6.17. Suppose that $X_1, X_2, \ldots$ are independent and $P[X_n = 0] = p$. Let L_n be the length of the run of 0's starting at the nth place: $L_n = k$ if $X_n = \cdots = X_{n+k-1} = 0 \neq X_{n+k}$. Show that $P[L_n \geq r_n \text{ i.o.}]$ is 0 or 1 according as $\sum_n p^{r_n}$ converges or diverges. Example 6.5 covers the case $p = \frac{1}{2}$.

SECTION 7. GAMBLING SYSTEMS

Let $X_1, X_2, \ldots$ be an independent sequence of random variables (on some $(\Omega, \mathscr{F}, P)$) taking on the two values $+1$ and -1 with probabilities $P[X_n = +1] = p$ and $P[X_n = -1] = q = 1 - p$. Throughout the section, X_n will be viewed as the gambler's gain on the nth of a series of plays at unit stakes. The game is favorable to the gambler if $p > \frac{1}{2}$, fair if $p = \frac{1}{2}$, and unfavorable if $p < \frac{1}{2}$. The case $p \leq \frac{1}{2}$ will be called the *subfair case*.

After the classical gambler's ruin problem has been solved, it will be shown that every gambling system is in certain respects without effect and that some gambling systems are in other respects optimal. Gambling problems of the sort considered here have inspired many ideas in the mathematical theory of probability, ideas that carry far beyond their origin.

Red-and-black will provide numerical examples. Of the 38 spaces on a roulette wheel, 18 are red, 18 are black, and 2 are green. In betting either on red or on black the chance of winning is $\frac{18}{38}$.

Gambler's Ruin

Suppose that the gambler enters the casino with capital a and adopts the strategy of continuing to bet at unit stakes until his fortune increases to c or his funds are exhausted. What is the probability of ruin, the probability that he will lose his capital, a? What is the probability he will achieve his goal, c? Here a and c are integers.

Let

$$S_n = X_1 + \cdots + X_n, \qquad S_0 = 0. \tag{7.1}$$

The gambler's fortune after n plays is $a + S_n$. The event

$$A_{a,n} = [a + S_n = c] \cap \bigcap_{k=1}^{n-1} [0 < a + S_k < c] \tag{7.2}$$

represents success for the gambler at time n, and

$$B_{a,n} = [a + S_n = 0] \cap \bigcap_{k=1}^{n-1} [0 < a + S_k < c] \tag{7.3}$$

represents ruin at time n. If $s_c(a)$ denotes the probability of ultimate success, then

$$s_c(a) = P\left(\bigcup_{n=1}^{\infty} A_{a,n}\right) = \sum_{n=1}^{\infty} P(A_{a,n}) \tag{7.4}$$

for $0 < a < c$.

Fix c and let a vary. For $n \geq 1$ and $0 < a < c$, define $A_{a,n}$ by (7.2), and adopt the conventions $A_{a,0} = \varnothing$ for $0 \leq a < c$ and $A_{c,0} = \Omega$ (success is impossible at time 0 if $a < c$ and certain if $a = c$), as well as $A_{0,n} = A_{c,n} = \varnothing$ for $n \geq 1$ (play never starts if a is 0 or c). By these conventions, $s_c(0) = 0$ and $s_c(c) = 1$.

Because of independence and the fact that the sequence $X_2, X_3, \ldots$ is a probabilistic replica of $X_1, X_2, \ldots$, it seems clear that the chance of success for a gambler with initial fortune a must be the chance of winning the first wager times the chance of success for an initial fortune $a + 1$, plus the chance of losing the first wager times the chance of success for an initial fortune $a - 1$. It thus seems intuitively clear that

$$s_c(a) = ps_c(a+1) + qs_c(a-1), \qquad 0 < a < c. \tag{7.5}$$

For a rigorous argument, define $A'_{a,n}$ just as $A_{a,n}$ but with $S'_n = X_2 + \cdots + X_{n+1}$ in place of S_n in (7.2). Now $P[X_i = x_i, i \leq n] = P[X_{i+1} = x_i, i \leq n]$ for each sequence $x_1, \ldots, x_n$ of $+1$'s and -1's, and therefore $P[(X_1, \ldots, X_n) \in H] = P[(X_2, \ldots, X_{n+1}) \in H]$ for $H \subset R^n$. Take H to be the set of $x = (x_1, \ldots, x_n)$ in R^n satisfying $x_i = \pm 1$, $a + x_1 + \cdots + x_n = c$, and $0 < a + x_1 + \cdots + x_k < c$ for $k < n$. It follows then that

$$P(A_{a,n}) = P(A'_{a,n}). \tag{7.6}$$

Moreover, $A_{a,n} = ([X_1 = +1] \cap A'_{a+1,n-1}) \cup ([X_1 = -1] \cap A'_{a-1,n-1})$ for $n \geq 1$ and $0 < a < c$. By independence and (7.6), $P(A_{a,n}) = pP(A_{a+1,n-1}) + qP(A_{a-1,n-1})$; adding over n now gives (7.5). Note that this argument involves the entire infinite sequence $X_1, X_2, \ldots$.

It remains to solve the difference equation (7.5) with the side conditions $s_c(0) = 0$, $s_c(c) = 1$. Let $\rho = q/p$ be the odds against the gambler. Then [A19] there exist constants A and B such that, for $0 \leq a \leq c$, $s_c(a) = A + B\rho^a$ if $p \neq q$ and $s_c(a) = A + Ba$ if $p = q$. The requirements $s_c(0) = 0$ and $s_c(c) = 1$ determine A and B, which gives the solution:

The probability that the gambler can before ruin attain his goal of c from an initial capital of a is

$$s_c(a) = \begin{cases} \dfrac{\rho^a - 1}{\rho^c - 1}, & 0 \leq a \leq c, \quad \text{if } \rho = \dfrac{q}{p} \neq 1, \\[2ex] \dfrac{a}{c}, & 0 \leq a \leq c, \quad \text{if } \rho = \dfrac{q}{p} = 1. \end{cases} \tag{7.7}$$

Example 7.1. The gambler's initial capital is \$900 and his goal is \$1000. If $p = \frac{1}{2}$, his chance of success is very good: $s_{1000}(900) = .9$. At red-and-black, $p = \frac{18}{38}$ and hence $\rho = \frac{20}{18}$; in this case his chance of success as computed by (7.7) is only about .00003. ■

Example 7.2. It is the gambler's desperate intention to convert his \$100 into \$20,000. For a game in which $p = \frac{1}{2}$ (no casino has one), his chance of success is 100/20,000 = .005; at red-and-black it is minute—about 3×10^{-911} ■

In the analysis leading to (7.7), replace (7.2) by (7.3). It follows that (7.7) with p and q interchanged (ρ goes to ρ^{-1}) and a and $c-a$ interchanged gives the probability $r_c(a)$ of ruin for the gambler: $r_c(a) = (\rho^{-(c-a)} - 1)/(\rho^{-c} - 1)$ if $\rho \neq 1$ and $r_c(a) = (c-a)/c$ if $\rho = 1$. Hence $s_c(a) + r_c(a) = 1$ holds in all cases: *The probability is* 0 *that play continues forever.*

For positive integers a and b, let

$$H_{a,b} = \bigcup_{n=1}^{\infty}\left\{[S_n = b] \cap \bigcap_{k=1}^{n-1}[-a < S_k < b]\right\}$$

be the event that S_n reaches $+b$ before reaching $-a$. Its probability is simply (7.7) with $c = a+b$: $P(H_{a,b}) = s_{a+b}(a)$. Now let

$$H_b = \bigcup_{a=1}^{\infty} H_{a,b} = \bigcup_{n=1}^{\infty}[S_n = b] = \left[\sup_n S_n \geq b\right]$$

be the event that S_n ever reaches $+b$. Since $H_{a,b} \uparrow H_b$ as $a \to \infty$, it follows that $P(H_b) = \lim_a s_{a+b}(a)$; this is 1 if $\rho = 1$ or $\rho < 1$, and it is $1/\rho^b$ if $\rho > 1$. Thus

$$P\left[\sup_n S_n \geq b\right] = \begin{cases} 1 & \text{if } p \geq q, \\ (p/q)^b & \text{if } p < q. \end{cases} \tag{7.8}$$

This is the probability that a gambler with unlimited capital can ultimately gain b *units.*

Example 7.3. The gambler in Example 7.1 has capital 900 and the goal of winning $b = 100$; in Example 7.2 he has capital 100 and b is 19,900. Suppose, instead, that his capital is infinite. If $p = \frac{1}{2}$, the chance of achieving his goal increases from .9 to 1 in the first example and from .005 to 1 in the second. At red-and-black, however, the two probabilities $.9^{100}$ and $.9^{19900}$ remain essentially what they were before (.00003 and 3×10^{-911}). ■

Selection Systems

Players often try to improve their luck by betting only when in the preceding trials the wins and losses form an auspicious pattern. Perhaps the gambler bets on the nth trial only when among $X_1, \ldots, X_{n-1}$ there are many more $+1$'s than -1's, the idea being to ride winning streaks (he is "in the vein"). Or he may bet only when there are many more -1's than $+1$'s, the idea being it is then surely time a $+1$ came along (the "maturity of the chances"). There is a mathematical theorem that, translated into gaming language, says all such systems are futile.

It might be argued that it *is* sensible to bet if among $X_1, \ldots, X_{n-1}$ there is an excess of $+1$'s, on the ground that it is evidence of a high value of p. But it is assumed throughout that statistical inference is not at issue: p is fixed—at $\frac{18}{38}$, for example, in the case of red-and-black—and is known to the gambler, or should be.

The gambler's strategy is described by random variables $B_1, B_2, \ldots$ taking the two values 0 and 1: If $B_n = 1$, the gambler places a bet on the nth trial; if $B_n = 0$, he skips that trial. If B_n were $(X_n + 1)/2$, so that $B_n = 1$ for $X_n = +1$ and $B_n = 0$ for $X_n = -1$, the gambler would win every time he bet, but of course such a system requires he be prescient—he must know the outcome X_n in advance. For this reason the value of B_n is assumed to depend only on the values of $X_1, \ldots, X_{n-1}$: there exists some function $b_n: R^{n-1} \to R^1$ such that

$$B_n = b_n(X_1, \ldots, X_{n-1}). \tag{7.9}$$

(Here B_1 is constant.) Thus the mathematics avoids, as it must, the question of whether prescience is actually possible.

Define

$$\begin{cases} \mathscr{F}_n = \sigma(X_1, \ldots, X_n), & n = 1, 2, \ldots, \\ \mathscr{F}_0 = \{\varnothing, \Omega\}. \end{cases} \tag{7.10}$$

The σ-field $\mathscr{F}_{n-1}$ generated by $X_1, \ldots, X_{n-1}$ corresponds to a knowledge of the outcomes of the first $n-1$ trials. The requirement (7.9) ensures that B_n is measurable $\mathscr{F}_{n-1}$ (Theorem 5.1) and so depends only on the information actually available to the gambler just before the nth trial.

For $n = 1, 2, \ldots$, let N_n be the time at which the gambler places his nth bet. This nth bet is placed at time k or earlier if and only if the number $\sum_{i=1}^k B_i$ of bets placed up to and including time k is n or more; in fact, N_n is the smallest k for which $\sum_{i=1}^k B_i = n$. Thus the event $[N_n \leq k]$ coincides with $[\sum_{i=1}^k B_i \geq n]$; by (7.9) this latter event lies in $\sigma(B_1, \ldots, B_k) \subset \sigma(X_1, \ldots, X_{k-1}) = \mathscr{F}_{k-1}$. Therefore,

$$[N_n = k] = [N_n \leq k] - [N_n \leq k-1] \in \mathscr{F}_{k-1}. \tag{7.11}$$

(Even though $[N_n = k]$ lies in $\mathscr{F}_{k-1}$ and hence in $\mathscr{F}$, N_n is, as a function on Ω, generally not a simple random variable, because it has infinite range. This makes no difference, because expected values of the N_n will play no role; (7.11) is the essential property.)

To ensure that play continues forever (stopping rules will be considered later) and that the N_n have finite values with probability 1, make the further assumption that

$$P[B_n = 1 \text{ i.o.}] = 1. \tag{7.12}$$

A sequence $\{B_n\}$ of random variables assuming the values 0 and 1, having the form (7.9), and satisfying (7.12) is a *selection system*.

Let Y_n be the gambler's gain on the nth of the trials at which he does bet: $Y_n = X_{N_n}$. It is only on the set $[B_n = 1 \text{ i.o.}]$ that all the N_n and hence all the Y_n are well defined. To complete the definition, set $Y_n = -1$, say, on $[B_n = 1 \text{ i.o.}]^c$; since this set has probability 0 by (7.12), it really makes no difference how Y_n is defined on it.

Now Y_n is a complicated function on Ω because $Y_n(\omega) = X_{N_n(\omega)}(\omega)$. Nonetheless,

$$[\omega: Y_n(\omega) = 1] = \bigcup_{k=1}^{\infty} \left([\omega: N_n(\omega) = k] \cap [\omega: X_k(\omega) = 1]\right)$$

lies in $\mathscr{F}$, and so does its complement $[\omega: Y_n(\omega) = -1]$. Hence Y_n is a simple random variable.

Example 7.4. An example will fix these ideas. Suppose that the rule is always to bet on the first trail, to bet on the second trial if and only if $X_1 = +1$, to bet on the third trial if and only if $X_1 = X_2$, and to bet on all subsequent trails. Here $B_1 = 1$, $[B_2 = 1] = [X_1 = +1]$, $[B_3 = 1] = [X_1 = X_2]$, and $B_4 = B_5 = \cdots = 1$. The table shows the ways the gambling can start out. A dot represents a value undetermined by X_1, X_2, X_3. Ignore the rightmost column for the moment.

X_1	X_2	X_3	B_1	B_2	B_3	N_1	N_2	N_3	N_4	Y_1	Y_2	Y_3	τ
−1	−1	−1	1	0	1	1	3	4	5	−1	−1	·	1
−1	−1	+1	1	0	1	1	3	4	5	−1	+1	·	1
−1	+1	−1	1	0	0	1	4	5	6	−1	·	·	1
−1	+1	+1	1	0	0	1	4	5	6	−1	·	·	1
+1	−1	−1	1	1	0	1	2	4	5	+1	−1	·	2
+1	−1	+1	1	1	0	1	2	4	5	+1	−1	·	2
+1	+1	−1	1	1	1	1	2	3	4	+1	+1	−1	3
+1	+1	+1	1	1	1	1	2	3	4	+1	+1	+1	·

In the evolution represented by the first line of the table, the second bet is placed on the third trial ($N_2 = 3$), which results in a loss because $Y_2 = X_{N_2} = X_3 = -1$. Since $X_3 = -1$, the gambler was "wrong" to bet. But remember that before the third trial he does not know $X_3(\omega)$ (much less ω itself); he knows only $X_1(\omega)$ and $X_2(\omega)$. See the discussion in Example 5.5. ∎

Selection systems achieve nothing because $\{Y_n\}$ has the same structure as $\{X_n\}$:

Theorem 7.1. *For every selection system, $\{Y_n\}$ is independent and $P[Y_n = +1] = p$, $P[Y_n = -1] = q$.*

PROOF. Since random variables with indices that are themselves random variables are conceptually confusing at first, the ω's here will not be suppressed as they have been in previous proofs.

Relabel p and q as $p(+1)$ and $p(-1)$, so that $P[\omega: X_k(\omega) = x] = p(x)$ for $x = \pm 1$. If $A \in \mathscr{F}_{k-1}$, then A and $[\omega: X_k(\omega) = x]$ are independent, and so $P(A \cap [\omega: X_k(\omega) = x]) = P(A)p(x)$. Therefore, by (7.11),

$$\begin{aligned} P[\omega: Y_n(\omega) = x] &= P[\omega: X_{N_n(\omega)}(\omega) = x] \\ &= \sum_{k=1}^{\infty} P[\omega: N_n(\omega) = k,\ X_k(\omega) = x] \\ &= \sum_{k=1}^{\infty} P[\omega: N_n(\omega) = k]\, p(x) \\ &= p(x). \end{aligned}$$

More generally, for any sequence $x_1, \ldots, x_n$ of ± 1's,

$$\begin{aligned} P[\omega: Y_i(\omega) = x_i,\ i \le n] &= P[\omega: X_{N_i(\omega)}(\omega) = x_i,\ i \le n] \\ &= \sum_{k_1 < \cdots < k_n} P[\omega: N_i(\omega) = k_i,\ X_{k_i}(\omega) = x_i,\ i \le n], \end{aligned}$$

where the sum extends over n-tuples of positive integers satisfying $k_1 < \cdots < k_n$. The event $[\omega: N_i(\omega) = k_i,\ i \le n] \cap [\omega: X_{k_i}(\omega) = x_i,\ i < n]$ lies in $\mathscr{F}_{k_n - 1}$ (note that there is no condition on $X_{k_n}(\omega)$), and therefore

$$\begin{aligned} P[\omega: Y_i(\omega) = x_i,\ i \le n] &= \sum_{k_1 < \cdots < k_n} P\big([\omega: N_i(\omega) = k_i,\ i \le n] \\ &\qquad \cap [\omega: X_{k_i}(\omega) = x_i,\ i < n]\big)\, p(x_n). \end{aligned}$$

Summing k_n over $k_{n-1}+1, k_{n-1}+2, \ldots$ brings this last sum to

$$\sum_{k_1<\cdots<k_{n-1}} P\left[\omega: N_i(\omega)=k_i,\ X_{k_i}(\omega)=x_i,\ i<n\right]p(x_n)$$
$$=P\left[\omega: X_{N_i(\omega)}(\omega)=x_i,\ i<n\right]p(x_n)$$
$$=P\left[\omega: Y_i(\omega)=x_i,\ i<n\right]p(x_n).$$

It follows by induction that

$$P\left[\omega: Y_i(\omega)=x_i,\ i\le n\right]=\prod_{i\le n}p(x_i)=\prod_{i\le n}P\left[\omega: Y_i(\omega)=x_i\right],$$

and so the Y_i are independent (see (5.9)). ■

Gambling Policies

There are schemes that go beyond selection systems and tell the gambler not only whether to bet but how much. Gamblers frequently contrive or adopt such schemes in the confident expectation that they can, by pure force of arithmetic, counter the most adverse workings of chance. If the wager specified for the nth trial is in the amount W_n and the gambler cannot see into the future, then W_n must depend only on $X_1,\ldots,X_{n-1}$. Assume therefore that W_n is a nonnegative function of these random variables: there is an $f_n: R^{n-1}\to R^1$ such that

$$W_n=f_n(X_1,\ldots,X_{n-1})\ge 0. \tag{7.13}$$

Apart from nonnegativity there are at the outset no constraints on the f_n, although in an actual casino their values must be integral multiples of a basic unit. Such a sequence $\{W_n\}$ is a *betting system*. Since $W_n=0$ corresponds to a decision not to bet at all, betting systems in effect include selection systems. In the double-or-nothing system, $W_n=2^{n-1}$ if $X_1=\cdots=X_{n-1}=-1$ ($W_1=1$) and $W_n=0$ otherwise.

The amount the gambler wins on the nth play is W_nX_n. If his fortune at time n is F_n, then

$$F_n=F_{n-1}+W_nX_n. \tag{7.14}$$

This also holds for $n=1$ if F_0 is taken as his initial (nonrandom) fortune. It is convenient to let W_n depend on F_0 as well as the past history of play and hence to generalize (7.13) to

$$W_n=g_n(F_0,X_1,\ldots,X_{n-1})\ge 0 \tag{7.15}$$

for a function $g_n: R^n\to R^1$. In expanded notation, $W_n(\omega)=g_n(F_0,X_1(\omega),\ldots,X_{n-1}(\omega))$. The symbol W_n does not show the dependence on ω or on F_0, either. For each *fixed* initial fortune F_0, W_n is a simple random variable; by (7.15) it is measurable $\mathscr{F}_{n-1}$. Similarly, F_n is a function of F_0 as well as of $X_1(\omega),\ldots,X_n(\omega)$: $F_n=F_n(F_0,\omega)$.

If $F_0=0$ and $g_n\equiv 1$, the F_n reduce to the partial sums (7.1).

Since $\mathscr{F}_{n-1}$ and $\sigma(X_n)$ are independent, and since W_n is measurable $\mathscr{F}_{n-1}$ (for each fixed F_0), W_n and X_n are independent. Therefore, $E[W_nX_n]=E[W_n]\cdot E[X_n]$. Now $E[X_n]=p-q\le 0$ in the subfair case ($p\le\frac{1}{2}$), with equality in the fair case ($p=\frac{1}{2}$). Since $E[W_n]\ge 0$, (7.14) implies that $E[F_n]\le E[F_{n-1}]$. Therefore,

$$F_0\ge E[F_1]\ge\cdots\ge E[F_n]\ge\cdots \tag{7.16}$$

in the subfair case, and

$$F_0=E[F_1]=\cdots=E[F_n]=\cdots \tag{7.17}$$

in the fair case. (If $p<q$ and $P[W_n>0]>0$, there is strict inequality in (7.16).) Thus no betting system can convert a subfair game into a profitable enterprise.

Suppose that in addition to a betting system, the gambler adopts some policy for quitting. Perhaps he stops when his fortune reaches a set target, or his funds are exhausted, or the auguries are in some way dissuasive. The decision to stop must depend only on the initial fortune and the history of play up to the present.

Let $\tau(F_0,\omega)$ be a nonnegative integer for each ω in Ω and each $F_0\ge 0$. If $\tau=n$, the gambler plays on the nth trial (betting W_n) and then stops; if $\tau=0$, he does not begin gambling in the first place. The event $[\omega: \tau(F_0,\omega)=n]$ represents the decision to stop just after the nth trial, and so, whatever value F_0 may have, it must depend only on $X_1,\ldots,X_n$. Therefore, assume that

$$[\omega: \tau(F_0,\omega)=n]\in\mathscr{F}_n,\qquad n=0,1,2,\ldots \tag{7.18}$$

A τ satisfying this requirement is a *stopping time*. (In general it has infinite range and hence is not a simple random variable; as expected values of τ play no role here, this does not matter.) It is technically necessary to let $\tau(F_0,\omega)$ be undefined or infinite on an ω-set of probability 0. This has no effect on the requirement (7.18), which must hold for each finite n. But it *is* assumed that τ *is finite with probability* 1: play is certain to terminate.

A betting system together with a stopping time is a *gambling policy*. Let π denote such a policy.

Example 7.5. Suppose that the betting system is given by $W_n=B_n$, with B_n as in Example 7.4. Suppose that the stopping rule is to quit after the first loss of a wager. Then $[\tau = n] = \bigcup_{k=1}^{n}[N_k = n, Y_1 = \cdots = Y_{k-1} = +1, Y_k = -1]$. For $j \le k \le n$, $[N_k = n, Y_j = x] = \bigcup_{m=1}^{n}[N_k = n, N_j = m, X_m = x]$ lies in $\mathscr{F}_n$ by (7.11); hence τ is a stopping time. The values of τ are shown in the rightmost column of the table. ■

The sequence of fortunes is governed by (7.14) until play terminates, and then the fortune remains for all future time fixed at F_τ (with value $F_{\tau(F_0,\omega)}(\omega)$). Therefore, the gambler's fortune at time n is

$$F_n^* = \begin{cases} F_n & \text{if } \tau \ge n, \\ F_\tau & \text{if } \tau \le n. \end{cases} \tag{7.19}$$

Note that the case $\tau = n$ is covered by both clauses here. If $n-1 < n \le \tau$, then $F_n^* = F_n = F_{n-1} + W_n X_n = F_{n-1}^* + W_n X_n$; if $\tau \le n-1 < n$, then $F_n^* = F_\tau = F_{n-1}^*$. Therefore, if $W_n^* = I_{[\tau \ge n]} W_n$, then

$$F_n^* = F_{n-1}^* + I_{[\tau \ge n]} W_n X_n = F_{n-1}^* + W_n^* X_n. \tag{7.20}$$

But this is the equation for a new betting system in which the wager placed at time n is W_n^*. If $\tau \ge n$ (play has not already terminated), W_n^* is the old amount W_n; if $\tau < n$ (play has terminated), W_n^* is 0. Now by (7.18), $[\tau \ge n] = [\tau < n]^c$ lies in $\mathscr{F}_{n-1}$. Thus $I_{[\tau \ge n]}$ is measurable $\mathscr{F}_{n-1}$, so that W_n^* as well as W_n is measurable $\mathscr{F}_{n-1}$, and $\{W_n^*\}$ represents a legitimate betting system. Therefore, (7.16) and (7.17) apply to the new system:

$$F_0 = F_0^* \ge E[F_1^*] \ge \cdots \ge E[F_n^*] \ge \cdots \tag{7.21}$$

if $p \le \frac{1}{2}$, and

$$F_0 = F_0^* = E[F_1^*] = \cdots = E[F_n^*] = \ldots \tag{7.22}$$

if $p = \frac{1}{2}$.

The gambler's ultimate fortune is F_τ. Now $\lim_n F_n^* = F_\tau$ with probability 1, since in fact $F_n^* = F_\tau$ for $n \ge \tau$. If

$$\lim_n E[F_n^*] = E[F_\tau], \tag{7.23}$$

then (7.21) and (7.22), respectively, imply that $E[F_\tau] \le F_0$ and $E[F_\tau] = F_0$. According to Theorem 5.4, (7.23) does hold if the F_n^* are uniformly bounded.

Call the policy *bounded by* M (M nonrandom) if

$$0 \le F_n^* \le M, \qquad n = 0, 1, 2, \ldots. \tag{7.24}$$

If F_n^* is not bounded above, the gambler's adversary must have infinite capital. A negative F_n^* represents a debt, and if F_n^* is not bounded below,

the gambler must have a patron of infinite wealth and generosity from whom to borrow and so must in effect have infinite capital. In case F_n^* is bounded below, 0 is the convenient lower bound—the gambler is assumed to have in hand all the capital to which he has access. In any real case, (7.24) holds and (7.23) follows. (There is a technical point that arises because the general theory of integration has been postponed: F_τ must be assumed to have finite range so that it will be a simple random variable and hence have an expected value in the sense of Section 5.[†]) The argument has led to this result:

Theorem 7.2. *For every policy, (7.21) holds if $p \le \frac{1}{2}$ and (7.22) holds if $p = \frac{1}{2}$. If the policy is bounded (and F_τ has finite range), then $E[F_\tau] \le F_0$ for $p \le \frac{1}{2}$ and $E[F_\tau] = F_0$ for $p = \frac{1}{2}$.*

Example 7.6. The gambler has initial capital a and plays at unit stakes until his capital increases to c ($0 \le a \le c$) or he is ruined. Here $F_0 = a$ and $W_n = 1$, and so $F_n = a + S_n$. The policy is bounded by c, and F_τ is c or 0 according as the gambler succeeds or fails. If $p = \frac{1}{2}$ and if s is the probability of success, then $a = F_0 = E[F_\tau] = sc$. Thus $s = a/c$. This gives a new derivation of (7.7) for the case $p = \frac{1}{2}$. The argument assumes however that play is certain to terminate. If $p \le \frac{1}{2}$, Theorem 7.2 only gives $s \le a/c$, which is weaker than (7.7). ■

Example 7.7. Suppose as before that $F_0 = a$ and $W_n = 1$, so that $F_n = a + S_n$, but suppose the stopping rule is to quit as soon as F_n reaches $a + b$. Here F_n^* is bounded above by $a + b$ but is not bounded below. If $p = \frac{1}{2}$, the gambler is by (7.8) certain to achieve his goal, so that $F_\tau = a + b$. In this case $F_0 = a < a + b = E[F_\tau]$. This illustrates the effect of infinite capital. It also illustrates the need for uniform boundedness in Theorem 5.4 (compare Example 5.7). ■

For some other systems (gamblers call them "martingales"), see the problems. For most such systems there is a large chance of a small gain and a small chance of a large loss.

Bold Play*

The formula (7.7) gives the chance that a gambler betting unit stakes can increase his fortune from a to c before being ruined. Suppose that a and c happen to be even and that at each trial the wager is two units instead of

† See Problem 7.11.

* This topic may be omitted.

one. Since this has the effect of halving a and c, the chance of success is now

$$\frac{\rho^{a/2}-1}{\rho^{c/2}-1} = \frac{\rho^{a}-1}{\rho^{c}-1}\,\frac{\rho^{c/2}+1}{\rho^{a/2}+1}, \qquad \frac{q}{p}=\rho\neq 1.$$

If $\rho>1$ ($p<\frac{1}{2}$), the second factor on the right exceeds 1: Doubling the stakes increases the probability of success in the unfavorable case $\rho>1$. In the case $\rho=1$, the probability remains the same.

There is a sense in which large stakes are optimal. It will be convenient to rescale so that the initial fortune satisfies $0\le F_0\le 1$ and the goal is 1. The policy of *bold play* is this: At each stage the gambler bets his entire fortune, unless a win would carry him past his goal of 1, in which case he bets just enough that a win would exactly achieve that goal:

$$W_n=\begin{cases} F_{n-1} & \text{if } 0\le F_{n-1}\le \frac{1}{2},\\ 1-F_{n-1} & \text{if } \frac{1}{2}\le F_{n-1}\le 1.\end{cases} \tag{7.25}$$

(It is convenient to allow even irrational fortunes.) As for stopping, the policy is to quit as soon as F_n reaches 0 or 1.

Suppose that play has not terminated by time $k-1$; under the policy (7.25), if play is not to terminate at time k, then X_k must be $+1$ or -1 according as $F_{k-1}\le\frac{1}{2}$ or $F_{k-1}\ge\frac{1}{2}$, and the conditional probability of this is at most $m=\max\{p,q\}$. It follows by induction that the probability that bold play continues beyond time n is at most m^n, and so play is certain to terminate (τ is finite with probability 1).

It will be shown that in the subfair case, bold play maximizes the probability of successfully reaching the goal of 1. This is the *Dubins–Savage theorem*. It will further be shown that there are other policies that are also optimal in this sense, and this maximum probability will be calculated. Bold play can be substantially better than betting at constant stakes. This contrasts with Theorems 7.1 and 7.2 concerning respects in which gambling systems are worthless.

From now on, consider only policies π that are bounded by 1 (see (7.24)). Suppose further that play stops as soon as F_n reaches 0 or 1 and that this is certain eventually to happen. Since F_τ assumes the values 0 and 1, and since $[F_\tau=x]=\bigcup_{n=0}^{\infty}[\tau=n]\cap[F_n=x]$ for $x=0$ and $x=1$, F_τ is a simple random variable. Bold play is one such policy π.

The policy π leads to success if $F_\tau=1$. Let $Q_\pi(x)$ be the probability of this for an initial fortune $F_0=x$:

$$Q_\pi(x)=P[F_\tau=1] \qquad \text{for } F_0=x. \tag{7.26}$$

Since F_n is a function $\psi_n(F_0,X_1(\omega),\dots,X_n(\omega))=\Psi_n(F_0,\omega)$, (7.26) in expanded notation is $Q_\pi(x)=P[\omega\colon \Psi_{\tau(x,\omega)}(x,\omega)=1]$. As π specifies that play stops at the boundaries 0 and 1,

$$\begin{aligned} &Q_\pi(0)=0,\quad Q_\pi(1)=1,\\ &0\le Q_\pi(x)\le 1,\qquad 0\le x\le 1.\end{aligned} \tag{7.27}$$

Let Q be the Q_π for bold play. (The notation does not show the dependence of Q and Q_π on p, which is fixed.)

Theorem 7.3. *In the subfair case, $Q_\pi(x)\le Q(x)$ for all π and all x.*

PROOF. Under the assumption $p\le q$, it will be shown later that

$$Q(x)\ge pQ(x+t)+qQ(x-t),\qquad 0\le x-t\le x\le x+t\le 1. \tag{7.28}$$

This can be interpreted as saying that the chance of success under bold play starting at x is at least as great as the chance of success if the amount t is wagered and bold play then pursued from $x+t$ in case of a win and from $x-t$ in case of a loss. Under the assumption of (7.28), optimality can be proved as follows.

Consider a policy π, and let F_n and F_n^* be the simple random variables defined by (7.14) and (7.19) *for this policy*. Now $Q(x)$ is a real function, and so $Q(F_n^*)$ is also a simple random variable; it can be interpreted as the conditional chance of success if π is replaced by bold play after time n. By (7.20), $F_n^*=x+tX_n$ if $F_{n-1}^*=x$ and $W_n^*=t$. Therefore,

$$Q(F_n^*)=\sum_{x,t} I_{[F_{n-1}^*=x,\,W_n^*=t]}\,Q(x+tX_n),$$

where x and t vary over the (finite) ranges of F_{n-1}^* and W_n^*, respectively.

For each x and t, the indicator above is measurable $\mathscr{F}_{n-1}$ and $Q(x+tX_n)$ is measurable $\sigma(X_n)$; since the X_n are independent, (5.25) and (5.17) give

$$E[Q(F_n^*)]=\sum_{x,t}P[F_{n-1}^*=x,\,W_n^*=t]\,E[Q(x+tX_n)] \tag{7.29}$$

By (7.28), $E[Q(x+tX_n)]\le Q(x)$ if $0\le x-t\le x\le x+t\le 1$. As it is assumed of π that F_n^* lies in $[0,1]$ (that is, $W_n^*\le\min\{F_{n-1}^*,1-F_{n-1}^*\}$), the probability in (7.29) is 0 unless x and t satisfy this constraint. Therefore,

$$\begin{aligned}E[Q(F_n^*)] &\le \sum_{x,t} P[F_{n-1}^* = x, W_n^* = t]Q(x) \\ &= \sum_x P[F_{n-1}^* = x]Q(x) = E[Q(F_{n-1}^*)].\end{aligned}$$

This is true for each n, and so $E[Q(F_n^*)] \le E[Q(F_0^*)] = Q(F_0)$. Since $Q(F_n^*) = Q(F_\tau)$ for $n \ge \tau$, Theorem 5.4 implies that $E[Q(F_\tau)] \le Q(F_0)$. Since $x = 1$ implies that $Q(x) = 1$, $P[F_\tau = 1] \le E[Q(F_\tau)] \le Q(F_0)$. Thus $Q_\pi(F_0) \le Q(F_0)$ for the policy π, whatever F_0 may be.

It remains to analyze Q and prove (7.28). Everything hinges on the functional equation

$$Q(x) = \begin{cases} pQ(2x), & 0 \le x \le \frac{1}{2}, \\ p + qQ(2x-1), & \frac{1}{2} \le x \le 1. \end{cases} \tag{7.30}$$

For $x = 0$ and $x = 1$ this is obvious because $Q(0) = 0$ and $Q(1) = 1$. The idea is this: Suppose that the initial fortune is x. If $x \le \frac{1}{2}$, the first stake under bold play is x; if the gambler is to succeed in reaching 1, he must win the first trial (probability p) and then from his new fortune $x + x = 2x$ go on to succeed (probability $Q(2x)$); this makes the first half of (7.30) plausible. If $x \ge \frac{1}{2}$, the first stake is $1 - x$; the gambler can succeed either by winning the first trial (probability p) or by losing the first trial (probability q) and then going on from his new fortune $x - (1 - x) = 2x - 1$ to succeed (probability $Q(2x - 1)$); this makes the second half of (7.30) plausible.

It is also intuitively clear that $Q(x)$ must be an increasing function of x $(0 \le x \le 1)$: the more money the gambler starts with, the better off he is. Finally, it is intuitively clear that $Q(x)$ ought to be a continuous function of the initial fortune x.

A formal proof of (7.30) can be constructed as for the difference equation (7.5). If $\beta(x)$ is x for $x \le \frac{1}{2}$ and $1 - x$ for $x \ge \frac{1}{2}$, then under bold play $W_n = \beta(F_{n-1})$. Starting from $f_0(x) = x$, recursively define

$$f_n(x; x_1, \dots, x_n) = f_{n-1}(x; x_1, \dots, x_{n-1}) + \beta(f_{n-1}(x; x_1, \dots, x_{n-1}))x_n.$$

Then $F_n = f_n(F_0; X_1, \dots, X_n)$. Now define

$$g_n(x; x_1, \dots, x_n) = \max_{0 \le k \le n} f_k(x; x_1, \dots, x_k).$$

If $F_0 = x$, then $T_n(x) = [g_n(x; X_1, \dots, X_n) = 1]$ is the event that bold play will by time n successfully increase the gambler's fortune to 1. From the recursive definition it

follows by induction on n that for $n \ge 1$, $f_n(x; x_1, \dots, x_n) = f_{n-1}(x + \beta(x)x_1; x_2, \dots, x_n)$ and hence that $g_n(x; x_1, \dots, x_n) = \max\{x, g_{n-1}(x + \beta(x)x_1; x_2, \dots, x_n)\}$. Since $x = 1$ implies $g_{n-1}(x + \beta(x)x_1; x_2, \dots, x_n) \ge x + \beta(x)x_1 = 1$, $T_n(x) = [g_{n-1}(x + \beta(x)X_1; X_2, \dots, X_n) = 1]$, and since the X_i are independent and identically distributed, $P(T_n(x)) = P([X_1 = +1] \cap T_n(x)) + P([X_1 = -1] \cap T_n(x)) = pP[g_{n-1}(x + \beta(x); X_2, \dots, X_n) = 1] + qP[g_{n-1}(x - \beta(x); X_2, \dots, X_n) = pP(T_{n-1}(x + \beta(x))) + qP(T_{n-1}(x - \beta(x)))$. Letting $n \to \infty$ now gives $Q(x) = pQ(x + \beta(x)) + qQ(x - \beta(x))$, which reduces to (7.30) because $Q(0) = 0$ and $Q(1) = 1$.

Suppose that $y = f_{n-1}(x; x_1, \dots, x_{n-1})$ is nondecreasing in x. If $x_n = +1$, then $f_n(x; x_1, \dots, x_n)$ is $2y$ if $0 \le y \le \frac{1}{2}$ and 1 if $\frac{1}{2} \le y \le 1$; if $x_n = -1$, then $f_n(x; x_1, \dots, x_n)$ is 0 if $0 \le y \le \frac{1}{2}$ and $2y - 1$ if $\frac{1}{2} \le y \le 1$. In any case, $f_n(x; x_1, \dots, x_n)$ is also nondecreasing in x, and by induction this is true for every n. It follows that the same is true of $g_n(x; x_1, \dots, x_n)$, of $P(T_n(x))$, and of $Q(x)$. Thus $Q(x)$ is nondecreasing.

Since $Q(1) = 1$, (7.30) implies that $Q(\frac{1}{2}) = pQ(1) = p$, $Q(\frac{1}{4}) = pQ(\frac{1}{2}) = p^2$, $Q(\frac{3}{4}) = p + qQ(\frac{1}{2}) = p + pq$. More generally, if $p_0 = p$ and $p_1 = q$, then

$$Q\left(\frac{k}{2^n}\right) = \sum \left[p_{u_1} \cdots p_{u_n} : \sum_{i=1}^n \frac{u_i}{2^i} < \frac{k}{2^n}\right], \qquad 0 < k \le 2^n, \quad n \ge 1, \tag{7.31}$$

the sum extending over n-tuples $(u_1, \dots, u_n)$ of 0's and 1's satisfying the condition indicated. Indeed, it is easy to see that (7.31) is the same thing as

$$Q(.u_1 \dots u_n + 2^{-n}) - Q(.u_1 \dots u_n) = p_{u_1} p_{u_2} \cdots p_{u_n} \tag{7.32}$$

for each dyadic rational $.u_1 \dots u_n$ of rank n. If $.u_1 \dots u_n + 2^{-n} \le \frac{1}{2}$, then $u_1 = 0$ and by (7.30) the difference in (7.32) is $p_0[Q(.u_2 \dots u_n + 2^{-n+1}) - Q(.u_2 \dots u_n)]$. But (7.32) follows inductively from this and a similar relation for the case $.u_1 \dots u_n \ge \frac{1}{2}$.

Therefore $Q(k2^{-n}) - Q((k-1)2^{-n})$ is bounded by $\max\{p^n, q^n\}$, and so by monotonicity Q is continuous. Since (7.32) is positive, it follows that Q is strictly increasing over $[0, 1]$.

Thus Q is continuous and increasing and satisfies (7.30). The inequality (7.28) is still to be proved. It is equivalent to the assertion that

$$\Delta(r, s) = Q(a) - pQ(s) - qQ(r) \ge 0$$

if $0 \le r \le s \le 1$, where a stands for the average: $a = \frac{1}{2}(r + s)$. Since Q is continuous, it suffices to prove the inequality for r and s of the form $k/2^n$, and this will be done by induction on n. Checking all cases disposes of $n = 0$. Assume that the inequality holds for a particular n, and that r and s have the form $k/2^{n+1}$. There are four cases to consider.

Case 1. $s \le \frac{1}{2}$. By the first part of (7.30), $\Delta(r, s) = p\Delta(2r, 2s)$. Since $2r$ and $2s$ have the form $k/2^n$, the induction hypothesis implies that $\Delta(2r, 2s) \ge 0$.

CASE 2. $\frac{1}{2} \le r$. By the second part of (7.30),

$$\Delta(r,s) = q\Delta(2r-1, 2s-1) \ge 0.$$

CASE 3. $r \le a \le \frac{1}{2} \le s$. By (7.30),

$$\Delta(r,s) = pQ(2a) - p[p + qQ(2s-1)] - q[pQ(2r)].$$

From $\frac{1}{2} \le s \le r+s = 2a \le 1$, follows $Q(2a) = p + qQ(4a-1)$; and from $0 \le 2a - \frac{1}{2} \le \frac{1}{2}$, follows $Q(2a-\frac{1}{2}) = pQ(4a-1)$. Therefore, $pQ(2a) = p^2 + qQ(2a - \frac{1}{2})$, and it follows that

$$\Delta(r,s) = q\left[Q(2a-\tfrac{1}{2}) - pQ(2s-1) - pQ(2r)\right].$$

Since $p \le q$, the right side does not increase if either of the two p's is changed to q. Hence

$$\Delta(r,s) \ge q \max[\Delta(2r, 2s-1), \Delta(2s-1, 2r)].$$

The induction hypothesis applies to $2r \le 2s-1$ or to $2s-1 \le 2r$, as the case may be, so one of the two Δ's on the right is nonnegative.

CASE 4. $r \le \frac{1}{2} \le a \le s$. By (7.30),

$$\Delta(r,s) = pq + qQ(2a-1) - pqQ(2s-1) - pqQ(2r).$$

From $0 \le 2a-1 = r+s-1 \le \frac{1}{2}$, follows $Q(2a-1) = pQ(4a-2)$; and from $\frac{1}{2} \le 2a - \frac{1}{2} = r + s - \frac{1}{2} \le 1$, follows $Q(2a - \frac{1}{2}) = p + qQ(4a-2)$. Therefore, $qQ(2a-1) = pQ(2a-\frac{1}{2}) - p^2$, and it follows that

$$\Delta(r,s) = p\left[q - p + Q(2a - \tfrac{1}{2}) - qQ(2s-1) - qQ(2r)\right].$$

If $2s - 1 \le 2r$, the right side here is

$$p[(q-p)(1-Q(2r)) + \Delta(2s-1, 2r)] \ge 0.$$

If $2r \le 2s-1$, the right side is

$$p[(q-p)(1-Q(2s-1)) + \Delta(2r, 2s-1)] \ge 0.$$

This completes the proof of (7.28) and hence of Theorem 7.3. ■

The equation (7.31) has an interesting interpretation. Let $Z_1, Z_2, \ldots$ be independent random variables satisfying $P[Z_n = 0] = p_0 = p$ and $P[Z_n = 1] = p_1 = q$. From $P[Z_n = 1 \text{ i.o.}] = 1$ and $\sum_{i>n} Z_i 2^{-i} \le 2^{-n}$ it follows that $P[\sum_{i=1}^{\infty} Z_i 2^{-i} \le k2^{-n}] \le P[\sum_{i=1}^{n} Z_i 2^{-i} < k2^{-n}] \le P[\sum_{i=1}^{\infty} Z_i 2^{-i} \le k2^{-n}]$. Since by (7.31) the middle term is $Q(k2^{-n})$,

$$Q(x) = P\left[\sum_{i=1}^{\infty} Z_i 2^{-i} \le x\right] \tag{7.33}$$

holds for dyadic rational x and hence by continuity holds for all x. In Section 31, Q will reappear as a continuous, strictly increasing function singular in the sense of Lebesgue. On p. 408 is a graph for the case $p_0 = .25$.

Note that $Q(x) \equiv x$ in the fair case $p = \frac{1}{2}$. In fact, for a bounded policy Theorem 7.2 implies that $E[F_\tau] = F_0$ in the fair case, and if the policy is to stop as soon as the fortune reaches 0 or 1, then the chance of successfully reaching 1 is $P[F_\tau = 1] = E[F_\tau] = F_0$. Thus in the fair case with initial fortune x, the chance of success is x for *every* policy that stops at the boundaries, and x is an upper bound even if stopping earlier is allowed.

Example 7.8. The gambler of Example 7.1 has capital \$900 and goal \$1000. For a fair game ($p = \frac{1}{2}$) his chance of success is .9 whether he bets unit stakes or adopts bold play. At red-and-black ($p = \frac{18}{38}$), his chance of success with unit stakes is .00003; an approximate calculation based on (7.31) shows that under bold play his chance $Q(.9)$ of success increases to about .88, which compares well with the fair case. ■

Example 7.9. In Example 7.2 the capital is \$100 and the goal \$20,000. At unit stakes the chance of successes is .005 for $p = \frac{1}{2}$ and 3×10^{-911} for $p = \frac{18}{38}$. Another approximate calculation shows that bold play at red-and-black gives the gambler probability about .003 of success, which again compares well with the fair case.

This example illustrates the point of Theorem 7.3. The gambler enters the casino knowing that he must by dawn convert his \$100 into \$20,000 or face certain death at the hands of criminals to whom he owes that amount. Only red-and-black is available to him. The question is not whether to gamble—he *must* gamble. The question is how to gamble so as to maximize the chance of survival, and bold play is the answer. ■

There are policies other than the bold one that achieve the maximum success probability $Q(x)$. Suppose that as long as the gambler's fortune x is less than $\frac{1}{2}$ he bets x for $x \le \frac{1}{4}$ and $\frac{1}{2} - x$ for $\frac{1}{4} \le x \le \frac{1}{2}$. This is, in effect, the bold-play strategy scaled down to the interval $[0, \frac{1}{2}]$, and so the chance he ever reaches $\frac{1}{2}$ is $Q(2x)$ for an initial fortune of x. Suppose further that if he does reach the goal of $\frac{1}{2}$, or if he starts with fortune at least $\frac{1}{2}$ in the first place, then he continues, but with ordinary bold play. For an initial fortune $x \geq \frac{1}{2}$, the overall chance of success is of course $Q(x)$, and for an initial fortune $x < \frac{1}{2}$, it is $Q(2x)Q(\frac{1}{2}) = pQ(2x) = Q(x)$. The success probability is indeed $Q(x)$ as for bold play, although the policy is different. With this example in mind, one can generate a whole series of distinct optimal policies.

Timid Play*

The optimality of bold play seems reasonable when one considers the effect of its opposite, timid play. Let the ϵ-*timid* policy be to bet $W_n = \min\{\epsilon, F_{n-1}, 1 - F_{n-1}\}$ and stop when F_n reaches 0 or 1. Suppose that $p < q$, fix an initial fortune $x = F_0$ with $0 \leq x < 1$, and consider what happens as $\epsilon \to 0$. By the strong law of large numbers, $\lim_n n^{-1}S_n = E[X_1] = p - q < 0$. There is therefore probability 1 that $\sup_k S_k < \infty$ and $\lim_n S_n = -\infty$. Given $\eta > 0$, choose ϵ so that $P[\sup_k(x + \epsilon S_k) < 1] > 1 - \eta$. Since $P(\bigcup_{n=1}^{\infty}[x + \epsilon S_n < 0]) = 1$, with probability at least $1 - \eta$ there exists an n such that $x + \epsilon S_n < 0$ and $\max_{k<n}(x + \epsilon S_k) < 1$. But under the ϵ-timid policy the gambler is in this circumstance ruined. If $Q_\epsilon(x)$ is the probability of success under the ϵ-timid policy, then $\lim_{\epsilon \to 0} Q_\epsilon(x) = 0$ for $0 \leq x < 1$. The law of large numbers carries the timid player to his ruin.†

PROBLEMS

7.1. A gambler with initial capital a plays until his fortune increases b units or he is ruined. Suppose that $\rho > 1$. The chance of success is multiplied by $1 + \theta$ if his initial capital is infinite instead of a. Show that $0 < \theta < (\rho^a - 1)^{-1} < (a(\rho - 1))^{-1}$; relate to Example 7.3.

7.2 As shown on p. 94, there is probability 1 that the gambler either achieves his goal of c or is ruined. For $p \neq q$, deduce this directly from the strong law of large numbers. Deduce it (for all p) via the Borel–Cantelli lemma from the fact that if play never terminates, there can never occur c successive $+1$'s.

7.3. 6.12↑ If V_n is the set of n-long sequences of ± 1's, the function b_n in (7.9) maps V_{n-1} into $\{0, 1\}$. A selection system is a sequence of such maps. Although there are uncountably many selection systems, how many have an *effective* description in the sense of an algorithm or finite set of instructions by means of which a deputy (perhaps a machine) could operate the system for the gambler? An analysis of the question is a matter for mathematical logic, but one can see that there can be only countably many algorithms or finite sets of rules expressed in finite alphabets.

Let $Y_1^{(\sigma)}, Y_2^{(\sigma)}, \ldots$ be the random variables of Theorem 7.1 for a particular system σ, and let C_σ be the ω-set where every k-tuple of ± 1's (k arbitrary) occurs in $Y_1^{(\sigma)}(\omega), Y_2^{(\sigma)}(\omega), \ldots$ with the right asymptotic relative frequency (in the sense of Problem 6.12). Let C be the intersection of C_σ over all effective selection systems σ. Show that C lies in $\mathcal{F}$ (the σ-field in the probability space $(\Omega, \mathcal{F}, P)$ on which the X_n are defined) and that $P(C) = 1$. A sequence $(X_1(\omega), X_2(\omega), \ldots)$ for ω in C is called a *collective*: a subsequence chosen by any of the effective rules σ contains all k-tuples in the correct proportions.

7.4. Let D_n be 1 or 0 according as $X_{2n-1} \neq X_{2n}$ or not, and let M_k be the time of the kth 1—the smallest n such that $\sum_{i=1}^n D_i = k$. Let $Z_k = X_{2M_k}$. In other words, look at successive nonoverlapping pairs (X_{2n-1}, X_{2n}), discard accordant $(X_{2n-1} = X_{2n})$ pairs, and keep the second element of discordant $(X_{2n-1} \neq X_{2n})$ pairs. Show that this process simulates a fair coin: $Z_1, Z_2, \ldots$ are independent and identically distributed and $P[Z_k = +1] = P[Z_k = -1] = \frac{1}{2}$, whatever p may be. Follow the proof of Theorem 7.1.

7.5. Suppose that a gambler with initial fortune 1 stakes a proportion θ $(0 < \theta < 1)$ of his current fortune: $F_0 = 1$ and $W_n = \theta F_{n-1}$. Show that $F_n = \prod_{k=1}^n (1 + \theta X_k)$ and hence that

$$\log F_n = \frac{n}{2}\left[\frac{S_n}{n}\log\frac{1+\theta}{1-\theta} + \log(1-\theta^2)\right].$$

Show that $F_n \to 0$ with probability 1 in the subfair case.

7.6. In "doubling," $W_1 = 1$, $W_n = 2W_{n-1}$, and the rule is to stop after the first win. For any positive p, play is certain to terminate. Here $F_\tau = F_0 + 1$, but of course infinite capital is required. If $F_0 = 2^k - 1$ and W_n cannot exceed F_{n-1}, the probability of $F_\tau = F_0 + 1$ in the fair case is $1 - 2^{-k}$. Prove this via Theorem 7.2 and also directly.

7.7. In "progress and pinch," the wager, initially some integer, is increased by 1 after a loss and decreased by 1 after a win, the stopping rule being to quit if the next bet is 0. Show that play is certain to terminate if and only if $p \geq \frac{1}{2}$. Show that $F_\tau = F_0 + \frac{1}{2}W_1^2 + \frac{1}{2}(\tau - 1)$. Infinite capital is required.

7.8. Here is a common martingale. Just before the nth spin of the wheel, the gambler has before him a pattern $x_1, \ldots, x_k$ of positive numbers (k varies with n). He bets $x_1 + x_k$, or x_1 in case $k = 1$. If he loses, at the next stage he uses the pattern $x_1, \ldots, x_k, x_1 + x_k$ (x_1, x_1 in case $k = 1$). If he wins, at the next stage he uses the pattern $x_2, \ldots, x_{k-1}$, unless k is 1 or 2, in which case he quits. Show

*This topic may be omitted.

†For each ϵ, however, there exist optimal policies under which the bet never exceeds ϵ; see Dubins & Savage.

that play is certain to terminate if $p > \frac{1}{3}$ and that the ultimate gain is the sum of the numbers in the initial pattern. Infinite capital is again required.

7.9. Suppose that $W_k = 1$, so that $F_k = F_0 + S_k$. Suppose that $p \ge q$ and τ is a stopping time such that $1 \le \tau \le n$ with probability 1. Show that $E[F_\tau] \le E[F_n]$, with equality in case $p = q$. Interpret this result in terms of a stock option that must be exercised by time n, where $F_0 + S_k$ represents the price of the stock at time k.

7.10. For a given policy, let A_n^* be the fortune of the gambler's adversary at time n. Consider these conditions on the policy: (i) $W_n^* \le F_{n-1}^*$; (ii) $W_n^* \le A_{n-1}^*$; (iii) $F_n^* + A_n^*$ is constant. Interpret each condition, and show that together they imply that the policy is bounded in the sense of (7.24).

7.11. Show that F_τ has infinite range if $F_0 = 1$, $W_n = 2^{-n}$, and τ is the smallest n for which $X_n = +1$.

7.12. Let u be a real function on $[0,1]$, $u(x)$ representing the *utility* of the fortune x. Consider policies bounded by 1; see (7.24). Let $Q_\pi(F_0) = E[u(F_\tau)]$; this represents the expected utility under the policy π of an initial fortune F_0. Suppose of a policy π_0 that

$$u(x) \le Q_{\pi_0}(x), \qquad 0 \le x \le 1, \tag{7.34}$$

and that

$$Q_{\pi_0}(x) \ge pQ_{\pi_0}(x+t) + qQ_{\pi_0}(x-t), \qquad 0 \le x-t \le x \le x+t \le 1. \tag{7.35}$$

Show that $Q_\pi(x) \le Q_{\pi_0}(x)$ for all x and all policies π. Such a π_0 is optimal.

Theorem 7.3 is the special case of this result for $p \le \frac{1}{2}$, bold play in the role of π_0, and $u(x) = 1$ or $u(x) = 0$ according as $x = 1$ or $x < 1$.

The condition (7.34) says that gambling with policy π_0 is at least as good as not gambling at all; (7.35) says that, although the prospects even under π_0 become on the average less sanguine as time passes, it is better to use π_0 now than to use some other policy for one step and then change to π_0.

7.13. The functional equation (7.30) and the assumption that Q is bounded suffice to determine Q completely. First, $Q(0)$ and $Q(1)$ must be 0 and 1, respectively, and so (7.31) holds. Let $T_0x = \frac{1}{2}x$ and $T_1x = \frac{1}{2}x + \frac{1}{2}$; let $f_0x = px$ and $f_1x = p + qx$. Then $Q(T_{u_1} \cdots T_{u_n}x) = f_{u_1} \cdots f_{u_n}Q(x)$. If the binary expansions of x and y both begin with the digits $u_1, \ldots, u_n$, they have the form $x = T_{u_1} \ldots T_{u_n}x'$ and $y = T_{u_1} \cdots T_{u_n}y'$. If K bounds Q and if $m = \max\{p, q\}$, it follows that $|Q(x) - Q(y)| \le Km^n$. Therefore, Q is continuous and satisfies (7.31) and (7.33).

SECTION 8. MARKOV CHAINS

As Markov chains illustrate in a clear and striking way the connection between probability and measure, their basic properties are developed here in a measure-theoretic setting.

Definitions

Let S be a finite or countable set. Suppose that to each pair i and j in S there is assigned a nonnegative number p_{ij} and that these numbers satisfy the constraint

$$\sum_{j \in S} p_{ij} = 1, \qquad i \in S. \tag{8.1}$$

Let $X_0, X_1, X_2, \ldots$ be a sequence of random variables whose ranges are contained in S. The sequence is a *Markov chain* or *Markov process* if

$$\begin{aligned} P[X_{n+1} = j \mid X_0 = i_0, \ldots, X_n = i_n] \\ = P[X_{n+1} = j \mid X_n = i_n] = p_{i_n j} \end{aligned} \tag{8.2}$$

for every n and every sequence $i_0, \ldots, i_n$ in S for which $P[X_0 = i_0, \ldots, X_n = i_n] > 0$. The set S is the *state space* or *phase space* of the process, and the p_{ij} are the *transition probabilities*. Part of the defining condition (8.2) is that the transition probability

$$P[X_{n+1} = j \mid X_n = i] = p_{ij} \tag{8.3}$$

does not vary with n.†

The elements of S are thought of as the possible states of a *system*, X_n representing the state at *time* n. The sequence or process $X_0, X_1, X_2, \ldots$ then represents the history of the system, which evolves in accordance with the probability law (8.2). The conditional distribution of the *next* state X_{n+1} given the *present* state X_n must not further depend on the *past* $X_0, \ldots, X_{n-1}$. This is what (8.2) requires, and it leads to a copious theory.

The *initial probabilities* are

$$\alpha_i = P[X_0 = i]. \tag{8.4}$$

The α_i are nonnegative and add to 1, but the definition of Markov chain places no further restrictions on them.

†Sometimes in the definition of the Markov chain $P[X_{n+1} = j \mid X_n = i]$ is allowed to depend on n. A chain satisfying (8.3) is then said to have *stationary transition probabilities*, a phrase that will be omitted here because (8.3) will always be assumed.