1 Descriptive Techniques

1.1 Linear Filtering

Definitions

$$\{a_r\} = \text{a set of weights} \tag{1}$$

Moving Average (MA)

$$\boxed{y_t = \mathrm{Sm}(x_t) = \sum_{r=-q}^{s} a_r x_{t+r} \quad \text{where } \sum a_r = 1} \tag{2}$$

- MA's are often symmetric with $s = q$ and $a_j = a_{-j}$
- Simple MA: $a_r = \frac{1}{2q+1}$

Exponential Smoothing

$$\boxed{\mathrm{Sm}(x_t) = \sum_{j=0}^{\infty} \alpha(1-\alpha)^j x_{t-j} \quad \text{with } 0 < \alpha < 1} \tag{3}$$

- Exponential Smoothing solves the end-effect problem

Residual Filter - Trend Removing

$$\boxed{\mathrm{Res}(x_t) = x_t - \mathrm{Sm}(x_t) = \sum_{r=-q}^{s} b_r x_{t+r}} \tag{4}$$

- where
- $b_0 = 1 - a_0$ and $b_r = -a_r$ for $r \neq 0$
- $\sum b_r = 0$

Sequential Filtering

$$\boxed{z_t = \sum_j b_j y_{t+j} = \sum_j b_j \sum_r a_r x_{t+j+r} = \sum_k c_k x_{t+k}} \tag{5}$$

$$\text{where by convolution } c_k = \sum_r a_r b_{k-r} = \{a_r\} * \{b_j\} \tag{6}$$

1.2 Basic Notions

Autocovariance

Autocovariance

Population acv.f:

$$\boxed{\gamma(k) = \mathrm{Cov}\left(X_t, X_{t+k}\right) = E\left\{(X_t - \mu)\left(X_{t+k} - \mu\right)\right\}} \tag{7}$$

Sample acv.f:

$$\boxed{c_k = \sum_{t=1}^{T-k} (x_t - \bar{x})\left(x_{t+k} - \bar{x}\right)} \tag{8}$$

Autocorrelation

Autocorrelation

Population ac.f:

$$\boxed{\mathrm{Cor}\left(X_t, X_{t+k}\right) = \frac{\mathrm{Cov}\left(X_t, X_{t+k}\right)}{\mathrm{SD}(X_t)\, SD\left(X_{t+k}\right)} = \frac{\mathrm{Cov}\left(X_t, X_{t+k}\right)}{\mathrm{Var}(X_t)}} \tag{9}$$

- The last equality follows from the series being stationary
Sample ac.f:

$$\boxed{r_k = \frac{\sum_{t=1}^{T-k} (x_t - \bar{x})\left(x_{t+k} - \bar{x}\right)}{\sum_{t=1}^{T} (x_t - \bar{x})^2} = \frac{c_k}{c_0}} \tag{10}$$

2 Some Time Series Models

2.1 Stochastic Processes

A stochastic process may be defined as a collection of random variables that are ordered in time and defined at a set of time points, which may be continuous or discrete.

Notation

- Random variable at time t is denoted by X_t, $t = 0, 1, 2, \ldots$
- Stochastic process is denoted by $\{X_t\}$
- Underlying probability mechanism is denoted by P
- Realization of the random variable at time t is denoted by x_t

Basic Statistics vs. Time Series Analysis

Situation in TSA:

- Can only observe one finite-length realization from the underlying (infinite) process $\{P\}$.
- Without model assumptions, we can not learn everything from $\{P\}$.

Moments of a Stochastic Process

Mean function:

$$\boxed{\mu_t = \mu(t) = E(X_t)} \tag{11}$$

Variance function:

$$\boxed{\sigma_t^2 = \sigma(t)^2 = Var(X_t)} \tag{12}$$

Autocovariance function:

$$\boxed{\gamma(t_1, t_2) = Cov(X_{t_1,t_2}) = E\{(X_{t_1} - \mu_{t_1})(X_{t_2} - \mu_{t_2})\}} \tag{13}$$

Clearly,

$$\boxed{\gamma(t, t) = \sigma_t^2(t)} \tag{14}$$

2.2 Stationary Processes

Strict Stationarity

$$\boxed{\text{joint D of } \{X_{t_1}, \ldots, X_{t_n}\} = \text{joint D of } \{X_{t_1+k}, \ldots, X_{t_n+k}\}}$$

Special Case $n = 1$:

- All Xt have the same distribution
- $\mu_t = \mu$ for all t
- $\sigma_t^2 = \sigma^2$ for all t

Special Case $n = 2$:

- $\gamma(t_1, t_2)$ only depends on the lag $k = t_2 - t_1$
- $\gamma(k) = Cov(X_t, X_{t+k}) = E\{(X_t - \mu)(X_{t+k} - \mu)\}$

Weak Stationarity

A process is called second-order stationary if its **mean is constant and its acv.f. only depends on the lag**:

- $E(X_t) = \mu$ for all t
- $Cov(X_t, X_{t+k}) = \gamma(k)$ for all t, k
- Let $k = 0$, this implies $Var(X_t) = \gamma(0) = \sigma^2$ for all t

Second-order stationary is actually equivalent to strict stationarity for normal processes.

2.3 Some Useful Models

Purely Random Process

Random Walk

A Purely Random Process $\{Z_t\}$ consists of a sequence of i.i.d. random variables.
The i.i.d. assumption implies:
- mean zero and variance σ_Z^2 - The process is strictly stationary - $\gamma(k) = \rho(k) = 0$ for $k \neq 0$

White Noise Process - Serially Uncorrelated

- $\gamma(k) = \rho(k) = 0$ for $k \neq 0$
- Hence, **uncorrelated** but there **may be dependence**

Random Walk

Random Walk

$$X_t = X_{t-1} + Z_t \tag{15}$$

Moving Average Process

Moving Average Process

$$\boxed{X_t = \sum_{i=0}^{q} \theta_i Z_{t-i} \quad \text{where the } \theta_i \text{ are constants, } \theta_0 = 1} \tag{16}$$

- $E(X_t) = 0$
- $\mathrm{Var}(X_t) = \sigma_Z^2 \sum_{i=0}^{q} \theta_i^2$

$$\gamma(k) = \begin{cases} 0 & k > q \\ \sigma_Z^2 \sum_{i=0}^{q-k} \theta_i \theta_{i+k} & k = 0, 1, \ldots, q \\ \gamma(-k) & k < 0 \end{cases} \tag{17}$$

$$\rho(k) = \frac{\gamma(k)}{\gamma(0)}$$

$$\rho(k) = \begin{cases} 0 & k > q \\ 1 & k = 0 \\ \sum_{i=0}^{q-k} \theta_i \theta_{i+k} / \sum_{i=0}^{q} \theta_i^2 & k = 1, \ldots, q \\ \rho(-k) & k < 0 \end{cases} \tag{18}$$

Invertibility of a Moving Average Process

TODO

Autoregressive Processes

Autoregressive Processes

$$\boxed{X_t = \sum_{i=1}^{p} \alpha_i X_{t-i} + Z_t \quad \text{where the } \alpha_i \text{ are constants}} \tag{19}$$

AR(1) Process

From Duality we have:

$$X_t = \sum_{j=0}^{\infty} \alpha^j Z_{t-j} \tag{20}$$

$$\begin{aligned} E(X_t) &= 0 \\ \operatorname{Var}(X_t) &= \sigma_Z^2 \sum_{j=0}^{\infty} \alpha^{2j} \end{aligned} \tag{21}$$

If $|\alpha| < 1$, the variance is finite with:

$$\operatorname{Var}(X_t) = \sigma_X^2 = \frac{\sigma_Z^2}{1-\alpha^2} \tag{22}$$

acv.f $\gamma(\cdot)$

$$\begin{aligned} \boxed{\gamma(\mathbf{k})} &= E\left(X_t X_{t+k}\right) \\ &= E\left(\sum_{j=0}^{\infty} \alpha^j Z_{t-j} \sum_{j=0}^{\infty} \alpha^j Z_{t+k-j}\right) \\ &\overset{k \geq 0}{=} \sigma_Z^2 \sum_{j=0}^{\infty} \alpha^j \alpha^{j+k} \\ &= \alpha^k \sigma_Z^2 \sum_{j=0}^{\infty} \alpha^{2j} \\ &\overset{|\alpha|<1}{=} \alpha^k \frac{\sigma_Z^2}{1-\alpha^2} \\ &\boxed{= \alpha^{\mathbf{k}} \sigma_{\mathbf{X}}^{\mathbf{2}}} \end{aligned} \tag{23}$$

Assuming $|\alpha| < 1$:

- The process is second-order stationary
- If the $\{Z_t\}$ are strictly stationary, so is $\{X_t\}$
- If the $\{Z_t\}$ are i.i.d. normal, then $\{X_t\}$ is a strictly stationary normal process

ac.f $\rho(\cdot)$

$$\rho(k) = \alpha^{|k|} \text{ for all } k \tag{24}$$

AR(p) Process

- by definitiion, invertible
- The process is stationary if the roots of the equation

$$\phi(x) = 1 - \sum_{i=1}^{p} \alpha_i x^i = 0 \quad (, x \text{ potentially complex}) \tag{25}$$

lie outside the unit circle

ac.f. - Yule-Walker equations

$$\rho(k) = \sum_{i=1}^{p} \alpha_i \rho(k-i) \quad \text{for all } k > 0 \tag{26}$$

ac.f. - General Solution

$$\rho(k) = \sum_{i=1}^{p} A_i \pi_i^{|k|} \tag{27}$$

where the $\{\pi_i\}$ are the roots of the so-called auxiliary equation

$$y^p - \sum_{i=1}^{p} \alpha_i y^{p-i} = 0 \tag{28}$$

Alternative (but equivalent) condition for stationarity: The roots $\{\pi_i\}$ of the auxiliary equation all satisfy $|\{\pi_i\}| < 1$

Duality between AR and MA processes

- **AR(1) Process:**

$$X_t = \alpha_1 X_{t-1} + Z_t \tag{29}$$

- By successive substitution we find $MA(\infty)$:

$$X_t = \sum_{j=0}^{\infty} \alpha^j Z_{t-j} \tag{30}$$

- Duality by Backshift operator:

$$\begin{aligned} X_t &= \alpha B X_t + Z_t \\ \implies (1-\alpha B) X_t &= Z_t \\ \implies X_t &= Z_t / (1-\alpha B) \end{aligned} \tag{31}$$

Autoregressive Processes

$$X_t = \sum_{i=1}^{p} \alpha_i X_{t-i} + Z_t + \sum_{i=1}^{q} \theta_i Z_{t-i} \tag{32}$$

equivalently

$$\phi(B) X_t = \theta(B) Z_t \tag{33}$$

with

$$\phi(B) = 1 - \sum_{i=1}^{p} \alpha_i B^i \text{ and } \quad \theta(B) = 1 + \sum_{i=1}^{q} \theta_i B^i \tag{34}$$

Condition for Stationarity

AR part $\phi(B)$ is stationary

Condition for Invertibility

MA part $\theta(B)$ is invertible

3 Fitting Time Series Models

Autoregressive Processes