Theorem (Ruzsa - Plünnecke)

If $A \subseteq G$ (additive abelian group) and $\exists\, B \subseteq G$ s.t. $|A+B| \leq C|B|$,

then $|kA - \ell A| \leq C^{k+\ell}|B|$ for all $\ell, k \in \mathbb{Z}$.

[In particular, $|kA - \ell A| \leq C^{k+\ell}|A|$ for all $\ell, k \in \mathbb{Z}$ if either $|A+A| \leq C|A|$ or $|A-A| \leq C|A|$.]

Proof (Petridis, 2011)

Let A & B be two sets and suppose $|A+B| \leq C|B|$.

Choose non-empty subset B' of B such that the quantity

$$\frac{|A+B'|}{|B'|}$$ is minimized. (call this number C')

Note:
- $|A+B'| = C'|B'|$
- $|A+Z| \geq C'|Z|$ for all $Z \subseteq B$.

Lemma: With A & B' as above, it follows that for any D,

$$|A+B'+D| \leq C'|B'+D|.$$

From this Lemma it follows that

$$|kA+B'| \leq C'|(k-1)A+B'| \leq \cdots \leq (C')^k|B'|. \quad (*)$$

[and since $C' \leq C$ & $B' \subseteq B$, that $|kA| \leq C^k|B|$ ✓ if $\ell = 0$.]

In order to deduce the main theorem from (*) we will use

<u>Ruzsa Δ-inequality</u>: $|U|\cdot|V-W| \le |U+V|\cdot|U+W|$

<u>Proof</u>: $v-w = (u+v)-(u+w)$, so each $v-w \in V-W$ has at least $|U|$ representations of the form $x+y$ with $(x,y) \in (U+V)\times(U+W)$

Combining this with (*) gives, for any $k, \ell \in \mathbb{Z}$

$$|B'|\cdot|kA-\ell A| \le |kA+B'|\cdot|\ell A+B'| \le (C')^{k+\ell}|B'|^2$$

$$\Rightarrow |kA-\ell A| \le C^{k+\ell}|B| \quad (\text{using } C' \le C \text{ and } B' \subseteq B). \quad \square$$

<u>Proof of Lemma</u>: (Induction on the size of D)

If $D=\{d\}$, then $|A+B'+D| = |A+B'| = C'|B'| = C'|B'+D|$. ✓

Now suppose we know that

$$|A+B'+D| \le C'|B'+D|$$

and want to show it for $D' = D \cup \{d\}$.

$$A+B'+D' = (A+B'+D) \cup (A+B'+d)$$

$$\Rightarrow |A+B'+D'| \le |A+B'+D| + |A+B'| \le C'\big(\underbrace{|B'+D| + |B'|}_{\overset{?}{\le}\, |B'+D'|}\big)$$

Only true if $B'+D$ & $B'+d$ disjoint!!

Being more careful...

$$A+B'+D' = (A+B'+D) \cup \left[(A+B'+d) \setminus (Z+A+d) \right]$$

where Z is all elements $b' \in B'$ s.t. $b'+A+d \subseteq B'+A+D$.

Using the fact that $|Z+A| \geq C'|Z|$ it follows that

$$|A+B'+D'| \leq |A+B'+D| + |A+B'| - |A+Z|$$

$$\leq C' \Big(\underbrace{|B'+D| + |B'| - |Z|}_{\underset{?}{\leq} |B'+D'|} \Big)$$

Well,

$$B'+D' = (B'+D) \cup ((B'+d) \setminus W+d)$$

disjoint & $W+d \subseteq B'+d$

where W is the set of all $b' \in B'$ s.t. $b'+d \in B'+D$.

$$\Rightarrow |B'+D'| = |B'+D| + |B'| - |W|$$

$$\geq |B'+D| + |B'| - |Z| \text{, since } W \subseteq Z \subseteq B'.$$

$\square$

$(w \in W \Rightarrow w+d \in B'+D \Rightarrow w+A+d \subseteq A+B'+D \Rightarrow w \in Z.)$