STRONGLY SINGULAR INTEGRAL OPERATORS ON $\mathbf{R}^d$ A NON-FOURIER TRANSFORM METHOD

NEIL LYALL

Contents

1. Introduction

These are convolution operators whose kernels are too singular at the origin to fall under the theory of Calderón and Zygmund. In order to extend to bounded operators on say $L^2(\mathbf{R}^d)$ we compensate for this strong singularity by introducing a highly oscillatory term. What is of interest here is the precise relationship between the size of the singularity and the size of the oscillation.

Let K_α be a distribution on $\mathbf{R}^d$ that away from the origin agrees with the function

$$K_\alpha(x) = |x|^{-d-\alpha} e^{i|x|^{-\beta}} \chi(|x|),$$

where $\beta > 0$ and χ is smooth and compactly supported in a small neighborhood of the origin (say where $|x| \leq \frac{1}{10}$). The distribution-valued function $\alpha \mapsto K_\alpha$, initially defined for $\operatorname{Re} \alpha < 0$, continues analytically to the entire complex plane[1].

Euclidean Result [7],[2]**.** *If $\alpha > 0$, then the (Euclidean) operator $T_E f(x) = f * K_\alpha(x)$, defined initially for test functions, extends to a bounded operator on $L^p(\mathbf{R}^d)$ whenever $\left|\frac{1}{p} - \frac{1}{2}\right| \leq \frac{1}{2} - \frac{\alpha}{d\beta}$.*

The counterexample showing that this result is sharp is due to Wainger [7] as is sufficiency up to the endpoints. The full theorem follows by an interpolation theorem of Fefferman and Stein [2] and duality once we have the L^2 and H^1 estimates below.

L^2 **Lemma.** *If $\alpha \leq \frac{d\beta}{2}$ then T_E extends to a bounded operator from $L^2(\mathbf{R}^d)$ to itself.*

H^1 **Lemma** [2]**.** *If $\alpha = 0$ then T_E extends to a bounded operator from $H^1(\mathbf{R}^d)$ to $L^1(\mathbf{R}^d)$.*

Date: November 14, 2004.

[1] Continue analytically the function $K_z^\varepsilon(x) = e^{-\varepsilon|x|^{-\beta}} K_z(x)$ via integration by parts and then let $\varepsilon \to 0$.

Fefferman [1] also showed that when $\alpha = 0$, T_E extends to an operator of weak type (1,1).

In this note we present proofs of the two lemmata above as well as an argument establishing the sharp necessary condition, in each case we do not make use of Fourier transform methods.

2. Proof of L^2 Lemma

We shall obtain this result by dyadically decomposing our operator and using almost orthogonality. Recall the following version of Cotlar's Lemma,

Cotlar's Lemma. *Suppose $\{T_i\}$ is a finite collection of bounded operators on L^2. If in addition these operators satisfy, for some $\delta > 0$, the almost orthogonality condition*

$$\|T_i^* T_j\|_{Op} + \|T_i T_j^*\|_{Op} \leq C 2^{-\delta|i-j|},$$

then

$$\Big\|\sum_i T_i\Big\|_{Op} \leq A,$$

where the constant A is independent of the number of these T_j.

For an elegant proof of this result see [5]. We now wish to decompose our operator $T_E = \sum_{j=0}^{\infty} T_j$. In order to do this we consider the following partition of unity; choose $\vartheta \in C_0^\infty(\mathbf{R})$ supported in $[\frac{1}{2}, 2]$ such that $\sum_{j=0}^{\infty} \vartheta_j(r) = 1$ for all $0 \leq r \leq 1$, where $\vartheta_j(r) = \vartheta(2^j r)$, and write

$$T_j f(x) = f * K_j(x) \quad \text{ where } \quad K_j(x) = \vartheta(2^j x) K_\alpha(x).$$

Lemma 1. *The operator norms of T_j are uniformly bounded whenever $\alpha \leq \frac{d\beta}{2}$, more precisely*

$$\|T_j f\|_{L^2(\mathbf{R}^d)} \leq C 2^{j(\alpha - \frac{d\beta}{2})} \|f\|_{L^2(\mathbf{R}^d)}.$$

The L^2 Lemma now follows from an application of Cotlar's lemma (and a standard limiting argument) once we have verified that our operators T_j are, in the following sense, almost orthogonal.

Lemma 2. *If $\alpha = \frac{d\beta}{2}$ then $\|T_i^* T_j\|_{Op} + \|T_i T_j^*\|_{Op} \leq C 2^{-\frac{d\beta}{2}|i-j|}$.*

Proof. This follows trivially from Lemma 1 whenever $|i - j| \leq 10$, since $\|T_i^* T_j\|_{Op} \leq \|T_i\|_{Op} \|T_j\|_{Op}$. We shall therefore, without loss of generality, assume that $j \gg i$. Now $T_i^* T_j$ has a kernel $L_{ij}(x) = K_j * \bar{K}_i(-x)$, and the same operator norm as the operator with kernel the L^1 dilate of L_{ij}, namely

$$\begin{aligned}
\widetilde{L}_{ij}(x) &= 2^{-jd} L_{ij}(2^{-j} x) \\
&= 2^{-jd} \int K_j(y) \bar{K}_i((2^{-j} x) - y) dy \\
&= 2^{-j2d} \int K_j(2^{-j} y) \bar{K}_i(2^{-j}(x - y)) dy \\
&= 2^{j2\alpha} \int\limits_{\substack{|y| \sim 1 \\ |x-y| \sim 2^{j-i}}} |y|^{-d-\alpha} |x - y|^{-d-\alpha} e^{i 2^{j\beta}[|y|^{-\beta} - |x-y|^{-\beta}]} dy.
\end{aligned}$$

Trivially we get the estimate $|\widetilde{L}_{ij}(x)| \leq C 2^{j2\alpha} 2^{(i-j)(d+\alpha)}$, while integration by parts d times gives

$$|\widetilde{L}_{ij}(x)| \leq C 2^{j(2\alpha - d\beta)} 2^{(i-j)(d+\alpha)} = 2^{(i-j)(d+\alpha)}.$$

This of course implies that $\|\widetilde{L}_{ij}\|_{L^1(\mathbf{R}^d)} \leq C 2^{(i-j)\alpha}$. □

We are left with the task of establishing Lemma 1. Now the operator norm of T_j is equal to that of

$$\begin{aligned} S_j f(x) &= 2^{-jd} \int_{\mathbf{R}^d} K_j(2^{-j}(x-y)) f(y) dy \\ &= 2^{j\alpha} \int_{\mathbf{R}^d} \vartheta(|x-y|) |x-y|^{-d-\alpha} e^{i 2^{j\beta} |x-y|^{-\beta}} f(y) dy. \end{aligned}$$

It therefore suffices to consider the operator S_j. The key to our argument will be the following proposition that can be thought of as a variable coefficient version of Plancherel's theorem.

Proposition 3. *Let Ψ be a smooth function of compact support in x and y and Φ be real-valued and smooth. If we assume that,*

$$\det\Big(\frac{\partial^2 \Phi}{\partial x_i \partial y_j}\Big) \neq 0,$$

on the support of Ψ, then

$$\Big\| \int_{\mathbf{R}^d} \Psi(x,y) e^{i\lambda \Phi(x,y)} f(y) dy \Big\|_{L^2(\mathbf{R}^d)} \leq C \lambda^{-\frac{d}{2}} \|f\|_{L^2(\mathbf{R}^d)}.$$

For a proof see [5], Chapter 9. Let us now see that the phase of our operator S_j is non-degenerate.

Lemma 4. *Let $\varphi(x,y) = |x-y|^\gamma$, then $\det\big(\frac{\partial^2 \varphi}{\partial x_i \partial y_j}\big) \neq 0$ whenever $\gamma \neq 1$.*

Proof. Recall that $\nabla |x|^\gamma = \gamma |x|^{\gamma-1} \frac{x}{|x|}$, it is then easy to see that

$$\partial_{x_i} \partial_{y_j} \varphi(x,y) = -\gamma |x-y|^{\gamma-2} \left(\delta_{ij} + (\gamma-2) u_i u_j \right),$$

where $u_i = \frac{(x-y)_i}{|x-y|}$. We therefore need to check that $I + (\gamma-2) u u^t$ is non-singular. To do this we shall denote by R the rotation matrix such that $Ru = e_1$, of course $\det R = 1$ and it is clear that

$$\det(I + (\gamma-2) u u^t) = \det\big(R(I + (\gamma-2) u u^t) R^t\big) = 1 + (\gamma - 2) = \gamma - 1.$$

□

In view of Lemma 4 we would like to apply Proposition 3 to our operator S_j, however our amplitude is only compactly supported about the diagonal, lets say where $|x-y| \leq 2$. Using proposition 3 we can however show that our operators are pseudo-local: that the main contribution of $S_j f$ in the unit ball about x_0 comes from the values of $f(x)$ for x near that ball. This psuedo-locality feature of our operator is a consequence of the following lemma.

Lemma 5. *For each $x_0 \in \mathbf{R}^d$,*

$$\int_{|x-x_0| \leq 1} |S_j f(x)|^2 dx \leq C 2^{j(2\alpha - d\beta)} \int_{|x-x_0| \leq 10} |f(x)|^2 dx.$$

Lemma 1 is an immediate consequence of Lemma 5 as an integration in x_0 clearly gives

$$\int_{\mathbf{R}^d} |S_j f(x)|^2 dx \leq C 2^{j(2\alpha - d\beta)} \int_{\mathbf{R}^d} |f(x)|^2 dx.$$

Proof of Lemma 5. Since our S_j are translation invariant we may, with no loss of generality, assume that $x_0 = 0$. Write $f = f_1 + f_2$, with f_1 is supported in $B(10)$, f_2 supported outside $B(9)$, f_1 and f_2 smooth, and with $|f_1|, |f_2| \leq |f|$. We fix $\chi \in C_0^\infty$ so that $\chi \equiv 1$ in $B(1)$.

Now, since S_j is compactly supported in $x - y$, it follows that χS_j is compactly supported in x and y, so applying proposition 3 we see that,

$$\begin{aligned}\int_{B(1)} |S_j f_1(x)| dx &= \int_{\mathbf{R}^d} |\chi(x) S_j f_1(x)| dx \\ &\leq C 2^{j(2\alpha - d\beta)} \int_{\mathbf{R}^d} |f_1(x)|^2 dx \\ &\leq C 2^{j(2\alpha - d\beta)} \int_{B(10)} |f(x)|^2 dx.\end{aligned}$$

However, if $|x| \leq 1$, and $|y| \geq 9$, it follows that $|x - y| \geq 8$ and hence $\chi S_j f_2 \equiv 0$. $\square$

3. Proof of H^1 Lemma

For any f in $H^1(\mathbf{R}^d)$ we have the atomic decomposition

$$f = \sum_Q \lambda_Q \, a_Q \qquad \text{where} \qquad \sum_Q |\lambda_Q| \sim \|f\|_{H^1(\mathbf{R}^d)},$$

and the individual atoms satisfy the following;

$$\text{(i)}\ \operatorname{supp} a_Q \subset Q \qquad \text{(ii)}\ \|a_Q\|_\infty \leq |Q|^{-1} \qquad \text{(iii)}\ \int a_Q(x) dx = 0.$$

As a consequence of this characterization of H^1 it suffices to check that for an individual atom a_Q

$$\int |T a_Q(x)| dx \leq C,$$

where C is independent of a_Q. Since T is translation–invariant, we assume that $a = a_Q$ is supported in a cube centered at the origin. Note that if $Q^* = 2Q$, then Cauchy–Schwarz and the L^2 Lemma imply that

$$\int_{Q^*} |Ta(x)| dx \leq C |Q^*|^{\frac{1}{2}} \|a\|_2 \leq C |Q^*|^{\frac{1}{2}} |Q|^{-\frac{1}{2}} \leq C.$$

Let $\ell = \ell(Q)$ denote the sidelength of Q. Now if $\ell \geq 1$ then it follows from the compact support of our kernel K that $\operatorname{supp} Ta \subset Q^*$ and from the argument above our result follows in this case. We may now assume that $\ell < 1$. It suffices to establish the following estimate for each dyadic operator.

Lemma 6. *If $\alpha = 0$, then*

$$\int |T_j a(x)| dx \leq C \min\{\ell 2^{j(\beta+1)}, (\ell 2^{j(\beta+1)})^{-\frac{d\beta}{2(\beta+1)}}\}.$$

The H^1 Lemma now follows, since

$$\int |Ta(x)|dx \leq \sum_{j=1}^{\infty} \int |T_j a(x)|dx \leq C\Big(\ell \sum_{2^j \leq \ell^{-\frac{1}{\beta+1}}} 2^{j(\beta+1)} + \ell^{-\frac{d\beta}{2(\beta+1)}} \sum_{2^{-j} \leq \ell^{\frac{1}{\beta+1}}} 2^{-j\frac{d\beta}{2}}\Big) \leq C.$$

Proof of Lemma 6. Our estimate naturally spits into two cases.

(i) If $2^j \leq \ell^{-\frac{1}{\beta+1}}$ then it is straightforward to see, using the cancellation of our atom a, that

$$\int |T_j a(x)|dx \leq \int |a(y)| \int |K_j(x-y) - K_j(x)|dx\, dy \leq C\ell \int \vartheta_j(|x|)|x|^{-d-\beta-1}dx \leq C\ell 2^{j(\beta+1)}.$$

(ii) If $2^{-j} \leq \ell^{\frac{1}{\beta+1}}$ then we shall as usual use an L^2 result, namely Lemma 1;

$$\int |T_j a(x)|dx \leq C\ell^{\frac{d}{2(\beta+1)}} \|T_j a\|_2 \leq C\ell^{\frac{d}{2(\beta+1)}} 2^{-j\frac{d\beta}{2}} \|a\|_2 \leq C(\ell 2^{j(\beta+1)})^{-\frac{d\beta}{2(\beta+1)}}.$$

□

4. Necessary Condition

It suffices to establish a necessary condition for the rescaled operators S_j to bounded uniformly on $L^p(\mathbf{R}^d)$ for $1 \leq p \leq 2$. Recall that

$$S_j f(x) = 2^{j\alpha} \int_{\mathbf{R}^d} \vartheta(|x-y|)|x-y|^{-d-\alpha} e^{i2^{j\beta}|x-y|^{-\beta}} f(y)\, dy.$$

We now choose a suitable test function, namely $f_0(x) = |x|^{-\frac{d}{p}}\chi(|x|)$.

At this point we also make the observation that $S_j f_0$ is a radial function, being the convolution of two radial functions, and as such we may with no loss in generality assume that $x = (|x|, 0, \ldots, 0)$.

Now making the change of variables $y = |x|s$ and noting that $\|f\|_p = 2^{j\frac{d}{p}} \|f(2^j\cdot)\|_p$, we see that we may again rescale our operator and consider

$$2^{j\frac{d}{p}} S_j f_0(2^j x) = |x|^{-\alpha-\frac{d}{p}} \vartheta(2^j|x|) \int_{\mathbf{R}^d} e^{i|x|^{-\beta}\varphi(s)} \psi(s)\, ds,$$

where

$$\varphi(s) = (1 - 2s_1 + |s|^2)^{-\frac{\beta}{2}} \quad \text{and} \quad \psi(s) = (1 - 2s_1 + |s|^2)^{-\frac{d+\alpha}{2}} \chi(|s|)|s|^{-\frac{d}{p}}.$$

We have therefore now reduced matters to the analysis of the oscillatory singular integral

$$I(\lambda) = \int_{\mathbf{R}^d} e^{i\lambda\varphi(s)} \psi(s)\, ds,$$

as $\lambda \to +\infty$. We now write

$$I(\lambda) = M(\lambda) + E_1(\lambda) + E_2(\lambda) + E_3(\lambda),$$

where

$$M(\lambda) = e^{i\lambda} \int_{\mathbf{R}^d} \chi(\lambda^{1-\epsilon}|s|) e^{ic\lambda s_1} |s|^{-\frac{d}{p}}\, ds,$$

$$E_1(\lambda) = \int_{\mathbf{R}^d} \chi(\lambda^{1-\epsilon}|s|)\big[e^{i\lambda\varphi(s)} - e^{i\lambda(1+cs_1)}\big]|s|^{-\frac{d}{p}}\,ds,$$
$$E_2(\lambda) = \int_{\mathbf{R}^d} \chi(\lambda^{1-\epsilon}|s|)e^{i\lambda\varphi(s)}\big[\psi(s) - |s|^{-\frac{d}{p}}\big]\,ds,$$

and

$$E_3(\lambda) = \int_{\mathbf{R}^d} \big[1 - \chi(\lambda^{1-\epsilon}|s|)\big]e^{i\lambda\varphi(s)}\psi(s)\,ds.$$

Let us first take care of the error terms. It is easy to verify that whenever $|s| \le \lambda^{-1+\epsilon}$ we have

$$|e^{i\lambda\varphi(s)} - e^{i\lambda(1+cs_1)}| \le C\lambda^{\epsilon}|s| \quad \text{and} \quad |\psi(s) - |s|^{-\frac{d}{p}}| \le C\lambda^{-1+\epsilon}|s|^{-\frac{d}{p}},$$

and hence that

$$|E_1(\lambda)| \le C\lambda^{\epsilon}\int_{|s|\le\lambda^{-1+\epsilon}} |s|^{-\frac{d}{p}+1}\,ds \le C\lambda^{\epsilon}\lambda^{-(\frac{d}{p'}+1)(1-\epsilon)} = C\lambda^{-\frac{d}{p'}}\lambda^{-1+\epsilon(2+\frac{d}{p'})},$$

while

$$|E_2(\lambda)| \le C\lambda^{-1+\epsilon}\int_{|s|\le\lambda^{-1+\epsilon}} |s|^{-\frac{d}{p}}\,ds \le C\lambda^{-1+\epsilon}\lambda^{-\frac{d}{p'}(1-\epsilon)} = C\lambda^{-\frac{d}{p'}}\lambda^{-1+\epsilon(1+\frac{d}{p'})}.$$

Now in the error integral $E_3(\lambda)$ it shall be advantageous to repeatedly apply integration by parts in the s_1 direction since $C\lambda^{-1+\epsilon} \le |s| \le \frac{1}{10}$. In fact it is clear that in this region

$$\partial_1\varphi(s) = \beta(1-s_1)(1-2s_1+|s|^2)^{-\frac{\beta+2}{2}} \ge C(\beta),$$

while

$$|\partial_1^{\ell}\varphi(s)| \le c_\ell \quad \text{and} \quad \big|\partial_1^{\ell}[1-\chi(\lambda^{1-\epsilon}|s|)]\psi(s)\big| \le c_\ell\big(\lambda^{(1-\epsilon)\ell}|s|^{-\frac{d}{p}}\vartheta(10\lambda^{1-\epsilon}|s|) + |s|^{-\frac{d}{p}-\ell}\big).$$

It therefore follows that after integrating by parts N times we obtain the estimate

$$|E_3(\lambda)| \le C\lambda^{-N}\big(\lambda^{(1-\epsilon)N}\int_{|s|\approx\lambda^{-1+\epsilon}} |s|^{-\frac{d}{p}}\,ds + \int_{|s|\ge\lambda^{-1+\epsilon}} |s|^{-\frac{d}{p}-N}\,ds\big) \le C\lambda^{-\frac{d}{p'}}\lambda^{-\epsilon(N-\frac{d}{p'})}.$$

It remains for us to show that $|M(\lambda)| \ge C\lambda^{-\frac{d}{p'}}$. Assuming this for the moment we see that it would then follow that $|I(\lambda)| \ge \lambda^{-\frac{d}{p'}}$ for large enough λ and hence that

$$\|S_jf_0\|_p^p \ge C\int_{\mathbf{R}^d} \vartheta(2^j|x|)|x|^{-(\alpha+\frac{d}{p}-\beta\frac{d}{p'})p}\,dx = C2^{j(\alpha p - d\beta(p-1))}.$$

Therefore if S_j were to extend to a bounded operator on $L^p(\mathbf{R}^d)$ for $1 \le p \le 2$ with operator norm independent of j we see that one must necessarily have the condition $\alpha \le \frac{d\beta}{p'}$.

The lower bound estimate for the main term will be an immediate consequence of the following, slightly more general lemma.

Lemma 7.

$$\int_{\mathbf{R}^d} \chi(|s|)e^{i\lambda s}|s|^{-\frac{d}{p}}\,ds = C\lambda^{-\frac{d}{p'}} + O(\lambda^{-\frac{d}{p'}-1})$$

Proof. We shall cheat and use the fact that this is merely a Fourier transform and as such

$$\int_{\mathbf{R}^d} \chi(|s|)e^{i\lambda s}|s|^{-\frac{d}{p}}\,ds = C\int_{\mathbf{R}^d} \widehat{\chi}(|\xi-\lambda|)|\xi|^{-\frac{d}{p'}}\,d\xi.$$

Now since χ is smooth and of compact support (and therefore in $\mathcal{S}$) it follows that $\widehat{\chi}$ is Schwartz and satisfies the inequality

$$|\widehat{\chi}(|\xi-\lambda|)| \le C_N(1+|\xi-\lambda|)^{-N},$$

for all $N \ge 0$. Using this standard estimate it is easy to see that whenever $|\lambda| \notin [\frac{1}{2}|\xi|, 2|\xi|]$ we have

$$\left| \int_{\mathbf{R}^d} \widehat{\chi}(|\xi-\lambda|)|\xi|^{-\frac{d}{p'}}\,d\xi \right| \le C|\lambda|^{-N+\frac{d}{p}}.$$

Now if $|\lambda| \in [\frac{1}{2}|\xi|, 2|\xi|]$, then

$$\begin{aligned}\int_{\mathbf{R}^d} \widehat{\chi}(|\xi-\lambda|)|\xi|^{-\frac{d}{p'}}\,d\xi &= |\lambda|^{-\frac{d}{p'}} \int_{\mathbf{R}^d} \widehat{\chi}(|\xi-\lambda|)\,d\xi + \int_{\mathbf{R}^d} \widehat{\chi}(|\xi-\lambda|)\big[|\xi|^{-\frac{d}{p'}} - |\lambda|^{-\frac{d}{p'}}\big]\,d\xi \\ &= |\lambda|^{-\frac{d}{p'}}\chi(0) + O\Big(\int_{|\xi|\approx|\lambda|} |\xi|^{-\frac{d}{p'}-1}\,d\xi\Big) \\ &= |\lambda|^{-\frac{d}{p'}} + O(|\lambda|^{-\frac{d}{p'}-1}).\end{aligned}$$

$\square$

References

1. Fefferman, C., *Inequalities for strongly singular convolution operators*, Acta Math. 124 (1970), 9–36.
2. Fefferman, C. and E. M. Stein, *H^p spaces of several variables*, Acta Math. 129 (1972), 137–193.
3. Hirschman, I. I., *Multiplier Transforms I*, Duke Math. J. 26 (1956), 222–242.
4. Seeger, A., C. D. Sogge and E. M. Stein, *Regularity properties of Fourier integral operators*, Ann. of Math. 134 (1991), 231–251.
5. Stein, E. M., *Harmonic Analysis: Real–Variable Methods, Orthogonality, and Oscillatory Integrals*, Princeton Univ. Press (1993).
6. Stein, E. M. and G. Weiss, *Introduction to Fourier Analysis on Euclidean Spaces*, Princeton Univ. Press (1971).
7. Wainger, S., *Special Trigonometric Series in k Dimensions*, Memoirs of the AMS #59, American Math. Society.