# Computational Geometry EX1 - Solution

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## 1 Question 1

**Definition 1.1.** *The orientation function:*

$$orient((p_x, p_y), (q_x, q_y), (r_x, r_y)) = sign\left(\begin{vmatrix} 1 & 1 & 1 \\ p_x & q_x & r_x \\ p_y & q_y & r_y \end{vmatrix}\right) = sing((q_x r_y - r_x q_y) - (p_x r_y - r_x p_y) + (p_x q_y - q_x p_y))$$

**Claim 1.2.** *All of the following holds:*

- $orient(p, q, r) = orient(q, r, p)$
- $orient(p, q, r) = orient(r, p, q)$
- $orient(p, q, r) = -orient(p, r, q)$
- $orient(p, q, r) = -orient(q, p, r)$
- $orient(p, q, r) = -orient(r, q, p)$

*Proof.* We prove each of the points.

- We reach it by taking the 'minus' out of the second and the third terms.

$$(q_x r_y - r_x q_y) - (p_x r_y - r_x p_y) + (p_x q_y - q_x p_y) = (r_x p_y - p_x r_y) - (q_x p_y - p_x q_y) + (q_x r_y - r_x q_y)$$

- We reach it by taking the 'minus' out of the first and the second terms.

$$(q_x r_y - r_x q_y) - (p_x r_y - r_x p_y) + (p_x q_y - q_x p_y) = (p_x q_y - q_x p_y) - (r_x q_y - q_x r_y) + (r_x p_y - p_x r_y)$$

- We observe that $orient(p, q, r) = -1 \cdot -1 \cdot orient(p, q, r)$

$$\begin{aligned} orient(p, q, r) &= (q_x r_y - r_x q_y) - (p_x r_y - r_x p_y) + (p_x q_y - q_x p_y) \\ &= -1 \cdot -1 \cdot ((q_x r_y - r_x q_y) - (p_x r_y - r_x p_y) + (p_x q_y - q_x p_y)) \\ &= -1 \cdot (-(q_x r_y - r_x q_y) + (p_x r_y - r_x p_y) - (p_x q_y - q_x p_y)) \\ &= -1 \cdot ((r_x q_y - q_x r_y) - (p_x q_y - q_x p_y) + (p_x r_y - r_x p_y)) \\ &= -1 \cdot orient(p, r, q) \end{aligned}$$

- We observe that $orient(p, q, r) = -1 \cdot -1 \cdot orient(p, q, r)$

$$\begin{aligned} orient(p, q, r) &= (q_x r_y - r_x q_y) - (p_x r_y - r_x p_y) + (p_x q_y - q_x p_y) \\ &= -1 \cdot -1 \cdot ((q_x r_y - r_x q_y) - (p_x r_y - r_x p_y) + (p_x q_y - q_x p_y)) \\ &= -1 \cdot (-(q_x r_y - r_x q_y) + (p_x r_y - r_x p_y) - (p_x q_y - q_x p_y)) \\ &= -1 \cdot ((p_x r_y - r_x p_y) - (q_x r_y - r_x q_y) + (q_x p_y - p_x q_y)) \\ &= -1 \cdot orient(q, p, r) \end{aligned}$$

- We observe that $orient(p, q, r) = -1 \cdot -1 \cdot orient(p, q, r)$

$$\begin{aligned} orient(p, q, r) &= (q_x r_y - r_x q_y) - (p_x r_y - r_x p_y) + (p_x q_y - q_x p_y) \\ &= -1 \cdot -1 \cdot ((q_x r_y - r_x q_y) - (p_x r_y - r_x p_y) + (p_x q_y - q_x p_y)) \\ &= -1 \cdot (-(q_x r_y - r_x q_y) + (p_x r_y - r_x p_y) - (p_x q_y - q_x p_y)) \\ &= -1 \cdot ((q_x p_y - p_x q_y) - (r_x p_y - p_x r_y) + (r_x q_y - q_x r_y)) \\ &= -1 \cdot orient(r, q, p) \end{aligned}$$

$\square$

**Claim 1.3.** *If $orient(p,x,q) = orient(q,x,r) = orient(r,x,p)$, then $orient(r,q,p)$ is also equal to them.*

Note. Before going further, for the ease of use we will defined that

$$pq = \begin{vmatrix} p_x & q_x \\ p_y & q_y \end{vmatrix} = p_x q_y - p_y q_x$$

and that $pq = -qp$. Therefore, by that definition we have that

$$orient(p,q,r) = sign(qr - pr + pq).$$

*Proof.*

$$\begin{aligned} orient(p,x,q) + orient(q,x,r) + orient(r,x,p) &= sign(xq - pq + px) + sign(xr - qr + qx) + sign(xp - rp + rx) \\ &\overbrace{=}^{(1)} sign(xq - pq + px + xr - qr + qx + xp - rp + rx) \\ &\overbrace{=}^{(2)} sign(-pq - qr - rp) \\ &\overbrace{=}^{(3)} sign(qp - rp + rq) \\ &= orient(r,q,p) \end{aligned}$$

Where:
(1) By assumption they have the same signe, therefore, we can take it out without affecting the final result.
(2) Algebra based on definition-observation above. For example, $xq + qx = 0$.
(3) Algebra based on definition-observation above. For instace, $-pq = qp$.

$\square$

**Claim 1.4.** *If the conditions of previous claim hold, then $x$ is a convex combination of $p$, $q$ and $r$.*

*Proof.* Let us begin by recalling the Cramer's rule and some basic linear algebra concepts. First, recall that the linear system of form $A \cdot x = b$, has solution if and only if $det(A) \neq 0$. And, we want to find sulution for the following system:

$$\overbrace{\begin{bmatrix} r_x & q_x & p_x \\ r_y & q_y & p_y \end{bmatrix}}^{A} \cdot \overbrace{\begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix}}^{\alpha} = \begin{bmatrix} x_x \\ x_y \end{bmatrix}$$

By previous lemma, it follows that $det(A) \neq 0$, therefore, there exists $\alpha \in \mathbb{R}^3$ such that $A \cdot \alpha = x$. All we left to do is: (1) Find such $\alpha$, and (2) show that $\alpha_1 + \alpha_2 + \alpha_3 = 1$. To do so, we use Cramer's rule.

*Theorem (Cramer).* Let $A \cdot \alpha = \beta$ be linear system, where $A = [\beta_1, \beta_2, ..., \beta_n]$ such that $\beta_i \in \mathbb{R}^k$, and $\beta \in \mathbb{R}^k$ as well. If $det(A) \neq 0$, then $\forall j \in [1...n]$ $\alpha_j = det(A_j)/deta(A)$. Where for each $j \in [1...n]$ we have that

$$A_j = [\beta_1, \beta_2, ..., \beta_{j-1}, \beta, \beta_{j+1}, ...\beta_n],$$

we replace the $\beta_j$ vector with $\beta$.

We use Cramer's theorem to conclude the following.

$$\alpha_1 = \frac{qr - xp + xq}{pq - rp + rq}, \quad \alpha_2 = \frac{xp - rp + rx}{pq - rp + rq}, \quad \alpha_3 = \frac{qx - rx + rp}{pq - rp + rq},$$

and from the previous lemma we conclude that $\alpha_1 + \alpha_2 + \alpha_3 = 1$ as required.

$\square$

**Claim 1.5.** *If $orient(p,x,q) = orient(p,x,r) = orient(p,x,s) = orient(q,x,r) = orient(r,x,s)$, then $orient(q,x,s)$ is also equal to them.*