Orthonormal sets & Characterization of Basis

Let H be a Hilbert space.

A countable subset $\{u_n\}_{n=1}^{\infty}$ is called orthonormal if

$$\langle u_n, u_m \rangle = 0 \text{ if } n \neq m \quad \& \quad \|u_n\| = \langle u_n, u_n \rangle^{1/2} = 1 \quad \forall n.$$

We say that $\{u_n\}_{n=1}^{\infty}$ forms an orthonormal basis for H if

$$\overline{\text{Span}\{u_n\}} = H.$$

i.e. if the collection of all finite linear combinations of elements from $\{u_n\}_{n=1}^{\infty}$ is dense in H...

Bessel's Inequality: If $\{u_n\}_{n=1}^{\infty}$ is an orthonormal set in H, then for any $x \in H$

$$\sum_{n=1}^{\infty} |\langle x, u_n \rangle|^2 \leq \|x\|^2 .$$

If this is true, then $\{u_n\}_{n=1}^{\infty}$ is a basis

$$\left(\text{i.e. } \{\langle x, u_n \rangle\}_{n=1}^{\infty} \in \ell^2(\mathbb{N}) \right).$$

In fact: We showed that for any fixed N, the best linear approximation $\sum_{n=1}^{N} a_n u_n$ to x in H is given when $a_n = \langle x, u_n \rangle$.

Proof: $0 \leq \|x - \sum_{n=1}^{N} \langle x, u_n \rangle u_n \|^2 = \|x\|^2 - \sum_{n=1}^{N} |\langle x, u_n \rangle|^2 \quad \forall N$ $\square$

Note:
$$\left(\sum_{n=1}^{\infty} |\langle x, u_n \rangle|^2 \right)^{1/2} = \|x\| \quad \forall x \in H \iff \lim_{N \to \infty} \|x - \sum_{n=1}^{N} \langle x, u_n \rangle u_n \| = 0$$

(Parseval's Identity) ("Fourier Series" converge in H)

Theorem (Riesz-Fischer) | $\boxed{\text{"} x \mapsto \{\langle x, u_n\rangle\}_{n=1}^{\infty} \text{ maps } H \text{ \underline{onto} } \ell^2(\mathbb{N})\text{"}}$

If $\{u_n\}_{n=1}^{\infty}$ is an orthonormal set in H & $\{a_n\}_{n=1}^{\infty} \in \ell^2(\mathbb{N})$, then

$$\exists\, x \in H \text{ such that } a_n = \langle x, u_n\rangle \ \forall n \in \mathbb{N}.$$

Moreover, x can be chosen such that $\|x\| = \left(\sum_{n=1}^{\infty} |a_n|^2\right)^{1/2}$.

Note: The choice of x is NOT unique unless $\{u_n\}_{n=1}^{\infty}$ is complete

$\left[\{u_n\}_{n=1}^{\infty} \text{ is \underline{complete} if } \langle x, u_n\rangle = 0 \ \forall n \in \mathbb{N} \Rightarrow x = 0\right]$

Proof

- Let $S_N = \sum_{n=1}^{N} a_n u_n$. It is easy to see that $\{S_N\}$ is Cauchy in H:

$$\left[\|S_N - S_M\|^2 = \Big\|\sum_{n=M+1}^{N} a_n u_n\Big\|^2 \underset{\text{Pyth.}}{=} \sum_{n=M+1}^{N} |a_n|^2 \to 0 \text{ as } N, M \to \infty\right]$$

Since H is complete it follows that $S_N \to \underline{x}$ (say) in H.

- Now $\langle x, u_n\rangle = \langle x - S_N, u_n\rangle - \langle S_N, u_n\rangle \quad \forall n \in \mathbb{N}$.

$$\langle x - S_N, u_n\rangle \to 0 \text{ as } N \to \infty$$

$$\langle S_N, u_n\rangle = a_n \text{ if } N \geq n.$$

(Since $|\langle x - S_N, u_n\rangle| \leq \|x - S_N\| \to 0$)

$\Rightarrow a_n = \langle x, u_n\rangle \ \forall n \in \mathbb{N}$.

- Finally, since $\|x - S_N\|^2 \underset{\text{Bessel (proof)}}{=} \|x\|^2 - \sum_{n=1}^{N} |\langle x, u_n\rangle|^2 \Rightarrow \sum_{n=1}^{\infty} |\langle x, u_n\rangle|^2 = \|x\|^2$. $\square$

Theorem (Characterization of Basis)

Let $\{u_n\}_{n=1}^{\infty}$ be an orthonormal set in a Hilbert space H. The following are equivalent & characterize when $\{u_n\}$ forms an orthonormal basis for H.

(i) $\overline{\mathrm{Span}\{u_n\}} = H$ (Finite linear comb. of elts of $\{u_n\}$ dense in H)

(ii) (Completeness) $\langle x, u_n \rangle = 0 \ \forall n \Rightarrow x = 0$

(iii) (Parseval) $\left(\sum_{n=1}^{\infty} |\langle x, u_n \rangle|^2\right)^{1/2} = \|x\| \quad \forall x \in H$.

(iv) ("Fourier Series" converge in H)

$$\lim_{N\to\infty} \left\| \sum_{n=1}^{N} \langle x, u_n \rangle u_n - x \right\| = 0.$$

Proof

(i) $\Rightarrow$ (ii): Let $\varepsilon > 0$ & $x \in H$. Suppose $\langle x, u_n \rangle = 0 \ \forall n$, then by assumption $\exists\, y \in \mathrm{Span}\{u_n\}$ s.t. $\|x - y\| < \varepsilon$.

Since $\langle x, u_n \rangle = 0 \ \forall n \Rightarrow \langle x, y \rangle = 0$

$$\Rightarrow \|x\|^2 = \langle x, x \rangle = \langle x, x - y \rangle$$
$$\leq \|x\| \|x - y\| < \varepsilon \|x\|$$
$$\Rightarrow \|x\| < \varepsilon \quad \forall \varepsilon > 0$$
$$\Rightarrow \|x\| = 0 \Leftrightarrow x = 0.$$

(ii) $\Rightarrow$ (iii): Bessel $\Rightarrow \sum |\langle x, u_n \rangle|^2 < \infty$

Riesz-Fischer $\Rightarrow \exists\, y \in H$ s.t. $\sum |\langle x, u_n \rangle|^2 = \|y\|^2$ & $\langle y, u_n \rangle = \langle x, u_n \rangle \ \forall n$

$\Rightarrow x = y$ by completeness. $\triangle$