CLASSICAL STRONGLY SINGULAR CONVOLUTION OPERATORS ON $\mathbf{R}^n$

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1. Introduction and Summary

We shall be concerned with convolution operators, formally given by

$$Tf(x) = \int_{\mathbf{R}^n} K_\alpha(x-y)f(y)dy, \tag{1}$$

where K_α is a distribution on $\mathbf{R}^n$ that away from the origin agrees with the function

$$K_\alpha(x) = |x|^{-n-\alpha}e^{i|x|^{-\beta}}\eta(x), \tag{2}$$

where $\beta > 0$ and η is a smooth, compactly supported, radial function equal to 1 in the unit ball. We shall assume in what follows that $\alpha \geq 0$. The following L^p mapping properties of this operator are due to Wainger.

Theorem 1. *The convolution operator $Tf = f * K_\alpha$, defined initially for test functions,*

(i) *extends to a bounded operator on $L^p(\mathbf{R}^n)$ whenver $|\frac{1}{p} - \frac{1}{2}| < \frac{1}{2} - \frac{\alpha}{n\beta}$,*

(ii) *is not bounded on $L^p(\mathbf{R}^n)$ if $|\frac{1}{p} - \frac{1}{2}| > \frac{1}{2} - \frac{\alpha}{n\beta}$.*

The question of what happens at the endpoints $|\frac{1}{p} - \frac{1}{2}| = \frac{1}{2} - \frac{\alpha}{n\beta}$ was settled later. When $\alpha = 0$, Fefferman showed that T extends to an operator of weak type (1,1), then Fefferman and Stein showed that T in fact extends to a bounded operator from $H^1(\mathbf{R}^n)$ to $L^1(\mathbf{R}^n)$, this in turn implies that T remains bounded on the critical L^p spaces, $p = \frac{n\beta}{n\beta-\alpha}$, and $p = \frac{n\beta}{\alpha}$ whenever $\alpha > 0$.

It is clear that out kernal K_α is integrable if and only if $\alpha < 0$, hence T will be bounded on $L^1(\mathbf{R}^n)$ if and only if $\alpha < 0$. The sufficient half of Theorem 1 follows by interpolating between this result and the $L^2(\mathbf{R}^n)$ result below.

Theorem 2. *T extends to a bounded operator on $L^2(\mathbf{R}^n)$ if and only if $\alpha \leq \frac{n\beta}{2}$.*

2. Interpolation Argument

Consider the analytic family of operators $\{R_z\}$ given by

$$R_z f = f * M_z,$$

where

$$M_z(x) = e^{z^2}|x|^{\frac{n\beta}{2}z - \frac{n\beta}{2} - \gamma - n}e^{i|x|^{-\beta}},$$

and γ satisfies $\alpha - \frac{n\beta}{2} \leq \gamma < 0$. We note that $T = R_z$ if $z = 1 - \frac{2\alpha}{n\beta} + \frac{2\gamma}{n\beta}$.

If the $\mathrm{Re}(z) = 1$, then $\mathrm{Re}(-\frac{n\beta}{2}z + \frac{n\beta}{2} + \gamma) = \gamma < 0$, which implies M_z is integrable and therefore

$$\|R_z f\|_{L^1(\mathbf{R}^n)} \leq C\|f\|_{L^1(\mathbf{R}^n)}.$$

If the $\mathrm{Re}(z)=0$, then $\mathrm{Re}(-\frac{n\beta}{2}z+\frac{n\beta}{2}+\gamma)=\frac{n\beta}{2}+\gamma<\frac{n\beta}{2}$, therefore Theorem 2 implies
$$\|R_z f\|_{L^2(\mathbf{R}^n)}\leq C\|f\|_{L^2(\mathbf{R}^n)}.$$
Analytic interpolation then implies that if $\mathrm{Re}(z)=\theta$, then
$$\|R_z f\|_{L^p(\mathbf{R}^n)}\leq C\|f\|_{L^p(\mathbf{R}^n)},$$
whenever
$$\frac{1}{p}=\frac{1-\theta}{2}+\theta.$$
Recalling that we are interested in the case where $\theta=1-\frac{2\alpha}{n\beta}+\frac{2\gamma}{n\beta}$, we obtain that
$$\|Tf\|_{L^p(\mathbf{R}^n)}\leq C\|f\|_{L^p(\mathbf{R}^n)},$$
whenever
$$\frac{1}{p}-\frac{1}{2}=\frac{1}{2}-\frac{\alpha}{n\beta}+\frac{\gamma}{n\beta}.$$
This holds for all $\gamma<0$, so by duality this proves Theorem 1.

3. Proof of Theorem 2

Our convolution operator, T, may be realised on the transform side as a Fourier multiplier,
$$\widehat{Tf}(\xi)=\widehat{f}(\xi)\cdot m(\xi),$$
where $m=\widehat{K_\alpha}$ is a function since K_α is a compactly supported distribution. Plancherel's theorem for the Fourier transform then implies that
$$\|Tf\|_{L^p(\mathbf{R}^n)}\leq A\|f\|_{L^p(\mathbf{R}^n)}\quad\text{if and only if}\quad|\widehat{K_\alpha}(\xi)|\leq A\ \text{ uniformly in }\xi.$$

We shall consider seperately the cases where $|\xi|$ is large and when $|\xi|$ is bounded.

3.1. *The case when $|\xi|$ is unbounded.*

Lemma 3. *For large $|\xi|$,*
$$m(\xi)=c_1|\xi|^{\frac{\alpha-\frac{n\beta}{2}}{\beta+1}}e^{ic_2|\xi|^{\frac{\beta}{\beta+1}}}+O\big(|\xi|^{\frac{\alpha-\frac{(n+1)\beta}{2}}{\beta+1}}\big).$$

Proof. Since K_α is radial it follows that
$$\begin{aligned}\widehat{K_\alpha}(\xi)&=2\pi|\xi|^{\frac{2-n}{2}}\int_0^\infty(K_\alpha)_0(r)J_{\frac{n-2}{2}}(r|\xi|)r^{\frac{n}{2}}\,dr\\&=2\pi|\xi|^{\frac{2-n}{2}}\int_0^\infty\eta_0(r)r^{-\frac{n}{2}-\alpha}e^{ir^{-\beta}}J_{\frac{n-2}{2}}(r|\xi|)dr,\end{aligned}$$
where $J_{\frac{n-2}{2}}$ is a Bessel function.

Let N_0 be a suitably large integer. Let $\psi\in C_0^\infty(\mathbf{R})$ be a cut-off function with the properties that $\psi(x)=1$ for $|x|\leq1$, and $\psi(x)=0$ for $|x|\geq2$. Writing $1=\psi(\frac{r|\xi|}{N_0})+(1-\psi(\frac{r|\xi|}{N_0}))$, we'll consider
$$\widehat{K_\alpha}(\xi)=\mu(\xi)+\nu(\xi),$$
where
$$\mu(\xi)=2\pi|\xi|^{\frac{2-n}{2}}\int_0^\infty\psi(\tfrac{r|\xi|}{N_0})\eta_0(r)r^{-\frac{n}{2}-\alpha}e^{ir^{-\beta}}J_{\frac{n-2}{2}}(r|\xi|)dr,$$

and

$$\nu(\xi) = 2\pi|\xi|^{\frac{2-n}{2}} \int_0^\infty (1-\psi(\tfrac{r|\xi|}{N_0}))\eta_0(r) r^{-\frac{n}{2}-\alpha} e^{ir^{-\beta}} J_{\frac{n-2}{2}}(r|\xi|)dr.$$

The multiplier $\mu(\xi)$

Here our Bessel function is not oscillating. We shall now introduce a dyadic decomposition, to this end we consider the following partition of unity; choose $\vartheta \in C_0^\infty(\mathbf{R})$ supported in $[\frac{1}{2}, 2]$ such that

$$\sum_{j=0}^\infty \vartheta(2^j r) = 1 \text{ for all } r.$$

We therefore have

$$\mu = \sum_{j=0}^\infty \mu_j,$$

where

$$\mu_j(\xi) = 2\pi|\xi|^{\frac{2-n}{2}} \int_0^\infty \vartheta(2^j r)\psi(\tfrac{r|\xi|}{N_0})\eta_0(r) r^{-\frac{n}{2}-\alpha} e^{ir^{-\beta}} J_{\frac{n-2}{2}}(r|\xi|)dr.$$

Rescaling this gives

$$\mu_j(\xi) = 2\pi 2^{j(\alpha+\frac{n-2}{2})}|\xi|^{\frac{2-n}{2}} \int_{\frac{1}{2}}^2 \vartheta(r)\psi(\tfrac{r2^{-j}|\xi|}{N_0})\eta_0(2^{-j}r) r^{-\frac{n}{2}-\alpha} e^{i2^{j\beta}r^{-\beta}} J_{\frac{n-2}{2}}(r2^{-j}|\xi|)dr.$$

We now wish to show

$$\sum_{j=0}^\infty \mu_j(\xi) \leq C,$$

uniformly in ξ. We note that j is large.

The phase in this integral is clearly never critical, we can therefore integrate by parts all day long and obtain

$$|\mu_j(\xi)| \leq C 2^{j(\alpha+\frac{n-2}{2}-N\beta)},$$

for all $N \geq 0$.

The multiplier $\nu(\xi)$

Here our Bessel function is oscillating. Recall that as $x \to \infty$,

$$J_m(x) = x^{-\frac{1}{2}}[\sigma_1(x)e^{ix} + \sigma_2(x)e^{-ix}],$$

where

$$|\sigma_i^{(k)}(x)| \leq C|x|^{-k} \quad \text{for } k = 0, 1, \ldots$$

Write

$$\nu(\xi) = \nu_1(\xi) + \nu_2(\xi),$$

where

$$\nu_1(\xi) = 2\pi|\xi|^{\frac{1-n}{2}} \int_0^\infty (1-\psi(\tfrac{r|\xi|}{N_0}))\eta_0(r) r^{-\frac{n+1}{2}-\alpha} e^{i(r^{-\beta}+r|\xi|)}\sigma_1(r|\xi|)dr,$$

$$\nu_2(\xi) = 2\pi|\xi|^{\frac{1-n}{2}} \int_0^\infty (1-\psi(\tfrac{r|\xi|}{N_0}))\eta_0(r) r^{-\frac{n+1}{2}-\alpha} e^{i(r^{-\beta}-r|\xi|)}\sigma_2(r|\xi|)dr,$$

Let us first consider ν_1, making the change of variables $r \mapsto |\xi|^{-\frac{1}{\beta+1}} r$ we see

$$\nu_1(\xi) = 2\pi|\xi|^{\frac{\alpha-\frac{n\beta}{2}}{\beta+1}}|\xi|^{\frac{\beta}{2(\beta+1)}} \int_0^\infty (1-\psi(\tfrac{r|\xi|^{\frac{\beta}{\beta+1}}}{N_0}))\eta_0(r|\xi|^{-\frac{1}{\beta+1}}) r^{-\frac{n+1}{2}-\alpha} e^{i|\xi|^{\frac{\beta}{\beta+1}}\varphi(r)}\sigma_1(r|\xi|^{\frac{\beta}{\beta+1}})dr,$$

where

$$\varphi(r) = r^{-\beta} + r.$$

Now

$$\varphi'(r) = 1 - \beta r^{-(\beta+1)},$$

hence our phase φ is critical at $r_0 = \beta^{\frac{1}{\beta+1}}$, while $\varphi''(r_0) \neq 0$. Let

$$I = \int_0^\infty (1 - \psi(\tfrac{r|\xi|^{\frac{\beta}{\beta+1}}}{N_0}))\eta_0(r|\xi|^{-\frac{1}{\beta+1}})r^{-\frac{n+1}{2}-\alpha}e^{i|\xi|^{\frac{\beta}{\beta+1}}\varphi(r)}\sigma_1(r|\xi|^{\frac{\beta}{\beta+1}})dr,$$

and write

$$I = \int_0^{\frac{r_0}{2}} + \int_{\frac{r_0}{2}}^{\frac{3r_0}{2}} + \int_{\frac{3r_0}{2}}^{\infty} = I_1 + I_2 + I_3.$$

In I_1 we have $|\xi|^{-\frac{\beta}{\beta+1}} \leq r \leq \frac{r_0}{2}$. It is easy to see that $\beta r^{-(\beta+1)} \geq 2^{\beta+1}$, and so

$$|\varphi'(r)| \geq Cr^{-(\beta+1)},$$

while $|\varphi^{(k)}(r)| \leq Cr^{-(\beta+k)}$. Integrating by parts N times gives

$$|I_1| \leq C|\xi|^{-N\frac{\beta}{\beta+1}} \int_0^{\frac{r_0}{2}} r^{-\frac{n+1}{2}-\alpha+N\beta}dr.$$

Hence for N large enough

$$|I_1| \leq C|\xi|^{-N\frac{\beta}{\beta+1}}.$$

In I_2 we will use the method of stationary phase. Notice that

$$|\varphi^{(k)}(r)| \leq C_k \quad \text{for} \quad k = 0, 1, ... \quad \text{and} \quad |\varphi''(r)| \geq C > 0.$$

Also if we let

$$\Psi(r) = (1 - \psi(\tfrac{r|\xi|^{\frac{\beta}{\beta+1}}}{N_0}))\eta_0(r|\xi|^{-\frac{1}{\beta+1}})r^{-\frac{n+1}{2}-\alpha}\sigma_1(r|\xi|^{\frac{\beta}{\beta+1}}),$$

then it is easy to see that

$$|\Psi^{(k)}(r)| \leq C_k \quad \text{for} \quad k = 0, 1, ...$$

Thus we shall write

$$\Psi(r) = \Psi_1(r) + \Psi_2(r),$$

where Ψ_1 is smooth and vanishes near the endpoints of $[\frac{r_0}{2}, \frac{3r_0}{2}]$, and Ψ_2 is supported in a small neighbourhood of these endpoints. Then,

$$\int_{\frac{r_0}{2}}^{\frac{3r_0}{2}} \Psi_1(r)e^{i|\xi|^{\frac{\beta}{\beta+1}}\varphi(r)}dr = C|\xi|^{-\frac{\beta}{2(\beta+1)}}e^{i\varphi(r_0)|\xi|^{\frac{\beta}{\beta+1}}} + O\big(|\xi|^{-\frac{3}{2}\frac{\beta}{\beta+1}}\big).$$

While Van der Corput's lemma ensures,

$$\left|\int_{\frac{r_0}{2}}^{\frac{3r_0}{2}} \Psi_2(r)e^{i|\xi|^{\frac{\beta}{\beta+1}}\varphi(r)}dr\right| = C|\xi|^{-\frac{\beta}{\beta+1}}.$$

Finally, in I_3, we have $|\varphi'(r)| \geq C > 0$, while φ'' is of one sign. Van der Corput's lemma therefore gives (as above) the estimate

$$|I_3| \leq C|\xi|^{-\frac{\beta}{\beta+1}}.$$

Bringing all of this together we see that

$$\nu_1(\xi) = C|\xi|^{\frac{\alpha-\frac{n\beta}{2}}{\beta+1}}e^{i\varphi(r_0)|\xi|^{\frac{\beta}{\beta+1}}} + O\big(|\xi|^{\frac{\alpha-\frac{n\beta}{2}}{\beta+1}}|\xi|^{-\frac{\beta}{2(\beta+1)}}\big).$$

To deal with ν_2, notice that its phase is never critical and so the same argument as was used for I_1 above gives

$$|\nu_2(\xi)| \leq C|\xi|^{\frac{\alpha-\frac{n\beta}{2}}{\beta+1}}|\xi|^{\frac{\beta}{2(\beta+1)}}|\xi|^{-N\frac{\beta}{\beta+1}},$$

for all $N \geq 0$.

This completes the proof of Lemma 3. $\square$

3.2. ***The case when $|\xi|$ is bounded.***

Lemma 4. *If $|\xi| \leq N_0$, then $|\widehat{K_\alpha}(\xi)| \leq C$.*

Proof. Recall that

$$\widehat{K_\alpha}(\xi) = 2\pi|\xi|^{\frac{2-n}{2}} \int_0^\infty \eta_0(r) r^{-\frac{n}{2}-\alpha} e^{ir^{-\beta}} J_{\frac{n-2}{2}}(r|\xi|)dr,$$

where $J_{\frac{n-2}{2}}$ is a Bessel function. Note that

$$\frac{d}{dr}[x^{-m}J_m(x)] = -x^m J_{m+1}(x),$$

and as $x \to 0$

$$|J_m(x)| \leq C|x|^m.$$

Choose $\vartheta \in C_0^\infty(\mathbf{R})$ supported in $[\frac{1}{2}, 2]$ such that $\sum_{j=0}^\infty \vartheta(2^j r) = 1$ for all r. We can therefore write

$$\widehat{K_\alpha} = \sum_{j=0}^\infty m_j,$$

where

$$\begin{aligned} m_j(\xi) &= 2\pi|\xi|^{\frac{2-n}{2}} \int_0^\infty \vartheta(2^j r)\eta_0(r) r^{-\frac{n}{2}-\alpha} e^{ir^{-\beta}} J_{\frac{n-2}{2}}(r|\xi|)dr \\ &= 2\pi 2^{j(\alpha+\frac{n-2}{2})}|\xi|^{\frac{2-n}{2}} \int_{\frac{1}{2}}^2 \vartheta(r)\eta_0(2^{-j}r) r^{-\frac{n}{2}-\alpha} e^{i2^{j\beta}r^{-\beta}} J_{\frac{n-2}{2}}(r2^{-j}|\xi|)dr. \end{aligned}$$

Notice that we care only for when $2^{-j}|\xi| \to 0$, and that the phase in this integral is clearly never critical, we can therefore integrate by parts all day long and obtain

$$|m_j(\xi)| \leq C 2^{j(\alpha+\frac{n-2}{2}-N\beta)},$$

for all $N \geq 0$. Hence for N large enough we have,

$$|\widehat{K_\alpha}(\xi)| \leq \sum_{j=0}^\infty |m_j(\xi)| \leq C. \qquad \square$$

From lemmas 3 & 4 we obtain the following.

Corollary 5. *The Fourier transform $\widehat{K_\alpha}$ is a function and satisfies the inequality,*

$$|\widehat{K_\alpha}(\xi)| \leq C(1+|\xi|)^{\frac{\alpha-\frac{n\beta}{2}}{\beta+1}}.$$

Theorem 2 follows immediately from this.