Oscillatory Integrals

Object of study,

$$I(\lambda) = \int_{\mathbb{R}^n} e^{i\pi\lambda\varphi(x)} \psi(x)\, dx$$

where $\lambda > 0$, $\varphi \in C^\infty(\mathbb{R}^n)$ and real-valued & $\psi \in C_c^\infty(\mathbb{R}^n)$.

In particular, we are interested in the (asymptotic) behaviour of $I(\lambda)$ as $\lambda \to \infty$

Propn 1 (Principle of Non-Stationary Phase)

If $\nabla\varphi \neq 0$ on the support of ψ, then $I(\lambda) = O(\lambda^{-N})$ for any N as $\lambda \to \infty$.

Thus, the asymptotic behaviour of $I(\lambda)$ is determined by those points where $\nabla\varphi(x_0) = 0$. If, at a critical point x_0, we also have $H\varphi(x_0) := \frac{\partial^2\varphi}{\partial x_i \partial x_j}(x_0)$ is invertible, we say the critical point is non-degenerate. It is easy to see that non-deg critical pts are isolated.

Propn 2 (Principle of Stationary Phase)

If φ has a non-deg critical pt at x_0 & ψ is supported in a sufficiently small neighbourhood of x_0, then

$$I(\lambda) = \lambda^{-\frac{n}{2}} e^{i\pi\sigma/4} \left[\frac{2^n}{|\det H\varphi(x_0)|}\right]^{1/2} \psi(x_0) e^{i\pi\lambda\varphi(x_0)} + O(\lambda^{-\frac{n+2}{2}})$$

as $\lambda \to \infty$, where $\sigma = r - (n-r) = 2r - n$ & $r = \#$ +ve eigenvalues of $H\varphi(x_0)$.

Important Observation: Any bound for the rate of decay of $I(\lambda)$ which is independent of ψ will be "diffeomorphism invariant".

Suppose $\varphi_1 = \varphi_2 \circ G$ where G is a smooth diffeomorphism.
Then

$$\int e^{i\pi\lambda\varphi_2(x)}\psi(x)\,dx = \int e^{i\pi\lambda\varphi_1(G^{-1}x)}\psi(x)\,dx$$

$$= \int e^{i\pi\lambda\varphi_1(y)}\psi(Gy)|J_G(y)|\,dy$$

where J_G is the Jacobian determinant. Note that the function

$$\tilde{\psi}(y) = \psi(Gy)|J_G(y)|$$

is again in C_c^∞.

Recall the following standard lemma, concerning the normal forms for a function near a regular pt or a non-deg critical point:

Lemma Let $\varphi : \mathbb{R}^n \to \mathbb{R}$ be smooth.

(i) [Straightening Lemma] If $\nabla\varphi(x_0) \neq 0$, then $\exists$ nbds U of x_0 & V of 0 and diffeo $G: V \to U$ with $G(0) = x_0$ s.t. $\varphi \circ G(x) = \varphi(x_0) + x_n$

(ii) [Morse Lemma] If x_0 is a non-deg critical pt of φ, then $\exists$ $1 \le r \le n$ and nbds U & V of x_0 & 0 resp. together with a diffeo $G: V \to U$ with $G(0) = x_0$ such that $\varphi \circ G(x) = \varphi(x_0) + x_1^2 + \cdots + x_r^2 - x_{r+1}^2 - \cdots - x_n^2$.

($r = \#$ +ve eigenvalues of $H\varphi(x_0)$).

This observation allows us to deduce Propns 1 & 2 from the following special model cases:

Propn 1$'$ If $J(\lambda)=\int_{\mathbb{R}^n} e^{i\pi\lambda x_n}\psi(x)\,dx$, then $J(\lambda)=O(\lambda^{-N})$ for any N as $\lambda\to\infty$

Propn 2$'$ If $K(\lambda)=\int_{\mathbb{R}^n} e^{i\pi\lambda(x_1^2+\cdots+x_r^2-x_{r+1}^2-\cdots-x_n^2)}\psi(x)\,dx$, then

$$K(\lambda)=\lambda^{-n/2}e^{i\pi\sigma/4}\psi(0)+O(\lambda^{-\frac{n+2}{2}}) \quad \text{as } \lambda\to\infty$$

where $\sigma=2r-n$.

Proof of Propn 1$'$: Immediate from fact that the Fourier transform of a Schwartz function is again a Schwartz function.

Proof of Propn 2$'$: We note first that (by Plancherel)

$$(*)\quad \int e^{-\pi z x^2}\psi(x)\,dx = z^{-1/2}\int e^{-\pi x^2/z}\hat{\psi}(x)\,dx \quad \text{for all } z>0.$$

Since both sides of $(*)$ are analytic in $\mathrm{Re}(z)>0$ & conts when $\mathrm{Re}(z)\geq 0$, $z\neq 0$ (Ex) it follows, by analytic continuation that $(*)$ in fact holds for all $\mathrm{Re}(z)\geq 0$, $z\neq 0$!

In particular,

$$(**)\quad \int e^{\pm i\pi\lambda x^2}\psi(x)\,dx=\lambda^{-1/2}e^{\pm i\pi/4}\int e^{\pm i\pi x^2/\lambda}\hat{\psi}(x)\,dx, \quad \lambda>0.$$

Using the fact that
$$\psi(0)=\int\hat{\psi}(x)\,dx$$
(a corollary of Fourier inversion), it follows from $(**)$ that

$$(***)\quad \int e^{\pm i\pi\lambda x^2}\psi(x)\,dx=\lambda^{-1/2}e^{\pm i\pi/4}\psi(0)+O(\lambda^{-3/2})$$

since $\int |e^{\pm i\pi x^2/\lambda}-1|\,|\hat{\psi}(x)|\,dx \leq C\lambda^{-1}\int |x|^2|\hat{\psi}(x)|\,dx\leq C\lambda^{-1}$.

Propn 2$'$ follow from $(***)$ & the fact that linear combinations of C_c^∞ tensor functions are dense in $C_c^\infty(\mathbb{R}^n)$ [Immediate from fact that Trig polys on $\mathbb{T}^n$ are dense in $C(\mathbb{T}^n)$.]

Proof of Propn 1

We first work locally.

Let $p_1 \in \mathrm{supp}(\psi)$ & U_1, V_1 be the neighbourhoods featured in Lemma (i) & $G_1 : V_1 \to U_1$ be the diffeomorphism.

Let $\psi_1 : \mathbb{R}^n \to \mathbb{R}$ be smooth and supported on U_1, then

$$\int_{U_1} e^{i\pi\lambda\varphi(x)}\psi_1(x)\,dx = e^{i\pi\lambda\varphi(p_1)}\int_{V_1} e^{i\pi\lambda x_n}\psi_1(Gx)\,|J_G(x)|\,dx$$

$$= O(\lambda^{-N}) \quad \text{by Propn } 1'.$$

Local → Global:

Choose a finite collection of points $p_1, \dots, p_m$ such that the corresponding neighborhoods U_j cover $\mathrm{supp}(\psi)$.
Take a partition of unity $\{\eta_j\}$ relative to $\{U_j\}$ so that

(i) $\eta_j \in C^\infty$
(ii) $\mathrm{supp}(\eta_j) \subseteq U_j$
(iii) $\sum_{j=1}^{m} \eta_j(x) = 1$ on $\bigcup_{j=1}^{m} U_j$

Set $\psi_j = \psi\eta_j$ & Propn 1 follows from the observation that

$$I(\lambda) = \sum_{j=1}^{m} \int_{U_j} e^{i\pi\lambda\varphi(x)}\psi_j(x)\,dx. \qquad \square$$

Proof of Propn 2

Lemma (ii) implies $\exists$ nbds $\mathcal{U}$ & V of x_0 & 0 respectively, together with a diffeomorphism $G: V \to \mathcal{U}$ with $G(0) = x_0$ such that

$$\phi(Gx) = \phi(x_0) + x_1^2 + \cdots + x_r^2 - x_{r+1}^2 - \cdots - x_n^2 .$$

Suppose ψ is supported on $\mathcal{U}$. Again

$$\int_{\mathcal{U}} e^{2\pi\lambda\phi(x)} \psi(x)\,dx = e^{i\pi\lambda\phi(x_0)} \int_V e^{i\pi\lambda(x_1^2+\cdots+x_r^2-x_{r+1}^2-\cdots-x_n^2)} \psi(Gx)|J_G(x)|\,d$$

$$= e^{i\pi\lambda\phi(x_0)} e^{i\pi\sigma/4} \lambda^{-n/2} \psi(x_0)\,|J_G(0)| + O\left(\lambda^{-\frac{n+2}{2}}\right)$$

$\uparrow$

Propn 2'.

We will therefore be done if $|J_G(0)| = \left[\dfrac{2^n}{|\det H\phi(x_0)|}\right]^{1/2}$.

This is a consequence of the Chain Rule & fact that $\nabla\phi(x_0) = 0$:

$$H(\phi\circ G)(0) = DG(0)^t \, H\phi(x_0)\, \underbrace{DG(0)}_{\text{Jacobian of } G} \qquad (*)$$

$$\Rightarrow |\det H(\phi\circ G)(0)| = |J_G(0)|^2 \, |\det H\phi(x_0)|$$

Result follows since $|\det H(\phi\circ G)(0)|$ clearly equals 2^n. $\square$

Note: $(*) \Rightarrow \sigma$ is a diffeomorphism invariant.