Math 8100 Exam 1

Thursday the 13th of October 2022

Answer any <u>THREE</u> of the following four problems

1. Let $E \subseteq \mathbb{R}^n$. Recall that the definition of the Lebesgue outer measure of E, $m_*(E)$ is given by
$$m_*(E) = \inf \sum_{j=1}^{\infty} |Q_j|$$
where the infimum is taken over all countable coverings $E \subseteq \bigcup_{j=1}^{\infty} Q_j$ by closed cubes.
   (a) Prove that for any $E \subseteq \mathbb{R}^n$ and $\varepsilon > 0$ there exists an open set G with $E \subseteq G$ and
$$m_*(E) \leq m_*(G) \leq m_*(E) + \varepsilon.$$
   Be sure to prove both of the inequalities above.
   (b) i. Prove that $m_*(A \cup B) \leq m_*(A) + m_*(B)$ for any subsets A and B of $\mathbb{R}^n$.
   ii. Give an example (no proofs required) of *disjoint* subsets A and B of $\mathbb{R}$ for which
$$m_*(A \cup B) \neq m_*(A) + m_*(B).$$

2. Let X be a set equipped with a σ-algebra $\mathcal{M}$.
   (a) Give the definition of a measure on $\mathcal{M}$.
   (b) Prove that if μ is a measure on $\mathcal{M}$ and $\{E_j\}_{j=1}^{\infty} \subseteq \mathcal{M}$ and $E_1 \subseteq E_2 \subseteq \cdots$, then
$$\mu\Big(\bigcup_{j=1}^{\infty} E_j\Big) = \lim_{j\to\infty} \mu(E_j).$$

3. (a) Prove that if $E \subseteq \mathbb{R}^n$ is Lebesgue measurable, then for any $\delta > 0$ the dilated set
$$\delta E := \{\delta x \,:\, x \in E\}$$
   is also Lebesgue measurable and satisfies $m(\delta E) = \delta^n m(E)$.
   (b) Carefully state Fatou's Lemma and deduce the Monotone Convergence Theorem from it.
   (c) Prove that if f is a non-negative measurable function on $\mathbb{R}^n$ and $\delta > 0$, then f_δ, defined by
$$f_\delta(x) = \delta^{-n} f(\delta^{-1}x)$$
   is also a non-negative measurable function and
$$\int f(x)\,dx = \int f_\delta(x)\,dx.$$

4. Let $(X, \mathcal{M}, \mu)$ be a measure space and $L^1 = L^1(X, \mu)$.
   (a) Prove that if $f \in L^1$, then $|f(x)| < \infty$ almost everywhere.
   (b) Prove that $\{f_k\}_{k=1}^{\infty}$ is a sequence in L^1 with $\sum_{k=1}^{\infty} \|f_k\|_1 < \infty$, then $\sum_{k=1}^{\infty} f_k$ converges in L^1.
   Hint: You may use, without proof, the fact that for any sequence $\{f_k\}_{k=1}^{\infty}$ in $L^+(X)$ one has
$$\int \sum_{k=1}^{\infty} f_k = \sum_{k=1}^{\infty} \int f_k.$$
   (c) Prove that if $\{f_k\}_{k=1}^{\infty}$ is a Cauchy sequence in L^1, then there exists a function $f \in L^1$ such that $f_k \to f$ in L^1.