Appendix C. Solving optimizaton problem in system Wolfram Mathematica.

1. We first solve equation (12) with respect to $\varphi = x$.

```
In[1]:= FullSimplify[Solve[s - 1 + 1 / x == (2 s + d - 1) / (s + (1 - x) (d - 2)) , x]]
```

$$\text{Out[1]=}\ \left\{\left\{x \to \left(5 + d\,(-3 + s) - 5\,s + s^2 - \sqrt{\left(4\,(-2 + d)\,(-1 + s)\,(-2 + d + s) + (5 + d\,(-3 + s) + (-5 + s)\,s)^2\right)}\right) \Big/ (2\,(-2 + d)\,(-1 + s))\right\}, \left\{x \to \left(5 + d\,(-3 + s) - 5\,s + s^2 + \sqrt{\left(4\,(-2 + d)\,(-1 + s)\,(-2 + d + s) + (5 + d\,(-3 + s) + (-5 + s)\,s)^2\right)}\right) \Big/ (2\,(-2 + d)\,(-1 + s))\right\}\right\}$$

2. Let us select the positive root. Note that the denominator 2(d - 2)(s - 1) is always positive in our case (d >= 3, s > 1). We will show that the only positive root is the greatest of these two routes. We will also show that this root lies in the segement [0.4, 0.65].

First show that 5 - 3d - 5s + ds + s^2 is always negative. Note that then there is at most one positive root. Solving maximization problems yields the maximum value is -4:

```
In[2]:=
    Maximize[{5 - 3 d - 5 s + d s + s^2, d ≥ 3, s > 1, s ≤ 2}, {s, d}]
```

$$\text{Out[2]=}\ \{-4, \{s \to 2, d \to 3\}\}$$

3. Set u to the value of this root:

```
In[3]:= u = (5 - 3 d - 5 s + d s + s^2 + √(-4 (2 - d - s) (2 - d - 2 s + d s) + (-5 + 3 d + 5 s - d s - s^2)^2)) /
        (2 (2 - d - 2 s + d s));
```

4. Let us find minimal and maximal values of u :

```
In[4]:=
    Maximize[{u, d ≥ 3, s > 1, s ≤ 2}, {s, d}]
```

$$\text{Out[4]=}\ \left\{-2 + \sqrt{7}, \{s \to 2, d \to 3\}\right\}$$

```
In[5]:= N[{-2 + √7, {s → 2, d → 3}}]
```

$$\text{Out[5]=}\ \{0.645751, \{s \to 2., d \to 3.\}\}$$

```
In[6]:=
    Minimize[{u, d ≥ 3, s > 1, s ≤ 2}, {s, d}]
```

Minimize::wksol :
Warning: there is no minimum in the region in which the objective function is defined and the constraints are satisfied; returning a result on the boundary. ≫

$$\text{Out[6]=}\ \left\{\frac{2}{5}, \{s \to 1, d \to 3\}\right\}$$

5. From the main system we have the worst B obtained at B = s - 1 + 1/u. Let us show what it is.

In[7]:=

```
b = FullSimplify[s - 1 + 1 / u]
```

Out[7]= $\frac{1}{2\ (-2+d+s)}\left(-1+d-s+d\,s+s^2+\sqrt{\left(4\ (-2+d)\ (-1+s)\ (-2+d+s)+(5+d\ (-3+s)+(-5+s)\ s)^2\right)}\right)$

6. Now run maximization problem to find the worst B. We obtain maximum at (s, d) = (2, +∞) with value 2.61803...

In[8]:=

```
FullSimplify[Maximize[{b, s > 1 , s ≤ 2, d >= 3}, {s, d}]]
```

Maximize::natt : The maximum is not attained at any point satisfying the given constraints. ≫

Out[8]= $\left\{\frac{1}{2}\left(3+\sqrt{5}\right),\ \{s\to 2,\ d\to \text{Indeterminate}\}\right\}$

In[9]:= $N\left[\frac{1}{2}\left(3+\sqrt{5}\right),\ 20\right]$

Out[9]= 2.6180339887498948482

7. Let us plot the graph:

In[10]:=

```
Plot3D[b, {s, 1, 2}, {d, 3, 100}]
```

Out[10]=