Lecture - 3

Propositional Equivalences

Introduction

Two logical expressions are said to be equivalent if they have the same truth value in all cases. Sometimes this fact helps in proving a mathematical result by replacing one expression with another equivalent expression, without changing the truth value of the original compound proposition.

Types of propositions based on Truth values

There are three types of propositions when classified according to their truth values

1. ***Tautology*** – A proposition which is always true, is called a tautology.
2. ***Contradiction*** – A proposition which is always false, is called a contradiction.
3. ***Contingency*** – A proposition that is neither a tautology nor a contradiction is called a contingency.

Tautologies

A proposition P is a tautology if it is true under all circumstances. It means it contains the only T in the final column of its truth table.

Example: Prove that the statement $(p \to q) \leftrightarrow (\neg q \to \neg p)$ is a tautology.

Solution: Make the truth table of the above statement:

p	q	$p \to q$	$\neg q$	$\neg p$	$\neg q \to \neg p$	$(p \to q) \leftrightarrow (\neg q \to \neg p)$
True	True	True	False	False	True	True
True	False	False	True	False	False	True
False	True	True	False	True	True	True
False	False	True	True	True	True	True

As the final column contains all T's, so it is a tautology.

Contradiction:

A statement that is always false is known as a contradiction.

Example: Show that the statement **p** $\wedge$ $\neg$**p** is a contradiction.

Solution:

p	**¬p**	**p ∧ ¬p**
True	False	False
False	True	False

Since, the last column contains all F's, so it's a contradiction.

Contingency:

A statement that can be either true or false depending on the truth values of its variables is called a contingency.

p	**q**	**p → q**	**p ∧ q**	**(p → q) → (p ∧ q)**
True	True	True	True	True
True	False	False	False	True
False	True	True	False	False
False	False	True	False	False

Exercise

1. Prove $[(A \rightarrow B) \wedge A] \rightarrow B$ is a tautology or not.
2. Prove $(A \vee B) \wedge [(\neg A) \wedge (\neg B)]$ is a tautology or not.
3. Prove $(A \vee B) \wedge (\neg A)$ a contradiction or contingency.
4. Prove $\neg(A \vee B)$ and $[(\neg A) \wedge (\neg B)$ are equivalent or not.
5. Prove that $p \leftrightarrow q$ is equivalent to $(p \rightarrow q) \wedge (q \rightarrow p)$.
6. Construct the truth table for the compound proposition below:

   $(p \rightarrow q) \wedge [(q \wedge \neg r) \rightarrow (p \vee r)]$

7. Show that the following is a tautology or not:

   $\neg(p \vee q) \leftrightarrow (\neg p \wedge \neg q)$

8. Construct the truth table $(p \wedge q) \vee \neg(p \rightarrow q)$.
9. Show that the following is a tautology or not:

   $[p \wedge (p \rightarrow q)] \rightarrow q$

10. $[(p \rightarrow q) \wedge (r \rightarrow s)] \rightarrow [(\neg q \vee \neg s) \rightarrow (\neg p \vee \neg r)]$ tautology or not.
11. Prove or Disprove

    I. $[p \rightarrow (q \rightarrow r)] \leftrightarrow [(p \rightarrow q) \rightarrow (p \rightarrow r)]$

    Il. $[(p \rightarrow q) \rightarrow r] \leftrightarrow [p \rightarrow (q \rightarrow r)]$

    Ill. $[(p \leftrightarrow q) \leftrightarrow r] \leftrightarrow [p \leftrightarrow (q \leftrightarrow r)]$

12. $p \rightarrow (\neg q \wedge \neg p) \leftrightarrow \neg q$ show the truth table.