Department of Atmospheric Sciences
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Chaos and Predictability / HW3

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2023 Fall Version

Question 1

The Lorenz 96 model can be written as,

$$\frac{dX_i}{dt} = X_{i-1}(X_{i+1} - X_{i-2}) - X_i + F \tag{1}$$

where X is periodic. (the leftmost is connected to the rightmost)

(a) Neglect damping term X_i and forcing term F in equation (1). Show that the total model energy is conserved (i.e., integrating $\frac{dX_i^2}{dt}$ over the entire domain equals 0)

(b) (1) Given F=8 and total i=100 (100 grid points), initialize the model with random white noise and plot the result. (2) Repeat similar simulations 500 times with slightly different initial conditions to generate ensemble simulations and plot the forecast error as a function of time. Estimate the error-doubling time. (hint:The error can be defined as the variance across all ensemble members.)

(c) Repeat (b) but with different forcing (e.g., $F = 1$). Discuss how the forcing amplitude influences the periodicity of X_i (i.e., whether the initial error grows or not?).

Question 2

For a dynamical system with third-order derivative,

$$x''' = P(t, x, x', x'') \tag{2}$$

where $x''' = \frac{dx''}{dt}$, we can find a Jacobian matrix J, which maps the initial states $x_{n_0} = x(t = 0, a, b, c)$ to the final states $x_{n_t} = x(t = n_t, a, b, c)$. The corresponding determinant can then be written as

$$\det(J) = \det\begin{pmatrix} \frac{dx}{da} & \frac{dx}{db} & \frac{dx}{dc} \\ \frac{dx'}{da} & \frac{dx'}{db} & \frac{dx'}{dc} \\ \frac{dx''}{da} & \frac{dx''}{db} & \frac{dx''}{dc} \end{pmatrix} \tag{3}$$

show that $\frac{d\det(J)}{dt} = \det(J)\frac{\partial P}{\partial x''}$. (Hint: using the chain rule $\frac{dx'''}{da} = \frac{\partial x'''}{\partial x}\frac{\partial x}{\partial a} + \frac{\partial x'''}{\partial x'}\frac{\partial x'}{\partial a} + \frac{\partial x'''}{\partial x''}\frac{\partial x''}{\partial a}$, also see Fig. 2 in https://www.ecmwf.int/en/elibrary/9271-liouville-equation-atmospheric-predictability). The LE suggests that the time evolution of $\det(J)$ only depends on the highest-order differentiation.