < n_k$ and consider the event that the system visits j at times $n_1 \ldots n_k$ but not in between; this event is determined by the conditions

$$\begin{array}{lll} X_1 \ne j, \ldots, & X_{n_1 - 1} \ne j, & X_{n_1} = j, \\ X_{n_1+1} \ne j, \ldots, & X_{n_2 - 1} \ne j, & X_{n_2} = j, \\ & \vdots & \\ X_{n_{k-1}+1} \ne j, \ldots, & X_{n_k - 1} \ne j, & X_{n_k} = j. \end{array}$$

Repeated application of (8.13) shows that under P_i the probability of this event is $f_{ij}^{(n_1)} f_{jj}^{(n_2 - n_1)} \cdots f_{jj}^{(n_k - n_{k-1})}$. Add this over the k-tuples $n_1, \ldots, n_k$: the P_i-probability that $X_n = j$ for at least k different values of n is $f_{ij} f_{jj}^{k-1}$. Letting $k \to \infty$ therefore gives

$$P_i[X_n = j \text{ i.o.}] = \begin{cases} 0 & \text{if } f_{jj} < 1, \\ f_{ij} & \text{if } f_{jj} = 1. \end{cases} \tag{8.16}$$

Recall that *i.o.* means *infinitely often*. Taking $i = j$ gives

$$P_i[X_n = i \text{ i.o.}] = \begin{cases} 0 & \text{if } f_{ii} < 1, \\ 1 & \text{if } f_{ii} = 1. \end{cases} \tag{8.17}$$

Thus $P_i[X_n = i \text{ i.o.}]$ is either 0 or 1; compare the zero–one law (Theorem 4.5), but note that the events $[X_n = i]$ here are not in general independent.†

†See Problem 8.35.

Theorem 8.2.

(i) *Transience of i is equivalent to $P_i[X_n = i \text{ i.o.}] = 0$ and to $\sum_n p_{ii}^{(n)} < \infty$.*

(ii) *Persistence of i is equivalent to $P_i[X_n = i \text{ i.o.}] = 1$ and to $\sum_n p_{ii}^{(n)} = \infty$.*

PROOF. By the first Borel–Cantelli lemma, $\sum_n p_{ii}^{(n)} < \infty$ implies $P_i[X_n = i \text{ i.o.}] = 0$, which by (8.17) in turn implies $f_{ii} < 1$. The entire theorem will be proved if it is shown that $f_{ii} < 1$ implies $\sum_n p_{ii}^{(n)} < \infty$.

The proof uses a first-passage argument: By (8.13),

$$\begin{aligned} p_{ij}^{(n)} = P_i[X_n = j] &= \sum_{s=0}^{n-1} P_i[X_1 \neq j, \ldots, X_{n-s-1} \neq j, X_{n-s} = j, X_n = j] \\ &= \sum_{s=0}^{n-1} P_i[X_1 \neq j, \ldots, X_{n-s-1} \neq j, X_{n-s} = j] P_j[X_s = j] \\ &= \sum_{s=0}^{n-1} f_{ij}^{(n-s)} p_{jj}^{(s)}. \end{aligned}$$

Therefore,

$$\begin{aligned} \sum_{t=1}^{n} p_{ii}^{(t)} &= \sum_{t=1}^{n} \sum_{s=0}^{t-1} f_{ii}^{(t-s)} p_{ii}^{(s)} \\ &= \sum_{s=0}^{n-1} p_{ii}^{(s)} \sum_{t=s+1}^{n} f_{ii}^{(t-s)} \le \sum_{s=0}^{n} p_{ii}^{(s)} f_{ii}. \end{aligned}$$

Thus $(1 - f_{ii}) \sum_{t=1}^{n} p_{ii}^{(t)} \le f_{ii}$, and if $f_{ii} < 1$, this puts a bound on the partial sums $\sum_{t=1}^{n} p_{ii}^{(t)}$. ∎

Example 8.6. Pólya's theorem. For the symmetric k-dimensional random walk (Example 8.4), all states are persistent if $k = 1$ or $k = 2$, and all states are transient if $k \ge 3$. To prove this, note first that the probability $p_{ii}^{(n)}$ of return in n steps is the same for all states i; denote this probability by $a_n^{(k)}$ to indicate the dependence on the dimension k. Clearly, $a_{2n+1}^{(k)} = 0$. Suppose that $k = 1$. Since return in $2n$ steps means n steps east and n steps west,

$$a_{2n}^{(1)} = \binom{2n}{n} \frac{1}{2^{2n}}.$$

By Stirling's formula, $a_{2n}^{(1)} \sim (\pi n)^{-1/2}$. Therefore, $\sum_n a_n^{(1)} = \infty$, and all states are persistent by Theorem 8.2.

In the plane, a return to the starting point in $2n$ steps means equal numbers of steps east and west as well as equal numbers north and south:

$$\begin{aligned} a_{2n}^{(2)} &= \sum_{u=0}^{n} \frac{(2n)!}{u!\,u!\,(n-u)!\,(n-u)!} \frac{1}{4^{2n}} \\ &= \frac{1}{4^{2n}} \binom{2n}{n} \sum_{u=0}^{n} \binom{n}{u} \binom{n}{n-u}. \end{aligned}$$

It can be seen on combinatorial grounds that the last sum is $\binom{2n}{n}$, and so $a_{2n}^{(2)} = (a_{2n}^{(1)})^2 \sim (\pi n)^{-1}$. Again, $\sum_n a_n^{(2)} = \infty$ and every state is persistent.

For three dimensions,

$$a_{2n}^{(3)} = \sum \frac{(2n)!}{u!\,u!\,v!\,v!\,(n-u-v)!\,(n-u-v)!} \frac{1}{6^{2n}},$$

the sum extending over nonnegative u and v satisfying $u + v \le n$. This reduces to

$$a_{2n}^{(3)} = \sum_{l=0}^{n} \binom{2n}{2l} \left(\frac{1}{3}\right)^{2n-2l} \left(\frac{2}{3}\right)^{2l} a_{2n-2l}^{(1)} a_{2l}^{(2)}, \tag{8.18}$$

as can be checked by substitution. (To see the probabilistic meaning of this formula, condition on there being $2n - 2l$ steps parallel to the vertical axis and $2l$ steps parallel to the horizontal plane.) It will be shown that $a_{2n}^{(3)} = O(n^{-3/2})$, which will imply that $\sum_n a_n^{(3)} < \infty$. The terms in (8.18) for $l = 0$ and $l = n$ are each $O(n^{-3/2})$ and hence can be omitted. Now $a_u^{(1)} \le K u^{-1/2}$ and $a_u^{(2)} \le K u^{-1}$, as already seen, and so the sum in question is at most

$$K^2 \sum_{l=1}^{n-1} \binom{2n}{2l} \left(\frac{1}{3}\right)^{2n-2l} \left(\frac{2}{3}\right)^{2l} (2n-2l)^{-1/2} (2l)^{-1}.$$

Since $(2n-2l)^{-1/2} \le 2n^{1/2}(2n-2l)^{-1} \le 4n^{1/2}(2n-2l+1)^{-1}$ and $(2l)^{-1} \le 2(2l+1)^{-1}$, this is at most a constant times

$$n^{1/2} \frac{(2n)!}{(2n+2)!} \sum_{l=1}^{n-1} \binom{2n+2}{2l-1} \left(\frac{1}{3}\right)^{2n-2l+1} \left(\frac{2}{3}\right)^{2l+1} = O(n^{-3/2}).$$

Thus $\sum_n a_n^{(3)} < \infty$, and the states are transient. The same is true for $k = 4, 5, \ldots$, since an inductive extension of the argument shows that $a_n^{(k)} = O(n^{-k/2})$. ∎

It is possible for a system starting in i to reach j ($f_{ij} > 0$) if and only if $p_{ij}^{(n)} > 0$ for some n. If this is true for all i and j, the Markov chain is *irreducible*.

Theorem 8.3. *If the Markov chain is irreducible, then one of the following two alternatives holds.*

(i) *All states are transient*, $P_i(\bigcup_j[X_n = j \text{ i.o.}]) = 0$ *for all* i, *and* $\sum_n p_{ij}^{(n)} < \infty$ *for all* i *and* j.

(ii) *All states are persistent*, $P_i(\bigcap_j[X_n = j \text{ i.o.}]) = 1$ *for all* i, *and* $\sum_n p_{ij}^{(n)} = \infty$ *for all* i *and* j.

The irreducible chain itself can accordingly be called persistent or transient. In the persistent case the system visits every state infinitely often. In the transient case it visits each state only finitely often, hence visits each finite set only finitely often, and so may be said to go to infinity.

PROOF. For each i and j there exist r and s such that $p_{ij}^{(r)} > 0$ and $p_{ji}^{(s)} > 0$. Now

$$p_{ii}^{(r+s+n)} \geq p_{ij}^{(r)} p_{jj}^{(n)} p_{ji}^{(s)}, \tag{8.19}$$

and from $p_{ij}^{(r)} p_{ji}^{(s)} > 0$ it follows that $\sum_n p_{ii}^{(n)} < \infty$ implies $\sum_n p_{jj}^{(n)} < \infty$: if one state is transient, they all are. In this case (8.16) gives $P_i[X_n = j \text{ i.o.}] = 0$ for all i and j, so that $P_i(\bigcup_j[X_n = j \text{ i.o.}]) = 0$ for all i. Since $\sum_{n=1}^{\infty} p_{ij}^{(n)} = \sum_{n=1}^{\infty}\sum_{\nu=1}^{n} f_{ij}^{(\nu)} p_{jj}^{(n-\nu)} = \sum_{\nu=1}^{\infty} f_{ij}^{(\nu)} \sum_{m=0}^{\infty} p_{jj}^{(m)} \leq \sum_{m=0}^{\infty} p_{jj}^{(m)}$, it follows that if j is transient, then (Theorem 8.2) $\sum_n p_{ij}^{(n)}$ converges for every i.

The other possibility is that all states are persistent. In this case $P_j[X_n = j \text{ i.o.}] = 1$ by Theorem 8.2, and it follows by (8.13) that

$$\begin{aligned} p_{ji}^{(m)} &= P_j([X_m = i] \cap [X_n = j \text{ i.o.}]) \\ &\leq \sum_{n>m} P_j[X_m = i, X_{m+1} \neq j, \ldots, X_{n-1} \neq j, X_n = j] \\ &= \sum_{n>m} p_{ji}^{(m)} f_{ij}^{(n-m)} = p_{ji}^{(m)} f_{ij}. \end{aligned}$$

There is an m for which $p_{ji}^{(m)} > 0$, and therefore $f_{ij} = 1$. By (8.16), $P_i[X_n = j \text{ i.o.}] = f_{ij} = 1$. If $\sum_n p_{ij}^{(n)}$ were to converge for some i and j, it would follow by the first Borel–Cantelli lemma that $P_i[X_n = j \text{ i.o.}] = 0$. ∎

Example 8.7. Since $\sum_j p_{ij}^{(n)} = 1$, the first alternative in Theorem 8.3 is impossible if S is finite: a finite, irreducible Markov chain is persistent. ∎

Example 8.8. The chain in Pólya's theorem is certainly irreducible. If the dimension is 1 or 2, there is probability 1 that a particle in symmetric random walk visits every state infinitely often. If the dimension is 3 or more, the particle goes to infinity. ∎

Example 8.9. Consider the unrestricted random walk on the line (Example 8.3). According to the ruin calculation (7.8), $f_{01} = p/q$ for $p < q$. Since the chain is irreducible, all states are transient. By symmetry, of course, the chain is also transient if $p > q$, although in this case (7.8) gives $f_{01} = 1$. Thus $f_{ij} = 1$ $(i \neq j)$ is possible in the transient case.[†]

If $p = q = \frac{1}{2}$, the chain is persistent by Pólya's theorem. If n and $j - i$ have the same parity,

$$p_{ij}^{(n)} = \binom{n}{\frac{n+j-i}{2}} \frac{1}{2^n}, \qquad |j-i| \leq n.$$

This is maximal if $j = i$ or $j = i \pm 1$, and by Stirling's formula the maximal value is of order $n^{-1/2}$. Therefore, $\lim_n p_{ij}^{(n)} = 0$, which always holds in the transient case but is thus possible in the persistent case as well (see Theorem 8.8). ∎

Another Criterion for Persistence

Let $Q = [q_{ij}]$ be a matrix with rows and columns indexed by the elements of a finite or countable set U. Suppose it is *substochastic* in the sense that $q_{ij} \geq 0$ and $\sum_j q_{ij} \leq 1$. Let $Q^n = [q_{ij}^{(n)}]$ be the nth power, so that

$$q_{ij}^{(n+1)} = \sum_\nu q_{i\nu} q_{\nu j}^{(n)}, \qquad q_{ij}^{(0)} = \delta_{ij}. \tag{8.20}$$

Consider the row sums

$$\sigma_i^{(n)} = \sum_j q_{ij}^{(n)}. \tag{8.21}$$

From (8.20) follows

$$\sigma_i^{(n+1)} = \sum_j q_{ij} \sigma_j^{(n)}. \tag{8.22}$$

Since Q is substochastic $\sigma_i^{(1)} \leq 1$, and hence $\sigma_i^{(n+1)} = \sum_j \sum_\nu q_{i\nu}^{(n)} q_{\nu j} = \sum_\nu q_{i\nu}^{(n)} \sigma_\nu^{(1)} \leq \sigma_i^{(n)}$. Therefore, the monotone limits

$$\sigma_i = \lim_n \sum_j q_{ij}^{(n)} \tag{8.23}$$

[†] But for each j there must be some $i \neq j$ for which $f_{ij} < 1$; see Problem 8.7.

exist. By (8.22) and the Weierstrass M-test [A28], $\sigma_i = \sum_j q_{ij}\sigma_j$. Thus the σ_i solve the system

$$(8.24)\qquad \begin{cases} x_i = \sum_{j \in U} q_{ij} x_j, & i \in U, \\ 0 \le x_i \le 1, & i \in U. \end{cases}$$

For an arbitrary solution, $x_i = \sum_j q_{ij} x_j \le \sum_j q_{ij} = \sigma_i^{(1)}$, and $x_i \le \sigma_i^{(n)}$ for all i implies $x_i \le \sum_j q_{ij}\sigma_j^{(n)} = \sigma_i^{(n+1)}$ by (8.22). Thus $x_i \le \sigma_i^{(n)}$ for all n by induction, and so $x_i \le \sigma_i$. Thus the σ_i give the *maximal* solution to (8.24):

Lemma 1. *For a substochastic matrix Q the limits* (8.23) *are the maximal solution of* (8.24).

Now suppose that U is a subset of the state space S. The p_{ij} for i and j in U give a substochastic matrix Q. The row sums (8.21) are $\sigma_i^{(n)} = \sum p_{ij_1} p_{j_1 j_2} \cdots p_{j_{n-1} j_n}$, where the $j_1, \ldots, j_n$ range over U, and so $\sigma_i^{(n)} = P_i[X_t \in U, t \le n]$. Let $n \to \infty$:

$$(8.25)\qquad \sigma_i = P_i[X_t \in U, t = 1, 2 \ldots], \qquad i \in U.$$

In this case, σ_i is thus the probability that the system remains forever in U, given that it starts at i. The following theorem is now an immediate consequence of Lemma 1.

Theorem 8.4. *For $U \subset S$ the probabilities* (8.25) *are the maximal solution of the system*

$$(8.26)\qquad \begin{cases} x_i = \sum_{j \in U} p_{ij} x_j, & i \in U, \\ 0 \le x_i \le 1, & i \in U. \end{cases}$$

The constraint $x_i \ge 0$ in (8.26) is in a sense redundant: Since $x_i \equiv 0$ is a solution, the maximal solution is automatically nonnegative (and similarly for (8.24)). And the maximal solution is $x_i \equiv 1$ if and only if $\sum_{j \in U} p_{ij} = 1$ for all i in U, which makes probabilistic sense.

Example 8.10. For the random walk on the line consider the set $U = \{0, 1, 2, \ldots\}$. The System (8.26) is

$$\begin{aligned} x_i &= p x_{i+1} + q x_{i-1}, \qquad i \ge 1, \\ x_0 &= p x_1. \end{aligned}$$

It follows [A19] that $x_n = A + An$ if $p = q$ and $x_n = A - A(q/p)^{n+1}$ if $p \ne q$. The only bounded solution is $x_n \equiv 0$ if $q \ge p$, and in this case there is probability 0 of staying forever among the nonnegative integers. If $q < p$, $A = 1$ gives the maximal solution $x_n = 1 - (q/p)^{n+1}$ (and $0 \le A < 1$ gives exactly the solutions that are not maximal). Compare (7.8) and Example 8.9. ∎

Now consider the system (8.26) with $U = S - \{i_0\}$ for an arbitrary single state i_0:

$$(8.27)\qquad \begin{cases} x_i = \sum_{j \ne i_0} p_{ij} x_j, & i \ne i_0, \\ 0 \le x_i \le 1, & i \ne i_0. \end{cases}$$

There is always the trivial solution—the one for which $x_i \equiv 0$.

Theorem 8.5. *An irreducible chain is transient if and only if* (8.27) *has a nontrivial solution.*

PROOF. The probabilities

$$(8.28)\qquad 1 - f_{ii_0} = P_i[X_n \ne i_0, n \ge 1], \qquad i \ne i_0,$$

are by Theorem 8.4 the maximal solution of (8.27). Therefore (8.27) has a nontrivial solution if and only if $f_{ii_0} < 1$ for some $i \ne i_0$. If the chain is persistent, this is impossible by Theorem 8.3(ii).

Suppose the chain is transient. Since

$$\begin{aligned} f_{i_0 i_0} &= P_{i_0}[X_1 = i_0] + \sum_{n=2}^{\infty} \sum_{i \ne i_0} P_{i_0}[X_1 = i, X_2 \ne i_0, \ldots, X_{n-1} \ne i_0, X_n = i_0] \\ &= p_{i_0 i_0} + \sum_{i \ne i_0} p_{i_0 i} f_{ii_0}, \end{aligned}$$

and since $f_{i_0 i_0} < 1$, it follows that $f_{ii_0} < 1$ for some $i \ne i_0$. ∎

Since the equations in (8.27) are homogeneous, the issue is whether they have a solution that is nonnegative, nontrivial, and *bounded*. If they do, $0 \le x_i \le 1$ can be arranged by rescaling.†

†See Problem 8.9.

Example 8.11. In the simplest of *queueing models* the state space is $\{0, 1, 2, \ldots\}$ and the transition matrix has the form

$$\begin{bmatrix} t_0 & t_1 & t_2 & 0 & 0 & 0 & \cdots \\ t_0 & t_1 & t_2 & 0 & 0 & 0 & \cdots \\ 0 & t_0 & t_1 & t_2 & 0 & 0 & \cdots \\ 0 & 0 & t_0 & t_1 & t_2 & 0 & \cdots \\ 0 & 0 & 0 & t_0 & t_1 & t_2 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{bmatrix}$$

If there are i customers in the queue and $i \geq 1$, the customer at the head of the queue is served and leaves, and then 0, 1, or 2 new customers arrive (probabilities t_0, t_1, t_2), which leaves a queue of length $i-1$, i, or $i+1$. If $i = 0$, no one is served, and the new customers bring the queue length to 0, 1, or 2. Assume that t_0 and t_2 are positive, so that the chain is irreducible.

For $i_0 = 0$ the system (8.27) is

$$\begin{aligned} x_1 &= t_1 x_1 + t_2 x_2, \\ x_k &= t_0 x_{k-1} + t_1 x_k + t_2 x_{k+1}, \qquad k \geq 2. \end{aligned} \tag{8.29}$$

Since t_0, t_1, t_2 have the form $q(1-t), t, p(1-t)$ for appropriate p, q, t, the second line of (8.29) has the form $x_k = p x_{k+1} + q x_{k-1}$, $k \geq 2$. Now the solution [A19] is $A + B(q/p)^k = A + B(t_0/t_2)^k$ if $t_0 \neq t_2$ $(p \neq q)$ and $A + Bk$ if $t_0 = t_2$ $(p = q)$, and A can be expressed in terms of B because of the first equation in (8.29). The result is

$$x_k = \begin{cases} B\big((t_0/t_2)^k - 1\big) & \text{if } t_0 \neq t_2, \\ Bk & \text{if } t_0 = t_2. \end{cases}$$

There is a nontrivial solution if $t_0 < t_2$ but not if $t_0 \geq t_2$.

If $t_0 < t_2$, the chain is thus transient, and the queue size goes to infinity with proability 1. If $t_0 \geq t_2$, the chain is persistent. For a nonempty queue the expected increase in queue length in one step is $t_2 - t_0$, and the queue goes out of control if and only if this is positive. ■

Stationary Distributions

Suppose that the chain has initial probabilities π_i satisfying

$$\sum_{i \in S} \pi_i p_{ij} = \pi_j, \qquad j \in S. \tag{8.30}$$

It then follows by induction that

$$\sum_{i \in S} \pi_i p_{ij}^{(n)} = \pi_j, \qquad j \in S, \quad n = 0, 1, 2, \ldots. \tag{8.31}$$

If π_i is the probability that $X_0 = i$, then the left side of (8.31) is the probability that $X_n = j$, and thus (8.30) implies that the probability of $[X_n = j]$ is the same for all n. A set of probabilities satisfying (8.30) is for this reason called a *stationary distribution*. The existence of such a distribution implies that the chain is very stable.

To discuss this requires the notion of periodicity. The state j has *period* t if $p_{jj}^{(n)} > 0$ implies that t divides n and if t is the largest integer with this property. In other words, the period of j is the greatest common divisor of the set of integers

$$\big[n: n \geq 1, \, p_{jj}^{(n)} > 0\big]. \tag{8.32}$$

If the chain is irreducible, then for each pair i and j there exist r and s for which $p_{ij}^{(r)}$ and $p_{ji}^{(s)}$ are positive, and of course

$$p_{ii}^{(r+s+n)} \geq p_{ij}^{(r)} p_{jj}^{(n)} p_{ji}^{(s)}. \tag{8.33}$$

Let t_i and t_j be the periods of i and j. Taking $n = 0$ in this inequality shows that t_i divides $r+s$; and now it follows by the inequality that $p_{jj}^{(n)} > 0$ implies that t_i divides $r+s+n$ and hence divides n. Thus t_i divides each integer in the set (8.32), and so $t_i \leq t_j$. Since i and j can be interchanged in this argument, i and j have the same period. One can thus speak of the period of the chain itself in the irreducible case. The random walk on the line has period 2, for example. If the period is 1, the chain is *aperiodic*

Lemma 2. *In an irreducible, aperiodic chain, for each i and j, $p_{ij}^{(n)} > 0$ for all n exceeding some $n_0(i,j)$.*

PROOF. Since $p_{jj}^{(m+n)} \geq p_{jj}^{(m)} p_{jj}^{(n)}$, if M is the set (8.32) then $m \in M$ and $n \in M$ together imply $m + n \in M$. But it is a fact of number theory [A21] that if a set of positive integers is closed under addition and has greatest common divisor 1, then it contains all integers exceeding some n_1. Given i and j, choose r so that $p_{ij}^{(r)} > 0$. If $n > n_0 = n_1 + r$, then $p_{ij}^{(n)} \geq p_{ij}^{(r)} p_{jj}^{(n-r)} > 0$. ■

Theorem 8.6. *Suppose of an irreducible, aperiodic chain that there exists a stationary distribution—a solution of (8.30) satisfying $\pi_i \geq 0$ and $\sum_i \pi_i = 1$. Then the chain is persistent,*

$$\lim_n p_{ij}^{(n)} = \pi_j \tag{8.34}$$

for all i and j, the π_j are all positive, and the stationary distribution is unique.

The main point of the conclusion is that the effect of the initial state wears off. Whatever the actual initial distribution $\{\alpha_i\}$ of the chain may be, if (8.34) holds, then it follows by the M-test that the probability $\sum_i \alpha_i p_{ij}^{(n)}$ of $[X_n = j]$ converges to π_j.

PROOF. If the chain is transient, then $p_{ij}^{(n)} \to 0$ for all i and j by Theorem 8.3, and it follows by (8.31) and the M-test that π_j is identically 0, which contradicts $\sum_i \pi_i = 1$. The existence of a stationary distribution therefore implies that the chain is persistent.

Consider now a Markov chain with state space $S \times S$ and transition probabilities $p(ij, kl) = p_{ik} p_{jl}$ (it is easy to verify that these form a stochastic matrix). Call this the *coupled* chain; it describes the joint behavior of a pair of independent systems, each evolving according to the laws of the original Markov chain. By Theorem 8.1 there exists a Markov chain (X_n, Y_n), $n = 0, 1, \ldots,$ having positive initial probabilities and transition probabilities

$$P[(X_{n+1}, Y_{n+1}) = (k,l) \| (X_n, Y_n) = (i,j)] = p(ij, kl).$$

For n exceeding some n_0 depending on i, j, k, l, the probability $p^{(n)}(ij, kl) = p_{ik}^{(n)} p_{jl}^{(n)}$ is positive by Lemma 2. Therefore, the coupled chain is *irreducible*. (This proof that the coupled chain is irreducible requires only the assumptions that the original chain is irreducible and aperiodic, a fact needed again in the proof of Theorem 8.7.)

It is easy to check that $\pi(ij) = \pi_i \pi_j$ forms a set of stationary initial probabilities for the coupled chain, which, like the original one, must therefore be *persistent*. It follows that, for an arbitrary initial state (i,j) for the chain $\{(X_n, Y_n)\}$ and an arbitrary i_0 in S, one has $P_{ij}[(X_n, Y_n) = (i_0, i_0)$ i.o.$] = 1$. If τ is the smallest integer such that $X_\tau = Y_\tau = i_0$, then τ is finite with probability 1 under P_{ij}. The idea of the proof is now this: X_n starts in i and Y_n starts in j; once $X_n = Y_n = i_0$ occurs, X_n and Y_n follow identical probability laws, and hence the initial states i and j will lose their influence.

By (8.13) applied to the coupled chain, if $m \le n$, then

$$\begin{aligned} P_{ij}[(X_n, Y_n) = (k,l), \tau = m] &= P_{ij}[(X_t, Y_t) \ne (i_0, i_0), t < m, (X_m, Y_m) = (i_0, i_0)] \\ &\quad \times P_{i_0 i_0}[(X_{n-m}, Y_{n-m}) = (k,l)] \\ &= P_{ij}[\tau = m] p_{i_0 k}^{(n-m)} p_{i_0 l}^{(n-m)}. \end{aligned}$$

Adding out l gives $P_{ij}[X_n = k, \tau = m] = P_{ij}[\tau = m] p_{i_0 k}^{(n-m)}$, and adding out k gives $P_{ij}[Y_n = l, \tau = m] = P_{ij}[\tau = m] p_{i_0 l}^{(n-m)}$. Take $k = l$, equate probabilities, and add over $m = 1, \ldots, n$:

$$P_{ij}[X_n = k, \tau \le n] = P_{ij}[Y_n = k, \tau \le n].$$

From this follows

$$\begin{aligned} P_{ij}[X_n = k] &\le P_{ij}[X_n = k, \tau \le n] + P_{ij}[\tau > n] \\ &= P_{ij}[Y_n = k, \tau \le n] + P_{ij}[\tau > n] \\ &\le P_{ij}[Y_n = k] + P_{ij}[\tau > n]. \end{aligned}$$

This and the same inequality with X and Y interchanged give

$$|p_{ik}^{(n)} - p_{jk}^{(n)}| = |P_{ij}[X_n = k] - P_{ij}[Y_n = k]| \le P_{ij}[\tau > n].$$

Since τ is finite with probability 1,

$$\lim_n |p_{ik}^{(n)} - p_{jk}^{(n)}| = 0. \tag{8.35}$$

(This proof of (8.35) goes through as long as the coupled chain is irreducible and persistent— no assumptions on the original chain are needed. This fact is used in the proof of the next theorem.)

By (8.31), $\pi_k - p_{jk}^{(n)} = \sum_i \pi_i (p_{ik}^{(n)} - p_{jk}^{(n)})$, and this goes to 0 by the M-test if (8.35) holds. Thus $\lim_n p_{jk}^{(n)} = \pi_k$. As this holds for each stationary distribution, there can be only one of them.

It remains to show that the π_j are all strictly positive. Choose r and s so that $p_{ij}^{(r)}$ and $p_{ji}^{(s)}$ are positive. Letting $n \to \infty$ in (8.33) shows that π_i is positive if π_j is; since some π_j is positive (they add to 1), all the π_i must be positive. ■

Example 8.12. For the queueing model in Example 8.11 the equations (8.30) are

$$\begin{aligned} \pi_0 &= \pi_0 t_0 + \pi_1 t_0, \\ \pi_1 &= \pi_0 t_1 + \pi_1 t_1 + \pi_2 t_0, \\ \pi_2 &= \pi_0 t_2 + \pi_1 t_2 + \pi_2 t_1 + \pi_3 t_0, \\ \pi_k &= \pi_{k-1} t_2 + \pi_k t_1 + \pi_{k+1} t_0, \qquad k \ge 3. \end{aligned}$$

Again write t_0, t_1, t_2, as $q(1-t), t, p(1-t)$. Since the last equation here is $\pi_k = q\pi_{k+1} + p\pi_{k-1}$, the solution is

$$\pi_k = \begin{cases} A + B(p/q)^k = A + B(t_2/t_0)^k & \text{if } t_0 \ne t_2, \\ A + Bk & \text{if } t_0 = t_2 \end{cases}$$

for $k \ge 2$. If $t_0 < t_2$ and $\sum \pi_k$ converges, then $\pi_k \equiv 0$, and hence there is no stationary distribution; but this is not new, because it was shown in Example 8.11 that the chain is transient in this case. If $t_0 = t_2$, there is again no stationary distribution, and this is new because the chain was in Example 8.11 shown to be persistent in this case.

If $t_0 > t_2$, then $\sum \pi_k$ converges, provided $A = 0$. Solving for π_0 and π_1 in the first two equations of the system above gives $\pi_0 = Bt_2$ and $\pi_1 = Bt_2(1 - t_0)/t_0$. From $\sum_k \pi_k = 1$ it now follows that $B = (t_0 - t_2)/t_2$, and the π_k can be written down explicitly. Since $\pi_k = B(t_2/t_0)^k$ for $k \geq 2$, there is small chance of a large queue length. ∎

If $t_0 = t_2$ in this queueing model, the chain is persistent (Example 8.11) but has no stationary distribution (Example 8.12). The next theorem describes the asymptotic behavior of the $p_{ij}^{(n)}$ in this case.

Theorem 8.7. *If an irreducible, aperiodic chain has no stationary distribution, then*

$$\lim_n p_{ij}^{(n)} = 0 \tag{8.36}$$

for all i and j.

If the chain is transient, (8.36) follows from Theorem 8.3. What is interesting here is the persistent case.

PROOF. By the argument in the proof of Theorem 8.6, the coupled chain is irreducible. If it is transient, then $\sum_n (p_{ij}^{(n)})^2$ converges by Theorem 8.2, and the conclusion follows.

Suppose, on the other hand, that the coupled chain is (irreducible and) persistent. Then the stopping-time argument leading to (8.35) goes through as before. If the $p_{ij}^{(n)}$ do not all go to 0, then there is an increasing sequence $\{n_u\}$ of integers along which some $p_{ij}^{(n)}$ is bounded away from 0. By the diagonal method [A14], it is possible by passing to a subsequence of $\{n_u\}$ to ensure that each $p_{ij}^{(n_u)}$ converges to a limit, which by (8.35) must be independent of i. Therefore, there is a sequence $\{n_u\}$ such that $\lim_u p_{ij}^{(n_u)} = t_j$ exists for all i and j, where t_j is nonnegative for all j and positive for some j. If M is a finite set of states, then $\sum_{j \in M} t_j = \lim_u \sum_{j \in M} p_{ij}^{(n_u)} \leq 1$, and hence $0 < t = \sum_j t_j \leq 1$. Now $\sum_{k \in M} p_{ik}^{(n_u)} p_{kj} \leq p_{ij}^{(n_u + 1)} = \sum_k p_{ik} p_{kj}^{(n_u)}$; it is possible to pass to the limit ($u \to \infty$) inside the first sum (if M is finite) and inside the second sum (by the M-test), and hence $\sum_{k \in M} t_k p_{kj} \leq \sum_k p_{ik} t_j = t_j$. Therefore, $\sum_k t_k p_{kj} \leq t_j$; if one of these inequalities were strict, it would follow that $\sum_k t_k = \sum_j \sum_k t_k p_{kj} < \sum_j t_j$, which is impossible. Therefore $\sum_k t_k p_{kj} = t_j$ for all j, and the ratios $\pi_j = t_j/t$ give a stationary distribution, contrary to the hypothesis. ∎

The limits in (8.34) and (8.36) can be described in terms of mean return times. Let

$$\mu_j = \sum_{n=1}^{\infty} n f_{jj}^{(n)}; \tag{8.37}$$

if the series diverges, write $\mu_j = \infty$. In the persistent case, this sum is to be thought of as the average number of steps to first return to j, given that $X_0 = j$.†

Lemma 3. *Suppose that j is persistent and that* $\lim_n p_{jj}^{(n)} = u$. *Then* $u > 0$ *if and only if* $\mu_j < \infty$, *in which case* $u = 1/\mu_j$.

Under the convention that $0 = 1/\infty$, the case $u = 0$ and $\mu_j = \infty$ is consistent with the equation $u = 1/\mu_j$.

PROOF. For $k \geq 0$ let $\rho_k = \sum_{n > k} f_{jj}^{(n)}$; the notation does not show the dependence on j, which is fixed. Consider the double series

$$\begin{array}{l} f_{jj}^{(1)} + f_{jj}^{(2)} + f_{jj}^{(3)} + \cdots \\ \qquad\; + f_{jj}^{(2)} + f_{jj}^{(3)} + \cdots \\ \qquad\qquad\quad + f_{jj}^{(3)} + \cdots \\ \qquad\qquad\qquad\qquad + \cdots \end{array}$$

The kth row sums to ρ_k ($k \geq 0$) and the nth column sums to $n f_{jj}^{(n)}$ ($n \geq 1$), and so [A27] the series in (8.37) converges if and only if $\sum_k \rho_k$ does, in which case

$$\mu_j = \sum_{k=0}^{\infty} \rho_k. \tag{8.38}$$

Since j is persistent, the P_j-probability that the system does not hit j up to time n is the probability that it hits j after time n, and this is ρ_n. Therefore,

$$\begin{aligned} 1 - p_{jj}^{(n)} &= P_j[X_n \neq j] \\ &= P_j[X_1 \neq j, \ldots, X_n \neq j] + \sum_{k=1}^{n-1} P_j[X_k = j, X_{k+1} \neq j, \ldots, X_n \neq j] \\ &= \rho_n + \sum_{k=1}^{n-1} p_{jj}^{(k)} \rho_{n-k}, \end{aligned}$$

and since $\rho_0 = 1$,

$$1 = \rho_0 p_{jj}^{(n)} + \rho_1 p_{jj}^{(n-1)} + \cdots + \rho_{n-1} p_{jj}^{(1)} + \rho_n p_{jj}^{(0)}.$$

Keep only the first $k + 1$ terms on the right here, and let $n \to \infty$; the result is $1 \geq (\rho_0 + \cdots + \rho_k)u$. Therefore $u > 0$ implies that $\sum_k \rho_k$ converges, so that $\mu_j < \infty$.

†Since in general there is no upper bound to the number of steps to first return, it is not a simple random variable. It does come under the general theory in Chapter 4, and its expected value is indeed μ_j (and (8.38) is just (5.29)), but for the present the interpretation of μ_j as an average is informal. See Problem 23.11.

Write $x_{nk} = \rho_k p_{jj}^{(n-k)}$ for $0 \le k \le n$ and $x_{nk} = 0$ for $n < k$. Then $0 \le x_{nk} \le \rho_k$ and $\lim_n x_{nk} = \rho_k u$. If $\mu_j < \infty$, then $\sum_k \rho_k$ converges and it follows by the M-test that $1 = \sum_{k=0}^{\infty} x_{nk} \to \sum_{k=0}^{\infty} \rho_k u$. By (8.38), $1 = \mu_j u$, so that $u > 0$ and $u = 1/\mu_j$. ■

The law of large numbers bears on the relation $u = 1/\mu_j$ in the persistent case. Let V_n be the number of visits to state j up to time n. If the time from one visit to the next is about μ_j, then V_n should be about n/μ_j: $V_n/n \approx 1/\mu_j$. But (if $X_0 = j$) V_n/n has expected value $n^{-1}\sum_{k=1}^{n} p_{jj}^{(k)}$, which goes to u under the hypothesis of Lemma 3 [A30].

Consider an irreducible, aperiodic, persistent chain. There are two possibilities. If there is a stationary distribution, then the limits (8.34) are positive, and the chain is called *positive persistent*. It then follows by Lemma 3 that $\mu_j < \infty$ and $\pi_j = 1/\mu_j$ for all j. In this case, it is not actually necessary to assume persistence, since this follows from the existence of a stationary distribution. On the other hand, if the chain has no stationary distribution, then the limits (8.36) are all 0, and the chain is called *null persistent*. It then follows by Lemma 3 that $\mu_j = \infty$ for all j. This, taken together with Theorem 8.3, provides a complete classification:

Theorem 8.8. *For an irreducible, aperiodic chain there are three possibilities:*

(i) *The chain is transient; then for all i and j, $\lim_n p_{ij}^{(n)} = 0$ and in fact $\sum_n p_{ij}^{(n)} < \infty$.*

(ii) *The chain is persistent but there exists no stationary distribution (the null persistent case); then for all i and j, $p_{ij}^{(n)}$ goes to 0 but so slowly that $\sum_n p_{ij}^{(n)} = \infty$, and $\mu_j = \infty$.*

(iii) *There exist stationary probabilities π_j and (hence) the chain is persistent (the positive persistent case); then for all i and j, $\lim_n p_{ij}^{(n)} = \pi_j > 0$ and $\mu_j = 1/\pi_j < \infty$.*

Since the asymptotic properties of the $p_{ij}^{(n)}$ are distinct in the three cases, these asymptotic properties in fact characterize the three cases.

Example 8.13. Suppose that the states are $0, 1, 2, \ldots$ and the transition matrix is

$$\begin{bmatrix} q_0 & p_0 & 0 & 0 & \cdots \\ q_1 & 0 & p_1 & 0 & \cdots \\ q_2 & 0 & 0 & p_2 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \end{bmatrix}$$

where p_i and q_i are positive. The state i represents the length of a success run, the conditional chance of a further success being p_i. Clearly the chain is irreducible and aperiodic.

A solution of the system (8.27) for testing for transience (with $i_0 = 0$) must have the form $x_k = x_1/p_1 \cdots p_{k-1}$. Hence there is a bounded, nontrivial solution, and the chain is transient, if and only if the limit α of $p_0 \cdots p_n$ is positive. But the chance of no return to 0 (for initial state 0) in n steps is clearly $p_0 \cdots p_{n-1}$; hence $f_{00} = 1 - \alpha$, which checks: the chain is persistent if and only if $\alpha = 0$.

Every solution of the steady-state equations (8.30) has the form $\pi_k = \pi_0 p_0 \cdots p_{k-1}$. Hence there is a stationary distribution if and only if $\sum_k p_0 \cdots p_k$ converges; this is the positive persistent case. The null persistent case is that in which $p_0 \cdots p_k \to 0$ but $\sum_k p_0 \cdots p_k$ diverges (which happens, for example, if $q_k = 1/k$ for $k > 1$).

Since the chance of no return to 0 in n steps is $p_0 \cdots p_{n-1}$, in the persistent case (8.38) gives $\mu_0 = \sum_{k=0}^{\infty} p_0 \cdots p_{k-1}$. In the null persistent case this checks with $\mu_0 = \infty$; in the positive persistent case it gives $\mu_0 = \sum_{k=0}^{\infty} \pi_k/\pi_0 = 1/\pi_0$, which again is consistent. ■

Example 8.14. Since $\sum_j p_{ij}^{(n)} = 1$, possibilities (i) and (ii) in Theorem 8.8 are impossible in the finite case: A finite, irreducible, aperiodic Markov chain has a stationary distribution. ■

Exponential Convergence*

In the finite case, $p_{ij}^{(n)}$ converges to π_j at an exponential rate:

Theorem 8.9. *If the state space is finite and the chain is irreducible and aperiodic, then there is a stationary distribution $\{\pi_i\}$, and*

$$\left| p_{ij}^{(n)} - \pi_j \right| \le A\rho^n,$$

where $A \ge 0$ and $0 \le \rho < 1$.

PROOF.† Let $m_j^{(n)} = \min_i p_{ij}^{(n)}$ and $M_j^{(n)} = \max_i p_{ij}^{(n)}$. By (8.10),

$$m_j^{(n+1)} = \min_i \sum_\nu p_{i\nu} p_{\nu j}^{(n)} \ge \min_i \sum_\nu p_{i\nu} m_j^{(n)} = m_j^{(n)},$$

$$M_j^{(n+1)} = \max_i \sum_\nu p_{i\nu} p_{\nu j}^{(n)} \le \max_i \sum_\nu p_{i\nu} M_j^{(n)} = M_j^{(n)}.$$

* This topic may be omitted.

† For other proofs, see Problems 8.18 and 8.27.

Since obviously $m_j^{(n)} \le M_j^{(n)}$,

$$0 \le m_j^{(1)} \le m_j^{(2)} \le \cdots \le M_j^{(2)} \le M_j^{(1)} \le 1. \tag{8.39}$$

Suppose temporarily that all the p_{ij} are positive. Let s be the number of states and let $\delta = \min_{ij} p_{ij}$. From $\Sigma_j p_{ij} \ge s\delta$ follows $0 < \delta \le s^{-1}$. Fix states u and v for the moment; let Σ' denote the summation over j in S satisfying $p_{uj} \ge p_{vj}$ and let Σ'' denote summation over j satisfying $p_{uj} < p_{vj}$. Then

$$\sum{}'(p_{uj} - p_{vj}) + \sum{}''(p_{uj} - p_{vj}) = 1 - 1 = 0. \tag{8.40}$$

Since $\Sigma' p_{vj} + \Sigma'' p_{uj} \ge s\delta$.

$$\sum{}'(p_{uj} - p_{vj}) = 1 - \sum{}'' p_{uj} - \sum{}' p_{vj} \le 1 - s\delta. \tag{8.41}$$

Apply (8.40) and then (8.41):

$$\begin{aligned} p_{uk}^{(n+1)} - p_{vk}^{(n+1)} &= \sum_j (p_{uj} - p_{vj}) p_{jk}^{(n)} \\ &\le \sum{}'(p_{uj} - p_{vj}) M_k^{(n)} + \sum{}''(p_{uj} - p_{vj}) m_k^{(n)} \\ &= \sum{}'(p_{uj} - p_{vj})\left(M_k^{(n)} - m_k^{(n)}\right) \\ &\le (1 - s\delta)\left(M_k^{(n)} - m_k^{(n)}\right). \end{aligned}$$

Since u and v are arbitrary,

$$M_k^{(n+1)} - m_k^{(n+1)} \le (1 - s\delta)\left(M_k^{(n)} - m_k^{(n)}\right).$$

Therefore, $M_k^{(n)} - m_k^{(n)} \le (1 - s\delta)^n$. It follows by (8.39) that $m_j^{(n)}$ and $M_j^{(n)}$ have a common limit π_j and that

$$\left| p_{ij}^{(n)} - \pi_j \right| \le (1 - s\delta)^n. \tag{8.42}$$

Take $A = 1$ and $\rho = 1 - s\delta$. Passing to the limit in $\Sigma_i p_{vi}^{(n)} p_{ij} = p_{vj}^{(n+1)}$ shows that the π_i are stationary probabilities. (Note that the proof thus far makes almost no use of the preceding theory.)

If the p_{ij} are not all positive, apply Lemma 2: Since there are only finitely many states, there exists an m such that $p_{ij}^{(m)} > 0$ for all i and j. By the case just treated, $M_j^{(mt)} - m_j^{(mt)} \le \rho^t$. Take $A = \rho^{-1}$ and then replace ρ by $\rho^{1/m}$. ■

Example 8.15. Suppose that

$$P = \begin{bmatrix} p_0 & p_1 & \cdots & p_{s-1} \\ p_{s-1} & p_0 & \cdots & p_{s-2} \\ \cdots & \cdots & \cdots & \cdots \\ p_1 & p_2 & \cdots & p_0 \end{bmatrix}.$$

The rows of P are the cyclic permutations of the first row: $p_{ij} = p_{j-i}$, $j - i$ reduced modulo s. Since the columns of P add to 1 as well as the rows, the steady-state equations (8.30) have the solution $\pi_i \equiv s^{-1}$. If the p_i are all positive, the theorem implies that $p_{ij}^{(n)}$ converges to s^{-1} at an exponential rate. If $X_0, Y_1, Y_2, \ldots$ are independent random variables with range $\{0, 1, \ldots, s-1\}$, if each Y_n has distribution $\{p_0, \ldots, p_{s-1}\}$, and if $X_n = X_0 + Y_1 + \cdots + Y_n$, where the sum is reduced modulo s, then $P[X_n = j] \to s^{-1}$. The X_n describe a random walk on a circle of points, and whatever the initial distribution, the positions become equally likely in the limit. ■

Optimal Stopping*

Assume throughout the rest of the section that S *is finite*. Consider a function τ on Ω for which $\tau(\omega)$ is a nonnegative integer for each ω. Let $\mathscr{F}_n = \sigma(X_0, X_1, \ldots, X_n)$; τ is a *stopping time* or a *Markov time* if

$$[\omega : \tau(\omega) = n] \in \mathscr{F}_n \tag{8.43}$$

for $n = 0, 1, \ldots$. This is analogous to the condition (7.18) on the gambler's stopping time. It will be necessary to allow $\tau(\omega)$ to assume the special value ∞, but only on a set of probability 0. This has no effect on the requirement (8.43), which concerns finite n only.

If f is a real function on the state space, then $f(X_0), f(X_1), \ldots$ are simple random variables. Imagine an observer who follows the successive states $X_0, X_1, \ldots$ of the system. He stops at time τ, when the state is X_τ (or $X_{\tau(\omega)}(\omega)$), and receives an reward or payoff $f(X_\tau)$. The condition (8.43) prevents prevision on the part of the observer. This is a kind of game, the stopping time is a strategy, and the problem is to find a strategy that maximizes the expected payoff $E[f(X_\tau)]$. The problem in Example 8.5 had this form; there $S = \{1, 2, \ldots, r+1\}$, and the payoff function is $f(i) = i/r$ for $i \le r$ (set $f(r+1) = 0$).

If $P(A) > 0$ and $Y = \Sigma_j y_j I_{B_j}$ is a simple random variable, the B_j forming a finite decomposition of Ω into $\mathscr{F}$-sets, the conditional expected value of Y

*This topic may be omitted.

given A is defined by

$$E[Y|A] = \sum_j y_i P(B_j|A).$$

Denote by E_i conditional expected values for the case $A = [X_0 = i]$:

$$E_i[Y] = E[Y|X_0 = i] = \sum_j y_j P_i(B_j).$$

The stopping-time problem is to choose τ so as to maximize simultaneously $E_i[f(X_\tau)]$ for all initial states i. If x lies in the range of f, which is finite, and if τ is everywhere finite, then $[\omega: f(X_{\tau(\omega)}(\omega)) = x] = \bigcup_{n=0}^{\infty}[\omega: \tau(\omega) = n, f(X_n(\omega)) = x]$ lies in $\mathscr{F}$, and so $f(X_\tau)$ is a simple random variable. In order that this always hold, put $f(X_{\tau(\omega)}(\omega)) = 0$, say, if $\tau(\omega) = \infty$ (which happens only on a set of probability 0).

The game with payoff function f has at i the *value*

$$v(i) = \sup E_i[f(X_\tau)], \tag{8.44}$$

the supremum extending over all Markov times τ. It will turn out that the supremum here is achieved: there always exists an optimal stopping time. It will also turn out that there is an optimal τ that works for all initial states i. The problem is to calculate $v(i)$ and find the best τ. If the chain is irreducible, the system must pass through every state, and the best strategy is obviously to wait until the system enters a state for which f is maximal. This describes an optimal τ, and $v(i) = \max f$ for all i. For this reason the interesting cases are those in which some states are transient and others are absorbing ($p_{ii} = 1$).

A function φ on S is *excessive* or *superharmonic*, if[†]

$$\varphi(i) \ge \sum_j p_{ij}\varphi(j), \qquad i \in S. \tag{8.45}$$

In terms of conditional expectation the requirement is $\varphi(i) \ge E_i[\varphi(X_1)]$.

Lemma 4. *The value function v is excessive.*

PROOF. Given ϵ, choose for each j in S a "good" stopping time τ_j satisfying $E_j[f(X_{\tau_j})] > v(j) - \epsilon$. By (8.43), $[\tau_j = n] = [(X_0, \ldots, X_n) \in I_{jn}]$ for some set I_{jn} of $(n+1)$-long sequences of states. Set $\tau = n + 1$ $(n \ge 0)$ on the set $[X_1 = j] \cap [(X_1, \ldots, X_{n+1}) \in I_{jn}]$; that is, take one step and then from the new state X_1 add on the "good" stopping time for that state. Then τ is a stopping time and

$$\begin{aligned} E_i[f(X_\tau)] &= \sum_{n=0}^{\infty}\sum_j\sum_k P_i[X_1 = j, (X_1, \ldots, X_{n+1}) \in I_{jn}, X_{n+1} = k]f(k) \\ &= \sum_{n=0}^{\infty}\sum_j\sum_k p_{ij}P_j[(X_0, \ldots, X_n) \in I_{jn}, X_n = k]f(k) \\ &= \sum_j p_{ij}E_j[f(X_{\tau_j})]. \end{aligned}$$

Therefore, $v(i) \ge E_i[f(X_\tau)] \ge \sum_j p_{ij}(v(j) - \epsilon) = \sum_j p_{ij}v(j) - \epsilon$. Since ϵ was arbitrary, v is excessive. ■

[†]Compare the conditions (7.28) and (7.35).

Lemma 5. *Suppose that φ is excessive.*

(i) *For all stopping times τ, $\varphi(i) \ge E_i[\varphi(X_\tau)]$.*
(ii) *For all pairs of stopping times satisfying $\sigma \le \tau$, $E_i[\varphi(X_\sigma)] \ge E_i[\varphi(X_\tau)]$.*

Part (i) says that for an excessive payoff function, $\tau \equiv 0$ represents an optimal strategy.

PROOF. To prove (i), put $\tau_N = \min\{\tau, N\}$. Then τ_N is a stopping time, and

$$\begin{aligned} E_i[\varphi(X_{\tau_N})] = \sum_{n=0}^{N-1}\sum_k P_i[\tau = n, X_n = k]\varphi(k) \\ + \sum_k P_i[\tau \ge N, X_N = k]\varphi(k). \end{aligned} \tag{8.46}$$

Since $[\tau \ge N] = [\tau < N]^c \in F_{N-1}$, the final sum here is by (8.13)

$$\begin{aligned} &\sum_k\sum_j P_i[\tau \ge N, X_{N-1} = j, X_N = k]\varphi(k) \\ &\quad = \sum_k\sum_j P_i[\tau \ge N, X_{N-1} = j]p_{jk}\varphi(k) \le \sum_j P_i[\tau \ge N, X_{N-1} = j]\varphi(j). \end{aligned}$$

Substituting this into (8.46) leads to $E_i[\varphi(X_{\tau_N})] \le E_i[\varphi(X_{\tau_{N-1}})]$. Since $\tau_0 = 0$ and $E_i[\varphi(X_0)] = \varphi(i)$, it follows that $E_i[\varphi(X_{\tau_N})] \le \varphi(i)$ for all N. But for $\tau(\omega)$ finite, $\varphi(X_{\tau_N(\omega)}(\omega)) \to \varphi(X_{\tau(\omega)}(\omega))$ (there is equality for large N), and so $E_i[\varphi(X_{\tau_N})] \to E_i[\varphi(X_\tau)]$ by Theorem 5.4.

The proof of (ii) is essentially the same. If $\tau_N = \min\{\tau, \sigma + N\}$, then τ_N is a stopping time, and

$$E_i[\varphi(X_{\tau_N})] = \sum_{m=0}^{\infty} \sum_{n=0}^{N-1} \sum_k P_i[\sigma = m, \tau = m+n, X_{m+n} = k]\varphi(k)$$
$$+ \sum_{m=0}^{\infty} \sum_k P_i[\sigma = m, \tau \ge m+N, X_{m+N} = k]\varphi(k).$$

Since $[\sigma = m, \tau \ge m+N] = [\sigma = m] - [\sigma = m, \tau < m+N] \in \mathscr{F}_{m+N-1}$, again $E_i[\varphi(X_{\tau_N})] \le E_i[\varphi(X_{\tau_{N-1}})] \le E_i[\varphi(X_{\tau_0})]$. Since $\tau_0 = \sigma$, part (ii) follows from part (i) by another passage to the limit. ∎

Lemma 6. *If an excessive function φ dominates the payoff function f, then it dominates the value function v as well.*

By definition, to say that g dominates h is to say that $g(i) \ge h(i)$ for all i.

PROOF. By Lemma 5, $\varphi(i) \ge E_i[\varphi(X_\tau)] \ge E_i[f(X_\tau)]$ for all Markov times τ, and so $\varphi(i) \ge v(i)$ for all i. ∎

Since $\tau \equiv 0$ is a stopping time, v dominates f. Lemmas 4 and 6 immediately characterize v:

Theorem 8.10. *The value function v is the minimal excessive function dominating f.*

There remains the problem of constructing the optimal strategy τ. Let M be the set of states i for which $v(i) = f(i)$; M, the *support set*, is nonempty, since it at least contains those i that maximize f. Let $A = \bigcap_{n=0}^{\infty}[X_n \notin M]$ be the event that the system never enters M. The following argument shows that $P_i(A) = 0$ for each i. As this is trivial if $M = S$, assume that $M \ne S$. Choose $\delta > 0$ so that $f(i) \le v(i) - \delta$ for $i \in S - M$. Now $E_i[f(X_\tau)] = \sum_{n=0}^{\infty}\sum_k P_i[\tau = n, X_n = k]f(k)$; replacing the $f(k)$ by $v(k)$ or $v(k) - \delta$ according as $k \in M$ or $k \in S - M$ gives $E_i[f(X_\tau)] \le E_i[v(X_\tau)] - \delta P_i[X_\tau \in S - M] \le E_i[v(X_\tau)] - \delta P_i(A) \le v(i) - \delta P_i(A)$, the last inequality by Lemmas 4 and 5. Since this holds for every Markov time, taking the supremum over τ gives $P_i(A) = 0$. Whatever the initial state, the system is thus certain to enter the support set M.

Let $\tau_0(\omega) = \min[n: X_n(\omega) \in M]$ be the *hitting time* for M. Then τ_0 is a Markov time, and $\tau_0 = 0$ if $X_0 \in M$. It may be that $X_n(\omega) \notin M$ for all n, in which case $\tau_0(\omega) = \infty$, but as just shown, the probability of this is 0.

Theorem 8.11. *The hitting time τ_0 is optimal: $E_i[f(X_{\tau_0})] = v(i)$ for all i.*

PROOF. By the definition of τ_0, $f(X_{\tau_0}) = v(X_{\tau_0})$. Put $\varphi(i) = E_i[f(X_{\tau_0})] = E_i[v(X_{\tau_0})]$. The first step is to show that φ is excessive. If $\tau_1 = \min[n: n \ge 1, X_n \in M]$, then τ_1 is a Markov time and

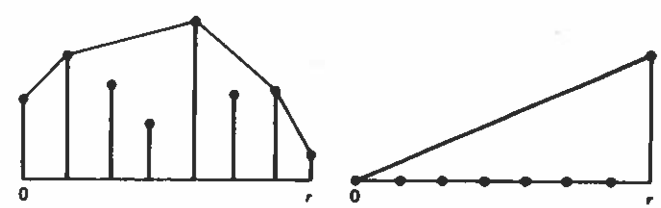
$$E_i[v(X_{\tau_1})] = \sum_{n=1}^{\infty} \sum_{k \in M} P_i[X_1 \notin M, \dots, X_{n-1} \notin M, X_n = k]v(k)$$
$$= \sum_{n=1}^{\infty} \sum_{k \in M} \sum_{j \in S} p_{ij} P_j[X_0 \notin M, \dots, X_{n-2} \notin M, X_{n-1} = k]v(k)$$
$$= \sum_j p_{ij} E_j[v(X_{\tau_0})].$$

Since $\tau_0 \le \tau_1$, $E_i[v(X_{\tau_0})] \ge E_i[v(X_{\tau_1})]$ by Lemmas 4 and 5.

This shows that φ is excessive. And $\varphi(i) \le v(i)$ by the definition (8.44). If $\varphi(i) \ge f(i)$ is proved, it will follow by Theorem 8.10 that $\varphi(i) \ge v(i)$ and hence that $\varphi(i) = v(i)$. Since $\tau_0 = 0$ for $X_0 \in M$, if $i \in M$ then $\varphi(i) = E_i[f(X_0)] = f(i)$. Suppose that $\varphi(i) < f(i)$ for some values of i in $S - M$, and choose i_0 to maximize $f(i) - \varphi(i)$. Then $\psi(i) = \varphi(i) + f(i_0) - \varphi(i_0)$ dominates f and is excessive, being the sum of a constant and an excessive function. By Theorem 8.10, ψ must dominate v, so that $\psi(i_0) \ge v(i_0)$, or $f(i_0) \ge v(v_0)$. But this implies that $i_0 \in M$, a contradiction ∎

The optimal strategy need not be unique. If f is constant, for example, all strategies have the same value.

Example 8.16. For the symmetric random walk with absorbing barriers at 0 and r (Example 8.2) a function φ on $S = \{0, 1, \dots, r\}$ is excessive if $\varphi(i) \ge \frac{1}{2}\varphi(i-1) + \frac{1}{2}\varphi(i+1)$ for $1 \le i \le r-1$. The requirement is that φ give a concave function when extended by linear interpolation from S to the entire interval $[0, r]$. Hence v thus extended is the minimal concave function dominating f. The figure shows the geometry: the ordinates of the dots are the values of f and the polygonal line describes v. The optimal strategy is to stop at a state for which the dot lies on the polygon.



If $f(r)=1$ and $f(i)=0$ for $i<r$, then v is a straight line; $v(i)=i/r$. The optimal Markov time τ_0 is the hitting time for $M=\{0,r\}$, and $v(i)=E_i[f(X_{\tau_0})]$ is the probability of absorption in the state r. This gives another solution of the gambler's ruin problem for the symmetric case. ∎

Example 8.17. For the selection problem in Example 8.5, the p_{ij} are given by (8.5) and (8.6) for $1\le i\le r$, while $p_{r+1,r+1}=1$. The payoff is $f(i)=i/r$ for $i\le r$ and $f(r+1)=0$. Thus $v(r+1)=0$, and since v is excessive,

$$v(i)\ge g(i)=\sum_{j=i+1}^{r}\frac{i}{j(j+1)}v(j),\qquad 1\le i<r. \tag{8.47}$$

By Theorem 8.10, v is the smallest function satisfying (8.47) and $v(i)\ge f(i)=i/r$, $1\le i\le r$. Since (8.47) puts no lower limit on $v(r)$, it follows that $v(r)=f(r)=1$, and r lies in the support set M. By minimality,

$$v(i)=\max\{f(i),g(i)\},\qquad 1\le i<r. \tag{8.48}$$

If $i\in M$, then $f(i)=v(i)\ge g(i)\ge\sum_{j=i+1}^{r}ij^{-1}(j-1)^{-1}f(j)=f(i)\sum_{j=i+1}^{r}(j-1)^{-1}$, and hence $\sum_{j=i+1}^{r}(j-1)^{-1}\le 1$. On the other hand, if this inequality holds and $i+1,\dots,r$ all lie in M, then $g(i)=\sum_{j=i+1}^{r}ij^{-1}(j-1)^{-1}f(j)=f(i)\sum_{j=i+1}^{r}(j-1)^{-1}\le f(i)$, so that $i\in M$ by (8.48). Therefore, $M=\{i_r,i_r+1,\dots,r,r+1\}$, where i_r is determined by

$$\frac{1}{i_r}+\frac{1}{i_r+1}+\cdots+\frac{1}{r-1}\le 1<\frac{1}{i_r-1}+\frac{1}{i_r}+\cdots+\frac{1}{r-1}. \tag{8.49}$$

If $i<i_r$, so that $i\notin M$, then $v(i)>f(i)$ and so, by (8.48),

$$\begin{aligned}v(i)=g(i)&=\sum_{j=i+1}^{i_r-1}\frac{i}{j(j-1)}v(j)+\sum_{j=i_r}^{r}\frac{i}{j(j-1)}f(j)\\&=\sum_{j=i+1}^{i_r-1}\frac{i}{j(j-1)}v(j)+\frac{i}{r}\left(\frac{1}{i_r-1}+\cdots+\frac{1}{r-1}\right).\end{aligned}$$

It follows by backward induction starting with $i=i_r-1$ that

$$v(i)=p_r=\frac{i_r-1}{r}\left(\frac{1}{i_r-1}+\cdots+\frac{1}{r-1}\right) \tag{8.50}$$

is constant for $1\le i<i_r$.

In the selection problem as originally posed, $X_1=1$. The optimal strategy is to stop with the first X_n that lies in M. The princess should therefore reject the first i_r-1 suitors and accept the next one who is preferable to all his predecessors (is dominant). The probability of success is p_r as given by (8.50). Failure can happen in two ways. Perhaps the first dominant suitor after i_r is not the best of all suitors; in this case the princess will be unaware of failure. Perhaps no dominant suitor comes after i_r; in this case the princess is obliged to take the last suitor of all and may be well aware of failure. Recall that the problem was to maximize the chance of getting the best suitor of all rather than, say, the chance of getting a suitor in the top half.

If r is large, (8.49) essentially requires that $\log r-\log i_r$ be near 1, so that $i_r\approx r/e$. In this case, $p_r\approx 1/e$.

Note that although the system starts in state 1 in the original problem, its resolution by means of the preceding theory requires consideration of all possible initial states. ∎

This theory carries over in part to the case of infinite S, although this requires the general theory of expected values, since $f(X_\tau)$ may not be a simple random variable. Theorem 8.10 holds for infinite S if the payoff function is nonnegative and the value function is finite.† But then problems arise: Optimal strategies may not exist, and the probability of hitting the support set M may be less than 1. Even if this probability is 1, the strategy of stopping on first entering M may be the worst one of all.‡

PROBLEMS

8.1. Prove Theorem 8.1 for the case of finite S by constructing the appropriate probability measure on sequence space S^∞: Replace the summand on the right in (2.21) by $\alpha_{u_1}p_{u_1u_2}\cdots p_{u_{n-1}u_n}$, and extend the arguments preceding Theorem 2.3. If $X_n(\cdot)=z_n(\cdot)$, then $X_1,X_2,\dots$ is the appropriate Markov chain (here time is shifted by 1).

8.2. Let $Y_0,Y_1,\dots$ be independent and identically distributed with $P[Y_n=1]=p$, $P[Y_n=0]=q=1-p$, $p\ne q$. Put $X_n=Y_n+Y_{n+1}$ (mod 2). Show that $X_0,X_1,\dots$ is not a Markov chain even though $P[X_{n+1}=j\mid X_{n-1}=i]=P[X_{n+1}=j]$. Does this last relation hold for all Markov chains? Why?

8.3. Show by example that a function $f(X_0),f(X_1),\dots$ of a Markov chain need not be a Markov chain.

8.4. Show that

$$f_{ij}\sum_{k=0}^{\infty}p_{jj}^{(k)}=\sum_{n=1}^{\infty}\sum_{m=1}^{n}f_{ij}^{(m)}p_{jj}^{(n-m)}=\sum_{n=1}^{\infty}p_{ij}^{(n)},$$

and prove that if j is transient, then $\sum_n p_{ij}^{(n)}<\infty$ for each i (compare Theorem 8.3(i)). If j is transient, then

$$f_{jj}=\sum_{n=1}^{\infty}p_{jj}^{(n)}\Bigg/\left(1+\sum_{n=1}^{\infty}p_{jj}^{(n)}\right).$$

†The only essential change in the argument is that Fatou's lemma (Theorem 16.3) must be used in place of Theorem 5.4 in the proof of Lemma 5.

‡See Problems 8.36 and 8.37.

Specialize to the case $i = j$: in addition to implying that i is transient (Theorem 8.2(i)), a finite value for $\sum_{n=1}^{\infty} p_{ii}^{(n)}$ suffices to determine f_{ii} exactly.

8.5. Call $\{x_i\}$ a *subsolution* of (8.24) if $x_i \le \sum_j q_{ij} x_j$ and $0 \le x_i \le 1$, $i \in U$. Extending Lemma 1, show that a subsolution $\{x_i\}$ satisfies $x_i \le \sigma_i$: The solution $\{\sigma_i\}$ of (8.24) dominates all subsolutions as well as all solutions. Show that if $x_i = \sum_j q_{ij} x_j$ and $-1 \le x_i \le 1$, then $\{|x_i|\}$ is a subsolution of (8.24).

8.6. Show by solving (8.27) that the unrestricted random walk on the line (Example 8.3) is persistent if and only if $p = \frac{1}{2}$.

8.7. (a) Generalize an argument in the proof of Theorem 8.5 to show that $f_{ik} = p_{ik} + \sum_{j \ne k} p_{ij} f_{jk}$. Generalize this further to

$$f_{ik} = f_{ik}^{(1)} + \cdots + f_{ik}^{(n)}$$
$$+ \sum_{j \ne k} P_i[X_1 \ne k, \ldots, X_{n-1} \ne k, X_n = j] f_{jk}.$$

(b) Take $k = i$. Show that $f_{ij} > 0$ if and only if $P_i[X_1 \ne i, \ldots, X_{n-1} \ne i, X_n = j] > 0$ for some n, and conclude that i is transient if and only if $f_{ji} < 1$ for some $j \ne i$ such that $f_{ij} > 0$.

(c) Show that an irreducible chain is transient if and only if for each i there is a $j \ne i$ such that $f_{ji} < 1$.

8.8. Suppose that $S = \{0, 1, 2, \ldots\}$, $p_{00} = 1$, and $f_{i0} > 0$ for all i.

(a) Show that $P_i(\bigcup_{j=1}^{\infty}[X_n = j \text{ i.o.}]) = 0$ for all i.

(b) Regard the state as the size of a population and interpret the conditions $p_{00} = 1$ and $f_{i0} > 0$ and the conclusion in part (a).

8.9. 8.5 ↑ Show for an irreducible chain that (8.27) has a nontrivial solution if and only if there exists a nontrivial, bounded sequence $\{x_i\}$ (not necessarily nonnegative) satisfying $x_i = \sum_{j \ne i_0} p_{ij} x_j$, $i \ne i_0$. (See the remark following the proof of Theorem 8.5.)

8.10. ↑ Show that an irreducible chain is transient if and only if (for arbitrary i_0) the system $y_i = \sum_j p_{ij} y_j$, $i \ne i_0$ (sum over *all* j), has a bounded, nonconstant solution $\{y_i, i \in S\}$.

8.11. Show that the P_i-probabilities of ever leaving U for $i \in U$ are the minimal solution of the system.

$$\begin{cases} z_i = \sum_{j \in U} p_{ij} z_j + \sum_{j \notin U} p_{ij}, & i \in U, \\ 0 \le z_i \le 1, & i \in U. \end{cases} \tag{8.51}$$

The constraint $z_i \le 1$ can be dropped: the minimal solution automatically satisfies it, since $z_i \equiv 1$ is a solution.

8.12. Show that $\sup_{ij} n_0(i, j) = \infty$ is possible in Lemma 2.

8.13. Suppose that $\{\pi_i\}$ solves (8.30), where it is assumed that $\sum_i |\pi_i| < \infty$, so that the left side is well defined. Show in the irreducible case that the π_i are either all positive or all negative or all 0. Stationary probabilities thus exist in the irreducible case if and only if (8.30) has a nontrivial solution $\{\pi_i\}$ ($\sum_i \pi_i$ absolutely convergent).

8.14. Show by example that the coupled chain in the proof of Theorem 8.6 need not be irreducible if the original chain is not aperiodic.

8.15. Suppose that S consists of all the integers and

$$p_{0,-1} = p_{0,0} = p_{0,+1} = \tfrac{1}{3},$$
$$p_{k,k-1} = q, \quad p_{k,k+1} = p, \qquad k \le -1,$$
$$p_{k,k-1} = p, \quad p_{k,k+1} = q, \qquad k \ge 1.$$

Show that the chain is irreducible and aperiodic. For which p's is the chain persistent? For which p's are there stationary probabilities?

8.16. Show that the period of j is the greatest common divisor of the set

$$[n: n \ge 1, f_{jj}^{(n)} > 0]. \tag{8.52}$$

8.17. ↑ *Recurrent events.* Let $f_1, f_2, \ldots$ be nonnegative numbers with $f = \sum_{n=1}^{\infty} f_n \le 1$. Define $u_1, u_2, \ldots$ recursively by $u_1 = f_1$ and

$$u_n = f_1 u_{n-1} + \cdots + f_{n-1} u_1 + f_n. \tag{8.53}$$

(a) Show that $f < 1$ if and only if $\sum_n u_n < \infty$.

(b) Assume that $f = 1$, set $\mu = \sum_{n=1}^{\infty} n f_n$, and assume that

$$\gcd[n: n \ge 1, f_n > 0] = 1. \tag{8.54}$$

Prove the *renewal theorem*: Under these assumptions, the limit $u = \lim_n u_n$ exists, and $u > 0$ if and only if $\mu < \infty$, in which case $u = 1/\mu$.

Although these definitions and facts are stated in purely analytical terms, they have a probabilistic interpretation: Imagine an event $\mathscr{E}$ that may occur at times $1, 2, \ldots$. Suppose f_n is the probability $\mathscr{E}$ occurs first at time n. Suppose further that at each occurrence of $\mathscr{E}$ the system starts anew, so that f_n is the probability that $\mathscr{E}$ next occurs n steps later. Such an $\mathscr{E}$ is called a *recurrent event*. If u_n is the probability that $\mathscr{E}$ occurs at time n, then (8.53) holds. The recurrent event $\mathscr{E}$ is called transient or persistent according as $f < 1$ or $f = 1$, it is called aperiodic if (8.54) holds, and if $f = 1$, μ is interpreted as the mean recurrence time.

8.18. (a) Let τ be the smallest integer for which $X_\tau = i_0$. Suppose that the state space is finite and that the p_{ij} are all positive. Find a ρ such that $\max_i(1 - p_{ii_0}) \le \rho < 1$ and hence $P_i[\tau > n] \le \rho^n$ for all i.

(b) Apply this to the coupled chain in the proof of Theorem 8.6: $|p_{ik}^{(n)} - p_{jk}^{(n)}| \le \rho^n$. Now give a new proof of Theorem 8.9.

8.19. A thinker who owns r umbrellas wanders back and forth between home and office, taking along an umbrella (if there is one at hand) in rain (probability p) but not in shine (probability q). Let the state be the number of umbrellas at hand, irrespective of whether the thinker is at home or at work. Set up the transition matrix and find the stationary probabilities. Find the steady-state probability of his getting wet, and show that five umbrellas will protect him at the 5% level against any climate (any p).

8.20. (a) A transition matrix is *doubly stochastic* if $\sum_i p_{ij} = 1$ for each j. For a finite, irreducible, aperiodic chain with doubly stochastic transition matrix, show that the stationary probabilities are all equal.

(b) Generalize Example 8.15: Let S be a finite group, let $p(i)$ be probabilities, and put $p_{ij} = p(j \cdot i^{-1})$, where product and inverse refer to the group operation. Show that, if all $p(i)$ are positive, the states are all equally likely in the limit.

(c) Let S be the symmetric group on 52 elements. What has (b) to say about card shuffling?

8.21. A set C in S is *closed* if $\sum_{j \in C} p_{ij} = 1$ for $i \in C$: once the system enters C it cannot leave. Show that a chain is irreducible if and only if S has no proper closed subset.

8.22. ↑ Let T be the set of transient states and define persistent states i and j (if there are any) to be equivalent if $f_{ij} > 0$. Show that this is an equivalence relation on $S - T$ and decomposes it into equivalence classes $C_1, C_2, \ldots$, so that $S = T \cup C_1 \cup C_2 \cup \cdots$. Show that each C_m is closed and that $f_{ij} = 1$ for i and j in the same C_m.

8.23. 8.11 8.21 ↑ Let T be the set of transient states and let C be any closed set of persistent states. Show that the P_i-probabilities of eventual absorption in C for $i \in T$ are the minimal solution of

$$\begin{cases} y_i = \sum_{j \in T} p_{ij} y_j + \sum_{j \in C} p_{ij}, & i \in T, \\ 0 \le y_i \le 1, & i \in T. \end{cases} \tag{8.55}$$

8.24. Suppose that an irreducible chain has period $t > 1$. Show that S decomposes into sets $S_0, \ldots, S_{t-1}$ such that $p_{ij} > 0$ only if $i \in S_\nu$ and $j \in S_{\nu+1}$ for some ν ($\nu + 1$ reduced modulo t). Thus the system passes through the S_ν in cyclic succession.

8.25. ↑ Suppose that an irreducible chain of period $t > 1$ has a stationary distribution $\{\pi_j\}$. Show that, if $i \in S_\nu$ and $j \in S_{\nu+\alpha}$ ($\nu + \alpha$ reduced modulo t), then $\lim_n p_{ij}^{(nt+\alpha)} = \pi_j$. Show that $\lim_n n^{-1} \sum_{m=1}^n p_{ij}^{(m)} = \pi_j / t$ for all i and j.

8.26. *Eigenvalues*. Consider an irreducible, aperiodic chain with state space $\{1, \ldots, s\}$. Let $r_0 = (\pi_1, \ldots, \pi_s)$ be (Example 8.14) the row vector of stationary probabilities, and let c_0 be the column vector of 1's; then r_0 and c_0 are left and right eigenvectors of P for the eigenvalue $\lambda = 1$.

(a) Suppose that r is a left eigenvector for the (possibly complex) eigenvalue λ: $rP = \lambda r$. Prove: If $\lambda = 1$, then r is a scalar multiple of r_0 ($\lambda = 1$ has geometric multiplicity 1). If $\lambda \ne 1$, then $|\lambda| < 1$ and $rc_0 = 0$ (the 1×1 product of $1 \times s$ and $s \times 1$ matrices).

(b) Suppose that c is a right eigenvector: $Pc = \lambda c$. If $\lambda = 1$, then c is a scalar multiple of c_0 (again the geometric multiplicity is 1). If $\lambda \ne 1$, then again $|\lambda| < 1$, and $r_0 c = 0$.

8.27. ↑ Suppose P is diagonalizable; that is, suppose there is a nonsingular C such that $C^{-1}PC = \Lambda$, where Λ is a diagonal matrix. Let $\lambda_1, \ldots, \lambda_s$ be the diagonal elements of Λ, let $c_1, \ldots, c_s$ be the successive columns of C, let $R = C^{-1}$, and let $r_1, \ldots, r_s$ be the successive rows of R.

(a) Show that c_i and r_i are right and left eigenvectors for the eigenvalue λ_i, $i = 1, \ldots, s$. Show that $r_i c_j = \delta_{ij}$. Let $A_i = c_i r_i$ ($s \times s$). Show that Λ^n is a diagonal matrix with diagonal elements $\lambda_1^n, \ldots, \lambda_s^n$ and that $P^n = C\Lambda^n R = \sum_{u=1}^s \lambda_u^n A_u, n \ge 1$.

(b) Part (a) goes through under the sole assumption that P is a diagonalizable matrix. Now assume also that it is an irreducible, aperiodic stochastic matrix, and arrange the notation so that $\lambda_1 = 1$. Show that each row of A_1 is the vector $(\pi_1, \ldots, \pi_s)$ of stationary probabilities. Since

$$P^n = A_1 + \sum_{u=2}^{s} \lambda_u^n A_u \tag{8.56}$$

and $|\lambda_u| < 1$ for $2 \le u \le s$, this proves exponential convergence once more.

(c) Write out (8.56) explicitly for the case $s = 2$.

(d) Find an irreducible, aperiodic stochastic matrix that is not diagonalizable.

8.28. ↑ (a) Show that the eigenvalue $\lambda = 1$ has geometric multiplicity 1 if there is only one closed, irreducible set of states; there may be transient states, in which case the chain itself is not irreducible.

(b) Show, on the other hand, that if there is more than one closed, irreducible set of states, then $\lambda = 1$ has geometric multiplicity exceeding 1.

(c) Suppose that there is only one closed, irreducible set of states. Show that the chain has period exceeding 1 if and only if there is an eigenvalue other than 1 on the unit circle.

8.29. Suppose that $\{X_n\}$ is a Markov chain with state space S, and put $Y_n = (X_n, X_{n+1})$. Let T be the set of pairs (i, j) such that $p_{ij} > 0$ and show that $\{Y_n\}$ is a Markov chain with state space T. Write down the transition probabilities. Show that, if $\{X_n\}$ is irreducible and aperiodic, so is $\{Y_n\}$. Show that, if π_i are stationary probabilities for $\{X_n\}$, then $\pi_i p_{ij}$ are stationary probabilities for $\{Y_n\}$.

8.30. 6.10 8.29 ↑ Suppose that the chain is finite, irreducible, and aperiodic and that the initial probabilities are the stationary ones. Fix a state i, let $A_n = [X_n = i]$, and let N_n be the number of passages through i in the first n steps. Calculate α_n and β_n as defined by (5.41). Show that $\beta_n - \alpha_n^2 = O(1/n)$, so that $n^{-1} N_n \to \pi_i$ with probability 1. Show for a function f on the state space that $n^{-1} \sum_{k=1}^n f(X_k) \to \sum_i \pi_i f(i)$ with probability 1. Show that $n^{-1} \sum_{k=1}^n g(X_k, X_{k+1}) \to \sum_{ij} \pi_i p_{ij} g(i, j)$ for functions g on $S \times S$.

8.31. 6.14 8.30↑ If $X_0(\omega) = i_0, \ldots, X_n(\omega) = i_n$ for states $i_0, \ldots, i_n$, put $p_n(\omega) = \pi_{i_0} p_{i_0 i_1} \cdots p_{i_{n-1} i_n}$, so that $p_n(\omega)$ is the probability of the observation observed. Show that $-n^{-1} \log p_n(\omega) \to h = -\sum_{ij} \pi_i p_{ij} \log p_{ij}$ with probability 1 if the chain is finite, irreducible, and aperiodic. Extend to this case the notions of source, entropy, and asymptotic equipartition.

8.32. A sequence $\{X_n\}$ is a Markov chain *of second order* if $P[X_{n+1} = j | X_0 = i_0, \ldots, X_n = i_n] = P[X_{n+1} = j | X_{n-1} = i_{n-1}, X_n = i_n] = p_{i_{n-1} i_n : j}$. Show that nothing really new is involved because the sequence of pairs (X_n, X_{n+1}) is an ordinary Markov chain (of first order). Compare Problem 8.29. Generalize this idea into chains of order r.

8.33. Consider a chain on $S = \{0, 1, \ldots, r\}$, where 0 and r are absorbing states and $p_{i,i+1} = p_i > 0$, $p_{i,i-1} = q_i = 1 - p_i > 0$ for $0 < i < r$. Identify state i with a point z_i on the line, where $0 = z_0 < \cdots < z_r$ and the distance from z_i to z_{i+1} is q_i/p_i times that from z_{i-1} to z_i. Given a function φ on S, consider the associated function $\hat{\varphi}$ on $[0, z_r]$ defined at the z_i by $\hat{\varphi}(z_i) = \varphi(i)$ and in between by linear interpolation. Show that φ is excessive if and only if $\hat{\varphi}$ is concave. Show that the probability of absorption in r for initial state i is t_{i-1}/t_{r-1}, where $t_i = \sum_{k=0}^{i} q_1 \cdots q_k / p_1 \cdots p_k$. Deduce (7.7). Show that in the new scale the expected distance moved on each step is 0.

8.34. Suppose that a finite chain is irreducible and aperiodic. Show by Theorem 8.9 that an excessive function must be constant.

8.35. *A zero-one law.* Let the state space S contain s points, and suppose that $\epsilon_n = \sup_{ij} |p_{ij}^{(n)} - \pi_j| \to 0$, as holds under the hypotheses of Theorem 8.9. For $a \le b$, let $\mathscr{G}_a^b$ be the σ-field generated by the sets $[X_a = u_a, \ldots, X_b = u_b]$. Let $\mathscr{T}_a = \sigma(\bigcup_{b=a}^{\infty} \mathscr{G}_a^b)$ and $\mathscr{T} = \bigcap_{a=1}^{\infty} \mathscr{T}_a$. Show that $|P(A \cap B) - P(A)P(B)| \le s(\epsilon_n + \epsilon_{b+n})$ for $A \in \mathscr{G}_0^b$ and $B \in \mathscr{G}_{b+n}^{b+m}$; the ϵ_{b+n} can be suppressed if the initial probabilities are the stationary ones. Show that this holds for $A \in \mathscr{G}_0^b$ and $B \in \mathscr{T}_{b+n}$. Show that $C \in \mathscr{T}$ implies that $P(C)$ is either 0 or 1.

8.36† Alter the chain in Example 8.13 so that $q_0 = 1 - p_0 = 1$ (the other p_i and q_i still positive). Let $\beta = \lim_n p_1 \cdots p_n$ and assume that $\beta > 0$. Define a payoff function by $f(0) = 1$ and $f(i) = 1 - f_{i0}$ for $i > 0$. If $X_0, \ldots, X_n$ are positive, put $\sigma_n = n$; otherwise let σ_n be the smallest k such that $X_k = 0$. Show that $E_i[f(X_{\sigma_n})] \to 1$ as $n \to \infty$, so that $v(i) \equiv 1$. Thus the support set is $M = \{0\}$, and for an initial state $i > 0$ the probability of ever hitting M is $f_{i0} < 1$.

For an arbitrary finite stopping time τ, choose n so that $P_i[\tau < n = \sigma_n] > 0$. Then $E_i[f(X_\tau)] \le 1 - f_{i+n,0} P_i[\tau < n = \sigma_n] < 1$. Thus no strategy achieves the value $v(i)$ (except of course for $i = 0$).

8.37. ↑ Let the chain be as in the preceding problem, but assume that $\beta = 0$, so that $f_{i0} = 1$ for all i. Suppose that $\lambda_1, \lambda_2, \ldots$ exceed 1 and that $\lambda_1 \cdots \lambda_n \to \lambda < \infty$; put $f(0) = 0$ and $f(i) = \lambda_1 \cdots \lambda_{i-1} / p_1 \cdots p_{i-1}$. For an arbitrary (finite) stopping time τ, the event $[\tau = n]$ must have the form $[(X_0, \ldots, X_n) \in I_n]$ for some set I_n of $(n+1)$-long sequences of states. Show that for each i there is at most one $n \ge 0$ such that $(i, i+1, \ldots, i+n) \in I_n$. If there is no such n, then $E_i[f(X_\tau)] = 0$. If there is one, then

$$E_i[f(X_\tau)] = P_i[(X_0, \ldots, X_n) = (i, \ldots, i+n)] f(i+n),$$

and hence the only possible values of $E_i[f(X_\tau)]$ are

$$0, \quad f(i), \quad p_i f(i+1) = f(i)\lambda_i, \quad p_i p_{i+1} f(i+2) = f(i)\lambda_i \lambda_{i+1}, \ldots.$$

Thus $v(i) = f(i)\lambda/\lambda_1 \cdots \lambda_{i-1}$ for $i \ge 1$; no strategy this value. The support set is $M = \{0\}$, and the hitting time τ_0 for M is finite, but $E_i[f(X_{\tau_0})] = 0$.

8.38. 5.12↑ Consider an irreducible, aperiodic, positive persistent chain. Let τ_j be the smallest n such that $X_n = j$, and let $m_{ij} = E_i[\tau_j]$. Show that there is an r such that $p = P_j[X_1 \ne j, \ldots, X_{r-1} \ne j, X_r = i]$ is positive; from $f_{jj}^{(n+r)} \ge p f_{ij}^{(n)}$ and $m_{jj} < \infty$, conclude that $m_{ij} < \infty$ and $m_{ij} = \sum_{n=0}^{\infty} P_i[\tau_j > n]$. Starting from $p_{ij}^{(t)} = \sum_{s=1}^{t} f_{ij}^{(s)} p_{jj}^{(t-s)}$, show that

$$\sum_{t=1}^{n} \left(p_{ij}^{(t)} - p_{jj}^{(t)} \right) = 1 - \sum_{m=0}^{n} p_{jj}^{(n-m)} P_i[\tau_j > m].$$

Use the M-test to show that

$$\pi_j m_{ij} = 1 + \sum_{n=1}^{\infty} \left(p_{ij}^{(n)} - p_{jj}^{(n)} \right).$$

If $i = j$, this gives $m_{jj} = 1/\pi_j$ again; if $i \ne j$, it shows how in principle m_{ij} can be calculated from the transition matrix and the stationary probabilities.

† The final three problems in this section involve expected values for random variables with infinite range.

SECTION 9. LARGE DEVIATIONS AND THE LAW OF THE ITERATED LOGARITHM*

It is interesting in connection with the strong law of large numbers to estimate the rate at which S_n/n converges to the mean m. The proof of the strong law used upper bounds for the probabilities $P[|S_n - m| \ge \alpha]$ for large α. Accurate upper and lower bounds for these probabilities will lead to the law of the iterated logarithm, a theorem giving very precise rates for $S_n/n \to m$.

The first concern will be to estimate the probability of large deviations from the mean, which will require the method of moment generating functions. The estimates will be applied first to a problem in statistics and then to the law of the iterated logarithm.

* This section may be omitted.