Singular Integrals

The Hilbert transform H, defined for $f \in S(\mathbb{R})$ by

$$Hf(x) = \lim_{\varepsilon \to 0} \frac{1}{\pi} \int_{|y| \geq \varepsilon} \frac{f(x-y)}{y}\, dy$$

arises naturally in a variety of contexts in mathematics. Perhaps most importantly, the Hilbert transform has long been understood to be a fundamental operator in Complex Analysis. More specifically, if f is an analytic function on $\{z \in \mathbb{C} : \operatorname{Im}(z) > 0\}$ with boundary values given by $u + iv$, where $u, v : \mathbb{R} \to \mathbb{R}$, & $u \in L^2(\mathbb{R})$, then H can be defined on u, and $v = Hu$.

Of more interest to us, is the Hilbert transforms connection with the theory of Fourier series & integrals in one dimension. Recall the "disc multiplier"

$$S_R f(x) = \int_{-R}^{R} \hat{f}(\xi)\, e^{2\pi i x \xi}\, d\xi .$$

By considering the multiplier representation of H,

$$\widehat{Hf}(\xi) = -i \operatorname{sgn}(\xi)\, \hat{f}(\xi)$$

one can easily verify that

$$S_R = \frac{i}{2} \{ M_{-R} H M_R - M_R H M_{-R} \} \qquad (\text{Exercise})$$

where $\quad M_R f(x) = e^{2\pi i R x} f(x)$.

Since M_R is clearly bounded on L^p, $1 \leq p \leq \infty$, with norm 1, it follows

$$S_R \text{ bounded on } L^p \iff H \text{ bounded on } L^p.$$

Calderón-Zygmund theory for singular integrals

Suppose $K: \mathbb{R}^n \setminus \{0\} \to \mathbb{C}$ satisfying

$$(*) \quad \int_{|x| \geq 2|y|} |K(x-y) - K(x)|\, dx \leq A$$

for all $y \neq 0$. Suppose T is a bounded operator on $L^2(\mathbb{R}^n)$ which commutes with translations and satisfies

$$Tf(x) = \int_{\mathbb{R}^n} K(y) f(x-y)\, dy$$

whenever $f \in S(\mathbb{R}^n)$ with $x \notin \operatorname{supp}(f)$. Such an operator is called as Calderón-Zygmund operator with Calderón-Zygmund kernel K.

Exercise: Verify that the Hilbert transform is a Calderón-Zygmund operator with Calderón-Zygmund kernel $K(x) = 1/\pi x$.

Remark:
- Condition $(*)$ is a smoothness condition on the kernel. In particular, if $|\nabla K(x)| \leq C|x|^{-n-1}$, then $(*)$ is satisfied.
- In order to make the conditions on T more explicit, we remark that the hypothesis of $L^2(\mathbb{R}^n)$ boundedness may be replaced with the size condition.

$$\text{(i)} \quad |K(x)| \leq C|x|^{-n}, \quad x \neq 0$$

along with the cancellation condition

$$\text{(ii)} \quad \sup_{0 < \varepsilon \leq N < \infty} \left| \int_{\varepsilon \leq |x| \leq N} K(x)\, dx \right| < \infty.$$

Theorem 1: If T is a Calderón-Zygmund operator, then T satisfies the weak-type $(1,1)$ inequality $\quad |\{x \in \mathbb{R}^n : |Tf(x)| > \alpha\}| \leq C \dfrac{\|f\|_1}{\alpha} \quad (**)$

for every $f \in S(\mathbb{R}^n)$ & extends to a bounded operator on $L^p(\mathbb{R}^n)$, $1 < p < \infty$.

The result here is the weak-type (1,1) estimate (**), since from this & the (assumed) boundedness on L^2 one immediately obtains the strong-type(p,p) estimates, for $1<p<\infty$, by invoking the Marcinkiewicz interpolation theorem and appealing to duality.

The proof of inequality (**), and hence Theorem 1 (by discussion above), relies on the following basic decomposition lemma due to Calderón & Zygmund for L^1 functions. The proof is an example of a "stopping time argument".

<u>Lemma</u>: Let $f \in L^1(\mathbb{R}^n)$ & $\alpha > 0$. Then one can decompose f as

$$f = g + b$$

where

(i) $|g(x)| \leq C\alpha$ a.e. x

and $b = \sum_Q b_Q$ where the sum runs over a collection $\mathcal{B} = \{Q\}$ of (essentially) disjoint "bad" cubes such that for each Q one has

$$\text{(ii)} \quad \operatorname{supp}(b_Q) \subseteq Q, \quad \frac{1}{|Q|}\int_Q |b_Q(x)|\,dx \leq C\alpha, \quad \int_Q b_Q(x)\,dx = 0.$$

Furthermore,

$$\text{(iii)} \quad \Big|\bigcup_{Q\in\mathcal{B}} Q\Big| \leq \frac{C}{\alpha}\|f\|_1 .$$

<u>Proof</u> Since $f \in L^1(\mathbb{R}^n)$, we may decompose $\mathbb{R}^n$ into a mesh of equal cubes Q', whose interiors are disjoint, and whose sides are large enough to ensure

$$\frac{1}{|Q'|}\int_{Q'} |f(x)|\,dx \leq \alpha$$

for every cube Q'. We now subject every cube Q' to the following process. By bisecting each of the sides of Q' we decompose it into 2^n subcubes Q''. If such a subcube satisfies

$$(***) \quad \frac{1}{|Q''|}\int_{Q'} |f(x)|\,dx > \alpha$$

we select it to be one of the cubes Q in collection $\mathcal{B}$.

Note that for such a subcube Q'',

$$\alpha < \frac{1}{|Q''|}\int_{Q''}|f(x)|\,dx \leq \frac{2^n}{|Q'|}\int_{Q'}|f(x)|\,dx \leq 2^n\alpha.$$

If a subcube Q'' fails to satisfy property $(***)$, we subdivide it again into 2^n further subcubes; and repeat the above selection process.
Continuing in the fashion produces a collection $\mathcal{B}=\{Q\}$ of cubes s.t

$$\alpha < \frac{1}{|Q|}\int_Q |f| \leq 2^n\alpha.$$

Consequently (iii) also holds, since

$$\Big|\bigcup_{Q\in\mathcal{B}} Q\Big| \leq \sum_{Q\in\mathcal{B}}|Q| < \sum_{Q\in\mathcal{B}}\frac{1}{\alpha}\int_Q |f| \leq \frac{1}{\alpha}\|f\|_1 .$$

Now let $x_0 \in \mathbb{R}^n \setminus \bigcup_{Q\in\mathcal{B}}\overline{Q}$. Then x_0 is contained in a decreasing sequence $\{Q_j\}$ of dyadic cubes, each of which satisfy

$$\frac{1}{|Q_j|}\int_{Q_j}|f(x)|\,dx \leq \alpha.$$

Hence, by the Lebesgue differentiation theorem, $|f(x_0)|\leq\alpha$ for a.e. such x_0.

Set

$$b_Q = \Big(f - \frac{1}{|Q|}\int_Q f\Big)\chi_Q$$

&

$$g = f - \sum_{Q\in\mathcal{B}} b_Q .$$

It follows that $|g(x)| \leq 2^n\alpha$ a.e x

& $\int_Q b_Q = 0 \quad \forall\, Q\in\mathcal{B}$.

& $\frac{1}{|Q|}\int_Q |b_Q| \leq 2^{n+1}\alpha$. $\square$

Proof of Theorem 1: Recall that our task is the verification that

$$|\{x \in \mathbb{R}^n : |Tf(x)| > \alpha\}| \leq C \frac{1}{\alpha} \|f\|_1$$

for every $f \in S(\mathbb{R}^n)$.

Let $f \in S(\mathbb{R}^n)$. We use our Calderón-Zygmund decomposition (Lemma) & write

$$f = g + \sum b_Q.$$

By the triangle inequality,

$$|\{x : |Tf(x)| > \alpha\}| \leq |\{x : |Tg(x)| > \tfrac{\alpha}{2}\}| + |\{x : |T(\sum b_Q)(x)| > \tfrac{\alpha}{2}\}|$$

so suffices to show that both terms on RHS are dominated by $C\|f\|_1/\alpha$.

- By Chebyshev, the L^2 boundedness of T and the fact that

  $$\|g\|_2^2 \leq 2^n \alpha \|f\|_1$$

  it follows that

  $$|\{x : |Tg(x)| > \tfrac{\alpha}{2}\}| \leq \left(\frac{2\|Tg\|_2}{\alpha}\right)^2 \leq C \frac{\|g\|_2^2}{\alpha^2} \leq C \frac{\|f\|_1}{\alpha}.$$

- Let y_Q denote the center of Q & $\Omega^* = \bigcup(2Q)$. Since

  $$|\Omega^*| \leq C 2^n \|f\|_1/\alpha \qquad \text{(by Lemma (iii))}$$

  it suffices to show

  $$|\{x \in \mathbb{R}^n \setminus \Omega^* : |T(\sum_Q b_Q)(x)| > \tfrac{\alpha}{2}\}| \leq C \|f\|_1/\alpha.$$

  Since

  Using $\int b_Q = 0$

  $$\int_{\mathbb{R}^n \setminus \Omega^*} |T(\sum_Q b_Q)(x)|\,dx = \int_{\mathbb{R}^n \setminus \Omega^*} \Big| \sum_Q \int_Q b_Q(y)\{K(x-y) - K(x-y_Q)\}\,dy \Big|\,dx$$

  $$\leq \sum_Q \int_Q |b_Q(y)| \int_{\mathbb{R}^n \setminus 2Q} |K(x-y) - K(x-y_Q)|\,dx\,dy$$

  $$\leq C\|f\|_1 \quad \text{(by Lemma (iii) \& (ii) \& } (*)\text{)}$$

  Chebyshev $\Rightarrow$ $|\{x : |T(\sum b_Q)(x)| \geq \tfrac{\alpha}{2}\}| \leq C\|f\|_1/\alpha$. $\square$

Further remarks on convergence of "partial Fourier integrals"

Let $S_{(0,\infty)}$ be the operator, defined for $f \in S(\mathbb{R}^*)$ by

$$\widehat{S_{(0,\infty)} f}(\xi) = \chi_{(0,\infty)}(\xi)\hat{f}(\xi).$$

As before, this has the equivalent expression: $S_{(0,\infty)} = \frac{I + iH}{2}$

It thus follows that $S_{(0,\infty)}$ is a bounded operator on $L^p(\mathbb{R})$, $1 < p < \infty$.

Exercise: Show that the L^p boundedness of $S_{(0,\infty)}$ implies the boundedness on $L^p(\mathbb{R}^n)$ of the operator S_+ defined by

$$\widehat{S_+ f}(\xi) = \chi_{\{\xi \in \mathbb{R}^n : \xi_n > 0\}}(\xi)\,\hat{f}(\xi).$$

Theorem: Let $P \subseteq \mathbb{R}^n$ be any convex polyhedron containing the origin. If we define, for $f \in S(\mathbb{R}^n)$ the operator S_P by

$$\widehat{S_P f}(\xi) = \chi_P(\xi)\hat{f}(\xi)$$

then it follows that S_P is bounded on $L^p(\mathbb{R}^n)$, $1 < p < \infty$.

⚠ In light of this observation, it is perhaps surprising that when $n > 1$, the disc multiplier $\widehat{S_B f} = \chi_B \hat{f}$, with B = unit ball, is only bounded on $L^p(\mathbb{R}^n)$ when $p = 2$ ⚠

Proof

One can clearly write χ_P as a finite product of characteristic functions of half spaces, each of which is simply a translation and rotation of the half space $\{\xi \in \mathbb{R}^n : \xi_n > 0\}$. Hence, $S_P = \prod_{j=1}^{m} R_j T_j S_+ T_j^{-1} R_j^{-1}$ where R_j & T_j are, for each $1 \le j \le m$, a specific rotation & translation. $\square$