Spring 2020 Topology Qual; General Topology Problems

$\boxed{J}$ #2) 2/10 They correctly point out that metrizable $\Rightarrow$ normal.
They prove the result for finite spaces: a trivial case, since a metrizable finite space is discrete. They say "similarly" if X is countably infinite.
This makes no sense. Nor is the normality being used in any helpful way.

#3) a) 1/5 It is assumed that a point p not lying in $\iota(X)$ must be an accumulation point of X. This need not be the case. The argument doesn't say so but seems to actually assume that p is an accumulation point of $\iota(X)$. But $\iota(X)$ is compact hence closed, so then certainly $p \in \iota(X)$.
It seems that only a trivial case is being done, and not very cleanly.

b) 4/5 Should mention that all closed balls in $\mathbb{R}^N$ w/ same radius are isometric.

5/10

$\boxed{K}$ #1) 10/10

#2) 2/10 Only does the trivial case of X finite.

$\boxed{L}$ #1) 10/10

#3) a) 0/5

b) 0/5 " Euclidean n-space is compact metric space." ☹

0/10

$\boxed{R}$ #1) 10/10

#2) 10/10 Use Tietze Extension w/o naming it, but this seems okay.

$\boxed{S}$ #1) 10/10 A mildly distressing jargon overlays the proof, but it is certainly correct.

#3) a) 0/5 Not attempted.

b) 0/5 Tried to extend isometry to one-point compactification. This makes no sense. ☹

0/10

J Be sure to write your letter on **each page** of your exam

Topology Qualification Exam, Spring 2020

Please attempt **8 out of 9** problems and clearly mark the one you do not want us to grade.

1. Let $f : X \to Y$ be a surjective, continuous map of topological spaces.

   a) Show: if f is an open map, then it is a quotient map.

   b) Show: if f is a closed map, then it is a quotient map.

2. Show that a connected metrizable space with at least two points is uncountably infinite. (You may use without proof that every metrizable space is normal.)

3. Let (X, d_X) and (Y, d_Y) be metric spaces. An **isometric embedding** $\iota : X \to Y$ is a map such that

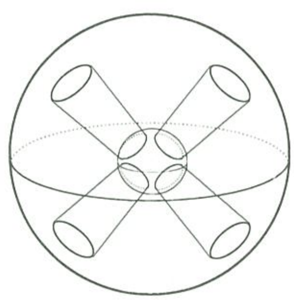
$$\forall x_1, x_2 \in X,\ d_Y(\iota(x_1), \iota(x_2)) = d_X(x_1, x_2).$$

   An **isometry** is a surjective isometric embedding.

   a) Show that every isometric embedding from a compact metric space to itself is an isometry.
   (You may use that a metric space is compact iff it is sequentially compact.)

   b) Show that every isometric embedding from Euclidean n-space to itself is an isometry.

4. Consider the solid S obtained by digging out the center of a 3-dimensional solid ball and 4 tunnels from the center to the boundary. What is the genus of the boundary surface $\Sigma = \partial S$? Justify your answer.



5. Let X be the topological space obtained by attaching a disk to $T^2 = S^1 \times S^1$ along the circle $S^1 \times \{p\}$ via the map $z \mapsto z^5$. Compute the fundamental group and the homology groups of X.

6. Classify the connected 2-fold covering spaces of the Klein bottle K.
   (You might want to consider K as the union of two Möbius bands.)

7. Show that every continuous map from $\mathbb{R}P^2 \times \mathbb{R}P^2$ to $T^4 = S^1 \times S^1 \times S^1 \times S^1$ is null-homotopic.

8. Let $X = \mathbb{R}P^5/\mathbb{R}P^1$, and let $f : X \to X$ be a continuous map that is homotopic to the identity. Show that f must have a fixed point.

9. Describe the CW structure of $X = \mathbb{C}P^2 \times \mathbb{R}P^2$ and use it to compute the homology groups of X.

(8) Let $X = \mathbb{RP}^5 / \mathbb{RP}^1$, and let $f: X \to X$ be a continuous map that is homotopic to the identity. Then there exists $F: X \times I \to X$ defined via $F([x], t) = (1-t) f([x]) + t[x]$ such that F is continuous, $F_t([x])$ is continuous for all $t \in I$, $F_0([x]) = f([x])$, and $F_1([x]) = [x]$ for all $[x] \in X$. Suppose f does not have any fixed point. Then $f([x]) \neq [x]$ for any $[x] \in X$.

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(2) Let (X, ρ) be a connected metrizable space with at least two points.

Since X is normal, for all disjoint closed subsets A and B of X, there exist open subsets U and V of X such that $A \subseteq U$, $B \subseteq V$, and $U \cap V = \emptyset$. We will use this to contradict the connectedness of X if X is countable.

So suppose X is countable. Then $X = \{x_n\}_{n=1}^{\infty}$ or $\{x_k\}_{k=1}^{N}$ for some $N \in \mathbb{N}$.

If $X = \{x_k\}_{k=1}^{N}$ then since each singleton $\{x_k\}$ is closed, the union of finitely many of them is closed. Then let $A = \{x_k\}_{k=1}^{N-1}$, $B = \{x_N\}$. Then by the normalness of X, X is disconnected, a contradiction. Similarly if $X = \{x_n\}_{n=1}^{\infty}$

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(3) (a) Let (X, ρ) be a compact metric space. Let $\iota: X \to X$ be an isometric embedding. Let, if possible, $y \in X$ is such that $\iota^{-1}(\{y\}) = \phi$. Let $\{x_n\}_{n=1}^{\infty}$ be a sequence of points in X with accumulation point y. Then there is a subsequence $\{x_{n_k}\}_{k=1}^{\infty}$ of $\{x_n\}_{n=1}^{\infty}$ such that $\lim_{n_k \to \infty} x_{n_k} = y$. Since ι preserves the distance between points, $\iota^{-1}(\{x_{n_k}\}_{n_k=1}^{\infty})$ is a sequence of points in X which also converges. Let $\lim_{n \to \infty} \iota^{-1}(\{x_{n_k}\}_{n_k=1}^{\infty}) = p$. Then for all $\varepsilon > 0$ $\rho(f(p), y) < \varepsilon$ $\Rightarrow f(p) = y$. So ι is surjective. Thus ι is an isometry.

(b) Let E^n denote Euclidean n-space, and let $\imath : E^n \to E^n$ be an isometric embedding. Since E^n can be expressed as the union of closed balls each of which is a compact metric space any isomectric embedding from E^n to E^n is an isometry.

(4) The solid S will have a deformation retraction to the center sphere with 4 open discs removed which will then deformation retract to the wedge of 4 circles. 

Hence the boundary surface is a genus 2 surface.

⑤

Let U be the subset of X consisting of disc along with an open collar in the torus. Then U will have a deformation retract to the disc. Let V be the subset of X consisting of the torus with an open collar in the disc. Then V will have a deformation retract to the torus. And $U \cap V$ will have a deformation retract to the S^1 along which the disc is glued. So by Van Kampen's Theorem, $\pi_1(X) \cong \dfrac{\langle a,b\rangle * \{0\}}{\langle a^5\rangle} \cong \langle a,b \mid a^5\rangle$

$\mathbb{Z} * \mathbb{Z}_5$

$$\begin{tikzcd}
& \overset{\langle a,b\rangle}{\overset{\shortparallel}{\pi_1(U)}} \arrow[dr] & \\
\pi_1(U\cap V) \arrow[ur] \arrow[dr] & & \pi_1(X) \\
& \underset{\{0\}}{\pi_1(V)} \arrow[ur] &
\end{tikzcd}$$

$i_{UV*}([\alpha]) = [a]^5$

$i_{VU*}([\alpha]) = \{0\}$

Consider the long exact sequence:

$$\cdots \xrightarrow{\delta_3} \overset{0}{\overset{\shortparallel}{H_2(U\cap V)}} \xrightarrow{\varphi_2} \overset{0}{\overset{\shortparallel}{H_2(U)}} \oplus \overset{\mathbb{Z}}{\overset{\shortparallel}{H_2(V)}} \xrightarrow{\psi_2} H_2(X)$$

$$\xrightarrow{\delta_2} \overset{\mathbb{Z}}{\overset{\shortparallel}{H_1(U\cap V)}} \xrightarrow{\varphi_1} \overset{0}{\overset{\shortparallel}{H_1(U)}} \oplus \overset{\mathbb{Z}\oplus\mathbb{Z}}{\overset{\shortparallel}{H_1(V)}} \xrightarrow{\psi_1} H_1(X)$$

$$\xrightarrow{\delta_1} \overset{0}{\overset{\shortparallel}{\tilde{H}_0(U\cap V)}} \xrightarrow{\varphi_0} \overset{0}{\overset{\shortparallel}{\tilde{H}_0(U)}} \to \overset{0}{\overset{\shortparallel}{\tilde{H}_0(V)}} \xrightarrow{\psi_0} \overset{0}{\overset{\shortparallel}{\tilde{H}_0(X)}} \xrightarrow{\delta_0} 0$$

Note that $\ker\delta_1 = H_1(X) = \operatorname{im}\psi_1 \Rightarrow \psi_1$ is surjective
And $\ker\psi_2 = \operatorname{im}\varphi_2 = 0 \Rightarrow \psi_2$ is injective. Hence by 1st isomorphis Thm $\mathbb{Z}/0 \cong H_2(X) \Rightarrow H_2(X) \cong \mathbb{Z}$. Now φ_1 is a map which sends the generator of $H_1(U\cap V)$ to 5 times one generator of

$H_1(V)$. Hence $\operatorname{im} \varphi_1 = 5\mathbb{Z}$. So $\ker \varphi_1 = 5\mathbb{Z}$.

Thus $\mathbb{Z} \oplus \mathbb{Z}/5\mathbb{Z} \cong H_1(X)$.

Hence

$$H_*(X) \cong \begin{cases} 0 & \text{otherwise} \\ \mathbb{Z} \oplus \mathbb{Z}/5\mathbb{Z} & \text{if } * = 1 \\ \mathbb{Z} & \text{if } * = 0, 2 \end{cases}$$

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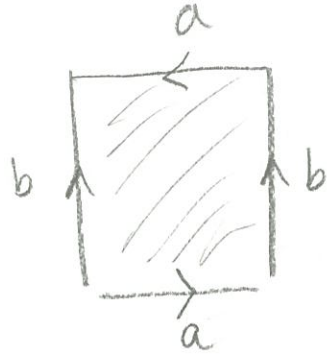
(6) The Klein bottle can be represented as a CW complex with one 0-cell, two 1-cells, and one 2-cell attached along the boundary $aba^{-1}b$. Hence



$$\pi_1(K) \cong \langle a, b \mid abab^{-1} \rangle \cong \mathbb{Z} * \mathbb{Z}_2$$

Then a 2-fold cover of K corresponds to an index 2 subgroup of $\mathbb{Z} * \mathbb{Z}_2$. Hence the connected 2-fold cover of K is $\mathbb{RP}^2$ which has fundamental group $\mathbb{Z}_2$.

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(7) Let $f : \mathbb{R}P^2 \times \mathbb{R}P^2 \longrightarrow T^4$.

If $f_*(\pi_1(\mathbb{R}P^2 \times \mathbb{R}P^2)) \leq p_*(\pi_1(\tilde{X}))$ where $\tilde{X}$ is a covering space of T^4, then there is a unique lift $\tilde{f}$ from $\mathbb{R}P^2 \times \mathbb{R}P^2 \longrightarrow \tilde{X}$. Then if $\tilde{X}$ is contractible, $\tilde{f}$ is nullhomotopic and, hence $p \circ \tilde{f} = f$ is nullhomotopic.

$$\begin{tikzcd} & \mathbb{R}P^2 \times \mathbb{R}P^2 \arrow[d, "p"] \\ \mathbb{R}P^2 \times \mathbb{R}P^2 \arrow[ur, "\tilde{f}"] \arrow[r, "f"'] & T^4 \end{tikzcd}$$

So let $\tilde{X} = \mathbb{R}P^2 \times \mathbb{R}P^2$. Then $f_*(\pi_1(\mathbb{R}P^2 \times \mathbb{R}P^2)) \leq p_*(\pi_1(\mathbb{R}P^2 \times \mathbb{R}P^2))$.

So we can define a homotopy

$\tilde{F} : \mathbb{R}P^2 \times \mathbb{R}P^2 \times I$ via $\tilde{F}(x, y, t) = (1-t)\tilde{f}(x,y) + t\, C_{(\tilde{x}_0, \tilde{y}_0)}$

where $C_{(\tilde{x}_0, \tilde{y}_0)} : \mathbb{R}P^2 \times \mathbb{R}P^2 \to \mathbb{R}P^2 \times \mathbb{R}P^2$ is defined via $C_{(\tilde{x}_0, \tilde{y}_0)}(x, y) = (\tilde{x}_0, \tilde{y}_0)$ for all $(x, y) \in \mathbb{R}P^2 \times \mathbb{R}P^2$.

Then $p \circ \tilde{F}$ is continuous, $(p \circ \tilde{F}_0)(x, y) = p \circ \tilde{f}(x, y) = f(x, y)$ for all $(x, y) \in \mathbb{R}P^2 \times \mathbb{R}P^2$ and $(p \circ \tilde{F}_1)(x, y) = p \circ C_{(\tilde{x}_0, \tilde{y}_0)} = C_{(x_0, y_0)}$ where $p(\tilde{x}_0, \tilde{y}_0) = (x_0, y_0)$.

So f is nullhomotopic.

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(9). $\mathbb{C}P^2$ can be thought of as a CW-complex consisting of one 0-cell, one 2-cell, and one 4-cell, while $\mathbb{R}P^2$ can be thought of as a CW-complex with one 0-cell, one 1-cell, and one 2-cell. Then $\mathbb{C}P^2 \times \mathbb{R}P^2$ can be represented by a CW-complex consisting of one 0-cell, one 1-cell, two 2-cells, one 3-cell, two 4-cells, one 5-cell, and one 6-cell.
Let e_{ij} be the $i+j$-cell generated by the i-cell from $\mathbb{C}P^2$ crossed with the j-cell from $\mathbb{R}P^2$.
Consider the chain complex

$$0 \xrightarrow{\partial_7} \mathbb{Z}[e_{42}] \xrightarrow{\partial_6} \mathbb{Z}[e_{41}] \xrightarrow{\partial_5} \mathbb{Z}[e_{40}, e_{22}] \xrightarrow{\partial_4} \mathbb{Z}[e_{21}]$$

$$\xrightarrow{\partial_3} \mathbb{Z}[e_{20}, e_{02}] \xrightarrow{\partial_2} \mathbb{Z}[e_{01}] \xrightarrow{\partial_1} \mathbb{Z}[e_{00}] \xrightarrow{\partial_0} 0$$

$\partial_6(e_{42}) = 0 + (1+(-1))e_{41} = 0$

$\partial_5(e_{41}) = 0 + (1+(-1)^0)e_{40} = 2e_{40}$

$\partial_4(e_{40}) = 0 + (1+(-1)^1)e_{41} = 0$

$\partial_4(e_{22}) = 0 + (1+(-1)^1)e_{21} = 0$

$\partial_3(e_{21}) = 0 + (1+(-1)^0)e_{20} = 2e_{20}$

$\partial_2(e_{20}) = 0$

$\partial_2(e_{02}) = 0 + (1+(-1)^1)e_{01} = 0$

$\partial_1(e_{00}) = 0$

Hence
$\operatorname{im}\partial_6 = \operatorname{im}\partial_4 = \operatorname{im}\partial_2 = \operatorname{im}\partial_1 = \operatorname{im}\partial_0 = 0.$

while
$\operatorname{im}\partial_5 = \mathbb{Z}[2e_{40}]$
$\operatorname{im}\partial_3 = \mathbb{Z}[2e_{20}]$

Then $\ker\partial_5 = 0$
$\ker\partial_3 = 0$

Hence $\ker \partial_6 / \operatorname{im} \partial_7 \cong \mathbb{Z}$

$\ker \partial_5 / \operatorname{im} \partial_6 \cong 0$

$\ker \partial_4 / \operatorname{im} \partial_5 \cong \mathbb{Z}[e_{40}, e_{22}] / \mathbb{Z}[2e_{40}] \cong \mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$

$\ker \partial_3 / \operatorname{im} \partial_4 \cong 0$

$\ker \partial_2 / \operatorname{im} \partial_3 \cong \mathbb{Z}[e_{20}, e_{02}] / \mathbb{Z}[2e_{20}] \cong \mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$

$\ker \partial_1 / \operatorname{im} \partial_2 \cong \mathbb{Z}$

$\ker \partial_0 / \operatorname{im} \partial_1 \cong \mathbb{Z}$

Hence $$H_*(\mathbb{C}P^2 \times \mathbb{R}P^2) \cong \begin{cases} 0 & \text{otherwise} \\ 0 & \text{if } * = 3, 5 \\ \mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} & \text{if } * = 2, 4 \\ \mathbb{Z} & \text{if } * = 0, 1, 6 \end{cases}$$

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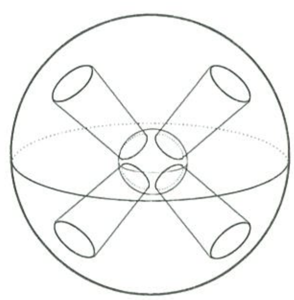

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9. Describe the CW structure of $X = \mathbb{C}P^2 \times \mathbb{R}P^2$ and use it to compute the homology groups of X.

[redacted] K

Question 1

Proof: We need to prove that V is an open set of Y iff $f^{-1}(V)$ is an open set of X

a) Since f is continuous, $f^{-1}(V)$ is open if V is open.

If $f^{-1}(V)$ is open, then $V = f(f^{-1}(V))$ is open since f is surjective and open map.

Thus f is a quotient map.

b) Since f is continuous, $f^{-1}(V)$ is open if V is open.

Suppose $f^{-1}(V)$ is open. Then $f^{-1}(V)^c$ is closed. $f^{-1}(V)^c = f^{-1}(V^c)$.

Thus $f^{-1}(V^c)$ is closed. $f(f^{-1}(V^c))$ is closed since f is a closed map.

Since f is surjective, $f(f^{-1}(V^c)) = V^c$. Then V^c is closed. Thus V is open.

f is a quotient map $\boxed{\text{//}}$

[redacted] K

QUESTION 2.

Proof: Suppose that X is a connected metrizable space with at most countable points.

Let U be an open set of X.

Suppose U consists of finite points. Since X is a metrizable space, then X satisfies T_1 axiom.

Thus U is a closed set. Since X is connected, then $U = X$. Thus X has finite points.

Let $x_1, x_2 \in X$ and $x_1 \neq x_2$. Since X satisfies T_2 axiom, there are open sets U_1 and U_2 s.t.

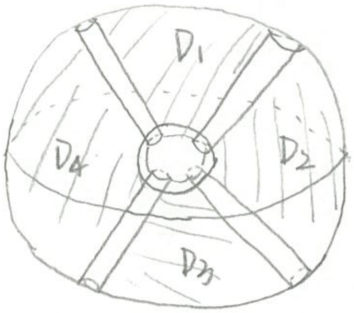
$x_1 \in U_1$, $x_2 \in U_2$ and $U_1 \cap U_2 = \phi$. But $U_1 = X$ and $U_2 = X$, which is a contradiction.

Thus U has countable points and X is an infinite set.

[redacted] K

Question 4.

Proof:



By cutting along the four disks D_i, we obtain two 3-balls.

Thus S is a handle body with genus 3.

Thus genus of the boundary surface is 3. $\square$

Consider D_1 as a polygon with 4 sides.

Two sides are on the two spheres respectively.

Two sides are on two tunnels respectively.

[redacted] K

Question 5.

Proof: Let D be the disk. $\pi_1(D)=\{1\}$. $\pi_1(T^2)=\mathbb{Z}\oplus\mathbb{Z}=\langle a\rangle\oplus\langle b\rangle$.

Let $L=D\cap T^2$ L is a circle. $\pi_1(L)=\mathbb{Z}=\langle c\rangle$.

$i: L\to D^2$, $j: L\to T^2$ are inclusions.

$i_*(c)=1$; $j_*(c)=a^5$

$\pi_1(X)=\pi_1(D)*\pi_1(T^2)/\langle j_*(c)\,i_*(c)^{-1}\rangle=\langle a\rangle\oplus\langle b\rangle/\langle a^5\rangle=\mathbb{Z}_5\oplus\mathbb{Z}$ by Van-kampen theorem

Since X is connected, then $H_0(X)=\mathbb{Z}$

Since $H_1(X)$ is the abelization of $\pi_1(X)$, then $H_1(X)=\mathbb{Z}_5\oplus\mathbb{Z}$

$$0\to \underset{0}{H_2(L)}\to \underset{\mathbb{Z}}{H_2(T^2)\oplus H_2(D)}\xrightarrow{f_1} H_2(X)\xrightarrow{f_2}\underset{\mathbb{Z}}{H_1(L)}\xrightarrow{f_3}\underset{\mathbb{Z}\oplus\mathbb{Z}}{H_1(T^2)\oplus H_1(D)}\to H_1(X)\to H_0(L)\to H_0(T^2)\oplus H_0(D)\to H_0(X)\to$$

Since $f_3(1)=(5,0)$, then f_3 is injective. $\ker f_3=0$. $\operatorname{Im} f_2=\ker f_3=0$

$0=\operatorname{Im} f_2=H_2(X)/\ker f_2=H_2(X)/\operatorname{Im} f_1$ Since f_1 is injective, then $0=H_2(X)/\mathbb{Z}\Rightarrow H_2(X)=\mathbb{Z}$ $\square$

[redacted] K

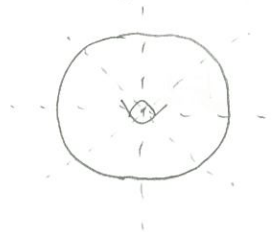
Question 6.

Proof: Let X be a 2-fold covering space of K.

Then $[\pi_1(K):\pi_1(X)]=2$. $\pi_1(X)$ is a normal subgroup of $\pi_1(K)$. $\pi_1(K)=\langle a,b \mid a^2b^2=1\rangle$.

X is a regular covering space of K.

T^2 is a 2-fold covering space of K.

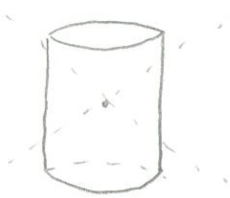


By identifying points symmetric with respect to the center, we obtain Klein bottle.

Consider K as the union of two Mobius bands.

Since $\pi_1(\text{Mobius band})=\mathbb{Z}$, there is exact one subgroup with index 2.

Thus there is exact one 2-fold covering space which is annulus.



By identifying points symmetric with respect to the center, we obtain Mobius band.

Thus the 2-fold covering spaces of K are the unions of two annuli.

Then the 2-fold covering spaces of K are T^2 or Klein bottle. $\boxed{//}$

[redacted] K.

Question 7.

Proof: $\mathbb{R}^4$ is the universial covering space of T^4. $p: \mathbb{R}^4 \to T^4$ is the covering map.

$\pi_1(\mathbb{R}P^2 \times \mathbb{R}P^2) = \pi_1(\mathbb{R}P^2) \times \pi_1(\mathbb{R}P^2) = \mathbb{Z}_2 \times \mathbb{Z}_2$ which is a finite group.

Let f be a continuous map from $\mathbb{R}P^2 \times \mathbb{R}P^2$ to T^4

Then $f_*: \pi_1(\mathbb{R}P^2 \times \mathbb{R}P^2) \to \pi_1(T^4)$ is a trivial map.

Then $f_*(\pi_1(\mathbb{R}P^2 \times \mathbb{R}P^2)) = p_*(\pi_1(\mathbb{R}^4))$. Thus there is a lift $\tilde{f}$ of f.

$$\begin{tikzcd} & \mathbb{R}^4 \arrow[d, "p"] \\ \mathbb{R}P^2 \times \mathbb{R}P^2 \arrow[ur, "\tilde{f}"] \arrow[r, "f"] & T^4 \end{tikzcd}$$

Since $\underbrace{\tilde{f} \text{ is null-homotopic}}_{\mathbb{R}^4 \text{ is convex.}}$, then $p\tilde{f}$ is null-homotopic. i.e. f is null-homotopic. ////

Question 8.

Proof: Since f is homotopic to identity, then $f_* = \mathrm{Id}_* : H_n(X) \to H_n(X)$ $\forall n \geq 0$.

By Lefschetz fixed point theorem. $\sum_{n=0}^{5} (-1)^n (\mathrm{tr} f_* : H_n(X) \to H_n(X)) = \sum_{n=0}^{5} (-1)^n (\mathrm{tr}\, \mathrm{Id}_* : H_n(X) \to H_n(X))$

$\mathbb{RP}^5$ has exact one n-cell for $n = 0, 1, 2, \cdots, 5$.

Then X has exact one n-cell for $n = 0, 2, 3, 4, 5$

$H_5(X^5, X^4) = \mathbb{Z}$ $\quad H_4(X^4, X^3) = \mathbb{Z}$ $\quad H_3(X^3, X^2) = \mathbb{Z}$ $\quad H_2(X^2, X^1) = \mathbb{Z}$. $\quad H_1(X^1, X^0) = 0$. $\quad H_0(X^0) = \mathbb{Z}$

$$0 \to H_5(X^5, X^4) \to H_4(X^4, X^3) \to H_3(X^3, X^2) \to H_2(X^2, X^1) \to H_1(X^1, X^0) \to H_0(X^0) \to 0.$$
$$0 \to \mathbb{Z} \xrightarrow{\mathrm{Id}} \mathbb{Z} \xrightarrow{\mathrm{Id}} \mathbb{Z} \xrightarrow{\mathrm{Id}} \mathbb{Z} \longrightarrow 0 \longrightarrow \mathbb{Z} \longrightarrow 0.$$

Thus $H_n(X) = 0$ for $n = 5, 4, 3, 2, 1$. $H_0(X) = \mathbb{Z}$.

Thus. $\sum_{n=0}^{5} (-1)^n (\mathrm{tr}\, \mathrm{Id}_* : H_n(X) \to H_n(X)) = \mathrm{tr}(\mathrm{Id}_* : H_0(X) \to H_0(X)) = 1 > 0$.

Thus f has a fixed point. $\boxed{//}$

Question 9.

Proof: $\mathbb{C}P^2$ has one 0-cell e_0, one 2-cell e_2, one 4-cell e_4.

$\mathbb{R}P^2$ has one 0-cell e_0', one 1-cell e_1', one 2-cell e_2'

Thus X has one 0-cell $e_0 \times e_0'$, one 1-cell $e_0 \times e_1'$, two 2-cells $e_0 \times e_2'$ and $e_2 \times e_0'$

one 3-cell $e_2 \times e_1'$ two 4-cells $e_4 \times e_0'$ and $e_2 \times e_2'$, one 5-cell $e_4 \times e_1'$

one 6-cell $e_4 \times e_2'$

$H_0(X^0) = \mathbb{Z}$ $H_1(X^1, X^0) = \mathbb{Z}$ $H_2(X^2, X^1) = \mathbb{Z} \oplus \mathbb{Z}$. $H_3(X^3, X^2) = \mathbb{Z}$. $H_4(X^4, X^3) = \mathbb{Z} \oplus \mathbb{Z}$.

$H_5(X^5, X^4) = \mathbb{Z}$. $H_6(X^6, X^5) = \mathbb{Z}$.

L Be sure to write your letter on **each page** of your exam

Topology Qualification Exam, Spring 2020

Please attempt **8 out of 9** problems and clearly mark the one you do not want us to grade.

1. Let $f : X \to Y$ be a surjective, continuous map of topological spaces.

   a) Show: if f is an open map, then it is a quotient map.

   b) Show: if f is a closed map, then it is a quotient map.

2. Show that a connected metrizable space with at least two points is uncountably infinite. (You may use without proof that every metrizable space is normal.)

3. Let (X, d_X) and (Y, d_Y) be metric spaces. An **isometric embedding** $\iota : X \to Y$ is a map such that

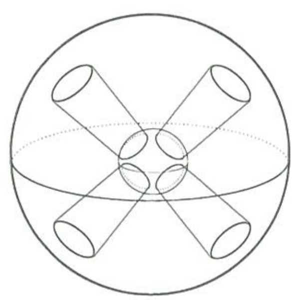
   $$\forall x_1, x_2 \in X,\ d_Y(\iota(x_1), \iota(x_2)) = d_X(x_1, x_2).$$

   An **isometry** is a surjective isometric embedding.

   a) Show that every isometric embedding from a compact metric space to itself is an isometry.
   (You may use that a metric space is compact iff it is sequentially compact.)

   b) Show that every isometric embedding from Euclidean n-space to itself is an isometry.

4. Consider the solid S obtained by digging out the center of a 3-dimensional solid ball and 4 tunnels from the center to the boundary. What is the genus of the boundary surface $\Sigma = \partial S$? Justify your answer.



Σ_3

5. Let X be the topological space obtained by attaching a disk to $T^2 = S^1 \times S^1$ along the circle $S^1 \times \{p\}$ via the map $z \mapsto z^5$. Compute the fundamental group and the homology groups of X.

6. Classify the connected 2-fold covering spaces of the Klein bottle K.
   (You might want to consider K as the union of two Möbius bands.)

7. Show that every continuous map from $\mathbb{R}P^2 \times \mathbb{R}P^2$ to $T^4 = S^1 \times S^1 \times S^1 \times S^1$ is null-homotopic.

8. Let $X = \mathbb{R}P^5/\mathbb{R}P^1$, and let $f : X \to X$ be a continuous map that is homotopic to the identity. Show that f must have a fixed point.

9. Describe the CW structure of $X = \mathbb{C}P^2 \times \mathbb{R}P^2$ and use it to compute the homology groups of X.

$e^0 \vee e^2 \vee e^4$ $\quad\quad$ $e^0 \vee e^1 \vee e^2$

$e^0 \quad e^2 \quad e^4$

$e^1 \quad e^3 \quad e^5$

$e^2 \quad e^4 \quad e^8$

L.

1. Def (Quotient map) A surjective map $f: X \to Y$ is called quotient map

if $f^{-1}(U)$: open in X $\Leftrightarrow$ U: open in Y.

a) It suffices to show ($\Rightarrow$) direction.

Let $f^{-1}(U)$ is open in X.

$\Rightarrow$ $f(f^{-1}(U)) = U$ be open in Y ($\because$ f is open map).

b) It suffices to show ($\Rightarrow$) direction.

Let $f^{-1}(U)$ be open in X.

$\Rightarrow$ $X - f^{-1}(U)$ is closed.

$\Rightarrow$ $f(X - f^{-1}(U)) = f(X) - f(f^{-1}(U)) = \underset{\underset{Y-U}{\parallel}}{f(X) - U}$ is closed. ($\because$ f is closed map)

$\Rightarrow$ U is open in Y. ▣

L

3. (a)

Let X be a compact metric space.

Let $f: X \longrightarrow X$ be a isometric embedding.

⊣ f is surjective.

Let $a \in X$.

$\Rightarrow \exists \{x_n\}$: sequence of points in X s.t $\{x_n\} \longrightarrow a$.

(b) Euclidean n-space is compact metric space.

So by (a), every isometric embedding from Euclidean n-space to itself

is an isometry.

L

4. We know $B_1^3 \cup B_2^3 = S^3$.

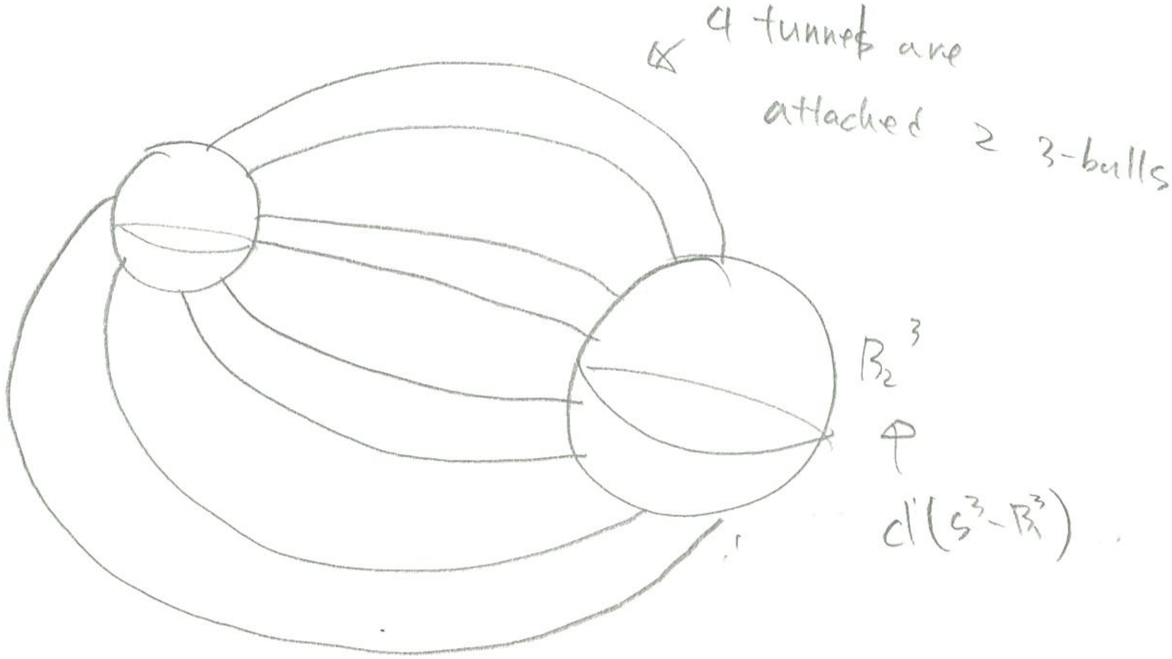
Let X be a digged part from B_1^3.

$$S^3 = B^3 \cup B^3 = \left(\overline{B_1^3 \setminus X}\right) \cup \left(\overline{B_2^3 \cup X}\right).$$

Let S be a space in the problem.

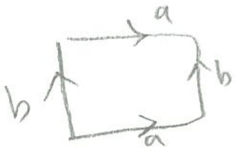
$$\Sigma = \partial S = \partial\left(\overline{B_1^3 \setminus X}\right) = \partial\left(\overline{B_2^3 \cup X}\right)$$

So

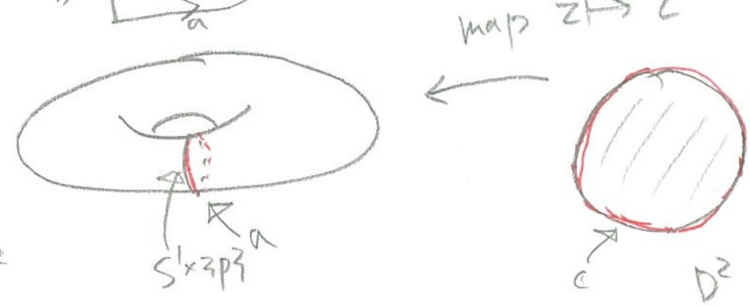


So Σ is genus 3-surface.

L



5.



$= X$

Let A be $T^2 \cup$ small regular neighborhood of $S^1 \times \{p\}$ in D^2

Let B be $D^2 \cup$ small regular neighborhood of ∂D^2 in T^2

$\Rightarrow$ A, B: open path connected and $A \cup B = X$ $A \cap B$: path connected and $A \cap B \simeq S^1$

$\Rightarrow$ $\pi_1(X) = \pi_1(A) *_{\pi_1(A \cap B)} \pi_1(B)$ by Van-kampen theorem

$= \langle a, b \mid aba^{-1}b^{-1}, a^5 \rangle$ ($\because$ $\pi_1(T^2) = \langle a, b \mid aba^{-1}b^{-1} \rangle$

$= \mathbb{Z}_5 \oplus \mathbb{Z}$ $\pi_1(D^2)$ is trivial.

$\pi_1(A \cap B) = \langle c \rangle$ and map $\underset{c \mapsto a^5}{z \mapsto z^5}$)

$\Rightarrow$ Clearly $H_1(X) = \pi_1(X)^{ab} = \mathbb{Z}_5 \oplus \mathbb{Z}$

Also $H_0(X) = \mathbb{Z}$ ($\because$ X is path connected)

Now we show $H_2(X) = \mathbb{Z}$

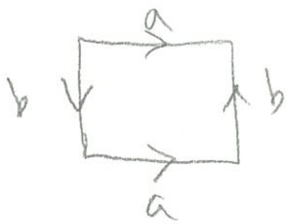
$$0 \to \underset{\langle A, B \rangle}{C_2^{cw}(X)} \xrightarrow{\Phi} \underset{\langle a, b \rangle}{C_1^{cw}(X)} \to C_0^{cw}(X) \to 0$$

2-cell in T^2 D^2

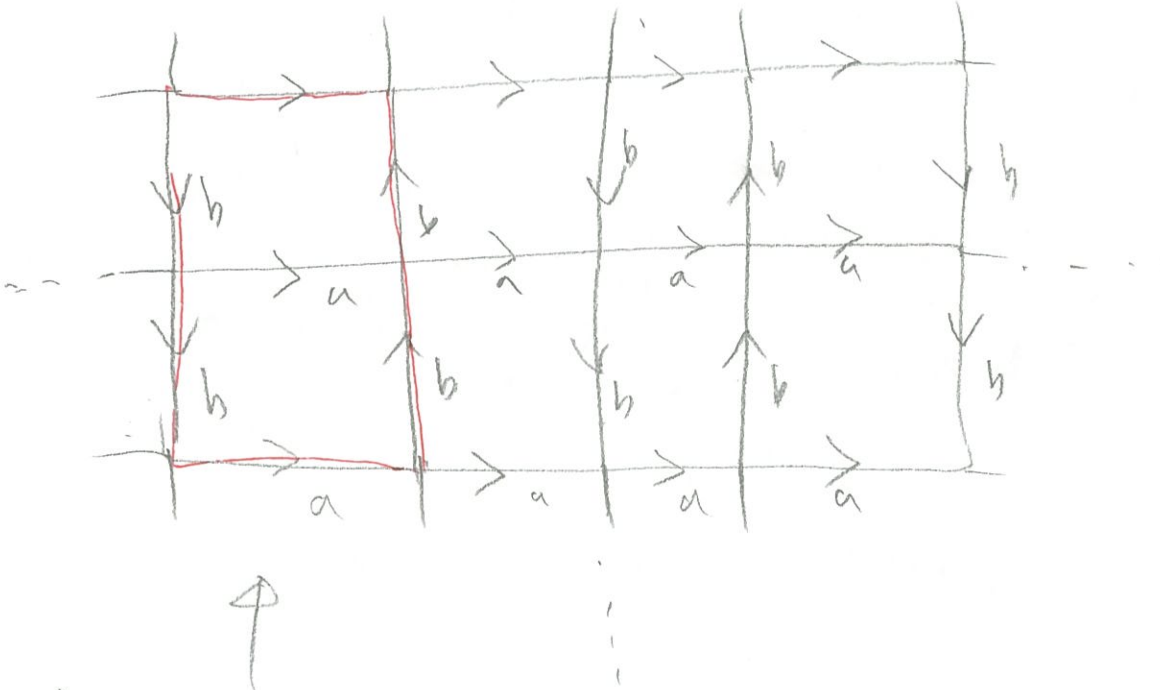
Here $\Phi(A) = a + b - a - b = 0$ so $H_2(X) = \overset{\ker \Phi}{\langle A, B \rangle} / \underset{\operatorname{Im}(0)}{\langle A \rangle} = \mathbb{Z}$.

$\Phi(B) = 5b$

6



$K \Rightarrow \pi_1(K) = \langle a, b \mid aba^{-1}b \rangle$



U: universal cover of K

We need to find proper group action G on universal cover so

$p: U \to U/G$ will be two sheeted cover.

Consider two groups $\langle a, 2b, aba^{-1}b \rangle$ and $\langle b, 2a, aba^{-1}b \rangle$.

$$\Rightarrow \frac{\pi_1(K)}{\langle a, 2a, aba^{-1}b \rangle} \cong \frac{\pi_1(K)}{\langle b, 2a, aba^{-1}b \rangle} \cong \mathbb{Z}_2.$$

$\Rightarrow U/\langle a, 2a, aba^{-1}b \rangle$ and $U/\langle b, 2a, aba^{-1}b \rangle$ are 2 fold covering spaces of K. ∎

$$\begin{array}{c} (\tilde{X}, \tilde{x}_0) \\ \Big\downarrow p \\ (X, x_0) \end{array}$$

$\Rightarrow$ ① $p_*: \pi_1(\tilde{X}, \tilde{x}_0) \longrightarrow \pi_1(X, x_0)$ is injective.

② $\rho : p^{-1}(x_0) \times \pi_1(X, x_0) \longrightarrow p^{-1}(x_0)$

transitive group action

$(\tilde{x}_0, [\alpha]) \longmapsto \tilde{\alpha}(1)$, where $\exists!$ $\tilde{\alpha}: I \to \tilde{X}$

s.t. $\tilde{\alpha}(0) = \tilde{x}_0$ and

$p \circ \tilde{\alpha} = \alpha$.

③ $p^{-1}(x_0) = \text{orbit}(\tilde{x}_0) = \{\tilde{\alpha}(1) \mid [\alpha] \in \pi_1(X, x_0) \text{ and } \tilde{\alpha}(0) = \tilde{x}_0\}$ $\forall \tilde{x}_0$

④ $\text{stab}(\tilde{x}_0) = p_*(\pi_1(\tilde{X}, \tilde{x}_0)) = \{\tilde{\alpha}(1) = \tilde{x}_0 \mid [\alpha] \in \pi_1(X, x_0)\}$ $\forall \tilde{x}_0$

⑤ $|p^{-1}(x_0)| = |\pi_1(\tilde{X}, \tilde{x}_0) : \text{stab}(\tilde{x}_0)|$

$= |\pi_1(\tilde{X}, \tilde{x}_0) : p_*(\pi_1(\tilde{X}, \tilde{x}_0))|$

⑥ $\text{Aut}(\tilde{X}) \cong N(p_*(\pi_1(\tilde{X}, \tilde{x}_0))) \big/ p_*(\pi_1(\tilde{X}, \tilde{x}_0))$

⑦ If $\tilde{X}$ is universal cover, $\text{Aut}(\tilde{X}) \cong \pi_1(X, x_0)$

L

1. Consider universal covering of S^1.

$$p: \mathbb{R} \to S^1$$
$$t \mapsto e^{2\pi i t}$$

$\Rightarrow \exists$ natural covering map $q: \mathbb{R}\times\mathbb{R}\times\mathbb{R}\times\mathbb{R} \to S^1\times S^1\times S^1\times S^1$ by product p.

Let $f: \mathbb{RP}^2\times\mathbb{RP}^2 \to T^4$ be a continuous map.

Here $\pi_1(\mathbb{RP}^2\times\mathbb{RP}^2) = \pi_1(\mathbb{RP}^2)\times\pi_1(\mathbb{RP}^2) = \mathbb{Z}_2\oplus\mathbb{Z}_2$

Consider following

$$\begin{tikzcd}
 & \mathbb{R}\times\mathbb{R}\times\mathbb{R}\times\mathbb{R} \arrow[d, "q"] \\
\mathbb{RP}^2\times\mathbb{RP}^2 \arrow[ur, "\tilde{f}"] \arrow[r, "f"'] & T^4
\end{tikzcd}$$

Here $f_*(\pi_1(\mathbb{RP}^2\times\mathbb{RP}^2)) \leq \pi_1(T^4) = \mathbb{Z}\oplus\mathbb{Z}\oplus\mathbb{Z}\oplus\mathbb{Z}$ ($\because f_*$ is homomorphism)

Here f_* should be finite and the only finite subgroup of $\pi_1(T^4)$ is trivial group.

$\Rightarrow f_*(\pi_1(\mathbb{RP}^2\times\mathbb{RP}^2)) = 0$.

Also $\mathbb{R}\times\mathbb{R}\times\mathbb{R}\times\mathbb{R}$ is simply connected $q_*(\pi_1(\mathbb{R}\times\mathbb{R}\times\mathbb{R}\times\mathbb{R})) = 0$.

$\Rightarrow f_*(\pi_1(\mathbb{RP}^2\times\mathbb{RP}^2)) \leq q_*(\pi_1(\mathbb{R}\times\mathbb{R}\times\mathbb{R}\times\mathbb{R}))$

$\Rightarrow \exists\ \tilde{f}: \mathbb{RP}^2\times\mathbb{RP}^2 \to \mathbb{R}\times\mathbb{R}\times\mathbb{R}\times\mathbb{R}$ s.t $q\circ\tilde{f} = f$.

$\Rightarrow \exists$ constant map c s.t $\tilde{f} \simeq c$ ($\because \mathbb{R}\times\mathbb{R}\times\mathbb{R}\times\mathbb{R}$ is contractible)

$\Rightarrow f = q\circ\tilde{f} = q\circ c$ (here $q\circ c$ is constant map) $\Rightarrow f$ is null-homotopic. ■

8

$$\mathbb{R}P^n = S^n/_{x\sim -x} = D^n/_{(x\sim -x \mid x\in \partial D^n)}$$

$\Rightarrow \mathbb{R}P^n = \mathbb{R}P^{n-1} \cup e^n$

$\Rightarrow \mathbb{R}P^0 = e^0$ and $\mathbb{R}P^n = e^0 \cup e^1 \cdots \cup e^n$

$\Rightarrow X = \mathbb{R}P^5/\mathbb{R}P^1 \underset{\uparrow}{\simeq} e^0 \cup e^2 \cup e^3 \cup e^4 \cup e^5 \quad (\because \mathbb{R}P^1 = e^0 \cup e^1)$

$e^0 \cup e^1 = \mathbb{R}P^1$ will be point.

$\Rightarrow \chi(X) = 1 - 0 + 1 - 1 + 1 - 1 = 1 \neq 0.$

When f is homotopic to the identity Letschetz number and Euler characteristic is same. i.e. $\tau(f) = \chi(X)$.

$\Rightarrow \tau(f) = \chi(X) = 1 \neq 0$

$\Rightarrow f$ has a fixed point ($\because$ by Letschez fixed point theorem) $\blacksquare$

$\underline{L}$

9. $\mathbb{C}P^n \subseteq \mathbb{C}P^{n+1}$ $\qquad$ $\mathbb{R}P^n \subseteq \mathbb{R}P^{n+1}$

$\mathbb{C}P^2 = e_c^0 \cup e_c^2 \cup e_c^4$ $\qquad$ $\mathbb{R}P^2 = e_R^0 \cup e_R^1 \cup e_R^2$

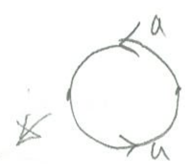
$$\Rightarrow \mathbb{C}P^2 \times \mathbb{R}P^2 = e_c^0 \times e_R^0 \cup e_c^2 \times e_R^0 \cup e_c^4 \times e_R^0$$
$$\cup\, e_c^2 \times e_R^1 \cup e_c^2 \times e_R^1 \cup e_c^2 \times e_R^1$$
$$\cup\, e_c^4 \times e_R^2 \cup e_c^4 \times e_R^2 \cup e_c^4 \times e_R^2$$
$$= e^0 \cup e^2 \cup e^4 \cup e^3 \cup e^3 \cup e^3 \cup e^6 \cup e^6 \cup e^8.$$

For $H_i(\mathbb{C}P^2)$,

$$0 \to \underset{\mathbb{Z}}{C_4^{CW}(\mathbb{C}P^2)} \xrightarrow{0} \underset{0}{C_3^{CW}(\mathbb{C}P^2)} \xrightarrow{0} \underset{\mathbb{Z}}{C_2^{CW}(\mathbb{C}P^2)} \xrightarrow{0} \underset{0}{C_1^{CW}(\mathbb{C}P^2)} \xrightarrow{0} \underset{\mathbb{Z}}{C_0^{CW}(\mathbb{C}P^2)} \to 0$$

$\Rightarrow H_4(\mathbb{C}P^2) = \mathbb{Z}$, $H_3(\mathbb{C}P^2) = 0$, $H_2(\mathbb{C}P^2) = \mathbb{Z}$, $H_1(\mathbb{C}P^2) = 0$, $H_0(\mathbb{C}P^2) = \mathbb{Z}$

For $H_i(\mathbb{R}P^2)$,



$$0 \to \underset{\mathbb{Z}}{C_2^{CW}(\mathbb{R}P^2)} \xrightarrow{\cdot 2} \underset{\mathbb{Z}}{C_1^{CW}(\mathbb{R}P^2)} \xrightarrow{0} \underset{\mathbb{Z}}{C_0^{CW}(\mathbb{R}P^2)} \to 0$$

$\Rightarrow H_2(\mathbb{R}P^2) = 0$, $H_1(\mathbb{R}P^2) = \mathbb{Z}_2$, $H_0(\mathbb{R}P^2) = \mathbb{Z}$

Now

I am not sure the name

$$H_i(\mathbb{C}P^2 \times \mathbb{R}P^2) = \sum_{i=0}^{8} H_i(\mathbb{C}P^2) \otimes H_{8-i}(\mathbb{R}P^2) \text{ by Hurwitz Theorem.}$$

R Be sure to write your letter on **each page** of your exam

Topology Qualification Exam, Spring 2020

Please attempt **8 out of 9** problems and clearly mark the one you do not want us to grade.

✓ 1. Let $f : X \to Y$ be a surjective, continuous map of topological spaces.

   a) Show: if f is an open map, then it is a quotient map.

   b) Show: if f is a closed map, then it is a quotient map.

✓ 2. Show that a connected metrizable space with at least two points is uncountably infinite. (You may use without proof that every metrizable space is normal.)

Not to grade

3. Let (X, d_X) and (Y, d_Y) be metric spaces. An **isometric embedding** $\iota : X \to Y$ is a map such that

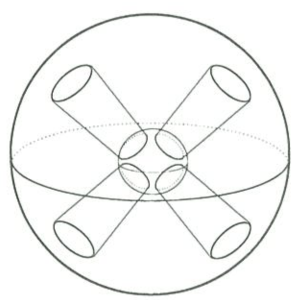
$$\forall x_1, x_2 \in X,\ d_Y(\iota(x_1), \iota(x_2)) = d_X(x_1, x_2).$$

   An **isometry** is a surjective isometric embedding.

   a) Show that every isometric embedding from a compact metric space to itself is an isometry.
   (You may use that a metric space is compact iff it is sequentially compact.)

   b) Show that every isometric embedding from Euclidean n-space to itself is an isometry.

✓ 4. Consider the solid S obtained by digging out the center of a 3-dimensional solid ball and 4 tunnels from the center to the boundary. What is the genus of the boundary surface $\Sigma = \partial S$? Justify your answer.



✓ 5. Let X be the topological space obtained by attaching a disk to $T^2 = S^1 \times S^1$ along the circle $S^1 \times \{p\}$ via the map $z \mapsto z^5$. Compute the fundamental group and the homology groups of X.

6. Classify the connected 2-fold covering spaces of the Klein bottle K. (You might want to consider K as the union of two Möbius bands.)

✓ 7. Show that every continuous map from $\mathbb{R}P^2 \times \mathbb{R}P^2$ to $T^4 = S^1 \times S^1 \times S^1 \times S^1$ is null-homotopic.

8. Let $X = \mathbb{R}P^5/\mathbb{R}P^1$, and let $f : X \to X$ be a continuous map that is homotopic to the identity. Show that f must have a fixed point.

[illegible]

9. Describe the CW structure of $X = \mathbb{C}P^2 \times \mathbb{R}P^2$ and use it to compute the homology groups of X.

1) ~~f open~~

a) Assume f is an open map. Suppose $f^{-1}(V)$ is an open set in X. Then $f(f^{-1}(V))$ is open in Y. Since f is surjective $f(f^{-1}(V)) = V$ Thus V is open.

Since f is continuous, for any V open in Y, $f^{-1}(V)$ is open in X. Hence V is open in Y iff $f^{-1}(V)$ is open in X.

So f is a quotient map.

b) If $f^{-1}(F)$ is closed in X, $f(f^{-1}(F)) = F$ is closed in Y, Since f is continuous, $f^{-1}(F)$ is closed if F is closed

Hence, a set $F \subseteq Y$ is closed in Y iff $f^{-1}(F)$ is closed in X. ~~Let U be open in Y, $F_u = Y \setminus U$ is closed,~~

~~Thus $f^{-1}(Y \setminus U)$ is closed. So $X \setminus f^{-1}(Y \setminus U)$ is open.~~

~~If $x \in f^{-1}(U)$, then $f(x) \in U$.~~

~~f is continuous so $f^{-1}(U)$ is open,~~ Suppose $f^{-1}(U)$ is open, Then $F = X \setminus f^{-1}(U)$ is closed. $f(F)$ is closed.

$Y \setminus f(F)$

$y \in Y \setminus f(F)$, then $y \neq f(x)$ for any $x \in X \setminus f^{-1}(U)$

f surjective $y = f(\tilde{x})$ for some $\tilde{x}$, $\tilde{x} \notin X \setminus f^{-1}(U)$

$\Rightarrow \tilde{x} \in f^{-1}(U) \Rightarrow y \in U$

$Y \setminus f(F) = U$ so U is open iff $f^{-1}(U)$ is open.

2) X normal, let $A = \{p, q\}$ for two points, define a continuous function $f: A \to [0,1]$ by

$f(p) = 0$ $\quad$ $f(q) = 1$. Since A is closed, X normal

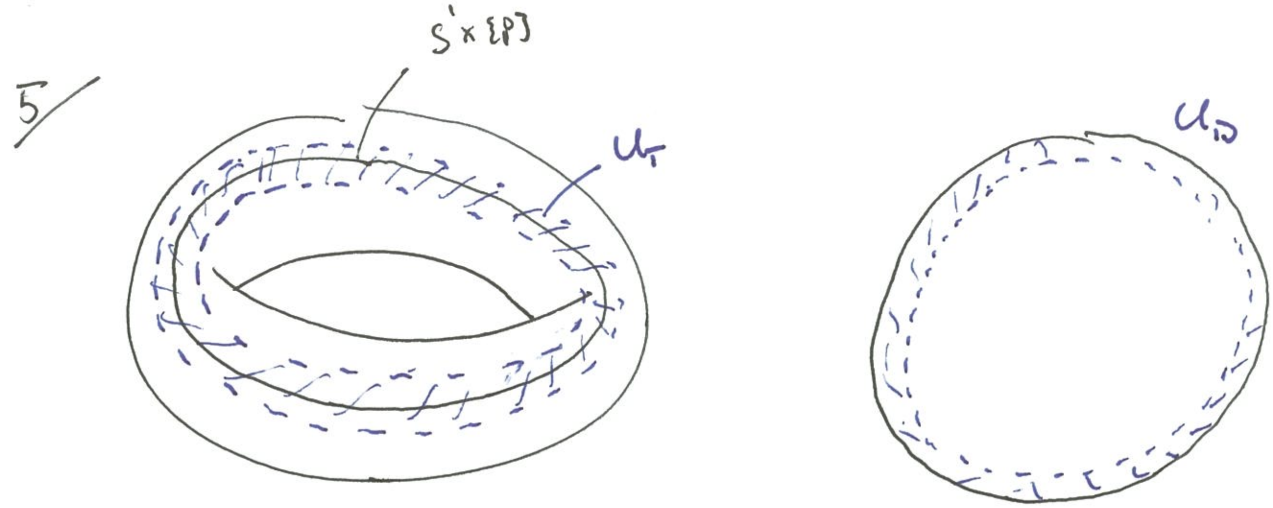
$\exists$ an extension $f: X \to [0,1]$ which is continuous.

Since X is connected, its image is connected.

Thus we have a surjection $X \to [0,1]$. Since $[0,1]$ is uncountable, X is uncountable.

4/ ∂S is a surface of genus 3. To see this, we start by considering B^3 with a solid B^3 removed from its interior. The Boundary of this space is two disjoint coppies of S^2. Thus The Boundary of this space is equivalent to the boundary of the space $B^3 \sqcup B^3$. By digging out the tunnels, we can take what was removed, $D^2 \times D^1$ and attach it to the $B^3 \sqcup B^3$. Thus digging tunnels out of S is the same as attaching 1-handles to $B^3 \sqcup B^3$. So the result is the result of attaching 4-1 handles to $B^3 \sqcup B^3$ with one end in each B^3. Now the core of the 1-handles intersects the co core of the 0-handles (The B^3's) geometrically once, Thus we may cacel a single 1-hadle and a single 0-hadle. The result is a solid hadle body who's boundry is the surface of genus 3.

5/



$U = T^2 \cup U_D \qquad V = D^2 \cup U_T$.

U and V are open in X. $U \cap V = U_D \cup U_T$ which deformation retracts onto $S^1 \times \{p\}$.

By Van Kampen $\pi_1(X) = \pi_1(T^2) * \pi_1(D^2) / i_*(\gamma) = i_*(\alpha)$

Here α is the generator for $\pi_1(S^1)$. Since $\pi_1(D^2) = 0$ ~~$i_*(\alpha) = \alpha$~~

$\pi_1(T^2) = \langle \alpha, \beta \mid \alpha\beta\alpha^{-1}\beta^{-1} = e \rangle$, $\pi_1(S^1) = \langle \alpha \rangle$, $i_*(\alpha) = \alpha^5$.

Thus $\pi_1(X) = \mathbb{Z} \oplus \mathbb{Z}_5$.

X is connected $\Rightarrow H_0(X) = \mathbb{Z}$

$\pi_1(X)$ is abelian $\Rightarrow H_1(X) = \pi_1(X) = \mathbb{Z} \oplus \mathbb{Z}_5$

$H_2(U \cap V) \to H_2(U) \oplus H_2(V) \to$

$0 \qquad \alpha_u \; a \qquad \alpha_v \; b$

$a + b$

If we split along $S_1 \times \{p\}$ $\{q \times S^1\}$, we have T^2 has a single 2-cell, D^2 has a single 2-cell

$C_2(X) = \langle A, B \rangle$

$\partial(A) = 0$

$\partial(B) = \alpha$

$\ker \partial =$

a

$H_2(X) = \mathbb{Z}_5$

7) Since $\pi_1(\mathbb{R}P^2) = \mathbb{Z}_2$, $\pi_1(S^1) = \mathbb{Z}$ we have that

$\pi_1(\mathbb{R}P^2) \times \pi_1(\mathbb{R}P^2) = \pi_1(\mathbb{R}P^2 \times \mathbb{R}P^2) = \mathbb{Z}_2 \times \mathbb{Z}_2$

$\pi_1(T^4) = \mathbb{Z}^4$.

Now $\mathbb{Z}^4$ is infinite ~~cycl~~ with no torsion. Hence it contains no subgroup of finite order. Thus every continuous map $f: \mathbb{R}P^2 \times \mathbb{R}P^2 \to T^4$ must induce the following map

$$f_* \pi_1(\mathbb{R}P^2 \times \mathbb{R}P^2) \longrightarrow \langle 0 \rangle \subset \mathbb{Z}^4.$$

Thus, by the lifting criterion, every map ~~f_*~~ $f: \mathbb{R}P^2 \times \mathbb{R}P^2 \to T^4$ lifts to the universal cover, $\mathbb{R}^4$. $\mathbb{R}^4$ is contractible, and hence every map $\tilde{f}: \mathbb{R}P^2 \times \mathbb{R}P^2 \to \mathbb{R}^4$ is homotopic to the constant map. Thus f is also null-homotopic.

8) Since f is homotopic to the ~~constant~~ identity map, it suffices to check the leftzetz # for the ~~cons~~ identity.

$\mathbb{R}P^5$ has a cell decomp of $\{e_0, e_1, e_2, e_3, e_4, e_5\}$, where e_i is the sigle i-cell. Since $X = \mathbb{R}P^5/\mathbb{R}P^1$ we have that X has a cell decomp $(e_0, e_2, e_3, e_4, e_5)$

Thus $\tau(f) = \tau(id) = 1+1-1+1-1 = 1 > 0$. So f, by the Leftshutz fixed point theorem, has a fixed point.

9) $\mathbb{C}P^2$ has a cell decomp $\{e_0, e_2, e_4\}$

$\mathbb{R}P^2$ has a decomp $\{e_0, e_1, e_2\}$

Here e_i denotes an i-cell.

So, we get a cell decomp of $\mathbb{C}P^2 \times \mathbb{R}P^2$ given by

$e_0 \times e_0$

$e_0 \times e_1$

$e_0 \times e_2$ $\quad e_2 \times e_0$

$e_2 \times e_1$

$e_4 \times e_0$ $\quad e_2 \times e_2$

$e_4 \times e_1$

$e_4 \times e_2$

6) We know $\pi_1(K) = \langle a, b \mid abab = e \rangle$. Take $H = \langle ab \rangle$. The order of H in $\pi_1(K)$ is 2. Let E be the covering space corresponding to H. Then E is a 2-fold cover of K. All other covering spaces ~~of~~ correspond to conjugates of H.

S Be sure to write your letter on **each page** of your exam

Topology Qualification Exam, Spring 2020

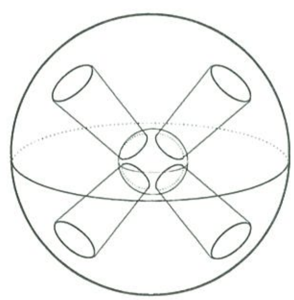
Please attempt **8 out of 9** problems and clearly mark the one you do not want us to grade.

1. Let $f : X \to Y$ be a surjective, continuous map of topological spaces.
   a) Show: if f is an open map, then it is a quotient map.
   b) Show: if f is a closed map, then it is a quotient map.

2. ~~Show that a connected metrizable space with at least two points is uncountably infinite.~~ ~~(You may use without proof that every metrizable space is normal.)~~

3. Let (X, d_X) and (Y, d_Y) be metric spaces. An **isometric embedding** $\iota : X \to Y$ is a map such that
$$\forall x_1, x_2 \in X,\ d_Y(\iota(x_1), \iota(x_2)) = d_X(x_1, x_2).$$
   An **isometry** is a surjective isometric embedding.
   a) Show that every isometric embedding from a compact metric space to itself is an isometry.
   (You may use that a metric space is compact iff it is sequentially compact.)
   b) Show that every isometric embedding from Euclidean n-space to itself is an isometry.

4. Consider the solid S obtained by digging out the center of a 3-dimensional solid ball and 4 tunnels from the center to the boundary. What is the genus of the boundary surface $\Sigma = \partial S$? Justify your answer.



5. Let X be the topological space obtained by attaching a disk to $T^2 = S^1 \times S^1$ along the circle $S^1 \times \{p\}$ via the map $z \mapsto z^5$. Compute the fundamental group and the homology groups of X.

6. Classify the connected 2-fold covering spaces of the Klein bottle K.
   (You might want to consider K as the union of two Möbius bands.)

7. Show that every continuous map from $\mathbb{R}P^2 \times \mathbb{R}P^2$ to $T^4 = S^1 \times S^1 \times S^1 \times S^1$ is null-homotopic.

8. Let $X = \mathbb{R}P^5/\mathbb{R}P^1$, and let $f : X \to X$ be a continuous map that is homotopic to the identity. Show that f must have a fixed point.

9. Describe the CW structure of $X = \mathbb{C}P^2 \times \mathbb{R}P^2$ and use it to compute the homology groups of X.

1. Let $f: X \to Y$ be a <u>surjective</u> <u>continuous</u> <u>map</u>. $\boxed{S}$

(a) Show that if f is an <u>open</u> map then f is a quotient map.

(b) Show that if f is a <u>closed</u> map then f is a quotient map.

Recall the universal property of quotient maps / the quotient topology: for X a space and $\sim$ an equivalence relation on X, the quotient topology on $X/\sim$ has open sets given by those subsets of $X/\sim$ which pull back to open sets of X under the canonical projection $X \to X/\sim$; in other words, the quotient topology on $X/\sim$ is the final topology on $X/\sim$ such that the canonical projection $X \to X/\sim$ is <u>continuous</u>.

Now let $f: X \to Y$ be a <u>surjective</u> <u>continuous</u> <u>map</u>. Define an equivalence relation $\sim$ on X, where for $x_0, x_1 \in X$, we have $x_0 \sim x_1$ precisely if $f(x_0) = f(x_1)$.

(a) Suppose $f: X \to Y$ is an open map, that is f sends open subsets $\mathcal{U}$ of X to open subsets $f(\mathcal{U})$ of Y. Take an open subset $\mathcal{U}$ of X so that $f(\mathcal{U}) = \mathcal{U}/\sim$ is open in Y. Then the set $f^{-1}(f(\mathcal{U})) = f^{-1}(\mathcal{U}/\sim) = \mathcal{U}$ is open in X. It follows that Y is identified with $X/\sim$, and Y has open sets given by those subsets of $X/\sim$ which pull back to open sets of X, therefore $f: X \to Y = X/\sim$ is a quotient map by the universal property.

(b) Dual to (a). Suppose $f: X \to Y$ is a closed map, that is f sends closed subsets Z of X to closed subsets $f(Z)$ of Y. Take a closed subset Z of X so that $f(Z) = Z/\sim$ is closed in Y. Then the set $f^{-1}(f(Z)) = f^{-1}(Z/\sim) = Z$ is closed in X. It follows that Y is identified with $X/\sim$, and Y has closed sets given by those subsets of $X/\sim$ which pull back to closed sets of X, therefore $f: X \to Y = X/\sim$ is a quotient map by (dual of) the universal property. $\boxtimes$

3: Let (X, d_X) and (Y, d_Y) be metric spaces. An isometric embedding is a map $i: X \to Y$ such that for all $x_0, x_1 \in X$, $d_Y(i(x_0), i(x_1)) = d_X(x_0, x_1)$. An isometry is a surjective isometric embedding.

$\boxed{5}$

(a) Show that every isometric embedding from a compact metric space X to itself is an isometry.

(b) Show that every isometric embedding from Euclidean space $\mathbb{R}^n$ to itself is an isometry

(we will take the result from (a) as a given and show (b) for partial credit).

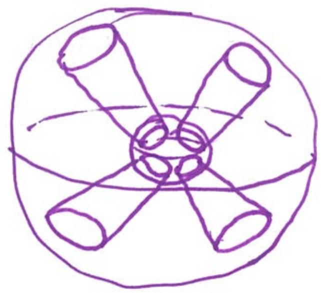
(b) Let $i: \mathbb{R}^n \to \mathbb{R}^n$ be an isometric embedding. Recall that the n-sphere S^n is the one point compactification of Euclidean space $\mathbb{R}^n$, and S^n is naturally a metric space as a one point compactification of $\mathbb{R}^n$. Then the isometric embedding $i: \mathbb{R}^n \to \mathbb{R}^n$ induces an isometric embedding $\bar{i}: S^n \to S^n$ of one point compactifications. By (a), since S^n is a compact metric space, $\bar{i}: S^n \to S^n$ is an isometry. But then removing the point at infinity from S^n, the isometry $\bar{i}: S^n \to S^n$ induces an isometry $i: \mathbb{R}^n \to \mathbb{R}^n$.

Hence every isometric embedding from Euclidean space $\mathbb{R}^n$ to itself is an isometry.

$\boxtimes$.

4. Consider the solid S obtained by digging out the center of a solid 3-dimensional ball and 4 tunnels from the center to the boundary. What is the <u>genus</u> of the boundary surface $\Sigma = \partial S$. ?

$\boxed{5}$

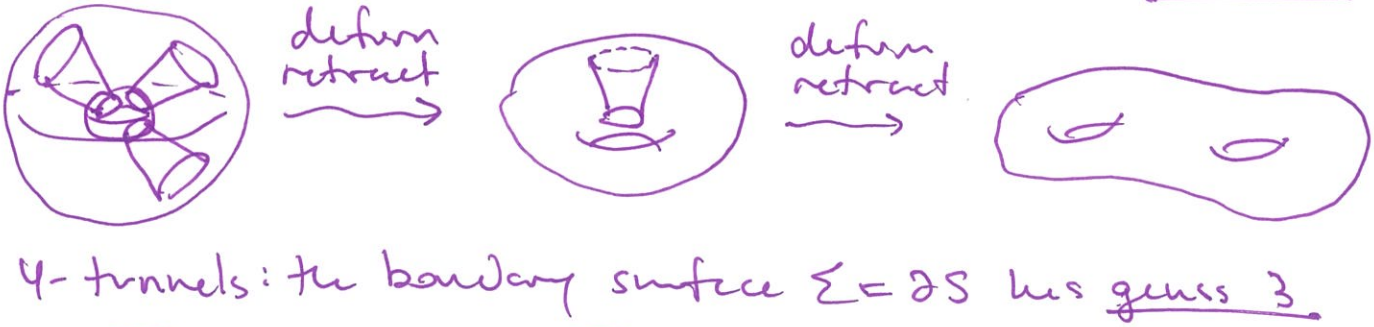


We can see this by some explicit deform-retracts.

- 1-tunnel: the boundary surface $\Sigma = \partial S$ has <u>genus 0</u>

- 2-tunnels: the boundary surface ~~h~~ $\Sigma = \partial S$ has <u>genus 1</u>

- 3-tunnels: the boundary surface $\Sigma = \partial S$ has <u>genus 2</u>



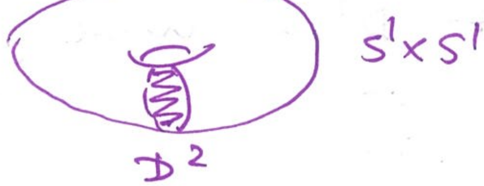
- 4-tunnels: the boundary surface $\Sigma = \partial S$ has <u>genus 3</u>

That is, once we have one tunnel we get a connected surface of genus 0, and then each subsequent tunnel adds one to the genus.

Rather for 4-tunnels, we are adding two handles to a surface of genus 1 to obtain a surface of genus 3. ☒

$\boxed{5}$

5. Let X be the topological space obtained by attaching a disk D^2 to $S^1 \times S^1$ along the circle $S^1 \times \{0\}$ via the map $z \mapsto z^5$. Compute the fundamental group and the (integral) homology of X.



We express the space X as a union of two subspaces $\mathcal{U}$ and V; $\mathcal{U}$ is the torus $S^1 \times S^1$, V is (a thickening of) the disk D^2, so that the intersection is homotopic to S^1: it is (a thickening of) the boundary of the disk D^2.

By the Van Kampen theorem we have the following pushout:

$$\begin{array}{ccc} \pi_1(\mathcal{U} \cap V) & \longrightarrow & \pi_1(\mathcal{U}) \\ \downarrow & \ulcorner & \downarrow \\ \pi_1(V) & \longrightarrow & \pi_1(X) \end{array}$$

that is, $\pi_1(X)$ is the amalgamated free product $\pi_1(X) = \pi_1(\mathcal{U}) *_{\pi_1(\mathcal{U} \cap V)} \pi_1(V)$

We have $\pi_1(\mathcal{U} \cap V) = \mathbb{Z}$, $\pi_1(\mathcal{U}) = \mathbb{Z} \times \mathbb{Z}$, and $\pi_1(V) = 0$, hence $\pi_1(X) = (\mathbb{Z} \times \mathbb{Z}) *_{\mathbb{Z}} 0$. But, the morphism $\pi_1(\mathcal{U} \cap V) \to \pi_1(\mathcal{U})$ sends a basis loop $\gamma \in \pi_1(\mathcal{U} \cap V) = \mathbb{Z}$ to $(5\gamma, 0) \in \pi_1(\mathcal{U}) = \mathbb{Z} \times \mathbb{Z}$ under the map $z \mapsto z^5$, so the amalgamated free product is really $(\mathbb{Z} \times \mathbb{Z}) *_{\mathbb{Z}} 0 = \mathbb{Z} \times \mathbb{Z}/5\mathbb{Z}$.

Hence $\boxed{\pi_1(X) = \mathbb{Z} \times \mathbb{Z}/5\mathbb{Z}}$

$$\mathcal{U} \cap V \hookrightarrow \begin{smallmatrix} \mathcal{U} \\ V \end{smallmatrix} \hookrightarrow \mathcal{U} \cup V = X$$

By the Mayer Vietoris long exact sequence,

$$\begin{aligned} 0 &\to H_2(\mathcal{U} \cap V, \mathbb{Z}) \to H_2(\mathcal{U}, \mathbb{Z}) \oplus H_2(V, \mathbb{Z}) \to H_2(X, \mathbb{Z}) \\ &\xrightarrow{\partial} H_1(\mathcal{U} \cap V, \mathbb{Z}) \to H_1(\mathcal{U}, \mathbb{Z}) \oplus H_1(V, \mathbb{Z}) \to H_1(X, \mathbb{Z}) \\ &\xrightarrow{\partial} H_0(\mathcal{U} \cap V, \mathbb{Z}) \to H_0(\mathcal{U}, \mathbb{Z}) \oplus H_0(V, \mathbb{Z}) \to H_0(X, \mathbb{Z}) \to 0 \end{aligned}$$

Since $\mathcal{U}$, V, $\mathcal{U} \cap V$, and $\mathcal{U} \cup V = X$ are connected we have $H_0(\mathcal{U} \cap V) = \mathbb{Z}$, $H_0(\mathcal{U}, \mathbb{Z}) \oplus H_0(V, \mathbb{Z}) = \mathbb{Z} \times \mathbb{Z}$, $H_0(X, \mathbb{Z}) = \mathbb{Z}$.

Since U is homotopic to $S^1 \times S^1$ we have

$$H_n(U,\mathbb{Z}) = \begin{cases} \mathbb{Z} & n=0 \\ \mathbb{Z}\times\mathbb{Z} & n=1 \\ \mathbb{Z} & n=2 \\ 0 & n>2. \end{cases}$$

Since V is homotopic to D^2 we have (since D^2 is contractible)

$$H_n(V,\mathbb{Z}) = \begin{cases} \mathbb{Z} & n=0 \\ 0 & n>0 \end{cases}$$

Since $U \cap V$ is homotopic to S^1 we have

$$H_n(U\cap V,\mathbb{Z}) = \begin{cases} \mathbb{Z} & n=0 \\ \mathbb{Z} & n=1 \\ 0 & n>1. \end{cases}$$

Hence the Mayer Vietoris long exact sequence is

$$\begin{aligned} 0 &\to 0 \to \mathbb{Z} \to H_2(X,\mathbb{Z}) \\ &\xrightarrow{\partial} \mathbb{Z} \to \mathbb{Z}\times\mathbb{Z} \to H_1(X,\mathbb{Z}) \\ &\xrightarrow{\partial} \mathbb{Z} \to \mathbb{Z}\times\mathbb{Z} \to H_0(X,\mathbb{Z}) = \mathbb{Z} \to 0 \end{aligned}$$

Since this is exact, reading off the connecting homomorphisms we obtain

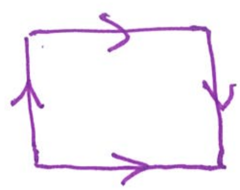
$$\boxed{H_n(X,\mathbb{Z}) = \begin{cases} \mathbb{Z} & n=0 \\ \mathbb{Z}\times\mathbb{Z}/5\mathbb{Z} & n=1 \\ \mathbb{Z} & n=2 \\ 0 & n>2. \end{cases}}$$

$\boxtimes$

6. Classify the 2-fold connected covering spaces of the Klein bottle K. $\boxed{5}$

It suffices to compute the fundamental group of the Klein bottle K using the Van Kampen theorem and then to apply the fundamental theorem of covering spaces.

First recall that the Klein bottle K is represented by the following gluing:



We express the Klein bottle K as a union of two mobius bands $\mathcal{U}$ and V, both homotopic to S^1, with intersection homotopic to S^1:

By the Van Kampen theorem we have the following pushout

$$\begin{tikzcd} \pi_1(\mathcal{U} \cap V) \arrow[r] \arrow[d] & \pi_1(\mathcal{U}) \arrow[d] \\ \pi_1(V) \arrow[r] \arrow[ur, phantom, "\ulcorner"] & \pi_1(K) \end{tikzcd}$$

that is, $\pi_1(K)$ is the amalgamated free product $\pi_1(K) = \pi_1(\mathcal{U}) *_{\pi_1(\mathcal{U} \cap V)} \pi_1(V)$

Since $\mathcal{U}$, V, and $\mathcal{U} \cap V$ are homotopic to S^1 we have $\pi_1(\mathcal{U}) = \pi_1(V) = \pi_1(\mathcal{U} \cap V) = \mathbb{Z}$, hence $\pi_1(K) \cong \mathbb{Z} *_{\mathbb{Z}} \mathbb{Z}$.

Now recall the fundamental theorem of covering spaces: for X a sufficiently nice (e.g. locally path connected semilocally simply connected) space X, monodromy gives a canonical bijection between isomorphism classes of connected covering spaces of X, and conjugacy classes of subgroups of $\pi_1(X)$; the degree of the covering space corresponds to the index of the (conjugacy class of) subgroup.

In the case of the Klein bottle K, we therefore have a canonical bijection between 2-fold connected covering spaces of K up to isomorphism, and index 2 subgroups of $\pi_1(K) = \mathbb{Z} *_{\mathbb{Z}} \mathbb{Z}$ up to conjugacy. $\boxtimes$

7. Show that every continuous map from $\mathbb{RP}^2 \times \mathbb{RP}^2$ to $S^1 \times S^1 \times S^1 \times S^1$ is null homotopic. $\boxed{S}$

It suffices to show that every continuous map f from $\mathbb{RP}^2 \times \mathbb{RP}^2$ to $S^1 \times S^1 \times S^1 \times S^1$ induces the zero homomorphism $f_*: \pi_n(\mathbb{RP}^2 \times \mathbb{RP}^2) \to \pi_n(S^1 \times S^1 \times S^1 \times S^1)$ on homotopy groups. To that end recall that

$$\pi_n(S^1) = \begin{cases} 0 & n=0 \\ \mathbb{Z} & n=1 \\ 0 & n>1 \end{cases} \qquad \pi_n(\mathbb{RP}^2) = \begin{cases} 0 & n=0 \\ \mathbb{Z}/2\mathbb{Z} & n=1 \\ \pi_n(S^2) & n>1 \end{cases}$$

Since π_n sends products to products we have

$$\pi_n(S^1 \times S^1 \times S^1 \times S^1) = \begin{cases} 0 & n=0 \\ \mathbb{Z}\times\mathbb{Z}\times\mathbb{Z}\times\mathbb{Z} & n=1 \\ 0 & n>1 \end{cases}$$

$$\pi_n(\mathbb{RP}^2 \times \mathbb{RP}^2) = \begin{cases} 0 & n=0 \\ \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} & n=1 \\ \pi_n(S^2) \times \pi_n(S^2) & n>1. \end{cases}$$

Now:

- $f_*: \pi_0(\mathbb{RP}^2 \times \mathbb{RP}^2) \to \pi_0(S^1 \times S^1 \times S^1 \times S^1)$ is the zero homomorphism since $\pi_0(\mathbb{RP}^2 \times \mathbb{RP}^2) = \pi_0(S^1 \times S^1 \times S^1 \times S^1) = 0$.
- $f_*: \pi_1(\mathbb{RP}^2 \times \mathbb{RP}^2) \to \pi_1(S^1 \times S^1 \times S^1 \times S^1)$ is the zero homomorphism since any group homomorphism $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \to \mathbb{Z}\times\mathbb{Z}\times\mathbb{Z}\times\mathbb{Z}$ is the zero homomorphism.
- $f_*: \pi_n(\mathbb{RP}^2 \times \mathbb{RP}^2) \to \pi_n(S^1 \times S^1 \times S^1 \times S^1)$ is the zero homomorphism for $n>1$ since $\pi_n(S^1 \times S^1 \times S^1 \times S^1) = 0$ for $n>1$.

Note that the vanishing of the higher homotopy groups of S^1 took care of the fact that the higher homotopy groups of S^2 are nonvanishing and <u>very</u> complicated. $\boxtimes$

8. Let $X = \mathbb{RP}^5/\mathbb{RP}^1$ and let $f: X \to X$ be a continuous map which is homotopic to the identity. Show that f must have a fixed point. $\boxed{3}$

First recall the Lefschetz fixed point theorem: let $f: X \to X$ be a continuous map. and let Λ_f be the Lefschetz number of f given by the alternating sum of traces of f acting on rational homology $H_*(X,\mathbb{Q})$:

$$\Lambda_f = \sum_{n \geq 0} (-1)^n \operatorname{tr}(f_* \mid H_n(X,\mathbb{Q})).$$

The Lefschetz fixed point theorem says that f has a fixed point if $\Lambda_f \neq 0$.

Now let us recall the integral and rational homology of $\mathbb{RP}^n$ from which we can deduce the rational homology of $\mathbb{RP}^5/\mathbb{RP}^1$. Recall that the integral homology is given

$$H_k(\mathbb{RP}^n;\mathbb{Z}) = \begin{cases} \mathbb{Z} & k=0 \\ \mathbb{Z} & k=n \text{ odd} \\ \mathbb{Z}/2\mathbb{Z} & k<n \text{ odd} \\ 0 & \text{otherwise} \end{cases}$$

X	$H_0(X,\mathbb{Z})$	$H_1(X,\mathbb{Z})$	$H_2(X,\mathbb{Z})$	$H_3(X,\mathbb{Z})$	$H_4(X,\mathbb{Z})$	$H_5(X,\mathbb{Z})$
$\mathbb{RP}^1$	$\mathbb{Z}$	$\mathbb{Z}$	0	0	0	0
$\mathbb{RP}^2$	$\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$	0	0	0	0
$\mathbb{RP}^3$	$\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$	0	$\mathbb{Z}$	0	0
$\mathbb{RP}^4$	$\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$	0	$\mathbb{Z}/2\mathbb{Z}$	0	0
$\mathbb{RP}^5$	$\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$	0	$\mathbb{Z}/2\mathbb{Z}$	0	$\mathbb{Z}$

recall the rational homology is given

$$H_k(\mathbb{RP}^n,\mathbb{Q}) = \begin{cases} \mathbb{Q} & k=0 \\ \mathbb{Q} & k=n \text{ odd} \\ 0 & \text{otherwise} \end{cases}$$

X	$H_0(X,\mathbb{Q})$	$H_1(X,\mathbb{Q})$	$H_2(X,\mathbb{Q})$	$H_3(X,\mathbb{Q})$	$H_4(X,\mathbb{Q})$	$H_5(X,\mathbb{Q})$
$\mathbb{RP}^1$	$\mathbb{Q}$	$\mathbb{Q}$	0	0	0	0
$\mathbb{RP}^2$	$\mathbb{Q}$	0	0	0	0	0
$\mathbb{RP}^3$	$\mathbb{Q}$	0	0	$\mathbb{Q}$	0	0
$\mathbb{RP}^4$	$\mathbb{Q}$	0	0	0	0	0
$\mathbb{RP}^5$	$\mathbb{Q}$	0	0	0	0	$\mathbb{Q}$

hence the rational homology of $X = \mathbb{RP}^5/\mathbb{RP}^1$ is given

$$H_n(\mathbb{RP}^5/\mathbb{RP}^1,\mathbb{Q}) = \begin{cases} \mathbb{Q} & n=0,1,5 \\ 0 & \text{otherwise} \end{cases}$$

X	$H_0(X,\mathbb{Q})$	$H_1(X,\mathbb{Q})$	$H_2(X,\mathbb{Q})$	$H_3(X,\mathbb{Q})$	$H_4(X,\mathbb{Q})$	$H_5(X,\mathbb{Q})$
$\mathbb{RP}^5/\mathbb{RP}^1$	$\mathbb{Q}$	$\mathbb{Q}$	0	0	0	$\mathbb{Q}$
tr	+1	−1	0	0	0	−1

Now suppose that $f: X \to X$ is a continuous map homotopic to the identity. Then its trace on $H_n(X,\mathbb{Q})$ is 0 when $H_n(X,\mathbb{Q}) = 0$ and 1 when $H_n(X,\mathbb{Q}) = \mathbb{Q}$. Hence:

$$\begin{aligned} \Lambda_f &= \sum_{n \geq 0} (-1)^n \operatorname{tr}(f_* \mid H_n(X,\mathbb{Q})) \\ &= \operatorname{tr}(f_* \mid H_0(X,\mathbb{Q})) - \operatorname{tr}(f_* \mid H_1(X,\mathbb{Q})) - \operatorname{tr}(f_* \mid H_5(X,\mathbb{Q})) \\ &= 1 - 1 - 1 = -1 \end{aligned}$$

Hence $\Lambda_f = -1 \neq 0$ so by the Lefschetz trace formula / fixed point theorem it follows that f has a fixed point. $\boxtimes$

9: Describe the CW structure of $X = \mathbb{C}P^2 \times \mathbb{R}P^2$ and use it to compute the (integral) homology of X. $\boxed{5}$

Recall that $\mathbb{C}P^2$ is formed by one 0-cell, one 2-cell, and one 4 cell, yielding the integral homology

$$H_n(\mathbb{C}P^2, \mathbb{Z}) = \begin{cases} \mathbb{Z} & n = 0,2,4 \\ 0 & \text{otherwise} \end{cases}$$

X	$H_0(X,\mathbb{Z})$	$H_1(X,\mathbb{Z})$	$H_2(X,\mathbb{Z})$	$H_3(X,\mathbb{Z})$	$H_4(X,\mathbb{Z})$	
$\mathbb{C}P^2$	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	--

Recall that $\mathbb{R}P^2$ is formed by one 0-cell, one 1-cell, and one 2-cell, yielding the integral homology (as before)

$$H_n(\mathbb{R}P^2, \mathbb{Z}) = \begin{cases} \mathbb{Z} & n = 0 \\ \mathbb{Z}/2\mathbb{Z} & n = 1 \\ 0 & \text{otherwise} \end{cases}$$

X	$H_0(X,\mathbb{Z})$	$H_1(X,\mathbb{Z})$	$H_2(X,\mathbb{Z})$	
$\mathbb{R}P^2$	$\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$	0	—

Now we apply the Künneth formula in integral homology: recall that for spaces X and Y the integral homology $H_*(X\times Y, \mathbb{Z})$ is given in terms of $H_*(X,\mathbb{Z})$ and $H_*(Y,\mathbb{Z})$ as:

$$H_n(X\times Y, \mathbb{Z}) = \bigoplus_{p+q=n} H_p(X,\mathbb{Z}) \otimes_{\mathbb{Z}} H_q(Y,\mathbb{Z}).$$

Applying this to the space $\mathbb{C}P^2 \times \mathbb{R}P^2$ with the above integral homology computations yields:

$$H_0(\mathbb{C}P^2\times \mathbb{R}P^2, \mathbb{Z}) = H_0(\mathbb{C}P^2, \mathbb{Z}) \otimes_{\mathbb{Z}} H_0(\mathbb{R}P^2, \mathbb{Z}) = \mathbb{Z} \quad \text{(connected)}$$
$$H_1(\mathbb{C}P^2\times \mathbb{R}P^2, \mathbb{Z}) = H_0(\mathbb{C}P^2, \mathbb{Z}) \otimes_{\mathbb{Z}} H_1(\mathbb{R}P^2, \mathbb{Z}) = \mathbb{Z}/2\mathbb{Z}$$
$$H_2(\mathbb{C}P^2\times \mathbb{R}P^2, \mathbb{Z}) = H_2(\mathbb{C}P^2, \mathbb{Z}) \otimes_{\mathbb{Z}} H_0(\mathbb{R}P^2, \mathbb{Z}) = \mathbb{Z}$$
$$H_3(\mathbb{C}P^2\times \mathbb{R}P^2, \mathbb{Z}) = H_2(\mathbb{C}P^2, \mathbb{Z}) \otimes_{\mathbb{Z}} H_1(\mathbb{R}P^2, \mathbb{Z}) = \mathbb{Z}/2\mathbb{Z}$$
$$H_4(\mathbb{C}P^2\times \mathbb{R}P^2, \mathbb{Z}) = H_4(\mathbb{C}P^2, \mathbb{Z}) \otimes_{\mathbb{Z}} H_0(\mathbb{R}P^2, \mathbb{Z}) = \mathbb{Z}$$
$$H_5(\mathbb{C}P^2\times \mathbb{R}P^2, \mathbb{Z}) = H_4(\mathbb{C}P^2, \mathbb{Z}) \otimes_{\mathbb{Z}} H_1(\mathbb{R}P^2, \mathbb{Z}) = \mathbb{Z}/2\mathbb{Z}$$

and 0 otherwise. We obtain

$$\boxed{H_n(\mathbb{C}P^2 \times \mathbb{R}P^2, \mathbb{Z}) = \begin{cases} \mathbb{Z} & n = 0,2,4 \\ \mathbb{Z}/2\mathbb{Z} & n = 1,3,5 \\ 0 & \text{otherwise} \end{cases}}$$

$\boxtimes$