# Math 3100 Assignment 5
# Infinite Series

*Due at 5:00 pm on Friday the 22nd of February 2019*

1. Suppose that $\sum_{k=1}^{\infty} a_k$ converges to $A$ and $\sum_{k=1}^{\infty} b_k$ converges to $B$.

   (a) Prove that $\sum_{k=1}^{\infty} (a_k + b_k)$ converges to $A + B$.

   (b) Must $\sum_{k=1}^{\infty} (a_k b_k)$ converge to $AB$? Give either a proof or counterexample.

2. Evaluate the following series (if they converge)

$$\text{(a)} \quad \sum_{n=1}^{\infty} \frac{1}{2^n} \qquad\qquad \text{(b)} \quad \sum_{n=2}^{\infty} \frac{3}{4^n} \qquad\qquad \text{(c)} \quad \sum_{n=3}^{\infty} \frac{7^{n-1}}{2^{n+1}}$$

3. Prove that omitting or changing a finite number of terms of a series does not affect its convergence.
   *Hint: One possible approach to this problem, but not the only one, is to use the Cauchy Criterion*

4. Let $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ be two sequences of positive real numbers. Prove the following:

   (i) If $\lim_{n\to\infty} \frac{a_n}{b_n} = c > 0$, then $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ either both converge or both diverge.

   (ii) If $\lim_{n\to\infty} \frac{a_n}{b_n} = 0$ and $\sum_{n=1}^{\infty} b_n$ converges, then $\sum_{n=1}^{\infty} a_n$ also converges.

   (iii) If $\lim_{n\to\infty} \frac{a_n}{b_n} = \infty$ and $\sum_{n=1}^{\infty} b_n$ diverges, then $\sum_{n=1}^{\infty} a_n$ also diverges.

5. Test the series for convergence or divergence.

$$\text{(a)} \ \sum_{n=1}^{\infty} \frac{\sqrt{n}}{n^2+3} \qquad \text{(b)} \ \sum_{n=0}^{\infty} \cos(n) \qquad \text{(c)} \ \sum_{n=1}^{\infty} \frac{2^n}{n3^{n+1}} \qquad \text{(d)} \ \sum_{n=1}^{\infty} \frac{n2^n}{3^{n+1}} \qquad \text{(e)} \ \sum_{n=3}^{\infty} \frac{(-1)^n}{(\log n)^2}$$

$$\text{(f)} \ \sum_{n=1}^{\infty} \frac{2n}{8n-5} \qquad \text{(g)} \ \sum_{n=3}^{\infty} \frac{2}{n(\log n)^3} \qquad \text{(h)} \ \sum_{n=1}^{\infty} \frac{3^n n^2}{n!} \qquad \text{(i)} \ \sum_{n=1}^{\infty} \frac{3^n}{5^n+n} \qquad \text{(j)} \ \sum_{n=1}^{\infty} \frac{n+5}{5^n}$$

6. Investigate the behavior (convergence or divergence) of $\sum_{n=1}^{\infty} a_n$ if

$$\text{(a)} \ a_n = \sqrt{n+1} - \sqrt{n} \qquad\qquad \text{(b)} \ a_n = \frac{\sqrt{n+1} - \sqrt{n}}{n}$$

# Math 3100 - Homework 5 - SOLUTIONS

1 (a) Claim

If $\sum_{k=1}^{\infty} a_k$ converges to $A$ and $\sum_{k=1}^{\infty} b_k$ converges to $B$, then

$$\sum_{k=1}^{\infty} (a_k+b_k) \text{ converges to } A+B.$$

Proof

Since $\sum_{k=1}^{\infty} a_k$ converges to $A$ we know that

$$\lim_{n\to\infty} (a_1+\cdots+a_n) = A$$

Since $\sum_{k=1}^{\infty} b_k$ converges to $B$ we know that

$$\lim_{n\to\infty} (b_1+\cdots+b_n) = B$$

since $((a_1+b_1)+\cdots+(a_n+b_n)) = (a_1+\cdots+a_n)+(b_1+\cdots+b_n)$ $\forall n\in\mathbb{N}$

It thus follows that

$$\lim_{n\to\infty} ((a_1+b_1)+\cdots+(a_n+b_n))$$

$$= \lim_{n\to\infty} ((a_1+\cdots+a_n)+(b_1+\cdots+b_n))$$

"sum limit law"

$$= \lim_{n\to\infty} (a_1+\cdots+a_n) + \lim_{n\to\infty} (b_1+\cdots+b_n) = A+B. \quad \square$$

(b) NO  Example: $\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{\sqrt{k}}$ converges, but

$$\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{\sqrt{k}}\cdot\frac{(-1)^{k+1}}{\sqrt{k}} = \sum_{k=1}^{\infty} \frac{1}{k} \text{ diverges.}$$

2. (a) $\sum_{n=1}^{\infty}\left(\frac{1}{2}\right)^n = \sum_{n=0}^{\infty}\left(\frac{1}{2}\right)^{n+1} = \frac{1}{2}\sum_{n=0}^{\infty}\left(\frac{1}{2}\right)^n = \frac{1}{2}\,\frac{1}{1-\frac{1}{2}} = \underline{\underline{1}}$

(b) $\sum_{n=2}^{\infty}\frac{3}{4^n} = 3\sum_{n=0}^{\infty}\left(\frac{1}{4}\right)^{n+2} = \frac{3}{16}\sum_{n=0}^{\infty}\left(\frac{1}{4}\right)^n = \frac{3}{16}\,\frac{1}{1-\frac{1}{4}} = \underline{\underline{\frac{1}{4}}}$

(c) $\sum_{n=3}^{\infty}\frac{7^{n-1}}{2^{n+1}}$ <u>DIVERGES</u> since $\lim_{n\to\infty}\frac{7^{n-1}}{2^{n+1}} \neq 0$ (actually $=\infty$).

[<u>A General Formula</u>:

If $|r|<1$, then $\sum_{n=m}^{\infty} r^n = \frac{r^m}{1-r}$ $\forall\, m \geq 0$. ]

3. <u>Claim</u>

Changing finitely many terms of a series does not affect its convergence

<u>Proof</u>

Suppose $\sum_{n=1}^{\infty} a_n$ converges and $\{b_n\}$ is a sequence with the property that $a_n \neq b_n$ for only finitely many $n \in \mathbb{N}$.

In particular $\exists\, N_1$ such that if $n > N_1$ then $a_n = b_n$.

Let $\varepsilon > 0$. Since $\sum_{n=1}^{\infty} a_n$ converges we know (Cauchy Criterion) that $\exists\, N_2$ such that $n > m > N_2$ implies $\left|\sum_{k=m+1}^{n} a_k\right| < \varepsilon$.

If $n > m > \max\{N_1, N_2\}$ then $\left|\sum_{k=m+1}^{n} b_k\right| = \left|\sum_{k=m+1}^{n} a_k\right| < \varepsilon$ $\square$

since $n>m>N_1$ — since $n>m>N_2$.

Cauchy Criterion $\Rightarrow \sum_{n=1}^{\infty} b_n$ converges.

4. Let $\{a_n\}$ and $\{b_n\}$ be two sequences of positive reals.

(a) Claim

If $\lim_{n\to\infty} \frac{a_n}{b_n} = c > 0$, then $\sum_{n=1}^{\infty} a_n$ & $\sum_{n=1}^{\infty} b_n$ either both converge or both diverge.

Proof

Since $\lim_{n\to\infty} \frac{a_n}{b_n} = c > 0$ $\exists N$ such that

$$\frac{c}{2} \le \frac{a_n}{b_n} \le 2c \quad \text{for all } n \ge N.$$

$$\Updownarrow$$

$$\underbrace{a_n \le (2c) b_n}_{1} \text{ and } \underbrace{a_n \ge \left(\tfrac{c}{2}\right) b_n}_{2} \quad \text{for all } n \ge N.$$

It follows from 1 and "Direct Comparison" that if $\sum_{n=1}^{\infty} b_n$ converges then $\sum_{n=1}^{\infty} a_n$ converges (since $\sum_{n=1}^{\infty} (2c) b_n$ conv)

It follow from 2 and "Direct comparison" that if $\sum_{n=1}^{\infty} b_n$ diverges (and hence $\sum_{n=1}^{\infty} \left(\frac{c}{2}\right) b_n$ diverges) then $\sum_{n=1}^{\infty} a_n$ diverges. $\square$

(b) Claim

If $\frac{a_n}{b_n} \to 0$ & $\sum_{n=1}^{\infty} b_n$ converges, then $\sum_{n=1}^{\infty} a_n$ converges

Proof Since $\frac{a_n}{b_n} \to 0$ $\exists N$ such that $a_n \le b_n$ $\forall n \ge N$.

Result then follow from "Direct Comparison". $\square$

(c) Claim

If $\lim_{n\to\infty} \frac{a_n}{b_n} = \infty$ & $\sum_{n=1}^{\infty} b_n$ diverges, then $\sum_{n=1}^{\infty} a_n$ diverges.

Proof

Since $\frac{a_n}{b_n} \to \infty$ we know $\exists N$ such that if $n \geq N$ then

$$\frac{a_n}{b_n} \geq 1 \iff a_n \geq b_n .$$

Result now follows by "Direct Comparison". $\square$

5. (a) $\sum_{n=1}^{\infty} \frac{\sqrt{n}}{n^2+3}$ CONVERGES (by "Direct Comparison")

since $\frac{\sqrt{n}}{n^2+3} \leq \frac{1}{n^{3/2}}$ $\forall n \in \mathbb{N}$ & $\sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$ converges. (p-series)

(b) $\sum_{n=0}^{\infty} \cos(n)$ DIVERGES since $\cos(n) \not\to 0$ as $n \to \infty$.

(c) $\sum_{n=\emptyset}^{\infty} \frac{2^n}{n\,3^{n+1}}$ CONVERGES (by "Direct Comparison")

since $\frac{2^n}{n\,3^{n+1}} \leq \left(\frac{2}{3}\right)^n$ $\forall n \in \mathbb{N}$ & $\sum_{n=1}^{\infty} \left(\frac{2}{3}\right)^n$ converges. (geometric)

⊗ One could also use the "Ratio Test" for this problem.

(d) $\sum_{n=1}^{\infty} \frac{n 2^n}{3^{n+1}}$ CONVERGES (by "Ratio Test")

since $\frac{(n+1)2^{n+1}}{3^{n+2}} \cdot \frac{3^{n+1}}{n\,2^n} = \frac{n+1}{n} \, \frac{2}{3} \longrightarrow \frac{2}{3} < 1$.

(e) $\sum_{n=3}^{\infty} \frac{(-1)^n}{(\log n)^2}$ CONVERGES (by "Alt. Series Test")

since $\left\{\frac{1}{(\log n)^2}\right\}$ is a decreasing sequence that converges to $0$.

(f) $\sum_{n=1}^{\infty} \frac{2n}{8n-5}$ DIVERGES since $\frac{2n}{8n-5} \not\to 0$ as $n \to \infty$.

(g) $\sum_{n=3}^{\infty} \frac{2}{n(\log n)^3}$ CONVERGES (by "Cauchy Condensation Test")

since $\left\{\frac{2}{n(\log n)^3}\right\}$ is a decreasing sequence and

$$\sum_{k=2}^{\infty} 2^k \frac{2}{2^k(\log 2^k)^3} = \frac{2}{(\log 2)^3} \sum_{k=2}^{\infty} \frac{1}{k^3} \text{ converges (p-series).}$$

(h) $\sum_{n=1}^{\infty} \frac{3^n n^2}{n!}$ CONVERGES (by "Ratio Test")

since $\frac{3^{n+1}(n+1)^2}{(n+1)!} \cdot \frac{n!}{3^n n^2} = \left(\frac{n+1}{n}\right)^2 \frac{1}{n+1} 3 \to (1)^2(0)(3) = 0 < 1$.

(i) $\sum_{n=1}^{\infty} \frac{3^n}{5^n+n}$ CONVERGES (by "Direct Comparison")

since $\frac{3^n}{5^n+n} \leq \left(\frac{3}{5}\right)^n \; \forall n \in \mathbb{N}$ & $\sum_{n=1}^{\infty} \left(\frac{3}{5}\right)^n$ converges (geometric)

(j) $\sum_{n=1}^{\infty} \frac{n+5}{5^n}$ CONVERGES (by "Ratio Test")

since $\frac{n+6}{5^{n+1}} \cdot \frac{5^n}{n+5} = \frac{n+6}{n+5} \frac{1}{5} \to \frac{1}{5} < 1$.

6.

(a) Let $a_n = \sqrt{n+1} - \sqrt{n}$.

Since $\sqrt{n+1} - \sqrt{n} = \dfrac{(\sqrt{n+1} - \sqrt{n})(\sqrt{n+1} + \sqrt{n})}{\sqrt{n+1} + \sqrt{n}} = \dfrac{1}{\sqrt{n+1}+\sqrt{n}} \geqslant \dfrac{1}{2\sqrt{n+1}}$

for all $n \in \mathbb{N}$ & $\sum_{n=1}^{\infty} \frac{1}{2\sqrt{n+1}}$ diverges (since $\sum_{n=2}^{\infty} \frac{1}{\sqrt{n}}$ diverges)

it follow by "Direct Comparison" that $\underline{\sum_{n=1}^{\infty} a_n \text{ DIVERGES}}$.

(b) Now let $a_n = \dfrac{\sqrt{n+1} - \sqrt{n}}{n}$.

Since $\sqrt{n+1} - \sqrt{n} = \dfrac{1}{\sqrt{n+1} + \sqrt{n}} \leq \dfrac{1}{\sqrt{n}} \quad \forall n \in \mathbb{N}$

$$\Rightarrow \quad \frac{\sqrt{n+1} - \sqrt{n}}{n} \leq \frac{1}{n^{3/2}} \quad \forall n \in \mathbb{N}.$$

and $\sum_{n \geq 1}^{\infty} \frac{1}{n^{3/2}}$ converges it follows by "Direct Comparison"

that $\underline{\sum_{n \geq 1}^{\infty} a_n \quad \text{CONVERGES}}$