Roth's Theorem Revisited

A key component of the "classical" Fourier analytic proof of Roth's theorem (actually the easy part! and essentially Lemm 1 from those notes, but ignoring "wrap-around issues" for this discussion) is the following:

<u>FACT</u>: If $A \subseteq \mathbb{Z}_N$ with $|A| = \delta N$ and $|\widehat{1}_A(\xi)| \leq \varepsilon$ for all $\xi \neq 0$, then

$$|AP_3(1_A, 1_A, 1_A) - \delta^3| \leq \delta\varepsilon .$$

"If $\underbrace{\widehat{1}_A(\xi) \text{ is small } \forall \xi \neq 0}_{A \text{ is Fourier-pseudorandom}}$, then A contains essentially the 'expected" number of 3AP's

A is Fourier-pseudorandom.

⚠ It is however, <u>not</u> true that if $\widehat{1}_A(\xi)$ is small $\forall \xi \neq 0$, then A will contain essentially the 'expected" number of 4AP's ⚠

<u>Example</u> (see supplement for details)

Let $A = \{a \in \mathbb{Z}_N : a^2 \equiv b \bmod N \text{ with } |b| \leq \frac{\delta N}{2}\}$.

It is not hard to see that

(i) $|A| \approx \delta N$

(ii) $|\widehat{1}_A(\xi)| \ll \frac{\log N}{\sqrt{N}} \;\; \forall \xi \neq 0$ — <u>very</u> small!

<u>But</u>, A contains many <u>more</u> 4AP's than "expected", namely $\approx \delta^4 N^2$.

It in fact has $\gg \delta^3 N^2$ 4AP's (many more!), this is a consequence of the magical identity: $a^2 - 3(a+d)^2 + 3(a+2d)^2 = (a+3d)^2$.

(<u>Note</u>: It is also the case that one can (nicely) express the operator $AP_4(f_1, f_2, f_3, f_4) = \frac{1}{N^2} \sum_{x \in \mathbb{Z}_N} \sum_{d \in \mathbb{Z}_N} f_1(x) f_2(x+d) f_3(x+2d) f_4(x+3d)$ on transform side.)

The purpose of this note is to present a proof of Roth's theorem, that gives slightly weaker quantitive bound than Roth's origninal, but lends itself more readily to generalization (to longer arith. progs).

We shall establish the following:

<u>Theorem</u>: $\dfrac{r_3(N)}{N} \ll \dfrac{1}{(\log\log N)^{1/5}}$.

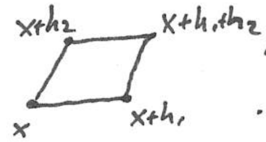
The key observation (made by Gowers) that leads to this approach and the subsequent proof of Szemerédi's theorem is the following

<u>Observation</u>: If $f : \mathbb{Z}_N \to \mathbb{D}$, then
unit ball in $\mathbb{C}$

$$(*) \quad \max_{\xi \in \mathbb{Z}_N} |\hat{f}(\xi)|^4 \leq \sum_{\xi \in \mathbb{Z}_N} |\hat{f}(\xi)|^4 \leq \max_{\xi \in \mathbb{Z}_N} |\hat{f}(\xi)|^2$$

and that for any $f : \mathbb{Z}_N \to \mathbb{C}$

$$(**) \quad \sum_{\xi \in \mathbb{Z}_N} |\hat{f}(\xi)|^4 = \frac{1}{N^3} \sum_{x, h_1, h_2 \in \mathbb{Z}_N} f(x)\overline{f(x+h_1)}\,\overline{f(x+h_2)}\,f(x+h_1+h_2).$$

"Average of f over parallelograms 

".

<u>Proof of (*)</u>: The 1^{st} inequality is immediate, the 2^{nd} follows from Plancherel and the fact that $|f(x)| \leq 1 \ \forall x \in \mathbb{Z}_N$.

<u>Proof of (**)</u>: Insert the definition of $\hat{f}(\xi)$ into LHS, multiply out the 4^{th} power and apply orthogonality (then relabel).

<u>Exercise 1</u>

For any $f: \mathbb{Z}_N \to \mathbb{C}$ we define its <u>Gowers U^2-norm</u> by

$$\|f\|_{U^2}^4 := \frac{1}{N^3} \sum_{x, h_1, h_2 \in \mathbb{Z}_N} f(x)\overline{f(x+h_1)}\,\overline{f(x+h_2)}f(x+h_1+h_2).$$

As in our presentation of Roth's original argument, we define, for functions $f_1, f_2, f_3 : \mathbb{Z}_N \to \mathbb{C}$, the operator

$$AP_3(f_1, f_2, f_3) = \frac{1}{N^2} \sum_{x, d \in \mathbb{Z}_N} f_1(x) f_2(x+d) f_3(x+2d).$$

We now observe the following:

<u>Lemma 1</u> (Generalized von-Neumann Theorem)

If $f_1, f_2, f_3 : \mathbb{Z}_N \to \mathbb{D}$, then $|AP_3(f_1, f_2, f_3)| \leq \|f_j\|_{U^2}$ $j = 1, 2, 3$.

(The proof of this is two application of Cauchy-Schwarz (see end of note))

and record the following consequence of the observation on the previous page

<u>Inverse Theorem for the U^2-norm</u>

If $f : \mathbb{Z}_N \to \mathbb{D}$ & $\|f\|_{U^2} \geq \varepsilon$, then $\exists\ \xi \neq 0$ s.t. $|\widehat{f}(\xi)| \geq \varepsilon^2$.

Recall the following result (from our presentation of Roth's original argument)

<u>Lemma 2</u> (Large non-zero Fourier coeff $\Rightarrow$ Arithmetic Structure)

If $\exists\ \xi \neq 0$ s.t. $|\widehat{1_A}(\xi)| \geq \varepsilon$, then $\exists$ <u>genuine</u> AP $P \subseteq \mathbb{Z}_N$ with

$$|P| \geq \left(\frac{\varepsilon N}{16\pi}\right)^{1/2} \text{ such that } |A \cap P| \geq (\delta + \varepsilon/8)|P|.$$

Here we are assuming that $A \subseteq \mathbb{Z}_N$ with density $\delta > 0$.

These 3 results allow us to estabish the following "dichotomy", from which the Theorem follows (via standard iterative argument).

Proposition (Dichotomy)

Let P be an arith. prog. of integers and $A \subseteq P$ with density $\delta > 0$. If $|P| \geq 1000\,\delta^{-10}$ (say), then either

(i) $\#$ 3AP's in A (inc. trivial) $\geq \dfrac{\delta^3 |P|^2}{32}$ ← in particular, at least one non-trivial 3AP.

OR

(ii) $\exists$ subprog. $P' \subseteq P$ with $|P'| \geq |P|^{1/3}$ such that

$$|A \cap P'| \geq \left(\delta + \frac{\delta^6}{2^{13}}\right)|P'| .$$

Exercise 2: Verify that Proposition $\Rightarrow$ Theorem.

Proof of Proposition: Recall (as in our presentation of Roth's original proof) that we can assume that $P = [1, N]$ and that $B := A \cap [N/3, 2N/3]$ satisfies $|B| \geq \frac{\delta}{4} N$. Now if (i) doesn't hold, then in particular

$$AP_3(1_B, 1_B, 1_A) < \delta^3/32$$

since $AP_3(1_B, 1_B, 1_A) \leq \quad \times \#$ (genuine) 3AP's in A (inc. trivial).

Let $f = 1_A - \delta$. It follows from Lemma 1 that

$$\|f\|_{U^2} \geq |AP_3(1_B, 1_B, f)| \geq \underbrace{AP_3(1_B, 1_B, \delta)}_{= (\frac{|B|}{N})^2 \delta \geq \delta^3/16} - \underbrace{AP_3(1_B, 1_B, 1_A)}_{< \delta^3/32} \geq \frac{\delta^3}{32} .$$

Since $\hat{f}(\xi) = \hat{1}_A(\xi)\ \forall\, \xi \neq 0$, it follows from the U^2-norm inverse theorem and Lemma 2, that $\exists$ arith prog. $P \in [1, N]$ with $|P| \geq N^{1/3}$ s.t. $\dfrac{|A \cap P|}{|P|} \geq \delta + \dfrac{\delta^6}{8(32)^2}$.

$\Box$

Proof of Lemma 1 (The Generalized von-Neumann Theorem)

Exercise 3: Prove that if $|B(x)| \leq 1 \ \forall x \in X$, then it follows from the Cauchy-Schwarz inequality that

$$\left| \frac{1}{|X|} \sum_{x \in X} \frac{1}{|Y|} \sum_{y \in Y} B(x) F(x,y) \right|^2 \leq \frac{1}{|X|} \sum_{x \in X} \frac{1}{|Y|^2} \sum_{y,y' \in Y} F(x,y) \overline{F(x,y')} .$$

Since

$$AP_3(f_1, f_2, f_3) = \frac{1}{N^2} \sum_{x,d} f_1(x) f_2(x+d) f_3(x+2d)$$

$$= \frac{1}{N^2} \sum_{x,y} f_1(x) f_2(y) f_3(2y-x) \qquad [N \text{ odd}]$$

and $|f_1(x)| \leq 1 \ \forall x$, it follows from Exercise 3, that

$$|AP_3(f_1,f_2,f_3)|^2 \leq \frac{1}{N^3} \sum_{x} \sum_{y,y'} f_2(y)\overline{f_2(y')} f_3(2y-x) \overline{f_3(2y'-x)}$$

$$= \frac{1}{N^2} \sum_{y,y'} \frac{1}{N} \sum_{x} \underbrace{f_2(y)\overline{f_2(y')}}_{|\cdot| \leq 1 \ \forall (y,y') \in \mathbb{Z}_N \times \mathbb{Z}_N} f_3(2y-x) \overline{f_3(2y'-x)}$$

Since $|f_2(y)\overline{f_2(y')}| \leq 1 \ \forall (y,y') \in \mathbb{Z}_N \times \mathbb{Z}_N$, it follows that

$$|AP_3(f_1,f_2,f_3)|^4 \leq \frac{1}{N^4} \sum_{y,y'} \frac{1}{N^2} \sum_{x,x'} \underset{\underset{a}{\parallel}}{f_3(2y-x)} \, \underset{\underset{a+h_1}{\parallel}}{\overline{f_3(2y'-x)}} \, \underset{\underset{a+h_2}{\parallel}}{\overline{f_3(2y-x')}} \, \underset{\underset{a+h_1+h_2}{\parallel}}{f_3(2y'-x')}$$

$$(h_1 = 2(y'-y)) \qquad (h_2 = x-x')$$

$$= \frac{1}{N^3} \sum_{a,h_1,h_2} f_3(a) \overline{f_3(a+h_1)}\, \overline{f_3(a+h_2)} f_3(a+h_1+h_2)$$

$$= \|f_3\|_{U^2}^4 .$$

□