Name,DoB: Answer Key

IT University of Copenhagen

Discrete Mathematics, MSc SD

Exam

10 March, 2021

Instructions (Read Carefully)

Contents: The exam contains 13 questions for a total of 100 points. The exam is divided into two parts: The first part has 9 multiple choice questions and the second part has 4 open ended questions.

What to check: In the multiple-choice questions, there is one and only one correct answer. You should only check 1 box.

Part I. Answer the following multiple choice questions.

1. (6 pts) Which of the following statements is **true**?

A $\{n \in \mathbb{Z} \mid n \leq 3 \vee n \geq 1\} = \{0, 1, 2, 3\} \cup \{1, 2, 3, 4\}$.

B $\{\emptyset\} \in \mathcal{P}(\{1, 2, 3\})$.

✔ $|\{1, 2\} \times \{-1, -2\}| = |\mathcal{P}(\{1, 2\})|$.

D $\emptyset \cap \{1, 2, 3\} = \{1, 2, 3\}$.

2. (6 pts) We would like to define a one-to-one function $f : A \to \mathbb{Z}$ such that

$$f(n) = \frac{n^2}{2} \quad \text{for all } n \in A$$

Only one of the choices for the set A given below makes f a one-to-one function. Which one?

A $A = \{-2, -1, 0, 1, 2\}$.

B $A = \{-4, -2, 0, 2, 4\}$.

✔ $A = \{-6, -2, 0, 4, 8\}$.

D $A = \{0, 1, 4, 9, 16\}$.

3. (6 pts) Let $f : \mathbb{Z} \to \mathbb{Z}$ be a function defined as follows:

$$f(n) = (2n) \bmod 6$$

What is the range of f?

A $\{-4, -2, 0, 2, 4\}$.

B $\{0, 1, 2, 3, 4, 5\}$.

✔ $\{0, 2, 4\}$.

D $\{0, 1, 2\}$.

4. (6 pts) ITU asks you to pick a new four-digit PIN. The PIN consists of four digits between 0 and 9, but you may **not** choose a PIN whose digits are all distinct and appear in increasing order. For example 0379 is not allowed, but 0337, 9730, and 1739 are allowed.
How many different allowed PINs can you choose from?

A 1210

[B] 5260

[C] 7824

[✔] 9790

5. (6 pts) The American smartphone manufacturer *Strawberry* ships their phones with two different chips, depending on whether the phone is sold in the US or Europe. 60% of their phones sold in the US use *Strawberry*'s own chips. The remaining 40% use chips supplied by a different company. By contrast, only 20% of their phones sold in European markets have *Strawberry*'s own chips. *Strawberry* sells 60% of their phones in Europe and the remaining 40% in the US.
You have recently bought a used *Strawberry* phone. After tinkering with the phone for a while, you notice that it uses a chip made by *Strawberry*. What is the probability that it was originally sold in the US?

[A] $\frac{1}{2}$

[B] $\frac{3}{5}$

[✔] $\frac{2}{3}$

[D] $\frac{3}{7}$

6. (6 pts) Consider the binary relation R on $\mathbb{Z}$ defined as follows for all $n, m \in \mathbb{Z}$:

$$R = \{(m, n) \mid \text{if } 4 \text{ divides } m, \text{ then } 4 \text{ divides } n\}$$

Which of the following statements is **true**?

[A] R is an equivalence relation.

[B] R is a partial order relation.

[C] R is symmetric but not reflexive

[✔] R is transitive, but not antisymmetric

7. (6 pts) Below you are given four languages described using Kleene closure (*), concatenation and union. Only one of these languages does **not** contain any strings where 1 occurs twice in a row. That is, it does not contain strings such as 01110 and 10110. Which language is that

[A] $\{10, 01\}^*$.

[B] $\{01\}^* \{10\}^*$.

[✔] $\{1, \lambda\} \{01\}^*$.

D $\{1\}^* \cup \{0\}^*$.

8. (6 pts) Let S be a set of integers defined recursively as follows:

- I. BASE: $2 \in S$.
- II. RECURSION: If $s \in S$ and $t \in S$, then $s - t \in S$.
- III. RESTRICTION: Nothing is in S other than objects defined in I and II above.

Which of the following statements is **true**?

A All elements in S are positive.

B All elements in S are even.

C S is finite.

D S contains all integers.

9. (6 pts) One of the compound propositions below is logically equivalent to the compound proposition $p \vee \sim q$. Which one?

A $(p \vee q) \rightarrow p$

B $(p \wedge q) \rightarrow \sim p$

C $(\sim q \vee p) \rightarrow p$

D $(p \vee q) \wedge \sim p$

Part II. Answer the following questions. Be brief but precise. Your correct use of mathematical notation is an important aspect of your answer.

10. (12 pts) Prove the following statement by mathematical induction:

$$\sum_{i=1}^{n} \frac{1}{i \cdot (i+1)} = \frac{n}{n+1} \qquad \text{for all } n \geq 1$$

Solution: We want to prove the statement

$$\underbrace{\sum_{i=1}^{n} \frac{1}{i \cdot (i+1)} = \frac{n}{n+1}}_{P(n)} \qquad \text{for all } n \geq 1$$

for all $n \geq 1$.

Basis step: Let $n = 1$. The left-hand side of $P(1)$ simplifies to

$$\sum_{i=1}^{1} \frac{1}{i \cdot (i+1)} = \frac{1}{1 \cdot 2} = \frac{1}{2}$$

and the right-hand side of $P(1)$ simplifies to

$$\frac{1}{1+1} = \frac{1}{2}$$

Inductive step:
Suppose that $k \geq 1$ and that $P(k)$ holds, that is,

$$\sum_{i=1}^{k} \frac{1}{i \cdot (i+1)} = \frac{k}{k+1}$$

We must show that $P(k+1)$ holds, that is,

$$\sum_{i=1}^{k+1} \frac{1}{i \cdot (i+1)} = \frac{k+1}{k+2}$$

The following derivation proves this equation:

$$
\begin{aligned}
& \sum_{i=1}^{k+1} \frac{1}{i \cdot (i+1)} \\
= & \sum_{i=1}^{k} \frac{1}{i \cdot (i+1)} + \frac{1}{(k+1)(k+2)} \\
= & \frac{k}{k+1} + \frac{1}{(k+1)(k+2)} && \text{(inductive hypothesis)} \\
= & \frac{(k+2)k+1}{(k+1)(k+2)} \\
= & \frac{k^2+2k+1}{(k+1)(k+2)} \\
= & \frac{(k+1)^2}{(k+1)(k+2)} && \text{(binomial theorem)} \\
= & \frac{k+1}{k+2} && \text{(cancel common factor } k+1\text{)}
\end{aligned}
$$

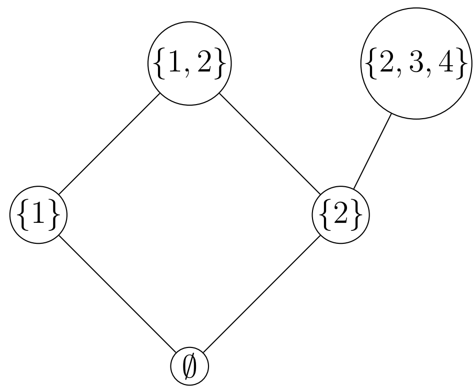
11. (12 pts) Consider the subset relation on the set $A = \{\emptyset, \{1\}, \{2\}, \{1,2\}, \{2,3,4\}\}$. That is, consider the relation R defined as follows:

$$R = \{(S,T) \in A \times A \mid S \subseteq T\}$$

The relation R is a partial order relation.

(a) Draw the Hasse diagram for R.

Solution:



(b) Find all minimal elements of R. Write 'None' if there aren't any.

Solution: $\emptyset$

(c) Find all maximal elements of R. Write 'None' if there aren't any.

Solution: $\{1,2\}, \{2,3,4\}$

(d) Find all least elements of R. Write 'None' if there aren't any.

Solution: $\emptyset$

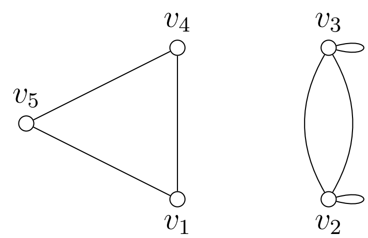
(e) Find all greatest elements of R. Write 'None' if there aren't any.

Solution: None

12. (12 pts) In the following, you are given two descriptions a graph. For each of these, you are asked whether a graph of that description exists. To answer the question, give either an example of a graph that satisfies the condition, or a reason why no such graph exists. In order to give an example, either draw the corresponding graph or give the triple (V, E, f) of vertices, edges and edge-endpoint function. Hint: Use the definitions and theorems about graphs and trees on pages 6–7 in the compendium.

(a) Does a graph G of the following description exist: G has no Euler circuit, and G has exactly 5 vertices. 3 of the vertices have degree 2, the other 2 vertices have degree 4.

Solution: Such a graph does exist:



(b) Does a graph G of the following description exist: G is a tree with 6 vertices, and each of the vertices has degree 2.

Solution: The tree has a total degree of 12 and hence must have exactly 6 edges according to the Handshake Theorem (Theorem 11). However, according to Theorem 14 a tree with 6 vertices has exactly 5 edges.

13. (10 pts) Construct a finite-state automaton A with input alphabet $\{0,1\}$ that recognises the set of strings with an even number of 0s but no occurrence of two consecutive 0s. That is, A must satisfy

$$L(A) = \{w \in \{0,1\}^* \mid |w|_0 \text{ even}\} - \{w00v \mid w, v \in \{0,1\}^*\}$$

Describe the automaton A using a next-state table *or* a transition diagram.

Solution: