Math 4110/6110

Problem Set 2: Sets of Measure Zero

1. Consider the sets
$$\mathfrak{N}_\delta = \left\{ t \in (0,1] \, : \, \lim_{n\to\infty} \frac{s_n(t)}{n^\delta} = 0 \right\}$$
where $s_n = r_1 + \cdots + r_n$ and r_k denotes, for each $1 \leq j \leq n$, the "jth" Rademacher function.

   Note that $\mathfrak{N}_1$ corresponds to the so-called *normal numbers* in $(0,1]$ and *Borel's Normal Number Theorem*, a special case of the *Strong Law of Large Numbers*, corresponds to the fact that the complement of $\mathfrak{N}_1$ in $(0,1]$ has measure zero.

   (a) Show that $\mathfrak{N}_1$ and its complement, the non-normal numbers in $(0,1]$, are both dense in $(0,1]$.

   (b) Revisit the proof of *Borel's Normal Number Theorem* given in class and show that "by tightening the slack" one can establish that the complement of $\mathfrak{N}_\delta$ in $(0,1]$ has measure zero for all $\delta > 3/4$.

   (c) Show that there exists some constant $C > 0$ such that
$$\int_0^1 s_n(t)^6 \, dt \leq Cn^3$$
and use this to establish that the complement of $\mathfrak{N}_\delta$ in $(0,1]$ has measure zero for all $\delta > 4/6$.

   (d) More generally, show that for any $k \in \mathbb{N}$ there exists some constant $C_k > 0$ such that
$$\int_0^1 s_n(t)^{2k} \, dt \leq C_k n^k$$
and use this to establish that the complement of $\mathfrak{N}_\delta$ in $(0,1]$ has measure zero for all $\delta > (k+1)/2k$ and hence for all $\delta > 1/2$.

2. (a) Given any irrational x one can show (using the pigeonhole principle, for example) that there exists infinitely many fractions a/q, with a and q relatively prime integers, such that
$$\left| x - \frac{a}{q} \right| \leq \frac{1}{q^2}.$$

   (b) Show that for any $\delta > 0$ the set of $x \in (0,1]$ such that there exists infinitely many fractions a/q, with a and q relatively prime integers, such that
$$\left| x - \frac{a}{q} \right| < \frac{1}{q^{2+\delta}}$$
is a set of measure zero.

3. Show that if $E \subseteq (0,1]$ has measure zero, then the set $E^2 = \{x^2 \mid x \in E\}$ also has measure zero.

4. The **Cantor set** $\mathcal{C}$ is the set of all $x \in [0,1]$ that have a ternary expansion $x = \sum_{k=1}^\infty a_k 3^{-k}$ with $a_k \neq 1$ for all k. Thus $\mathcal{C}$ is obtained from $[0,1]$ by removing the open middle third $(\frac{1}{3}, \frac{2}{3})$, then removing the open middle thirds $(\frac{1}{9}, \frac{2}{9})$ and $(\frac{7}{9}, \frac{8}{9})$ of the two remaining intervals, and so forth.

   (a) Find a real number x belonging to the Cantor set which is not the endpoint of one of the intervals used in its construction.

   (b) Prove that $\mathcal{C}$ is compact, nowhere dense, totally disconnected, and perfect.

   (c) Prove that $\mathcal{C}$ has measure zero.

   (d) Prove that $\mathcal{C}$ is uncountable by showing that the function $f(x) = \sum_{k=1}^\infty b_k 2^{-k}$ where $b_k = a_k/2$, maps $\mathcal{C}$ onto $[0,1]$.