Discrete Mathematics
Peergrade assignment 3

1. Let $f : \mathbb{Z} \to \mathbb{Z}$ be defined as $f(n) = 2n - 5$ for all integers n.

   (a) List three elements that are in the range of f

   (b) Is f one-to-one? Justify your answer

   (c) Is f onto? Justify your answer

   (d) Write an explicit formula for the composition $f \circ f$

   Note: to justify your answers in (b) and (c), if the answer is positive you need to provide a proof, and if it is negative, give a counterexample.

2. Let $a_k = 2k - 5$ and $b_k = 2 - k$. Simplify each expression to only use a single summation (Σ) or product (Π) using the properties of summations and products listed below. List intermediate steps.

   (a) $\sum_{k=m}^{n} a_k - 3 \cdot \sum_{k=m}^{n} b_k$

   (b) $\prod_{k=m}^{n} a_k \cdot \prod_{k=m}^{n} b_k$

   Theorem 5.1.1

   If $a_m, a_{m+1}, a_{m+2}, \ldots$ and $b_m, b_{m+1}, b_{m+2}, \ldots$ are sequences of real numbers and c is any real number, then the following equations hold for any integer $n \geq m$:

   1. $\displaystyle\sum_{k=m}^{n} a_k + \sum_{k=m}^{n} b_k = \sum_{k=m}^{n} (a_k + b_k)$

   2. $\displaystyle c \cdot \sum_{k=m}^{n} a_k = \sum_{k=m}^{n} c \cdot a_k$ generalized distributive law

   3. $\displaystyle\left(\prod_{k=m}^{n} a_k\right) \cdot \left(\prod_{k=m}^{n} b_k\right) = \prod_{k=m}^{n} (a_k \cdot b_k).$

   For instance, given sequences $a_k = k$ and $b_k = 2k + 1$, simplifying an expression $\sum_{k=m}^{n} a_k + 2 \cdot \sum_{k=m}^{n} b_k$ could be done in the following way:

$$\begin{aligned}
\sum_{k=m}^{n} a_k + 2 \cdot \sum_{k=m}^{n} b_k &= \sum_{k=m}^{n} k + 2 \cdot \sum_{k=m}^{n} (2k+1) && \text{by substitution} \\
&= \sum_{k=m}^{n} k + \sum_{k=m}^{n} 2 \cdot (2k+1) && \text{by (2)} \\
&= \sum_{k=m}^{n} k + \sum_{k=m}^{n} (4k+2) && \text{by algebraic simplification} \\
&= \sum_{k=m}^{n} (k + (4k+2)) && \text{by (1)} \\
&= \sum_{k=m}^{n} (5k+2) && \text{by algebraic simplification}
\end{aligned}$$

3. Prove, using mathematical induction, that 3 divides $n^3 + 5n - 6$ for all integers $n \geq 0$.

   Hint: You can use the fact that given $a \,|b$ and $a \,|c$, we can conclude $a \,|(b+c)$ for all integers a, b, c. Moreover, you can use the binomial theorem for exponent 3, which states that $(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$ for all real numbers a, b.