Using Szemerédi's Regularity Lemma to Prove Roth's Theorem

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In this paper we use Szemerédi's Regularity Lemma to show that we can take A, a subset of the integer lattice, of some density and show that it can be approximated by random graphs of the same density. Once we can think of these graphs as somewhat random, we can then prove that for N sufficiently large, one can always find a right isosceles triangle in A which can then be used to imply Roth's Theorem. These ideas were borrowed from notes taken by Tom Sanders (March 14, 2004) on a course given by Tim Gowers titled Topics in Combinatorics.

To begin, here are some definitions that will be used extensively in the following arguments.

Definition 1 (Density)**.** *Let G be a graph and let $A,B \subset V(G)$. The* density *of the pair (A,B), with edge set $E(A,B)$ is defined to be*

$$d(A,B) := \frac{|E(A,B)|}{|A||B|}$$

Definition 2 (Epsilon Regular)**.** *Let $\varepsilon > 0$, then the pair (A,B) is said to be ε* -regular *if whenever $A' \subset A$ with $|A'| \geq \varepsilon|A|$ and $B' \subset B$ with $|B'| \geq \varepsilon|B|$ then*

$$|d(A',B') - d(A,B)| \leq \varepsilon$$

Definition 3 (Epsilon Regular Partitions)**.** *For a graph G with N vertices we shall say that a partition $\chi := \{X_1, ..., X_j\}$ of $V(G)$ is ε -regular if*

$$\sum_{(i,j):(X_i,X_j) \text{ is not } \varepsilon-regular} \frac{|X_i||X_j|}{N^2} \leq \varepsilon$$

Theorem 1 (Roth's Theorem)**.** *Let $\delta > 0$. Then there is $N_0 \in \mathbb{N}$ such that if $N \geq N_0$ and $A \subset [N]$ with $|A| \geq \delta N$ then A contains a non-trivial three term arithmetic progression, i.e. a triple $(x-d, x, x+d)$ with $d \neq 0$.*

Theorem 2 (Corners Theorem)**.** *Let $\varepsilon > 0$. Then there exists $N_0 \in \mathbb{N}$ such that if $N \geq N_0$ and $A \subset [N]^2$ with $|A| \geq \varepsilon N^2$ then A contains a non-degenerate isosceles right angled triangle i.e. a triple of the form $((x,y),(x+d,y),(x,y+d))$ with $d \neq 0$.*

NOTE: by $[N]^2$ we mean the Cartesian product of the set $\{1,\ldots,N\}$ with itself.

Theorem 3 (Triangle Removal)**.** *Let $\varepsilon > 0$. Then there is a $\delta = \delta(\varepsilon) > 0$ such that if G is any graph with n vertices with at most δn^3 triangles then it is possible to remove fewer than εn^2 edges from G to make it triangle-free.*

Theorem 4 (Szemerédi's Regularity Lemma)**.** *Let $\varepsilon > 0$. Then there is a K=$K(\varepsilon)$ such that for ever graph G with N vertices there is an ε-regular partition $\chi := \{X_1, ..., X_k\}$ of $V(G)$ with $k \leq K$.*

For a proof of Szemerédi's Regularity Lemma, we refer the reader to *Topics in Combinatorics*, the lecture notes from a course given by Tim Gowers.

Proof of Theorem 1. Let $B := \{(x,y) \in [2N]^2 : y - x \in A\}$; it is the collection of lines of gradient 1 that intersect the y-axis at elements of A. If $x \leq N$ then all elements of $x + A$ are at most $2N$ so that $|B| \geq |A|N \geq \delta N^2 = \frac{\delta}{4}(2N)^2$. Thus, by Theorem 3.8 we can find a non-degenerate isosceles right triangle,$((x,y),(x+d,y),(x,y+d))$, in B (taking $\varepsilon = \frac{\delta}{4}$ in the statement of that theorem). By the definition of B, $y-x, y-x-d, y+d-x \in A$, so our progression is $((y-x)-d, y-x, (y-x)+d)$.

To get a lower bound for how large N must be, we must use the results from the theorems stated about. First, we know that, given $\varepsilon > 0$, the $\delta = \delta(\varepsilon)$ whose existence is guaranteed by Theorem 3 is smaller than $\frac{\varepsilon^6}{2^{13}M^3}$, where $M = M(\varepsilon)$ is the M from Szemerédi's Regularity Lemma. Next, we use the result shown in Theorem 2 that the $N_0 = N_0(\epsilon)$ from Theorem 2 must be larger than $\frac{1}{64\delta}$. Finally, to get a lower bound for N, we apply Theorem 2 to $2N$ set $\varepsilon = \frac{\delta}{4}$. From this we find that N must be larger than $\frac{2^{48}M(\frac{\delta}{2^7})^3}{\delta^6}$. And, from Szemerédi's Regularity Lemma, we know that $M(\frac{\delta}{2^7}) > 2^{2^{\cdot^{\cdot^{\cdot^{2}}}}}$ where the tower of 2s is of height $3(\frac{\delta}{2^7})$. So $N > \frac{2^{48}}{\delta^6}\left(2^{2^{\cdot^{\cdot^{\cdot^{2}}}}}\right)^3$ with height $3(\frac{2^{35}}{\delta^5})$ which means $N > \frac{2^{48}}{\delta^6}\left(2^{3*2^{2^{\cdot^{\cdot^{\cdot^{2}}}}}}\right)$ with height $3(\frac{2^{35}}{\delta^5})$

$\square$

We now provide some definitions and lemmas that provide the background necessary to prove Theorem 2, and conclude with the proof of Theorem 2.

Theorem 5 (Counting Lemma)**.** *Let G be a graph and let X,Y,Z be disjoint subsets of the vertices of G. Suppose that the bipartitions (X,Y), (Y,Z) and (X,Z) are $\varepsilon - regular$ with densities α, β, and γ respectively, and that $\alpha, \beta, \gamma \leq 2\varepsilon$. Then the number of triples $(x,y,z) \in X \times Y \times Z$ that form triangles in G is at least*

$$(1-2\varepsilon)(\alpha-\varepsilon)(\beta-\varepsilon)(\gamma-\varepsilon)|X||Y||Z|.$$

Proof. First, for $x \in X$, let N_x denote the neighbors of x (the other vertices with which x shares an edge). Now, for each $x \in X$, define $d_Y(x) := |N_x \cap Y|$ and $d_Z(x) := |N_x \cap Z|$ as the sizes of the neighbor sets of x in Y and Z respectively. Then first consider $|X'| := |\{x \in X : d_Y(x) < (\alpha - \varepsilon)|Y|\}|$. We want to show that $|X'| < \varepsilon|X|$. So suppose $|X'| \geq \varepsilon|X|$. Thus, by $\varepsilon - regularity$ of X,

$$\begin{aligned} \big|d(X',Y) - d(X,Y)\big| &\leq \varepsilon \text{ (since } |Y| \geq \varepsilon|Y|) \\ \Rightarrow \left|\frac{|E(X',Y)|}{|X'||Y|} - \alpha\right| &\leq \varepsilon \\ \Rightarrow \left|\frac{1}{|X'||Y|}\sum_{x \in X'} d_Y(x) - \alpha\right| &\leq \varepsilon \end{aligned}$$

Thus, $\varepsilon \geq \alpha - \frac{1}{|X'||Y|} \sum_{x \in X'} d_Y(x)$. Then note that since each $x \in X'$ has $d_Y(x) < (\alpha - \varepsilon)|Y|$, then

$$\sum_{x \in X'} d_Y(x) < \sum_{x \in X'} (\alpha - \varepsilon)|Y| = |X'|(\alpha - \varepsilon)|Y|.$$

$$\text{Thus } \varepsilon \geq \alpha - \frac{1}{|X'||Y|} \sum_{x \in X'} d_Y(x) > \alpha - \frac{1}{|X'||Y|}|X'|(\alpha - \varepsilon)|Y| = \alpha - (\alpha - \varepsilon) = \varepsilon.$$

Which is a contradiction because $\varepsilon \not> \varepsilon$ so it must not be true that $|X'| \geq \varepsilon|X|$ thus it must be true that $|X'| < \varepsilon|X|$. Notice that Y wasn't special, so we could likewise show that $|\widetilde{X}'| := |\{x \in \ X : d_Z(x) < (\gamma - \varepsilon)|Z|\}| < \varepsilon|X|$ too.

Recall that $\alpha, \gamma \geq 2\varepsilon$. Let $Y' := N_x \cap Y$ and $Z' := N_x \cap Z$. Suppose that $|Y'| = d_Y(x) \geq (\alpha - \varepsilon)|Y|$ and $|Z'| = d_Z(x) \geq (\gamma - \varepsilon)|Z|$. In this case, $|Y'| \geq (\alpha - \varepsilon)|Y| \geq (2\varepsilon - \varepsilon)|Y| = \varepsilon|Y|$ and $|Z'| \geq (\gamma - \varepsilon)|Z| \geq (2\varepsilon - \varepsilon)|Z| = \varepsilon|Z|$. So by ε-regularity we have

$$\begin{aligned}
&\big|d(Y', Z') - d(Y, Z)\big| \leq \varepsilon \\
\Rightarrow &\left|\frac{|E(Y', Z')|}{|Y'||Z'|} - \beta\right| \leq \varepsilon \\
&\quad \Rightarrow |E(Y', Z')| \geq (\beta - \varepsilon)|Y'||Z'| \\
&\quad \Rightarrow |E(Y', Z')| \geq (\beta - \varepsilon)d_Y(x)d_Z(x)
\end{aligned}$$

Thus, in this case, the number of edges between $N_x \cap Y$ and $N_x \cap Z$ is at least $(\varepsilon + \beta)d_Y(x)d_Z(x)$.

Now we can see that the number of triples $(x, y, z) \in X \times Y \times Z$ that form triangles in G is

$$\sum_{x \in X} (N_x \cap Y, N_x \cap Z) \geq \sum_{\substack{x \in X: \\ d_Y(x) \geq (\alpha - \varepsilon)|Y| \ \& \\ d_Z(x) \geq (\gamma - \varepsilon)|Z|}} (\beta - \varepsilon)d_Y(x)d_Z(x)$$

and because there are no more than $2\varepsilon|X|$ vertices x in X that satisfy $d_Y(x) < (\alpha - \varepsilon)|Y|$ and $d_Z(x) < (\gamma - \varepsilon)|Z|$ then there are at least $|X| - 2\varepsilon|X| = (1 - 2\varepsilon)|X|$ vertices in X that are counted in this sum. Thus, the value of the sume is at least

$$(1 - 2\varepsilon)|X|(\beta - \varepsilon)(\alpha - \varepsilon)|Y|(\gamma - \varepsilon)|Z|.$$

□

Proof. First, note that if $\epsilon \geq 1$ then the result is trivial since there are no more than n^2 edges in a graph on n vertices.

So suppose $\epsilon < 1$. Apply Szemeredi's Regularity Lemma to G with $\frac{\varepsilon}{4}$-regularity and let $\chi = \{X_1, ..., X_K\}$ be the given $\frac{\varepsilon}{4}$-regular partition of $V(G)$ into $K = K(\frac{\varepsilon}{4})$ sets. We will remove the edge xy from G if any of the following conditions apply

1. $(x, y) \in X_i \times X_j$ with (X_i, X_j) not $\frac{\varepsilon}{4}$-regular.
2. $(x, y) \in X_i \times X_j$ with $d(X_i, X_j) < \frac{\varepsilon}{2}$.
3. $x \in X_i$ with $|X_i| < \frac{\varepsilon n}{4K}$.

Now we count the maximum number of edges that could have been removed from G according to these conditions

1. In the first case we removed at most $\displaystyle\sum_{\substack{(i,j):\\ (X_i,X_j)\text{is not}\frac{\varepsilon}{4}-regular}} |X_i||X_j| \leq \frac{\varepsilon n^2}{4}$ edges (by Theorem 1).

2. In the second case we remove at most $\displaystyle\sum_{i,j} \frac{\varepsilon}{2}|X_i||X_j| \leq \frac{\varepsilon n^2}{2}$ edges.

3. In the third case we remove at most $K\left(\frac{\varepsilon n}{4K}\right)n = \frac{\varepsilon n^2}{4}$ because there are K sets in χ and each vertex in G can be adjacent to n vertices.

So the total number we removed was at most εn^2 edges. Now let's suppose that after we deleted all of these edges we still have a triangle $(x, y, z) \in X_i \times X_j \times X_k$. Then each of the pairs (X_i, X_j), (X_i, X_k), and (X_j, X_k) must each have a density of at least $\frac{\varepsilon}{2}$ and are $\frac{\varepsilon}{4}$-regular. In addition, each of the sets X_i, X_j, and X_k has at least $\frac{\varepsilon n}{4K}$ elements. Applying Theorem 3 to X_i, X_j, and X_k, we see that we have at least

$$\left(1-\frac{\varepsilon}{2}\right)\left(\frac{\varepsilon}{2}-\frac{\varepsilon}{4}\right)\left(\frac{\varepsilon}{2}-\frac{\varepsilon}{4}\right)\left(\frac{\varepsilon}{2}-\frac{\varepsilon}{4}\right)|X_i||X_j||X_k| \geq \left(1-\frac{1}{2}\right)\left(\frac{\varepsilon}{4}\right)^3\left(\frac{\varepsilon n}{4K}\right)^3 = \frac{\varepsilon^6 n^3}{2^{13}K^3} \tag{1}$$

triangles in G. Hence if we have $\delta < \frac{\varepsilon^6}{2^{13}K^3}$ then we have a contradiction and we couldn't have had a triangle after those edges were removed. $\square$

Thus, we now know that for a given graph G on n vertices and a given $\varepsilon > 0$, if G has ηn^3 triangles where η is smaller than the δ we know exists from Theorem 3, then we can make G triangle-free by removing fewer than εn^2 edges from G .

In the proof of Theorem 2, we will construct a tripartite graph. The following example illustrates the constructs, which are the following: Let $X = [1, N]$, $Y = [1, N]$, and $Z = [2, 2N]$. X will be the set of vertical lines through A, Y the set of horizontal ones and Z the diagonal lines of gradient -1 (starting with $z = 1$ as the diagonal joining $(1, 0)$ to $(0, 1)$). In G, join $x \in X$ to $y \in Y$ iff $(x, y) \in A$, join $x \in X$ to $z \in Z$ iff $(x, z - x) \in A$ and join $y \in Y$ to $z \in Z$ iff $(z - y, y) \in A$.

Below is an example of what an isosceles triangle in A would look like in G for $N = 5$. Note that this is assuming there *is* an isosceles triangle in A which is *not* what we were assuming is true. This is merely to demonstrate the construction.

Also this construction relies on restricting A to just be within $[1, N]^2$ of the $[0, N]^2$ grid below, then for the remainder of this paper, $|A| \leq N^2$ which is easier to work with than if

$|A| \leq (N+1)^2$. But we want to include the rest of the grid so the numbering of the diagonals makes more sense, rather than start to number the first diagonal you can see as 3.

In this example, $A = (3,1), (1,1), (1,3)$. The elements in A are coloured red, blue, and green respectively, and corresponds to red, blue, and green degenerate right triangles in G. A triangle in G is considered *degenerate* if it's edges are constructed from an element in A. The right isosceles triangle's sides in A are made up of the first vertical line, the first horizontal line, and the second diagonal line. This corresponds to the black triangle defined by the edges connecting $1 \in X$ to $1 \in Y$ to $4 \in Z$ and back to $1 \in X$ in G, as shown below. Notice that the black triangle in G is a consequence of the construction of G from including the points that make up the isosceles triangle in A.

Proof of Theorem 2. Assume that A has no triples $((x,y),(x+d,y),(x,y+d))$ with $d \neq 0$ and construct a tripartite graph G as follows. Let $X = [1,N]$, $Y = [1,N]$, and $Z = [2,2N]$. X will be the set of vertical lines through A, Y the set of horizontal ones and Z the diagonal lines of gradient -1 (starting with $z = 1$ as the diagonal joining $(1,0)$ to $(0,1)$). In G, join $x \in X$ to $y \in Y$ iff $(x,y) \in A$, join $x \in X$ to $z \in Z$ iff $(x, z-x) \in A$ and join $y \in Y$ to $z \in Z$ iff $(z-y, y) \in A$.

Observe that if G contains a triangle then we have $(x,y), (x, y+(z-x-y)), (x+(z-x-y), y) \in A$, so unless $z = x+y$ we have a non-degenerate triangle, which would contradict our original assumption that G contains no degenerate triangles. Since $A \subset [N]^2$, G has at most $N^2 = \frac{1}{64N}(4N)^3$ triangles (note that G has $4N$ vertices if we let Z have $2N$ vertices, but 0 and 1 won't be used). Now note that by Theorem 3, the Edge Removal Lemma, there exists $\delta(\frac{\varepsilon}{32}) > 0$ such that if G has no more than $\delta(4N)^3$ triangles then it is possible to remove fewer than $\frac{\epsilon(4N)^2}{32} = \frac{\varepsilon N^2}{2}$ edges from G to make it triangle free. So if $N > \frac{1}{64\delta}$ then we can make G triangle free by removing no more than $\frac{\varepsilon N^2}{2}$ edges. However, the degenerate triangles are all edge disjoint and there are $|A| \geq \varepsilon N^2$ of them. This contradicts the fact that we removed them all using only $\frac{\varepsilon N^2}{2}$ edges. $\square$

References

T. Gowers, Topics in Combinatorics (Lecture Notes), 2004