Lecture 6

The Brun Sieve

It was with the aim of improving the efficiency of the sieve of Eratosthenes that the Norwegian mathematician Viggo Brun invented the theory of the combinatorial sieve between 1917 and 1924.

We saw last time that Eratosthenes' sieve relies fundamentally on the identity

$$\sum_{d|n} \mu(d) = \begin{cases} 1 & \text{if } n=1 \\ 0 & \text{if } n>1 \end{cases}.$$

Recall the proof of this identity:

distinct prime divisors of n.

Trivial for $n=1$. If $n>1$ and $k=\omega(n)$, then

$$\sum_{d|n} \mu(d) = \sum_{j=0}^{k} \binom{k}{j}(-1)^j = (1-1)^k = 0.$$

Exercise 1: Show that for any $k \in \mathbb{N}$, the alternating series

$$\sum_{j=0}^{m} \binom{k}{j}(-1)^j = (-1)^m \binom{k-1}{m} \begin{cases} \geq 0 & \text{if } m \text{ even} \\ \leq 0 & \text{if } m \text{ odd}. \end{cases}$$

It follows from this exercise that

$$\sum_{\substack{d|n \\ \omega(d)\leq m}} \mu(d) = \sum_{j=0}^{m} \binom{k}{j}(-1)^j \begin{cases} =1 & \text{if } k=0 \\ \geq 0 & \text{if } k\geq 1 \ \&\ m \text{ even} \\ \leq 0 & \text{if } k\geq 1 \ \&\ m \text{ odd}. \end{cases}$$

In particular, for every $h \in \mathbb{N}$

$$\sum_{\substack{d|n \\ \omega(d) \leq 2h-1}} \mu(d) \leq \sum_{d|n} \mu(d) \leq \sum_{\substack{d|n \\ \omega(d) \leq 2h}} \mu(d) \quad . \quad (*)$$

Recall that for a given finite sequence $\mathcal{A} = \{a_i\}$ of natural numbers and subset of the primes $\mathcal{P}$ we defined for $z > 0$

$$S(\mathcal{A}, \mathcal{P}, z) := \# \{a \in A : (a, P(z)) = 1\} = \sum_{d | P(z)} \mu(d) A_d$$

where $\displaystyle P(z) = \prod_{\substack{p \in \mathcal{P} \\ p \leq z}} p$ and $A_d = \# \{a \in \mathcal{A} : d | a\}$.

It follows immediately from $(*)$ that for any $h \in \mathbb{N}$

$$\sum_{\substack{d | P(z) \\ \omega(d) \leq 2h-1}} \mu(d) A_d \leq S(\mathcal{A}, \mathcal{P}, z) \leq \sum_{\substack{d | P(z) \\ \omega(d) \leq 2h}} \mu(d) A_d$$

General Form of Brun's Sieve

For applications the following is more immediately useful:

Theorem 1 (Brun's Sieve)

If there exists a non-negative multiplicative function v such that

$$A_d =: X \frac{v(d)}{d} + r_d \qquad (d | P(z))$$

with $X \approx$ size of $\mathcal{A}$, then for every $h \in \mathbb{N}$

$$S(\mathcal{A}, \mathcal{P}, z) = X \prod_{\substack{p \in \mathcal{P} \\ p \leq z}} \left(1 - \frac{v(p)}{p}\right) + O\Big(\sum_{\substack{d | P(z) \\ \omega(d) \leq 2h}} |r_d|\Big) + O\Big(X \sum_{\substack{d | P(z) \\ \omega(d) \geq 2h}} \frac{v(d)}{d}\Big) .$$

Proof of Theorem 1

It follows immediately from the "general form of Brun's sieve" that

$$S(\mathcal{A},\mathcal{P},z) = \sum_{\substack{d \mid P(z) \\ w(d) \le 2h}} \mu(d) A_d + O\Big(\sum_{\substack{d \mid P(z) \\ w(d) = 2h}} A_d \Big)$$

Using the fact that $A_d = X \frac{\nu(d)}{d} + r_d$ for all $d \mid P(z)$ we see that

- $$\sum_{\substack{d \mid P(z) \\ w(d) \le 2h}} \mu(d) A_d = X \sum_{\substack{d \mid P(z) \\ w(d) \le 2h}} \mu(d) \frac{\nu(d)}{d} + \sum_{\substack{d \mid P(z) \\ w(d) \le 2h}} \mu(d) r_d$$

$$= \underbrace{X \sum_{d \mid P(z)} \mu(d) \frac{\nu(d)}{d}}_{= X \prod_{\substack{p \in \mathcal{P} \\ p \le z}} \left(1 - \frac{\nu(p)}{p}\right)} + O\Big(X \sum_{\substack{d \mid P(z) \\ w(d) > 2h}} \frac{\nu(d)}{d} \Big) + O\Big(\sum_{\substack{d \mid P(z) \\ w(d) \le 2h}} |r_d| \Big).$$

- $$\sum_{\substack{d \mid P(z) \\ w(d) = 2h}} A_d \le X \sum_{\substack{d \mid P(z) \\ w(d) = 2h}} \frac{\nu(d)}{d} + \sum_{\substack{d \mid P(z) \\ w(d) = 2h}} |r_d| .$$

The result then follows immediately. □

Applications

Theorem 2 (Primes in an interval)

If $\mathcal{A} = \{x_0 < n \le x_0 + x\}$ and $\mathcal{P} = \{\text{all primes}\}$, then

$$S(\mathcal{A}, \mathcal{P}, z) = x \frac{e^{-\gamma}}{\log z} + O\left(\frac{x}{(\log z)^2}\right)$$

provided $z \le x^{1/8\log\log x}$ (and sufficiently large). Since for any x_0

$$\pi(x_0 + x) - \pi(x_0) \le z + S(\mathcal{A}, \mathcal{P}, z)$$

$$\Rightarrow \quad \pi(x_0 + x) - \pi(x_0) \ll \frac{x \log\log x}{\log x}.$$

Proof: Recall that $A_d = x\frac{1}{d} + r_d$, with $|r_d| \le 1$ for all $d \mid P(z)$.

Since

$$\prod_{p \le z}\left(1 - \frac{1}{p}\right) = \frac{e^{-\gamma}}{\log z} + O\left(\frac{1}{(\log z)^2}\right)$$

for all $z \ge 2$, it follows from Theorem 1 that we need only show:

$$\text{(i)} \quad \sum_{\substack{d \mid P(z) \\ \omega(d) \ge 2h}} 1/d = O\left(\frac{1}{(\log z)^3}\right)$$

and

$$\text{(ii)} \quad \sum_{\substack{d \mid P(z) \\ \omega(d) \le 2h}} 1 = O\left(\frac{x}{(\log z)^3}\right)$$

for some $h \in \mathbb{N}$, provided $z \le x^{1/8\log\log x}$ (and suff. large).

Proof of (i): For any $u \geq 1$

$$\sum_{\substack{d \mid P(z) \\ \omega(d) \geq 2h}}' \frac{1}{d} \leq \sum_{d \mid P(z)}' \frac{1}{d} u^{\omega(d)-2h} = u^{-2h} \prod_{p \leq z} \left(1 + \frac{u}{p}\right) \leq \exp\left(-2h \log u + u \sum_{p \leq z} \frac{1}{p}\right).$$

For the optimal choice $h := \frac{1}{2} u \sum_{p \leq z} \frac{1}{p}$ it follows, recalling that

$$\sum_{p \leq z} \frac{1}{p} = \log\log z + O(1)$$

that

$$\sum_{\substack{d \mid P(z) \\ \omega(d) \geq 2h}}' \frac{1}{d} \leq \exp\left((u - u\log u) \sum_{p \leq z} \frac{1}{p}\right) \ll_u (\log z)^{u - u\log u}.$$

Note that if $u > 5$, then $u - u\log u < -3$. Moreover, provided z is sufficiently large, it is easy to see that there exist $u \in (5,6)$ such that

$$u \sum_{p \leq z} \frac{1}{p} \in 2\mathbb{Z}.$$

Proof of (ii): $$\sum_{\substack{d \mid P(z) \\ \omega(d) \leq 2h}} 1 \leq \sum_{k=0}^{2h} \binom{\pi(z)}{k} \leq \sum_{k=0}^{2h} \pi(z)^k \leq 2\,\pi(z)^{2h} \leq z^{2h}.$$

For our choice of h, namely $h := \frac{1}{2} u \sum_{p \leq z} \frac{1}{p}$ (with $u \in (5,6)$), we have

$$z^{2h} \leq z^{7\log\log z} \leq \frac{x}{(\log x)^3} \quad \left(\ll \frac{x}{(\log z)^3}\right)$$

provided $$z \leq \left(\frac{x}{(\log x)^3}\right)^{1/7\log\log x}.$$

$\square$

It is a famous(ly open) conjecture that there are infinitely many twin primes. Let

$$J := \{p : p, p+2 \text{ both prime}\}$$

$$\pi_2(x) := \#\{1 \le n \le x : n \in J\}.$$

Using Brun's sieve we can establish the following:

<u>Theorem 3</u>: $\pi_2(x) \ll \dfrac{x(\log\log x)^2}{(\log x)^2}$.

<u>Corollary (Brun)</u>: $\displaystyle\sum_{p\in J} \frac{1}{p} < \infty$.

<u>Proof of Corollary</u>: By summation by parts

$$\sum_{\substack{p\in J\\ p\le x}} \frac{1}{p} = \frac{\pi_2(x)}{x} + \int_1^x \frac{\pi_2(t)}{t^2}\,dt$$

$$\ll \underbrace{\left(\frac{\log\log x}{\log x}\right)^2}_{\to 0 \text{ as } x\to\infty} + \underbrace{\int_1^\infty \frac{(\log\log t)^2}{t(\log t)^2}\,dt}_{<\infty}. \qquad \square$$

<u>Remark</u>: For historical reasons, in place of $\sum_{p\in J}\frac{1}{p}$, one usually considers

$$\sum_{p\in J}\left(\frac{1}{p}+\frac{1}{p+2}\right).$$

Of course this series is also convergent & its value B is known as Brun's constant. All we know presently about B is $1.830 \le B \le 2.347$!

Proof of Theorem 3:

Recall that is $\mathcal{A} = \{n(n+2) : n \le x\}$ and $\mathcal{P} = \{\text{all primes}\}$, then

$$\pi_2(x) \le z + S(\mathcal{A}, \mathcal{P}, z)$$

for all $z > 0$ and for each $d \mid P(z)$:

$$A_d = x\,\frac{v(d)}{d} + r_d\,, \quad |r_d| \le v(d)$$

where v is multiplicative with $v(p) = \begin{cases} 1 & \text{if } p = 2 \\ 2 & \text{if } p \ge 3 \end{cases}$.

In light of Theorem 1 and the fact that

$$\prod_{p \le z}\left(1 - \frac{v(p)}{p}\right) = \frac{1}{2}\, e^{-\sum_{3 \le p \le z} \log\left(1 - \frac{2}{p}\right)^{-1}} \le \frac{1}{2}\, e^{-2\sum_{3 \le p \le z} \frac{1}{p}} \ll \frac{1}{(\log z)^2}$$

it suffices to show that

$$\text{(i)} \quad \sum_{\substack{d \mid P(z) \\ \omega(d) \ge 2h}} \frac{v(d)}{d} = O\left(\frac{1}{(\log z)^3}\right)$$

and

$$\text{(ii)} \quad \sum_{\substack{d \mid P(z) \\ \omega(d) \le 2h}} v(d) = O\left(\frac{x}{(\log z)^3}\right)$$

for some $h \in \mathbb{N}$ whenever $\underline{z \le x^{1/12\log\log x}}$ (and suff. large).

Proof of (i): For any $u \geq 1$

$$\sum_{\substack{d \mid P(z) \\ \omega(d) \geq 2h}} \frac{\nu(d)}{d} \leq \sum_{d \mid P(z)} \frac{\nu(d)}{d} u^{\omega(d)-2h} = u^{-2h} \prod_{p \leq z} \left(1 + u \frac{\nu(p)}{p}\right)$$

$$\leq \exp\left(-2h \log u + 2u \sum_{p \leq z} \frac{1}{p}\right).$$

Letting $h := u \sum_{p \leq z} \frac{1}{p}$ it follows that

$$\sum_{\substack{d \mid P(z) \\ \omega(d) \geq 2h}} \frac{\nu(d)}{d} \leq \exp\left(2(u - u\log u) \sum_{p \leq z} 1/p\right) \ll_u (\log z)^{2(u - u\log u)}.$$

Note that if $u > 4$, then $2(u - u\log u) < -3$. Moreover, provided z is sufficiently large, there exist $u \in (4,5)$ such that $u \sum_{p \leq z} \frac{1}{p} \in \mathbb{Z}$.

Proof of (ii):

$$\sum_{\substack{d \mid P(z) \\ \omega(d) \leq 2h}} \nu(d) \leq \sum_{k=0}^{2h} 2^k \binom{\pi(z)}{k} \leq 2\,(2\pi(z))^{2h} \leq 2\, z^{2h}.$$

For our choice of h, namely $h := u \sum_{p \leq z} \frac{1}{p}$ (with $u \in (4,5)$), we have

$$z^{2h} \leq z^{11 \log\log z} \leq \frac{x}{(\log x)^3} \leq \frac{x}{(\log z)^3}.$$

provided $z \leq \left(\frac{x}{(\log x)^3}\right)^{1/12 \log\log x}$. $\square$

Exercise 2: Use the Brun Sieve to obtain an upper bound on

$$\pi_{\square + 1}(x) := \#\{n \leq x : n^2 + 1 \text{ prime}\}.$$