Math 3100
Sample Exam 2 – Version 1

No calculators. Show your work. Give full explanations. Good luck!

1. (15 points)

   (a) Carefully state what it means to say that $\sum_{n=1}^{\infty} a_n$ converges to A and prove that if this indeed the case, then $\sum_{n=1}^{\infty}(10a_n)$ converges to $10A$.

   (b) Prove that if $b_n > 0$ for all $n \in \mathbb{N}$ and $\sum_{n=1}^{\infty} b_n$ converges, then $\sum_{n=1}^{\infty} b_n^2$ also converges.

   (c) Prove that if a series converges absolutely, then it is convergent.

2. (15 points)

   (a) Determine if the following series are absolutely convergent, conditionally convergent, or divergent. Justify your answers.

   $$\text{(i)} \quad \sum_{n=1}^{\infty} \frac{(-1)^n n}{n^2+1} \qquad\qquad \text{(ii)} \quad \sum_{n=1}^{\infty} \frac{\log n}{n^{3/2}}$$

   (b) Use the "Cauchy Condensation Test" to determine the convergence or divergence of

   $$\sum_{n=2}^{\infty} \frac{1}{n \log n}$$

   (c) Find all $x \in \mathbb{R}$ for which $\sum_{n=1}^{\infty} \frac{(-2)^n x^{2n}}{n}$ converges.

3. (20 points) Let $f : \mathbb{R} \to \mathbb{R}$.

   (a) Carefully state the ε-δ definition of what it means for f to be *continuous* at x_0 and conclude that if f is continuous at x_0 with $f(x_0) = 2$, then there exists $\delta > 0$ such that $f(x) \geq 1$ whenever $|x - x_0| < \delta$.

   (b) Use the definition from part (a) to prove that $f(x) = \frac{1}{x}$ is continuous at $x_0 = 1$.

   (c) Prove that f is continuous at x_0 if and only if $\lim_{n\to\infty} f(x_n) = f(x_0)$ for all sequences with $\lim_{n\to\infty} x_n = x_0$.

Math 3100 - Sample Exam 2 (Version 1) - SOLUTIONS

1. (a) We say that $\sum_{n=1}^{\infty} a_n$ converges to A iff

$$\lim_{n\to\infty} \underbrace{(a_1+a_2+\cdots+a_n)}_{=: s_n \text{ ("}n^{th}\text{ partial sum")}} \text{ exists \& equals } A.$$

Claim

If $\sum_{n=1}^{\infty} a_n$ converges to A, then $\sum_{n=1}^{\infty} (10a_n)$ converges to $10A$.

Proof

We know that $\lim_{n\to\infty}(a_1+\cdots+a_n)=A$. It follows that

$$\lim_{n\to\infty} \underbrace{(10a_1+10a_2+\cdots+10a_n)}_{\text{This is the } n^{th} \text{ partial sum for the series } \sum_{n=1}^{\infty} 10a_n.} \overset{\text{"constant times limit law for sequences."}}{=} 10\lim_{n\to\infty}(a_1+a_2+\cdots+a_n) = 10A. \quad \square$$

(b) Claim: If $b_n>0$ for all $n\in\mathbb{N}$ and $\sum_{n=1}^{\infty} b_n$ converges, then $\sum_{n=1}^{\infty} b_n^2$ converges also.

Proof. Since $\sum_{n=1}^{\infty} b_n$ converges we know that $\lim_{n\to\infty} b_n = 0$ and hence that $\exists N$ such that $0<b_n\le 1$ for all $n>N$.

$$\Rightarrow 0<b_n^2\le b_n \text{ for all } n>N.$$

It thus follows by "Direct Comparison" that $\sum_{n=1}^{\infty} b_n^2$ converges $\square$.

(c) Claim:

If $\sum_{n=1}^{\infty} |a_n|$ converges, then $\sum_{n=1}^{\infty} a_n$ converges also.

Proof 1 (Using the "Cauchy Criterion for Series").

Recall: Cauchy Criterion for Series

$\sum_{n=1}^{\infty} b_n$ converges $\Leftrightarrow$ $\forall \varepsilon > 0\ \exists N$ such that if $n > m > N$, then $\left|\sum_{k=m+1}^{n} b_k\right| < \varepsilon$.

Let $\varepsilon > 0$. Since $\sum_{n=1}^{\infty} |a_n|$ converges we know that $\exists N$ such that if $n > m > N$, then $\left|\sum_{k=m+1}^{n} |a_k|\right| = \sum_{k=m+1}^{n} |a_k| < \varepsilon$.

Using direction $\Rightarrow$ of CC.

Since $\left|\sum_{k=m+1}^{n} a_k\right| = |a_{m+1} + a_{m+2} + \cdots + a_n|$

$\leq |a_{m+1}| + |a_{m+2}| + \cdots + |a_n| = \sum_{k=m+1}^{n} |a_k|$

Δ-inequality

it follows that if $n > m > N$, then $\left|\sum_{k=m+1}^{n} a_k\right| < \varepsilon$ and hence by direction $\Leftarrow$ of the Cauchy Criterion for Series that

$$\sum_{n=1}^{\infty} a_n \text{ converges.} \qquad \square$$

Proof 2 (Using "Direct Comparison")

Since $-|a_n| \leq a_n \leq |a_n|$ for all $n \in \mathbb{N}$

$\Rightarrow 0 \leq a_n + |a_n| \leq 2|a_n|$ for all $n \in \mathbb{N}$.

Since $\sum_{n=1}^{\infty} (2|a_n|)$ converges it follows by "Direct Comparison" that $\sum_{n=1}^{\infty} (a_n + |a_n|)$ also converges.

Since $a_n = (a_n + |a_n|) - |a_n|$ and both

$$\sum_{n=1}^{\infty} (a_n + |a_n|) \;\&\; \sum_{n=1}^{\infty} |a_n| \text{ converge}$$

it follows that $\sum_{n=1}^{\infty} a_n$ also converges. (by "Difference Limit Law")

$\square$.

2.

(a) (i) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n n}{n^2+1}$ CONVERGES CONDITIONALLY.

since $\displaystyle\sum_{n=1}^{\infty} \frac{n}{n^2+1}$ diverges. (since $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$ diverges and $\displaystyle\lim_{n\to\infty} \frac{\frac{n}{n^2+1}}{\frac{1}{n}} = \lim_{n\to\infty} \frac{n^2}{n^2+1} = 1$. by "Limit Comparison".)

But $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n n}{n^2+1}$ converges

[By Alt. Series Test, since $\frac{n}{n^2+1} \searrow 0$.]

(ii) $\displaystyle\sum_{n=1}^{\infty} \frac{\log n}{n^{3/2}}$ CONVERGES ABSOLUTELY.

since $\log n \le n^{1/4}$ for all sufficiently large n implies

$$\frac{\log n}{n^{3/2}} \le \frac{n^{1/4}}{n^{3/2}} = \frac{1}{n^{5/4}} \text{ "eventually" } \& \sum_{n=1}^{\infty} \frac{1}{n^{5/4}} \text{ converges.}$$

(b) Since $\frac{1}{n\log n} \searrow 0$ and

$$\sum_{k=1}^{\infty} 2^k \frac{1}{2^k \log(2^k)} = \sum_{k=1}^{\infty} \frac{1}{(\log 2) k} \text{ diverges} \quad \text{(right ?)}$$

it follows from the "Cauchy Condensation Test" that

$$\sum_{n=2}^{\infty} \frac{1}{n\log n} \text{ diverges also.}$$

(c) Claim $\sum_{n=1}^{\infty} \frac{(-2)^n x^{2n}}{n}$ converges $\Leftrightarrow$ $x \in \left[-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right]$.

Proof Let $a_n = \frac{(-2)^n x^{2n}}{n}$.

Since $\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{(-2)^{n+1} x^{2n+2}}{n+1} \cdot \frac{n}{(-2)^n x^{2n}}\right|$

$$= (2)\left(\frac{n}{n+1}\right)|x|^2 \longrightarrow 2|x|^2$$

it follows from the "Ratio Test" that $\sum a_n$ converges absolutely if $|x| < \frac{1}{\sqrt{2}}$ and diverges if $|x| > \frac{1}{\sqrt{2}}$.

If $|x| = \frac{1}{\sqrt{2}}$, then $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ which converges by the Alt. Series test since $\frac{1}{n} \searrow 0$. $\square$

3. (a) A function $f: \mathbb{R} \to \mathbb{R}$ is <u>continuous at a point</u> $x_0 \in \mathbb{R}$ if

$\forall \varepsilon > 0 \; \exists \delta > 0$ such that if $|x - x_0| < \delta$, then $|f(x) - f(x_0)| < \varepsilon$.

<u>Claim</u> If $f: \mathbb{R} \to \mathbb{R}$ is continuous at x_0 and $f(x_0) = 2$, then

$\exists \delta > 0$ such that if $|x - x_0| < \delta$, then $f(x) \geq 1$.

<u>Proof</u>

Since f is conts at x_0 & $f(x_0) = 2$ it follows (with $\varepsilon = 1$)

that $\exists \delta > 0$ such that if $|x - x_0| < \delta$, then

$$|f(x) - f(x_0)| < 1$$

$$\Updownarrow \quad \leftarrow \text{since } f(x_0) = 2$$

$$1 < f(x) < 3$$

□

(b) <u>Claim</u> $f(x) = \frac{1}{x}$ is continuous at $x_0 = 1$.

<u>Proof</u> Let $\varepsilon > 0$ and set $\delta = \min\{\frac{1}{2}, \frac{\varepsilon}{2}\}$.

If $|x - 1| < \delta$ it follows that

since $|x-1| < \varepsilon/2$

$$\left|\frac{1}{x} - \frac{1}{1}\right| = \frac{|x-1|}{|x|} < 2|x-1| < 2\left(\frac{\varepsilon}{2}\right) = \varepsilon.$$

□

since $|x-1| < \frac{1}{2}$

$\Rightarrow |x| > \frac{1}{2} \Leftrightarrow \frac{1}{|x|} < 2$

(c) Claim

$f: \mathbb{R} \to \mathbb{R}$ conts at x_0 $\Longleftrightarrow$ $\lim_{n\to\infty} f(x_n) = f(x_0)$ for all seqs $x_n \to x_0$.

Proof

($\Rightarrow$) Let $\{x_n\}$ be a sequence with $\lim_{n\to\infty} x_n = x_0$ and $\varepsilon > 0$.

Since f conts at x_0 we know $\exists\ \delta > 0$ such that if

$|x - x_0| < \delta$ then $|f(x) - f(x_0)| < \varepsilon$.

Since $x_n \to x_0$ we know $\exists N$ such that if $n > N$ then

$|x_n - x_0| < \delta$

and hence $|f(x_n) - f(x_0)| < \varepsilon$ as required.

($\Leftarrow$) [Contrapositive]

Suppose f is not continuous at x_0. This means $\exists\ \varepsilon_0 > 0$ such that for every $\delta > 0$ there exists $x \in \mathbb{R}$ with $|x - x_0| < \delta$, but

$|f(x) - f(x_0)| \geq \varepsilon_0$.

In particular, $\exists\ x_1 \in \mathbb{R}$ with $|x_1 - x_0| < 1$ and $|f(x_1) - f(x_0)| \geq \varepsilon_0$

$\exists\ x_2 \in \mathbb{R}$ with $|x_2 - x_0| < \frac{1}{2}$ and $|f(x_2) - f(x_0)| \geq \varepsilon_0$

$\vdots$

In fact, $\forall\, n \in \mathbb{N}\ \exists\ x_n \in \mathbb{R}$ with $|x_n - x_0| < \frac{1}{n}$ but $|f(x_n) - f(x_0)| \geq \varepsilon_0$

Clearly, $\lim_{n\to\infty} x_n = x_0$ (by "Baby Squeeze"), but $f(x_n) \not\to f(x_0)$.

$\square$