On Representations of Trigonometric Functions for Multiples of Real Numbers

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Abstract

In this article, we analyze the polynomials $T_n(x)$ that satisfy $T_n(\cos(x)) = \cos(nx)$ and $S_n(\cos(x)) = \frac{\sin(nx)}{\sin(x)}$. After establishing a recurrence relation for both of these polynomials, we can find interesting properties, such as closed forms for the terms, patterns and relationships between neighboring coefficients, factorizations for the polynomials, representations of the coefficients in $\mathbb{Z}_p[x]$, and generalize.

1 Introduction

For centuries, mathematicians have attempted to devise formulas for trigonometric functions due to their computational difficulty. Some of these have involved the Taylor series. In this paper we will attempt to explore a mechanism by which we can express trigonometric functions for the multiples of a value in terms of the original value. More specifically, we investigate formulas for $\cos(n\alpha)$, $\sin(n\alpha)$, and other trigonometric functions in terms of $\cos(\alpha)$, focusing on when these expressions yield polynomial functions.

First, we define the polynomial function that expresses cosine in terms of cosine.

Definition. $T_n(x)$ is the polynomial function for which $T_n(\cos(\alpha)) = \cos(n\alpha)$ for all non-negative integers n.

We also define a polynomial to express sine in terms of sine similarly.

Next, we express a polynomial that defines sine in terms of cosine.

Definition. $H_n(x)$ is a function for which $H_n(\cos(\alpha)) = \sin(n\alpha)$ for all non-negative integers n and α in the first or third quadrants.

Though, as we will later see, this function is not a polynomial, so we create a new function, this time dividing by $sin(x)$, to make sure that we get a polynomial function for sine.

Definition. $S_n(x)$ is the polynomial function for which $S_n(\cos(\alpha)) = \frac{\sin(n\alpha)}{\sin(\alpha)}$ for all non-negative integers n and α in the first or third quadrants.

Once we have functions for sine and cosine, we also define similar functions for tangents and cotangents.

Definition. $A_n(x)$ is the function for which $A_n(\cos(\alpha)) = \tan(n\alpha)$ for all positive integers n and and α in the first or third quadrants.

Definition. $C_n(x)$ is the function for which $C_n(\cos(\alpha)) = \cot(n\alpha)$ for all non-negative integers n and and α in the first or third quadrants.

2 Polynomial Values

2.1 Programming

Since we have found a recursion for the polynomials, we can use it to create a table of polynomials using programming. In a table, we can store the coefficients of the polynomial in a list, and we can make a list of the lists to store all of the polynomials. We can also using dynamic programming to generate the next polynomial in the sequence because we already have the previous two state stored, so the recursion helps us get the next polynomial in the sequence. Since the recursions for T_n and S_n differ only by their starting terms, we only need to modify that to create tables for both sets of polynomials. The code for both T_n and S_n can be found in `https://github.com/nekr3/promystrig`.

2.2 Tables

Here are tables of initial values of T_n and S_n.

n	T_n
0	1
1	x
2	$2x^2 - 1$
3	$4x^3 - 3x$
4	$8x^4 - 8x^2 + 1$
5	$16x^5 - 20x^3 + 5x$
6	$32x^6 - 48x^4 + 18x^2 - 1$
7	$64x^7 - 112x^5 + 56x^3 - 7x$
8	$128x^8 - 256x^6 + 160x^4 - 32x^2 + 1$
9	$256x^9 - 576x^7 + 432x^5 - 120x^3 + 9x$
10	$512x^{10} - 1280x^8 + 1120x^6 - 400x^4 + 50x^2 - 1$
11	$1024x^{11} - 2816x^9 + 2816x^7 - 1232x^5 + 220x^3 - 11x$
12	$2048x^{12} - 6144x^{10} + 6912x^8 - 3584x^6 + 840x^4 - 72x^2 + 1$

n	$S_n(x)$
0	0
1	1
2	$2x$
3	$4x^2 - 1$
4	$8x^3 - 4x$
5	$16x^4 - 12x^2 + 1$
6	$32x^5 - 32x^3 + 6x$
7	$64x^6 - 80x^4 + 24x^2 - 1$
8	$128x^7 - 192x^5 + 80x^3 - 8x$
9	$256x^8 - 448x^6 + 240x^4 - 40x^2 + 1$
10	$512x^9 - 1024x^7 + 672x^5 - 160x^3 + 10x$
11	$1024x^{10} - 2304x^8 + 1792x^6 - 560x^4 + 60x^2 - 1$
12	$2048x^{11} - 5120x^9 + 4608x^7 - 1792x^5 + 280x^3 - 12x$

3 Recursion

3.1 Recurrence Relation for T_n

Let $T_{n-1}(x)$ be the polynomial such that $T_{n-1}(\cos(\alpha)) = \cos((n-1)\alpha)$ and $T_{n-2}(x)$ be the polynomial such that $T_{n-2}(\cos(\alpha)) = \cos((n-2)\alpha)$.

Theorem 3.1. $T_n(x) = 2xT_{n-1}(x) - T_{n-2}(x)$

Proof.

$$T_n(x) = \cos(n\alpha)$$

$$T_n(x) = \cos((n-1)\alpha + \alpha)$$

$$T_n(x) = \cos(\alpha)\cos((n-1)\alpha) - \sin(\alpha)\sin((n-1)\alpha)$$

$$T_n(x) = \cos(\alpha)\cos((n-1)\alpha) - \frac{\cos((n-2)\alpha) - \cos(n\alpha)}{2}$$

$$2T_n(x) = 2\cos(\alpha)\cos((n-1)\alpha) - \cos((n-2)\alpha) + \cos(n\alpha)$$

$$2T_n(x) = 2xT_{n-1}(x) - T_{n-2}(x) + T_n(x)$$

$$T_n(x) = 2xT_{n-1}(x) - T_{n-2}(x)$$

□

3.2 Recurrence Relation for S_n

Let $H_{n-1}(x)$ be the polynomial such that $H_{n-1}(\cos(\alpha)) = \sin((n-1)\alpha)$ and $H_{n-2}(x)$ be the polynomial such that $H_{n-2}(\cos(\alpha)) = \sin((n-2)\alpha)$. We can find a similar recurrence relation for H_n.

Lemma 3.2.1. $H_n(x) = 2xH_{n-1}(x) - H_{n-2}(x)$

$$H_n(x) = \sin(n\alpha)$$
$$H_n(x) = \sin((n-1)\alpha + \alpha)$$
$$H_n(x) = \sin((n-1)\alpha)\cos(\alpha) + \cos((n-1)\alpha)\sin(x)$$
$$H_n(x) = \sin((n-1)\alpha)\cos(\alpha) + \frac{\sin(nx) + \sin((-n+2)x)}{2}$$
$$2H_n(x) = 2\sin((n-1)\alpha)\cos(\alpha) + \sin(nx) - \sin((n-2)x)$$
$$2H_n(x) = 2xH_{n-1}(x) + H_n(x) - H_{n-2}(x)$$
$$H_n(x) = 2xH_{n-1}(x) - H_{n-2}(x)$$

Therefore, we can derive a similar recurrence relation for S_n from this statement.

Theorem 3.2. $S_n(x) = 2xS_{n-1}(x) - S_{n-2}(x)$

$$H_n(x) = 2xH_{n-1}(x) - H_{n-2}(x)$$
$$H_n(\cos(\alpha)) = 2\cos(\alpha)Hn - 1(\cos(\alpha)) - H_{n-2}(\cos(\alpha))$$
$$\frac{H_n(\cos(\alpha))}{\sin(\alpha)} = \frac{2\cos(\alpha)Hn-1(\cos(\alpha)) - H_{n-2}(\cos(\alpha))}{\sin(\alpha)}$$
$$\frac{H_n(\cos(\alpha)}{\sin(\alpha)} = \frac{2\cos(\alpha)Hn-1(\cos(\alpha))}{\sin(\alpha)} - \frac{H_{n-2}(\cos(\alpha))}{\sin(\alpha)}$$
$$\frac{H_n(\cos(\alpha)}{\sin(\alpha)} = \frac{2\cos(\alpha)Hn-1(\cos(\alpha))}{\sin(\alpha)} - \frac{H_{n-2}(\cos(\alpha))}{\sin(\alpha)}$$
$$\frac{\sin(n\alpha)}{\sin(\alpha)} = 2\cos(\alpha)\frac{\sin((n-1)\alpha)}{\sin(\alpha)} - \frac{\sin((n-2)\alpha)}{\sin(\alpha)}$$
$$S_n(\cos(\alpha)) = 2\cos(\alpha)S_{n-1}(\cos(\alpha)) - S_{n-2}(\cos(\alpha))$$
$$S_n(x) = 2xS_{n-1}(x) - S_{n-2}(x)$$

3.3 Characterization as Polynomials

With these recursions, we can finally prove that T_n and S_n are always polynomials.

Theorem 3.3. *T_n is always a polynomial.*

Proof. We show this inductively. We know that T_0 and T_1 are polynomials since 1 and x are polynomials. We also know that $\mathbb{Z}[x]$ is closed under both addition and multiplication as well as has multiplicative inverses since it is a ring. Thus, we show that if T_n and T_{n+1} are polynomials, then T_{n+2} is a polynomial.

$$T_{n+2}(x) = 2xT_{n+1}(x) - T_n(x)$$

$T_{n+2}(x) = 2xT_{n+1}(x) + (-T_n(x))$, which we can do since we know $\mathbb{Z}[x]$ has additive inverses that are also polynomials since they are also in $\mathbb{Z}[x]$.

$2, x$, and $T_{n+1}(x)$ are polynomials, so $2xT_{n+1}(x)$ is a polynomial by multiplicative closure.

$2xT_{n+1}(x)$ and $-T_n(x)$ are polynomials, so $T_{n+2}(x)$ is a polynomial by additive closure. □

Theorem 3.4. *S_n is always a polynomial.*

Proof. We can perform a similar inductive proof as that for T_n. We know that S_0 and S_1 are polynomials since 0 and 1 are polynomials. From here, the proof follows the same as that for T_n. □

Though, as a result, we can notice that H_n is never a polynomial for $n > 0$.

Theorem 3.5. *H_n is never a polynomial for $n > 0$.*

Proof. We know that S_n is always a polynomial and that

$$\tfrac{H_n(\cos(\alpha))}{\sin(\alpha)} = S_n(\cos(\alpha))$$

$$H_n(\cos(\alpha)) = S_n(\cos(\alpha))\sin(\alpha)$$

$$H_n(\cos(\alpha)) = S_n(\cos(\alpha))\sqrt{1-\cos^2(\alpha)}$$

$$H_n(x) = S_n(x)\sqrt{1-x^2}$$

We know that $S_n(x)$ is a polynomial with integer coefficients, so multiplication by $\sqrt{1-x^2}$ gives it coefficients of an integer times $\sqrt{1-x^2}$ on every term, which is not a polynomial. Thus, H_n is not a polynomial for $n > 0$ since when $S_n(x) = 0$ then $H_n(x)$ is a polynomial since the $\sqrt{1-x^2}$ term is cancelled. Since this only happens when $n = 0$, we exclude this case.

□

Since H_n is never a polynomial but S_n is very similar yet is a polynomial, this gives us reasonable motivation to continue using S_n primarily for the rest of this paper.

3.4 Cross-relations between sine and cosine

We can also find recursive formulas for S_n and T_n in terms of each other.

Proposition 3.1. $S_n(x) = xS_{n-1}(x) + T_{n-1}(x)$

Proof.

$$\sin(n\alpha) = \sin((n-1)\alpha + \alpha)$$

$$\sin(n\alpha) = \sin((n-1)\alpha)\cos(\alpha) + \cos((n-1)\alpha)\sin(\alpha)$$

$$\tfrac{\sin(n\alpha)}{\sin(\alpha)} = \tfrac{\sin((n-1)\alpha)}{\sin(\alpha)}\cos(\alpha) + \cos((n-1)\alpha)\tfrac{\sin(\alpha)}{\sin(\alpha)}$$

$$S_n(\cos(\alpha)) = \cos(\alpha)S_{n-1}(\cos(\alpha)) + \cos((n-1)\alpha)$$

$$S_n(\cos(\alpha)) = \cos(\alpha)S_{n-1}(\cos(\alpha)) + T_{n-1}(\cos(\alpha))$$

$$S_n(x) = xS_{n-1}(x) + T_{n-1}(x)$$

$\square$

We can find a similar formula to express T_n.

Proposition 3.2. $T_n(x) = x^2 S_{n-1}(x) + xT_{n-1}(x) - S_{n-1}(x)$

Proof.

$$\cos(n\alpha) = \cos((n-1)\alpha + \alpha)$$

$$\cos(n\alpha) = \cos((n-1)\alpha)\cos(\alpha) - \sin((n-1)\alpha)\sin(\alpha)$$

$$\cos(n\alpha) = \cos((n-1)\alpha)\cos(\alpha) - \tfrac{\sin((n-1)\alpha)}{\sin(\alpha)}\sin^2(\alpha)$$

$$T_n(\cos(\alpha)) = \cos(\alpha)T_{n-1}(\cos(\alpha)) - \sin^2(\alpha)S_n(\cos(\alpha))$$

$$T_n(\cos(\alpha)) = \cos(\alpha)T_{n-1}(\cos(\alpha)) - (1-\cos^2(\alpha))S_n(\cos(\alpha))$$

$$T_n(\cos(\alpha)) = \cos^2(\alpha)S_n(\cos(\alpha)) + \cos(\alpha)T_{n-1}(\cos(\alpha)) - S_n(\cos(\alpha))$$

$$T_n(x) = x^2 S_{n-1}(x) + xT_{n-1}(x) - S_{n-1}(x)$$

$\square$

3.5 Recursions for tangent based functions

Using our previous recursions, we can derive one for C_n and A_n.

Corollary 3.5.1. $A_n(x) = 2xA_{n-1}(x) - A_{n-2}(x)$

Proof. Using the recursions for $H_n(x)$ and $T_n(x)$ and the fact that $\tan(n\alpha) = \frac{\sin(n\alpha)}{\cos(n\alpha)} \Rightarrow A_n(x) = \frac{H_n(x)}{T_n(x)}$, this statement becomes trivial. $\square$

Corollary 3.5.2. $C_n(x) = 2xC_{n-1}(x) - C_{n-2}(x)$

Proof. Similarly, since $H_n(x)$ and $T_n(x)$ also obey this same recursion and $\cot(n\alpha) = \frac{\cos(n\alpha)}{\sin(n\alpha)} \Rightarrow C_n(x) = \frac{T_n(x)}{H_n(x)}$, this statement is also obvious. $\square$

4 Coefficients

To contextualize our investigation of these polynomials, we first attempt to determine the coefficients of the polynomial through mathematical formulas.

Definition. $f_M(n, d)$ is equal to the coefficient of x^d in the polynomial $M_n(x)$.

4.1 Properties

Proposition 4.1. *The coefficients of $T_n(x)$ for all integers $n \geq 0$ are integers.*

Proof. We can prove this using induction. For the base case, we have $T_0(x) = 1$ and $T_1(x) = x$, which both have only integer coefficients. Now, assume that all the coefficients of $T_n(x)$, where $n < k$, are integers. We have $T_k(x) = 2kT_{k-1}(x) + T_{k-2}$. For each degree, we also get that $f_T(k, d) = 2f_T(k-1, d-1) - f_T(k-2, d)$. Because we assume that T_{k-1} and T_{k-2} are made up of only integral coefficients, we know that $f_T(k-1, d-1), f_T(k-2, d) \in \mathbb{Z}$, so $f_T(k, d) \in \mathbb{Z}$ as well. Therefore, T_{k-2} and T_{k-1} are made up of only integral coefficients, then T_k is only made of up integral coefficients. By induction, we get that T_n only has integers for coefficients for all integers $n \geq 0$. $\square$

Proposition 4.2. *In T_n, a x^d has a nonzero coefficient if and only $d \equiv n \pmod 2$ and $d \leq n$.*

Proof. First of all, we note that if $d > n$, then since the degree of $T_n(x)$ is n, the coefficient of the term must be 0. Now, for the base cases $T_0(x)$ and $T_1(x)$, we can clearly see that our claim holds true. Now, supposed that our claim holds true for $T_{k-2}(x)$ and $T_{k-1}(x)$. We know that, by the recursion, $f_T(k, d) = 2f_T(k-1, d-1) - f_T(k-2, d)$. If $k \equiv d \pmod 2$, then $2f_T(k-1, d-1)$ is nonzero by the inductive hypothesis, so $f_T(k, d)$ is nonzero as well. However, if $k \not\equiv d \pmod 2$, then we get $f_T(k, d) = 2(0) - 0 = 0$, so the coefficient is equal to 0 in this case. Therefore, by induction, our claim in the proposition is true. $\square$

Proposition 4.3. *In S_n, a x^d has a nonzero coefficient if and only $d \equiv n + 1 \pmod 2$ and $d < n$.*

Proof. First of all, we note that if $d \geq n$, then since the degree of $S_n(x)$ is $n - 1$, the coefficient of the term must be 0. Now, for the base cases $S_0(x)$ and $S_1(x)$, we can clearly see that our claim holds true. Now, supposed that our claim holds true for $S_{k-2}(x)$ and $S_{k-1}(x)$. We know that, by the recursion, $f_S(k, d) = 2f_S(k-1, d-1) - f_S(k-2, d)$. If $k \not\equiv d \pmod 2$, then $2f_S(k-1, d-1)$ is nonzero by the inductive hypothesis, so $f_S(k, d)$ is nonzero as well. However, if $k \equiv d \pmod 2$, then we get $f_S(k, d) = 2(0) - 0 = 0$, so the coefficient is equal to 0 in this case. Therefore, by induction, our claim in the proposition is true. $\square$

Therefore, we only define f_T and f_S on degrees that satisfy these above conditions. Now we discuss some other patterns of the coefficients.

Proposition 4.4. *The leading coefficient in $T_n(x)$ is 2^{n-1}, for all positive integers n.*

Proof. We prove this by induction on n. We start with $n = 1$. We know that $T_1(x) = x$, which has leading coefficient 1, which is indeed $2^{n-1} = 2^0$. Now suppose this is true for $n = k$, and we wish to show it is true for $n = k + 1$. So we have that the leading coefficient in $T_k(x)$ is 2^{k-1}. But, we know that $T_{k+1}(x) = 2xT_k(x) - T_{k-1}(x)$. Then, because the degree of $T_n(x)$ is always n, we have that that the degree of $T_{k+1}(x)$ is $k + 1$, and then we see from the recursion that the x^{k+1} term has coefficient $2 * 2^{k-1} = 2^k = 2^{(k+1)-1}$, so the inductive step is complete. Thus the claim is proven. $\square$

Proposition 4.5. *The constant term in $T_n(x)$, if it exists, is $(-1)^{\frac{n}{2}}$ for all positive integers n.*

Proof. We prove this by induction on n. We know that $T_n(x)$ has a constant term if and only if $n \equiv 0 \pmod 2$, so our base case is $n = 2$. We have that $T_2(x) = 2x^2 - 1$, so it has constant term -1, which is the same as $(-1)^{\frac{2}{2}}$, so the base case is proven. Now we suppose this is true for $n = k$, we wish to show that it is true for $n = k+2$. But we know that $T_{k+2}(x) = 2xT_{k+1}(x) - T_k(x)$, so that the constant term is the negative of the constant term in $T_k(x)$, which is $-(-1)^{\frac{k}{2}} = (-1)^{\frac{k+2}{2}} = (-1)^{\frac{n}{2}}$, so the inductive step is complete. Thus the claim is proven. □

Proposition 4.6. *For all positive integers d and n, $2^{d-1} | f_T(n, d)$.*

Proof. We prove this by fixing n and inducting on d. We first prove the base case, $d = 1$. But if $d = 1$, then $2^{d-1} = 1$, and then we must of course have that $1 | f_T(n, d)$ because $f_T(n, d)$ is an integer. Thus the base case is proven. Now suppose the claim is proven for $d = d_0$, we must show it is true for $d = d_0 + 1$. But for $d = d_0 + 1$, we know that, by the recursion established in Section 3.1,

$$\begin{aligned} f_T(n, d) &= f_T(n, d_0 + 1) \\ &= 2f_T(n-1, d_0) - f_T(n-2, d_0+1) \end{aligned}$$

Therefore, we have $f_T(n, d_0 + 1) + f_T(n-2, d_0+1) = 2f_T(n-1, d_0)$. But we know that $f_T(n-1, d_0)$, by the inductive hypothesis, is divisible by $2^{d_0 - 1}$, so that

$$f_T(n, d_0 + 1) + f_T(n-2, d_0+1) \equiv 0 \pmod{2^{d_0}}$$

and thus

$$f_T(n, d_0 + 1) \equiv -f_T(n-2, d_0+1) \pmod{2^{d_0}}$$

But this argument applies for all choices of n, so that we must have, modulo 2^{d_0}, that

$$\begin{aligned} f_T(n, d_0 + 1) &\equiv -f_T(n-2, d_0+1) \\ &\equiv f_T(n-4, d_0+1) \\ &\vdots \\ &\equiv (-1)^k f_T(n-2k, d_0+1) \end{aligned}$$

where k is the quotient of n divided by 2, i.e., by the division algorithm, $n = 2k + r$ where $0 \le r < 2$, so that $r = 0$ or $r = 1$. But if $n - 2k = 0$, then clearly we have $f_T(n-2k, d_0+1) = 0$ because $d_0 + 1$ is a positive integer, so it is greater than 0, but the only nonzero coefficient in $T_0(x)$ is the coefficient of x^0. Also, if $n - 2k = 1$, then we also have $f_T(n-2k, d_0+1) = 0$, because the only nonzero coefficient in $T_1(x)$ is that of x^1, and we know that $d_0 > 0$, so that $d_0 + 1 > 1$. Therefore, we must in fact have that

$$f_T(n, d_0 + 1) \equiv 0 \pmod{2^{d_0}}$$

so that $2^{(d_0+1)-1} | f_T(n, d_0 + 1)$, completing the inductive step. Thus the claim is proven. □

Conjecture 4.1. *For positive integers d and n,*

$$\left| \frac{f_T(n, d)}{2^{d-1}} \right| = \sum_{k=0}^{n-1} \left| \frac{f_T(k, d-1)}{2^{d-2}} \right|$$

Conjecture 4.2. *The quantity*

$$f_T(n, n-2) - f_T(n+1, n-1)$$

equals the total number of 1's in all possible compositions of n*.*

Remark 1. For example, if we let $n = 4$, we have $f_T(4,2) - f_T(5,3) = 12$, and we have the following compositions of 4:

$$\begin{aligned} 4 &= 4 \\ &= 1+3 \\ &= 3+1 \\ &= 1+1+2 \\ &= 1+2+1 \\ &= 2+1+1 \\ &= 1+1+1+1 \\ &= 2+2 \end{aligned}$$

where we notice that there are 12 1's in total throughout these compositions. This is a fascinating pattern that we discovered, but we have not yet discovered how to prove it.

4.2 Closed Forms

4.2.1 Sum-Based Formulas

Theorem 4.3. *The coefficient of the* x^d *term in the polynomial representation of* $T_n(x)$ *(symbolized by the function* $f_T(n,d)$*) is, if* $d \equiv n$ *(mod 2),*

$$f_T(n,d) = (-1)^{\frac{n-d}{2}} \sum_{k=\frac{n-d}{2}}^{[\frac{n}{2}]} \left(\binom{n}{2k} \binom{k}{\frac{n-d}{2}} \right)$$

Proof. From the definition of T_n, we have $T_n(\cos(\theta)) = \cos(n\theta)$. Additionally, we have by Euler's identity that

$$\begin{aligned} \cos(n\theta) &= Re[\cos(n\theta) + i\sin(n\theta)] \\ &= Re[e^{i(n\theta)}] \\ &= Re[(e^{i\theta})^n] \\ &= Re[(\cos(\theta) + i\sin(\theta))^n] \end{aligned}$$

Thus we wish to expand $(\cos(\theta) + i\sin(\theta))^n$. By the binomial theorem, we see that

$$(\cos(\theta) + i\sin(\theta))^n = \sum_{k=0}^{n} \left(\binom{n}{k} (\cos(\theta))^{n-k} (i\sin(\theta))^k \right)$$

Therefore, we see that an i^k coefficient appears in each term, so that terms in the above summation with odd k have imaginary coefficients, and thus we must limit the summation to even coefficients. We then have the following:

$$\begin{aligned}
\text{Re}\left[\sum_{k=0}^{n}\left(\binom{n}{k}(\cos(\theta))^{n-k}(i\sin(\theta))^{k}\right)\right] &= \sum_{k=0}^{[\frac{n}{2}]}\binom{n}{2k}(\cos(\theta))^{n-2k}(i\sin(\theta))^{2k})\\
&= \sum_{k=0}^{[\frac{n}{2}]}\left((-1)^k\binom{n}{2k}(\cos(\theta))^{n-2k}(\sin^2(\theta))^{k}\right)\\
&= \sum_{k=0}^{[\frac{n}{2}]}\left((-1)^k\binom{n}{2k}(\cos(\theta))^{n-2k}(\sin^2(\theta))^{k}\right)\\
&= \sum_{k=0}^{[\frac{n}{2}]}\left((-1)^k\binom{n}{2k}(\cos(\theta))^{n-2k}(1-\cos^2(\theta))^{k}\right)
\end{aligned}$$

(Note that the final summation above proves that $\cos(n\theta)$ can indeed be represented as a polynomial in $\cos(\theta)$, and for this reason, we will now replace $\cos(\theta)$ by the variable x). Now we expand the factor $(1-x^2)^k$ in the above summation, finding that the summation becomes

$$\sum_{k=0}^{[\frac{n}{2}]}((-1)^k\binom{n}{2k}x^{n-2k}\sum_{i=0}^{k}(\binom{k}{i}(-x^2)^i))$$

which equals

$$\sum_{k=0}^{[\frac{n}{2}]}\left((-1)^k\binom{n}{2k}x^{n-2k}\sum_{i=0}^{k}\left((-1)^i\binom{k}{i}x^{2i}\right)\right)$$

Now we wish to focus on a specific term in our polynomial, the x^d term. To do this, we must find all possible ways to have $n-2k+2i=d$, because each term created in the summation above has x to the power of $n-2k+2i$. Therefore, we want $k-i=\frac{n-d}{2}$. We know that k can go from 0 to $[\frac{n}{2}]$, while i can go from 0 to k. Thus, to have $k-i=\frac{n-d}{2}$, we must have k from $\frac{n-d}{2}$ to $[\frac{n}{2}]$, and $i=k-\frac{n-d}{2}$. Then we see that the coefficient of the x^d term is (assuming the parity of d and n are the same)

$$\begin{aligned}
\sum_{k=\frac{n-d}{2}}^{[\frac{n}{2}]}\left((-1)^k(-1)^{k-\frac{n-d}{2}}\binom{n}{2k}\binom{k}{k-\frac{n-d}{2}}\right) &= \sum_{k=\frac{n-d}{2}}^{[\frac{n}{2}]}\left((-1)^{\frac{n-d}{2}}\binom{n}{2k}\binom{k}{\frac{n-d}{2}}\right)\\
&= (-1)^{\frac{n-d}{2}}\sum_{k=\frac{n-d}{2}}^{[\frac{n}{2}]}\left(\binom{n}{2k}\binom{k}{\frac{n-d}{2}}\right)
\end{aligned}$$

proving the claim. □

Remark 2. Note that if the parity of d and n are different, then the term including $\cos^d(\theta)$ in $T_n(x)$ will be inseparable from a single sine function, and thus not appear in our polynomial as there will be an i in the coefficient of this term.

4.2.2 Product-based Formulas

Theorem 4.4.

$$f_T(n,d) = (-1)^{\frac{n-d}{2}} \cdot \frac{\prod_{k=1}^{d-1}(n-d+2k)}{d!} n$$

Proof. By the recursion $T_n(x) = 2T_{n-1}(x) - T_{n-2}(x)$, we have $f_T(n,d) = 2f_T(n-1,d-1) - f_T(n-2,d)$. Assume this function holds for $f_T(n-1,d-1)$ and $f_T(n-2,d)$. We now must show that:

$$f_T(n,d) = 2f_T(n-1,d-1) - f_T(n-2,d)$$

$$(-1)^{\frac{n-d}{2}} \cdot \frac{\prod_{k=0}^{d-1}(n-d+2k)}{d!} n =$$

$$2(-1)^{\frac{(n-1)-(d-1)}{2}} \cdot \frac{\prod_{k=0}^{d-2}((n-1)-(d-1)+2k)}{(d-1)!}(n-1) - (-1)^{\frac{n-d+2}{2}} \cdot \frac{\prod_{k=0}^{d-1}(n-2-d+2k)}{d!}(n-2) =$$

$$2(-1)^{\frac{(n-d}{2}} \cdot \frac{\prod_{k=0}^{d-2}(n-d+2k)}{d!} d(n-1) - (-1)^{\frac{n-d}{2}+1} \cdot \frac{\prod_{k=0}^{d-1}(n-d+2(k-1))}{d!}(n-2)$$

We divide out $(-1)^{\frac{n-d}{2}} \cdot \frac{\prod_{k=1}^{d-2}(n-d+2k)}{d!}$ from both sides and are left with

$$\begin{aligned} (n-d+2(d-1))n &= 2d(n-1) + (n-d)(n-2) \\ n^2 + dn - 2n &= 2dn - 2d + n^2 - 2n - dn + 2d \\ n^2 + dn - 2n &= n^2 + dn - 2n \end{aligned}$$

It is trivial to show that this holds for $f_T(0,0)$, $f_T(1,0)$, and $f_T(1,1)$. Then, we can induct on n and d to prove it for all n, d. □

Theorem 4.5.

$$f_S(n,d) = (-1)^{\frac{n-d-1}{2}} \cdot \frac{\prod_{k=0}^{d-1}(n-d+2k+1)}{d!}$$

Proof. By the recursion $S_n(x) = 2S_{n-1}(x) - S_{n-2}(x)$, we have $f_S(n,d) = 2f_S(n-1,d-1) - f_S(n-2,d)$. Assume this function holds for $f_S(n-1,d-1)$ and $f_S(n-2,d)$. We now must show that:

$$f_S(n,d) = 2f_S(n-1,d-1) - f_S(n-2,d)$$

$$(-1)^{\frac{n-d-1}{2}} \cdot \frac{\prod_{k=1}^{d-1}(n-d+2k+1)}{d!} =$$

$$2(-1)^{\frac{(n-1)-(d-1)-1}{2}} \cdot \frac{\prod_{k=1}^{d-2}((n-1)-(d-1)+2k+1)}{(d-1)!} - (-1)^{\frac{n-d+2-1}{2}} \cdot \frac{\prod_{k=1}^{d-1}(n-2-d+2k+1)}{d!} =$$

$$2(-1)^{\frac{(n-d-1)}{2}} \cdot \frac{\prod_{k=1}^{d-2}(n-d+2k+1)}{d!} d - (-1)^{\frac{n-d-1}{2}+1} \cdot \frac{\prod_{k=1}^{d-1}(n-d+2(k-1)+1)}{d!}$$

We divide out $(-1)^{\frac{n-d-1}{2}} \cdot \frac{\prod_{k=1}^{d-2}(n-d+2k+1)}{d!}$ from both sides and are left with

$$n-d+2(d-1)+1 = 2d+(n-d-2+1)$$
$$n+d-1 = n+d-1$$

It is trivial to show that this holds for $f_S(0,0)$, $f_S(1,0)$, and $f_S(1,1)$. Then, we can induct on n and d to prove it for all n, d. □

4.2.3 Relationships Between Sine and Cosine

Theorem 4.6.

$$f_T(n,d) = \frac{n}{d} f_S(n,d-1)$$

Proof.

$$f_T(n,d) = (-1)^{\frac{n-d}{2}} \cdot \frac{\prod_{k=1}^{d-1}(n-d+2k)}{d!} n$$

$$f_T(n,d) = \tfrac{n}{d}(-1)^{\frac{n-d}{2}} \cdot \frac{\prod_{k=1}^{d-1}(n-d+2k)}{(d-1)!}$$

$$f_T(n,d) = \tfrac{n}{d}(-1)^{\frac{n-(d-1)-1}{2}} \cdot \frac{\displaystyle\prod_{k=1}^{d-1}(n-(d-1)-1+2k)}{(d-1)!}$$

$$f_T(n,d) = \tfrac{n}{d}(-1)^{\frac{n-(d-1)-1}{2}} \cdot \frac{\displaystyle\prod_{k=1-1}^{(d-1)-1}(n-(d-1)-1+2(k+1))}{(d-1)!}$$

$$f_T(n,d) = \tfrac{n}{d}(-1)^{\frac{n-(d-1)-1}{2}} \cdot \frac{\displaystyle\prod_{k=0}^{(d-1)-1}(n-(d-1)+1+2k))}{(d-1)!}$$

$$f_T(n,d) = \tfrac{n}{d} f_S(n,d-1)$$

□

4.3 Neighboring Coefficients

Proposition 4.7. *The absolute values of two consecutive nonzero coefficients, namely that of x^d and x^{d-2}, are equal in T_n if and only if $d = \frac{5+\sqrt{8n^2-7}}{4}$, or that $8n^2 - 7 = k^2$ where $k \in \mathbb{N}$ such that $k \equiv 3 \pmod 4$, and $d \equiv n \pmod 2$.*

Proof. By Theorem 4.4, the absolute value of two nonzero coefficients of x^d and x^{d-2} are equal in T_n if and only if

$$\left|(-1)^{\frac{n-d}{2}} \cdot \frac{\displaystyle\prod_{k=1}^{d-1}(n-d+2k)}{d!} n\right| = \left|(-1)^{\frac{n-d-2}{2}} \cdot \frac{\displaystyle\prod_{k=1}^{d-3}(n-d-2+2k)}{(d-2)!} n\right|$$

$$\frac{\displaystyle\prod_{k=1}^{d-1}(n-d+2k)}{d!} n = \frac{\displaystyle\prod_{k=1}^{d-3}(n-d-2+2k)}{(d-2)!} n$$

$$\prod_{k=1}^{d-1}(n-d+2k) = d(d-1)\prod_{k=1}^{d-3}(n-d-2+2k)$$

$$\prod_{k=1}^{d-1}(n-d+2k) = d(d-1)\prod_{k=1}^{d-3}(n-d+2(k-1))$$

$$\prod_{k=1}^{d-1}(n-d+2k) = d(d-1)\prod_{k=2}^{d-2}(n-d+2k)$$

$$(n-d+2)(n-d+2(d-1)) = d(d-1)$$

$$(n-d+2)(n+d-2) = d(d-1)$$

$$n^2 - d^2 + 4d - 4 = d^2 - d$$
$$2d^2 - 5d - n^2 + 4 = 0$$
$$d = \frac{5 \pm \sqrt{5^2 - 4 \cdot 2 \cdot (-n^2 + 4)}}{2 \cdot 2}$$
$$d = \frac{5 \pm \sqrt{8n^2 - 7}}{4}$$

Since d must be a positive integer, $5 \pm \sqrt{8n^2 - 7} \geq 0$. If $d = 5 - \sqrt{8n^2 - 7}$, then we can check cases and notice that there are no solutions for $n \leq 3$ and for $n > 3$, $d = 5 - \sqrt{8n^2 - 7}$ is negative and d is not a positive integer. Thus,

$$d = \frac{5 + \sqrt{8n^2 - 7}}{4}$$

It follows that $\sqrt{8n^2 - 7}$ must be an integer since d is an integer. Let $k = \sqrt{8n^2 - 7}$. It follows that $k \equiv 3 \pmod 4$ since $\frac{5+k}{4}$ is an integer. Finally, by Proposition 4.2, $d \equiv n \pmod 2$.

Now, assume that there is some $k \in \mathbb{N}$ with $k^2 = 8n^2 - 7$ for some $n \in \mathbb{Z}$. Consider $d = \frac{5+k}{4}$. We now see the following:

d is a solution to the quadratic equation $d^2 - \frac{5}{2}d - \frac{n^2}{2} + 2 = 0 \implies 2d^2 - 5d - n^2 + 4 = 0$

$$n^2 - d^2 + 4d - 4 = d^2 - d$$
$$(n - d + 2)(n + d - 2) = d(d - 1)$$
$$\prod_{k=1}^{d-1}(n - d + 2k) = d(d-1)\prod_{k=2}^{d-2}(n - d + 2k)$$
$$\frac{\prod_{k=1}^{d-1}(n - d + 2k)}{d!}n = \frac{\prod_{k=1}^{d-3}(n - d - 2 + 2k)}{(d-2)!}n$$
$$\left|(-1)^{\frac{n-d}{2}} \cdot \frac{\prod_{k=1}^{d-1}(n - d + 2k)}{d!}n\right| = \left|(-1)^{\frac{n-d-2}{2}} \cdot \frac{\prod_{k=1}^{d-3}(n - d - 2 + 2k)}{(d-2)!}n\right|$$
$$|f_T(n,d)| = |f_T(n,d-2)|$$

Finally, we know that these coefficients is nonzero because of Proposition 4.2.

Thus, for $n, d \in \mathbb{Z}_{\geq 0}$, $f_T(n,d) = f_T(n,d-2) \iff d = \frac{5+\sqrt{8n^2-7}}{4} \iff \exists k \in \mathbb{N}$ such that $8n^2 - 7 = k^2, k \equiv 3 \pmod 4, d \equiv n \pmod 2$

□

Proposition 4.8. *The absolute values of two consecutive nonzero coefficients, namely that of x^d and x^{d-2}, are equal in S_n if and only if $d = \frac{3+\sqrt{8n^2+1}}{4}$, or that $8n^2 + 1 = k^2$ where $k \in \mathbb{N}$ such that $k \equiv 1 \pmod 4$, and $d \equiv n + 1 \pmod 2$.*

Proof. By Theorem 4.5, the absolute value of two nonzero coefficients of x^d and x^{d-2} are equal in S_n if and only if

$$\left|(-1)^{\frac{n-d-1}{2}} \cdot \frac{\prod_{k=0}^{d-1}(n-d+2k+1)}{d!}\right| = \left|(-1)^{\frac{n-d-1-2}{2}} \cdot \frac{\prod_{k=0}^{d-3}(n-d-2+2k+1)}{(d-2)!}\right|$$

$$\frac{\prod_{k=0}^{d-1}(n-d+2k+1)}{d!} = \frac{\prod_{k=0}^{d-3}(n-d-2+2k+1)}{(d-2)!}$$

$$\prod_{k=0}^{d-1}(n-d+2k+1) = d(d-1)\prod_{k=0}^{d-3}(n-d-2+2k+1)$$

$$\prod_{k=0}^{d-1}(n-d+2k+1) = d(d-1)\prod_{k=0}^{d-3}(n-d+2(k-1)+1)$$

$$\prod_{k=0}^{d-1}(n-d+2k+1) = d(d-1)\prod_{k=1}^{d-2}(n-d+2k+1)$$

$$(n-d+1)(n-d+2(d-1)+1) = d(d-1)$$

$$(n-d+1)(n+d-1) = d(d-1)$$

$$n^2-d^2+2d-1 = d^2-d$$

$$2d^2-3d-n^2+1 = 0$$

$$d = \frac{3 \pm \sqrt{3^2-4\cdot 2\cdot(-n^2+1)}}{2\cdot 2}$$

$$d = \frac{3 \pm \sqrt{8n^2+1}}{4}$$

Since d must be a positive integer, $3 \pm \sqrt{8n^2+1} \geq 0$. If $d = 3-\sqrt{8n^2+1}$, then we can check cases and notice that there are no solutions for $n \leq 1$ and for $n > 1$, $d = 3-\sqrt{8n^2+1}$ is negative and d is not a positive integer. Thus,

$$d = \frac{3+\sqrt{8n^2+1}}{4}$$

It follows that $\sqrt{8n^2+1}$ must be an integer since d is an integer. Let $k = \sqrt{8n^2+1}$. It follows that $k \equiv 1 \pmod 4$ since $\frac{3+k}{4}$ is an integer. Finally, by Proposition 4.3, $d \equiv n+1 \pmod 2$.

Now, assume that there is some $k \in \mathbb{N}$ with $k^2 = 8n^2+1$ for some $n \in \mathbb{Z}$. Consider $d = \frac{3+k}{4}$. We now see the following:

d is a solution to the quadratic equation $d^2 - \frac{3}{2}d - \frac{n^2}{2} + \frac{1}{2} = 0 \implies 2d^2 - 3d - n^2 + 1 = 0$

$$n^2 - d^2 + 2d - 1 = d^2 - d$$

$$(n - d + 1)(n + d - 1) = d(d - 1)$$

$$\prod_{k=0}^{d-1}(n - d + 2k + 1) = d(d - 1)\prod_{k=1}^{d-2}(n - d + 2k + 1)$$

$$\frac{\prod_{k=0}^{d-1}(n - d + 2k + 1)}{d!} = \frac{\prod_{k=0}^{d-3}(n - d - 2 + 2k + 1)}{(d - 2)!}$$

$$\left|(-1)^{\frac{n-d-1}{2}} \cdot \frac{\prod_{k=0}^{d-1}(n - d + 2k + 1)}{d!}\right| = \left|(-1)^{\frac{n-d-1-2}{2}} \cdot \frac{\prod_{k=0}^{d-3}(n - d - 2 + 2k + 1)}{(d - 2)!}\right|$$

$$|f_S(n, d)| = |f_S(n, d - 2)|$$

Finally, we know that these coefficients are nonzero because of Proposition 4.3.

Thus, for $n, d \in \mathbb{Z}_{\geq 0}$, $f_S(3n, d) = f_S(n, d - 2) \iff d = \frac{3+\sqrt{8n^2+1}}{4} \iff \exists k \in \mathbb{N}$ such that $8n^2 + 1 = k^2, k \equiv 1 \pmod 4, d \equiv n + 1 \pmod 2$ $\square$

4.3.1 Pell's Equation

Theorem 4.7. *The integers d for which $f_S(n, d) = f_S(n, d - 2)$ (for some positive integer n) are precisely $d = y^2 + 1$ where y is an integer such that there exists $x \in \mathbb{Z}$ such that $x^2 - 2y^2 = 1$.*

Proof. As we have previously seen, we have $f_S(n, d) = f_S(n, d - 2)$ if and only if

$$d = \frac{3 + \sqrt{8n^2 + 1}}{4}$$

Therefore, rearranging, we have that

$$\begin{aligned} n &= \sqrt{2d^2 - 3d + 1} \\ &= \sqrt{(2d - 1)(d - 1)} \end{aligned}$$

But we know that any factor of $d - 1$ is a factor of $2(d - 1) = 2d - 2$, and so if some integer is a factor of both $2d - 1$ and $d - 1$, it must be a factor of $(2d - 1) - (2d - 2) = 1$, and thus a unit. Therefore, $d - 1$ and $2d - 1$ are relatively prime for any integer d. Thus if $(2d - 1)(d - 1)$ is a perfect square, then both $2d - 1$ and $d - 1$ must be perfect squares. We shall let

$$\begin{aligned} d - 1 &= y^2 \\ 2d - 1 &= x^2 \end{aligned}$$

for some integers x and y. But then we must have

$$y^2 + 1 = d = \frac{x^2 + 1}{2}$$

and thus we have

$$x^2 + 1 = 2y^2 + 2$$

so that

$$x^2 - 2y^2 = 1$$

Thus all such d where $f_S(n, d) = f_S(n, d - 2)$ for some positive integers n must be $y^2 + 1$ for some y where there exists an integer x such that $x^2 - 2y^2 = 1$. Now, to show that the set of desired d's is precisely equal to the set of all integers $y^2 + 1$, we must show that all integers $y^2 + 1$ provide us with a d such that for some n, $f_S(n, d) = f_S(n, d - 2)$. So suppose that we have a positive integer y such that $x^2 - 2y^2 = 1$ for some positive integer x. Then we set $d = y^2 + 1$ and see that

$$\begin{aligned} \sqrt{2d^2 - 3d + 1} &= \sqrt{(2d - 1)(d - 1)} \\ &= \sqrt{(2y^2 + 1)(y^2)} \\ &= \sqrt{x^2 y^2} \\ &= xy \end{aligned}$$

so that we have, setting $n = xy$ (note that n is then a positive integer because $\mathbb{N}$ is closed under multiplication). So we have $f_S(n, d) = f_S(n, d - 2)$, as desired. $\square$

5 Tangent and Cotangent

We can use our findings so far to attempt to generalize to A_n and C_n. Though, using our previous findings we can show that both of these are never polynomials.

Theorem 5.1. *A_n is never a polynomial for $n > 0$.*

Proof. As shown earlier, $A_n = \frac{H_n}{T_n}$. We already showed that for $n > 0$, H_n is never a polynomial. Since $\mathbb{Z}[x]$ is closed under multiplication, then a non-polynomial divided by a polynomial cannot yield a polynomial due to the fact that a polynomial times a polynomial cannot yield a non-polynomial. Thus, A_n is never a polynomial. $\square$

For C_n, since $\sin(0) = 0$ then C is not defined for $n = 0$. Thus, we can make an even stronger statement.

Theorem 5.2. *C_n is never a polynomial.*

Proof. We know that T_n and S_n are polynomials and that

$$H_n(\cos(\alpha)) = S_n(\cos(\alpha)) \cdot \sin(\alpha)$$

$$H_n(\cos(\alpha)) = S_n(\cos(\alpha)) \cdot \sqrt{1 - \cos^2(\alpha)}$$

Therefore,

$$C_n(\cos(\alpha)) = \frac{T_n(\cos(\alpha))}{H_n(\cos(\alpha))}$$

$$C_n(\cos(\alpha)) = \frac{T_n(\cos(\alpha))}{S_n(\cos(\alpha))\sqrt{1-\cos^2(\alpha)}}$$

$$C_n(\cos(\alpha)) = \frac{T_n(\cos(\alpha))\sqrt{1-\cos^2(\alpha)}}{S_n(\cos(\alpha))(1-\cos^2(\alpha))}$$

$$C_n(x) = \frac{T_n(x)\sqrt{1-x^2}}{S_n(x)(1-x^2)}$$

$1-x^2$ is a polynomial, so $S_n(x)(1-x^2)$ is a polynomial by closure. Though, $\sqrt{1-x^2}$ is not a polynomial, but since $T_n(x)$ is a polynomial then $T_n(x)\sqrt{1-x^2}$ must not be a polynomial since it will always include the $\sqrt{1-x^2}$ term. Therefore, since the numerator of this fraction is not a polynomial but its denominator is, $C_n(x)$ cannot be a polynomial. □

6 Factorizations

Proposition 6.1. $T_n(x) = 2^{n-1} \prod_{k=1}^{n} \left(x \cos\left(\frac{2k-1}{2n}\pi\right)\right)$.

Proof. We know that $T_n(\cos(x)) = \cos(nx)$, so if $\cos(nx) = 0$, then $\cos(x)$ is a root of the polynomial. We note that $\cos\left(\frac{2k-1}{2n}\pi\right)$ for $1 \leq k \leq n$ is distinct. Additionally, we know that each of these is a root because $\cos\left(n \cdot \frac{2k-1}{2n}\pi\right) = \cos\left(\frac{2k-1}{2}\pi\right) = 0$, which is true because $\frac{2k-1}{2}\pi$ is a multiple of $\frac{\pi}{2}$ but not π, which is the condition for $\cos(x) = 0$.

We have found n distinct roots for $T_n(x)$. However, have have proven earlier that $T_n(x)$ has a degree of n. We have found n roots of $T_n(x)$, so there cannot be any more. Therefore, $T_n(x) = a \prod_{k=1}^{n} \left(x \cos\left(\frac{2k-1}{2n}\pi\right)\right)$, where a is a constant. To find this constant, we not that the leading term of the polynomial is $2^{n-1}x^n$, and the leading term of the product is ax^n. Therefore, we get $a = 2^{n-1}$, so the factorization is $T_n(x) = 2^{n-1} \prod_{k=1}^{n} \left(x \cos\left(\frac{2k-1}{2n}\pi\right)\right)$. □

Proposition 6.2. $S_n(x) = 2^{n-1} \prod_{k=1}^{n-1} \left(x - \cos\left(\frac{k}{n}\pi\right)\right)$.

Proof. We know that $S_n(\cos(x)) = \frac{\sin(nx)}{\sin(x)}$, so if $\sin(nx) = 0$ and $\sin(x) \neq 0$, then $\cos(x)$ is a root of the polynomial. We note that $\sin\left(\frac{k}{n}\pi\right)$ for $1 \leq k \leq n$ is distinct. Additionally, we know that each of these is a root because $\sin\left(n \cdot \frac{k}{n}\pi\right) = \sin(k\pi) = 0$, which is true because $k\pi$ is a multiple of π, which is the condition for $\sin(x) = 0$.

We have found $n-1$ distinct roots for $S_n(x)$. However, have have proven earlier that $S_n(x)$ has a degree of $n-1$. We have found $n-1$ roots of $S_n(x)$, so there cannot be any more. Therefore,

$T_n(x) = a \prod_{k=1}^{n-1} \left(x - \cos\left(\frac{k}{n}\pi\right)\right)$, where a is a constant. To find this constant, we not that the leading term of the polynomial is $2^{n-1}x^{n-1}$, and the leading term of the product is ax^{n-1}. Therefore, we get $a = 2^{n-1}$, so the factorization is $S_n(x) = 2^{n-1} \prod_{k=1}^{n-1} \left(x - \cos\left(\frac{k}{n}\pi\right)\right)$. □

7 Fractals

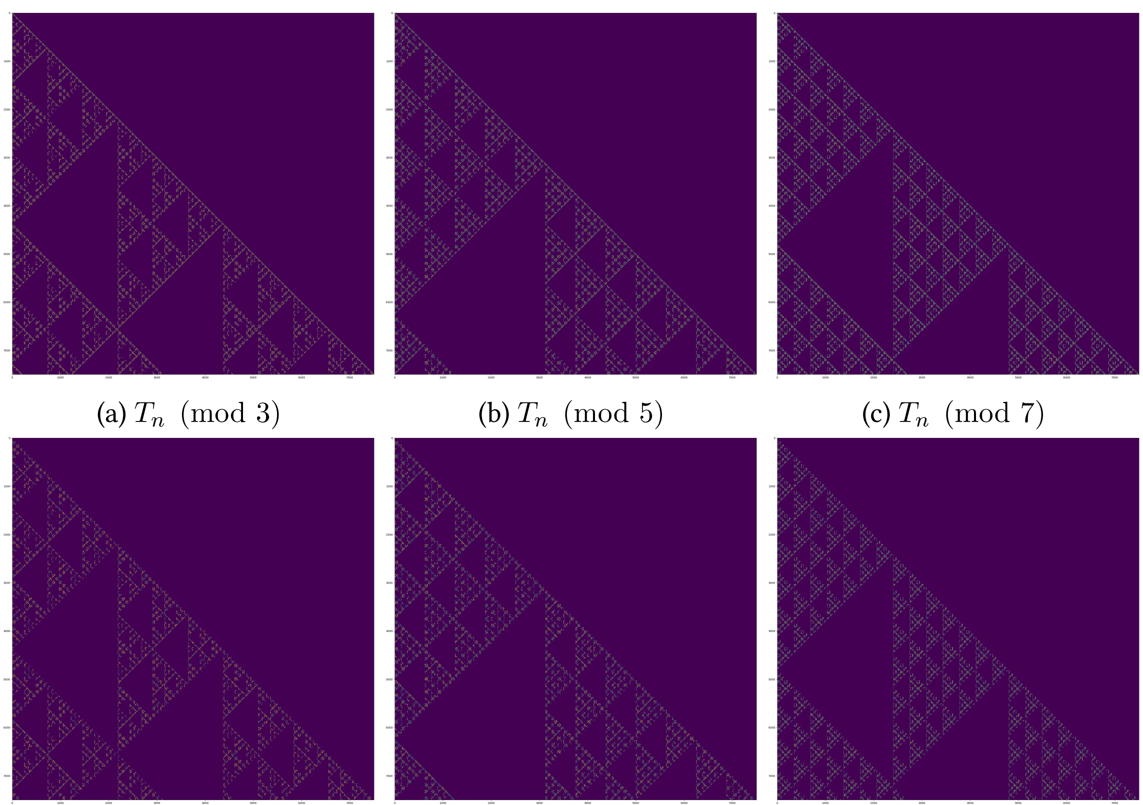
By taking the coefficients of T_n and S_n modulo various natural numbers, notable patterns emerged with intriguing properties.

7.1 Methodology

A python script was written to generate T_n and S_n for large n. The residues of the coefficients in a certain modulus were then plotted on a graph using the library 'Matplotlib' with darker colors representing residues closer to 0 and purple representing a residue of exactly 0.

7.2 Prime Moduli

First, we look at the graphs for prime moduli.



(a) $T_n \pmod 3$ (b) $T_n \pmod 5$ (c) $T_n \pmod 7$

(d) $S_n \pmod 3$ (e) $S_n \pmod 5$ (f) $S_n \pmod 7$

Figure 1: Graphs of T_n and S_n in prime moduli

We notice that a sort of regularity begins to emerge, namely a Sierpinski Triangle like figure. Though, rather than forming one additional "white" triangle in each "black" triangle, for a prime modulus p we have the $p-1^{\text{th}}$ triangular number "white" triangles added into each "black" triangle. We then try to formalize this concept mathematically.

Definition. The figure generated for a prime modulus p is the p-Sierpinski Triangle.

Conjecture 7.1 (The Prime Sierpinski Conjecture). *The p-Sierpinski Triangle always contains exactly t_{p-1} blank triangles and t_p colored triangles, where t_k denotes the kth triangular number. Within each colored triangle, another p-Sierpinski Triangle exists. This continues on forever but cannot be observed forever due to the limitations of the precision of this representation of the p-Sierpinski Triangles.*

Furthermore, throughout this section we ignore all even numbers because the graph for modulo 2, as shown earlier in the paper, would just end up being all coefficients except exactly one of the last two being zero. Furthermore, due to the Chinese Remainder Theorem then all even numbers' graphs just simplify down to that of the greatest odd number that divides them, ignoring these two columns. Therefore, we decide to ignore investigating the properties of even numbers.

7.3 Moduli That Are the Product of Distinct Primes

We then investigate moduli that are the product of distinct primes.

The graph of $p_1 p_2 \dots p_n$ appears to be p_1-Sierpinski Triangle overlaid with the p_2-Sierpinski Triangle overlaid with …overlaid with the p_n-Sierpinski Triangle. We formalize this and prove it.

Theorem 7.2. *The graph for the product of distinct primes p_k for all $k \in \mathbb{Z}$ and $0 < k \leq n$ is the overlaying of the p_k-Sierpinski Triangles when taking as either zero or nonzero.*

Proof. Call this product x. Using the Chinese Remainder Theorem and the fact that all these prime factors are relatively prime to one another, we know that k has a unique nonzero residue modulo x if and only if it has a nonzero residue modulo some p_k. As a result, for every tile in the graph, it is not purple if and only if it is not purple in some p_k-Sierpinski Triangle. Thus, this graph is the overlay of each p_k-Sierpinski Triangle when taken as either zero or not zero. $\square$

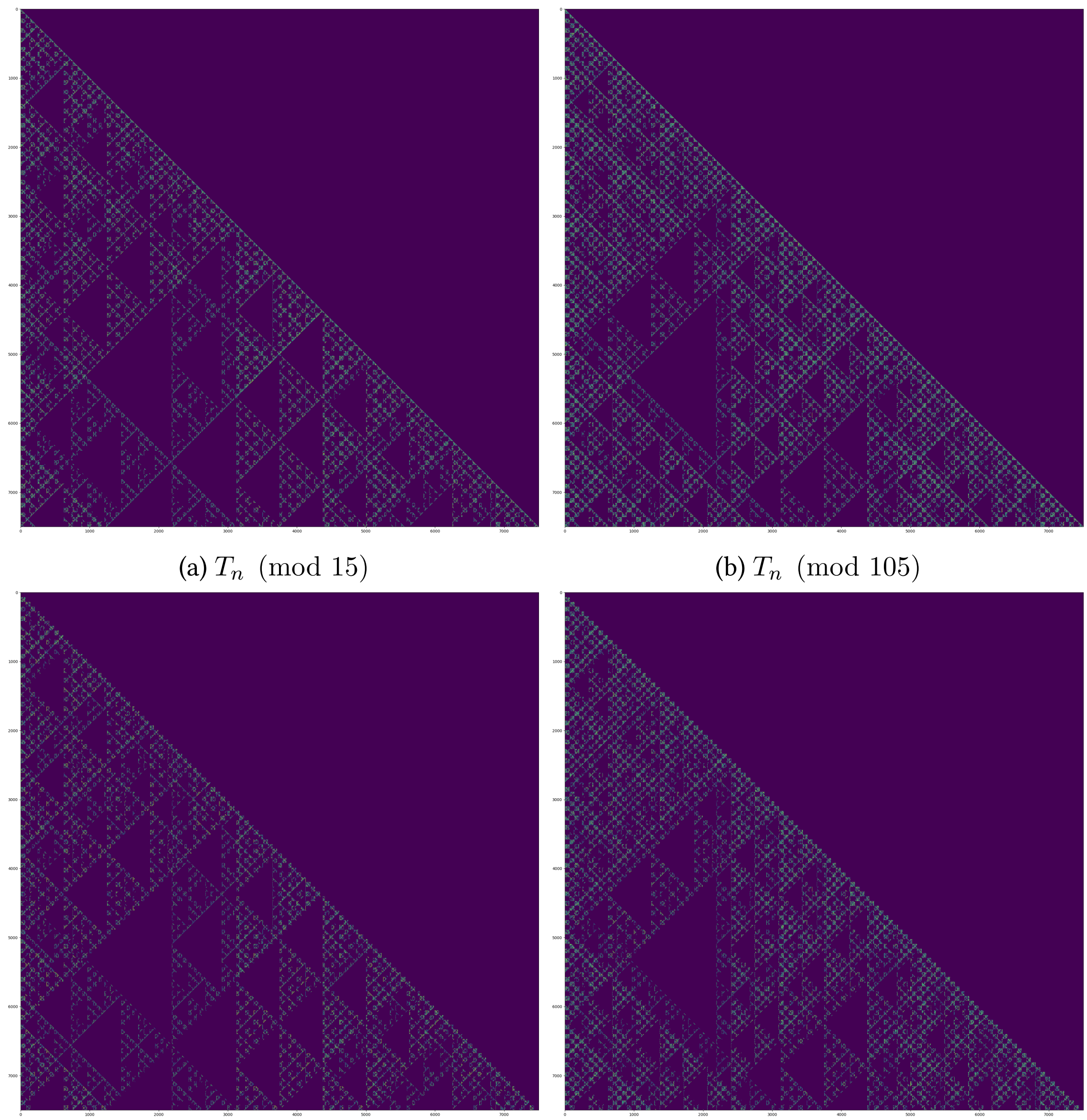
7.4 Square Moduli

From observing the fractals generated for p^2, we see that every formerly blank triangle is now filled with the $p-1^{\text{th}}$ triangular number p-Sierpinski Triangles.

We also see in 5^2 that all the residues are contained in alternating 16 by 16 squares that tile the graph.

Definition. A graph is square-contained if all the nonzero residues are contained in 16 by 16 boxes that border other boxes only at their corners and tile the entire graph.

Definition. A graph is square-tiled if all the nonzero residues are contained in 16 by 16 boxes that border other boxes only at their corners and tile the entire graph and each box is filled in with greater than 50% of points greater than zero.



(a) $T_n \pmod{15}$ (b) $T_n \pmod{105}$

(c) $S_n \pmod{15}$ (d) $S_n \pmod{105}$

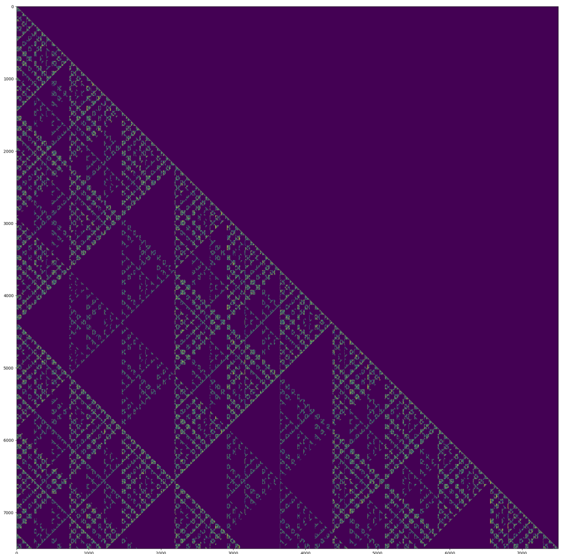
Figure 2: Graphs of T_n and S_n in moduli that are the product of distinct primes

Conjecture 7.3. *All graphs for p^2 with some prime $p > 3$ are square-tiled.*

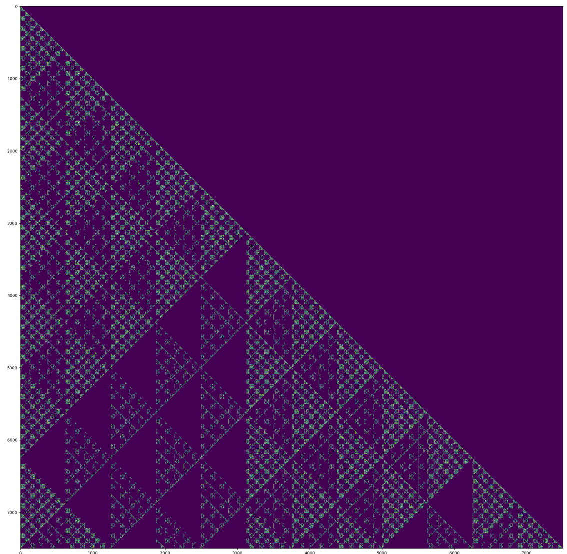
For the graphs of composite numbers squared, we see that they are still the overlay of other graphs. More specifically, we can extend our Chinese Remainder Theorem argument to encompass all possible cases of the product of relatively prime integers then show this as a trivial corollary.

Theorem 7.4 (The Overlay Theorem)**.** *The graph of n where n is the product of relatively prime integers is the overlay of the graphs of each of those relatively prime factors up to being zero or nonzero.*

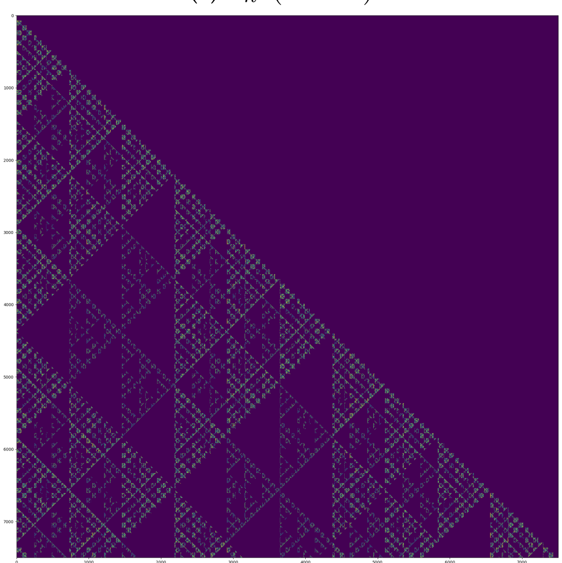
Proof. We can extend our Chinese Remainder Theorem argument. Let $n = j_0 j_1 \dots j_m$ where if



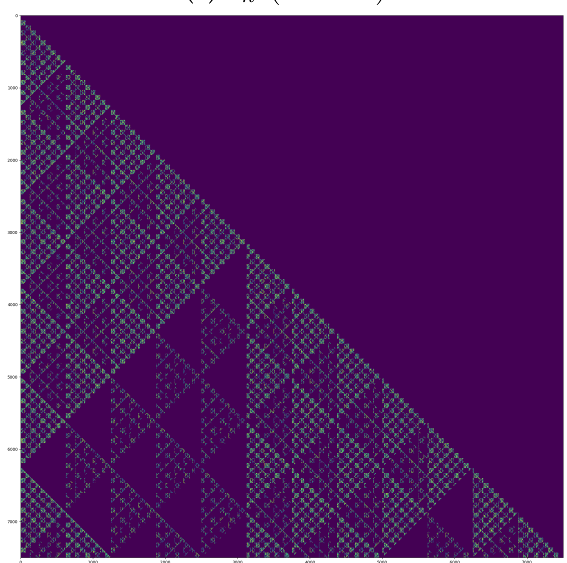
(a) $T_n \pmod 9$



(b) $T_n \pmod{25}$



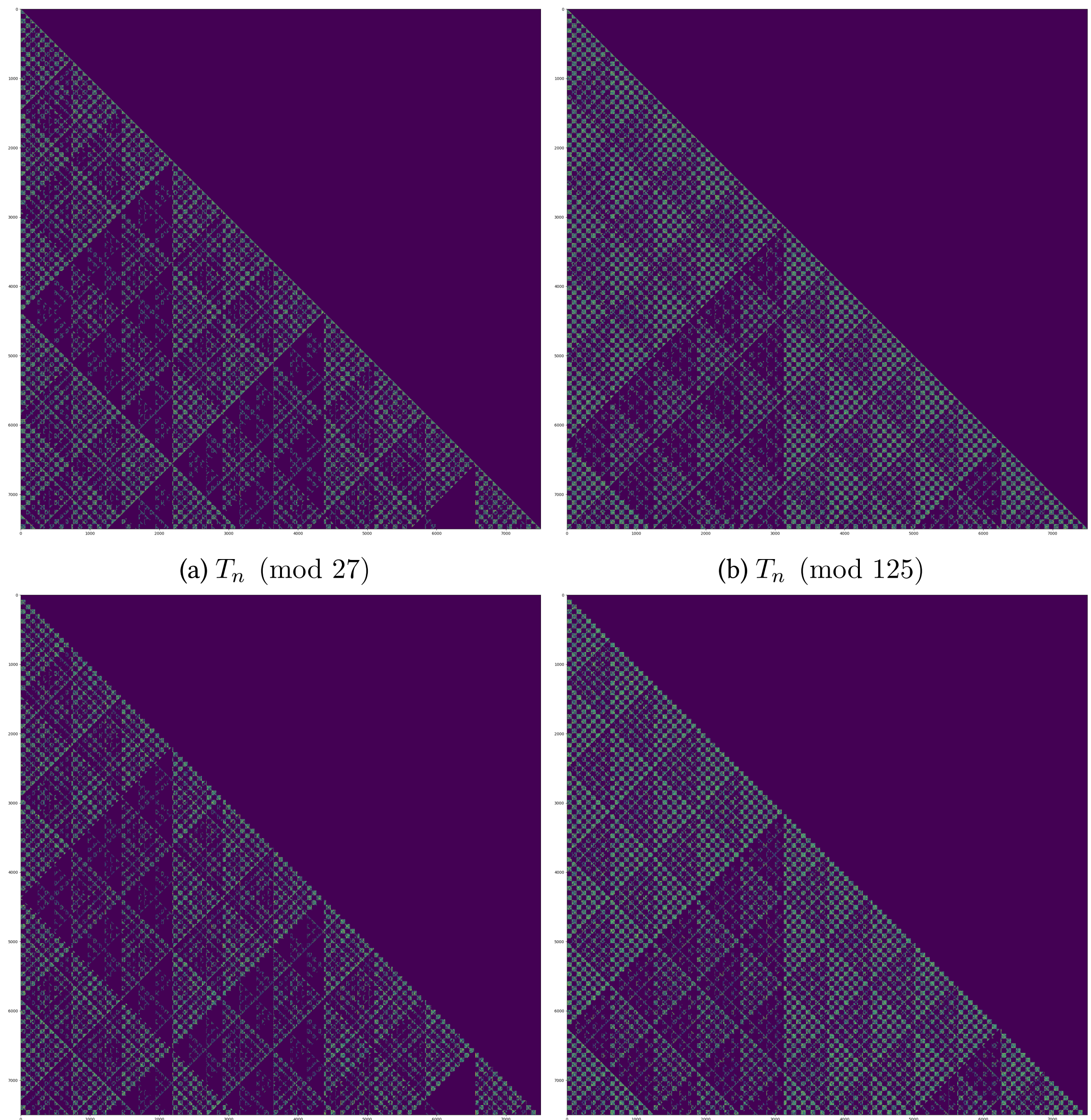
(c) $S_n \pmod 9$



(d) $S_n \pmod{25}$

Figure 3: Graphs of T_n and S_n in squares of prime moduli

$(j_x, j_y) > 1$ then $x = y$. Thus, all these factors are relatively prime to each other. Using the Chinese Remainder Theorem and the fact that all these prime factors are relatively prime to one another, we know that k has a unique nonzero residue modulo n if and only if it has a nonzero residue modulo some j where j is one of these relatively prime factors of n. As a result, for every tile in the graph, it is not purple if and only if it is not purple in the graph of some j_i. Thus, this graph is the overlay of each j_i-Sierpinski Triangle when taken as either zero or not zero. $\square$



(a) $T_n \pmod{27}$

(b) $T_n \pmod{125}$

(c) $S_n \pmod{27}$

(d) $S_n \pmod{125}$

Figure 4: Graphs of T_n and S_n in cubes of prime moduli

7.5 Cube Moduli

For the graphs of cubes of primes, we first see that a generalization of our patterns for squares holds in 3^3. We see that each previously blank triangle is filled with the 2nd triangular number amount of 3-Sierpinski Triangles. Though, due to square-tiling in primes higher than 3, we see that we begin to lose intricacy in structure as p increases. Though, the affected regions become "brighter" (have a higher proportionate residue modulo p^3) as the power of p increases in the affected regions. As a result, we can form a generalized conjecture that seems to hold for the graphs generated for power of primes.

Conjecture 7.5 (The Power Conjecture)**.** *The graph of p^n is that of p^{n-1} with the $p-1^{th}$ triangular* *number amount of evenly spaced and tiled p-Sierpinski Triangles in every triangular region of the graph of p^{n-1} that is completely filled with zeroes, all up to square tiling.*

We can then see that the Overlay Theorem seems to hold as well, which is a good sign. Furthermore, since the square tiles always line up, the composite graphs are also square tiled. Though, we conjecture when graphs are square tiled.

Conjecture 7.6. *All graphs for p^3 with some prime p are square-tiled.*

We can then try to form a more general conjecture for all cases when graphs are square-tiled.

Conjecture 7.7 (The Square Tiling Conjecture)**.** *The graph of n is square tiled if n is divisible by some p^k where $p > 3$ is a prime and $n \geq 2$ or $p = 3$ and $n \geq 3$.*

This makes some sense from the Overlay Theorem when considering composite numbers, but it becomes more interesting when considering the power of prime cases. Even when considering composite numbers, it's not immediately obvious that non-square tiled graphs end up fitting into the squares when overlaid. Though, this leads us into another conjecture, that

Conjecture 7.8 (The Square Containing Conjecture)**.** *Every graph for modulo an odd number is square-contained.*

7.5.1 Brightness

Definition 1. The structure of the graph of n is it reduced to either being zero or nonzero modulo n.

We notice when comparing T_n and S_n that S_n has the same structure as T_n but shifted down by one tile.

Conjecture 7.9. *S_n has the same structure as T_n shifted down by one tile.*

Furthermore, it also becomes obvious from this definition that once the graph is square-tiled, then its structure is the same no matter what it is multiplied by.

We also see that T_n is always "brighter" than S_n, or that the remainders as a whole modulo k are higher in T_n than S_n. We conjecture this as well.

Definition 2. A square is "brighter" than another square if on average it has higher residues modulo k in the graph for modulo k. A graph is "brighter" than another graph if on whole, the tiles in one are usually brighter than the tiles of another.

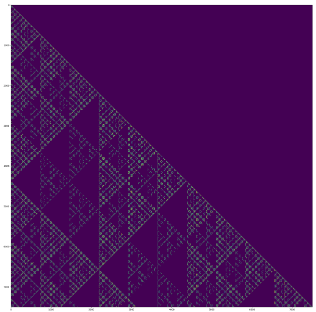
Conjecture 7.10. *T_n is always "brighter" than S_n using the definition of brighter provided above.*

Conjecture 7.11. *When structure stagnates as n increases due to square tiling in higher powers of p^n, squares become "brighter" instead in affected regions.*

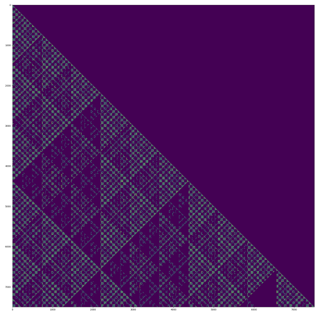
7.6 Moduli That Are Powers of 3

We can observe that in each successive graph of a modulus that is a power of 3, the previously blank space begins to be filled in with more 3-Sierpinski triangles. Numerically, this is because as we take the coefficients to a larger power of 3, less of the coefficients will be divisible by this larger power. These triangles should continue to form forever, but their intricate structures become more difficult to observe as larger powers are taken.

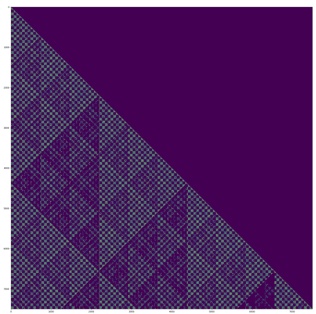
(a) $T_n \pmod 3$



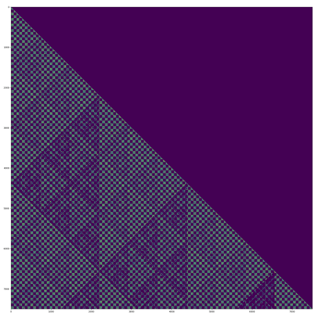
(b) $T_n \pmod 9$



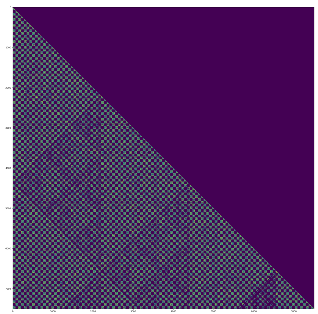
(c) $T_n \pmod{27}$



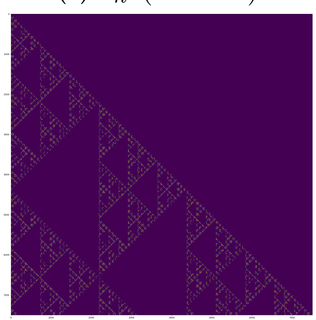
(d) $T_n \pmod{81}$



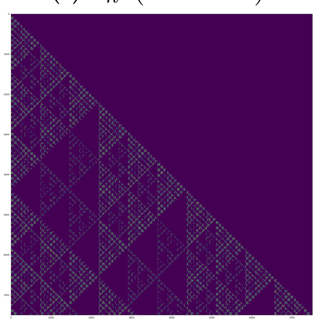
(e) $T_n \pmod{243}$



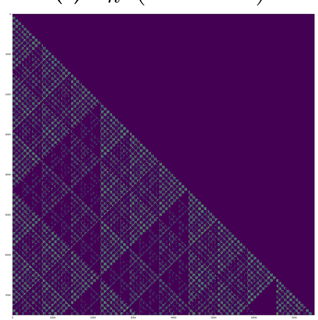
(f) $T_n \pmod{729}$



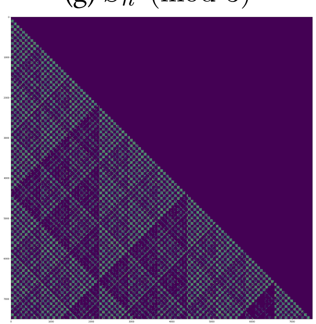
(g) $S_n \pmod 3$



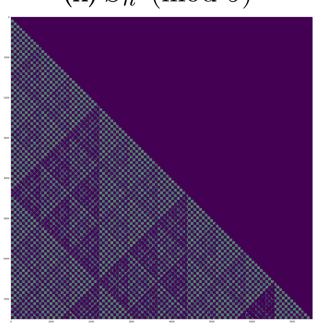
(h) $S_n \pmod 9$



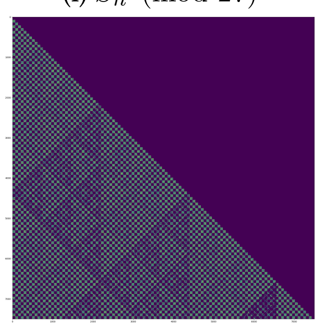
(i) $S_n \pmod{27}$



(j) $S_n \pmod{81}$



(k) $S_n \pmod{243}$



(l) $S_n \pmod{729}$

Figure 5: Graphs of T_n and S_n in moduli that are powers of 3

8 Future Work

8.1 Polynomials in $\mathbb{Z}_p[x]$

We want to see what happens if we take the expression in $\mathbb{Z}_p[x]$, where p is an odd prime. Doing this, we notice that:

$$\begin{aligned}T_3(x) &= x^3\\ T_9(x) &= x^9\\ T_{27}(x) &= x^{27}\\ T_{81}(x) &= x^{81}\end{aligned}$$

This sort of pattern holds for other odd primes p as well. Looking into why this is true and how this relates to the fractals that we found is something that we can look to do moving forward.

Another interesting pattern that we noticed in $\mathbb{Z}_p[x]$ was when $x \in \mathbb{Z}_p$ was a root of the expression. We noticed that when $p = 3$ and $p = 5$, only $x = 0$ was ever a root of the expression and that was only for T_{2k+1}, where $k \in \mathbb{Z}$, where $x = 0$ was only a single root. We can use this to establish what types of algebraic numbers have equivalent expressions in $\mathbb{Z})_p$.

8.2 Derivatives

We let $D_n(x)$ be the polynomial which is the derivative of $T_n(x)$. Then, clearly, from the power rule, we have that

$$f_D(n, d) = (d + 1)f_T(n, d + 1)$$

Thus, using the sum-based formula established in Section 4.2.1, we have

$$f_D(n, d) = (d + 1)(-1)^{\frac{n-d-1}{2}} \sum_{k=\frac{n-d-1}{2}}^{[\frac{n}{2}]} \left(\binom{k}{\frac{n-d-1}{2}} \binom{n}{2k} \right)$$

and, using the product-based formula established in Section 4.2.2, we have that

$$f_D(n, d) = (-1)^{\frac{n-d-1}{2}} \cdot \frac{\prod_{k=1}^{d} (n - d - 1 + 2k)}{d!} n$$

In future investigations of these polynomials, we could study some properties of the coefficients, factorizations, roots, and other topics about the derivatives of $T_n(x)$.

8.3 Tangent in terms of Tangent

Even though there is no polynomial $A_n(x)$ such that $A_n(\cos(x)) = \tan(nx)$. However, we know that $\tan(2x) = \frac{2\tan(x)}{1-\tan^2(x)}$, so if $A_2'(x)$ is the function such that $A_2'(\tan(x)) = \tan(2x)$, then we get $A_2'(x) = \frac{2x}{1-x^2}$. This is a rational function. In the future, it would be worth investigating if $A_n'(x)$ is always a rational function and any interesting properties it might have.

8.4 Crossing-Over Points

Definition. We define $cross(x)$ to be the least $n \in \mathbb{Z}$ such that $|f_T(n, n-2x+2)| < |f_T(n, n-2x)|$. Some examples of this are:

$$
\begin{aligned}
cross(1) &= 5 \\
cross(2) &= 12 \\
cross(3) &= 18 \\
cross(4) &= 25 \\
cross(5) &= 32 \\
cross(6) &= 39
\end{aligned}
$$

If we take the difference between consecutive terms of this sequence, we get the sequence $7, 6, 7, 7, 7, 7, 7, 7, 6, 7, 7, 7, 7, 7, 7, 6, 7, 7, 7, 7, 7, 7, 6, 7, 7, 7, 7, \ldots$. Up until the numbers that we have tested, which is until the 1464th crossing number, we get that the differences are always 7 or 6, which 7 being much more common. In the future, we can try to see why this pattern holds true.

9 Acknowledgements

We would like to thank the PROMYS Program for providing us with the opportunity to perform this lab, our parents for facilitating our ability to participate in this program, and the PROMYS Foundation for providing funding for the program and thus our ability to perform this exploration.