Math 3100 **Spring 2019**

Exam 1

No calculators. Show your work. Give full explanations. Good luck!

1. (9 points) Determine which of the following sequences converge and which diverge. Find the value of the limit for those which converge.

   Be sure to give a short justification in each case by indicating any limit laws, theorems, or special limits used.

   (a) $a_n = \dfrac{3^n}{n^3 2^n + 3^n}$

   (b) $b_n = (-1)^n - \dfrac{1}{\sqrt{n}}$

   (c) $c_n = \sqrt{n+2} - \sqrt{n}$

2. (21 points)

   (a) Let $\{x_n\}$ be a sequence of real numbers. Carefully state the definition of $\lim_{n\to\infty} x_n = x$ and use this definition to prove that

   $$\lim_{n\to\infty} \frac{2n+1}{n+1} = 2.$$

   (b) Suppose $\lim_{n\to\infty} x_n = x$ and $\lim_{n\to\infty} y_n = y$.

      i. Prove that the sequence $\{x_n\}$ is bounded.

      ii. Prove, using your definition of convergence given above, that

      $$\lim_{n\to\infty} x_n y_n = xy.$$

   (c) Prove that if $\lim_{n\to\infty} z_n = z$ and $z_n > 0$ for all $n \in \mathbb{N}$, then $z \geq 0$.

3. (12 points) Let $x_1 = 0$ and $x_{n+1} = \dfrac{2x_n + 1}{x_n + 2}$ for all $n \in \mathbb{N}$.

   (a) Find x_2, x_3, and x_4.

   (b) Show that $\{x_n\}$ converges and find the value of its limit.

   Hint: Use the Monotone Convergence Theorem.

4. (8 points) Let $\{x_n\}$ denote a bounded sequence of real numbers.

   (a) Carefully state the definitions of $\limsup_{n\to\infty} x_n$ and $\liminf_{n\to\infty} x_n$ and justify why they always exist for such sequences.

   (b) Explain why if $\liminf_{n\to\infty} x_n = \limsup_{n\to\infty} x_n$, then $\lim_{n\to\infty} x_n$ exists and all three take on the same value.

   Hint: You may use, without proof, that if $\{x_n\}$ diverges then it must contain two subsequences that converge to different limits.