The Collatz Cycle Theorem: There Are No Cycles Other Than (4,2,1)

Nelson Abdiel Colón Vargas

Leverhulme Center for the Future of Intelligence, University of Cambridge, Cambridge, UK.

Abstract

We explore the Collatz Conjecture, a long-standing problem that suggests every positive integer, when subjected to the Collatz iteration (halving if even, tripling and adding one if odd), will eventually reach the cycle $\mathbf{(4, 2, 1)}$. In this work, we prove by contradiction that if a sequence generated by the Collatz function enters a cycle, then it must necessarily be the trivial one. We establish this result by analyzing the inherent properties of potential cycles and demonstrating that any such cycle must necessarily include either **1** or **2**, thereby confirming the uniqueness of the trivial cycle.

Keywords: Collatz Conjecture, Number Theory

1 Introduction

The Collatz Conjecture, also known as the $3n+1$ conjecture, stands as one of the most tantalizingly simple, yet unsolved problems in mathematics. Posed in 1937 by Lothar Collatz, a German mathematician, the conjecture proposes a recursive sequence that can be applied to any positive integer. The sequence follows two simple rules: if the number is even, divide it by two; if the number is odd, triple it and add one. The conjecture asserts that no matter which positive integer you start with, the sequence will always reach the cycle of $(1, 2, 4)$. Since its inception, the conjecture has been the subject of extensive study and numerous computational experiments. Mathematicians have verified the conjecture for very large numbers, with computational efforts extending the verification to numbers as large as 2^{68} [Ba20].

Definition 1 (Collatz Function)**.** *Let n be an arbitrary positive integer. Define the function $f : \mathbb{Z}^+ \to \mathbb{Z}^+$ by the following cases:*

$$f(n) = \begin{cases} \frac{n}{2} & \textit{if } n \textit{ is even,} \\ 3n + 1 & \textit{otherwise.} \end{cases}$$

Consider the sequence $\{a_i\}_{i=0}^{\infty}$ generated by iteratively applying the function f to n, where the sequence is defined as:

$$a_i = \begin{cases} n & \text{if } i = 0, \\ f(a_{i-1}) & \text{if } i > 0. \end{cases}$$

Thus, a_i represents the result of applying the function f to n recursively i times, i.e., $a_i = f^i(n)$.

Conjecture (Collatz Conjecture)**.** *For any $n \in \mathbb{Z}^+$, $\exists m \in \mathbb{N}$ such that $f^m(n) = 1$.*

The Collatz conjecture posits that for any initial positive integer n, the sequence $\{a_i\}$ will eventually reach the value 1. In other words, for each n, there exists some i such that $a_i = 1$. If the conjecture is false, it can only be because there is some starting number which gives rise to a sequence that does not contain 1. Such a sequence would either enter a repeating cycle that excludes 1, or increase without bound. The main result of this paper is that there are no repeating cycles that exclude 1.

Main Theorem (Collatz Cycle Theorem)**.** *For all $n \in \mathbb{Z}^+$, if the sequence $\{f^i(n)\}_{i=0}^{\infty}$ enters a cycle then it must be the trivial one $(4, 2, 1)$.*

In the context of studying the Collatz conjecture, since $f(x)$ results in an even value when x is odd, it is often useful to represent the sequence in terms of a function that accounts for two steps in the Collatz process when x is odd:

Definition 2 ([BS78])**.** *The function $u(x)$ is defined as follows:*

$$u(x) = \begin{cases} \frac{x}{2} & \textit{if } x \textit{ is even,} \\ \frac{3x+1}{2} & \textit{otherwise.} \end{cases}$$

Böhm and Sontacchi showed that the elements of a cycle generated by u have a common factor:

Proposition 1 ([BS78])**.** *$\forall x \in \mathbb{Z}$ we have: $\exists l \in \mathbb{Z}^+$ such that $u^l(x) = x$ if and only if $\exists \langle v_0, v_1, \cdots, v_m \rangle \in \mathbb{N}^{m+1}$ with $v_{i-1} < v_i$ for $1 \leq i < m$ and $v_m = l$ such that*

$$x = \frac{\sum_{k=0}^{m-1} 3^{m-k-1} \cdot 2^{v_k}}{2^l - 3^m}.$$

Remark 1. *Notice that on the previous proposition l represents the minimum amount of times we apply f in order return to the initial value, and m refers to the amount of times we specifically applied $\frac{3x+1}{2}$, as such, $m \leq l$.*

We will show that the elements of a non-trivial cycle must have 1 as its greatest common divisor (gcd). Then with the help of the previous proposition, along with the following theorem from Mihăilescu's proof of the Catalan's Conjecture [M02] will show that any cycle generated by f contains 1 or 2 giving rise to a contradiction.

Theorem 2 (Mihăilescu's Theorem [M02])**.** *The only solution in the natural numbers of the equation $x^a - y^b = 1$ for $a, b > 1$ is $x = 3$, $a = 2$, $y = 2$, and $b = 3$.*

2 On the Uniqueness of the (4,2,1) Cycle

It has been shown by [G81, E93, SdW05] that if a non-trivial cycle exists then it contains thousands of terms. To prove the main theorem we only need to understand the behavior of 6 consecutive terms in the sequence, hence we assume that the length of a cycle is at least 42.

Lemma 3. *Let $\{x_i\}_{i=0}^k := \{f^i(x_0)\}_{i=0}^k$ be a non-trivial cycle of length $k+1 \geq 42$, where k is the smallest positive integer such that $f(x_k) = x_0$ and $x_l \neq x_j$ for all $l \neq j$ when $0 \leq l, j \leq k$. Without loss of generality, and from now on, suppose that x_0 is the largest element in the cycle. Then the following statements hold:*

1. *x_0 is divisible by 4, i.e., $4 \mid x_0$. This follows from the fact that x_0 is the max thus x_1 must be of the form $\frac{x_0}{2}$ and $x_2 = \frac{x_1}{2}$. Otherwise $x_2 = 3x_1 + 1 = 3\frac{x_0}{2} + 1 > x_0$.*
2. *$x_k < x_0$ then we must have $x_0 = 3x_k + 1$, implying that x_k is odd.*
3. *Since x_k is odd, then $x_{k-1} = 2x_k$.*
4. *Furthermore, $x_{k-1} = 3x_{k-2} + 1$, implying x_{k-2} is odd. Otherwise, if $x_{k-1} = \frac{x_{k-2}}{2}$, then $x_{k-2} = 2x_{k-1} = 4x_k > 3x_k + 1 = x_0$.*

See Figure 1 below for reference.

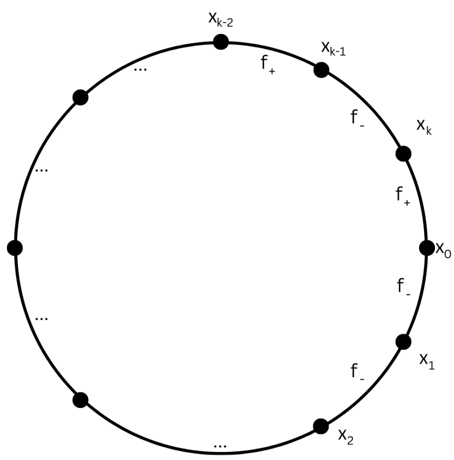

Fig. 1 Lemma 3 visualized. f_+ corresponds to a $3x+1$ and f_- to a $x/2$ step.

Proposition 4. *For x_k defined by the relations $x_{k-1} = 3x_{k-2} + 1$ and $x_{k-1} = 2x_k$, it holds that $x_k = 12n + 5$ for some integer n.*

Proof. **Part I:** Show that $x_k \equiv 1 \pmod 4$
By Lemma 3 we know that x_k is odd and $3x_k + 1 = x_0 = 4m$ for some integer m. We start by showing that x_k must also be of the form $x_k = 4n + 1$ for some integer n. Starting with the equation $3x_k + 1 = 4m$, we subtract 1 from both sides:

$$3x_k = 4m - 1$$

Next, we divide both sides by 3:

$$x_k = \frac{4m - 1}{3}$$

For x_k to be an integer, $4m - 1$ must be divisible by 3. We check the congruence modulo 3:

$$4m - 1 \equiv 0 \pmod 3$$

This simplifies to:

$$4m \equiv 1 \pmod 3$$

Since $4 \equiv 1 \pmod 3$, we have:

$$m \equiv 1 \pmod 3$$

This implies that m can be written as:

$$m = 3j + 1$$

for some integer j. Substituting this into the expression for x_k, we get:

$$x_k = \frac{4(3j + 1) - 1}{3}$$

Expanding and simplifying:

$$x_k = \frac{12j + 4 - 1}{3} = \frac{12j + 3}{3} = 4j + 1$$

Thus, $x_k = 4j + 1$ for some integer j. Therefore, x_k is of the form $x_k = 4n + 1$ for some integer n, which completes the first part of the proof.

Part II: Show that $x_k = 12n + 5$

Next, consider the equation $2x_k = 3x_{k-2} + 1$ modulo 3:

$$2x_k \equiv 3x_{k-2} + 1 \equiv 1 \pmod 3.$$

Since 2 is relatively prime to 3, we multiply both sides by the modular inverse of 2 modulo 3, which is also 2:

$$x_k \equiv 2 \cdot 1 \equiv 2 \pmod 3.$$

We have determined:

$$x_k \equiv 2 \pmod 3.$$

By *Part I* we also had:

$$x_k \equiv 1 \pmod 4.$$

We now need to find a number that satisfies both congruences. We solve the system of congruences:

$$x_k \equiv 1 \pmod 4 \quad \text{and} \quad x_k \equiv 2 \pmod 3.$$

Let $x_k = 4m + 1$ for some integer m. Substituting this into the second congruence:

$$4m + 1 \equiv 2 \pmod 3.$$

Simplifying modulo 3:

$$4m \equiv 1 \pmod 3.$$

Since $4 \equiv 1 \pmod 3$, this reduces to:

$$m \equiv 1 \pmod 3.$$

Thus, $m = 3n + 1$ for some integer n. Substituting back, we find:

$$x_k = 4(3n + 1) + 1 = 12n + 4 + 1 = 12n + 5.$$

Therefore, x_k must be of the form $12n + 5$ for some integer n.

□

Proposition 5. *For x_{k-2} defined by the relations $x_{k-1} = 3x_{k-2} + 1$ and $x_{k-1} = 2x_k$, it holds that $x_{k-2} = 8a + 3$ for some integer a.*

Proof. From Proposition 4 above, we know that $x_k = 12n + 5$ for some integer n. The recurrence relations are given by:

$$x_{k-1} = 3x_{k-2} + 1$$

and

$$x_{k-1} = 2x_k.$$

Substituting $x_{k-1} = 2x_k$ into the first equation, we have:

$$2x_k = 3x_{k-2} + 1.$$

Substituting $x_k = 12n + 5$ from the previous proposition:

$$2(12n + 5) = 3x_{k-2} + 1.$$

Expanding and simplifying:

$$24n + 10 = 3x_{k-2} + 1.$$

Subtracting 1 from both sides:

$$24n + 9 = 3x_{k-2}.$$

Now, divide both sides by 3:

$$8n + 3 = x_{k-2}.$$

□

Example 1. *The following values satisfy the relations described so far in this section:* $x_{k-2} = 3$*,* $x_{k-1} = 10$*,* $x_k = 5$*,* $x_0 = 16$*,* $x_1 = 8$*, and* $x_2 = 4$*. From smallest to largest we have:*

$$x_{k-2} = 3 < x_2 = 4 < x_k = 5 < x_1 = 8 < x_{k-1} = 10 < x_0 = 16$$

Lemma 6. *Let* $x_{k-2} = 3 + 8m$ *for some integer* m*, and define the sequence* $x_{k-2}, x_{k-1}, x_k, x_0, x_1, x_2$ *based on the relationships from Lemma 3:*

$$x_{k-1} = 3x_{k-2} + 1, \quad x_k = \frac{x_{k-1}}{2}, \quad x_0 = 3x_k + 1, \quad x_1 = \frac{x_0}{2}, \quad x_2 = \frac{x_1}{2}.$$

Then, the inequality $x_{k-1} < x_2 < x_k < x_1 < x_{k-1} < x_0$ *holds for all integers* m*.*

Proof. We begin by substituting $x_{k-2} = 3 + 8m$ into the given relationships:

$$\begin{aligned}
x_{k-1} &= 3x_{k-2} + 1 = 3(3 + 8m) + 1 = 9 + 24m + 1 = 10 + 24m, \\
x_k &= \frac{x_{k-1}}{2} = \frac{10 + 24m}{2} = 5 + 12m, \\
x_0 &= 3x_k + 1 = 3(5 + 12m) + 1 = 15 + 36m + 1 = 16 + 36m, \\
x_1 &= \frac{x_0}{2} = \frac{16 + 36m}{2} = 8 + 18m, \\
x_2 &= \frac{x_1}{2} = \frac{8 + 18m}{2} = 4 + 9m.
\end{aligned}$$

Next, we verify the inequality $x_{k-2} < x_2 < x_k < x_1 < x_{k-1} < x_0$ by comparing the expressions for each x_i:

- $x_{k-2} < x_2$:
$$3 + 8m < 4 + 9m$$
- $x_2 < x_k$:
$$4 + 9m < 5 + 12m$$
- $x_k < x_1$:
$$5 + 12m < 8 + 18m$$

- $x_1 < x_{k-1}$:
$$8 + 18m < 10 + 24m$$
- $x_{k-1} < x_0$:
$$10 + 24m < 16 + 36m$$

Since each of these inequalities holds for all integers m, we conclude that the inequality $x_1 < x_6 < x_3 < x_5 < x_2 < x_4$ remains valid for all values of m. $\square$

Proposition 7. *Let $\{x_i\}_{i=0}^k$ be a non-trivial cycle of length $k + 1 \geq 42$, where k is the smallest positive integer such that $x_{k+1} = x_0$ and $x_l \neq x_j$ for all $l \neq j$ when $0 \leq l, j \leq k$, and x_0 is the largest element in the cycle, then $\gcd(\{x_i\}_{i=0}^k) = 1$.*

Proof. It suffices to show that $\gcd(x_{k-2}, x_k) = 1$. From the proof of Lemma 6 we know that $x_{k-2} = 8n + 3$, and $x_k = 12n + 5$ for some $n \in \mathbb{N}$. Subtract $2 \times (12n + 5)$ from $3 \times (8n + 3)$ to eliminate the n terms:

$$3 \times (8n + 3) - 2 \times (12n + 5) = 24n + 9 - 24n - 10 = -1.$$

Since the greatest common divisor divides both $8n+3$ and $12n+5$, it must also divide any linear combination of them, including -1. Hence, $\gcd(8n + 3, 12n + 5) = 1$, and consequently the $\gcd(\{x_i\}_{i=0}^k) = 1$. $\square$

Lemma 8. *Let $\{x_i\}_{i=0}^k$ be a non-trivial cycle as defined in Proposition 7. Then x_1, x_k, and x_{k-2} belong to the minimal cycle $\{y_i\}_{i=0}^l$ reached by applying u iteratively on any element of $\{x_i\}_{i=0}^k$.*
See Figure 2 below for reference.

Proof. This simply follows from the definition of u and the fact that x_{k-2} and x_k are odd. $\square$

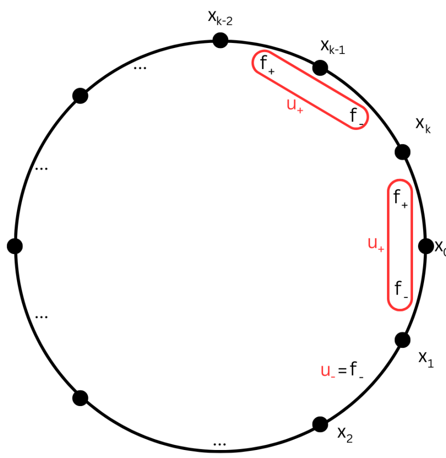

Fig. 2 Lemma 8 visualized.

Lemma 9. *Let $\{y_i\}_{i=0}^{l}$ be the minimal cycle reached by applying u to any element of $\{x_i\}_{i=0}^{k}$, then $\gcd(\{y_i\}_{i=0}^{l}) = 1$.*

Proof. Since $x_{k-2}, x_k \in \{y_i\}_{i=0}^{l}$, then $\gcd(\{y_i\}_{i=0}^{l})$ is also 1 by Proposition 7. □

Remark 2. *Notice from Figure 2 and the lemmas above that x_0 is not in the corresponding cycle generated by u but it is one step away as $u(x_0) = x_1$ which is in the cycle.*

Collatz Cycle Theorem. *For all $n \in \mathbb{Z}^+$, if the sequence $\{f^i(n)\}_{i=0}^{\infty}$ enters a cycle then it must be the trivial one $(4, 2, 1)$.*

Proof. We will prove this by contradiction. Let $n \in \mathbb{Z}^+$. Suppose $\{f^i(n)\}_{i=0}^{\infty}$ enters a non-trivial cycle, $\{x_i\}_{i=0}^{j}$. By Proposition 7 we know that $\gcd(\{x_i\}_{i=0}^{j}) = 1$. Furthermore, let $\{\hat{y}_i\}_{i=0}^{l}$ be the minimal cycle obtained from applying u to x_0, and remember from Remark 2 that $\{\hat{y}_i\}_{i=0}^{l}$ may not contain x_0. By Lemma 9 we also have that $\gcd(\{y_i\}_{i=0}^{l}) = 1$.

By Proposition 1 we know that for $\hat{y} \in \{y_i\}_{i=0}^{j}$:

$$\hat{y} = \frac{\sum_{k=0}^{m-1} 3^{m-k-1} \cdot 2^{v_k}}{2^l - 3^m}.$$

for some $m, l \in \mathbb{Z}^+$, $m \leq l$. Since the y_i's are positive integers and the $\gcd(\{y_i\}_{i=0}^{j}) = 1$ then $2^l - 3^m = 1$. By Mihănescu's theorem (2) we know $2^l - 3^m = 1$ does not have another solution other than $l = 2$ and $m = 1$ in $l, m \in \mathbb{Z}^+$. The proposition hence simplifies to:

$$\hat{y} = \frac{\sum_{k=0}^{0} 3^{1-k-1} \cdot 2^{v_k}}{2^2 - 3^1} = \frac{2^{v_0}}{4 - 3} = 2^{v_0}.$$

Since v_0 is a natural number (i.e., $v_0 \in \mathbb{N}$), the possible values of 2^{v_0} are powers of 2. Additionally, because $v_1 = l = 2$ by definition, the largest value v_0 can take is 1. Thus, the possible values of x are:

$$\hat{y} = 2^0 = 1, \text{ or } \quad \hat{y} = 2^1 = 2.$$

Then $\hat{y} \in \{y_i\}_{i=0}^{j} \subset \{x_i\}_{i=0}^{j}$. But $\{x_i\}_{i=0}^{j}$ was a non-trivial cycle and, as such, cannot contain either $\hat{y} = 1$ or $\hat{y} = 2$.

$$\Rightarrow\Leftarrow$$

Therefore if $\{f^i(n)\}_{i=0}^{\infty}$ enters a cycle, then it must be the trivial one $(4, 2, 1)$.

□

References

[BS78] Böhm C., Sontacchi, G. "On the existence of cycles of given length in integer sequences like $x_{n+1} = x_n/2$ if x_n even, and $x_{n+1} = 3x_n + 1$". *Atti della Accademia Nazionale dei Lincei. Classe di Scienze Fisiche, Matematiche e Naturali.* Rendiconti, Serie 8, Vol. 64, n.3, p. 260-264, 1978. Accademia Nazionale dei Lincei http://www.bdim.eu/item?id=RLINA_1978_8_64_3_260_0

[Ba20] Barina, D., "Convergence verification of the Collatz problem". *The Journal of Supercomputing.* 77 (3), 2681-2688, 2020. https://link.springer.com/article/10.1007/s11227-020-03368-x

[G81] Garner, L. (1981). "On the Collatz 3n + 1 algorithm". *Proceedings of the American Mathematical Society.* 82 (1): 19-22, 1981. https://www.ams.org/journals/proc/1981-082-01/S0002-9939-1981-0603593-2/.

[E93] Shalom Eliahou, S., "The 3x+1 problem: new lower bounds on nontrivial cycle lengths". *Discrete Mathematics*, Volume 118, Issues 1-3, 1993, Pages 45-56, https://doi.org/10.1016/0012-365X(93)90052-U.

[M02] Mihănescu, P. "Primary cyclotomic units and a proof of Catalan's conjecture". *From the journal Journal für die reine und angewandte Mathematik* Math.572(2004), 167-195. https://doi.org/10.1515/crll.2004.048

[SdW05] Simons, J., de Weger, B., "Theoretical and computational bounds for m-cycles of the 3n + 1 problem". *Acta Arithmetica.* 117 (1): 51-70, 2005. Archived from the original on 2022-03-18. Retrieved 2023-03-28. https://www.impan.pl/pl/wydawnictwa/czasopisma-i-serie-wydawnicze/acta-arithmetica/all/117/1/82263/theoretical-and-computational-bounds-for-m-cycles-of-the-3n-1-problem