Math 3100 Assignment 2

Sequences: Boundedness, Monotonicity, and Convergence

Due at the beginning of class on Friday the 26th of January 2018

1. Which of the sequences below are increasing, strictly increasing, decreasing, strictly decreasing, or none of the above? Justify your answers. Which are bounded above, or bounded below; which are bounded? Give an upper bound and/or lower bound when applicable.

   (a) $a_n = n^2 - n$

   (b) $b_n = \dfrac{1}{n+1}$

   (c) $c_n = \dfrac{(-1)^n}{n^3}$

   (d) $x_{n+1} = x_n + \dfrac{1}{(n+1)^2}$, for $n \in \mathbb{N}$ and $x_1 = 1$

   (e) $y_n = 17$ for all $n \in \mathbb{N}$

   Challenge: Can you show that the sequence defined by $x_{n+1} = x_n + \dfrac{1}{n+1}$, *for* $n \in \mathbb{N}$ *and* $x_1 = 1$ *is strictly increasing and* <u>*not*</u> *bounded above.*

2. (a) Let $\{a_n\}$ be a sequence given recursively by $a_{n+1} = \dfrac{3a_n + 2}{a_n + 2}$ with $a_1 = 1$.
   Prove that $\{a_n\}$ is increasing and satisfies $a_n \leq 2$ for all $n \in \mathbb{N}$.
   Hint: Depending on your approach it may help to also verify that $a_n \geq 0$ *for all* $n \in \mathbb{N}$.

   (b) Let $\{b_n\}$ be a sequence given recursively by $b_{n+1} = \dfrac{b_n}{2} + \dfrac{1}{b_n}$ with $b_1 = 2$.
   Use induction to prove that $\{b_n\}$ satisfies both $b_n > 0$ and $b_n^2 - 2 \geq 0$ for all $n \in \mathbb{N}$. Use this to establish that $\{b_n\}$ is a decreasing sequence.

3. (a) Let $q \neq 0$ be rational and x be irrational. Prove that $q + x$ and qx are both irrational.

   (b) Give examples of the following:
      i. A sequence $\{x_n\}$ of irrational numbers whose limit is a rational number.
      ii. A sequence $\{q_n\}$ of rational numbers whose limit is an irrational number.

4. Verify, using the definition of convergence of a sequence, that the following sequences converge to the proposed limit.

   (a) $\displaystyle\lim_{n\to\infty} \frac{1}{n^{1/3}} = 0$ (b) $\displaystyle\lim_{n\to\infty} \frac{3n+1}{2n+5} = \frac{3}{2}$ (c) $\displaystyle\lim_{n\to\infty} \frac{1}{6n^2+1} = 0$

5. Determine the value of the following limits, and then prove your claims using the definition of convergence of a sequence.

   (a) $\displaystyle\lim_{n\to\infty} \frac{n}{n^2+1}$ (b) $\displaystyle\lim_{n\to\infty} \frac{4n+3}{7n-5}$ (c) $\displaystyle\lim_{n\to\infty} \frac{\sin n}{n^{1/2}}$

Math 3100 – Homework 2 – SOLUTIONS

1. (a) Claim: $a_n = n^2 - n$ is strictly increasing, bounded below by 0, but not bounded above.

Proof

- Since $[(n+1)^2-(n+1)]-[n^2-n] = 2n > 0 \ \forall n \in \mathbb{N}$

$\Leftrightarrow \quad a_{n+1} - a_n > 0 \ \forall n \in \mathbb{N} \Leftrightarrow a_{n+1} > a_n \ \forall n \in \mathbb{N}$.

- Since $n^2 \geq n \ \forall n \in \mathbb{N} \Rightarrow n^2 - n \geq 0 \ \forall n \in \mathbb{N}$.

- Suppose $\exists M \geq 0$ such that $\underbrace{n^2-n}_{=n(n-1)} \overset{(*)}{\leq} M \ \forall n \in \mathbb{N}$ and seek a contradiction.
 It would follow from $(*)$ that $(n-1)^2 \leq M \ \forall n \in \mathbb{N}$, (since $n-1 \leq n$) and hence that $n \leq \sqrt{M}+1 \ \forall n \in \mathbb{N}$
 But this contradicts the fact that $\mathbb{N}$ is unbounded!

$\square$

(b) Claim: $b_n = \frac{1}{n+1}$ is strictly decreasing and bounded, explicitly bounded above by $\frac{1}{2}$ & below by 0.

Proof

- Since $\frac{1}{n+2} - \frac{1}{n+1} = -\frac{1}{(n+1)(n+2)} < 0 \ \forall n \in \mathbb{N}$

$\Rightarrow b_{n+1} - b_n < 0 \ \forall n \in \mathbb{N} \Leftrightarrow b_{n+1} < b_n \ \forall n \in \mathbb{N}$.

- If $n \geq 1 \Rightarrow 0 \leq \frac{1}{n+1} \leq \frac{1}{2} \Rightarrow 0 \leq b_n \leq \frac{1}{2} \ \forall n \in \mathbb{N}$.

(this is just the fact that if $x > 0 \Rightarrow \frac{1}{x} > 0$)

$\square$

(c) Claim: $c_n = \frac{(-1)^n}{n^3}$ is neither increasing or decreasing, but it is bounded, in particular $|c_n| \leq 1$ ($\Leftrightarrow -1 \leq c_n \leq 1$) $\forall n \in \mathbb{N}$.

Proof

- $c_1 = -1$, $c_2 = \frac{1}{8}$ & $c_3 = -\frac{1}{27}$

  Since $c_1 < c_2$ & $c_2 > c_3$ $\{c_n\}$ is neither increasing or decreasing.

- $\left|\frac{(-1)^n}{n^3}\right| = \frac{1}{n^3} \leq 1 \quad \forall n \in \mathbb{N}$

$\square$

(d) Claim: $x_{n+1} = x_n + \frac{1}{(n+1)^2}$ $\forall n \in \mathbb{N}$ & $x_1 = 1$ defines a strictly increasing bounded sequence with $1 \leq x_n \leq 2$ $\forall n \in \mathbb{N}$.

Proof

- Since $x_{n+1} - x_n = \frac{1}{(n+1)^2} > 0$ $\forall n \in \mathbb{N} \Rightarrow \{x_n\}$ strictly inc.

- Easy to see that $x_n = 1 + \frac{1}{4} + \frac{1}{9} + \cdots + \frac{1}{n^2}$ $\forall n \in \mathbb{N}$.

  We proved in class (when reviewing induction) that $x_n \leq 2 - \frac{1}{n}$ $\forall n \in \mathbb{N} \Rightarrow x_n \leq 2$ $\forall n \in \mathbb{N}$.

  Since $\{x_n\}$ is increasing & $x_1 = 1 \Rightarrow x_n \geq 1$ $\forall n \in \mathbb{N}$ $\square$

(e) Claim: $y_n = 17$ $\forall n \in \mathbb{N}$ defines a constant sequence

(*) Every constant sequence is clearly both increasing and decreasing. It is also clearly bounded.

In this example any $U \geq 17$ is an upper bound & any $L \leq 17$ is a lower bound.

$\square$

2.

(a) <u>Claim</u>: If $a_{n+1} = \dfrac{3a_n+2}{a_n+2}$ & $a_1 = 1$, then

(i) $0 \le a_n \le 2 \quad \forall n \in \mathbb{N}$

(ii) $\{a_n\}$ increasing

<u>Proof</u>

<u>(i)</u>: We will show that $\overbrace{-2 \le a_n - 2 \le 0}^{(*)}$ by Induction:

Base case ($n=1$): $-2 \le \underbrace{1-2}_{=-1} \le 0$ ✓

Suppose $(*)$ holds for some $n \in \mathbb{N}$, then

$$a_{n+1} - 2 = \frac{3a_n+2}{a_n+2} - 2 = \frac{a_n-2}{a_n+2}$$

Since the <u>Ind. Hyp</u> implies that $2 \le a_n + 2 \le 4$ it follows that $\underbrace{\frac{-2}{4}}_{=-1/2} \le \frac{a_n-2}{a_n+2} \le \underbrace{\frac{0}{2}}_{=0} \Rightarrow -2 \le a_{n+1} - 2 \le 0$.

<u>(ii)</u>: Since $a_{n+1} - a_n = \dfrac{3a_n+2}{a_n+2} - a_n$

All terms ≥ 0

$$= \frac{(3a_n+2)-(a_n^2+2a_n)}{a_n+2}$$

$$= -\frac{(a_n^2 - a_n + 2)}{a_n+2} = -\frac{(a_n-2)(a_n+1)}{a_n+2}$$

≤ 0 by (i). □

* For an alternative (better?) proof see "Lecture Notes" on "Two examples of proving a seq in montone & bounded".

(b) <u>Claim</u> If $b_1 = 2$ & $b_{n+1} = \frac{b_n}{2} + \frac{1}{b_n}$ $\forall n \geq 1$, then

(i) $b_n > 0$ & $b_n^2 - 2 \geq 0$ $\forall n \in \mathbb{N}$

(ii) $\{b_n\}$ decreasing

<u>Proof</u>

<u>(i)</u>: <u>Proof that $b_n > 0$ $\forall n \in \mathbb{N}$</u>: (Induction)

Base Case ($n=1$): $b_1 = 2 > 0$ ✓

Suppose $b_n > 0$ ~~for~~ for some $n \in \mathbb{N}$, then $\frac{b_n}{2} > 0$ & $\frac{1}{b_n} > 0$

and hence $b_{n+1} = \frac{b_n}{2} + \frac{1}{b_n} > 0$.

<u>Proof that $b_n^2 - 2 \geq 0$ $\forall n \in \mathbb{N}$</u>: (Actually <u>not</u> induction!)

If $n=1$, then $b_1^2 - 2 = 2^2 - 2 = 2 \geq 0$ ✓

If $n \in \mathbb{N}$, then $b_{n+1}^2 - 2 = \left(\frac{b_n}{2} + \frac{1}{b_n}\right)^2 - 2$

$$\left[\begin{array}{l} \left(\frac{b_n}{2} + \frac{1}{b_n}\right)^2 = \left(\frac{b_n}{2}\right)^2 + 1 + \left(\frac{1}{b_n}\right)^2 \\ \Rightarrow \left(\frac{b_n}{2} + \frac{1}{b_n}\right)^2 - 2 = \left(\frac{b_n}{2}\right)^2 - 1 + \left(\frac{1}{b_n}\right)^2 \end{array}\right]$$

$$= \left(\frac{b_n}{2} - \frac{1}{b_n}\right)^2 \geq 0.$$

(ii) Since $b_{n+1} - b_n = \frac{b_n}{2} + \frac{1}{b_n} - b_n$

$$= \frac{1}{b_n} - \frac{b_n}{2}$$

$$= \frac{2 - b_n^2}{2b_n} \leq 0 \quad \text{since } b_n^2 - 2 \geq 0 \ \& \ 2b_n > 0$$

$\Rightarrow \{b_n\}$ is decreasing

□

3. (a) Claim

If $q \neq 0$ is rational and x is irrational, then

(i) $q + x$ irrational

(ii) qx irrational.

Proof

(i) We use that fact that the difference of two rationals is always rational.

Suppose $q+x$ were rational, then

$$x = (q+x) - q \in \mathbb{Q} \quad \text{↯ Contradiction} \quad \square$$

(ii) We use the fact that if $r \in \mathbb{Q}$ & $q \in \mathbb{Q}$ and $\neq 0$ then $r/q \in \mathbb{Q}$.

Suppose qx were rational, then

$$x = (qx)/q \in \mathbb{Q} \quad \text{↯ Contradiction} \quad \square$$

(b) (i) Note that $\frac{\sqrt{2}}{n}$ is irrational (by (ii) above) $\forall n \in \mathbb{N}$ and $\lim_{n\to\infty} \frac{\sqrt{2}}{n} = 0 \in \mathbb{Q}$.

(ii) Note that it follows from the density of $\mathbb{Q}$ in $\mathbb{R}$ that for every $n \in \mathbb{N}$ $\exists$ rational q_n such that
$\sqrt{2} - \frac{1}{n} < q_n < \sqrt{2} + \frac{1}{n} \iff |q_n - \sqrt{2}| < \frac{1}{n}$

It follows that $\lim_{n\to\infty} q_n = \sqrt{2}$ (why?)

4. (a) Claim: $\lim_{n\to\infty} \frac{1}{n^{1/3}} = 0$.

Proof Let $\varepsilon > 0$ and set $N = 1/\varepsilon^3$.

If $n > N$ it follows that

$$\left| \frac{1}{n^{1/3}} - 0 \right| = \frac{1}{n^{1/3}} < \varepsilon \quad \left(\text{since } \frac{1}{n^{1/3}} < \varepsilon \Leftrightarrow n^{1/3} > \frac{1}{\varepsilon} \Leftrightarrow n > \frac{1}{\varepsilon^3}\right)$$

$\square$

(b) Claim: $\lim_{n\to\infty} \frac{3n+1}{2n+5} = \frac{3}{2}$

Proof Let $\varepsilon > 0$ and set $N = \frac{13}{4\varepsilon}$.

If $n > N$ it follows that

$$\left| \frac{3n+1}{2n+5} - \frac{3}{2} \right| = \left| \frac{-13}{4n+10} \right| \leq \frac{13}{4n} < \varepsilon \text{ since } n > \frac{13}{4\varepsilon}.$$

$\square$

(c) Claim: $\lim_{n\to\infty} \frac{1}{6n^2+1} = 0$

Proof Let $\varepsilon > 0$ & set $N = \frac{1}{\sqrt{6\varepsilon}}$.

If $n > N$ it follows that

$$\left| \frac{1}{6n^2+1} - 0 \right| = \frac{1}{6n^2+1} \leq \frac{1}{6n^2} < \varepsilon \text{ since } n^2 > \frac{1}{6\varepsilon}$$

$\square$

5. (a) <u>Claim</u>

$$\lim_{n\to\infty} \frac{n}{n^2+1} = 0$$

<u>Proof</u>

Let $\varepsilon > 0$ & set $N = \varepsilon^{-1}$.

If $n > N$ it follows that

$$\left| \frac{n}{n^2+1} - 0 \right| = \frac{n}{n^2+1} \leq \frac{n}{n^2} = \frac{1}{n} < \varepsilon \text{ since } n > \varepsilon^{-1} \quad \square$$

(b) <u>Claim</u>

$$\lim_{n\to\infty} \frac{4n+3}{7n-5} = \frac{4}{7}$$

<u>Proof</u>

Let $\varepsilon > 0$ & set $N = 1/14\varepsilon$.

If $n > N$ it follows that

$$\left| \frac{4n+3}{7n-5} - \frac{4}{7} \right| = \frac{1}{7(7n-5)} \leq \frac{1}{7(7n-5n)} = \frac{1}{14n} < \varepsilon$$

since $n > \frac{1}{14\varepsilon}$ $\square$.

(c) <u>Claim</u>

$$\lim_{n\to\infty} \frac{\sin(n)}{n^{1/2}} = 0$$

<u>Proof</u> Let $\varepsilon > 0$ & set $N = 1/\varepsilon^2$.

If $n > N$ it follows that

$$\left| \frac{\sin(n)}{n^{1/2}} - 0 \right| = \frac{|\sin(n)|}{n^{1/2}} \leq \frac{1}{n^{1/2}} < \varepsilon \text{ since } n > \frac{1}{\varepsilon^2} \quad \square$$