Basic Theory of L^p Spaces

Given $f:\mathbb{R}^n \to \mathbb{C}$ measurable and $0<p<\infty$, we define

$$\|f\|_p := \left(\int_{\mathbb{R}^n} |f(x)|^p\,dx\right)^{1/p}$$

(allowing the possibility that $\|f\|_p=\infty$), and we define

$$L^p(\mathbb{R}^n) = \{f:\mathbb{R}^n\to\mathbb{C} : f \text{ measurable and } \|f\|_p<\infty\}.$$

More generally, given $X\subseteq\mathbb{R}^n$ measurable we define

$$L^p(X) = L^p(\mathbb{R}^n)\cap\{f:X\to\mathbb{C} : f \text{ measurable}\}.$$

We will abbreviate $L^p(X)$ by simply L^p when this causes no confusion. As we have done with L^1, we consider two functions to define the same element of L^p when equal almost everywhere.

- L^p is a vector space: If $f,g\in L^p$, then $f+g\in L^p$ since

$$|f+g|^p \le (2\max\{|f|,|g|\})^p \le 2^p(|f|^p+|g|^p).$$

- Our notation suggests that $\|\cdot\|_p$ is a norm, is it?

Obvious that (i) $\|f\|_p=0 \iff f=0$ a.e.

(ii) $\|cf\|_p = |c|\,\|f\|_p$.

¿ But what about the Δ-inequality?

⚠ Only true if $p \geq 1$ ⚠

Minkowski's Inequality

If $1 \leq p < \infty$ and $f, g \in L^p$, then $\|f+g\|_p \leq \|f\|_p + \|g\|_p$.

Key to proving this result is the incredibly useful and important

Hölder's Inequality: Let $1 < p < \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$ (that is $q = p/p-1$). If f and g are measurable functions on $X \subseteq \mathbb{R}^n$. then

$$\|fg\|_1 \leq \|f\|_p \|g\|_q . \qquad (*)$$

In particular, if $f \in L^p$ and $g \in L^q$, then $fg \in L^1$, and in this case equality holds above iff $\alpha |f|^p = \beta |g|^q$ a.e. for some $\alpha, \beta \in \mathbb{C}$. (with $\alpha\beta \neq 0$.)

The proof of Hölder's inequality relies on the following generalization of the usual arithmetic - geometric mean inequality:

Lemma: If $a, b \geq 0$ and $0 < \lambda < 1$, then

$$a^\lambda b^{1-\lambda} \leq \lambda a + (1-\lambda) b ,$$

with equality iff $a = b$.

Proof of Lemma : Result obvious if $b=0$; otherwise, by dividing both sides by b and setting $t = a/b$ we see that matters reduces to showing that $\quad t^\lambda \leq \lambda t + (1-\lambda) \quad$ with equality iff $t=1$.

By elementary calculus, $t^\lambda - \lambda t$ is strictly increasing for $t<1$ and strictly decreasing for $t>1$, so its minimum value, namely $1-\lambda$, occurs at $t=1$. $\square$

Proof of Hölder's Inequality.

It suffice to establish that (*) holds when $\|f\|_p = \|g\|_q = 1$ (Check!) with equality iff $|f|^p = |g|^q$. To this end we apply the Lemma above with $a = |f(x)|^p$ and $b = |g(x)|^q$ and $\lambda = 1/p$ to obtain

$$|f(x)||g(x)| \leq \frac{|f(x)|^p}{p} + \frac{|g(x)|^q}{q}.$$

Integrating both sides yields

$$\|fg\|_1 \leq \frac{1}{p}\int |f|^p + \frac{1}{q}\int |g|^q = \frac{1}{p} + \frac{1}{q} = 1 = \|f\|_p \|g\|_q . \quad \square$$

* The condition $\frac{1}{p} + \frac{1}{q} = 1$ occurs frequently in L^p theory. If $1<p<\infty$, then the number $q = p/p-1$ is called the conjugate exponent to p.

Proof of Minkowski's Inequality.

- Result obvious if $p=1$ or if $f+g=0$ a.e.
- Otherwise we write

$$|f+g|^p \leq (|f|+|g|)|f+g|^{p-1}$$

and apply Hölder's inequality, noting that $(p-1)q = p$ when q is the conjugate exponent to p:

$$\int |f+g|^p \leq \|f\|_p \, \| |f+g|^{p-1} \|_q + \|g\|_p \, \| |f+g|^{p-1} \|_q$$

$$= \left(\|f\|_p + \|g\|_p\right)\left(\int |f+g|^p\right)^{1/q}.$$

Therefore,

$$\|f+g\|_p = \left(\int |f+g|^p\right)^{1-1/q} \leq \|f\|_p + \|g\|_p.$$

$\square$.

Minkowski's Inequality shows that, for $p \geq 1$, L^p is a normed vector space. More is true:

Theorem. For $1 \leq p < \infty$, L^p is a complete normed vector space (i.e. a Banach Space).

Proof (essentially the same as for $p=1$).

- Let $\{f_n\}_{n=1}^{\infty}$ be a Cauchy sequence in L^p, and consider a subsequence $\{f_{n_k}\}_{k=1}^{\infty}$ of $\{f_n\}$ with the property that $\|f_{n_{k+1}} - f_{n_k}\|_p \leq 2^{-k}$ for all $k \geq 1$.

- We now consider the following series (whose convergence is shown below)

$$f(x) := f_{n_1}(x) + \sum_{k=1}^{\infty} \left(f_{n_{k+1}}(x) - f_{n_k}(x) \right)$$

truncated at K.

and

$$g(x) := |f_{n_1}(x)| + \sum_{k=1}^{\infty} |f_{n_{k+1}}(x) - f_{n_k}(x)|.$$

and the corresponding partial sums $S_K(f)(x)$ & $S_K(g)(x)$.

- Minkowski's inequality implies that

$$\|S_K(g)\|_p \leq \|f_{n_1}\|_p + \sum_{k=1}^{K} \|f_{n_{k+1}} - f_{n_k}\|_p \leq \|f_{n_1}\|_p + \sum_{k=1}^{K} 2^{-k}.$$

Letting $K \to \infty$ and applying the MCT gives that $\int g^p < \infty$, and therefore that the series defining g (and hence also the series defining f) converges almost everywhere and $f \in L^p$.

- We now show that f is the desired limit of our sequence $\{f_n\}$.

Since, by the construction of the telescoping series,

$$S_{K-1}(f) = f_{n_K}. \Rightarrow f_{n_K} \to f \text{ a.e. as } K \to \infty.$$

To prove that $f_{n_k} \to f$ in L^p as well, we observe that

$$|f(x) - f_{n_k}(x)|^p \leq 2^p \max\{|f(x)|^p, |S_{k+1}(f)|^p\}$$
$$\leq 2^{p+1} |g(x)|^p \qquad \text{for all } k \geq 1.$$

DCT $\Rightarrow$ $f_{n_k} \to f$ in L^p as $k \to \infty$.

- Finally, we have to show that $f_n \to f$ in L^p as $n \to \infty$. We use the fact that $\{f_n\}$ was assumed to be Cauchy:

  Given $\varepsilon > 0$, $\exists$ N s.t for all $n, m > N$ we have

  $$\|f_m - f_n\|_p < \varepsilon/2.$$

  If n_k is chosen so that $n_k > N$ <u>and</u> $\|f_{n_k} - f\|_p < \varepsilon/2$, it follows that

  $$\|f_n - f\|_p \leq \|f_n - f_{n_k}\|_p + \|f_{n_k} - f\|_p < \varepsilon$$

  whenever $n > N$. $\square$

The Space L^∞

To complete the picture of L^p spaces, we introduce a space corresponding to the limiting value $p=\infty$.

If f is a measurable function on $\mathbb{R}^n$, we define

$\|f\|_\infty$ is called the essential supremum of $|f|$

$$\|f\|_\infty := \inf\{a \geq 0 : m(\{x : |f(x)| > a\}) = 0\},$$

with the convention that $\inf \phi = \infty$.

($\|f\|_\infty$ is the smallest $M \in \overline{\mathbb{R}}$ such that $|f(x)| \leq M$ a.e.)

We now define

$$L^\infty(\mathbb{R}^n) = \{f : \mathbb{R}^n \to \mathbb{C} : f \text{ measurable and } \|f\|_\infty < \infty\}.$$

with usual convention that two functions that are equal a.e. define the same element of L^∞ and given $X \subseteq \mathbb{R}^n$ measurable

$$L^\infty(X) = L^\infty(\mathbb{R}^n) \cap \{f : X \to \mathbb{C} : f \text{ measurable}\}.$$

The results we proved for $1 \leq p < \infty$ extend easily to the case $p=\infty$:

In light of this and "$\frac{1}{1} + \frac{1}{\infty} = 1$" we shall regard 1 & ∞ as conj. exponents of each other

Theorem

- (Hölder) If f, g m'ble, then $\|fg\|_1 \leq \|f\|_1 \|g\|_\infty$
- $\|\cdot\|_\infty$ is a norm & L^∞ a Banach Space.

Proof (Exercise).

The fact that L^∞ is a limiting case of L^p as $p \to \infty$ can be understood as follows:

<u>Theorem</u>: If $X \subseteq \mathbb{R}^n$ with $m(X) < \infty$, then $\|f\|_p \to \|f\|_\infty$ as $p \to \infty$.

<u>Proof</u>

Let $M = \|f\|_\infty$. If $L < M$, then $m(\underbrace{\{x : |f(x)| > L\}}_{=E}) > 0$.

Hence

$$\|f\|_p \geq \left(\int_E |f|^p\right)^{1/p} \geq L\, m(E)^{1/p} \to L \quad \text{as } p \to \infty.$$

Therefore

$$\liminf_{p \to \infty} \|f\|_p \geq M.$$

Since

$$\|f\|_p \leq \left(\int_X M^p\right)^{1/p} = M\, m(X)^{1/p} \to M \quad \text{as } p \to \infty$$

$$\Rightarrow \quad \limsup_{p \to \infty} \|f\|_p \leq M. \qquad \square$$

Note that this is simply the proof of the essentially trivial

<u>Tchebychev's Inequality</u> (L^p-version)

If $0 < p < \infty$ and $f \in L^p(\mathbb{R}^n)$, then for any $\alpha > 0$

$$m(\{x : |f(x)| > \alpha\}) \leq \frac{1}{\alpha^p} \|f\|_p^p.$$