Math 4110/6110

Problem Set 5: The Lebesgue Integral

1. (a) Give an example of a continuous function f in $L^1(\mathbb{R})$ for which $f(x) \nrightarrow 0$ as $|x| \to \infty$.
   (b) Prove that if $f \in L^1(\mathbb{R})$ and uniformly continuous, then $\lim_{|x|\to\infty} f(x) = 0$.

2. Suppose $f \geq 0$, and let $E_{2^k} = \{x : f(x) > 2^k\}$ and $F_k = \{x : 2^k < f(x) \leq 2^{k+1}\}$. If f is finite almost everywhere, then $\bigcup_{k=-\infty}^{\infty} F_k = \{f(x) > 0\}$, and the sets F_k are disjoint. Prove that

$$\int |f(x)| < \infty \iff \sum_{k=-\infty}^{\infty} 2^k m(F_k) < \infty \iff \sum_{k=-\infty}^{\infty} 2^k m(E_{2^k}) < \infty.$$

3. Let $\{f_n\}$ be a sequence of measurable functions on $\mathbb{R}$ such that $\lim_{n\to\infty} f_n(x) = g(x)$ a.e. in $\mathbb{R}$,

$$\lim_{n\to\infty} \int |f_n(x)|\, dx = A \qquad \text{and} \qquad \int |g(x)|\, dx = B.$$

   (a) Prove that

$$\lim_{n\to\infty} \int |f_n(x) - g(x)|\, dx = A - B.$$

   (b) Give an example of a sequence $\{f_n\}$ of such functions for which $A \neq B$.

4. Suppose $\{f_k\}$ is a sequence in L^1 and $f \in L^1$ and $f_k \to f$ almost everywhere. Prove that

$$\int |f - f_k| \to 0 \quad \iff \quad \int |f_k| \to \int |f|.$$

5. (a) Suppose that $f(x)$ and $xf(x)$ are both integrable functions on $\mathbb{R}$. Prove that the function

$$F(t) = \int_{\mathbb{R}} f(x)\cos(tx)\, dx.$$

   is differentiable at every t and find a formula for $F'(t)$.
   (b) Giving complete justification, evaluate

$$\lim_{t\to 0} \int_0^1 \frac{e^{t\sqrt{x}} - 1}{t}\, dx.$$

6. A sequence $\{f_k\}$ of integrable functions on $\mathbb{R}^n$ is said to *converge in measure* to f if for every $\varepsilon > 0$,

$$\lim_{k\to\infty} m(\{x \in \mathbb{R}^n : |f_k(x) - f(x)| \geq \varepsilon\}) = 0.$$

   (a) Prove that if $f_k \to f$ in L^1 then $f_k \to f$ in measure.
   (b) Give an example to show that the converse of Question 6a is false.
   (c) Prove that if we make the additional assumption that there exists an integrable function g such that $|f_k| \leq g$ for all k, then $f_k \to f$ in measure implies that
      i. * (Bonus points) $f \in L^1$
      Hint: First show that $\{f_k\}$ contains a subsequence which converges to f almost everywhere.
      ii. $f_k \to f$ in L^1.
      Hint: Try using absolute continuity and "small tails property" of the Lebesgue integral.