Lab 1: Filtering operations

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Question 1: Repeat this exercise with the coordinates p and q set to (5, 9), (9, 5), (17, 9), (17, 121), (5, 1) and (125, 1) respectively. What do you observe?

Answer:

The observations made are as follows

In the Fourier Domain all the six figures follow the similar pattern as it has the same value of the Amplitude. When we take the inverse Fast Fourier transform, We get

$$f(a,b) = \frac{1}{\sqrt{MN}} * \sum \quad \sum F(k,l) * e^{2*\pi*i\left(\frac{ka}{M}+\frac{lb}{N}\right)}$$

Here the image is a square Image M = N and also the Fast Fourier transform value is 1 at k=p and l=q points across the image. Thus, F(k,l) =1. Hence, we see that all the plots have the same amplitude value and also the frequency value is same and the real and imaginary part has the phase difference of $\pi/2$.

Considering the wavelength as we know that the wavelength λ is given as,

$$\lambda = (1/\sqrt{u^2 + v^2})$$

And we have u=p and v=q , we can conclude that the wavelength for every p and q will be inversely proportional to the distance of (p,q) from the point (0,0) in Fourier domain.

Question 2: Explain how a position (p; q) in the Fourier domain will be projected as a sine wave in the spatial domain. Illustrate with a MATLAB figure.

Answer:

Technically any Image can be split as a sum of sinusoidal waves of various frequencies in the spatial domain. In this particular case we can illustrate using an example. Let’s take p= 17 and q = 9

Thus, for any image we can technically obtain the sinusoidal in 3d in spatial domain. As we observe the Fast Fourier Inverse Transform as

$$f(a,b) = \frac{1}{\sqrt{MN}} * \sum \quad \sum F(k,l) * e^{2*\pi*i\left(\frac{ka}{M}+\frac{lb}{N}\right)}$$

As there exist only value for F(k,l) = 1 there for the exponential term will give to the real term characterised by sine and the imaginary term with the cosine having the same frequency as we can see it after plugging values like k=p,l=q and M=N=128. The figure below illustrates the plots when p = 17 and q= 9.

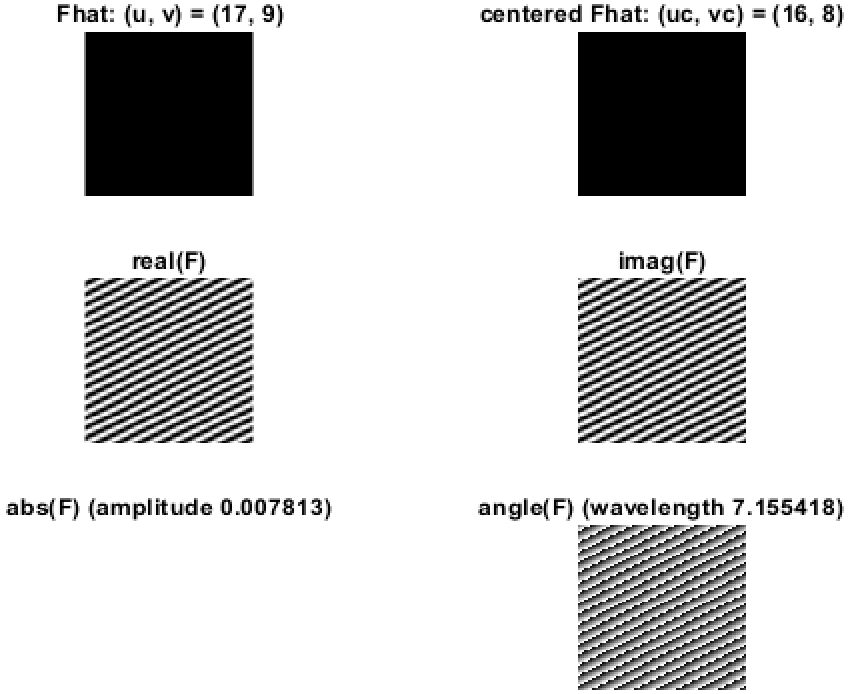


Figure 1: Plots for the inverse fast Fourier transform considering p=17 and q=9

Making a surface plot for the real F we observe that it is a sine plot as shown below.

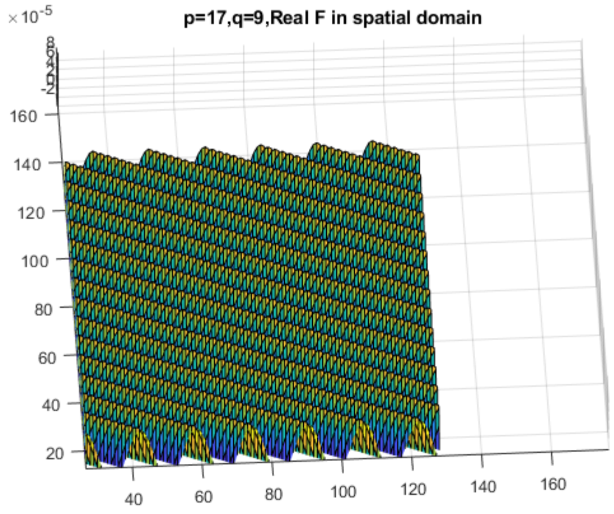


Figure 2 : The surface plot of the real F when p =17 and q=9.

Question 3: How large is the amplitude? Write down the expression derived from Equation (4) in these notes. Complement the code (variable amplitude) accordingly.

Answer:

We obtain the inverse fast fourier transform by using the following equation:

$$f(a,b) = \frac{1}{\sqrt{MN}} * \sum \quad \sum F(k,l) * e^{2*\pi*i\left(\frac{ka}{M}+\frac{lb}{N}\right)}$$

Amplitude is derived in the following way:

$$A = \frac{1}{\sqrt{MN}} * \max\big(F(k,l)\big) = 1/128$$

The expression for amplitude in the code will look like this:

```
w1 = 2 * pi * uc / sz;
w2 = 2 * pi * vc / sz;
wavelength = 2 * pi / sqrt(w1^2 + w2^2); % wavelength
Amplitude = max(Fhat(:)) / sz; % amplitude
```

Question 4 : How does the direction and length of the sine wave depend on p and q? Write down the explicit expression that can be found in the lecture notes. Complement the code (variable wavelength) accordingly.

Answer:

As mentioned in the lecture notes, wavelength formula is as follows:

$$\text{wavelength} = 2 * \pi \;/(\sqrt{((\text{w1})^2 + (\text{w2})^2})$$

$$w1 = 2 * \pi * \frac{uc}{sz}$$

$$w2 = 2 * \pi * \frac{vc}{sz}$$

The expression for wavelength in the code will look like this:

```
w1 = 2 * pi * uc / sz;
w2 = 2 * pi * vc / sz;
wavelength = 2 * pi / sqrt(w1^2 + w2^2); % wavelength
Amplitude = max(Fhat(:)) / sz; % amplitude
```

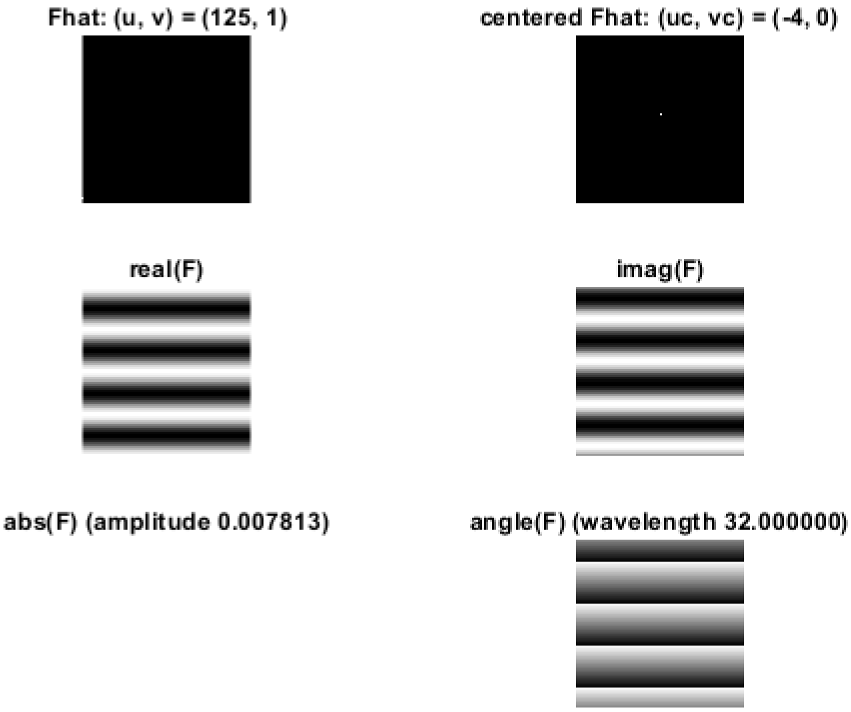
In our example uc =p and vc =q thus we see that wavelength is inversely proportional to the distance of (p,q) from the origin in frequency domain.

The direction is found by a line from origin to the point (p,q) as it follows a traversing wave property.

Question 5 : What happens when we pass the point in the center and either p or q exceeds half the image size? Explain and illustrate graphically with Matlab!

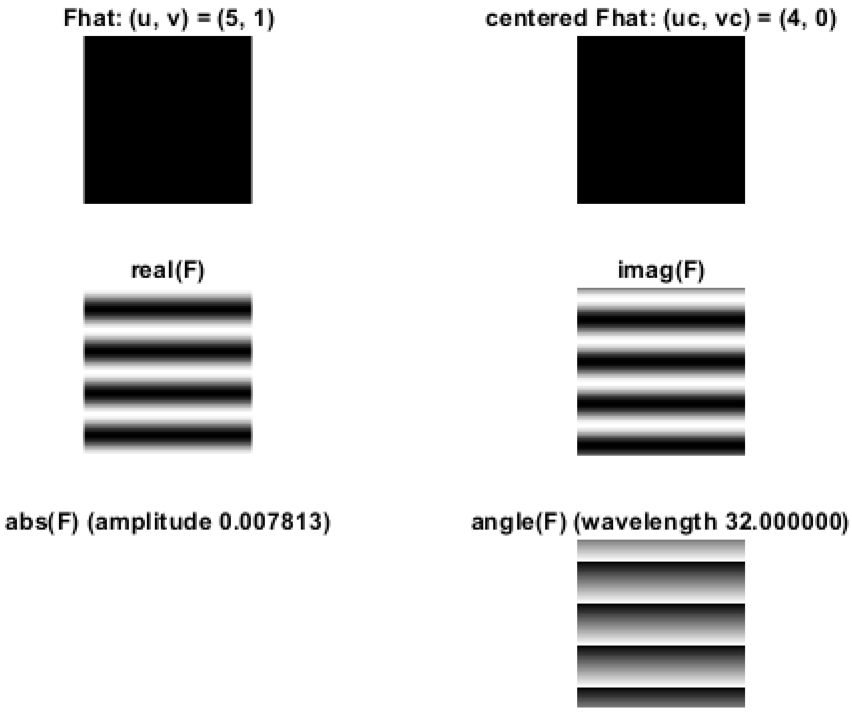
Answer :

Let us compare the difference in choosing p,q within the range and out of the range we take p=125,q=1



The above figure is the case of (p,q) taken to be (125, 1) giving center as (-4,0)

Now we consider a case where (p,q) is taken as (5,1)



Comparing both the only difference is when we take the Fourier transform for both cases the real part remains the same where as the imaginary part has a phase shift of 180 degrees.

Question 6 : What is the purpose of the instructions following the question What is done by these instructions? in the code?

Answer:

The code basically converts the given coordinates (p,q) to match the new scale [-π,π] from the original scale [0,2π] in which the image is formed.

Question 7 : Why are these Fourier spectra concentrated to the borders of the images? Can you give a mathematical interpretation? Hint: think of the frequencies in the source image and consider the resulting image as a Fourier transform applied to a 2D function. It might be easier to analyze each dimension separately!

Answer:

As we see for the matrix F = [zeros(56, 128); ones(16, 128); zeros(56, 128)] and the general Fourier
Transform can be calculated as:

$$F(a,b) = \frac{1}{N} * \sum \sum f(u,v) * e^{-2*\pi*\frac{i(au+bv)}{N}}$$

Now, solving for the case described for this question we have the following result

$$F(a,b) = \sum_{u=56}^{u=72} e^{-2*\pi*\frac{i(au)}{N}} * direct\_delta(b)$$

As the direct delta(b) gives a non-zero value only when b=0 which tells us why the Fourier spectra is concentrated in the borders for F in the left border and for G in the upper border.

Question 8 : Why is the logarithm function applied?

Answer:

Logarithm Function is applied for image enhancement in order to widen the range in pixel values so that range of low pixel Values gets expanded and the range of high pixel values gets compressed, thus enhancing contrast in Fourier domain.

Question 9 : What conclusions can be drawn regarding linearity? From your observations can you derive a mathematical expression in the general case?

Answer

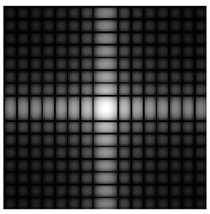
The images in Fourier domain can be found by taking Fourier transform of each and superimposing them together thus we can conclude the following equation:

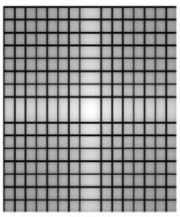
$$F[a * f1(m,n) + b * f2(m,n)] = a * F[f1(m,n)] + b * F[f2(m,n)]$$

Question 10 : Are there any other ways to compute the last image? Remember what multiplication in Fourier domain equals to in the spatial domain! Perform these alternative computations in practice.

Answer

We can verify the property that convolution in Fourier domain is multiplication in spatial domain and vice versa.





Question 11:

What conclusions can be drawn from comparing the results with those in the previous exercise? See how the source images have changed and analyze the effects of scaling.

Answer:

As the scaling is done as follows:

$$f1(x, y) = f2\,(2x, \frac{y}{2})$$

In Fourier domain we will have the following result:

$$F1(w1, w2) = F2(\frac{w1}{2}, 2w2)$$

Thus, we see that an expansion in Fourier domain is same as compression in spatial domain and vice versa.

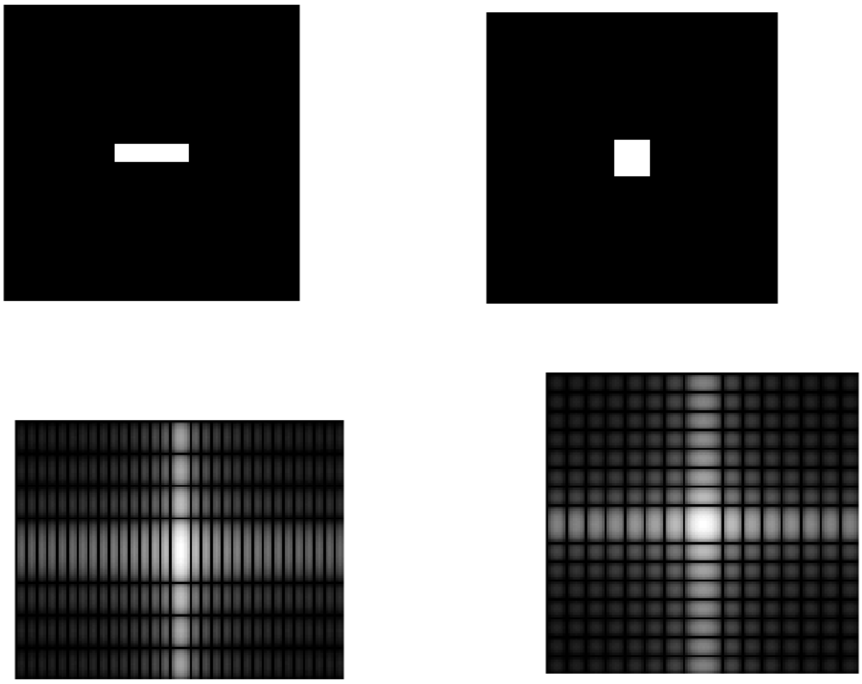


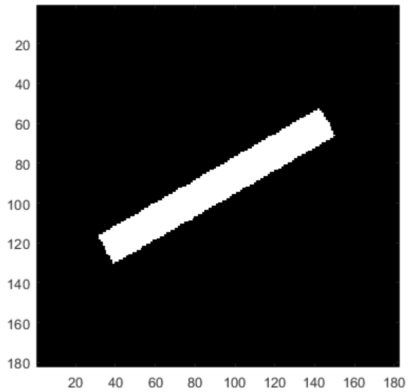
Figure 4: The images in spatial domain and their corresponding Fourier transforms vertically ordered.

Question 12:

What can be said about possible similarities and differences? Hint: think of the frequencies and how they are affected by the rotation.

Answer:

Rotation of Image in spatial domain is similarly transformed to the same rotation in the Fourier domain, It can be proven mathematically as well as we can see it here



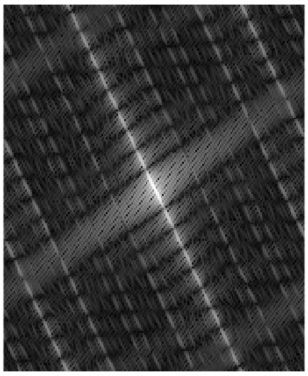


Figure 5: The above figure in the left is 30 degrees inclined previous F and its fourier transform. (right)

Question 13:

What information is contained in the phase and in the magnitude of the Fourier transform?

Answer:

Phase defines how waveforms are shifted along its direction Where edges will end up in the image. Magnitude defines how large the waveforms are What grey-levels are on either side of edge. Basically, magnitude carries the brightness or intensity of image.

Question 14:

Show the impulse response and variance for the above-mentioned t-values. What are the variances of your discretized Gaussian kernel for t = 0.1, 0.3, 1.0, 10.0 and 100.0?

Answer:

The covariance matrices are as follows

1. For t=0.1, t=0.3, t=1, t=10, t=100

0.0132967251852263	1.95594111278971e-14
1.95594111278971e-14	0.0132967251855921

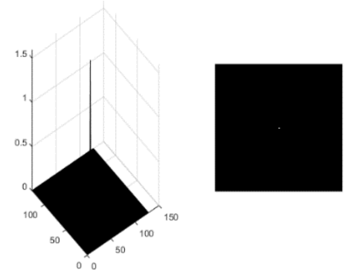
0.281053830089380	1.75961603661103e-15
1.75961603661103e-15	0.281053830089784

0.999999788773478	2.14436875313702e-15
2.14436875313702e-15	0.999999788773594

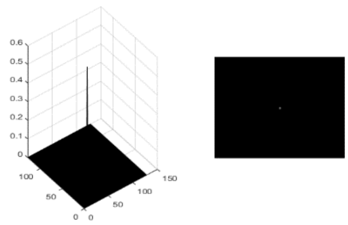
10.0000000000013	1.06791235017787e-15
1.06791235017787e-15	10.0000000000014

99.9999993282295   8.69911019992506e-17
8.69911019992506e-17   99.9999993282299

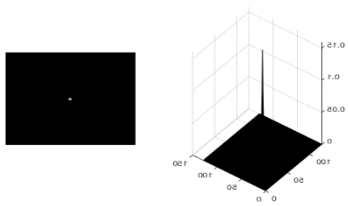
And their plots are as follows:



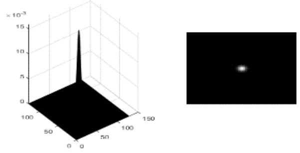
Delta function gaussian t=0.1



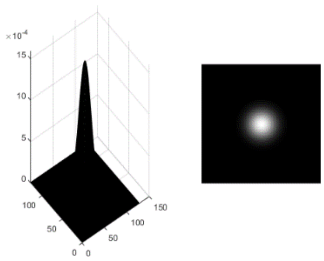
Delta function gaussian t=0.3



Delta function gaussian t=1



Delta function gaussian t=10



Delta function gaussian t=100

Question 15:

Are the results different from or similar to the estimated variance? How does the result correspond to the ideal continuous case? Lead: think of the relation between spatial and Fourier domains for different values of t.

Answer:

We observe that the estimated covariance and the calculated covariance has a huge error when the variance is below 1 and has very minimal error after the variance 1, because when the variance is small the gaussian filter kernel after sampling has a tendency to deviate from Gaussian behaviour as the sampling points is less and information obtained is less and hence higher error is obtained.

Question 16:

Convolve a couple of images with Gaussian functions of different variances (like t =1.0, 4.0, 16.0, 64.0 and 256.0) and present your results. What effects can you observe?

Answer:

As the variance increases, the images become more and more blur. This can be concluded from the gaussian kernel as it becomes bulgy and acts more and more like a low pass filter and thus high frequency information is weakly weighted in the filter leading to blur images and blur edges.

Question 17:

What are the positive and negative effects for each type of Filter? Describe what you observe and name the effects that you recognize. How do the results depend on the Filter parameters?
Illustrate with Matlab figure(s).

Answer:

The positive and negative effects of the given three filters are as follows:

Gaussian Smoothing

Positive : Very good if the required information has lower frequencies and also the fourier transform of a gaussian results into a gaussian

Negative : Images blur for high variance and thus if the information is in the higher frequency gaussian shows poor results for large variance

Median Filter

Positive: Doesn’t blur sharp edges and also removes local extrema values, Best suited for salt and pepper noise

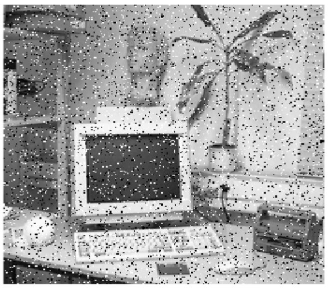
Negative: There is no error propagation and difficult to analytically visualise the process as it is a non linear filter.

Ideal low-pass filter

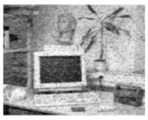
Positive : Easy to visualise and operate

Negative: blocks all high frequency and ringing effect creates a problem

In this experiment an input image with gaussian noise and with salt and pepper noise was compared with gaussian filter, median filter and ideal low pass filter. And the following best results are presented here



The images shown above (left) have gaussian noise added and (right) salt and pepper noise added.



Gaussian filter to salt and pepper noise image

Gaussian noise and Gaussian smoothing

Low pass filter and gaussian noise

Low pass filter and salt and pepper noise

Median filter and gaussian noise

Median filter and salt and pepper noise added to image

Question 18:

What conclusions can you draw from comparing the results of the respective methods?

Answer:

Conclusions are as follows:

For Gaussian smoothing, when the variance increases, the image obtained is blur. Median filter preserves edges and works best when it comes to salt and pepper noise, in comparison to the two other filters. Ideal low pass filter preserves low frequency components of image and cancels out the high frequencies, also the disadvantage is the ringing effect.

Question 19:

What effects do you observe when subsampling the original image and the smoothed variants? Illustrate both filters with the best results found for iteration i = 4.

Answer:

We observe that subsampled images are much better when processed with smoothing before subsampling. However, as the iteration improves, we observe the ringing effect due to sine ways in the spatial domain. The ringing effect increases as the filter order increases

Question 20:

What conclusions can you draw regarding the effects of smoothing when combined with subsampling? Hint: think in terms of frequencies and side effects.

Answer:

If we think in terms of frequencies, smoothing an image basically alters the highest frequency component and, in most cases,, it decreases that component and thus Nyquist frequency which is the minimum frequency required to sample the entire image without losing information decreases. Hence subsampling without loosing information can be done on lower frequency.