A Somewhat Soft Proof of Roth's Theorem (via Varnavides)

Recall

$$r_3(N) := \max_{A \subseteq \{1,\dots,N\}} \{ |A| : A \text{ contains } \underline{\text{no}} \text{ 3AP's} \}$$

and

Lemma 1 (Quantitative Varnavides)

For any $1 \leq M \leq N$ and $B \subseteq \{1,\dots,N\}$,

$$\#\,\text{3AP's in } B \geq \left(\frac{\frac{|B|}{N} - \frac{r_3(M)+2}{M}}{M^4} \right) N^2 .$$

In this lecture we shall use this lemma (and a little Fourier analysis) to prove:

Theorem 1: Let $M, N \in \mathbb{N}$. There exists constant $C > 0$ such that

$$\frac{r_3(N)}{N} \leq \frac{4}{5} \frac{r_3(M)}{M}$$

provided $N \geq \exp(CM^{10})$ and M is sufficiently large.

* It immediately follow from this that $\limsup_{N \to \infty} \frac{r_3(N)}{N} = 0$, thus gives yet another (although quantitatively rather weak) proof of Roth's theorem.

"Tower-type bounds".

Proof of Theorem 1

Let $A \subseteq \{1, \dots, N\}$ with no 3AP's and $|A| = r_3(N)$. Key to this argument is to construct, from this extremal set A, a new set

$$B \subseteq \{1, \dots, N\}$$

with the following properties:

"B much bigger than A"

"B still has few 3AP's".

(i) $|B| \geq \frac{4}{3}|A|$

& (ii) $\#\text{ 3AP's in } B \ll \left(\frac{\log\log N}{\log N}\right)^{1/2} N^2$.

* This construction will amount to defining $B = A \cup (A+t) \cup (A-t)$, for some appropriately chosen value of t (as large as we can choose it!)

Given the existence of such a set B, Theorem 1 follows by combining this with Lemma 1: Indeed,

$$\frac{\frac{|B|}{N} - \frac{r_3(M)+1}{M}}{M^4} \underset{\substack{\uparrow \\ \text{Lemma 1}}}{\leq} \frac{\#\text{ 3AP's in } B}{N^2} \underset{\substack{\uparrow \\ \text{(ii)}}}{\leq} C\left(\frac{\log\log N}{\log N}\right)^{1/2}$$

$$\Rightarrow \frac{|B|}{N} \leq \frac{r_3(M)}{M} + \frac{2}{M} + \underbrace{C\left(\frac{\log\log N}{\log N}\right)^{1/2} M^4}_{\leq \frac{1}{M} \text{ if } M \ll \left(\frac{\log\log N}{\log N}\right)^{-1/10}} \leq \frac{r_3(M)}{M} + \frac{3}{M}$$

$$\Rightarrow \frac{r_3(N)}{N} = \frac{|A|}{N} \underset{\substack{\uparrow \\ \text{(i)}}}{\leq} \frac{3}{4}\frac{|B|}{N} \leq \frac{3}{4}\frac{r_3(M)}{M} + \frac{1}{4M}$$

$$\leq \frac{4}{5}\frac{r_3(M)}{M} \text{ if } M \text{ suff. large.}$$

Recall $r_3(M) \gg M^{1/2}$ (say)

Construction of B

We initially define, for a parameter $0<t<N^{1/2}$ to be determined,

$$B' = A \cup (A+t) \cup (A-t) \subseteq \{1-t, \dots, N+t\}.$$

Since A contains no 3AP's we know that

$$1_{B'}(x) \leq 1_A(x) + 1_A(x+t) + 1_A(x-t) \leq 2\, 1_{B'}(x)$$

$$\Rightarrow \quad |B'| \geq \frac{3}{2}|A|.$$

Now define
$$B := B' \cap \{1, \dots, N\}.$$

$|A| = r_3(N) \gg N^{1/2}$

Note:

- $|B| \geq |B'| - 2t \geq \frac{3}{2}|A| - 2N^{1/2} \geq \frac{4}{3}|A|$ ✓
- $\#$ 3AP's in $B \leq \#$ 3AP's in B'.

* Matter thus reduce to showing that $\#$ 3AP's in $B' \ll \left(\frac{\log\log N}{\log N}\right)^{1/2} N^2$.

- We know $\exists$ prime $2N < p \leq 4N$, and embed

$$\{1, \dots, N\} \subseteq \left\{\frac{1-p}{2}, \dots, \frac{p+1}{2}\right\} \simeq \mathbb{Z}_p.$$

Recall, given $f_1, f_2, f_3 : \mathbb{Z}_p \to \mathbb{C}$

$$AP_3(f_1, f_2, f_3) = \frac{1}{p^2} \sum_{x,d \in \mathbb{Z}_p} f_1(x) f_2(x+d) f_3(x+2d)$$
$$= \sum_{\xi \in \mathbb{Z}_p} \hat{f}_1(\xi) \hat{f}_2(-2\xi) \hat{f}_3(\xi).$$

<u>Note</u>: Since $A \subseteq \{1, \dots, N\}$ has <u>no</u> 3AP's and $2N < p \leq 4N$, it follows that

$$AP_3(1_A, 1_A, 1_A) = \frac{r_3(N)}{p^2} \qquad \text{(only trivial 3AP's)}.$$

Now define

$$f(x) := \frac{1}{3}\left(1_A(x) + 1_A(x+t) + 1_A(x-t)\right).$$

Since $f(x) \geq \frac{1}{3} 1_{B'}(x)$, it follows that

$$\underbrace{AP_3(1_{B'}, 1_{B'}, 1_{B'})}_{\#\,3AP\text{'s in } B'} \leq 27\, AP_3(f, f, f)$$

Furthermore,

$$|AP_3(f,f,f) - AP_3(1_A, 1_A, 1_A)|$$
$$\leq \left| \frac{1}{p^2} \sum_{x,d} \left\{ f(x) f(x+d) f(x+2d) - 1_A(x) 1_A(x+d) 1_A(x+2d) \right\} \right|$$
$$\leq \left| \frac{1}{p^2} \sum_{x,d} \left\{ f(x) - 1_A(x) \right\} \left\{ f(x+d) f(x+2d) \right\} \right|$$
$$+ \left| \frac{1}{p^2} \sum_{x,d} \left\{ f(x+d) - 1_A(x+d) \right\} 1_A(x) f(x+2d) \right|$$
$$+ \left| \frac{1}{p^2} \sum_{x,d} \left\{ f(x+2d) - 1_A(x+2d) \right\} 1_A(x) 1_A(x+d) \right|$$
$$\leq \left| \sum_{\xi} \left(\hat{f}(\xi) - \hat{1}_A(\xi) \right) \hat{f}(-2\xi) \hat{f}(\xi) \right|$$
$$+ \left| \sum_{\xi} \left(\hat{f}(-2\xi) - \hat{1}_A(-2\xi) \right) \hat{1}_A(\xi) \hat{f}(\xi) \right|$$
$$+ \left| \sum_{\xi} \left(\hat{f}(\xi) - \hat{1}_A(\xi) \right) \hat{1}_A(\xi) \hat{1}_A(-2\xi) \right| \leq 3 \|\hat{f} - \hat{1}_A\|_\infty$$

Now consider,

Easy Exercise.

$$|\hat{f}(\xi) - \hat{1}_A(\xi)| = \frac{1}{3}|\hat{1}_A(\xi)|\,|(e^{2\pi i t\xi/p}-1)+(e^{-2\pi i t\xi/p}-1)|$$

$$\leq \frac{4}{3}\varepsilon\delta \quad \text{for all } \underline{t\in B(\mathrm{Spec}_\varepsilon(A),2\varepsilon)}$$

where $\delta = |A|/p$, $\underline{\text{for all } \xi\in\mathbb{Z}_p}$.

Thus,

$$AP_3(f,f,f) \leq AP_3(1_A,1_A,1_A) + 3\,\|\hat{f}-\hat{1}_A\|_\infty \ll \varepsilon\delta.$$

¿ <u>But how small can we take $\varepsilon\delta$</u> ?

Plancherel

By pigeonhole principle $\exists\ 1\leq t\leq (2\varepsilon)^{-d}$ with $d = |\mathrm{Spec}_\varepsilon(A)| \ll \varepsilon^{-2}\delta^{-1}$ such that $t\in B(\mathrm{Spec}_\varepsilon(A),2\varepsilon)$. Recall we <u>need</u> $t\leq N^{1/2}$, so we "need"

$$(2\varepsilon)^{-\varepsilon^{-2}\delta^{-1}} \leq \exp\underbrace{\left((\varepsilon\delta)^{-2}\log(2\varepsilon\delta)^{-1}\right) \leq N^{1/2}}$$

$$\Leftrightarrow (\varepsilon\delta)^{-2}\log(2\varepsilon\delta)^{-1} \ll \log N$$

Hence as long as

$$\underline{\underline{\varepsilon\delta \geq C\left(\frac{\log\log N}{\log N}\right)^{1/2}}} \quad \text{for suff. large } C>0$$

it follows that $\exists\ 1\leq t\leq N^{1/2}$ s.t. $t\in B(\mathrm{Spec}_\varepsilon(A),2\varepsilon)$.

$\square$.