Math 3100
Sample Exam 1 – Version 1

No calculators. Show your work. Give full explanations. Good luck!

1. (6 points) Determine which of the following sequences converge and which diverge. Find the value of the limit for those which converge.

   Be sure to give a short justification in each case by indicating any limit laws, theorems, or special limits used.

   (a) $a_n = \dfrac{\sin(n)}{n^3}$

   (b) $b_n = \dfrac{1}{(1+3^{1/n})^5}$

   (c) $c_n = (-1)^n - \dfrac{n}{2^n}$

2. (24 points)

   (a) Let $\{x_n\}$ be a sequence of real numbers. Carefully state the definition of the following:

   i. $\lim_{n\to\infty} x_n = x$

   ii. $\lim_{n\to\infty} x_n = \infty$.

   (b) Use the definition given in (i) to prove that $\lim_{n\to\infty} \dfrac{2n+1}{n-3} = 2$.

   (c) Use the definitions given above to prove that if $\lim_{n\to\infty} x_n = 2$, then

   i. $\{x_n\}$ is bounded

   ii. $\lim_{n\to\infty} \dfrac{1}{x_n} = \dfrac{1}{2}$

   iii. $\lim_{n\to\infty} (x_n + y_n) = \infty$ whenever $\lim_{n\to\infty} y_n = \infty$

3. (10 points)

   (a) Carefully state the *Monotone Convergence Theorem.*

   (b) Let $x_1 = 1$ and $x_{n+1} = \left(\dfrac{n}{n+1}\right) x_n^2$ for all $n \in \mathbb{N}$.

   i. Find x_2, x_3, and x_4.

   ii. Show that $\{x_n\}$ converges and find the value of its limit.

4. (10 points) Let $\{x_n\}$ be a bounded sequence of real numbers.

   (a) Carefully state the definition of $\limsup_{n\to\infty} x_n$ and justify why it always exists for such sequences.

   (b) Prove that if $\alpha = \limsup_{n\to\infty} x_n$ and $\beta > \alpha$, then there exists an N such that $x_n < \beta$ whenever $n > N$.

Math 3100 - Sample Exam 1 (Version 1) - SOLUTIONS

1. (a) $a_n = \dfrac{\sin(n)}{n^3}$ converges to 0.

Justification

Since $\left|\dfrac{\sin(n)}{n^3}\right| \le \dfrac{1}{n^3}$ $\forall n \in \mathbb{N}$ & $\lim\limits_{n\to\infty} \dfrac{1}{n^3} = 0$

it follows from "Baby Squeeze" that $\dfrac{\sin(n)}{n^3} \to 0$.

(b) $b_n = \dfrac{1}{(1+3^{1/n})^5}$ converges to $\dfrac{1}{32}$.

Justification

Since $3^{1/n} \to 1$ (special limit) it follow from "limit laws"

that

$$\lim_{n\to\infty} \frac{1}{(1+3^{1/n})^5} = \frac{1}{(\lim\limits_{n\to\infty} 1+3^{1/n})^5} = \frac{1}{2^5} = \frac{1}{32}.$$

since $\lim\limits_{n\to\infty} 1+3^{1/n} = 2 \neq 0$

(c) $c_n = (-1)^n - \dfrac{n}{2^n}$ DIVERGES.

Justification 1

$\dfrac{n}{4^n} \to 0$ $\qquad \dfrac{2n-1}{2^{2n-1}} \to 0$

Since $c_{2n} = 1 - \dfrac{2n}{4^n} \to 0$ & $c_{2n-1} = -1 - \dfrac{2n-1}{2^{2n-1}} \to -1$

we have two subsequence that converge to different limits.

Justification 2

for some $L \in \mathbb{R}$ $\qquad$ since $\frac{n}{2^n} \to 0$

If $c_n \to L$, then $(-1)^n = c_n + \dfrac{n}{2^n} \to L + 0 = L$,

contradicting the fact that $(-1)^n$ is divergent.

② (a)

(i) $\lim_{n\to\infty} x_n = x \iff \forall \varepsilon>0\ \exists N$ s.t $n>N \Rightarrow |x_n - x| < \varepsilon$

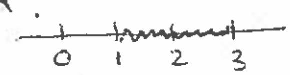
(ii) $\lim_{n\to\infty} x_n = \infty \iff \forall M>0\ \exists N$ s.t. $n>N \Rightarrow x_n > M$

(b) <u>Claim</u> $\lim_{n\to\infty} \frac{2n+1}{n-3} = 2$

<u>Proof</u> Let $\varepsilon>0$ & set $N = 3 + 7/\varepsilon$. If $n>N$, then

$$\left|\frac{2n+1}{n-3} - 2\right| = \left|\frac{7}{n-3}\right| \underset{\text{since } n>3}{=} \frac{7}{n-3} < \varepsilon \quad \text{since } n > 3 + 7/\varepsilon \qquad \square$$

(c) (i) <u>Claim</u>: If $\lim_{n\to\infty} x_n = 2$, then $\{x_n\}$ is bounded.



<u>Proof</u> Since $\lim_{n\to\infty} x_n = 2$ $\exists N$ s.t $n>N \Rightarrow |x_n - 2| < 1$

$\Rightarrow 1 < x_n < 3$ (if $n>N$)

Thus $|x_n| \leq \max\{|x_1|, \ldots, |x_N|, 3\}$ for all $n \in \mathbb{N}$. $\square$

(ii) <u>Claim</u> If $\lim_{n\to\infty} x_n = 2$, then $\lim_{n\to\infty} \frac{1}{x_n} = \frac{1}{2}$

<u>Proof</u> Recall from above that since $\lim_{n\to\infty} x_n = 2$ $\exists N_1$ s.t.

$n > N_1 \Rightarrow |x_n| > 1$.

Let $\varepsilon>0$. Since $\lim_{n\to\infty} x_n = 2$ $\exists N_2$ s.t $n>N_2 \Rightarrow |x_n - 2| < 2\varepsilon$

and hence if $n > \max\{N_1, N_2\} \Rightarrow \left|\frac{1}{x_n} - \frac{1}{2}\right| = \frac{|x_n - 2|}{2|x_n|} \underset{\text{Since } |x_n|>1}{<} \frac{|x_n-2|}{2} \underset{\text{since } |x_n-2|<2\varepsilon}{<} \varepsilon.$ $\square$

(iii) <u>Claim</u>: If $\lim_{n\to\infty} x_n = 2$ & $\lim_{n\to\infty} y_n = \infty$, then $\lim_{n\to\infty}(x_n + y_n) = \infty$.

<u>Proof</u>

Recall that since $\lim_{n\to\infty} x_n = 2$ $\exists N_1$ s.t $n>N_1 \Rightarrow x_n > 1 > 0$.

Let $M>0$. Since $\lim_{n\to\infty} y_n = \infty$ $\exists N_2$ s.t $n > N_2 \Rightarrow y_n > M$

Hence if $n > \max\{N_1, N_2\} \Rightarrow x_n + y_n > y_n > M$. $\square$

(3) (a) Monotone Convergence Theorem (MCT)

If $\{x_n\}$ is a bounded monotone sequence of reals, then

$\lim_{n\to\infty} x_n$ exists.

(b) Let $x_1 = 1$ & $x_{n+1} = \left(\frac{n}{n+1}\right) x_n^2$ $\forall n \in \mathbb{N}$.

(i) $x_2 = \frac{1}{2}$, $x_3 = \frac{1}{6}$, $x_4 = \frac{1}{48}$.

(ii) Claim: $\{x_n\}$ is decreasing and bounded below.

Proof

- $x_1 = 1 \geq 0$ & $x_n = \left(\frac{n}{n+1}\right) x_n^2 \geq 0$ $\forall n \in \mathbb{N}$.

  $\longrightarrow$ $\{x_n\}$ bounded below by 0.

- Subclaim: $x_n \leq 1$ $\forall n \in \mathbb{N}$

  Pf: $x_1 = 1 \leq 1$ ✓

  Suppose $x_n \leq 1$ for some $n \in \mathbb{N}$.

  It follows that $x_{n+1} = \left(\frac{n}{n+1}\right) x_n^2 \leq x_n^2 \leq 1$ □

  $\rightsquigarrow$ In particular, $\underbrace{x_{n+1} \leq x_n^2 \leq x_n \quad \forall n \in \mathbb{N}}_{\Rightarrow \text{ decreasing}}$ □

It now follows from MCT that $\lim_{n\to\infty} x_n = x$ for some $x \in \mathbb{R}$.

Since $x_{n+1} = \left(\frac{n}{n+1}\right) x_n^2$ $\forall n \in \mathbb{N}$ it follows from limit laws (& uniqueness) that

$x = (1)x^2 \Rightarrow \underline{\underline{x = 0}}$ or $\cancel{1}$ ← since $\{x_n\}$ dec & $x_2 = \frac{1}{2}$. □

(4) See Homework 4 Question 5