THE HARDY-LITTLEWOOD METHOD AND SUMS OF SQUARES

NEIL LYALL

1. Introduction

For $d, N \in \mathbb{N}$ we define

$$r_d(N) = \#\{(m_1, \dots, m_d) \in \mathbb{N}^d \,:\, m_1^2 + \cdots + m_d^2 = N\}.$$

The main objective of this note is to establish the following result.

Theorem 1.1. *If $d \geq 5$ then there exists a constant $c_0 > 0$ such that $r_d(N) \leq c_0 N^{(d-2)/2}$.*

In actual fact $r_d(N) \sim N^{(d-2)/2}$ whenever $d \geq 5$. A secondary objective of this note is to establish the general form of the asymptotic formula for $r_d(N)$ required to prove this, giving the required analysis of the error term, but deferring the careful analysis of the main term for another time, see Section 7.

We first make the important observation that we can turn this number-theoretic problem into an analytic problem of estimating (and eventually finding an asymptotic expansion for) an integral involving Weyl sums.

Proposition 1.2.

$$r_d(N) = \int_0^1 S_M(\alpha)^d e^{-2\pi i \alpha N} d\alpha$$

where

$$S_M(\alpha) = \sum_{m=1}^{M} e^{2\pi i m^2 \alpha} \quad \textit{with} \quad M = \lfloor \sqrt{N} \rfloor.$$

Proof. It follows immediately from the orthogonality relation

$$\int_0^1 e^{2\pi i n \alpha} d\alpha = \begin{cases} 1 \text{ if } n = 0 \\ 0 \text{ if } n \in \mathbb{Z}\backslash\{0\} \end{cases}$$

that

$$r_d(N) = \sum_{(m_1,\dots,m_d)\in[1,M]^d} \int_0^1 e^{2\pi i(m_1^2+\cdots+m_k^2-N)\alpha} d\alpha = \int_0^1 S_M(\alpha)^d e^{-2\pi i \alpha N} d\alpha. \qquad \square$$

2. Additive quadruples of squares

Before setting to work on proving Theorem 1.1 we note that it is easy to see that for any given $\eta > 0$ we must have $r_d(N) \leq c_\eta N^{(d-2)/2+\eta}$. Key to this observation (and our latter arguments on the minor arcs) is the following estimate for the number of additive quadruples of squares (a special case of Hua's lemma).

Lemma 2.1. *For any $\eta > 0$ there exists a constant $c_\eta > 0$ such that*

$$\int_0^1 |S_M(\alpha)|^4 \, d\alpha \leq c_\eta M^{2+\eta}.$$

Since we trivially have that $|S_M(\alpha)| \leq M$ it follows immediately from Lemma 2.1 that whenever $d \geq 4$ we have

$$r_d(N) \leq \int_0^1 |S_M(\alpha)|^d \, d\alpha \leq M^{d-4} \int_0^1 |S_M(\alpha)|^4 \, d\alpha \leq c_\eta N^{(d-2)/2+\eta}$$

for every $\eta > 0$.

Proof of Lemma 2.1. We note that

$$\int_0^1 |S_M(\alpha)|^4\,d\alpha = \#\{(x_1,x_2,y_1,y_2)\in[1,M]^4 \mid x_1^2-y_1^2=x_2^2-y_2^2\}.$$

It follows that

$$\int_0^1 |S_M(\alpha)|^4\,d\alpha = \sum_{\ell=-M^2}^{M^2} \#\{(x,y)\in[1,M]^2 \mid x^2-y^2=\ell\}^2 \leq 2M^2+4\sum_{\ell=1}^{M^2} d(\ell)^2,$$

where $d(\ell)$ denotes the number of divisors of ℓ. The result then follows once we recall the basic fact that for every fixed $\eta>0$,

$$\lim_{\ell\to\infty}\frac{d(\ell)}{\ell^\eta}=0.$$

This is easy to verify; since $f(\ell)=d(\ell)/\ell^\eta$ is multiplicative it suffice to prove that $\lim_{p^k\to\infty} f(p^k)=0$ as p^k runs through the sequence of all prime powers. The details can be found in any good elementary book on number theory, for example Nathanson's book *Elementary Methods in Number Theory.* □

In order to obtain the desired stronger result (Theorem 1.1) we will make use of estimates, on specific major and minor arcs, for the Weyl sum $S_M(\alpha)$.

3. The major and minor arcs

Informally one refers to the points in $[0,1]$ that are "close" to rationals a/q with "small" denominators as the major arcs and refer to the remaining points as the minor arcs.

In order to motivate the precise definition of these major and minor arcs we recall the Weyl inequality.

Proposition 3.1 (The Weyl inequality for quadratic monomials)**.** *If* $|\alpha-a/q|\leq q^{-2}$ *and* $(a,q)=1$*, then*

$$|S_M(\alpha)|\leq 20M\log M(1/q+1/M+q/M^2)^{1/2}.$$

This gives a non-trivial estimate whenever $M^\mu\leq q\leq M^{2-\mu}$ for some $0<\mu<1$. We will take $\mu=1/10$.

We now make our informal definition of the major and minor arcs more precise.

Definition 3.2 (Major arcs)**.** The major arcs are defined to be

$$\mathfrak{M} = \bigcup_{1\leq q\leq M^{1/10}} \bigcup_{\substack{1\leq a<q\\(a,q)=1}} \mathbf{M}_{a/q} \quad\cup\quad \mathbf{M}_{0/1}$$

where for $1\leq a<q$ with $(a,q)=1$ (and $a=0$, $q=1$) we define

$$\mathbf{M}_{a/q}=\left\{\alpha\,:\,\left|\alpha-\frac{a}{q}\right|\leq\frac{1}{M^{2-1/10}}\right\}.$$

Definition 3.3 (Minor arcs)**.** The minor arcs $\mathfrak{m}$ are simply defined to be $[0,1]\backslash\mathfrak{M}$.

It is easy to see that $|\mathfrak{M}|\leq M^{-2+3/10}$. We further make the observation that the major arcs are in fact a union of (necessarily short) pairwise disjoint intervals.

Lemma 3.4. *If* $a/q\neq a'/q'$ *with* $1\leq q,q'\leq M^{1/10}$*, then* $\mathbf{M}_{a/q}\cap\mathbf{M}_{a'/q'}=\emptyset$*.*

Proof. Suppose that $\mathbf{M}_{a/q}\cap\mathbf{M}_{a'/q'}\neq\emptyset$. Using the fact that $aq'-a'q\neq0$, we see that

$$\frac{2}{M^{2-1/10}}\geq\left|\frac{a}{q}-\frac{a'}{q'}\right|=\left|\frac{aq'-a'q}{qq'}\right|\geq\frac{1}{qq'}\geq\frac{1}{M^{1/5}},$$

a contradiction. □

4. Proof of Theorem 1.1

Theorem 1.1 follows immediately from Corollaries 4.2 and 4.5 below.

4.1. **The minor arc estimates.**

Proposition 4.1 (Minor arc estimate for S_M). *Let $M \in \mathbb{N}$. If $\alpha \in \mathfrak{m}$, then $|S_M(\alpha)| \leq CM^{1-1/40}$.*

Corollary 4.2. *Let $d, M \in \mathbb{N}$. If $d \geq 5$, then*

$$\int_{\mathfrak{m}} |S_M(\alpha)|^d d\alpha \leq CM^{d-2}M^{-1/80}.$$

Proof. It then follows Proposition 4.1 and Lemma 2.1, with $\eta = 1/80$, that

$$\begin{aligned}\int_{\mathfrak{m}} |S_M(\alpha)|^d d\alpha &\leq \sup_{\alpha\in\mathfrak{m}} |S_M(\alpha)|^{d-4} \int_0^1 |S_M(\alpha)|^4 \, d\alpha \\ &\leq CM^{d-4}M^{-(d-4)/40}M^2M^{1/80} \\ &\leq CM^{d-2}M^{-1/80}. \end{aligned}$$

□

Proof of Proposition 4.1. It follows from the Dirichlet principle[1] and the fact that $\alpha \in \mathfrak{m}$ that there exists a reduced fraction a/q with

$$M^{1/10} \leq q \leq M^{2-1/10}$$

such that $|\alpha - a/q| \leq q^{-2}$. It therefore follows from the Weyl inequality that

$$|S_M(\alpha)| \leq 60M^{1-1/20} \log M \leq CM^{1-1/40}.$$

□

4.2. **The major arc estimates.**

Proposition 4.4 (Major arc estimate for S_M). *If $\alpha \in \mathbf{M}_{a/q}$ with $1 \leq q \leq M^{1/10}$, then*

$$|S_M(\alpha)| \leq CMq^{-1/2}(1 + M^2|\alpha - a/q|)^{-1/2}.$$

Corollary 4.5. *If $\alpha \in \mathfrak{M}$ and $d \geq 5$, then*

$$\int_{\mathfrak{M}} |S_M(\alpha)|^d \, d\alpha \leq CM^{d-2}.$$

Proof. It follows from Propositon 4.4 that on a fixed major arc

$$\begin{aligned}\int_{\mathbf{M}_{a/q}} |S_M(\alpha)|^d \, d\alpha &\leq CM^dq^{-d/2} \int_{|\beta|\leq 1/qM^{2-1/10}} (1 + M^2|\beta|)^{-d/2} \, d\beta \\ &\leq CM^{d-2}q^{-d/2} \int_{-\infty}^{\infty} (1 + |\beta|)^{-d/2} \, d\beta \\ &\leq CM^{d-2}q^{-d/2}.\end{aligned}$$

Therefore

$$\int_{\mathfrak{M}} |S_M(\alpha)|^d \, d\alpha \leq CM^{d-2} \sum_{q=1}^{M^{1/10}} \sum_{a=0}^{q-1} q^{-d/2} \leq CM^{d-2} \sum_{q=1}^{\infty} q^{-(d-2)/2} \leq CM^{d-2}.$$

□

We are thus left with the task of proving Proposition 4.4.

---

[1] Recall that it follows immediately from Dirichlet's "principle of the pigeons" (Lemma 4.3 below) that for every $\alpha \in [0, 1]$ there exists $1 \leq q \leq M^{2-1/10}$ and $1 \leq a < q$ with $(a, q) = 1$ such that $|\alpha - a/q| \leq 1/qM^{2-1/10}$ $(\leq 1/q^2)$.

Lemma 4.3. *Let $\alpha \in \mathbb{R}$ and $Q \in \mathbb{N}$. Then there exists $1 \leq q \leq Q$ and $1 \leq a < q$ with $(a, q) = 1$ such that $|\alpha q - a| \leq 1/Q$.*

Proof. With loss of generality we shall assume that $\alpha > 0$. Of the reals $\alpha, 2\alpha, \ldots, (Q + 1)\alpha$, two clearly lie within Q^{-1} of each other (mod 1). Thus there must exists $j, k \in \mathbb{N}$ with $k > j$ and $a \in \mathbb{N} \cup \{0\}$ such that $|(k - j)\alpha - a| \leq Q^{-1}$. □

5. Proof of Proposition 4.4

We will now take a closer look at the behaviour of the Weyl sum on the major arcs. The following fundamental result, which gives a more precise approximation to S_M on $\mathfrak{M}$, will key to both the proof of Proposition 4.4 and later results.

Proposition 5.1. *If $\alpha \in \mathbf{M}_{a/q}$ with $1 \leq q \leq M^{1/10}$, then*

$$S_M(\alpha) = q^{-1}S(a,q)v(\alpha - a/q) + O(M^{1/5}), \tag{1}$$

where

$$S(a,q) := \sum_{r=0}^{q-1} e^{2\pi i r^2 a/q} \quad \text{and} \quad v(\beta) := \int_0^M e^{2\pi i x^2\beta}dx.$$

Proof. We can write $\alpha = a/q + \beta$ where $|\beta| \leq 1/M^{2-1/10}$ and $1 \leq q \leq M^{1/10}$. We can also write each $1 \leq m \leq M$ uniquely as $m = nq + r$ with $0 \leq r < q$ and $0 \leq n \leq M/q$. It then follows that

$$\begin{aligned} S_M(\alpha) &= \sum_{r=0}^{q-1}\sum_{n=0}^{M/q} e^{2\pi i(a/q+\beta)(nq+r)^2} + O(q) \\ &= \sum_{r=0}^{q-1} e^{2\pi i r^2 a/q} \sum_{n=0}^{M/q} e^{2\pi i(nq+r)^2\beta} + O(q). \end{aligned}$$

Since

$$\left|e^{2\pi i(nq+r)^2\beta} - e^{2\pi i n^2q^2\beta}\right| \leq \left|e^{2\pi i(2nqr+r^2)\beta} - 1\right| \leq C\frac{M}{q}q^2\frac{1}{M^{2-1/10}} \leq CM^{-1+1/5},$$

and

$$\begin{aligned} \left|\sum_{n=0}^{M/q} e^{2\pi i n^2q^2\beta} - \int_0^{M/q} e^{2\pi i x^2q^2\beta}dx\right| &\leq \sum_{n=0}^{M/q}\int_n^{n+1}\left|e^{2\pi i n^2q^2\beta} - e^{2\pi i x^2q^2\beta}\right|dx \\ &\leq \sum_{n=0}^{M/q} 2\pi(2n+1)q^2|\beta| \\ &\leq 20M^{1/10}, \end{aligned}$$

it follows that

$$\left|S_M(\alpha) - \frac{1}{q}S(a,q)v(\beta)\right| \leq CM^{1/5}. \qquad \square$$

Proposition 4.4 then follows immediately given the two basic lemmas below.

Lemma 5.2 (Gauss sum estimate)**.** *If $(a,q) = 1$, then $|S(a,q)| \leq \sqrt{2q}$. More precisely,*

$$|S(a,q)| = \begin{cases} \sqrt{q} & \text{if } q \text{ odd} \\ \sqrt{2q} & \text{if } q \equiv 0 \mod 4 \\ 0 & \text{if } q \equiv 2 \mod 4 \end{cases}.$$

Lemma 5.3 (Oscillatory integral estimate)**.** *For any $\lambda \geq 0$*

$$\left|\int_0^1 e^{2\pi i\lambda x^2}dx\right| \leq C(1+\lambda)^{-1/2}.$$

Proof of Proposition 4.4. Lemmas 5.2 and 5.3 imply that the main term in (1)

$$q^{-1}\left|S(a,q)v(\alpha - a/q)\right| \leq Mq^{-1/2}(1 + M^2|\alpha - a/q|)^{-1/2},$$

and since $q^{-1/2} \geq M^{-1/20}$ and $M^2(|\alpha - a/q| \leq M^{1/10}$, it follows that

$$Mq^{-1/2}(1 + M^2|\alpha - a/q|)^{-1/2} \geq M^{9/10} \gg M^{1/5}. \qquad \square$$

6. Proof of Lemmas 5.2 and 5.3

Proof of Lemma 5.2. Squaring-out $S(a,q)$ we obtain

$$|S(a,q)|^2 = \sum_{s=0}^{q-1}\sum_{r=0}^{q-1} e^{2\pi i a(r^2-s^2)/q}.$$

Letting $r = s+t$ and using the fact that $(a,q)=1$ and

$$\sum_{s=0}^{q-1} e^{2\pi i a(2st)/q} = \begin{cases} q & \text{if} \quad 2at \equiv 0 \mod q \\ 0 & \text{otherwise} \end{cases}$$

it follows that

$$|S(a,q)|^2 = \sum_{t=0}^{q-1} e^{2\pi i a t^2/q} \sum_{s=0}^{q-1} e^{2\pi i a(2st)/q} = \begin{cases} q & \text{if } \ q \text{ odd} \\ q\left(e^{2\pi i a(q/4)}+1\right) & \text{if } \ q \text{ even} \end{cases}. \qquad \square$$

Proof of Lemma 5.3. We need only consider the case when $\lambda \geq 1$. We write

$$\int_0^1 e^{2\pi i\lambda x^2}dx = \int_0^{\lambda^{-1/2}} e^{2\pi i\lambda x^2}dx + \int_{\lambda^{-1/2}}^1 e^{2\pi i\lambda x^2}dx =: I_1 + I_2.$$

It is then easy to see that $|I_1| \leq \lambda^{-1/2}$, while integration by parts gives that

$$\begin{aligned} |I_2| &= \left|\int_{\lambda^{-1/2}}^1 \frac{1}{4\pi i\lambda x}\left(\frac{d}{dx}e^{2\pi i\lambda x^2}\right)dx\right| \\ &\leq \frac{1}{4\pi\lambda}\left|\left[\frac{1}{x}\,e^{2\pi i\lambda x^2}\right]_{\lambda^{-1/2}}^1 + \int_{\lambda^{-1/2}}^1 \frac{1}{x^2}\,e^{2\pi i\lambda x^2}dx\right| \\ &\leq C\lambda^{-1/2}. \end{aligned} \qquad \square$$

7. The Asymptotic Formula of Hardy and Littlewood

We define the *singular series* for $r_d(N)$ to be the arithmetic function

$$\mathfrak{S}(N) = \sum_{q=1}^{\infty} A_N(q)$$

where

$$A_N(q) = \sum_{\substack{a=1 \\ (a,q)=1}}^{q} \left(q^{-1}S(a,q)\right)^d e^{-2\pi i Na/q}$$

and the *singular integral* for $r_d(N)$ to be the integral

$$J(N) = \int_{-\infty}^{\infty} v(\beta)^d e^{-2\pi i N\beta}\,d\beta.$$

Using Proposition 5.1 (and some analysis of additional error terms) one can establish the following.

Theorem 7.1 (Hardy-Littlewood)**.**

$$r_d(N) = \mathfrak{S}(N)J(N) + O(N^{d/4})$$

Proposition 7.2.

$$J(N) = \frac{\pi^{d/2}}{2^d\Gamma(d/2)}N^{(d-2)/2}$$

Proposition 7.3. *If $d \geq 5$, then there exists positive constant $c_1 = c_1(d), c_2 = c_2(d)$ such that*

$$c_1 \leq \mathfrak{S}(N) \leq c_2$$