Limit Laws and more examples

Proposition (Limit Laws)

Suppose $\lim_{n\to\infty} a_n = A$ and $\lim_{n\to\infty} b_n = B$, then

1. $\lim_{n\to\infty} (a_n + b_n) = A + B$.

2. $\lim_{n\to\infty} (a_n b_n) = AB$.

   [In particular $\lim_{n\to\infty} k a_n = kA$ for any $k \in \mathbb{R}$]

3. $\lim_{n\to\infty} \left(\frac{a_n}{b_n}\right) = \frac{A}{B}$ provided $B \neq 0$.

Applications:

① Claim: $\lim_{n\to\infty} \frac{5n+1}{3n-2} = \frac{5}{3}$

Using limit laws 1, 2 (with $k=-2$) and 3 (since $3-2(0)\neq 0$).

Proof

$$\frac{5n+1}{3n-2} = \frac{5+\frac{1}{n}}{3-2\frac{1}{n}} \to \frac{5+0}{3-2(0)} = \frac{5}{3}.$$

Proof of Limit Laws:

Proof of 1: Let $\varepsilon > 0$. Since $a_n \to A$ we know $\exists\, N_1$ such that $n > N_1$ implies $|a_n - A| < \varepsilon/2$ (since $\frac{\varepsilon}{2} > 0$)

Similarly, since $b_n \to B$ we know $\exists\, N_2$ such that $n > N_2 \Rightarrow |b_n - B| < \frac{\varepsilon}{2}$.

Let $N = \max\{N_1, N_2\}$. If $n > N$, then

$$|(a_n+b_n)-(A+B)| = |(a_n-A)+(b_n-B)|$$

since $n > N_1$ & $n > N_2$.

$$\leq |a_n - A| + |b_n - B| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \quad \square$$

Proof of 2: Let $\varepsilon > 0$.

Since $\{b_n\}$ converges we know $\exists M > 0$ such that $|b_n| \le M \ \forall n \in \mathbb{N}$.

Since $b_n \to B$ we know $\exists N_1$ such that

$$|b_n - B| < \frac{\varepsilon}{2M} \quad \forall n > N_1 .$$

Since $a_n \to A$ we know $\exists N_2$ such that

$$|a_n - A| < \frac{\varepsilon}{2(|A|+1)} \quad \forall n > N_2 .$$

Set $N = \max\{N_1, N_2\}$. If $n > N$ it follows that

$$\begin{aligned}
|a_n b_n - AB| &= |b_n a_n - b_n A + b_n A - BA| \\
&\le |b_n||a_n - A| + |A||b_n - B| \\
&\le M|a_n - A| + (|A|+1)|b_n - B| \\
&< \underbrace{M\left(\frac{\varepsilon}{2M}\right)}_{=\varepsilon/2} + \underbrace{(|A|+1)\left(\frac{\varepsilon}{2(|A|+1)}\right)}_{=\varepsilon/2.} \\
&= \varepsilon .
\end{aligned}$$

Since $n > N_1$ & $n > N_2$

$\square$

Proof of 3: Since $\frac{a_n}{b_n} = a_n\left(\frac{1}{b_n}\right)$ it suffices in light of "limit law 2" above to establish:

Claim: If $b_n \to B$ with $B \ne 0$, then $\frac{1}{b_n} \to \frac{1}{B}$.

Proof of Claim: Let $\varepsilon > 0$. Since $b_n \to B$ we know $|b_n| \to |B|$.

Since $|B| > 0$ we know $\exists N_1$ such that $|b_n| > \frac{|B|}{2} \ \forall n > N_1$.

Since $b_n \to B$ we also know $\exists N_2$ such that $|b_n - B| < \frac{|B|^2}{2}\varepsilon \ \forall n > N_2$.

Set $N = \max\{N_1, N_2\}$. If $n > N$ it follows that

$$\left|\frac{1}{b_n} - \frac{1}{B}\right| = \frac{|b_n - B|}{|b_n||B|} \underset{\text{since } n>N_1}{<} \frac{|b_n - B|}{(|B|^2/2)} \underset{\text{since } n>N_2}{<} \frac{1}{(|B|^2/2)}\left(\frac{|B|^2}{2}\varepsilon\right) = \varepsilon .$$

$\square$