Dense Subspaces of $L^p(\mathbb{R}^n)$

Theorem 1: Let $1 \le p \le \infty$. The collection of all simple functions

$$\varphi = \sum_{j=1}^{N} a_j \chi_{E_j} \quad \text{with } m(E_j) < \infty \quad (1 \le j \le N)$$

is dense in $L^p(\mathbb{R}^n)$.

Proof: We know $\exists$ seq of simple functions $\{\varphi_n\}$ such that

$$\varphi_n(x) \to f(x) \text{ a.e. } x \quad \text{and} \quad |\varphi_n| \le |f|.$$

Since $|\varphi_n - f|^p \le \underbrace{2^p |f|^p}_{\in L^1}$, the result follows from the DCT.

Note that if $\varphi_n = \sum_{j=1}^{N} a_j \chi_{E_j}$ with a_j's all distinct ($\& \ne 0$) & E_j's all disjoint

then $m(E_j) < \infty$ for $1 \le j \le N$, since

$$|\varphi_n|^p = \sum_{j=1}^{N} |a_j|^p \chi_{E_j}$$

$$\Rightarrow \int |\varphi_n|^p = \sum_{j=1}^{N} |a_j|^p m(E_j) \le \int |f|^p < \infty.$$

This completes the proof for $1 \le p < \infty$. $\square$

Exercise: Prove the $p = \infty$ case of the above theorem.

Theorem 2

Let $1 \le p < \infty$, then continuous functions with compact support are dense in $L^p(\mathbb{R}^n)$, i.e. for any $f \in L^p(\mathbb{R}^n)$ and $\varepsilon > 0$, $\exists\, g \in C_c(\mathbb{R}^n)$ s.t.

$$\|f - g\|_p < \varepsilon.$$

Proof:

Let $f \in L^p(\mathbb{R}^n)$ and $\varepsilon > 0$. We have just shown that $\exists$ simple function

$$\varphi = \sum_{j=1}^{N} a_j \chi_{E_j} \quad \text{with } a_j \neq 0$$

such that

$$\int |f - \varphi|^p < \varepsilon^p.$$

- We now show that "step functions" are dense (wrt L^p-norm) in the space of all simple functions (and hence in $L^p(\mathbb{R}^n)$ also).

Note that for each j,

$$m(E_j) = \frac{1}{|a_j|^p} \int_{E_j} |\varphi|^p \le \frac{1}{|a_j|^p} \int_{\mathbb{R}^n} |f| < \infty$$

Now, by Question 1 from Homework 3, we know $\exists$ a set A_j that is a finite union of closed cubes such that

$$m(\underbrace{E_j \Delta A_j}_{= E_j \setminus A_j \cup A_j \setminus E_j}) < \varepsilon \quad (1 \le j \le N) \quad \& \ A_j\text{'s } \underline{\text{disjoint}}$$

¿Can we really do this?

Now let $\tilde{\varphi} = \sum_{j=1}^{N} a_j \chi_{A_j}$

$$\Rightarrow \int |\tilde{\varphi} - \varphi|^p \leq \sum_{j=1}^{N} |a_j|^p \int |\chi_{A_j} - \chi_{E_j}|^p$$
$$= \sum_{j=1}^{N} |a_j|^p \, m(A_j \Delta E_j)$$
$$< \varepsilon \sum_{j=1}^{N} |a_j|^p \qquad \checkmark$$

- To finish we need only show that if $f = \chi_Q$ with Q a closed cube in $\mathbb{R}^n$ and $\varepsilon > 0$, then $\exists\, g \in C_c(\mathbb{R}^n)$ such that

$$\int |f - g|^p < \varepsilon .$$

We know $\exists$ open set $G \subseteq \mathbb{R}^n$ such that $Q \subseteq G$ and

$$m(G \setminus Q) < \varepsilon .$$

Simply let g be any continuous function with

(i) $0 \leq g \leq 1$

(ii) $g(x) = \begin{cases} 1 & \text{if } x \in Q \\ 0 & \text{if } x \in G^c, \end{cases}$

for then $$\int |f - g| \leq \int_{G \setminus Q} 1 = m(G \setminus Q) < \varepsilon . \qquad \square$$

<u>Exercise</u>: Show that Theorem 2 fails for $p = \infty$.