Distance between two lines in space

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Consider two lines in space ℓ_1 and ℓ_2 such that ℓ_1 passes through point P_1 and is parallel to vector $\vec{v}_1$ and ℓ_2 passes through P_2 and is parallel to $\vec{v}_2$. We want to compute the smallest distance D between the two lines.

If the two lines intersect, then it is clear that $D = 0$. If they do not intersect and are parallel, then D corresponds to the distance between point P_2 and line ℓ_1 and is given by

$$D = \frac{\|\overrightarrow{P_1P_2} \times \vec{v}_1\|}{\|\vec{v}_1\|}.$$

Assume the lines are not parallel and do not intersect (skew lines) and let $\vec{n} = \vec{v}_1 \times \vec{v}_2$ be a vector perpendicular to both lines. The norm of the projection of vector $\overrightarrow{P_1P_2}$ over $\vec{n}$ will give us D, i.e.,

$$D = \frac{|\overrightarrow{P_1P_2} \cdot \vec{n}|}{\|\vec{n}\|}.$$

Example

Consider the two lines $\ell_1 : x = 0,\ y = -t,\ z = t$ and $\ell_2 : x = 1 + 2s,\ y = s,\ z = -3s$. It is easy to see that the two lines are skew. Let $P_1 = (0, 0, 0)$, $\vec{v}_1 = (0, -1, 1)$, $P_2 = (1, 0, 0)$, and $\vec{v}_2 = (2, 1, -3)$. Then, $\overrightarrow{P_1P_2} = (1, 0, 0)$ and $\vec{n} = \vec{v}_1 \times \vec{v}_2 = (2, 2, 2)$. We then get

$$D = \frac{|\overrightarrow{P_1P_2} \cdot \vec{n}|}{\|\vec{n}\|} = \frac{1}{\sqrt{3}}.$$

Observe that the problem can also by solved with Calculus. Consider the problem of minimizing the Euclidean distance between two points on ℓ_1 and ℓ_2. Let $Q_1 = (x_1, y_1, z_1)$ and $Q_2 = (x_2, y_2, z_2)$ be arbitrary points on ℓ_1 and ℓ_2, and let

$$\begin{aligned} F(s,t) &= (x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2 \\ &= (1 + 2s)^2 + (s + t)^2 + (-3s - t)^2 \\ &= 14s^2 + 2t^2 + 8st + 4s + 1. \end{aligned}$$

Note that $F(s, t)$ corresponds to the square of the Euclidean distance between Q_1 and Q_2. Let's find the critical points of F.

$$\begin{aligned} F_s(s,t) &= 28s + 8t + 4 = 0 \\ F_t(s,t) &= 4t + 8s = 0 \end{aligned}$$

By solving the linear system, we find that the unique critical point is $(s_0, t_0) = (-1/3, 2/3)$. Since the Hessian matrix of F,

$$H = \begin{bmatrix} F_{ss} & F_{st} \\ F_{ts} & F_{tt} \end{bmatrix} = \begin{bmatrix} 28 & 8 \\ 8 & 4 \end{bmatrix},$$

is positive definite, the critical point corresponds to the absolute minimum of F over all $(s, t) \in \mathbf{R}^2$. The minimal distance between the two lines is then

$$D = \sqrt{F(s_0, t_0)} = \frac{1}{\sqrt{3}}.$$

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