# Math 8100 Assignment 2
# Lebesgue measure and outer measure

*Due date: Thursday the 8th of September 2022*

1. Prove that if $E \subseteq \mathbb{R}$ with $m_*(E) = 0$, then $E^2 := \{x^2 \mid x \in E\}$ also has Lebesgue outer measure zero.
   *Hint: First consider the case when $E$ is a bounded subset of $\mathbb{R}$.*

2. Prove that if $E_1$ and $E_2$ are measurable subsets of $\mathbb{R}^n$, then
$$m(E_1 \cup E_2) + m(E_1 \cap E_2) = m(E_1) + m(E_2).$$

3. Suppose that $A \subseteq E \subseteq B$, where $A$ and $B$ are Lebesgue measurable subsets on $\mathbb{R}^n$.
   (a) Prove that if $m(A) = m(B) < \infty$, then $E$ is measurable.
   (b) Give an example showing that the same conclusion does not hold if $A$ and $B$ have infinite measure.

4. Suppose $A$ and $B$ are a pair of compact subsets of $\mathbb{R}^n$ with $A \subseteq B$, and let $a = m(A)$ and $b = m(B)$. Prove that for any $c$ with $a < c < b$, there is a compact set $E$ with $A \subseteq E \subseteq B$ and $m(E) = c$.
   *Hint: As a warm-up example, consider the one dimensional example where $A$ a compact measurable subset of $B := [0,1]$ and the quantity $m(A) + t - m(A \cap [0,t])$ as a function of $t$.*

5. Let $\mathcal{N}$ denote the non-measurable subset of $[0,1]$ that was constructed in lecture.
   (a) Prove that if $E$ is a measurable subset of $\mathcal{N}$, then $m(E) = 0$.
   (b) Show that $m_*([0,1] \setminus \mathcal{N}) = 1$
   [*Hint: Argue by contradiction and pick an open set $G$ such that $[0,1] \setminus \mathcal{N} \subseteq G \subseteq [0,1]$ with $m_*(G) \leq 1 - \varepsilon$.*]
   (c) Conclude that there exists *disjoint* sets $E_1 \subseteq [0,1]$ and $E_2 \subseteq [0,1]$ for which
$$m_*(E_1 \cup E_2) \neq m_*(E_1) + m_*(E_2).$$

6. (a) **The Borel-Cantelli Lemma.** Suppose $\{E_j\}_{j=1}^{\infty}$ is a countable family of measurable subsets of $\mathbb{R}^n$ and that
$$\sum_{j=1}^{\infty} m(E_j) < \infty.$$
   Let
$$E = \limsup_{j\to\infty} E_j := \{x \in \mathbb{R}^n \,:\, x \in E_j, \text{ for infinitely many } j\}.$$
   Show that $E$ is measurable and that $m(E) = 0$. *Hint: Write $E = \cap_{k=1}^{\infty} \cup_{j \geq k} E_j$.*
   (b) Given any irrational $x$ one can show (using the pigeonhole principle, for example) that there exists infinitely many fractions $a/q$, with $a$ and $q$ relatively prime integers, such that
$$\left| x - \frac{a}{q} \right| \leq \frac{1}{q^2}.$$
   However, show that the set of those $x \in \mathbb{R}$ such that there exists infinitely many fractions $a/q$, with $a$ and $q$ relatively prime integers, such that
$$\left| x - \frac{a}{q} \right| \leq \frac{1}{q^3}$$
   is a set of Lebesgue measure zero.

**Extra Challenge Problems**

*Not to be handed in with the assignment*

1. Prove that any $E \subset \mathbb{R}$ with $m_*(E) > 0$ necessarily contains a non-measurable set.

2. The **outer Jordan content** $J_*(E)$ of a set $E$ in $\mathbb{R}$ is defined by

$$J_*(E) = \inf \sum_{j=1}^{N} |I_j|,$$

where the infimum is taken over every *finite* covering $E \subseteq \cup_{j=1}^{N} I_j$, by intervals $I_j$.

   (a) Prove that $J_*(E) = J_*(\bar{E})$ for every set $E$ (here $\bar{E}$ denotes the closure of $E$).

   (b) Exhibit a countable subset $E \subseteq [0,1]$ such that $J_*(E) = 1$ while $m_*(E) = 0$.

3. If $I$ is a bounded interval and $\alpha \in (0,1)$, let us call the open interval with the same midpoint as $I$ and length equal to $\alpha$ times the length of $I$ the "open middle $\alpha$th" of $I$. If $\{\alpha_j\}_{j=1}^{\infty}$ is any sequence of numbers in $(0,1)$, then, we can define a decreasing sequence $\{K_j\}$ of closed sets as follows: $K_0 = [0,1]$, and $K_j$ is obtained by removing the the open middle $\alpha_j$th from each of the intervals that make up $K_{j-1}$. The resulting limiting set $K = \bigcap_{j=1}^{\infty} K_j$ is called a **generalized Cantor set**.

   (a) Suppose $\{\alpha_j\}_{j=1}^{\infty}$ is any sequence of numbers in $(0,1)$.

      i. Prove that $\prod_{j=1}^{\infty}(1-\alpha_j) > 0$ if and only if $\sum_{j=1}^{\infty} \alpha_j < \infty$.

      ii. Given $\beta \in (0,1)$, exhibit a sequence $\{\alpha_j\}$ such that $\prod_{j=1}^{\infty}(1-\alpha_j) = \beta$.

   (b) Given $\beta \in (0,1)$, construct an open set $G$ in $[0,1]$ whose boundary has Lebesgue measure $\beta$.
   *Hint: Every closed nowhere dense set is the boundary of an open set.*