Aside on Minkowski Dimension

Let E be a non-empty bounded subset of $\mathbb{R}^n$. For $\delta>0$ define

$N_\delta(E) =$ smallest number of balls of radius δ needed to cover E.

The upper & lower Minkowski dimensions of E are defined by

$$\overline{\dim}_M(E) := \inf\{\alpha \geq 0 : \overline{\lim_{\delta\to 0^+}} N_\delta(E)\delta^\alpha = 0\}$$

OR
$\inf\{\alpha : \overline{\lim} < \infty\}$
$\sup\{\alpha : \overline{\lim} = \infty\}$
$\sup\{\alpha : \overline{\lim} > 0\}$,
etc.

and

$$\underline{\dim}_M(E) := \inf\{\alpha \geq 0 : \underline{\lim_{\delta\to 0^+}} N_\delta(E)\delta^\alpha = 0\}.$$

Note: It follows immediately from the definitions that

$$\dim_H(E) \leq \underline{\dim}_M(E) \leq \overline{\dim}_M(E) \leq n.$$

Exercise 4: Show that

$$\underline{\dim}_M(\{0\}\cup\{j^{-1} : j=1,2,3,\dots\}) = \frac{1}{2}.$$

Exercise 5: (Equivalent Definitions). For any non-empty bounded $E\subset\mathbb{R}^n$:

1. $\underline{\dim}_M(E) = \underline{\lim_{\delta\to 0^+}} \dfrac{\log N_\delta(E)}{\log \delta^{-1}}$ & $\overline{\dim}_M(E) = \overline{\lim_{\delta\to 0^+}} \dfrac{\log N_\delta(E)}{\log \delta^{-1}}$.

2. $\underline{\dim}_M(E) = \sup\{\alpha \geq 0 : m_n(\{x\in\mathbb{R}^n : \operatorname{dist}(x,E)<\delta\}) \geq C\delta^{n-\alpha}\ \forall \delta\in(0,1]\}$

   $\overline{\dim}_M(E) = \sup\{\alpha\geq 0 : m_n(\{x\in\mathbb{R} : \operatorname{dist}(x,E)<\delta\}) \geq C\delta^{n-\alpha}$ for some sequence of δ's converging to $0\}$

⋮

$$\boxed{\circledast\ \textbf{Idea}: \dim_M(E)=\alpha \iff \text{"}N_\delta(E)\approx\delta^{-\alpha}\text{"}\ \forall \text{ small } \delta>0.\ \circledast}$$

$N_\delta(E)\delta^\beta \to \infty$ if $\beta < \dim_M(E)$
& $N_\delta(E)\delta^\beta \to 0$ if $\beta > \dim_M(E)$