QUICKSORT IS OPTIMAL

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MOTIVATION

MOORE'S LAW: Processing Power Doubles every 18 months
but also:

- memory capacity doubles every 18 months
- problem size expands to fill memory

Sedgewick's Corollary: Need Faster Sorts every 18 months!
(annoying to wait longer, even to sort twice as much, on new machine)

old: $N \lg N$

new: $(2N \lg 2N)/2 = N \lg N + N$

Other compelling reasons to study sorting

- cope with new languages, machines, and applications
- rebuild obsolete libraries
- intellectual challenge of basic research

Simple fundamental algorithms: the ultimate portable software

Quicksort

```
void quicksort(Item a[], int l, int r)
{ int i = l-1, j = r; Item v = a[r];
  if (r <= l) return;
  for (;;)
    {
      while (a[++i] < v) ;
      while (v < a[--j]) if (j == l) break;
      if (i >= j) break;
      exch(a[i], a[j]);
    }
  exch(a[i], a[r]);
  quicksort(a, l, i-1);
  quicksort(a, i+1, r);
}
```

Detail (?): How to handle keys equal to the partitioning element

Partitioning with equal keys

How to handle keys equal to the partitioning element?

METHOD A: Put equal keys all on one side?

4	4	4	4	4	4	4	4	4	4	4	4	4
4	4	4	4	4	4	4	4	4	4	4	4	4

NO: quadratic for n=1 (all keys equal)

METHOD B: Scan over equal keys? (linear for n=1)

1	4	1	1	4	4	4	1	4	1	1	4	4
1	1	1	1	4	4	4	1	4	1	4	4	4

NO: quadratic for n=2

METHOD C: Stop both pointers on equal keys?

4	9	4	4	1	4	4	4	9	4	4	1	4
1	4	4	4	1	4	4	4	9	4	9	4	4

YES: NlgN guarantee for small n, no overhead if no equal keys

Partitioning with equal keys

How to handle keys equal to the partitioning element?

METHOD C: Stop both pointers on equal keys?

4	9	4	4	1	4	4	4	9	4	4	1	4
1	4	4	4	1	4	4	4	9	4	9	4	4

YES: NlgN guarantee for small n, no overhead if no equal keys

METHOD D (3-way partitioning): Put all equal keys into position?

4	9	4	4	1	4	4	4	9	4	4	1	4
1	1	4	4	4	4	4	4	4	4	4	9	9

yes, BUT: early implementations cumbersome and/or expensive

Quicksort common wisdom (last millennium}

1. Method of choice in practice
   - tiny inner loop, with locality of reference
   - NlogN worst-case "guarantee" (randomized)
   - but use a radix sort for small number of key values
2. Equal keys can be handled (with care)
   - NlogN worst-case guarantee, using proper implementation
3. Three-way partitioning adds too much overhead
   - "Dutch National Flag" problem
4. Average case analysis with equal keys is intractable
   - keys equal to partitioning element end up in both subfiles

Changes in Quicksort common wisdom

1. Equal keys abound in practice.
   - never can anticipate how clients will use library
   - linear time required for huge files with few key values
2. 3-way partitioning is the method of choice.
   - greatly expands applicability, with little overhead
   - easy to adapt to multikey sort
   - no need for separate radix sort
3. Average case analysis already done!
   - Burge, 1975
   - Sedgewick, 1978
   - Allen, Munro, Melhorn, 1978

Bentley-McIlroy 3-way partitioning

Partitioning invariant

equal	less		greater	equal

- move from left to find an element that is not less
- move from right to find an element that is not greater
- stop if pointers have crossed
- exchange
- if left element equal, exchange to left end
- if right element equal, exchange to right end

Swap equals to center after partition

less	equal	greater

KEY FEATURES

- always uses N-1 (three-way) compares
- no extra overhead if no equal keys
- only one "extra" exchange per equal key

Quicksort with 3-way partitioning

```
void quicksort(Item a[], int l, int r)
{ int i = l-1, j = r, p = l-1, q = r; Item v = a[r];
  if (r <= l) return;
  for (;;)
    {
      while (a[++i] < v) ;
      while (v < a[--j]) if (j == l) break;
      if (i >= j) break;
      exch(a[i], a[j]);
      if (a[i] == v) { p++; exch(a[p], a[i]); }
      if (v == a[j]) { q--; exch(a[j], a[q]); }
    }
  exch(a[i], a[r]); j = i-1; i = i+1;
  for (k = l; k < p; k++, j--) exch(a[k], a[j]);
  for (k = r-1; k > q; k--, i++) exch(a[i], a[k]);
  quicksort(a, l, j);
  quicksort(a, i, r);
}
```

Information-theoretic lower bound

Definition: **An $(x_1, x_2, \ldots, x_n)$-file has**

$N = x_1 + x_2 + \ldots + x_n$ **keys,**

n distinct key values, with

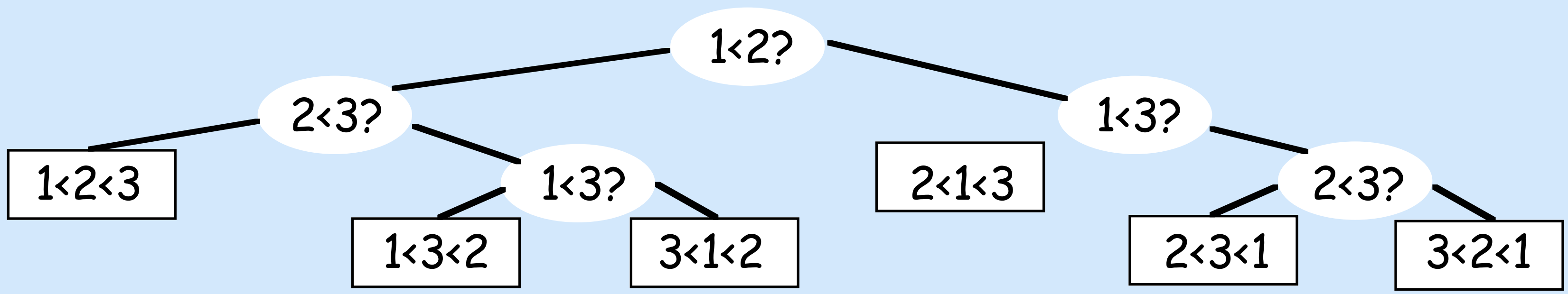
$x_i \equiv$ **number of occurences of the i-th smallest key**

$p_i \equiv x_i / N$

THEOREM. **Any sorting method uses at least $NH - N$ compares (where $H = -\sum_{1 \le k \le n} p_k \lg p_k$ is the entropy) to sort an $(x_1, x_2, \ldots, x_n)$-file, on the average.**

Information-theoretic lower-bound proof

DECISION TREE describes all possible sequences of comparisons



Number of leaves must exceed number of possible files

$$\binom{N}{x_1 x_2 \ldots x_n} = \frac{N!}{x_1! x_2! \ldots x_n!}$$

Avg. number of compares is minimized when tree is balanced

$$C > \lg \frac{N!}{x_1! x_2! \ldots x_n!} = \lg N! - \lg x_1! - \lg x_2! - \ldots - \lg x_n!$$

By Stirling's approximation,

$$\begin{aligned} C &> N \lg N - N - x_1 \lg x_1 - x_2 \lg x_2 - \ldots - x_n \lg x_n \\ &= (x_1 + \ldots + x_n) \lg N - N - x_1 \lg x_1 - x_2 \lg x_2 - \ldots - x_n \lg x_n \\ &= NH - N \end{aligned}$$

Analysis of Quicksort with equal keys

1. Define $C(x_1,\ldots,x_n) \equiv C(1,n)$ to be the mean # compares to sort the file

$$C(1,n) = N - 1 + \frac{1}{N}\sum_{1\le j\le n} x_j\big(C(1,j-1) + C(j+1,n)\big)$$

2. Multiply both sides by $N = x_1 + \ldots + x_n$

$$NC(1,n) = N(N-1) + \sum_{1\le j\le n} x_j C(1,j-1) + \sum_{1\le j\le n} x_j C(j+1,n)$$

3. Subtract same equation for $x_2,\ldots,x_n$ and let $D(1,n) \equiv C(1,n) - C(2,n)$

$$(x_1 + \ldots + x_n)D(1,n) = x_1^2 - x_1 + 2x_1(x_2 + \ldots + x_n) + \sum_{2\le j\le n} x_j D(1,j-1)$$

4. Subtract same equation for $x_1,\ldots,x_{n-1}$

$$(x_1 + \ldots + x_n)D(1,n) - (x_1 + \ldots + x_{n-1})D(1,n-1) = 2x_1x_n + x_nD(1,n-1)$$

Analysis of Quicksort with equal keys (cont.)

$$(x_1 + \ldots + x_n)D(1,n) - (x_1 + \ldots + x_{n-1})D(1,n-1) = 2x_1x_n + x_nD(1,n-1)$$

5. Simplify, divide both sides by $N = x_1 + \ldots + x_n$

$$D(1,n) = D(1,n-1) + \frac{2x_1x_n}{x_1 + \ldots + x_n}$$

6. Telescope (twice)

$$C(1,n) = N - n + \sum_{1 \le k < j \le n} \frac{2x_kx_j}{x_k + \ldots + x_j}$$

THEOREM. Quicksort (with 3-way partitioning, randomized) uses

$N - n + 2QN$ compares (where $Q = \sum_{1 \le k < j \le n} \frac{p_kp_j}{p_k + \ldots + p_j}$, with $p_i = x_i/N$)

to sort an $(x_1,\ldots,x_n)$-file, on the average .

Basic properties of quicksort "entropy"

$$Q = \sum_{1 \le k < j \le n} \frac{p_k p_j}{p_k + \ldots + p_j} \quad \text{with } p_i = x_i / N$$

Example: all frequencies equal ($p_i = 1/n$)

$$Q = \sum_{1 \le k < n} \frac{1}{n} \sum_{k < j \le n} \frac{1}{j - k + 1} = \ln n + O(1)$$

Conjecture: Q maximized when all keys equal?

NO:

$Q = .4444\ldots$ for $x_1 = x_2 = x_3 = N/3$

$Q = .4453\ldots$ for $x_1 = x_3 = .34N$, $x_2 = .32N$

Upper bound on quicksort "entropy"

$$Q = \sum_{1 \le k < j \le n} \frac{p_k p_j}{p_k + \ldots + p_j}$$

1. Separate double sum

$$Q = \sum_{1 \le k < n} p_k \sum_{k < j \le n} \frac{p_j}{p_k + \ldots + p_j}$$

2. Substitute $q_{ij} = (p_i + \ldots + p_j)/p_i$ (note: $1 = q_{ii} \le q_{i(i+1)} \le \ldots \le q_{in} < 1/p_i$)

$$Q = \sum_{1 \le k < n} p_k \sum_{k < j \le n} \frac{q_{kj} - q_{k(j-1)}}{q_{kj}}$$

3. Bound with integral

$$Q = \sum_{1 \le k < n} p_k \int_{q_{kk}}^{q_{kn}} \frac{1}{x} dx < \sum_{1 \le k < n} p_k \ln q_{kn} < \sum_{1 \le k \le n} p_k (-\ln p_k) = H \ln 2$$

Quicksort is optimal

The average number of compares per element C/N is always within a constant factor of the entropy H

lower bound: $C > NH - N$ (information theory)

upper bound: $C < 2\ln 2NH + N$ (Burge analysis, Melhorn bound)

No comparison-based algorithm can do better.

Conjecture: With sampling, $C / N \to H$ as sample size increases.

Extensions and applications

Optimality of Quicksort

- underscores intrinsic value of algorithm
- resolves basic theoretical question

Analysis shows Quicksort to be sorting method of choice for

- randomly ordered keys, abstract compare
- small number of key values

Extension 1: Adapt for varying key length`

Multikey Quicksort

SORTING method of choice: (Q/H)NlgN byte accesses

Extension 2: Adapt algorithm to searching

Ternary search trees (TSTs)

SEARCHING method of choice: (Q/H)lgN byte accesses

Both conclusions validated by

- Flajolet, Clèment, Valeé analysis
- practical experience

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