Definition: Let a probability distribution be symmetric if there exists a value x_0 such that its probability density function $f(\cdot)$ exhibits the following property:

$$f(x_0 - \delta) = f(x_0 + \delta) \;\; \forall \delta \in \mathbb{R}$$

Further, the median and mean of a symmetric distribution are both equal to x_0.

Theorem (Two-Player Variance Independence): Let X_1, X_2 be two independent random variables with symmetric probability density functions $f_1(\cdot)$ and $f_2(\cdot)$ and expectation $\mathbb{E}[X_1] = \mathbb{E}[X_2] = \mu$. Then, $\mathbb{P}(X_1 > X_2) = \mathbb{P}(X_2 > X_1) = {}^1\!/\!_2$.

Proof: WLOG, let $\mu = 0$. Now, $f_{1-2}(z)$, the probability density function of $X_1 - X_2$, is given by:

$$f_{1-2}(z) = \int_{-\infty}^{\infty} f_1(z + x) f_2(x)\, dz$$

Now, I assert that $f_{1-2}(\cdot)$ is symmetric about zero. To see this, use symmetry of $f_1(\cdot)$ and $f_2(\cdot)$:

$$f_{1-2}(z) = \int_{-\infty}^{\infty} f_1(-z - x) f_2(-x)\, dz$$

Let $y = -x$. Substituting this:

$$f_{1-2}(z) = \int_{-\infty}^{\infty} f_1(-z + y) f_2(y)\, dy$$

This is simply equal to $f_{1-2}(-z)$. Thus, $f_{1-2}(z) = f_{1-2}(-z) \;\; \forall z$, so f_{1-2} is symmetric about zero. The mass above and below zero must be equal. As a result, we have:

$$\mathbb{P}(X_1 > X_2) = \int_{0}^{\infty} f_{1-2}(x)\, dx = {}^1\!/\!_2$$

and

$$\mathbb{P}(X_2 > X_1) = \int_{-\infty}^{0} f_{1-2}(x)\, dx = {}^1\!/\!_2$$

which proves our result. ■