Probability and Statistics 101

Can we ever beat the Casino?

Objectives

- **Basic elements of Probability**

  The world is full of randomness. It is hard to predict what will exactly happen next. However, we can describe the randomness using probability. We will use a simple game to encapsulate the basic elements of probability: a sample space, events and probability.

- **Basic concepts of Statistics**

  We learn and infer the world using what we have observed.

- Gambling and probability

  Gambling shows that there has been an interest in quantifying the ideas of probability for millennia.

Table of Content

100 times

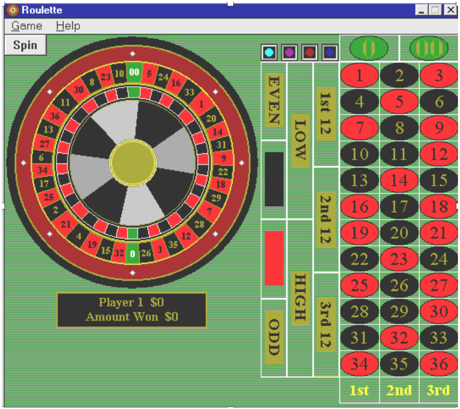
lost $26

Roulette Game

Event : Collections of outcome

step1

Sample space : {0, 00, 1, 2, ... 36}



- A wheel
  - 0, 00, 1, ..., 36
  - 18 numbers: red
  - 18 numbers: black
  - 0, 00: green
- A ball

Event 1: Red occurs

{1, 3, 5, 7, ... 36}

Spin the wheel in one direction and spin the ball in the opposite direction. Observe where the ball lands.

18

Event 2: 1 occurs

Def. (*) $P(\text{Event 1 occurs}) = 18/38$

step3

$P(\text{Event 2}) = \frac{1}{38}$

step 2

Claim 1: A losing game

There are different ways to bet.

- Bet on one single number ✗
- Bet on red or black ✓

Claim 1

One will be for sure losing all the money in hands if playing the Roulette game MANY times.

Claim 2: An unfair game

I once went to a casino and played Red-Black games

- 100 times
- Each time bet \$1.00
- I lost \$28 at the end (Same as lost \$.28 on average)

Claim 2

The roulette table is not a fair one!

How to prove the claim?

We need the concept of **probability** and **statistics**.

Probablity

- In a roulette game, you can not predict where the ball is going to land. (Randomness)
- But . . . We know the probability of events
  - Probability of seeing a 20 is $\frac{1}{38}$
  - Probability of seeing a red is $\frac{18}{38} = 0.47 < 1/2$
- What does Prob (seeing a 20) $= \frac{1}{38}$ mean?

1000 times : $\frac{1}{38} \times 1000 = 26$

long term frequency.

10 times :

Probability

What does Prob (seeing a 20) $= \frac{1}{38}$ mean?

One way: if one plays 1000 times, 20 will roughly appear $1000 \times \frac{1}{38} = 26$ times

Probability of a random event: a long term frequency.

Key elements:

- a sample space
- events
- probability

Random Variables (R.V.)

$X=35 = \{10 \text{ occurs}\}$ $P(X=35)=\frac{1}{38}$
$X=-1 = \{10 \text{ didn't occure}\}$ $P(X=-1) = 37/38$

- A single number game (straight bet): Odds paid 35 to 1 (Put one dollar on a number (say 10) and you will win 35 (and get back your original \$1) if 10 appears; or you will lose \$1)
- Let X be the money won for one dollar bet, it is called a random variable.
- What are the possible values and corresponding prob?

$$X = 35 \quad \text{or} \quad X = -1$$

r.v.	dis^n
X	P
35	1/38
-1	37/38

- Random variables are functions of the sample space.

Distributions

- The possible values together with their probabilities is called the distribution
  - If we win: $X = 35$ with prob $\frac{1}{38}$
  - If we lose: $X = -1$ with prob $1 - \frac{1}{38} = \frac{37}{38}$
- On average how much do you expect that we will win?

$X = 35$ or -1

Def : expected value of X

: $E(X)$

Expected Value (Probability)

- On average how much do you expect that we will win?

$P(X=35) = 1/38$

long term freq

x	p
35	1/38
-1	37/38

$$E(X) = 35 \times \frac{1}{38} + (-1) \times \frac{37}{38}$$
$$= \frac{35}{38} - \frac{37}{38} = -\frac{2}{38} = -.0526$$

- Jargon: $-.0526$ is called the **expected** value of X. It is the weighted average of X and is denoted by $E(X)$.
- Question: What does -.0526 tell us?

Another game: Red-Black | Odds paid 1 to 1

- Put one dollar on one color, say red. If any of the red numbers appears you win \$1, otherwise you lose \$1
- Let Y be the money won for one dollar bet.
  - If we win: $Y = 1$ with prob $\frac{18}{38}$
  - If we lose: $Y = -1$ with prob $1 - \frac{18}{38} = \frac{20}{38}$
- The expected winning is now

$$E(\underset{Y}{X}) = 1 \times \frac{18}{38} + (-1) \times \frac{20}{38} = -\frac{2}{38} = -.0526$$

- *This is same as the expected winning of one number game!!!!!*

$E(Y) = -.0526$

$E(X) = -.0526$

HW: $E(\text{win if bet 1st/ region}) \overset{?}{=}$

$-.0526$.

Interpretation of Expected Value

- When we play Red-Black games on one dollar bet, we expect to win -0.0526, that is, on average we are going to lose 5.26 cents.
- Let us see what does -0.0526 mean.

  I was in Las Vegas not too long ago and I played Red-Black game 200 times. I only bet one dollar each time.

Interpretation of Expected Value

$\{ Y_1 = 1,\ Y_2 = 1,\ Y_3 = -1,\ Y_4 = 1\ \ldots\ Y_{200} = 1 \}$

Statistics $n = 200$, sample $n = 200$

- Here is the summary of the 200 Red-Black games:

	Actual : Sample	Expected Prob
Lost	105 times	$200 \times \frac{20}{38} = 105.3$
Won	95 times	$200 \times \frac{18}{38} = 94.7$
Average Winning	$\bar{Y}_{200} = \frac{Y_1 + \ldots + Y_{200}}{200} = (-105 + 95)/200 = -0.050$ $= \bar{Y}_{200}$	$\boxed{-0.0526}$ fixed $\approx E(Y) = -.0526$

Sample mean
a statistic !!

Are you surprised to see this?

Behavior of Long Term Frequency

\frametitle{Behavior
of
Long
Term
Fre-
quency$}

```
r n
<-
200
win_prob
<-
18/38
#
winning
event
set_seed(2021)
```

Behavior of Sample Mean $\bar{Y}_n$

```r
# expected gain
expected_gain <- win_prob - (1-win_prob)

# if win: +1; if lose: -1
gain_vec <- win_vec*2-1

# sample() function:
# x: elements to choose; size: repeat how many times;
# replace: sample with replacement; prob: probability to choose each x
gain_vec <- sample(x = c(1,-1),
                   size = n,
                   replace = T,
                   prob = c(win_prob, 1- win_prob))

gain_vec
```

```
##   [1]  1  1 -1 -1 -1 -1  1 -1 -1  1 -1 -1  1 -1 -1  1  1  1 -1 -1 -1 -1 -1 -1 -1
##  [26] -1  1 -1 -1 -1  1  1 -1  1 -1 -1  1  1  1  1 -1 -1  1 -1  1 -1  1  1 -1 -1
##  [51]  1  1  1 -1  1  1  1  1  1 -1 -1 -1 -1  1  1 -1  1  1  1  1 -1  1  1 -1  1
##  [76]  1  1  1  1 -1 -1  1 -1 -1 -1 -1  1 -1 -1  1 -1  1 -1 -1 -1 -1 -1  1  1 -1
## [101]  1  1 -1  1 -1 -1  1  1  1 -1 -1 -1  1 -1  1  1 -1 -1 -1 -1 -1  1 -1 -1  1
## [126]  1 -1  1  1 -1 -1  1  1  1  1  1  1 -1  1 -1  1  1  1 -1  1 -1 -1 -1  1 -1
## [151] -1  1 -1 -1 -1 -1  1 -1  1 -1 -1  1  1 -1  1 -1 -1  1  1 -1 -1 -1 -1  1  1
## [176]  1  1  1 -1  1 -1 -1  1 -1  1 -1 -1 -1  1  1  1 -1 -1 -1 -1 -1  1 -1 -1  1
```

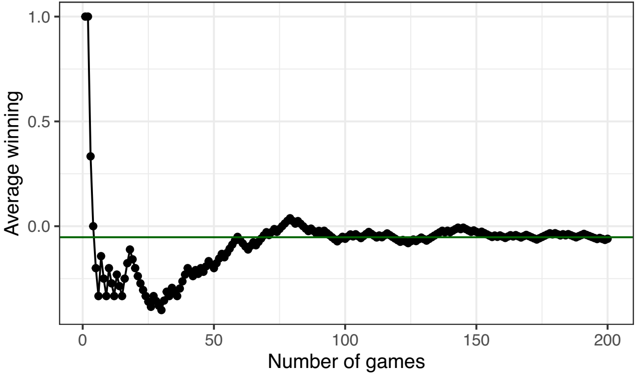
Behavior of $\bar{Y}_n$

```r
# average gain
ave_gain <- cumsum(gain_vec)/1:n

# data.frame
cum_gain_df <- data.frame(num_of_games = 1:n,
                          ave_gain = ave_gain)
```

```r
# plot
ggplot(cum_gain_df,
     aes(x = num_of_games, y = ave_gain)) +
geom_line() + geom_point() +
geom_hline(yintercept = expected_gain, col = "darkgreen") +
xlab("Number of games") +
ylab("Average winning") +
theme_bw()
```

Behavior of $\bar{Y}_n$



Law of Large Numbers : Theorem

- The expected winning for Red-Black game is -0.0526
- Long term Average ≈ expected value

$$\boxed{\bar{Y}_n \to \mu \text{ or } E(Y) \text{ (Expected value)}}$$

n is large

$\approx$ $-.0526$

$\bar{Y}_{200} \approx -.0526$

Q1. • $\bar{Y}_{10} \overset{?}{\approx} -.05$

• $\bar{Y}_{10} \overset{?}{\approx} -.05$

$\bar{Y}_{10}$ is new rv :

$E(\bar{Y}_{10}) = ?$ $\bar{Y}_{10}$, dis^n ?

Which game is better?

- The expected winning for Red-Black game is -0.0526 $\approx E(Y_i) =$
- Recall that the expected winning for Single number bet is also -0.0526 $E(X_i)$ //
- Both games have the same expected values.

Which game should we play to make money?

B/R : $P(\bar{Y}_{1000} > 0)$

Single: $P(\bar{X}_{1000} > 0)$

large ?

Dis" $\begin{cases} \bar{Y}_{100} \\ \bar{X}_{100} \end{cases}$ needed!!

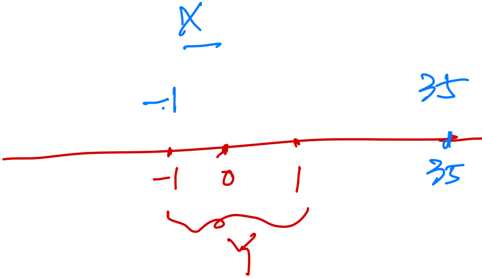
Jonathan claim: $P(\bar{X}_{100} > 0)$ larger!!

Risk measurement: Variance

HOW? Little long stories!

$$X = \begin{cases} 35 \\ -1 \end{cases}$$

$$Y = \begin{cases} 1 \\ -1 \end{cases}$$



Variability: Variance

- X=winning on a single number bet: It can be 35 or -1 with prob 1/38 or 37/38. The expected winning is -0.0526
- Variance: the expected squared difference of the winning from the expected winning $=E(X-\mu)^2 = \sigma^2 = VAR(X)$:

$\mu = E(X)$
$\sigma^2 = Var(X)$

$$\sigma_X^2 = (35-(-0.0526))^2 \times \frac{1}{38} + (-1-(-0.0526))^2 \times \frac{37}{38} = 33.208$$

- Standard Deviation:

$$\sigma = \sigma_X = \sqrt{\sigma_X^2} = \sqrt{33.208} = 5.76$$

Notice: Expected values and Variances are theoretical quantities. They are different from sample means and sample variances.

$Y: \quad EY = -.0526 \qquad \sigma_Y^2 = ?$

Standard Deviation for Y, the winning for Red-Black game?

- Y takes value 1 and -1 with prob. 18/38 and 20/38
- $$Var(Y) = (1-(-0.0526))^2 \times \frac{18}{38} + (-1-(-0.0526))^2 \times \frac{20}{38} = \boxed{0.997}$$
- $$\sigma_Y = \sqrt{0.997} = 0.998 \approx 1$$
- The variability of winning from a single number game (SD=5.76) is much larger than that of Red-Black (SD=0.998)
- How do Variances help us to determine which game to play?

heading

dist^n $\bar{Y}_{200}$?

$\bar{X}_{200}$

Behavior of the average winning

normal dis^n ?

Q: dis^n

1. $\bar{Y}_{10}$ dis^n
2. $\boxed{\bar{Y}_{100}}$ dis^n
3. $\bar{Y}_{10,000}$ dis^n

(Sample of size 10, 100, 10000 vs. the population)

We all play Red-Black game, bet one dollar each time

- $\bar{Y}_{10}$, each person play 10 times 48 of us
- $\bar{Y}_{100}$, each person play 100 times 48 of us
- $\bar{Y}_{10,000}$, each person play 10,000 times 48 of us

A2: $\bar{Y}_{100}$ dis^n

Two ways: 1. Theory

2. Take large number of $\bar{Y}_{100}$

$\{\bar{Y}^{P_1}_{100}, \bar{Y}^{P_2}_{100} \ldots \bar{Y}^{P_{48}}_{100}\}$ 48 $\bar{Y}_{100}$'s

hist (48 $\bar{Y}$'s)

Behavior of the average winning

```
n_times <- 48
win_prob <- 18/38

# create a data frame
## 10 times
set.seed(1)
avg_winning_df_10 <-
  data.frame(id = 1:n_times,
             n = 10,
             num_win = rbinom(n_times, 10, win_prob))
## 100 times
avg_winning_df_100 <-
  data.frame(id = 1:n_times,
             n = 100,
             num_win = rbinom(n_times, 100, win_prob))

# 10000 times
avg_winning_df_10000 <-
  data.frame(id = 1:n_times,
             n = 10000,
             num_win = rbinom(n_times, 10000, win_prob))

avg_winning_df <- rbind(avg_winning_df_10, avg_winning_df_100, avg_winning_df_10000)

avg_winning_df <-
  avg_winning_df %>%
  mutate(avg = (num_win - (n-num_win))/n )

## another way
# times <- c(10, 100, 10000)
# ns <- rep(times, each = n_times)
# avg_winning_df <-
#   data.frame(id = rep(1:n_times, 3),
#              n = ns,
#              num_win = unlist(lapply(times,
```

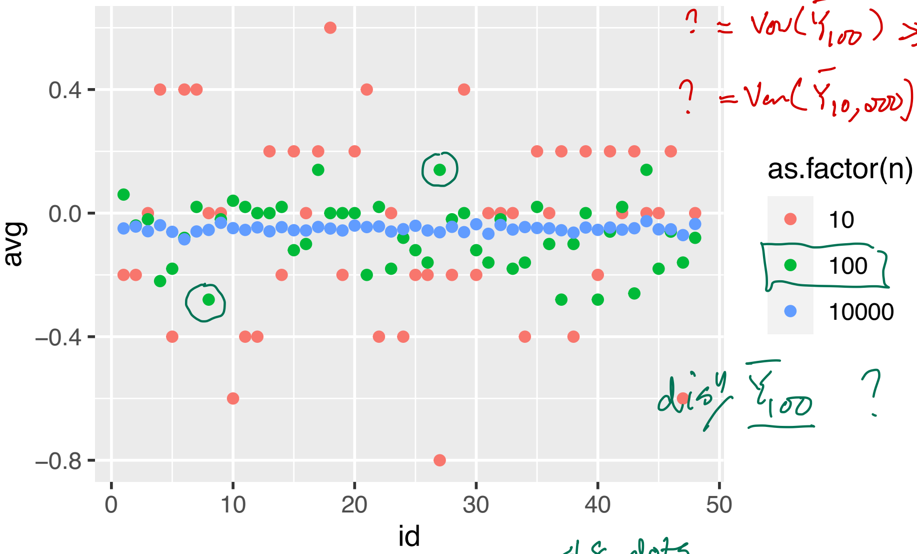
48 90

Behavior of the average winning

obs: 1^o

? $\approx Var(\bar{Y}_{100}) \gg$

? $= Var(\bar{Y}_{10,000})$



dis'n $\bar{Y}_{100}$?

48 dots

Behavior of the average winning

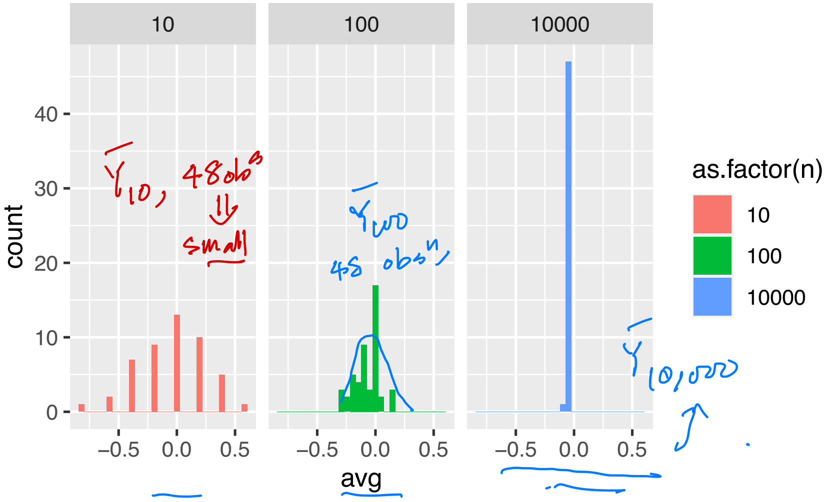
```
avg_winning_df %>%
  group_by(n) %>%
  summarize(mean = mean(avg),
            sd = sd(avg),
            total = sum(avg*n))
```

```
## # A tibble: 3 x 4
##       n    mean     sd  total
##   <dbl>   <dbl>  <dbl>  <dbl>
## 1    10 -0.0417 0.303     -20
## 2   100 -0.0704 0.109    -338
## 3 10000 -0.0510 0.0104 -24466
```

Behavior of the average winning

```
ggplot(avg_winning_df, aes(x = avg, fill = as.factor(n))) +
  geom_histogram() +
  facet_wrap(~n, nrow = 1)
```

```
## `stat_bin()` using `bins = 30`. Pick better value with `binwidth`.
```



Central Limit Theorem (CLT)

① $\bar{Y}_{100} \overset{approx}{\approx}$ Normal $\Leftarrow$ CLT

- When a large number of games are played
  - The average amount each person wins (lost in this case) tends to be close to the center = "expectation'' (-0.0526)
  - The distribution is also approximately a bell curve!
- The Central Limit Theorem
  - $\bar{Y}_n$ has a normal distribution
  - $E[\bar{Y}_n] = \mu/n$
  - $Var(\bar{Y}_n) = \sigma^2/n$
- Almost for sure each one of us will lose all the money if we keep playing!

② $E(\bar{Y}_{100})$

$$= E\left(\frac{Y_1+Y_2+\cdots+Y_{100}}{100}\right) = \frac{E(Y_1)+E(Y_2)+\cdots+E(Y_{100})}{100}$$

$$= \frac{-.0526-.0526 - \cdots - .0526}{100}$$

$$= \frac{100(-.0526)}{100} = -.0526 = E(Y)$$

$\Downarrow$ next page $Var(\bar{Y}_{100})$

Single number games

$$\mathrm{Var}(\bar{Y}_{100}) = \mathrm{Var}\left(\frac{Y_1 + Y_2 + \cdots + Y_{100}}{100}\right) = \frac{\mathrm{Var}(Y_1 + Y_2 + \cdots + Y_{100})}{(100)^2}$$

$$= \frac{\mathrm{Var}(Y_1) + \mathrm{Var}(Y_2) \cdots + \mathrm{Var}(Y_{100})}{100^2} = \frac{100 \times \mathrm{Var}(Y)}{100^2}$$

$$= \frac{\mathrm{Var}(Y)}{100} = \frac{\sigma^2}{100} = \frac{.997}{100} = .00997$$

What about instead we have all played single number games?

$$\mathrm{Var}(Y_1 + Y_2) \overset{?}{=} \mathrm{Var}(Y_1) + \mathrm{Var}(Y_2)$$

Provided Y_1 & Y_2 are $\underline{\text{ind}}$!!

Summary: $\bar{Y}_{100} \sim N\left(\mu = -.0526,\ \sigma^2_{\bar{Y}_{100}} = \frac{\sigma^2_Y}{100}\right)$

$$= N\left(-.0526,\ \sigma^2_{\bar{Y}_{100}} = \frac{.997}{100}\right)$$

Single number game

```
# winning probability
win_prob = 1/38
# number of game
n_times <- 48
# number of trials each game
times <- c(10, 100, 10000)
ns <- rep(times, each = n_times)
# number of win
num_win <- c(sapply(times,
                 function(trial) rbinom(n_times, trial, win_prob)))

avg_winning_df <- data.frame(id = rep(1:n_times, 3),
                             n = ns,
                             num_win = num_win)

avg_winning_df <-
  avg_winning_df %>%
  mutate(avg = (num_win*35 - (n-num_win))/n )
```

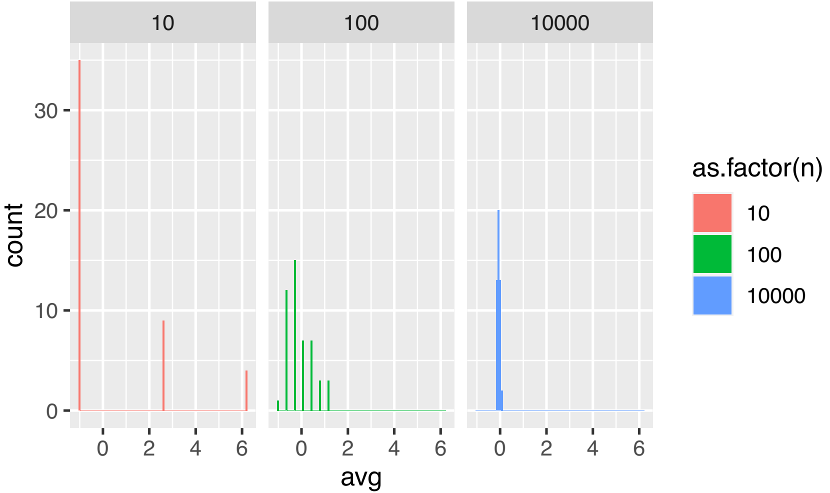
Single number game

```
avg_winning_df %>%
  group_by(n) %>%
  summarize(mean = mean(avg),
            sd = sd(avg),
            total = sum(avg*n))
```

```
## # A tibble: 3 x 4
##       n    mean     sd  total
##   <dbl>   <dbl>  <dbl>  <dbl>
## 1    10  0.275  2.29      132
## 2   100 -0.07   0.550    -336
## 3 10000 -0.0674 0.0568 -32340
```

Single number game

```r
ggplot(avg_winning_df, aes(x = avg, fill = as.factor(n))) +
  geom_histogram(bins = 100) +
  facet_wrap(~n, nrow = 1)
```



Summary of two games: Single number vs Red-Black

- The expected winning is same: -.0526 on one dollar
- Single number:
  - One may have chance to win large amount
  - BUT one may also lose a lot
  - On average you come out the same as Red-Black
- Red-Black:
  - Much more conservative
  - If you want to kill time you may choose this game

After all: Almost for sure to lose money if one plays many times

Take away:

- You can not tell for sure what will happen for a random event.
- Probability tells us on average how often the event will occur.
- A random number changes
  - The center: expected value
  - The spread: standard deviation
- An average of random sample follows a bell curve
  - It tends to the expected value
  - The variability is much smaller when sample size is larger

Normal Random Variable

X = value drawn randomly from a normal population with mean μ and standard deviation σ.

- Often abbreviated as $X \sim N(\mu, \sigma^2)$.
- Density:

$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp - \frac{(x-\mu)^2}{2\sigma^2}$$

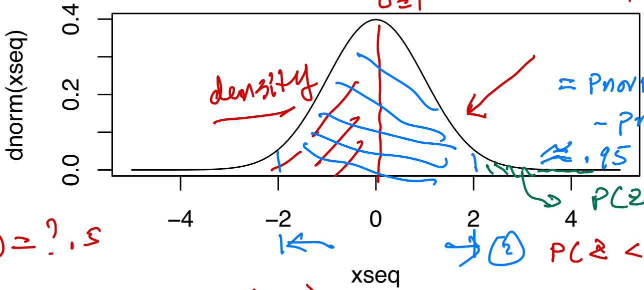
$\mu = E(X)$

$\sigma^2 = Var(X)$

$Z \sim N(0, \sigma^2 = 1)$

standard norm

$\sigma^2 = 1$ $\mu = 1$



(3) $P(-2 < Z < 2)$

$= \text{Pnorm}(2) - \text{Pnorm}(-2)$

$\approx .95$

$P(Z > 2)$

(1) $P(Z < 0) = ?\ .5$

$= \text{Pnorm}(0, \text{mean}=0, \text{sd}=1) = .5$

(2) $P(Z < 2) = \text{Pnorm}(2, 0, 1)$

The Standard Normal Variable Z

④ $P(Z>2)$
$= 1 - \text{pnorm}(2) \approx .025$

- $\mu = 0$ and $\sigma = 1$
- Example: find

$$P(-1 \le Z \le 1) = P(Z \le 1) - P(Z < -1) = .842 - .159 \approx 68\%$$

Q: generate the following r.n:

$$P(-1.96 \le Z \le 1.96) = .95$$

2 2

$-4, 0, 2, -4, 3, 5$

$$\boxed{P(-3 \le Z \le 3) \approx 1}$$

pnorm(3) - pnorm(-3) $\approx .999$

Are those numbers familiar?

$1.2, 1.3, -4, 0$ $n=10$

Q: can this be Z ?

A Normal Variable X

- If $X \sim N(\mu, \sigma^2)$, let $Z = \frac{x-\mu}{\sigma}$, then $Z \sim N(0,1)$
- So

$$P(a \leq X \leq b) = P(\frac{a-\mu}{\sigma} \leq Z \leq \frac{b-\mu}{\sigma})$$

-

$$P(\mu - 1\sigma \leq X \leq \mu + 1\sigma) = P(-1 \leq Z \leq 1) = 68\%$$

$$P(\mu - 2\sigma \leq X \leq \mu + 2\sigma) = P(-2 \leq Z \leq 2) = 95\%$$

$$P(\mu - 3\sigma \leq X \leq \mu + 3\sigma) = P(-3 \leq Z \leq 3) = 100\%$$

Distribution, mean and variance of $\bar{Y}_n$

Example: If we play Red and Black games 100 times, we agree that the average winning $\bar{Y}_{100}$ follows a normal distribution with mean being

$$E(\bar{Y}_{100}) = \mu = -.0526$$

and a variance of

$$Var(\bar{Y}_{100}) = 0.997/100 \approx 0.01$$

$$\sigma_{\bar{Y}_{100}} = \sqrt{0.01} = .1$$

So

$$\bar{Y}_{100} \sim N(-.0526, 0.01)$$

Distribution, mean and variance of $\bar{X}_n$

Example: If we play a single number game 100 times, we agree that the average winning $\bar{X}_{100}$ follows a normal distribution with mean being

$$E(\bar{X}_{100}) = \mu = -.0526$$

and a variance of

$$\sigma_{\bar{X}_{100}} = 5.76/\sqrt{100} = .576$$

So

$$\bar{X}_{100} \sim N(-.0526, 0.576^2)$$

Comparison of two games

- 95% of time
  - $\bar{Y}_{100}$ will be within $-.0526 \pm 2 \times .1 = (-.25, .147)$
  - $\bar{X}_{100}$ will be within $-.0526 \pm 2 \times .576 = (-1.2, 1.09)$
- The chance for $\bar{Y}_{100} > .147$ is same as $\bar{X}_{100} > 1.09$, being 2.5%

Again, which game will you play?

More detailed calculations:

We can also find out:

a) Prob (positive winning)=Prob($\bar{Y}_{100} > 0$)
b) Prob (losing money)=Prob($\bar{Y}_{100} \leq 0$)
c) Prob($-.2 \leq \bar{Y}_{100} \leq -.1$)

Red and Black games 100 times

Recall $\bar{Y}_{100} \sim N(-.0526, 0.01)$.

a) Prob (positive winning)=Prob($\bar{Y}_{100} > 0$)

$$P(\bar{Y}_{100} \geq 0) = P\left(Z \geq \frac{0 - (-.0526)}{.1}\right)$$
$$= P\left(Z \geq .526\right) = .3$$

```
pnorm(.526, lower.tail = F)
```

```
## [1] 0.2994441
```

```
# pnorm(0, mean = -.0526, sd = .1, lower.tail = F)
```

Red and Black games 100 times

(b) Prob (losing money)=Prob($\bar{Y}_{100} \leq 0$) = 1- Prob($\bar{Y}_{100} > 0$) =1-.3=.7

On average the chance to lose money is 70%.

c) Prob($-.2 \leq \bar{Y}_{100} \leq -.1$)

$$P(-.2 \leq \bar{Y}_{100} \leq -.1) = P\left(\frac{-.2-(-.0526)}{.1} \leq Z \leq \frac{-.1-(-.0526)}{.1}\right)$$
$$= P\left(-1.474 \leq Z \leq -.474\right) = .32 - .07 = .25$$

```
pnorm(-.474) - pnorm(-1.474)
```

```
## [1] 0.2475092
```

The chance of loosing between 10 and 20 cents on average is 25%

Single number game 100 times

Recall $\bar{X}_{100} \sim N(-.0526, 0.576^2)$.

Prob(losing money):

$$\begin{aligned} P(\bar{X}_{100} < 0) &= P\left(Z < \frac{0 - (-.0526)}{.576}\right) \\ &= P\left(Z \geq .0913\right) = .536 \end{aligned}$$

On average the chance to lose money is 53.6%!

Is the Casino being honest?

Case: Linda played roulette 100 times

- \$1 bet each time on Red/Black
- She lost \$28
- She knew the roulette table is a biased one. HOW????

95% Confidence interval

$\bar{X}$ has a normal distribution with μ and $sd = \frac{\sigma}{\sqrt{100}} = \frac{.998}{10} \approx .1$

Which means 95% of time

$$|\bar{X} - \mu| < 1.96 \times .1$$

This is same to say 95% time the mean μ should be in

$$\bar{X} \pm 1.96 \times \frac{\sigma}{\sqrt{100}} = (\bar{X} - .2, \bar{X} + .2)$$

Apply to our data, we have a 95% confidence interval (z):

$$-.28 \pm 2 \times .1 = (-.48, -.08)$$

Conclusion: The roulette is not fair. 95% CI does not contain -.0526.

t-Confidence interval

- σ is not known either, we estimate σ by $s = .965$
- We will have a t-interval:

$$\bar{X} \pm t_{df} \times \frac{s}{\sqrt{100}} = -.28 \pm 1.98 \times \frac{.965}{\sqrt{100}} = (-.471, -.089)$$

- We have the same conclusion that the wheel is not a fair one since the true mean -.0526 in not in the interval.
- t intervals are wider than z intervals

Hypotheses testing

- We may ask is it possible that$\mu = -.0526$?
- $H_0 : \mu = -.0526 \quad \text{vs.} \quad H_1 : \mu \neq -.0526$
- Testing statistics

$$Z = \frac{\bar{X} - (-.0526)}{s/\sqrt{n}} = \frac{-.28 + .0526}{.998//sqrt100} = -2.28$$

- p-value $= P(|Z| > 2.28) = .022$ if $\mu = -.0526$
- Conclusion: Since p-value is so small, we reject H_0.

Bernoulli Distribution

The success of each bet X of the single number game or the Red-Black game follows a Bernoulli distribution. Denote success as 1.

- Single number game

$$X = \begin{cases} 1 & \text{with probability (w.p.) } 1/38 \\ 0 & \text{w.p. } 37/38 \end{cases}$$

- Red-Black game

$$X = \begin{cases} 1 & \text{w.p. } 18/38, \\ 0 & \text{w.p. } 20/38 \end{cases}$$

Bernoulli Distribution

Simulate 100 trials. Use `rbinom()` to generate random samples.

```
##   [1] 1 0 0 1 1 0 0 0 1 1 1 1 0 0 0 0 1 0 0 1 1 0 0 0 1 0 0 0 0
##  [38] 0 1 0 1 1 0 0 1 0 0 1 0 1 1 1 0 0 0 0 1 0 0 0 0 1 0 1 1 0
##  [75] 0 1 1 1 1 1 0 0 1 0 1 0 1 1 1 1 1 0 0 1 0 1 0 1 0 1
```

Binomial Distribution

If we bet 100 times, or say we draw 100 samples from the Bernoulli distribution, the total number of success among these 100 times Y follow binomial distribution.

$$Y \sim Binomial(n, p)$$

where n is the total number of trials and p is the probability of success of each trial.

- Single number game

$$X = Binomial(100, 1/38)$$

- Red-Black game

$$X = Binomial(100, 18/38)$$

Binomial Distribution

The probability of success k times among 100 trials is

$$Prob(Y = k) = \left(\begin{array}{c} 100 \\ k \end{array} \right) p^k (1 - p)^{n-k}$$

```
## [1] 0.7349765
```

Binomial Distribution

Simulate total number of success among 100 trials. Use `rbinom()` to generate random samples.

```
## [1] 50
```

Covariances: $Cov(X_R, X_B)$

- $X_R =$ Winning over one dollar bet on Red
- $X_B =$ Winning over one dollar bet on Black
- X_R and X_B are related: if $X_R = 1$, then $X_B = -1$
- We use covariance to measure the relationship

$$COV(X_R, X_B) = E(X_R - E(X_R)(X_B - E(X_B))$$

$$COV(X_R, X_B) = -.8975$$

- Or Correlation

$$\rho = \frac{COV(X_R, X_B)}{SD(X_R)SD(X_B)} = \frac{-.8975}{.998 \times 998} = -.9011$$

Correlation

$$\rho = \frac{COV(X_R, X_B)}{SD(X_R)SD(X_B)} = \frac{-.8975}{.998 \times 998} = -.9011$$

- Correlation captures linear relationship between XR and XB
- $-1 < \rho < 1$
- The larger $|\rho|$ is, the stronger of the relationship
- The sign of ρ reflects the direction of associations

$E(X_R + X_B)$ and $VAR(X_R + X_B)$

- $E(X_R + X_B) = E(X_R) + E(X_B)$
- $VAR(X_R + X_B) = VAR(X_R) + VAR(X_B) + 2COV(X_R, X_B)$
- $VAR(aX_R + bX_B) = a^2 VAR(X_R) + b^2 VAR(X_B) + 2abCOV(X_R, X_B)$
- If X and Y are independent $COV(X, Y) = 0$
  $VAR(aX + bY) = a^2 VAR(X) + b^2 VAR(Y)$
- That is why $Var(\bar{X}_n) = \frac{\sigma^2}{n}$