A Modular Completeness Proof for the Superposition Calculus

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1 Background

Framework paper: Waldmann et al. [6]

Formalized and extended in: Tourret [5], Blanchette and Tourret [4]

Superposition calculus: Bachmair and Ganzinger [3], Bachmair and Ganzinger [2]

Rewriting basics: Baader and Nipkow [1]

2 Preliminaries

We assume a first-order *signature* $\Sigma = (\Xi, \Omega, \Pi)$, where Ξ is a set of *sorts*, Ω is set of *function symbols*, and Π is set of *predicate symbols*. (All sets are disjoint.) Every function symbol $f \in \Omega$ and predicate symbol $P \in \Pi$ has a unique declaration $f : \xi_1 \ldots \xi_n \to \xi_0$ and $P : \xi_1 \ldots \xi_n$, $n \geq 0$, $\xi_j \in \Xi$. Furthermore let X be a Ξ-sorted set of *variables* (disjoint from Ξ, Ω, Π). Then $\mathcal{T}_\Sigma(X)$ is the set of (well-sorted) *terms* over Σ and X.

A (well-sorted) *equation* is an unordered pair (s, t) of terms with the same sort, usually written as $s \approx t$. A (well-sorted) *non-equational atom* over Σ and X has the form $P(t_1, \ldots, t_n)$ with $P \in \Pi$ and $t_i \in \mathcal{T}_\Sigma(X)$. An *atom* is an equation or a non-equational atom. A *literal* is an atom A or a negated atom $\neg A$. We usually write $s \not\approx t$ instead of $\neg(s \approx t)$. A *clause* is a multiset of literals, usually written as a disjunction. The symbol $\bot$ denotes the empty clause, that is, false.

In the sequel, all terms, atoms, literals, equations, substitutions are assumed to be well-sorted.

To simplify the presentation, we will assume from now on that $\Pi = \emptyset$ and that predicate symbols $P : \xi_1 \ldots \xi_n$ are replaced by function symbols $f_P : \xi_1 \ldots \xi_n \to \mathsf{bool}$, so that non-equational literals $[\neg]\, P(t_1, \ldots, t_n)$ are encoded as equations $[\neg]\, f_P(t_1, \ldots, t_n) \approx \mathsf{true}$.

We assume that $\succ$ is a reduction ordering on terms that is total on ground terms and has the subterm property on ground terms i.e., $t[s]_p \succ s$ if $p \neq \varepsilon$. (In the single-sorted case, the subterm property follows from totality on ground terms, compatibility with contexts, and well-foundedness, but in the multi-sorted case, we have to require it explicitly.)

3 (Ground) Inference System

Let a fixed first-order signature $\Sigma = (\Xi, \Omega, \Pi)$ be given. We define $\mathbf{G}$ as the set of ground first-order clauses over Σ and $\mathbf{G}_\bot$ as the subset $\{\bot\} \subseteq \mathbf{G}$. We denote the first-order (Tarski) entailment relation between subsets of $\mathbf{G}$ by $\models$. This relation satisfies properties (C1)–(C4) of Waldmann et al. [6]).

Note that for *ground* formulas, Tarski entailment (i.e., entailment w.r.t. *all* Σ-models) agrees with Herbrand entailment (i.e., entailment w.r.t. *term-generated* Σ-models).

To avoid duplication, we present ordering extensions, selection functions, and inference rules for general clauses, even though we currently need them only for ground clauses.

The term ordering $\succ$ is extended to a literal ordering and a clause ordering in the following way: To every positive literal $s \approx t$, we assign the multiset $\{s, t\}$, to every negative literal $s \not\approx t$, we assign the multiset $\{s, s, t, t\}$. The literal ordering $\succ_{\mathsf{L}}$ compares these multisets using the multiset extension of $\succ$. The clause ordering $\succ_{\mathsf{C}}$ compares clauses by comparing their multisets of literals using the multiset extension of $\succ_{\mathsf{L}}$. We say that a literal L is maximal (strictly maximal) in a clause C, if there is no other literal in C that is greater (greater or equal) than L w.r.t. $\succ_{\mathsf{L}}$.

Note that $s \approx t$ and $t \approx s$ are mapped to the same multiset (and analogously for negative literals), so $\succ_{\mathsf{L}}$ is well-defined.

The multiset extension of an ordering that is stable under substitutions is again stable under substitutions, so $\succ_{\mathsf{L}}$ is also stable under substitutions.

A selection function S maps every clause to a submultiset of its negative literals. We assume that S is stable under renaming substitutions (bijective mappings from variables to variables) that is, $S(C\rho) = S(C)\rho$ for every clause C and every renaming ρ.

For clauses C and D let $\mathrm{rename}(D, C)$ be an arbitrary but fixed renaming substitution ρ such that $D\rho$ and C are variable-disjoint. (In particular, if C and D are variable-disjoint, $\mathrm{rename}(D, C)$ is the identity substitution).

Inference rules for superposition with ordering $\succ$ and selection function S. Recall that we consider equations as *unordered* pairs of terms, so that all inference rules are to be read modulo symmetry of the equality symbol.

In the non-ground case, we apply a renaming substitution ρ to the first premise of a binary inference to ensure that the premises become variable-disjoint.

Pos. Superposition:

$$\frac{D' \vee t \approx t' \qquad C' \vee s[u] \approx s'}{(D'\rho \vee C' \vee s[t'\rho] \approx s')\sigma}$$

where $\rho = \text{rename}(D' \vee t \approx t',\, C' \vee s[u] \approx s')$,
$\sigma = \text{mgu}(t\rho, u)$ and u is not a variable,
$(D' \vee t \approx t')\rho\sigma \not\succeq_{\mathsf{C}} (C' \vee s[u] \approx s')\sigma$,
no literal is selected by S in the premises,
$(t \approx t')\rho\sigma$ is strictly maximal in $(D' \vee t \approx t')\rho\sigma$,
$(s[u] \approx s')\sigma$ is strictly maximal in $(C' \vee s[u] \approx s')\sigma$,
$t\rho\sigma \not\preceq t'\rho\sigma$,
$s[u]\sigma \not\preceq s'\sigma$.

Neg. Superposition:

$$\frac{D' \vee t \approx t' \qquad C' \vee s[u] \not\approx s'}{(D'\rho \vee C' \vee s[t'\rho] \not\approx s')\sigma}$$

where $\rho = \text{rename}(D' \vee t \approx t',\, C' \vee s[u] \not\approx s')$,
$\sigma = \text{mgu}(t\rho, u)$ and u is not a variable,
$(D' \vee t \approx t')\rho\sigma \not\succeq_{\mathsf{C}} (C' \vee s[u] \not\approx s')\sigma$,
no literal is selected by S in the left premise,
$(t \approx t')\rho\sigma$ is strictly maximal in $(D' \vee t \approx t')\rho\sigma$,
either $s[u] \not\approx s'$ is selected by S in the right premise
or no literal is selected by S in the right premise
and $(s[u] \not\approx s')\sigma$ is maximal in $(C' \vee s[u] \not\approx s')\sigma$,
$t\rho\sigma \not\preceq t'\rho\sigma$,
$s[u]\sigma \not\preceq s'\sigma$.

Equality Resolution:

$$\frac{C' \vee s \not\approx s'}{C'\sigma}$$

where $\sigma = \text{mgu}(s, s')$,
either $s \not\approx s'$ is selected by S in the premise
or no literal is selected by S in the premise
and $(s \not\approx s')\sigma$ is maximal in $(C' \vee s \not\approx s')\sigma$.

Equality Factoring:

$$\frac{C' \vee t \approx t' \vee s \approx s'}{(C' \vee s' \not\approx t' \vee s \approx t')\sigma}$$

where $\sigma = \text{mgu}(s, t)$,
no literal is selected by S in the premise,
$(s \approx s')\sigma$ is maximal in $(C' \vee t \approx t' \vee s \approx s')\sigma$,
$s\sigma \not\preceq s'\sigma$.

(These are the most commonly found restrictions for negative superposition and equality resolution, but not the strongest ones. To strengthen the restrictions, one can replace "L is selected by S in C" by "L is selected by S in C and $L\sigma$ is maximal in $C^S\sigma$, where C^S is the subclause of C consisting of the literals selected by S".)

The $\mathbf{G}$-inference system $\mathit{GInf}^{\succ,S}$ consists of all ground inferences (C_2, C_1, C_0) and (C_1, C_0) of the superposition calculus that satisfy the side conditions for $\succ$ and S. The formulas $C_n, \ldots, C_1$ are called *premises* of an inference ι, C_0 is called the *conclusion* of ι, denoted by $\mathit{concl}(\iota)$. If $N \subseteq \mathbf{G}$, we write $\mathit{GInf}^{\succ,S}(N)$ for the set of all inferences in $\mathit{GInf}^{\succ,S}$ whose premises are contained in N.

We define the redundancy criterion $\mathit{Red}^{\succ,S} = (\mathit{Red}_{\mathrm{I}}^{\succ,S}, \mathit{Red}_{\mathrm{F}}^{\succ,S})$ for $\mathit{GInf}^{\succ,S}$ as follows:

- Let $N \subseteq \mathbf{G}$. An inference $\iota \in \mathit{GInf}^{\succ,S}$ is contained in $\mathit{Red}_{\mathrm{I}}^{\succ,S}(N)$ if $M \models \mathit{concl}(\iota)$, where M is the set of all clauses in N that are smaller than the right (or only) premise of ι.
- Let $N \subseteq \mathbf{G}$. A clause $C \in \mathbf{G}$ is contained in $\mathit{Red}_{\mathrm{F}}^{\succ,S}(N)$ if $M \models C$, where M is the set of all clauses in N that are smaller than C.

By compactness of first-order logic, this is equivalent to "…where M is *some finite* set of clauses in N that are smaller than the right (or only) premise of ι / that are smaller than C".

Inferences in $\mathit{Red}_{\mathrm{I}}^{\succ,S}(N)$ and formulae in $\mathit{Red}_{\mathrm{F}}(N)^{\succ,S}$ are called *redundant w.r.t. N*.

$\mathit{Red}^{\succ,S}$ satisfies properties (R1)–(R4) of Waldmann et al. [6]. Since we keep $\succ$ fixed in the sequel, we will usually omit the superscript $\succ$ and write GInf^S and Red^S instead of $\mathit{GInf}^{\succ,S}$ and $\mathit{Red}^{\succ,S}$.

4 Ground Refutational Completeness

A set $N \subseteq \mathbf{G}$ is *saturated* w.r.t. GInf^S and Red^S if $\mathit{GInf}^S(N) \subseteq \mathit{Red}_{\mathrm{I}}^S(N)$. The pair $(\mathit{GInf}^S, \mathit{Red}^S)$ is *statically refutationally complete* w.r.t. $\models$ if $\bot \in N$ for every saturated set $N \subseteq \mathbf{G}$ with $N \models \{\bot\}$.

For any set E of ground equations, $\mathcal{T}_\Sigma(\emptyset)/E$ is an E-interpretation (or E-algebra) with universe $\{\, [t]_E \mid t \in \mathcal{T}_\Sigma(\emptyset) \,\}$, where $[t]_E = \{\, t' \in \mathcal{T}_\Sigma(\emptyset) \mid E \models t \approx t' \,\}$ is the E-congruence class of $t \in \mathcal{T}_\Sigma(\emptyset)$.

One can show (similar to the proof of Birkhoff's Theorem) that for every *ground* equation $s \approx t$ we have $E \models s \approx t$ if and only if $\mathcal{T}_\Sigma(\emptyset)/E \models s \approx t$ if and only if $s \leftrightarrow_E^* t$.

In particular, if E is a convergent set of rewrite rules R and $s \approx t$ is a ground equation, then $\mathcal{T}_\Sigma(\emptyset)/R \models s \approx t$ if and only if $s \downarrow_R t$ (i.e., $s \rightarrow_R^* u \leftarrow_R^* t$ for some u). By abuse of terminology, we say that an equation or clause is valid (or true) in R if and only if it is true in $\mathcal{T}_\Sigma(\emptyset)/R$.

Our refutational completeness proof follows Bachmair and Ganzinger [2, 3]: Given a subset $N \subseteq \mathbf{G}$ with $\bot \notin N$, we first construct a *candidate interpretation*, that is, a convergent set of rewrite rules R_∞. Afterwards we use well-founded induction to show that for a saturated set N, R_∞ is actually a model of N.

Let $N \subseteq \mathbf{G}$ be a set of clauses not containing $\bot$. Using induction on the clause ordering we define sets of rewrite rules E_C and R_C for all $C \in N$ as follows:

Assume that E_D has already been defined for all $D \in N$ with $D \prec_{\mathsf{C}} C$. Then $R_C = \bigcup_{D \prec_{\mathsf{C}} C} E_D$. The set E_C contains the rewrite rule $s \to t$, if

(a) $C = C' \vee s \approx t$.
(b) no literal is selected in C.
(c) $s \approx t$ is strictly maximal in C.
(d) $s \succ t$.
(e) C is false in R_C.
(f) C' is false in $R_C \cup \{s \to t\}$.
(g) s is irreducible w.r.t. R_C.

In this case, C is called *productive*. Otherwise $E_C = \emptyset$. Finally, we define $R_\infty = \bigcup_{D \in N} E_D$.

Lemma 1. *If $E_C = \{s \to t\}$ and $E_D = \{u \to v\}$, then $s \succ u$ if and only if $C \succ_{\mathsf{C}} D$.*

Proof. ($\Rightarrow$): By condition (c), $s \approx t$ is strictly maximal in C and $u \approx v$ is strictly maximal in D, and since the literal ordering is total on ground literals, this implies that all other literals in C or in D are actually smaller than $s \approx t$ or $u \approx v$, respectively.

Moreover, $s \succ t$ and $u \succ v$ by condition (d). Therefore $s \succ u$ implies $\{s,t\} \succ_{\mathsf{mul}} \{u,v\}$. Hence $s \approx t \succ_{\mathsf{L}} u \approx v \succeq_{\mathsf{L}} L$ for every literal L of D, and thus $C \succ_{\mathsf{C}} D$.

($\Leftarrow$): Let $C \succ_{\mathsf{C}} D$, then $E_D \subseteq R_C$. By condition (g), s must be irreducible w.r.t. R_C, so $s \neq u$.

Assume that $s \not\succ u$. By totality, this implies $s \preceq u$, and since $s \neq u$, we obtain $s \prec u$. But then $C \prec_{\mathsf{C}} D$ can be shown in the same way as in the ($\Rightarrow$)-part, contradicting the assumption. □

Corollary 2. *The rewrite systems R_C and R_∞ are convergent (i.e., terminating and confluent).*

Proof. By condition (d), $s \succ t$ for all rules $s \to t$ in R_C and R_∞, so R_C and R_∞ are terminating.

Furthermore, it is easy to check that there are no critical pairs between any two rules: Assume that there are rules $u \to v$ in E_D and $s \to t$ in E_C such that u is a subterm of s. As $\succ$ is a reduction ordering that is total on ground terms, we get $u \prec s$ and therefore $D \prec_{\mathsf{C}} C$ and $E_D \subseteq R_C$. But then s would be reducible by R_C, contradicting condition (g).

Now the absence of critical pairs implies local confluence, and termination and local confluence imply confluence. □

Lemma 3. *If $D \preceq_{\mathsf{C}} C$ and $E_C = \{s \to t\}$, then $s \succ u$ for every term u occurring in a negative literal in D and $s \succeq u$ for every term u occurring in a positive literal in D.*

Proof. If $s \preceq u$ for some term u occurring in a negative literal $u \not\approx v$ in D, then $\{u, u, v, v\} \succ_{\mathsf{mul}} \{s, t\}$. So $u \not\approx v \succ_{\mathsf{L}} s \approx t \succeq_{\mathsf{L}} L$ for every literal L of C, and therefore $D \succ_{\mathsf{C}} C$.

Similarly, if $s \prec u$ for some term u occurring in a positive literal $u \approx v$ in D, then $\{u, v\} \succ_{\mathsf{mul}} \{s, t\}$. So $u \approx v \succ_{\mathsf{L}} s \approx t \succeq_{\mathsf{L}} L$ for every literal L of C, and therefore $D \succ_{\mathsf{C}} C$. □

Corollary 4. *If $D \in N$ is true in R_D, then D is true in R_∞ and R_C for all $C \succ_{\mathsf{C}} D$.*

Proof. If a positive literal $s \approx t$ of D is true in R_D, then $s \downarrow_{R_D} t$. Since $R_D \subseteq R_C$ and $R_D \subseteq R_\infty$, we have $s \downarrow_{R_C} t$ and $s \downarrow_{R_\infty} t$, so $s \approx t$ is true in R_C and R_∞.

Otherwise, some negative literal $s \not\approx t$ of D must be true in R_D, hence $s \not\downarrow_{R_D} t$. As the rules in $R_\infty \setminus R_D$ have left-hand sides that are larger than s and t, they cannot be used in a rewrite proof of $s \downarrow t$, hence $s \not\downarrow_{R_C} t$ and $s \not\downarrow_{R_\infty} t$. □

Corollary 5. *If $D = D' \vee u \approx v$ is productive, then D' is false and D is true in R_∞ and R_C for all $C \succ_{\mathsf{C}} D$.*

Proof. Obviously, D is true in R_∞ and R_C for all $C \succ_{\mathsf{C}} D$.

Since all negative literals of D' are false in R_D, it is clear that they are false in R_∞ and R_C. For the positive literals $u' \approx v'$ of D', condition (f) ensures that they are false in $R_D \cup \{u \to v\}$. Since $u' \preceq u$ and $v' \preceq u$ and all rules in $R_\infty \setminus R_D$ have left-hand sides that are larger than u, these rules cannot be used in a rewrite proof of $u' \downarrow v'$, hence $u' \not\downarrow_{R_C} v'$ and $u' \not\downarrow_{R_\infty} v'$. □

Theorem 6 ("Model Construction"). *Let N be a set of clauses that is saturated and does not contain the empty clause. Then we have for every ground clause $C \in N$:*

(i) $E_C = \emptyset$ if and only if C is true in R_C.

(ii) C is true in R_∞ and in R_D for every $D \in N$ with $D \succ_{\mathsf{C}} C$.

Proof. We use induction on the clause ordering $\succ_{\mathsf{C}}$ and assume that (i) and (ii) are already satisfied for all clauses in N that are smaller than C. Note that the "if" part of (i) is obvious from the construction and that condition (ii) follows immediately from (i) and Corollaries 4 and 5. So it remains to show the "only if" part of (i).

Case 1: *C contains selected literals or a maximal negative literal.* Suppose that $C = C' \vee s \not\approx s'$, where $s \not\approx s'$ is a selected literal of C if C has selected literal, and where $s \not\approx s'$ is a maximal literal of C if C does not have selected literals. If $s \approx s'$ is false in R_C, then C is clearly true in R_C and we are done. So assume that $s \approx s'$ is true in R_C, that is, $s \downarrow_{R_C} s'$. Without loss of generality, $s \succeq s'$.

Case 1.1: $s = s'$. If $s = s'$, then there is an *equality resolution* inference

$$\frac{C' \vee s \not\approx s'}{C'}\ .$$

As N is saturated, this inference is contained in $Red_{\mathrm{I}}^{S}(N)$. So $M \models C'$, where M is the set of all clauses in N that are smaller than C. By the induction hypothesis, all clauses in M are true in R_C, therefore C' and C are true in R_C.

Case 1.2: $s \succ s'$. By definition, $s \downarrow_{R_C} s'$ means $s \rightarrow^{*}_{R_C} u \leftarrow^{*}_{R_C} s'$ for some term u. Since $s' \rightarrow^{*}_{R_C} u$, we know that $s' \succeq u$. If $s \succ s'$, then the derivation $s \rightarrow^{*}_{R_C} u$ cannot be empty, so it has the form $s = s[t] \rightarrow_{R_C} s[t'] \rightarrow^{*}_{R_C} u$, where $t \rightarrow t'$ is a rule in $E_D \subseteq R_C$ for some $D \in N$ with $D \prec_{\mathsf{C}} C$. Let $D = D' \vee t \approx t'$ with $E_D = \{t \rightarrow t'\}$. By property (b), no literal in D may be selected. Consequently, there is a *negative superposition* inference

$$\frac{D' \vee t \approx t' \qquad C' \vee s[t] \not\approx s'}{D' \vee C' \vee s[t'] \not\approx s'}$$

from D and C. As N is saturated, this inference is contained in $Red_{\mathrm{I}}^{S}(N)$. So its conclusion $D' \vee C' \vee s[t'] \not\approx s'$ is entailed by the set M of all clauses in N that are smaller than C. By the induction hypothesis, all clauses in M are true in R_C, therefore the conclusion is true in R_C. Since D is productive, D' is false in R_C by Cor. 5. Moreover, $s[t'] \rightarrow^{*}_{R_C} u \leftarrow^{*}_{R_C} s'$, so $s[t'] \not\approx s'$ is also false in R_C. Since D' and $s[t'] \not\approx s'$ are false in R_C, C' must be true, and therefore C is also true in R_C.

Case 2: *C contains neither selected literals nor a maximal negative literal.* If C does not fall into Case 1, it must have the form $C' \vee s \approx s'$, where $s \approx s'$ is a maximal literal of C. If $E_C = \{s \rightarrow s'\}$ or C' is true in R_C or $s = s'$, then there is nothing to show, so assume that $E_C = \emptyset$ and that C' is false in R_C. Without loss of generality, $s \succ s'$.

Case 2.1: *$s \approx s'$ is maximal in C, but not strictly maximal.* If $s \approx s'$ is maximal in C, but not strictly maximal, then C can be written as $C'' \vee s \approx s' \vee s \approx s'$. In this case, there is a *equality factoring* inference

$$\frac{C'' \vee s \approx s' \vee s \approx s'}{C'' \vee s' \not\approx s' \vee s \approx s'}$$

As in Case 1, saturation implies that the conclusion is true in R_C. Since $s' = s'$ implies $s' \downarrow_{R_C} s'$, we know that $s' \not\approx s'$ is false in R_C. So $C'' \vee s \approx s'$ must be true, and therefore C is also true in R_C.

Case 2.2: *$s \approx s'$ is strictly maximal in C and s is reducible.* Suppose that $s \approx s'$ is strictly maximal in C and s is reducible by some rule in $E_D \subseteq R_C$. Let $D = D' \vee t \approx t'$ and $E_D = \{t \rightarrow t'\}$. By property (b), no literal in D may be selected. Consequently, there is a *positive superposition* inference

$$\frac{D' \vee t \approx t' \qquad C' \vee s[t] \approx s'}{D' \vee C' \vee s[t'] \approx s'}$$

from D and C. Again, saturation implies that the conclusion is true in R_C. Since D is productive, D' is false in R_C by Cor. 5. Since D' and C' are false in R_C, $s[t'] \approx s'$ must be true in R_C, that is, $s[t'] \rightarrow^{*}_{R_C} u \leftarrow^{*}_{R_C} s'$. On the other hand, $s[t] \rightarrow_{R_C} s[t']$, so $s[t] \downarrow_{R_C} s'$, which means that $s[t] \approx s'$ and C are true in R_C.

Case 2.3: *$s \approx s'$ is strictly maximal in C and s is irreducible.* Suppose that $s \approx s'$ is strictly maximal in C and s is irreducible by R_C. Then conditions (a)–(d) and (g) for productivity are satisfied. If C is productive, there is nothing to show. If C is not productive, then either property (e) or (f) must be violated. If (e) is violated, that is, if C is true in R_C, there is again nothing to show. Let us therefore assume that (e) holds but (f) does not hold, that is, C (and hence C') is false in R_C but C' is true in $R_C \cup \{s \to s'\}$. Clearly any negative literal that is true in $R_C \cup \{s \to s'\}$ is also true in R_C. So C' must have the form $C' = C'' \vee t \approx t'$, where the positive literal $t \approx t'$ is true in $R_C \cup \{s \to s'\}$ and false in R_C. In other words, $t \downarrow_{R_C \cup \{s \to s'\}} t'$, but not $t \downarrow_{R_C} t'$. Consequently, there is a rewrite proof of $t \to^* u \leftarrow^* t'$ by $R_C \cup \{s \to s'\}$ in which the rule $s \to s'$ is used at least once. Without loss of generality we assume that $t \succeq t'$. If t were strictly smaller than s, it would be impossible to use $s \to s'$ in the rewrite proof. If t were strictly larger than s (which is again larger than s'), then $s \approx s' \prec_{\mathsf{L}} t \approx t'$, contradicting the assumption that $s \approx s'$ is strictly maximal in C. So we have $s = t$. Moreover, since $s \approx s'$ is strictly maximal in C, we must have $t = s \succ s' \succ t'$. From $t \succ t'$ we conclude that the rewrite proof of $t \to^* u \leftarrow^* t'$ has the form $t \to t'' \to^* u \leftarrow^* t'$. Since $t \succ t''$ and $t \succ t'$, every left-hand side of a rule used in $t'' \to^* u \leftarrow^* t'$ must be strictly smaller than t. Because $s \to s'$ must be used at least once in $t \to t'' \to^* u \leftarrow^* t'$ and cannot be used after the first step, the rewrite proof has the form $t = s \to s' \to^* u \leftarrow^* t'$, where the first step uses $s \to s'$ and all other steps use rules from R_C. Consequently, $s' \approx t'$ is true in R_C. Now observe that there is an *equality factoring* inference

$$\frac{C'' \vee t \approx t' \vee s \approx s'}{C'' \vee s' \not\approx t' \vee t \approx t'}$$

whose conclusion is true in R_C by saturation. Since the literal $s' \not\approx t'$ must be false in R_C, the rest of the clause must be true in R_C, and therefore C must be true in R_C, contradicting our assumption. This concludes the proof of the theorem. □

Corollary 7 ("Static Refutational Completeness"). *The pair $(\mathit{GInf}^S, \mathit{Red}^S)$ is* statically refutationally complete *w.r.t. $\models$ for every selection function S.*

Proof. Let N be a subset of $\mathbf{G}$ that does not contain $\bot$. By part (ii) of the model construction theorem, the interpretation R_∞ (that is, $\mathcal{T}_\Sigma(\emptyset)/R_\infty$) is a model of all clauses in N. □

5 Lifting

We will now lift the completeness result for ground first-order clauses to a completeness result for general first-order clauses.

Let $\mathbf{F}$ be the set of first-order clauses over Σ; let $\mathbf{F}_\bot$ be the subset $\{\bot\} \subseteq \mathbf{F}$. The $\mathbf{F}$-inference system $\mathit{FInf}^{\succ,S}$ consists of all inferences (C_2, C_1, C_0) and (C_1, C_0) of the superposition calculus that satisfy the side conditions for $\succ$ and

S. The formulas $C_n, \ldots, C_1$ are called *premises* of an inference ι, C_0 is called the *conclusion* of ι, denoted by $concl(\iota)$. If $N \subseteq \mathbf{F}$, we write $\mathit{FInf}^{\succ,S}(N)$ for the set of all inferences in $\mathit{FInf}^{\succ,S}$ whose premises are contained in N.

A selection function T for $\mathbf{G}$ is a grounding of a selection function S for $\mathbf{F}$, if for every $D \in \mathbf{G}$ there exists some $C \in \mathbf{F}$ such that $D = C\theta$ and $T(D) = (S(C))\theta$, that is, if for every clause $D \in \mathbf{G}$ the literals selected by T in D correspond to the literals selected by S in some $C \in \mathbf{F}$ such that $D = C\theta$. The set of all groundings of a selection function S is denoted by $\mathrm{gs}(S)$.

If $\iota = (C_1, C_0)$ is an inference in $\mathit{FInf}^{\succ,S}$ and θ is a substitution such that $C_1\theta, C_0\theta \in \mathbf{G}$, then $(C_1\theta, C_0\theta)$ is called a pre-instance of ι. Analogously, if $\iota = (C_2, C_1, C_0)$ is an inference in $\mathit{FInf}^{\succ,S}$, $\rho = \mathrm{rename}(C_2, C_1)$, and θ is a substitution such that $C_2\rho\theta, C_1\theta, C_0\theta \in \mathbf{G}$, then $(C_2\rho\theta, C_1\theta, C_0\theta)$ is called a pre-instance of ι.

Let ι be an inference (C_1, C_0) or (C_2, C_1, C_0) in $\mathit{FInf}^{\succ,S}$, let ι' be a pre-instance $(C_1\theta, C_0\theta)$ or $(C_2\rho\theta, C_1\theta, C_0\theta)$ of ι, and let $T \in \mathrm{gs}(S)$. Then ι' is called a T-ground instance of ι if it is a $\mathbf{G}$-inference in $\mathit{GInf}^{\succ,T}$ and $T(C_1\theta) = (S(C_1))\theta$ (and $T(C_2\rho\theta) = (S(C_2))\rho\theta$).

Note that the definition of a T-ground instance depends implicitly on S. In fact, it is possible that ι is an inference in both $\mathit{FInf}^{\succ,S_1}$ and $\mathit{FInf}^{\succ,S_2}$ and $T \in \mathrm{gs}(S_1) \cap \mathrm{gs}(S_2)$, and that the pre-instance ι' is a T-ground instance of ι, if we consider ι as an $\mathit{FInf}^{\succ,S_1}$ inference, but not, if we consider ι as an $\mathit{FInf}^{\succ,S_2}$ inference. In a formalized proof, one should rather talk about (S,T)-ground instances instead of T-ground instances.

Let S be a selection function for $\mathbf{F}$, Let $T \in \mathrm{gs}(S)$ be a selection function for $\mathbf{G}$. We define the grounding function $\mathcal{G}^T$ as follows:

- For a clause $C \in \mathbf{F}$, $\mathcal{G}^T(C) = \{\, C\theta \mid C\theta \in \mathbf{G} \,\}$ is the set of all ground instances of C.
- For an $\mathbf{F}$-inference $\iota \in \mathit{FInf}^{\succ,S}$, $\mathcal{G}^T(\iota)$ is the set of all T-ground instances of ι.

The function $\mathcal{G}$ is extended to sets $\mathcal{S}$ of formulas or inferences by defining $\mathcal{G}(\mathcal{S}) = \bigcup_{x\in\mathcal{S}} \mathcal{G}(x)$.

$\mathcal{G}^T$ satisfies properties (G1)–(G3) of grounding functions of Waldmann et al. [6]; in fact, it also satisfies (G3′), which implies (G3).

For $T \in \mathrm{gs}(S)$, the $\mathcal{G}^T$-*lifting* of $\models$ is the relation $\models^{\mathcal{G}^T} \subseteq \mathcal{P}(\mathbf{F}) \times \mathcal{P}(\mathbf{F})$ defined by $N_1 \models^{\mathcal{G}^T} N_2$ if and only if $\mathcal{G}^T(N_1) \models \mathcal{G}^T(N_2)$. Note that for sets N of clauses, $\mathcal{G}^T(N)$ is independent of T, which implies that $\models^{\mathcal{G}^T}$ is also independent of T. In fact, $\models^{\mathcal{G}^T}$ is the Herbrand entailment relation $\models_{\mathsf{H}}$ for sets of first-order clauses (which in turn is equivalent to the standard (Tarskian) entailment relation $\models_{\mathsf{T}}$ for sets of first-order clauses as long as we are only interested in refutations, i.e., $N \models_{\mathsf{H}} \bot$ holds if and only if $N \models_{\mathsf{T}} \bot$ holds). Trivially, the intersection $\models^{\cap}$ of all $\models^{\mathcal{G}^T}$ for $T \in \mathrm{gs}(S)$ is again $\models_{\mathsf{H}}$.

We define the $\mathcal{G}^T$-*lifting* $\mathit{Red}^{\mathcal{G}^T} = (\mathit{Red}_{\mathrm{I}}^{\mathcal{G}^T}, \mathit{Red}_{\mathrm{F}}^{\mathcal{G}^T})$ of Red^T with $\mathit{Red}_{\mathrm{I}}^{\mathcal{G}^T} : \mathcal{P}(\mathbf{F}) \to \mathcal{P}(\mathit{FInf}^{\succ,S})$ and $\mathit{Red}_{\mathrm{F}}^{\mathcal{G}^T} : \mathcal{P}(\mathbf{F}) \to \mathcal{P}(\mathbf{F})$ by $\iota \in \mathit{Red}_{\mathrm{I}}^{\mathcal{G}^T}(N)$ if and only if $\mathcal{G}^T(\iota) \subseteq \mathit{Red}_{\mathrm{I}}^T(\mathcal{G}^T(N))$ and $C \in \mathit{Red}_{\mathrm{F}}^{\mathcal{G}^T}(N)$ if and only if $\mathcal{G}^T(C) \subseteq \mathit{Red}_{\mathrm{F}}^T(\mathcal{G}^T(N))$.

For every $T \in \mathrm{gs}(S)$, $\mathit{Red}^{\mathcal{G}^T}$ is a redundancy criterion for $\mathit{FInf}^{\succ,S}$ and $\models_{\mathsf{H}}$ by Thm. 30 of Waldmann et al. [6]. Moreover, the intersection $\mathit{Red}^{\cap} = \bigcap_{T \in \mathrm{gs}(S)} \mathit{Red}^{\mathcal{G}^T}$ is a redundancy criterion for $\mathit{FInf}^{\succ,S}$ and $\models_{\mathsf{H}}$ by Thm. 24 of [6].

Lemma 8. *Let $C = C' \vee s \not\approx s' \in \mathbf{F}$, let θ be a substitution such that $C\theta \in \mathbf{G}$. Let S be a selection function for $\mathbf{F}$, let $T \in \mathrm{gs}(S)$ be a selection function for $\mathbf{G}$ such that $T(C\theta) = (S(C))\theta$. Let*

$$\iota = \frac{C'\theta \vee s\theta \not\approx s'\theta}{C'\theta}$$

with $s\theta = s'\theta$ be an equality resolution inference in $\mathit{GInf}^{\succ,T}$ from $C\theta$. Then ι is a T-ground instance of an equality resolution inference in $\mathit{FInf}^{\succ,S}$ from C.

Proof. Since $s\theta = s'\theta$, the terms s and s' are unifiable. Let σ be an idempotent most general unifier of s and s' such that $\theta = \sigma \circ \tau$. Note that by idemotence, $\sigma \circ \theta = \sigma \circ \sigma \circ \tau = \sigma \circ \tau = \theta$.

Suppose that ι is an equality resolution inference in $\mathit{GInf}^{\succ,T}$ from $C\theta$. Then either $s\theta \not\approx s'\theta$ is selected in $C\theta$ by T or no literal is selected in $C\theta$ by T and $s\theta \not\approx s'\theta$ is maximal in $C\theta$. Since $T(C\theta) = (S(C))\theta$, in the first case $s \not\approx s'$ is selected in C by S, and in the second case no literal is selected in C by S. Moreover, in the second case, $s\sigma \not\approx s'\sigma$ must be a maximal literal in $C\sigma$ (if it were not maximal, then $L\sigma \succ_{\mathsf{L}} s\sigma \not\approx s'\sigma$ for some other literal $L\sigma$ in $C\sigma$, hence $L\theta = L\sigma\tau \succ_{\mathsf{L}} s\sigma\tau \not\approx s'\sigma\tau = s\theta \not\approx s'\theta$ for a literal $L\theta$ in $C\theta$, contradicting the maximality of $s\theta \not\approx s'\theta$). Therefore

$$\iota' = \frac{C' \vee s \not\approx s'}{C'\sigma}$$

is an equality resolution inference in $\mathit{FInf}^{\succ,S}$ from C. Moreover $C'\sigma\theta = C'\theta$, and by assumption $T(C\theta) = (S(C))\theta$, so $\iota \in \mathcal{G}^T(\iota')$. □

Lemma 9. *Let $C = C' \vee t \approx t' \vee s \approx s' \in \mathbf{F}$, let θ be a substitution such that $C\theta \in \mathbf{G}$. Let S be a selection function for $\mathbf{F}$, let $T \in \mathrm{gs}(S)$ be a selection function for $\mathbf{G}$ such that $T(C\theta) = (S(C))\theta$. Let*

$$\iota = \frac{C'\theta \vee t\theta \approx t'\theta \vee s\theta \approx s'\theta}{C'\theta \vee s'\theta \not\approx t'\theta \vee s\theta \approx t'\theta}$$

with $s\theta = t\theta$ be an equality factoring inference in $\mathit{GInf}^{\succ,T}$ from $C\theta$. Then ι is a T-ground instance of an equality factoring inference in $\mathit{FInf}^{\succ,S}$ from C.

Proof. Since $s\theta = s'\theta$, the terms s and s' are unifiable. Let σ be an idempotent most general unifier of s and s' such that $\theta = \sigma \circ \tau$. Note that by idemotence, $\sigma \circ \theta = \sigma \circ \sigma \circ \tau = \sigma \circ \tau = \theta$.

Suppose that ι is an equality factoring inference in $\mathit{GInf}^{\succ,T}$ from $C\theta$. Then no literal is selected in $C\theta$ by T, and since $T(C\theta) = (S(C))\theta$, this implies that no literal is selected in C by S. Furthermore $s\theta \approx s'\theta$ must be a maximal literal in $C\theta$, which implies that $s\sigma \approx s'\sigma$ must be a maximal literal in $C\sigma$. Finally, we know that $s\theta \succ s'\theta$, hence $s\sigma \not\preceq s'\sigma$ (since $s\sigma \preceq s'\sigma$ would imply $s\theta = s\sigma\tau \preceq s'\sigma\tau = s'\theta$). Therefore

$$\iota' = \frac{C' \vee t \approx t' \vee s \approx s'}{(C' \vee s' \not\approx t' \vee s \approx t')\sigma}$$

is an equality resolution inference in $FInf^{\succ,S}$ from C. Moreover $(C' \vee s' \not\approx t' \vee s \approx t')\sigma\theta = (C' \vee s' \not\approx t' \vee s \approx t')\theta$, and by assumption $T(C\theta) = (S(C))\theta$, so $\iota \in \mathcal{G}^T(\iota')$. $\square$

Lemma 10. *Let $D = D' \vee t \approx t'$ and $C = C' \vee [\neg]\, s \approx s'$ be two clauses in* $\mathbf{F}$*; let θ_1 and θ_2 be substitutions such that $D\theta_2 \in \mathbf{G}$ and $C\theta_1 \in \mathbf{G}$. Let S be a selection function for $\mathbf{F}$, let $T \in \mathrm{gs}(S)$ be a selection function for $\mathbf{G}$ such that $T(C\theta_1) = (S(C))\theta_1$ and $T(D\theta_2) = (S(D))\theta_2$. Let*

$$\iota = \frac{D'\theta_2 \vee t\theta_2 \approx t'\theta_2 \qquad C'\theta_1 \vee [\neg]\, s\theta_1[v]_p \approx s'\theta_1}{D'\theta_2 \vee C'\theta_1 \vee [\neg]\, s\theta_1[t'\theta_2]_p \approx s'\theta_1}$$

with $t\theta_2 = v = s\theta_1|_p$ be a positive or negative superposition inference in $GInf^{\succ,T}$ from $D\theta_2$ and $C\theta_1$. If p is a position of s and $s|_p$ is not a variable, then ι is a T-ground instance of a superposition inference in $FInf^{\succ,S}$ from D and C.

Proof. We consider the case of negative superposition inferences, the proof for positive superposition inferences is similar.

Let $\rho = \mathrm{rename}(D, C)$, then $D\rho$ and C are variable-disjoint. Define the substitution θ by $x\theta = x\theta_1$ if x is a variable of C and $x\theta = x\rho^{-1}\theta_2$ if x is a variable of $D\rho$. Clearly $C\theta = C\theta_1$ and $D\rho\theta = D\rho\rho^{-1}\theta_2 = D\theta_2$.

Assume that p is a position of s and that $s|_p$ is not a variable. Let $u = s|_p$. Then $t\rho\theta = t\theta_2 = v = u\theta$ and we have $s\theta = s\theta[v]_p = s\theta[u\theta]_p = (s[u]_p)\theta$ and $s\theta[t'\rho\theta]_p = (s[t'\rho]_p)\theta$. Since $t\rho\theta = u\theta$, the terms $t\rho$ and u are unifiable. Let σ be an idempotent most general unifier of $t\rho$ and u such that $\theta = \sigma \circ \tau$. Note that by idemotence, $\sigma \circ \theta = \sigma \circ \sigma \circ \tau = \sigma \circ \tau = \theta$.

Suppose that ι is a negative resolution inference in $GInf^{\succ,T}$ from $D\theta$ and $C\theta$. Then either $s\theta \not\approx s'\theta$ is selected in $C\theta$ by T or no literal is selected in $C\theta$ by T and $s\theta \not\approx s'\theta$ is maximal in $C\theta$. Since $T(C\theta) = (S(C))\theta$, in the first case $s \not\approx s'$ is selected in C by S, and in the second case no literal is selected in C by S, and moreover, in the second case, $s\sigma \not\approx s'\sigma$ must be a maximal literal in $C\sigma$.

Similarly, no literal may be selected in $D\rho\theta$ by T and $t\rho\theta \approx t'\rho\theta$ must be strictly maximal in $D\rho\theta$. Since $T(D\rho\theta) = (S(D))\rho\theta$, this implies that zno literal is selected in D by S and that $t\rho\sigma \approx t'\rho\sigma$ must be a strictly maximal literal in $D\rho\sigma$.

Finally, $t\rho\theta \succ t'\rho\theta$, $s[u]\theta \succ s'\theta$, and $D\rho\theta \not\succeq_{\mathsf{C}} C\theta$, from which we conclude that $t\rho\sigma \not\preceq t'\rho\sigma$, $s[u]\sigma \not\preceq s'\sigma$, and $D\rho\sigma \not\succeq_{\mathsf{C}} C\sigma$.

Therefore

$$\iota' = \frac{D' \vee t \approx t' \qquad C' \vee s[u]_p \not\approx s'}{(D'\rho \vee C' \vee s[t'\rho]_p \not\approx s')\sigma}$$

is a negative superposition inference in $\mathit{FInf}^{\succ,S}$ from D and C. Moreover $D\rho\theta = D\theta_2$ and thus $T(D\rho\theta) = T(D\theta_2) = (S(D))\theta_2 = (S(D))\rho\theta$, $C\theta = C\theta_1$ and thus $T(C\theta) = T(C\theta_1) = (S(C))\theta_1 = (S(C))\theta$, and $(D'\rho \vee C' \vee s[t'\rho] \not\approx s')\sigma\theta = (D'\rho \vee C' \vee s[t'\rho] \not\approx s')\theta = (D'\theta_2 \vee C'\theta_1 \vee s\theta_1[t'\theta_2] \not\approx s'\theta_1)$, so $\iota \in \mathcal{G}^T(\iota')$. $\square$

Lemma 11. *Let $N \subseteq \mathbf{F}$. Let S be a selection function for $\mathbf{F}$ and let $T \in \mathrm{gs}(S)$ be a selection function for $\mathbf{G}$, such that for every $D \in \mathcal{G}^T(N)$ there exists some $C \in N$ and a some substitution θ such that $D = C\theta$ and $T(D) = (S(C))\theta$. Then $\mathit{GInf}^{\succ,T}(\mathcal{G}^T(N)) \subseteq \mathcal{G}^T(\mathit{FInf}^{\succ,S}(N)) \cup \mathit{Red}_{\mathrm{I}}^T(\mathcal{G}^T(N))$.*

Proof. Let ι be an inference in $\mathit{GInf}^{\succ,T}$ from premises $D_n, \ldots, D_1 \in \mathcal{G}^T(N)$ (with $n \in \{1,2\}$). By assumption, there exist $C_n, \ldots, C_1 \in N$ and substitutions θ_i such that $D_i = C_i\theta_i$ and $T(D_i) = (S(C_i))\theta_i$.

If ι is an equality resolution inference with premise $D_1 = D' \vee v \not\approx v'$, then C_1 must have the form $C' \vee s \not\approx s'$ with $C'\theta_1 = D'$, $s\theta_1 = v$, and $s'\theta_1 = v'$. By Lemma 8, $\iota \in \mathcal{G}^T(\iota')$ for some $\iota' \in \mathit{FInf}^{\succ,S}(N)$.

If ι is an equality factoring inference with premise $D_1 = D' \vee v \approx v' \vee v \approx v''$, then C_1 must have the form $C' \vee t \approx t' \vee s \approx s'$ with $C'\theta_1 = D'$, $s\theta_1 = t\theta_1 = v$, $t'\theta_1 = v'$, and $s'\theta_1 = v''$. By Lemma 9, $\iota \in \mathcal{G}^T(\iota')$ for some $\iota' \in \mathit{FInf}^{\succ,S}(N)$.

Otherwise, ι is a positive or negative superposition inference with premises $D_2 = D_2' \vee v \approx v'$ and $D_1 = D_1' \vee [\neg]\, u[v]_p \approx u'$. Then C_2 and C_1 must have the form $C_2 = C_2' \vee t \approx t'$ and $C_1 = C_1' \vee [\neg]\, s \approx s'$ with $C_2'\theta_2 = D_2'$, $t\theta_2 = v$, $t'\theta_2 = v'$, and $C_1'\theta_1 = D_1'$, $s\theta_1 = u[v]_p$, $s'\theta_1 = u'$.

If p is a position of s and $s|_p$ is not a variable, then $\iota \in \mathcal{G}^T(\iota')$ for some $\iota' \in \mathit{FInf}^{\succ,S}(N)$ by Lemma 10.

Otherwise let p' be the longest prefix of p that is a position of s and let $p = p'p''$. The subterm $s|_{p'}$ must be a variable x. Define the substitution θ_1' by $x\theta_1' = x\theta_1[v']_{p''}$ and $y\theta_1' = y\theta_1$ for every variable y of C_1 different from x. Clearly, $C_1\theta_1' \in \mathcal{G}^T(N)$, and since $v \succ v'$ we have $x\theta_1 = x\theta_1[v]_{p''} \succ x\theta_1[v']_{p''} = x\theta_1'$ and thus $D_1 = C_1\theta_1 \succ_{\mathsf{C}} C_1\theta_1'$. Furthermore, we already know from the definition of superposition inferences that $D_1 \succ_{\mathsf{C}} D_2$. We will show that $C_1\theta_1'$ and D_2 entail the conclusion $D_2' \vee D_1' \vee [\neg]\, u[v']_p \approx u'$ of ι: Suppose that $C_1\theta_1'$ and $D_2 = D_2' \vee v \approx v'$ hold in some interpretation. If D_2' holds, then the conclusion holds trivially. Otherwise, $v \approx v'$ holds in that interpretation, then $x\theta_1[v]_{p''} \approx x\theta_1[v']_{p''}$ holds by congruence, and since $x\theta_1 = x\theta[v]_{p''}$ and $x\theta_1' = x\theta_1[v']_{p''}$, we know that $x\theta_1 \approx x\theta_1'$ holds. Moreover, $y\theta_1 \approx y\theta_1'$ holds for every variable y different from x. Since $C_1\theta_1'$ holds in the interpretation by assumption, congruence implies that $C_1\theta_1 = D_1' \vee [\neg]\, u[v]_p \approx u'$ holds in the interpretation. Once more by congruence, $D_1' \vee [\neg]\, u[v']_p \approx u'$ holds, so the conclusion of ι must hold as well. Therefore, the conclusion of ι is entailed by $C_1\theta_1'$ and D_2; both clauses are contained in $\mathcal{G}^T(N)$ and smaller than the right premise of ι; so $\iota \in \mathit{Red}_{\mathrm{I}}^T(\mathcal{G}^T(N))$. $\square$

By Lemma 31 of Waldmann et al. [6], we get immediately:

Lemma 12. *Let $N \subseteq \mathbf{F}$. Let S be a selection function for $\mathbf{F}$ and let $T \in \mathrm{gs}(S)$ be a selection function for $\mathbf{G}$, such that for every $D \in \mathcal{G}^T(N)$ there exists some $C \in N$ and a some substitution θ such that $D = C\theta$ and $T(D) = (S(C))\theta$. If N*

is saturated w.r.t. $FInf^{\succ,S}$ *and* $Red^{\mathcal{G}^T}$*, then* $\mathcal{G}^T(N)$ *is saturated w.r.t.* $GInf^{\succ,T}$ *and* Red^T*.*

From this, static refutational completeness of $(FInf^{\succ,S}, Red^{\cap})$ follows by choosing an appropriate selection function $T \in \mathrm{gs}(S)$ for a given saturated set N:

Theorem 13. $(FInf^{\succ,S}, Red^{\cap})$ *is statically refutationally complete.*

Proof. Let $N \subseteq \mathbf{F}$ be saturated w.r.t. $FInf^{\succ,S}$ and $Red^{\cap}$. Suppose that $\bot \notin N$. We define a selection function $T \in \mathrm{gs}(S)$ as follows: If there exists a $C \in N$ such that $D = C\theta$ for some θ, we set $T(D) = (S(C))\theta$ for some $C \in N$ with this property; otherwise we set $T(D) = (S(C))\theta$ for an arbitrary $C \in \mathbf{F}$ such that $D = C\theta$. Since N is saturated w.r.t. $FInf^{\succ,S}$ and $Red^{\cap}$, N is saturated w.r.t. $FInf^{\succ,S}$ and $Red^{\mathcal{G}^T}$. So, by the previous lemma, $\mathcal{G}^T(N)$ is saturated w.r.t. $GInf^{\succ,T}$ and Red^T. Since $\bot \notin \mathcal{G}^T(N)$, the static refutational completeness of $GInf^{\succ,T}$ implies $\mathcal{G}^T(N) \not\models \{\bot\} = \mathcal{G}^T(\{\bot\})$, hence $N \not\models^{\mathcal{G}^T} \{\bot\}$, and hence $N \not\models_{\mathsf{H}} \bot$. □

The quantification over all $T \in \mathrm{gs}(S)$ in the definition of $Red^{\cap}$ looks rather complicated. We observe, however, that $Red_{\mathrm{F}}^{\mathcal{G}^T}$ doesn't depend on T anyhow, so that $Red_{\mathrm{F}}^{\cap}$ agrees with $Red_{\mathrm{F}}^{\mathcal{G}^T}$ for an arbitrary T. For $Red_{\mathrm{I}}^{\cap}$, restricting to a single $T \in \mathrm{gs}(S)$ *globally* does not work, but we can at least restrict ourselves to a single $T \in \mathrm{gs}(S)$ *per pre-instance*. We need two simple lemmas:

Lemma 14. *Let* ι *be an inference in* $FInf^{\succ,S}$*, let* ι' *be a pre-instance* $(C_1\theta, C_0\theta)$ *or* $(C_2\rho\theta, C_1\theta, C_0\theta)$ *of* ι*. Let* $T, T' \in \mathrm{gs}(S)$ *be two groundings of* S *that agree on* $C_1\theta$ *(or* $C_2\rho\theta$ *and* $C_1\theta$*, respectively). Then* $\iota' \in \mathcal{G}^T(\iota)$ *if and only if* $\iota' \in \mathcal{G}^{T'}(\iota)$*.*

Proof. Since T and T' agree on $C_1\theta$ (and $C_2\rho\theta$), we know that $\iota' \in \mathcal{G}^T(\iota)$ if and only if $\iota' \in GInf^{\succ,T}$ and analogously $\iota' \in \mathcal{G}^{T'}(\iota)$ if and only if $\iota' \in GInf^{\succ,T'}$. The side conditions of all ground inferences depend only on the ordering (which is the same for $GInf^{\succ,T}$ and $GInf^{\succ,T'}$) and the selected literals *in the premise(s)* $C_1\theta$ (and $C_2\rho\theta$), which are the same by assumption. So ι' is an inference in $GInf^{\succ,T}$ if and only if it is an inference in $GInf^{\succ,T'}$. □

Lemma 15. *Let* $N \subseteq F$*, let* ι *be an inference in* $FInf^{\succ,S}(N)$*, let* ι' *be a pre-instance of* ι*. Let* $T, T' \in \mathrm{gs}(S)$ *be two groundings of* S *such that* ι' *is contained in both* $\mathcal{G}^T(\iota)$ *and* $\mathcal{G}^{T'}(\iota)$*. Then* $\iota' \in Red_{\mathrm{I}}^T(\mathcal{G}^T(N))$ *if and only if* $\iota' \in Red_{\mathrm{I}}^{T'}(\mathcal{G}^{T'}(N))$*.*

Proof. First, we observe that the definition of $\mathcal{G}^T(N)$ does not depend on T, so that $\mathcal{G}^T(N) = \mathcal{G}^{T'}(N)$. So the set M of all clauses in $\mathcal{G}^T(N)$ that are smaller than the right (or only) premise of ι' agrees with the set M' of all clauses in $\mathcal{G}^{T'}(N)$ that are smaller than the right (or only) premise of ι'. Consequently $\iota' \in Red_{\mathrm{I}}^T(\mathcal{G}^T(N))$ if and only if $M \models concl(\iota')$ if and only if $M' \models concl(\iota')$ if and only if $\iota' \in Red_{\mathrm{I}}^{T'}(\mathcal{G}^{T'}(N))$. □

We can now show that $Red_{\mathrm{I}}^{\cap}$ is in fact equivalent to the classical definition from [3] (in a slightly rephrased form):

Theorem 16. *For any inference $\iota \in \mathit{FInf}^{\succ,S}$ of the form (C_1, C_0) or (C_2, C_1, C_0) and for any pre-instance ι' of ι of the form $(C_1\theta, C_0\theta)$ or $(C_2\rho\theta, C_1\theta, C_0\theta)$ let* $\mathrm{inherit}(S, \iota, \iota')$ *be an arbitrary but fixed selection function $T \in \mathrm{gs}(S)$ such that $T(C_1\theta) = (S(C_1))\theta$ (and $T(C_2\rho\theta) = (S(C_2))\rho\theta$), if such a selection function exists. Let* $\mathrm{inherit}(S, \iota, \iota')$ *be undefined otherwise.*

Then an inference $\iota = (C_1, C_0) \in \mathit{FInf}^{\succ,S}$ is contained in $\mathit{Red}_{\mathrm{I}}^{\cap}(N)$ if and only if every pre-instance ι' is contained in $\mathit{Red}_{\mathrm{I}}^{T}(\mathcal{G}^T(N))$ whenever $T = \mathrm{inherit}(S, \iota, \iota')$ is defined and ι' is a T-ground instance of ι.

Proof. Let $\iota \in \mathit{FInf}^{\succ,S}$ be contained in $\mathit{Red}_{\mathrm{I}}^{\cap}(N) = \bigcap_{T' \in \mathrm{gs}(S)} \mathit{Red}_{\mathrm{I}}^{\mathcal{G}^{T'}}(N)$. Then $\iota \in \mathit{Red}_{\mathrm{I}}^{\mathcal{G}^{T'}}(N)$ for every $T' \in \mathrm{gs}(S)$. Now assume that $T = \mathrm{inherit}(S, \iota, \iota')$ is defined and that ι' is a T-ground instance of ι. Then $T \in \mathrm{gs}(S)$, so $\iota \in \mathit{Red}_{\mathrm{I}}^{\mathcal{G}^{T}}(N)$, and hence $\iota' \in \mathcal{G}^T(\iota) \subseteq \mathit{Red}_{\mathrm{I}}^{T}(\mathcal{G}^T(N))$.

Conversely assume that every pre-instance ι' is contained in $\mathit{Red}_{\mathrm{I}}^{T}(\mathcal{G}^T(N))$ whenever $T = \mathrm{inherit}(S, \iota, \iota')$ is defined and ι' is a T-ground instance of ι. We have to show that $\iota \in \mathit{Red}_{\mathrm{I}}^{\cap}(N)$, which is equivalent to $\iota \in \mathit{Red}_{\mathrm{I}}^{\mathcal{G}^{T'}}(N)$ for every $T' \in \mathrm{gs}(S)$. Choose $T' \in \mathrm{gs}(S)$ arbitrarily. We now have to show that $\mathcal{G}^{T'}(\iota) \subseteq \mathit{Red}_{\mathrm{I}}^{T'}(\mathcal{G}^{T'}(N))$, which means by definition that every T'-ground instance of ι is contained in $\mathit{Red}_{\mathrm{I}}^{T'}(\mathcal{G}^{T'}(N))$. Let ι'' be a T'-ground instance of ι, then $T'(C_1\theta) = (S(C_1))\theta$ (and $T'(C_2\rho\theta) = (S(C_2))\rho\theta$), so $T = \mathrm{inherit}(S, \iota, \iota'') \in \mathrm{gs}(S)$ is defined. Since $T(C_1\theta) = T'(C_1\theta) = (S(C_1))\theta$ (and $T(C_2\rho\theta) = T'(C_2\rho\theta) = (S(C_2))\rho\theta$), we know by Lemma 14 that ι'' is a T-ground instance of ι. By assumption, $\iota'' \in \mathit{Red}_{\mathrm{I}}^{T}(\mathcal{G}^T(N))$, so by Lemma 15, $\iota'' \in \mathit{Red}_{\mathrm{I}}^{T'}(\mathcal{G}^{T'}(N))$ as required. $\square$

The function $\mathrm{inherit}(S, \iota, \iota')$ is undefined if the selections of the premises of ι are contradictory for the pre-instance ι'. For instance, consider the clauses $C_2 = (\neg\, f(a) \approx b \vee f(y) \approx y)$ and $C_1 = (\neg\, f(x) \approx b \vee f(x) \approx a)$, where S selects the first literal in C_1 and nothing in C_2. Then there is a Negative Superposition inference ι

$$\frac{\neg\, f(a) \approx b \vee f(y) \approx y \qquad \neg\, f(x) \approx b \vee f(x) \approx a}{\neg\, f(a) \approx b \vee f(x) \approx a \vee \neg\, x \approx b}$$

in which the maximal second literal of C_2 and the selected first literal of C_1 are overlapped.

For the pre-instance ι'

$$\frac{\neg\, f(a) \approx b \vee f(a) \approx a \qquad \neg\, f(a) \approx b \vee f(a) \approx a}{\neg\, f(a) \approx b \vee f(a) \approx a \vee \neg\, a \approx b}$$

of ι there is no selection function T such that $T(C_1\theta) = (S(C_1))\theta$ and $T(C_2\rho\theta) = (S(C_2))\rho\theta$, since $C_1\theta = C_2\rho\theta$, but $(S(C_1))\theta \neq (S(C_2))\rho\theta$. (In fact, ι' violates the ordering restrictions of Negative Superposition, so it is not an inference for any selection function.)

To see that restricting to a single $T \in \mathrm{gs}(S)$ *globally* does not work, consider the clauses $C_2 = (\neg\, f(x) \approx g(a, x') \vee f(f(x)) \approx g(a, x'))$ and $C_1 = (\neg\, f(y) \approx g(y', a) \vee f(f(y)) \approx g(y', a))$, where S selects the first literal in C_1 and nothing in C_2 and the term ordering $\succ$ is an LPO with precedence $f > g > a$. There is a Negative Superposition inference ι

$$\frac{\neg\, f(x) \approx g(a, x') \vee f(f(x)) \approx g(a, x') \qquad \neg\, f(y) \approx g(y', a) \vee f(f(y)) \approx g(y', a)}{\neg\, f(x) \approx g(a, x') \vee \neg\, g(a, x') \approx g(y', a) \vee f(f(f(x))) \approx g(y', a)}$$

in which the maximal second literal of C_2 and the selected first literal of C_1 are overlapped.

Now consider the pre-instances ι_1

$$\frac{\begin{array}{l}\neg\, f(a) \approx g(a, a) \vee f(f(a)) \approx g(a, a) \\ \qquad\qquad\qquad\qquad \neg\, f(f(a)) \approx g(a, a) \vee f(f(f(a))) \approx g(a, a)\end{array}}{\neg\, f(a) \approx g(a, a) \vee \neg\, g(a, a) \approx g(a, a) \vee f(f(f(a))) \approx g(a, a)}$$

and ι_2

$$\frac{\begin{array}{l}\neg\, f(f(a)) \approx g(a, a) \vee f(f(f(a))) \approx g(a, a) \\ \qquad\qquad\qquad\qquad \neg\, f(f(f(a))) \approx g(a, a) \vee f(f(f(f(a)))) \approx g(a, a)\end{array}}{\neg\, f(f(a)) \approx g(a, a) \vee \neg\, g(a, a) \approx g(a, a) \vee f(f(f(f(a)))) \approx g(a, a)}$$

of ι. The ground clause $D = (\neg\, f(f(a)) \approx g(a, a) \vee f(f(f(a))) \approx g(a, a))$ occurs in ι_1 as a ground instance of C_1 and in ι_2 as a ground instance of C_2. Since the selection in the ground instances of C_1 and C_2 must correspond to the selection in C_1 and C_2 themselves, this means that for ι_1 the first literal in D must be selected and for ι_2 no literal in D must be selected. Consequently, the selection functions for ι_1 and ι_2 must be different. (In fact, ι_i is a T_i-instances of ι with $T_i = \mathrm{inherit}(S, \iota, \iota_i)$ for both $i = 1, 2$.)

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