

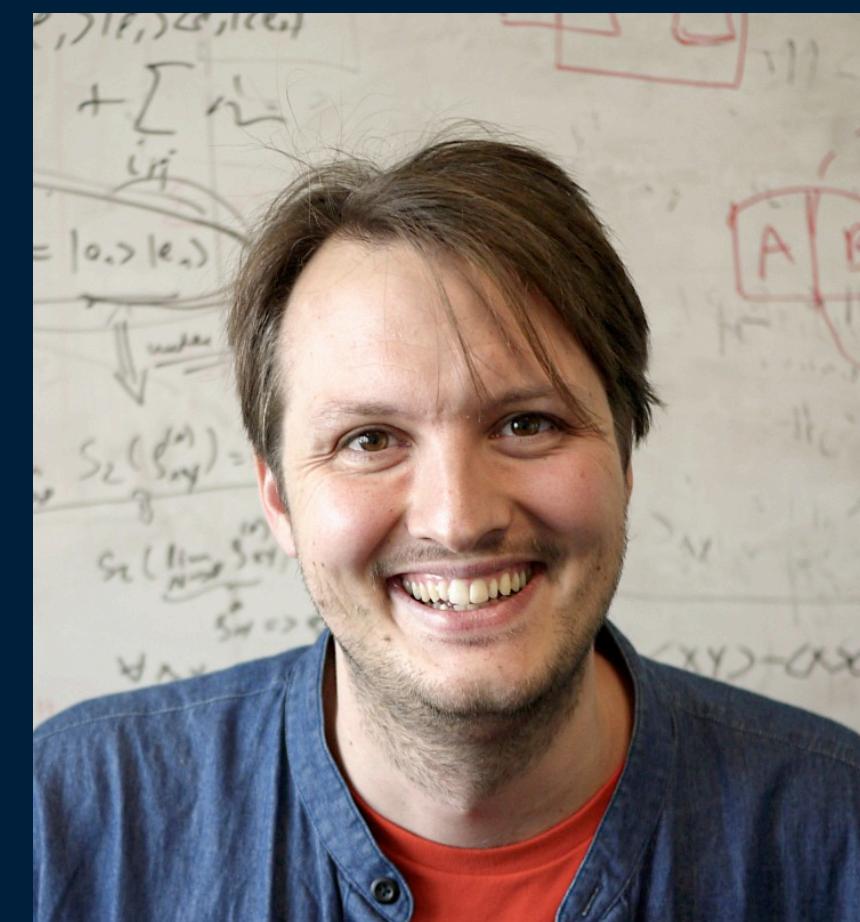
Catalysis in Quantum Information Theory

Nelly Ng, IPS 2023



arXiv:2306.00798

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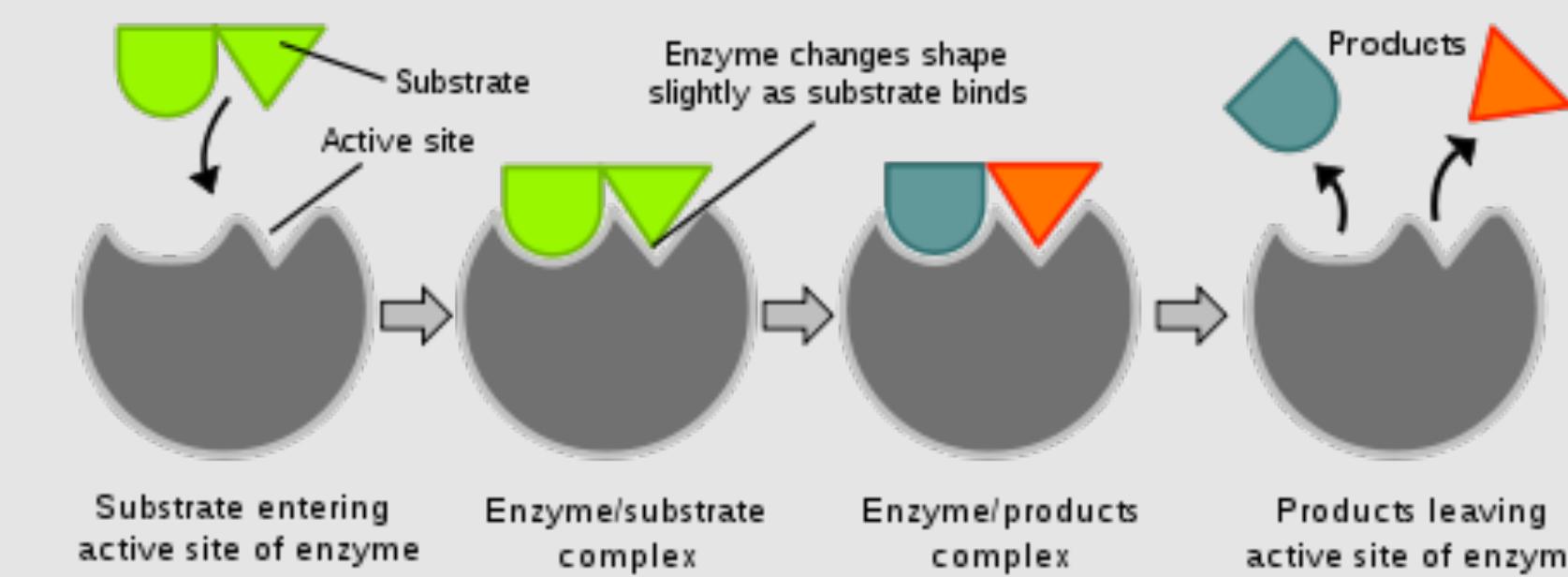
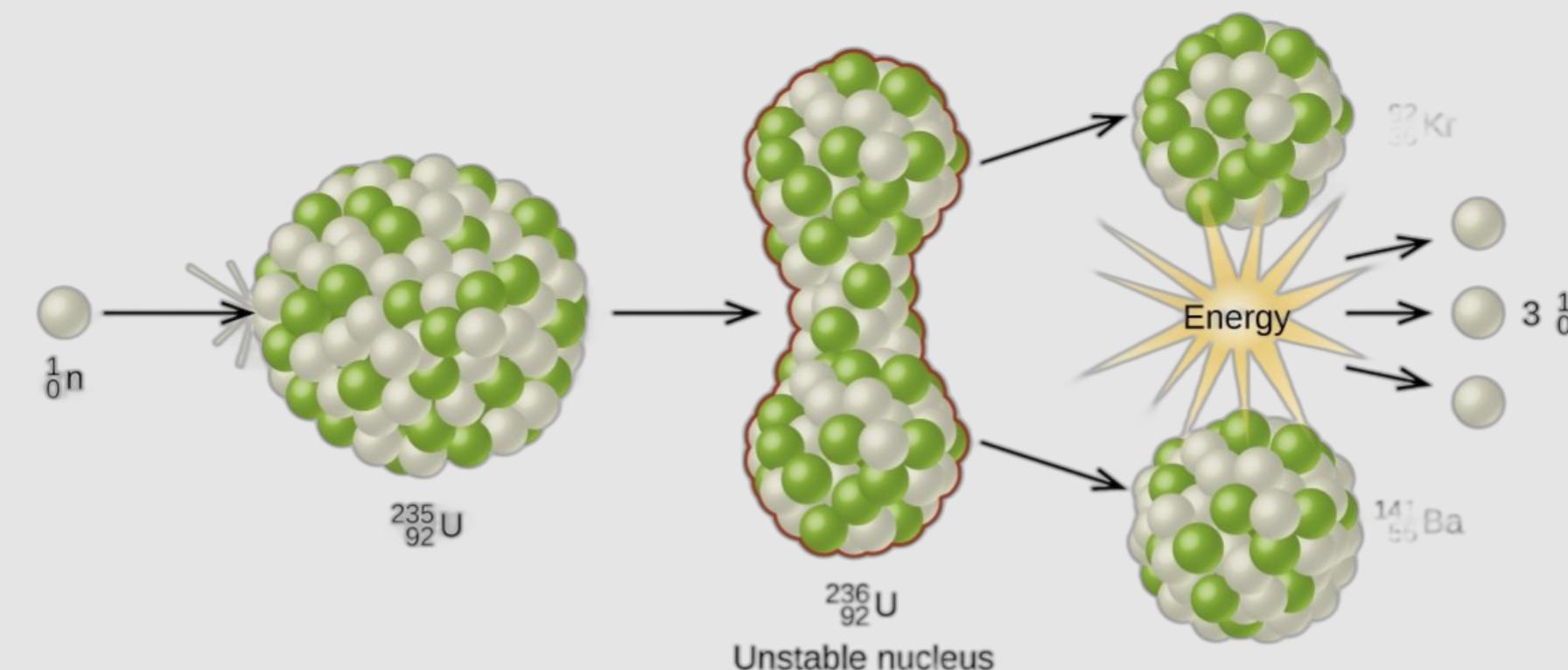
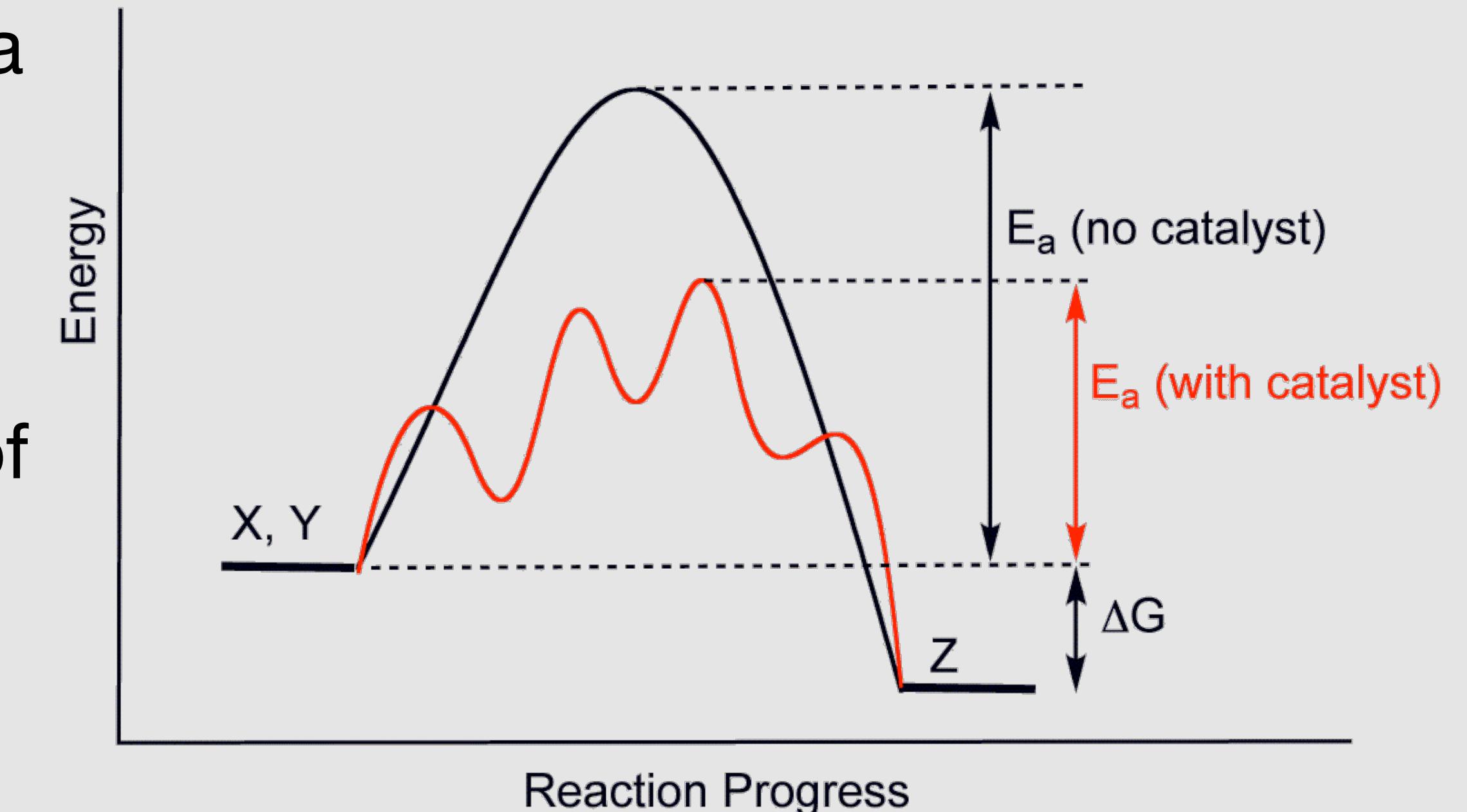
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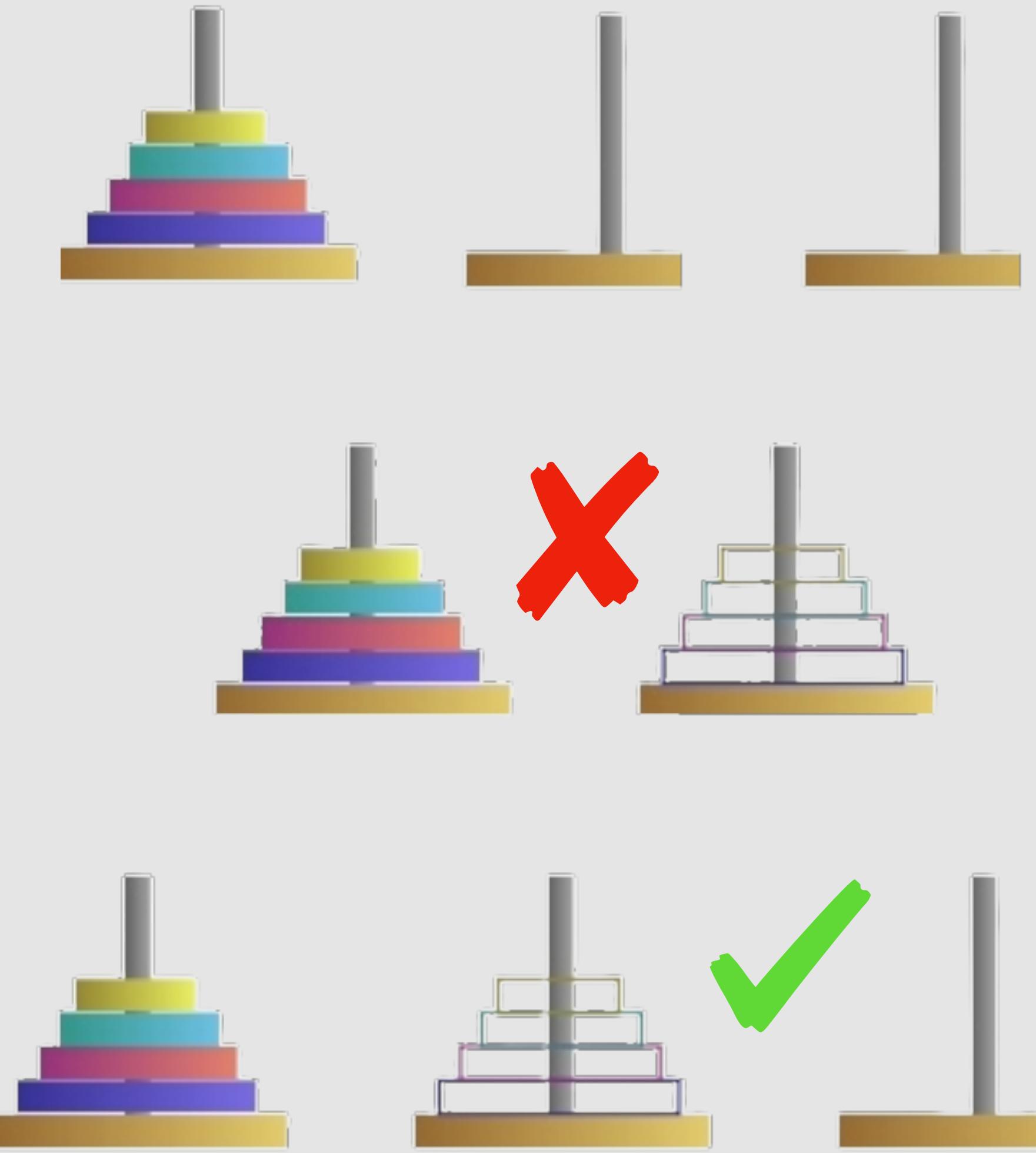
Catalysis in biological, chemical and physical processes

- ❖ Usage of additional substances to facilitate a process in such a way that
 - ❖ they remain intact during a process,
 - ❖ the reaction rates get a significant boost
- ❖ Mechanism: lowering the activation energy of a process, catalyst can be reused
- ❖ Chemical production, Enzyme biology, photosynthesis, nuclear processes etc



The tower of Hanoi as a catalytic puzzle

- ❖ k rods and n disks of different sizes
- ❖ Initially, disks arranged on a single rod, sorted by size, with the smallest disk on top.
- ❖ Goal: find the minimal number of moves needed to transfer the entire stack of disks from one rod to another, given that disks can only be placed on top of larger disks.
- ❖ Minimally, need $k = 3$;
- ❖ number of optimal moves required is known to be $2^n - 1$
- ❖ Number of optimal moves in general n, k is an open problem



Catalysis in quantum information processing

- ❖ Goal: create entanglement between Atom A & B
- ❖ Tools: local preparations of the atoms + energy-preserving interactions with a resonant cavity F

❖ Protocol:

- ❖ Prepare $| \uparrow \rangle_A | 0 \rangle_F | \downarrow \rangle_B$
- ❖ Jaynes-Cummings interaction for suitable durations,

$$U_{FA} | 0 \rangle_F | \uparrow \rangle_A = \frac{1}{\sqrt{2}} (| 0 \rangle_F | \uparrow \rangle_A - | 1 \rangle_F | \downarrow \rangle_A)$$

$$U_{FB} | 1 \rangle_F | \downarrow \rangle_B = | 0 \rangle_F | \uparrow \rangle_B$$

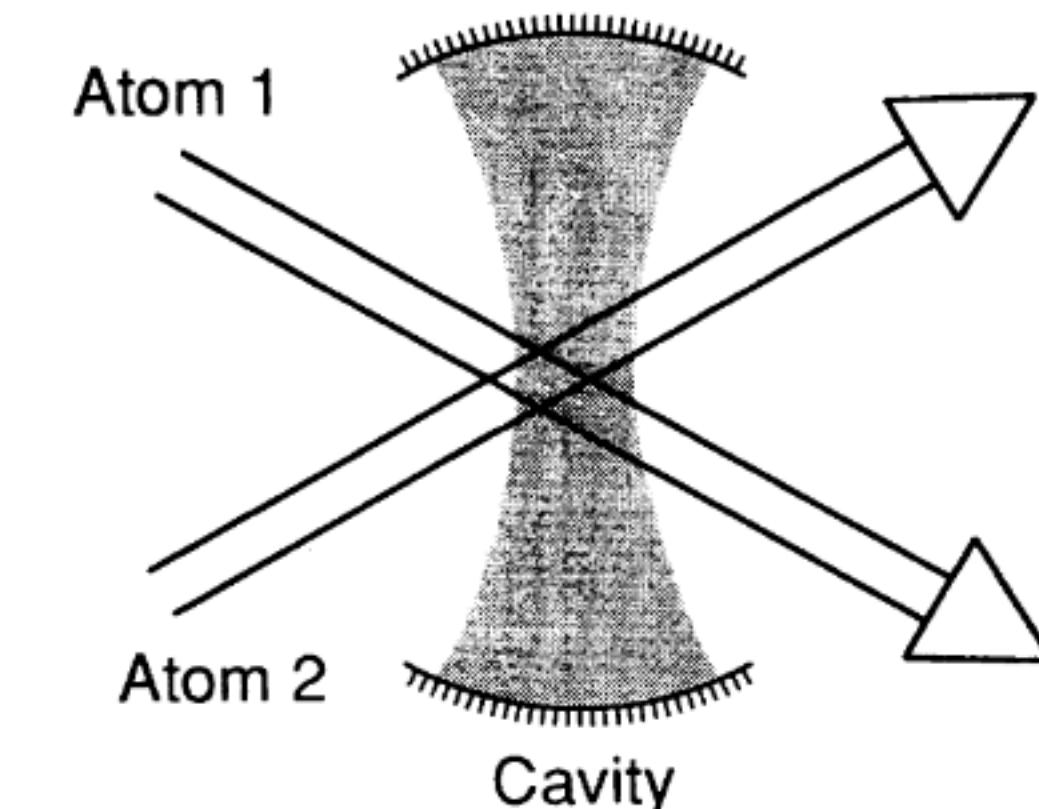


FIG. 1. Proposed configuration for the preparation of the entangled state (singlet state) of two two-level atoms. Initially the cavity is in the vacuum state. Atom 1 enters the cavity in its excited state and atom 2 in its ground state. After crossing the cavity, the atoms become entangled and the cavity is left in its original state: the vacuum.

J. I. Cirac and P. Zoller, PRA 50, R2799(R). 1994

$$| \uparrow \rangle_A | 0 \rangle_F | \downarrow \rangle_B \rightarrow | 0 \rangle_F \otimes \frac{| \downarrow \uparrow \rangle_{AB} + | \uparrow \downarrow \rangle_{AB}}{\sqrt{2}}$$

Also other states, e.g. GHZ

Catalysis in entanglement LOCC

Entanglement-Assisted Local Manipulation of Pure Quantum States

Daniel Jonathan and Martin B. Plenio

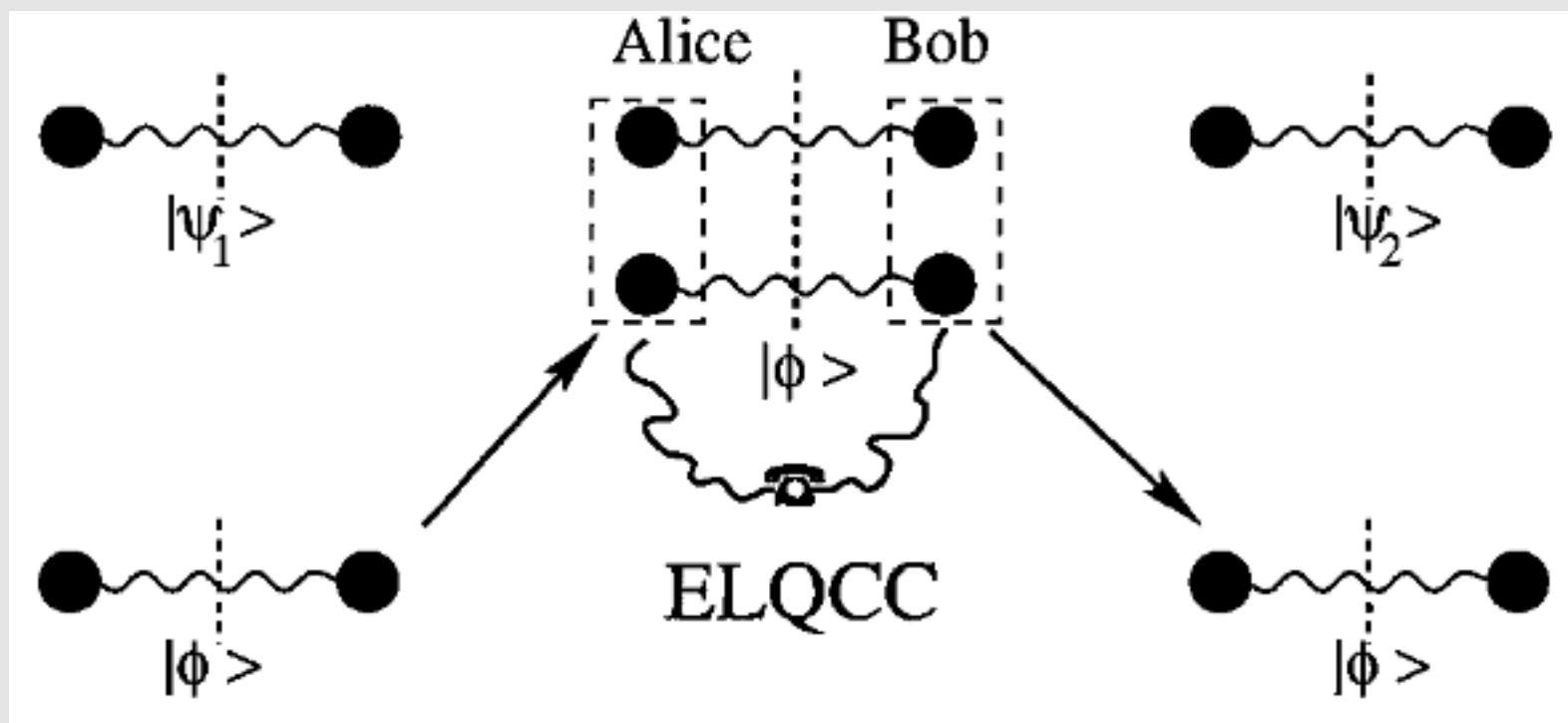
Blackett Laboratory, Imperial College, London SW7 2BZ, United Kingdom

(Received 24 May 1999)

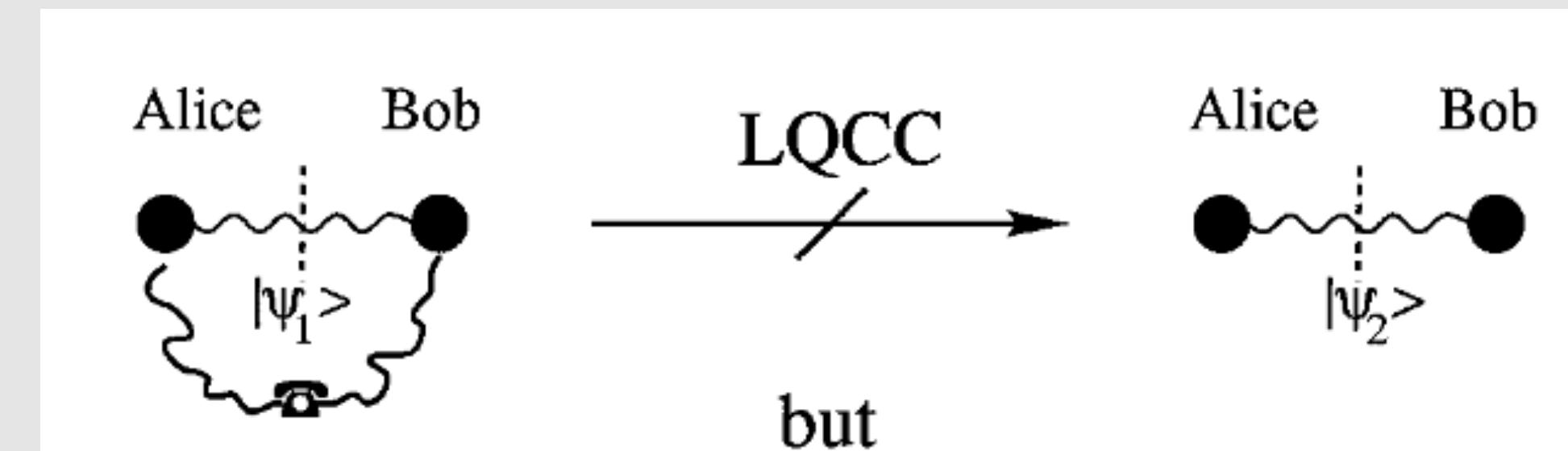
We demonstrate that local transformations on a composite quantum system can be enhanced in the presence of certain entangled states. These extra states act much like *catalysts* in a chemical reaction: they allow otherwise impossible local transformations to be realized, without being consumed in any way. In particular, we show that this effect can considerably improve the efficiency of entanglement concentration procedures for finite states.

If we compare the entanglement entropy, $|\psi_1\rangle$ might have more entanglement than $|\psi_2\rangle$, but still LOCC doesn't enable the transition.

- Reverse is also not possible — impasse.
- Alternatives: taking more copies of ψ_1 ...



Alice and Bob share an entangled state $|\psi_1\rangle$. They would like to create a target state $|\psi_2\rangle$... let's assume that they can perform any local quantum operations, and also communication freely (classical).

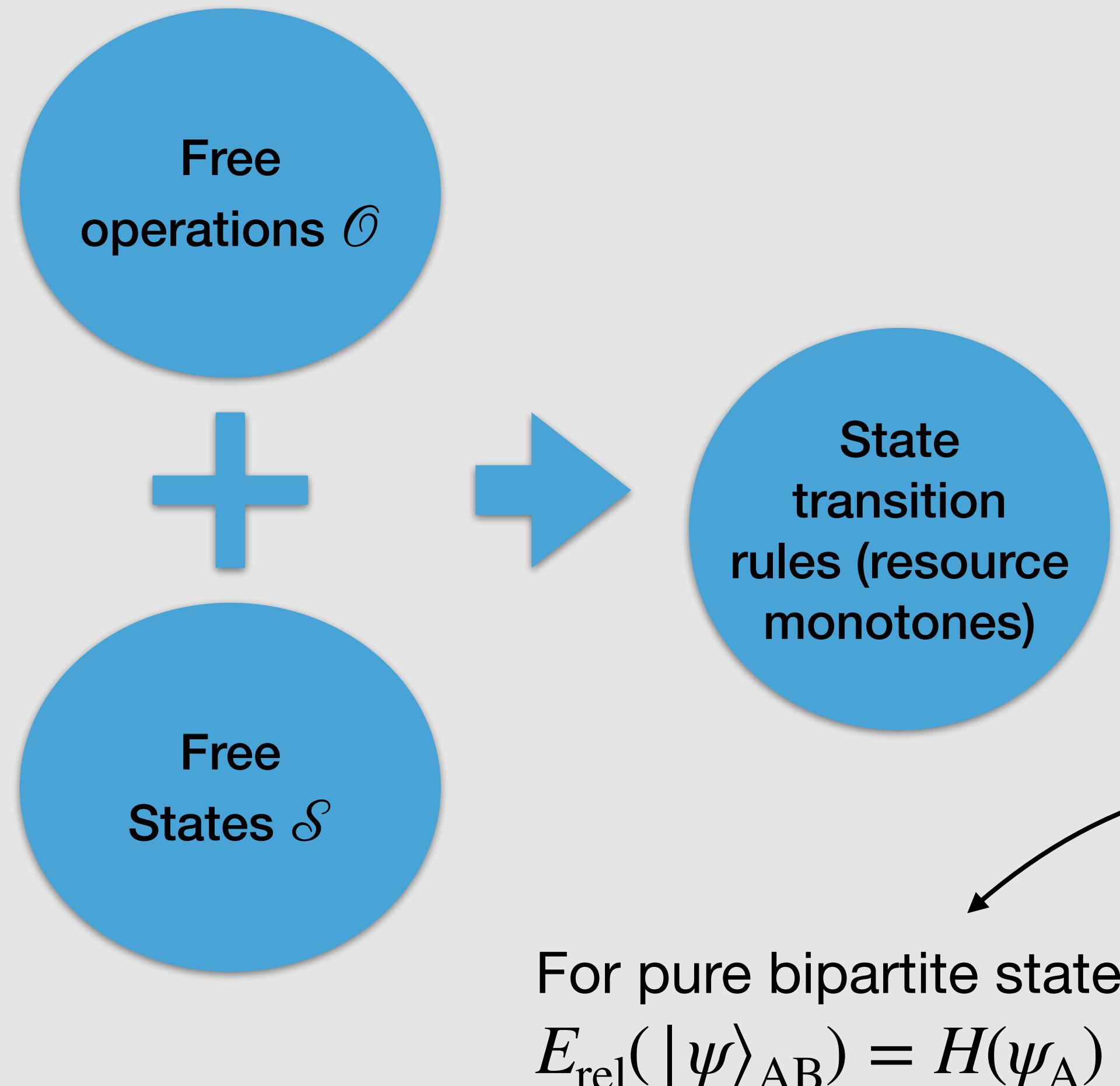


However, sometimes Alice and Bob can actually do it if they are given access to an additional bipartite entangled state $|\phi\rangle$! After the process, $|\phi\rangle$ is not consumed — and therefore can be used repeatedly.

Questions:

- ❖ **how can we decide if a desired process is catalytically feasible?**
 - ❖ The reason for being able to uncover catalysis in LOCC easily, was because we could **mathematically identify complete state transition rules** for pure bipartite quantum states — majorization [Nielsen, PRL 83, 436-439 (1999)]
 - ❖ Systematically expanding the mathematical structure of majorization gives us a tool to investigate catalysis (shift focus from particular protocol designs to state trans. rules)
- ❖ **do we find a useful catalyst (and the corresponding process)?**
- ❖ **what are the properties of catalyst states that make them useful?**
- ❖ **can we find lower/upper bounds to the required dimensionality of the catalyst?**
- ❖

Resource theories



Ex:

$\mathcal{O}_{\text{LOCC}}$: any protocol composed of

- i) preparing local quantum states, applying local unitaries and measurements,
- ii) exchanging classical messages,
- iii) discarding physical subsystems.

$\mathcal{S}_{\text{LOCC}}$: any state that can be prepared via LOCC, i.e.
all separable states

State transition rules: **LOCC monotones**

Ex: relative entropy of entanglement

$$E_{\text{rel.}}(\rho_{AB}) = \inf_{\gamma_{AB} \in \mathcal{S}_{\text{LOCC}}(AB)} D(\rho_{AB} \parallel \gamma_{AB})$$

$$\rho_{AB} \rightarrow \rho'_{AB} \Rightarrow E_{\text{rel.}}(\rho_{AB}) \geq E_{\text{rel.}}(\rho'_{AB})$$

For pure bipartite states,
 $E_{\text{rel.}}(|\psi\rangle_{AB}) = H(\psi_A)$

Gaussianity

\mathcal{O} : Gaussian unitaries (*or subsets thereof*)

\mathcal{S} : Gaussian states

Generic quantum information

\mathcal{O} : set of all unital maps (*or subsets thereof*)
 \mathcal{S} : maximally mixed states

Thermodynamics

\mathcal{O} : Gibbs-state preserving operations (*or subsets thereof*)

\mathcal{S} : thermal Gibbs states

Others:

non-Markovianity, Complexity, etc

Simulation of quantum computers

\mathcal{O} : Clifford gates + measurement feed forward (*or subsets thereof*)
 \mathcal{S} : stabiliser states

The essential resource-theoretic question

Given a fixed set of free channels \mathcal{O} , and free states \mathcal{S} , when is $\rho \xrightarrow{\mathcal{O}} \rho'$ possible?

The essential catalytic question

Given a fixed set of free channels \mathcal{O} , and free states \mathcal{S} , and access to catalysts, when is $\rho \xrightarrow{\mathcal{O}} \rho'$ possible? (State transition conditions will be relaxed)

The basic lemma of quantum catalysis

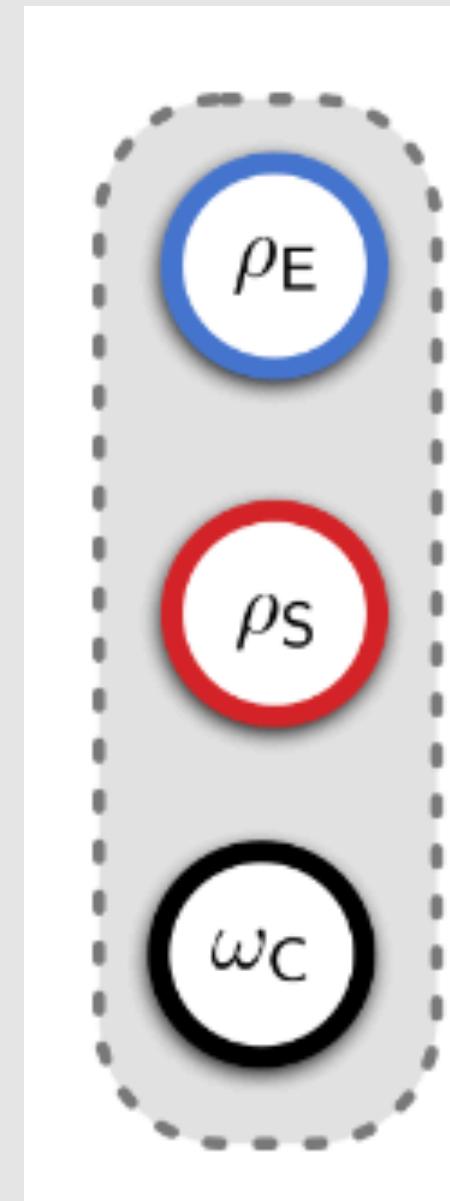
Let U be a unitary on the Hilbert space of \mathcal{H}_{SEC} , and consider the initial quantum state $\rho_S \otimes \rho_E \otimes \omega_C$.

If $U(\rho_S \otimes \rho_E \otimes \omega_C) U^\dagger = \sigma_{\text{SE}} \otimes \omega_C$, for some σ_{SE} ,

then there exists

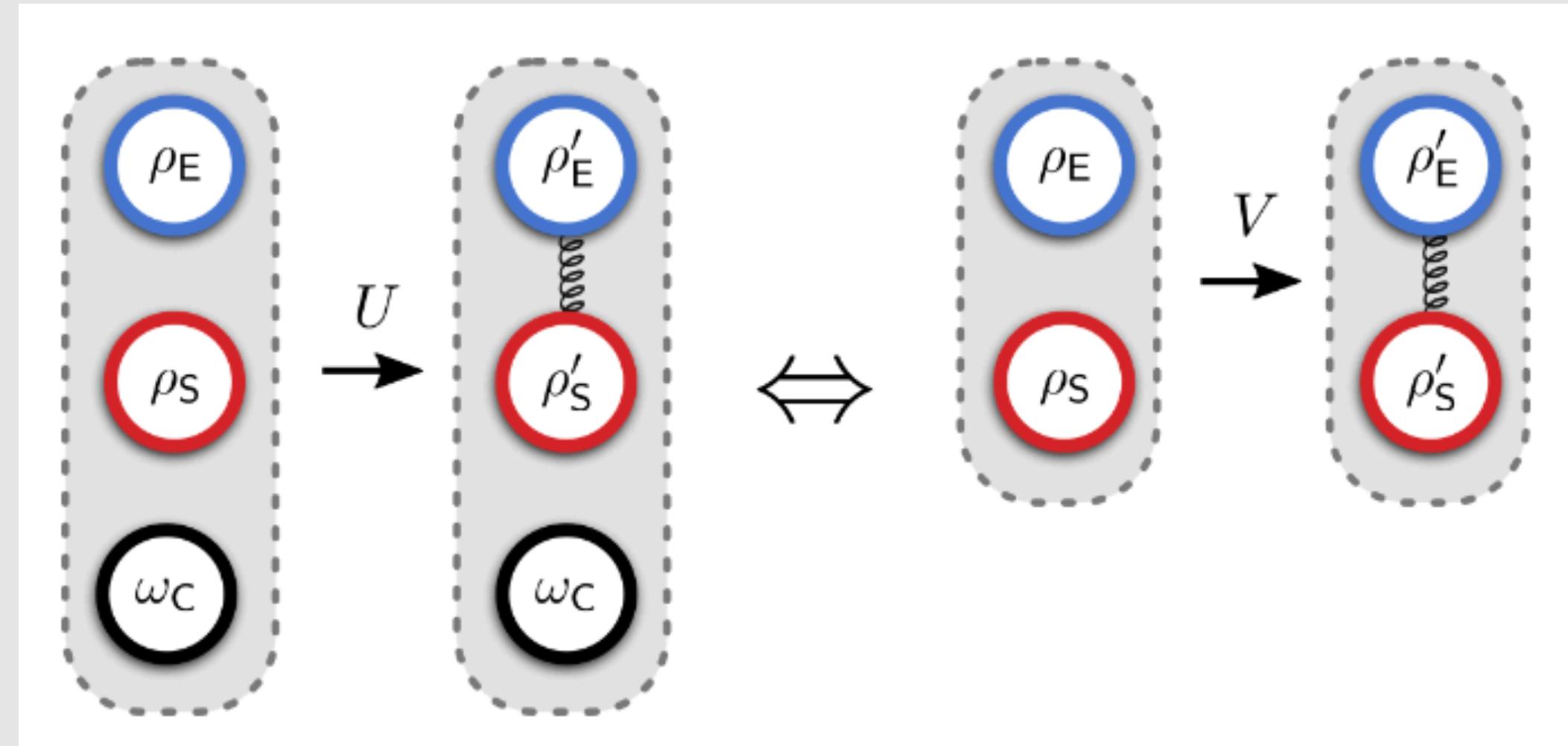
a unitary operator V_{SE} such

that $\sigma_{\text{SE}} = V\rho_{\text{SE}}V^\dagger$.



Implications of the basic lemma

Situations where a catalyst is useful



(1) **Catalysis is not exact**,
 $\text{Tr}_{\text{SE}} [\mathcal{U} [\rho_{\text{SE}} \otimes \omega_{\text{C}}]] \neq \omega_{\text{C}}$.

Therefore the state on C changes
(at least a bit).

(2) The final catalyst C is **correlated with SE**, i.e.

$\mathcal{U} [\rho_{\text{S}} \otimes \rho_{\text{E}} \otimes \omega_{\text{C}}] = \sigma_{\text{SEC}} \neq \sigma_{\text{SE}} \otimes \omega_{\text{C}}$,
although $\text{Tr}_{\text{SE}} [\sigma_{\text{SEC}}] = \omega_{\text{C}}$.

(3) The set of **implementable operations** \mathcal{O} **is restricted** (either by practical/experimental reasons, or fundamental restrictions e.g. conservation laws). While the unitary V exists, the channel it induces on S is not in \mathcal{O} .

This was why the earlier LOCC example works!

An overview of catalytic types

Possible :: \mathcal{O} is sufficiently
(but not too) restricted

Strict catalysis

Arbitrarily strict
catalysis

Possible :: correlations created
between catalyst and other systems

**Correlated
catalysis**

Marginally
correlated catalysis

Possible :: catalysis is not exact

**Approximate
catalysis**

Embezzlement

**Relaxation of state transition
conditions is qualitatively
different across types!**

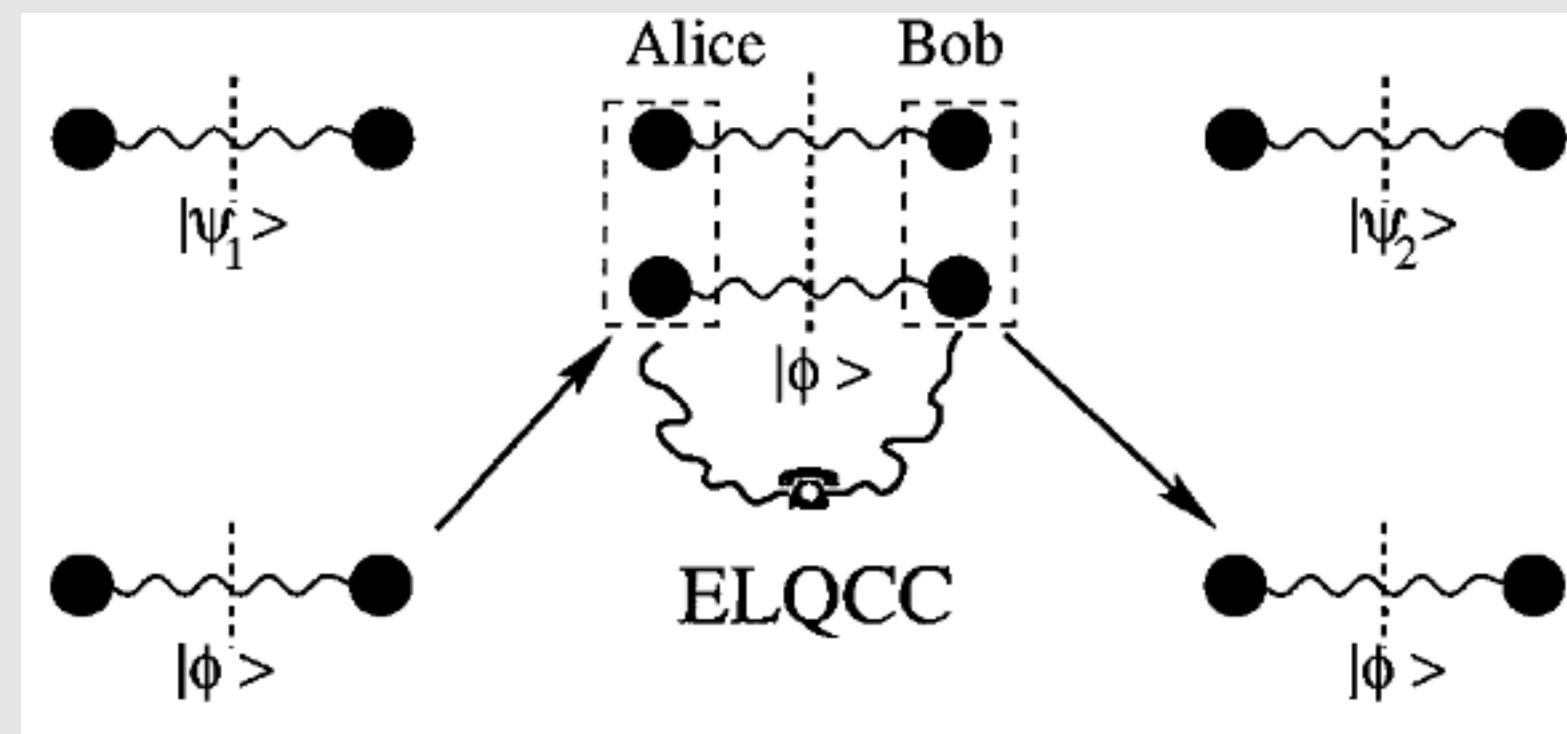
**Infinite-
dimensional
catalysis**

**State-
independent
catalysis**

Example of difference between catalytic types

Strict catalysis: $|\psi_1\rangle_{AB} \otimes |\phi\rangle_{C_A C_B} \xrightarrow{\text{LOCC}} |\psi_2\rangle_{AB} \otimes |\phi\rangle_{C_A C_B}$

Strict catalysis



Entanglement entropy insufficient to guarantee LOCC transition!

What other monotones remain when strict catalysis is allowed?

The above is possible iff $H_\alpha(\psi_{1_A}) \geq H_\alpha(\psi_{2_A}), \alpha \in \mathbb{R}$!

Any intuition behind this?

Renyi entanglement entropies

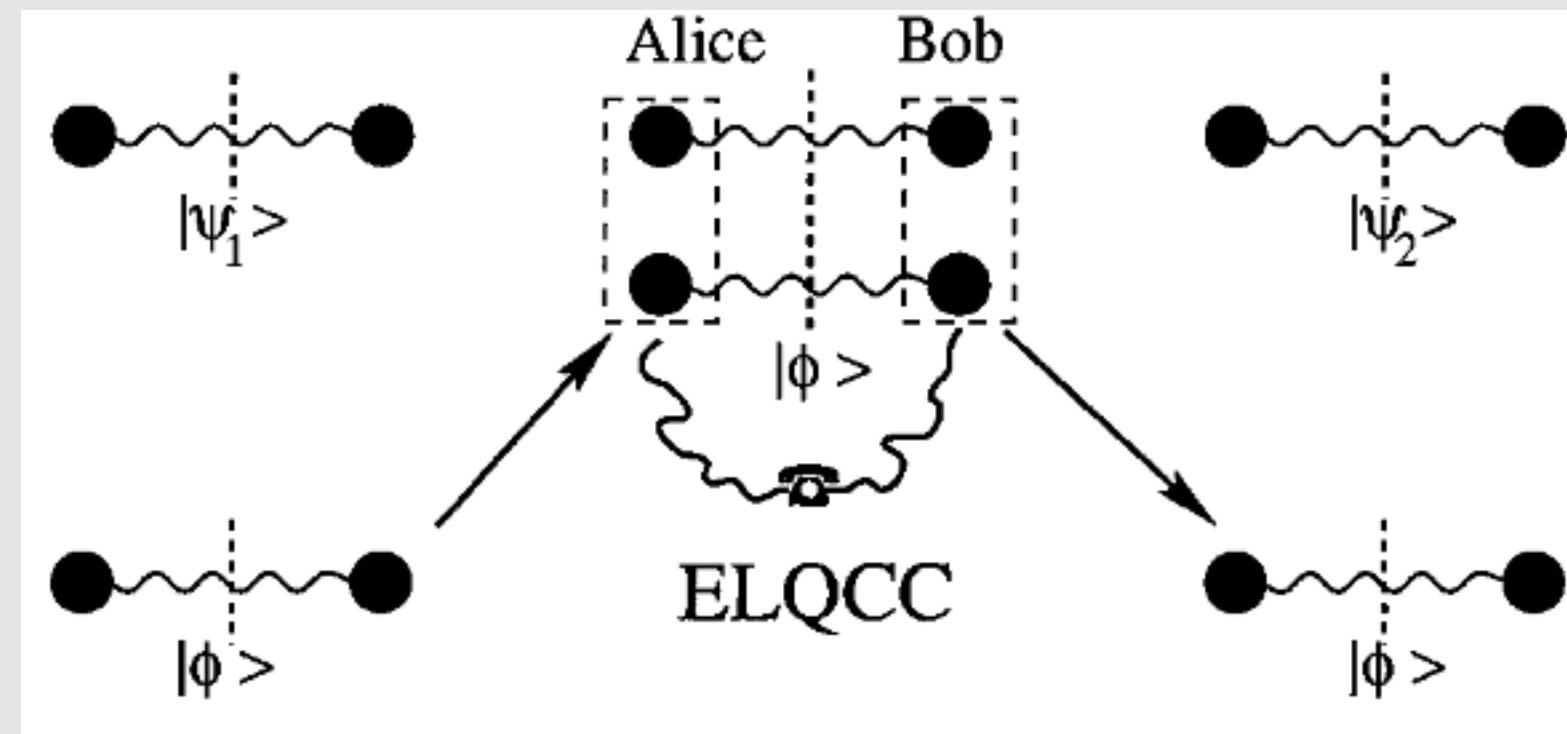
- additive under tensor product
- $\lim_{\alpha \rightarrow 1} H_\alpha(\rho) = H(\rho)$, i.e. the entanglement entropy is a special instance



Example of difference between catalytic types

Strict catalysis: $|\psi_1\rangle_{AB} \otimes |\phi\rangle_{C_A C_B} \xrightarrow{\text{LOCC}} |\psi_2\rangle_{AB} \otimes |\phi\rangle_{C_A C_B}$

Strict catalysis



Entanglement entropy insufficient to guarantee LOCC transition!

Monotones for s.c. are additive under tensor product

What other monotones remain when strict catalysis is allowed?

- 1) f is an **LOCC monotone**, i.e. $\rho_{AB} \xrightarrow{\text{LOCC}} \rho'_{AB}$ implies $f(\rho_{AB}) \geq f(\rho'_{AB})$, and
- 2) f is **additive under tensor product**, i.e. $f(\rho_1 \otimes \rho_2) = f(\rho_1) + f(\rho_2)$,

Then if $\rho \xrightarrow{\text{s.c.-LOCC}} \rho'$, this means that $f(\psi_{1A} \otimes \phi_A) \geq f(\psi_{2A} \otimes \phi_A)$, meaning that f remains a monotone for LOCC even when strict catalysis is allowed!

Example of difference between catalytic types

PHYSICAL REVIEW A 67, 060302(R) (2003)

Universal entanglement transformations without communication

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(Received 10 January 2002; published 11 June 2003)

Embezzlement

No monotones :O

$$\text{Embezzling: } |\psi_1\rangle_{AB} \otimes |\phi\rangle_{C_A C_B} \xrightarrow{\text{LOCC}} |\psi_2\rangle_{AB} \otimes |\phi'\rangle_{C_A C_B}, \langle\phi|\phi'\rangle \geq 1 - \varepsilon$$

What monotones remain when embezzling is allowed?

If errors are allowed on the catalyst, then it is possible to construct a family of catalyst states $\{|\mu_n\rangle\}_n$, such that for any $\varepsilon > 0$ and any bipartite state $|\varphi_{AB}\rangle$,

$$|\mu(n)\rangle \otimes |00_{AB}\rangle \mapsto |\mu(n)\rangle \otimes |\varphi_{AB}\rangle$$

can be accomplished with fidelity better than $1 - \varepsilon$, for all sufficiently large n **without any communication (LO)**.

How do we make sense of embezzling?

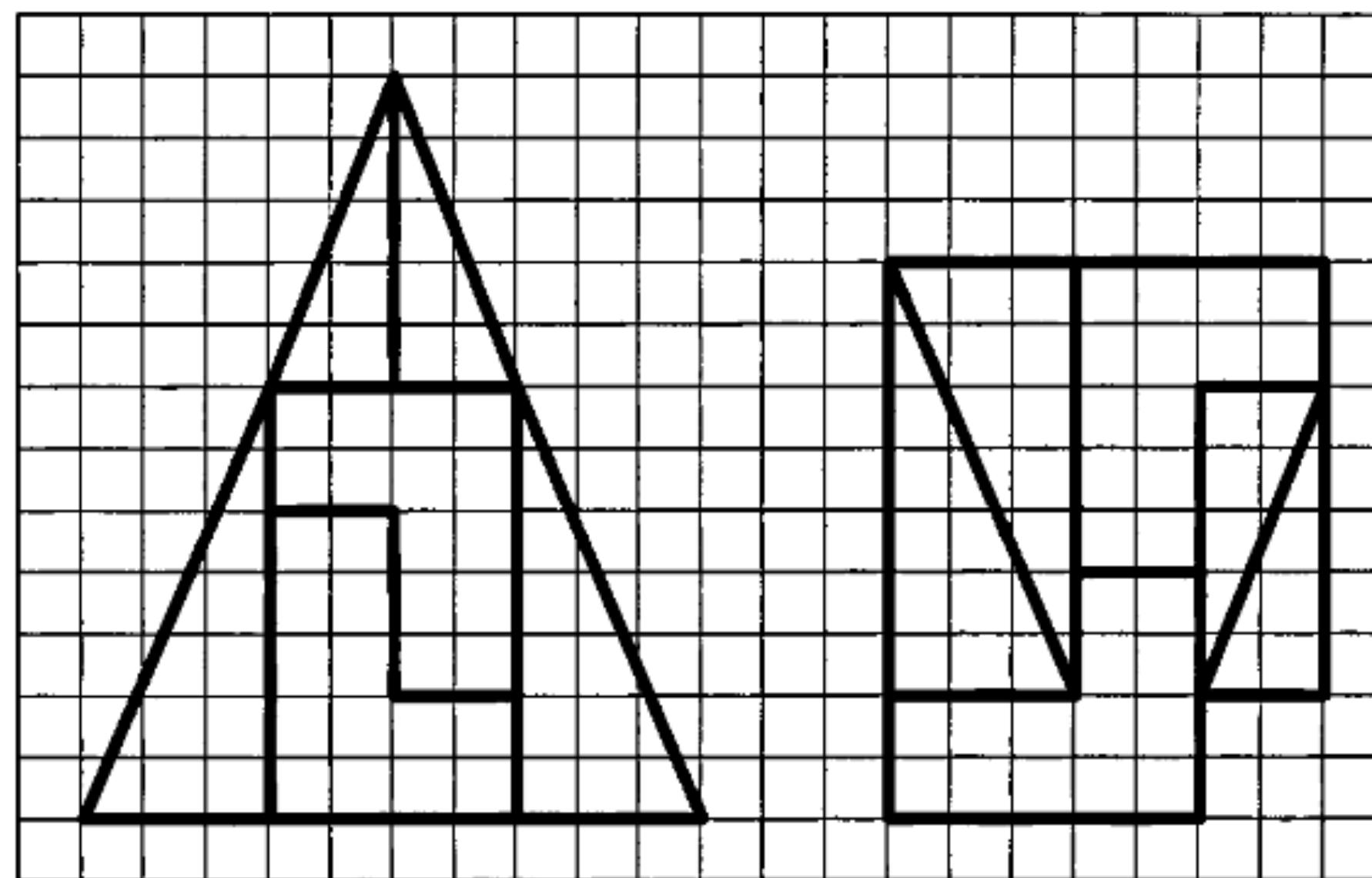


FIG. 1. An illustration of the “embezzlement effect.” By a well-chosen rearrangement, we can create the suggestion that the six pieces of the rightmost figure, with area size 59, can also be used to cover the triangle on the left with its surface of 60 units. A similar phenomenon is described in this paper for the entanglement of a distributed quantum state. It is shown, how we can reorder the amplitudes of an embezzling state μ such that we get a very close approximation of an enlarged state $\mu \otimes \varphi$, which appears to have significantly more entanglement than the original μ .

W. van Dam, P. Hayden, PRA 67, 060302 (2003)

Embezzlement

results from the fact that fidelity is not sensitive enough to capture the difference in **resource content** between two quantum states, especially when the states are high dimensional.

Embezzling effects can be bounded:

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PAPER

Limits to catalysis in quantum thermodynamics

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Example of difference between catalytic types

PHYSICAL REVIEW LETTERS 127, 150503 (2021)

Catalytic Transformations of Pure Entangled States

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(Received 23 April 2021; accepted 9 September 2021; published 5 October 2021)

Correlating
catalysis

Correlated catalysis:

$$|\psi_1\rangle_{AB} \otimes \omega_{C_A C_B} \xrightarrow{\text{LOCC}} \rho_{ABC_A C_B},$$

1. $\rho_{ABC_A C_B} \approx_{\varepsilon} |\psi_2\rangle_{AB} \otimes \omega_{C_A C_B}$
2. $\rho_{C_A C_B} = \omega_{C_A C_B}$

What other monotones remain when correlating catalysis is allowed?

The above is possible iff $H(\psi_{1_A}) \geq H(\psi_{2_A})!$

Just one monotone! :O

T&C: high-dimensional catalyst required, i.e. $\dim(C) \rightarrow \infty$ when $\varepsilon \rightarrow 0$.

Example of difference between catalytic types

$$|\psi_1\rangle_{AB} \otimes \omega_{C_A C_B} \xrightarrow{\text{LOCC}} \rho_{ABC_A C_B}, \quad \begin{aligned} 1. \quad & \rho_{ABC_A C_B} \approx_{\epsilon} |\psi_2\rangle_{AB} \otimes \omega_{C_A C_B} \\ 2. \quad & \rho_{C_A C_B} = \omega_{C_A C_B} \end{aligned}$$

Correlating
catalysis

What monotones remain when correlating catalysis is allowed?

If:

- 1) f is an LOCC monotone, i.e. $\rho \xrightarrow{\text{LOCC}} \rho'$ implies $f(\rho) \geq f(\rho')$, and
- 2) f is additive under tensor product, i.e. $f(\rho_1 \otimes \rho_2) = f(\rho_1) + f(\rho_2)$,
- 3) f is super-additive, i.e. $f(\rho'_{12}) \geq f(\rho'_1) + f(\rho'_2)$,

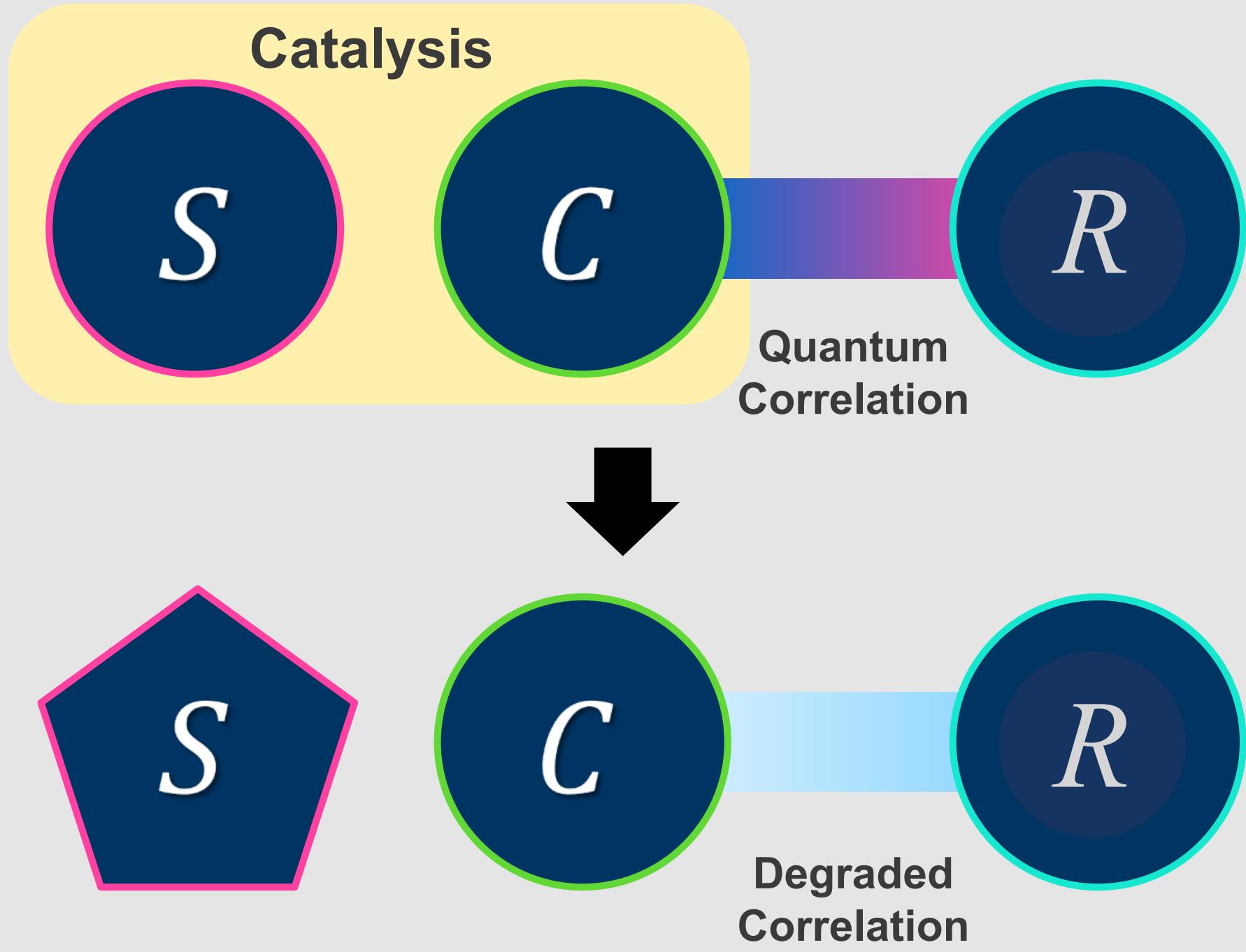
Then if $\rho \xrightarrow{\text{corr.-LOCC}} \rho'$, this means that $f(|\psi_1\rangle_{AB} \otimes \omega_{C_A C_B}) \geq f(\rho_{ABC_A C_B})$.

But that in turn means that $f(|\psi_1\rangle_{AB}) + f(\omega_{C_A C_B}) \gtrsim f(|\psi_2\rangle_{AB}) + f(\omega_{C_A C_B})$

f remains a monotone for LOCC even when correlating catalysis is allowed!

A generic remark on degradation of quantum correlations

- ❖ What if catalyst is correlated with an external system R ?
 - ❖ It turns out non-trivial (state-independent) catalysis cannot be done without the cost of degrading existing quantum correlations
 - ❖ Catalysts look locally unaltered, but a reference system that has some “quantum memory” of catalyst can always detect the difference
 - ❖ Existing classical correlations, however, can in general persist



Well, at least I now know that I should identify which catalytic type I want, and check certain monotones to see if a state transition is possible. But how do I identify the suitable catalyst state, or determine the actual transformation?



You raise a fair point. In general, this is not easy....



But there are a handful of useful tips!

Generic constructions of catalyst states

Multi-copy transformations to strict catalysis

Suppose that:

- 1) you can freely control on classical randomness, e.g. whenever we have a set of allowed/free operations $\{\mathcal{F}_i\}_i$, then

$$\rho_S \otimes \omega_A \mapsto \sum_i \mathcal{F}_i [\rho_S] \otimes |i\rangle\langle i| \omega_A |i\rangle\langle i|$$

Catalyst dimension is gonna be large, though...

is also easily implementable, and

- 2) we know that for sufficiently large n , $\rho_S^{\otimes n} \xrightarrow{\mathcal{O}} \sigma_S^{\otimes n}$



Then, we have a recipe to construct ω_C such that $\rho_S \otimes \omega_C \xrightarrow{\mathcal{O}} \sigma_S \otimes \omega_C$
(strict catalysis)

Generic constructions of catalyst states

Finding a catalyst when input state ρ_S and channel \mathcal{E}_{SC} is fixed

The effective channel acting on C can be written as $\mathcal{E}_C(\cdot) := \text{Tr}_S \mathcal{E} [\rho_S \otimes (\cdot)_C]$

Finding a correlated-catalytic transformation means finding a density operator ω_C such that $\mathcal{E}_C(\omega_C) = \omega_C$. This is equivalent to finding fixed points of the map \mathcal{E}_C — which exists (due to Brouwer's fixed point theorem)!

SDP formulation:

(Can be generalized
for e.g. approximate
catalysis)

A feasible solution
 X would be a catalyst

$$\begin{aligned} & \min_{X} && 0 \\ & \text{subject to } && \mathcal{E}_C[X] = X \\ & && X \geq 0, \text{Tr } X = 1. \end{aligned}$$

More in
review...

Snapshots of catalysis for thermodynamics

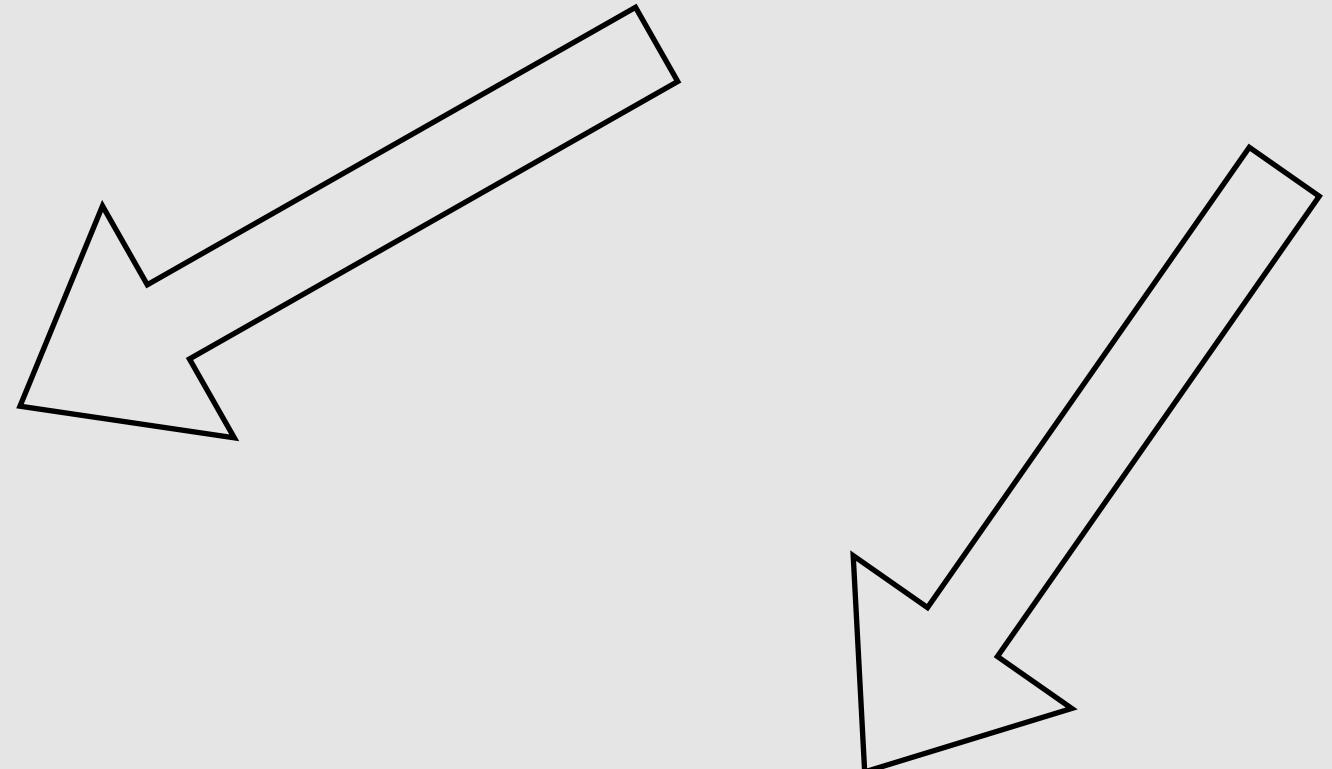
Catalysis in thermal operations

\mathcal{O}_{TO} : any channel that can be written as

- i) preparing a thermal state of a fixed inv. temperature β ,
- ii) applying a global energy-preserving unitary,
- iii) discarding physical subsystems

\mathcal{S}_{TO} : Thermal states of inv. temperature β

**Limits to cooling,
formulation of the
third law for
quantum thermo**



**Efficiency of small
heat engines**

For the case of energy-incoherent target states:

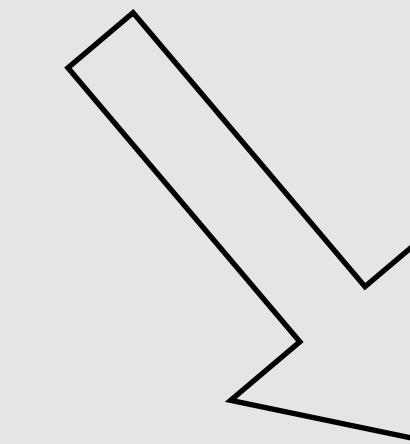


Full set of monotones for strict, arb.
strict, correlating catalysis

For the case of energy-coherent target states:

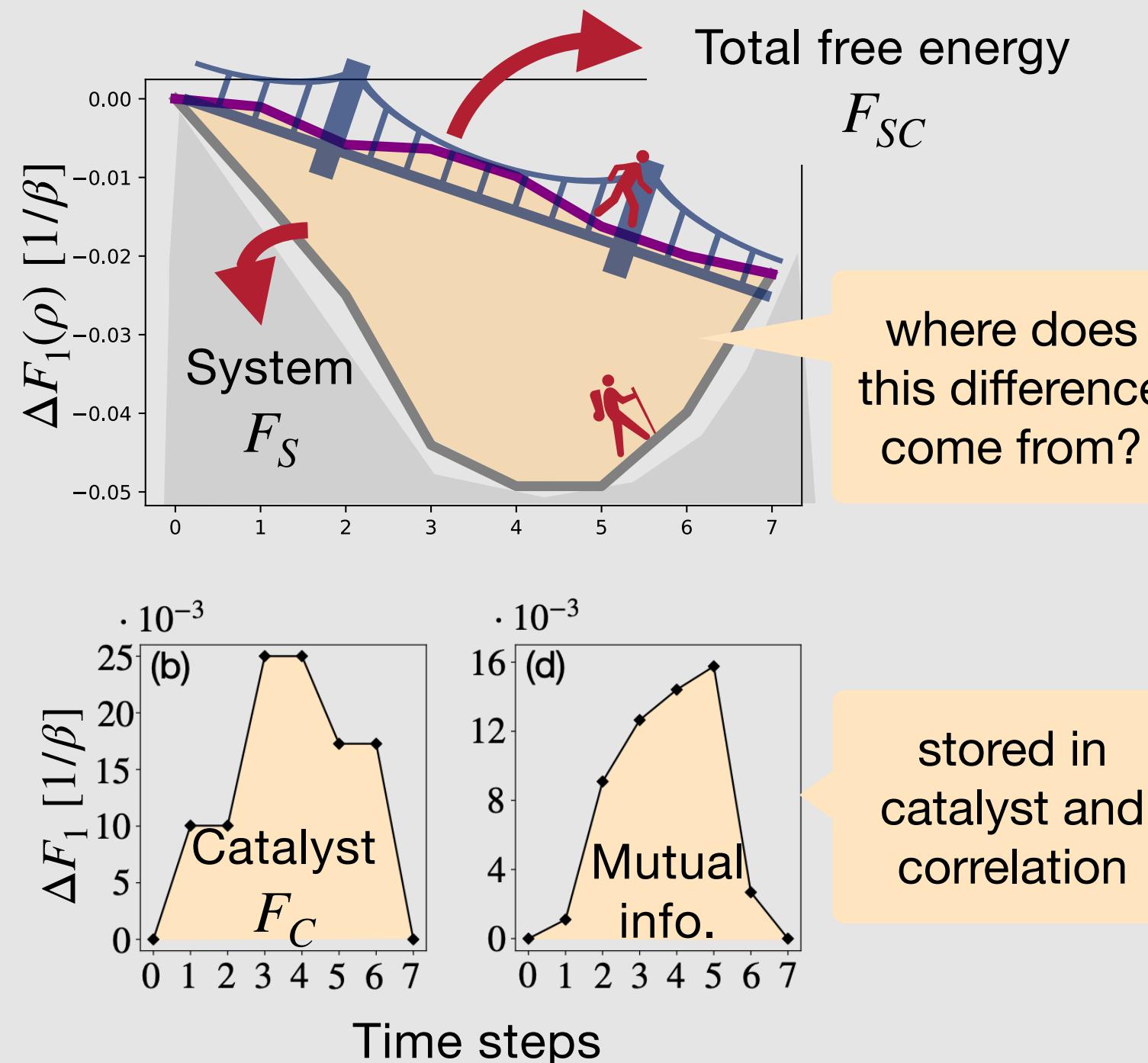


Partial set of monotones



**Usage of correlations
in bypassing
Jarzynski equality**

Catalysis in thermodynamics: non-Markovian boost



Elementary Thermal Operations (ETO):
combinations & concatenations of 2-lvl TOs

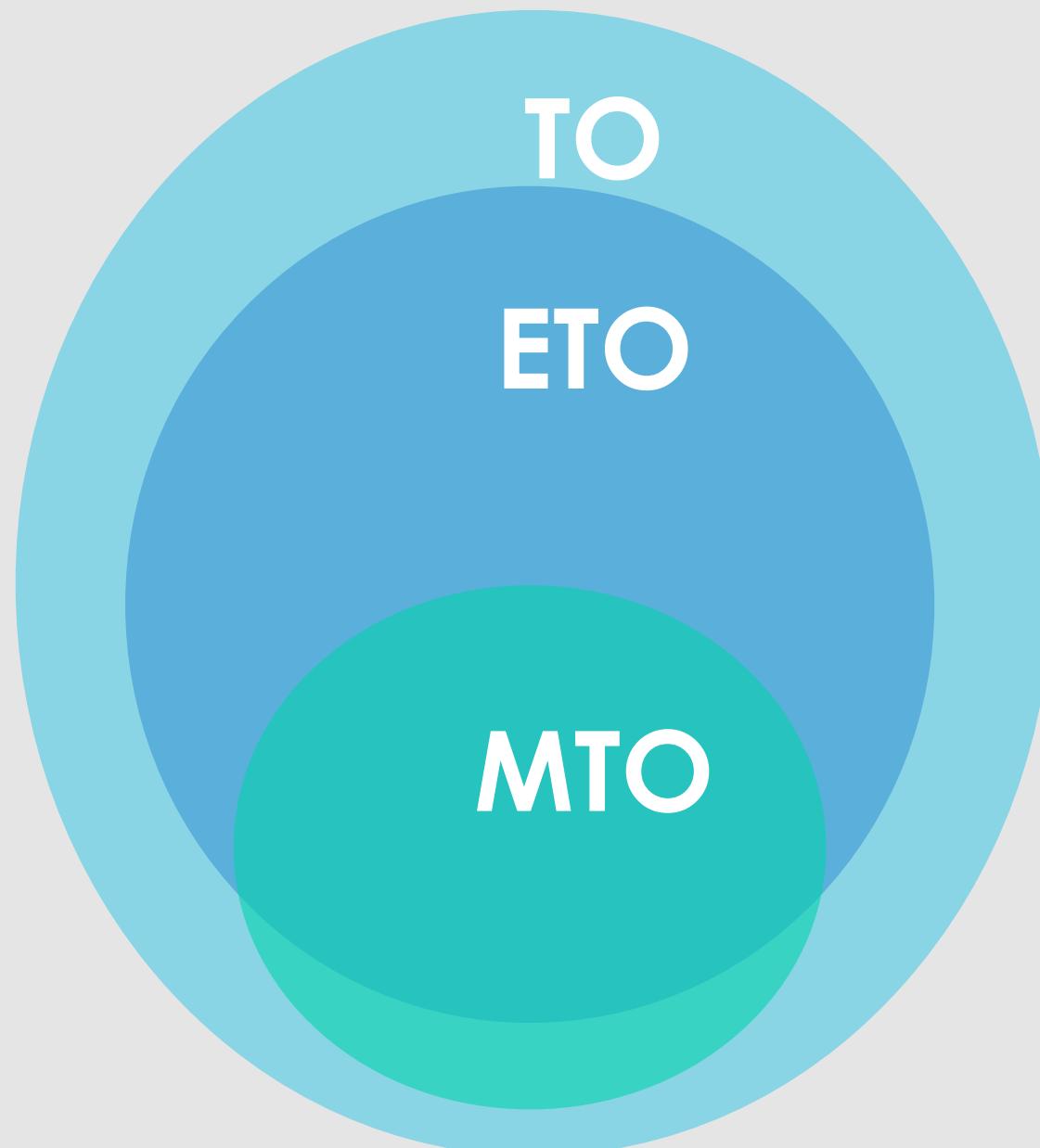
- potentially simpler to implement
- natural notion of time given by number of steps
(in contrast to thermal operations)
- innate Markovianity makes **ETO weaker than TO**
- state transition conditions inefficient to compute

- Mechanism: catalysts **store & release free energies** to enable such transitions
- Even **small catalysts** can provide substantial boost

→ What if the catalyst size is unbounded?

Catalysis in thermodynamics: non-Markovian boost

Energy coherent inputs



Lostaglio & Korzekwa,
PRA **106**, 012426 (2022)

Energy incoherent inputs



Under catalysis, for
energy-coherent
inputs



[Submitted on 1 Jun 2023]

Catalysis in Quantum Information Theory

[Patryk Lipka-Bartosik](#), [Henrik Wilming](#), [Nelly H. Y. Ng](#)

Catalysts open up new reaction pathways which can speed up chemical reactions while not consuming the catalyst. A similar phenomenon has been discovered in quantum information science, where physical transformations become possible by utilizing a (quantum) degree of freedom that remains unchanged throughout the process. In this review, we present a comprehensive overview of the concept of catalysis in quantum information science and discuss its applications in various physical contexts.

Comments: Review paper; Comments and suggestions welcome!

Subjects: [Quantum Physics \(quant-ph\)](#); Mathematical Physics (math-ph); Atomic Physics (physics.atom-ph)

Cite as: [arXiv:2306.00798 \[quant-ph\]](#)

(or [arXiv:2306.00798v1 \[quant-ph\]](#) for this version)

<https://doi.org/10.48550/arXiv.2306.00798> 

If you're interested, read more in our review!

Thank you