Freiman's Theorem for Torsion Groups

Theorem (Ruzsa)

Suppose $A \subseteq G$, that every element of G has order at most r, and that $|A+A| \leq K|A|$, then the subgroup generated by A (which obviously contains A) has size at most $K^2 r^{K^4} |A|$.

This result follows quickly and easily from Plünnecke's inequality and the following "covering lemma":

Lemma (Ruzsa's Covering Lemma)

If A and B are finite subsets of an abelian group and $|A+B| \leq K|A|$, then B may be covered by at most K translates of $A-A$.

Proof (of Lemma)

Choose $X \subseteq B$ maximal with property that $\{A+x : x \in X\}$ are disjoint. The union of these sets contain exactly $|A||X|$ elements all of which are contained in $A+X \subseteq A+B$. Thus $|X| \leq K$.

Now if $b \in B$, then maximality of X ensures that $\exists\, x \in X$ s.t

$$(A+b) \cap (A+x) \neq \emptyset$$

$$\Rightarrow \quad b \in x + A - A. \qquad \square$$

Proof of Theorem

Since $|A+A| \le K|A|$ it follows from Plünnecke's inequality that

$$|A-A| \le K^2|A| \qquad \underline{\text{and}} \qquad |3A-A| \le K^4|A|.$$

Now Ruzsa's covering lemma, applied to $B = 2A-A$, implies that $\exists\, X \subseteq 2A-A$ with $|X| \le K^4$ such that

$$2A-A \subseteq X + A - A.$$

Adding A gives

$$3A-A \subseteq X + 2A - A \subseteq 2X + A - A.$$

Continuing we see that for all $m \ge 1$

$$mA - A \subseteq \langle X \rangle + A - A$$

↑ subgroup gen. by X.

Since $\langle A \rangle \subseteq \bigcup_{m \ge 1} (mA - A)$ it follows that

$$\langle A \rangle \subseteq \langle X \rangle + A - A$$

and hence

$$|\langle A \rangle| \le |\langle X \rangle||A-A| \le r^{|X|} K^2 |A| \le K^2 r^{K^4} |A|$$

$\square$.