Math 3100 Assignment 5
Infinite Series

Due at 5:00 pm on Friday the 22nd of February 2019

1. Suppose that $\sum_{k=1}^{\infty} a_k$ converges to A and $\sum_{k=1}^{\infty} b_k$ converges to B.

   (a) Prove that $\sum_{k=1}^{\infty} (a_k + b_k)$ converges to $A + B$.

   (b) Must $\sum_{k=1}^{\infty} (a_k b_k)$ converge to AB? Give either a proof or counterexample.

2. Evaluate the following series (if they converge)

$$\text{(a)} \sum_{n=1}^{\infty} \frac{1}{2^n} \qquad \text{(b)} \sum_{n=2}^{\infty} \frac{3}{4^n} \qquad \text{(c)} \sum_{n=3}^{\infty} \frac{7^{n-1}}{2^{n+1}}$$

3. Prove that omitting or changing a finite number of terms of a series does not affect its convergence.
   Hint: One possible approach to this problem, but not the only one, is to use the Cauchy Criterion

4. Let $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ be two sequences of positive real numbers. Prove the following:

   (i) If $\lim_{n\to\infty} \frac{a_n}{b_n} = c > 0$, then $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ either both converge or both diverge.

   (ii) If $\lim_{n\to\infty} \frac{a_n}{b_n} = 0$ and $\sum_{n=1}^{\infty} b_n$ converges, then $\sum_{n=1}^{\infty} a_n$ also converges.

   (iii) If $\lim_{n\to\infty} \frac{a_n}{b_n} = \infty$ and $\sum_{n=1}^{\infty} b_n$ diverges, then $\sum_{n=1}^{\infty} a_n$ also diverges.

5. Test the series for convergence or divergence.

$$\text{(a)} \sum_{n=1}^{\infty} \frac{\sqrt{n}}{n^2+3} \qquad \text{(b)} \sum_{n=0}^{\infty} \cos(n) \qquad \text{(c)} \sum_{n=1}^{\infty} \frac{2^n}{n3^{n+1}} \qquad \text{(d)} \sum_{n=1}^{\infty} \frac{n2^n}{3^{n+1}} \qquad \text{(e)} \sum_{n=3}^{\infty} \frac{(-1)^n}{(\log n)^2}$$

$$\text{(f)} \sum_{n=1}^{\infty} \frac{2n}{8n-5} \qquad \text{(g)} \sum_{n=3}^{\infty} \frac{2}{n(\log n)^3} \qquad \text{(h)} \sum_{n=1}^{\infty} \frac{3^n n^2}{n!} \qquad \text{(i)} \sum_{n=1}^{\infty} \frac{3^n}{5^n+n} \qquad \text{(j)} \sum_{n=1}^{\infty} \frac{n+5}{5^n}$$

6. Investigate the behavior (convergence or divergence) of $\displaystyle\sum_{n=1}^{\infty} a_n$ if

$$\text{(a) } a_n = \sqrt{n+1} - \sqrt{n} \qquad \text{(b) } a_n = \frac{\sqrt{n+1} - \sqrt{n}}{n}$$