The Multiplier Problem for the Ball

We are concerned with the operator S, defined for $f \in L^2(\mathbb{R}^n)$ by

$$Sf(x) = \int_{|\xi|<1} \hat{f}(\xi) e^{2\pi i x\cdot\xi} \, d\xi .$$

<u>Theorem</u> (C. Fefferman) If $n \geq 2$ & $p \neq 2$, then S, initially defined on $L^2 \cap L^p$, does <u>not</u> extend to a bounded operator from $L^p(\mathbb{R}^n)$ to itself.

* Recall that if $P \subseteq \mathbb{R}^n$ is any convex <u>polyhedron</u> containing the origin, and

$$S_P f(x) = \int_{\xi \in P} \hat{f}(\xi) e^{2\pi i x\cdot\xi} \, d\xi ,$$

then S_P <u>does</u> extend to a bounded operator from $L^p(\mathbb{R}^n)$ to itself *

<u>Notation</u>: • For any ball $B \subseteq \mathbb{R}^n$, denote by S_B the multiplier operator associated to the ball B:

$$S_B f(x) = \int_B \hat{f}(\xi) e^{2\pi i x\cdot\xi} \, d\xi .$$

• Similarly, if u is a unit vector in $\mathbb{R}^n$, denote by S_u the multiplier operator associated to half-space in $\mathbb{R}^n$ whose normal direction is u:

$$S_u f(x) = \int_{\xi\cdot u > 0} \hat{f}(\xi) e^{2\pi i x\cdot\xi} \, d\xi .$$

We shall show that if an inequality of the form

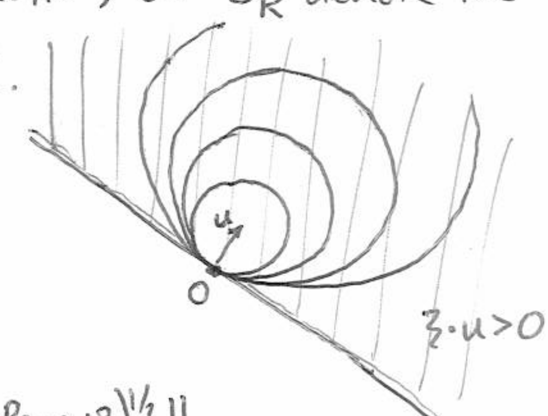
$$\|Sf\|_p \le A_p \|f\|_p \quad \leftarrow \text{Assumption } (*)$$

holds for all $f \in L^2 \cap L^p$, then so will corresponding vector-valued inequalities involving the S_B's & S_u's, to which we will exhibit a counterexample. In particular, we will prove

<u>Lemma 1</u>: Suppose that $\|Sf\|_p \le A_p \|f\|_p$ for all $f \in L^2 \cap L^p$ for some p, $1 \le p \le \infty$. Let $f_1, \dots, f_M \in L^2 \cap L^p$ & $u_1, \dots, u_M$ be unit vectors in $\mathbb{R}^n$, then

$$\Big\| \Big(\sum_{j=1}^{M} |S_{u_j} f_j|^2 \Big)^{1/2} \Big\|_p \le A_p \Big\| \Big(\sum_{j=1}^{M} |f_j|^2 \Big)^{1/2} \Big\|_p .$$

<u>Proof</u> Idea is to replace S_{u_j} with operators more closely related to the unit ball. Given a unit vector $u \in \mathbb{R}^n$, let B_R^u denote the ball of radius R centered at uR.



<u>Note</u>: $S_{B_R^u} = M_{uR} S_{B_R} M_{-uR}$

where $M_{uR} f(x) = e^{2\pi i uR \cdot x} f(x)$. Thus

$$\Big\| \Big(\sum_{j=1}^{M} |S_{B_R^{u_j}} f_j|^2 \Big)^{1/2} \Big\|_p = \Big\| \Big(\sum_{j=1}^{M} |S_{B_R}(e^{-2\pi i u_j R \cdot} f_j)|^2 \Big)^{1/2} \Big\|_p .$$

$$\le A_p \Big\| \Big(\sum_{j=1}^{M} |f_j|^2 \Big)^{1/2} \Big\|_p \qquad (**)$$

by Lemma 2 (coming up!) since $\|S_{B_R} f\|_p \le A_p \|f\|_p$ for all $f \in L^2 \cap L^p$.

Having established the conclusion of Lemma 1 for the operators $S_{B_R^{u_j}}$, the result follows by letting $R \to \infty$.

Note that $\chi_{B_R^{u_j}}(\xi) \longrightarrow \chi_{\{\xi : \xi \cdot u_j > 0\}}(\xi)$ for every $\xi \in \mathbb{R}^n$ as $R \to \infty$.

and hence it is easy to see (using Plancherel) that

$$S_{B_R^{u_j}} f_j \longrightarrow S_{u_j} f_j \quad \text{in } L^2(\mathbb{R}^n) \quad \text{as } R \to \infty$$

and consequently that an appropriate subsequence converges to $S_{u_j} f_j$ almost everywhere. Since from this it follows that

abusing notation

$$\left(\sum_{j=1}^{M} |S_{B_R^{u_j}} f_j(x)|^2 \right)^{p/2} \longrightarrow \left(\sum_{j=1}^{M} |S_{u_j} f_j(x)|^2 \right)^{p/2}$$

for a.e. x as $R \to \infty$, it follows from Fatou's lemma that

$$\int_{\mathbb{R}^n} \left(\sum_{j=1}^{M} |S_{u_j} f_j(x)|^2 \right)^{p/2} dx \leq \liminf_{R \to \infty} \int_{\mathbb{R}^n} \left(\sum_{j=1}^{M} |S_{B_R^{u_j}} f_j(x)|^2 \right)^{p/2}$$

$$\leq A_p^p \int_{\mathbb{R}^n} \left(\sum_{j=1}^{M} |f_j(x)|^2 \right)^{p/2} dx$$

by (**), as required. $\square$

Recall that the validity of (**) followed from Lemma 2, an as yet unstated result. We now briefly digress from the main argument to state and prove this basic result.

Lemma 2 Let $1 \le p \le \infty$. If T is a bounded linear operator from $L^p(\mathbb{R}^n)$ to itself that satisfies $\|Tf\|_p \le A_p \|f\|_p$ for all $f \in L^p(\mathbb{R}^n)$, then given any $f_1, \dots, f_M \in L^p(\mathbb{R}^n)$

$$\Big\| \Big(\sum_{j=1}^{M} |Tf_j|^2 \Big)^{1/2} \Big\|_p \le A_p \Big\| \Big(\sum_{j=1}^{M} |f_j|^2 \Big)^{1/2} \Big\|_p .$$

Proof Let $f = (f_1, \dots, f_M)$ & $Tf := (Tf_1, \dots, Tf_M)$.

For any given $w \in \mathbb{C}^M$ with $|w| = 1$ we define

$$T_w f = \langle Tf, w \rangle = \sum_{j=1}^{M} \overline{w}_j \, Tf_j$$

$$\& \quad f_w = \langle f, w \rangle = \sum_{j=1}^{M} \overline{w}_j \, f_j$$

Notice that $T_w(f) = T(f_w)$ and hence

$$\int_{\mathbb{R}^n} |T_w f(x)|^p \, dx \le A_p^p \int_{\mathbb{R}^n} |f_w(x)|^p \, dx$$

* Given any $z \in \mathbb{C}^M$, $\langle z, w \rangle = |z| \, \varphi(z, w)$ where $\varphi(z,w) = \langle \frac{z}{|z|}, w \rangle$. *

$$\Rightarrow \int_{\mathbb{R}^n} \Big(\sum_{j=1}^{M} |Tf_j(x)|^2 \Big)^{p/2} |\varphi(Tf(x), w)|^p \, dx$$

$$(\ast\ast\ast) \qquad \le A_p^p \int_{\mathbb{R}^n} \Big(\sum_{j=1}^{M} |f_j(x)|^2 \Big)^{p/2} |\varphi(f(x), w)|^p \, dx$$

Now integrate both sides of $(\ast\ast\ast)$ <u>with respect to w</u> (before integrating in x)

<u>Exercise</u>: Show that $\displaystyle\int_{|w|=1} |\varphi(z,w)|^p \, dw = \gamma_p \neq 0$ <u>independent of z</u>

$$\Rightarrow \int_{\mathbb{R}^n} \Big(\sum_{j=1}^{M} |Tf_j(x)|^2 \Big)^{p/2} dx \le A_p^p \int_{\mathbb{R}^n} \Big(\sum_{j=1}^{M} |f_j(x)|^2 \Big)^{p/2} dx . \qquad \square$$

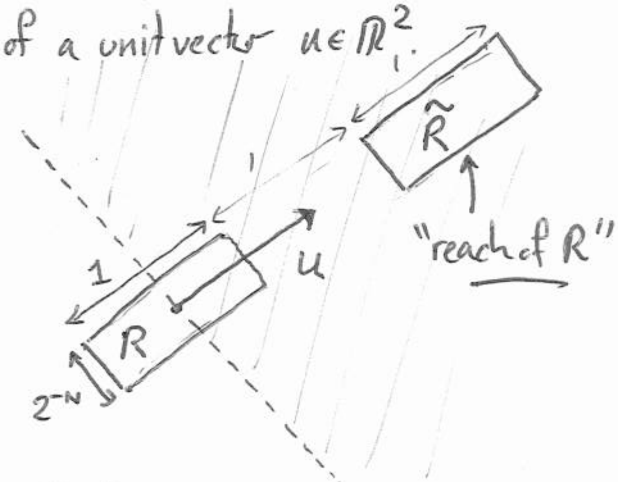
Back to the proof of the theorem...

We are charged with the task of constructing a counterexample to the conclusion of Lemma 1. Towards this end we note.

<u>Lemma 3</u> (in the plane $\mathbb{R}^2$)

Let R be a rectangle whose center is the origin in $\mathbb{R}^2$, has width 2^{-N} and length 1 and points in the direction of a unit vector $u \in \mathbb{R}^2$.

Then



$$(\ast\ast\ast\ast) \quad |S_u \chi_R(x)| \geq \frac{1}{10\pi} \chi_{\tilde{R}}(x)$$

for all $x \in \mathbb{R}^2$.

<u>Proof</u>

By applying a rotation we may assume that $u = e_2$ & hence

$$R = [-2^{-N-1}, 2^{-N-1}] \times [-\tfrac{1}{2}, \tfrac{1}{2}].$$

<u>Note</u>: If $F(x_1,x_2) = f_1(x_1) f_2(x_2)$, then (since the Fourier transform act independently in each variable) it follows that

$$S_u F(x_1,x_2) = f_1(x_1)\, S_{(0,\infty)} f_2(x_2)$$

$$\Rightarrow \quad S_u \chi_R(x_1,x_2) = \chi_{[-2^{-N-1}, 2^{-N-1}]}(x_1) \, \frac{I + iH}{2} \, \chi_{[-\frac{1}{2},\frac{1}{2}]}(x_2)$$

• <u>Analyze $H\chi_{[-\frac{1}{2},\frac{1}{2}]}(x)$ when $\frac{3}{2} \leq |x| \leq 5/2$</u>: Since $|x| \geq 3/2$ it follows that

$$H\chi_{[-\frac{1}{2},\frac{1}{2}]}(x) = \frac{1}{\pi}\int_{-1/2}^{1/2} \frac{1}{x-y}\, dy \geq \frac{1}{\pi}\int_{-1/2}^{1/2} \frac{1}{2|x|}\, dy = \frac{1}{2\pi|x|} \geq \underline{\frac{1}{5\pi}}$$

Since if $|x| \geq 3/2$ & $|y| \leq \frac{1}{2}$, then $x - y \leq |x-y| \leq |x| + |y| \leq 2|x|$. $\square$

We are now in a position to obtain our contradiction, using the following

<u>Lemma 4</u> [Theorem 1 in Handout on the Besicovitch set].

Given any $\varepsilon>0$, $\exists\ N=N(\varepsilon)$, and 2^N rectangles $R_1,\dots,R_{2^N}$, each having side lengths 1 & 2^{-N}, so that

(i) $\left|\bigcup_{j=1}^{2^N} R_j\right| < \varepsilon$

& (ii) the "reaches of R_j", $\widetilde{R}_j$ are mutually disjoint, $j=1,\dots,2^N$, and so

$$\left|\bigcup_{j=1}^{2^N} \widetilde{R}_j\right| = 1.$$

<u>Proof</u> (See Handout)

Let $\varepsilon>0$ & take $R_1,\dots,R_{2^N}$ to be the collection of rectangles from Lemma 4. We will obtain a contradiction with assumption $(*)$, first for $p<2$ & $n=2$.

As we have seen, assumption $(*)$ implies the conclusion of Lemma 1. If we take $f_j=\chi_{R_j}$ & $M=2^N$, then $(****)$ shows that the LHS of the conclusion of Lemma 1:

$$\Big\|\Big(\sum_{j=1}^{2^N} |S_{u_j}\chi_{R_j}|^2\Big)^{1/2}\Big\|_p \geq \frac{1}{10\pi} \qquad \text{since } \left|\bigcup \widetilde{R}_j\right| = 1.$$

However, by Hölder, the RHS of the conclusion:

$$A_p\Big\|\Big(\sum_{j=1}^{2^N} |\chi_{R_j}|^2\Big)^{1/2}\Big\|_p \leq A_p\Big(\underbrace{\int\Big(\sum_j |\chi_{R_j}|^2\Big)dx}_{=\sum|R_j|=1}\Big)^{1/2}\cdot\Big(\underbrace{\int_{\cup R_j} dx}_{=|\cup R_j|<\varepsilon}\Big)^{1/p-1/2}$$

$$\Rightarrow \frac{1}{10\pi} \leq A_p\,\varepsilon^{\frac{1}{p}-\frac{1}{2}} \quad ⚡ \text{ if } \varepsilon>0 \text{ is small enough.}$$

- When $n>2$, we split coordinates as $x=(x_1,x_2,x')$ & take $f_j(x)=\chi_{R_j}(x_1,x_2)f(x')$.
- The result for $p>2$ follows from $p<2$ by duality, since $\langle Sf,g\rangle=\langle f,Sg\rangle$ if $f,g\in L^2$ □