Theorem ($\mathbb{Q}$ is dense in $\mathbb{R}$)

Given any two real numbers x & y with $x<y$, there exists a rational q such that $x<q<y$.

Proof

Since $y-x>0$ it follow from the "Archimedean Property" that $\exists\, n \in \mathbb{N}$ with $\frac{1}{n} < y-x \Leftrightarrow yn - xn > 1$.

Since the real numbers yn & xn are more than a distance 1 apart there must exist an integer m such that

$$xn < m < yn$$

But this implies that $x < \underbrace{\frac{m}{n}}_{= q} < y$ $\square$.

(*) In HW 1 Q3(b) you are asked to use this result to deduce that the irrationals are also dense in $\mathbb{R}$.