Szemerédi's Proof of Roth's Theorem

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Theorem 1. *(Roth's Theorem) Let* $0 < \delta < 1$*. For* N *sufficiently large, any* $A \subseteq [1, N]$ *with* $\mid A \mid = \delta N$ *necessarily contains a non-trivial 3-term arithmetic progression.*

The purpose of this paper is to provide the Szemerédi's proof of Roth's theorem on the existence of 3-term arithmetic progressions in large sets. We will rely on much of the method used in the Fourier analytic proof to show that if a set $A \subseteq [1, N]$ with $|A| = \delta N$ does not contain any 3-term arithmetic progressions, then there exists a "long" arithmetic progression on which the (relative) density of A increases, to say $\delta + \frac{\delta^2}{20}$. Provided N is sufficiently large, the density of A will eventually exceed 1 on some progression after a number of iterations, and this is the contradiction that we are seeking for in our argument.

Before we begin the proof, it is important to note that the convenient notation $\log N$ will replace $\log_2 N$.

One of the most important tools in this proof of Roth's theorem is the Cube Lemma, which will be employed later in the paper.

Lemma 2. *(Cube Lemma) Define a* k*-dimensional cube to be a set of the form*

$$Q(a, d_1, ..., d_k) = \{a + \varepsilon_1 d_1 + ... + \varepsilon_k d_k : \varepsilon_i = 0 \text{ or } 1 \text{ for all } 1 \leq i \leq k\} \text{ where } a, d_1, ..., d_k \in \mathbb{N}$$

For $\delta > 0$ *and* $k \in \mathbb{N}$*, if* $N \geq (3/\delta)^{2^k}$ *and* $A \subseteq [1, N]$ *with* $|A| = \delta N$ *then* A *must contain a* k*-dimensional cube.*

Fix $A \subseteq [1, N]$ with $|A| = \delta N$. Within A, we will have $\binom{\delta N}{2} = \frac{(\delta N)(\delta N - 1)}{2}$ pairs of points a and $b \in A$ such that $a \neq b$. Also there are $(N - 1)$ different possible values for $\mid a - b \mid$. Let d_1 be the most common value for $\mid a - b \mid$, the number of pairs that share the same difference d_1 would be

$$k \geq \frac{(\delta N)(\delta N - 1)}{2(N - 1)} \geq \frac{(\delta N)(\delta N - 1)}{2N} = \frac{(\delta)(\delta N - 1)}{2} = \frac{\delta^2 N - \delta}{2} \geq \frac{\delta^2 N - 1}{2} \geq \frac{\delta^2 N}{2} - \frac{1}{2}$$

$$\frac{3k}{2} = k + \frac{k}{2} \geq k + \frac{1}{2} \geq \frac{\delta^2 N}{2} \text{ which implies that } k \geq \frac{\delta^2 N}{3}$$

In other words, $d_1 = b_1 - a_1 = ... = b_k - a_k$ where $a_i, b_i \in A$ for $1 \leq i \leq k$. Let $A_1 = \{a_1, ..., a_k\}$ and $\delta_1 = \frac{\delta^2}{3}$, then $\mid A_1 \mid = k \geq \delta_1 N$ and $A_1 \cup (A_1 + d_1) \subset A$. Note that $(A_1 + d_1) = \{b_1 = a_1 + d_1, ..., b_k = a_k + d_1\}$. We will proceed to show that A contains a k-dimensional cube by induction for $N \geq (3/\delta)^{2^k}$.

For the base case, as long as A contains at least 2 points, A has a 1-dimensional cube. Assume that $N \geq (3/\delta)^{2^{k-1}}$ and $A \subseteq [1, N]$ with $|A| = \delta N$ then A must contain a $k - 1$-dimensional cube. Because $N \geq (\frac{3}{\delta})^{2^k} = (\frac{9}{\delta^2})^{2^{k-1}} = (\frac{3}{\delta_1})^{2^{k-1}}$ and since $\mid A_1 \mid = \delta_1 N$, then A_1 will have a $k - 1$ - dimensional cube, say $Q(a, d_2, ..., d_k)$, thus this $k - 1$ - dimensional cube is also contained in A. Also, $Q(a, d_2, ..., d_k) + d_1 \subset A_1 + d_1 \subset A$. Thus the k- dimensional cube $Q(a, d_1, ..., d_k) = Q(a, d_2, ..., d_k) \cup (Q(a, d_2, ..., d_k) + d_1) \subset A$ so A contains a k - dimensional cube.

The next step is to show that $\delta > 0$, $N \geq 2^{(\log(6/\delta))^2}$ and $A \subseteq [1, N]$ with $|A| = \delta N$ then A must contain a k-dimensional cube with $k \geq \frac{1}{2}\log\log N$.

Suppose $N = (3/\delta)^{2^{k'}}$

$$\log N = 2^{k'} \log(3/\delta)$$

$$2^{k'} = \frac{\log N}{\log(3/\delta)}$$

$$k' \log 2 = \log\left(\frac{\log N}{\log(3/\delta)}\right) = \log\log N - \log\log(3/\delta)$$

$$k' = \log\log N - \log\log(3/\delta) \text{ where } k' \text{ may/ may not be an integer}$$

Thus, we want $k \in \mathbb{Z}$ such that $k \geq k' = \log\log N - \log\log(3/\delta)$ and $N \geq (3/\delta)^{2^k}$ in order for A to have a k - dimensional cube by the Cube Lemma. If $N \geq 2^{(\log(6/\delta))^2} \geq 2^{(\log(3/\delta))^2} \geq (3/\delta)^{2^k}$, then A contains a k - dimensional cube with $k \geq \frac{\log\log N}{2}$. Indeed, for $N \geq 2^{(\log(3/\delta))^2}$, we'll have

$$\log N \geq \log 2^{(\log(3/\delta))^2} = (\log(3/\delta))^2$$

$$\log\log N \geq 2\log\log(3/\delta)$$

$$\frac{\log\log N}{2} \geq \log\log(3/\delta)$$

So $k \geq \log\log N - \log\log(3/\delta) = \frac{\log\log N}{2} + [\frac{\log\log N}{2} - \log\log(3/\delta)] \geq \frac{\log\log N}{2}$ because $\frac{\log\log N}{2} - \log\log(3/\delta) \geq 0$

We next let $A, P \subseteq [1, N]$ and $P = P_1 \cup ... \cup P_k$ be a partition. With some manipulation, we can rewrite $\frac{|A \cap P|}{|P|}$ as $\sum_{j=1}^{k} \frac{|P_j|}{|P|} \frac{|A \cap P_j|}{|P_j|}$

Indeed, by noting that $P_1, ..., P_k$ are disjoint subsets of P so $(A \cap P_1), ..., (A \cap P_k)$ are also disjoint and thus $| A \cap (P_1 \cup ... \cup P_k) | = | A \cap P_1 | + ... + | A \cap P_k |$, we have

$$\begin{aligned}
\frac{| A \cap P |}{| P |} &= \frac{| A \cap (P_1 \cup ... \cup P_k) |}{| P |} \\
&= \frac{| (A \cap P_1) \cup ... \cup (A \cap P_k) |}{| P |} \\
&= \frac{| A \cap P_1 | + ... + | A \cap P_k |}{| P |} \\
&= \frac{1}{| P |}\left(| P_1 | \frac{| A \cap P_1 |}{| P_1 |} + ... + | P_k | \frac{| A \cap P_k |}{| P_k |} \right) \\
&= \frac{1}{| P |}\sum_{j=1}^{k} | P_j | \frac{| A \cap P_j |}{| P_j |} \\
&= \sum_{j=1}^{k} \frac{| P_j |}{| P |}\frac{| A \cap P_j |}{| P_j |}
\end{aligned}$$

This equality allows us to conclude that if $A \subseteq P$ with $| A | = \delta N$ and $| P | \leq (1 - \delta/10)N$ then there must exist $1 \leq j \leq k$ such that

$$\frac{| A \cap P_j |}{| P_j |} \geq \delta + \delta^2/20 \text{ and } | P_j | \geq \frac{\delta^3}{20}\frac{N}{k}$$

Indeed, we have

$$\frac{| A \cap P |}{| P |} = \frac{| A |}{| P |} \geq \frac{\delta N}{(1-\delta/10)N} = \frac{\delta}{1-\delta/10} = \frac{10\delta}{10-\delta} \geq \delta + \frac{\delta^2}{10}$$

$$\begin{aligned}
\delta + \frac{\delta^2}{10} \leq \frac{| A \cap P |}{| P |} &= \sum_{j=1}^{k} \frac{| P_j |}{| P |}\frac{| A \cap P_j |}{| P_j |} \\
&= \sum_{\{j : |P_j| \geq \frac{\delta^3}{20}\frac{N}{k}\}}^{k} \frac{| P_j |}{| P |}\frac{| A \cap P_j |}{| P_j |} + \sum_{\{j : |P_j| < \frac{\delta^3}{20}\frac{N}{k}\}}^{k} \frac{| P_j |}{| P |}\frac{| A \cap P_j |}{| P_j |} \\
&\leq \sum_{\{j : |P_j| \geq \frac{\delta^3}{20}\frac{N}{k}\}}^{k} \frac{| P_j |}{| P |}\frac{| A \cap P_j |}{| P_j |} + \sum_{\{j : |P_j| < \frac{\delta^3}{20}\frac{N}{k}\}}^{k} \frac{| P_j |}{| A |}\frac{| A \cap P_j |}{| P_j |} \\
&< \sum_{\{j : |P_j| \geq \frac{\delta^3}{20}\frac{N}{k}\}}^{k} \frac{| P_j |}{| P |}\frac{| A \cap P_j |}{| P_j |} + k\frac{\frac{\delta^3}{20}\frac{N}{k}}{\delta N} \\
&\leq \sum_{\{j : |P_j| \geq \frac{\delta^3}{20}\frac{N}{k}\}}^{k} \frac{| P_j |}{| P |}\frac{| A \cap P_j |}{| P_j |} + \frac{\delta^2}{20}
\end{aligned}$$

Subtracting $\frac{\delta^2}{20}$ from both sides of the inequality, we obtain the following inequality

$$
\begin{aligned}
\delta+\frac{\delta^2}{20} \leq \sum_{\{j:|P_j|\geq\frac{\delta^3}{20}\frac{N}{k}\}}^{k} \frac{\mid P_j\mid}{\mid P\mid}\frac{\mid A\cap P_j\mid}{\mid P_j\mid} &\leq \sum_{\{j:|P_j|\geq\frac{\delta^3}{20}\frac{N}{k}\}}^{k} \frac{\mid P_j\mid}{\mid P\mid} \max_{\{j:|P_j|\geq\frac{\delta^3}{20}\frac{N}{k}\}} \frac{\mid A\cap P_j\mid}{\mid P_j\mid} \\
&= \max_{\{j:|P_j|\geq\frac{\delta^3}{20}\frac{N}{k}\}} \frac{\mid A\cap P_j\mid}{\mid P_j\mid} \sum_{\{j:|P_j|\geq\frac{\delta^3}{20}\frac{N}{k}\}}^{k} \frac{\mid P_j\mid}{\mid P\mid} \\
&= \max_{\{j:|P_j|\geq\frac{\delta^3}{20}\frac{N}{k}\}} \frac{\mid A\cap P_j\mid}{\mid P_j\mid}(1) \\
&= \max_{\{j:|P_j|\geq\frac{\delta^3}{20}\frac{N}{k}\}} \frac{\mid A\cap P_j\mid}{\mid P_j\mid}
\end{aligned}
$$

Proposition 3. *If $\delta > 0$, $\mid A\mid = \delta N$ and we know that A can be covered by a union of disjoint progressions $P_1, ..., P_k$ with $k \leq \frac{4N}{\log\log N}$ whose union P satisfies $\mid P\mid \leq (1-\frac{\delta}{10})N$, then there must exist $1 \leq j \leq k$ such that*

$$\frac{\mid A\cap P_j\mid}{\mid P_j\mid} \geq \delta+\delta^2/20 \text{ and } \mid P_j\mid \geq \frac{\delta^3}{80}\log\log N$$

where the lower bound for $\mid P_j\mid$ is obtained by substituting $\frac{4N}{\log\log N}$ into k for $\mid P_j\mid \geq \frac{\delta^3}{20}\frac{N}{k}$ in the above conclusion.

This proposition is crucial to our proof of Roth's theorem and in order to apply it to the proof, it is necessary to introduce the notion of partitioning sets into progressions of a common difference d into the paper. We start with the following definition and proposition,

Definition 1. *Let $A \subseteq [1, N]$ and $d \in \mathbb{N}$. We will say that $a, b \in A$ are equivalent, and write $a \sim b$, if there exists $L \in \mathbb{N}$ such that either $\{a, a+d, ..., a+(L-1)d = b\} \subseteq A$ or $\{b, b+d, ..., b+(L-1)d = a\} \subseteq A$*

Proposition 4. *One has*
(a)The relation $\sim$ is an equivalence relation whose equivalence classes are (maximal) progressions of step size d.
(b)The number of equivalence classes k, satisfies $k = \mid (A+d)\backslash A\mid$.
(c)The complement of A can be partitioned into at most $k+d$ progressions with step size d, that is
$[1, N]\backslash A = P_1 \cup ... \cup P_l$ *where each P_j is a progression with step size d and $l \leq k+d$.*

Proof. (a)$\sim$ is an equivalence relation for the following reasons:
$\sim$ is reflexive because $a \sim a$, in other words, a is in its own (maximal) progression with step size d.
$\sim$ is symmetric because $a \sim b$ means a is in the same (maximal) progression as b with step size d, which then implies that b is in the same (maximal) progression as a with step size d

or $b \sim a$.
$\sim$ is transitive since $a \sim b$ implies that a is in the same (maximal) progression as b with step size d and $b \sim c$ implies that b is in the same (maximal) progression as c with step size d, then a must be in the same (maximal) progression as c with step size d or $a \sim c$.

It is obvious that each element in A will belong to some (maximal) progression of step size d (there might be some 1-element progressions). Elements of A that are in the same progression can be viewed as being in the same equivalence class so each (maximal) progression of step size d can be made an equivalence class in A.

(b) For every maximal progression with step size d (No matter what the length of each progression is), we will have exactly 1 point in $A+d$ that is not in A, namely the last point in the original progression $+d$. Thus, there is a bijection between these points and the maximal progressions of step size d. As a result,

$$\text{No. equivalence classes } k \;=\; \text{No. (maximal) progressions of step size } d \;=\mid (A+d)\backslash A \mid$$

(c) Let k_i denote the number of progressions in A congruent to $i(\text{mod} d)$. We can conclude that each progression in A belongs to a congruence class (mod$d$) in $[1, N]$ and thus $[1, N]$ has d congruence classes. We have

$$k_1 + ... + k_d = k = \text{ No. of (maximal) progressions in } A$$

Just like the progressions in A, each (maximal) progression in A^c of step size d belongs to a congruence class (mod$d$) in $[1, N]$. It is easy to observe that A^c has at most $k_i + 1$ progressions that are congruent to $i(\text{mod} d)$ for each i from $1 \leq i \leq d$ (In other words, if viewed on a number line from 1 to N the progressions of A congruent to $i(\text{mod} d)$) will be interspersed with the progressions of A^c congruent to $i(\text{mod} d)$. Thus each each progression of A^c congruent to $i(\text{mod} d)$) is assigned to a progression of A congruent to $i(\text{mod} d)$). The $+1$ accounts for a possible additional progression of A^c congruent to $i(\text{mod} d)$) in the case where there are progressions of A^c congruent to $i(\text{mod} d)$) at both ends of the interval of the number line on which the progressions of A congruent to $i(\text{mod} d)$) lie. So the total number of progressions that partition A^c is bounded above by

$$\sum_{i=1}^{d} k_1 + 1 = d + \sum_{i=1}^{d} k_i = d + k$$

□

Now that we have the proper tools necessary, we can apply them to our proof of Roth's theorem where we have our set $N \geq 2^{2(\log(6/\delta))^2}$, $A \subseteq [1, N]$, and $\mid A \mid= \delta N$ with $\delta > 0$. Set $B := A \cap [N/3, 2N/3]$ and suppose $\mid B \mid \geq \delta N/6$.

Before we reach the final stage of our proof, it is desirable to establish an upper bound for the number of progressions that partition B_j^c. This is when our Cube Lemma comes into the picture. We need to show that B must contain a k-dimensional cube $Q(a, d_1, ..., d_k)$ with $k \geq \frac{1}{2} \log \log N - \frac{1}{2}$ in order to prove that $1 \leq d_1 + ... + d_k \leq \sqrt{N}$, which then supplies us with a useful fact that $d_i \leq \sqrt{N}$ for all $1 \leq i \leq k$.

Decompose $[N/3, 2N/3]$ into intervals of length $\sqrt{N}$. The density of B relative to the interval $[N/3, 2N/3]$ is

$$\frac{\mid B \mid}{\mid [N/3, 2N/3] \mid} \geq \frac{\frac{\delta N}{6}}{\frac{N}{3}} = \frac{\delta}{2}.$$

This means that at least one of the intervals I with length $\sqrt{N}$ has density $\geq \frac{\delta}{2}$ or $\frac{A \cap I}{I} \geq \frac{\delta}{2}$. Let $\delta_1 = \frac{\delta}{2}$, we will have $\mid I \mid = 2^{(\log(3/\delta_1))^2}$ and $\mid A \cap I \mid = \delta_1 \mid I \mid$. By the Cube Lemma, $A \cap I$ will contain a k-dimensional cube for a $k \geq \frac{1}{2} \log\log(\mid I \mid) = \frac{1}{2} \log\log(\sqrt{N}) = \frac{1}{2} \log\log N - \frac{1}{2}$ since $\log\log\sqrt{N} = \log(\frac{1}{2}\log N) = \log\frac{1}{2} + \log\log N = -1 + \log\log N$. As a result, B must contain a k-dimensional cube $Q(a, d_1, ..., d_k)$ with $k \geq \frac{1}{2}\log\log N - \frac{1}{2}$.

Since$A \cap I$ contains a k-dimensional cube, I has to contain a k-dimensional cube. In other words, $(a + d_1 + ... + d_k) \in I$ where $a \in I$ so $1 \leq d_1 + ... + d_k = \mid (a + d_1 + ... + d_k) - a \mid \leq \sqrt{N}$.

Next, we set $Q_j := Q(a, d_1, ..., d_j)$ for $1 \leq j \leq k$, $Q_0 := \{a\}$ and introduce the sets

$$B_j := \{x \in [1, N] : x = 2z - y, z \in Q_j \text{ and } y \in B\}$$

For each $0 \leq j \leq k$ the sets B_j will satisfy the followings:

i. $B_j \cap A = \emptyset$ if A contains no 3-term arithmetic progressions.

ii. $\mid B_j \mid \geq \mid B \mid \geq \delta N/6$

iii. $B_j \cup (B_j + 2d_{j+1}) \subset B_{j+1}$ (for $0 \leq j \leq k-1$)

Proof. i. We know that $B_j := \{x \in [1, N] : x = 2z - y, z \in Q_j \subseteq B \subseteq A \text{ and } y \in B \subseteq A\}$ so in order for A to contain no 3-term arithmetic progression, for all $x \in B_j, x \notin A$. As a result, $B_j \cap A = \emptyset$.

ii. For each fixed z, we will have $\mid B \mid$ different values for $x \in B_j$ so

$$\mid B_j \mid := \mid \{x \in [1, N] : x = 2z - y, z \in Q_j \text{ and } y \in B\} \mid \geq \mid \{y \in B\} \mid = \mid B \mid \geq \frac{\delta N}{6}$$

iii. We want to show that for $x \in B_j \cup (B_j + 2d_{j+1})$, $x \in B_{j+1}$ (for $0 \leq j \leq k-1$) so we start by showing that for $x \in B_j$ or $x \in (B_j + 2d_{j+1})$, $x \in B_{j+1}$.

$x \in B_j$ implies $x = 2z - y$ where $z \in Q_j$ and $y \in B$ but $z \in Q_j$ implies that $z \in Q_{j+1}$ also. Hence, we will have $x = 2z - y$ where $z \in Q_{j+1}$ and $y \in B$, which implies $x \in B_{j+1}$ or $B_j \subseteq B_{j+1}$

$x \in (B_j + 2d_{j+1})$ implies $x = x' + 2d_{j+1}$ where $x' \in B_j$ or $x' = 2z - y$ where $z \in Q_j$ and $y \in B$. Thus, we have

$$\begin{aligned} x &= (2z - y) + 2d_{j+1} \\ &= 2(z + d_{j+1}) - y \\ &= 2z' - y \text{ where } z' = z + d_{j+1} \text{ which is clearly in } Q_{j+1} \end{aligned}$$

As a result, $x \in B_{j+1}$ and $(B_j + 2d_{j+1}) \subseteq B_{j+1}$ We have just proved that $B_j \cup (B_j + 2d_{j+1})B_{j+1}$ (for $0 \leq j \leq k-1$) $\square$

Now we can incorporate all the details that we just proved into the following conclusion that the complement of the set B_j can be partitioned into at most l progressions with step size

$d = 2d_{j+1}$ where

$$l \leq d + \mid B_{j+1} \backslash B_j \mid \leq 2\sqrt{N} + \frac{3N}{\log\log N} \leq \frac{4N}{\log\log N}$$

Note that $(B_{j+1}\backslash B_j) \cap (B_{i+1}\backslash B_i) = \emptyset$ for distinct i and j. In particular, for $i < j$, $B_{i+1} \subset B_j$, so $(B_{i+1}\backslash B_i) \subset B_j$. So $(B_{j+1}\backslash B_j)$ for $0 \leq j \leq k-1$ are disjoint subsets of $[1, N]$ and hence, $\sum_{j=0}^{k-1} \mid B_{j+1}\backslash B_j \mid \leq N$. Therefore, $\frac{1}{k}\sum_{j=0}^{k-1} \mid B_{j+1}\backslash B_j \mid \leq \frac{1}{k}N \leq \frac{2N}{\log\log N - 1} \leq \frac{3N}{\log\log N}$

From earlier, $B_j + 2d_{j+1} \subseteq B_{j+1}$ so $(B_j + 2d_{j+1}) \cap (B_j)^c \subseteq B_{j+1} \cap (B_j)^c$ and this gives us the result that $\mid (B_j + 2d_{j+1})\backslash B_j \mid \leq \mid B_{j+1}\backslash B_j \mid$. The set B_j can be partitioned into $\mid (B_j + 2d_{j+1})\backslash B_j \mid$ progressions with step size $2d_{j+1}$ so by Proposition 3(b), the number of equivalence classes $k = \mid (B_j + 2d_{j+1})\backslash B_j \mid \leq \mid B_{j+1}\backslash B_j \mid$.

Let $(B_j)^c$ be the complement of B_j, $d = 2d_{j+1}$. By Proposition 3(c), $(B_j)^c$ can be partitioned into at most $k + d$ progressions with step size d, i.e $(B_j)^c = P_1 \cup ... \cup P_l$ where each P_j for $1 \leq j \leq l$ is a progression with step size d and $l \leq k + d$, or $l \leq \mid (B_j + 2d_{j+1})\backslash B_j \mid + d \leq \mid B_{j+1}\backslash B_j \mid + 2d_{j+1} \leq \frac{3N}{\log\log N} + 2\sqrt{N} \leq \frac{4N}{\log\log N}$. Note that $d_{j+1} \leq \sqrt{N}$ because from 4(a) $1 + d_1 + ... + d_k \leq \sqrt{N}$ so each $d_i \leq \sqrt{N}$ for $1 \leq i \leq k$.

The final step in this paper is to to show that for $A \subseteq [1, N]$ such that $\delta = \mid A \mid /N$, if A contains no non-trivial 3-term arithmetic progressions, then either $N < 2^{2(\log(6/\delta))^2}$ or there exists an arithmetic progression P with $\mid P \mid \geq c\delta^3 \log\log N$ such that

$$\frac{\mid A \cap P \mid}{\mid P \mid} \geq \delta + \frac{\delta^2}{20}$$

If A contains no non-trivial 3-term arithmetic progressions, then we have two different scenarios:

- N is too small, or $N < 2^{2(\log(6/\delta))^2}$ or
- N is sufficiently large, or $N \geq 2^{2(\log(6/\delta))^2}$. For N sufficiently large, we have to consider two additional cases: when $\mid B \mid \geq \delta N/6$ and when $\mid B \mid < \delta N/6$

When $\mid B \mid \geq \delta N/6$, we know from previously that B_j^c can be partitioned into at most $l \leq \frac{4N}{\log\log N}$ progressions. We also can deduce that $A \subseteq B_j^c$ because A has no 3-term arithmetic progressions and hence $B_j \cap A = \emptyset$ by the way B_j is defined. Since $\mid B_j^c \mid = N - \mid B_j \mid \leq N - \delta N/6 = (1 - \frac{\delta}{6})N \leq (1 - \frac{\delta}{10})N$, we can conclude that A is covered by a union of disjoint progressions $P_1, ..., P_l$ (i.e $\cup_1^l P_i = B_j^c$) with $l \leq \frac{4N}{loglogN}$ for $1 \leq i \leq l$ whose $\cup_1^l P_i = B_j^c$ satisfies $\mid B_j^c \mid \leq (1 - \frac{\delta}{10})N$. Then, by Proposition 2 again, there must exists $1 \leq j \leq l$ such that

$$\frac{\mid A \cap P_j \mid}{\mid P_j \mid} \geq \delta + \delta^2/20 \text{ and } \mid P_j \mid \geq \frac{\delta^3}{80}\log\log N = c\delta^3 \log\log N$$

If $\mid B \mid < \delta N/6$, then in one of the other two intervals $[0, N/3]$ and $[N/3, 2N/3]$, call it P, we will have $\mid A \cap P \mid \geq \frac{5\delta}{12}N$ (by applying Pigeonhole Principle using the fact that $\mid A \mid = \delta N$ and $A \cap [N/3, 2N/3] < \delta N/6$). As a result,

$$\frac{\mid A \cap P \mid}{\mid P \mid} \geq \frac{\frac{5\delta}{12}N}{\frac{N}{3}} = \frac{15\delta}{12} = \delta + \frac{\delta}{4} \geq \delta + \frac{\delta^2}{20} \text{ and } \mid P \mid = \frac{N}{3} \geq \log\log N \geq \frac{\delta^3}{80}\log\log N$$

We will iterate this argument to prove Roth's Theorem. We want to assume that $N \geq 2^{2(\log(6/\delta))^2}$ and A contains no non-trivial 3-term arithmetic progressions. For N large enough, the following proof of Roth's theorem based on density increment will lead to a contradiction, which concerns the fact that the relative density of A will exceed 1 after a number of incrementing steps.

We begin the proof by denoting $A = A_0, N = N_0$, and $\delta = \delta_0$. Based on our conclusion earlier, there must exist a P_0 such that $\mid P_0 \mid \geq c\delta_0^3 \log\log N$ and $\frac{|A \cap P_0|}{|P_0|} \geq \delta_0 + \delta_0^2/20$. We now define $A_{j+1} \subseteq [1, N_{j+1}]$ to be the set obtained from translation and dilation of $A_j \cap P_j$ such that $A_{j+1} = \delta_{j+1} N_{j+1}$ where $N_{j+1} \geq c\delta_j^3 \log\log N_j$ and $\delta_{j+1} \geq \delta_j + \delta_j^2/20$. By this definition, we will have the followings (which can be proven by a simple induction proof)

$$\delta_1 \geq \delta_0 + \frac{\delta_0^2}{20}$$

$$\delta_2 \geq \delta_1 + \frac{\delta_1^2}{20} \geq \delta_0 + \frac{\delta_0^2}{20} + \frac{(\delta_0 + \frac{\delta_0^2}{20})^2}{20} \geq \delta_0 + 2\frac{\delta_0^2}{20}$$

...

$$\delta_k \geq \delta_{k-1} + \frac{\delta_{k-1}^2}{20} \geq \delta_0 + k\frac{\delta_0^2}{20}$$

Substituting $20/\delta_0$ into k gives $\delta_k \geq 2\delta_0$, which implies that if we iterate the argument above $k = 20/\delta_0$ times, we will have successfully doubled the density δ_0. Now we want to establish a general formula for the number of steps it will take to increase δ_0 by 4 times, 8 times, and so on...

From above, we know that

$$\delta_{k+1} \geq \delta_k + \frac{\delta_k^2}{20} \geq 2\delta_0 + \frac{(2\delta_0)^2}{20}$$

$$\delta_{k+2} \geq \delta_{k+1} + \frac{\delta_{k+1}^2}{20} \geq 2\delta_0 + 2\frac{(2\delta_0)^2}{20}$$

... with a simple proof by induction we can prove that $\delta_{k+l} \geq 2\delta_0 + l\frac{(2\delta_0)^2}{20}$

It then comes to our attention that if we substitute l by $20/2\delta_0$, we will obtain $\delta_{k+l} \geq 2\delta_0 + l\frac{(2\delta_0)^2}{20} = 4\delta_0$. This implies that after $k + l = \frac{20}{\delta_0} + \frac{10}{\delta_0}$ steps, δ_0 increases to $4\delta_0$. By iterating this way, we can conclude that a density of $2^j\delta_0$ can be achieved in $\leq \frac{20}{\delta_0}\sum_{i=0}^{j-1}\frac{1}{2^i}$ steps. As a result, for no more than $\frac{20}{\delta_0}\sum_{i=0}^{\infty}\frac{1}{2^i} = \frac{40}{\delta_0}$ steps, we will obtain a density whose value surpasses the maximum value of density, i.e 1. Thus, we have reached a contradiction in our argument and we can conclude that for N large enough, the set $A \subseteq [1, N]$ with $\mid A \mid = \delta N$ contains at least one 3-term arithmetic progression.

Now that we have proved Roth's theorem for N sufficiently large, we want to find a specific lower bound for the value of N that would allow us to use the process of iteration in our proof and reach a contradiction in our argument.

In order to have no non-trivial 3-term arithmetic progressions in the set A, it is required that

$$
\begin{aligned}
2^{2(\log(6/\delta))^2} \geq N_j &\geq c\delta_{j-1}^3 \log\log N_{j-1} \\
&\geq c\delta_{j-2}^3 \log\log(c\delta_{j-2}^3 \log\log N_{j-2}) \\
&... \\
&\geq c\delta_1^3 \log\log(\delta_1^3 \log\log ...(\delta_1^3 \log\log N)) \\
&\geq c \underbrace{\delta_0^3 \log\log(\delta_0^3 \log\log ...(\delta_0^3 \log\log N))}\text{ where } N = N_{j-j} = N_0 \\
&\text{j nested } \delta_0^3 \log\log' s
\end{aligned}
$$

By exponentiating, we can obtain a tower bound for N:

$$
\begin{aligned}
N \leq &\underbrace{2^{2^{c(\delta_0)2^{2^{c(\delta_0)...2^{2(\log(6/\delta))^2}}}}}} \\
&\text{j stacked } 2^{2^{c(\delta_0)}} \text{ where } c(\delta_0) \\
&\text{is a constant that depends on } \delta_0 \\
&\text{and } j = \frac{40}{\delta_0}
\end{aligned}
$$

As a result, if $N \geq 2^{2^{c(\delta_0)2^{2^{c(\delta_0)...2^{2(log(6/\delta))^2}}}}}$, then A will have a non-trivial 3-term arithmetic progression.

References

[1] A. Magyar, "Roth's Theorem. The Combinatorial Approach," (2005) University of Georgia http://www.math.uga.edu/ lyall/REU/szem.pdf