Math 8100 Assignment 7
Hilbert Spaces

Due date: Thursday 17th of November 2022

1. (a) Prove that $\ell^2(\mathbb{N})$ is complete.

   Recall that $\ell^2(\mathbb{N}) := \{x = \{x_j\}_{j=1}^\infty : \|x\|_{\ell^2} < \infty\}$, *where* $\|x\|_{\ell^2} := \Big(\sum_{j=1}^\infty |x_j|^2\Big)^{1/2}$.

   (b) Let H be a Hilbert space. Prove the so-called *polarization identity*, namely that for any $x, y \in H$,

   $$\langle x, y\rangle = \frac{1}{4}\left(\|x+y\|^2 - \|x-y\|^2 + i\|x+iy\|^2 - i\|x-iy\|^2\right)$$

   and conclude that any invertible linear map from H to $\ell^2(\mathbb{N})$ is *unitary* if and only if it is *isometric*.

   Recall that if H_1 and H_2 are Hilbert spaces with inner products $\langle\cdot,\cdot\rangle_1$ and $\langle\cdot,\cdot\rangle_2$, then a mapping $U : H_1 \to H_2$ is said to be **unitary** *if it is an invertible linear map that preserves inner products, namely $\langle Ux, Uy\rangle_2 = \langle x, y\rangle_1$, and an* **isometry** *if it preserves "lengths", namely $\|Ux\|_2 = \|x\|_1$.*

2. Let H be a Hilbert space.

   (a) Let $x \in H$ and $\{u_1, \dots, u_N\}$ be a orthonormal set in H. Prove that the best approximation to x in H by an element of the form $\sum_{n=1}^N c_n u_n$, with $c_1, \dots, c_N \in \mathbb{C}$, is given when $c_n = \langle x, u_n\rangle$.

   (b) Conclude from part (a), or otherwise, that finite dimensional subspaces of H are always closed.

3. In $L^2([0,1])$ let $e_0(x) = 1$, $e_1(x) = \sqrt{3}(2x-1)$ for all $x \in (0,1)$.

   (a) Show that e_0, e_1 is an orthonormal system in $L^2(0,1)$.

   (b) Show that the polynomial of degree 1 which is closest with respect to the norm of $L^2(0,1)$ to the function $f(x) = x^2$ is given by $g(x) = x - 1/6$. What is $\|f - g\|_2$?

4. (a) Verify that the following systems are orthogonal in $L^2([0,1])$:

   i. $\{1/\sqrt{2}, \cos(2\pi x), \sin(2\pi x), \dots, \cos(2\pi kx), \sin(2\pi kx), \dots\}$

   ii. $\{e^{2\pi ikx}\}_{k=-\infty}^\infty$

   (b) i. Show that $L^2([0,1]) \subseteq L^1([0,1])$.

   ii. Show that $L^2([0,1])$ is in fact dense in $L^1([0,1])$.

   iii. Prove the so-called *Riemann-Lebesgue lemma*: If $f \in L^1([0,1])$, then

   $$\lim_{k\to\infty} \int_0^1 f(x)e^{-2\pi ikx}\,dx = 0.$$

5. (a) The first three Legendre polynomials are

   $$P_0(x) = 1, \quad P_1(x) = x, \quad P_2(x) = (3x^2-1)/2.$$

   Show that the orthonormal system in $L^2([-1,1])$ obtained by applying the Gram-Schmidt process to $1, x, x^2$ are scalar multiples of these.

(b) Compute

$$\min_{a,b,c} \int_{-1}^{1} |x^3 - a - bx - cx^2|^2 \, dx$$

(c) Find

$$\max \int_{-1}^{1} x^3 g(x) \, dx$$

where g is subject to the restrictions

$$\int_{-1}^{1} g(x) \, dx = \int_{-1}^{1} x g(x) \, dx = \int_{-1}^{1} x^2 g(x) \, dx = 0 \, ; \quad \int_{-1}^{1} |g(x)|^2 \, dx = 1.$$

6. Let

$$\mathcal{C} = \left\{ f \in L^2([0,1]) \, : \, \int_0^1 f(x) \, dx = 1 \text{ and } \int_0^1 x f(x) \, dx = 2 \right\}$$

(a) Let $g(x) = 18x^2 - 5$. Show that $g \in \mathcal{C}$ and that

$$\mathcal{C} = g + \mathcal{S}^\perp$$

where $\mathcal{S}^\perp$ denotes the orthogonal complement of $\mathcal{S} = \mathrm{Span}\left(\{1, x\}\right)$.

(b) Find *the* function $f_0 \in \mathcal{C}$ for which

$$\int_0^1 |f_0(x)|^2 dx = \inf_{f \in \mathcal{C}} \int_0^1 |f(x)|^2 dx.$$

Extra Challenge Problems

Not to be handed in with the assignment

1. Prove that every closed convex set K in a Hilbert space has a unique element of minimal norm.
2. **The Mean Ergodic Theorem:** *Let U be a unitary operator on a Hilbert space H.*

   Prove that if $M = \{x \, : \, Ux = x\}$ and $S_N = \frac{1}{N} \sum_{n=0}^{N-1} U^n$, then $\lim_{N \to \infty} \|S_N x - Px\| = 0$ for all $x \in H$, where Px denotes the orthogonal projection of x onto M.