STRONGLY SINGULAR CONVOLUTION OPERATORS ON $\mathbf{R}^d$ ASSOCIATED TO NORMS WITH WELL CURVED LEVEL HYPERSURFACES

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1. Introduction

Let $\rho(x)$ be a quasi-norm on $\mathbf{R}^d$ with the property that the level hypersurface $\rho(x) = 1$ has everywhere non-vanishing Gaussian curvature and $K_{\alpha,\beta}$ be a distribution[1] on $\mathbf{R}^d$ that away from the origin agrees with the function

$$K_{\alpha,\beta}(x) = \rho(x)^{-d-\alpha} e^{i\rho(x)^{-\beta}} \chi(\rho(x)), \tag{1}$$

where $\beta > 0$ and χ is smooth and compactly supported in a small neighborhood of the origin.

Theorem 1. *If $\alpha \leq \frac{d\beta}{2}$ then $Tf = f * K_{\alpha,\beta}$ extends to a bounded operator from $L^2(\mathbf{R}^d)$ to itself.*

A model case for operators of this type would be when we take $\rho(x) = |x|$, operators of this type were first studied by Hirschman [1] in the case $d = 1$ and then in higher dimensions by Wainger [4].

In tackling the model case it is efficient to use Fourier transform methods. Since $K_{\alpha,\beta}$ is radial it is well known that its Fourier transform is given by

$$m(\xi) = (2\pi)^{d/2} \int_0^\infty \chi(r) r^{-1-\alpha} e^{ir^{-\beta}} J_{d/2-1}(r|\xi|)(r|\xi|)^{1-d/2}\, dr, \tag{2}$$

where $J_{d/2-1}$ is a Bessel function; see [3]. Using Plancherel's theorem and the asymptotics of Bessel functions it is then straightforward to establish Theorem 1 in this case.

The argument above can be modified to prove Theorem 1 in full generality, we choose however to present a proof which is independent of Fourier transform methods.

2. Proof of Theorem 1

We now wish to decompose our operator $T = \sum_{j=0}^\infty T_j$. In order to do this we consider the following partition of unity; choose $\vartheta \in C_0^\infty(\mathbf{R})$ supported in $[\frac{1}{2}, 2]$ such that $\sum_{j=0}^\infty \vartheta(2^j r) = 1$ for all $0 \leq r \leq 1$, and write

$$T_j f(x) = f * K_j(x) \quad \text{where} \quad K_j(x) = \vartheta(2^j \rho(x)) K_{\alpha,\beta}(x).$$

Theorem 2. *The operator norms of T_j are uniformly bounded whenever $\alpha \leq \frac{d\beta}{2}$, more precisely*

$$\int_{\mathbf{R}^d} |T_j f(x)|^2 dx \leq C 2^{j(2\alpha - d\beta)} \int_{\mathbf{R}^d} |f(x)|^2 dx. \tag{3}$$

[1] The distribution-valued function $\alpha \mapsto K_{\alpha,\beta}$, initially defined for $\operatorname{Re}\alpha < 0$, continues analytically to all of $\mathbf{C}$.

We note that as the operator norms of T_j are equal to that of

$$S_j f(x) = 2^{j\alpha} \int_{\mathbf{R}^d} \vartheta(\rho(x-y))\rho(x-y)^{-d-\alpha} e^{i2^{j\beta}\rho(x-y)^{-\beta}} f(y)dy,$$

to prove Theorem 2 it suffices to establish estimate (3) for the operators S_j.

Key to establishing this result is the following proposition of Hörmander, which may be thought of as a variable coefficient version of Plancherel's theorem. See [5], Chapter 7 or [2], Chapter IX.

Proposition 3. *Let Ψ be a smooth function supported on the set $\{(x,y) \in \mathbf{R}^d \times \mathbf{R}^d : \rho(x-y) \leq C\}$ and Φ be real-valued and smooth on the support of Ψ. If we assume that all partial derivatives of Ψ and Φ are bounded and that*

$$\det\Big(\frac{\partial^2 \Phi}{\partial x_i \partial y_j}\Big) \neq 0 \tag{4}$$

on the support of Ψ, then

$$\Big\| \int_{\mathbf{R}^d} \Psi(x,y) e^{i\lambda\Phi(x,y)} f(y)dy \Big\|_{L^2(\mathbf{R}^d)} \leq C\lambda^{-\frac{d}{2}} \|f\|_{L^2(\mathbf{R}^d)}.$$

It is clear that estimate (3) for the operators S_j will be an immediate consequence of Proposition 3 once we have established that the phase of its kernel is non-degenerate (in the sense of (4) above).

Lemma 4. *Let $\Phi(x,y) = \rho(x-y)^{-\beta}$, then $\det\big(\frac{\partial^2\Phi}{\partial x_i \partial y_j}\big) \neq 0$ whenever $x \neq y$ and $\beta \neq -1$.*

Proof. It clearly suffice to verify that if $\rho(x) = 1$ and $\beta \neq -1$, then

$$\det H\rho^{-\beta}(x) \neq 0$$

where $H\rho^{-\beta}(x)$ denotes the (pure) Hessian matrix of the phase function $\rho(x)^{-\beta}$. Now an easy calculation shows that

$$H\rho^{-\beta}(x) = -\beta\, \rho(x)^{-(\beta+2)} \left[\rho(x)\, H\rho(x) - (\beta+1)\nabla\rho(x)\nabla\rho(x)^{\mathrm{T}}\right].$$

Matters therefore reduce to showing that if $\beta \neq -1$ and

$$H\rho(x)u = (\beta+1)\langle \nabla\rho(x), u\rangle \nabla\rho(x), \tag{5}$$

then $u = 0$. Since $\langle \nabla\rho(x), x\rangle = \rho(x) \neq 0$ we may write $u = \lambda x + v$ where v is a vector perpendicular to $\nabla\rho(x)$.

It follows from Euler's homogeniety relations that

$$\langle \nabla\rho(x), x\rangle = \rho(x) \quad \text{while} \quad H\rho(x)x = 0, \tag{6}$$

and hence that (5) is equivalent to the identity

$$H\rho(x)v = \lambda(\beta+1)\nabla\rho(x).$$

It then follows from (6) and the symmetry of $H\rho(x)$ that $\lambda = 0$. The result then follows from the curvature condition, since from this it follows that if $v \neq 0$ is perpendicular to $\nabla\rho(x)$, then

$$\langle H\rho(x)v, v\rangle \neq 0. \tag{7}$$

□

Theorem 1 now follows from Theorem 2 and an application of Cotlar's lemma (plus a standard limiting argument) once we have verified that the T_j are, in the following sense, almost orthogonal.

Lemma 5. *If* $\alpha = \frac{d\beta}{2}$ *then* $\|T_i^* T_j\|_{Op} + \|T_i T_j^*\|_{Op} \leq C 2^{-\frac{d\beta}{2}|i-j|}$.

Proof. This follows trivially from Theorem 2 whenever $|i-j| \leq 10$, since $\|T_i^* T_j\|_{Op} \leq \|T_i\|_{Op} \|T_j\|_{Op}$. We shall therefore, without loss of generality, assume that $j \geq i + 10$. Now $T_i^* T_j$ has a kernel

$$L_{ij}(x) = K_j * \bar{K}_i(-x),$$

and the same operator norm as the operator with kernel

$$\begin{aligned} \widetilde{L}_{ij}(x) &= 2^{-jd} L_{ij}(2^{-j}x) \\ &= 2^{-j2d} \int K_j(2^{-j}y) \bar{K}_i(2^{-j}(x-y)) dy \\ &= 2^{j2\alpha} \int_{\substack{\rho(y)\sim 1 \\ \rho(x-y)\sim 2^{j-i}}} \rho(y)^{-d-\alpha} \rho(x-y)^{-d-\alpha} e^{i2^{j\beta}[\rho(y)^{-\beta} - \rho(x-y)^{-\beta}]} dy. \end{aligned}$$

We trivially have the estimate $|\widetilde{L}_{ij}(x)| \leq C 2^{j2\alpha} 2^{(i-j)(d+\alpha)}$. It is easy to verify, by homogeneity, that

$$|\nabla_y [\rho(y)^{-\beta} - \rho(x-y)^{-\beta}]| \geq C_0,$$

thus there is always a direction in which we may integrating by parts, in doing so d times we obtain

$$|\widetilde{L}_{ij}(x)| \leq C 2^{j(2\alpha - d\beta)} 2^{(i-j)(d+\alpha)} = 2^{(i-j)(d+\alpha)}.$$

This of course implies that

$$\int |\widetilde{L}_{ij}(x)| \, dx \leq C 2^{(i-j)\alpha}. \qquad \square$$

References

[1] I. I. Hirschman, *Multiplier Transforms I*, Duke Math. J., 26 (1956), pp. 222–242.
[2] E. M. Stein, *Harmonic Analysis: Real–Variable Methods, Orthogonality, and Oscillatory Integrals*, Princeton Univ. Press, Princeton, 1993.
[3] E. M. Stein and G. Weiss, *Introduction to Fourier Analysis on Euclidean Spaces*, Princeton Univ. Press, Princeton, 1971.
[4] S. Wainger, *Special Trigonometric Series in k Dimensions,* Memoirs of the AMS ***59***, American Math. Soc., 1965.
[5] T. H. Wolff, *Lectures on Harmonic Analysis,* University Lecture Series ***29***, American Math. Soc., 2003.