OSCILLATORY INTEGRALS IN ONE DIMENSION

NEIL LYALL

In this note we give an overview of the theory of oscillatory integrals in one dimension, which gives an essentially complete description of the behavior of integrals of the form

$$I(\lambda) = \int_a^b e^{i\lambda\Phi(x)}\Psi(x)dx, \qquad \lambda > 0,$$

as $\lambda \to \infty$, where Φ and Ψ are smooth functions. The behavior of $I(\lambda)$ is governed by three basic principles: *localization*, *scaling*, and *asymptotics*. We shall present these respective principles as three propositions: the first of these can be thought of as a principle of non–stationary phase, the second is one of van der Corput's lemmas, and the third is a formulation of the method of staionary phase; for proofs see [3] and [2].

Proposition 1. *Suppose Ψ has compact support in (a, b) and Φ' is never vanishes, then for all $N \geq 0$ we have*

$$|I(\lambda)| \leq C_{N,\Phi,\Psi}\lambda^{-N}.$$

Remark. If we do not assume that Ψ vanishes near the endpoints of the interval $[a, b]$ then the best estimate we can obtain for $I(\lambda)$ is $O(\lambda^{-1})$. However, in the "periodic" case, i.e, if we have $\Phi^{(k)}(a) = \Phi^{(k)}(b)$ and $\Psi^{(k)}(a) = \Psi^{(k)}(b)$, we again, as in Proposition 1, obtain the rapid decrease of $I(\lambda)$.

Proposition 2. *Suppose Φ is real–valued and $|\Phi^{(k)}(x)| \geq 1$ for all $x \in (a, b)$, then*

$$|I(\lambda)| \leq kC_k\lambda^{-\frac{1}{k}}\Big[|\Psi(b)| + \int_a^b |\Psi'(x)|dx\Big],$$

whenever (i) $k = 1$ and $\Phi''(x)$ has at most one zero, or (ii) $k \geq 2$.

Remark. We note that the constants in Proposition 2 are $C_1 = 2$, while $C_k \leq 2^{\frac{5}{3}}$ for all $k \geq 2$ and $C_k \to 4/e$ as $k \to \infty$; see [1].

Of course if Φ is completely stationary then the best one can do is $|I(\lambda)| \leq (b - a)\|\Psi\|_\infty$.

Proposition 3. *Suppose Φ is real–valued, $\Phi'(x_0) = 0$, while $\Phi''(x_0) \neq 0$. If Ψ is supported in a sufficiently small neighborhood of x_0, then*

$$I(\lambda) = a_0\lambda^{-\frac{1}{2}}e^{i\lambda\Phi(x_0)} + O(\lambda^{-\frac{3}{2}})$$

as $\lambda \to \infty$, where $a_0 = e^{i\frac{\pi}{4}}\left(\frac{2\pi}{\Phi''(x_0)}\right)^{\frac{1}{2}}\Psi(x_0)$, and the bounds occuring in the error term depend on upper bounds for finitely many derivatives of Φ and Ψ on the $\operatorname{supp}\Psi$*, the size of this support, and on a lower bound for $|\Phi''(x_0)|$.*

Remark 4. If we merely assume that Φ is real–valued, $\Phi'(x_0) = 0$ and $\Phi'(x) \neq 0$ on $\operatorname{supp}\Psi \setminus \{x_0\}$. Then if $\Phi''(x_0) \neq 0$ we may conclude that

$$I(\lambda) = e^{i\lambda\Phi(x_0)}\sigma(\lambda),$$

where σ is a symbol of order $-\frac{1}{2}$, that is $|\sigma^{(\ell)}(\lambda)| \leq c_\ell(1+\lambda)^{-\frac{1}{2}-\ell}$; see [4].

The constant c_ℓ depends on the $C^{\ell+1}$ norms of Φ and Ψ on the $\operatorname{supp}\Psi$, the size of this support, and on a lower bound for $|\Phi''(x_0)|$.

Example. The *Bessel functions*, defined for $k \in \mathbf{Z}^+$ by

$$J_k(\lambda) = \frac{1}{2\pi}\int_0^{2\pi} e^{i\lambda\sin\theta}e^{ik\theta}d\theta,$$

are a model case for these oscillatory integrals. It follows from Proposition 3 and the remarks following Propostion 1, that for $\lambda \gg 1$

$$J_k(\lambda) = \left(\frac{2}{\pi}\right)^{\frac{1}{2}}\lambda^{-\frac{1}{2}}\cos(\lambda - \tfrac{\pi k}{2} - \tfrac{\pi}{4}) + O(\lambda^{-\frac{3}{2}}). \tag{1}$$

The Bessel function can also be defined for real $k > -\frac{1}{2}$ by the formula

$$J_k(\lambda) = (\pi^{\frac{1}{2}}\Gamma(k+\tfrac{1}{2}))^{-1}\left(\frac{\lambda}{2}\right)^k\int_{-1}^{1} e^{i\lambda t}(1-t^2)^{k-\frac{1}{2}}dt.$$

These two definitions agree when k is a positive integer and the asymptotic expression (1) is still valid. One can in fact show that

$$J_k(\lambda) = \sigma_1(\lambda)e^{i\lambda} + \sigma_2(\lambda)e^{-i\lambda}, \tag{2}$$

where $|\sigma_i^{(\ell)}(\lambda)| \leq c_\ell(1+\lambda)^{-\frac{1}{2}-\ell}$. See Remark 4.

References

[1] K. Rogers, *Sharp van der Corput estimates and minimal divided differences.* preprint, 2004.
[2] C. D. Sogge, *Fourier Integrals in Classical Analysis* Cambridge tracts in Mathematics ***105***, Cambridge Univ. Press, Cambridge, 1993.
[3] E. M. Stein, *Harmonic Analysis: Real–Variable Methods, Orthogonality, and Oscillatory Integrals*, Princeton Univ. Press, Princeton, 1993.
[4] T. H. Wolff, *Lectures on Harmonic Analysis,* University Lecture Series ***29***, American Math. Soc., 2003.