Strongly Singular Integrals along Curves in $\mathbb{R}^d$

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joint work with:

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Taking, for example, $\gamma(t) = (t, |t|^2, \ldots, |t|^d)$ it is easy to see that this operator is unbounded on $L^2(\mathbb{R}^d)$.

Instead consider the following **strongly singular** analogues

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- We shall say a smooth mapping $\gamma(t)$ is *well-curved* if $\gamma(0) = 0$ and

$$\left.\frac{d^k\gamma(t)}{dt^k}\right|_{t=0}, \quad k = 1, 2, \ldots, \ \text{spans } \mathbb{R}^d.$$

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- To every smooth *well-curved* mapping $\gamma(t)$ there exists a constant nonsingular matrix M such that

$$\widetilde{\gamma}(t) = M\gamma(t),$$

is of *standard type*, that is

$$\widetilde{\gamma}_k(t) = t^{a_k}/a_k! + \text{ higher order terms}$$

with $1 \leq a_1 < a_2 < \cdots < a_d$.

Strategy: Consider the rescaled dyadic operators

$$T_j f(x) = 2^{j\alpha} \int \vartheta(t) t^{-1} |t|^{-\alpha} e^{i 2^{j\beta} |t|^{-\beta}} f(x - \gamma(2^{-j} t))\, dt$$

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- <u>Dyadic Estimate</u>

$$\|T_j f\|_{L^2(\mathbb{R}^d)} \leq C 2^{j(\alpha - \beta/(d+1))} \|f\|_{L^2(\mathbb{R}^d)}$$

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- <u>Almost Orthogonality</u> If $\alpha \leq \beta/(d+1)$, then

$$\|T_j^* T_{j'}\|_{L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)} \leq C 2^{-\delta|j-j'|},$$

for some $\delta > 0$.

Establishing L^2 estimates for T_j is equivalent to establishing *uniform* bounds for the multipliers

$$m_j(\xi) = 2^{j\alpha} \int \vartheta(t) t^{-1} |t|^{-\alpha} e^{i2^{j\beta}[|t|^{-\beta} - 2^{-j\beta}\gamma(2^{-j}t)\cdot\xi]} dt$$

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  $$\varphi(t) = |t|^{-\beta} - (\mu_1 t^{a_1} + \cdots + \mu_d t^{a_d}),$$

  where

  $$\mu = 2^{-j} \circ \xi = (2^{-j(\beta+a_1)}\xi_1, \dots, 2^{-j(\beta+a_d)}\xi_d)$$

Lemma. *Let* $\epsilon \le a < b \le \epsilon^{-1}$*, for some* $\epsilon > 0$*, and*

$$\varphi(t) = t^{b_0} + \mu_1 t^{b_1} + \cdots + \mu_n t^{b_n}$$

where $b_0, b_1, \ldots, b_n$ *are distinct nonzero reals, then*

$$\left| \int_a^b e^{i\lambda\varphi(t)} dt \right| \le C\lambda^{-1/(n+1)},$$

with C *independent of* $\mu_1, \ldots, \mu_n$*, and* λ*.*

Lemma. *Let $\epsilon \leq a < b \leq \epsilon^{-1}$, for some $\epsilon > 0$, and*

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where $b_0, b_1, \ldots, b_n$ are distinct <u>nonzero</u> reals, then

$$\left| \int_a^b e^{i\lambda\varphi(t)} dt \right| \leq C\lambda^{-1/(n+1)},$$

with C <u>independent</u> of $\mu_1, \ldots, \mu_n$, and λ.

- Key to proof: Show $\exists\, C_1$ so that for each $t \in [a, b]$

$$|\varphi^{(k)}(t)| \geq C_1 t^{b_0 - k}$$

for at least one $k = 1, \ldots, n + 1$.

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applying the Lemma (with $b_0 = -\beta$ and $b_k = a_k$ for $k = 1, \ldots, d$) gives that

$$|m_j(\xi)| = 2^{j\alpha} \left| \int \psi(t)\, e^{i2^{j\beta}[|t|^{-\beta} - (2^{-j} \circ \xi)\cdot\gamma(t)]} dt \right|$$
$$\leq C 2^{j(\alpha - \beta/(d+1))}$$

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- <u>Necessity:</u> $\exists$ constants $c_1, \dots, c_d$, such that

$$m_j(c_1\xi_1, c_2\xi_1^{\frac{\beta+a_2}{\beta+a_1}}, \dots, c_d\xi_1^{\frac{\beta+a_d}{\beta+a_1}}) \sim 2^{j(\alpha-\beta/(d+1))}$$

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- If $|2^{-j} \circ \xi| \gg 1$, then $|t|^{-\beta}$ is subordinate and

$$|m_j(\xi)| \le C2^{j(\alpha-\beta/d)}|2^{-j} \circ \xi|^{-1/d}$$

(apply Lemma with $\varphi(t) = \sum_{k=1}^{d} \mu_k t^{a_k}$)

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with $1 \leq a_1 < a_2 < \cdots < a_d$.
- The corresponding phase function is then

$$\varphi(t) = |t|^{-\beta} + \max_{1\leq k\leq d} 2^{-j(\beta+a_k)}|\xi_k| \sum_{k=1}^{d} \mu_k t^{a_k}(1+O(2^{-j}t))$$

with $|\mu_k| \leq 1$ for all $k = 1, \ldots, d$.

Proposition (Dyadic estimates). *If* $\gamma(t)$ *is a curve of* standard type, *then*

(i) *for all* $\xi \in \mathbb{R}^d$

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Proposition (Almost Orth). *If* $\alpha \le \beta/(d+1)$, *then*

$$\|T_j^* T_{j'}\|_{L^2(\mathbb{R}^d)\to L^2(\mathbb{R}^d)} \le C2^{-\delta|j-j'|},$$

where $\delta = (\beta + a_1)/d$.

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and hence

$$\begin{aligned} |m_j(\xi)| &\leq C2^{j(\alpha-\beta/d)}|2^{-j} \circ \xi|^{-1/d} \\ &\leq C|2^{j'-j}2^{-j'} \circ \xi|^{-1/d} \\ &\leq C2^{-(j'-j)(\beta+a_1)/d}|2^{-j'} \circ \xi|^{-1/d} \\ &\leq C\epsilon^{-1/d}2^{-(j'-j)(\beta+a_1)/d} \end{aligned}$$

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In particular we have that $\|T_j\|_{1\to 1} \leq C$ if $\alpha = 0$ and by interpolation it then follows that for $1 < p < \infty$

$$\left| \frac{1}{p} - \frac{1}{2} \right| < \frac{\beta - (d+1)\alpha}{2\beta} \implies T : L^p(\mathbb{R}^d) \to L^p(\mathbb{R}^d)$$