An application of the Bolzano–Weierstrass Theorem

Theorem

Every bounded divergent sequence of real numbers contains at least two subsequences that converge to different limits.

Before giving the short proof of this Theorem we recall:

1. The Bolzano–Weierstrass Theorem ensures that every bounded sequence contains at least one convergent subsequence.
2. If a sequence $\{x_n\}$ does not converge to x, then for some $\varepsilon > 0$ there must be infinitely many x_n's lying outside $(x-\varepsilon, x+\varepsilon)$.

Proof of Theorem

Let $\{x_n\}$ be a divergent sequence for which $\exists\, M > 0$ such that

$$|x_n| \leq M \quad \text{for all } n \in \mathbb{N}.$$

Bolzano–Weierstrass $\Rightarrow$ $\exists$ a subsequence of $\{x_n\}$ which converges to some limit x.

Since $\{x_n\}$ is divergent, we know that $x_n \not\to x$ & hence for some $\varepsilon > 0$ we have infinitely many x_n's satisfying

$$x_n \in [-M, x-\varepsilon] \cup [x+\varepsilon, M]$$

(making M larger if necessary).

other case handled similarly.

Suppose there are infinitely many x_n's in $[x+\varepsilon, M]$, then BW implies $\{x_n\}$ contains a subsequence in $[x+\varepsilon, M]$ that converges to a limit in $[x+\varepsilon, M]$ (by order limit laws). $\square$